A General Relativistic study of the neutrino path and calculation of minimum photosphere for different stars

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Abstract

A detailed general relativistic (GR) calculation of the neutrino path for a general metric describing a rotating star is studied. We have calculated the neutrino path along a plane, with the consideration that the neutrino does not at any time leave the plane. The expression for the minimum photosphere radius (MPR) is obtained and matched with the Schwarzschild limit. The MPR is calculated for the stars with two different equations of state (EOS) each rotating with two different velocities. The results shows that the MPR for the hadronic star is much greater than the quark star and the MPR increases as the rotational velocity of the star decreases. The MPR along the polar plane is larger than that along the equatorial plane.

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I. INTRODUCTION

Gamma Ray Bursts (GRB), the possible engines for GRBs and its connection with the neutrino production is a field of high current interest. It was proposed that the neutrino-antineutrino annihilation to electron-positron pairs in compact stars is a possible and important candidate to explain the energy source of GRBs. The previous calculations of the reaction $\nu \bar{\nu} \rightarrow e^+e^-$ in the vicinity of a neutron star have been based on Newtonian gravity, i.e. $(2GM/c^2R)<<1$. The effect of gravity was incorporated in refs. [3, 4], but only for a static star.

Neutron stars are objects formed in the aftermath of supernova. The central density of these stars can be as high as 10 times that of normal nuclear matter. At such high density, any small perturbation, e.g. spin down of the star, may trigger the phase transition from nuclear to quark matter system. As a result, the neutron star may fully convert to a quark star or a hybrid star with a quark core. It has been shown [5] that such a phase transition [6] produces a large amount of high energy neutrinos. These neutrinos (and antineutrinos) could annihilate and give rise to electron-positron pairs through the reaction $\nu \bar{\nu} \rightarrow e^+e^-$. These $e^+e^-$ pairs may further give rise to gamma rays which may provide a possible explanation of the observed GRB. Furthermore, the rotating neutron star has been shown [7] to produce the observed beaming effect. At present, it is necessary to have a better understanding of the energy deposition in the neutrino annihilation to $e^+e^-$ in the realistic neutron star environment.

We would like to study the $\nu + \bar{\nu} \rightarrow e^+ + e^-$ energy deposition rate near a rotating compact star. This reaction is important for the study of gamma ray bursts. The General Relativistic (GR) effect increase the efficiency of the process immensely, but the inclusion of the rotational effect is yet to be incorporated and studied. The geodesic of neutrinos are also important in the study of pulse shapes and accretion disc illumination [8] to name a few. Therefore the path of neutrino (or generally of massless particle) is of immense importance and needs a detailed study. The neutrino path for Schwarzschild metric along the equatorial plane can be found in text books [9] and different papers [3]. Asano and Fukuyama [10, 11] did the same calculation near a thin accretion disc using Kerr metric. Prasanna and Shrubbabati [12] studied it for a slowly rotating star. To address all the above problems we present a detailed GR study of the neutrino path for a most general metric.
describing a rotating star \[13\] along a plane. We have made our calculation using two
different EOS, one quark and the other hadronic.

In this paper first we will discuss about the metric, the EOS and the star structure. Next
we will present the detailed GR calculation of the neutrino path and minimum photosphere.
Finally we will present our results for the two EOS and have a brief discussion.

II. THE STAR

The structure of the star is described by Cook-Shapiro-Teukolsky (CST) metric \[13\]
\[ds^2 = -e^{\gamma+\rho}dt^2 + e^{2\alpha}(dr^2 + r^2d\theta^2) + e^{\gamma-\rho}r^2\sin^2\theta(d\phi - \omega dt)^2.\] (1)

Accurate models of rotating neutron stars for tabulated EOS can be computed numerically
using the ‘rns’ code \[14, 15, 16\]. This computer code computes the metric functions \(\alpha, \gamma, \rho\)
and \(\omega\) appearing in the axisymmetric metric, and these metric functions depends only on the
coordinates \(\theta\) and \(r\). The metric function \(\omega\) is the term responsible for the frame dragging
effect and would vanish if the rotational velocity (\(\Omega\)) is zero. The coordinate \(r\) is related to
the standard radial coordinate that appears in the Schwarzschild metric, \(r_s\), by \(r_s = re^{(\gamma-\rho)/2}\)
\[17\]. In the limit of zero rotation, the following combination of metric functions are

\[\lim_{\omega \to 0} re^{-\rho} = \frac{r_s}{\sqrt{1 - \frac{2M}{r_s}}}\] (2)

\[\lim_{\omega \to 0} e^{(\gamma+\rho)/2} = \sqrt{1 - \frac{2M}{r_s}}\] (3)

\[\lim_{\omega \to 0} e^{\alpha - [(\gamma+\rho)/2]} dr = \frac{dr_s}{1 - \frac{2M}{r_s}}.\] (4)

