\textbf{q-CRYSTALS AND q-CONNECTIONS}

ANDRE CHATZISTAMATIOU

\textbf{Abstract.} We study how the category of $q$-connections depends on the choice of coordinates. We exploit Bhatt’s and Scholze’s $q$-crystalline site, which is based on a coordinate free formulation of $q$-PD structures, in order to relate $q$-crystals and $q$-connections in the $p$-adic setting. This yields a natural equivalence between the categories of $q$-connections for different choices of coordinates in the $p$-adic setting. The equivalence can be described explicitly in terms of differential operators.

In order to obtain a global equivalence, we patch the $p$-adic differential operators to create a global one. The process is not entirely formal, and we are only able to obtain a global equivalence after inverting 2.

\textbf{Introduction}

The subject of this paper are Scholze’s conjectures about $q$-connections [Sch]. In their simplest form $q$-connections are modules, satisfying a completeness condition, over a certain subalgebra of the ring of differential operators.

For example, let $R_q = \mathbb{Z}[x][q^{-1}]$, and let $R_q\{\nabla_{x,q}\}$ be the $(q-1)$-completion of the non commutative $R_q$-algebra generated by $\nabla_{x,q}$ and satisfying $\nabla_{x,q} \cdot x = q \cdot x \cdot \nabla_{x,q} = 1$. This is a $q$-deformation of the Weyl algebra $\mathbb{Z}[x]\{\nabla_x\}$. It is an arithmetic phenomenon that this deformation depends on the coordinate $x$. Replacing $\mathbb{Z}$ by $\mathbb{Q}$ would yield the trivial deformation of the Weyl algebra.

The category of $(q-1)$-(derived) complete $R_q\{\nabla_{x,q}\}$-modules is the category of $q$-connections over $\mathbb{Z}[x]$ with respect to the coordinate $x$. According to the conjectures of [Sch], this category does not depend on the choice of the coordinate $x$. For example, there should be a natural equivalence between the category of $(q-1)$-(derived) complete $R_q\{\nabla_{x,q}\}$-modules and $(q-1)$-(derived) complete $R_q\{\nabla_{x-1,q}\}$-modules, where $\nabla_{x-1,q} \cdot (x-1) = q \cdot (x-1) \cdot \nabla_{x-1,q} = 1$.

Our paper exploits the notion of $q$-crystals introduced in [BS]. Our first equivalence is between $q$-crystals and $q$-HPD stratifications (Theorem $1.3.3$). Both notions specialize to the respective classical notions in crystalline theory for $q = 1$, and the proof of the equivalence follows from the crystalline formalism. The main observation, which allows us to run the crystalline formalism, is the existence of local retractions in the $q$-crystalline site (Proposition $1.1.2$).

By definition, $q$-HPD stratifications depend on the choice of a “lifting”. Independence of this choice is established via the identification with $q$-crystals.

The second equivalence is between $(p,q-1)$-completely flat $q$-HPD stratifications and $(p,q-1)$-completely flat quasi-nilpotent $q$-connections (Proposition $2.1.4$). Indirectly, it shows the independence of the choice of coordinates for $(p,q-1)$-completely flat quasi-nilpotent $q$-connections. By using differential operators, we can make this independence explicit and extend it to all $q$-connections (Proposition $2.3.2$). All this takes place in the $p$-adic setting and works for all primes.
In order to briefly explain how differential operators are used to transform from coordinates $x$ to coordinates $y$, let us suppose that $q$-connections for $x$ (and $y$) are $A_{\mathcal{p},x}$-modules (and $A_{\mathcal{p},y}$-modules, respectively). Moreover, $A_{\mathcal{p},x}$ and $A_{\mathcal{p},y}$ are subalgebras of $D_{\mathcal{p}}$, (a completion of) the ring of differential operators. We can show that there is a distinguished free left $A_{\mathcal{p},x}$-module and free right $A_{\mathcal{p},y}$-module $M_{\mathcal{p},x}$ contained in $D_{\mathcal{p}}$. A choice of a common generator $D \in M_{\mathcal{p},x}$ yields an isomorphism $\psi_D : A_{\mathcal{p},x} \to A_{\mathcal{p},y}$ defined by $a \cdot D = D \cdot \psi_D(a)$. This isomorphism is well-defined up to inner automorphisms, and induces an equivalence between $A_{\mathcal{p},x}$-modules and $A_{\mathcal{p},y}$-modules up to natural transformation.

To obtain a global result, we use a natural condition to identify the differential operators $M_{\mathcal{p},x}^{\text{global}} \subset M_{\mathcal{p},x}$ that are the image of global ones. This approach does not work for $p = 2$, because $M_{\mathcal{p},x}^{\text{global}} = \emptyset$ in general.

Finally, after inverting $2$, we patch $\{M_{\mathcal{p},x}^{\text{global}} \}_{p > 2}$ together to obtain a free left $A_{\mathcal{p},x}$-module and free right $A_{\mathcal{p},y}$-module $M_{\mathcal{p},x}$ contained in the ring of global differential operators $D$.

1. $q$-CRYSTALS

This section heavily relies on [BS] §16]. Bhatt and Scholze introduce the notion of a $q$-divided power algebra [BS Definition 16.2], a $q$-divided power envelope [BS Lemma 16.10] and the $q$-crystalline site [BS Definition 16.12]. These are major innovations which allows them to glue a $q$-de Rham complex, prove some of the conjectures in [Sch], and to relate their previous work on $\mathfrak{sl}$-complexes to prismatic cohomology.

1.1. The $q$-crystalline site. We fix a $q$-PD pair $(D, I)$ and a $p$-completely smooth and $p$-complete $D/I$-algebra $R$.

**Definition 1.1.1** (The $q$-crystalline site). (See [BS Definition 16.12]) The $q$-crystalline site of $R$ relative to $D$, denoted $(R/D)_{q-\text{crys}}$, is the category of $q$-PD thickenings of $R$ relative to $D$, i.e. the category of $q$-PD pairs $(E, J)$ over $(D, I)$ equipped with an isomorphism $R \to E/J$ of $D/I$-algebras.

An important property of the classical crystalline site and the infinitesimal site for a smooth scheme is the existence of local retractions (see [Gro68 §4.2]). This property is used to relate crystals to more explicit objects like stratifications and connections.

**Proposition 1.1.2** (Existence of retractions). Suppose there is $(\hat{R}, I\hat{R})$ in $(R/D)_{q-\text{crys}}$ such that a $(p, [p]_q)$-étale map $\psi : D[x_1, \ldots, x_n]^\wedge \to \hat{R}$ exists.

For every $(S, J)$ in $(R/D)_{q-\text{crys}}$, the coproduct of $(S, J)$ and $(\hat{R}, I\hat{R})$ in $(R/D)_{q-\text{crys}}$ exists. We denote it by $(\hat{S}, \hat{J}) = (S, J) \otimes_{q-\text{crys}} (\hat{R}, I\hat{R})$. The natural morphism

$$(S, J) \xrightarrow{I} (\hat{S}, \hat{J})$$

is $(p, [p]_q)$-faithfully flat. Moreover, there is a section $\hat{S} \to S$ in the category of $S$-modules.

**Proof.** We note that $A := D[x_1, \ldots, x_n]^\wedge$, $\hat{R}$, and $S$ are classically $(p, [p]_q)$-complete [BS Lemma 3.7(1)].
We can find a morphism of $D$-algebras $\tau': A \to S$ lifting the map $A/I \xrightarrow{\gamma} \tilde{R}/I = R \to S/J$. As a first step, we will extend $\tau'$ to $\tilde{R}$.

We can find a presentation $\tilde{R} = A[y_1, \ldots, y_d]/(f_1, \ldots, f_d)$ such that det$(\partial f_j/\partial y_i)$ is invertible. For $B = A[y_1, \ldots, y_d][z_1, \ldots, z_d]/(f_1 - z_1, \ldots, f_d - z_d)$, there is a ring homomorphism $B \to S$ extending $\tau': A \to S$, mapping $z_i$ to $J$, and reducing to $R \to S/J$. Indeed, for every $a \in J$, we have $a^p \in (p, [p]_q)$. After mapping all $y_i$ to some lifts, we map $z_i$ to the image of $f_i$.

Since $A[y_1, \ldots, y_d]/(f_1, \ldots, f_d)$ is étale over $A$, we can find a morphism of $A$-algebras $A[y_1, \ldots, y_d]/(f_1, \ldots, f_d) \to B$ such that the composition with $B \to B/(z_1, \ldots, z_d)$ is the identity. The composition with $B \to S$ yields the desired lift $\tau': \tilde{R} \to S$ after completion.

We note that $\tau'$ is in general not compatible with the $\delta$-structures. However, there is a $\delta$-structure on $S[[\epsilon_1, \ldots, \epsilon_n]]$ such that $\tau: \tilde{R} \to S[[\epsilon_1, \ldots, \epsilon_n]]$ defined by $x_i \mapsto \tau'(x_i) + \epsilon_i$ is a morphism of $\delta$-rings. In order to define this $\delta$-structure we need to solve $\delta(\tau(x_i)) = (\tau(\delta(x_i)))$. Since

$$\delta(x_i) = x_i \mapsto \sum_I \tau'((\delta(x_i)(I)) \cdot \epsilon_1^{i_1} \cdot \cdots \epsilon_n^{i_n},$$

where the second equation is the Taylor expansion and $\delta(x_i)(I) = (\prod_{j=1}^n \frac{\partial^{i_j}}{\partial x_j^j}) \delta(x_i)$, we have to set

$$\delta(\epsilon_i) = \sum_I \tau'((\delta(x_i)(I)) \cdot \epsilon_1^{i_1} \cdot \cdots \epsilon_n^{i_n} - \delta(x_i)) - \sum_{i=1}^{\text{deg} f} \frac{1}{p} \tau'(x_i)^{p^{i-1}} \cdot \epsilon_1^{i_1} \cdot \cdots \epsilon_n^{i_n}.$$

This extends to a $\delta$-structure on $S[[\epsilon_1, \ldots, \epsilon_n]]$ with the desired properties.

Finally, applying the $q$-PD envelope [HS Lemma 16.10] construction over $(S, J)$ to the $(p, [p]_q)$-completely flat $\delta$-S-algebra $S[[\epsilon_1, \ldots, \epsilon_n]]$ and the regular sequence $\epsilon_1, \ldots, \epsilon_n$, we obtain $(\tilde{S}, \tilde{J})$. We denote by $f : (S, J) \to (\tilde{S}, \tilde{J})$ and $\tau : (R, IR) \to (\tilde{S}, \tilde{J})$ the induced morphisms in $(R/D)_{q\text{-crys}}$.

In order to show that $(\tilde{S}, \tilde{J})$ is the coproduct of $(S, J)$ and $(\tilde{R}, IR)$, we can use the universal property of a $q$-PD envelope. For morphisms $g_S : S \to T$ and $g_{\tilde{R}} : \tilde{R} \to T$ in $(R/D)_{q\text{-crys}}$, it suffices to show that there is a unique morphism of $\delta$-rings $g : S[[\epsilon_1, \ldots, \epsilon_n]] \to T$ such that $g \circ \tau = g_{\tilde{R}}$, $g \circ f = g_S$, and which maps the ideal $(\epsilon_1, \ldots, \epsilon_n)$ into the $q$-PD ideal of $T$. Only the morphism of $S$-algebras induced by $\epsilon_i \mapsto g_{\tilde{R}}(x_i) - g_S(\tau'(x_i))$ satisfies these properties.

We still need to show the existence of a section $\tilde{S} \to S$ in the category of $S$-modules. Let $\gamma_p(a) = \frac{\delta(a)}{\partial a} - \delta(a)$, we have $\gamma_p(\tilde{J}) \subset \tilde{J}$. We set $\gamma_1 = \text{id}_S, \gamma_p^k = \gamma_p \circ \gamma_p^{k-1}$, for all $k \geq 1$, and for any positive integer $i$ with $p$-adic expansion $i = \sum_{k=0}^{\infty} i_k p^k$, we set $\gamma_i(a) = \prod_{k=0}^{\infty} \gamma_p^k(a)^{i_k}$. This defines maps $\gamma_i : \tilde{J} \to \tilde{J}$.

