THE SATURATION PROBLEM FOR REFINED LITTLEWOOD-RICHARDSON COEFFICIENTS

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Abstract. Given partitions $\lambda, \mu, \nu$ with at most $n$ nonzero parts and a permutation $w \in S_n$, the refined Littlewood-Richardson coefficient $c^\nu_{\lambda \mu} (w)$ is the multiplicity of the irreducible $GL_n \mathbb{C}$ module $V(\nu)$ in the so-called Kostant-Kumar submodule $K(\lambda, w, \mu)$ of the tensor product $V(\lambda) \otimes V(\mu)$. We derive a hive model for these coefficients and prove that the saturation property holds if $w$ is 312-avoiding, 231-avoiding or a commuting product of such elements. This generalizes the classical Knutson-Tao saturation theorem.

Keywords: hives, saturation, refined Littlewood-Richardson coefficients

1. Introduction

The main results of this paper were announced in [13]. The Schur polynomials $s_\lambda(x)$ form a basis of the ring of symmetric polynomials $\mathbb{C}[x]^S_n$ in the $n$ variables $x = (x_1, x_2, \cdots, x_n)$ as $\lambda$ varies over the set $P[n]$ of partitions with at most $n$ parts. The Littlewood-Richardson coefficients are the structure constants of this basis:

$$ s_\lambda(x) s_\mu(x) = \sum_{\nu \in P[n]} c^\nu_{\lambda \mu} s_\nu(x) $$

Arguably among the most celebrated numbers in all of algebraic combinatorics, the $c^\nu_{\lambda \mu}$ can be explicitly computed by the Littlewood-Richardson rule (and its numerous equivalent formulations). They have been generalized in many directions over the years and in this article, we undertake a closer study of one such generalization.

To define our main objects of study, we recall the Demazure operators $T_i : \mathbb{C}[x] \to \mathbb{C}[x]$ given by:

$$ (T_i f)(x) = \frac{x_i f(x_1, x_2, \cdots, x_n) - x_{i+1} f(x_1, \cdots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \cdots, x_n)}{x_i - x_{i+1}} $$

for $1 \leq i \leq n-1$. For $w \in S_n$, let $T_w = T_{i_1} T_{i_2} \cdots T_{i_k}$ where $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ is any reduced expression for $w$ as a product of simple transpositions $s_i = (i, i+1)$. The $T_i$ satisfy the braid relations and $T_w$ is independent of the chosen decomposition. Further if $w_0$ denotes the longest element of $S_n$, then $T_{w_0}$ is given by

$$ T_{w_0}(f) = \sum_{w \in S_n} \text{sgn}(w) w(x^w f) \frac{1}{\prod_{1 \leq i < j \leq n} (x_i - x_j)} \quad (1) $$

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where \( \rho = (n-1, n-2, \cdots, 1, 0) \) is the staircase partition. The map \( T_{w_0} : \mathbb{C}[x] \to \mathbb{C}[x]^{S_n} \) is \( \mathbb{C}[x]^{S_n} \)-linear and surjective, with \( T_{w_0}(x^\mu) = s_\mu(x) \) for \( \mu \in \mathcal{P}[n] \). Here we use the standard notation \( x^\alpha = \prod x_i^{\alpha_i} \) for an \( n \)-tuple \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \) of non-negative integers.

For \( w \in S_n \) and \( \mu \in \mathcal{P}[n] \), the Demazure character (or key polynomial) is \( \chi_{w,\mu}(x) := T_w(x^\mu) \). The Demazure characters form a basis of \( \mathbb{C}[x] \) as \( w, \mu \) vary.

**Definition 1.** Given \( w \in S_n \) and \( \lambda, \mu, \nu \in \mathcal{P}[n] \), the \( w \)-refined Littlewood-Richardson coefficient \( c_{\lambda \mu}^\nu(w) \) is the coefficient of \( s_\nu(x) \) in the Schur basis expansion

\[
T_{w_0}(x^\lambda \chi_{w,\mu}(x)) = \sum_{\nu \in \mathcal{P}[n]} c_{\lambda \mu}^\nu(w) s_\nu(x). \quad (2)
\]

In particular, \( c_{\lambda \mu}(1) = \delta_{\lambda+\mu, \nu} \). From (2) and the \( \mathbb{C}[x]^{S_n} \)-linearity of \( T_{w_0} \), it follows that \( c_{\lambda \mu}^\nu(w_0) = c_{\lambda \mu}^\nu \). In this article, we will be interested in the saturation problem for the \( w \)-refined Littlewood-Richardson coefficients.

**Definition 2.** A permutation \( w \in S_n \) is said to have the saturation property if the following holds for all \( \lambda, \mu, \nu \in \mathcal{P}[n] \):

\[
c_{k\lambda \nu}^{k\mu}(w) > 0 \text{ for some } k \geq 1 \text{ implies } c_{\lambda \mu}^\nu(w) > 0 \quad (3)
\]

Both \( w = 1 \) and \( w = w_0 \) have the saturation property. While the former is a trivial consequence of the definition, the latter is exactly Klyachko’s classical saturation conjecture for the \( c_{\lambda \mu}^\nu \), established by Knutson-Tao [8] using the honeycomb model. To state our main theorem, we recall the definition of pattern-avoidance (in our limited context).

**Definition 3.** Fix \( n \geq 1 \). A permutation \( w \in S_n \) is said to be 312-avoiding if there do not exist \( 1 \leq i < j < k \leq n \) such that \( w(j) < w(k) < w(i) \). Likewise, we say that \( w \) is 231-avoiding if there do not exist \( 1 \leq i < j < k \leq n \) such that \( w(k) < w(i) < w(j) \).

Our main result is the following sufficiency condition for saturation.

**Theorem 1.** Fix \( n \geq 1 \).

1. Let \( w \in S_n \) be either 312-avoiding or 231-avoiding. Then \( w \) has the saturation property.

2. More generally, let \( H = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_p} \subseteq S_n \) be a Young subgroup and \( w = w_1 w_2 \cdots w_p \in H \) such that each \( w_i \in S_{n_i} \) is either 312- or 231-avoiding. Then \( w \) has the saturation property.

We observe that \( w_0 \) is 312-avoiding (and also 231-avoiding). Theorem [1] thus generalizes the Knutson-Tao saturation theorem.

Permutations satisfying the conditions of Theorem [1] appear in work of Postnikov-Stanley, where explicit formulas for the degree polynomials of the corresponding Schubert varieties were established [18] Theorem 13.4, Corollary 13.5, Remark 15.5. For \( n = 1, 2, 3 \), all permutations in \( S_n \) are of the form of the theorem. For \( n = 4 \), our theorem establishes saturation for all except the following four permutations of \( S_4 \) (written in one-line notation): 2413, 3142, 3412, 4231 (see §5.3).
In section 2, we explain the representation theoretic underpinnings of the $w$-refined Littlewood-Richardson coefficients. The rest of the sections are devoted to deriving a hive description of the $c_{\lambda\mu}^\nu(w)$ and proving Theorem 1.

