Well-posedness of the free boundary problem in incompressible MHD with surface tension

Changyan Li 1 · Hui Li 2

Received: 30 April 2021 / Accepted: 19 July 2022 / Published online: 2 August 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
In this paper, we study the two phase flow problem with surface tension in the ideal incompressible magnetohydrodynamics. First, we prove the local well-posedness of the two phase flow problem with surface tension. Second, for the initial data satisfying a Syrovatskij type stability condition, we prove that as surface tension tends to zero, the solution of the two phase flow problem with surface tension converges to the solution of the two phase flow problem without surface tension.

Mathematics Subject Classification 35Q35 · 76W05

1 Introduction

1.1 Presentation of the problem
In this paper, we consider the two phase flow problem with surface tension in the ideal incompressible MHD. The incompressible MHD system can be written as

\[
\begin{align*}
\rho \frac{\partial}{\partial t} \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{h} \cdot \nabla \mathbf{h} + \nabla p &= 0, \\
\text{div} \mathbf{u} &= 0, \\
\text{div} \mathbf{h} &= 0, \\
\frac{\partial}{\partial t} \mathbf{h} + \mathbf{u} \cdot \nabla \mathbf{h} - \mathbf{h} \cdot \nabla \mathbf{u} &= 0,
\end{align*}
\]

where \( \rho \) is the density of fluids, \( \mathbf{u} = (u_1, u_2, u_3) \) is the fluids velocity, \( \mathbf{h} = (h_1, h_2, h_3) \) is the magnetic field, \( p \) denotes the pressure.

Communicated by L. Caffarelli.

✉ Hui Li
lihui92@zju.edu.cn
Changyan Li
lcy941024@pku.edu.cn

1 School of Mathematical Sciences, Peking University, Beijing 100871, China
2 Department of Mathematics, Zhejiang University, Hangzhou 310027, China
We study weak solutions of (1.1) which are smooth on each side of a smooth interface \( \Gamma_f \) in a domain \( \Omega \), and satisfy suitable jump conditions on the interface. More precisely, we let

\[
\Omega = T^2 \times [-1, 1], \quad \Gamma_f = \{ x \in \Omega | x_3 = f(t, x'), x' = (x_1, x_2) \in T^2 \},
\]

\[
\Omega_f^\pm = \{ x \in \Omega | x_3 \geq f(t, x'), x' \in T^2 \}, \quad \mathcal{Q}_T^\pm = \bigcup_{t \in (0, T)} \{ t \} \times \Omega_f^\pm,
\]

and \( \rho|_{\Omega_f^\pm} = \rho^\pm \), where \( \rho^\pm \) are two constants. The solution

\[
\mathbf{u}^\pm := \mathbf{u}|_{\Omega_f^\pm}, \quad \mathbf{h}^\pm := \mathbf{h}|_{\Omega_f^\pm}, \quad p^\pm := p|_{\Omega_f^\pm},
\]

is smooth in \( \mathcal{Q}_T \) and satisfy

\[
\begin{aligned}
\rho^\pm \partial_t \mathbf{u}^\pm + \rho^\pm \mathbf{u}^\pm \cdot \nabla \mathbf{u}^\pm - \mathbf{h} \cdot \nabla \mathbf{h}^\pm + \nabla p^\pm &= 0 \quad \text{in} \quad \mathcal{Q}_T^\pm, \\
&\text{div} \mathbf{u}^\pm = 0, \quad \text{div} \mathbf{h}^\pm = 0 \quad \text{in} \quad \mathcal{Q}_T^\pm, \\
\partial_t \mathbf{h}^\pm + \mathbf{u}^\pm \cdot \nabla \mathbf{h}^\pm - \mathbf{h}^\pm \cdot \nabla \mathbf{u}^\pm &= 0 \quad \text{in} \quad \mathcal{Q}_T^\pm.
\end{aligned}
\]  

(1.2)

On the moving interface \( \Gamma_f \), we impose the following boundary conditions:

\[
[p] := p^+ - p^- = \sigma H(f) = \sigma \nabla x' \cdot \left( \frac{\nabla x' f}{\sqrt{1 + |\nabla x' f|^2}} \right),
\]

\[
\mathbf{u}^\pm \cdot \mathbf{N} = \partial_t f, \quad \mathbf{h}^\pm \cdot \mathbf{N} = 0,
\]

(1.3)

(1.4)

where \( \sigma \) is the surface tension coefficient, \( H(f) \) is the mean curvature of the surface, \( \mathbf{N} = (-\partial_1 f, -\partial_2 f, 1) \) is the normal vector of the surface. Condition (1.3) means that there is surface tension acting on the free boundary. Condition (1.4) means that the free boundary is moving with the fluid, and the magnetic will not pass through the free boundary.

On the artificial boundary \( \Gamma_f^\pm = T^2 \times \{ \pm 1 \} \), we also assume that

\[
u_3^\pm = 0, \quad h_3^\pm = 0 \quad \text{on} \quad \Gamma_f^\pm.
\]

(1.5)

The system (1.2) is supplement with the initial data:

\[
f(0, x') = f_0(x') \in T^2, \quad \mathbf{u}^\pm(0, x) = \mathbf{u}_0^\pm(x), \quad \mathbf{h}^\pm(0, x) = \mathbf{h}_0^\pm(x) \quad \text{in} \quad \Omega_{f_0}^\pm,
\]

(1.6)

which satisfies

\[
\begin{aligned}
\text{div} \mathbf{u}_0^\pm &= 0, \quad \text{div} \mathbf{h}_0^\pm = 0 \quad \text{in} \quad \Omega_{f_0}^\pm, \\
\mathbf{u}_0^\pm \cdot \mathbf{N}_0 &= \mathbf{u}_0^- \cdot \mathbf{N}_0, \quad \mathbf{h}_0^\pm \cdot \mathbf{N}_0 &= \mathbf{h}_0^- \cdot \mathbf{N}_0 = 0 \quad \text{on} \quad \Gamma_{f_0}, \\
\mathbf{u}_0^\pm &= 0, \quad h_0^\pm = 0 \quad \text{on} \quad \Gamma_f^\pm.
\end{aligned}
\]  

(1.7)

The system (1.2)–(1.7) is called the two-phase flow problem for incompressible MHD. One of main goals in this paper is to study the local well-posedness and the zero surface tension limit of this system.

We remark that the divergence-free restriction on \( \mathbf{h}^\pm \) is a compatibility condition. Applying the divergence operator to the third equation of (1.2), we have

\[
\partial_t \text{div} \mathbf{h}^\pm + \mathbf{u}^\pm \cdot \nabla \text{div} \mathbf{h}^\pm = 0.
\]

Therefore, if \( \text{div} \mathbf{h}_0^\pm = 0 \), the solution of (1.2)–(1.3) will satisfies \( \text{div} \mathbf{h}^\pm = 0 \) for \( \forall t > 0 \). A similar argument can be applied to yield that \( \mathbf{h}^\pm \cdot \mathbf{N} = 0 \) if \( \mathbf{h}_0^\pm \cdot \mathbf{N}_0 = 0 \).
1.2 Background and related works

In inviscid flow, a surface across which there is a discontinuity in fluid velocity is called a vortex sheet. In the absence of surface tension and magnetic field, it is well known that the vortex sheet problem of incompressible fluids is ill-posed due to the Kelvin-Helmholtz instability [30]. During the past several decades, researches have found that such instability can be stabilized by surface tension. For irrotational flow, Ambrose [5] and Ambrose-Masmoudi [7] proved the local well-posedness of vortex sheets with surface tension for in two and three dimensions respectively. For general problem with vorticity, Shatah-Zeng [41] established a priori estimates in a geometric approach, and Cheng-Coutand-Shkoller [14, 15] proved the local well-posedness of the three dimensional problems. For other results about the vortex sheet problems, we refer the readers to [10, 12, 51].

For electrically conducting fluids, e.g. plasmas and liquid metals, there may be discontinuities in both the velocity field and magnetic field. Two phase flow problem plays an important role in researches of tokamak devices [34], magnetosphere [25], and solar flares [37]. In the mid-twentieth century, based on the normal modes analysis, Syrovatskij [39] and Axford [1] found a necessary and sufficient stability condition for the planar incompressible current-vortex sheet, and showed that the magnetic field has a stabilization effect on the Kelvin-Helmholtz instability. Michael [33] proved similar results for the 2D problem. The Syrovatskij stability condition can be expressed as (see also Section 71 of [26]):

\[
[|u|]^2 \leq 2(|h^+|^2 + |h^-|^2), \quad \text{on } \Gamma_f,
\]

\[
||u| \times h^+|^2 + ||u| \times h^-|^2 \leq 2|h^+ \times h^-|^2, \quad \text{on } \Gamma_f,
\]

where \([u] = (u^+ - u^-)\).

In the recent decades, great progress has been made in studying the stabilizing effect of the Syrovatskij condition (1.8). Morando-Trakhinin-Trebeschi [36] proved a priori estimates with a loss of derivatives for the linearized system. Furthermore, under a strong stability condition

\[
\max(||u| \times h^+, |u| \times h^-|) < |h^+ \times h^-|, \quad \text{on } \Gamma_f,
\]

Trakhinin [44] proved an a priori estimate for the linearized problem without loss of derivative. For the nonlinear current-vortex sheet problem, Coulombel-Morando-Secchi-Trebeschi [17] proved an a priori estimate under the strong stability condition (1.9). Recently, Sun-Wang-Zhang [40] gave the first rigorous confirmation of the stabilizing effect of the magnetic field on Kelvin-Helmholtz instability under the Syrovatskij stability condition (1.9). We also refer to some related works [13, 45, 46, 48] on the compressible problem and works [19, 21, 22, 38, 47] on the plasma-vacuum problem.

The aim of this paper is to show the local well-posedness for the two phase flow problem with surface tension. That is to say, the magnetic field do not destroy the stabilization effect of surface tension. Under additional assumption that the Syrovatskij condition holds, we also show that, as surface tension tends to zero, the solution of the two phase flow problem with surface tension converges to the solution of the two phase flow problem without surface tension.

The water wave problem is another free boundary problem of inviscid flow where there is only one fluid. For this problem, instead of the Kelvin-Helmholtz instability, the Rayleigh-Taylor instability may occur. There are a lot of remarkable literatures studying such problems [3, 16, 18, 27, 42, 49, 50, 52].
For the one fluid free boundary problem in incompressible magnetohydrodynamics with surface tension, Luo-Zhang [29] gave an a priori estimate. From a mathematical point of view, the elastodynamics have similar structures to the magnetohydrodynamics. In a very recent work, Gu-Lei [20] proved the local well-posedness of the free-boundary in incompressible elastodynamics with surface tension.

1.3 Main results

We first introduce some notations used throughout this paper. We denote by $C(\cdot, \cdot)$ a positive constant or a positive nondecreasing function depending only on its variables which may be different from line to line. We use $x = (x_1, x_2, x_3)$ to denote the coordinates in the fluid region, and use $x' = (x_1, x_2)$ to denote the natural coordinates on the free boundary $\Gamma_f$ or on the top/bottom boundary $\Gamma^\pm$. In addition, we will use the Einstein summation notation where a summation from 1 to 2 is implied over repeated index (i.e. $a_i b_i = a_1 b_1 + a_2 b_2$).

For a function $g : \Omega^+_f \to \mathbb{R}$, we denote $\nabla g = (\partial_1 g, \partial_2 g, \partial_3 g)$, and for a function $\eta : \mathbb{T}^2 \to \mathbb{R}$, $\nabla \cdot \eta = (\partial_1 \eta, \partial_2 \eta)$. For a function $g : \Omega^+_f \to \mathbb{R}$, we denote its trace on $\Gamma_f$ by $g(x')$. Thus, for $i = 1, 2$,

$$\partial_i g(x') = \partial_1 g(x', f(x')) + \partial_2 g(x', f(x')) \partial_i f(x').$$

We denote by $\| \cdot \|_{H^s(\Omega^+_f)}$, $\| \cdot \|_{H^s}$ the Sobolev norm on $\Omega^+_f$ and $\mathbb{T}^2$ respectively. Moreover, for an operator $P$ defined on $H^s(\mathbb{T}^2)$, we denote its operator norm by

$$\| P \|_{H^s \to H^k} = \sup_{\| f \|_{H^s} \leq 1} \| P f \|_{H^k}.$$

Now, let us state our main results.

**Theorem 1.1** Let $s \geq 6$ be an integer and $\sigma, \rho^+, \rho^- > 0$ be constants. Assume the initial data $(f_0, u_0, h_0)$ satisfies (1.7) and

$$f_0 \in H^{s+1}(\mathbb{T}^2), \quad u_0^\pm, h_0^\pm \in H^s(\Omega^\pm_{f_0}).$$

Moreover we assume that there exists $c_0 \in (0, \frac{1}{2})$ so that

$$-(1 - 2c_0) \leq f_0 \leq (1 - 2c_0).$$

Then there exists a time $T > 0$ that depends on $(c_0, \sigma, \rho^+, \rho^-)$ such that system (1.2)–(1.7) admits a unique solution $(f, u, h)$ in $[0, T]$ satisfying

1. $\sup_{t \in [0, T]} \left( \| f \|_{H^{s+1}} + \| \partial_t f \|_{H^{s-1}} \right)(t)$
   $$\leq C(c_0, \sigma, \rho^+, \rho^-, \| f_0 \|_{H^{s+1}}, \| u_0^\pm \|_{H^s(\Omega^\pm_{f_0})}, \| h_0^\pm \|_{H^s(\Omega^\pm_{f_0})}),$$
2. $\sup_{t \in [0, T]} \left( \| u^\pm \|_{H^s(\Omega^\pm_f)} + \| \partial_t u^\pm \|_{H^{s-1}(\Omega^\pm_f)} + \| h^\pm \|_{H^s(\Omega^\pm_f)} + \| \partial_t h^\pm \|_{H^{s-1}(\Omega^\pm_f)} \right)(t)$
   $$\leq C(c_0, \sigma, \rho^+, \rho^-, \| f_0 \|_{H^{s+1}}, \| u_0^\pm \|_{H^s(\Omega^\pm_{f_0})}, \| h_0^\pm \|_{H^s(\Omega^\pm_{f_0})},$$
3. $-(1 - c_0) \leq f \leq (1 - c_0)$ for $t \in [0, T], x' \in \mathbb{T}^2$.

Before state the result of zero surface tension limit, we introduce a Syrovatskij type stability condition:

$$\Lambda(h^\pm, [u]) = \inf_{x' \in \Gamma_f \cap \{ \varphi_1 = 1 \}} \inf_{\varphi_2 = 1} \frac{1}{\rho^+ + \rho^-} (h_1^+ \varphi_1 + h_2^+ \varphi_2)^2 + \frac{1}{\rho^+ + \rho^-} (h_1^- \varphi_1 + h_2^- \varphi_2)^2$$
\[-(v_1\varphi_1 + v_2\varphi_2)^2 \geq c_0 > 0, \quad (1.10)\]

where \(v_i = \sqrt{\rho^+ \rho^-} [u_i]\).

With such stability condition, Sun-Wang-Zhang [40] prove the local well-posedness of current-vortex sheet problem without surface tension for the case \(\rho^+ = \rho^- = 1\) and we [28] get similar results for the general case \(\rho^+, \rho^- > 0\).

Under the assumption that the initial data satisfies the stability condition (1.10), we prove that as \(\sigma\) tends to 0, the solution of the two-phase flow problem got in [28] is the limit of the solutions got in Theorem 1.1. Indeed, we have the following result.

**Theorem 1.2** Let \(s \geq 6\) be an integer and \(\sigma, \rho^+, \rho^- > 0\) be constants. Assume the initial data \((f_0, u_0, h_0)\) satisfies (1.7) and

\[f_0 \in H^{s+1}(\mathbb{T}^2), \quad u_0^\pm, h_0^\pm \in H^s(\Omega^\pm_{f_0}).\]

Moreover we assume that there exists \(c_0 \in (0, \frac{1}{2})\) so that

1. \((-1 - 2c_0) \leq f_0 \leq (1 - 2c_0),\]
2. \(\Lambda(h_0^\pm, [u_0]) \geq 2c_0.\)

Then there exist \(T > 0\) that depends on \((c_0, \rho^+, \rho^-)\) but not on \(\sigma\) such that system (1.2)–(1.7) admits a unique solution \((f^\sigma, u^\sigma, h^\sigma)\) in \([0, T]\) satisfying

1. \(\sup_{t \in [0, T]} \left( \| f^\sigma \|_{H^{s+\frac{1}{2}}} + \| \partial_t f^\sigma \|_{H^{s-\frac{1}{2}}} \right) (t) \leq C(c_0, \rho^+, \rho^-, \| f_0 \|_{H^{s+1}}, \| u_0^\pm \|_{H^s(\Omega^\pm_{f_0})}, \| h_0^\pm \|_{H^s(\Omega^\pm_{f_0})}),\]
2. \(\sup_{t \in [0, T]} \left( \| u^\sigma \|_{H^s(\Omega^\pm _f)} + \| \partial_t u^\sigma \|_{H^{s-1}(\Omega^\pm _f)} + \| h^\sigma \|_{H^s(\Omega^\pm _f)} + \| \partial_t h^\sigma \|_{H^{s-1}(\Omega^\pm _f)} \right) (t) \leq C(c_0, \rho^+, \rho^-, \| f_0 \|_{H^{s+1}}, \| u_0^\pm \|_{H^s(\Omega^\pm_{f_0})}, \| h_0^\pm \|_{H^s(\Omega^\pm_{f_0})}),\]
3. \((-1 - c_0) \leq f^\sigma (t) \leq (1 - c_0)\) for \(t \in [0, T], x' \in \mathbb{T}^2,\)
4. \(\inf_{t \in [0, T]} \Lambda(h^\sigma, [u^\sigma]) \geq c_0.\)

Moreover, as \(\sigma\) tends to 0, the solution \((f^\sigma, u^\sigma, h^\sigma)\) converges to the solution \((f, u, h)\) of the system (1.2)–(1.7) with \(\sigma = 0.\) More precisely, let

\[u_\sigma^\pm = u^\sigma \circ \Phi_{f_0}^\pm, \quad h_\sigma^\pm = h^\sigma \circ \Phi_{f}^\pm,\]

where \(\Phi_{f}^\pm : \Omega^\pm_{f_0} \to \Omega^\pm_{f}\) is defined in (2.1). For any positive sequence \(\{\sigma_k\}\) that tends to zero, there exists a subsequence of \(\{(f_{\sigma_k}, u_{\sigma_k}^{\pm}, h_{\sigma_k}^{\pm})\}\) that converges to \((f, u \circ \Phi_{f}^\pm, h \circ \Phi_{f}^\pm)\) in

\[C([0, T], H^{s-\frac{1}{2}}(\mathbb{T}^2)) \times C([0, T], H^{s-1}(\Omega^\pm_{f_0})) \times C([0, T], H^{s-1}(\Omega^\pm_{f_0})).\]

**Remark 1.3** Our method is also applicable to establish the local well-posedness of the one fluid problem where there is no fluid and no magnetic in the upper domain. The key steps can be found in Sect. 7.

### 1.4 Main ideas and sketch of the proof

The framework we used in this paper is developed by Sun-Wang-Zhang in [40] studying the local well-posedness of the current-vortex sheet problem.

We first derive a new system which is equivalent to the original system (1.2)–(1.5). The new system consists of the evolution equations of the the following quantities: free interface
the vorticity $\omega$, the current $h$, and the average of the tangential part of the velocity and the magnetic field on the top and bottom fixed boundary $(a^+_{1}, b^+_{1})$. From the above quantities, we can recover the vorticity and magnetic field by solving the Div-Curl system.

**Evolution equation of free interface.**

In the new system, the evolution equation of the free interface plays a crucial role. And the main innovation of this paper is that we derive the following evolution equation for the free interface:

$$D_t^2 f = \left(-v_i v_j + \frac{1}{\rho^+ + \rho^-} (h^+_i h^+_j + h^-_i h^-_j) \right) \partial_i \partial_j f$$

$$+ \frac{\sigma}{(\rho^+ + \rho^-)^2} (\rho^+ N^+_f + \rho^- N^-_f) H(f) + l.o.t.,$$

(1.11)

where $D_t = \partial_t + w_1 \partial_1 + w_2 \partial_2, w_i = \frac{1}{\rho^+ + \rho^-} (\rho^+ u^+_l + \rho^- u^-_l)$, $v_i = \frac{\sqrt{\rho^+ \rho^-}}{\rho^+ + \rho^-} (u^+_l - u^-_l)$, $N^\pm_f$ is the Dirichlet-Neumann operator which is a positive operator of order 1 (see Sect. 2), and $l.o.t.$ denotes the lower order terms.

In (1.11), $-\frac{\sigma}{(\rho^+ + \rho^-)^2} (\rho^+ N^+_f + \rho^- N^-_f) H(\cdot)$ is a positive operator of order 3,

$$- \left(-v_i v_j + \frac{1}{\rho^+ + \rho^-} (h^+_i h^+_j + h^-_i h^-_j) \right) \partial_i \partial_j$$

(1.12)

is a non-negative operator of order 2 (if the stability condition (1.10) holds, it is a positive operator of order 2), and the order of the lower order terms are at most $\frac{3}{2}$. Thus (1.11) is a strictly hyperbolic equation, which allows us to close the energy estimate.

**Principle in establishing local well-posedness.**

To explain how to establish the local well-posedness from the hyperbolic evolution equation, we introduce the following toy model:

$$\partial_t^2 f(t, x') - \tilde{H}(f)(t, x') + (\partial_t f)^2(t, x') = 0, \quad x' \in \mathbb{T}^2,$$

(1.13)

where $-\tilde{H}(f) = -\nabla \cdot \left(\frac{\nabla f}{\sqrt{1 + |f|^2}}\right)$ is a two order positive operator, and $(\partial_t f)^2$ is a one order error term. We call $\tilde{H}$ the regularized curvature operator whose coefficient is more regular than the curvature operator $H$. This model contains only the evolution equation for $f$ and is easy to study.

In order to obtain the existence of solutions to the nonlinear equation, we need to introduce the linearization equation.

For given function $f^*$ such that

$$\sup_{t \in [0, T]} \|f^*\|_{L^\infty} \leq \tilde{L}_0, \quad \sup_{t \in [0, T]} \left(\|f^*\|_{H^{s+1}} + \|\partial_t f^*\|_{H^s}\right) \leq \tilde{L}_1,$$

(1.14)

where $s \geq 6$ is a constant. We introduce the following linearized equation:

$$\partial_t^2 f(t, x') - \tilde{H}_{f^*}(f)(t, x') + (\partial_t f^*)^2(t, x') = 0,$$

(1.15)

where

$$\tilde{H}_{f^*}(f) = (\partial_1, \partial_2) \left(\begin{array}{cc} \frac{1}{\sqrt{1 + |f^*|^2}} & 0 \\ 0 & \frac{1}{\sqrt{1 + |f^*|^2}} \end{array}\right) \left(\begin{array}{c} \partial_1 f \\ \partial_2 f \end{array}\right)$$

(1.16)

is the linear-regularized curvature operator. It is obvious that $-\tilde{H}_{f^*}(f)$ is a positive operator of order 2.
By taking inner product with $\partial_t f$ in $\dot{H}^s(\mathbb{T}^2)$, one can get that

$$\frac{1}{2} \partial_t \left( \| |\nabla|^s \partial_t f \|^2_{L^2} + \frac{\| |\nabla|^{s+1} f \|_{L^2}^2}{(1 + |f^*|^2)^{\frac{s}{2}}} \right) \leq C(\tilde{L}_1)(\| \partial_t f^* \|_{L^\infty} \| |\nabla|^{s+1} f \|^2_{L^2} + \| [|\nabla|^{s}, \tilde{H}_{f^*}] f \|_{L^2} \| |\nabla|^s \partial_t f \|^2_{L^2}$$

$$+ \| f^* \|^2_{H^{s+1}} \| |\nabla|^s \partial_t f \|^2_{L^2}) \leq C(\tilde{L}_1)(\| |\nabla|^s \partial_t f \|^2_{L^2} + \| f^* \|^2_{H^{s+1}} + \| \partial_t f^* \|^2_{H^{s}} + \| f^* \|^2_{H^{s+1}}).$$

Here $[|\nabla|^s, \tilde{H}_{f^*}] = |\nabla|^s \tilde{H}_{f^*} - \tilde{H}_{f^*} |\nabla|^s$ denotes the commutator, and we use Lemma A.8 to get

$$\| [|\nabla|^s, \tilde{H}_{f^*}] f \|_{L^2} \leq C \| f^* \|_{H^{s+1}} \| f \|_{H^{s+1}}.$$  (1.17)

One can easily check that

$$\frac{1}{2} \partial_t \left( \| \partial_t f \|^2_{L^2} + \| f \|^2_{L^2} \right) \leq C(\tilde{L}_1)(\| \partial_t f \|^2_{H^{s}} + \| f \|^2_{H^{s+1}} + \| \partial_t f^* \|^2_{H^{s}} + \| f^* \|^2_{H^{s+1}}).$$

Then, it follows from Gronwall’s inequality that

$$\sup_{t \in [0, T]} (\| f \|_{H^{s+1}} + \| \partial_t f \|_{H^{s}}) \leq C(\tilde{L}_0)(\| f_0 \|_{H^{s+1}} + \| \partial_t f_0 \|_{H^{s}} + T \tilde{L}_1) e^{C(\tilde{L}_1)T}. \quad (1.18)$$

Here the constant $C(\tilde{L}_0)$ comes from the equivalence that

$$C(\tilde{L}_0) \left( \left\| \frac{|\nabla|^{s+1} f}{(1 + |f^*|^2)^{\frac{s}{2}}} \right\|_{L^2} + \| f \|_{L^2} \right) \leq \| f \|_{H^{s+1}} \leq C(\tilde{L}_0) \left( \left\| \frac{|\nabla|^{s+1} f}{(1 + |f^*|^2)^{\frac{s}{2}}} \right\|_{L^2} + \| f \|_{L^2} \right).$$

This equivalence is depends only on the lower order norm of $f$, this is important in the construction of the iteration map.

