Light-cone effect and relaxation after a quantum quench from excited states in the Ising chain

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Abstract. The time evolution and stationary behaviour of local observables is studied after a quantum quench of the magnetic field in the transverse field Ising chain, when the initial state is chosen as an excited state of the pre-quench Hamiltonian. Relaxation to the Generalized Gibbs Ensemble is proved. Equal-time longitudinal two point function and entanglement entropy are analytically derived for a class of excited states which confirm the light-cone spreading of correlations also starting from excited states.

1. Introduction

Recent advances in the technology of ultra-cold atoms [1] have made it possible to experimentally realize quantum systems which represent nearly ideal testing grounds for the study of non-equilibrium isolated many-body quantum systems. This has led to the possibility of addressing long standing theoretical questions concerning equilibration and thermalisation [2], the role of integrability and universality [3] in out-of-equilibrium many-body quantum systems.

The simplest way of driving a system out of equilibrium is through a sudden quantum quench [4]; it consists in preparing a system in an initial pure state, that is an eigenstate of the pre-quench Hamiltonian $H$, then suddenly varying a parameter of the Hamiltonian and letting the system evolve unitarily, i.e. without any coupling to the external environment, according to a post-quench Hamiltonian $H'$, with $[H,H'] \neq 0$.

The two main results derived from both experimental and theoretical investigations are (a) the existence of a maximum value for the velocity of spreading of correlations in a lattice system considered in a purely quantum mechanical framework [5] and (b) the long-time relaxation of local observables to time independent values, when the thermodynamic limit is taken over the whole system.

Interestingly enough, the stationary values of the subsystem can be predicted using a statistical ensemble formulation, as if the density matrix of the whole system be in a mixed state and tracing out the degrees of freedom external to the subsystem [6]. The general picture is that non integrable systems thermalise [2], i.e. they can be described by the Gibbs or microcanonical ensemble which retains only very few information of the initial state, while integrable ones relax to the Generalized Gibbs Ensemble (GGE) [7], which takes into account all local mutually commuting conserved charges. This scenario has been widely verified in numerical, experimental and theoretical analysis, although some exceptions have been found for some specific initial states.

For some insights in the definition of quantum integrability see for instance [8].
clearly, the most convincing evidence supporting it comes from the exactly solvable models, whose dynamics can be worked out explicitly. The role of initial states has also proved to be crucial for the time evolution following a quantum quench [10] but I would like to point out that it has always been taken for granted that the initial state was the ground state of the pre-quench Hamiltonian. In this work I set up to thoroughly study, for the first time to my knowledge, stationary and dynamical behaviour of local observables after a quench in an exactly solvable model, starting with initial states that are highly excited state of the pre-quench Hamiltonian. Its interest should rely on the radically different behaviour of entanglement entropy for ground states of local Hamiltonians (following an area law, with at most multiplicative logarithmic corrections [11]) compared to that of highly excited states (extensive in subsystem’s size [12]).

In the next sections I will present the model studied and show that in the long-time limit, taken after the thermodynamic one, the system can be locally described by the GGE; I will then address the time evolution of some relevant observables, namely the transverse magnetisation, the longitudinal two-point function and entanglement entropy, highlighting the differences obtained with respect to the ground-state case.

2. Model

In order to shed some light on this problem I will consider here, among the exactly solvable models, the transverse field Ising chain which, despite its simplicity, represents a crucial paradigm for quantum critical behaviour. Its Hamiltonian is

$$H(h) = -\frac{1}{2} \sum_{j=1}^{N} [\sigma_j^x \sigma_{j+1}^x + h \sigma_j^z],$$

where $\sigma^\alpha_j$, $\alpha = x, y, z$ are the Pauli matrices at site $j$ of the chain of length $N$, $h$ is the transverse field and periodic boundary conditions are imposed. The model can be mapped to spinless free fermions through the non-local Jordan–Wigner transformation and diagonalised by a further Bogoliubov transformation in momentum space, yielding [13]

