DUALITY FOR JACOBI GROUP ORBIT SPACES AND ELLIPTIC SOLUTIONS OF THE WDVV EQUATIONS

ANDREW RILEY\(^{(1)}\) AND IAN A. B. STRACHAN\(^{(2)}\)

Abstract. From any given Frobenius manifold one may construct a so-called ‘dual’ structure which, while not satisfying the full axioms of a Frobenius manifold, shares many of its essential features, such as the existence of a pre-potential satisfying the WDVV equations of associativity. Jacobi group orbit spaces naturally carry the structures of a Frobenius manifold and hence there exists a dual prepotential. In this paper this dual prepotential is constructed and expressed in terms of the elliptic polylogarithm function of Beilinson and Levin.

1. Introduction

The Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equation

\[
\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^\eta^\mu - \frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\delta} \eta^\nu^\lambda \frac{\partial^3 F}{\partial t^\delta \partial t^\gamma \partial t^\alpha} = 0, \quad \alpha, \beta, \gamma, \delta = 1 \ldots , n
\]

originally appeared in the theory of 2D topological quantum field theories, with the geometric structure behind this equation being formalized by Dubrovin into the concept of a Frobenius manifold.

Definition 1. An algebra \((A, \circ, \eta)\) over \(\mathbb{C}\) is a Frobenius algebra if:

- the algebra \(\{A, \circ\}\) is commutative, associative with unity \(e\);
- the multiplication is compatible with a \(\mathbb{C}\)-valued bilinear, symmetric, non-degenerate inner product
  \[
  \eta : A \times A \to \mathbb{C}
  \]
  in the sense that
  \[
  \eta(a \circ b, c) = \eta(a, b \circ c)
  \]
  for all \(a, b, c \in A\).

With this structure one may defined a Frobenius manifold \([5]\):

Definition 2. \((M, \circ, e, \eta, E)\) is a Frobenius manifold if each tangent space \(T_pM\) is a Frobenius algebra varying smoothly over \(M\) with the additional properties:

- the inner product is a flat metric on \(M\) (the term ‘metric’ will denote a complex-valued quadratic form on \(M\).)
- \(\nabla e = 0\), where \(\nabla\) is the Levi-Civita connection of the metric;
- the tensor \((\nabla_{W\circ})(X, Y, Z)\) is totally symmetric for all vectors \(W, X, Y, Z \in TM\);

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• a vector field $E$ exists such that

$$\nabla(\nabla E) = 0$$

and that the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of the metric and by rescalings on the Frobenius algebras $T_p M$.

Since the metric $\eta$ is flat there exists a distinguished coordinate system of flat coordinates $\{t^\alpha, \alpha = 1, \ldots, n\}$ in which the components of the metric are constant. From the various symmetry properties of tensors $\circ$ and $\nabla \circ$ it then follows that there exists a function $F$, the prepotential, such that

$$c_{\alpha\beta\gamma} = \eta \left( \frac{\partial}{\partial t^\alpha} \circ \frac{\partial}{\partial t^\beta} \circ \frac{\partial}{\partial t^\gamma} \right),$$

$$= \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma},$$

and the associativity condition then implies that $F$ satisfies the WDVV-equation.

On such a Frobenius manifold there exists another metric $g$ - the intersection form - defined by the formula

$$g^{-1}(x, y) = E(x \circ y)$$

where $x, y \in T^* M$ and the metric $\eta$ has been used to extend the multiplication from the tangent bundle to the cotangent bundle. This metric is also flat and hence there exists another distinguished coordinate system $\{p^i, i = 1, \ldots, n\}$ in which the components of the intersection form are constant. Explicitly, the notation

$$G^{ij} = \frac{\partial p^i}{\partial t^\alpha} \frac{\partial p^j}{\partial t^\beta} g^{\alpha\beta}(t)$$

will be used to denote the components of the intersection form in its own flat coordinate system.

