On the Finite Temperature Effective Potential in Scalar QED with N Flavors.

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Abstract

The effective potential of scalar quantum electrodynamics with N flavors of complex scalar fields is studied, by performing a self consistent $1/N$ expansion up to next to leading order in $1/N$. Starting from the broken phase at zero temperature, the theory exhibits a phase transition to the symmetric phase at some finite temperature $T_c$. We work in general covariant gauges and demonstrate the gauge invariance of both, the critical temperature $T_c$ and the minimization condition at any finite temperature $T$. Furthermore, the only minimum of the potential is at zero scalar vacuum expectation value for any temperature $T \geq T_c$ and varies continuously to nonzero values for temperatures below $T_c$, implying the existence of a second order phase transition.
1 Introduction

In the last years, there has been a great interest in understanding the nature of the finite temperature symmetry restoration phase transition in the Standard Model (SM) [1] - [12]. Recently, the main motivation arose from the realization that the rate for baryon number violation in the SM at finite temperature is much larger than what it was originally assumed [1] - [3]. In fact, once the temperature is much higher than the height of the energy barrier between the different topologically distinct vacua [13], the exponential suppression of the anomalous baryon number violation processes, present at low temperature, disappears [4]-[5]. Thus, at sufficiently high temperatures, the anomalous baryon number violation processes could wash out the primordial baryogenesis generated, in some grand unified scenarios, at the very early stages of the universe [14]. In such cases, the generation of baryon number could have occurred through non-equilibrium processes in the transition from the symmetric to the broken $SU(2)_L \times U(1)_Y$ phase [4]. Quite generally, models of electroweak baryogenesis [15] fulfill Sakharov’s three requirements [16] for the generation of baryon asymmetry. Besides the nonconservation of B the underlying dynamics must involve nonequilibrium processes and violate CP. The violation of CP is present in the SM and its extensions, although in the SM it is probably too weak in order to generate the required baryon asymmetry [13]. To satisfy the nonequilibrium condition, these models rely on the presence of a first order phase transition, sufficiently strong in order to imply the suppression of baryon number violation processes in the broken phase.

The nature of the electroweak phase transition is a fundamental question which needs to be carefully explored and requires a detailed study of the effective potential of gauged scalar theories at finite temperature. A one loop analysis of the abelian theory was first done by Dolan and Jackiw [2]. The one loop effective potential shows the existence of a first order phase transition, whose strength depends on the values of the electroweak gauge and the scalar quartic coupling constants [1]. However, the naive one loop definition of the effective potential is known to have serious infrared problems, such as the appearance of a complex effective potential even at temperatures above
the critical temperature, $T_c$, at which the transition from the broken to the symmetric phase takes place. Moreover, it is found that the critical temperature itself is a gauge dependent quantity. It was realized in Ref. [2] that both - the infrared and the gauge dependence - problems do not appear, if only the dominant terms in the high temperature expansion are kept. Within such an approximation, a definition of a gauge invariant critical temperature can be achieved by assuming the existence of a second order phase transition. However, in order to study the nature of the phase transition, higher order terms in the effective potential need to be computed.

Quite recently, an improved analysis of the one loop effective potential has been considered by several authors [7] -[12]. The improved one loop effective potential is generically obtained by the inclusion of the finite temperature one loop corrections, at zero external momentum, to the boson propagators. When this process is carefully done, in order to avoid double counting of diagrams, it is equivalent to the inclusion of the so called ring diagrams in the one loop computation [10]. Such resummation of diagrams solves several problems inherent to the one loop expansion and leads to a first order phase transition, although weaker than that one found within the one loop expansion [11]. However, in the pure scalar case the ring expansion breaks down at temperatures close to the critical one, $|T - T_c| \simeq \tilde{\lambda} T_c$, with $\tilde{\lambda}$ the scalar quartic coupling [12]. Hence, it can not be used to analyze the behavior of the effective potential at $T \simeq T_c$. In addition, after the inclusion of the gauge fields, for values of the squared gauge coupling $g^2$ of the order of the quartic coupling it is not possible to conclude that the phase transition is first order based solely on the ring expansion. This is due to the fact that, within the ring approximation, and for $g^2 \simeq \tilde{\lambda}$, the difference between the transition temperature $T_0$, at which the curvature at the origin vanishes, and the critical temperature $T_c$ is of order $g^2 T_c$, but the ring expansion breaks down for $|T - T_c| \leq g^2 T_c$ [12]. Thus, an analysis beyond the improved one loop approximation is necessary in order to study the nature of the phase transition for $g^2 / \tilde{\lambda} \simeq O(1)$, which is the phenomenologically interesting region in which the Higgs mass is of the order of the gauge boson mass.

As it has been explained in the seminal work by Dolan and Jackiw [2], for the pure scalar case a self consistent resummation of the diagrams contributing to the effective
potential can be done, by extending the theory to include N flavors of self interacting scalar fields and performing a 1/N study of this model. The 1/N expansion provides information which does not appear at any fixed order of perturbation theory and, hence, it is a more appropriate tool to study the nature of the phase transition, which is in itself a nonperturbative phenomenon. Studying the behaviour of the effective squared mass, by doing a self consistent 1/N expansion, Dolan and Jackiw \cite{2} showed that the infrared problems inherent to the pure O(N) scalar theory disappear. The explicit form of the effective potential at finite temperature was, however, not derived in this analysis, and it was studied in Refs. \cite{17} and \cite{18}. A similar resummation of diagrams has been recently performed in Ref. \cite{19}, while an analysis of the $N$ component - $\phi^4$ theory, by a method based on average fields, has been considered in Ref.\cite{21}.

The validity of the finite temperature computations in the scalar O(N) linear model was questioned, when it was realized that, in the continuum limit, the effective scalar potential at zero temperature was either complex, unstable or did not allow the spontaneous breakdown of the O(N) symmetry \cite{21} - \cite{23}. This issue was clarified by Bardeen and Moshe Moshe \cite{24}, who explained that the problems appearing in the large N computations were not associated with the 1/N expansion but with a more fundamental nonperturbative property of the O(N) scalar model in four dimensions, which is the issue of triviality. In fact, for positive bare quartic coupling, the continuum limit of the four dimensional O(N) theory is trivial, in the sense that the renormalized coupling goes to zero when the cutoff is removed. This implies that, to work with a nontrivial theory, one has either to take a negative bare quartic coupling, rendering the theory unstable at finite temperature \cite{24}, or treat the theory as an effective one, by keeping a finite effective cutoff. For a large effective cutoff, the 1/N expansion at zero and finite temperature can be defined in a consistent way, avoiding the problems mentioned above. Thus, in the following, we shall assume that we are dealing with an effective theory valid up to an energy scale of the order of an effective cutoff $\Lambda$, at which, quite generally, new physics should appear.

It must be mentioned, that the nature of the phase transition for scalar QED with $N$ flavors and a scalar Higgs heavier than the gauge boson, have been also studied by using
a $4 - \epsilon$ expansion \cite{25}-\cite{28}. Within such approximation, the theory may be analyzed by performing a renormalization group study for small $\epsilon$. The high temperature theory may be treated as an effective three dimensional theory and thus, to obtain physical results, the expansion parameter must be taken to be $\epsilon = 1$. In this framework, the critical value for the number of flavors, $N_c$, below which the phase transition becomes first order, was found to be surprisingly large, $N_c = 183$. Based also on this analysis, Halperin, Lubensky and Ma \cite{25} predicted a first order phase transition for the smectic-A to nematic phase transition in liquid crystals. However, when contrasted with experiment, this transition was found to be second order \cite{27}. In addition, for $\epsilon \to 1$, a $(2 + \epsilon)$ expansion gives results which are at variance with those ones coming from the $(4 - \epsilon)$ expansion \cite{25},\cite{28}. In particular, in the case in which the Higgs particle is heavier than the gauge boson, the $(2 + \epsilon)$ expansion indicates a much lower critical value of $N$, if any, for the three dimensional theory \cite{28}.

This paper is organized as follows: In section 2 we present the model, a scalar theory with $N$ flavors of complex scalar fields coupled to an abelian gauge field. In section 3 we derive the general expression of the effective potential up to next to leading order in $1/N$. In section 4 we concentrate on the leading order results, and we show that a second order phase transition takes place in this case. In section 5 we carry out the analysis of the critical temperature and the nature of the phase transition up to next to leading order in $1/N$. We start discussing three related aspects of the theory, which are the renormalization of couplings, the triviality of the theory and the absence of tachyons. Then, we analyze the radiative corrections at finite temperature to finally perform the minimization of the effective potential up to next to leading order in $1/N$. We demonstrate that a gauge invariant expression for the position of the minimum is obtained. In addition, the position of the minimum is not modified by next to leading order corrections and, hence, the phase transition remains second order. A comparison of our large $N$ results with those previously derived within the improved one loop method is performed in section 6, where we analyze the source of the discrepancy between the results obtained in both approaches. We reserve section 7 for our conclusions.
The Lagrangian density for a scalar theory with $N$ flavors of complex scalar fields in interaction with an abelian gauge field is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_{\mu} A^{\mu})^2 + \frac{1}{2} (D_{\mu} \phi^a)^\dagger (D^{\mu} \phi^a) - \frac{\bar{\lambda}}{4!} (|\phi|^2 - v^2)^2.$$  \hspace{1cm} (1)

In the above expression, $a$ is a flavor index taking values from 1 to $N$, $\phi^a$ is a complex scalar field, $|\phi|^2 = \phi^\dagger_a \phi_a$, $A_{\mu}$ is an abelian gauge field, and $D_{\mu}$ is the covariant derivative associated with it. In addition, we have included a gauge fixing term with $\alpha$ being the gauge fixing parameter ($\alpha = 0$ is the Landau gauge). Note that, at vanishing gauge coupling, this model reduces to the $O(2N)$ linear sigma model.

