THE ALGEBRAIC BRAUER GROUP OF A REDUCTIVE GROUP OVER
A NONARCHIMEDEAN LOCAL FIELD

DYLON CHOW

ABSTRACT. We show that for nonarchimedean local fields $F$, the pairing from the algebraic part of the Brauer group of a reductive group $G$ characterizes all continuous homomorphisms from $G(F)$ into $\mathbb{Q}/\mathbb{Z}$. This generalizes results of Loughran and Loughran-Tanimoto-Takloo-Bighash.

INTRODUCTION

Let $F$ be a non-Archimedean local field of characteristic 0 and $X$ an algebraic variety defined over $F$. The set $X(F)$ of $F$-rational points on $X$ acquires a natural analytic topology from $F$. Each element of the Brauer group $\text{Br}(X)$ of $X$ defines a locally constant map from $X(F)$ into $\mathbb{Q}/\mathbb{Z}$.

Let $X$ be a $F$-variety and let $\text{Br}X$ denote the Brauer group of $X$. If $L$ is a $k$-algebra and $x \in X(L)$, then $x : \text{Spec } L \to X$ induces a homomorphism $\text{Br } X \to \text{Br } L$. By composition with the invariant map $\text{Br}(F) \to \mathbb{Q}/\mathbb{Z}$ of local class field theory, each element $x \in X(F)$ defines a homomorphism $\text{Br } X \to \mathbb{Q}/\mathbb{Z}$. Similarly, each element of $\text{Br}X$ defines a map $X(F) \to \mathbb{Q}/\mathbb{Z}$.

Let $F$ be any field and let $\overline{F}$ be a separable closure of $F$. Let $\text{Br}_1X$ be the kernel of the homomorphism $\text{Br } X \to \text{Br } X_{\overline{F}}$, and let $\text{Br}_0X$ denote the image of $\text{Br } F \to \text{Br } X$. The algebraic part of $\text{Br}(X)$ is defined to be the quotient $\text{Br}_aX = \text{Br}_1X/\text{Br}_0X$. For an algebraic group $G$ over $F$, the morphism $e : \text{Spec } F \to G$ associated with the identity element $e \in G(F)$ induces a homomorphism $\text{Br } G \to \text{Br } F$. Let $\text{Br}_eG$ be the intersection of $\text{Br}_1G$ with the kernel of $\text{Br } G \to \text{Br } F$. The quotient homomorphism $\text{Br}_1G \to \text{Br}_aG$ restricts to an isomorphism $\text{Br}_eG \cong \text{Br}_aG$. Elements of $\text{Br}_eG$ define continuous homomorphisms from $G(F)$ into $\text{Br}(F)$ [San81, Lemme 6.9].

We consider the algebraic part $\text{Br}_a(G)$ of the Brauer group of a connected reductive $F$-group $G$. To state our main result, let $G^{\text{sc}}$ be the simply connected cover of the derived group $G^{\text{der}}$ of $G$ and let

$$\rho : G^{\text{sc}} \to G$$

be the natural morphism $G^{\text{sc}} \to G^{\text{der}} \to G$. The main purpose of this paper is to prove the following:

**Theorem 0.1.** Let $F$ be a non-archimedean local field of characteristic 0. Let $G$ be a connected reductive group defined over $F$ and let $\rho : G^{\text{sc}} \to G$ be the natural map. The pairing

$$\text{Br}_eG \times G(F) \to \mathbb{Q}/\mathbb{Z}$$

is...
induces an isomorphism

\[ \text{Br}_e G \cong \text{Hom}_{\text{cont}}(G(F)/\rho(G^{sc}(F)), \mathbb{Q}/\mathbb{Z}). \]

This generalizes a theorem of Loughran [Lou18], who proved it for tori, and Loughran, Takloo-Bighash, and Tanimoto [LTBT20], who proved it for semisimple groups. We will address the situation of a number field in future work.

1. Notation and conventions

1.1. We use \( F \) to denote a field. Let \( \overline{F} \) be an algebraic closure of \( F \) and write \( F^s \) for the separable closure of \( F \) in \( \overline{F} \). We let \( \Gamma = \Gamma_F \) denote the Galois group of \( F^s \) over \( F \); it is a profinite topological group with the Krull topology.

1.2. If \( G \) is a connected reductive group defined over a field \( F \) and \( K \) is a field extension of \( F \), we write \( G_K \) for the \( K \)-group obtained from \( G \) by extension of scalars. Let \( \mathbb{G}_m \) be the multiplicative group scheme \( GL_1 \). For an algebraic group \( G \), let \( X(G) \) denote the group of characters of \( G \), i.e. the group of algebraic group homomorphisms \( G \to \mathbb{G}_m \). We let \( X^*(G) = X(G_{F^s}) \). In other words, \( X^*(G) \) consists of the characters of \( G \) defined over \( F^s \). The group \( \Gamma \) acts continuously on \( X^*(G) \).

