Applications of Quantum Group to Fractional Quantum Hall Effect

Guang-Hong Chen and Mo-Lin Ge

Theoretical Physics Division, Nankai Institute of Mathematics,
Tianjin 300071, People’s Republic of China

Abstract

We show that there exists quantum group symmetry $sl_q(2)$ in the fractional quantum Hall effect (FQHE) and this symmetry governs the degeneracy of ground-state level. Under the periodic boundary condition, the degree of degeneracy is related to the cyclic representation of $sl_q(2)$. We also discuss the influence of impurity by using quantum group technique and give the energy correction due to the impurity potential.

PACS number(s): 73.20H, 02.20.+b
1 Introduction

In the past ten years, quantum groups (including Yangian [1] and quantum algebras [2]) and their representation theories were intensively studied from the point of view of mathematical physics [3,4]. The simplest quantum algebra is $sl_q(2)$ which can be viewed as a q-deformation of classical Lie algebra $sl(2)$ through the q-deformed commutation relations:

\[
[J_+, J_-] = [2J_3]_q \tag{1}
\]

\[
q^{J_3}J_\pm q^{-J_3} = q^{\pm 1}J_\pm \tag{2}
\]

and the related co-products, where we have used the notation:

\[
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{3}
\]

and $q$ stands for a complex deformation parameter. The representation for Eqs.(1) and (2) are strongly dependent on whether $q$ at root of unity or not. For the case where $q^p \neq 1$ ($p = 2, 3, \ldots$), the representations are similar to those of $sl(2)$ algebra with the only difference that Clebsch-Gordan (C-G) coefficients should be replaced by the q-deformed ones and there are still the highest weight and the lowest weight [5] as appeared in the usual Lie algebras. However, it is not the case for $q^p = 1$ whose representation does no longer preserve the corresponding Lie algebraic structure [6,7,8]. Or rather, it allows some new types of representations forbidden by classical Lie algebras. The typical one among them is the cyclic representation which has neither the highest weight nor the lowest weight. In physics if a moving particle experiences an external magnetic field, then the deformation parameter $q$ is often related to the applied flux $\Phi$ through:

\[
q = e^{i\Phi} \tag{4}
\]

The case with $q$ at root of unity means that the magnetic flux $\Phi$ is quantized.

As is known that the quantum algebras shed a new light on the new symmetry to the quantum integrable systems in physics [9]. More attractively, they can be related to some interesting physical models. Among them Wiegmann-Zabrodin [10], Faddeev-Kashaev [11] and Hatsugai-Kohmoto-Wu [12] have accomplished their remarkable works in this respect. In the Ref. [10], the Azbel-Hofstadter-Wannier problem of two-dimension Bloch electrons in magnetic field was rediscussed by using the techniques of quantum group.
Furthermore, in Ref. [11] the idea was extended to the Chiral Potts model. However, it is still very interesting to find other physical examples which can be described in terms of $sl_q(2)$, especially, its cyclic representation.

In this paper, we would like to apply the cyclic representation to fractional quantum Hall effect (FQHE) which has extensively attracted attention of physicists for many years [13,14]. An important feature of FQHE is the degeneracy of ground state [15,16] and the degeneracy can be removed by impurity [17,18]. In the present paper we shall show the following points:

1. The degeneracy of quantum states for FQHE can be described in terms of the cyclic representations of $sl_q(2)$. We find a proper quantum number $k$ which can labels the degenerate quantum states;

2. The potential for weak impurity can be reduced to a special case of the model which has been detailly studied by L. D. Faddeev and R. M. Kashaev in Ref. [11] and Bethe Ansatz in Ref. [11] can be used to calculate the energy correction due to the weak impurity potential. This is made through translating the method given by Tao-Haldane [17] into quantum algebraic language. However, it is of interest since the results of Faddeev-Kashaev is concerned with the anisotropic extension of Wiegmann’s approach for Hofstadter model and so far it has not been related to any ”real physics”.

This paper is organized as follows. For self-containing we first introduce the cyclic representation of quantum algebra through a simple physical realization. In Sec.3 we show how to explicitly define the quantum group symmetry in single particle system and many-body system. In sec.4 we give the relation between irreducible cyclic representation of quantum algebra and degeneracy for ground state of FQHE. In Sec.5 the removing of degeneracy due to impurity will be discussed. The final concluding remarks will be made in Sec.6.

2 Cyclic Representation of Quantum Algebras

A simple physical realization of cyclic representation can be made through the following example. We go along the line of Pegg-Barnett(PB) theory on the quantization of phase [19]. One can consider a harmonic oscillator in finite dimension Hilbert space and the set $|n\rangle$, $(n = 0, 1, \ldots, s)$ is the basis of the space. $\hat{N}$ is the ”number” operator in above
Hilbert space. Then one can define phase states as:

\[ |\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta_m} |n\rangle \]  

(5)

where

\[ \theta_m = \theta_0 + \frac{2m\pi}{s+1} \]  

(6)

and \( \theta_0 \) is an arbitrary constant. It is easy to check that orthogonality and completeness hold:

\[ \langle \theta_p | \theta_m \rangle = \delta_{pm}, \quad \sum_{m=0}^{s} |\theta_m\rangle \langle \theta_m| = 1 \]  

(7)

The PB phase operator can be defined as[19]:

\[ \hat{\Phi}_\theta = \sum_{m=0}^{s} \theta_m |\theta_m\rangle \langle \theta_m| \]  

