INVARIA NCE RESULTS FOR PAIRINGS WITH ALGEBRAIC $K$-THEORY

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Abstract

To each algebra over the complex numbers we associate a sequence of abelian groups in a contravariant functorial way. In degree $(m - 1)$ we have the $m$-summable Fredholm modules over the algebra modulo stable $m$-summable perturbations. These new finitely summable $K$-homology groups pair with cyclic homology and with algebraic $K$-theory. In the case of cyclic homology the pairing is induced by the Chern-Connes character. The pairing between algebraic $K$-theory and finitely summable $K$-homology is induced by the Connes-Karoubi multiplicative character.
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1. Introduction

The aim of this paper is to study the invariance properties of the Connes-Karoubi multiplicative pairing between finitely summable Fredholm modules and algebraic $K$-theory,

$$\mathcal{M} : \text{Ell}^{m-1}(A) \times K_m(A) \to \mathbb{C}/(2\pi i)^{[m/2]}\mathbb{Z} \quad m \in \mathbb{N}$$

Here $\text{Ell}^{m-1}(A)$ denotes the abelian group of unitary equivalence classes of $m$-summable Fredholm modules over a $\mathbb{C}$-algebra $A$ together with a natural choice of inverses. The abelian group $K_m(A)$ is the algebraic $K$-theory of $A$. The multiplicative pairing was constructed by A. Connes and M. Karoubi in [7]. The associated invariants of algebraic $K$-theory

$$\mathcal{M}_\mathcal{F} : K_m(A) \to \mathbb{C}/(2\pi i)^{[m/2]}\mathbb{Z} \quad \mathcal{F} \in \text{Ell}^{m-1}(A)$$

can be interpreted as higher determinants induced by $m$-summable Fredholm modules. Indeed, when $m = 1$, the multiplicative character is constructed from the application

$$(g, h) \mapsto \det(gh^{-1}) \quad g, h \in \mathcal{L}(H)^*, \ g - h \in \mathcal{L}^1(H)$$

where $\det : GL_1(\mathcal{L}^1(H)) \to \mathbb{C}^*$ denotes the Fredholm determinant. See [7, §5]. When $m = 2$, the multiplicative character can be described in terms of the determinant invariant of L. Brown. This invariant is essentially given by the homomorphism

$$\delta := (\det \circ \partial) : K_2(\mathcal{L}(H)/\mathcal{L}^1(H)) \to K_1(\mathcal{L}^1(H), \mathcal{L}(H)) \to \mathbb{C}^*$$

Here the boundary map comes from the short exact sequence of $\mathbb{C}$-algebras

$$0 \longrightarrow \mathcal{L}^1(H) \longrightarrow \mathcal{L}(H) \longrightarrow \mathcal{L}(H)/\mathcal{L}^1(H) \longrightarrow 0$$

and the homomorphism $\det : K_1(\mathcal{L}^1(H), \mathcal{L}(H)) \to \mathbb{C}^*$ is the Fredholm determinant on relative algebraic $K$-theory. See [2]. The precise relation between the multiplicative character and the determinant invariant can be found in [14], see also [7, §5]. The determinant invariant has found an interesting application in the work of R. Carey and J. Pincus on generalizations of the Szegö limit theorem to the case of non-zero winding numbers, see [3, 4]. When $m \geq 3$ and $A$ is a commutative Banach algebra, the multiplicative character can be calculated on higher Loday products of exponentials. The result is expressed in terms of the index cocycle applied to permutations of logarithms. The precise formula reads

$$\mathcal{M}_\mathcal{F}(\ldots \ast [e^{a_0}] \ast \ldots [e^{a_{m-1}}]) = \sum_{s \in \Sigma_{m-1}} (q \circ \tau_\mathcal{F})(c_{m-1} \text{sgn}(s)b_0 \otimes b_{s(1)} \otimes \ldots \otimes b_{s(m-1)})$$

Here $\tau_\mathcal{F} : C^*_m(A) \to \mathbb{C}$ denotes the index cocycle of the $m$-summable Fredholm module $\mathcal{F} \in \text{Ell}^m(A)$, $q : \mathbb{C} \to \mathbb{C}/(2\pi i)^{[m/2]}\mathbb{Z}$ is the quotient map and $c_{m-1} \in \mathbb{Q}$ is a rational constant. The notation $\ast : K_n(A) \times K_k(A) \to K_{n+k}(A)$ refers to the interior Loday product. The elements $b_0, \ldots, b_{m-1} \in A$ are any logarithms of the exponentials $e^{a_0}, \ldots, e^{a_{m-1}}$,

$$e^{b_i} = e^{a_i} \quad \text{for all} \quad i \in \{0, \ldots, m-1\}$$

See [15].

Having these computational results for the multiplicative character in mind, it seems desirable to answer the following question:
Which reasonable equivalence relations can be put on the abelian groups of finitely summable Fredholm modules without changing the multiplicative pairing \((1.1)\)?

As the formula \((1.2)\) suggests, an important part of the multiplicative character is the index cocycle \(\tau_F : C^\lambda_{m-1}(A) \to \mathbb{C}\) associated with the \(m\)-summable Fredholm module. In fact, it turns out that the study of the invariance properties of the multiplicative character essentially amounts to a study of the invariance properties of the Chern character

\[
\text{Ch} : \text{Ell}^{m-1}(A) \to HC^{m-1}(A) \quad \text{Ch}(F) = [\tau_F]
\]

Here \(HC^{m-1}(A)\) denotes the cyclic cohomology of the \(\mathbb{C}\)-algebra \(A\) and \([\tau_F]\) denotes the class of the index cocycle. It should be noticed that the target for the above Chern character is one of the non-periodic versions of cyclic cohomology. If the target is replaced by periodic cyclic cohomology then there are good invariance results available. Here periodic cyclic cohomology, \(HP^0(A)\) and \(HP^1(A)\), is defined as the inductive limit of cyclic cohomology under the periodicity operators \(S : HC^*(A) \to HC^{*+2}(A)\). For example, it has been proved by A. Connes that the "topological" Chern character

\[
\text{Ch} : \text{Ell}^{m-1}(A) \to HP^j(A) \quad j \text{ parity as } m-1
\]

is invariant under appropriate differentiable fields of \(m\)-summable Fredholm modules, see [5 I 5 Lemma 1] for a precise statement of this result. A study of these homotopy invariance properties in the bivariant case has been carried out by V. Nistor, see [19]. In general, the Chern character from smooth homotopy classes of Fredholm modules to periodic cyclic cohomology is constructed to analyze data coming from topological \(K\)-theory, or more specifically for index-theoretical considerations. When dealing with the more subtle algebraic \(K\)-theory it is however crucial to avoid any application of the periodicity operator. For example, when \(A\) is commutative, the cyclic cycle given by the sum over all permutations

\[
x = \sum_{s \in \Sigma_{m-1}} \text{sgn}(s)b_0 \otimes b_{n(1)} \otimes \ldots \otimes b_{n(m-1)} \in Z^\lambda_{m-1}(A) \quad b_0, \ldots, b_{m-1} \in A
\]

lies in the kernel of the periodicity operator in cyclic homology. But this specific element shows up in our explicit formula \((1.2)\) for the multiplicative character. In order to analyze the information produced by secondary invariants of algebraic \(K\)-theory it is therefore necessary to obtain invariance results for the more refined Chern character

\[
\text{Ch} : \text{Ell}^{m-1}(A) \to HC^{m-1}(A)
\]

and not only for its topological sibling. The similarity between finitely summable Fredholm modules and generators of analytic \(K\)-homology suggests that the appropriate equivalence relation on \(\text{Ell}^{m-1}(A)\) has a \(K\)-homological analogue. Furthermore, we should choose the equivalence relation with the most algebraic flavour. A good candidate could thus be a finitely summable version of stable compact perturbation. See [11 Definition 17.2.4]. The next Theorem, which we prove in Section 2.4, indicates that this is an appropriate choice:

**Theorem 1.1.** Let \(F = (\pi, H, F)\) and \(G = (\pi, H, G)\) be two \(m\)-summable Fredholm modules over a \(\mathbb{C}\)-algebra \(A\). Assume that the difference \(G - F \in L^m(H)\) lies in the \(m\)th Schatten ideal. Then the Chern characters of \(F\) and \(G\) coincide in cyclic cohomology,

\[
\text{Ch}(F) = \text{Ch}(G) \in HC^{m-1}(A)
\]
Let us give a few comments on the proof of this result. We apply the language of generalized chains and generalized cycles together with the explicit formulas for their Chern characters in the \((b, B)\)-complex, due to A. Gorokhovsky, see [11]. For any generalized chain \(\Omega\) with boundary \(\partial \Omega\) we have the identity

\[(b + B)(\text{Ch}(\Omega)) = S(\text{Ch}(\partial \Omega))\]

in the \((b, B)\)-complex, \(B^* (A)\). Thus, the Chern character of the boundary of a generalized chain is a cyclic coboundary after application of the periodicity operator. This formula only seems to give information on the vanishing of Chern characters in periodic cyclic cohomology. However, when the toppart of the cyclic cochain \(\text{Ch}(\Omega) = (\varphi^m, \varphi^{m-2}, \ldots) \in B^m (A)\) vanishes, we actually get that the Chern character of the boundary \(\text{Ch}(\partial \Omega) \in B^{m-1} (A)\) is a cyclic coboundary. This observation was used by A. Gorokhovsky to give a proof of an invariance result similar to Theorem 1.1 in the even case and with slightly stronger summability conditions, see [11, Remark 4.1]. One should also remark the invariance result of A. Connes in the odd case, see [5, I.7 Proposition 4]. In order to prove Theorem 1.1 we construct a generalized chain \(\Omega_T\) such that

1. The toppart of the Chern character \(\text{Ch}(\Omega_T)\) vanishes.
2. The Chern character of the boundary

\[\text{Ch}(\partial \Omega_T) = \frac{1}{(m-1)!} (\tau_G - \tau_F)\]

is the difference of the index cocycles of the Fredholm modules \(G\) and \(F\) up to a constant.

Encouraged by the achievement of Theorem 1.1 we define a new \(K\)-homology type functor. As mentioned earlier, the main idea consists of replacing the compactness condition on Kasparov modules by the condition of finite summability. Furthermore, the equivalence relation is given by a finitely summable version of stable compact perturbation. We also distinguish between even and odd by means of grading operators. Thus, for any algebra \(A\) over \(\mathbb{C}\) we define a sequence of abelian groups

\[FK^{m-1}(A) := \text{Ell}^{m-1}(A)/\sim_{fc} \quad m \in \mathbb{N}\]

In dimension \((m - 1)\) we have the \(m\)-summable Fredholm modules modulo the equivalence relation of stable \(m\)-summable perturbation. Because of the resemblance with the \(K\)-homology of \(C^*\)-algebras we call this abelian group the \(m\)-summable \(K\)-homology of \(A\). Notice that we do not consider operator homotopic finitely summable Fredholm modules to be equal. We expect that such an equivalence relation would be to weak for the appropriate pairing with algebraic \(K\)-theory to exist. See Theorem 1.2.

