Abstract. In this paper we extend Bourgain’s double recurrence result to the Wiener–
Wintner averages. Let \((X, \mathcal{F}, \mu, T)\) be a standard ergodic system. We will show that for
any \(f_1, f_2 \in L^\infty(\mu)\), the double recurrence Wiener–Wintner average
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i nt}
\]
converges off a single null set of \(X\) independent of \(t\) as \(N \to \infty\). Furthermore, we will
show a uniform Wiener–Wintner double recurrence result: if either \(f_1\) or \(f_2\) belongs to the
orthogonal complement of the Conze–Lesigne factor, then there exists a set of full measure
such that the supremum on \(t\) of the absolute value of the averages above converges to 0.

1. Historical background
In 1990, Bourgain proved the result on double recurrence [9], which is stated as follows.

**Theorem 1.1.** (Bourgain [9]) Let \((X, \mathcal{F}, \mu, T)\) be an ergodic system, and \(T_1, T_2\) be
powers of \(T\). Then, for \(f_1, f_2 \in L^\infty(\mu)\),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_1(T_1^n x) f_2(T_2^n x)
\]
exists for \(\mu\)-almost every \(x \in X\).

In [9], the theorem was proven for the case \(T_1 = T = T_2^{-1}\). Bourgain’s proof relies on
the uniform Wiener–Wintner theorem, which is stated as follows (see, for example, [2] for
a proof).
THEOREM 1.2. Let \((X, F, \mu, T)\) be an ergodic system, and let \(f\) be a function in the orthogonal complement of the Kronecker factor of \((X, T)\). Then there exists a set of full measure \(X_f\) such that
\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^n x) e^{2\pi i nt} \right| = 0
\]
for all \(x \in X_f\).

In 2001, the second author worked on an extended result of Bourgain in his PhD thesis [12], and proved the double recurrence Wiener–Wintner result for the case where \(T\) is totally ergodic (i.e. \(T^a\) is ergodic for any \(a \in \mathbb{Z}\)).

THEOREM 1.3. Let \((X, F, \mu, T)\) be a standard ergodic dynamical system (i.e. a compact metrizable space, \(F\) is a Borelian \(\sigma\)-algebra, \(\mu\) is a probability Borel measure, and \(T\) is a self-homeomorphism), where \(T\) is a totally ergodic map. Suppose that \(f_1\) and \(f_2\) belong to \(L^2(X)\). Let \(CL\) be the maximal isometric extension of the Kronecker factor of \(T\). Let
\[
W_N(f_1, f_2, x, t) = \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an} x) f_2(T^{bn} x) e^{2\pi i nt}.
\]

(1) (Double uniform Wiener–Wintner theorem) If either \(f_1\) or \(f_2\) belongs to \(CL^\perp\), then there exists a set of full measure \(X_{f_1 \otimes f_2}\) such that, for all \(x \in X_{f_1 \otimes f_2}\),
\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} |W_N(f_1, f_2, x, t)| = 0.
\]

(2) (General convergence) If \(f_1, f_2 \in CL\), then \(W_N(f_1, f_2, x, t)\) converges for \(\mu\)-almost every (a.e.) \(x \in X\) for all \(t \in \mathbb{R}\), provided that the cocycle associated with \(CL\) is affine.

Theorem 1.3 was proved in several stages. For (1), first one identifies the pointwise limit of the double recurrence averages as an integral with respect to a particular Borel measure (disintegration). Then one uses Wiener’s lemma on the continuity of spectral measures and van der Corput’s inequality to show that the double recurrence average converges to 0. For (2), one first shows that the total ergodicity of \(T\) asserts that \(CL\) is the same for every integer power of \(T\), which allows one to assume that both functions lie in the same factor of \(L^2(X, \mu)\). Furthermore, the assumption that the measurable cocycle associated with \(CL\) is affine allows one to use the homomorphism property to simplify the computations.

A little was known about characteristic factors back then, especially for pointwise convergence. Originally in [12], the factor \(CL\) was referred to as the ‘Conze–Lesigne’ factor, having first appeared in work by Conze and Lesigne (see, for example, [10, 11] for details), and named so by Rudolph [21]. But with the work of Host and Kra in [17], the definition of the Conze–Lesigne factor was updated when the Host–Kra–Ziegler factors emerged in 2005. It is noted that the updated Conze–Lesigne factor \(Z_2\), the second Host–Kra–Ziegler factor, is smaller than \(CL\), so more work is needed to prove the uniform double recurrence Wiener–Wintner theorem for the case whether either \(f_1, f_2 \in Z_2^\perp\) since \(CL^\perp \subset Z_2^\perp\).
2. Introduction

In this paper we will prove the uniform Wiener–Wintner result for the case \( f_1 \in Z_2^1 \) using the seminorms that characterize these related factors. These characteristic factors and seminorms were developed in the work of Host and Kra \([17]\) (the characteristic factors were also developed independently by Ziegler \([22]\) without the use of seminorms).

**Definition 2.1.** Let \((X, \mathcal{F}, \mu, T)\) be an ergodic dynamical system on a probability measure space. The factors \(Z_k\) are defined in terms of seminorms as follows.

- The factor \(Z_0\) is the trivial \(\sigma\)-algebra.
- The factor \(Z_1\) can be characterized by the seminorm \(\| \cdot \|_2^4\) which is defined as
  \[
  \| f \|_2^4 = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \left| \int f \cdot f \circ T^h \, d\mu \right|^2,
  \]
  i.e. a function \(f\) belongs to \(Z_1^1\) if and only if \(\| f \|_2^4 = 0\).
- The factor \(Z_2\) is the Conze–Lesigne factor. Functions in this factor are characterized by the seminorm \(\| \cdot \|_3^8\) such that
  \[
  \| f \|_3^8 = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \| f \cdot f \circ T^h \|_2^4,
  \]
  i.e. a function \(f\) belongs to \(Z_2^1\) if and only if \(\| f \|_3^8 = 0\).
- More generally, for each positive integer \(k\), we have
  \[
  \| f \|_{k+1}^{2k+1} = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \| f \cdot f \circ T^h \|_2^{2k},
  \]
  with the condition that \(f\) belongs to \(Z_{k-1}^1\) if and only if \(\| f \|_{k}^{2k} = 0\).

Note that these are similar to the seminorms introduced by Gowers in \([15]\). In particular, \(\| f \|_{2k}^{2k}(X) = \| f \|_{2k}^{2k}\). In this paper, we chose to use the notation \(\| \cdot \|_{2k}^{2k}\) merely for the sake of readability.

In 2012, the first author and Presser published an update \([8]\) of their earlier unpublished work \([7]\) on characteristic factors and the multiterm return times theorem.

**Definition 2.2.** Let \((X, \mathcal{F}, \mu, T)\) be an ergodic dynamical system on a probability measure space. We define factors \(A_k\) in the following inductive way.

- The factor \(A_0\) is the trivial \(\sigma\)-algebra \(\{X, \emptyset\}\).
- The factor \(A_1\) is the Kronecker factor of \(T\). We denote \(N_1(f) = \| E(f | A_1) \|_2\).
- For \(k \geq 1\), the factor \(A_{k+1}\) is characterized by the following: a function \(f \in A_{k+1}^1\) if and only if
  \[
  N_{k+1}(f)^4 := \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \| E(f \cdot f \circ T^h | A_k) \|^2_2 = 0.
  \]

It was proven that the quantities \(N_k(f)\) are well defined in \([1]\), and they characterize factors \(A_k\) of \(T\) which are successive maximal isometric extensions. These successive factors turned out to be the \(k\)-step distal factors introduced by Furstenberg in \([13]\).
In [8], it was shown that given an ergodic system \((X, \mathcal{F}, \mu, T)\) and \(f_1 \in L^\infty(\mu)\), there exists a set of full measure \(X_f\) such that for any \(x \in X_f\) and for any measure-preserving system \((Y, \mathcal{G}, \nu, S)\) and \(f_2 \in L^\infty(\nu)\) such that \(\|f_2\|_{L^\infty(\nu)} \leq 1\), the average
\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(S^n y) e^{2\pi i nt} \right| \leq CN_3(f_1)^2
\]
converges for \(\nu\text{-a.e. } y \in Y\) for some absolute constant \(C\) independent of \(f_1, f_2, S, \) and \(y\).

It is known that \(Z_k \subset A_k\) for each \(k\) (in fact, \(Z_0\) equals \(A_0\) and \(Z_1\) equals \(A_1\), but \(Z_2 \subsetneq A_2\)), so \(Z_k \supseteq A_k^\perp\). In [8], it was proven that \(Z_k\) and \(A_k\) are both pointwise characteristic for the \(k\)-term return times averages.

In this paper we will update Theorem 1.3 in the following ways:
- We will only assume that \(T\) is ergodic, rather than totally ergodic.
- We will show that \(Z_2\) (and \(A_2\)) is a characteristic factor for this Wiener–Wintner average, i.e. we will prove the uniform double Wiener–Wintner result for the case either \(f_1 \in Z_2^\perp\) or \(f_2 \in Z_2^\perp\) rather than \(CL^\perp\).
- We will show that the convergence holds in general for the case where \(f_1, f_2 \in Z_2\).

In other words, we will prove the following theorem.

**Theorem 2.3.** Let \((X, \mathcal{F}, \mu, T)\) be a standard ergodic dynamical system, and \(f_1, f_2 \in L^2(X)\). Let
\[
W_N(f_1, f_2, x, t) = \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an} x) f_2(T^{bn} x) e^{2\pi i nt}.
\]

1. **(Double uniform Wiener–Wintner theorem)** If either \(f_1\) or \(f_2\) belongs to \(Z_2^\perp\), then there exists a set of full measure \(X_{f_1 \otimes f_2}\) such that, for all \(x \in X_{f_1 \otimes f_2}\),
\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} |W_N(f_1, f_2, x, t)| = 0.
\]
2. **(General convergence)** If \(f_1, f_2 \in Z_2\), then for \(\mu\text{-a.e. } x \in X\), \(W_N(f_1, f_2, x, t)\) converges for all \(t \in \mathbb{R}\).

We will use Bourgain’s double recurrence theorem and the seminorms mentioned above to prove (1). Some useful inequalities are introduced in §3. We will first show that (1) holds for the case \(f_1 \in Z_2^\perp\) when \(a = 1\) and \(b = 2\) in §4. Complication arises when \(|b - a| > 1\), and we will prove the case for general \(a, b \in \mathbb{Z}\) in §5. In §6, we will prove that there is a pointwise estimate for the limit of the double recurrence Wiener–Wintner average for the case \(a = 1\) and \(b = 2\) using the seminorm \(N_2(\cdot)\). Finally, in §7, we will show (2) of Theorem 2.3 using Leibman’s convergence result from [19].

Throughout this paper we will without loss of generality assume that the functions \(f_1\) and \(f_2\) are real-valued, and \(\|f_1\|_{L^\infty}, \|f_2\|_{L^\infty} \leq 1\), unless specified otherwise. Note that this implies that, for any sub-\(\sigma\)-algebra \(\mathcal{G}\) of \(\mathcal{F}\), \(\|\mathbb{E}(f_i | \mathcal{G})\|_{L^\infty} \leq 1\) for both \(i = 1, 2\).