(8)

which satisfies:

\[ e^{i\hat{\Phi}_\theta} |n\rangle = |n-1\rangle \quad (n \neq 0), \quad e^{i\hat{\Phi}_\theta} |0\rangle = e^{i(s+1)\theta_0} |s\rangle \]  

(9)

\[ e^{-i\hat{\Phi}_\theta} |n\rangle = |n+1\rangle \quad (n \neq s), \quad e^{-i\hat{\Phi}_\theta} |s\rangle = e^{-i(s+1)\theta_0} |0\rangle \]  

(10)

Denoting

\[ q = e^{i\frac{2\pi}{s+1}}, \quad (q^{s+1} = 1) \]  

(11)

and defining

\[ q^N |n\rangle = q^{n+\eta} |n\rangle \quad (n = 0, 1, \ldots, s) \]  

(12)

where \( \eta \) stands for an arbitrary phase factor, one can prove that the following equations hold:

\[ q^N e^{i\hat{\Phi}_\theta} = q^{-1} e^{i\hat{\Phi}_\theta} q^N \]  

(13)

\[ q^N e^{-i\hat{\Phi}_\theta} = q e^{-i\hat{\Phi}_\theta} q^N \]  

(14)

\[ e^{i\hat{\Phi}_\theta} e^{-i\hat{\Phi}_\theta} = e^{-i\hat{\Phi}_\theta} e^{i\hat{\Phi}_\theta} \]  

(15)

Therefore, if one introduces:

\[ b^\dagger b = [\hat{N}] = \frac{q^N - q^{-\hat{N}}}{q - q^{-1}} \]  

(16)
\[ e^{i\hat{\Phi}_0} = b[\hat{N}]^{-\frac{1}{2}}, \quad e^{-i\hat{\Phi}_0} = [\hat{N}]^{-\frac{1}{2}}b^\dagger \]  

(17)

then by making use of Eqs.(16) and (17) we find that Eqs.(13),(14) and (15) can be recast into

\[ bb^\dagger - q^\pm b^\dagger b = q^\pm \hat{N} \]  

(18)

\[ q^\pm \hat{N} b^\dagger = q^\pm b^\dagger q^\pm \hat{N} \]  

(19)

\[ q^\pm \hat{N} b = q^\mp bq^\pm \hat{N} \]  

(20)

Eqs.(18), (19) and (20) were known as the deformed boson commutators of quantum algebra \( sl_q(2) \) [20].

Eqs.(9) and (10) will become [20]:

\[ b|n\rangle = \sqrt{[n+\eta]} |n-1\rangle \quad (n \neq 0), \quad b|0\rangle = \sqrt{[\eta]}e^{i(s+1)\theta_0} |s\rangle \]  

(21)

\[ b^\dagger |n\rangle = \sqrt{[n+\eta+1]} |n+1\rangle \quad (n \neq s), \quad b^\dagger |s\rangle = \sqrt{[\eta]}e^{-i(s+1)\theta_0} |0\rangle \]  

(22)

where the notation \([x] = \frac{q^x - q^{-x}}{q-q^{-1}}\) has been used. Above properties of state vectors are called the cyclic representation of quantum algebra \( sl_q(2) \) and can be illustrated by Fig.1. There is another kind of realization of Eqs.(13)-(15) that will be useful for the later discussion. In order to make it explicit one can define:

\[ q^\hat{N} = X, \quad e^{i\hat{\Phi}_0} = Z \]  

(23)
Then Eq.(13) becomes
\[ ZX = qXZ \quad (q^{s+1} = 1) \] (24)
which is Heisenberg-Weyl algebra. Based on it one is able to construct the translation operators by setting:
\[ T_x = ZX, \quad T_y = ZX^{-1} \] (25)

It is easy to check:
\[ T_y T_x = q^2 T_x T_y \quad T_{-y} T_x = q^2 T_x T_{-y} \] (26)
\[ T_{-y} T_x = q^{-2} T_x T_{-y} \quad T_y T_x = q^{-2} T_{-x} T_y \] (27)

which is identical with quantum algebra denoted by the commutation relations Eqs.(18), (19) and (20) through defining:
\[ T_x + T_y = i(q - q^{-1}) J_- \quad T_{-x} + T_{-y} = i(q - q^{-1}) J_+ \] (28)
\[ T_{-x} T_y = q K^{-2} \quad T_{-y} T_x = q^{-1} K^{+2} \] (29)

We then have
\[ [J_+, J_-] = \frac{K^{+2} - K^{-2}}{q - q^{-1}} \] (30)
\[ K^+ J_\pm K^- = q^{\pm 1} J_\pm \] (31)

By straightforward calculation it is easy to find:
\[ K^+ = q^{\tilde{N} + \frac{i}{2}} \quad K^- = q^{-\tilde{N} - \frac{i}{2}} \] (32)

and
\[ K^\pm |n\rangle = q^{\pm (n + \eta + \frac{i}{2})} |n\rangle \] (33)
\[ J_+ |n\rangle = -\frac{\cos [\gamma (n + \eta + 1)]}{\sin \gamma} |n + 1\rangle \quad (n \neq s) \] (34)
\[ J_+ |s\rangle = -\frac{\cos \gamma \eta}{\sin \gamma} e^{-i(s + 1)\theta_0} |0\rangle \] (35)
\[ J_- |n\rangle = -\frac{\cos [\gamma (n + \eta)]}{\sin \gamma} |n - 1\rangle \quad (n \neq 0) \] (36)
\[ J_- |0\rangle = -\frac{\cos \gamma \eta}{\sin \gamma} e^{i(s + 1)\theta_0} |s\rangle \] (37)

where the notations \( \gamma = \frac{2\pi}{s+1} \) and \( q = e^{i\gamma} \) have been used. This realization of quantum algebra \( sl_q(2) \) is very useful and will be used in the following discussion.
3 Magnetic Translation Invariance and Quantum Group Symmetry