2. Kostant-Kumar modules

2.1. Let $V(\lambda)$ denote the finite-dimensional irreducible representation of $\mathfrak{g} := \mathfrak{sl}_n \mathbb{C}$ corresponding to the partition $\lambda \in \mathcal{P}[n]$. This remains irreducible when restricted to $\mathfrak{sl}_n \mathbb{C}$.

Given $\lambda, \mu \in \mathcal{P}[n]$ and $w \in S_n$, let $v_\lambda$ denote the highest weight vector of $V(\lambda)$ and $v'_{w\mu}$ a nonzero vector of weight $w\mu$ in $V(\mu)$. Let $\mathfrak{U}\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. The cyclic $\mathfrak{g}$-submodule of the tensor product $V(\lambda) \otimes V(\mu)$ generated by $v_\lambda \otimes v'_{w\mu}$ is called a Kostant-Kumar module $\mathbf{11, 12}$ and is denoted $K(\lambda, w, \mu)$:

$$K(\lambda, w, \mu) = \mathfrak{U}\mathfrak{g}(v_\lambda \otimes v'_{w\mu}) \subseteq V(\lambda) \otimes V(\mu)$$

We now recall the following properties of Kostant-Kumar modules from $\mathbf{12}$, in the context of the simple Lie algebra $\mathfrak{sl}_n \mathbb{C}$.

We fix $n \geq 2$ and let $W := S_n$. For $\lambda \in \mathcal{P}[n]$, let $W_\lambda$ denote the stabilizer of $\lambda$ in $W$.

Proposition 1. (\cite{11, 12}) Let $w, w' \in W$ and $\lambda, \mu \in \mathcal{P}[n]$.

1. $K(\lambda, w, \mu) = K(\lambda, w', \mu)$ if $W_\lambda w W_\mu = W_\lambda w' W_\mu$.
2. $K(\lambda, w, \mu) \subseteq K(\lambda, w', \mu)$ if $W_\lambda w W_\mu \leq W_\lambda w' W_\mu$ in the Bruhat poset $W_\lambda \backslash W / W_\mu$ of double cosets.
3. $K(\lambda, w, \mu) = K(\mu, w^{-1}, \lambda)$.
4. $K(\lambda, 1, \mu) \cong V(\lambda + \mu)$ and $K(\lambda, w_0, \mu) = V(\lambda) \otimes V(\mu)$, where $w_0$ is the longest element in $W$.

The character of the Kostant-Kumar module was computed by Kumar $\mathbf{11}$ (for any finite dimensional simple Lie algebra $\mathfrak{g}$):

Theorem 2. (Kumar) Fix $n \geq 2$. Let $w \in S_n$ and $\lambda, \mu \in \mathcal{P}[n]$. Then $\text{char } K(\lambda, w, \mu) = T_{w_0}(x^\lambda \chi_{w, \mu}(x))$, where $w_0$ is the longest element of $S_n$.

We conclude from this theorem and $\mathbf{2}$ that

$$c_{\lambda\mu}^\nu(w) = \text{multiplicity of } V(\nu) \text{ in } K(\lambda, w, \mu) \quad (4)$$

2.2. We can now prove the key properties of the $c_{\lambda\mu}^\nu(w)$.

Proposition 2. Fix $n \geq 2$. Let $\lambda, \mu, \nu \in \mathcal{P}[n]$. Then

(a) $c_{\lambda\mu}^\nu(w) \in \mathbb{Z}_+$ for all $w \in S_n$
(b) $c_{\lambda\mu}^\nu(w) = c_{\lambda\mu}^\nu(w')$ if $W_\lambda w W_\mu = W_\lambda w' W_\mu$.
(c) $c_{\lambda\mu}^\nu(w) \leq c_{\lambda\mu}^\nu(w')$ if $w \leq w'$ in the Bruhat order on $S_n$
(d) $c_{\lambda\mu}^\nu(w) = c_{\mu\lambda}^\nu(w^{-1})$ for all $w \in S_n$
(e) $c_{\lambda\mu}^\nu(w_0) = c_{\lambda\mu}^\nu$ and $c_{\lambda\mu}^\nu(1) = \delta_{\lambda+\mu, \nu}$.
Figure 1. Values of $c_{\lambda\mu}^{\nu}(w)$ (superimposed on the Bruhat graph of $S_4$) for $n = 4, \lambda = (13, 7, 4), \mu = (13, 7, 2), \nu = (21, 12, 9, 4)$.

**Proof.** As remarked already, (e) follows directly from (2). Equation (4) implies (a), while proposition 1 establishes properties (b)-(d). □

A bijective proof of (d) using the hive model occurs below in §6.

Proposition 2 shows that for fixed $\lambda, \mu, \nu$, the map $w \mapsto c_{\lambda\mu}^{\nu}(w)$ is an increasing function of posets $S_n \to \mathbb{Z}_+$. Figure 1 shows an example for $n = 4$, with the values $c_{\lambda\mu}^{\nu}(w)$ superimposed on the Bruhat graph of $S_4$.

Finally, we remark that the $c_{\lambda\mu}^{\nu}(w)$ are also related to the *combinatorial excellent filtrations* of Demazure modules and have descriptions in terms of Lakshmibai-Seshadri paths [14] or crystals [4, 5]. We will return to this point of view in section 3.

### 3. A Hive Model for $c_{\lambda\mu}^{\nu}(w)$

Putting together recent results of Fujita [2] and those of [5, 14], one obtains a combinatorial model for $c_{\lambda\mu}^{\nu}(w)$ as a certain subset of integer points in the Gelfand-Tsetlin (GT) polytope. We describe this first, followed by our more succinct reformulation in terms of hives. A word on notation: if $P$ is a (not necessarily bounded) polytope, or a face thereof, then $P_\mathbb{Z}$ will denote the set of integer points in $P$.

Given a partition $\mu \in \mathcal{P}[n]$, let $\text{Tab}(\mu)$ denote the set of semistandard Young tableaux of shape $\mu$ with entries in $1, 2, \cdots, n$. To each $T \in \text{Tab}(\mu)$ we associate its reverse row word $b_T$ obtained by reading the entries of $T$ from right to left and top to bottom (in English notation), for example,
\[
T = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3
\end{bmatrix} \text{ and } b_T = 32132
\] (5)

We recall the crystal operators \( e_i \) and \( f_i \) (1 \( \leq \) \( i < n \)). (we refer to [16, Chapter 5] for more details). The crystal raising and lowering operators \( e_i, f_i \) (1 \( \leq \) \( i < n \)) act on the set \( \text{Tab}(\mu) \), and more generally on words in the alphabet \( \{1, 2, \cdots, n\} \).

There exists a unique tableau \( T^\circ_\mu \) in \( \text{Tab}(\mu) \) such that \( e_i T^\circ_\mu = 0 \) for all \( i \) and a unique tableau \( T^*_\mu \) such that \( f_i T^*_\mu = 0 \) for all \( i \). The weight of a word \( u \) in the alphabet \( \{1, 2, \cdots, n\} \).

