ABSTRACT

We consider torsional oscillations that are trapped in a layer of spherical-hole (bubble) nuclear structure, which is expected to occur in the deepest region of the inner crust of a neutron star. Because this layer intervenes between the phase of slab nuclei and the outer core of uniform nuclear matter, torsional oscillations in the bubble phase can be excited separately from usual crustal torsional oscillations. We find from eigenmode analyses for various models of the equation of state of uniform nuclear matter that the fundamental frequencies of such oscillations are almost independent of the incompressibility of symmetric nuclear matter, but strongly depend on the slope parameter of the nuclear symmetry energy $\Lambda$. Although the frequencies are also sensitive to the entrainment effect, i.e., what portion of nucleons outside bubbles contribute to the oscillations, by having such a portion fixed, we can successfully fit the calculated fundamental frequencies of torsional oscillations in the bubble phase inside a star of specific mass and radius as a function of $\Lambda$. By comparing the resultant fitting formula to the frequencies of quasi-periodic oscillations (QPOs) observed from the soft-gamma repeaters, we find that each of the observed low-frequency QPOs can be identified either as a torsional oscillation in the bubble phase or as a usual crustal oscillation, given generally accepted values of $\Lambda$ for all the stellar models considered here.

Key words: stars: neutron – equation of state – stars: oscillations
liquid crystals, possible observations of global free oscillations from neutron stars could be useful for obtaining information about elastic and superfluid properties of neutron star interiors (Passamonti & Andersson 2012). This technique is known as asteroseismology, which is essentially the same as seismology in the case of the Earth and helioseismology in the case of the Sun. In fact, it has been suggested that the neutron star’s properties such as the mass and radius, the equation of state (EOS) of matter therein, and the magnetic properties would be possible to obtain via the spectra of the star’s oscillations (see, e.g., Van Horn et al. (1993)).

Neutron star asteroseismology is unique in the sense that in addition to electromagnetic waves, gravitational waves radiating from the star are expected to provide us with information about the star’s global oscillations (Andersson & Kokkotas 1996; Sotani, Tominaga & Maeda 2001; Sotani, Kohri & Harada 2004; Sotani et al. 2011; Doneva et al. 2013). Direct gravitational wave detections from neutron stars, which have yet to be done, would be highly promising in the near future.

Meanwhile, there are X-ray observational evidences for neutron star oscillations. In fact, quasi-periodic oscillations (QPOs) were discovered in the X-ray afterglow of giant flares from soft-gamma repeaters (SGRs) (Israel et al. 2005; Strohmayer & Watts 2005, 2006; Huppenkothen et al. 2014), which are supposed to be strongly magnetized neutron stars (Kouveliotou et al. 1998; Hurley et al. 1999). Although there are still many uncertainties in understanding of the mechanism of the giant flares and the subsequent QPOs, it is generally accepted that the QPOs arise from global oscillations of the neutron stars. The observed QPO frequencies are in the range of tens Hz up to kHz, while typical frequencies of neutron star acoustic oscillations are around kHz (Van Horn et al. 1995). Particularly, identification of the QPO frequencies lower than $\sim 100$ Hz is not straightforward but could significantly constrain the possible origin of the QPOs. Basically, candidates for the corresponding global oscillations are crustal torsional oscillations, magnetic oscillations, and coupled oscillations between these two.

