On the local existence of solutions to the Navier-Stokes-wave system with a free interface

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Abstract. We address a system of equations modeling a compressible fluid interacting with an elastic body in dimension three. We prove the local existence and uniqueness of a strong solution when the initial velocity belongs to the space $H^{2+\varepsilon}$ and the initial structure velocity is in $H^{1.5+\varepsilon}$, where $\varepsilon \in (0, 1/2)$.

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1. Introduction

The purpose of this paper is to establish the local-in-time existence of solutions for the free boundary fluid-structure interaction model with low regularity assumptions on the initial data. The model describes the interaction problem between an elastic structure and a viscous compressible fluid in which the structure is immersed. Mathematically, the dynamics of the fluid are captured by the compressible Navier-Stokes equations in the velocity and density variables $(u, \rho)$, while the elastic dynamics are described by a second-order elasticity equation (we replace it with a wave equation to simplify the presentation) in the vector variables $(w, w_t)$ representing the displacement and velocity of the structure. The interaction between the structure and the fluid is mathematically described by velocity and stress matching boundary conditions at the moving interface separating the solid and fluid regions. Since the interface position evolves with time and is not known a priori, this is a free-boundary problem. Important features of the problem are a mismatch between parabolic and hyperbolic regularity and the complexity of the stress-matching condition on the free boundary.

The well-posedness and local-in-time existence were first obtained in [CS1, CS2] in 2005, where the authors used the Lagrangian coordinate system to fix the domain and the Tychonoff fixed point theorem to construct a local-in-time solution given an initial fluid velocity $u_0 \in H^3$ and structural velocity $w_1 \in H^3$. In [KT1, KT2], the authors established a priori estimates for the local existence of solutions using direct estimates given $u_0 \in H^3$ for the initial velocity and $w_1 \in H^{5/2+r}$, where $r \in (0, (\sqrt{2} - 1)/2)$, for the initial structural velocity. A key ingredient in obtaining the result was the hidden regularity trace theorem for the wave equations established...
in [LLT, BL, L1, L2, S, T]. A wealth of literature on wave-heat coupled systems on a non-moving domain was instrumental in further understanding the heat-wave interaction phenomena (cf. [ALT, AT1, AT2, DGHL, BGLT1, BGLT2, KTZ1, KTZ2, KTZ3, LL1]). Recently, a sharp regularity result for the case when the initial domain is a flat channel was obtained by Raymond and Vanninathan [RV]. The authors study the system in the Lagrangian coordinate setting and obtain local-in-time solutions for the 3D model when the initial velocity \( u_0 \in H^{1+\alpha} \) and the initial structural velocity \( w_1 \in H^{1/2+\alpha+\beta} \), where \( \alpha \in (1/2, 1) \) and \( \beta > 0 \). More recently, Boulakia, Guerrero, and Takahashi obtained in [BGT] a unique local-in-time solution given initial data \( u_0 \in H^2 \) and \( w_1 \in H^{9/8} \) for the case of a general domain.

Concerning compressible fluids, the global existence of weak solutions for the interaction with a rigid structure was obtained in [DEGL] for the density law \( p = \rho^\gamma \) with \( \gamma \geq 2 \) and in [F] with \( \gamma > d/2 \). In [BG1], the authors obtained the existence and uniqueness for the initial density \( \rho_0 \) belonging to \( H^3 \), the velocity \( u_0 \) in \( H^4 \), and the structure displacement and velocity \( (w, w_t) \) in \( H^{3} \times H^{2} \). A similar result was later obtained by Kukavica and Tuffaha [KT3] with less regular initial data \( (\rho_0, u_0, w_1) \in H^{3/2+r} \times H^{3} \times H^{3/2+r} \), where \( r \in (0, (\sqrt{2} - 1)/2) \). In [BG2], the existence of a regular global solution is proved for small initial data. In a recent work [BG3], the authors proved the existence of a unique local-in-time strong solution of the interaction problem between a compressible fluid and elastic structure, which is modeled by the Saint-Venant Kirchhoff system. For some other works on fluid-structure models, cf. [AL, B, BuL, BZ1, BZ2, BTZ, DEGL, F, GH, GGCC, GGCL, IKLT1, IKLT2, KKL+, KMT, KOT, LL1, LL2, LT, LTr1, LTr2, MC1, MC2, MC3, SST].

In this paper, we provide a natural proof of the existence of a unique local-in-time solution to the system under a low regularity assumptions \( u_0 \in H^{2+\epsilon} \) and \( w_1 \in H^{1.5+\epsilon} \), where \( \epsilon \in (0, 1/2) \), in the case of the flat initial configuration. Our proof relies on the existence of unique local-in-time solution for the nonhomogeneous linear parabolic problem with Neumann type conditions on the fluid-structure interface, in addition to the hidden regularity theorems (cf. Lemma 3.3–3.4) for the wave equation. The time regularity of the solution is obtained using the energy estimates, which, combined with the elliptic regularity, yield the spatial regularity of the solutions. An essential ingredient of the proof of the main results is a trace inequality

\[
\|u\|_{H^s((0,T),L^2(\Gamma))} \lesssim \|u\|_{L^2((0,T),H^s(\Omega))}^{1/2s} \|u\|_{L^2((0,T),H^{s/2-1/2}(\Gamma))}^{(2s-1)/2s} + \|u\|_{L^2((0,T),H^s(\Omega))},
\]

for functions which are Sobolev in the time variable and square integrable on the boundary (cf. Lemma 3.1 and (3.7) below). This is used essentially in the proof of the existence for the nonlinear parabolic-wave system, Theorem 5.2, and the proof of the main result, Theorem 2.1.

The construction of a unique solution for the fluid-structure problem is obtained via the Banach fixed point theorem. The scheme involves solving the nonlinear parabolic-wave system with the variable coefficients treated as given forcing perturbations. The iteration scheme then requires solving the wave equation with Dirichlet data using the integrated velocity matching conditions and appealing to the hidden trace regularity Lemma 3.4 for the normal derivative. Note that the configuration we adopt, (2.8) with the periodic boundary conditions in the \( y_1 \) and \( y_2 \) directions, is needed only in Lemma 3.4. In addition, the solution in each iteration step is used to prescribe the new variable coefficients for the next iteration step. The contracting property of the Navier-Stokes-wave system is maintained by taking a sufficiently short time to ensure closeness of the Jacobian and the inverse matrix of the flow map to their initial states.

The paper is organized as follows. In Section 2 we introduce the fluid-structure model and state our main result. In Section 3 we provide the trace inequality, interpolation, and hidden regularity lemmas. Section 4 provides the maximal regularity for the nonhomogeneous parabolic problem, which is needed in the proof of local existence to the nonlinear parabolic-wave system in Section 5. In Section 6, we prove the main result, Theorem 2.1, using the local existence obtained in Section 5 and construct a unique solution using the Banach fixed point theorem.

2. The model and main results

We consider the fluid-structure problem for a free boundary system involving the motion of an elastic body immersed in a compressible fluid. Let \( \Omega_\text{f}(t) \) and \( \Omega_\text{c}(t) \) be the domains occupied by the fluid and the solid body at
time $t$ in $\mathbb{R}^3$, whose common boundary is denoted by $\Gamma_c(t)$. The fluid is modeled by the compressible Navier-Stokes equations, which in Eulerian coordinates reads

$$
\rho_t + \text{div} (\rho u) = 0 \quad \text{in } (0, T) \times \Omega_f(t),
$$

$$
\rho u_t + \rho (u \cdot \nabla)u - \lambda \text{div} (\nabla u + (\nabla u)^T) - \mu \nabla \text{div} u + \nabla p = 0 \quad \text{in } (0, T) \times \Omega_f(t),
$$

where $\rho = \rho(t, x) \in \mathbb{R}_+$ is the density, $u = u(t, x) \in \mathbb{R}^3$ is the velocity, $p = p(t, x) \in \mathbb{R}_+$ is the pressure, and $\lambda, \mu > 0$ are physical constants. (We remark that the condition for $\lambda$ and $\mu$ can be relaxed to $\lambda > 0$ and $3\lambda + 2\mu > 0$.) The system (2.1)–(2.2) is defined on $\Omega_f(t)$ which set to $\Omega_f = \Omega_f(0)$ and evolves in time. The dynamics of the coupling between the compressible fluid and the elastic body are best described in the Lagrangian coordinates. Namely, we introduce the Lagrangian flow map $\eta(t, \cdot): \Omega_f(0) \rightarrow \Omega_f(t)$ and rewrite the system (2.1)–(2.2) as

$$
R_t - Ra_kj \partial_k v_j = 0 \quad \text{in } (0, T) \times \Omega_f,
$$

$$
\partial_t v_j - \lambda Ra_kj \partial_k (a_{m} \partial_m v_j + a_{m} \partial_m v_i) - \mu Ra_kj \partial_k (a_{m} \partial_m v_i) + Ra_kj \partial_k (q(R^{-1})) = 0 \quad \text{in } (0, T) \times \Omega_f,
$$

for $j = 1, 2, 3$, where $R(t, x) = \rho^{-1}(t, \eta(t, x))$ is the reciprocal of the Lagrangian density, $v(t, x) = u(t, \eta(t, x))$ is the Lagrangian velocity, $a(t, x) = (\nabla \eta(t, x))^{-1}$ is the inverse matrix of the flow map and $q$ is a given function of the density. The system (2.3)–(2.4) is expressed in terms of Lagrangian coordinates and posed in a fixed domain $\Omega_c$.

On the other hand, the elastic body is modeled by the wave equation in Lagrangian coordinates, which is posed in a fixed domain $\Omega_c$ as

$$
w_{tt} - \Delta w = 0 \quad \text{in } (0, T) \times \Omega_c,
$$

where $(w, w_t)$ are the displacement and the structure velocity. The interaction boundary conditions are the velocity and stress matching conditions, which are formulated in Lagrangian coordinates over the fixed common boundary $\Gamma_c = \Gamma_c(0)$ as

$$
v_j = \partial_t w_j \quad \text{on } (0, T) \times \Gamma_c,
$$

$$
\partial_t w_j \nu^k = \lambda J a_k j \partial_k (a_{m} \partial_m v_j + a_{m} \partial_m v_i) \nu^k + \mu J a_k j a_{m} \partial_m v_i \nu^k - J a_k j q \nu^k \quad \text{on } (0, T) \times \Gamma_c,
$$

for $j = 1, 2, 3$, where $J(t, x) = \text{det}(\nabla \eta(t, x))$ is the Jacobian and $\nu$ is the unit normal vector to $\Gamma_c$, which is outward with respect to $\Omega_c$. In the present paper, we consider the reference configurations $\Omega = \Omega_f \cup \Omega_e \cup \Gamma_c, \Omega_f$, and $\Omega_e$ given by

$$
\Omega = \{ y = (y_1, y_2, y_3) \in \mathbb{R}^3 : (y_1, y_2) \in \mathbb{T}^2, 0 < y_3 < L_3 \},
$$

$$
\Omega_f = \{ y = (y_1, y_2, y_3) \in \mathbb{R}^3 : (y_1, y_2) \in \mathbb{T}^2, 0 < y_3 < L_1 \text{ or } L_2 < y_3 < L_3 \},
$$

$$
\Omega_e = \{ y = (y_1, y_2, y_3) \in \mathbb{R}^3 : (y_1, y_2) \in \mathbb{T}^2, L_1 < y_3 < L_2 \},
$$

where $0 < L_1 < L_2 < L_3$ and $\mathbb{T}^2$ is the two-dimensional torus with the side $2\pi$. Thus, the common boundary is expressed as

$$
\Gamma_c = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1, y_2, y_3) \in \Omega, y_3 = L_1 \text{ or } y_3 = L_2 \},
$$

while the outer boundary is represented by

$$
\Gamma_f = \{ y \in \Omega : y_3 = 0 \} \cup \{ y \in \Omega : y_3 = L_3 \}.
$$

To close the system, we impose the homogeneous Dirichlet boundary condition

$$
v = 0 \quad \text{on } (0, T) \times \Gamma_f,
$$

on the outer boundary $\Gamma_f$ and the periodic boundary conditions for $w$, $\rho$, and $u$ on the lateral boundary, i.e.,

$$
w(t, \cdot), \rho(t, \eta(t, \cdot)), u(t, \eta(t, \cdot)) \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions.}
$$

Note that the inverse matrix of the flow map $a$ satisfies the ODE system

$$
\frac{\partial a}{\partial t}(t, x) = -a(t, x) \nabla v(t, x) a(t, x) \quad \text{in } [0, T] \times \Omega_f,
$$

where $v$ is the velocity of the fluid.
while the Jacobian satisfies the ODE system
\begin{align}
\frac{\partial J}{\partial t}(t, x) &= J(t, x) \alpha \kappa (t, x) \partial_k v_j (t, x) \quad \text{in } [0, T] \times \Omega_t, \\
J(0) &= 1 \quad \text{in } \Omega_t.
\end{align}
\tag{2.13}
(\ref{2.14})

The initial data of the system \((2.3)\)–\((2.5)\) is given as
\begin{align}
(R, v, w, w_1)(0) &= (R_0, v_0, w_0, w_1) \quad \text{in } \Omega_t \times \Omega_t \times \Omega_c \times \Omega_e, \\
(R_0, v_0, w_0, w_1) &= \text{periodic in the } y_1 \text{ and } y_2 \text{ directions},
\end{align}
\tag{2.15}

where \(w_0 = 0\). Denote
\[H^{r,s}((0, T) \times \Omega_c) = H^r((0, T), L^2(\Omega_c)) \cap L^2((0, T), H^s(\Omega_c)),\]
with the corresponding norm
\[\|f\|_{H^{r,s}((0, T) \times \Omega_c)} = \|f\|_{H^r((0, T), L^2(\Omega_c))} + \|f\|_{L^2((0, T), H^s(\Omega_c))}.\]

We write \(H_{\Gamma_c}^{r,s}\) for the analogous space for functions defined on \((0, T) \times \Gamma_c\). For simplicity of notation, we frequently write
\[K^s = H^{s/2,s} = H^{s/2,s}((0, T) \times \Omega_t).\]

Our main result states the local-in-time existence of solution to the system \((2.3)\)–\((2.5)\) with the mixed boundary conditions \((2.6)\)–\((2.10)\) and the initial data \((2.15)\).

**Theorem 2.1.** Let \(s \in (2, 2 + \epsilon_0)\) where \(\epsilon_0 \in (0, 1/2)\). Assume that \(R_0 \in H^s(\Omega_t), R_0^{-1} \in H^s(\Omega_t), v_0 \in H^s(\Omega_t), w_1 \in H^{s-1/2}(\Omega_c),\) and \(w_0 = 0\), with the compatibility conditions
\begin{align}
w_{1j} &= v_{0j} \quad \text{on } \Gamma_c, \\
v_{0j} &= 0 \quad \text{on } \Gamma_f, \\
\lambda(\partial_k v_{0j} + \partial_j v_{0k}) v^k + \mu \partial_k v_{0k} v^j - q(R_0^{-1}) v^j &= 0 \quad \text{on } \Gamma_c, \\
\lambda \partial_k (\partial_j v_{0j} + \partial_j v_{0k}) + \mu \partial_k \partial_j v_{0k} - \partial_k q(R_0^{-1}) &= 0 \quad \text{on } \Gamma_f,
\end{align}
for \(j = 1, 2, 3\). Then the system \((2.3)\)–\((2.5)\) with the boundary conditions \((2.6)\)–\((2.10)\) and the initial data \((2.15)\) admits a unique solution
\begin{align}
v &\in K^{s+1}((0, T) \times \Omega_t), \\
R &\in H^1((0, T), H^s(\Omega_t)), \\
w &\in C([0, T], H^{s+1/4-\epsilon_0}(\Omega_c)), \\
w_t &\in C([0, T], H^{s-3/4-\epsilon_0}(\Omega_c)),
\end{align}
for some \(T > 0\), where the corresponding norms are bounded by a function of the norms of the initial data.

**Remark 2.2.** We assume \(v_0 \in H^s(\Omega_t)\) where \(s \in (2, 2 + \epsilon_0)\) with \(\epsilon_0 \in (0, 1/2)\) since the elliptic regularity for \(\|v\|_{L_t^2 H^1_x}\) in \((4.28)\) requires that \(R^{-1} \in L^\infty H^2((0, T) \times \Omega_t)\). From the density equation \((2.3)\), we deduce that the regularity for the initial velocity must be at least in \(H^2(\Omega_t)\), showing the optimality of the range \(s \geq 2\). It would be interesting to find whether the statement of the theorem holds for the borderline case \(s = 2\).

The proof of the theorem is given in Section 5 below. For simplicity, we present the proof for the pressure law \(q(R) = R\), noting that the general case follows completely analogously.
3. Space-time trace, interpolation, and hidden regularity inequalities

In this section, we provide four auxiliary results needed in the fixed point arguments. The first lemma provides an estimate for the trace in a space-time norm and is an essential ingredient when constructing solutions to the nonlinear parabolic-wave system in Section 5 below.

**Lemma 3.1.** Let \( r > 1/2 \) and \( \theta \geq 0 \). If \( u \in L^2((0, T), H^r(\Omega_f)) \cap L^2((0, T), L^2(\Omega_f)) \), then \( u \in H^0((0, T), L^2(\Gamma_c)) \), and for all \( \epsilon \in (0, 1) \) we have the inequality

\[
\| u \|_{H^0((0, T), L^2(\Gamma_c))} \leq \epsilon \| u \|_{H^{2r/(2r-1)}((0, T), L^2(\Omega_f))} + C \| u \|_{L^2((0, T), H^r(\Omega_f))},
\]

where \( C \geq 0 \) is a constant depending on \( \epsilon \).

