Algebraic Quantum Codes: 
Linking Quantum Mechanics and Discrete Mathematics

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ABSTRACT
We present a general framework of quantum error-correcting codes (QECCs) as a subspace of a complex Hilbert space and the corresponding error models. Then we illustrate how QECCs can be constructed using techniques from algebraic coding theory. Additionally, we discuss secondary constructions for QECCs, leading to propagation rules for the parameters of QECCs.

KEYWORDS
quantum error-correction, stabilizer codes, algebraic coding theory, puncture code, quantum Construction X

1. Introduction

Quantum error-correcting codes (QECCs) are essential for the realization of information processing using the principles of quantum mechanics. About 25 years ago, Peter Shor presented the first scheme to reduce errors in a quantum computer [22]. This was followed by more general constructions by Calderbank and Shor [3], as well as Steane [23], now referred to as CSS codes.

The ground for the theory of so-called stabilizer codes was laid by the work of Gottesman [6] from the physics point of view, and by Calderbank, Rains, Shor, and Sloane [2] taking more the point of view of algebraic codes. In a nutshell, the theory of stabilizer codes allows the construction of quantum error-correcting codes using classical codes that are self-orthogonal with respect to a certain symplectic inner product.

A survey on constructions of stabilizer codes from various families of classical codes can be found in [13]. A collection of quantum codes with the best known parameters is available online [8]. Currently, the table covers only so-called qubit codes, but it is planned to extend them in the near future.

In this article, we focus on the basic principles of general quantum error-correcting codes and establish the link between quantum mechanics and algebraic coding theory. Additionally, we discuss secondary constructions of quantum codes, i.e., how one can derive new quantum codes from a given one.
2. General Quantum Error-Correcting Codes

2.1. Axiomatic Quantum Mechanics

In order to establish the framework for quantum error-correcting codes, we introduce the required concepts of quantum mechanics in an axiomatic way. A more detailed description can, for example, be found in the book by Nielsen and Chuang [20].

2.1.1. Pure quantum states

To every quantum mechanical system, we associate a complex Hilbert space $\mathcal{H}$. In our context, the dimension of the Hilbert space is finite, i.e., $\mathcal{H} = \mathbb{C}^d$ for some positive integer $d$. The basis states of the Hilbert space correspond to perfectly distinguishable states of the quantum mechanical system. For the smallest example $d = 2$, the two basis states may correspond to an atom or ion being in its ground or exited state, a quantum mechanical spin being aligned parallel or anti-parallel with an external magnetic field, or two orthogonal polarization directions of a photon. When we have complete knowledge of the state of the quantum mechanical system, the system is in a pure state that can be described by a unit-norm vector in the Hilbert space $\mathcal{H}$.

Looking again at the case $d = 2$, the state of a so-called qubit is given by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$  \hspace{1cm} (1)

with complex coefficients $\alpha, \beta \in \mathbb{C}$. Here we have used the ket-notation for the column vectors of an orthonormal basis of $\mathcal{H}$, i.e.,

$$|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (2)

When both coefficients in (1) are non-zero, $\alpha \neq 0 \neq \beta$, the state is referred to as a superposition of the basis states $|0\rangle$ and $|1\rangle$.

Correspondingly, the state of a so-called qudit, a $d$-dimensional quantum system, is given by

$$|\psi\rangle = \sum_{i=0}^{d-1} c_i |i\rangle, \quad \text{with} \quad \sum_{i=0}^{d-1} |c_i|^2 = 1,$$  \hspace{1cm} (3)

where the states $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ form an orthonormal basis of $\mathcal{H} = \mathbb{C}^d$. When the dimension $d = p^m = q$ is a prime power, we can label the basis states by elements of the finite field $\mathbb{F}_q$ instead of integers:

$$|\psi\rangle = \sum_{x \in \mathbb{F}_q} c_x |x\rangle, \quad \text{with} \quad \sum_{x \in \mathbb{F}_q} |c_x|^2 = 1.$$  \hspace{1cm} (4)

One may think of the state $|\psi\rangle$ as an element of the group algebra $\mathbb{C}[\mathbb{F}_q]$.

The state space of composite quantum systems is given by the tensor product of the state spaces of the individual quantum systems, i.e., for a bipartite quantum system, we have $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$. When a system is composed of $n$ qudits, each of dimension $q$, the dimension of the composite system is $q^n$, i.e., it grows exponentially in the number
of systems. A basis for the composite systems is given by the tensor product of the bases for the component systems. The basis vectors

$$|\mathbf{x}\rangle = |x_1x_2\ldots x_n\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$$  \hspace{1cm} (5)

can be labeled by strings \(\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n\) of length \(n\) over the finite field.

The state of a quantum register with \(n\) qudits is given by

$$|\psi\rangle = \sum_{\mathbf{x} \in \mathbb{F}_q^n} c_{\mathbf{x}} |\mathbf{x}\rangle,$$

with \(\sum_{\mathbf{x} \in \mathbb{F}_q^n} |c_{\mathbf{x}}|^2 = 1\).  \hspace{1cm} (6)

Again, one may think of the state \(|\psi\rangle\) being an element of the group algebra \(\mathbb{C}[\mathbb{F}_q^n]\).

2.1.2. Quantum measurements and quantum operations

From the physics point of view, a quantum measurement is associated with a Hermitian (self-adjoint) operator \(A\). Being self-adjoint, the eigenvalues \(\lambda_i\) of the observable \(A\) are real and correspond to some physical quantity, for example energy. As already mentioned, the two basis states of a qubit may correspond to two distinct energy states of an atom or ion. More abstractly, any Hermitian operator \(A\) on a Hilbert space of dimension \(d\) possesses a spectral decomposition of the form

$$A = \sum_{i=0}^{d-1} \lambda_i |\psi_i\rangle \langle \psi_i|,$$  \hspace{1cm} (7)

where the eigenvalues are \(\lambda_i\), and \(|\psi_i\rangle \langle \psi_i|\) denotes the orthogonal projection onto the eigenspace spanned by the corresponding eigenvector \(|\psi_i\rangle\). Here \(|\psi_i\rangle\) denotes a bra-vector that is the row vector formed by the complex conjugate coefficients of the column vector \(|\psi_i\rangle\). When the spectrum of \(A\) is degenerate, i.e., the eigenvalues are not all distinct, the spectral decomposition (7) can be rewritten as

$$A = \sum_{i=0}^{m-1} \lambda_i P_i = \sum_{i=0}^{m-1} \lambda_i \sum_{j=1}^{\mu_i} |\psi_i^{(j)}\rangle \langle \psi_i^{(j)}|,$$  \hspace{1cm} (8)

Here we assume that there are \(m\) distinct eigenvalues \(\lambda_i\) with multiplicities \(\mu_i\). The vectors \(|\psi_i^{(j)}\rangle\) form an orthonormal basis of the corresponding eigenspace.

