Fast Consensus Protocols in the Asynchronous Poisson Clock Model with Edge Latencies

Gregor Bankhamer¹, Robert Elsässer¹, Dominik Kaaser², and Matjaž Krnc³

1 University of Salzburg, Austria
gregor.stefan.bankhamer@stud.sbg.ac.at, elsa@cosy.sbg.ac.at
2 Universität Hamburg, Germany
dominik.kaaser@uni-hamburg.de
3 University of Primorska, Slovenia
matjaz.krnc@sbg.ac.at

Abstract
We study the problem of distributed plurality consensus among n nodes, each of which initially holds one of k opinions. The goal is to eventually agree on the initially dominant opinion. We first describe our algorithm for the synchronous case and then extend it to the following asynchronous model. Every node is equipped with a random Poisson clock that ticks at constant rate. Upon a tick, the node may open constantly many communication channels to other nodes. The time for establishing a channel is exponentially distributed with constant mean. Once the channel is established, nodes may exchange messages and update their opinions. A simple clustering algorithm is used and each cluster contains a designated leader. These cluster leaders coordinate the progress of the distributed computation. We show that for $k < n^{1/2 - \varepsilon}$, $\varepsilon$ constant, and initial multiplicative bias $\alpha \geq 1 + \frac{k \log n}{\log k}$ all but an $1/\log O(1)$ fraction of nodes converge towards the initial plurality opinion in $O(\log \log n, k \cdot \log k + \log \log n)$ time whp., and all nodes have the initial plurality opinion after additional $O(\log n)$ time whp. To achieve this result, we first derive an algorithm for a system with a designated leader, and then extend our approach to a distributed system without a leader.

1 Introduction

Reaching consensus is a fundamental problem in distributed computing. It has a multitude of applications, from distributed databases [Gif79] to game theory [DP94], network analysis [FKW13, Pel14, LM15] and distributed community detection [CG10, KPS13, RAK07].

In our work, we are given a complete graph $K_n$ of n nodes. Each node starts with one initial opinion from a set of k possible opinions. We will also refer to opinions as colors. Nodes interact with each other, based on some model of communication, and update their opinions accordingly. If eventually all nodes agree on one opinion, we say this opinion wins, and the process converges. We are interested in plurality consensus, that is, the opinion with the largest initial support should win with high probability¹, assuming a sufficiently large bias towards the initial plurality opinion. In this paper, we consider two different communication models and are interested in simple protocols that converge in these models as fast as possible.

¹ The expression with high probability (whp.) refers to a probability of at least $1 - n^{-O(1)}$. 
1.1 Related Work

Synchronized Protocols: From Pull Voting to Plurality Consensus. Plurality consensus in synchronized models has one of its roots in randomized rumor spreading. Two papers by Hassin and Peleg [HP01] and Nakata et al. [NIY99] have considered the so-called pull voting process. This process runs in synchronous rounds during which each player contacts a neighbor uniformly at random and adopts its opinion. They show that in the setting where each node is initially assigned one of two possible opinions the probability for one opinion to win is proportional to the number of edges incident at nodes supporting this opinion. Furthermore, Hassin and Peleg [HP01] have shown that the number of rounds until the two-opinion pull voting process converges on general graphs can only be bounded by $O(n^3 \log n)$. Tighter bounds have been shown in [CEOR13, BGKM16, KMS16].

The expected convergence time for pull voting is at least $\Omega(n)$ on many graph classes. However, for many fundamental problems in distributed computing, such as information dissemination [KSSV00] or aggregate computation [KDG03], much more efficient solutions are known on these graphs. Therefore, Cooper et al. [CER14] introduced the two choices voting process. In this process, every node is allowed to contact two neighbors chosen uniformly at random. If the neighbors' opinions coincide, this opinion is adopted, otherwise the node keeps its opinion. They show in random $d$-regular graphs a convergence time of $O(\log n)$ whp., provided the initial bias is large enough. In [CER+15], these results have been extended to general expander graphs. In [CRRS17], more than two initial opinions have been considered and the following bound on the convergence time in regular expanders was shown. If the initial additive bias between the largest and second largest opinion is at least $Cn \max\{\sqrt{\log n}/c_1, \lambda_2\}$ where $\lambda$ is the absolute second eigenvalue of the matrix $\text{Adj}(G)/d$ and $C$ is a suitable constant, then the largest opinion wins in $O(n/c_1 \log((c_1/(c_1 - c_2)) + \log n)$ rounds whp.

In addition to the results described above, further variants of pull voting have been studied. See, e.g., the work by Abdullah and Draief [AD15] on five-sample voting or the more general analysis of multi-sample voting by Cruise and Ganesh [CG14] on the complete graph.

Making the step from pull voting to plurality consensus, Becchetti et al. [BCN+14] described a simple dynamics on the clique for $k$ opinions. In their protocol, each node samples three opinions and adopts the majority, breaking ties uniformly at random. They need $O(\log k)$ memory bits and prove a tight running time of $\Theta(k \cdot \log n)$ for this protocol, given a sufficiently large absolute bias. More recently, a detailed study and comparison of the two-choices process and the $3$-majority processes has been performed by Berenbrink et al. [BCE+17]. They proved a separation in the running time of these two processes when the number of initial opinions is high. In another recent paper, Becchetti et al. [BCN+15] adopt the $3$-state population protocol from [AAE08] and generalize the protocol to $k$ opinions. They provide a bound on the running time in terms of the so-called monochromatic distance, a measure that depends on initial opinion distribution.

In [BFGK16] the authors propose two plurality consensus protocols. Both assume a complete graph and realize communication via the random phone call model. The first protocol is very simple and, whp., achieves plurality consensus within $O(\log(k) \cdot \log \log \gamma n + \log \log n)$ rounds using $\Theta(\log \log k)$ bits of additional memory. The second, more sophisticated protocol achieves plurality consensus within $O(\log(n) \cdot \log \log \gamma n)$ rounds using only $4$ overhead bits. Here, $\gamma$ denotes the initial ratio between the plurality opinion and the second-largest opinion. They require an initial absolute gap of $\omega(\sqrt{\log^2 n})$. At the heart of their protocols lies the use of the undecided state, originally introduced by Angluin et al. [AAE08]. In [EFK+16], a similar protocol is presented. The authors combine the two choices protocol with push-pull
rumor spreading and achieve essentially the same run time. Another similar protocol for plurality consensus is introduced in [GP16], which achieves similar time and memory bounds.

Towards Asynchronous Protocols: From Population Protocols to Poisson Clocks. Population protocols are a model of asynchronous distributed computations. In the basic variant, nodes are modeled as finite state machines. The protocol runs in discrete time steps, where in each step a pair of nodes is chosen (adversarially or randomly) to interact. The interacting nodes update their states according to a simple deterministic rule. See [AAER07, AR07] for a detailed model description.

Angluin et al. [AAE08] proposed a three state population protocol to solve majority, which is consensus for two opinions. If the larger opinion initially has size at least \( n/2 + \omega(\sqrt{n \log n}) \), their protocol converges after \( O(n \log n) \) interactions. To allow for an easier comparison with the synchronous model, the run time of asynchronous algorithms is commonly divided by \( n \) [AGV15]. To make this explicit, we sometimes refer to this as parallel time. This parallel time is based on the intuition that, in expectation, each node interacts once within \( n \) time steps.

Draief and Vojnović [DV10] and Mertzios et al. [MNRS14] analyzed two similar four-state majority protocols. Their protocols solve exact majority in general graphs and always return the majority opinion, independently of the initial bias, however, require \( O(n^2 \cdot \log n) \) interactions in the clique. More recently, upper and lower bounds for exact majority have been considered in [AGV15, AAE+17, AAG18]. The currently best known protocol from [AAG18] requires \( O(\log n) \) states and \( O(\log^2 n) \) parallel time, which is tight w.r.t. the states and almost tight w.r.t. the run time.

There is a multitude of further related models which differ in various criteria, such as, e.g., the consensus requirement, the time model, or the underlying graph. The following is merely an overview over some variants. In one common variant of the voter model [AF02, CEOR13, DW83, HP01, HL75, LN07, Lig12, BCE+17], the authors are interested in the time it takes for the nodes to agree on any arbitrary opinion. The 3-state protocol from [AAE08] for two opinions in the complete graph has been adapted to a continuous time model [PVV09]. In [AD15], majority on special graphs given by a degree sequence has been considered. Further protocols such as [DV12] guarantee convergence to the majority opinion. Cooper et al. [CDFR16] considered Discordant voting processes, where they assume that only pairs of nodes which have different opinions are selected for an interaction. Berenbrink et al. [BFK+16] solve plurality consensus in general graphs and for general bias using load balancing. However, when the initial bias is small and the number of opinions is polynomial in \( n \), their run time becomes substantial.

1.2 Models and Results

In this paper we present several algorithms for plurality consensus in synchronous and asynchronous models. The algorithms are simple and fast; however, in the asynchronous case we have to elect some leader nodes first in order to coordinate the actions between the nodes. In the following, we describe our communication models.

Synchronous Model. In this standard model, we assume that every node has access to a global clock which ticks exactly once per time step. Whenever the clock ticks, each node is allowed to contact a constant number of nodes chosen independently and uniformly at random. Then, the node updates its own opinion according to the information provided by the sampled nodes. We assume that contacting a node and acquiring information from it is
Fast Consensus Protocols

an atomic operation which does not require additional time. Thus, every node samples a set of constantly many nodes, processes the received information, and updates its opinion within one time step in parallel. The algorithm we present in this model as well as its time complexity are very similar to the ones in [BFGK16, GP16, EFK+16]. However, we think that in order to understand the principle behind our algorithms in the asynchronous case, it is helpful to provide the corresponding algorithm in the synchronous case first.

Asynchronous Model. In this model we assume that every node is equipped with a random Poisson clock that ticks in expectation once per time step (cf. [BGPS06, FPS12]). Then, the node may choose up to a constant number of nodes, either chosen randomly or from a set of addresses known locally, and open communication channels to these nodes. In contrast to the synchronous case, we assume that after initiating a call, it requires time to build up a connection to the sampled node. This time is – in our case – an exponentially distributed random variable with constant parameter $\lambda$. Once the channel is established, the nodes can exchange messages, and we assume in our model that for such an exchange of messages no additional time is required. This reflects the fact that under some circumstances the time required for opening a communication channel may dominate the time required for the entire communication. E.g., in peer-to-peer networks, random walks are commonly used to select random nodes (see, e.g., [VF06]), which requires substantially more time than the actual (direct) communication with the selected node. Also in classical networking the initial three-way handshake and, possibly, name lookup or key-exchange for encryption may cause a long initial delay in comparison to the time required for passing messages over an established channel. In Section 5, we briefly discuss how to extend this model such that some results carry over to the case where the time required for the actual communication is also a random variable.

Our Results – Synchronous Model. We present an algorithm that solves plurality consensus whp. in time $O(\log(k) \cdot \log \log \alpha k + \log \log n)$, where $k \leq n^{1/2-\epsilon}$, for some constant $\epsilon > 0$ is the number of different opinions at the beginning and $\alpha > 1 + k \cdot \log n \cdot \log k/\sqrt{n}$ is the initial ratio between the largest and second largest opinion. Throughout the execution of our algorithm, the nodes go through several stages we call generations, and at any time step each node is in one of these generations. There is a sequence of predefined time steps $\{t_i\}_{i \geq 0}$ at which each node $v$ is allowed to perform a so-called two choices step. That is, if the two sampled nodes are in the same generation (denoted by $i$), this generation is at least as high as $v$’s generation, and they have the same opinion, then $v$ adopts this opinion and sets its generation to $i+1$. At all other time steps (or if the condition for performing two choices steps is not fulfilled) the node just adopts the opinion and the generation of the sampled node with higher generation (if both are in the same generation, then one of them is chosen arbitrarily). We call such a step a propagation step.

At every $t_i$, $i \geq 1$ a new generation $i$ is born with high probability. The sequence of time steps $\{t_i\}_{i \geq 0}$ is chosen such that throughout the steps $t_i, t_i + 1, \ldots, t_{i+1} - 1$ the latest generation created at $t_i$ grows to a linear number of nodes. Then, at time $t_{i+1}$ the two choices step guarantees that whp. a new generation $i + 1$ is born and a certain number of nodes switches to this generation. Also, one can show that the ratio between the largest and second largest opinion becomes higher and higher in later generations. These two properties together imply the result of the synchronous case.
Our Results – Asynchronous Model. If in the asynchronous case one aims for an $O(\log n)$ algorithm (in which the nodes have to pass multiple phases), then it is not possible to strictly synchronize the actions of the nodes within different phases of the algorithm as long as the number of phases is $\omega(1)$. This follows from the fact that synchronization within one phase usually requires $\Theta(\log n)$ time, since the clock of a node may not tick at all within a time frame of $\epsilon \log n$, $\epsilon$ small, with reasonably high probability. In [EFK+17] this problem has been approached by using so-called weak synchronization; i.e., only $n(1 - o(1))$ nodes have been synchronized within one phase, and the remaining nodes have been taken care of at the end, when almost all nodes agreed on one opinion. However, the initial ratio between the first and second opinion that could be handled by the algorithm was quite high (i.e., not less than $1 + 1/\log O(1) n$) and several gadgets were used to achieve this weak synchronization, which resulted in a complex algorithm. Here, we adopt the approach described above for the synchronous case. Again, we use generations to group the nodes that already went through a certain number of stages.

The difference between [EFK+17] and this paper is two-fold. First, using the generation approach, we think that the algorithm becomes simpler, and we can tolerate a much smaller initial bias. Second, in the model of this paper we assume time delays (we call latencies) when a communication channel is to be established between two nodes. While in [EFK+17] the authors dealt with Poisson clocks only, and thus could sequentialize the process in the analysis – due to the memoryless property of these clocks, this sequentialization is not possible in the model of this paper anymore. Note that due to the edge latencies, the delays between the actions of the nodes do not follow an exponential distribution, and the memoryless property is also not given.

We first present an algorithm where we assume that there is one leader in the system. This leader has a restricted amount of memory and if a node sends a request to this node, then it answers with the values stored in this memory. More precisely, the leader has a value for the highest generation allowed to be created in the system (initially set to 1), and it stores a bit which indicates whether the nodes should perform two-choices or propagation steps (initially allowing two choices, see synchronous case). We show that for $k \leq n^{1/2 - \epsilon}$ opinions, where $\epsilon$ can be any small constant, and bias (i.e., the ratio between the largest and second largest opinion) $\alpha > 1 + k \log n \log k / \sqrt{n}$ in time $O(\log \log \alpha k \log k + \log \log n)$ all but a $1/polylog n$ fraction of the nodes has the initial dominant opinion whp. Consensus can be reached after $O(\log n)$ additional time whp.

Our algorithm basically mimics the synchronous procedure. When a node ticks (and the node is not blocked by another communication attempt from a previous tick) then it contacts two randomly chosen nodes and the leader. If the leader’s bit allows two-choices and the generation provided by the leader is higher than the generation of the node, then this node performs a two-choices step – as described in the synchronous case – provided that the information sent by the leader the last time they communicated was the same. The last condition ensures that for any newly created generation all two-choices steps are performed before any propagation steps occur. Once the leader allows the creation of a new generation and thus sets its bit to two-choices, it starts counting the number of so-called incoming signals sent out by some nodes. After a linear number of signals have been received, it sets its bit to propagation. This ensures that for a constant time frame the nodes promote themselves to a new generation only using the two-choices mechanism and thus a new generation of a certain size is created by the two-choices mechanism only.

If a node receives a bit from the leader which allows propagation, it performs a propagation step – as described in the synchronous case – if the information provided by the leader the
last time they communicated was the same. Again, this ensures that no two-choices steps are executed within the propagation stage. When a node contacts a leader, it sends its generation number to it so that the leader has a good approximation of the nodes being in the highest generation created so far. Once a certain (linear) number of nodes are in the highest generation created so far, the leader allows the nodes to promote themselves to a higher generation and the bit is again set such that two-choices steps are allowed to be performed by the nodes. These alternating two choices/propagation stages are repeated until it can be guaranteed that the last generation created is monochromatic whp.

Finally, the algorithm described above is extended to a distributed system without a leader. First, we partition almost all nodes in clusters of size \(\text{polylog } n\). During this procedure, in all these clusters leaders emerge. Then, these leaders act in a distributed manner to coordinate the actions of the nodes, and we derive an algorithm that mimics the procedure designed for the single leader case. At the end, we obtain the same result as in the previous case – however, without assuming the existence of a leader.

