On the Polyhedrality of Cutting-plane Closure

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Abstract. We study the equivalent condition for the closure of one particular family of cutting-planes being polyhedral, from the perspective of convex geometry. Based on this result we propose a new method to show the polyhedrality of a general closure. We further use it to prove one of the problems left in [4], namely the polyhedrality of aggregation closure for general covering polyhedron. This method also enables us to show some results about the Chvátal-Gomory closure for some type of unbounded irrational polyhedron, and general convex set. We believe this general approach can also be used to tackle the polyhedrality of many other different closures.

Keywords: Closure · Polyhedral · Convex Cone · Cutting-planes.

1 Introduction

Cutting-plane method is one of the central areas of mixed integer programming. In general, a cutting plane for a polyhedron \( P \) is an inequality that is satisfied by all integer points in \( P \) and, when added to the polyhedron \( P \), typically yields a stronger relaxation of its integer hull. In theory or practice there are usually infinitely many cutting-planes generated from one specific cut generation method, e.g., Chvátal-Gomory cuts, split cuts etc. Researchers in integer programming are always interested in the feasible region given by all possible cutting-planes of one type, the structural property of that corresponding region can usually reveal important information about that specific family of cuts. In the terminology of the cutting-plane theory, such region is usually referred as closure \([2]\). We give its formal definition in the following:

**Definition 1.** For a family of half-spaces \( \{H^i\}_{i \in I} \) in \( \mathbb{R}^n \), each of those half-space \( H^i \) is given by an inequality \((\alpha^i)^T \cdot x \leq \beta^i\), then the closure for that family of half-spaces is:

\[
\mathcal{F} := \bigcap_{i \in I} H^i = \bigcap_{i \in I} \{x \in \mathbb{R}^n \mid (\alpha^i)^T \cdot x \leq \beta^i\}.
\]

For the ease of notation, we would normally use upper index to refer vectors, and lower index to refer numbers, e.g., \( \alpha^i \in \mathbb{R}^n \) for some \( n \), \( \beta^i \in \mathbb{R} \). We would
also skip the transpose symbol in an inequality, and use $\alpha x \leq \beta$ rather than $\alpha^T \cdot x \leq \beta$.

Later for any $A \subseteq \mathbb{R}^{n+1}$, we would refer “a family of cuts given by $A$” to the family of cuts given by $\alpha x \leq \beta$, for all $(\alpha, \beta) \in A$. Here $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}$. We would also denote the corresponding closure to be:

$$\mathcal{F}(A) := \bigcap_{(\alpha, \beta) \in A} \{ x \in \mathbb{R}^n \mid \alpha x \leq \beta \}. \quad (1)$$

Clearly this closure is convex and closed, but whenever $A$ is an infinite set, this closure is obtained by intersecting infinitely many half-spaces. One natural question then arises: what is the necessary and (or) sufficient condition for that closure also being polyhedral? In this paper we are going to answer that general question. People have been looking at such polyhedrality of many different specific closures in literature, see \cite{10, 19} e.t.c. But as far as we were aware, our work is the first attempt trying to derive characterization for totally general closures. Our equivalent characterization is the following:

**Theorem 1.** Given $A \subseteq \mathbb{R}^{n+1}$ which contains $(0, \ldots, 0, 1)$. If $\mathcal{F}(A)$ is full-dimensional, then $\mathcal{F}(A)$ is polyhedral iff $\text{cl \ cone } A$ has finitely many different extreme rays.

Here for any set $K$, cone $K$ is the conical hull of $K$, which is defined as the set of all conical combination of finitely many points in $K$:

$$\text{cone } K := \{ \sum_{i=1}^k \alpha_i x^i \mid x^i \in K, \alpha_i \geq 0, i, k \in \mathbb{N} \}.$$  

For any set $S$, we use $\text{cl } S$ to denote the the smallest closed set containing $S$, which is also called closure in topology. To avoid confusion, we would only use $\text{cl } S$ to refer the topological closure, and whenever we say closure in this paper, it refers to Definition \ref{def:closure} or \ref{def:polyhedron}.

Moreover, in Theorem \ref{thm:closure} all the extreme rays of $\text{cl \ cone } A$ can be characterized by points in $A$, see Lemma \ref{lem:extreme_rays} for details. Due to the lack of specificity of cutting-planes in $A$, this result seems rather intuitive and can hardly be of any use. But with this result in hand, in order to show the polyhedrality of any closure, we can now start the argument by assuming for contradiction: $\mathcal{F}(A)$ is non-polyhedral. In mathematics, proof by contradiction is a really powerful tool. Often times coupled with the celebrated Bolzano-Weierstrass theorem, we can derive contradiction during the argument of proof. We would be able to see this more clearly in later context. Originally without such equivalent characterization, assuming the closure is non-polyhedral would usually go nowhere.

In order to explore the power of this characterization, we show the polyhedrality of aggregation closure for general covering polyhedron, proposed by \cite{4}. We were informed that recently this problem was independently proved by \cite{14}, and the more general version of its counterpart problem was proved in \cite{21}. We contain the detailed definition of this closure and covering polyhedron in Sect. \ref{sect:aggregation}.

**Theorem 2.** For a covering polyhedron $Q$, it’s aggregation closure $\mathcal{A}(Q)$ is also a covering polyhedron.
The other result we showed is the Chvátal-Gomory (CG) closure for some general polyhedron. A series of work have been done in showing the polyhedrality of CG closure for many different sets, even non-polyhedral sets. See [17, 11, 7, 10, 5] e.t.c. One of the most important results among these work is about showing the CG closure for irrational polytope is also a rational polytope. This question was proposed by [17] in 1980, and solved by independent work of [11,7], and a follow-up short proof by [5]. All those current results do not include the case of unbounded irrational polyhedron. In fact, it’s a consensus that for unbounded irrational polyhedron, its CG closure cannot always be expected to be rational polyhedron. Here we showed that for some special irrational polyhedron, its CG closure is still polyhedral:

**Theorem 3.** Given a polyhedron \( P = \{ x \in \mathbb{R}^n \mid Mx \leq d \} \) with full-dimensional integer hull, \( M \in \mathbb{R}^{m \times n}, d \in (\mathbb{Z} \setminus \{0\})^m \). If each row vector \( M_i \) satisfies at least one of the following:

1. There exists \( r \in \mathbb{R} \) such that \( rM_i \in \mathbb{Q}^n \);
2. \( M_i^T \cdot x = 1 \) has no rational solution;
3. There exists \( v \in V_{M_i} \) such that \( vx < 0 \) is valid to \( \{ x \in P \mid M_i x = d_i \} \).

Furthermore, if facet \( \{ x \in P \mid M_i x = d_i \} \) is unbounded, there exists \( r \in \mathbb{R} \) such that \( rM_i \in \mathbb{Q}^n \). Then, \( P^{CG} \) is a rational polyhedron.

The \( V_{M_i} \) in this theorem is a linear subspace of \( \mathbb{R}^n \) uniquely defined by vector \( M_i \), which we would introduce in Sect. 4.

Also for the CG closure of general convex set, the following result can be immediately derived from Theorem 1:

**Theorem 4.** For any convex set \( S \), if \( S^{CG} \) does not meet the boundary of \( S \), then \( S^{CG} \) is polyhedral.

This paper is organized as follows: In Sect. 2 we show Theorem 1, give the characterization for extreme rays of cl cone \( A \), and introduced a special class of valid inequalities for general convex set, which is a generalization for facet defining inequality of polyhedron. We also showed the one-to-one correspondence between such inequality to the extreme ray of cl cone \( A \). As applications to those general statements in Sect. 2, we contained the proof of Theorem 2, Theorem 3, and Theorem 4 in Sect. 3 and Sect. 4.

## 2 Equivalent Condition for the Polyhedrality of Closure

First, from our definition in (1), We notice the following chain of equations:

\[
\mathcal{I}(A) = \bigcap_{(\alpha, \beta) \in A} \{ x \in \mathbb{R}^n \mid \alpha x \leq \beta \} = \{ x \in \mathbb{R}^n \mid (\alpha, \beta)(x, -1) \leq 0, \forall (\alpha, \beta) \in A \} = \{ x \in \mathbb{R}^n \mid (\alpha, \beta)(x, -1) \leq 0, \forall (\alpha, \beta) \in \text{cone} A \} = \text{proj}_{[n]}((\text{cone} A)^\circ \cap \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1 \}) = \text{proj}_{[n]}((\text{cl cone} A)^\circ \cap \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1 \}).
\]
Here for a non-empty cone \( K \subseteq \mathbb{R}^n \), \( K^\circ \) denotes the polar cone of \( K : K^\circ := \{ x^* \in \mathbb{R}^n \mid \forall x \in K, \langle x, x^* \rangle \leq 0 \} \); \([n]\) denotes the set \{1, \ldots, n\}, \( x_{[n]} := (x_1, \ldots, x_n) \), and \( \text{proj}_{[n]} \) denotes the orthogonal projection onto components in \( x_{[n]} \).

Also, since \( K \), . . . , By assumption, \( (0, \ldots, 0, 1) \) is a sufficient condition for (cl \( K \) cone \( A \)).

First we prove the following equation:

**Proof.**

Equation (3)

By assumption, \( (0, \ldots, 0, 1) \in K \), which means for any \( x \in K^\circ \), there is \( x_{n+1} \leq 0 \).

Hence \( K^\circ \subseteq \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} \leq 0 \} \). We have:

\[
\text{cl}(K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} < 0\}) \subseteq \text{cl}(K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} \leq 0\})
\]

\[
= \text{cl}(K^\circ)
\]

\[
= K^\circ.
\]

Also, since \( K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} < 0\} \) is non-empty, we know \( K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1\} \) is also non-empty. Pick \( x^* \in K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1\} \), and arbitrarily pick \( x \in K^\circ \), denote \( x^* := x + \frac{1}{n}x^* \). Because \( x \in K^\circ \subseteq \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1 \} \) is non-empty we }

Before showing the main result Theorem \[1\] we would need the following intermediate result:

**Proposition 1.** Given \( A \subseteq \mathbb{R}^n \) containing \((0, \ldots, 0, 1)\). Closure \( \mathcal{I}(A) \) is polyhedral iff \( \text{cl cone} A \) is polyhedral cone.

We should remark that, in order to utilize this result to show the polyhedrality of \( \mathcal{I}(A) \), we have to obtain a good understanding about when a closed convex cone is also a polyhedral cone. Normally inner description (finite extreme points, extreme rays) and outer description (finite facet defining inequalities) are two characterizations for a polyhedron. Obviously given the way this convex cone \( \text{cl cone} A \) is defined, inner description is much more preferable. As simple as this proposition seems, the main reason for us focusing on our main Theorem \[1\] is that, when \( \mathcal{I}(A) \) is full-dimensional, those extreme rays of \( \text{cl cone} A \) can be exactly characterized by points in \( A \) as we would see later. On the other hand, \( \text{cl cone} A \) itself being a polyhedral cone does not even guarantee the existence of extreme rays, which would make this characterization result in Proposition \[1\] be of little practical use.

Now we are going to show this Proposition \[1\]. Observe from (2), one sufficient condition for \( \mathcal{I}(A) \) being polyhedral is \( (\text{cl cone} A)^\circ \) being polyhedral cone, and one sufficient condition for \( (\text{cl cone} A)^\circ \) being polyhedral cone is \( \text{cl cone} A \) being polyhedral cone \[16\]. Therefore it suffices to show \( \text{cl cone} A \) being polyhedral cone is also a necessary condition for \( \mathcal{I}(A) \) being polyhedral. The next lemma would be crucial to show this necessary condition, and would also be helpful for later discussion in this section.

**Lemma 1.** For any set \( K \in \mathbb{R}^{n+1} \) containing \((0, \ldots, 0, 1)\), and \( K^\circ \cap \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} < 0 \} \neq \emptyset \), there is \( \text{cl cone} (K^\circ \cap \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1 \}) = K^\circ \).

**Proof.** First we prove the following equation:

\[
\text{cl}(K^\circ \cap \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} < 0 \}) = K^\circ.
\]

By assumption, \((0, \ldots, 0, 1) \in K \), which means for any \( x \in K^\circ \), there is \( x_{n+1} \leq 0 \).