According to the postulates of quantum mechanics, when performing a measurement of the observable \(A\) on a quantum system in state \(|\psi\rangle\), one will observe a randomly chosen eigenvalue \(\lambda_i\). The probability to observe \(\lambda = \lambda_i\) is given by

$$\text{Pr}_{|\psi\rangle}(\lambda = \lambda_i) = \|P_i|\psi\rangle\|^2 = \langle \psi | P_i |\psi\rangle.$$  \hspace{1cm} (9)

The post-measurement state is given by the re-normalized projection of the state onto the corresponding eigenspace, i.e.,

$$|\psi(\lambda = \lambda_i)\rangle = \frac{P_i|\psi\rangle}{\|P_i|\psi\|}.$$  \hspace{1cm} (10)
In the context of quantum information processing, we are usually not interested in the physical quantity corresponding to the eigenvalues $\lambda_i$, but only in the index $i$ of the eigenvalue and the associate eigenspace, given by the projection $P_i$. We may, for example, talk about measuring a quantum system in the standard basis when the eigenspaces are the one-dimensional spaces defined by the basis of the Hilbert space. For a qubit in the state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ (cf. (11)), we will obtain the outcomes “0” and “1” corresponding to the basis states $|0\rangle$ and $|1\rangle$, respectively, with probability $|\alpha|^2$ and $|\beta|^2$, respectively. The fact that the probabilities have to sum to unity explains why the vector describing a pure quantum state has to be normalized. In principle, one can also start with a decomposition of the whole Hilbert space into mutually orthogonal subspaces and define an observable that has the corresponding eigenspaces.

Transformations of the state of a closed quantum system are described by linear operators. As the operations have to preserve normalization, the admissible transformations are unitary transformations $U \in U(d)$. Note that the probability (9) of observing a particular eigenvalue $\lambda_i$ of an observable $A$ does not change when multiplying the state vector by complex number $e^{i\phi}$ of modulus 1. Therefore, states that differ by a global phase factor $e^{i\phi}$ can not be distinguished. Accordingly, the effective group of transformations of quantum states are all elements of the special unitary group $SU(d)$.

For composite quantum systems, operations that act on the individual subsystems are termed local operations. For an $n$-qudit system, the group of local unitary transformations is given by the $n$-fold tensor product $SU(d)^{\otimes n} = SU(d) \otimes \ldots \otimes SU(d)$.

Similarly, one can perform local measurements on the individual subsystems. Given an observable $A$ on a single qudit, the operator $A^{(1)} = A \otimes I_d \otimes \ldots \otimes I_d$ (where $I_d$ denotes a $d \times d$ identity matrix) is the Hermitian operator corresponding to the measurement of only the first qudit. As an example, consider the state of a two-qubit system given by

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle = \alpha |0\rangle \otimes (c'_{00}|0\rangle + c'_{01}|1\rangle) + \beta |1\rangle \otimes (c'_{10}|0\rangle + c'_{11}|1\rangle).$$

The coefficients in the second form are chosen such that the states of the second qubit are normalized, i.e., $|c'_{00}|^2 + |c'_{01}|^2 = |c'_{10}|^2 + |c'_{11}|^2 = 1$, which implies $|\alpha|^2 + |\beta|^2 = 1$. Hence the probability to obtain the result “0” or “1” when measuring the first qubit is given by $|\alpha|^2$ and $|\beta|^2$, respectively. The corresponding post-measurements states are

$$|\psi(y = 0)\rangle = |0\rangle \otimes (c'_0|0\rangle + c'_{01}|1\rangle)$$

and

$$|\psi(y = 1)\rangle = |1\rangle \otimes (c'_{10}|0\rangle + c'_{11}|1\rangle).$$

For the maximally entangled state

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle,$$

the probability for each outcome is $1/2$, regardless which of the qubits is measured in the standard basis. The possible post-measurement states are $|00\rangle$ and $|11\rangle$. Hence, measuring the second qubit after the first qubit has been measured, the result will always agree with the outcome of the first measurement. We will not discuss this phenomenon of entanglement in more detail here, but it has to be noted that measuring
one subsystem affects the state of the other subsystem.

2.1.3. Mixed quantum states and reduced quantum states

When performing a measurement on quantum systems, each of the post-measurement states occurs with some probability. When we ignore the measurement result, we have an ensemble of quantum states \( \{ |\psi_i \rangle \} \) with corresponding probabilities \( p_i \). Such an ensemble can be described by the density matrix

\[
\rho = \sum_i p_i |\psi_i \rangle \langle \psi_i |, \tag{16}
\]

where \( |\psi_i \rangle \langle \psi_i | \) is the orthogonal projection onto the state \( |\psi_i \rangle \). This concept holds more generally, e.g., when we prepare different quantum states \( |\psi_i \rangle \) with some probability \( p_i \) or when a source emits quantum states with certain probabilities. Different ensembles of quantum states may result in the same density matrix \( \rho \), but the density matrix contains all information that can be obtained from the ensemble. When we start with a uniform distribution over the states of a basis of the Hilbert space, the corresponding density matrix will be proportional to the identity matrix:

\[
\rho = \frac{1}{d} I. \tag{17}
\]

This maximally mixed state is invariant under unitary transformations, i.e., it is uniformly random with respect to any basis. Another special case is an ensemble with just a single pure state \( |\psi \rangle \). The corresponding density matrix equals the projection onto the one-dimensional space spanned by the state \( |\psi \rangle \), i.e.,

\[
\rho = |\psi \rangle \langle \psi |. \tag{18}
\]

As we have seen, measuring a subsystem of a composite quantum system has also an effect on the subsystems that we have not measured. If we ignore the measured subsystem and the measurement result, we arrive at an ensemble of quantum states for the remaining subsystems, described by the reduced density matrix. To make this more precise, consider a bipartite quantum system of dimension \( d_1 d_2 \). A mixed state of the composite system can be written as

\[
\rho_{12} = \sum_{i,i'=0}^{d_1-1} \sum_{j,j'=0}^{d_2-1} c_{i,j,i',j'} |i,j \rangle \langle i',j'|. \tag{19}
\]

Measuring the first system in the standard basis and discarding both the measurement result and the first system, the reduced quantum state of the second system is given by

\[
\rho_2 = \text{Tr}_1(\rho) = \sum_{j,j'=0}^{d_2-1} \sum_{i=0}^{d_1-1} c_{i,j,i,j'} |j \rangle \langle j'|. \tag{20}
\]

The matrix \( \rho_2 \) is referred to as the partial trace of \( \rho \) with respect to system 1, or reduced density matrix of system 2. It can be obtained by summing the \( d_1 \) non-overlapping
submatrices of size $d \times d$ on the diagonal of $\rho$, when we interpret the first index $i$ as labeling the block and the second index $j$ as labeling the position within the block. Note that from the state of the second subsystem only, one cannot deduce whether a measurement has been performed on the first system or not. The reduced density matrix (20) is independent of that measurement.

While the partial trace discards all information of the first system, one can define an inverse of the operation (20) in the following sense. For any mixed state $\rho$ on a $d$-dimensional Hilbert space $\mathcal{H}_1$, there exists a pure state $|\psi\rangle_{12}$ on the joint Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\dim \mathcal{H}_2 = d$ such that

$$\rho_1 = \text{Tr}_2(|\psi\rangle\langle\psi|).$$

(21)

This purification $|\psi\rangle_{12}$ of the mixed state $\rho_1$ is unique up to a unitary transformation on the Hilbert space $\mathcal{H}_2$.