2 Synchronous case

2.1 Main Results

We first present a randomized distributed protocol that efficiently solves plurality consensus in the synchronous model. We are given a complete graph on \(n\) nodes, where initially every node is assigned one of \(k\) opinions. We define the initial bias \(\alpha\) as the initial ratio among the first and the second dominant opinion. If the number of opinions is not too large and the initial bias is not too small, we guarantee fast convergence in the following theorem. The protocol is formally specified in Algorithm 1.

\[\begin{aligned}
&\text{Theorem 1.} \quad \text{Let } k \text{ be the number of opinions such that } 2 \leq k \leq n^\varepsilon \text{ for any } 0 < \varepsilon < 1/2, \\
&\text{and let } \alpha > 1 + \frac{k \log n}{\sqrt{n}} \cdot \log k. \quad \text{For a complete graph } K_n \text{ and } k \text{ opinions with initial bias } \alpha, \quad \text{Algorithm 1 converges towards the initially dominant opinion in } O(\log \log k \cdot \log k + \log \log n) \text{ steps whp.}
\end{aligned}\]

In the statement of Theorem 1, the restrictions on the number of opinions and the size of initial bias depend on each other and both affect the running time. In related work, the initial bias is sometimes rather measured as an absolute initial difference among the two most-dominant opinions, however, one can easily express one notion in terms of the other.

By imposing additional restrictions on parameters \(\alpha\) and \(k\), one may give more specific running times. For instance, if the bias from Theorem 1 is increased to \(\alpha > 1 + \frac{\log \log n}{\sqrt{n}} \cdot \log k\), the theorem gives us a running time \(O(\log \log n \cdot \log k)\). Taking this further, if \(k \leq \exp\left(\frac{\log \log n}{\log \log \log n}\right)\) and \(\alpha > 1 + \frac{1}{\log \log n \cdot 1}\), we get a running time of order \(O(\log \log n)\).

A crucial concept introduced by our approach is the notion of generation \(\text{gen}(v)\) of a node \(v\), initially set to \(\text{gen}(v) = 0\) for all nodes. Intuitively, for the nodes of higher generation the probability of having the initial majority opinion will be higher. In fact, we will show that after some threshold generation value \(G^*\), any node of generation at least \(G^*\) will have the same opinion, and this opinion is the initial largest opinion, whp. For most of the time steps of the execution of Algorithm 1, the algorithm uses simple pull voting to overwrite the

\[\text{Note that our analysis also holds for } \alpha > 1 + \frac{A k \log n}{\sqrt{n}} \cdot \log k \text{ for a sufficiently large } A. \quad \text{For simplicity, we used here the slightly weaker version.}\]
Algorithm 1 The basic synchronous procedure.

Require: a vertex \( v \in V(G) \)
1: sample neighbors \( v' \) and \( v'' \) u.a.r.
2: wlog. assume \( \text{gen}(v') \geq \text{gen}(v'') \)
3: if \( t \in \{t_i\}_{i \in \mathbb{N}} \) and \( \text{gen}(v) \leq \text{gen}(v') = \text{gen}(v'') \) and \( \text{col}(v') = \text{col}(v'') \) then
   4: \( \text{gen}(v) \leftarrow \text{gen}(v') + 1 \)
   5: \( \text{col}(v) \leftarrow \text{col}(v') \)
   6: else if \( \text{gen}(v') > \text{gen}(v) \) then
      7: \( \text{gen}(v) \leftarrow \text{gen}(v') \)
      8: \( \text{col}(v) \leftarrow \text{col}(v') \)

current node’s opinion with the sampled one, as long as the sampled generation is higher than the current node’s generation (see Line 6, cf. [BFGK16, GP16, EFK+16]). That said, note that there is a predefined sequence of time steps \( \{t_i\}_{i \in \mathbb{N}} \) in which the algorithm (see Line 3 of Algorithm 1) allows the nodes to perform a so-called two-choices steps, introducing a higher generation to the system whp.

2.2 Model, Definitions, and Conventions

We have \( k \) opinions, where \( 2 \leq k \leq n^\varepsilon \) with \( \varepsilon < \frac{1}{2} \), and we assume that the initial bias \( \alpha \) is bounded by \( \alpha > 1 + \frac{k \log n}{\sqrt{n \log k}} \). The analysis is parametrized by a positive constant \( \gamma \in [0,1] \) which determines the threshold of the generation density needed to create the next generation. The value of \( \gamma \) may be considered a fixed constant throughout the analysis. Empirical data show that the value \( \frac{1}{2} \) works well for reasonable input sizes, while too high values increase the time, and too small values decrease the stability.

Let \( g_i(i) \) be the fraction of nodes of generation \( i \) at time \( t \), let \( c_{j,i,t} \) correspond to the fraction of vertices of color \( j \) inside some fixed generation \( i \), at a fixed time \( t \), and let \( \alpha_{i,t} = \frac{c_{a,i,t}}{c_{b,i,t}} \) be the ratio between a dominant and a second-dominant color in given generation \( i \), where \( a \) and \( b \) are the dominating two colors in generation \( i \) at time \( t \). Also define \( p_{i,t} = \sum_j c_{j,i,t}^2 \) as the probability that two distinct vertices of the same generation \( i \) are at time \( t \) of the same color. Note \( \frac{1}{k} \leq p_{i,t} \leq 1 \), but we will also use the following, slightly better lower-bound.

\[ p_{i,t} = \sum_j c_{j,i,t}^2 = \frac{k - 1}{(\alpha_{i,t} + k - 1)^2} + \left( \frac{\alpha_{i,t}}{\alpha_{i,t} + k - 1} \right)^2. \]

Set \( X_i = \frac{2 \ln (\alpha_{a,i}^{k-1} + k-1) - \ln (\alpha_{a,i}^k + k-1) - \ln \gamma}{\ln(2 - \gamma)} + 2 \). Intuitively, \( X_i \) corresponds to the length of a life-cycle of an \( i \)-th generation, i.e., the number of steps needed whp. from the time when the \( i \)-th generation is created until it populates a \( \gamma \)-fraction of the total number of nodes. As we show in Section 2.4, after a new \( i \)-th generation is born, in the subsequent next \( X_i - 1 \) steps, whp. it grows to be of size at least \( \gamma n \). For \( i \geq 1 \) define a time \( t_i = \sum_{j=0}^{i-1} X_i + 1 \), which corresponds to the time when \( i \)-th generation is born, i.e., the smallest time with \( g_{t_i}(i) \neq 0. \)
Note that for any \( i \), the value of \( X_i \) is bounded by \( O(\log k) \). The total number of generations is set to \( G^* = \left\lfloor \log \frac{\log n}{\log k} \right\rfloor \).

Unless the base of logarithms throughout the text is explicitly given, \( \log n = \log_2 n \) while \( \ln n = \log e \). We say \( a \ll b \) if \( \exists \epsilon > 0 : a \cdot n^\epsilon \leq b \). We say that \( a \sim b \) if \( a \ll b \) and \( a \gg b \).

### 2.3 Concentration of Opinions

In this section we show that for any \( t \in [t_i, t_{i+1} - 1] \), the value of \( \alpha_{i,t} \) is close to \( \alpha_0^2 \). Among the relevant time values, Lemma 4 is focused on the initial value of the bias \( \alpha_{i,t} \) at the birth of \( i \)-th generation. But first we demonstrate the bias-concentration in the first time step.

**Example 3.** For time \( t_1 = 1 \), both \( c_{a,1,1} \) and \( c_{b,1,1} \) are modeled by \( B(n, c_a^2) \) and \( B(n, c_b^2) \), respectively. For big enough \( \delta = o(1) \), it follows from Chernoff bounds that \( \operatorname{whp} \).

\[
\alpha_{1,1} = \frac{B(n, c_{a,0,0}^2)}{B(n, c_{b,0,0}^2)} \operatorname{whp} \frac{n \cdot c_{a,0,0}^2(1 - \delta)}{n \cdot c_{b,0,0}^2(1 + \delta)} \geq \alpha_{0,0}^2 \cdot (1 - 2\delta).
\]

**Example 3** nicely shows the reason of why the bias is squared in each subsequent generation, as well as why we should expect to be careful with small values of the second-dominant color. In the next lemma, we show the growth of the bias for the cases when \( c_{b,\cdot} \) is reasonably large.

**Lemma 4** \((t = t_i - 1 \rightarrow t_{i+1})\). Whenever the bias is not very big at the creation of generation \( i \), its value \( \alpha_{i,t} \) is very close to \( \alpha_{i-1,t_i-1}^2 \). In particular, if \( \alpha_{i-1,t_i-1} \ll \sqrt{n} \), then we have

\[
\alpha_{i,t} \geq \alpha_{i-1,t_i-1}^2 - (1 - 2\delta),
\]

where

\[
\delta = \sqrt{\frac{6 \log n}{n}} \cdot \max(k, \alpha_{i-1,t_i-1}).
\]

**Proof.** For easier notation, instead of writing \( \alpha_{i,t}, \alpha_{i-1,t_i-1}, \alpha_{i-1,t_i-1}(i-1), c_{a,i-1,t_i-1}, c_{b,i-1,t_i-1} \), in this proof we use letters \( \alpha', \alpha, g, c_a, c_b \), respectively. For generation \( i \), similarly as in Example 3, we model \( c_{a,i,t} \) and \( c_{b,i,t} \) by \( \frac{1}{n} \cdot B(n, (g \cdot c_a)^2) \) and \( \frac{1}{n} \cdot B(n, (g \cdot c_b)^2) \), respectively.

First assume that \( \alpha < k \), which implies \( c_a \geq c_b \geq \frac{1}{\alpha + k - 1} = \Theta \left( \frac{1}{k} \right) \), so that in this case \( \delta = k \sqrt{\frac{6 \log n}{n}} \). Together with the assumption \( g \geq \gamma \), it follows

\[
\alpha' = \frac{B(n, (gc_a)^2)}{B(n, (gc_b)^2)} \operatorname{whp} \frac{c_a^2(1 - \delta)}{c_b^2(1 + \delta)} > \alpha^2 \cdot \left( 1 - \frac{2\delta}{1 + \delta} \right) \geq \alpha^2 \cdot (1 - 2\delta).
\]

where the probability that (1) holds is, by the union bound, bounded to

\[
1 - \exp \left( -\frac{n(\delta gc_a)^2}{2} \right) - \exp \left( -\frac{n(\delta gc_b)^2}{3} \right) \geq 1 - \frac{2}{n^\gamma}.
\]

Now assume \( k \leq \alpha \ll \sqrt{n} \). In this case, we have \( c_a \geq c_b \geq \frac{1}{\alpha + k - 1} = \Omega \left( \frac{1}{k} \right) \), and also \( \delta = \alpha \sqrt{\frac{6 \log n}{n}} \). The statement follows very similarly along the lines of (2).
While the lemma above correctly bounds the variance from expectation from below, it allows the choice of an error term $\delta \approx n^{-\varepsilon}$ in a strong way, i.e. by choosing too small value of $\varepsilon$, the distance from expectation $\alpha_{i-1,t_i-1}^2 \cdot 2n^{-\varepsilon}$ could possibly become non-negligible compared to $\alpha_{i-1,t_i-1}^2$. This effect is particularly apparent in the very beginning of our process, where our bias have the smallest value. In the analysis below, we will hence carefully select appropriate value of $\delta$, whenever using the lemma above.

The next lemma deals with the initial bias of $\alpha_{i,t_i}$ in the case when $\alpha_{i-1,t_i-1}$ is asymptotically similar to $\sqrt{n}$.

Lemma 5. If $\alpha_{i-1,t_i-1} \sim \sqrt{n}$, then we have $\sqrt{n} \ll \alpha_{i,t_i}$. Furthermore if $\alpha_{i-1,t_i-1} \gg \sqrt{n}$, then all nodes in generation $i$ are of the same color $a$, whp.

Proof. We use notions of $\alpha$ and $c_b$ as in Lemma 4. In both cases we use the fact that, for a given bias $\alpha$, the value of $c_b$ is maximized whenever all non-dominant colors are of the same cardinality, i.e.

$$c_b \leq \frac{1}{\alpha}. \quad (4)$$

For the first statement we assume that $\alpha \sim \sqrt{n}$ and notice that the number of nodes of color $b$ in generation $i$ at time $t_i$ may be modeled by $nc_{b,i,t_i} \sim B(n, c_b^2)$, with expectation $\mu < 1$. Setting the parameter $\delta = n^\varepsilon$ for $0 < \varepsilon < \frac{1}{2}$, the Chernoff bound whp. implies $B(n, c_b^2) \leq 1 + \delta$, with probability at least

$$1 - \exp\left(-\frac{n^{1+\varepsilon}}{3\alpha^2}\right) \geq 1 - \exp\left(n^{\varepsilon/2}\right).$$

The bound on $\alpha_{i,t_i}$ may be derived accordingly

$$\alpha_{i,t_i} \geq \frac{1-c_{b,i,t_i}(k-1)}{c_{b,i,t_i}} \geq \frac{1}{c_{b,i,t_i}} - k + 1 \geq \frac{n}{1+n^{\varepsilon}} - k + 1 \sim n^{1-\varepsilon}.$$

For the proof of the second statement, assume now $\alpha_{i-1,t_i-1} \geq n^{1/2+\varepsilon}$ for some $\varepsilon > 0$ and again notice that the number of nodes of color $b$ in generation $i$ at time $t_i$ may be modeled by $nc_{b,i,t_i} \sim B(n, c_b^2)$, with expectation bounded as

$$\mu = nc_b^2 \leq \frac{n}{\alpha^2} \sim n^{-2\varepsilon},$$

where we used (4). In this case, setting $\delta = n^{-\varepsilon'}$ with $\varepsilon < \varepsilon' < 2\varepsilon$ assures that $\mu(1+\delta) \ll 1$, with probability at least

$$1 - \exp\left(-\frac{n^{2\varepsilon'-2\varepsilon}}{3}\right)$$

which is tending to zero with the desired speed.

The above two lemmas only describe the growth of the bias upon the creation of generation. Below, we argue that after we establish the value of the initial bias for any generation, that value will not change significantly during time interval $[t_i + 1, t_{i+1} - 1]$. While studying the bias propagation through the time interval $[t_i + 1, t_{i+1} - 1]$, we first consider what happens in an arbitrary time step.
10 Fast Consensus Protocols

Lemma 6 (One step of the growth phase). At a fixed time $t$ let $i$ denote the current highest generation and assume that $t \neq t_i$. Then $\alpha_{i,t+1}$ remains close to $\alpha_{i,t}$, in particular, whp.

$$\alpha_{i,t+1} \geq \alpha_{i,t} \cdot (1 - \delta),$$

where again

$$\delta = \sqrt{\frac{6 \log n}{n}} \cdot \max(k, \alpha).$$

Proof. For easier notation, instead of writing $\alpha_i, \alpha_{i,t}, \alpha_{i,t+1}, v_i, r_i, c_a, c_b, c_t$, we use letters $\alpha, \alpha', x, c_a, c_b$. First assume that $\alpha < k$, which implies $c_a \geq c_b \geq \frac{1}{\alpha} = \Theta\left(\frac{1}{\alpha}\right)$, so that in this case $\delta = k \sqrt{\frac{6 \log n}{n}}$. Notice that an increase of nodes of opinion $j$ in time $t \rightarrow t+1$ may be modeled by $B(n(1-x), x \cdot c_{i,t})$. In particular, $\alpha'$ may be expressed as

$$\alpha_{i,t+1} = \frac{c_{a,i,t+1}}{c_{b,i,t+1}} = \frac{n \cdot x \cdot c_a + B(n(1-x), x \cdot c_a)}{n \cdot x \cdot c_b + B(n(1-x), x \cdot c_b)} \quad \text{whp.}$$

$$\geq \frac{c_a + (1-x)c_a(1-\delta)}{c_b + (1-x)c_b(1+\delta)} = \alpha_{i,t} \cdot \left(1 - \frac{(1-x)\delta}{2 + \delta}\right) > \alpha_{i,t} \cdot (1 - \delta). \quad (5)$$

Observe that from $x \geq \frac{2}{\alpha}$ and $c_b = \Omega\left(\frac{1}{\alpha}\right)$, it follows that

$$\delta^2 n x c_a \geq \delta^2 n x c_b = \Omega(\log n),$$

and the second line holds with probability at least

$$1 - \exp\left(\frac{-n(1-x) \cdot x \cdot c_a \delta^2}{2}\right) = 1 - \exp\left(\frac{-n(1-x) \cdot x \cdot c_b \delta^2}{3}\right) > 1 - \frac{2}{n^2}.$$

Now assume $k \leq \alpha \ll \sqrt{n}$. In this case, we have $c_a \geq c_b \geq \frac{1}{\alpha + k - 1} = \Theta\left(\frac{1}{\alpha}\right)$, and also $\delta = \alpha \sqrt{\frac{6 \log n}{n}}$. The proof is done along the lines of (5), where one need to observe that in this case we have

$$\delta^2 n x c_a \geq \delta^2 n x c_b \geq \alpha^2 \cdot \frac{1}{k} \cdot \frac{1}{\alpha + k - 1} = \Omega(\log n). \quad \boxdot$$

Concluding Concentration Statements

For arbitrary but fixed $i$, we now compare the bias $\alpha_{i,t_i}$ with $\alpha_{i,t_{i+1}-1}$. In some sense we accumulate the error from Lemma 6 $(X_i - 1)$ times. The Corollary 7 bounds the overall error from Lemma 6 and Lemma 4.

