Hawking Emission and Black Hole Thermodynamics *

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1 Introduction

Black holes are perhaps the most perfectly thermal objects in the universe, and yet their thermal properties are not fully understood. They are described very accurately by a small number of macroscopic parameters (e.g., mass, angular momentum, and charge), but the microscopic degrees of freedom that lead to their thermal behavior have not yet been adequately identified.

Strong hints of the thermal properties of black holes came from the behavior of their macroscopic properties that were formalized in the (classical) four laws of black hole mechanics [1], which have analogues in the corresponding four laws of thermodynamics:

The zeroth law of black hole mechanics is that the surface gravity $\kappa$ of a stationary black hole is constant over its event horizon [2, 1]. This is analogous to the zeroth law of thermodynamics, that the temperature $T$ is constant for a system in thermal equilibrium.

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The first law of black hole mechanics expresses the conservation of energy by relating the change in the black hole mass $M$ to the changes in its area $A$, angular momentum $J$, and electric charge $Q$ in the following way:

$$\delta M = \frac{1}{8\pi} \kappa \delta A + \Omega \delta J + \Phi \delta Q,$$

where an extended form of the zeroth law implies that not only the surface gravity $\kappa$, but also the angular velocity $\Omega$ and the electrostatic potential $\Phi$ are constant over the event horizon of any stationary black hole. This first law is essentially the same as the first law of thermodynamics.

The second law of black hole mechanics is Hawking’s area theorem [3], that the area $A$ of a black hole horizon cannot decrease. This is obviously analogous to the second law of thermodynamics, that the entropy $S$ of a closed system cannot decrease.

The third law of black hole mechanics is that the surface gravity $\kappa$ cannot be reduced to zero by any finite sequence of operations [4]. This is analogous to the weaker (Nernst) form of the third law of thermodynamics, that the temperature $T$ of a system cannot be reduced to absolute zero in a finite number of operations. However, the classical third law of black hole mechanics is not analogous to the stronger (Planck) form of the third law of thermodynamics, that the entropy of a system goes to zero when the temperature goes to zero.

Thus the four laws of black hole mechanics are analogous to the four laws of thermodynamics if one makes an analogy between temperature $T$ and some multiple of the black hole surface gravity $\kappa$, and between entropy $S$ and some inversely corresponding multiple of the black hole area $A$. That is, one might say that $T = \epsilon \kappa$ and $S = \eta A$, with $8\pi \epsilon \eta = 1$, so that the $\kappa \delta A/(8\pi)$ term in the first law of black hole mechanics becomes the heat transfer term $T \delta S$ in the first law of thermodynamics.

Nevertheless, by a quite independent line of reasoning that was not directly motivated by Bekenstein’s proposal that he had rejected [1], Hawking made the remarkable discovery that black holes are not
completely black but instead emit radiation [5, 6]. Once he found that the radiation had a thermal spectrum, he realized that it did make Bekenstein’s idea consistent, of a finite black hole entropy proportional to area, though not Bekenstein’s conjectured value for $\eta$. In fact, Hawking found that the black hole temperature was $T = \kappa / (2\pi)$, so $\epsilon = 1/(2\pi)$ and hence $\eta = 1/4$. This gives the famous Bekenstein-Hawking formula for the entropy of a black hole:

$$S_{bh} = S_{BH} \equiv \frac{1}{4} A.$$  \hfill (2)

Here the subscript $bh$ stands for “black hole,” and the subscript $BH$ stands for “Bekenstein-Hawking.”

### 2 Hawking Emission Formulae

For the Kerr-Newman metrics [7, 8], which are the unique asymptotically flat stationary black holes in Einstein-Maxwell theory [9, 10, 11, 12, 13], one can get explicit expressions [14] for the area $A$, surface gravity $\kappa$, angular velocity $\Omega$, and electrostatic potential $\Phi$ of the black hole horizon in terms of the macroscopic conserved quantities of the mass $M$, angular momentum $J \equiv Ma \equiv M^2 a_*$, and charge $Q \equiv MQ_*$ of the hole, using the value $r_+$ of the radial coordinate $r$ at the event horizon as an auxiliary parameter:

$$r_+ = M + (M^2 - a^2 - Q^2)^{1/2} = M[1 + (1 - a_*^2 - Q_*^2)^{1/2}],$$
$$A = 4\pi r_+^2 = 4\pi M^2 [2 - Q_*^2 + 2(1 - a_*^2 - Q_*^2)^{1/2}],$$
$$\kappa = \frac{4\pi (r_+ - M)}{A} = \frac{1}{2} M^{-1}[1 + (1 - \frac{1}{2} Q_*^2)(1 - a_*^2 - Q_*^2)^{-1/2}]^{-1},$$
$$\Omega = \frac{4\pi a_*}{A} = a_* M^{-1}[2 - Q_*^2 + 2(1 - a_*^2 - Q_*^2)^{1/2}]^{-1},$$
$$\Phi = \frac{4\pi Q r_+}{A} = Q_* \frac{1 + (1 - a_*^2 - Q_*^2)^{1/2}}{2 - Q_*^2 + 2(1 - a_*^2 - Q_*^2)^{1/2}}.$$  \hfill (3)

