On an integrable discretization
of the modified Korteweg-de Vries equation

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Abstract. We find time discretizations for the two ”second flows” of the Ablowitz–Ladik hierarchy. These discretizations are described by local equations of motion, as opposed to the previously known ones, due to Taha and Ablowitz. Certain superpositions of our maps allow a one–field reduction and serve therefore as valid space–time discretizations of the modified Korteweg-de Vries equation. We expect the performance of these discretizations to be much better than that of the Taha–Ablowitz scheme. The way of finding interpolating Hamiltonians for our maps is also indicated, as well as the solution of an initial value problem in terms of matrix factorizations.
1 Introduction

Already in the early period of the soliton theory it was realised that by the numerical simulation of soliton equations it is highly desirable that the difference schemes inherit the integrability property [1], [2], [3]. These papers contained a full (space and time) discretization of such soliton equations as the nonlinear Schrödinger equation (NLS), Korteweg–de Vries equation (KdV) and the modified Korteweg–de Vries equation (mKdV). This was done in two steps. On the first step the space discretization was performed [1]. To this end the authors discretized the auxiliary linear problem related to the corresponding soliton equation. Concerning as an example the mKdV equation,

\[ q_t = q_{xxx} \mp 6q_xq^2 \]  

(1.1)

the associated linear problem is the Zakharov–Shabat problem

\[ \Psi_x = \begin{pmatrix} i\zeta & q \\ \pm q & -i\zeta \end{pmatrix} \Psi. \]  

(1.2)

Its discretization chosen in [1] is

\[ \Psi_{k+1} = \begin{pmatrix} \lambda & q_k \\ \pm q_k & \lambda^{-1} \end{pmatrix} \Psi_k, \]  

(1.3)

and the space discretization of (1.1) found in [1] is

\[ \dot{q}_k = (1 \mp q_k^2) \left( q_{k+2} - 2q_{k+1} + 2q_{k-1} - q_{k-2} \right) + q_{k+1}(q_{k+2}q_{k+1} + q_{k+1}q_k) \pm q_{k-1}(q_kq_{k-1} + q_{k-1}q_{k-2}) \]  

(1.4)

(to perform the corresponding continuous limit, one has to assume in (1.4) \( q_k = \epsilon q(\epsilon k) \), to rescale the time \( t \mapsto t/(2\epsilon^3) \), and then to send \( \epsilon \to 0 \)).

The second step of discretization – the time discretization – was performed in [2] (for NLS) and in [3] (for KdV and mKdV). The approach to this second step of the discretization process was fundamentally different: the linear problem (1.3) was not modified any more, and only a suitable choice of a (discrete)–time evolution of the wave function \( \Psi_k \) was imposed. The outcome of [3] was an excellent numerical scheme for the mKdV, though not free of some drawbacks, the main one being the non-locality (which means that \( \bar{q}_k \), the discrete time update of \( q_k \), depends explicitly on all \( q_j \)’s and \( \bar{q}_j \)’s with \( j \leq k \)). This property is unpleasant from the aesthetical point of view as well as from the practical one (since it implies a large amount of computations by numerical realization).
In a more modern language, time discretizations in \([2], [3]\) were sought in the same hierarchies to which the underlying continuous time systems belong. Recently this approach was pushed forward as a systematic procedure of obtaining integrable discretizations \([4]–[7]\). The results of \([2]\) were re-considered in \([8]\), where a significant amendment was achieved. Namely, the non-locality of the schemes in \([2]\) for NLS was overcome. In the present paper we do an analogous work for the time discretization of \((1.4)\) from \([3]\).

## 2 Ablowitz–Ladik hierarchy

To deal with the equation \((1.1)\) in a slightly more systematic way, one has to consider it as a particular case of the following, more general system:

\[
\begin{align*}
q_t &= q_{xxx} - 6q_xqr, \\
r_t &= r_{xxx} - 6qr_xr
\end{align*}
\]  

(2.1)

under the reduction

\[
r = \pm q.
\]  

(2.2)

Analogously, the space discretization \((1.4)\) arises by the same reduction from the more general system

\[
\begin{align*}
\dot{q}_k &= (1 - q_k r_k) \left( q_{k+2} - 2q_{k+1} + 2q_{k-1} - q_{k-2} \right) \\
&\quad - q_{k+1} (q_{k+2} r_{k+1} + q_{k+1} r_k + q_k r_{k-1}) \\
&\quad + q_{k-1} (q_{k-2} r_{k-1} + q_{k-1} r_k + q_k r_{k+1}) \\
\dot{r}_k &= (1 - q_k r_k) \left( r_{k+2} - 2r_{k+1} + 2r_{k-1} - r_{k-2} \right) \\
&\quad - r_{k+1} (q_{k+1} r_{k+2} + q_{k+1} r_{k+1} + q_k r_{k+1}) \\
&\quad + r_{k-1} (q_{k-1} r_{k-2} + q_{k-1} r_{k-1} + q_k r_{k})
\end{align*}
\]  

(2.3)

Below we consider this system either on an infinite lattice \((k \in \mathbb{Z})\) under the boundary conditions of a rapid decay \((|q_k|, |r_k| \to 0 \text{ as } k \to \pm \infty)\), or on a finite lattice \((1 \leq k \leq N)\) under the periodic boundary conditions \((q_0 \equiv q_N, r_0 \equiv r_N, q_{N+1} \equiv q_1, r_{N+1} \equiv r_1)\). In any case we denote by \(q\) (resp. \(r\)) the (infinite- or finite-dimensional) vector with the components \(q_k\) (resp. \(r_k\)).

From the modern point of view, the system \((2.3)\) is a representative of a whole hierarchy of commuting Hamiltonian flows, the Ablowitz–Ladik hierarchy. An object of the principle importance in the description of this hierarchy is the \(2 \times 2\) Lax matrix

\[
L_k = L_k(q, r) = \begin{pmatrix} \lambda & q_k \\ r_k & \lambda^{-1} \end{pmatrix}
\]  

(2.4)
depending on the variables $q, r$ and on the additional (spectral) parameter $\lambda$. Each flow of the hierarchy allows a commutation representation (semi-discrete version of a zero–
curvature representation)

$$\dot{L}_k = M_{k+1}L_k - L_kM_k$$

(2.5)

with some $2 \times 2$ matrix $M_k = M_k(q, r, \lambda)$.

The Hamiltonians of the commuting flows are the coefficients in the Laurent expansion of the trace $\text{tr} \ T_N(q, r, \lambda)$ where $T_N$ is the monodromy matrix

$$T_N = L_N \cdot L_{N-1} \cdots \cdot L_2 \cdot L_1,$$

(2.6)

supplied by the function

$$H_0(q, r) = \log \det T_N = \sum_{k=1}^N \log(1 - q_k r_k).$$

(2.7)

The Poisson bracket on the phase space is given by

$$\{q_k, r_j\} = (1 - q_k r_k)\delta_{jk}, \quad \{q_k, q_j\} = \{r_k, r_j\} = 0.$$

(2.8)

The involutivity of all integrals of motion follows from the fundamental $r$–matrix relation:

$$\{L(\lambda) \otimes L(\mu)\} = [L(\lambda) \otimes L(\mu), \rho(\lambda, \mu)],$$

(2.9)

where

$$\rho(\lambda, \mu) = \begin{pmatrix}
\frac{1}{2} \lambda^2 + \mu^2 & 0 & 0 & 0 \\
0 & \frac{1}{2} \lambda \mu \lambda^2 - \mu^2 & 0 & 0 \\
0 & \frac{\lambda \mu}{\lambda^2 - \mu^2} & -\frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} \lambda^2 + \mu^2
\end{pmatrix}.$$  

(2.10)

We refer the reader to [9] for the general formula expressing the matrix $M_k$ in terms of the corresponding Hamiltonian function $H(q, r)$ and the $r$–matrix $\rho(\lambda, \mu)$.

