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Commuting difference operators of rank two

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In this note we investigate one-point commuting difference operators of rank two which correspond to hyperelliptic spectral curves.

If two difference operators $L_k = \sum_{j=N}^{N_+} u_j(n)T^j$ and $L_s = \sum_{j=M_+}^{M_+} v_j(n)T^j$, $n \in \mathbb{Z}$, of orders $k = N_+ + N_-$ and $s = M_+ + M_-$, respectively, commute, where $T$ is the shift operator, then there exists a non-zero polynomial $F(z, w)$ such that $F(L_k, L_s) = 0$ [1]. This polynomial $F$ defines the spectral curve $\Gamma = \{(z, w) \in \mathbb{C}^2 \mid F(z, w) = 0\}$ of the pair $L_k, L_s$. The spectral curve parametrizes the eigenvalues in the following sense: if $L_k\psi = z\psi$ and $L_s\psi = w\psi$, then $(z, w) \in \Gamma$. The rank $l$ of the pair $L_k, L_s$ is defined to be the dimension $l = \dim\{\psi : L_k\psi = z\psi, L_s\psi = w\psi\}$ of the space of common eigenfunctions for fixed eigenvalues, where it is assumed that the point $(z, w)$ is in general position on $\Gamma$. The maximal commutative ring of difference operators that contains $L_k$ and $L_s$ is isomorphic to the ring of meromorphic functions on the spectral curve which have poles at some distinguished points $q_1, \ldots, q_m$ (see [2]). Such operators are called $m$-point operators. Difference operators of rank one were found in [1] and [3]. The problem of classifying $m$-point difference operators of rank $l$ was solved in [2], but nevertheless, finding such operators for $l > 1$ is an open problem. One-point operators of rank two that correspond to elliptic spectral curves were found in [2], and operators in this class with polynomial coefficients were found in [4].

Here we take one-point operators $L_4 = \sum_{i=-2}^{2} u_i(n)T^i$ and $L_{4g+2} = \sum_{i=-(2g+1)}^{2g+1} v_i(n)T^i$ of rank two with $u_2 = v_{2g+1} = 1$ which correspond to a hyperelliptic spectral curve $\Gamma$ of genus $g$ given by the equation

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \cdots + c_0,$$

and $L_4\psi = z\psi$ and $L_{4g+2}\psi = w\psi$ for $\psi = \psi(n, P)$, where $P = (z, w) \in \Gamma$. The common eigenfunctions of $L_4$ and $L_{4g+2}$ satisfy the equation

$$\psi(n+1, P) = \chi_1(n, P)\psi(n-1, P) + \chi_2(n, P)\psi(n, P)$$

(see [2]), where the functions $\chi_1(n, P)$ and $\chi_2(n, P)$ are rational on $\Gamma$ and have $2g$ simple poles which depend on $n$. The function $\chi_2(n, P)$ has, in addition, a simple pole at the point at infinity. Let $\sigma$ be the involution $\sigma(z, w) = (z, -w)$ on $\Gamma$. The main results of this note are Theorems 1–4.

Theorem 1. If $\chi_1(n, P) = \chi_1(n, \sigma(P))$ and $\chi_2(n, P) = -\chi_2(n, \sigma(P))$, then $L_4$ has the form

$$L_4 = (T + V_nT^{-1})^2 + W_n,$$

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and furthermore, \( \chi_1 = -V_n Q_{n+1}/Q_n \) and \( \chi_2 = w/Q_n \), where the functions \( V_n, W_n \), and \( Q_n \) satisfy the equation

\[
F_g(z) = Q_{n-1} Q_{n+1} V_n + Q_n (Q_{n+2} V_{n+1} + Q_{n+1} (z - V_n - V_{n+1} - W_n)).
\]

Remarkably, equation (1) can be linearized. Namely, if in (1) we replace \( n \) by \( n + 1 \) and subtract (1) from the equation obtained, then the result is divisible by \( Q_{n+1} (z) \). In the end we arrive at a linear equation with respect to \( Q_n (z) \).

**Corollary 1.** The functions \( Q_n (z) \), \( V_n \), and \( W_n \) satisfy the equation

\[
Q_{n-1} V_n + Q_n (z - V_n - V_{n+1} - W_n) - Q_{n+2} (z - V_{n+1} - V_{n+2} - W_{n+1}) - Q_{n+3} V_{n+2} = 0.
\]

For \( g = 1 \) equation (1) makes it possible to express \( V_n \) and \( W_n \) in terms of an arbitrary functional parameter \( \gamma_n \).

**Corollary 2.** The operator \( L_4 = (T + V_n T^{-1})^2 + W_n \), where

\[
V_n = \frac{F_1(\gamma_n)}{(\gamma_n - \gamma_n-1)(\gamma_n - \gamma_{n+1})}, \quad W_n = -c_2 - \gamma_n - \gamma_{n+1},
\]

and \( F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0 \), commutes with the operator

\[
L_6 = T^3 + (V_n + V_{n+1} + V_{n+2} + W_n - \gamma_{n+2})T
+ V_n (V_{n-1} + V_n + V_{n+1} + W_n - \gamma_{n-1}) T^{-1} + V_{n-2} V_{n-1} V_n T^{-3}.
\]

The spectral curve of the pair \( L_4, L_6 \) is given by \( u^2 = F_1(z) \).

Using Theorem 1, we can effectively construct examples of commuting operators.

**Theorem 2.** The operator \( L_4 = (T + (r_3 n^3 + r_2 n^2 + r_1 n + r_0) T^{-1})^2 + g(g+1) r_3 n, r_3 \neq 0, \) commutes with the corresponding difference operator \( L_{4g+2} \).

Theorems 1 and 2 resemble in appearance some similar results for differential operators in [5], but we could not deduce them directly from [5].

**Theorem 3.** The operator \( L_4 = (T + (r_1 a^n + r_0) T^{-1})^2 + r_1 (a^{2g+1} - a^{g+1} - a^g + 1) a^{n-g}, \) where \( r_1 \neq 0, a \neq 0, \) and \( a^{2g+1} - a^{g+1} - a^g + 1 \neq 0 \), commutes with the corresponding difference operator \( L_{4g+2} \).

**Theorem 4.** The operator

\[
L_4 = (T + (r_1 \cos n + r_0) T^{-1})^2 - 4 r_1 \sin \frac{g}{2} \sin \left( \frac{g+1}{2} \right) \cos \left( n + \frac{1}{2} \right), \quad r_1 \neq 0,
\]

commutes with the corresponding difference operator \( L_{4g+2} \).

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