POLYNOMIALS WHOSE ROOTS AND CRITICAL POINTS ARE INTEGERS

JEAN-CLAUDE EVARD

Abstract. Polynomials whose coefficients, roots, and critical points lie in the ring of rational integers are called nice polynomials. In this paper, we present a general method for investigating such polynomials. We extend our results from the ring of rational integers to rings of algebraic integers that are unique factorization domains, with special interest in the ring of Gaussian integers. We apply our method to establish strong properties of nice polynomials whose degree is a prime power. We present a considerable reduction of the system of equations for nice polynomials of arbitrary degree with three roots. We establish properties of nice antisymmetric polynomials, and properties of the averages of the roots of the derivatives of nice polynomials. Finally, we present new examples of nice polynomials obtained with the help of a computer, after a considerable simplification of the computation by our method.

1. Introduction

The problem of finding properties, characterizations, and methods of construction of polynomials with coefficients, roots, and critical points in the ring of rational integers is on the list of unsolved problems published by Richard Nowakowski [10] in The American Mathematical Monthly in 1999. Such polynomials are called nice polynomials. The earliest paper on this subject was published by M. Chapple [5] in 1960. The most important paper was published by Ralph Buchholz and James MacDougall [2] in 2000. Their paper contains a comprehensive bibliography on the subject, where we just need to add the most recent continuation of their work published by Eugene Flynn [6] in 2001, and
the paper published by Johann Walter [12] in 1987. In the present paper, we establish a new method to deal with nice polynomials, and we present new properties and examples obtained with our method. While there is still a long way to go to find a complete description of the set of nice polynomials, it seems that many more results can be obtained in the near future with our method.

In Section 2, we present equivalences of nice polynomials, which reduce the search for nice polynomials to the search of just one representative in each equivalence class. In Section 3, we establish the key relations to deal with nice polynomials, namely, the system of equations (3.2), which are relations between the roots and the critical points of a polynomial. All of the results that follow are consequences of these relations. In Section 4, we establish properties of the averages of the roots of the derivatives of nice polynomials, with an application to nice cubics. In Section 5, we deal with nice symmetric and antisymmetric polynomials. In Section 6, we present the main result of this paper, namely, Theorem 6.3, which is about nice polynomials whose degree is a prime power, with an application to nice quartics. In Section 7, we present an important reduction of the system of equations (3.2) in the case of nice polynomials having three roots of any multiplicities. In Section 8, we present new examples of nice polynomials obtained by using our method with the help of a computer.

Notation, definitions, and terminology. We denote the set of rational integers 0, ±1, ±2, . . . by \( \mathbb{Z} \), the set of positive rational integers by \( \mathbb{N} \), the set of non-negative rational integers by \( \mathbb{N}_0 \), the set of Gaussian integers \( a + bi \), with \( a, b \in \mathbb{Z} \), \( i^2 = -1 \), by \( \mathbb{G} \), and the set of rational numbers by \( \mathbb{Q} \). For the sake of simplicity, we use the term polynomial for both, polynomial and polynomial function. Let \( \mathfrak{R} \) be a ring. We denote by \( \mathfrak{R}[x] \) the ring of polynomials in one indeterminate \( x \) with coefficients in \( \mathfrak{R} \). If \( \mathfrak{R} \) is commutative, then we say that a polynomial \( p \in \mathfrak{R}[x] \) of degree \( d > 0 \) splits over \( \mathfrak{R} \) if \( p \) has the form \( p(x) = c \prod_{i=1}^{d} (x - x_i) \), where \( c, x_1, \ldots, x_d \) belong to \( \mathfrak{R} \), and \( c \neq 0 \).

Let \( \mathfrak{R} \) be a ring of characteristic zero possessing a multiplicative identity element \( 1_{\mathfrak{R}} \). Let \( \Phi \) denote the ring homomorphism from \( \mathbb{Z} \) into \( \mathfrak{R} \) determined by \( \Phi(1) = 1_{\mathfrak{R}} \). Since \( \mathfrak{R} \) is of characteristic zero, the homomorphism \( \Phi \) gives a ring isomorphism from \( \mathbb{Z} \) onto \( \Phi(\mathbb{Z}) = \mathbb{Z} \cdot 1_{\mathfrak{R}} \), so that \( \mathbb{Z} \) can be considered as a subring of \( \mathfrak{R} \). For the sake of simplicity, we use the same notation \( n \) for an element \( n \) in \( \mathbb{Z} \) and its corresponding element \( \Phi(n) = n \cdot 1_{\mathfrak{R}} \) in \( \mathfrak{R} \); for example \( \pm 1 = \pm 1_{\mathfrak{R}}, 2 = 1_{\mathfrak{R}} + 1_{\mathfrak{R}}, \ldots \).

Let \( \mathfrak{D} \) be an integral domain of characteristic zero. Let \( p \in \mathfrak{D}[x] \) be a polynomial of degree \( d > 0 \) with coefficients \( c_0, \ldots, c_d \in \mathfrak{D} \), \( c_d \neq 0 \), that
is, \( p(x) = \sum_{k=0}^{d} c_k x^k \). We define the derivative \( p' \in D[x] \) of \( p \) as \( p'(x) = \sum_{k=1}^{d} c_k \Phi(k) x^{k-1} \), which we simply write \( p'(x) = \sum_{k=1}^{d} c_k k x^{k-1} \) with the above convention. Since \( D \) is of characteristic zero, we have \( d \neq 0 \) in \( D \). Since \( D \) is an integral domain, it follows that \( c_d \cdot d \neq 0 \) in \( D \), so that \( p' \) is of degree \( d - 1 \).

Since \( D \) is an integral domain, \( D \) possesses a (unique) field of fractions that we denote by \( \overline{Q(D)} \). If \( d \geq 2 \), then both \( p \) and \( p' \) split in that algebraic closure:

\[
p(x) = c \prod_{i=1}^{d} (x - x_i) \quad \text{and} \quad p'(x) = c' \prod_{j=1}^{d-1} (x - x'_j),
\]

where \( c, c' \) are in \( D \), and \( x_1, \ldots, x_d, x'_1, \ldots, x'_{d-1} \) are in \( \overline{Q(D)} \). Since \( \overline{Q(D)} \) is a field, we have that the ring \( \overline{Q(D)}[x] \) of polynomials over this field is an Euclidean domain, and therefore, it is a unique factorization domain, which we abbreviate as UFD. Therefore, the two decompositions (1.1) are unique up to a change of the order of the factors and multiplication of the factors by units. Consequently, the roots \( x_1, \ldots, x_d \) of \( p \) and the roots \( x'_1, \ldots, x'_{d-1} \) of \( p' \) are unique in the unique field \( \overline{Q(D)} \). Because of this, we can talk about the roots of \( p \) and the roots of \( p' \). For the sake of simplicity, we will talk about the \( d \) roots of \( p \) and the \( d - 1 \) roots of \( p' \), even if some of them are equal, that is, we will omit to specify repeated according to their multiplicities. As usual, we call the roots of \( p' \) the critical points of \( p \). We say that \( p \) is nice if and only if both \( p \) and \( p' \) split over \( D \), that is to say, iff the \( d \) roots and the \( d - 1 \) critical points of \( p \) are in \( D \). We say that \( p \) is totally nice iff \( p, p', \ldots, p^{(d-1)} \) split over \( D \).

2. Nicety preserving transformations

When we establish lists of nice polynomials, we do not want to write several times the "same" polynomial. We say that two nice polynomials are equivalent if one of them can be obtained from the other by one of the transformations described in Proposition 2.1 below. These transformations were already mentioned in [2] and [3] in a different setting. In Proposition 2.1 we present transformations which transform nice
polynomials into nice polynomials. In Corollary 22, we consider transformations which transform all nice polynomials into nice polynomials, and which have an inverse transformation with the same property.

Proposition 2.1. [Nicety preserving transformations]. Let $\mathcal{D}$ be an integral domain of characteristic zero. Let $p \in \mathcal{D}[x]$ be a nice polynomial of degree $d \geq 2$. Then we obtain another nice polynomial $q \in \mathcal{D}[x]$ of degree $d$ from $p$ in the following three cases:

(a) $q(x) = ap(ux - b)$, when $a, b, u \in \mathcal{D}$, $a \neq 0$, and $u$ is a unit.
(b) $q(x) = a^dp(a^{-1}x)$, when $a$ is a nonzero element of $\mathcal{D}$, and $a^{-1}$ denotes its multiplicative inverse in $\mathbb{Q}(\mathcal{D})$.
(c) $q(x) = p(ax)$, when $a \in \mathcal{D}$ is a common divisor of the roots and critical points of $p$.