Global magnetic oscillations in neutron stars depend crucially on the magnetic field strength and structure therein (Gabler et al. 2013), but those are still poorly known, particularly in superconducting materials, as well as the EOS in matter. In order to avoid such uncertainties, in this paper we simply consider the observed low-lying QPOs as crustal torsional oscillations. In fact, within such identifications, one can obtain information about the crustal properties by fitting the calculated eigenfrequencies to the observed QPO frequencies (Samuelsson & Andersson 2007; Steiner & Watts 2009; Gearheart et al. 2011; Sotani et al. 2012, 2013a,b; Sotani, Iida & Oyamatsu 2016). Although there are still many uncertainties in understanding of the mechanism of the giant flares and the subsequent QPOs, it is generally accepted that the QPOs arise from global oscillations of the neutron stars. The observed QPO frequencies are in the range of tens Hz up to kHz, while typical frequencies of neutron star acoustic oscillations are around kHz (Van Horn et al. 1995). Particularly, identification of the QPO frequencies lower than $\sim 100$ Hz is not straightforward but could significantly constrain the possible origin of the QPOs. Basically, candidates for the corresponding global oscillations are crustal torsional oscillations, magnetic oscillations, and coupled oscillations between these two.

So far, several calculations of the eigenfrequencies of crustal torsional oscillations have been done by including the effect of superfluidity, but the effect of the possible existence of the pasta structure has been neglected in most of such calculations: an artificial shear modulus has been at most taken into consideration for the pasta phases (Sotani 2011; Passamonti & Pons 2016). Since the crystalline structure in the bubble phase is presumably the same as that in the low density region composed of spherical nuclei, however, one can likewise calculate the eigenfrequencies of the torsional oscillations in the bubble phase. As we shall see, furthermore, the smectic-A liquid-crystalline properties in the phase with slab-shaped nuclei (Pethick & Potekhin 1998) do not allow torsional shear oscillations to occur in linear analysis, which leads to the conclusion that the torsional oscillations in the bubble phase can be excited separately from those in the low density regime. Bearing this in mind, we search for a better fitting to the observed low-lying QPO frequencies while keeping the value of $L$ reasonable.

In Sec. the equilibrium configuration of a neutron star crust is constructed. Section is devoted to eigenmode analyses of torsional shear oscillations within the bubble phase. The resultant eigenfrequencies are compared with the observed QPO frequencies in Sec. Concluding remarks are given in Sec. We use units in which $c = G = 1$, where $c$ and $G$ denote the speed of light and the gravitational constant, respectively.

2 CRUST IN EQUILIBRIUM

We start with description of the equilibrium configuration of a neutron star crust. In this description, we need the EOS of equilibrated crustal matter. For simplicity, we assume that the temperature of the matter is zero. This is a very good approximation in analyzing the crust’s equilibrium configuration and eigenfrequencies of its torsional oscillations, but is not necessarily so in describing damping of such oscillations, which is beyond the scope of the present analysis.

As discussed in Lattimer (1981), the bulk energy per nucleon of uniform nuclear matter at zero temperature can be generally expanded as a function of baryon number density $n_b$ and neutron excess $\alpha$:

$$w = w_0 + \frac{K_0}{18w_0^2} (n_b - n_0)^2 + \left[ S_0 + \frac{L}{3n_0} (n_b - n_0) \right] \alpha^2. \quad (1)$$

Here $w_0$, $n_0$, and $K_0$ denote the saturation energy, saturation density, and incompressibility of symmetric nuclear matter ($\alpha = 0$), while $S_0$ and $L$ are associated with the density dependent symmetry energy $S(n_b)$, i.e., $S_0 \equiv S(n_0)$ and...
We remark that the crust-core boundary is set to the position where the phase transition occurs from spherical-hole nuclei into uniform matter in the present analysis, while, in the previous studies (Sotani et al. 2012, 2013a,b; Sotani, Iida & Oyamatsu 2016), being simply set to the position where the phase transition occurs from spherical nuclei into cylindrical nuclei.

