ON THE LOCAL WELLPOSEDNESS OF FREE BOUNDARY PROBLEM FOR THE NAVIER-STOKES EQUATIONS IN AN EXTERIOR DOMAIN

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Abstract. This paper deals with the local well-posedness of free boundary problems for the Navier-Stokes equations in an exterior domain. The problem is formulated as follows. Let \( \Omega_t \) be a domain in \( \mathbb{R}^N \) occupied at time \( t \) by an incompressible viscous fluid of density \( \rho \) and viscosity coefficient \( \mu \), where \( \rho \) and \( \mu \) are positive constants. Assume that \( \Omega_t = \mathbb{R}^N \setminus \Omega_t \) is a bounded domain. Let \( \Gamma_t \) and \( n_t \) be the boundary of \( \Omega_t \) and the unit outer normal to \( \Gamma_t \), respectively. We consider the free boundary problem of the Navier-Stokes equations formulated by

\[
\begin{align*}
\rho (\partial_t u + u \cdot \nabla u) - \text{Div} (\mu D(u) - pI) &= 0, & \text{div} u &= 0 \quad \text{in} \quad \cup_{0 < t < T} \Omega_t \times \{t\}, \\
(\mu D(u) - pI)n_t &= 0, & V_{\Gamma_t} &= \text{n}_t \cdot u \quad \text{on} \quad \cup_{0 < t < T} \Gamma_t \times \{t\}, \\
u|_{t=0} &= u_0, & \Omega_t|_{t=0} &= \Omega_0.
\end{align*}
\]

(1)

Here, \( u = (u_1, \ldots, u_N) \) is an \( N \)-vector of functions describing the velocity field, where \( \text{T} M \) denotes the transposed \( M \), \( p \) a scalar function describing the pressure field, and \( u_0 = (u_{01}, \ldots, u_{0N}) \) the prescribed initial data. Further, \( D(u) = \nabla u + \text{T} \nabla u \) is the doubled deformation tensor, \( I \) is the \( N \times N \) identity matrix, and \( V_{\Gamma_t} \) is the evolution speed of \( \Gamma_t \) along \( n_t \). Moreover, for any matrix field \( K \) with \( (i,j)^{th} \) component \( K_{ij} \), the quantity \( \text{Div} K \) is an \( N \)-vector whose \( i^{th} \) component is \( \sum_{j=1}^{N} \partial_j K_{ij} \) \( (\partial_j = \partial / \partial x_j) \), and for any \( N \)-vector \( w = (w_1, \ldots, w_N) \), let \( \text{div} w = \sum_{j=1}^{N} \partial_j w_j \) and the quantity \( w \cdot \nabla w \) be an \( N \)-vector with \( i^{th} \) component \( \sum_{j=1}^{N} w_j \partial_j w_i \). The unknowns are \( \Omega_t \), \( u \) and \( p \). We write \( \Omega_0 \), \( \Gamma_0 \) and \( n_0 \) as \( \Omega \), \( \Gamma \) and \( n \), respectively, which are given.

1. Introduction. This paper deals with the local well-posedness of free boundary problems for the Navier-Stokes equations in an exterior domain. The problem is formulated as follows. Let \( \Omega_t \) be a domain in \( \mathbb{R}^N \) occupied at time \( t \) by an incompressible viscous fluid of density \( \rho \) and viscosity coefficient \( \mu \), where \( \rho \) and \( \mu \) are positive constants. Assume that \( \Omega_t = \mathbb{R}^N \setminus \Omega_t \) is a bounded domain. Let \( \Gamma_t \) and \( n_t \) be the boundary of \( \Omega_t \) and the unit outer normal to \( \Gamma_t \), respectively. We consider the free boundary problem of the Navier-Stokes equations formulated by

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2000 Mathematics Subject Classification. Primary: 35Q35; Secondary: 76D07.

Key words and phrases. Exterior domains, Navier-Stokes equations, free boundary problem, without surface tension, local well-posedness, maximal \( L^p-L^q \) regularity, weak Dirichlet problem.

Partially supported by JSPS@Grant-in-aid for Scientific Research (A) - 17H0109, Top Global University Project, and JSPS program of the Japanese-German Graduate Externship.
As $\Omega_t$ is unknown, $\Omega_t$ is usually transformed to some fixed domain. One method is to use the Hanzawa transform (cf. [16, p.34]). However, this method is not applicable in our case. In fact, from the maximal regularity point of view, the $W_{q}^{3-1/q}$ regularity of a height function defining a free boundary is required, and such regularity is not guaranteed in our case. This regularity is, for example, guaranteed by the surface tension, but we do not take surface tension into account. Another known method is to use the Lagrange transformation (cf. Solonnikov, [30] and Shibata [21]). However, our final goal is to prove the global well-posedness, so this transformation does not fit our purpose. Let $\mathbf{v}$ be a velocity field in the Lagrangian coordinate. Nonlinear terms of the form: first derivatives of $\mathbf{v}$ appear in the equations transformed from Eq. (1). As the reference domain $\Omega$ is unbounded, solutions of the linearized equations have only a polynomial decay rate (cf. Shibata - Shimizu [27]), which is not sufficient to control the term: first derivatives of $\int_0^t \mathbf{v}(\xi, s) \, ds$ times the second derivatives of $\mathbf{v}$.

To overcome this difficulty, our idea is to use the Lagrange transformation only near the boundary. Let $R$ be a fixed positive number such that $\mathcal{O} = \mathcal{O}_0 \subset BR/2$, where $B = \{ x \in \mathbb{R}^N \mid |x| < L \}$, and let $\kappa$ be an element in $C_\infty^\infty(\mathbb{R}^N)$ such that $\kappa(x) = 1$ for $|x| \leq R$ and $\kappa(x) = 0$ for $|x| \geq 2R$. Let $\mathbf{v}(\xi, t)$ be a velocity field in the Lagrange coordinate $\xi = (\xi_1, \ldots, \xi_N)$ with

$$\mathbf{v} \in H^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), H^2_q(\Omega)).$$

As (cf. Tanabe [33, p.10])

$$H^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), H^2_q(\Omega)) \subset BUC([0,T], B^{2(1-1/p)}_{q,p}(\Omega)), \quad (2)$$

where $BUC$ denotes the set of all $B^{2(1-1/p)}_{q,p}(\Omega)$ valued bounded uniformly continuous functions on $[0,T]$ and the inclusion is continuous, the map

$$x = X_\mathbf{v}(\xi, t) := \xi + \int_0^t \kappa(\xi)\mathbf{v}(\xi, s) \, ds$$

has $C^{1+\delta}(\overline{\Omega} \times [0,T])$ regularity for some small $\delta > 0$ provided that $N < q < \infty$, $2 < p < \infty$ and $2/p + N/q < 1$. Assuming that

$$\int_0^T \|\kappa(\cdot)\mathbf{v}(\cdot, s)\|_{H^1_p(\Omega)} \, ds \leq \sigma \quad (3)$$

for small $\sigma > 0$ yields that the Jacobian matrix $\partial x/\partial \xi = I + \int_0^t \nabla(\kappa(\xi)\mathbf{v}(\xi, s)) \, ds$ is invertible. Thus, that $\Omega_t = \{ x = X_\mathbf{v}(\xi, t) \mid \xi \in \Omega \}$ is $C^{1+\delta}$ diffeomorphic to $\Omega$ is guaranteed by the kinetic condition: $V_\mathbf{T}_t = n_t \cdot \mathbf{u}$ on $\Gamma_t$. Moreover, writing the inverse matrix of $\partial x/\partial \xi$ by $A_\mathbf{v}$, we have

$$A_\mathbf{v} = I + \mathbf{V}_0(\int_0^t \nabla(\kappa\mathbf{v}) \, ds), \quad (4)$$

where $\mathbf{V}_0(k)$ is an $N \times N$ matrix of $C^\infty$ functions with respect to $k$ for $|k| < 2\sigma$ such that $V_0(0) = 0$, where $k = \{ k_{ij} \mid i, j = 1, \ldots, N \}$ is an $N^2$ real vector and $k_{ij}$ are the variables corresponding to $\int_0^t \partial \mathbf{v} / \partial \xi_j \, ds$. By (3)

$$\|A_\mathbf{v}\|_{L_\infty(\Omega \times (0,T))} \leq C \quad (5)$$
for some constant $C > 0$ independent of $v$. As $(\partial x/\partial \xi)(\partial \xi/\partial x) = I$, we have

\[
(I + V_0(\int_0^t \nabla(\kappa v) \ ds))(I + \int_0^t \nabla(\kappa v) \ ds) = (I + \int_0^t \nabla(\kappa v) \ ds)(I + V_0(\int_0^t \nabla(\kappa v) \ ds)) = I.
\] (6)

Let $\tilde{n}$ be an $N$-vector of $C^2$ functions defined on $\mathbb{R}^N$ such that $\tilde{n} = n$ on $\Gamma$ and $||\tilde{n}||_{H^2_\omega(\mathbb{R}^N)} \leq C$. In the following, we write $\tilde{n}$ as $n$. Let

\[
u(x, t) = v(\xi, t) = (v_1(\xi, t), \ldots, v_N(\xi, t)), \quad p(x, t) = q(\xi, t),
\]

\[\Omega_t = \{x = X_v(\xi, t) \mid \xi \in \Omega\}.
\]

Let $V_{0ij}(k)$ be the $(i, j)^{th}$ component of the $N \times N$ matrix $V_0(k)$, and let $\nabla_x = \nabla_x^\top(\partial/\partial x_1, \ldots, \partial/\partial x_N)$ and $\nabla_\xi = \nabla_\xi^\top(\partial/\partial \xi_1, \ldots, \partial/\partial \xi_N)$. Then, we have

\[
\nabla_x = (I + V_0(k)) \nabla_\xi, \quad \frac{\partial}{\partial x_i} = \sum_{j=1}^N (\delta_{ij} + V_{0ij}(k)) \frac{\partial}{\partial \xi_j},
\] (7)

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. Let $J$ be the Jacobian of the transformation: $x = X_v(\xi, t)$. Choosing $\sigma > 0$ smaller if necessary, we may write $J = J(k) = 1 + J_0(k)$, where $J_0(k)$ is a $C^\infty$ function defined for $|k| < 2\sigma$ such that $J_0(0) = 0$. Then, we have

\[
D(u) = D(v) + D_D(v), \quad J \text{div} u = \text{div} v - G(v) = \text{div} v - \text{div} G(v),
\]

where $D_D(v)$ is an $N \times N$ matrix whose $(i, j)^{th}$ component $D_D(v)_{ij}$ is given by

\[
D_D(v)_{ij} = \sum_{k=1}^N (V_{0jk}(k) \frac{\partial v_j}{\partial \xi_k} + V_{0ik}(k) \frac{\partial v_i}{\partial \xi_k});
\] (8)

$G(v)$ is a scalar function given by

\[
G(v) = -\sum_{j,k=1}^N V_{0jk}(k) \frac{\partial v_j}{\partial \xi_k};
\] (9)

and $G(v)$ is an $N$-vector whose $k^{th}$ component $G(v)_k$ is given by

\[
G(v)_k = -(\sum_{j=1}^N V_{0jk}(k)v_j + J_0(k) \sum_{j=1}^N (\delta_{jk} + V_{0jk}(k))v_j).
\] (10)

Moreover, we have

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = \frac{\partial v}{\partial t} + (1 - \kappa)F_1(v), \quad \text{Div} D(u) = \text{Div} D(v) + F_2(v)
\]

where $F_1(v)$ is an $N$-vector whose $i^{th}$ component $F_1(v)_i$ is given by

\[
F_1(v)_i = \sum_{j,k=1}^N (\delta_{jk} + V_{0jk}(k))v_j \frac{\partial v_i}{\partial \xi_k};
\] (11)

and $F_2(v)$ is an $N$-vector whose $i^{th}$ component $F_2(v)_i$ is given by

\[
F_2(v)_i = \sum_{j,k=1}^N (\delta_{jk} + V_{0jk}(k)) \frac{\partial}{\partial \xi_k} D_D(v)_{ij}.
\] (12)
As
\[ n_t = \frac{\tau A_\nu \tilde{n}}{|\tau A_\nu \tilde{n}|}, \]
the boundary condition: 
\[ (\mu D(u) - pI)n_t = 0 \]
is transformed to
\[ 0 = (\mu(D(v) + D_D(v)) - qI)^\top (I + V_0(k))\tilde{n}, \]
and so we have
\[ \mu(I + \tau k)(D(v) + D_D(v))(I + \tau V_0(k))\tilde{n} - q\tilde{n} = 0 \]
on \(\Gamma \times (0, T)\).

Thus, noting (6), we see that Eq. (1) is transformed to
\[ G \]
where \(N\) vectors given by
\[
\begin{array}{l}
G \\
\end{array}
\]
possessing the estimate:
\[ (3) \]
for some constant \(C > 0\).

For small \(\sigma > 0\), the transformation: 
\[ x = \xi + \int_0^t \kappa(\xi) v(\xi, s) ds = X_\nu(\xi, t) \]
is injective. Thus, problem (1) admits unique solutions:
\[ \Omega_t = \{x = X_\nu(\xi, t) \mid \xi \in \Omega\}, \]
\[ u(x, t) = v(\xi, t), \quad p(x, t) = q(\xi, t). \]
To explain the reason why we use a partial Lagrange transform $x = X_\nu(\xi, t)$ with a cut-off function $\kappa$, we explain our strategy to prove the global well-posedness, which is the final goal of our study of Eq. (1). The global well-posedness is proved by the prolongation of local (in time) solutions of problem (13) to any time interval with the help of uniform estimates of several different $L_p - L_q$ space-time norms of local (in time) solutions. To derive uniform estimates, we divide $\int_0^T \nabla (\kappa v) \, ds$ into two parts as follows:

$$\int_0^t \nabla (\kappa v) \, ds = \int_0^T \nabla (\kappa v) \, ds - \int_t^T \nabla (\kappa v) \, ds,$$

because the second term of the right hand side gives the decay properties of nonlinear terms. By Taylor’s formula, we divide $V_{0ij}$ into two parts as follows:

$$V_{0ij} (\int_0^T \nabla (\kappa v) \, ds) = a_{ij}(T) + b_{ij}(t)$$

with

$$a_{ij}(T) = V_{0ij} (\int_0^T \nabla (\kappa v) \, ds),$$

$$b_{ij}(t) = - \sum_{l,m=1}^N \int_0^1 \nabla V_{0ij} (\theta) \nabla (\kappa v) \, d\theta \int_0^T \nabla (\kappa v) \, ds \theta \int_t^T \nabla (\kappa v) \, ds \theta \int_0^T \frac{\partial}{\partial \xi^l} (\kappa v_m) \, ds.$$  

Let $\tilde{S}(v, q)$ and $\tilde{\nabla} v$ be operators obtained from $\mu D(u) - p I$ and $\tilde{\nabla} u$ by the change of variables: $x = \xi + \int_0^T \tilde{\nabla} (\kappa v) \, ds$, and then Eq. (13) is transformed into

$$\begin{align*}
\partial_t v - J^{-1} \text{Div} \tilde{S}(v, q) &= \tilde{F}(v) & \text{in } \Omega \times (0, T), \\
\tilde{\nabla} v &= \tilde{G}(v) = \tilde{\nabla} \tilde{G}(v) & \text{in } \Omega \times (0, T), \\
\tilde{S}(v, q) \mathbf{n} &= \tilde{H}(v) & \text{on } \Gamma \times (0, T), \\
v |_{\xi=0} &= u_0 & \text{in } \Omega.
\end{align*}$$  

(16)

Here, $\tilde{F}(v)$, $\tilde{G}(v)$, $\tilde{G}(v)$ and $\tilde{H}(v)$ denote nonlinear terms defined similarly to $F(v)$, $G(v)$, $G(v)$, and $H(v)$, respectively. The important point is that these nonlinear terms consist of $\int_0^T \tilde{\nabla} (\kappa v) \, ds N(v)$ multiplied by second derivatives of $v$ or $\partial_t v$, and $\int_0^T \tilde{\nabla}^2 (\kappa v) \, ds N(v)$ multiplied by the first derivatives of $v$, where $N(v)$ is some function with respect to $\int_0^T \tilde{\nabla} (\kappa v) \, ds - \theta \int_t^T \tilde{\nabla} (\kappa v) \, ds$ ($0 \leq \theta \leq 1$).

To obtain uniform estimates of local (in time) solutions, it is essential to prove the so-called $L_p - L_q$ decay estimates: for any $t > 0$,

$$\begin{align*}
|w|_{L_p(\Omega)} &\leq Ct^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{q})} \|w_0\|_{L_q(\Omega)}, \\
\|\nabla w\|_{L_p(\Omega)} &\leq Ct^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{q})} \|w_0\|_{L_q(\Omega)},
\end{align*}$$  

(17)

hold for any solutions of the Stokes equations:

$$\begin{align*}
\partial_t w - J^{-1} \text{Div} \tilde{S}(w, r) &= 0, & \tilde{\nabla} w &= 0 & \text{in } \Omega \times (0, \infty), \\
\tilde{S}(w, r) \mathbf{n} &= 0 & \text{on } \Gamma \times (0, \infty), \\
w |_{\xi=0} &= w_0 & \text{in } \Omega,
\end{align*}$$  

(18)

provided that $1 < q \leq p \leq \infty$ and $q \neq \infty$.

One reason for using the cut-off function $\kappa$ is that $\partial_t w - J^{-1} \text{Div} \tilde{S}(w, r) = \partial_t w - \text{Div} (\mu D(w) - r I)$ and $\tilde{\nabla} w = \tilde{\nabla} w$ outside of $B_R$, so the system of equations in (18)
can be regarded as a compact perturbation from the usual Stokes equations with a free boundary condition. The $L_p$-$L_q$ decay properties corresponding to solutions of (17) have been proved by Shibata and Shimizu [27]. In Shibata [25], it is shown that choosing $\sigma > 0$ to be sufficiently small gives (17).

Another reason for using $\kappa$ is that the nonlinear terms can be estimated as follows:

$$
\| \int_t^T \nabla(\kappa v) ds \partial_t v, \nabla^2 v \|_{L_r(\Omega)} \leq C_R \int_t^T \| \nabla(\kappa v) \|_{L_\infty(\Omega)} ds \| (\partial_t v, \nabla^2 v) \|_{L_q(\Omega)},
$$

$$
\| \int_t^T \nabla^2(\kappa v) ds \nabla v \|_{L_r(\Omega)} \leq C_R \int_t^T \| \kappa v \|_{H^2_\| (\Omega)} ds \| \nabla v \|_{L_\infty(\Omega)}
$$

for any $r$ with $1 \leq r \leq q$, where $C_R$ is a constant depending on $R$, $r$ and $q$. Here, the fact that the support of $\kappa$ is contained in $B_{2R}$ is essential. From these considerations, we finally drive a sufficiently uniform estimate of the local (in time) solutions to prove the global well-posedness of problem (13) [24].

The local and global well-posedness of free boundary problems of the Navier-Stokes equations have been studied in the bounded domain, half-space, exterior domains, and layer cases by many authors: (e.g., [1, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 18, 21, 23, 30, 31, 34, 35, 36], and references therein). However, none of the papers mentioned above deal with global well-posedness in the exterior domain case.

To prove the global well-posedness of the quasilinear parabolic equations in unbounded domains such as the whole space or half space, where only polynomial decay properties can be obtained, it is essential to treat the problem in an $L_p$-$L_q$ maximal regularity framework (cf. [18], [19], [24]). This is because $L_p$ integrability on the whole time interval $(0, \infty)$ is guaranteed by choosing a suitably large index $p$ with the help of the pointwise time decay property of solutions in the $L_q$ space. From this point of view, the maximal $L_p$-$L_q$ regularity theorem for the Stokes equations with free boundary conditions is another subjects of this paper. This theorem is proved by combining the existence of $\mathcal{R}$ bounded solution operators for the corresponding generalized resolvent problem [20, 22] with the Weis operator valued Fourier multiplier theorem [37].

The present paper is organized as follows. In Sect. 2, we give several preparatory results concerning $\mathcal{R}$ bounded solution operators, Bessel potential spaces, some rapidly decaying functions that attain given functions in $B^2_{q,p}(\Omega)$ and $B^{1,2}_{q,p}(\Omega)$, and the maximal $L_p$-$L_q$ regularity theorem for the free boundary problem of the Stokes equations in a uniform $C^2$ domain. In Sect. 3, the maximal $L_p$-$L_q$ regularity theorem for the Stokes equations with free boundary conditions is proved in the exterior domain case. The key is to prove the unique existence theorem for solutions to the weak Dirichlet problem. In Sect. 4, the local well-posedness is proved. In Appendix A, some comments about the uniqueness of solutions of the weak Dirichlet problems are given, because there is a counter example for the uniqueness theorem in the strong Dirichlet problem case. In Appendix B, the unique existence theorem for the strong Dirichlet-Neumann problem (Lemma 3.4) is studied in the bounded domain case. Lemma 3.4 seems to be well-known, but the author could not find any proof, thus Lemma 3.4 is proved, because it gives an important step in our proof.

Notation. Finally, we explain symbols used in this paper. We denote the sets of all complex numbers, real numbers and natural numbers by $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$, respectively.
Let $N_0 = N \cup \{0\}$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in N_0^N$ we set $\partial_\alpha h = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N} h$ with $\partial_i = \partial / \partial x_i$. In particular, for scalars, $\theta$, and $N$-vectors, $u = (u_1, \ldots, u_N)$, functions, and $n \in N_0$, we set $\nabla^n \theta = (\partial^{\alpha} \theta \mid |\alpha| = n)$ and $\nabla^n u = (\nabla^n u_j \mid j = 1, \ldots, N)$. In particular, $\nabla^0 \theta = \theta$, $\nabla^1 \theta = \nabla \theta$, $\nabla^0 u = u$ and $\nabla^1 u = \nabla u$. We use bold lowercase letters to denote $N$-vectors and bold capital letters to denote $N \times N$ matrices.