**Proof of Lemma 3.1.** It is sufficient to prove the inequality for \( u \in C^\infty_0(\mathbb{R} \times \mathbb{R}^3) \) with the trace taken on the set \( \Gamma = \{(t, x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{R}^3 : x_3 = 0\} \), the general case is settled by the partition of unity and straightening of the boundary. Since it should be clear from the context, we usually do not distinguish in notation between a function and its trace. Denoting by \( \hat{u} \) the Fourier transform of \( u \) with respect to \( (t, x_1, x_2, x_3) \), we have

\[
\| u \|_{H^0((0, T), L^2(\Gamma_c))} \lesssim \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( 1 + \tau^2 \right)^{\theta} \left| \int_{-\infty}^{\infty} \hat{u}(\xi_1, \xi_2, \xi_3, \tau) \, d\xi_3 \right|^2 \, d\tau \, d\xi_1 \, d\xi_2.
\]

Denote by

\[
\gamma = \frac{2r-1}{2\theta}
\]

the quotient between the exponents \( r \) and \( 2r/(2r-1) \) in (3.1). Then, with \( \lambda > 0 \) to be determined below, we have

\[
\| u \|_{H^0((0, T), L^2(\Gamma_c))} \lesssim \int_{\mathbb{R}^3} (1 + \tau^2)^{\theta} \left| \int_{-\infty}^{\infty} \hat{u}(\xi_1, \xi_2, \xi_3, \tau) \, d\xi_3 \right|^2 \, d\tau \, d\xi_1 \, d\xi_2
\]

\[
\lesssim \int_{\mathbb{R}^3} (1 + \tau^2)^{\theta} \left( \int_{-\infty}^{\infty} \left( 1 + (\xi_1^2 + \xi_2^2)^{\gamma} + e^{-2\xi_3^{2\gamma}} + \tau^2 \right)^{\lambda/2} |\hat{u}| \, d\xi_3 \right)^2 \, d\tau \, d\xi_1 \, d\xi_2
\]

\[
\lesssim \int_{\mathbb{R}^3} (1 + \tau^2)^{\theta} \left( \int_{-\infty}^{\infty} \left( 1 + (\xi_1^2 + \xi_2^2)^{\gamma} + e^{-2\xi_3^{2\gamma}} + \tau^2 \right)^{\lambda} |\hat{u}|^2 \, d\xi_3 \right)^2 \, d\tau \, d\xi_1 \, d\xi_2
\]

\[
\times \left( \int_{-\infty}^{\infty} \frac{d\xi_3}{1 + (\xi_1^2 + \xi_2^2)^{\gamma} + e^{-2\xi_3^{2\gamma}} + \tau^2} \right) \, d\xi_1 \, d\xi_2,
\]

where we used the Cauchy-Schwartz inequality in \( \xi_3 \). Note that, using a substitution,

\[
\int_{-\infty}^{\infty} \frac{dx}{(A^2 + e^{-2x^2})^{1/2}} \leq e^{1/2} A^{1/2}, \quad A, \epsilon > 0,
\]

provided \( \lambda \) satisfies \( 2\gamma\lambda > 1 \), which is by (3.2) equivalent to

\[
\lambda > \frac{\theta}{2r - 1}.
\]

Note that \( 2\gamma\lambda > 1 \) implies \( 1/\gamma - 2\lambda < 0 \) for the exponent of \( A \) in (3.3). Now we use (3.3) for the integral in \( \xi_3 \) with \( A = (1 + (\xi_1^2 + \xi_2^2)^{\gamma} + \tau^2)^{1/2} \), while noting that

\[
(1 + \tau^2)^{\theta} A^{1/\gamma - 2\lambda} = \frac{(1 + \tau^2)^{\theta}}{(1 + (\xi_1^2 + \xi_2^2)^{\gamma} + \tau^2)^{\lambda - 1/2\gamma}} \leq (1 + \tau^2)^{\theta - \lambda + 1/2\gamma} \leq (1 + (\xi_1^2 + \xi_2^2)^{\gamma} + e^{-2\xi_3^{2\gamma}} + \tau^2)^{\theta - \lambda + 1/2\gamma},
\]

provided \( \lambda - 1/2\gamma \leq \theta \), i.e.,

\[
\lambda \leq \frac{2r\theta}{2r - 1}.
\]
Under the condition (3.5), we thus obtain
\[
\|u\|^2_{H^\theta((0,T),L^2(\Gamma_c))} \lesssim \epsilon^{1/\gamma} \int_{\mathbb{R} \times \mathbb{R}^3} (1 + (\xi_1^2 + \xi_2^2 + \xi_3^2)\gamma + \epsilon^{-2} + \tau^2)^{\theta+1/2\gamma} |\hat{u}|^2 d\xi_1 d\xi_2 d\xi_3 d\tau
\]
\[
\lesssim \epsilon^{1/\gamma} \int_{\mathbb{R} \times \mathbb{R}^3} (1 + \epsilon^{-2} + \tau^2)^{\theta+1/2\gamma} |\hat{u}|^2 d\xi_1 d\xi_2 d\xi_3 d\tau
\]
\[
\lesssim \epsilon^{-2\theta} \|u\|^2_{L^2((0,T),H^\theta+1/2(\Omega))} + \epsilon^{1/\gamma} \|u\|^2_{H^{\theta+1/2}(0,T),L^2(\Omega))},
\]
for all \( \epsilon \in (0, 1) \). Using (3.2), we get
\[
\|u\|^2_{H^\theta((0,T),L^2(\Gamma_c))} \lesssim \epsilon^{-2\theta} \|u\|^2_{L^2((0,T),H^r(\Omega))} + \epsilon^{2\theta/(2r-1)} \|u\|^2_{H^{2\theta/(2r-1)}(0,T),L^2(\Omega))},
\]
(3.6)
for all \( \epsilon \in (0, 1) \). Finally, note that \( \lambda = 2r\theta/(2r-1) \) satisfies (3.4) and (3.5) under the condition \( r > 1/2 \).

Optimizing \( \epsilon \in (0, 1] \) in (3.6) by using
\[
\epsilon = \left( \frac{\|u\|_{L^2((0,T),H^r(\Omega))}}{\|u\|_{L^2((0,T),H^\theta(\Omega))} + \|u\|_{H^{2\theta/(2r-1)}(0,T),L^2(\Omega))}} \right)^{(2r-1)/2r\theta},
\]
we obtain a trace inequality
\[
\|u\|^2_{H^\theta((0,T),L^2(\Gamma_c))} \lesssim \|u\|^2_{L^2((0,T),H^\theta(\Omega))}(2r-1)^{2r\theta},
\]
(3.7)
which is a more explicit version of (3.1).

The second lemma provides a space-time interpolation inequality needed in several places in Sections 5 and 6 below.

**Lemma 3.2.** Let \( \alpha, \beta > 0 \). If \( u \in H^\alpha((0,T), L^2(\Omega)) \cap L^2((0,T), H^\beta(\Omega)) \), then \( u \in H^\theta((0,T), H^\lambda(\Omega)) \) and for all \( \epsilon > 0 \), we have the inequality
\[
\|u\|_{H^\theta((0,T),H^\lambda(\Omega))} \leq \epsilon \|u\|_{H^\alpha((0,T),L^2(\Omega))} + C_\epsilon \|u\|_{L^2((0,T),H^\beta(\Omega))},
\]
for all \( \theta \in (0, \alpha) \) and \( \lambda \in (0, \beta) \) such that
\[
\frac{\theta}{\alpha} + \frac{\lambda}{\beta} \leq 1,
\]
where \( C_\epsilon > 0 \) is a constant depending on \( \epsilon \).

**Proof of Lemma 3.2.** Using a partition of unity, straightening of the boundary, and a Sobolev extension, it is sufficient to prove the inequality in the case \( \Omega = \mathbb{R}^3 \) and \( u \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3) \). Then, using the Parseval identity and the definition of the Sobolev, we only need to prove
\[
(1 + |\tau|^{2\theta})(1 + |\xi|^{2\lambda}) \leq \epsilon^2 (1 + |\tau|^{2\alpha}) + C_\epsilon (1 + |\xi|^{2\beta}),
\]
(3.8)
for \( \tau \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \). Finally, (3.8) follows from Young’s inequality.

In the last part of this section, we address the regularity for the wave equation. We first recall the hidden regularity result for the wave equation
\[
w_{tt} - \Delta w = 0 \text{ in } (0,T) \times \Omega_e,
\]
(3.9)
\[
w = \psi \text{ on } (0,T) \times \Gamma_e,
\]
(3.10)
\[
w \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions},
\]
(3.11)
and the initial data
\[
(w, w_t)(0, \cdot) = (w_0, w_1)
\]
(3.12)
(cf. [LLT]).
Lemma 3.3 ([LLT]). Assume that \((u_0, u_1) \in H^\beta(\Omega_e) \times H^{\beta-1}(\Omega_e)\), where \(\beta \geq 1\), and 
\[ \psi \in C([0, T], H^{\beta-1/2}(\Gamma_e)) \cap H^\beta((0, T) \times \Gamma_e). \]
Then there exists a solution \((w, w_1) \in C([0, T], H^\beta(\Omega_e) \times H^{\beta-1}(\Omega_e))\) of (3.9)–(3.12), which satisfies the estimate
\[ \|w\|_{C([0, T], H^\beta(\Omega_e))} + \|w_1\|_{C([0, T], H^{\beta-1}(\Omega_e))} + \left\| \frac{\partial w}{\partial \nu} \right\|_{H^{\beta-1}((0, T) \times \Gamma_e)} \]
\[ \lesssim \|w_0\|_{H^\beta(\Omega_e)} + \|w_1\|_{H^{\beta-1}(\Omega_e)} + \|\psi\|_{H^\beta((0, T) \times \Gamma_e)}, \]
where the implicit constant depends on \(\Omega_e\). 

In the final lemma of this section, we recall an essential trace regularity result for the wave equation from [RV].

Lemma 3.4 ([RV]). Assume that \((u_0, u_1) \in H^\beta(\Omega_e) \times H^{\beta+1}(\Omega_e)\), where \(0 < \beta < 5/2\), and 
\[ \psi \in L^2((0, T), H^{\beta+2}(\Gamma_e)) \cap H^{\beta/2+1}((0, T), H^{\beta/2+1}(\Gamma_e)), \]
with the compatibility condition \(\partial_t \psi|_{t=0} = u_1\). Then there exists a solution \(w\) of (3.9)–(3.12) such that
\[ \left\| \frac{\partial w}{\partial \nu} \right\|_{L^2((0, T), H^{\beta+2}(\Gamma_e))} \lesssim \|w_0\|_{H^\beta(\Omega_e)} + \|w_1\|_{H^{\beta+1}(\Omega_e)} + \|\psi\|_{L^2((0, T), H^{\beta+2}(\Gamma_e))} + \|\psi\|_{H^{\beta/2+1}((0, T), H^{\beta/2+1}(\Gamma_e))}, \]
where the implicit constant depends on \(\Omega_e\).

4. The nonhomogeneous parabolic problem

In this section, we consider the parabolic problem
\[ \partial_t u - \lambda R \text{div}(\nabla u + (\nabla u)^T) - \mu R \text{div} u = f \quad \text{in} \quad (0, T) \times \Omega, \]
with the nonhomogeneous boundary conditions and the initial data
\[ \lambda(\partial_k u^j + \partial_j u_k)\nu^k + \mu \partial_k u_k \nu^j = h_j \quad \text{on} \quad (0, T) \times \Gamma, \]
\[ u = 0 \quad \text{on} \quad (0, T) \times \Gamma, \]
\[ u \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \]
\[ u(0) = u_0 \quad \text{in} \quad \Omega, \]
for \(j = 1, 2, 3\). To state the maximal regularity for (4.1)–(4.5), we consider the homogeneous version when (4.2)–(4.5) is replaced by
\[ \lambda(\partial_k u^j + \partial_j u_k)\nu^k + \mu \partial_k u_k \nu^j = 0 \quad \text{on} \quad (0, T) \times \Gamma, \]
\[ u = 0 \quad \text{on} \quad (0, T) \times \Gamma, \]
\[ u \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \]
\[ u(0) = 0 \quad \text{in} \quad \Omega, \]
for \(j = 1, 2, 3\).

Lemma 4.1. Assume that
\[ (R, R^{-1}) \in L^\infty([0, T], H^2(\Omega_e)) \times L^\infty([0, T], H^2(\Omega_e)) \cap H^1([0, T], L^\infty(\Omega_e)), \]
for some \(T > 0\). Then the parabolic problem (4.1) with the boundary conditions and the initial data (4.6)–(4.9) admits a solution \(u\) satisfying
\[ \|u\|_{K^2((0, T) \times \Omega)} \lesssim \|f\|_{K^0((0, T) \times \Omega)}, \]
\[ \|u\|_{K^4((0, T) \times \Omega)} \lesssim \|f\|_{K^2((0, T) \times \Omega)}, \]
where the implicit constants depend on the norms of \(R\) and \(R^{-1}\) in (4.10).
PROOF. Analogously to [LM, Theorem 3.2], the parabolic problem (4.1) admits a solution \( u \in K^2(0, T) \times \Omega_t \) if \( f \in K^0((0, T) \times \Omega_t) \) and \( u \in K^4((0, T) \times \Omega_t) \) if \( f \in K^2((0, T) \times \Omega_t) \). In the reminder of the proof we shall prove the regularity. Taking the \( L^2 \)-inner product of (4.1) with \( u \), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |u|^2 - \lambda \int_{\Omega} Ru_j \partial_k (\partial_k u_j + \partial_j u_k) - \mu \int_{\Omega_t} Ru_j \partial_j \partial_k u_k = f u. \tag{4.13}
\]

For the second and third terms on the left side of (4.13), we integrate by parts with respect to \( \partial_k \) and \( \partial_j \) to get

\[
-\lambda \int_{\Omega_t} Ru_j \partial_k (\partial_k u_j + \partial_j u_k) = \lambda \int_{\Gamma_x} Ru_j (\partial_k u_j \partial_j u_k) + \lambda \int_{\Omega_t} R \partial_k u_j (\partial_k u_j + \partial_j u_k)
\]

and

\[
-\mu \int_{\Omega_t} Ru_j \partial_j \partial_k u_k = \mu \int_{\Gamma_x} Ru_j \partial_k u_j + \mu \int_{\Omega_t} u_j \partial_j R \partial_k u_k. \tag{4.14}
\]

Inserting (4.14) into (4.13) and appealing to (4.6)–(4.7), we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |u|^2 + \lambda \int_{\Omega_t} R \partial_k u_j (\partial_k u_j + \partial_j u_k) + \mu \int_{\Omega_t} R \partial_j u_j \partial_k u_k = f u + \lambda \int_{\Omega} Ru_j (\partial_k u_j \partial_j u_k) + \mu \int_{\Omega_t} u_j \partial_j R \partial_k u_k.
\]

where the last inequality follows from Hölder’s and Young’s inequalities. Note that for any \( v \in H^1(\Omega_t) \), using the Sobolev and Young’s inequalities, we have

\[
\|v\|_{L^4} \lesssim \|v\|^3_{H^1} \|v\|_{L^2}^{1/2} \lesssim \epsilon \|v\|_{H^1} + C_\epsilon \|v\|_{L^2}, \tag{4.17}
\]

for any \( \epsilon \in (0, 1] \), where \( C_\epsilon > 0 \) is a constant depending on \( \epsilon \). We integrate (4.16) in time from 0 to \( t \) and use

\[
(\partial_k u_j + \partial_j u_k) \partial_k u_j = \frac{1}{2} \sum_{j,k=1}^{3} (\partial_k u_j + \partial_j u_k)^2, \tag{4.18}
\]

obtaining

\[
\|u(t)\|_{L^2}^2 + \frac{3}{2} \sum_{j,k=1}^{3} \int_{0}^{t} \int_{\Omega_t} R (\partial_k u_j + \partial_j u_k)^2 + \int_{0}^{t} \int_{\Omega} R |\partial_k u_k|^2 \lesssim \|f\|_{L^2}^2 + \epsilon \int_{0}^{t} \|u\|_{H^1}^2 + C_\epsilon \int_{0}^{t} \|u\|_{L^2} \|u\|_{H^1}, \tag{4.19}
\]

for any \( \epsilon \in (0, 1] \), where we used (4.17) and Young’s inequality. By Korn’s inequality, we get

\[
\int_{0}^{t} \|u\|_{H^1}^2 \lesssim \sum_{j,k=1}^{3} \int_{0}^{t} \int_{\Omega_t} R (\partial_k u_j + \partial_j u_k)^2 + \int_{0}^{t} \|u\|_{L^2}^2. \tag{4.20}
\]