It is easy to see that the following functions belong to the involutive family generated by $\text{tr} \ T_N$:

$$H_1(q, r) = \sum_{k=1}^N q_{k+1} r_k, \quad H_{-1}(q, r) = \sum_{k=1}^N q_k r_{k+1},$$

(2.11)

$$H_2(q, r) = \sum_{k=1}^N q_{k+2} r_k - \sum_{k=1}^N q_{k+2} q_{k+1} r_{k+1} r_k - \frac{1}{2} \sum_{k=1}^N q_k^2 r_k^2,$$

(2.12)

$$H_{-2}(q, r) = \sum_{k=1}^N q_k r_{k+2} - \sum_{k=1}^N q_k q_{k+1} r_{k+1} r_{k+2} - \frac{1}{2} \sum_{k=1}^N q_k^2 r_{k+1}^2.$$

(2.13)
The corresponding Hamiltonian flows are described by the differential equations
\[
\mathcal{F}_1: \begin{cases} 
\dot{q}_k = q_{k+1}(1 - q_k r_k) \\
\dot{r}_k = -r_{k-1}(1 - q_k r_k)
\end{cases} \quad (2.14)
\]
\[
\mathcal{F}_{-1}: \begin{cases} 
\dot{q}_k = q_{k-1}(1 - q_k r_k) \\
\dot{r}_k = -r_{k+1}(1 - q_k r_k)
\end{cases} \quad (2.15)
\]
\[
\mathcal{F}_2: \begin{cases} 
\dot{q}_k = (q_{k+2} - q_{k+1}(q_{k+2}r_{k+1} + q_{k+1}r_k + q_k r_{k-1}))(1 - q_k r_k) \\
\dot{r}_k = -(r_{k-2} - r_{k-1}(q_{k-1}r_{k-2} + q_k r_{k-1} + q_{k+1}r_k))(1 - q_k r_k)
\end{cases} \quad (2.16)
\]
\[
\mathcal{F}_{-2}: \begin{cases} 
\dot{q}_k = (q_{k-2} - q_{k-1}(q_{k-2}r_{k-1} + q_{k-1}r_k + q_k r_{k+1}))(1 - q_k r_k) \\
\dot{r}_k = -(r_{k+2} - r_{k+1}(q_{k+1}r_{k+2} + q_k r_{k+1} + q_{k-1}r_k))(1 - q_k r_k)
\end{cases} \quad (2.17)
\]

The flow (2.3) is an obvious superposition of these more fundamental and simple flows, namely
\[
\mathcal{F}_2(t) \circ \mathcal{F}_1(-2t) \circ \mathcal{F}_{-2}(-t) \circ \mathcal{F}_{-1}(2t)
\]

In what follows we will concentrate on the flows
\[
\mathcal{F}_2(t) \circ \mathcal{F}_1(-ct) \quad (2.18)
\]
\[
\mathcal{F}_{-2}(t) \circ \mathcal{F}_{-1}(-ct), \quad (2.19)
\]
having in mind that in the application to the mKdV case the value \( c = 2 \) is of interest.

The matrix \( M_k \) corresponding to the flow (2.18) is given by the formula
\[
M_k^{(2)}(c) = \begin{pmatrix} 
\lambda^3 - \lambda^2 \mathcal{A}_k^{(2)} - \mathcal{A}_k^{(0)} & \lambda^3 q_k + \lambda \mathcal{B}_k^{(1)} \\
\lambda^3 r_{k-1} + \lambda \mathcal{C}_k^{(1)} & \lambda^2 q_k r_{k-1}
\end{pmatrix} \quad (2.20)
\]
with
\[
\mathcal{A}_k^{(2)} = c + q_k r_{k-1}
\]
\[
\mathcal{B}_k^{(1)} = q_{k+1} - q_k(c + q_{k+1} r_k + q_k r_{k-1})
\]
\[
\mathcal{C}_k^{(1)} = r_{k-2} - r_{k-1}(c + q_{k-1} r_{k-2} + q_k r_{k-1})
\]
\[
\mathcal{A}_k^{(0)} = q_{k+1} r_{k-1} + q_k r_{k-2} - q_k r_{k-1}(c + q_{k+1} r_k + q_k r_{k-1} + q_{k-1} r_{k-2}).
\]

Similarly, the matrix \( M_k \) corresponding to the flow (2.19) is given by
\[
M_k^{(-2)}(c) = \begin{pmatrix} 
-\lambda^{-2} q_{k-1} r_k & -\lambda^{-3} q_{k-1} - \lambda^{-1} \mathcal{B}_k^{(-1)} \\
-\lambda^{-3} r_k - \lambda^{-1} \mathcal{C}_k^{(-1)} & -\lambda^{-4} + \lambda^{-2} \mathcal{D}_k^{(-2)} + \mathcal{D}_k^{(0)}
\end{pmatrix} \quad (2.21)
\]
with
\[
\begin{align*}
\mathcal{D}_k^{(-2)} &= c + q_{k-1}r_k \\
\mathcal{B}_k^{(-1)} &= q_{k-2} - q_{k-1}(c + q_{k-2}r_{k-1} + q_{k-1}r_k) \\
\mathcal{C}_k^{(-1)} &= r_{k+1} - r_k(c + qr_{k+1} + q_{k-1}r_k) \\
\mathcal{D}_k^{(0)} &= q_{k-2}r_k + q_{k-1}r_{k+1} - q_{k-1}r_k(c + q_{k-2}r_{k-1} + q_{k-1}r_k + qr_{k+1}).
\end{align*}
\]

Obviously, one has:
\[
M_k^{(2)}(c) = M_k^{(2)}(0) - cM_k^{(1)}, \quad M_k^{(-2)}(c) = M_k^{(-2)}(0) - cM_k^{(-1)},
\]
where the matrices
\[
M_k^{(1)} = \begin{pmatrix}
\lambda^2 - qr_{k-1} & \lambda q_k \\
\lambda r_{k-1} & 0
\end{pmatrix}, \quad (2.22)
\]
\[
M_k^{(-1)} = \begin{pmatrix}
0 & -\lambda^{-1}q_{k-1} \\
-\lambda^{-1}r_k & -\lambda^{-2} + q_{k-1}r_k
\end{pmatrix}, \quad (2.23)
\]
correspond to the flows \( F_{\pm 1} \).

The matrix \( M_k \) for the system (2.3) is equal to
\[
M_k = M_k^{(2)}(2) - M_k^{(-2)}(2).
\]

3 General remarks about the time discretization

In [3] Taha and Ablowitz constructed a time discretizations of the system (2.3), thus achieving a full discretization of the system (2.1). The basic feature of the time discretization in [3] is following: it admits a discrete analog of the zero–curvature representation,
\[
\tilde{L}_k V_k = V_{k+1} L_k
\]
with the same matrix \( L_k \) as the underlying continuous time system. (In (3.1) and below we use the tilde to denote the \( h \)–shift in the discrete time \( h\mathbb{Z} \)). In a more modern language, the maps generated by the discretizations in [3] belong to the same integrable hierarchy as the continuous time system (2.3).

We shall not need concrete expressions for the entries of the matrix \( V_k \) and the corresponding evolution equations obtained in [3]. However, the details of the derivation seem to be never published, therefore we present them (in a slightly amended form) in

5
the Appendix. Here we restrict ourselves with some general remarks following from the work of Taha and Ablowitz and necessary for the following presentation.

They considered difference equations allowing a commutation representation (3.1) with the matrix $L_k$ given by (2.4) and assumed that the entries of the matrix

\[
V_k = \begin{pmatrix}
A_k & B_k \\
C_k & D_k
\end{pmatrix}
\]

have the following $\lambda$–dependence:

\[
\begin{align*}
A_k(\lambda) &= 1 + h\lambda^4 A_k^{(4)} + h\lambda^2 A_k^{(2)} + h A_k^{(0)} + h\lambda^{-2} A_k^{(-2)} + h\lambda^{-4} A_k^{(-4)} \\
D_k(\lambda) &= 1 + h\lambda^4 D_k^{(4)} + h\lambda^2 D_k^{(2)} + h D_k^{(0)} + h\lambda^{-2} D_k^{(-2)} + h\lambda^{-4} D_k^{(-4)} \\
B_k(\lambda) &= h\lambda^3 B_k^{(3)} + h\lambda B_k^{(1)} + h\lambda^{-1} B_k^{(-1)} + h\lambda^{-3} B_k^{(-3)} \\
C_k(\lambda) &= h\lambda^3 C_k^{(3)} + h\lambda C_k^{(1)} + h\lambda^{-1} C_k^{(-1)} + h\lambda^{-3} C_k^{(-3)}.
\end{align*}
\]

They showed that each such difference equation may be completely characterized (in the case of rapidly decaying boundary conditions) by the limit values

\[
\alpha^{(j)} = \lim_{k \to \pm\infty} A_k^{(j)}, \quad \delta^{(j)} = \lim_{k \to \pm\infty} D_k^{(j)} \quad (j = 4, 2, 0, -2, -4)
\]

Under the condition

\[
\alpha^{(j)} = \delta^{(-j)} \quad (j = 4, 2, 0, -2, -4)
\]

the corresponding difference equation allows the reduction

\[
r = \pm q.
\]

The last statement is easy to see directly. Indeed, the discrete zero–curvature equation (3.1) is equivalent to the following four equations:

\[
\begin{align*}
A_{k+1}(\lambda) - A_k(\lambda) &= \lambda^{-1} \big( \bar{q}_k C_k(\lambda) - r_k B_{k+1}(\lambda) \big) \\
D_{k+1}(\lambda) - D_k(\lambda) &= \lambda \big( \bar{r}_k B_k(\lambda) - q_k C_{k+1}(\lambda) \big) \\
\lambda^{-1} B_{k+1}(\lambda) - \lambda B_k(\lambda) &= \bar{q}_k D_k(\lambda) - q_k A_{k+1}(\lambda) \\
\lambda C_{k+1}(\lambda) - \lambda^{-1} C_k(\lambda) &= \bar{r}_k A_k(\lambda) - r_k D_{k+1}(\lambda)
\end{align*}
\]

Obviously, these equations allow the reduction

\[
r = \pm q, \quad A(\lambda) = D(\lambda^{-1}), \quad B(\lambda) = \pm C(\lambda^{-1}).
\]
But in this reduction, obviously, the condition (3.4) is satisfied. Since the difference equation is completely characterised by the quantities (3.3), the condition (3.4) is also sufficient for the above reduction to be admissible.

