Dark energy interacting with dark matter and a third fluid: Possible EoS for this component

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A cosmological model of dark energy interacting with dark matter and another general component of the universe is considered. The equations for the coincidence parameters r and s, which represent the ratios between dark energy and dark matter and the other cosmic fluid respectively, are analyzed in terms of the stability of stationary solutions. The obtained general results allow to shed some light on the equations of state of the three interacting fluids, due to the constraints imposed by the stability of the solutions. We found that for an interaction proportional to the sum of the dark energy density and the third fluid density, the hypothetical fluid must have positive pressure, which leads naturally to a cosmological scenario with radiation, unparticle or even some form of warm dark matter as the third interacting fluid.

Inspired by these previous investigation we have recently formulated an effective model where DE is decaying in DM and another hypothetical fluid [9]. In the framework of the holographic DE, and using the Hubble radius as infrared cutoff, we have shown that our scenario leads naturally, for a flat universe, to a more suitable approach to the cosmic coincidence problem in which the ratio between the energy densities of DM and DE, r, can be variable during the cosmic evolution. Our model has been discussed as a possible approach to solve the triple coincidence problem in [13], where it was assumed that the third fluid is radiation. Nevertheless, although matter and radiation are almost non-interacting fluids, since the decoupling era, they could interact with DE. In [14] the dynamical behavior when DE is coupling to DM and unparticle in the flat FRW cosmology was investigated.

The goal of this Letter is to investigate further the model of DE interacting with DM and another hypothetical fluid. Despite the interesting results found in [13,14] where this third fluid was specifically identified with radiation and unparticle, we do not identify this third component with radiation, unparticle or even neutrinos, which are the most expectable physically relevant candidates. Since we are interested in obtaining useful information about this fluid from the interacting equations, we only assume at this stage that this unknown component has an equation of state with ω constant. Studying the stationary solutions of the evolution equation for r and s, where r is the ratio between the DM energy density and the DE energy density and s is the ratio between the energy density of the third fluid and DE energy density, we
expect to find suitable constraints for the equation of state of the unknown fluid. Our purpose is to shed some light on the nature of this fluid going further only with an analysis of the evolution equations for the parameters and .

In this Letter we choose three different coupling terms to investigate the dynamical behavior of the models of DE interacting with DM and a third unknown fluid.

Our Letter is organized as follows. In Section 2 we present the model of a universe filled with dark matter, dark energy and another fluid. We shall impose that the interacting terms δ and δ′ which appear in the conservation equations are different. In Section 3 we study the stationary solutions and their stability for the equations of evolution of the ratios between the DM and DE and the third fluid and DE. We present the constraints on the equations of state for DE and the hypothetical third fluid. In Section 4 we discuss our results. In Appendix A the conditions for the stability of the stationary solutions, corresponding to the three different couplings, are discussed.

2. Interacting dark energy

In the following we modelled the universe made of CDM with an energy density , a dark energy component, and a third unknown fluid with an energy density . For a flat FRW universe the sourced Friedmann equation is then given by

\[ 3H^2 = \rho_D + \rho_m + \rho_X. \] (1)

We will assume that the three fluids are interacting between them, so their continuity equations take the form

\[ \dot{\rho}_D + 3H(1 + \omega_D)\rho_D = -\delta, \] (2)
\[ \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = \delta', \] (3)
\[ \dot{\rho}_X + 3H(1 + \omega_X)\rho_X = \delta - \delta', \] (4)

where δ and δ′ are the interactions terms, which are different in order to include the scenario in which the mutual interaction between the two principal components of the universe leads to some loss in other forms of cosmic constituents. We assume that the interactions terms, δ and δ′ are of the form 3Hδ′, where \( \Gamma = \Gamma'(\rho_D, \rho_m, \rho_X) \). Since we will study a universe with dark energy decaying into dark matter and another third fluid, we have from the beginning \( \delta > 0 \) and \( \delta' > 0 \).

