CP Violation due to Flavor Mixing in Gauge-Higgs Unification

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Abstract

We discuss CP violation in the five dimensional $SU(3) \otimes SU(3)_{\text{color}}$ gauge-Higgs unification scenario in which the fifth dimension is compactified on an orbifold $S^1/Z_2$. It is shown that CP-violating phase appears even in the two generation case in contrast with the fact that at least three generations are required to break CP symmetry in the Standard Model. As our prediction, we obtain the phenomenological constraint on the compactification scale by comparing the typical CP-violating observables, namely $\varepsilon_K$ parameter and the mass difference of two neutral $K$ mesons $\Delta m_K$ with experimental data.
1 Introduction

Gauge-Higgs unification (GHU) [1] opens an new avenue to go beyond the standard model (SM) since the hierarchy problem is solved without supersymmetry by identifying Higgs scalar field with an extra spatial component of gauge fields. In other words, the quantum correction to the Higgs boson mass becomes finite due to the higher dimensional gauge symmetry though the theory is non-renormalizable. The finiteness of the Higgs mass has been verified in various models [2, 3].\(^1\) It is interesting to find other finite observables similar to the Higgs mass in this scenario. In this regard, several studies were done concerning about the $S$ and $T$ parameters, gluon-fusion interaction and fermion anomalous magnetic moment and electric dipole moment [5–7].

On the other hand, the flavor mixing and CP violation in this scenario is non-trivial issue since the Yukawa and the gauge interaction are unified into higher dimensional gauge interaction. In this context, the Yukawa coupling becomes “real” and weak gauge eigenstate is essentially equal to mass eigenstates, it seems to be difficult to incorporate CP violation and flavor mixing. We have already discussed the flavor mixing in the GHU [8–10] and pointed out that the flavor mixing is realized by an interplay between the non-degenerate fermion bulk mass term and non-trivial brane-localized mass term. As a result, the flavor changing neutral current (FCNC) via non-zero Kaluza-Klein (KK) gauge boson exchange occurs at tree level even in QCD sector.

Focussing on the CP violation, several types of CP violation in the GHU have been already investigated [11,12]. In the paper [11], the Higgs boson behaves as a CP odd scalar and CP symmetry breaks down spontaneously with electroweak symmetry breaking. In the paper [12], a complex structure can be embedded into the compactified space and the CP violation is obtained by incompatibility with the orbifolding parity. Both of the above CP violation are specific to higher dimensional gauge theory, however, the Kobayashi-Maskawa (KM) type CP violation is not studied precisely.

In this paper we address the issue of CP violation related to flavor mixing in the GHU scenario. It is highly non-trivial problem to explain the CP violation in addition to the variety of fermion masses and flavor mixings in this scenario, since Yukawa coupling is the real and universal gauge couplings. To see how CP symmetry breaks in such models, we need to mention the mechanism to realize flavor mixing. In the $SU(3) \otimes SU(3)_{\text{color}}$ model, $n$ set of $\mathbf{3}$ and $\bar{\mathbf{6}}$ representation of $SU(3)$ should be introduced to reproduce $n$ generations of quark.\(^2\) Since each representation has two doublets $Q_3$ and $Q_6$, namely two massless

\(^1\)For the case of gravity-gauge-Higgs unification, see [4].

\(^2\)More precisely, $\mathbf{15}$ representation should be introduced to embedding top quark, similar argument can be expanded.
quark doublets appear per generation, we identify the SM quark doublet $Q_{\text{SM}}$ as follows.

$$
\begin{bmatrix}
U_1 & U_3 \\
U_2 & U_4
\end{bmatrix}
\begin{bmatrix}
Q_{HL}(x) \\
Q_{\text{SML}}(x)
\end{bmatrix}
= 
\begin{bmatrix}
Q_{3L}(x) \\
Q_{6L}(x)
\end{bmatrix}
\tag{1.1}
$$

where $U_3$, $U_4$ are $n \times n$ matrices which indicate that the SM quark doublets are given by what kind of mixture of $Q_{3L}(x)$ and $Q_{6L}(x)$ and compose of a $2n \times 2n$ unitary matrix together with $U_1$ and $U_2$. The eigenstate $Q_H$ becomes massive and decouples from the low energy processes by the brane-localized mass term, while $Q_{\text{SM}}$ remains massless at this stage and is identified with the SM quark doublet. $U_3$ and $U_4$ should satisfy the following unitarity condition:

$$U_3^\dagger U_3 + U_4^\dagger U_4 = 1_{n \times n}. \tag{1.2}$$

Now we discuss the counting of physical CP-violating phases. Note that $U_3$ and $U_4$ generically have complex components, they potentially violate CP symmetry. These $n \times n$ complex matrices $U_3$ and $U_4$ are known to be written in a product of an unitary matrix and a diagonal matrix:

$$
\begin{cases}
U_3 = P_3 \mathcal{U} P_3' \text{ diag}\left(c_{a_1}, c_{a_2}, \cdots, c_{a_n}\right)
\vspace{1em}
U_4 = P_4 \mathcal{V} P_4' \text{ diag}\left(s_{a_1}, s_{a_2}, \cdots, s_{a_n}\right)
\end{cases}
\tag{1.3}
$$

where $s_{a_i} \equiv \sin a_i$, $c_{a_i} \equiv \cos a_i$. In the above expression, we use the fact that the arbitrary unitary matrices $V$ can be always decomposed into some phase matrices $P$ and $P'$ with $n$ and $n - 1$ phases respectively : $V = PP'$. Then the $\mathcal{U}$ and $\mathcal{V}$ have $\frac{(n-1)(n-2)}{2}$ phases, respectively.