Corollary 7. If $\alpha_{i,t_i} \ll \sqrt{n}$, then whp. $\alpha_{i+1,t_{i+1}}$ is very close to its expected value $\alpha_{i,t}^2$, in particular there exists $\epsilon > 0$ such that whp. we have

$$\alpha_{i+1,t_{i+1}} \geq \alpha_{i,t_i}^2 \cdot (1 - n^{-\epsilon}).$$

Proof. Set $\delta = \max(k, \alpha_{i,t_i}) \cdot \sqrt{\frac{6 \log n}{n}}$, as required by Lemma 4 and Lemma 6, and observe that the former lemma implies

$$\alpha_{i,t_{i+1}-1} \leq \sqrt{\frac{\alpha_{i+1,t_{i+1}}}{(1-2\delta)^2}}.$$
On the other hand, by applying Lemma 6 precisely $X_i - 1$ times, we get

$$
\alpha_{i,t_{i+1}} - \alpha_{i,t_i} \geq \alpha_{i,t_{i+1}-1} - \alpha_{i,t_i} \geq \alpha_{i,t_{i+1}}(1-\delta)^{X_i-1} > \alpha_{i,t_i}(1-2\delta)^{(X_i-1)/2}.
$$

Note that Lemma 6 is used at most

$$
\max_i X_i - 1 \leq \ln k - \ln \gamma \leq \ln(2\gamma) - 1 < \ln k
$$
times, hence it follows

$$
\alpha_{i+1,t_i+1} > \alpha_{i,t_i}^2 (1-2\delta)^{(X_i+1)/2} > \alpha_{i,t_i}^2 (1-\ln k) > \alpha_{i,t_i}^2 \left( 1 - \max(k,\alpha) \cdot \sqrt{\frac{6\log n}{n}} \log k \right).
$$

The claim follows by the fact that both $k \ll \sqrt{n}$ and $\alpha_{i,t_i} \ll \sqrt{n}$, hence

$$
\max(k,\alpha) \cdot \sqrt{\frac{6\log n}{n}} \log k \ll 1.
$$

Putting everything together, we observe that the bias is guaranteed to be well concentrated throughout the certain fraction of $\log n$ generations, however this fraction depends on $k$. We formalize the statement in the following proposition, while the case when the bias grows over the threshold $k$ even sooner is considered separately in Lemma 11.

**Proposition 8.** Set $\delta = k \cdot \sqrt{\frac{6\log n}{n}} \log k$. Then either $\alpha_{i-1,t_{i-1}} > k$ or we have

$$
\alpha_{i,t_i} > (\alpha_{0,0} \cdot (1-\delta))^2.
$$

**Proof.** Assume $\alpha_{i-1,t_{i-1}} \leq k$. Using Corollary 7 observe

$$
\alpha_{i,t_i} \geq \alpha_{i-1,t_{i-1}}^2 \cdot (1-\delta)
\geq \alpha_{0,0}^2 \cdot \prod_{j=0}^{i-1} (1-\delta)^{2^j} = \alpha_{0,0}^2 \cdot (1-\delta)^{2^{i-1}}
> (\alpha_{0,0} \cdot (1-\delta))^{2^i}.
$$

We may now determine the number of generations needed, such that the bias increases till $k$. In other words, this is a number $i$, such that $(\alpha_{0,0} \cdot (1-\delta))^{2^i} > k$. Setting $\delta$ as in proposition above, this requires at most

$$
\log \frac{\log k}{\log(\alpha_{0,0} \cdot (1-\delta))}
$$
of generations to increase the bias till $k$.

It is important to point out that, in order for the Proposition 8 to make sense, we need the error-term not to overweight our bias. To deal with this issue, notice that by our restriction on the initial bias we have that

$$
\alpha_{0,0} \cdot (1-\delta) > \left( 1 + \frac{k \log n}{\sqrt{n}} \cdot \log k \right) \cdot \left( 1 - k \cdot \sqrt{\frac{6\log n}{n}} \log k \right)
= 1 + \Theta \left( \frac{k \log n}{\sqrt{n}} \cdot \log k \right),
$$

which again resembles to our initial bound on $\alpha_{0,0}$. 

2.4 Time to Increase a Generation

In this section we upper-bound the time needed to increase the number of nodes in a generation to \( \gamma n \). For the \( i \)-th generation we denote this time by \( X_i \). The content of this section is summarized by the following 1.

▶ Proposition 9. We have \( g_{t_i,1-1}(i) \geq \gamma \) for any \( i \), whp.

Proof. Let \( \varepsilon < \frac{1 - \log_2 k}{2} \) be a small positive constant, i.e. \( z = \sqrt{\frac{\varepsilon}{n}} \cdot n^\varepsilon = o(1) \). We prove the statement by induction on \( i \). Since \( g_0(0) = 1 \), the statement clearly holds for \( i = 0 \).

To prove the induction step, we first focus on the initial cardinality of the generation \( i \). Clearly, by definition \( g_{t_i-1}(i) = 0 \) and at time \( t_i - 1 \) any node of generation \( i - 1 \) will be promoted as a result of Line 3 of Algorithm 1, hence \( B \left(n, (g_{t_i-1}(i-1))^2 \cdot p_{t_i-1, t_i-1}\right) \). Keep in mind that \( p_{t_i-1, t_i-1} \) corresponds to the probability that the two nodes sampled in a time-step just before \( i \)-th generation appeared are of the same color, conditioned by an event that they are both of generation \( i - 1 \). We first show that if \( g_{t_i-1}(i) \geq \gamma \), then whp. we have \( g_{t_i}(i) \geq \gamma^2 p_{t_i-1, t_i-1} (1 - z) \). Indeed,

\[
n \cdot g_{t_i}(i) \sim B \left(n, (g_{t_i-1}(i-1))^2 \cdot p_{t_i-1, t_i-1}\right)
\geq B \left(n, \gamma^2 p_{t_i-1, t_i-1}\right)
\geq n \gamma^2 p_{t_i-1, t_i-1} (1 - z)
\]

where in the second line we used the induction hypothesis \( g_{t_i-1}(i-1) \geq \gamma \). Note that \( p_{t_i-1, t_i-1} > \frac{1}{2} \) implies \( n \gamma^2 p_{t_i-1, t_i-1} \cdot z^2 \geq \gamma^2 n^2 \varepsilon^2 \) and the last line is true with probability at least

\[
1 - \exp \left( -\frac{\gamma^2 n^2 \varepsilon^2}{2} \right) \geq 1 - \exp(-n^\varepsilon),
\]

for \( n \) large enough.

We proceed by showing that, as long as \( x \leq \gamma \), the growth in each additional step is bounded below by a multiplicative factor of \( (2 - \gamma) \). The analysis is similar as in pull broadcasting, in particular, for any \( j \in [t_i, t_{i+1}] \), setting \( x = g_{j}(i) \) and \( x' = g_{j+1}(i) \), we will show that whp.

\[
x' \geq (2 - \gamma) \cdot x \cdot \left(1 - n^{-\frac{1}{2} \cdot \varepsilon}\right).
\]

(7)

To obtain this result observe that the nodes that are promoted to a generation \( i + 1 \) as a result of Line 6 only, which may be modeled by \( X \sim B(n(1 - x), x) \). Assuming \( x \in \left[\frac{1 - \varepsilon}{2}, \gamma\right] \), we bound \( X \) using Chernoff bounds with \( \mu = (1 - x)nx \). We have

\[
P(X < \mu(1 - z)) \leq \exp \left( -\left(\frac{\mu z^2}{2} \right) \right)
\leq \exp \left( -\left(1 - x\right) x kn^2 \varepsilon \right)
\leq \exp \left( -\left(1 - \gamma\right) \gamma^2 n^2 \varepsilon \right)
\leq \exp(-n^\varepsilon) = o(1),
\]
for big enough $n$, and the statement follows by

\[
x' \sim x + \frac{1}{n} B(n(1 - x), x)
\]

whp. $\geq x + \frac{n(1 - \delta)}{n}$

\[
\geq (2 - x) \cdot x \cdot (1 - z),
\]

with probability at least $1 - \exp(-n^\varepsilon)$, where in the first line we crudely assume that the vertices of generation less then $i$ which ticked in time $t$ sampled a node of generation $i$ with probability $x < 2x - x^2$. Since we are looking at the expression (8) on the interval $\leftATE\right\rangle$, only the statement (7) follows.

It remains to show that, after initial step, $X_t - 1 = \frac{\ln \left(\sum_{i=1}^{n} \gamma^i \right)}{\ln(2\gamma)} + 1$ steps suffices for the exponential growth from (7) to increase the cardinality of arbitrary fixed generation $i$ from the value close to $\gamma^2 p_{t-1,i-1}$ at time $t_i$, to the value greater then $\gamma$, at time $t_i + X_t - 1$. Now, using Proposition 8 we may derive

\[
p_{t-1,i-1} \geq \frac{\alpha_0^2 \gamma^i + k - 1}{\alpha_0^2 + k - 1} \sim \frac{\gamma^2 p_{t-1,i-1}}{2 + \ln \frac{\ln(2\gamma)}{\ln(2\gamma)}},
\]

so in fact $X_t \leq 2 - \frac{\ln \gamma p_{t-1,i-1}}{\ln(2\gamma)}$. Let us now redefine $x = g_{t,i}(t) \geq \gamma^2 p_{t-1,i-1}$ and $x' = g_{t+1,i}(t)$. Indeed whp.

\[
x' \geq (2 - \gamma) X_i - 1 \cdot x
\]

\[
\geq (2 - \gamma) \cdot \exp(-\ln \gamma p_{t-1,i-1}) x (1 - z) X_i - 1
\]

\[
\geq (2 - \gamma) \frac{\gamma^2 p_{t-1,i-1}}{\gamma p_{t-1,i-1}} \cdot (1 - z) X_i - 1
\]

\[
\geq (2 - \gamma) \gamma (1 - z) X_i
\]

\[
\geq (2 - \gamma) \gamma \left( 1 - \left( 2 + \frac{\ln k - \ln \gamma}{\ln(2\gamma)} \right) z \right),
\]

from where the result follows by the fact that there exists a fixed $C > 0$ such that

\[
\frac{x'}{\gamma} = (2 - \gamma) \left( 1 - O(n^{-C} \log n) \right) \to 2 - \gamma > 1
\]

as $n \to \infty$.

Throughout the $X_t$ time steps, our concentration analysis was shown to be accurate with probability at least $1 - \exp(-n^\varepsilon)$. Hence, we have that $g_{i,X}(i) \geq \gamma n$ for any $i$ is accurate with probability at least

\[
1 - \left( 2 + \frac{\ln k - \ln \gamma}{\ln(2\gamma)} \right) \cdot \exp(-n^\varepsilon) = 1 - o(1).
\]

\[\square\]

### 2.5 Required Number of Generations

Since the bias is well concentrated within initial $O(\log n)$ generations, this means that the generations are indeed a reliable measure of how close some node is to the consensus, as long as we do not need more then $O(\log n)$ of them. In this section, we determine how many
generations is needed in order for the bias to grow enough so that we claim that the graph is in consensus. The analysis is parametrized by the initial bias.

We start by an easy application of Proposition 8, which describes how many generations are needed to reach a bias of \( n \).

\[ \text{Corollary 10.} \] Let \( i \) be the smallest integer such that \( \alpha_{i,t} > k \). Then, for an initial bias \( \alpha > 1 + \frac{4\log n}{\sqrt{n}} \), we have

\[ i \leq 1 + \log \log_\alpha k \]
generations, whp.

\[ \text{Proof.} \] Set \( G = 1 + \log \log_\alpha k \) and set \( \delta = k \cdot \sqrt{\frac{6\log n}{n}} \log k \), as in Proposition 8. Note that by (6), the error term \( 1 - \delta \) is negligible compared to the initially required bias, i.e.

\[ \Omega(\alpha_0 \cdot (1 - \delta)) = \Omega(\alpha_0). \]

Also notice that by Proposition 8, after \( G \) generations we have

\[ \alpha_G > (\alpha \cdot (1 - \delta))^{\frac{k^2}{1 - k}} = k^2 (1 - \delta)^{2 \log_\alpha n} > k, \]

which concludes the proof of our claim. \( \square \)

It remains to ask how many generations we need after reaching a bias of order \( k \), which we consider in a separate lemma.

\[ \text{Lemma 11.} \] Let \( i \) be the smallest integer such that \( \alpha_{i,t} \geq k \). Then, after an additional \( j = \log \log_k n \) generations, all nodes in generation \( i + j \) are of the same color.

\[ \text{Proof.} \] From now onwards, in order to bound number of remaining generations needed from above, we only look at the fraction of the dominant color. In this case, the dominant color is not shrinking and it is easy to see that the concentration problems from above may not occur here, hence we rather give a simple proof concerning the expected value of the dominant color fraction and omit the concentration constants here.

We denote our color by \( a \), while it’s index represents a generation. Clearly, we have

\[ a_{i+1} \geq \frac{a_i^2}{\sum_{t=1}^i a_t^2} \geq \frac{a_i^2}{a_i^2 + (1 - a_i)^2}, \]

and the dominant color is minimized in the next step if and only if all non-dominant colors, except the second-dominant, already disappeared in generation \( i \). For any integer \( l > i \), we hence have

\[ a_{j+1} \geq \frac{a_i^2}{a_j^2 + (1 - a_i)^2}, \]

and in particular \( a_i \geq \frac{k}{k+1} \). But then, by (9), it follows

\[ a_{i+1} \geq \frac{k^2}{k+1}, \]

and by induction we easily get

\[ a_{i+j} \geq 1 - \frac{1}{1 + k^2j}, \]

and the claim follows. \( \square \)

2.6 Overall Running Time Analysis

Supposing that \( G^* = \log \log_{\alpha_0} n \) generations suffice for our consensus, as a consequence of the above sections, one may accurately predict that after \( T = \sum_{i=0}^{\log \log_{\alpha_0} n} X_i \) steps, at least a \( \gamma \) fraction of nodes will be of high generation (and hence of good color). In the next lemma we will show that the rest of the \( n \cdot \sum_{j<i} g_\ell(j) \) vertices propagate to a generation of at least \( i \) in the consecutive \( O(\log \log n) \) time steps.
Lemma 12. Suppose for some time $T$ that $g_T(i) \geq \gamma$. Then whp. $\sum_{j \geq t} g_T + s(j) = 1$ whenever

$$s \geq \frac{\log \gamma}{\log n} + \log \log n$$

Although this proof follows the general ideas of pull broadcasting, we include it here for the sake of completeness.

Proof. Let $R_t := \sum_{j \geq t} g_T(j)$ and $S_t = \sum_{j < t} g_T(j)$, let $s_1$ be the minimal number of steps needed so that $R_{T+s_1} > \frac{1}{2}$ and let $s_2$ be the minimal time needed so that $S_{T+s_1+s_2} < \frac{1}{n}$. To prove that $\sum_{j \geq t} g_T + s(j) = 1$ it is enough to show that $s_1 + s_2 \leq \frac{\log n}{\log n} + \log \log n$, whp.

Notice that for $t \in [T, T + s_1)$, we have $R_{t+1} \sim R_t + 1/nB(nS_t, R_t)$, hence $E(R_{t+1}) = R_t(2 - R_t)$ where whp. it holds that

$$R_{t+1} \geq R_t(2 - R_t)(1 - n^{-\varepsilon_0})$$

$$\geq \frac{3}{2} \cdot R_t(1 - n^{-\varepsilon_0}),$$

which implies that $s_1 \leq \frac{\log \gamma}{\log n}$, whp.

Now suppose that $t \in [T + s_1, T + s_2]$ and consider the value of $s_2$ such that $S_{T+s_1+s_2} < \frac{1}{n}$. Similarly, the value of $S_{t+1}$ may be modeled by $S_{t+1} \sim \frac{1}{n}B(nS_t, R_t)$, hence $E(S_{t+1}) = S_t^2$, in particular $S_{t+1} \leq S_t^2 \cdot (1 - n^{\varepsilon_0})$, whp. Using also the fact that initially $S_{T+s_1} < \frac{1}{2}$, the value of $s_2$ may be upper-bounded by $s_2 \leq \log \log n$. ▶

Using all the above mentioned pieces, the proof of the main theorem follows along the lines below.