Hence \( K^\circ \subseteq \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} \leq 0 \} \). We have:

\[
\text{cl}(K^\circ \cap \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} < 0 \}) \subseteq \text{cl}(K^\circ \cap \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} \leq 0 \})
\]

\[
= \text{cl}(K^\circ)
\]

\[
= K^\circ.
\]
Obtain that $K, \ldots, K$ because the inequality associated with this vector is 0
$K$ as the conical combination of $K$ we know that $K$ further implies $K$
$polyhedral cone$. From Lemma 1, we know $K$
no extreme ray, no matter it’s polyhedral or not. So in that case $cl cone$
pointed polyhedral cone would be really hard to characterize by inner description. For
As we remarked above, a polyhedral cone may not even have any extreme ray. Recall that a ray $0 \neq r \in K$ is called an extreme ray, if $r$ can be written as the conical combination of $r^1, \ldots, r^k$ for $r^i \in K, i \in [k], k \in \mathbb{N}$ with non-zero coefficients would imply that $r^i \in cone\{r\}$ for all $i \in [k]$. In fact, when a convex cone is not pointed, which we would define later, such convex cone must have no extreme ray, no matter it’s polyhedral or not. So in that case $cl cone A$ being polyhedral cone would be really hard to characterize by inner description. For that reason, we focus ourselves on the pointed case: Given a closed convex cone

$$x_{n+1} \leq 0, x^i \in K^o, x^i_{n+1} = -1,$$ we know $x^i \in K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} < 0\}.$
Hence we obtain that, for any $x \in K^o$, there exists $\{x^i\} \subseteq K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} < 0\}$ such that $x^i \rightarrow x$, which implies $K^o \subseteq cl(K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} < 0\}),$ and we conclude the proof of equation (3).
Therefore, we have:

$$K^o = cl cone(K^o)$$
$$\supseteq cl cone(K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} = -1\})$$
$$\supseteq cl \left( \bigcup_{\lambda > 0} \lambda (K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} = -1\}) \right)$$
$$= cl \left( \bigcup_{\lambda > 0} \lambda K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} = -\lambda \} \right)$$
$$= cl \left( K^o \cap \bigcup_{\lambda > 0} \{x \in \mathbb{R}^{n+1} | x_{n+1} = -\lambda \} \right)$$
$$= cl(K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} < 0\})$$
$$= K^o.$$

The first equation is because the polar cone of any cone is a closed, convex cone; The last equation is from [3]. Hence every terms in (5) are equal to each other, and we complete the proof. □

Now we are proceeding to the proof of Proposition 1.

Proof (Proof of Proposition 1). As we analyzed above, we only have to show:
If $\mathcal{J}(A)$ is polyhedral, then $cl cone A$ is a polyhedral cone. W.l.o.g. assuming $\mathcal{J}(A) \neq \emptyset$, and denote $cl cone A := K$, a closed convex cone.

From equation (2), we know $\mathcal{J}(A) \times \{-1\} = K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} = -1\}$. $\mathcal{J}(A)$ being polyhedral would imply that $K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} = -1\}$ is also a non-empty polyhedron. Hence $cl cone (K^o \cap \{x \in \mathbb{R}^{n+1} | x_{n+1} = -1\})$ is a polyhedral cone. From Lemma 1, we know $K^o$ is also a polyhedral cone, which further implies $K^{oo}$ is also a polyhedral cone. Moreover, from convex analysis we know that $K^{oo} = K$ for any non-empty closed convex cone $K$, therefore we obtain that $K$ is a polyhedral cone, which is just $cl cone A$.

We should note that the assumption of $(0, \ldots, 0, 1) \in A$ can be made w.l.o.g., because the inequality associated with this vector is $0^T \cdot x \leq 1$, which trivially holds.

As we remarked above, a polyhedral cone may not even have any extreme ray. Recall that a ray $0 \neq r \in K$ is called an extreme ray, if $r$ can be written as the conical combination of $r^1, \ldots, r^k$ for $r^i \in K, i \in [k], k \in \mathbb{N}$ with non-zero coefficients would imply that $r^i \in cone\{r\}$ for all $i \in [k]$. In fact, when a convex cone is not pointed, which we would define later, such convex cone must have no extreme ray, no matter it’s polyhedral or not. So in that case $cl cone A$ being polyhedral cone would be really hard to characterize by inner description. For that reason, we focus ourselves on the pointed case: Given a closed convex cone
$K$, we say $K$ is pointed if $K \cap (-K) = \{0\}$. It is well-known that when $K \neq \{0\}$ is pointed, it must have extreme rays, and the number of these extreme rays is finite if and only if $K$ is a pointed polyhedral cone. We state them formally as the follows:

**Lemma 2 ([18]).** A closed convex cone $K$ has extreme rays if and only if it is pointed and contains a non-zero vector $0 \neq x \in K$.

**Lemma 3 ([18]).** A pointed closed convex cone is a polyhedral cone if and only if it has finitely many different extreme rays.

Here we say two extreme rays $r^1$ and $r^2$ are different, if $r^1 \notin \text{cone}(\{r^2\})$. For the ease of notation we denote $\text{cone}(r)$, instead of $\text{cone}(\{r\})$, to represent the set $\{\lambda r \mid \lambda \geq 0\}$. It is therefore an immediate consequence to obtain the next corollary, from Lemma 3 and Proposition 1.

**Corollary 1.** Given $A \subseteq \mathbb{R}^{n+1}$ containing $(0, \ldots, 0, 1)$. If $\text{cl cone} A$ is pointed, then $\mathcal{F}(A)$ is polyhedral iff $\text{cl cone} A$ has finitely many different extreme rays.

### 2.1 Finitely-irredundant inequality of Convex Set

In this section we want to introduce a new concept for general convex set, namely the finitely-irredundant inequality, which can be seen as a natural generalization of facet defining inequality for polyhedron. Before giving the formal definition, we want to first look at the relationship between valid inequalities of $\mathcal{F}(A)$ and points in $\text{cl cone} A$.

**Lemma 4.** Given $A \subseteq \mathbb{R}^{n+1}$ containing $(0, \ldots, 0, 1)$. Then $ax \leq \beta$ is a valid inequality to $\mathcal{F}(A)$ if and only if $(\alpha, \beta) \in \text{cl cone} A$.

**Proof.** Denote $K := \text{cl cone} A$. First we show the “if” direction. Given $(\alpha, \beta) \in K$, we want to show $ax \leq \beta$ is a valid inequality to $\mathcal{F}(A)$. Arbitrarily pick $\bar{x} \in \mathcal{F}(A)$, from equation (2), we know $\mathcal{F}(A) \times \{1\} = K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}$, then $(\bar{x}, -1) \in K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1\}$, so $(\bar{x}, -1) \in K^\circ$. From the definition of polar cone, and $(\alpha, \beta) \in K$, then we get: $(\alpha, \beta)(\bar{x}, -1) \leq 0$. Since here $\bar{x} \in \mathcal{F}(A)$ is arbitrary, we know $ax \leq \beta$ would be a valid inequality to $\mathcal{F}(A)$.

Next we show the “only if” direction. Assume $ax \leq \beta$ is valid to $\mathcal{F}(A)$, which means $\mathcal{F}(A) \subseteq \{x \in \mathbb{R}^n \mid (\alpha, \beta)(x, -1) \leq 0\}$. Hence $\mathcal{F}(A) \times \{1\} \subseteq \{x \in \mathbb{R}^{n+1} \mid (\alpha, \beta)x \leq 0\}$. By equation (2), $\mathcal{F}(A) \times \{1\} = K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1\}$, so we have $K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1\} \subseteq \{x \in \mathbb{R}^{n+1} \mid (\alpha, \beta)x \leq 0\}$. Taking cl(·) and conic hull operator cone(·) on both sides, we have: $\text{cl cone}(K^\circ \cap \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = -1\}) \subseteq \{x \in \mathbb{R}^{n+1} \mid (\alpha, \beta)x \leq 0\}$. By Proposition 1 we obtain: $K^\circ \subseteq \{x \in \mathbb{R}^{n+1} \mid (\alpha, \beta)x \leq 0\}$. Taking the polar cone of both sides, we obtain: $\{(\alpha, \beta)\} \subseteq K^{\text{co}} = K$, which is just saying $(\alpha, \beta) \in K$. \hfill \Box

From this lemma, we can simply obtain the following corollary:
Corollary 2. Given \( A \subseteq \mathbb{R}^{n+1} \) containing \((0, \ldots, 0, 1)\). \( \mathcal{I}(A) \) is full-dimensional iff \( \text{cl cone}(A) \) is a pointed closed convex cone.

Proof. Denote \( K := \text{cl cone}(A) \), first we assume \( \mathcal{I}(A) \) is full-dimensional, we want to show \( K \) is pointed. Assume for contradiction, which means \( K \cap (-K) \supseteq \{0\} \). We arbitrarily pick \( 0 \neq (\alpha^*, \beta^*) \in K \cap (-K) \). From equation (2), we have:

\[
\mathcal{I}(A) = \left\{ x \in \mathbb{R}^n \mid (\alpha, \beta)(x, -1) \leq 0, \forall (\alpha, \beta) \in K \right\} \\
\subseteq \left\{ x \in \mathbb{R}^n \mid (\alpha^*, \beta^*)(x, -1) \leq 0, (\alpha^* - \beta^*)(x, -1) \leq 0 \right\} \\
= \left\{ x \in \mathbb{R}^n \mid \alpha^* x = \beta^* \right\}.
\]

Hence \( \mathcal{I}(A) \) is contained in a hyperplane, contradict the assumption.

Next we are going to show, if \( \mathcal{I}(A) \) is not full-dimensional, then \( K \) would be non-pointed. Since \( \mathcal{I}(A) \) is not full-dimensional, we know that it could be contained in a hyperplane, say \( \mathcal{I}(A) \subseteq \{ x \in \mathbb{R}^n \mid \alpha^* x = \beta^* \} \). In particular, we know \( \alpha^* x \leq \beta^* \) and \( -\alpha^* x \leq -\beta^* \) would both be valid inequality to \( \mathcal{I}(A) \). From Lemma 1, we get: \( (\alpha^*, \beta^*) \in K \cap (-K) \), which means \( K \) is non-pointed.

Therefore, Theorem 1 automatically follows from Corollary 2 and Corollary 1.

Now, for general convex set \( S \), we define the class of finitely-irredundant inequalities to \( S \) as:

Definition 2. Given a convex set \( S \subseteq \mathbb{R}^n \), and a valid inequality \( \alpha x \leq \beta \). If any finitely many valid inequalities which are different from \( \alpha x \leq \beta \): \( \alpha^i x \leq \beta_i, i \in [k], k \in \mathbb{N} \), we always have:

\[
\bigcap_{i \in [k]} \left\{ x \in \mathbb{R}^n \mid \alpha^i x \leq \beta_i \right\} \not\subseteq \left\{ x \in \mathbb{R}^n \mid \alpha x \leq \beta \right\}.
\]

Then we say \( \alpha x \leq \beta \) is a finitely-irredundant inequality (FII) of \( S \).

Note that FII is defined for any convex set, and for \( S \) being a polyhedron, it degenerates to facet defining inequality. There are also cases where there’s no facet, but any supporting half-space is a FII. One simple example is a ball: \( S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \), any FII of \( S \) has the form \( x^* \cdot x + y^* \cdot y \leq 1 \) where \( (x^*)^2 + (y^*)^2 = 1 \).

As we have shown before, all valid inequalities to \( \mathcal{I}(A) \) corresponding to some points in \( \text{cl cone}(A) \). The next lemma further shows, all FII corresponding to the extreme rays in \( \text{cl cone}(A) \).

Lemma 5. Given a full-dimensional \( \mathcal{I}(A) \) for some \( A \subseteq \mathbb{R}^{n+1} \) which contains \((0, \ldots, 0, 1)\), and a valid inequality \( \alpha x \leq \beta \) to \( \mathcal{I}(A) \). Then \( \alpha x \leq \beta \) is a FII iff \( (\alpha, \beta) \) is an extreme ray of \( \text{cl cone}(A) \).

Proof. Denote \( K := \text{cl cone} A \). First we are showing: if \( \alpha x \leq \beta \) is not a FII to \( \mathcal{I}(A) \), then \( (\alpha, \beta) \) is not an extreme ray of \( K \). By the definition of FII, we know there must exists finitely many valid inequalities of \( \mathcal{I}(A) \) which are different from \( \alpha x \leq \beta \): \( \alpha^i x \leq \beta_i, i \in [k], k < \infty \), such that \( \bigcap_{i \in [k]} \{ x \in \mathbb{R}^n \mid \alpha^i x \leq \beta_i \} \not\subseteq \left\{ x \in \mathbb{R}^n \mid \alpha x \leq \beta \right\} \).
The main results in this subsection is saying, this sequence \( \{\alpha^i, \beta_i\} \subseteq \{x \in \mathbb{R}^n \mid \alpha x \leq \beta\} \). In other words, \( \alpha x \leq \beta \) is a valid inequality to \( \mathcal{I}(A) \) where \( A := \{\{\alpha^i, \beta_i\}\}_{i \in [k]} \cup \{(0, \ldots, 0, 1)\} \subseteq K \). From Lemma 4 we know \((\alpha, \beta) \in \text{cl cone } \bar{A} = \text{cone } A\), here the equation is because \( \bar{A} \) is a finite set. Since \((\alpha, \beta)\) is an extreme ray of \( K \), and it can be written as the conical combination of finitely many points \( \bar{A} \) in \( K \), by the definition of extreme ray, we know there must exist some vector among \( \bar{A} \) being the same extreme ray as \((\alpha, \beta)\), meaning some inequality \( \alpha^i x \leq \beta_i \) is the same as \( \alpha x \leq \beta \), which gives the contradiction.