Next, we focus on the counting of CP-violating phases in this model. Since the phase matrices $P_3$ and $P_4$ in the left-hand side of $\mathcal{U}$ and $\mathcal{V}$ act on the right-handed up- and down-type quark, they can be eliminated by the re-definition of singlet quarks. Since a similar discussion on phase matrices in the right-hand side of $\mathcal{U}$ and $\mathcal{V}$ can be applied, they can be eliminated by the re-definition of the $Q_{\text{SM}}$. However, both phases matrix denoted by $P_3'$ and $P_4'$ commonly act on the $Q_{\text{SML}}$ and thus only $n - 1$ phases of $P_3'$ and $P_4'$ can be absorbed by the $Q_{\text{SML}}$. Then the remaining physical phases are given by the sum of the phases in the $\mathcal{U}$, $\mathcal{V}$ and $P_3'$ or $P_4'$:

$$
\frac{(n-1)(n-2)}{2} + \frac{(n-1)(n-2)}{2} + (n-1) = (n-1)^2.
\tag{1.4}
$$

Since there are FCNC vertices in the strong interaction as we have discussed before \cite{8}, the CP-violating phase appears in the interactions between the zero-mode fermion and non-zero KK gluons even in the two generation scheme in our model. Note that the above argument is consistent with the KM theory, the phases in the Yukawa interactions are completely removed by the ordinary field re-definition.
This paper is organized as follows. The next section after this introduction, we construct the model in the two generation scheme as a simplest example of CP-violating model. As an application of the CP violation due to the flavor mixing discussed in section 3, we calculate the Wilson coefficient caused by the $\Delta S = 2$ process in section 4, $K^0 - \bar{K}^0$ mixing via non-zero KK gluon exchange at the tree level in order to compare $\varepsilon_K$ parameter which is known as a typical CP-violating observables in our model with the experimental result. We also obtain the lower bound for the compactification scale. Section 5 is devoted to our conclusions.

2 The Model

Although the model we consider in this paper is the same as the one taken in [8, 9], we briefly describe the model for completeness. The model taken in this paper is a five dimensional (5D) $SU(3) \otimes SU(3)_{\text{color}}$ GHU model compactified on an orbifold $S^1/\mathbb{Z}_2$ with a radius $R$ of $S^1$. As matter fields, we introduce two generations of bulk fermion in the 3 and the $\bar{6}$ dimensional representations of $SU(3)$ gauge group denoted by a column vector and a $3 \times 3$ matrix, $\psi^i(3)$ and $\psi^i(\bar{6})$ ($i = 1, 2$) [13].

The bulk lagrangian is given by

$$
\mathcal{L} = -\frac{1}{2} \text{Tr}(F_{MN}F^{MN}) - \frac{1}{2} \text{Tr}(G_{MN}G^{MN})
+ \bar{\psi}^i(3)\left\{i \slashed{\partial}_3 - M_i \epsilon(y)\right\}\psi^i(3)
+ \frac{1}{2} \text{Tr}\left[\bar{\psi}^i(\bar{6})\left\{i \slashed{\partial}_6 - M_i \epsilon(y)\right\}\psi^i(\bar{6})\right] \tag{2.1}
$$

where

$$
F_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N], \tag{2.2a}
$$

$$
G_{MN} = \partial_M G_N - \partial_N G_M - ig_s[G_M, G_N], \tag{2.2b}
$$

$$
\slashed{\partial}_3 \psi^i(3) = \Gamma^M(\partial_M - ig A_M - ig_s G_M)\psi^i(3), \tag{2.2c}
$$

$$
\slashed{\partial}_6 \psi^i(\bar{6}) = \Gamma^M \left[\partial_M \psi^i(\bar{6}) + ig\left\{A^*_M \psi^i(\bar{6}) + \psi^i(\bar{6})(A_M)\right\} - ig_s G_M \psi^i(\bar{6})\right], \tag{2.2d}
$$

with $G_M$ being understood to act on the color index, not explicitly written here. The gauge fields $A_M$ and $G_M$ are written in a matrix form, e.g. $A_M = A_M^a \lambda^a$ in terms of Gell-Mann matrices $\lambda^a$. $M, N = 0, 1, 2, 3, 5$ and the 5D gamma matrices are given by $\Gamma^5 = (\gamma^\mu, i\gamma^5)$ ($\mu = 0, 1, 2, 3$). $g$ and $g_s$ are 5D gauge coupling constants of $SU(3)$ and $SU(3)_{\text{color}}$, respectively. $M_i$ are generation dependent bulk mass parameters of the fermions accompanied by the sign function $\epsilon(y)$. As was discussed in the introduction, here we take the base where the bulk mass term is flavor-diagonal.

