Topological excitations around the vacuum of quantum gravity I: The symmetries of the vacuum

Artem Starodubtsev*

Department of Physics, University of Waterloo
200 University ave W, Waterloo, ON, Canada N2L 3G1
and
Perimeter Institute for Theoretical Physics
35 King st N, Waterloo, ON, Canada N2J 2W9

Abstract

This is the first paper in a series in which an attempt is made to formulate a perturbation theory around the the Chern-Simons state of quantum gravity discovered by Kodama. It is based on an extension of the theory of ’t Hooft Deser and Jackew describing point particles in 3D gravity to four spacetime dimensions. General covariance now requires the basic excitations to be extended in one spatial dimension rather than pointlike. As a consequence the symmetry of the Kodama state, which is the (anti)deSitter symmetry, is realized ’holigraphically’ on a timelike boundary, the generators of this symmetry being related to quasilocal energy-momenta. As GR induces a Chern-Simons theory on the boundary, the deSitter symmetry of the vacuum must be q-deformed with the deformation parameter related to the cosmological constant. It is proposed to introduce excitations around this vacuum by putting punctures on the boundary to each of which is associate a vector in some representation of deSitter group projected to the boundary. By equations of motion those punctures must be connected by continuous lines of fluxes of an SO(3,1) gauge field. It is also argued that quasilocal masses and spins of these excitations must satisfy a relation of Regge type, which may point on a possible relation between non-perturbative quantum gravity and string theory.

1 Introduction

The main problem of loop quantum gravity (see [1] for review) is the problem of the classical limit. Soon after the hamiltonian constraint of quantum gravity was defined on spin network states [2], and a wide class of solutions to it was found, it has become apparent that none of those solutions represent continuous spacetime geometry in the classical limit. The problem manifests itself in the fact that the hamiltonian constraint acts on distinct vertices of a spin network independently allowing for no possibility to send

*email: astarodu@astro.uwaterloo.ca
a signal between them [3]. Also, the algebra of the hamiltonian constraints appears to be abelian and therefore not isomorphic to the classical constraints algebra [4].

There are two possible strategies to address this problem. First one can try to approximate states of the type that have been proved to have a good classical limit in ordinary QFT by complicated combinations of loop or spin network solutions. This includes the coherent states program [5] and approximating states of the Fock space [6]. Second one can extend the range of our search to include other states which are not of loop or spin network type.

An exact state of quantum gravity which has a good classical limit does exist and this is the Chern-Simons state discovered by Kodama back in 1990 [7].

\[ \Psi_K = e^{3\Lambda \Lambda} S_{CS}(A), \]  

where \( S_{CS}(A) \) is the Chern-Simons action for the Ashtekar connection [8], and \( \Lambda \) is the cosmological constant. Although we don’t have a strict definition of the physical space of states the Kodama state belongs to and, therefore, we cannot calculate the expectation values of physical observables to show that they are peaked around a classical solution, there is some indirect evidence that the Kodama state has a good classical limit. On each spacial slice it satisfies a self-dual set of initial conditions (electric field is proportional to magnetic). These initial conditions are preserved by Einstein’s equations and result in the deSitter or anti-deSitter spacetime depending on the sign of the cosmological constant. This means that (anti)deSitter spacetime must be the classical limit of the Kodama state. For more details about the Kodama state, its relation to loop states and an outline of the program of how to solve the problem of classical limit in loop quantum gravity by using it see [9].

The problem with the Kodama state is that there is only one state of this kind known. The question is then: Can we define states which can be considered as perturbations around the Kodama state, the later playing the role of the vacuum? More precisely: Given that the classical limit of the Kodama state is the deSitter spacetime can we define states the classical limit of which would be gravitons propagating against the deSitter background?

This is not an easy question to answer because to define these excitations explicitly we need some spacetime structure to build them on. But we don’t know a priori what quantum spacetime is. One can setup a semiclassical framework [9] in which excitations around the Kodama state are introduced via expanding around classical deSitter spacetime. This has already led to some nontrivial predictions such as modified energy-momentum relations which cannot be obtained from studying purely classical spacetime. The next step would be to develop a formalism that does not rely on any classical spacetime structure at all. One may hope that such completely background independent consideration may reveal more information on what quantum spacetime is. This paper is about a particular way of approaching this.