Next we are going to show: if \((\alpha, \beta)\) is not an extreme ray of \( K \), then \( \alpha x \leq \beta \) is not a FII to \( \mathcal{I}(A) \). Since \((\alpha, \beta)\) is not an extreme ray of \( K \), then we can find \( \{(\alpha^i, \beta_i)\}_{i \in [k]} \subseteq K, k \in \mathbb{N} \) all different from ray \((\alpha, \beta)\), such that \((\alpha, \beta) \in \text{cone}(\{(\alpha^i, \beta_i)\}_{i \in [k]} \cup \{(0, \ldots, 0, 1)\}) \). By Lemma 4 we know \( \alpha x \leq \beta \) is a valid inequality to \( \mathcal{I}(\{(\alpha^i, \beta_i)\}_{i \in [k]} \cup \{(0, \ldots, 0, 1)\}) \), meaning: \( \bigcap_{i \in [k]} \{x \in \mathbb{R}^n \mid \alpha^i x \leq \beta_i\} \subseteq \{x \in \mathbb{R}^n \mid \alpha x \leq \beta\} \). Since each \( \alpha^i x \leq \beta_i \) is valid inequality to \( \mathcal{I}(A) \) and all different from \( \alpha x \leq \beta \), we obtain that \( \alpha x \leq \beta \) is not a FII to \( \mathcal{I}(A) \). □

In Theorem 4 of [3], the authors wrote:

"Assume \( P_Q \) is full-dimensional. The inequality \( \alpha x \geq \beta \) defines a facet of \( P_Q \) if and only if \((\alpha, \beta)\) is an extreme ray of the cone \( P_Q^* \)."

Here \( P_Q \) is the convex hull of a disjunctive set, and \( P_Q^* \) is the reverse polar cone of \( P_Q \). Easy to observe this theorem can be treated as a special case of our Lemma.

So far, we have obtained that, when \( \mathcal{I}(A) \) is full-dimensional, it is polyhedral if and only if it has finitely many FII, and each FII corresponding to some extreme ray of \( \text{cl cone } A \). In the next section we are going to give a full characterization for those extreme rays, using points in \( A \).

### 2.2 Characterization of Extreme Rays for \( \text{cl cone } A \)

**Definition 3.** Given \( \{\alpha^i\} \subseteq \mathbb{R}^n, \alpha^* \in \mathbb{R}^n \), if there exists \( \{\lambda_i\} > 0 \) such that \( \lim_{i \to \infty} \lambda_i \alpha^i = \alpha^* \), then we say \( \{\alpha^i\} \) **conically converges** to \( \alpha^* \), or \( \alpha^i \xrightarrow{c} \alpha^* \).

For any extreme ray \( r \in K \), there are simply two cases: \( r \in \text{cone } A \), or \( r \in \text{cl cone } A \setminus \text{cone } A \). For the first case, as we would see later, it is equivalent to \( r \in \text{cone}(a) \) for some \( a \in A \). While for the second case, we know \( r \) can be expressed as the limit of a convergent sequence in \( \text{cone } A \), using the definition of conical convergence, we know there exists \( \{r^i\} \subseteq \text{conv } A \) such that \( r^i \xrightarrow{c} r \). One of the main results in this subsection is saying, this sequence \( \{r^i\} \) can be further picked from \( A \) instead of \( \text{conv } A \). We state them as the following lemma:

**Lemma 6.** Given \( A \subseteq \mathbb{R}^n \) with \( \text{cl cone } A \) being a pointed closed convex cone, \((0, \ldots, 0, 1) \in A \) and \( 0 \notin A \). For any extreme ray \( r \in \text{cl cone } A \), at least one of the following is true:

1. \( r \in \text{cone}(a) \) for some \( a \in A \);
2. There exists \( \{r^i\} \subseteq A \) such that \( r^i \xrightarrow{c} r \).
Before we proceeding to the proof, the following results and definitions would be needed. In convex geometry, a point \( p \in K \) (closed convex set) is called an exposed point if there is an \( n-1 \) dimensional hyperplane whose intersection with \( K \) is \( p \) alone.

**Lemma 7 (Strasziewicz [12]).** Let \( K \subset \mathbb{R}^n \) be a closed convex set. Then the set of exposed points is dense in the set of extreme points.

Now we present the analogous version of Lemma 6 for extreme points in convex hull:

**Lemma 8.** Given \( S \subseteq \mathbb{R}^n \) and \( \alpha^* \in \text{cl conv} S \) being its extreme point. If \( \alpha^* \notin S \), then there exists \( \{ \alpha^i \} \subseteq S \) such that \( \alpha^i \to \alpha^* \).

Note that \( \text{cl}(\cdot) \) and \( \text{conv}(\cdot) \) are not commutative, and it would be trivial if we switch \( \text{cl}(\cdot) \) and \( \text{conv}(\cdot) \) by each other.

**Proof.** We discuss according to whether or not \( \alpha^* \) is an exposed point of the closed convex set \( \text{cl conv} S \). In the following proof, we denote \( B_\epsilon(x^*) := \{ x \in \mathbb{R}^n \mid \| x - x^* \|_2 \leq \epsilon \} \) for \( x^* \in \mathbb{R}^n \).

- If \( \alpha^* \) is an exposed point of \( \text{cl conv} S \): According to the definition of exposed point, we know there exists \( h \in \mathbb{R}^n \), such that \( \text{cl conv} S \cap \{ x \in \mathbb{R}^n \mid hx = 1 \} = \{ \alpha^* \} \). Here w.l.o.g we assume the hyperplane doesn’t pass through the origin. Clearly this hyperplane \( hx = 1 \) is also a supporting hyperplane, and we assume \( \text{cl conv} S \subseteq \{ x \in \mathbb{R}^n \mid hx \leq 1 \} \).

**Claim.** For any \( \epsilon > 0 \), there exists \( \delta > 0 \), such that \( \{ x \in \text{cl conv} S \mid hx \geq 1 - \delta \} \subseteq B_\epsilon(\alpha^*) \).

**Proof (Proof of Claim).** Assuming for contradiction: there exists \( \epsilon > 0 \), such that for all \( \delta > 0 \), there is \( x_\delta \in \{ x \in \text{cl conv} S \mid hx \geq 1 - \delta \} \) and \( \| x_\delta - \alpha^* \|_2 > \epsilon \). Then within the line segment between \( \alpha^* \) and \( x_\delta \), we can find one point \( y_\delta \) such that \( y_\delta \in \{ x \in \text{cl conv} S \mid hx \geq 1 - \delta \} \) while \( \| y_\delta - \alpha^* \|_2 = \epsilon \). Pick \( \delta = \frac{1}{N}, N \in \mathbb{N} \). Then we consider \( y_{\frac{1}{N}} \), it satisfies: \( y_{\frac{1}{N}} \in \text{cl conv} S, 1 - \frac{1}{N} \leq hy_{\frac{1}{N}} \leq 1 \), and \( \| y_{\frac{1}{N}} - \alpha^* \|_2 = \epsilon \). Since \( B_\epsilon(\alpha^*) \) is a compact set, by Bolzano-Weierstrass theorem we know in this sequence \( \{ y_{\frac{1}{N}} \}_{N \in \mathbb{N}} \), there exists a convergent subsequence, the limit point \( y^* \) must also satisfy \( \| y^* - \alpha^* \|_2 = \epsilon \) and \( y^* \in \text{cl conv} S \). W.l.o.g we still assume the convergent subsequence to be \( \{ y_{\frac{1}{N}} \}_{N \in \mathbb{N}} \), since \( 1 - \frac{1}{N} \leq hy_{\frac{1}{N}} \leq 1 \), then the limit point \( y^* \) must satisfy: \( hy^* = 1 \). Hence we get a point \( y^* \) satisfy \( \| y^* - \alpha^* \|_2 = \epsilon \), \( y^* \in \text{cl conv} S \) and \( hy^* = 1 \), which contradicts to the assumption at the beginning: \( \text{cl conv} S \cap \{ x \in \mathbb{R}^n \mid hx = 1 \} = \{ \alpha^* \} \). \( \diamond \)

**Claim.** For any \( \delta > 0 \), there exists \( \alpha_\delta \in S \) such that \( h\alpha_\delta \geq 1 - \delta \).

**Proof (Proof of Claim).** Assuming for contradiction: there exists \( \delta > 0 \), such that for all \( \alpha \in S \), \( h\alpha < 1 - \delta \). Then clearly we would have \( \text{cl conv} S \subseteq \{ x \in \mathbb{R}^n \mid hx \leq 1 - \delta \} \), which contradicts to the fact that \( hx \leq 1 \) is the supporting hyperplane of \( \text{cl conv} S \). \( \diamond \)
Now we proceed to the proof of this lemma in this case. For any $\epsilon > 0$, by Claim 2.2 we know there exists $\delta > 0$ such that $\{x \in \text{cl conv } S \mid hx \geq 1 - \delta\} \subseteq B_\epsilon(\alpha^*)$. Also by Claim 2.2 we know there exists $\alpha_\delta \in S$ such that $\alpha_\delta \in \{x \in \text{cl conv } S \mid hx \geq 1 - \delta\}$. Therefore $\alpha_\delta \in B_\epsilon(\alpha^*)$, meaning $\|\alpha_\delta - \alpha^*\|_2 \leq \epsilon$. Here each $\alpha_\delta \neq \alpha^*$, because $\alpha^* \notin S$. Since this is argued for any $\epsilon > 0$, and $\alpha_\delta \in S$, we obtain a sequence in $S$ which converges to $\alpha^*$.

- If $\alpha^*$ is not the exposed point of $\text{cl conv } S$: Since $\alpha^*$ is an extreme point of $\text{cl conv } S$, and by Lemma 7 we know $\alpha^*$ can be approached by a sequence of exposed points in $\text{cl conv } S$. Together with the result from the last bullet we know the statement of the lemma also holds in this case.

$\square$

The next result would be used as a bridge to derive Lemma 6 from Lemma 8.

**Lemma 9 (A Supporting Hyperplane Theorem for Pointed Cones).**

Let $K \subseteq \mathbb{R}^n$ be a non-degenerate closed, convex, pointed cone. Then it is strictly supported at the origin: there is $h \in \mathbb{R}^n$ such that if $k \in K$ and $k \neq 0$ then $hk > 0$.

We are now ready to prove the main Lemma 6.

**Proof (Proof of Lemma 6).** We want to show: If $r \notin \{\lambda a \mid a \in A, \lambda \geq 0\}$, then there exists $\{r^i\} \subseteq A$ such that $r^i \to r$. Denote $K := \text{cl cone } A$.

For the pointed closed convex cone $K$, from Lemma 9 we can find a supporting hyperplane $hx = 0$ such that for all $0 \neq a \in K$, $ha > 0$. We denote $\tilde{A} := \{\frac{a}{ha} \mid a \in A\}$, which is well defined since $0 \notin A$ and for all $0 \neq a \in A$ there is always $ha > 0$.

**Claim.** $\{x \in \mathbb{R}^n \mid hx = 1\} \cap K \subseteq \text{cl conv } \tilde{A}$.

**Proof (Proof of Claim).** First we show $\{x \in \mathbb{R}^n \mid hx = 1\} \cap K \subseteq \text{cl conv } \tilde{A}$. Arbitrarily pick $\alpha^* \in \{x \in \mathbb{R}^n \mid hx = 1\} \cap K$, we have $\alpha^* = 1$, and there exists $\{\alpha^i\} \subseteq \text{cone } A$ such that $\alpha^i \to \alpha^*$. Denote $\beta^i = \frac{\alpha^i}{ha}$. Since $\alpha^i \to \alpha^*$, and $ha = 1$, we know $\beta^i \to 1$. Hence we also have $\beta^i \to \alpha^*$, and here $\beta^i \in \{x \in \mathbb{R}^n \mid hx = 1\} \cap \text{cone } A$. In the following, we show: $\{x \in \mathbb{R}^n \mid hx = 1\} \cap \text{cone } A \subseteq \text{conv } \tilde{A}$, which would imply that $\alpha^* \in \text{conv } \tilde{A}$.