The periodic boundary condition is imposed along $S^1$ and $Z_2$ parity assignments are taken for gauge fields as

$$
A_\mu(-y) = PA_\mu(y)P^{-1}, \quad A_y(-y) = -PA_y(y)P^{-1}, \tag{2.3a}
$$

with $p = (1, 0)$ for the KK modes of the bulk fermion.
\[ G_\mu(-y) = G_\mu(y) \quad , \quad G_y(-y) = -G_y(y) \]  

(2.3b)

where the orbifolding matrix is defined as \( P = \text{diag}(-, -, +) \) and operated in the same way at the fixed points \( y = 0, \pi R \). We can see that the gauge symmetry \( SU(3) \) is explicitly broken to \( SU(2) \times U(1) \) by the boundary conditions. The gauge fields with \( Z_2 \) odd parity and even parity are expanded by use of mode functions,

\[ S_n(y) = \frac{1}{\sqrt{\pi R}} \sin \frac{n y}{R}, \quad C_n(y) = \frac{1}{\sqrt{2^{n,0} \pi R}} \cos \frac{n y}{R}, \]  

(2.4)

respectively.

A chiral theory is realized in the zero-mode sector by \( Z_2 \) orbifolding. The fermions are also expanded by an orthonormal set of mode functions. Here we will focus on the zero-mode sector which are necessary for the argument of flavor mixing:

\[ \Psi^i(3) \supset Q^i_L f^i_L(y) \oplus d^i_R f^i_R(y), \]  

(2.5a)

\[ \Psi^i(6) \supset \Sigma^i_R f^i_R(y) \oplus Q^i_L f^i_L(y) \oplus u^i_R f^i_R(y) \]  

(2.5b)

where the mode function for the zero mode of each chirality is given in [11]:

\[ f^i_L(y) = \sqrt{\frac{M_i}{1 - e^{-2\pi R M_i}}} e^{-M_i |y|}, \quad f^i_R(y) = \sqrt{\frac{M_i}{e^{2\pi R M_i} - 1}} e^{M_i |y|}. \]  

(2.6)

We notice that there are two left-handed quark doublets \( Q_{3L} \) and \( Q_{6L} \) per generation in the zero-mode sector, which are massless before electroweak symmetry breaking. In the one generation case, for instance, one of two independent linear combinations of these doublets should correspond to the quark doublet in the SM, but the other one should be regarded as an exotic state. Moreover, we have an exotic fermion \( \Sigma_R \). We therefore introduce brane localized four dimensional (4D) Weyl spinors to form \( SU(2) \times U(1) \) invariant brane localized Dirac mass terms in order to remove these exotic massless fermions from the low-energy effective theory [13,14].

Some comments on this model are in order. The predicted Weinberg angle of this model is not realistic, \( \sin^2 \theta_W = \frac{3}{4} \). Possible modification is to introduce an extra \( U(1) \) [10] or the brane localized gauge kinetic term [15]. However, the wrong Weinberg angle does not affect our argument, since our interest is \( K^0 - \bar{K}^0 \) mixing via KK gluon exchange in the QCD sector, whose amplitude is independent of the Weinberg angle.

3 CP violation due to flavor mixing

As was mentioned in introduction, the CP phase remains even in the two generation scheme in our model. For an illustrative purpose to confirm the mechanism of CP violation due to flavor mixing, we will see how the realistic quark masses and mixing are reproduced.
Then, $2 \times 2$ matrices $U_3$ and $U_4$ can be written without loss of generality as,

$$
U_3 = \begin{bmatrix} 
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta 
\end{bmatrix} \begin{bmatrix} 
c_{a_1} & 0 \\
0 & c_{a_2} 
\end{bmatrix}, \quad U_4 = \begin{bmatrix} 
\cos \theta' & -\sin \theta' \\
\sin \theta' & \cos \theta' 
\end{bmatrix} \begin{bmatrix} 
s_{a_1} & 0 \\
0 & s_{a_2} e^{i \gamma} 
\end{bmatrix}
$$

(3.1)

where the CP-violating phases $\gamma$ do not need to appear in $U_4$ but may appear in $U_3$ if we wish. It is a matter of convention.

In this case, Yukawa couplings are read off from the higher dimensional gauge interaction of $A_y$, whose zero mode is the Higgs field $H(x)$:

$$
- \frac{g_4}{2} \left\{ \left( H^\dagger \right) \bar{d}_R(x) I_R^{(0)} U_3^{ij} Q_s(x) + \sqrt{2} \left( H^{\dagger} \right) i \sigma^2 \bar{u}_R(x) I_R^{(0)} U_4^{ij} Q_s(x) \right\} + \text{h.c.}
$$

(3.2)

where $g_4 \equiv \frac{g}{\sqrt{2} \pi R}$ and $I_R^{(0)}$ is an overlap integral of mode functions of fermions with matrix elements $(I_R^{(0)})_{ij} = \delta_{ij} I_R^{(0)}$:

$$
I_R^{(0)} \equiv \int_{-\pi}^{\pi} dy f_i f_R = \frac{\pi R M_i}{\sinh(\pi R M_i)},
$$

(3.3)

which behaves as $2 \pi R M_i e^{-\pi R M_i}$ for $\pi R M_i \gg 1$, thus realizing the hierarchical small quark masses without fine tuning of $M_i$. We thus know that the matrices of Yukawa coupling $\frac{g_4 Y_u}{2}$ and $\frac{g_4 Y_d}{2}$ are given as

$$
\frac{g_4}{2} Y_u = \frac{g_4}{2} \sqrt{2} I_R^{(0)} U_4, \quad \frac{g_4}{2} Y_d = \frac{g_4}{2} I_R^{(0)} U_3
$$