In the following we will investigate interactions that are linear combinations of the dark sector densities (see, for example, [11]). For other types of interaction which include products or powers of the energy densities see [12]. In the case of only two interacting dark fluids, a general form of interaction has been taken into account in [10], which is given by

\[ \Gamma = \lambda_D \rho_D + \lambda_m \rho_m. \] (5)

Since we are introducing a third fluid that could be also interacting with the dark sector, a straightforward generalization is to assume an interaction of the form

\[ \Gamma = \lambda_D \rho_D + \lambda_m \rho_m + \lambda_X \rho_X = (\lambda_D + \lambda_m r + \lambda_X s) \rho_D. \] (6)

Introducing this general expression for the interaction in the conservation equations (2), (3) and (4) we obtain

\[ \dot{\rho}_D + 3H(1 + \omega_D)\rho_D = -3H(\lambda_D \rho_D + \lambda_m \rho_m + \lambda_X \rho_X), \] (7)
\[ \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = 3H(\lambda_D' \rho_D + \lambda_m' \rho_m + \lambda_X' \rho_X), \] (8)
\[ \dot{\rho}_X + 3H(1 + \omega_X)\rho_X = 3H[\lambda_D - \lambda_D' \rho_D + (\lambda_m - \lambda_m') \rho_m + (\lambda_X - \lambda_X') \rho_X]. \] (9)

Instead of choosing the λ and λ′ as the free parameters, we will chose their differences as new parameters denoted by λπ, which allows to rewrite Eq. (9) as

\[ \dot{\rho}_X + 3H(1 + \omega_X)\rho_X = 3H[\lambda_D^\pi \rho_D + \lambda_m^\pi \rho_m + \lambda_X^\pi \rho_X]. \] (10)

In order to study the evolution of the densities of these three fluids, we construct the differential equations for the coincidence parameters and . These equations take the following expressions

\[ r' = \frac{\dot{r}}{H} = \frac{\rho_m - \rho_D}{\rho_m - \rho_D}, \] (11)
\[ s' = \frac{\dot{s}}{H} = \frac{\rho_X - \rho_D}{\rho_X - \rho_D}. \] (12)

Introducing in Eqs. (11) and (12), the expressions for the continuity equations given in (7), (8) and (10), we obtain the evolution equations for the parameters and

\[ r' = 3r[(\lambda_D + \lambda_m r + \lambda_X s)(1 + 1/r) + (\lambda_D^\pi + \lambda_m^\pi r + \lambda_X^\pi s) + \omega_D - \omega_m]. \] (13)
\[ s' = 3s[(\lambda_D + \lambda_m r + \lambda_X s + (1 + 1/s)(\lambda_D^\pi + \lambda_m^\pi r + \lambda_X^\pi s) + \omega_D - \omega_X]. \] (14)

3. Stationary solutions

Before looking for the stationary solutions of particular cases corresponding to Eqs. (13) and (14), let us briefly discuss the assumptions for the equations of state of the three interacting cosmic fluids. At this stage, we only consider that one fluid, with an energy density , and equation of state , is decaying into the other two fluids. Although, it is reasonable to take at the beginning, since we are thinking in the dark matter fluid, we shall postpone this election until obtaining the constraint derived from the study of the stationary solutions of Eqs. (13) and (14) and its stability. For simplicity, the three equations of state are taken to be constant.

Since we are looking for stationary solutions of Eqs. (13) and (14) we set \( r' = s' = 0 \), obtaining a systems of algebraic equations in terms of the variables \( r \) and \( s \), with the parameters \( \lambda_D, \lambda_m, \lambda_X, \lambda_D^\pi, \lambda_m^\pi, \lambda_X^\pi, \omega_D, \omega_X \). In the following analysis we will consider only three particular cases of the general expression given in Eq. (6), which are previously considered in the literature. The first case correspond to an interaction which is only proportional to the DM energy density. In the second one the interaction is only proportional to the DM energy density. Both cases have been widely discussed in the literature without the inclusion of a third fluid. The third case corresponds to a linear combination of the DE energy density and the unknown fluid density.