Equating coefficients by different powers of \( \lambda \) in (3.6)–(3.9), one obtains 20 equations. In [3], Taha and Ablowitz assumed a special role of evolution equations for \( q_k, r_k \) to two of these equations, using the other 18 to determine the 18 unknown functions \( A_k^{(j)}, D_k^{(j)} (j = 4, 2, 0, -2, -4) \) and \( B_k^{(j)}, C_k^{(j)} (j = 3, 1, -1, -3) \). This way of dealing with the problem (inherited from the continuous time case) resulted in highly nonlocal expressions. This feature of the resulting difference scheme made its numerical realization extremely time consuming. Nevertheless, even despite the drawback of non-locality, this difference scheme proved to be the best among the numerical methods tested in [3]. The reason for this lies undoubtedly in the integrable nature of this scheme.

The goal of the present paper is to demonstrate how the feature of nonlocality may be overcome. This is achieved with the help of two basic ideas. First, we factorize the difference scheme into the product of several simpler ones, corresponding to the fundamental and simple flows of the Ablowitz–Ladik hierarchy. Second, by dealing with these simpler schemes we consider all the 20 equations on an equal footing. This allows us to derive simple local schemes, which we expect to exceed the original one due to Taha and Ablowitz.

### 4 Local discretizations of the flows \( \mathcal{F}_{\pm 2} \)

In the present Section we introduce two maps which serve as time discretizations of the flows (2.18), (2.19).

**Theorem 1.** Consider the map \( \mathcal{T}_2(h;c) : (q, r) \mapsto (\tilde{q}, \tilde{r}) \) defined by the following equations of motion:

\[
\mathcal{T}_2(h;c) : \begin{cases}
(q_k - q_k)/h &= \frac{(q_{k+2} - q_{k+1}P_{k+1}) (1 - q_k \tilde{r}_k)}{(1 - hq_{k+1}\tilde{r}_{k-1})}, \\
(\tilde{r}_k - r_k)/h &= -\frac{(\tilde{r}_{k-2} - \tilde{r}_{k-1}P_k) (1 - q_k \tilde{r}_k)}{(1 - hq_{k+1}\tilde{r}_{k-1})}
\end{cases}
\]

where the local quantity \( P_k = P(q_{k+1}, q_k, \tilde{q}_{k-1}, \tilde{r}_k, \tilde{r}_{k-1}, \tilde{r}_{k-2}; h; c) \) is defined by

\[
(P_k - hq_{k+1}\tilde{r}_{k-2}) (1 - hq_kq_{k-1}\tilde{r}_{k-1}\tilde{r}_{k-2}) (1 - hq_{k+1}q_k\tilde{r}_k\tilde{r}_{k-1}) (1 - hq_{k+1}\tilde{r}_{k-1})
\]

\[
- (q_{k+1}\tilde{r}_k + q_k\tilde{r}_{k-1} + q_{k-1}\tilde{r}_{k-2} - hq_kq_{k-1}\tilde{r}_k\tilde{r}_{k-1}\tilde{r}_{k-2}) = c,
\]

\[\text{(4.2)}\]
so that
\[ P_k = c + q_{k+1} \ddot{r}_k + q_k \ddot{r}_{k-1} + q_{k-1} \ddot{r}_{k-2} + O(h). \]
These equations approximate the flow (2.18) and have the commutation representation (3.1) with the matrix
\[
V_k^{(2)} = \begin{pmatrix}
1 + h \lambda^4 - h \lambda^2 A_k^{(2)} - h A_k^{(0)} & h \lambda^3 q_k + h \lambda B_k^{(1)} \\
1 + h \lambda^3 \ddot{r}_{k-1} + h \lambda C_k^{(1)} & 1 + h \lambda^2 q_k \ddot{r}_{k-1}
\end{pmatrix},
\]
(4.3)
Here
\[
A_k^{(2)} = P_k - q_{k+1} r_k - \ddot{q}_{k-1} \ddot{r}_{k-2}
\]
(4.4)
\[
B_k^{(1)} = q_{k+1} - q_k (P_k - \ddot{q}_{k-1} \ddot{r}_{k-2})
\]
(4.5)
\[
C_k^{(1)} = \ddot{r}_{k-2} - \ddot{r}_{k-1} (P_k - q_{k+1} r_k)
\]
(4.6)
\[
A_k^{(0)} = q_{k+1} \ddot{r}_{k-1} + q_k \ddot{r}_{k-2} - q_k \ddot{r}_{k-1} P_k
\]
(4.7)

**Theorem 2.** Consider the map \( \mathcal{T}_2(h; c) : (\mathbf{q}, \mathbf{r}) \mapsto (\dddot{\mathbf{q}}, \dddot{\mathbf{r}}) \) defined by the following equations of motion:
\[
\mathcal{T}_2(h; c) : \begin{cases}
(\dddot{q}_k - q_k) / h = \frac{(\dddot{q}_{k-2} - \dddot{q}_{k-1} S_k) (1 - \dddot{q}_k r_k)}{(1 + h \dddot{q}_{k-1} r_{k+1})}, \\
(\dddot{r}_k - r_k) / h = -\frac{(r_{k+2} - r_{k+1} S_{k+1}) (1 - \dddot{q}_k r_k)}{(1 + h \dddot{q}_{k-1} r_{k+1})}
\end{cases},
\]
(4.8)
where the local quantity \( S_k = S(\dddot{q}_{k-2}, \dddot{q}_{k-1}, \dddot{q}_k, r_{k+1}, r_k, r_{k+1}; h; c) \) is defined by
\[
(S_k + h \dddot{q}_{k-2} r_{k+1}) \frac{(1 + h \dddot{q}_{k-2} \dddot{q}_{k-1} r_{k-1} r_k)}{(1 + h \dddot{q}_{k-2} r_k)(1 + h \dddot{q}_{k-1} r_{k+1})} - (\dddot{q}_k r_{k+1} + \dddot{q}_{k-1} r_k + \dddot{q}_{k-2} r_{k-1} + h \dddot{q}_{k-2} \dddot{q}_{k-1} \dddot{q}_{k-1} r_{k-1} r_k + 1) = c,
\]
(4.9)
so that
\[ S_k = c + \dddot{q}_k r_{k+1} + \dddot{q}_{k-1} r_k + \dddot{q}_{k-2} r_{k-1} + O(h). \]
These equations approximate the flow (2.19) and have the commutation representation (3.1) with the matrix
\[
V_k^{(-2)} = \begin{pmatrix}
1 - h \lambda^{-2} \dddot{q}_{k-1} r_k & -h \lambda^{-3} \dddot{q}_{k-1} - h \lambda^{-1} B_k^{(-1)} \\
-h \lambda^{-3} r_k - h \lambda^{-1} C_k^{(-1)} & 1 - h \lambda^{-4} + h \lambda^{-2} D_k^{(-2)} + h D_k^{(0)}
\end{pmatrix}
\]
(4.10)
Here

\begin{align*}
D^{(-2)}_k &= S_k - q_k r_{k+1} - \bar{q}_{k-2} \bar{r}_{k-1} \\
B^{(-1)}_k &= \bar{q}_{k-2} - \bar{q}_{k-1} (S_k - q_k r_{k+1}) \\
C^{(-1)}_k &= r_{k+1} - r_k (S_k - \bar{q}_{k-2} \bar{r}_{k-1}) \\
D^{(0)}_k &= \bar{q}_{k-1} r_{k+1} + \bar{q}_{k-2} r_k - \bar{q}_{k-1} r_k S_k
\end{align*}

(4.11) (4.12) (4.13) (4.14)

**Proof.** Since the two Theorems are proved similarly, we give here only the proof of the Theorem 2. Substituting the ansatz \([1.10]\) for the matrix \(V^{(-2)}_k\) into \([1.1]\), one sees that the following 6 equations have to be satisfied:

\begin{align*}
D^{(-2)}_{k+1} - D^{(-2)}_k &= q_k r_{k+1} - \bar{q}_{k-1} \bar{r}_k \\
D^{(0)}_{k+1} - D^{(0)}_k &= q_k C^{(-1)}_{k+1} - \bar{r}_k B^{(-1)}_k \\
B^{(-1)}_{k+1} &= \bar{q}_{k-1} - \bar{q}_k (q_k r_{k+1} + D^{(-2)}_k) \\
C^{(-1)}_k &= r_{k+1} - r_k (\bar{q}_{k-1} \bar{r}_k + D^{(-2)}_k) \\
\bar{q}_k - q_k &= h B^{(-1)}_k - h \bar{q}_k D^{(0)}_k \\
\bar{r}_k - r_k &= -h C^{(-1)}_{k+1} + h r_k D^{(0)}_k
\end{align*}

(4.15) (4.16) (4.17) (4.18) (4.19) (4.20)

As indicated above, the approach by Taha and Ablowitz to these equations would be as follows: consider the last 2 equations as the equations of motion, where the (non-local) expressions for the quantities \(B^{(-1)}_k\), \(C^{(-1)}_k\), \(D^{(0)}_k\) follow directly from the first 4 equations. The crucial feature of our approach to the solution of these equations is that we use all 6 equations to derive *local* expressions. In other words, we do not assume that the last two of them play a special role.