Following the method first proposed by Jackiw \cite{29}, to compute the effective potential $V(\hat{\phi})$ for this theory, we shift the fields $\phi_a \rightarrow \hat{\phi}_a + \phi_a$, where $\hat{\phi}_a$ is the vacuum expectation value of the scalar field and $\phi_a$ stands for its quantum fluctuations. The Lagrangian must be truncated by dropping all terms which are linear in the $\phi_a$ quantum fluctuations. The new Lagrangian, $\mathcal{L}(\hat{\phi}_a + \phi_a, A_{\mu})$, may be decomposed into a term quadratic in the $A_{\mu}$ and $\phi_a$ fields and an interaction term. The effective potential is given by the sum of the tree level potential plus the one loop contributions, obtained from the bilinear part of the shifted Lagrangian, plus higher order loop contributions, given by the sum over all connected one-particle irreducible vacuum graphs for the theory described by the shifted Lagrangian density $\mathcal{L}(\hat{\phi}_a + \phi_a, A_{\mu})$.

The quartic interactions pose a disadvantage for the diagrammatic analysis of the theory. Therefore, following Refs. \cite{21} - \cite{22}, we shall eliminate the scalar quartic term through the introduction of an auxiliary field $\chi$. The modified Lagrangian reads

$$\mathcal{L}(\phi_a, A_{\mu}, \chi) = \mathcal{L}(\phi_a, A_{\mu}) + \frac{3}{2\bar{\lambda}} \left[ \chi - \frac{\bar{\lambda}}{6} (|\phi|^2 - v^2) \right]^2.$$  \hspace{1cm} (2)

The Euler-Lagrange equation for $\chi$ is a constraint equation relating $\chi$ with the scalar field $\phi$. In fact, the effective potential for the modified theory $V(\hat{\chi}, \hat{\phi}_a)$ reduces to the one of the original theory if $\hat{\chi}$ satisfies the requirement

$$\frac{\partial V}{\partial \hat{\chi}} = 0,$$  \hspace{1cm} (3)
which defines the vacuum expectation value \( \hat{\chi} \) as a function of \( \hat{\phi} \).

A self consistent \( 1/N \) expansion can be defined if the gauge and quartic coupling constants \( g \) and \( \tilde{\lambda} \), respectively, depend on \( N \) so that in the limit of \( N \to \infty \), \( g^2 N \) and \( \tilde{\lambda} N \) go to constant values. Hence, for the purpose of our study it is better to define the new coupling constants \( e \equiv g \sqrt{N} \) and \( \lambda \equiv \tilde{\lambda} N \). Once the field \( \chi \) is introduced the Lagrangian density of the theory reads

\[
L(\phi_a, A_\mu, \chi) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \frac{1}{2} \partial_\mu \phi_{a,i} \partial^\mu \phi_{a,i} - \frac{e}{\sqrt{N}} A^\mu \epsilon^{ij} \phi_{a,j} \partial_\mu \phi_{a,i} \\
+ \frac{e^2}{2N} A_\mu A^\mu \phi_{a,i} \phi_{a,i} + \frac{3N}{2\lambda} \chi^2 - \frac{1}{2} \left(|\phi|^2 - v^2\right),
\]

where \( i = 1, 2, \phi_{a,1} \) and \( \phi_{a,2} \) are the real and imaginary parts of the field \( \phi_a \), respectively, and summation over \( a \) and \( i \) is understood. Although the introduction of the field \( \chi \) does not alter the dynamics of the full theory, it does lead to a new perturbation series, in which the \( 1/N \) expansion has a simpler diagrammatic interpretation. After considering the shifted fields, which now include the shift for the auxiliary field \( \chi \),

\[
\phi_a \to \sqrt{N} \hat{\phi}_a + \phi_a, \quad \chi \to \hat{\chi} + \chi
\]

and ignoring the linear terms in the quantum fluctuations \( \chi \) and \( \phi_a \), the shifted Lagrangian reads

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \frac{e^2}{2} A^\mu \hat{\phi}_{a,i} \hat{\phi}_{a,i} - e A^\mu \epsilon^{ij} \hat{\phi}_{a,j} \partial_\mu \phi_{a,i} \\
- \frac{e}{\sqrt{N}} A^\mu \epsilon^{ij} \phi_{a,j} \partial_\mu \phi_{a,i} + \frac{e^2}{2N} A^\mu A^\nu \phi_{a,i} \phi_{a,i} + \frac{e^2}{\sqrt{N}} A^\mu A^\nu \hat{\phi}_{a,i} \phi_{a,i} + \frac{1}{2} \partial_\mu \phi_{a,i} \partial^\mu \phi_{a,i} \\
+ \frac{3N}{2\lambda} \chi^2 - \frac{1}{2} \hat{\chi} \phi_{a,i} \phi_{a,i} - \sqrt{N} \hat{\phi}_{a,i} \chi \phi_{a,i} - \frac{1}{2} \phi_{a,i} \phi_{a,i} \chi - V_{\text{tree}},
\]

where the tree level potential is given by,

\[
V_{\text{tree}} = -\frac{3N}{2\lambda} \chi^2 + \frac{N}{2} \hat{\chi} \left(|\hat{\phi}|^2 - v^2\right).
\]

In the above, we have followed Root \[24\] in rescaling the vacuum expectation value \( \hat{\phi}_a \) and, consistently, \( v \) by a factor \( \sqrt{N} \). This simplifies the \( 1/N \) counting. In fact, if this rescaling were not done, the vacuum expectation value would be of order \( N \) in the natural scales of the theory, \( \hat{\phi}^2 = NF(\mu^2, T^2) \), where \( F \) is a function of the bare mass.
\[ \mu^2 = -\bar{\lambda}v^2/6, \] the temperature \( T \) and the couplings of the theory. This extra factor of \( N \) would have to be properly considered while doing the expansion at next to leading order. The physical results, of course, would remain unchanged.

The above Lagrangian may be decomposed into a quadratic and an interacting part. The expression of the quadratic Lagrangian reads

\[
L_{\text{quad}} = \frac{1}{2} A^\mu \left[ (\partial^2 + e^2 \hat{\phi}^2)g_{\mu\nu} - \partial_\mu \partial_\nu \left( 1 - \frac{1}{\alpha} \right) \right] A^\nu - \frac{1}{2} \phi_{a,i} \left( \partial^2 + \hat{\chi} \right) \phi_{a,i} \\
+ \frac{3N}{2\lambda} \chi^2 - \sqrt{N} \hat{\phi}_{a,i} \chi \phi_{a,i} - eA^\mu \epsilon^{ij} \hat{\phi}_{a,j} \partial_\mu \phi_{a,i}
\]

from which it is easy to deduce the inverse propagator matrices relevant for the computation of the effective potential. The interacting part of the Lagrangian \( L_I \), on the other hand, determines the vertices of the boson fields interactions. For simplicity, we will assume in the following that

\[ \hat{\phi}_{1,1} = \hat{\phi}, \]  

while all other vacuum expectation values are zero. In fact, it is not important which state is chosen as the ground state of the theory. Independent of such a choice the original global U(N) symmetry is spontaneously broken to U(N-1) and 2N-1 Goldstone bosons are generated, one of which is eaten by the gauge field through the usual Higgs mechanism.

We obtain two decoupled inverse propagator matrices. One that mixes the gauge field \( A_\mu \) with \( \phi_{1,2}, \mathcal{M}(A_\mu, \phi_{1,2}) \), and the other one which mixes the auxiliary field \( \chi \) with \( \phi_{1,1}, \mathcal{M}(\chi, \phi_{1,1}) \). Thus, in momentum space representation, the nonzero components of the inverse propagator matrices read,

\[
iD^{-1}_{\mu\nu} = - \left[ g_{\mu\nu} \left( k^2 - e^2 \hat{\phi}^2 \right) - k_\mu k_\nu \left( 1 - \frac{1}{\alpha} \right) \right] \\
iD^{-1}_{a,i,b,j} = \delta_{ab} \delta_{ij} (k^2 - \hat{\chi}) \\
iD^{-1}_{\chi\chi} = \frac{3N}{\lambda} \\
iD^{-1}_{1,1,\chi} = - \hat{\phi} \sqrt{N} \\
iD^{-1}_{1,2,\mu} = iek_\mu \hat{\phi}.
\]
Considering the above expressions it is straightforward to compute the determinant of the inverse propagator mass matrices, which in Euclidean space are given by

\[
\det \mathcal{M}(A_\mu, \phi_{1,2}) = \frac{(k^2 + \hat{\chi})(k^2 + e^2 \hat{\phi}^2)^3}{\alpha} \left[ k^2 + \frac{e^2 \hat{\phi}^2 \hat{\chi} \alpha}{(k^2 + \hat{\chi})} \right],
\]

(11)

\[
\det \mathcal{M}(\chi, \phi_{1,1}) = \frac{-3N}{\lambda} \left[ k^2 + \hat{\chi} + \frac{\lambda \hat{\phi}^2}{3} \right].
\]

(12)

Furthermore, the propagators of the scalars and gauge bosons, which are relevant for the computation of the effective potential up to next to leading order in \(1/N\) may be then obtained from Eqs. (10), (11) and (12), and in Euclidean space are given by

\[
iD_{\mu\nu} = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2 + e^2 \hat{\phi}^2} + \frac{k_\mu k_\nu}{k^2} \frac{\alpha (k^2 + \hat{\chi})}{[k^4 + \hat{\chi}k^2 + \alpha \hat{\chi}e^2 \hat{\phi}^2]}
\]

(13)

\[
iD_{\chi\chi} = \left( \frac{k^2 + \hat{\chi}}{k^2 + \hat{\chi} + \lambda \hat{\phi}^2/3} \right)
\]

(14)

3 The Effective Potential

As we mentioned above, in order to analyze which Feynman diagrams contribute to the effective potential up to next to leading order in \(1/N\), it is necessary to consider the behaviour in \(1/N\) of the propagators and of the interaction vertices. From Eqs. (10)-(14) we observe that a factor \(1/N\) is associated with \(D_{\chi\chi}\) and a factor \(1/\sqrt{N}\) is associated with \(D_{1,1;\chi}\). Moreover, a factor of order \(N\) is associated with any closed loop of \(\phi_a\) fields, due to the summation over all possible internal fields. From the interaction part of the Lagrangian in Eq.(3), we observe that a factor \(1/\sqrt{N}\) is associated with the derivative coupling of the scalar fields \(\phi_a\) to the gauge bosons, while a factor \(1/N\) appears in any quartic vertex \(A^2 \phi_{a,i} \phi_{a,i}\). Note that the vertex \(\chi \phi_{a,i} \phi_{a,i}\) has, instead, a factor 1 in the coupling.

