Compacted binary trees admit a stretched exponential

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Abstract

A compacted binary tree is a directed acyclic graph encoding a binary tree in which
common subtrees are factored and shared, such that they are represented only once. We
show that the number of compacted binary trees of size $n$ is asymptotically given by
$$\Theta \left( n! \, 4^n e^{3a_1 n^{1/3}} n^{3/4} \right),$$
where $a_1 \approx -2.3381$ is the largest root of the Airy function. Our method involves a new
two parameter recurrence which yields an algorithm of quadratic arithmetic complexity
for computing the number of compact trees of a given size. We use empirical methods
to estimate the values of all terms defined by the recurrence, then we prove by induction
that these estimates are sufficiently accurate for large $n$ to determine the asymptotic form
of the number of compacted trees.

Keywords: Airy function, asymptotics, directed acyclic graphs, lattice paths, bijection,
stretched exponential, Dyck paths, compacted trees.

1 Introduction

Compacted binary trees are a special class of directed acyclic graphs that appear as a model
for data structures in the compression of XML documents [3]. Given a rooted binary tree of
size $n$, its compacted form can be computed in expected and worst time $O(n)$ with expected
compacted size $O(n/\log(n))$ [8]. Recently, Genitrini, Gittenberger, Kauers and Wallner solved
the reversed question on the asymptotic number of compacted trees under certain height
restrictions [9], however the asymptotic number in the unrestricted case remained elusive.
They also solved this problem for a simpler class of trees known as relaxed trees under the
same height restrictions. In this paper we show that the counting sequences $(c_n)_{n \in \mathbb{N}}$ of
(unrestricted) compacted binary trees and $(r_n)_{n \in \mathbb{N}}$ of (unrestricted) relaxed binary trees each
admit a stretched exponential:

**Theorem 1.1.** The number of compacted and relaxed binary trees satisfy for $n \to \infty$
$$c_n = \Theta \left( n! \, 4^n e^{3a_1 n^{1/3}} n^{3/4} \right) \quad \text{and} \quad r_n = \Theta \left( n! \, 4^n e^{3a_1 n^{1/3}} n \right),$$
where $a_1 \approx -2.3381$ is the largest root of the Airy function $\text{Ai}(x)$ defined as the unique function
satisfying $\text{Ai}''(x) = x \text{Ai}(x)$ and $\lim_{n \to \infty} \text{Ai}(x) = 0$. 
We believe that there are constants $\gamma_c$ and $\gamma_r$ such that
\[ c_n \sim \gamma_c n! 4^n e^{3a_1 n^{1/3}} n^{3/4} \quad \text{and} \quad r_n \sim \gamma_r n! 4^n e^{3a_1 n^{1/3}} n, \]
however we have been unable to find the exact values of these constants or even prove that they exist. Despite this, our empirical analysis yields what we believe to be very accurate estimates for $\gamma_r$ and $\gamma_c$, namely $\gamma_c \approx 173.12670485$ and $\gamma_r \approx 166.95208957$.

The presence of a stretched exponential term in a combinatorial sequence remains a fairly rare occurrence, although it is not without precedent. One simple example is that of pushed Dyck paths, where Dyck paths of maximum height $h$ are given a weight $y^{-h}$ for some $y < 1$. In this case Brendan McKay and Nick Beaton determined the weighted number $d_n$ of paths of length $2n$ up to and including the constant term to be asymptotically given by
\[ d_n \sim A^{-y y^{2/3} \exp \left( -C (-\log(y))^{2/3} n^{1/3} \right)} n^{-5/6}, \]
where $A = 2^{5/3} \pi^{5/6}/\sqrt{3}$ and $C = 3(\pi/2)^{2/3}$; see [10]. For the analogous problem of counting pushed self avoiding walks, Beaton et al. [2] gave a (non-rigorous) probabilistic argument for the presence of a stretched exponential of the form $e^{-cn^{3/7}}$ for some $c > 0$. In each of these cases, a stretched exponential appears as part of a compromise between the large height regime in which most paths occur and the small height regime in which the weight is maximised. We will see that a similar compromise occurs in this paper. Another situation in which stretched exponentials have appeared is in cogrowth sequences in groups [6], that is, paths on Cayley graphs which start and end at the same point. In particular, Revelle [13] showed that in the lamplighter group the number $c_n$ of these paths of length $2n$ behaves like
\[ c_n \sim c \cdot 9^n \kappa^{n^{1/3}} n^{1/6}. \]

In the group $\mathbb{Z}/\mathbb{Z}$, Pittet and Saloff-Coste showed that the asymptotics of the cogrowth series contains the slightly more complicated term $\kappa^{-n \log n} [12]$. Another example comes from the study of pattern avoiding permutations, where Conway, Guttmann and Zinn-Justin [4,5] have given compelling numerical evidence that the number $p_n$ of 1324-avoiding permutations of length $n$ behaves like
\[ p_n \sim B \mu^n \mu_1^{\sqrt{n}} n^{\mu}, \]
with $\mu \approx 11.600$, $\mu_1 \approx 0.0400$, $g \approx -1.1$. Despite these few examples, it is generally quite difficult to prove that a sequence has a stretched exponential in its asymptotics. Part of the difficulty is that a sequence which has a stretched exponential cannot be very nice. In particular, the generating function cannot be algebraic, and can only be $D$-finite if it has an irregular singularity [7]. On the other hand some explicit examples of $D$-finite generating series with a stretched exponential are known, see eg. [15–17].

A well-known method for proving the presence of a stretched exponential is the saddle point method [7, Chapter VIII]. To apply this method, one needs to meticulously check various analytic conditions on the generating function, or to bound related integrals in a delicate way. These tasks can be highly non-trivial and require a good knowledge on the analytic properties of the generating function. In the cases that we present, such detailed information is not known or needed, and a simple recurrence relation suffices. This recurrence
relation corresponds to a complicated partial differential equation, to which the saddle point method cannot be readily applied. We present a method, working directly with the two-parameter recurrence relation for \( e_{n,k} \) where we are interested in the asymptotics of \( e_{n,0} \), as \( n!e_{2n,0} \) is the number of compacted trees of size \( n \). We find two explicit families \( X_{n,k} \) and \( Z_{n,k} \) such that

\[
X_{n,k} \leq e_{n,k} \leq Z_{n,k},
\]

for all \( k \) and all \( n \) large enough. The idea is that \( X_{n,k} \) and \( Z_{n,k} \) satisfy the same recurrence as \( e_{n,k} \) with the equality sign exchanged by a less than or equal and a larger than or equal sign, respectively. From these, we inductively prove (1) up to multiplication by constants. In order to find appropriate sequences \( X_{n,k} \) and \( Z_{n,k} \) we start by performing a heuristic analysis to conjecture the asymptotic shape of \( e_{n,k} \) for large \( n \). We then prove that the required recursive inequalities hold for sufficiently large \( n \) using adapted Newton polygons.

If applied as is, our method only works for recurrences with positive terms, for instance that of relaxed binary trees. However, for some recurrences involving negative terms, for example that of compacted binary trees, it is possible to construct a sandwiching pair of recurrences with positive terms, to which our method applies. In this way, our method can be extended to general recurrences. This is one of the reason why we expect this method to be applicable to a wide range of recurrence relations with two or more parameters, many of which may involve a stretched exponential. Therefore, our method could possibly be applied to combinatorial problems with such recurrences, notably but not limited to models based on lattice paths.

We now outline the contents in the rest of the article. In Section 2 we introduce compacted binary trees and the related relaxed binary trees in detail and we derive a bijection to Dyck paths with weights on their horizontal steps. In Section 3 we show a heuristic method of how to conjecture the asymptotics and in particular the appearance of a stretched exponential term. Building on these heuristics, we prove exponentially and polynomially tight bounds for the recurrence of relaxed binary trees in Section 4 and compacted binary trees in Section 5.

Notation. In this paper we define a rooted binary tree to be a directed graph with a distinguished node called the root in which all nodes have out-degree either 0 or 2 and all nodes other than the root have in-degree 1, while the root has in-degree 0, moreover, for each vertex with out-degree 2 the out going edges are distinguished as a right edge and a left edge. Nodes with out-degree 0 are called leaves, nodes with out-degree 2 are called internal nodes. All trees in this paper will be rooted and we omit this term in the future.

2 A two-parameter recurrence relation

Originally, compacted binary trees arose in a compression procedure in [8] which computes the number of unique fringe subtrees. Relaxed binary trees are then defined by relaxing the uniqueness conditions on compacted binary trees. As we will not need this algorithmic point of view, we directly give the following definition adapted from [9, Definition 3.1 and Proposition 4.3].
Definition 2.1 (Relaxed binary tree). A relaxed binary tree (or simply relaxed tree) of size $n$ is a directed acyclic graph obtained from a binary tree with $n$ internal nodes, called its spine, by keeping the left-most leaf and turning other leaves into pointers, with each one pointing to a node (internal ones or the left-most leaf) preceding it in postorder.

Using the class of relaxed trees it is then easy to define the set of compacted trees by requiring the uniqueness of subtrees.

Definition 2.2 (Compacted binary tree). Given a relaxed tree, to each node $u$ we can associate a binary tree $B(u)$. We proceed by postorder. If $u$ is the left-most leaf, we define $B(u) = u$. Otherwise, $u$ has two children $v, w$, then $B(u)$ is the binary tree with $B(v)$ and $B(w)$ as left and right sub-trees, respectively. A compacted binary tree, or simply compacted tree of size $n$ is a relaxed tree with $B(u) \neq B(v)$ for all pairs of distinct nodes $u, v$.

In Figure 1 all relaxed (and compacted) trees of size $n = 0, 1, 2$ are shown. In Figure 2 we see the smallest relaxed tree that is not a compacted tree.

As a corollary of our main result Theorem 1.1, we directly get that the proportion of compacted trees among relaxed trees is given by

$$\frac{c_n}{r_n} = \Theta(n^{-1/4}).$$

In [9, Corollary 3.5] Genitrini, Gittenberger, Kauers and Wallner showed an analogous result for compacted and relaxed trees of bounded right height. The right height of a tree is the maximal number of right edges from the root to a leaf. Let $c_{k,n}$ (resp. $r_{k,n}$) be the number of compacted (resp. relaxed) trees of right height at most $k$. Then [9, Corollary 3.5] states that for fixed $k$,

$$\frac{c_{k,n}}{r_{k,n}} = \lambda_k n^{-\frac{1}{4} \left( 1 - \frac{1}{k+1} \right)} \cos \left( \frac{\pi}{k+3} \right) = o \left( n^{-1/4} \right),$$

for a constant $\lambda_k$ independent of $n$. As $k$ approaches $\infty$, we see that the exponent of $n$ approaches $-1/4$. It is therefore not surprising that the exponent in the unbounded case is also $-1/4$.

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Figure 1: All relaxed (and also compacted) binary trees of size 0, 1, 2, where internal nodes are shown by circles and the unique leaf is drawn as a square.

In [9, Theorem 5.1 and Corollary 5.4] recurrence relations are derived for the number of compacted and relaxed binary trees. Using these recurrences, computing the first $n$ terms of the sequence requires $O(n^3)$ arithmetic operations. In this section we give an alternative recurrence with only one auxiliary parameter other than the size $n$, which leads to an algorithm of arithmetic complexity $O(n^2)$ to compute the first $n$ terms of the sequence. The construction is motivated by the new structural insights from the recent bijection [14].
2.1 Relaxed binary trees and horizontally decorated paths

For the subsequent construction we need the following type of lattice paths.