3. **Some inequalities**

Throughout this paper, we will refer to the following inequalities repetitively. The proofs are given in the appendix.
One of the key ingredients of our proofs is van der Corput’s inequality, which is the following.

**Lemma 3.1.** (van der Corput) If \((a_n)\) is a sequence of complex numbers and if \(H\) is an integer between 0 and \(N - 1\), then

\[
\left| \frac{1}{N} \sum_{n=0}^{N-1} a_n \right|^2 \leq \frac{N + H}{N^2(H + 1)} \sum_{n=0}^{N-1} |a_n|^2 + \frac{2(N + H)}{N^2(H + 1)^2} \sum_{h=1}^{H}(H + 1 - h) \text{Re} \left( \sum_{n=0}^{N-h-1} a_n \overline{a}_{n+h} \right). \tag{1}
\]

The following inequalities can be derived directly from (1).

**Lemma 3.2.**
- There exists an absolute constant \(C\) such that, for any sequence of complex numbers \((a_n)\) such that \(\sup_n |a_n| \leq 1\) and any positive integer \(N\),

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} a_n \right|^2 \leq \frac{C}{H} + \frac{C}{(H+1)^2} \sum_{h=1}^{H}(H + 1 - h) \text{Re} \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n \overline{a}_{n+h} \right) \tag{2}
\]

for any \(H \in \mathbb{N}\).
- There exists an absolute constant \(C\) such that, for any sequence of complex numbers \((a_n)\) such that \(\sup_n |a_n| \leq 1\) and any positive integer \(N\),

\[
\sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} a_n e^{2\pi i nt} \right|^2 \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H}(H + 1 - h) \text{Re} \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n \overline{a}_{n+h} \right) \tag{3}
\]

for \(1 \leq H \leq N\).
- There exists an absolute constant \(C\) such that, for any sequence of complex numbers \((a_n)\) such that \(\sup_n |a_n| \leq 1\) and any positive integer \(N\),

\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} a_n e^{2\pi i nt} \right|^2 \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N-h} a_n \overline{a}_{n+h} \right| \tag{4}
\]

for all \(H \in \mathbb{N}\).

The following inequality is sometimes known as the reverse Fatou lemma.

**Lemma 3.3.** Let \((X, \mathcal{F}, \mu)\) be a measure space. Suppose that \((f_n)\) is a sequence of integrable, real-valued functions such that \(\sup_n f_n \leq F\) for some integrable function \(F\). Then

\[
\limsup_{n \to \infty} \int f_n \, d\mu \leq \int \limsup_{n \to \infty} f_n \, d\mu. \tag{5}
\]

Finally, here is an inequality that can be used to control the average along the cubes. This inequality is similar to [3, Lemma 5].
Lemma 3.4. Let \( a_n, b_n, \) and \( c_n, n \in \mathbb{N} \) be three complex-valued sequences, the norm of each bounded above by 1. Then for each positive integer \( N \),

\[
\left| \frac{1}{N(N+1)^2} \sum_{m,n=0}^{N-1} (N+1-m) a_n \cdot b_m \cdot c_{n+m} \right|^2 \leq \sup_t \left| \frac{1}{N} \sum_{m'=1}^{2(N-1)} c_{m'} e^{2\pi i m'} \right|^2. \tag{6}
\]

4. When \( f_1 \in \mathbb{Z}_2^\perp, a = 1, b = 2 \)

In this section we will prove the uniform Wiener–Wintner theorem for the case where \( f_1 \) belongs to \( \mathbb{Z}_2^\perp \). We will prove this special case since the fact that \(|b - a| = 1\) simplifies the proofs tremendously, since for any \( f, g \in L^2(\mu) \),

\[
\int f(Tx)g(T^2x) \, d\mu(x) = \int f(x)g(Tx) \, d\mu(x).
\]

Theorem 4.1. Let \((X, \mathcal{F}, \mu, T)\) be an ergodic dynamical system, and \( f_1, f_2 \in L^\infty(X) \), and \( \| f_i \|_2 \leq 1 \) for both \( i = 1, 2 \). If \( f_1 \in \mathbb{Z}_2^\perp \), then

\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) e^{2\pi i nt} \right| = 0
\]

for \( \mu \)-a.e. \( x \in X \).

We will present two proofs; the first is more direct and concise than the second, while the second will be similar to the proof for the general case (when \( a, b \in \mathbb{Z} \)).

In the first proof, we first find the upper bound for the limit supremum for the \( L^2 \)-norm of the average of the sequence \( G_1(T^n x)G_2(T^{2n} x) \) for any \( G_1, G_2 \in L^\infty(\mu) \) using inequality (2)—this upper bound turns out to be a constant multiple of \( \| f_1 \|_2^2 \). Then we use this upper bound as well as the double recurrence theorem and inequality (4) to find the upper bound for the norm of the limit supremum for the average of the sequence \( f_1(T^n x) f_2(T^{2n} x) e^{2\pi i nt} \), which turns out to be a constant multiple of \( \| f_1 \|_2^2 \).

For the second proof, we first apply inequality (3) by setting \( a_n = f_1(T^n x) f_2(T^{2n} x) \) pointwise, and then apply inequality (1) to the new average. After noticing that the average after the van der Corput trick converges almost everywhere (by the double recurrence theorem), we take the limit supremum of the first average and integrate. Using the ergodic decomposition, Wiener’s lemma, inequality (6), and an inequality found in [8], we can conclude that the original average converges to zero \( \mu \)-almost everywhere.

**First proof of Theorem 4.1.** First, we would like to show that for any two functions \( G_1, G_2 \in L^\infty(\mu) \), where \( \| G_i \|_{L^\infty(\mu)} \leq 1 \) for \( i = 1, 2 \), the following estimate holds:

\[
\limsup_{N \to \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N} G_1(T^n x) G_2(T^{2n} x) \right|^2 \, d\mu \leq C \| G_1 \|_2^2. \tag{7}
\]
We apply inequality (2) by setting $a_n = G_1(T^n x)G_2(T^{2n} x)$ pointwise to obtain

$$
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} G_1(T^n x)G_2(T^{2n} x) \leq \frac{C}{H} + \frac{C}{(H+1)^2} \sum_{h=1}^{H} (H + 1 - h) \times \Re \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (G_1 \cdot G_1 \circ T^h)(T^n x)(G_2 \cdot G_2 \circ T^{2h})(T^{2n} x) \right).
$$

Then we integrate both sides of this inequality and apply (5) to obtain

$$
\limsup_{N \to \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N} G_1(T^n x)G_2(T^{2n} x) \right|^2 d\mu \leq \int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} G_1(T^n x)G_2(T^{2n} x) \right|^2 d\mu \leq \frac{C}{H} + \frac{C}{(H+1)^2} \sum_{h=1}^{H} (H - h + 1) \times \Re \left( \int \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (G_1 \cdot G_1 \circ T^h)(T^n x)(G_2 \cdot G_2 \circ T^{2h})(T^{2n} x) d\mu \right). \quad (8)
$$

Note that $\lim_{N \to \infty} (1/N) \sum_{n=1}^{N} (G_1 \cdot G_1 \circ T^h)(T^n x)(G_2 \cdot G_2 \circ T^{2h})(T^{2n} x)$ exists for $\mu$-a.e. $x \in X$ by the double recurrence theorem. Hence, the dominated convergence theorem tells us that (8) becomes

$$
\limsup_{N \to \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N} G_1(T^n x)G_2(T^{2n} x) \right|^2 d\mu \leq \frac{C}{H} + \frac{C}{(H+1)^2} \sum_{h=1}^{H} (H - h + 1) \times \Re \left( \lim_{N \to \infty} \int \frac{1}{N} \sum_{n=1}^{N} (G_1 \cdot G_1 \circ T^h)(T^n x)(G_2 \cdot G_2 \circ T^{2h})(T^{2n} x) d\mu \right).
$$

Using the fact that $T$ is measure-preserving, we can apply the mean ergodic theorem to obtain

$$
\limsup_{N \to \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N} G_1(T^n x)G_2(T^{2n} x) \right|^2 d\mu \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \lim_{N \to \infty} \int (G_1 \cdot G_1 \circ T^h)(x) \frac{1}{N} \sum_{n=1}^{N} (G_2 \cdot G_2 \circ T^{2h})(T^n x) d\mu \int (G_2 \cdot G_2 \circ T^{2h})(y) d\mu(y)
$$

$$
= \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \int (G_1 \cdot G_1 \circ T^h)(x) d\mu(x) \int (G_2 \cdot G_2 \circ T^{2h})(y) d\mu(y).
$$
Because $\|G_2\|_{L^\infty(\mu)} \leq 1$, we know that $|\int (G_2 \cdot G_2 \circ T^{2h})(y) \, d\mu(y)| \leq 1$. Hence, we can apply the Cauchy–Schwarz inequality and let $H \to \infty$ to obtain (7).

We are now ready to prove the theorem. Our goal is to show that there exists a universal constant $C$ such that

$$
\int \limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right|^2 \, d\mu \leq C\|f_1\|_3^2.
$$

Using inequality (4) by setting $a_n = f_1(T^n x) f_2(T^{2n} x)$ pointwise, we obtain

$$
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right|^2 \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \left( \limsup_{N \to \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right|^2 \, d\mu \right)^{1/2}.
$$

Applying (7) by setting $G_1 = F_{1,h}$ and $G_2 = F_{2,h}$ on the right-hand side while letting $H \to \infty$, we obtain the desired upper bound; we have

$$
\int \limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right|^2 \, d\mu \leq \limsup_{H \to \infty} \frac{C}{H} \sum_{h=1}^{H} \|f_1 \cdot f_1 \circ T^h\|_2
$$

$$
\leq C \left( \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \|f_1 \cdot f_1 \circ T^h\|^4 \right)^{1/4} \leq C \|f_1\|_3^2.
$$

Since $f_1$ belongs to $Z_+^2$, we know that $\|f_1\|_3 = 0$, which completes the proof. \hfill \square

**Second proof.** We denote $F_{1,h}(x) = f_1(x) f_1 \circ T^h(x)$ and $F_{2,h}(x) = f_2(x) f_2 \circ T^{2h}(x)$. We apply inequality (4) by setting $a_n = f_1(T^n x) f_2(T^{2n} x)$ pointwise, and the Cauchy–Schwarz inequality to obtain the following estimate for any $H \in \mathbb{N}$:

$$
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right|^2 \leq \frac{C}{H} + \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right| \leq \frac{C}{H} + \left( \frac{C}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right|^2 \right)^{1/2}.
$$

(9)
Now we apply inequality (2) on the average inside the absolute value by setting $a_n = F_{1,h}(T^n x)F_{2,h}(T^{2n} x)$ pointwise so that, for any $K \in \mathbb{N}$,

$$\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^n x)F_{2,h}(T^{2n} x) \right|^2 \leq \frac{C}{K} + \frac{C}{(K+1)^2} \sum_{k=1}^{K} (K-k+1) \times \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (F_{1,h} \cdot F_{1,h} \circ T^k)(T^n x)(F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^{2n} x) \right).$$  