Additionally, in Table 1 we list the values of the phase transition density from spherical to cylindrical nuclei, from cylindrical-hole to spherical-hole nuclei, and from spherical-hole nuclei to uniform matter. We remark that all the pasta structures are predicted to appear for all the EOS parameter sets shown in this table except the cases of \((K_0, L) = (360, 76.4)\) and \((360, 146.1)\) in MeV. In practice, no pasta structures appear for the case of \((K_0, L) = (360, 146.1)\), i.e., spherical nuclei transform directly to uniform matter at \(n_0 = \frac{0.06680 \text{ fm}^{-3}}{}\), while only the bubble structure is absent for \((K_0, L) = (360, 76.4)\), i.e., cylindrical-hole nuclei transform to uniform matter at \(n_0 = \frac{0.07918 \text{ fm}^{-3}}{}\). Thus, the maximum value of \(L\) for the bubble structure to appear in neutron stars is predicted to be \(L \approx 75 \text{ MeV}^1\) Since we focus on torsional oscillations confined in the bubble phase in this work, we shall consider only the cases with \(L \lesssim 75 \text{ MeV}\).

The equilibrium configuration of the crust of a spherically symmetric neutron star is constructed by integrating the Tolman-Oppenheimer-Volkoff (TOV) equations inward from the star’s surface down to the crust-core boundary in combination with the crust EOS (Iida & Sato 1997), in such a way that we do not have to use the core EOS, which is uncertain, explicitly. We remark that the crust-core boundary is set to the position where the phase transition occurs from spherical-hole nuclei into uniform matter in the present analysis, while, in the previous studies (Sotani et al. 2012, 2013a,b; Sotani, Iida & Oyamatsu 2016), being simply set to the position where the phase transition occurs from spherical nuclei into cylindrical nuclei.

\(^1\) The maximum value of \(L\) for any pasta structures to appear, which is of order 100 MeV, has already been discussed in Oyamatsu & Iida (2007) in terms of fission-like instability of spherical nuclei as well as proton clustering instability in uniform nuclear matter at subnuclear densities.

\[ n_s = n_0 - \frac{3n_0 L}{K_0} \frac{\alpha^2}{3}, \]

\[ w_s = w_0 + S_0 \alpha^2. \]
nuclei or uniform matter, depending on whether or not the phase with cylindrical nuclei can be energetically favorable. Under spherical symmetry, the metric can be obtained in terms of the spherical polar coordinates \( r, \theta, \) and \( \phi \) as
\[
    ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]
where \( \Phi \) and \( \Lambda \) are the metric functions that depend only on \( r \).

According to the solutions to the TOV equations, the thickness of the bubble phase for typical neutron star models with mass \( M \) and radius \( R \) depends sensitively on \( L \) and is at most \( \sim 10 \) m. In Fig. 2, the thickness of the bubble phase is shown for the stellar models with \( M = 1.4 M_\odot, 1.8 M_\odot \) and \( R = 10, 12, 14 \) km. We remark that the spherical layer of bubbles is located only within \( 1.5 \) km from the star’s surface for all the stellar models considered here. It is not the thickness but the radius of this layer that is essential to the fundamental frequencies of torsional oscillations trapped therein. The overtone frequencies are sensitive to the thickness, but are high enough to be beyond the scope of this paper.

The shear modulus of crustal matter is one of the most important properties in describing crustal torsional oscillations. For example, the shear modulus of a bcc lattice of spherical nuclei has been calculated as a function of the charge number \( Z \), the number density of nuclei \( n_i \), and the Wigner-Seitz radius \( a \) as (Ogata & Ichimaru 1990; Strohmayer et al. 1991)
\[
    \mu = 0.1194 \frac{n_i(Ze)^2}{a},
\]
by assuming that each nucleus is a point particle. Although modifications of the shear modulus by electron screening and polycrystalline nature have also been considered by Kobyakov & Pethick (2013, 2015), we adopt the traditional formula for the shear modulus [Eq. (5)] for simplicity in the present analysis. Since the crystalline structure in the bubble phase is presumably

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**Table 1.** The SP-C, CH-SH, and SH-U transition densities obtained for each EOS model, which is characterized by \( K_0 \) and \( L \). The asterisk at the value of \( K_0 \) denotes the EOS model by which the spherical-hole phase is not predicted to appear. In this case, the SH-U transition density should read the density at which the system melts into uniform matter.