Proof. Since $p$ is nice, we have that $p$ and $p'$ split over $\mathcal{D}$, that is

$$p(x) = c \prod_{i=1}^{d} (x - x_i) \quad \text{and} \quad p'(x) = c' \prod_{j=1}^{d-1} (x - x'_j),$$

where $c, c', x_1, \ldots, x_d, x'_1, \ldots, x'_{d-1}$ are in $\mathcal{D}$.

(a) Since $u$ is unit of $\mathcal{D}$, $u$ possesses a multiplicative inverse $u^{-1} \in \mathcal{D}$. Then

$$q(x) = ap(ux - b) = ac \prod_{i=1}^{d} [(ux - b) - x_i]$$

$$= ac \prod_{i=1}^{d} \left[ u \left[ x - u^{-1}(b + x_i) \right] \right] = acu \prod_{i=1}^{d} \left[ x - u^{-1}(b + x_i) \right].$$

By the well-known properties of the algebraic derivative, we get

$$q'(x) = ap'(ux - b)u = auc \prod_{j=1}^{d-1} [(ux - b) - x'_j]$$

$$= auc \prod_{j=1}^{d-1} \left[ u \left[ x - u^{-1}(b + x'_j) \right] \right] = au^d c \prod_{j=1}^{d-1} \left[ x - u^{-1}(b + x'_j) \right].$$

Thus both $q$ and $q'$ split over $\mathcal{D}$, that is to say, $q$ is nice.

(b) We have

$$q(x) = a^d c \prod_{i=1}^{d} (a^{-1}x - x_i) = a^d c \prod_{i=1}^{d} \left[ a^{-1}(x - ax_i) \right] = c \prod_{i=1}^{d} (x - ax_i),$$

Thus, $q(x)$ is a polynomial of degree $d$ in $\mathcal{D}[x]$ which is nice.
which shows that \( q \in \mathcal{D}[x] \), and \( q \) splits over \( \mathcal{D} \). By the well-known properties of the algebraic derivative, we get
\[
q'(x) = adp'(a^{-1}x)a^{-1} = ad^{-1}c' \prod_{j=1}^{d-1}(a^{-1}x - x_j')
\]
\[
= ad^{-1}c' \prod_{j=1}^{d-1} [a^{-1}(x - ax'_j)] = c' \prod_{j=1}^{d-1}(x - ax'_j).
\]
Thus \( q' \in \mathcal{D}[x] \), and \( q' \) splits over \( \mathcal{D} \).

\( \textbf{(c)} \) By hypothesis, there exist \( y_1, \ldots, y_d, y'_1, \ldots, y'_{d-1} \) in \( \mathcal{D} \) such that \( x_1 = ay_1, \ldots, x_d = ay_d, x'_1 = ay'_1, \ldots, x'_{d-1} = ay'_{d-1} \). Consequently,
\[
q(x) = c \prod_{i=1}^{d}(ax - x_i) = c \prod_{i=1}^{d} [a(x - y_i)] = adc \prod_{i=1}^{d}(x - y_i).
\]
By the well-known properties of the algebraic derivative, we get
\[
q'(x) = p'(ax)a = ac' \prod_{j=1}^{d-1}(ax - x'_j) = ac' \prod_{j=1}^{d-1} [a(x - y'_j)] = adc' \prod_{j=1}^{d-1}(x - y'_j).
\]
Thus both \( q \) and \( q' \) split over \( \mathcal{D} \). \( \blacksquare \)

**Corollary 2.2.** [Nicety preserving transformations whose inverses are also nicety preserving]. Let \( \mathcal{D} \) be an integral domain of characteristic zero, let \( p \in \mathcal{D}[x] \) be a polynomial of degree \( d \geq 2 \), let \( u_1 \) and \( u_2 \) be units of \( \mathcal{D} \), let \( a \in \mathcal{D} \), and let \( q(x) = u_1p(u_2x + a) \). Then \( p \) is nice iff \( q \) is nice. In particular, this equivalence holds when \( q(x) = p(-x) \).

\( \textbf{Proof.} \) By hypothesis, \( u_1 \) and \( u_2 \) possess multiplicative inverses \( u_1^{-1} \) and \( u_2^{-1} \) in \( \mathcal{D} \). Let \( y = u_2x + a \). Then \( x = u_2^{-1}(y - a) \), so that \( q(x) = u_1p(u_2x + a) \) gives \( p(y) = u_1^{-1}q(u_2^{-1}y - u_2^{-1}a) \), and the conclusion follows by Proposition 2.1(a). \( \blacksquare \)

## 3. Relations between the roots and the critical points of a polynomial

In Proposition 3.1, we recall the well-known relations between the roots and the coefficients of a polynomial. In Corollary 3.2, we deduce a system of relations between the roots and the critical points of a polynomial, namely, Equations (3.2), which is the key tool for dealing with nice polynomials.
Notation. Let $\mathfrak{R}$ be a commutative ring. For every positive integer $k$, we define the $k^{\text{th}}$ elementary symmetric polynomial $s_k$ in the $m$ indeterminates $x_1, \ldots, x_m$ by
\[
s_k(x_1, \ldots, x_m) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} x_{i_1} \cdots x_{i_k},
\]
and we define $s_0(x_1, \ldots, x_m) = 1$.

Let us recall the following well-known proposition, which is due to François Viète (1540–1603) for polynomials of low degrees, and to Albert Girard (1595–1632) for the general case [11, Equation (1.2.3) p. 6, and p. 63].

**Proposition 3.1.** [Expression of the coefficients of a polynomial in terms of its roots]. Let $\mathfrak{R}$ be a commutative ring. Let $p \in \mathfrak{R}[x]$ be a polynomial of nonzero degree $n$ which splits over $\mathfrak{R}$, that is
\[
p(x) = \sum_{k=0}^{n} c_k x^k = c_n \prod_{i=1}^{n} (x - x_i),
\]
where $c_0, \ldots, c_n, x_1, \ldots, x_n$ are in $\mathfrak{R}$, and $c_n \neq 0$. Then for every $k \in \{0, \ldots, n\}$, we have
\[
c_k = (-1)^{n-k} c_n s_{n-k}(x_1, \ldots, x_n).
\]

**Corollary 3.2.** [Relations between the roots and the critical points of a polynomial]. Let $\mathfrak{D}$ be an integral domain of characteristic zero. Let $p$ and $q \in \mathfrak{D}[x]$ be polynomials of degrees $d \geq 2$ and $d - 1$, respectively, that split over $\mathfrak{D}$, that is
\[
p(x) = \sum_{k=0}^{d} c_k x^k = c_d \prod_{i=1}^{d} (x - x_i),
\]
\[
q(x) = \sum_{k=0}^{d-1} c'_k x^k = c'_{d-1} \prod_{i=1}^{d-1} (x - x'_i),
\]
where $c_0, \ldots, c_d, c'_0, \ldots, c'_{d-1}, x_1, \ldots, x_d, x'_1, \ldots, x'_{d-1}$ are in $\mathfrak{D}$, and $c_d \neq 0, c'_{d-1} \neq 0$. Then $q$ is the derivative of $p$ if and only if $c'_{d-1} = dc_d$, and for every $k \in \{1, \ldots, d-1\}$,
\[
(d - k)s_k(x_1, \ldots, x_d) = ds_k(x'_1, \ldots, x'_{d-1}).
\]

**Proof.** By substituting $k = j - 1$ into (3.1), we get that the relation $q = p'$ is equivalent to $\sum_{j=1}^{d} c'_{j-1} x^{j-1} = \sum_{j=1}^{d} c_j j x^{j-1}$, which implies that
\[
c'_{j-1} = jc_j, \quad \forall j \in \{1, \ldots, d\}.
\]
For every $k \in \{0, \ldots, d - 1\}$, let
\[
S_k = s_k(x_1, \ldots, x_d) \quad \text{and} \quad S'_k = s_k(x'_1, \ldots, x'_{d-1}).
\]
By applying Proposition 3.1 to both sides of Equation (3.3), observing that $(d - 1) - (j - 1) = (d - j)$, we obtain that $q = p'$ if and only if
\[
(-1)^{d-j} c'_{d-1} S'_{d-j} = j (-1)^{d-j} c_d S_{d-j}, \quad \forall j \in \{1, \ldots, d\},
\]
that is, with $k = d - j$,
\[
c'_{d-1} S'_k = (d - k) c_d S_k, \quad \forall k \in \{0, \ldots, d - 1\},
\]
that is, when $k = 0$, $c'_{d-1} = dc_d$, and otherwise
\[
dc_d S'_k = (d - k) c_d S_k, \quad \forall k \in \{1, \ldots, d - 1\}.
\]
Since $c_d \neq 0$ and $\mathfrak{D}$ is an integral domain, we can simplify both sides of this relation by $c_d \neq 0$, which gives the conclusion. \qed

4. Averages of the roots

In Proposition 4.1 we present properties and relations between the averages of the roots of polynomials and their derivatives of all orders, and in Corollary 4.2 we present an application to nice cubics.