(10)

Note that the average

$$\frac{1}{N} \sum_{n=1}^{N} (F_{1,h} \cdot F_{1,h} \circ T^k)(T^n x)(F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^{2n} x)$$

converges as $N \to \infty$ by the double recurrence theorem. Now we combine (9) and (10), integrate both sides, and apply Hölder’s inequality as well as the dominated convergence theorem to obtain

$$\int \limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right|^2 d\mu \leq \frac{C}{H} + \int \left( \frac{C}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^n x)F_{2,h}(T^{2n} x) \right|^2 \right)^{1/2} d\mu$$

$$\leq \frac{C}{H} + \left( \frac{C}{H} \sum_{h=1}^{H} \int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^n x)F_{2,h}(T^{2n} x) \right|^2 d\mu \right)^{1/2}$$

$$\leq \frac{C}{H} + \left( \frac{C}{H} \sum_{h=1}^{H} \left( \frac{C}{K} + \frac{C}{(K+1)^2} \sum_{k=1}^{K} (K+1-k) \times \lim_{N \to \infty} \int \left( \frac{1}{N} \sum_{n=1}^{N} (F_{1,h} \cdot F_{1,h} \circ T^k)(T^n x)(F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^{2n} x) d\mu(x) \right) \right)^{1/2} \right).$$

Using the fact that $T$ is measure-preserving, we can evaluate the limit in the last term by applying the mean ergodic theorem: we have

$$\lim_{N \to \infty} \int \frac{1}{N} \sum_{n=1}^{N} (F_{1,h} \cdot F_{1,h} \circ T^k)(T^n x)(F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^{2n} x) d\mu(x)$$

$$= \lim_{N \to \infty} \int (F_{1,h} \cdot F_{1,h} \circ T^k)(x) \frac{1}{N} \sum_{n=1}^{N} (F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^n x) d\mu(x)$$

$$= \int \int (F_{1,h}(x)F_{1,h} \circ T^k)(x) F_{2,h}(y)(F_{2,h} \circ T^{2k})(y) d\mu(x) d\mu(y)$$

$$= \int \int f_1 \otimes f_2(x, y) f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y)) \times f_1 \otimes f_2(U^{h+k}(x, y)) d\mu \otimes \mu(x, y),$$
where $U = T \otimes T^2$ is a measure-preserving transformation on $X^2$. If we take the ergodic decomposition of $\mu \otimes \mu$ with respect to $U$, then the integral becomes

$$\int \int f_1 \otimes f_2(x, y) f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y))$$

$$\times f_1 \otimes f_2(U^{h+k}(x, y)) d(\mu \otimes \mu)(x, y) d\mu(c).$$

Let $H = K$. Note that, on the system $(X^2, (\mu \otimes \mu)_c, U)$ for almost every $c \in X$, inequality (6) tells us that

$$\limsup_{H \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{H} \int_2^{H+1} (H + 1 - k) f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y))$$

$$\times f_1 \otimes f_2(U^{h+k}(x, y)) \right|^2 \leq \sup_{t \in \mathbb{R}} \left| \frac{1}{H} \int_2^{H+1} f_1 \otimes f_2(U^h(x, y)) e^{2\pi i ht} \right|^2. \quad (11)$$

Note that the proof is complete if we can show that $\limsup_{H \to \infty} \sup_{t \in \mathbb{R}} |(1/H) \sum_{h=1}^{H} f_1 \otimes f_2(U^h(x, y)) e^{2\pi i ht}| = 0$ for $(\mu \otimes \mu)_c$-a.e. $(x, y) \in X^2$, for $\mu$-a.e. $c \in X$. In other words, we would like to show that if $f_1$ belongs to $Z_2$, then $f_1 \otimes f_2$ belongs to the orthogonal complement of the Kronecker factor with respect to the transformation $U$ and the measure $(\mu \otimes \mu)_c$ for $\mu$-a.e. $c \in X$, so that we can apply the uniform Wiener–Wintner theorem (Theorem 1.2).

To show that this is indeed the case, we first prove the following lemma.

**Lemma 4.2.** Suppose that $(Y, \mathcal{G}, \nu, U)$ is a measure-preserving system, and $f \in L^\infty(X)$ such that

$$\int \limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f(U^n y) e^{2\pi i nt} \right| \, d\nu = 0.$$

If $\sigma_f$ is the spectral measure of $f$ with respect to $U$, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\hat{\sigma}_f(n)|^2 = 0.$$

**Proof.** By Wiener’s lemma, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\hat{\sigma}_f(n)|^2 = \sum_{t} |\sigma_f((t))|^2.$$

Observe that, by the spectral theorem,

$$|\sigma_f((t))| = \left| \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \hat{\sigma}_f(n) e^{2\pi i nt} \right| = \left| \lim_{N \to \infty} \int f(y) \frac{1}{N} \sum_{n=1}^{N} f(U^n y) e^{2\pi i nt} \, d\nu(y) \right|. $$
Since \( \lim_{N \to \infty} (1/N) \sum_{n=1}^{N} f(U^n y) e^{2\pi i nt} \) exists by the Wiener–Wintner pointwise ergodic theorem, we can apply the dominated convergence theorem to show that

\[
\left| \sigma_f(t) \right| = \left| \int f(y) \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(U^n y) e^{2\pi i nt} \, dv(y) \right|
\]

\[
\leq \| f \|_\infty \int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(U^n y) e^{2\pi i nt} \right| \, dv(y) = 0. \quad \square
\]

The following lemma will complete the proof of this theorem.

**Lemma 4.3.** Suppose that \( f_1 \in \mathcal{L}_2^\perp \). Then \( f_1 \otimes f_2 \) belongs to the orthogonal complement of the Kronecker factor of \( U = T \times T^2 \) with respect to measure \( (\mu \otimes \mu)_c \) for \( \mu \)-a.e. \( c \).

**Proof.** Equivalently, we would like to show that if \( \sigma_{f_1 \otimes f_2} \) is the spectral measure with respect to \( U \), then

\[
\frac{1}{N} \sum_{n=1}^{N} |\hat{\sigma}_{f_1 \otimes f_2}(n)|^2 \to 0,
\]

i.e. we would like to show that \( \sigma_{f_1 \otimes f_2} \) is a continuous measure. By Lemma 4.2, this can be done by showing that

\[
\int \limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f_1 \otimes f_2(U^n(x, y)) e^{2\pi i nt} \right\|_2 \, d(\mu \otimes \mu)_c(x, y) = 0 \quad (12)
\]

for \( \mu \)-a.e. \( c \in X \). Because of the ergodic decomposition and Fubini’s theorem, we have

\[
\int \limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f_1 \otimes f_2(U^n(x, y)) e^{2\pi i nt} \right\|_2 \, d(\mu \otimes \mu)_c(x, y) \, d\mu(c)
\]

\[
= \int \left( \int \limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} y) e^{2\pi i nt} \right\|_2 \, d\mu(y) \right) \, d\mu(x).
\]

The inner integral has an upper bound, as is stated and proved in [8, Lemma 8]: there exists a set of full measure \( X_{f_1} \) such that, for all \( x \in X_{f_1} \),

\[
\int \limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} y) e^{2\pi i nt} \right\|_2 \, d\mu(y) \leq C \| f_1 \|_3^2 = 0
\]

for \( \mu \)-a.e. \( y \in X \). Therefore,

\[
\int \limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f_1 \otimes f_2(U^n(x, y)) e^{2\pi i nt} \right\|_2 \, d(\mu \otimes \mu)_c(x, y) \, d\mu(c) = 0. \quad (13)
\]

Since

\[
\int \limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f_1 \otimes f_2(U^n(x, y)) e^{2\pi i nt} \right\|_2 \, d(\mu \otimes \mu)_c(x, y)
\]

is a \( \mu \)-almost everywhere non-negative function of \( c \) that is measurable with respect to \( \mu \), we can deduce from (13) that (12) holds. \[ \square \]
Because of Lemma 4.3, we can show that the average (11) converges to 0 almost everywhere, which proves Theorem 4.1. □

**Remark.** A result similar to Lemma 4.3 was proven by Rudolph in [21]. In his work, the Conze–Lesigne algebra referred is the maximal isometric extension of the Kronecker factor, which is $CL$ in Theorem 1.3 of this paper.

5. **Case $f_1 \in \mathbb{Z}_2^\perp$, when $a, b \in \mathbb{Z}$**

Here, we will prove the uniform Wiener–Wintner double recurrence property for any $a, b \in \mathbb{Z}$, which is stated precisely as follows.

**Theorem 5.1.** Let $(X, \mathcal{F}, \mu, T)$ be an ergodic dynamical system, and $f_1, f_2 \in L^\infty(X)$, and $\|f_2\|_\infty = 1$. If $f_1 \in \mathbb{Z}_2^\perp$, then

$$\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i nt} \right| = 0$$

for $\mu$-a.e. $x \in X$, and for any pair of integers $a$ and $b$.

Unlike the case in Theorem 4.1, $T^{b-a}$ may no longer be ergodic. We will first prove various lemmas to overcome this obstacle.

The following lemma will show that for any positive integer $s$, any $T^s$-invariant function $f$ can be expressed in terms of an integral kernel (that does not depend on $f$). The kernel first appeared in the work of Furstenberg and Weiss [14, Theorem 2.1]; we will present a detailed proof here†. This kernel will be useful to characterize various conditional expectations.

**Lemma 5.2.** Let $T$ be an ergodic map, and $s$ be a positive integer. Then there exist a disjoint partition of $T^s$-invariant sets $A_1, \ldots, A_l$ such that every $T^s$-invariant function $f$ can be expressed as an integral with respect to the kernel

$$K(x, y) = l \sum_{k=1}^{l} \mathbb{1}_{A_k}(x) \mathbb{1}_{A_k}(y).$$

**Proof.** If $T^s$ is ergodic, we are done, since $f$ is a constant. If not, suppose that $A$ is a $T^s$-invariant subset of $X$ such that $0 < \mu(A) < 1$. Define a function

$$f_A := \mathbb{1}_A + \mathbb{1}_{T^{-1}A} + \mathbb{1}_{T^{-2}A} + \cdots + \mathbb{1}_{T^{-(s-1)}A}.$$

Observe that $f_A$ is $T$-invariant, and since $T$ is ergodic, $f_A$ must be a constant. Therefore,

$$\mathbb{1}_A + \mathbb{1}_{T^{-1}A} + \mathbb{1}_{T^{-2}A} + \cdots + \mathbb{1}_{T^{-(s-1)}A} = \int f_A \, d\mu = s\mu(A).$$

Note that $f_A \neq 0$, since $\mu(A) \neq 0$. Similarly, $f_A \neq s$, since $\mu(A) \neq 1$. If $f_A = 1$, then for $\mu$-a.e. $x \in X$, $\mathbb{1}_A + \mathbb{1}_{T^{-1}A} + \mathbb{1}_{T^{-2}A} + \cdots + \mathbb{1}_{T^{-(s-1)}A} = 1$, which implies that

† The authors recently had an opportunity to discuss this proof with Benjamin Weiss, during the Ergodic Theory Workshop at UNC-Chapel Hill (April 3–6, 2014). The proof given here is longer than that provided by Weiss, but provides some information about the number $l$. 
$\mu(T^{-i}A \cap T^{-j}A) = 0$ for any $0 \leq i < j \leq s - 1$. Hence, $A, T^{-1}A, \ldots, T^{-(s-1)}A$ are disjoint, and furthermore, $\mu(X) = \sum_{k=0}^{s-1} \mu(T^{-k}A) = 1$, so $A, T^{-1}A, \ldots, T^{-(s-1)}A$ is a partition of $X$.