For an $N$ vector $a$, $a_i$ denotes the $i$th component of $a$ and for an $N \times N$ matrix $A$, $A_{ij}$ denotes the $(i,j)^{th}$ component of $A$, and moreover, the $N \times N$ matrix whose $(i,j)^{th}$ component is $K_{ij}$ is written as $(K_{ij})$. Let $\delta_{ij}$ be the Kronecker delta symbol, that is $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. In particular, $I = (\delta_{ij})$ is the $N \times N$ identity matrix. For any $N$-vectors $a$ and $b$, let $a \cdot b = \sum_{i=1}^{N} a_i b_j$. For any $N$-vector $a$, let $a_r = a - <a,n> n$. Given $1 < q < \infty$, let $\beta_q = q/(q-1)$. For $L > 0$, let $B_L = \{x \in \mathbb{R}^N \mid |x| < L\}$ and $S_L = \{x \in \mathbb{R}^N \mid |x| = L\}$. Throughout this paper, $R > 0$ is a fixed positive number such that $\varnothing = \mathbb{R}^N \setminus \Omega \subset B_{R/2}$ and $\kappa$ is a fixed function in $C_0^\infty(\mathbb{R}^N)$ such that $\kappa(x) = 1$ for $x \in B_R$ and $\kappa(x) = 0$ for $x \notin B_2R$. For $0 < R \leq L$, let $\Omega_L = \Omega \cap B_L$; for $R \leq L_1 < L_2$, let $D_{L_1,L_2} = \{x \in \mathbb{R}^N \mid L_1 \leq |x| \leq L_2\}$. For any domain $G$ in $\mathbb{R}^N$, let $L_q(G)$, $H^{N}_q(G)$, and $B^*_q(G)$ be the standard Lebesque, Sobolev, and Besov spaces on $G$, and let $\| \cdot \|_{L_q(G)}$, $\| \cdot \|_{H^{N}_q(G)}$, and $\| \cdot \|_{B^*_q(G)}$ denote their respective norms. We write $L_q(G)$ as $H^0_q(G)$, and $B^*_q(G)$ as simply $W^*_q(G)$. For a Banach space $X$ with norm $\| \cdot \|_X$, let $X^d = \{(f_1, \ldots, f_d) \mid f_i \in X \ (i = 1, \ldots, d)\}$, and write the norm of $X^d$ as simply $\| \cdot \|_X$, which is defined by $\|f\|_X = \sum_{j=1}^{d} \|f_j\|_X$ for $f = (f_1, \ldots, f_d) \in X^d$.

Let $(u,v)_G = \int_G u \cdot v \, dx$ and let $(u,v)_{\partial G} = \int_{\partial G} \partial u \cdot \nu d\omega$, where $d\omega$ denotes the surface element on $\partial G$. For $1 \leq p \leq \infty$, $L_p((a,b),X)$ and $H^{N}_p((a,b),X)$ denote the standard Lebesque and Sobolev spaces of $X$-valued functions defined on an interval $(a,b)$, and $\| \cdot \|_{L_p((a,b),X)}$, $\| \cdot \|_{H^{N}_p((a,b),X)}$ denote their respective norms. Let $C_\infty^\infty(G)$ be the set of all $C^\infty$ functions whose supports are compact and contained in $G$. For two Banach spaces $X$ and $Y$, $X + Y = \{x + y \mid x \in X, y \in Y\}$, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from $X$ into $Y$ and $\mathcal{L}(X,Y)$ is written simply as $\mathcal{L}(X)$. For a domain $U$ in $\mathcal{C}$, $\text{Hol}(U,\mathcal{L}(X,Y))$ denotes the set of all $\mathcal{L}(X,Y)$-valued holomorphic functions defined on $U$. Let $\mathcal{R}(\mathcal{L}(X,Y))$ be the $\mathcal{R}$ norm of the operator family $\mathcal{F}(\lambda) \in \text{Hol}(U,\mathcal{L}(X,Y))$.

Let $\Sigma_{\varepsilon} = \{\lambda \in \mathcal{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \varepsilon\}$, $\Sigma_{\varepsilon,\lambda_0} = \{\lambda \in \Sigma_{\varepsilon} \mid |\lambda| \geq \lambda_0\}$. Moreover, the letter $C$ denotes a generic constant and $C_{a,b,c,\ldots}$ denotes that the constant $C_{a,b,c,\ldots}$ depends on $a$, $b$, $c$, $\ldots$. The value of $C$ and $C_{a,b,c,\ldots}$ may change from line to line.
2. Preliminaries. In this section, $\Omega$ is assumed to be a uniform $C^2$ domain and $n$ denotes the unit outer normal to the boundary $\Gamma$ of $\Omega$.

2.1. $\mathcal{R}$ bounded solution operators. In this subsection, we quote a result given by Shibata [20, 22] concerning the $\mathcal{R}$ bounded solution operators for the generalized resolvent problem of the Stokes equations with free boundary conditions formulated as follows:

$$
\begin{aligned}
\lambda u - \text{Div} (\mu Du - pI) &= f, & \text{div } u &= g & \text{in } \Omega, \\
(\mu Du - pI)n &= h & \text{on } \Gamma.
\end{aligned}
$$

To state results concerning the $\mathcal{R}$-bounded solution operators, we first consider the weak Dirichlet problem:

$$
(\nabla u, \nabla \varphi)_\Omega = (f, \nabla \varphi)_\Gamma \quad \text{for any } \varphi \in \dot{H}^{1,0}_q(\Omega),
$$

where $\dot{H}^{1,0}_r(\Omega) = \{u \in L_r,\text{loc}(\Omega) \mid \nabla u \in L_r(\Omega)^N, \ u|_{\Gamma} = 0\}$ for $r \in (1, \infty)$.

**Definition 2.1.** Let $1 < q < \infty$. We say that the weak Dirichlet problem is uniquely solvable with an exponent $q \in (1, \infty)$, if the following assertion holds: For any $f \in L_q(\Omega)^N$ and $g \in W^{1,1/q}_q(\Gamma)$, there exists a unique $u \in H^1_q(\Omega) + \dot{H}^{1,0}_q(\Omega)$ that satisfies the variational equation (20) and the estimate: $\|\nabla \vartheta\|_{L_q(\Omega)} \leq C_q \|f\|_{L_q(\Omega)}$ for some constant $C_q$ independent of $f$, $\vartheta$ and $\varphi$.

**Remark 2.** If the weak Dirichlet problem is uniquely solvable with an exponent $q \in (1, \infty)$, then given $f \in L_q(\Omega)^N$ and $g \in W^{1,1/q}_q(\Gamma)$, there exists a unique $u \in H^1_q(\Omega) + \dot{H}^{1,0}_q(\Omega)$ that satisfies the variational equation (20) subject to $u = g$ on $\Gamma$, where $H^1_q(\Omega) + \dot{H}^{1,0}_q(\Omega) = \{p_1 + p_2 \mid p_1 \in H^1_q(\Omega), p_2 \in \dot{H}^{1,0}_q(\Omega)\}$.

Next, we introduce the $\mathcal{R}$ boundedness and the Weis operator valued Fourier Multiplier Theorem [37].

**Definition 2.2.** Let $X$ and $Y$ be two Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is said to be $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $q \in [1, \infty)$ such that for each $n \in \mathbb{N}$, $\{T_j\}^n_{j=1} \subset \mathcal{T}$, $\{f_j\}^n_{j=1} \subset X$ and for all sequences $\{r_j(u)\}^n_{j=1}$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, the following inequality holds:

$$
\int_0^1 \left\| \sum_{j=1}^n r_j(u)T_jf_j \right\|_Y^q \, du \leq C \int_0^1 \left\| r_j(u)f_j \right\|_X^q \, du. \tag{21}
$$

The smallest such $C$ is called the $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

The following theorem was given by Weis [37].

**Theorem 2.3.** Let $X$ and $Y$ be two UMD Banach spaces and $1 < p < \infty$. Let $M$ be a function in $C^\ell([\mathbb{R}\setminus\{0\}], \mathcal{L}(X,Y))$ such that

$$
\mathcal{R}_{\mathcal{L}(X,Y)}\{((\tau \frac{d}{d\tau})^\ell M(\tau) \mid \tau \in \mathbb{R}\setminus\{0\})\} \leq \kappa < \infty \quad (\ell = 0, 1)
$$

for some constant $\kappa$. Then, the operator

$$
T_M[f] = \mathcal{F}^{-1}[M(\cdot)\mathcal{F}[f](\cdot)] \quad f \in \mathcal{S}(\mathbb{R}, X)
$$

is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by $T_M$, we have

$$
\|T_M\|_{\mathcal{L}(L_p(\mathbb{R}, X), L_p(\mathbb{R}, Y))} \leq C\kappa
$$
for some positive constant $C$ depending on $p$, $X$ and $Y$. Here, $F$ and $F^{-1}$ denote the Fourier transform and inverse Fourier transform, and $S(\mathbb{R}, X)$ is the Schwartz space of $X$-valued functions defined on $\mathbb{R}$.

**Remark 3.** For the definition of UMD space, we refer to a book written by Amann [2, p.141]. For $1 < q < \infty$, $L_q(\Omega)$ and $H^m_q(\Omega)$ $(m \in \mathbb{N})$ are both UMD spaces.

Finally, we consider the divergence equation: $\text{div} \, u = g$. Let $DI_q(\Omega)$ be a space defined by

$\text{div} \, g = g \in \Omega$ \quad \text{such that} \quad (g, \varphi)_\Omega = -(G, \nabla \varphi)_\Omega \quad \text{for any} \quad \varphi \in H^1_{q,0}(\Omega), \tag{22}$

where $H^1_{q,0}(\Omega) = \{ \varphi \in H^1_q(\Omega) \mid \varphi|_{\partial \Omega} = 0 \}$. Let $\mathcal{G}(g) = \{ H \in L_q(\Omega)^N \mid \text{div} \, G = \text{div} \, H \} \quad \text{and} \quad \mathcal{G}(g)$ denotes the representative elements of the set $\mathcal{G}(g)$. Where there is no possibility of confusion, we write $\mathcal{G}(g)$ as $\mathcal{G}(g)$ for simplicity. $DI_q(\Omega)$ is the data space for the divergence equation: $\text{div} \, u = g \in \Omega$. As $C^\infty_0(\Omega) \subset H^1_{q,0}(\Omega)$, $\text{div} \, \mathcal{G}(g) = g$ in a distribution sense. Moreover, if $g = \text{div} \, G$ for some $G \in L_q(\Omega)^N$, then $G = \mathcal{G}(g)$.

In this paper, we say that $u \in H^1_q(\Omega)^N$ satisfies $\text{div} \, u = g$ in $\Omega$, if

$$(u, \nabla \varphi)_\Omega = (\mathcal{G}(g), \nabla \varphi)_\Omega \quad \text{for any} \quad \varphi \in \dot{H}^1_{q,0}(\Omega). \tag{23}$$

The following theorem, which has been proved by Shibata [20, 22], states the existence of an $\mathcal{R}$ bounded solution operator for problem (19).

**Theorem 2.4.** Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Assume that $\Omega$ is a uniform $C^2$ domain and that the weak Dirichlet problem is uniquely solvable with exponents $q$ and $q'$. Let

$X_q(\Omega) = \{ (f, g, h) \mid f \in L_q(\Omega)^N, \quad g \in DI_q(\Omega), \quad h \in H^1_q(\Omega) \},$

$X_q(\Omega) = \{ (f_1, \ldots, f_6) \mid f_1, f_2, f_3 \in L_q(\Omega)^N, \quad f_3 \in L_q(\Omega), \tag{24}$

$\quad f_4 \in H^1_q(\Omega), \quad f_6 \in H^1_q(\Omega)^N \}.$

Then, there exist a constant $\lambda_0 \geq 1$ and operator families: $\textbf{A}(\lambda)$ and $\textbf{P}(\lambda)$ with

$\textbf{A}(\lambda) \in \text{Hol} \left( \Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q(\Omega), H^2_q(\Omega)^N) \right),$

$\textbf{P}(\lambda) \in \text{Hol} \left( \Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q(\Omega), H^1_q(\Omega) + H^1_{4,0}(\Omega)) \right)$

such that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ and $(f, g, h) \in X_q(\Omega)$, $u = \textbf{A}(\lambda)F_\lambda(f, g, h)$ and $p = \textbf{P}(\lambda)F_\lambda(f, g, h)$, where $F_\lambda(f, g, h) = (f, \lambda \mathcal{G}(g), \lambda^{1/2}g, g, \lambda^{1/2}h, h)$, are unique solutions to (18), and

$\mathcal{R}_{\mathcal{L}(X_q(\Omega), H^2_q(\Omega)^N)}(\{ (\tau \partial_{\tau})^j (\lambda^{1/2} \textbf{A}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0} \} \leq \gamma_* \quad (j = 0, 1, 2),$

$\mathcal{R}_{\mathcal{L}(X_q(\Omega), H^1_q(\Omega) + H^1_{4,0}(\Omega))}(\{ (\tau \partial_{\tau})^j (\nabla \textbf{P}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0} \} \leq \gamma_* \quad (j = 0, 1),$ with a constant $\gamma_* > 0$ for $\ell = 0, 1$. Here and in the following, $\lambda$ represents a complex number with $\lambda = \gamma + i\tau \in \mathbb{C}$.

**Remark 4.** In Theorem 2.4, $F_1, F_2, F_3, F_4, F_5,$ and $F_6$ are variables corresponding to $f, \lambda \mathcal{G}(g), \lambda^{1/2}g, g, \lambda^{1/2}h,$ and $h$, respectively. The norms $\| \cdot \|_{X_q(\Omega)}$ and $\| \cdot \|_{X_q(\Omega)}$ of the spaces $X_q(\Omega)$ and $X_q(\Omega)$ are defined by

$\| (f, g, h) \|_{X_q(\Omega)} = \| f \|_{L_q(\Omega)} + \| g \|_{H^1_q(\Omega)} + \| \mathcal{G}(g) \|_{L_q(\Omega)} + \| h \|_{W^1_q(\Omega)},$

$\| (F_1, \ldots, F_6) \|_{X_q(\Omega)} = \| (F_1, F_2, F_3, F_5) \|_{L_q(\Omega)} + \| (F_4, F_6) \|_{H^1_q(\Omega)}. \tag{25}$
2.2. About $H_p^{1/2}(\mathbb{R}, H^1_q(\Omega))$. Let $X$ be a UMD Banach space. For any $1 < p < \infty$ and $\alpha \in \mathbb{R}$, let
\[
H^\alpha_p(\mathbb{R}, X) = \{ f \in \mathcal{S}'(\mathbb{R}, X) \mid \| f \|_{H^\alpha_p(\mathbb{R}, X)} = \| \Lambda^\alpha f \|_{L_p(\mathbb{R}, X)} < \infty \},
\]
where $\Lambda^\alpha f = \mathcal{F}^{-1}[(1 + s^2)^{\alpha/2}\mathcal{F}[f](s)]$. Using the Weis operator valued Fourier multiplier theorem and employing the same argument as in Calderón [7], we can extend the properties of Bessel potential spaces on $\mathbb{R}$ to the $X$-valued case. For example, we have
\[
H^m_p(\mathbb{R}, X) = \{ f \in L_p(\mathbb{R}, X) \mid \| \partial^j f \|_{L_p(\mathbb{R}, X)} < \infty \text{ for any } j = 1, \ldots, m \} \quad (m \in \mathbb{N})
\]
in which $\| f \|_{H^m_p(\mathbb{R}, X)}$ and $\| f \|_{L_p(\mathbb{R}, X)} + \sum_{j=1}^m \| \partial^j f \|_{L_p(\mathbb{R}, X)}$ are equivalent norms, where $\partial^j$ denotes the $j$-th derivative of $f$. If $\alpha < \beta$, then
\[
\| f \|_{H^\alpha_p(\mathbb{R}, X)} \leq C_{\alpha, \beta} \| f \|_{H^\beta_p(\mathbb{R}, X)}
\]
for any $f \in H^\alpha_p(\mathbb{R}, X)$ with some constant $C_{\alpha, \beta}$. If $f \in H^\alpha_p(\mathbb{R}, X)$ with $\alpha > 1/p$, then $f \in C^{\alpha - 1/p - \epsilon}(\mathbb{R}, X)$ for any $\epsilon > 0$ with $\alpha - 1/p > \epsilon$. Moreover, in the case where $\alpha = s + \sigma$ with $s \in \mathbb{N}_0$ and $0 < \theta < 1$,
\[
H^{s+\theta}_p(\mathbb{R}, X) = (H^s_p(\mathbb{R}, X), H^{s+1}_p(\mathbb{R}, X))_{[\theta]},
\]
where $(\cdot, \cdot)_{[\theta]}$ denotes a complex interpolation functor.

**Proposition 1.** Let $1 < p, q < \infty$ and let $\Omega$ be a uniformly $C^2$ domain in $\mathbb{R}^N$. Then, there exists a constant $C$ such that for any $u \in H^1_p(\mathbb{R}, L^q(\Omega)) \cap L^p(\mathbb{R}, H^2_q(\Omega))$ we have
\[
\| u \|_{H^{1/2}_p(\mathbb{R}, H^1_q(\Omega))} \leq C \left( \| u \|_{H^1_p(\mathbb{R}, L^q(\Omega))} + \| u \|_{L^p(\mathbb{R}, H^2_q(\Omega))} \right).
\]

In the following, we prove Proposition 1. A lemma given by Enomoto-Shibata [8, Theorem 3.3] is used to prove the $\mathcal{R}$-boundedness of operator families.

**Lemma 2.5.** Let $1 < q < \infty$ and let $\Sigma$ be a set in $\mathbb{C}$. Let $m(\lambda, \xi)$ be a function defined on $\Sigma \times (\mathbb{R}^N \setminus \{0\})$ such that for any multi-index $\alpha \in \mathbb{N}_0^N$, there exists a constant $C_\alpha$ depending on $\alpha$ and $\Sigma$ such that
\[
|\partial_\xi^\alpha m(\lambda, \xi)| \leq C_\alpha |\xi|^{-|\alpha|}
\]
for any $(\lambda, \xi) \in \Sigma \times (\mathbb{R}^N \setminus \{0\})$. Let $K_\ell f = \mathcal{F}^{-1}[m(\lambda, \xi)\mathcal{F}[f](\xi)]$. Then,
\[
\mathcal{R}_{\mathcal{L}(L^q(\mathbb{R}^N))}(\{ K_\ell \mid \ell \in \Sigma \}) \leq C \max_{|\alpha| \leq N+2} C_\alpha
\]
for some constant $C$ depending solely on $q$ and $N$.

To prove Proposition 1, in view of Theorem 2.3, it is sufficient to prove that there exist $\Phi_1^\ell \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(L^q(\Omega)))$ and $\Phi_2^\ell \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(H^2_q(\Omega), L^q(\Omega)))$ such that
\[
\begin{align*}
&(1 + \lambda^2)^{\ell/2} g = \Phi_1^\ell(\lambda)(1 + \lambda^2)^{\ell/2} g + \Phi_2^\ell(\lambda) g \quad \text{for any } g \in H^2_q(\Omega), \quad (28) \\
&\mathcal{R}_{\mathcal{L}(L^q(\Omega))}(\{ (\lambda \partial_\lambda)^\ell \Phi_1^\ell(\lambda) \mid \lambda \in \mathbb{R} \setminus \{0\} \}) \leq \gamma \quad (\ell = 0, 1), \quad (29) \\
&\mathcal{R}_{\mathcal{L}(H^2_q(\Omega), L^q(\Omega))}(\{ (\lambda \partial_\lambda)^\ell \Phi_2^\ell(\lambda) \mid \lambda \in \mathbb{R} \setminus \{0\} \}) \leq \gamma \quad (\ell = 0, 1) \quad (30)
\end{align*}
\]
for some constant $\gamma$. In fact, setting $g = \mathcal{F}[f](x, \lambda) = \int_{\mathbb{R}} e^{-i\lambda t} f(x, t) dt$ for $f = f(x, t)$ in (28) and using Theorem 2.3 gives
\[
\| f \|_{H^1_p(\mathbb{R}, H^2_q(\Omega))} \leq \| f \|_{H^1_p(\mathbb{R}, L^q(\Omega))} + \sum_{k=1}^N \| \mathcal{F}^{-1}[(1 + \lambda^2)^{\ell/2} \mathcal{F}[\partial_\lambda f]] \|_{L^p(\mathbb{R}, L^q(\Omega))}.
\]
\[
\leq C\|f\|_{H^1_\ell(R,L^q(\Omega))} + \sum_{k=1}^N \left\{ \|\mathcal{F}^{-1}[\Phi_k^\ell(\lambda)\mathcal{F}[\Lambda^1 f](\lambda)]\|_{L^p(R,L^q(\Omega))} + \|\mathcal{F}^{-1}[\Phi_k^\ell(\lambda)\mathcal{F}[f](\lambda)]\|_{L^p(R,L^q(\Omega))} \right\}
\]

\[
\leq (C + \gamma)\|f\|_{H^1_\ell(R,L^q(\Omega))} + \gamma\|f\|_{L^p(R,H^1_\ell(\Omega))},
\]

which shows Proposition 1.