We can see that the energy could be closed because the error term $(\partial_1 f^*)^2$ in (1.15) is of order 1, and 1 is half of the order of the main positive operator $-\tilde{H}_{f^*}$. Moreover, in (1.16), the coefficient $\frac{1}{\sqrt{1+ |f^*|^2}}$ is 1 order more regular than the function $(\partial_1 f, \partial_2 f)$, and this allows us to have the commutator estimate (1.17).

To get the existence of solution to (1.13), we use a fixed point argument. (The existence of solution to the linear hyperbolic equation is classical).

Given initial data $(f_0, \partial_t f_0)$ such that

$$\| f_0 \|_{H^{s+1}} + \| \partial_t f_0 \|_{H^{s}} \leq \frac{1}{2} \tilde{L}_0.$$

Let $\mathcal{X}(T, \tilde{L}_1)$ be the collection of $f$ satisfying

$$f(0) = f_0, \quad \partial_t f(0) = \partial_t f_0, \quad \sup_{t \in [0, T]} (\| f \|_{H^{s+1}} + \| \partial_t f \|_{H^{s}}) (t) \leq \tilde{L}_1.$$

It is clear that $\mathcal{X}(T, \tilde{L}_1)$ is a compact subset of $C([0, T], H^{s}(\mathbb{T}^2))$. For any $\tilde{L}_1$, by taking $T$ small enough, we can see that it holds for $f \in \mathcal{X}(T, \tilde{L}_1)$ that

$$\sup_{t \in [0, T]} \| f \|_{L^\infty} \leq \| f_0 \|_{L^\infty} + T \sup_{t \in [0, T]} \| \partial_t f \|_{L^\infty} \leq \frac{1}{2} \tilde{L}_0 + T \tilde{L}_1 \leq \tilde{L}_0. \quad (1.19)$$
Next, we define the iteration map: \( \mathcal{F}(f^*) \overset{\text{def}}{=} f \), where \( f \) is the solution of the linear equation (1.15). We need to show that \( \mathcal{F}(X(T, \widetilde{L}_1)) \subset X(T, \widetilde{L}_1) \). For any \( f^* \in X(T, \widetilde{L}_1) \), we can see from (1.19) that \( f^* \) satisfies (1.14) in a short time \( T \) which depends on \( \widetilde{L}_1 \). Thus the energy estimate (1.18) holds (this is why we introduce both \( L_0 \) and \( L_1 \) at the beginning of Sect. 4). By taking \( \widetilde{L}_1 \) big enough and then taking \( T \) small enough, we have \( \mathcal{F} : X(T, \widetilde{L}_1) \to X(T, \widetilde{L}_1) \).

At last, we show the contraction of \( \mathcal{F} \). Assume that
\[
 f^A = \mathcal{F}(f^{*A}), \quad f^B = \mathcal{F}(f^{*B}),
\]
then \( f^D = f^A - f^B \) satisfies
\[
\partial_t^2 f^D(t, x') - \tilde{H}_{f^*A}(f^D)(t, x') + \tilde{H}_{f^*B}(f^B)(t, x') + (\partial_t f^{*A} + \partial_t f^{*B})\partial_1 f^D(t, x') = 0. \tag{1.20}
\]
Compare (1.20) to (1.15), the error terms become \( \tilde{H}_{f^*A}(f^B)(t, x') - \tilde{H}_{f^*B}(f^B) \) and \( (\partial_t f^{*A} + \partial_t f^{*B})\partial_1 f^D \). Here all the error terms depend linearly on \( f^D \). Indeed, we have
\[
\tilde{H}_{f^*A}(f^B) - \tilde{H}_{f^*B}(f^B) = (\partial_1, \partial_2) \left( \frac{(f^{*B} - f^{*A})(f^{*B} + f^{*A})}{\sqrt{1 + |f^{*A}|^2} \sqrt{1 + |f^{*B}|^2}} \right) (1, 0, 0, 1)
\]
\[
= (\partial_t f^B)^2 + (\partial_2 f^B)^2,
\]
thus, we have
\[
\left\| \tilde{H}_{f^*A}(f^B) - \tilde{H}_{f^*B}(f^B) \right\|_{H^{s-1}} \leq C \left\| f^{*D} \right\|_{H^s} \left\| f^B \right\|_{H^{s+1}}.
\]
Different to (1.15), when we apply inner product to (1.20), we can not move the derivative from \( f^B \) to \( \partial_1 f^{*D} \). As a result, all the derivative of \( \tilde{H} \) could apply on \( f^B \), that is why in Proposition 5.4 we prove the contraction in a space with lower regularity. Actually, in this error term, \( f^B \) plays the role of coefficient. Then we have
\[
\left\| f^D \right\|_{H^s}^2 (t) + \left\| \partial_t f^D \right\|_{H^{s-1}}^2 (t)
\]
\[
\leq C \left( \left\| f^0_{D} \right\|_{H^s}^2 + \left\| \partial_t f^0_{D} \right\|_{H^{s-1}}^2 + \int_0^t \left( \left\| f^D \right\|_{H^s}^2 + \left\| \partial_t f^D \right\|_{H^{s-1}}^2 \right) dt \right) + \left\| f^{*D} \right\|_{H^s}^2 + \left\| \partial_t f^{*D} \right\|_{H^{s-1}}^2 (t') dt' + \left( \left\| f^D \right\|_{H^s} + \left\| \partial_t f^D \right\|_{H^{s-1}} \right) (t') dt'.
\]
Recalling that the initial data is \( f^0_{D} = 0, \partial_t f^0_{D} = 0 \), by using Gronwall’s inequality and taking \( T \) small enough, we deduce
\[
\sup_{t \in [0, T]} \left( \left\| f^D \right\|_{H^s} + \left\| \partial_t f^D \right\|_{H^{s-1}} \right) \leq C T e^{C T} \sup_{t \in [0, T]} \left( \left\| f^{*D} \right\|_{H^s} + \left\| \partial_t f^{*D} \right\|_{H^{s-1}} \right)
\]
\[
\leq \frac{1}{2} \sup_{t \in [0, T]} \left( \left\| f^{*D} \right\|_{H^s} + \left\| \partial_t f^{*D} \right\|_{H^{s-1}} \right).
\]
Thus there exist an unique \( f \) such that \( f = \mathcal{F}(f) \), which means that \( f \) solves the nonlinear system (1.13) and satisfies

\[
f(0) = f_0, \quad \partial_t f(0) = \partial_t f_0, \quad \sup_{t \in [0,T]} (\| f \|_{H^{\frac{3}{2}}} + \| \partial_t f \|_{H^{\frac{1}{2}}}) (t) \leq L_1.
\]

**Technical difficulties and paralinearization.**

Now, we turn back to the original evolution equation (1.11). We could not linearize \( H(f) \) in the same way to \( H(f) \). In \( H(f) \) the regularity of the coefficient is too low, and the commutator estimate (1.17) no longer holds. The usual way to deal with this problem is to take derivative of \( H(f) \):

\[
\partial_t H(f) = H^{(1)}(\partial_t f) = (\partial_1, \partial_2) \left( \frac{1 + \partial_2 f^2}{(1 + |\nabla f|^2)^{\frac{3}{2}}}, -\frac{\partial_1 f \partial_2 f}{(1 + |\nabla f|^2)^{\frac{3}{2}}} \right) \left( \frac{\partial_1}{\partial_2} \right) \partial_t f,
\]

and introduce the following linearize operator

\[
H^1_{f*}(\partial_t f) = (\partial_1, \partial_2) \left( \frac{1 + (\partial_2 f^* f^*)^2}{(1 + |\nabla f^*|^2)^{\frac{3}{2}}}, -\frac{\partial_1 f^* \partial_2 f^*}{(1 + |\nabla f^*|^2)^{\frac{3}{2}}} \right) \left( \frac{\partial_1}{\partial_2} \right) \partial_t f.
\]

Here \( -H^1_{f*} \) is a positive operator of order 2. We can see that the coefficients \( \left( \frac{1 + (\partial_2 f^* f^*)^2}{(1 + |\nabla f^*|^2)^{\frac{3}{2}}} \right) \) is 1 order more regular than \( \nabla \partial_t f \).

However, this technical is not suitable for our problem. For \( \rho^+ = \rho^- = 1 \), the main term is \( (N_f^+ + N_f^-) H(f) \). After taking derivative, we get

\[
\partial_t (N_f^+ + N_f^-) H(f) = (N_f^+ + N_f^-) H^{(1)}(\partial_t f) + [\partial_t, (N_f^+ + N_f^-)] H(f),
\]

Here \( -(N_f^+ + N_f^-) H^{(1)} \) is a positive operator of order 3. Meanwhile, \([\partial_t, (N_f^+ + N_f^-)] H(f)\) is an error term that can be regarded as a operator of order 2 applying on \( \partial_t f \). However, from the analysis of toy model, we can see that the order of the error term could be at most half of the main positive operator. In this problem, the order of the error term should be at most \( \frac{3}{2} \), which means we can not get the energy estimate with an error term of order 2.

To overcome this difficulty, we use the paralinearization. A key observation of Bony [11] is that one can replace nonlinear expressions by paradifferential expressions. By using Proposition A.7, we can write \( H(f) = -T_l f + r_2 \), where \( T_l \) is a positive operator of order 2 and \( \| r_2 \|_{H^{\frac{3}{2} - \frac{3}{2}}} \leq C(\| f \|_{H^{\frac{3}{2}}}) \). In other word, the error term could be much more regular than \( f \) if \( f \) is smooth enough. This solve the problem about lack of regularity in the linearization of \( H(f) \). Here \( T_l \) is a paradifferential operator, and \( l \) is the corresponding symbol, see Definition A.1. An advantage of paradifferential operator is that we do not have to worry about the regularity of the symbols, see Proposition A.2.

Combining the paralinearization of \( N_f^\pm \), we write

\[
(N_f^+ + N_f^-) H(f) = -(T_{\lambda +} + T_{\lambda +}) T_l f + R(f),
\]

where \( (T_{\lambda +} + T_{\lambda +}) T_l \) is a positive operator of order 3 and \( R(f) \) is an error term of order \( \frac{3}{2} \).

Then, by using the method of symmetrizers, we establish a priori estimate for the linearized system in a function space equivalent to the Sobolev space. With the energy estimate, we can prove the local well-posedness in the same way as we study the toy model.
This paralinearization approach was established by Alazard-Burq-Zuliy [2] in studying the water-wave equations with surface tension.

**Zero surface tension limit.**

If the stability condition (1.10) holds, the two order operator (1.12) will be strictly positive. Thus, even if \( \sigma = 0 \), the evolution equation (1.11) is still strictly hyperbolic. With this property, we can get a \( \sigma \)-independent energy estimate and the solution \( (f^{\sigma}, u^{\sigma}, h^{\sigma}) \) got in Theorem 1.1 can be extended to a lifespan \( \bar{T} \) independent of \( \sigma \). At last, we use the Aubin-Lions Lemma to show that, as \( \sigma \) tends to 0, the solution \( (f^{\sigma}, u^{\sigma}, h^{\sigma}) \) converges to \( (f, u, h) \) which is the solution of the problem without surface tension. This approach is standard, see [6, 8, 35].

### 2 Reference domain, harmonic coordinate and Dirichlet-Neumann Operator

In this section, we recall some fundamental lemmas on the harmonic coordinate and Dirichlet-Neumann operators.

To solve the free boundary problem, we introduce a fixed reference domain. Let \( \Gamma_* \) be a fixed graph given by

\[
\Gamma_* = \{(y_1, y_2, y_3) : y_3 = f_*(y_1, y_2)\},
\]

where \( f_* \) satisfies \( \int_{\mathbb{T}^2} f_*(y')dy' = 0 \). The reference domain is given by

\[
\Omega_* = \mathbb{T}^2 \times (-1, 1), \quad \Omega_*^\pm = \{y \in \Omega_* | y_3 \gtrless f_*(y_1, y_2), y' \in \mathbb{T}^2\}.
\]

We will look for a free boundary that lies close to the reference domain. For this purpose, we define

\[
\Upsilon(\delta, k) := \{f \in H^k(\mathbb{T}^2) : \|f - f_*\|_{H^k(\mathbb{T}^2)} \leq \delta\}.
\]

For \( f \in \Upsilon(\delta, k) \), we define \( \Gamma_f, \Omega_f^\pm, \Omega_f^- \) by

\[
\Gamma_f := \{x \in \Omega_f | x_3 = f(t, x'), x' \in \mathbb{T}^2\}, \quad \Omega_f^\pm = \{x \in \Omega_f | x_3 \gtrless f(t, x'), x' \in \mathbb{T}^2\}.
\]

We denote by \( \mathbf{N} = (N_1, N_2, N_3) := (-\partial_1 f, -\partial_2 f, 1) \) the outward normal vector of \( \Omega_f^- \) on \( \Gamma_f \), and \( \mathbf{n} = (n_1, n_2, n_3) := \mathbf{N}/\sqrt{1 + |\nabla f|^2} \). Then we need to introduce the harmonic coordinate. For given \( f \in \Upsilon(\delta, k) \), we define a map \( \Phi_f^\pm : \Omega_f^\pm \rightarrow \Omega_f^\pm \) by the harmonic extension:

\[
\begin{align*}
\Delta_y \Phi_f^\pm &= 0, \quad y \in \Omega_f^\pm, \\
\Phi_f^\pm(y', f_*(y')) &= (y', f(y')) \quad y' \in \mathbb{T}^2, \\
\Phi_f^\pm(y', \pm 1) &= (y', \pm 1) \quad y' \in \mathbb{T}^2.
\end{align*}
\]

For each \( \Gamma_* \), there exists \( \delta_0 = \delta_0(\|f_*\|_{W^{1,\infty}}) > 0 \) so that \( \Phi_f^\pm \) is a bijection whenever \( \delta \leq \delta_0 \). Then, there exists an inverse map \( \Phi_f^{\pm-1} : \Omega_f^\pm \rightarrow \Omega_f^\pm \) such that

\[
\Phi_f^{\pm-1} \circ \Phi_f^\pm = \Phi_f^\pm \circ \Phi_f^{\pm-1} = Id.
\]

We list some properties of the harmonic coordinate (see Section 2 of [40] for example):
Lemma 2.1  Let \( f \in \Upsilon(\delta_0, s - \frac{1}{2}) \) for \( s \geq 3 \). Then there exists a constant \( C \) depending only on \( \delta_0 \) and \( \|f^*_s\|_{H^{s - \frac{1}{2}}} \) so that

1. If \( u \in H^\sigma(\Omega_\pm_f^s) \) for \( \sigma \in [0, s] \), then
   \[
   \|u \circ \Phi^\pm_f\|_{H^\sigma(\Omega_\pm_f^s)} \leq C \|u\|_{H^\sigma(\Omega_\pm_f^s)}.
   \]

2. If \( u \in H^\sigma(\Omega_\pm_a^s) \) for \( \sigma \in [0, s] \), then
   \[
   \|u \circ \Phi^\pm_{a-1}\|_{H^\sigma(\Omega_\pm_a^s)} \leq C \|u\|_{H^\sigma(\Omega_\pm_a^s)}.
   \]

3. If \( u, v \in H^\sigma(\Omega_\pm_a^s) \) for \( \sigma \in [2, s] \), then
   \[
   \|uv\|_{H^\sigma(\Omega_\pm_a^s)} \leq C \|u\|_{H^\sigma(\Omega_\pm_a^s)} \|v\|_{H^\sigma(\Omega_\pm_a^s)}.
   \]

Now we introduce the Dirichlet-Neumann operator which maps the Dirichlet boundary value of a harmonic function to its Neumann boundary value. For any \( g(x') \in H^k(\mathbb{T}^2) \), we denote by \( H^\pm_f g \) the harmonic extension from \( \Gamma_f \) to \( \Omega_\pm_f^s \):

\[
\begin{cases}
\Delta H^\pm_f g = 0 & x \in \Omega_\pm_f^s, \\
(H^\pm_f g)(x', f(x')) = g(x') & x' \in \mathbb{T}^2, \\
\partial_3 H^\pm_f g(x', \pm1) = 0 & x' \in \mathbb{T}^2.
\end{cases}
\]

(2.2)

Then we define the Dirichlet-Neumann operator:

\[
N^\pm_f g \overset{\text{def}}{=} \mp \n \cdot (\nabla H^\pm_f g) |_{\Gamma_f}.
\]

We will use the following properties from \([3, 40]\).

Lemma 2.2  It holds that

1. \( N^\pm_f \) is a self-adjoint operator:
   \[
   (N^\pm_f \psi, \phi) = (\psi, N^\pm_f \phi), \quad \forall \phi, \psi \in H^\frac{1}{2}(\mathbb{T}^2);
   \]

2. \( N^\pm_f \) is a positive operator:
   \[
   (N^\pm_f \phi, \phi) = \|\nabla H^\pm_f \phi\|_{L^2(\Omega_\pm_f^s)}^2 \geq 0, \quad \forall \phi \in H^\frac{1}{2}(\mathbb{T}^2);
   \]

   Especially, if \( \int_{\mathbb{T}^2} \phi(x') dx' = 0 \), there exists \( c > 0 \) depending on \( c_0, \|f\|_{W^{1,\infty}} \) such that
   \[
   (N^\pm_f \phi, \phi) \geq c\|H^\pm_f \phi\|_{H^1(\Omega_\pm_f^s)}^2 \geq c\|\phi\|_{H^\frac{1}{2}}^2, \quad \forall \phi \in H^\frac{1}{2}(\mathbb{T}^2).
   \]

3. \( N^\pm_f \) is a bijection from \( H^k_0(\mathbb{T}^2) \) to \( H^k_0(\mathbb{T}^2) \) for \( k \geq 0 \), where
   \[
   H^k_0(\mathbb{T}^2) := H^k(\mathbb{T}^2) \cap \{\phi \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} \phi(x') dx' = 0\}.
   \]
3 Reformulation of the problem

In this section, we derive a new system that is equivalent to the original system (1.2)–(1.5). The new system consists of the following quantities:

- The height function of the interface: \( f \);
- The scaled normal velocity on the interface: \( \theta = u^\pm \cdot N_f \);
- The vorticity and current in the fluid region: \( \omega = \nabla \times u, \xi = \nabla \times h \);
- The average of the tangential part of the velocity and the magnetic field on the top and bottom fixed boundary:

\[
\begin{align*}
a_i^\pm(t) &= \int_{\Gamma^2} u_i^\pm(t, x', \pm 1) dx', \\
b_i^\pm(t) &= \int_{\Gamma^2} h_i^\pm(t, x', \pm 1) dx' (i = 1, 2). 
\end{align*}
\]

3.1 Evolution of the Scaled Normal Velocity

Let

\[
\theta(t, x') \overset{\text{def}}{=} u^\pm(t, x', f(t, x')) \cdot N_f(t, x'),
\]

we have

\[
\partial_t f(t, x') = \theta(t, x').
\]

In this subsection, we will derive the evolution equation of \( \theta \). To this end, we need the following elementary lemma, which can be proved by direct calculation.

**Lemma 3.1** [40] For \( u = u^\pm, h^\pm \), we have

\[
(u \cdot \nabla u) \cdot N_f - \partial_3 u_j N_j(u \cdot N_f)_{|x_3 = f(t, x')} = u_1 \partial_1 (u_j N_j) + u_2 \partial_2 (u_j N_j) + \sum_{i, j=1,2} u_i u_j \partial_i \partial_j f.
\]

With the help of Lemma 3.1, we deduce from (1.2) that

\[
\begin{align*}
\partial_t \theta &= (\partial_t u^+ + \partial_3 u^+ \partial_t f) \cdot N_f + u^+ \cdot \partial_t N_f |_{x_3 = f(t, x')} \\
&= (-u^+ \cdot \nabla u^+ + \frac{1}{\rho^+} h^+ \cdot \nabla h^+ - \frac{1}{\rho^+} \nabla p^+ + \partial_3 u^+ \partial_t f) \cdot N \\
&- u^+ \cdot (\partial_1 \partial_t f, \partial_2 \partial_t f, 0)_{|x_3 = f(t, x')} \\
&= ((-u^+ \cdot \nabla) u^+ + \partial_3 u^+ (u^+ \cdot N)) \cdot N + \frac{1}{\rho^+} (h^+ \cdot \nabla) h^+ \cdot N \\
&- \frac{1}{\rho^+} N \cdot \nabla p^+ - u^+ \cdot (\partial_1 \theta, \partial_2 \theta, 0)_{|x_3 = f(t, x')} \\
&= -2(u_1^+ \partial_1 \theta + u_2^+ \partial_2 \theta) - \frac{1}{\rho^+} N \cdot \nabla p^+ - \sum_{i, j=1,2} u_i^+ u_j^+ \partial_i \partial_j f \\
&+ \frac{1}{\rho^+} \sum_{i, j=1,2} h_i^+ h_j^+ \partial_i \partial_j f.
\end{align*}
\]
and similarly,
\[ \partial_t \theta = -2(u_1^- \partial_1 \theta + u_2^- \partial_2 \theta) - \frac{1}{\rho^-} \mathbf{N} \cdot \nabla p^- - \sum_{i,j=1,2} u_i^- u_j^- \partial_i \partial_j f + \frac{1}{\rho^-} \sum_{i,j=1,2} h_i^- h_j^- \partial_i \partial_j f. \]

Therefore, it holds that
\[ 2(u_1^+ \partial_1 \theta + u_2^+ \partial_2 \theta) + \frac{1}{\rho^+} \mathbf{N} \cdot \nabla p^+ + \sum_{i,j=1,2} u_i^+ u_j^+ \partial_i \partial_j f - \frac{1}{\rho^+} \sum_{i,j=1,2} h_i^+ h_j^+ \partial_i \partial_j f = 2(u_1^- \partial_1 \theta + u_2^- \partial_2 \theta) + \frac{1}{\rho^-} \mathbf{N} \cdot \nabla p^- + \sum_{i,j=1,2} u_i^- u_j^- \partial_i \partial_j f - \frac{1}{\rho^-} \sum_{i,j=1,2} h_i^- h_j^- \partial_i \partial_j f. \]

From the first equation of (1.2) and the boundary condition (1.5), we get
\[ \Delta p^\pm = \text{tr}(\nabla h^\pm)^2 - \rho^\pm \text{tr}(\nabla u^\pm)^2 \quad \text{in } \Omega_f^\pm, \]
and
\[ \partial_3 p^\pm = 0 \quad \text{on } \Gamma^\pm. \]

Recalling the definition of harmonic extension \( \mathcal{H}_f^\pm \), we have the following representation for the pressure \( p^\pm \):
\[ p^\pm = \mathcal{H}_f^\pm p^\pm + \rho^\pm p_{u^\pm,u^\pm} - p_{h^\pm,h^\pm}, \]
where \( p_{v^\pm,v^\pm} \) denotes the solution of the elliptic equation
\[
\begin{aligned}
\Delta p_{v^\pm,v^\pm} = -\text{tr}(\nabla v^\pm \nabla v^\pm) & \quad \text{in } \Omega_f^\pm, \\
p_{v^\pm,v^\pm} = 0 & \quad \text{on } \Gamma_f, \\
e_3 \cdot \nabla p_{v^\pm,v^\pm} = 0 & \quad \text{on } \Gamma^\pm. 
\end{aligned}
\]

Thus, we infer from (3.4) that
\[
\frac{1}{\rho^+} \mathbf{N} \cdot \nabla \mathcal{H}_f^+ p^+ - \frac{1}{\rho^-} \mathbf{N} \cdot \nabla \mathcal{H}_f^- p^-
= -\left[ 2(u_1^+ \partial_1 \theta + u_2^+ \partial_2 \theta) + \mathbf{N} \cdot \nabla (p_{u^+,u^+} - \frac{1}{\rho^+} p_{h^+,h^+}) \right]
+ \sum_{i,j=1,2} (u_i^+ u_j^+ - \frac{1}{\rho^+} h_i^+ h_j^+) \partial_i \partial_j f
+ \left[ 2(u_1^- \partial_1 \theta + u_2^- \partial_2 \theta) + \mathbf{N} \cdot \nabla (p_{u^-,u^-} - \frac{1}{\rho^-} p_{h^-,h^-}) \right]
+ \sum_{i,j=1,2} (u_i^- u_j^- - \frac{1}{\rho^-} h_i^- h_j^-) \partial_i \partial_j f
= \Delta -g^+ + g^-.
\]

Recalling the definition of Dirichlet-Neumann operator, we rewrite the above equality as
\[ -\frac{1}{\rho^+} \mathcal{N}_f^+ p^+ - \frac{1}{\rho^-} \mathcal{N}_f^- p^- = -g^+ + g^-.
\]
As $p^+ - p^- = \sigma H(f)$ on $\Gamma_f$, we have

$$p^\pm = \mathcal{N}_f^{-1} (g^+ - g^- \pm \frac{1}{\rho^\pm} \mathcal{N}_f^\pm \sigma H(f))$$

where

$$\mathcal{N}_f \overset{\text{def}}{=} \frac{1}{\rho^+} \mathcal{N}_f^+ + \frac{1}{\rho^-} \mathcal{N}_f^-. $$

