NEW EXACT RESULTS ON DENSITY MATRIX FOR XXX SPIN CHAIN

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ABSTRACT. Using the fermionic basis we obtain the expectation values of all $\mathfrak{s}\mathfrak{l}_2$-invariant and $C$-invariant local operators on 10 sites for the anisotropic six-vertex model on a cylinder with generic Matsubara data. This is equivalent to the generalised Gibbs ensemble for the XXX spin chain. In the case when the $\mathfrak{s}\mathfrak{l}_2$ and $C$ symmetries are not broken this computation is equivalent to finding the entire density matrix up to 10 sites. As application, we compute the entanglement entropy without and with temperature, and compare the results with CFT predictions.

1. INTRODUCTION

Since remarkable works by Boos and Korepin [1] it became clear that all the expectation values of local operators for XXX antiferromagnet must be expressible in terms of values $\zeta$-function at odd positive integer arguments. This statement was proved in the paper [2]. Methods of this paper were used by Takahshi et al. [3] to compute the correlation functions of spins up to 8 sites, and the density matrix up to 6 sites. The latter computation allowed one to find the entanglement entropy.

In the paper [4] the computation of expectation values is put in rather general framework. The main ingredient used in this paper is the fermionic basis. It is shown that this basis allows one to compute the expectation values on a cylinder with arbitrary Matsubara data. This circumstance was used in the paper [5] in order to find an analog of OPE on the lattice: the coefficients expressing a local operator in terms of the fermionic basis. The expectation values for the latter are simple.

In the present paper we apply the methods of [5] to the expectation values of all the $\mathfrak{s}\mathfrak{l}_2$-invariant and $C$-invariant operators for subchains of up to 10 sites. Namely, we decompose all of them in the fermionic basis. Then the expectation values for any Matsubara data are easy to compute.

When the $\mathfrak{s}\mathfrak{l}_2$-symmetry and $C$-invariance are not broken by the Matsubara eigenvector (antiferromagnet with temperature, but without magnetic field) our results are sufficient to derive entire density matrix. As application we compute the entanglement entropy for zero temperature and for small temperatures different from zero, and compare the results with the CFT predictions. The agreement is good, so, $n = 10$ seems to be already a large number.

The paper consists of six sections and one Appendix. In Section 2 we give some information about the fermionic basis. In Section 3 we explain how to compute efficiently the expectation values of operators with small Matsubara lattices. The computation is based on Slavnov formula [8] for scalar product and some basic formulae of QISM [6, 7]. We solve the combinatorial problem of expressing the results in terms of Schur polynomials. Section 4 summarises the computational procedure. In Section 5 we compute the density matrix and entanglement entropy for zero temperature. In Section 6 we explain how to compute efficiently at non-zero temperature the basic object of our method which is the function $\omega$. In Section 7 we present results for the entanglement entropy at small temperatures different from zero and compare them with the CFT prediction. In Appendix we present the eigenvalues of the density matrix at zero temperature with the precision $10^{-11}$.
2. Fermionic basis

Fermionic basis for the case of \( sl_2 \)-invariant and \( C \)-invariant operators is explained in details in \[5\]. So, we shall be brief here. We have two sets of fermionic operators \( b_j, b_j^*, c_j, c_j^* \), \((j = 1, 2, 3, \cdots)\) with canonical commutation relations, and use notations \( b_j^*, c_j^* \) for products, \( J \) being a strictly ordered multi-index \( \{j_1, \cdots, j_k\} \). For two multi-indices of the same length we write \( I \preceq J \) if \( i_p \leq j_p \) for all \( p \). We denote by \(|I|\) the sum of elements in \( I \). Our fermionic operators act on the space of local fields, role of vacuum is played by the unit operator \( I \).

Consider the space \( V \) of fermionic basis for the case of \( 2 \), \( \beta \) stands for equality of expectation values on a cylinder with arbitrary Matubara data as \( \beta \). We define the subspace \( V \) of \( \tilde{V} \) by

\[
\mathcal{V} = \{ v \in \tilde{V} | Q_m v \in M \tilde{V} \} \quad (m = n + 1, n + 2, \cdots) .
\]

It is easy to see that \( Q_m \mathcal{V} = 0 \) for \( m > 2n - 1 \), so the actual number of requirements is finite.

Denoting basis of \( \mathcal{V} \) by \( v_\alpha \) we have \( F = |F_{\alpha,\{I, J\}}| \), the first one of several matrices used below:

\[
v_\alpha = \sum_{\#(I) = \#(J), \max(I \cup J) \leq n} F_{\alpha,\{I, J\}} b_I^* c_J^* \cdot I .
\]

From now on we shall demonstrate complexity of computation by the most difficult case to be considered in this paper, which is \( n = 10 \). In that case the dimension of \( \tilde{V} \) equals 12041 while the dimension of \( \mathcal{V} \) is 1141 (reasonably small).

On the other hand consider the space \( \mathcal{H} \subset \text{End}(\mathbb{C}^2)^{\otimes n} \) of \( sl_2 \)-invariant and \( C \)-invariant (invariant under simultaneous change of sign for all \( \sigma_j^a, a = 1, 2, 3 \)) operators located on \( n \) sites of the spin chain. We require also that the operators cannot be reduced to smaller interval, formal definition is given in Section 3. Let us denote a basis of this space by \( O_a \). The main statement is the relations

\[
O_a \equiv v_\alpha X_{\alpha,a}(n) ,
\]

where \( \equiv \) stands for equality of expectation values on a cylinder with arbitrary Matubara data as we are going to explain. This expectation value is denoted by \( \langle \cdot \rangle_{M \beta d} \). For \( n = 10 \) the dimension of \( \mathcal{H} \) is 4286. So, our main problem is to define the matrix \( X(n) \) which for \( n = 10 \) is a 1141 \( \times 4286 \) matrix.

The Matubara data consist of a positive integer \( L \), the coefficients \( \{a_1, \cdots, a_L\}, \{d_1, \cdots, d_L\} \) and Bethe numbers \( \beta_1, \cdots, \beta_m (m \leq L/2) \), which satisfy the Bethe equations

\[
a(\beta_j)Q(\beta_j + 1) + d(\beta_j)Q(\beta_j - 1) = 0 , \quad j = 1, \cdots, m .
\]
where
\[ a(\lambda) = \lambda^L + \sum_{j=1}^{L} a_j \lambda^{L-j}, \quad d(\lambda) = \lambda^L + \sum_{j=1}^{L} d_j \lambda^{L-j}, \quad Q(\lambda) = \prod_{j=1}^{m} (\lambda - \beta_j). \]

The matrix \( X(n) \) does not depend on the Matsubara data. Hence the main idea: to take a set of simple unphysical Matsubara data \( \{M_d\} \) in order to fix \( X(n) \) through the linear equations
\[ \langle O_a \rangle_{M_d} = X_{a,\alpha}(n) \langle v_\alpha \rangle_{M_d}, \tag{3} \]
and then apply it to physically relevant cases.

We construct the unphysical data as follows. Take the input data
\[ \text{input} = \{ \beta_1, \ldots, \beta_m, a_{m+1}, \ldots, a_L, d_1, \ldots, d_L \}, \tag{4} \]
and find the remaining \( a_1, \ldots, a_m \) solving the Bethe equations which are linear for these unknowns. In practice we take the input data as random integers, so, the procedure is very fast.

The expectation value \( \langle O_a \rangle_{M_d} \) in the left hand side of (3) is easy to compute using QISM. In order to compute \( \langle v_\alpha \rangle_{M_d} \) in the right hand side, we begin with defining a symmetric function of two variables \( \omega(\lambda, \mu) \) for given Matsubara data [12].