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Definition 4. A word \( u = u_1 u_2 \cdots u_k \) in the alphabet \( \{1, 2, \cdots, n\} \) is said to be dominant (or a ballot sequence) if every left subword \( u^t = u_1 u_2 \cdots u_t \) of \( u \) for \( 1 \leq t \leq k \), contains more occurrences of \( i \) than \( i + 1 \) for all \( 1 \leq i < n \), or equivalently, if \( e_i u = 0 \) for all \( 1 \leq i < n \).

Definition 5. Given \( w \in S_n \), fix a reduced decomposition \( w = s_{i_1} s_{i_2} \cdots s_{i_k} \). The set

\[
\text{Dem}(\mu, w) := \{ f_{i_1}^{m_{i_1}} f_{i_2}^{m_{i_2}} \cdots f_{i_k}^{m_{i_k}} T^\circ_\mu : m_j \geq 0 \}
\]

is called a Demazure crystal.

Definition 6. Given \( w \in S_n \), fix a reduced decomposition \( w = s_{i_1} s_{i_2} \cdots s_{i_k} \). The set

\[
\text{Dem}(\mu, w)^{op} := \{ e_{i_1}^{m_{i_1}} e_{i_2}^{m_{i_2}} \cdots e_{i_k}^{m_{i_k}} T^*_\mu : m_j \geq 0 \}
\]

is called an opposite Demazure crystal.

The following theorem gives a way to calculate refined Littlewood-Richardson coefficients via Demazure crystals, and is essentially due to Joseph [5].

Theorem 3. \( c^\nu_\lambda(\mu) \) is the cardinality of the set

\[
\text{Dem}^\nu_\lambda(\mu, w) := \{ T \in \text{Dem}(\mu, w) : b^\nu_\lambda * b_T \text{ is a dominant word of weight } \nu \}
\]

Here * denotes concatenation of words.

Proof. We know that, \( c^\nu_\lambda(\mu) \) is the multiplicity of \( V(\nu) \) in the decomposition of the Kostant-Kumar module \( K(\lambda, w, \mu) \). We recall from [12] the Joseph decomposition rule for Kostant-Kumar modules in terms of Lakshmibai-Seshadri (LS) paths.

Recall that a LS path \( \pi \) of shape \( \mu \) consists of a sequence \( w_1 > w_2 > \cdots > w_l \) such that \( w_i \in W/W_\mu \) together with rational numbers \( 0 = a_0 < a_1 < \cdots < a_l = 1 \) subject to some integrality conditions (see [15]). We say \( w_1 \) is the initial direction of \( \pi \). Let \( P_\mu \) denotes the set of LS paths of shape \( \mu \). A LS path \( \pi \) is called \( \lambda \)-dominant if \( \lambda + \pi(t) \) in the closure of the dominant Weyl chamber for \( 0 \leq t \leq 1 \).
The decomposition rule for Kostant-Kumar modules $K(\lambda, w, \mu)$ in terms of LS paths (see [12]) is:

$$K(\lambda, w, \mu) = \bigoplus_{\pi \in P^\lambda_{\mu}(w)} V(\lambda + \pi(1))$$  \hspace{1cm} (6)

where $P^\lambda_{\mu}(w) = \{\lambda\text{-dominant LS paths of shape } \mu \text{ whose initial direction } \leq wW\mu\}$ and $\pi(1)$ is the endpoint of $\pi$. We recall that there is a crystal isomorphism $\varphi : P_\mu \rightarrow \text{Tab}(\mu)$. By restriction, we obtain a weight-preserving bijection

$$\varphi : P^\lambda_{\mu}(w) \rightarrow \text{Dem}_\lambda(\mu, w)$$  \hspace{1cm} (7)

where $\text{Dem}_\lambda(\mu, w) := \{T \in \text{Dem}(\mu, w) : b^\lambda \ast b_T \text{ is a dominant word} \}$. Then we can write the decomposition rule of equation (6) in terms of tableaux as follows:

$$K(\lambda, w, \mu) = \bigoplus_{T \in \text{Dem}_\lambda(\mu, w)} V(\lambda + \text{weight}(T))$$  \hspace{1cm} (8)

This concludes the proof. \hfill \Box

**Remark 1.** We note that we could replace $b^\lambda_\mu$ with any other dominant word $b^+$ of weight $\mu$ (these are Knuth equivalent [16]), define $\text{Dem}(\mu, w) := \{f^{m_1}_{i_1} f^{m_2}_{i_2} \cdots f^{m_k}_{i_k} b^+ : m_j \geq 0\}$, and the theorem still holds, appropriately modified. We will use this in §5.2.

### 3.1. Kogan faces of GT polytopes.

A GT pattern of size $n$ is a triangular array $A = (a_{ij})_{n \geq i \geq j \geq 1}$ of real numbers (figure 2) satisfying the following (“North-East” and “South-East”) inequalities for all $i > j$: $\text{NE}_{ij} = a_{ij} - a_{i-1,j} \geq 0$ and $\text{SE}_{ij} = a_{i-1,j} - a_{i,j+1} \geq 0$. For $\mu \in \mathcal{P}[n]$, the GT polytope $\text{GT}(\mu)$ is the set of all GT patterns with $a_{ni} = \mu_i$ for $1 \leq i \leq n$. For example let $\mu = (6, 4, 2, 2, 1)$, following is a GT pattern of shape $\mu$. 

![Gelfand-Tsetlin array for $n = 5$. The red edges $a_{ij} \rightarrow a_{i-1,j}$ are labelled by $s_{i-j}$](image-url)
We have the standard bijection \( A \mapsto \Gamma(A) \) from \( GT_\mathbb{Z}(\mu) \) to \( Tab(\mu) \), with the tableau \( \Gamma(A) \) uniquely specified by the condition that for all \( i \geq j \), the number of \( i \) in row \( j \) equals \( NE_{ij} \) (with \( a_{i-1,i} := 0 \)). For example, letting \( A \) be the above GT pattern of shape \( \mu = (6, 4, 2, 2, 1) \), then \( \Gamma(A) \) is:

\[
\begin{array}{ccccc}
1 & 1 & 2 & 3 & 3 & 5 \\
2 & 2 & 3 & 4 \\
3 & 4 \\
4 & 5 \\
6 \\
\end{array}
\] (9)

Fix a subset \( F \subseteq \{(i, j) : n \geq i > j \geq 1\} \). Consider the face of \( GT(\mu) \) obtained by setting \( NE_{ij} = 0 \) for \((i, j) \in F\) and leaving all other inequalities untouched. We call this the Kogan face \( K(\mu, F) \). To each pair \( i > j \), associate the simple transposition \( s_{i-j} \in S_n \). We list the elements of \( F \) in lexicographically increasing order: \((i, j) \) precedes \((i', j') \) \( \Leftrightarrow \) either \( i < i' \), or \( i = i' \) and \( j < j' \). Denote the product of the corresponding \( s_{i-j} \) in this order by \( \sigma(F) \). If \( \text{len} \sigma(F) = |F| \), i.e., this word is reduced, we say that \( F \) is reduced and set \[ \text{Definition 5.1} \] :

\[
\varpi(F) = w_0 \sigma(F) w_0
\]

For \( w \in S_n \), let \( K(\mu, w) := \cup K(\mu, F) \), the union over reduced \( F \) for which \( \varpi(F) = w \). We can now state \[ \text{Corollary 5.19} \] :

**Proposition 3.** The bijection \( \Gamma : GT_\mathbb{Z}(\mu) \to Tab(\mu) \) restricts to a bijection \( K_\mathbb{Z}(\mu, w_0w) \to Dem(\mu, w) \).

