Penrose’s Weyl curvature hypothesis and conformally-cyclic cosmology

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Abstract. I discuss two ideas of Roger Penrose and the relation between them: his Weyl curvature hypothesis from thirty years ago and his recent idea for a conformally-cyclic universe, that is a cosmological model with positive cosmological constant in which the conformal metric is cyclic - it can be continued through the future null infinity of one aeon to the big bang of the following aeon.

1. Introduction
Penrose’s Weyl curvature hypothesis was proposed by him in [1] thirty years ago. Since that time, it has provided the motivation for a series of works in mathematical general relativity, so that one has a good knowledge of cosmological models satisfying the hypothesis. More recently, he has described his idea of a conformally-cyclic cosmology in numerous lectures over the past five years or so, with the fullest account of this circle of ideas so far in his recent article [2]. In this talk, I shall sketch a review of both sets of ideas, emphasising how the second fills a gap in the first.

2. The Weyl Curvature Hypothesis
The starting point for the WCH is the observation that the Big Bang singularity was very special. Penrose argues that the universe must have been in a low entropy state initially in order for there now to be a second law of thermodynamics. Assuming, as is commonly done, that the matter content of the universe was in thermal equilibrium near to the bang, and therefore in a state of high entropy, one needs the contribution to the entropy from the rest of physics, which means the contribution form gravity or equivalently geometry, to be low. That is, the geometry must be highly ordered.

Penrose gives a quantitative form to this argument as follows: the entropy of the contents of the past light cone of an observer now can be estimated as $10^{88}$ (this estimate from the earlier article [1] is revised up to $10^{101}$ in [2] to take account of the large black holes now known to exist in galactic centres); however if all the matter in this past light cone was in black holes the entropy would rise to $10^{123}$, which is probably the maximum possible; interpreting entropy as the logarithm of phase-space volume, this means that the actual universe lies in an unimaginably small proportion $10^{101}/10^{123}$ of the available phase-space volume, which is very special indeed.

There is no generally accepted definition of gravitational entropy but Penrose argues that low gravitational entropy must mean small Weyl tensor. Why the Weyl tensor? One answer
is that this is the ‘free gravitational field’ since, in GR, the Ricci tensor is fixed point-wise by the matter. Another is that, according to the BKL picture, singularities of the vacuum Einstein equations, which are necessarily singularities of the Weyl tensor, can be extraordinarily complicated, probably space-time chaotic. To avoid this chaos the Big Bang, while being a cosmological or curvature singularity, needs to have a non-singular Weyl tensor.

In [1], as a result of these arguments, Penrose proposed that the Weyl tensor vanishes at the Big Bang singularity. A weaker hypothesis which he allowed is simply that the Weyl tensor is finite there. In either case one has the problem of identifying finiteness of some components of the Riemann tensor, namely those corresponding to the Weyl tensor, simultaneously with other components, belonging to the Ricci tensor, being singular.

One way to deal with this problem is to make a stronger hypothesis, based on the conformal rescaling properties of the curvature. If two metrics \( \tilde{g}_{ab} \) and \( g_{ab} \) are conformally related:

\[
\tilde{g}_{ab} = \tilde{\Omega}^2 g_{ab},
\]

then their Weyl tensors are related by

\[
\tilde{C}^{\cdot \cdot \cdot \cdot}_{abc} = C^{\cdot \cdot \cdot \cdot}_{abc}.
\]

Now suppose that \( \tilde{\Omega} = 0 \) on a smooth space-like hypersurface \( \Sigma \) in the manifold \( M \) with metric \( g_{ab} \), then the metric \( \tilde{g}_{ab} \) on the submanifold \( \tilde{M} \) of \( M \) on which \( \tilde{\Omega} > 0 \) will have a curvature singularity at \( \Sigma \), but one at which the Weyl tensor is finite, by virtue of (2). We shall call a cosmological singularity with this character a \textit{conformal gauge singularity} [3] since the conformal metric is regular and the singularity is due to the choice of metric in the conformal class. Singularities like this have also been called \textit{isotropic} [4] and \textit{conformally compactifiable} [5].