Another class of solutions to the WDVV equations (though not satisfying the full axioms of a Frobenius manifold) appeared in Seiberg-Witten theory [11]. The connection between these two class has been elucidated by Dubrovin [6]. Given a Frobenius manifold $M$ one may define a dual multiplication $\star : T^* M \to T^* M$ by

$$X \star Y = E^{-1} \circ X \circ Y, \quad \forall X, Y \in T^* M,$$

where $E^{-1}$ is the vector field defined by the equation $E^{-1} \circ E = e$ and $M = M \setminus \Sigma$, where $\Sigma$ is the (discriminant) submanifold where $E^{-1}$ is undefined. This new multiplication is clearly commutative and associative with the Euler field playing the role of the unity field for the new multiplication. Furthermore the multiplication is compatible with the intersection form

$$g(X \star Y, Z) = g(X, Y \star Z).$$

Further properties are inherited from the original Frobenius structure on $M$, in particular:
Theorem 3. There exists a function $\hat{F}(p)$ such that

$$\frac{\partial^3 \hat{F}(p)}{\partial p^i \partial p^j \partial p^k} = G_{ia} G_{jb} \frac{\partial^a}{\partial p^i} \frac{\partial^b}{\partial p^j} \frac{\partial}{\partial p^k} \gamma_{\alpha\beta\gamma}(t)$$

and which satisfies the WDVV-equations in the $\{p^i\}$ coordinates.

Thus given a specific Frobenius manifold one may construct a ‘dual’ solution to the WDVV-equations by constructing the flat-coordinates of the intersection form and using the above result. The aim of this paper is to perform such calculations for the Frobenius manifolds which appear naturally on the Jacobi group orbit space $\Omega/J(g)$ in the specific case where $W_g = A_n$. The dual prepotential is given in terms elliptic polylogarithm introduced by Beilinson and Levin [1, 10]. This functions has also appeared in the theory of Frobenius manifolds and the enumeration of curves by Jacobi forms [9]. Before this, the similar calculation of the dual prepotential on the orbit space $\mathbb{C}^n/A_n$, will be described, both for completeness and for comparison.

2. A Landau-Ginzburg/Superpotential construction

Given a superpotential $\lambda(w)$ the formulae for the various tensors are given by the following theorem:

Theorem 4.

$$\eta(\partial', \partial'') = -\sum_{d\lambda=0} \text{res} \frac{\partial'(\lambda(v) dv)\partial''(\lambda(v) dv)}{d\lambda(v)},$$

$$c(\partial', \partial'', \partial''') = -\sum_{d\lambda=0} \text{res} \frac{\partial'(\lambda(v) dv)\partial''(\lambda(v) dv)\partial'''(\lambda(v) dv)}{d\lambda(v)},$$

$$g(\partial', \partial'') = -\sum_{d\lambda=0} \text{res} \frac{\partial'(\log \lambda(v) dv)\partial''(\log \lambda(v) dv)}{d\log \lambda(v)},$$

$$\ast(\partial', \partial'', \partial''') = -\sum_{d\lambda=0} \text{res} \frac{\partial'(\log \lambda(v) dv)\partial''(\log \lambda(v) dv)\partial'''(\log \lambda(v) dv)}{d\log \lambda(v)}.$$
where the ‘special points’ are points like zero or infinity, where the residues may be easily calculated. Thus the residues, and hence the various tensors, in Theorem 4 may be calculated very simply.

We first consider the orbit space $\mathbb{C}^n/A_n$ where the function $\lambda$ is just a traceless polynomial and the Riemann surface is just the Riemann sphere before studying the main example of the orbit space $\Omega/J(g)$ where the function $\lambda$ is elliptic and the corresponding Riemann surface an elliptic curve.

3. The orbit space $\mathbb{C}^n/A_n$

Given a Coxeter group $W$ one may study the algebra of $W$-invariant functions over the vector space $\mathbb{C}$. The orbit space $\mathbb{C}^n/W$ is in fact a manifold and it was shown by Dubrovin (and earlier by K.Saito) that this manifold inherits the structure of a Frobenius manifold, and as such, there exists an associated superpotential. When $W = A_n$ the superpotential is particularly simple:

$$\lambda(v) = \prod_{i=0}^{n}(v - z^i)$$

with the constraint $\sum_{i=0}^{n} z^i = 0$ which ensure the polynomial $\lambda$ is traceless. The metric is easily calculated via the residue theorem:

$$g = \frac{n}{4} \left| \sum_{i=0}^{n} (dz^i)^2 \right|_{\sum_{j=0}^{n} z^j = 0}.$$

Similarly, the structure constants may be calculated in an identical manner and integrated up to yield the dual prepotential

$$F^* = \frac{1}{4} \sum_{i \neq j} (z_i - z_j)^2 \log(z_i - z_j)^2.$$

The same result may be obtained via the geometry such Coxeter orbit spaces. This example may be generalised in a number of different ways:

- By studying other Coxeter groups. This leads to dual prepotential akin to $F^*$ but where the sum is over the root systems of the corresponding Coxeter group.
- The orbit space $\mathbb{C}^n/A_n$ is isomorphic to the genus-zero Hurwitiz space $H_{0,n+1}(n+1)$. One may easily generalize to an arbitrary genus-zero Hurwitz space $H_{0,N}(k_1, \ldots, k_l)$. This generalization includes the special case $H_{0,n+1}(k, n+1-k)$ which corresponds to the orbit space $\mathbb{C}^n/W^{(k)}(A_{n-1})$ where $W^{(k)}(A_{n-1})$ is an extended affine Weyl group corresponding to the Coxeter group $A_{n-1}$.
- More generally still, one may study the induced ‘dual’ structure on the discriminant strata of either Coxeter groups or Hurwitz spaces.

This last case reduces to using the superpotential

$$\lambda(v) = \prod_{i=0}^{n}(v - z^i)^{k_i}, \quad k_i \in \mathbb{Z}$$

with constraints

$$\sum_{i=1}^{n} k_i z^i = 0.$$
The corresponding ‘dual’ prepotential is
\[ F = \frac{1}{4} \sum_{i \neq j} k_i k_j (z_i - z_j)^2 \log(z_i - z_j)^2 \]
(prepotentials of this form and their connection with deformed root systems where first studied in [4]). Note that while the restriction of the intersection form of a Frobenius manifold to a discriminant submanifold is flat, the restriction of the metric \( \eta \) to a discriminant submanifold is not flat. While the tensors induced from \( \circ \) and \( \nabla \circ \) are still totally symmetric, the curvature obstructs the existence of an induced prepotential [12]. Hence there is no prepotential to which the above ‘dual’ prepotential is dual. This may also be rewritten as a sum over a ‘deformed root system’. This extends the result of Feigin and Veselov [8] to negative values of the parameters \( k_i \).

4. The orbit space \( \Omega / J(A_n) \)

Naively, the Jacobi group orbit spaces may be regarded as an elliptic generalization of Coxeter group orbit spaces. The full definition will not be required here and the reader is referred to [2,7] and [15] for details. The Jacobi group \( J(\mathfrak{g}) \) (where \( \mathfrak{g} \) is a complex finite dimensional simple Lie algebra of rank \( l \) with Weyl group \( W \)) acts on the space
\[ \Omega = \mathbb{C} \oplus \mathfrak{h} \oplus \mathbb{H} \]
where \( \mathfrak{h} \) is the complex Cartan subalgebra of \( \mathfrak{g} \) and \( \mathbb{H} \) is the upper-half-plane, and this leads to the study of invariant functions - the Jacobi forms. Analogous to the Coxeter case, the orbit space
\[ \Omega / J(\mathfrak{g}) \]
is a manifold and carries the structure of a Frobenius manifold. Again, the case \( W = A_n \) (where we abuse notation and write \( \Omega / J(A_n) \) for the orbit space) is particularly simple with the superpotential being given by
\[ \lambda(v) = e^{2\pi i u} \prod_{i=0}^{l} \frac{\vartheta_1(v - z_i | \tau)}{\vartheta_1(v | \tau)} \]
with the constraint \( \sum_{i=0}^{l} z_i = 0 \) and \((u, z, \tau) \in \mathbb{C} \oplus \mathfrak{h} \oplus \mathbb{H}\). The details of the construction of this superpotential is given in [2].

Lemma 5. With the above [2] superpotential the intersection form for the orbit space \( \Omega / J(A_l) \) is
\[ g = 2\pi^2 du d\tau - \sum_{i=0}^{l} (dz_i)^2 \bigg|_{\sum_{j=0}^{n} z_j = 0} \]
and the dual prepotential is
\[ \hat{F} = 2\pi i \left[ \frac{1}{2} \pi^2 \tau u^2 - u \sum_{i=0}^{l} (z_i)^2 \right] + \hat{F}_{\text{quantum}}(\mathfrak{g}|\tau). \]
where this function is evaluated on the plane \( \sum z_i = 0 \).
Proof. The calculation of the intersection form in these coordinate appears in [2]. The normalization of the \( \vartheta_1 \)-function follows [16] which accounts for various numerical factors. The main device is the use of the elliptic connection, which again is described in [2]. With the normalizations used here the elliptic connection is

\[
D^* f(v|\tau) = (\nabla^* f)(v|\tau) + \frac{\pi i}{2} \frac{\vartheta'_1(v|\tau)}{\vartheta_1(v|\tau)} f'(v|\tau)
\]

where

\[
(\nabla^*_\tau f)(v|\tau) = (\partial_\tau f)(v|\tau) + \frac{\pi i k}{6} \frac{\vartheta''''_1(0|\tau)}{\vartheta_1(0|\tau)} f(v|\tau).
\]