The invariance result of Theorem 1.1 entails that the Chern-Connes character of finitely summable Fredholm modules descends to a natural homomorphism of degree 0,

\[\text{Ch} : FK^{m-1}(A) \to HC^{m-1}(A) \quad \text{Ch}([F]) = [\tau_F]\]

In particular, we get a well-defined pairing

\[(1.3) \quad \tau : FK^*(A) \times HC_*(A) \to \mathbb{C}\]

between finitely summable \(K\)-homology and cyclic homology.
Furthermore, as hinted at earlier, the well-definedness of the pairing \(1.3\) essentially entails a factorization result for the Connes-Karoubi multiplicative pairing. This is the main result of the present paper and the primary reason for the introduction of finitely summable \(K\)-homology.

**Theorem 1.2.** The multiplicative character associated to \(m\)-summable Fredholm modules over \(A\) induces a pairing
\[
\mathcal{M} : FK^{m-1}(A) \times K_m(A) \to \mathbb{C}/(2\pi i)^{\lceil m/2 \rceil} \mathbb{Z}
\]
between the \(m\)-summable \(K\)-homology group and the \(m\)th algebraic \(K\)-group.

The finitely summable \(K\)-homology groups can be equipped with some interesting homomorphisms. First of all there are the natural periodicity homomorphisms
\[
S : FK^*(A) \to FK^{*+2}(A)
\]
which are induced by the forgetful map related to the inclusion of Schatten ideals \(L^*(H) \subseteq L^{*-2}(H)\). The periodicity homomorphisms in finitely summable \(K\)-homology are paralleled by the periodicity homomorphisms in cyclic cohomology.

Furthermore, when \(A\) is a pre-\(C^*\)-algebra there are natural comparison homomorphisms
\[
\alpha : FK^{m-1}(A) \to K^j(\overline{A}) \quad j \in \{0, 1\} \text{ parity as } m - 1
\]
from finitely summable \(K\)-homology to analytic \(K\)-homology. Here \(\overline{A}\) denotes the \(C^*\)-algebra closure of \(A\). This comparison homomorphism comes from the inclusion of the Schatten ideals into the compact operators.

It is not unlikely that the comparison homomorphism becomes an isomorphism from a certain degree for many interesting pre-\(C^*\)-algebras. In this respect, compare with the case of cyclic cohomology and periodic cyclic cohomology of the smooth functions on a manifold.

Let us finish this introduction by making some remarks on an earlier definition which is related to our finitely summable \(K\)-homology.

In the papers, \([22, 23]\), X. Wang deals with contravariant functors from a category of smooth algebras to abelian semi groups. These functors are denoted by \(\text{Ext}_\tau\) and depend on the choice of a certain operator ideal \(\mathcal{K}_\tau\). The purpose is to classify smooth extensions of smooth algebras by these operator ideals. When \(\mathcal{K}_\tau = \mathcal{L}^m(H)\) and \(m = 2k\) is even, the definition of the functor \(\text{Ext}_\tau\) is strongly related to the definition of the \(m\)-summable \(K\)-homology functor, \(FK^{m-1}\), which we will introduce in the sequel. This provides a link between the present paper and the work on smooth extensions carried out by R. Douglas, D. Voiculescu, G. Gong and others, see \([8, 9, 10]\), for example.

We have organized our material as follows.

In the first section we consider the Chern-Connes character which maps a finitely summable Fredholm to the class of its index cocycle in cyclic cohomology. We are then able to prove that this homomorphism is invariant under finitely summable perturbations of the finitely summable Fredholm module.

In the second section we start by introducing the finitely summable \(K\)-homology groups. We then discuss the invariance properties of the secondary invariants which are associated to finitely summable Fredholm modules. This includes proving the existence of a pairing between finitely
summable $K$-homology and algebraic $K$-theory which is induced by the Connes-Karoubi multiplicative character.

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2. Finite summable Chern characters and perturbations

In this section we study the invariance properties of the Chern-Connes character

$$\text{Ch} : \text{Ell}^{m-1}(A) \to HC^{m-1}(A) \quad \text{Ch}(\mathcal{F}) = [\tau_{\mathcal{F}}]$$

which sends an $m$-summable Fredholm module to the class of its index cocycle in cyclic cohomology times a constant. Our main result is that the Chern-Connes characters of two $m$-summable Fredholm modules $\mathcal{F} = (\pi, H, F)$ and $\mathcal{G} = (\pi, H, G)$ coincide whenever the difference $G - F \in \mathcal{L}^m(H)$ lies in the $m^{th}$ Schatten ideal,

$$\text{Ch}(\mathcal{F}) = [\tau_{\mathcal{F}}] = [\tau_{\mathcal{G}}] = \text{Ch}(\mathcal{G}) \in HC^{m-1}(A)$$

We construct an explicit cyclic cochain with coboundary equal to the difference of the index cocycles.

In the first three subsections we review the following material:

1. Basic definitions in cyclic cohomology.
2. The explicit formulas in the $(b, B)$-bicomplex for Chern characters of generalized chains and generalized cycles.
3. The universal differential graded algebra associated to a graded algebra.

In the last subsection we prove the invariance of the Chern character of an $m$-summable Fredholm module under $m$-summable perturbations. The proof relies heavily on the material covered in the first three subsections.

2.1. Cyclic cohomology. This section contains a short review of the construction of cyclic cohomology by means of the $(b, B)$-bicomplex. For more details on these matters we refer to the book of A. Connes, [6].

Let $A$ be a unital $\mathbb{C}$-algebra. The Hochschild cochains of degree $m \in \mathbb{N}_0$, $C^m(A)$, consist of the multilinear maps of $(m + 1)$ variables

$$C^m(A) := \text{Hom}_\mathbb{C}(A \otimes A^{\otimes m}, \mathbb{C})$$

The Hochschild cochains can be equipped with two differentials: The Hochschild coboundary $b : C^m(A) \to C^{m+1}(A)$ and the Connes coboundary $B : C^m(A) \to C^{m-1}(A)$. Both of these operators have explicit descriptions and satisfy the relations

$$b^2 = 0 \quad B^2 = 0 \quad bB = -Bb$$
We can thus form the bicomplex
\[ \begin{array}{c}
\vdots \\
C^m(A) \xrightarrow{B} C^{m-1}(A) \xrightarrow{B} \ldots \\
C^{m-1}(A) \xrightarrow{B} C^{m-2}(A) \xrightarrow{B} \ldots \\
C^1(A) \xrightarrow{B} C^0(A) \\
C^0(A)
\end{array} \]
which we will refer to as the \((b, B)\)-bicomplex. The totalization of the \((b, B)\)-bicomplex is denoted by \(B^\ast(A)\) and the cyclic cohomology is the cohomology of this complex,
\[ HC^\ast(A) = H^\ast(B^\ast(A)) \]
A cyclic \(m\)-cochain \(\varphi \in B^m(A)\) is thus given by a sequence of homomorphisms
\[ \varphi = (\varphi^m, \varphi^{m-2}, \ldots) \]
and it is a cyclic \(m\)-cocycle precisely when
\[ (b + B)\varphi = (b\varphi^m, B\varphi^m + b\varphi^{m-2}, \ldots) = 0 \]
The cyclic cohomology comes equipped with periodicity operators,
\[ S : HC^m(A) \to HC^{m+2}(A) \quad S(\varphi^m, \varphi^{m-2}, \ldots) = (0, \varphi^m, \varphi^{m-2}, \ldots) \]
which are induced by a shift in the \((b, B)\)-bicomplex.

The definition of cyclic cohomology can be extended to a general \(\mathbb{C}\)-algebra \(A\). Let \(\tilde{A}\) denote the unitalization of \(A\). The Hochschild cochains over \(\tilde{A}\) can be restricted to Hochschild cochains over \(\mathbb{C}\) by means of the inclusion \(i : \mathbb{C} \to \tilde{A}\). This operation induces a cochain homomorphism
\[ i^* : B^\ast(\tilde{A}) \to B^\ast(\mathbb{C}) \]
By a slight abuse of notation we let \(B^\ast(A) := \text{Ker}(i^*)\) denote the kernel complex. The cyclic cohomology of \(A\) is then defined as the cohomology of \(B^\ast(A)\), \(HC^\ast(A) := H^\ast(B^\ast(A))\). When \(A\) has a unit, the groups obtained by this more general definition are isomorphic to the cyclic cohomology groups defined above.

The cyclic cohomology is a contravariant functor from the category of algebras over \(\mathbb{C}\) to the category of vector spaces over \(\mathbb{C}\).
On some occasions in the sequel we will make use of a continuous version of cyclic cohomology. Let \( A \) be a unital locally convex topological algebra with topology defined by a fundamental system of seminorms \( \{ p_i \}_{i \in I} \). The continuous Hochschild \( m \)-cochains \( C^m_{\text{cont}}(A) \) consists of the Hochschild \( m \)-cochains \( \varphi \in C^m(A) \) which are continuous in the sense that there exists an \( i \in I \) and a constant \( C \in [0, \infty) \) with

\[
|\varphi(a_0, \ldots, a_m)| \leq C p_i(a_0) \cdot \ldots \cdot p_i(a_m) \text{ for all } a_0, \ldots, a_m \in A
\]

The continuous cyclic cohomology is then defined precisely as the algebraic version of cyclic cohomology except that the Hochschild cochains are replaced by continuous Hochschild cochains. The continuous cyclic cohomology is denoted by \( HC^*_\text{cont}(A) \). The definition of continuous cyclic cohomology is extended to non-unital locally convex topological algebras in the same way as algebraic cyclic cohomology. The continuous cyclic cohomology is a contravariant functor from locally convex topological algebras to vector spaces over \( \mathbb{C} \).

2.2. Generalized chains and cobordisms. In this section we will review the notions of generalized chains and generalized cycles as well as the explicit formulas for their Chern characters. The concrete formulas which we present was introduced by A. Gorokhovsky, \[11\], whereas the general scheme of ideas is due to A. Connes, \[5, 6\]. Our exposition will follow \[11\] closely, but see also \[20\].

Let \( A \) be a unital algebra over \( \mathbb{C} \). By a generalized chain \( \Omega = (\Omega, \partial \Omega, \nabla, \partial \nabla, \int) \) of dimension \( m \in \mathbb{N}_0 \) over \( A \) we shall understand the following data:

(1) Two unital graded algebras \( \Omega = \bigoplus_{n=0}^{\infty} \Omega_n \) and \( \partial \Omega = \bigoplus_{n=0}^{\infty} \partial \Omega_n \) together with a unital surjective graded algebra homomorphism \( r: \Omega \rightarrow \partial \Omega \) and a unital algebra homomorphism \( \rho: A \rightarrow \Omega_0 \).