Moreover, it’s easy to see

$$\mathcal{N}_f^+ = \left( \frac{1}{\rho^+} + \frac{1}{\rho^-} \right)^{-1} (\mathcal{N}_f^+ + \frac{1}{\rho^-} (\mathcal{N}_f^+ - \mathcal{N}_f^-)), $$

$$\mathcal{N}_f^- = \left( \frac{1}{\rho^+} + \frac{1}{\rho^-} \right)^{-1} (\mathcal{N}_f^- - \frac{1}{\rho^+} (\mathcal{N}_f^+ - \mathcal{N}_f^-)), $$

and

$$\frac{1}{\rho^+} \mathcal{N}_f^+ \mathcal{N}_f^{-1} g^- + \frac{1}{\rho^-} \mathcal{N}_f^- \mathcal{N}_f^{-1} g^+ = \frac{\rho^+ g^+ + \rho^- g^-}{\rho^+ + \rho^-} - \frac{1}{\rho^+ + \rho^-} (\mathcal{N}_f^+ - \mathcal{N}_f^-) \mathcal{N}_f^{-1} (g^+ - g^-).$$

$$\mathcal{N}_f^+ \mathcal{N}_f^{-1} \mathcal{N}_f^- = \frac{\rho^+ \rho^-}{(\rho^+ + \rho^-)^2} (\mathcal{N}_f^+ + \rho^- \mathcal{N}_f^-)$$

$$- \frac{\rho^+ \rho^-}{(\rho^+ + \rho^-)^2} (\mathcal{N}_f^+ - \mathcal{N}_f^-) \mathcal{N}_f^{-1} (\mathcal{N}_f^+ - \mathcal{N}_f^-).$$

Accordingly, we obtain that

$$\partial_t \theta = \frac{1}{\rho^+} \mathcal{N}_f^+ p^+ - g^+ = \frac{1}{\rho^+} \mathcal{N}_f^+ \mathcal{N}_f^{-1} (g^+ - g^- + \frac{1}{\rho^-} \mathcal{N}_f^- \sigma H(f)) - g^+$$

$$= - \frac{\rho^+ g^+ + \rho^- g^-}{\rho^+ + \rho^-} + \frac{1}{\rho^+ + \rho^-} (\mathcal{N}_f^+ - \mathcal{N}_f^-) \mathcal{N}_f^{-1} (g^+ - g^-)$$

$$+ \frac{\sigma}{(\rho^+ + \rho^-)^2} (\mathcal{N}_f^+ + \rho^- \mathcal{N}_f^-) H(f)$$

$$- \frac{\sigma}{(\rho^+ + \rho^-)^2} (\mathcal{N}_f^+ - \mathcal{N}_f^-) \mathcal{N}_f^{-1} (\mathcal{N}_f^+ - \mathcal{N}_f^-) H(f)$$

$$= \frac{2}{\rho^+ + \rho^-} ((\rho^+ u^+_i + \rho^- u^-_i) \partial_i \theta + (\rho^+ u^+_2 + \rho^- u^-_2) \partial_2 \theta)$$

$$- \frac{1}{\rho^+ + \rho^-} \sum_{i,j=1,2} (\rho^+ u^+_i u^+_j - h^+_i h^+_j + \rho^- u^-_i u^-_j - h^-_i h^-_j) \partial_i \partial_j f$$

$$- \frac{\sigma}{(\rho^+ + \rho^-)^2} (\mathcal{N}_f^+ - \mathcal{N}_f^-) \mathcal{N}_f^{-1} (\mathcal{N}_f^+ - \mathcal{N}_f^-) H(f)$$

$$+ \frac{2}{\rho^+ + \rho^-} (\mathcal{N}_f^+ - \mathcal{N}_f^-) \mathcal{N}_f^{-1} \mathcal{P}((u^+_i - u^-_i) \partial_i \theta + (u^+_2 - u^-_2) \partial_2 \theta).$$
It follows from (1.2) by direct calculation that

\[ u = \text{a result, it holds that} \]

\[ \rho^+ + \rho^- (N_j^+ - N_j^-) \mathcal{N}_f^{-1} \mathcal{P} \left( \sum_{i,j=1,2} (u_i^+ u_j^-) \right) \]

\[ - \frac{1}{\rho^+ + \rho^-} \left( u_i^+ h_j^- + u_j^- h_i^- \right) \mathcal{J} \partial_j f \]

\[ - \frac{1}{\rho^+ + \rho^-} \mathbf{N} \cdot \left( \nabla (\rho^+ p_{u-, h}^+ - p_{h, h}^+) + \nabla (\rho^- p_{u-, h}^- - p_{h, h}^-) \right) \]

\[ + \frac{1}{\rho^+ + \rho^-} (N_j^- - N_j^+) \mathcal{N}_f^{-1} \mathcal{P} \mathbf{N} \cdot \left( \nabla (\rho^+ p_{u+, h}^+ - p_{h, h}^+) \right) \]

\[ - \nabla (\rho^- p_{u-, h}^- - p_{h, h}^-) \right) \tag{3.6} \]

Here \( \mathcal{P} : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2) \) denotes the projection operator such that

\[ \mathcal{P} g = g - \langle g \rangle \]

where \( \langle g \rangle := \int_{\mathbb{T}^2} g dx' \). We can apply the operator \( \mathcal{P} \) to some of the terms in (3.6) for the same reason as in [40], since it does not change the formulation of this system by the fact that \( \mathcal{P} g^\pm = g^\pm \).

### 3.2 Equations for the Vorticity and Current

Now we derive the equations for

\[ \omega^\pm = \nabla \times \mathbf{u}^\pm, \quad \xi^\pm = \nabla \times \mathbf{h}^\pm. \tag{3.7} \]

It follows from (1.2) by direct calculation that \((\omega^\pm, \xi^\pm)\) satisfies

\[ \rho^\pm \partial_t \omega^\pm + \rho^\pm \mathbf{u}^\pm \cdot \nabla \omega^\pm - \mathbf{h}^\pm \cdot \nabla \xi^\pm = \rho^\pm \omega^\pm \cdot \nabla \mathbf{u}^\pm - \xi^\pm \cdot \nabla \mathbf{h}^\pm \text{in} \ \Omega_f^\pm, \]

\[ \partial_t \xi^\pm + \mathbf{u}^\pm \cdot \nabla \xi^\pm - \mathbf{h}^\pm \cdot \nabla \omega^\pm = \xi^\pm \cdot \nabla \mathbf{u}^\pm - \omega^\pm \cdot \nabla \mathbf{h}^\pm - 2 \sum_{i=1}^{3} \nabla u_{i}^\pm \times \nabla h_{i}^\pm \text{in} \ \Omega_f^\pm. \tag{3.8} \]

### 3.3 Tangential velocity and magnetic field on \( \Gamma^\pm \)

As in [40], we need to derive the evolution equations of the following quantities:

\[ a_i^\pm(t) = \int_{\mathbb{T}^2} u_i^\pm(t, x', \pm1) dx', \quad b_i^\pm(t) = \int_{\mathbb{T}^2} h_i^\pm(t, x', \pm1) dx'. \tag{3.9} \]

From the fact that \( u_3^\pm(t, x', \pm1) \equiv 0 \), we deduce that for \( i = 1, 2 \)

\[ \rho^\pm \partial_t u_i^\pm + \rho^\pm u_j^\pm \partial_j u_i^\pm - h_j^\pm \partial_j h_i^\pm - \partial_i p^\pm = 0 \quad \text{on} \ \Gamma^\pm. \]

As a result, it holds that

\[ \partial_t a_i^\pm + \int_{\Gamma^\pm} (u_j^\pm \partial_j u_i^\pm - \frac{1}{\rho^\pm} h_j^\pm \partial_j h_i^\pm) dx' = 0, \]

or equivalently

\[ a_i^\pm(t) = a_i^\pm(0) - \int_0^t \int_{\Gamma^\pm} (u_j^\pm \partial_j u_i^\pm - \frac{1}{\rho^\pm} h_j^\pm \partial_j h_i^\pm(x', t') dx' dt'. \tag{3.10} \]
Similarly, we have
\[ b^\pm_i(t) = b^\pm_i(0) - \int_0^t \int_{\Gamma^\pm} (u_j^\pm \partial_j h_i^\pm - h_j^\pm \partial_j u_i^\pm)(x', t') dx' dt'. \] (3.11)

### 3.4 Solvability Conditions for the Div-Curl System

In order to recover the divergence-free velocity field or magnetic field from its curled part, we need to solve the following div-curl system:
\[
\begin{align*}
\begin{cases}
\text{curl} u^\pm = \omega^\pm, & \text{in } \Omega_f^\pm, \\
u^\pm \cdot N = \theta & \text{on } \Gamma_f, \\
u^\pm \cdot e_3 = 0, & \int_{\Gamma^\pm} u_i dx' = a_i^\pm (i = 1, 2) \text{ on } \Gamma^\pm.
\end{cases}
\end{align*}
\] (3.12)

The solvability of the above system was obtained in Section 5 of [40] under the following compatibility conditions:

C1. div $\omega^\pm = 0$ in $\Omega_f^\pm$,
C2. $\int_{\Gamma^\pm} \omega^\pm dx' = 0$,
C3. $\int_{\Gamma_f} \omega^\pm dx = \int_{\Gamma_f} g^\pm dx$.

**Proposition 3.2** [40] Let $\sigma \in [2, s]$ be an integer, $c_0 \in (0, 1)$. Given $f \in H^{s+\frac{1}{2}}(\mathbb{T}^2)$, $\omega^\pm, g^\pm \in H^{s-1}(\Omega_f^\pm)$, $\theta \in H^{s-\frac{1}{2}}(\Gamma_f)$ satisfying:
\[
\mp \int_{\Omega_f^\pm} g^\pm dx = \int_{\Gamma_f} \theta ds, \quad \int_{\Gamma^\pm} \omega^\pm dx' = 0,
\]
\[
\text{div} \omega^\pm = 0 \text{ in } \Omega_f^\pm, \quad -(1 - c_0) \leq f \leq (1 - c_0) \text{ on } \Gamma_f.
\]

Then there exists a unique $u^\pm \in H^s(\Omega_f^\pm)$ of the div-curl system (3.12) so that
\[
\|u^\pm\|_{H^s(\Omega_f^\pm)} \leq C(c_0, \|f\|_{H^{s+\frac{1}{2}}(\mathbb{T}^2)}) \left(\|\omega^\pm\|_{H^{s-1}(\Omega_f^\pm)} + \|g^\pm\|_{H^{s-1}(\Omega_f^\pm)} + \|\theta\|_{H^{s-\frac{1}{2}}(\Gamma_f)} + |a_1^\pm| + |a_2^\pm|\right).
\]

### 4 Energy Estimates for the Linearized System

In this section, we linearize the equivalent system derived in Sect. 3 around given functions $(f, u^\pm, h^\pm)$ and give the energy estimates for the linearized system. We assume that there exists $T > 0$ such that for any $t \in [0, T]$, there holds
\[
\begin{align*}
\|\langle u^\pm, h^\pm \rangle\|_{W^{1, \infty}}(t) + \|f\|_{W^{2, \infty}}(t) & \leq L_0, \\
\sigma \|f\|_{H^{s+\frac{1}{2}}(t)} + \|f\|_{H^{s+\frac{1}{2}}(t)} + \|\partial_t f\|_{H^{s-\frac{1}{2}}(t)} + \|u^\pm\|_{H^s(\Omega_f^\pm)}(t) + \|h^\pm\|_{H^s(\Omega_f^\pm)}(t) & \leq L_1, \\
\|(\partial_t u^\pm, \partial_t h^\pm)\|_{W^{2, \infty}}(t) & \leq L_2, \\
\|f - f_0\|_{H^{s+\frac{1}{2}}(t)} & \leq \delta_0, \\
-(1 - c_0) & \leq f(t, x') \leq (1 - c_0),
\end{align*}
\]
and
\[
\begin{align*}
\text{div} u^± &= \text{div} h^± = 0 \quad \text{in } \Omega^±, \\
h^± \cdot N &= 0, \quad u^± \cdot N = \partial_t f \quad \text{on } \Gamma_f, \\
u_3^± &= h_3^± = 0 \quad \text{on } \Gamma^±.
\end{align*}
\]

Here \(s \geq 6\) is an integer and \(L_0, L_1, L_2, c_0, \delta_0\) are positive constants.

### 4.1 Paralinearization of \(\mathcal{N}_f^±\) and \(H\)

The third order term \(\frac{\sigma}{\rho+\rho^-} (\rho^+ \mathcal{N}_f^+ + \rho^- \mathcal{N}_f^-) H(f)\) in (3.6) is a fully nonlinear term of \(f\), and is difficult to linearize by conventional methods. To overcome this difficulty, we use the paralinearization approach developed in [2, 4]. Here we follow the presentation by Métivier in [31].

**Definition 4.1** \(\forall m \in \mathbb{R}\), we say that a symbol \(a \in \Sigma^m\) if and only if \(a\) has the form

\[
a = a^{(m)} + a^{(m-1)}
\]

with

\[
a^{(m)}(t, x, \xi) = F(\nabla f(t, x, \xi)), \\
a^{(m-1)}(t, x, \xi) = \sum_{|\alpha|=2} G_\alpha(\nabla f(t, x, \xi)) \partial_\xi^{\alpha} f(t, x),
\]

such that:

- \(T_a\) maps real-valued functions to real-valued functions;
- \(F \in C^\infty\) is a real-valued function of \((\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)\), and homogeneous of order \(m\) in \(\xi\), with a continuous function \(C = C(\zeta) > 0\) such that \(F(\zeta, \xi) \geq C(\zeta) \|\xi\|^m\) for \(\forall (\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)\);
- \(G_\alpha\) is a \(C^\infty\) complex-valued function of \((\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)\), homogeneous of order \(m-1\) in \(\xi\).

**Definition 4.2** Let \(m \in \mathbb{R}\), and consider two families of operators of order \(m\),

\[
[A(t) : t \in [0, T]], \quad \{B(t) : t \in [0, T]\}.
\]

We shall say that \(A \sim B\) if \(A - B\) is an operator of order \(m-2\) (see Definition A.1) and satisfies the following estimate: for all \(\mu \in \mathbb{R}\), it holds for all \(t \in [0, T]\),

\[
\|A(t) - B(t)\|_{H^\mu \rightarrow H^{\mu-m+2}} \leq C(\|f\|_{W^{4, \infty}}).
\]

The above two definitions are consistent with Definition 4.1 and Definition 4.2 of [2]. We first list some important properties.

**Proposition 4.3** [2] Let \(m, m' \in \mathbb{R}\). Then

1. If \(a \in \Sigma^m\) and \(b \in \Sigma^{m'}\), then \(T_a T_b \sim T_{a\ast b}\) where \(a\ast b \in \Sigma^{m+m'}\) is given by

\[
a\ast b = a^{(m)} b^{(m')} + a^{(m-1)} b^{(m')} + a^{(m)} b^{(m'-1)} + \frac{1}{i} \partial_\xi a^{(m)} \partial_x b^{(m')},
\]

where \(\partial_\xi a^{(m)} \cdot \partial_x b^{(m')} = \sum_{j=1,2} \partial_{x_j} a^{(m)} \partial_{x_j} b^{(m')}\).
(2) If \( a \in \Sigma^m \), then \((T_a)^* \sim T_b\) where \( b \in \Sigma^m \) is given by
\[
b = a^{(m)} + a^{(m-1)} + \frac{1}{i} (\partial_\xi \cdot \partial_x) a^{(m)}.
\]

**Proof** This proposition an imitation of Proposition 4.3 in [2]. From Proposition A.4 with \( \rho = 2 \), we can see that
\[
\|T_a(a(m)T_b(m') - T_a(a(m)b(m') + \frac{1}{i} \partial_\xi a^{(m)}, \partial_x b(m')) H^\mu \mapsto H^{\mu - m - m'} + 2 \| \leq C \| \nabla f \| W^{2,\infty}.
\]

Also, Proposition A.4 applied with \( \rho = 1 \) implies that
\[
\|T_a(a(m)T_b(m' - 1) - T_a(a(m)b(m' - 1)) H^\mu \mapsto H^{\mu - m - m'} + 2 \| \leq C \| \nabla f \| W^{2,\infty},
\]
\[
\|T_a(a(m - 1)T_b(m') - T_a(a(m - 1)b(m')) H^\mu \mapsto H^{\mu - m - m'} + 2 \| \leq C \| \nabla f \| W^{2,\infty}.
\]

Moreover, it follows from Proposition A.2 that
\[
\|T_a(a(m - 1)T_b(m' - 1)) H^\mu \mapsto H^{\mu - m - m'} + 2 \| \leq C \| \nabla f \| W^{1,\infty}.
\]

The desired conclusion of the first point comes from Definition 4.2. Furthermore, it also shows that \( a(m)b \in \Sigma^{m + m'} \).

Similarly, the second point follows from Proposition A.5. \( \square \)

**Corollary 4.4** If \( a \in \Sigma^m \) satisfying
\[
\text{Im} a^{(m-1)} = -\frac{1}{2} (\partial_\xi \cdot \partial_x) a^{(m)},
\]
then \((T_a)^* \sim T_a\).

**Proof** As \( \text{Im} a^{(m-1)} = -\frac{1}{2} (\partial_\xi \cdot \partial_x) a^{(m)} \), we have
\[
a = a^{(m)} + a^{(m-1)} = a^{(m)} + \text{Re} a^{(m-1)} - \frac{i}{2} (\partial_\xi \cdot \partial_x) a^{(m)} = a^{(m)} + a^{(m-1)}
\]
\[
+ \frac{1}{i} (\partial_\xi \cdot \partial_x) a^{(m)}.
\]

The result of this corollary follows from (2) of Proposition 4.3 and the above equality. \( \square \)

We have the following result about the elliptic property of symbols in \( \Sigma^m \).

**Proposition 4.5** [2] Let \( m \in \mathbb{R} \) and let \( \mu \in \mathbb{R} \). Then there exists a function \( C \) such that for all \( a \in \Sigma^m \), \( u \in H^{m+\mu}(\mathbb{T}^2) \), and all \( t \in [0, T] \), we have
\[
\|u\|_{H^{m+\mu}} \leq C(L_0) \left( \|T_a u\|_{H^\mu} + \|u\|_{L^2} \right).
\]

**Proof** This proposition an imitation of Proposition 4.6 in [2]. We represent the proof here for completeness.

Let \( b = 1/a^{(m)} \in \Sigma^{-m} \). By applying Proposition A.4 with \( \rho = 1 \) we can see that \( T_b T_a = \text{Id} + r \), where \( \text{Id} \) is the identity map, \( r \) is of order \(-1\) and satisfies
\[
\|ru\|_{H^{\mu - 1}} \leq C(\|f\|_{W^{2,\infty}}) \|u\|_{H^\mu}.
\]

Then
\[
u = T_b T_a u - ru = T_b T_a u - ru - T_b T_a^{m-1} u.
\]

\( \square \) Springer
Denoting by \( \widetilde{R} = -r - T_b T_{a^{-1}} \), we have
\[
\| \widetilde{R} u \|_{H^{\mu-1}} \leq C(\| f \|_{W^{2,\infty}}) \| u \|_{H^\mu}.
\]
We write
\[
(\text{Id} + \widetilde{R} + \cdots + \widetilde{R}^N)(\text{Id} - \widetilde{R})u = (\text{Id} + \widetilde{R} + \cdots + \widetilde{R}^N)T_b T_a u,
\]
then
\[
u = (\text{Id} + \widetilde{R} + \cdots + \widetilde{R}^N)T_b T_a u + \widetilde{R}^{N+1} u.
\]
Taking \( N + 1 \geq \mu + m \), we obtain that
\[
\| u \|_{H^{\mu+m}} \leq \left\| (\text{Id} + \widetilde{R} + \cdots + \widetilde{R}^N)T_b T_a u \right\|_{H^{\mu+m}} + \left\| \widetilde{R}^{N+1} u \right\|_{H^{\mu+m}}
\leq C(\| f \|_{W^{2,\infty}}) \left( \| T_a u \|_{H^\mu} + \| u \|_{H^{\mu+m-N-1}} \right) \leq C(L_0) \left( \| T_a u \|_{H^\mu} + \| u \|_{L^2} \right).
\]

Next, we show the paralinearization of the Dirichlet-Neumann operator and the mean curvature operator.

**Lemma 4.6** [2] Assume that \((f, \psi) \in H^{s+1}(\mathbb{T}^2) \times H^{s+\frac{1}{2}}(\mathbb{T}^2)\), then
\[
\mathcal{N}_f^+ \psi = T_{\lambda^+} \psi + R_{1+}^+ (f, \psi) + r_{1+}^+ (f, \psi), \quad \mathcal{N}_f^- \psi = T_{\lambda^-} \psi + R_{1-}^- (f, \psi) + r_{1-}^- (f, \psi).
\]
Here the symbols \( \lambda^\pm = \lambda^{\pm(1)} + \lambda^{\pm(0)} \) are given by
\[
\lambda^{-(1)} = \lambda^{+(1)} = \sqrt{1 + |\nabla f|^2} |\xi|^2 - (\nabla f \cdot \xi)^2,
\lambda^{-(0)} = -\lambda^{+(0)} = \frac{1 + |\nabla f|^2}{2\lambda^{-(1)}} \left( \text{div}_x (\alpha^{(1)} \nabla f) + i \partial_\xi \lambda^{-(1)} \cdot \nabla \alpha^{(1)} \right).
\]
with
\[
\alpha^{(1)} = \frac{1}{1 + |\nabla f|^2} (\lambda^{-(1)} + i \nabla f \cdot \xi).
\]
Moreover, we have the estimates
\[
\| R_{1+}^+ (f, \psi) \|_{H^{s-\frac{1}{2}}} + \| R_{1-}^- (f, \psi) \|_{H^{s-\frac{1}{2}}} \leq C(\| f \|_{W^{2,\infty}}, \| \psi \|_{H^1}) \| f \|_{H^{s+\frac{1}{2}}},
\]
\[
\| r_{1+}^+ (f, \psi) \|_{H^{s-\frac{1}{2}}} + \| r_{1-}^- (f, \psi) \|_{H^{s-\frac{1}{2}}} \leq C(\| f \|_{H^{s-\frac{1}{2}}}) \| \nabla \psi \|_{H^{s-2}}.
\]

**Proof** It is well known that the Dirichlet-Neumann operator is an elliptic operator of order 1, and the expression of its principal symbol \( \lambda^{(1)} \) and its subprincipal symbol \( \lambda^{(0)} \) is given in [24]. We claim that the Dirichlet-Neumann operator \( \mathcal{N}_f^- \) can be reformulated as
\[
\mathcal{N}_f^- \psi = T_{\lambda^-} (\psi - T_B f) - T_V \cdot \nabla f + r_{1-}^- (f, \psi),
\]
which satisfies
\[
\| r_{1-}^- (f, \psi) \|_{H^{s-\frac{1}{2}}} \leq C(\| f \|_{H^{s-\frac{1}{2}}}) \| \nabla \psi \|_{H^{s-2}}.
\]
Here
\[
B := \frac{\nabla f \cdot \nabla \psi + \mathcal{N}_f^- \psi}{1 + |\nabla f|^2}, \quad V := \nabla \psi - B \nabla f.
\]
For the proof of the above claim, we refer the readers to Proposition 3.14 of [2].

Then, we let $R^-_1(f, \psi) = -T_{\lambda} - T_B f - T_V \cdot \nabla f$. By using Proposition A.2 with $m = 0, 1$, one can see that

$$
\|T_{\lambda} - T_B f\|_{H^{r-\frac{1}{2}}} \leq C(\|f\|_{W^{2,\infty}}) \|T_B f\|_{H^{r+\frac{1}{2}}} \leq C(\|f\|_{W^{2,\infty}}, \|\psi\|_{H^3}) \|f\|_{H^{r+\frac{1}{2}}},
$$

$$
\|T_V \cdot \nabla f\|_{H^{r-\frac{1}{2}}} \leq C(\|f\|_{H^3}, \|\psi\|_{H^3}) \|\nabla f\|_{H^{r-\frac{1}{2}}} \leq C(\|f\|_{W^{2,\infty}}, \|\psi\|_{H^3}) \|f\|_{H^{r+\frac{1}{2}}},
$$

which means that $\|R^-_1(f, \psi)\|_{H^{r-\frac{1}{2}}} \leq C(\|f\|_{W^{2,\infty}}, \|\psi\|_{H^3}) \|f\|_{H^{r+\frac{1}{2}}}.

The proof for $\mathcal{N}_f^+$ is similar. $\square$

It is easy to check that

$$
\text{Im}\lambda^{-\langle 0 \rangle} = \text{Im}\lambda^{+\langle 0 \rangle} = -\frac{1}{2}(\partial_\xi \cdot \partial_\lambda)\lambda^{-\langle 1 \rangle} = -\frac{1}{2}(\partial_\xi \cdot \partial_\lambda)\lambda^{+\langle 1 \rangle}. \quad (4.2)
$$

See (3.13) of [2]. Then, it follows from Corollary 4.4 that

$$(T_{\lambda}^-)^* \sim T_{\lambda}^-, \quad (T_{\lambda}^+)^* \sim T_{\lambda}^+,$$

which reflects that Dirichlet-Neumann operators are symmetric operators.