(3.4)

These matrices are diagonalized by bi-unitary transformations as in the SM and Cabibbo-Kobayashi-Maskawa (CKM) matrix is defined in a usual way [16].

$$
\begin{align*}
\hat{Y}_d &= \text{diag}(\hat{m}_d, \cdots) = V_{dR}^\dagger Y_d V_{dL} \\
\hat{Y}_u &= \text{diag}(\hat{m}_u, \cdots) = V_{uR}^\dagger Y_u V_{uL}
\end{align*}
$$

where all the quark masses are normalized by the W-boson mass as $\hat{m}_f = \frac{m_f}{M_W}$. A remarkable point is that the Yukawa couplings $\frac{g_4 Y_u}{2}$ and $\frac{g_4 Y_d}{2}$ are mutually related by the unitarity condition eq. (1.2), on the contrary those are completely independent in the SM.

Now physical observables $\hat{m}_u, \hat{m}_c, \hat{m}_d, \hat{m}_s$ and the Cabibbo angle $\theta_c$ are all written in terms of $a_i, b_i (\equiv I_R^{(0)})$, $\theta, \theta'$ and $\gamma$. Namely trivial relations

$$
\hat{Y}_d^\dagger \hat{Y}_u = \text{diag}(\hat{m}_u^2, \cdots) \quad , \quad \hat{Y}_u^\dagger \hat{Y}_d = \text{diag}(\hat{m}_d^2, \cdots).
$$

(3.6)

Let us note that some phases appear in $(2 \times 2)$ CKM matrix in this parametrization. So we change the base and eliminate any phases from $(2 \times 2)$ CKM here in order to fit the phase convention of the SM. We can achieve it by the following rephrasing.

$$
\begin{align*}
u &\to P_u \nu \\
d &\to P_d d
\end{align*}
$$

while

$$
\begin{align*}
P_u &= \text{diag} \left( e^{i(\gamma - \theta_2)}, e^{i(\gamma - \theta_1)} \right) \\
P_d &= \text{diag} \left( e^{i(\gamma - \theta_1 - \theta_2)}, 1 \right)
\end{align*}
$$

(3.7)
where the phases $\theta_1$ and $\theta_2$ are given as

\[
\tan \theta_1 = \frac{\sin \gamma \sin \theta_{aL} \sin \theta_{dL}}{\cos \theta_{aL} \cos \theta_{dL} + \cos \gamma \sin \theta_{aL} \sin \theta_{dL}}, \quad (3.8a)
\]

\[
\tan \theta_2 = \frac{\sin \gamma \cos \theta_{aL} \cos \theta_{dL}}{\sin \theta_{aL} \cos \theta_{dL} - \cos \gamma \cos \theta_{aL} \cos \theta_{dL}}. \quad (3.8b)
\]

The $\theta_{dL}$, $\theta_{uL}$ are angles parametrizing $V_{dL}$, $V_{uL}$, respectively:

\[
\tan 2\theta_{uL} = \frac{2s_{a1}s_{a2}(b_2^2 - b_1^2)\sin 2\theta'}{(s_{a1}^2 - s_{a2}^2)(b_1^2 + b_2^2) - (s_{a1}^2 + s_{a2}^2)(b_2^2 - b_1^2)\cos 2\theta'}, \quad (3.9a)
\]

\[
\tan 2\theta_{dL} = \frac{2c_{a1}c_{a2}(b_2^2 - b_1^2)\sin 2\theta}{(c_{a1}^2 - c_{a2}^2)(b_1^2 + b_2^2) - (c_{a1}^2 + c_{a2}^2)(b_2^2 - b_1^2)\cos 2\theta'}. \quad (3.9b)
\]

Then the Cabibbo angle $\theta_c$ is given as

\[
\cos 2\theta_c = \cos 2\theta_{aL} \cos 2\theta_{dL} + \cos \gamma \sin 2\theta_{aL} \sin 2\theta_{dL}. \quad (3.10)
\]

Note that 5 physical observables are written in terms of 7 parameters, $a_i$, $b_i$ ($i = 1, 2$), $\theta$, $\theta'$ and $\gamma$ in this case. So our theory has 2 degrees of freedom, which cannot be determined by the observables. We choose $\theta'$ and $\gamma$ as free parameters here. Then once we choose the values of $\theta'$ and $\gamma$, other 5 parameters can be completely fixed by the observables, by solving eqs. (3.6) and (3.10) numerically for $a_i$, $b_i$ and $\theta$.

Thus we have confirmed that observed quark masses and flavor mixing angle can be reproduced in our model of GHU. Let us note that in eq. (3.10) Cabibbo angle $\theta_c$ vanishes in the limit of universal bulk mass, i.e. $M_1 = M_2$ leads to $b_1 = b_2$ as is expected.

### 4 Constraint from $\Delta S = 2$ process

In this section, we apply the results of the previous section to a representative CP-violating FCNC process, $K^0 - \bar{K}^0$ mixing in the down-type quark sector caused by the non-zero KK mode gluon exchange at the tree level as the dominant contribution to this FCNC process, and we also estimate the lower bound on the compactification scale $R^{-1}$ from $K^0 - \bar{K}^0$ mixing responsible for the mass difference $\Delta m_K$ of two neutral $K$ mesons and the parameter $\varepsilon_K$.