Here $a_* = a/M = J/M^2$ and $Q_* = Q/M$ are the dimensionless angular momentum and charge parameters in geometrical units. For a nonrotating uncharged stationary black hole (described by the
Schwarzschild metric), \( a_* = Q_* = 0 \), so \( r_+ = 2M \), \( A = 16\pi M^2 \),\n\( \kappa = M/r_+^2 = 1/(4M) \), \( \Omega = 0 \), and \( \Phi = 0 \).

Then Hawking’s black hole emission calculation [5, 6] for free fields gives the expected number of particles of the \( j \)th species with charge \( q_j \) emitted in a wave mode labeled by frequency or energy \( \omega \), spheroidal harmonic \( l \), axial quantum number or angular momentum \( m \), and polarization or helicity \( p \) as

\[
N_{j\omega lmp} = \Gamma_{j\omega lmp} \{ \exp[2\pi\kappa^{-1}(\omega - m\Omega - q_j\Phi)] \mp 1 \}^{-1}.
\]  

Here the upper sign (minus above) is for bosons, and the lower sign (plus above) is for fermions, and \( \Gamma_{j\omega lmp} \) is the absorption probability for an incoming wave of the mode being considered.

From the mean number \( N_{j\omega lmp} \) and the entropy \( S_{j\omega lmp} \) per mode, one can sum and integrate over modes to get the emission rates of energy, angular momentum (the component parallel to the black hole spin axis), charge, and entropy by the black hole:

\[
\frac{dE_{\text{rad}}}{dt} = -\frac{dM}{dt} = \frac{1}{2\pi} \sum_{j,l,m,p} \int \omega N_{j\omega lmp} d\omega,
\]  

\[
\frac{dJ_{\text{rad}}}{dt} = -\frac{dJ}{dt} = \frac{1}{2\pi} \sum_{j,l,m,p} \int m N_{j\omega lmp} d\omega,
\]  

\[
\frac{dQ_{\text{rad}}}{dt} = -\frac{dQ}{dt} = \frac{1}{2\pi} \sum_{j,l,m,p} \int q_j N_{j\omega lmp} d\omega,
\]  

\[
\frac{dS_{\text{rad}}}{dt} = \frac{1}{2\pi} \sum_{j,l,m,p} \int S_{j\omega lmp} d\omega.
\]

Here \( M, J, \) and \( Q \) (without subscripts) denote the black hole’s energy, angular momentum, and charge. By the conservation of the total energy, angular momentum, and charge, the black hole loses these quantities at the same rates that the radiation gains them.

This is not so for the total entropy, which generically increases. The black hole entropy changes at the rate

\[
\frac{dS_{\text{bh}}}{dt} = \frac{1}{2\pi} \sum_{j,l,m,p} \int [2\pi\kappa^{-1}(\omega - m\Omega - q_j\Phi)] N_{j\omega lmp} d\omega,
\]
and by using Eq. (4), one can show that the total entropy $S = S_{bh} + S_{rad}$ (black hole plus radiation) changes at the rate

$$\frac{dS}{dt} = \frac{1}{2\pi} \sum_{j,l,m,p} \int d\omega \left[ \pm \ln \left( 1 \pm N_{j\omega lmp} \right) + N_{j\omega lmp} \ln \left( 1 + \frac{1 - \Gamma_{j\omega lmp}}{\Gamma_{j\omega lmp} \pm N_{j\omega lmp}} \right) \right].$$

For the emission of $n_s$ species of two-polarization massless particles of spin $s$ from a Schwarzschild black hole (nonrotating and uncharged) into empty space, numerical calculations [14, 15, 16] gave

$$\frac{dE_{rad}}{dt} = -\frac{dM}{dt} = 10^{-5}M^{-2}(8.1830n_{1/2} + 3.3638n_1 + 0.3836n_2),$$

$$\frac{dS_{rad}}{dt} = 10^{-3}M^{-1}(3.3710n_{1/2} + 1.2684n_1 + 0.1300n_2),$$

$$\frac{dS_{bh}}{dt} = -10^{-3}M^{-1}(2.0566n_{1/2} + 0.8454n_1 + 0.0964n_2).$$

### 3 The Generalized Second Law

Even if a black hole is not emitting into empty space, there are strong arguments that the total entropy of the black hole plus its environment cannot decrease. This is the Generalized Second Law (GSL). Bekenstein first conjectured it when he proposed that black holes have finite entropy proportional to their area [17], and he gave various arguments on its behalf, though it would have been violated by immersing a black hole in a heat bath of sufficiently low temperature if the black hole could not emit radiation [1].