We have previously mentioned that tabulated EOS are needed to compute the code
numerically. In this paper we have computed for two different EOS, the quark EOS and the
hadronic EOS. The hadronic EOS has been evaluated using the nonlinear Walecka model
\[18\]. The Lagrangian density in this model is given by:
\[
\mathcal{L} = \sum_i \bar{\psi}_i (i\gamma^\mu \partial_\mu - m_i + g_{\sigma i} \sigma + g_{\omega i} \omega_\mu \gamma^\mu - g_{\rho_\mu^a}^a \gamma^\mu T_a) \psi_i
\]
\[- \frac{1}{4} \omega^{\mu \nu} \omega_{\mu \nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu + \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2)
\]
\[- \frac{1}{4} \rho_{\mu^a \nu} \rho^{\mu^a \nu} + \frac{1}{2} m_\rho^2 \rho_{\mu^a}^a \rho^{\mu^a} - \frac{1}{3} b m_N (g_{\sigma N} \sigma)^3 - \frac{1}{4} c (g_{\sigma N} \sigma)^4
\]
\[+ \bar{\psi}_e (i\gamma^\mu \partial_\mu - m_e) \psi_e \]

The Lagrangian in eqn. 5 includes nucleons (neutrons and protons), electrons, isoscalar scalar, isoscalar vector and isovector vector mesons denoted by \(\psi_i, \psi_e, \sigma, \omega^\mu\) and \(\rho^{a, \mu}\), respectively. The Lagrangian also includes cubic and quartic self interaction terms of the \(\sigma\) field. The parameters of the nonlinear Walecka model are meson-baryon coupling constants, meson masses and the coefficient of the cubic and quartic self interaction of the \(\sigma\) mesons (b and c, respectively). The meson fields interact with the baryons through linear coupling. The \(\omega\) and \(\rho\) meson masses have been chosen to be their physical masses. The rest of the parameters, namely, nucleon-meson coupling constants \((\frac{g_{\sigma}}{m_\sigma}, \frac{g_{\omega}}{m_\omega}, \frac{g_{\rho}}{m_\rho})\) and the coefficients of cubic and quartic terms of the \(\sigma\) meson self interaction (b and c, respectively) are determined by fitting the nuclear matter saturation properties, namely, the binding energy/nucleon (-16 MeV), baryon density \((\rho_0=0.17 \text{ fm}^{-3})\), symmetry energy coefficient (32.5 MeV), Landau mass (0.83 \(m_n\)) and nuclear matter incompressibility (300 MeV). We have used a stable three-flavour quark matter EOS, obtained from the standard Bag model with \(B^{1/4} = 145\text{ MeV}\).

The shape of a fast rotating neutron star becomes oblate spheroid \[13\]. The star gets compressed along the z-axis and along x and y-axes, it bulges by equal amounts, the polar radius is thus smaller than equatorial radius.

III. GR CALCULATION

The metric is independent of \(t'\) and \(\phi'\), the coordinates are cyclic, hence the corresponding covariant generalized momenta is constant \[2\], i.e
\[p_t = p_0 = \text{const.} = -E\]
\[p_\phi = p_3 = \text{const} = L\]
The magnitude of the 4 vector energy momentum is given by

\[ g_{ij}p^ip^j + \mu^2 = 0, \tag{7} \]

where \( \mu \) is the rest mass of the particle. Writing it explicitly we have

\[ g_{00}p^0p^0 + g_{11}p^1p^1 + \mu^2 = 0. \tag{8} \]

These are contravariant momenta, and to find the contravariant momenta \( p^i \), we calculate the inverse matrix \( g_{\mu\nu} \), of the metric given in eqn. 1.

\[
\begin{pmatrix}
-e^{-(\gamma+\rho)} & 0 & 0 & -\omega e^{-(\gamma+\rho)} \\
0 & e^{-2\alpha} & 0 & 0 \\
0 & 0 & \frac{1}{r^2}e^{-2\alpha} & 0 \\
-\omega e^{-(\gamma+\rho)} & 0 & 0 & -\left(\omega^2 e^{-(\gamma+\rho)} - \frac{e^{(\gamma-\rho)}}{r^2 \sin^2 \theta}\right) \\
\end{pmatrix}
\]