We focus on the FCNC processes of zero-mode down-type quarks due to gauge boson exchange at the tree level. We derive the 4D effective strong interaction vertices with respect to the zero-modes of down-type quarks relevant for our calculation:

\[
\mathcal{L}_s \supset \frac{g_s}{2\sqrt{2}\pi R} G_{\mu}^{\alpha} \left( \tilde{d}_R^\alpha \lambda^\alpha \gamma^\mu d_R^i + \tilde{d}_L^\alpha \lambda^\alpha \gamma^\mu d_L^i \right) + \sum_{n=1}^{\infty} \frac{g_s}{2} G_{\mu}^{(n)}(\tilde{d}_R^\alpha \lambda^\alpha \gamma^\mu d_R^i) \left( P_d^i V_d^\dagger f^{(00)}_{dR} V_{dR} P_d \right)_{ij} \\
+ \sum_{n=1}^{\infty} \frac{g_s}{2} G_{\mu}^{(n)}(\tilde{d}_L^\alpha \lambda^\alpha \gamma^\mu d_L^i) \left\{ P_d^i V_d^\dagger \left( U_3^i f^{(00)}_{dL} U_3^\dagger + U_4^i f^{(00)}_{dL} U_4^\dagger \right) V_{dL} P_d \right\}_{ij}
\]

\footnote{For the studies of $K^0 - \bar{K}^0$ mixing in other new physics models, see for instance [17,18].}
\[ \frac{g_\alpha}{2\sqrt{2\pi R}} \left( \bar{d}_{L}\gamma^\mu d_{R} + \bar{d}_{L}\gamma^\mu d_{R} \right) \]
\[ - \frac{g_\alpha e^{i\gamma'}}{4} \sin 2\theta_{dR} \sum_{n=1}^{\infty} (I_{RR}^{(0n)} - I_{RR}^{(2n)}) \cdot G^a_{\mu} \tilde{s}_{R}\gamma^\mu d_{R} \]
\[ - \frac{g_\alpha e^{i\gamma'}}{4} (\alpha_d + i\beta_d) \sum_{n=1}^{\infty} (-1)^n (I_{RR}^{(0n)} - I_{RR}^{(2n)}) \cdot G^a_{\mu} \tilde{s}_{L}\gamma^\mu d_{L} \]

(4.1)

where \( d' = (d, s) \). The phase \( \gamma' \equiv \gamma - (\theta_1 + \theta_2) \) and the parameters \( \alpha_d \) and \( \beta_d \) are defined as follows:

\[ \alpha_d \equiv \text{Re} \left\{ V_{dd}^\dagger U_3^\dagger \sigma_3 U_3 + U_4^\dagger \sigma_3 U_4 \right\} \]
\[ = \frac{c_{a_1}^2 + c_{a_2}^2}{2} \sin 2\theta_{dL} \cos 2\theta + c_{a_1}c_{a_2} \cos 2\theta_{dL} \sin 2\theta \]
\[ + \frac{s_{a_1}^2 + s_{a_2}^2}{2} \sin 2\theta_{dL} \cos 2\theta' + s_{a_1}s_{a_2} \cos 2\theta_{dL} \sin 2\theta' \cos \gamma \]

(4.2a)

\[ \beta_d \equiv \text{Im} \left\{ V_{dd}^\dagger U_3^\dagger \sigma_3 U_3 + U_4^\dagger \sigma_3 U_4 \right\} \]
\[ = -s_{a_1}s_{a_2} \sin 2\theta' \sin \gamma. \]

(4.2b)

The \( \theta_{dR} \) in the rotation matrix \( V_{dR} \) to diagonalize \( I_{RL}^{(00)} U_3 U_3^\dagger I_{RL}^{(00)} \):

\[ \tan 2\theta_{dR} = \frac{2(c_{a_1}^2 - c_{a_2}^2)b_1b_2 \sin 2\theta}{(c_{a_1}^2 + c_{a_2}^2)(b_1^2 - b_2^2) + (c_{a_1}^2 - c_{a_2}^2)(b_1^2 + b_2^2) \cos 2\theta}. \]

(4.3)

\( I_{RR}^{(0n)} \) and \( I_{LL}^{(0n)} \) is an overlap integral relevant to gauge interaction,

\[ I_{RR}^{(0n)} \equiv \frac{1}{\sqrt{\pi R}} \int_{\pi R}^{\pi R} (f_{RR}^i)^2 \cos \frac{n}{R} y = \frac{1}{\sqrt{\pi R}} \frac{(2RM_i)^2}{(2RM_i)^2 + n^2} e^{2\pi RM_i} - 1, \]

(4.4a)

\[ I_{LL}^{(0n)} = I_{RR}^{(0n)} \bigg|_{M_i \rightarrow -M_i} = (-1)^n I_{RR}^{(0n)} \]

(4.4b)

since the chirality exchange corresponds to the exchange of two fixed points. We can see from (4.1) that the FCNC appears in the couplings of non-zero KK gluons due to the fact that \( I_{RR}^{(0n)} \) is not proportional to the unit matrix in the generation space, while the coupling of the zero-mode gluon is flavor conserving, as we expected.