We now show that $A$ (and similarly, $T^{-1}A, \ldots, T^{-(s-1)}A$) is an atom (of a collection of $T^s$-invariant sets). If $B \subset A$ and $B$ is $T^s$-invariant, then

$$f_B = \mathbb{1}_B + \mathbb{1}_{T^{-1}B} + \mathbb{1}_{T^{-2}B} + \cdots + \mathbb{1}_{T^{-(s-1)}B} = s\mu(B) \leq s\mu(A) = 1.$$  

The above holds only when $\mu(B) = 0$ or $\mu(B) = 1/s = \mu(A)$, which implies that $B = A$ $\mu$-almost everywhere. For $k > 0$, we note that if $B \subset T^{-k}A$ is $T^s$-invariant, then $T^kB \subset A$ is also $T^s$-invariant, so if $\mu(B) \neq 0$, then $\mu(B) = \mu(T^kB) = \mu(A) = \mu(T^{-k}A)$, which proves that $T^{-k}A$ is also an atom for $k > 0$.

If $f$ is a $T^s$-invariant function, then we claim that

$$f = \sum_{k=0}^{s-1} \left( \frac{f_{T^{-k}A} f d\mu}{\mu(T^{-k}A)} \right) \mathbb{1}_{T^{-k}A} = s \sum_{k=0}^{s-1} \left( \int_{T^{-k}A} f d\mu \right) \mathbb{1}_{T^{-k}A}. \tag{15}$$

First, we note that $\mathcal{S}$, the $\sigma$-algebra generated by the sets $A, T^{-1}A, \ldots, T^{-(s-1)}A$, is a collection of finite union of sets $A, T^{-1}A, \ldots, T^{-(s-1)}A$. We also know that $f$ is $\mathcal{S}$-measurable, since

$$\{f > \lambda\} = \bigcup_{k=0}^{s-1} (\{f > \lambda\} \cap T^{-k}A),$$

and we note that $\{f > \lambda\} \cap T^{-k}A$ is $T^s$-invariant. Since $T^{-k}A$ is an atom for each $k$, we know $\{f > \lambda\} \cap T^{-k}A$ equals either $T^{-k}A$ or the empty set. This implies that $\{f > \lambda\} \in \mathcal{S}$.

Since we know that $f$ is $\mathcal{S}$-measurable, we note that $f$ can be expressed as the expression above (a fact regarding conditional expectation). This proves (15), and if we denote $T^{-k}A = A_k$, then we have

$$f \circ T^s(x) = f(x) = \int s \sum_{k=0}^{s-1} \mathbb{1}_{A_k}(y) \mathbb{1}_{A_k}(x) f(y) d\mu(y) = \int f(y) K(x, y) d\mu(y),$$

which proves the lemma for the case $f_A = 1$.

Now suppose that $f_A = k$ for $2 \leq k \leq s - 1$. Let $B = T^{-l_1}A \cap T^{-l_2}A \cap \cdots T^{-l_k}A$, where $0 \leq l_1 < l_2 < \cdots < l_k \leq s - 1$, and $\mu(B) > 0$ (we know such $B$ exists since $f_A = k$). Define

$$f_B = \mathbb{1}_B + \mathbb{1}_{T^{-1}B} + \cdots + \mathbb{1}_{T^{-(s-1)}B}.$$  

Note that $f_B$ is $T$-invariant, so it must be a constant that equals to $s\mu(B)$. Since $\mu(B) > 0$, we know that $f_B > 0$.

Also note that each $T^{-j}B$ is disjoint for $0 \leq j \leq s - 1$. Assume that it is not. Then, for some $0 \leq i < j \leq s - 1$, there exists $x \in T^{-i}B \cap T^{-j}B$ such that $f_A(x) > k$, which is a contradiction. Therefore, we must have $f_B \leq 1$, and we can conclude that $f_B = 1$. By letting $A_i = T^{-i}B$, we have proved the lemma. \hfill \Box

The next lemma will provide a simple yet useful comparison between $\limsup_{H \to \infty} (1/H) \sum_{h=1}^H \|f \circ T^{ah}\|_k^{2^k} \cdot \|f\|_{k+1}^{2^k+1}$. 


LEMMA 5.3. Let \((X, \mathcal{F}, \mu, T)\) be an ergodic dynamical system, and \(a \in \mathbb{Z}\). Then, for any positive integer \(k\),

\[
\limsup_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \| f \cdot f \circ T^{ah} \|_{k}^{2k} \leq |a| \| f \|_{k+1}^{2k+1}.
\]

Proof. Note that

\[
\frac{1}{H} \sum_{h=1}^{H} \| f \cdot f \circ T^{ah} \|_{k}^{2k} \leq \frac{1}{H} \sum_{h=1}^{H} \| f \cdot f \circ T^{h} \|_{k}^{2k} = |a| \left( \frac{1}{|a|H} \sum_{h=1}^{H} \| f \cdot f \circ T^{h} \|_{k}^{2k} \right).
\]

The sequence \((1/|a|) \sum_{h=1}^{H} \| f \cdot f \circ T^{h} \|_{k}^{2k})_{H}\) is a subsequence of \((1/H) \sum_{h=1}^{H} \| f \cdot f \circ T^{h} \|_{k}^{2k})_{H}\), which converges to \(\| f \|_{k+1}^{2k+1}\). By taking the limit supremum on both sides of the inequality above, we get

\[
\limsup_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \| f \cdot f \circ T^{ah} \|_{k}^{2k} \leq |a| \left( \lim_{H \to \infty} \frac{1}{|a|H} \sum_{h=1}^{H} \| f \cdot f \circ T^{h} \|_{k}^{2k} \right) = |a| \| f \|_{k+1}^{2k+1}.
\]

\(\square\)

The proof of Bourgain’s double recurrence theorem in [9] relies on the classical uniform Wiener–Wintner theorem, which holds for the case where \(T\) is ergodic. Here, we prove the uniform Wiener–Wintner theorem that holds for the case where the measure-preserving transformation is a power of an ergodic map. This allows us to use Bourgain’s double recurrence theorem without assuming that \(T\) is a totally ergodic map.

THEOREM 5.4. Let \((X, \mathcal{F}, \mu, T)\) be an ergodic system. Suppose that \(f \in \mathcal{L}^{1}_{\perp}\). Then there exists a set of full measure \(X_{f}\) such that, for any \(x \in X_{f}\) and for any integer \(a\),

\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^{an}x)e^{2\pi int} \right| = 0.
\]

Proof. To show that the uniform convergence holds, we apply inequality (4) for \(a_{n} = f(T^{an}x)\) pointwise, and use the pointwise ergodic theorem and the Cauchy–Schwarz inequality to obtain

\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^{an}x)e^{2\pi int} \right|^{2} \leq C H + C H \sum_{h=1}^{H} \left| \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (f \cdot f \circ T^{ah})(T^{n}x) \right|
\]

\[
= C H + C H \sum_{h=1}^{H} |\mathbb{E}_{a}(f \cdot f \circ T^{ah})(x)|
\]

\[
\leq C H + \left( \frac{C H}{H} \sum_{h=1}^{H} |\mathbb{E}_{a}(f \cdot f \circ T^{ah})(x)|^{2} \right)^{1/2},
\]

(16)
where $\mathbb{E}_a$ is the conditional expectation operator with respect to the $\sigma$-algebra of $T^a$-invariant sets. Let $\gamma_x$ be a measure on $\mathbb{R}$ such that $\hat{\gamma}_x(h) = \mathbb{E}_a(f \cdot f \circ T^{ah})(x)$. We would like to show that $\gamma_x$ is a continuous measure, since that would tell us that the limit above converges to 0 by Wiener’s lemma:

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} |\hat{\gamma}_x(h)|^2 = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} |\mathbb{E}_a(f \cdot f \circ T^{ah})|^2 = 0.$$ 

To show this, we use the integral kernel from Lemma 5.2 so that, for some positive integer $l$,

$$\mathbb{E}_a(f \cdot f \circ T^{ah})(x) = l \int \mathbb{1}_{A_i}(y) f(y) f(T^{ah} y) \, d\mu(y)$$

where $A_i$ is one of the sets of the partition of $X$ given in Lemma 5.2 such that $x \in A_i$. Set $g(y) = \mathbb{1}_{A_i}(y) f(y)$. Then we notice that

$$\int g(y) f(T^{ah} y) \, d\mu(y) = \hat{\sigma}_{f,g,T^a}(h),$$

where $\sigma_{f,g,T^a}$ is the spectral measure of the functions $f$ and $g$ with respect to the transformation $T^a$. Note that $\sigma_{f,g,T^a}$ is absolutely continuous with respect to $\sigma_{f,T^a}$ (see, for example, [20, Proposition 2.4] for a proof). We claim that $\sigma_{f,T^a}$ is a continuous measure. Since $f \in Z_1$ and $\sigma_{f,T}$ is a continuous measure, Wiener’s lemma tells us that $\lim_{H \to \infty} (1/H) \sum_{h=1}^{H} |\hat{\sigma}_{f,T}(h)|^2 = 0$. Since $((1/|a|H) \sum_{h=1}^{H} |\hat{\sigma}_{f,T}(h)|^2)_H$ is a subsequence of $((1/H) \sum_{h=1}^{H} |\hat{\sigma}_{f,T}(h)|^2)_H$, we have

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} |\hat{\sigma}_{f,T^a}(h)|^2 = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} |\hat{\sigma}_{f,T}(ah)|^2 
\leq \lim_{H \to \infty} |a| \left( \frac{1}{|a|H} \sum_{h=1}^{H} |\hat{\sigma}_{f,T}(h)|^2 \right) = 0,$$

and again, by Wiener’s lemma, $\sigma_{f,T^a}$ is a continuous measure. Hence, $\sigma_{f,g,T^a}$ is continuous, so

$$0 = \sigma_{f,g,T^a}([-\tau]) = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} e^{-2\pi iht} \int g(y) f(T^{ah} y) \, d\mu(y)$$

$$= \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} e^{-2\pi iht} \mathbb{E}_a(f \cdot f \circ T^{ah})(x) = \gamma_x([-\tau]). \quad (17)$$

Hence, $\gamma_x$ is a continuous measure. The proof is concluded by letting $H \to \infty$ in (16). \qed

Here we introduce seminorms that are similar to the ones introduced in Definition 2.1. These seminorms hold for any measure-preserving system.