This connection takes elliptic modular functions of weight \( k \) to elliptic modular functions of weight \((k+2)\). It follows from the simple dependence of \( \lambda \) on the coordinate \( u \) that

\[
\hat{c}_{uij} = 2\pi \sqrt{-1} G_{ij}
\]

and this fixes the above \( u \)-dependence of the dual prepotential.

To calculate the term \( \hat{F}^{\text{quantum}} \) it is necessary to calculated the remaining components of the tensor \( \hat{c}^* \). We begin with the components \( \hat{c}_{z_iz_jz_k} \).

**Proposition 6.** The dual structure functions \( \hat{c}_{z_iz_jz_k} \) are given in terms of the third derivatives of a function \( \hat{F}^\text{temp} \),

\[
\hat{c}_{z_iz_jz_k} = \frac{\partial^3 \hat{F}^\text{temp}}{\partial z_i^3 \partial z_j^3 \partial z_k^3}, \quad i,j,k = 1, \ldots, l,
\]

where\(^2\)

\[
\hat{F}^\text{temp} = \frac{1}{8} \sum_{i \neq j}' A_3(z^i - z^j) - \frac{(l+1)}{4} \sum_i' A_3(z^i).
\]

Hence

\[
\hat{F}^{\text{quantum}} = \hat{F}^\text{temp} + \frac{1}{2} \sum_{i,j} A_{ij}(\tau)z^i z^j + \sum_i B_i(\tau)z^i + C(\tau).
\]

Proof. Using Theorem one may calculate these structure functions in an exactly analogous manner to the calculation of the \( A_n \)-structure functions. For example

\[
\hat{c}_{z_iz_jz_k} = \sum_{i \neq 0} \frac{\partial^3}{\partial 1^3} (z^0 - z^i |\tau) - (l+1) \frac{\partial^3}{\partial 1^3} (z^0 |\tau)
\]

\[+ \frac{\partial^3}{\partial 1^3} (z^0 - z^j |\tau) + \frac{\partial^3}{\partial 1^3} (z^0 - z^j |\tau) + \frac{\partial^3}{\partial 1^3} (z^0 - z^k |\tau)\]

Note, \( \sum_i' \) includes the term \( i = 0 \).
and similar formulae may be derived for $\star c_{z_i z_j}$ and $\star c_{z_i z_j z_k}$. Using the formulae presented in the appendix one may integrate these equations in terms of the function $\Lambda_3$. Hence by construction

$$\frac{\partial^3}{\partial z^i \partial z^j \partial z^k} (\hat{F}_{\text{quantum}} - \hat{F}_{\text{temp}}) = 0, \quad i, j, k = 1, \ldots, l,$$

from which the general form of $\hat{F}_{\text{quantum}}$ follows.

To find the three functions $A_{ij}(\tau), B_i(\tau)$ and $C(\tau)$ requires the calculation of, respectively, the structure functions $\star c_{z_i z_j}, \star c_{z_i z_j z_k}$ and $\star c_{z_i z_j z_k}$. However, by using the modularity property of these functions at the point $z = 0$ one may find these three functions without having to calculate these structure functions exactly.

**Theorem 7.**

$$\hat{F}_{\text{quantum}} = -\frac{1}{8} \sum'_{i \neq j} \left\{ L_{i3}[q^2, e^{2i(z_i - z_j)}] - L_{i3}[q^2, 1] \right\} + \frac{(l + 1)}{4} \sum'_{i} \left\{ L_{i3}[q^2, e^{2iz_i}] - L_{i3}[q^2, 1] \right\}.$$