For simplicity we first consider a spinless particle which moves in a plane and experiences an uniform external magnetic field along \( z \)-direction, \( \vec{B} = B\hat{e}_z \). The Hamiltonian of system can be written as
\[
H_0 = \frac{1}{2m}(\vec{p} + e\vec{A})^2
\]
where \( m,e \) are mass and charge of particle, respectively. \( \vec{A} \) is the vector potential satisfying:
\[
\nabla \times \vec{A} = B\hat{e}_z
\]

Above problem can easily be solved in proper gauge \([21,22]\). We shall discuss the gauge-independent case. It is well known that in this system there is not translation invariance, however, it exhibits magnetic translation invariance generated by the magnetic translation operator defined by
\[
t(\vec{a}) = \exp\left[i\frac{\bar{\hbar}}{\hbar}\vec{a} \cdot (\vec{p} + e\vec{A} + e\vec{r} \times \vec{B})\right]
\]
where \( \vec{a} = a_x\hat{e}_x + a_y\hat{e}_y \) is an arbitrary two-dimensional vector. The magnetic translation operator \( t(\vec{a}) \) satisfies the following group property \([23,24]\):
\[
t(\vec{a})t(\vec{b}) = \exp\left[-i\frac{\hat{e}_z \cdot (\vec{a} \times \vec{b})}{a_0^2}\right]t(\vec{b})t(\vec{a})
\]
where \( a_0 \equiv \frac{\hbar}{\sqrt{\epsilon B}} \) is the magnetic length.

Let
\[
\vec{\kappa} = \vec{p} + e\vec{A} + e\vec{r} \times \vec{B}
\]

It is easy to prove that
\[
[t(\vec{a}), H_0] = 0, \quad [\vec{\kappa}, H_0] = 0
\]
i.e. the system under consideration is invariant under the magnetic translation transformation, and \( \vec{\kappa} \) is a conservative quantity.

With the help of the magnetic translation operator, one can construct the following operators \([10]\):
\[
J_+ = \frac{1}{q-q^{-1}}[t(\vec{a}) + t(\vec{b})], \quad J_- = \frac{-1}{q-q^{-1}}[t(-\vec{a}) + t(-\vec{b})]
\]
\[ q^{2J_3} = t(b - a), \quad q^{-2J_3} = t(a - b) \] (45)

with

\[ q = \exp(i2\pi \frac{\Phi}{\Phi_0}) \] (46)

where \( \Phi = \frac{1}{2} \vec{B} \cdot (\vec{a} \times \vec{b}) \) is magnetic flux through the triangle enclosed by vectors \( \vec{a} \) and \( \vec{b} \) and \( \Phi_0 = \frac{\hbar}{e} \) is magnetic flux quanta. It turns out that the operators \( J_+, J_- \) and \( J_3 \) satisfy the algebraic relations of the \( sl_q(2) \) [2] as shown by Eqs.(1), (2) and (46).

From Eqs.(43)-(45) it follows that:

\[ [J_\pm, H_0] = 0 \] (47)

\[ [q^{\pm J_3}, H_0] = 0 \] (48)

which indicates that \( J_\pm \) and \( J_3 \) are conservative quantities of the system. Therefore, there is the \( sl_q(2) \) structure in the Landau problem under our consideration. This structure still holds for many-body system. We assume that a system contains \( N_e \) electrons and the interaction among particles is pair-potential \( V(|\vec{r}_i - \vec{r}_j|) \), where \( \vec{r}_i \) is the coordinate of the \( i-th \) electron. The Hamiltonian of system in the absence of impurities can be written as:

\[ H = \sum_{j=1}^{N_e} \frac{1}{2m}(\vec{p}_j + e\vec{A}_j)^2 + \sum_{i \neq j} V(|\vec{r}_i - \vec{r}_j|) \] (49)

In this case one can construct the following generators of magnetic translation:

\[ T(\vec{a}) = \prod_{j=1}^{N_e} t_j(\vec{a}) = \exp(\frac{i}{\hbar}N_e\vec{\kappa}_c \cdot \vec{a}) \] (50)

where

\[ \vec{\kappa}_c = \frac{1}{N_e} \sum_{j=1}^{N_e} \vec{\kappa}_j \] (51)

is the pseudo-momentum of mass centre, \( N_e \) stands for the number of electrons and \( \vec{\kappa}_j \) is the pseudo-momentum of the \( j-th \) electron. It is easy to know:

\[ T_{\vec{a}}T_{\vec{b}} = \exp[-iN_e\vec{\epsilon}_z \cdot (\vec{a} \times \vec{b})/a_0^2]T_{\vec{b}}T_{\vec{a}} \] (52)

Similarly, by defining

\[ J_+ = \frac{1}{q - q^{-1}}[T(\vec{a}) + T(\vec{b})], \quad J_- = \frac{-1}{q - q^{-1}}[T(-\vec{a}) + T(-\vec{b})] \] (53)
\[ q^{2J_3} = T(b - a), \quad q^{-2J_3} = T(a - b) \]  
\[(54)\]

with
\[ q = \exp\left[iNe\hat{e}_z \cdot (\vec{a} \times \vec{b})/2a_0^2\right] \]  
\[(55)\]

We still obtain
\[ [J_+, J_-] = [2J_3]_q \]
\[ q^{J_3}J_\pm q^{-J_3} = q^{\pm 1}J_\pm \]  
\[(56)\]

In other words, we can also define a quantum group describing the global behavior of a many-body system.