Definition 2.3. A horizontally decorated path \( P \) is a lattice path starting from \((0,0)\) with steps \( H = (1,0) \) and \( V = (0,1) \) confined to the region \( 0 \leq y \leq x \), where each horizontal step \( H \) is decorated by a number in \( \{1, \ldots, k+1\} \) with \( k \) its \( y \)-coordinate. If \( P \) ends at \( (n,n) \), we call it a horizontally decorated Dyck path.

We denote by \( D_n \) the set of horizontal decorated Dyck paths of length \( 2n \).

Remark 2.4. Horizontally decorated Dyck paths can also be interpreted as classical Dyck paths, where below every horizontal step a box given by a unit square between the horizontal step and the line \( y = -1 \) is marked, see Figure 3. This gives an interpretation connecting these paths with the heights of Dyck paths, which we will exploit later.

Theorem 2.5. There exists a bijection \( \text{Dyck} \) between relaxed binary trees of size \( n \) and the set \( D_n \) of horizontally decorated Dyck paths of length \( 2n \).

Proof. Let \( C \) be a relaxed binary tree of size \( n \), and \( C_* \) its spine. For convenience, we identify the internal nodes in \( C \) and \( C_* \), and pointers in \( C \) with leaves (not the left-most one) in \( C_* \).

We now give a recursive procedure transforming \( C_* \) into a horizontally decorated Dyck path \( P \). First, we take \( C_* \) and label its internal nodes and the left-most leaf with the postorder from 1 to \( n+1 \). Next, we define the following function \( \text{Path} \) that transforms \( C_* \) into a lattice path in \( H \) and \( V \). Given a binary tree \( T \), it either consists of two sub-trees \( (T_1, T_2) \), or it is a leaf \( \varepsilon \). We thus define \( \text{Path} \) recursively by

\[
\text{Path}((T_1, T_2)) = \text{Path}(T_1)\text{Path}(T_2)V, \quad \text{Path}(\varepsilon) = H.
\]

It is clear that \( \text{Path}(C_*) \) starts with \( H \) for the left-most leaf. Let \( P_0 \) be \( \text{Path}(C_*) \) with its starting \( H \) removed. Note that \( \text{Path} \) performs a postorder traversal on \( C_* \) where leaves are matched with \( H \) and internal nodes with \( V \). Then, \( \text{Path}(C_*) \) ends at \( (n+1,n) \) and stays always strictly below \( y = x \) because every binary (sub-)tree has one more leaf than internal nodes, and each initial segment of \( \text{Path}(C_*) \) corresponds to a collection of sub-trees of \( C_* \). Hence, \( P_0 \) is a Dyck path. Observe that the \( i \)-th step \( V \) in \( P_0 \) corresponds to the \( (i+1) \)-st node in postorder, as the left-most leaf is labeled 1. Finally, for each step \( H \) in \( P_0 \), we label it by the label of the internal node (or the left-most leaf) to which its corresponding leaf in \( C_* \) points in \( C \). We thus obtain a Dyck path \( P \) with labels on the horizontal steps, and we define \( \text{Dyck}(C_*) = P \).
We have seen that the Dyck path $P_0$ is in bijection with the spine $C_\ast$. To see that the labeling condition on horizontally decorated Dyck paths is equivalent to the condition on relaxed binary trees, we take a pointer $p$ pointing to a node $u$ with label $\ell$ that corresponds to a step $H$ with a certain coordinate $k$. By construction of the Dyck path, $p$ comes after $u$ in postorder if and only if the step $H$ from $p$ comes after the step $V$ from $u$, which is equivalent to $\ell \leq k + 1$, as the node with label 1 is the left-most leaf and is not recorded as a step $H$. We thus have the equivalence of the two conditions, so Dyck is indeed a bijection as claimed. 

\[ \text{Figure 3: Example of the bijection Dyck between relaxed trees and horizontally decorated Dyck paths. It transforms internal nodes into vertical steps and pointers into vertical steps.} \]

The big advantage of horizontally decorated paths is that they are easier to count. The following result gives the claimed quadratic time algorithm to count and generate such paths, and therefore relaxed binary trees.

**Proposition 2.6.** Let $r_{n,m}$ be the number of horizontally decorated paths ending at $(n, m)$. Then,

\[
\begin{align*}
    r_{n,m} &= r_{n,m-1} + (m+1)r_{n-1,m}, & \text{for } n, m \geq 1 \text{ and } n \geq m, \\
    r_{n,m} &= 0, & \text{for } n < m, \\
    r_{n,0} &= 1, & \text{for } n \geq 0.
\end{align*}
\]

The number of relaxed binary trees of size $n$ is equal to $r_{n,n}$.

**Proof.** Let us start with the boundary conditions. First of all, no such path is allowed to cross the diagonal $y = x$, thus $r_{n,m} = 0$ for $n < m$. Second, the paths consisting only of horizontal steps stay at altitude 0 allowing just one label.

For the recursion let us consider how a path can jump to $(n, m)$. It either uses a step $V$ from $(n, m - 1)$ or it uses a step $H$ from $(n - 1, m)$. In the second case, there are $m + 1$ possible decorations as the path is currently at altitude $m$.

**Remark 2.7** (Compacted trees of bounded right height). In [9] the class of compacted and relaxed trees with bounded right height was asymptotically enumerated. The right height of a compacted tree is the maximal number of internal right edges that need to be traversed when moving in the spine from the root to a leaf. This restriction naturally translates relaxed binary trees of right height at most $k$ into horizontally decorated Dyck paths of height at most $k + 1$, where height is the maximal normal distance rescaled by $\sqrt{2}$ from a node to the diagonal. In other words, these paths are restricted to a strip given by the diagonal and a line translated to the right parallel to the diagonal by $k + 1$ unit steps.
2.2 Compacted binary trees

Given a relaxed tree \( C \), an internal node \( u \) is called a *cherry* if its children in the spine are both leaves and none of them is the left-most one. According to the discussion at the end of Section 4 in [9], the only obstacle for a relaxed tree to be a compacted tree is a cherry with badly chosen pointers. For the convenience of the reader, we now recall and formalize this observation in the following proposition.

**Proposition 2.8.** A relaxed tree \( C \) is a compacted tree if and only if, for each cherry \( v \), and for any internal node \( u \) preceding \( v \) in the postorder, we have \( B(u) \neq B(v) \) with \( B(u) \) defined in Definition 2.1.

**Proof.** The “only if” part follows directly from the definition. We now focus on the “if” part. Suppose that \( C \) is not a compacted tree, which means there is at least a pair of internal nodes \( u, v \) such that \( u \) precedes \( v \) and \( B(u) = B(v) \). Now we want to show that there is one such pair with \( v \) a cherry. We take such a pair \( u, v \). If \( v \) is a cherry, then we are done. Otherwise, without loss of generality, we suppose that the left child \( v' \) of \( v \) is not a leaf. Let \( u_\ell \) be the left child of \( u \). If \( u_\ell \) is an internal node, we take \( u' = u_\ell \). Otherwise, we take \( u' \) to be the internal node pointed to by \( u_\ell \). By definition, we have \( B(u') = B(v') \), and clearly \( u' \) precedes \( v' \) in the postorder. We thus obtain a new pair with the same conditions but in greater depth in the spine. However, since the spine has finite depth, this process cannot continue forever. As it only stops with \( v \) is a cherry, we have the existence of such a pair \((u, v)\) with \( v \) a cherry. \( \square \)

This restriction also has an analogue in the class of horizontally decorated paths:

We label every \( V \)-step with its final altitude, which corresponds to its row number in the interpretation with marked boxes, and also the label of its internal node in the relaxed tree. Recall that each \( H \)-step is already labelled. For any step \( S \), let \( L(S) \) be its label. We associate to every step \( V \) a pair of integers \((v_1, v_2)\), which correspond to the labels of its left and right child. First, let \( S' \) be the step before \( V \) and set \( v_2 = L(S') \). Next, draw a line parallel to the diagonal from the ending point of \( V \). Let \( S'' \) be the last step before \( V \) that ends on this line (if there is no such step, set \( v_1 = 1 \)). Then set \( v_1 = L(S'') \).

**Definition 2.9.** A C-decorated path \( P \) is a horizontally decorated path where the decorations \( h_1 \) and \( h_2 \) of each pattern of consecutive steps HHV fulfill \((h_1, h_2) \neq (v_1, v_2)\) for all preceding steps \( V \).

**Proposition 2.10.** The map Dyck bijectively sends the set of compacted trees of size \( n \) to the set of C-decorated Dyck paths of length \( 2n \) are in bijection.

**Proof.** Recall from Theorem 2.5 that the map Dyck is a bijection sending relaxed trees of size \( n \) to the set of horizontally decorated Dyck paths of size \( 2n \). C-decorated paths are defined precisely so that their corresponding relaxed trees satisfy the condition of Proposition 2.8. Therefore, Dyck forms a bijection between C-decorated paths and compacted trees. \( \square \)

Note that for the counting result the crucial observation is that exactly one pair of labels \((h_1, h_2)\) is avoided for each preceding step \( V \) of a consecutive pattern HHV. Applying this classification to the previous result we get a similar quadratic-time recurrence relation for compacted binary trees.
Proposition 2.11. Let $c_{n,m}$ be the number of C-decorated paths ending at $(n,m)$. Then,

\[
c_{n,m} = c_{n,m-1} + (m+1)c_{n-1,m} - (m-1)c_{n-2,m-1}, \quad \text{for } n,m \geq 1,
\]

\[
c_{n,m} = 0, \quad \text{for } n < m,
\]

\[
c_{n,0} = 1, \quad \text{for } n \geq 0.
\]

The number of compacted binary trees of size $n$ is equal to $c_{n,n}$.

Proof. In the first case, the term $(m+1)c_{n-1,m}$ counts the paths ending with a $H$-step while $c_{n,m-1} - (m-1)c_{n-2,m-1}$ counts the paths ending with a $V$-step. The term $-(m-1)c_{n-2,m-1}$ occurs because, for each C-decorated path ending at $(n-2,m-1)$, there are exactly $m-1$ paths formed by adding an additional $HHV$ that are not C-decorated paths. \qed

Note that one might also count the following simpler class which is in bijection with C-decorated paths, albeit without an explicit bijection.

Definition 2.12. A H-decorated path $P$ is a horizontally decorated path where the decorations $h_1$ and $h_2$ of each pattern of consecutive steps $HHV$ fulfill $h_1 \neq h_2$ except for $h_1 = h_2 = 1$.

In terms of marked boxes, this constraint translates to the fact that below the horizontal steps in each consecutive pattern $HHV$ the marks must be in different rows except possibly for the lowest row.

2.3 Weighted Dyck meanders

We now propose another model of lattice paths. A Dyck meander (or simply a meander) $M$ is a lattice path consisting of up steps $U = (1,1)$ and down steps $D = (1,-1)$ while never falling below $y = 0$. It is clear that Dyck paths of length $2n$ are in bijection with Dyck meanders of length $2n$ ending on $y = 0$ with the transcription $H \rightarrow U, V \rightarrow D$. This bijection can also be viewed geometrically as the linear transformation $x' = x + y, y' = x - y$. This transformation will simplify the following analysis. We can consider Dyck meanders as initial segments of Dyck paths.