2.2. Characterizing Quantum Codes

2.2.1. Quantum channels

The notion of error correction requires to fix an error model. In quantum information processing, the two main sources of errors are limited precision of the desired operations and unwanted interaction of the system with its environment. Imperfect operations can be modeled as operations that depend on the state of the environment.

As described above, using the purification of mixed states we can always choose the Hilbert space of the system of sufficiently large dimension so that the state of the system $|\psi\rangle_{\text{sys}}$ is pure. Similarly, we can assume that the environment is initially in the pure state $|\varepsilon\rangle_{\text{env}}$. Additionally, we make the assumption that system and environment have not yet interacted, and hence the initial state is described by the tensor product

$$|\Psi\rangle_{\text{in}} = |\psi\rangle_{\text{sys}} \otimes |\varepsilon\rangle_{\text{env}}.$$ (22)

The interaction of system and environment is given by a unitary transformation $U_{\text{sys/env}}$ on the joint Hilbert space, resulting in the state

$$|\Psi\rangle_{\text{out}} = U_{\text{sys/env}}(|\psi\rangle_{\text{sys}} \otimes |\varepsilon\rangle_{\text{env}}).$$ (23)

As we have no access to the environment, we take the partial trace over the environment and obtain the reduced density matrix of the system

$$\rho_{\text{out}} = \text{Tr}_{\text{env}} \left( U_{\text{sys/env}} (|\psi\rangle\langle\psi|_{\text{sys}} \otimes |\varepsilon\rangle\langle\varepsilon|_{\text{env}}) U_{\text{sys/env}}^{-1} \right).$$ (24)

Fixing the initial state $|\varepsilon\rangle_{\text{env}}$ of the environment and the unitary interaction $U_{\text{sys/env}}$, one can rewrite (24) in the form

$$\rho_{\text{out}} = \sum_i E_i |\psi\rangle\langle\psi|_{\text{sys}} E_i^\dagger,$$ (25)

where $E_i^\dagger$ denotes the adjoint (complex conjugate transpose) of the matrix $E_i$. Clearly, the matrices $E_i$ depend on $|\varepsilon\rangle_{\text{env}}$ and $U_{\text{sys/env}}$. 

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Definition 2.1 (quantum channel). A quantum channel $Q$ is a linear transformation on mixed quantum states of the form

$$\rho \mapsto Q(\rho) = \sum_i E_i \rho E_i^\dagger.$$  \hfill (26)

The operators $E_i$ are referred to as error operators or Kraus operators of the channel.

The analog of the binary symmetric channel—or more generally, the uniform symmetric channel or the additive white Gaussian noise (AWGN) channel—is the depolarizing channel given by

$$Q_{\text{depol}}(\rho) = (1 - p)\rho + p \frac{1}{d} I.$$  \hfill (27)

With probability $p$, the depolarizing channel replaces the input state by the maximally mixed state, and with probability $1 - p$, the input state is not changed. Like its classical counterpart, the depolarizing channel is in some sense related to the worst case assumption that the channel either transmits the input error-free or replaces it with a completely random state.

Another important example is the quantum erasure channel \[^9\] for which the dimension of the Hilbert spaces for the input and output differ. With probability $p$, the input is replaced by a state $|\bot\rangle$ that is orthogonal to all states of the input Hilbert space, and with probability $1 - p$ the state is transmitted without error:

$$Q_{\text{erasure}}(\rho) = (1 - p)\rho + p |\bot\rangle\langle\bot|.$$  \hfill (28)

The state $|\bot\rangle$ indicates that the quantum information has been erased.

The combination of two independent channels is modeled as follows:

Definition 2.2 (product channel). Given two quantum channels $Q_1$ and $Q_2$ with error operators $E_i$ and $F_j$, respectively, the action of the product channel $Q_1 \otimes Q_2$ on a mixed state $\rho_{12}$ of the composite system (cf. (19)) is given by

$$\rho_{12} \mapsto (Q_2 \otimes Q_2)(\rho_{12}) = \sum_{i,j} (E_i \otimes F_j) \rho_{12} (E_i^\dagger \otimes F_j^\dagger).$$  \hfill (29)

A particular example is the channel $Q^{\otimes n}$ which models $n$ independent uses of the memoryless channel $Q$. Loosely speaking, this requires that the Hilbert space of the environment is different for each use of the channel.

2.2.2. Necessary and sufficient conditions for quantum error correction

While a classical error-correcting code uses only a subset of all possible messages, a quantum error-correcting code (QECC) uses only states in a suitably chosen subspace of the full Hilbert space. Given a quantum channel $Q$ with error operators $E_i$, we have the following necessary and sufficient conditions for perfect error correction, also known as the Knill-Laflamme conditions \[^{15}\]:

Theorem 2.3. A subspace $C$ with orthonormal basis $\{|c_i\rangle\}$ of a Hilbert space $\mathcal{H}$ is a quantum error-correcting code for a quantum channel with error operators $\{E_k\}$ if and
only if the following conditions hold for all \(i, j, k, \ell\):

(i) \(\langle c_i | E_k^\dagger E_\ell | c_j \rangle = 0 \quad \text{for } i \neq j \) \hfill (30)

(ii) \(\langle c_i | E_k^\dagger E_\ell | c_i \rangle = \langle c_j | E_k^\dagger E_\ell | c_j \rangle = \alpha_{k\ell} \) \hfill (31)

In terms of the orthogonal projection \(P_C = \sum |c_i\rangle \langle c_i|\) onto the code \(C\), the conditions are equivalent to

(iii) \(P_C E_k^\dagger E_\ell P_C = \alpha_{k\ell} P_C \). \hfill (32)

The first condition (30) requires that orthogonal states in the code—which can be perfectly distinguished by a suitable measurement—remain orthogonal under the action of the channel. Loosely speaking, the second condition (31) requires that errors transform all states of the code in the same way.

First we note without proof that the Knill-Laflamme conditions are bi-linear in the error operators \(E_k\):

**Lemma 2.4.** If the conditions (30) and (31) hold for error operators \(E_k\) and \(E_\ell\), then they hold for any linear combination \(\mu E_k + \nu E_\ell\).

This implies the following.

**Corollary 2.5.** It is sufficient that the conditions (31) and (30) hold for a basis of the linear space of operators spanned by the error operators \(\{E_k\}\) of the quantum channel.

Since we are considering only finite dimensional Hilbert spaces here, it turns out that we are dealing with a finite number of errors that have to be corrected, while the error operators of the channel might be parameterized by continuous parameters. It can be shown that one can replace the original error operators by linear combinations such that the Knill-Laflamme conditions have a particularly nice form (see, e.g., [7]):

**Theorem 2.6.** Let \(C\) with orthonormal basis \(\{|c_i\}\) be a QECC for the channel \(Q\) with errors operators \(\{E_k\}\). Then there are error operators \(\{\widetilde{E}_k\}\) such that

\[ \langle c_i | \widetilde{E}_k^\dagger \widetilde{E}_\ell | c_j \rangle = \delta_{ij} \delta_{k\ell}, \] \hfill (33)

and the linear span of \(\{E_k|c_i\}\) equals the linear span of \(\{\widetilde{E}_k|c_i\}\).