Note first of all that due to \([4.13]\), the equations \([4.17]\), \([4.18]\) may be equivalently re-written as

\begin{align*}
B^{(-1)}_{k+1} &= \bar{q}_{k-1} - \bar{q}_k (\bar{q}_{k-1} \bar{r}_k + D^{(-2)}_{k+1}) \\
C^{(-1)}_k &= r_{k+1} - r_k (q_k r_{k+1} + D^{(-2)}_k)
\end{align*}

(4.21) (4.22)

Now we introduce the auxiliary quantity \(S_k\) by the formula

\[ S_k = D^{(-2)}_k + q_k r_{k+1} + \bar{q}_{k-2} \bar{r}_{k-1}. \]

(4.23)

which immediately gives \([4.11]\). With the help of this quantity we re-write the equations \([4.21]\), \([4.22]\) as \([4.12]\) and \([4.13]\), respectively. Substituting \([4.12]\), \([4.13]\) into \([4.16]\), we obtain the equation which may be written as

\begin{align*}
D^{(0)}_{k+1} - D^{(0)}_k &= \bar{q}_k r_{k+2} - \bar{q}_{k-2} r_k - \bar{q}_k r_{k+1} S_{k+1} = \bar{q}_{k-1} r_k S_k \\
&-(\bar{q}_k - q_k) (r_{k+2} - r_{k+1} S_{k+1}) - (\bar{r}_k - r_k) (\bar{q}_{k-2} - \bar{q}_{k-1} S_k).
\end{align*}

(4.24)
Similarly, substituting (4.12), (4.13) into (4.19), (4.20) we obtain two equations which may be put down as

\[(1 + h\tilde{q}_{k-1}r_{k+1})(\tilde{q}_k - q_k) = h(\tilde{q}_{k-2} - \tilde{q}_{k-1}S_k) - h\tilde{q}_k \left(D^{(0)}_k - \tilde{q}_{k-1}r_{k+1}\right)\] (4.25)

\[(1 + h\tilde{q}_{k-1}r_{k+1}r_{k+1})(\tilde{r}_k - r_k) = -h(r_{k+2} - r_{k+1}S_{k+1}) + hr_k \left(D^{(0)}_{k+1} - \tilde{q}_{k-1}r_{k+1}\right)\] (4.26)

Multiplying (4.24) by \((1 + h\tilde{q}_{k-1}r_{k+1})\) and using on the right–hand side the expressions (4.25), (4.26), we obtain an equation, which after some identical re-arrangements may be presented as

\[\left(1 + hD^{(0)}_{k+1}\right) \left(1 + h\tilde{q}_{k-1}r_{k+1} + h\tilde{q}_{k-2}r_k - h\tilde{q}_{k-1}r_kS_k\right)\]

\[\quad = \left(1 + hD^{(0)}_k\right) \left(1 + h\tilde{q}_kr_{k+2} + h\tilde{q}_{k-1}r_{k+1} - h\tilde{q}_kr_{k+1}S_{k+1}\right).\]

So, the quantity

\[\left(1 + hD^{(0)}_k\right) / \left(1 + h\tilde{q}_{k-1}r_{k+1} + h\tilde{q}_{k-2}r_k - h\tilde{q}_{k-1}r_kS_k\right)\]

does not depend on \(k\), and, taking the limit \(k \to \pm \infty\), we see that it is equal to 1, which gives (4.14). Substituting this result back in (4.25), (4.26), we obtain (4.8).

Now to finish the proof it remains to obtain the local expression (4.9). To this end we first of all re-write the equation (4.15) in terms of \(S_k\) and find the following difference equation:

\[S_{k+1} - S_k = q_{k+1}r_{k+2} - \tilde{q}_k - \tilde{r}_{k-1}.\] (4.27)

As it stands, it does not allow a local solution. In order to achieve our goal, we have to once more switch on the equations of motion (4.8). Putting on the right–hand side of (4.27)

\[q_{k+1} = \tilde{q}_{k+1} - h \frac{(\tilde{q}_{k-1} - \tilde{q}_kS_{k+1})(1 - \tilde{q}_{k+1}r_{k+1})}{(1 + h\tilde{q}_kr_{k+2})},\]
\[\tilde{r}_{k-1} = r_{k-1} - h \frac{(r_{k+1} - r_kS_k)(1 - \tilde{q}_{k-1}r_{k-1})}{(1 + h\tilde{q}_{k-2}r_k)},\]

we obtain after some re-arrangements:

\[\left(S_{k+1} + h\tilde{q}_{k-1}r_{k+2}\right) \frac{(1 + h\tilde{q}_k\tilde{q}_{k+1}r_{k+1}r_{k+2})}{(1 + h\tilde{q}_kr_{k+2})} - \left(S_k + h\tilde{q}_{k-2}r_{k+1}\right) \frac{(1 + h\tilde{q}_{k-2}\tilde{q}_{k-1}r_{k-1}r_k)}{(1 + h\tilde{q}_{k-2}r_k)}\]

\[= (\tilde{q}_{k+1}r_{k+2} - \tilde{q}_{k-2}r_{k-1})(1 + h\tilde{q}_{k-1}r_{k+1}).\]

Multiplying both sides by \(\frac{(1 + h\tilde{q}_{k-1}\tilde{q}_kr_{k+1})}{(1 + h\tilde{q}_{k-1}r_{k+1})}\),
we arrive at the identity which may be read in the following way: the left-hand side of (4.9) does not depend on k. The value of this constant may be determined from the $k \to \pm \infty$ limit. ■

**Remark 1.** It might be desirable to have the expressions for all entries of the matrix $V_k^{(2)}$ in terms of $(q, \mathbf{r})$ solely, and of the matrix $V_k^{(-2)}$ – in terms of $(\mathbf{q}, \mathbf{r})$. Notice that the quantities $A_k^{(0)}, D_k^{(0)}$ already have the desired form. For the quantities $B_k^{(1)}, C_k^{(1)}$ it is possible to derive from (4.2), (4.1) the following nice expressions:

$$P_k - q_{k+1}r_k = (c + q_k\tilde{r}_{k-1} + q_{k-1}\tilde{r}_{k-2}) \frac{(1 - h q_k \tilde{r}_{k-2})}{(1 - h q_k q_{k-1} \tilde{r}_{k-1} \tilde{r}_{k-2})},$$  \hspace{1cm} (4.28)

$$P_k - \tilde{q}_{k-1}\tilde{r}_{k-2} = (c + q_{k+1}\tilde{r}_{k-1} + q_k \tilde{r}_{k-1}) \frac{(1 - h q_{k+1} \tilde{r}_{k-1})}{(1 - h q_{k+1} q_k \tilde{r}_{k-1} \tilde{r}_{k-1})},$$  \hspace{1cm} (4.29)

Analogously, for the quantities $B_k^{(-1)}, C_k^{(-1)}$ it follows from (4.9), (4.8):

$$S_k - q_k r_{k+1} = (c + \tilde{q}_{k-1}r_k + \tilde{q}_{k-2}r_{k-1}) \frac{(1 + h \tilde{q}_{k-2}r_k)}{(1 + h \tilde{q}_{k-2}q_k r_{k-1} r_{k-1})},$$  \hspace{1cm} (4.30)

$$S_k - \tilde{q}_{k-2}\tilde{r}_{k-1} = (c + \tilde{q}_k r_{k+1} + \tilde{q}_{k-1}r_{k}) \frac{(1 + h \tilde{q}_{k-1}r_{k+1})}{(1 + h \tilde{q}_{k-1}q_k r_{k+1} r_{k+1})}.$$  \hspace{1cm} (4.31)

Unfortunately, analogous expressions for $A_k^{(2)},$ resp. $D_k^{(-2)}$ are complicated and non-elegant.