From (4.19)–(4.20) it follows that

\[
\|u(t)\|_{L^2}^2 + \|u\|_{L^2}^2 \lesssim \|f\|_{L^2}^2 + \int_{0}^{t} \|u\|_{L^2}^2, \tag{4.21}
\]

by choosing \( \epsilon, \tilde{\epsilon} > 0 \) sufficiently small. By Gronwall’s inequality, we obtain

\[
\|u(t)\|_{L^2} \lesssim \|f\|_{L^2}, \tag{4.22}
\]

for any \( t \geq 0 \).
and then, after using (4.22) in (4.21), we arrive at
\[ \| u \|^2 \| L^2 \Omega \| H^1 \| L^2 \Omega \| \lesssim \| f \|^2 \| L^2 \Omega \|. \] (4.23)

Next, we take the \( L^2 \)-inner product of (4.1) with \( u_t \), obtaining
\[ \int_{\Omega_t} |u_t|^2 - \lambda \int_{\Omega_t} R u_j \partial_k (\partial_k u_j + \partial_j u_k) - \mu \int_{\Omega_t} R u_j \partial_j \partial_k u_k = \int_{\Omega_t} f u_t. \] (4.24)

Then, proceeding as in (4.14)–(4.15), we get
\[-\lambda \int_{\Omega_t} R u_j \partial_k (\partial_k u_j + \partial_j u_k) = \lambda \int_{\Gamma_t} R u_j (\partial_k u_j + \partial_j u_k) u^k + \lambda \int_{\Omega_t} R \partial_k u_j (\partial_k u_j + \partial_j u_k) + \lambda \int_{\Omega_t} u_t \partial_j R (\partial_k u_j + \partial_j u_k) \] (4.25)
and
\[-\mu \int_{\Omega_t} R u_j \partial_j \partial_k u_k = \mu \int_{\Gamma_t} R u_j \partial_k u_j + \mu \int_{\Omega_t} u_t \partial_j R \partial_k u_k. \] (4.26)

Inserting (4.25)–(4.26) into (4.24) and appealing to (4.6)–(4.7), we arrive at
\[ \int_{\Omega_t} |u_t|^2 + \frac{\lambda}{2} \frac{d}{dt} \int_{\Omega_t} R \partial_k u_j (\partial_k u_j + \partial_j u_k) + \frac{\mu}{2} \frac{d}{dt} \int_{\Omega_t} R \partial_j u_j \partial_k u_k \] (4.27)
\[ \leq C \| f \|^2 \| L^2 \Omega \| + \epsilon \| u_t \|^2 \| L^2 \Omega \| + \| \nabla u \| \| L^2 \Omega \| \| u_t \| \| L^2 \Omega \| + \| R \| \| L^\infty \| \| \nabla u \|^2 \| L^2 \Omega \| , \]
for any \( \epsilon \in (0, 1] \), where we used Hölder’s and Young’s inequalities. Integrating in time from 0 to \( t \) and using the Young, Sobolev, and Korn inequalities with (4.17)–(4.18), we get
\[ \| u_t \|^2 \| L^2 \Omega \| + \| u(t) \|^2 \| H^1 \| \lesssim C \| f \|^2 \| L^2 \Omega \| + \epsilon \| u_t \|^2 \| L^2 \Omega \| + \int_0^t (\epsilon \| u \| \| H^2 \| + C \| u \| \| H^1 \| ) \| u_t \| \| L^2 \| + \int_0^t \| R \| \| L^\infty \| \| u \|^2 \| H^1 \| , \] (4.28)
for any \( \epsilon, \tilde{\epsilon}, \tilde{\epsilon} \in (0, 1] \). For the space regularity, note that \( u \) is the solution of the elliptic problem
\[- \lambda \text{div}(\nabla u + (\nabla u)^T) - \mu \text{div} u = - \frac{\partial_t u}{R} + \frac{f}{R} \quad \text{in} \ (0, T) \times \Omega_t, \] (4.29)
with the boundary conditions
\[ \lambda (\partial_k u_j + \partial_j u_k) u^k + \mu \partial_k u_j u^j = 0 \quad \text{on} \ (0, T) \times \Gamma_c, \] (4.30)
for \( j = 1, 2, 3 \). From the elliptic regularity for (4.28)–(4.30) it follows that
\[ \| u \| \| H^2 \| \lesssim \left\| \frac{1}{R} u_t \right\| \| L^2 \Omega \| + \left\| \frac{1}{R} f \right\| \| L^2 \Omega \| \lesssim \| u_t \| \| L^2 \Omega \| + \| f \| \| L^2 \Omega \| , \] (4.31)
from where
\[ \| u \|^2 \| L^2 \Omega \| \lesssim \| u_t \|^2 \| L^2 \Omega \| + \| f \|^2 \| L^2 \Omega \|. \] (4.32)
Combining (4.22)–(4.23), (4.27), and (4.32), we obtain
\[
\|u_t\|^2_{L^2_t H^2_x} + \|u(t)\|^2_{H^2_x} \lesssim \|f\|^2_{L^2_t L^2_x} + \int_0^t \|R_t\|_{L^\infty} \|u\|^2_{H^3_x}, \tag{4.33}
\]
by taking \(\epsilon, \tilde{\epsilon} > 0\) sufficiently small. Using Gronwall’s inequality, we arrive at
\[
\|u(t)\|^2_{H^1_x} \leq C\|f\|^2_{L^2_t L^2_x} \exp\left(\int_0^t \|R_t\|_{L^\infty} \, dt\right) \leq C\|f\|^2_{L^2_t L^2_x},
\]
and thus (4.33) implies
\[
\|u\|^2_{H^1_t L^2_x} \lesssim \|f\|^2_{L^2_t L^2_x}. \tag{4.34}
\]
From (4.32) and (4.34) it follows that
\[
\|u\|_{K^2} \lesssim \|u\|_{L^2_t H^2_x} + \|u\|_{H^1_t L^2_x} \lesssim \|f\|_{L^2_t L^2_x},
\]
completing the proof of (4.11).

Differentiating (4.1) in time and taking the \(L^2\)-inner product with \(u_t\), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |u_t|^2 + \lambda \int_{\Omega_t} Ru_{t,ij} \partial_k (\partial_k u_{t,j} + \partial_j u_{t,k}) - \mu \int_{\Omega_t} Ru_{t,ij} \partial_k \partial_k u_{t,k} = \int_{\Omega_t} f_t u_t + \lambda \int_{\Omega_t} Ru_{t,ij} \partial_k (\partial_k u_{t,j} + \partial_j u_{t,k}) + \mu \int_{\Omega_t} R u_{t,ij} \partial_k \partial_k u_{t,k}. \tag{4.35}
\]
We proceed as in (4.14)–(4.15) to obtain
\[
-\lambda \int_{\Gamma_t} Ru_{t,ij} \partial_k (\partial_k u_{t,j} + \partial_j u_{t,k}) = \lambda \int_{\Gamma_t} Ru_{t,ij} (\partial_k u_{t,j} + \partial_j u_{t,k}) \nu^k + \lambda \int_{\Omega_t} u_{t,ij} \partial_k R (\partial_k u_{t,j} + \partial_j u_{t,k}) \tag{4.36}
\]
and
\[
-\mu \int_{\Gamma_t} Ru_{t,ij} \partial_k u_{t,k} = \mu \int_{\Omega_t} Ru_{t,ij} \partial_k u_{t,k} \nu^j + \mu \int_{\Omega_t} u_{t,ij} \partial_j R \partial_k u_{t,k} + \mu \int_{\Omega_t} R \partial_j u_{t,ij} \partial_k u_{t,k}. \tag{4.37}
\]
Inserting (4.36)–(4.37) into (4.35), we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |u_t|^2 + \lambda \int_{\Omega_t} Ru_{t,ij} \partial_k (\partial_k u_{t,j} + \partial_j u_{t,k}) + \mu \int_{\Omega_t} R u_{t,ij} \partial_k \partial_k u_{t,k} \lesssim \|f_t\|^2_{L^2} + \|u_t\|^2_{L^2} + \|R_t\|_{L^\infty} \|u_t\|_{L^2} + \|R_t\|_{L^\infty} \|\nabla R\|_{L^1} \|u_t\|_{H^1_x},
\]
where we used Young’s, Hölder’s, and Sobolev inequalities. Integrating in time from 0 to \(t\) and using the Young’s and Korn’s inequality and (4.17)–(4.18), we obtain
\[
\|u_t(t)\|^2_{L^2} + \|u_t\|^2_{L^2_t H^2_x} \lesssim \|f\|^2_{H^1_t L^2_x} + \int_0^t \|R_t\|_{L^\infty} \|u_t\|^2_{L^2} + \int_0^t \|R_t\|_{L^\infty} \|u_t\|^2_{L^2} + \int_0^t \|R_t\|_{L^\infty} \|f\|^2_{L^2}
+ C_{\epsilon, \tilde{\epsilon}} \|u_t\|^2_{L^2_t H^2_x},
\]
for any \(\epsilon, \tilde{\epsilon} \in (0, 1)\). From the elliptic regularity (4.31) and (4.34) it follows that
\[
\|u_t(t)\|^2_{L^2} + \|u_t\|^2_{L^2_t H^2_x} \lesssim \|f\|^2_{H^1_t L^2_x} + \|u_t\|^2_{L^2_t L^2_x} + \int_0^t \|R_t\|_{L^\infty} \|u_t\|^2_{L^2} + \int_0^t \|R_t\|_{L^\infty} \|f\|^2_{L^2} \tag{4.38}
\]
by taking \(\epsilon, \tilde{\epsilon} > 0\) sufficiently small, where in the last inequality we used \(\|f\|_{L^\infty_t L^2_x} \lesssim \|f\|_{H^1_t L^2_x}\). Appealing to Gronwall’s inequality, (4.38) implies
\[
\|u_t(t)\|^2_{L^2} \lesssim \|f\|_{K^2}^2, \tag{4.39}
\]
and then, after using (4.39) in (4.38), we arrive at
\[ \|u_t\|_{L^2_t H^1_x}^2 \lesssim \|f\|_{H^2}^2. \]

Differentiating (4.1) in time and taking the \(L^2\)-inner product with \(u_{tt}\), we obtain
\[
\int_{\Omega_t} |u_{tt}|^2 + \frac{\lambda}{2} \frac{d}{dt} \int_{\Omega_t} R \partial_k u_{tj} (\partial_k u_{tj} + \partial_j u_{tk}) + \frac{\mu}{2} \frac{d}{dt} \int_{\Omega_t} R \partial_j u_{tj} \partial_k u_{tk} \\
= \int_{\Omega_t} f_t u_t - \lambda \int_{\Omega_t} u_{ttj} \partial_k R (\partial_k u_{tj} + \partial_j u_{tk}) - \mu \int_{\Omega_t} u_{ttj} \partial_j R (\partial_k u_{tj} + \partial_j u_{tk}) \\
+ \frac{\lambda}{2} \int_{\Omega_t} R \partial_k u_{tj} (\partial_k u_{tj} + \partial_j u_{tk}) + \frac{\mu}{2} \int_{\Omega_t} R \partial_j u_{tj} \partial_k u_{tk} + \lambda \int_{\Omega_t} u_{ttj} R \partial_k (\partial_k u_{tj} + \partial_j u_{tk}) \\
+ \mu \int_{\Omega_t} u_{ttj} R \partial_j u_{tk},
\]
where we have integrated by parts in spatial variables. We proceed as in (4.35)–(4.38) to get
\[
\|u_{tt}\|_{L^2_t L^2_x}^2 + \|u_t(t)\|_{H^1_x}^2 \lesssim C_\varepsilon \|f\|_{H^1_x}^2 + \|u_t(t)\|_{L^2_x}^2 + \varepsilon \int_0^t \|u_t\|_{H^2_x} \|u_{tt}\|_{L^2_x} + C_\varepsilon \int_0^t \|R_t\|_{L^\infty_x} \|f\|_{L^2_x}^2 \\
+ (\varepsilon C_\varepsilon + \varepsilon \varepsilon) \|u_{tt}\|_{L^2_t L^2_x}^2 + C_{\varepsilon, \varepsilon} \|u_t\|_{L^2_t H^1_x}^2 + C_\varepsilon \int_0^t (1 + \|R_t\|_{L^\infty_x}^2) \|u_t\|_{H^1_x}^2,
\]
for any \(\varepsilon, \varepsilon, \varepsilon \in (0, 1]\), where we used Young’s, Hölder’s, Sobolev, and Korn inequalities. Note that \(u_t\) is the solution of the elliptic problem
\[-\lambda \div (\nabla u_t + (\nabla u_t)^T) - \mu \nabla \div u_t = -R^{-1} u_{tt} + R^{-2} u_t R_t + R^{-1} f_t - R^{-2} f_t f \quad \text{in } (0, T) \times \Omega_t,
\]
with the boundary conditions
\[\lambda (\partial_k u_{tj} + \partial_j u_{tk}) v^k + \mu \partial_k u_{tj} v^j = 0 \quad \text{in } (0, T) \times \Gamma_c,
\]
\[u_{tj} = 0 \quad \text{in } (0, T) \times \Gamma_t,
\]
for \(j = 1, 2, 3\). The elliptic regularity implies that
\[
\|u_t\|_{H^2_x} \lesssim \|u_t\|_{L^2_x} + \|u_t R_t\|_{L^2_x} + \|f_t\|_{L^2_x} + \|R_t f\|_{L^2_x} \\
\lesssim \|u_t\|_{L^2_x} + \|u_t\|_{L^2_x} \|R_t\|_{L^\infty_x} + \|f_t\|_{L^2_x} + \|R_t\|_{L^\infty_x} \|f\|_{L^2_x},
\]
(4.40)
where we used Hölder’s inequality. From (4.39)–(4.40), we obtain
\[
\|u_{tt}\|_{L^2_t L^2_x}^2 + \|u_t(t)\|_{H^1_x}^2 \lesssim \|f\|_{H^2_x}^2 + \|u_t(t)\|_{L^2_x}^2 + \int_0^t (1 + \|R_t\|_{L^\infty_x}^2) \|u_t\|_{H^1_x}^2 + \int_0^t \|R_t\|_{L^\infty_x} \|f\|_{L^2_x}^2 \\
\lesssim \|f\|_{H^2_x}^2 + \int_0^t (1 + \|R_t\|_{L^\infty_x}^2) \|u_t\|_{H^1_x}^2,
\]
by taking \(\varepsilon, \varepsilon, \varepsilon > 0\) sufficiently small, where we used \(\|f\|_{L^\infty_x L^2_x} \lesssim \|f\|_{H^1_x L^2_x}\). Appealing to Gronwall’s inequality, we arrive at
\[\|u_t(t)\|_{H^2_x}^2 \lesssim \|f\|_{H^2_x}^2,
\]
whence
\[\|u_{tt}\|_{L^2_t L^2_x}^2 \lesssim \|f\|_{H^2_x}^2.
\]
(4.41)
From the \(H^4\) regularity of the elliptic problem (4.28)–(4.30) and (4.40) it follows that
\[
\|u\|_{H^4_x} \lesssim \|R^{-1} u_t\|_{H^2_x} + \|R^{-1} f\|_{H^2_x} \\
\lesssim \|u_t\|_{L^2_x} + \|R_t\|_{L^\infty_x} \|u_t\|_{L^2_x} + \|R_t\|_{L^\infty_x} \|f\|_{L^2_x} + \|f_t\|_{L^2_x} + \|f\|_{H^2_x},
\]
(4.42)
since $H^2$ is an algebra. We combine (4.39) and (4.41)–(4.42) to get
\[
\|u\|_{K^1} = \|u\|_{L^2 H^2_t} + \|u\|_{H^2_t L^2_x} \\
\lesssim \|u_t\|_{L^2 H^2_t} + \|R_t\|_{L^2 L^2_y} \|u_t\|_{L^2_t L^2_x} + \|R_t\|_{L^2_t L^2_y} \|f\|_{H^1_t L^2_x} + \|f\|_{K^2} \lesssim \|f\|_{K^2},
\]
completing the proof of (4.12). \qed

The following lemma provides a maximal regularity for the parabolic system (4.1)–(4.5).

**Lemma 4.2.** Let $s \in (2, 2 + \epsilon_0)$ where $\epsilon_0 \in (0, 1/2)$. Assume the compatibility conditions
\begin{align}
h_j(0) &= \lambda(\partial_k u_{0j} + \partial_j u_{0k})u^k + \mu\partial_k u_{0k}u^j \quad \text{on } \Gamma_c, \quad (4.43) \\
u_{0j} &= 0 \quad \text{on } \Gamma_f, \quad (4.44)
\end{align}
for $j = 1, 2, 3$, and suppose that
\begin{align*}
(R, R^{-1}, h, f) &\in L^\infty([0, T], H^2(\Omega)) \times L^\infty([0, T], H^2(\Omega)) \cap H^1([0, T], L^\infty(\Omega)) \\
&\quad \times H^{s/2-1/4, s-1/2}((0, T) \times \Gamma_c) \times K^{s-1}((0, T) \times \Omega_f),
\end{align*}
for some $T > 0$. Then the system (4.1)–(4.5) admits a solution $u$ satisfying
\[
\|u\|_{K^{s+1}}(0, T) \lesssim \|h\|_{H^{s/2-1/4, s-1/2}(0, T) \times \Gamma_c} + \|u_0\|_{H^s} + \|f\|_{K^{s-1}(0, T) \times \Omega_f},
\]
where the implicit constant depends on the norms of $R$ and $R^{-1}$ in (4.45).

