An exponential approximation algorithm in linear programming

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Abstract
The algorithm determines a non-iteration procedure for computing the optimal solution of a linear programming problem. We consider approximating system of linear equations to determine the optimal basis variables of the linear programming problem. The usefulness of the algorithm is illustrated by a numerical example.

Keywords: Linear programming; non-iteration algorithm; exponential approximation; approximating system

1. Introduction
The algorithm implements the exponential approximation method for solving linear programming problems. The method is based on replacing the linear programming problem with constraints by the unconstrained nonlinear optimization problem with parameter. The objective function of this problem is the strictly concave function. As parameter tends to zero the extreme values of the objective function and the variables converge to the optimal solution of the original linear programming problem. We use the explicit functional relationship between the primal and the dual approximating variables. The power series expansion of the exponential function of bond between the primal and the dual approximating variables allows transforming the system of necessary and sufficient conditions for an extremum to a system of linear equations. As the parameter tends to zero, we obtain the approximating linear system of equations. Solving this system we obtain the approximate values of variables and constraints of the original linear problem. These values define the active constraints and the optimal basis variables. We find the

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optimal values of the objective function and variables of the original problem from the original system of active constraints with the basis optimal variables. In conclusion for comparing numerical example is solved in two ways by non-iteration algorithm and by simplex method.

2. Data preparation and transformation of the original problem

Linear programming problem will be considered as follows:

$$\sum_{j=1}^{n} \hat{c}_j \hat{x}_j \rightarrow \min,$$

$$\sum_{j=1}^{n} \hat{a}_{ij} \hat{x}_j \geq \hat{b}_i, \; i = 1, \ldots, m, \; \hat{x}_j \geq 0, \; j = 1, \ldots, n > m,$$

where $\hat{x}_j \geq 0, \; j = 1, \ldots, n$ are the variables of the original problem, $\hat{c}_j, \; j = 1, \ldots, n$ are the coefficients of the linear form, $\hat{a}_{ij}, \; i = 1, \ldots, m, \; j = 1, \ldots, n$ are the coefficients of the constraints and $\hat{b}_i, \; i = 1, \ldots, m$ are the right-hand sides of the constraints. It is assumed that (1) has a solution. The corresponding dual problem is

$$\sum_{i=1}^{m} \hat{b}_i \hat{\lambda}_i \rightarrow \max,$$

$$\sum_{i=1}^{m} \hat{a}_{ij} \hat{\lambda}_i \leq \hat{c}_j, \; j = 1, \ldots, n, \; \hat{\lambda}_i \geq 0, \; i = 1, \ldots, m,$$

where $\hat{\lambda}_i, \; i = 1, \ldots, m$ are the values of the original dual problem. Introducing additional non-negative variables $\hat{\lambda}_i \geq 0, \; i = m + 1, \ldots, n$, we transform the problem (2) to the canonical form:

$$\sum_{i=1}^{m} \hat{b}_i \hat{\lambda}_i \rightarrow \max,$$

$$\sum_{i=1}^{m} \hat{a}_{ij} \hat{\lambda}_i + \hat{\lambda}_i \hat{\alpha}_{i,j} = \hat{c}_j, \; j = 1, \ldots, n,$$

$$\hat{\lambda}_i \geq 0, \; i = 1, \ldots, m + n.$$

The coefficients of the objective function, the coefficients of the constraints and the right-hand sides of constraints become positive as a result of equivalent transformations:

$$\sum_{i=1}^{m+n} \hat{b}_i \hat{\lambda}_i \rightarrow \max,$$

$$\sum_{i=1}^{m+n} \hat{a}_{ij} \hat{\lambda}_i = \hat{c}_j, \; \hat{\bar{a}}_{ij} \geq 0, \; i = 1, \ldots, m+n, \; j = 1, \ldots, n,$$