1.3. If \( A \) is an abelian group with the discrete topology on which a profinite group \( \Gamma \) acts as a group of automorphisms, then \( A \) is called a \( \Gamma \)-module if the action map \( \Gamma \times A \to A \), \( (\sigma, a) \mapsto \sigma a \) is continuous. Equivalently, \( A \) is a \( \Gamma \)-module if for all \( a \in A \) the stabilizer \( \{\sigma \in \Gamma | \sigma a = a\} \) of \( a \) is open in \( \Gamma \).

1.4. Let \( k \) be a field, \( k_s \) a separable closure of \( k \) and \( G \) an algebraic \( k \)-group. Then \( H^i(k, H) \) denotes the \( i \)-th cohomology set of the Galois group \( \text{Gal}(k_s/k) \) of \( k_s \) over \( k \), with coefficients in \( H(k_s) \) (\( i = 0, 1 \)) and, if \( G \) is commutative, the \( i \)-th cohomology group of \( \text{Gal}(k_s/k) \) in \( G(k_s) \) for all \( i \in \mathbb{N} \).

1.5. If \( F' \) is a finite field extension of \( F \) and \( G \) is an algebraic group over \( F' \), the Weil restriction of \( G \) is the algebraic group \( G_{F'/k} \) over \( k \) such that for all \( k \)-algebras \( R \), \( G_{F'/F}(R) = G(F' \otimes R) \). By an induced \( \Gamma \)-module we mean a \( \Gamma \)-module that has a finite \( \Gamma \)-stable \( \mathbb{Z} \)-basis. We say that an \( F \)-torus \( T \) is induced if \( X^*(T) \) is an induced \( \Gamma \)-module. Equivalently, an \( F \)-torus \( T \) is induced if it is a finite product of tori of the form \( (\mathbb{G}_m)_{k'/F} \) with \( k' \) a finite separable extension of \( F \).

1.6. As usual, \( \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) will denote respectively the fields of rational, real, and complex numbers; \( \mathbb{Z} \) denotes the ring of rational integers.

1.7. Sometimes our characters have values in \( \mathbb{Q}/\mathbb{Z} \), in which case we use the exponential mapping \( x \mapsto \exp(2\pi ix) \) from \( \mathbb{Q}/\mathbb{Z} \) to \( \mathbb{C}^\times \) to view them as complex-valued characters.

1.8. Let \( G_{\text{der}} \) denote the derived group of \( G \), \( G_{\text{sc}} \) the simply connected cover of \( G_{\text{der}} \), and \( G_{\text{ad}} \) the adjoint group of \( G \), i.e., \( G_{\text{ad}} = G/Z_G \) where \( Z_G \) is the center of \( G \). Let \( \rho : G^{sc} \to G \) be the natural morphism. Given a maximal \( F \)-torus \( T \) of \( G \), let \( T_{sc} = \rho^{-1}(T) \).
2. Preliminaries

2.1. Let $F$ be a local field or a number field and let $X$ be a $F$-variety. Let $\text{Br}(X)$ be the Brauer group of $X$. If $L$ is a $k$-algebra and $x \in X(L)$, then $x : \text{Spec } L \to X$ induces a homomorphism $\text{Br}(X) \to \text{Br}(L)$. By composition with the invariant map $\text{Br}(F) \to \mathbb{Q}/\mathbb{Z}$ of local class field theory, each element $x \in X(F)$ defines a homomorphism $\text{Br}(X) \to \mathbb{Q}/\mathbb{Z}$. Similarly, each element of $\text{Br}(X)$ defines a map $X(F) \to \mathbb{Q}/\mathbb{Z}$.

2.2. Let $F$ be any field and let $\overline{F}$ be a separable closure of $F$. Let $\text{Br}^1(X)$ be the kernel of the homomorphism $\text{Br}(X) \to \text{Br}(X)(\overline{F})$, and let $\text{Br}^0(X)$ denote the image of $\text{Br}(F) \to \text{Br}(X)$. The algebraic part of $\text{Br}(X)$ is defined to be the quotient $\text{Br}^a(X) = \text{Br}^1(X)/\text{Br}^0(X)$. For an algebraic group $G$ over $F$, the morphism $e : \text{Spec } F \to G$ associated with the identity element $e \in G(F)$ induces a homomorphism $\text{Br}(G) \to \text{Br}(F)$. Let $\text{Br}_e(G)$ be the intersection of $\text{Br}^1(G)$ with the kernel of $\text{Br}(G) \to \text{Br}(F)$. The quotient homomorphism $\text{Br}^1(G) \to \text{Br}^a(G)$ restricts to an isomorphism $\text{Br}_e(G) \cong \text{Br}_a(G)$. Elements of $\text{Br}_e(G)$ define continuous homomorphisms from $G(F)$ into $\text{Br}(F)$.