**Lemma 4.7** [2] Assume that $f \in H^{s+\frac{1}{2}}(\mathbb{T}^2)$, we shall paralinearize the $H(f) = \text{div}(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}})$ as $H(f) = -T_I f + r_2$, where $l = l^{(2)} + l^{(1)}$ is given by

$$
l^{(2)} = (1 + |\nabla f|^2)^{-\frac{1}{2}}(|\xi|^2 - \frac{(\nabla f \cdot \xi)^2}{1 + |\nabla f|^2}),
$$

$$
l^{(1)} = -i\frac{1}{2}(\partial_\xi \cdot \partial_\lambda)l^{(2)},
$$

and $r_2 \in H^{2s-3}(\mathbb{T}^2)$ satisfying

$$
\|r_2\|_{H^{2s-3}} \leq C(\|f\|_{H^{s+\frac{1}{2}}}). \quad (4.4)
$$

**Proof** See Lemma 3.25 of [2]. $\square$

**Remark 4.8** From the expression of $\lambda^\pm$ and $l$, one can see that $\lambda^\pm \in \Sigma^1$ and $l \in \Sigma^2$, and both of them are elliptic symbols.

Based on the above results, we have

$$
\mathcal{N}_f^+(H(f)) = -T_{\lambda} + T_I f + T_{\lambda}^+ r_2 + R_f^+(f, H(f)) + r^+_I(f, H(f)),
$$

$$
\mathcal{N}_f^-(H(f)) = -T_{\lambda} + T_I f + T_{\lambda}^- r_2 + R_f^-(f, H(f)) + r^-_I(f, H(f)). \quad (4.5)
$$

By using Lemma 4.6 and Lemma 4.7, one can see that

$$
\|T_{\lambda} \pm r_2\|_{H^{r-\frac{1}{2}}} + \|T_{\lambda}^\pm - r_2\|_{H^{r+\frac{1}{2}}} \leq C(\|f\|_{W^{2,\infty}}) \|r_2\|_{H^{r+\frac{1}{2}}} \leq C(\|f\|_{H^{r+\frac{1}{2}}}.
$$

$$
\|R^+_f(f, H(f))\|_{H^{r-\frac{1}{2}}} + \|R^-_f(f, H(f))\|_{H^{r+\frac{1}{2}}} \leq C(\|f\|_{H^3}) \|f\|_{H^{r+\frac{1}{2}}},
$$

$$
\|r^+_I(f, H(f))\|_{H^{r-\frac{1}{2}}} + \|r^-_I(f, H(f))\|_{H^{r+\frac{1}{2}}} \leq C(\|f\|_{H^3}) \|f\|_{H^{r+1}}.
$$

Accordingly, we rewrite the three order term as

$$
\frac{\sigma}{(\rho^+ + \rho^-)^2}(\rho^+ \mathcal{N}_f^+ + \rho^- \mathcal{N}_f^-)H(f) = -\frac{\sigma}{(\rho^+ + \rho^-)^2}T_{\lambda} T_I f + \frac{\sigma}{(\rho^+ + \rho^-)^2}R, \quad (4.6)
$$

\(\square\) Springer
where
\begin{equation}
\lambda = \frac{\rho^+ \lambda^{(1)} + \rho^- \lambda^{-(1)} + \rho^+ \lambda^{(0)} + \rho^- \lambda^{-(0)}}{\lambda^{(1)}} ,
\end{equation}
and
\begin{align*}
R &= \rho^+ (T_{\lambda^+} r_2 + R_1^+ (f, H(f))) + r_1^+ (f, H(f)) + \\
&= \rho^- (T_{\lambda^-} r_2 + R_1^- (f, H(f)) + r_1^- (f, H(f)))
\end{align*}
satisfying
\begin{equation}
\|R\|_{H^{s-\frac{1}{2}}} \leq C (\|f\|_{H^{s+\frac{1}{2}}} + \|f\|_{H^{s+1}}).
\end{equation}
We also remark that,
\begin{align*}
\Re \lambda^{(0)} &= (\rho^- - \rho^+) \Re \lambda^{-(0)}, \\
\Im \lambda^{(0)} &= (\rho^- + \rho^+) \Im \lambda^{-(0)}.
\end{align*}

Next, we symmetrize the above paradifferential operator $T_{\lambda} T_l$.

**Proposition 4.9** Let $q \in \Sigma^0$ and $\gamma \in \Sigma^{\frac{3}{2}}$ be defined as follows:
\begin{align*}
q &= (1 + |\nabla f|^2)^{-\frac{1}{2}}, \\
\gamma &= \sqrt{\lambda^{(1)}} \left( 1 + \frac{1}{2} \sqrt{\lambda^{(1)}} \Re \lambda^{(0)} + \frac{1}{2i} \partial_\xi \cdot \partial_x \lambda^{(1)} \right),
\end{align*}
then $T_q T_{\lambda} T_l \sim T_{\gamma} T_{\gamma} T_q$ and $T_{\gamma} \sim (T_{\gamma})^*$.

**Proof** It is clear that
\[
\Im \gamma^{(\frac{1}{2})} = -\frac{1}{2} (\partial_\xi \cdot \partial_x) \gamma^{(\frac{1}{2})},
\]
then by Corollary 4.4, we have $T_{\gamma} \sim (T_{\gamma})^*$.

From (1) of Proposition 4.3, one can see that proving $T_q T_{\lambda} T_l \sim T_{\gamma} T_{\gamma} T_q$ is equivalent to showing that
\begin{align*}
q \not(\lambda \not l) &= q (\lambda \not l) + \frac{1}{i} \partial_\xi q \cdot \partial_x (l^{(2)} \lambda^{(1)}) \\
&= \gamma^{(\frac{1}{2})} q = (\gamma^{(\frac{1}{2})} \gamma) q + \frac{1}{i} \partial_\xi (\gamma^{\frac{3}{2}} \gamma^{\frac{1}{2}}) \cdot \partial_x q.
\end{align*}
Direct symbolic calculation shows that
\begin{align*}
q \not(\lambda \not l) &= q (\lambda \not l) + \frac{1}{i} \partial_\xi q \cdot \partial_x (l^{(2)} \lambda^{(1)}) \\
&= q \left( l^{(2)} \lambda^{(1)} + l^{(1)} \lambda^{(1)} + l^{(2)} \lambda^{(0)} + \frac{1}{i} \partial_\xi \lambda^{(1)} \cdot \partial_x l^{(2)} \right),
\end{align*}
and
\begin{align*}
(\gamma^{(\frac{1}{2})} \gamma) q &= (\gamma^{(\frac{1}{2})} \gamma) q + \frac{1}{i} \partial_\xi (\gamma^{\frac{3}{2}} \gamma^{\frac{1}{2}}) \cdot \partial_x q \\
&= q \left( (\gamma^{\frac{1}{2}})^2 + 2 \gamma^{\frac{1}{2}} \gamma^{\frac{1}{2}} + \frac{1}{i} \partial_\xi \gamma^{\frac{1}{2}} \cdot \partial_x \gamma^{\frac{1}{2}} \right) + \frac{1}{i} \partial_\xi (\gamma^{\frac{3}{2}} \gamma^{\frac{1}{2}}) \cdot \partial_x q.
\end{align*}
Then we have

\[
\text{Proof} \quad \text{From (1) of Proposition 4.3, we have}
\]

\[
\text{where} \quad \beta \gamma \end{equation}
\]

and (1.1), (4.2), (4.3), and (4.7), we can see that

\[
\text{At the end of this subsection, we present some properties that will be useful in proving}
\]

\[
\text{It is clear that} \quad T \in \text{an elliptic operator, whose commutator with} \quad T \text{is an operator of order}
\]

\[
\text{Therefore, we have}
\]

\[
\text{This completes the proof.}
\]

The choice of \( q \) and \( \gamma \) is inspired by Proposition 4.8 of [2].

Next, we introduce the paradifferential operator \( T_\beta \) with the symbol

\[
\beta := (\gamma^{(\frac{1}{2}))} \frac{2s-1}{2s} \in \Sigma^{s-1/2}.
\]

\textbf{Lemma 4.10} For all \( \mu \in \mathbb{R} \), there exists an non-decreasing function \( C \), such that

\[
\| [T_\beta, T_\gamma] \|_{H^{\mu+1-1}} \leq C(L_1).
\]

\textbf{Proof} From (1) of Proposition 4.3, we have

\[
T_\beta T_\gamma \sim T_{\beta \gamma}, \quad T_\gamma T_\beta \sim T_{\gamma \beta},
\]

where \( \beta \gamma, \gamma \beta \in \Sigma^{s+1} \). It follows from the definition of \( \beta \) that

\[
\beta \gamma = \beta \gamma^{(\frac{1}{2})} + \beta \gamma^{(\frac{1}{2})} + \frac{1}{i} \partial_\xi \beta \cdot \partial_\gamma^{(\frac{1}{2})} = \gamma^{(\frac{1}{2})} \beta + \gamma^{(\frac{1}{2})} \beta + \frac{1}{i} \partial_\xi \gamma^{(\frac{1}{2})} \cdot \partial_\gamma \beta = \gamma \beta.
\]

Therefore, we have \( T_\beta T_\gamma \sim T_\gamma T_\beta \), and \( T_\beta T_\gamma - T_\gamma T_\beta \) is an operator of order \( s - 1 \).
From the definition of the paradifferential operator $T_a$, one can easily see that

$$[\partial_t, T_a]f = T_{\partial_t}a f, \quad [\partial_i, T_a]f = T_{\partial_i}a f.$$  

The following estimates hold.

**Lemma 4.11** For all $\mu \in \mathbb{R}$, it holds that

$$\|T_{\partial_t}a\|_{H^\mu \to H^\mu} + \|T_{\partial_t}b\|_{H^\mu \to H^\mu + \frac{1}{2}} \leq C(L_0),$$

$$\|T_{\partial_t}a\|_{H^\mu \to H^\mu} + \|T_{\partial_t}b\|_{H^\mu \to H^\mu + \frac{1}{2}} \leq C(L_1, L_2),$$

$$\|T_{\partial_t}a\|_{H^\mu \to H^{\mu - \frac{3}{2}}} + \|T_{\partial_t}b\|_{H^\mu \to H^{\mu - \frac{3}{2}}} \leq C(L_1).$$

**Proof** Recalling the expression of $q$, $\beta$, and $\gamma$, one can see that $q \in \Gamma_0^0$, $\beta \in \Gamma_0^{s-\frac{1}{2}}$, and $\gamma \in \Gamma_0^{\frac{3}{2}}$. The definition of $\Gamma_0^{s-\frac{1}{2}}$ is given in Definition A.1. In other word, $T_q, T_\beta, T_\gamma$ are operators of order $0, s - \frac{1}{2}$, and $\frac{3}{2}$ respectively. So what we should do is to estimate the seminorm of these symbols. We only give the estimate for the most complex one, and the other ones can be treated in the same way.

With the help of Proposition A.2 and Definition A.1, we have

$$\|T_{\partial_t}a\|_{H^\mu \to H^{\mu - \frac{3}{2}}} \leq C M_0 \sup_{\|\alpha\| \leq 4, \|\xi\| \geq \frac{1}{2}} \sup_{\|\gamma\| \leq 4} \|T_{\partial_t}a\|_{L^\infty}.$$  

From (4.10), we can see that $\gamma^{\frac{1}{2}}$ and $\gamma^{\frac{3}{2}}$ are polynomials of $\xi$ with coefficients defined on $\nabla f$ and $\nabla^2 f$. Moreover, taking derivative of $\gamma$ does not generate singularity, because $(1 + |\nabla f|^2)$ and $\sqrt{(1 + |\nabla f|^2)|\xi|^2 - (\nabla f \cdot \xi)^2}$ will not approach to 0 for $|\xi| \geq \frac{1}{2}$. Therefore, one can easily check that

$$M_0 \sup_{\|\alpha\| \leq 4, \|\xi\| \geq \frac{1}{2}} \sup_{\|\gamma\| \leq 4} \|T_{\partial_t}a\|_{L^\infty} \leq C \left( \|\nabla f\|_{L^\infty}, \|\nabla^2 f\|_{L^\infty}, \|\nabla^2 \partial_t f\|_{L^\infty}, \|\nabla^2 \partial_t f\|_{L^\infty} \right).$$

Here the most troublesome term is $\|\nabla^2 \partial_t^2 f\|_{L^\infty}$. For $s \geq 6$, we have

$$\|\nabla^2 \partial_t^2 f\|_{L^\infty} \leq \|\nabla^2 \partial_t (u^\pm \cdot N)\|_{L^\infty} \leq C \left( \|\partial_t u^\pm\|_{H^{s-\frac{3}{2}}}, \|u^\pm\|_{H^{s+\frac{3}{2}}} \right) \leq C(L_1, L_2).$$

This gives the estimate of $\|T_{\partial_t}a\|_{H^{\mu - \frac{3}{2}}}$, and the other estimates can be obtained in the same way. \hfill \square

**Lemma 4.12** For all function $a \in H^{s-\frac{1}{2}}(\mathbb{T}^2)$ and $\psi \in H^{s-\frac{3}{2}}(\mathbb{T}^2)$, it holds that

$$\|T_{\beta}[T_q, a]\psi\|_{L^2} \leq C(L_1) \|a\|_{H^{s-\frac{3}{2}}} \|\psi\|_{H^{s-\frac{1}{2}}},$$

$$\|[T_{\beta}, a]T_q\psi\|_{L^2} \leq C(L_1) \|a\|_{H^{s-\frac{3}{2}}} \|\psi\|_{H^{s-\frac{1}{2}}},$$

$$\|[T_{\gamma}, a]\psi\|_{L^2} \leq C(L_1) \|a\|_{H^{s-\frac{3}{2}}} \|\psi\|_{H^{s-\frac{1}{2}}}.$$  

**Proof** From Proposition A.2, we have

$$\|T_{\beta}[T_q, a]\psi\|_{L^2} \leq C(L_1) \|[T_q, a]\psi\|_{H^{s-\frac{3}{2}}}.$$
By using Bony’s decomposition, we rewrite $[T_q, a] \psi$ as

$$
[T_q, a] \psi = T_q(a \psi) - a T_q \psi = T_q T_a \psi + T_q T_\gamma a + T_q R_B(a, \psi) - T_\gamma T_q a - R_B(a, T_\gamma \psi).
$$

We start with the estimate of $[T_q, T_a] \psi$. By taking $\rho = 1$ in Proposition A.4, we have $a_2 q = q a_2 = a q$ as both $a$ and $q$ are functions, and then

$$
\|T_a T_q - T_q T_a\|_{H^{s-\frac{3}{2}} \to H^{s-\frac{3}{2}}} \leq \|T_a T_q - T_q T_a\|_{H^{s-\frac{3}{2}} \to H^{s-\frac{3}{2}}} + \|T_q T_a - T_a q\|_{H^{s-\frac{3}{2}} \to H^{s-\frac{3}{2}}} \leq CM_1^2(q) M_1^0(a) \leq C(\|f\|_{W^{2, \infty}}) \|a\|_{W^{1, \infty}}.
$$

It follows that

$$
\|[T_q, T_a] \psi\|_{H^{s-\frac{1}{2}}} \leq C(\|f\|_{W^{2, \infty}}) \|a\|_{W^{1, \infty}} \|\psi\|_{H^{s-\frac{3}{2}}} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{s-\frac{3}{2}}}.
$$

With the help of Proposition A.2 and Lemma A.3, one can deduce that

$$
\|T_q T_\gamma a\|_{H^{s-\frac{1}{2}}} \leq C(\|f\|_{W^{1, \infty}}) \|\psi\|_{L^\infty} \|a\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{s-\frac{3}{2}}},
$$

$$
\|T_\gamma T_q a\|_{H^{s-\frac{1}{2}}} \leq C \|T_q \psi\|_{L^\infty} \|a\|_{H^{s-\frac{1}{2}}} \leq C \|T_q \psi\|_{H^2} \|a\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{s-\frac{3}{2}}},
$$

$$
\|T_\gamma R_B(a, \psi)\|_{H^{s-\frac{1}{2}}} \leq C(\|f\|_{W^{1, \infty}}) \|R_B(a, \psi)\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{s-\frac{3}{2}}},
$$

$$
\|R_B(a, T_\gamma \psi)\|_{H^{s-\frac{1}{2}}} \leq C(\|f\|_{W^{1, \infty}}) \|R_B(a, T_\gamma \psi)\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{s-\frac{3}{2}}}.
$$

Combining the above estimates yields

$$
\|T_B [T_q, a] \psi\|_{L^2} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{s-\frac{1}{2}}}.
$$

Similarly, we rewrite $[T_\gamma, a] \psi$ as

$$
[T_\gamma, a] \psi = [T_\gamma, T_a] \psi + T_\gamma T_q a - T_\gamma T_q a + T_\gamma R_B(a, \psi) - R_B(a, T_\gamma \psi).
$$

Taking $\rho = 1$ Proposition A.4 we have $\gamma_2 a = \gamma a = a_2 \gamma$, and

$$
\|[T_\gamma, T_a] \psi\|_{L^2} \leq \|(T_\gamma T_a - T_\gamma T_q a) \psi\|_{L^2} + \|(T_q T_\gamma a - T_\gamma T_q a) \psi\|_{L^2} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{s-\frac{3}{2}}}.
$$

By using Lemma A.6 with $m = \frac{1}{2}$, we have

$$
\|T_\gamma \psi a\|_{L^2} \leq C(L_1) \|T_\psi a\|_{H^{\frac{s}{2}}} \leq C(L_1) \|\psi\|_{H^{s-\frac{1}{2}}} \|a\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{\frac{s}{2}}}.
$$

Then by using Lemma A.6 with $m = 2$, we have

$$
\|T_\gamma \psi a\|_{L^2} \leq C \|T_\gamma \psi\|_{H^{-1}} \|a\|_{H^2} \leq C(L_1) \|\psi\|_{H^{\frac{s}{2}}} \|a\|_{H^2} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{\frac{s}{2}}}.
$$

From Proposition A.2, Lemma A.3, and Lemma A.6 that

$$
\|T_\gamma R_B(a, \psi)\|_{L^2} \leq C(L_1) \|R_B(a, \psi)\|_{H^{\frac{s}{2}}} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{\frac{s}{2}}},
$$

$$
\|R_B(a, T_\gamma \psi)\|_{L^2} \leq C \|R_B(a, T_\gamma \psi)\|_{H^1} \leq C \|a\|_{H^{s-\frac{1}{2}}} \|T_\gamma \psi\|_{H^1} \leq C(L_1) \|a\|_{H^{s-\frac{1}{2}}} \|\psi\|_{H^{\frac{s}{2}}}.
$$
As a conclusion, it holds that

\[ \| [ T_\gamma, a ] \psi \|_{L^2} \leq C(L_1) \| a \|_{H^{1-\frac{1}{2}}} \| \psi \|_{H^{1/2}}. \]

The estimate of \( [ T_\beta, a ] T_q \psi \|_{L^2} \) can be proved in a similar way. \( \Box \)

### 4.2 Linearized System of \((f, \theta)\)

In this subsection, we linearize the system of \((f, \theta)\), and give the energy estimates. From (3.6) and (4.6), we derive the following linearized system for \((\bar{f}, \bar{\theta})\):

\[
\partial_t \bar{f} = \bar{\theta},
\]

\[
\partial_t \bar{\theta} = -\frac{\sigma}{(\rho^+ + \rho^-)^2} (T_\lambda T_1 \bar{f})
- \frac{2}{\rho^+ + \rho^-} ((\rho^+ u^+_1 + \rho^- u^-_1) \partial_1 \bar{\theta} + (\rho^+ u^+_2 + \rho^- u^-_2) \partial_2 \bar{\theta})
- \frac{1}{\rho^+ + \rho^-} \sum_{i,j=1,2} (\rho^+ u^+_i u^+_j - h^+_i h^+_j + \rho^- u^-_i u^-_j - h^-_i h^-_j) \partial_i \partial_j \bar{f}
+ g + \frac{\sigma}{(\rho^+ + \rho^-)^2} R,
\]

where

\[
g = -\frac{\sigma}{(\rho^+ + \rho^-)^2} (N^+_f - N^-_f) \bar{N}^{-1}_f (N^+_f - N^-_f) H(f)
+ \frac{2}{\rho^+ + \rho^-} (N^+_f - N^-_f) \bar{N}^{-1}_f \mathcal{P}((\rho^+ u^+_1 + \rho^- u^-_1) \partial_1 \theta + (\rho^+ u^+_2 + \rho^- u^-_2) \partial_2 \theta)
+ \frac{1}{\rho^+ + \rho^-} (N^+_f - N^-_f) \bar{N}^{-1}_f \mathcal{P} \left( \sum_{i,j=1,2} (u^+_i u^+_j - \frac{1}{\rho^+} h^+_i h^+_j - u^-_i u^-_j) \right)
+ \frac{1}{\rho^-} h^-_i h^-_j \partial_i \partial_j f
- \frac{1}{\rho^+ + \rho^-} \mathbf{N} \cdot \left( \nabla (\rho^+ p_{u^+} u^+ - p_{h^+} h^+) + \nabla (\rho^- p_{u^-} u^- - p_{h^-} h^-) \right)
+ \frac{1}{\rho^+ + \rho^-} \mathcal{P} \mathbf{N} \cdot \left( \nabla (\rho^+ p_{u^+} u^+ - p_{h^+} h^+) \right)
- \mathcal{P} \mathbf{N} \cdot \left( \nabla (\rho^- p_{u^-} u^- - p_{h^-} h^-) \right)
= g_1 + g_2 + g_3 + g_4 + g_5,
\]

and

\[
R = \rho^+ (T^+_{\lambda} r_2 + T^+_1 (f, H(f)) + r^+_1 (f, H(f)))
+ \rho^- (T^-_{\lambda} r_2 + T^-_1 (f, H(f)) + r^-_1 (f, H(f)) ) .
\]

We emphasize that all the paradifferential operators \( T_{\lambda}^\pm, T_1 \), and the remainders \( r^\pm_1, \ R^\pm_1, \ r^\pm_2 \) here are defined by the given function \( f \).
Defining \( \mathbf{w}_i = \frac{1}{\rho^+ + \rho^-} (\rho^+ u^+_i + \rho^- u^-_i) \), and \( v_i = \frac{\sqrt{\rho^+ \rho^-}}{\rho^+ + \rho^-} (u^+_i - u^-_i) \), we rewrite the linearized system as:

\[
\begin{align*}
\partial_t^2 \ddot{f} &= -\frac{\sigma}{(\rho^+ + \rho^-)^2} (T_\gamma T_\beta \ddot{f}) - 2 \sum_{i,j=1,2} \mathbf{w}_i \partial_i \partial_j \ddot{f} \\
&\quad + \left( -\mathbf{w}_j \mathbf{w}_j - v_i v_j + \frac{1}{\rho^+ + \rho^-} (h^+_i h^+_j + h^-_i h^-_j) \right) \partial_i \partial_j \ddot{f} + g + \frac{\sigma}{(\rho^+ + \rho^-)^2} R.
\end{align*}
\]

(4.12)

We remark that \( \int_{\Sigma_2} \partial_t \ddot{f} d\mathbf{x}' \) may not vanish since we have performed the linearization.