The only diagram contributing to the potential at leading order in \(1/N\) \([17],[18]\), \(V_{l.o.}\), is the closed loop involving the 2(N-1) Goldstone bosons which do not mix with either \(A_\mu\) or \(\chi\),

\[
V_{l.o.} = \frac{2(N-1)}{2} \int_k \ln \left( k^2 + \hat{\chi} \right).
\]

(15)
We perform the finite temperature computations in the imaginary time formalism: After
a Wick rotation to Euclidean space we impose periodic boundary conditions on the time
direction of length \( L = T^{-1} \equiv \beta \). For simplicity of notation, we have defined
\[
\int_k f(k) \equiv \frac{1}{\beta} \sum_{n=-\infty}^{n=\infty} \int \frac{d^3k}{(2\pi)^3} f(\omega_n, \vec{k}),
\]
with \( \omega_n = 2\pi n/\beta \).

The computation of the effective potential at next to leading order in \( 1/N \) involves
two type of contributions. There are one loop contributions, involving the two nontrivial
propagator matrices,
\[
V_{n,1,0}^{1,l.} = \frac{1}{2} \int_k \left[ \ln \det (\mathcal{M}(A_{\mu}, \phi_{1,2})) + \ln \det (\mathcal{M}(\chi, \phi_{1,1})) \right],
\]
and there are those coming from multiloop one particle irreducible graphs, depicted
in Fig.1. Observe that any multiloop contribution involving the mixing of \( \chi \) and \( A_{\mu} \),
like those depicted in Fig.2, vanish in the regularized theory. After some work, the
resummation of all the multiloop diagrams contributing to next to leading order in \( 1/N \) gives
\[
V_{n,1,0}^{n-\text{loop}} = \sum_{n=1}^{\infty} \frac{1}{2n} \int_k \ln \left[ \frac{\lambda (k^2 + \hat{\chi})}{3 \left( k^2 + \hat{\chi} + \lambda \hat{\phi}^2/3 \right)} \right]^n B_n(k^2, \hat{\chi})
- \sum_{n=1}^{\infty} \frac{1}{2n} \int_k Tr \left[ ie^2 D_{\mu\nu} \Pi^{\alpha} \right]^n,
\]
which can be rewritten as
\[
V_{n,1,0}^{n-\text{loop}} = \frac{1}{2} \int_k \ln \left[ 1 + \frac{\lambda (k^2 + \hat{\chi})}{3 \left( k^2 + \hat{\chi} + \lambda \hat{\phi}^2/3 \right)} \right] B(k^2, \hat{\chi})
+ 1/2 \int_k Tr \ln \left( \delta_{\mu}^{\alpha} - ie^2 D_{\mu\nu} \Pi^{\alpha} \right),
\]
where
\[
B(k^2, \hat{\chi}) = \int_p \frac{1}{(p^2 + \hat{\chi}) \left( (k + p)^2 + \hat{\chi} \right)}
\]
and
\[
\Pi^{\alpha}(k, \hat{\chi}) = \int_p \frac{(2p + k)^{\alpha}(2p + k)^{\alpha}}{(p + k)^2 + \hat{\chi}} - 2\delta^{\alpha\nu} \int_p \frac{1}{p^2 + \hat{\chi}},
\]
is the finite temperature vacuum polarization contribution. Furthermore, using the properties
\[
\det \mathcal{M}(\phi_{1,2}, A_\mu) = (k^2 + \hat{\chi}) \det [-iD^{-1}_{\mu\nu}]
\]
and
\[
\det \mathcal{M}(\phi_{1,1}, \chi) = (k^2 + \hat{\chi}) D^{-1}_{\chi\chi},
\]
we arrive to the final formal expression for the effective potential up to next to leading order in $1/N$,
\[
V(\hat{\phi}, \hat{\chi}) = V_{\text{tree}} + V_{\text{l.o.}} + V_{n.l.o.}^{1,\text{loop}} + V_{n.l.o.}^{n,\text{loop}}
\]
\[
= -\frac{3N}{2\lambda} \hat{\chi}^2 + \frac{N}{2} \hat{\chi} (|\hat{\phi}|^2 - v^2) + \frac{(2N)}{2} \int_k \ln (k^2 + \hat{\chi})
\]
\[
+ \frac{1}{2} \int_k \ln \det \left[-iD^{-1}_{\mu\nu}(k, \hat{\chi}, \hat{\phi}) - e^2 \Pi_{\mu\nu}(k, \hat{\chi})\right]
\]
\[
+ \frac{1}{2} \int_k \ln \left[(k^2 + \hat{\chi}) \left(1 + \lambda B(\hat{\chi}, k^2)/3 + \lambda \hat{\phi}^2/3 \right) \right].
\]

As we discussed above, the expression for the effective potential of the original theory may be obtained from the above equation by imposing the condition
\[
\frac{\partial V(\hat{\phi}, \hat{\chi})}{\partial \hat{\chi}} = 0,
\]
which determines the expression of $\hat{\chi}$ as a function of $\hat{\phi}^2$. Furthermore, as it was first noticed in Ref. [22], in order to obtain the effective potential up to next to leading order in $1/N$, it is sufficient to solve the gap equation, Eq. (25), up to leading order in $1/N$. In fact, calling $\hat{\chi}(\hat{\phi}^2)$ the exact solution to the gap equation and $\bar{\chi}(\hat{\phi}^2)$ the leading order solution, then,
\[
\hat{\chi}(\hat{\phi}^2) - \bar{\chi}(\hat{\phi}^2) = \mathcal{O}(1/N).
\]
Expanding the effective potential around $\hat{\chi}(\hat{\phi}^2) = \bar{\chi}(\hat{\phi}^2)$, we have
\[
V(\hat{\phi}, \hat{\chi}) = V(\hat{\phi}, \bar{\chi}) + \frac{\partial V(\hat{\phi}, \bar{\chi})}{\partial \hat{\chi}} (\hat{\chi} - \bar{\chi}) + \mathcal{O}(1/N).
\]
However, since $\bar{\chi}(\hat{\phi})$ is the solution to the gap equation at leading order, it follows that \(\partial V_{\text{l.o.}}(\hat{\phi}, \hat{\chi})/\partial \hat{\chi} = 0\). Therefore, the second term in the above equation is also of order $1/N$,
\[
V(\hat{\phi}, \hat{\chi}(\hat{\phi})) = V(\hat{\phi}, \bar{\chi}(\hat{\phi})) + \mathcal{O}(1/N).
\]
Thus, in order to compute the effective potential up to next to leading order in $1/N$ we just need to consider Eq. (24) with $\hat{\chi}$ given by its leading order expression, $\hat{\chi}(\hat{\phi}) = \bar{\chi}(\hat{\phi})$.

Moreover, for the extent of this work we shall concentrate in computing the location of the extrema of the effective potential, which are derived from the condition,

$$ \frac{dV(\hat{\phi}, \bar{\chi})}{d\hat{\phi}^2} = 0. \quad (29) $$

In fact, the number and location of the extrema of the effective potential provide sufficient information to study the nature of the phase transition. In addition, as we will show below, a gauge independent expression for the solutions to Eq. (29) is found. Observe that if Eq. (29) is not fulfilled for any nontrivial value of the scalar field, the only minimum would be at the origin and the global $U(N)$ symmetry, together with the local $U(1)$ symmetry of the theory would be preserved. From Eq. (24), we find the relation $\partial V_{l.o.}/\partial \hat{\phi}^2 = N\bar{\chi}/2$, and recalling the fact that $\partial V_{l.o.}/\partial \bar{\chi} = 0$, we obtain

$$ \frac{dV}{d\hat{\phi}^2} = \frac{N\bar{\chi}(\hat{\phi}^2)}{2} + \left( \frac{\partial}{\partial \hat{\phi}^2} + \frac{\partial \bar{\chi}}{\partial \hat{\phi}^2} \frac{\partial}{\partial \bar{\chi}} \right) \left\{ \frac{1}{2} \int_k \ln \det \left[ -iD^{-1}_{\mu\nu}(k, \bar{\chi}, \hat{\phi}) - e^2 \Pi_{\mu\nu}(k, \bar{\chi}) \right] \right. $$

$$ + \left. \frac{1}{2} \int_k \ln \left[ (k^2 + \bar{\chi}) \left( 1 + \lambda B(\bar{\chi}, k^2)/3 \right) + \lambda \hat{\phi}^2/3 \right] \right\}, \quad (30) $$

which is the formal expression from which, after proper integration and renormalization procedures, we shall be able to determine the critical temperature and the order of the phase transition up to next to leading order in $1/N$.