We now consider the following weight on $U$-steps in a meander $M$. If $U$ starts from $(a,b)$, then its weight is $(a-b+2)/(a+b+2)$, and the weight of $M$ is the product of the weights of its up steps. Let $d_{n,m}$ denote the weighted sum of meanders ending at $(n,m)$. We have the following recurrence for $d_{n,m}$.

Proposition 2.13. The weighted sum $d_{n,m}$ defined above for meanders ending at $(n,m)$ satisfies the recurrence

\[
\begin{align*}
d_{n,m} &= \frac{n-m+2}{n+m}d_{n-1,m-1} + d_{n-1,m+1}, \quad \text{for } n > 0, m \geq 0, \\
0 &= 0, \quad \text{for } m > 0, \\
0 &= 0, \quad \text{for } n \geq 0, \\
1 &= 1.
\end{align*}
\]

Proof. We concentrate on the first case, as other cases are border cases directly from the definition of meanders. Given a meander ending at $(n,m)$ with $n > 0$, the last step may be an up step or a down step. The contribution of the former case is $\frac{n-m+2}{n+m}d_{n-1,m-1}$, with the weight of the last up step taken into account. The contribution of the latter case is simply $d_{n-1,m+1}$. We thus have the recurrence. \qed
Corollary 2.14. For integers \( m, n \) of the same parity, we have
\[
d_{n,m} = \frac{1}{((n+m)/2)!^2(n-m)/2}.
\]
When \( m, n \) are not of the same parity, we have \( d_{n,m} = 0 \).

In particular, the number of relaxed trees of size \( n \) is given by \( n!d_{2n,0} \).

Proof. It is clear that meanders can only end on points \( (n, m) \) for \( n, m \) of the same parity. In this case, it suffices to compare Proposition 2.6 with Proposition 2.13 under the proposed equality.

For some simple cases of \( d_{n,m} \), elementary computations show that \( d_{n,m} = 0 \) for \( n > m \), \( d_{n,n} = \frac{1}{n!} \), \( d_{n,n-2} = \frac{2^{n-1}-1}{(n-1)!} \) and \( d_{n,n-4} = \frac{21 \cdot 3^{n-4} - 2^{n+1}}{2(n-2)!} \).

3 Heuristic analysis

In this section, we will explain briefly some heuristics and ansatz that we will be applying later to get the asymptotic behavior of \( r_n \) and \( c_n \). These heuristics are closely related to the asymptotic behavior of Dyck paths and the Airy function.

3.1 An intuitive explanation of the stretched exponential

We can consider \( r_n \) as a weighted sum of Dyck paths, where each Dyck path \( P \) has a weight \( w(P) \) that is the number of horizontally decorated Dyck paths that it gives rise to. There is thus a balance of the number of total paths and their weights for the weighted sum \( r_n \). On one hand, most paths have an (average) height of \( O(\sqrt{n}) \). On the other hand, their weight is maximal if their height is \( O(1) \), i.e., they are close to the \( x \)-axis. In other words, typical Dyck paths are numerous but with small weight, and Dyck paths atypically close to the \( x \)-axis are few but with enormous weight. The asymptotic behavior of the weighted sum of Dyck paths that we consider should be a result of a compromise between these two forces. We will now make this more explicit by analysing Dyck paths with height approximately \( n^{\alpha} \) for some \( \alpha \in (0, 1/2) \).

Given a Dyck path \( P \) with steps \( H = (1, 0) \) and \( V = (0, 1) \) as in Definition 2.3, let \( m_i \) be the \( y \)-coordinate of the \( i \)-th step \( H \). We have the following result from [11] for the number of Dyck paths with \( m_i \) bounded uniformly.

**Proposition 3.1** (Theorem 3.3 of [11]). For a Dyck path \( P \) of length \( 2n \) chosen uniformly at random, let \( m_i \) be the \( y \)-coordinate of the \( i \)-th step \( H \). For \( \alpha < 1/2 \), we have
\[
\log \mathbb{P} \left( \max_{1 \leq i \leq n} (i - m_i) < n^\alpha \right) \sim -\pi^2 n^{1-2\alpha}.
\]

Let \( w(P) \) the number of horizontally decorated Dyck paths whose unlabeled version is the Dyck path \( P \). For a randomly chosen Dyck path \( P \) of length \( 2n \) with \( i - m_i \) bounded uniformly by \( n^\alpha \), we heuristically expect most values of \( i - m_i \) to be in the order \( \Theta(n^\alpha) \), with \( i \) of order \( \Theta(n) \), which leads to the following approximation:
\[
\log \frac{w(P)}{n!} = \sum_{1 \leq i \leq n} \log (m_i/i) = \sum_{1 \leq i \leq n} \log \left( 1 - \frac{i - m_i}{i} \right) \approx cn \cdot \left( \frac{n^\alpha}{n} \right) = -cn^\alpha.
\]
Here, $c > 0$ is some constant depending on $\alpha$. This approximation is only heuristically justified and very hard to prove. The contribution of Dyck paths with $i - m_i$ uniformly bounded by $n^\alpha$ should thus roughly be $n^{1/4} \exp(-(1 + o(1))c'n^{p(\alpha)})$, with $p(\alpha) = \min(\alpha, 1 - 2\alpha)$ and $c' > 0$ a constant depending on $\alpha$. Here, $d^n$ comes from the growth constant of Dyck paths. The function $p(\alpha)$ takes its minimum at $\alpha = 1/3$, which maximizes the contribution, leading to the following heuristic guess that the number of relaxed binary trees $r_n$ should satisfy

$$\log \frac{r_n}{n!4^n} \xrightarrow{n \to \infty} -an^{1/3}$$

for some constant $a > 0$. Furthermore, we know that the main contribution should come from horizontally decorated Dyck paths with $i - m_i$ mostly of order $\Theta(n^{1/3})$. Since most such $i$'s should be of order $\Theta(n)$, we can even state the condition above as $x - y = \Theta(y^{1/3})$ for most endpoints $(x, y)$ of horizontal steps. This heuristic is the starting point of our analysis.

### 3.2 Analytic approximation of weighted Dyck meanders

The heuristic in Section 3.1 suggests that the main weight of $d_{n,m}$ comes from the region $m = \Theta(n^{1/3})$. It thus suggests an approximation of $d_{n,m}$ of the form

$$d_{n,m} \sim f(n^{-1/3}(m + 1))h(n)$$

for some functions $f$ and $h$, where we expect $h(n) \approx \frac{2^n}{\rho} n^{1/3}$ for some $\rho$. The idea is that $h(n)$ describes how the total weight for a fixed $n$ grows, and $f(\kappa)$ describes the rescaled weight distribution in the main region $m = \Theta(n^{1/3})$.

Let $s(n)$ be the ratio $\frac{h(n)}{h(n-1)}$. Suppose that $m = n^{1/3}\kappa - 1$, the recurrence relation becomes

$$f(\kappa)s(n) = \frac{n - n^{1/3}\kappa + 1}{n + n^{1/3}\kappa - 1} f \left( (n - 1)^{-1/3}(n^{1/3}\kappa - 2) \right) + f \left( (n - 1)^{-1/3}n^{1/3}\kappa \right).$$

Now, since we expect $h(n) \approx \frac{2^n}{\rho} n^{1/3}$, we postulate that the ratio $s(n)$ to behave as

$$s(n) = 2 + cn^{-2/3} + O(n^{-1})$$

and that $f$ is analytic. Using these assumptions, we can expand (4) as a Puiseux series in $1/n$. Moving all terms to the right-hand side yields

$$0 = ((c + 2k)f(\kappa) - f''(\kappa))n^{-2/3} + O(n^{-1}).$$

Solving the differential equation $(c + 2k)f(\kappa) - f''(\kappa) = 0$ under the condition $f(\kappa) \to 0$ when $\kappa \to \infty$ yields the unique solution (up to multiplication by a constant)

$$f(\kappa) = b\text{Ai} \left( \frac{c + 2\kappa}{2^{2/3}} \right).$$

The condition on the behavior of $f(\kappa)$ near $\infty$ is motivated by the experimental observation that $d_{n,m}$ is quickly decaying for $m$ close to $n$. We also insist that $f(0) = 0$ as $d_{n,-1} = 0$, which implies that $c = 2^{2/3}a_1$ where $a_1 \approx -2.3381$ is the first root of the Airy function $\text{Ai}(x)$. Now, using this conjectural value of $c$, it follows that (ignoring polynomial terms)

$$h(n) \approx 2^n \exp(3a_1(n/2)^{1/3}).$$
This suggests that the number of relaxed trees \( r_n = n!d_{2n,0} \) behaves like
\[
r_n \approx n!4^n \exp(3a_1 n^{1/3}),
\]
which is compatible with what we want to prove.

We observe that (4) can be expanded into a Puiseux series of \( n^{1/3} \) by taking appropriate series expansions of \( f(\kappa) \) and \( s(n) \). Therefore, to refine the analysis above, it is natural to look at expansion of \( s(n) \) in (5) to more subdominant terms, and to postulate a more refined ansatz of \( d_{n,m} \) than (3), probably as a series in \( n^{1/3} \). Indeed, if we take
\[
d_{n,m} \sim \left(f(n^{-1/3}(m + 1)) + n^{-1/3}g(n^{-1/3}(m + 1))\right)h(n)
\]
and
\[
s(n) = 2 + cn^{-2/3} + dn^{-1} + O(n^{-4/3}),
\]
then using the same method we can reach the polynomial part of the asymptotic behavior of \( r_n \) as
\[
r_n \sim \gamma_r n!4^n \exp(3a_1 n^{1/3})n.
\]
Here, \( \gamma_r > 0 \) is some constant. In general, we can postulate
\[
d_{n,m} \approx h(n) \sum_{j=0}^{k} f_j(n^{-1/3}(m + 1))n^{-j/3},
\]
and
\[
s(n) = 2 + \gamma_2 n^{-2/3} + \gamma_3 n^{-1} + \ldots + \gamma_{k+2} n^{-k-2} + o(n^{-k-2}).
\]
The proof of our main result on relaxed binary trees is based on choosing the cutoff appropriately, and using perturbations of that truncation to bound \( r_n \).

### 3.3 Discussion on the constants

One of the first steps in our method for proving the constant terms involves taking the ratio of successive terms \( h(n)/h(n-1) \), or equivalently \( r_n/r_{n-1} \). From the leading asymptotic behaviour of these ratios we can deduce the exact asymptotic form up to the constant term. Unfortunately, however, this method makes it impossible to exactly determine the constant term \( \gamma_r \). In this section we give estimates of the constant terms: we believe that there are constants \( \gamma_c \approx 166.95208957 \) and \( \gamma_c \approx 173.12670485 \) such that
\[
c_n \sim \gamma_c n!4^n e^{3a_1 n^{1/3}} n^{3/4} \quad \text{and} \quad r_n \sim \gamma_r n!4^n e^{3a_1 n^{1/3}} n.
\]

Based on the analysis in Section 3.2, we expect the ratios \( r_n/r_{n-1} \) to behave like
\[
\frac{r_n}{r_{n-1}} = 9 + \sum_{j=0}^{k-1} \beta_j n^{1-j/3} + O(n^{1-k/3}),
\]
for any positive integer \( k \), with the sequence \( \beta_0, \beta_1, \ldots \) beginning with the terms 4, 0, 4\( a_1 \), 4. This is equivalent to the existence of a sequence \( \delta_0, \delta_1, \ldots \) such that \( r_n \) behaves like
\[
r_n = n!4^n \exp(3a_1 n^{1/3})n \left( \sum_{j=0}^{k-1} \delta_j n^{-j/3} + O(n^{-k/3}) \right),
\]

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for any positive integer $k$. In this equation, $\delta_0 = \gamma$ is the constant term that we aim to approximate. A simple way to approximate $\gamma_r$ is to write

$$u_n = \frac{r_n}{n!4^n \exp(3a_1n^{1/3})}. $$

Then the graph of the values of $u_n$ plotted against $n^{-1/3}$ should be roughly linear (see Figure 4), and the point where it crosses the y-axis can be taken as an approximation for $\gamma_r$.