In the following, we construct \( \Phi_1^k \) and \( \Phi_2^k \). First, we consider the case where \( \Omega = \mathbb{R}^N \). Let \( \psi \) be a function in \( C_0^\infty(\mathbb{R}) \) such that \( \psi(\ell) = 1 \) for \( |\ell| \leq 1 \) and \( \psi(\ell) = 0 \) for \( |\ell| \geq 2 \). Let \( \varphi_0(\lambda, \xi) = \psi((1 + |\xi|^2)/(1 + \lambda^2)^{1/2}) \) and \( \varphi_\infty(\lambda, \xi) = 1 - \psi((1 + |\xi|^2)/(1 + \lambda^2)^{1/2}) \). Note that

\[
|\partial^\alpha_\xi \varphi_m(\lambda, \xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad |\partial_\lambda \partial^\alpha_\xi \varphi_m(\lambda, \xi)| \leq C_\alpha |\lambda|^{-1}|\xi|^{-|\alpha|}
\]

for any \( \alpha \in \mathbb{N}_0^N \) and \( m = 0, \infty \). We write

\[
(1 + \lambda^2)^{\frac{1}{2}} \partial_\ell f = \mathcal{F}_\xi^{-1}[(1 + \lambda^2)^{\frac{1}{2}} i\xi_\ell \mathcal{F}[f](\xi)] = \Xi_1^k(\lambda)[(1 + \lambda^2)^{\frac{1}{2}} f] + \Xi_2^k(\lambda)[(1 - \Delta) f]
\]

with

\[
\Xi_1^k(\lambda)[g] = \mathcal{F}_\xi^{-1}[(1 + \lambda^2)^{\frac{1}{2}} i\xi_\ell \varphi_0(\lambda, \xi)\mathcal{F}[g](\xi)],
\]

\[
\Xi_2^k(\lambda)[g] = \mathcal{F}_\xi^{-1}[(1 + \lambda^2)^{\frac{1}{2}}(1 + |\xi|^2)^{-\frac{1}{2}} i\xi_\ell \varphi_\infty(\lambda, \xi)\mathcal{F}[g](\xi)],
\]

where \( \Delta f = \sum_{j=1}^N \partial^2_\ell f \), and \( \mathcal{F} \) and \( \mathcal{F}_\xi^{-1} \) denote the Fourier transform and inverse Fourier transform on \( \mathbb{R}^N \) defined by

\[
\mathcal{F}[g](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} g(x) \, dx, \quad \mathcal{F}_\xi^{-1}[h] = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} h(\xi) \, d\xi.
\]

As

\[
|\partial^\alpha_\xi \{(1 + \lambda^2)^{\frac{1}{2}} i\xi_\ell \varphi_0(\lambda, \xi)\}| \leq C_\alpha |\xi|^{-|\alpha|},
\]

\[
|\partial^\alpha_\xi \partial_\lambda \{(1 + \lambda^2)^{\frac{1}{2}} i\xi_\ell \varphi_0(\lambda, \xi)\}| \leq C_\alpha |\lambda|^{-1}|\xi|^{-|\alpha|},
\]

\[
|\partial^\alpha_\xi \{(1 + \lambda^2)^{\frac{1}{2}}(1 + |\xi|^2)^{-\frac{1}{2}} i\xi_\ell \varphi_\infty(\lambda, \xi)\}| \leq C_\alpha |\xi|^{-|\alpha|},
\]

\[
|\partial^\alpha_\xi \partial_\lambda \{(1 + \lambda^2)^{\frac{1}{2}}(1 + |\xi|^2)^{-\frac{1}{2}} i\xi_\ell \varphi_\infty(\lambda, \xi)\}| \leq C_\alpha |\lambda|^{-1}|\xi|^{-|\alpha|}
\]

for \( \ell = 0, 1 \), Lemma 2.5 implies that

\[
\mathcal{R}_\mathcal{L}(L^p(\mathbb{R}^N))(\{(\lambda \partial_\lambda)^\ell \Xi_1^k(\lambda) \mid \lambda \in \mathbb{R} \setminus \{0\}\}) \leq \gamma_0,
\]

\[
\mathcal{R}_\mathcal{L}(L^p(\mathbb{R}^N))(\{(\lambda \partial_\lambda)^\ell \Xi_2^k(\lambda) \mid \lambda \in \mathbb{R} \setminus \{0\}\}) \leq \gamma_0
\]

(31)

for some constant \( \gamma_0 \).

Next, we consider the case where \( \Omega = \mathbb{R}^N_N = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\} \). Given \( f \) defined on \( \mathbb{R}^N_N \), let \( \mathcal{L}[f] \) be a Lions extension of \( f \), that is

\[
\mathcal{L}[f](x', x_N) = \begin{cases} f(x', x_N) & x_N > 0, \\ 3f(x', -x_N) - 2f(x', -2x_N) & x_N < 0, \end{cases}
\]

where \( x' = (x_1, \ldots, x_{N-1}) \). It is easily shown that \( \|\mathcal{L}[f]\|_{H^1_\ell(\mathbb{R}^N)} \leq C\|f\|_{H^1_\ell(\mathbb{R}^N)} \) for \( j = 0, 1, 2 \). Moreover,

\[
(1 + \lambda^2)^{\frac{1}{2}} \partial_\ell \mathcal{L} f = (1 + \lambda^2)^{\frac{1}{2}} \partial_\ell \mathcal{L}[f] = \Xi_1^k(\lambda)[(1 + \lambda^2)^{\frac{1}{2}} \mathcal{L}[f]] + \Xi_2^k(\lambda)[(1 - \Delta) \mathcal{L}[f]]
\]
on $\mathbb{R}^N_+$. Thus, let

$$\Xi_2^k(\lambda)[f] = \Xi_1^k(\lambda)[\mathcal{L}[f]], \quad \Xi_2^k(\lambda)[f] = \Xi_2^k(\lambda)[(1-\Delta)\mathcal{L}[f]],$$

and then by (31)

$$\mathcal{R}_{\mathcal{L}(H^2_1(\mathbb{R}^N_+), L^q_1(\mathbb{R}^N_+))}\left(\{\lambda \partial \lambda \}^2 \Xi_2^k(\lambda) \mid \lambda \in \mathbb{R} \setminus \{0\}\right) \leq \gamma_1 \ (\ell = 0, 1),$$

$$\mathcal{R}_{\mathcal{L}(H^2_1(\mathbb{R}^N_+), L^q_1(\mathbb{R}^N_+))}\left(\{\lambda \partial \lambda \}^2 \Xi_2^k(\lambda) \mid \lambda \in \mathbb{R} \setminus \{0\}\right) \leq \gamma_1 \ (\ell = 0, 1), \tag{32}$$

where $\gamma_1 = C\gamma_0$ for some constant $C > 0$. In fact, the second formula in (31) gives

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u) \Xi_2^k(\lambda_j)[f_j] \right\|_{L^q_0(\mathbb{R}^N_+)}^q \, du$$

$$\leq \int_0^1 \left\| \sum_{j=1}^n r_j(u) \Xi_2^k(\lambda_j)[(1-\Delta)\mathcal{L}[f_j]] \right\|_{L^q_0(\mathbb{R}^N)}^q \, du$$

$$\leq \gamma_0 \int_0^1 \left\| (1-\Delta)\left\{ \sum_{j=1}^n r_j(u)\mathcal{L}[f_j] \right\} \right\|_{L^q_0(\mathbb{R}^N)}^q \, du$$

$$\leq C\gamma_0 \int_0^1 \left\| \sum_{j=1}^n r_j(u)f_j \right\|_{L^q_{\mathcal{L}}(\mathbb{R}^N_+)}^q \, du,$$

which yields the second formula with $\ell = 0$ in (32). Other formulas in (32) can be obtained analogously.

Finally, we consider the case where $\Omega$ is a uniform $C^2$ domain. We need the following lemma, which can be proved by employing the same argument as that in the appendix of [8].

**Proposition 2.** Let $\Omega$ be a uniform $C^2$ domain in $\mathbb{R}^N$. Then, there exist constants $M > 0, d > 0, c_0 > 0$, an open set $U \subset \Omega$, at most countably many $N$-vectors of functions $\Psi_j \in C^2(\mathbb{R}^N)^N$ and points $x_j \in \Gamma$ such that the following assertions hold:

(i) The maps: $\mathbb{R}^N \ni x \mapsto \Psi_j(x) \in \mathbb{R}^N$ are bijective.

(ii) $\Omega = U \cup \left( \bigcup_{j=1}^\infty (\Psi_j(\mathbb{R}^N_+)^n \cap B_j) \right)$, $\Psi_j(\mathbb{R}^N_+)^n \cap B_j = \Omega \cap B_j$, $\Psi_j(\mathbb{R}^N_0) \cap B_j = \Gamma \cap B_j$ ($j = 1, 2, \ldots$), where $B_j = \{ x \in \mathbb{R}^N \mid |x - x_j| < d \}$, $\mathbb{R}^N_+ = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0 \}$, and $\mathbb{R}^N_0 = \{ x = (x_1, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N \mid x_N = 0 \}$.

(iii) There exist $C^\infty$ functions $\zeta_j$ and $\tilde{\zeta}_j$ ($j = 0, 1, 2, \ldots$) such that

$$0 \leq \zeta_j, \ \tilde{\zeta}_j \leq 1, \ ||\zeta_j||_{H^2_\infty(\mathbb{R}^N)} = ||\tilde{\zeta}_j||_{H^2_\infty(\mathbb{R}^N)} \leq c_0, \ \tilde{\zeta}_j = 1 \text{ on supp} \zeta_j (j = 0, 1, 2, \ldots); \quad \sum_{j=0}^\infty \tilde{\zeta}_j = 1 \text{ on } \Gamma; \quad \sum_{j=1}^\infty \zeta_j = 1 \text{ on } \Omega.$$

(iv) $\nabla \Psi_j$, $\nabla \Psi_j^{-1} \in C^1$, $||\nabla \Psi_j^{-1}, \det \Psi_j, \det \Psi_j^{-1}||_{H^2_\infty(\mathbb{R}^N)} \leq M$ ($j = 1, 2, 3, \ldots$).

There exists an integer $L \geq 2$ such that any $L + 1$ distinct sets of $\{B_j \mid j = 1, 2, 3, \ldots \}$ have an empty intersection, thus for any $r \in [1, \infty)$ there exists a constant $C_{r,L}$ such that

$$\left[ \sum_{j=1}^\infty ||f||_{L_r(\mathbb{R} \cap B_j)}^r \right]^{\frac{1}{r}} \leq C_{r,L} ||f||_{L_r(\Omega)} \quad \text{for any } f \in L_r(\Omega). \tag{33}$$
Let $\Psi_j^{-1} = (a_j, \ldots, a_j N)$, and then

$$(1 + \lambda^2)\frac{\partial \Phi}{\partial k} = (1 + \lambda^2)\frac{\partial}{\partial k} \left\{ (1 + \lambda^2)\frac{\partial}{\partial y} (\Psi_j \circ \Phi) \right\} \circ \Psi_j^{-1}$$

Moreover, by (31) and (32),

$$\int_0^1 \left\| \sum_{m=1}^n \zeta_0 \Xi^k (\lambda_m) (\zeta_0 f_m) \right\|_{L^q(\Omega)}^q du \leq \int_0^1 \left\| \sum_{m=1}^n r_m (u) \Xi^k (\lambda_m) (\zeta_0 f_m) \right\|_{L^q(\mathbb{R}^N)}^q du$$

and then, $(1 + \lambda^2)\frac{\partial \Phi}{\partial k} = \Phi^k (\lambda) (1 + \lambda^2)\frac{\partial \Phi}{\partial k} + \Phi^k f$. Moreover, by (31) and (32),

$$\int_0^1 \left\| \sum_{m=1}^n r_m (u) \Xi^k (\lambda_m) (\zeta_0 f_m) \right\|_{L^q(\mathbb{R}^N)}^q du \leq \gamma_0 \int_0^1 \left\| \sum_{m=1}^n r_m (u) \Xi^k (\lambda_m) (\zeta_0 f_m) \right\|_{L^q(\mathbb{R}^N)}^q du$$

By the Minkowski inequality and (33),

$$\left( \int_0^1 \left\| \sum_{m=1}^n r_m (u) \Xi^k (\lambda_m) f_m \right\|_{L^q(\Omega)}^q du \right)^{1/q}$$

Finally, by (31) and (32),

$$\int_0^1 \left\| \sum_{m=1}^n r_m (u) \Xi^k (\lambda_m) (\zeta_0 f_m) \right\|_{L^q(\mathbb{R}^N)}^q du$$

By the Minkowski inequality and (33),

$$\left( \int_0^1 \left\| \sum_{m=1}^n r_m (u) \Xi^k (\lambda_m) f_m \right\|_{L^q(\Omega)}^q du \right)^{1/q}$$

Finally, by (31) and (32),

$$\int_0^1 \left\| \sum_{m=1}^n r_m (u) \Xi^k (\lambda_m) (\zeta_0 f_m) \right\|_{L^q(\mathbb{R}^N)}^q du$$

By the Minkowski inequality and (33),

$$\left( \int_0^1 \left\| \sum_{m=1}^n r_m (u) \Xi^k (\lambda_m) f_m \right\|_{L^q(\Omega)}^q du \right)^{1/q}$$
\[ + \sum_{j=1}^{\infty} \sum_{k=1}^{N} \int_{0}^{1} \| r_{m}(u) \partial_{x_{j}} \{ \frac{\partial}{\partial x_{k}} \Xi_{x_{j}}(\lambda_{3}) [ ( \tilde{G} f_{m} ) \circ \Psi_{j} ] \} \circ \Psi_{j}^{-1} \|_{L_{q}(\Omega)}^{q} \, du \]

\[ \leq \gamma_{0}^{q} \int_{0}^{1} \| r_{m}(u) f_{m} \|_{L_{q}(\Omega)}^{q} \, du \]

\[ + N M^{q+2} q_{1}^{q} \int_{0}^{1} \| \sum_{m=1}^{n} r_{m}(u) f_{m} \|_{L_{q}(\Omega \cap B_{j})}^{q} \, du \]

\[ \leq (\gamma_{0}^{q} + C_{q,M}^{q+2} N \gamma_{1}^{q}) \int_{0}^{1} \| r_{m}(u) f_{m} \|_{L_{q}(\Omega)}^{q} \, du, \]

which shows that

\[ \mathcal{R}_{L(\Lambda_{q}(\Omega))}(\{ \Phi_{i}^{\lambda}(\lambda) \mid \lambda \in \mathbb{R} \setminus \{ 0 \} \}) \leq (\gamma_{0}^{q} + C_{q,M}^{q+2} N \gamma_{1}^{q})^{1/q}. \]

Analogously, we can show that the rest of (29) and (30) hold, which completes the proof of Proposition 1.

2.3. An extension map defined on \( B_{q,p}^{1-2/p}(\Omega) \) and \( B_{q,p}^{2(1-1/p)}(\Omega) \). Let \( 1 < p, q < \infty \). We prove the existence of an operator \( T(t) : B_{q,p}^{1-2/p}(\Omega) \to H_{p}^{1/2}(\mathbb{R}, L_{q}(\Omega)) \cap L_{p}(\mathbb{R}, H_{q}^{1}(\Omega)) \) such that for any \( g \in B_{q,p}^{1-2/p}(\Omega) \), \( T(0)g = g \in \Omega \) and

\[ \| T(t)g \|_{H_{p}^{1/2}(\mathbb{R}, L_{q}(\Omega))} + \| T(t)g \|_{L_{p}(\mathbb{R}, H_{q}^{1}(\Omega))} \leq C \| g \|_{B_{q,p}^{1-2/p}(\Omega)}. \tag{34} \]

For a given \( g \in B_{q,p}^{1-2/p}(\Omega) \), there exists a \( \tilde{g} \in B_{q,p}^{1-2/p}(\mathbb{R}^{n}) \) such that \( \tilde{g} = g \) on \( \Omega \) and \( \| \tilde{g} \|_{B_{q,p}^{1-2/p}(\mathbb{R}^{n})} \leq C \| g \|_{B_{q,p}^{1-2/p}(\Omega)} \). Thus, it suffices to prove (34) in the case where \( \Omega = \mathbb{R}^{n} \). Let

\[ T(t)g = e^{T(1)dt}g = F_{\xi}^{-1}[e^{-(\xi^{2}+1)|\xi|^{2}|t|}]\mathcal{F}[g](\xi)(x). \]

Noting that

\[(|\xi|^{2} + 1)e^{-(1/2)|\xi|^{2}|t|} \leq Ct^{-1}e^{-(1/2)(1+|\xi|^{2})|t|},\]

by the Fourier multiplier theorem we have

\[ \| T(t)g \|_{L_{q}(\mathbb{R}^{n})} \leq C \| g \|_{L_{q}(\mathbb{R}^{n})}, \]

\[ \| \partial_{t}T(t)g \|_{L_{q}(\mathbb{R}^{n})} \leq C \| g \|_{H_{q}^{1/2}(\mathbb{R}^{n})}, \]

\[ \| \partial_{t}T(t)g \|_{L_{q}(\mathbb{R}^{n})} \leq C \| g \|_{L_{q}(\mathbb{R}^{n})}, \]

\[ \| \partial_{t}T(t)g \|_{L_{q}(\mathbb{R}^{n})} \leq C \| g \|_{L_{q}(\mathbb{R}^{n})}. \tag{35} \]

We observe that

\[ \| T(t)g \|_{L_{q}(\mathbb{R}^{n})} \leq \left( 2 \sum_{j=-\infty}^{\infty} \int_{2^{j}}^{2^{j+1}} \| T(t)g \|_{L_{q}(\mathbb{R}^{n})}^{p} \right)^{1/p} \]

\[ \leq 2^{2/p} \left( \sum_{j=-\infty}^{\infty} \left( 2^{j/p} b_{j} \right)^{p} \right)^{1/p}, \]

where we have set \( b_{j} = \sup_{2^{j} \leq t \leq 2^{j+1}} \| T(t)g \|_{L_{q}(\mathbb{R}^{n})} \). By (35),

\[ (b_{j})_{\ell_{p}^{q}} \leq C \| g \|_{L_{q}(\mathbb{R}^{n})}, \quad (b_{j})_{\ell_{p}^{\infty}} \leq C \| g \|_{H_{q}^{1/2}(\mathbb{R}^{n})}, \]

As

\[ (b_{j})_{\ell_{p}^{q}} = (b_{j})_{\ell_{p}^{q}, \ell_{p}^{q}} \leq C \| g \|_{L_{q}(\mathbb{R}^{n})}, \quad (b_{j})_{\ell_{p}^{\infty}} \leq C \| g \|_{B_{q,p}^{1-2/p}(\mathbb{R}^{n})}, \]
we have
\[
\|T(\cdot)g\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \leq C\|g\|_{B^{2-2/p}_{q,p}(\mathbb{R}^n)}.
\] (36)

Analogously, by (35) we have
\[
\|\partial_t T(\cdot)g\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \leq C\|g\|_{(H^2_q(\mathbb{R}^n), L^q(\mathbb{R}^n))_{1/p,p}} \leq C\|g\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^n)}.
\] (37)

Because \(\|g\|_{B^{2-2/p}_{q,p}(\mathbb{R}^n)} \leq C\|g\|_{B^{(1-1/p)}_{q,p}(\mathbb{R}^n)}\), (36) gives
\[
\|T(\cdot)g\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \leq C\|g\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^n)},
\]
which, combined with (37), leads to
\[
\|T(\cdot)g\|_{H^2_q(\mathbb{R}^n)} \leq C\|g\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^n)}.
\]

As \(H^{1/2}_p(\mathbb{R}, L^q(\mathbb{R}^n)) = (L^p(\mathbb{R}, L^q(\mathbb{R}^n)), H^1_p(\mathbb{R}, L^q(\mathbb{R}^n)))_{1/2}\), Proposition in [17, p.86] implies that
\[
\|T(\cdot)g\|_{H^{1/2}_p(\mathbb{R}, L^q(\mathbb{R}^n))} \leq C\|g\|_{(B^{2-2/p}_{q,p}(\mathbb{R}^n), B^{2(1-1/p)}_{q,p}(\mathbb{R}^n))_{1/2}} \leq C\|g\|_{B^{1-2/p}_{q,p}(\mathbb{R}^n)}.
\]

Analogously, because
\[
\|T(t)g\|_{H^2_q(\mathbb{R}^n)} \leq C\|g\|_{H^2_q(\mathbb{R}^n)}, \quad \|T(t)g\|_{H^2_q(\mathbb{R}^n)} \leq C|t|^{-1}\|g\|_{L^q(\mathbb{R}^n)},
\]
and
\[
L^p(\mathbb{R}, H^1_q(\mathbb{R}^n)) = (L^p(\mathbb{R}, L^q(\mathbb{R}^n)), L^p(\mathbb{R}, H^2_q(\Omega)))_{1/2},
\]
we have
\[
\|T(\cdot)g\|_{L^p(\mathbb{R}, H^1_q(\mathbb{R}^n))} \leq C\|g\|_{(B^{2-2/p}_{q,p}(\mathbb{R}^n), B^{2(1-1/p)}_{q,p}(\mathbb{R}^n))_{1/2}} \leq C\|g\|_{B^{1-2/p}_{q,p}(\mathbb{R}^n)}.
\]

Thus, we have (34).

In the same manner, we can also see that
\[
T : B^{2(1-1/p)}_{q,p}(\Omega) \to H^1_p(\mathbb{R}, L^q(\Omega)) \cap L^p(\mathbb{R}, H^2_q(\Omega)),
\]
and that for \(g \in B^{2(1-1/p)}_{q,p}(\Omega)\)
\[
\|T(\cdot)g\|_{H^1_p(\mathbb{R}, L^q(\Omega))} \| T(\cdot)g\|_{H^{1/2}_p(\mathbb{R}, H^1_q(\Omega))} + \| T(\cdot)g\|_{L^p(\mathbb{R}, H^2_q(\Omega))} \leq C\|g\|_{B^{2(1-1/p)}_{q,p}(\Omega)}.
\] (38)

2.4. Maximal \(L^p_q\)-\(L^q\) regularity. In this subsection, we consider non-stationary Stokes equations with free boundary conditions. Throughout this subsection, we assume that the weak Dirichlet problem is uniquely solvable. First, we consider the Cauchy problem:
\[
\begin{aligned}
\partial_t u - \text{Div} (\mu D(u) - p\mathbf{I}) &= 0, & \text{div } u &= 0 & \text{in } \Omega \times (0, \infty), \\
(\mu D(u) - p\mathbf{I})\mathbf{n} &= 0 & \text{on } \Gamma \times (0, \infty), \\
\mathbf{u}|_{t=0} &= \mathbf{u}_0 & \text{in } \Omega.
\end{aligned}
\] (39)

Let \(J_q(\Omega)\) be the solenoidal space defined by
\[
J_q(\Omega) = \{ u \in L^q(\Omega)^N | \langle u, \nabla \varphi \rangle_{\Omega} = 0 \text{ for any } \varphi \in \dot{H}^1_{q,0}(\Omega) \}. \] (40)

We consider problem (39) in an analytic semi-group setting. Given \(u \in H^2_q(\Omega)^N\), let \(K(u)\) be a unique solution of the variational problem:
\[
(\nabla K(u), \nabla \varphi)_{\Omega} = (\text{Div} (\mu D(u)) - \nabla \text{div } u, \nabla \varphi)_{\Omega} \text{ for any } \varphi \in \dot{H}^1_{q,0}(\Omega),
\] (41)
subject to $K(u) = \langle \mu D(u) n, n \rangle > -\text{div} u$ on $\Gamma$. As the weak Dirichlet problem (20) is assumed to be uniquely solvable, $K(u)$ exists uniquely in $H^1_q(\Omega) + H^1_{q,0}(\Omega)$ satisfying the estimate: $\|\nabla K(u)\|_{L^q(\Omega)} \leq \|\nabla u\|_{H^1_q(\Omega)}$.