Pick $\beta \in \{x \in \mathbb{R}^n \mid hx = 1\} \cap \text{cone } A$, we can write it as: $\beta = \sum_{i=1}^{k} \lambda_i b^i$ for some $\lambda_i > 0, b^i \in A, i \in [k], k \in \mathbb{N}$. Here because $\beta \in \{x \in \mathbb{R}^n \mid hx = 1\}$, we know $\sum_{i=1}^{k} \lambda_i h b^i = 1$. Therefore, we can also write $\beta$ as:

$$\beta = \sum_{i=1}^{k} (\lambda_i h b^i) \cdot \frac{b^i}{hb^i}, \text{ here } \frac{b^i}{hb^i} \in \tilde{A}, \sum_{i=1}^{k} \lambda_i h b^i = 1.$$

we get $\beta \in \text{conv } (\tilde{A})$, which concludes $\{x \in \mathbb{R}^n \mid hx = 1\} \cap \text{cone } A \subseteq \text{conv } \tilde{A}$.

Next, we show $\{x \in \mathbb{R}^n \mid hx = 1\} \cap K \subseteq \text{cl conv } \tilde{A}$. By definition, $\tilde{A} \subseteq \{x \in \mathbb{R}^n \mid hx = 1\}$, which implies $\text{cl conv } \tilde{A} \subseteq \{x \in \mathbb{R}^n \mid hx = 1\}$. On the other hand, clearly $\tilde{A} \subseteq \text{cone } A$, so $\text{cl conv } \tilde{A} \subseteq \text{cl cone } A$, and we complete the proof. $\diamond$
Given an extreme ray \( r \in K \), w.l.o.g. we assume \( hr = 1 \). From the above claim, we know \( r \in \text{cl conv} \, A \). Next we show \( r \) is an extreme point of \( \text{cl conv} \, A \). If not, then we can write \( r \) as: \( r = \sum_{i=1}^{k} \lambda_i a^i \), \( \sum_{i=1}^{k} \lambda_i = 1, \lambda_i > 0, r \neq a^i \in \text{cl conv} \, A \). From the definition of \( \bar{A} \) we also have \( ha^i = 1, a^i \in K \). Since \( r \) is the extreme ray of \( K \), while it can be written as conical combination (convex combination is also conical combination) of other points in \( K \), hence we know there exists \( \gamma_i > 0 \) such that \( a^i = \gamma_i r \). So \( ha^i = \gamma_i hr \), which implies that \( \gamma_i = 1 \), meaning \( a^i = r \) for all \( i \in [k] \), contradicting the assumption that \( a^i \neq r \). Hence we conclude that \( r \) is an extreme point of \( \text{cl conv} \, \bar{A} \). Also by assumption \( r \notin \lambda \{a \in A, \lambda \geq 0\} \), we have \( r \notin \bar{A} \), together with Lemma 8 we know there exists \( \{r^i\} \subseteq \bar{A} \) such that \( r^i \to r \). According to the definition of \( \bar{A} \) and conical convergence, we complete the proof. \( \square \)

As applications of the above characterization, in the next two sections we are going to show the polyhedrality of two different closures.

## 3 Aggregation Closure for Covering Polyhedron

A \textit{packing polyhedron} is of the form \( \{ x \in \mathbb{R}^n_+ \mid Mx \leq d \} \) where \( M \in \mathbb{R}^{m \times n} \), \( d \in \mathbb{R}^m_+ \) for some \( m, n \in \mathbb{N} \). Similarly a \textit{covering polyhedron} is a polyhedron of the form \( \{ x \in \mathbb{R}^n_+ \mid Mx \geq d \} \) where all the data \( (M, d) \) are also non-negative.

We also define \textit{integer packing set} in \( \mathbb{R}^n \) to be the set \( S \subseteq \mathbb{N}^n \) with the property: \( \forall x \in S \), if \( \mathbb{N} \ni x' \leq x \) component-wisely, then \( x' \in S \). See [21] for detailed discussion.

In [4] the authors present a new class of closure, namely \textit{aggregation closure}, which is defined particularly for the pure integer hull of packing and covering polyhedron. For a covering polyhedron \( Q = \{ x \in \mathbb{R}^n_+ \mid Mx \geq d \} \), its \textit{aggregation closure} is defined as:

\[
A(Q) := \bigcap_{\lambda \in \mathbb{R}^n_+} \text{conv}(\{ x \in \mathbb{N}^n \mid \lambda Mx \geq \lambda d \}).
\]

This can be easily seen to be a tighter relaxation for \( \text{conv}(\{ x \in \mathbb{N}^n \mid Mx \geq d \}) \) than the classic Chvatál-Gomory closure. For a packing polyhedron \( Q = \{ x \in \mathbb{R}^n_+ \mid Mx \leq d \} \) its aggregation closure can be defined through almost identical way. Given such closure is obtained by intersecting infinitely many polyhedra, people wondering whether this closure is still polyhedral or not. The authors of [4] showed that if the matrix \( M \) is fully dense then the closure is still polyhedral. For general case they state this question as an open problem.

As our recent paper [21] showed, the polyhedrality of the aggregation closure on packing polyhedron can be automatically derived from the structural study of general integer packing sets, which is not possessed by covering case. Therefore it’s our goal to prove the other counterpart in this section, namely the aggregation closure is still polyhedral for covering polyhedron, as stated in Theorem 2. Note that we were recently informed that an independent and almost simultaneous proof was given by [14] through induction.
3.1 Outline of the Proof

For each \( \lambda \in \mathbb{R}^n_+ \), denote \( Q^I_\lambda := \text{conv}(\{ x \in \mathbb{N}^n | \lambda M x \geq \beta d \}) \). It turns out that this polyhedron is also a covering polyhedron, as stated next:

**Lemma 10.** For any \( \lambda \in \mathbb{R}^n_+ \), \( Q^I_\lambda \) is a full-dimensional covering polyhedron with integer extreme points.

We would make use of the well-known Gordan-Dickson Lemma for its proof. Note that the \( \leq \) between vectors are component-wisely.

**Lemma 11 (Gordan-Dickson lemma).** For any \( X \subseteq \mathbb{N}^n \), there exists \( X' \subseteq X \) with \(|X'| < \infty\), such that every \( x \in X \) satisfies \( x' \leq x \) for some \( x' \in X' \).

**Proof (Proof of Lemma 11).** Denote \( C_\lambda := \{ x \in \mathbb{N}^n | \lambda M x \geq \beta d \} \). Note that \( Q^I_\lambda = \text{conv}(C_\lambda) \). Since \( C_\lambda \subseteq \mathbb{N}^n \), by Gordan-Dickson lemma, we know there exists \( C'_\lambda \subseteq C_\lambda \) with \(|C'_\lambda| < \infty \), such that every \( x \in C_\lambda \) satisfies \( x' \leq x \) for some \( x' \in C'_\lambda \). Next we show: \( \text{conv}(C'_\lambda) + \mathbb{R}^n_+ = \text{conv}(C_\lambda) = Q^I_\lambda \), then from the finiteness of \( C'_\lambda \) and \( C_\lambda \subseteq \mathbb{N}^n \), we would finish the proof.

Since \( C'_\lambda \subseteq C_\lambda \subseteq Q^I_\lambda \), we know \( \text{conv}(C'_\lambda) + \mathbb{R}^n_+ \subseteq Q^I_\lambda + \mathbb{R}^n_+ = Q^I_\lambda \). On the other hand, for any \( x \in Q^I_\lambda \), it can be written as the convex combination of points in \( C_\lambda \), meaning there exists \( t_j \geq 0, y_j \in C_\lambda, j \in J, \sum_{j \in J} t_j = 1 \), such that \( x = \sum_{j \in J} t_j \cdot y_j \). From the property of \( C'_\lambda \) we know for each \( y_j \in C_\lambda, j \in J \), we have \( y_j' \leq y_j \) for some \( y_j' \in C'_\lambda \). So: \( \sum_{j \in J} t_j \cdot y_j' \leq \sum_{j \in J} t_j \cdot y_j = x \). Hence \( x \in \text{conv}(C'_\lambda) + \mathbb{R}^n_+ \), which implies \( Q^I_\lambda \subseteq \text{conv}(C'_\lambda) + \mathbb{R}^n_+ \), and we conclude the proof. \( \square \)

Following the same idea as we discussed in previous sections, first thing we want to do is defining the corresponding “family of cuts” \( A \). From the definition of aggregation closure, we observe that all it’s inequalities come from the facet defining inequalities of every \( Q^I_\lambda \), and the trivial inequalities \( x_i \geq 0, i \in [n] \). And as we’ve shown in Lemma 11, every facet defining inequality \( \alpha x \geq \beta \) would have \((\alpha, \beta) \geq 0 \). For that reason, we define \( \mathcal{F}_\lambda := \{ a \in \mathbb{R}^n_+ | ax \geq 1 \} \), denote the family of cuts \( A := \{ (-c_i, 0) \}_{i \in [n]} \cup \{ (-a, -1) \}_{a \in \bigcup_{\lambda \in \mathbb{R}^n, \mathcal{F}_\lambda} \} \cup \{ (0, \ldots, 0, 1) \} \), and we have the following:

\[
\mathcal{A}(Q) = \mathbb{R}^n_+ \bigcap_{a \in \bigcup_{\lambda \in \mathbb{R}^n_+, \mathcal{F}_\lambda}} \{ x \in \mathbb{R}^n | ax \geq 1 \} = \mathcal{F}(A).
\]

As in our previous discussion, we also denote \( K := \text{cl} \text{one} A \). Since it’s clear to see \( \mathcal{A}(Q) \) is full-dimensional (as a matter of fact it’s recession cone is \( \mathbb{R}^n_+ \)), therefore \( K \) here is a pointed closed convex cone (Corollary 2), and it’s our goal here to show \( K \) having only finitely many different extreme rays. W.L.o.g. we can assume \((-r, -1)\) being the extreme ray, where \( r \geq 0 \). From Lemma 1 we know that \( r \) has to satisfy one of the following:

1. \( r \in \mathcal{F}_\lambda \) for some \( \lambda \in \mathbb{R}^n_+ \);
2. \( r^i \rightarrow r \) for \( r^i \in Q^I_{\lambda^i}, \lambda^i \in \mathbb{R}^n_+ \), \( i \in \mathbb{N} \).
In the following two subsections, the first step is to show $r$ has to satisfy (1):

**Proposition 2.** For $K$ defined as above, if $(-r, -1)$ is an extreme ray of $K$, then $r \in \mathcal{F}_\lambda$ for some $\lambda \in \mathbb{R}^n_+$. 

And the second step is proving the number of $r$ satisfying (1) is finite:

**Proposition 3.** For $K$ defined as above, the number of different $r \in \bigcup_{\lambda \in \mathbb{R}^n_+} \mathcal{F}_\lambda$ with $(-r, -1)$ being extreme ray of $K$ is finite.

In the end, Theorem 2 would follow from Theorem 1 and these two propositions.

### 3.2 The First Step

For $\alpha \in \mathbb{R}^n$, denote $\text{supp}(\alpha)$ to be the support of this vector $\alpha$; For polyhedron $P$, $\text{rec}(P)$ refers to its recession cone; For $M \in \mathbb{R}^{m \times n}$, $I \subseteq [n]$, denote $M_I$ to be the column matrix of $M$ with indices in $I$.

**Proof (Proof of Proposition 2).** We prove by induction on dimension. For one-dimensional case the result can be trivially verified. Next we assume the statement of this proposition is true when dimension is at most $n-1$, and we consider the case of dimension equals $n$.

Assuming $(-r, -1) \in K$ is an extreme ray, and $r \notin \mathcal{F}_\lambda$ for any $\lambda \in \mathbb{R}^n_+$. By Lemma 1 we know there exists $r^i \to r$ for different $r^i \in Q^i_\lambda$, $\lambda^i \in \mathbb{R}^n_+$. Next we argue by two cases, depending on whether or not $\text{supp}(r) = [n]$. In either case we want to form the contradiction. Recall that $r \geq 0$.