Proof of Theorem 1. Given an initial bias $\alpha_0 \geq 1 + \frac{k \log n}{\sqrt{n}}$, log $k$, let

$$X_i = \frac{2 \ln \left( \alpha_0^{2^{i-1}} + k - 1 \right) - \ln \left( \alpha_0^{2^i} + k - 1 \right) - \ln \gamma}{\ln(2 - \gamma)} + 2$$

and let $G^* = \log \log n$ be the number of generations needed so that $\alpha_{G^*} > n - 1$ whp. In addition, let $A = \frac{\log \gamma}{\log n} + \log \log n$ be the value from Lemma 12. Then, using Corollary 10, Proposition 9, and Lemma 12 we have after

$$T \leq \sum_{i=0}^{G^*} X_i + A$$

time steps that $g_T(G^*) = 1$, whp. Clearly, as $i$ increases, $X_i$ decreases. To simplify the further notions let

$T_1$ be the time needed for the bias to reach value $k$,

$T_2$ correspond to the time needed for the bias to reach value close to $n$, and

$A$ be the time needed for all remaining nodes to reach generation $\log \log_{n} n$.

Since $\log \log_{n} n = \log \log_{n} k + \log \log_{k} n$, we crudely estimate $T_1 + T_2 = \sum_{i=0}^{G^*} X_i$ as

$$T_1 + T_2 < \sum_{i=1}^{\left\lfloor \log \log_{n} k \right\rfloor} X_0 + \sum_{i=1}^{\left\lfloor \log \log_{k} n \right\rfloor} X_{\log \log_{n} k},$$
where whp. we have
\[ X_0 = O(\log k), \] (10)
while
\[ X_{\log \log \alpha} \approx \frac{\ln 4 - \ln \gamma}{\ln(2 - \gamma)} + 2 = O(1). \] (11)

The total time equals to the sum of above mentioned quantities, each of which is upper-bounded by the following complexity classes.
\[ T_1 = O(\log k \cdot \log \log \alpha k), \] see (10),
\[ T_2 = O(1 \cdot \log \log n), \] see (11) and Lemma 11,
\[ A = O(\log \log n), \] see Lemma 12.

Since clearly \( T_2 < A \), we safely omit it from the final asymptotic estimate, which implies the claim from the main theorem. ▶

3 Asynchronous Model with a Centralized Leader

The main difference in the asynchronous case is that we cannot predict precisely when the new generation is born. Also, there is no predefined time when the nodes should switch between two-choices and propagation steps. This is now governed by the states of a leader. This leader determines the time of the creation of a new generation.

In the asynchronous Poisson clock model full convergence cannot be achieved in \( o(\log n) \) time, since there always exist nodes which do not tick even once in \( o(\log n) \) time with constant probability. However, if we only aim for partial convergence, this bound can be avoided. We therefore investigate algorithms which satisfy the so called \( \epsilon \)-convergence. A system is said to have \( \epsilon \)-converged, if at least an \((1 - \epsilon)\) fraction of nodes have the initially dominant opinion. Recall that we say \( a \ll b \) if \( \exists \varepsilon > 0: a \cdot n^\varepsilon \leq b \), and \( a \sim b \) if \( a \ll b \) and \( a \gg b \).

\[ \textbf{Theorem 13.} \text{ Let } k \text{ be the number of opinions such that } 2 \leq k \ll \sqrt{n} \text{ and let } \alpha > 1 + \frac{k \log n}{\sqrt{n}} \cdot \log k. \text{ For a complete graph } K_n \text{ and } \epsilon = \frac{1}{\log^{O(1)} n}, \text{ Algorithm 2 guarantees } \epsilon \text{-convergence towards the initially dominant opinion in time } O(\log \log_{\alpha} k \cdot \log k + \log \log n) \text{ whp. Within additional } O(\log n) \text{ time, all nodes have the initially dominant opinion whp.} \]

3.1 Edge Latencies

Recall that each node is equipped with a random Poisson clock, and upon a tick, the node may perform an operation. Nodes are labeled by addresses, and a node may establish a communication channel to either a random node from a network, or to a known address (e.g., the leader). The leader is predefined and all nodes know its address. Each established communication channel is bi-directional and allows sending or receiving small packets of at most \( O(\log n) \) bits. We assume that the computation is executed instantly and atomically (except for opening communication channels), either when a node ticks or when it receives a message through a communication channel. After the communication is finished, the channel is closed. All nodes are aware of \( n \) and they are able to throw a biased random coin.

Time in our model is measured continuously, by using two related units. The \textit{time step} corresponds to the basic measure of time, and the \textit{time unit} defined below consists of several, but constantly many time steps. Observe that we take into account two types of
asynchronicity, the waiting time between two local operations (ticking time) and the length of the local operation (latency time). In the following, we give an overview over the variables used in our analysis.

- We denote the random variable for the Poisson clock of a node as $T_1$. W.l.o.g., $T_1$ ticks once per time step in expectation.
- The waiting time for establishing communication between two nodes (also called latency) is determined by an exponentially distributed random variable $T_2$ with parameter $\lambda$. Apart from opening communication channels, we assume that the other operations are executed instantly and atomically. While waiting for a communication channel to be established, nodes are locked and do not perform any actions, except for sending out a 0-signal (see Lines 1 and 2 of Algorithm 2). If a non-locked node ticks, we say that this tick is good.
- In any execution of our algorithm any node waits for up to three connections, two for the random samples and one for the leader. Note that the signal sent to the leader in Line 1 of Algorithm 2 does not need a confirmation and thus has no waiting time (however, it needs time for the signal to reach the destination).
- At each good tick a node opens a connection to the leader after having received responses of two random nodes$^3$. We denote the accumulated latency time by $T_3 = \max(T_2, T_2) + T_2$. Observe that $T_3 \leq T_2 + T_2 + T_2 = \lambda \Gamma(3, 1)$. For a fixed node, the waiting time between two consecutive good ticks is distributed as $T_1 + T_2$. Define $T_3$ to be the total waiting time between two good ticks together with the time needed to establish the three communication channels after the second tick, and let $F : (0, \infty) \mapsto [0, 1]$ be a CDF of $T_3$. In particular $T_3 \sim T_2 + T_1 + T_2 \leq \lambda \Gamma(6, 1) + \Gamma(1, 1) \leq \Gamma(7, \beta)$, where $\beta = \min(1, \lambda)$, and $F(t) = P(T_3 < t)$. Note that, by our assumptions on $T_1$ and $T_2$, also $T_3$ has finite mean and variance.
- For the purposes of this analysis we define a time unit to consist of $C_1 = F^{-1}(0.9)$ time steps. We then get that for any time interval $T$ of length $t$ the complete procedure is executed within that interval $T$ with probability at least 0.9.

\[ \begin{align*}
\textbf{Remark 14.} \ & \text{Let } \beta = \min(1, \lambda). \ & \text{Then the length of a time unit } C_1 \ & \text{is at most } C_1 < \frac{10}{3\sqrt{\pi}}. \\
\textbf{Proof.} \ & \text{By definition, CDF of gamma distribution } \Gamma(\alpha, \beta) \text{ with integral shape is} \\
F(x, \alpha, \beta) = e^{-\beta x} \sum_{i=0}^{\infty} \frac{(\beta x)^i}{i!} = \frac{(\beta x)^\alpha}{\alpha^\alpha}. \ & (12) \\
\text{It is clear that } F(x, \alpha, \beta) > 0.9 \text{ is satisfied by plugging } x \leq \frac{\sqrt{\pi} \cdot 0.9}{\beta} \text{ into (12). Since in our case } T_3 \text{ is majorized by a gamma distribution } \Gamma(7, \beta), \text{ it is enough to set } C_1 \text{ to at most } \frac{\sqrt{\pi} \cdot 0.9}{\beta} < \frac{10}{3\sqrt{\pi}}. \ & \end{align*} \]

The length of time unit assures that within any particular time unit a big fraction of nodes perform Lines 1-15 in Algorithm 2, while at the same time the signals to the leader from a large fraction of nodes have already been committed. The described model is very general, so we make an example for the case when both $T_1$ and $T_2$ are governed by Poisson clocks.

$^3$ In our analysis, we assume that the channels to the two random nodes and the leader are established one after the other, but the communication with these three nodes is performed concurrently after the three channels have been established. A slightly more detailed analysis has to be applied if the information of the two random nodes is pulled before the leader is contacted.
Figure 1 The values of $F^{-1}(0.9)$, parametrized by the expected latency waiting time $1/λ$, in case when $T_2$ is distributed exponentially.

Example 15. Assuming $E(α)$ stands for exponentially distributed r.v. with parameter $α$, we have $T_1 = E(1)$ and we set $T_2 = E(λ)$. In this case, one may calculate

$$E(T_3) = 1 + \frac{3}{λ}$$

and calculating $F(t)$ shows that the value $F^{-1}(0.9)$ grows linearly with $1/λ$, as expected. While the precise expression $F(t)$, i.e., CDF of a waiting time $T_3$ is not very informative to include here as an expression, Figure 1 shows the values of $F^{-1}(0.9)$, parametrized by the expected latency waiting time $1/λ$.

Additional Notions and Differences from the Synchronous Case. In order to avoid repeating Section 2.5, the constants in the asynchronous case are set such that we may use the analysis of the number of needed generations in similar way as in the synchronous case. The number of the needed generations is again parametrized by an initial bias, as described by Corollary 10.

We use the same notation as in the synchronous case, however in what follows, the time is continuous and expressed in terms of time units. We define $g_t(i)$ to be the fraction of nodes of generation $i$ at time $t$, let $c_{j,i,t}$ correspond to the fraction of vertices of color $j$ inside some fixed generation $i$, at a fixed time $t$, and let $α_{i,t} = \frac{c_{a,i,t}}{c_{b,i,t}}$ be the ratio between a dominant and a second-dominant color in given generation $i$, where $a$ and $b$ are the dominating two colors in generation $i$ at time $t$. Also define $p_{i,t} = \sum_{j} c_{j,i,t}^2$ as the probability that two distinct vertices of the same generation $i$ are at time $t$ of the same color, and let $t_i(χ)$ correspond to the point in time, when generation $i$ globally reached cardinality at least $χ · n$. Define $t_i$ as the number of time units when generation $i$ first appeared.

3.2 Our Procedure

We analyze the procedure defined in Algorithm 2, which is run atomically at each node, asynchronously. While the idea of the algorithm is the same as in the synchronous case, we introduce a couple of additions which allow us to use similar ideas in the asynchronous setting. Since some of these additions are far from realistic, we relax them in the next section.

The leader does not operate based on any clock, but just receives signals from nodes. On each request, the leader performs a simple incrementation-based operation described in
Algorithm 2. The basic asynchronous procedure of a non-leader node.

Require: a non-leader vertex \( v \in V(G) \) ticked
1: send 0-signal to the leader
2: if not locked, then locked ← True, else break!
3: contact \( v', v'' \), sampled u.a.r \( \triangleright \) takes time
4: contact leader, read his \( \text{gen} \) and \( \text{prop} \), \( \triangleright \) takes time
5: if \( l.\text{gen} \) and \( l.\text{prop} \) coincide with \( \text{gen} \) and \( \text{prop} \) from leader then
6: if \( \text{gen} - 1 = \text{gen}(v') = \text{gen}(v'') \) and \( \text{col}(v') = \text{col}(v'') \) and \( \neg \text{prop} \) then
7: \( \text{col}(v) ← \text{col}(v') \)
8: \( \text{gen}(v) ← \text{gen} \)
9: else if \( \exists\bar{v} \in \{v', v''\} : \text{gen}(v) < \text{gen}(\bar{v}) \) and \( (\text{gen}(\bar{v}) < \text{gen} \text{ or } \text{prop}) \) then
10: \( \text{col}(v) ← \text{col}(\bar{v}) \)
11: \( \text{gen}(v) ← \text{gen}(\bar{v}) \)
12: if \( \text{gen}(v) \) has increased, notify \( \text{leader} \) with \( v.\text{gen}-\text{signal} \)
13: else
14: update own \( \text{1.gen} \) and \( \text{1.prop} \) with \( \text{gen} \) and \( \text{prop} \)
15: locked ← False

Algorithm 3. The purpose of the leader is to keep the relevant values of \( \text{gen} \) and \( \text{prop} \), which are always publicly available to anyone. In essence, number \( \text{gen} \) represents the currently highest allowed generation in the system, initially set to 1, while a Boolean variable \( \text{prop} \), initially set to false, contains information regarding whether or not nodes are allowed to propagate their opinion to \( \text{gen}-\text{th} \) generation in a pull-based approach.

The basic procedure of a non-leader node, described in Algorithm 2, is similar to the algorithm for the synchronous case, with modified conditions for both types of generation-promotions\(^4\). Instead of specifying the time steps in Line 9 and Line 6 of Algorithm 2 (as it was done in the synchronous case in Line 6 and Line 3 in Algorithm 1), we use the state of the leader for the nodes to decide when to perform a two-choices or a propagation step. To avoid an interleaving of the two different promotion mechanisms (two-choices and propagation), the nodes store the last-seen state of the leader (\( \text{1.prop} \) and \( \text{1.gen} \)) and compare it with the current state.

Whenever a node opens a communication channel (which might take some time), it is locked until it has finished the execution of Lines 1-15. In that way, no other lines except Line 1 (sending a signal to the leader) can be executed if further ticks occur. Although contacting a node and establishing a communication channel implies a waiting time of \( T_2 \), sending a signal (Line 1 and Line 12) is different. When we say that a node \( v \) sends a signal to its leader, \( v \) does not wait for the reply (nothing is locked), but carries on with the next instruction. Such a signal will still need a time of \( T_2 \) to reach the destination, however, this time does not affect the execution of the algorithm by \( v \).

In the execution of our algorithm, we assume that the values \( \alpha_0 \) and \( k \) are known to the nodes. If the nodes are only aware of certain bounds on these values, then in the corresponding lines of the algorithm these bounds are used instead.

The following invariants can be observed from Algorithm 2.

- In any stage of the algorithm, no node can achieve the generation value higher than \( \text{gen} \)

\(^4\) In the algorithm, \( C_3 \) is a large enough constant to be specified later
Algorithm 3 The behavior of the leader node.

Require: an $i$-signal
1: if $i = 0$, increment $t$
2: if $t = C \cdot n$ then $\triangleright$ allow propagation
3: $\text{prop} \leftarrow \text{True}$
4: if $i = \text{gen}$ then $\triangleright$ from the leader (see Line 6 and Line 9).
5: increment $\text{gen}_\text{size}$$\triangleright$ A necessary condition for a promotion of an arbitrary node $v$ via Line 9 is that $\text{prop} = \text{True}$ before the relevant node $\bar{v}$ was sampled (cf. Line 5).
6: if $\text{gen}_\text{size} \geq \left\lceil \frac{n}{2} \right\rceil$ and $\text{gen} < \left\lceil \log \frac{\log n}{\alpha_0 - 1} \right\rceil$ then $\triangleright$ allow next generation $\triangleright$ Suppose that node $v$ is promoted to a generation $i$ as a consequence of Line 6. Then the promotion occurred during the time the leader was in $\text{prop} = \text{False}$ and $\text{gen} = i$.
7: increment $\text{gen}$$\triangleright$ Whenever the leader sets $\text{prop}$ to $\text{True}$ while at $\text{gen} = i$, no node may be promoted to generation $i$ as a consequence of Line 6 anymore (cf. Line 5).
8: $t \leftarrow 0$
9: $\text{prop} \leftarrow \text{False}$

In this section we analyze the algorithm. We first bound the time needed for a newly created generation to reach a certain size using two-choices steps only. Then, we bound the time needed for the algorithm to increase the size of this generation to $n/2$ by propagation steps and the color fractions in generation $i - 1$ during the time the new generation $i$ is growing by two-choices steps. After that we show that the color fractions in the new generation are highly concentrated during the propagation steps. These results are then used to show that whp, from one generation to the next the ratio $\alpha$ of the largest and second largest opinion becomes squared (up to some small error term). From this, we can compute (Corollary 24) how many generations have to be traversed in order to obtain the first monochromatic generation whp. In Lemma 25, we conclude the proof.

In our setting, the r.v. $p_{i,t}$, $c_{j,i,t}$, and $\alpha_{i,t}$ are well concentrated around their expectation (see further below). Hence, we will use them without the notion of time (but subject to the conditions mentioned in the statements below). Also, as in the synchronous case, we will treat the case when the bias is large separately.