4 Critical Temperature and Nature of the Phase Transition at Leading Order in $1/N$

In section 3 we derived the expression of the effective potential up to leading order in $1/N$ to be,

$$ V(\bar{\chi}, \hat{\phi}) = -\frac{3N}{2\lambda_0} \bar{\chi}^2 + \frac{N}{2} \bar{\chi} \left( \hat{\phi}^2 - v_0^2 \right) + N \int_k \ln \left( k^2 + \bar{\chi} \right), \quad (31) $$

where we have introduced the subscript zero to denote unrenormalized quantities. Thus, at leading order in $1/N$, the gap equation, which determines the expression of $\bar{\chi}$ as a
function of $\phi^2$, reads

$$
\frac{\partial V}{\partial \bar{\chi}} = -\frac{3N}{\lambda_0} \bar{\chi} + \frac{N}{2} (\phi^2 - v_0^2) + N \int_k \frac{1}{k^2 + \bar{\chi}} = 0. \quad (32)
$$

In order to explicitly evaluate the temperature dependent integrals, we shall use the relation

$$
\int f(k_0, \vec{k}) = \int \frac{d^4k}{(2\pi)^4} f(k) + \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left[ f(k_0, \vec{k}) + f(-k_0, \vec{k}) \right],
$$

which allows a decomposition in the zero temperature and the finite temperature contributions - the first and second terms in the right hand side of the above equation, respectively. For the particular case of the integral appearing in Eq.(32), we obtain,

$$
\int_k \frac{1}{k^2 + \bar{\chi}} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + \bar{\chi})} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + \bar{\chi}}} \left[ \frac{1}{\exp(\beta \sqrt{k^2 + \bar{\chi}}) - 1} \right],
$$

where the $k_0$ dependence in the last integral was evaluated by performing a contour integration in the complex $k_0$ plane. At large temperatures, $T \gg \bar{\chi}$, the above integral may be expanded in a way first derived by Dolan and Jackiw

$$
\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + \bar{\chi}}} \left[ \frac{1}{\exp(\beta \sqrt{k^2 + \bar{\chi}}) - 1} \right] = \frac{1}{12\beta^2} - \frac{\sqrt{\bar{\chi}}}{4\pi\beta} + \frac{\bar{\chi}}{16\pi^2} \ln \left( \frac{cT^2}{\bar{\chi}} \right) + h.o.(\beta^2 \bar{\chi})
$$

where $c = 16\pi^2 \exp(1 - 2\gamma)$, with $\gamma \approx 0.577$.

Moreover, the zero temperature contribution is quadratically divergent and needs to be regularized. Introducing a momentum cutoff $\Lambda$, so that

$$
\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + \bar{\chi})} = \frac{1}{16\pi^2} \left[ \Lambda^2 - \bar{\chi} \ln \left( \frac{\Lambda^2}{\bar{\chi}} \right) \right],
$$

we then absorb the quadratic and logarithmic divergences by defining the renormalized quantities

$$
v^2 = v_0^2 - \frac{\Lambda^2}{8\pi^2} \quad (37)
$$

and

$$
\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{1}{48\pi^2} \ln \left[ \frac{\Lambda^2}{M^2} \right],
$$

where $\lambda = \lambda_0 + \frac{\Lambda^2}{M^2}$. \quad (38)
where $M^2$ is a renormalization scale. Observe that, for any positive value of the unrenormalized quartic coupling $\lambda_0$, the renormalized coupling $\lambda$ vanishes at $\Lambda \to \infty$, and the theory becomes trivial in the continuum limit. If the unrenormalized quartic coupling is taken to be negative, the theory becomes unstable [24]. Therefore, the only consistent definition of the theory is when it is considered as an effective theory with an effective finite cutoff $\Lambda$.

We can further define a temperature dependent coupling by

$$\frac{1}{\lambda_T} = \frac{1}{\lambda} - \frac{1}{48\pi^2} \ln \left[ \frac{cT^2}{M^2} \right]$$

(39)

to finally rewrite the gap equation as

$$\tilde{\chi} - \frac{\lambda_T}{6} \left( \hat{\phi}^2 - v^2 + \frac{1}{6\beta^2} \right) + \frac{\lambda_T}{12\pi\beta} \sqrt{\tilde{\chi}} = 0.$$

(40)

The above expression gives a quadratic equation in $\sqrt{\tilde{\chi}}$, which can be easily solved to find [2], [18]

$$\sqrt{\tilde{\chi}} = \frac{-\lambda_T}{24\pi\beta} \pm \sqrt{\left( \frac{\lambda_T}{24\pi\beta} \right)^2 + \frac{\lambda_T}{6} \left( \hat{\phi}^2 - v^2 + \frac{1}{6\beta^2} \right)},$$

(41)

where the plus sign has been chosen in order to obtain positive values of $\sqrt{\tilde{\chi}}$, which is a precondition for the validity of the gap equation derived above.

As we have shown in section 3, the minimization condition up to leading order in $1/N$ reads,

$$2 \frac{dV(\hat{\phi})}{d\hat{\phi}^2} \equiv N \tilde{\chi}(\hat{\phi}^2) = 0.$$

(42)

Considering the explicit expression for $\tilde{\chi}(\hat{\phi}^2)$, given in Eq.(11), this yields the relation

$$\hat{\phi}^2 - v^2 + \frac{1}{6\beta^2} = 0,$$

(43)

which determines the value of $\hat{\phi}$ at the minimum. Eq. (13) has no solution for temperatures above the critical value

$$T_c^2 \equiv \frac{1}{\beta_c^2} = 6v^2,$$

(44)
and, hence, the minimum is at $\hat{\phi}_{\text{min}} = 0$ in such region of $T$. On the other hand, at temperatures below the critical one, the vacuum expectation value is given by

$$\hat{\phi}_{\text{min}}^2 = \frac{1}{6} \left( \frac{1}{\beta_c^2} - \frac{1}{\beta^2} \right). \quad (45)$$

Equation (45) shows that the vacuum expectation value varies continuously from zero to nonzero values, signaling the presence of a second order phase transition within the leading order in $1/N$ solution of the model. It is worth to remark that, for values of $\hat{\phi}^2$ at the left of the minimum, $\hat{\phi}^2 < (v^2 - 1/6\beta^2)$, $\sqrt{\chi}$ takes negative or even complex values, showing that, as was first discussed by Coleman, Jackiw and Politzer [21], the effective potential can not be defined in a sensible way in that region.

5 Critical Temperature and Phase Transition: Analysis up to Next to Leading Order in $1/N$.

In terms of the renormalized quantities defined in section 4, and within the framework of the high temperature expansion, we can rewrite the effective potential up to next to leading order in $1/N$, Eq.(24), as

$$V(\hat{\phi}, \bar{\chi}) = -\frac{3N}{2\lambda_T} \chi^2 + \frac{N}{2} \bar{\chi} (\hat{\phi}^2 - v^2) + \frac{N \bar{\chi}}{12 \beta^2} - \frac{N \bar{\chi}^{3/2}}{6\pi \beta} + \frac{1}{2} \int_k \ln \det \left[ -iD_{\mu\nu}^{-1}(k, \bar{\chi}, \hat{\phi}) - e_0^2 \Pi_{\mu\nu}(k, \bar{\chi}) \right] + \frac{1}{2} \int_k \ln \left[ \frac{(k^2 + \bar{\chi}) (1 + \lambda_0 B(\bar{\chi}, k)/3) + \lambda_0 \hat{\phi}^2/3}{k^2 + \bar{\chi}} \right]. \quad (46)$$

As we shall explicitly show in section 5.1, $B(\bar{\chi}, k^2)$ and $\Pi_{\mu\nu}(\bar{\chi}, k^2)$ are logarithmically divergent quantities. However, these logarithmic divergences can be naturally absorbed in the renormalization of the gauge and quartic couplings entering in the integrands of Eq.(46). The remaining divergences, arising from the zero temperature contributions to the integrals, may be absorbed in the renormalization of $\lambda$ and $v^2$ at next to leading order in $1/N$.

\footnote{Observe that there is a difference in a factor 2 with respect to the results of Refs. [2], [17], [18] due to the presence of $2N$, instead of $N$, real scalar bosons in the theory.}

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From Eq.(30) the minimization condition is given by

$$\bar{\chi} = -\frac{1}{N} \left\{ \left[ \frac{\partial}{\partial \phi^2} + \frac{\partial \bar{\chi}}{\partial \phi^2} \right] \frac{\partial}{\partial \bar{\chi}} \right\} \left( \int_k \ln \det \left[ -iD^{-1}_{\mu\nu}(k, \bar{\chi}, \hat{\phi}) - \epsilon_0^2 \Pi_{\mu\nu}(k, \bar{\chi}) \right] + \int_k \ln \left[ \frac{(k^2 + \bar{\chi}) (1 + \lambda_0 B(\bar{\chi}, k)/3 + \lambda_0 \hat{\phi}^2/3)}{k^2 + \bar{\chi}} \right] \right\} \equiv 0. \quad (47)$$

Thus, since the minimum is obtained for $\bar{\chi} \approx O(1/N)$, we can safely set $\bar{\chi} = 0$ in the right hand side of Eq.(47), under the assumption that no infrared divergences arise in this process.