**Proof.** From [LM, Theorem 2.3] and the compatibility conditions (4.43)–(4.44) it follows that there exists $v \in K^{s+1}$ satisfying the boundary conditions and the initial data (4.2)–(4.5) with
\[
\|v\|_{K^{s+1}} \lesssim \|h\|_{H^{s/2-1/4, s-1/2}(0, T) \times \Gamma_c} + \|u_0\|_{H^s},
\]
since $s > 1/2$. Now we consider the homogeneous parabolic problem
\[
\partial_t w - \lambda R \operatorname{div}(\nabla w + (\nabla w)^T) - \mu R \nabla \operatorname{div} w = F \quad \text{in } (0, T) \times \Omega_f,
\]
with the homogeneous boundary conditions and the initial data
\begin{align}
\lambda(\partial_k w_j + \partial_j w_k)u^k + \mu\partial_k w_ku^j &= 0 \quad \text{on } (0, T) \times \Gamma_c, \quad (4.49) \\
w &= 0 \quad \text{on } (0, T) \times \Gamma_f, \quad (4.50) \\
w &= 0 \quad \text{in } \Omega_f, \quad (4.51) \\
w(0) &= 0 \quad \text{in } \Omega_f, \quad (4.52)
\end{align}
for $j = 1, 2, 3$, where
\[
F = -f + \partial_t v - \lambda R \operatorname{div}(\nabla v + (\nabla v)^T) - \mu R \nabla \operatorname{div} v.
\]
By Lemma 4.1, there exists a solution $w$ to the system (4.48)–(4.53) satisfying
\[
\|w\|_{K^2} \lesssim \|F\|_{K^0}
\]
and
\[
\|w\|_{K^1} \lesssim \|F\|_{K^2}.
\]
From [LM, Theorem 6.2] and (4.54)–(4.55) it follows that
\[
\|w\|_{K^{s+1}} \lesssim \|F\|_{K^{s-1}},
\]
for $s \neq \text{integer} + 1/2$ and $s/2 \neq \text{integer}$. From (4.53), we get
\[
\|F\|_{K^{s-1}} \lesssim \|f\|_{K^{s-1}} + \|v_t\|_{K^{s-1}} + \|R(\nabla^2 v)\|_{K^{s-1}}.
\]
For the second term on the right side of (4.57), we obtain
\[
\|v_t\|_{K^{s-1}} \lesssim \|v_t\|_{L^2_t H^{s-1}_x} + \|v_t\|_{L^2_{t,2} H^{s-1}_x} \lesssim \|v\|_{K^{s+1}},
\]
where we used Lemma 3.2. Regarding the last term on the right side of (4.57), we appeal to Hölder’s and Sobolev inequalities, yielding
\[
\|R \nabla^2 v\|_{L^2_t H^{s-1}_x} \lesssim \|R\|_{L^\infty_t H^s_x} \|\nabla^2 v\|_{L^2_t H^{s-1}_x} \lesssim \|v\|_{K^{s+1}}
\]
and
\[
\|R \nabla^2 v\|_{H^{(s-1)/2}_t L^2_x} \lesssim \|R\|_{H^{(s-1)/4}_t L^2_x} \|\nabla^2 v\|_{L^4_t L^4_x} + \|R\|_{L^\infty_t H^s_x} \|\nabla^2 v\|_{H^{(s-1)/2}_t L^2_x} \lesssim \|R\|_{H^{(s-1)/2}_t L^2_x} \|\nabla^2 v\|_{H^{(s-1)/2}_t L^2_x} + \|v\|_{K^{s+1}} \lesssim \|v\|_{K^{s+1}},
\]
(4.58)
since \(3/2 < s < 5/2\). Note that from (4.48)–(4.53) we infer that the difference \(u = v - w\) is a solution of the system (4.1)–(4.5). From (4.47) and (4.56)–(4.58) it follows that
\[
\|u\|_{K^{s+1}} \lesssim \|w\|_{K^{s+1}} + \|v\|_{K^{s+1}} \lesssim \|\nabla u\|_{H^{(s-1)/2-1/4,s-1/2}_t} + \|u_0\|_{H^s} + \|f\|_{K^{s-1}},
\]
concluding the proof of (4.46). □

5. Solution to a parabolic-wave system

In this section, we consider the coupled parabolic-wave system
\[
\begin{align*}
\partial_t v - \lambda R \div (\nabla v + (\nabla v)^T) - \mu R \nabla \div v + R \nabla (R^{-1}) &= f \quad \text{in } (0, T) \times \Omega_t, \\
R_t - R \div v &= 0 \quad \text{in } (0, T) \times \Omega_t, \\
\partial_t w - \Delta w &= 0 \quad \text{in } (0, T) \times \Omega_e,
\end{align*}
\]
(5.1)
(5.2)
(5.3)
with the boundary conditions
\[
\begin{align*}
v &= \partial_t w \quad \text{on } (0, T) \times \Gamma_c, \\
\lambda (\partial_k v_j + \partial_j v_k) \nu^k + \mu \partial_k v_k \nu^j &= \partial_k w_j \nu^k + R^{-1} \nu^j + h_j \quad \text{on } (0, T) \times \Gamma_c, \\
v, w \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \\
v &= 0 \quad \text{on } (0, T) \times \Gamma_t,
\end{align*}
\]
(5.4)
(5.5)
(5.6)
(5.7)
for \(j = 1, 2, 3\), and the initial data
\[
\begin{align*}
(v, R, w, w_t)(0) &= (v_0, R_0, w_0, w_t) \quad \text{in } \Omega_t \times \Omega_t \times \Omega_e \times \Omega_e, \\
(v_0, R_0, w_0, w_t) &= \text{periodic in the } y_1 \text{ and } y_2 \text{ directions}, \\
w_0 &= 0.
\end{align*}
\]
(5.8)

To provide the maximal regularity for the system (5.1)–(5.8), we state the following necessary a priori density estimates.

**Lemma 5.1.** Let \(s \in (2, 2 + \epsilon_0]\) where \(\epsilon_0 \in (0, 1/2)\). Consider the ODE system
\[
\begin{align*}
R_t - R \div v &= 0 \quad \text{in } (0, T) \times \Omega_t, \\
R(0) &= R_0 \quad \text{on } \Omega_t.
\end{align*}
\]
(5.9)
(5.10)
Assume \((R_0, R_0^{-1}) \in H^s(\Omega_t) \times H^s(\Omega_t)\) and \(\|v\|_{K^{s+1}} \leq M\) for some constants \(T > 0\) and \(M > 1\). Then there exists a constant \(T_0 \in (0, T)\) depending on \(M\) such that we have
\[
\begin{align*}
(i) \quad &\|R\|_{L^\infty_t L^\infty_x} \lesssim 1 \quad \text{and} \quad \|R^{-1}\|_{L^\infty_t L^\infty_x} \lesssim 1 \quad \text{in } [0, T_0] \times \Omega_t, \\
(ii) \quad &\|R\|_{L^\infty_t H^s_x} + \|R\|_{H^s_t L^\infty_x} + \|R^{-1}\|_{L^\infty_t H^s_x} + \|R^{-1}\|_{H^s_t L^\infty_x} \lesssim 1, \\
(iii) \quad &\|R\|_{H^s_t H^s_x} \lesssim M.
\end{align*}
\]
We emphasize that the implicit constants in the above inequalities are independent of \(M\).
**Proof of Lemma 5.1.** (i) The solution of the ODE system (5.9)–(5.10) reads
\[ R(t, x) = R_0(x) e^{\int_0^t \text{div} v(\tau) d\tau} \text{ in } [0, T] \times \Omega, \]
from which it follows that
\[ \| R \|_{L^\infty_t L^\infty_x} \lesssim \| R_0 \|_{H^s} e^{\int_0^t \| \text{div} v(\tau) \|_{L^\infty_x} d\tau} \lesssim e^{T^{1/2} M} \lesssim 1 \]
and
\[ \| R^{-1} \|_{L^\infty_t L^\infty_x} \lesssim \| R_0^{-1} \|_{H^{-s}} e^{\int_0^t \| \text{div} v(\tau) \|_{L^\infty_x} d\tau} \lesssim e^{T^{1/2} M} \lesssim 1, \]
by taking \( T_0 > 0 \) sufficiently small.
(ii) From (5.11) we have
\[ \| R \|_{L^\infty_t H^s_x} \lesssim \| R_0 \|_{H^s} e^{\int_0^t \| \text{div} v(\tau) \|_{L^\infty_x} d\tau} \lesssim 1 \]
and
\[ \| R^{-1} \|_{L^\infty_t H^s_x} \lesssim \| R_0^{-1} \|_{H^{-s}} e^{\int_0^t \| \text{div} v(\tau) \|_{L^\infty_x} d\tau} \lesssim 1, \]
by taking \( T_0 > 0 \) sufficiently small. From (5.9), using Hölder’s and Sobolev inequalities, we obtain
\[ \| R_\epsilon \|_{L^2_t L^\infty_x} \lesssim \| R \|_{L^\infty_t L^\infty_x} \| v \|_{L^2_t L^\infty_x} \lesssim \epsilon \| v \|_{H^s_{x+1/2}} + C \| v \|_{L^2_t L^2_x}, \]
for any \( \epsilon \in (0, 1] \), since \( s \geq 2 \). It then follows
\[ \| R_\epsilon \|_{L^2_t L^\infty_x} \lesssim 1 + T^{1/2} Q(M) \| v \|_{L^\infty_t L^2_x} \lesssim 1 + T^{1/2} Q(M), \]
and
\[ \| (R^{-1})_\epsilon \|_{L^2_t L^\infty_x} \lesssim \| R \|_{L^\infty_t L^\infty_x} \lesssim 1 + T^{1/2} Q(M), \]
by taking \( \epsilon = 1/M \), since \( \| v \|_{L^2_t L^\infty_x} \lesssim T^{1/2} \| v \|_{L^\infty_t L^\infty_x} \lesssim T^{1/2} \| v \|_{H^{s+1/2}/L^\infty_x}. \) In (5.13)–(5.14) and below, \( Q \) denotes a generic nondecreasing function which may vary from inequality to inequality. Combining (5.12)–(5.14), we conclude the proof of (iii) by taking \( T_0 > 0 \) sufficiently small.
(iii) From (5.9) and (5.12), we have
\[ \| R_\epsilon \|_{L^2_t H^s_x} \lesssim \| R \|_{L^\infty_t H^s_x} \| v \|_{L^\infty_t H^s_x} \lesssim \| v \|_{L^2_t H^{s+1}} \lesssim M, \]
where we used Hörmander’s inequality.

The following theorem provides the local existence for the parabolic-wave system (5.1)–(5.8).

**Theorem 5.2.** Let \( s \in (2, 2 + \epsilon_0) \) where \( \epsilon_0 \in (0, 1/2) \). Assume the compatibility conditions
\[
\begin{align*}
 w_{1,i} &= v_{0,j} & & \text{on } \Gamma_c, \\
 v_{0,j} &= 0 & & \text{on } \Gamma_f, \\
 \lambda(\partial_k v_{0,j} + \partial_{ij} v_{0,k}) \nu^k + \mu \partial_k v_{0,j} \nu^j - R_0^{-1} \nu^j - \partial_k w_{0,j} \nu^k &= 0 & & \text{on } \Gamma_c, \\
 \lambda \partial_k (\partial_k v_{0,j} + \partial_{ij} v_{0,k}) + \mu \partial_{ij} v_{0,k} - \partial_k (R_0^{-1}) &= 0 & & \text{on } \Gamma_f.
\end{align*}
\]
Suppose that
\[
(v_0, w_0, w_1, R_0^{-1}, R_0) \in H^s(\Omega_f) \times H^{s+1/2}(\Omega_e) \times H^{s-1/2}(\Omega_e) \times H^{s}(\Omega_f) \times H^{s}(\Omega_f),
\]
and the nonhomogeneous terms satisfy
\[
(f, h) \in K^{s-1}((0, T) \times \Omega_f) \times H^{s-1/2} s/2+1/4((0, T) \times \Gamma_c),
\]
where \( T > 0 \). Then the system (5.1)–(5.8) admits a unique solution
\[
(v, R, w, w_1) \in K^{s+1}((0, T_0] \times \Omega_f) \times H^{1} r(0, T_0] \times H^{s}(\Omega_f))
\]
\[
\times C((0, T_0], H^{s+1/4-\epsilon_0}(\Omega_e)) \times C((0, T_0], H^{s-3/4-\epsilon_0}(\Omega_e)),
\]
for some constant \( T_0 \in (0, T) \), where the corresponding norms are bounded by a function of the initial data.
We define
\[ Z_T = \{ v \in K^{s+1} : v(0) = v_0 \text{ in } \Omega_t, v = 0 \text{ on } (0, T) \times \Gamma_t, \]
\[ v \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions, and } \|v\|_{K^{s+1}} \leq M \}, \quad (5.15) \]
where \( M > 1 \) and \( T \in (0, 1) \) are constants, both to be determined below. For any \( v \in Z_T \), we first obtain the solution \( R \) by using the explicit formula (5.11) with the initial data \( R(0) = R_0 \). Then we solve the wave equation (5.3) for \( w \) with the boundary condition and the initial data
\[ w(t) = w(0) + \int_0^t v(\tau)d\tau \quad \text{on } (0, T) \times \Gamma_c, \quad (5.16) \]
\[ (w, w_t)(0) = (w_0, w_1) \quad \text{in } \Omega_c. \quad (5.17) \]
With \( (R, w) \) constructed, we define a mapping
\[ \Lambda : v(\in Z_T) \mapsto \bar{v}, \]
where \( \bar{v} \) is the solution of the nonhomogeneous parabolic problem
\[ \partial_t \bar{v} - \lambda R \text{ div}(\nabla \bar{v} + (\nabla \bar{v})^T) - \mu R \nabla \text{ div } \bar{v} = f - R \nabla R^{-1} \in (0, T) \times \Omega_t, \quad (5.18) \]
with the boundary conditions and the initial data
\[ \lambda (\partial_k \bar{v}_j + \partial_j \bar{v}_k) \nu^k + \mu \partial_k \bar{v}_k \nu^j = \partial_k w_j \nu^k + R^{-1} \nu^j + h_j \quad \text{on } (0, T) \times \Gamma_c, \quad (5.19) \]
\[ \bar{v} = 0 \quad \text{on } (0, T) \times \Gamma_t, \quad (5.20) \]
\[ \bar{v} \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions,} \quad (5.21) \]
\[ \bar{v}(0) = v_0 \quad \text{in } \Omega_t, \quad (5.22) \]
for \( j = 1, 2, 3 \). We shall prove that \( \Lambda \) is a contraction mapping and use the Banach fixed-point theorem.

5.1. Uniform boundedness of the iterative sequence. In this section we show that the mapping \( \Lambda \) is well-defined from \( Z_T \) to \( Z_T \), for some sufficiently large constant \( M > 1 \) and sufficiently small constant \( T_0 \in (0, 1) \). We emphasize that the implicit constants below in this section are independent of \( M \). From Lemmas 4.2 and 5.1 it follows that
\[ \|\bar{v}\|_{K^{s+1}} \lesssim \left\| \frac{\partial w}{\partial \nu} \right\|_{H^{s-1/2}_{\Gamma_t}} + \|h\|_{H^{s-1/2}_{\Gamma_t}} + \|v_t\|_{H^s} + \|f\|_{K^{s+1}} + \|R^{-1} \nabla R\|_{K^{s-1}} \quad (5.23) \]
for a sufficiently small constant \( T_0 > 0 \) depending on \( M \).