Because \( T(a) \) is the magnetic translation operator of mass centre, it can not change the relative distance among particles, namely,
\[ [T(a), V(|r_i - r_j|)] = 0 \]  
\[(57)\]

Similarly, one can also check
\[ [T(a), \sum_{j=1}^{N_e} \frac{1}{2m} (p_j + eA_j)^2] = 0 \]  
\[(58)\]

We then obtain
\[ [T(a), H] = 0 \]  
\[(59)\]

Eqs.(53), (54) and (59) implies that:
\[ [J_\pm, H] = 0, \quad [q^{\pm J_3}, H] = 0 \]  
\[(60)\]

Therefore, the interacting electrons in magnetic field also exhibit a hidden symmetry—quantum algebra \( sl_q(2) \). The similar result has been found in interacting anyon system [25]. However, after review the above results one would like to ask what kind of effect can be caused by above symmetry. The answer will be given in following section.

4 Cyclic Representation and Degeneracy of Ground State for FQHE

As is known that the fractional quantum Hall system can be described by the Hamiltonian Eq.(49) and the pair-potential \( V(|r_i - r_j|) \) is just the Coloumb interaction \( e^2/|r_i - r_j| \).
Let $\Psi$ be wave function of system in the Schrödinger picture. Following the basic idea of Wen and Niu [18], we employ the periodic boundary condition (PBC):

$$t_j(\vec{L}_1)\Psi = \Psi, \quad t_j(\vec{L}_2)\Psi = \Psi$$  \hspace{1cm} (61)

where $\vec{L}_1 = L_1 \hat{e}_x$, $\vec{L}_2 = L_2 \hat{e}_y$, and $j = 1, 2, \ldots, N_e$. This boundary condition means that particles are confined in a rectangular area of size $L_1 \times L_2$. From Eq.(61) it follows that the operators $t_j(\vec{L}_1)$ and $t_j(\vec{L}_2)$ commute with each other. That is,

$$t_j(\vec{L}_1)t_j(\vec{L}_2) = t_j(\vec{L}_2)t_j(\vec{L}_1)$$  \hspace{1cm} (62)

however, from Eq.(41) we have

$$t_j(\vec{L}_1)t_j(\vec{L}_2) = \exp\left[-i\hat{e}_z \cdot (\vec{L}_1 \times \vec{L}_2) / a_0^2\right]t_j(\vec{L}_2)t_j(\vec{L}_1)$$  \hspace{1cm} (63)

Combining Eq.(62) with Eq.(63) yields:

$$\exp(i2\pi \Phi / \Phi_0) = 1$$  \hspace{1cm} (64)

where $\Phi = \frac{1}{2} BL_1 L_2$ is magnetic flux through the triangle enclosed by $\vec{L}_1$ and $\vec{L}_2$. Eq.(64) implies that

$$\Phi = N_s \Phi_0$$  \hspace{1cm} (65)

where $N_s$ is a positive integer. Therefore, the periodic boundary condition (61) is equivalent to magnetic flux quantization. It is well known that the Landau filling factor $\nu$ satisfies

$$\nu = N_e / N_s = P / Q$$  \hspace{1cm} (66)

where $P$ and $Q$ are two mutual prime integers. One should notice that when the boundary condition Eq.(61) is taken, not all the translation operators $T(\vec{a})$ leaves Eq.(61) invariant. In other words,

$$t_j(\vec{L}_i)T(\vec{a})\Psi = T(\vec{a})\Psi$$  \hspace{1cm} (i = 1, 2)  \hspace{1cm} (67)

can not be satisfied by an arbitrary magnetic translation $T(\vec{a})$. However, if we define two primitive magnetic translation operators in the following way [18]:

$$T_x \equiv T(\frac{\vec{L}_1}{N_s}), \quad T_y \equiv T(\frac{\vec{L}_2}{N_s})$$  \hspace{1cm} (68)
then only $T_x, T_y$ and their integer powers can make Eq.(67) hold.

By a straightforward calculation it can be checked that the following relations hold

$$T_y T_x = \exp(i2\pi \frac{P}{Q}) T_x T_y,$$  
$$T_y T_x = \exp(i2\pi \frac{P}{Q}) T_x T_y$$  

$$T_{-y} T_x = \exp(-i2\pi \frac{P}{Q}) T_x T_{-y},$$  
$$T_{-y} T_x = \exp(-i2\pi \frac{P}{Q}) T_x T_{-y}$$  

$$T_{-x} T_x = T_{-y} T_y = 1$$  

(69)  

(70)  

(71)

It is easy to see that the generators $T_x, T_y$ are not enough to describe the specified physical problem. However, similar to Ref. [10] by making use of the operators $T_{\pm x}, T_{\pm y}$ and the above commutation relations we can construct the generators of quantum group as follows:

$$\hat{J}_+ = \frac{-i}{q-q^{-1}} (T_x + T_y),$$  
$$\hat{J}_- = \frac{-i}{q-q^{-1}} (T_x + T_y)$$  

$$\hat{K}^{+2} = q T_{-y} T_x,$$  
$$\hat{K}^{-2} = q^{-1} T_{-x} T_y$$  

(72)  