For each $I = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n$, we set

$$(1.1.1) \quad \Gamma_I := \prod_{j=1}^n \gamma_{i_j}(\epsilon_j),$$
and claim that

\[(1.1.2) \quad \left( \bigoplus_{I \in \mathbb{Z}_p^*} S \cdot \Gamma_I \right)^\wedge \to \tilde{S}, \]

is an isomorphism of \( S \)-modules, where the left hand side is the derived \((p, [p]_q)\)-completion. Indeed, \((p, [p]_q)\)-completion is the same as \((p, q-1)\)-completion, and by the derived Nakayama lemma it suffices to show the isomorphism after derived base change \( \otimes_S S/(p,q-1) \). Both modules are \((p, [p]_q)\)-completely flat, hence derived base change reduces to simple base change \( \otimes_S S/(p,q-1) \). After reduction modulo \( q-1 \) the map already becomes an isomorphism. Indeed, \( \tilde{S}/q-1 \) is the \( p \)-completion of the \( \mathrm{pd} \)-envelope of \( S/(q-1)[\epsilon_1, \ldots, \epsilon_n] \) along \( (\epsilon_1, \ldots, \epsilon_n) \), and

\[\Gamma_I \equiv \prod_j \epsilon_j^i \mod q-1,\]

where \( [n]_p \) denotes the largest \( p \)-power dividing \( n \).

By using (1.1.2) and projecting to the summand corresponding to \( I = 0 \), we obtain the desired section. \( \square \)

1.2. \( q \)-crystals. For a \( D \)-algebra \( E \), we have the abelian category of derived \((p, [p]_q)\)-complete \( E \)-modules \( \text{Mod}^\wedge_E \) at our disposal (see Appendix 3.1.1). For a morphism of \( D \)-algebras \( f : E \to E' \), there is a right exact base change functor \( f^* : \text{Mod}^\wedge_E \to \text{Mod}^\wedge_{E'} \), \( M \mapsto E'^{\otimes} \otimes_{E} M \).

**Definition 1.2.1** (\( q \)-crystals). A \( q \)-crystal on \((R/D)_{q\text{-crys}}\) is a derived \((p, [p]_q)\)-complete \( E \)-module \( M_E \) for each \((E, J) \in (R/D)_{q\text{-crys}}\) together with isomorphisms

\[M_f : f^* M_E \to M_{E'},\]

for each \( f : (E, J) \to (E', J') \) in \((R/D)_{q\text{-crys}}\) that satisfy

\[M_{g \circ f} = M_g \circ g^*(M_f)\]

for all \( f : (E, J) \to (E', J') \) and \( g : (E', J') \to (E'', J'') \).

A morphism of \( q \)-crystals \( u \) consists of a morphism of derived \((p, [p]_q)\)-complete \( E \)-modules

\[u_E : M_E \to M'_E\]

for every \((E, J) \in (R/D)_{q\text{-crys}}\) such that

\[u_{E'} \circ M_f = M'_f \circ f^*(u_E)\]

for all \( f : (E, J) \to (E', J') \).

We denote the category of \( q \)-crystals by \( q\text{-Cris}(R/D) \).

A \( q \)-crystal \( M \) is called \((p, [p]_q)\)-completely flat if \( M_E \) is \((p, [p]_q)\)-completely flat for every \((E, J) \in (R/D)_{q\text{-crys}}\) (see Appendix 3.1.4 for the definition of \((p, [p]_q)\)-completely flat modules).
1.3. Let \( \hat{R} \) be as in Proposition 1.1.2. As a first step towards understanding \( q \)-\textit{Cris}(\( R/D \)), we will outline which objects \( q \)-crystals induce on \( \hat{R} \). This will lead to the definition of \( q \)-HPD stratifications. The main result in this section will be an equivalence of categories between \( q \)-crystals and \( q \)-HPD stratifications if coordinates exist. This is analogous to the equivalence for the crystalline theory [BO78, §6].

Consider the diagram

\[
\begin{array}{c}
\hat{R} & \xrightarrow{\delta_0^1} & \hat{R}[\epsilon_1, \ldots, \epsilon_n] & \xrightarrow{\delta_2^1} & \hat{R}[\epsilon_1, \ldots, \epsilon, \tau_1, \ldots, \tau_n],
\end{array}
\]

where \( \delta_0^1(x_i) = x_i + \epsilon_i \), \( \delta_1^1(x_i) = x_i \), and

\[
\begin{align*}
\delta_0^2(x_i) &= x_i + \epsilon_i \\
\delta_1^2(x_i) &= x_i + \tau_i \\
\delta_2^2(x_i + \epsilon_i) &= x_i + \epsilon_i.
\end{align*}
\]

We equip \( \hat{R}[\epsilon_1, \ldots, \epsilon_n] \) and \( \hat{R}[\epsilon_1, \ldots, \epsilon, \tau_1, \ldots, \tau_n] \) with the \( \delta \)-structure that makes all maps in the diagram to maps of \( \delta \)-rings (see the proof of Proposition 1.1.2 for how to make this work).

Next, we apply the \( q \)-PD envelope [BS, Lemma 16.10] construction over \( (\hat{R}, I \hat{R}) \) to the \( (p, [p]_q) \)-completely flat \( \delta \)-\( R \)-algebras \( \hat{R}[\epsilon_1, \ldots, \epsilon_n] \) and \( \hat{R}[\epsilon_1, \ldots, \epsilon, \tau_1, \ldots, \tau_n] \) for the regular sequences \( \epsilon_1, \ldots, \epsilon_n \) and \( \epsilon_1, \ldots, \epsilon_n, \tau_1, \ldots, \tau_n \), respectively. We denote the resulting objects in \( (R/D)_q \)-crystals by \( \hat{R}^{(2)} \) and \( \hat{R}^{(3)} \). By using the universal property of a \( q \)-PD envelope we obtain an induced diagram

\[
\begin{array}{c}
\hat{R} & \xrightarrow{\delta_0^1} & \hat{R}^{(2)} & \xrightarrow{\delta_1^1} & \hat{R}^{(3)},
\end{array}
\]

We have \( \hat{R}^{(2)} = \hat{R} \otimes_{q-\text{crys}} \hat{R} \) and \( \hat{R}^{(3)} = \hat{R} \otimes_{q-\text{crys}} \hat{R} \otimes_{q-\text{crys}} \hat{R} \).

**Definition 1.3.1.** A \( q \)-HPD stratifications on \( (\hat{R}, I \hat{R}) \) is a \( (p, [p]_q) \)-derived complete \( R \)-module \( M \) together with an isomorphism of \( \hat{R}^{(2)} \)-modules

\[
e : M \otimes_{\hat{R}, \delta_0^1} \hat{R}^{(2)} \to M \otimes_{\hat{R}, \delta_1^1} \hat{R}^{(2)}
\]

such that the following cocycle condition is satisfied:

\[
e \otimes_{\hat{R}^{(2)}, \delta_2^1} \hat{R}^{(3)} \circ e \otimes_{\hat{R}^{(2)}, \delta_0^1} \hat{R}^{(3)} = e \otimes_{\hat{R}^{(2)}, \delta_1^1} \hat{R}^{(3)}.
\]

We form the category of \( q \)-HPD stratifications in the usual way.

A \( q \)-HPD stratification \( (M, e) \) is called \( (p, [p]_q) \)-completely flat if \( M \) is a \( (p, [p]_q) \)-completely flat \( R \)-module (see Appendix 3.1.3 for the definition of \( (p, [p]_q) \)-completely flat modules).

**Remark 1.3.2.** It follows from the cocycle condition and the existence of an inverse for \( e \) that \( \text{id}_M = e \otimes_{\hat{R}^{(2)}, m} \hat{R} \), where \( m : \hat{R} \otimes_{q-\text{crys}} \hat{R} \to \hat{R} \) is induced by the identity on each factor.

We have a functor

\[
q-\text{Cris}(R/D) \to (q \text{-HPD stratifications on } (\hat{R}, I \hat{R})),
\]
defined by

\[ M = (E \mapsto ME, f \mapsto Mf) \mapsto (M_{\tilde{R}}, M_{\tilde{R}}^{-1} \circ M_{\tilde{R}}). \]

**Theorem 1.3.3.** Let \((\tilde{R}, I \tilde{R})\) be as in Proposition 1.1.2. The functor \(1.3.3\) from \(q\)-crystals to \(q\)-HPD stratifications on \(\tilde{R}\) is an equivalence of categories.

**Proof.** For \((S, J)\) in \((R/D)_{q-crys}\), Proposition 1.1.2 yields morphisms

\[ (S, J) \xrightarrow{\tilde{f}} (\tilde{S}, \tilde{J}) \xleftarrow{\tilde{\delta}} (\tilde{R}, I \tilde{R}) \]

in \((R/D)_{q-crys}\). We define the diagram

\[ (1.3.5) \]

\[
\begin{array}{c}
\tilde{S} \\
\downarrow \delta_1^1 \quad \downarrow \delta_2^1 \\
\tilde{S} \otimes_S \tilde{S} \\
\downarrow \delta_1^2 \quad \downarrow \delta_2^2 \\
\tilde{S} \otimes_S \tilde{S} \otimes_S \tilde{S}
\end{array}
\]

in the usual way, that is, \(\delta_1^1(s) = 1 \otimes s\), \(\delta_2^1(s) = s \otimes 1\), and

\[ \delta_3^1(s_1 \otimes s_2) = 1 \otimes s_1 \otimes s_2, \quad \delta_4^1(s_1 \otimes s_2) = s_1 \otimes 1 \otimes s_2, \quad \delta_5^1(s_1 \otimes s_2) = s_1 \otimes s_2 \otimes 1. \]

Note that \(\tilde{S} \otimes_S \tilde{S}\) and \(\tilde{S} \otimes_S \tilde{S} \otimes_S \tilde{S}\) are objects in \((R/D)_{q-crys}\). They are \(q\)-PD thickenings of \(S/J\).

Next, we extend \(\tau : \tilde{R} \rightarrow \tilde{S}\) to morphisms \(\tau^{(2)} : \tilde{R}^{(2)} \rightarrow \tilde{S} \otimes_S \tilde{S}\) and \(\tau^{(3)} : \tilde{R}^{(3)} \rightarrow \tilde{S} \otimes_S \tilde{S} \otimes_S \tilde{S}\) in \((R/D)_{q-crys}\). We want

\[ \tau^{(2)} \circ \delta_1^1 = \delta_1^1 \circ \tau, \quad \tau^{(2)}(\epsilon_i) = 1 \otimes \tau(x_i) - \tau(x_i) \otimes 1, \]

\[ \tau^{(3)} \circ \delta_1^2 = \delta_1^2 \circ \tau^{(2)}, \quad \tau^{(3)}(\tau_i) = 1 \otimes 1 \otimes \tau(x_i) - 1 \otimes \tau(x_i) \otimes 1. \]

This induces well-defined maps, because \(\tilde{S}/\tilde{J} = S/J\) implies \(1 \otimes \tau(x_i) - \tau(x_i) \otimes 1 \in \tilde{J} \otimes_S \tilde{S} + \tilde{S} \otimes_S \tilde{J}\).

By using \(\tau, \tau^{(2)}\), and \(\tau^{(3)}\), we get a morphism of diagrams \((1.3.2) \rightarrow (1.3.5)\). And this yields a functor from \(q\)-HPD stratifications to descent data

\[ (q\text{-HPD stratifications on } (\tilde{R}, I \tilde{R})) \rightarrow DD_{\tilde{S}/S}^\wedge, \]

(see Appendix 3.1.1). Proposition 1.1.2 guarantees a section \(\tilde{S} \rightarrow S\) in the category of \(S\)-modules. Proposition 3.1.2 implies \(\text{Mod}^\wedge_{\tilde{S}/S} \cong DD_{\tilde{S}/S}^\wedge\). We will denote by \((f, \tau)^*\) the resulting functor

\[ (q\text{-HPD stratifications on } (\tilde{R}, I \tilde{R})) \rightarrow \text{Mod}^\wedge_{\tilde{S}/S}. \]

The next step is to define natural isomorphisms

\[ (1.3.6) \]

\[ T_{(f_0, \tau_0)^*, (f_1, \tau_1)^*} : (f_0, \tau_0)^* \cong (f_1, \tau_1)^* \]

for two different choices for the retractions in \((1.3.4)\). We will assume that both choices have a section \(\tilde{S}_i \rightarrow S\), \(i = 0, 1\). Taking \(\tilde{S} = \tilde{S}_1 \otimes_S \tilde{S}_0\), \(f = f_1 \otimes 1 = 1 \otimes f_0\), and identifying \(DD^\wedge_{\tilde{S}_1/S} \cong DD^\wedge_{\tilde{S}_0/S} \cong DD^\wedge_{\tilde{S}_0/S}\) we reduce to constructing

\[ (f, 1 \otimes \tau_0)^* \cong (f, \tau_1 \otimes 1)^*. \]

To simplify the notation, we will simply write \(\tau_1\) for \(\tau_1 \otimes 1\), and similarly for \(\tau_0\).