For the first term on the right side of (5.23), we appeal to Lemma 3.4 and obtain
\[ \left\| \frac{\partial w}{\partial \nu} \right\|_{L^2_{\Gamma_t}H^{s-1/2}(\Gamma_c)} \lesssim \|w_0\|_{H^{s+1/2}} + \|w_t\|_{H^{s-1/2}} + \|w\|_{L^2 H^{s+1/2}(\Gamma_c)} + \|w\|_{H^{s+1/2} H^{s+3/2}(\Gamma_c)} \quad (5.24) \]
where we used the trace inequality. Regarding the third term on the far right side, using (5.16), we get
\[ \|w\|_{L^2_{\Gamma_t}H^{s+1/2}} \lesssim T^{1/2} \left( \int_0^t \|v\|_{H^{s+1/2}} \right) \lesssim T \int_0^t \|v\|_{L^2_{\Gamma_t}H^{s+1/2}} \lesssim T M, \quad (5.25) \]
while for the fourth term, we appeal to Lemma 3.2, obtaining
\[ \|v\|_{H^{s-3/4}_{\Gamma_t} H^{s+3/4}_{\Gamma_c}} \lesssim \epsilon_1 \|v\|_{H^{s+1/2}_{\Gamma_c}} + C_{\epsilon_1} \|v\|_{L^2_{\Gamma_t}H^{s+1/2}(\Gamma_c)} \quad (5.26) \]
for any $\epsilon_1, \epsilon_2 \in (0, 1]$, since $s < 7/2$ and $\|v\|_{L^2_t L^2} \lesssim T^{1/2} \|v\|_{H^{s+1/2}_t L^2}$. From (5.24)–(5.26) it follows that

$$\left\| \frac{\partial w}{\partial \tau} \right\|_{L^2_t H^{s-1/2}(\Gamma_\tau)} \lesssim \|w_0\|_{H^{s+1/2}} + \|w_1\|_{H^{s-1/2}} + (\epsilon_1 + C_\epsilon \epsilon_2 + C_\epsilon T^{1/2}) M. \tag{5.27}$$

For the time component of the first term on the right side of (5.23), we use the trace inequality and arrive at

$$\left\| \frac{\partial w}{\partial \tau} \right\|_{H^{s+1/2} - 1/4 L^2(\Gamma_\tau)} \lesssim \|\nabla v\|_{L^2 L^2(\Gamma_\tau)} + \left\| \frac{\partial w}{\partial \tau} \right\|_{L^2 L^2(\Gamma_\tau)} \lesssim \|v\|_{L^2_t H^{s+1/2 + 3/4}} + \left\| \frac{\partial w}{\partial \tau} \right\|_{L^2_t H^{s-1/2}} \tag{5.28}$$

since $1/2 < s < 5/2$, where we used (5.26)–(5.27). From (5.27)–(5.28), we have

$$\left\| \frac{\partial w}{\partial \tau} \right\|_{H^{s+1/2, -1/4, -1/2}} \lesssim \|w_0\|_{H^{s+1/2}} + \|w_1\|_{H^{s-1/2}} + (\epsilon_1 + C_\epsilon \epsilon_2 + C_\epsilon T^{1/2}) M. \tag{5.29}$$

On the other hand, using Lemma 3.3, we have the interior regularity estimate

$$\|w\|_{C([0, T], H^{s+1/4-\epsilon_0}(\Omega))} + \|w_t\|_{C([0, T], H^{s-3/4-\epsilon_0}(\Omega))} \lesssim \|w_0\|_{H^{s+1/4-\epsilon_0}} + \|w_1\|_{H^{s-3/4-\epsilon_0}} + \|w\|_{H^{1/4-\epsilon_0, s+1/4-\epsilon_0}}. \tag{5.30}$$

For the last term on the right side, we use Lemma 3.1 to get

$$\|w\|_{H^{s+1/4-\epsilon_0, s+1/4-\epsilon_0}} \lesssim \|v\|_{H^{s+1/4-\epsilon_0, 0}(\Gamma_\tau)} + \|w_t\|_{L^2 L^2(\Gamma_\tau)} + \|w\|_{L^2_t H^{s+1/4-\epsilon_0}} \lesssim (\epsilon_1 + \epsilon_2 C_\epsilon + C_\epsilon T^{1/2}) M, \tag{5.31}$$

since $s - 3/4 - \epsilon_0 \leq s/2 + 1/4$ and $s \leq 2 + 2\epsilon_0$, where we used (5.25). From (5.30)–(5.31), it follows that

$$\|w\|_{C([0, T], H^{s+1/4-\epsilon_0}(\Omega))} + \|w_t\|_{C([0, T], H^{s-3/4-\epsilon_0}(\Omega))} \lesssim \|w_0\|_{H^{s+1/4}} + \|w_1\|_{H^{s-3/4}} + (\epsilon_1 + \epsilon_2 C_\epsilon + C_\epsilon T^{1/2}) M. \tag{5.32}$$

For the space component of the norm of the fifth term on the right side of (5.23), using Lemma 5.1, we arrive at

$$\|R^{-1} \nabla R\|_{L^2_t H^{s-1}} \lesssim T^{1/2} \|R^{-1} \|_{L^\infty_t H^{s-1}} \|\nabla R\| \lesssim T^{1/2} Q(M), \tag{5.33}$$

where we appealed to H"{o}lder’s inequality. For the time component, we get

$$\|R^{-1} \nabla R\|_{H^{s+1/2} L^2} \lesssim \|R^{-1} \nabla R\|_{H^{1/2} L^2} \lesssim \|R^{-1} \nabla R\lfloor_{L^2_t L^2} + \|R^{-1} \nabla R\|_{L^2_t L^2}, \tag{5.34}$$

since $s < 3$. Regarding the first term on the far right side of (5.34), using H"{o}lder’s and Sobolev inequalities, together with an application of Lemma 5.1, we obtain

$$\langle (R^{-1} \nabla R)_{t} \rangle_{L^2_t L^2} \lesssim \|R_0 \nabla R\|_{L^2_t L^2} + \|\nabla R_t\|_{L^2_t L^2} \lesssim \|\nabla v\|_{L^2_t L^2} \lesssim \|R_0 \|_{L^\infty_t L^2} \|v\|_{L^\infty_t L^2} \|L^\infty_t L^2 \| \int_0^t \|\nabla v\|_{L^\infty_t L^2} \|L^\infty_t L^2 \| d\tau \tag{5.35}$$

for any $\epsilon \in (0, 1]$.

For the space component of the last term on the right side of (5.23), we use the trace inequality to obtain

$$\|R^{-1} v\|_{L^2_t H^{s-1/2}(\Gamma_\tau)} \lesssim \|R^{-1} \|_{L^2_t H^{s}} \lesssim T^{1/2}, \tag{5.36}$$
where the last inequality follows from Lemma 5.1, while for the time component, we get
\[
\norm{R^{-1} v}_{H^s \mathbb{C}^{-1/4} L^2} \lesssim \norm{R^{-1} v}_{H^s H^s} \lesssim \norm{R^{-1} v}_{L^2 H^s} + \norm{R^{-2} R v}_{L^2 H^s} \lesssim T^{1/2} \norm{R^{-1} v}_{L^\infty H^s} + \norm{R^{-1} v}_{L^\infty H^s} \\text{div} v_{L^2 H^s} \tag{5.37}
\]

since \( s/2 - 1/4 \leq 1 \).

Combining (5.23), (5.29), and (5.32)–(5.37), we arrive at
\[
\norm{\tilde{v}}_{K, s+1} + \norm{w}_{C([0, T], H^{s+1\rho - 1/2})(\Omega_{\epsilon})} + \norm{w_t}_{C([0, T], H^{s+1\rho - 1/2})(\Omega_{\epsilon})} \\
\lesssim \norm{h}_{H^{s+1/2, s/2-1/4}} + \norm{v_0}_{H^s} + \norm{f}_{K, s+1} + \norm{u_0}_{H^{s+1/2}} + \norm{u_1}_{H^{s-1/2}} \\
+ (\epsilon + \epsilon_1 + \epsilon_2 C_{\epsilon_1} + C_{\epsilon_1, \epsilon_2} T^{1/2})Q(M),
\tag{5.38}
\]

for any \( \epsilon, \epsilon_1, \epsilon_2 \in (0, 1] \). Taking \( \epsilon, \epsilon_1, \epsilon_2 \), and \( T_0 > 0 \) sufficiently small, it follow that
\[
\norm{\tilde{v}}_{K, s+1} \leq M, \tag{5.39}
\]

by allowing \( M \geq 1 \) sufficiently large. Thus, we have shown that the mapping \( \Lambda : v \mapsto \tilde{v} \) is well-defined from \( Z_T \) to \( Z_T \), for some \( M \geq 1 \) as in (5.39) and some sufficiently small \( T_0 > 0 \).

### 5.2. Contracting property

In this section we shall prove
\[
\norm{\Lambda(v_1) - \Lambda(v_2)}_{K, s+1} \leq \frac{1}{2} \norm{v_1 - v_2}_{K, s+1}, \quad v_1, v_2 \in Z_T, \tag{5.40}
\]

where \( M \geq 1 \) is fixed as in (5.39) and \( T_0 \in (0, 1) \) is sufficiently small to be determined below. We emphasize that the implicit constants below are allowed to depend on \( M \). Let \( v_1, v_2 \in Z_T \) and \( (R_1, \xi_1, \tilde{v}_1, \bar{v}_1) \) and \( (R_2, \xi_2, \tilde{v}_2, \bar{v}_2) \) be the corresponding solutions of (5.2)–(5.3), (5.16)–(5.17), and (5.18)–(5.21) with the same initial data \( (R_0, u_0, w_0, v_0, v_0) \) and the same nonhomogeneous terms \( (f, h) \). We denote \( \tilde{V} = \tilde{v}_1 - \bar{v}_2, \tilde{v} = v_1 - v_2, \tilde{R} = R_1 - R_2 \), and \( \tilde{\xi} = \xi_1 - \xi_2 \). The difference \( \tilde{V} \) satisfies
\[
\lambda (\partial_k \tilde{V}_j + \partial_j \tilde{V}_k) \nu^j + \mu \partial_k \tilde{V}_k \nu^j = \partial_k \tilde{\xi} \nu^k - R_1^{-1} R_2^{-1} \tilde{R} \nu^j \quad \text{on} \ (0, T) \times \Gamma_c, \]
\[
\tilde{V} = 0 \quad \text{on} \ (0, T) \times \Gamma_t, \]
\[
\tilde{V} \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \]
\[
\tilde{V}(0) = 0 \quad \text{in} \ \Omega_t, \]

for \( j = 1, 2, 3 \), where
\[
g = -R_1 \nabla R_1^{-1} + R_2 \nabla R_2^{-1} + \lambda \tilde{R} \div (\nabla \bar{v}_2 + (\nabla \bar{v}_2)^T) + \mu \tilde{R} \nabla \div \bar{v}_2. \tag{5.41}
\]

**Proof of Theorem 5.2.** We proceed as in (5.23) to obtain
\[
\norm{\tilde{V}}_{K, s+1} \lesssim \norm{\frac{\partial \tilde{\xi}}{\partial v}}_{H^{s+1/4, s+1/2}} + \norm{\tilde{R} R_1^{-1} R_2^{-1} \nabla R_2}_{K, s+1} + \norm{R_1^{-1} \nabla \tilde{R}}_{K, s+1} + \norm{\tilde{R} \nabla^2 \bar{v}_2}_{K, s+1} \\
+ \norm{R_1^{-1} R_2^{-1} \tilde{R} \nu}_{H^{s+1/4, s+1/2}} \tag{5.42}
\]

where the last inequality follows from (5.41). The difference \( \tilde{\xi} \) satisfies the wave equation
\[
\tilde{\xi}_{tt} - \Delta \tilde{\xi} = 0 \quad \text{in} \ (0, T) \times \Omega_c,
\]

with the boundary condition and the initial data
\[
\tilde{\xi}(t) = \tilde{\xi}(0) + \int_0^t \bar{v}(\tau) d\tau \quad \text{on} \ (0, T) \times \Gamma_c,
\]
\[(\tilde{\zeta}, \tilde{\zeta}_t)(0) = (0, 0) \quad \text{in} \quad \Omega_t.\]

For the first term on the right side of (5.42), we proceed as in (5.24)–(5.29) to obtain
\[
\left\| \frac{\partial \zeta}{\partial \nu} \right\|_{H^{s-1/4, s-1/2}_r} \lesssim (\epsilon_1 + \epsilon_2 C\epsilon_1 + C\epsilon_1 T^{1/2}) \left\| \tilde{v} \right\|_{K^{s+1}},
\]
for any \(\epsilon_1, \epsilon_2 \in (0, 1] \). Since the difference \(\tilde{R}\) satisfies the ODE system
\[
\tilde{R}_t - \tilde{R} \text{ div } v_2 = R_1 \text{ div } \tilde{v} \quad \text{in} \quad (0, T) \times \Omega_t,
\]
\[
\tilde{R}(0) = 0 \quad \text{in} \quad \Omega_t,
\]
the solution is given by
\[
\tilde{R}(t, x) = e^{\int_0^t \text{div } v_2(\tau) d\tau} \int_0^t e^{-\int_0^s \text{div } v_2(\tau) d\tau} R_1(\tau) d\tau.
\]

For the second term on the right side of (5.42), we obtain
\[
\left\| \tilde{R} R^{-1}_1 R^{-1}_2 \nabla R_2 \right\|_{L^2_t H^{s-1}_r} \lesssim T^{1/2} \left\| \tilde{R} \right\|_{L^\infty_t H^s_r} \left\| R^{-1}_1 \right\|_{L^\infty_t H^s_r} \left\| R^{-1}_2 \right\|_{L^\infty_t H^s_r} \left\| R_2 \right\|_{L^\infty_t H^s_r} \lesssim T^{1/2} \left\| \tilde{v} \right\|_{K^{s+1}},
\]
and
\[
\left\| (\tilde{R} R^{-1}_1 R^{-1}_2 \nabla R_2)_t \right\|_{L^2_t L^2_r} \lesssim \left\| \tilde{R} \right\|_{L^\infty_t L^\infty_r} \left\| \nabla R_2 \right\|_{L^2_t L^2_r} \left\| \nabla R_2 \right\|_{L^2_t L^2_r} + \left\| \tilde{R} R^{-1}_1 R^{-1}_2 \nabla R_2 \right\|_{L^2_t L^2_r} \left\| \nabla R_2 \right\|_{L^2_t L^2_r} + \left\| \tilde{R} R^{-1}_1 R^{-1}_2 \nabla R_2 \right\|_{L^2_t L^2_r} \left\| \nabla R_2 \right\|_{L^2_t L^2_r} \lesssim (\epsilon + C\epsilon T^{1/2}) \left\| \tilde{v} \right\|_{K^{s+1}},
\]
for any \(\epsilon \in (0, 1] \), where we used Hölder’s inequality, Lemma 5.1, (5.44), and (5.46).

For the third term on the right side of (5.42), we get
\[
\left\| R_1 \nabla \tilde{R} \right\|_{L^2_t H^{s-1}_r} \lesssim T^{1/2} \left\| R_1 \right\|_{L^\infty_t H^s_r} \left\| \tilde{R} \right\|_{L^\infty_t H^s_r} \lesssim T^{1/2} \left\| \tilde{v} \right\|_{K^{s+1}}
\]
and
\[
\left\| (R_1 \nabla \tilde{R})_t \right\|_{L^2_t L^2_r} \lesssim \left\| R_1 \nabla \tilde{R} \right\|_{L^2_t L^2_r} \left\| \nabla \tilde{R} \right\|_{L^2_t L^2_r} + \left\| R_1 \nabla \tilde{R} \right\|_{L^2_t L^2_r} \left\| \nabla \tilde{R} \right\|_{L^2_t L^2_r} \lesssim (\epsilon + C\epsilon T^{1/2}) \left\| \tilde{v} \right\|_{K^{s+1}},
\]
where we appealed to Lemma 5.1, (5.44), and (5.46).

Regarding the fourth term on the right side of (5.42), it follows that
\[
\left\| \tilde{R} \nabla^2 \tilde{v}_2 \right\|_{L^2_t H^{s-1}_r} \lesssim \left\| \tilde{R} \right\|_{L^\infty_t H^s_r} \left\| \nabla \tilde{v}_2 \right\|_{L^2_t H^{s+1}_r} \lesssim T^{1/2} \left\| \tilde{v} \right\|_{K^{s+1}}
\]
and
\[
\left\| \tilde{R} \nabla^2 \tilde{v}_2 \right\|_{H^{s-1}_r(\gamma)} \lesssim \left\| \tilde{R} \right\|_{W^{1, \infty}_{t, 2} L^\infty_r} \left\| \nabla \tilde{v}_2 \right\|_{H^{s-1}_r(\gamma)} + \left\| \tilde{R} \right\|_{W^{1, \infty}_{t, 2} L^\infty_r} \left\| \nabla \tilde{v}_2 \right\|_{H^{s-1}_r(\gamma)} \lesssim (\epsilon + C\epsilon T^{1/2}) \left\| \tilde{v} \right\|_{K^{s+1}},
\]
since \(s/2 - 1/4 \leq 1\) and \(1/4 \leq s/2 - 1/2\).

For the last term on the right side of (5.42), using the trace inequality, we arrive at
\[
\left\| R^{-1}_1 R^{-1}_2 \tilde{R} \right\|_{L^2_t H^{s-1/2}(\gamma)} \lesssim \left\| \tilde{R} \right\|_{L^2_t H^s_r} \lesssim T^{1/2} \left\| \tilde{v} \right\|_{K^{s+1}}
\]
(5.49)
and
\[ \| R_1^{-1} R_2^{-1} \dot{R}\|_{H^s R^2} \lesssim \| R_1^{-1} R_2^{-1} \dot{R}\|_{L^2 R^2} \lesssim \| (R_1^{-1} R_2^{-1} \dot{R})\|_{L^2 R^2} + \| R_1^{-1} R_2^{-1} \dot{R}\|_{L^2 R^2}, \]
(5.50)

since \( s/2 - 1/4 \leq 1 \). For the first term on the right side of (5.50), we proceed as in (4.48) to obtain
\[ \| (R_1^{-1} R_2^{-1} \dot{R})\|_{L^2 R^2} \lesssim \| R_{1t} \dot{R}\|_{L^2 R^2} + \| R_{2t} \dot{R}\|_{L^2 R^2} \lesssim \| \tilde{v}\|_{K^{s+1}}. \]
(5.51)
The second term on the right side of (5.50) is estimated analogously as in (5.49).