**Definition 5.5.** Suppose that $(Y, \mathcal{Y}, \nu, U)$ is a measure-preserving system, and $f \in L^\infty(\nu)$. We define seminorms $\|\langle \cdot \rangle\|_2$ and $\|\langle \cdot \rangle\|_3$ on $L^2(\nu)$ such that

$$\|\langle f \rangle\|_2 = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \left| \int f \cdot f \circ U^h \, d\nu \right|^2$$
and
\[
\|\langle f \rangle \|_3^8 = \lim_{H \to \infty} \left( \frac{1}{H} \sum_{h=1}^{H} \| f \cdot f \circ U^h \|_2^4 \right).
\]

Certainly, if \( U \) is ergodic, then \( \|\langle f \rangle \|_k = \| f \|_k \) for \( k = 2, 3 \). We can easily verify that \( \|\langle \cdot \rangle \|_2 \) and \( \|\langle \cdot \rangle \|_3 \) are indeed seminorms. For example, we can show that \( \|\langle \cdot \rangle \|_2 \) is a positive semidefinite function by using the dominated convergence theorem and the pointwise convergence theorem:

\[
\|\langle f \rangle \|_2^4 = \lim_{H \to \infty} \left( \frac{1}{H} \right) \left( \int f \cdot f \circ U^h \, dv \right)^2
= \lim_{H \to \infty} \left( \frac{1}{H} \int (f \cdot f \circ U^h)(x) \, dv(x) \int (f \cdot f \circ U^h)(y) \, dv(y) \right)
= \int \int (f \otimes f)(x, y) \, d\mu \otimes \mu(x, y)
\times \lim_{H \to \infty} \left( \frac{1}{H} \sum_{h=1}^{H} (f \otimes f)((U \otimes U)^h(x, y)) \right) \, d\mu \otimes \mu(x, y)
= \int \int (f \otimes f)(x, y) \mathbb{E}(f \otimes f|\mathcal{I}^2)(x, y) \, dv \otimes v(x, y)
= \int \int \mathbb{E}(f \otimes f|\mathcal{I}^2)^2(x, y) \, dv \otimes v(x, y) \geq 0,
\]

where \( \mathcal{I}^2 \) is the \(\sigma\)-algebra generated by \( U \times U \)-invariant sets.

Before we proceed to prove the Theorem 5.1, we will prove the following preliminary lemmas.

**Lemma 5.6.** Suppose that \((Y, \mathcal{Y}, \nu, U)\) is a measure-preserving system, and \( f \in L^\infty(\nu) \). Then

\[
\int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(U^n y) \right|^2 \, dv \leq C \|\langle f \rangle \|_2^2
\]

for \(\nu\)-a.e. \( y \in Y \).

**Proof.** We denote \( F_h(x) = f(x) f \circ U^h(x) \). We apply inequality (2) by setting \( a_n = f(U^n y) \) pointwise and the pointwise ergodic theorem to obtain

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(U^n y) \right|^2 \leq \frac{C}{H} + \frac{C}{(H+1)^2} \sum_{h=1}^{H} (H+1-h) \left( \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} F_h(U^n y) \right)
= \frac{C}{H} + \frac{C}{(H+1)^2} \sum_{h=1}^{H} (H+1-h) \mathbb{E}(f \cdot f \circ U^h|\mathcal{I})(y)
\]

where \( \mathcal{I} \) is the \(\sigma\)-algebra generated by \( U \)-invariant sets. Note that \( \int \mathbb{E}(f \cdot f \circ U^h|\mathcal{I}) \, dv = \int f \cdot f \circ U^h \, dv \). So if we take the integral on both sides of inequality (18), we would obtain the following after applying the Cauchy–Schwarz inequality to the second term:

\[
\int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(U^n y) \right|^2 \, dv \leq \frac{C}{H} + C \left( \frac{1}{H} \sum_{h=1}^{H} \int f \circ f \cdot U^h \, dv \right)^{1/2}.
\]
Now we let $H \to \infty$ to obtain
\[
\int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(U^n y) \right|^2 \, d\nu \\
\leq C \left( \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \left\| f \cdot f \circ U^h \, d\nu \right\|^2 \right)^{1/2} = C\|f\|^2_2.
\]

Lemma 5.7. Suppose that $(X, \mathcal{F}, \mu, T)$ is an ergodic dynamical system, and $f_1, f_2 \in L^\infty(\mu)$. Then, for any integers $a$ and $b$,\[
\int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an} x) f_2(T^{bn} y) \right|^2 \, d\mu \otimes \mu(x, y) \leq C|a|\|f_1\|^2_2
\]
for $\mu$-a.e. $x, y \in X$.

Proof. We denote $F_{1,h}(x) = f_1(x) f_1 \circ T^{ah}(x)$ and $F_{2,h}(x) = f_2(x) f_2 \circ T^{bh}(x)$. If $U = T^a \times T^b$, then $(X^2, \mathcal{F} \otimes \mathcal{F}, \mu \otimes \mu, U)$ is a measure-preserving system. Hence, we can use (19) in Lemma 5.6 to obtain, for any $H \in \mathbb{N}$,
\[
\int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f_1 \otimes f_2(U^n(x, y)) \right|^2 \, d\mu \otimes \mu(x, y) \\
\leq C + C \left( \frac{1}{H} \sum_{h=1}^{H} \left\| F_{1,h}(x) F_{2,h}(y) \, d\mu \otimes \mu(x, y) \right\|^2 \right)^{1/2} \\
\leq C + C \left( \frac{1}{H} \sum_{h=1}^{H} \left\| f_1 \cdot f_1 \circ T^{ah} \, d\mu \right\|^2 \right)^{1/2}.
\]
As we let $H \to \infty$, we obtain the desired result by Lemma 5.3. \qed

Lemma 5.8. Suppose that $(X, \mathcal{F}, \mu, T)$ is an ergodic system, and $f_1, f_2 \in L^\infty(\mu)$. Then, for any integers $a$ and $b$,
\[
\int \limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an} x) f_2(T^{bn} y)e^{2\pi int} \right|^2 \, d\mu \otimes \mu(x, y) \leq C|a|^{1/2}\|f_1\|^2_3
\]
for $\mu$-a.e. $x, y \in X$.

Proof. We denote $F_{1,h}(x) = f_1(x) f_1 \circ T^{ah}(x)$ and $F_{2,h}(x) = f_2(x) f_2 \circ T^{bh}(x)$. By applying inequality (4) for $a_n = f_1(T^{an} x) f_2(T^{bn} y)$ pointwise, we obtain
\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an} x) f_2(T^{bn} y)e^{2\pi int} \right|^2 \\
\leq C + C \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^{an} x) F_{2,h}(T^{bn} y) \right|.
\]
Again, if we set $U = T^a \times T^b$, then $(X^2, \mathcal{F} \otimes \mathcal{F}, \mu \otimes \mu, U)$ is a measure-preserving system. Hence, Birkhoff’s pointwise ergodic theorem asserts that the average $(1/N)\]
\[ \sum_{n=1}^{N} F_{1,h} \otimes F_{2,h}(U^n(x, y)) \] converges \( \mu \otimes \mu \)-almost everywhere. By the Cauchy–Schwarz inequality, we have

\[
\int \limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}y)e^{2\pi int} \right|^2 d\mu \otimes \mu (x, y)
\]

\[
\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}y) \right| d\mu \otimes \mu (x, y)
\]

\[
\leq \frac{C}{H} + \frac{C}{H} \left( \frac{1}{H} \sum_{h=1}^{H} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}y) \right|^2 d\mu \otimes \mu (x, y) \right)^{1/2}.
\]

By Lemma 5.7, we know that

\[
\int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}y) \right|^2 d\mu \otimes \mu (x, y) \leq C |a||\|F_{1,h}\|_2^2.
\]

Hence,

\[
\int \limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}y)e^{2\pi int} \right|^2 d\mu
\]

\[
\leq \frac{C}{H} + C \left( \frac{|a|}{H} \sum_{h=1}^{H} \|f_1 \cdot f_1 \circ T^{ah}\|_2^2 \right)^{1/2}
\]

\[
\leq \frac{C}{H} + C \left( \frac{|a|^2}{H} \sum_{h=1}^{H} \|f_1 \cdot f_1 \circ T^{ah}\|_2^4 \right)^{1/4}.
\]

Let \( H \to \infty \), and apply Lemma 5.3 to obtain the desired result. \( \square \)

We are now ready to prove Theorem 5.1. The beginning of the proof is very similar to the second proof of Theorem 4.1, where we apply inequalities (4) and (2), as well as the double recurrence theorem and the mean ergodic theorem. Then we use the integral kernel from the Lemma 5.2 to obtain the integral expression for the upper bound, and then we use inequality (6) and Lemma 5.8 to obtain the desired result.

**Proof of Theorem 5.1.** We denote \( F_{1,h}(x) = f_1(x) f_1 \circ T^{ah}(x) \) and \( F_{2,h}(x) = f_2(x) f_2 \circ T^{bh}(x) \). Applying inequality (4) by setting \( a_n = f_1(T^{an}x) f_2(T^{bn}x) \), we obtain the following for all \( H \in \mathbb{N} \):

\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x)e^{2\pi int} \right|^2
\]

\[
\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}x) \right|
\]

\[
\leq \frac{C}{H} + \left( \frac{C}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}x) \right|^2 \right)^{1/2}.
\]
where the second inequality is the consequence of the Cauchy–Schwarz inequality. Note that we can apply inequality (2) on the average \(|(1/N) \sum_{n=1}^{N} F_{1,h}(T^{an}x)F_{2,h}(T^{bn}x)|^2\) to obtain the following bound for \(0 < K < N\):

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^{an}x)F_{2,h}(T^{bn}x) \right|^2
\]

\[
\leq \frac{C}{K} + \frac{C}{(K+1)^2} \sum_{k=1}^{K} (K + 1 - k)
\]

\[
\times \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (F_{1,h} \cdot F_{1,h} \circ T^{ak})(T^{an}x)(F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{bn}x) \right).
\]

Note that the average

\[
\frac{1}{N} \sum_{n=1}^{N} (F_{1,h} \cdot F_{1,h} \circ T^{ak})(T^{an}x) \cdot (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{bn}x)
\]

converges as \(N \to \infty\) by Bourgain’s almost everywhere double recurrence theorem in [9]. Therefore,

\[
\int \limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x)f_2(T^{bn}x)e^{2\pi int} \right|^2 d \mu(x)
\]

\[
\leq \frac{C}{H} + \frac{C}{H} \int \left( \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^{an}x)F_{2,h}(T^{bn}x) \right|^2 \right)^{1/2} d \mu(x)
\]

\[
\leq \frac{C}{H} + \frac{C}{H} \left( \sum_{h=1}^{H} \int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1,h}(T^{an}x)F_{2,h}(T^{bn}x) \right|^2 d \mu(x) \right)^{1/2}
\]