We consider the resolvent problem:

$$
\begin{aligned}
\lambda u - \text{Div} (\mu D(u) - K(u)I) &= f \quad \text{in } \Omega, \\
(\mu D(u) - K(u)I) n &= 0 \quad \text{on } \Gamma.
\end{aligned}
$$

To solve problem (42), we consider the auxiliary problem given by

$$
\begin{aligned}
\lambda u - \text{Div} (\mu D(u) - pI) &= f, \quad \text{div} u = 0 \quad \text{in } \Omega, \\
(\mu D(u) - pI) n &= 0 \quad \text{on } \Gamma.
\end{aligned}
$$

As defined in (23), $\text{div} u = 0$ means that $u \in J_q(\Omega)$. By Theorem 2.4, for any $\epsilon > 0$ there exists a $\lambda_0 > 0$ such that for any $f \in J_q(\Omega)$ and $\lambda \in \Sigma_{\epsilon, \lambda_0}$, problem (43) admits unique solutions $u \in H^2_q(\Omega) \cap J_q(\Omega)$ and $p \in H^1_q(\Omega) + H^1_{q,0}(\Omega)$ satisfying the estimate:

$$
|\lambda| \|u\|_{L^q(\Omega)} + |\lambda|^{1/2} \|u\|_{H^1_q(\Omega)} + \|u\|_{H^2_q(\Omega)} \leq C \|f\|_{L^q(\Omega)}.
$$

For any $\varphi \in \tilde{H}^1_{q,0}(\Omega)$

$$
0 = (f, \nabla \varphi) = \lambda (u, \nabla \varphi) + (\text{div} u, \nabla \varphi) + (\nabla (p - K(u)), \nabla \varphi) \\
= (\nabla (p - K(u)), \nabla \varphi).
$$

Moreover, $p - K(u) = \langle \mu D(u) n, n \rangle - K(u) = \text{div} u = 0$ on $\Gamma$, which yields that $p = K(u)$, which means that problem (42) admits a unique solution $u \in H^2_q(\Omega) \cap J_q(\Omega)$ satisfying (44).

Let $\mathcal{D}_q(\Omega)$ and $\mathcal{A}$ be a domain and an operator defined by

$$
\mathcal{D}_q(\Omega) = \{ u \in H^2_q(\Omega) \cap J_q(\Omega) \mid (\mu D(u) - K(u)I) n = 0 \text{ on } \Gamma \},
$$

$$\mathcal{A} u = \text{Div} (\mu D(u) - K(u)I) \quad \text{for } u \in \mathcal{D}_q(\Omega).$$

Then, $\mathcal{A}$ generates a $C_0$ analytic semi-group $\{T(t)\}_{t \geq 0}$ on $J_q(\Omega)$. As

$$
\langle (\mu D(u) - K(u)I) n, n \rangle = \text{div} u = 0 \quad \text{on } \Gamma,
$$

$\mathcal{D}_q(\Omega)$ is given by

$$
\mathcal{D}_q(\Omega) = \{ u \in H^2_q(\Omega) \cap J_q(\Omega) \mid (\mu D(u)n)_\tau = 0 \text{ on } \Gamma \},
$$

where $d_\tau = d - <d, n>n$ for any $N$-vector $d$, that is $d_\tau$ denotes the tangential vector of $d$ along $n$.

Let $u = T(t)u_0$ and $p = K(u)$. Then, $u$ and $p$ are unique solutions of Eq. (39) possessing the estimates:

$$
\|u(\cdot, t)\|_{L^q(\Omega)} \leq C e^{\gamma t} \|u_0\|_{L^q(\Omega)},
$$

$$
\|\partial_t u(\cdot, t)\|_{L^q(\Omega)} \leq C e^{\gamma t} \|u_0\|_{H^1_q(\Omega)},
$$

$$
\|\partial_t u(\cdot, t)\|_{L^q(\Omega)} \leq C t^{-1} e^{\gamma t} \|u_0\|_{L^q(\Omega)}
$$

for any $t > 0$ with some constants $C$ and $\gamma$. Thus, by a real interpolation method similar to that in Subsec. 2.3 we have the following theorem (cf. [28]).
**Theorem 2.6.** Let $1 < p, q < \infty$ and assume that the weak Dirichlet problem is uniquely solvable for an index $q$. Let $D_{p,q}(\Omega) = (J_q(\Omega), D_q(\Omega))_{1-1/p,p}$. Then, for any $u_0 \in D_{p,q}(\Omega)$, problem (39) admits unique solutions $u$ and $p$ with

\[
\begin{align*}
  u &\in L_p((0,T), H^2_q(\Omega)^N) \cap H^1_p((0,T), L_q(\Omega)^N), \\
p &\in L_p((0,T), H^1_q(\Omega) + \dot{H}^1_{q,0}(\Omega))
\end{align*}
\]

for any $T > 0$ satisfying the estimate:

\[
\|e^{-\gamma t} u\|_{L_p((0,\infty), H^2_q(\Omega))} + \|e^{-\gamma t} \partial_t u\|_{L_p((0,\infty), L_q(\Omega))} \leq C\|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}
\]

for any $\gamma \geq \gamma_0$ for some $\gamma_0$, where the constant $C$ is independent of $\gamma$ as far as $\gamma \geq \gamma_0$.

**Remark 5.** $D_{p,q}(\Omega)$ is a closed subspace of $B^{2(1-1/p)}_{q,p}(\Omega)^N \cap J_q(\Omega)$ such that

\[
D_{q,p}(\Omega) = \{ u_0 \in B^{2(1-1/p)}_{q,p}(\Omega)^N \cap J_q(\Omega) \mid (D(u_0)n)_\tau = 0 \text{ on } \Gamma \} \quad \text{for } 1 - \frac{2}{p} > \frac{1}{q},
\]

\[
D_{q,p}(\Omega) = B^{2(1-1/p)}_{q,p}(\Omega)^N \cap J_q(\Omega) \quad \text{for } 1 - \frac{2}{p} < \frac{1}{q}.
\]

Next, we consider the initial-boundary value problem:

\[
\begin{align*}
  \partial_t u - \text{Div}(\mu D(u) - p I) &= f, \quad \text{Div} u = g \quad \text{in } \Omega \times (0,T), \\
  (\mu D(u) - p I) n &= h \quad \text{on } \Gamma \times (0,T), \\
  u|_{t=0} &= u_0 \quad \text{in } \Omega.
\end{align*}
\]

We then have the following theorem.

**Theorem 2.7.** Let $1 < p, q < \infty$ with $2/p + 1/q \neq 1$ and $0 < T < \infty$. Assume that the weak Dirichlet problem is uniquely solvable for an index $q$. Let

\[
\begin{align*}
  u_0 &\in B^{2(1-1/p)}_{q,p}(\Omega)^N, \\
f &\in L_p((0,T), L_q(\Omega)^N), \\
g &\in L_p(\mathbb{R}, DL_q(\Omega)) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega)), \\
h &\in H^{1/2}_p(\mathbb{R}, H^1_q(\Omega)) \cap L_p(\mathbb{R}, H^1_q(\Omega))
\end{align*}
\]

which satisfy the compatibility condition:

\[
|u_0 - G(g)|_{t=0} \in J_q(\Omega) \cap B^{2(1-1/p)}_{q,p}(\Omega)
\]

(47)

and, in addition,

\[
(\mu D(u_0)n - h|_{t=0})_\tau = 0 \quad \text{on } \Gamma
\]

(48)

for $2/p + 1/q < 1$. Then, problem (45) admits unique solutions $u$ and $p$ with

\[
\begin{align*}
  u &\in L_p((0,T), H^2_q(\Omega)^N) \cap H^1_p((0,T), L_q(\Omega)^N), \\
p &\in L_p((0,T), H^1_q(\Omega) + \dot{H}^1_{q,0}(\Omega))
\end{align*}
\]

satisfying the estimates

\[
\begin{align*}
  \|u\|_{L_p((0,T), H^2_q(\Omega))} + \|\partial_t u\|_{L_p((0,T), L_q(\Omega))} &
\leq C_{\gamma} \gamma T \left[ \|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|f\|_{L_p((0,T), L_q(\Omega))} + \|(g,h)\|_{L_p(\mathbb{R}, H^1_q(\Omega))} + \|G(g)\|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))} \right]
\end{align*}
\]

(50)

for some positive constants $C$ and $\gamma$. 


Remark 6. (1) Implicitly, we assume that \( \mathcal{G}(g)|_{t=0} \in B_{q,p}^{2(1-1/p)}(\Omega) \).
(2) In the case where \( 2/p + 1/q < 1 \), \( \mu \mathbf{D}(u_0) \in B_{q,p}^{-2/p}(\Omega) \) and \( 1 - 2/p > 1/q \), and so \( \mu \mathbf{D}(u_0)|_{\Gamma} \) exists. However, \( 1/p < 1/2 \) and \( h \in H_{p}^{1/2}(\mathbb{R}, L_{q}(\Omega)N) \) implies that \( h \) is continuous with respect to \( t \in \mathbb{R} \) in the \( L_{q}(\Omega) \) topology, and so \( h|_{t=0} \) exists as an element in \( L_{q}(\Omega) \), but we do not know whether the trace of \( h|_{t=0} \) to \( \Gamma \) exists. Thus, we implicitly assume the existence of the trace of \( h|_{t=0} \) to \( \Gamma \) in (48).

Proof. Let \( \psi(t) \) be a function in \( C^{\infty}(\mathbb{R}) \) such that \( \psi(t) = 1 \) for \( t > -1 \) and \( \psi(t) = 0 \) for \( t < -2 \). Let \( F \) be the zero extension of \( f \), that is \( \mathbf{F}(\cdot, t) = f(\cdot, t) \) for \( t \in (0, T) \) and \( \mathbf{F}(\cdot, t) = 0 \) for \( t \not\in (0, T) \). Let \( G = \psi(t)g \) and \( H = \psi(t)h \), then for any \( \gamma > 0 \)

\[
\begin{align*}
\| e^{-\gamma t} F \|_{L_p(\mathbb{R}, L_q(\Omega))} & \leq \| f \|_{L_p((0, T), L_q(\Omega))}, \\
\| e^{-\gamma t} G \|_{L_p(\mathbb{R}, H^1_q(\Omega))} & \leq (1 + \gamma)e^{2\gamma t}(\| g \|_{L_p(\mathbb{R}, H^1_q(\Omega))} + \| g \|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))}), \\
\| e^{-\gamma t} (G) \|_{H^1_p(\mathbb{R}, L_q(\Omega))} & \leq C(1 + \gamma)e^{2\gamma t}\| g \|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))}, \\
\| e^{-\gamma t} H \|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))} & \leq (1 + \gamma)(\| h \|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))} + \| h \|_{L_p(\mathbb{R}, H^1_q(\Omega))}),
\end{align*}
\]

(51)

because \( \mathcal{G}(G) = \psi(t)\mathcal{G}(g) \). We consider the equations:

\[
\begin{cases}
\partial_t v - \text{Div}(\mu \mathbf{D}(v) - q I) = F, & \text{div } v = G \quad \text{in } \Omega \times \mathbb{R}, \\
(\mu \mathbf{D}(v) - q I)n = H \quad \text{on } \Gamma \times \mathbb{R}.
\end{cases}
\]

(52)

Let \( A(\lambda) \) and \( P(\lambda) \) be the operators given in Theorem 2.4, and let

\[
\begin{align*}
\mathbf{v} &= \mathcal{L}^{-1}[A(\lambda)F(\lambda) + L[G(\lambda)] + L[H(\lambda)]], \\
q &= \mathcal{L}^{-1}[P(\lambda)F(\lambda) + L[G(\lambda)] + L[H(\lambda)]](t)
\end{align*}
\]

where \( \mathcal{L} \) and \( \mathcal{L}^{-1} \) denote the Laplace transform and the inverse Laplace transform defined by

\[
\mathcal{L}[g](\lambda) = \int_{\mathbb{R}} e^{-(\gamma + i\tau)g(t)}dt, \quad \mathcal{L}^{-1}[h(\lambda)](t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{(\gamma + i\tau)t}h(\lambda)d\tau
\]

with \( \lambda = \gamma + i\tau \in \mathbb{C} \). As \( A(\lambda) \) and \( P(\lambda) \) are the \( \mathcal{R} \) bounded operator families stated in Theorem 2.4, we have from Theorem 2.3 that there exists some \( \gamma_0 \) such that for any \( \gamma \geq \gamma_0 \) the following holds:

\[
\begin{align*}
\| e^{-\gamma t} \mathbf{v} \|_{L_p(\mathbb{R}, H^2_q(\Omega))} & + \| e^{-\gamma t} \mathbf{v} \|_{H^{1/2}_p(\mathbb{R}, H^1_q(\Omega))} + \| e^{-\gamma t} \mathbf{v} \|_{H^1_p(\mathbb{R}, L_q(\Omega))} \\
& + \| e^{-\gamma t} \nabla q \|_{L_p(\mathbb{R}, L_q(\Omega))} \\
\leq & C\| e^{-\gamma t} F \|_{L_p(\mathbb{R}, L_q(\Omega))} + \| e^{-\gamma t} G \|_{L_p(\mathbb{R}, H^1_q(\Omega))} + \| \mathcal{G}(G) \|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))} + \| e^{-\gamma t} H \|_{L_p(\mathbb{R}, H^1_q(\Omega))} + \| e^{-\gamma t} H \|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))}.
\end{align*}
\]

Putting this and (51) together gives

\[
\begin{align*}
\| \mathbf{v} \|_{L_p((0, T), H^2_q(\Omega))} & + \| \mathbf{v} \|_{H^{1/2}_p((0, T), L_q(\Omega))} + \| \nabla q \|_{L_p((0, T), L_q(\Omega))} \\
& \leq C\gamma e^{2\gamma t}\left[\| F \|_{L_p((0, T), L_q(\Omega))} + \| (g, h) \|_{L_p(\mathbb{R}, H^1_q(\Omega))} + \| (g, h) \|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))} \right] \\
& + \| \mathcal{G}(g) \|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))}
\end{align*}
\]

(53)
for any $S > 0$ with some constant $C_s$ that depends on $\gamma$ and is independent of $S > 0$. As $F = f$ on $(0, T)$, $v$ and $q$ satisfy Eq. (45), except for the initial condition. Thus, we have to compensate the intial condition. Let $w$ and $r$ be solutions of the Cauchy problem:

$$
\begin{align*}
\frac{\partial_t u}{\partial t} - \text{div} (\mu D(u) - p I) &= 0, &\text{div} u &= 0 \quad &\text{in} \quad \Omega \times (0, T), \\
(\mu D(u) - p I) n &= 0 &\text{on} \quad \Gamma \times (0, T),
\end{align*}
$$

By (2) and (53) with $S = 1$, we know that $v|_{t=0} \in B_{q,p}^{2(1-1/p)}(\Omega)^N$ and

$$
\|v|_{t=0}\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq C\|v\|_{L_p((0,1),H^2_q(\Omega))} + \|v\|_{H^1_q((0,1),L_q(\Omega))}.
$$

By the compatibility condition in (47),

$$
\|v|_{t=0}\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq C\|v\|_{L_p((0,1),H^2_q(\Omega))} + \|v\|_{H^1_q((0,1),L_q(\Omega))}.
$$

By the compatibility condition in (47),

$$
\|v|_{t=0}\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq C\|v\|_{L_p((0,1),H^2_q(\Omega))} + \|v\|_{H^1_q((0,1),L_q(\Omega))}.
$$

Moreover, in the case where $2/p + 1/q < 1$, by the compatibility condition in (48)

$$
(\mu D|_{t=0} - \mu u)|_{t=0} = (h|_{t=0} - \mu D|_{t=0}) = 0 \quad \text{on} \quad \Gamma.
$$

Thus, $u_v - v|_{t=0} \in D_{q,p}(\Omega)$, and so by Theorem 2.4, there exist $w$ and $r$ with

$$
\begin{align*}
\|e^{-\gamma t} w|_{t=0}\|_{L_p((0,\infty),H^2_q(\Omega))} + \|e^{-\gamma t} \partial_t w|_{t=0}\|_{L_p((0,\infty),L_q(\Omega))} \leq C\|v|_{t=0}\|_{B_{q,p}^{2(1-1/p)}(\Omega)}.
\end{align*}
$$

Summing up, $u = v + w$ and $p = q + r$ satisfy Eq. (45) and the estimate in (50). The uniqueness follows from the existence of solutions to the dual problem (cf. [28]), and so the proof of Theorem 2.7 is completed.

3. **Maximal $L_p$-$L_q$ regularity in exterior domains.** In this section, let $\Omega$ be an exterior domain whose boundary $\Gamma$ is a compact hypersurface of $C^2$ class. We consider the non-stationary Stokes equations with free boundary conditions:

$$
\begin{align*}
\frac{\partial_t u}{\partial t} - \text{div} (\mu D(u) - p I) &= f, &\text{div} u &= g \quad &\text{in} \quad \Omega \times (0, T), \\
(\mu D(u) - p I) n &= h &\text{on} \quad \Gamma \times (0, T),
\end{align*}
$$

Then, we will show the following theorem.

**Theorem 3.1.** Let $1 < p, q < \infty$ with $2/p + 1/q \neq 1$ and $0 < T < \infty$. Let $u_0$, $f$, $g$, $\mathcal{G}(g)$ and $h$ be initial data and right members for (55) satisfying (46), (47) and (48). Then, problem (55) admits unique solutions $u$ and $p$ satisfying the regularity condition (49) and possessing the estimate (50).

In view of Theorem 2.7, to prove Theorem 3.1, it is sufficient to prove the unique solvability of the weak Dirichlet problem for any exponent $q \in (1, \infty)$. Thus, in the following, we prove the unique existence theorem of the weak Dirichlet problem:

$$
(\nabla u, \nabla \varphi)_{\Omega} = (f, \nabla \varphi)_{\Omega} \quad \text{for any} \quad \varphi \in H^1_{q',0}(\Omega),
$$

(56)
where \( \hat{H}^1_q(\Omega) = \{ \varphi \in L^q_{q',\text{loc}}(\Omega) \mid \nabla \varphi \in L^q_q(\Omega)^N, \varphi|_\Gamma = 0 \} \), namely, we will prove the following theorem.

**Theorem 3.2.** Let \( 1 < q < \infty \). Then, for any \( f \in L^q_q(\Omega)^N \) problem (56) admits a unique solution \( u \in \hat{H}^1_q(\Omega) \) possessing the estimate: \( \|\nabla u\|_{L^q_q(\Omega)} \leq C\|f\|_{L^q_q(\Omega)} \).

**Remark 7.** Theorem 3.2 was proved by Simader and Sohr [29, p.45 Theorem 1.2] and also recently by Pruess and Simonette [16, p.354, Theorem 7.4.3]. The proofs in [29] and [16] depend on functional analytic method.

Another approach to prove Theorem 3.2 is to use the uniform estimate for the following resolvent problem for the Stokes operator with free boundary conditions:

\[
\begin{cases}
\lambda u - \text{Div} (\mu D(u)) + \nabla p = f, & \text{in } \Omega, \\
(\mu D(u) - p I)u = 0 & \text{on } \Gamma.
\end{cases}
\]

In fact, Shibata and Shimizu [26] proved that for any \( \varepsilon \in (0, \pi/2) \) there exists a \( \lambda_0 > 0 \) such that for any \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \) and \( f \in L^q_q(\Omega)^N \), problem (57) admits unique solutions \( u_\lambda \in \hat{H}^2_q(\Omega)^N \) and \( p_\lambda \in \hat{H}^1_q(\Omega) + \hat{H}^1_q(\Omega) \) satisfying the estimate:

\[
\|\lambda\|\|u_\lambda\|_{L^q_q(\Omega)} + \|u_\lambda\|_{L^q_q(\Omega)} + \|p_\lambda\|_{L^q_q(\Omega)} \leq C\|f\|_{L^q_q(\Omega)}
\]

with some constant \( C \) independent of \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \). From (57) it follows that for any \( \varphi \in \hat{H}^2_q(\Omega) \)

\[
(f, \nabla \varphi)_{\Omega} = -\lambda(\text{div} u_\lambda, \varphi)_{\Omega} - (\mu \text{Div} D(u_\lambda), \nabla \varphi)_{\Omega} + (\nabla p_\lambda, \nabla \varphi)_{\Omega}
\]

and \( p_\lambda = <\mu D(u_\lambda), n> \) on \( \Gamma \). Letting \( \lambda \to \infty \), by (58) \( u_\lambda \to 0 \) weakly in \( H^2_q(\Omega) \), \( \|p_\lambda\|_{L^q_q(\Omega)} \leq C\|f\|_{L^q_q(\Omega)} \) and \( p_\lambda \to 0 \) on \( \Gamma \). Thus, there exists a \( p \in \hat{H}^1_q(\Omega) \) that is a solution of problem (56) possessing the estimate \( \|\nabla p\|_{L^q_q(\Omega)} \leq C\|f\|_{L^q_q(\Omega)} \). The uniqueness follows from the existence theorem for the dual problem. This proves Theorem 3.2.