Introduce the kernel and “half-kernel” functions:
\[ K(\lambda) = \frac{2}{\lambda^2 - 1}, \quad H(\lambda) = \frac{1}{(\lambda - 1)\lambda}, \]
and the measure
\[ dm(\lambda) = \frac{d\lambda}{1 + a(\lambda)}, \quad a(\lambda) = \frac{a(\lambda)Q(\lambda + 1)}{d(\lambda)Q(\lambda - 1)}. \]

We need an auxiliary function defined by the integral equation
\[ G(\eta, \mu) = H(\eta - \mu) + \frac{1}{2\pi i} \oint_{\Gamma} K(\eta - \sigma)G(\sigma, \mu)dm(\sigma), \]
where the contour \( \Gamma \) goes around the Bethe roots \( \beta_1, \ldots, \beta_m \) and the point \( \sigma = \mu \). For a finite Matsubara chain we have a finite number of Bethe roots for which the equation above reduces to a linear system for \( G(\beta_j, \mu) \). Then solving it we obtain \( G(\beta_j, \mu) \), and \( G(\eta, \mu) \) itself as well.

The function \( \omega(\lambda, \mu) \) is given by
\[ \omega(\lambda, \mu) = \frac{1}{2\pi i} \oint_{\Gamma'} H(\eta - \lambda)G(\eta, \mu)dm(\eta) + \frac{1}{4}K(\lambda - \mu), \]
with \( \Gamma' \) containing one more point: \( \eta = \lambda \).

The Taylor series of \( \omega(\lambda, \mu) \) define an half-infinite matrix \( ||\omega_{i,j}|| \):\[ \omega(\lambda, \mu) = \sum_{i,j=1}^{\infty} \lambda^{i-1} \mu^{j-1} \omega_{i,j}. \tag{5} \]

Then for two multi-indices \( I, J \) of length \( l \) define
\[ \omega_{I,J} = \det ||\omega_{i,p,j}||_{p,q=1,\ldots,l}. \]

Then
\[ \langle b_I^* c_J^* \rangle_{M_d} = \omega_{I,J}, \]
consequently
\[ \langle v_\alpha \rangle_{M_d} = \sum F_{\alpha,I,J} \omega_{I,J}. \]

We take Matsubara data numerical, so, all these computations are very fast.
3. DIRECT COMPUTATION OF EXPECTATION VALUES

First of all we define the Matubara monodromy matrix

\[ T_j(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \],

where \( j \) counts the tensor components in the space direction.

We use the notation \( A = A(0) \) etc. Our goal is to compute

\[ \langle O \rangle_{\mathrm{Md}} = \frac{\langle \Psi | \text{Tr}_{[1,n]} \left( O T_1 \cdots T_n \right) | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \]

where \( |\Psi\rangle \) is the Bethe vector considered in the previous section; for given Matsubara data we write

\[ \langle \Psi | = \langle \beta_1, \cdots, \beta_m | = \langle \downarrow | B(\beta_1) \cdots B(\beta_m) \rangle. \]

The normalization is provided by the Gaudin formula below.

The Slavnov formula for the scalar product of the Bethe covector \( \langle \beta_1, \cdots, \beta_m | \) with an off-shell vector \( |\mu_1, \cdots, \mu_m \rangle = C(\mu_1) \cdots C(\mu_m) |\downarrow \rangle \), \( \mu_j \) being arbitrary is

\[ \langle \beta_1, \cdots, \beta_m | \mu_1, \cdots, \mu_m \rangle = m \prod_{j=1}^m d(\beta_j) d(\mu_j) \prod_{i,j=1}^m (\beta_j - \mu_i + 1) \prod_{i<j} (\mu_i - \mu_j) \prod_{i<j} (\beta_j - \beta_i) \det(N), \]

where the matrix \( N \) has entries

\[ N_{i,j} = \frac{1}{(\mu_i - \beta_j)} \left( \frac{1}{(\mu_i - \beta_j - 1)} - \frac{1}{(\mu_i - \beta_j + 1)} a(\mu_i) \right). \]

The Slavnov formula (6) obviously from the right hand side, is a polynomial in the variables \( \mu_1, \cdots, \mu_m \). In the below \( N \) denotes the symmetric polynomial of the variables \( \mu_1, \cdots, \mu_m \) given by the Slavnov formula.

Using the L’Hospital rules in (6) we get the Gaudin formula for normalisation. Explicitly, we have

\[ \langle \beta_1, \cdots, \beta_m | \beta_1, \cdots, \beta_m \rangle = m \prod_{j=1}^m a(\beta_j) d(\beta_j) \prod_{i \neq j} (\beta_i - \beta_j + 1) \beta_i - \beta_j \] \[ \det(G), \]

where the matrix \( G \) has entries

\[ G_{k,l} = \frac{\partial}{\partial \beta_l} \log a(\beta_k), \quad k, l = 1, \cdots, m. \]

We want to compute

\[ \langle \beta_1, \cdots, \beta_m | X_1 X_2 \cdots X_N | \beta_1, \cdots, \beta_m \rangle, \]

where \( X_j \) is one of \( A, B, C, D \). Note that this quantity is zero unless \( \sharp \{ j | X_j = B \} = \sharp \{ j | X_j = C \} \). Starting with the Slavnov formula, we begin the following computation. Introduce the notations \( v(\lambda) = 1/\lambda, \ u(\lambda) = v(\lambda) + 1. \)

Let \( \langle \Phi | = \langle \Psi | X_1 X_2 \cdots X_N \) for some \( X_1, \cdots, X_N \) and set

\[ f(\Phi)(\mu_1, \cdots, \mu_q) = \langle \Phi | \mu_1, \cdots, \mu_q \rangle. \]
Note that this is zero unless \( \sharp \{j | X_j = B \} - \sharp \{j | X_j = C \} = q \geq 0 \). The commutation relation RTT = TTR implies

\[
\begin{align*}
    f_{\langle \Phi | A (\mu_1, \cdots, \mu_q)} &= a(0) \prod_{j=1}^{q} u(-\mu_j) f_{\langle \Phi | (\mu_1, \cdots, \mu_q)} \\
    &\quad - \sum_{j=1}^{q} a(\mu_j) v(-\mu_j) \prod_{r \neq j}^{q} u(\mu_j - \mu_r) f_{\langle \Phi | (\mu_1, \cdots, \hat{\mu_j}, \cdots, \mu_q, 0)}, \\
    f_{\langle \Phi | D (\mu_1, \cdots, \mu_q)} &= d(0) \prod_{j=1}^{q} u(\mu_j) f_{\langle \Phi | (\mu_1, \cdots, \mu_q)} \\
    &\quad - \sum_{j=1}^{q} d(\mu_j) v(\mu_j) \prod_{r \neq j}^{q} u(\mu_r - \mu_j) f_{\langle \Phi | (\mu_1, \cdots, \hat{\mu_j}, \cdots, \mu_q, 0)}, \\
    f_{\langle \Phi | B (\mu_1, \cdots, \mu_q)} &= \sum_{j=1}^{q} \left( a(0) d(\mu_j) v(-\mu_j) \prod_{r \neq j}^{q} u(-\mu_r)(\mu_r - \mu_j) \\
    &\quad + d(0) a(\mu_j) v(\mu_j) \prod_{r \neq j}^{q} u(\mu_r - \mu_r) \right) f_{\langle \Phi | (\mu_1, \cdots, \hat{\mu_j}, \cdots, \mu_q)} \\
    &\quad + \sum_{j>i} \left( d(\mu_i) a(\mu_j) v(-\mu_i) v(\mu_j) u(\mu_j - \mu_i) \prod_{r \neq i,j}^{q} u(\mu_r - \mu_i)(\mu_r - \mu_i) \\
    &\quad + a(\mu_i) d(\mu_j) v(-\mu_j) v(\mu_i) u(\mu_i - \mu_j) \prod_{r \neq i,j}^{q} u(\mu_r - \mu_r)(\mu_r - \mu_j) \right) \\
    &\quad \times f_{\langle \Phi | (\mu_1, \cdots, \hat{\mu_i}, \cdots, \hat{\mu_j}, \cdots, \mu_q, 0)}. \\
    f_{\langle \Phi | C (\mu_1, \cdots, \mu_q)} &= f_{\langle \Phi | (\mu_1, \cdots, \mu_q, 0)}.
\end{align*}
\]

Notice that the fact that we are doing with \( A(0), B(0), C(0), D(0) \) simplifies the general formulae available, for example, in \([7]\).