**Figure 4:** Plot of approximations $u_n$ for the constant term $\gamma_r$ of relaxed trees vs. $10n^{-1/3}$.

**Figure 5:** Plot of $\hat{v}_n = v_n - 166.95208957$ vs. $10^{18}n^{-6}$ for $800 \leq n \leq 1000$, where the terms $v_n$ approximate the constant term $\gamma_r$ for relaxed trees.

**Figure 6:** Plot of approximations $u_n$ for the constant term $\gamma_c$ of compacted trees vs. $10n^{-1/3}$.

**Figure 7:** Plot of $\hat{v}_n = v_n - 173.1267048$ vs. $10^{18}n^{-6}$ for $800 \leq n \leq 1000$, where the terms $v_n$ approximate the constant term $\gamma_c$ for compacted trees.
This yields \( \gamma_r \approx 160 \). We get a more precise estimate as follows: Fix \( k \) to be some positive integer. Then, for each \( n \), consider the integers \( m \in [n, n + k] \). For each such \( m \) we expect the equation
\[
u_m \approx \sum_{j=0}^{k-1} \delta_j m^{-j/3}
\]
to be approximately true. We then solve this system of equations for \( \delta_0, \ldots, \delta_{k-1} \) as though the equations were exact, using known, exact values of \( u_m \). This yields approximations for \( \delta_0, \ldots, \delta_{k-1} \). Denote the approximation thus obtained for \( \gamma_r = \delta_0 \) by \( v_n \). Note that this is equivalent to writing \( v_n \) as a weighted sum of the numbers \( u_m \), which cancels the terms \( n^{-j/3} \) for \( 1 \leq j < k \). For example, if \( k = 2 \) then \( v_n = ((n + 1)^{1/3} u_n - n^{1/3} u_{n+1})/((n + 1)^{1/3} - n^{1/3}) \). Hence, if our assumptions are correct then \( v_n = \gamma_r + O(n^{-k/3}) \). Taking \( k = 18 \) and plotting \( v_n \) against \( 10^{18} n^{-6} \) as in Figure 5 yields the approximation \( \gamma_r \approx 166.95208957 \), where we expect the quoted digits to be correct. In Figures 6 and 7 we show a similar analysis of the counting sequence for compacted trees, yielding the approximation \( \gamma_c \approx 173.12670485 \).

## 4 Inductive proof of upper and lower bounds

Suppose that \( \{s_n\}_{n \geq 1} \) and \( \{X_{n,m}\}_{n \geq m \geq 0} \) are sequences of non-negative real numbers satisfying
\[
X_{n,m} s_n \leq \frac{n - m + 2}{n + m} X_{n-1,m-1} + X_{n-1,m+1}
\]
for all sufficiently large \( n \) and all \( m \in [0, n] \). We define the sequence \( \{h_n\}_{n \geq 0} \) by \( h_0 = 1 \) and \( h_n = s_n h_{n-1} \). By induction on \( n \), for some constant \( b_0 \), the following inequality holds for all sufficiently large \( n \) and all \( m \geq 0 \):
\[
X_{n,m} h_n \leq \left( \begin{array}{c}
\frac{n - m + 2}{n + m} X_{n-1,m-1} h_{n-1} + X_{n-1,m+1} h_{n-1} \\
\text{IS} \\
\text{(2)}
\end{array} \right)
\]
\[
\leq \frac{n - m + 2}{n + m} b_0 d_{n-1,m-1} + b_0 d_{n-1,m+1}
\]
\[
\leq b_0 d_{n,m}.
\]
Here (IS) marks the “Induction Step”. Similarly, if we can show the opposite of (7), it will imply that
\[
X_{n,m} h_n \geq b_1 \cdot d_{n,m}
\]
for all sufficiently large \( n \) and all \( m \in [0, n] \).

Comparing to the heuristic analysis in Section 3.2, we see that \( X_{n,m} \) acts as the function \( f(\kappa) \), and \( s_n \) as \( s(n) \). Therefore, we should expect \( X_{n,m} \) to be close to (6), and \( s_n \) to be a slight deviation of (5).

In Lemma 4.2 we will prove that certain explicit \( X_{n,m} \) and \( s_n \) satisfy (7), which will lead to a lower bound on the numbers \( d_{n,m} \). Similarly, in Lemma 4.4, we will show that another explicit \( X_{n,m} \) and \( s_n \) satisfy the opposite of (7), which therefore yields an upper bound on the numbers \( d_{n,m} \). Together, these two bounds determine the exact asymptotic form of the numbers \( d_{n,0} \) up to the constant term.
To prove these bounds with the explicit expressions of $X_{n,m}$ and $s_n$, we will consider the difference between the left and the right-hand side of (7). To prove that this difference is positive, we start by expanding the involved Airy function and its derivative in the neighborhood of an appropriate point $\alpha$, leading to a sum of the form

$$p_{n,m} \text{Ai}(\alpha) + p'_{n,m} \text{Ai}'(\alpha),$$

where $p_{n,m}$ and $p'_{n,m}$ can be expressed as Puiseux series in $n$ with fractional polynomials in $m$ as coefficients. By looking at the “Newton polygon” of these Puiseux series, we can pick out the dominant term at different regimes of $n$ and $m$, leading to a proof of (7) (or the reverse direction).

![Figure 8: (Left) The Airy function $\text{Ai}(a_1 + z)$, (Centre) its derivative $\text{Ai}'(a_1 + z)$, and (Right) the quotient $z \frac{\text{Ai}'(a_1 + z)}{\text{Ai}(a_1 + z)}$ on the positive real line.](image)

The following Lemma summarizes some elementary results on the relation between the Airy function $\text{Ai}$ and its derivative $\text{Ai}'$. We will use these results in Lemmas 4.2 and 4.4 to bound the subsequently defined auxiliary sequence $\tilde{X}_{n,m}$.

**Lemma 4.1.** The functions

$$\Phi(x) = x \frac{\text{Ai}'(a_1 + x)}{\text{Ai}(a_1 + x)}, \quad \Psi(x) = \frac{\text{Ai}'(a_1 + x)}{\text{Ai}(a_1 + x)}$$

are infinitely differentiable and monotonically decreasing on $x > 0$ with $\Phi(0) = 1$.

**Proof.** First, by l’Hospital’s rule it is easy to see that $\Phi(0) = 1$. Second, as $a_1$ is the largest root of the Airy function, $\Phi(x)$ and $\Psi(x)$ are infinitely differentiable as compositions of differentiable functions. It remains to prove the monotonicity. A local expansion at $x = 0$ shows that the functions are initially decreasing. The same holds for large $x$ due to the approximation $\text{Ai}(x) \sim \exp\left(-\frac{2}{3}x^{3/2}\right) / 2\sqrt{\pi x^{1/4}}$, see [1, Equation 10.5.49] giving

$$\Psi(x) \sim -\sqrt{a_1 + x},$$

for $x \to \infty$. We will show that $\Phi'(x)$ and $\Psi'(x)$ are always negative for $x > 0$. Note that $\Phi(x)$ and $\Psi(x)$ will change sign only once at $x_0 \approx 0.91$. 

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We present the following argument for the monotonicity of $\Phi(x)$. Assume that there exists an $x_+$ such that $\Phi'(x_+) > 0$. Then, as $\Phi(x)$ is initially and finally decreasing, there must exist $y_1 < x_+ < y_2$ such that $\Phi'(y_1) = \Phi'(y_2) = 0$ and $\Phi''(y_1) \geq 0 \geq \Phi''(y_2)$.

The second derivatives are equal to

$$\Phi''(x) = 2a_1 + 3x - \frac{2}{x} \Phi(x) \Phi'(x).$$

These lead to the contradictions $y_1 \geq 0 \geq y_2$.

The argument for the monotonicity of $\Psi(x)$ is analogous, except that the second derivative is now

$$\Psi''(x) = 1 - 2\Psi(x) \Psi'(x),$$

leading to the contradiction $\Psi''(y_1) = \Psi''(y_2) = 1$.

4.1 Lower bound

**Lemma 4.2.** For all $n, m \geq 0$ let

$$\tilde{X}_{n,m} := \left(1 - \frac{2m^2}{3n} + \frac{m}{2n}\right) \text{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right) \quad \text{and}$$

$$\tilde{s}_n := 2 + \frac{2^{2/3}a_1}{n^{2/3}} + \frac{8}{3n} - \frac{1}{n^{7/6}}.$$

Then, for any $\varepsilon > 0$, there exists an $\tilde{n}_0$ such that

$$\tilde{X}_{n,m} \tilde{s}_n \leq \frac{n - m + 2}{n + m} \tilde{X}_{n-1,m-1} + \tilde{X}_{n-1,m+1}$$

for all $n \geq \tilde{n}_0$ and for all $0 \leq m < n^{2/3-\varepsilon}$.

**Proof.** First, define the following sequence

$$P_{n,m} := -X_{n,m}s_n + \frac{n - m + 2}{n + m} X_{n-1,m-1} + X_{n-1,m+1},$$

where

$$s_n := \sigma_0 + \frac{\sigma_1}{n^{1/3}} + \frac{\sigma_2}{n^{2/3}} + \frac{\sigma_3}{n} + \frac{\sigma_4}{n^{7/6}}, \quad X_{n,m} := \left(1 + \frac{\tau_2m^2 + \tau_1m}{n}\right) \text{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right)$$

with $\sigma_i, \tau_j \in \mathbb{R}$.

Then, the inequality is equivalent to $P_{n,m} \geq 0$ with $\sigma_0 = 2$, $\sigma_1 = 0$, $\sigma_2 = 2^{2/3}a_1$, $\sigma_3 = 8/3$, and $\sigma_4 = -1$ as well as $\tau_0 = 0$, $\tau_1 = 1/2$, and $\tau_2 = -2/3$. Next, we expand $\text{Ai}(z)$ in a neighborhood of

$$\alpha = a_1 + \frac{2^{1/3}m}{n^{1/3}},$$

and we get

$$P_{n,m} = p_{n,m} \text{Ai}(\alpha) + p'_{n,m} \text{Ai}'(\alpha),$$
where \( p_{n,m} \) and \( p'_{n,m} \) are functions of \( m \) and \( n^{-1} \) and may be expanded as power series in \( m \) and \( n^{-1/6} \). As long as \( 1 < n \) and \( m < n \), this series converges absolutely because the Airy function is entire and so all functions expanded are analytic in the region defined by \( 1 < |n| > |m| \).