From (4.21)–(4.43) and (5.47)–(5.51) it follows that
\[ \| \tilde{V}\|_{K^{s+1}} \lesssim (\varepsilon + \varepsilon_1 + \varepsilon_2 + \varepsilon T^{1/2})|\tilde{v}|_{K^{s+1}}. \]
Taking \( \varepsilon, \varepsilon_1, \varepsilon_2 \), and \( T_0 > 0 \) sufficiently small, we conclude the proof of (5.40). Thus, the mapping \( \Lambda \) is contracting and from (5.38) and Lemma 5.1 it follows that there exists a unique solution
\[ (v, R, w, w_t) \in K^{s+1}(0, T_0) \times \Omega_t \times H^1((0, T_0), H^s(\Omega_t)) \times C((0, T_0), H^{s+1/4-\varepsilon}(\Omega_\varepsilon)) \times C((0, T_0), H^{s-3/4-\varepsilon}(\Omega_\varepsilon)), \]
for some \( T_0 > 0 \).

\[ \square \]

6. Solution to the Navier-Stokes-wave system

In this section, we provide the local existence for the coupled Navier-Stokes-wave system (2.3)–(2.5) with the boundary conditions (2.6)–(2.10) and the initial data (2.15). Let \( v \in Z_T \) where \( Z_T \) is as in (5.15), with constants \( M > 1 \) and \( T \in (0, 1) \) both to be determined below. Let \( \eta(t, x) = x + \int_0^t v(\tau, x) d\tau \) be the corresponding Lagrangian flow map and \( a(t, x) = (\nabla \eta(t, x))^{-1} \) be the inverse matrix of the flow map, while \( J(t, x) = \det(\nabla \eta(t, x)) \) denotes the Jacobian. First, we solve (2.3) for \( R \) with the initial data \( R(0) = R_0 \), obtaining
\[ R(t, x) = R_0(x) e^{\int_0^t a_{i,j}(\tau) \partial_i \eta_j(\tau) d\tau} \quad \text{ in } [0, T] \times \Omega_t. \]
Then we solve the wave equation (2.5) for \( w \) with the boundary condition
\[ w(t, x) = w(0) + \int_0^t v(\tau, x) d\tau \quad \text{ on } (0, T) \times \Gamma_\varepsilon \]
and the initial data
\[ (w, w_t)(0) = (w_0, w_1) \quad \text{ in } \Omega_\varepsilon. \]
With \( (R, w, \eta, J, a) \) constructed, we define a mapping
\[ \Pi : v \in Z_T \mapsto \tilde{v}, \]
where \( \tilde{v} \) is the solution of the nonhomogeneous parabolic problem
\[
\begin{align*}
    \partial_t v_j - \lambda R \partial_k (\partial_j v_k + \partial_k v_j) - \mu R \partial_j \partial_k v_k &= f_j \quad \text{in } (0, T) \times \Omega_t, \\
    \lambda (\partial_k v_j + \partial_j v_k) \nu^k + \mu \partial_k v_k \nu^j &= \partial_l w_j \nu^l + h_j \quad \text{in } (0, T) \times \Gamma_\varepsilon, \\
    \tilde{v} \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \\
    \tilde{v}(0) &= v_0 \quad \text{ in } \Omega_\varepsilon,
\end{align*}
\]
for \( j = 1, 2, 3 \). In (6.1)–(6.2), we denote
\[
\begin{align*}
f_j &= \lambda R b_k (b_{m,k} \partial_m v_j + b_{m,j} \partial_m v_k) + \lambda R b_k \partial_k (b_{m,l} \partial_m v_j + b_{m,j} \partial_m v_l) + \lambda R b_k \partial_k (\partial_l \tilde{v} + \partial_j \tilde{v}_l) \\
    &\quad + \mu R \partial_j (b_{m,l} \partial_m v_l) + \mu R b_{k,j} \partial_k (b_{m,l} \partial_m v_l) + \mu R b_{k,j} \partial_k \partial_l \tilde{v}_l - R b_{k,j} \partial_k R^{-1} - R \partial_j R^{-1} \\
    &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\end{align*}
\]
and
\[
\begin{align*}
h_j &= \lambda(1 - J)(\partial_t \tilde{v}_j + \partial_j \tilde{v}_k)\nu^k + \mu(1 - J)\partial_k \tilde{v}_j \nu^j - \lambda Jb_{kl}(b_{ml}\partial_m \tilde{v}_j + b_{mj}\partial_m \tilde{v}_l)\nu^k \\
&\quad + Jb_{kl}R^{-1}\nu^k + (J - 1)R^{-1}\nu^j - \lambda J(b_{mk}\partial_m \tilde{v}_j + b_{mj}\partial_m \tilde{v}_k)\nu^k - \lambda Jb_{kl}(\partial_t \tilde{v}_j + \partial_j \tilde{v}_l)\nu^k \\
&\quad - \mu Jb_{kl}b_{mj}\partial_m \tilde{v}_l \nu^j - \mu Jb_{mk}\partial_m \tilde{v}_l \nu^j + R^{-1}\nu^j \\
&=: K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8 + K_9 + K_{10} + K_{11},
\end{align*}
\]
for \( j = 1, 2, 3 \), where \( b_{ml} = a_{ml} - \delta_{ml} \) for \( m, l = 1, 2, 3 \).

Before we bound the terms in (6.5)–(6.6) and construct a contraction mapping as in Section 5, we provide some necessary a priori estimates on the variable coefficients.

### 6.1. The Lagrangian flow map, Jacobian matrix, and density estimates.

We start with the necessary Jacobian and the inverse matrix of the flow map estimates.

**Lemma 6.1.** Let \( v \in K^{s+1} \) with \( \eta(t, x) = x + \int_0^t v(\tau, x) \, d\tau \) the associated Lagrangian map. Denote \( b = a - I_3 \) and \( J = \det(\nabla \eta) \) where \( a = (\nabla \eta)^{-1} \). Suppose that \( |v|_{K^{s+1}} \leq M \) for some constants \( T > 0 \) and \( M > 1 \). Then for any \( \epsilon \in (0, 1] \), there exists a constant \( T_0 \in (0, T) \) depending on \( M > 1 \) such that the following statements hold:

(i) \( \|b\|_{L^\infty_t H^s} + \|b\|_{H^1_t L^\infty_x} + \|b\|_{H^1_t H^s} \lesssim \epsilon \),

(ii) \( \|b\|_{H^1_t H^s} \lesssim M \),

(iii) \( \|1 - J\|_{L^\infty_t H^s} \lesssim \epsilon \),

(iv) \( J(t, x) \lesssim 1 \) and \( J(t, x)^{-1} \lesssim 1 \), for any \( (t, x) \in [0, T_0] \times \Omega \),

(v) \( \|J\|_{L^\infty_t H^s} + \|J\|_{H^1_t L^\infty_x} + \|J^{-1}\|_{L^\infty_t H^s} + \|J^{-1}\|_{H^1_t L^\infty_x} \lesssim 1 \), and

(vi) \( \|J(1 - J)\|_{H^1_t H^1} \lesssim M \).

We emphasize that the implicit constants in the above inequalities are independent of \( M \).

**Proof of Lemma 6.1.** (i) From (2.11) we have
\[
-b_t = b(\nabla v)b + b \nabla v + (\nabla v)b + \nabla v.
\]
Note that \( b(0) = 0 \). From the fundamental theorem of calculus, it follows that
\[
\begin{align*}
\|b(t)\|_{H^s} &\lesssim \int_0^t \|b\|_{H^s}^2 \|\nabla v\|_{H^s} \, d\tau + \int_0^t \|b\|_{H^s} \|\nabla v\|_{H^s} \, d\tau + \int_0^t \|\nabla v\|_{H^s} \, d\tau \\
&\lesssim \epsilon_1 \int_0^t \|v\|_{H^{s+1}}^2 \, d\tau + C_\epsilon \int_0^t (\|b\|_{H^s}^4 + \|b\|_{H^s}^2) \, d\tau + T^{1/2} \int_0^t \|v\|_{H^{s+1}}^2 \, d\tau,
\end{align*}
\]
for any \( \epsilon_1 \in (0, 1] \). Using Gronwall’s inequality and taking \( \epsilon_1, T_0 > 0 \) sufficiently small, we arrive at
\[
\|b\|_{L^\infty_t H^s_x} \lesssim \epsilon. \tag{6.8}
\]
From (6.7), we use Hölder’s inequality to obtain
\[
\begin{align*}
\|b_t\|_{L^2_t L^\infty_x} &\lesssim \|\nabla v\|_{L^2_t L^\infty_x} \|b\|_{L^\infty_t L^\infty_x} + \|\nabla v\|_{L^2_t L^\infty_x} \|b\|_{L^\infty_t L^\infty_x} + \|\nabla v\|_{L^2_t L^\infty_x} \\
&\lesssim (\epsilon_1 + C_\epsilon T^{1/2})M \lesssim \epsilon
\end{align*}
\]
and
\[
\begin{align*}
\|b_t\|_{L^2_t H^s_x} &\lesssim \|\nabla v\|_{L^2_t H^s_x} \|b\|_{L^\infty_t H^s_x} + \|\nabla v\|_{L^2_t H^s_x} \|b\|_{L^\infty_t H^s_x} + \|\nabla v\|_{L^2_t H^s_x} \\
&\lesssim (\epsilon_1 + C_\epsilon T^{1/2})M \lesssim \epsilon, \tag{6.9}
\end{align*}
\]
by taking \( \epsilon_1 \) and \( T_0 > 0 \) sufficiently small. Combining (6.8)–(6.9), we conclude the proof of (i).

(ii) From (6.7) it follows that
\[
\|b_t\|_{L^2_t H^s_x} \lesssim \|\nabla v\|_{L^2_t H^s_x} \|b\|_{L^\infty_t H^s_x} + \|\nabla v\|_{L^2_t H^s_x} \|b\|_{L^\infty_t H^s_x} + \|\nabla v\|_{L^2_t H^s_x} \lesssim M,
\]
completing the proof of (ii).
(iii) Since the Jacobian matrix $J$ satisfies the ODE system (2.13)–(2.14), the solution is given by the explicit formula as

$$J(t, x) = e^{\int_0^t a_k \partial \phi_k \partial \phi_k \, dt}.$$ 

Thus, using the nonlinear estimate of the Sobolev norm, we arrive at

$$\|1 - J\|_{L^p H^s} \lesssim T^{1/2} Q(M) \lesssim \epsilon,$$

by taking $T_0 > 0$ sufficiently small.

(iv), (v), and (vi) are analogous to the proof of Lemma 5.1 using (i)–(iii). Thus we omit the details.

The following lemma provides the necessary a priori estimates.

**Lemma 6.2.** Let $s \in (2, 2 + \epsilon_0]$ where $\epsilon_0 \in (0, 1/2)$. Consider the ODE system

$$R_t - R a_{kj} \partial_k v_j = 0 \quad \text{in } (0, T) \times \Omega_f,$$

$$R(0) = R_0 \quad \text{on } \Omega_f.$$

Let $b_{kj} = a_{kj} - \delta_{kj}$ for $k, j = 1, 2, 3$. Assume

$$(R_0, R_0^{-1}, b) \in H^{s}(\Omega_f) \times H^{s}(\Omega_f) \times L^\infty([0, T], H^{s}(\Omega_f)),$$

and $\|v\|_{K,s+1} \leq M$, for some $T > 0$ and $M > 1$. Then there exists a constant $T_0 \in (0, T)$ depending on $M$ such that the following statements hold:

1. $\|R\|_{L^p H^s} \lesssim 1$ and $\|R^{-1}\|_{L^p H^s} \lesssim 1$ in $[0, T_0] \times \Omega_f$,
2. $\|R\|_{L^p H^s} + \|R\|_{H^s L^\infty} + \|R^{-1}\|_{L^p H^s} + \|R^{-1}\|_{H^s L^\infty} \lesssim 1$, and
3. $\|R\|_{H^s L^\infty} \lesssim M$.

We emphasize that the implicit constants in the above inequalities are independent of $M$.

The proof of Lemma 6.2 is analogous to the proof of Lemma 5.1. The modifications needed are the estimates of $b_{kj}$ which are provided in Lemma 6.1. Thus we omit the details.

**6.2. Uniform boundedness of the iterative sequence.** In this section we shall prove that the mapping $\Pi$ is well-defined from $Z_T$ to $Z_T$, for some sufficiently large $M > 1$ and sufficiently small $T \in (0, 1)$. From Lemma 4.2 and 6.2 it follows that

$$\|\tilde{v}\|_{K,s+1} \lesssim \|\partial v\|_{H^{s+1/2, -1/4, -1/2}} + \|h\|_{H^{s+1/2, -1/2, -1/4}} + \|v_0\|_{H^{s}} + \|f\|_{K^{s-1},}$$

(6.10)

for some constant $T \in (0, 1)$ depending on $M$, where $f$ and $h$ are as in (6.5)–(6.6). We emphasize that the implicit constants in this section are independent of $M$.

For the first term on the right side of (6.10), we proceed as in (5.23)–(5.32) to obtain

$$\|\tilde{v}\|_{K^{s+1}} + \|w\|_{C([0, T_0], H^{s+1/4,-\alpha_0}(\Omega_f))} + \|w_t\|_{C([0, T_0], H^{s+1/4,-\alpha_0}(\Omega_f))} \lesssim \|h\|_{H^{s+1/4,-1/2, -1/2}} + \|v_0\|_{H^{s}} + \|f\|_{K^{s-1}} + \|w_0\|_{H^{s+1/2}}$$

(6.11)

$$+ \|w_1\|_{H^{s+1/2}} + (\epsilon_1 + \epsilon_2 C_{\epsilon_1} + C_{\epsilon_1} T^{1/2}) M,$$

for any $\epsilon_1, \epsilon_2 \in (0, 1]$.

Next, we estimate the $K^{s+1}$ norm of the terms on the right side of (6.5) for $j = 1, 2, 3$. For the term $I_1$, we use Hölder’s inequality, we get

$$I_1 \|_{L^p H^{s+1}} \lesssim \|R \nabla b \nabla \tilde{v}\|_{L^p H^{s+1}} + \|R b \nabla^2 \tilde{v}\|_{L^p H^{s+1}} \lesssim \|R\|_{L^p H^s} \|b\|_{L^p H^s} \|\tilde{v}\|_{L^p H^{s+1}} \lesssim \epsilon \|\tilde{v}\|_{K^{s+1}}$$

(6.12)
and
\[
\|I_1\|_{H_t^{s-1/2}L_x^2} \lesssim \|R\nabla b \nabla \bar{v}\|_{H_t^{s-1/2}L_x^2} + \|Rb \nabla^2 \bar{v}\|_{H_t^{s-1/2}L_x^2} + \|R\|_{L_x^\infty L_t^\infty} \|\nabla \bar{v}\|_{H_t^{s-1/2}L_x^2} + \|R\|_{L_x^\infty L_t^{2/3}} \|\nabla \bar{v}\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4}
\]
\[
+ \|R\|_{L_x^\infty L_t^\infty} \|\nabla \bar{v}\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4} + \|R\|_{L_x^\infty L_t^\infty} \|b\|_{L_t^4 L_x^4} \|\nabla^2 \bar{v}\|_{H_t^{s-1/2}L_x^2}
\]
\[
+ \|R\|_{L_x^\infty L_t^{2/3}} \|b\|_{L_t^4 L_x^4} \|\nabla^2 \bar{v}\|_{L_t^4 L_x^4} + \|R\|_{L_x^\infty L_t^\infty} \|b\|_{L_t^4 L_x^4} \|\nabla^2 \bar{v}\|_{L_t^4 L_x^4}
\]
(6.13)
where we used Lemmas 6.1–6.2. From Lemma 3.2 and the Sobolev inequality it follows that
\[
\|\nabla \bar{v}\|_{L_t^4 L_x^4} \lesssim \|\bar{v}\|_{H_t^{s+1/2}H_x^{s+1/2}} \lesssim \epsilon \|\bar{v}\|_{H_t^{s-1/2}L_x^2} + C_\epsilon \|\bar{v}\|_{L_t^2 H_x^2} \lesssim \epsilon (1 + C_\epsilon T^{1/2}) \|\bar{v}\|_{K^{s+1}},
\]
for any \(\epsilon \in (0, 1]\). Combining (6.12)–(6.14), we arrive at
\[
\|I_1\|_{K^{s+1}} \lesssim M(\epsilon + C_\epsilon T^{1/2}) \|\bar{v}\|_{K^{s+1}},
\]
For the term \(I_2\), we use Hölder’s and Sobolev inequalities, obtaining
\[
\|I_2\|_{L_t^4 H_x^{s+1/2}} \lesssim \|Rb \nabla \bar{v}\|_{L_t^4 H_x^{s+1/2}} + \|Rb \nabla^2 \bar{v}\|_{L_t^4 H_x^{s+1/2}} \lesssim \|R\|_{L_t^\infty H_x^2} \|b\|_{L_t^\infty H_x^2} \|\nabla \bar{v}\|_{L_t^4 H_x^4} \|\nabla \bar{v}\|_{L_t^4 H_x^4} \lesssim \epsilon \|\bar{v}\|_{K^{s+1}}
\]
and
\[
\|I_2\|_{H_t^{s-1/2}L_x^2} \lesssim \|Rb \nabla \bar{v}\|_{H_t^{s-1/2}L_x^2} + \|Rb \nabla^2 \bar{v}\|_{H_t^{s-1/2}L_x^2} + \|R\|_{L_x^\infty L_t^\infty} \|\nabla \bar{v}\|_{H_t^{s-1/2}L_x^2} + \|R\|_{L_x^\infty L_t^{2/3}} \|b\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4}
\]
\[
+ \|R\|_{L_x^\infty L_t^\infty} \|b\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4} + \|R\|_{L_x^\infty L_t^\infty} \|b\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4}
\]
\[
+ \|R\|_{L_x^\infty L_t^{2/3}} \|b\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4} \|\nabla \bar{v}\|_{L_t^4 L_x^4} \lesssim M(\epsilon + C_\epsilon T^{1/2}) \|\bar{v}\|_{K^{s+1}},
\]
where the last inequality follows from Lemmas 6.1–6.2, (6.14), and the inequality
\[
\|\nabla^2 \bar{v}\|_{L_t^4 L_x^4} \lesssim \|\bar{v}\|_{H_t^{s+1/2}H_x^{s+1/2}} \lesssim \epsilon \|\bar{v}\|_{H_t^{s-1/2}L_x^2} + C_\epsilon \|\bar{v}\|_{L_t^2 H_x^2} \lesssim \epsilon (1 + C_\epsilon T^{1/2}) \|\bar{v}\|_{K^{s+1}}.
\]
The terms \(I_3, I_4, I_5,\) and \(I_6\) are estimated analogously as \(I_1\) and \(I_2\), and we get
\[
\|I_3\|_{K^{s+1}} + \|I_4\|_{K^{s+1}} + \|I_5\|_{K^{s+1}} + \|I_6\|_{K^{s+1}} \lesssim M(\epsilon + C_\epsilon T^{1/2}) \|\bar{v}\|_{K^{s+1}}.
\]
For the term \(I_7\), we obtain
\[
\|I_7\|_{L_t^4 H_x^{s+1/2}} \lesssim \|R^{-1} b \nabla R\|_{L_t^4 H_x^{s+1/2}} \lesssim \|R^{-1} \|L_t^\infty H_x^2\|b\|_{L_t^\infty H_x^2} \|R\|_{L_t^\infty H_x^2} \lesssim \epsilon
\]
and
\[
\|I_7\|_{H_t^{s-1/2}L_x^2} \lesssim \|R^{-1} b \nabla R\|_{H_t^{s-1/2}L_x^2} \lesssim \|R^{-1} \|H_t^1 L_x^2\|b\|_{L_t^\infty L_x^2} \|\nabla R\|_{L_t^\infty L_x^2} + \|R^{-1} \|L_t^\infty L_x^\infty\|b\|_{L_t^\infty L_x^\infty} \|\nabla R\|_{L_t^\infty L_x^\infty}
\]
\[
+ \|R^{-1} \|L_t^\infty L_x^\infty\|b\|_{H_t^1 L_x^2} \|\nabla R\|_{L_t^\infty L_x^\infty} \lesssim \epsilon M
\]
where we appealed to Lemmas 6.1–6.2. We proceed as in (5.33)–(5.35) to estimate the term \(I_8\), obtaining
\[
\|I_8\|_{K^{s+1}} \lesssim (\epsilon + C_\epsilon T^{1/2}) Q(M).
\]
Using the estimates on \(I_1–I_8\) in (6.5), it follows that
\[
\|f\|_{K^{s+1}} \lesssim (\epsilon + C_\epsilon T^{1/2}) Q(M) \|\bar{v}\|_{K^{s+1}} + (\epsilon + C_\epsilon T^{1/2}) Q(M).
\]
Next we estimate the \(H_t^{s-1/2}L_x^{2/3} \lesssim \|\nabla \bar{v}\|_{L_t^4 H_x^4} \|\nabla \bar{v}\|_{L_t^4 H_x^4} \|\nabla \bar{v}\|_{L_t^4 H_x^4} \|\nabla \bar{v}\|_{L_t^4 H_x^4}
\]
\[
|K_1|_{L_t^2 H_x^{s-1/2}L_x^2} \lesssim \|J \nabla \bar{v}\|_{L_t^2 H_x^2} \|J \nabla \bar{v}\|_{L_t^2 H_x^{s+1}} \lesssim \epsilon \|\bar{v}\|_{K^{s+1}}
\]
(6.15)
and