We can define \(\mu : \tilde{R}^{(2)} \rightarrow \tilde{S}\) in \((R/D)_{q-crys}\) such that \(\mu \circ \delta_0^1 = \tau_0\) and \(\mu \circ \delta_1^2 = \tau_1\). This gives a natural isomorphism

\[ \epsilon \otimes_{\tilde{R}, \mu} \tilde{S} : M \otimes_{\tilde{R}, \tau_0} \tilde{S} \rightarrow M \otimes_{\tilde{R}, \tau_1} \tilde{S}. \]
for all \(q\)-HPD stratifications \((M, \epsilon)\). We claim that this induces an isomorphism of descent data. In other words, we have to prove the equality

\[(\hat{\otimes}_{R(D)}^2, \delta^2_{\mu}) \circ (\hat{\otimes}_{R(D)}^2, \delta^2_{\mu} \circ \hat{\otimes}_{S}^2) = (\hat{\otimes}_{R(D)}^2, \delta^2_{\mu} \circ \hat{\otimes}_{S}^2) \circ (\hat{\otimes}_{R(D)}^2, \delta^2_{\mu} \circ \hat{\otimes}_{S}^2).\]

We can define morphisms in \((R/D)\): 
\[
\begin{array}{c}
\hat{R}((3) \quad \mu_0 \quad \mu_1 \quad \hat{S} \hat{S} \\
\end{array}
\]
such that \(\mu_0 \circ \delta^2_0 = \tau^2_{0}, \mu_0 \circ \delta^2_2 = \delta^1_1 \circ \mu_1, \mu_1 \circ \delta^2_2 = \tau^2_{1}, \mu_1 \circ \delta^2_0 = \delta^1_0 \circ \mu_1, \) and \(\mu_0 \circ \delta^2_2 = \mu_1 \circ \delta^2_2.\) After applying \(\otimes_{\hat{R}(3), \mu_0} \hat{S} \hat{S} \hat{S} \) and \(\otimes_{\hat{R}(3), \mu_1} \hat{S} \hat{S} \hat{S}\) to the cocycle condition, we obtain \((1.3.3).\)

At this point we have constructed the natural isomorphisms \((1.3.6).\) Next, we would like to show that

\[(1.3.8) \quad T_{(f_1, \tau_1)^*} \circ T_{(f_0, \tau_0)^*} = T_{(f_1, \tau_1)^*} \circ T_{(f_0, \tau_0)^*} \circ T_{(f_0, \tau_0)^*} \circ (f_1, \tau_1)^*.\]

Again, by considering \(\hat{S} = \hat{S}_2 \hat{S} \hat{S}_1 \hat{S} \hat{S}_0\), we may reduce to the case where the \(\tau_i\) have the same target. Let \(\mu_{(i,j)} : \hat{R}^2 \to \hat{S}\) for \((i,j) = (0,1), (1,2), (0,2),\) be such that \(\mu_{(i,j)} \circ \delta^1_{0} = \tau_i\) and \(\mu_{(i,j)} \circ \delta^1_{1} = \tau_j.\) Let \((M, \epsilon)\) be a \(q\)-HPD stratifications. Showing \((1.3.3)\) for \(M\) is equivalent to showing

\[e \hat{\otimes}_{R(3), \mu_{(i,2)}} \hat{S} \circ e \hat{\otimes}_{R(3), \mu_{(i,1)}} \hat{S} = e \hat{\otimes}_{R(3), \mu_{(i,2)}} \hat{S}.\]

To prove this we define \(\rho : \hat{R}^3 \to \hat{S}\) with \(\rho \circ \delta^2_0 = \mu_{(0,1)}, \rho \circ \delta^2_1 = \mu_{(0,2)}, \) and \(\rho \circ \delta^2_2 = \mu_{(1,2)}\). Applying \(\hat{\otimes}_{R(3), \rho^*} \hat{S}\) to the cocycle condition implies the claim.

We will also need the compatibility of the isomorphisms \((1.3.6)\) with base change. Let \(u : S \to P\) be a morphism in \((R/D)\), we write \(P = S \hat{S} \hat{S}\) and denote by \(u' : \hat{S} \to \hat{P}\) and \(f' : P \to \hat{P}\) the base change of \(u\) and \(f\), respectively. Let \(u^*\) be the base change functor \(\hat{\otimes}_{S} P.\) We can identify \(u^* \circ (f, \tau)^*\) with \((f', u' \circ \tau)^*\). The equality

\[(1.3.9) \quad u^* T_{(f_0, \tau_0)^*} \circ (f_1, \tau_1)^* = T_{u^* \circ (f_0, \tau_0)^*} \circ u^* \circ (f_1, \tau_1)^*\]

follows immediately from the construction of \(T.\)

Finally, we define a quasi-inverse to the functor \((1.3.3).\) For each object \(S\) in \((R/D)\), make a choice \((f_s, \tau_s).\) For a \(q\)-HPD stratification \((M, \epsilon)\) we set \(M_S = (f_s, \tau_s)^*(M).\) For each \(u : P \to S\) we define \(M_u : u^* M_S \to M_P\) by \(M_u := T_{u^* \circ (f_0, \tau_0)^*} \circ (f_1, \tau_1)^*\). By using \((1.3.8)\) and \((1.3.9)\) this defines a functor to \(q\)-crystals. For a different choice of \((f_s, \tau_s)\) we can use \(T\) to construct a natural isomorphism between the functors.

For a \(q\)-crystal \(M,\) the isomorphism

\[\hat{S} \hat{\otimes}_{f, S} M_S \xrightarrow{M} M_S \xrightarrow{M_S^{-1}} \hat{S} \hat{\otimes}_{f, R} M_R\]

is compatible with the trivial descent datum on \(\hat{S} \hat{\otimes}_{S} M_S\) and the descent datum induced by the associated \(q\)-HPD stratification \((M_R, M_R^{-1} \circ M_{R_{\delta}})\) via \(\tau.\) It induces a natural isomorphism.

On the other hand, starting with \(q\)-HPD stratifications, we may simplify the situation by choosing \(R = \hat{S}, f_{\hat{R}} = \text{id},\) and \(\tau = \text{id}_{\hat{R}}.\) For a \(q\)-HPD stratification
which takes a $(p, \epsilon)$ with associated $q$-crystal $(S \mapsto M_S, u \mapsto M_u)$, we get $M = M_R$ and $\epsilon = M_{\delta^{-1}} \circ M_{\delta^1}$.

This shows the equivalence of categories.

\[\square\]

**Proposition 1.3.4.** The equivalence of categories of Theorem 1.3.3 induces an equivalence between the $(p, [p]_q)$-completely flat objects.

**Proof.** We only need to show that $M$ is a $(p, [p]_q)$-completely flat $q$-crystal if and only if $M_R$ is a $(p, [p]_q)$-completely flat $R$-module. Then use Proposition 1.1.2 together with Lemma 3.1.6, Lemma 3.1.7, and Lemma 3.1.8. \[\square\]

2. $q$-connections

In this section, we let $\tilde{R}$ be as in Proposition 1.1.2. We will use the notation from §1.3. This section will borrow from [BO78, §4].

**Definition 2.0.1.** For $(p, [p]_q)$-derived complete $\tilde{R}$-modules $M$ and $N$, we define the $q$-HPD differential operators from $M$ to $N$ by

$$q{-}\text{HPD} \tilde{R}(M, N) = \text{Hom}_{\tilde{R}}((M \otimes_{\tilde{R}, \delta^1} \tilde{R}^{(2)} )_{\delta^1}, N),$$

where $(.)_{\delta^1}$ means that we consider it as an $\tilde{R}$-module via $\delta^1$.

We would like to make $(p, [p]_q)$-derived complete $\tilde{R}$-modules together with $q$-HPD differential operators as morphisms into a category. In order to define the composition, we set

$$V = \tilde{R}^{(2)} \otimes_{\delta^1, \tilde{R}, \delta^1} \tilde{R}^{(2)},$$

which is an object in $(R/D)_{q{-}\text{cryst}}$, and note that there is a unique morphism $\theta : \tilde{R}^{(3)} \to V$ in $(R/D)_{q{-}\text{cryst}}$ such that $\theta \circ \delta^2_0 = \text{id}_{\tilde{R}^{(3)} \otimes 1}$ and $\theta \circ \delta^2_1 = 1 \otimes \text{id}_{\tilde{R}^{(2)}}$.

Now, let $f \in q{-}\text{HPD} \tilde{R}(M, N)$ and $g \in q{-}\text{HPD} \tilde{R}(N, P)$, we can form the composition

\[\text{(2.0.1)}\]

$$M \otimes_{\tilde{R}, \delta^1} \tilde{R}^{(2)} \xrightarrow{id_M \otimes \theta \delta^2_0} M \otimes_{\tilde{R}, \delta^1} \tilde{R}^{(2)} \otimes_{\delta^1} \tilde{R}^{(2)} \otimes_{\delta^1} \tilde{R}^{(2)} \xrightarrow{f \otimes \text{id}_{\tilde{R}^{(2)}}} N \otimes_{\tilde{R}, \delta^1} \tilde{R}^{(2)} \xrightarrow{g} P,$$

which is $\tilde{R}$-linear if we consider the source as an $\tilde{R}$-module via $\delta^1$. We define $g \circ f$ as the composition \[\text{(2.0.1)}\].

In order to define $\text{id}_M \in q{-}\text{HPD} \tilde{R}(M, M)$, we note that there is a unique morphism $\text{mult} : \tilde{R}^{(2)} \to \tilde{R}$ in $(R/D)_{q{-}\text{cryst}}$ such that $\text{mult} \circ \delta^3_0 = \text{id}_{\tilde{R}} = \text{mult} \circ \delta^3_1$. We take

$$M \otimes_{\tilde{R}, \delta^1} \tilde{R}^{(2)} \xrightarrow{id_M \otimes \text{mult}} M$$

as the identity. This finishes the definition of the category $q{-}\text{HPD} \tilde{R}$.

Next, we define a functor

\[\text{(2.0.2)}\]

$$q{-}\text{HPD} \tilde{R} \to \text{Mod}_D,$$

which takes a $(p, [p]_q)$-derived complete $\tilde{R}$-module to a $(p, [p]_q)$-derived complete $D$-module by restriction via $D \to \tilde{R}$. For $f \in q{-}\text{HPD} \tilde{R}(M, N)$ we simply define

$$M \xrightarrow{m \cdot m \delta^1} M \otimes_{\tilde{R}, \delta^1} \tilde{R}^{(2)} \xrightarrow{f} N$$

as the corresponding morphism.
The functor defined in (2.0.2) is not faithful (see [BO78, §4.4]). However, the map
\begin{equation}
q \text{-HPDiff}\hat{\mathcal{R}}(M, N) \to \text{Hom}_D(M, N)
\end{equation}
is injective if \( N \) is \([p]_q\)-torsion free.

2.1. Let \((M, \epsilon)\) be a \(q\)-HPD stratification on \((\hat{\mathcal{R}}, I\hat{\mathcal{R}})\). To simplify the notation, we set \(q\text{-HPDiff}(N) := q\text{-HPDiff}(N, N)\).

