Toy model for a two-dimensional accretion disk
dominated by Poynting flux

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Abstract

We discuss the effect of the Poynting flow on the magnetically dominated thin accretion disk, which is simplified to a two-dimensional disk on the equatorial plane. It is shown in the relativistic formulation that the Poynting flux by the rotating magnetic field with Keplerian angular velocity can balance the energy and angular momentum conservation of a steady accretion flow.

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I. INTRODUCTION

The Poynting flux model [1] [2] suggested for an accretion disk with the ordered magnetic fields has been considered to be one of the viable models for the astrophysical jets [3] [4]. In contrast to the hydrodynamic jets in which the energy and angular momentum are carried by the kinetic flux of the matter, the Poynting flux is characterized by the outflow of energy and angular momentum carried predominantly by the electromagnetic field.

In the non-relativistic formulation, Blandford [2] suggested an axisymmetric and steady solution for the Poynting outflow assuming a force-free magnetosphere surrounding an accretion disk. The poloidal field configuration for a black hole in a force-free magnetosphere has been discussed recently by Ghosh [5] in the relativistic formulation, in which the possible forms of the poloidal magnetic fields are suggested. The developments of the ordered magnetic field and the Poynting outflow on the disk have been studied by many authors [6] [7] [8] and recently Ustyugova et al. [4] perform a axisymmetric magnetohydrodynamical simulation to show that the quasi-stationary and approximately force-free Poynting jet from the inner part of the accretion disk is possible.

The Poynting flux in a system of black hole-accretion disk recently has also been studied in connection to the gamma ray bursts [9] [10] [11]. One of the main reasons is that the Poynting flux carries very small baryonic component, which is essential for powering the gamma ray bursts [12]. The evolution of the system is found to be largely depend on the Poynting flux on the disk [13] [14]. The relativistic effects on the accreting flows close to a black hole has been discussed by Lasota [13], Abramowicz et al. [16] and recently by Gammie and Popham [17] in which the relativistic effects on the slim-disk with the vertical structures averaged are discussed in detail with viscous stresses. However the effect of relativity on the accreting flow dominated by the Poynting flux has not been discussed well in depth so far.

The purpose of this work is to study the effect of the Poynting flux on the accretion flow in the relativistic formulation. In this work, we consider a toy model for a magnetically dominated thin accretion disk, which is assumed to be a two-dimensional disk located on the
equatorial plane with a black hole at the center. To see the magnetic effect transparently it is also assumed that there is no viscous stress tensor in the disk and there is no radiative transfer from the disk. We develop a relativistic description of the two-dimensional model for a magnetically dominated thin accretion disk in the background Kerr geometry. It is shown that the energy balance of the accretion disk for a stationary accretion flow can be maintained by the Poynting flux, provided that the poloidal magnetic field is rotating with the same Keplerian angular velocity $\Omega_K$ on the disk.

II. TWO-DIMENSIONAL ACCRETION FLOW

The stress-energy tensor $T^{\mu\nu}_m$ of the matter in the disk can be given in the general form

$$T^{\mu\nu}_m = (\rho_m + p + \Pi)u^\mu u^\nu + pg^{\mu\nu} + S^{\mu\nu} + u^\mu q^\nu + u^\nu q^\mu,$$

(1)

where $\rho_m$, $\Pi$, and $p$ are the rest-mass density, internal energy and pressure respectively. $S^{\mu\nu}$ is the viscous tensor and $q^\mu$ is the radiative energy flux [16].