Once Hawking found that black holes radiate [5, 6], he showed that the GSL held for a black hole immersed in a heat bath of arbitrary temperature, assuming that the radiation thermalized to the temperature of the heat bath. Zurek and Thorne [18], and Thorne, Zurek, and Price [19], gave more general arguments for the GSL without this last assumption. Their arguments were later fleshed out in a mathematical proof of the GSL for any process involving a quasistationary semiclassical black hole [20]. Other proofs of the GSL have also been given [21, 22, 23, 24].
With some exceptions [22, 24], these proofs so far generally have two key assumptions: (1) The black hole is assumed to be quasistationary, changing only slowly during its interaction with an environment. It has been conjectured [19] that the GSL also holds, using the Bekenstein-Hawking $A/4$ formula for the black hole entropy, even for rapid changes in the black hole, but this has not been rigorously proved.

(2) The semiclassical approximation holds, so that the black hole is described by a classical metric which responds only to some average or expectation value of the quantum stress-energy tensor. This allows the black hole entropy to be represented by $A/4$ of its classical horizon.

Other reviews [25, 26, 27, 28, 29, 30, 31, 32, 33, 34] have given more details of Hawking radiation and black hole thermodynamics.

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References

[1] J. M. Bardeen, B. Carter, and S. W. Hawking, *Comm. Math. Phys.* **31**, 161 (1973).

[2] B. Carter, in *Black Holes*, eds. C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973), p. 57.

[3] S. W. Hawking, *Phys. Rev. Lett.* **26**, 1344 (1971).

[4] W. Israel, *Phys. Rev. Lett.* **57**, 397 (1971).

[5] S. W. Hawking, *Nature* **248**, 30 (1974).

[6] S. W. Hawking, *Comm. Math. Phys.* **43**, 199 (1975).

[7] R. P. Kerr, *Phys. Rev. Lett.* **11**, 237 (1963).

[8] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, *J. Math. Phys.* **6**, 918 (1965).

[9] W. Israel, *Phys. Rev.* **164**, 1776 (1967).
[10] B. Carter, *Phys. Rev. Lett.* **26**, 331 (1971).

[11] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973).

[12] D. C. Robinson, *Phys. Rev. Lett.* **34**, 905 (1975).

[13] P. O. Mazur, *J. Phys.* **A15**, 3173 (1982); hep-th/0101012.

[14] D. N. Page, *Phys. Rev.* **D13**, 198 (1976).

[15] D. N. Page, *Phys. Rev.* **D14**, 3260 (1976).

[16] D. N. Page, *Phys. Rev. Lett.* **50**, 1013 (1983).

[17] J. D. Bekenstein, Ph.D. thesis, Princeton University (1972); *Lett. Nuovo Cimento* **4**, 737 (1972); *Phys. Rev.* **D7**, 2333 (1973); *Phys. Rev.* **D9**, 3292 (1974).

[18] W. H. Zurek and K. S. Thorne, *Phys. Rev. Lett.* **54**, 2171 (1985).

[19] K. S. Thorne, W. H. Zurek, and R. H. Price, in *Black Holes: The Membrane Paradigm*, eds. K. S. Thorne, R. H. Price, and D. A. MacDonald (Yale University Press, New Haven, 1986), p. 280.

[20] V. P. Frolov and D. N. Page, *Phys. Rev. Lett.* **71**, 3902 (1993).

[21] R. D. Sorkin, in *Tenth International Conference on General Relativity and Gravitation, Padua, 4-9 July, 1983, Contributed Papers*, vol. II, eds. B. Bertotti, F. de Felice, and A. Pascolini (Roma, Consiglio Nazionale Delle Ricerche, 1983), p. 734.

[22] R. D. Sorkin, *Phys. Rev. Lett* **56**, 1885 (1986).

[23] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).

[24] R. D. Sorkin, in *Black Holes and Relativistic Stars*, ed. R. M. Wald (University of Chicago Press, Chicago, 1998), p. 177; gr-qc/9705006.
[25] R. M. Wald, *Living Rev. Rel.* **4**, 6 (2001).

[26] P. Majumdar, *Pramana* **55**, 511 (2000), hep-th/0009008; hep-th/0011284; hep-th/0110198; gr-qc/0604026.

[27] C. Kiefer, in *The Galactic Black Hole*, eds. H. Falcke and F. W. Hehl (IOP Publishing, Bristol, 2002), astro-ph/0202032.

[28] W. Israel, *Lect. Notes Phys.* **617**, 15 (2003).

[29] T. Jacobson, gr-qc/0308048.

[30] T. Damour, hep-th/0401160.

[31] S. Das, *Pramana* **63**, 797 (2004), hep-th/0403202.

[32] D. V. Fursaev, *Phys. Part. Nucl.* **36**, 81 (2005), gr-qc/0404038.

[33] D. N. Page, *New J. Phys.* **7**, 203 (2005), hep-th/0409024.

[34] S. F. Ross, hep-th/0502195.