Let us introduce the energy functions \( E_1, E_2, E_1, \) and \( E_2 \):

\[
E_1 = \frac{1}{2(\rho^+ + \rho^-)^2} \| T_\gamma T_\beta T_q \ddot{f} \|_{L^2}^2,
\]

\[
E_2 = \| (\partial_t + \mathbf{w}_i \partial_i) T_\beta T_q \ddot{f} \|_{L^2}^2 - \frac{1}{2} \| v_i \partial_i T_\beta T_q \ddot{f} \|_{L^2}^2 + \frac{1}{2(\rho^+ + \rho^-)} (\| h^+_i \partial_i T_\beta T_q \ddot{f} \|_{L^2}^2
\]

\[
+ \| h^-_i \partial_i T_\beta T_q \ddot{f} \|_{L^2}^2),
\]

and

\[
E_1(t) = \| \ddot{f} \|_{H^{r+1}}^2, \quad E_2(t) = \| \ddot{f} \|_{H^{r+1}}^2 + \| \partial_t \ddot{f} \|_{H^{r-1/2}}^2.
\]

By Proposition A.2 and Lemma 4.12, one can easily see that

\[
E_1 = \frac{1}{4} \| T_\gamma T_\beta T_q \ddot{f} \|_{L^2}^2 \leq C(L_0) \| T_\beta T_q \ddot{f} \|_{H^r}^2 \leq C(L_0) \| T_\beta T_q \ddot{f} \|_{H^{r+1}} \leq C(L_0) \| \ddot{f} \|_{H^{r+1}},
\]

\[
\| (\partial_t + \mathbf{w}_i \partial_i) T_\beta T_q \ddot{f} \|_{L^2}^2 \leq C(\| T_\beta T_q \ddot{f} \|_{L^2} + \| T_\beta T_q \ddot{f} \|_{L^2} + \| \mathbf{u} \|_{W^{1.\infty}}(t), \| f \|_{W^{2,\infty}}(\| \ddot{f} \|_{H^{r+1}} + \| \partial_t \ddot{f} \|_{H^{r-1/2}})
\]

\[
\leq C(\| \mathbf{u} \|_{W^{1.\infty}}(t), \| f \|_{W^{2,\infty}}(\| \ddot{f} \|_{H^{r+1}} + \| \partial_t \ddot{f} \|_{H^{r-1/2}})
\]

\[
\leq C(L_0)(\| \ddot{f} \|_{H^{r+1}} + \| \partial_t \ddot{f} \|_{H^{r-1/2}}),
\]

\[
\frac{1}{2} \| v_i \partial_i T_\beta T_q \ddot{f} \|_{L^2}^2 + \frac{1}{2(\rho^+ + \rho^-)} (\| h^+_i \partial_i T_\beta T_q \ddot{f} \|_{L^2}^2
\]

\[
+ \| h^-_i \partial_i T_\beta T_q \ddot{f} \|_{L^2}^2 \leq C(L_0) \| \ddot{f} \|_{H^{r+1}}^2,
\]

which means that

\[
\sigma E_1 + E_2 \leq C(L_0)(\sigma E_1 + E_2).
\]

(4.13)

On the other hand, as \( T_\gamma, T_\beta, \) and \( T_q \) are all elliptic operators with \( \gamma \in \Sigma^{1,1}, \beta \in \Sigma^{s-\frac{1}{2}}, \) and \( q \in \Sigma^0 \). Then, it follows from Proposition 4.5, Proposition A.2, and the Gagliardo-Nirenberg interpolation inequality that

\[
\sigma \| \ddot{f} \|_{H^{r+1}} \leq C(L_0) \sigma (\| T_\beta T_q \ddot{f} \|_{H^{r+1}} + \| \ddot{f} \|_{L^2}^2)
\]

\[
\leq C(L_0) \sigma (\| T_\beta T_q \ddot{f} \|_{H^{r+1}} + \| \ddot{f} \|_{L^2}^2) \leq C(L_0) \sigma (\| T_\beta T_q \ddot{f} \|_{H^{r+1}} + \| \ddot{f} \|_{L^2}^2),
\]

\[
\| \ddot{f} \|_{H^{r+1}}^2 \leq C(S) \| \ddot{f} \|_{H^{r+1}} + C(S) \| \ddot{f} \|_{L^2}^2 \leq C(L_0)(\sigma \| T_\beta T_q \ddot{f} \|_{L^2}^2 + C(\sigma) \| \ddot{f} \|_{L^2}^2),
\]

\[
\| \partial_t \ddot{f} \|_{H^{r-1/2}}^2 \leq C(L_0)(\| T_\beta T_q \partial_t \ddot{f} \|_{L^2}^2 + \| \partial_t \ddot{f} \|_{L^2}^2)
\]

\[
\square \text{ Springer}
\]
\[
\begin{align*}
&\leq C(L_0)( \| \partial_t + w_i \partial_i \| T \| T \dot{f} \|_{L^2}^2 + \| \ddot{f} \|_{H^{s+1}}^2 + \| \partial_t \ddot{f} \|_{L^2}^2) \\
&\leq C(L_0)( \| \partial_t + w_i \partial_i \| T \| T \dot{f} \|_{L^2}^2 + \| \ddot{f} \|_{L^2}^2 + \sigma \| T \| T \ddot{f} \|_{L^2}^2 \\
&\quad + C(\sigma) \| \dddot{f} \|_{L^2}^2 + \| \partial_t \dddot{f} \|_{L^2}^2). \\
\end{align*}
\]

Thus
\[
\sigma E_1 + E_2 \leq C(L_0, \sigma)(\sigma E_1 + E_2 + \| \dddot{f} \|_{L^2}^2 + \| \partial_t \dddot{f} \|_{L^2}^2).
\]

\textbf{Remark 4.13} Here we add an extra \( C(\sigma) \| \dddot{f} \|_{L^2}^2 \) to ensure that
\[
\sigma \| T \dddot{f} \|_{L^2}^2 + C(\sigma) \| \dddot{f} \|_{L^2}^2 \geq \| \dddot{f} \|_{H^{s+\frac{1}{2}}}^2.
\]

Here \( C(\sigma) \) will get larger as \( \sigma \) gets smaller, which leads to the above estimates depending on \( \sigma \). If the stability condition (1.10) holds, we have
\[
- \frac{1}{2} \| \partial_t \dddot{f} \|_{L^2}^2 + \frac{1}{2}(\rho^+ + \rho^-)(\| \dddot{f} \|_{L^2}^2 + \| \partial_t \dddot{f} \|_{L^2}^2) \geq C(c_0) \| \dddot{f} \|_{L^2}^2.
\]

In this case, we no longer need to introduce \( C(\sigma) \), and the energy estimate will not depend on \( \sigma \). We will discuss this kind of problems in Sect. 6.

Based on these energy functions, we have the following energy estimate.

\textbf{Proposition 4.14} Assume \( s \geq 6 \), given initial data \((\tilde{\theta}_0, f_0) \in H^{s-\frac{1}{2}} \times H^{s+1}(\mathbb{T}^2)\), there exists a unique solution \((\tilde{\theta}, \tilde{f}) \in C\left([0, T]; H^{s-\frac{1}{2}} \times H^{s+1}(\mathbb{T}^2)\right)\) to the system (4.11) from \((\tilde{\theta}_0, f_0)\) so that
\[
\sup_{t \in [0, T]} (\| \partial_t \ddot{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \ddot{f} \|_{H^{s+\frac{1}{2}}}^2 + \sigma \| \dddot{f} \|_{H^{s+1}}^2) \leq C(L_0, L_1, L_2, \sigma, L_1, L_2, T). \\
\int_0^T \| g \|_{H^{s-\frac{1}{2}}}^2 + \| R \|_{H^{s-\frac{1}{2}}}^2 d\tau \leq C(L_1, L_2, T).
\]

\textbf{Proof} We only present the uniform estimates, which ensure the existence and uniqueness of the solution. For convenience, we put all the terms can be bounded by \( C(L_0, L_1, L_2) \| \ddot{f} \|_{H^{s+1}}^2 \) in \( R_1 \), and terms that can be bounded by \( C(L_0, L_1, L_2) \| \ddot{f} \|_{H^{s+\frac{1}{2}}}^2 + \| \partial_t \ddot{f} \|_{H^{s-\frac{1}{2}}}^2 \) in \( R_2 \). We start the energy estimates from \( \frac{1}{2} \partial_t \langle (\partial_t + w_i \partial_i) T \dot{f} \rangle, (\partial_t + w_i \partial_i) T \dot{f} \rangle = \langle \partial_t \dot{f} \dot{f} \rangle, (\partial_t + w_i \partial_i) T \dot{f} \rangle + \langle w_i \partial_i \dot{f} \ddot{f} \rangle, (\partial_t + w_i \partial_i) T \dot{f} \rangle + \langle (\partial_t w_i \partial_i) T \dot{f} \rangle, (\partial_t + w_i \partial_i) T \dot{f} \rangle \]
\[
\def \textstyle \sum \limits \end{align*}
\]

It follows from Proposition A.2 that
\[
III \leq C(L_1, L_2) \| \ddot{f} \|_{L^2}^2 + \| \partial_t \ddot{f} \|_{L^2}^2.
\]
From (4.12), we deduce by using Lemma 4.11 and Proposition A.2 that

\[ I = \langle T_{\beta} T_{q} \sigma_{j} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f} \rangle + \langle T_{\partial_{t} \beta} T_{q} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f} \rangle \\
+ 2 \langle T_{\beta} T_{q} \partial_{t} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f} \rangle \\
= (T_{\beta} T_{q} \sigma_{j} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f}) + R_{2} \\
= - \frac{\sigma}{(\rho^{+} + \rho^{-})^{2}} (T_{\beta} T_{q} T_{i} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f}) \\
+ \langle T_{\beta} T_{q} (-2w_{i} \partial_{t} \bar{f}), (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f} \rangle \\
+ \langle T_{\beta} T_{q} \{(-w_{i}w_{j} - v_{i}v_{j} + \frac{1}{\rho^{+} + \rho^{-}}(h_{i}^{+}h_{j}^{+} + h_{i}^{-}h_{j}^{-})) \partial_{t} \partial_{j} \bar{f}), (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f} \rangle \\
+ \langle T_{\beta} T_{q} (\theta + \frac{\sigma}{(\rho^{+} + \rho^{-})^{2}} R), (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f} \rangle + R_{2} \\
\text{def} I_{1} + I_{2} + I_{3} + I_{4}.

With the help of Proposition 4.9, Lemma 4.10, and Lemma 4.12, we have

\[ I_{1} = - \frac{\sigma}{(\rho^{+} + \rho^{-})^{2}} (T_{\beta} T_{q} T_{i} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f}) \\
= - \frac{\sigma}{(\rho^{+} + \rho^{-})^{2}} (T_{\gamma}^{*} T_{\gamma} T_{\beta} T_{q} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f}) \\
\quad - \frac{\sigma}{(\rho^{+} + \rho^{-})^{2}} ((T_{\gamma} - (T_{\gamma})^{*}) T_{\beta} T_{\gamma} + T_{\gamma} T_{\beta} T_{\gamma}) \\
\quad + [T_{\beta}, T_{\gamma}] T_{\beta} T_{q} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f}) \\
\quad - \frac{\sigma}{(\rho^{+} + \rho^{-})^{2}} (T_{\beta} (T_{q} T_{\beta} T_{q} - T_{\gamma} T_{\gamma} T_{q}) \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f}) \\
\quad - \frac{\sigma}{(\rho^{+} + \rho^{-})^{2}} (T_{\gamma} T_{\beta} T_{q} \bar{f}, T_{\gamma} T_{\beta} T_{q} \bar{f} + R_{2}) \\
\quad + \frac{\sigma}{2(\rho^{+} + \rho^{-})^{2}} (T_{\gamma} T_{\beta} T_{q} \bar{f}, (\partial_{t} w_{i}) T_{\gamma} T_{\beta} T_{q} \bar{f}) + R_{2} \\
\quad - \frac{\sigma}{2(\rho^{+} + \rho^{-})^{2}} \partial_{t} \langle T_{\gamma} T_{\beta} T_{q} \bar{f}, T_{\gamma} T_{\beta} T_{q} \bar{f} \rangle + \sigma R_{1} + R_{2}.

Similarly, it follows from Proposition 4.9, Lemma 4.11, and Lemma 4.12 that

\[ I_{2} = - 2 \langle T_{\beta} T_{q} w_{i} \partial_{t} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f} \rangle \\
= - 2 \langle w_{i} \partial_{t} \partial_{t} T_{\beta} T_{q} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f} \rangle + R_{2}.

Therefore, we have

\[ I_{2} + II = - \langle w_{i} \partial_{t} \partial_{t} T_{\beta} T_{q} \bar{f}, (\partial_{t} + w_{i} \partial_{t}) T_{\beta} T_{q} \bar{f} \rangle + R_{2}, \\
= - \langle w_{i} \partial_{t} \partial_{t} T_{\beta} T_{q} \bar{f}, w_{i} \partial_{t} T_{\beta} T_{q} \bar{f} \rangle - \langle w_{i} \partial_{t} \partial_{t} T_{\beta} T_{q} \bar{f}, \partial_{t} T_{\beta} T_{q} \bar{f} \rangle + R_{2} \\
= - \langle \partial_{t} (w_{i} \partial_{t} T_{\beta} T_{q} \bar{f}), w_{i} \partial_{t} T_{\beta} T_{q} \bar{f} \rangle + \frac{1}{2} \langle \partial_{t} w_{i} \partial_{t} T_{\beta} T_{q} \bar{f} \rangle + R_{2} \\
+ \langle \partial_{t} w_{i} \partial_{t} T_{\beta} T_{q} \bar{f}, \partial_{t} T_{\beta} T_{q} \bar{f} \rangle + R_{2}.
By using Proposition A.11, Proposition A.12, and classical elliptic estimate, we have

\[ -\frac{1}{2} \frac{d}{dt} \|w_i \partial_t T_q \tilde{f}\|_{L^2}^2 + R_2. \]

For the same reason, let \( a_i = w_i, v_i, h_i^+, h_i^- \), one can deduce that

\[
\begin{align*}
&T_q a_i a_j \partial_t \partial_j \tilde{f}, (\partial_t + w_k \partial_k) T_q \tilde{f}) \\
&= \langle a_i a_j \partial_t \partial_j T_q \tilde{f}, (\partial_t + w_k \partial_k) T_q \tilde{f} \rangle + R_2 \\
&= -\langle a_j \partial_j T_q \tilde{f}, a_i \partial_t (\partial_t + w_k \partial_k) T_q \tilde{f} \rangle + R_2 \\
&= -\langle a_j \partial_j T_q \tilde{f}, \partial_t (a_i \partial_t T_q \tilde{f}) \rangle - \langle a_j \partial_j T_q \tilde{f}, w_k \partial_k (a_i \partial_t T_q \tilde{f}) \rangle + R_2 \\
&= -\frac{1}{2} \frac{d}{dt} \|a_i \partial_t T_q \tilde{f}\|_{L^2}^2 + R_2.
\end{align*}
\]

It follows that

\[
I_3 = \frac{1}{2} \frac{d}{dt} \left( \|w_i \partial_t T_q \tilde{f}\|_{L^2}^2 + \|v_i \partial_t T_q \tilde{f}\|_{L^2}^2 - \frac{1}{\rho^+ + \rho^-} (\|h_i^+ \partial_t T_q \tilde{f}\|_{L^2}^2 + \|h_i^- \partial_t T_q \tilde{f}\|_{L^2}^2) \right) + R_2.
\]

And obviously, it holds that

\[
I_4 \leq \|g\|_{H^{s-\frac{1}{2}}}^2 + \sigma^2 \|R\|_{H^{s-\frac{1}{2}}}^2 + C(L_1, L_2)(\|\tilde{f}\|_{H^{s+\frac{1}{2}}}^2 + \|\partial_t \tilde{f}\|_{H^{s-\frac{1}{2}}}^2).
\]

Putting the above estimates together, we arrive at

\[
\frac{d}{dt} (\sigma E_1(t) + E_2(t)) \leq C(L_1, L_2)(\sigma E_1(t) + E_2(t)) + \|g\|_{H^{s-\frac{1}{2}}}^2 + \sigma^2 \|R\|_{H^{s-\frac{1}{2}}}^2. \tag{4.15}
\]

It is easily seen that

\[
\frac{d}{dt} \left( C(\sigma) \|\tilde{f}\|_{L^2}^2 + \|\partial_t \tilde{f}\|_{L^2}^2 \right) \leq C(\sigma, L_0)(\|\tilde{f}\|_{H^{s+\frac{1}{2}}}^2 + \|\partial_t \tilde{f}\|_{H^{s-\frac{1}{2}}}^2) + \|g\|_{L^2}^2 + \sigma^2 \|R\|_{L^2}^2.
\]

Thus we get by (4.15), (4.13), and (4.14) that

\[
\sup_{\tau \in [0, T]} \{\sigma E_1(\tau) + E_2(\tau)\} \leq C(\sigma, L_0) \left\{ \sigma \|\tilde{f}_0\|_{H^{s+\frac{1}{2}}}^2 + \|\tilde{f}_0\|_{H^{s+\frac{1}{2}}}^2 + \|\tilde{f}_0\|_{H^{s-\frac{1}{2}}}^2 \\
+ \int_0^T \|g\|_{H^{s-\frac{1}{2}}}^2 + \sigma^2 \|R\|_{H^{s-\frac{1}{2}}}^2 d\tau \\
+ C(L_1, L_2) \int_0^T \sigma E_1(\tau) + E_2(\tau)d\tau \right\}.
\]

One can get the desired estimates by Gronwall’s inequality. □

**Lemma 4.15** It holds that

\[
\|g\|_{H^{s-\frac{1}{2}}}^2 + \sigma^2 \|R\|_{H^{s-\frac{1}{2}}}^2 \leq C(L_1).
\]

**Proof** It follows from (4.8) that

\[
\sigma \|R\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \sigma \|f\|_{H^{s+\frac{1}{2}}} \leq C(L_1).
\]

By using Proposition A.11, Proposition A.12, and classical elliptic estimate, we have

\[
\|g\|_{H^{s-\frac{1}{2}}} \leq \sigma C(\|f\|_{H^{s+\frac{1}{2}}}^2) \|N_f^{-1}(N_f^+ - N_f^-) H(f)\|_{H^{s-\frac{1}{2}}}.
\]
\[ g_2 \leq C(L_1) \| \frac{u^\pm}{H^{\frac{1}{2}}} \|_{H^{-\frac{1}{2}}} \| \theta \|_{H^{-\frac{1}{2}}} \leq C(L_1), \]
\[ g_3 \leq C(L_1) \| \frac{u^\pm + h^\pm}{H^{\frac{1}{2}}} \|_{H^{-\frac{1}{2}}} \| \frac{u^\pm}{H^{\frac{1}{2}}} \|_{H^{\frac{1}{2}}} \leq C(L_1), \]
\[ (g_4, g_5) \leq C(L_1) \| \frac{\nabla (p u^+, u^+ - p h^+, h^+)}{H^{\frac{1}{2}}} + \| \nabla (p u^-, u^- - p h^-, h^-) \|_{H^{-\frac{1}{2}}} \]
\[ \leq C(L_1) \| \nabla (p u^+, u^+ - p h^+, h^+) \|_{H^1(\Omega_1^+)} + \| \nabla (p u^-, u^- - p h^-, h^-) \|_{H^1(\Omega_1^-)} \]
\[ \leq C(L_1) \| (u^\pm, h^\pm) \|_{H^1(\Omega^\pm)} \leq C(L_1). \]

This ends the proof. \[ \square \]

### 4.3 The Linearized System of \((\omega, \xi)\)

In this subsection, we show the existence of solution of system (3.8), and give the energy estimates. These results are given in Section 4.2 of [28]. We represent the proof here for completeness.

From (3.8), we introduce the following linearized system:

\[
\begin{aligned}
\begin{cases}
\partial_t \tilde{\omega}^\pm + u^\pm \cdot \nabla \tilde{\omega}^\pm - \frac{1}{\rho^\pm} h^\pm \cdot \nabla \tilde{\xi}^\pm = \tilde{\omega}^\pm \cdot \nabla u^\pm - \frac{1}{\rho^\pm} \tilde{\xi}^\pm \cdot \nabla h^\pm, \\
\partial_t \tilde{\xi}^\pm + u^\pm \cdot \nabla \tilde{\xi}^\pm - h^\pm \cdot \nabla \tilde{\omega}^\pm = \tilde{\xi}^\pm \cdot \nabla u^\pm - \tilde{\omega}^\pm \cdot \nabla h^\pm - 2 \sum_{i=1}^3 \nabla u^\pm_i \times \nabla h^\pm_i.
\end{cases}
\end{aligned}
\]

(4.16)

which gives

\[
\partial_t (\sqrt{\rho^\pm} \tilde{\omega}^\pm + \tilde{\xi}^\pm) + u^\pm \cdot \nabla (\sqrt{\rho^\pm} \tilde{\omega}^\pm + \tilde{\xi}^\pm) - \frac{1}{\sqrt{\rho^\pm}} h^\pm \cdot \nabla (\sqrt{\rho^\pm} \tilde{\omega}^\pm + \tilde{\xi}^\pm)
\]

\[ = (\sqrt{\rho^\pm} \tilde{\omega}^\pm + \tilde{\xi}^\pm) \cdot \nabla u^\pm - \frac{1}{\sqrt{\rho^\pm}} (\sqrt{\rho^\pm} \tilde{\omega}^\pm + \tilde{\xi}^\pm) \cdot \nabla h^\pm - 2 \sum_{i=1}^3 \nabla u^\pm_i \times \nabla h^\pm_i. \]

Therefore, we introduce \(\omega^\pm = \sqrt{\rho^\pm} \tilde{\omega}^\pm + \tilde{\xi}^\pm\) which satisfies

\[
\partial_t \omega^\pm + (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm) \cdot \nabla \omega^\pm = \omega^\pm \cdot \nabla (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm) - 2 \nabla u^\pm_i \times \nabla h^\pm_i. \quad (4.17)
\]

We define

\[
\begin{cases}
\frac{dx^\pm(t, x)}{dt} = (u - \frac{1}{\sqrt{\rho^\pm}} h)\pm(t, x^\pm(t, x)), \quad x \in \Omega^\pm_x, \\
x^\pm(0, x) = \text{Id}, \quad x \in \Omega^\pm_{x_0}.
\end{cases}
\]

(4.18)

Recalling that \(\underline{h}^\pm \cdot N = 0\), one can see that \(x^\pm(t, \cdot)\) is a flow map from \(\Omega^\pm_{x_0}\) to \(\Omega^\pm_{x(t)}\). Then we have

\[
\frac{d\sigma^\pm(t, x^\pm(t, x))}{dt} = (\sigma^\pm \cdot \nabla (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm))
\]

\[ - 2 \nabla u^\pm_i \times \nabla h^\pm_i(t, x^\pm(t, x)), \quad x \in \Omega^\pm_{x_0}. \]

Springer
This is a linear ODE system, and the existence of \( \sqrt{\rho \pm \omega} \pm \xi \) follows immediately. So do \( \sqrt{\rho \pm \omega} - \xi \). Next, we give the energy estimates for \((\omega^\pm, \xi^\pm)\).

**Proposition 4.16** It holds that

\[
\sup_{t \in [0, T]} \left( \| \omega^\pm(t) \|^2_{H^{-1}(\Omega^+_f)} + \| \xi^\pm(t) \|^2_{H^{-1}(\Omega^+_f)} \right) \\
\leq (1 + \| \omega_0^\pm \|^2_{H^{-1}(\Omega^+_0)} + \| \xi_0^\pm \|^2_{H^{-1}(\Omega^+_0)}) e^{C(L_1)T}.
\]

**Proof** Using the fact that \( \mathbf{u}^\pm \cdot \mathbf{N} = \partial_f f \) and \( \mathbf{h}^\pm \cdot \mathbf{N} = 0 \), we deduce from (4.17) that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^+_f} |\nabla^s \sigma^\pm(t, x)|^2 dx \\
= \int_{\Gamma_f^+} \nabla^s \sigma^\pm \cdot \nabla \sigma^\pm dt + \frac{1}{2} \int_{\Gamma_f} |\nabla^s \sigma^\pm|^2 (u^\pm \cdot \mathbf{n}) d\sigma \\
\leq \int_{\Omega^+_f} |\nabla^s \sigma^\pm \cdot \nabla^s (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm) \cdot \nabla \sigma^\pm | dx + \frac{1}{2} \int_{\Gamma_f} |\nabla^s \sigma^\pm|^2 (u^\pm \cdot \mathbf{n}) d\sigma \\
+ C(L_1)(1 + \| \omega^\pm(t) \|^2_{H^{-1}(\Omega^+_f)} + \| \xi^\pm(t) \|^2_{H^{-1}(\Omega^+_f)}) \\
\leq \frac{1}{2} \int_{\Omega^+_f} (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm) \cdot \nabla \left(|\nabla^s \sigma^\pm|^2\right) dx + \frac{1}{2} \int_{\Gamma_f} |\nabla^s \sigma^\pm|^2 (u^\pm \cdot \mathbf{n}) d\sigma \\
+ C(L_1)(1 + \| \omega^\pm(t) \|^2_{H^{-1}(\Omega^+_f)} + \| \xi^\pm(t) \|^2_{H^{-1}(\Omega^+_f)}) \\
= -\frac{1}{2} \int_{\Omega^+_f} \text{div}(u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm)|\nabla^s \sigma^\pm|^2 dx \\
+ C(L_1)(1 + \| \omega^\pm(t) \|^2_{H^{-1}(\Omega^+_f)} + \| \xi^\pm(t) \|^2_{H^{-1}(\Omega^+_f)}) \\
\leq C(L_1)(1 + \| \omega^\pm(t) \|^2_{H^{-1}(\Omega^+_f)} + \| \xi^\pm(t) \|^2_{H^{-1}(\Omega^+_f)}).
\]

Similarly, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^+_f} |\nabla^s (\sqrt{\rho^\pm \omega} - \xi)|^2 dx \leq C(L_1)(1 + \| \omega^\pm(t) \|^2_{H^{-1}(\Omega^+_f)} + \| \xi^\pm(t) \|^2_{H^{-1}(\Omega^+_f)}).
\]

The desired estimate follows from Gronwall’s inequality.

To solve velocity and magnetic field from the vorticity and current \((\omega^\pm, \xi^\pm)\), we need to verify the following compatibility conditions.

**Lemma 4.17** It holds that

\[
\frac{d}{dt} \int_{\Gamma^+} \omega^\pm x' dx' = 0, \quad \frac{d}{dt} \int_{\Gamma^+} \xi^\pm x' dx' = 0.
\]

**Proof** The proof is straightforward, we refer the readers to Lemma 6.4 of [40].
5 Construction and contraction of the Iteration Map

Assume that

\[ f_0 \in H^{s+1}(\mathbb{T}^2), \quad \mathbf{u}_0^\pm, \mathbf{h}_0^\pm \in H^s(\Omega_{f_0}^\pm), \]

which satisfy

\[ - (1 - 2c_0) \leq f(x') \leq (1 - 2c_0); \]

for some constant \( c_0 \in (0, \frac{1}{2}) \).