In this work we simplify the accretion disk to be non-viscous ($S^{\mu\nu} = 0$), cool ($p = 0, \Pi = 0$) and non-radiative ($q^\mu = 0$) disk to investigate the effect of the magnetic field transparently. It is also assumed that there is negligible mass flow in the direction perpendicular to the disk: $u^\theta = 0$. Then the stress-energy tensor is given simply by

$$T^{\mu\nu}_m = \rho_m u^\mu u^\nu.$$

(2)

The energy flux is obtained by

$$E^{\mu}_{(m)} = -\rho_m u_0 u^\mu,$$

(3)

and the angular momentum flux by

$$L^{\mu}_{(m)} = \rho_m u_\phi u^\mu.$$

(4)

For an idealized thin disk on two-dimensional plane we take
\[ \rho_m = \frac{\sigma_m}{\rho} \delta(\theta - \pi/2), \]  

(5)

where \(\sigma_m\) is the surface rest-mass density. In this work, the background geometry is assumed to be Kerr metric (see Appendix A) with rotating black hole at the center and we adopt the natural unit \(G = c = 1\).

The rate of the rest-mass flow crossing the circle of radius \(r\) is given by

\[ \int r \alpha \rho_m u r^\rho \overline{\rho} \sqrt{\Delta} \, d\theta d\phi = 2\pi \sigma_m \rho u r, \]  

(6)

which defines the mass accretion rate \(\dot{M}_+\) by

\[ \dot{M}_+ = -2\pi \sigma_m \rho u r. \]  

(7)

This expression is identical to that derived in [16] and [17], in which the vertical structures are averaged. The stationary accretion flow implies that \(\dot{M}_+\) is \(r\)-independent.

Using eqs. (3) and (4) the radial flow of the matter energy at \(r\) can be given by

\[ \int r \alpha \mathcal{E}_{(r)} r^\rho \overline{\rho} \sqrt{\Delta} \, d\theta d\phi = -2\pi \sigma_m \rho u r \, u_0 = u_0 \dot{M}_+, \]  

(8)

and the radial flow of the matter angular momentum by

\[ \int r \alpha \mathcal{L}_{(r)} r^\rho \overline{\rho} \sqrt{\Delta} \, d\theta d\phi = 2\pi \sigma_m \rho u r \, u_\phi = -u_\phi \dot{M}_+. \]  

(9)

For the magnetosphere outside the accretion disk, there are electromagnetic currents, \(J^\mu\), which can flow through the central object at the center and also along the magnetic field lines anchored on the disk. In this work we suppose a situation in which the currents flow into the inner edge of the disk through the central object and the currents flows out along the magnetic field lines from the disk as discussed by Blandford and Znajek [2] [18]. The continuity of the currents (current conservation) necessarily requires surface currents in radial direction on the disk. Similar consideration has been applied to the black hole horizon as discussed by Thorne et al. [13].

In general the structure of the electromagnetic field with discontinuity at \(\theta = \pi/2\) plane can be reproduced by assigning the surface charge and current on the plane in addition to
the bulk charge and current distributions which terminate on the disk. The surface charge density $\sigma_e$ and surface current density $K^i$ can be defined systematically using the procedure suggested by Damour [20] (See Appendix B for details):

$$\sigma_e = -\frac{E^\theta}{4\pi},$$  \hspace{1cm} (10)

$$K^r = -\frac{1}{4\pi} B^\phi, \quad K^\phi = \frac{1}{4\pi} B^r.$$  \hspace{1cm} (11)

### III. POYNTING FLUX

The energy and angular momentum of the disk can be carried out by Poynting flux along the magnetic field lines anchored on the disk. In our simple model, Poynting flux is the main driving force for the accretion flow. We calculate the Poynting flux on a two-dimensional disk assuming a force-free magnetosphere around the accretion disk.

Using the Killing vector in $t$-direction,

$$\xi^\mu = (1, 0, 0, 0),$$  \hspace{1cm} (12)

we can define the energy flux $E^\mu$ from the energy momentum tensor, $T^{\mu\nu}$

$$E^\mu = -T^{\mu\nu}\xi_\nu = (\alpha^2 - \omega^2 \beta)T^{\mu0} - \omega^2 \beta T^{\mu\phi},$$  \hspace{1cm} (13)

$$E^\mu\cdot T^\mu = 0,$$  \hspace{1cm} (14)

where

$$T^{\mu\nu} = \frac{1}{4\pi}(F^\mu_\rho F^{\nu\rho} - \frac{1}{4}g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}).$$  \hspace{1cm} (15)