To investigate the existence of cosmological models with conformal gauge singularities, one may seek a Cauchy problem for the unphysical metric \( g_{ab} \) and conformal factor \( \tilde{\Omega} \) with data at \( \Sigma \), whose solution gives a solution \( \tilde{g}_{ab} \) of the Einstein equations with a suitable matter source in \( \tilde{M} \). This programme has been carried out with the following results: such a Cauchy problem is well-posed for

- perfect fluids with equation of state \( p = (\gamma - 1)\mu \), \( 1 < \gamma \leq 2 \), and for dust assuming also spatial homogeneity [5] (see the earlier work [6] for \( \gamma = 4/3 \)); the data consist of just the unconstrained 3-metric of \( \Sigma \); if the space-time Weyl tensor vanishes at \( \Sigma \) then it vanishes throughout the space-time, which is then FLRW;

- massless Einstein-Vlasov [7] (see [8] for the spatially-homogeneous case); the data consist of just the initial distribution function, a non-negative function of six variables subject to an integral ‘vanishing dipole’ condition; now the space-time Weyl tensor can vanish at \( \Sigma \) but be nonzero in the space-time;

- spatially-homogeneous metrics with cosmological constant and either perfect fluids with equation of state \( p = (\gamma - 1)\mu \), \( 1 \leq \gamma \leq 2 \), or massive Einstein-Vlasov; data as before [9].

There is also some evidence, without a complete existence proof, for well-posedness with other matter models including scalar fields and Einstein-Yang-Mills-Vlasov [10].

This work provides many examples of cosmological models which satisfy the Weyl curvature hypothesis, based on the stronger assumption of a cosmological singularity which is a conformal gauge singularity. One may therefore ask for conditions on a cosmological singularity for it to be a conformal gauge singularity. This is a difficult question but there exist partial answers [3], [11]. One needs more tools from conformal geometry, in particular the notions of conformal geodesic and a conformally-invariant propagation along conformal geodesics. The theory can be succinctly expressed in terms of the conformally-invariant tractor calculus, [12]. The result is as follows: given an incomplete conformal geodesic \( \gamma \) running into a space-time singularity which
is not ‘conformally-infinitely remote’ (in a way which can be made precise), then there will exist
a conformal rescaling of a neighbourhood $U$ of a final segment of $\gamma$ for which the rescaled metric
extends, with any desired degree of smoothness, to a larger neighbourhood \textit{provided} the tractor
curvature and its tractor derivatives up to a certain order are bounded. The tractor curvature
essentially consists of the Weyl curvature and its divergence, so that the theorem takes the
form that a conformal extension of a neighbourhood of $\gamma$ is possible given suitable boundedness
conditions on the Weyl tensor and its derivatives.

This is a \textit{local} extension theorem. One would like to have a \textit{global} extension theorem, yielding
an extension through all space-time singular points at once, given some boundedness conditions
on the Weyl tensor and its derivatives. This has been done in the case of spatial homogeneity
[11] but is outstanding in general.

3. Conformally-cyclic cosmology
The starting point for the second set of ideas under review is the emerging consensus that
the expansion of the universe is accelerating in a way consistent with a positive cosmological
constant, $\Lambda > 0$. Assuming there is no recollapse, the universe will be asymptotically-de Sitter
in the remote future in the sense that the physical space-time metric, which we shall now write
as $\hat{g}_{ab}$ with the physical space-time as $\hat{M}$, will have a conformal rescaling

$$\hat{g}_{ab} = \hat{\Omega}^2 g_{ab}$$

in terms of an unphysical metric $g_{ab}$ on unphysical $M$, and a conformal factor $\hat{\Omega}$ which grows
exponentially with physical proper-time $t$,

$$\hat{\Omega} \sim \exp H t, \text{ where } \Lambda = 3H^2.$$ 

The unphysical metric is smooth at future null infinity $I^+$, which is the space-like surface at
$t = \infty$ or at $\Omega^{-1} = 0$ in $M$. It necessarily follows that the Weyl tensor $C_{abcd}$ of $M$ is zero at
$I^+$, but the ‘spin-two field’ $K_{abcd} = \hat{\Omega} C_{abcd}$ has a nonzero limit there, representing gravitational
radiation in the remote future.