By using
\[
M \hat{\otimes}_{R, \delta_1} \hat{\mathcal{R}}^{(2)} \to \text{Hom}_\hat{\mathcal{R}}(q\text{-HPDiff}(\hat{\mathcal{R}}), M)
\]
we get a map
\begin{equation}
(2.1.1)
\text{Hom}_{\hat{\mathcal{R}}^{(2)}}(M \hat{\otimes}_{R, \delta_0} \hat{\mathcal{R}}^{(2)}, M \hat{\otimes}_{R, \delta_1} \hat{\mathcal{R}}^{(2)}) \to \text{Hom}_{\hat{\mathcal{R}}}(q\text{-HPDiff}(\hat{\mathcal{R}}), q\text{-HPDiff}(M)).
\end{equation}
We denote by \(\nabla\) the image of \(\epsilon\). Explicitly, \(\nabla(\xi)(m \otimes t) = (\text{id}_M \hat{\otimes}\xi)(\epsilon(m \otimes t))\).

**Lemma 2.1.1.** The map \(\nabla\) respects compositions. That is, for any \(\zeta, \xi \in q\text{-HPDiff}(\hat{\mathcal{R}})\) we have \(\nabla(\zeta \circ \xi) = \nabla(\zeta) \circ \nabla(\xi)\).

**Proof.** Recall that \(V = \hat{\mathcal{R}}^{(2)} \hat{\otimes}_{\delta_1} \mathcal{R}^{(2)}\) and we have \(\theta : \hat{\mathcal{R}}^{(3)} \to V\). As in (2.0.1), we consider
\begin{equation}
(2.1.2)
M \hat{\otimes}_{R, \delta_0} \hat{\mathcal{R}}^{(2)} \hat{\otimes}_{\delta_1} \mathcal{R}^{(2)} \xrightarrow{\nabla(\xi) \hat{\otimes}\text{id}_{\hat{\mathcal{R}}^{(2)}}} M \hat{\otimes}_{R, \delta_0} \hat{\mathcal{R}}^{(2)} \xrightarrow{\nabla(\xi)} M.
\end{equation}
This map can be rewritten as follows
\begin{equation}
(2.1.3)
M \hat{\otimes}_{R, \theta \circ \delta_0 \circ \delta_0} V \xrightarrow{(\theta \circ \delta_0^2)^\ast(\epsilon)} M \hat{\otimes}_{R, \theta \circ \delta_0 \circ \delta_0} V \xrightarrow{(\theta \circ \delta_0^2)^\ast(\epsilon)} M \hat{\otimes}_{R, \theta \circ \delta_0 \circ \delta_0} V \xrightarrow{\text{id}_M \hat{\otimes}\zeta \circ (\xi \hat{\otimes}\text{id}_{\hat{\mathcal{R}}^{(2)}})} M,
\end{equation}
where \(f^\ast(\epsilon) := e \hat{\otimes}_{\hat{\mathcal{R}}^{(2)}, f} V\), and the last arrow is induced by the last two arrows in (2.0.1) for \(f = \xi\) and \(g = \zeta\).

We use the cocycle condition to identify the composition of the first two arrows in (2.1.3) with \((\theta \circ \delta_0^2)^\ast(\epsilon)\). Then the claim follows from the commutativity of the following diagram:
\[
\begin{array}{ccc}
M \hat{\otimes}_{R, \delta_0} \hat{\mathcal{R}}^{(2)} & \xrightarrow{\epsilon} & M \hat{\otimes}_{R, \delta_1} \hat{\mathcal{R}}^{(2)} \\
\text{id}_M \hat{\otimes}\theta \circ \delta_0^2 & & \text{id}_M \hat{\otimes}\theta \circ \delta_0^2 \\
M \hat{\otimes}_{R, \theta \circ \delta_0 \circ \delta_0} V & \xrightarrow{(\theta \circ \delta_0^2)^\ast(\epsilon)} & M \hat{\otimes}_{R, \theta \circ \delta_0 \circ \delta_0} V \\
\text{id}_M \hat{\otimes}\theta \circ \delta_0^2 & & \text{id}_M \hat{\otimes}\theta \circ \delta_0^2 \\
\end{array}
\]
\[
\begin{array}{ccc}
M \hat{\otimes}_{R, \delta_0} \hat{\mathcal{R}}^{(2)} & \xrightarrow{\epsilon} & M \hat{\otimes}_{R, \delta_1} \hat{\mathcal{R}}^{(2)} \\
\text{id}_M \hat{\otimes}\theta \circ \delta_0^2 & & \text{id}_M \hat{\otimes}\theta \circ \delta_0^2 \\
M \hat{\otimes}_{R, \theta \circ \delta_0 \circ \delta_0} V & \xrightarrow{(\theta \circ \delta_0^2)^\ast(\epsilon)} & M \hat{\otimes}_{R, \theta \circ \delta_0 \circ \delta_0} V \\
\text{id}_M \hat{\otimes}\theta \circ \delta_0^2 & & \text{id}_M \hat{\otimes}\theta \circ \delta_0^2 \\
\end{array}
\]
\[
\begin{array}{ccc}
M \hat{\otimes}_{R, \delta_0} \hat{\mathcal{R}}^{(2)} & \xrightarrow{\epsilon} & M \hat{\otimes}_{R, \delta_1} \hat{\mathcal{R}}^{(2)} \\
\text{id}_M \hat{\otimes}\theta \circ \delta_0^2 & & \text{id}_M \hat{\otimes}\theta \circ \delta_0^2 \\
M \hat{\otimes}_{R, \theta \circ \delta_0 \circ \delta_0} V & \xrightarrow{(\theta \circ \delta_0^2)^\ast(\epsilon)} & M \hat{\otimes}_{R, \theta \circ \delta_0 \circ \delta_0} V \\
\text{id}_M \hat{\otimes}\theta \circ \delta_0^2 & & \text{id}_M \hat{\otimes}\theta \circ \delta_0^2 \\
\end{array}
\]

\end{proof}

**Definition 2.1.2.** A \(q\text{-connection}\) on \(\hat{\mathcal{R}}\) is a \((p, [p]_q)\)-derived complete \(D\)-module \(M\) together with morphism of \(D\)-algebras \(\nabla \in \text{Hom}_D(q\text{-HPDiff}(\hat{\mathcal{R}}), \text{Hom}_D(M, M))\).

Morphisms of \(q\text{-connections}\) are morphisms of \(D\)-modules that are compatible with \(\nabla\) in the obvious way. This defines the category of \(q\text{-connections}\) \(q\text{-Conn}(\hat{\mathcal{R}}, I)\).
Naturally, every \( q \)-connection is an \( \hat{R} \)-module and every morphism of \( q \)-connections is a morphism of \( \hat{R} \)-modules. A \( q \)-connection is called \((p, [p]_q)\)-completely flat if \( M \) is a \((p, [p]_q)\)-completely flat \( \hat{R} \)-module.

Lemma 2.1.4 shows that \( (M, \epsilon) \mapsto (M, \nabla) \) from (2.1.1) (and using (2.0.2)) yields a functor

\[
\text{(2.1.4)} \quad \text{\((q\)-HPD stratifications on \( (\hat{R}, I\hat{R}) \)) \rightarrow q-\text{Conn}_{(\hat{R}, I)}.}
\]

Our next goal is to show an equivalence between the full subcategories of \((p, [p]_q)\)-completely flat \( q \)-HPD stratifications and \((p, [p]_q)\)-completely flat \( q \)-connections induced by this functor. We will start by finding a nice basis for \( \hat{R}^{(2)} \) lifting the standard basis (given by the coordinates) modulo \( q - 1 \).

Recall that we have \( \hat{R}[\epsilon_1, \ldots, \epsilon_n] \rightarrow \hat{R}^{(2)} \). For \( I \in \mathbb{Z}^n_{\geq 0} \) we set \( \epsilon_I := \prod_{k=1}^n \epsilon_k^{l_k} \), and \([I]_p \) is defined as the largest \( p \)-power dividing \( \prod_{k=1}^n l_k ! \).

As in the proof of Proposition 2.1.2, we can find a sequence \((\Gamma_I)_{I \in \mathbb{Z}^n_{\geq 0}}\) of elements in \( \hat{R}^{(2)} \) such that \([I]_p \cdot \Gamma_I \equiv \epsilon_I \mod (q - 1) \). And we obtain an isomorphism

\[
\left( \bigoplus_{I \in \mathbb{Z}^n_{\geq 0}} \delta_I^1(\hat{R}) \cdot \Gamma_I \right)^{\wedge} \xrightarrow{\approx} \hat{R}^{(2)},
\]

where the source is the derived \((p, [p]_q)\)-completion of the direct sum \( \bigoplus_{I \in \mathbb{Z}^n_{\geq 0}} \delta_I^1(\hat{R}) \cdot \Gamma_I \), but turns out to be the classical \((p, [p]_q)\)-completion, and is automatically the direct sum in the category of \((p, [p]_q)\)-derived complete modules. In particular, if \((\xi_I)_{I \in \mathbb{Z}^n_{\geq 0}}\) denotes the dual basis then

\[
q-\text{HPDiff}(\hat{R}) = \prod_{I \in \mathbb{Z}^n_{\geq 0}} \hat{R} : \xi_I.
\]

**Definition 2.1.3.** A \((p, [p]_q)\)-completely flat \( q \)-connection \((M, \nabla)\) is called quasi-nilpotent if for each \( m \in M \), we have \( \lim_I \nabla(\xi_I)(m) = 0 \) in the \((p, [p]_q)\)-adic sense.

**Proposition 2.1.4.** The functor (2.1.4) induces an equivalence between \((p, [p]_q)\)-completely flat \( q \)-HPD stratifications and \((p, [p]_q)\)-completely flat quasi-nilpotent \( q \)-connections.

**Proof.** Let \( M \) be a \((p, [p]_q)\)-derived complete and \((p, [p]_q)\)-completely flat \( \hat{R} \)-module. Then

\[
\left( \bigoplus_{I \in \mathbb{Z}^n_{\geq 0}} M \otimes \Gamma_I \right)^{\wedge} \xrightarrow{\approx} M \hat{\otimes}_{\hat{R}, \delta_1} \hat{R}^{(2)}
\]

is an isomorphism and the left hand side is the same as the classical completion. Therefore

\[
M \hat{\otimes}_{\hat{R}, \delta_1} \hat{R}^{(2)} \xrightarrow{(\xi_I)_I} \prod_{I \in \mathbb{Z}^n_{\geq 0}} M
\]

is injective and the image equals \( \{(m_I)_I \mid \lim_I m_I = 0\} \), where the limit is in the \((p, [p]_q)\)-adic sense.

Now let \((M, \epsilon)\) be a \( q \)-HPD stratification with associated \( q \)-connection \((M, \nabla)\). By definition, \( \nabla(\xi_I)(m) = \xi_I(\epsilon(m)) \), which implies \( \lim_I \nabla(\xi_I)(m) = 0 \), hence \((M, \nabla)\) is quasi-nilpotent.
Given a quasi-nilpotent \((M, \nabla)\), we can construct \(\epsilon\) as follows. We want
\[
\epsilon(m \otimes 1) = \sum_I \nabla(\xi_I)(m) \otimes \Gamma_I,
\]
which will give a well-defined morphism after showing that
\[
\sum_I \nabla(\xi_I)(a \cdot m) \otimes \Gamma_I = \sum_I \nabla(\xi_I)(m) \otimes \Gamma_I \cdot \delta_0^I(a)
\]
for all \(a \in \tilde{R}\). This is equivalent to
\[
\xi_K(a \cdot m) = \sum_I \xi_K(\Gamma_I \cdot \delta_0^I(a)) \cdot \nabla(\xi_I)(m)
\]
for all \(K\), and follows from
\[
\xi_K \circ a = \sum_I \xi_K(\Gamma_I \cdot \delta_0^I(a)) \cdot \xi_I
\]
in \(-q\text{-HPDiff}(\tilde{R})\). But this just means
\[
(\xi_K \circ a)(\Gamma_I) = \xi_K(\Gamma_I \cdot \delta_0^I(a))
\]
for all \(I\), and holds by definition of the composition in \(-q\text{-HPDiff}(\tilde{R})\).