**Proof.** Since

$$\star c_{z_i z_j} = \frac{\partial^3 \hat{F}_{\text{temp}}}{\partial \tau \partial z^i \partial z^j} + A'_{ij}(\tau)$$

and $\hat{F}_{\text{temp}}$ is known explicitly, it follows that

$$A'_{ij}(\tau) = \star c_{z_i z_j}(0|\tau).$$

Similarly

$$B''_i(\tau) = \star c_{z_i z_j}(0|\tau),$$

$$C'''(\tau) = \star c_{z_i z_j z_k}(0|\tau) - \frac{\partial^3 \hat{F}_{\text{temp}}}{\partial \tau^3}\bigg|_{z=0},$$

$$= \star c_{z_i z_j z_k}(0|\tau) + \frac{i\pi^3}{120} (l + 1)(l + 2) E_4(\tau).$$

Hence it suffices to calculate the remaining structure functions at the special point $z = 0$. Using the quasi-modular property

$$\frac{\partial}{\partial \tau} \left( \frac{z}{\tau} - \frac{1}{\tau} \right) = \frac{2iz}{\pi} + \frac{\partial}{\partial \tau} \left( \psi_1(z|\tau) \right)$$

and its differential consequences one may derive the following modularity properties of the structure functions directly from Theorem 4 without having to calculate them explicitly:

$$\star c_{z_i z_j}(0|\tau) - \tau^{-1} = \tau^2 \star c_{z_i z_j}(0|\tau),$$

$$\star c_{z_i z_j z_k}(0|\tau) - \tau^{-1} = \tau^3 \star c_{z_i z_j z_k}(0|\tau),$$

$$\star c_{z_i z_j z_k}(0|\tau) - \tau^{-1} = \tau^4 \star c_{z_i z_j z_k}(0|\tau).$$

As $q \to 0$ (or equivalently, as $\tau \to i\infty$) the three structure functions vanish. Hence they must be analytic in $q$ with vanishing constant term. This, with the modularity
property, implies they are cusp-forms of degrees 2, 3 and 4. However there are no
no-zero cusp-forms of these degree so
\[ c_{\tau z_i z_j}(0|\tau) = 0, \]
\[ c_{\tau \tau z_i}(0|\tau) = 0, \]
\[ c_{\tau \tau \tau}(0|\tau) = 0. \]

Hence \( A_{ij} = 0 \) and \( B_i = 0 \) (ignoring quadratic terms in the prepotential) and \( C \)
is the triple integral of the Eisenstein series \( E_4 \) which may be evaluated using the
elliptic polylogarithm function (see Appendix). Hence the result. □

Comments

As in the Coxeter case, this basic example may be generalised in a number of
different directions:

• By studying other Weyl groups. This should lead to dual prepotential akin
to that in Theorem 7 (the \( B_n \)-case may be done very easily using the
\( A_{2n+1} \) to \( B_n \) reduction). A problem that this immediately generates is
this: it follows from the general theory of dual Frobenius manifolds that
a prepotential exists and satisfies the WDVV equations. However a direct
verification would be considerably harder, presumably involving various
\( \vartheta \)-function identities and their modular counterparts. This should also gen-
erate as a by-product elliptic Dunkl-type operators, since the flatness of the
dual Dubrovin connection is directly related, at least in the Coxeter case,
to the properties of the classical Dunkl operators.

• The orbit space \( \mathbb{C} \oplus \mathfrak{h} \oplus \mathbb{H}/J(A_n) \) is isomorphic to the genus-one Hurwitz
space \( H_{1,n+1}(n+1) \). One may easily generalize to an arbitrary genus-one
Hurwitz space \( H_{1,N}(k_1, \ldots, k_l) \).

• More generally still, one may study the induced ‘dual’ structure on the
discriminant strata in these spaces.

Some of these points will be addressed in 11 and others are under investiga-
tion. One final question is whether these elliptic solutions to the WDVV equa-
tions have any use in Seiberg-Witten theory and Calogero-Moser systems and their
generalizations 3.

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Appendix A. The integration of \( \log \vartheta_1 \) and elliptic polylogarithms

The definition of the polylogarithm functions is, for positive integers \( N \),
\[ Li_N(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^N}. \]
The series converges for \(|z| < 1\), but the functions may be analytically continued elsewhere as well as extended to non-integer values of \(N\) via integral representations of the function. Immediate consequences of the definitions are:

\[
Li_0(z) = \frac{z}{1 - z},
\]

\[
\frac{d}{dz} Li_n(z) = \frac{1}{z} Li_{n-1}(z).
\]