In section 2 it is suggested that the structure of a TQFT which the Kodama state is the only solution of can be used to define excitations. This can be done by analogy with ’t Hooft, Deser, and Jackew’s formulation of 2+1 gravity with point particles [10]. The new features arising in 3+1 dimensions are described.
Among those features is a 'holographic' realization of the basic symmetries and observables of the theory. This requires a specific formulation of GR in the presence of a (timelike) boundary. This is done in section 3.

In section 4 the boundary theory is studied in more details. The algebra of boundary observables is derived and the generators of that algebra are related to quasilocal energy momentum and spin. The formalism fixes the coupling constant of the boundary Chern-Simons theory which leads to a q-deformation of the symmetry of the ground state.

In section 5 all the constraint of general relativity are recast in a form of a local conservation law. This form of constraints suggests how the boundary observables should be continued into the bulk.

In the absence of a strict regularization scheme some generic properties of the boundary observables are studied in section 6. In particular it is shown that the Kodama state is the ground state.

In section 7 a general way of introducing excitations around the Kodama state is discussed. In some simple cases the spectra of the boundary observables can be evaluated. In particular it is argued that quasilocal masses and spins must satisfy a relation of the Regge type.

2 Point particles in 2+1 gravity: how to extend to 3+1 dimensions

It is remarkable that in 2+1 gravity there is a way to define excitations around the vacuum which does not refer to any structure of classical spacetime. This is a specific way of describing point particles moving in 2+1 dimensional spacetime. Below for definiteness we consider the case of positive cosmological constant.

The action of 2+1 gravity is

\[ S = \int (e \wedge F(A) + \Lambda e \wedge e \wedge e), \]  

where the connection \( A \) and the metric \( e \) take the values on 2+1 dimensional \( \gamma \)-matrices and \( F(A) \) is the curvature of the connection \( A \). The action (2) can also be considered as the Chern-Simons action for \( SO(3,1) \) group. The equations of motion of (2)

\[ F(A) + \Lambda e \wedge e = 0, \]  

and

\[ \nabla \wedge e = 0 \]  

simply mean that the curvature is constant everywhere and is determined by the cosmological constant. Put in the Chern-Simons language this means that \( SO(3,1) \)-curvature is zero. Therefore there is no intrinsic dynamics in the theory.

Dynamics can be introduced by adding matter to the action. The simplest example of matter is point particles having a certain mass. In 2+1 dimensions point particles do not create a Newtonian potential around them – the \( SO(3,1) \) curvature is non-zero only at
the points of the location of the particles. Classically these particles can be represented as conical singularities with the deficit angles proportional to their masses moving along certain trajectories in the deSitter space.

Quantum mechanically, on the other hand, there are no definite trajectories and the particle is described by a vector in a certain representation of the $SO(3,1)$ group. Therefore from the Chern-Simons point of view they can be thought of as point charges of the $SO(3,1)$ gauge group, or topological punctures. It is important to notice that $SO(3,1)$-symmetry is now understood as the gauge symmetry, the symmetry generated by the constraints \(3\), and not as the global symmetry of a classical background manifold.

This is an example of how the same object can be given two equivalent descriptions, one of which is background dependent and the other is background independent. On one side we have (quantum) particles propagating against a classical deSitter background. On the other side we have a topological field theory on a punctured manifold with all the physical information encoded in the vectors associated to the punctures and the positions of the punctures being irrelevant.

It is tempting to try to extend the above picture to 3+1 dimensions which would give rise to excitations around the Kodama state. The Kodama state is a solution not only of quantum general relativity but also of a four dimensional topological field theory called BF-theory defined by the following action

\[
S = \int B \wedge F(A) + \frac{\Lambda}{2} B \wedge B,
\]

where $B$ is a 2-form field. The constraints of \(3\)

\[
F(A) + \Lambda B = 0
\]

have a unique solution, which is the Kodama state. So perturbations around the Kodama state can be considered as addition of external sources of curvature to the BF theory, in the same way as matter can be added to 2+1 gravity. The difference is that these sources can still be intrinsic for 3+1 general relativity, i.e. still satisfy the vacuum Einstein equations. This is why pure 3+1 gravity has nontrivial intrinsic dynamics.