(2) Two graded derivations of degree one (termed connections) \( \nabla: \Omega_\ast \rightarrow \Omega_{\ast+1} \) and \( \partial \nabla: \partial \Omega_\ast \rightarrow \partial \Omega_{\ast+1} \) together with an element \( \theta \in \Omega_2 \) such that

\[
r \circ \nabla = \partial \nabla \circ r \quad \nabla^2(\omega) = \theta \omega - \omega \theta \quad \nabla(\theta) = 0
\]

The element \( \theta \in \Omega_2 \) is called the curvature of the connection \( \nabla \).

(3) A graded trace \( \int: \Omega_m \rightarrow \mathbb{C} \) satisfying

\[
\int \nabla \omega = 0 \quad \forall \omega \in \Omega_{m-1} \text{ with } r(\omega) = 0
\]

A generalized chain with \( \partial \Omega = 0 \) will be called a generalized cycle. A generalized chain with curvature \( \theta = 0 \) will be called a chain. A generalized chain with \( \partial \Omega = 0 \) and curvature \( \theta = 0 \) will be called a cycle.

By the boundary of the generalized chain \( \Omega = (\Omega, \partial \Omega, \nabla, \partial \nabla, \int) \) we shall understand the generalized cycle \( \partial \Omega = (\partial \Omega, \partial \nabla, \int') \) over \( A \) of dimension \((m-1)\). Here the graded trace \( \int': \partial \Omega_{m-1} \rightarrow \mathbb{C} \) is given by

\[
\int' \sigma = \int \nabla \omega \quad \text{for any } \omega \in \Omega_{m-1} \text{ with } r(\omega) = \sigma
\]

The unital algebra homomorphism \( \partial \rho: A \rightarrow \partial \Omega_0 \) is defined as the composition \( \partial \rho = r \circ \rho \). The curvature of the boundary is given by \( \partial \theta = r(\theta) \).
Definition 2.1. We say that two generalized cycles over $A$, $\Omega^0$ and $\Omega^1$, are cobordant if there exists a generalized chain $\Omega$ with boundary $\Omega^1 \oplus \Omega^0$ such that $r \circ \rho = (\rho_1, \rho_0)$. Here $\Omega^0$ is obtained from $\Omega^0$ by changing the sign of the graded trace $\int^0$.

The relation of cobordancy is an equivalence relation between generalized cycles, it will be denoted by $\sim_{\text{co}}$.

To each generalized chain $\Omega$ of dimension $m$ we associate a cyclic cochain

$$\text{Ch}(\Omega) = (\text{Ch}(\Omega)^m, \text{Ch}(\Omega)^{m-2}, \ldots) \in \mathcal{B}^m(A)$$

The cochain $\text{Ch}(\Omega)$ is called the Chern character of the generalized chain and is given by the following JLO-type formula

$$\text{Ch}(\Omega)^{m-2k}(a_0, \ldots, a_{m-2k}) = (-1)^k/(m-k)! \sum_{i_0 + \ldots + i_{m-2k} = k} \int \rho(a_0)\theta^{i_0} \nabla(\rho(a_1))\theta^{i_1} \ldots \nabla(\rho(a_{m-2k}))\theta^{i_{m-2k}}$$

$$k = 1, \ldots, \lfloor m/2 \rfloor$$

This explicit description is due to A. Gorokhovsky, [11]. The next theorem clarifies the relation between the Chern character of a generalized chain and the Chern character of its boundary.

Theorem 2.2. [11, Theorem 2.1] For each generalized chain $\Omega$ of dimension $m$ over the unital $\mathbb{C}$-algebra $A$ with boundary $\partial \Omega$ we have the identity

$$(B + b)\text{Ch}(\Omega) = S\text{Ch}(\partial \Omega)$$

in $\mathcal{B}^{m+1}(A)$.

In particular, the Chern character of a generalized cycle over $A$ is a cyclic cocycle in $\mathcal{B}^*(A)$. Furthermore, the difference of the Chern characters associated with cobordant generalized cycles lies in the kernel of the periodicity operator. In fact, as we shall see in the next theorem, this characterizes the relation of cobordancy between generalized cycles.

Theorem 2.3. Any two generalized cycles $\Omega^0$ and $\Omega^1$ of dimension $m - 1$ over the unital $\mathbb{C}$-algebra $A$ are cobordant if and only if the difference of their Chern characters in cyclic cohomology lies in the kernel of the periodicity operator. That is

$$\Omega^0 \sim_{\text{co}} \Omega^1 \Leftrightarrow [\text{Ch}(\Omega^0)] - [\text{Ch}(\Omega^1)] \in \text{Ker}(S)$$

Proof. This follows from [11, Theorem 2.6] and [6, III 1.3 Theorem 21].

2.3. The universal differential graded algebra of a graded algebra. We will now recall the construction of universal differential graded algebras of graded algebras. This is a natural adaption to the graded case of the universal differential graded algebras considered in [6, 16, 17], for example. We will find use for this construction in Section 2.4 where we build an explicit generalized chain which implements the cobordism between the cycles associated with Fredholm modules which are perturbations of each other.
Let \( A = \bigoplus_{i=0}^{\infty} A_i \) be a unital graded algebra over \( \mathbb{C} \). We let \( \overline{A} = A/\mathbb{C} \) denote the vector space quotient. Let \( k \in \mathbb{N}_0 \). To each multi index \( I = (i_0, \ldots, i_k) \in \mathbb{N}_0^{k+1} \) we associate the \( \mathbb{C} \)-vector space

\[
\Omega(I(A)) = \begin{cases} 
A_{i_0} \otimes_{\mathbb{C}} \overline{A}_{i_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \overline{A}_{i_k} & \text{for } k \geq 1 \\
A_{i_0} & \text{for } k = 0
\end{cases}
\]

The degree of the elements in \( \Omega(I(A)) \) is given by \( |I| = i_0 + \ldots + i_k + k \).

For each \( n \in \mathbb{N}_0 \) we have the \( \mathbb{C} \)-vector space \( \Omega_n(A) \) defined as the direct sum

\[
\Omega_n(A) = \bigoplus_{k=0}^{n} \bigoplus_{I \in \mathbb{N}_0^{k+1}, |I| = n} \Omega(I(A))
\]

As a \( \mathbb{C} \)-vector space the universal differential graded algebra of the unital graded \( \mathbb{C} \)-algebra \( A \) is given by the direct sum

\[
\Omega(A) = \bigoplus_{n=0}^{\infty} \Omega_n(A)
\]

The differential on \( \Omega(A) \) is given by

\[
d : \Omega_n(A) \to \Omega_{n+1}(A) \quad \text{and} \quad d(\omega_0 \otimes \ldots \otimes \omega_k) = 1_A \otimes \omega_0 \otimes \ldots \otimes \omega_k
\]

Clearly we then have \( d^2 = 0 \).

We endow \( \Omega(A) \) with the structure of an \( A \)-bimodule by defining the following products

\[
\omega \cdot (\omega_0 \otimes \ldots \otimes \omega_k) = (\omega \omega_0) \otimes \omega_1 \otimes \ldots \otimes \omega_k \quad \text{and} \quad (\omega_0 \otimes \ldots \otimes \omega_k) \cdot \omega = \omega_0 \otimes \ldots \otimes \omega_k \otimes (\omega_k \omega)
\]

\[
+ \sum_{i=0}^{k-1} (-1)^{|\omega_{i+1}|+\ldots+|\omega_k|+k-i} \omega_0 \otimes \ldots \otimes (\omega_i \omega_{i+1}) \otimes \ldots \otimes \omega_k \otimes \omega
\]

of homogeneous elements \( \omega_0 \otimes \ldots \otimes \omega_k \in \Omega_n(A) \) and elements \( \omega \in A \).

The graded multiplicative structure on \( \Omega(A) \) is given by the product

\[
(\omega_0 \otimes \ldots \otimes \omega_k) \cdot (\omega_0' \otimes \omega_1' \otimes \ldots \otimes \omega_l') = ((\omega_0 \otimes \ldots \otimes \omega_k) \cdot \omega_0') \otimes \omega_1' \otimes \ldots \otimes \omega_l'
\]

of homogeneous elements \( \omega_0 \otimes \ldots \otimes \omega_k \in \Omega_n(A) \) and \( \omega_0' \otimes \ldots \otimes \omega_l' \in \Omega_m(A) \).

It can be proved by a straightforward computation that the differential \( d : \Omega_n(A) \to \Omega_{n+1}(A) \) is a graded derivation.

We let \( \rho : A \to \Omega(A) \) denote the unital homomorphism of graded algebras which is given by inclusion. Our differential graded algebra \( \Omega(A) \) satisfies a universal property, justifying its name:

**Theorem 2.4.** Let \( A = \bigoplus_{i=0}^{\infty} A_i \) be a unital graded algebra over \( \mathbb{C} \) and let \( \Omega = \bigoplus_{n=0}^{\infty} \Omega_n \) be a unital differential graded algebra over \( \mathbb{C} \). Suppose that we have a unital homomorphism of graded algebras \( \phi : A \to \Omega \). Then there is a unique unital homomorphism of differential graded algebras \( \Omega(\phi) : \Omega(A) \to \Omega \) such that

\[
\Omega(\phi) \circ \rho = \phi
\]
The unital homomorphism of differential graded algebras $\Omega(\phi) : \Omega(A) \to \Omega$ is given by
$$\Omega(\phi)(\omega_0 \otimes \ldots \otimes \omega_k) = \phi(\omega_0)d_\Omega(\phi(\omega_1))\ldots d_\Omega(\phi(\omega_k)).$$
Here $d_\Omega : \Omega_* \to \Omega_{*+1}$ denotes the differential on $\Omega$. Note that $d_\Omega$ is a graded derivation for the multiplicative structure on $\Omega$ by assumption.

The uniqueness of the construction is immediate. □

### 2.4. Perturbations of Fredholm modules and cobordisms

In this section we will prove that the Chern-Connes character of $m$-summable Fredholm modules with values in the cyclic cohomology group $HC^{m-1}(A)$ is invariant under $m$-summable perturbations. This is a stronger version of a result proved by A. Gorokhovsky, [11, Remark 4.1]. This section also provides the motivation for the definition of the finitely summable $K$-homology groups which we give in Section 3. Indeed, Theorem 2.11 is really what ensures us, that we obtain a (non-trivial) multiplicative pairing between algebraic $K$-theory and these new $K$-homology type groups.

Let $A$ be an algebra over $\mathbb{C}$ and let $m \in \mathbb{N}$ be a positive integer. For a Hilbert space $H$ we will denote the $m$th Schatten ideal by $L^m(H)$.  