| \( K_0 \) (MeV) | \( L \) (MeV) | SP-C (fm\(^{-3}\)) | CH-SH (fm\(^{-3}\)) | SH-U (fm\(^{-3}\)) |
|----------------|-------------|-------------------|-------------------|-------------------|
| 180            | 5.7         | 0.06000           | 0.12925           | 0.13489           |
| 180            | 17.5        | 0.05849           | 0.10206           | 0.10321           |
| 180            | 31.0        | 0.05887           | 0.09000           | 0.09068           |
| 180            | 52.2        | 0.05957           | 0.09817           | 0.09899           |
| 230            | 7.6         | 0.05816           | 0.12364           | 0.12736           |
| 230            | 23.7        | 0.06238           | 0.08604           | 0.08637           |
| 230            | 42.6        | 0.06421           | 0.09379           | 0.09414           |
| 360            | 40.9        | 0.06743           | 0.09379           | 0.09414           |
| *360           | 76.4        | 0.06680           | —                 | 0.09181           |

Figure 1. (Color online) The baryon number density \( n_b \) (left) and its combination with \( n_0 (n_0 n_b)^{1/2} \) (right), at the structural phase transitions, as plotted for eleven EOS models which are classified by the value of \( L \). Here, the circles, squares, diamonds, double circles, and double squares represent the transition densities from spherical to cylindrical nuclei (SP-C), from cylindrical to slablike nuclei (C-S), from slablike to cylindrical-hole nuclei (S-CH), from cylindrical-hole to spherical-hole (bubble) nuclei (CH-SH), and from spherical-hole nuclei to uniform matter (SH-U), respectively. The data are extracted from Oyamatsu & Iida (2007).
the same as that in the phase composed of spherical nuclei, we will consider torsional oscillations in the bubble phase by using the shear modulus given by Eq. (5) with appropriate replacements. For the bubble phase, we reinterpret $n_i$ as the number density of bubbles and, as in Watanabe & Iida (2003), replace $Z$ with the effective charge number $Z_{\text{bubble}}$ of a bubble. Here, the effective charge number is given by $Z_{\text{bubble}} = n_Q V_{\text{bubble}}$, where $V_{\text{bubble}}$ denotes the volume of the bubble, and $n_Q$ is the effective charge number density inside the bubble. $n_Q$ can be calculated as $n_Q = -n_e - (n_p - n_e) = -n_p$ with the number density of protons outside the bubble, $n_p$, and the number density of a uniform electron gas, $n_e$, because, in the bubble phase, the background (outside the bubble) charge number density is $n_p - n_e$, while the charge number density inside the bubble is $-n_e$.

The elastic properties in the pasta phases of cylindrical and slab nuclei have also been discussed in terms of liquid crystals (Pethick & Potekhin 1998). In fact, these phases can be regarded as a columnar phase and a smectic A, respectively. For these liquid crystals, the elastic properties and the propagating modes are well known (Landau & Lifshitz 1986; de Gennes & Prost 1993). Pethick & Potekhin (1998) utilized an incompressible liquid-drop model for pasta nuclei to derive the relation between the elastic constants involved and the Coulomb energy density. We remark that the elastic properties in the cylindrical-hole phase is essentially the same as that in the cylindrical phase. The important aspect of global torsional oscillations in the crust is that a restoring force due to the shear stress is responsible for the propagation of the oscillations. To linear order in displacements of the pasta nuclei, there is no such restoring force in the slab phase, while nonzero shear modulus in the cylindrical-hole phase plays a role in propagation of a torsional shear wave, often referred to as a “third sound.” Thus, the torsional oscillations that are excited within the cylindrical-hole and bubble phases are expected to be separable from those within the spherical and cylindrical phases, although there could be nonlinear coupling between these two. For simplicity, in the next section, we will consider the torsional oscillations that propagate only in the bubble phase; a “third sound” in the cylindrical hole phase and its connection with the torsional oscillations in the bubble and cylindrical phases will be allowed for elsewhere.