Since we are interested in $\theta$-direction (normal to the disk on the equatorial plane), we consider $E^\theta$ for which the second term in eq.(13) vanishes:

$$E^\theta = \frac{1}{4\pi}(\alpha^2 - \omega^2 \beta) F^\theta_\rho F^{\rho\theta} - \frac{1}{4\pi} \omega^2 \beta F^\theta_\rho F^{\rho\phi}.$$  \hspace{1cm} (16)

Using the identities given in Appendix A, it can be rewritten in terms of electric and magnetic fields as given by
\[ \mathcal{E}^\theta = \frac{1}{4\pi \rho}[\alpha \hat{E} \times \hat{B}]_\theta + \beta \varpi (E^\theta E^\phi + B^\phi B^\theta)]. \]  

(17)

The power measured at infinity can be obtained using eq.(17) by

\[ P_{\text{energy}}^\theta = \int_{\theta=\pi/2}^{\theta=\pi} \alpha \mathcal{E}^\theta d\Sigma_\theta = \int \frac{\alpha}{4\pi} [\alpha (\vec{E} \times \vec{B})_\theta + \varpi \beta (E^\theta E^\phi + B^\phi B^\theta)] \frac{\rho \varpi}{\sqrt{\Delta}} d\theta d\phi. \]  

(18)

When the tangential components of the electromagnetic field are multiplied by the laps function \( \alpha \) as in the membrane paradigm \([19]\), we can hide \( \alpha \) in the integrand.

For the steady and axisymmetric case which we are interested in, \( E^\phi = 0 \), we get

\[ \mathcal{E}^\theta = \frac{1}{4\pi \rho}(-\alpha E^r B^\phi + \beta \varpi B^\phi B^\theta). \]  

(19)

The first term in the integrand in eq.(18) can be rewritten in terms of the surface current density: Using eq.(71) we get

\[ \frac{1}{4\pi} (\vec{E} \times \vec{B})_\theta = \vec{E} \cdot \vec{K}. \]  

(20)

For the current in the direction of tangential electric field on the disk there is a energy dissipation into the disk surface. This is what one can expect on the black hole horizon \([19]\). However for the current in the opposite direction this term corresponds to the electro-motive force and it is the case for the accretion disk on the equatorial plane discussed in Blandford and Znajek \([2\) \([18]\) as well as in this work.

Similarly the third term in eq.(18)(equivalently the second term in eq.(19)) can be rewritten in terms of the surface current using eq.(71) as given by

\[ \frac{1}{4\pi} \omega \beta B^\phi B^\theta = \omega \varpi K^\phi B^\theta. \]  

(21)

We can see that this is the magnetic braking power on the rotating body with an angular velocity \( \omega = -\beta \).

Using the Killing vector in \( \phi \)-direction for the axial symmetric case,

\[ \eta^\mu = (0, 0, 0, 1), \]  

(22)

we can define the angular momentum flux \( \mathcal{L}^\mu \) from the energy momentum tensor, \( T^{\mu\nu} \).
\[ \mathcal{L}^\mu = T^{\mu\nu} \eta_\nu = \omega^2 \beta T^{\mu 0} + \omega^2 T^{\mu \phi}, \] (23)

\[ \mathcal{L}^\mu ;_{\mu} = 0, \] (24)

Then we get the flux in \( \theta \)-direction (normal to the disk on the equatorial plane), \( \mathcal{L}^\theta \),

\[ \mathcal{L}^\theta = -\frac{\omega}{4\pi \rho} \hat{B}^\theta \hat{B}^\phi = \frac{1}{\rho} \omega K^i \hat{B}^i, \] (25)

which is nothing but a torque applied on the surface current density \( K^r \). Eq.(17) can be written as

\[ \mathcal{E}^\theta = \frac{1}{4\pi \rho} \alpha \hat{E} \times \hat{B} |_\theta - \beta \mathcal{L}^\theta. \] (26)