**Definition 2.5.** By an $m$-summable Fredholm module, $\mathcal{F} = (\pi, H, F)$, over $A$ we will understand the given of a separable Hilbert space $H$, an algebra homomorphism $\pi : A \to L(H)$ and a bounded operator $F \in L(H)$ such that

1. $F^2 - 1 = 0$.
2. $F - F^* = 0$.
3. $[F, \pi(a)] \in L^m(H)$.

for all $a \in A$. In the case where $(m - 1)$ is even we will also assume the existence of a $\mathbb{Z}/(2\mathbb{Z})$ grading operator $\gamma \in L(H)$ which anticommutes with $F$ and commutes with all elements in $\pi(A)$. In order to unify the notation we will use the convention that $\gamma = \text{Id}$ when $(m - 1)$ is odd.

When $A$ is a locally convex topological algebra we will say that the $m$-summable Fredholm module $\mathcal{F} = (\pi, H, F)$ is continuous if the maps
$$a \mapsto \pi(a) \in L(H) \quad \text{and} \quad a \mapsto [F, \pi(a)] \in L^m(H)$$
are continuous.

Let $\mathcal{F} = (\pi, H, F)$ be an $m$-summable Fredholm module over $A$. We recall from [6, Chapter IV] that there is an associated index cocycle
$$\tau_\mathcal{F} = (\tau_\mathcal{F}, 0, \ldots) \in B^{m-1}(A)$$
given by
$$\tau_\mathcal{F}(x_0, \ldots, x_{m-1}) = \frac{1}{2}\text{Tr}(\gamma^m F[x_0] \ldots [F, x_{m-1}]) \quad x_0, \ldots, x_{m-1} \in \tilde{A}$$
Here $\tilde{A}$ denotes the unitalization of $A$. We will refer to the class of the index cocycle in cyclic cohomology
$$\text{Ch}(\mathcal{F}) := [\tau_\mathcal{F}] \in HC^{m-1}(A)$$
as the Chern-Connes character of $\mathcal{F}$. 


When $A$ is a locally convex topological algebra and $\mathcal{F}$ is continuous the index cocycle determines a class
\[
\text{Ch}^\text{cont}(\mathcal{F}) := [\tau^\text{cont}_F] \in HC^{m-1}_{\text{cont}}(A)
\]
in continuous cyclic cohomology. We will call this class the \textit{continuous Chern-Connes} character of $\mathcal{F}$.

**Definition 2.6.** We say that two $m$-summable Fredholm modules $\mathcal{F} = (\pi, H, F)$ and $\mathcal{G} = (\pi, H, G)$ over $A$ are $m$-summable perturbations of each other if the difference of operators
\[
G - F \in \mathcal{L}^m(H)
\]
lies in the $m$th Schatten ideal. When the dimension, $(m - 1)$, is even we will also require that the grading operators of the two Fredholm modules agree.

Let $\mathcal{F} = (\pi, H, F)$ and $\mathcal{G} = (\pi, H, G)$ be two $m$-summable Fredholm modules which are $m$-summable perturbations of each other. The aim of the present section is to prove that the Chern-Connes characters of $\mathcal{F}$ and $\mathcal{G}$ determine the same class in cyclic cohomology. Thus, we will show that
\[
\text{Ch}(\mathcal{F}) = [\tau^\mathcal{F}] = [\tau^\mathcal{G}] = \text{Ch}(\mathcal{G}) \in HC^{m-1}(A)
\]
This result should be compared with [11, Theorem 4.1] and [20, Theorem 2.11].

Let $T = G - F \in \mathcal{L}^m(H)$ denote the difference of $G$ and $F$. We will constantly make use of the following identity.

**Lemma 2.7.**
\[
FT + TF + T^2 = 0
\]

**Proof.** We have that
\[
F^2 + T^2 + FT + TF = (F + T)^2 = 1 = F^2
\]
and the result follows immediately. \qed

Let us add an element $\tau$ of order 1 to the unitalization $\tilde{A}$. That is, we let $\tilde{A}_\tau$ denote the unital graded algebra over $\mathbb{C}$ which as a vector space is given by
\[
\tilde{A}_\tau := \bigoplus_{j=1}^\infty \tilde{A}_{\otimes^c j}
\]
The degree of an element $x = x_0 \otimes \ldots \otimes x_n \in \tilde{A}_\tau$ is defined as $|x| = n$. The multiplication on $\tilde{A}_\tau$ is determined by the rule,
\[
(x_0 \otimes \ldots \otimes x_n) \cdot (y_0 \otimes \ldots \otimes y_m) = (x_0 \otimes \ldots \otimes x_{n-1}) \otimes (x_n y_0) \otimes (y_1 \otimes \ldots \otimes y_m)
\]
for $x_0, \ldots, x_n, y_0, \ldots, y_m \in \tilde{A}$ on simple tensors. We will use the special notation $\tau$ for the element
\[
\tau := 1_{\tilde{A}} \otimes 1_{\tilde{A}} \in (\tilde{A}_\tau)_1
\]
of degree 1.
Let us consider the unital differential graded algebra $\Omega_\tau$ defined as the universal differential graded algebra of $\tilde{A}_\tau$ modulo the relation $d(\tau) + \tau^2 \sim 0$. That is, we let

$$\Omega_\tau := \Omega(A_\tau)/\langle d(\tau) + \tau^2 \sim 0 \rangle$$

with differential induced by the differential on $\Omega(A_\tau)$. In this respect, notice that

$$d(d(\tau) + \tau^2) = d(\tau^2) + \tau d(\tau) + \tau^2 - \tau d(\tau) + \tau^2$$

We will use $\Omega_\tau$ to construct a generalized chain $\Omega_T$ over $\tilde{A}$ which provides a cobordism between two generalized cycles $\hat{\Omega}_F$ and $\hat{\Omega}_G$ over $\tilde{A}$. We shall then see in Lemma 2.9 that the Chern characters of these two generalized cycles coincides with the index cocycles of $F$ and $G$ up to a constant.

Furthermore, the nature of the generalized chain $\Omega_T$ together with the explicit formula for the associated cyclic cochain will allow us to conclude that the difference of cyclic cocycles

$$\text{Ch}(\hat{\Omega}_G) - \text{Ch}(\hat{\Omega}_F) \in B^{m-1}(\tilde{A})$$

is a cyclic coboundary without any application of the periodicity operator. This is thus a stronger result than what is initially provided by the cobordism relation of Theorem 2.2.

The proof of the desired invariance result will then follow from some straightforward argumentation concerning the restriction cochain homomorphism $i^*: B^*(\tilde{A}) \to B^*(\tilde{\mathbb{C}})$.

Let us continue with the construction of the generalized chain $\Omega_T$ over $\tilde{A}$. We start by extending the algebra homomorphism $\pi: A \to \mathcal{L}(H)$ to a unital algebra homomorphism $\pi: \Omega_\tau \to \mathcal{L}(H)$. We do this in the following way:

(1) To begin with, we define the unital algebra homomorphism $\pi: \tilde{A}_\tau \to \mathcal{L}(H)$ by

$$\pi: \tau \mapsto T \in \mathcal{L}(H) \quad \text{and} \quad (a, \lambda) \mapsto \pi(a) + \lambda \in \mathcal{L}(H)$$

(2) We then define the unital algebra homomorphism $\pi: \Omega(\tilde{A}_\tau) \to \mathcal{L}(H)$ by

$$\pi(\omega_0 \otimes \ldots \otimes \omega_k) = \pi(\omega_0)[F, \pi(\omega_1)] \ldots [F, \pi(\omega_k)]$$

Here the commutators $[F, \pi(\omega_i)]$ are graded, thus

$$[F, \pi(\omega_i)] = F\pi(\omega_i) + (-1)^{\mid\omega_i\mid+1}\pi(\omega_i)F$$

Note that this is well-defined even when the Hilbert space $H$ is trivially graded.

(3) Finally, it follows from the identity $FT + TF + T^2 = 0$ of Lemma 2.7 that the unital algebra homomorphism $\pi: \Omega(\tilde{A}_\tau) \to \mathcal{L}(H)$ descends to the desired unital algebra homomorphism $\pi: \Omega_\tau \to \mathcal{L}(H)$.

In order to define the graded trace of the generalized chain $\Omega_T$ of dimension $m$, we need to show that the unital algebra homomorphism $\pi: \Omega_\tau \to \mathcal{L}(H)$ maps homogeneous elements of degree $m$ to trace class operators.

**Lemma 2.8.** For each element $\omega \in (\Omega_\tau)_m$ the operator $\pi(\omega) \in \mathcal{L}^1(H)$ is of trace class.
Proof. We recall that the Schatten ideals satisfy the multiplication rule
\[ \mathcal{L}^{m/p}(H) \cdot \mathcal{L}^{m/q}(H) \subseteq \mathcal{L}^{m/(p+q)}(H) \]
for any \( p, q \in \{1, \ldots, m\} \) with \( p + q \in \{1, \ldots, m\} \).

Suppose that \( \sigma \in \tilde{A}_r \) is a homogeneous element of order \( i \in \{0, \ldots, m\} \) of the form
\[ \sigma = x_0 \tau^{i_0} \ldots x_k \tau^{i_k} \]
for some \( x_0, \ldots, x_k \in \tilde{A} \) and \( i_0 + \ldots + i_k = i \). Since \( T \in \mathcal{L}^m(H) \) by assumption we get that
\[ \pi(\sigma) = \tilde{\pi}(x_0) T^{i_0} \ldots \tilde{\pi}(x_k) T^{i_k} \in \mathcal{L}^{m/i}(H) \]
Here \( \tilde{\pi}(a, \lambda) = \pi(a) + \lambda \) for any \( (a, \lambda) \in \tilde{A} \).

Next, assume that the order of \( \sigma \) is less than or equal to \( m - 1 \). Let \( j \in \{0, \ldots, k\} \). Assume that \( i_j = 2l + 1 \) is odd. We evolve on the graded commutator
\[ [F, \tilde{\pi}(x_j) T^{i_j}] = [F, \tilde{\pi}(x_j)] T^{i_j} + \tilde{\pi}(x_j) (FT + TF) T^{2l} \]
\[ = [F, \tilde{\pi}(x_j)] T^{i_j} - \tilde{\pi}(x_j) T^{2l+2} \]
Here we have used the identity \( FT + TF = T^2 \) of Lemma 2.7. Assume that \( i_j = 2l \) is even. The graded commutator is then given by
\[ [F, \tilde{\pi}(x_j) T^{i_j}] = [F, \tilde{\pi}(x_j)] T^{i_j} \]
In both cases we get that \( [F, \tilde{\pi}(x_j) T^{i_j}] \in \mathcal{L}^{m/(i_j+1)}(H) \). Since the graded commutator \([F, \cdot]\) is a graded derivation we conclude that \([F, \pi(\sigma)] \in \mathcal{L}^{m/(i_j+1)}(H)\).