3 BUBBLE TORSIONAL OSCILLATIONS

Let us now calculate the eigenfrequencies of torsional oscillations in the bubble phase that is embedded in the spherically symmetric equilibrium configuration of a neutron star crust as constructed in the previous section. We start with the case in which all the matter components participate in the oscillations. Since torsional oscillations are incompressible, i.e., the oscillations do not involve the density variation, one can determine their frequencies with high accuracy even within the relativistic Cowling approximation in which the metric is fixed during the oscillations. The perturbation equation that governs the torsional oscillations can be derived from the linearized equation of motion as (Schumaker & Thorne 1983)

$$\ddot{\mathcal{Y}}'' + \left(\frac{4}{r} + \Phi' - \Lambda'\right) \dot{\mathcal{Y}}' + \left[\frac{H}{\mu^2} \right] \ddot{\mathcal{Y}} + \left[\frac{H}{\mu} \omega^2 e^{-2\phi} - \frac{(\ell + 2)(\ell - 1)}{r^2}\right] e^{2\lambda} \mathcal{Y} = 0,$$

where $\mathcal{Y}$ denotes the Lagrangian displacement in the $\phi$ direction, $H$ is the enthalpy density defined as $H \equiv p + \varepsilon$ with the pressure $p$ and energy density $\varepsilon$, and $\ell$ is the angular index. $\mathcal{Y}$ is associated with the $\phi$ component of the perturbed four

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2 Even in the incompressible limit, there is a propagating transverse mode, often referred to as a “second sound” (Landau & Lifshitz 1986; de Gennes & Prost 1993). This mode involves spatially varying interlayer compression, which in turn couples with spatially varying fluid pressure. A similar kind of “second sound” wave can occur in the columnar phase. All of these modes are beyond the scope of the present analysis.
velocity as $\delta u^{(e)} = e^{-\Phi} \partial_t \mathcal{Y}(t,r)(\sin \theta)^{-1} \partial_\theta P_l(\cos \theta)$, where $P_l(\cos \theta)$ is the $l$-th order Legendre polynomial. As mentioned in the previous section, we assume the situation in which torsional oscillations occur solely in the bubble phase. In terms of the boundary conditions, this situation conforms to the zero-traction conditions, i.e., $\mathcal{Y}' = 0$ at the inner and outer boundaries of the bubble phase.

It is well known that the frequencies of torsional oscillations are proportional to the shear velocity defined as $u_s = (\mu/H)^{1/2}$ (Hansen & Cioffi 1980). Thus, not only $\mu$, but also the enthalpy density plays a role in determining the frequencies of torsional oscillations. As in the case of spherical nuclei in a neutron superfluid (Sotani et al. 2013a), we here consider the reduction of the enthalpy density by superfluidity of nuclear matter outside the bubbles. If the whole nuclear matter behaves as a superfluid and only a neutron gas inside the bubbles participate in the oscillations, then, the effective enthalpy density $\tilde{H}$ that contributes to the oscillations would be minimal. In a real world, however, a portion of nuclear matter outside the bubbles comove nondissipatively with the bubbles by undergoing Bragg scattering off the bcc lattice of the bubbles. This effect, often denoted by the entrainment effect, was originally considered for a neutron gas dripped out of the spherical nuclei (Chamel 2013), and hence the effective enthalpy density $\tilde{H}$ could be quantitatively estimated from similar band calculations. Instead of performing such calculations, we mainly analyze the two extreme cases in which the effective enthalpy density is maximal, i.e., $\tilde{H} = H$, and minimal, corresponding to the minimum and maximum frequencies of torsional oscillations in the bubble phase. We remark in passing that in the minimal $\tilde{H}$ case, neutrons inside the bubbles are assumed to oscillate without escaping from the bubble surface or inviting neutrons outside the surface to come in.