3.1. Case \( \Gamma = \lambda_D \rho_D \)

In this case we are choosing \( \lambda_m = \lambda_X = \lambda_D^\pi = \lambda_X^\pi = 0 \). The interactions terms are proportional to the dark energy density. This type of interaction has been investigated in [14–16]. The condition \( r' = s' = 0 \) leads to the algebraic system

\[ f(r, s)|_{r=r_s, s=s_s} = \lambda_D(1 + r_s) + \lambda_m^\pi r_s + (\omega_D - \omega_m)r_s = 0. \] (15)
\[ g(r, s)|_{r=r_s, s=s_s} = \lambda_D s_s + \lambda_m^\pi (1 + s_s) + (\omega_D - \omega_X)s_s = 0. \] (16)
Notice that the assumed interaction gives two simple non-coupled linear equations in the variables \( r_s \) and \( s_s \), which are the stationary solutions given by

\[
\begin{align*}
\dot{r}_s &= -\frac{\lambda_D}{\lambda_D + \lambda_D^\pi + \omega_D - \omega_m}, \\
\dot{s}_s &= -\frac{\lambda_D^\pi}{\lambda_D + \lambda_D^\pi + \omega_D - \omega_X}.
\end{align*}
\]

and

\[
\begin{align*}
\frac{\omega_D}{\omega_m} &= \left(\lambda_D + \lambda_D^\pi\right), \\
\frac{\omega_X}{\omega_D} &= \left(\lambda_D + \lambda_D^\pi\right).
\end{align*}
\]

Since \( r_s \) and \( s_s \) are positive quantities, the denominators of the above both equations must be negative, so we obtain the following inequalities

\[
\omega_D < \omega_m - \left(\lambda_D + \lambda_D^\pi\right),
\]

(19)

and

\[
\omega_X > \omega_D + \left(\lambda_D + \lambda_D^\pi\right).
\]

(20)

We find (see Appendix A) that the condition of stability is the same just contained in Eqs. (19) and (20). Therefore, if for a given \( \omega_D, \omega_m, \omega_X, \lambda_D, \lambda_D^\pi \) there are positive stationary solutions, they are also stable.

For the case of dark matter with negligible pressure, i.e., \( \omega_m = 0 \), Eq. (19) implies that the dark energy must necessarily have an equation of state with \( \omega_D < 0 \).

A rough estimation of the value of the sum \( \lambda_D + \lambda_D^\pi \) can be obtained from the equation for the acceleration

\[
\frac{\dot{a}}{a} = -\frac{1}{6}(1 + 3\omega)\rho,
\]

(21)

where \( \rho \equiv \rho_D(1 + r + s) \) and \( \omega \equiv (\omega_D + \omega_m r + \omega_X s)/(1 + r + s) \). If an accelerated phase is required we obtain the following inequality

\[
\omega_D < -\frac{1}{3}\left[1 + (1 + 3\omega_m)r + (1 + 3\omega_X)s\right].
\]

(22)

In terms of the parameters \( \Omega_D \) and \( \Omega_X \) (for \( \omega_m = 0 \)) we obtain

\[
\omega_D < -\frac{1}{3\Omega_D}(1 + 3\omega_X\Omega_X).
\]

(23)

Taking the right-hand side of the expressions (19) and (23) as equal, and assuming that \( \Omega_X \ll 1 \), as we expect for the present era for any other fluid different from the dark sector, we obtain (for \( \Omega_D = 0.7 \))

\[
\lambda_D + \lambda_D^\pi \simeq 0.5,
\]

(24)

and hence for \( \omega_D \lesssim -0.5 \) the three interacting fluids lead to stationary and stable solutions for \( r \) and \( s \). From the inequalities (19) and (20) we obtain \( \omega_X \lesssim \omega_m \). So for \( \omega_m = 0 \) the third fluid could be a normal fluid or even an exotic fluid. Nevertheless, from the expression for the ratio \( \frac{\dot{s}_s}{s_s} \), given by

\[
\frac{s_s}{r_s} = \frac{\lambda_D^\pi}{\lambda_D \lambda_D^\pi} \left(1 - \frac{\omega_X}{\omega_D + \omega_D^\pi + \lambda_D^\pi}\right)^{-1}
\]

(25)

we can conclude, assuming \( \frac{s_s}{r_s} \ll 1 \), that a scenario with \( \omega_X > 0 \) is more suitable taking \( \lambda_D \gg \lambda_D^\pi \), from Eq. (17) and taking \( r_s \approx 0.3 \), we obtain for the realistic case \( \lambda_D \gg \lambda_D^\pi \), that \( \lambda_D \simeq 0.3 \).