In our analysis, we consider in this version the case $k \ll n^{1/4}$ and $\alpha_{0,0} = 1 + \omega(k^2 \log n \log k/\sqrt{n})$. For the other cases of $k$ and $\alpha_{0,0}$ tighter bounds are needed in the error terms of Lemma 20, Corollary 21, and Lemma 23. To obtain these bounds, the exact probability for specific loads in a Pólya-Eggenberger urn has to be considered (see e.g. Theorem 3.1 in [Mah08]), and the tighter bounds lead then to the statement of the theorem in the omitted case.

Time to Increase a Generation. We first show that for an arbitrary but fixed generation $i$ we have $t_{i+1} - t_i = O(- \log p_{i,t})$ whp. Starting from time $t_i$, in Proposition 16 we show that within constant time units the condition in Line 2 of Algorithm 3 becomes satisfied, and by that time $\text{gen}_\text{size} \geq \frac{n}{2} n$ whp. Then, we argue that $t'' = O(\log k)$ steps are needed such that the $i$-th generation exceeds $\frac{n}{2}$, whp.
Proposition 16. For a fixed generation $i$, let $t' > 0$ be the time (i.e., the number of time units) required after time $t_i$ for the counter in Line 2 of Algorithm 3 to reach $C_3n$ ticks. The following statements are true:

a) The two-choices phase lasts roughly two time units such that we have whp.

$$2 < t' < 2 \cdot \left(1 + \frac{\log n}{\sqrt{n}}\right).$$

(13)

b) Whp., we have $t_i + t' \geq t_i \left(\frac{\log n}{n}\right)\).$

Proof. We prove the claims separately.

a) Let $t'$ be the amount of time units needed for the leader to count to $C_3n$ signals, where $C_3 = C_1 \left(2 + \frac{\log n}{\sqrt{n}}\right)$, and let $T = t'C_1$ be the corresponding number of time steps.\(^5\) By the displacement theorem (see for instance the book by Kingman [Kin93]), we may assume that the 0-signals arriving to the leader are distributed by a perfect Poisson clock with rate $n$.

Let $T' = C_3n$ be the total number of the received ticks. Alongside the lines of [BGPS06, Lemma 1], we have

$$P\left(\left| T - T' \right| > \delta T' \right) < 2 \exp\left(-\frac{\delta^2 T'}{2}\right),$$

where we will use $\delta = 2\sqrt{\frac{\log n}{n}}$, so that the probability of bad event is at most $\frac{1}{n^2}$. This implies that whp.

$$T \in \left[\frac{T'}{n}(1-\delta), \frac{T'}{n}(1+\delta)\right],$$

with the lower bound

$$T \text{ whp} > C_1 \left(2 + \frac{\log n}{\sqrt{n}}\right) \left(1 - 2\sqrt{\frac{\log n}{n}}\right) > 2C_1.$$

For the upper-bound we obtain

$$T < C_1 \left(2 + \frac{\log n}{\sqrt{n}}\right) \left(1 + 2\sqrt{\frac{\log n}{n}}\right) = 2C_1 \cdot \left(1 + \frac{\log n}{\sqrt{n}}\right)$$

and the expression (13) follows readily.

b) We want to show that the counting time of our leader (i.e., $t'$ time units) suffices for the generation $i$ to grow to contain at least $\frac{p(i)}{n}$-th fraction of nodes. To this end, define a modified process, where we set $t'$ to precisely two time units. Since this is a lower bound on the actual value, and since in the time interval $t \in [t_i, t_i + t']$ the value of $g_t(i)$ can only grow monotonously, it is enough to show that the modified process achieves the desired value of $g_t(i)$ at time $t_i + 2$.

Similarly as in the original process, at time $t < t_i$ clearly $g_t(i) = 0$, however for our modified process during the time interval $[t_i, t_i + 2]$, any node of generation $i$ is guaranteed to be promoted to $i$ as a result of two-choices condition only. For the mentioned time interval we note the following statements, each of which is easy to show.

\(^5\) Note that the scaling factor between time units and time steps is by definition $C_1$. 
i) At each such execution of Algorithm 2, we have $g_t(i-1) \geq \frac{1}{2} - \frac{\rho_{i,t}}{9}$.

ii) Hence the probability that any fixed node $v$ first spends one execution to refresh the value of $gen$, and uses next execution to sample two nodes of the same opinion from generation precisely $i - 1$, is equal to at least $0.81 \cdot p_{i,t} \left( \frac{1}{2} - \frac{\rho_{i,t}}{9} \right)^2 > \frac{\rho_{i,t}}{8.2}$.

iii) Considering all this, the value of $g_{t'}(i)$ may be minorized by

$$\frac{1}{n}B\left(n, \frac{p_{i,t}}{8.2}\right) \text{ whp.} > \frac{p_{i,t}}{8.2} - \frac{\log n}{n},$$

which concludes our claim.

The next proposition deals with the time required in Algorithm 3 once the condition in Line 2 is fulfilled until the condition in Line 6 is fulfilled. For the sake of the analysis, we assume that after the two-choices phase an extra time unit is used, which allows roughly 9/10 of the nodes to update their leader information accordingly. For the remaining of this phase, it is enough to just consider these informed nodes.

**Proposition 17.** For arbitrary generation $i$, let $t_i + t'$ be the time when the two-choices phase was concluded. Then after additional $t'' = \log \frac{9}{2\rho_{i,t}}/\log 1.4$ time units, the cardinality of the $i$-th generation exceeds $n/2$ whp., i.e., $g_{t_i + t' + t''} (i) \geq \frac{1}{2}$.

**Proof.** Define $t'$ similarly as in Proposition 16 and first observe the growth in arbitrary time unit. By the construction, for any $t \in [t_i + t', t_i(\frac{1}{2})]$, the $i$-th generation will only grow by propagation-steps. We define $x = g_{t,i}(i)$ and $x' = g_{t+1,i}(i)$, where by Proposition 16 we have that $x \geq \frac{p_{i,t}}{9}$. If during the time interval $[t, t + 1]$, an arbitrary node $v$ (i) arrived from generation at most $i - 1$, (ii) had an updated leader-information, (iii) sampled a node from generation $i$, and (iv) executed a complete operation in the mentioned time-unit, then surely $v$ increased its generation to $i$. In fact, it is enough to only consider such promotions which may be modeled directly as

$$x' \geq x + \frac{1}{n}B(n, (1 - x) \cdot 0.9 \cdot x \cdot 0.9) \text{ whp.}$$

$$> x + x(0.81 - 0.81 \cdot x) \left( 1 - \frac{\log n}{\sqrt{n}} \right)$$

$$> 1.4 \cdot x,$$

where in the first line we crudely assume that the vertices of generation less than $i$ which ticked twice within a time interval $[t, t + 1]$ sampled a node of generation $i$ only with probability $x$. To prove that $g_{t_i + t''} (i) \geq 0.5$, it is enough to iterate the above process $t''$ times. Indeed,

$$g_{t_i + t''} (i) \geq 1.4^{t''} \cdot \frac{p_{i,t}}{9} \geq \frac{1}{2}.$$

Together, Proposition 16 and Proposition 17 yield the following corollary which states that the time needed to globally increment a generation is of the same order as in the synchronous model.

**Corollary 18.** Whp., $t_{i+1} \leq t_i + 2 + \frac{2\log n}{\sqrt{n}} + \log 1.4 \frac{9}{2\rho_{i,t}} = t_i + O(-\log p_{i,t})$.

**Concentration Results.** We now show that in our setting the random variables $p_{i,t,c_j,i,t}$, and $\alpha_{i,t}$ are well concentrated around their expectation. We therefore use them without the notion of time, as long as we use them with respect of the conditions mentioned in the statements below. Similarly to the synchronous model we treat the case when the bias is
large separately. We therefore assume throughout this section that $c_b > \frac{\log n}{\sqrt{n}}$ (unless stated otherwise).

The main difference to the asynchronous case is that we cannot predict precisely when the new generation is born. The birth is defined by the cardinality of the previous-largest generation, rather than by a global time given in advance. Another difference is that during the time units of two-choices, the ratio may change slightly (unlike in the synchronous case). For these reasons, the analysis of concentration of opinion requires a modified approach. We start with the concentration of r.v. $c_{j,i}$, for arbitrary color $j$.

**Concentration of Color Fractions.** As usual, define $t'$ such that $t_i + t'$ is the time whenever the leader allows propagation, see Line 2. In order to claim that the distribution of opinions is well-concentrated during the generation life-cycle, we proceed with the following lemma.

**Lemma 19.** For any color $j$ and time $t \in [t_i, t_i + t']$, we have

$$c_{j,i-1,t} \in [c_{j,i-1,t_i} - z, c_{j,i-1,t_i} + z],$$

with $z = 2e^{-3C_i} \sqrt{\frac{\log n}{n}}$ and probability at least $1 - \frac{1}{n^2}$.

**Proof.** At time $t \in [t_i, t_i + t']$, some nodes from generation $i - 1$ appear, and some leave, potentially changing the values $c_{a,i-1,t_i}$ and $c_{b,i-1,t_i}$, and affecting the newly created members of generation $i$. In this analysis we show that the deviation of values $c_{a,i-1,t_i}$ and $c_{b,i-1,t_i}$ throughout the time interval $[t_i, t_i + t']$ remains small. The main ingredients for the Azuma concentration results are the following:

- We need to estimate the minimal possible cardinality of generation $i - 1$, at any time $t \in [t_i, t_i + t']$. For the purposes of this bound it is enough to crudely use the fact that during $t'C_1$ time steps, some nodes did not tick at all. In particular, the probability for arbitrary node not to tick even once is at least

$$\exp(-C_1 t') > e^{-2C_1} \left(1 - C_1 \frac{4\log n}{\sqrt{n}}\right),$$

which gives us a lower estimate of at least $e^{-3C_1 n}$.

- The number of nodes from generations at most $i - 2$ that is added to generation $i - 1$ is at most $n/2$.

- For arbitrary color, all color fractions in generation $i - 1$ change at most $n$ times, while the sequence of these fractions for any fixed color, and also the corresponding sequence of biases form a martingale. Hence, the length $l$ of our martingale is at most $3n/2$.

- The difference $c$ of any two consecutive values of color fraction (for any color) cannot exceed $c \leq \frac{1}{e^{-3C_1 n}}$.

For any color $j$, the above assumptions allow us to use Azuma inequality in order to upper-bound $\Delta_i := [c_{j,i-1,t_i} - c_{j,i-1,t_i + t'}]$. In particular, we have the following

$$P(\Delta_i \geq z) \leq 2 \exp\left(\frac{-z^2}{2l \cdot c^2}\right) \leq 2 \exp\left(\frac{-z^2 e^{-6C_2 n}}{2}\right)$$

$$= 2 \exp(-2 \log n) \leq \frac{2}{n^2}. \quad \blacktriangleleft$$

**Lemma 20.** For any color $j$ and time $t \in [t_i + t', t_{i+1}]$, we have

$$c_{j,i,t} \in [c_{j,i,t_i + t'} - z, c_{j,i,t_i + t'} + z],$$

with $z = 18k \sqrt{\frac{\log n}{n}}$ and probability at least $1 - \frac{1}{n^2}$.
Fast Consensus Protocols

Proof. The proof follows from Azuma inequality, similarly as the proof of the Lemma 19, with the following particulars.

- Throughout time $t \in [t_i + t', t_i + t']$, nodes are only added to the generation $i$. The change of the value $c_{i,j,t}$ is determined by whether the newly promoted node is of color $j$ or not.
- The minimal possible cardinality of generation $i$ is its initial cardinality at time $t_i + t'$, which is guaranteed to be at least $\frac{p_{i-1,t_i-1}}{9} t_i - 1$-th fraction of $n$.
- The number of nodes from generations at most $i - 1$ that is promoted to generation $i$ during our interval is at most

$$n(1 - g_{t_i + t'}(i)) < n,$$

which corresponds to the length $l$ of our martingale.
- The difference $c$ of any two consecutive values of color fraction (for any color) cannot exceed $c \leq \frac{9}{p_{i-1,t_i-1} n} \leq \frac{9k}{n}$.

For any color $j$, the above assumptions allow us to use Azuma inequality in order to upper-bound $\Delta_j := |c_{j,i-1,t_i} - c_{j,i-1,t_i + t'}|$. In particular, we have the following

$$P(\Delta_j \geq z) \leq 2 \exp\left(\frac{-z^2}{2l \cdot c^2}\right) \leq 2 \exp\left(\frac{-z^2 n}{162 \cdot k^2}\right) = 2 \exp(-2 \log n) \leq \frac{2}{n^2}. \quad \Box$$

Concentration of $p_{i,t}$. The following corollary follows easily.

**Corollary 21.** Let $z$ be as defined in Lemma 20. For any color $j$ and time $t \in [t_{i-1} + t', t_i + t']$, we have

$$p_{j,i-1,t} \in [p_{j,i-1,t} - 3z, p_{j,i-1,t} + 3z],$$

with probability at least $1 - \frac{1}{n^2}$.

Concentration of the Bias. Using the fact that the colors are well concentrated, we show that for any fixed generation $i$ and $t \in [t_i + t', t_i + t']$, the value of $\alpha_{i,t}$ is concentrated around its expected value $\alpha_{i,t}^0$. We first estimate the value of $\alpha_{i,t_i + t'}$ in Lemma 22, which describes the bias of generation $i$ at the time when the propagation phase for the $i$-th generation begins

**Lemma 22 (Time $t_i \to t_i + t'$).** Let $t_i + t'$ correspond to the time when the propagation phase for $i$-th generation begins. Then the value of $\alpha_{i,t_i + t'}$ is very close to $\alpha_{i-1,t_i}^2$. In particular,

$$\alpha_{i,t_i + t'} \geq \alpha_{i-1,t_i}^2 \cdot \left(1 - 5 \cdot \max(k, \alpha_{i-1,t_i}) \sqrt{\frac{\log n}{n}}\right).$$

Proof. For $\alpha_{i,t_i + t'}$, we upper bound the number of $a$-colored nodes and lower bound number of $b$-colored nodes, in generation $i - 1$, at time $t_i + t'$. By the above lemma, $c_{a,i,t_i + t'}$ may be minorized by $B\left(\mathcal{N}_{(a_t_i, i, t_i - z)^2}/(ng_{i,t_i + t'}(i))\right)$ while $c_{b,i,t_i + t'}$ may be majorized by $B\left(\mathcal{N}_{(a_t_i, i, t_i + z)^2}/(ng_{i,t_i + t'}(i))\right)$, where $\mathcal{N} = O(n)$ is the number of events where some node from generation less then $i$ ticked during $[t_i, t_i + t']$ time interval and sampled both vertices from generation $i - 1$. Also, we have $n \cdot c_{b,i-1,t_i} \geq \frac{n}{\alpha_{i-1,t_i} + k - 1}$ which is asymptotically
Let 

\[
\Theta\left(\frac{n}{\max(K_n, \alpha, \alpha_{1,t_i})}\right),
\]

as in Lemma 6. Setting accordingly 

\[z' = \max(k, \alpha_{1,t_i}) \sqrt{\frac{n \log n}{n}}\]

and 

\[z = \frac{2}{e^{2\pi^2}} \sqrt{\frac{\log n}{n}},\]

it follows that 

\[
\alpha_{i,t_i+1} \geq \frac{B(N_i(c_{i,t_i} - z))}{B(N_i(c_{i,t_i} + z))} \geq \frac{(c_{i,t_i} - z)}{1 + z} \cdot \frac{1 - z'}{1 + z'}
\]

where the second line follows by 

\[c_{i,t_i} > \frac{\log n}{n}\]

and high-probability statement in the first line is true, by the union bound, with probability at least 

\[1 - \frac{1}{n}.\]

We now consider the propagation phase. The concentration result is now given in Lemma 23. It follows from repeating the process 

\[t'' = \log \frac{9}{2\delta} / \log 1.4\]

times.

**Lemma 23.** Assume 

\[k \geq \alpha_{1,t_i}.\]

Whp., \(\alpha_{i,t_{i+1}}\) is close to its expected value \(\alpha_{i,t_i}^2\), i.e.,

\[
\alpha_{i,t_{i+1}} \geq \alpha_{i,t_i}^2 \left(1 - O\left(k^2 \sqrt{\frac{\log n}{n}}\right)\right).
\]

**Proof.** Set \(t'_i = t_i + t'\) and 

\[z = 18k \sqrt{\frac{\log n}{n}}\]

as in Lemma 20 and note that

\[
\alpha_{i,t_{i+1}} \geq \alpha_{i,t_i}^2 \left(1 - \frac{2z}{c_{i,t_i} + z}\right) \geq \alpha_{i,t'_i} \left(1 - 36k^2 \sqrt{\frac{\log n}{n}}\right).
\]

The case for \(\alpha_{i-1,t_i} > k\) is similar, but yields a less restrictive bound, and we omit the proof. We now show that the bias is well-concentrated in roughly the first \(\log \sqrt{n}\) generations.