As the unique solvability of the weak Dirichlet problem is the core of the maximal \( L^q \)-\( L^q \) theorem for the Stokes equations with free boundary conditions, in the following we give a new proof of Theorem 3.2, whose idea is completely different than that in all the above studies.

**A proof of Theorem 3.2.** First, we consider the variational problem:

\[
(\nabla u, \nabla v)_{\mathbb{R}^N} = (f, \nabla v)_{\mathbb{R}^N} \quad \text{for any } v \in \hat{H}^1_q(\mathbb{R}^N).
\]

To construct a solution of the variational problem (59), we may assume that \( f \in C^\infty_0(\mathbb{R}^N) \), because the space \( C^\infty_0(\mathbb{R}^N) \) is dense in \( L^q_q(\mathbb{R}^N) \). Let

\[
u = \Delta^{-1} \text{div } f,
\]

and then \( u \) satisfies (59) by the divergence theorem of Gauß. Moreover, adding some constant, if necessary, we see that \( u \) satisfies the estimate:

\[
\|\nabla u\|_{L^q_q(\mathbb{R}^N)} + \|u\|_{L^q_q(B_{2R})} \leq C\|f\|_{L^q_q(\mathbb{R}^N)}
\]

by the Fourier multiplier theorem of the Marcinkiewicz-Miklhin-Hörmander type and Poincaré’s inequality. Let \( S_\infty \) be an operator acting on \( f \) defined by \( S_\infty f = u \), and then we have the following lemma.

**Lemma 3.3.** Let \( 1 < q < \infty \). Then, there exists an operator \( S_\infty : L^q_q(\mathbb{R}^N)^N \to \hat{H}^1_q(\mathbb{R}^N) \) such that for any \( f \in L^q_q(\mathbb{R}^N)^N \), \( u = S_\infty f \) is a unique solution of the variational equation (59) possessing the estimate (61).
Next, we consider the variational equation:
\[(\nabla u, \nabla v)_{\Omega_{2R}} = (f, \nabla v)_{\Omega_{2R}} \text{ for any } v \in \hat{H}^1_{q,0}(\Omega_{5R}),\]
where \(\hat{H}^1_{q,0}(\Omega_{5R}) = \{v \in H^1_{q}(\Omega_{5R}) \mid v|_{\Gamma} = 0\}\), and then we have

**Lemma 3.4.** Let \(1 < q < \infty\). Then, there exists an operator \(S_0 : L_q(\Omega_{5R})^N \to H^1_{q,0}(\Omega_{5R})\) such that for any \(f \in L_q(\Omega_{5R})^N\), \(u = S_0f\) is a unique solution of the variational equation (62) possessing the estimate
\[\|S_0f\|_{H^1_{q,0}(\Omega_{5R})} \leq C\|f\|_{L_q(\Omega_{5R})}.\]

**Remark 8.** For the sake of completeness of the paper as much as possible, we give a sketch of proof of Lemma 3.4 in Appendix B below.

Let \(\varphi_0\) be a function in \(C_0^\infty(\mathbb{R}^N)\) such that
\[\varphi_0(x) = \begin{cases} 1 & \text{for } x \in B_{3R}, \\ 0 & \text{for } x \notin B_{4R}. \end{cases}\]
Let \(\varphi_{\infty} = 1 - \varphi_0\), and then \(\varphi_{\infty}(x) = 1\) for \(x \in \mathbb{R}^N \setminus B_{4R}\) and \(\varphi_{\infty}(x) = 0\) for \(x \in B_{3R}\). Let \(\psi_0\) be a function in \(C_0^\infty(\mathbb{R}^N)\) such that \(\psi_0(x) = 1\) for \(x \in B_{(4+1/3)R}\) and \(\psi_0(x) = 0\) for \(x \in \mathbb{R}^N \setminus B_{(4+2/3)R}\), and let \(\psi_{\infty}\) be a function in \(C^\infty(\mathbb{R}^N)\) such that \(\psi_{\infty}(x) = 1\) for \(x \in \mathbb{R}^N \setminus B_{(3-1/3)R}\) and \(\psi_{\infty}(x) = 0\) for \(x \in B_{(3-2/3)R}\).

Obviously,
\[\psi_0 = 1 \text{ on supp } \varphi_0, \quad \psi_{\infty} = 1 \text{ on supp } \varphi_{\infty}.\]

Let \(S\) be an operator acting on \(f \in L_q(\Omega)^N\) defined by
\[Sf = \psi_{\infty}S_\infty(\varphi_{\infty}f) + \psi_0S_0(\varphi_0f).\]

By Lemma 3.3 and Lemma 3.4, \(Sf \in \hat{H}^1_{q,0}(\Omega)\) and
\[\|\nabla Sf\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}.\]

Since
\[\nabla Sf = \psi_{\infty}\nabla S_\infty(\varphi_{\infty}f) + (\nabla \psi_{\infty}) \cdot S_\infty(\varphi_{\infty}f) + \psi_0\nabla S_0(\varphi_0f) + (\nabla \psi_0) \cdot S_0(\varphi_0f),\]
for any \(v \in \hat{H}^1_{q,0}(\Omega)\)
\[(\nabla Sf, \nabla v)_{\Omega} = (\nabla S_\infty(\varphi_{\infty}f), \nabla (\psi_{\infty}v))_{\Omega} - (\nabla S_\infty(\varphi_{\infty}f), (\nabla \psi_{\infty})v)_{\Omega} + (\nabla S_0(\varphi_0f), \nabla (\psi_0v))_{\Omega} \]
\[= (f, \nabla v)_{\Omega} + (Rf, v)_{\Omega}\]
with
\[Rf = -2(\nabla \psi_{\infty}) \cdot \nabla S_\infty(\varphi_{\infty}f) + 2(\nabla \psi_0) \cdot \nabla S_0(\varphi_0f)\]
\[+ (\Delta \psi_{\infty})S_\infty(\varphi_{\infty}f) + (\Delta \psi_0)S_0(\varphi_0f).\]

Here, we have used the fact:
\[(\varphi_{\infty}f, \nabla (\psi_{\infty}v))_{\Omega} + (\varphi_0f, \nabla (\psi_0v))_{\Omega} = (\varphi_{\infty}f, \nabla v)_{\Omega} + (\varphi_0f, \nabla v)_{\Omega} = (f, \nabla v)_{\Omega},\]
which follows from (65) and the fact that \(\varphi_{\infty} + \varphi_0 = 1\). Obviously,
\[\text{supp } Rf \subset D_{R_1,R_2}, \quad Rf \in L_q(\mathbb{R}^N), \quad \|Rf\|_{L_q(\mathbb{R}^N)} \leq C\|f\|_{L_q(\Omega)},\]
where \(R_1 = (3 - 2/3)R\) and \(R_2 = (4 + 2/3)R\).
On the other hand, for any \( v \in \dot{H}^1_q(\mathbb{R}^N) \) we have

\[
(Rf, v)_{\mathbb{R}^N} = -\left(2(\nabla \psi_\infty) \cdot \nabla S_\infty(\varphi_\infty f) + 2(\nabla \psi_0) \cdot \nabla S_0(\varphi_0 f) + (\Delta \psi_\infty) S_\infty(\varphi_\infty f) + (\Delta \psi_0) S_0(\varphi_0 f), v\right)_{\mathbb{R}^N}
\]

\[
= -((\nabla \psi_\infty) \cdot \nabla S_\infty(\varphi_\infty f), \varphi_\infty f)_{\mathbb{R}^N} - ((\nabla \psi_0) \cdot \nabla S_0(\varphi_0 f), \varphi_\infty f)_{\mathbb{R}^N} - ((\nabla \psi_\infty) \cdot \nabla S_0(\varphi_0 f), \varphi_\infty f)_{\mathbb{R}^N}
\]

\[
= -((\nabla \psi_\infty) \cdot \nabla S_\infty(\varphi_\infty f), \varphi_\infty f)_{\mathbb{R}^N} + ((\nabla \psi_\infty) \cdot \nabla S_\infty(\varphi_\infty f), \varphi_\infty f)_{\mathbb{R}^N}
\]

\[
= -((\nabla \psi_\infty) \cdot \nabla S_\infty(\varphi_\infty f), \varphi_\infty f)_{\mathbb{R}^N} + ((\nabla \psi_\infty) \cdot \nabla S_\infty(\varphi_\infty f), \varphi_\infty f)_{\mathbb{R}^N}
\]

where \([\nabla S_0(\varphi_0 f)]_0 = \nabla S_0(\varphi_0 f)\) for \( x \in \Omega_{5R} \) and \([\nabla S_0(\varphi_0 f)]_0 = 0\) for \( x \notin \Omega_{5R} \).

Since \((1 - \psi_0)v \in \dot{H}^1_q(\Omega_{5R})\) and supp \( \varphi_\infty \cap \text{supp}(1 - \psi_0) = 0 \), we have

\[
(Rf, v)_{\mathbb{R}^N} = ((1 - \psi_0)v, \varphi_\infty f)_{\Omega_{5R}} = 0.
\]

Moreover, \(-(\varphi_\infty f, \nabla (\psi_\infty v))_{\mathbb{R}^N} = -((\varphi_\infty f, \nabla v))_{\mathbb{R}^N}\), because \( \psi_\infty = 1 \) on \( \text{supp}\varphi_\infty \).

Thus, letting

\[
\mathcal{G}f = -\varphi_\infty f - \psi_\infty \nabla S_\infty(\varphi_\infty f) - (1 - \psi_0)[\nabla S_0(\varphi_0 f)]_0
\]

\[
+ (\nabla \psi_\infty) \cdot \nabla S_\infty(\varphi_\infty f) + (\nabla \psi_0) \cdot \nabla S_0(\varphi_0 f),
\]

we have

\[
(Rf, v)_{\mathbb{R}^N} = (\mathcal{G}f, \nabla v)_{\mathbb{R}^N} \quad \text{for any } v \in \dot{H}^1_q(\mathbb{R}^N), \tag{69}
\]

Obviously,

\[
\mathcal{G}f \in L_q(\mathbb{R}^N), \quad \text{supp} \mathcal{G}f \subset \mathbb{R}^N \setminus B_R, \quad \|\mathcal{G}f\|_{L_q(\mathbb{R}^N)} \leq C\|f\|_{L_q(\Omega)} \tag{70}
\]

For \( v \in \dot{H}^1_q(\mathbb{R}^N) \), let \( \{v\} = \{v + c \mid c \in \mathbb{R}\} \). Let \( \dot{H}^1_q(\mathbb{R}^N) \) be the set of all \( \{v\} \) with \( v \in \dot{H}^1_q(\mathbb{R}^N) \). \( \dot{H}^1_q(\mathbb{R}^N) \) is a Banach space with norm \( \|v\|_{\dot{H}^1_q(\mathbb{R}^N)} = \|\nabla v\|_{L_q(\mathbb{R}^N)} \).

Let \( \dot{H}^{-1}_q(\mathbb{R}^N) \) be the dual space of \( \dot{H}^1_q(\mathbb{R}^N) \). For \( I \in \dot{H}^{-1}_q(\mathbb{R}^N) \) we set

\[
\|I\|_{\dot{H}^{-1}_q(\mathbb{R}^N)} = \sup\{\|I(\varphi)\|/\|\nabla \varphi\|_{L_q(\mathbb{R}^N)} \mid \varphi \in \dot{H}^1_q(\mathbb{R}^N), \nabla \varphi \neq 0\}.
\]

Let

\[
\mathcal{H}_q(\mathbb{R}^N) = \{f \in L_q(\mathbb{R}^N) \cap \dot{H}^{-1}_q(\mathbb{R}^N) \mid \text{supp } f \subset D_{R_1, R_2}, \quad (f, 1)_{\mathbb{R}^N} = 0\}.
\]

For any \( f \in \mathcal{H}_q(\mathbb{R}^N) \), \( \varphi \in \dot{H}^1_q(\mathbb{R}^N) \) and \( c \in \mathbb{R} \), \( (f, \varphi + c)_{\mathbb{R}^N} = (f, \varphi)_{\mathbb{R}^N} \), and so we define an operator \( I_f \) acting on \( \varphi \in \dot{H}^1_q(\mathbb{R}^N) \) by

\[
I_f(\varphi) = (f, \varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in \dot{H}^1_q(\mathbb{R}^N).
\]

Since

\[
(f, \varphi)_{\mathbb{R}^N} = (f, \varphi - [D_{R_1, R_2}])^{-1}\int_{D_{R_1, R_2}} \varphi \, dx_{\mathbb{R}^N},
\]

where \( |D_{R_1, R_2}| \) means the Lebesgue measure of \( D_{R_1, R_2} \), by Poincaré’s inequality, we have

\[
(f, \varphi)_{\mathbb{R}^N} \leq \|f\|_{L_q(D_{R_1, R_2})}\|\varphi - [D_{R_1, R_2}]^{-1}\int_{D_{R_1, R_2}} \varphi \, dx\|_{L_q(D_{R_1, R_2})},
\]

where \( |D_{R_1, R_2}| \) means the Lebesgue measure of \( D_{R_1, R_2} \), by Poincaré’s inequality, we have

\[
(f, \varphi)_{\mathbb{R}^N} \leq \|f\|_{L_q(D_{R_1, R_2})}\|\varphi - [D_{R_1, R_2}]^{-1}\int_{D_{R_1, R_2}} \varphi \, dx\|_{L_q(D_{R_1, R_2})},
\]
\[ \leq C_R \| f \|_{L_q(\mathbb{R}^N)} \| \nabla \varphi \|_{L_{q'}(\mathbb{R}^N)}, \]

which yields that
\[ \| I_f \|_{\dot{H}^{-1}_q(\mathbb{R}^N)} \leq C_R \| f \|_{L_q(\mathbb{R}^N)}. \]  

Let \( f \) be an element of \( L_q(\mathbb{R}^N) \) such that \( \text{supp} \ f \subset D_{R_1, R_2} \). If there exists a \( f \in L_q(\mathbb{R}^N) \) such that
\[ (f, \varphi)_{\mathbb{R}^N} = (f, \nabla \varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in \dot{H}^1_q(\mathbb{R}^N), \]  

then \( f \in \mathcal{H}_q(\mathbb{R}^N) \). In fact, since \( 1 \in \dot{H}^1_q(\mathbb{R}^N) \), by (72), \( (f, c)_{\mathbb{R}^N} = 0 \), and so
\[ (f, \psi)_{\mathbb{R}^N} = (f, \nabla \varphi)_{\mathbb{R}^N} \quad \text{for any } \psi \in [\varphi]. \]  

In this case,
\[ I_f([\varphi]) = (f, \varphi)_{\mathbb{R}^N} = (f, \nabla \varphi)_{\mathbb{R}^N} \quad \text{for any } [\varphi] \in \dot{H}^1_q(\mathbb{R}^N), \]  

and so
\[ \| I_f \|_{\dot{H}^{-1}_q(\mathbb{R}^N)} \leq \| f \|_{L_q(\mathbb{R}^N)}. \]  

In view of (68), (69), (70), and (72), \( \mathcal{R} f \in \mathcal{H}_q(\mathbb{R}^N) \) and
\[ \| \mathcal{R} f \|_{L_q(\mathbb{R}^N)} + \| I_{\mathcal{R} f} \|_{\dot{H}^{-1}_q(\mathbb{R}^N)} \leq C \| f \|_{L_q(\Omega)} \]  

for any \( f \in L_q(\Omega) \).

In the following, for any \( f \in \mathcal{H}_q(\mathbb{R}^N) \) we will look for \( u \in \dot{H}^1_{q,0}(\Omega) \) that is a solution of the variational equation:
\[ (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \text{for any } v \in \dot{H}^1_{q,0}(\Omega), \]  

possessing the estimate:
\[ \| \nabla u \|_{L_q(\Omega)} \leq C \| f \|_{L_q(\mathbb{R}^N)} \]  

for some constant \( C > 0 \).

Let \( E_N(x) \) be a fundamental solution of \( -\Delta \), that is
\[ E_N(x) = \begin{cases} -(2\pi)^{-1} \log |x| & N = 2, \\ ((N - 2)\omega_N)^{-1} |x|^{-(N-2)} & N \geq 3, \end{cases} \]  

where \( \omega_N \) is the area of the unit sphere \( S_1 = \{ \omega \in \mathbb{R}^N \mid |\omega| = 1 \} \). Note that \( \omega_N = 2\pi^{N/2}/\Gamma(N/2) \) with the gamma function \( \Gamma(s) = \int_0^\infty e^{-x}x^{s-1} \, dx \) \( (s > 1) \).

For \( f \in \mathcal{H}_q(\mathbb{R}^N) \), let
\[ [\mathcal{T}_\infty f](x) = -\int_{\mathbb{R}^N} E_N(x - y)f(y) \, dy, \]  

and then, \( -\Delta [\mathcal{T}_\infty f] = f \) in \( \mathbb{R}^N \). Moreover,
\[ \| [\mathcal{T}_\infty f] \|_{L_q(B_R)} + \sup_{|x| \geq 6R} \frac{|x|^{N-1}|[\mathcal{T}_\infty f](x)|}{|x|^N} \leq C \| f \|_{L_q(\mathbb{R}^N)}, \]  

\[ \sup_{|x| \geq 6R} |x|^N |\nabla [\mathcal{T}_\infty f](x)| + \| \nabla [\mathcal{T}_\infty f] \|_{L_q(\mathbb{R}^N)} \leq C \| f \|_{L_q(\mathbb{R}^N)}, \]  

\[ \| \nabla^2 [\mathcal{T}_\infty f] \|_{L_q(\mathbb{R}^N)} \leq C \| f \|_{L_q(\mathbb{R}^N)}. \]  

In fact, by Sobolev’s theorem for the weak singular operator (cf. [9, Theorem 9.2 and Theorem 9.3 in Sect. II.9]), we have \( \| [\mathcal{T}_\infty f] \|_{L_q(B_R)} \leq C_R \| f \|_{L_q(\mathbb{R}^N)} \). Since \( (f, 1)_{\mathbb{R}^N} = 0 \), we have
\[ [\mathcal{T}_\infty f](x) = \int_{\mathbb{R}^N} f(y)(E_N(x) - E_N(x - y)) \, dy, \]  

for some constant \( C > 0 \).
which yields that
\[ |\mathcal{T}_\infty f(x)| \leq C_R |x|^{-(N+1)} \|f\|_{L_q(\mathbb{R}^n)}, \quad |\nabla [\mathcal{T}_\infty f](x)| \leq C_R |x|^{-N} \|f\|_{L_q(\mathbb{R}^n)} \]
for \( x \not\in B_{6R} \) because \( \text{supp} f(x) \subset B_{5R} \). For any \( g \in C^0_0(\mathbb{R}^n)^N \),
\[ (\nabla \mathcal{T}_\infty f, g)_{\mathbb{R}^n} = -(f, \mathcal{T}_\infty (\text{div} g))_{\mathbb{R}^n}. \]
Since \( \tilde{T}_\infty (\text{div} g) = \text{div} (\tilde{T}_\infty g) \in L_{q,\text{loc}}(\mathbb{R}^n) \) as follows from Sobolev’s theorem for the weak singular operator and since \( \|\nabla \mathcal{T}_\infty (\text{div} g)\|_{L_q(\mathbb{R}^n)} \leq C \|g\|_{L_q(\mathbb{R}^n)} \) as follows from the boundedness of the singular integral operator due to Calderón and Zygmund (cf. [9, Theorem 9.4 in Sect. II.9]), we have
\[ \|\nabla [\tilde{T}_\infty f, g]\|_{L_q(\mathbb{R}^n)} \leq C \|f\|_{L_q(\mathbb{R}^n)} \|g\|_{L_q(\mathbb{R}^n)}, \]
which leads to \( \|\nabla [\tilde{T}_\infty f]\|_{L_q(\mathbb{R}^n)} \leq C \|f\|_{L_q(\mathbb{R}^n)}. \) By the boundedness of the singular integral operator, we also have
\[ \|\nabla^2 \tilde{T}_\infty f\|_{L_q(\mathbb{R}^n)} \leq C \|f\|_{L_q(\mathbb{R}^n)}. \]
Using the facts that \( C^0_0(\mathbb{R}^n) \) is dense in \( H^1_q(\mathbb{R}^n) \), that \( (f,1)_{\mathbb{R}^n} = 0 \) and that \( -\Delta \tilde{T}_\infty f = f \), we have
\[ (\nabla \tilde{T}_\infty f, \nabla v)_{\mathbb{R}^n} = (f, v)_{\mathbb{R}^n} \quad \text{for any } v \in H^1_q(\mathbb{R}^n). \]

Next, let \( u \in H^2_q(\Omega_{5R}) \) be a unique solution of the strong Dirichlet-Neumann problem:
\[ -\Delta u = f \quad \text{in } \Omega_{5R}, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial \omega}|_{S_{5R}} = 0, \quad (83) \]
where \( \frac{\partial u}{\partial \omega} = (x/|x|) \cdot \nabla u. \) Let \( \mathcal{T}_0 \) be an operator defined by \( \mathcal{T}_0 f = u \), and then we have
\[ (\nabla \mathcal{T}_0 f, \nabla v)_{\Omega_{5R}} = (f, v)_{\Omega_{5R}} \quad \text{for any } v \in \dot{H}^1_q(\Omega_{5R}), \]
\[ \|\mathcal{T}_0 f\|_{H^2_q(\Omega_{5R})} \leq C \|f\|_{L_q(\Omega_{5R})}. \]

For \( f \in \mathcal{H}_q(\mathbb{R}^n) \), let \( c_f \) be a constant such that
\[ \int_{\Omega_{5R}} (\tilde{T}_\infty f + c_f - \mathcal{T}_0 f) \, dx = 0. \]
Let \( \mathcal{T}_\infty f = \tilde{T}_\infty f + c_f \), and then
\[ \int_{\Omega_{5R}} (\mathcal{T}_\infty f - \mathcal{T}_0 f) \, dx = 0. \]
Moreover, by (80) and (84),
\[ \|\mathcal{T}_\infty f\|_{L_q(B_L)} \leq C_L \|f\|_{L_q(\mathbb{R}^n)}. \]
Let
\[ \mathcal{T} f = \varphi_\infty \mathcal{T}_\infty f + \varphi_0 \mathcal{T}_0 f \quad \text{for } f \in \mathcal{H}_q(\mathbb{R}^n). \]
Then,
\[ (\nabla \mathcal{T} f, \nabla v)_{\Omega} = (f + Mf, v)_{\Omega} \quad \text{for any } v \in \dot{H}^1_q(\Omega), \]
with
\[ Mf = -\{2(\nabla \varphi_\infty) \cdot \nabla (\mathcal{T}_\infty f) + 2(\nabla \varphi_0) \cdot \nabla (\mathcal{T}_0 f) + (\Delta \varphi_\infty) \mathcal{T}_\infty f + (\Delta \varphi_0) \mathcal{T}_0 f \}. \]
Let
\[ \dot{H}^2_q(\Omega) = \{ u \in \dot{H}^1_q(\Omega) \mid \nabla u \in H^1_q(\Omega)^N \}. \]
By (64), (80), and (84), we have $T f \in \dot{H}^2_{q,0}(\Omega),$
\[
\|Tf\|_{L_q(\Omega)} + \|\nabla T f\|_{H^1_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}, \tag{89}
\]
sup \mathbf{M} f \subset D_{R_1,R_2}, \quad \|\mathbf{M} f\|_{H^1_q(\mathbb{R}^N)} \leq C\|f\|_{L_q(\mathbb{R}^N)}
for any $f \in \mathcal{H}_q(\mathbb{R}^N)$.