The result of the Lemma is a consequence of these two observations and the multiplication rule for Schatten ideals.

We are now ready to define the generalized chain \( \Omega_T \) of dimension \( m \) over \( \tilde{A} \). It is given by the following data:

1. The unital graded algebras
\[ \Omega^*([0,1]) \hat{\otimes} \Omega_T \quad \text{and} \quad \partial(\Omega_T) = \Omega_T \oplus \Omega_T \]
Here \( \Omega^*([0,1]) \) denotes the unital graded algebra of differential forms on the unit interval and \( \hat{\otimes} \) denotes the graded tensor product over \( \mathbb{C} \) of graded algebras.

The unital surjective homomorphism of graded algebras
\[ (r_1, r_0) : \Omega^*([0,1]) \hat{\otimes} \Omega_T \rightarrow \Omega_T \oplus \Omega_T \]
given by the restriction to 1 and 0 respectively.

The unital algebra homomorphism
\[ \rho : x \mapsto 1 \hat{\otimes} x \in C^\infty([0,1]) \hat{\otimes} \tilde{A} \quad \text{for} \quad x \in \tilde{A} \]

2. The graded derivations of degree one
\[ \nabla_T = d \hat{\otimes} 1 + 1 \hat{\otimes} d + t \hat{\otimes} [\tau, \cdot] \quad \text{and} \quad \partial \nabla_T = (d + [\tau, \cdot]) \oplus d \]
with curvatures
\[ \theta_T = dt \hat{\otimes} \tau + t \hat{\otimes} d(\tau) + t^2 \hat{\otimes} \tau^2 \in (\Omega_T)_2 \quad \text{and} \quad \partial \theta = (d(\tau) + (\tau)^2, 0) = (0,0) \]
Here the connection on $\Omega^*([0,1]) \hat{\otimes} \Omega_\tau$ should be understood in the following sense
\[ \nabla_\tau(\alpha \hat{\otimes} \omega) = d\alpha \hat{\otimes} \omega + (-1)^{|\alpha|} \alpha \hat{\otimes} d\omega + (-1)^{|\alpha|} t\alpha \hat{\otimes} (\tau \omega + (-1)^{|\omega|+1} \omega \tau) \]
for homogeneous elements $\alpha \in \Omega^*([0,1])$ and $\omega \in \Omega_\tau$.

(3) The graded trace $\int_\tau: (\Omega_\tau)_m \to \mathbb{C}$ is given by
\[ \int_\tau(\alpha \hat{\otimes} \omega) = \begin{cases} \frac{1}{2} \left( \int_0^1 \alpha \right) \cdot \text{Tr}(\gamma^m F_\pi d\omega) & \text{for } \alpha \in \Omega^1([0,1]) \\ 0 & \text{for } \alpha \in \Omega^0([0,1]) \end{cases} \]
We leave it to the reader to verify that $\Omega_\tau$ satisfies the generalized chain conditions given in Section 2.2. See also [6, IV.1.α Proposition 1], [111 Section 4] and [20, Theorem 2.11].

Let $\hat{\Omega}_G$ and $\hat{\Omega}_F$ be the cycles of dimension $(m-1)$ over $\tilde{A}$ given by
\[ \hat{\Omega}_G := (\Omega_\tau, d + [\tau, \cdot], f) \quad \hat{\Omega}_F := (\Omega_\tau, d, f) \]
Here $f: (\Omega_\tau)_{m-1} \to \mathbb{C}$ is the graded trace defined by
\[ \int f = \frac{1}{2} \text{Tr}(\gamma^m F_\pi d\omega) \]
The unital algebra homomorphism $\rho: \tilde{A} \to (\Omega_\tau)_0 = \tilde{A}$ is the identity homomorphism. The boundary of the generalized chain $\Omega_\tau$ then equals the direct sum of cycles
\[ \partial \Omega_\tau = \hat{\Omega}_G \oplus \hat{\Omega}_F \]
Here $\hat{\Omega}_F$ is obtained from $\hat{\Omega}_F$ by changing the sign of the graded trace. This means that $\Omega_\tau$ defines a cobordism between $\hat{\Omega}_F$ and $\hat{\Omega}_G$. In particular we have the cobordism relation
\[ (b + B)(\text{Ch}(\Omega_\tau)) = S(\text{Ch}(\hat{\Omega}_G)) - S(\text{Ch}(\hat{\Omega}_F)) \]
at the level of cyclic cochains in $B^{m+1}(\tilde{A})$, see Theorem 2.2.

As mentioned earlier, the Chern characters of the cycles $\hat{\Omega}_G$ and $\hat{\Omega}_F$ coincides with the index cocycles of the Fredholm modules $G$ and $F$ up to a constant.

**Lemma 2.9.** We have the identities
\[ (m-1)! \text{Ch}(\hat{\Omega}_F) = \tau_F \quad \text{and} \quad (m-1)! \text{Ch}(\hat{\Omega}_G) = \tau_G \]
at the level of cyclic cocycles in $B^{m-1}(\tilde{A})$.

**Proof.** The first identity is clear. The proof of the second identity amounts to the following calculation
\[
\text{Tr}(\gamma^m x_0[G, x_1] \ldots [G, x_{m-1}] + (-1)^m \gamma^m F x_0[G, x_1] \ldots [G, x_{m-1}])
= \text{Tr}(\gamma^m x_0[G, x_1] \ldots [G, x_{m-1}] + (-1)^m \gamma^m G x_0[G, x_1] \ldots [G, x_{m-1}])
- (-1)^m \text{Tr}(\gamma^m T x_0[G, x_1] \ldots [G, x_{m-1}] (F + T))
- (-1)^m \text{Tr}(\gamma^m F x_0[G, x_1] \ldots [G, x_{m-1}])
= \text{Tr}(\gamma^m G [G, x_0] \ldots [G, x_{m-1}])
\]
which is valid for all $x_0, \ldots, x_{m-1} \in \tilde{A}$. Here we have used that the operator $FT + TF + T^2$ vanishes, see Lemma 2.7. \qed
The result of Lemma 2.9 improves the equality of (2.1), which now reads

\[(b + B)(\text{Ch}(\Omega_T)) = \frac{1}{(m-1)!} S(\tau_G - \tau_F)\]

The explicit description of the involved cyclic cochains allows us to carry out a further improvement of the above relation. In this respect, we should refer to the proof of [11, Lemma 2.7] where the main idea of the next Lemma appears.

**Lemma 2.10.** The difference of cyclic cocycles

\[\tau_G - \tau_F \in \mathcal{B}^{m-1}(\tilde{A})\]

is a cyclic coboundary.

**Proof.** Let \((y_{m+1}, y_{m-1}, \ldots) \in \mathcal{B}_{m+1}(\tilde{A})\). From the identity (2.2) we get that

\[\tau_G(y_{m-1}, y_{m-3}, \ldots) - \tau_F(y_{m-1}, y_{m-3}, \ldots) = (m-1)! \text{Ch}(\Omega_T)(by_{m+1} + By_{m-1}, by_{m-1} + By_{m-3}, \ldots)\]

Now, let \(\text{Ch}(\Omega_T)^m\) denote the top component of \(\text{Ch}(\Omega_T)\). We then have

\[\text{Ch}(\Omega_T)^m(x_0, \ldots, x_m) = \frac{1}{m!} \int_{\tau} (1 \hat{\otimes} x_0) \nabla_\tau (1 \hat{\otimes} x_1) \ldots \nabla_\tau (1 \hat{\otimes} x_m)\]

Here \(x_0, \ldots, x_m \in \tilde{A}\) are any elements in the unitalization. However by definition of the graded trace

\[\int_{\tau} : (\Omega^*([0, 1]) \hat{\otimes} \Omega_\tau)_m \rightarrow \mathbb{C}\]

the above quantity is zero, since

\[(1 \hat{\otimes} x_0) \nabla_\tau (1 \hat{\otimes} x_1) \ldots \nabla_\tau (1 \hat{\otimes} x_m) \in \Omega^0([0, 1]) \hat{\otimes} (\Omega_\tau)_m\]

It follows that

\[\tau_G(y_{m-1}, y_{m-3}, \ldots) - \tau_F(y_{m-1}, y_{m-3}, \ldots) = (m-1)! \text{Ch}(\Omega_T)^{m-2}(by_{m-1} + By_{m-3}) + (m-1)! \text{Ch}(\Omega_T)^{m-4}(by_{m-3} + By_{m-5}) + \ldots\]

But this proves the lemma. \(\square\)

We are now in position to prove the invariance result which lies at the core of the present paper.

**Theorem 2.11.** Let \(\mathcal{F} = (\pi, H, F)\) and \(\mathcal{G} = (\pi, H, G)\) be two \(m\)-summable Fredholm modules over a \(\mathbb{C}\)-algebra \(A\). Suppose that \(\mathcal{F}\) and \(\mathcal{G}\) are \(m\)-summable perturbations of each other. Then their Chern-Connes characters

\[\text{Ch}(\mathcal{F}) = \text{Ch}(\mathcal{G}) \in HC^{m-1}(A)\]

agree in the cyclic cohomology group of degree \((m - 1)\).
Proof. Let \( i^* : B^*(\tilde{A}) \to B^*(\mathbb{C}) \) denote the cochain homomorphism induced by the inclusion \( i : \mathbb{C} \to \tilde{A} \).

Suppose that \( m = 2k + 1 \) is odd. By Lemma 2.10 we have the identity

\[
(b + B)(\text{Ch}(\Omega_T)^{2k-1}, \ldots, \text{Ch}(\Omega_T)^1) = \frac{1}{(m-1)!}\tau_G - \frac{1}{(m-1)!}\tau_F
\]

in \( B^{2k}(\tilde{A}) \). The desired result follows by noting that all the involved cyclic cochains lie in the kernel of the restriction \( i^* \).

Let \( p : \tilde{A} \to \mathbb{C} \) denote the unital algebra homomorphism given by \( p(a, \lambda) = \lambda \).

Suppose that \( m = 2k \) is even. Let \( \psi \in B^{m-2}(\tilde{A}) \) denote the cyclic cochain given by

\[
\psi(y_{2k-2}, \ldots, y_0) = (\text{Ch}(\Omega_T)^{2k-2}, \ldots, \text{Ch}(\Omega_T)^0) (y_{2k-2}, \ldots, y_0) - \text{Ch}(\Omega_T)^0(p(y_0))
\]

We then have that \( i^*(\psi) = 0 \). Furthermore, the coboundary of \( \psi \) is the desired difference of Chern characters

\[
(b + B)(\psi) = (b + B)(\text{Ch}(\Omega_T)^{2k-2}, \ldots, \text{Ch}(\Omega_T)^0) = \frac{1}{(m-1)!}\tau_G - \frac{1}{(m-1)!}\tau_F
\]

This proves the theorem in the even case as well. \( \square \)

On some occasions in the sequel we will also need a continuous version of Theorem 2.11.