3.2. Case \( \Gamma = \lambda_m \rho_m \)

In this case we are choosing \( \lambda_D = \lambda_X = \lambda_D^\pi = \lambda_D^m = 0 \). The interaction terms are proportional only to the dark matter density. This type of interaction was investigated for models of interacting phantom dark energy with dark matter [14,17–19] and also in [16, 20,21]. Observational constraints on \( \lambda \) for this type of interaction have been investigated in [23]. A general case of this type of interaction where \( \lambda_m \) is a function of a scalar field is studied in [22]. The condition \( r' = 0 \) leads to the algebraic system

\[
\begin{align*}
\frac{f(r, s)_{|r=r_s, s=s_s}}{g(r, s)_{|r=r_s, s=s_s}} &= r_s^2(\lambda_m + \lambda_X^m) + r_s(\lambda_m + \omega_D - \omega_m) = 0, \\
&= s_s r_s(\lambda_m + \lambda_X^m) + \lambda_m^\pi r_s + (\omega_D - \omega_X)s_s = 0.
\end{align*}
\]

(26)

In this case the assumed interaction yields two coupled non-linear equations in the variables \( r_s \) and \( s_s \). Eq. (26) has the following non-zero solution

\[
r_s = -\frac{\lambda_m + \omega_D - \omega_m}{\lambda_m + \lambda_m^\pi}.
\]

(27)

Imposing the condition \( r_s > 0 \) we obtain the constraint

\[
\lambda_m + \omega_D - \omega_m < 0.
\]

(29)

Introducing the value for \( r_s \), given by Eq. (28), in Eq. (27) yields

\[
\frac{\omega_D}{\omega_m} = \frac{\lambda_m^\pi}{\lambda_m + \omega_m - \omega_X} \left(\frac{\lambda_m + \omega_D - \omega_m}{\lambda_m + \lambda_m^\pi}\right).
\]

(30)

Using the constrain given in Eq. (29) in the expression for \( s_s \), we obtain that \( s_s > 0 \) implies

\[
-\lambda_m + \omega_m - \omega_X < 0.
\]

(31)

The condition of stability of these equations (see Appendix A) tells us that a positive \( r_s \) is not stable. Then for this type of interaction it is not possible to have stable solutions for three cosmic interacting fluids. Nevertheless, this situation is a consequence of choosing \( \lambda_m > 0 \) and \( \lambda_m^\pi > 0 \) from the beginning. It is straightforward to prove that for \( \lambda_m + \lambda_m^\pi < 0 \) and \( \lambda_m^\pi > 0 \), or equivalently \( \lambda_m < 0 \) and \( |\lambda_m| > |\lambda_m^\pi| \), the fixed point of the system is stable. The constraints for the equations of state are the following

\[
\lambda_m + \omega_D - \omega_m > 0,
\]

(32)

and

\[
-\lambda_m + \omega_D - \omega_X < 0,
\]

(33)

which for \( \omega_m = 0 \) yields \( \lambda_m + \omega_D > 0 \) and \( \lambda_m + \omega_X > 0 \). Since \( \lambda_m < 0 \) the above conditions implies a third fluid with \( \omega_X > 0 \) and also a DE with \( \omega_D > 0 \).

Briefly, in the approach of DE interacting with two fluids, interaction terms proportional to the dark matter density drive stable solutions but a cosmic evolution without acceleration, which is a non-physically desirable scenario.