**Remark 2.** Notice that from the Theorems 1,2 we can recover the maps from the previous paper [8], together with the corresponding matrices $V_k$. Indeed, rescaling $h \mapsto h/c$ and then sending $c \to \infty$, we find from the Theorem 1 the map

$$\mathcal{T}_i(-h) : \begin{cases} (\tilde{q}_k - q_k)/h = -q_{k+1}(1 - q_k \tilde{r}_k), \\ (\tilde{r}_k - r_k)/h = \tilde{r}_{k-1}(1 - q_k \tilde{r}_k) \end{cases}.$$  \hspace{1cm} (4.32)

and the matrix

$$V_k^{(1)}(\mathbf{q}, \mathbf{r}, -h) = \begin{pmatrix} 1 - h \lambda^2 + h q_k \tilde{r}_{k-1} & -h \lambda q_k \\ -h \lambda \tilde{r}_{k-1} & 1 \end{pmatrix}.$$  \hspace{1cm} (4.33)

and from the Theorem 2 the map

$$\mathcal{T}_{-i}(-h) : \begin{cases} (q_k - q_k)/h = -q_{k-1}(1 - q_k r_k), \\ (\tilde{r}_k - r_k)/h = r_{k+1}(1 - q_k r_k) \end{cases}.$$  \hspace{1cm} (4.34)
along with the corresponding matrix

\[
V_k^{(-1)}(\vec{q}, r, -h) = \begin{pmatrix}
1 & h\lambda^{-1}\vec{q}_{k-1} \\
h\lambda^{-1}r_k & 1 + h\lambda^{-2} - h\vec{q}_{k-1}r_k
\end{pmatrix}.
\] (4.35)

**Corollary 1.** Consider the map \( T_2^{-1}(-h; c) : (q, r) \mapsto (\vec{q}, \vec{r}) \). This map is defined by the equations of motion

\[
\begin{align*}
(\vec{q}_k - q_k)/h &= \frac{(\vec{q}_{k+2} - \vec{q}_{k+1}Q_{k+1}) (1 - \vec{q}_k r_k)}{(1 + h\vec{q}_{k+1}r_{k-1})}, \\
(\vec{r}_k - r_k)/h &= -\frac{(r_{k-2} - r_{k-1}Q_k) (1 - \vec{q}_k r_k)}{(1 + h\vec{q}_{k+1}r_{k-1})}
\end{align*}
\] (4.36)

where

\[
(Q_k + h\vec{q}_{k+1}r_{k-2}) \frac{(1 + h\vec{q}_k\vec{q}_{k-1}r_{k-1}r_{k-2}) (1 + h\vec{q}_{k+1}\vec{q}_k r_{k-1})}{(1 + h\vec{q}_{k+1}r_{k-1})} = c.
\]

It approximates the flow (2.18) and has a commutation representation (3.4) with the role of \( V_k \) played by the matrix

\[
W_k^{(2)} = \frac{1}{1 + hA_k^{(0)}} \begin{pmatrix}
1 - h\lambda^2\vec{q}_k r_{k-1} & -h\lambda^3\vec{q}_k - h\lambda B_k^{(1)} \\
-h\lambda^3 r_{k-1} - h\lambda C_k^{(1)} & 1 + hA_k^{(0)} + h\lambda^2 A_k^{(2)} - h\lambda^4
\end{pmatrix}.
\] (4.37)

Here the quantities \( A_k^{(2)}, B_k^{(1)}, C_k^{(1)}, A_k^{(0)} \) are obtained from the quantities \( A_k^{(2)}, B_k^{(1)}, C_k^{(1)}, A_k^{(0)} \) by the change \( h \) to \(-h\), \( q \) to \( \vec{q} \), and \( r \) to \( \vec{r} \).

**Corollary 2.** Consider the map \( T_2^{-1}(-h; c) : (q, r) \mapsto (\vec{q}, \vec{r}) \). This map is defined by the equations of motion

\[
\begin{align*}
(\vec{q}_k - q_k)/h &= \frac{(q_{k-2} - q_{k-1}R_k) (1 - q_k \vec{r}_k)}{(1 - h\vec{q}_{k-1}\vec{r}_{k+1})}, \\
(\vec{r}_k - r_k)/h &= -\frac{(\vec{r}_{k+2} - \vec{r}_{k+1}R_{k+1}) (1 - q_k \vec{r}_k)}{(1 - h\vec{q}_{k-1}\vec{r}_{k+1})}
\end{align*}
\] (4.38)

where

\[
(R_k - h\vec{q}_{k-2}\vec{r}_{k+1}) \frac{(1 - h\vec{q}_{k-2}\vec{r}_{k-1}\vec{r}_{k+1}) (1 - h\vec{q}_{k-1}q_k \vec{r}_{k+1})}{(1 - h\vec{q}_{k-2}\vec{r}_k) (1 - h\vec{q}_{k-1}\vec{r}_{k+1})}
\]
\[-(q_k r_{k+1} + q_{k-1} \tilde{r}_k + q_{k-2} r_{k-1} - h q_{k-2} q_{k-1} q_k \tilde{r}_{k-1} r_{k+1}) = c.\]

It approximates the flow (2.19) and has a commutation representation (3.1) with the role of \(V_k\) played by the matrix

\[
W_k^{-2} = \frac{1}{1 - hD_k^{(0)}} \begin{pmatrix}
1 - hD_k^{(0)} - h\lambda^{-2}D_k^{(-2)} + h\lambda^{-4} h\lambda^{-3} q_{k-1} + h\lambda^{-1} B_k^{(-1)} & h\lambda^{-3} \tilde{r}_k + h\lambda^{-1} C_k^{(-1)} \\
h\lambda^{-3} \tilde{r}_k + h\lambda^{-1} C_k^{(-1)} & 1 + h\lambda^{-2} q_{k-1} \tilde{r}_k
\end{pmatrix}.
\]

Here the quantities \(D_k^{(-2)}, B_k^{(-1)}, C_k^{(-1)}, D_k^{(0)}\) are obtained from the quantities \(D_k^{(-2)}, B_k^{(-1)}, C_k^{(-1)}, D_k^{(0)}\) by the change \(h\) to \(-h\), \(\tilde{q}\) to \(q\), and \(r\) to \(\tilde{r}\).

**Proof.** Since the both Corollaries 1,2 are proved analogously, we demonstrate again only the second one. The map \(T_{-2}^{-1}(-h; c)\) obviously allows the commutation representation (3.1) with the matrix \(\left(V_k^{-2}(\tilde{q}, r, -h)\right)^{-1}\) playing the role of \(V_k\). The Corollary 2 will be proved, if we show that

\[
W_k^{(-2)} = (1 + h\lambda^{-4}) \left(V_k^{(-2)}(\tilde{q}, r, -h)\right)^{-1}.
\]

But from the expressions given in the Theorem 2 it is possible to derive that

\[
\det V_k^{(-2)}(\tilde{q}, r, h) = (1 - h\lambda^{-4}) \left(1 + hD_k^{(0)}\right),
\]

so that

\[
\det V_k^{(-2)}(\tilde{q}, r, -h) = (1 + h\lambda^{-4}) \left(1 - hD_k^{(0)}\right),
\]

which implies the above statement. □

5 Relation to the Taha–Ablowitz schemes

From the results of the previous Section it is easy to see that our maps are indeed some particular cases of the Taha–Ablowitz scheme. Namely:

- The map \(T_2(h; c)\) coincides with the Taha–Ablowitz difference scheme with the only nonzero parameters \(\alpha^{(4)} = 1, \alpha^{(2)} = -c\).

- The map \(T_{-2}(h; c)\) coincides with the Taha–Ablowitz difference scheme with the only nonzero parameters \(\delta^{(-4)} = -1, \delta^{(-2)} = c\).

- The map \(T_2^{-1}(-h; c)\) coincides with the Taha–Ablowitz difference scheme with the only nonzero parameters \(\delta^{(4)} = -1, \delta^{(2)} = c\).
− The map \( T_{-2}^{-1}(-h; c) \) coincides with the Taha–Ablowitz difference scheme with the only nonzero parameters \( \alpha^{(-4)} = 1, \alpha^{(-2)} = -c \).

These schemes, obviously, do not satisfy the condition (3.4) (they certainly should not, because the underlying continuous time flows (2.18), (2.19) do not allow the reduction (2.2)). However, certain compositions of our maps, approximating the flow

\[
F_2(t) \circ F_1(-ct) \circ F_{-2}(-t) \circ F_{-1}(ct),
\]

(5.1)
do have the desired property (3.4), making them suitable for numerical integration of the space discretized mKdV equation (1.4).

Namely, taking into account that the composition of maps corresponds to the multiplication of the corresponding \( V_k \) matrices, we see immediately that the following two statements hold.

**Theorem 3.** The map

\[
T_2(h; c) \circ T_{-2}(-h; c)
\]

(5.2)

approximating the flow (5.1), coincides with the Taha–Ablowitz difference scheme with the only nonzero parameters \( \alpha^{(4)} = \delta^{(-4)} = 1, \alpha^{(2)} = \delta^{(-2)} = -c \), and allows the reduction (3.5).

**Theorem 4.** The map

\[
T_2^{-1}(-h; c) \circ T_{-2}^{-1}(h; c)
\]

(5.3)

approximating the flow (5.1), coincides with the Taha–Ablowitz difference scheme with the only nonzero parameters \( \alpha^{(-4)} = \delta^{(4)} = -1, \alpha^{(-2)} = \delta^{(2)} = c \), and allows the reduction (3.5).