Using (33), it turns out that the images

\[ \mathcal{V}_k = \widetilde{E}_k C = \text{span}\{\widetilde{E}_k|c_i\}\] \hfill (34)

of the code \(C\) under the error operators \(\widetilde{E}_k\) are mutually orthogonal. Moreover, for fixed \(k\) the states \(\{\widetilde{E}_k|c_i\}\) constitute an orthonormal basis of \(\mathcal{V}_k\), and all those spaces have the same dimension as the code \(C\). Hence there are isometries mapping the space \(\mathcal{V}_k\) to the code \(C\). There is a measurement whose eigenspaces are the spaces \(\mathcal{V}_k\). Performing that measurement yields information onto which of the spaces \(\mathcal{V}_k\) the output of the quantum channel has been projected. Applying the corresponding isometry, the state can be mapped back to the code \(C\), i.e., perfectly corrected.
2.2.3. Local errors and error bases

For classical error correcting-codes, the Hamming weight of an error equals the number of positions that are changed by the error. We can define a similar notion for quantum codes $C$ for the $n$-fold use of a quantum channel $Q$ acting on a Hilbert space $H = C^q$. In this case, the error operators of the product channel $Q^\otimes n$ are tensor products of the error operators of the channel $Q$ (see Definition 2.2). We distinguish whether an error acts trivially on a certain subsystem, i.e., is proportional to identity, or not. We extend this notion by linearity and obtain:

Definition 2.7 (error weight). The weight $\text{wgt}(E)$ of an error operator $E$ acting on the Hilbert space $(C^q)^\otimes n = C^q \otimes \ldots \otimes C^q$ equals the number of subsystems on which it acts non-trivially.

If $E = I_q^{n-t} \otimes E'$, where $I_q^{n-t}$ is an identity matrix of size $q^{n-t} \times q^{n-t}$, the weight of $E$ is at most $t$, the number of subsystems on which $E'$ acts. In particular, when $E = E_1 \otimes E_2 \otimes \ldots \otimes E_n$, the weight of $E$ equals the number of tensor factors that are not proportional to identity.

Definition 2.8. A quantum error-correcting code $C = ((n,K,d)_q$ is a subspace of dimension $K > 1$ of the $n$-fold tensor product $(C^q)^\otimes n$ such that the Knill-Laflamme conditions (20) and (21) hold for all pairs of errors $E_k, E_\ell$ with $\text{wgt}(E_k^\dagger E_\ell) < d$.

When the true minimum distance of the code $C$ equals $d$, then there is an operator $E = E_k^\dagger E_\ell$ of weight $d$ such that for two orthonormal states $|c_i \rangle$ and $|c_j \rangle$ of the code, the image of $|c_j \rangle$ under $E$ overlaps with $|c_i \rangle$, i.e., $\langle c_i | E | c_j \rangle \neq 0$, or the action of $E$ relative to $|c_i \rangle$ and $|c_j \rangle$ differs, i.e., $\langle c_i | E | c_i \rangle \neq \langle c_j | E | c_j \rangle$. The latter situation occurs, for example, when both states are eigenstates of $E$, but with different eigenvalues. On the other hand, there might be errors $E$ of weight $0 < \text{wgt}(E) < d$ that act trivially on the code, i.e., the restriction of $E$ to the code subspace is proportional to identity (see (32)). Clearly, such an “error” $E$, having no effect on the code, can not be detected.

When we only distinguish whether an error operator $E$ on a subsystem of dimension $q$ is proportional to identity or not, the possible errors are all linear operators. The dimension of the vector space of those operators is $q^2$. According to Corollary 2.5, it is sufficient to consider a basis of $q^2$ operators. Naturally, the basis is chosen to contain identity. Recall that when the dimension $q$ of the subsystem is a prime power $q = p^m$, we can label the basis states by elements of the finite field $\mathbb{F}_q$.

Lemma 2.9 (local error basis). On the Hilbert space $C^q$ with orthonormal basis $\{|x\rangle : x \in \mathbb{F}_q\}$ we define the operators

$$X^a = \sum_{x \in \mathbb{F}_q} |x + a\rangle \langle x|,$$

$$Z^b = \sum_{y \in \mathbb{F}_q} \omega_p^{\text{tr}(by)} |y\rangle \langle y|,$$

where $\omega_p = \exp(2\pi i/p)$ is a complex primitive $p$-th root of unity and $\text{tr}(y)$ denotes the absolute trace from $\mathbb{F}_q$ to the prime field $\mathbb{F}_p$. Then the $q^2$ operators $\{X^a Z^b : a, b \in \mathbb{F}_q\}$ form a basis of all linear operators on $C^q$, containing the operator $I = X^0 Z^0$.

\footnote{For simplicity, we exclude the case $K = 1$ here.}
Proof. In order to show that the operators form a basis, one checks that they are orthogonal with respect to $\text{Tr}(A^\dagger B)$.

The operators $X^a$ correspond to permutations of the basis states according to the addition of $a$. The operators $Z^b$ are their diagonalized version, and the operators $X^a$ and $Z^b$ are related by the Fourier transformation for the additive group of $\mathbb{F}_q$. Taking tensor products of the elements of the local error basis with no more than $t$ operators different from identity, one obtains a basis of all errors up to weight $t$. Generalizations of error bases with nice properties can be found in [14].

In summary, Theorem 2.3 characterizes quantum error-correcting codes for arbitrary channels. Considering $n$ independent uses of the same channel, one can introduce the notion of the weight of an error and arrives at the more special case of a quantum error-correcting code $\mathcal{C} = ((n, K, d))_q$ being a linear subspace of $(\mathbb{C}^q)^\otimes n$ with $\dim \mathcal{C} = K$. Such a code is able to

- correct all errors $E$ of weight $\text{wgt}(E) < d/2$, or
- correct up to $t < d$ erasures, or
- detect all errors $E$ of weight $\text{wgt}(E) < d$ that act non-trivially on the code.

Operationally, the parameter $d$ plays the same role as the minimum distance of classical codes. A discussion of the assumption of local errors can be found, e.g., in [16].

3. Stabilizer Codes

The general theory of quantum error-correcting codes described in the previous section does not provide information on how to construct good codes. In this section we will establish the link between quantum and classical codes.