In this work we assume that there is a sufficient ambient plasma around the disk to maintain the force-free magnetosphere \[2\] \[4\]. The force-free condition for the magnetosphere with the current density \( J^\mu \) is given by

\[ F_{\mu\nu} J^\mu = 0, \] (27)

The magnetic flux, \( \Psi \), through a circuit encircling \( \phi = 0 \rightarrow 2\pi \),

\[ \Psi = \oint A_\phi d\phi = 2\pi A_\phi, \] (28)

defines a magnetic surface on which \( A_\phi(r, \theta) \) is constant and therefore is characterized by the magnetic flux \( \Psi \) contained inside it. From the force-free condition it can be shown that \( A_0 \) is also constant along the magnetic field lines and the electric field is always perpendicular to the magnetic surface. We can define a function \( \Omega_F(r, \theta) \),

\[ dA_0 = -\Omega_F dA_\phi, \] (29)

which is also constant along the magnetic surface. \( \Omega_F \) can be identified as an angular velocity of magnetic field line on a magnetic surface \[18\] \[19\].

Then we get from the force-free condition

\[ \bar{E} = -\omega \alpha (\Omega_F + \beta) \hat{\phi} \times \hat{B}, \] (30)
and the Poynting flux perpendicular to the disk can be written as

$$\mathcal{E}^\theta = -\frac{\Omega_F \varpi}{4\pi \rho} B^\theta B^\phi = \Omega_F \mathcal{L}_\phi^\theta, \tag{31}$$

as in [18].

**IV. ENERGY AND ANGULAR MOMENTUM CONSERVATION**

In the idealized thin disk considered in this work, it is assumed that there is no radiative transfer or viscous interaction which balance the radial flow of the energy and angular momentum of the accreting matter. Hence the electromagnetic field anchored on the disk is responsible for the conservation of the total energy and angular momentum. Since we assume the idealized two-dimensional disk, there is no radial flow of electromagnetic energy and angular momentum unless there is any singular structure of electromagnetic field on the disk.

Let us consider a circular strip at $r$ with infinitesimally small width $\delta r$. Using eq. (25) the angular momentum flux of electromagnetic field into the $\theta$-direction is given by

$$\Delta L = 2 \int_{r}^{r+\delta r} \alpha \mathcal{L}_\theta d\Sigma_\theta = -B^\theta B^\phi \rho \varpi \delta r, \tag{32}$$

where the factor 2 is introduced to take account of both sides of the disk. It is balanced by the change of the angular momentum of matter given by

$$\Delta L_m = -u_\phi \dot{M}_+ |_{r+\delta r} \rightarrow -\frac{du_\phi}{dr} \dot{M}_+ \delta r. \tag{33}$$

Then we get

$$\dot{M}_+ = \frac{B^\theta B^\phi \rho \varpi}{du_\phi/dr}. \tag{34}$$

In the non-relativistic limit $u_\phi$ can be identified as non-relativistic Keplerian angular momentum: $\Omega_K = \sqrt{M/r^3}$ and we get

$$\dot{M}_+ \rightarrow \frac{B^\theta B^\phi r^2}{r \Omega_K^2/2} = 2r \frac{B^\theta B^\phi}{\Omega_K}, \tag{35}$$
which is identical to that used in [13] and [4].

Similarly the energy flux of electromagnetic field in $\theta$-direction is given by

$$\Delta E = 2 \int_{r}^{r+\delta r} \alpha \mathcal{E}^\theta d\Sigma_\theta = -\Omega_F B^\theta B^\phi \rho \varpi \delta r$$

$$= \Omega_F \Delta L,$$

where factor 2 is introduced for the same reason as in the angular momentum flux. It is balanced by the change of the energy of the matter given by

$$\Delta E_m = -(u_0) \dot{M}_+ |_{r+dr} \rightarrow \frac{du_0}{dr} \dot{M}_+ \delta r,$$

which is possible with $\Omega_F$ given by

$$\Omega_F = -\frac{du_0/dr}{du_\phi/dr},$$

where eq.(34) has been used.