These two maps (taken by \( c = 2 \)) are the first candidates to the role of valid difference schemes for numerical integration of the mKdV equation. Just as the scheme tested in [3], they are integrable, i.e. they have infinitely many integrals of motion (in the rapidly decaying case; in the \( N \)-periodic case the number of integrals reduces to \( N \)), and all the integrals are in involution with respect to the Poisson bracket (2.8). Additionally, they have an advantage of locality. More precisely, each step of time integration by the corresponding numerical scheme consists of solving two local systems of nonlinear equations for the updates of \( q, r \). We expect that the savings of the amount of computations due to locality will let these schemes to exceed the original Taha–Ablowitz scheme, even despite the fact that the latter is of the second order in \( h \), while (5.2), (5.3) are only of the first order in \( h \).

In this connection it should be mentioned that the maps (5.2), (5.3) are suitable building blocks for applying the Ruth–Yoshida–Suzuki techniques [10], which in principle allow to construct the schemes of an arbitrary high order in \( h \). The first step in this direction is given by the following statement.
Theorem 5. The composition

\[ T_2(h/2; c) \circ T_2(-h/2; c) \circ T_2^{-1}(-h/2; c) \circ T_2^{-1}(h/2; c) \]  

approximates the flow (5.1) up to the second order in \( h \). It coincides with the Taha–Ablowitz difference scheme with the parameters

\[ \alpha^{(4)} = \delta^{(-4)} = -\alpha^{(-4)} = -\delta^{(4)} = \frac{1}{2\Delta}, \]

\[ \alpha^{(2)} = \delta^{(-2)} = \frac{c(1 - h/2)}{2\Delta}, \quad \alpha^{(-2)} = \delta^{(2)} = \frac{c(1 + h/2)}{2\Delta}, \]

\[ \alpha^{(0)} = \delta^{(0)} = 0, \]

where

\[ \Delta = 1 - h^2(1 + c^2)/4 \]

In particular, this composition allows the reduction (3.5).

Proof. The map (5.4), as a composition of two Taha–Ablowitz schemes, allows a commutation representation (3.1) with the matrix \( V_k \) whose \( A_k, D_k \) entries containing, in principle, terms with \( \lambda^j, j = 0, \pm 2, \pm 4, \pm 6, \pm 8 \), while the \( B_k, C_k \) entries contain the terms with \( j = \pm 1, \pm 3, \pm 5, \pm 7 \). A careful inspection will convince that only the terms with \(|j| \leq 4\) are actually present in this matrix. To this end we need the following two statements.

Lemma 1. The entries of the matrix \( V_k \) for the map (5.2) have the following \( \lambda \)-dependence:

\[ A(\lambda) = 1 + h\lambda^4 + h\lambda^2A_k^{(2)} + hA_k^{(0)} + h\lambda^{-2}\bar{q}_{k-1}r_k \]

\[ D(\lambda) = 1 + h\lambda^2q_k\bar{r}_{k-1} + hD_k^{(0)} + h\lambda^{-2}D_k^{(-2)} + h\lambda^{-4} \]

\[ B(\lambda) = h\lambda^3q_k + h\lambda B_k^{(1)} + h\lambda^{-1}B_k^{(-1)} + h\lambda^{-3}\bar{q}_{k-1} \]

\[ C(\lambda) = h\lambda^3r_k + h\lambda C_k^{(1)} + h\lambda^{-1}C_k^{(-1)} + h\lambda^{-3}\bar{r}_k \]

Lemma 2. The entries of the matrix \( V_k \) for the map (5.2) have the following \( \lambda \)-dependence:

\[ A(\lambda) = \Lambda_k \left( 1 - h\lambda^{2}\bar{q}_k r_{k-1} + hA_k^{(0)} + h\lambda^{-2}A_k^{(-2)} - h\lambda^{-4} \right) \]

\[ D(\lambda) = \Lambda_k \left( 1 - h\lambda^4 + h\lambda^2D_k^{(2)} + hD_k^{(0)} - h\lambda^{-2}q_{k-1}\bar{r}_k \right) \]

\[ B(\lambda) = \Lambda_k \left( h\lambda^3q_k + h\lambda B_k^{(1)} + h\lambda^{-1}B_k^{(-1)} + h\lambda^{-3}q_{k-1} \right) \]

\[ C(\lambda) = \Lambda_k \left( h\lambda^3r_{k-1} + h\lambda C_k^{(1)} + h\lambda^{-1}C_k^{(-1)} + h\lambda^{-3}\bar{r}_k \right) \]
(The exact expressions for the unspecified functions play no role in the following reasonings; therefore we denoted carelessly different objects in the above statements by one and the same symbol, such as $A_k^{(0)}$). The proof of these two lemmas are similar, so we give only the proof of the lemma 1. Let

$$T_2(-h; c) : (q, r) \mapsto (\tilde{q}, \tilde{r}), \quad T_2(h; c) : (\tilde{q}, \tilde{r}) \mapsto (q, r)$$

It follows from the Theorems 1,2 that the the matrix $V_k$ for the composition map (5.2) is equal to

$$V_k = V_k^{(2)}(q, \tilde{r}, h) V_k^{(-2)}(\tilde{q}, r, -h).$$

The general $\lambda$–dependence of this matrix follows directly from (4.3), (4.10), and we have only to prove the explicit expressions for $A_k^{\pm 3}, B_k^{\pm 3}, C_k^{\pm 3}, D_k^{(2)}$ given in the formulation of the Lemma. The expressions $C_k^{(3)} = \tilde{r}_{k-1}, C_k^{(-3)} = r_k$ follow immediately, the rest is derived as follows:

$$B_k^{(3)} = \tilde{q}_k (1 - hD_k^{(0)}) + hB_k^{(1)} = q_k$$

$$D_k^{(2)} = \tilde{q}_k \tilde{r}_{k-1} (1 - hD_k^{(0)}) + h \tilde{r}_{k-1} B_k^{(-1)} = q_k \tilde{r}_{k-1}$$

$$B_k^{(-3)} = \tilde{q}_{k-1} (1 - hA_k^{(0)}) + hB_k^{(1)} = \tilde{q}_{k-1}$$

$$A_k^{(-2)} = \tilde{q}_{k-1} r_k (1 - hA_k^{(0)}) + hr_k B_k^{(1)} = \tilde{q}_{k-1} r_k$$

Here the first two expressions follow from (4.19) (of course, with the change $h$ to $-h$ and $q$ to $\tilde{q}$), and the last two expressions follow from an analogous ”evolution equation for $q_{k-1}$” for the map $T_2$. ■

Expressions given in the Lemmas 1,2 imply that in the product of the corresponding matrices $V_k$ the following terms vanish identically: the terms with $\lambda^{\pm 8}, \lambda^{\pm 6}$ in the entries $A_k, D_k$, the terms with $\lambda^7, \lambda^{-5}$ in the entry $B_k$, and the terms with $\lambda^{-7}, \lambda^5$ in the entry $C_k$. After this the commutation representation (4.1) shows that the last obstacles, namely the term with $\lambda^5$ in $B_k$ and the term with $\lambda^{-5}$ in $C_k$, also must vanish. So, the matrix $V_k$ corresponding to the composition (5.2) has the desired $\lambda$–dependence. Now it remains to observe that the limit values of its entries by $k \to \pm \infty$ are:

$$\lim_{k \to \pm \infty} A_k(\lambda) = \left(1 - h \lambda^{-4}/2 + h \lambda^{-2} c/2\right) \left(1 + h \lambda^{-4}/2 - h \lambda^2 c/2\right)$$

$$= \Delta \left(1 + h \lambda^4 \alpha^{(4)} + h \lambda^2 \alpha^{(2)} + h \lambda^{-2} \alpha^{(-2)} + h \lambda^{-4} \alpha^{(-4)}\right),$$

$$\lim_{k \to \pm \infty} D_k(\lambda) = \left(1 - h \lambda^4/2 + h \lambda^2 c/2\right) \left(1 + h \lambda^{-4}/2 - h \lambda^{-2} c/2\right)$$

$$= \Delta \left(1 + h \lambda^4 \delta^{(4)} + h \lambda^2 \delta^{(2)} + h \lambda^{-2} \delta^{(-2)} + h \lambda^{-4} \delta^{(-4)}\right)$$

with the values $\alpha^{(j)}, \delta^{(j)}$ given in the Theorem. ■
6 Conclusion

In the present paper we re-considered the discretizations of the modified Korteweg-de Vries equation due to Taha and Ablowitz. We demonstrated that by some choice of parameters their highly non-local scheme may be factorized into the product of much more simple (local) ones, each of them approximating a more simple and fundamental flow of the Ablowitz–Ladik hierarchy. These local schemes may be studied exhaustively. In particular, by the same change of variables as in the previous paper [8], we can establish a relation to the pair of ”second” discrete time flows of the relativistic Toda hierarchy, which gives a simple way to determine interpolating Hamiltonian flows for our maps and to solve them in terms of a factorization problem in a loop group (the relation between the Ablowitz–Ladik and the relativistic Toda hierarchies is due to [11]). We guess that also in the practical computations our variant of the difference scheme will exceed considerably the old one. It would be interesting and important to carry out the corresponding numerical experiments.