3.1. The Generalized n-qudit Pauli Group

We return to the local error basis of Lemma 2.9 and notice that the operators $X^a$ and $Z^b$ generate a matrix group $\mathcal{P} = \langle X^a Z^b : a, b \in \mathbb{F}_q \rangle$ which is sometimes referred to as the generalized Pauli group. The order of the group is $|\mathcal{P}| = pq^2$, and the center consists of the multiples of identity $Z(\mathcal{P}) = \{\exp(k2\pi i/p)I : k = 0, \ldots, p - 1\}$. Conjugating an element $Z^b$ by an element $X^a$ results in a phase factor:

$$X^{-a}Z^bX^a = \left( \sum_{x \in \mathbb{F}_q} |z - a\rangle \langle x| \right) \left( \sum_{y \in \mathbb{F}_q} \omega_{p}^{\text{tr}(by)} |y\rangle \langle y| \right) \left( \sum_{x \in \mathbb{F}_q} |x + a\rangle \langle x| \right) \quad (37)$$

$$= \sum_{x,y,z \in \mathbb{F}_q} \omega_{p}^{\text{tr}(by)} |z - a\rangle \langle z| |y\rangle \langle y| |x + a\rangle \langle x| \quad (38)$$

$$= \sum_{x \in \mathbb{F}_q} \omega_{p}^{\text{tr}(b(x+a))} |x\rangle \langle x| = \omega_{p}^{\text{tr}(ab)} Z^b \quad (39)$$

Here we have used the linearity of the trace and the fact that the basis states are orthonormal, i.e., $\langle x|y\rangle = \delta_{x,y}$. Eq. (39) implies the relation

$$Z^bX^a = X^a \left( X^{-a}Z^bX^a \right) = \omega_{p}^{\text{tr}(ab)} X^a Z^b, \quad (40)$$
and further

\[ X^a Z^b X'^a Z'^b = \omega_p^{\text{tr}(a'b')} X^a X'^a Z^b Z'^b = \omega_p^{\text{tr}(a'b')} X^a X^b Z^b Z'^b. \] (41)

Forming \( n \)-fold tensor products of the local error operators, we get the generalized \( n \)-qudit Pauli group \( \mathcal{P}_n \) of order \( |\mathcal{P}_n| = pq^{2n} \). The elements of the group can be written as

\[ \mathcal{P}_n = \{ \omega_p^\gamma X^{a_1} Z^{b_1} \otimes \cdots \otimes X^{a_n} Z^{b_n} : a_i, b_i \in \mathbb{F}_q, \gamma = 0, \ldots, p - 1 \}. \] (43)

Modulo its center, the group \( \mathcal{P}_n \) is isomorphic to the vector space \( \mathbb{F}_q^{2n} \) considered as an additive group via the homomorphism

\[ \phi: \omega_p^\gamma X^{a_1} Z^{b_1} \otimes \cdots \otimes X^{a_n} Z^{b_n} \mapsto (a_1, \ldots, a_n|b_1, \ldots, b_n) = (a|b). \] (44)

What is more, using the commutator relation (42) for single-qudit operators, we obtain the commutator relation

\[ X^a Z^b X'^a Z'^b = \omega_p^{\text{tr}(a'b'-ab')} X'^a Z'^b X^a Z^b. \] (45)

Here we have used the shorthand \( X^a = X^{a_1} \otimes \cdots \otimes X^{a_n} \) and \( Z^b = Z^{b_1} \otimes \cdots \otimes Z^{b_n} \). The exponent of the phase factor in (43) is determined by the \( \mathbb{F}_p \)-valued symplectic form

\[ (a|b) \ast (a'|b') = \text{tr}(a' \cdot b - a \cdot b') = \text{tr} \left( \sum_{i=1}^n a'_i b_i - a_i b'_i \right). \] (46)

on \( \mathbb{F}_q^{2n} = \mathbb{F}_q^n \times \mathbb{F}_q^n \).

The homomorphism \( \phi \) in (44) is compatible with the weight of an operator in Definition 2.7:

\[ \text{wgt} \left( X^a Z^b \right) = |\{ i : i = 1, \ldots, n | (a_i, b_i) \neq (0, 0) \}|. \] (47)

The right-hand side of (47) equals the Hamming weight of the vector \( (a|b) = \phi(X^a Z^b) \) when considered as a vector of length \( n \) over \( \mathbb{F}_q \times \mathbb{F}_q \), i.e., \( ((a_1|b_1), \ldots, (a_n|b_n)) \).

### 3.2. Stabilizer Codes and Classical Codes

With this preparation, we are ready to define the class of stabilizer codes.

**Definition 3.1** (stabilizer code). A stabilizer quantum error-correcting code \( \mathcal{C} = ((n, K, d))_q \) is the common eigenspace (with eigenvalue +1) of an Abelian subgroup \( \mathcal{S} \) of the \( n \)-qudit Pauli group \( \mathcal{P}_n \) that does not contain a non-trivial multiple of identity. The dimension of the code is \( K = \dim \mathcal{C} = q^n/|\mathcal{S}| \).

Note that a stabilizer code is a complex vector space. In the literature, the term “stabilizer code” is quite often used for the corresponding classical additive code obtained via the homomorphism \( \phi \).
Definition 3.2 (classical stabilizer code). The image $C = \phi(S)$ of the stabilizer subgroup $S$ is an additive ($\mathbb{F}_p$-linear) code of length $n$ over the alphabet $\mathbb{F}_q \times \mathbb{F}_q$ respectively an additive code of length $2n$ over $\mathbb{F}_q$.

As the stabilizer subgroup $S$ is Abelian, we have

Lemma 3.3. The classical stabilizer code $C = \phi(S)$ is self-orthogonal with respect to the symplectic form $\omega_p$.

There is a one-to-one correspondence between the Abelian subgroups of $\mathcal{P}_n$ having trivial intersection with the center $Z(\mathcal{P}_n)$ and the additive codes that are self-orthogonal with respect to the symplectic form $\omega_p$. Special cases include $\mathbb{F}_q$-linear codes and $\mathbb{F}_{q^2}$-linear codes when choosing a suitable basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$ and identifying $\mathbb{F}_{q^2}$ with $\mathbb{F}_q^2$. For $\mathbb{F}_q$-linear codes, the codes are self-orthogonal with respect to the symplectic form $(a|b) \ast (a'|b') = a' \cdot b - a \cdot b'$, omitting the trace. For $\mathbb{F}_{q^2}$-linear codes, the codes are self-orthogonal with respect to the Hermitian form $x \ast y = \sum_{i=1}^{n} x_i y_i^q$ for vectors $x, y \in \mathbb{F}_{q^2}^n$. More details can be found in [1, 13].

The stabilizer subgroup $S$ does not only define the stabilizer code $C$ itself, it also gives rise to an orthogonal decomposition of the whole Hilbert space $(\mathbb{C}^q)^\otimes n$ into spaces of equal dimension $K$ labeled by the characters of the group $S$. Fixing a set of $\kappa$ independent generators $s_i$ of the group $S$ of size $p^\kappa$, the characters correspond to the tuple of eigenvalues $(\lambda_1, \ldots, \lambda_\kappa)$ of the generators. As mentioned above, one can find a measurement with the corresponding eigenspaces. Although the operators $s_i$ are in general unitary and not self-adjoint operators, one finds the terminology “measuring the stabilizer (generators) $s_i$” in the literature for the corresponding measurement.