There are some other differences between the dynamics of topological matter in 2+1 and in 3+1 dimensions. Let us first ask why the natural extrinsic sources in 2+1 gravity are point particles or 0+1 dimensional objects. What makes the point particles preferred over singular sources of other dimensions? The reason is that this is the only possible objects that can be defined in a general-covariant way. Point particles are sources of curvature and the curvature is a 2-form. For the source to be singular it must contain a $\delta$-function and the $\delta$-function is a density. Density in turn results from the duality operation acting on forms, and so the volume form of the source must be dual to the curvature form. Therefore the objects which source the curvature must be of codimension 2, which are point particles or punctures. These arguments are similar to those according to which the electric charge in 11-dimensional M-theory must be carried by M2-branes and magnetic charge by M5-branes. The difference is that what is distributional now is not a divergence of a curvature but the curvature itself.
The above logic extends directly to 3+1 dimensions but now objects of codimension 2 are 1+1 dimensional objects. Classically the natural candidate for such objects would be cosmic strings, which are conical singularities extended in one spacial dimension. Being stretched in flat space along \( z \)-axis they can be described by the following metric

\[
\text{ds}^2 = -dt^2 + dz^2 + dr^2 + (1 - \alpha)^2 r^2 d\phi^2, \tag{7}
\]

where \( \alpha \) is the deficit angle. This metric solves the Einstein’s equations with the following stress-energy tensor

\[
T^{\mu \nu} = \text{diag}(\alpha, 0, 0, -\alpha) \delta(r), \tag{8}
\]

which in particular means that it solves all the constraints except the Hamiltonian. This means that general configurations of cosmic strings are not purely gravitational objects as they do not solve the vacuum Einstein’s equations. Yet there may be some specific configurations of them which solve the vacuum Einstein’s equations at least approximately in a weak coupling limit. The example is the graviton mode of a string.

In this paper we will focus mostly on quantum mechanical background independent description of the excitations. The natural candidates for these are Wilson loops

\[
W(\Gamma) = Pe^{iA_a dx^a}. \tag{9}
\]

They were shown to solve the gauge and the diffeomorphism constraints and also the Hamiltonian constraints for a certain factor ordering \([11], [12]\). However the Kodama state is a solution to the Hamiltonian constraint for a different factor ordering. Therefore, as excitations around the Kodama state loops do not solve all the constraints of GR. But, again, there may exist specific configurations of loops which solve all of them. They can be formally defined via expansion of the projector on the physical states or in spin foam models \([13]\).

Finally, the most profound difference between 2+1 and 3+1 TQFT is that in the constraint algebra. The constraints of 2+1 gravity form an \( SO(3,1) \)-algebra, so the symmetry of the only classical solution is present in the canonical structure of the theory. On the other hand the algebra of the constraints of 3+1 dimensional TQFT is abelian, while the symmetry of the only classical solution of the theory which is the deSitter spacetime is \( SO(4,1) \). So the symmetry of the "vacuum" solution is not represented in the constraint algebra. Behind this difference is the "holographical" nature of 3+1 dimensional gravity. This is a consequence of the fact that the basic excitations of the theory are non-local.

Let us expand the last point in a bit more detail. If the constraints form the symmetry algebra of the vacuum classical solution then they can be understood as operators of energy momentum and angular momentum of the theory. Therefore in 2+1 gravity, as it is described here, the operators of the above quantities can be introduced locally. Indeed, energy, momentum, and spin can be assigned to each point particle, and because there is no Newtonian potential, and no gravitational waves in the theory, there is nothing like gravitational energy which generally cannot be defined locally. In 3+1 dimensions on the other hand excitations which do not produce Newtonian potential and can in principle
represent the gravitational radiation themselves are strings. But these objects cannot be localized on spacelike 3-slice and do not allow us to define energy, momentum, and angular momentum locally. This is why the symmetry of the vacuum solution cannot be realized as the algebra of bulk constraints of the theory. It can however be realized in a "holographic" sense. If we pick out a closed 2-surface in the spacial slice and impose a certain boundary conditions on it, then strings can be localized at this surface at the intersection points. On the surface can be defined a field theory, the constraints of which form (a subalgebra of) the deSitter algebra.