(73)

where the deformation parameter is given by

$$q = \exp(i\pi \frac{P}{Q})$$  

(74)

It is easy to check that these generators obey the standard commutation relations of the quantum group $sl_q(2)$ as shown by Eqs.(30), (31) and (74). We can also find that the generators $\hat{J}_\pm$ and $\hat{K}_\pm$ commute with Hamiltonian

$$[\hat{J}_\pm, H] = 0, \quad [\hat{K}^{\pm}, H] = 0$$  

(75)

Above analysis indicates that $sl_q(2)$ is the basic symmetry in our system. Furthermore, according to the fundamental principle of quantum mechanics, Eqs.(30), (31) and (75) imply the degeneracy of ground state in the FQHE.

Let us discuss the relationship between degeneracy and cyclic representation of $sl_q(2)$.

Since $P, Q$ are integers in Eq.(74), we should discuss the following two cases:

(i) $P = \text{even}$

We have:

$$q^Q = 1$$  

(76)

In this case the representation of quantum group has so-called cyclic representation and the dimension of the irreducible representation is $Q$ [7].
Furthermore, without loss of generality, according to Eq.(75) we can simultaneously diagonalize $H$ and $\hat{K}^\pm$. In other words, one can choose a set of basis vectors $|n, k\rangle = |n\rangle \otimes |k\rangle$ to be eigenvectors of operators $H$ and $\hat{K}^\pm$, i.e.,

$$H |n, k\rangle = E_n |n, k\rangle$$ (77)

and

$$\hat{K}^\pm |n, k\rangle = q^{\pm(k+\frac{\eta}{2})} |n, k\rangle$$ (78)

where $n = 0, 1, ..., \infty$ is the symbols of the energy level, and $k = 0, 1, ..., Q - 1$ is the new quantum numbers which label the different quantum states in the same degenerate energy level. The cyclic representation for this case is shown by Fig.2. According to Eqs.(33)-(37) and acting the $sl_q(2)$ generators on these basis vectors yield:

$$\hat{J}_- |n, k\rangle = -\frac{\cos \left[\gamma(n + \eta)\right]}{\sin \gamma} |n, k - 1\rangle \quad (k \neq 0)$$ (79)

$$\hat{J}_- |n, 0\rangle = -\frac{\cos (\gamma \eta)}{\sin \gamma} e^{iQ\theta_0} |n, s\rangle$$ (80)

$$\hat{J}_+ |n, k\rangle = -\frac{\cos \left[\gamma(k + \eta + 1)\right]}{\sin \gamma} |n, k + 1\rangle \quad (k \neq Q - 1)$$ (81)

$$\hat{J}_+ |n, s\rangle = -\frac{\cos \gamma \eta}{\sin \gamma} e^{-iQ\theta_0} |n, 0\rangle$$ (82)

where $\eta, \theta_0$ are arbitrary constants, $q = e^{i\gamma}$ and $\gamma = \frac{\pi P}{Q}$.

Since the dimension of the irreducible cyclic representation space $|n, k\rangle$ is $Q$, from Eq.(79)-(82) we can see that the degree of degeneracy of the ground state is just $Q$,
which is in accordance with the statement of the current literatures [15-18]. This is one of the main conclusions of this paper.

(ii) \( P = odd \)

This case is different from case (i). We have:

\[
q^{2Q} = 1
\]  

(83)

Similar to case (i) we can draw a similar conclusion about the degeneracy, but the degree of degeneracy is \( 2Q \). The properties of transformation among the degenerate states are similar to case (i) except that \( Q \) should be replaced by \( 2Q \).

Briefly speaking, the degree of degeneracy of ground state depends on the Landau filling factor \( \nu = \frac{P}{Q} \). There is \( Q \)-fold degeneracy for the ground state for \( P = even \), whereas \( 2Q \)-fold degeneracy for \( P = odd \). The latter is somewhat different from the previous discussion [17,18,26]. Our results concerning the degeneracy of energy are independent of the ground state or the excited states of the system. It allows the mixture of different levels. Besides, we do not know how to eliminate the case (ii) from the point of view of symmetry-breaking.

5 Influence of Impurity on the Degeneracy

Although FQHE can often be observed in the high mobility sample, the impurity can not be avoided at any case. It is interesting to investigate the influence of impurity potential. We shall concentrate on the behavior of weak impurity. In this section we first derive the effective impurity potential in terms of the generators of quantum group \( sl_q(2) \), then we shall explore how the degeneracy of ground state is removed by means of perturbation theory. A generalized Bethe-Ansatz equation will be used for deriving the energy correction.

Generally speaking, the impurity potential can be written as [17]:

\[
U = \sum_{j=1}^{N_e} V(r_j) = \sum_{\vec{k}} \tilde{V}(\vec{k}) \sum_{j=1}^{N_e} e^{i\vec{k} \cdot \vec{r}_j}
\]  

(84)

where \( \tilde{V}(\vec{k}) \) is the Fourier transform of \( V(r_j) \) and \( \vec{k} \) is the wave vector of Fourier transform which is defined as:

\[
k_x = \frac{2\pi}{L_1} n_1, \quad k_y = \frac{2\pi}{L_2} n_2
\]  

(85)
where \( n_1 \) and \( n_2 \) are integers.