In the following, we sketch how the correction of quantum errors is related to the classical stabilizer code. We consider an error operator $E = X^{a'} Z^{b'} \in \mathcal{P}_n$. Furthermore, we fix a generator $s_i = X^a Z^b$ of the stabilizer group $S$. For any pure state $|c\rangle \in C$ of the code, we compute

$$E|c\rangle = Es_i|c\rangle = X^{a'} Z^{b'} X^a Z^b |c\rangle = \omega_p^{(a'|b')\ast(a|b)} X^{a'} Z^{b'} X^a Z^b |c\rangle = \omega_p^{(a'|b')\ast(a|b)} s_i E|c\rangle.$$  

(48)

Hence the states $E|c\rangle$ are eigenvectors of the stabilizer generator $s_i$ with eigenvalue $\omega_p^{(a'|b')\ast(a|b)}$. Therefore, an error $E$ for which $\phi(E) \ast \phi(s_i) \neq 0$ for some $s_i \in S$ can be detected, as the spaces $C$ and $EC$ correspond to different eigenvalues of $s_i$ and are hence orthogonal to each other. On the other hand, undetectable errors that act non-trivially on the code are exactly those with $\phi(E) \ast \phi(s) = 0$ for all $s \in S$, but $E \notin S$. The errors $E$ with $\phi(E) \ast \phi(s) = 0$ form a group, the centralizer of $S$ in $\mathcal{P}_n$.

In this particular case, the centralizer and the normalizer agree, and usually the term normalizer group $N$ is used.

The homomorphism $\phi$ captures this situation as well:

Definition 3.4 (classical normalizer code). The image of the normalizer group $N$ associated with a stabilizer code with stabilizer group $S$ under the homomorphism $\phi$ is an additive code $C^\ast$ that is the dual code of the classical stabilizer code $C = \phi(S)$ with respect to the symplectic form $\omega_p$, i.e.,

$$C^\ast = \{(a'|b'): (a'|b') \ast (a|b) = 0 \text{ for all } (a|b) \in C\}.$$  

(49)

$^2$For simplicity, we assume that the phase factor $\omega_p^0 = 1$. 

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In summary, we can express the minimum distance of a stabilizer code $\mathcal{C}$ in terms of the associated classical codes:

**Theorem 3.5.** The minimum distance $d$ of a stabilizer code $\mathcal{C} = ([n, K, d])_q$ with associated classical stabilizer code $\phi(\mathcal{S})$ is given by

$$d = \min\{\text{wgt}(a|b): (a|b) \in C^* \setminus C\} \geq \min\{\text{wgt}(a|b): (a|b) \in C^*\} = d_{\min}(C^*),$$

(50)

where we use the Hamming weight for codes over the alphabet $\mathbb{F}_q \times \mathbb{F}_q$ (see (47)). A code with $d = d_{\min}(C^*)$ is called pure, otherwise impure.

As outlined above, error correction is based on performing a measurement that yields the tuple of eigenvalues $(\lambda_1, \ldots, \lambda_\kappa)$ of a set of $\kappa$ independent generators $s_i$ of the stabilizer group $\mathcal{S}$. For an error $E$, the eigenvalues are

$$\lambda_i = \omega_p^{\phi(E)*\phi(s_i)}.$$

(51)

The vectors $\phi(s_i)$ are generators of the additive code $C$ and hence give rise to check equations for the dual code $C^*$. The task is to determine the error $E$ using the classical syndrome vector

$$(\phi(E) * \phi(s_1), \ldots, \phi(E) * \phi(s_\kappa)) \in \mathbb{F}_p^\kappa,$$

(52)

which can be solved by a classical decoding algorithm for the code $C^*$. Note that given $\phi(E)$, the error operator $E$ is only determined up to a phase factor $\omega_p$, but this is irrelevant.

Most stabilizer codes in the literature are constructed from codes that are not just additive, but linear over $\mathbb{F}_q$ or $\mathbb{F}_{q^2}$. In this case, the size of the stabilizer group is an integral power of $q$. Then the dimension $K$ of the stabilizer code $\mathcal{C} = ([n, K, d])_q$ is an integral power $K = q^K$ of $q$ as well, and the notation $\mathcal{C} = [n, k, d]_q$ is used. The parameter $k$ is referred to as the number of encoded or logical qudits of the code.

### 4. Secondary Constructions for Quantum Codes

#### 4.1. Trivial Constructions

First, we consider some trivial propagation rules for the parameters of quantum error-correcting codes. For qubit codes, such rules have been stated in [2, Theorem 6]. The first set of rules apply to all QECCs.

**Theorem 4.1.** Assume that a quantum code $\mathcal{C} = ([n, K, d])_q$ with $d > 1$ exists. Then the following QECCs exist as well:

1. $\mathcal{C'} = ([n, K', d])_q$ for all $1 < K' \leq K$ (subcode)
2. $\mathcal{C'} = ([n', K, d])_q$ for all $n' \geq n$ (lengthening)
3. $\mathcal{C'} = ([n - 1, K, d - 1])_q$ (puncturing)

**Proof.**

1. Any subspace $\mathcal{C'}$ of dimension $K'$ of the code $\mathcal{C}$ has clearly at least the same minimum distance.
(2) In order to increase the length of a code, one simply takes the tensor product of any state \( |c\rangle \in \mathcal{C} \) of the code with a fixed quantum state \( |\psi_0\rangle \) with \( n' - n \) qudits. When decoding the code \( \mathcal{C}' \), one can just discard the last \( n' - n \) qudits and decode the first \( n \) qudits with respect to the original code \( \mathcal{C} \). Then one takes the tensor product with the known fixed state \( |\psi_0\rangle \). The resulting code \( \mathcal{C}' \) will be impure, as there are error operators of weight less than \( d \) that act only on the last \( n' - n \) qudits and that stabilize the code.

(3) Fix a pure state \( |\psi_0\rangle \) of a single qudit and consider the projection operator \( P_0 = I_{q^{n-1}} \otimes |\psi_0\rangle\langle \psi_0| \). As \( \text{wgt}(P_0) = 1 < d \), eq. (31) implies that \( \langle c_i|P_0|c_i\rangle = \alpha \) for all basis states \( |c_i\rangle \) of the code \( \mathcal{C} \). We can choose the state \( |\psi_0\rangle \) such that \( \alpha \neq 0 \). After renormalization, the images of the basis states \( |c_i\rangle \) under the projection \( P_0 \) have the form \( |c'_i\rangle \otimes |\psi_0\rangle \). The new code \( \mathcal{C}' \) is spanned by the states \( |c'_i\rangle \) on \( n - 1 \) qudits.

Assume that an error \( E' \) acts on the punctured code \( \mathcal{C}' \). Considering the tensor product of the received state with the fixed state \( |\psi_0\rangle \) is equivalent to the error \( E = E' \otimes |\psi_0\rangle\langle \psi_0| \) acting on the original code \( \mathcal{C} \). For the original code, the Knill-Laflamme conditions hold for all errors with \( \text{wgt}(E) < d \). As \( \text{wgt}(E) = \text{wgt}(E') + 1 \), for the punctured code \( \mathcal{C}' \) the Knill-Laflamme conditions will hold for all errors with \( \text{wgt}(E') < d - 1 \), i.e., the minimum distance of the punctured code is at least \( d - 1 \).