**Definition.** Let $\mathfrak{D}$ be an integral domain of characteristic zero. Let $p \in \mathfrak{D}[x]$ be a polynomial of degree $n \geq 1$. Then $p$ splits over $\overline{Q(\mathfrak{D})}$, that is, $p(x) = c \prod_{i=1}^{n} (x - x_i)$, where $c \in \mathfrak{D}$, and $x_1, \ldots, x_n$ are in $\overline{Q(\mathfrak{D})}$. Since $\mathfrak{D}$ is of characteristic zero, we have that $n \neq 0$ in $\mathfrak{D}$, which allows us to define the average $A(p)$ of the roots of $p$, as
\[
A(p) = \frac{x_1 + x_2 + \cdots + x_d}{n} \in \overline{Q(\mathfrak{D})}.
\]

Another version of the following proposition, valid for totally nice polynomials, can be found in [4].

**Proposition 4.1.** [The average of the roots of any polynomial is equal to the average of the roots of each of its derivatives, this common average is rational even if some of the roots are irrational, and if the polynomial is nice, then this common average is an integer.] Let $\mathfrak{D}$ be an integral domain of characteristic zero. Let $p \in \mathfrak{D}[x]$ be a polynomial of degree $d \geq 2$. Let $r \in \overline{Q(\mathfrak{D})}$ denote the only root of $p^{(d-1)}$. Then
\[
A(p) = A(p') = A(p'') = \cdots = A(p^{(d-1)}) = r \in \overline{Q(\mathfrak{D})}.
\]
Furthermore, if $\mathfrak{D}$ is a UFD and $p$ is nice, then $r \in \mathfrak{D}$. 
Proof. Since \( p \) and \( p' \) split in \( \overline{\mathbb{Q}(D)} \), they have the form

\[
p(x) = c \prod_{i=1}^{d} (x - x_i), \quad p'(x) = c' \prod_{i=1}^{d-1} (x - x'_i),
\]

where \( c, c' \in D \), and \( x_1, \ldots, x_d, x'_1, \ldots, x'_{d-1} \in \overline{\mathbb{Q}(D)} \). Let

\[
S_1 = s_1(x_1, \ldots, x_d) \quad \text{and} \quad S'_1 = s_1(x'_1, \ldots, x'_{d-1}).
\]

By Equation (3.2), we have \((d-1)S_1 = dS'_1\), which gives \( S_1 = \frac{S'_1}{d} \), that is \( A(p) = A(p') \). Thus, the averages of the roots of a polynomials and the roots of its first derivative are equal. It follows by induction that the averages of the roots of all of the derivatives of \( p \) are equal. Since the polynomial \( p^{(d-1)} \in \mathbb{D}[x] \) is of degree one, it has the form \( p(x) = ax + b \), with \( a, b \in \mathbb{D} \) and \( a \neq 0 \). Consequently, \( A(p^{(d-1)}) = r = -\frac{b}{a} \in \mathbb{Q}(D) \).

Assume that \( \mathbb{D} \) is a UFD and \( p \) is nice. Then \( x_1, \ldots, x_d, x'_1, \ldots, x'_{d-1} \) are in \( \mathbb{D} \), and so are \( S_1 \) and \( S'_1 \). Since every common divisor of \( d \) and \( (d-1) \) in \( \mathbb{D} \) divides the difference \( d - (d-1) = 1 \), we have that \( d \) and \( (d-1) \) do not have any common prime divisors in \( \mathbb{D} \). This together with the equation \((d-1)S_1 = dS'_1 \) in \( \mathbb{D} \) imply that all of the prime factors of the prime decomposition of \( d \) are in the prime decomposition of \( S_1 \). Consequently, \( d \) divides \( S_1 \), so that \( r = A(p) = \frac{S_1}{d} \in \mathbb{D} \).

**Corollary 4.2.** Let \( \mathbb{D} \) be a UFD of characteristic zero. Then every nice cubic over \( \mathbb{D} \) is totally nice.

**Proof.** Let \( p \) be a nice cubic over \( \mathbb{D} \). Then by definition, \( p \) and \( p' \) split over \( \mathbb{D} \). Furthermore, by Proposition the only root of \( p'' \) is in \( \mathbb{D} \), that is to say, \( p'' \) also splits over \( \mathbb{D} \), so that \( p \) is totally nice.

### 5. Nice symmetric and antisymmetric polynomials

This section deals with symmetric and antisymmetric nice polynomials. It is restricted to polynomials with roots in \( \mathbb{Z} \), because we use Rolle’s theorem, considering such polynomials as real-valued functions of a real variables. For example, if \( p \) is a symmetric polynomial of degree \( d \geq 2 \), with real roots, and center \( C \) of the set of its roots, and if \( C \) is not a root of \( p \), then \( p' \) is antisymmetric with center \( C \), consequently, \( C \) is a root of \( p' \), and the multiplicity of this root is one because of Rolle’s theorem and the fact that \( p' \) is of degree \( (d-1) \). This is not true for a symmetric polynomial with complex roots (See Example B4 in Section 8). Theorem 5.1 gives necessary conditions for the existence of nice antisymmetric polynomials. Corollary 5.2 deals with the special case where the center of a nice antisymmetric polynomial is a root.
of multiplicity one of this polynomial. Corollary 5.3 shows that such polynomials cannot be totally nice.

**Definitions.** Let \( \mathfrak{R} \) be a ring, and let \( p \in \mathfrak{R}[x] \) be a non-constant polynomial. We say that \( p \) is **symmetric** iff there exists \( C \in \mathfrak{R} \) such that \( p(C - a) = p(C + a) \) for all \( a \in \mathfrak{R} \). We call \( C \) the **center** of \( p \). We say that \( p \) is **antisymmetric** iff there exists \( C \in \mathfrak{R} \) such that \( p(C - a) = -p(C + a) \) for all \( a \in \mathfrak{R} \). We call \( C \) the center of \( p \).

**Theorem 5.1.** Let \( p \in \mathbb{Z}[x] \) be a nice antisymmetric polynomial of degree \( d \geq 3 \) and center \( C \) with more than one root. Let \( m_0 \) denote the multiplicity of the root \( C \) of \( p \). Let \( g = \gcd(m_0, d) \). Then the integers \( m_0 \cdot \frac{d}{g} \) and \( \frac{d}{g} \) are odd squares.

**Proof.** Let \( q(x) = p(x + C) \). Then \( q \) is a nice over \( \mathbb{Z} \) by Proposition 2.11 (a), and it is easy to check that \( q \) is antisymmetric with center zero. Consequently, \( r \in \mathbb{Z} \) is a root of \( q \) iff \(-r\) is also a root of \( q \) with same multiplicity. Since \( C \) is a root of \( p \) of multiplicity \( m_0 \), we have that zero is a root of \( q \) of multiplicity \( m_0 \). Since \( p \) has more than one root, we have that the number \( N \) of positive roots of \( q \) is greater than zero. It follows that \( q \) has the form

\[
q(x) = cx^{m_0} \prod_{i=1}^{N} (x^2 - x_i^2)^{m_i},
\]

where \( c \in \mathbb{Z}, x_1, \ldots, x_N \in \mathbb{N} \) are distinct, and \( m_1, \ldots, m_N \) are such that \( d = m_0 + 2m_1 + \cdots + 2m_N \). Since \( q \) is an odd function, we know from Taylor polynomials that all of the terms of \( q \) are of odd degrees, which implies that \( d \) and \( m_0 \) are odd. Furthermore, all of the terms of \( q' \) are of even degree, so that \( q' \) is symmetric with center zero. Consequently, \( r' \in \mathbb{Z} \) is a root of \( q' \) iff \(-r'\) is also a root of \( q' \) with the same multiplicity. It is well known from Algebra that each root of \( q \) of multiplicity \( m \) gives a root of \( q' \) of multiplicity at least \((m - 1)\). In addition, since \( q \) has \((2N + 1)\) distinct roots in \( \mathbb{Z} \), we have by virtue of Rolle’s theorem that \( q' \) has \( 2N \) additional roots. Taking into account that the degree of \( q' \) is \((d - 1)\), we obtain that \( q' \) has the form

\[
q'(x) = cdx^{m_0-1} \prod_{i=1}^{N} (x^2 - x_i^2)^{m_i-1} \prod_{j=1}^{N} (x_j^2 - x_j'^2),
\]

where \( x_1', \ldots, x_N' \in \mathbb{N} \) are distinct. By applying Equation (3.2) with \( k = d - m_0 \), we obtain

\[
m_0 \cdot \prod_{i=1}^{N} x_i^{m_i} \cdot (-x_i)^{m_i} = d \cdot \prod_{i=1}^{N} x_i^{m_i-1} \cdot (-x_i)^{m_i-1} \cdot \prod_{j=1}^{N} x_j' \cdot (-x_i').
\]
Let \( \bar{m} = \frac{m_0}{g} \) and \( \bar{d} = \frac{d}{g} \). Then the above equation becomes

\[
\bar{m} \left( \prod_{i=1}^{N} x_i^{m_i} \right) = \bar{d} \left( \prod_{i=1}^{N} x_i^{m_i-1} \prod_{j=1}^{N} x'_i \right).
\]