(by Hölder’s inequality)

\[
\leq \frac{C}{H} + \frac{C}{H} \left( \sum_{h=1}^{H} \left( \frac{C}{K} + \frac{C}{(K+1)^2} \sum_{k=1}^{K} (K + 1 - k)
\right.
\]

\[
\times \left( \lim_{N \to \infty} \int \frac{1}{N} \sum_{n=1}^{N} (F_{1,h} \cdot F_{1,h} \circ T^{ak})(T^{an}x)
\]

\[
\times (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{bn}x) d \mu(x) \right) \right)^{1/2}.
\]

Since \(T^a\) is a measure-preserving transformation, we can apply the mean ergodic theorem to obtain

\[
\lim_{N \to \infty} \int \frac{1}{N} \sum_{n=1}^{N} (F_{1,h} \cdot F_{1,h} \circ T^{ak})(T^{an}x)(F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{bn}x) d \mu(x)
\]

\[
= \lim_{N \to \infty} \int (F_{1,h} \cdot F_{1,h} \circ T^{ak})(x) \left( \frac{1}{N} \sum_{n=1}^{N} (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{(b-a)n}x) \right) d \mu(x)
\]

\[
= \int (F_{1,h} \cdot F_{1,h} \circ T^{ak})(x) \mathbb{E}(F_{2,h} \cdot F_{2,h} \circ T^{bk}[I_{b-a}](x)) d \mu(x),
\]
where \( I_{h-a} \) is the \( \sigma \)-algebra generated by \( T^{b-a} \)-invariant sets. By Lemma 5.2, there exist a positive integer \( l_{b-a} \) and partition \( A_1, \ldots, A_{l_{b-a}} \) of \( X \) such that

\[
\mathbb{E}(F_{2,h} \cdot F_{2,h} \circ T^{bk}\mid I_{b-a})(x) = \int (F_{2,h} \cdot F_{2,h} \circ T^{bk})(y) K_{b-a}(x, y) \, d\mu(y),
\]

where \( K_{b-a}(x, y) = l_{b-a} \sum_{i=1}^{l_{b-a}} 1_{A_i}(x) 1_{A_i}(y) \). Note that

\[
\int \int (F_{1,h}(x) \cdot F_{1,h} \circ T^{ak})(x)(F_{2,h} \cdot F_{2,h} \circ T^{bk})(y) K_{b-a}(x, y) \, d\mu(x) \, d\mu(y)
\]

\[
= \int \int f_1 \otimes f_2(x, y) K_{b-a}(x, y)
\]

\[
\times [f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y)) f_1 \otimes f_2(U^{h+k}(x, y))] \, d\mu \otimes \mu(x, y),
\]

where \( U = T^a \times T^b \) is a measure-preserving transformation on \( X^2 \). Let \( H = K \). Note that, on the system \( (X^2, \mu \otimes \mu, U) \), inequality (6) tells us that we have

\[
\frac{1}{H(H + 1)^2} \left| \sum_{h,k=0}^{H-1} (H + 1 - k) f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y))
\]

\[
\times f_1 \otimes f_2(U^{h+k}(x, y)) e^{2\pi iht} \right|^2 
\]

\[
\leq \sup_{t \in \mathbb{R}} \frac{1}{H} \sum_{h=1}^{2(H-1)} f_1 \otimes f_2(U^h(x, y)) e^{2\pi iht} \right|^2.
\]

(20)

By Lemma 5.8, we know that

\[
\int \limsup_{H \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{H} \sum_{h=1}^{H} f_1 \otimes f_2(U^h(x, y)) e^{2\pi iht} \right|^2 \, d\mu \otimes \mu(x, y)
\]

\[
= \int \limsup_{H \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{H} \sum_{h=1}^{H} f_1(T^{ah}x) f_2(T^{bh}y) e^{2\pi iht} \right|^2 \, d\mu \otimes \mu(x, y)
\]

\[
\leq C|a|^{1/2} \|f_1\|_2^2 = 0.
\]

Hence, by letting \( H \to \infty \), we obtain

\[
\int \limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi imt} \right|^2 \, d\mu(x)
\]

\[
\leq \int \int f_1 \otimes f_2(x, y) K(x, y) \cdot \lim_{H \to \infty} \frac{1}{H(H + 1)^2} \sum_{h,k=0}^{H-1} (H + 1 - k)
\]

\[
\times (f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y)) f_1 \otimes f_2(U^{h+k}(x, y))) \, d\mu \otimes \mu(x, y)
\]

\[= 0. \quad \square \]

6. Case where \( f_1 \in A_{1^+} \), \( a = 1, b = 2 \)

In this section we will show that we can obtain a pointwise estimate to the Wiener–Wintner double recurrence averages using the seminorm of \( A_2 \). This means that we can bound the
double recurrence averages using the seminorm $N_2(\cdot)$ without taking the integral of the norm of the averages. This was not the case when we used the Host–Kra seminorm $\| \cdot \|_3$, where we obtained the norm bound
\[
\int \limsup_{N \to \infty} \max_{t \in \mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) e^{2\pi int} \right| d\mu \leq C \| f_1 \|_3^2.
\]

We recall that $(X, \mathcal{F}, \mu, T)$ is an ergodic system, and that $f_1, f_2 \in L^\infty(\mu)$ such that $\| f_i \|_\infty \leq 1$ for both $i = 1, 2$.

**Theorem 6.1.** Let $(X, \mathcal{F}, \mu, T)$ be an ergodic dynamical system. Then there exists a universal constant $C$ such that
\[
\limsup_{N \to \infty} \max_{t \in \mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) e^{2\pi int} \right| \leq C[N_2(f_1)]^2
\]
for $\mu$-a.e. $x \in X$.

**Proof.** We first apply inequality (4) to the sequence $a_n = f_1(T^n x) f_2(T^{2n} x)$ pointwise to obtain
\[
\limsup_{N \to \infty} \max_{t \in \mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) e^{2\pi int} \right|^2 \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} (f_1 \cdot f_1 \circ T^h)(T^n x) (f_2 \cdot f_2 \circ T^{2h})(T^{2n} x) \right|.
\]

Our main task is to show that
\[
\limsup_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} (f_1 \cdot f_1 \circ T^h)(T^n x) (f_2 \cdot f_2 \circ T^{2h})(T^{2n} x) \right| \leq [N_2(f_1)]^2
\]
for $\mu$-a.e. $x \in X$. We will first prove the following lemma, which would allow us to take conditional expectations of $f_i \cdot f_i \circ T^h$ for $i = 1, 2$ given the Kronecker factor $A_1$, i.e. it would suffice to show that
\[
\limsup_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(f_1 \cdot f_1 \circ T^h | A_1)(T^n x) \mathbb{E}(f_2 \cdot f_2 \circ T^{2h} | A_1)(T^{2n} x) \right| \leq [N_2(f_1)]^2.
\]

**Lemma 6.2.** Suppose that $F_1, F_2 \in L^\infty(X)$ such that $\| F_1 \|_\infty, \| F_2 \|_\infty \leq 1$. If $F_1 \in A_1^\perp$, then for $\mu$-a.e. $x \in X$,
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F_1(T^n x) F_2(T^{2n} x) = 0.
\]

Thus, $A_1$ is a pointwise characteristic factor of this average, i.e.
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F_1(T^n x) F_2(T^{2n} x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(F_1 | A_1)(T^n x) \mathbb{E}(F_2 | A_1)(T^{2n} x).
\]
Proof. Since \( |(1/N) \sum_{n=1}^{N} F_{1}(T^{n}x)F_{2}(T^{2n}x)| \) is non-negative, we can prove this lemma by showing that

\[
\int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1}(T^{n}x)F_{2}(T^{2n}x) \right| \, d\mu(x) = 0
\]

where the first equality holds by Bourgain’s double recurrence theorem and Lebesgue’s dominated convergence theorem. Note that the Cauchy–Schwarz inequality asserts that

\[
\int \left| \frac{1}{N} \sum_{n=1}^{N} F_{1}(T^{n}x)F_{2}(T^{2n}x) \right| \, d\mu(x) \leq \left( \int \left| \frac{1}{N} \sum_{n=1}^{N} F_{1}(T^{n}x)F_{2}(T^{2n}x) \right|^2 \, d\mu(x) \right)^{1/2}.
\]

We will proceed by applying inequality (2) to \( a_{n} = F_{1}(T^{n}x)F_{2}(T^{2n}x) \) pointwise. Observe that, by inequality (5),

\[
\limsup_{N \to \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N} F_{1}(T^{n}x)F_{2}(T^{2n}x) \right|^2 \, d\mu(x)
\]

\[
\leq \int \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} F_{1}(T^{n}x)F_{2}(T^{2n}x) \right|^2 \, d\mu(x)
\]

\[
\leq \frac{C}{H} + \frac{C}{(H + 1)^2} \sum_{h=1}^{H} (H + 1 - h)
\]

\[
\times \int \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (F_{1} \cdot F_{1} \circ T^{h})(T^{n}x)(F_{2} \cdot F_{2} \circ T^{2h})(T^{2n}x) \, d\mu
\]

\[
\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \int \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (F_{1} \cdot F_{1} \circ T^{h})(T^{n}x)(F_{2} \cdot F_{2} \circ T^{2h})(T^{2n}x) \, d\mu.
\]

Note that the limit inside the integral exists by the double recurrence theorem. Hence, the dominated convergence theorem tells us

\[
\limsup_{N \to \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N} F_{1}(T^{n}x)F_{2}(T^{2n}x) \right|^2 \, d\mu(x)
\]

\[
\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \int \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (F_{1} \cdot F_{1} \circ T^{h})(T^{n}x)(F_{2} \cdot F_{2} \circ T^{2h})(T^{2n}x) \, d\mu
\]

\[
= \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \int (F_{1} \cdot F_{1} \circ T^{h})(x) \frac{1}{N} \sum_{n=1}^{N} (F_{2} \cdot F_{2} \circ T^{2h})(T^{n}x) \, d\mu.
\]
Then we use the mean ergodic theorem and the Cauchy–Schwarz inequality to obtain
\[
\limsup_{N \to \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N} F_1(T^n x) F_2(T^{2n} x) \right|^2 d\mu(x)
\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \|F_2\|_2^2 \left| \int (F_1 \cdot F_1 \circ T^h)(x) \, d\mu \right|
\leq \frac{C}{H} + C \left( \frac{1}{H} \sum_{h=1}^{H} \left| \int (F_1 \cdot F_1 \circ T^h)(x) \, d\mu \right|^2 \right)^{1/2}
= \frac{C}{H} + C \left( \frac{1}{H} \sum_{h=1}^{H} |\hat{\sigma}_{F_1}(h)|^2 \right)^{1/2},
\]
where \(\sigma_{F_1}\) is the spectral measure of \(F_1\) with respect to the transformation \(T\). Now we let \(H \to \infty\) to obtain
\[
\limsup_{N \to \infty} \int \frac{1}{N} \sum_{n=1}^{N} |F_1(T^n x) F_2(T^{2n} x)|^2 \, d\mu(x) \leq C \left( \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} |\hat{\sigma}_{F_1}(h)|^2 \right)^{1/2},
\]
and because \(F_1 \in \mathcal{A}^+_1\), the spectral measure \(\sigma_{F_1}\) is continuous, so Wiener’s lemma implies the limit of the right-hand side of the inequality above equals 0. \(\square\)