1. $\text{supp}(r) = [n]$: In this case, we observe that $\{x \in \mathbb{R}^n_+ \mid r x \in \left[\frac{1}{2}, \frac{3}{2}\right]\}$ is a bounded set, so $\{x \in \mathbb{N}^n \mid r x \in \left[\frac{1}{2}, \frac{3}{2}\right]\}$ is a finite set. Since $r^i \to r$, we know there exists $N \in \mathbb{N}$, such that when $i \geq N$, $\{x \in \mathbb{N}^n \mid r^i x = 1\} \subseteq \{x \in \mathbb{N}^n \mid r x \in \left[\frac{1}{2}, \frac{3}{2}\right]\}$, and $\{x \in \mathbb{R}^n_+ \mid r^i x = 1\}$ also being a bounded set. By the Pigeonhole Principle, we know there must exist infinitely many different $r^i$ with the same $\{x \in \mathbb{N}^n \mid r^i x = 1\}$. Since $\{x \in \mathbb{R}^n_+ \mid r^i x = 1\} \cap Q^i_\lambda$, defines a bounded facet of $Q^i_\lambda$, which is a full-dimensional polyhedron with integral extreme points, therefore, we can find $n$ linearly independent integer points from $\{x \in \mathbb{N}^n_+ \mid r x = 1\}$. We compose them as row vectors to get a non-singular square matrix, we get a linear equation with that nonsingular square matrix being the coefficient matrix, and the right hand side vector being $(1, \ldots, 1)^T$. Then, all those infinitely many different $r^i$ would be solutions to this linear equation, which is impossible.

2. $\text{supp}(r) \subset [n]$: W.l.o.g. we assume $\text{supp}(r) = [k]$ for $k < n$.

**Claim.** For covering polyhedron $Q$ in $\mathbb{R}^n_+$ and $k < n, k \in \mathbb{N}$, there exists another covering polyhedron $Q'$ in $\mathbb{R}^n_+$ such that $\text{proj}_{[k]} A(Q) = A(Q')$. Furthermore, for any $Q^i_\gamma$, there exists $Q^i_\lambda$ such that $Q^i_\gamma = \text{proj}_{[k]} Q^i_\lambda$.

**Proof (Proof of Claim).** First, we show for any $P, Q \subseteq \mathbb{R}^n_+$ with $\text{rec}(P) = \text{rec}(Q) = \mathbb{R}^n_+$, there is

$$\text{proj}_{[k]} (P \cap Q) = \text{proj}_{[k]} P \cap \text{proj}_{[k]} Q.$$ (8)
The first $\subseteq$ direction is well-known, we only need to show: $\forall x \in \text{proj}_I P \cap \text{proj}_I Q$, there is $x \in \text{proj}_I(P \cap Q)$. By assumption, we know there exists $y, z \in \mathbb{R}^{n-k}_+$, such that $(x, y) \in P, (x, z) \in Q$. Since $\text{rec}(P) = \mathbb{R}^n_+, z \in \mathbb{R}^{n-k}_+$, we get $(x, y + z) \in P$. Similarly, we also get $(x, z + y) \in Q$. Thus $(x, y + z) \in P \cap Q$, which implies $x \in \text{proj}_I(P \cap Q)$.

Since for any $\lambda \in \mathbb{R}^n_+$, $Q^I_\lambda$ has $\text{rec}(Q^I_\lambda) = \mathbb{R}^n_+$, then from equation (9), we get:

$$\text{proj}_I A(Q) = \bigcap_{\lambda \in \mathbb{R}^n_+} \text{proj}_I Q^I_\lambda.$$  \hspace{1cm} (9)

Also, because $Q^I_\lambda = \text{conv}(\{x \in \mathbb{N}^n \mid \lambda Mx \geq \lambda d\})$, and for any set $S \subseteq \mathbb{R}^n_+$, $\text{proj}_I(S) = \text{conv}(\text{proj}_I(S))$, we can get $\text{proj}_I Q^I_\lambda = \text{conv}(\text{proj}_I Q^I_\lambda \{x \in \mathbb{N}^n \mid \lambda Mx \geq \lambda d\})$. If $\lambda M_{[n]}[k] = 0$, then $\text{proj}_I Q^I_\lambda \{x \in \mathbb{N}^n \mid \lambda Mx \geq \lambda d\} = \{x \in \mathbb{N}^n \mid \lambda M[k]x \geq \lambda d\}$; If $\lambda M_{[n]}[k] \neq 0$, then $\text{proj}_I Q^I_\lambda \{x \in \mathbb{N}^n \mid \lambda Mx \geq \lambda d\} = \mathbb{N}^k$. Denote $A := \{\lambda \in \mathbb{R}^n_+ \mid \lambda M_{[n]}[k] = 0\}$, then from the above analysis we would have:

$$\text{proj}_I A(Q) = \bigcap_{\lambda \in \mathbb{R}^n_+} \text{proj}_I Q^I_\lambda$$

$$= \bigcap_{\lambda \in A} \text{proj}_I Q^I_\lambda$$

$$= \bigcap_{\lambda \in A} \text{conv}(\{x \in \mathbb{N}^k \mid \lambda M[k]x \geq \lambda d\}). \hspace{1cm} (10)$$

Here $A$ is a cone in $\mathbb{Q}^m_+$, from Minkowski-Weyl theorem we can express $A = \{\gamma R \mid \gamma \in \mathbb{R}^n_+\}$, here $R \in \mathbb{Q}^{d \times m}_+$ is the matrix constructed by the extreme rays of $A$. Hence:

$$\bigcap_{\lambda \in A} \text{conv}(\{x \in \mathbb{N}^k \mid \lambda M[k]x \geq \lambda d\}) = \bigcap_{\gamma \in \mathbb{R}^n_+} \text{conv}(\{x \in \mathbb{N}^k \mid \gamma RM[k]x \geq \gamma Rd\})$$

$$= \bigcap_{\gamma \in \mathbb{R}^n_+} \text{conv}(\{x \in \mathbb{N}^k \mid \gamma M'x \geq \gamma d'\})$$

$$= A(Q'). \hspace{1cm} (11)$$

Where $M' := RM[k], d' = Rd$, and $Q' := \{x \in \mathbb{R}^k_+ \mid M'x \geq d'\}$. Easy to verify such $Q'$ is also a rational covering polyhedron. Combining equation (10) and (11) we have $\text{proj}_I A(Q) = A(Q')$.

Moreover, for each $\gamma \in \mathbb{R}^n_+, Q^I_\gamma = \text{conv}(\{x \in \mathbb{N}^k \mid \gamma RM[k]x \geq \gamma Rd\})$, denote $\lambda = \gamma R$, then we would have $Q^I_\gamma = \text{proj}_I Q^I_\lambda$.

Since $(-r, -1)$ is an extreme ray of $K$, by Lemma 5, we know that $rx \geq 1$ is a FII of $A(Q)$. Because $\text{supp}(r) = [k]$, it’s easy to see that $r[k]x \geq 1$ is also a FII to $\text{proj}_I A(Q)$, since otherwise we can find finitely different valid inequalities to $\text{proj}_I A(Q)$ which dominate $r[k]x \geq 1$, then the extension of
these inequalities to $\mathbb{R}^n$ would also dominate $rx \geq 1$, and this contradicts to the FII assumption. By Claim 2 we have $r[k]x \geq 1$ is a FII to $\mathcal{A}(Q^\prime)$ for some covering polyhedron $Q^\prime$ in $\mathbb{R}^k$. By inductive hypothesis, we know there exists $\gamma^*$ such that $r[k]x \geq 1$ is a facet defining inequality of $Q^\prime$. Still from Claim 2 there exists $\lambda^* \in \mathbb{R}^m_+$ with $Q^\prime_{\gamma^*} = \text{proj}_{[k]} Q^\prime_{\lambda^*}$, so $r[k]x \geq 1$ is valid to $\text{proj}_{[k]} Q^\prime_{\lambda^*}$. Since $r = (r[k], 0, \ldots, 0)$, we further have $rx \geq 1$ is valid to $Q^\prime_{\lambda^*}$. Together with the assumption that $rx \geq 1$ is a FII to $\mathcal{A}(Q)$, we know $rx \geq 1$ must be a facet defining inequality to $Q^\prime_{\lambda^*}$, which contradicts to the uppermost assumption $r \notin \mathcal{F}_{\lambda}$ for any $\lambda \in \mathbb{R}^m_+$.

Therefore, when the dimension is $n$, the statement of this proposition also holds. By induction we conclude the proof. $\square$

### 3.3 The Second Step

In order theory, given a poset $\mathcal{O}$ with order $\leq$, the downset of a subset $S \subseteq \mathcal{O}$ is defined to be:

$$\mathcal{D}(S) := \{ x \in \mathcal{O} \mid \exists s \in S, \text{ s.t. } x \leq s \}. $$

Downsets are defined with respect to a specific poset. Here for the ease of illustration, without further mentioning we would refer $\mathcal{D}(S)$ as $\{ x \in \mathbb{N}^n \mid \exists s \in S, \text{ s.t. } x \leq s \}$ for some $n \in \mathbb{N}, S \subseteq \mathbb{N}^n$, which is the downset with respect to $(\mathbb{N}^n, \leq)$. And when it’s clear from the context, we do not distinguish them between different dimension $n$.

Given a valid inequality to $\mathcal{A}(Q)$, next we give a sufficient condition for that valid inequality not being a FII to $\mathcal{A}(Q)$, this would be a crucial tool for our later proof to lead to contradiction. Here for polyhedron $P$, we use $\text{ext}(P)$ to denote the set of all extreme points of $P$.

**Lemma 12.** For $\lambda^1, \lambda^2 \in \mathbb{R}^m_+$, and $a^1 \in \mathcal{F}_{\lambda^1}, a^2 \in \mathcal{F}_{\lambda^2}$ with $\text{supp}(a^1) = \text{supp}(a^2) := I, a^1 \neq a^2$. If $\mathcal{D}(\text{proj}_I \text{ext}(F^1)) \subseteq \mathcal{D}(\text{proj}_I \text{ext}(F^2))$, where $F^i = \{ x \in \mathbb{R}^n \mid a^i x = 1 \} \cap Q^\prime_{\lambda^i}$, for $i \in [2]$. Then $a^1 x \geq 1$ is not a FII to $\mathcal{A}(Q)$.

**Proof.** Since $\mathcal{A}(Q) \subseteq Q^\prime_{\lambda^1} \cap Q^\prime_{\lambda^2}$, it’s sufficient for us to show: $a^1 x \geq 1$ is not a facet defining inequality to $Q^\prime_{\lambda^1} \cap Q^\prime_{\lambda^2}$. It’s equivalent to showing: $\text{dim} \{ x \in \mathbb{R}^n \mid a^1 x = 1 \} \cap (Q^\prime_{\lambda^1} \cap Q^\prime_{\lambda^2}) < n - 1$, where:

$$\{ x \in \mathbb{R}^n \mid a^1 x = 1 \} \cap (Q^\prime_{\lambda^1} \cap Q^\prime_{\lambda^2}) = (\{ x \in \mathbb{R}^n \mid a^1 x = 1 \} \cap Q^\prime_{\lambda^1}) \cap (Q^\prime_{\lambda^2}) = F^1 \cap \bigcap_{\lambda^2} Q^\prime_\lambda \subseteq F^1 \cap \{ x \in \mathbb{R}^n \mid a^2 x \geq 1 \}. $$

Easy to see: for $i \in [2], F^i = \text{conv}(\text{ext}(F^i)) + \text{cone}(\{ \epsilon_i \mid \epsilon \in [n] \setminus I \}$, where $I = \text{supp}(a^1) = \text{supp}(a^2)$. We arbitrarily pick $f^1 \in \text{ext}(F^1)$, from $\mathcal{D}(\text{proj}_I \text{ext}(F^1)) \subseteq \mathcal{D}(\text{proj}_I \text{ext}(F^2))$, we know there exists $f^2 \in F^2$ such that $\text{proj}_I f^1 \leq \text{proj}_I f^2$. 
Fig. 1. In this case $\mathcal{D}(\{f_1, g_1\}) \subseteq \mathcal{D}(\{f_2, g_2\})$, and $a^i x = 1$ are facet defining inequality of $Q^I_{\lambda_i}$ for $i = 1, 2$. Here $a^i x \geq 1$ cannot be a FII to $A(Q)$. Since supp$(a^2) = I$, we know for any $x \in \mathbb{R}^n, a^2 x = \text{proj}_f a^2 \cdot \text{proj}_f x$. Therefore:

$$a^2 \cdot f^1 = \text{proj}_f a^2 \cdot \text{proj}_f f^1 \leq \text{proj}_f a^2 \cdot \text{proj}_f f^2 = a^2 f^2 = 1.$$ (13)

Here the first $\leq$ is because $a^2 \in \mathbb{R}^n_+$ and $\text{proj}_f f^1 \leq \text{proj}_f f^2$, the last equation is because $f^2 \in F^2$.