All the above fits very well with the well known fact that energy, momentum and angular momentum cannot be defined locally in general relativity. They however can be defined quasilocally on a surface on which some boundary conditions are satisfied. This is done in a form most suitable for our purposes in the next section.

3 Action principle for GR with a boundary: the choice of the boundary action

As was mentioned in the previous section, the only context in which the excitations can be studied from the point of view of the symmetry of the vacuum is when spacetime has a boundary. The boundary doesn’t have to surround the whole spacetime, it may be finite, but a certain kind of condition on the boundary must be satisfied.

In general the variation of an action for general relativity in a bounded region contains a surface term which must vanish for the variational principle to be well defined. This can be achieved by imposing some boundary conditions plus adding a surface term to the action.

There are many possible choices of the surface term in the action. Different boundary conditions require different surface terms. But they do not determine them completely. The surface term also depends on the choice of canonical variables. When we perform a canonical transformation in the theory, this is equivalent to the addition of a total derivative term to the Lagrangian. This term doesn’t change the equations of motion in the bulk, but it may have a nontrivial effect in the boundary. This is another piece of ambiguity in the choice of the surface term.

Here we will follow the proposal of Smolin [14, 15] relating quantum gravity in the bulk region with topological quantum field theory on the boundary. The self-dual boundary conditions considered there were shown to be satisfied on black hole horizons [16] and therefore were extremely useful for studying black hole mechanics. However they are not very suitable for our purposes. They do not allow us to define energy and momentum and as a consequence the boundary theory does not completely capture the symmetry of the vacuum.

Therefore we choose another set of boundary conditions. To have a sensible definition of energy and momentum we need to fix metric on the boundary. We must also choose what canonical variables to use and this is motivated by our interest in the Kodama state. First, we must choose the Ashtekar variables in their original form, with the Immirzi parameter equal to \( i \), as only in this variables the constraints take a simple form which
the Kodama state is a solution of. Second, it is well known that General relativity can be obtained from breaking $SO(4, 1)$-symmetry in topological field theory down to $SO(3, 1)$ \[17\]. The resulting form of GR action has definite implications as to what the boundary theory should be. Given that $SO(4, 1)$ is the symmetry of the Kodama state it is natural to choose this form of the boundary action.

Here it is worth reviewing how a breaking of $SO(4, 1)$ symmetry in TQFT leads to General Relativity, and what kind of action it results in. For convenience, we will use spinorial notations. Our starting point will be a topological field theory for $Sp(4)$ group which is locally isomorphic to $SO(4, 1)$. The action depends on $Sp(4)$-connection $A^\beta_\alpha$, where $\alpha, \beta = 0, \ldots, 3$.

\[
S = \frac{1}{8\pi GA} \int Tr F_\beta^\alpha P_\gamma^\beta \wedge F_\delta^\gamma P_\alpha^\delta.
\]  

(10)

Here $F_\beta^\alpha = dA_\beta^\alpha + A_\gamma^\alpha \wedge A_\beta^\gamma$ is the $Sp(4)$-curvature 2-form and $P_\beta^\alpha$ is a fixed symmetry breaking 0-form matrix.

$Sp(4)$ quantities can be decomposed by using $SL(2, C)$ indices notation $A = 0, 1, A' = 0, 1$. The $Sp(4)$ connection can be represented as

\[
A^\beta_\alpha = \begin{pmatrix} A^B_A & \frac{1}{l} e^B_A \wedge e^A_{A'} \\ \frac{1}{l} e^A_{A'} \wedge e^B_A \\ A^{B'}_{A'} & \end{pmatrix}.
\]  

(11)

where $A^B_A$ and $A^{B'}_{A'}$ are left-handed and right-handed $SL(2, C)$-connections respectively, $e^B_A$ is a tetrad and $l$ is a fixed parameter of the dimension of length. Similarly one can decompose $Sp(4)$-curvature

\[
F^\beta_\alpha = \begin{pmatrix} F_{AB}^A & F_{A'}^A \\ F_{B'}^A & F_{A'}^{B'} \\ \end{pmatrix}.
\]  

(12)

Here

\[
F_{AB}^A = f_{AB}^A + \frac{1}{l^2} e^A_A \wedge e^B_A,
\]  

(13)

where $f_{AB}^A$ is $SU(2)_L$-curvature of the connection $A^B_A$, and

\[
F_{A'}^{B'} = \nabla \wedge e^B_{A'}
\]  

(14)

is the torsion.