Since \( \bar{m} \) and \( \bar{d} \) have no common prime factor, the above relation implies that all of the prime factors of the prime decompositions of \( \bar{m} \) and \( \bar{d} \) are raised to even powers, which implies that \( \bar{m} \) and \( \bar{d} \) are squares. Since \( m_0 = gm \) and \( d = gd \) are odd, we have that \( \bar{m} \) and \( \bar{d} \) are odd too. \( \square \)

**Corollary 5.2.** (a) Let \( p \in \mathbb{Z}[x] \) be a nice antisymmetric polynomial of degree \( d \geq 3 \) whose center \( C \in \mathbb{Z} \) is a root of \( p \) of multiplicity one. Then \( d \) is an odd square.

(b) Let \( p \in \mathbb{Z}[x] \) be a symmetric polynomial of degree \( d \geq 4 \) whose center \( C \in \mathbb{Z} \) is not a root of \( p \), and whose derivative is nice. Then \( (d - 1) \) is an odd square.

**Proof.** (a) This is a direct consequence of Theorem 5.1 applied with \( m_0 = 1 \).

(b) By using that the derivative of a symmetric polynomial is antisymmetric, using the hypothesis that \( C \) is not a root of \( p \), using that the center is a root of all antisymmetric polynomials, using that the degree of \( p' \) is \( d - 1 \), and using Rolle’s theorem, we deduce that \( p' \) is antisymmetric, and its center is a root of \( p' \) of multiplicity one, which implies (b) by (a). \( \square \)

**Existence of non-trivial nice antisymmetric polynomials:** In Example A 2 of Section 8, we provide examples of nice antisymmetric polynomials of degree \( d \), as described in Corollary 5.2 (a), for every odd square \( d \geq 9 \). In Example A 3, we provide examples of non-trivial nice symmetric polynomials of degree \( n^2 + 1 \) whose derivative is also nice for every odd positive integer \( n \).

**Corollary 5.3.** (a) Let \( p \in \mathbb{Z}[x] \) be a nice antisymmetric polynomial of degree \( d \geq 5 \) whose center \( C \) is a root of \( p \) of multiplicity one. Then \( p'' \) is not nice.

(b) Let \( p \in \mathbb{Z}[x] \) be a symmetric polynomial of degree \( d \geq 6 \) whose center \( C \) is not a root of \( p \), and whose derivative is nice. Then \( p''' \) is not nice.

**Proof.** (a) Since \( p \) is antisymmetric of degree \( d \) with center \( C \), we have that \( p' \) is symmetric of degree \( d - 1 \) with same center, and \( p'' \) is antisymmetric of degree \( d - 2 \) with same center. It is easy to check using
Rolle’s theorem that the hypothesis implies that \( C \) is not a root of \( p' \), and \( C \) is a root of \( p'' \) of multiplicity one. If \( p'' \) were nice, then both \( d \) and \( d - 2 \) would be squares by Corollary 5.2 (a), which is impossible, because the distance between two nonzero distinct squares is at least three. Therefore, \( p'' \) is not nice.

(b) The hypothesis implies that \( p' \) is a nice antisymmetric polynomial of degree \( d \geq 5 \) whose center \( C \in \mathbb{Z} \) is a root of \( p \) of multiplicity one. This implies the conclusion by (a).

\[ \Box \]

6. Nice polynomials whose degree is a prime-power

In this section, we present the main result of this paper, which is about nice polynomials whose degree is a prime-power. This result holds for polynomials whose roots and critical points belong to a UFD \( \mathcal{D} \) of characteristic zero such that \( \mathcal{D} \cap \mathbb{Q} = \mathbb{Z} \) in \( \mathbb{Q}(\mathcal{D}) \). Proposition 6.1 shows that this condition is satisfied by every free integral domain, and in particular, by every ring of algebraic integers of an algebraic number field. Lemma 6.2 is the key lemma to extend our main theorem from the ring \( \mathbb{Z} \) of rational integers to rings of algebraic integers that are UFDs. Theorem 6.3 about nice polynomials of degree \( d = b^e \), where \( b \) is a prime, is the main result of this paper, with a stronger conclusion when the base \( b \) divides the exponent \( e \). Corollary 6.4 presents an important application to nice quartics.

Definitions. Let \( \mathcal{D} \) be an integral domain. We say that a subset \( \mathcal{B} \) of \( \mathcal{D} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{D} \) iff every element \( a \) of \( \mathcal{D} \) can be expressed in a unique way as a finite \( \mathbb{Z} \)-linear combination of elements of \( \mathcal{B} \), that is, iff for every \( a \in \mathcal{D} \), there exists a unique family \( (a_\beta)_{\beta \in \mathcal{B}} \) of elements of \( \mathbb{Z} \), with only a finite number of nonzero elements, such that \( a = \sum_{\beta \in \mathcal{B}} a_\beta \beta \). We say that \( \mathcal{D} \) is free iff \( \mathcal{D} \) possesses a \( \mathbb{Z} \)-basis.

Let \( \mathcal{D} \) be an integral domain of characteristic zero. Then the ring homomorphism \( \Phi \) from \( \mathbb{Z} \) into \( \mathcal{D} \) determined by \( \Phi(1) = 1_\mathcal{D} \) can be extended to a field homomorphism from \( \mathbb{Q} \) to \( \mathcal{Q}(\mathcal{D}) \). Furthermore, since \( \mathcal{D} \) is of characteristic zero, the homomorphism \( \Phi \) gives a ring isomorphism from \( \mathbb{Z} \) onto \( \Phi(\mathbb{Z}) = \mathbb{Z} \cdot 1_\mathcal{D} \), and a field isomorphism from \( \mathbb{Q} \) onto \( \Phi(\mathbb{Q}) = \mathbb{Q} \cdot 1_\mathcal{D} \). Consequently, \( \mathbb{Z} \) can be considered as a subring of \( \mathcal{D} \), \( \mathbb{Q} \) can be considered as a subfield of \( \mathcal{Q}(\mathcal{D}) \), and both \( \mathbb{Z} \) and \( \mathbb{Q} \) can be considered as subrings of \( \mathcal{Q}(\mathcal{D}) \).

Proposition 6.1. Every free integral domain \( \mathcal{D} \) is a ring of characteristic zero satisfying the condition \( \mathcal{D} \cap \mathbb{Q} = \mathbb{Z} \) in \( \mathcal{Q}(\mathcal{D}) \). In particular,
every ring of algebraic integers of an algebraic number field has these two properties.

Proof. Let $\mathcal{D}$ be an integral domain with a $\mathbb{Z}$-basis $\mathcal{B}$. Assume that $\mathcal{D}$ has nonzero characteristic $n \in \mathbb{N}$. Then $na = 0$ for all in $a \in \mathcal{D}$. Since $1 \neq 0$ in the integral domain $\mathcal{D}$, the basis $\mathcal{B}$ has at least one element $\beta_0$. Then $\beta_0 = 1\beta_0 = (1 + n)\beta_0$, so that $\beta_0$ can be expressed in two different ways as a finite $\mathbb{Z}$-linear combination of elements of the basis $\mathcal{B}$. This contradicts the definition of a basis. Therefore $\mathcal{D}$ has characteristic zero.

Clearly $\mathbb{Z} \subseteq \mathcal{D} \cap \mathbb{Q}$. Let us prove that $\mathcal{D} \cap \mathbb{Q} \subseteq \mathbb{Z}$. Let $c \in \mathcal{D} \cap \mathbb{Q}$. Then there exist $a, b \in \mathbb{Z}$ with $b \neq 0$ such that $c = \frac{a}{b}$. By the expression of $(c\beta_0)$ in the $\mathbb{Z}$-basis $\mathcal{B}$, there exists a unique family $(n_\beta)_{\beta \in \mathcal{B}}$ of elements of $\mathbb{Z}$, with only a finite number of nonzero elements, such that $c\beta_0 = \sum_{\beta \in \mathcal{B}} n_\beta \beta$. Since $a, b, n_{\beta_0}$ are in $\mathbb{Z}$, it follows that $a = bn_{\beta_0}$, that is $bc = bn_{\beta_0}$. Since $\mathcal{D}$ is of characteristic zero and $b \neq 0$ in $\mathbb{Z}$, we have that $b \neq 0$ in $\mathcal{D}$. Consequently, $bc = bn_{\beta_0}$ implies that $c = n_{\beta_0} \in \mathbb{Z}$. Thus, $\mathcal{D} \cap \mathbb{Q} = \mathbb{Z}$.