Notice that for any $i \notin I, a^2 e_i = 0$, also because $F^1 = \text{conv}(\text{ext}(F^1)) + \text{cone}\{e_i\}_{i \in [n] \setminus I}$, and for any $f^1 \in \text{ext}(F^1)$, there is $a^2 \cdot f^1 \leq 1$. So for any $f \in F^1$, we always have $a^2 f \leq 1$, which implies

$$F^1 \cap \{x \in \mathbb{R}^n \mid a^2 x \geq 1\} = F^1 \cap \{x \in \mathbb{R}^n \mid a^2 x = 1\}.$$ (14)

Together with (12), we would get: $\{x \in \mathbb{R}^n \mid a^1 x = 1\} \cap (Q^I_{\lambda_1} \cap Q^I_{\lambda_2}) \subseteq F^1 \cap \{x \in \mathbb{R}^n \mid a^2 x = 1\}$, which is further contained in $\{x \in \mathbb{R}^n \mid a^1 x = 1\} \cap \{x \in \mathbb{R}^n \mid a^2 x = 1\}$, with dimension $n - 2$ because $a^1 \neq a^2$. Hence we complete the proof.

The last pieces of results we would need are from [21]. Recall that an antichain is a subset of a poset where no two elements are comparable.

**Proposition 4 (Proposition 2 [21]).** For any $n \in \mathbb{N}$, there is no infinite antichain in poset given by all integer packing sets in $\mathbb{R}^n$, ordered by inclusion.
Lemma 13 (Lemma 8 [21]). Let \((\mathcal{O}, \preceq)\) be a poset with no infinite-sized antichain. Let \(I\) be a set with \(|I| = \infty\) and \(k \in \mathbb{N}\), let \(a_j^i \in \mathcal{O}\) for each \(i \in I\) and \(j \in [k]\). Assume that, for every \(i \in I\) and \(j \in [k]\), the downset \(\mathcal{D}(\{a_j^i\})\) in the poset \((\mathcal{O}, \preceq)\) has finite size. Then there exist \(t_1, t_2 \in I\) such that \(a_{t_1}^i \preceq a_{t_2}^j\) for every \(j \in [k]\).

Now we are ready to prove Proposition 3.

Proof (Proof of Proposition 3). Assume for contradiction: \(a^i x \geq 1, i \in \mathbb{N}\) to be a sequence of infinitely many FIs of \(\mathcal{A}(Q)\), here \(a^i \in \mathcal{F}_{\lambda^i}\) for some \(\lambda^i \in \mathbb{R}^n_+\).

Claim. \(\|a^i\|_\infty \leq 1\) for all \(i \in \mathbb{N}\).

Proof (Proof of Claim). We know \(Q_{\lambda^i}\) is a full-dimensional polyhedron, with integral extreme points and \(\text{rec}(Q_{\lambda^i}) = \mathbb{R}^n_+\). Therefore, for each facet defining inequality which doesn’t pass the origin, we can find \(n\) linearly independent integer points on that corresponding facet. If for some \(i \in \mathbb{N}, j \in [n], a_j^i > 1\), then: \(Q_{\lambda^i} \cap \{x \in \mathbb{R}^n \mid a^i x = 1\} \cap \mathbb{Z}^n \subseteq \{x \in \mathbb{N}^n \mid ax = 1\} \subseteq \{x \in \mathbb{N}^n \mid x_j = 0\}\), so we cannot find \(n\) linearly independent integer points on facet \(Q_{\lambda^i} \cap \{x \in \mathbb{R}^n \mid a^i x = 1\}\), which gives the contradiction.

Since the \(\infty\)-norm unit ball is a compact set, by Bolzano-Weierstrass theorem, now we know there must exist a convergent subsequence of \(\{a^i\}_{i \geq 1}\). W.l.o.g. we still assume this convergent subsequence to be \(\{a^i\}_{i \geq 1}\), and \(a^i \to \alpha\). As our proof in Proposition 2 here we also discuss according to \(\text{supp}(\alpha)\):

1. \(\text{supp}(\alpha) = [n]\): The proof in this case is exact the same as the corresponding part in Proposition 2 and we omit it here.
2. \(\text{supp}(\alpha) \subseteq [n]\): W.l.o.g. we assume \(\text{supp}(\alpha) = [k], k < n\). Since \(a^i \to \alpha\), we know there exists \(N_1\), such that for any \(i \geq N_1\), we have \([k] = \text{supp}(\alpha) \subseteq \text{supp}(a^i) \subseteq [n]\).

Since here \([n]\) is a finite set, and there are infinitely many \(\{a^i\}_{i \geq N_1}\), we know there must exist infinitely many vectors among \(\{a^i\}_{i \geq N_1}\) with the same support. W.l.o.g. we still denote this infinitely many vectors to be \(\{a^i\}_{i \geq 1}\), and the support of those vectors to be \([t], t \geq k\). Hence we have:

For any \(i \in \mathbb{N}\), \([k] = \text{supp}(\alpha) \subseteq \text{supp}(a^i) = [t]\).

Here we denote \(T := \text{proj}_{[k]}\{x \in \mathbb{N}^n \mid ax \leq \frac{1}{2}\}\), \(T^i := \text{proj}_{[k]}\{x \in \mathbb{N}^n \mid a^i x = 1\}\). Because \(a^i \to \alpha, \text{supp}(a^i) = [t] \supseteq [k] = \text{supp}(\alpha)\), we know there exists \(N_2\), such that for all \(i \geq N_2\), \(T^i \subseteq T\). Clearly here \(T\) is a finite set, and because \(T^i \subseteq T\) for all \(i \geq N_2\), by the Pigeonhole principle, we know there are infinitely many \(a^i\) with the same \(T^i\). W.l.o.g we still assume this infinitely many \(a^i\) to be \(\{a^i\}_{i \geq 1}\), and denote \(T^* = T^i\) for all \(i \in \mathbb{N}\). Here \(T^* \subseteq T\).
Recall what we have established so far: For any $i \in \mathbb{N}$, there is

$$[k] = \text{supp}(\alpha) \subseteq \text{supp}(a_i) = [t], \quad T^i = T^*.$$  \hfill (15)

Now we denote $F^i$ to be the corresponding facet in $Q^i_{\lambda}$, for facet defining inequality $a^i x \geq 1$, namely, $F^i := Q^i_{\lambda} \cap \{x \in \mathbb{R}^n \mid a^i x = 1\}$. Also for each $j \in T^*$, $i \in \mathbb{N}$, we denote $A_j^i := \text{proj}_{[t] \setminus [k]} \{f \mid f \in \text{ext}(F^j)\}$ with $\text{proj}_{[k]} f = j \subseteq \mathbb{N}^{t-k}$ (it’s defined to be $\emptyset$ if $t = k$). Here $A_j^i$ is a finite set.

**Claim.** For any $i \in \mathbb{N}$, $\text{proj}_{[t]} \text{ext}(F^i) = \bigcup_{j \in T^*} \{(j) \times A_j^i\}$.

**Proof (Proof of Claim).** First, from the definition of $A_j^i$, we clearly have $(j) \times A_j^i \subseteq \text{proj}_{[t]} \text{ext}(F^i)$, for any $j \in T^*$, $i \in \mathbb{N}$. So we have $\text{proj}_{[t]} \text{ext}(F^i) \supseteq \bigcup_{j \in T^*} \{(j) \times A_j^i\}$.

On the other hand, for any point $f \in \text{ext}(F^i)$, denote $j := f_{[k]}$, which belongs to $T^i = T^*$. Obviously $f_{[t] \setminus [k]} \in A_j^i$, so $f \in \bigcup_{j \in T^*} \{(j) \times A_j^i\}$.

Next, the goal is to find $p, q \in \mathbb{N}$ with $\mathcal{D}(\text{proj}_{[t]} \text{ext}(F^p)) \subseteq \mathcal{D}(\text{proj}_{[t]} \text{ext}(F^q))$. Then by Lemma 12 we would imply that $a^p x \geq 1$ is not a FII of $\mathcal{A}(Q)$, which gives us the contradiction to our initial assumption.

We are going to apply Lemma 13 here. Pick the poset of all integer packing sets in $\mathbb{R}^{t-k}$ with partial order $\subseteq$ to be the poset $(\mathcal{O}, \subseteq)$ in Lemma 13, $I$ is picked to be $\mathbb{N}$, $j \in [k]$ in Lemma 13 is picked to be $j \in T^*$, and $a^i_j \in \mathcal{O}$ in Lemma 13 is picked to be $\mathcal{D}(A_j^i)$, which is an integer packing set in $\mathbb{R}^{t-k}$. Here the downset $\mathcal{D}(\cdot)$ is with respect to poset $(\mathbb{N}^{t-k}, \subseteq)$. Since $A_j^i$ has finite size, we know $\mathcal{D}(A_j^i)$ is a finite set in $\mathbb{N}^{t-k}$. Denote $\tilde{\mathcal{D}}(\cdot)$ to be the downset with respect to the poset $(\mathcal{O}, \subseteq)$. Hence $\tilde{\mathcal{D}}(\mathcal{D}(A_j^i))$ represents the collection of all possible integer packing sets in $\mathbb{R}^{t-k}$ which are also subset of $\mathcal{D}(A_j^i)$. Since $\mathcal{D}(A_j^i)$ is a finite set, therefore the number of its subsets is also finite. Thus the conditions in Lemma 13 are all satisfied, and this gives us: there exists $p, q \in \mathbb{N}$ with $\mathcal{D}(A_p^i) \subseteq \mathcal{D}(A_q^i)$ for all $j \in T^*$.

Note that

$$\mathcal{D}(\text{proj}_{[t]} \text{ext}(F^p)) = \mathcal{D}\left(\bigcup_{j \in T^*} \{(j) \times A_j^p\}\right) = \bigcup_{j \in T^*} \left(\mathcal{D}\{(j)\} \times \mathcal{D}(A_j^p)\right).$$

Here the first equation is from the last claim, the second equation is from the definition of downsets. Since $\mathcal{D}(A_j^p) \subseteq \mathcal{D}(A_j^q)$ for each $j \in T^*$, we obtain: $\mathcal{D}(\text{proj}_{[t]} \text{ext}(F^p)) \subseteq \mathcal{D}(\text{proj}_{[t]} \text{ext}(F^q))$, and this concludes the proof, as we remarked before.

All in all, there are at most finitely many FIIs $ax \geq 1$ with $a \in \bigcup_{\lambda \in \mathbb{R}_{\mathbb{N}}^n} \mathcal{F}_\lambda$. \hfill \Box

From Proposition 2, Proposition 3 and Theorem 1, we concluded the proof of Theorem 2.
A Chvátal-Gomory cut (CG cut) for a polyhedron $P$ is an inequality of the form $cx \leq \delta$, where $c$ is an integer vector and $cx \leq \delta$ is valid for $P$. Then the CG closure of $P$ is the intersection of all half-spaces defined by such inequalities. Schrijver [17] showed that for a rational polyhedron, the CG closure is again a rational polyhedron, and asking whether the CG closure of an arbitrary polytope is a rational polytope. This longstanding open problem was proved by [11, 7] individually. Among their 20 pages and 28 pages proof, they shared similar idea and both were using the induction argument on dimension. Even though CG cuts were originally introduced for polyhedra, they have lately been applied to other convex sets as well. In this general setting, a series of studies have been conducted to the polyhedrality of CG closure of many different convex sets. In [10], the authors showed that the CG closure of a bounded full-dimensional convex set is a rational polytope. This longstanding open problem was proved by [11, 7]

**Proof (Proof of Theorem 4).**

We present its proof in the following:

The first result we can obtain from our general characterization Theorem 1 and Lemma 6 is Theorem 4. We present its proof in the following:

1. First, we want to show there are at most finitely many different extreme rays $(r, 1)$ of $K$ which does not belong to $\{\lambda a \mid \lambda \geq 0, a \in A \}$. By Lemma 6 we know there exists $(c^i, [\sigma_P(c^i)]) \subseteq A$ such that $(c^i, [\sigma_P(c^i)]) \sim (r, r_0)$; there exists $\gamma_i : (c^i, [\sigma_P(c^i)]) \rightarrow (r, r_0)$ for $\gamma_i > 0$. Since $c^i \in \mathbb{Z}^n$, we know $\|c^i\| \rightarrow \infty$. So $\gamma_i \rightarrow 0$. Also because $\gamma_i[\sigma_P(c^i)] \rightarrow 1$, and $\gamma_i[\sigma_P(c^i)] - 1 < \gamma_i[\sigma_P(c^i)] \leq \gamma_i[\sigma_P(c^i)]$, we know that $\gamma_i[\sigma_P(c^i)] \rightarrow r_0$. Due to the smoothness of linear programming objective value with respect to objective vector, for $\gamma_i c^i \rightarrow r$, we have $\sigma_P(r) = \sigma_P(\lim\gamma_i c^i) = \lim\sigma_P(\gamma_i c^i)$, which is just $\lim\gamma_i[\sigma_P(c^i)] = r_0$. Hence extreme ray $(r, r_0)$ of $K$ just corresponding to a valid inequality of $P$, which has finitely many facet defining inequalities.