For the stellar models with $M = 1.4M_\odot$ and $R = 12$ km that are constructed from the EOS with various sets of $K_0$ and $L$, the $\ell = 2$ fundamental frequencies of torsional oscillations in the bubble phase are calculated for the two extreme cases of the enthalpy density noted above. The numerical results are shown in Fig. 3 where the left and right panels correspond to the results for the maximal and minimal enthalpy that contributes to the oscillations, respectively. One can observe that the frequencies of torsional oscillations in the bubble phase are almost independent of the values of $K_0$, but strongly depend on the value of $L$. This $L$ dependence arises mainly from the fact that the proton density outside the bubbles and hence $Z_{\text{bubble}}$ decreases with $L$. In fact, the frequencies for the maximal and minimal enthalpies can be well fitted as a function of $L$ via

\begin{equation}
\omega_2 = d_2^{(0)} + d_2^{(1)} + d_2^{(2)} L,
\end{equation}

where $d_2^{(0)}$, $d_2^{(1)}$, and $d_2^{(2)}$ are the adjustable parameters. The resultant fitting lines are also shown in Fig. 3 for $L \leq 75$ MeV. We remark that the fundamental frequencies of torsional oscillations in the crustal region composed of spherical nuclei can also be well fitted as a function of $L$, but the functional form is different from Eq. (7) (Sotani et al. 2012, 2013a). We also remark that in the case of the minimal enthalpy (no entrainment), the eigenfrequencies in the bubble phase are higher than those in the phase of spherical nuclei by a factor that increases with decreasing $L$. This is mainly because the gas density is smaller than the liquid density, while decreasing with decreasing $L$.

\section{Comparison with the QPO Frequencies}

We proceed to compare the observed QPO frequencies from SGRs with the calculated frequencies of torsional oscillations in the bubble phase, together with those of usual torsional oscillations in the crustal region composed of spherical nuclei. We have already discussed the possibility of identifying the observed lowest three QPOs (18, 30, 92.5 Hz) except the 26 Hz QPO in the giant flare of SGR 1806–20 as the $\ell = 2, 3, 10$ fundamental crustal torsional oscillations (Sotani et al. 2013a). In Fig.
Figure 4. (Color online) Comparison of the observed QPO frequencies in SGR 1806–20 with the crustal torsional oscillations calculated for the stellar model with $M = 1.4M_{\odot}$ and $R = 12$ km. The horizontal lines denote the observed QPO frequencies, where the dot-dash lines correspond to 18, 26, 30, and 92.5 Hz discovered in the giant flare, and the broken line corresponds to 57 Hz found from the shorter and less energetic recurrent 30 bursts. The solid lines represent the calculated $\ell = 2$, 3, 6, and 10 fundamental frequencies of torsional oscillations in the crustal region composed of spherical nuclei. The vertical solid line denotes the most suitable value of $L$, 73.5 MeV, for explaining the observed QPOs except 26 Hz in terms of the crustal torsional oscillations. The shaded area denotes the evaluated frequency range of the $\ell = 2$ fundamental oscillation in the bubble phase, which covers the allowed values of the participant ratio $H/H$. The dotted line is the result for the 50 % participant ratio.

We show such an identification in SGR 1806–20 for the stellar models with $M = 1.4M_{\odot}$ and $R = 12$ km. One can find from this figure that the most suitable value of $L$ for explaining the QPOs in terms of the crustal torsional oscillations that have the entrainment effect included by following Chamel (2012) is $L = 73.5$ MeV. We remark that the QPO frequency (57 Hz) discovered from the shorter and less energetic recurrent 30 bursts (Huppenkothen et al. 2014) can also be identified as the $\ell = 6$ fundamental crustal torsional oscillation with the same value of $L$. With all these successful identifications, there is a problem with explaining the remaining QPO frequency, i.e., 26 Hz, because the interval between 26 and 30 Hz is too small to explain by the fundamental crustal torsional oscillations with neighboring $\ell$ as long as the lowest QPO is identified as the lowest crustal torsional mode with $\ell = 2$ (Sotani, Kokkotas & Stergioulas 2007). Note that if the lowest QPO is identified as the crustal torsional mode with $\ell = 3$, one can explain all the low-lying QPOs in terms of the crustal torsional modes only when the value of $L$ is assumed to be of order or even larger than 100 MeV (Sotani et al. 2013; Sotani, Iida & Oyamatsu 2016), being significantly larger than the values deduced from experiments (Tsang et al. 2012).