**Corollary 24.** Let \(\alpha_i = \alpha_{i,t_{i+1}}\) for \(i > 0\) be the bias at the time when generation \(i\) concluded the two-choices phase, let 

\[\delta = O\left(k^2 \sqrt{\frac{\log n}{n}}\right)\]

and assume \(\alpha_i \leq k\). Then for

\[i \leq \log \log \alpha_0 k\]

we have \(\alpha_i > (\alpha_0 \cdot (1 - \delta))^{2^i}\) whp.

Furthermore, let \(i\) be the smallest generation such that \(\alpha_i > k\). Then 

\[i \leq 1 + \log \log \alpha_0 k\]

whp.

**Proof.** Using Lemma 23 observe

\[
\alpha_i \geq \alpha_{i-1}^2 \cdot (1 - \delta) \geq \cdots \geq \alpha_0^2 \prod_{j=0}^{i-1} (1 - \delta)^{2^j}
\]

\[= (\alpha_0 \cdot (1 - \delta))^{2^i},\]

which concludes the first part of our claim.
Now set $G = 1 + \log \log_{\alpha_0} k$ and note that, similarly as in the Synchronous case, the error term $1 - \delta$ is negligible compared to the initially required bias, and by what we showed above it follows that after $G$ generations we have

$$\alpha_G > (\alpha \cdot (1 - \delta))^2 G = k^2 (1 - \delta)^2 \log_{\alpha_0} k > k,$$

which concludes the proof of our claim.  

Putting everything together, we first established that, as long as the bias is smaller then roughly $\sqrt{n}$, the time needed to increase the generation by one is similar to the one in the synchronous model, i.e., around $O(-\log p_i) = O(\log k)$. Furthermore, we have shown that the variance of the bias through the generations cannot hurt our result.

Now we consider the case when there exists one large dominating color while even the second-dominant color has a very small cardinality. While this case is easy to resolve, one cannot use the same approach as before. However, in order to show the following lemma, it turns out that it is enough to track the cardinality of the dominant color.

▶ **Lemma 25.** If $c_{b,i-1,t_i} < \log^2 n / \sqrt{n}$, then within the following $\lceil \frac{1}{2 - \log_{\alpha_0} k} \rceil = O(1)$ of subsequent generations, we obtain a monochromatic generation whp.

**Proof.** So we have $c_b < \log^2 n / \sqrt{n}$ and $c_a > 1 - k \log^2 n / \sqrt{n} = 1 - O(n \delta)$ for some $\delta > 0$. Since we don’t have the concentration results for $c_b$ or any other color, we estimate the movement of $c_a$ directly.

For two consequent generations, let $a = c_{a,i-1,t_i} = 1 - \varepsilon$ and let $a'$ be the expected value of $c_{a,i,t_{i+1}}$. We clearly have

$$a' > \frac{a^2}{1 + 2a^2 - 2a} = \frac{1 - 2\varepsilon + \varepsilon^2}{1 - 2\varepsilon + 2\varepsilon^2} > \frac{1 - 2\varepsilon}{1 - 2\varepsilon + \varepsilon^2}$$

$$= \frac{1}{1 - \varepsilon} - \frac{\varepsilon}{(1 - \varepsilon)^2} = 1 - \left(\frac{\varepsilon}{1 - \varepsilon}\right)^2,$$

and by setting $\varepsilon'$ such that $a' = 1 - \varepsilon'$, assuming $\varepsilon < \frac{1}{2}$ we get

$$\varepsilon' < \left(\frac{\varepsilon}{1 - \varepsilon}\right)^2 < \frac{\varepsilon^2}{1 - 2\varepsilon} < \varepsilon^2 (1 + 4\varepsilon) < 2\varepsilon^2.$$

Starting from $\varepsilon_0 < k \log^2 n / \sqrt{n}$, in expectation, after another $i > \log \frac{\log n - 1}{\log k}$ generations we have $\varepsilon_i \leq \frac{(2\varepsilon_0)^{2i}}{2} \leq \frac{1}{n}$. Note that $k \ll n^{1/2}$ implies $k \log^2 n / \sqrt{n} \ll 1$, hence $\log \frac{n - 1}{\log k}$ is a constant, converging towards $\frac{1}{2 - \log_{\alpha_0} k}$.

Once a generation is monochromatic, any later generation is monochromatic as well. Assuming half of the nodes are in this monochromatic generation, it is easy to see that after additional $O(\log \log n)$ time units, all but $n^{\log \log n}$ nodes will be promoted to the monochromatic generation, and within the subsequent $O(\log n)$ time steps full consensus is reached.

### 4 Decentralized System with Multiple Leaders

In the previous section, we assumed the presence of a designated leader node and every node was aware of the address of this leader. While there exist distributed protocols to efficiently elect a leader, this centralized approach violates the distributed computing paradigm and
has several drawbacks. Most notably, a huge number of requests is induced on the leader in each time step and thus the leader becomes the bottleneck of the execution of the protocol. Furthermore, the system becomes highly vulnerable against attacks, since an adversary can compromise the entire computation by taking over the leader. Clearly, this single leader case can be viewed as a large shared memory system, in which every node can access the content of the shared memory. However, in many distributed applications there is no shared memory. Moreover, if the majority of the nodes try to access the memory within short time frames, then the bottleneck issue mentioned above still remains. We therefore propose the following approach to resolve these problems.

**Multiple Leaders.** Instead of having one pre-defined leader, we use a set of cluster leaders. We assume that the execution of the protocol runs in two parts, where in the first part we use a distributed clustering algorithm (see Section 4.1) to cluster the nodes into groups of roughly poly log n nodes each. In each of these clusters, one node becomes the cluster leader, and all other nodes know their cluster leader’s address. In that way, we no longer have one designated leader, but O(n/poly log n) decentralized cluster leaders. Note that the use of multiple leaders is beneficial in many ways. Most importantly, the strengths of a distributed approach such as resilience against limited adversarial attacks and there is an upper bound on the communication load of the nodes. Furthermore, the load is fairly balanced among the cluster leaders. Once the clustering is done, nodes may perform a similar (simple) algorithm as in the single-leader case.

**Our Result.** The communication model (Poisson clock at each node, latency time for establishing a communication channel between two nodes) is the same as in the single leader case. As in the previous section, it holds that within o(log n) time we can only achieve partial convergence. We show that the approach from the single-leader case carries over to the model with n/polylog n cluster leaders, and the algorithm in the multi-leader case mimics the algorithm in the single leader case. Furthermore, all interactions of nodes in Algorithm 4 are based on cluster leaders sampled uniformly at random from the entire system, and hence there do not exist correlations between the opinions of nodes and the states of their own cluster leaders. This allows us to apply the same analysis as in the single-leader case to the distributed system with multiple cluster leaders. Formally, we show the following result.

> **Theorem 26.** Let k be the number of opinions such that 2 ≤ k ≪ \sqrt{n} and let α > 1 + \frac{k \log n}{\sqrt{n}} \cdot \log k. For a complete graph \( K_n \), our algorithm 1/log\(^{(1)}\) n-converges towards the initially dominant opinion in O(log log \( k \cdot \log k + \log \log n \)) steps whp. After O(log n) additional steps, all nodes have the initially dominant opinion whp.

### 4.1 Clustering Algorithm

In the first part, all but an O(1/polylog n)-fraction of nodes are partitioned into clusters, each containing a distinguished node which we also call a leader. Similarly as in Section 3, the task of a leader is to receive signals from its cluster and act upon them.

The clustering works as follows. At the beginning, each node flips a coin and with probability 1/log\(^{c}\) n, the node becomes a leader, where c is a sufficiently large constant. The other nodes are followers. Whenever the clock of a node ticks, this node establishes communication channels to its own leader (if any), and to three other nodes chosen uniformly
at random. These neighbors send the address of their leaders to the node they were contacted by, and then one of these leaders is called that node. If a follower (not assigned to a cluster so far) contacts a leader, then it joins the cluster of that leader as long as the cluster has size less than $\log^{c-1} n$. The leader nodes keep track of the size of their clusters, and if a follower joins the cluster of some leader, then this leader notifies the follower that his request to join was successful (recall that establishing a communication channel requires time, but the exchange of messages is instant). The nodes in a cluster keep sending $0$-signals to their leader at each tick of their individual Poisson clocks, which enables the leader to count the time (similar to the single leader case).

Once the size $\log^{c-1} n$ is reached, the leader starts counting $0$ signals, and rejects any further request until its counter reaches value $c^2 \log^{c-1} n \cdot \log \log n$. As soon as the $c^2 \log^{c-1} n \cdot \log \log n$’th signal is received, the leader starts accepting further followers to its cluster. However, only the first $\log^{c-1} n$ nodes keep sending $0$-signals. After additional $c^2 \log^{c-1} n \log \log n$ received $0$-signals, the node switches to the consensus protocol (i.e., the leader changes in consensus mode), and informs all leaders about this fact (see next subsection for broadcasting among leaders). If a leader with a cluster of size at least $\log^{c-1} n$ receives such a message, then it switches to consensus mode, and broadcasts this message further for constant many time steps. After the consensus protocol has started, and the constant time elapsed, the node will disregard any such message. If followers of clusters with size less than $\log^{c-1} n$ meet a node in consensus mode, then they notify their leader about this fact, and none of the nodes of this cluster switches to consensus mode until they are informed by the others to do so (at the time when almost all nodes reached consensus). Thus, only the leaders that have a cluster of size $\log^{c-1} n$ by the time they receive the message to switch to consensus mode will participate in the consensus process described in Section 4.4.

As we show in the next subsection, broadcasting any information to leaders with clusters of size at least $\log^{c-1} n$ takes only constant time, provided that almost all nodes are contained in such a cluster. Our clustering algorithm yields the following theorem.

**Theorem 27.** In a complete graph of size $n$, after $C \log \log n$ time steps there are at least $n - n/\log^{C'} n$ nodes that are contained in clusters of size at least $\log^{c-1} n$, where $C$ and $C'$ are constants, and all the leaders of these clusters are in consensus mode by that time whp. Furthermore, if we denote by $t_f$ the time at which the first such leader switched to consensus mode and by $t_l$ the last such leader, then $t_f - t_l = O(1)$ whp. The other leaders (and their followers) will not participate in the consensus protocol described in Section 4.4 whp.

**Proof.** As described in the algorithm, every node flips a coin and becomes a leader with some probability $1/\log^c n$. Using simple Chernoff bounds, it follows that there will be $n(1 \pm o(1))/\log^c n$ leaders whp. We assume in this proof that all nodes flip their coins at the beginning, and flipping a coin is not related to the ticks of the clocks; however, this could also be relaxed by assuming that the nodes flip their coins at their first tick, and the result of the theorem would not change. Let the set of leaders be denoted by $L$.

First, we show that within $c \log \log n$ time there will be at least $|L|(1 - 1/\log^{2C'} n)$ leaders having at least $c' \log \log n$ members in its cluster whp, where $C'$ and $c'$ are constants depending on $c$. As in the single leader case, we call a time unit the period of time in which a node performs a complete execution of one clustering step with probability $9/10$. That is,

---

6 It would be enough, to just contact one randomly selected node. However, in order to select the same number of nodes as in the algorithm, we allow here the selection of three randomly chosen neighbors as well.
in this case a time unit is the time needed for a node to perform a good tick and to establish connections to a leader and two randomly chosen nodes with probability 9/10. We know that a time unit has constant length. We divide now the time frame of length $c \log \log n$ into a sequence of non-overlapping time units. Having in mind that for a time frame of length at least $c(1 - o(1)) \log \log n$ no leader will have more than $\log^{c-1} n$ members in its cluster, there will be whp $9n/10 \cdot (1 - o(1))$ nodes communicating with another node in a time unit of the sequence of time units defined above. Thus, a leader is contacted with probability at least

$$1 - \left(1 - \frac{1}{n}\right)^{9n/10 \cdot (1 - o(1))} = 1 - e^{-9(1 - o(1))/10}.$$  

Using Chernoff bounds, we obtain that in $\Theta(\log \log n)$ time units, all but $|L|(1 - 1/\log^{C'} n)$ leaders have been contacted by at least $c' \log \log n$ other nodes whp, where the constant hidden in $\Theta(\log \log n)$ governs $C'$ and $c'$. Thus, choosing $c$ accordingly we obtain our claim.

We consider now the next $(c^2 - c) \log \log n$ time steps and, again, we divide the time into a sequence of time units. As long as no cluster has larger size than $\log^{c-1} n$, in each time unit $9n/10 \cdot (1 - o(1))$ nodes try to join a cluster. Let $L_{v}$ be the cluster of a leader $v$, and we assume that $|L_{v}| \geq c' \log \log n$ at the beginning of the sequence of time units defined above. We call a time unit successful, if the size of the cluster grows by a factor of $9$ within a time unit or the cluster has size $\log^{c-1} n$ at the end of the time unit. As before, we know that within a time unit, a node of the cluster is contacted with probability at least

$$1 - \left(1 - \frac{1}{n}\right)^{9n/10 \cdot (1 - o(1))} = 1 - e^{-9(1 - o(1))/10}.$$  

Using simple Chernoff bounds, we obtain that a time unit is successful with probability at least $1 - 1/\log^{2C'} n$, where $C'$ depends on the size of that cluster at the beginning of the time unit, i.e., in the first time unit of the sequence, $C'$ depends on $c'$. Hence, if there are enough time units in the sequence of length $\Theta(\log \log n)$, then there will be $(c - 1) \log \log n$ successful time units for $L_{v}$, with probability at least $1 - 1/\log^{2C'-1} n$. Thus, the expected number of clusters, for which the number of successful time units is less than $(c - 1) \log \log n$, is less than $|L|/\log^{2C'-2} n$. Note that these events are not independent between clusters. However, applying the method of bounded differences, we obtain that at most $|L|/\log^{C'} n$ clusters have size less than $\log^{c-1} n$ at the end of this sequence of time units, whp, provided the constant $c$ is large enough.

During the next $c^3 \log \log n$ time steps all but at most a $1/\log^{C'+1} n$-fraction of the nodes are placed among clusters (note that all the clusters with sizes at least $\log^{c-1} n$ nodes start accepting further nodes after additional $O(c^2 \log \log n)$ steps).

Now we consider the time when the first cluster switches to the consensus algorithm. We show that within constant time, a cluster has either size $\log^{c-1} n$ and participates in the consensus process, or the cluster will not participate in the clustering process till the point in time when the last generation is created whp. By this time, at most $|L|/\log^{C'} n$ clusters have size less than $\log^{c-1} n$ whp. Assume that $c$ is chosen such that clusters of size at least $\log^{c-2} n$ receive the broadcast message in constant time whp, according to Theorem 28. We consider all the clusters of size less than $\log^{c-2} n$ at the time the first leader switches to the consensus mode faulty clusters. Clearly, whp it requires time $\Theta(\log \log n)$ for a faulty cluster to increase its size to $\log^{c-1} n$ [KSSV00], and if we denote by $f$ the number of faulty clusters, then all but $O(f)$ clusters become informed (i.e., the leader knows that a cluster leader switched to the consensus mode) within $O(1)$ time whp [KSSV00]. Now, by pull all non-faulty clusters become informed within additional $O(1)$ time whp (see last paragraph
in the proof of Theorem 28). This non-faulty clusters either will not participate in the consensus procedure (if their size is less than \(\log^{c^1} n\)), or they switch to consensus mode. After additional \(O(1)\) time all clusters stop sending the message according to the description of our algorithm. The clusters, which were faulty at the time the first cluster leader switched to consensus mode, will whp not be able to increase their size to \(\log^{c^1} n\) within constant time, and hence none of these clusters will switch to consensus mode during this broadcast procedure.

\[\square\]

### 4.2 Broadcasting among Clusters

The broadcasting procedure is very similar to the asynchronous variant of push and pull considered by Fountoulakis et al. [FPS12]. In this paper it is shown that in the Chung-Lu model almost all nodes become informed within constant time whp., if the nodes perform push-pull operations at each tick of a local Poisson clock with rate 1. In [FPS12], the authors make use of the structure of the Chung-Lu model to show their result. Here we use the structure of the clusters to obtain a very similar result.