2. Then, we are showing there are at most finitely many different extreme ray of $K$ which also belong to $A$. Assume for contradiction: there exists $r^i x \leq [\sigma_P(r^i)]$, $r^i \in C$, $i \in \mathbb{N}$, which are all FIIs to $S^{CG}$. Here $(r^i, [\sigma_P(r^i)])$ are all extreme rays of $K$. By Bolzano-Weierstrass theorem, any infinite sequence contains a conically convergent subsequence to a non-zero point (scaling those infinitely vectors to a unit 1-norm ball, for example). We still denote this subsequence to be $i \in \mathbb{N} : \gamma_i(r^i, [\sigma_P(r^i)]) \rightarrow (r, r_0)$. From the same discussion as in the last case, here we have $\gamma_i[\sigma_P(r^i)] \rightarrow \sigma_P(r)$. By condition, $S^{CG}$ doesn’t meet any face of $S$, and strictly contained in $S$, so for
any \( r \in \mathbb{R}^n \) with \( \sigma_S(r) < \infty, \sigma_S(r) \geq \sigma_{SCG}(r) \). If there are infinitely many \( i \in \mathbb{N} \) with \( \gamma_i [\sigma_S(r^i)] \leq \sigma_{SCG}(\gamma_i r^i) \), then taking the limit, we would get \( \sigma_S(r) \leq \sigma_{SCG}(r) \), which is not true. Therefore, we have infinitely many \( i \in \mathbb{N} \) with \( \gamma_i [\sigma_S(r^i)] \leq \sigma_{SCG}(\gamma_i r^i) \). Notice that \( r^i x \leq \sigma_{SCG}(r^i) \) is valid to \( SCG \), which is \( S(A) \). By Lemma 4 we know \( (r^i, \sigma_{SCG}(r^i)) \in K \). However, \( \sigma_{SCG}(r^i) < [\sigma_S(r^i)] \), and \( (0, \ldots, 0, 1) \in K \), we know \( (r^i, [\sigma_S(r^i)]) \) can be written as the conical combination of \( (0, \ldots, 0) \) and \( (r^i, \sigma_{SCG}(r^i)) \), giving contraction to the assumption that \( (r^i, [\sigma_S(r^i)]) \) is an extreme ray of \( K \).

\[ \square \]

All current results about CG closure so far were essentially based on assumption of either compactness or rational data. In fact, for unbounded irrational polyhedron it’s believed the CG closure is not necessarily still polyhedral. Bear that in mind, we want to further understand what can be a more general sufficient condition to guarantee such polyhedrality, rather than rational data? It is our goal to partially answer this question in this section.

We would need the following well-known simultaneous diophantine approximation theorem due to Kronecker. Note that the version we used here is very similar to the one used by [5].

Lemma 14 ([13, 20, 5]). Let \( n, N_0 \in \mathbb{N} \) and \( \pi \in \mathbb{R}^n \) with \( \pi \neq 0 \). Then \( \mathbb{Z}^n - \pi \mathbb{Z}_{\geq N_0} \) contains a dense subset of a linear subspace \( V \) of \( \mathbb{R}^n \).

In fact, as shown in the proof of Lemma 2 [5], we can construct such linear space \( V \) for each \( \pi \in \mathbb{R}^n \): assume \( \{1, \pi_i \text{ for } i \in I\} \) is a linear basis of \( \{1, \pi_1, \ldots, \pi_n\} \) over \( \mathbb{Q} \), here \( I \subseteq [n] \). For \( j \notin I \), we can find positive integer \( m \) and integers \( n_{j,i}, i \in \{0\} \cup I \), such that \( m \pi_j = n_{j,0} + \sum_{i \in I} n_{j,i} \pi_i \), for \( j \notin I \). Then \( V \) is defined by: \( m x_j = \sum_{i \in I} n_{j,i} x_i, j \notin I \). Easy to see, for different basis of \( \{1, \pi_1, \ldots, \pi_n\} \) over \( \mathbb{Q} \), the constructed linear subspace would always be the same. Hence we can denote such linear subspace of vector \( \pi \) to be \( V_\pi \). For \( \pi \neq 0 \), from this above construction of \( V_\pi \), we note that if \( \pi \in V_\pi \), then for any \( q \in \mathbb{Q}^n, \pi q \) can never be non-zero rational number. In fact the reverse is also true. We state this a little differently in the following:

Lemma 15. For \( 0 \neq \pi \in \mathbb{R}^n, \pi \in V_\pi \) iff \( \pi x = 1 \) has no rational solution.

Proof. First, we show: if \( \pi \in V_\pi \), then \( \pi x = 1 \) has no rational solution. Arbitrarily pick a set of linear basis \( \{1, \pi_i \text{ for } i \in I\} \) of \( \{1, \pi_1, \ldots, \pi_n\} \) over \( \mathbb{Q} \). Since \( \pi \in V_\pi \), from the construction of \( V_\pi \), we know there exists positive integer \( m \) and integers \( n_{j,i}, i \in I \) for \( j \notin I \), such that \( m \pi_j = \sum_{i \in I} n_{j,i} \pi_i \). Therefore, \( \pi^T \cdot x = \pi^T \cdot Tx \) for some \( T \in \mathbb{Q}^{|I| \times n} \). Because here \( \{1, \pi_i \text{ for } i \in I\} \) are linearly independent over \( \mathbb{Q} \), we know there does not exist \( x \in \mathbb{Q}^n \) such that \( \pi^T \cdot Tx \in \mathbb{Q} \), which is saying \( \pi x \notin \mathbb{Q} \) for \( x \in \mathbb{Q}^n \), and it’s equivalent to saying \( \pi x = 1 \) has no rational solution.

Next, we show that: if \( \pi \notin V_\pi \), then we can find \( x \) in \( \mathbb{Q}^n \) with \( \pi x = 1 \). Arbitrarily pick a set of linear basis \( \{1, \pi_i \text{ for } i \in I\} \) of \( \{1, \pi_1, \ldots, \pi_n\} \) over \( \mathbb{Q} \). From assumption \( \pi \notin V_\pi \), we know there exists some \( j \notin I \), such that \( \pi_j = q_j + \sum_{i \in I} q_{j,i} \pi_i \) for some \( 0 \neq q_j \in \mathbb{Q}, q_{j,i} \in \mathbb{Q}, i \in I \). Then, define \( x \) to be:
Lemma 17 (Sticky face lemma [15]). Before giving the proof to this lemma, we would also need the following result:

Proof (Proof of Lemma 16). This lemma can be seen as the polyhedral version of Lemma 1 in [5].

1. $M^T \cdot x = 1$ has no rational solution;
2. There exists $v \in V_M$ such that $v \cdot x < 0$ is valid to $\{ x \in P \mid M_i x = d_i \}$.

Then, for any extreme point $p$ on facet $\{ x \in P \mid M_i x = d_i \}$, there exists $\{ c^i \}_{i \in [k]} \subseteq \mathbb{Z}^n, \{ \tau_i \}_{i \in [k]} \subseteq \mathbb{R}^n$ for some $k \in \mathbb{N}$, such that $M_i = \sum_{i \in [k]} \tau_i c^i$, and $\sum_{i \in [k]} \tau_i (c^i p - \max \{ c^i x \mid x \in P \}) > 0$.

Before giving the proof to this lemma, we would also need the following result:

Lemma 17 (Sticky face lemma [15]). If $P$ is a polyhedron in $\mathbb{R}^n$, $x_0^*$ is a point of $\mathbb{R}^n$ and $F$ is the set of maximizers of $\langle x_0^*, \cdot \rangle$ on $P$ (a face of $P$). Then for any $x^*$ close enough to $x_0^*$, the maximizers of $\langle x^*, \cdot \rangle$ on $P$ are just its maximizers on $F$.

This lemma can be seen as the polyhedral version of Lemma 1 in [5].

Proof (Proof of Lemma 17). Denote $F := \{ x \in P \mid M_i x = d_i \}$, $E$ to be the set of extreme points on $F$, and $U := \max \{ 1, \| e \|_2 \}$ for all $e \in E$. Since $F$ is a bounded facet, there is $x = \text{conv} E$. Easy to see, for any $q \in \mathbb{Q}$, $M_i$ satisfying any of those two conditions if and only if $qM_i$ satisfying the same condition. Therefore w.l.o.g. we assume $d_i = 1$. From Lemma 17 we know there exists $\epsilon > 0$, such that when $\| c - M_i \|_2 \leq \epsilon$, there is $\max \{ cx \mid x \in P \} = \max \{ cx \mid x \in F \}$. Pick $N_0 \in \mathbb{N}$ and $N_0 > \frac{1}{\epsilon}$.

1. If $M_i x = 1$ has no rational solution, by Lemma 15 we know $M_i \in V_M$. Since $\frac{M_i}{2} \in V_M$, we can find a small simplex in $V_M$ containing $\frac{M_i}{2}$ in its relative interior. From Lemma 14 we have $\mathbb{Z}^n - M_i N > N_0$ containing a dense subset in $V_M$, so we can also find $\{ c^i \}_{i \in [k]} \subseteq \mathbb{Z}^n, \{ n_i \}_{i \in [k]} \subseteq \mathbb{N} > N_0$, $k \in \mathbb{N}$, such that $\frac{M_i}{2} = \sum_{i \in [k]} \lambda_i (c^i - n_i M_i), \| c^i - (n_i + \frac{1}{2}) M_i \|_2 < \frac{1}{2U}$.

Here $\lambda_i > 0, \sum_{i \in [k]} \lambda_i = 1, i \in [k]$. So we have: $M_i = \sum_{i \in [k]} \frac{\lambda_i \lambda_n}{n_i + \frac{1}{2}} c^i$, and we denote $\tau_i := \frac{\lambda_i \lambda_n}{n_i + \frac{1}{2}} \in \mathbb{R}_+$. From $\| c^i - (n_i + \frac{1}{2}) M_i \|_2 < \frac{1}{2U}$, here $n_i > N_0 = \frac{1}{\epsilon}$, clearly we have $\| \frac{c^i - M_i}{n_i + \frac{1}{2}} \|_2 < \epsilon$, therefore we also have $\max \{ c^i x \mid x \in P \} = \max \{ c^i x \mid x \in E \}$. Now, for any $p \in E$, notice that $\| c^i p - (n_i + \frac{1}{2}) p \| = \| c^i p - (n_i + \frac{1}{2}) M_i p \| \leq U \| c^i - (n_i + \frac{1}{2}) M_i \|_2 < \frac{1}{\epsilon}$, hence: $\max \{ c^i x \mid x \in P \} = \max \{ c^i x \mid x \in E \} = n_i$, and $c^i p > n_i$, for $p \in E$. Therefore, we also obtained: $\sum_{i \in [k]} \tau_i (c^i p - \max \{ c^i x \mid x \in P \} ) > 0$. 

$x_i = \frac{a_i}{q_i}$ for each $i \in I, x_j = \frac{1}{q_j}$, and $x_k = 0$ for $k \notin I \cup \{ j \}$. Easy to check $x \in \mathbb{Q}^n$, and $\pi x = 1$. Hence we complete the proof. □
2. Now assuming there exists \( v \in V_M \), such that \( vx < 0 \) is valid to facet \( F \). Since \( V_M \) is a linear space, and for any \( \epsilon > 0, \epsilon v \cdot x < 0 \) is still valid to \( F \), so here we assume \( ||v||_2 < \frac{1}{\epsilon} \). Since \( v, -v \in V_M \), and \( \mathbb{Z}^n - M \mathbb{N}_{>0} \) contains a dense subset in \( V_M \), we can find points in \( \mathbb{Z}^n - M \mathbb{N}_{>0} \), and sufficiently close to \( v, -v \), such that 0 is contained in its relative interior. Namely, there exists \( \{c_i\}_{i \in [k]} \subseteq \mathbb{Z}^n, \{n_i\}_{i \in [k]} \subseteq \mathbb{N}_{>0}, 1 \leq t < k \in \mathbb{N} \), such that: 0 = \( \sum_{i \in [k]} \lambda_i (c_i - n_i M_i) \), \( (c_i - n_i M_i) x > 0 \) valid to \( F \) for all \( i \in [t] \), \( (c_i - n_i M_i) x < 0 \) valid to \( F \) for all \( j \in [k] \setminus [t] \), and \( ||c_i - n_i M_i|| < \frac{1}{2} \) for all \( i \in [k] \).