Making use of the well-known formula:

\[
e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A}, \hat{B}]} \tag{86}
\]

where \( \hat{A} \) and \( \hat{B} \) are two operators with \([\hat{A}, \hat{B}]\) being constant, we obtain:

\[
e^{i\vec{k} \cdot \vec{r}_j} = e^{-\frac{1}{2}k^2a_0^2} e^{\frac{k_y n_j}{2a_0}} e^{i \frac{k_y n_j - k_x d_j}{2a_0}} e^{-\frac{k_y n_j}{2a_0}} \tag{87}
\]

where \( k_\pm \) and \( \Pi_{j\pm} \) are defined by:

\[
k_\pm = k_x \pm i k_y, \quad \Pi_{j\pm} = \Pi_{jx} \pm i \Pi_{jy} \tag{88}
\]

The operators \( \Pi_{j\pm} \) can raise or lower the Landau levels [17]. Therefore, if the influence of impurity on the ground state is mainly taken into account, we only need to discuss the projection of \( V(\vec{r}_i) \) at the lowest Landau level, that is:

\[
U_0 = \sum_{\vec{k}} V(\vec{k}) e^{-\frac{1}{2}k^2a_0^2} \sum_{j=1}^{N_0} e^{i \frac{k_y n_j - k_x d_j}{2a_0}} \tag{89}
\]

Besides, from Eqs.(69) and (70) we know that \( T_{i\pm}^Q, T_{\pm y}^Q \) are the centre elements of the \( sl_q(2) \). They can then be taken as constants, i.e.,

\[
T_{i\pm}^Q e^{i\vec{k} \cdot \vec{r}} = e^{i\vec{k} \cdot \vec{r}} T_{i\pm}^Q \tag{90}
\]

where \( i = \pm x, \pm y \). However, through the detail calculation one can find that:

\[
T_{i\pm}^Q e^{i\vec{k} \cdot \vec{r}} = e^{i\vec{k} \cdot \vec{r}} T_{i\pm}^Q e^{\pm 2\pi \frac{2n Q}{N_s}}
\]

\[
T_{\pm y}^Q e^{i\vec{k} \cdot \vec{r}} = e^{i\vec{k} \cdot \vec{r}} T_{\pm y}^Q e^{\pm 2\pi \frac{2n Q}{N_s}} \tag{91}
\]

In comparison to Eq.(90) one obtains:

\[
n_1 = \frac{l_1 N_s}{Q}, \quad n_2 = \frac{l_2 N_s}{Q} \tag{92}
\]

where \( l_1 \) and \( l_2 \) are other new integers.

Substituting Eqs.(85) and (92) into Eq.(89) one has:

\[
U_0 = \frac{N_q}{Q} \sum_{l_1, l_2} \tilde{V}(l_1 m_1, l_2 m_2) e^{-\frac{1}{2}((\frac{l_1}{Q a_0})^2 + (\frac{l_2}{Q a_0})^2)} t(\frac{l_2}{Q} \natural L_1 - \frac{l_1}{Q} \natural L_2) \tag{93}
\]

where constants \( m_1 \) and \( m_2 \) are defined as:

\[
m_1 = \frac{L_2}{Q a_0^2}, \quad m_2 = \frac{L_1}{Q a_0^2} \tag{94}
\]
and the magnetic translation operator \( t(\frac{L_1}{Q} \vec{L}_1 - \frac{L_2}{Q} \vec{L}_2) \) act on electrons only.

As is known that the Gaussian factor appeared in Eq.(93) will make \( U_0 \) rapidly decay, therefore we can only take the leading term in the expansion of \( U_0 \), i.e., to the lowest order of \( e^{-\frac{(\vec{L}_1)^2}{4Q^2}} \) and \( e^{-\frac{(\vec{L}_2)^2}{4Q^2}} \), we the get:

\[
U_0 = u_1 t(\frac{L_1}{Q}) + u_2 t(\frac{L_2}{Q}) + h.c. \tag{95}
\]

where we have neglected the constant term and defined:

\[
u_1 = N_e \tilde{V}(0, m_2)e^{-\frac{1}{2} (\frac{L_1}{Q})^2} \tag{96}
\]

\[
u_2 = N_e \tilde{V}(-m_1, 0)e^{-\frac{1}{2} (\frac{L_2}{Q})^2} \tag{96}
\]

we should bear in mind that the set \( T_\pm x \) and \( T_\pm y \) is complete and closed for \( sl_q(2) \). By tedious calculation one finds:

\[
[T_x^{-r}t(\frac{L_1}{Q}), T_x] = 0, \quad [T_y^{-r}t(\frac{L_2}{Q}), T_y] = 0 \tag{97}
\]

and

\[
T_x^{-r}t(\frac{L_1}{Q})T_y = T_y T_x^{-r}t(\frac{L_1}{Q})e^{i2\pi \frac{Pr}{Q}^{-1}}
\]

\[
T_y^{-r}t(\frac{L_2}{Q})T_x = T_x T_y^{-r}t(\frac{L_2}{Q})e^{i2\pi \frac{Pr}{Q}^{-1}}
\]

\[
T_x^{-r}t(\frac{L_1}{Q})T_{-y} = T_{-y} T_x^{-r}t(\frac{L_1}{Q})e^{-i2\pi \frac{Pr}{Q}^{-1}}
\]

\[
T_y^{-r}t(\frac{L_2}{Q})T_{-x} = T_{-x} T_y^{-r}t(\frac{L_2}{Q})e^{-i2\pi \frac{Pr}{Q}^{-1}} \tag{98}
\]