Let $\mathfrak{F}$ be an algebraic number field. Let $\mathcal{D}_\mathfrak{F}$ denote the ring of algebraic integers of $\mathfrak{F}$. Then $\mathcal{D}_\mathfrak{F}$ is an integral domain, because it is a subring of the field $\mathfrak{F}$. Furthermore, $\mathcal{D}_\mathfrak{F}$ is free by [8, Theorem 1.69, p.39], and the conclusion follows by the first part of this proof.

**Lemma 6.2.** [Key lemma for the extension of the main theorem from the ring $\mathbb{Z}$ to rings of algebraic integers] Let $\mathcal{D}$ be an integral domain of characteristic zero satisfying $\mathcal{D} \cap \mathbb{Q} = \mathbb{Z}$. Let $a, b, e \in \mathbb{N}$ be such that $a < b^e$. Let $h \in \mathbb{N}_0$ denote the largest exponent such that $b^h$ divides $a$ in $\mathcal{D}$. Then $h < e$.

**Proof.** By definition of $h$, there exists $c \in \mathcal{D}$ such that $a = b^h c$ in $\mathcal{D}$. Then $c = \frac{a}{b^h} \in \mathcal{D} \cap \mathbb{Q} = \mathbb{Z}$, so that $c \in \mathbb{Z}$. Since the ring $\mathcal{D}$ is of characteristic zero, the relation $a = b^h c$ in $\mathbb{Z} \cdot 1_{\mathcal{D}}$ implies that the same relation holds in $\mathbb{Z}$. Since $a$ and $b$ are positive integers, the relation $a = b^h c$ in $\mathbb{Z}$ implies that the integer $c$ is positive, which implies that $c \geq 1$. Consequently, we have $b^h \leq b^h c = a < b^e$ in $\mathbb{N}$, which implies the conclusion.

**Theorem 6.3.** [Main theorem: Properties of nice polynomials whose degree is a power of a prime] Let $\mathcal{D}$ be a UFD of characteristic zero satisfying $\mathcal{D} \cap \mathbb{Q} = \mathbb{Z}$. Let $b, e \in \mathbb{N}$ be such that $b$ is prime in $\mathcal{D}$. Let $d = b^e$. Let $p \in \mathcal{D}[x]$ be a nice polynomial of degree $d$ with a root at zero. Then the roots and critical points of $p$ have the following properties:
(a) All of the roots of $p$ are multiples of $b$.

(b) At most one of the critical points of $p$ are not multiples of $b$.

(c) In the special case where zero is a root of $p$ of multiplicity one, and $b$ divides $e$ in $\mathcal{D}$, we have the following additional two properties:

(c1) At least one nonzero root of $p$ is a multiple of $b^2$.

(c2) At most $e - 1$ of the critical points of $p$ are not multiples of $b$.

Proof. By hypothesis, $p$ and $p'$ have the form

$$p(x) = cx \prod_{i=1}^{d-1} (x - x_i), \quad p'(x) = c' \prod_{j=1}^{d-1} (x - x'_j),$$

where $c, c' \in \mathcal{D}$, the roots $x_1, \ldots, x_{d-1} \in \mathcal{D}$ of $p$ are numbered in such a way that there exists $m \in \{0, \ldots, d - 1\}$ such that for all $i \in \{1, \ldots, d - 1\}$, $x_i$ is a multiple of $b$ if and only if $i \leq m$, and the critical points $x'_1, \ldots, x'_{d-1} \in \mathcal{D}$ of $p$ are numbered in such a way that there exists $n \in \{0, \ldots, d - 1\}$ such that for all $j \in \{1, \ldots, d - 1\}$, $x'_j$ is a multiple of $b$ if and only if $j \leq n$. For every $k \in \{1, \ldots, d - 1\}$, let

$$S_k = s_k(x_1, \ldots, x_{d-1}, 0) = s_k(x_1, \ldots, x_{d-1}), \quad S'_k = s_k(x'_1, \ldots, x'_{d-1}).$$

Then by Equation (6.2),

$$(d - k)S_k = dS'_k, \quad \forall k \in \{1, \ldots, d - 1\}. \quad (6.1)$$

When $k = d - 1$, this gives

$$x_1 \cdots x_{d-1} = b^e x'_1 \cdots x'_{d-1}. \quad (6.2)$$

Since $b$ is prime in the UFD $\mathcal{D}$, the above equation implies that $b$ divides at least one of the roots of $p$, that is, $m \geq 1$. By definition of $m$, there exist $a_1, \ldots, a_m \in \mathcal{D}$ such that

$$x_1 = ba_1, \ldots, x_m = ba_m. \quad (6.3)$$

(a) Let us prove that $m = d - 1$. Assume that $m < d - 1$. Let $h \in \mathbb{N}_0$ be the highest exponent such that $b^h$ divides $(m + 1)$ in $\mathcal{D}$, and let $r = e - h$. Since $m + 1 < d = b^r$, we have by Lemma 6.2 that $h < e$, so that $r > 0$. Furthermore, by definition of $h$ there exists $c \in \mathcal{D}$ such that $(m + 1) = b^hc$ and $b$ does not divide $c$. By applying Equation (6.1) with $k = d - 1 - m$, we obtain $(m + 1)S_k = b^e S'_k$, that is $b^h c S_k = b^e S'_k$, that is $c S_k = b^e S'_k$, which gives

$$c \left[ x_1 s_{k-1}(x_2, \ldots, x_{d-1}) + x_2 s_{k-1}(x_3, \ldots, x_{d-1}) + \cdots + x_m s_{k-1}(x_{m+1}, \ldots, x_{d-1}) + s_k(x_{m+1}, \ldots, x_{d-1}) \right] = b^e S'_k. \quad (6.4)$$
Since $k = d - 1 - m$, we have $s_k(x_{m+1}, \ldots, x_{d-1}) = x_{m+1} \cdots x_{d-1}$. Substituting this and (6.3) into (6.4) gives

$$c x_{m+1} \cdots x_{d-1} = b \left[ b^{r-1} S_k^{(r)} - c a_1 s_{k-1}(x_2, \ldots, x_{d-1}) \right]$$

Since $b$ does not divide $c$, and $r$ is a positive integer, it follows that $b$ divides at least one of the roots $x_{m+1}, \ldots, x_{d-1}$, which contradicts the definition of $m$. Therefore $m < d - 1$ is impossible, so that we do have $m = d - 1$, which implies (a).

(b) If $n = d - 1$, then (b) is true. Assume that $n \leq d - 2$. Let $s = d - 1 - e$. Let us prove that $n \geq s$. Assume that $n < s = d - 1 - e$. Then $d > n + 1 + e \geq 0 + 1 + 1 = 2$, so that $d \geq 3$. Substituting (6.3) into Equation (6.2), knowing from (a) that $m = d - 1$, gives

$$b^{d-1} a_1 \cdots a_{d-1} = b^n x_1' \cdots x_{d-1}'$$

that is

$$b^n a_1 \cdots a_{d-1} = x_1' \cdots x_{d-1}'.$$  \hspace{1cm} (6.5)

Let us show that $s \geq 1$. If $e = 1$, then $s = d - 1 - e \geq 3 - 1 - 1 = 1$. If $e \geq 2$, then $s = d - 1 - e = b^e - 1 - e \geq 2^e - 1 - e \geq 1$, by induction on $e$, starting from $e = 2$. Thus $s \geq 1$ in all cases, and Equation (6.5) implies that $b$ divides at least one of the critical points of $p$, that is, $n \geq 1$. Then by definition of $n$, there exist $a'_1, \ldots, a'_n \in \mathcal{D}$ such that

$$x_1' = ba'_1, \ldots, x_n' = ba'_n.$$  \hspace{1cm} (6.6)

Let $k = d - 1 - n$ and $t = k - e$. Since $n \leq d - 2$, we have that $k \geq 1$. Let $A_k = s_k(a_1, \ldots, a_{d-1})$. By (6.1), we have $(n+1)S_k = b^n S_k'$, that is, $(n+1)b^k A_k = b^n S_k'$, which gives

$$(n+1)b^k A_k = x'_1 s_{k-1}(x_2', \ldots, x_{d-1}') + \cdots + x'_n s_{k-1}(x_{n+1}', \ldots, x_{d-1}') + s_k(x_{n+1}', \ldots, x_{d-1}').$$  \hspace{1cm} (6.7)

Since $k = d - 1 - n$, we have $s_k(x_{n+1}', \ldots, x_{d-1}') = x_{n+1}' \cdots x_{d-1}'$. Substituting this and (6.6) into (6.7) gives

$$x_{n+1}' \cdots x_{d-1}' = b \left[ (n+1)b^{-1} A_k - a'_1 s_{k-1}(x_2', \ldots, x_{d-1}') \right]$$

$$- \cdots - a'_n s_{k-1}(x_{n+1}', \ldots, x_{d-1}').$$

We have $t = k - e = d - 1 - n - e = s - n > 0$, so that $t \geq 1$, and the above equation implies that $b$ divides at least one of the critical points $x_{n+1}', \ldots, x_{d-1}'$. But this contradicts the definition of $n$. Therefore, $n < s$ is impossible, we do have $n \geq s = d - 1 - e$, which implies (b).
(c) Assume that zero is a root of \( p \) of multiplicity one, and that \( b \) divides \( e \) in \( \mathcal{O} \). Let us prove (c2) before proving (c1).

