The wave-particle duality of the qudit-based quantum space demonstrated by the wave-like quantum functionals

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Abstract: The wave-particle duality of the qudit-based quantum space is demonstrated by the wave-like quantum functionals and the particle-like quantum states. The quantum functionals are quantum objects generated by the basis qudit functionals, which are the duals of the basis qudit states. The relation between the quantum states and quantum functionals is analogous to the relation between the position and momentum in fundamental quantum physics. In particular, the quantum states and quantum functionals are related by a Fourier transform, and the quantum functionals have wave-like interpretations. The quantum functionals are not just mathematical constructs but have clear physical meanings and quantum circuit realizations. Any arbitrary qudit-based quantum state has dual characters of wave and particle, and its wave character can be evaluated by an observable and realized by a quantum circuit.

1. Introduction

Quantum information and quantum computation has emerged as a burgeoning new field [1-6] that holds the potential to revolutionize information processing and computing technologies [7-12]. In particular, numerous quantum algorithms have been developed for solving a variety of problems in the last few decades: e.g. the phase estimation algorithm [13], Shor’s factorization algorithm [14], the Harrow-Hassidim-Lloyd algorithm for linear systems [15], the hybrid classical-quantum algorithms [16, 17], the quantum machine learning algorithms [18-20], and quantum algorithms for open quantum dynamics [21-24].

Despite the diverse selection of quantum algorithm designs, there is the need for a theory [25-28] to systematically understand the structures and behaviors of the quantum circuits used to implement the quantum algorithms. Qudits are the fundamental building blocks of quantum information and quantum computation – usually the 2-level qubits are used, but recently there are growing interests in generalizing to $d$-level qudit systems [29-33]. In a quantum computer, the creation, manipulation and measurement of quantum states are all based on qudits. Therefore, a deeper understanding of the qudit-based quantum space is essential to developing a systematic theory for designing efficient quantum circuits.

In this fundamental study of the qudit-based quantum space, we propose a new “quantum functional space” that is generated by the basis set consisting of the “qudit functionals”. By the mathematical theory of Pontryagin duality, the qudit functionals are duals of the basis states of the usual quantum state space, and therefore the quantum functionals and the quantum states are related by a Fourier transform – this is the same as the relation between the momentum and position representations commonly known in fundamental quantum physics. It is well known that the duality between the momentum and position representations leads to the fundamental concept of
wave-particle duality, and here we propose that the quantum functionals and quantum states are respectively the wave representation and particle representation of the same qudit-based quantum object, and thus there is also a wave-particle duality for the qudit-based quantum space. Indeed, we find the basis of the quantum functionals, i.e. the qudit functionals, can be interpreted as planewaves in the state-representation where the wavenumber \( k \) describes the fluctuations of the wavefunction in the state space. We then show the wave-like quantum functionals are not just mathematical constructs but physical objects, as they can be physically interpreted as partitions of the qudit-based quantum state space, and a physical realization on the quantum circuit can be constructed for the quantum functional associated with any arbitrary quantum state. Any arbitrary qudit-based quantum state has dual characters of wave and particle, and its wave character can be evaluated by an observable and realized by a quantum circuit. In particular, this observable \( \hat{k} \) for the wavenumber of the quantum functionals can be related to the operator \( \hat{T} \) that acts as a translation on the state space – this is similar to the relation between the momentum operator and translation operator in the conventional momentum-position duality. The wave-particle duality as demonstrated by the wave-like quantum functionals is a fundamental physical reality of the qudit-based quantum space. The partition interpretation of quantum functionals relates them to the effects of quantum gates, and thus further studies on quantum functionals may benefit the design of efficient quantum circuits.