We can also trivially combine two codes.

**Theorem 4.2.** Assume that quantum error-correcting codes \( \mathcal{C}_1 = ((n_1, K_1, d_1))_q \) and \( \mathcal{C}_2 = ((n_2, K_2, d_2))_q \) exist. Then a code \( \mathcal{C}' = ((n_1 + n_2, K_1 K_2, \min\{d_1, d_2\}))_q \) exist as well.

**Proof.** The code \( \mathcal{C}' \) is the tensor product of the codes \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \). The total number of qudits is \( n_2 + n_2 \), and the dimension of the tensor product is \( K_1 K_2 \). Decoding the two blocks independently implies that the minimum distance of the tensor product is at least the minimum of \( d_1 \) and \( d_2 \).

Using the isomorphism between \( \mathbb{C}^q \otimes^m \) and \( (\mathbb{C}^q)^{\otimes m} \), one can obtain codes for subsystems of dimension \( q \) from codes for subsystems of dimension \( q^m \):

**Theorem 4.3.** Assume that a quantum error-correcting code \( \mathcal{C} = ((n, K, d))_q \) exists. Then a code \( \mathcal{C}' = ((mn, K, d') \geq d))_q \) exist as well.

**Proof.** The code \( \mathcal{C}' \) is the same complex vectors space as the original code \( \mathcal{C} \), but considered as a subspace of \( (\mathbb{C}^q)^{\otimes mn} \). The weight of an error does not decrease under this expansion, and hence the minimum distance \( d' \) of the expanded code is not smaller than the original minimum distance \( d \).

### 4.2. Shortening the Classical Stabilizer Code

The list of trivial propagation rules does not include the analog of shortening of classical block codes which reads \( C = (n, M, d)_q \Rightarrow C' = (n - 1, M/q, d)_q \). The shortened code \( \mathcal{C}' \) is obtained by first taking all codewords of \( C \) with a fixed symbol at the last position (usually the symbol \( 0 \)) and then discarding that symbol. When we apply this operation to the classical stabilizer code, we obtain the following.
Theorem 4.4. Assume that a pure stabilizer code \( C = ([n,K,d])_q \) with \( d > 1 \) and \( q = p^m \) exists. Then a stabilizer code \( C' = ([n-1,K,d-1])_q \) exists as well.

**Proof.** Let \( C = \phi(S) = ([n,p^k,\delta])_q \) be the classical stabilizer code associated with \( C \). We have \( K = q^n/p^k \). Shortening the code \( C \) yields an additive code \( C' = ([n-1,p^k,\delta])_q \) which remains self-orthogonal with respect to the symplectic form. The size of the shortened code depends on the number of different symbols at the shortened position. As the minimum distance of the symplectic dual code \( C^* \) equals \( d > 1 \), all \( q^2 \) pairs \( (a,b) \in \mathbb{F}_q \times \mathbb{F}_q \) occur at every position \( C \). Assuming the contrary, the code \( C_1 \) of length 1 obtained by puncturing the code \( C \) at all but one position has size \( p^\mu \) with \( \mu < 2m \). Then the symplectic dual \( C_1^* \) of size \( 2m - \mu \) contains a non-zero codeword. The code \( C_1^* \) equals the code obtained by shortening \( C^* \) at all but the corresponding position. Hence, the minimum distance of \( C^* \) would be 1, a contradiction. Therefore, the size of the code shortened code \( C' \) is \( p^{\kappa - 2m} \). The minimum distance of \( (C')^* \) is at least \( d - 1 \). For the dimension \( K' \) of the new stabilizer code \( C' \) we compute

\[
K' = \frac{q^{n-1}}{p^{k-2m}} = \frac{p^{2m}}{q} = qK.
\]

\( \square \)

While we have stated the result only for stabilizer codes, this propagation rule can be applied to all quantum codes [12]. For \( \mathbb{F}_q \)-linear codes, the propagation rule reads \([n,k,d]_q \Rightarrow [n-1,k+1,d-1]_q \). So the quantity \( n + k \) is preserved. The minimum distance is decreased by one while the number of logical qudits is increased by one when reducing the length by one.

### 4.3. Shortening Stabilizer Codes

In certain cases, a propagation rule similar to that of shortening classical codes is possible. For stabilizer codes, one would shorten the classical normalizer code, reducing the length while preserving the minimum distance. Shortening of the classical normalizer code corresponds to puncturing of the classical stabilizer code. In general, the punctured code is no longer self-orthogonal. The idea is to multiply the coordinates of the punctured code by suitably chosen scalars to make the code self-orthogonal. The construction was stated for prime dimensions \( q = p \) in [21].

**Definition 4.5** (Rains puncture code). Let \( C = (n,p^\kappa)_q \) be an additive code over \( (\mathbb{F}_q \times \mathbb{F}_q)^n \). The puncture code \( P(C) \) is the Euclidean dual of the additive code over \( \mathbb{F}_p \) obtained by evaluating the symplectic form \( \langle \rangle \) on each coordinate of \( C \) for all pairs of codewords, i.e.,

\[
P(C) = \langle (\text{tr}(a_1b'_1 - a'_1b_1), \ldots, \text{tr}(a_nb'_n - a'_n b_n)) : (a|b), (a'|b') \in C \rangle ^\perp.
\]

**Theorem 4.6.** Assume that the puncture code \( P(C) \) of an additive code \( C = (n,p^\kappa)_q \) contains a codeword of weight \( r \). Then one can derive an additive self-orthogonal code \( C' = (r,p^{\kappa'})_q \) with \( \kappa' \leq \kappa \) from \( C \). The minimum distance of \( (C')^* \) is not smaller than the minimum distance of \( C^* \).

**Proof.** Let \( c \) be a codeword of \( P(C) \leq \mathbb{F}_p^m \) of weight \( r \). Without loss of generality assume that the first \( r \) coordinates of \( c \) are non-zero. Consider the code \( \overline{C} \) obtained
by multiplying the first part of each pair of coordinate symbols \((a_i|b_i)\) by \(c_i\), i.e.,
\[
\bar{C} = \{(c_1a_1, \ldots, c_na_n|b_1, \ldots, b_n) : (a|b) \in C\}.
\] (55)

The code \(C'\) is obtained by puncturing \(\bar{C}\) at the last \(n-r\) positions. Using the definition of the puncture code and the \(\mathbb{F}_p\)-linearity of the trace, it follows that both \(\bar{C}\) and \(C'\) are self-orthogonal. The size of \(C'\) is not larger than the size of the original code \(C\). Puncturing the code \(C\) corresponds to shortening the symplectic dual code, which implies the lower bound on the minimum distance. \(\square\)

**Corollary 4.7.** Assume that the puncture code of a stabilizer code \(C = (n,K,d)_q\) contains a word of weight \(r = n-s\). Then a stabilizer code \(C' = ((n-s,q^{-s}K,d)_q\) exists as well.