It is thus interesting to search for the possibility of explaining the 26 Hz QPO in terms of torsional oscillations in the bubble phase in the same stellar model. Recall that these modes can coexist with crustal torsional oscillations in the phase of spherical nuclei, thanks to vanishing shear modulus in the slab phase, and that the $\ell = 2$ fundamental frequency of torsional oscillations in the bubble phase is expected to lie in a range between the frequencies shown in Fig. 4 in the cases of the maximal and minimal enthalpy that contributes to the torsional oscillations. This range corresponds to the shaded region in Fig. 4. Interestingly, 26 Hz is in the middle of this range at the optimal $L$. In fact, to identify the 26 Hz QPO as the fundamental torsional oscillation in the bubble phase, the participant ratio $H/H$ should be close to 50 %, as can be seen from Fig. 4. Here, the participant ratio is closely related to the entrainment effect; to deduce what portion of nucleons outside bubbles are locked to the motion of the bubbles, we have only to know the participant ratio in the absence of the entrainment effect, which is of order 5–20 %. Note that the $L$ values at which the bubble phase is predicted to occur marginally contain the optimal value of 73.5 MeV. This suggests that neutron stars of $M \geq 1.4M_{\odot}$ and $R \geq 12$ km are favored by the present scenario that requires the presence of bubbles, because the calculated eigenfrequencies and hence the optimal value of $L$ tend to decrease with increasing $R$ and/or $M$.

To demonstrate that the above scenario works more reasonably for larger and heavier neutron stars, we have repeated the same analysis for the stellar model with $M = 1.8M_{\odot}$ and $R = 14$ km; the results are exhibited in Fig. 5. This figure shows that the same identifications of the observed QPOs work again, while the optimal values of $L$ and $H/H$ are now $\sim 54$ MeV and $\sim 70$ %, respectively. We remark in passing that the optimal participant ratio in the bubble phase changes with the stellar model, which may open up the way of constraining the star’s mass and radius once the entrainment effect is known from band calculations.

5 CONCLUSION

In summary, we have examined torsional oscillations in the bubble phase located just above the crust-core boundary of neutron stars. The corresponding eigenfrequencies of the fundamental modes have been calculated for various models of the crust EOS, for various values of the star’s mass and radius, as well as for various values of the participant ratio that reflects...
Figure 5. (Color online) Same as Fig. 4 for $M = 1.8M_\odot$ and $R = 14$ km. In this case, the dotted line is the result for the 70% participant ratio.

the entrainment effect, i.e., what portion of nucleons outside the bubbles comove with the oscillating bubbles. The resultant eigenfrequencies in the bubble phase are appreciably higher than the ones in the phase of spherical nuclei. This feature allows one to search for the possibility of reproducing the low-lying QPO frequencies observed from SGRs by appropriately identifying the low-lying QPOs either as a torsional oscillation in the bubble phase or as a usual crustal oscillation in the phase of spherical nuclei, as well as by keeping the value of $L$ reasonable. By simplified calculations, we have succeeded in finding out such a possibility. To make better estimates, however, many questions remain. It would be significant to examine the entrainment effect in the bubble phase based on band calculations (Chamel 2012). For completeness, possible coupling with propagating shear modes in the cylindrical, slab, and cylindrical-hole phases should be allowed for. Magnetic fields, shell and pairing effects on bubbles, electron screening, polycrystalline nature, etc. have been also ignored, but would play a role in modifying the eigenfrequencies in the bubbles phase.

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