We now conclude the proof of Theorem 6.1. Set \(F_{1,h} = f_1 \cdot f_1 \circ T^h\) and \(F_{2,h} = f_2 \cdot f_2 \circ T^{2h}\). Denote
\[
P_N(F_{1,h}, F_{2,h}) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(F_{1,h} | A_1) \circ T^n \cdot \mathbb{E}(F_{2,h} | A_1) \circ T^{2n}.
\]
By (22), it suffices to prove that \(\limsup_{H \to \infty} (1/H) \sum_{h=1}^{H} \limsup_{N \to \infty} P_N(F_{1,h}, F_{2,h})(x) \leq [N_2(f_1)]^2\) for \(\mu\)-a.e. \(x \in X\) in order to show (21). Let \(\{e_j\}\) be an eigenbasis of \(\mathcal{A}_1\), where \(\lambda_j\) is a corresponding eigenvalue of \(e_j\). Then we would have
\[
\mathbb{E}(F_{1,h} | A_1) \circ T^n = \sum_{j=0}^{\infty} \left( \int F_{1,h} e_j \, d\mu \right) \lambda_j^n e_j,
\]
\[
\mathbb{E}(F_{2,h} | A_1) \circ T^{2n} = \sum_{l=0}^{\infty} \left( \int F_{2,h} e_l \, d\mu \right) \lambda_l^{2n} e_l
\]
in the \(L^2\)-norm. Hence,
\[
\lim_{N \to \infty} P_N(F_{1,h}, F_{2,h}) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{j=0}^{\infty} \sum_{l=1}^{\infty} \left( \int F_{1,h} e_j \, d\mu \right) \left( \int F_{2,h} e_l \, d\mu \right) \lambda_j^n \lambda_l^{2n} e_j e_l
\]
in the \(L^2\)-norm. Note that for each \(j\) and \(l\),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \lambda_j^n \lambda_l^{2n} = \begin{cases} 1 & \text{if } \lambda_j = \frac{\lambda_l}{\lambda_j}, \\ 0 & \text{otherwise}. \end{cases}
\]
Hence, if we denote \( R = \{(j, l_j) \in \mathbb{N}^2 : \lambda_j = \frac{2}{l_j}\} \), then
\[
\lim_{N \to \infty} P_N(F_{1,h}, F_{2,h}) = \sum_{(j, l_j) \in R} \left( \int F_{1,h} e_j \, d\mu \right) \left( \int F_{2,h} e_{l_j} \, d\mu \right) e_j e_{l_j}
\]
in the \( L^2 \)-norm. Note that the sequence
\[
B_J = \left( \sum_{(j, l_j) \in R, j \leq J} \left( \int F_{1,h} e_j \, d\mu \right) \left( \int F_{2,h} e_{l_j} \, d\mu \right) e_j e_{l_j} \right)_J
\]
converges to \( \lim_{N \to \infty} P_N(F_{1,h}, F_{2,h}) \) in the \( L^2 \)-norm as \( J \to \infty \). Therefore, there exists a subsequence \( (B_{J_k})_k \) that converges to \( \lim_{N \to \infty} P_N(F_{1,h}, F_{2,h})(x) \) for \( \mu \)-a.e. \( x \in X \). Thus,
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(F_{1,h} | A_1)(T^n x) \mathbb{E}(F_{2,h} | A_1)(T^{2n} x)
\]
\[
= \lim_{k \to \infty} \sum_{(j, l_j) \in R, j \leq J_k} \left( \int F_{1,h} e_j \, d\mu \right) \left( \int F_{2,h} e_{l_j} \, d\mu \right) e_j(x) e_{l_j}(x)
\]
\[
\leq \lim_{k \to \infty} \left( \sum_{(j, l_j) \in R, j \leq J_k} \left| \int F_{1,h} e_j \, d\mu \right|^2 \right)^{1/2} \left( \sum_{(j, l_j) \in R, j \leq J_k} \left| \int F_{2,h} e_{l_j} \, d\mu \right|^2 \right)^{1/2}
\]
(Cauchy–Schwarz inequality)
\[
\leq \left( \sum_{j=1}^{\infty} \left| \int F_{1,h} e_j \, d\mu \right|^2 \right)^{1/2} \left( \sum_{l=1}^{\infty} \left| \int F_{2,h} e_{l_j} \, d\mu \right|^2 \right)^{1/2}
\]
\[
= \| \mathbb{E}(F_{1,h} | A_1) \|_2 \| \mathbb{E}(F_{2,h} | A_1) \|_2
\]
\[
\leq \| \mathbb{E}(F_{1,h} | A_1) \|_2,
\]
since \( \| f_2 \|_\infty \leq 1 \). Therefore,
\[
\limsup_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (f_1 \cdot f_1 \circ T^h)(T^n x) (f_2 \cdot f_2 \circ T^{2h})(T^{2n} x)
\]
\[
\leq \limsup_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \| \mathbb{E}(f_1 \cdot f_1 \circ T^h | A_1) \|_2
\]
\[
\leq \left( \limsup_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \| \mathbb{E}(f_1 \cdot f_1 \circ T^h | A_1) \|_2^2 \right)^{1/2} = [N_2(f_1)]^2,
\]
where the second inequality holds by the Cauchy–Schwarz inequality. \( \square \)

7. Case where both \( f_1, f_2 \in \mathcal{Z}_2 \)

Here we prove the convergence of double recurrence Wiener–Wintner averages for the case where \( f_1, f_2 \in \mathcal{Z}_2 \). To do so, we will use the structural properties of nilsystems, which we shall discuss briefly.

Let \( (X, \mathcal{F}, \mu, T) \) be an ergodic system. Recall that \( X \) is called a \( k \)-step nilsystem if \( X \) is a homogeneous space of a \( k \)-step nilpotent Lie group \( G \) (such a manifold is called a nilmanifold). Let \( \Lambda \) be a discrete cocompact subgroup of \( G \) such that \( X = G/\Lambda \). The outline of the proof of the following theorem is given in \([16]\).
Theorem 7.1. (Host and Kra [16]) If $X$ is a Conze–Lesigne system, then it is the inverse limit of a sequence of 2-step nilsystems.

In the outline of the proof, $X$ is reduced to the case where $X$ is a group extension of the Kronecker factor $Z_1$ and torus $U$, with cocycle $\rho : Z_1 \to U$. A group $G$ is defined to be a family of transformations of $X = Z_1 \times U$, where $U$ is a finite-dimensional torus and $Z_1$ is the Kronecker factor of $X$ that has the structure of compact abelian Lie group. If $g \in G$, $(z, u) \in X$, then

$$g \cdot (z, u) = (sz, uf(z))$$

where $s \in Z_1$ and $f : Z_1 \to U$ satisfy the Conze–Lesigne equation

$$\rho(sz)\rho(z)^{-1} = f(Rz)f(z)^{-1}c$$

for some constant $c \in U$. It can be easily verified that $G$ is a 2-step nilpotent group, and $T$ corresponds to $(\beta, \rho) \in G$, where $\beta \in Z_1$ such that if $\pi_1 : Z_2 \to Z_1$ is a factor map, then $\pi_1(Tx) = \beta \pi_1(x)$. Furthermore, if $G$ is given a topology of convergence in probability, then we know that $G$ is a Lie group.

The outline of the proof given in [16] concludes by stating that $G$ acts on $X$ transitively, and $X$ can be identified with the nilmanifold $G/3$, where $3$ is a stabilizer group of a point $x_0 \in X$ (hence it is a discrete cocompact subgroup of $G$). Furthermore, $\mu$ is a Haar measure on $X$, and $T$ is a translation by the element $(\beta, \rho) \in G$. Hence, $T$ acts on $X$ by translation. We will use this fact to prove the convergence of the double recurrence Wiener–Wintner average for the case where $f_1, f_2 \in \mathcal{Z}_2$.

The following convergence result of Leibman will be used. We say that $\{g(n)\}_{n \in \mathbb{Z}}$ is a polynomial sequence if $g(n) = a_{p_1(n)} \cdots a_{p_m(n)}$, where $a_1, \ldots, a_m \in G$, and $p_1, \ldots, p_m$ are polynomials taking on integer values on the integers.

Theorem 7.2. (Leibman [19]) Let $X = G/\Lambda$ be a nilmanifold and $\{g(n)\}_{n \in \mathbb{Z}}$ be a polynomial sequence in $G$. Then, for any $x \in X$ and continuous function $F$ on $X$, the average

$$\frac{1}{N} \sum_{n=1}^{N} F(g(n)x)$$

converges as $N \to \infty$.

Theorem 7.3. Let $(X, \mathcal{F}, \mu, T)$ be an ergodic dynamical system. Suppose that $f_1, f_2 \in \mathcal{Z}_2$ are both continuous functions on $X$. Then the average

$$\frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x)f_2(T^{bn}x)e^{2\pi int}$$

converges, as $N \to \infty$, off a single null set that is independent of $t$.

Proof. In this proof, we will consider two cases: the case where $t$ is rational, and the case where $t$ is irrational.

Case I: When $t$ is rational. Fix $t \in \mathbb{Q}$. Let $S_t$ be a rotation on $\mathbb{T}$ by $e^{2\pi it}$. Let $(X \times \mathbb{T}, \mu \otimes m, U)$ be a measure-preserving system, where $m$ is the Lebesgue measure on $\mathbb{T}$,
and $U = T \otimes S_t$. Define $F_1(x, y) = f_1(x)e^{2\pi i x_1 y}$ and $F_2(x, y) = f_2(x)e^{2\pi i x_2 y}$, where $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 a + \alpha_2 b = 1$. Then

$$\frac{1}{N} \sum_{n=1}^{N} F_1(U^{an}(x, y))F_2(U^{bn}(x, y)) = \frac{e^{2\pi iy}}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x)e^{2\pi int}.$$  \hfill (23)

Note that the average on the left-hand side of (23) converges $\mu \otimes m$-almost everywhere as $N \to \infty$ by Bourgain’s double recurrence theorem [9]. So there exists a set of full measures $V_t \subset X \times \mathbb{T}$ such that the average in (23) converges for all $(x, y) \in V_t$. If $V = \bigcup_{t \in \mathbb{Q}} V_t$, then $V$ is a set of full measures such that the average on (23) converges for all $(x, y) \in V$ for all $t \in \mathbb{Q}$. This implies that the claim holds for $\mu$-a.e. $x \in X$ when $t \in \mathbb{Q}$.

**Case II: When $t$ is irrational.** Without loss of generality, we let $X = \mathbb{Z}_2$, the Conze–Lesigne system. Let $\beta \in Z_1$ be an element such that $T(z, u) = (\beta z, u \rho(z))$ for any $(z, u) \in Z_1 \times U = X$. In other words, $T$ acts on $Z_1$ as a rotation by $\beta$ (here, we let $Z_1$ be a multiplicative abelian group). Then note that $B = \langle \beta \rangle$, the cyclic subgroup generated by $\beta$, is dense in the Kronecker factor $Z_1$. Define a character $\phi_t : B \to \mathbb{T}$ such that $\phi_t(\beta) = e^{2\pi it}$. Such group homomorphism exists since $t$ is irrational, and $e^{2\pi it}$ generates a dense cyclic subgroup in $\mathbb{T}$.