As a first step we compute the possible range of the powers in \( m \) and \( n \). We will start by showing that \( [m' n^j] P_{n,m} = 0 \) for \( i + j > 1 \), \( i, j \in \mathbb{Q} \). The expansions of the three involved Airy functions only give terms of the form \( \mathcal{O}(m^j n^{-j} (n^{-1/3})^k) \) \( \text{Ai}^{(k)}(\alpha) \), with \( j, k \geq 0 \). Due to the differential equation \( \text{Ai}'(\alpha) = \alpha \text{Ai}(\alpha) \), the term \( \text{Ai}^{(k)}(\alpha) \) takes the form \( \mathcal{O}(\alpha^{1/2}) \text{Ai}(\alpha) + \mathcal{O}(\alpha^{i(k-1/2)}) \text{Ai}'(\alpha) \). Hence, all terms in the expansion of the Airy function are of the form \( \mathcal{O}(m^j n^{-j}) \text{Ai}(\alpha) \) or \( \mathcal{O}(m^j n^{-j-1/3}) \text{Ai}'(\alpha) \) for some \( j \geq 0 \). Due to the factor \( m^2 n^{-1} \) in the definition of \( X_{n,m} \), this implies that \( [m' n^j] P_{n,m} = 0 \) for \( i + j > 1 \). Additionally, it also implies that the coefficients of \( \text{Ai}'(\alpha) \) are equal to 0 for \( i + j > 2/3 \).

Next, we strengthen this result by choosing suitable values \( a, b, c, d \) and \( e \) in the definition of \( s_n \) in order to eliminate more initial coefficients. Then, we will show that the remaining terms satisfy \( P_{n,m} \geq 0 \). We performed this tedious task in Maple. The results are summarized in Figures 9 where the initial non-zero coefficients are shown. A diamond at \( (i,j) \) is drawn if and only if the coefficient \( [m' n^j] P_{n,m} \) is non-zero. It is an empty diamond if a suitable choice of \( \sigma_i \) or \( \tau_j \) makes it disappear, whereas it is a solid diamond if it remains non-zero.

The convex hull is given by the three lines

\[
L_1 : j = \frac{7}{6} - \frac{7i}{18}, \\
L_2 : j = \frac{1}{3} - \frac{2i}{3}, \\
L_3 : j = 1 - i.
\]

Next, we distinguish between the contributions arising from \( p_{n,m} \) and \( p'_{n,m} \). The non-zero coefficients are shown in Figure 10. The expansions for \( n \) tending to infinity start as follows, where the elements on the convex hull are written in colour.

\[
P_{n,m} = \text{Ai}(\alpha) \left( \frac{-\frac{\sigma_4}{n^7/6} - \frac{25/3 a_1 m}{3 n^{5/3}} - \frac{41 m^2}{9 n^2} - \frac{28/3 a_1 m^3}{3 n^{8/3}} - \frac{34 m^4}{9 n^3} - \frac{62 m^5}{135 n^4} + \ldots}{n^{7/6}} \right) + \text{Ai}'(\alpha) \left( \frac{2^{1/3}(2\tau_1 - 1)}{n^{4/3}} + \frac{2^{1/3}}{3 n^{2}} - \frac{8 a_1 m}{9 n^2} + \frac{2^{1/3} (24 \tau_1 - 31) m^2}{9 n^{7/3}} - \frac{2^{13/3} m^3}{9 n^{7/3}} - \frac{5}{9 n^{10/3}} + \ldots \right).
\]

We now choose \( \sigma_4 = -1 \) which leads to a positive term \( \text{Ai}(\alpha)n^{-7/6} \) and \( \tau_1 = 1/2 \) which eliminates the \( \text{Ai}'(\alpha)n^{-1/3} \) term. Next, for fixed (large) \( n \) we prove that for all \( m \) the dominant contributions in \( P_{n,m} \) are positive. Therefore, we consider three different regimes. Let \( x_0 \) be the unique positive root of \( \Psi(x) \) from Lemma 4.1.

1. Consider the range of small values of \( m \) given by \( m \leq x_0(n/2)^{1/3} \). In this range \( \text{Ai}(\alpha) \) and \( \text{Ai}'(\alpha) \) are both positive. Moreover, the (red) coefficients of \( \text{Ai}(\alpha) \) are dominated by \( n^{-7/6} \) for large \( n \), while the (blue) coefficients of \( \text{Ai}'(\alpha) \) apart from the term \( \nu = -\frac{2^{13/3} m^3}{9 n^{7/3}} \text{Ai}'(\alpha) \) are dominated by \( 2^{1/3} n^{-3/2} \). By Lemma 4.1 we have

\[
\frac{2^{1/3} m}{n^{1/3}} \text{Ai}'(\alpha) - \text{Ai}(\alpha) < 0.
\]
Figure 9: (Left) Non-zero coefficients of $P_{n,m} = \sum a_{i,j} m^i n^j$ shown by diamonds for $s_n := \sigma_0 + \sigma_1 x_0^{1/3} + \sigma_2 x_0^{2/3} + \sigma_3 x_0^{7/18}$ and $X_{n,m} := (1 + \tau_2 m^2 + \tau_1 m) \text{Ai} \left( a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}} \right)$. There are no terms in the blue dashed area. The blue terms vanish for $\sigma_0 = 2$, the red terms vanish for $\sigma_1 = 0$, the green terms vanish for $\sigma_2 = 2^{2/3} a_1$, and the yellow term vanishes for $\sigma_3 = 8/3$ and $\tau_2 = -2/3$. The black lines represent the two parts $L_1$ and $L_2$ of the convex hull. (Right) The solid gray diamonds are decomposed into the coefficients $p_{m,n}$ of $\text{Ai}(\alpha)$ (red boxes) and $p'_{m,n}$ of $\text{Ai}'(\alpha)$ (blue diamonds).

Hence, $\nu > -\frac{16m^2}{9n^2} \text{Ai}(\alpha)$, and it can therefore be treated as if it belonged to the coefficients of $\text{Ai}(\alpha)$. Thus, as the dominating terms are positive, there is some $N_0$ such that $P_{n,m} > 0$ whenever $n > N_0$ and $m \leq x_0(n/2)^{1/3}$.

2. Next, consider the central range $x_0(n/2)^{1/3} < m \leq n^{7/18}$. Here, we have $\text{Ai}'(\alpha) < 0$. On the one hand, as seen in the left part of Figure 10, the (red) coefficients of $\text{Ai}(\alpha)$ are still dominated by $n^{-7/6}$ (which holds up to $m = \Theta(n^{5/12})$). On the other hand, in this range the term $\nu = -\frac{2^{2/3} m^3}{9n^3} \text{Ai}'(\alpha)$ dominates all other (blue) coefficients of $\text{Ai}'(\alpha)$. Since $\nu > 0$ in this range, this implies that there is some (sufficiently large) $N_1$ such that $P_{n,m} > 0$ whenever $n > N_1$ and $x_0(n/2)^{1/3} < m \leq n^{7/18}$.

3. Finally, consider the range of large values $n^{7/18} < m < n^{2/3-\epsilon}$. By the reasoning on $\Psi(x)$ in Lemma 4.1 we see that $-\text{Ai}'(\alpha) > \text{Ai}(\alpha) > 0$. Therefore the (blue) term $\nu$ dominates all of the (red) terms of $\text{Ai}(\alpha)$ as well as all other (blue) terms of $\text{Ai}'(\alpha)$. Hence there is some $N_2$ such that $P_{n,m} > 0$ whenever $n > N_2$ and $n^{7/18} < m < n^{2/3-\epsilon}$.

Choosing $\tilde{n}_0 = \max\{N_0, N_1, N_2\}$ completes the proof. \qed

Remark 4.3. The previous result could be strengthened to hold up to $m \leq n^{1-\epsilon}$ by (9) as will be shown in the proof of Lemma 4.4. However, we will not need this result in the sequel.

Now to complete the lower bound we define the sequence $X_{n,m}$ by

$$X_{n,m} := \begin{cases} \tilde{X}_{n,m}, & \text{if } 2m^2 < 3n, \\ 0, & \text{if } 2m^2 \geq 3n. \end{cases}$$
Equivalently, \( X_{n,m} = \max\{\tilde{X}_{n,m}, 0\} \). Then, for \( m < \sqrt{3n/2} \) we have
\[
X_{n,m} \tilde{s}_n \leq \frac{n - m + 2}{n + m} \tilde{X}_{n-1,m-1} + \tilde{X}_{n-1,m+1} \leq \frac{n - m + 2}{n + m} X_{n-1,m-1} + X_{n-1,m+1},
\]
by Lemma 4.2. For \( n \geq m \geq \sqrt{3n/2} \) we have
\[
X_{n,m} \tilde{s}_n = 0 \leq \frac{n - m + 2}{n + m} X_{n-1,m-1} + X_{n-1,m+1}.
\]

Writing \( \tilde{h}_n = \tilde{s}_n \tilde{h}_{n-1} \) it then follows by induction that \( d_{n,m} \geq b \tilde{h}_n X_{n,m} \) for some constant \( b > 0 \), all sufficiently large \( n \) and all \( m \in [0,n] \). In particular, it follows that the number \( r_n = n!d_{2n,0} \) of relaxed trees of size \( n \) is bounded below by
\[
r_n \geq \gamma n!4^n e^{3a_1 n^{1/3}}/n,
\]
for some constant \( \gamma > 0 \). In the next section we will show an upper bound with the same asymptotic form, but with a different constant \( \gamma \).

### 4.2 Upper bound

Next, we consider a similar auxiliary sequence \( \tilde{X}_{n,m} \) which will give rise to an upper bound on the number of relaxed binary trees.

**Lemma 4.4.** Choose \( \eta > 2/9 \) fixed and for all \( n, m \geq 0 \) let
\[
\tilde{X}_{n,m} := \left( 1 - \frac{2m^2}{3n} + \frac{m}{2n} + \eta \frac{m^4}{n^2} \right) \text{Ai} \left( a_1 + \frac{2^{1/3}(m + 1)}{n^{1/3}} \right) \quad \text{and}
\]
\[
\tilde{s}_n := 2 + \frac{2^{2/3}a_1}{n^{2/3}} + \frac{8}{3n} + \frac{1}{n^{7/6}}.
\]
Then, for any \( \varepsilon > 0 \), there exists a constant \( \tilde{n}_0 \) such that
\[
\tilde{X}_{n,m} \tilde{s}_n \geq \frac{n - m + 2}{n + m} \tilde{X}_{n-1,m-1} + \tilde{X}_{n-1,m+1},
\]
for all \( n \geq \tilde{n}_0 \) and all \( 0 \leq m < n^{1-\varepsilon} \).
Proof. The proof follows the same lines as the one of Lemma 4.2. Therefore we only focus on
the small modifications. As a first step we define the following sequence
\[ Q_{n,m} := \hat{X}_{n,m} \hat{s}_n - \frac{n - m + 2}{n + m} \hat{X}_{n-1,m-1} - \hat{X}_{n-1,m+1}. \]
Then the inequality is equivalent to \( Q_{n,m} \geq 0 \). Again, we expand \( \text{Ai}(z) \) in a neighborhood of \( \alpha = a_1 + \frac{2^{1/3} n}{m^{1/3}} \), and we get
\[ Q_{n,m} = q_{n,m} \text{Ai}(\alpha) + q'_{n,m} \text{Ai}'(\alpha), \]
where \( q_{n,m} \) and \( q'_{n,m} \) are functions of \( m \) and \( n^{-1} \) and may again be expanded as power series
in \( m \) and \( n^{-1/6} \). In this case it is easy to see that \( |m^n| Q_{n,m} = 0 \) for \( i + j > 2 \). The shift
by one compared to the lower bound arises due to the factor \( n^{-2} \). The initial non-zero
coefficients are shown in Figure 11. The four lines (black, red, green, blue) of the convex hull are
\[ \hat{L}_1 : j = -\frac{7}{6} - \frac{7i}{18}, \]
\[ \hat{L}_2 : j = -\frac{5}{6} - \frac{i}{2}, \]
\[ \hat{L}_3 : j = -\frac{2i}{3}, \]
\[ \hat{L}_4 : j = 2 - i. \]