Therefore, if we choose

\[
Pr + Qs = 1 \tag{99}
\]

where \( s \) is an integer, then the operator \( T_x^{-r}t(\frac{L_1}{Q}) \) can commute with any operator of the set \( T_\pm x \) and \( T_\pm y \). The same results can be derived for the operator \( T_y^{-r}t(\frac{L_2}{Q}) \). Taking Shur’s lemma into account the operators \( T_x^{-r}t(\frac{L_1}{Q}) \) and \( T_y^{-r}t(\frac{L_2}{Q}) \) are nothing but constants. Without loss of generality we can choose:

\[
T_x^{-r}t(\frac{L_1}{Q}) = e^{i\phi_1} \quad T_y^{-r}t(\frac{L_2}{Q}) = e^{i\phi_2} \tag{100}
\]

i.e., we have

\[
t(\frac{L_1}{Q}) = e^{i\phi_1} T_x \quad t(\frac{L_2}{Q}) = e^{i\phi_2} T_y \tag{101}
\]
where $\phi_1$ and $\phi_2$ are two constants. Furthermore, we can prove based on the number theory that the solution of Eq.(99) is unique if we impose a constraint on the value of $r$

$$|r| < \frac{Q}{2}$$  \hspace{1cm} (102)

That is to say, we can uniquely express $t(\frac{\vec{L}}{Q})$ and $t(\frac{\vec{L}}{Q})$ in terms of $T_x^r$ and $T_y^r$. This is an interesting result. Substituting Eq.(101) into Eq.(95) we obtain:

$$U_0 = u_1 e^{i\phi_1} T_x^r + u_2 e^{i\phi_2} T_y^r + c.c.$$  \hspace{1cm} (103)

Because the impurity potential $U_0$ is only related to operators $T_x$ and $T_y$, but not to $t_j(\vec{a})$, so we have to find the effective change of boundary conditions, or equivalently, we should find an effective operator $U_0$ which leaves the wave function unchanged. A simple form of the operator can be chosen as [18]:

$$U_0 = u_1 e^{i\phi_1} e^{i\alpha_1 L_1 r} \frac{\vec{P}}{\vec{Q}} T_x^r + u_2 e^{i\phi_2} e^{i\alpha_2 L_2 r} \frac{\vec{P}}{\vec{Q}} T_y^r + c.c.$$  \hspace{1cm} (104)

or can be simply rewritten as:

$$U_0 = u_1 \alpha T_x^r + v_1 \beta T_y^r + u_1 \alpha^{-1} T_x^{-r} + v_1 \beta^{-1} T_y^{-r}$$  \hspace{1cm} (105)

where $\alpha$ and $\beta$ are defined by

$$\alpha = e^{i\phi_1} e^{i\alpha_1 L_1 r} \frac{\vec{P}}{\vec{Q}}$$

$$\beta = e^{i\phi_2} e^{i\alpha_2 L_2 r} \frac{\vec{P}}{\vec{Q}}$$  \hspace{1cm} (106)

Taking

$$T_x^r = T^r(\frac{\vec{L}_1}{N_s}) = T(\frac{r \vec{L}_1}{N_s}) = \tilde{T}_x$$

$$T_y^r = T^r(\frac{\vec{L}_2}{N_s}) = T(\frac{r \vec{L}_2}{N_s}) = \tilde{T}_y$$  \hspace{1cm} (107)

into account and noting the fact that $\tilde{T}_{\pm x}$ and $\tilde{T}_{\pm y}$ form another algebra similar to that of $T_{\pm x}$ and $T_{\pm y}$, without loss of generality, we obtain the following form for impurity potential $U_0$:

$$U_0 = u_1 \alpha T_x + v_1 \beta T_y + u_1 \alpha^{-1} T_x^{-1} + v_1 \beta^{-1} T_y^{-1}$$  \hspace{1cm} (108)

Making use of the perturbation theory for degenerate case, we have the secular equation:

$$U_0 \Psi_{n,k} = \varepsilon \Psi_{n,k}$$  \hspace{1cm} (109)
where \( \varepsilon \) is the first order correction of energy and \( \Psi_{n,k} \) is the degenerate wave function.

A more general case has been discussed by L. D. Faddeev and R. M. Kashaev [11]. Eq.(108) is nothing but a special case of Ref.[11] by setting \( \rho = 0 \) in their paper. Therefore, we can directly quote their results:

\[
\varepsilon = -u_1 u_2 (q - q^{-1}) \sum_{m=1}^{Q-1} z_m + (u_1 + u_2)(q^{\frac{1}{2}} + q^{-\frac{1}{2}})
\]

where \( z_m \) can be determined by the generalized Bethe-Ansatz Equation:

\[
q^{\frac{1}{2}}(u_1 z_l + \frac{1}{2})(u_2 z_l + \frac{1}{2}) = \prod_{m=1, m \neq l}^{Q-1} \frac{q z_l - z_m}{z_l - z_m}
\]

where \( l = 1, 2, ..., Q - 1 \). The integer number \( Q \) is just the number appearing in \( \nu = \frac{p}{Q} \).

From the above discussion we can see that the energy is splitted into many subbands in the presence of weak impurity potential and can be described in terms of the theory presented in Ref.[11]. This is the main result of this paper.