Using the original \( g_{\mu\nu} \) and the inverse \( g_{\mu\nu} \) matrix we calculate the contravariant momenta, which are given by

\[
\begin{align*}
p^0 &= g^{00}p^0 + g^{03}p_3 = e^{-(\gamma+\rho)}(E - \omega L) = e^{-(\gamma+\rho)}B(\omega) \\
p^3 &= g^{30}p^0 + g^{33}p^3 = e^{-(\gamma+\rho)}[\omega B(\omega) + \frac{e^{2\rho}}{r^2 \sin^2 \theta}L] \tag{9}
\end{align*}
\]

We consider the particle motion is in a particular \( \theta = \text{const} \) plane, and orient the coordinate system such that the particle lies in the equatorial plane for \( \theta = \frac{\pi}{2} \). The particle has at start, and continues to have zero momentum in the given plane i.e \( p^\theta = p^2 = 0 \).

Finally, substituting these values in the above eqn. 8 we get

\[ -e^{-(\gamma+\rho)}B^2 + \frac{L^2}{r^2 \sin^2 \theta}e^{\rho-\gamma} + e^{2\alpha} \left( \frac{dr}{d\lambda} \right)^2 + \mu^2 = 0. \tag{10} \]

if, \( \lambda = \frac{\tau}{\mu} \) i.e propertime per unit rest mass

\[
dr d\lambda = \frac{dr}{d\tau} \frac{d\tau}{d\lambda} = \mu \frac{dr}{d\tau} \\
dr d\lambda = \mu \left( \frac{dr}{d\phi} \frac{d\phi}{d\tau} \right) = \mu \left( \frac{dr}{d\phi} \frac{d\lambda}{d\tau} \right) \frac{d\tau}{d\phi} \\
dr d\lambda = \left( \frac{dr}{d\phi} \right) p^\phi. \tag{11}
\]

We define \( \overline{E} = \frac{E}{\mu} \) and \( \overline{L} = \frac{L}{\mu} \). As the particle is massless (neutrino), therefore we define

\[ \text{Lim}_{\mu \rightarrow 0} \frac{\overline{L}}{\overline{E}} = b \tag{12} \]
where \( b \) is the impact parameter. Substituting this in eqn. 10, we have

\[
e^{2\alpha \left( \frac{dr}{d\phi} \right)^2 [\omega (1 - \omega b) + \frac{b \epsilon^{2\rho}}{r^2 \sin^2 \theta} ]^2 - e^{(\gamma + \rho)} (1 - \omega b)^2 + \frac{b^2}{r^2 \sin^2 \theta} e^{\gamma + 3\rho} = 0. \tag{13}
\]

The lagrangian of the system we are considering is given by

\[
L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \tag{14}
\]

where, \( \dot{x}^i = \frac{dx^i}{d\lambda} \), \( \lambda \) the affine parameter. Therefore writing it explicitly, we get

\[
L = (-e^{\gamma + \rho} e^{\gamma - \rho} \omega^2 r^2 \sin^2 \theta \dot{t}^2 + e^{2\alpha} \dot{r}^2 + e^{\gamma - \rho} r^2 \sin^2 \theta \dot{\phi}^2 - 2e^{\gamma - \rho} \omega r^2 \sin^2 \theta \dot{t} \dot{\phi}) = 0. \tag{15}
\]

Using the above lagrangian we can write the covariant momenta, and they are given in terms of total energy and total angular momentum

\[
p_0 = p_t = \frac{\partial L}{\partial \dot{t}} = (-e^{\gamma + \rho} e^{\gamma - \rho} \omega^2 r^2 \sin^2 \theta) \dot{t} - e^{\gamma - \rho} \omega r^2 \sin^2 \theta \dot{\phi} = -E
\]

\[
p_3 = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = e^{\gamma - \rho} r^2 \sin^2 \theta \dot{t} \dot{\phi} - e^{\gamma - \rho} \omega r^2 \sin^2 \theta \dot{t} = L. \tag{16}
\]

Having written the momenta in terms of total energy and total angular momentum it is quite simple to solve for \( \dot{t} \) and \( \dot{\phi} \). They are given by,

\[
\dot{t} = \frac{E - \omega L}{e^{\gamma + \rho}}, \quad \dot{\phi} = \frac{L}{e^{\gamma - \rho} r^2 \sin^2 \theta} + \omega \frac{E - \omega L}{e^{\gamma + \rho}}. \tag{17}
\]

