Title: Stabilization Properties of Symmetric Chains of Ideals

Outline: ① Sym-invariant chains of ideals
② Free resolutions + Betti tables
③ Stabilization of Betti tables

Reference: "Asymptotic Behavior of Symmetric Ideals: A Brief Survey"
Martha Juhnke-Kubitzke, Dinh Van Le, Tim Römer
A (commutative) Noetherian ring

- satisfies ACC on ideals: For every chain \( I_1 \subset I_2 \subset \cdots \) in \( \mathbb{N} \) s.t. \( I_n = I_{n+1} = \cdots \)
- has every ideal finitely generated.

Example of a non-Noetherian ring: \( R = k[x_1, x_2, x_3, \ldots] \), \( k \) a field

- The chain \( 0 \subset \langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \cdots \) doesn't stabilize.
- The ideal \( I = \langle x_1, x_2 \rangle \cdots \) has no finite generating set.

\( \text{Sym}(\infty) \) to the rescue!

\[ \text{Sym}(\infty) = \bigcup_{n \geq 0} \text{Sym}(n), \quad \text{where if } m \geq n, \quad \text{Sym}(n) \subset \text{Sym}(m) \text{ stabilizes } \{n+1, \ldots, m\} \]
Sym(\infty) \cong R: \ \sigma \cdot x_i := x_{\sigma(i)}

Sym\, e.g. \ (58) \cdot (x_1^4 x_5 + x_8 x_7) = x_1^4 x_8 + x_5^2 \cdot x_7

We can write \ I = \langle \text{Sym} (x_1) \rangle = \langle \sigma \cdot x_i \mid \sigma \in \text{Sym} (\infty) \rangle

- I is \underline{Sym-invariant}: \text{Sym}(I) \subseteq I
- I is generated by finitely many \underline{Sym-orbits}.

Another: \ I = \langle \text{Sym} (\infty) \cdot (x_1^n + \ldots + x_n^n) \mid n \geq 4 \rangle

= \langle \text{Sym} (\infty) \cdot x_1^4 \rangle \ \text{if char}(k) \neq 2

**Theorem**: Cohen (1987) \ R is "\underline{Sym-Noetherian}":

Any \underline{Sym-invariant ideal} \ I \subset R \ has \ I = \langle \text{Sym}(f_1), \text{Sym}(f_2), \ldots, \text{Sym}(f_k) \rangle

for some \ k < \infty.
A more general setup

Fix $c \geq 1$. \[ R := k[x_{ij} \mid 1 \leq i \leq c, \; j \geq 1] \]
\[ R_n := k[x_{ij} \mid 1 \leq i \leq c, \; 1 \leq j \leq n] \]

\text{sym}(R) \otimes R \text{ via }

\[ \sigma \cdot x_{ij} := x_{i, \sigma(j)} \]

(13) \quad \begin{align*}
2^2 x_{2,1} x_{8,2} x_{7,3}^5 &= 2^2 x_{2,3} x_{8,2} x_{7,1}^2
\end{align*}

$R$ is also $\text{Sym}$-Noetherian (Hillar–Sullivant, 2012).

Example:

\[ I = \langle 2 \times 2 \text{ minors of a } 3 \times 6 \text{ generic matrix} \rangle \]

\[
\begin{bmatrix}
\text{X1,1} & \text{X1,2} & X_{1,3} & \cdots \\
X_{2,1} & X_{2,2} & X_{2,3} & \cdots \\
X_{3,1} & X_{3,2} & X_{3,3} & \cdots
\end{bmatrix}
= \langle \text{Sym} (\text{minors from 1st 2 columns}) \rangle
\]

Row set $\{1,3\}$

Col set $\{i,j\}$

1 $\mapsto$ i

2 $\mapsto$ j
Can consider truncations \( \text{In} := I \cap \mathbb{R}^n \)

\( \text{In} = \langle 2 \times 2 \text{ minors of a } 3 \times n \text{ generic matrix } \rangle \)

\[
\begin{align*}
\text{Sym}(1) & \subseteq \text{Sym}(2) \subseteq \text{Sym}(3) \subseteq \cdots, \\
\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} & \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}
\end{align*}
\]

This is a \textit{Sym-invariant chain}: \( \text{Sym}(m)(\text{In}) \subseteq \text{Im} \) for each \( m \geq n \).

\[
\begin{align*}
\left\{ \text{Sym-invariant chains} \right\} \quad \left\{ \text{Sym-invariant ideals} \right\} \\
\{ \text{(In)}_{n \geq 0} \} \quad \{ I \subseteq \mathbb{R} \}
\end{align*}
\]
Motivating Question What can we say about asymptotic behavior of sym-invariant chains?