We claim that there exists a multiplicative character $\overline{\phi}_t : Z_1 \to \mathbb{T}$ such that $\overline{\phi}_t|_B = \phi_t$. Since $B$ is dense in $Z_1$, for any $z \in Z_1$, there exists a sequence $(\beta^n_i)_k$ such that $\lim_{k \to \infty} \beta^{n_k} = z$. So we define

$$\overline{\phi}_t(z) = \lim_{k \to \infty} \phi_t(\beta^{n_k}).$$

We must show that this limit converges, which would show that $\overline{\phi}_t$ is well defined by the continuity of $\phi$. Note that $\mathbb{T}$ is compact, so there exists a converging subsequence $(\phi_t(\beta^{n_k})) \in \mathbb{T}$ such that $\lim_{k \to \infty} \phi_t(\beta^{n_k}) = \gamma$ for some $\gamma \in \mathbb{T}$. We will show that $\lim_{k \to \infty} \phi_t(\beta^{n_k}) = \gamma$. Assuming the contrary, suppose that there exists a subsequence $(\phi_t(\beta^{n_{km}}))_m$ such that $|\phi_t(\beta^{n_{km}}) - \gamma| > \epsilon$ for all $m \in \mathbb{N}$. This implies that, for sufficiently large $l$, we have $|\phi_t(\beta^{n_{kl}}) - \phi_t(\beta^{n_{il}})| > \epsilon/2$. This, however, contradicts the continuity of $\phi_t$, since if $d_{Z_1}$ is the metric on $Z_1$, then $d_{Z_1}(\beta^{n_{kl}}, \beta^{n_{il}}) \to 0$ as $l, m \to \infty$, because both $\beta^{n_{kl}}$ and $\beta^{n_{km}}$ converge to the same limit $z$. This proves that $\overline{\phi}_t$ is well defined for all $z \in Z_1$. The fact that $\overline{\phi}_t$ is a multiplicative character is obvious from the way $\overline{\phi}_t$ is defined in terms of $\phi_t$.

We define a continuous function $f_t := \overline{\phi}_t \circ \pi_1$, where $\pi_1 : Z_2 \to Z_1$ is the factor map. We note that

$$f_t(T^n x) = \overline{\phi}_t(\pi_1(T^n x)) = \overline{\phi}_t(\pi_1(x) \beta^n) = f_t(x) \phi_t(\beta^n) = f_t(x) e^{2\pi int}.$$ 

Therefore,

$$\frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x) f_3(T^n x) = \frac{f_t(x)}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi int}.$$

To show the convergence of this average, let $F(x_1, x_2, x_3) = f_1(x_1) f_2(x_2) f_3(x_3)$ be a function on $X^3 = G^3 / H^3$. Let $T_1 = T \times \text{Id} \times \text{Id}$, $T_2 = \text{Id} \times T \times \text{Id}$, and $T_3 = \text{Id} \times \text{Id} \times T$. Note that an action of $T_1$ on $X^3$ corresponds to $g_1 = ((\beta, \rho), e, e) \in G^3$ (where $e$ is the
identity element of $G$, and similarly, $T_2$ corresponds to $g_2 = (e, (\beta, \rho), e) \in G^3$, and $T_3$ corresponds to $g_3 = (e, e, (\beta, \rho)) \in G^3$. Thus,

$$g(n) = g_1^n g_2^n g_3^n$$

is a polynomial sequence. Furthermore, if $\bar{x} = (x, x, x) \in X^3$, then

$$1/N \sum_{n=1}^{N} F(g(n) \bar{x}) = 1/N \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x) f_3(T^{cn}x)$$

converges by Theorem 7.2.

□

A. Appendix

Here we provide the proofs of the inequalities mentioned in §3.

Proof of Lemma 3.1. One can find the proof of van der Corput’s lemma in many different sources; see [18], for example.

□

Proof of Lemma 3.2. To show (2), we take the limit supremum (as $N \to \infty$) on both sides of (1). Then we obtain

$$\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} a_n \right|^2 \leq \frac{1}{H} + \frac{2}{(H+1)^2} \sum_{h=1}^{H} (H+1-h) \text{Re} \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-h-1} a_n \bar{a}_{n+h} \right).$$

Let $u_n$ be another sequence of complex numbers norms bounded by 1. Then, for fixed $h$, we have

$$\frac{1}{N} \sum_{n=0}^{N-h-1} u_n = \frac{1}{N} \sum_{n=0}^{N} u_n - \frac{1}{N} \sum_{n=N-h}^{N} u_n.$$

Since $|u_n| \leq 1$, we know that for fixed $h$,

$$\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=N-h}^{N} u_n \right| \leq \limsup_{N \to \infty} \frac{h}{N} = 0.$$

Therefore,

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-h-1} u_n = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} u_n.$$

(24)

Now applying (24) to $u_n = a_n \bar{a}_{n+h}$, we obtain

$$\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} a_n \right|^2 \leq \frac{1}{H} + \frac{2}{(H+1)^2} \sum_{h=1}^{H} (H+1-h) \text{Re} \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} a_n \bar{a}_{n+h} \right),$$

so set $C > 2$, and the claim holds.

To show (3), we recall that [2, Corollary 2.1] states that

$$\sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} a_n e^{2\pi int} \right|^2 \leq \frac{2}{NH} \sum_{n=1}^{N} |a_n|^2 + \frac{4}{H} \sum_{h=1}^{H} \frac{1}{N} \sum_{n=1}^{N} a_n \bar{a}_{n+h}.$$
Since \( \sup_n |a_n|^2 \leq 1 \), we have
\[
\frac{2}{NH} \sum_{n=1}^{N} |a_n|^2 \leq \frac{2}{H}.
\]
Choose \( C > 4 \), and we obtain the desired inequality.

To show (4), we apply the limit supremum (as \( N \to \infty \)) to both sides of (3), which gives us
\[
\limsup_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} a_n e^{2 \pi i nt} \right|^2 \leq \frac{C}{H} \left[ \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} a_n \overline{a}_{n+h} \right| \right].
\]
Applying (24) to \( u_n = a_n \overline{a}_{n+h} \), we obtain the desired inequality. \( \square \)

**Proof of Lemma 3.3.** Note that
\[
\limsup_{n \to \infty} f_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n.
\]
So if we set \( g_k = \sup_{n \geq k} f_n \), and since \( g_k \) is decreasing, its limit exists pointwise (i.e. \( \limsup_n f_n \)), and \( g_1 = \sup_{n \geq 1} f_n \leq F \), we apply the dominated convergence theorem to obtain
\[
\int \limsup_{n \to \infty} f_n \, d\mu = \lim_{k \to \infty} \int g_k \, d\mu = \lim_{k \to \infty} \int \sup_{n \geq k} f_n \, d\mu.
\]
Of course, \( f_i \leq \sup_{n \geq k} f_n \) for all \( i \geq k \), and we know that \( \int f_i \, d\mu \leq \int \sup_{n \geq k} f_n \, d\mu \). So in particular, \( \sup_{n \geq k} \int f_n \leq \int \sup_{n \geq k} f_n \). Hence,
\[
\int \limsup_{n \to \infty} f_n \, d\mu \geq \limsup_{k \to \infty} \int f_n = \limsup_{n \to \infty} \int f_n \, d\mu. \quad \square
\]

**Proof of Lemma 3.4.** This proof is a small modification of the proof provided in [3, Lemma 5]. By the Cauchy–Schwarz inequality,
\[
\left| \frac{1}{H(H+1)^2} \sum_{h,k=0}^{H-1} (H+1-k)a_h b_k c_{h+k} \right|^2 \leq \|a\|_\infty^2 \left( \frac{1}{H} \sum_{h=0}^{H-1} \left| \frac{1}{H+1} \sum_{k=0}^{H-1} (H+1-k)b_k c_{h+k} \right|^2 \right).
\]
Set \( B_k = b_k (H+1-k)/(H+1) \), and the inequality above becomes
\[
\left| \frac{1}{H(H+1)^2} \sum_{h,k=0}^{H-1} (H+1-k)a_h b_k c_{h+k} \right|^2 \leq \|a\|_\infty^2 \left( \frac{1}{H} \sum_{h=0}^{H-1} \left| \frac{1}{H+1} \sum_{k=0}^{H-1} B_k c_{h+k} \right|^2 \right) \leq \|a\|_\infty^2 \left( \frac{1}{H} \sum_{h=0}^{H-1} \left( \sum_{k=0}^{H-1} B_k e^{-2 \pi i k t} \right) \left( \frac{1}{H+1} \sum_{k=0}^{2(H-1)} c_k e^{2 \pi i k t} \right) e^{-2 \pi i h t} \, dt \right)^2.
\]
We apply Parseval’s inequality to the integral above to obtain
\[
\left| \frac{1}{H(H+1)^2} \sum_{h,k=0}^{H-1} (H + 1 - k)a_k b_k c_{H+k} \right|^2 \\
\leq \|a\|_\infty^2 \frac{1}{H} \sum_{h=0}^{H-1} \left| \sum_{k=0}^{H-1} B_k e^{-2\pi i h t} \right|^2 \frac{1}{H+1} \sum_{k'=0}^{2(H-1)} c_{k'} e^{2\pi i k't} dt \\
\leq \|a\|_\infty^2 \sup_{t \in \mathbb{R}} \left| \frac{1}{H+1} \sum_{k=0}^{2(H-1)} c_k e^{2\pi i k t} \right|^2 \frac{1}{H} \sum_{h=0}^{H-1} \left| \sum_{k=0}^{H-1} B_k e^{-2\pi i h t} \right|^2 dt \\
\leq \|a\|_\infty^2 \sup_{t \in \mathbb{R}} \left| \frac{1}{H+1} \sum_{k=0}^{2(H-1)} c_k e^{2\pi i k t} \right|^2 \frac{1}{H+1} \sum_{h=0}^{H-1} |B_k|^2.
\]
Since $|B_k| < 1$, we know that $(1/(H + 1)) \sum_{h=0}^{H-1} |B_k|^2 \leq 1$. Thus, (6) holds. \hfill \Box

Remarks.

1. In [5], the first and the third authors showed that Theorem 2.3 can be extended to averages of the form
\[
\frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i p(n)t},
\]
i.e. given a degree-$k$ polynomial with real coefficients $p$, we can find a set of full measure in $X$ for which these averages converge. The first and the third authors also showed that
\[
\frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i p(n)}
\]
converges off a single null set independent of $p$ [6]. These two preprints [5, 6] were combined and submitted.

2. The first and the third authors are currently preparing the extension of Theorem 2.3 to show that the sequence $u_n = f_1(T^{an}x) f_2(T^{bn}x)$ is a good universal weight for the pointwise ergodic theorem [4].

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