Theorem 2.12. Let \( \mathcal{F} = (\pi, H, G) \) and \( \mathcal{G} = (\pi, H, G) \) be two continuous \( m \)-summable Fredholm modules over \( A \). Assume that \( \mathcal{F} \) and \( \mathcal{G} \) are \( m \)-summable perturbations of each other. Then their continuous Chern-Connes characters

\[
\text{Ch}^{\text{cont}}(\mathcal{F}) = \text{Ch}^{\text{cont}}(\mathcal{G}) \in HC^{m-1}_{\text{cont}}(A)
\]

agree in the continuous cyclic cohomology group of dimension \( (m-1) \).

Proof. This follows by noting that the cyclic cochain \( \text{Ch}(\Omega_T) \in B^m(\tilde{A}) \), which is used for the proof of Theorem 2.11 satisfies the relevant continuity properties. \( \square \)

3. Finitely summable \( K \)-homology and pairings with algebraic \( K \)-theory

We will define a variant of analytic \( K \)-homology which has a more algebraic flavour. The main idea is simply to replace the compact operators in the definition of analytic \( K \)-homology with Schatten ideals. The most "algebraic" equivalence relations in the \( C^* \)-algebraic setup are unitary equivalence and perturbation by compact operators. We will thus replace the usual definitions with the corresponding finitely summable versions. Our main result is then that the new \( K \)-homology type groups which we obtain actually pair with algebraic \( K \)-theory by means of the Connes-Karoubi multiplicative character.

The results in this section rely on the invariance under perturbations of the Chern character which we proved in Section 2.

The material is organized as follows:
In subsection 3.1 we introduce the finitely summable $K$-homology together with natural periodicity and comparison homomorphisms. The definitions also make sense in a continuous setup.

In subsection 3.2 we show that the continuous version of finitely summable $K$-homology pairs with M. Karoubi’s relative $K$-groups. The pairing is induced by the “additive" character of continuous finitely summable Fredholm modules.

In subsection 3.3 we show that finitely summable $K$-homology pairs with algebraic $K$-theory. The pairing is induced by the Connes-Karoubi multiplicative character of finitely summable Fredholm modules. We review the construction of the multiplicative character also.

3.1. Finitely summable Fredholm modules and their equivalence relations. In this subsection we will introduce the finitely summable $K$-homology groups of an algebra over the complex numbers. These groups are equipped with periodicity homomorphisms which correspond to the periodicity homomorphisms in cyclic cohomology. Furthermore, for pre-$C^*$-algebras, there are interesting comparison homomorphisms from finitely summable $K$-homology to analytic $K$-homology. To begin with, let us give some relevant definitions concerning finitely summable Fredholm modules.

Let $A$ be an algebra over $\mathbb{C}$ and let $m \in \mathbb{N}$ be a positive integer. Let $\mathcal{F} = (\pi, H, F)$ be an $m$-summable Fredholm module over $A$.

We will make the following standard assumption: When $(m - 1)$ is even we will assume that the eigenspaces of the grading operator are infinite dimensional. When $(m - 1)$ is odd we will always assume that the eigenspaces of the selfadjoint unitary $F$ are infinite dimensional. This mild condition is necessary for the construction in Subsection 3.3 of the Connes-Karoubi multiplicative pairing between finitely summable $K$-homology and algebraic $K$-theory. See also [7, §1].

Definition 3.1. We say that $\mathcal{F}$ is degenerate if all the relations in Definition 2.5 are exactly satisfied. That is, they should not only be satisfied modulo the $m$th Schatten ideal.

We let $\mathcal{F}^{-1}$ denote the $m$-summable Fredholm module over $A$ given by

\begin{equation}
\mathcal{F}^{-1} := \left\{ \begin{array}{ll}
(\pi, H, -F) & \text{for } (m - 1) \text{ odd} \\
(\pi, H^{\text{op}}, -F) & \text{for } (m - 1) \text{ even}
\end{array} \right.
\end{equation}

Here $H^{\text{op}}$ denotes the Hilbert space with grading operator $-\gamma \in \mathcal{L}(H)$ whenever $H$ is a $\mathbb{Z}/(2\mathbb{Z})$-graded Hilbert space with grading operator $\gamma \in \mathcal{L}(H)$.

We will think of $\mathcal{F}^{-1}$ as a specific choice of an inverse to $\mathcal{F}$. This choice is motivated by the standard choice of an inverse in $K$-homology, see [11 Proposition 17.3.3] and [13 Proposition 8.2.10] for example. It should also be noted that the Chern-Connes character of the inverse is given by minus the Chern-Connes character of the original element.

Let $\mathcal{F}_1 = (\pi_1, H_1, F_1)$ and $\mathcal{F}_2 = (\pi_2, H_2, F_2)$ be two $m$-summable Fredholm modules over $A$.

Definition 3.2. By the direct sum of $\mathcal{F}_1$ and $\mathcal{F}_2$ we will understand the $m$-summable Fredholm module

$$\mathcal{F}_1 \oplus \mathcal{F}_2 = (\pi_1 \oplus \pi_2, H_1 \oplus H_2, F_1 \oplus F_2)$$
over $A$.

In the case where $(m - 1)$ is even, the grading on $H_1 \oplus H_2$ is given by the direct sum of the grading operators.

We will now concentrate on the appropriate equivalence relations.

**Definition 3.3.** We say that $F_1$ and $F_2$ are unitarily equivalent if there exists a unitary operator $u : H_1 \to H_2$ such that

$$\pi_1 = u^* \pi_2 u \text{ and } F_1 = u^* F_2 u$$

In the case where $(m - 1)$ is even we will also require that $\gamma_1 = u^* \gamma_2 u$.

The notation $\sim_m$ will refer to the equivalence relation on $m$-summable Fredholm modules generated by unitary equivalence and $m$-summable perturbations in the sense of Definition 2.6.

**Definition 3.4.** We say that $F_1$ and $F_2$ are stable $m$-summable perturbations of each other when there exist $m$-summable Fredholm modules $G_1, G_2$ and $\mathcal{H}$ together with degenerate $m$-summable Fredholm modules $D_1$ and $D_2$ such that

$$F_1 \oplus G_1 \oplus G_1^{-1} \oplus D_1 \oplus \mathcal{H} \sim_m F_2 \oplus G_2 \oplus G_2^{-1} \oplus \mathcal{H}$$

We will denote the equivalence relation of stable $m$-summable perturbations by $\sim_{mc}$.

We are now ready to define the finitely summable $K$-homology of a $\mathbb{C}$-algebra. This is the main definition of this section.

**Definition 3.5.** By the $m$-summable $K$-homology of $A$ we will understand the abelian group given by the $m$-summable Fredholm modules over $A$ modulo the equivalence relation of stable $m$-summable perturbation.

The group operation is given by the direct sum of $m$-summable Fredholm modules. The $m$-summable $K$-homology of $A$ is denoted by $FK_{m-1}(A)$.

Let $A$ be a locally convex topological algebra. The above definitions apply to continuous Fredholm modules over $A$ as well. This gives rise to a continuous version of finitely summable $K$-homology.

**Definition 3.6.** By the continuous $m$-summable $K$-homology of $A$ we will understand the abelian group given by the $m$-summable continuous Fredholm modules over $A$ modulo the equivalence relation of stable $m$-summable perturbation.

The group operation is given by the direct sum of $m$-summable Fredholm modules. The continuous $m$-summable $K$-homology of $A$ is denoted by $FK_{\text{cont}}^{m-1}(A)$.

Let $A$ and $B$ be algebras over $\mathbb{C}$ and let $\varphi : A \to B$ be an algebra homomorphism. To each $m$-summable Fredholm module, $F = (\pi, H, F)$, over $A$ we can associate an $m$-summable Fredholm module over $B$,

$$FK(\varphi)(F) := (\pi \circ \varphi, H, F)$$

In this way we obtain a group homomorphism $FK(\varphi) : FK_{m-1}(A) \to FK_{m-1}(B)$ for each $m \in \mathbb{N}$. This turns finitely summable $K$-homology into a contravariant functor from the category of algebras over $\mathbb{C}$ to the category of abelian groups.
Likewise, the continuous finitely summable $K$-homology becomes a contravariant functor from the category of locally convex topological algebras to the category of abelian groups.

Let us describe some interesting forgetful homomorphisms.

Let $\mathcal{F}$ be an $m$-summable Fredholm module over a $C^*$-algebra $A$. Since the $m$th Schatten ideal is contained in the $(m + 2)$th Schatten ideal, $\mathcal{F}$ is also an $(m + 2)$-summable Fredholm module. This forgetful map induces a natural periodicity homomorphism

$$S : FK^{m-1}(A) \to FK^{m+1}(A)$$

The construction applies in the continuous case as well. Thus, for each locally convex topological algebra, $A$, and each $m \in \mathbb{N}$ there is a natural periodicity homomorphism

$$S : FK^{m-1}_{\text{cont}}(A) \to FK^{m+1}_{\text{cont}}(A)$$

Let $A$ be a pre-$C^*$-algebra in the sense of [6, IV.1] and let $\mathcal{F} = (\pi, H, F)$ be an $m$-summable Fredholm module over $A$. The algebra homomorphism $\pi : A \to \mathcal{L}(H)$ extends uniquely to a continuous algebra homomorphism $\overline{\pi} : \overline{A} \to \mathcal{L}(H)$. Here $\overline{A}$ denotes the $C^*$-algebra closure of $A$ and the topology on $\mathcal{L}(H)$ is defined by the operator norm. Since the compact operators agrees with the closure of the $m$th Schatten ideal in $\mathcal{L}(H)$ we get a Fredholm module $(\overline{\pi}, H, F)$ over $\overline{A}$. This operation induces a comparison homomorphism

$$\alpha : FK^{m-1}(A) \to K^j(\overline{A})$$

for each $m \in \mathbb{N}$. Here $K^*(\overline{A})$ denotes the analytic $K$-homology of the $C^*$-algebra $\overline{A}$.

### 3.2. The additive character and pairings with relative $K$-theory

In this subsection we will show that continuous finitely summable $K$-homology pairs with M. Karoubi’s relative $K$-theory. It was already noted in [7] that each continuous finitely summable Fredholm module yields an "additive" character on relative $K$-theory. We use the invariance under perturbations of the Chern-Connes character of continuous finitely summable Fredholm modules to establish invariance properties of this additive character. To begin with, let us show that finitely summable $K$-homology pairs with cyclic homology.

Let $A$ be an algebra over $\mathbb{C}$ and let $m \in \mathbb{N}$ be a positive integer. Let $\mathcal{F}$ be an $m$-summable Fredholm module over $A$.

We recall from Section 2.4 that $\text{Ch}(\mathcal{F}) = [\tau_\mathcal{F}]$ denotes the class of the index cocycle in cyclic cohomology.