The angle \( \theta_r \) between the particle trajectory and the tangent vector to the orbit can be derived by constructing the local lorentz tetrad \( k_\mu \), for our metric

\[
\begin{array}{cccc}
e^{(\gamma + \rho)/2} & 0 & 0 & 0 \\
0 & e^\alpha & 0 & 0 \\
0 & 0 & re^\alpha & 0 \\
-e^{(\gamma - \rho)/2} \omega r \sin \theta & 0 & 0 & e^{(\gamma - \rho)/2} r \sin \theta \\
\end{array}
\]

The angle \( \theta_r \) is given by

\[
tan \theta_r = \frac{V^1}{V^2} = \frac{k^1_r V^r}{k^3_t V^t + k^3_\phi V^\phi} \tag{18}
\]

where, \( V^r = \frac{\dot{t}}{\dot{t}} \) and \( V^\phi = \frac{\dot{\phi}}{\dot{t}} \) are the local velocities. Using eqn. 17 \( V^\phi \) can be written as

\[
V^\phi = \frac{L}{E - \omega L} \frac{e^{2\rho}}{r^2 \sin^2 \theta} + \omega = A(r, \theta) + \omega. \tag{19}
\]
Therefore the angle $\theta_r$ is

$$\tan\theta_r = \frac{e^\alpha}{e^{(\gamma-\rho)/2r\sin\theta}} \cdot \frac{V^\phi}{V^\phi - \omega} \left(\frac{dr}{d\phi}\right)$$

$$= \frac{e^\alpha}{e^{(\gamma-\rho)/2r\sin\theta}} \cdot \frac{A(r, \theta) + \omega}{A(r, \theta)} \left(\frac{dr}{d\phi}\right)$$

(20)

Squaring the above equation and writing as

$$\left(\frac{dr}{d\phi}\right)^2 = \left[\frac{A}{A + \omega}\right]^2 e^{(\gamma-\rho)r^2\sin^2\theta} e^{2\alpha} \left\{\tan^2\theta_r \cdot \frac{\omega(1 - \omega b)}{r^2\sin^2\theta} + \frac{b^2}{r^2\sin^2\theta} e^{(\gamma+3\rho)}\right\} = 0.$$  

(21)

This equation can be solved using the potentials obtained from the 'rns' code to obtain a minimum radius $r = R$, the minimum photosphere radius, below which a massless particle (neutrino) emitted tangentially to the stellar surface ($\theta_R = 0$) would be gravitationally bound.

In the limit in which CST metric reduces to Schwarzschild metric, i.e

$$\lim_{\omega \to 0} r e^{-\rho} = \frac{r_s}{\sqrt{1 - \frac{2M}{r_s}}}$$

eqn. 22 (for $\theta = \pi/2$) reduces to the equation for $b$ obtained by Salmonson and Wilson, i.e

$$b = \frac{r_s\cos\theta_r}{\sqrt{1 - \frac{2M}{r_s}}}.$$  

IV. RESULTS

The minimum photosphere is calculated solving eqn. 22 using the potential functions obtained from the 'rns' code. Starting our calculation by choosing the central energy density of the star to be $1 \times 10^{15} \text{gm/cm}^3$. Fig. 1, 2, 3, 4 and 5 gives the nature of the equations of state used. In the figures $\chi = \cos\theta$. Fig. 1 shows that the quark matter EOS, considered here is much steeper than the hadronic matter EOS. Fig. 2 (for quark
matter EOS) shows that at the centre of the star the pressure is maximum, and as we go outside it falls off, and becomes zero outside the star. Along the pole the pressure falls off in a much steeper way than along the equator, as it is of much shorter length. As the the rotational velocity decreases the equatorial radius of the star decreases but the polar radius increases (although still less than equatorial radius). For the keplerian velocity the star is maximally deformed and as the rotational velocity of the star decreases the star regains a much spherical shape. Fig. 3 show the same nature for a hadronic star. In fig. 4 and 5 the variation of energy density is shown for the quark and hadronic star simultaneously, and its nature is more or less similar to that of the behaviour of pressure discussed above.
FIG. 2: Variation of pressure along the radial direction of the star for two different rotational velocities each with two different values of $\chi$ for the quark matter EOS.

FIG. 3: Variation of pressure along the radial direction of the star for two different rotational velocities each with two different values of $\chi$ for the hadronic matter EOS.
FIG. 4: Variation of energy density along the radial direction of the star for two different rotational velocities each with two different values of $\chi$ for the quark matter EOS.