$$H(h) = \sum_k \epsilon_k \left( b_k^\dagger b_k - \frac{1}{2} \right),$$

where $b_k$, $b_k^\dagger$ are the annihilation and creation operator for the fermionic quasi-particles of momentum $k$, satisfying $\{b_k, b_{k'}^\dagger\} = \delta_{kk'}$, with one-particle dispersion relation given by

$$\epsilon_k = (h - \cos p_k)^2 + \sin^2 (p_k),$$

where $p_k = \frac{2\pi k}{N}$.

I will focus on quenches when at $t = 0$ the transverse magnetic field is suddenly switched from $h_0$ to $h$. As initial state I will consider an excited state of the pre-quench Hamiltonian $H(h_0)$ which is neither an eigenstate nor a finite superposition of eigenstates of $H(h)$ of the form

$$|\Psi_0\rangle = \prod_k (b_k^\dagger)^{m_k} |0\rangle,$$

where $m_k = 0, 1$ is the fermionic initial occupation number of the $k$-mode (i.e. the excitation profile characterizing the initial state $|\Psi_0\rangle$) while $|0\rangle$ is the ground state of the pre-quench Hamiltonian. In the thermodynamic limit, $m_k$ becomes an arbitrary function $m(\varphi) \in [0, 1]$, with $\varphi \in [-\pi, \pi]$ and represents a coarse-grained version of $m_k$ [12]. I would like to point out that Eq. (3) is not the most generic excited state, since I have focused only on states that are a basis for the many-body Hilbert space, and not linear superposition of them.

$^2$ From now on, primed quantities will be used to denote pre-quench operators, while non-primed quantities for post-quench ones.
The analytic expression for the transverse magnetisation 

\[ m^z(t) = \langle \psi_0(t)|\sigma^z|\psi_0(t) \rangle \]  

with 

\[ |\psi_0(t)\rangle = e^{iH(t)}|\psi_0\rangle, \]  
in the thermodynamic limit is [16] 

\[ m^z(t) = \int_{-\pi}^{\pi} \frac{dk}{4\pi} e^{i\theta_k} m^S_k \cos \Delta_k - i \sin \Delta_k \cos(2\epsilon_k t), \]  

where \( \theta_k \) and \( \Delta_k \) are the angle of the Bogoliubov rotation that respectively diagonalises the Hamiltonian and connects pre-quench with post-quench variables 

\[ \tan \theta_k = \frac{\sin (2\pi k/N)}{\cos (2\pi k/N) - h} \quad \cos \Delta_k = \frac{hh_0 - (h + h_0) \cos \varphi_k + 1}{\sqrt{1 + h^2 - 2h \cos \varphi_k} \sqrt{1 + h_0^2 - 2h_0 \cos \varphi_k}}; \]  

\]  

Figure 1. Transverse magnetisation minus its stationary value as a function of time for quenches from \( h_0 = 12 \) to \( h = 2 \) starting from (a) \( m(k) = \theta(k - \pi/2) \) and (b) \( m(k) = (k + \pi)/(4\pi) \).
obtain analytical results [16] in the scaling limit (i.e. the determinant approach for all initial excited states analysed [16]; I have also been able to calculate the time dependence of the two-point function \( \langle \rho_{XX}(t) \rangle \) for quenches from the ground state the power law decay is \( t^{-2/3} \), while for \( m(k) = (k + \pi)/(4\pi) \), Fig.1b, it goes like \( t^{-2} \), as for the ground state. Thus, my conclusion is that while for quenches starting from the ground state the power law decay is \( t^{-3/2} \), this is not true in general for excited states (see Fig.1b) which display a much slower relaxation \( \propto t^{-1} \); actually the excited state for which that happens is the most physical one among those studied since it has all modes larger than a given one occupied with a fermion.