Next, we distinguish between the contributions arising from \( q_{n,m} \) and \( q'_{n,m} \). The non-zero
coefficients are shown in Figure 12. The expansions for \( n \) tending to infinity start as follows,
Figure 12: Non-zero coefficients $q_{k,l}$ (red) and $q'_{k,\ell}$ (blue) of the expansion for $Q_{n,m}$. The coefficient of $n^{-4/3}$ in the right picture depicted as a solid blue circle disappears for $\tau_1 = 1/2$, where the elements on the convex hull are written in color.

$$Q_{n,m} = \text{Ai}(\alpha) \left( \frac{\sigma_4}{n^{7/6}} + \frac{25/3 a_1 m}{3 n^{3/3}} + \frac{m^2 (41 - 108 \eta)}{9 n^2} + \frac{28/3 a_1 m^3 (1 - 6 \eta)}{3 n^{8/3}} + \frac{2m^4 (17 - 132 \eta)}{9 n^3} \right.$$

$$- \frac{25/3 a_1 m^5 \eta}{n^{11/3}} - \frac{17m^6 \eta}{3 n^4} - \frac{31m^7 \eta}{45 n^5} + \ldots + \left. \frac{\text{Ai}'(\alpha)}{n^{4/3}} \left( \frac{2^{1/3}(1 - 2 \tau_1)}{n^{1/3}} + \frac{2^{1/3}}{n^{3/2}} + \frac{8a_1 m}{9 n^2} + \frac{2^{1/3}m^2 (31 - 108 \eta - 24 \tau_1)}{9 n^{7/3}} + \frac{2^{10/3}m^3 (2 - 9 \eta)}{9 n^{7/3}} \right.$$

$$+ \frac{5m^4 2^{1/3}(2 - 27 \eta)}{9 n^{10/3}} - \frac{2^{10/3}m^5 \eta}{3 n^{10/3}} - \frac{5m^6 2^{1/3} \eta}{3 n^{13/3}} - \frac{89m^7 2^{1/3} \eta}{45 n^{16/3}} + \ldots \right) \right).$$

Let $x_0$ be again the unique positive root of $\Psi(x)$ from Lemma 4.1. In this case we consider the following four regions:

1. $m \leq x_0(n/2)^{1/3}$,
2. $x_0(n/2)^{1/3} < m \leq n^{7/18}$,
3. $n^{7/18} < m \leq n^{1/2}$,
4. $n^{1/2} < m \leq n^{1-\varepsilon}$.

The treatment of the first 3 cases is analogous to the lower bound except for 2 differences. First, in the second and third regime we need an additional variable $\eta$ to make the dominant term $-2^{10/3}m^{9/3} \eta^2 \text{Ai}'(\alpha)$ positive. Second, in the third regime an additional dominant term $-2^{10/3}m^{7/3} \eta^2 \text{Ai}'(\alpha)$ appears for $m = \Theta(n^{1/2})$ which is positive anyway. Then, in the fourth regime for $m = \Theta(n^{1/2+\varepsilon})$ only this factor dominates everything. However, for $m = \Theta(n^{2/3})$ the term $-17m^6 \eta \text{Ai}(z)$ seems to be of the same order of magnitude and is negative. Yet, due to (9) we know that for $m/n^{1/3}$ tending to infinity, $\text{Ai}'(\alpha) \sim -2^{1/6}m^{1/2} \pi^{1/6} \text{Ai}(\alpha)$. Hence, the blue term $-2^{10/3}m^{7/3} \eta^2 \text{Ai}'(\alpha)$ continues to dominate up to $m = n^{1-\varepsilon}$. \qed
To finish the proof of the upper bound, we will choose some constant $N > 0$ and define a sequence $\tilde{d}_{n,m}$ by the same rules as $d_{n,m}$ except that $\tilde{d}_{n,m} = 0$ whenever $m > n^{3/4}$ and $n > N$. Then, writing $\hat{h}_n = s_n \hat{h}_{n-1}$, we can use the lemma above to show by induction that the numbers $\tilde{d}_{n,m}$ satisfy the inequality

$$b_0 \tilde{d}_{n,m} \leq \hat{h}_n X_{n,m},$$

for some constant $b_0$ and all sufficiently large $n$; compare (8). In particular, the numbers $\tilde{d}_{2n,0}$ are bounded above by

$$\tilde{d}_{2n,0} \leq \gamma 4^n e^{3a_1 n^{1/3}}$$

for some constant $\gamma > 0$. The rest of this section is dedicated to proving that there is some choice of $N$ such that $\tilde{d}_{2n,0} \geq d_{2n,0}/2$ for all $n$.

In order to finish our proof of the upper bound for the weighted number of Dyck paths $d_{2n,0}$, it will be useful to have an upper bound on the number of these paths which pass through a certain point $(2x, 2y)$ as a proportion of the total weighted number of paths. Let $p_{\ell,m,2n}$ denote the weighted number of paths from $(\ell, m)$ to $(2n, 0)$; see Figure 13. Then the proportion $s_{x,y,n}$ of the $d_{2n,0}$ weighted Dyck paths that pass through $(2x, 2y)$ is

$$s_{x,y,n} = \frac{d_{2x,2y} p_{2x,2y,2n}}{d_{2n,0}}.$$

The following lemma yields an upper bound on the number $p_{2x,2y,2n}$.

**Figure 13:** Proportion of meanders of length $2n$ passing through the point $(2x, 2y)$ showing one example path contributing to $s_{x,y,n}$.

**Lemma 4.5.** The numbers $p_{\ell,m,2n}$ satisfy the inequality

$$\frac{p_{\ell,j,2n}}{j+1} \geq \frac{p_{\ell,k,2n}}{k+1},$$

for integers $0 \leq j < k \leq \ell \leq 2n$ satisfying $2|k - j$.

**Proof.** First we note that the numbers $p_{\ell,m,2n}$ are determined by the recurrence relation

$$p_{\ell,m,2n} = p_{\ell+1,m-1,2n} + \frac{\ell - m + 2}{\ell + m + 2} p_{\ell+1,m+1,2n}.$$
along with the initial conditions $p_{2n,m,2n} = \delta_{m,0}$ and $p_{l,-1,2n} = 0$. We will now prove the statement of the lemma by reverse induction on $\ell$. Our base case is $\ell = 2n$, for which the inequality clearly holds. For the inductive step, we assume that the inequality holds for $\ell + 1$ and all $m$, and we will prove that it holds for $\ell$. It suffices to prove that for $m \geq 1$ the following inequality holds
\[
\frac{p_{\ell,m-1,2n}}{m} - \frac{p_{\ell,m+1,2n}}{m + 2} \geq 0.
\]
Let $D$ denote the left-hand side of this inequality. Using the recurrence relation we can rewrite $D$ as
\[
D = \frac{p_{\ell+1,m-2,2n}}{m} + \frac{(\ell - m + 3)p_{\ell+1,m,2n}}{(\ell + m + 1)m} - \frac{p_{\ell+1,m+2,2n}}{m + 2} - \frac{(\ell - m + 1)p_{\ell+1,m+2,2n}}{(\ell + m + 3)(m + 2)}.
\]
Now, by the inductive assumption, $p_{\ell+1,m+2,2n} \leq \frac{m+3}{m+1}p_{\ell+1,m,2n}$ and $p_{\ell+1,m-2,2n} \geq \frac{m-1}{m+1}p_{\ell+1,m,2n}$, where the latter inequality even holds for $m = 1$ as then both sides are 0. It follows that
\[
D \geq \frac{(m-1)p_{\ell+1,m,2n}}{(m+1)m} + \frac{(\ell - m + 3)p_{\ell+1,m,2n}}{(\ell + m + 1)m} - \frac{p_{\ell+1,m,2n}}{m + 2} - \frac{(m+3)(\ell - m + 1)p_{\ell+1,m,2n}}{(m+1)(\ell + m + 3)(m + 2)}
\]
\[
= \frac{4(3 + m + \ell + m\ell)p_{\ell+1,m,2n}}{m(m+2)(1+m+\ell)(3+m+\ell)} \geq 0
\]
as desired. This completes the induction, which proves the inequality for $\ell \in [0, 2n]$. \hfill \Box

In particular, it follows from this lemma that $p_{2x,2y,2n} \leq (2y + 1)p_{2x,0,2n}$.

Moreover, note that the proportion $s_{x,0,n}$ of weighted paths passing through $(2x, 0)$ satisfies $s_{x,0,n} \leq 1$. Hence the proportion $s_{x,y,n}$ satisfies
\[
s_{x,y,n} = \frac{p_{2x,2y,2n}d_{2x,2y}}{d_{2n,0}} \leq \frac{(2y + 1)p_{2x,0,2n}d_{2x,2y}}{d_{2n,0}s_{x,0,n}} = \frac{(2y + 1)d_{2x,2y}}{d_{2x,0}}.
\]
From the lower bound at the end of Section 4.1 we have
\[
d_{2x,0} \geq 4x^3e^{3a_1x^{1/3}},
\]
so we now desire an upper bound for $d_{2x,2y}$. It will suffice to use the upper bound
\[
d_{2x,2y} \leq \frac{2x}{x+y},
\]
which holds because the right-hand side is the number of (unweighted) paths from $(0,0)$ to $(2x,2y)$. We are now ready to prove the following lemma

**Lemma 4.6.** There exists a constant $N > 0$ with the following property: Recall that $d_{n,m}$ is the weighted number of paths ending at $(n,m)$. Let $\tilde{d}_{n,m}$ be the number of these paths such that no intermediate point $(2x,2y)$ on the path satisfies $x > N$ and $y > x^{3/4}$. Then $d_{2n,0} \leq 2\tilde{d}_{2n,0}$ for all $n > 0$. 

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Proof. We can rewrite the desired inequality as

\[ 1 - \frac{\tilde{d}_{2n,0}}{d_{2n,0}} \leq \frac{1}{2}. \]

Note that the left-hand side is equal to the proportion of weighted paths with at least one intermediate point \((2x, 2y)\) satisfying \(x > N\) and \(y > x^{3/4}\). The proportion \(s_{x,y,n}\) of weighted paths which go through any one of these points \((2x, 2y)\) is bounded above by

\[ s_{x,y,n} \leq \frac{(2y+1)}{4^x e^{3a_1 x^{1/3}}} \left( \frac{2x}{x+y} \right) \leq 4^{-x} e^{\frac{3a_1 x^{1/3}}{3}} x^{-1} \frac{(2y+1) \Gamma(2x+1)}{\Gamma(x+x^{3/4}+1) \Gamma(x-x^{3/4}+1)} . \]

The right-hand side of this inequality behaves like

\[ \Theta \left( e^{-x^{1/2} + O(x^{1/3})} \right) \]  

for large \(x\). Hence, there is some constant \(c\) such that

\[ s_{x,y,n} \leq c \cdot 2^{-x^{1/2}} \]

for all \(x, y, n\) satisfying \(y > x^{3/4}\).