To get an action the canonical form of which is Ashtekar’s one has to restrict it to purely self-dual $SL(2, C)$-connection, which means that we should choose $P_\beta^\alpha$ in (10) to be

\[
P^\beta_\alpha = \begin{pmatrix} \delta^B_A & 0 \\ 0 & 0 \\ \end{pmatrix}.
\]  

(15)

The resulting bulk action is then

\[
S_{Bulk} = \frac{1}{8\pi GA} \int (f_{AB}^A + \Lambda e^A_A \wedge e^B_A) \wedge (f_{B'}^{A'} + \Lambda e^{A'}_{B'} \wedge e^A_A),
\]  

(16)
where $\Lambda = \frac{1}{l^2}$ is the cosmological constant. This will be the basic bulk action for the rest of the paper.

As we mentioned before in order to be able to define energy and momentum we choose to fix the metric on the boundary

$$\delta e^A_A |_S = 0. \quad (17)$$

We can now calculate the variation of the action (16) subject to this condition.

$$\delta S_{Bulk} = \frac{1}{4\pi G} \int \left( f^B_A + \Lambda e^A_A \wedge e^B_B \wedge \delta e^A_A - \frac{1}{4\pi G} \int \nabla \wedge (f^B_A + \Lambda e^A_A \wedge e^B_B) \wedge \delta A^B_B \right) + \frac{1}{4\pi G} \int_S \left( f^B_A + \Lambda e^A_A \wedge e^B_B \right) \wedge \delta A^A_A. \quad (18)$$

For the variation principle to be well defined the boundary term in (18) has to vanish which can be achieved by adding a boundary term to the action (16):

$$S = S_{Bulk} + S_S, \quad (19)$$

where

$$S_S = \frac{1}{6\pi G} S_{CS}[A] + \frac{1}{4\pi G} \int_S e^A_A \wedge A^A_B \wedge e^B_B + S[e], \quad (20)$$

where $S_{CS}[A]$ is Chern-Simons action of the connection $A$ and $S[e]$ is a surface action which depends purely of metric. It can be checked that the the variation of the action (19) subject to the condition (17) has no surface term for arbitrary choice of $S[e]$ in (20). The later can be fixed by the requirement that the total action (19) be covariant, i.e. that the gauge and diffeomorphism invariance at the boundary be broken by the boundary conditions and not by the action itself. However diffeomorphism and gauge symmetries are partially broken due to the presence of the boundary. We can keep only invariance with respect to diffeomorphisms tangent to the boundary and with respect to rotations in the tangent space of the boundary. To make it explicit let us introduce an arbitrary unit vector field on the boundary $s^\mu$ and its spinorial representation $s^\mu_A = s^\mu e^A_A$, $s^\mu_A s^\mu_B = \delta^A_B$.

By using it one can parametrize the self-dual part of the tetrad $e^A_A \wedge e^A_A$ by a traceless triad $\sigma^A_A = e^A_A s^A_A - 1/2e^A_A s^B_B \delta^A_B$ as

$$e^A_A \wedge e^A_A = \sigma^A_C \wedge \sigma^C_B + s \wedge \sigma^A_B. \quad (21)$$

The second term in the r.h.s. of (20) can then be rewritten in the form

$$\int_S e^A_A \wedge A^A_B \wedge e^B_B = \int_S \sigma^B_A \wedge A^A_B \wedge \sigma^B_A + \int_S s \wedge A^A_B \wedge \sigma^B_A. \quad (22)$$

The second term in the r.h.s of (22) does not admit a covariant extension and has to be removed. This can be done by choosing the vector field $s$ on the boundary to be the unit normal to this boundary, which makes the above term disappear automatically. This
means that the triad $\sigma^A_B$ is chosen to be the projection of the tetrad $e_A^A$ on the surface $S$. The remaining term in the r.h.s of (22) by a specific choice of $S[e]$ can be completed to the term with a covariant derivative of $\sigma$. The covariant form of the boundary action is thus

$$S_S = \frac{1}{6\pi G} S_{CS}[A] + \frac{1}{4\pi G} \int_S \sigma^B_A \wedge \nabla \wedge \sigma^A_B$$