5.1 Tachyon Poles and Triviality of the Gauged $O(2N)$ Model at Zero $T$

At zero temperature, it is possible to compute the radiative corrections $B(p, \bar{\chi})$ and $\Pi_{\mu\nu}(p, \bar{\chi})$ by making use of a gauge invariant regularization scheme, like Pauli - Villars. This gives the result

$$B_{T=0}(p^2, \bar{\chi}(\hat{\phi})) = \frac{1}{16\pi^2} \left\{ \ln \left( \frac{\Lambda^2}{\bar{\chi}} \right) - 2 \left[ \left( \frac{4\bar{\chi} + p^2}{4p^2} \right)^{1/2} \ln \left[ \frac{p^2}{\bar{\chi}} \left\{ \left( \frac{4\bar{\chi} + p^2}{4p^2} \right)^{1/2} + \frac{1}{2} \right\}^2 \right] - 1 \right] \right\} \quad (48)$$

while

$$\Pi_{T=0}^{\mu\nu}(p^2, \bar{\chi}(\hat{\phi}^2)) = \frac{(\delta^{\mu\nu}p^2 - p^{\mu}p^{\nu})}{48\pi^2} \left\{ - \ln \frac{\Lambda^2}{\bar{\chi}} - \frac{2}{3} \right\} \quad (49)$$

$$+ 2 \left( \frac{4\bar{\chi} + p^2}{p^2} \right) \left[ \left( \frac{4\bar{\chi} + p^2}{4p^2} \right)^{1/2} \ln \left[ \frac{p^2}{\bar{\chi}} \left\{ \left( \frac{4\bar{\chi} + p^2}{4p^2} \right)^{1/2} + \frac{1}{2} \right\}^2 \right] - 1 \right]$$

The divergence associated with $B(p^2, \bar{\chi})$ may be absorbed in the renormalization of $\lambda$ [22]. In fact, the expression

$$F(\bar{\chi}, p^2) \equiv \left( p^2 + \bar{\chi} \right) \left( 3/\lambda_0 + B_{T=0}(\bar{\chi}, p^2) \right) + \hat{\phi}^2, \quad (50)$$

appearing in the potential, Eq.(24), is proportional to the inverse propagator of the massive scalar particle $\sigma \equiv \phi_{1,1}$ and may be rewritten as

$$F(\bar{\chi}, p^2) = \left( p^2 + \bar{\chi} \right) \left( 3/\lambda + B_{T=0}(\bar{\chi}, p^2) \right) + \hat{\phi}^2, \quad (51)$$
where

\[ B_{T=0}(p, \bar{\chi}(\hat{\phi}^2)) = \frac{1}{16\pi^2} \left\{ \ln \left( \frac{M^2}{\bar{\chi}} \right) + 2 \left[ \left( \frac{4\bar{\chi} + p^2}{4p^2} \right)^{1/2} \ln \left[ \frac{p^2}{\bar{\chi}} \left\{ \left( \frac{4\bar{\chi} + p^2}{4p^2} \right)^{1/2} + \frac{1}{2} \right\}^2 \right] - 1 \right] \right\} \] (52)

and \( \lambda \) is the renormalized quartic coupling at leading order in \( 1/N \), which is given in Eq.(38). Observe that, as first noticed in Ref. [21], the renormalized theory seems to be spoiled by the presence of tachyons. In fact, since we are working in Euclidean space, a tachyon pole in the \( \phi_1, 1 \) two point function will appear if, for some positive value of \( p^2 \), \( F(\bar{\chi}, p^2) = 0 \). Considering the limit \( \bar{\chi} = \hat{\phi} = 0 \), this implies an equation for \( p^2 \), which reads

\[ \frac{3}{\lambda} + \frac{1}{16\pi^2} \ln \left( \frac{M^2}{p^2} \right) + \frac{1}{8\pi^2} = 0. \] (53)

For a finite value of the renormalized coupling \( \lambda \), the above equation has a solution at sufficiently large values of \( p^2 \), which could be interpreted as the presence of a tachyon in the spectrum. However, rewriting Eq.(53) in terms of the unrenormalized quartic coupling, we obtain,

\[ \frac{3}{\lambda_0} + \frac{1}{16\pi^2} \ln \left( \frac{\Lambda^2}{p^2} \right) + \frac{1}{8\pi^2} = 0. \] (54)

Therefore, for any \( \lambda_0 > 0 \), the potentially dangerous pole appears at momentum above the cutoff scale \( \Lambda > 0 \),

\[ p^2 \simeq \Lambda^2 \exp \left( \frac{48\pi^2}{\lambda_0} \right). \] (55)

Hence, it has no physical consequences. (Observe that at nonvanishing values of \( \hat{\phi} \) and \( \bar{\chi} \) the pole would be at even larger values of \( p^2 \).) The apparent discrepancy between the results obtained in terms of the renormalized and unrenormalized couplings is related to the issue of triviality. In fact, while naively analyzing the existence of a tachyon pole using Eq.(53), one overlooks the fact that the renormalized quartic coupling \( \lambda \) goes to zero in the continuum limit \( \Lambda \to \infty \), and for this reason, a finite effective cutoff is needed in order to have a finite renormalized coupling.

An analogous situation occurs in the gauge sector of the theory. In fact, calling

\[ iD^{\mu\nu} = D_T \left( \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) + D_L \frac{p^\mu p^\nu}{p^2} \] (56)
and
\[ \Pi_{\mu\nu}^{T=0}(\bar{\chi}, p^2) = \left( \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \Pi_{T=0}(\bar{\chi}, p^2), \]  
(57)
a tachyon pole in the current - current correlation function would appear if
\[ G \left( p^2, \bar{\chi} \right) \equiv \frac{1}{e_0^2} D_T^{-1} \left( \bar{\chi}, \phi, p^2 \right) - \Pi_{T=0} \left( \bar{\chi}, p^2 \right) \]  
(58)
had a zero in Euclidean space. At \( \bar{\chi} = \hat{\phi} = 0 \), the condition \( G(p^2, \bar{\chi}) = 0 \) leads to the equation
\[ \frac{1}{e_0^2} + \frac{1}{48\pi^2} \left[ \ln \left( \frac{\Lambda^2}{p^2} \right) + \frac{8}{3} \right] = 0. \]  
(59)
Apart from a change from the gauge to the quartic coupling, the above equation is equivalent to that one found in the previous case, Eq. (54). The would be tachyon pole appears at momentum above the physical cutoff of the theory
\[ p^2 \simeq \Lambda^2 \exp(48\pi^2/e_0^2) \]  
(60)
and, hence, has no physical implications.

The renormalization of the gauge coupling proceeds in similar way to that one of the quartic coupling. The logarithmic divergences in \( \Pi_{\mu\nu}^{T=0} \) are absorbed in the definition of the renormalized gauge coupling \( e \), which is given by
\[ \frac{1}{e^2} = \frac{1}{e_0^2} + \frac{1}{48\pi^2} \ln \left( \frac{\Lambda^2}{M^2} \right). \]  
(61)
We can therefore define the renormalized vacuum polarization, \( \bar{\Pi}_{T=0} \), by replacing \( \ln(\Lambda^2/\bar{\chi}) \) by \( \ln(M^2/\bar{\chi}) \) in \( \Pi_{T=0} \). Moreover, the expression of the renormalized gauge coupling, Eq. (61), shows the triviality of the gauged theory in the continuum limit.

5.2 Radiative Corrections at Finite Temperature

We shall now analyze the structure of \( B(p, \chi) \) and \( \Pi_{\mu\nu}(p, \bar{\chi}) \) at finite temperature, which, by making use of Eq. (33), may be decomposed into zero temperature contributions, \( B_{T=0} \) and \( \Pi_{\mu\nu}^{T=0} \), already analyzed in section 5.1, and the corresponding finite temperature parts to be computed in the following. The radiative correction to the renormalized \( \chi \)
propagator, $\bar{B}_T(p, \bar{\chi})$, has the expression
\[
\bar{B}_T(p, \bar{\chi}) - \bar{B}_{T=0}(p, \bar{\chi}) = \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{[k^2 + \bar{\chi}][(k+p)^2 + \bar{\chi}]} \right\}
\]
\[
+ \frac{1}{[\bar{k}^2 + \bar{\chi}][(\bar{k}+\bar{p})^2 + \bar{\chi}]} \frac{1}{\exp(-i\beta k_0) - 1}
\]
(62)
where $\bar{k} = (-k_0, \bar{k})$. The $k_0$ integral may be performed by making a contour integration in the complex $k_0$ plane, leading to
\[
\bar{B}_T(p, \bar{\chi}) = \bar{B}_{T=0}(p, \bar{\chi}) + 2 \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{\sqrt{k^2 + \bar{\chi}}} \left\{ \exp \left( \beta \sqrt{k^2 + \bar{\chi}} \right) - 1 \right\} \right.
\]
\[
\times \left. \frac{((\bar{k}+\bar{p})^2 - \bar{k}^2 + p_0^2)}{\left\{ [((\bar{k}+\bar{p})^2 - \bar{k}^2 + p_0^2)^2 + 4p_0^2(\bar{k}^2 + \bar{\chi})} \right\} \right] .
\]
(63)