Let $\mathcal{G}$ be another $m$-summable Fredholm module over $A$ and let $\mathcal{D}$ be an $m$-summable degenerate Fredholm module over $A$. Recall from Section 3.1 that $\mathcal{F}^{-1}$ denotes the inverse Fredholm module of $\mathcal{F}$. We have the following well known properties

(3.2) $\text{Ch}(\mathcal{F} \oplus \mathcal{G}) = \text{Ch}(\mathcal{F}) + \text{Ch}(\mathcal{G})$

(3.3) $\text{Ch}(\mathcal{D}) = 0$

(3.4) $\text{Ch}(\mathcal{F}) + \text{Ch}(\mathcal{F}^{-1}) = 0$

(3.5) $\text{Ch}(\mathcal{F}) = \text{Ch}(\mathcal{G})$ whenever $\mathcal{F} \sim_u \mathcal{G}$
for the Chern-Connes characters. Together with the invariance result of Theorem [2,11] these properties imply that there is a well defined homomorphism
\[
\text{Ch} : FK^{m-1}(A) \to HC^{m-1}(A) \quad \text{Ch}([F]) = \text{Ch}(F)
\]
In particular we get a pairing
\[
\tau : FK^{m-1}(A) \times HC_{m-1}(A) \to \mathbb{C} \quad ([F], [x]) \mapsto \tau_F([x])
\]
between finitely summable K-homology and cyclic homology.

Let \( A \) be a locally convex topological algebra and let \( F \) be a continuous \( m \)-summable Fredholm module over \( A \).

We recall from Section 2.4 that \( \text{Ch}_{\text{cont}}(F) = [\tau_{\text{cont}}^F] \in HC_{m-1}^{\text{cont}}(A) \) denotes the class in continuous cyclic cohomology of the index cocycle. For the same reasons as above we get a well defined homomorphism
\[
\text{Ch}_{\text{cont}} : FK^{m-1}_{\text{cont}}(A) \to HC_{m-1}^{\text{cont}}(A) \quad \text{Ch}_{\text{cont}}([F]) = \text{Ch}_{\text{cont}}(F)
\]
in the continuous case as well. See Theorem [2.12]. In particular, we have a pairing
\[
(3.6) \quad \tau_{\text{cont}} : FK^{m-1}_{\text{cont}}(A) \times HC_{m-1}^{\text{cont}}(A) \to \mathbb{C} \quad ([F], [x]) \mapsto \tau_{\text{cont}}^F([x])
\]
between continuous finitely summable \( K \)-homology and continuous cyclic homology. Here \( \tau_{\text{cont}}^F : HC_{m-1}^{\text{cont}}(A) \to \mathbb{C} \) denotes the character on continuous cyclic homology induced by the class \( \text{Ch}_{\text{cont}}(F) \in HC_{m-1}^{\text{cont}}(A) \) in continuous cyclic cohomology.

Remark that both of the Chern-Connes characters which we have defined above are well-behaved with respect to the periodicity operators in (continuous) finitely summable \( K \)-homology and (continuous) cyclic cohomology. This follows by [6, IV 1,β Proposition 2].

Let us proceed with some short reminders on relative \( K \)-theory and the additive character.

In the book [17] M. Karoubi constructs a sequence of covariant functors from the category of Fréchet algebras to the category of abelian groups. For each positive integer \( m \in \mathbb{N} \) and each Fréchet algebra \( A \) the associated abelian group is denoted by \( K_m^{\text{rel}}(A) \) and termed the \( m \)-th relative \( K \)-group of \( A \). These abelian groups measure the difference between algebraic and topological \( K \)-theory.

**Theorem 3.7.** For each Fréchet algebra \( A \) there is a long exact sequence
\[
\ldots \to \partial K_m^{\text{rel}}(A) \to \partial K_m(A) \to K_m^{\text{top}}(A) \to \partial K_m^{\text{rel}}(A) \to \ldots
\]
Here the notation \( K_*(A) \) and \( K_*^{\text{top}}(A) \) refers to the algebraic and topological \( K \)-theory of the Fréchet algebra, respectively.

The relative \( K \)-theory is linked to continuous cyclic homology by means of a relative Chern character.

**Theorem 3.8.** [7, 17] For each Fréchet algebra \( A \) there is a natural homomorphism of degree minus one
\[
\text{ch}^{\text{rel}} : K_m^{\text{rel}}(A) \to HC_{m-1}^{\text{cont}}(A)
\]
from relative \( K \)-theory to continuous cyclic homology. This homomorphism is termed the relative Chern character.
In the above definition the relative Chern character is normalized as in [15]. That is, we have
\[ \text{ch}^{\text{rel}} := (-1)^m (m - 1)! \text{ch}^{\text{rel}} : K^r_m(A) \to HC^\text{cont}_{m-1}(A) \]
where \( \text{ch}^{\text{rel}} \) denotes the Chern character as defined in [7, §4]. The additional constant ensures that the relative Chern character is both multiplicative in an appropriate sense and induced by a chain map. See also [21].

Let \( F \) be a continuous \( m \)-summable Fredholm module over \( A \). We can compose the relative Chern character with the character on continuous cyclic homology induced by \( F \). In this way we obtain the additive character on relative \( K \)-theory,
\[ \mathcal{A}_F : K^r_m(A) \to \mathbb{C} \quad \mathcal{A}_F = c_{m-1} \cdot (r^\text{cont}_F \circ \text{ch}^{\text{rel}}) \]
Here \( c_{m-1} \in \mathbb{Q} \) is the rational constant given by
\[ c_{m-1} = \begin{cases} (-1)^{k+1} \frac{k!}{(2k)!} & \text{for } m - 1 = 2k \text{ even} \\ -\frac{1}{2^{k-1}(k-1)!} & \text{for } m - 1 = 2k - 1 \text{ odd} \end{cases} \]
This constant is chosen in such a way that the multiplicative character which we will define in the next section agrees with the multiplicative character defined in [7].

The existence of the pairing (3.6) implies the following result.

**Theorem 3.9.** For each Fréchet algebra \( A \) the map
\[ \mathcal{A} : FK^r_{\text{cont}}(A) \times K^r_m(A) \to \mathbb{C} \quad \mathcal{A}([F], [\sigma]) \mapsto \mathcal{A}_F([\sigma]) \]
yields a well-defined pairing between continuous finitely summable \( K \)-homology and relative \( K \)-theory.

3.3. The multiplicative character and pairings with algebraic \( K \)-theory. In this subsection we will show that the finitely summable \( K \)-homology groups pair with algebraic \( K \)-theory. It was already noted by A. Connes and M. Karoubi in [7] that each finitely summable Fredholm module gives rise to a multiplicative character on algebraic \( K \)-theory. An important ingredient in their construction is the additive character which we considered in the last section. The well-definedness of the pairing given in Theorem 3.9 will therefore aid us in defining our pairing between finitely summable \( K \)-homology and algebraic \( K \)-theory. Let us start by giving some details on the Connes-Karoubi multiplicative character. To this end, we will need the concept of "universal Fredholm algebras".

Let \( H \) be a fixed separable infinite dimensional Hilbert space and let \( m \in \mathbb{N} \) be a positive integer.

Suppose that \( (m - 1) \) is even. We let \( \mathcal{M}^{m-1} \subseteq \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \) denote the \( \mathbb{C} \)-subalgebra of diagonal operators, \( d(x, y) \), with difference \( x - y \in \mathcal{L}^m(\mathcal{H}) \) in the \( m^{\text{th}} \) Schatten ideal. This \( \mathbb{C} \)-algebra becomes a unital Banach algebra when equipped with the norm
\[ \|x\| = \|x\|_\infty + \|[F_{m-1}, x]\|_m \quad x \in \mathcal{M}^{m-1} \]
Here \( F_{m-1} \) denotes the bounded operator
\[ F_{m-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \]
The notations $\| \cdot \|_\infty$ and $\| \cdot \|_m$ refer to the operator norm and the norm on the $m^{\text{th}}$ Schatten ideal.

Suppose that $(m-1)$ is odd. We let $\mathcal{M}^{m-1} \subseteq \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ denote the $\mathbb{C}$-subalgebra of operators with antidiagonal in the $m^{\text{th}}$ Schatten ideal. This $\mathbb{C}$-algebra becomes a unital Banach algebra when equipped with the norm

$$\| x \| = \| x \|_\infty + \|[F_{m-1}, x]\|_m \quad x \in \mathcal{M}^{m-1}$$

Here $F_{m-1}$ denotes the bounded operator

$$F_{m-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$$

In both the even and the odd case there is a universal $m$-summable Fredholm module over $\mathcal{M}^{m-1}$ given by

$$F_{m-1} = (i, \mathcal{H} \oplus \mathcal{H}, F_{m-1})$$

Here $i : \mathcal{M}^{m-1} \to \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is the inclusion. The grading operator is given by the diagonal matrix $\gamma = d(1, -1)$ when $(m-1)$ is even. Remark that the universal $m$-summable Fredholm module is continuous in the sense of Definition 2.5.

The notation

$$A_{m-1} : K^\text{rel}_m(\mathcal{M}^{m-1}) \to \mathbb{C}$$

will refer to the additive character induced by the universal $m$-summable Fredholm module. See Section 3.2.

The topological $K$-groups of the unital Banach algebras $\mathcal{M}^{m-1}$ are computed by A. Connes and M. Karoubi.

**Lemma 3.10.** [7, Théorème 2.8] For any $m \in \mathbb{N}$ the topological $K$-theory of $\mathcal{M}^{m-1}$ is given by

$$K^\text{top}_0(\mathcal{M}^{m-1}) = \begin{cases} \{0\} & \text{for } (m-1) \text{ odd} \\ \mathbb{Z} & \text{for } (m-1) \text{ even} \end{cases}$$

$$K^\text{top}_1(\mathcal{M}^{m-1}) = \begin{cases} \mathbb{Z} & \text{for } (m-1) \text{ odd} \\ \{0\} & \text{for } (m-1) \text{ even} \end{cases}$$

In the case of the unital Banach algebra $\mathcal{M}^{m-1}$ the long exact sequence of $K$-groups from Theorem 3.7 then reads

$$\ldots \longrightarrow K_{m+1}(\mathcal{M}^{m-1}) \longrightarrow \mathbb{Z} \longrightarrow K^\text{rel}_m(\mathcal{M}^{m-1}) \longrightarrow K_m(\mathcal{M}^{m-1}) \longrightarrow 0$$

In particular, the homomorphism $\theta : K^\text{rel}_m(\mathcal{M}^{m-1}) \to K_m(\mathcal{M}^{m-1})$ is surjective. Furthermore, since both the boundary map and the universal additive character are homomorphisms we get that

$$\text{Im}(A_{m-1} \circ \partial) = c\mathbb{Z} \subseteq \mathbb{C}$$

where $c \in \mathbb{C}$ is some constant. In [7] this constant is calculated to be $c = (2\pi i)^{\lceil m/2 \rceil}$. These results allow us to define the universal multiplicative character.
Definition 3.11. By the universal $m$-summable multiplicative character we will understand the homomorphism

$$\mathcal{M}_{m-1} : K_m(M^{m-1}) \to \mathbb{C}/(2\pi i)^{[m/2]}\mathbb{Z}$$

given by

$$\mathcal{M}_{m-1}(\theta(y)) = [A_{m-1}(y)]$$

Here $[\cdot] : \mathbb{C} \to \mathbb{C}/(2\pi i)^{[m/2]}\mathbb{Z}$ denotes the quotient map.