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7 Appendix: difference scheme by Taha and Ablowitz

The twenty equations, following from (3.6)–(3.9), read:

\[ A^{(4)}_{k+1} - A^{(4)}_k = 0 \] (7.1)
\[ A^{(-4)}_{k+1} - A^{(-4)}_k = \tilde{q}_k C^{(-3)}_k - r_k B^{(-3)}_{k+1} \] (7.2)
\[ A^{(2)}_{k+1} - A^{(2)}_k = \tilde{q}_k C^{(3)}_k - r_k B^{(3)}_{k+1} \] (7.3)
\[ A^{(-2)}_{k+1} - A^{(-2)}_k = \tilde{q}_k C^{(-1)}_k - r_k B^{(-1)}_{k+1} \] (7.4)
\[ A^{(0)}_{k+1} - A^{(0)}_k = \tilde{q}_k C^{(1)}_k - r_k B^{(1)}_{k+1} \] (7.5)

\[ D^{(-4)}_{k+1} - D^{(-4)}_k = 0 \] (7.6)
\[ D^{(4)}_{k+1} - D^{(4)}_k = \tilde{r}_k B^{(3)}_k - q_k C^{(3)}_{k+1} \] (7.7)
\[ D^{(-2)}_{k+1} - D^{(-2)}_k = \tilde{r}_k B^{(-3)}_k - q_k C^{(-3)}_{k+1} \] (7.8)
\[ D^{(2)}_{k+1} - D^{(2)}_k = \tilde{r}_k B^{(1)}_k - q_k C^{(1)}_{k+1} \] (7.9)
\[ D^{(0)}_{k+1} - D^{(0)}_k = \tilde{r}_k B^{(-1)}_k - q_k C^{(-1)}_{k+1} \] (7.10)

\[ B^{(3)}_k = q_k A^{(4)}_{k+1} - \tilde{q}_k D^{(4)}_k \] (7.11)
\[ B^{(-3)}_{k+1} = \tilde{q}_k D^{(-4)}_k - q_k A^{(-4)}_{k+1} \] 
\[ B^{(1)}_k = q_k A^{(2)}_{k-1} - \tilde{q}_k D^{(2)}_k + B^{(3)}_{k+1} \] 
\[ B^{(-1)}_{k+1} = \tilde{q}_k D^{(-2)}_k - q_k A^{(-2)}_{k+1} + B^{(-3)}_k \]  
\[ C^{(-3)}_k = r_k D^{(-4)}_{k+1} - \tilde{r}_k A^{(-4)}_k \] 
\[ C^{(3)}_{k+1} = -\tilde{r}_k A^{(4)}_k - r_k D^{(4)}_{k+1} \] 
\[ C^{(-1)}_k = r_k D^{(-2)}_{k+1} - \tilde{r}_k A^{(-2)}_k + C^{(-3)}_{k+1} \] 
\[ C^{(1)}_{k+1} = -\tilde{r}_k A^{(2)}_k - r_k D^{(2)}_{k+1} + C^{(3)}_k \] 
\[ \tilde{q}_k - q_k = \left\{ q_k A^{(0)}_{k+1} - \tilde{q}_k D^{(0)}_k + B^{(1)}_{k+1} - B^{(-1)}_k \right\} \] 
\[ \tilde{r}_k - r_k = \left\{ r_k D^{(0)}_{k+1} - \tilde{r}_k A^{(0)}_k + C^{(-1)}_{k+1} - C^{(1)}_k \right\} \] 

Taha and Ablowitz interpret the first 18 equations (7.1)–(7.18) as the defining relations for the 18 coefficients \( A^{(j)}_k - D^{(j)}_k \), and the last two equations (7.19), (7.20) as the evolution equations for \( q_k, r_k \). This leads to a highly nonlocal scheme. Let us indicate how do Taha and Ablowitz solve the first 18 equations (with some amendments).

First of all, (7.1), (7.6) imply
\[ A^{(4)}_k = \alpha^{(4)}, \quad D^{(4)}_k = \delta^{(-4)} \]  
\[ \text{(7.21)} \]
where \( \alpha^{(4)}, \delta^{(-4)} \) are some constants (and the same holds for other \( \alpha^{(j)}, \delta^{(j)} \) appearing further on). Substituting (7.11), (7.16) into (7.7), and (7.12), (7.15) into (7.2), we get upon use of (7.21) the following difference equations:
\[ D^{(4)}_{k+1} (1 - q_k r_k) = D^{(4)}_k (1 - \tilde{q}_k \tilde{r}_k), \quad A^{(-4)}_{k+1} (1 - q_k r_k) = A^{(-4)}_k (1 - \tilde{q}_k \tilde{r}_k). \]  
\[ \text{(7.22)} \]
Their solution may be put into the form
\[ D^{(4)}_k = \delta^{(4)} \Lambda_k, \quad A^{(-4)}_k = \alpha^{(-4)} \Lambda_k, \]  
\[ \text{(7.23)} \]
where
\[ \Lambda_k = \prod_{j=-\infty}^{k-1} \frac{1 - \tilde{q}_j \tilde{r}_j}{1 - q_j r_j} \]  
\[ \text{(7.24)} \]
is for all \( k \) of order \( 1 + O(h) \) and tends to 1 as \( k \to \pm \infty \) (due to the infinite-dimensional analog of the integral of motion (2.7)).
The back substitution \((7.21), (7.23)\) into \((7.11), (7.16), (7.12), (7.13)\) results in
\[
D_k^{(3)} = \alpha^{(4)}q_k - \delta^{(4)}\bar{q}_k\Lambda_k, \quad (7.25)
\]
\[
C_k^{(-3)} = \delta^{(-4)}\bar{r}_k - \alpha^{(-4)}\bar{r}_k\Lambda_k, \quad (7.26)
\]
\[
B_k^{(-3)} = \delta^{(-4)}\bar{q}_{k-1} - \alpha^{(-4)}q_{k-1}\Lambda_k, \quad (7.27)
\]
\[
C_k^{(3)} = \alpha^{(4)}\bar{r}_{k-1} - \delta^{(4)}r_{k-1}\Lambda_k. \quad (7.28)
\]
Substituting the last formulas into \((7.3), (7.8)\), we arrive at the following difference equations:
\[
A_{k+1}^{(2)} - A_k^{(2)} = \alpha^{(4)}(q_k\bar{r}_{k-1} - q_{k+1}\bar{r}_k) + \delta^{(4)}(q_{k+1}\bar{r}_k\Lambda_{k+1} - \bar{q}_k r_{k-1}\Lambda_k), \quad (7.29)
\]
\[
D_{k+1}^{(-2)} - D_k^{(-2)} = \delta^{(-4)}(\bar{q}_{k-1}\bar{r}_k - q_k r_{k+1}) + \alpha^{(-4)}(q_{k+1}\bar{r}_{k+1}\Lambda_{k+1} - q_{k-1}\bar{r}_k\Lambda_k). \quad (7.30)
\]
Their solutions are given by
\[
A_k^{(2)} = \alpha^{(2)} - \alpha^{(4)}(q_k r_{k-1} + \Phi_k) + \delta^{(4)}\bar{q}_k r_{k-1}\Lambda_k, \quad (7.31)
\]
\[
D_k^{(-2)} = \delta^{(-2)} - \delta^{(-4)}(q_k r_{k-1} + \Psi_k) + \alpha^{(-4)}q_{k-1}\bar{r}_k\Lambda_k, \quad (7.32)
\]
where the quantities
\[
\Phi_k = \sum_{j=-\infty}^{k-1} (q_j r_{j-1} - \bar{q}_j \bar{r}_{j-1}), \quad \Psi_k = \sum_{j=-\infty}^{k-1} (q_{j-1} r_j - \bar{q}_{j-1} \bar{r}_j) \quad (7.33)
\]
are of order \(O(h)\) and vanish by \(k \to \pm\infty\) (the statement for \(k \to +\infty\) follows from the fact that the infinite dimensional analogs of the quantities \(H_{\pm 1}(q,r)\) from \((2.11)\) are the integrals of motion).