This gives us \( M_i = \sum_{i \in [k]} \lambda_{i,j} c_i \), and we denote \( \tau_i = \frac{\lambda_i}{\sum_{j \in [k]} \lambda_{i,j}} > 0 \).

For \( i \in [t] \), we have \( (c_i - n_i M_i)p > 0 \), which means \( c_i p > n_i \). Also, for any \( \rho' \in E \), we also have \( |c_i \rho' - n_i| = |(c_i - n_i M_i)p'| \leq U ||c_i - n_i M_i|| < 1 \), which implies \( c_i \rho' < n_i + 1 \) for all \( \rho' \in E \). Also since \( ||\frac{c_i}{n_i} - M_i|| < \frac{1}{\epsilon} < \epsilon \), we know \( \max\{c_i x \mid x \in P\} = \max\{c_i x \mid x \in F\} = \max\{c_i x \mid x \in E\} \). Hence: \( c_i \rho - \max\{c_i x \mid x \in P\} = c_i \rho - n_i > 0 \). Similarly, for \( j \in [k] \setminus [t] \), we can do the same discussion, and obtain: \( c_i \rho - \max\{c_i x \mid x \in P\} = 1 + c_i \rho - n_j > 0 \).

Therefore, we also have \( \sum_{i \in [k]} \tau_i (c_i p - \max\{c_i x \mid x \in P\}) > 0 \).

\[ \square \]

We should remark that, without any assumption on \( M_i \), we can still derive almost the same result, except now we can only guarantee \( \sum_{i \in [k]} \tau_i (c_i p - \max\{c_i x \mid x \in P\}) \geq 0 \). It turns out the strictly positivity is the key for our later proof.

Now we are ready to prove our Theorem\( \Box \) We reiterate it in the following:

**Theorem**\( \Box \) Given a polyhedron \( P = \{x \in \mathbb{R}^n \mid Mx \leq d\} \) with full-dimensional integer hull, \( M \in \mathbb{R}^{m \times n}, d \in \{\mathbb{Z} \setminus \{0\}\}^m \). If each row vector \( M_i \) satisfies at least one of the following:

1. There exists \( r \in \mathbb{R} \) such that \( r M_i \in \mathbb{Q}^n \);
2. \( M_i^T \cdot x = 1 \) has no rational solution;
3. There exists \( v \in V_M \) such that \( vx < 0 \) is valid to \( \{x \in P \mid M_i x = d_i\} \).

Furthermore, if facet \( \{x \in P \mid M_i x = d_i\} \) is unbounded, there exists \( r \in \mathbb{R} \) such that \( r M_i \in \mathbb{Q}^n \). Then, \( P^{CG} \) is a rational polyhedron.

**Proof** (Proof of Theorem\( \Box \)). For \( a \in \mathbb{R}^n \), denote \( \sigma_P(a) := \sup_{x \in P} (a, x) \), and \( C := \{c \in \mathbb{Z}^n \mid \sigma_P(c) < \infty\} \). All CG cut are given by \( c x \leq \sigma_P(c) \) for \( c \in C \). Denote \( A := \{(c, \sigma_P(c))\}_{c \in C} \cup \{(0, \ldots, 1)\} \), and \( K := \text{cl cone } A \). So \( P^{CG} = \mathcal{F}(A) \).

Since \( P \) has full-dimensional integer hull, then \( P^{CG} \) is also full-dimensional. By Theorem\( \Box \) it suffices for us to show \( K \) only have finitely many extreme rays. From Lemma\( \Box \) we know any extreme ray of \( K \) is either in \( A \), or can be conically converged by elements in \( A \). Follows the exact same argument as in the proof of case 1 in Theorem\( \Box \) it suffices for us to assume all extreme rays of \( K \) belong to \( A \), and we want to show there are at most finitely many of them.

Assume for contradiction: there exists \( r^i x \leq \sigma_P(r^i) \), \( r^i \in C, i \in \mathbb{N} \), which are all FIs to \( P^{CG} \). Here \( \sigma_P(r^i) = r^i p^i \) for some extreme point \( p^i \) of \( P \). Since \( P \) only have finitely many extreme points, by Pigeonhole principle we know there must be infinitely many \( r^i \) with \( \sigma_P(r^i) = r^i p^i \) for one specific extreme
point. W.l.o.g. we assuming for all \( i \in \mathbb{N} \), \( \sigma_p(r^i) = r^i p \). Because of the strong duality of linear programming and Farkas Lemma, it’s well-known that \( cx \leq P \) is a valid inequality to \( P \) if and only \( c \in \text{cone}(M_1, \ldots, M_t) \), here for each \( i \in [t], M_i x \leq M_i p \) is a facet defining inequality of \( P \) which is also active at extreme point \( p \). According to the condition of this theorem, assuming \( \beta^1 := \kappa_1 M_1, \ldots, \beta^t := \kappa_t M_t \in \mathbb{Z}^n \), here \( t_1 \leq t, \kappa_1, \ldots, \kappa_t \in \mathbb{R}_+ \), and for \( i \in [t \setminus [t_1] \), by Lemma 16 we can find \( \{c^{i,j}\}_{j \in \{k_i \}} \subseteq \mathbb{Z}^n \), \( \{\tau_{i,j}\}_{j \in \{k_i \}} \subseteq \mathbb{R}_+ \), \( k_i \in \mathbb{N} \), such that \( M_t = \sum_{j \in \{k_i \}} \tau_{i,j} c^{i,j}, \) and \( \sum_{j \in \{k_i \}} \tau_{i,j} \{c^{i,j} p - [\sigma_p(c^{i,j})]\} > 0 \). Therefore, for each \( r^i \), there exists infinitely many \( \gamma_{i,j} \geq 0 \) for \( j \in [t] \) such that

\[
 r^i = \sum_{j=1}^{t_1} \gamma_{i,j} M_j = \sum_{j=1}^{t_1} \frac{\gamma_{i,j}}{k_j} \beta^j + \sum_{j=t_1+1}^{t} \gamma_{i,j} \sum_{l \in [k_j]} \tau_{j,l} c^{j,l}. \tag{16}
\]

Here \( \beta^j, c^{j,l} \in \mathbb{Z}^n \), \( \gamma_{i,j}, \kappa_j, \tau_{j,l} \geq 0 \).

Claim. For each \( j \in \{t \setminus [t_1], \{\gamma_{i,j}\}_{i \in \mathbb{N}} \) is a bounded set.

Proof (Proof of Claim). Consider these CG cuts: \( \beta^j x \leq [\beta^j p] \) for \( j \in [t_1], c^{j,l} x \leq [\sigma_p(c^{j,l})] \) for \( j \in [t \setminus [t_1], l \in [k_j] \). Hence, from equation (16), the following is a valid inequality to \( P^{CG} \):

\[
 r^i x \leq \sum_{j=1}^{t_1} \frac{\gamma_{i,j}}{k_j} [\beta^j p] + \sum_{j=t_1+1}^{t} \gamma_{i,j} \sum_{l \in [k_j]} \tau_{j,l} [\sigma_p(c^{j,l})].
\]

For any \( j \in [t \setminus [t_1] \), we have \( \gamma_{i,j} \sum_{l \in \{k_j \}} \tau_{j,l} [\sigma_p(c^{j,l})] < \gamma_{i,j} \sum_{l \in \{k_j \}} \tau_{j,l} c^{j,l} p - 1 \) as long as \( \gamma_{i,j} > \frac{1}{\sum_{l \in \{k_j \}} \tau_{j,l} c^{j,l} p - [\sigma_p(c^{j,l})]} \). In that case, \( r^i x \leq r^i p - 1 \) is also valid to \( P^{CG} \), which contradicts to the assumption that \( r^i x \leq [r^i p] \) is a FII to \( P^{CG} \).

For each \( r^i \), it can be split into two parts: \( r^i = \bar{r}^i + \underline{r}^i \), where:

\[
 \bar{r}^i := \sum_{j=1}^{t_1} \frac{\gamma_{i,j}}{k_j} \beta^j + \sum_{j=t_1+1}^{t} \sum_{l \in \{k_j \}} \gamma_{i,j} \tau_{j,l} c^{j,l}, \tag{17}
\]

\[
 \underline{r}^i := \sum_{j=1}^{t_1} \left( \frac{\gamma_{i,j}}{k_j} \left[ \beta^j - \frac{\gamma_{i,j}}{k_j} \right] \right) + \sum_{j=t_1+1}^{t} \sum_{l \in \{k_j \}} \left( \left[ \gamma_{i,j} \tau_{j,l} - \frac{\gamma_{i,j}}{k_j} \right] \right) c^{j,l}. \tag{18}
\]

Because all \( \beta^j \) and \( c^{j,l} \) are integer vectors, we know \( \bar{r}^i \) is integer vector, therefore \( \underline{r}^i \) is also an integer vector. Notice that the coefficients of all \( \beta^j \) and \( c^{j,l} \) in \( \underline{r}^i \) is bounded in \([0, 1]\), so \( \underline{r}^i \) can only have finitely many possibilities. Therefore there exists infinitely many \( r^i \) with the same \( \underline{r}^i \). We still assuming this infinite sequence to be \( \{r^i\}_{i \in \mathbb{N}} \).

Now, we take a look at the coefficients of \( \bar{r}^i \), which are all non-negative integer numbers. For \( r^i \), we denote \((u^i, v^i)\) to represent the coefficients of \( r^i \) in (17), here \( u^i := (\frac{\gamma_{i,1}}{k_1}, \ldots, \frac{\gamma_{i,t_1}}{k_{t_1}}) \), and \( v^i := (\gamma_{i,t_1+1}, \tau_{t_1+1,1}, \ldots, \gamma_{i,t_1+1,t_{t_1}}) \).
From the last claim, \( \{ v^i \}_{i \in \mathbb{N}} \) is a bounded set. Since each \( v^i \) is an integer vector, we know there must exist infinitely many \( i \in \mathbb{N} \) with the same \( v^i \) vector. We still assuming all \( i \in \mathbb{N} \) have the same \( v^i \). In this infinite set \( \{ u^i \}_{i \in \mathbb{N}} \), which is a subset of \( \mathbb{N}^t \), by Gordan-Dickson lemma 11, we know there exists \( a, b \in \mathbb{N} \), such that \( u^a \leq u^b \). Therefore, we find \( a, b \in \mathbb{N} \), satisfying the following:

\[
L^a = L^b, v^a = v^b, u^a \leq u^b.
\]

Consider vector \( r^b - r^a \). It can be written as:

\[
r^b - r^a = (r^b - r^a) + (L^b - L^a) = r^b - r^a = \sum_{j=1}^{t_1} (u^b_j - u^a_j) \beta^j + \sum_{j=t_1+1}^t \sum_{i \in [k_j]} (v^b_{j,i} - v^a_{j,i}) c_{j,i}.
\]

Since \( u^a \leq u^b \), which means \( r^b - r^a \) can be written as the conical combination of \( \{ \beta^j \}_{j \in [t_1]} \). Because for each \( \beta^j, \beta^j x \leq \beta^j p \) is valid to \( P \), we have \( (r^b - r^a) x \leq (r^b - r^a) p \) is also valid to \( P \). Therefore \( (r^b - r^a) x \leq \lfloor (r^b - r^a) p \rfloor \) is also a CG cut. Note that \( \lfloor (r^b - r^a) p \rfloor + \lfloor r^a p \rfloor \leq |r^b p| \), so CG cut \( r^b x \leq |r^b p| \) is dominated by \( (r^b - r^a) x \leq \lfloor (r^b - r^a) p \rfloor \) and \( r^a x \leq \lfloor r^a p \rfloor \), which contradicts to the FII assumption.

\[\square\]

5 Epilogue

In this paper we propose a novel way of showing the polyhedrality of general cutting-plane closures, by characterizing the number of extreme rays in one of its related closed convex cone. Due to the lack of specificity, this is not a one-size-fits-all approach, in most cases further necessary argument is needed. Because of that, the sufficient condition we provided in Theorem 3 for CG closure is nowhere near being complete, for instance. Meanwhile our proof of Theorem 4 is rather neat, and is very representative for proofs using Theorem 1 and Lemma 6. That being said, we do believe this approach can enable us to tackle most polyhedrality problems from a different angle, and further investigation might be of independent interest.

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