In [2], Penrose presents a picture of the very remote future with positive $\Lambda$ as a physical world
in which proper-time plays no role. He remarks that all stars will have completed their evolution
and either collapsed to form black holes or been swallowed up by the massive black holes at the
centres of galaxies. Black holes themselves will eventually decay by the Hawking process and
the content of the universe will very largely be just electromagnetic and gravitational radiation,
both of them massless fields. To complete the picture of a world from which proper-time has
vanished, Penrose hypothesises that all massive particles eventually either decay to radiation
or lose their mass in some unspecified way. Of course according to some theories protons do
decay, but electrons present a challenge to this view. However, this doesn’t seem to me to be
crucial for the next step, which is Penrose’s ‘outrageous’ proposal (his adjective): the stronger
version of the Weyl curvature hypothesis proposes that the space-time metric can be rescaled to
extend the conformal structure through the Big Bang, which is then a surface, say $\Sigma$, at which
the Weyl curvature vanishes; a positive $\Lambda$ leads to a universe in which the space-time metric
can be rescaled to extend the conformal metric through $I^+$, which is then another surface, $\Sigma'$
at which the Weyl curvature vanishes; these surfaces, with their conformal metrics, can then be
identified. One arrives at a description of the universe in terms of a conformal manifold with a
conformal metric which is cyclic – a conformally-cyclic cosmology. The physical or space-time
metric is not cyclic in the sense of being smooth through $\Sigma$, as the conformal factor giving the
physical metric from the unphysical cyclic metric runs from zero at the Big Bang to infinity at $I^+$,
returning to zero again for the next cycle, which Penrose calls an aeon.
Note that the electromagnetic and gravitational radiation present in the remote future in one aeon propagate through $\Sigma$ as primordial radiation in the next aeon. There will also be a scalar field after the Big Bang. This is mentioned in [2] but the following argument may not be exactly what Penrose had in mind: if the physical metrics before and after are $\hat{g}_{ab}$ and $\check{g}_{ab}$, while the unphysical cyclic metric is $g_{ab}$ so that

$$\hat{g}_{ab} = \hat{\Omega}^2 g_{ab}, \quad \check{g}_{ab} = \check{\Omega}^2 g_{ab},$$

then the corresponding physical (before and after) Ricci tensors are related by

$$\Phi^2 \hat{R}_{ab} = \Phi^2 \check{R}_{ab} + 2\Phi \nabla_a \Phi_b - 4\Phi_a \Phi_b + \check{g}_{ab}(\Phi \square \Phi + |\nabla \Phi|^2), \quad (3)$$

where $\Phi = \hat{\Omega}\check{\Omega}^{-1}$. The trace of (3) gives

$$\square \Phi + \frac{1}{6} \check{R} \Phi = \frac{1}{6} \check{R} \Phi^3, \quad (4)$$

so that, assuming constant $\check{R}$ and $\hat{R}$, $\Phi$ satisfies the conformally-invariant wave equation (and diverges at $\Sigma$).

Now suppose the matter content in $\hat{M}$ before $\Sigma$ is radiation plus cosmological constant so that

$$\hat{\check{R}}_{ab} = -\hat{T}_{ab} + \frac{1}{4} \hat{\check{R}}_{ab},$$

where $\hat{R}$ is constant, and

$$\check{T}_{ab} = \frac{1}{3} \check{\rho}(4\hat{u}_a \hat{u}_b - \check{g}_{ab}).$$

The rescaling which preserves the conservation equation is

$$\check{T}_{ab} = \Phi^2 \check{T}_{ab},$$

which again has the radiation form with, in particular, $\check{\rho} = \Phi^4 \hat{\rho}$. Then (3) becomes

$$-\check{T}_{ab} = \Phi^2 \hat{R}_{ab} + 2\Phi \nabla_a \Phi_b - 4\Phi_a \Phi_b + \check{g}_{ab}(|\nabla \Phi|^2 - \frac{1}{6} \check{R} \Phi^2 - \frac{1}{12} \hat{R} \Phi^4). \quad (5)$$

With our assumptions, this is the Einstein field equation in $\check{M}$, where $\Phi$ satisfies the conformally-invariant wave equation (4), and then the right-hand-side in (5) is recognisable as the ‘new improved energy-momentum tensor’ for such a $\Phi$ [13].

Whether or not this is exactly Penrose’s route to $\Phi$, part of his circle of ideas is that the dark matter in $\hat{M}$ is described by a scalar field $\Phi$ arising after the Bang.

One observational consequence of conformally-cyclic cosmology (CCC) would be a calculation of the spectrum of density perturbations to be expected after the Bang. This hasn’t been done, but it is likely to be scale-free as in the calculation from inflation since, in a sense, one can think of CCC in terms of inflation happening ‘before the bang’. Penrose [2] has drawn attention to another consequence of CCC which gives the possibility of a crucial test: approaching $I^+$, late on in conformal time in $\hat{M}$ (rather than in proper time, which has an infinite range so that there is no meaning to ‘late on’) the principal physical events will be black hole mergers; these will give a characteristic gravitational radiation pattern of a burst concentrated near a single light cone, which will in turn give characteristic spherical ripples on $\Sigma$; these ripples should appear as circles of perturbation $\Delta T$ in CMB maps, the detailed distribution of which would depend on the late-time history of the previous aeron.
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