Remark Let \( P_q \) be the space of symmetric polynomials of \( q \) variables. The right hand sides of \((8), (10), (11), (9)\) define respectively actions of the operators \( A, B, C, D \) on the space \( \bigoplus_{q \geq 0} P_q \); \( A \) and \( D \) from \( P_q \) to itself, \( B \) from \( P_{q-1} \) to \( P_q \) and \( C \) \( P_{q+1} \) to \( P_q \).

Using the formulae above we compute inductively

\[
f_{\langle \Phi | X_1 X_2 \cdots X_N (\mu_1, \cdots, \mu_q)} = \langle \beta_1, \cdots, \beta_m | X_1 X_2 \cdots X_N | \mu_1, \cdots, \mu_m \rangle,
\]

and then set \( \mu_j = \beta_j, j = 1, \cdots, m \). However direct application of this procedure to computer calculation may be very time consuming. Indeed, in order to arrive at symmetric polynomial we have to factorise the right hand sides of \((8), (9), (10)\). For operators considered in \([5]\) this is not very hard: the worst expression we had there is \( B T T \cdots T C \) where \( T = A + D \).

For this expression we have to go up to \( m + 1 \) variables, \( m \) is not large (for \( n = 10 \) it suffices to consider \( m = 2 \) at most). That is why the direct procedure works and the improvement which we explain in what follows only accelerates it. But when computing the expectation values for all the operators it is simply impossible to manage for \( n = 10 \) because we have, for example, expressions like \( B \cdots B C \cdots C \) for which the number of variables in the middle
becomes \( m + N \). Hence the rewriting of the procedure in terms of Schur polynomials which was mentioned only briefly in [5] becomes crucial.

Consider Young diagrams \( Y_\lambda \) where \( \lambda = (\lambda_1, \ldots, \lambda_n), \lambda_i \geq \lambda_{i+1} > 0 \) is a partition. We set \( \#(\lambda) = n \). It is called the length of \( Y_\lambda \). We work in the space \( H_q \) whose elements are

\[
Y = \sum_{\#(\lambda) \leq q} c_\lambda Y_\lambda.
\]

In the below we will identify \( Y_\lambda \) with \( \lambda \). The symbol \( \emptyset \) denotes the empty diagram. Define the operation \( \text{cut}_q \) which acts from \( H_{q'} \) with \( q' > q \) to \( H_q \) erasing all the terms with \( \#(\lambda) > q \). Consider the Grassmann space \( F_q \) with the basis \( \psi_{k_1}^* \cdots \psi_{k_q}^* (k_1 > \cdots > k_q \geq 0) \). We have the usual isomorphism between the spaces \( H_q \) and \( F_q \):

\[
\psi_{k_1}^* \psi_{k_2}^* \cdots \psi_{k_q}^* \mapsto (k_1 - (q - 1), k_2 - (q - 2), \ldots, k_q),
\]

\[
(\lambda_1, \ldots, \lambda_n) \mapsto \psi_{\lambda_1+q-1}^* \cdots \psi_{\lambda_n+q-n}^* \psi_{q-n-1}^* \cdots \psi_0^*, \text{ where } n \leq q.
\]

In the above, \( ()_0 \) means removing all entries equal to 0. Schur polynomial \( s_\lambda(x_1, \ldots, x_q) \) is the symmetric polynomial

\[
s_\lambda(x_1, \ldots, x_q) = \frac{\det ||x_j^{\lambda_i + q - i}||}{\det ||x_j^{q-i}||}.
\]

The above formula gives an isomorphism between \( H_q \) and \( P_q \).

For a given polynomial of one variable \( P(x) = \sum_{j=0}^d p_j x^j \) we define the operator \( P \land F_{q-1} \subset F_q \) multiplying by \( \sum_{j=0}^d p_j \psi_j^* \), this operator is defined as \( P \land H_{q-1} \subset H_q \) by the isomorphism [12]. We shall also need the simplest Littlewood-Richardson formula for multiplication of a Schur polynomial by elementary symmetric function \( \sigma_j \), which translates as action on \( H_q \)

\[
\sigma_j \circ (\lambda_1, \ldots, \lambda_n) = \sum_{j} \left( (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) + e_j \right)_{\text{order}},
\]

where \( e_j \) are all vectors of dimension \( n + \min(j, q - n) \) with \( j \) elements equal to 1 other elements being 0, “order” means that we have to drop all the tables in which elements happen to be not ordered, and we also drop all zeros in the final table.

The Slavnov formula [6] gives the symmetric polynomial \( N \) in variables \( \mu_1, \ldots, \mu_m \) which can be written as follows. Define

\[
P_j(x) = \frac{x}{x - \beta_j} \left( a(x) \frac{Q(x + 1)}{x - \beta_j + 1} - d(x) \frac{Q(x - 1)}{x - \beta_j - 1} \right)
\]

then

\[
N = (-1)^{1/2m(m-1)} \prod_{i<j} \frac{d(\beta_i)}{(\beta_i - \beta_j) \cdot P_1 \land P_2 \land \cdots \land P_m \land \emptyset} \in H_m.
\]

For us \( \beta_1, \ldots, \beta_m \) are numbers, so, the computation of \( N \) is extremely fast.

In what follows we shall need the operation \( \text{cut}_q(Y) \) which erases all the Young diagrams in \( Y \) with lengths greater than \( q \).

Now we have to translate the action of the operators \( A, B, C, D \). The operators \( A, D \) act from \( H_q \) to itself; we have

\[
AY = \text{cut}_q \left( \sum_{k=1}^{q+1} \sigma_{k-1} \circ A_k \land Y \right), \quad DY = \text{cut}_q \left( \sum_{k=1}^{q+1} \sigma_{k-1} \circ D_k \land Y \right).
\]
where
\[ A_k(x) = (-1)^{k-1}(x + 1)^{q+1-k}a(x) , \quad D_k(x) = (-1)^{k-1}(x - 1)^{q+1-k}d(x) . \]
The operator \( B \) is more complicated. It acts from \( H_{q-1} \) to \( H_q \). For a polynomial of two variables \( R(x, y) = \sum_{i,j=0}^{d} R_{i,j} x^i y^j \) we define the operator \( R = H_{q-2} \subset H_q \) multiplying by \( \sum_{i,j=0}^{d} R_{i,j} \psi_i^\ast \psi_j^\ast \). Then
\[
(13) \quad B Y = \text{cut}_q \left( \sigma_q^{-1} \circ \left[ \sum_{p=0}^{q} \sum_{r=0}^{q-1} \sigma_r \circ \sigma_p \circ \left( P_{p,r} \cap \text{cut}_{q-1}(Y) + R_{p,r} \cap \sigma_{q-2} \circ \text{cut}_{q-2}(Y) \right) \right] \right),
\]
where \( P_{p,r} \) and \( R_{p,r} \) are polynomials of one and two variables, respectively:
\[
P_{p,r}(x) = (-1)^r \sum_{s=r}^{q-1} [d(0)a(x)(x + 1)^{q-p}x^{s-r} - a(0)d(x)(1 - x)^{q-p}x^{s-r}]
\]
\[
R_{p,r}(x, y) = d(x)a(y)(-1)^{p+r}(y + 1)^{q-p}y^{s-r}.
\]
It can be shown that the expression inside the square brackets in (13) consists of Young diagrams of length \( q \), not shorter. Hence the \( \sigma_q^{-1} \) is applicable: we just subtract 1 from all entries of Young diagrams, and drop zeros.