3.3. Case \( \Gamma = \lambda_X (\rho_D + \rho_X) \)

In this case we have taken \( \lambda_D = \lambda_X = \lambda_m = \lambda_m^\pi = 0 \), and \( \lambda_X^m = \lambda_D^m \). The interaction is then proportional to \( \rho_D + \rho_X \). A coupling term which includes a different fluid from those of the dark sector was already introduced in [14], but throughout expressions like \( \rho_D \rho_X \) and \( \rho_m \rho_X \), where \( \rho_X \) was identified with the energy density of the unparticle. The condition \( r' = 0 \) leads to the following algebraic system

\[
\begin{align*}
\frac{f(r, s)_{|r=r_s, s=s_s}}{g(r, s)_{|r=r_s, s=s_s}} &= r_s^2(\lambda_m + \lambda_m^\pi) + r_s(\lambda_m + \omega_D - \omega_m) = 0, \\
&= s_s r_s(\lambda_m + \lambda_m^\pi) + \lambda_m^\pi r_s + (\omega_D - \omega_X)s_s = 0.
\end{align*}
\]
Table 1
Critical points, stability conditions and the corresponding cosmological scenario for the three interactions considered. The expressions for the parameters $B$ and $C$ are given in Eq. (36).

| $r$ | Critical points | Condition of stability | Cosmological scenario |
|-----|-----------------|------------------------|-----------------------|
| $\lambda_D \rho_D$ | $r_s = \frac{\lambda_D \rho_D}{\lambda_D \rho_D + \lambda_D \rho_D}$ | Stable if $\omega_D < 0$ for $\lambda_D > 0$ and $\lambda_D^* > 0$. | DE must necessarily have an equation of state $\omega_D < 0$. |
| $\lambda_m \rho_m$ | $r_s = \frac{\lambda_m \rho_m}{\lambda_m \rho_m + \lambda_m \rho_m}$ | Unstable for $\lambda_m > 0$ and $\lambda_m^* > 0$. | The universe contains DM decaying most into non-exotic DE and in a lesser amount into a unknown fluid with positive pressure. |
| $\lambda_X (\rho_s + \rho_X)$ | $r_s = \frac{\lambda_X}{\lambda_X + 1 + \frac{\lambda_X}{\omega_X}}$ | Stable if $\omega_X < 0$ and $\omega_X > 0$ for $\lambda_X > 0$ and $\lambda_X^* > 0$. | DE is an exotic fluid decaying into DM and another fluid with positive pressure (radiation, unparticle or warm DM). |

$$f(r, s) |_{r=r_s, s=s} = \lambda_X (1 + r_s) (1 + s) + \lambda_X^* r_s (1 + s) + (\omega_D - \omega_m) r_s = 0, \quad (34)$$

$$g(r, s) |_{r=r_s, s=s} = \lambda_X s (1 + s) + \lambda_X^* (1 + s)^2 + (\omega_D - \omega_X) s = 0. \quad (35)$$

Solving first Eq. (35), which is a quadratic equation of the form $x^2 + Bx + C = 0$, where the coefficients $B$ and $C$ are given by

$$B = \frac{\lambda_X + 2 X}{\lambda_X + X}, \quad C = \frac{\omega_X}{\lambda_X + X}. \quad (36)$$

The solutions of Eq. (35) have the form

$$s_s = \frac{B}{2} \left( 1 \pm \sqrt{1 - \frac{4C}{B^2}} \right). \quad (37)$$

Since $s_s$ is a positive and real number, we need to impose the two constraints $B < 0$ and $B^2 > 4C$. The first one implies that $\lambda_X + 2 X < 0$ and the second one, $\lambda_X + X < 0$, which gives the following range for $\omega_D - \omega_X < 0$

$$-\left( \lambda_X + 2 X \right) + 2 \sqrt{\frac{\omega_X}{\lambda_X + X} \left( \lambda_X + X \right)} < \omega_D - \omega_X < -\left( \lambda_X + 2 X \right). \quad (38)$$

Introducing the two solutions of Eq. (35), which we denote by $s_{s+}$ and $s_{s-}$, in Eq. (34), we obtain two solutions for $r_s$, $r_{s+}$ and $r_{s-}$, given by

$$r_{s+} = -\frac{\lambda_X}{\lambda_X + X + \omega_D - \omega_m^{\pm}}, \quad (39)$$

Since $r_s > 0$, Eq. (39) gives the following constraint

$$\left( \lambda_X + X \right) (1 + s_{s+}) + \omega_D - \omega_m < 0, \quad (40)$$

therefore $\omega_D$ must satisfy

$$\omega_D < \omega_m - \left( \lambda_X + X \right) (1 + s_{s+}). \quad (41)$$

The conditions which are necessary holding in order to have positive solutions for $r_{s+}$ and $s_{s+}$ are then the inequalities given by Eqs. (38) and (40).