As we have shown above, in order to study the minimization condition up to next to leading order in $1/N$, it is sufficient to study the behaviour of $\bar{B}_T$ for $\bar{\chi} \simeq 0$. From the expression of $\bar{B}_T(p, \bar{\chi})$, doing an expansion in the neighborhood of $\bar{\chi} = 0$ at finite external momentum, we obtain
\[
\bar{B}_T(p_0, \bar{p}, \bar{\chi}) = \bar{B}_{T=0}(p_0, \bar{p}, \bar{\chi} = 0) + K \frac{\sqrt{\bar{\chi}}}{\beta p^2} + \mathcal{O}(\bar{\chi}),
\]
(64)
where $K$ is a coefficient of order one, which can be evaluated by analyzing the infrared divergences associated with $\partial \bar{B}_T / \partial \bar{\chi}$ as $\bar{\chi} \to 0$. From Eq.(63) it is possible to prove that
\[
\frac{\partial \bar{B}_T(p, \bar{\chi})}{\partial \sqrt{\bar{\chi}}} \to -\frac{1}{2\pi \beta (p_0^2 + \bar{p}^2)}
\]
(65)
as $\bar{\chi} \to 0$ and, hence, $K = -1/2\pi$. Furthermore, the last term in the Eq.(64) involves higher order contributions in $\bar{\chi}$ which are not relevant to solve the minimization condition. This is due to the fact that, from the gap equation up to leading order in $1/N$, Eq.(41),
\[
\frac{\partial \bar{\chi}(\phi^2)}{\partial \phi^2} = \frac{4\pi \beta \sqrt{\bar{\chi}}}{1 + 4\pi \beta \sqrt{\bar{\chi}}/\lambda_T} \bigg|_{\bar{\chi} \to 0} \to 4\pi \beta \sqrt{\bar{\chi}}.
\]
(66)
Therefore, the derivative operator involved in the minimization condition, Eq.(47), only receives contributions from the infrared dominant part of $\partial V_{n.l.o} / \partial \bar{\chi}$. 19
An analogous procedure may be used to analyze the behaviour of the vacuum polarization contribution $\Pi_{\mu\nu}^T$.

$$
\Pi_{\mu\nu}^T - \Pi_{\mu\nu}^{T=0} = \int_{-\infty+ie}^{\infty+ie} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{(2k+p)^\mu(2k+p)^\nu}{[k^2 + \bar{\chi}][(k+p)^2 + \bar{\chi}]} + \frac{4\delta^{\mu\nu}}{k^2 + \bar{\chi}} \right\} \frac{1}{\exp(-i\beta k_0) - 1} \tag{67}
$$

At finite temperature it is easy to demonstrate the transversality of $\Pi_{\mu\nu}^T$ by making use of the $p_0$ quantization, $p_0 \equiv \omega_n = 2\pi n/\beta_0$. However, general covariance is explicitly lost and, hence, the vacuum polarization takes the general form

$$
\Pi_{\mu\nu}^T(p, \bar{\chi}) = \Pi_1^T(p, \bar{\chi}) \left( \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) + \Pi_2^T(p, \bar{\chi}) \left( \delta^{ij} - \frac{p^i p^j}{p^2} \right), \tag{68}
$$

Recalling the general form of the gauge boson propagator, Eq.(56), it is straightforward to show that

$$
\det \left( -iD_{\mu\nu}^{-1} - e_0^2\Pi_{\mu\nu}^T \right) = \left[ \frac{1}{D_T(p^2, \hat{\phi})} - e_0^2 \left( \Pi_1^T(p, \bar{\chi}) + \Pi_2^T(p, \bar{\chi}) \right) \right]^2 \left[ \frac{1}{D_T(p^2, \hat{\phi})} - e_0^2 \Pi_1^T(p, \bar{\chi}) \right] \frac{1}{D_L(p^2, \bar{\chi}, \hat{\phi})}, \tag{69}
$$

where, from Eq.(13), we have,

$$
\frac{1}{D_T(k^2, \phi)} = k^2 + e_0^2 \phi^2
$$

$$
\frac{1}{D_L(k^2, \bar{\chi}, \hat{\phi})} = \frac{\left[ k^2 (k^2 + \bar{\chi}) + \alpha e_0^2 \phi^2 \bar{\chi} \right]}{\alpha (k^2 + \bar{\chi})}. \tag{70}
$$

Therefore,

$$
\det \left( -iD_{\mu\nu}^{-1} - e_0^2\Pi_{\mu\nu}^T \right) = \left[ k^2 + e_0^2 \phi^2 - e_0^2 \left( \Pi_1^T(k, \bar{\chi}) + \Pi_2^T(k, \bar{\chi}) \right) \right]^2 \left[ k^2 + e_0^2 \phi^2 - e_0^2 \Pi_1^T(k, \bar{\chi}) \right] \frac{1}{D_L}. \tag{71}
$$

This expression is equivalent to that one found by Carrington [10] in the limit of small external momentum ($p_0 = 0, \vec{p} \approx 0$), while computing the ring diagrams contribution to the effective potential in the Landau gauge. Observe that, at zero temperature, $\Pi_1^{T=0} = \Pi_{T=0}$, while $\Pi_2^{T=0} = 0$ and all logarithmic divergences may be absorbed in the
The definition of the renormalized gauge coupling, which implies that \( \Pi^T_{T=0} = 0 \), turns into its renormalized expression \( \bar{\Pi}^T_{T=0} = 0 \). Consequently, defining \( \bar{\Pi}^T_{T=0} \) as the finite temperature renormalized quantity, the logarithm of the determinant takes the form

\[
\ln \det \left[ -iD^{-1}_{\mu\nu}(k, \hat{\phi}, \bar{\chi}) \right] = 2 \ln \left[ k^2 + \epsilon^2 \hat{\phi}^2 - \epsilon^2 \left( \bar{\Pi}^T_{T=0} + \Pi^T_{00} \right) \right] + \ln \left[ \frac{k^2 (k^2 + \bar{\chi}) + \alpha \epsilon^2 \hat{\phi}^2 \bar{\chi}}{k^2 + \bar{\chi}} \right].
\]

The first two terms in the right hand side of the equation above are gauge independent and, for \( \Pi^T = 0 \), they give the well known one loop contribution to the effective potential.

The last term contains all the gauge dependence and its contribution to the effective potential only vanishes in the Landau gauge, \( \alpha = 0 \). We shall return to the issue of the gauge dependence in section 5.3.

Since \( \Pi^T_{\mu\nu} \), Eq. (68), has only two independent components, we can obtain the functions \( \bar{\Pi}^T_{T=0} \) by computing the expressions of \( \Pi^T_{00} \) and \( \Pi^T_{\mu\mu} \). In fact,

\[
\Pi^T_{00} - \Pi^T_{00, T=0} = \left( \bar{\Pi}^T_{1} - \bar{\Pi}^T_{1, T=0} \right) \frac{\vec{p}^2}{p^2},
\]

\[
\Pi^T_{\mu\mu} - \Pi^T_{\mu\mu, T=0} = 3 \left( \bar{\Pi}^T_{1} - \bar{\Pi}^T_{1, T=0} \right) + 2\Pi^T_{2}.
\]

Moreover, using Eqs. (49) and (67), we have

\[
\begin{align*}
\Pi^T_{00}(k, \bar{\chi}) - \Pi^T_{00, T=0}(k, \bar{\chi}) & = I_1(k, \bar{\chi}) - 2I_2(k, \bar{\chi}) \quad (74) \\
\Pi^T_{\mu\mu}(k, \bar{\chi}) - \Pi^T_{\mu\mu, T=0}(k, \bar{\chi}) & = -4I_2(k, \bar{\chi}) - \left( k^2 + 4\bar{\chi} \right) \left( \bar{B}_{T}(k, \bar{\chi}, \hat{\phi}) - \bar{B}_{T=0}(k, \bar{\chi}, \hat{\phi}) \right),
\end{align*}
\]

with

\[
I_1(p, \bar{\chi}) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + \bar{\chi}} \left\{ \frac{(2k_0 + p_0)^2}{(k^2 + \bar{\chi}) (k^2 + p^2 + \bar{\chi})} \right\} \left( \exp(-i\beta k_0) - 1 \right)
\]

\[
+ \frac{(2k_0 - p_0)^2}{(k^2 + \bar{\chi}) (k^2 + p^2 + \bar{\chi})} \right\} \left( \exp(-i\beta k_0) - 1 \right)
\]

and

\[
I_2(p, \bar{\chi}) = 2 \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + \bar{\chi}} \left\{ \frac{1}{\exp(-i\beta k_0)} - 1 \right\}.
\]

(76)
The integrals $I_i$ may be computed by performing a contour integration in the complex $k_0$ plane. We obtain,

$$I_1 = 2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + \bar{\chi}}} \left[ \exp \left( \beta \sqrt{k^2 + \bar{\chi}} \right) - 1 \right]$$

$$\times \left[ p_0^2 - 4 (\bar{k}^2 + \bar{\chi}) \right] \left[ p_0^2 + (\bar{k} + \bar{p})^2 - \bar{k}^2 \right] + 8p_0^2 (\bar{k}^2 + \bar{\chi})$$

$$\left[ p_0^2 + (\bar{k} + \bar{p})^2 - \bar{k}^2 \right]^2 + 4p_0^2 (\bar{k}^2 + \bar{\chi})$$

and

$$I_2 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + \bar{\chi}}} \left( \exp \left( \beta \sqrt{k^2 + \bar{\chi}} \right) - 1 \right).$$

(77)

It is first interesting to analyze the behaviour of $I_i(p_0, \bar{p})$ at $\bar{\chi} = 0$, $p_0 = 0$ and $|\bar{p}| \ll T$. The integral $I_2$ has already been considered in section 4, and its high temperature expansion result, Eq.(35), in the limit we are studying, gives $I_2(\bar{\chi} = 0) = 1/12\beta^2$. Furthermore, from Eq.(77), it is straightforward to show that

$$I_1(\bar{\chi} = 0, p_0 = 0, \bar{p} \simeq 0) = -\frac{1}{\pi^2} \int_0^{\infty} d|\vec{k}| \frac{|\vec{k}|}{(\exp(\beta|\vec{k}|) - 1)} = -\frac{1}{6\beta^2}. \quad (79)$$

The above results imply that, in this momentum regime,

$$\bar{\Pi}_1^T - \bar{\Pi}_1^{T=0} = -\bar{\Pi}_2^T = -\frac{1}{3\beta^2} \quad (80)$$

Observe that this contribution gives a static mass term to the abelian gauge field. In fact, recalling Eq.(71) we obtain that, since $\bar{\Pi}_1^T$ and $\bar{\Pi}_2^T$ come with opposite signs, only the longitudinal mode receives a contribution to the mass term equal to

$$\mathcal{M}_D^2 = \frac{e^2}{3\beta^2}, \quad (81)$$

which is the well known Debye screening mass term [30]. It has been recently pointed out in the literature [10]-[12], that this term is important in order to determine the strength of the phase transition within the framework of the improved one loop effective potential. Within such approximation, the phase transition appears to be first order but, due to the Debye screening suppression of the longitudinal mode, its strength is about two thirds of the one obtained using the naive one loop approach. We shall now
demonstrate that, when using the self consistent 1/N expansion up to next to leading order in 1/N, and computing the value of $\Pi^\mu_\nu$ at finite external momentum, extra terms appear in the effective potential, which change not only the strength but also the order of the phase transition.