It was previously shown in \[ 6 \] (for regular \( \mu \)) and \[ 18 \] (for \( w \) 312-avoiding) that \( K_\mathbb{Z}(\mu, w_0w) \) and \( Dem(\mu, w) \) have the same character. This weaker statement is however inadequate for our present purposes.

**Remark 2.** We can also put a different total order on the set \( \{(i, j) : n \geq i > j \geq 1\} \). Let \((i, j) \) precede \((i', j') \) if and only if \( j < j' \), or \( j = j' \) and \( i < i' \). List the \((i, j) \in F \) in increasing order relative to this new total order and denote the product of the corresponding \( s_{i-j} \) by \( \sigma'(F) \). It can be easily checked that \( \sigma(F) = \sigma'(F) \) in \( S_n \).
3.2. Hives. We begin with a quick overview. The big hive triangle \( \Delta \) is the array of Figure 3, with \((n + 1)\) vertices on each boundary edge, and \((n - 2)(n - 1)/2\) interior vertices. We note that there are 3 types of rhombi in \( \Delta \) (figure 3), the NE slanted (in red), the SE slanted (in green) and the vertical diamonds (in blue). Given \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \), let \( |\lambda| = \sum \lambda_i \).

**Definition 7.** For \((\lambda, \mu, \nu) \in (\mathbb{R}^n)^3\) with \(|\lambda| + |\mu| = |\nu|\), the hive polytope \( \text{Hive}(\lambda, \mu, \nu) \) is the set of all labellings of the vertices of \( \Delta \) with real numbers such that:

1. The boundary labels are \( \lambda \) (left edge, top to bottom), \( |\lambda| + \mu \) (bottom edge, left to right) and \( \nu \) (right edge, top to bottom) (figure 3).
2. The content\((R) \geq 0\) for each rhombus \( R \) in \( \Delta \), where content\((R) \) is the sum of the labels on the obtuse angled vertices of \( R \) minus the sum of labels on its acute angled vertices.

For example, consider the following rhombus \( R \):

\[
R \quad \begin{array}{c}
\lambda_1 \\
\sum_{i=1}^{2} \lambda_i \\
\sum_{i=1}^{3} \lambda_i \\
\sum_{i=1}^{4} \lambda_i \\
|\lambda| \\
\sum_{i=1}^{\mu_1} \lambda_i \\
\sum_{i=1}^{\mu_2} \lambda_i \\
\sum_{i=1}^{\mu_3} \lambda_i \\
|\nu| = |\lambda| + |\mu| \\
\end{array}
\]

Then content\((R) = (b + d) - (a + c)\).

**Definition 8.** A hive is an element of \( \text{Hive}(\lambda, \mu, \nu) \) for some \( \lambda, \mu, \nu \).

We now fix \( \lambda, \mu, \nu \in \mathcal{P}[n] \) with \( |\lambda| + |\mu| = |\nu| \). Since each \( h \in \text{Hive}(\lambda, \mu, \nu) \) is an \( \mathbb{R} \)-labelled triangular array (of size \( n + 1 \)), its horizontal sections (marked in blue in figure 4) form a
sequence of vectors $h_0, h_1, \ldots, h_n$ (listed from top to bottom), with $h_i \in \mathbb{R}^{i+1}$. Consider the (“row-wise successive differences”) map $h \mapsto \partial h$ where $\partial h$ is the sequence of vectors $\partial h_1, \partial h_2, \ldots, \partial h_n$, thought of again as a triangular array (of size $n$ this time) (figure 4). One sees immediately that $\partial h \in \text{GT} (\mu)$ [1, Appendix], [19].

Figure 4. The hive on the left maps under $\partial$ to the GT pattern on the right (example borrowed from [19]).

Note that each NE edge difference $NE_{ij}$ of $\partial h$ (§3.1) equals the content of a corresponding NE-slanted rhombus $R_{ij}$ of $h$ (figure 5). Thus $NE_{ij} = 0$ (in $\partial h$) if and only if content($R_{ij}$) = 0 (in $h$). A rhombus with zero content is said to be flat.

Proposition 4. Let $\lambda, \mu, \nu \in P[n]$ with $|\lambda| + |\mu| = |\nu|$. Then

1. $\partial : \text{Hive}(\lambda, \mu, \nu) \to \text{GT}(\mu)$ is an injective, linear map.
2. $h \in \text{Hive}_Z(\lambda, \mu, \nu) \iff \partial h \in \text{GT}_Z(\mu)$.
3. $\Gamma \circ \partial$ is a bijection between $\text{Hive}_Z(\lambda, \mu, \nu)$ and $\text{Tab}_x^{\lambda}(\mu)$, where $\text{Tab}_x^{\lambda}(\mu) := \{ T \in \text{Tab}(\mu) : b_0^{\lambda} \ast b_T \text{ is a dominant word of weight } \nu \}$.

Proof. (1) Consider the map $\partial$ on the ambient space $\partial : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$, where $\ell = \frac{(n+1)(n+2)}{2}$ and $\ell' = \frac{n(n+1)}{2}$. It is clear from the definition that $\partial$ is linear. We will show that its

Figure 5. (a) Labelling of North-East slanted rhombi (shown for $n = 5$). (b) A typical configuration of rhombi in $F_w$. 
Thus, denote the number of occurrences of \( k \) induction on \( \leq h \) then \( \partial h = (a_{ij}) \) such that \( a_{ij} = h_{i(j+1)} - h_{ij} \) for \( 1 \leq j \leq i \leq n \).

restriction to Hive(\( \lambda, \mu, \nu \)) is injective. Let \( h = (h_0, h_1, ..., h_n) \), \( h' = (h'_0, h'_1, ..., h'_n) \) be two elements in Hive(\( \lambda, \mu, \nu \)), where \( h_i, h'_i \in \mathbb{R}^{i+1} \) for \( 0 \leq i \leq n \). Let \( h_i = (h_{i1}, h_{i2}, ..., h_{i(i+1)}) \) and \( h'_i = (h'_{i1}, h'_{i2}, ..., h'_{i(i+1)}) \). We have \( h_{01} = h'_{01} = 0 \), \( h_{i1} = \sum_{j=1}^{i} \lambda_j = h'_{i1} \) for \( 1 \leq i \leq n \).