As in Section 3.1, where the interaction is proportional only to the DE density, if $\omega_m = 0$ the above inequality implies that the dark energy must necessarily have an equation of state with $\omega_D < 0$. A rough estimation for the range of the values that the parameters $\lambda_X$ and $\lambda_X^*$ can take, may also be done for this case using Eq. (23) with $\lambda_X < 1$ and $\lambda_X^* = 0.7$. Equating the right-hand side of the expressions (41) and (23) and since $1 + s_{s+} > 1$

$$\lambda_X + \lambda_X^* < 0.5. \quad (42)$$

The analysis of stability (see Appendix A) yields the fact that the equation of state of the non-decaying fluids satisfies

$$\omega_X > \omega_m. \quad (43)$$

If $\omega_m = 0$, which is the equation of state for the dark matter fluid, the above result indicates us that the exigency of stability for the stationary solutions of the evolution equations (13) and (14), imposes an unknown interacting fluid with non-null pressure.

4. Discussion

In the present investigation we have considered a cosmological scenario where the dark energy is decaying into the dark matter and another component of the universe, which we do not identify explicitly. We have assumed that each of this three fluids have an equation of state with $\omega$ constant. We have chosen three different coupling terms, and we have analyzed the stationary solutions of the evolution equation for parameters $r$ and $s$. Our main results are summarized in Table 1, indicating the allowed cosmological scenarios for stable solutions, when cold DM, i.e., $\omega_m = 0$ is assumed.

When the coupling is proportional to the DM energy only, we have found that those which are the conditions for the stationary solutions to be positive are the same of those to be stable. For dark matter with negligible pressure we obtain that the dark energy must necessarily have an equation of state with $\omega_D < 0$. We have also showed that a third fluid with positive pressure is favored.

When the coupling is only proportional to the DM energy density, it is not possible to obtain stable solutions for the three interacting fluids, if $\lambda_m > 0$ and $\lambda_m^* > 0$, which guarantee that DE is decaying in the other fluids. Nevertheless, relaxing this condition and taking $\lambda_m + \lambda_m^* < 0$ and $\lambda_m^* > 0$, we obtain stationary solutions which are stable. Notice that this physically corresponds to a DM decaying in DE and in the third fluid. It is interesting to mention that the usual case of DE interacting only with DM, which have been discussed for this coupling in [23], showed that the data slightly favored a DM decaying in DE and $\omega_D < -1$. In our approach, this scenario implies that the unknown fluid and DE have positive pressure, leading to a decelerated expansion.

In the third studied case, the coupling considers a different fluid from those of the dark sector, taking a term proportional to the sum of the DE density and the third fluid density. We have found two fixed points and from the constraints derived from the condition of stability we have obtain that $\omega_D > \omega_m$, which means an
interacting third fluid with positive pressure. This type of coupling can then accommodate a scenario with radiation [13], unparticle [14] or even some form of warm DM [24] as the third interacting fluid.

As a summary, we can point out that for a universe filled with three interacting fluids, in which the interactions are linear combinations of the energy densities, the study of the stationary solutions of the coincidence parameters \( r \) and \( s \), shed some light on the possible nature of the unknown third fluid (constraining its equation of state) assumed to interact with the dark sector.