(c2) Assume that \( n = d - 1 - e \). Since by hypothesis \( b \) divides \( e \) in \( \mathcal{O} \), there exists \( f \in \mathcal{O} \) such that \( e = bf \). Let \( g = b^{e-1} - f \), and let \( A_e = s_e(a_1, \ldots, a_{d-1}) \). From \( 2^e \leq b^e = d < 2^d \), we get that \( e \leq d - 1 \). Hence, by Equation (6.1), we have \((a) \) and (c2) that \( \) (c1) Substituting (6.3) and (6.6) into Equation (6.2), knowing from (c) that zero is a root of \( p \) is zero is not a loss of generality, because if \( x_0 \) denotes a root of a polynomial \( p_1 \in \mathcal{O}[x] \), then zero is a root of \( p_2(x) = p_1(x + x_0) \), and by Corollary 2.2, \( p_1 \) is nice if and only if \( p_2 \) is nice.

(c1) Substituting (6.3) and (6.6) into Equation (6.2), knowing from (a) and (c2) that \( m = d - 1 \) and \( n \geq d - e \), gives
\[
b^{d-1}a_1 \cdots a_{d-1} = b^e b^{d-e} a_1' \cdots a'_{d-e} x'_{d-e+1} \cdots x'_{d-1},
\]
that is
\[
a_1 \cdots a_{d-1} = ba_1' \cdots a'_{d-e} x'_{d-e+1} \cdots x'_{d-1}.
\]
This implies that \( b \) divides at least one of the elements \( a_1, \ldots, a_{d-1} \) of \( \mathcal{O} \), that is, there exists \( i \in \{1, \ldots, d - 1\} \) and \( b_i \in \mathcal{O} \) such that \( a_i = bb_i \). Consequently, \( x_i = ba_i = b^2 b_i \), so that (c1) is true.

Remarks. (1) In Theorem (6.3), the hypothesis that at least one root of \( p \) is zero is not a loss of generality, because if \( x_0 \) denotes a root of a polynomial \( p_1 \in \mathcal{O}[x] \), then zero is a root of \( p_2(x) = p_1(x + x_0) \), and by Corollary 2.2, \( p_1 \) is nice if and only if \( p_2 \) is nice.

(2) If we don’t make this shift, then the conclusion of Theorem (6.3) (a) and (b) is that there exists at least \( d - 1 - e \) critical points of \( p \) such that if \( \mathcal{S} \) denotes the union of these critical points with the roots of \( p \), then all of the differences of two elements of \( \mathcal{S} \) are multiples of \( d \).

(3) By Proposition 6.1, a ring of algebraic integers of an algebraic number field satisfies the hypotheses of Theorem (6.3) iff it is a UFD.
For more information about rings of algebraic integers that are UFDs, see [9, pp. 121–123, p. 140, and pp. 185–186].

(4) In the case of a computer search for nice polynomials whose degree \( d \) is a power \( e \) of a prime \( b \) in \( \mathbb{Z}[x] \), we may look for integers \( x_1, \ldots, x_{d-1}, x_d = 0, x'_1, \ldots, x'_{d-1} \) satisfying (3.2). Then by Theorem 6.3, we have to consider only systems such that \( d - 1 \) roots and \( d - 1 - e \) critical points are multiples of \( b \), which reduces the number of choices for each of them by a factor \( b \). Thus Theorem 6.3 makes such a search \( b^{d-1} \cdot b^{d-1-e} = b^{2d-2-e} \) times faster.

**Corollary 6.4.** [Application to nice quartics]. Let \( p \in \mathbb{Z}[x] \) be a nice quartic having zero as a root of multiplicity one. Then:

(a) All of the roots of \( p \) are multiples of 2.

(b) At least one nonzero root of \( p \) is a multiple of 4.

(c) At most 1 of the critical points of \( p \) is not a multiple of 2.

**Proof.** The conclusion is a direct consequence of Theorem 6.3 applied with \( b = e = 2 \).

**Remark.** Corollary 6.4 is not valid if we replace \( \mathbb{Z} \) by \( G \), because \( 2 = (1 + i)(1 - i) \) is not prime in \( G \).

### 7. Nice polynomials with three roots

In this section, we present an important reduction of the system of equations for the roots and critical points of nice polynomials of arbitrary degree with exactly three roots. This is done in Theorem 7.1. For the proof of this theorem, it is essential to extend the definition of the binomial coefficients, in order to avoid serious problems with finite sums over different ranges with too many cases to consider. In order to work on a common range, we have to replace all of these finite sums by infinite sums over all of the integers, where all of these infinite sums have only a finite number of nonzero terms.

**Extended binomial coefficients.** Using a small part of [7], we extend the definition of the binomial coefficients by defining \( \binom{n}{k} = 0 \), for all \( n \in \mathbb{N}_0 \) and \( k \in \mathbb{Z} \) such that \( k < 0 \), or \( k > n \). By [7], these extended binomial coefficients still satisfy the basic identities:

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}, \quad \binom{n-1}{k-1} = \frac{k}{n} \binom{n}{k},
\]

for all \( n \in \mathbb{N}, k \in \mathbb{Z} \).
In the following theorem, we consider a nice polynomial \( p \) with three roots \( r_0, r_1, \) and \( r_2 \) of multiplicities \( m_0, m_1, \) and \( m_2. \) By Corollary 2.2, one of the roots of \( p, \) say \( r_0, \) can be chosen as zero. It is well known that \( p' \) has the same three roots with multiplicities at least equal to \( m_0 - 1, \) \( m_1 - 1, \) and \( m_2 - 1. \) Since the degree of \( p' \) is \( d - 1, \) \( p' \) must have two additional roots \( r'_1 \) and \( r'_2, \) not necessarily distinct from \( r_0 = 0, r_1, \) and \( r_2. \) By Corollary 3.2, the \((m_1 + m_2)\) nonzero roots \( r_1 \) and \( r_2 \) of \( p, \) and the \((m_1 + m_2)\) critical points \( r_1, r_2, r'_1, \) and \( r'_2 \) of \( p, \) counted with their multiplicities, satisfy a system of \((m_1 + m_2)\) equations of degrees 1, 2, \( \ldots, m_1 + m_2. \) Theorem 7.1 shows that we can reduce this system to a system of two equations of degrees one and two for the four variables \( r_1, r_2, r'_1, \) and \( r'_2. \)

Theorem 7.1. [Reduction of the system of equations (3.2) for nice polynomials with three roots to a system of two equations of degrees one and two.] Let \( \mathcal{O} \) be an integral domain of characteristic zero. Let \( r_1 \) and \( r_2 \) be nonzero elements of \( \mathcal{O}. \) Let \( r'_1 \) and \( r'_2 \) be elements of \( \mathcal{O}. \) Let \( m_0, m_1, m_2 \in \mathbb{N}. \) Let \( d = m_0 + m_1 + m_2. \) Let

\[
\begin{align*}
p(x) &= x^{m_0}(x - r_1)^{m_1}(x - r_2)^{m_2}; \\
q(x) &= dx^{m_0-1}(x - r_1)^{m_1-1}(x - r_2)^{m_2-1}(x - r'_1)(x - r'_2).
\end{align*}
\]

Then \( q = p' \) iff

\[
\begin{align*}
(d - m_1)r_1 + (d - m_2)r_2 &= d(r'_1 + r'_2), \quad (7.2) \\
m_0r_1r_2 &= dr'_1r'_2. \quad (7.3)
\end{align*}
\]