The broadcasting protocol works as follows. Assume, we have \(n - n/\log^{c'} n\) nodes organized in clusters of size at least \(\log^{c^1} n\). We call such nodes active. Furthermore, assume that an (arbitrary but fixed) leader has a message to be distributed to all leaders of active nodes. In the sequel we call a leader informed if it has this message.

If an active node ticks, then it contacts its own leader and two additional nodes. From these nodes, it requests the addresses of their leaders (if any), which are then contacted. If one of the three leaders is informed, then the other two become informed as well. Remember, establishing a communication channel also requires some time. We use here that if the clusters have sizes at least \(\log^{c^1} n\), where \(c\) is sufficiently large, then whp. at each time in every (active) cluster there will be at least one node that contacts three leaders and exchanges all the necessary information between them within some time \(1/\log^{c'} n\), where \(c' < c\) is some constant depending on \(c\). This, however, can be seen as a standard pull algorithm between leaders with an accelerated rate. Since a message is spread in a complete graph of size \(n\) in (synchronous or asynchronous) time \(O(\log n)\) whp., the result follows. However, it is necessary here to ensure that at each time there are enough active nodes. Thus, we obtain the following theorem.

\[\blacktriangleleft\]

**Theorem 28.** If there are \(n - n/\log^{c'} n\) nodes organized in clusters of size at least \(\log^{c^1} n\), \(c\) large enough, and one of the leaders of these clusters has a message, then this message can be broadcasted to all the leaders of these clusters in \(O(1)\) time whp.

**Proof.** As described above, we first show that in a cluster of size \(\log^{c^1} n\) there will be at any time a node which exchanges the information between the three contacted leaders within a time frame of \(1/\log n\) whp. Then, we can apply the result of [KSSV00] in the asynchronous case (see e.g. [EFK+16]) with an accelerated rate, and obtain the result.

To show the first statement, we proceed as in the single leader case. The accumulated latency for establishing the communication channel to the own leader, the two randomly sampled nodes, and then to the leaders of these nodes is described by \(T'_2 + T'_1\). Then, the waiting time between two good ticks is distributed as \(T'_1 + T'_2\). Then, at an arbitrary but fixed time \(t\), let \(T'_2\) be the random variable describing the time needed after \(t\) for a fixed

---

\(^7\) The broadcasting also works if always just one leader is provided (if the chosen node(s) are clustered, otherwise clearly no leader is sent).
node $v$ to have a good tick, and establish connections to all nodes needed for the exchange of information between the three leaders. Then,

$$T_2^w \leq T_2^u + T_1 + T_2^{''} \leq T_1 + 10T_2$$

Since $T_1$ and $T_2$ have exponential distribution with constant mean, $P[T_2^w < 1/\log n] = \Omega(1/\log n)$. Applying Chernoff bounds over the nodes of a cluster of size $\log c^{-1} n$ with $c$ large enough, we obtain the first statement.

For the second statement, consider the graph, in which the nodes are the clusters (nodes that are not clustered are also nodes of this graph), and there are edges between all pairs of clusters. A cluster is chosen for communication with probability proportional to its size. If a synchronous push-pull algorithm is executed on this graph with $f$ faulty nodes, then in time $O(\log n)$ all except $O(f)$ nodes become informed whp. [KSSV00]. Here, all clusters of size less than $\log c^{-1} n$ (or non-clustered nodes from the original graph) are considered faulty nodes. Dividing now the time into time frames of length $1/\log n$, and assuming that each non-faulty cluster connects to exactly one other cluster in an $1/\log n$ time frame, we obtain that all except $O(f)$ non-faulty clusters receive the message within constant time. If $c$ is large enough, then in a time frame of $1/\log n$ some $\Theta(\log n)$ nodes of a non-faulty cluster establish connections to their own leader and some other leaders. This implies that after additional constant time, all non-faulty clusters are informed whp. ◀

4.3 Adapted Definitions

The conditions in the multi-leader algorithm are adapted from the single-leader case such that they allow us to re-use most of the analysis from Section 3. Each node is equipped with the following variables:

- **self** (containing its unique address),
- the current generation $\text{gen}$ and color $\text{col}$,
- a leader’s address $\text{leader}$, initially set to null,
- a flag $\text{finished}$, indicating a local awareness of some nodes containing the final opinion.

A leader node, when acting as a regular node, is aware of the fact that it is a leader, i.e., it behaves like a regular node and just has $\text{leader}$ set to its own address. However, when reacting to a signal, a leader node has access to additional public variables $\text{gen}$, $\text{state}$ and $\text{finished}$. Furthermore, it privately stores

- $\text{card} \in \mathbb{N}$, the precise cardinality of the cluster, initially set to 1, of size $O(\log \log n)$ bits
- $\text{gen} - \text{size} \in \mathbb{N}$, the cardinality of the latest generation in the cluster, of size at most $\log(\text{card}) = O(\log \log n)$ bits.

Let $G^\ast = \lceil \log \log_\alpha n \rceil$ be the number of the required generations, parametrized by the initial bias $\alpha$. As in the single-leader case, we fix $\gamma = 0.5$.

The notions of $t_i$ or $t_i(j)$ in this section are defined analogously to the single-leader case, and they are always used in the context of a particular cluster in order to resolve ambiguity. Furthermore, we will use the following variables analogously to the single-leader case.

- $\lambda$ – the rate of the exponentially distributed r.v. governing the latency time $T_2$
- $n'$ – the size of the current cluster (the cluster is clear from the context). Note that $n' > \log^2 n$ according to Section 4.1.
- $\lambda$ – the rate-parameter of the latency Poisson clock for r.v. $T_2$
4.4 Our Procedure After the Clustering

The main difference between the single leader and multi leader case is in the way how $\Theta(n/\log n)$ leaders of different clusters with limited "range of view" work together and emulate the algorithm given in Section 3.

The algorithm works as follows. At each tick of the clock of a node $v$, this node contacts its leader and samples three other nodes (we call $v_1$, $v_2$, and $v_3$) chosen uniformly at random. Two of these nodes are asked for their current opinion and generation, and $v_3$ is asked for the address of its leader; then – after receiving the required information from all three nodes – the leader of $v_3$ is also contacted. As in the single leader case, the actions of $v$ depend on the generation number and propagation bit of this leader sampled randomly (but not necessarily uniformly at random). If the information provided by $v_1$ and $v_2$ together with the state of the leader of $v_3$ and the state obtained by $v$ at the previous communication with a leader allows two choices, then a two choices step is performed as in Algorithm 2. That is, if the following conditions are fulfilled

- $v_1$ and $v_2$ are in the same generation $i$ and have the same opinion
- the highest generation allowed by the leader of $v_3$ is $i+1$ and this leader allows two choices
- the value $\text{gen}$ stored by $v$ from the previous communication with a leader is also $i+1$ and two choices were also allowed by this leader

then $v$ takes the opinion of $v_1$ and $v_2$, and sets its generation number to $i+1$. If according to the information received from these nodes a propagation step is to be performed, then $v$ executes a propagation step (cf. again Algorithm 2). Also, a 0-signal and an eventual notice about the incrementation of $v$’s generation number are sent to the own leader.

Similarly to Algorithm 3, if the number of nodes in the highest allowed generation in the cluster of a leader exceeds a certain value, then this leader increases the generation number by 1 and starts allowing two choices steps. Moreover, this information is then propagated to all leaders in the system using the broadcast procedure as described in the previous subsection. Also, the leader starts counting the received 0-signals to measure time. After a certain (constant) time elapsed, where this constant is significantly larger than the constant needed to broadcast a message among leaders, the leader switches to a sleeping mode for some constant time. In this time frame, the leader just counts the 0-signals, but will not allow two choices or propagation to the highest generation for the nodes which contact this leader during this time (see description of the procedure above for non-leader nodes). Then, the leader starts allowing propagation, and the nodes which contact it will act as described in the paragraph above. The broadcasting procedure initiated after increasing the generation number and the sleeping period after the constant two choices period guarantee that whp all leaders of clusters above a certain size will be mostly synchronized during the time the nodes execute two choices or propagation steps, respectively.

This algorithm basically mimics the algorithm of the single leader case, and it leads to Theorem 26.

The following sub-procedure resets the parameters whenever leader’s generation is increased, depending on the received signal. These signals may be of type $(i, j), 0 \leq j \leq 3,$
Algorithm 4 The basic procedure of a node after clustering. The $\text{in\_sync}(\cdot, \cdot)$ checks whether the appropriate public variables stored in both nodes correspond.

**Require:** a vertex $v$ with leader $\neq$ null ticked

1. send $(0,3,\cdot)$-signal to leader
2. if not locked then locked $\leftarrow$ True else break!
3. contact $v_1, v_2, v_3$, sampled u.a.r., inquire respective parameters $\triangleright$ takes time
4. contact own leader, and $l$ (leader of $v_3$), and inquire gen and state $\triangleright$ takes time
5. if finished, then propagate col and also set finished to all samples.
6. if $\exists v'$ among the sampled nodes, such that finished($v'$), then
   1. col $\leftarrow$ col($v'$) and finished $\leftarrow$ True
   2. if gen($l$) = 0, unlock and break! $\triangleright$ non-active cluster sampled
   3. if $\exists i \in \{1,2\} : \text{gen} < \text{gen}(v_i) = \text{gen}(l)$ and $l.$state $= 3$ then and in_sync($l,v_i)$
   4. col $\leftarrow$ col($v_i$)
   5. gen $\leftarrow$ gen($v_i$)
   6. send (gen,3,True)-signal to own leader
   7. else $\triangleright$ generation not changed
      1. send (gen($l$),l.state, False)-signal to own leader
      2. update $v.$tmp.state and $v.$tmp_gen, corresponding to the own leader
    8. if gen $= \lceil \log \log_\lambda n \rceil$ then finished $\leftarrow$ True
9. locked $\leftarrow$ False

$i \in \mathbb{Z}^+$, which intuitively describe the leaders expected state and generation. Depending on the type of the signal, the behavior of the leader is described in Algorithm 5.

Note that a $(0,3)$-signal may only be sent in Line 1 of Algorithm 4, and observe that such signal is sent at any tick of $T_l$, hence distributed as Poisson$(1)$. By the memoryless property of the Poisson distribution, we conclude that also the arrival times at the corresponding leader are distributed accordingly, i.e. as Poisson($\frac{1}{\text{card}}$).

Finally, observe that in this extended algorithm with multiple leaders, three leaders are contacted. We adjust the definitions and bounds regarding the latency time and the definition of time unit accordingly such that $T_2 = \text{max}(T_2, T_2, T_2) + \text{max}(T_2, T_2)$, and $T_3 \sim T_2 + T_1 + T_2 \lesssim \Gamma(6,1) + \Gamma(1,1) \lesssim \Gamma(7, \beta),$ where we still assume $\beta = \min(1, \lambda)$.

**Dealing With Synchronicity.** For any fixed generation, each cluster goes through the following phases, see also Figure 2: a) the two-choices phase, b) the sleeping phase, and c) the propagation phase. While the nodes may be highly dis-synchronized in a given time-step (a node may wait $\log n$ time units before ticking), this is not the case for the leaders. Indeed, each leader is contacted whenever any of its (at least $\log \log_\lambda n$) followers ticks, and therefore we expect the leaders to be much better synchronized.

The following lemma follows from the fast broadcasting among the
Algorithm 5 The description of the signals, received by leader.

Require: an \((i, s, \text{hasChanged})\)-signal

1: if \((i, s) > (\text{gen, state})\) then \(\triangleright \) if member ahead
2: \((\text{gen, state}) \leftarrow (i, s)\)
3: if \(s = 1\) then \(t \leftarrow 0\) else \(t \leftarrow C_1 \cdot \text{card} \cdot C_s\)
4: if \(i = 0\) then
5: increment \(t\)
6: if \(t = C_1 \cdot \text{card} \cdot C_2\) then \(\triangleright \) start sleeping
7: \(\text{state} \leftarrow 2\)
8: else if \(t = C_1 \cdot \text{card} \cdot C_3\) then \(\triangleright \) allow propagation
9: \(\text{state} \leftarrow 3\)
10: if \(i = \text{gen}\) and \(\text{hasChanged} = \text{True}\) then
11: increment \(\text{gen}\_\text{size}\)
12: if \(0 < \text{gen} \leq \lceil \log \log n \rceil\) and \(\text{gen\_size} = \left\lfloor \text{card} \cdot \left(\frac{1}{2} + \frac{1}{\sqrt{\log n}}\right)\right\rfloor\) then
13: increment \(\text{gen}\)
14: \(t \leftarrow 0, \text{gen\_size} \leftarrow 0\)
15: \(\text{state} \leftarrow 1\)

clusters, and basically allows us to synchronize all leaders whenever a new generation is formed.

Lemma 29. Let \(t'\) be the time when the slowest cluster allows generation \(i\). Whp., \(t' - t_i \leq C_{\text{br}}, \) where \(C_{\text{br}} = O(1)\).

While this result seems surprising, it follows from the fact that in our asynchronous setting with cluster leaders broadcasting messages can be done rapidly. Indeed, in order to inform a cluster, it suffices that one single informed node contacts any node in that cluster, which then informs its leader. Note that a similar observation has been made by Fountoulakis et al. [FPS12] for a related model.

We now consider the deviance in synchronization due to tick-counters.

Proposition 30. For a fixed leader in generation \(i\) in a cluster of size \(n'\), let \(t > 0\) be the number of time units required after time \(t_i\) until the counter in Line 6 of Algorithm 5 reaches \(C \cdot n' \cdot C_1\) ticks. Then

\[
C - \frac{2}{C_1} < t' < C \left(1 + \frac{1}{\sqrt{\log n}}\right).
\]

Proof. Let \(T = t' C_1\). We focus on the time \(\frac{T}{C_1}\) time units, when our leader counts to \(C \cdot n' \cdot C_1\). Note that the leader is counting all the ticks, including the ones that arise from the time when the counter was not yet initiated. When lower-bounding the counting-time we will crudely subtract all these ticks. Note that the total number of such ticks is at most \(n'\). Let the total number good ticks be denoted by \(T'\). Clearly, we have

\[
C_1 C - \frac{2}{C_1} \leq \frac{T'}{n'} \leq C_1 C.
\]

Alongside the lines of [BGPS06, Lemma 1], we have

\[
P\left(\left|T - \frac{T'}{n'}\right| > \delta \frac{T'}{n'}\right) < 2 \exp\left(-\frac{\delta^2 T'}{2}\right),
\]
where we will use $\delta = \frac{1}{\sqrt{n}}$, so that the probability of bad event is at most

$$2 \exp\left(-\frac{T'}{2\sqrt{n}}\right) < 2 \exp\left(-\frac{2 \log^2 n}{\log n}\right) = \frac{2}{n^2}.$$

The lower bound follows from $\frac{T'}{n} > C_1 C - \frac{1}{C_1}$, which implies

$$T > \frac{T'}{n} - \delta \frac{T'}{n},$$

$$> CC_1 - \frac{1}{C_1} - \frac{CC_1 - \frac{1}{C_1}}{\sqrt{n}}$$

$$> CC_1 - \frac{2}{C_1}.$$

For the upper-bound we use $\frac{T'}{n} < C_1 C$, and obtain

$$T' < CC_1(1 + \delta)$$

and the expression (14) follows readily.

Several corollaries follow, as we may now calculate maximal time difference of clusters allowing sleeping, as well as propagation, for arbitrary fixed generation.

**Proposition 31.** Fix a generation $i$ and observe the following statements regarding the flow of Algorithm 5. Note that all the states mentioned correspond to generation $i$.

a) When the fastest leader starts sleeping, every cluster have been doing two choices for at least one time unit.

b) Let $t''$ correspond to the number of time units from the time when the first leader dozed off, till the time when all of them were asleep. Then $t'' \leq C_{br} + \frac{2}{C_1} + \frac{1}{\sqrt{\log n}} = O(1)$.

c) The first leader does not wake up before every other leader (already) starts sleeping.

**Proof.** We prove the statements separately.

a) **Proposition 30**, together with $C_2 = C_{br} + 1 + \frac{2}{C_1}$ implies that for arbitrary leader, the two-choices phase lasts at least $C_{br} + 1$ time units. From **Lemma 29** it follows that all leaders started with their two-choices phase within a difference of at most $C_{br}$ time units, hence the conclusion.
b) From Proposition 30 we know that the difference in the length of the two-choices phase may not exceed \( \frac{2}{C_1} + \frac{1}{\sqrt{\log n}} \) time units. Also, the synchronization which is initiated at time \( t_i \) lasts \( C_{br} \) time units only. Summing the two gives the desired bound.

c) Since the time difference from b) is bounded above by \( C_{br} + \frac{2}{C_1} + \frac{1}{\sqrt{\log n}} \), we must guarantee that the sleeping time of any node is bounded below by the same number of time units. To check whether the constant \( C_3 \) is big enough, it is hence enough to multiply value above with \( (1 + 1/\sqrt{\log n}) \), and add it to \( C_2 \).