**Equal time two point longitudinal correlation function and entanglement entropy**

The time dependence of the two-point function \( \rho_{XX}(t) \) has been computed numerically within the determinant approach for all initial excited states analysed [16]; I have also been able to obtain analytical results [16] in the scaling limit (i.e. \( \ell \to \infty, t \to \infty \), with \( \ell/t \) fixed) for quenches within the ferromagnetic phase and for initial excitation profiles preserving \( k \to -k \) symmetry.

Calculations have been carried out for many excited states, with excitation profile \( m(k) \) of linear, quadratic, or well shape. All the data obtained from these states show a quite general behaviour, exemplified in Fig. 2a: for \( t < t_F \) the correlation function decays exponentially, while for \( t > t_F \) it shows a slow relaxation towards the GGE value. This is a manifestation of the light-cone spreading of correlations [4] also for quenches from excited states. Still there is a fundamental difference with respect to the ground-state initial case: since the initial state is \( Z_2 \)-invariant, the one point function \( \langle \sigma_x^2 \rangle \) is always vanishing, explaining why the initial value of the two-point functions in Fig. 2 is vanishing for increasing values of \( \ell \), while it is close to 1 for the ground-state quench.

For one of the excited states, namely \( m(k) = \theta(k - \pi/2) \), I found an unusual manifestation of the light-cone behaviour in the two-point function; as can be seen in Fig. 2b, after an initial decay

\[ m_k^S \text{ and } m_k^A \text{ are respectively the symmetrized and antisymmetrized part of } m(k), \text{ namely } m_k^S = m_{-k} + m_k - 1 \text{ and } m_k^A = m_{-k} - m_k. \]

The approach to stationary values can be evaluated by a stationary phase approximation and exhibits a different power-law decay depending on the analyticity of the \( m(k) \) excitation profile. In Fig. 1 I report the time dependent part of the transverse magnetisation for different prototypical excited states (figures are taken from [16]): exact results obtained from the numerical evaluation of the integral in Eq. (5) (points) are compared with its stationary phase approximation (line), and it can be seen that the agreement is excellent. For an initial excitation profile given by \( m(k) = k^2/(2\pi)^2 \) for \( \ell = 60 \); (b) \( m(k) = \theta(k - \pi/2) \) for \( \ell = 90 \).

\[ m_k^S \text{ and } m_k^A \text{ are respectively the symmetrized and antisymmetrized part of } m(k), \text{ namely } m_k^S = m_{-k} + m_k - 1 \text{ and } m_k^A = m_{-k} - m_k. \]
the correlation function displays a sort of plateaux and at \( t \sim t_F \) it sets around the GGE value which is reached in an oscillating manner. An intuitive explanation of this uncommon behaviour is presently lacking, even because the analytical method developed for \( k \to -k \) symmetric states does not straightforwardly generalize to the state under consideration.

The entanglement entropy for a block of \( \ell \) contiguous spins, defined as the Von Neumann entropy of the reduced density matrix of the subsystem, can be related to the eigenvalues of a \( 2\ell \times 2\ell \) block Toeplitz matrix constructed with the fermionic two point functions [17]. I evaluated its time dependent behaviour in a similar way as done for \( \rho_{xx}(\ell,t) \); the result, common to all excited states analysed (for figures see [16]), shows a light-cone behaviour, i.e. a linear growth for \( t < t_F \) followed by a slow saturation. The main difference with the ground state case is the fact that the entanglement entropy in the initial state is extensive.

5. Conclusions

I have considered the stationary and dynamical behaviour of several observables after a sudden quantum quench of the transverse magnetic field in the Ising chain, starting from an initial excited state. While still having relaxation towards the GGE, I have found some significant differences with respect to the initial ground state case, among which a different power law decay of the transverse magnetisation, and an unusual manifestation of the light-cone effect for a specific initial state. The role of excitations in truly interacting integrable models will be investigated in future works.

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