Moreover, noting that $\nabla \varphi_\infty = -\nabla \varphi_0$, $\Delta \varphi_\infty = -\Delta \varphi_0$, $\Delta T_\infty f = \Delta T_0 f = -f,$
and using (83), for any $v \in \dot{H}^1_q(\mathbb{R}^N)$ we have
\[
(M f, v)_{\mathbb{R}^N} = -2(\langle \nabla \varphi_\infty \rangle \cdot (\nabla T_\infty f) + (\Delta \varphi_\infty) T_\infty f, v)_{\mathbb{R}^N} \tag{90}
\]
and then we have
\[
(M f, v)_{\mathbb{R}^N} = (f, v)_{\mathbb{R}^N} - (N f, \nabla v)_{\mathbb{R}^N} \tag{91}
\]
for any $v \in \dot{H}^1_q(\mathbb{R}^N)$.

Since $1 \in \dot{H}^1_q(\mathbb{R}^N)$ and $f \in \mathcal{H}_q(\mathbb{R}^N)$, we have $(M, 1)_{\mathbb{R}^N} = 0$, and so $M f \in \mathcal{H}_q(\mathbb{R}^N)$.

Moreover, $M$ is a compact operator on $\mathcal{H}_q(\mathbb{R}^N)$. In fact, let $\{f_j\}_{j=1}^\infty$ be a bounded sequence in $\mathcal{H}_q(\mathbb{R}^N)$, that is $f_j \in \mathcal{H}_q(\mathbb{R}^N)$ and $\|f_j\|_{L_q(\mathbb{R}^N)} \leq K$ with some constant $K$. By (89) and the Rellich compactness theorem, passing to a subsequence and writing it $\{j\}$ again, we see that there exists an $m f \in \dot{H}^1_q(\mathbb{R}^N)$ such that
\[
M f_j \to m f \quad \text{in } W^* \text{ topology of } H^1_q(\mathbb{R}^N), \quad \lim_{j \to \infty} \|M f_j - m f\|_{L_q(B_{R_2})} = 0. \tag{92}
\]

Since $\text{supp} M f_j \subset D_{R_1,R_2}$, $\text{supp} m f \subset D_{R_1,R_2}$ and $0 = \lim_{j \to \infty} (M f_j, 1)_{\mathbb{R}^N} = (m f, 1)_{\mathbb{R}^N}$. Thus, $m f \in \mathcal{H}_q(\mathbb{R}^N)$. Moreover, since $\text{supp} M f_j \subset D_{R_1,R_2}$, by (92) $M f_j \to m f$ strongly in $L_q(\mathbb{R}^N)$ as $j \to \infty$. Thus, $M$ is a compact operator on $\mathcal{H}_q(\mathbb{R}^N)$.

The following lemma is a key to prove the solvability of the variational equation (76).

**Lemma 3.5.** Let $1 < q < \infty$. Let $T$ be the operator defined by (87). Assume that if $f \in \mathcal{H}_q(\mathbb{R}^N)$ satisfies $f + M f = 0$, then $T f = 0$. Then, for any $f \in \mathcal{H}_q(\mathbb{R}^N)$ problem (76) admits a solution $u \in H^2_{q,0}(\Omega)$ satisfying the estimate:
\[
\|\nabla u\|_{H^1_q(\Omega)} \leq C\|f\|_{L_q(\mathbb{R}^N)}. \tag{93}
\]

**Proof.** In view of (88), to prove Lemma 3.5 it suffices to prove the existence of the inverse operator $(I + M)^{-1}$, which is a bounded linear operator on $\mathcal{H}_q(\mathbb{R}^N)$. Since $M$ is a compact operator on $\mathcal{H}_q(\mathbb{R}^N)$, in view of the Riesz-Schauder theorem, in
particular the Fredholm alternative principle, it suffices to prove that the kernel of the operator \(I + M\) is trivial. Thus, let \(f\) be an element in \(H_q(\mathbb{R}^N)\) such that \((I + M)f = 0\). Our task is to prove that \(f = 0\). By the assumption, we have \(Tf = 0\), which implies that

\[
(1 - \varphi_0)T_\infty f + \varphi_0 T_0 f = 0 \quad \text{in } \Omega, \tag{94}
\]

where we have used the definition: \(\varphi_\infty = 1 - \varphi_0\). By (64) and (94), \(T_\infty f = 0\) for \(x \not\in B_{4R}\) and \(T_0 f = 0\) for \(x \in B_{3R} \cap \Omega\). Let

\[
w = \begin{cases} T_0 f & \text{for } x \in \Omega_{5R} \cap (\mathbb{R}^N \setminus B_{3R}), \\ 0 & \text{for } x \in B_{3R}, \end{cases}
\]

and then noting (83) and (84), we see that \(w \in H_0^2(B_{5R})\) and that \(w\) satisfies the Neumann problem:

\[
\Delta w = -f \quad \text{in } B_{5R}, \quad \frac{\partial w}{\partial \omega}|_{\partial B_{5R}} = \frac{\partial T_0 f}{\partial \omega}|_{\partial B_{5R}} = 0. \tag{95}
\]

On the other hand, what \(T_\infty f = 0\) for \(x \not\in B_{4R}\) implies that \(T_\infty f \in H_0^2(B_{3R})\) and \(T_\infty f\) satisfies the Neumann problem (95), which, combined with the uniqueness, leads to \(T_\infty f = T_0 f + c\) in \(\Omega\) with some constant \(c\). However, (85) holds, and so \(c = 0\). Namely, \(T_\infty f = T_0 f\) on \(\Omega_{5R}\).

Thus, by (94) we have \(0 = T_\infty f + \varphi_0 (T_0 f - T_\infty f) = T_\infty f\) in \(\Omega\), and so \(f = \Delta T_\infty f = 0\) in \(\Omega\). But, supp \(f \subseteq D_{R_1,R_2} \subseteq \Omega\), and so \(f = 0\) in \(\mathbb{R}^N\). This implies the existence of the inverse operator \((I + M)^{-1} \in \mathcal{L}(H_q(\mathbb{R}^N))\) of \(I + M\) and then, by (88) \(u = T(I + M)^{-1}f\) is a solution of (76). Moreover, by (89), \(u\) satisfies (93), which completes the proof of Lemma 3.5. \(\square\)

**Lemma 3.6.** Let \(2 \leq q < \infty\). Then, for any \(f \in L_q(\Omega)^N\), problem (59) admits a solution \(u \in H^1_{q,0}(\Omega)\) satisfying the estimate:

\[
\|\nabla u\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}. \tag{96}
\]

**Proof.** First, we solve (76). In view of Lemma 3.5, it is sufficient to prove that \(Tf = 0\) provided that \(f \in H_q(\mathbb{R}^N)\) and \(f + Mf = 0\). Let \(\omega\) be a function in \(C_0^\infty(\mathbb{R}^N)\) such that \(\omega(x) = 1\) for \(|x| \leq 1\) and \(\omega(x) = 0\) for \(|x| \geq 2\), and let \(\omega_L(x) = \omega(x/L)\). Since \(2 \leq q < \infty\), by (89) \(\hat{T}f \in H^2_{q,loc}\), which, combined with \(\hat{T}f|_\Gamma = 0\), leads to \(\omega_L \hat{T}f \in \hat{H}^1_{q,0}(\Omega)\). Thus, by (88)

\[
0 = ((I + M)f, \omega_L \hat{T}f)|_\Omega = (\nabla \hat{T}f, \nabla (\omega_L \hat{T}f))|_\Omega \\
= (\omega_L \nabla \hat{T}f, \nabla \hat{T}f)|_\Omega + ((\nabla \omega_L) \nabla \hat{T}f, \hat{T}f)|_\Omega.
\]

Let \(L > 7R\), and then supp \(\nabla \varphi_\infty \cap \text{supp } \nabla \omega_L = \emptyset\). Recalling that

\[
Tf = \varphi_\infty(\hat{T}_\infty f + c_f) + \varphi_0 T_0 f,
\]

by (80) we have

\[
|((\nabla \omega_L) \nabla \hat{T}f, \hat{T}f)|_\Omega \leq |((\nabla \omega_L) \nabla \hat{T}_\infty f, c_f)|_{\mathbb{R}^N} + |((\nabla \omega_L) \nabla \hat{T}_\infty f, \hat{T}_\infty f)|_{\mathbb{R}^N} \\
\leq C_R \sup |\omega'| L^{-1} \int_{D_L \cup 2L} (|x|^{-N}|c_f| + |x|^{-2N+1}) \, dx \to 0
\]
as \(L \to \infty\), which leads to \(\|\nabla \hat{T}f\|_{L_2(\Omega)} = 0\). Thus, \(\hat{T}f\) is a constant. However, \(\hat{T}f|_\Gamma = 0\), and so \(\hat{T}f = 0\). Namely, in the case \(2 \leq q < \infty\), the assumption in
Lemma 3.5 is satisfied, and so for any $f \in \mathcal{H}_q(\mathbb{R}^N)$, problem (76) admits a solution $z \in \hat{H}^2_{q,0}(\Omega)$ possessing the estimate:

$$\|\nabla z\|_{H^1_q(\Omega)} \leq C(\|f\|_{\mathcal{H}^{-1}_q(\mathbb{R}^N)} + \|f\|_{L_q(\mathbb{R}^N)}).$$

Next, we consider (67). For $f \in L_q(\Omega)$, by (75) $\mathcal{R}f \in \mathcal{H}_q(\mathbb{R}^N)$ and $\|\mathcal{R}f\|_{\mathcal{H}_q(\mathbb{R}^N)} \leq C\|f\|_{L_q(\mathbb{R}^N)}$. Thus, there exists a $\psi \in \hat{H}^2_{q,0}(\Omega)$ such that $\psi$ satisfies the variational equation:

$$(\nabla \psi, \nabla v)_\Omega = (\mathcal{R}f, v)_\Omega \quad \text{for any } v \in \hat{H}^1_{q',0}(\Omega),$$

and the estimate:

$$\|\nabla \psi\|_{H^1_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}. \quad (97)$$

In view of (67), $u = \mathcal{S}f + \psi$ solves the variational equation:

$$(\nabla u, \nabla \varphi)_\Omega = (f, \nabla \varphi)_\Omega \quad \text{for any } \varphi \in \hat{H}^1_{q',0}(\Omega).$$

Moreover, by (66) and (97)

$$\|\nabla u\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}.$$

This completes the proof of Lemma 3.6.

**Lemma 3.7.** Let $1 < q < 2$. If $u \in \hat{H}^1_{q,0}(\Omega)$ satisfies the homogeneous equation:

$$(\nabla u, \nabla v)_\Omega = 0 \quad \text{for any } v \in \hat{H}^1_{q',0}(\Omega), \quad (98)$$

then $u = 0$.

**Proof.** For any $f \in L_q(\Omega)$, let $v \in \hat{H}^1_{q',0}(\Omega)$ be a solution of the variational equation:

$$(\nabla v, \nabla \psi)_\Omega = (f, \nabla \psi)_\Omega \quad \text{for any } \psi \in \hat{H}^1_{q',0}(\Omega).$$

Since $1 < q < 2$, $2 < q' < \infty$, and so Lemma 3.6 guarantees the existence of such $v$. Since $u \in \hat{H}^1_{q,0}(\Omega)$, we have $0 = (\nabla u, \nabla v)_\Omega = (\nabla u, f)_\Omega$. Since $f \in L_q(\Omega)$ is chosen arbitrarily, we have $\nabla u = 0$, which implies that $u$ is a constant. However, $u|_{\Gamma} = 0$, and so $u = 0$, which completes the proof of Lemma 3.7.

**Lemma 3.8.** Let $1 < q < 2$. Then, for any $f \in L_q(\Omega)^N$, problem (59) admits a unique solution $u \in \hat{H}^1_{q,0}(\Omega)$ satisfying the estimate:

$$\|\nabla u\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}. \quad (99)$$

**Proof.** Let $f$ be an element in $\mathcal{H}_q(\mathbb{R}^N)$ such that $f + \mathcal{M}f = 0$. Then, by (88) $\mathcal{T}f \in \hat{H}^1_{q,0}(\Omega)$ satisfies the homogeneous equation:

$$(\nabla \mathcal{T}f, \nabla v)_\Omega = 0 \quad \text{for any } v \in \hat{H}^1_{q',0}(\Omega).$$

Thus, by Lemma 3.7, we have $\mathcal{T}f = 0$, which, combined with Lemma 3.5, leads to the existence of solutions of the variational problem (76), which satisfy the estimate (93). Employing the same argument as that in the proof of Lemma 3.6, we have Lemma 3.8.

Since the uniqueness for problem (59) follows from the existence theorem for the dual problem, we have completed the proof of Theorem 3.2.
4. Local well-posedness. In this section, we prove Theorem 1.1. Let $T$ and $L$ be positive numbers determined later and let
\[
\mathcal{I}_T = \{v \in L_p((0, T), H^1_q(\Omega)^N) \cap H^1_p((0, T), L_q(\Omega)^N) \mid v|_{t=0} = u_0, \|v\|_{L^p(0, T), H^1_q(\Omega)} + \|\partial_t v\|_{L^p((0, T), L_q(\Omega))} \leq L, \int_0^T \|\kappa(\cdot)v(\cdot, s)\|_{H^1_q(\Omega)} ds \leq \sigma \}.
\]
Since $T > 0$ is chosen small enough eventually, we may assume that $0 < T \leq 1$.
Given $w \in \mathcal{I}_T$, let $v$ be a solution of linear equations:
\[
\begin{align*}
\frac{\partial_t v - \text{Div} (\mu D(v) - q I)}{\text{Div} v = G(w) = \text{Div} G(w)} & \quad \text{in } \Omega \times (0, T), \\
\partial_t v - \text{Div} (\mu D(v) - q I) & \quad \text{in } \Omega \times (0, T), \\
\mu D(v) - q I & \quad \text{on } \Gamma \times (0, T), \\
|v|_{t=0} = u_0 & \quad \text{in } \Omega.
\end{align*}
\tag{100}
\]
To solve (100), we use Theorem 2.7. To this end, we have to extend the right hand side of the equations (100) for all $t \in \mathbb{R}$. Let $h$ be a function defined on $(0, T)$ such that $h|_{t=0} = 0$, and then an operator $e_T$ acting on $h$ is defined by
\[
[e_T h](\cdot, t) = \begin{cases}
0 & t < 0, \\
h(\cdot, t) & 0 < t < T, \\
h(\cdot, 2T - t) & T < t < 2T, \\
0 & t > 2T.
\end{cases}
\tag{101}
\]
Since $h|_{t=0}$, we have
\[
[\partial_t (e_T h)](\cdot, t) = \begin{cases}
0 & t < 0, \\
(\partial_t h)(\cdot, t) & 0 < t < T, \\
(\partial_t h)(\cdot, 2T - t) & T < t < 2T, \\
0 & t > 2T.
\end{cases}
\tag{102}
\]
Let $T(t)$ be the operator given in Subsect. 2.3, and let $\psi(t)$ be a function in $C^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $t > -1$ and $\psi(t) = 0$ for $t < -2$. Let $w \in \mathcal{I}_T$, and let
\[
r_T w = e_T (w - \psi(t) T(t)|_{t=0}) + \psi(t) T(t)|_{t=0}.
\]
Note that $r_T w(\cdot, t) = w(\cdot, t)$ for $0 < t < T$. For simplicity, we write $G(w)$, $G(w)$ and $H(w)$ given in Eq. (13) symbolically as
\[
G(w) = -\sum_{j,k=1}^N V_{0jk} \int_0^t \nabla(\kappa w) ds \frac{\partial w_j}{\partial x_k} := \nabla v_1 \left( \int_0^t \nabla(\kappa w) ds \right), \nabla w > 1,
\]
\[
G(w)|_k = -\sum_{j=1}^N V_{0jk} \int_0^t \nabla(\kappa w) ds w_j
\]
\[
+ J_0 \int_0^t \nabla(\kappa w) ds \nabla V_{0jk} \left( \int_0^t \nabla(\kappa w) ds \right) w_j
\]
\[
:= \nabla v_{2k} \left( \int_0^t \nabla(\kappa w) ds \right), \ nabla w > 2,
\]
respectively. We may think that $v_k|_{\Omega}$ for $v_k < 2\sigma$ where $31 = f(0) = 0$, $v_k$ and $3N = h(0) = 0$, $v_1 = 0$, $v_2(0) = 0$ and $v_3(0) = 0$. Let $F(w) = \nabla \int_0^t \nabla (2\sigma) \, ds, \nabla w > 1, \quad G(w) = \nabla \int_0^t \nabla (2\sigma) \, ds \, w, \quad H(w) = \nabla \int_0^t \nabla (2\sigma) \, ds \, \nabla w$ where $< \cdot, \cdot >_1$ and $< \cdot, \cdot >_2$ denote the standard inner products in $\mathbb{R}^{N^2}$ and $\mathbb{R}^N$, respectively. We may think that $v_1(k)$ is an $N^2$ vector of some smooth functions for $|k| < 2\sigma$, $v_2(k)$ an $N \times N$ matrix of some smooth functions for $|k| < 2\sigma$, and $v_3(k)$ an $N \times N^2$ matrix of smooth functions for $|k| < 2\sigma$, and we may assume that $v_1(0) = 0$, $v_2(0) = 0$ and $v_3(0) = 0$. Let $f = \begin{cases} F(w) & 0 < t < T, \\
 0 & t \notin (0, T), \end{cases} \quad g = e_T(G(w)), \quad h = e_T(H(w)).$ In particular, we have $g = e_T[< v_1(\int_0^t \nabla (2\sigma) \, ds, \nabla (r_T w) >_1], \quad h = e_T[v_3(\int_0^t \nabla (2\sigma) \, ds) \nabla (r_T w)]]$. Since

$$g(\cdot, t) = \begin{cases} 0 & t < 0, \\
[G(w)](\cdot, t) & 0 < t < T, \\
[G(w)](\cdot, 2T - t) & T < t < 2T, \\
0 & t > 2T, \end{cases}$$

$$g(\cdot, t) = \begin{cases} 0 & t < 0, \\
[G(w)](\cdot, t) & 0 < t < T, \\
[G(w)](\cdot, 2T - t) & T < t < 2T, \\
0 & t > 2T, \end{cases}$$
as follows from \( r_T w = w \) for \( 0 < t < T \) and since \( \text{div} \, G(w) = G(w) \) for \( 0 < t < T \), we have \( \text{div} \, g = g \) for any \( t \in \mathbb{R} \), that is \( G(g) = g \).

Let \( v \) and \( q \) be solutions of the linear equations:

\[
\begin{aligned}
\partial_t v - \text{Div}(\mu D(v) - qI) &= f \quad \text{in } \Omega \times (0, T), \\
\text{div} v &= g = \text{div} \, g \quad \text{in } \Omega \times (0, T), \\
(\mu D(v) - qI)n &= h \quad \text{on } \Gamma \times (0, T), \\
v|_{t=0} &= u_0 \quad \text{in } \Omega,
\end{aligned}
\tag{103}
\]

and then \( v \) and \( q \) are also solutions of the equations (100), because \( f = F(w) \), \( g = G(w) \) and \( h = H(w) \) for \( t \in (0, T) \). In the following, using the Banach fixed point theorem, we prove that there exists a unique \( v \in \mathcal{D}_T \) such that \( v = w \), which is a required solution of Eq. (13).