For clarity, the flow of the algorithm is depicted in Figure 2, in which

\[
\begin{align*}
\hat{t}_1 &< \hat{t}_0 + C_{br} \\
\hat{t}_2 &> \hat{t}_0 + C_{br} + 1 \\
\hat{t}_3 &< \hat{t}_0 + C_{br} + 1 + \frac{2}{C_1} + \frac{1}{\sqrt{\log n}} \\
\hat{t}_4 &> \hat{t}_0 + C_{br} + 1 + \frac{3}{C_1} \\
\hat{t}_5 &< \hat{t}_0 + 2C_{br} + 1 + \frac{5}{C_1} + \frac{1}{\sqrt{\log n}}
\end{align*}
\]

**Time to Go Through a Generation-Cycle.** From the proposition above we satisfied some important invariants which allow us to use similar analysis as in Section 3. Indeed, while the vertices may be far from synchronized, the leaders behave quite synchronized in several aspects, in particular:

- For any fixed generation \( i \), whp. the leaders of all clusters will be allowing two-choices phase for at least one time unit, at the same time.
- For any fixed generation \( i \), whp. no node may be promoted to \( i \) as a result of two-choices after the first node have been promoted to \( i \) as a result of propagation.
- The time interval from whenever the fastest leader allowed generation \( i \), till the time when the slowest leader allowed propagation is bounded above by \( 3 + \frac{5}{C_1} + \frac{1}{\sqrt{\log n}} < 4 = O(1) \) time units, after which time all leaders allow propagation.

Any time some node gets new color and generation, this outcome is determined by the global generation/color distribution and the same statements regarding the concentration of color fractions \( c_{i,j,t} \), probability \( p_{i,t} \) or bias \( \alpha_{i,t} \) hold, by the same arguments.

**Corollary 32** (Concentration statements as in Section 3.3). Fix a generation \( i \), let and let \( z = 2e^{-3C_1}\sqrt{\frac{\log n}{n}} \), and let \( t_i + t' \) be the time when the last leader woke up. Then, for any time \( t \in [t_i, t_i + t'] \), with probability at least \( 1 - \frac{1}{n^2} \), we have

\[
p_{j,i-1,t} \in [p_{j,i-1,t_i} - 3z, p_{j,i-1,t_i} + 3z]
\]

and

\[
c_{j,i-1,t} \in [c_{j,i-1,t_i} - z, c_{j,i-1,t_i} + z],
\]

for any color \( j \). In addition, if \( c_{b,i-1,t_i} > \log^2 n \sqrt{n} \), then

\[
\alpha_{i,t_i+t'} \geq \alpha_{i-1,t_i}^2 \left( 1 - \frac{8k}{e^{3C_1}} \frac{\log n}{n} \right).
\]
Using precisely the same arguments as in Proposition 16, we have the following proposition.

**Proposition 33.** Fix a generation $i$ and let $t''$ be the (global) time, such that in interval $[t'', t'' + 1]$, all leaders allowed two-choices. Then we (globally) have $t'' + 1 \geq t_i(\frac{p_i}{9})$, whp.

Since any leader went to sleep only after $t''$ (as defined above), we reach the same situation as in a single-leader case.

**Corollary 34.** Fix a generation $i$ and arbitrary leader $l$, and let $\hat{t}''$ correspond to the time when $l$ started sleeping. Then $g_o(i) \geq \frac{p_i}{9}$. In particular, no promotion to $i$ as a result of propagation is possible, before globally $\frac{p_i}{9}$-th fraction of nodes is in generation $i$.

We are now ready to proceed with the speed of propagation. Note that, for every cluster, the propagation phase at least $4$ time units after the introduction of the corresponding generation. The analysis of the growth in the propagation phase is described by Proposition 17 in Section 3.3. Since we have the same initial guarantee, the proof is precisely the same, so we omit it here, and just restate its corollary.

**Corollary 35.** For $X_i = 4 + \log \frac{\log_{\log \log n}}{\log \alpha_0 - 1} = O(\log \frac{1}{\alpha_0})$, it holds that $t_{i+1} \leq t_i + X_i$, whp.

Taking now into account that Lemmas 19-25 also hold in this decentralized case, by using the same arguments as in the previous section, we obtain the theorem.

### 4.5 Complexity Parameters of the Decentralized System

Recall that the algorithm in the multi-leader case mimics the algorithm from the single-leader case, provided the distributed clustering has been successful. Therefore, with some minor adjustments, the analysis from Section 3 carries over to the multi-leader case, and in particular to Algorithm 4.

**Communication and Memory Complexity of our Procedure.** In what follows we argue various complexity properties of our procedure. In particular, we will argue that the drawbacks from the single leader case (e.g., the large communication load on the leader) are effectively mitigated.

- **memory complexity:** The total memory per node is bounded by $O(\log n)$ bits.
- **message complexity:** In general, the messages have length at most $O(\log n)$ bits.
  - In the partitioning phase, we need to transmit addresses in the network, which are of size $O(\log n)$ bits.
  - Leader messages consist of the current generation and the current state. While the latter is a constant, the former is of order $O(\log \log \frac{\log n}{\alpha_0 - 1})$.
  - Non-leader messages when reporting the membership in a cluster are of constant size.
  - Non-leader messages when reporting the promotion to a new generation are of size $O(\log \log \frac{\log n}{\alpha_0 - 1})$ bits.
- **congestion:** The number of requests per node in a given time unit does not asymptotically exceed $O(\text{poly log } n)$. 
5 Summary and Conclusion

We analyzed the problem of distributed plurality consensus in a complete graph, in which initially every node holds one of \( k \) distinct opinions. In a first step we considered the standard synchronous model, and derived a fast algorithm to solve plurality consensus in this model. Then, we assumed that every node is equipped with a random Poisson clock with rate 1 and for establishing communication channels between nodes delays occur, which follow an exponential distribution with constant mean. In this model we first assumed that the system has a leader, and derived an efficient algorithm under these conditions. Then, we extended our single leader approach to a distributed system without a leader.

There are a number of open questions left. In our analysis, we assumed that establishing a communication channel takes time, but exchanging information through an established channel is instant. This can easily be relaxed in the single leader case by contacting the leader after each potential update of opinions and generation number, and the updates are committed only, if the state of the leader has not been changed in the meantime. However, it is not clear whether this idea can directly be applied to the system without a leader. It would also be interesting to see whether it is possible to consider a more general asynchronous model instead of the Poisson clocks and the exponential distribution of the delays, while preserving the results of our theorems.
References

[AAE+17] Dan Alistarh, James Aspnes, David Eisenstat, Rati Gelashvili, and Ronald L. Rivest: Time-Space Trade-offs in Population Protocols. In Proc. SODA ’17, 2017, pages 2560–2579.

[AAE08] Dana Angluin, James Aspnes, and David Eisenstat: A simple population protocol for fast robust approximate majority. In Distributed Computing, volume 21(2), 2008, pages 87–102.

[AAER07] Dana Angluin, James Aspnes, David Eisenstat, and Eric Ruppert: The computational power of population protocols. In Distributed Computing, volume 20(4), 2007, pages 279–304.

[AAG18] Dan Alistarh, James Aspnes, and Rati Gelashvili: Space-Optimal Majority in Population Protocols. In Proc. SODA ’18, 2018, pages 2221–2239.

[AD15] Mohammed Amin Abdullah and Moez Draief: Global majority consensus by local majority polling on graphs of a given degree sequence. In Discrete Applied Mathematics, volume 180, 2015, pages 1–10.

[AF02] David Aldous and James Allen Fill: Reversible Markov Chains and Random Walks on Graphs. Unpublished. http://www.stat.berkeley.edu/~aldous/RWG/book.html. 2002.

[AGV15] Dan Alistarh, Rati Gelashvili, and Milan Vojnović: Fast and Exact Majority in Population Protocols. In Proc. PODC ’15, 2015, pages 47–56.

[AR07] James Aspnes and Eric Ruppert: An Introduction to Population Protocols. In Bulletin of the EATCS, volume 93, 2007, pages 98–117.

[BCE+17] Petra Berenbrink, Andrea E. F. Clementi, Robert Elsässer, Peter Kling, Frederik Mallmann-Trenn, and Emanuele Natale: Ignore or Comply?: On Breaking Symmetry in Consensus. In Proc. PODC ’17, 2017, pages 335–344.

[BCN+14] Luca Becchetti, Andrea Clementi, Emanuele Natale, Francesco Pasquale, Riccardo Silvestri, and Luca Trevisan: Simple Dynamics for Plurality Consensus. In Proc. SPAA ’14, 2014, pages 247–256.

[BCN+15] Luca Becchetti, Andrea Clementi, Emanuele Natale, Francesco Pasquale, and Riccardo Silvestri: Plurality Consensus in the Gossip Model. In Proc. SODA ’15, 2015, pages 371–390.

[BFGK16] Petra Berenbrink, Tom Friedetzky, George Giakkoupis, and Peter Kling: Efficient Plurality Consensus, or: The benefits of cleaning up from time to time. In Proc. ICALP ’16, 2016.

[BFK+16] Petra Berenbrink, Tom Friedetzky, Peter Kling, Frederik Mallmann-Trenn, and Chris Wastell: Plurality Consensus via Shuffling: Lessons Learned from Load Balancing. In CoRR, volume abs/1602.01342, 2016.

[BGKM16] Petra Berenbrink, George Giakkoupis, Anne-Marie Kermarrec, and Frederik Mallmann-Trenn: Bounds on the Voter Model in Dynamic Networks. In Proc. ICALP ’16, 2016.

[BGPS06] Stephen Boyd, Arpita Ghosh, Balaji Prabhakar, and Devavrat Shah: Randomized Gossip Algorithms. In IEEE Transactions on Information Theory, volume 52(6), 2006, pages 2508–2530.

[CFR16] Colin Cooper, Martin Dyer, Alan Frieze, and Nicolás Rivera: Discordant Voting Processes on Finite Graphs. In Proc. ICALP ’16. Volume 55, 2016, pages 145:1–145:13.
[CEOR13] Colin Cooper, Robert Elsässer, Hirotaka Ono, and Tomasz Radzik: Coalescing Random Walks and Voting on Connected Graphs. In SIAM Journal on Discrete Mathematics, volume 27 (4), 2013, pages 1748–1758.

[CEOR+15] Colin Cooper, Robert Elsässer, Tomasz Radzik, Nicolás Rivera, and Takeharu Shiraga: Fast Consensus for Voting on General Expander Graphs. In Proc. DISC ’15, 2015, pages 248–262.

[CER14] Colin Cooper, Robert Elsässer, and Tomasz Radzik: The Power of Two Choices in Distributed Voting. In Proc. ICALP ’14, 2014, pages 435–446.

[CG10] Gennaro Cordasco and Luisa Gargano: Community Detection via Semi-Synchronous Label Propagation Algorithms. In Proc. BASNA, 2010, pages 1–8.

[CG14] James Cruise and Ayalvadi Ganesh: Probabilistic consensus via polling and majority rules. In Queueing Systems, volume 78 (2), 2014, pages 99–120.

[CRRS17] Colin Cooper, Tomasz Radzik, Nicolas Rivera, and Takeharu Shiraga: Fast Plurality Consensus in Regular Expanders. In Proc. DISC ’17, 2017, pages 13:1–13:16.

[DP94] Xiaotie Deng and Christos Papadimitriou: On the Complexity of Cooperative Solution Concepts. In Mathematics of Operations Research, volume 19 (2), 1994, pages 257–266.

[DV10] Moez Draief and Milan Vojnovic: Convergence Speed of Binary Interval Consensus. In Proc. INFOCOM, 2010, pages 1792–1800.

[DV12] Moez Draief and Milan Vojnovic: Convergence speed of binary interval consensus. In SIAM Journal on Control and Optimization, volume 50 (3), 2012, pages 1087–1109.

[DW83] Peter Donnelly and Dominic Welsh: Finite particle systems and infection models. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 94 (1), 1983, pages 167–182.

[EFK+16] Robert Elsässer, Tom Friedetzky, Dominik Kaaser, Frederik Mallmann-Trenn, and Horst Trinker: Efficient k-Party Voting with Two Choices. In CoRR, volume abs/1602.04667, 2016.

[EFK+17] Robert Elsässer, Tom Friedetzky, Dominik Kaaser, Frederik Mallmann-Trenn, and Horst Trinker: Brief Announcement: Rapid Asynchronous Plurality Consensus. In Proc. PODC ’17, see [EFK+16] for the full version, 2017, pages 363–365.

[FKW13] Silvio Frischknecht, Barbara Keller, and Roger Wattenhofer: Convergence in (Social) Influence Networks. In Proc. DISC, 2013, pages 433–446.

[FPS12] Nikolaos Fountoulakis, Konstantinos Panagiotou, and Thomas Sauerwald: Ultra-fast rumor spreading in social networks. In Proc. SODA, 2012, pages 1642–1660.

[Gif79] David Gifford: Weighted Voting for Replicated Data. In Proc. SOSP, 1979, pages 150–162.

[GP16] Mohsen Ghaffari and Merav Parter: A Polylogarithmic Gossip Algorithm for Plurality Consensus. In Proc. PODC ’16, 2016.

[HL75] Richard Holley and Thomas Liggett: Ergodic Theorems for Weakly Interacting Infinite Systems and the Voter Model. In The Annals of Probability, volume 3 (4), 1975, pages 643–663.
REFERENCES

[HP01] Yehuda Hassin and David Peleg: *Distributed Probabilistic Polling and Applications to Proportionate Agreement*. In *Information and Computation*, volume 171 (2), 2001, pages 248–268.

[KDG03] David Kempe, Alin Dobra, and Johannes Gehrke: *Gossip-Based Computation of Aggregate Information*. In *Proc. FOCS ’03*, 2003, pages 482–491.

[Kin93] John Frank Charles Kingman: *Poisson processes*. Wiley Online Library, 1993.

[KMS16] V. Kanade, F. Mallmann-Trenn, and T. Sauerwald: *On coalescence time in graphs—When is coalescing as fast as meeting?* In *CoRR*, volume abs/1611.02460, 2016.

[KPS13] Kishore Kothapalli, Sriram Pemmaraju, and Vivek Sardeshmukh: *On the Analysis of a Label Propagation Algorithm for Community Detection*. In *Proc. ICDCN*, 2013, pages 255–269.

[KSSV00] Richard Karp, Christian Schindelhauer, Scott Shenker, and Berthold Vöcking: *Randomized Rumor Spreading*. In *Proc. FOCS ’00*, 2000, pages 565–574.

[Lig12] Thomas Liggett: *Interacting particle systems*. Springer Science & Business Media, 2012.

[LM15] Yuezhou Lv and Thomas Moscibroda: *Local Information in Influence Networks*. In *Proc. DISC*, 2015, pages 292–308.

[LN07] Nicolas Lanchier and Claudia Neuhauser: *Voter model and biased voter model in heterogeneous environments*. In *Journal of Applied Probability*, volume 44(3), 2007, pages 770–787.

[Mah08] Hosam M. Mahmoud: *Pólya Urn Models*. CRC Press, 2008.

[MNRS14] George B. Mertzios, Sotiris E. Nikoletseas, Christoforos Raptopoulos, and Paul G. Spirakis: *Determining Majority in Networks with Local Interactions and Very Small Local Memory*. In *Proc. ICALP*, 2014, pages 871–882.

[NY99] Toshio Nakata, Hiroshi Imahayashi, and Masafumi Yamashita: *Probabilistic Local Majority Voting for the Agreement Problem on Finite Graphs*. In *Proc. COCOON ’99*, 1999, pages 330–338.

[Pe14] David Peleg: *Immunity against Local Influence*. In *Language, Culture, Computation. Computing - Theory and Technology*, volume 8001, LNCS. Springer, 2014, pages 168–179.

[PVV09] Etienne Perron, Dinkar Vasudevan, and Milan Vojnović: *Using Three States for Binary Consensus on Complete Graphs*. In *Proc. INFOCOM*, 2009, pages 2527–2535.

[RAK07] Usha Raghavan, Réka Albert, and Soundar Kumara: *Near linear time algorithm to detect community structures in large-scale networks*. In *Physical Review E*, volume 76 (3), 2007, pages 036106.

[VF06] Vivek Vishnumurthy and Paul Francis: *On Heterogeneous Overlay Construction and Random Node Selection in Unstructured P2P Networks*. In *Proc. INFOCOM*, 2006, pages 1–12.