Let $A$ be an algebra over $\mathbb{C}$ and let $\mathcal{F}$ be an $m$-summable Fredholm module over $A$. Suppose that $(m-1)$ is odd and let $P = (F+1)/2$. Since the Hilbert space $PH$ and $(1-P)H$ are infinite dimensional by assumption, we can identify both of them with the "universal" Hilbert space $\mathcal{H}$. In this way we obtain an algebra homomorphism

$$\pi_{\mathcal{F}} : A \to \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$$

which is well defined up to conjugation by a diagonal unitary operator.

Suppose that $(m-1)$ is even. By assumption both of the eigenspaces $H^+$ and $H^-$ for the grading operator $\gamma \in \mathcal{L}(H)$ are infinite dimensional. We can thus identify each of them with the "universal" Hilbert space $\mathcal{H}$. In this way we obtain a unital algebra homomorphism

$$\pi_{\mathcal{F}} : A \to \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$$

which is well defined up to conjugation by a diagonal unitary operator.

In both the even and the odd case the condition on the commutator $[\mathcal{F}, \pi(a)] \in \mathcal{L}^m(H)$ ensures that the algebra homomorphism $\pi_{\mathcal{F}}$ factorizes through the Banach algebra $M^{m-1}$. The algebra homomorphism $A \to M^{m-1}$ will also be denoted by $\pi_{\mathcal{F}}$. Remark that this algebra homomorphism is continuous whenever $A$ is a locally convex topological algebra and $\mathcal{F}$ is continuous.

Remark 3.12. The terminology "universal", which we have used for the $m$-summable Fredholm module $\mathcal{F}_{m-1}$, is justified by the following identity in $FK^{m-1}(A)$,

$$FK(\pi_{\mathcal{F}})([\mathcal{F}_{m-1}]) = [\mathcal{F}]$$

There is a similar identity in the continuous case as well.

We are now in position to give the general definition of the multiplicative character.

Definition 3.13. By the multiplicative character of $\mathcal{F}$ we will understand the homomorphism

$$\mathcal{M}_\mathcal{F} : K_m(A) \to \mathbb{C}/(2\pi i)^{[m/2]}\mathbb{Z}$$

$$\mathcal{M}_\mathcal{F} = \mathcal{M}_{m-1} \circ (\pi_{\mathcal{F}})_*$$

Here $(\pi_{\mathcal{F}})_* : K_m(A) \to K_m(M^{m-1})$ denotes the homomorphism on algebraic $K$-theory induced by the algebra homomorphism $\pi_{\mathcal{F}} : A \to M^{m-1}$.

Let $\mathcal{F}$ and $\mathcal{G}$ be two $m$-summable Fredholm modules over $A$ and let $[x] \in K_m(A)$. The following properties for the multiplicative character are proved in [7],

$$\mathcal{M}_{\mathcal{F} \oplus \mathcal{G}}([x]) = \mathcal{M}_{\mathcal{F}}([x]) + \mathcal{M}_{\mathcal{G}}([x])$$

$$\mathcal{M}_{\mathcal{F}^{-1}}([x]) = -\mathcal{M}_{\mathcal{F}}([x])$$

$$\mathcal{M}_{\mathcal{F}}([x]) = \mathcal{M}_{\mathcal{G}}([x])$$

whenever $\mathcal{F} \sim_u \mathcal{G}$.
We would like to show that the multiplicative character is invariant under stable \( m \)-summable perturbations. See Definition 3.4. Indeed, this would ensure us that the map

\[
\mathcal{M} : FK^{m-1}(A) \times K_m(A) \to \mathbb{C}/(2\pi i)^{m/2}\mathbb{Z} \quad \mathcal{M}([F], [x]) = \mathcal{M}_F([x])
\]
yields a well-defined pairing between finitely summable \( K \)-homology and algebraic \( K \)-theory.

In view of the above relations we will only need to consider the behaviour of the multiplicative character with respect to finitely summable perturbations and degeneracies. The case of degeneracies is carried out by the next lemma.

**Lemma 3.14.** Suppose that \( D \) is an \( m \)-summable degenerate Fredholm module over \( A \). Then the homomorphism of abelian groups

\[(\pi_D)_* : K_m(A) \to K_m(\mathcal{M}^{m-1})\]

vanishes identically. In particular, the multiplicative character associated to \( D \) is trivial.

**Proof.** Suppose that \((m - 1)\) is odd. The induced algebra homomorphism

\[
\pi_D : A \to \mathcal{M}^{m-1}
\]
is then diagonal. In particular it factorizes through the direct sum of bounded operators \( \mathcal{L}(\mathcal{H}) \oplus \mathcal{L}(\mathcal{H}) \). But this unital ring has trivial algebraic \( K \)-theory by [18].

In the even case the induced algebra homomorphism factorizes through \( \mathcal{L}(\mathcal{H}) \) so the same argument applies. \( \square \)

Suppose that we have two \( m \)-summable Fredholm modules \( F_1 = (\pi, H, F_1) \) and \( F_2 = (\pi, H, F_2) \) over \( A \) which are \( m \)-summable perturbations of each other. In order to show that their multiplicative character coincide we spell out the effect of the perturbation at the level of the associated algebra homomorphisms \( \pi_{F_1} : A \to \mathcal{M}^{m-1} \) and \( \pi_{F_2} : A \to \mathcal{M}^{m-1} \).

**Lemma 3.15.** There exists a continuous \( m \)-summable Fredholm module \( T \) over \( \mathcal{M}^{m-1} \) such that

1. The diagram of algebra homomorphisms

\[
\begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
\pi_{F_1} & & \pi_{F_2} \\
\downarrow & & \downarrow \\
\mathcal{M}^{m-1} & \xrightarrow{\pi_T} & \mathcal{M}^{m-1}
\end{array}
\]

is commutative.

2. The continuous \( m \)-summable Fredholm module \( T \) is an \( m \)-summable perturbation of the universal \( m \)-summable Fredholm module.

**Proof.** By assumption the difference \( F_2 - F_1 = T \in \mathcal{L}^m(H) \) lies in the \( m \)th Schatten ideal.

Suppose that \((m - 1)\) is odd. Let \( P_1 = \frac{F_1 + 1}{2} \) and let \( P_2 = \frac{F_2 + 1}{2} \). Let \( u_1 : P_1 H \oplus (1 - P_1)H \to \mathcal{H} \oplus \mathcal{H} \) and \( u_2 : P_2 H \oplus (1 - P_2)H \to \mathcal{H} \oplus \mathcal{H} \) denote some diagonal unitary operators. The algebra homomorphisms

\[
\pi_{F_1} : A \to \mathcal{M}^{m-1} \quad \text{and} \quad \pi_{F_2} : A \to \mathcal{M}^{m-1}
\]
are then given by
\[ \pi_{\mathcal{F}_1} = u\pi u^* \quad \text{and} \quad \pi_{\mathcal{F}_2} = v\pi v^* \]
We define the continuous \( m \)-summable Fredholm module \( \mathcal{T} \) over \( \mathcal{M}^{m-1} \) by \( \mathcal{T} := (i, \mathcal{H} \oplus \mathcal{H}, uF_2 u^*) \). Clearly \( \mathcal{T} \) is an \( m \)-summable perturbation of the universal \( m \)-summable Fredholm module. Furthermore, the algebra homomorphism associated to \( \mathcal{T} \) is given by
\[ \pi_{\mathcal{T}} : \mathcal{M}^{m-1} \rightarrow \mathcal{M}^{m-1} \quad \pi_{\mathcal{T}}(x) = vu^*xuv^* \]
This proves the odd case of the lemma. The even case follows by similar considerations. \( \square \)

We are now in position to prove the main result of this paper.

**Theorem 3.16.** For each positive integer \( m \in \mathbb{N} \) there is a well-defined pairing
\[ FK^{m-1}(A) \times K_m(A) \rightarrow \mathbb{C}/(2\pi i)^{[m/2]}\mathbb{Z} \]
given by the formula
\[ ([\mathcal{F}], [x]) \mapsto \mathcal{M}_{\mathcal{F}}([x]) \quad [\mathcal{F}] \in FK^{m-1}(A), [x] \in K_m(A) \]

**Proof.** As mentioned earlier, we will only need to prove that the multiplicative character is invariant under \( m \)-summable perturbations. Thus, let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two \( m \)-summable Fredholm modules over \( A \) which are \( m \)-summable perturbations of each other. By Lemma 3.15 we have a continuous \( m \)-summable Fredholm module \( \mathcal{T} \) which is an \( m \)-summable perturbation of the universal \( m \)-summable Fredholm module. Furthermore, the diagram (3.8) of algebra homomorphisms is commutative. Now, let \([x] \in K_m(A)\). By definition of the multiplicative character associated with \( \mathcal{F}_1 \) we have that
\[ \mathcal{M}_{\mathcal{F}_1}([x]) = \mathcal{M}_{m-1}((\pi_{\mathcal{F}_1})_*(x)) = [A_{m-1}([y])] \in \mathbb{C}/(2\pi i)^{[m/2]}\mathbb{Z} \]
Here \([y] \in K_m^\text{rel}(\mathcal{M}^{m-1})\) is any element such that
\[ \theta([y]) = (\pi_{\mathcal{F}_1})_*(x) \in K_m(\mathcal{M}^{m-1}) \]
By naturality of the homomorphism \( \theta : K_m^\text{rel}(\mathcal{M}^{m-1}) \rightarrow K_m(\mathcal{M}^{m-1}) \) and the commutativity of the diagram (3.8) we then get that
\[ (\theta \circ (\pi_{\mathcal{T}})_*)([y]) = (\pi_{\mathcal{F}_2})_*([x]) \]
We can thus deduce that
\[ \mathcal{M}_{\mathcal{F}_2}([x]) = [A_{m-1}((\pi_{\mathcal{T}})_*[y])] = [A_T([y])] \in \mathbb{C}/(2\pi i)^{[m/2]}\mathbb{Z} \]
But \( \mathcal{T} \) was an \( m \)-summable perturbation of the universal \( m \)-summable Fredholm module. The desired result therefore follows from Theorem 3.9. \( \square \)

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