It turns out that the non-zero KK gluon vertex of the left-handed type contains the CP-violating phase in any base of phase, since it gets contributions from both of \( U_3 \) and \( U_4 \). The non-zero KK gluon exchange diagrams, which give the dominant contribution to the process of \( K^0 \rightarrow \bar{K}^0 \) mixing, are depicted in figure 1. These diagrams are expected to give the imaginary part in the amplitude of \( K^0 \rightarrow \bar{K}^0 \) mixing through non-zero KK gluon exchange at the tree level, and therefore to the parameter \( \varepsilon_K \). The contributions from each type diagram of the \( K^0 \rightarrow \bar{K}^0 \) mixing in figure 1 is written in the form of effective four-Fermi lagrangian,

\[ (\text{LL type}) \sim -\frac{\pi\alpha_d e^{i\gamma'}}{2} \frac{(\alpha_d + i\beta_d)^2 S_{KK}^{LL}}{R^2} (\tilde{s}_L\gamma^\mu d_L)(\tilde{s}_L\gamma^\mu d_L), \]

(4.5a)
The diagrams of $K^0 - \bar{K}^0$ mixing via non-zero KK gluon exchange

\[(RR \text{ type}) \sim -\frac{\pi \alpha_s}{2} e^{2i\gamma'} \sin^2 2\theta d_{RR}^{\alpha_R \gamma}\mu d_R \left(\bar{s}_R \gamma^\alpha d_R\right) \right], \quad (4.5b) \]

\[(LR \text{ type}) \sim -\frac{\pi \alpha_s}{2} e^{2i\gamma'(\alpha_d + i\beta_d)} \sin 2\theta d_{RR}^{\alpha_R \gamma}\mu d_R \left(\bar{s}_L \gamma^\alpha d_L\right) \right), \quad (4.5c)\]

where four-dimensional $\alpha_s$ is defined by $\alpha_s = \frac{(g_4^D)^2}{4\pi}$ with $g_4^D \equiv \frac{g_s}{\sqrt{2\pi R}}$ and the mode sums are defined as

\[S_{KK}^{LL} = S_{KK}^{RR} \equiv \pi R \sum_{n=1}^{\infty} \frac{1}{n^2} \left(I_{RR}^{(0n0)} - I_{RR}^{(2n0)}\right)^2, \quad (4.6a)\]

\[S_{KK}^{LR} \equiv \pi R \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(I_{RR}^{(0n0)} - I_{RR}^{(2n0)}\right)^2. \quad (4.6b)\]

The sum over the integer $n$ is convergent and the coefficients of the effective lagrangian (4.5) are suppressed by the compactification scale as $1/M_c^2$ where $M_c = R^{-1}$.

Comparing the calculation of (4.5) with the experimental data, we can obtain a lower bound on the compactification scale. The most general effective Hamiltonian for $\Delta S = 2$ processes due to some “New Physics” at a high scale $\Lambda_{NP} \gg M_W$ can be written as follows;

\[\mathcal{H}^{\Delta S=2}_{\text{eff}} = \frac{1}{\Lambda_{NP}^2} \left(\sum_{i=1}^{5} z_i Q_i + \sum_{i=1}^{3} \tilde{z}_i \tilde{Q}_i\right) \quad (4.7)\]

where

\[Q_1 = \bar{s}_L^\alpha \gamma_\mu d_L^\beta \bar{s}_L^\gamma d_L^\beta, \quad Q_2 = \bar{s}_R^\alpha p_\mu L \bar{s}_R^\beta d_L^\beta, \quad Q_3 = \bar{s}_R^\alpha d_\mu L \bar{s}_R^\beta d_L^\beta, \quad Q_4 = \bar{s}_R^\alpha d_\mu L \bar{s}_R^\beta d_R^\beta, \quad Q_5 = \bar{s}_R^\alpha d_\mu L \bar{s}_R^\beta d_R^\beta, \quad (4.8)\]

Indices $\alpha, \beta$ stand for the color degrees of freedom. The operators $\tilde{Q}_{1,2,3}$ are obtained from the $Q_{1,2,3}$ by the chirality exchange $L \leftrightarrow R$. If we assume one of these possible operators gives dominant contribution to the mixing, each coefficient is independently constrained as follows, with the constraints for $\tilde{z}_i$ are the same with those for $z_i$ ($i = 1, 2, 3$) [20];

\[\text{Re } z_1 \leq [-9.6, 9.6] \times 10^{-7} \left(\Lambda_{NP}/1\text{TeV}\right)^2, \]

\[\text{Re } z_2 \leq [-1.8, 1.9] \times 10^{-8} \left(\Lambda_{NP}/1\text{TeV}\right)^2, \]

\[\text{Im } z_1 \leq [-1.8, 1.9] \times 10^{-7} \left(\Lambda_{NP}/1\text{TeV}\right)^2, \]

\[\text{Im } z_2 \leq [-1.8, 1.9] \times 10^{-8} \left(\Lambda_{NP}/1\text{TeV}\right)^2. \]
\[ \text{Re } z_3 \leq [-6.0, 5.6] \times 10^{-8} (\Lambda_{\text{NP}}/1\text{TeV})^2, \] (4.9a)
\[ \text{Re } z_4 \leq [-3.6, 3.6] \times 10^{-9} (\Lambda_{\text{NP}}/1\text{TeV})^2, \]
\[ \text{Re } z_5 \leq [-1.0, 1.0] \times 10^{-8} (\Lambda_{\text{NP}}/1\text{TeV})^2 \]