Now, the proportion \(1 - \frac{\tilde{d}_{2n,0}}{d_{2n,0}}\) of weighted paths passing through at least one point \((2x, 2y)\) is no greater than the sum of the proportions of paths going through each such point. Hence

\[ 1 - \frac{\tilde{d}_{2n,0}}{d_{2n,0}} \leq \sum_{x=N+1}^{\infty} \sum_{x \leq y > x^{3/4}} s_{x,y,n} \leq \sum_{x=N+1}^{\infty} \sum_{x \leq y > x^{3/4}} c \cdot 2^{-x^{1/2}} < \sum_{x=N+1}^{\infty} c x \cdot 2^{-x^{1/2}} . \]

Since the sum on the right converges, it converges to a value less than 1/2 for sufficiently large \(N\). This completes the proof of the lemma.

Remark 4.7. Choosing \(y > x^{2/3}\) instead of \(y > x^{2/3}\) one can show that (13) behaves like \(O(\frac{3a_1 x^{1/3}}{x^{2/3}})\). Hence, any \(\beta > 2/3\) gives the same result, yet \(\beta = 2/3\) is not sufficient.

Finally, we define \(\tilde{d}_{n,m}\) as in the above lemma. Then it follows from Lemma 4.4 that there is some constant \(\gamma' > 0\) such that

\[ \tilde{d}_{2n,0} \leq \gamma' 4^n e^{3a_1 n^{1/3}} \]

for all \(n\). Hence

\[ r_n = n! \frac{\tilde{d}_{2n,0}}{d_{2n,0}} \leq 2\gamma' n! 4^n e^{3a_1 n^{1/3}} , \]

completing the proof of the upper bound. We have now proven upper and lower bounds for the number \(r_n\) of relaxed trees, which differ only in the constant term. Therefore,

\[ r_n = \Theta \left( n! 4^n e^{3a_1 n^{1/3}} \right) . \]
5 A stretched exponential for compacted binary trees

We start by transforming the terms $c_{n,k}$ counting compacted trees to a sequence $e_{n,m}$ using the equation

$$e_{n,m} = \frac{1}{((n+m)/2)!} \left( c_{(n+m)/2,(n-m)/2} - \frac{n-m+2}{2} c_{(n+m-2)/2,(n-m)/2} \right).$$

Then the terms $e_{n,m}$ are determined by the recurrence

$$
\begin{align*}
    e_{n,m} &= \frac{n-m+2}{n+m} e_{n-1,m-1} + \frac{n-m-2}{n-m} e_{n-1,m+1} + \frac{n-m-4}{n-m-2} \left( e_{n-2,m+2} + \frac{2}{n+m} e_{n-3,m+1} \right), \\
    e_{0,m} &= 0, \\
    e_{n,-1} &= 0, \\
    e_{0,0} &= 1.
\end{align*}
$$

(14)

and the number of compacted trees of size $n$ is $n! e_{2n,0}$.

We are unable to directly apply the same method to this recurrence as there is a negative term on the right-hand side. We solve this problem using the following lemma:

**Lemma 5.1.** The recurrence $e_{n,m}$ for compacted binary trees satisfies the following bounds for all $n \geq 3$ and $m \geq 2$:

$$
\frac{n-m+2}{n+m} e_{n-1,m-1} + \frac{n-m-2}{n-m} e_{n-1,m+1} + \frac{n-m-4}{n-m-2} \left( e_{n-2,m+2} + \frac{2}{n+m} e_{n-3,m+1} \right) \\
\leq e_{n,m} \leq \frac{n-m+2}{n+m} e_{n-1,m-1} + \frac{n-m-2}{n-m} e_{n-1,m+1} + \frac{2}{n-m} e_{n-2,m+2} + \frac{2}{n+m} e_{n-3,m+1}.
$$

**Proof.** We start by proving the upper bound. In order to prove this we will compute successively stronger upper bounds. We start with the trivial upper bound

$$e_{n,m} \leq \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1}.$$

Applying this bound to $e_{n-1,m+1}$ then $e_{n-2,m}$ we find that

$$e_{n-1,m+1} \leq \frac{n-m}{n+m} \left( e_{n-3,m+1} + e_{n-3,m+1} \right) + e_{n-2,m+2}.$$

Adding $2/(n-m)$ times this inequality to the original equation yields

$$e_{n,m} \leq \frac{n-m+2}{n+m} e_{n-1,m-1} + \frac{n-m-2}{n-m} e_{n-1,m+1} + \frac{2}{n-m} e_{n-2,m+2} + \frac{2}{n+m} e_{n-3,m+1}.$$

For the lower bound, we start with the inequality

$$e_{n,m} \geq \frac{n-m+2}{n+m} e_{n-1,m-1}.$$

This is clear for $m \in \{0, 1\}$, and for $m \geq n$ it is an equality. We can then deduce this inequality for all $n, m$ using induction: Assume that the statement is true for all $n < N$ and all $m \in [0, n]$. Then for $m \in [2, n-2]$ and $n = N$,

$$\frac{1}{n-m} e_{n-1,m+1} \geq \frac{1}{n+m} e_{n-2,m} \geq \frac{n-m}{(n+m)(n-m-2)} e_{n-3,m+1}.$$
Hence,
\[ e_{n,m} = \frac{n - m + 2}{n + m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{2(n - m)}{(n + m)(n + m - 2)} e_{n-3,m-1} \]
\[ \geq \frac{n - m + 2}{n + m} e_{n-1,m-1} + \left(1 - \frac{2}{n - m}\right) e_{n-1,m+1}. \]
\[ \geq \frac{n - m + 2}{n + m} e_{n-1,m-1}. \]

This completes the induction. Moreover, it shows that 
\[ e_{n,m} \geq \frac{n - m + 2}{n + m} e_{n-1,m-1} + \left(1 - \frac{2}{n - m}\right) e_{n-1,m+1}, \]
for \( m \in [2, n - 2] \). It is easy to see that this also holds for \( m \in \{0, 1\} \) and \( m \geq n \). Applying this inequality to \( e_{n-1,m+1} \) then \( e_{n-2,m} \) yields
\[ \frac{1}{n - m} e_{n-1,m+1} \geq \frac{1}{n + m} e_{n-2,m} + \frac{n - m - 4}{(n - m)(n - m - 2)} e_{n-2,m+2} \]
\[ \geq \frac{1}{n + m} \left(\frac{n - m}{n + m - 2} e_{n-3,m-1} + \frac{n - m - 4}{n - m - 2} e_{n-3,m+1}\right) + \frac{n - m - 4}{(n - m)(n - m - 2)} e_{n-2,m+2}. \]

Finally, combining this with the inequality 
\[ e_{n,m} \geq \frac{n - m + 2}{n + m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{2(n - m)}{(n + m)(n + m - 2)} e_{n-3,m-1} \]
yields the desired result. \( \square \)

The advantage of the bounds in the lemma above is that all terms are positive, so we can derive the asymptotics using the same techniques as for relaxed binary trees. Note, that the behaviour stays the same in the process of deriving the Newton polygons and leads to the same pictures as shown in Figures 9 and 10.

### 5.1 Lower bound

The following result is analogous to Lemma 4.2. The first two inequalities are needed to deal with the special behavior of the recurrence (14) for \( m = 0 \) and \( m = 1 \).

**Lemma 5.2.** For all \( n, m \geq 0 \) let
\[ \tilde{Y}_{n,m} := \left(1 - \frac{2m^2}{3n} + \frac{m}{4n}\right) \text{Ai} \left( a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}} \right) \]
and
\[ \tilde{s}_n := 2 + \frac{2^{2/3}a_1}{n^{2/3}} + \frac{13}{6n} - \frac{1}{n^{7/6}}. \]

Then, for any \( \varepsilon > 0 \), there exists a constant \( \tilde{n}_0 \) such that
\[ \tilde{Y}_{n,0} \tilde{s}_n \leq \tilde{Y}_{n-1,1}, \]
\[ \tilde{Y}_{n,1} \tilde{s}_n \leq \tilde{Y}_{n-1,0} + \tilde{Y}_{n-1,2}, \]
\[ \tilde{Y}_{n,m} \tilde{s}_n \tilde{s}_{n-1} \tilde{s}_{n-2} \leq \frac{n - m + 2}{n + m} \tilde{Y}_{n-1,m-1} \tilde{s}_{n-1} \tilde{s}_{n-2} + \frac{n - m - 2}{n - m} \tilde{Y}_{n-1,m+1} \tilde{s}_{n-1} \tilde{s}_{n-2} + \frac{n - m - 4}{n - m} \left( \frac{2}{n - m} \tilde{Y}_{n-2,m+2} \tilde{s}_{n-2} + \frac{2}{n + m} \tilde{Y}_{n-3,m+1} \right) \]
(15)
for all \( n \geq \tilde{n}_0 \) and all \( 0 \leq m < n^{2/3 - \varepsilon} \).
In this expansion we choose $n$ for all $m$. The following result is analogous to Lemma 4.4, except that in the first factor of $5.2$ Upper bound get by induction 

Therefore the pictures are identical to the case of relaxed trees shown in the Figures 9 and 10.

Note that any other suitable choice for $\varepsilon$ is also fine, as in contrast to the lower bound of relaxed trees, the factor $1 - 2m^2/n + \eta m^4/n^2$ is always positive. Finally, defining $\tilde{h}_n = \hat{s}_n\tilde{h}_{n-1}$ we get by induction

$$e_{n,m} \geq \kappa_0 \tilde{h}_n Y_{n,m}.$$  

### 5.2 Upper bound

The following result is analogous to Lemma 4.4, except that in the first factor of $\hat{Y}_{n,m}$ the term $m^3/n^3$ is omitted. This is necessary for the second inequality at $m = 1$ to hold.

**Lemma 5.3.** Choose $\eta > 2/9$ fixed and for all $n, m \geq 0$ let

$$\hat{Y}_{n,m} := \left(1 - 2m^2/3n + \eta m^4/n^2\right) \text{Ai} \left(a_1 + \frac{11/3}{m^{1/3}}\right)$$

and

$$\hat{s}_n := 2 + \frac{2^{2/3}a_1}{n^{2/3}} + \frac{13}{6n} + \frac{1}{n^{1/6}}.$$

Then, for any $\varepsilon > 0$, there exists a constant $\hat{n}_0$ such that

$$\hat{Y}_{n,0}\hat{s}_n \geq \hat{Y}_{n-1,1},$$

$$\hat{Y}_{n,1}\hat{s}_n \geq \hat{Y}_{n-1,0} + \hat{Y}_{n-1,2},$$

$$\hat{Y}_{n,m}\hat{s}_n\hat{s}_{n-1}\hat{s}_{n-2} \geq \frac{n - m + 2}{n + m}\hat{Y}_{n-1,m-1}\hat{s}_{n-1}\hat{s}_{n-2} + \frac{n - m - 2}{n + m}\hat{Y}_{n-1,m+1}\hat{s}_{n-1}\hat{s}_{n-2}$$

$$+ \frac{2}{n + m}\hat{Y}_{n-2,m+2}\hat{s}_{n-2} + \frac{2}{n + m}\hat{Y}_{n-3,m+1}$$

for all $n \geq \hat{n}_0$ and all $0 \leq m < n^{1-\varepsilon}$. 