(23)

In [15] two possible sets of boundary conditions dual to each other were studied. One could either fix connection on the boundary $\delta A^B = 0$ or choose self-dual boundary conditions $F^B_A - \Lambda e^A_A \wedge e^B_B$ and leave the connection loose. Similarly, instead of fixing metric on the boundary the action principle (19) by imposing the following set of conditions

$$\nabla \wedge \sigma^A_B = 0$$

(24)

can be made consistent. As in the case of free varying laps and shift functions we cannot define energy and momentum, the condition (24) must imply that energy and momentum is zero. In the next section we will see that this is indeed the case.

4 Quasilocal quantities and the algebra of boundary observables

In this section we will study the boundary theory defined by the action (23) in more details and relate its observables with quasilocal energy, momenta, and angular momenta of the bulk theory.

A theory of the form (23) was considered by Witten [18] along with the action of 2+1 gravity. It can be rewritten as a Chern-Simons action

$$S_S = \frac{1}{6\pi G} S_{CS}(a)$$

(25)

for SO(3,1)-connection

$$a = A_i J^i + \sqrt{\Lambda} \sigma_i P^i$$

(26)

where

$$[J^i, J^j] = \epsilon^{ijk} J_k, \quad [J^i, P^j] = \epsilon^{ijk} P_k, \quad [P^i, P^j] = \epsilon^{ijk} J_k$$

(27)

are the generators of the SO(3,1) group. This means that the constraints of the theory (23) form an SO(3,1) algebra with respect to the boundary symplectic form. In this the theory is similar to 2+1 dimensional gravity. Also its constraints have the same form

$$C^A_B = \epsilon^{\alpha \beta} (F^A_{\alpha \beta} + \Lambda \sigma^A_{\alpha C} \sigma^C_{\beta B})$$

$$H^A_B = \epsilon^{\alpha \beta} \nabla_{\alpha} \sigma^A_{\beta B}.$$  

(28)

were indices $\alpha, \beta = 1, 2$ are two-dimensional spacial manifold indices and $\epsilon^{\alpha \beta}$ is completely antisymmetric tensor. It differs however form (2+1) gravity by the fact that the gauge
and the diffeomorphism constraints have traded places. Also different are the canonical commutation relations between basic variables. Now they are

\[
\{ A^A_{aB}, A^C_{bD} \} = 3 \pi G \epsilon_{\alpha\beta}(\delta^A_B \delta^C_D + \epsilon^{AC} \epsilon_{BD})
\]

\[
\{ \sigma^A_{aB}, \sigma^C_{bD} \} = 2 \pi G \epsilon_{\alpha\beta}(\delta^A_B \delta^C_D + \epsilon^{AC} \epsilon_{BD})
\]

\[
\{ A^A_{aB}, \sigma^C_{bD} \} = 0.
\]

The fact that the boundary theory is a Chern-Simons theory for $\text{SO}(3,1)$ group with the coupling constant $\kappa = \frac{6 \pi G}{\Lambda}$ means that the symmetry group of the vacuum is now q-deformed with $q = \exp\left(\frac{1}{\kappa+2}\right)$. This means that particles inserted in punctures of the boundary theory will “propagate” in q-deformed spacetime.

In the rest of this section we will relate the constraints (28) with quasilocal observables of the bulk theory. This relation will involve projection of spinors on surfaces which may be spacelike or timelike. For this some useful notations are introduced below.

Let $\Sigma$ be an arbitrary surface, which may be spacelike or timelike. Let $n_a$ be the unit normal vector to this surface, $n_a n^a = \pm 1$, and $n^A_a = n^a e^A_a$, its spinorial representation. $n^A_a$ can be considered as an Hermitian metric for spinors on $\Sigma$, which allow us to introduce an operation of Hermitian conjugation for such spinors

\[
\mu^A = n^A \bar{\mu}^A,
\]

where bar means complex conjugation. This operation is involutive $(\mu^A)^\dagger = \pm \mu_A$, where ‘+’ stays for a timelike surface and ‘-’ for a spacelike one.