Suppose that \( \partial h = \partial h' \) that \( \partial h_i = \partial h'_i \) for \( 1 \leq i \leq n \). We will prove \( h_{ik} = h'_{ik} \) by induction on \( k \), for each fixed \( i \). For the base case we have \( h_{i1} = h'_{i1} \). Let \( p \geq 2 \) and suppose that \( h_{ik} = h'_{ik} \) for \( k < p \). Since \( \partial h_i = \partial h'_i \), we have \( h_{i(k+1)} - h_{ik} = h'_{i(k+1)} - h'_{ik} \) for all \( 1 \leq k \leq i \). Since \( h_{ik} = h'_{ik} \) for \( k < p \), this implies that \( h_{ip} = h'_{ip} \). Hence we have \( h_i = h'_i \) for \( 0 \leq i \leq n \).

(2) This is clear by the definition of \( \partial h \).

(3) Let \( h \) be an element of Hive(\( \lambda, \mu, \nu \)) then it is clear that \( T = \Gamma(\partial h) \) is in Tab(\( \mu \)). We want to show that \( T \in \text{Tab}_X(\mu) \). Consider \( h = (h_0, h_1, ..., h_n) \) where \( h_i = (h_{i1}, h_{i2}, ..., h_{i(i+1)}) \); then \( \partial h = (a_{ij}) \) for \( 1 \leq j \leq i \leq n \) where \( a_{ij} = h_{i(j+1)} - h_{ij} \) (see figure \( \ref{fig:6} \)). From \$ \text{§ 3.1} \$ it follows that the number of in row \( j \) of \( T \) is \( a_{ij} - a_{(i-1)j} \) for \( j \leq i \) (setting \( a_{i-1,i} = 0 \)).

We want to show that \( b_X^j * b_T \) is a dominant word. Let \( b_T = b_{T1} * b_{T2} * ... * b_{Tn} \) where \( b_{Tk} \) is the reverse reading word of the \( k^{th} \)-row of \( T \). For each \( 1 \leq i \leq n \) and \( 0 \leq k \leq i \), let \( N_{i,k} \) denote the number of occurrences of \( i \) in the word \( w_k := b_X^k * b_{T1} * b_{T2} * ... * b_{Tk} \). It follows easily from definition \$ \text{§ 4} \$ that \( b_X^k * b_T \) is a dominant word if and only if \( N_{i,k} \geq N_{i+1,k+1} \) for all \( 1 \leq i < n \) and \( 0 \leq k \leq i \).

We have

\[
N_{i,k} = \lambda_i + (a_{i1} - a_{i-1,1}) + (a_{i2} - a_{i-1,2}) + ... + (a_{i,k} - a_{(i-1),k})
\]

Using the definition of \( \partial h \) to rewrite this in terms of \( h \), we get:

\[
N_{i,k} = h_{ik} - h_{(i-1)k}
\]

Thus,

\[
N_{i,k} - N_{i+1,k+1} = (h_{ik} - h_{(i-1)k}) - (h_{(i+1)(k+1)} - h_{i(k+1)})
\]
which is \( \geq 0 \) by the corresponding vertical rhombus inequality in \( h \). This proves that \( b^\nu_\lambda \ast b_T \) is a dominant word.

Next we will check the weight of \( b^\nu_\lambda \ast b_T \). The number of times \( i \) appears in the \( b^\nu_\lambda \ast b_T \) is:

\[
\lambda_i + (a_{i1} - a_{(i-1)1}) + (a_{i2} - a_{(i-1)2}) + \ldots + (a_{i(i-1)} - a_{(i-1)(i-1)}) + a_{ii}
\]

\[
= \lambda_i + (h_{i2} - h_{(i-1)2}) - (h_{i1} - h_{(i-1)1}) + \ldots + h_{(i+1)i} - h_{ii}
\]

\[
= \lambda_i + (h_{i2} - h_{(i-1)2}) - \lambda_i + \ldots + h_{i(i+1)} - h_{ii} = h_{i(i+1)} - h_{i(i-1)i} = \nu_i
\]

This shows that the weight of the word \( b^\nu_\lambda \ast b_T \) is \( \nu \). Thus \( \Gamma \circ \partial \) is an injective map from \( \text{Hive}_Z(\lambda, \mu, \nu) \) to \( \text{Tab}_h^\nu(\mu) \).

The surjectivity follows from similar arguments. Let \( T \in \text{Tab}_h^\nu(\mu) \); then \( \Gamma^{-1}(T) = (a_{ij}) \) for \( 1 \leq j \leq i \leq n \) where \( a_{ij} \) are defined recursively for \( i \geq j \) by \( a_{ij} = (\# \blacksquare \text{ in the } j^{th}\text{-row of } T) + a_{(i-1)j} \) and \( a_{(i-1)i} = 0 \). Let \( \partial^{-1}(\Gamma^{-1}(T)) = (0, h_1, \ldots, h_n) \) where \( h_i = (h_{i1}, h_{i2}, \ldots, h_{i(i+1)}) \) for \( 1 \leq i \leq n \) and \( h_{ik} \) is defined recursively such that \( h_{i1} = \sum_{j=1}^{i} \lambda_j \) and \( h_{i(j+1)} = a_{ij} + h_{ij} \) for \( 1 \leq j \leq i \). We consider \( h = (0, h_1, h_2, \ldots, h_n) \) as vertex labels of the big hive triangle \( \Delta \) with \( n+1 \) vertices in each side, with \( h_i \) being the \( (i+1)^{th}\) edge of \( \Delta \) (see figure 6). We want to show that \( h \in \text{Hive}_Z(\lambda, \mu, \nu) \).

The NE slanted and SE slanted rhombi inequalities holds since \( \Gamma^{-1}(T) \) is a Gelfand-Tsetlin pattern. The vertical rhombus inequalities holds from equation 12 since \( b^\nu_\lambda \ast b_T \) is a dominant word. We can easily verify that the boundary labels are \( \overline{\lambda} \) (left edge, top to bottom), \( |\lambda| + \overline{\nu} \) (bottom edge, left to right) and \( \overline{\nu} \) (right edge, top to bottom).

\( \square \)

Note that the last assertion of proposition 4 implies that \( |\text{Hive}_Z(\lambda, \mu, \nu)| = c^\nu_{\lambda \mu} \) (and is a variation of proofs in [1]. [17]).