Applying Theorem 2.7 gives that

\[
[v]_T \leq C\{\|u_0\|_{B^{1(\frac{N-1}{p})}_p(\Omega)} + \|f\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|(g, h)\|_{L_p(\mathbb{R}, H^1_q(\Omega))} \\
+ \|(g, h)\|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))} + \|\partial_t g\|_{L_p(\mathbb{R}, L_q(\Omega))}\},
\tag{104}
\]

provided that the right hand side in (104) is finite. In the following, \( C \) denotes generic constants independent of \( T \) and \( L \). Note that

\[
[w]_T := \|w\|_{L_p((0, T), H^2_p(\Omega))} + \|\partial_t w\|_{L_p((0, T), L_q(\Omega))} \leq L,
\int_0^T \|\kappa(\cdot, s)w(\cdot, s)\|_{H^2_p(\Omega)} ds \leq \sigma.
\tag{105}
\]

Recalling the definition of \( F_1 \) and \( F_2 \) given in (11) and (12), by (105) and (5) we have

\[
\|F(w)\|_{L_q(\Omega)} \leq C\left\{ \int_0^T \|w(\cdot, s)\|_{H^2_p(\Omega)} ds \|\nabla^2 w, \partial_t w\|_{L_q(\Omega)} \right\}
\tag{106}
\]

By the Hölder inequality, the Sobolev inequality and the assumption: \( N < q < \infty \),

\[
\int_0^T \|w(\cdot, s)\|_{H^2_p(\Omega)} ds \leq C\left( \int_0^T \|w(\cdot, s)\|_{H^2_p(\Omega)}^p ds \right)^{1/p} T^{1/p'} \leq CT^{1/p'} L,
\tag{107}
\]

Moreover, by a real interpolation theorem,

\[
\sup_{0 \leq t \leq T} \|w(\cdot, t)\|_{H^{2(\frac{N-1}{p})}_p(\Omega)} \leq C(\|u_0\|_{B^{1(\frac{N-1}{p})}_p(\Omega)} + [w]_T).
\tag{108}
\]

In fact, let \( z \) and \( \tau \) be solutions of the shifted Stokes equations:

\[
\begin{aligned}
\partial_t z + \lambda_0 z - \text{Div}(\mu D(z) - \tau I) &= 0 \quad \text{in } \Omega \times (0, \infty), \\
\text{div} z &= 0 \quad \text{in } \Omega \times (0, \infty), \\
(\mu D(z) - \tau I)n &= 0 \quad \text{on } \Gamma \times (0, \infty), \\
z|_{t=0} &= u_0 \quad \text{in } \Omega
\end{aligned}
\tag{109}
\]
with large $\lambda_0 > 0$. Then, employing the same argument as that in proving Theorem 2.6 gives the unique existence of $z$ possesses the estimate:

$$\|z\|_{L^p((0,\infty),H^1_p(\Omega))} + \|\partial_n z\|_{H^1_p((0,\infty),L^q(\Omega))} \leq C \|u_0\|_{B^{2(1-1/p)}_q(\Omega)}, \quad (110)$$

because $u_0 \in D_{p,q}(\Omega)$. Since $(w - z)_{|t=0} = 0$, we consider $e_T(w - z) + z$, and then $e_T(w - z) + z = w$ for $t \in (0, T)$. By (102) and (110) we have

$$\|e_T(w - z)\|_{L^p((0,\infty),H^2_p(\Omega))} + \|e_T(w - z)\|_{H^1_p((0,\infty),L^q(\Omega))} \leq C(\|w\|_T + \|u_0\|_{B^{2(1-1/p)}_q(\Omega)}). \quad (111)$$

By (2) with $T = \infty$, (110) and (111),

$$\sup_{0 \leq t \leq T} \|w(\cdot, t)\|_{B^{2(1-1/p)}_q(\Omega)} \leq \sup_{t \in (0, \infty)} \|e_T(w - z)\|_{B^{2(1-1/p)}_q(\Omega)} + \sup_{t \in (0, \infty)} \|z\|_{B^{2(1-1/p)}_q(\Omega)} \leq C(\|w\|_T + \|u_0\|_{B^{2(1-1/p)}_q(\Omega)}),$$

which shows (108). By (108) and the Sobolev inequality,

$$\sup_{0 \leq t \leq T} \|w(\cdot, t)\|_{L^\infty(\Omega)} \|\nabla w(\cdot, t)\|_{L^q(\Omega)} \leq \sup_{0 < t < T} \|w(\cdot, t)\|^2_{H^2_p(\Omega)} \leq \sup_{0 < t < T} \|w(\cdot, t)\|^2_{B^{2(1-1/p)}_q(\Omega)} \leq CL^2 \quad (112)$$

because $2(1 - 1/p) > 1$ as follows from $p > 2$. Combining (106), (107) and (112), we have

$$\|f\|_{L^p(\mathbb{R},L^q(\Omega))} = \|F(w)\|_{L^p(0,T),L^q(\Omega))} \leq CL^2(T^{1/p'} + T^{1/p}). \quad (113)$$

Next, we consider the estimate of $g$ and $h$. Since $v_1(0) = 0$, by (101), (105), the Sobolev inequality and the assumption: $N < q < \infty$, we have

$$\|g(\cdot, t)\|_{H^1_p(\Omega)} \leq C \begin{cases} \int_0^T \|w(\cdot, s)\|_{H^2_p(\Omega)} ds \|w(\cdot, t)\|_{H^2_p(\Omega)} & t < 0, \\
\int_0^{2T-t} \|w(\cdot, s)\|_{H^2_p(\Omega)} ds \|w(\cdot, 2T-t)\|_{H^2_p(\Omega)} & 0 \leq t < T, \\
0 & t > 2T. \end{cases} \quad (114)$$

Thus,

$$\|g(\cdot, t)\|_{H^1_p(\Omega)} \leq C \begin{cases} \int_0^T \|w(\cdot, t)\|_{H^2_p(\Omega)} ds \|w(\cdot, t)\|_{H^2_p(\Omega)} & t < 0, \\
T^{1/p'}[w]_T \|w(\cdot, t)\|_{H^2_p(\Omega)} & 0 \leq t < T, \\
T^{1/p'}[w]_T \|w(\cdot, 2T-t)\|_{H^2_p(\Omega)} & T \leq t < 2T, \\
0 & t > 2T, \end{cases}$$

which, combined with (105), gives that

$$\|g\|_{L^p(\mathbb{R},H^1_p(\Omega))} \leq CL^2T^{1/p'}. \quad (114)$$

Analogously,

$$\|h\|_{L^p(\mathbb{R},H^1_p(\Omega))} \leq CL^2T^{1/p'}. \quad (115)$$

To estimate the $H^{1/2}_p(\mathbb{R},L^q(\Omega))$ norm of $g$ and $h$, we use the following lemma.
Lemma 4.1. Let $N < q < \infty$, $1 < p < \infty$ and $T > 0$. Let $f \in H^1_q((0, T), L_q(\Omega)) \cap H^1_p((0, T), H^1_q(\Omega))$ and $g \in H^{1/2}_p(\mathbb{R}, L_q(\Omega)) \cap \mathcal{L}_p(\mathbb{R}, H^1_{2q}(\Omega))$. Assume that $f|_{t=0} = 0$. Then,

$$
\| \mathcal{F}(fg) \|_{H^{1/2}_p((0, \infty), L_q(\Omega))} + \| \mathcal{F}(fg) \|_{\mathcal{L}_p((0, \infty), H^{1/2}_q(\Omega))} \leq C \left\{ T^{1/p'} \| \partial_t f \|_{L_p((0, T), H^1_q(\Omega))} + T^{q/N} \| \partial_t f \|_{L_p((0, T), H^1_q(\Omega))} \| \partial_t f \|_{L^\infty((0, T), L_q(\Omega))} \right\} \\
\times \left( \| g \|_{H^{1/2}_p(\mathbb{R}, L_q(\Omega))} + \| g \|_{\mathcal{L}_p(\mathbb{R}, H^{1/2}_q(\Omega))} \right).
$$

(116)

Proof. We write $H^1_{p,q}(\mathbb{R}, L_q(\Omega)) \cap \mathcal{L}_p(\mathbb{R}, H^1_q(\Omega))$ and $H^1_{p,q}(\mathbb{R}, L_q(\Omega)) + \| h \|_{\mathcal{L}_p(\mathbb{R}, H^1_q(\Omega))}$ simply by $H^1_{p,q}$ and $\| h \|_{H^1_{p,q}}$, respectively. We use the fact that

$$
H^{1/2}_{p,q/2} = (L_p(\mathbb{R}, L_q(\Omega)), H^{1,1}_{p,q}[1/2]).
$$

(117)

By (102),

$$
\partial_t [\mathcal{F}(fg)] = \begin{cases} \\
(\partial_t f)(\cdot, t) g(\cdot, t) + f(\cdot, t) (\partial_t g)(\cdot, t) & 0 < t < T, \\
-(\partial_t f)(\cdot, 2T-t) g(\cdot, 2T-t) - f(\cdot, 2T-t) (\partial_t g)(\cdot, 2T-t) & T < t < 2T, \\
0 & t \notin [0, 2T].
\end{cases}
$$

and so by the Hölder inequality and the Sobolev inequality,

$$
\| \partial_t [\mathcal{F}(fg)] \|_{L_p(\mathbb{R}, L_q(\Omega))} \leq 2( I_1^{1/p} + I_2^{1/p})
$$

with

$$
I_1 = \int_0^T \| \partial_t f(\cdot, t) \|_{L^2_q(\Omega)}^p \| g(\cdot, t) \|_{L^p(\Omega)}^p \, dt, \\
I_2 = \int_0^T \| f(\cdot, t) \|_{H^1_q(\Omega)}^p \| \partial_t g(\cdot, t) \|_{L^p_q(\Omega)}^p \, dt.
$$

First, we observe that

$$
I_1 \leq \| f \|_{L^\infty((0, T), H^1_q(\Omega))} \| g \|_{H^1_p(\mathbb{R}, L_q(\Omega))} \leq T^{p/p'} \| \partial_t f \|_{L^p((0, T), H^1_q(\Omega))} \| g \|_{H^1_p(\mathbb{R}, L_q(\Omega))}.
$$

In fact, using the fact that $f|_{t=0} = 0$, we have

$$
\| f \|_{L^\infty((0, T), H^1_q(\Omega))} \leq T^{1/p'} \| \partial_t f \|_{L^p((0, T), H^1_q(\Omega))}.
$$

(118)

Next, by the Sobolev imbedding theorem and complex interpolation theory,

$$
\| h(\cdot, t) \|_{L^2_q(\Omega)} \leq C \| h(\cdot, t) \|_{H^1_q(\Omega)} \| h(\cdot, t) \|_{L^2_q(\Omega)} \leq C \| h(\cdot, t) \|_{L^2_q(\Omega)}^{1-\frac{q}{N}} \| h(\cdot, t) \|_{H^1_q(\Omega)}^{\frac{q}{N}}.
$$

Thus, by the Hölder inequality,

$$
I_2 \leq \int_0^T \left( \| \partial_t g(\cdot, t) \|_{H^1_q(\Omega)} \| g(\cdot, t) \|_{H^1_q(\Omega)} \right)^{\frac{q}{N}} \left( \| \partial_t f(\cdot, t) \|_{L^\infty((0, T), L_q(\Omega))} \| g(\cdot, t) \|_{L^\infty((0, \infty), L_q(\Omega))} \right)^{1-\frac{q}{N}} \, dt
$$

$$
\leq \| \partial_t f \|_{L^\infty((0, T), L_q(\Omega))} \| g \|_{L^\infty((0, \infty), L_q(\Omega))}
$$
Summing up, we have obtained

\[ \|\partial_t f\|_{L_p((0,T),H^1_p(\Omega))} \|g\|_{L_p((0,T),H^1_p(\Omega))} \left( \int_0^T dt \right)^{\frac{N}{N-1}}. \]

Since

\[ L_p((0,\infty),E_1) \cap H^1_p((0,\infty),E_0) \subset BUC((0,\infty),(E_0,E_1)_{1-1/p,p}) \]

and since this imbedding is continuously (cf. Tanabe [33, p.10, (1.19)]), where \( E_0 \) and \( E_1 \) are two Banach spaces such that \( E_1 \) is continuously imbedded into \( E_0 \), setting \( E_0 = L_q(\Omega) \) and \( E_1 = H^1_q(\Omega) \) we have

\[ \|g\|_{L_{\infty}((0,\infty),L_q(\Omega))} \leq \|g\|_{L_{\infty}((0,\infty),B^{1-1/p}_{1,q}(\Omega))} \leq C\|g\|_{H^{1;1}_{p,q}((0,\infty),L_q(\Omega))} \]

because \( 1 - 1/p > 0 \). Summing up, we have obtained

\[ \|\partial_t e_T(fg)\|_{L_p(\mathbb{R},L_q(\Omega))} \leq CMT\|g\|_{H^{1;1}_{p,q}(\Omega)} \]

with

\[ M_T = T^\frac{N}{p} \|\partial_t f\|_{L_p((0,T),H^1_p(\Omega))} + T^\frac{N}{p-1} \|\partial_t f\|_{L_p((0,T),H^1_p(\Omega))} \|\partial_t f\|_{L_{\infty}((0,T),L_q(\Omega))} \cdot \]

By the Sobolev inequality and (101) and (118),

\[ \|e_T[fg]\|_{L_p(\mathbb{R},H^1_p(\Omega))} \leq C\|f\|_{L_{\infty}((0,\infty),H^1_p(\Omega))} \|g\|_{L_p(\mathbb{R},H^1_p(\Omega))} \]

\[ \leq CMT\|g\|_{H^{1;1}_{p,q}(\Omega)}. \]

Summing up, we have obtained

\[ \|e_T[fg]\|_{H^{1;1}_{p,q}((0,\infty),L_q(\Omega))} \leq CMT\|g\|_{H^{1;1}_{p,q}(\Omega)}. \]

Thus, by (117), we have Lemma 4.1. □

Applying Lemma 4.1 to \( g = e_T G(w) = e_T[< v_1(\int_0^t \nabla(kw)ds),\nabla(r_T w)>_1] \) gives

\[ \|g\|_{H^{1/2}_{p,q}(\mathbb{R},L_q(\Omega))} \leq C(T^\frac{N}{p} + T^\frac{N}{p-1}) \]

\[ \leq C\left( T^\frac{N}{p} \|w\|_{L_p((0,T),H^1_p(\Omega))} + T^\frac{N}{p-1} \|w\|_{L_p((0,T),H^1_p(\Omega))} \right) \]

\[ \times (\|\nabla(r_T w)\|_{L^{1/2}_{p,q}(\mathbb{R},L_q(\Omega))} + \|\nabla(r_T w)\|_{L_p(\mathbb{R},H^{1/2}_{p,q}(\Omega))}). \]

By (110), Proposition 1, (101) and (102)

\[ \|\nabla(r_T w)\|_{L^{1/2}_{p,q}(\mathbb{R},L_q(\Omega))} + \|\nabla(r_T w)\|_{L_p(\mathbb{R},H^{1/2}_{p,q}(\Omega))} \]

\[ \leq C(\|r_T w\|_{H^1_p(\mathbb{R},L_q(\Omega))} + \|r_T w\|_{L_p(\mathbb{R},H^1_p(\Omega))}) \]

\[ \leq C(\|w\|_{H^1_p(0,T),L_q(\Omega)} + \|w\|_{L_p((0,T),H^1_p(\Omega))} + \|u_0\|_{H^{2(1-1/p)}_{2,p}(\Omega)}), \]

which, combined with (119) and (108), leads to

\[ \|g\|_{H^{1/2}_{p,q}(\mathbb{R},L_q(\Omega))} \leq C(L + S)(T^\frac{N}{p} + T^\frac{N}{p-1}). \]

Analogously,

\[ \|h\|_{H^{1/2}_{p,q}(\mathbb{R},L_q(\Omega))} \leq C(L + S)(T^\frac{N}{p} + T^\frac{N}{p-1}). \]
Finally, we estimate $\partial_t g$. To this end, we write

$$
\begin{aligned}
\partial_t g &=
\begin{cases}
\nu_2 \int_0^T \nabla (\kappa w) \, ds (\partial_t r_T w)(\cdot, t) \\
+ \nu_2' \int_0^T \nabla (\kappa w) \, ds (\nabla (\kappa w)(\cdot, t)) (r_T w)(\cdot, t) \\
- \nu_2 \int_0^{2T-t} \nabla (\kappa w) \, ds (\partial_t r_T w)(\cdot, 2T - t) \\
- \nu_2' \int_0^{2T-t} \nabla (\kappa w) \, ds (\nabla (\kappa w)(\cdot, 2T - t)) (r_T w)(\cdot, 2T - t) \\
0
\end{cases}
\end{aligned}
$$

where $\nu_2'(k) = \nabla_k \nu_2(k)$. By (108),

$$
\|\partial_t g\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C(T^{\frac{1}{p'}} + T^{\frac{1}{p}}) L (L + S),
$$

which, combined with (105), (113), (114), (115), (120) and (121), leads to

$$
[v]_T \leq C \{ S + L (L + S) (T^{\frac{1}{p'}} + T^{\frac{1}{p}} + T^{\frac{2 q - N}{p q}}) \}.
$$

Choosing $T > 0$ so small that

$$
C (L + S) (T^{\frac{1}{p'}} + T^{\frac{1}{p}} + T^{\frac{2 q - N}{p q}}) \leq 1/2,
$$

we have $[v]_T \leq CS + L/2$. Thus, choosing $L = 2CS$, we have

$$
[v]_T \leq L.
$$

Moreover, we have

$$
\int_0^T \|\kappa(\cdot) v(\cdot, s)\|_{H^1_q(\Omega)} \, ds \leq C_q \|\kappa\|_{H^1_q(\Omega)} T^{\frac{1}{p'}} \left( \int_0^T \|v(\cdot, s)\|_{H^2_q(\Omega)}^p \, ds \right)^{1/p} \leq C_q \|\kappa\|_{H^1_q(\Omega)} LT^{1/p'},
$$

and so choosing $T > 0$ in such a way that $C_q \|\kappa\|_{H^1_q(\Omega)} LT^{1/p'} \leq \sigma$, we have

$$
\int_0^T \|\kappa(\cdot) v(\cdot, s)\|_{H^1_q(\Omega)} \, ds \leq \sigma.
$$

Thus, $v \in \mathcal{I}_T$. Let $Q$ be a map defined by $Qw = v$, and then $Q$ maps $\mathcal{I}_T$ into itself. Analogously, we can show that for any $w_1, w_2 \in \mathcal{I}_T$,

$$
[Qw_1 - Qw_2]_T \leq C (L + S) (T^{\frac{1}{p'}} + T^{\frac{1}{p}} + T^{\frac{2 q - N}{p q}}) |w_1 - w_2|_T
$$

holds. Choosing $T$ smaller if necessary, we may assume that $C (L + S) (T^{\frac{1}{p'}} + T^{\frac{1}{p}} + T^{\frac{2 q - N}{p q}}) \leq 1/2$, and so $Q$ is a contraction map on $\mathcal{I}_T$. By the Banach fixed point theorem, there exists a unique $v \in \mathcal{I}_T$ such that $Qv = v$, which is a required unique solution of problem (13). This completes the proof of Theorem 1.1.

Analogously, we can prove the following theorem, which is used to prove the global well-posedness.

**Theorem 4.2.** Let $2 < p < \infty$, $N < q < \infty$ and $T > 0$. Let $\Omega$ be an exterior domain in $\mathbb{R}^N$ $(N \geq 2)$, whose boundary $\Gamma$ is a $C^2$ compact hypersurface. Assume that $2/p + N/q < 1$. Then, there exists an $\epsilon_0 > 0$ depending on $T$ such that if initial data $u_0 \in B_{2,p}^{2(1-1/p)}(\Omega)^N$ satisfies $\|u_0\|_{B_{2,p}^{2(1-1/p)}(\Omega)} \leq \epsilon_0$ and the compatibility condition (15), then problem (13) admits a unique solution $(v, q)$ with

$$
v \in L_p((0, T), H^1_q(\Omega)^N) \cap H^1_q((0, T), L_q(\Omega)^N),$$

$$
q \in L_p((0, T), H^1_q(\Omega) + \dot{H}^1_{q,0}(\Omega)).$$
possessing the estimate:
\[ \|v\|_{L^1((0,T),H^2(\Omega))} + \|\partial_t v\|_{L^p(0,T),L^q(\Omega)} \leq C\epsilon_0, \quad \int_0^T \|\kappa(\cdot)v(\cdot,s)\|_{H^2(\Omega)} \leq \sigma. \]

Here, \( C \) is a constant independent of \( \epsilon_0 \) and \( T \).

**Appendix A. Remark on the uniqueness of the weak Dirichlet problem.**

Let \( \Omega = \mathbb{R}^N \setminus B_1 \), and then \( \Gamma = \{x \in \mathbb{R}^N \mid |x| = 1\} \). Let
\[ f(x) = \begin{cases} \log |x| & N = 2, \\ |x|^{-(N-2)} - 1 & N \geq 3. \end{cases} \]  
(122)

Then, \( f(x) \) satisfies the strong Dirichlet problem \( \Delta f = 0 \) in \( \Omega \) and \( f|\Gamma = 0 \). Since \( |\nabla f(x)| \leq C|x|^{-(N-1)} \), \( f \in H^1_{0,0}(\Omega) \) provided that \( q > N/(N-1) \). Thus, there may be a possibility that \( f \) violates the uniqueness of the weak Dirichlet problem. But, from the following consideration, \( f \) does not satisfies the homogeneous weak Dirichlet problem:
\[ (\nabla f, \nabla \varphi)_\Omega = 0 \]  
(123)

In fact, let \( \varphi_0(r) \) be a function in \( C^\infty(\mathbb{R}) \) such that \( \varphi_0(r) = 0 \) for \( r \leq 2 \) and \( \varphi_0(r) = 1 \) for \( r \geq 3 \), and let \( \varphi(x) = \varphi_0(|x|) \). Let \( \psi(r) \) be a function in \( C^\infty(\mathbb{R}) \) such that \( \psi(r) = 1 \) for \( r \leq 1 \) and \( \psi(r) = 0 \) for \( r \geq 2 \), and let \( \psi(x) = \psi(|x|/R) \). Let \( c_N \) be a constant depending on \( N \) defined by \( c_N = 1 \) for \( N = 2 \) and \( c_N = -(N-2) \) for \( N \geq 3 \). We see that
\[ \nabla f \cdot \nabla \psi_R = c_N r^{-(N-1)} \psi'(r/R) R^{-1}. \]  
(124)

Since \( \varphi \in \dot{H}^{1}_{q',0}(\Omega) \), we have
\[ (\nabla f, \nabla \varphi)_\Omega = \lim_{R \to \infty} (\psi_R \nabla f, \nabla \varphi)_\Omega = -\lim_{R \to \infty} ((\nabla \psi_R) \cdot \nabla f) \varphi_\Omega = I + II \]
with
\[ I = -\lim_{R \to \infty} ((\nabla \psi_R) \cdot (\nabla f), 1)_\Omega, \quad II = \lim_{R \to \infty} ((\nabla \psi_R) \cdot (\nabla f), 1 - \varphi)_\Omega, \]
because \( \Delta f = 0 \) in \( \Omega \). By (124),
\[ I = -\lim_{R \to \infty} c_N \int_{|\omega|=1} d\omega \int_1^\infty \frac{d}{dr} \psi(r/R) dr = c_N \int_{|\omega|=1} d\omega \psi(0) = c_N \int_{|\omega|=1} d\omega \neq 0. \]

And also, from (124) it follows that \( |\nabla f \cdot \nabla \psi_R| \leq |c_N| \sup |\psi'| R^{-1} r^{-(N-1)} \), and so
\[ |II| \leq \lim_{R \to \infty} |c_N| \sup |\psi'| R^{-1} \int_{|\omega|=1} d\omega \int_0^\infty |1 - \varphi_0(r)| dr = 0. \]

Thus, we have
\[ (\nabla f, \nabla \varphi)_\Omega = c_N \int_{|\omega|=1} d\omega \neq 0. \]

Namely, we can conclude that \( f \) does not satisfy the homogeneous weak Dirichlet problem (123).