Now that we have constructed \(\epsilon\), we need to show the cocycle condition. One easily computes
\[
(\delta_2^* \circ \delta_0^*) \circ \epsilon(m \otimes 1) = \sum_{I,J} \nabla(\xi_J)(\nabla(\xi_I)(m)) \otimes \delta_2^I(\Gamma_J) \cdot \delta_0^I(\Gamma_I)
\]
\[
\delta_1^* \circ \epsilon(m \otimes 1) = \sum_K \nabla(\xi_K)(m) \otimes \delta_1^I(\Gamma_K).
\]
We have
\[
\bigg( \bigoplus_{I,J} \delta_2^I(\tilde{R}) \cdot \delta_0^I(\Gamma_I) \delta_2^J(\Gamma_J) \bigg) \wedge \cong \tilde{R}^{(3)},
\]
and the "dual" basis is given by
\[
\tilde{R}^{(3)} \xrightarrow{\delta} V \xrightarrow{\xi_I \circ \text{id}_{\tilde{R}^{(2)}}} \tilde{R}^{(2)} \xrightarrow{\xi_J} \tilde{R}.
\]
Let us write
\[
\delta_2^I(\Gamma_K) = \sum_{I,J} \delta_2^I(t_{I,J}(K)) \cdot \delta_0^I(\Gamma_I) \delta_2^J(\Gamma_J),
\]
which automatically implies \(\lim_{I,J} t_{I,J}(K) = 0\). Then we get
\[
\xi_J \circ \xi_I = \sum_K t_{I,J}(K) \cdot \xi_K
\]
in \(-q\text{-HPDiff}(\tilde{R})\) by definition of the composition. This proves
\[
(\delta_2^* \circ \delta_0^*)(\epsilon)(m \otimes 1) = \delta_1^*(\epsilon)(m \otimes 1),
\]
and the cocycle condition.

The functors are inverse to each other on the nose. \(\square\)
2.2. Our next goal is to understand how the categories of $q$-connections for two different liftings $\tilde{R}_1$ and $\tilde{R}_2$ are related. Theorem 1.3.8 and Proposition 2.1.4 tell us that the full subcategories of $(p, [p]_q)$-completely flat and quasi-nilpotent $q$-connections are equivalent.

Recall that we work in $(R/D)_{q\text{-crys}}$, with base the $q$-PD pair $(D, I)$. With lifting we mean $(\tilde{R}_i, I\tilde{R}_i)$ are objects in $(R/D)_{q\text{-crys}}$ such that $\tilde{R}_i/I\tilde{R}_i = R$. In particular, $\tilde{R}_i$ is a $D-\delta$-algebra.

By using the arguments of Proposition 1.1.2 one can find an isomorphism of $D$-algebras $\tau' : \tilde{R}_1 \to \tilde{R}_2$ inducing the identity modulo $I$. However, this isomorphism is not compatible with the $\delta$-structures, hence not a morphism in $(R/D)_{q\text{-crys}}$. In general, finding an isomorphism in $(R/D)_{q\text{-crys}}$ is not possible.

Let us consider the classical crystalline situation $q - 1 = 0$ for a moment. This simplifies the situation, because we can drop $\delta$ and use PD-ideals only. In this case, $\tilde{R}_i$ is $p$-torsion free and $I\tilde{R}_i$ is a PD-ideal. Moreover, $q$–$\text{HPDiff}(\tilde{R}_i)$ becomes the non-commutative $\tilde{R}_i$-algebra formally generated by $\partial x_1, \ldots, \partial x_n$ for some coordinates $x_1, \ldots, x_n$ (whose existence we assume). Explicitly,

$$q\text{–HPDiff}(\tilde{R}_i) = \prod_{I \in \mathbb{Z}_{\geq 0}} \tilde{R}_i \cdot \partial x_1^I \cdots \partial x_n^I,$$

(to see the independence of the choice of coordinates, we have to use that $\tilde{R}_i$ is classically $p$-complete.) In this case, we can simply use $\tau'$ to construct an isomorphism of $D$-algebras

$$\tau' : q\text{–HPDiff}(\tilde{R}_1) \cong q\text{–HPDiff}(\tilde{R}_2),$$

which will induce an equivalence of categories $(\tau')^*$ between the $q$-connections on $\tilde{R}_1$ and $\tilde{R}_2$, respectively. For another choice of an isomorphism $\tau''$, there is a natural isomorphism $(\tau')^* \to (\tau'')^*$

Let us go back to the general case (where maybe $q - 1 \neq 0$). Then $q$–$\text{HPDiff}(\tilde{R}_i)$ depends on the $\delta$-structure of $\tilde{R}_i$, and $\tau'$ cannot be used. However, it is possible to use the construction from Proposition 1.1.2 to define an isomorphism of $D$-algebras

$$s : q\text{–HPDiff}(\tilde{R}_1) \cong q\text{–HPDiff}(\tilde{R}_2),$$

inducing an equivalence of categories between the categories of $q$-connections. Again, $s$ is not unique, but two choices are naturally isomorphic. When restricted to $(p, [p]_q)$-completely flat and quasi-nilpotent $q$-connections this equivalence is compatible with the equivalence from Theorem 1.3.8.

2.3. Let us recall the constructions from Proposition 1.1.2 for $\tilde{R} = \tilde{R}_1$, $S = \tilde{R}_2$. We obtain

$$\xymatrix{ \tilde{S} \ar[rr]^-f \ar[dr]_-\tau' & & \tilde{R}_2 \ar[dl]^-\tau \ar[rr] & & \tilde{R}_1 \ar@{.>}[rr] & & \tilde{R}_2 }$$

where the dotted arrow is an isomorphism, but is not compatible with the $\delta$-structure. Moreover, $\tilde{S}$ is the $q$-PD-envelope of $\tilde{R}_2[[x_1, \ldots, x_n]]$ for the regular sequence $\epsilon_1, \ldots, \epsilon_n$, and $\tau$ and $f$ factor over $\tilde{R}_2[[x_1, \ldots, x_n]]$. Explicitly, we have $\tau(x_i) = \tau'(x_i) + \epsilon_i$, for coordinates $x_1, \ldots, x_n$, and $f(a) = a$. However, $\tilde{S}$ does not depend on the coordinates $(x_i)$. Even better, it does not depend on $\tau'$. It is the
such that $\rho_{t,s}$ is associative and compatible with the composition on $q$.

We will modify the constructions which were used to define the algebra structure on $q$-HPDiff($\hat{R}$) in order to obtain

\[(2.3.1) \quad \text{Hom}_{\hat{R}_3} (\hat{R}_2 \otimes_{q\text{-crys}} \hat{R}_3, \hat{R}_3) \times \text{Hom}_{\hat{R}_2} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_2, \hat{R}_2) \rightarrow \text{Hom}_{\hat{R}_3} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_3, \hat{R}_3),\]

which we will write as composition $(t, s) \mapsto t \circ s$.

By using the universal property we get

$$\rho : \hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_3 \rightarrow (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_2) \otimes_{\delta_1, \hat{R}_2, \delta_0} (\hat{R}_2 \otimes_{q\text{-crys}} \hat{R}_3)$$

such that $\rho \circ \delta_0 = \delta_0 \circ 1$ and $\rho \circ \delta_1 = 1 \circ \delta_1$. The composition \[(2.3.1)\] is defined by

$$\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_3 \xrightarrow{\rho} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_2) \otimes_{\delta_1, \hat{R}_2, \delta_0} (\hat{R}_2 \otimes_{q\text{-crys}} \hat{R}_3) \xrightarrow{\hat{\delta} \circ \text{id}} \hat{R}_2 \otimes_{q\text{-crys}} \hat{R}_3 \xrightarrow{\hat{\tau}_0} \hat{R}_3.$$

It is associative and compatible with the composition on $q$-HPDiff($\hat{R}$) introduced in \[(2.0.1)\]. In particular, $\text{Hom}_{\hat{R}_2} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_2, \hat{R}_2)$ is a right $q$-HPDiff($\hat{R}_1$)-module and a left $q$-HPDiff($\hat{R}_2$)-module.

We define

$$\text{SHom}_{\hat{R}_2} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_2, \hat{R}_2) = \{s \in \text{Hom}_{\hat{R}_2} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_2, \hat{R}_2) \mid s(1) = 1, (s \circ \delta_0) \otimes_D D/I = \text{id}_R\}.$$

Note that $s(1) = 1$ implies $s \circ \delta_1 = \text{id}_{\hat{R}_2}$, in other words, $s$ is a section.

We know that $\text{SHom}_{\hat{R}_2} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_2, \hat{R}_2)$ is not empty, because the section constructed in the proof of Proposition \[1.1.2\] satisfies the requirements (indeed, it satisfies $s \circ \delta_0 \otimes_D D/(q - 1) = \tau' \otimes_D D/(q - 1)$, and $\tau' \otimes_D D/I = \text{id}_R$ holds by definition).

The composition \[(2.3.1)\] respects $\text{SHom}$, because $(t \circ s) \circ \delta_0 = (t \circ \delta_0) \circ (s \circ \delta_0)$. Moreover, $\text{SHom}_{\hat{R}_2} (\hat{R} \otimes_{q\text{-crys}} \hat{R}, \hat{R})$ is a multiplicative subgroup of $q$-HPDiff($\hat{R}$), because $I^p \hat{R} \subset (p, [p]_q) \hat{R}$ and the injectivity of the map \[(2.0.3)\] $q$-HPDiff($\hat{R}$) $\rightarrow$ Hom$_D (\hat{R}, \hat{R})$, $\xi \mapsto \xi \circ \delta_1$.

\textbf{Lemma 2.3.1.} For $s \in \text{SHom}_{\hat{R}_2} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_2, \hat{R}_2)$ there exists a unique $t \in \text{SHom}_{\hat{R}_1} (\hat{R}_2 \otimes_{q\text{-crys}} \hat{R}_1, \hat{R}_1)$ such that $s \circ t = 1$ and $t \circ s = 1$.

\textit{Proof.} We know that $\text{SHom}_{\hat{R}_1} (\hat{R}_2 \otimes_{q\text{-crys}} \hat{R}_1, \hat{R}_1)$ is non empty. So we can find some $t'$. Then $t' \circ s \in \text{SHom}_{\hat{R}_1} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_1, \hat{R}_1)$, and we set $t = (t' \circ s)^{-1} \circ t'$. Now that we have $t \circ s = 1$, we play the same game with $t$ to get $u$ with $u \circ t = 1$, hence $u = u \circ t \circ s = s$. \hfill \Box

\textbf{Proposition 2.3.2.} Every $s \in \text{SHom}_{\hat{R}_2} (\hat{R}_1 \otimes_{q\text{-crys}} \hat{R}_2, \hat{R}_2)$ induces an isomorphism

$$\psi_s : q\text{-HPDiff}(\hat{R}_1) \rightarrow q\text{-HPDiff}(\hat{R}_2).$$
For a second choice $s' \in \text{SHom}_{\tilde{R}_2}(\tilde{R}_1 \otimes_{q-\text{crys}} \tilde{R}_2, \tilde{R}_2)$ there is a $\xi \in \text{SHom}_{\tilde{R}_2}(\tilde{R}_2 \otimes_{q-\text{crys}} \tilde{R}_2, \tilde{R}_2)$ such that $\psi_{s'}(\zeta) = \xi \cdot \psi_s(\zeta) \cdot \xi^{-1}$ for all $\zeta$. In other words, $\psi_s$ is up to conjugation by elements in $\text{SHom}_{\tilde{R}_2}(\tilde{R}_2 \otimes_{q-\text{crys}} \tilde{R}_2, \tilde{R}_2)$ independent of the choice of $s$.

Proof. We can use Lemma 2.3.1 to find $t$, and define $\psi_s(\zeta) = s \circ \zeta \circ t$.

Making another choice $s'$, we define $\xi := s' \circ t$. Then $s' = \xi \circ s$ and $\psi_{s'}(\zeta) = \xi \cdot \psi_s(\zeta) \cdot \xi^{-1}$ for all $\zeta$. \hfill $\square$

Corollary 2.3.3. Let $(\tilde{R}_i, I \tilde{R}_i)$ in $(R/D)_{q-\text{crys}}$, for $i = 0, 1, 2, 3$, be such that $\tilde{R}_i/I \tilde{R}_i = R$, and suppose that $(p, [p]_q)$-etale maps $h_i : D[x_1, \ldots, x_n] \rightarrow R_i$ exist.