3. Non-archimedean local fields

We prove the theorem in two stages. In the first stage we start from the case of tori and generalize the result for only those $G$ whose derived group is simply connected.

3.1. Tori.

**Lemma 3.1.** Let $T$ be a torus over a local field $F$ of characteristic $0$. The bilinear pairing
\[ \text{Br}_e(T) \times T(F) \to \text{Br}(F) \subset \mathbb{Q}/\mathbb{Z} \]
is perfect, i.e., the induced map
\[ \text{Br}_e(T) \to \text{Hom}(T(F), \mathbb{Q}/\mathbb{Z}) \]
is an isomorphism of abelian groups.

**Proof.** See [Lou18, Theorem 4.3].

3.2. Groups with simply connected derived group. Let $F$ be a $p$-adic field. Now assume that $G$ is such that $G^{\text{der}} = G^{\text{sc}}$. Define $T = G/G^{\text{der}}$. We have an exact sequence
\[ 1 \to G^{\text{der}} \to G \to T \to 1. \]
We get an exact sequence
\[ 1 \to G^{\text{der}}(F) \to G(F) \to T(F) \to 1, \]
and thus an isomorphism
\[ G(F)/j(G^{\text{der}}(F)) \cong T(F). \]
Since $\text{Pic}(G) = 0$, we have (San81 Lemme 6.9 (i)) canonical isomorphisms $H^2(F, X^*(G)) \cong \text{Br}_aG$ and $H^2(F, X^*(T)) \cong \text{Br}_aT$. The projection $G \to T$ yields a commutative diagram.
The vertical arrows are isomorphisms. The first row is part of a long exact sequence coming from the exact sequence $1 \to G^{sc} \to G \to T \to 1$: ([San81, Corollaire 6.11])

... $\to \text{Pic}(G^{sc}) \to \text{Br}_a T \to \text{Br}_a G \to \text{Br}_a G^{sc}$.

Since $\text{Pic}(G^{sc}) = 1$ and $\text{Br}_a(G^{sc}) = 1$ ([San81, Lemme 9.4 (iv)]), the horizontal arrow is an isomorphism. We get a commutative diagram

\[
\begin{array}{ccc}
\text{Br}_a T & \longrightarrow & \text{Br}_a G \\
\downarrow & & \downarrow \\
H^2(F, X^*(T)) & \longrightarrow & H^2(F, X^*(G)).
\end{array}
\]

This proves the result for groups whose derived group is simply connected.

3.3. **General reductive groups.** In the second stage we use the following result, which allows one to reduce to the case where the derived group is simply connected.

**Lemma 3.2.** For any connected reductive $F$-group $G$ split by $K$, there exists an extension

\[1 \to Z \to \tilde{G} \to G \to 1\]

such that

- $Z$ is a central torus in $\tilde{G}$,
- $Z$ is obtained from Weil restriction of scalars from a split $K$-torus, and
- $\tilde{G}^{\text{der}}$ is simply connected.

Such an extension is called a $z$-extension. We proceed with the proof of the general case. A similar result appears in [LM15, Lemma A.1, Appendix]. Consider a $z$-extension as above. We get two more exact sequences. First, since $\text{Pic}(Z) = 0$, we get from [San81, Corollary 6.11] an exact sequence of abelian groups

\[1 \to \text{Br}_e G \to \text{Br}_e \tilde{G} \to \text{Br}_e Z.\]

Since $Z$ is an induced torus, $H^1(F, Z) = 0$, and so we get another exact sequence

\[1 \to Z(F) \to \tilde{G}(F) \to G(F) \to 1.\]

Applying $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$ we get an exact sequence

\[1 \to \text{Hom}(G(F), \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(\tilde{G}(F), \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(Z(F), \mathbb{Q}/\mathbb{Z}) \to 1.\]

This induces an exact sequence

\[1 \to \text{Hom}(G(F)/G^{\text{der}}(F), \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(\tilde{G}(F)/\tilde{G}^{\text{der}}(F), \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(Z(F), \mathbb{Q}/\mathbb{Z}).\]

We get the following commutative diagram with exact rows:
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1 \rightarrow \operatorname{Br}_e G \rightarrow \operatorname{Br}_e G' \rightarrow \operatorname{Br}_e Z \rightarrow 1

The two vertical arrows are the isomorphisms constructed above. We define a homomorphism $\operatorname{Br}_e G \rightarrow \operatorname{Hom}(G(F)/G^\text{der}(F), \mathbb{Q}/\mathbb{Z})$ to be the unique homomorphism that makes the diagram commute – it is an isomorphism, which can be seen to be induced from the Brauer pairing. This completes the proof.

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