It is not a trivial task to find the solution of the full relativistic MHD equations satisfying eq.(7) and (34) which leads to

$$B^\theta B^\phi = -2\pi \sigma_m \rho u^r \frac{du_\phi}{dr},$$

together with eq.(39). Without solving the full MHD equation, the magnetic fields and its angular velocity $\Omega_F$ can be considered to be unknown parameters as well as $u_i$.

As a first trial, we assume $u_0$ and $u_\phi$ to be those of the stable orbit of a test particle around a Kerr black hole [21]:

$$-u_0 = \frac{r^2 - 2Mr + a\sqrt{Mr}}{r\sqrt{r^2 - 3Mr + 2a\sqrt{Mr}}},$$

$$u_\phi = \frac{\sqrt{Mr}(r^2 - 2a\sqrt{Mr} + a^2)}{r\sqrt{r^2 - 3Mr + 2a\sqrt{Mr}}},$$

although it is natural to expect non-negligible effect of the magnetic field on the stable orbit. The straightforward calculation, eq.(39), shows a very interesting result that the angular velocity $\Omega_F$ of the magnetic field is the same as the Keplerian angular velocity $\Omega_K(\equiv u^\phi/u^0)$.
\[ \Omega_F = \Omega_K, \]  

(42)

where

\[ \Omega_K = \sqrt{\frac{M}{r^3}} \frac{1}{1 + a \sqrt{M/r^3}}. \]  

(43)

In fact one can derive eq. (42) formally without using the explicit form of eq. (41), since it can be shown that

\[ u^0 \frac{du_0}{dr} + u^\phi \frac{du_\phi}{dr} = 0 \]  

(44)

holds in general for a stable circular orbit on the equatorial plane. This shows that the energy balance can be realized by the magnetic field rotating rigidly with Keplerian angular velocity in this simplified two-dimensional disk model.

It is also possible to show that eq. (4.2) in [2] can be obtained from eq. (34) in the non-relativistic limit. Hence for a Newtonian limit or for the outer radius of the thin disk, the configuration of the electromagnetic field suggested by Blandford [2] is consistent with our result when the the magnetic field angular velocity \( \Omega_F \) is given by \( \Omega_K \).

V. POWER OUT OF A MAGNETICALLY DOMINATED THIN ACCRETION DISK

The total power out of the disk, \( P_{\text{disk}} \), can be calculated by integrating eq. (36) from the inner most stable point \( r_{\text{in}} \) to outer most edge of the disk with Poynting flux \( r_{\text{out}} \),

\[ P_{\text{disk}} = - \int_{r_{\text{in}}}^{r_{\text{out}}} (-\Omega_F B^\theta \dot{B}^\rho \rho \omega) dr, \]  

(45)

where \(-\) sign is referring the outward direction on the disk.

Using the accretion rate eq. (34), which is independent of \( r \) for a stationary accretion flow, and the energy balance condition eq. (34), we get

\[ P_{\text{disk}} = \int_{r_{\text{in}}}^{r_{\text{out}}} \Omega_F \dot{M}_+ \frac{d\dot{u}_\phi}{dr} dr = \dot{M}_+ \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{d(-u_0)}{dr} dr, \]  

(46)

\[ = [-u_0(r_{\text{out}})] \dot{M}_+ - [-u_0(r_{\text{in}})] \dot{M}_+. \]  

(47)
The first term corresponds to the energy rate into the disk at the outer most edge of the
disk and the second term corresponds to the energy accretion rate at the inner edge of the
disk to the central object. This is the expected result from the energy conservation.