Proof. For every \( i \in \{1, \ldots, d\}, \) let

\[
x_i = \begin{cases} 
  r_1, & \text{if } i \leq m_1, \\
  r_2, & \text{if } m_1 + 1 \leq i \leq m_1 + m_2, \\
  0, & \text{if } i \geq m_1 + m_2 + 1.
\end{cases}
\]

For every \( j \in \{1, \ldots, d - 1\}, \) let

\[
x'_j = \begin{cases} 
  r_1, & \text{if } j \leq m_1 - 1, \\
  r_2, & \text{if } m_1 \leq j \leq m_1 + m_2 - 2, \\
  r'_1, & \text{if } j = m_1 + m_2 - 1, \\
  r'_2, & \text{if } j = m_1 + m_2, \\
  0, & \text{if } j \geq m_1 + m_2 + 1.
\end{cases}
\]
For every \( k \in \{1, \ldots, d - 1\} \), let
\[
S_k = s_k(x_1, \ldots, x_d) = s_k(x_1, \ldots, x_{m_1 + m_2}),
\]
\[
S'_k = s_k(x'_1, \ldots, x'_{d-1}) = s_k(x'_1, \ldots, x'_{m_1 + m_2}).
\]
Thus, we have by Corollary 3.2 that
\[
E(k) = [(d - k)S_k = dS'_k].
\]
Since \( S_k = S'_k = 0 \) for all \( k \in \{1, \ldots, d - 1\} \) such that \( k > m_1 + m_2 \), we have by Corollary 3.2 that
\[
(q = p') \iff [E(k), \ \forall k \in \{1, \ldots, m_1 + m_2\}] \quad (7.4)
\]
Since
\[
S_1 = x_1 + \cdots + x_{m_1 + m_2} = m_1r_1 + m_2r_2,
\]
\[
S'_1 = x'_1 + \cdots + x'_{m_1 + m_2} = (m_1 - 1)r_1 + (m_2 - 1)r_2 + r'_1 + r'_2,
\]
we have that \( E(1) \) is equivalent to
\[
(d - 1)(m_1r_1 + m_2r_2) = d[(m_1 - 1)r_1 + (m_2 - 1)r_2 + r'_1 + r'_2],
\]
which is equivalent to \( (7.2) \). Since \( d - m_1 - m_2 = m_0 \), and
\[
S_{m_1 + m_2} = x_1 \cdots x_{m_1 + m_2} = r_1^{m_1}r_2^{m_2},
\]
\[
S'_{m_1 + m_2} = x'_1 \cdots x'_{m_1 + m_2} = r_1^{m_1-1}r_2^{m_2-1}r'_1r'_2,
\]
we have that \( E(m_1 + m_2) \) is equivalent to
\[
m_0r_1^{m_1}r_2^{m_2} = dr_1^{m_1-1}r_2^{m_2-1}r'_1r'_2,
\]
which is equivalent to \( (7.3) \). Thus
\[
E(1) \iff (7.2) \quad \text{and} \quad E(m_1 + m_2) \iff (7.3).
\]
This implies by \( (7.4) \) that the proof is complete if \( m_1 + m_2 = 2 \), and otherwise it remains to prove that \( (7.2) \) and \( (7.3) \) imply \( E(k) \) for all \( k \in \{2, \ldots, m_1 + m_2 - 1\} \).

Assume that \( m_1 + m_2 > 2 \), and that \( (7.2) \) and \( (7.3) \) hold. Let \( k \in \{2, \ldots, m_1 + m_2 - 1\} \). Let us prove that \( E(k) \) is true. With the above extended definition of the binomial coefficients, we have
\[
dS'_k = d \sum_{i \in \mathbb{Z}} \binom{m_1 - 1}{i} \binom{m_2 - 1}{k - i} r_1^i r_2^{k-i}
\]
\[
+ d \sum_{i \in \mathbb{Z}} \binom{m_1 - 1}{i} \binom{m_2 - 1}{k - 1 - i} r_1^i r_2^{k-1-i}(r'_1 + r'_2)
\]
\[
+ d \sum_{i \in \mathbb{Z}} \binom{m_1 - 1}{i} \binom{m_2 - 1}{k - 2 - i} r_1^i r_2^{k-2-i}r'_1r'_2,
\]
which gives by (7.2) and (7.3)
\[ dS'_k = d \sum_{i \in \mathbb{Z}} \binom{m_1 - 1}{i} \binom{m_2 - 1}{k - i} r_1^i r_2^{k-i} \]
\[ + \sum_{i \in \mathbb{Z}} \binom{m_1 - 1}{i} \binom{m_2 - 1}{k - 1 - i} r_1^i r_2^{k-1-i} [(d - m_1)r_1 + (d - m_2)r_2] \]
\[ + m_0 \sum_{i \in \mathbb{Z}} \binom{m_1 - 1}{i} \binom{m_2 - 1}{k - 2 - i} r_1^i r_2^{k-2-i} r_1 r_2. \]

Using \( m_0 = d - m_1 - m_2 \), grouping terms with factor \( d, m_1, \) and \( m_2 \), simplifying with (7.1) gives
\[ dS'_k = d \sum_{i \in \mathbb{Z}} \binom{m_1}{i} \binom{m_2}{k - i} r_1^i r_2^{k-i} \]
\[ - m_1 \sum_{i \in \mathbb{Z}} \frac{i}{m_1} \binom{m_1}{i} \binom{m_2}{k - i} r_1^i r_2^{k-i} \]
\[ - m_2 \sum_{i \in \mathbb{Z}} \binom{m_1}{i} \frac{k - i}{m_2} \binom{m_2}{k - i} r_1^i r_2^{k-i}. \]
\[ = (d - k) \sum_{i \in \mathbb{Z}} \binom{m_1}{i} \binom{m_2}{k - i} r_1^i r_2^{k-i} = (d - k)S_k. \]

Thus \( E(k) \) is true, and the theorem is proved.

Applications. 1. While the three root case is not solved yet, Theorem 7.1 makes it easy to find examples of nice polynomials with three roots of any given multiplicities with the help of a computer.

2. We have established the formulas A2, A5, A6, A7, A8, and A10 of Section 8 below by applying Theorem 7.1.

Conjecture. Theorem 7.1 suggests that nice polynomials with one root at zero and \( n \) nonzero roots may be determined by a reduced system of \( n \) equations of degrees 1, 2, \ldots, \( n \) for the \( n \) nonzero roots and the \( n \) nonzero critical points.

8. Examples and open problems

In this section, we present examples of nice polynomials whose coefficients, roots, and critical points are rational integer or Gaussian integers, with special interest in the case where the coefficients are real, while the roots and critical points can be real or complex. For each type of nice polynomials, we have tried to find the smallest one, and to
present it in its most reduced form, except for the symmetric ones. All of the examples of this section have been checked with Maple through Maple methods totally different from the methods of this paper.

**Definitions.** As usual, we call diameter of a compact subset \( \mathcal{S} \) of the complex plane the largest distance between two points of \( \mathcal{S} \). We define the diameter \( \text{diam}(p) \) of a polynomial \( p \in \mathbb{G}[x] \) as the diameter of the set of the roots of \( p \) in the complex plane. We say that a polynomial \( p \in \mathbb{G}[x] \) is smaller than a polynomial \( q \in \mathbb{G}[x] \) iff \( \text{diam}(p) < \text{diam}(q) \). We say that a nice polynomial \( p \in \mathbb{G} \) is in reduced form iff \( p \) cannot be transformed into a smaller nice polynomial by using the transformations of Proposition 2.1. We say that a nice polynomial \( p \in \mathbb{G} \) is in most reduced form iff \( p \) is in reduced form, zero is a root of \( p \), \( p \) cannot be transformed into a nice polynomial \( q \) with \( \text{diam}(p) = \text{diam}(q) \) and \( |\mathcal{A}(q)| < |\mathcal{A}(p)| \) by using the transformations of Proposition 2.1 and when all of the roots of \( p \) are real, none of them is negative. For example, \( p(x) = x^2(x - 7)^5 \) is a nice polynomial in reduced form, and \( q(x) = x^5(x - 7)^2 \) is the most reduced form of \( p \).

**A. A few general formulas.** Here, we present general formulas that are not in final form. These formulas may give several times the same polynomials, and they do not separate the cases where some roots are equal. In the following formulas, \( m \) and \( n \) are rational integers with no common factors.

**A 1.** For every integral domain \( \mathfrak{D} \) of characteristic zero, and for every positive integer \( n \) greater than one, the polynomial \( p(x) = x^n \) is a totally nice polynomial over \( \mathfrak{D} \). It is symmetric iff \( n \) is even and antisymmetric iff \( n \) is odd.