To proceed further, we substitute \((7.13), (7.18)\) into \((7.9)\). Using \((7.25), (7.28), \) and \((7.29)\), we derive from \((7.9)\) the following difference equation:
\[
(D_{k+1}^{(2)} - \alpha^{(4)}q_{k+1}\bar{r}_k)(1 - q_k r_k) - (D_k^{(2)} - \alpha^{(4)}q_k \bar{r}_{k-1})(1 - \bar{q}_k \bar{r}_k) =
\]
\[
-\delta^{(4)}(\bar{q}_{k+1}\bar{r}_k(1 - q_k r_k)\Lambda_{k+1} - q_k r_{k-1}(1 - \bar{q}_k \bar{r}_k)\Lambda_k).
\]
Analogously, substituting \((7.14), (7.17)\) into \((7.4)\) and using \((7.27), (7.26), \) and \((7.30)\), we obtain:
\[
(A_{k+1}^{(-2)} - \delta^{(-4)}\bar{q}_{k+1} r_{k+1})(1 - q_k r_k) - (A_k^{(-2)} - \delta^{(-4)}\bar{q}_{k-1} r_k)(1 - \bar{q}_k \bar{r}_k) =
\]
\[
-\alpha^{(-4)}(\bar{q}_k \bar{r}_{k+1}(1 - q_k r_k)\Lambda_{k+1} - q_{k-1} r_k(1 - \bar{q}_k \bar{r}_k)\Lambda_k).
\]
A little trick (apparently missed in [8]), based on the formula $(1-q_k r_k)\Lambda_{k+1} = (1-\tilde{q}_k \tilde{r}_k)\Lambda_k$, allows to solve these difference equations as

$$D_k^{(2)} = \delta^{(2)} \Lambda_k + \alpha^{(4)} q_k \tilde{r}_{k-1} - \delta^{(4)} (\tilde{q}_k \tilde{r}_{k-1} - \Phi_k) \Lambda_k, \quad (7.34)$$

$$A_k^{(-2)} = \alpha^{(-2)} \Lambda_k + \delta^{(-4)} \tilde{q}_{k-1} r_k - \alpha^{(-4)} (\tilde{q}_{k-1} \tilde{r}_k - \Psi_k) \Lambda_k. \quad (7.35)$$

Now we have determined all the quantities from the right-hand sides of (7.13), (7.14), (7.18), (7.17). Collecting all the results, we find:

$$B_k^{(1)} = \alpha^{(2)} q_k - \delta^{(2)} \tilde{q}_k \Lambda_k + \alpha^{(4)} \left( q_{k+1} - q_{k} (q_{k+1} r_k + q_k r_{k-1} + \Phi_k) \right) - \delta^{(4)} \left( \tilde{q}_{k+1} - \tilde{q}_k (\tilde{q}_{k+1} \tilde{r}_k + \tilde{q}_k \tilde{r}_{k-1} - \Phi_k) \right) \Lambda_k \quad (7.36)$$

$$B_k^{(-1)} = \delta^{(-2)} \tilde{q}_{k-1} - \alpha^{(-2)} q_{k-1} \Lambda_k + \delta^{(-4)} \left( \tilde{q}_{k-2} - \tilde{q}_{k-1} (q_{k-2} r_k + \tilde{q}_{k-2} \tilde{r}_{k-1} + \Psi_k) \right) - \alpha^{(-4)} \left( q_{k-2} - q_{k-1} (q_{k-2} r_{k-1} + \tilde{q}_{k-2} \tilde{r}_{k-1} - \Psi_k) \right) \Lambda_k \quad (7.37)$$

$$C_k^{(1)} = \alpha^{(2)} \tilde{r}_{k-1} - \delta^{(2)} \tilde{r}_{k-1} \Lambda_k + \alpha^{(4)} \left( \tilde{r}_{k-2} - \tilde{r}_{k-1} (q_{k-2} r_{k-1} + \tilde{q}_{k-2} \tilde{r}_{k-2} + \Phi_k) \right) - \delta^{(4)} \left( r_{k-2} - r_{k-1} (q_{k-2} r_{k-2} + \tilde{q}_{k-2} \tilde{r}_{k-1} - \Phi_k) \right) \Lambda_k \quad (7.38)$$

$$C_k^{(-1)} = \delta^{(-2)} r_k - \alpha^{(-2)} \tilde{r}_k \Lambda_k + \delta^{(-4)} \left( r_{k+1} - r_k (q_r r_{k+1} + q_{k-1} r_k + \Psi_k) \right) - \alpha^{(-4)} \left( \tilde{r}_{k+1} - \tilde{r}_k (\tilde{q}_r \tilde{r}_{k+1} + \tilde{q}_{k-1} \tilde{r}_k - \Psi_k) \right) \Lambda_k \quad (7.39)$$

Finally, these expressions, being substituted in (7.5), (7.10), result in difference equations whose solutions may be represented as

$$A_k^{(0)} = \alpha^{(0)} - \alpha^{(2)} (q_k r_{k-1} + \Phi_k) + \delta^{(2)} \tilde{q}_k r_{k-1} \Lambda_k - \alpha^{(4)} (q_{k+1} r_{k-1} + q_k r_{k-2} - \tilde{q}_k q_k \tilde{r}_{k-1} r_{k-1} - \tilde{q}_k q_{k-1} \tilde{r}_{k-1} r_{k-2} + \Xi_k) + \delta^{(4)} (\tilde{q}_{k+1} r_{k-1} + \tilde{q}_k r_{k-2} - \tilde{q}_k q_{k-1} \tilde{r}_{k-1} r_{k-1} + \tilde{q}_k q_{k-1} \tilde{r}_{k-1} r_{k-2} - \Phi_k) \Lambda_k \quad (7.40)$$

$$D_k^{(0)} = \delta^{(0)} - \delta^{(-2)} (q_{k-1} r_k + \Psi_k) + \alpha^{(-2)} q_{k-1} \tilde{r}_k \Lambda_k - \delta^{(-4)} (q_{k-1} r_{k+1} + q_k r_{k-2} - \tilde{q}_k q_{k-1} \tilde{r}_{k+1} r_{k+1} - \tilde{q}_k q_k \tilde{r}_{k-1} r_{k-1} - \tilde{q}_k q_{k-1} \tilde{r}_{k-1} r_{k-2} + \Omega_k) + \alpha^{(-4)} (q_{k-1} \tilde{r}_{k+1} + q_k \tilde{r}_{k-2} - q_{k-1} \tilde{r}_k - \tilde{q}_k q_{k-1} \tilde{r}_{k+1} r_{k+1} + \tilde{q}_k q_k \tilde{r}_{k-1} r_{k-1} + \tilde{q}_k q_{k-1} \tilde{r}_{k-1} r_{k-2} - \Psi_k) \Lambda_k \quad (7.41)$$
Here

\[ \Xi_k = \sum_{j=-\infty}^{k-1} \left( q_j r_{j+2} - \tilde{q}_j \tilde{r}_{j+2} - q_{j+1} r_j \Phi_{j+3} + \tilde{q}_j \tilde{r}_{j+2} \Phi_{j+1} \right) \]

\[ \Omega_k = \sum_{j=-\infty}^{k-1} \left( q_{j-2} r_j - \tilde{q}_{j-2} \tilde{r}_j - q_{j+1} r_{j+1} \Psi_{j+3} + \tilde{q}_{j-2} \tilde{r}_j \Psi_{j+1} \right) \]

These quantities, just as \( \Phi_k, \Psi_k \), are of order \( O(h) \) and vanish in both limits \( k \to \pm \infty \). The see that this is true for \( k \to +\infty \), one has to use the partial summation. For example, for \( \Xi_k \) one arrives at the following expression:

\[ \Xi_k = q_k r_{k-1} \Phi_{k+2} - \frac{1}{2} \left( q_k r_{k-1} \phi_{k-1} + \tilde{q}_k \tilde{r}_{k-1} \tilde{\phi}_{k-1} - 2 \tilde{q}_k \tilde{r}_{k-1} \phi_{k-1} \right) \]

\[ + \sum_{j=-\infty}^{k-1} \left( q_j r_{j+2} - \frac{1}{2} q_j r_{j-1} (q_{j+1} r_j + q_j r_{j-1} + q_{j-1} r_{j-2}) \right) \]

\[ - \sum_{j=-\infty}^{k-1} \left( \tilde{q}_j \tilde{r}_{j+2} - \frac{1}{2} \tilde{q}_j \tilde{r}_{j-1} (\tilde{q}_{j+1} \tilde{r}_j + \tilde{q}_j \tilde{r}_{j-1} + \tilde{q}_{j-1} \tilde{r}_{j-2}) \right) , \]

where

\[ \phi_k = \sum_{j=-\infty}^{k-1} q_j r_{j-1} , \quad \text{so that} \quad \Phi_k = \phi_k - \tilde{\phi}_k . \]

The obtained expression for \( \Xi_k \) clearly vanishes at \( k \to +\infty \) due to the fact that an infinite-dimensional analog of (2.12) is an integral of motion.

Now the Taha–Ablowitz difference scheme for the mKdV is given by (7.19), (7.20) with the expressions (7.36)– (7.41). Due to the appearance of the quantities \( \Lambda_k, \Phi_k, \Psi_k, \Xi_k, \Omega_k \) this scheme is not only implicit, but also highly non-local. More precisely, they consider only the reduced case \( r = \pm q \), assuming \( \delta^{(j)} = \alpha^{(-j)} \). In this reduction \( A_k^{(j)} = D_k^{(-j)} \), \( B_k^{(j)} = \pm C_k^{(-j)} \), and, of course, \( \Phi_k = \Psi_k, \Xi_k = \Omega_k \), so that the number of the nonlocal ingredients is reduced by two.