This function may be used to integrate \(\log \vartheta_1(z)\). Using the infinite product representation

\[
\vartheta_1(z) = 2G q^{1/4} \sin z \prod_{n=1}^{\infty} (1 - q^{2n} e^{2iz}) \prod_{n=1}^{\infty} (1 - q^{2n} e^{-2iz})
\]

(where \(q = e^{i\pi \tau}\)) one may write

\[
\log \vartheta_1(z) = \log(iG q^{1/4}) + \left\{ -iz - \sum_{n=0}^{\infty} Li_1(q^{2n} e^{2iz}) + \sum_{n=1}^{\infty} Li_1(q^{2n} e^{-2iz}) \right\}.
\]

In what follows it is convenient to define a new function \(\Lambda_N(z, q)\) as

\[
\Lambda_N(z, q) = -\frac{1}{2} \frac{(2iz)^N}{N!} - \sum_{n=0}^{\infty} Li_N(q^{2n} e^{2iz}) + (-1)^{N+1} \sum_{n=1}^{\infty} Li_1(q^{2n} e^{-2iz})
\]

so

\[
\frac{d}{dz} \Lambda_N(z, q) = 2i \Lambda_{N-1}(z, q)
\]

and

\[
\Lambda_0(z, q) = \frac{1}{2i} \vartheta_1(z),
\]

\[
\Lambda_1(z, q) = \log \vartheta_1(z) - \log(iG q^{1/4}).
\]

Recall the inversion formula

\[
(-1)^{N-1} Li_N(\zeta^{-1}) = Li_N(\zeta) + \sum_{j=0}^{N} \frac{B_j(2\pi \sqrt{-1})^j}{(N - j)! j!} - (\log \zeta)^{N-j}
\]

where \(B_j\) are the Bernoulli numbers. From this it follows that

\[
\Lambda_N(-z, q) + (-1)^N \Lambda_N(z, q) = (-1)^N \sum_{j=1}^{N} \frac{B_j(2\pi \sqrt{-1})^j}{(N - j)! j!} (2iz)^{N-j},
\]

\[
= \text{polynomial of degree } (N - 1).
\]

Remarkably, the function \(\Lambda_N(z, q)\) is closely related to the elliptic polylogarithm introduced by Beilinson and Levin \[1, 10\], following the real-valued version introduced by Zagier \[17\]. By definition,

\[
Li_r(q, \zeta) = \sum_{n=0}^{\infty} Li_r(q^n \zeta) + \sum_{n=1}^{\infty} Li_r(q^n \zeta^{-1}) - \chi_r(q, \zeta), \quad r \text{ odd},
\]

where

\[
\chi_r(q, \zeta) = \sum_{n=0}^{\infty} \frac{B_{j+1}}{(r - j)! (j + 1)!} (\log \zeta)^{(r-j)} (\log q)^{j}.
\]
Hence, for example,
\[ A_3(z, q) = -LI_3(q^2, e^{2iz}) + \frac{1}{3}(\log q)z^2 + \frac{1}{90}(\log q)^3. \]

These formulae enable the equation
\[ \star c_{ijk} = \sum \textnormal{terms involving } \frac{\vartheta'_1}{\vartheta_1} \]

to be integrate in closed form, leaving the answer in terms of the elliptic trilogarithmic function. Note that these functions are multivalued, so have non-trivial monodromy group \([13]\). However such transformations change the elliptic trilogarithm by quadratic terms in the flat \(\{p^i\}\)-variables and hence leave the physical structure functions unchanged.

It also follows from these formulae that
\[
\begin{align*}
\frac{\partial^3}{\partial\tau^3}LI_3(q^2, e^{2iz}) & \bigg|_{z=0} = -\frac{i\pi^3}{15}E_4(\tau), \\
\frac{\partial^3}{\partial\tau^2\partial z}LI_3(q^2, e^{2iz}) & \bigg|_{z=0} = 0, \\
\frac{\partial^3}{\partial\tau\partial z^2}LI_3(q^2, e^{2iz}) & \bigg|_{z=0} = \frac{2i\pi^3}{3}E_2(\tau),
\end{align*}
\]

where \(E_2(\tau), E_4(\tau)\) are the normalised Eisenstein series.

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(1) Department of Mathematics, University of Hull, Hull HU6 7RX, U.K.  
E-mail address: a.riley@math.hull.ac.uk

(2) Department of Mathematics, University of Glasgow, Glasgow G12 8QQ, U.K.  
E-mail address: i.strachan@maths.gla.ac.uk