**A 2.** For every positive integer \( n \), there exists a nice antisymmetric polynomial \( p \in \mathbb{Z}[x] \) of degree \( (2n + 1)^2 \) with three roots, given by

\[
p(x) = x \left[ x^2 - (2n + 1)^2 \right]^{2n(n+1)},
\]

\[
p'(x) = (2n + 1)^2(x^2 - 1) \left[ x^2 - (2n + 1)^2 \right]^{2n(n+1) - 1}.
\]

**A 3.** For every positive integer \( n \), there exists a nice symmetric polynomial \( p \in \mathbb{Z}[x] \) of degree \( (2n + 1)^2 + 1 \) with two roots, whose derivative is also nice, which is given by

\[
p(x) = \left[ x^2 - (2n + 1)^2 \right]^{2n(n+1) + 1},
\]

\[
p'(x) = 2(2n^2 + 2n + 1)x \left[ x^2 - (2n + 1)^2 \right]^{2n(n+1)},
\]

\[
p''(x) = 2(2n + 1)^2(2n^2 + 2n + 1)(x^2 - 1) \left[ x^2 - (2n + 1)^2 \right]^{2n(n+1) - 1}.
\]
A 4. Nice polynomials with two roots in reduced form are given by
\[ p(x) = x^{ab}(x - c)^{b(c-a)}, \]
\[ p'(x) = bcx^{ab-1}(x - c)^{b(c-a)-1}(x - a), \]
where \(a, b, c \in \mathbb{N}\) are such that \(c > a\), and \(\gcd(a, c) = 1\).

A 5. Nice Cubics with three real roots: This case was solved by M. Chapple, [5] in 1960, Karl Zuser, [13] in 1963, and Tom Bruggeman and Tom Gush [1] in 1980. Those cubics are given by the formula
\[ p(x) = x[x - 3m(m + 2n)] [x - 3n(2m + n)], \]
\[ p'(x) = 3(x - 3mn) [x - (m + 2n)(2m + n)]. \]
Karl Zuser observed that when \(mn \equiv 1 \pmod{3}\), these polynomials can be made three times smaller by using Proposition 2.1(c).

A 6. Nice Cubics with one real root, and two complex conjugate roots with even imaginary part are given by
\[ p(x) = (x^2 + 36m^2n^2) [x - 3(m^2 + n^2)], \]
\[ p'(x) = 3(x - 2m^2)(x - 6n^2). \]

A 7. Nice Cubics with one real root, and two complex conjugate roots with odd imaginary part are given by
\[ p(x) = [x^2 + 9(2m + 1)^2(2n + 1)^2] \cdot [x - 6[m(m + 1) + 3n(n + 1) + 1]], \]
\[ p'(x) = 3[x - (2m + 1)^2][x - 3(2n + 1)^2]. \]

A 8. Nice quartics with a double root are given by
\[ p(x) = x^2 [x - 2(2m^2 - mn - n^2)] [x - 2(2m^2 + mn - n^2)], \]
\[ p'(x) = 4x [x - 2(m^2 - n^2)] [x - (4m^2 - n^2)], \]
and by
\[ p(x) = x [x - 2(n^2 + mn - 2m^2)]^2 (x - 4mn), \]
\[ p'(x) = 4 [x - (2mn + n^2)] [x - 2(n^2 + mn - 2m^2)] [x - 2m(n - m)]. \]

A 9. Nice symmetric quartics with four real roots. This case was solved by Chris Caldwell [3] in 1990. These quartics are given by
\[ p(x) = [x^2 - (n^2 - m^2 - 2mn)^2] [x^2 - (n^2 - m^2 + 2mn)^2], \]
\[ p'(x) = 4x [x^2 - (m^2 + n^2)^2]. \]
A 10. Nice quintics with a triple root at zero, and a critical point that
is a multiple of 15 are given by
\[ p(x) = x^3 \left[ x - 5m(m - 4n) \right] \left[ x - 5n(4m - 15n) \right], \]
\[ p'(x) = 5x^2 \left[ x - 15n(m - 4n) \right] \left[ x - m(4m - 15n) \right]. \]

Other nice quintics with a triple root at zero are given by
\[ p(x) = x^3 \left[ x - 5m(4n - 5m) \right] \left[ x - 5n(3n - 4m) \right], \]
\[ p'(x) = 5x^2 \left[ x - 5m(3n - 4m) \right] \left[ x - 3n(4n - 5m) \right]. \]

A 11. Nice symmetric sextics with at least two double roots are given
by
\[ p(x) = \left[ x^2 - \left( m^2 + 4mn - 2n^2 \right)^2 \right] \left[ x^2 - \left( m^2 - 2mn - 2n^2 \right)^2 \right] \]
\[ p'(x) = 6x \left[ x^2 - \left( 2n^2 - 4mn - m^2 \right)^2 \right] \left[ x^2 - \left( m^2 + 2n^2 \right)^2 \right]. \]

B. A few numerical examples.
B 1. The smallest nice cubic with three distinct real roots:
\[ p(x) = x(x - 9)(x - 24), \quad p'(x) = 3(x - 4)(x - 18). \]

B 2. The smallest nice cubic with two complex conjugate roots:
\[ p(x) = (x - 6)(x^2 + 9), \quad p'(x) = 3(x - 1)(x - 3). \]

B 3. The smallest nice cubic with roots at unequal distances from each
other in the complex plane:
\[ p(x) = x \left[ x - 3(1 + 4i) \right] \left[ x - 15 \right], \quad p'(x) = 3 \left[ x - 3(1 + 2i) \right] \left[ x - (9 + 2i) \right]. \]

B 4. The smallest nice symmetric quartic with four distinct complex
roots is \( p(x) = x^4 - 1 \), which gives \( p'(x) = 4x^3 \).

B 5. The smallest nice symmetric quartic with four distinct real roots:
\[ p(x) = (x^2 - 1)(x^2 - 49), \quad p'(x) = 4x(x^2 - 25). \]

B 6. Concerning the case of nice non-symmetric quartics with four dis-
tinct real roots, the first five examples were published by Chris Caldwell
[3] in 1990. Since then, it was not known whether other examples ex-
est. After considerable simplification of the computation with the
results of this paper, and formulas that are not in publishable form
yet, we have found 358 other examples with the help of a computer.
The five examples of [3] are in the positions 88, 107, 181, 247, and 321
on our list of examples ordered in increasing order. The smallest of our
examples is:

\[ p(x) = x(x - 50)(x - 176)(x - 330), \]
\[ p'(x) = 4(x - 22)(x - 120)(x - 275). \]

The largest example that we have obtained, in position 363 on our list, after two weeks of uninterrupted computation with a computer, is

\[ p(x) = x(x - 33408)(x - 44100)(x - 138040), \]
\[ p'(x) = 4(x - 11760)(x - 38976)(x - 110925). \]

**B 7.** The only nice quintic with five distinct real roots that we have found is

\[ p(x) = x(x - 180)(x - 285)(x - 460)(x - 780), \]
\[ p'(x) = 5(x - 60)(x - 230)(x - 390)(x - 684). \]

**B 8.** We have found only two nice quintics with five distinct complex roots. The first one has a double critical point:

\[ p(x) = x(x - 595)(x - 1020)[(x - 220)^2 + 40,000], \]
\[ p'(x) = 5(x - 170)^2(x - 420)(x - 884), \]
\[ p(x) = x(x - 585)(x - 1040)[(x - 270)^2 + 44,100], \]
\[ p'(x) = 5(x - 130)(x - 312)(x - 390)(x - 900). \]

**B 9.** The smallest nice antisymmetric polynomial with more than one root is \( p(x) = x(x^2 - 9)^4 \), with \( p'(x) = 9(x^2 - 1)(x^2 - 9)^3 \).

**B 10.** The smallest nice symmetric polynomial with more than one critical point, and whose first derivative is also nice, is:

\[ p(x) = (x^2 - 9)^5, \quad p'(x) = 10x(x^2 - 9)^4, \quad p''(x) = 90(x^2 - 1)(x^2 - 9)^3. \]

**Open problems.**

1. With the equivalence relation defined in Section 2, find the number of equivalence classes for any set of nice polynomials of a given type (For example, symmetric with four roots of any multiplicities).
2. Generalize Theorem 6.3 from polynomials whose degree is a prime power to polynomials of any degree. A first step would be to consider polynomials whose degree is the product of two primes.
3. Prove or disprove the conjecture of the end of Section 7.
4. Find a nice sextic with six distinct roots.
5. Find a nice antisymmetric polynomial with more than three roots.
6. Find a general formula for nice polynomials with three roots by
using the system of two equations (7.2) and (7.3).

7. Find a general formula for nice non-symmetric quartics.

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Jean-Claude Evard
Department of Mathematics
Western Kentucky University
Bowling Green, KY 42101–3576

E-mail: Jean-Claude.Evard@wku.edu