5.3 Minimization of the Effective Potential

The full effective potential up to next to leading order in 1/N, Eq.(24) may be rewritten as

$$V(\bar{\chi}, \hat{\phi}) = V_{\text{tree}} + V_{\text{l.o.}} + \frac{1}{2} \int_k \left[ \ln \left( \frac{k^2 + \bar{\chi} + \lambda \bar{B}_T(\bar{\chi}, k)/3 + \lambda \hat{\phi}^2/3}{k^2 + \bar{\chi}} \right) \right]$$

$$+ \frac{1}{2} \int_k \left\{ 2 \ln \left[ k^2 + e^2 \hat{\phi}^2 - e^2 \left( \bar{\Pi}^T_1 + \bar{\Pi}^T_2 \right) \right] 
+ \ln \left[ k^2 + e^2 \hat{\phi}^2 - e^2 \bar{\Pi}^T_1 \right] \right\} + V_{\text{g.d.}}, \quad (82)$$

where $e$ and $\lambda$ are the renormalized gauge and quartic couplings, respectively, and $\bar{B}_T$ and $\bar{\Pi}^T_i$ are the full, temperature dependent radiative corrections contributions with their logarithmic divergences removed through the definition of the renormalized couplings, in the way explained above. Moreover, we can rewrite the gauge dependent term, $V_{\text{g.d.}}$, coming from Eq.(72) like

$$V_{\text{g.d.}} = \frac{1}{2} \int_k \left[ \ln \left( k^2 + R^2_1 \right) + \ln \left( k^2 + R^2_2 \right) - \ln \left( k^2 + \bar{\chi} \right) \right] \quad (83)$$

where

$$R^2_{1,2} = \bar{\chi} \pm \sqrt{\bar{\chi}^2 - 4 \alpha e^2 \hat{\phi}^2 \bar{\chi}}. \quad (84)$$

Similarly to the computation of $\bar{B}_T$, from Eq.(17) it follows that, in order to evaluate the minimum condition up to next to leading order in 1/N, we only need to study the behaviour of $\bar{\Pi}^T_i$ for $\bar{\chi} \approx 0$. As we have just remarked, the computation of the vacuum polarization expression for finite external momentum is crucial in obtaining the correct behaviour of the effective potential at small values of $\bar{\chi}$. Recalling Eqs.(49),(67), (74),(77) and (78), it is straightforward to find that, for $\bar{\chi} \to 0$, $p^\mu \neq 0$,

$$\Pi^\mu_\nu(p_0, \vec{p}, \bar{\chi}) = \Pi^\mu_\nu(p_0, \vec{p}, \bar{\chi} = 0) + \frac{3\sqrt{\bar{\chi}}}{2\pi \beta} + h.o.(\bar{\chi}), \quad (85)$$

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and
\[ \Pi_{T}^{00}(p_0, \vec{p}, \bar{\chi}) = \Pi_{T}^{00}(p_0, \vec{p}, \bar{\chi} = 0) + \frac{\sqrt{\bar{\chi}}}{2\pi\beta} \frac{\vec{p}^2}{(p_0^2 + \vec{p}^2)} + h.o.(\bar{\chi}). \tag{86} \]
Note that, as implied by transversality, the expression of \( \Pi_{T}^{00} \) vanishes in the limit \( \vec{p} \to 0 \) for \( p_0 \neq 0 \). From the above expressions it follows that, at finite momentum transfer and in the small \( \bar{\chi} \) limit, the vacuum polarization components read,
\[ \bar{\Pi}_{T}^{1}(p_0, \vec{p}, \bar{\chi}) = \bar{\Pi}_{T}^{1}(p_0, \vec{p}, \bar{\chi} = 0) + \frac{\sqrt{\bar{\chi}}}{2\pi\beta} \bar{\chi} + h.o. \bar{\chi}. \tag{87} \]
\[ \Pi_{T}^{2}(p_0, \vec{p}, \bar{\chi}) = \Pi_{T}^{2}(p_0, \vec{p}, \bar{\chi} = 0) + h.o. \bar{\chi}. \tag{87} \]
It is important to emphasize that, this is the correct behaviour for \( \bar{\chi} \to 0 \) at any finite \( |p| = \sqrt{p_0^2 + \vec{p}^2} \), which is the relevant regime for the computation of the integrals contributing to the minimum of the effective potential. Using the above expressions, Eqs.\((64),(87)\), we can rewrite the minimization condition as
\[ \bar{\chi} = -\frac{1}{N} \left[ \frac{\partial}{\partial \phi^2} + \frac{\partial \bar{\chi}}{\partial \phi^2} \frac{\partial}{\partial \bar{\chi}} \right] \{ V_{g.d.} \}
\]
\[ + \int \ln \left[ k^2 + e^2 \phi^2 - e^2 \left( \bar{\Pi}_{T}^{1}(k, \bar{\chi} = 0) + \frac{\sqrt{\bar{\chi}}}{2\pi\beta} \right) \right] \]
\[ + 2 \int \ln \left[ k^2 + e^2 \phi^2 - e^2 \left( \bar{\Pi}_{T}^{1}(k, \bar{\chi} = 0) + \bar{\Pi}_{T}^{2}(k, \bar{\chi} = 0) + \frac{\sqrt{\bar{\chi}}}{2\pi\beta} \right) \right] \]
\[ + \int \ln \left[ \frac{(k^2 + \bar{\chi}) \left( 3/\lambda + \bar{B}_{T}(k, \bar{\chi} = 0) - \sqrt{\bar{\chi}/(2\pi\beta k^2)} \right)}{k^2 + \bar{\chi}} \right] \bar{\chi} = 0 \tag{88} \]
The first term in Eq.\((88)\) involves all the possible gauge dependent contributions to the minimum of the effective potential. However, explicitly applying the total derivative to the logarithmic expression of \( V_{g.d.} \),
\[ \left( \frac{\partial}{\partial \phi^2} + \frac{\partial \bar{\chi}}{\partial \phi^2} \frac{\partial}{\partial \bar{\chi}} \right) V_{g.d.} \bigg|_{\bar{\chi} = 0} = \frac{1}{2} \int k^2 \frac{1}{\phi^2} \left[ \frac{\partial}{\partial \phi^2} \right] \left( R_1^2 + R_2^2 \right) \bar{\chi} = 0 \tag{89} \]
and using the fact that \( (R_1^2 + R_2^2 - \bar{\chi}) = 0 \), we observe that \( V_{g.d.} \) does not give any contribution to the minimization condition up to this order in \( 1/N \). (Observe that, since we are working with regularized integrals with a finite cutoff, we can safely interchange derivatives by integrals in the expression above.) Therefore, although the expression
of the effective potential is gauge dependent, the minimization condition gives a gauge independent result.

Furthermore, from Eq. (88), it comes to notice that, as we discussed above, for the minimization of the effective potential, the only relevant contributions from the finite temperature radiative corrections come from the infrared dominant terms of $\bar{B}_T(p, \bar{\chi})$ and $\bar{\Pi}_T^i(p, \bar{\chi})$. In addition, recalling the fact that $\partial \bar{\chi}/\partial \hat{\varphi}^2 \to \sqrt{\chi}4\pi\beta$ as $\bar{\chi} \to 0$, it readily follows that the minimization condition reads,

$$\bar{\chi} = -\frac{1}{N} \left\{ 2e^2 \int_k \left[ \frac{1 - 2\sqrt{\chi}\partial(\sqrt{\chi})/\partial \bar{\chi}}{k^2 + e^2\hat{\varphi}^2 - e^2 \left( \bar{\Pi}_T(k, \bar{\chi} = 0) + \bar{\Pi}_T^2(k, \bar{\chi} = 0) \right)} \right] \\
+ e^2 \int_k \left[ \frac{1 - 2\sqrt{\chi}\partial(\sqrt{\chi})/\partial \bar{\chi}}{k^2 + e^2\hat{\varphi}^2 - e^2 \left( \bar{\Pi}_T(k, \bar{\chi} = 0) \right)} \right] \\
+ \int_k \left[ \frac{1 - 2\sqrt{\chi}\partial(\sqrt{\chi})/\partial \bar{\chi}}{k^2 \left( 3/\lambda + B_T(k, \bar{\chi} = 0) \right)} \right] \right\}_{\chi=0}$$

(90)

The above equation implies that, at next to leading order in $1/N$ the only minimum is at

$$\bar{\chi}_{\text{min}}(\hat{\varphi}) = 0.$$  

(91)

Thus, the value of $\bar{\chi}(\hat{\varphi})$ at the minimum remains unchanged after the inclusion of the next to leading order contributions. As we discuss in section 4, Eq. (91) leads to a nontrivial solution for the vacuum expectation value of the scalar field, $\hat{\varphi}_{\text{min}}^2 = v^2 - 1/(6\beta^2)$, for temperatures below the critical temperature, $T_c^2 = 6v^2$. Hence, the vacuum expectation value of the scalar field varies continuously at the critical temperature, characterizing a second order phase transition.