$$\hat{\lambda}_i \geq 0, \; \hat{b}_i > 0, \; i = 1, \ldots, m+n.$$

Further, the constants and the variables are replaced with some new ones as follows: divide the coefficients, the
constraints, and their right-hand sides by \( \hat{c}_j \) and obtain \( \sum_{j=1}^{m+n} (\hat{a}_{ij} / \hat{c}_j) \hat{\lambda}_j = 1, \ j = 1,..., n \). Now we replace variables \( \hat{\lambda}_j \) with \( \hat{x}_j = \hat{\lambda}_j / (e^{\lambda_j^{\text{max}}}) \), \( \lambda_j^{\text{max}} = \max_j (\hat{c}_j / \hat{a}_{ij}) \), \( \hat{a}_{ij} \neq 0 \), \( i = 1,..., m+n \). After this, the coefficients of the objective function and the constraints are: \( \bar{a}_i = (e\hat{a}_i \lambda_i^{\text{max}}) \hat{c}_i \), \( \bar{b}_j = e\hat{b}_j \lambda_j^{\text{max}} \). We replace the coefficients of the objective function with \( b_i = b_i / b_j^{\text{max}} \), \( i = 1,..., m+n \), \( b_j^{\text{max}} = \max_i b_i \). Next, we replace the variables \( \hat{x}_j \) with \( \lambda_i = b_i / b_j \) and obtain: \( a_j = \bar{a}_i / b_i \), \( i = 1,..., m+n \), \( j = 1,..., n \). Finally, the problem (4) can be written in the following form:

\[
\begin{align*}
\sum_{i=1}^{m+n} \lambda_i & \rightarrow \text{min}, \\
0 \leq \lambda_i & \leq e^{-1}, \ i = 1,..., m+n, \\
\sum_{j=1}^{m+n} a_j \lambda_i & = 1, \ j = 1,..., n.
\end{align*}
\]

The corresponding dual problem is

\[
\sum_{j=1}^{n} x_j \rightarrow \max, \ x_j \geq 0, \ j = 1,..., n,
\]

\[
y_i = \sum_{j=1}^{n} a_j x_j - 1 \leq 0, \ i = 1,..., m+n.
\]

The linear programming problem (5) is replaced with the exponential type approximating function with the positive parameter \( \varepsilon \) :

\[
\phi(\hat{x}, \varepsilon) = \sum_{j=1}^{n} \hat{x}_j - \varepsilon \sum_{j=1}^{n} \exp(\sum_{j=1}^{n} a_j \hat{x}_j - 1 - \varepsilon) \varepsilon^{-1} \rightarrow \max,
\]

\[
\hat{x}_j(\hat{x}, \varepsilon) = \exp(\sum_{j=1}^{n} a_j \hat{x}_j - 1 - \varepsilon) \varepsilon^{-1}, \ i = 1,..., m+n, \ \varepsilon > 0.
\]

The exponential transformation of linear programming problems (1) with proofs and examples are described in details in the publications1,2. In this paper, the exponential transformation is applied to the dual problem with respect to the problem (1). As \( \varepsilon \rightarrow 0 \) the optimal value of the approximating function \( \phi(\hat{x}, \varepsilon) \), the variables \( \hat{x}_j \) and \( \hat{x}_j \) converges to the optimal solutions of the primal (5) and the dual (6) linear problems: \( \sum_{i=1}^{m+n} \hat{x}_j \rightarrow \sum_{i=1}^{m+n} \lambda_i^*, \sum_{j=1}^{n} \hat{x}_j \rightarrow \sum_{j=1}^{n} x_j^*, \hat{x}_j \rightarrow x_j^*, \ \lambda_i^*, \ \hat{x}_j^*, \ i = 1,..., m+n \) are the optimal solutions of the primal (5) and dual (6) linear problems. Since \( \phi(\hat{x}, \varepsilon) \) is strictly concave, the first order necessary and sufficient conditions for \( \hat{x} \) to be a maximum can be written as follows:

\[
\nabla \phi(\hat{x}, \varepsilon) / \hat{x} \hat{\lambda} = \sum_{i=1}^{m+n} a_{ij} \hat{x}_j(\hat{x}, \varepsilon) - 1 = 0, \ j = 1,..., n.
\]

Here the variables \( \hat{x}_j(\hat{x}, \varepsilon) \) are the exponential functions
of the variables $\tilde{x}_j$. The system of the necessary and sufficient conditions can be simplified. Substituting the first order power series expansion of $\tilde{\lambda}_i(\tilde{x},\varepsilon)$ with respect to $((\sum_{j=1}^n a_{ij}\tilde{x}_j-1)-\varepsilon)$, we get

$$\sum_{i=1}^n \tilde{x}_i \sum_{j=1}^m a_{ij} a_{ij} - \sum_{j=1}^n a_{ij} - \varepsilon = 0, \quad j = 1, \ldots, n.$$ 