\[\|K_1\|_{H_t^{1/2-1/4}L^2(G_\Gamma)} \lesssim \| (1 - J) \nabla \tilde{v} \|_{H^{1/2}L^2_{x_t}} + \| (1 - J) \nabla \tilde{v} \|_{L^2H^2_t} \lesssim \| \nabla \tilde{v} \|_{H^{1/2}L^2_{x_t}} \| 1 - J \|_{L^\infty L^\infty_x} + \| \nabla \tilde{v} \|_{L^\infty L^2_x} \| 1 - J \|_{H^{1/2}L^\infty_x} + \| (1 - J) \nabla \tilde{v} \|_{L^2H^2_t}\]  

(6.16)

where we used Lemmas 3.1 and 6.1. From (2.13) and Lemma 6.1 it follows that

\[\|J\|_{H_t^{1/2-1/4}L^2(G_\Gamma)} \lesssim \| J b \nabla \tilde{v} \|_{H^{1/2-1/4}L^2_{x_t}} + \| J \div \tilde{v} \|_{H^{1/2-1/4}L^2_{x_t}} \lesssim \| J b \nabla \tilde{v} \|_{H^{1/2}L^\infty_x} \| \nabla \tilde{v} \|_{L^2L^\infty_x} + \| J \|_{L^\infty L^\infty_x} \| b \|_{W^{1/2-1/4, \infty} L^\infty_x} \| \nabla \tilde{v} \|_{L^2L^\infty_x} \] 

(6.17)

where we used Lemma 6.1 to bound the factor \( 1 - J \|_{L^1 L^\infty_x} \). The term \( K_2 \) is estimated analogously to \( K_1 \), and we arrive at

\[\|K_2\|_{H_t^{1/2-1/4,1/2}L^2(G_\Gamma)} \lesssim \| \epsilon \|_{H^{1/2}L^2_t} + C_t \| \tilde{v} \|_{L^2H^{1/2}t} \lesssim (\epsilon + C_t T^{1/2}) Q(M),\]  

(6.18)

Combining (6.15)–(6.17), we have

\[\|K_1\|_{H_t^{1/2-1/4}L^2(G_\Gamma)} \lesssim \| \nabla \tilde{v} \|_{H^{1/2}L^2_{x_t}} \| 1 - J \|_{L^\infty L^\infty_x} + \| \nabla \tilde{v} \|_{L^\infty L^2_x} \| 1 - J \|_{H^{1/2}L^\infty_x} + \| (1 - J) \nabla \tilde{v} \|_{L^2H^2_t} \lesssim (\epsilon + C_t T^{1/2}) Q(M) \] 

\[\|K_3\|_{H_t^{1/2-1/4}L^2(G_\Gamma)} \lesssim \| J b b \nabla \tilde{v} \|_{L^2H^2_t} \lesssim (\epsilon + C_t T^{1/2}) Q(M) \| \tilde{v} \|_{K^{1/2}} \]  

(6.19)

For the term \( K_3 \), we use the trace inequality to obtain

\[\|K_3\|_{H_t^{1/2-1/4}L^2(G_\Gamma)} \lesssim \| J b b \nabla \tilde{v} \|_{L^2H^2_t} \lesssim \| J \|_{L^\infty L^\infty_x} \| b \|_{L^2 H^{1/2}t} \| \tilde{v} \|_{L^2 H^{1/2}t} \lesssim \epsilon \| \tilde{v} \|_{K^{1/2}} \] 

and

\[\|K_3\|_{H_t^{1/2-1/4}L^2(G_\Gamma)} \lesssim \| J b b \nabla \tilde{v} \|_{H^{1/2}L^2_{x_t}} + \| J b b \nabla \tilde{v} \|_{L^2H^2_t} \lesssim \| J \|_{H^{1/2}L^\infty_x} \| b \|_{L^2 L^\infty_x} \| \nabla \tilde{v} \|_{L^2 L^\infty_x} + \| J \|_{L^\infty L^\infty_x} \| b \|_{H^{1/2}L^\infty_x} \| \nabla \tilde{v} \|_{L^2 L^\infty_x} \] 

(6.20)

where we appealed to Lemma 6.1. From (6.7) it follows that

\[\|K_3\|_{H_t^{1/2-1/4}L^2(G_\Gamma)} \lesssim \| J b b \nabla \tilde{v} \|_{H^{1/2}L^2_{x_t}} + \| J b b \nabla \tilde{v} \|_{H^{1/2}L^2_{x_t}} + \| \nabla \tilde{v} \|_{H^{1/2}L^2_{x_t}} \lesssim (\epsilon + C_t T^{1/2}) Q(M) \] 

(6.21)

where we used Lemma 6.1 and (6.18). Combining (6.17) and (6.20)–(6.21), we arrive at

\[\|K_3\|_{H_t^{1/2-1/4}L^2(G_\Gamma)} \lesssim (\epsilon + C_t T^{1/2}) Q(M) \| \tilde{v} \|_{K^{1/2}} \] 

(6.22)

where we used Lemma 3.2. Regarding the terms \( K_4 \) and \( K_5 \), we proceed as in (5.36)–(5.37), obtaining

\[\|K_4\|_{H_t^{1/2-1/4}L^2(G_\Gamma)} \lesssim \| J b R^{-1} \|_{L^2H^2_t} + \| J b R^{-1} \|_{H_t^{1/2}H^2_t} \] 

\[\lesssim \| J b R^{-1} \|_{L^2H^2_t} \| \tilde{v} \|_{L^\infty H^2_t} \| R^{-1} \|_{L^\infty H^2_t} + \| J \|_{L^2H^2_t} \| b \|_{L^\infty H^2_t} \| R^{-1} \|_{L^\infty H^2_t} \] 

\[+ \| J \|_{L^2H^2_t} \| b \|_{L^\infty H^2_t} \| R^{-2} \|_{L^2H^2_t} \] 

\[\lesssim (\epsilon + C_t T^{1/2}) M \]
and
\[
\|K_5\|_{H^{s-1/2}} \lesssim \|(J - 1)R^{-1}\|_{L^2} + \|(J - 1)R^{-1}\|_{H^s}
\]
\[
\lesssim T^{1/2}\|J - 1\|_{L^\infty} R^{-1}\|L^2\|_{H^s} + \|J\|_{L^2} R^{-1}\|L^2\|_{H^s}
\]
\[
+ \|J\|_{L^\infty} R^{-2} R_t\|L^2\|_{H^s}
\]
\[
\lesssim (\varepsilon + C\varepsilon T^{1/2})M,
\]
where the we appealed to Lemmas 6.1 and 6.2. The terms \(K_7, K_8, K_9,\) and \(K_{10}\) are estimated analogously to \(K_3,\) and we have
\[
\|K_7\|_{H^{s-1/2}} \lesssim \|K_8\|_{H^{s-1/2}} + \|K_9\|_{H^{s-1/2}} + \|K_{10}\|_{H^{s-1/2}}
\]
\[
\lesssim (\varepsilon + C\varepsilon T^{1/2})Q(M)\|\bar{v}\|_{K^{s-1}}.
\]
For the term \(K_{11},\) we proceed as in (5.36)–(5.37) to obtain
\[
K_{11} \lesssim (\varepsilon + C\varepsilon T^{1/2})Q(M).
\]
Collecting the above estimates, we arrive at
\[
\|\bar{v}\|_{K^{s-1}} \lesssim \|v_0\|_{H^s} + \|w_0\|_{H^{s+1/2}} + \|w_1\|_{H^{-1/2}} + (\varepsilon + C\varepsilon T^{1/2})Q(M)
\]
\[
+ (\varepsilon + C\varepsilon T^{1/2})Q(M)\|\bar{v}\|_{K^{s-1}}.
\]
Taking \(M > 1\) sufficiently large and \(\varepsilon, T > 0\) sufficiently small we have
\[
\|\bar{v}\|_{K^{s-1}} \leq M.
\]
Thus, the mapping \(\Pi: v \mapsto \bar{v}\) is well-defined from \(Z_T\) to \(Z_T,\) for some \(M > 1\) as in (6.22) and some sufficiently small \(T > 0.\)

6.3. Contracting property. In this section we shall prove
\[
\|\Pi(v_1) - \Pi(v_2)\|_{K^{s-1}} \leq \frac{1}{2}\|v_1 - v_2\|_{K^{s-1}}, \quad v_1, v_2 \in Z_T,
\]
where \(M > 1\) is fixed as in (6.22) and \(T \in (0, 1)\) is sufficiently small to be determined below. Note that the implicit constants below are allowed to depend on \(M.\)

Let \(v_1, v_2 \in Z_T\) and \((\eta_1, J_1, a_1)\) and \((\eta_2, J_2, a_2)\) be the corresponding Lagrangian flow map, Jacobian, and the inverse matrix of the flow map. First we solve the for \((R_1, R_2)\) from (2.3) with the same initial data \(R_0.\) Then we solve for \((\xi_1, \xi_{1t})\) and \((\xi_2, \xi_{2t})\) from (2.5) with the boundary conditions (2.6)–(2.7) and (2.10) and the same initial data \((w_0, w_1)\). To obtain the next iterate \((\bar{v}_1, \bar{v}_2),\) we solve (2.4) with the boundary conditions (2.6) and (2.9) and with the same initial data \(v_0.\) Denote \(b_1 = a_1 - I_3, b_2 = a_2 - I_3, \tilde{b} = b_1 - b_2, \tilde{V} = \bar{v}_1 - \bar{v}_2, \tilde{v} = v_1 - v_2, \tilde{R} = R_1 - R_2, \tilde{\xi} = \xi_1 - \xi_2, \tilde{\eta} = \eta_1 - \eta_2,\) and \(J = J_1 - J_2.\) The difference \(\tilde{V}\) satisfies
\[
\partial_t \tilde{V}_j - \lambda \tilde{R} \partial_k (\partial_j \tilde{V}_k + \partial_k \tilde{V}_j) - \mu \tilde{R} \partial_j \partial_k \tilde{V}_k = \tilde{f}_j \quad \text{in } (0, T) \times \Omega_t, \]
\[
\lambda (\partial_j \tilde{V}_j + \partial_j \tilde{V}_k) \nu^j + \mu \partial_k \tilde{V}_k \nu^j = \partial_t \tilde{\xi}_j \nu^j + \tilde{h}_j \quad \text{in } (0, T) \times \Gamma_c, \]
\[
\tilde{V} \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions},
\]
\[
\tilde{V}(0) = 0 \quad \text{in } \Omega_t,
\]
where

\[ \tilde{f}_j = \lambda \tilde{R} \partial_h (\partial_j \tilde{v}_{2k} + \partial_k \tilde{v}_{1j}) + \mu \tilde{R} \partial_j \partial_k \tilde{v}_{2k} + \lambda \tilde{R} \partial_h (b_{1mk} \partial_m \tilde{v}_{1j} + b_{1mj} \partial_m \tilde{v}_{1k}) \\
+ \lambda R_{2k} \partial_h (b_{1mk} \partial_m \tilde{V}_j + b_{1mj} \partial_m \tilde{V}_k) + \lambda R_{2k} \partial_h (b_{mkl} \partial_m \tilde{V}_{1j} + b_{mlj} \partial_m \tilde{V}_{1k}) \\
+ \lambda R_{2k} \partial_h (b_{1mk} \partial_m \tilde{V}_{1j} + b_{1mj} \partial_m \tilde{V}_{1k}) + \lambda R_{2k} \partial_h (b_{mkl} \partial_m \tilde{V}_{1j} + b_{mlj} \partial_m \tilde{V}_{1k}) \\
+ \lambda R_{2k} \partial_h (\partial_j \tilde{v}_{1j} + \partial_k \tilde{v}_{1k}) + \lambda R_{2k} \partial_h (\partial_j \tilde{v}_{1j} + \partial_k \tilde{v}_{1k}) + \lambda R_{2k} \partial_h (\partial_j \tilde{V}_j + \partial_k \tilde{V}_k) \\
+ \mu \tilde{R} \partial_j (b_{1mi} \partial_m \tilde{v}_{1j}) + \mu \tilde{R} \partial_j (b_{2m} \partial_m \tilde{V}_i) \\
+ \mu R_{1kj} \partial_h (b_{1mj} \partial_m \tilde{v}_{1j}) + \mu R_{2k} \partial_h (b_{1mj} \partial_m \tilde{v}_{1j}) + \mu R_{2k} \partial_h (b_{2m} \partial_m \tilde{V}_i) \\
+ \mu R_{2k} \partial_h (b_{2m} \partial_m \tilde{V}_i) + \mu R_{1kj} \partial_h (b_{1mj} \partial_m \tilde{v}_{1j}) + \mu R_{2k} \partial_h (b_{2m} \partial_m \tilde{V}_i) \\
- R^{-1}_{1} R^{-1} \tilde{R} b_{1kj} \partial_h \tilde{R} + R^{-1}_{2} b_{2kj} \partial_h \tilde{R} - R^{-1}_{2} R^{-1} \tilde{R} \partial_h \tilde{R} \\
+ R^{-1}_{2} \partial_j \tilde{R} \]

and

\[ \tilde{h}_j = -\lambda \tilde{J} (\partial_h \tilde{v}_{1j} + \partial_j \tilde{v}_{1k}) \nu^k + \lambda (1 - J_2) (\partial_k \tilde{V}_j + \partial_j \tilde{V}_k) \nu^k - \mu \tilde{J} \partial_h \tilde{v}_{1k} \nu^j + \mu (1 - J_2) \partial_h \tilde{V}_i \nu^j \\
+ \lambda \tilde{J} b_{1mj} (b_{1mi} \partial_m \tilde{v}_{1j} + b_{1mj} \partial_m \tilde{v}_{1k}) \nu^k + \lambda \tilde{J} b_{2k} (b_{2m} \partial_m \tilde{V}_i + b_{2m} \partial_m \tilde{V}_i) \nu^k \\
+ \lambda \tilde{J} b_{2k} (b_{2m} \partial_m \tilde{V}_i + b_{2m} \partial_m \tilde{V}_i) \nu^k + \lambda \tilde{J} b_{2k} (b_{2m} \partial_m \tilde{V}_i + b_{2m} \partial_m \tilde{V}_i) \nu^k \\
+ \lambda \tilde{J} b_{1mj} \tilde{v}_{1k} \nu^k - \lambda \tilde{J} b_{1kj} \tilde{v}_{1k} \nu^k - \lambda \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \lambda \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k \\
+ \lambda \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \lambda \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \lambda \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \lambda \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k \\
- \mu \tilde{J} b_{1mj} \tilde{v}_{1k} \nu^k - \mu \tilde{J} b_{1mj} \tilde{v}_{1k} \nu^k - \mu \tilde{J} b_{1mj} \tilde{v}_{1k} \nu^k - \mu \tilde{J} b_{1mj} \tilde{v}_{1k} \nu^k \\
- \mu \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \mu \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \mu \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \mu \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k \\
- \mu \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \mu \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \mu \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k - \mu \tilde{J} b_{2k} \tilde{v}_{1k} \nu^k \\
(6.25) \\
\text{for } j = 1, 2, 3. \]

Before we bound the terms on the right sides of (6.24) and (6.25), we provide necessary a priori estimates for the differences of the density, Jacobian, inverse matrix of the flow map.