FIG. 5: Variation of energy density along the radial direction of the star for two different rotational velocities each with two different values of $\chi$ for the hadronic matter EOS.
Using the quark matter EOS on the 'rns' code, the keplerian velocity of the quark star comes out to be $0.89 \times 10^4 \text{s}^{-1}$. For comparison we have also computed the 'rns' code with rotational velocity of $0.5 \times 10^4 \text{s}^{-1}$. The same treatment is done also for the hadronic EOS where the keplerian velocity is $0.61 \times 10^4 \text{s}^{-1}$ and for comparison the other rotational velocity was chosen to be $0.4 \times 10^4 \text{s}^{-1}$. The code solves the metric for the given EOS and gives the different potential functions as a function of $r$ and $\theta$. Solving eqn. 22 with these values of potential functions for different $\theta$ we obtain the value of minimum photosphere for different planes. Table 1. sums up all our results in a compact form.

Let us now analyze the table given above. It points out the fact that as the rotational velocity decreases the mass of the star also decreases. For the same central energy density, the mass of the quark star is much greater than that of the hadronic star but the radius is much smaller. It signifies that the quark matter EOS, considered in our work is much steeper than that of hadronic matter EOS as pointed out previously in the figures. As the rotational velocity decreases the equatorial radius decreases but the polar radius increases. It shows that the star is maximally deformed for the keplerian velocity and as the rotational velocity of the star decreases it tries to regain a more spherical shape. A static star is of spherical shape, where polar and equatorial radius are same.

| EOS   | $\Omega$ | Mass in $M_\odot$ | $r_e, r_p$ in Km | $\chi$ | MPR in Km |
|-------|----------|-------------------|------------------|--------|-----------|
| Quark | 0.89     | 2.8               | 12, 5.5          | 0      | 2.74      |
|       | 0.89     | 2.8               | 12, 5.5          | 0.5    | 2.74      |
|       | 0.89     | 2.8               | 12, 5.5          | 0.99   | 3.11      |
|       | 0.5      | 2.2               | 9, 8             | 0      | 3.6       |
|       | 0.5      | 2.2               | 9, 8             | 0.5    | 3.7       |
|       | 0.5      | 2.2               | 9, 8             | 0.99   | 4.96      |
| hadron| 0.61     | 2                 | 16, 9            | 0      | 3.9       |
|       | 0.61     | 2                 | 16, 9            | 0.5    | 4        |
|       | 0.61     | 2                 | 16, 9            | 0.99   | 5.5       |
|       | 0.4      | 1.7               | 12, 10           | 0      | 4.85      |
|       | 0.4      | 1.7               | 12, 10           | 0.5    | 4.96      |
|       | 0.4      | 1.7               | 12, 10           | 0.99   | 5.85      |
In the above table we have tabulated the minimum photosphere radius (MPR) for three different values of $\chi$, i.e. for three planes. Along the equator ($\chi = 0$), along the pole ($\chi = 0.99$) and along a plane lying at $\chi = 0.5$. The MPR is minimum along the equator and maximum along the pole. For $\chi = 0.5$ it lies somewhere in between these two values. The MPR is minimum for the quark star rotating with keplerian velocity and is maximum for the hadronic star rotating with $0.4 \times 10^4 \text{s}^{-1}$ velocity. The MPR is much greater for the hadronic star than the quark star. As the rotational velocity decreases the MPR shifts outward from the centre of the star toward the surface.

V. SUMMARY AND DISCUSSION

In this paper we have addressed the problem of path of the neutrino and the radius of minimum photosphere. We have done a complete GR calculation of the neutrino path for the most general metric describing a rotating star, and have obtained its geodesic equation along a given plane. We have calculated the MPR for four cases, i.e stars with two different EOS and both rotating with two different velocities. Previous calculation of the neutrino path was either done for a static star [3, 4] or for a slowly rotating star [12] and only along the equatorial plane. We have shown that our results also matches very well with the previous findings [3] for the Schwarzschild limit. We have found that the MPR is maximum along the pole and minimum along the equator. The MPR is much greater for the hadronic star than that of the quark star. As the rotational velocity decreases the MPR increases and is maximum for the static star. Prasanna and Shrubabati [12] had showed that the MPR is inversely proportional to the rotational velocity of the star. Salmonson and Wilson [3] had shown that for the static star the MPR limit is $R = 3M$, and that is very close to the surface. So our results are at par with the previous findings in those limits and goes beyond them.

Finally we would like to mention that this calculation of neutrino path is very important in the sense that this forms the heart of different problem like GRB central engine, pulse shape and accretion disc illumination. This path is the general path followed by any massless particle (photon) in the vicinity of a compact object. So the calculation might also be important to those problems. Currently we are trying to address other problems related to neutrino path.
Acknowledgments

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