The operation of Hermitian conjugation allows one to define a new type of connection on the surface $\Sigma$. In four dimensions the only relevant completely covariant connection is the torsion free one. We will denote it simply by $\nabla$:

\[
\nabla \wedge e^A_A = de^A_A + A^B_A \wedge e^A_B - e^B_A \wedge A^+_{B'} A' = 0.
\]

Here $A^B_A$ and $A^+_{B'} A'$ are anti-self-dual and self-dual parts of the torsion-free connection which act on unprimed (left-handed) and primed (right-handed) spinors respectively. They are related to each other by complex conjugation:

\[
A^+_{B'} A' = \bar{A}^{-A'}_{B'}.
\]

Along with the torsion-free covariant derivative one can define purely anti-self-dual and purely self-dual covariant derivatives. Let $\mu^{AA'}$ be an arbitrary spinor with one primed and one unprimed indices. Then we define

\[
\nabla^-_a \mu^{AA'} = \partial_a \mu^{AA'} + A^-_{aB} \mu^{BA'}
\]

\[
\nabla^+_a \mu^{AA'} = \partial_a \mu^{AA'} + A^+_{aB'} \mu^{AB'}.
\]

These “covariant” derivatives are not completely covariant. The first of them restricts the gauge covariance to anti-self dual transformations and the second to self-dual. However they may give rise to fully covariant derivatives when projected on the surface.
Below, unless otherwise stated, $\Sigma$ is a spacelike slice of spacetime, $n_{A'B}$ is the spinorial representation of timelike unit normal vector and $e^A_B = e^{AA'}n_{A'B} - \frac{1}{2}e^{CA'}n_{A'C}\delta_B^A$ is the triad on $\Sigma$.

Let us first show that the torsion of (anti)self-dual connection defined by (33) projected on a timelike surface $S$ is ADM energy-momentum (generally $S$ is supposed to be taken to infinity although it doesn’t necessarily have to). Let us consider the second term in the r.h.s of (20) which is the only term dependent on metric needed for consistency of the action principle if we don’t care about covariance. If we take a spacial slice $\Sigma$ and make a 3+1 decomposition this action will read

$$S_S = \ldots + \int dt \int e^{A'}_A \wedge A^A_B \wedge e^{B}_{A'}.$$  

(34)

ADM energy and momentum are coefficients in front of lapse and shift functions in the above integral, and given that $e^{A'}_A n_{A'}^A = N$ and $e^{A'}_{iA} e_{iA} = N_i$ they are

$$E_{ADM} = \frac{1}{4\pi G} \int_{S \cap \Sigma} e^A_B \wedge A^B_A,$$

$$\left( P_{ADM} \right)^A_B = \frac{1}{4\pi G} \int_{S \cap \Sigma} (e^A_C \wedge A^C_B - \frac{1}{2}e^D_C \wedge A^D_C \delta^A_B),$$  

(35)

where the symmetric pare of spinorial indices $A, B$ labels 3 spacelike directions. The same expressions written in Ashtekar canonical variables can be found e.g. in [19]. Now taking into account that the total connection is torsion-free it is easy to see that the expressions entering the first and the second integral in (35) are components of the torsion of the connection $\nabla + \wedge e^A_A$, projected on $n_{A'}^A$ and orthogonal to $n_{A'}^A$ respectively.

The above expressions for energy and momenta are simple and have been proved to be zero in the vacuum, however they are not covariant and do not form any algebraic structure from the point of view of boundary theory. Below we consider covariant expressions for energy and momenta given by the boundary constraints (28).

First, let us notice that the constraints from the first line in (28) define quasilocal angular momenta of the bulk theory.

$$J^A_B = \frac{1}{4\pi G} \int_{\Sigma \cap S} \left( \frac{1}{\Lambda} F^A_B + \sigma^A_C \wedge \sigma^C_B \right).$$  

(36)

Indeed $C^A_B$ in (28) are boundary terms resulting from the variation of the Gaussian constraints of the bulk theory, generating local Lorentz transformations. One can fix a tetrad on the boundary so that it include gauge condition aligning intrinsic Lorentz frame with the global basis of boundary spacetime. So intrinsic Lorentz transformation identified with global ones and the operator generating them becomes the angular momentum of the theory.