Eq. (47) indicates that the power out of the disk in this toy model is not dependent
strongly on the details of the electromagnetic field configuration. The power can be cal-
culated once $\dot{M}_+$ is known at one point $r$ on the disk. For example, suppose that the
exact solution of the relativistic equation can be approximated at a large distance $r_0$ by the
configuration suggested by Blandford [2], then we can use the relation, for example,

$$B^{\phi} = 2\Omega_F r B^{\theta}, \quad (48)$$

to get the accretion rate expressed in terms of the magnetic field component perpendicular
to the disk,

$$\dot{M}_+ = 4(B^{\theta}(r_0)r_0)^2. \quad (49)$$

Then the total power for a disk with Poynting flux extended to $r_{out} = \infty$ can be given by

$$P_{disk} = 4(1 - [-u_0(r_{in})])(B^{\theta}(r_0)r_0)^2, \quad (50)$$

where we take $-u_0(\infty) = 1$.

VI. DISCUSSION

In this work, we discuss a toy model for the magnetically dominated thin accretion disk.
Assuming a two-dimensional disk dominated by the Poynting flux where the viscous stress
and radiative transfer are ignored, the accretion flow in two-dimension is discussed in the
background of Kerr geometry. We have demonstrated that the stationary accretion can be
possible with the co-rotating magnetic field with the same Keplerian angular velocity as that
of matters in the disk in the stable orbit. Also it is observed that the solution proposed by
Blandford [2] is consistent with our result in the non-relativistic limit( or at a large distance
$r$ from the center with sufficiently small $a/r$ and $a/M$). It is shown in this toy model that
the total Poynting power out of the disk depends on the accretion rate and the energy at the inner most stable orbit but not on the details of the electromagnetic field configuration.

Related issues to be discussed in the future are the possible solutions of magnetic field configuration (not only the poloidal component of the magnetic field discussed by Ghosh [3] but also the toroidal component) analogous to [2] in the non-relativistic limit and the effect of the magnetic field on the energy \((-u_0)\) and angular momentum \((u_\phi)\) of a particle in the stable orbit which has not been taken into account in this work.

We have thus far discussed only one specific aspect, Poynting flux, of the accretion disk. For this two-dimensional disk model to be realistic and viable one, there are many obstacles to overcome. For example the viscous stress and radiation transfer should be included, which naturally leads to the questions on the steady state [1] and the thin disk approximation assumed in this work. Moreover recent works [22] [23] [24] on the accretion flow toward a black hole as a central object and on the magnetic coupling [14] between the disk and the black hole seem to indicate that the physics is more complex than the simplified two-dimensional model discussed in this work particularly near the inner region of the disk.

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Appendix A : Kerr geometry

In this Appendix, the useful identities expressed in terms of explicit orthonormal components defined by ZAMO [13] are listed. Using the Boyer-Lindquist coordinates [25] in the natural unit \(G = c = 1\), the metric tensor for Kerr Geometry [26], \(g_{\mu\nu}\), is given by
\[
(g_{\mu\nu}) = \begin{pmatrix}
-(\alpha^2 - \varpi^2\beta^2) & 0 & 0 & \varpi^2\beta \\
0 & \frac{\rho^2}{\Delta} & 0 & 0 \\
0 & 0 & \rho^2 & 0 \\
\varpi^2\beta & 0 & 0 & \varpi^2
\end{pmatrix},
\]

where

\[
\alpha = \frac{\rho \sqrt{\Delta}}{\Sigma}, \quad \beta = \frac{g_{\theta\phi}}{g_{\phi\phi}}, \quad \tilde{\omega} = \frac{\Sigma}{\rho} \sin \theta,
\]

\[
\Delta = r^2 + a^2 - 2Mr, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta,
\]

ZAMO’s four velocity (a time-like unit vector orthogonal to the t-constant surface: \(dx^\alpha U_\alpha = 0\)) is given by

\[
U_\mu = -\alpha(1, 0, 0, 0).
\]