Given \( F \subseteq \{ (i, j) : n \geq i > j \geq 1 \} \), recall that \( K(\mu, F) \) is the face of \( \text{GT}(\mu) \) on which \( NE_{ij} \) vanishes for all \( (i, j) \in F \). The inverse image \( \partial^{-1}K(\mu, F) \) is thus the face \( \{ h \in \text{Hive}(\lambda, \mu, \nu) : R_{ij} \text{ is flat in } h \text{ for all } (i, j) \in F \} \). We denote this (“hive Kogan”) face by \( K^\text{Hive}(\lambda, \mu, \nu, F) \), and term it reduced if \( F \) is. For \( w \in S_n \), define \( K^\text{Hive}(\lambda, \mu, \nu, w) := \partial^{-1}(K(\mu, w)) \). Putting together Theorem 3 and Propositions 3.4, we obtain our hive description of the \( c^\nu_{\lambda \mu}(w) \):

**Theorem 4.** \( c^\nu_{\lambda \mu}(w) = |(\Gamma \circ \partial)^{-1}(\text{Dem}_Z(\mu, w))| = \#K^\text{Hive}(\lambda, \mu, \nu, w_0 w) \).

3.3. Hive Kogan faces for 312-avoiding permutations. Let \( w \in S_n \) be 312-avoiding. Then, \( w_0 w \) is 132-avoiding and there exists a unique reduced \( F_w \subseteq \{ (i, j) : n \geq i > j \geq 1 \} \) such that \( w_0 (F_w) = w_0 w \) (§3.1). Further, it has the following form \( F_w = \{ (i, j) : p \leq i \leq n, 1 \leq j \leq m_i \} \) for some \( 2 \leq p \leq n, 1 \leq m_p \leq m_{p+1} \leq \cdots \leq m_n \) with \( m_i < i \) for all \( i \) [10]. [18]. Pictorially, the union of the rhombi \( R_{ij}, (i, j) \in F_w \) forms a left-and-bottom justified region in the big hive triangle \( \Delta \) (figure 5). Thus, for such \( w \), \( K^\text{Hive}(\lambda, \mu, \nu, w_0 w) \) is just the single Kogan face \( K^\text{Hive}(\lambda, \mu, \nu, F_w) \) on which the \( R_{ij} \) are flat for all \( (i, j) \in F_w \).
4. Increasable subsets for hives

Let $\lambda, \mu, \nu \in \mathbb{R}^n$ with $|\lambda| + |\mu| = |\nu|$ and let $h \in \text{Hive}(\lambda, \mu, \nu)$. A subset $S$ of the interior vertices of $\Delta$ is said to be increasable for $h$ if those vertex labels of $h$ can be simultaneously increased by some $\epsilon > 0$ to obtain another element of $\text{Hive}(\lambda, \mu, \nu)$. Formally, let $I_S$ denote the indicator function of $S$ (1 on $S$ and 0 elsewhere); then $S$ is increasable if there is an $\epsilon > 0$ such that $h' = h + \epsilon I_S \in \text{Hive}(\lambda, \mu, \nu)$. This notion is one of the central ideas of Knutson-Tao’s proof of the saturation conjecture via hives $[7, 1]$:

**Proposition 5.** (Knutson-Tao) Let $\lambda, \mu, \nu \in \mathbb{R}^n$ be regular (i.e., $\lambda_i \neq \lambda_j$ if $i \neq j$, and likewise for $\mu, \nu$) with $|\lambda| + |\mu| = |\nu|$. Let $h$ satisfy the following properties: (i) $h$ is a vertex of the hive polytope $\text{Hive}(\lambda, \mu, \nu)$, (ii) $h$ has no increasable subsets. Then each interior label of $h$ is an integral linear combination of its boundary labels. In particular, if $\lambda, \mu, \nu \in \mathcal{P}[n]$, then $h \in \text{Hive}_\mathbb{Z}(\lambda, \mu, \nu)$.

4.1. Increasable subsets for hives in 312-avoiding Kogan faces. Let $w$ be 312-avoiding and let $F_w$ be as in $[3, 3]$. Let $\lambda, \mu, \nu \in \mathbb{R}^n$ with $|\lambda| + |\mu| = |\nu|$. The following simple observation is a crucial step in extending the Knutson-Tao method to our problem.

**Lemma 1.** Let $h \in K^{\text{Hive}}(\lambda, \mu, \nu, F_w)$ and let $S$ be an increasable subset for $h$, say $h' = h + \epsilon I_S \in \text{Hive}(\lambda, \mu, \nu)$ for some $\epsilon > 0$. Then $h' \in K^{\text{Hive}}(\lambda, \mu, \nu, F_w)$.

**Proof.** We will show that $S$ is disjoint from the set of vertices of the rhombi $R_{ij}$ for $(i, j) \in F_w$. This would imply that the $R_{ij}$ remain flat in $h'$, which is the desired conclusion. This is trivial if $F_w$ is empty. If $F_w$ is non-empty, then $(n, 1) \in F_w$. The rhombus $R_{n1}$ has three vertices on the boundary, and these cannot be in $S$. The fourth vertex is acute-angled, and if it belongs to $S$, then $\text{content}(R_{n1}) < 0$ in $h'$, a contradiction. Moving on to the next rhombus $R_{n2}$ (if $(n, 2) \in F_w$), again three of its vertices cannot be in $S$ since they are either on the boundary or shared with $R_{n1}$. Neither can its fourth vertex, since it is acute-angled as before. Proceed in this fashion, left-to-right along the rows, from the bottom row to the top. \[\square\]

**Remark 3.** The following converse holds too. If $K^{\text{Hive}}(\lambda, \mu, \nu, F)$ is a hive Kogan face for which the conclusion of Lemma 1 holds for all $\lambda, \mu, \nu \in \mathbb{R}^n$, then $F = F_w$ for some 312-avoiding permutation $w$.

5. Proof of the main theorem

With Lemma 1 in place, we can use Knutson-Tao’s arguments to complete the proof of Theorem 1 for $w$ 312-avoiding.

Consider the set of all $\mathbb{R}$-labellings of vertices of the big hive triangle $\Delta$ (with the boundary labels also allowed to vary) subject to (i) the inequalities: $\text{content}(R) \geq 0$ for all rhombi $R$ in $\Delta$, and (ii) the equalities: $\text{content}(R_{ij}) = 0$ for all $(i, j) \in F_w$. This set forms a polyhedral cone, denoted $K^{\text{Hive}}(-, w_0w)$. Given $h \in K^{\text{Hive}}(-, w_0w)$, consider the projection $\pi : K^{\text{Hive}}(-, w_0w) \rightarrow (\mathbb{R}^n)^3$ defined by $\pi(h) = (\lambda, \mu, \nu)$, where the boundary labels of $h$ are
The image of \( \pi \) is a polyhedral cone in \( \mathbb{R}^{3n} \) [lecture 1], which we denote by \( \text{Horn}(w_0w) \) (adapting the notation of [7]). For \( w = w_0 \), \( \text{Horn}(1) \) is the cone of spectra of triples \((A, B, C)\) of \( n \times n \) Hermitian matrices with \( C = A + B \) [9].