2. The wave-particle duality of the qudit-based quantum space.

2.1 The quantum functional space.

For an \( n \)-qudit system with \( d \) levels on each qudit, an arbitrary quantum state can be written as a linear combination \( |\psi\rangle = \sum_{j=0}^{d^n-1} a_j |j\rangle \) of the basis states \(|j\rangle\) that are essentially strings of qudit values: e.g. any 2-qutrit \( (d=3) \) state vector has \( 3^2 = 9 \) basis states corresponding to 9 qudit-value-strings: \( |00\rangle \sim (0,0) \), \( |01\rangle \sim (0,1) \), \( |02\rangle \sim (0,2) \), \( |10\rangle \sim (1,0) \), ..., \( |22\rangle \sim (2,2) \). For the following discussion until Section 2.4 we assume \( d \) is a prime number \( p \), then mathematically the complete collection of the qudit-value-strings (i.e. basis states) of any \( n \)-qudit quantum space forms a linear space \( Q \) over the field \( \mathbb{Z}_p \), which means any vector \( q \) in \( Q \) is:

\[
q = (q_1, q_2, \ldots, q_n) = \sum_{j=1}^{n} q_j^{(j)}
\]

\( q_j = 0, 1, 2, \ldots, p-1 \) (i.e. values from \( \mathbb{Z}_p \)); the addition is digit-wise addition modulo \( p \). Using the 2-qutrit example again, any vector \( q \) in \( Q \) can be expressed as a linear combination of the 2 basis vectors \( (1,0) \) and \( (0,1) \): \( q = (q_1, q_2) = q_1 (1,0) + q_2 (0,1) \), where the coefficients \( q_1 \) and \( q_2 \) take integer values 0, 1 or 2 (i.e. values from \( \mathbb{Z}_3 \)). By the mathematical theory of linear spaces, all the linear
functionals defined on $Q$ form another linear space $Q^*$ that is the dual space of $Q$. For reasons that will become obvious later, we label a vector in $Q^*$ by $k$, which can be written as:

$$k = (k_1, k_2, \ldots, k_n) = k_1 q_1 \oplus k_2 q_2 \oplus \cdots \oplus k_n q_n \quad k_j = 0, 1, 2, \ldots, p-1$$

(2)

where the scalars $k_j$’s take integer values from 0 to $p-1$; $\oplus$ is addition modulo $p$. Clearly, $k$ is a linear combination of the basis functionals (basis vectors of the functional space $Q^*$):

$$f^{(j)}(q) = q_j.$$

Some examples of the linear functionals defined on the 2-qutrit space are $(0,1) = q_2$, $(1,2) = q_1 \oplus 2q_2$, $(2,1) = 2q_1 \oplus q_2$, and there are totally $3^2 = 9$ of these linear functionals in $Q^*$ – matching the number of qudit-value-strings in $Q$. Here the linear functionals in $Q^*$ are essentially functionals of qudit values, and in the following we will call them the “qudit functionals”.

Now we notice that the vectors $q$ in $Q$ are basis states that generate the usual quantum state space such that any quantum state is written as $|\psi\rangle = \sum_{q \in Q} a_q |q\rangle$, where the summation goes over all $q$ in $Q$ and $a_q$ are complex numbers. Then naturally, the dual vectors of $q$, i.e. the qudit functionals can also be basis states, or more appropriately “basis functionals”, to generate the “quantum functional space” where an arbitrary quantum functional is written as:

$$|\phi\rangle = \sum_{k \in Q^*} b_k |k\rangle, \quad b_k \text{ is a complex number that } \sum_{k \in Q^*} |b_k|^2 = 1$$

(3)

Here $|\phi\rangle = \sum_{k \in Q^*} b_k |k\rangle$ has the same quantum interpretation as $|\psi\rangle = \sum_{q \in Q} a_q |q\rangle$ such that $b_k$ represents the probability amplitude of the corresponding qudit functional $|k\rangle$. In fact, as the dual space $Q^*$ has all the mathematical properties of $Q$ itself, the quantum functional space generated by $k$ in $Q^*$ also has all the mathematical properties of the usual quantum state space generated by $q$ in $Q$: this means there are also quantum entities like superposition and entanglement in the quantum functional space. Again using the 2-qutrit case for example, the quantum functional $|\phi\rangle = (b_0 |0\rangle + b_1 |1\rangle) \otimes |2\rangle$ is a “superposition functional” for which the $k_1$ entry of $|k\rangle = |k_1 k_2\rangle = |k_1\rangle \otimes |k_2\rangle = (k_1, k_2) = k_1 q_1 \oplus k_2 q_2$ is in the superposition of 0 and 1, and thus the total functional is in the superposition of $(0,2) = 2q_2$ and $(1,2) = q_1 \oplus 2q_2$, with the probability amplitudes $b_0$ and $b_1$ respectively. Similarly, the quantum functional $|\phi\rangle = b_0 |12\rangle + b_1 |21\rangle$ is an “entangled functional” such that the $k_1$ and $k_2$ entries are correlated in a quantum way with no classical equivalent.
2.2 The Fourier transform and the wave-particle duality.