and
\[ \text{Im } z_1 \leq [-4.4, 2.8] \times 10^{-9} (\Lambda_{\text{NP}}/1\text{TeV})^2, \]
\[ \text{Im } z_2 \leq [-5.1, 9.3] \times 10^{-11} (\Lambda_{\text{NP}}/1\text{TeV})^2, \]
\[ \text{Im } z_3 \leq [-3.1, 1.7] \times 10^{-10} (\Lambda_{\text{NP}}/1\text{TeV})^2, \]
\[ \text{Im } z_4 \leq [-1.8, 0.9] \times 10^{-11} (\Lambda_{\text{NP}}/1\text{TeV})^2, \]
\[ \text{Im } z_5 \leq [-5.2, 2.8] \times 10^{-11} (\Lambda_{\text{NP}}/1\text{TeV})^2. \]

where \( \Lambda_{\text{NP}} \) is regarded as the compactification scale \( R^{-1} \) in our case. All we have to do is to represent (4.5) by use of (4.8) and to utilize these constraints (4.9).

We can rewrite the each type effective lagrangian (4.5) in terms of effective Hamiltonian by using the Fierz transformation and the completeness condition for Gell-Mann matrices;

\[ H_{\text{eff,LL}}^{\Delta S=2} = \frac{z_1 Q_1}{R^{-2}}, \quad H_{\text{eff,RR}}^{\Delta S=2} = \frac{\tilde{z}_1 \tilde{Q}_1}{R^{-2}}, \quad H_{\text{eff,LR}}^{\Delta S=2} = \frac{z_4 Q_4 + z_5 Q_5}{R^{-2}} \] (4.10)

where Wilson coefficients \( z_i \) are obtained as

\[ \text{Re } z_1 = \frac{2\pi\alpha_s}{3} \left\{ \left( \alpha_d^2 - \beta_d^2 \right) \cos 2\gamma' - 2\alpha_d\beta_d \sin 2\gamma' \right\} S_{KK}^{LL}, \] (4.11a)
\[ \text{Re } \tilde{z}_1 = \frac{2\pi\alpha_s}{3} \sin^2 2\theta_{dR} \cos 2\gamma' S_{KK}^{RR}, \] (4.11b)
\[ \text{Re } z_4 = -\frac{2\pi\alpha_s}{3} \sin 2\theta_{dR} \left( \alpha_d \cos 2\gamma' - \beta_d \sin 2\gamma' \right) S_{KK}^{LR}, \] (4.11c)
\[ \text{Re } z_5 = \frac{2\pi\alpha_s}{3} \sin 2\theta_{dR} \left( \alpha_d \cos 2\gamma' - \beta_d \sin 2\gamma' \right) S_{KK}^{LR}. \] (4.11d)

and

\[ \text{Im } z_1 = \frac{2\pi\alpha_s}{3} \left\{ \left( \alpha_d^2 - \beta_d^2 \right) \sin 2\gamma' + 2\alpha_d\beta_d \cos 2\gamma' \right\} S_{KK}^{LL}, \] (4.12a)
\[ \text{Im } \tilde{z}_1 = \frac{2\pi\alpha_s}{3} \sin^2 2\theta_{dR} \sin 2\gamma' S_{KK}^{RR}, \] (4.12b)
\[ \text{Im } z_4 = -\frac{2\pi\alpha_s}{3} \sin 2\theta_{dR} \left( \alpha_d \sin 2\gamma' + \beta_d \cos 2\gamma' \right) S_{KK}^{LR}, \] (4.12c)
\[ \text{Im } z_5 = \frac{2\pi\alpha_s}{3} \sin 2\theta_{dR} \left( \alpha_d \sin 2\gamma' + \beta_d \cos 2\gamma' \right) S_{KK}^{LR}. \] (4.12d)

The constant \( \alpha_s \) should be estimated at the scale \( \mu_K = 2.0 \text{ GeV} \) where the \( \Delta S = 2 \) process is actually measured [20]. So we have to take into account the renormalization group effect from the weak scale down to \( \mu_K \) and we obtain \( \alpha_s \approx 0.268 \) [8, 9].

Combining these results, we obtain the lower bounds for the compactification scale \( R^{-1} \) from the constraint (4.9). For that purpose, we have to consider the “weight” for the contribution of each type diagram from lattice QCD calculations of the matrix elements
with non-perturbative renormalization. As is discussed precisely in the ref [20], it is known that an analytic formula for the contribution to the $K^0 - \bar{K}^0$ mixing amplitudes induced by a given New Physics scale coefficient, denoted by $\langle \bar{K}^0 | \mathcal{H}_{\text{eff}} | K^0 \rangle_i$:

$$\langle \bar{K}^0 | \mathcal{H}_{\text{eff}} | K^0 \rangle_i = \sum_{j=1}^{5} \sum_{r=1}^{5} \left( b_j^{(r,i)} + \eta c_j^{(r,i)} \right) \eta^2 j \frac{z_j (A_{\text{NP}})}{A_{\text{NP}}^2} R_r \langle \bar{K}^0 | Q_1 | K^0 \rangle ,$$

(4.13)

where $\eta \equiv \alpha_s (A_{\text{NP}}) / \alpha_s (m_t)$ ($m_t$: top quark mass), the “magic numbers” $a_j$, $b_j^{(r,i)}$ and $c_j^{(r,i)}$ can be found in ref. [19]. $R_r$ are the ratios of the matrix elements of the New Physics operators $Q_i$ over the SM one [20]:

$$R_1 = 1 , \quad R_2 \simeq -12.9 , \quad R_3 \simeq 3.98 , \quad R_4 \simeq 20.8 , \quad R_5 \simeq 5.2 .$$