We note that the saturation property [3] is equivalent to the statement that:

\[
K_Z^{\text{Hive}}(\lambda, \mu, \nu) = \emptyset \quad \text{for all } (\lambda, \mu, \nu) \in \text{Horn}_Z(w_0w).
\]

This follows from Theorem 4 and the scaling property:

\[
K^{\text{Hive}}(p\lambda, p\mu, p\nu, w_0w) = pK^{\text{Hive}}(\lambda, \mu, \nu, w_0w) \quad \text{for all positive real numbers } p.
\]

### 5.1. The largest lift map.

Following [7, 1], choose a functional \( \zeta \) on the cone \( K^{\text{Hive}}(\lambda, \mu, \nu) \) which maps each hive \( h \) to a generic positive linear combination of its vertex labels. Then, for each \( (\lambda, \mu, \nu) \in \text{Horn}(w_0w) \), the maximum value of \( \zeta \) on \( \pi^{-1}(\lambda, \mu, \nu) \) is attained at a unique point; this point will be called its largest lift. The map \( \ell : \text{Horn}(w_0w) \to K^{\text{Hive}}(\lambda, \mu, \nu) \mapsto \) largest lift of \( (\lambda, \mu, \nu) \), is continuous and piecewise-linear.

It is also clear that \( \ell(\lambda, \mu, \nu) \) is a vertex of \( K^{\text{Hive}}(\lambda, \mu, \nu, w_0w) \), thereby satisfying the first condition of Proposition 5. We claim that it also satisfies the second condition there, i.e., that \( h = \ell(\lambda, \mu, \nu) \) has no increasing subsets. For if \( S \) is an increasing subset, let \( h' = h + \epsilon S \in \text{Hive}(\lambda, \mu, \nu) \) for some \( \epsilon > 0 \). By Lemma 1, \( h' \in K^{\text{Hive}}(\lambda, \mu, \nu, w_0w) \). But \( \zeta(h') > \zeta(h) \), violating maximality of \( \zeta(h) \).

So Proposition 5 implies that for \( \lambda, \mu, \nu \) regular, each label of \( \ell(\lambda, \mu, \nu) \) is an integer linear combination of the \( \lambda_i, \mu_i, \nu_i \), \( 1 \leq i \leq n \). As in [1, §4] and [7], by the continuity of \( \ell \), it follows that each piece of \( \ell \) is a linear function of \( (\lambda, \mu, \nu) \in \mathbb{R}^{3n} \) with \( \mathbb{Z} \)-coefficients. As a corollary:

\[
\ell(\text{Horn}_Z(w_0w)) \subseteq K_Z^{\text{Hive}}(\lambda, \mu, \nu, w_0w)
\]

This proves Theorem 1 for \( w \) 312-avoiding.

### 5.2. Completing the proof.

We now complete the proof of Theorem 1. If \( w \) is 231-avoiding, then \( w^{-1} \) is 312-avoiding. Proposition 2(d) finishes the argument in this case.

To handle the remaining \( w \), note that only the \( p = 2 \) case needs to be established (in the notation of Theorem 1), with induction doing the rest. We sketch the contours of the argument. It is more natural to work with the Lie algebra \( \mathfrak{sl}_n \mathbb{C} \) here. Suppose \( I^1, I^2 \subseteq \{1, 2, \ldots, n-1\} \) are such that \( s_i \) and \( s_j \) commute for all \( i \in I^1, j \in I^2 \). Let \( I^0 = \{1, 2, \ldots, n-1\} - \bigcup_{r=1}^{2} I^r \). Let \( W^r \) be the subgroup of \( S_n \) generated by the \( \{s_i : i \in I^r\} \) for \( r = 1, 2 \). Let \( \mu \) be a dominant integral weight \( \mu = \sum_{i=1}^{n-1} c_i \Lambda_i \) of \( \mathfrak{sl}_n \mathbb{C} \), where \( \Lambda_i \) are its fundamental weights. Define \( \mu = \mu^0 + \mu^1 + \mu^2 \) where \( \mu^r = \sum_{i \in I^r} c_i \Lambda_i \) for \( r = 0, 1, 2 \). Let \( b(\mu^r) \) denote the reverse row word of the highest weight tableau \( T^\mu_r \) of shape \( \mu^r \) for \( r = 0, 1, 2 \). The concatenation \( \eta = b(\mu^0) * b(\mu^1) * b(\mu^2) \) is a dominant word of weight \( \mu \) (§3).

Given \( w^r \in W^r \) for \( r = 1, 2 \) and \( w = w^1 w^2 \), consider the Demazure crystal (cf. remarks following Theorem 3). \( \text{Dem}(\mu, w) := \{\sum_{i} m_i f_{s_i} \cdots f_{s_1} \eta : m_i \geq 0, \, \text{where } s_{i_1} s_{i_2} \cdots s_{i_k} \text{ is a reduced word of } w \text{ obtained by concatenating reduced words of } w^1 \text{ and } w^2 \}. \) It follows from
the hypotheses and the properties of the crystal operators \[16\] that:

\[
\text{Dem}(\mu, w) = b(\mu^0) \ast \text{Dem}(\mu^1, w^1) \ast \text{Dem}(\mu^2, w^2)
\]

Given dominant weights \(\lambda, \nu\) of \(\mathfrak{sl}_n\mathbb{C}\), we decompose them likewise into \(\lambda^r, \nu^r\) for \(r = 0, 1, 2\). Let \(\pi \in \text{Dem}(\mu, w)\), say \(\pi = b(\mu^0) \ast \pi^1 \ast \pi^2\) with \(\pi^r \in \text{Dem}(\mu^r, w^r)\) for \(r = 1, 2\). Then, \(\pi \in \text{Dem}_k(\mu, w) \iff \lambda^0 + \mu^0 = \nu^0\) and \(\pi^r \in \text{Dem}_k(\mu^r, w^r)\) for \(r = 1, 2\). We have thus proved that \(c^\nu_{\lambda\mu}(w) = \delta^\lambda_{\lambda^0} \ast \nu^0 \ast c^\nu_{\lambda^1\mu^1}(w^1) \ast c^\nu_{\lambda^2\mu^2}(w^2)\). It is easy to see that this equation establishes that if \(w^1\) and \(w^2\) have the saturation property, then so does \(w\). This concludes the proof of Theorem \[1\] \(\square\)

5.3. For \(n = 4\), the only permutations in \(S_4\) that are not of the form of Theorem \[1\] are 3412, 3142, 2413, 4231 (in one-line notation). For \(w = 3142\), we have \(w_0w = 2413\) with reduced decompositions \(s_3s_1s_2 = s_1s_3s_2\). There is a unique reduced \(F\) such that \(\varpi(F) = 2413\), but the rhombi \(R_{ij}\), \((i, j) \in F\) are not left-and-bottom justified, and Lemma \[1\] fails (Remark \[3\]). For the other three \(w\), there exist two reduced faces each. In these cases, \(K^\text{Hive}(\cdot, w_0w)\) is a union of two polyhedral cones.