For every $s_{2,1} \in \text{SHom}_{\tilde{R}_2}(\tilde{R}_1 \otimes_{q-\text{crys}} \tilde{R}_2, \tilde{R}_2)$ we have an equivalence of categories

$$F_{s_{2,1}} : q-\text{Conn}(\tilde{R}_2, I \tilde{R}_2) \rightarrow q-\text{Conn}(\tilde{R}_1, I \tilde{R}_1)$$

$$(M, \nabla) \mapsto (M, \nabla \circ \psi_s).$$

Between two equivalences of this form there is a natural isomorphism:

$$u_\xi : F_s \rightarrow F_{s'}, \quad u_\xi(M, \nabla)(m) := \nabla(\xi)(m),$$

where $\xi$ is defined by $s' = \xi \circ s$.

For $s_{3,1} \in \text{SHom}_{\tilde{R}_3}(\tilde{R}_1 \otimes_{q-\text{crys}} \tilde{R}_3, \tilde{R}_3)$ and $s_{3,2} \in \text{SHom}_{\tilde{R}_2}(\tilde{R}_2 \otimes_{q-\text{crys}} \tilde{R}_3, \tilde{R}_3)$, we have $F_{s_{2,1}} \circ F_{s_{3,2}} = F_{s_{3,2} \circ s_{2,1}}$. In particular, we have a natural isomorphism

$$t_{3,2,1} : F_{s_{2,1}} \circ F_{s_{3,2}} = F_{s_{3,2} \circ s_{2,1}} \xrightarrow{u_\zeta} F_{s_{3,1}},$$

where $\zeta$ is such that $s_{3,1} = \zeta \circ s_{3,2} \circ s_{2,1}$.

Moreover, for all $(M, \nabla)$ the following diagram is commutative:

$$F_{s_{1,0}}(F_{s_{2,1}} \circ F_{s_{3,2}}(M, \nabla)) \xrightarrow{F_{s_{1,0}}(t_{3,2,1})} F_{s_{1,0}}(F_{s_{3,1}}(M, \nabla))$$

$$F_{s_{2,0}}(F_{s_{3,2}}(M, \nabla)) \xrightarrow{t_{3,2,0}} F_{s_{3,0}}(M, \nabla).$$

Proof. This follows immediately from Proposition 2.3.2. \hfill $\square$

Proposition 2.3.4. Assumptions as in Corollary 2.3.3. The following diagram of functors commutes up to natural transformation

$$q-\text{Cris}(R/D)$$

$$\xrightarrow{q-\text{HPD str. on } (\tilde{R}_2, I \tilde{R}_2)}$$

$$\xrightarrow{q-\text{Conn}(\tilde{R}_1, I \tilde{R}_1)}$$

$$q-\text{Conn}(\tilde{R}_2, I \tilde{R}_2)$$

$$\xrightarrow{(q-\text{HPD str. on } (\tilde{R}, I \tilde{R}_1))}$$

$$\xrightarrow{(q-\text{HPD str. on } (\tilde{R}, I \tilde{R}_1))}$$

$$q-\text{Conn}(\tilde{R}_1, I \tilde{R}_1)$$
2.4. The sections in \( \text{SHom}_{\tilde{R}_2}(\tilde{R}_1 \otimes_{q\text{-crys}} \tilde{R}_2, \tilde{R}_2) \) used to define the functor between the categories of \( q \)-connections are \( p \)-adic in nature. For the next section, where we will prove a global version of Corollary 2.3.3, we need to isolate those sections that come from global ones. This will not work for \( p = 2 \).

Recall the setup from Subsection 2.3. The \( D \)-algebra \( \tilde{R}_2[\epsilon_1, \ldots, \epsilon_n] \) together with the ideal \( I + (\epsilon_1, \ldots, \epsilon_n) \) and the morphisms \( f : \tilde{R}_2 \to \tilde{R}_2[\epsilon_1, \ldots, \epsilon_n] \) and \( \tau : \tilde{R}_1 \to \tilde{R}_2[\epsilon_1, \ldots, \epsilon_n] \) does not depend on the choice of the coordinates or \( \tau' \) or the \( \delta \)-structures. More conceptually, it is the coproduct of \( \tilde{R}_1 \) and \( \tilde{R}_2 \) in a suitable category of infinitesimal thickenings of \( R \). We will write

\[
\tilde{R}_1 \otimes_{\text{inf}} \tilde{R}_2 := \tilde{R}_2[\epsilon_1, \ldots, \epsilon_n], \quad J := I \cdot \tilde{R}_2[\epsilon_1, \ldots, \epsilon_n] + (\epsilon_1, \ldots, \epsilon_n),
\]

and \( \delta_1 := f \) and \( \delta_0 := \tau \). Moreover, if \( K \subset \tilde{R}_2 \) is an ideal, then we will use the notation

\[
\text{Diff}_K^\wedge(\tilde{R}_1, \tilde{R}_2) := \{ s \in \text{Hom}_{\tilde{R}_2}(\tilde{R}_1 \otimes_{\text{inf}} \tilde{R}_2, \tilde{R}_2) \mid \lim_k s(J^k) = 0 \quad \text{\( K \)-adically} \}
\]

Actually, we will only consider \( K = (p, [p]_q) \) and \( K = I \). Since \( I^p \subset (p, [p]_q) \), we have an injective map

\[
\text{Diff}_K^\wedge(\tilde{R}_1, \tilde{R}_2) \to \text{Diff}_{(p, [p]_q)}^\wedge(\tilde{R}_1, \tilde{R}_2).
\]

As in (2.3.1), we get composition maps

\[
\text{Diff}_K^\wedge(\tilde{R}_2, \tilde{R}_3) \times \text{Diff}_K^\wedge(\tilde{R}_1, \tilde{R}_2) \to \text{Diff}_K^\wedge(\tilde{R}_1, \tilde{R}_3)
\]

for \( K = (p, [p]_q) \) and \( K = I \). This turns \( \text{Diff}_K^\wedge(\tilde{R}_1, \tilde{R}_2) \) into an algebra.

We set

\[
\text{SDiff}_K^\wedge(\tilde{R}_1, \tilde{R}_2) = \{ s \in \text{Diff}_K^\wedge(\tilde{R}_1, \tilde{R}_2) \mid s(1) = 1, (s \circ \delta_0) \otimes_D D/I = \text{id}_R \}.
\]

**Proposition 2.4.1.** The map \( \text{Hom}_{\tilde{R}_2}(\tilde{R}_1 \otimes_{q\text{-crys}} \tilde{R}_2, \tilde{R}_2) \to \text{Hom}_{\tilde{R}_2}(\tilde{R}_1 \otimes_{\text{inf}} \tilde{R}_2, \tilde{R}_2), \) induced by \( \tilde{R}_1 \otimes_{\text{inf}} \tilde{R}_2 \to \tilde{R}_1 \otimes_{q\text{-crys}} \tilde{R}_2 \), factors through \( \text{Diff}_{(p, [p]_q)}^\wedge(\tilde{R}_1, \tilde{R}_2) \).

**Proof.** This follows immediately from \( J^p \subset (p, [p]_q) \), where \( J \subset \tilde{R}_1 \otimes_{q\text{-crys}} \tilde{R}_2 \) is the \( q\text{-PD} \) ideal. \( \square \)

**Proposition 2.4.2.** Suppose \( p > 2 \) and \( (p, [p]_q) \) is a regular sequence in \( D \). Then there exists \( s \in \text{SHom}_{\tilde{R}_2}(\tilde{R}_1 \otimes_{q\text{-crys}} \tilde{R}_2, \tilde{R}_2) \) with image in \( \text{SDiff}_K^\wedge(\tilde{R}_1, \tilde{R}_2) \).

Moreover, if \( \tau' : \tilde{R}_1 \to \tilde{R}_2 \) is an isomorphism of \( D \)-algebras inducing the identity modulo \( I \), then we can find an \( s \) such that \( (s \circ \delta_0) \otimes_D D/(q-1)^p-1 = \tau' \otimes_D D/(q-1)^p-1 \).

**Proof.** Let \( m = p - 1 \), there is \( u \in \mathbb{Z}_p[q - 1] \) such that \( u \equiv 1 \mod q - 1 \) and \( u \cdot [p]_q - p \in (q - 1)^m \mathbb{Z}_p[q - 1] \); we set \( d = u \cdot [p]_q \) and \( x = q - 1 \). Note that \( d - p \in (x^m) \) and \( \phi(x) \subset (x \cdot d) \).

Let \( J \subset \tilde{R}_1 \otimes_{q\text{-crys}} \tilde{R}_2 \) be the \( q\text{-PD} \) ideal. We will define maps \( \gamma_{pk} : J \to J \) by induction on \( k \). For \( k = 0 \), we take \( \gamma_1 = \text{id}_J \). These maps will satisfy \( \phi(\gamma_{pk}(J)) \subset d^{m^k}(\tilde{R}_1 \otimes_{q\text{-crys}} \tilde{R}_2) \).

We define

\[
\gamma_{pk}(a) = -\delta(\gamma_{pk-1}(a)) + \frac{d^{m^k-1} - (d - p)^{m^k-1}}{p} \cdot \phi(\gamma_{pk-1}(a)) \frac{d^{m^k-1}}{d^{m^k-1}}.
\]

\[
\gamma_{pk}(a) = \frac{1}{p} \left( \gamma_{pk-1}(a)^p - (d - p)^{m^k-1} \cdot \phi(\gamma_{pk-1}(a)) \right),
\]

where \( u \) is a unit of the \( D \)-algebra and \( \delta : J \to J \).
and have to show that \( \phi(\gamma_p^k(J)) \subset d^{m^k} \) and \( \gamma_p^k(J) \subset J \). Now, \( \phi(\gamma_p^k(J)) \subset d^{m^k} \)
follows from \(2.4.3\), \( d-p \in (x^m) \), and \((p,d)\) are a regular sequence. For \( \gamma_p^k(J) \subset J \),
we note that by using \( d^{m^{k-1}} - (d-p)^{m^{k-1}} \& \equiv d^{m^{k-1} - 1} \) mod \( x \) and \( x \in J \), we only have
to show \(- \delta(a) + \frac{\phi(a)}{a} \in J\) for all \( a \in J \). Since \( u \equiv 1 \mod q - 1 \), this follows from
the definition of a \( q \)-PD pair.

Find coordinates \( x_1, \ldots, x_n \) for \( R_1 \) and write \( R_1 \otimes_{\text{inf}} R_2 = R_2[\epsilon_1, \ldots, \epsilon_n] \),
with \( \delta_0(x_i) = \tau'(x_i) + \epsilon_i \). We can construct a basis \((\Gamma_I)\) for \( R_1 \otimes_{q-\text{crys}} R_2 \)
and a section \( s \) as was done in the proof of Proposition \(1.1.2\) and by using the newly defined \( \gamma_p^k \).

Evidently, \((s \circ \delta_0) \otimes_D D/x^m = \tau' \otimes_D D/x^m \) holds. Moreover, it is not hard to
find \( n_I \in \mathbb{Z}_p[x] \) such that \( n_I \) is power of \( p \) modulo \( x \), \( n_I \cdot \Gamma_I \subset R_2[\epsilon_1, \ldots, \epsilon_n] \), and
\( n_I \cdot \Gamma_I = \epsilon^I \) modulo \( x^m \), where we used the notation \( \epsilon^I = \epsilon_1^I \ldots \epsilon_n^I \).

Writing \( n_I \Gamma_I = \sum_J B_{J, I} \cdot \epsilon^J \), we get \( B_{J, I} = 1, B_{J, J} \in (x^m) \) for \( I \neq J \), and, by
using \(2.4.3\), \( \lim_I B_{J, I} = 0 \) for the \( x \)-adic topology. Now, writing \( \epsilon^J = \sum_{J} A_{J, I} n_I \Gamma_J \),
it is not hard to conclude \( \lim_I A_{J, I} = 0 \) for all \( J \). Since \( s(\epsilon^I) = A_{0, I} \), we are
done. \( \square \)

2.5. A global approach. In this subsection we are going to patch the functors from Corollary \(2.3.3\) for various primes \( p \)
together in order to obtain a global result on \( q \)-connections. By definition, the category of \( q \)-connections depends on the choice of
coordinates. The study of their independence was initiated by Scholze \( \text{[Sch]} \). Our approach will not be able to include the prime \( 2 \),
but we have no doubt that other approaches can prove the results in full generality.