(4.14)

Utilizing these values, the contribution of LR type diagram ($z_4$ and $z_5$) can be written as just like that of LL type ($z_1$);

$$\mathcal{H}_{\text{eff, LL} + R + LR} = \frac{Z_1 Q_1}{R^2} \quad \text{where} \quad Z_1 \simeq z_1 + \tilde{z}_1 - 294 \cdot z_4 - 97.2 \cdot z_5$$

(4.15)

Note that we use the fact $\langle \bar{K}^0 | Q_1 | K^0 \rangle = \langle \bar{K}^0 | \tilde{Q}_1 | K^0 \rangle$ due to the parity symmetry of strong interaction. Then we can get lower bound on the compactification scale $R^{-1}$ by use of the upper bound on the relevant coefficients $z_1$ given in (4.9):

$$\text{Real part : } \frac{1}{R} \gtrsim \sqrt{\frac{|\text{Re } Z_1|}{9.6 \times 10^{-7} [\text{TeV}]}} ,$$

(4.16a)

$$\text{Imaginary part : } \frac{1}{R} \gtrsim \begin{cases} \sqrt{\frac{|\text{Im } Z_1|}{2.8 \times 10^{-9} [\text{TeV}]}} \quad \text{for } \text{Im } Z_1 > 0 \\ \sqrt{\frac{|\text{Im } Z_1|}{4.4 \times 10^{-9} [\text{TeV}]}} \quad \text{for } \text{Im } Z_1 < 0 \end{cases} .$$

(4.16b)

We can get the lower bound on $R^{-1}$ from $\Delta S = 2$ process in our model by combining these constraints. Since our theory has two free parameters, say $\theta'$ and $\gamma$, the lower bound on $R^{-1}$ depends on it. The obtained numerical result is displayed in figure 2, where the lower bound on $R^{-1}$ is plotted as a function of $(\sin \theta', \sin \gamma)$. The phenomenological constraint on $R^{-1}$ is from a few TeV to about 900 TeV from this result.

5 Summary

In this paper, we discussed the Kobayashi-Maskawa type CP violation in the context of 5D $SU(3) \otimes SU(3)_{\text{color}}$ gauge-Higgs unification. In this model, a pair of $\mathbf{3}$ and $\mathbf{\bar{6}}$ representation of $SU(3)$ should be introduced to reproduce a generation, and then, an extra quark doublet per generation appears. So we identify the quark doublets corresponding Standard Model $Q_{SM}$ as some combination of them. These combinations generically contains complex components, it potentially violates CP symmetry.
As was mentioned in introduction, the $n$ generation model is considered and the Yukawa coupling has $(n - 1)^2$ phases after re-phasing. Then the similar discussion of KM theory can be proceeded and CP symmetry breaks down more than the three generation at zero-mode sector as is in the SM. On the other hand, the FCNC vertices in the strong interaction between the zero-mode fermion and non-zero KK mode gluon exist in this model and thus CP-violating phase appears in the strong interaction.

In fact, one can show that the CP-violating phases disappear when the flavor symmetry is restored, which corresponds to the case of degenerate fermion bulk mass term. In the main text, we construct the two generation model and it can be easily extended to the $n$ generation model. In this case, the CP-violating phases included in the coefficients of FCNC vertices in the strong interaction (4.1) vanishes in the limit of universal bulk masses $M_1 = M_2 = \cdots$ by use of the unitarity condition (1.2):

$$P_d^\dagger V_{dR}^\dagger I_{RR}^{(0m0)} V_{dR} P_d \xrightarrow{M_1=M_2=\cdots} P_d^\dagger V_{dR}^\dagger V_{dR} P_d I_{RR}^{(0m0)} = I_{RR}^{(0m0)}$$

(5.1a)

and

$$P_d^\dagger V_{dL}^\dagger \left( U_3^\dagger I_{RR}^{(0m0)} U_3 + U_4^\dagger I_{RR}^{(0m0)} U_4 \right) V_{dL} P_d \xrightarrow{M_1=M_2=\cdots} P_d^\dagger V_{dL}^\dagger \left( U_3^\dagger U_3 + U_4^\dagger U_4 \right) V_{dL} P_d I_{RR}^{(0m0)} = I_{RR}^{(0m0)}.$$  

(5.1b)

Then, one can see that the CP violation is raised together with flavor violation.

From the above argument, the CP violation generally occurs in the even two generation, we construct the two generation model as a simplest example of CP-violating model. Thereupon, the neutral kaon system is discussed in this context and the rate of $K^0 - \bar{K}^0$ mixing is estimated. We calculate the Wilson coefficient caused by the $\Delta S = 2$ process and obtain the lower bound of the compactification scale $R^{-1}$ by comparing the

\[\text{Figure 2: The lower bound for } R^{-1} \text{ from } \Delta S = 2 \text{ process.}\]
mass difference $\Delta m_K$ and $\varepsilon_K$ which is known as a typical CP-violating observables. The LL type diagrams give stringent constraint in most cases, we put the phenomenological constraint on $R^{-1}$ as $\mathcal{O}(1)$ TeV $\sim \mathcal{O}(10^3)$ TeV.

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