So far we have shown the quantum functional space is generated by the qudit functionals $|k\rangle$, and the latter are dual vectors of the basis states $|q\rangle$ of the usual quantum state space. It turns out the relation between $q$ and $k$ is analogous to the relation between the position variable $x$ and the momentum variable $k$ in fundamental quantum physics. In particular, $x$ and $k$ are also dual vectors and quantum wavefunctions can be expressed in either the position representation $\psi(x)$ or the momentum representation $\phi(k)$. Mathematically, both the $x$ - $k$ duality and the $q$ - $k$ duality are examples of the Pontryagin duality [34] that guarantees a Fourier transform between wavefunctions expressed in the dual representations. For the $x$ - $k$ pair we have the Fourier transform:

$$\psi(x) = \int_{k\text{-space}} \phi(k) e^{2\pi i k x} dk \quad \phi(k) = \int_{x\text{-space}} \psi(x) e^{-2\pi i k x} dx$$  \hspace{1cm} (4)$$

where $\psi(x)$ is the probability amplitude evaluated at $|x\rangle$ in the position space, and $\phi(k)$ is the probability amplitude evaluated at $|k\rangle$ in the momentum space. Similarly for the $q$ - $k$ pair we have the Fourier transform:

$$\psi(q) = \frac{1}{\sqrt{p^n}} \sum_k \phi(k) e^{2\pi i k q/p} \quad \phi(k) = \frac{1}{\sqrt{p^n}} \sum_q \psi(q) e^{-2\pi i q k/p}$$  \hspace{1cm} (5)$$

where $p$ is the dimension of one qudit, $n$ is the number of qudits, $k \cdot q = k_1 q_1 \oplus k_2 q_2 \oplus \ldots \oplus k_n q_n$ ($\oplus$ is addition modulo $p$), $\psi(q)$ is the probability amplitude evaluated at $|q\rangle$ in the $Q$ space, and $\phi(k)$ is the probability amplitude evaluated at $|k\rangle$ in the $Q^*$ space: clearly $\psi(q) = a_q$ corresponds to $|\psi\rangle = \sum_{q \in \mathcal{Q}} a_q |q\rangle$ and $\phi(k) = h_k$ corresponds to $|\phi\rangle = \sum_{k \in \mathcal{Q}^*} h_k |k\rangle$. A graphical illustration of the $q$ - $k$ duality and the Fourier transform between $\psi(q)$ and $\phi(k)$ is shown in Figure 1 with a 2-qutrit example.
Figure 1. Graphical illustration of the $q$-$k$ duality and the Fourier transform between the wavefunctions $\psi(q)$ and $\phi(k)$, using a 2-qutrit system as an example. The $q$ and $k$ vectors in the middle are duals of each other. The particle-like quantum state generated by $q$ and the wave-like quantum functional generated by $k$ are related by a Fourier transform.

The striking similarity between the Fourier transforms in Equations (4) and (5) clearly shows the equivalence between the $q$-$k$ duality and the $x$-$k$ duality. In particular, the $x$-$k$ duality is commonly interpreted as the wave-particle duality where $\psi(x)$ is the particle representation that describes the spread of the quantum state in the position space, and $\phi(k)$ is the wave representation that describes the spread of the quantum state in the momentum space. Here we make the logical claim that:

The $q$-$k$ duality can be interpreted as the wave-particle duality of the qudit-based quantum space where $\psi(q)$ is the particle representation that describes the spread of the quantum state in the basis state $q$ space, and $\phi(k)$ is the wave representation that describes the spread of the quantum state in the qudit functional $k$ space.
To better see the wave character of \( \phi(k) \), consider the basis functional \( |k\rangle \): its wavefunction in the \( k \)-representation is \( \phi(k) = \delta_{k,j} \) (\( \delta_{k,j} \) is the Kronecker delta), so \( |k\rangle \) in the \( q \)-representation is obtained by the Fourier transform in Equation (5):

\[
\langle q|k \rangle = \psi(q) = \frac{1}{\sqrt{p^n}} \sum_j \delta_{k,j} e^{2\pi i q_j/p} = \frac{1}{\sqrt{p^n}} e^{2\pi i q/k}
\]

where \( \langle q|k \rangle \) has the form of the planewave similar to the usual \( e^{2\pi i k \cdot x} \), and \( k \cdot q = k_1 q_1 + k_2 q_2 + \ldots + k_n q_n \) can be interpreted as the dot product between the wavenumber vector and the position vector. Now we see that labeling the qudit functionals by the symbol \( k \) is appropriate as it indeed plays the role of the wavenumber vector that specifies the fluctuations of the wavefunction in the \( q \) space: each component \( k_j \) is the wavenumber on the direction of \( q_j \).

In a similar manner we find \( \langle k|q \rangle = \frac{1}{\sqrt{p^n}} e^{-2\pi i k \cdot q/p} \) and thus the Fourier transform in Equation (5) can be understood as taking the inner product of the wavefunction with the basis of the dual space:

\[
\psi(q) = \langle q|\phi \rangle = \sum_k \left( \frac{1}{\sqrt{p^n}} e^{-2\pi i k \cdot q/p} \right)^* \phi(k) \quad \phi(k) = \langle k|\psi \rangle = \sum_q \left( \frac{1}{\sqrt{p^n}} e^{2\pi i k \cdot q/p} \right)^* \psi(q)
\]

\[
\langle f(k)|g(k) \rangle = \sum_k f^*(k) \cdot g(k) \quad \langle f(q)|g(q) \rangle = \sum_q f^*(q) \cdot g(q)
\]

where the inner product in each representation is defined below the corresponding Fourier transform. It follows that any quantum functional \( |\phi\rangle = \sum_k b_k |k\rangle \) is indeed the linear combination of the planewaves \( \frac{1}{\sqrt{p^n}} e^{2\pi i k \cdot q/p} \) and thus \( |\phi\rangle \) is a wave-like quantum object.

2.3 The physical reality of the qudit functionals and quantum functionals.

The qudit functionals \( |k\rangle \), the quantum functionals \( |\phi\rangle = \sum b_k |k\rangle \), and their roles in the wave-particle duality of the qudit-based quantum space have been established by the mathematical theory of Pontryagin duality. In this section we show that the qudit functionals and quantum functionals are not only mathematical constructs but physical objects with clear physical meanings and realizations.

2.3.1 The realization of the qudit functionals as partitions of the quantum state space. As shown in Equation (2), the qudit functional \( k \) is defined as the linear functional of the basis state \( q \), and therefore any \( k \) can be interpreted as a “partition” of the quantum state space. Using the 2-qutrit example again, say a qudit functional \( k_1 = (0,1) = q_2 \), then it defines the “qudit conditions”
"q_2 = 0", "q_2 = 1", and "q_2 = 2", which respectively specify the subspace spanned by \{\{00\},\{10\},\{20\}\}, \{\{01\},\{11\},\{21\}\}, and \{\{02\},\{12\},\{22\}\}: this is a partition of the total quantum state space into three equal subspaces. Similarly, \(k_2 = (2,1) = 2q_1 \oplus q_2\) defines the qudit conditions "2q_1 \oplus q_2 = 0", "2q_1 \oplus q_2 = 1", and "2q_1 \oplus q_2 = 2", which respectively specify the subspaces spanned by \{\{00\},\{11\},\{22\}\}, \{\{01\},\{20\},\{12\}\}, and \{\{10\},\{21\},\{02\}\}: this is a different partition of the total quantum state space into three different equal subspaces. It is simple to use the partition of \(k_1 = (0,1) = q_2\) on a quantum circuit: just use the 2\textsuperscript{nd} qutrit \(q_2\) as the control, and then apply to the target the gate \(U_0\) when \(q_2 = |0\rangle\), \(U_1\) when \(q_2 = |1\rangle\), and \(U_2\) when \(q_2 = |2\rangle\). We see that a controlled-gate can be understood as having its effect conditional on the value of the functional held by the control qudit, and in the \(k_1 = (0,1) = q_2\) case we do not need to do anything to \(q_2\) before using it as the control. In the \(k_2 = (2,1) = 2q_1 \oplus q_2\) case however, we need to first create the functional on some qudit before using it as the control. This can be done by applying the following gate:

\[
CU_{1\rightarrow 2}^{(2)} = \begin{pmatrix}
U_0 & U_1 \\
& U_2
\end{pmatrix}, \quad U_0 = I, \quad U_1 = \begin{pmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} \tag{8}
\]

where \(CU_{1\rightarrow 2}^{(2)}\) is a controlled-gate using \(q_1\) as the control and \(q_2\) as the target, and \(U_1\) and \(U_2\) are the “translations” that advance the value of \(q_2\) by 2 and 1 respectively. Clearly, the \(CU_{1\rightarrow 2}^{(2)}\) in Equation (8) adds \(2q_1\) to \(q_2\) modulo 3, and thus \(k_2 = (2,1) = 2q_1 \oplus q_2\) is now stored on \(q_2\): i.e. using \(q_2\) as the control for any subsequent controlled-gate will now use the partition defined by \(k_2 = (2,1) = 2q_1 \oplus q_2\) to determine its effects on the target qudit. Now \(CU_{1\rightarrow 2}^{(2)}\) can be easily generalized to \(n\)-qudit with \(p\) levels for each qudit:

\[
CU_{1\rightarrow m}^{(k_1)} = \begin{pmatrix}
U_0 & U_1 & \cdots & \\
& U_1 & \cdots & \\
& & \ddots & U_{p-1}
\end{pmatrix}, \quad U_j = T_m(k_{1j}) \quad j = 0, 1, ..., p-1 \tag{9}
\]

where \(T_m(k_{1j})\) is the translation that advances the value of \(q_m\) by the product \(k_{1j}\), and clearly \(CU_{1\rightarrow m}^{(k_1)}\) adds \(k_{1j}q_1\) to \(q_m\) and creates \(k_3 = (k_1,0,...,0,k_m = 1,0,...,0) = k_{1j}q_1 \oplus q_m\) on \(q_m\) if \(q_m\) is in its original functional. With some additional quantum gate algebra we can construct the gate \(CU_{1\rightarrow m}^{(k_1)}\) that adds any \(k_i\) multiples of any \(q_i\) to any \(q_m\) and thus any qudit functional \(k\) can be created on any qudit.
2.3.2 The realization of the quantum functionals on the quantum circuit. Having seen the physical interpretation and realization of the qudit functionals, we now construct the quantum circuit to realize any quantum functionals \( \phi = \sum_k b_k |k\rangle \):

![Quantum Circuit Diagram]

Figure 2. The quantum circuit for realizing any arbitrary 2-qudit quantum functional. In the two 2-qudit-controlled gates, the handling qudits \( q_{h1} \) and \( q_{h2} \) control the values \( k_1 \) and \( k_2 \) that specify respectively how many multiples of \( q_1 \) and \( q_2 \) are added to the functional holder \( q_f \) by the translations \( T(k_1,j_1) \) and \( T(k_2,j_2) \). Any arbitrary 2-qudit quantum functional \( |\phi\rangle = \sum_{k_1,k_2=0}^{n-1} b_{k_1,k_2} |k_1 k_2\rangle \) defined on the space of \( q_1 \) and \( q_2 \) can be created on \( q_f \) by initializing \( |\phi\rangle \) as a 2-qudit state on the handling qudits \( q_{h1} \) and \( q_{h2} \) and then applying the circuit.