Moreover, as is seen in Proposition 3 below, \( C^\infty(\Omega) \) is not dense in \( \dot{H}^{1}_{q',0}(\Omega) \) for \( 1 < q' < N \).
Proposition 3. If $1 < q < N$, then $C_0^\infty(\Omega)$ is not dense in $\tilde{H}^1_{q,0}(\Omega)$ with gradient norm $\| \nabla \cdot \|_{L_q(\Omega)}$.

Proof. The proposition is proved by a contradiction argument. Assume that $C_0^\infty(\Omega)$ is dense in $\tilde{H}^1_{q,0}(\Omega)$. Since $1 < q < N$, there exists a constant $C$ such that
\[ \left\| \frac{\varphi}{|x|} \right\|_{L_q(\Omega)} \leq C \| \nabla \varphi \|_{L_q(\Omega)} \text{ for any } \varphi \in C_0^\infty(\Omega) \] (125)
(cf. Galdi [9]). Let $g$ be any element in $\tilde{H}^1_{q,0}(\Omega)$ and let $\{ \varphi_j \}_{j=1}^\infty$ be a sequence in $C_0^\infty(\Omega)$ such that
\[ \| \nabla(\varphi_j - g) \|_{L_q(\Omega)} = 0. \] (126)
Let $L$ be any large number such that $L > R$, and then the Poincaré inequality yields that there exists a number $C_L$ depending on $L$ such that
\[ \| \varphi_j - g \|_{L_q(\Omega)} \leq C_L \| \nabla(\varphi_j - g) \|_{L_q(\Omega)}, \] (127)
because $(\varphi_j - g)\big|_{|x|=0} = 0$. We may assume that there exists a number $c_0 > 0$ such that $\mathbb{R}^N \setminus \Omega \supset B_{c_0}$, and so $|x| \geq c_0$ for any $x \in \Omega$. Thus, by (125), (126) and (127),
\begin{align*}
\left\| \frac{g}{|x|} \right\|_{L_q(\Omega)} &\leq \left\| \frac{g}{|x|} - \frac{\varphi_j}{|x|} \right\|_{L_q(\Omega)} + \left\| \frac{\varphi_j}{|x|} \right\|_{L_q(\Omega)} \\
&\leq c_0^{-1} |g - \varphi_j|_{L_q(\Omega)} + C \| \nabla \varphi_j \|_{L_q(\Omega)} \\
&\leq c_0^{-1} C_L \| \nabla(\varphi_j - g) \|_{L_q(\Omega)} + C \| \nabla(\varphi_j - g) \|_{L_q(\Omega)} + C \| \nabla g \|_{L_q(\Omega)}.
\end{align*}
Letting $j \to \infty$, we have
\[ \left\| \frac{g}{|x|} \right\|_{L_q(\Omega)} \leq C \| \nabla g \|_{L_q(\Omega)}. \] (128)
Thus, letting $L \to \infty$, we have
\[ \left\| \frac{g}{|x|} \right\|_{L_q(\Omega)} \leq C \| \nabla g \|_{L_q(\Omega)} \] for any $g \in \tilde{H}^1_{q,0}(\Omega)$.

Let $g$ be an element of $C^\infty(\mathbb{R}^N)$ such that $g(x) = 1$ for $|x| \geq 3R$ and $g(x) = 0$ for $|x| \leq 2R$. Since $\nabla g(x) = 0$ for $|x| \geq 3R$, $g \in \tilde{H}^1_{q,0}(\Omega)$. By (128),
\[ \left\| |x|^{-1} \right\|_{L_q(\mathbb{R}^N \setminus B_{3R})} \leq \left\| \frac{g}{|x|} \right\|_{L_q(\Omega)} \leq C \| \nabla g \|_{L_q(\Omega)} \leq C \left( \sup_{x \in \Omega} |\nabla g(x)| \right) \| B_{3R} \|^{1/q} < \infty, \] where $|B_{3R}|$ denotes the volume of $B_{3R}$. But, in the case where $1 < q < \infty$, $\left\| |x|^{-1} \right\|_{L_q(\mathbb{R}^N \setminus B_{3R})} = \infty$, which leads to the contradiction. Thus, $C_0^\infty(\Omega)$ is not dense in $\tilde{H}^1_{q,0}(\Omega)$ provided that $1 < q < N$. \hfill \Box

If we define $D_q(\Omega)$ by the closure of $C_0^\infty(\Omega)$ with respect to the gradient norm $\| \nabla \cdot \|_{L_q(\Omega)}$ on $\Omega$, then by Proposition 3, $D_q(\Omega) \subset \tilde{H}^1_{q,0}(\Omega)$ and $D_q(\Omega) \neq \tilde{H}^1_{q,0}(\Omega)$ provided that $1 < q < N$. As was known (cf. Galdi [9]), the weak Dirichlet problem:
\[ (\nabla u, \nabla \varphi)_{\Omega} = (f, \nabla \varphi)_{\Omega} \] for any $\varphi \in D_q(\Omega)$
admits a unique solution $u \in D_q(\Omega)$ for any $f \in L_q(\Omega)^N$ provided that $N/(N-1) \leq q \leq N$.

Let $f$ be the function defined in (122). We see that $D_q(\Omega) = \tilde{H}^1_{q,0}(\Omega)$ provided that $N < q < \infty$, and so $f \in D_q(\Omega)$ for $N < q < \infty$. Moreover, $f$ satisfies the equations:
\[ (\nabla f, \nabla \varphi)_{\Omega} = 0 \] for any $\varphi \in D_q(\Omega). \] (129)
Because, for any \( \varphi \in C_0^\infty(\Omega) \) it follows \( (\nabla f, \nabla \varphi)_\Omega = (\Delta f, \varphi)_\Omega = 0 \) from the divergence theorem of Gauß, and so the denseness of \( C_0^\infty(\Omega) \) in \( D'_q(\Omega) \) yields (129).

Thus, we should take \( H^1_{q,0}(\Omega) \) to obtain the unique existence theorem of the weak Dirichlet problem, which is a different situation from the strong Dirichlet problem.

**Appendix B. A proof of Lemma 3.4.** In this appendix, we prove

**Theorem B.1.** Let \( 1 < q < \infty \). Then, for any \( f \in L_q(\Omega_5R)^N \), problem (62) admits a unique solution \( u \in \dot{H}^1_{q,0}(\Omega_5R) \) possessing the estimate:

\[
\|u\|_{\dot{H}^1_{q,0}(\Omega_5R)} \leq C\|f\|_{L_q(\Omega_5R)}.
\] (130)

For this purpose, we start with proving the unique existence theorem of the following auxiliary problem:

\[
\lambda(u, \varphi)_{\Omega_5R} + (\nabla u, \nabla \varphi)_{\Omega_5R} = (f, \nabla \varphi)_{\Omega_5R} + (f, \varphi)_{\Omega_5R} \quad \text{for any } \varphi \in H^1_{q,0}(\Omega_5R).
\] (131)

**Theorem B.2.** Let \( 1 < q < \infty \) and \( 0 < \epsilon < \pi/2 \). Then, there exists a \( \lambda_0 > 0 \) such that for any \( \lambda \in \Sigma_{\epsilon,\lambda_0}, f \in L_q(\Omega_5R)^N, \) and \( f \in L_q(\Omega_5R), \) problem (131) admits a unique solution \( u \in \dot{H}^1_{q,0}(\Omega_5R) \) possessing the estimate:

\[
|\lambda|^{1/2}\|u\|_{L_q(\Omega_5R)} + \|u\|_{H^1_{q,0}(\Omega_5R)} \leq C\{\|\lambda\|_{L_q(\Omega_5R)} + |\lambda|^{-1/2}\|f\|_{L_q(\Omega_5R)}\}.
\] (132)

**Proof.** We first consider (131) in \( \mathbb{R}^N \). Given \( \lambda \in \Sigma_{\epsilon,1}, f \in C_0^\infty(\mathbb{R}^N)^N \) and \( f \in C_0^\infty(\mathbb{R}^N), \) let

\[
u = F^{-1}_\xi\left[\frac{-i\xi \cdot \mathcal{F}[f]\xi + \mathcal{F}[f]\xi}{\lambda + |\xi|^2}\right],
\]

where \( \mathcal{F}[g] \) denotes the Fourier transform of \( g \) and \( \mathcal{F}^{-1}_\xi[h(\xi)] \) the inverse Fourier transform of \( h(\xi) \) with \( \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \). Obviously, \( u \) satisfies the equation:

\[
(\lambda - \Delta)u = -\div f + f \quad \text{in } \mathbb{R}^N,
\]

and possesses the estimate:

\[
|\lambda|^{1/2}\|u\|_{L_q(\mathbb{R}^N)} + \|u\|_{H^1_{q}(\mathbb{R}^N)} \leq C\{\|f\|_{L_q(\Omega)} + |\lambda|^{-1/2}\|f\|_{L_q(\mathbb{R}^N)}\}.
\] (133)

By the divergence theorem of Gauß, \( u \) satisfies the equation:

\[
\lambda(u, \varphi)_{\mathbb{R}^N} + (\nabla u, \nabla \varphi)_{\mathbb{R}^N} = (f, \nabla \varphi)_{\mathbb{R}^N} + (f, \varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in H^1_{q}(\mathbb{R}^N).
\] (134)

Since \( C_0^\infty(\mathbb{R}^N) \) is dense in \( L_q(\mathbb{R}^N) \), by the density argument we can construct \( u \in H^1_{q}(\mathbb{R}^N) \) that is a unique solution of problem (134) possessing the estimate (133) provided that \( \lambda \in \Sigma_{\epsilon,1}, f \in L_q(\mathbb{R}^N)^N \) and \( f \in L_q(\mathbb{R}^N) \).

Next, we consider the case where \( \mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\} \). Given function \( f \) defined on \( \mathbb{R}^N_+ \), \( f^o \) and \( f^e \) denote the odd and even extension of \( f \) to \( x_N < 0 \), respectively, that is

\[
f^o(x) = \begin{cases} f^o(x) & (x_N > 0), \\ f^o(x) = -f(x', -x_N) & (x_N < 0). \end{cases}
\]

\[
f^e(x) = \begin{cases} f^e(x) & (x_N > 0), \\ f^e(x) = f(x', -x_N) & (x_N < 0). \end{cases}
\]
with $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. Given $\lambda \in \Sigma_{c,1}$, $f \in C_0^\infty(\mathbb{R}^+_N)$ and $f \in C_0^\infty(\mathbb{R}^+_N)$, let

$$u = \mathcal{F}_\xi^{-1} \left[ \frac{-\mathcal{F}[\text{div } f]|(\xi) + \mathcal{F}[f]|(\xi)}{\lambda + |\xi|^2} \right],$$

and then $u$ satisfies the equations:

$$(\lambda - \Delta)u = -\text{div } f + f \text{ in } \mathbb{R}^+_N, \quad u|_{x_N = 0} = 0. \quad (135)$$

Moreover, noting that $\mathcal{F}[\text{div } f]|(\xi) = \sum_{j=1}^{N-1} i \xi_j \mathcal{F}[f_j]|(\xi) + i \xi_N \mathcal{F}[f_N]|(\xi)$, we have

$$|\lambda|^{1/2} \|u\|_{L_4(\mathbb{R}^+_N)} + \|u\|_{H^1_4(\mathbb{R}^+_N)} \leq C \{ \|f\|_{L_4(\mathbb{R}^+_N)} + |\lambda|^{-1/2} \|f\|_{L_4(\mathbb{R}^+_N)} \}. \quad (136)$$

Using the divergence theorem of Gauß, from (135) we have

$$\lambda(u, \varphi)_{R^N} + (\nabla u, \nabla \varphi)_{R^N} = (f, \varphi)_{R^N} \quad (137)$$

for any $\varphi \in H^1_{q',0}(\mathbb{R}^+_N) = \{ \varphi \in H^1_{q'}(\mathbb{R}^+_N) \mid \varphi|_{x_N = 0} = 0 \}$. Since $C_0^\infty(\mathbb{R}^+_N)$ is dense in $L_q(\mathbb{R}^+_N)$, for any $\lambda \in \Sigma_{c,1}$, $f \in L_q(\mathbb{R}^+_N)$ and $f \in L_q(\mathbb{R}^+_N)$, problem (137) admits a unique solution $u \in H^1_{q',0}(\mathbb{R}^+_N)$ possessing the estimate (136).

Analogously, given $\lambda \in \Sigma_{c,1}$, $f \in C_0^\infty(\mathbb{R}^+_N)$ and $f \in C_0^\infty(\mathbb{R}^+_N)$, let

$$u = \mathcal{F}_\xi^{-1} \left[ \frac{-\mathcal{F}[\text{div } f]|(\xi) + \mathcal{F}[f]|(\xi)}{\lambda + |\xi|^2} \right],$$

and then $u$ satisfies the equations:

$$(\lambda - \Delta)u = -\text{div } f + f \text{ in } \mathbb{R}^+_N, \quad \frac{\partial u}{\partial x_N}|_{x_N = 0} = 0. \quad (138)$$

Moreover, noting that $\mathcal{F}[\text{div } f]|(\xi) = \sum_{j=1}^{N-1} i \xi_j \mathcal{F}[f_j]|(\xi) + i \xi_N \mathcal{F}[f_N]|(\xi)$, we have

$$|\lambda|^{1/2} \|u\|_{L_4(\mathbb{R}^+_N)} + \|u\|_{H^1_4(\mathbb{R}^+_N)} \leq C \{ \|f\|_{L_4(\mathbb{R}^+_N)} + |\lambda|^{-1/2} \|f\|_{L_4(\mathbb{R}^+_N)} \}. \quad (139)$$

Using the divergence theorem of Gauß, from (138) we have

$$\lambda(u, \varphi)_{R^N} + (\nabla u, \nabla \varphi)_{R^N} = (f, \varphi)_{R^N} \quad (140)$$

for any $\varphi \in H^1_{q'}(\mathbb{R}^+_N)$. Since $C_0^\infty(\mathbb{R}^+_N)$ is dense in $L_q(\mathbb{R}^+_N)$, for any $\lambda \in \Sigma_{c,1}$, $f \in L_q(\mathbb{R}^+_N)$ and $f \in L_q(\mathbb{R}^+_N)$, problem (140) admits a unique solution $u \in H^1_{q',0}(\mathbb{R}^+_N)$ possessing the estimate (139).

The results in the half-space can be extended to the bent half space case. Thus, by constructing a parametrix with the help of partition of unity, we can prove Theorem B.2, which completes the proof of Theorem B.2.

**A Proof of Theorem B.1.** Let $H^{-1}_q(\Omega_{\delta R})$ be the dual space of $\hat{H}^{-1}_{q,0}(\Omega_{\delta R})$. For $f \in L_q(\Omega_{\delta R})$, $[f] = \{ g \in L_q(\Omega) \mid (g - f, \nabla \varphi)_{\Omega_{\delta R}} = 0 \text{ for any } \varphi \in \hat{H}^{-1}_{q',0}(\Omega_{\delta R}) \}$, and then let $\hat{L}_q(\Omega) = \{ [f] \mid f \in L_q(\Omega_{\delta R}) \}$. In view of the Hahn-Banach theorem, $H^{-1}_q(\Omega_{\delta R})$ is identified with $\hat{L}_q(\Omega_{\delta R})$ by the operation: $(f, \nabla \varphi)_{\Omega_{\delta R}} = \langle F_f, \varphi \rangle$ for any $\varphi \in \hat{H}^{-1}_{q',0}(\Omega_{\delta R})$, where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H^{-1}_q(\Omega_{\delta R})$ and $\hat{H}^{-1}_{q',0}(\Omega_{\delta R})$. Let $A$ be an operator acting on $u \in \hat{H}^{-1}_{q',0}(\Omega_{\delta R})$ defined by

$$\langle Au, v \rangle = -\langle \nabla u, \nabla v \rangle_{\Omega_{\delta R}} \text{ for any } \varphi \in \hat{H}^{-1}_{q',0}(\Omega_{\delta R}).$$

Let $F_f$ be an element in $H^{-1}_q(\Omega_{\delta R})$ defined by $\langle F_f, \varphi \rangle = \langle f, \nabla \varphi \rangle_{\Omega_{\delta R}}$ for any $\varphi \in \hat{H}^{-1}_{q',0}(\Omega_{\delta R})$ and $f \in L_q(\Omega_{\delta R})$. Problems (62) and (131) with $f = 0$ are rewritten as $-Au = F_f$ and $(\lambda - A)u = F_f$ in $H^{-1}_q(\Omega_{\delta R})$. Since $-A = \lambda - A - \lambda = \lambda_A$.
(I − λ(λ − A)^{-1})(λ − A), where I is the identity operator, formally (−A)^{-1} = (λ − A)^{-1}(I − λ(λ − A)^{-1})^{-1}. In view of Theorem B.2, for large λ, (λ − A)^{-1} exists, and so it suffices to prove the existence of (I − λ(λ − A)^{-1})^{-1}. First, we prove that (λ − A)^{-1} is a compact operator on H_q^{-1}(Ω_{5R}). In fact, let \{F_j\}_{j=1}^\infty be a bounded sequence in H_q^{-1}(Ω_{5R}), say \|F_j\|_{H_q^{-1}(Ω_{5R})} ≤ K for j ∈ N. By Theorem B.2 with f = 0 and F = F_f with some \( f \) ∈ L_q(Ω_{5R}), we have

\[ \|((λ − A)^{-1}F\|_{H_q^1(Ω_{5R})} ≤ C\|F\|_{H_q^{-1}(Ω_{5R})} ≤ CK. \]

Since \( \hat{H}_q^1(Ω_{5R}) \) is compactly embedded in L_q(Ω_{5R}) as follows from Rellich compactness theorem, (λ − A)^{-1} is a compact operator from H_q^{-1}(Ω_{5R}) into L_q(Ω_{5R}). Moreover, \( L_q(Ω_{5R}) \) is continuously embedded into H_q^{-1}(Ω_{5R}). In fact, for any u ∈ L_q(Ω_{5R}), by the Poincaré inequality we have

\[ |< u, ϕ >| = |< u, ϕ >|_{Ω_{5R}} |≤ ∥u∥_{L_q(Ω_{5R})} ∥ϕ∥_{L_q(Ω_{5R})} ≤ C\|u∥_{L_q(Ω_{5R})} ∥∇ϕ∥_{L_q(Ω_{5R})} \]

for any ϕ ∈ \( \hat{H}_q^1(Ω_{5R}) \), which leads to \( ∥u∥_{H_q^{-1}(Ω_{5R})} ≤ C\|u∥_{L_q(Ω_{5R})} \) for any u ∈ L_q(Ω_{5R}). Therefore, (λ − A)^{-1} is a compact operator on H_q^{-1}(Ω_{5R}). In view of the Riesz-Schauder theorem, what the kernel of the map I − λ(λ − A)^{-1} is trivial yields that the existence of the inverse operator (I − λ(λ − A)^{-1})^{-1}. So, let F ∈ H_q^{-1}(Ω_{5R}) be an element such that (I − λ(λ − A)^{-1})F = 0, and then F = λ(λ − A)^{-1}F ∈ \( \hat{H}_q^1(Ω) \) and −AF = 0. If we show that F = 0, then (−A)^{-1} exists, which proves Theorem B.1. First, we consider the case where 2 ≤ q < ∞. In this case, F ∈ \( \hat{H}_q^1(Ω_{5R}) \) ⊂ \( \hat{H}_{q',0}^1(Ω_{5R}) \), and so

\[ 0 = < -AF, F > = (∇F, ∇F)_{Ω_{5R}}, \]

which yields that F is a constant. However, F|_Γ = 0, and so F = 0. Thus, we have Theorem B.1 in the case where 2 ≤ q < ∞. Next, we consider the case 1 < q < 2. Let G be any element in H_q^{-1}(Ω_{5R}). Since 2 < q' < ∞, there exists a v ∈ \( \hat{H}_{q',0}^1(Ω_{5R}) \) such that \( G = -Av \), that is v = (−A)^{-1}G. Thus,

\[ < G, F > = < -Av, F > = (∇v, ∇F)_{Ω_{5R}} = < -AF, v > = 0, \]

which leads to F = 0, because of the arbitrary choice of G ∈ H_q^{-1}(Ω_{5R}). This completes the proof of Theorem B.1.

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Received February 2017; revised July 2017.

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