**Proof.** Using Theorem 4.6, we only have to compute the dimension of the stabilizer code derived from \(C'\). As \(|C'| \leq p^n = q^n/K\), we calculate
\[
K' = \frac{q^{n-s}}{|C'|} \geq Kq^{-s}.
\] (56)

The remaining problem in algebraic coding theory is to determine which weights occur in the puncture code.

**Remark 1.** As it can be seen from the proof of Theorem 4.6, the definition of the puncture code can be generalized as follows:
\[
\tilde{P}(C) = \{(c_1, \ldots, c_n) \in \mathbb{F}_q^n : \sum_{i=1}^n \text{tr} \left(c_i(a_i b'_i - a'_i b_i)\right) = 0 \text{ for all } (a|b), (a'|b') \in C\}.
\] (57)

In comparison, the original definition (54) can be expressed as
\[
P(C) = \{(c_1, \ldots, c_n) \in \mathbb{F}_p^n : \sum_{i=1}^n c_i \text{tr} \left((a_i b'_i - a'_i b_i)\right) = 0 \text{ for all } (a|b), (a'|b') \in C\}.
\] (58)

This generalization has yet to be investigated.

### 4.4. Extending Additive Codes

Another possibility to obtain self-orthogonal additive codes is by extension, i.e., increasing the length of a code \(C = (n,p^k)_q\) by adding new symbols such that the resulting code \(C' = (n',p^k)_q\) becomes self-orthogonal. This approach has been used in [24] to obtain stabilizer codes with improved parameters. The main problem is, however, to control the minimum distance of the symplectic dual \((C')^*\) of the extended code and hence the minimum distance of the resulting stabilizer code.

We have found a new code \(C = [96,50,10]_2\). The classical normalizer code is an \(\mathbb{F}_4\)-linear code \(C^* = [96,73,10]_4\) containing its Hermitian dual. The code \(C^*\) is obtained
by a three-step variant of Construction X (see, e.g., [14]) using a cyclic code $C_1 = [93, 73, 8]_4$ with generator polynomial

$$g_0(x) = (x^5 + x^2 + \alpha)(x^5 + x^3 + x^2 + x + 1)$$

$$(x^5 + ax^4 + x^2 + ax + a^2)(x^5 + ax^4 + a^2 x^3 + x^2 + a^2),$$

(59)

where $\alpha$ is a generator of $\mathbb{F}_4$, and subcodes with generator polynomial $g(x)(x-1)^a(x-\alpha)^b(x-\alpha^3)^c$, $a, b, c \in \{0, 1\}$. It should be noted that shortening the stabilizer of the new code $C = [96, 50, 10]_2$ three times yields a code $C' = [93, 53, 8]_2$ whose classical normalizer code is the code $C_1$.

4.5. Quantum Construction X

Lisonek and Singh [13] proposed a construction of Hermitian self-orthogonal $\mathbb{F}_{q^2}$-linear codes applying Construction X to an arbitrary linear code of rate at most $1/2$ over $\mathbb{F}_{q^2}$ using a trivial code of distance one as auxiliary code. While Lisonek and Singh focused on the case $q = 2$, the construction has natural extensions to linear codes over $\mathbb{F}_{q^2}$ [3] as well as quadratic extension fields $\mathbb{F}_{q^2}$ in general [5]. In [17] the construction was extended to additive codes over $\mathbb{F}_4$. We present the main result in the version of [5, Theorem 2]:

**Theorem 4.8.** For a linear code $C = [n, k]_{q^2}$, let $e := k - \dim_{\mathbb{F}_{q^2}}(C \cap C^*)$. Then there exists a stabilizer code $C = [n+e, n-2k+e, d]_q$ with $d \geq \min\{d(C^*), d(C+C^*)+1\}$.

**Proof.** We sketch the main idea. The code $C \cap C^*$ is a linear subcode of $C$ of co-dimension $e$ that is Hermitian self-orthogonal. One can show that it is possible to find a basis of the complement $\bar{C}$ of $C \cap C^*$ in $C$ such that Construction X applied to $C$ and $C \cap C^*$ using a trivial auxiliary code $C_{\text{aux}} = [e, e, 1]_{q^2}$ yields a linear Hermitian self-orthogonal code $C' = [n+e, k]_{q^2}$. The co-dimension of $C^*$ in $(C \cap C^*)^* = C+C^*$ is $e$ as well. It turns out that the Hermitian dual code $(C')^*$ is spanned by the codewords of $\bar{C}$ with $e$ zeros appended, as well as the vectors corresponding to the juxtaposition of $\bar{C}$ and $C_{\text{aux}}$. This implies the lower bound on the minimum distance. ☐

Note that the linear code $C \cap C^*$ of dimension $k-e$ in the proof yields a stabilizer code $C_0 = [n, n-2k+2e, d_0]_q$ where $d_0 \geq d(C+C^*)$. For $e = 1$, the parameters of the code $C_0$ are exactly those which one obtains by shortening the classical stabilizer code of the code $C$ obtained by quantum Construction X. Hence, one can consider quantum Construction X in some sense as the inverse of shortening the classical stabilizer code.

Starting with a quantum code $C = [n, k, d]_q$, one considers the classical normalizer code $C^*$. Then one seeks a subcode $C^*_1$ of $C^*$ of small co-dimension, but with a minimum distance that is at least $d+1$. In general, the code $C^*_1$ will not contain its symplectic dual $C_1$. Applying quantum Construction X to $C_1$, one will obtain a quantum code $C' = [n+e, k-e, d+1]_q$ for some $e$.

5. Conclusions

Starting with the characterization of general quantum error-correcting codes, we have discussed how algebraic coding theory fits into that picture via the theory of stabilizer
codes. Classical codes that are self-orthogonal with respect to a symplectic form correspond to Abelian subgroups of the generalized Pauli group, defining quantum codes via their joint eigenspaces. At the same time, using a local error model for quantum codes naturally corresponds to the Hamming metric for classical codes. By now, there are numerous publications from the algebraic coding theory community on the construction self-orthogonal classical codes that give rise to good quantum codes. There have also been attempts to adopt specific concepts from algebraic coding theory—like codes for different metrics than the Hamming metric—to quantum codes, missing to relate the concepts to quantum mechanics. On the other hand, we have only considered the depolarizing channel and the quantum erasure channel here, together with their $n$-fold uses. For other quantum channels, one has to investigate what their correspondence in algebraic coding theory is and how one can find good codes for that situation.

Complementing the direct construction of classical self-orthogonal codes leading to stabilizer codes with good parameters, we presented the notion of the puncture code in Section IV.3 and quantum Construction X in Section IV.5. The puncture code has been used to show the existence of many quantum MDS codes [10]. In general it is an open question how one can determine the weights in the puncture code for certain classes of codes. The generalization mentioned in Remark 4 is worth to be further explored.

For quantum Construction X, the main question is how to find nested classical codes that result in quantum codes with good parameters. Another promising concept that we have not discussed here is generalized concatenation for quantum codes [11]. Concatenation can also be a method to design good codes for specific quantum channels.

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