The electromagnetic field tensor can be expressed by the electric and magnetic fields as given by

\[
F_{\theta\phi} = \frac{\varpi}{\rho} B^\phi, \quad F_{\theta r} = -\frac{1}{\sqrt{\Delta}} B^\phi, \quad F_{\theta 0} = \frac{1}{\rho^2} (\alpha \rho E^\phi + \beta \varpi \varpi B^r),
\]

\[
F_{0 r} = \sqrt{\Delta} \frac{\alpha \rho}{\varpi} E^\phi, \quad F_{0 \phi} = \frac{1}{\alpha \varpi} E^\phi, \quad F_{\phi r} = -\frac{\beta \sqrt{\Delta}}{\alpha \rho} E^\phi + \frac{\sqrt{\Delta}}{\varpi \rho} B^\phi.
\]

Appendix B : Surface current on a two-dimensional accretion Disk

Consider a conserved current \(J^\mu\) defined by

\[
J^\mu = J^\mu Y(\theta - \pi/2) + j^\mu,
\]

where the conserved current \(J^\mu\) is the bulk current density satisfying the Maxwell equation,

\[
F_{\mu \nu} = 4\pi J^\mu, \quad J^{\mu}_{\ ; \mu} = 0
\]

and \(Y(\theta - \pi/2)\) satisfies

\[
Y(\theta - \pi/2)_{,\mu} = -\delta_{\theta \mu} \delta(\theta - \pi/2)
\]
From the conservation of the current,

\[ \mathcal{J}_\mu = 0, \quad (59) \]

we get,

\[ j^\mu = \frac{1}{4\pi} F^{\theta\mu} \delta(\theta - \pi/2). \quad (60) \]

Then we can identify

\[ j^\mu = \frac{1}{4\pi} F^{\theta\mu} \delta(\theta - \pi/2), \quad (61) \]

since \( F^{\theta\theta} \) vanishes identically. It is an analogous expression as in [20].

Now the charge density defined by

\[ \tilde{\rho}_e = \alpha j^0 = \alpha \frac{F^{\theta0}}{4\pi} \delta(\theta - \pi/2), \quad (62) \]

can be rewritten as surface charge density \( \sigma_e \):

\[ \int \tilde{\rho}_e dV \equiv \int \sigma_e \rho \sqrt{\Delta} dr d\phi. \quad (63) \]

Then we get the surface charge density in terms of the electric field given in Appendix A:

\[ \sigma_e = -\frac{E^\theta}{4\pi}, \quad (64) \]

It is Gauss' law on the disk plane:

\[ E^\phi = -4\pi \sigma_e. \quad (65) \]

Similarly using the current density defined by

\[ \tilde{j}^i = j^i - j^0(0, 0, -\beta), \quad i = r, \theta, \phi \quad (66) \]

we get

\[ \tilde{j}^r = -\frac{1}{4\pi} \frac{B^\phi \sqrt{\Delta}}{\rho^2} \delta(\theta - \pi/2), \quad \tilde{j}^\phi = \frac{1}{4\pi} \frac{B^r}{\rho \cos \theta} \delta(\theta - \pi/2), \quad (67) \]
where $j^\theta = 0$ by construction. The radial surface current density $K^r$ and the surface current density in $\phi$-direction $K^\phi$ on the disk can be defined by

$$
\int \alpha j^r d\Sigma_r = -\frac{1}{4\pi} \int \alpha B^\phi \varpi d\phi \equiv \int \alpha K^r \varpi d\phi, \quad (68)
$$

$$
\int \alpha j^\phi d\Sigma_\phi = \frac{1}{4\pi} \int \alpha B^r \frac{\rho}{\sqrt{\Delta}} dr \equiv \int \alpha K^\phi \frac{\rho}{\sqrt{\Delta}} dr, \quad (69)
$$

to get

$$
K^r = -\frac{1}{4\pi} B^\phi, \quad K^\phi = \frac{1}{4\pi} B^r, \quad (70)
$$

where $d\Sigma_i$ is the corresponding surface element.

This result can be summarized by Ampere’s law on the disk surface:

$$
\vec{B} = -4\pi \vec{K} \times \hat{\theta}. \quad (71)
$$
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