---

**Proof.** The first two inequalities for $m = 0$ and $m = 1$ are follow from elementary considerations by the expansions for large $n$. The proof of the last inequality is analogous to the case of relaxed trees. The expansions for $n$ tending to infinity start as follows, where the elements on the convex hull are written in color.

$$P_{n,m} = \text{Ai}(\alpha) \left( -\frac{4\sigma_4}{n^{7/6}} - \frac{2^{11/3}a_1m}{3n^{5/3}} - \frac{164m^2}{9n^2} - \frac{2^{14/3}a_1m^3}{3n^{8/3}} - \frac{136m^4}{9n^3} - \frac{248m^5}{135n^4} + \ldots \right) +$$

$$\text{Ai}'(\alpha) \left( \frac{2^{4/3}(4\tau_1 - 1)}{n^{1/3}} + \frac{2^{7/3}}{n^{3/2}} - \frac{32a_1m}{9n^2} + \frac{2^{10/3}m^2(12\tau_1 - 17)}{9n^{7/3}} - \frac{2^{19/3}m^3}{9n^{17/3}} + \ldots \right).$$

In this expansion we choose $\sigma_4 = -1$ which leads to a positive term $\text{Ai}(\alpha)n^{-7/6}$ and $\tau_1 = 1/4$ which kills the leading coefficient of $\text{Ai}'(\alpha)$ for small $m = o(n^{1/3})$. Then, the behavior and therefore the pictures are identical to the case of relaxed trees shown in the Figures 9 and 10. Hence, the proof follows exactly the same lines as the one of Lemma 4.2. 

Then we choose $\varepsilon = 1/12$ and set

$$Y_{n,m} = \begin{cases} \hat{Y}_{n,m}, & \text{if } m < n^{7/12}, \\ 0, & \text{if } m \geq n^{7/12}. \end{cases}$$

Note that any other suitable choice for $\varepsilon$ is also fine, as in contrast to the lower bound of relaxed trees, the factor $1 - 2m^2/n + \eta m^4/n^2$ is always positive. Finally, defining $\tilde{h}_n = \hat{s}_n\tilde{h}_{n-1}$ we get by induction

$$e_{n,m} \geq \kappa_0 \tilde{h}_n Y_{n,m}.$$
Proof. The first two inequalities for \( m = 0 \) and \( m = 1 \) are again elementary consequences of the expansion at \( n \to \infty \). Note that for \( m = 1 \) it is crucial that the term \( \frac{121}{27} \) is not present in the factor of \( \hat{Y}_{n,m} \). The proof of the last inequality changes then only for small \( m \). The expansions for \( n \) tending to infinity start as follows, where the elements on the convex hull are written in color.

\[
Q_{n,m} = \text{Ai}(\alpha) \left( \frac{4\sigma_4}{n^{7/6}} + \frac{4m^2(41 - 108\eta) + 214/3 a_1 m^3 (1 - 6\eta)}{3n^{8/3}} + \frac{8m^4(17 - 132\eta)}{9n^3} \right.
\]

\[
- \frac{211/3 a_1 m^{5} \eta}{n^{1/3}} - \frac{68 m^6 \eta}{3n^2} - \frac{124 m^7 \eta}{45n^3} + \ldots \right) +
\]

\[
\text{Ai}'(\alpha) \left( \frac{24^4/3 (1 - 2\tau_1)}{n^{4/3}} + \frac{9m^5}{9n^2} + \frac{324 m^4 (17 - 54\eta - 12\tau_1)}{9n^{7/3}} + \frac{216/3 m^3 (2 - 9\eta)}{9n^{7/3}} \right.
\]

\[
+ \frac{24^4/3 m^4 (20 - 27\eta)}{9n^{10/3}} - \frac{216/3 m^5 \eta}{3n^{10/3}} - \frac{5m^6 27/3 \eta}{3n^{13/3}} - \frac{8m^7 27/3 \eta}{45n^{16/3}} + \ldots \right).
\]

In this expansion we choose \( \sigma_4 = 1 \) which leads to a positive term \( \text{Ai}(\alpha) n^{-7/6} \). Then, as mentioned above we set \( \tau_1 = 0 \) instead of \( \tau_1 = 1/4 \). Hence, the dominant term is \( \frac{24^4/3 n^{-4/3}}{n} \) for the blue coefficients of \( \text{Ai}'(\alpha) \). This means that the solid blue point in Figure 12 is present. However, as it is still positive the same method yields the result. The proof of the other cases follows the same lines as the one of Lemma 4.4, because the general behavior is still the same as the one for relaxed trees shown in the Figures 11.

As in the relaxed tree case we still need to do some more work to deduce the desired upper bound. In order to use the lemma, we define a new sequence \( \hat{e}_{n,m} \) by the recurrence relation

\[
\begin{align*}
\hat{e}_{n,m} &= \frac{u - m + 2}{n + m} \hat{e}_{n-1,m-1} + \frac{u - m - 2}{n - m} \hat{e}_{n-1,m+1} + \frac{2}{n + m} \hat{e}_{n-2,m+2} + \frac{2}{n - m} \hat{e}_{n-3,m+1}, & \text{for } n > 0, m \geq 2, \\
\hat{e}_{n,m} &= \frac{u - m + 2}{n + m} \hat{e}_{n-1,m-1} + \hat{e}_{n-1,m+1}, & \text{for } n > 0, m \in \{0, 1\}, \\
\hat{e}_{0,m} &= 0, & \text{for } m > 0, \\
\hat{e}_{n,-1} &= 0, & \text{for } n \geq 0, \\
\hat{e}_{0,0} &= 1.
\end{align*}
\]

Then it follows from Lemma 5.1 that \( \epsilon_{n,m} \leq \hat{e}_{n,m} \) for all \( n, m \). We will frequently use the fact that

\[
\hat{e}_{n,m} \leq \frac{n - m + 2}{n + m} \hat{e}_{n-1,m-1} + \hat{e}_{n-1,m+1},
\]

which is easy to prove from the recurrence. Note in particular that this implies that \( \hat{e}_{n,m} \leq d_{n,m} \). We also choose some \( N > 0 \) and define a second sequence \( \hat{e}_{n,m} \) by the same rules as \( \hat{e}_{n,m} \) except that \( \hat{e}_{n,m} = 0 \) whenever \( m > n^{3/4} \) and \( n > N \). Then, using Lemma 5.3 and defining \( \hat{h}_n = \hat{s}_n \hat{h}_{n-1} \), we can show by induction that there is some constant \( \kappa_1 \) such that

\[
\hat{e}_{n,m} \leq \kappa_1 \hat{h}_n \hat{Y}_{n,m}.
\]

Hence it suffices to prove that there is some choice of \( N \) such that \( \hat{e}_{n,0} \leq 2\hat{e}_{2n,0} \) for all \( n \).

We define a class of weighted paths with steps set \( \{(1,1), (1,-1), (2,-2), (3,-1)\} \) and weights corresponding to the recurrence defining \( \hat{e}_{n,m} \), such that \( \hat{e}_{n,m} \) is the weighted number of paths from \( (0,0) \) to \( (n,m) \). We start with the following lemma, which is analogous to Lemma 4.5.
Lemma 5.4. Let $q_{\ell,m,2n}$ denote the weighted number of paths from $\ell, m$ to $(2n, 0)$. Then the numbers $q_{\ell,m,2n}$ satisfy the inequality
\[
\frac{q_{\ell,j,2n}}{j + 1} \geq \frac{q_{\ell,k,2n}}{k + 1},
\]
for integers $0 \leq j < k \leq \ell \leq 2n$ satisfying $2|k - j$.

Proof. The proof is along the same lines as the proof of Lemma 4.5. As in that case, it suffices to prove that
\[
\frac{q_{\ell,m-1,2n}}{m} - \frac{q_{\ell,m+1,2n}}{m+2} \geq 0,
\]
(18) for all $m \geq 1$. We proceed by reverse induction on $\ell$, with base case $\ell = 2n$. For the inductive step, note that $q$ satisfies the following recurrence for $\ell < 2n$:
\[
\begin{aligned}
q_{\ell,m,2n} &= q_{\ell+1,m-1,2n} + \ell-m+2q_{\ell+1,m+1,2n}, & \text{for } m \in \{0, 1, 2\},
q_{\ell,3,2n} &= \ell-3q_{\ell+1,2,2n} + \ell-1q_{\ell+1,4,2n} + \frac{2}{\ell+5}q_{\ell+3,2n},
q_{\ell,m,2n} &= \ell-m+q_{\ell+1,m-1,2n} + \ell-m+2q_{\ell+1,m+1,2n} + \frac{2}{\ell-m+2}q_{\ell+3,m-2,2n}, & \text{for } m \geq 4.
\end{aligned}
\]
Now in order to prove (18), we expand the left-hand side $L(\ell, m, n)$ using the recurrence relation above. We then use the inductive assumption, which says that (18) holds for all larger values of $\ell$ and all $m$, to show that
\[
L(\ell, m, n) \geq R_1(\ell, m)q_{\ell+1,m+2n} + R_2(\ell, m)q_{\ell+2,m-1,2n} + R_3(\ell, m)q_{\ell+3,m-2,2n},
\]
for some explicit rational functions $R_1$, $R_2$ and $R_3$. Due to the nature of the functions $R_1$, $R_2$ and $R_3$, we can prove that the right-hand side above is positive using the inequalities
\[
q_{\ell+1,m,2n} \geq \frac{\ell - m + 1}{\ell - m + 3}q_{\ell+2,m,2n-1,2n} \quad \text{and} \quad q_{\ell+2,m,2n-1,2n} \geq \frac{\ell - m + 3}{\ell - m + 5}q_{\ell+3,m,2n-2,2n},
\]
which follow directly from the recurrence relation.

Now, amongst the $\hat{e}_{2n,0}$ weighted paths starting at $(0, 0)$ and ending at $(2n, 0)$, the proportion which pass through some point $(2x, 2y)$ is
\[
\frac{\hat{e}_{2x,2y}q_{2x,2y,2n}}{\hat{e}_{2n,0}} \leq \frac{\hat{e}_{2x,2y}q_{2x,2y,2n}}{\hat{e}_{2x,0}q_{2x,0,2n}} \leq (2y + 1)\frac{\hat{e}_{2x,2y}}{\hat{e}_{2x,0}} \leq (2y + 1)\frac{d_{2x,2y}}{d_{2x,0}} \leq \frac{2y + 1}{4e^{3\alpha_1 x^{1/3}} x^{3/4}} \left( \frac{2x}{x + y} \right).
\]
we can finish in exactly the same way as in Lemma 4.6 for relaxed trees, thereby showing that there is some choice for $N$ such that $\hat{e}_{2n,0} \leq 2\hat{e}_{2n,0}$ for all $n$.

Recall that $\hat{e}_{2n,0} \leq \hat{e}_{2n,0}$ and there is some constant $\kappa_1$ such that $\hat{e}_{n,m} \leq \kappa_1 \hat{n}_n \hat{Y}_{n,m}. This implies that
\[
c_n = n!c_{2n,0} \leq 2\kappa_1 n! \hat{n}_n \hat{Y}_{2n,0}.
\]
The right-hand side behaves asymptotically like $\Theta(n!4^n e^{3\alpha_1 n^{1/3}} n^{3/4})$, hence there is some constant $\gamma''$ such that
\[
c_n \leq \gamma'' n!4^n e^{3\alpha_1 n^{1/3}} n^{3/4}
\]
for all $n$. This completes the upper bound. Indeed, since we have now proven both the upper and lower bounds, and these differ only in the constant term, this implies that
\[
c_n = \Theta\left( n!4^n e^{3\alpha_1 n^{1/3}} n^{3/4} \right).
\]
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