Finally \( C \) act from \( H_{q+1} \) to \( H_q \) simply as
\[
(14) \quad C Y = \text{cut}_q(Y)
\]
Now we are ready to compute the Matsubara expectation value of the right hand side of (2).
Consider operators \( O \) localised on the interval \([1, n]\). We realise them as linear combinations of tensor products of \( I, \sigma^\pm, \sigma^3 \). We have to take into account symmetries. First of them is the translational invariance. The operator \( O \) may contain terms of the form
\[
\underbrace{I \otimes I \otimes O'}_{k} \otimes \underbrace{I \otimes I \otimes I}_{l},
\]
with \( O' \) localised on \( n - k - l \) sites. The expectation value for such operator can be computed using our procedure for this number of sites. We shall denote by \( O^{(n)}_A \) the basis of operators on \( n \) sites irreducible in that way. It consists of the tensor products which do not contain \( I \) neither at the left nor on the right end. Further, we require \( \#(\sigma^+) = \#(\sigma^-) \) to have zero total charge, \( \#(\sigma^3) \equiv 0(\mod 2) \) for \( C \)-invariance. Then for any operator \( O \) localised on \( n \) sites we have the reduction due to the translational invariance:
\[
(15) \quad T(O) = \{ O^{(0)}, O^{(2)}, \ldots , O^{(n-1)}, O^{(n)} \},
\]
where \( O^{(k)} \) are translationally irreducible and \( C \)-invariant operators on \( k \) sites.
First impression is rather discouraging even after the serious acceleration of the procedure discussed above. It has been said that for \( n = 10 \) we are interested in \( 4286 \) \( sl_2 \)-invariant, \( C \)-invariant and translationally irreducible operators. We choose the basis of such operators \( O^{(n)}_a \) in certain simplest possible way. So, we have a matrix
\[
O^{(n)}_a = L^{(n)}_a O^{(n)}_A
\]
The problem is that our procedure does not allow one to compute directly for \( O^{(n)}_a \), but rather for \( O^{(n)}_A \). For \( n = 10 \) the number of the latter is horrifying: 50354. Fortunately we do not need to compute for all of them independently. Our computation goes from the left to the right, and,
for example in $BBBBCCCCBC$ and $BBBBCCCCCB$ the pieces $BBBBCCCC$ coincide, so, we have to organise the computation in order not to do the same computation twice. This can be done making the total computation reasonably fast.

4. SUMMARY OF COMPUTATION PROCEDURE

Let us summarise. Consider translationally irreducible operators. First important point is that the equation

\[(16)\]

\[\langle O_{a}^{(n)} \rangle_{Md} = X_{a,a}(n)\langle v_{a} \rangle_{Md},\]

holds for any Matsubara data $Md$. So, in principle we have an infinite overdetermined system of equations for the coefficients $X_{a,a}(n)$.

Consider our favourite case $n = 10$. We begin with the simplest case $L = 1, m = 0$, and take $Md_{j}$ ($j = 1, \ldots, 20$) with 20 random integer input data $|4\rangle$. The rank of the matrix

\[||\langle v_{a} \rangle_{Md_{j}} ||_{j,a} = || F_{a,\{I,J\}} \omega_{I,J} (Md_{j}) ||_{j,a,j},\]

is 15. We can add as many $L = 1, m = 0$ equation as we wish, the rank will not change. So, we proceed to $L = 2, m = 0$ taking 200 equations the rank raises by almost 200 and stabilises. Then we take in addition 300 eqs with $L = 3, m = 0, 90$ eqs with $L = 4, m = 0, 10$ eqs with $L = 5, m = 0$. Adding any other equation with $m = 0$ changes nothing, so we proceed to “one-particle” case taking 10 eqs with $L = 2, m = 1, 200$ eqs with $L = 3, m = 1, 325$ eqs with $L = 4, m = 1, 100$ eqs with $L = 5, m = 1$, more “one-particle” equations add nothing to rank, and we have to take several “two-particle” ones (fortunately not too many because the computation for them is getting longer). We take 10 eqs with $L = 4, m = 2, 35$ eqs with $L = 5, m = 2, 7$ eqs with $L = 6, m = 2$. Altogether we have 1307 equations and the rank is 1141. So, we can proceed computing the left hand side of $(16)$ for all these Matsubara data.

Now we proceed as follows. Construct the matrix $1307 \times 1141$ matrix

\[A = ||F_{a,\{I,J\}} \omega_{I,J} (Md_{j})||_{j=1,\ldots,1307},a=1,\ldots,1141\]

and the $1307 \times 4286$ matrix

\[B = ||\langle O_{a}^{(n)} \rangle_{Md_{j}} ||_{j=1,\ldots,1307},a=1,\ldots,4286.\]

Put them together

\[||A,B||.\]

By Gaussian procedure which multiplies $GL(1307)$ from the left we bring the matrix $||A,B||$ to the form

\[||I \quad X(10)|| \quad 0 \quad 0||,\]

where $X(10)$ is the matrix of transformation to the fermionic basis defined above. The fact that the first 1141 columns become in this form and all the rows starting from 1142-th one vanish is a crucial check of our entire procedure. It shows that the vectors $\langle v_{a} \rangle$ are linearly independent, and, more importantly, that all the expectation values of our invariant operators are expressible as linear combinations of $\langle v_{a} \rangle$.

We took some simplest basis of sl$_{2}$-invariant and C-invariant operators $O_{a}$. The price to pay for the simplicity is that we did not input the orthogonality from the very beginning, and now we have to find the operators $\overline{O}_{a}^{(n)}$ such that

\[\text{Tr}_{[1,n]}(O_{a}^{(n)} \overline{O}_{b}^{(n)}) = \delta_{a,b}.\]

Introduce $D(n)_{I,J}$ as

\[D(n)_{I,J}(\bullet) = F_{a,\{I,J\}}(n)\text{Tr}_{[1,n]}(\overline{O}_{a}^{(n)})\].
For any Matsubara data we construct $\omega_{i,j}$, and the expectation value of any $\mathfrak{sl}_2$-invariant and $C$-invariant translationally irreducible operator $O^{(n)}$ is

$$\langle O^{(n)} \rangle = \omega_{I,J} D^{(n)}_{I,J}(O^{(n)}). \quad (17)$$

We can drop the requirement of translational irreducibility applying to any operator $O$ located on $n$-sites the operator $T$ $^{(15)}$, and further acting by the block-diagonal operators composed of $D^{(0)}_{I,J}, D^{(2)}_{I,J}, \ldots, D^{(n)}_{I,J}$.

The expressions for $D^{(n)}_{I,J}$ become long for $n > 6$, so, we cannot present them here, but they are available at

https://www.dropbox.com/sh/l363gixrrgsm95d/AACtqHLdUz7Qj8mD3NSVawMwa?dl=0

The only Mathematica notebook in this directory gives necessary explanations (hopefully sufficient) for application.

If the symmetries are not broken by the Matsubara data (as it happens for the antiferromagnetic chain at any temperature, but in absence of magnetic field) we obtain an entire density matrix. Let us redo everything in more conventional way. Density matrix $D^{(n)}$ is defined by

$$\langle O \rangle = \text{Tr}_{[1,n]}(D^{(n)}O),$$

for operators located on $n$ sites. In the next section we shall consider the entanglement entropy which is defined by

$$s(n) = -\text{Tr}(D^{(n)} \log D^{(n)}).$$

So, in order to compute it we have to diagonalise the density matrix.