Let \( R \) be smooth over \( \mathbb{Z} \). We set \( R_q = R[[q - 1]] \), and denote by \( R_p \) and \( R_{q,p} \) the
\( p \)-adic completions. If \( \psi : \mathbb{Z}[x_1, \ldots, x_n] \to R \) is étale, then we denote by \( R_{q,p,\psi} \) the \( \mathbb{Z}_p[[q - 1]] \)-algebra \( R_{q,p} \)
together with the \( \delta \)-structure induced by \( \phi(\psi(x_i)) = x_i^p \). We note that \((R_{q,p,\psi}, (q - 1))\) is an object in \((R_p/\mathbb{Z}_p[[q - 1]])_{q-\text{crys}} \).

As in the \( p \)-adic setup, we set \( R_q \otimes_{\text{inf}} R_q = R_q[\epsilon_1, \ldots, \epsilon_n] \), \( \delta_1(a) = a \), for all \( a \in R_q \), and \( \delta_0 \) is induced by \( \delta_0(\psi(x_i)) = \psi(x_i) + \epsilon_i \) for some \( \psi \). Again, we
note that \( R_q \otimes_{\text{inf}} R_q \) together with the ideal \( J = (q - 1, \epsilon_1, \ldots, \epsilon_n) \), \( \delta_0 \), and \( \delta_1 \),
is independent of \( \psi \). It is the coproduct of \( R_q \) with itself in a suitable category of
infinitesimal thickenings of \( R \) over \( \mathbb{Z}[q - 1] \).

We set
\[
\text{Diff}^{\wedge}_{(q-1)}(R_q, R_q) = \{ s \in \text{Hom}_{R_q}((R_q \otimes_{\text{inf}} R_q)_{\delta_1}, R_q) \mid \lim_n s(J^n) = 0 \quad q - 1 \text{-adically} \},
\]
and note that
\[
\text{Diff}^{\wedge}_{(q-1)}(R_q, R_q) \to \text{Hom}_{\mathbb{Z}[q - 1]}(R_q, R_q), \quad s \mapsto s \circ \delta_0
\]
is injective.

Depending on \( \psi \), we have the differential operators \( \nabla_{x,q} \in \text{Diff}^{\wedge}_{(q-1)}(R_q, R_q) \)
from \( \text{[Sch]} \). We will use the notations \( \nabla_{x,q}^I := \prod_{k=1}^n \nabla_{x,k,q}^I \) and
\[
A_{\psi} := \{ \sum_I a_I \cdot \nabla_{x,q}^I \mid a_I \in R_q, \lim_I a_I = 0 \quad q - 1 \text{-adically} \}.
\]
It is a subalgebra of \( \text{Diff}^{\wedge}_{(q-1)}(R_q, R_q) \).

We have the \( p \)-adic completion map
\[
\text{Diff}^{\wedge}_{(q-1)}(R_q, R_q) \to \text{Diff}^{\wedge}_{(q-1)}(R_{q,p}, R_{q,p}),
\]
and define \( A_{\psi,p} \) in a similar way.
Lemma 2.5.1. We have

\[
q^{-}\text{HPDiff}(R_{q,p,\psi}) = \prod_{l \in \mathbb{Z}^+_{p,0}} R_{q,p} \cdot \nabla_{\psi,q}^l,
\]

(2.5.2)

\[
\mathcal{A}_{\psi,p} = \{ s \in q^{-}\text{HPDiff}(R_{q,p,\psi}) \mid s \in \text{Diff}^{\wedge}_{(q-1)}(R_{q,p},R_{q,p}) \}.
\]

Proof. We know \([p]q \cdot x_i^{p-1} \cdot \phi \circ (\nabla_{x_i,q} \circ \delta_0) = (\nabla_{x_i,q} \circ \delta_0) \circ \phi\). This implies

\[
(2.5.3)
\]

as equality of maps \(R_{q,p} \otimes_{\text{inf}} R_{q,p} \rightarrow R_{q,p}\). From (2.5.3) we see that \(\nabla_{x_i,q}\) factors over \(R_{q,p,\psi} \otimes_{q^{-}\text{-crys}} R_{q,p,\psi}\). Since \(\nabla_{x_i,q} \equiv \nabla_{x_i} \mod q - 1\), we have proved (2.5.1).

In fact, it is not hard to compute the basis \((\Gamma_{I})\) dual to \((\nabla_{x_i,q}^{I})\). For \([I]q! := \prod_{k=1}^{n}[I_k]_q!\), we get \([I]q! \cdot \Gamma_I \in R_{q,p} \otimes_{\text{inf}} R_{q,p}\) and \([I]q! \cdot \Gamma_I \in J \sum_{k=1}^{n} I_k\). This proves (2.5.2).

Lemma 2.5.2. The map

\[
\text{Diff}^{\wedge}_{(q-1)}(R_q,R_q)/\mathcal{A}_\psi \rightarrow \bigoplus_{p \text{ prime}} \text{Diff}^{\wedge}_{(q-1)}(R_{q,p},R_{q,p})/\mathcal{A}_{\psi,p},
\]

where \(\bigoplus^{\wedge}\) means the \(q - 1\)-adic completion of the direct sum, is bijective.

Proof. Let us consider the differential operators of finite rank:

\[
\text{Diff}^{\wedge}_{(q-1)}(R_q,R_q) = \left\{ \sum_{I} a_I \cdot \nabla_{\psi,q}^{I} \mid \text{the sum is finite} \right\},
\]

\[
\mathcal{A}_{\psi}^{\text{finite}} = \left\{ \sum_{I} a_I \cdot \nabla_{\psi,q}^{I} \mid \text{the sum is finite} \right\},
\]

and similarly for the \(p\)-adic analogs. Then \(\text{Diff}^{\wedge}_{(q-1)}(R_q,R_q)\) is the \(q - 1\)-adic completion of \(\text{Diff}^{\wedge}_{(q-1)}(R_q,R_q)\), and \(\mathcal{A}_{\psi}^{\text{finite}}\) is the \(q - 1\)-adic completion of \(\mathcal{A}_{\psi}^{\text{finite}}\). And similarly for the \(p\)-adic analogs.

It is clear that

\[
\text{Diff}^{\wedge}_{(q-1)}(R_q,R_q)/\mathcal{A}_{\psi}^{\text{finite}} \rightarrow \bigoplus_{p \text{ prime}} \text{Diff}^{\wedge}_{(q-1)}(R_{q,p},R_{q,p})/\mathcal{A}_{\psi,p}^{\text{finite}}
\]

is bijective. Thus the statement follows by \(q - 1\)-adic completion after observing that \(\text{Diff}^{\wedge}_{(q-1)}(R_q,R_q)/\mathcal{A}_{\psi}^{\text{finite}}\) has no \(q - 1\)-torsion.

Following our convention, we define \(\text{SX} = \{ s \in X \mid s \circ \delta_0 \equiv \text{id} \mod q - 1 \}\), where \(X = \text{Diff}^{\wedge}_{(q-1)}(R_q,R_q)\) or \(X = \mathcal{A}_{\psi}\) etc. We note that \(\text{SX}\) is a group via the multiplication on \(X\).

Corollary 2.5.3. The map

\[
\text{SDiff}^{\wedge}_{(q-1)}(R_q,R_q)/\text{SA}_{\psi} \rightarrow \bigoplus_{p \text{ prime}} \text{SDiff}^{\wedge}_{(q-1)}(R_{q,p},R_{q,p})/\text{SA}_{\psi,p},
\]

where \(\bigoplus^{\wedge}\) means that each element \((s_p)_p\) has only finitely many components \(\neq 1\) after reduction modulo \((q - 1)^m\) for every \(m\).
**Definition 2.5.4.** A $q$-connection on $(R_q, \psi)$ is a $q - 1$-derived complete $\mathbb{Z}[q - 1]$-module $M$ together with a morphism of $\mathbb{Z}[q - 1]$-algebras
\[ \nabla \in \text{Hom}_{\mathbb{Z}[q - 1]}(A_\psi, \text{Hom}_{\mathbb{Z}[q - 1]}(M, M)). \]

Morphisms of $q$-connections are morphisms of $\mathbb{Z}[q - 1]$-modules that are compatible with $\nabla$ in the obvious way. This defines the category of $q$-connections $\mathbb{C}$.

Automatically, every $q$-connection is an $R_q$-module and every morphism of $q$-connections is a morphism of $R_q$-modules.

**Proposition 2.5.5.** Suppose $\frac{1}{2} \in R$. Let $\psi_i : \mathbb{Z}[x_1, \ldots, x_n] \to R$ be two étale maps. There is a unique $s \in \text{SDiff}^{(q - 1)}(R_q, R_q)$ such that
\[ s : \text{SHom}_{R_{p,q},\psi_2}(R_{q,p,\psi_1} \otimes_{q-crys} R_{q,p,\psi_2}, R_{q,p}) \cap \text{SDiff}^{(q - 1)}(R_{q,p}, R_{q,p}), \]
for each prime $p$ (with $R_p \neq 0$). Moreover, $sA_{\psi_1} s^{-1} = A_{\psi_2}$.

**Proof.** We know that
\[ \text{SHom}_{R_{p,q},\psi_2}(R_{q,p,\psi_1} \otimes_{q-crys} R_{q,p,\psi_2}, R_{q,p}) \cap \text{SDiff}^{(q - 1)}(R_{q,p}, R_{q,p}) \neq \emptyset \]
by Proposition [2.4.2] and $\frac{1}{2} \in R$.

It is clear that $\text{SHom}_{R_{p,q},\psi_2}(R_{q,p,\psi_1} \otimes_{q-crys} R_{q,p,\psi_2}, R_{q,p}) \cap \text{SDiff}^{(q - 1)}(R_{q,p}, R_{q,p})$ is a right $\text{SA}_\psi$-torsor hence defines an element in $\text{SDiff}^{(q - 1)}(R_{q,p}, R_{q,p})/\text{SA}_\psi$, which is also trivial modulo $(q - 1)^p - 1$ by Proposition [2.4.2]. We can use Corollary [2.5.3] to obtain $s$ and its uniqueness.

Moreover, $sA_{\psi_1} s^{-1} = A_{\psi_2}$ follows from $sA_{\psi_1} s = A_{\psi_2}$ for every $p$. □

This implies the analog of Proposition [2.5.6].

**Corollary 2.5.6.** Every $s$ as in Proposition [2.5.6] induces an isomorphism
\[ \psi_s : A_{\psi_1} \to A_{\psi_2}, \quad \zeta \mapsto s \cdot \zeta \cdot s^{-1}. \]

For a second choice $s'$ there is a $\xi \in A_{\psi_2}$ such that $\psi_{s'}(\zeta) = \xi \cdot \psi_s(\zeta) \cdot \xi^{-1}$ for all $\zeta$.

Corollary [2.5.6] implies the analog of Corollary [2.3.3]. This yields a natural equivalence between the categories of $q$-connections for different coordinates.

3. **Appendix**

3.1. **Derived complete modules.** Our reference for derived completion is [Sta20 Tag 091N].

Let $A$ be a commutative ring and $I$ a finitely generated ideal. We denote the category of $I$-derived complete modules by $\text{Mod}^I_A$. It is a weak Serre subcategory of the category of $A$-modules $\text{Mod}_A$ [Sta20 Tag 091U]. In particular, it is an abelian category and $\text{Mod}^I_A \subset \text{Mod}_A$ is fully faithful and exact. Every classically $I$-complete $A$-module is automatically $I$-derived complete [Sta20 Tag 091T].

For $M, N \in \text{Mod}^I_A$ the $A$-module $\text{Hom}_{\text{Mod}^I_A}(M, N) = \text{Hom}_A(M, N)$ is $I$-derived complete $\text{Mod}^I_A$ [Sta20 Tag 091E]. The functor $N \mapsto \text{Hom}_R(M, N)$ has a left adjoint functor $N \mapsto N \otimes^I_A M$, which is given by $N \mapsto H^0((M \otimes^I_A N)\wedge)$ with $K \mapsto K\wedge$ being the derived completion [Sta20 Tag 091V]. Indeed, this follows from
\[ \text{Hom}(N, R\text{Hom}(M, P)) = \text{Hom}(N \otimes^L M, P) = \text{Hom}((N \otimes^L M)\wedge, P) \]
\[ = \text{Hom}((N \otimes^L M)\wedge, P) = \text{Hom}(N \otimes^I M, P), \]
where we used that derived completion behaves like a left derived functor [Sta20, Tag 0AAJ].