As $\varepsilon \to 0$ we get an approximating system of linear equations to calculate the approximate variables $\tilde{x}_j$:

$$\sum_{i=1}^n \tilde{x}_i \sum_{j=1}^m a_{ij} a_{ij} = \sum_{j=1}^n a_{ij}, \quad j = 1, \ldots, n. \quad (7)$$

The variables $\tilde{y}_i = \sum_{j=1}^m a_{ij} \tilde{x}_j - 1 \leq 0$ are the approximation of the constraints $y_i \leq 0$ of the problem (6). If the constraint $y_i$ is active, then $\tilde{y}_i$ reaches its maximum value. Accordingly largest variables $\tilde{y}_i$ define $n$ the optimal basis variables $\tilde{\lambda}_i$ of the problem (5). On the other hand, smallest variables $\tilde{y}_i$ define $m$ the optimal non-basis variables $\tilde{\lambda}_i = 0$ of the problem (5) and $\hat{\lambda}_i = 0$ of problems (4) and (3). These variables determine the active constraints of the problem (3). The active constraints in turn determine the optimal basis variables of the problem (1).

3. The description of the algorithm

The algorithm consists the following steps: 1) Convert the original problem (1) to (5); 2) Solve the approximating system of linear equations (7) to determine the variables $\tilde{x}_j$; 3) Calculate the approximating values of constraints $\tilde{y}_i$; 4) Identify the active constraints and the optimal basis variables of the original problem (1); 5) Calculate the values of the optimal basis variables of the problem (1).

4. A numerical example

Application of the algorithm is illustrated by the following numerical example:

$$8\tilde{x}_1 + 2\tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4 \rightarrow min,$$

$$\tilde{x}_1 + 3\tilde{x}_2 + 0,2\tilde{x}_3 + \tilde{x}_4 \geq 2,$$

$$4\tilde{x}_1 - 2\tilde{x}_2 + \tilde{x}_3 - 0,5\tilde{x}_4 \geq 4,$$

$$\tilde{x}_j \geq 0, \quad j = 1, \ldots, 4. \quad (8)$$

The solution of the original linear problem is $\sum_{j=1}^4 \tilde{c}_j \tilde{x}_j = 9,7143, \quad (\tilde{x}_j^*) = (1,1429 \quad 0,2857 \quad 0,0 \quad 0,0)$.

The corresponding dual problem is

$$2\hat{\lambda}_1 + 4\hat{\lambda}_2 \rightarrow max,$$

$$\hat{\lambda}_1 + 4\hat{\lambda}_2 \leq 8,$$

$$3\hat{\lambda}_1 - 2\hat{\lambda}_2 \leq 2,$$

$$0,2\hat{\lambda}_1 + \hat{\lambda}_2 \leq 2, \quad (9)$$

The corresponding dual problem is
\begin{align*}
\hat{\lambda}_i - 0.5\hat{\lambda}_i & \leq 1, \\
\hat{\lambda}_i & \geq 0, \quad i = 1, 2.
\end{align*}

Now we introduce the additional non-negative variables \( \hat{\lambda}_i \geq 0, \ i = 3, \ldots, 6 \) to transform the problem to the canonical form:

\begin{align*}
2\hat{\lambda}_1 + 4\hat{\lambda}_2 & \rightarrow \text{max}, \quad (10) \\
\hat{\lambda}_1 + 4\hat{\lambda}_2 + \hat{\lambda}_3 & = 8, \\
3\hat{\lambda}_1 - 2\hat{\lambda}_2 + \hat{\lambda}_4 & = 2, \\
0.2\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_5 & = 2, \\
\hat{\lambda}_1 - 0.5\hat{\lambda}_2 + \hat{\lambda}_6 & = 1, \\
\hat{\lambda}_i & \geq 0, \quad i = 1, 2.
\end{align*}