We have to take into account the $\mathfrak{sl}_2$-symmetry of the density matrix. Let $\epsilon = 0$ or $1/2$ where $2\epsilon = n \pmod{2}$. We have the orthogonal decomposition

$$V = (\mathbb{C}^2)^{\otimes n} = \bigoplus_{j=\epsilon}^{n/2} (M_j \otimes V_j),$$

where $V_j$ is the spin $j$ irreducible representation of $\mathfrak{sl}_2$ and $M_j$ is the space of multiplicities counted by Bratteli diagrams. The $\mathfrak{sl}_2$-invariant density matrix acts on

$$M = \bigoplus_{j=\epsilon}^{n/2} M_j,$$

but computing the spectrum we have to take into account that the eigenvalues come with multiplicity $2j + 1$. The dimension of $M_j$ equals

$$\binom{n}{n/2 - j} - \binom{n}{n/2 - j - 1}.$$ 

So, for $n = 10$ the maximal dimension is that of $M_1$, it is equal to 90 which is quite appropriate for the computer diagonalisation.

The density matrix is obtained from the formulae of the previous section. We recalculate it in the new basis. Since we are interested in universal formulae, applicable to any Matsubara data, we compute everything keeping the indices $I,J$ for fermions.
5. Entanglement entropy at zero temperature

Now we can proceed to the diagonalisation of the density matrix. We begin with the antiferromagnetic at zero temperature. In that case the function \( \omega(\lambda, \mu) \) is known explicitly:

\[
\omega(\lambda, \mu) = \omega(\lambda - \mu),
\]

\[
\omega(\lambda) = -\frac{1}{2} + 2 \log 2 + \sum_{k=1}^{\infty} \lambda^{2k} \left( 2\zeta(2k+1)(1 - 2^{-2k}) - \frac{1}{2} \right).
\]

With these data we diagonalise the density matrix. The eigenvalues decrease with the spin \( j \). The most striking example is given by \( j = n/2 \), corresponding block is \( 1 \times 1 \), it coincides with the vacuum formation probability. The numerical values:

\[
P(2) = \frac{10.228427314684897}{10^{22}},
P(3) = \frac{0.00762415812490254761}{10^{22}},
P(4) = \frac{0.000206270046519527063}{10^{22}},
P(5) = \frac{2.011725958884905 \times 10^{-6}}{10^{22}},
P(6) = \frac{7.06812753309203896 \times 10^{-9}}{10^{22}},
P(7) = \frac{8.93090684226941650 \times 10^{-12}}{10^{22}},
P(8) = \frac{4.0574950525538289 \times 10^{-15}}{10^{22}},
P(9) = \frac{6.62359212493539014 \times 10^{-19}}{10^{22}},
P(10) = \frac{3.88481154904260358 \times 10^{-23}}{10^{22}},
\]

are in very good agreement with asymptotics \[9\] which looks as follows:

\[
P(n) \approx A n^{-\frac{1}{12}} \left( \frac{\Gamma^2(1/4)}{\pi \sqrt{2\pi}} \right)^{-n^2} (n \to \infty).
\]

The constant \( A \) is unknown, in \[9\] it is estimated as \( A = 0.841 \). Our data show that

\[
X(n) = \log P(n) + \log \left( \frac{\Gamma^2(1/4)}{\pi \sqrt{2\pi}} \right) n^2 + \frac{1}{12} \log n,
\]

slightly oscillates around \( \sim \log(0.841) \). So, we ask a question whether the next correction to the asymptotics is purely oscillating or there is a non-oscillating part. To answer this question we compute:

\[
\exp \left( \frac{1}{4} (X(10) + 2X(9) + X(8)) \right) = \frac{0.8412645021372811}{10^{22}},
\]

\[
\exp \left( \frac{1}{4} (X(9) + 2X(8) + X(7)) \right) = \frac{0.8412642481617325}{10^{22}}.
\]

This computation convinces us that the power corrections are purely oscillating, and that the good approximation for \( A \) is

\[
A = 0.841264(5).
\]

In the Appendix we give the eigenvalues of the density matrix with 11 digits accuracy (with this accuracy the eigenvalues disappear for high spins).
Here are entanglement entropies with 24 digits

\[ s(2) = 0.95367162656978945738557 \]
\[ s(3) = 1.09690078367655639608404 \]
\[ s(4) = 1.19547447383418925567332 \]
\[ s(5) = 1.27102739309231825158036 \]
\[ s(6) = 1.33247760568637557112695 \]
\[ s(7) = 1.38430489902101253089084 \]
\[ s(8) = 1.42913854287157243504956 \]
\[ s(9) = 1.46864496929391162170464 \]
\[ s(10) = 1.50396085818734543200735 \]

To verify that the first five numbers agree with those of [3] one has to pass from natural logarithms to binary ones.

The CFT predicts [10] that

\[ s(n) \simeq \frac{1}{3} \log n + C. \]

The following figure shows that we are rather close to the conformal limit

6. Computation of \( \omega(\lambda, \mu) \) with temperature

At finite temperature the function \( \omega(\lambda, \mu) \) can be computed only numerically. For temperature equal to \( T \) we shall denote it by \( \omega_T(\lambda, \mu) \), in particular, the function (18) will be denoted by \( \omega_0(\lambda, \mu) \) from now on. We compute 10 Taylor coefficients in each variable of the function \( \omega_T(\lambda, \mu) \). The main problem here is that we need to know it with very high precision: our answers contain the sign changing sums with huge rational coefficients of determinants made of the Taylor coefficients of \( \omega \) with sizes up to \( 5 \times 5 \). So, we have to find a good and controllable way of computation.

We follow the definitions of the Section 2 but now our goal is different: we are interested in quite special Matsubara data, staggering inhomogeneities and limit \( L \to \infty \). It will be convenient to change the variables to \( \lambda = ix, \mu = iy, etc \). We do not go into details which are
well-known from [11] presenting directly the equation for \( a(x) \):

\[
\log a(x) = \frac{h_0(x)}{T} + \int_C K(x-y) \log (1 + a(y)) \, dy , \quad h_0(x) = \frac{1}{x(x+i)} ,
\]

which holds for the ground state which is of interest to us. We slightly change the definition of \( K \):

\[
K(x) = -\frac{1}{\pi(x^2 + 1)} .
\]

The contour \( C \) goes around the Bethe roots, and the change of variables was performed in order to make them real. The Bethe roots are situated symmetrically with respect to the point \( x = 0 \), and accumulate at the point \( x = 0 \) where the function \( a(x) \) has essential singularity. The maximal Bethe root, \( \beta_{\text{max}} \) grows logarithmically with \( 1/T \). We shall take \( C \) as ellipse

\[
x(\phi) = -R \cos(\phi) - it \sin \phi , \quad 0 \leq \phi < 2\pi .
\]

We shall denote by \( C_- \) the part of \( C \) situated in lower half plane, and by \( C_+ \) the part of \( C \) in the upper half plane with reversed orientation. The parameter \( R \) must be bigger than \( \beta_{\text{max}} \) while \( 0 < t < 1 \). We shall take \( t = 2/5 \). For \( R \) there is a simple check: for given \( T \) solve the equation

\[
\log a(R)/i < \pi .
\]

In order to make the iterative procedure for (19) efficient we use Destri-DeVega trick. By Schwarz principle

\[
a(x) = \frac{1}{a(\bar{x})}
\]

For sufficiently small temperatures \(|a(x)| < 1\) holds for \( x \in \mathbb{C}^+ \), and it gets very small when \( x \) is close to 0, which is the point of essential singularity. So, we rewrite (19) as

\[
\log a(x) = \frac{h(x)}{T} - \int_{C_+} R(x-y) \log (1 + a(y)) \, dy + \int_{C_-} R(x-y) \log \left(1 + \overline{a(y)}\right) \, dy ,
\]

where \( R(x) \) is the resolvent of the operator \( I - K \) on the interval \([-R,R]\),

\[
h = (I + R)h_0 .
\]

So, our first task is to solve with good precision the equation

\[
R(x,y) = K(x-y) + \int_{-R}^R K(x-z)R(z,y)dz .
\]

Simple experiments show that in order to go to temperatures as low as \( 1/200 \) we need \( R = 2 \). Then for temperatures higher than \( 1/10 \) we can switch to \( R = 1 \). These are two cases which we shall consider. The main problem here is at the ends of integration; simple-minded discretisation gives very bad results for finite intervals. In order to avoid this problem we use the double exponential method [14]. To integrate a function \( f(x) \) from \( -R \) to \( R \) we introduce

\[
g(t) = -1 + \frac{4}{\pi} \arctan(\exp(c \sinh(t))) ,
\]

and use the approximation

\[
\int_{-R}^R f(x)dx \simeq hR \sum_{k=-N}^N f(Rg(hk))g'(hk) .
\]
Actually, the function (23) is different from the ones used traditionally, it was introduced rather recently [15]. It makes the numerical integration procedure rather fast and marvellously precise. We always take the parameter $c$ equal to $1/10$. For the rest of parameters we take

\[ h = 1/20, \quad N = 200, \quad \text{for } R = 1, \]
\[ h = 1/25, \quad N = 250, \quad \text{for } R = 2. \]

This gives astonishingly good precision of 70 digits for the functions of the type of $K(x)$.