6 Comparison with the One Loop Approximation

The results of section 5 show that, in the presence of a large number of scalars and in the region of parameters $e^2/\lambda \ll N$, in which the scalar loop contributions are enhanced in comparison to the gauge loop ones, the phase transition is second order. In spite of that, the gauge contributions to the effective potential at next to leading order in $1/N$ resemble those ones contributing to the effective potential in the improved one
loop approximation. In fact, for small $\hat{\phi}$, and at $T = T_0$, at which the quadratic term in $\hat{\phi}$ vanishes, one would expect that due to the presence of the gauge contributions, a dominant negative cubic term would appear, leading to a nontrivial minimum in the effective potential and, hence, to a first order phase transition. However, the fundamental difference comes from considering the finite external momentum contributions to the radiative corrections at finite temperature.

Quite generally, as we have already said, the integral

$$I_T = \frac{1}{2} \int_k \ln(k^2 + m^2)$$

may be evaluated in the high temperature regime to give,

$$I_T = I_{T=0} + \frac{m^2}{24\beta^2} - \frac{m^{3/2}}{12\pi\beta} + h.o.(m^2\beta^2).$$

Therefore, since the zero temperature gauge boson mass is equal to $m_g^2 = e^2\hat{\phi}^2$ one would naively expect that the inclusion of the gauge field would induce the generation of a quadratic term in $\hat{\phi}$, which would modify the definition of the transition temperature, as well as the appearance of a cubic term, which would lead to a first order phase transition. In the next to leading order in $1/N$ analysis, these two effects are not present, and, hence, some sort of finite temperature screening, which modifies the nature of the phase transition, should occur. One type of screening, which is already contained within the improved one loop approximation, is that one induced by the generation of an effective temperature dependent mass for the longitudinal gauge boson mode. As we already discussed above, when compared with the naive one loop result, the Debye screening causes the reduction in two thirds of the value of the coefficient of the cubic term. In our analysis, however, the preservation of the value of the critical temperature and of the nature of the phase transition after considering the next to leading order in $1/N$ effects is a result that would be obtained even in the case of a cancellation of the $\bar{\chi}(\hat{\phi})$-independent radiative corrections contributions to the photon and scalar propagators. In fact, within the $1/N$ expansion, there is a different source of screening, which can only be obtained by considering the $\hat{\phi}$ dependent, finite momentum contributions to the radiative corrections.
For values of \( \hat{\phi} \) close to the minimum of the effective potential (\( \bar{\chi} \approx 0 \)), and at momentum \( p^2 \gg \bar{\chi} \), the evaluation of the finite temperature radiative corrections to the photon and scalar propagators have as outcome the replacement of the functional dependence on \( \hat{\phi}^2 \) by a dependence on the combination \( [\hat{\phi}^2 - \sqrt{\bar{\chi}}/(2\pi\beta)] \). In fact, as it is clearly seen from Eqs.\((82)-(87)\), in this momentum regime, the finite temperature effective mass of the transverse gauge boson mode is given by

\[
m_T^2 = e^2 \left( \phi^2 - \sqrt{\chi}/(2\pi\beta) \right) + h.o.(\bar{\chi}) \tag{94}\]

To easily understand the relevance of this effect, let us first analyze the behaviour of \( \bar{\chi} \) at \( T = T_c \),

\[
\sqrt{\bar{\chi}} = -\frac{\lambda_{T_c}}{24\pi^2\beta_c} + \sqrt{\left( \frac{\lambda_{T_c}}{24\pi^2\beta_c} \right)^2 + \frac{\lambda_{T_c}\hat{\phi}^2}{6}}. \tag{95}\]

Expanding Eq.\((95)\) in the neighborhood of the origin we obtain

\[
\sqrt{\bar{\chi}} = 2\pi\beta_T\hat{\phi}^2 \left[ 1 - \frac{24\pi^2}{\lambda_{T_c}} \left( \frac{\phi}{T_c} \right)^2 \right]. \tag{96}\]

From Eq.\((96)\) it follows that at \( T = T_c \) and for the momentum regime under consideration, the effective mass of the photon transverse mode in the neighborhood of the origin is given by

\[
m_T^2 = \hat{\phi}^2 \frac{24\pi^2 e^2}{\lambda_{T_c}} \left( \frac{\phi}{T_c} \right)^2 + h.o.(\bar{\chi}) \tag{97}\]

Thus, since for the same range of parameters,

\[
\bar{\chi} \simeq 4\pi^2\hat{\phi}^2 \left( \frac{\hat{\phi}}{T_c} \right)^2, \tag{98}\]

the effective squared mass of the photon transverse mode behaves like \( \hat{\phi}^4/T^2 \) rather than the expected \( \hat{\phi}^2 \) dependence. Within the region of parameters considered above, the radiatively corrected squared mass of the \( \sigma \) particle also behaves as \( \hat{\phi}^4/T^2 \).

The dependence of \( m_T^2 \) in the neighborhood of the origin and at \( T = T_c \) has two effects. First, it explains the cancellation of the quadratic term in \( \hat{\phi} \) at \( T = T_c \) and hence the preservation of the value of the critical temperature at next to leading order in \( 1/N \). Second, it implies that, in the neighborhood of the origin, and independently
of the existence of \( \hat{\phi} \) - independent radiative corrections contributions, no cubic term is induced and thus, at \( T = T_c \) the finite temperature effective potential is well described by a positive, quartic term.

It is important to remark that the expressions we obtain for the radiatively corrected photon and scalar propagators are only valid for \( T, k^2 \gg \bar{\chi} \). Moreover, the cubic term arising in the integral, Eq.(92), receives the most relevant contributions from the momentum regime \( k_0 = 0, \vec{k}^2 = \mathcal{O}(m^2) \). Thus, the present analysis would fail if in the neighborhood of the origin the relation \( \hat{\phi}^2 \gg \bar{\chi} \) were not fulfilled. However, as readily seen from Eq.(98), \( \bar{\chi} \ll \phi^2 \) and the effective potential close to the origin is well described by our approach.

### 7 Conclusions

In this article we have analyzed the finite temperature phase transition of scalar electrodynamics with \( N \) flavors of complex scalar fields, by means of a large \( N \) expansion, up to next to leading order in \( 1/N \). We have shown that the effective potential takes a compact and simple form, which, although gauge dependent, leads to gauge invariant results for the critical temperature as well as for the extrema of the potential, which are well defined physical quantities. At leading order in \( 1/N \), the effective potential coincides with the one of the \( \text{O}(2N) \) vector model, already studied in the literature. At the critical temperature \( T_c = 6v^2 \), the system develops a symmetry restoration phase transition, with an order parameter which varies continuously from zero to nonzero values. The phase transition is hence second order.

At next to leading order in the \( 1/N \) expansion, the effective potential receives contributions from the radiatively corrected \( \sigma \) and gauge field propagators. The gauge field contribution is essentially that one found at one loop, but with the photon corrected by vacuum polarization effects. The \( \hat{\phi} \) dependence of the vacuum polarization effects is crucial in order to define a correct \( 1/N \) expansion. We have shown that, when the radiative corrections are considered at finite external momentum, their effects lead to an effective screening of the gauge boson mass, which change the nature of the gauge boson loop
contributions to the effective potential. Due to this screening effect, no extra $\phi^2$ term is induced in the neighborhood of the origin at the leading order critical temperature and, therefore, the critical temperature remains the same as in the leading order analysis. More generally, the position of the minima of the effective potential is not modified by next to leading order effects and, in addition, no new minimum of the effective potential appears at next to leading order. Hence, the order parameter has the same behaviour as in leading order in $1/N$ and the phase transition remains second order.

The results of this work show that in the presence of a large number of flavors of complex scalar fields, the symmetry restoration phase transition of scalar QED becomes second order. Within the context of this study, however, we can not rule out the presence of a critical value of $N$, below which the transition is first order. In general, we expect the range of validity of the large $N$ approximation to depend on the relation between the gauge and quartic couplings. More specifically, the expansion considered in the present work assumes the dominance of the scalar loops contributions to the effective potential. Therefore, it is rigorously valid for values of $e^2/\lambda \ll N$. Thus, for low values of $N$, the expansion becomes more reliable for a scalar Higgs mass, $m^2 = \lambda\hat{\phi}^2/3$ larger than the gauge boson mass, $m_g^2 = e^2\hat{\phi}^2$, $m^2/m_g^2 \geq 1$, for which the fluctuations of the scalar fields are enhanced.

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FIGURE CAPTIONS

Fig.1. Multiloop diagrams which contribute to the effective potential at next to leading order in $1/N$. Solid and dashed lines denote the propagators of the scalar fields, $\phi_{a,i}$ and $\chi$, respectively, while the curved lines denote the gauge field propagator.

Fig.2. Same as in Fig. 1, but considering those diagrams which involve a mixing of $\chi$ and $A_\mu$. 
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