Next, using equivalent transformations, we transform the problem to the form:

\begin{align*}
8.2\hat{\lambda}_1 + 18.5\hat{\lambda}_2 + 6\hat{\lambda}_3 + \hat{\lambda}_4 + \hat{\lambda}_5 + \hat{\lambda}_6 & \rightarrow \text{min}, \quad (11) \\
\hat{\lambda}_1 + 4\hat{\lambda}_2 + \hat{\lambda}_3 & = 8, \\
4\hat{\lambda}_1 + 2\hat{\lambda}_2 + \hat{\lambda}_4 + \hat{\lambda}_5 & = 10, \\
0.2\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_6 & = 2, \\
2\hat{\lambda}_1 + 3.5\hat{\lambda}_2 + \hat{\lambda}_5 + \hat{\lambda}_6 & = 9, \\
\hat{\lambda}_i & \geq 0, \quad i = 1, \ldots, 6.
\end{align*}

Finally, replacing of constants and variables, we transform the problem to the form:

\begin{align*}
\sum_{i=1}^{6} \hat{\lambda}_i & \rightarrow \text{min}, \quad 0 \leq \hat{\lambda}_i \leq e^{-1}, \quad i = 1, \ldots, 6, \quad (12) \\
3.8329\hat{\lambda}_1 + 6.7957\hat{\lambda}_2 + 5.2384\hat{\lambda}_3 & = 1, \\
12.2654\hat{\lambda}_1 + 2.7183\hat{\lambda}_2 + 4.1907\hat{\lambda}_3 + 25.1441\hat{\lambda}_4 & = 1, \\
3.0664\hat{\lambda}_1 + 6.7957\hat{\lambda}_2 + 125.7205\hat{\lambda}_4 & = 1, \\
6.8141\hat{\lambda}_1 + 5.2855\hat{\lambda}_2 + 4.6563\hat{\lambda}_3 + 27.9379\hat{\lambda}_6 & = 1.
\end{align*}

The approximating system (7) has the form:

\begin{align*}
88.3134\tilde{x}_1 + 87.4376\tilde{x}_2 + 57.9348\tilde{x}_3 + 86.4286\tilde{x}_4 & = 15,867, \\
87.4376\tilde{x}_1 + 807.6175\tilde{x}_2 + 56.0828\tilde{x}_3 + 117.4588\tilde{x}_4 & = 44,3185, \\
57.9348\tilde{x}_1 + 56.0828\tilde{x}_2 + 15861.2369\tilde{x}_3 + 56.8135\tilde{x}_4 & = 135,5826.
\end{align*}
The solution is $(\hat{x}_i) = (0, 1.022, 0.0381, 0.0079, 0.0353)$. The approximating values of the constraints are $(\hat{y}_j) = (0, 1.241, 0.0384, -0.1406, -0.0412, -0.0051, -0.0141)$. We will put these values in the descending order: $\hat{y}_1 = 0.1241, \hat{y}_2 = 0.0384, \hat{y}_3 = 0.0051, \hat{y}_4 = -0.0412, \hat{y}_5 = -0.1406$. The optimal basis variables of the problems (10), (11), (12) correspond to largest $\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4$. The optimal non-basis variables correspond to smallest $\hat{y}_5, \hat{y}_6$. These variables are additional for the first two inequalities of problem (10). The values of $\hat{y}_3 = 0, \hat{y}_4 = 0$ indicate that the first two inequalities are the active constraints of the problem (9). These inequalities transformed to a system of two equations with two variables $\hat{x}_1, \hat{x}_2$. The variables $\hat{x}_1, \hat{x}_2$ of the problem (8) correspond to the two equations of the problem (9). These variables need to be determined by solving the system of two equations with two variables. This is possible if the variables $\hat{x}_3, \hat{x}_4$ are the optimal non-basis variables, that are $\hat{x}_3 = 0, \hat{x}_4 = 0$. Finally, we find the optimal values of the variables of the problem (8) by solving the system:

$\hat{x}_1 + 3\hat{x}_2 = 2,$
$4\hat{x}_1 - 2\hat{x}_2 = 4.$

The solution is $\sum_{j=1}^{4} \hat{x}_j \hat{x}_j^* = 9.7143, \quad (\hat{x}_j^*) = (1, 1.429, 0.2857, 0, 0, 0)$. Thus, we find the variables of the optimal basis solution by solving the approximating system of the linear equations with four constraints and four variables. To calculate the optimal values of the basic variables we solve the system of the linear equations with two variables.