The functor $N \mapsto N \hat{\otimes}_A M$ is right exact. We have $N \hat{\otimes}_A M = M \hat{\otimes}_A N$, and $(N \hat{\otimes}_A M) \hat{\otimes}_A P = N \hat{\otimes}_A (M \hat{\otimes}_A P)$.

3.1.1. **Descent.** Let $B$ be a derived $I$-complete $A$-algebra. Then $M \mapsto M \hat{\otimes}_A B$ defines a functor $\text{Mod}^\wedge_A \to \text{Mod}^\wedge_B$ that is left adjoint to the forgetful functor $\text{Mod}^\wedge_B \to \text{Mod}^\wedge_A$. Note that the derived completion functor commutes with the forgetful functor from $B$-modules to $A$-modules [Sta20, Tag 09ZQ]. In particular, a $B$-module $M$ is $IB$-derived complete if and only if it is $I$-derived complete as an $A$-module. In order to see that $M \mapsto M \hat{\otimes}_A B$ is the left adjoint, we use

$$\text{Hom}_B(M \hat{\otimes}_A B, N) = \text{Hom}_B(M \otimes_A B, N) = \text{Hom}_A(M, N).$$

We define descent data in the usual way, that is, descent data correspond to $(M, \theta)$, where $M$ is an $IB$-derived complete $B$-module and $\theta$ is a morphism of $B$-modules

$$M \to M \hat{\otimes}_A B,$$

where $M \hat{\otimes}_A B$ is a $B$-module via the right factor. Since $M \hat{\otimes}_A B$ is also a $B$-module via the action on $M$, we get an induced morphism

$$M \hat{\otimes}_A B \to M \hat{\otimes}_A B.$$

We require that this map is an isomorphism and the cocycle condition is satisfied. The cocycle condition is

$$(\theta \hat{\otimes} \text{id}_B) \circ \theta = (\text{id}_M \hat{\otimes} \delta_0^1) \circ \theta,$$

as equality of morphism $M \to M \hat{\otimes}_A B \hat{\otimes}_A B$ with $\delta_0^1 : B \to B \hat{\otimes} B$ induced by $b \mapsto 1 \otimes b$. We will denote the category of derived complete descent data by $\text{DD}^\wedge_{B/A}$.

As usual, we have the base extension functor

$$(3.1.1) \quad \text{Mod}^\wedge_A \to \text{DD}^\wedge_{B/A}. \quad M \mapsto (M \hat{\otimes}_A B, \text{id}_M \hat{\otimes} \delta_0^1).$$

It has a right adjoint functor given by

$$\text{DD}^\wedge_{B/A} \to \text{Mod}^\wedge_A, \quad (M, \theta) \mapsto \ker(\theta - \iota)$$

where $\iota : M \to M \hat{\otimes}_A B$ is induced by $m \mapsto m \otimes 1$.

**Proposition 3.1.2.** If there is a section $s : B \to A$ in the category of $A$-modules then base extension $\text{Mod}^\wedge_A \to \text{DD}^\wedge_{B/A}, M \mapsto M \hat{\otimes}_A B$ is an equivalence of categories.

A quasi-inverse is given by $M \mapsto \ker(\theta - \iota)$.

**Proof.** The strategy of the proof is taken from [Sta20, Tag 08WE].

Let $(M, \theta)$ be a descent datum. We set $f = \theta$, $g_1 = \theta \hat{\otimes} \text{id}_B$ and $g_2 = \text{id}_M \hat{\otimes} \delta_0^1$. We claim that

$$(3.1.2) \quad M \xrightarrow{f} M \hat{\otimes}_A B \xrightarrow{g_1} M \hat{\otimes}_A B \hat{\otimes}_A B \xrightarrow{g_2} M \hat{\otimes}_A B \hat{\otimes}_A B$$

is a split equalizer (see [Sta20, Tag 08WH] for the definition). Indeed, we can take $h : M \hat{\otimes}_A B \xrightarrow{\theta^{-1}} M \hat{\otimes}_A B \to M$, where the last arrow uses the $B$-module
structure, and \(i : M \otimes_A B \otimes_A B \to M \hat{\otimes}_A B\) is induced by \(\text{id}_M\) and the multiplication \(B \otimes_A B \to B\), in order to split the equalizer. This means

\[
(3.1.3) \quad h \circ f = \text{id}_M, \quad f \circ h = i \circ g_1, \quad i \circ g_2 = \text{id}_{M \hat{\otimes}_A B}.
\]

Furthermore, we claim that the equalizer

\[
\ker(\theta - \iota) \xrightarrow{\iota} M \xrightarrow{\theta} M \hat{\otimes}_A B
\]

is split (in the category \(\text{Mod} \hat{\otimes}_A\)). We note that \(M \xrightarrow{\theta} M \hat{\otimes}_A B \xrightarrow{\text{id}_M \hat{\otimes}_A s} M\), which we denote by \(h'\), factors through \(\ker(\theta - \iota)\). Indeed, we have a commutative diagram

\[
\begin{array}{ccc}
M \hat{\otimes}_A B & \xrightarrow{g_1} & M \hat{\otimes}_A B \\
\text{id}_M \hat{\otimes}_A s & \downarrow & \downarrow \text{id}_M \hat{\otimes}_A s \\
M & \xrightarrow{\theta} & M \hat{\otimes}_A B.
\end{array}
\]

Therefore we may take \(h'\) as section \(M \to \ker(\theta - \iota)\). We use \(\text{id}_M \hat{\otimes}_A s : M \hat{\otimes}_A B \to M\) as the second splitting. The required identities are obvious.

Since split equalizers remain equalizers after application of the functor \(\hat{\otimes}_A\), the map \(\ker(\theta - \iota) \hat{\otimes}_A B \to M\) is an isomorphism. This shows that base extension is essentially surjective.

Now suppose \((M, \theta)\) is the base extension of an \(A\)-module \(M'\). By inspection of \(h'\), we conclude that the natural map \(M' \to \ker(\theta - \iota)\) is surjective. It is also injective, because \(M' \to M' \hat{\otimes}_A B, m' \mapsto m' \hat{\otimes} 1\), has \(\text{id}_{M'} \hat{\otimes}_A s\) as section. Hence it is an isomorphism. This finishes the proof.

\[\square\]

**Remark 3.1.3.** The functor \(\hat{\otimes}_A B\) is not exact in general, even if \(B\) is an \(I\)-completely flat \(A\)-algebra. We do not know whether there is descent for a \(I\)-completely faithfully flat \(A\)-algebra.

### 3.1.4. \(I\)-completely flat modules.

Recall from [BS], that a complex \(M\) of \(A\)-modules is \(I\)-completely flat if for any \(I\)-torsion \(A\)-module \(N\), the derived tensor product \(M \hat{\otimes}_A N\) is concentrated in degree 0. This implies in particular that \(M \hat{\otimes}_A^L A/I\) is concentrated in degree 0 and a flat \(A\)-module.

In this section we will only consider the case where \(I\) is generated by two elements. For more general results we refer to [Yek20]. The next proposition may very well follow from [Yek20].

**Proposition 3.1.5.** Suppose \(I\) is generated by two elements \(p, d\). And suppose that \(d\) is a non-zero divisor of \(A\). If \(M\) is a \((p, d)\)-completely flat complex of \(A\)-modules then the derived completion \(M^\wedge\) is \((p, d)\)-completely flat.

**Proof.** As a first step, suppose that \(A = A/d\) and \(I\) is principal and generated by \(p\). We claim that if \(M\) is a \(p\)-completely flat complex then \(M^\wedge\) is \(p\)-completely flat.

Let \(\text{cone}(p)\) be the cone of the multiplication by \(p\) endomorphism of \(A\). Because its cohomology is \(p\)-torsion, \(M \hat{\otimes}^L \text{cone}(p)\) is derived \(p\)-complete. Therefore we get

\[
(3.1.4) \quad \text{cone}(M^\wedge \overset{p}{\to} M^\wedge) \cong M \hat{\otimes}^L \text{cone}(p).
\]
Let $T$ be a $p$-torsion module. The complex $\text{cone}(p) \otimes^L T$ has only cohomology in degree $0, -1$. Since $M$ is $p$-completely flat, the same holds for $\text{cone}(p) \otimes^L T \otimes^L M$. Now (3.1.5) implies $H^i(M^\wedge \otimes^L T)$ for all $i \neq 0$, because $T$ is $p$-torsion.

Let us consider the case $I = (p, d)$ now. The $(p, d)$-complete flatness of $M$ implies the $p$-complete flatness of $M \otimes^L_A A/d$ as complex of $A/d$-modules. We already know that the derived $p$-completion $(M \otimes^L_A A/d)^{\wedge}$ is $p$-completed flat as complex of $A/d$-modules. Since $(M \otimes^L_A A/d)^{\wedge}$ is $d$-torsion, it is automatically $(p, d)$-complete. Therefore we have an isomorphism

(3.1.5) $\text{cone}(M^\wedge \stackrel{d}{\to} M^\wedge) \cong (M \otimes^L_A A/d)^{\wedge}$.

Let $T$ be a $(p, d)$-torsion $A$-module. We have

$$(M \otimes^L_A A/d)^{\wedge} \otimes^L_A T = (M \otimes^L_A A/d)^{\wedge} \otimes^L_A A/d \otimes^L_A/d T =\big((M \otimes^L_A A/d)^{\wedge} [-1] \oplus (M \otimes^L_A A/d)^{\wedge}\big) \otimes^L_A/d T,$$

and this complex has cohomology in degree $0, -1$ only. In view of (3.1.5), we conclude $H^i(M^\wedge \otimes^L_A T)$ for all $i \neq 0$.

Lemma 3.1.6. Let $A$ and $B$ be bounded prisms. Let $M$ be a derived $(p, d)$-complete and $(p, d)$-completely flat $A$-module. Then $M \otimes^L_A B$ is a $(p, d)$-completely flat $B$-module and it is the classical $(p, d)$-completion of $M \otimes^L_A B$.

Proof. The complex of $B$-modules $M \otimes^L_A B$ is $(p, d)$-completely flat, hence $(M \otimes^L_A B)^{\wedge}$ is $(p, d)$-completely flat by Proposition 3.1.6. Now, [BS] Lemma 3.7(2) can finish the proof. □

The next lemma has essentially the same proof.

Lemma 3.1.7. Let $A$ be a bounded prism. Let $M, N$ be derived $(p, d)$-complete and $(p, d)$-completely flat $A$-modules. Then $M \otimes^L_A N$ is $(p, d)$-completely flat and it is the classical $(p, d)$-completion of $M \otimes^L_A N$.

Lemma 3.1.8. Let $B$ be a derived $(p, d)$-complete and $(p, d)$-completely flat $A$-algebra. Suppose there is a section $B \to A$ in the category of $A$-modules. Let $M$ be a derived $(p, d)$-complete $A$-module. If $M \otimes^L_A B$ is $(p, d)$-completely flat as $B$-module then $M$ is $(p, d)$-completely flat.

Proof. Since $M$ is a direct summand of $M \otimes^L_A B$, it suffices to consider $(M \otimes^L_A B) \otimes^L_A T = (M \otimes^L_A B) \otimes^L_B (B \otimes^L_A T)$ for $(p, d)$-torsion $A$-modules $T$. Now, $B \otimes^L_A T \cong B \otimes_A T$ is a $(p, d)$-torsion $B$-module and we are done. □

References

[BO78] Pierre Berthelot and Arthur Ogus. Notes on Crystalline Cohomology. Princeton University Press and University of Tokyo Press, 1978.

[BS] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. arXiv:1905.08229.

[Gro68] A. Grothendieck. Crystals and the de Rham cohomology of schemes. In Dix exposés sur la cohomologie des schémas, volume 3 of Adv. Stud. Pure Math., pages 306–358. North-Holland, Amsterdam, 1968. Notes by I. Coates and O. Jussila.

[Sch] Peter Scholze. Canonical $q$-deformations in arithmetic geometry. arXiv:1606.01796v1.

[Sta20] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2020.

[Yek20] Amnon Yekutieli. Weak prorerunary, derived completion, adic flatness, and prisms, 2020. arXiv:2002.04901.