While our methods do not apply to a general \(w \in S_n\) (beyond those covered by Theorem \[1\]), we do not know if the saturation property fails there. In particular, a preliminary search using \textit{Sage} for \(n = 4, 5\) and small \(\lambda, \mu, \nu, k\) did not turn up any counter-examples.

6. Symmetry of the \(c^\nu_{\lambda\mu}(w)\)

The symmetry \(c^\nu_{\lambda\mu} = c^\nu_{\mu\lambda}\) was first studied via hives in \[3\]. There is another point-of-view stemming from Proposition \[4\] which leads to a bijective proof of the general symmetry property

\[
c^\nu_{\lambda\mu}(w) = c^\nu_{\mu\lambda}(w^{-1})
\]

First we recall some definitions and notations. Consider the “North-Easterly” version \(\partial^{\text{NE}}\) of \(\partial\), which takes successive differences of labels along the \(NE - SW\) direction (red edges of Figure \[4\]) (see \[19\] Example 2.8 and \[1\] Appendix), whose hive-drawing conventions differ from ours and from each other!). Consider \(h \in \text{Hive}(\lambda, \mu, \nu)\) then \(\partial^{\text{NE}}(h)\) is a GT pattern of shape \(\lambda\), which can be interpreted as a \textit{contetraouble} \(T^\dagger\) of shape \(\lambda\) \[1\]. The map \(\partial^{\text{NE}}\) is injective, as follows by arguments similar to those establishing proposition \[4\].

Fix a subset \(F \subseteq \{(i, j) : n \geq i \geq j \geq 2\}\). Consider the face of \(GT(\mu)\) obtained by setting \(a_{(i-1)(j-1)} - a_{ij} = 0\) for \((i, j) \in F\) and leaving all other inequalities untouched. We call this the \textit{dual Kogan face} \(\overline{K}(\mu, F)\). To each pair \((i, j) \in \{(i, j) : n \geq i \geq j \geq 2\}\), associate the simple transposition \(s_{j-1} \in S_n\). We consider the total order on pairs \((i, j)\) defined by \((i, j)\) precedes \((i', j')\) \(\iff\) either \(i < i'\), or \(i = i'\) and \(j > j'\). We list the elements of \(F\) in increasing order relative to this total order. Denote the product of the corresponding \(s_{j-1}\) in this order by \(\overline{\sigma}(F)\). If \(\text{len}(\overline{\sigma}(F)) = |F|\), i.e., this word is reduced, we say that \(F\) is \textit{reduced} and set \[2\] Definition 5.1:

\[
\overline{\sigma}(F) = w_0 \overline{\sigma}(F) w_0
\]
For $w \in S_n$, let $K(\mu, w) := \bigcup K(\mu, F)$, the union over reduced $F$ for which $\varpi(F) = w$. We can now state the following result of Fujita [2 Corollary 5.2]:

**Proposition 6.** There is a bijection between $K_Z(\mu, w_0w_0w)$ and $\text{Dem}(\mu, w)^{\text{op}}$.

We also recall from Fujita [2, §2] that there is an involution $\eta_\mu : \text{Tab}(\mu) \to \text{Tab}(\mu)$ such that:

$$\eta_\mu(\text{Dem}(\mu, w)) = \text{Dem}(\mu, w_0w)^{\text{op}}; \quad \eta_\mu(\text{Dem}(\mu, w)^{\text{op}}) = \text{Dem}(\mu, w_0w). \quad (14)$$

Putting together proposition 3, proposition 6, and equation 14, we get the following:

$$\eta_\mu(K_Z(\mu, w_0w)) = K_Z(\mu, w_0w)^{\text{op}}; \quad \eta_\mu(K_Z(\mu, w_0w_0w)) = K_Z(\mu, w)^{\text{op}} \quad (15)$$

**Lemma 2.** Let $h \in K_Z^{\text{Hive}}(\lambda, \mu, \nu, w_0w)$ then $\partial h \in K_Z(\mu, w_0w)$ and $\partial^{NE} h \in K_Z(\lambda, w^{-1}w_0)$.

**Proof.** Clearly $\partial h \in K_Z(\mu, w_0w)$ by the definition of $K_Z^{\text{Hive}}(\lambda, \mu, \nu, w_0w)$. This means that $\partial h \in K(\mu, F)$ for some reduced face $F$ of $\text{GT}(\mu)$ such that $\sigma(F) = w_0w$. Fix such a reduced face $F$, then $h \in K_Z^{\text{Hive}}(\lambda, \mu, \nu, F_0)$, where $F_0$ is the hive reduced face such that $R_{ij}$ is flat for all $(i, j) \in F$. Observe that $\partial^{NE} h \in K_Z(\lambda, F_0)$, where $F_0$ is thought as a face of $\text{GT}(\lambda)$.

By remark 2 we know that $\sigma(F) = \sigma'(F)$. Observe that $\bar{\sigma}(F_0) = \sigma'^{-1}(F)$ since $\bar{\sigma}(F_0)$ is the product of $s_i$’s in the reverse order of the product of $s_i$’s in $\sigma'(F)$. Then we have $\bar{\sigma}(F_0) = w_0w^{-1}$ and $\varpi(F_0) = w^{-1}w_0$. This shows that $\partial^{NE} h \in K_Z(\lambda, w^{-1}w_0)$.

We will now construct a bijective map $\Psi : K_Z^{\text{Hive}}(\lambda, \mu, \nu, w_0w) \to K_Z^{\text{Hive}}(\mu, \lambda, \nu, w_0w^{-1})$. Let $h \in K_Z^{\text{Hive}}(\lambda, \mu, \nu, w_0w)$ then by lemma 2 we have $\partial h \in K_Z(\mu, w_0w)$ and $\partial^{NE} h \in K_Z(\lambda, w^{-1}w_0)$. Since $\eta_\lambda(K_Z(\lambda, w^{-1}w_0)) = K_Z(\lambda, w_0w^{-1})$ from equation 15 then we have $\eta_\lambda(\partial^{NE} h) \in K_Z(\lambda, w_0w^{-1})$.

From [1, Appendix A] we know that $\eta_\lambda(\partial^{NE} h)$ is in $\text{Tab}_\mu(\lambda)$, which implies that $\eta_\lambda(\partial^{NE} h)$ is in the image of the injective map $\partial : K_Z^{\text{Hive}}(\mu, \lambda, \nu, w_0w^{-1}) \to K_Z(\lambda, w_0w^{-1})$. Denote the preimage of $\eta_\lambda(\partial^{NE} h)$ under $\partial$ by $h^*$, that is $h^* = \partial^{-1}(\eta_\lambda(\partial^{NE} h))$.

Now we define $\Psi(h) = h^*$. Clearly the map $\Psi$ is a well defined injective map since $\Psi = \partial^{-1} \circ \eta_\lambda \circ \partial^{NE}$ is a composition of injective maps. The inverse map of $\Psi$ can be easily defined in a similar way such that $\Psi^{-1} = (\partial^{NE})^{-1} \circ \eta_\lambda \circ \partial$.

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