In Figure 2 we have constructed the quantum circuit to realize any quantum functional that can be defined on the 2-qudit system of \( q_1 \) and \( q_2 \). There are two 2-qudit-controlled-gates that are qudit generalizations of the Toffoli gates. The 1st gate uses \( q_1 \) and the handling qudit (the qudit used to handle the quantum functionals) \( q_{h1} \) as the controls and the functional holder qudit \( q_f \) as the target. The circle with \( k_1 \) inside indicates \( q_{h1} \) controls the value of \( k_1 \) in the translation \( T(k_1,j_1) \), and the circle with \( j_1 \) inside indicates \( q_1 \) controls the value of \( j_1 \) in \( T(k_1,j_1) \). The same mechanism applies to how the 2nd gate is controlled by \( q_{h2} \) and \( q_2 \). \( q_f \) is initialized to \( |0\rangle \) so its value does not contribute to the functional created on it. Now if the handling qudits \( q_{h1} \) and \( q_{h2} \) take definite values \( |k_1\rangle \) and \( |k_2\rangle \), then the two 2-qudit-controlled-gates are just implementations of the \( CU_{l=m}(k_l) \) gate discussed above for adding \( k_l \) multiples of \( q_l \) to \( q_m \), and thus the circuit creates the qudit functional \( |k\rangle = |k_1 k_2\rangle = k_l q_l \oplus k_2 q_2 \) on \( q_f \): this means any controlled-gate using \( q_f \) as the control and either \( q_1 \) or \( q_2 \) as the target will use the partition of \( |k\rangle \) to determine its effect on the 2-qudit system of \( q_1 \) and \( q_2 \). However, if the handling qudits are set to be quantum states like
\( q_{h1} = \sum_{j=0}^{p-1} a_j |j\rangle \) and \( q_{h2} = \sum_{j=0}^{p-1} b_j |j\rangle \), then the circuit will create the quantum functional
\[
|\phi\rangle = \left( \sum_{k_{i}=0}^{p-1} a_{k_{i}} |k_{i}\rangle \right) \otimes \left( \sum_{k_{2}=0}^{p-1} b_{k_{2}} |k_{2}\rangle \right)
\]
on \( q_f \) : which means any controlled-gate using \( q_f \) as the control and either \( q_1 \) or \( q_2 \) as the target will use the partition of \( |k\rangle = |k_1 k_2\rangle = k_1 q_1 \oplus k_2 q_2 \) with the probability of \( |a_{k_1}|^2 \cdot |b_{k_2}|^2 \) – this is a typical quantum interpretation. Indeed, one can verify that any arbitrary 2-qudit quantum functional \( |\phi\rangle = \sum_{k_{1},k_{2}} b_{k_{1}k_{2}} |k_{1}k_{2}\rangle \) can be created on \( q_f \) by first initializing \( |\phi\rangle \) as a 2-qudit state on the handling qudits \( q_{h1} \) and \( q_{h2} \), and then applying the circuit in Figure 2. It is also straightforward to extend the 2-qudit case to any \( n \)-qudit case.

The wave-particle duality means any qudit-based quantum object has dual characters of particle and wave, and therefore any arbitrary particle-like quantum state \( |\psi\rangle = \sum_{q} a_{q} |q\rangle \) in the \( q \)-representation is inherently associated with a wave-like quantum functional \( |\phi\rangle = \sum_{k} b_{k} |k\rangle \) in the \( k \)-representation. Now given \( |\psi\rangle = \sum_{q} a_{q} |q\rangle \), we can easily perform the Fourier transform in Equation (5) to obtain the associated \( |\phi\rangle = \sum_{k} b_{k} |k\rangle \), and then realize \( |\phi\rangle \) with the quantum circuit in Figure 2 (as generalized to \( n \)-qudit). This means the quantum functional inherently associated with any quantum state can be realized and utilized with a simple quantum circuit setup.