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Appendix A

In Section 3.1, when the interactions terms are proportional to the dark energy density, the algebraic system is given by Eqs. (15) and (16). We need to evaluate the eigenvalues of the matrix \( \mathbf{M} \), given by

\[ \mathbf{M} = \left( \begin{array}{c} \frac{\partial f(r,s)}{\partial r} \\ \frac{\partial g(r,s)}{\partial s} \end{array} \right), \]

whose elements are evaluated at the critical point \((r_c, s_c)\). From the equation for the eigenvalues, \( \det[\mathbf{M} - \eta \mathbf{I}] \), and since \( \frac{\partial f(r,s)}{\partial r} = \frac{\partial g(r,s)}{\partial s} = 0 \), we obtain

\[ \left[ \frac{\partial f(r,s)}{\partial r} \right]_{r=r_c, s=s_c} - \eta \left[ \frac{\partial g(r,s)}{\partial s} \right]_{r=r_c, s=s_c} = 0. \]

The eigenvalues are then

\[ \eta_1 = \left( \frac{\partial f(r,s)}{\partial r} \right)_{r=r_c, s=s_c} = \lambda + \lambda \pi + \omega_D - \omega_m, \]

and

\[ \eta_2 = \left( \frac{\partial g(r,s)}{\partial s} \right)_{r=r_c, s=s_c} = \lambda + \lambda \pi + \omega_D - \omega_m. \]

Notice that the condition of stability, \( \eta_1 < 0 \) and \( \eta_2 < 0 \) gives us, for \( \eta_1 < 0 \) the following constraint

\[ \lambda + \omega_D - \omega_m > 0, \]

which cannot be allowed if Eq. (29) is satisfied.

In Section 3.3, when the interactions terms are proportional to the sum of the dark matter density and the third fluid density, the algebraic system is given by Eqs. (34) and (35). Evaluating the elements of the matrix \( \mathbf{M} \) at the critical points \((r_{c+}, s_{c+})\) and \((r_{c-}, s_{c-})\), we obtain from the equation for the eigenvalues, \( \det[\mathbf{M} - \eta \mathbf{I}] \), and since \( \frac{\partial g(r,s)}{\partial r} = 0 \), that

\[ \left[ \left( \frac{\partial f(r,s)}{\partial r} \right)_{r=r_{c\pm}, s=s_{c\pm}} - \eta \right] \left[ \left( \frac{\partial g(r,s)}{\partial s} \right)_{r=r_{c\pm}, s=s_{c\pm}} - \eta \right] = 0. \]

The eigenvalues are then

\[ \eta_1 = \left( \frac{\partial f(r,s)}{\partial r} \right)_{r=r_{c\pm}, s=s_{c\pm}} = (\lambda + \lambda \pi)(1 + s_{c\pm}) + \omega_D - \omega_m, \]

and

\[ \eta_2 = \left( \frac{\partial g(r,s)}{\partial s} \right)_{r=r_{c\pm}, s=s_{c\pm}} = (\lambda + \lambda \pi)(1 + s_{c\pm}) + \omega_D - \omega_m. \]

The condition of stability, \( \eta_1 < 0 \) and \( \eta_2 < 0 \) implies that the following constraints must hold

\[ \omega_D - \omega_m < -(\lambda + \lambda \pi)(1 + s_{c\pm}), \]

and

\[ \omega_D - \omega_X < -(\lambda + \lambda \pi)(1 + s_{c\pm}) + \lambda s_{c\pm}. \]

Notice that the constraint given by Eq. (55) is the same obtained in Eq. (40). Nevertheless, we need to look for the range of \( \omega_D - \omega_X \) which can accommodate the constraint given by Eqs. (38) and (56). Choosing Eq. (56) as the constraint for the upper limit of \( \omega_D - \omega_X \), the upper limit indicated in Eq. (38) is also satisfied. We can impose the condition

\[ -\left(\lambda + 2\lambda \pi \right) + 2\sqrt{\lambda \pi (\lambda + \lambda \pi)} \]

\[ < -\left[\lambda + 2\lambda \pi \right](1 + s_{c\pm}) + \lambda s_{c\pm} \].

which leads to the following condition for \( s_{c\pm} \)

\[ s_{c\pm} < \sqrt{\frac{\lambda \pi}{\lambda + \lambda \pi}}. \]

Since it is physically reasonable to assume \( \lambda \gg \lambda \pi \), the last expression indicates that \( s_{c\pm} \ll 1 \) for a late time evolution. It is straightforward to check from Eqs. (55) and (56) that, independently of the critical point considered, the equation of state of the non-decaying fluids satisfy \( \omega_X > \omega_m \).
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