Let \( f_0 = f_0 \), and \( \Omega_{f_0}^\pm = \Omega_{f_0}^\pm \) be the reference region. The initial data \((f_I, (\partial_t f)_I, \omega_{sI}^\pm, \xi_{sI}^\pm, a_{iI}^\pm, b_{iI}^\pm)\) for the equivalent system is defined as follows:

\[ f_I = f_0, \quad (\partial_t f)_I = \mathbf{u}_0^\pm(x', f_0(x')) \cdot (-\partial_1 f_0, -\partial_2 f_0, 1); \]

\[ \omega_{sI}^\pm = \text{curl}\mathbf{u}_0^\pm, \quad \xi_{sI}^\pm = \text{curl}\mathbf{h}_0^\pm; \]

\[ a_{iI}^\pm = \int_{\mathbb{T}^2} u_{0i}^\pm(x', \pm 1)dx', \quad b_{iI}^\pm = \int_{\mathbb{T}^2} h_{0i}^\pm(x', \pm 1)dx', \]

which satisfy

\[ \sigma^{1/2} \| f_I \|_{H^{s+1}} + \| f_I \|_{H^{s+1/2}} + \| (\omega_{sI}^\pm, \xi_{sI}^\pm) \|_{H^{s-1}(\Omega_{f_0}^\pm)} + \| (\partial_t f)_I \|_{H^{s-1/2}} + |a_{iI}^\pm| + |b_{iI}^\pm| \leq \frac{M_0}{2}, \]

for some constant \( M_0 > 0 \). Here we remark that, as \((f_0, \mathbf{u}_0^\pm)\) satisfies (1.7), it follows from

\[ \text{div}\mathbf{u}_0^\pm = 0 \text{ in } \Omega_{f_0}^\pm, \quad u_{03}^\pm = 0 \text{ on } \Gamma^\pm \]

that

\[ \int_{\mathbb{T}^2} (\partial_t f)_I(x')dx' = \int_{\mathbb{T}^2} \mathbf{u}_0^\pm(x', f_0(x')) \cdot (-\partial_1 f_0, -\partial_2 f_0, 1(x'))dx' = 0. \quad (5.1) \]

Then we define the following function set.

**Definition 5.1** Let \( s \geq 6 \) be a integer. Given two constant \( M_1, M_2 > 0 \) with \( M_1 > 2M_0 \), we define the set \( \mathcal{X}_s = \mathcal{X}_s(T, M_1, M_2) \) be the collection of \((f, \omega_\pm^s, \xi_\pm^s, a_\pm^i, b_\pm^i)\) that satisfies

\[ (f(0), \partial_t f(0), \omega_\pm^s(0), \xi_\pm^s(0), a_\pm^i(0), b_\pm^i(0)) = (f_I, (\partial_t f)_I, \omega_{sI}^\pm, \xi_{sI}^\pm, a_{iI}^\pm, b_{iI}^\pm), \]

\[ \| f(t, \cdot) - f_\ast \|_{H^{s-1/2}} \leq \delta_0, \]

\[ \sup_{t \in [0, T]} \left( \sigma^{1/2} \| f \|_{H^{s+1}} + \| f \|_{H^{s+1/2}}(t) \right. \]

\[ + \| (\omega_\pm^s, \xi_\pm^s) \|_{H^{s-1}(\Omega_{f_0}^\pm)}(t) + \| \partial_t f \|_{H^{s+1/2}}(t) + |a_\pm^i(t)| \right. \]

\[ + \| b_\pm^i(t) \| \leq M_1, \]

\[ \sup_{t \in [0, T]} \left( \| (\partial_t \omega_\pm^s, \partial_t \xi_\pm^s) \|_{H^{s-2}(\Omega_{f_0}^\pm)}(t) + \| \nabla_\pm f \|_{H^{s-2}}(t) + |\partial_t a_\pm^i(t)| + |\partial_t b_\pm^i(t)| \right) \leq M_2, \]

\[ \int_{\mathbb{T}^2} \partial_t f(t, x')dx' = 0. \]

It is clear that \( \mathcal{X}_s(T, M_1, M_2) \) is nonempty. Let \( \tilde{f} \) be the solution of

\[ \partial_t^2 \tilde{f}(t, x') + |\nabla|^2 \tilde{f}(t, x') = 0, \]

\[ \tilde{f}(0, x') = f_I(x'), \quad \partial_t \tilde{f}(0, x') = (\partial_t f)_I(x'). \]
As \( \Omega^\pm \), we introduce the projection operator \( P \) then, we define \( u^\pm \). It follows from Proposition 3.2 that \( (f, \partial_t f, \omega_*^\pm, \xi_*^\pm, \tilde{a}_i^\pm, \tilde{b}_i^\pm) \) belongs to \( \chi_\sigma(T, M_1, M_2) \) for some suitable constants \( T, M_1, M_2 \).

Next, we will construct an iteration map
\[
\mathcal{F}_\sigma : \chi_\sigma(T, M_1, M_2) \to \chi_\sigma(T, M_1, M_2)
\]
with suitable constants \( T, M_1, M_2 \).

### 5.1 Recover the bulk region, velocity and magnetic field

We define
\[
\tilde{\omega}^\pm \overset{\text{def}}{=} P_f^\text{div} (\omega_*^\pm \circ \Phi_f^{-1}), \quad \tilde{\xi}^\pm \overset{\text{def}}{=} P_f^\text{div} (\xi_*^\pm \circ \Phi_f^{-1}),
\]
where \( \Phi_f^\pm : \Omega_*^\pm \to \Omega_f^\pm \) is the harmonic coordinate map, and \( P_f^\text{div} \omega^\pm = \omega^\pm - \nabla \phi^\pm \) with
\[
\begin{align*}
\Delta \phi^\pm &= \text{div} \omega^\pm \quad \text{in } \Omega_f^\pm, \\
\partial_3 \phi^\pm &= 0 \quad \text{on } \Gamma^\pm, \\
\phi^\pm &= 0 \quad \text{on } \Gamma_f.
\end{align*}
\]
We introduce the projection operator \( P_f^\text{div} \) to ensure that \( (\tilde{\omega}^\pm, \tilde{\xi}^\pm) \) satisfy conditions (C1) and (C2) defined in Sect. 3.4. It is obvious that
\[
\begin{align*}
\| (\tilde{\omega}^\pm, \tilde{\xi}^\pm) (t) \|_{H^{s-1}(\Omega_f^\pm)} &\leq C(M_1), \\
\| (\partial_3 \tilde{\omega}^\pm, \partial_3 \tilde{\xi}^\pm) (t) \|_{H^{s-2}(\Omega_f^\pm)} &\leq C(M_1, M_2).
\end{align*}
\]
Then, we define \( u^\pm \) and \( h^\pm \) as the solution of
\[
\begin{align*}
\text{curl} u^\pm &= \tilde{\omega}^\pm, \quad \text{div} u^\pm = 0 \quad \text{in } \Omega_f^\pm, \\
u^\pm \cdot N &= \partial_t f \quad \text{on } \Gamma_f, \\
u^\pm \cdot e_3 &= 0, \quad \int_{\Gamma^\pm} u_i dx' = a_i^\pm (i = 1, 2) \quad \text{on } \Gamma^\pm, \\
\text{curl} h^\pm &= \tilde{\xi}^\pm, \quad \text{div} h^\pm = 0 \quad \text{in } \Omega_f^\pm, \\
h^\pm \cdot N &= 0 \quad \text{on } \Gamma_f, \\
h^\pm \cdot e_3 &= 0, \quad \int_{\Gamma^\pm} h_i dx' = b_i^\pm (i = 1, 2) \quad \text{on } \Gamma^\pm.
\end{align*}
\]
As \( \Omega_*^\pm = \Omega_f^\pm \), it holds that
\[
u^\pm (0) = u_0^\pm, \quad h^\pm (0) = h_0^\pm.
\]
It follows from Proposition 3.2 that
\[
\begin{align*}
\| u^\pm \|_{H^s(\Omega_f^\pm)} &\leq C(M_1)(\| \tilde{\omega}^\pm \|_{H^{s-1}(\Omega_f^\pm)} + \| \partial_t f \|_{H^{s-1/2}} + |a_1^\pm (t)| + |a_2^\pm (t)|) \leq C(M_1), \\
\| h^\pm \|_{H^s(\Omega_f^\pm)} &\leq C(M_1)(\| \tilde{\xi}^\pm \|_{H^{s-1}(\Omega_f^\pm)} + \| b_1^\pm (t) \| + \| b_2^\pm (t) \|) \leq C(M_1).
\end{align*}
\]
Using the same argument to treat $\partial_t u^\pm$ and $\partial_t h^\pm$, we deduce that
$$\|\partial_t u^\pm\|_{H^{r-1}(\Omega_T)} \leq C(M_1, M_2), \quad \|\partial_t h^\pm\|_{H^{r-1}(\Omega_T)} \leq C(M_1, M_2),$$
which implies
$$\|u^\pm\|_{W^{1,\infty}(t)} \leq \|u^\pm_0\|_{W^{1,\infty}} + \int_0^t \|\partial_t u^\pm\|_{W^{1,\infty}(t')}dt' \leq \frac{M_0}{2} + TC(M_1, M_2),$$
$$\|h^\pm\|_{W^{1,\infty}(t)} \leq \frac{M_0}{2} + TC(M_1, M_2).$$
Similar argument shows that
$$\|f\|_{W^{2,\infty}(t)} \leq \frac{M_0}{2} + TC(M_1).$$
Besides, it holds that
$$\|f(t) - f_0\|_{L^\infty} \leq \|f(t) - f_0\|_{H^{r-\frac{1}{2}}} \leq T\|\partial_t f\|_{H^{r-\frac{1}{2}}} \leq T M_1.$$
By choosing $T$ small enough, we have
$$T M_1 \leq \min\{\delta_0, c_0\}, \quad TC(M_1) + TC(M_1, M_2) \leq \frac{M_0}{2}, \quad TC(M_1, M_2) \leq c_0.$$ 
Taking $L_0 = M_0, L_1 = M_1, L_2 = C(M_1, M_2)$, we conclude that for $\forall t \in [0, T]$: 
1. $- (1 - c_0) \leq f(t, x') \leq (1 - c_0)$,
2. $\|(u^\pm, h^\pm)\|_{W^{1,\infty}(t)} + \|f\|_{W^{2,\infty}(t)} \leq L_0$,
3. $\|f\|_{H^{r-\frac{1}{2}}(t)} \leq \delta_0$,
4. $\sigma^{1/2}\|f\|_{H^{r+1}(t)} + \|f\|_{H^{r+\frac{1}{2}}(t)} + \|\partial_t f\|_{H^{r-\frac{1}{2}}(t)} + \|u^\pm\|_{H^{r}(\Omega_T)}(t)$
   $\quad + \|h^\pm\|_{H^{r}(\Omega_T)}(t) \leq L_1$,
5. $\|\partial_t u^\pm, \partial_t h^\pm\|_{W^{2,\infty}(t)} \leq L_2$.

5.2 Defining the Iteration Map

Given $(f, u^\pm, h^\pm)$ which is constructed from $(f, \omega^\pm, \xi^\pm, a^\pm, b^\pm)$. Let $\tilde{f}_1$ and $(\tilde{\omega}^\pm, \tilde{\xi}^\pm)$ be the solutions of the linearized systems (4.11) and (4.16) with initial data
$$(\tilde{f}_1(0), \tilde{\omega}(0), \tilde{\omega}^\pm(0), \tilde{\xi}^\pm(0)) = (f_1, (\partial_t f)_I, \omega^\pm, \xi^\pm).$$

We define
$$\tilde{\omega}^\pm = \tilde{\omega}^\pm \circ \Phi_f^\pm, \quad \tilde{\xi}^\pm = \tilde{\xi}^\pm \circ \Phi_f^\pm,$$
$$\tilde{a}_i^\pm(t) = a_i^\pm(0) - \int_0^t \int_{\Gamma^H} \sum_{j=1}^3 (u_j^\pm \partial_j u_i^\pm - h_j^\pm \partial_j h_i^\pm)(x', t')dx'dt',$$
$$\tilde{b}_i^\pm(t) = b_i^\pm(0) - \int_0^t \int_{\Gamma^H} \sum_{j=1}^3 (u_j^\pm \partial_j h_i^\pm - h_j^\pm \partial_j u_i^\pm)(x', t')dx'dt'.$$
Then we have the iteration map $\mathcal{F}_\sigma$ as follows
\[
\mathcal{F}_\sigma : \mathcal{X}_0(T, M_1, M_2) \to \mathcal{X}_0(T, M_1, M_2),
\]
\[
\mathcal{F}_\sigma(f, \omega^\pm, \xi^\pm, a^\pm_i, b^\pm_i) \overset{\text{def}}{=} (\tilde{f}, \tilde{\omega}^\pm, \tilde{\xi}^\pm, \tilde{a}^\pm_i, \tilde{b}^\pm_i),
\]
where $\tilde{f}(t, x') = \mathcal{P}\tilde{f}_1(t, x') + (f_I)$. Hence, $\langle \tilde{f}, f \rangle = \langle f_I \rangle$ and $\int_{T^2} \partial_t \tilde{f}(t, x') dx' = 0$ for $t \in [0, T]$.

**Remark 5.2** As $\tilde{f}_1(0, x') = f_I(x')$, we have $\tilde{f}_1(0, x') = \mathcal{P}\tilde{f}_1(0, x')$ and $(f_I) = f_I(x')$. Then we have $\Phi^\mp = \text{Id}$, it follows that $\omega^\pm(0, x) = \omega^\pm(x)$ and $\xi^\pm(0, x) = \xi^\pm(x)$. From (5.1), we also have $\partial_t \tilde{f}(0, x') = \mathcal{P}\partial_t \tilde{f}_1(0, x') = (\partial_t f)_I(x')$.

**Proposition 5.3** There exists $M_1, M_2 > 0$ depending on $\delta_0, M_0, \sigma$ so that $\mathcal{F}_\sigma$ is a map from $\mathcal{X}_0(T, M_1, M_2)$ to itself.

**Proof** From Remark 5.2, we can see that the initial conditions in Definition 5.1 are automatically satisfied. Indeed, we have
\[
(\tilde{f}(0), \partial_t \tilde{f}(0), \tilde{\omega}^\pm(0), \tilde{\xi}^\pm(0), \tilde{a}^\pm_i(0), \tilde{b}^\pm_i(0)) = (f_I, (\partial_t f)_I, \omega^\pm, \xi^\pm, a^\pm_i, b^\pm_i).
\]
As $\tilde{f}(t, x') = \mathcal{P}\tilde{f}_1(t, x') + (f_I)$, it holds that $\partial_t \tilde{f}(t, x') = \mathcal{P}\partial_t \tilde{f}_1(t, x')$, and then $\int_{T^2} \partial_t \tilde{f}(t, x') dx' = 0$.

It follows from Proposition 4.14 and Proposition 4.16 that
\[
\sup_{t \in [0, T]} \left( \frac{1}{2} \| \tilde{f}_I \|_{H^{s+1}(t)} + \| \tilde{f}_I \|_{H^{s+1}(\Omega^+_x)}(t) + \| \tilde{\omega}^\pm \|_{H^{s-1}(\Omega^+_x)}(t) + \| \tilde{\xi}^\pm \|_{H^{s-1}(\Omega^+_x)}(t) \right) \leq C(\epsilon, \sigma, M_0)e^{C(\sigma, M_1, M_2)T}.
\]

We first take $M_1 = 2C(\epsilon, \sigma, M_0)$, then let $T < \frac{1}{C(\epsilon, M_1, M_2)}$ which is still to be determined. It can be derived directly from (4.11), (4.16) and the above estimate that
\[
\sup_{t \in [0, T]} \left( \| \partial_t^s \tilde{f}_I \|_{H^{s-2}(\Omega^+_x)}(t) + \| \partial_t \tilde{\omega}^\pm \|_{H^{s-2}(\Omega^+_x)}(t) + \| \partial_t \tilde{\xi}^\pm \|_{H^{s-2}(\Omega^+_x)}(t) \right) \leq C(M_1).
\]

It is clear that
\[
|a^\pm_i(t)| + |b^\pm_i(t)| \leq \frac{M_0}{2} + TC(M_1),
\]
\[
|\partial_t^s \tilde{\omega}^\pm(t)| + |\partial_t^s \tilde{\xi}^\pm(t)| \leq C(M_1),
\]
\[
\| \tilde{f}(t) - f_I \|_{H^{s-1}} \leq \int_0^t \| (\partial_t \tilde{f})(t') \|_{H^{s-1}} dt' \leq TC(M_1).
\]

We take $M_2 = C(M_1)$ and then take $T$ sufficiently small depending only on $\epsilon, \delta_0, M_0$. One can see that all the conditions in Definition 5.1 are satisfied. This complete the proof. \hfill \Box

### 5.3 Contraction of the iteration map

Now, we show that $\mathcal{F}_\sigma$ is contract in $\mathcal{X}_0(T, M_1, M_2)$. Suppose $(f^A, \omega^A, \xi^A, a^A, b^A)$ and $(f^B, \omega^B, \xi^B, a^B, b^B)$ are two elements in $\mathcal{X}_0(T, M_1, M_2)$ and
\[
(\tilde{f}^C, \tilde{\omega}^C, \tilde{\xi}^C, \tilde{a}^C, \tilde{b}^C) = \mathcal{F}(f^C, \omega^C, \xi^C, a^C, b^C)
\]
for $C = A, B$. We denote by $g^D$ the difference $g^A - g^B$. \hfill \square
Proposition 5.4 There exists $T = T(c_0, \sigma, \delta_0, M_0) > 0$ so that

$$\bar{E}^D \overset{\text{def}}{=} \sup_{t \in [0, T]} \left( \sigma \frac{1}{2} \| \bar{f}^D \|_{H^{r-1}}(t) + \| \bar{f}^D \|_{H^{r-\frac{3}{2}}}(t) \right)
$$

$$+ \| \tilde{\omega}^D_\ast \|_{H^{r-3}(\Omega^+)_\ast}(t) + \| \tilde{\xi}^D_\ast \|_{H^{r-3}(\Omega^-)_\ast}(t)
$$

$$+ \|(\partial_t \bar{f}^D)\|_{H^{r-\frac{5}{2}}}(t) + |\alpha^D_i|(t) + |\beta^D_i|(t) \right)
$$

$$\leq \frac{1}{2} \sup_{t \in [0, T]} \left( \sigma \frac{1}{2} \| f^D \|_{H^{r-1}}(t) + \| f^D \|_{H^{r-\frac{3}{2}}}(t) \right)
$$

$$+ \| \omega^D_\ast \|_{H^{r-3}(\Omega^+)_\ast}(t) + \| \xi^D_\ast \|_{H^{r-3}(\Omega^-)_\ast}(t)
$$

$$+ \|(\partial_t f^D)\|_{H^{r-\frac{5}{2}}}(t) + |\alpha^D_i|(t) + |\beta^D_i|(t) \right)$$

$$\overset{\text{def}}{=} E^D.$$

Proof By elliptic estimates, we have

$$\| \Phi^\pm_{fA} - \Phi^\pm_{fB} \|_{H^{s-2}(\Omega^\pm)} \leq C(M_1) \| f^A - f^B \|_{H^{r-\frac{3}{2}}} \leq C E^D.$$

For $C = A, B$ we define

$$u^\pm_C = u^\pm \circ \Phi^\pm_{fC}, \quad h^\pm_C = h^\pm \circ \Phi^\pm_{fC},$$

and claim that

$$\| u^\pm_C \|_{H^{s-2}(\Omega^\pm)} + \| h^\pm_C \|_{H^{s-2}(\Omega^\pm)} \leq C E^D.$$

Indeed, for a vector field $v^\pm_\ast$ defined on $\Omega^\pm_\ast$, we define

$$\text{curl}_C v^\pm_\ast = \left( \text{curl}(v^\pm_\ast \circ (\Phi^\pm_{fC})^{-1}) \right) \circ \Phi^\pm_{fC},$$

$$\text{div}_C v^\pm_\ast = \left( \text{div}(v^\pm_\ast \circ (\Phi^\pm_{fC})^{-1}) \right) \circ \Phi^\pm_{fC}.$$

Thus, for $C = A, B$, it holds that

$$\begin{cases}
\text{curl}_C u^\pm_C = \tilde{\omega}^\pm_\ast & \text{in } \Omega^\pm_\ast, \\
\text{div}_C u^\pm_C = 0 & \text{in } \Omega^\pm_\ast, \\
u^\pm_C \cdot N_{fC} = \partial_t f^C & \text{on } \Gamma^\pm_\ast, \\
u^\pm_C \cdot e_3 = 0, \quad \int_{\Gamma^\pm} u^\pm_C \, dx' = \tilde{a}^\pm C (i = 1, 2) & \text{on } \Gamma^\pm_\ast.
\end{cases}$$

Accordingly, we deduce that

$$\begin{cases}
\text{curl}_A u^\pm_D = \tilde{\omega}^\pm_D + (\text{curl}_B - \text{curl}_A) u^\pm_B & \text{in } \Omega^\pm_\ast, \\
\text{div}_A u^\pm_D = (\text{div}_B - \text{div}_A) u^\pm_B & \text{in } \Omega^\pm_\ast, \\
u^\pm_D \cdot N_{fA} = \partial_t f^D + u^\pm_B \cdot (N_{fB} - N_{fA}) & \text{on } \Gamma^\pm_\ast, \\
u^\pm_D \cdot e_3 = 0, \quad \int_{\Gamma^\pm} u^\pm_D \, dx' = \tilde{a}^\pm D (i = 1, 2) & \text{on } \Gamma^\pm_\ast.
\end{cases}$$
Direct calculation shows that

\[ \| (\text{curl}_B - \text{curl}_A) u_\pm^B \|_{H^{3/2}(\Omega_\pm^*)} \leq C \| \Phi_{fA}^\pm - \Phi_{fB}^\pm \|_{H^{3/2}(\Omega_\pm^*)} \leq C \| f^D \|_{H^{1/2}} \leq C E^D, \]

\[ \| (\text{curl}_B - \text{curl}_A) u_\pm^B \|_{H^{3/2}(\Omega_\pm^*)} \leq C E^D, \]

\[ \| u_\pm^B \cdot (N_{fB} - N_{fA}) \|_{H^{1/2}} \leq C E^D. \]

Then, we get by Proposition 3.2 that

\[ \| u_\pm^D \|_{H^{3/2}(\Omega_\pm^*)} \leq C (\| \tilde{\omega}_\pm^D \|_{H^{3/2}(\Omega_\pm^*)} + \| \partial_t f^D \|_{H^{1/2}} + E^D) \leq C E^D. \]

Similarly, we have

\[ \| h_\pm^D \|_{H^{3/2}(\Omega_\pm^*)} \leq C E^D. \]

Recalling (4.11), we deduce that

\[
\partial_t \tilde{f}_1^D = \tilde{\theta}^D, \\
\partial_t \tilde{\theta}^D = -\frac{\sigma}{(\rho^+ + \rho^-)^3} (T_{\Lambda K} T_{\Lambda^*} \tilde{f}_1^D) - \frac{2}{\rho^+ + \rho^-} ((\rho^+ u_1^A + \rho^- u_1^- A) \partial_1 \tilde{\theta}^D + (\rho^+ u_2^A + \rho^- u_2^- A) \partial_2 \tilde{\theta}^D) \\
- \frac{1}{\rho^+ + \rho^-} \sum_{i,j=1,2} \left( \rho^+ u_i^+ A u_j^+ A - h_i^+ A h_j^+ A \right) \partial_i \partial_j \tilde{f}_1^D + \mathcal{R},
\]

where

\[ \mathcal{R} = -\frac{\sigma}{(\rho^+ + \rho^-)^3} (T_{\Lambda K} T_{\Lambda^*} - T_{\Lambda B} T_{\Lambda^*}) \tilde{f}_1^B \\
- \frac{2}{\rho^+ + \rho^-} \left( (\rho^+ u_1^+ D + \rho^- u_1^- D) \partial_1 \tilde{\theta}^B + (\rho^+ u_2^+ D + \rho^- u_2^- D) \partial_2 \tilde{\theta}^B \right) \\
- \frac{1}{\rho^+ + \rho^-} \sum_{i,j=1,2} \left[ (\rho^+ u_i^+ A u_j^+ A - h_i^+ A h_j^+ A + \rho^- u_i^- A u_j^- A - h_i^- A h_j^- A) \\
- (\rho^+ u_i^+ B u_j^+ B - h_i^+ B h_j^+ B + \rho^- u_i^- B u_j^- B - h_i^- B h_j^- B) \right] \partial_i \partial_j \tilde{f}_1^B \\
+ g^A - g^B + \frac{\sigma}{(\rho^+ + \rho^-)^3} (R^A - R^B), \]
and for $C = A, B$,
\[
g^C = -\frac{\sigma}{(\rho^+ + \rho^-)^2} (N_{c}^+ - N_{fc}^-) (N_{c}^- - N_{fc}^+) H(f^C) \\
+ \frac{2}{\rho^+ + \rho^-} (N_{f}^+ c - N_{fc}^-) (N_{f}^- c - N_{fc}^+) P\left((u_{1}^C - u_{1}^-)^2 \partial_1 \theta^C + (u_{2}^C - u_{2}^-)^2 \partial_2 \theta^C\right) \\
+ \frac{1}{\rho^+ + \rho^-} (N_{f}^+ c - N_{fc}^-) (N_{f}^- c - N_{fc}^+) P\left(\sum_{i,j=1,2} (u_{1,i}^+ u_{j}^C - u_{i}^- u_{j}^-)^2 \rho^C\right) \\
+ \frac{1}{\rho^+ + \rho^-} (N_{f}^+ c - N_{fc}^-) (N_{f}^- c - N_{fc}^+) P\left(\sum_{i,j=1,2} \frac{1}{\rho^+ + \rho^-} h_{i}^- h_{j}^- \partial_i \partial_j f^C\right) \\
- \frac{1}{\rho^+ + \rho^-} (N_{f} c \cdot \nabla (\rho^+ p_{u}^C - p_{h}^+ c, h^+) - N_{f} c \cdot \nabla (\rho^- p_{u}^- c - p_{h}^- c, h^-)) \\
+ \frac{1}{\rho^+ + \rho^-} (N_{f}^+ c - N_{fc}^-) (N_{f}^- c - N_{fc}^+) P\left(\sum_{i,j=1,2} \frac{1}{\rho^+ + \rho^-} h_{i}^- h_{j}^- \partial_i \partial_j f^C\right) \\
\cdots
\]