Who cares?
- Representation theorists
- Commutative algebraists
- Algebraic statisticians
- Chemists?? chirality

Some results:
(Nagel–Römer, 2017)
- $\dim(\mathcal{R}_n/\mathcal{I}_n)$ is eventually a linear function of $n$
- $\deg(\mathcal{I}_n)$ exponential

Murai (2019), Reich (2019)
- Homological properties stabilize in $c=1$ case.
Free Resolutions + Betti Tables

Fix $c=1$. $R = k[x_1, x_2, ...]$ $R_n = k[x_1, ..., x_n]$

For homogeneous ideals $J \subseteq R_n$, we can compute a graded free resolution of $J$:

$$0 \leftarrow J \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$$

$J = \langle x_1^5 x_2, x_1 x_2^5, x_1^2 x_2^2 \rangle \subset R_2 = S$

$X_{i_1} \cdot x_1^5 x_2 - x_1^3 \cdot x_1^2 x_2^2 = 0$

$S(-6) \leftarrow \begin{bmatrix} x_2 & 0 \\ -x^3 & -x^3 \\ 0 & x_1 \end{bmatrix} \leftarrow S(-7) \leftarrow S(-7)$
Hilbert's Syzygy theorem

Minimal free resolutions of finitely-generated
$k[x_1,\ldots,x_n]$-modules are finite (of length ≤ n).

\[
0 \leftarrow J \leftarrow F_0 \leftarrow \cdots \leftarrow F_i \leftarrow \cdots \leftarrow F_{\ell} \leftarrow 0
\]

\[
= \bigoplus_{j} S(-j)^{\beta_{ij}}
\]

\[l = \text{pdim} (J)\]

Arrange the $\beta_{ij}$ into a Betti table $[\beta_{i,i+j}]$.

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 |   |   |   |   |   |   |
| 1 |   |   |   |   |   |   |
| 2 |   |   |   |   |   |   |
| 3 |   |   |   |   |   |   |
| 4 |   | 1 |   |   |   |   |
| 5 |   |   |   |   |   |   |
| 6 | 2 | 2 |   |   |   |   |

\[\text{pdim (J)} = 1\]

\[\text{reg (J)} = 6\]

\[\text{deg shift}\]

\[\text{homological deg.}\]

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | $\beta_{00}$ | $\beta_{11}$ | $\beta_{22}$ |   |   |   |   |
| 1 | $\beta_{01}$ | $\beta_{12}$ | $\beta_{23}$ |   |   |   |   |
| 2 | $\beta_{02}$ | $\beta_{13}$ | $\beta_{24}$ |   |   |   |   |
|   |   |   |   |   |   |   |   |   |
| 0 |   |   |   |   |   |   |   |   |
Betti tables for $I_n = \langle \text{Sym}(n)(x_1^5x_2), \text{Sym}(n)(x_1^2x_2^3) \rangle = \langle \text{Sym}(n)(x_1^3), \text{Sym}(n)(x_2^3) \rangle$ (Murai)

| $J_2$ | 0 | 1 |
|-------|---|---|
| 4     | 1 |
| 5     |   |
| 6     | 2 | 2 |

| $J_3$ | 0 | 1 | 2 |
|-------|---|---|---|
| 4     | 3 | 2 |
| 5     |   |   |
| 6     | 6 | 9 |
| 7     |   |   | 3 |

| $J_4$ | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| 4     | 6 |   |   |   |
| 5     |   | 8 |   |   |
| 6     | 2 | 2 | 7 |
| 7     |   |   | 12|
| 8     |   |   | 4 |

Theorem (Murai, 2019): If $I = \langle \text{Sym}(x^3: x \in L) \rangle$ is a Sym-invariant monomial ideal, the Betti table is non-vanishing on a union of line segments of the same length.
Corollaries: \[ \text{pdim}(I_n) = n - D \quad \text{for } n \gg 0 \]
\[ \text{reg}(I_n) = W_n + C \]

For arbitrary \( c \), it is known:
- (Nagel-Römer) For any \( p > 0 \), the \( p \)th column of has stable non-zero entries for \( n \gg 0 \).