Then we continue the resolvent to $C_\pm$ by virtue of the equation (22) and its transposition (the operators are self-adjoint), and apply the same double exponential trick for the integrals over $\phi$ in (21) with the parametrisation (20). For these integrals we shall use other parameters (mostly for computations to follow, which need higher precision):

\[ h = 1/30, \quad N = 300, \quad \text{for } R = 1 \]
\[ h = 1/40, \quad N = 400, \quad \text{for } R = 2. \]

We begin with equations for $\omega(x, y, T) = \omega_T(ix, iy)$, which is the result of a procedure, similar to that we used to modify the equation for $\log a$ [13]. We have

\[ \omega(x, y, T) = \omega_1(x, y) + \omega_2(x, y, T). \]

The first term does not depend on temperature, but it depends on $R$ which has to be chosen for a given range of temperatures as has been explained. We have

\[ \omega_1(x, y) = \frac{1}{2\pi} \int_{C_-} f(z - x) F(z, y) dz + \frac{\pi}{2} K(x - y), \]

where

\[ F(x, y) = f(x - y) + \int_{C_-} R(x, z) f(z - y) dz, \quad f(x) = \frac{i}{x(x + i)}, \]

For $\omega_2(x, y, T)$ we have

\[ \omega_2(x, y, T) = \frac{1}{\pi} \left( \int_{C_+} F(z, x) G(z, y) dm(z) + \int_{C_-} F(z, x) G(z, y) dm(z) \right), \]

where the measure is as before

\[ dm(x) = \frac{dx}{1 + a(x)}, \]

and the auxiliary function satisfying the equation

\[ G(x, y) = F(x, y) - \int_{C_+} R(x - z) G(x, y) dm(z) - \int_{C_-} R(x - z) G(x, y) dm(z) \]

We need not the function $\omega(x, y, T)$, but rather its Taylor coefficients

\[ \omega(x, y, T) = \sum_{j,k=1}^{\infty} \omega_{j,k} x^{j-1} y^{k-1}, \]

for $1 \leq j, k \leq 10$. To get them we begin with the functions $f_k(x)$ which are Taylor coefficients of $f(x - y)$ in $y$, and define $F_k(x), G_k(x)$ in obvious way. Then we plug $f_k(x), F_k(x), G_k(x)$ into the definitions of $\omega_1, \omega_2$ getting directly $\omega_{j,k}(T)$. The trouble here is that the functions $f_k(x)$, and consequently $F_k(x), G_k(x)$, have poles of order $k - 1$ at $x = 0, -i$. These poles are
close to the integration contour which makes the integrands rather sharp. Numerical integration of such functions needs too much of precision. This concerns especially the function \( \omega_1(x, y) \) where the singularities coming doubly from two multipliers. It is not so bad, but still unpleasant for \( F(x, y) \). Finally, for small enough temperature this problem does not concern \( G \) and \( \omega_2 \); the measure \( dm(x) \) is very small near \( \Re(x) = 0 \), so, the contribution of singularities is dumped by it. Let us explain how to treat this problem for \( F \) and \( \omega_1 \).

Fortunately, we have an explicit solutions for \( T = 0 \). The corresponding function \( F_0(x, y) \) satisfying

\[
F_0(x - y) - \int_{-\infty-i0}^{\infty-i0} K(x - z) F_0(z - y) = f(x - y) ,
\]

is simply

\[
F_0(x) = \frac{\pi}{\sinh(\pi x)} .
\]

Certainly the singularities at \( x = y \) cancel in

\[
\Delta F(x, y) = F(x, y) - F_0(x - y) .
\]

For this function one immediately derives

\[
\Delta F(x, y) = d(x, y) + \int_{-R}^{R} R(x, z) d(z, y) dz ,
\]

\[
d(x, y) = -\left( \int_{-\infty}^{-R} + \int_{R}^{\infty} \right) K(x - z) F_0(z - y) dz .
\]

Now we rewrite the definition of \( \omega_1 \):

\[
\omega_1(x, y) = \omega(i(x - y)) - \frac{1}{2\pi} \left( \int_{-R}^{\infty} f(z - x) F_0(z - y) dz + \frac{1}{2\pi} \int_{C_-} f(z - x) \Delta F(z, y) dz ,
\]

where \( \omega(\lambda) \) is defined in (18). In the last integral singularities close to the contour of integration remain in \( f(z - x) \), but they do not double with the singularities of \( F(z, y) \), and we can arrive at good precision.

Let us summarise our procedure. For given \( R \) (we take \( R = 1, 2 \)) we first solve the equation for \( R \) (22) by iterations, with great precision (60 digits). Then we find \( R(x, y) \) with \( x, y \in C_\pm \) using the equation (22). Then we find \( \Delta F \) from (27) and \( \omega_1 \) from (28). Now we start to work with temperature. First we solve the equation for \( \log \alpha \) (19) and verify that \( 1/i \log \alpha(R) < \pi \). Now we solve by iterations the equation for \( F \) (27), finally we find \( \omega_2 \) (25).

Let us mention checks which we have performed. Our numerical integration over the real line and the ellipse for the resolvent can be checked by the Cauchy theorem:

\[
R(x, y) = K(x, y) - \int_{C_-} K(x, z) R(z, y) dz ,
\]

More crucial is to check that \( \omega_1(T)_{i,j} \) vanish for \( i + j \) odd. This is really nontrivial when we apply (28), and if the precision is lost somewhere it is immediately felt. Finally, we take \( R = 2 \) starting from \( T = 1/200 \) than from \( T = 1/10 \) we can switch to \( R = 1 \) which is more economic.
for the computer time. The check is to see that for $T = 1/10$ both $R = 1$ and $R = 2$ give the same result.

7. Entanglement entropy at finite temperature

We shall consider temperatures from $1/200$ to $3/8$. With temperature the fates of different expectation values differ. For example, the correlation function $-\langle \sigma_1^3 \sigma_{10}^3 \rangle$ obviously decays while the vacuum formation probability grows: it is more probable to find a piece of ferromagnetic chain when the antiferromagnetic order is destroyed by temperatures. This is illustrated on the figures below:

We compute the entanglement entropy $s(n, T)$ for low temperatures (up to $T = 3/8$). We shall be interested in the difference $s(n, T) - s(n, 0)$ for which we want to verify two things. First, is it true that for $n = 8, 9, 10$ we are approaching the scaling limit which means that the difference in question becomes a function of $nT$? Second, is it true that we are not far from the CFT [16, 17] which predicts

$$s(n, T) - s(n, 0) \approx \frac{1}{3} \log \left( \frac{\sinh(nT)}{nT} \right).$$