5. Conclusion

To compare, we solve the problem (8) by the simplex method, which is an iterative algorithm. The introduction of additional variables and the transition to maximization leads to the canonical form:

$-8\hat{x}_1 - 2\hat{x}_2 - 2\hat{x}_3 - \hat{x}_4 \rightarrow \max,$
$\hat{x}_1 + 3\hat{x}_2 + 0.2\hat{x}_3 + \hat{x}_4 - \hat{x}_5 = 2,$
$4\hat{x}_1 - 2\hat{x}_2 + \hat{x}_3 - 0.5\hat{x}_4 - \hat{x}_6 = 4,$
$\hat{x}_j \geq 0, \quad j = 1, \ldots, 4.$

The maximum number of iterations for solving the simplex method is calculated by formula $c^* = 6/\left(2!1(6-2)!\right) = 15$. This number depends essentially on the size of the problem. The actual number of iterations is a priori unknown. Let us take variables $\hat{x}_1, \hat{x}_2$ as the starting basis. We transform the problem so that the coefficients of these variables would form the unit matrix.
Table 1. The first iteration

| \( \hat{x}_j \) | \( \hat{c}_j \) | \( \hat{b}_j \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \hat{x}_3 \) | -2              | 4,455           | 0,0909         | 0,0909         | 0,0909         | 1,0818         | 0,0             | 0,0             |
| \( \hat{x}_4 \) | -1              | 1,0909          | 0,0889         | 0,0889         | 0,0889         | 1,0818         | 0,0             | 0,0             |

Criterion K

|\( \frac{\hat{b}_j}{\hat{a}_{ij}} \)|
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0,3636          | 0,2222          | 0,0857          | 0,0714          | 1,7143          | 1,5714          |

The initial basis solution is \( \hat{x}_j = (0, 0, 0, 4, 4555, 1, 0909, 0, 0, 0) \). The first value of the objective function is \( \sum_{j=1}^{4} \hat{c}_j \hat{x}_j = -10,1818 \). The minimum negative evaluation is \( K = -0,3636 \), therefore the variable \( x_1 \) is introduced into the basis. The first column is the key column. The minimum ratio \( \frac{\hat{b}_j}{\hat{a}_{ij}} \) corresponds to the first row. Hence the first row is the key row. The variable \( \hat{x}_3 \) is derived from the basis variables. Now we divide the elements of the key row for the key element \( \hat{a}_{11} = 4,0909 \) and add to the second row elements of the transformed key row, multiplied by \( -1 \(\hat{a}_{21} = 0,1818 \). New basis solution is obtained.

Table 2. The second iteration

| \( \hat{x}_j \) | \( \hat{c}_j \) | \( \hat{b}_j \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \hat{x}_3 \) | -8              | 0,0909          | 0,0909         | 1,0818         | 0,0889         | 1,0818         | 0,0             | 0,0             |
| \( \hat{x}_4 \) | -1              | 0,0889          | 1,0818         | 0,0889         | 1,0818         | 0,0889         | 1,0818         | 0,0889         |

Criterion K

|\( \frac{\hat{b}_j}{\hat{a}_{ij}} \)|
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0,1111          | 0,2222          | 0,0857          | 0,0714          | 1,7143          | 1,5714          |

The solution is \( \sum_{j=1}^{4} \hat{c}_j \hat{x}_j = -9,7778 \), \( \hat{x}_j = (1,1111, 0,0, 0,0, 0,0, 0,0, 0,0, 0,0, 0,0, 0,0) \).

Table 3. The third iteration

| \( \hat{x}_j \) | \( \hat{c}_j \) | \( \hat{b}_j \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \hat{x}_3 \) | -8              | 0,0909          | 1,0818         | 0,0             | 0,0             | 0,0             | 0,0             | 0,0             |
| \( \hat{x}_4 \) | -1              | 0,0889          | 1,0818         | 0,0             | 0,0             | 0,0             | 0,0             | 0,0             |

Criterion K

|\( \frac{\hat{b}_j}{\hat{a}_{ij}} \)|
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0,0857          | 0,0714          | 1,7143          | 1,5714          | 0,0857          | 0,0714          | 1,7143          | 1,5714          |

All the values \( K \) are non-negative here. The optimal solution is \( \sum_{j=1}^{4} \hat{c}_j \hat{x}_j^* = -9,7143 \), \( \hat{x}_j^* = (1,1429, 0,2857, 0,0, 0,0, 0,0, 0,0) \). The advantage of the non-iteration algorithm is that the amount of computation is defined a priori by the dimension of the problem.

References

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