It is easy to check that
\[
\|\Re\|_{H^{\frac{1}{2}}} \leq CE^D.
\]

The key point is to write each error term in the form of
\[
Oh(f^D, u^\pm A, h^\pm B),
\]
where $Op$ is an operator with coefficients defined on $(f^1 \cdot f^2 B, \tilde{\theta}^A, \tilde{\theta}^B, f^A, f^B, u^\pm A, u^\pm B, h^\pm A, h^\pm B)$. We give the estimate of $\frac{\sigma}{(\rho^+ + \rho^-)^2} (T_{h^A} T_{l^A} - T_{h^B} T_{l^B}) \tilde{f}^1_B$ for example, and the other terms can be treated in a similar way.

\[
(T_{h^A} T_{l^A} - T_{h^B} T_{l^B}) \tilde{f}^1_B = (T_{h^A(1)} - T_{h^B(1)}) T_{l^A} \tilde{f}^1_B + (T_{h^A(0)} - T_{h^B(0)}) T_{l^A} \tilde{f}^1_B
\]

Recalling that $\lambda^{(1)} = \sqrt{(1 + |\nabla f|^2)\xi^2 - (\nabla f \cdot \xi)^2}$, we have
\[
\lambda^{(1)A} - \lambda^{(1)B} = \sqrt{(1 + |\nabla f^A|^2)\xi^2 - (\nabla f^A \cdot \xi)^2 - (1 + |\nabla f^B|^2)\xi^2 - (\nabla f^B \cdot \xi)^2}
\]

It follows from Proposition A.2 that
\[
\| (T_{h^A(1)} - T_{h^B(1)}) T_{l^A} \tilde{f}^1_B \|_{H^{\frac{1}{2}}} \leq C \| \nabla f^A - \nabla f^B \|_{L^\infty} \| T_{l^A} \tilde{f}^1_B \|_{H^{\frac{1}{2}}} 
\]

\[
\leq C \| \tilde{f}^1_B \|_{H^{\frac{1}{2}}} \| \nabla f^D \|_{L^\infty} \leq CE^D.
\]
In this way, one can deduce that
\[ \|(T_{\lambda A} T_I A - T_{\lambda B} T_I B) \tilde{f}^B \|_{H^{s - \frac{5}{2}}} \leq C E^D. \]

Now, we define
\[
\tilde{F}_A (\partial_t \tilde{f}_1^D, \tilde{f}_1^D) \overset{\text{def}}{=} \frac{\sigma}{2(\rho^+ + \rho^-)^2} \| T_{\gamma A} T_{\rho A} \tilde{f}_1^D \|_{L^2}^2 + \| (\partial_t + \bar{w}_i^A \partial_i) T_{\rho A} \tilde{f}_1^D \|_{L^2}^2
- \frac{1}{2} \| \bar{v}_t^A \partial_t T_{\rho A} \tilde{f}_1^D \|_{L^2}^2
+ \frac{1}{2(\rho^+ + \rho^-)} (\| h_\beta^A \partial_t T_{\rho A} \tilde{f}_1^D \|_{L^2}^2 + \| h_{\beta}^- A \partial_t T_{\rho A} \tilde{f}_1^D \|_{L^2}^2),
\]
where \( \beta^A := (\gamma^0 (-1))^\frac{2s - 5}{s} \in \Sigma^{s - \frac{5}{2}}, \) by following the proof of Proposition 4.14, one can see that
\[
\frac{d}{dt} \left( \tilde{F}_A (\partial_t \tilde{f}_1^D, \tilde{f}_1^D) + C(\sigma) \| \tilde{f}_1^D \|_{L^2}^2 + \| \partial_t \tilde{f}_1^D \|_{L^2}^2 \right) \leq C (E^D + \tilde{E}_1^D),
\]
where
\[
\tilde{E}_1^D = \sigma \| \tilde{f}_1^D \|_{H^{s - 1}}^2 + \| \tilde{f}_1^D \|_{H^{s - \frac{1}{2}}}^2 + \| \partial_t \tilde{f}_1^D \|_{H^{s - \frac{5}{2}}}^2.
\]
As \((f^A, \bar{w}_i^A, \xi^A, \alpha_i^A, \tilde{b}_i^A) \in \mathcal{X}(T, M_1, M_2),\) it holds that
\[
\tilde{E}_1^D \leq C \left( \tilde{F}_A (\partial_t \tilde{f}_1^D, \tilde{f}_1^D) + C(\sigma) \| \tilde{f}_1^D \|_{L^2}^2 + \| \partial_t \tilde{f}_1^D \|_{L^2}^2 \right).
\]
Note that \( \tilde{f}_1^D(0) = \tilde{f}_1^A(0) - \tilde{f}_1^B(0) = 0.\) Applying Gronwall’s inequality, we have
\[
\sup_{t \in [0, T]} \left( \sigma \| \tilde{f}_1^D(t) \|_{H^{s - 1}} + \| \tilde{f}_1^D(t) \|_{H^{s - \frac{1}{2}}} + \| \partial_t \tilde{f}_1^D(t) \|_{H^{s - \frac{5}{2}}} \right) \leq C T e^{C T} E^D,
\]
which implies
\[
\sup_{t \in [0, T]} \left( \sigma \| \tilde{f}_1^D(t) \|_{H^{s - 1}} + \| \tilde{f}_1^D(t) \|_{H^{s - \frac{1}{2}}} + \| \partial_t \tilde{f}_1^D(t) \|_{H^{s - \frac{5}{2}}} \right) \leq C T e^{C T} E^D.
\]
Similar to the proof of Proposition 4.16, one can get
\[
\sup_{t \in [0, T]} \left( \| \bar{w}_i^D \|_{H^{s - 3}(\Omega^\pm_\delta)} + \| \xi_i^D \|_{H^{s - 3}(\Omega^\pm_\delta)} \right) \leq C T e^{C T} E^D.
\]
From the equation
\[
\bar{a}_i^\pm C(t) = \bar{a}_i^\pm C(0) - \int_0^t \int_{\tau \pm} \sum_{j=1}^3 (u_j^\pm C \partial_j u_i^\pm C - h_j^\pm C \partial_j h_i^\pm C) dx' d\tau,
\]
we have
\[
| \bar{a}_i^\pm D(t) | \leq T C E^D.
\]
Similarly,
\[
| \bar{b}_i^\pm D(t) | \leq T C E^D.
\]
As a conclusion, we arrive at

\[ \tilde{E}^D \leq C T e^{CT} E^D. \]

One can achieve the result by taking \( T \) small enough. \( \square \)

### 5.4 The limit system

Proposition 5.4 shows that there exists a unique fixed point 
\((f, \omega^\pm, \xi^\pm, a^\pm_i, b_i^\pm)\) of the map \(F\) in \(\mathcal{X}(T, M_1, M_2)\). Now, we will finish the proof of Theorem 1.1, and show that one can recover \((u^\pm, h^\pm, p^\pm)\) from \((f, \omega^\pm, \xi^\pm, a^\pm_i, b_i^\pm)\) which is the unique solution to the original system (1.2)–(1.3). We call \((f, \omega^\pm, \xi^\pm, a^\pm_i, b_i^\pm)\) is the solution system of (1.2)–(1.5).

From the construction of \(F\), the fixed point

\[(f, \omega^\pm, \xi^\pm, a^\pm_i, b_i^\pm) = (f, \omega^\pm \circ \Phi_f^{-1}, \xi^\pm \circ \Phi_f^{-1}, a^\pm_i, b_i^\pm)\]

satisfies

\[
\partial_t f = \mathcal{P} \theta, \\
\partial_t \theta = \frac{\sigma}{(\rho^+ + \rho^-)^2} (N_f^+ H(f) + N_f^- H(f)) \\
- \frac{\sigma}{(\rho^+ + \rho^-)^2} (N_f^+ - N_f^-) \tilde{\mathcal{N}}_f^{-1} (N_f^+ H(f) - N_f^- H(f)) \\
- \frac{2}{\rho^+ + \rho^-} \left( (\rho^+ u_i^+ + \rho^- u_i^-) \partial_i \theta + (\rho^+ u_i^+ + \rho^- u_i^-) \partial_2 \theta \right) \\
+ \frac{1}{\rho^+ + \rho^-} \left( (\rho^+ u_i^+ u_j^+ - h_i^+ h_j^+ + \rho^- u_i^- u_j^- - h_i^- h_j^-) \partial_i \partial_j f \right) \\
+ \frac{2}{\rho^+ + \rho^-} (N_f^+ - N_f^-) \tilde{\mathcal{N}}_f^{-1} \mathcal{P} \left( (\rho^+ u_i^+ + \rho^- u_i^-) \partial_i \theta + (\rho^+ u_i^+ + \rho^- u_i^-) \partial_2 \theta \right) \\
- \frac{1}{\rho^+ + \rho^-} \left( u_i^+ u_j^+ - \rho^- h_i^+ h_j^+ - \rho^+ h_i^- h_j^- \right) \partial_i \partial_j f \\
- \frac{1}{\rho^+ + \rho^-} \mathbf{N} \cdot \left( \nabla (\rho^+ p_{u^+} u^+ - p_{h^+} h^+) + \nabla (\rho^- p_{u^-} u^- - p_{h^-} h^-) \right) \\
+ \frac{1}{\rho^+ + \rho^-} (N_f^+ - N_f^-) \tilde{\mathcal{N}}_f^{-1} \mathcal{P} \mathbf{N} \cdot \left( \nabla (\rho^+ p_{u^+} u^+ - p_{h^+} h^+) \right. \\
\left. - \nabla (\rho^- p_{u^-} u^- - p_{h^-} h^-) \right).\]
Here \(p_{\pm, v_{\pm}}\) is defined in (3.5), and \((u^\pm, h^\pm)\) is the solution to

\[
\begin{align*}
\text{curl}u^\pm &= P^\pm f, \quad \text{div}u^\pm = 0 \quad \text{in} \quad \Omega_f^\pm; \\
u^\pm \cdot N &= \partial_t f \quad \text{on} \quad \Gamma_f; \\
u^\pm \cdot e_3 &= 0, \quad \int_{\Gamma^\pm} u_i dx' = a_i^\pm (i = 1, 2) \quad \text{on} \quad \Gamma^\pm; \\
\partial_t a_i^\pm &= -\int_{\Gamma^\pm} \sum_{j=1}^{3} (u^\pm_j \partial_j u^\pm_i - h^\pm_j \partial_j h^\pm_i) dx',
\end{align*}
\]

and

\[
\begin{align*}
\text{curl}h^\pm &= P^\pm g, \quad \text{div}h^\pm = 0 \quad \text{in} \quad \Omega_f^\pm; \\
h^\pm \cdot N &= 0 \quad \text{on} \quad \Gamma_f; \\
h^\pm \cdot e_3 &= 0, \quad \int_{\Gamma^\pm} h_i dx' = b_i^\pm (i = 1, 2) \quad \text{on} \quad \Gamma^\pm; \\
\partial_t b_i^\pm &= -\int_{\Gamma^\pm} \sum_{j=1}^{3} (u^\pm_j \partial_j h^\pm_i - h^\pm_j \partial_j u^\pm_i) dx',
\end{align*}
\]

and \((\omega^\pm, \xi^\pm)\) satisfies

\[
\rho^\pm \partial_t \omega^\pm + \rho^\pm u^\pm \cdot \nabla \omega^\pm - h^\pm \cdot \nabla \xi^\pm = \rho^\pm \omega^\pm \cdot \nabla u^\pm - \xi^\pm \cdot \nabla h^\pm,
\]

\[
\partial_t \xi^\pm + u^\pm \cdot \nabla \xi^\pm - h^\pm \cdot \nabla \omega^\pm = \xi^\pm \cdot \nabla u^\pm - \omega^\pm \cdot \nabla h^\pm - 2 \sum_{i=1}^{3} \nabla u^\pm_i \times \nabla h^\pm_i.
\]

Next, we will show that the above system is equivalent to the origin system (1.2)–(1.5). We introduce the pressure \(p^\pm\) by

\[
p^\pm = \mathcal{H}^\pm p^\pm + p_{u^\pm, u^\pm} - p_{h^\pm, h^\pm},
\]

where

\[
p^\pm = -\mathcal{N}^{-1}_f (g^+ - g^- \pm \mathcal{N}_f^\pm \sigma H(f)),
\]

with

\[
g^\pm = 2(u_i^\pm \partial_1 \theta + u_2^\pm \partial_2 \theta) + \nabla((p_{u^\pm, u^\pm} - p_{h^\pm, h^\pm}) + \sum_{i,j=1,2} (u_j^\pm u_i^\pm - h_j^\pm h_i^\pm)) \partial_i \partial_j f.
\]

Then, for

\[
k^\pm \overset{\text{def}}{=} \partial_t u^\pm + u^\pm \cdot \nabla u^\pm - h^\pm \cdot \nabla h^\pm + \nabla p^\pm,
\]

or

\[
k^\pm \overset{\text{def}}{=} \partial_t h^\pm + u^\pm \cdot \nabla h^\pm - h^\pm \cdot \nabla u^\pm,
\]

one can check by following the proof in Section 9 of [40] that

\[
\begin{align*}
\text{curl}k^\pm &= 0, \quad \text{div}k^\pm = 0 \quad \text{in} \quad \Omega_f^\pm; \\
k^\pm \cdot N &= 0 \quad \text{on} \quad \Gamma_f; \\
k^\pm \cdot e_3 &= 0, \quad \int_{\Gamma^\pm} k_i dx' = 0 (i = 1, 2) \quad \text{on} \quad \Gamma^\pm.
\end{align*}
\]
which means that $k^\pm \equiv 0$, and $(f, u^\pm, h^\pm, p^\pm)$ is the unique solution to the original system (1.2)–(1.5).

From Definition 5.1, we can see that
\[
\sup_{t \in [0,T]} \left( \|f\|_{H^{r+1}} + \|\partial_t f\|_{H^{r-\frac{1}{2}}} \right) \leq M_1,
\]
\[-(1-c_0) \leq f \leq (1-c_0) \text{ for } t \in [0,T], \ x' \in \mathbb{T}^2.
\]

Then by using Proposition 3.2, we have
\[
\sup_{t \in [0,T]} \left( \|u^\pm\|_{H^r(\Omega^\pm_f)} + \|\partial_t u^\pm\|_{H^{r-1}(\Omega^\pm_f)} + \|h^\pm\|_{H^r(\Omega^\pm_q)} + \|\partial_t h^\pm\|_{H^{r-1}(\Omega^\pm_q)} \right) (t)
\]
\[\leq C(M_1, M_2).
\]

This completes the proof of Theorem 1.1.

6 Zero surface tension limit

In the previous section, we have showed that if the initial data $(f_0, u_0, h_0)$ satisfies the assumption of Theorem 1.1, there is a unique solution $(f^\sigma, u^\sigma, h^\sigma)$ of system (1.2)–(1.7) in time $[0, T^\sigma]$. To study the zero surface tension limit, we need to show that if $(f_0, u_0, h_0)$ additionally satisfies stability condition (1.10) with
\[
\Lambda(h^\pm_0, [u_0]) \geq 2c_0,
\]
the solution $(f^\sigma, u^\sigma, h^\sigma)$ can be extended to a $\sigma$ independent time $\bar{T}$.

Defining energy functions
\[
G_1^\sigma = \frac{1}{2(\rho^+ + \rho^-)^2} \|T_{\gamma^\sigma} T_{\beta^\sigma} T_{\theta^\sigma} f^\sigma\|_{L^2}^2,
\]
\[
G_2^\sigma = \|\partial_t + w^\sigma_{ri} \partial_i\| T_{\beta^\sigma} T_{\theta^\sigma} f^\sigma\|_{L^2}^2 - \frac{1}{2} \|v^\sigma_{ri} \partial_i T_{\beta^\sigma} T_{\theta^\sigma} f^\sigma\|_{L^2}^2
\]
\[+ \frac{1}{2(\rho^+ + \rho^-)^2} \|\partial_t + w^\sigma_{ri} \partial_i\| T_{\beta^\sigma} T_{\theta^\sigma} f^\sigma\|_{L^2}^2 + \|\partial_t - \partial_i T_{\beta^\sigma} T_{\theta^\sigma} f^\sigma\|_{L^2}^2,
\]
and
\[
G_1^\sigma (t) = \|f^\sigma\|_{H^{r+1}}^2, \quad G_2^\sigma (t) = \|f^\sigma\|_{H^{r+1}}^2 + \|\partial_t f^\sigma\|_{H^{r-\frac{1}{2}}}^2,
\]
\[
G^\sigma (t) = \|\partial_t f^\sigma\|_{H^{r-\frac{1}{2}}}^2 + \|f^\sigma\|_{H^{r+\frac{1}{2}}}^2 + \|f^\sigma\|_{H^{r+1}}^2 + \|u^\sigma\|_{H^r(\Omega^\pm_q)}^2 + \|h^\sigma\|_{H^r(\Omega^\pm_q)}^2,
\]
we give the following uniform a priori estimate.

Proposition 6.1 Assume $(f^\sigma, u^\sigma, h^\sigma)$ is the solution of system (1.2)–(1.7) from initial data $(f_0, u_0, h_0)$ in $[0, T]$ such that the stability condition holds on $[0, T]$
\[
\inf_{t \in [0,T]} \Lambda(h^\pm_0, [u^\sigma]) (t) \geq c_0,
\]
then, it holds that
\[
\sup_{t \in [0,T]} G^\sigma (t) \leq C G^\sigma (0) e^{CT},
\]
where $C$ is a constant independent on $\sigma$.
With the assumption (6.1), we can see that
\[
\frac{d}{dt}(\sigma G_1^\sigma + G_2^\sigma + ||f^\sigma||_L^2 + ||\partial_t f^\sigma||_L^2 \\
+ ||\omega^\sigma\pm||_H^{1-1}(\Omega^\pm) + ||\xi^\sigma\pm||_H^{1-1}(\Omega^\pm) + |a_i^\sigma\pm| + |b_i^\sigma\pm|(t) \leq C(G^\sigma)(t) \tag{6.3}
\]
Here \( C(\cdot) \) is a function whose coefficients are independent on \( \sigma \).

Let
\[
G^\sigma = \sigma G_1^\sigma + G_2^\sigma + ||\omega^\sigma\pm||_H^{1-1}(\Omega^\pm) + ||\xi^\sigma\pm||_H^{1-1}(\Omega^\pm) + |a_i^\sigma\pm| + |b_i^\sigma\pm|.
\]

It is clear that
\[
G^\sigma \leq C(G_2^\sigma, ||u^\sigma\pm||_H^1(\Omega^\pm), ||h^\sigma\pm||_H^1(\Omega^\pm))G^\sigma.
\]

Next we will show that \( G^\sigma \) is equivalent to \( G^\sigma \).

By Proposition 3.2 we have
\[
||u^\sigma\pm||_H^1(\Omega^\pm) + ||h^\sigma\pm||_H^1(\Omega^\pm)
\leq C(c_0, ||f^\sigma||_H^{1+1}(\Omega^\pm), ||\partial_t f^\sigma||_H^{1-1}(\Omega^\pm) + ||\omega^\sigma\pm||_H^{1-1}(\Omega^\pm) + ||\xi^\sigma\pm||_H^{1-1}(\Omega^\pm) + |a_i^\sigma\pm| + |b_i^\sigma\pm|).
\]

With the assumption (6.1), we can see that \( G_2^\sigma \) is positive. Indeed, it holds that
\[
\leq C(c_0, ||f^\sigma||_H^{1+1}(\Omega^\pm), ||\partial_t f^\sigma||_H^{1-1}(\Omega^\pm) + ||\omega^\sigma\pm||_H^{1-1}(\Omega^\pm) + ||\xi^\sigma\pm||_H^{1-1}(\Omega^\pm) + |a_i^\sigma\pm| + |b_i^\sigma\pm|)
\]

Using Lemma 4.15, we can easily deduce that
\[
\partial_t f^\sigma \leq ||\omega^\sigma\pm||_H^{1-1}(\Omega^\pm) + ||\xi^\sigma\pm||_H^{1-1}(\Omega^\pm) + |a_i^\sigma\pm| + |b_i^\sigma\pm|
\]

Remark 6.2 From Proposition 6.1, we can get uniform estimates for \( ||\partial_t u^\sigma\pm||_H^{1-1}(\Omega^\pm) \) and \( ||\partial_t h^\sigma\pm||_H^{1-1}(\Omega^\pm) \), which determine the value of \( \Lambda(h^\sigma\pm, [u^\sigma]) \). Note that \( \Lambda(h^\sigma\pm, [u^\sigma]) \geq 2c_0 \). Then, by using the continuous argument, one can easily show that the solution \( (f^\sigma, u^\sigma, h^\sigma) \) got in Theorem 1.1 can be extended to a lifespan \( \tilde{T} \) independent of \( \sigma \).

Now, we give the proof of Theorem 1.2.

The initial data \( (f_0, u_0^\pm, h_0^\pm) \) satisfies all the assumption in Theorem 1.1, then there exist a unique solution \( (f^\sigma, u^\sigma, h^\sigma) \) of the system (1.2)–(1.5) in \( T^\sigma \). Moreover, from the assumption that
\[
\Lambda(h_0^\pm, [u_0]) \geq 2c_0.
\]
one can see from Proposition 6.1 and Remark 6.2 that the solutions can be extended to the one with a lifespan $T$ independent of $\sigma$. We also denote by $(f^\sigma, u^\sigma, h^\sigma)$ the extended solutions in $[0, T]$ that satisfy

1. $\sup_{t \in [0, T]} \left( \| f^\sigma \|_{H^{s+\frac{1}{2}}} + \| \partial_t f^\sigma \|_{H^{s-\frac{1}{2}}} \right)(t) 
   \leq C(c_0, \rho^+, \rho^-, \| f_0 \|_{H^{s+1}}, \| u_0^\sigma \|_{H^s(\Omega^\pm_0)}, \| h_0^\sigma \|_{H^s(\Omega^\pm_0)}),$

2. $\sup_{t \in [0, T]} \left( \| u^\sigma\pm \|_{H^s(\Omega^\pm_0)} + \| \partial_t u^\sigma\pm \|_{H^{s-1}(\Omega^\pm_0)} + \| h^\sigma\pm \|_{H^s(\Omega^\pm_0)} + \| \partial_t h^\sigma\pm \|_{H^{s-1}(\Omega^\pm_0)} \right)(t)
   \leq C(c_0, \rho^+, \rho^-, \| f_0 \|_{H^{s+1}}, \| u_0^\sigma \|_{H^s(\Omega^\pm_0)}, \| h_0^\sigma \|_{H^s(\Omega^\pm_0)}),$

3. $- (1 - c_0) \leq f^\sigma(t) \leq (1 - c_0)$ for $t \in [0, T], x' \in \mathbb{T}^2,$

4. $\inf_{t \in [0, T]} \Lambda(h^\sigma\pm, [u^\sigma]) \geq c_0.$

Recalling that $f_\ast = f_0$, and $\Omega^\pm_0 = \Omega^\pm_{f_0}$, we introduce

$$u^\sigma\pm_0 = u^\sigma\pm \circ \Phi^\pm_0, \quad h^\sigma\pm_0 = h^\sigma\pm \circ \Phi^\pm_0,$$

where $\Phi^\pm_0 : \Omega^\pm_0 \to \Omega^\pm_f$ is defined in (2.1). It holds that

$$\partial_t u^\sigma_\ast = (\partial_t u^\sigma) \circ \Phi^\pm_0 + \partial_t \Phi^\pm_0 (\partial_3 u^\sigma) \circ \Phi^\pm_0,$$

$$\partial_t h^\sigma_\ast = (\partial_t h^\sigma) \circ \Phi^\pm_0 + \partial_t \Phi^\pm_0 (\partial_3 h^\sigma) \circ \Phi^\pm_0,$$

and $\partial_t \Phi^\pm_0$ is the solution of

$$\begin{cases}
   \Delta_y \partial_t \Phi^\pm_0(t) = 0 & \text{if } y \in \Omega^\pm_0, \\
   \partial_t \Phi_\ast^\pm(t)(y', f_\ast(y')) = \partial_t f_\ast(t, y') & \text{if } y' \in \mathbb{T}^2, \\
   \partial_t \Phi^\pm_0(t)(y', \pm 1) = 0 & \text{if } y' \in \mathbb{T}^2.
   \end{cases}$$

Then we have

$$\| \partial_t u^\sigma_\ast \|_{H^{s-1}(\Omega^\pm_0)}^2 \leq C(\| f^\sigma \|_{H^{s+\frac{1}{2}}}^2, \| \partial_t f^\sigma \|_{H^{s-\frac{1}{2}}}^2 (\| \partial_t u^\sigma_\ast \|_{H^{s-1}(\Omega^\pm_0)}^2 + \| u^\sigma_\ast \|_{H^s(\Omega^\pm_0)}^2),$$

$$\| \partial_t h^\sigma_\ast \|_{H^{s-1}(\Omega^\pm_0)}^2 \leq C(\| f^\sigma \|_{H^{s+\frac{1}{2}}}^2, \| \partial_t f^\sigma \|_{H^{s-\frac{1}{2}}}^2 (\| \partial_t h^\sigma_\ast \|_{H^{s-1}(\Omega^\pm_0)}^2 + \| h^\sigma_\ast \|_{H^s(\Omega^\pm_0)}^2).$$

It follows that

$$\sup_{t \in [0, T]} \left( \| f^\sigma \|_{H^{s+\frac{1}{2}}}^2 + \| \partial_t f^\sigma \|_{H^{s-\frac{1}{2}}}^2 + \| u^\sigma_\ast \|_{H^s(\Omega^\pm_0)}^2 + \| \partial_t u^\sigma_\ast \|_{H^{s-1}(\Omega^\pm_0)}^2 \right)(t) \leq C.$$

Let $\{\sigma_k\}_k$ be a positive sequence that tends to zero. We introduce a sequence $\{(f^{\sigma_k}, u^{\sigma_k}_\ast, u^{\sigma_k}_{\pm\ast})\}_k$. By the Aubin-Lions lemma, we can see that $\{(f^{\sigma_k}, u^{\sigma_k}_\ast, u^{\sigma_k}_{\pm\ast})\}_k$ is relatively compact in

$$C([0, T], H^{s+\frac{1}{2}}(\mathbb{T}^2)) \times C([0, T], H^{s-1}(\Omega^\pm)) \times C([0, T], H^{s-1}(\Omega^\pm_0)).$$

Thus there exist a subsequence of $\{(f^{\sigma_k}, u^{\sigma_k}_\ast, u^{\sigma_k}_{\pm\ast})\}_k$ that converges to some $(f, u_\ast, h_\ast)$ in

$$C([0, T], H^{s-\frac{1}{2}}(\mathbb{T}^2)) \times C([0, T], H^{s-1}(\Omega^\pm)) \times C([0, T], H^{s-1}(\Omega^\pm)).$$
Let $u^\pm = u_s^\pm \circ \Phi_f^{\pm -1}$ and $h^\pm = h_s^\pm \circ \Phi_f^{\pm -1}$. By a standard compactness argument, we can prove that $(f, u, h)$ is a solution of the system (1.2)–(1.5) with $\sigma = 0$, which is the solution get in [28].