In the right hand side we took into account all necessary normalisations. The answers to both question are in the following table in which $s(n, T) - s(n, 0)$ are given for $n = 8, 9, 10$ with $nT$ varying from .05 to 2 with step .05. We see that the values of $s(n, T) - s(n, 0)$ are close for $n = 8, 9, 10$, and the last one is reasonably close to the CFT prediction. Certainly, the difference grows for large $nT$. It is interesting to notice that for small $nT$ the values of $s(n, T) - s(n, 0)$ are lower than the CFT prediction, around $nT = 1.2$ they cross the CFT prediction, and start to be a little larger.
| $nT$ | $n = 8$ | $n = 9$ | $n = 10$ | CFT |
|------|---------|---------|---------|-----|
| 0.05 | 0.00013718326 | 0.00013758335 | 0.00013786523 | 0.00013887732 |
| 0.10 | 0.00054888502 | 0.00055046054 | 0.00055156807 | 0.0005537049 |
| 0.15 | 0.00123508732 | 0.00123856920 | 0.00124101147 | 0.00124906384 |
| 0.20 | 0.00219552716 | 0.002201589066 | 0.00220583162 | 0.00221926676 |
| 0.25 | 0.00342971960 | 0.00343896461 | 0.00344541993 | 0.00346501700 |
| 0.30 | 0.00493696730 | 0.00494991449 | 0.00495893249 | 0.00498508514 |
| 0.35 | 0.006716367454 | 0.00673343866 | 0.006745297245 | 0.00677798038 |
| 0.40 | 0.008766818592 | 0.0087832527 | 0.00880322110 | 0.00884195738 |
| 0.45 | 0.01108702761 | 0.01113119736 | 0.011175024250 | 0.01121863885 |
| 0.50 | 0.01367551758 | 0.01370632378 | 0.01372751381 | 0.013774951538 |
| 0.55 | 0.01653063593 | 0.01656602169 | 0.01659026149 | 0.01663282089 |
| 0.60 | 0.01965056377 | 0.01969027193 | 0.01971374418 | 0.01976534178 |
| 0.65 | 0.02303323512 | 0.02307692586 | 0.02310648849 | 0.023150250959 |
| 0.70 | 0.02667679930 | 0.026723676529 | 0.02675525456 | 0.026790936485 |
| 0.75 | 0.030578728022 | 0.03062806972 | 0.030661047258 | 0.03069414322 |
| 0.80 | 0.03473673076 | 0.03478751545 | 0.03482112784 | 0.03486452504 |
| 0.85 | 0.039148314771 | 0.03919929988 | 0.03923262593 | 0.03927484423 |
| 0.90 | 0.04381088783 | 0.04386059748 | 0.04389255176 | 0.043843922307 |
| 0.95 | 0.04872177091 | 0.04876848353 | 0.04879878064 | 0.04871520796 |
| 1.00 | 0.05387821125 | 0.05391994665 | 0.05394520545 | 0.05381520524 |
| 1.05 | 0.05927739556 | 0.05931190153 | 0.05933146930 | 0.059344661603 |
| 1.10 | 0.06491646359 | 0.06494120167 | 0.064953257954 | 0.064970199641 |
| 1.15 | 0.07079252225 | 0.07080465212 | 0.07080717234 | 0.07048090530 |
| 1.20 | 0.07690265901 | 0.076899022173 | 0.07688976878 | 0.07647710404 |
| 1.25 | 0.08324395685 | 0.08322105804 | 0.08319757106 | 0.08268626146 |
| 1.30 | 0.08981350845 | 0.08976749539 | 0.089727082245 | 0.08910400872 |
| 1.35 | 0.09660843075 | 0.09653507178 | 0.09647496262 | 0.09572595123 |
| 1.40 | 0.10362587957 | 0.103520539027 | 0.10343720911 | 0.10254767960 |
| 1.45 | 0.110863063859 | 0.11072067531 | 0.11061082984 | 0.109567477991 |
| 1.50 | 0.11831725958 | 0.11813229713 | 0.11799219117 | 0.11677284346 |
| 1.55 | 0.12598582287 | 0.125752207925 | 0.12557785975 | 0.12416747595 |
| 1.60 | 0.13386620216 | 0.13357752430 | 0.1333644609 | 0.13174430595 |
| 1.65 | 0.14195594901 | 0.14160505680 | 0.14134861406 | 0.13949899283 |
| 1.70 | 0.15025272729 | 0.149831949990 | 0.149527090012 | 0.14742723403 |
| 1.75 | 0.15875432052 | 0.15825337676 | 0.15789667142 | 0.15552477175 |
| 1.80 | 0.16745863710 | 0.16687260977 | 0.16645423495 | 0.16378739896 |
| 1.85 | 0.17636371328 | 0.17568102870 | 0.17519674406 | 0.17221096492 |
| 1.90 | 0.18546771377 | 0.18467812640 | 0.18412155818 | 0.18079138004 |
| 1.95 | 0.19476892996 | 0.19386153653 | 0.19322492710 | 0.18952462022 |
| 2.00 | 0.204265775830 | 0.20322892251 | 0.20250501998 | 0.19840673068 |
For better visualisation we compare the $n = 10$ results (dashed line) with the CFT curve up to $nT = 3$. We observe a reasonable agreement.

![Graph showing comparison between the $n = 10$ results and the CFT curve.]

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8. **Appendix**

In this appendix we give the eigenvalues of the density matrix for $T = 0$ with accuracy $10^{-11}$. For high spins the eigenvalues become too small, and therefore we do not write them.

- $n = 2$, $j = 0, 1$
  - $\{0.69314718056\}$,
  - $\{0.10228427315\}$.

- $n = 3$, $j = 1/2, 3/2$
  - $\{0.450771338685, 0.03398034507\}$,
  - $\{0.007624158125\}$.

- $n = 4$, $j = 0, 1, 2$
  - $\{0.61451589297, 0.00365561121\}$,
  - $\{0.12071380424, 0.00552473720, 0.00069384043\}$,
  - $\{0.000206270047\}$.