**Lemma 6.3.** Let \( v_1, v_2 \in Z_T \). Suppose \( \| v_1 \|_{K^{s+1}} \leq M \) and \( \| v_2 \|_{K^{s+1}} \leq M \) for some \( T > 0 \), where \( M > 1 \) is fixed as in (6.22). Then for every \( \epsilon \in (0, 1] \), there exists a sufficiently small \( T_0 \in (0, 1) \) depending on \( M \) such that the following statements hold:

(i) \( \| \tilde{b} \|_{L^\infty H^s_2} + \| \tilde{b} \|_{H^s_1 L^\infty} \leq \epsilon \| \tilde{v} \|_{K^{s+1}} \),
(ii) \( \| \tilde{R} \|_{L^\infty H^s_2} + \| \tilde{R} \|_{H^s_1 L^\infty} \leq \epsilon \| \tilde{v} \|_{K^{s+1}} \),
(iii) \( \| \tilde{J} \|_{L^\infty H^s_2} \leq \epsilon \| \tilde{v} \|_{K^{s+1}} \), and
(iv) \( \| \tilde{R} \|_{H^s_1 H^s_2} + \| \tilde{b} \|_{H^s_1 H^s_2} + \| \tilde{J} \|_{H^s_1 H^s_2} \leq \| \tilde{v} \|_{K^{s+1}} \).

**Proof of Lemma 6.3.** (i) The difference \( \tilde{b} \) satisfies the ODE

\[ -\tilde{b}_t = \tilde{b}(\nabla v_1) b_1 + b_2(\nabla \tilde{v}) b_1 + b_2(\nabla v_2) \tilde{b} + (\nabla \tilde{v}) b_1 + (\nabla v_2) \tilde{b} + \tilde{b}(\nabla v_1) \\
+ b_2(\nabla \tilde{v}) + \nabla \tilde{v} \quad \text{in } (0, T) \times \Omega, \]

(6.26)
with the initial data \( b(0) = 0 \). From the fundamental theorem of calculus it follows that
\[
\|\tilde{b}(t)\|_{H^s} \lesssim \int_0^t \|\tilde{b}\|_{H^s} \|\nabla v_1\|_{H^s} \|b_1\|_{H^s} + \int_0^t \|b_2\|_{H^s} \|\nabla \tilde{v}\|_{H^s} \|b_1\|_{H^s} + \int_0^t \|\nabla \tilde{v}\|_{H^s}
\]
\[
+ \int_0^t \|b_2\|_{H^s} \|\nabla v_2\|_{H^s} \|\tilde{b}\|_{H^s} + \int_0^t \|\tilde{b}\|_{H^s} \|\nabla v_1\|_{H^s} + \int_0^t \|b_2\|_{H^s} \|\nabla \tilde{v}\|_{H^s}
\]
\[
\lesssim \int_0^t \|\nabla \tilde{v}\|_{H^s} + \int_0^t \|\tilde{b}\|_{H^s} (\|\nabla v_1\|_{H^s} + \|\nabla v_1\|_{H^s}),
\]
where the last inequality follows from Lemma 6.1. Using Gronwall’s inequality, we arrive at
\[
\|\tilde{b}\|_{L^\infty_t L^2_x} \lesssim \epsilon \|\tilde{v}\|_{K^{s+1}}. \tag{6.27}
\]
From (6.26) and Hölder’s inequality, we have
\[
\|\tilde{b}\|_{L^2_t L^\infty_x} \lesssim \|\tilde{b}\|_{L^2_t L^\infty_x} \|\nabla v_1\|_{L^2_t L^\infty_x} \|b_1\|_{L^\infty_t L^2_x} + \|b_2\|_{L^\infty_t L^2_x} \|\nabla \tilde{v}\|_{L^2_t L^2_x} \|b_1\|_{L^\infty_t L^2_x} + \|b_2\|_{L^\infty_t L^2_x} \|\nabla v_2\|_{L^2_t L^2_x} \|\tilde{b}\|_{L^2_t L^\infty_x}
\]
\[
+ \|\nabla v_2\|_{L^2_t L^\infty_x} \|b_2\|_{L^\infty_t L^2_x} \|\nabla \tilde{v}\|_{L^2_t L^2_x} \|b_1\|_{L^\infty_t L^2_x} \|\tilde{b}\|_{L^2_t L^\infty_x}
\]
\[
\lesssim (\epsilon + C_\epsilon T^{1/2}) \|\tilde{v}\|_{K^{s+1}} \lesssim \epsilon \|\tilde{v}\|_{K^{s+1}}, \tag{6.28}
\]
by taking \( T_0 > 0 \) sufficiently small.

(ii) Since the difference \( \tilde{R} \) satisfies the ODE
\[
\begin{align*}
\tilde{R}_t - \tilde{R}(\text{div } v_2 + b_{1\ell} \partial_\ell v_{1j}) &= R_1 \text{ div } \tilde{v} + R_2 b_{\ell k} \partial_\ell v_{1j} + R_2 b_{2k\ell} \partial_\ell \tilde{v}_j \quad \text{in } (0, T) \times \Omega_t, \tag{6.29} \\
\tilde{R}(0) &= 0 \quad \text{in } \Omega_t, \tag{6.30}
\end{align*}
\]
thus the solution is given as
\[
\tilde{R}(t, x) = e^{\int_0^t (\text{div } v_2 + b_{1\ell} \partial_\ell v_{1j})} \int_0^t e^{-\int_0^\tau (\text{div } v_2 + b_{1\ell} \partial_\ell v_{1j})} \times \text{ (6.26)} \times \text{ (6.27)} \tag{6.31}
\]
It follows that
\[
\|\tilde{R}\|_{L^\infty_t H^s_x} \lesssim \int_0^t \|\nabla \tilde{v}\|_{H^s} + \int_0^t \|\tilde{b}\|_{H^s} \|\nabla v_1\|_{H^s} \lesssim \epsilon \|\tilde{v}\|_{K^{s+1}}, \tag{6.31}
\]
where we used (6.27). Using (6.29), the Hölder’s, and Sobolev inequalities, we obtain
\[
\|\tilde{R}\|_{L^2_t L^\infty_x} \lesssim \|\tilde{R}\|_{L^\infty_t L^\infty_x} \|\nabla v_2\|_{L^2_t L^\infty_x} + \|\tilde{R}\|_{L^\infty_t L^2_x} \|b_1\|_{L^\infty_t L^2_x} \|\nabla \tilde{v}\|_{L^2_t L^2_x}
\]
\[
+ \|R_1\|_{L^\infty_t L^\infty_x} \|\nabla \tilde{v}\|_{L^2_t L^\infty_x} + \|R_2\|_{L^\infty_t L^2_x} \|b_2\|_{L^\infty_t L^2_x} \|\nabla v_1\|_{L^2_t L^2_x}
\]
\[
\lesssim (\epsilon + C_\epsilon T^{1/2}) \|\tilde{v}\|_{K^{s+1}} \lesssim \epsilon \|\tilde{v}\|_{K^{s+1}}, \tag{6.32}
\]
by taking \( T > 0 \) sufficiently small. Thus we conclude the proof of (ii) by combining (6.31)–(6.32).

(iii) The difference \( \tilde{J} \) satisfies the same ODE system (6.29)–(6.30) and from (ii) it follows that
\[
\|\tilde{J}\|_{H^s_x} \lesssim \epsilon \|\tilde{v}\|_{K^{s+1}}.
\]
(iv) The proof is analogous to (i)–(iii) using (6.28) and (6.32). Thus we omit the details. \qed
PROOF OF THEOREM 2.1. From Lemma 4.2 and 6.2 it follows that
\[ \|\hat{V}\|_{K^{s+1}} \lesssim \left\| \frac{\partial \hat{\varepsilon}}{\partial \nu} \right\|_{H_{x}\Gamma^{s-1/4, s-1/2}} + \|\hat{h}\|_{H_{x}\Gamma^{s-1/4, s-1/2}} + \|\hat{f}\|_{K^{s+1}}, \]  
(6.33)
where \(\hat{f}\) and \(\hat{h}\) are as in (6.24)–(6.25), for \(j = 1, 2, 3\).

We proceed as in (5.24)–(5.29) to bound the first term on the right side of (6.33), obtaining
\[ \left\| \frac{\partial \hat{\varepsilon}}{\partial \nu} \right\|_{H_{x}\Gamma^{s-1/4, s-1/2}} \lesssim (\epsilon_1 + \epsilon_2 C_\epsilon + C_{\epsilon_1} T^{1/2}) \|\hat{\varepsilon}\|_{K^{s+1}}, \]
for any \(\epsilon_1, \epsilon_2 \in (0, 1]\).

Next we estimate the \(K^{s-1}\) norm of terms on the right side of (6.24) for \(j = 1, 2, 3\). The term \(\tilde{R}b_{1kj}\partial_k(b_{1mi}\partial_m\tilde{v}_{1i})\) is bounded as
\[ \|\tilde{R}b_{1kj}\partial_k(b_{1mi}\partial_m\tilde{v}_{1i})\|_{L^2_x H^{s-1/2}_x} \lesssim \|\tilde{R}\|_{L^\infty_x L^\infty_t} \|b_{1}\|_{L^\infty_x L^\infty_t} \|\tilde{V}\|_{L^2_x H^s_x} \lesssim \epsilon \|\tilde{V}\|_{K^{s+1}}, \]
where we used Lemma 6.3. The term \(\mu R_2 b_{2kj}\partial_k \partial_t\tilde{V}_t\) is estimated as
\[ \|\mu R_2 b_{2kj}\partial_k \partial_t\tilde{V}_t\|_{L^2_x H^{s-1}_x} \lesssim \|R_2\|_{L^\infty_x L^\infty_t} \|b_{2}\|_{L^\infty_x L^\infty_t} \|\tilde{V}\|_{L^2_x H^s_x} \lesssim \epsilon \|\tilde{V}\|_{K^{s+1}}, \]
and
\[ \|\mu R_2 b_{2kj}\partial_k \partial_t\tilde{V}_t\|_{H^{s-1/2}_x L^2_x} \lesssim \|R_2\|_{W^{s-1/4, s-1/2}_x L^\infty_t} \|b_{2}\|_{L^\infty_x L^\infty_t} \|\tilde{V}\|_{L^4_x L^2_t} \lesssim \epsilon \|\tilde{V}\|_{K^{s+1}}, \]
where we used Lemma 6.1–6.2. Other terms on the right side of (6.24) are treated analogously as in Theorem 5.2 using Lemma 6.1–6.3, and we arrive at
\[ \|\hat{f}\|_{K^{s+1}} \lesssim \epsilon \|\hat{\varepsilon}\|_{K^{s+1}} + \epsilon \|\tilde{V}\|_{K^{s+1}}, \]
(6.34)
by taking \(T > 0\) sufficiently small.

Next we estimate the \(H^{s-1/4, s-1/2}_x\) norm of the terms on the right side of (6.25) for \(j = 1, 2, 3\). The term \(\lambda(1 - J_2)\partial_k\tilde{V}_k\mu^k\) is estimated as
\[ \|\lambda(1 - J_2)(\partial_k\tilde{V}_j + \partial_j\tilde{V}_k)\mu^k\|_{L^2_x H^{s-1/2}(\Gamma_1)} \lesssim \|1 - J_2\|_{L^\infty_x L^\infty_t} \|\tilde{V}\|_{L^2_x H^s_x} \lesssim \|\tilde{V}\|_{K^{s+1}}, \]
and
\[ \| \lambda (1 - J_2) (\partial_t \tilde{V}_j + \partial_j \tilde{V}_k) \|_{H_t^{1/2-1/4} L^2(\Gamma_\varepsilon)} \leq \| (1 - J_2) (\partial_t \tilde{V}_j + \partial_j \tilde{V}_k) \|_{H_t^{1/2-1/4} H_{t+1}^{1/2}} \]
\[ \leq \| 1 - J_2 \|_{H_t^1 H_t^1} \| \nabla \tilde{V} \|_{L^\infty_t H_t^{1/2}} + \| 1 - J_2 \|_{L^\infty_t H_t^1} \| \nabla \tilde{V} \|_{H_t^{1/2-1/4} H_{t+1}^{1/2}} \]
\[ \leq (\epsilon + C T^{1/2}) \| \tilde{V} \|_{K_t^{s+1}}, \]
where we used the trace inequality. The term $\mu J_2 b_{2kj} \tilde{b}_{mi} \partial_m \tilde{v}_{1i} \nu_k$ is estimated as
\[ \| \mu J_2 b_{2kj} \tilde{b}_{mi} \partial_m \tilde{v}_{1i} \nu_k \|_{H_t^{1/2-1/4} L^2(\Gamma_\varepsilon)} \]
\[ \leq \| J_2 b_{2kj} \tilde{b}_{mi} \partial_m \tilde{v}_{1i} \|_{H_t^{1/2-1/4} L^2(\Gamma_\varepsilon)} \]
\[ \leq \| J_2 b_{2kj} \tilde{b}_{mi} \partial_m \tilde{v}_{1i} \|_{L^2_t H_{t+1/2}^s(\Gamma_\varepsilon)} \]
\[ \leq \| J_2 b_{2kj} \tilde{b}_{mi} \partial_m \tilde{v}_{1i} \|_{L^2_t H_{t+1/2}^s(\Gamma_\varepsilon)} \]
\[ \leq \epsilon \| \tilde{v} \|_{K_t^{s+1}}, \]
and
\[ \| \mu J_2 b_{2kj} \tilde{b}_{mi} \partial_m \tilde{v}_{1i} \nu_k \|_{H_t^{1/2-1/4} L^2(\Gamma_\varepsilon)} \]
\[ \leq \| J_2 b_{2kj} \tilde{b}_{mi} \partial_m \tilde{v}_{1i} \|_{H_t^{1/2-1/4} L^2(\Gamma_\varepsilon)} \]
\[ \leq \| J_2 b_{2kj} \tilde{b}_{mi} \partial_m \tilde{v}_{1i} \|_{L^2_t H_{t+1/2}^s(\Gamma_\varepsilon)} \]
\[ \leq \| J_2 b_{2kj} \tilde{b}_{mi} \partial_m \tilde{v}_{1i} \|_{L^2_t H_{t+1/2}^s(\Gamma_\varepsilon)} \]
\[ \leq \epsilon \| \tilde{v} \|_{K_t^{s+1}}. \]

Other terms on the right side of (6.25) are treated analogously as in Theorem 5.2 using Lemma 6.1–6.3, and we arrive at
\[ \| \tilde{b} \|_{H_t^{1/2-1/4, s-1/2}} \leq \epsilon \| \tilde{v} \|_{K_t^{s+1}} + \epsilon \| \tilde{V} \|_{K_t^{s+1}}, \] (6.35)
by taking $T > 0$ sufficiently small.

Since the terms involving $\| \tilde{V} \|_{K_t^{s+1}}$ on the right side of (6.34)–(6.35) are absorbed to the left side (6.33) by taking $\epsilon > 0$ sufficiently small, we obtain from (6.33)–(6.35) that
\[ \| \tilde{V} \|_{K_t^{s+1}} \leq \epsilon \| \tilde{v} \|_{K_t^{s+1}}, \]
completing the proof of (6.23) by taking $\epsilon > 0$ sufficiently small. From (6.11) and Lemma 6.2 it follows that the system (2.3)–(2.10) admits a unique solution
\[ (v, R, w, w_t) \in K_t^{s+1} \times H^1(0, T_0) \times H^1(0, T_0) \times H^1(0, T_0) \times H^1(0, T_0) \]
\[ \times C([0, T_0], H^{s+1/4-\epsilon_0}(\Omega_t)) \times C([0, T_0], H^{s-3/4-\epsilon_0}(\Omega_t)), \]
with the corresponding norms bounded by a function of the initial data. \qed

\section*{Acknowledgments}

IK was supported in part by the NSF grant DMS-1907992, while LL was supported in part by the NSF grants DMS-2009458 and DMS-1907992. The work was undertaken while the authors were members of the MSRI program “Mathematical problems in fluid dynamics” during the Spring 2021 semester (NSF DMS-1928930).

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