7 Remark on the one fluid problem

In this paper, we study the two phase flow problem with surface tension in the ideal incompressible magnetohydrodynamics. We give a proof of local well-posedness and zero surface tension limit for the case $\rho^+, \rho^- > 0$. The method developed in this paper still works for one fluid problem. For the case there is no fluid and no magnetic in the upper domain, the evolution equation of $f$ is

$$
\partial_t^2 f = -2(u^- \partial_1 \theta + u^- \partial_2 \theta) + \frac{\sigma}{\rho^-} N_f^- H(f) - \frac{1}{\rho^-} \nabla \cdot (\rho^- p_+ u^- - p_- h^-)
$$

For the three order term $\frac{\sigma}{\rho^-} N_f^- H(f)$, it holds that

$$
\frac{\sigma}{\rho^-} N_f^- H(f) = -\frac{\sigma}{\rho^-} T_{\gamma^-} T_{\gamma^-} f + \frac{\sigma}{\rho^-} R^-,
$$

and $T_q T_{\gamma^-} T_{\gamma^-} \sim T_{\gamma^-} T_{\gamma^-} T_q$ and $T_{\gamma^-} \sim (T_{\gamma^-})^*$, where

$$
\gamma^- = \sqrt{I(2) \lambda^- (1)} + \frac{1}{2} \sqrt{I(2) \lambda^- (1)} \Re(\lambda^- (0)) + \frac{1}{2} i (\partial_x \cdot \partial_y) \sqrt{I(2) \lambda^- (1)}.
$$

Then one can get local well-posedness of the one fluid problem by using the method developed in this paper.

Acknowledgements The authors wish to express their thanks to Prof. Zhifei Zhang and Prof. Wei Wang for suggesting the problem and for many helpful discussions. This work was supported by NSF of China under Grant No. 11871424.

Appendix A

A.1 Paradifferential Operator

In this subsection we will introduce some notations and results about Bony’s paradifferential calculus. Here we follow the presentation by Métivier in [31], for the general theory we refer to [11, 23, 31, 32] and [43].

For $\rho \in \mathbb{N}$, we denote $W^{\rho, \infty}(\mathbb{T}^d)$ the space of continuous and bounded functions on $\mathbb{T}^d$ such that

$$
[u]_\rho = \sup \frac{|u(x) - u(y)|}{|x - y|^\rho} < +\infty.
$$
For \( \rho \in (0, \infty)/\mathbb{N} \), denoting by \([\rho]\) the greatest integer \(< \rho\), the space \(W^{\rho, \infty}(\mathbb{T}^d)\) is the space of functions in \(W^{[\rho], \infty}(\mathbb{T}^d)\) such that their derivatives of order \([\rho]\) belong to \(W^{\rho-[\rho], \infty}(\mathbb{T}^d)\). This is consistent with Definition 4.12 of [31].

**Definition A.1** Given \( \rho \geq 0 \) and \( m \in \mathbb{R} \), denote by \( \Gamma^m_\rho(\mathbb{T}^d) \) the space of locally bounded functions \( a(x, \xi) \) on \( \mathbb{T}^d \times \mathbb{R}^d/\{0\} \), which are \( C^\infty \) with respect to \( \xi \) for \( \xi \neq 0 \) and such that, for all \( \alpha \in \mathbb{N}^d \) and all \( \xi \neq 0 \), the function \( x \to \partial_\xi^\alpha a(x, \xi) \) belongs to \( W^{\rho, \infty} \) and there exists a constant \( C_\alpha \) such that

\[
\| \partial_\xi^\alpha a(x, \xi) \|_{W^{\rho, \infty}} := C_\alpha (1 + |\xi|)^{m-|\alpha|} \quad \forall |\xi| \geq \frac{1}{2}
\]

The seminorm of the symbol is defined by

\[
M^m_\rho(a) := \sup_{|\alpha| \leq m} \sup_{|\xi| \geq \frac{1}{2}} \| (1 + |\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot, \xi) \|_{W^{\rho, \infty}}.
\]

If \( a \) is a function independent of \( \xi \), then

\[
M^m_\rho(a) = \| a \|_{W^{\rho, \infty}}.
\]

Given a symbol \( a \), the paradifferential operator \( T_a \) is defined by

\[
\widehat{T_a u}(\xi) := (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \psi(\eta) \hat{u}(\eta) d\eta,
\]

where \( \hat{a}(x, \xi) \) is the Fourier transform of \( a \) with respect to the first variable, \( \chi(\theta, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) is an admissible cutoff function: there exists \( \varepsilon_1, \varepsilon_2 \) small enough such that \( 0 \leq \varepsilon_1 < \varepsilon_2 \psi \) and

\[
\chi(\theta, \eta) = 1 \text{ if } |\theta| \leq \varepsilon_1|\eta|, \quad \chi(\theta, \eta) = 0 \text{ if } |\theta| \geq \varepsilon_2|\eta|.
\]

and such that for any \((\theta, \xi) \in \mathbb{R}^d \times \mathbb{R}^d\),

\[
|\partial_\theta^\alpha \partial_\eta^\beta \chi(\theta, \xi)| \leq C_{\alpha, \beta}(1 + |\eta|)^{-|\alpha|-|\beta|}.
\]

The cutoff function \( \psi(\eta) \in C^\infty(\mathbb{R}^d) \) satisfies

\[
\psi(\eta) = 0 \text{ for } |\eta| \leq 1, \quad \psi(\eta) = 1 \text{ for } |\eta| \geq 2,
\]

Here we will take the admissible cutoff function \( \chi(\theta, \xi) \)

\[
\chi(\theta, \xi) = \sum_{k=0}^{\infty} \xi_k - 3(\theta) \phi_k(\eta),
\]

where \( \xi(\theta) = 1 \) for \(|\theta| \leq 1.1 \), \( \xi(\theta) = 0 \) for \(|\theta| \geq 1.9 \), and

\[
\left\{
\begin{array}{ll}
\xi_k(\theta) = \xi (2^{-k}\theta) & \text{for } k \in \mathbb{Z} \\
\varphi_0 = \xi, \varphi_k = \xi_k - \xi_{k-1} & \text{for } k \geq 1
\end{array}
\right.
\]

We also introduce the Littlewood-Paley operators \( \Delta_k \), \( S_k \) defined by

\[
\Delta_k u = \mathcal{F}^{-1} \left( \varphi_k(\xi) \hat{u}(\xi) \right) \text{ for } k \geq 0, \quad \Delta_k u = 0 \text{ for } k < 0,
\]

\[
S_k u = \sum_{\ell \leq k} \Delta_{\ell} u \text{ for } k \in \mathbb{Z}.
\]
In the case when the function $a$ depends only on the first variable $x$ in $T_a u$, we take $\psi = 1$. Then $T_a u$ is just the usual Bony’s paraproduct defined by

$$T_a u = \sum_k S_k a \Delta_k u.$$ 

We have the following well-known Bony’s decomposition (see [9]):

$$a u = T_a u + T_a a + R_B(u, a),$$

where the remainder term $R_B(u, a)$ is defined by

$$R_B(u, a) = \sum_{|k - \ell| \leq 2} \Delta_k a \Delta_\ell u.$$ 

The above definition is consistent with Definition A.1 of [40] and Definition 3.1, 3.2, 3.3 of [2].

We list the main features of symbolic calculus for paradifferential operators.

**Proposition A.2** Let $m \in \mathbb{R}$. If $a \in \Gamma^m_0(\mathbb{T}^d)$, then $T_a$ is of order $m$. Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $C$ such that

$$\|T_a\|_{H^\mu \to H^{\mu - m}} \leq C M^m_\rho (a).$$

See Theorem 3.6 of [2] or Proposition A.3 of [40].

**Lemma A.3** If $s > 0$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 + s_2 = s + d/2$, then we have

$$\|R_B(u, a)\|_{H^s} \leq C \|a\|_{H^{s_1}} \|u\|_{H^{s_2}}.$$ 

See (3.5.13) of [43] or Lemma A.2 of [40].

**Proposition A.4** Let $m \in \mathbb{R}$, and let $\rho > 0$. If $a \in \Gamma^m_\rho(\mathbb{T}^d)$, $b \in \Gamma^{m'}_\rho(\mathbb{T}^d)$, then $T_a T_b - T_{a \star b}$ is of order $m + m' - \rho$ where

$$a \star b = \sum_{|\alpha| < \rho} \frac{1}{i|\alpha|!} \partial_\xi^\alpha a \partial_\xi^\beta b.$$ 

Furthermore, $\forall \mu \in \mathbb{R}$ there exists a constant $C$ such that

$$\|T_a T_b - T_{a \star b}\|_{H^{\mu - m} \to H^{\mu - m + \rho}} \leq C M^m_\rho (a) M^{m'}_\rho (b).$$

See Theorem 3.7 of [2] or Theorem 6.1.4 of [31].

**Proposition A.5** Let $m \in \mathbb{R}$, let $\rho > 0$, and let $a \in \Gamma^m_\rho(\mathbb{T}^d)$. Denote by $(T_a)^*$ the adjoint operator of $T_a$ and by $\bar{a}$ the complex conjugate of $a$. Then $(T_a)^* - T_a^*$ is of order $m - \rho$ where

$$a^* = \sum_{|\alpha| < \rho} \frac{1}{i|\alpha|!} \partial_\xi^\alpha \partial_\xi^\beta \bar{a}.$$ 

Furthermore, $\forall \mu \in \mathbb{R}$ there exists a constant $C$ such that

$$\|(T_a)^* - T_a^*\|_{H^{\mu - m + \rho}} \leq C M^m_\rho (a).$$

See Theorem 3.10 of [2] or Theorem 6.2.1 of [31].
Lemma A.6 Let $m > 0$. If $a \in H^{\frac{d}{2} - m}(\mathbb{T}^d)$ and $u \in H^\mu(\mathbb{T}^d)$, then $T_a u \in H^{\mu - m}(\mathbb{T}^d)$. Moreover,

$$\|T_a u\|_{H^{\mu - m}} \leq C \|a\|_{H^{\frac{d}{2} - m}} \|u\|_{H^\mu},$$

where the constant $C$ is independent of $a$ and $u$.

See Lemma 3.11 of [2] or Lemma A.2 of [40].

Proposition A.7 Let $\alpha \in \mathbb{R}$ such that $\alpha > \frac{d}{2}$. For all $F \in C^\infty$, if $\forall a \in H^\alpha(\mathbb{T}^d)$, then

$$F(a) - F(0) - T_{F'(a)} a \in H^{2\alpha - \frac{d}{2}}(\mathbb{T}^d).$$

(A.1)

See Theorem 3.12 of [2] or Theorem 5.2.4 of [31].

Lemma A.8 If $s > 1 + \frac{d}{2}$, then we have

$$\|[(a, (\nabla)^s]u\|_{L^2} \leq C \|a\|_{H^{s}} \|u\|_{H^{s-1}}.$$  

(A.2)

See Lemma A.4 of [40].

A.2 Elliptic estimate in a strip

Let $S_f \overset{\text{def}}{=} \{(x, y) : x \in \mathbb{T}^2, -1 < y < f(x)\}$ be a strip, where $f(x)$ satisfies

$$1 + f(x) \geq c_0 > 0 \text{ for } x \in \mathbb{T}^2.$$

We consider the elliptic boundary value problem in $S_f$ :

$$\begin{cases}
\Delta_{x,y} \Phi = 0 & \text{in } S_f, \\
\Phi(x, f(x)) = \psi(x) & \text{for } x \in \mathbb{T}^2, \\
\partial_y \Phi(x, -1) = 0 & \text{for } x \in \mathbb{T}^2.
\end{cases}$$

(A.3)

The Lax-Milgram theorem ensures that for $\phi(x) \in H^{\frac{1}{2}}(\mathbb{T}^2)$, there exists a unique weak solution $\Phi(x, z) \in H^1(S_f)$ satisfying

$$\|\Phi\|_{H^1(S_f)} \leq C \|\phi\|_{H^{\frac{1}{2}}},$$

where the constant $C$ depends on $c_0$ and $\|f\|_{W^{1,\infty}}$.

Proposition A.9 Let $\Phi \in H^1(S_f)$ be a weak solution of (A.3). Assume that $f \in H^{s+\frac{1}{2}}(\mathbb{T}^d)$ for $s > \frac{d}{2} + \frac{1}{2}$. Then for any integer $\sigma \in [0, s]$, it holds that

$$\|\Phi\|_{H^{\sigma+1}(S_f)} \leq C \left(c_0, \|f\|_{H^{\sigma+\frac{1}{2}}}, \|\psi\|_{H^{\sigma+\frac{1}{2}}} \right).$$

See Proposition A.5 of [40].

A.3 Sobolev estimates of DN operator

In what follows, we denote by $K_{s,f}$ a constant depending on $c_0$ and $\|f\|_{H^{\sigma}}$ which may be different from line to line.
Proposition A.10 If \( f \in H^{s + \frac{1}{2}}(\mathbb{T}^2) \) for \( s > \frac{5}{2} \), then we can write
\[
\mathcal{N}_f^\pm \psi = T_{\pm}^{\pm(1)} \psi + R_f^\pm \psi.
\]
It holds that for any \( \sigma \in \left[ \frac{1}{2}, s - \frac{1}{2} \right] \),
\[
\| R_f^\pm \psi \|_{H^\sigma} \leq K_{s + \frac{1}{2}, f} \| \psi \|_{H^\sigma}.
\] (A.4)

See Lemma A.9 of [40].

Proposition A.11 If \( f \in H^{s + \frac{1}{2}}(\mathbb{T}^2) \) for \( s > \frac{5}{2} \), then it holds that for any \( \sigma \in \left[ -\frac{1}{2}, s - \frac{1}{2} \right] \),
\[
\| \mathcal{N}_f^\pm \psi \|_{H^\sigma} \leq K_{s + \frac{1}{2}, f} \| \psi \|_{H^{\sigma + 1}}.
\] (A.5)

Moreover, it holds that for any \( \sigma \in \left[ \frac{1}{2}, s - \frac{1}{2} \right] \),
\[
\| (\mathcal{N}_f^+ - \mathcal{N}_f^-) \psi \|_{H^\sigma} \leq K_{s + \frac{1}{2}, f} \| \psi \|_{H^\sigma},
\] (A.6)

where \( K_{s + \frac{1}{2}, f} \) is a constant depending on \( c_0 \) and \( \| f \|_{H^s} \).

See Proposition A.10 of [40].

Proposition A.12 If \( f \in H^{s + \frac{1}{2}}(\mathbb{T}^2) \) for \( s > \frac{5}{2} \), \( \rho^+, \rho^- > 0 \) are two constants, then it holds for any \( \sigma \in \left[ -\frac{1}{2}, s - \frac{1}{2} \right] \) that
\[
\| \mathcal{N}_f^{-1} \psi \|_{H^{\sigma + 1}} \leq K_{s + \frac{1}{2}, f} \| \psi \|_{H^\sigma},
\] (A.7)

where \( \mathcal{N}_f \overset{\text{def}}{=} \frac{1}{\rho^+} \mathcal{N}_f^+ + \frac{1}{\rho^-} \mathcal{N}_f^- \).

Proof This proposition is an imitation of Proposition A.11 in [40]. One can get the estimate by following the proof in [40]. \( \square \)

References
1. Axford, W.I.: Note on a problem of magnetohydrodynamic stability. Canad. J. Phys. 40, 654–655 (1962)
2. Alazard, T., Burq, N., Zuily, C.: On the water-wave equations with surface tension. Duke Math. J. 158, 413–499 (2011)
3. Alazard, T., Burq, N., Zuily, C.: On the Cauchy problem for gravity water waves. Invent. Math. 198, 71–163 (2014)
4. Alazard, T., Métivier, G.: Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves. Comm. Partial Differ. Equ. 34, 1632–1704 (2009)
5. Ambrose, D.M.: Well-posedness of vortex sheets with surface tension. SIAM J. Math. Anal. 35, 211–244 (2003)
6. Ambrose, D.M., Masmoudi, N.: The zero surface tension limit of two-dimensional water waves. Comm. Pure Appl. Math. 58, 1287–1315 (2005)
7. Ambrose, D.M., Masmoudi, N.: Well-posedness of 3D vortex sheets with surface tension. Comm. Math. Sci. 5, 391–430 (2007)
8. Ambrose, D.M., Masmoudi, N.: The zero surface tension limit of three-dimensional water waves. Indiana U. Math. J. 58, 479–521 (2009)
9. Bahouri, H., Chemin, J.Y., Danchin, R.: Fourier analysis and nonlinear partial differential equations. Springer, Heidelberg (2011)
10. Beale, J.T., Hou, T.Y., Lowengrub, J.S.: Growth rates for the linearized motion of fluid interfaces away from equilibrium. Comm. Pure Appl. Math. 46, 1269–1301 (1993)
11. Bony, J.M.: Calcul symbolique et propagation des singularites pour les equations aux derivees partielles non lineaires. Ann. Sci. Ecole Norm. Sup. (4) 14, 209–246 (1981)
12. Caflisch, R.E., Orellana, O.F.: Long time existence for a slightly perturbed vortex sheet. Comm. Pure Appl. Math. 39, 807–838 (1986)
13. Chen, G.-Q., Wang, Y.-G.: Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics. Arch. Ration. Mech. Anal. 187, 369–408 (2008)
14. Cheng, C.A., Coutand, D., Shkoller, S.: On the motion of vortex sheets with surface tension in three-dimensional Euler equations with vorticity. Comm. Pure. Appl. Math. 61, 1715–1752 (2008)
15. Cheng, C., Coutand, D., Shkoller, S.: On the limit as the density ratio tends to zero for two perfect incompressible fluids separated by a surface of discontinuity. Comm. Partial Differ. Equ. 35, 817–845 (2010)
16. Christodoulo, D., Lindblad, H.: On the motion of the free surface of a liquid. Comm. Pure Appl. Math. 53, 1536–1602 (2000)
17. Coulombel, J.-F., Morando, A., Secchi, P., Trebeschi, P.: A priori estimates for 3D incompressible current-vortex sheets. Comm. Math. Phys. 311, 247–275 (2012)
18. Coutand, D., Shkoller, S.: Well-posedness of the free-surface incompressible Euler equations with or without surface tension. J. A. Math. Soc. 20, 829–930 (2007)
19. Gu, X., Wang, Y.: On the construction of solutions to the free-surface incompressible ideal magnetohydrodynamic equations. J. Math. Pure. Appl. 128, 1–41 (2019)
20. Gu, X., Lei, Z.: Local well-posedness of the free boundary incompressible elastodynamics with surface tension, arXiv:2008.13354
21. Hao, C.: On the motion of free interface in ideal incompressible MHD. Arch. Ration. Mech. Anal. 224, 515–553 (2017)
22. Hao, C., Luo, T.: A priori estimates for free boundary problem of incompressible inviscid magnetohydrodynamic flows. Arch. Ration. Mech. Anal. 212, 805–847 (2014)
23. Hormander, L.: Lectures on nonlinear hyperbolic differential equations, vol. 26. Springer-Verlag, Berlin (1997)
24. Iooss, G., Plotnikov, P.I.: Small divisor problem in the theory of three-dimensional water gravity waves. Mem. Amer. Math. Soc. 200, 427–433 (2009)
25. La Belle-Hamer, A.L., Fu, Z.F., Lee, L.C.: A mechanism for patchy reconnection at the dayside magnetopause. Geophys. Res. Lett. 15, 152–155 (1988)
26. Landau, L.D., Lifshitz, E.M., Pitaevskii, L.P.: Electrodynamics of continuous media, 2nd edn. Pergamon Press, Oxford (1984)
27. Lannes, D.: Well-posedness of the water-waves equations. J. A. Math. Soc. 18, 605–654 (2005)
28. Li, C., Li, H.: Well-posedness of the two-phase flow problem in incompressible MHD. Discret. Contin. Dyn. Syst. 41, 5609–5632 (2021)
29. Luo, C., Zhang, J.: A priori estimates for the incompressible free-boundary magnetohydrodynamics equations with surface tension. SIAM J. Math. Anal. 53, 2595–2630 (2021)
30. Majda, A., Bertozzi, A.: Vorticity and incompressible flow. Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge (2002)
31. Metivier, G.: Para-differential calculus and applications to the Cauchy problem for nonlinear systems. Edizioni della Normale, Pisa, (5) (2008)
32. Meyer, Y.: Remarques sur un théorème de J.-M. Bony. Rend. Circ. Mat. Palermo 2, 1–20 (1981)
33. Michael, D.H.: The stability of a combined current and vortex sheet in a perfectly conducting fluid. Proc. Cambridge Phil. Soc. 51, 528–532 (1955)
34. Miloshevsky, G.V., Hassanein, A.: Modelling of Kelvin-Helmholtz instability and splashing of melt layers from plasma-facing components in tokamaks under plasma impact. Nucl. Fusion 15, 115005 (2010)
35. Ming, M., Zhang, Z.: Well-posedness of the water-wave problem with surface tension. J. Math. Pures. Appl. 92, 429–455 (2009)
36. Morando, A., Trakhinin, Y., Trebeschi, P.: Stability of incompressible current-vortex sheets. J. Math. Anal. Appl. 347, 502–520 (2008)
37. Ofman, L., Chen, X.L., Morrison, P.J., et al.: Resistive tearing mode instability with shear flow and viscosity. Phys. Fluids B: Plasma Phys. 3, 1364 (1991)
38. Secchi, P., Trakhinin, Y.: Well-posedness of the plasma-vacuum interface problem. Nonlinearity 27, 105–169 (2014)
39. Syrovatskij, S.I.: The stability of tangential discontinuities in a magnetohydrodynamic medium. Z. Eksperim. Teoret. Fiz. 24, 622–629 (1953)
40. Sun, Y., Wang, W., Zhang, Z.: Nonlinear stability of the current-vortex sheet to the incompressible MHD equations. Comm. Pure Appl. Math. 71, 356–403 (2018)
41. Shatah, J., Zeng, C.: Geometry and a priori estimates for free boundary problems of the Euler’s equation. Comm. Pure Appl. Math. 61, 698–744 (2008)
42. Shatah, J., Zeng, C.: A priori estimates for fluid interface problems. Comm. Pure Appl. Math. 61, 848–876 (2008)
43. Taylor, M.E.: Pseudodifferential operators and nonlinear PDE, p. 213. Birkhäuser Boston Inc, Boston, MA (1991)
44. Trakhinin, Y.: On the existence of incompressible current-vortex sheets: study of a linearized free boundary value problem. Math. Methods Appl. Sci. 28, 917–945 (2005)
45. Trakhinin, Y.: Existence of compressible current-vortex sheets: variable coefficients linear analysis. Arch. Ration. Mech. Anal. 177, 331–366 (2005)
46. Trakhinin, Y.: The existence of current-vortex sheets in ideal compressible magnetohydrodynamics. Arch. Ration. Mech. Anal. 191, 245–310 (2009)
47. Trakhinin, Y.: On the well-posedness of a linearized plasma-vacuum interface problem in ideal compressible MHD. J. Differ. Equ. 249, 2577–2599 (2010)
48. Wang, Y.-G., Yu, F.: Stabilization effect of magnetic fields on two-dimensional compressible current-vortex sheets. Arch. Ration. Mech. Anal. 208, 341–389 (2013)
49. Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 2-D. Invent. Math. 130, 39–72 (1997)
50. Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 3-D. J. Amer. Math. Soc. 12, 445–495 (1999)
51. Wu, S.: Mathematical analysis of vortex sheets. Comm. Pure Appl. Math. 59, 1065–1206 (2006)
52. Zhang, P., Zhang, Z.: On the free boundary problem of three-dimensional incompressible Euler equations. Comm. Pure Appl. Math. 61, 877–940 (2008)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.