### 6 Concluding Remarks

In the above discussions we have pointed out that the degeneracy for FQHE can be described in terms of the cyclic representation of quantum algebra. As is shown in Ref.[7], when \( q^Q = 1 \), the dimension of irreducible cyclic representation of quantum algebra associated with \( sl_q(2) \) should be \( Q \). Therefore our conclusion is consistent with the general results of De Concini and Kac [7]. The quantum algebra is introduced based on the relations Eq.(72) and Eq.(73), which is nothing but the quantum plane defined in Ref. [27]. Therefore, the degeneracy properties can be read from the geometry on the quantum plane and described in terms of the \( q \)-boson operators shown by Eq.(18)-(20). The determination of dimension is complicated and some of discussions had been made in Refs.[6,7,20]. In this paper we have discussed the weak impurity, namely, only leading term of the potential \( U \) is survived, which removes the degeneracy for FQHE. Fortunately, the leading term \( U_0 \) can be expressed through \( T^r_{\pm x} \) and \( T^r_{\pm y} \). We then are able to quote all the results in Ref.[11]. It is also attractive to calculate the overlapping between the ground state and the first excited state due to strong impurity. To do this, it may be beyond the present quantum algebraic structure. A new approach including more complicated algebra may be deserved.
Acknowledgements

The authors acknowledge Prof. Y. S. Wu, Prof. L. Yu and Drs F. Caitan, H. C. Fu, L. M. Kuang for valuable discussions, They also thank Miss Y. J. Wu for reading and typing the paper. This research is partly supported by the National Natural Science Foundation of China.

References

[1] V. Drinfeld, *Soviet Math. Dokl.,* Vol.32, (1985);

[2] V. Drinfeld, Proceedings ICM, Berkley, P.798, (1986); M. Jimbo, *Lett. Math. Phys.*, 10, P.63-69, (1985); L. D. Faddeev , N. Yu. Reshetikhin and L. A. Takhtajian, *Algebraic Analysis, Vol.1*, P.178, (1988);

[3] L. C. Beidenharn and M. D. Gould, *Proceedings of XXI International Conference on Diff. Geometric Methods in Theoretic Physics*, World Scientific, Singapore, P.3-12, (1992);

[4] V. Chari and A. Pressley, *Commn. Math. Phys.,* 142, 261, (1991)

[5] T. L. Curtright, *Quantum Groups*, Ed. by T. L. Curtright, D. Fairlie and C. Zachos, p.72, World Scientific, Singapore, (1990)

[6] G. Lusztig, *Adv. Math.,* 70, 237, (1988); *Contemp. Math.,* 82, 59, (1989);

[7] C. De concini and V. G. Kac, Representations of Quantum Groups at root of unity, *Colique Dixmier, Progress in Math.,* 92, 471, (1989), Birkhäuser, (1990);

[8] M. Rosso, *Commn. Math. Phys.,* 117, 581, (1988); *ibid, 124, 307, (1989)

[9] P. B. Wiegmann, *in Proceedings of Nobel Symposium 73*, [ Phys. Scr., T27, (1988)]; F. D. M. Haldane et. al, *Phys. Rev. Lett.,* 69, 2021, (1992);

[10] P. B. Wiegmann and A.Z. Zabrodin, *Phys. Rev. Lett.,* 72, 1890, (1994);

[11] L. D. Faddeev, R. M. Kashaev, *Generalized Bethe Ansatz Equations for Hofstadter Problem*, University of Helsinki report, (to be published)

[12] Y. Hasuhiro, M. Kohmoto and Y. S. Wu, *Phys. Rev. Lett.,* 73, 1134, (1994);
[13] K. von Klitzing, G. Dorda and M. Pepper, *Phys. Rev. Lett.*, **45**, 494, (1980);

[14] D. C. Tsui, H. L. Störmer and A. C. Gossard, *Phys. Rev. Lett.*, **48**, 1599, (1982);

[15] D. Yoshioka, *Phys. Rev.*, **B29**, 6833, (1984);

[16] W. P. Su, *Phys. Rev.*, **B30**, 1069, (1984);

[17] R. Tao and F. D. M. Haldane, *Phys. Rev.*, **B33**, 3844, (1986);

[18] W. G. Wen and Q. Niu, *Phys. Rev.*, **B41**, 9377, (1990);

[19] D. T. Pegg and S. M. Barnett, *Phys. Rev.*, **A39**, 1665, (1989);

[20] C. P. Sun and M. L. Ge, *Quantum Group and Quantum Integrable systems*, P.133-229, World Scientific, Singapore, (1991)

[21] L. D. Landau and E. M. Lifshitz, Quantum Mechanics, 3rd edn. (Pergman Press Inc., 1981) ch.XV section 112;

[22] R. Laughlin, *Phys. Rev.*, **B27**, 3383, (1983);

[23] E. Brown, *Phys. Rev.*, **133,4A**, 1038, (1964);

[24] J. Zak, *Phys. Rev.*, **134,6A**, 1602, (1964);

[25] D. F. Wang and C. Gruber, A Remark on Interaction Anyons in Magnetic Field, ITP-EPFL, Preprint, (1994);

[26] Q. Niu, D. J. Thouless, Yong-Shi Wu, *Phys. Rev.*, **B31**, 3372, (1985);

[27] Yu. I. Manin, *Quantum Groups and Non-Commutative Geometry*, by Publ. C. R. M., Univ. Montreal, (1988);