- $n = 5$, $j = 1/2, 3/2, 5/2$
  - $\{0.42478947699, 0.04837782416, 0.00132787973, 0.00016215953, 0.00002079330\}$,
  - $\{0.01220782094, 0.00041374155, 0.00003079567, 5.55739 \cdot 10^{-6}\}$,
  - $\{2.01173 \cdot 10^{-6}\}$. 
\[
n = 6, \ j = 0, 1, 2, 3
\{
0.5722507209, 0.00689732739, 0.00012390859, 0.00001153518, 2.1124 \cdot 10^{-7},
0.1281080844, 0.00963410772, 0.00146363784, 0.00020810707, 0.00003475259,
2.69435 \cdot 10^{-6}, 1.59341 \cdot 10^{-6}, 2.7386 \cdot 10^{-7}, 5.023 \cdot 10^{-8},
0.00045834467, 0.00001216336, 6.7394 \cdot 10^{-7}, 7.206 \cdot 10^{-8}, 1.690 \cdot 10^{-8},
7.07 \cdot 10^{-9}
\}.
\]
\[
n = 7, \ j = 1/2, 3/2, 5/2
\{
0.40741354415, 0.05661439956, 0.00274447210, 0.00041094696, 0.0000605511,
0.00004663152, 5.80502 \cdot 10^{-6}, 1.09218 \cdot 10^{-6}, 3.7937 \cdot 10^{-7}, 6.181 \cdot 10^{-8}, 3.019 \cdot 10^{-8},
4.47 \cdot 10^{-9}, 8.4 \cdot 10^{-10}, 1.6 \cdot 10^{-10},
0.01533056579, 0.00089067320, 0.00008573919, 0.00001697604, 0.00001524263,
1.72604 \cdot 10^{-6}, 3.4884 \cdot 10^{-7}, 6.306 \cdot 10^{-8}, 1.710 \cdot 10^{-8}, 1.208 \cdot 10^{-8}, 1.46 \cdot 10^{-9},
9.1 \cdot 10^{-10}, 2.0 \cdot 10^{-10}, 5 \cdot 10^{-11},
6.30299 \cdot 10^{-6}, 1.3831 \cdot 10^{-7}, 5.91 \cdot 10^{-9}, 4.8 \cdot 10^{-10}, 7 \cdot 10^{-11}, 2 \cdot 10^{-11}
\}.
\]
\[
n = 8, \ j = 0, 1, 2, 3
\{
0.54407108951, 0.009518029040, 0.00031430987, 0.00003722014, 4.34242 \cdot 10^{-6},
8.6837 \cdot 10^{-7}, 4.4767 \cdot 10^{-7}, 2.109 \cdot 10^{-8}, 1.263 \cdot 10^{-8}, 5.6 \cdot 10^{-10}, 2.0 \cdot 10^{-10}, 3 \cdot 10^{-11},
0.13192740945, 0.01273100394, 0.00217334341, 0.00049770442, 0.00009211769,
9.33699 \cdot 10^{-6}, 7.06799 \cdot 10^{-6}, 5.77728 \cdot 10^{-6}, 1.29977 \cdot 10^{-6}, 1.10141 \cdot 10^{-6}, 2.1627 \cdot 10^{-7},
1.1066 \cdot 10^{-7}, 9.385 \cdot 10^{-8}, 1.705 \cdot 10^{-8}, 5.46 \cdot 10^{-9}, 3.62 \cdot 10^{-9}, 2.72 \cdot 10^{-9}, 6.6 \cdot 10^{-10},
3.4 \cdot 10^{-10}, 1.4 \cdot 10^{-10}, 7 \cdot 10^{-11}, 4 \cdot 10^{-11},
0.00070629696, 0.00003306502, 2.45652 \cdot 10^{-6}, 4.7394 \cdot 10^{-7}, 3.0235 \cdot 10^{-7}, 7.450 \cdot 10^{-8},
4.116 \cdot 10^{-8}, 5.43 \cdot 10^{-9}, 1.33 \cdot 10^{-9}, 1.19 \cdot 10^{-9}, 1.8 \cdot 10^{-10}, 5 \cdot 10^{-11}, 3 \cdot 10^{-11}, 1 \cdot 10^{-11},
3.159 \cdot 10^{-8}, 5.9 \cdot 10^{-10}, 2 \cdot 10^{-11}
\}.
$n = 9, \ j = 1/2, 3/2, 5/2, 7/2$

\{0.39446858225, 0.06203960539, 0.00404495595, 0.00068660207, 0.00012681903, \\
0.0001130269, 0.00001809518, 3.66388 \cdot 10^{-6}, 1.57011 \cdot 10^{-6}, 1.53455 \cdot 10^{-6}, \\
2.8038 \cdot 10^{-7}, 2.1435 \cdot 10^{-7}, 1.4717 \cdot 10^{-7}, 4.352 \cdot 10^{-8}, 2.421 \cdot 10^{-8}, 1.612 \cdot 10^{-8}, \\
4.91 \cdot 10^{-9}, 2.99 \cdot 10^{-9}, 1.98 \cdot 10^{-9}, 9.6 \cdot 10^{-10}, 6.1 \cdot 10^{-10}, 3.2 \cdot 10^{-10}, 1.1 \cdot 10^{-10}, \\
6. \cdot 10^{-11}, 5. \cdot 10^{-11}, 1. \cdot 10^{-11}, 0. \cdot 10^{-11}\},$

\{0.01764053114, 0.00135269525, 0.00015298092, 0.00004270988, 0.00003235274, \\
5.57512 \cdot 10^{-6}, 1.20176 \cdot 10^{-6}, 5.2612 \cdot 10^{-7}, 2.8393 \cdot 10^{-7}, 8.094 \cdot 10^{-8}, 6.731 \cdot 10^{-8}, \\
5.960 \cdot 10^{-8}, 1.449 \cdot 10^{-8}, 8.14 \cdot 10^{-9}, 5.22 \cdot 10^{-9}, 4.09 \cdot 10^{-9}, 1.23 \cdot 10^{-9}, 9.8 \cdot 10^{-10}, \\
7.0 \cdot 10^{-10}, 3.0 \cdot 10^{-10}, 1.1 \cdot 10^{-10}, 1. \cdot 10^{-10}, 8. \cdot 10^{-11}, 3. \cdot 10^{-11}, 2. \cdot 10^{-11}, 2. \cdot 10^{-11}\},$

\{0.00001225502, 4.8365 \cdot 10^{-7}, 2.849 \cdot 10^{-8}, 5.91 \cdot 10^{-9}, 2.72 \cdot 10^{-9}, 4.2 \cdot 10^{-10}, \\
4.2 \cdot 10^{-10}, 1.2 \cdot 10^{-10}, 4. \cdot 10^{-11}, 0. \cdot 10^{-11}\},$

\{6. \cdot 10^{-11}\}.

$n = 10, \ j = 0, 1, 2, 3$

\{0.5232247016, 0.01165676559, 0.00053353501, 0.00007341532, 0.00001374860, \\
2.03723 \cdot 10^{-6}, 1.66763 \cdot 10^{-6}, 1.4508 \cdot 10^{-7}, 1.0705 \cdot 10^{-7}, 5.471 \cdot 10^{-8}, 1.737 \cdot 10^{-8}, \\
3.36 \cdot 10^{-9}, 1.36 \cdot 10^{-9}, 9.6 \cdot 10^{-10}, 5.5 \cdot 10^{-10}, 2.2 \cdot 10^{-10}, 4. \cdot 10^{-11}, 3. \cdot 10^{-11}, \\
3. \cdot 10^{-11}, 2. \cdot 10^{-11}\},$

\{0.13415188237, 0.01516080455, 0.00280724821, 0.00081234228, 0.00016161568, \\
0.0002156505, 0.00001938501, 0.00001238378, 4.24831 \cdot 10^{-6}, 2.54338 \cdot 10^{-6}, \\
5.2611 \cdot 10^{-7}, 4.3033 \cdot 10^{-7}, 3.7137 \cdot 10^{-7}, 2.3056 \cdot 10^{-7}, 7.255 \cdot 10^{-8}, 4.556 \cdot 10^{-8}, \\
2.913 \cdot 10^{-8}, 1.613 \cdot 10^{-8}, 1.522 \cdot 10^{-8}, 4.47 \cdot 10^{-9}, 3.92 \cdot 10^{-9}, 3.90 \cdot 10^{-9}, 2.20 \cdot 10^{-9}, \\
8.6 \cdot 10^{-10}, 7.5 \cdot 10^{-10}, 5.1 \cdot 10^{-10}, 2.8 \cdot 10^{-10}, 2.7 \cdot 10^{-10}, 1.9 \cdot 10^{-10}, 1.7 \cdot 10^{-10}, 7. \cdot 10^{-11}, \\
4. \cdot 10^{-11}, 3. \cdot 10^{-11}, 2. \cdot 10^{-11}, 1. \cdot 10^{-11}\},$

\{0.00093840865, 0.00005918266, 5.29807 \cdot 10^{-6}, 1.58240 \cdot 10^{-6}, 7.2140 \cdot 10^{-7}, \\
1.8468 \cdot 10^{-7}, 1.6292 \cdot 10^{-7}, 2.367 \cdot 10^{-8}, 1.706 \cdot 10^{-8}, 6.67 \cdot 10^{-9}, 6.05 \cdot 10^{-9}, 1.73 \cdot 10^{-9}, \\
1.09 \cdot 10^{-9}, 3.1 \cdot 10^{-10}, 2.5 \cdot 10^{-10}, 2.0 \cdot 10^{-10}, 9. \cdot 10^{-11}, 7. \cdot 10^{-11}, 6. \cdot 10^{-11}, 2. \cdot 10^{-11}, \\
2. \cdot 10^{-11}, 1. \cdot 10^{-11}\},$

\{7.919 \cdot 10^{-8}, 2.69 \cdot 10^{-9}, 1.3 \cdot 10^{-10}, 3. \cdot 10^{-11}, 1. \cdot 10^{-11}\}

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