2.3.3 The wavenumber \( k \) as a physical observable. The wave-particle duality means any qudit-based quantum object has dual characters of particle and wave, and therefore any quantum state in the \( q \)-representation can also be evaluated by an observable \( \hat{k} \) to get the expectation value of \( k \) inherently associated to the state. Focusing on a single dimension \( \hat{k} \) first, in the \( k \)-representation it is clearly a diagonal matrix:
\[
K_{k} = \text{diag}(0,1,..., p-1)
\]
and its form in the \( q \)-representation can be obtained by the Fourier-transform such that:
\[
K_{q} = F K_{k} F^{-1}, \quad F = \frac{1}{\sqrt{p}} \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & \omega_{p} & \cdots & \omega_{p}^{p-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \omega_{p}^{p-1} & \cdots & \omega_{p}^{(p-1)p}
\end{pmatrix}, \quad \omega_{p}^{p} = 1 \quad (11)
\]
where $F$ is the matrix form of the single-dimension Fourier transform from $k$ to $q$ as shown in Equation (5), and $\omega_p$ is the $p$-th root of 1. Clearly $K_q = F K_k F^{-1}$ gives the matrix form of the observable $\hat{k}$ in any single dimension of $\hat{k} = (\hat{k}_1, \hat{k}_2, ..., \hat{k}_n)$, and now given any quantum state $|\psi\rangle = \sum_q a_q |q\rangle = \sum_q a_q |q_1\rangle \otimes |q_2\rangle \otimes ... \otimes |q_n\rangle$, just apply $K_q$ to a qudit $q_j$ to evaluate $\langle \hat{k}_j \rangle$, and $\langle \hat{k} \rangle = (\langle \hat{k}_1 \rangle, \langle \hat{k}_2 \rangle, ..., \langle \hat{k}_n \rangle)$ will be evaluated after obtaining each $\langle \hat{k}_j \rangle$. Here we have implicitly used the fact that the commutator $[\hat{k}_j, \hat{k}_k] = 0$ so all the $\hat{k}_j$ can be evaluated simultaneously. Now if we consider the observable $\hat{q}_j$ that evaluates the expectation value of $q_j$ of a quantum state, then in the $q$-representation the matrix form of $\hat{q}_j$ is $Q_q = diag(0,1, ..., p-1)$, and we have:

$$[\hat{q}_j, \hat{k}_j] = [Q_q, K_k] \neq 0$$

(12)

which is similar to the fact that $[\hat{x}_j, \hat{p}_j] \neq 0$ in the usual position-momentum duality, and thus we can conclude that $\hat{q}_j$ and $\hat{k}_j$ are incompatible variables that cannot be measured simultaneously. Indeed, by the Fourier transform in Equation (5), if a wavefunction $\phi(k)$ is concentrated on a single $k$ then its $q$-representation $\psi(q)$ is uniformly distributed over $q$; similarly if a wavefunction $\psi(q)$ is concentrated on a single $q$ then its $k$-representation $\phi(k)$ is uniformly distributed over $k$. This leads to an entropic uncertainty principle [35-37] such that:

$$H_q + H_k > 0$$

$$H_q = -\sum_q |\psi(q)|^2 \log \left( |\psi(q)|^2 \right)$$

$$H_k = -\sum_k |\phi(k)|^2 \log \left( |\phi(k)|^2 \right)$$

(13)

where $H_q$ and $H_k$ are the information entropies for measuring into the quantum state and quantum functional respectively. Equation (13) means for any qudit-based quantum object, the sum of the information entropies for measuring into the $q$-representation and into the $k$-representation is strictly greater than zero. A tighter lower bound of $H_q + H_k$ can be derived in a future study and should depend on both the qudit dimension $p$ and the number of qudits $n$. The entropic perspective in Equation (13) may have potential applications in quantum encryption using quantum states as the ciphertexts [38], where maximizing $H_q + H_k$ or balancing $H_q$ and $H_k$ may increase the security of the quantum encryption method.

In the usual position-momentum $x$ - $k$ duality, the momentum operator $\hat{k}$ is related to the translation operator $\hat{T}(x)$ by $\hat{T}(x) = e^{-\imath x \hat{k}}$, and $\hat{k} = -\imath \nabla$ is obtained as the infinitesimal generator of $\hat{T}(x)$. The $q$ - $k$ spaces are discrete so there can be neither derivative nor infinitesimal
generator, but we recognize that on any single dimension of \( \hat{k} \), the observable \( \hat{k} \) is still related to the translation operator \( \hat{T}(q) \) by:

\[
\hat{T}(q) = e^{-\frac{2\pi i \hat{k}q}{\hat{p}}}, \quad \hat{T}(q)|q_0\rangle = |q_0 + q\rangle
\]

where \( \hat{T}(q) \) translates any state \( |q_0\rangle \) to \( |q_0 + q\rangle \), similar to the usual translation \( \hat{T}(x)|x_0\rangle = |x_0 + x\rangle \). In the \( k \)-representation the matrix of \( \hat{T}(q) \) is

\[
T_k(q) = \text{diag}(1, e^{-\frac{2\pi i q}{\hat{p}}}, e^{-\frac{2\pi i 2q}{\hat{p}}}, \ldots, e^{-\frac{2\pi i (p-1)q}{\hat{p}}})
\]

and in Equation (10) \( K_k = \text{diag}(0,1,\ldots,p-1) \), thus Equation (14) is correct, and for multiple dimensions we also have \( \hat{T}(q) = e^{-\frac{2\pi i \hat{k}q}{\hat{p}}} \). Although there cannot be any infinitesimal generator on a discrete space, \( \hat{k} \) can still be interpreted as inducing a translation in the \( q \) space.

2.4 The case of \( d \) not prime.

In all the discussions from the start of Section 2.1 we have assumed the dimension of a single qudit \( d \) is a prime number \( p \). Now when \( d \) is not a prime number, \( \mathbb{Z}_d \) is not a field. Mathematically, a field requires that any element of the field except 0 has a multiplicative inverse. In \( \mathbb{Z}_d \), if the non-prime \( d \) can be factored as \( p_1 \cdot p_2 \), then \( p_1 \cdot p_2 = d \mod d = 0 \) and thus neither \( p_1 \) nor \( p_2 \) can have an inverse. Now the consequence of \( \mathbb{Z}_d \) not being a field is although we can still define \( q = (q_1, q_2, \ldots, q_n) \) and \( k = (k_1, k_2, \ldots, k_n) = k_1 q_1 \oplus k_2 q_2 \oplus \ldots \oplus k_n q_n \) with \( q_j \) and \( k_j \) taking integer values from 0 to \( d - 1 \) and \( \oplus \) being addition modulo \( d \) – in the same way as Equations (1) and (2) – we can no longer consider \( q \) and \( k \) as vectors. This is because vector spaces must have scalars from a field. However, perhaps the most surprising thing is, other than \( q \) and \( k \) no longer being vectors, the entire discussion from Section 2.1 through Section 2.3 still holds when \( d \) is not a prime! This includes \( k \) being duals of \( q \), \( k \) generating its own quantum space, the Fourier transform between \( q \) and \( k \), the plane-wave interpretation of \( |k\rangle \), the partition interpretation of \( |k\rangle \), the physical realization of \( |k\rangle \) and \( |\phi\rangle = \sum_k b_k |k\rangle \), and the physical observable of \( \hat{k} \). The reason for this is the mathematical foundation of the entire discussion is the Pontryagin duality [34] that can be defined on any abelian group (i.e. a set with an addition operation defined on it): although \( \mathbb{Z}_d \) is not a field, it is nonetheless an abelian group and thus all the discussions from Section 2.1 through Section 2.3 follow because of Pontryagin duality.

Is the difference between \( d \) being prime and non-prime purely mathematical, or rather it could lead to certain physical consequences? This is an interesting question for a future study.
3. Conclusions

In this fundamental study, we have proposed the wave-particle duality on the qudit-based quantum space, with the quantum functionals acting as the wave-representation of any quantum state. The quantum functionals are quantum objects generated by the basis consisting of qudit functionals, which are the duals of the usual basis states of qudit-based quantum states. The relation between the quantum states and quantum functionals is analogous to the relation between the position and momentum in fundamental quantum physics. In particular, there is a Fourier transform between the quantum functionals and quantum states, and the quantum functionals have wave-like interpretations. Physically, the quantum functionals can be interpreted as partitions of the state space, and a quantum circuit has been designed to realize and utilize the quantum functionals. Any arbitrary qudit-based quantum state has dual characters of wave and particle, and an observable has been defined to evaluate the wavenumber of any given quantum state.

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