The rest of constraints (28) are components of torsion of purely self-dual connection the triad $\sigma^A_B$ which is the tetrad $e^{AA'}$ projected on $S$. The connection entering this torsion can be equally understood as the four-dimensional torsion-free connection:

$$\nabla^+ \wedge \sigma^A_B = \nabla \wedge \sigma^A_B.$$  

(37)
From the fact that the four-dimensional covariant derivative annihilates the spacetime tetrad $e^{AA'}$, it follows that only the derivative of the normal vector $n_{A'B'}$ contributes to the torsion. This means that the torsion is related to the extrinsic curvature of $\Sigma$.

\[
\nabla \wedge \sigma_B^A = \nabla(n_{A'B'}) \wedge e^{AA'} = (dn_{A'B'} + A_C^{A'} n_{A'C} - A_C^{A'} n_{C'B'}) n_{D'}^A \wedge \sigma^D
\]

(38)

Here in the last line we introduced a self-dual connection acting on left-handed spinors

\[
A_B^{+A} = n_A^A dn_{A'}^B - n_B^B A_B^{+B'} n_C^C = n_A^A \nabla^+ n_B^A
\]

(39)

There are some simple relations between self-dual and anti-self-dual connections acting on spinors of the same chirality. First, it follows from (38) that

\[
A_B^{+A} - A_C^{+A} = iK_B^A,
\]

(40)

where $K_B^A$ is the tensor of extrinsic curvature of $\Sigma$. On the other hand the usual reality condition for Ashtekar variables means that

\[
A_B^{+A} + A_C^{-A} = \Gamma_B^A,
\]

(41)

where $\Gamma_B^A$ is the connection which is torsion-free on $\Sigma$:

\[
\nabla \Gamma \wedge \sigma_B^A = 0.
\]

(42)

Similarly one can introduce an anti-self-dual connection acting on right-handed spinors

\[
A_B^{-A'} = n_A^{A'} \nabla^- n_B^{A'}.
\]

(43)

Now we should relate $H_B^A$ in (28) to energy and momenta. Let $\tau_{0A'}$, $\mu = 0, 1, 2$ be 3 generator of $\text{SL}(2, \mathbb{R})$ group, which is the restriction of $\text{SL}(2, \mathbb{C})$ on timelike slice, $\mu = 0$ corresponding to rotation and $\mu = 1, 2$ corresponding to boosts. By using (38) and (40) it is easy to show that

\[
\begin{align*}
\left. (\nabla \wedge \sigma_B^A) \right|_{\Sigma \cap S} &= \tau_{0A}^B \left. (K_C^A \wedge \sigma_B^C) \right|_{\Sigma \cap S} = \det(\sigma_{\Sigma \cap S}) K_a^a \\
\left. (\nabla \wedge \sigma_A^B) \right|_{\Sigma \cap S} &= \tau_{iA}^B \left. (K_C^A \wedge \sigma_B^C) \right|_{\Sigma \cap S} = \det(\sigma_{\Sigma \cap S}) K_i^a,
\end{align*}
\]

(44)

where $*$ denotes hodge dual with respect to the volume form on $\Sigma \cap S$. In the r.h.s. of the first line of (45) we recognize the density which when integrated over $\Sigma \cap S$ give rise to the Brown-York quasilocal energy and in the second line we find quasilocal momentum. Thus we can write down the relation between constraints $H_B^A$ from (28) and quasilocal energy-momentum as follows

\[
\begin{align*}
E &= i\tau_{0A}^B \frac{1}{4\pi G} \int \nabla \wedge \sigma_B^A, \\
P_i &= i\tau_{iA}^B \frac{1}{4\pi G} \int \nabla \wedge \sigma_B^A.
\end{align*}
\]

(45)
The expressions are not simply related to the basic canonical variables of the bulk theory and they do not vanish in the vacuum. However they can be related to simple ADM expressions \(E_{ADM} = E - E_{ref}\)

\[(P_{ADM})^A_B + B^A_i = P_i - (P_i)_{ref},\]  

(46)

where the subscript \(ref\) means 'calculated in a reference spacetime'. Thus simple form can be restored at the cost of covariance. The advantage of this form is that quantities with subscript \(ref\) are non-dynamical (they are c-numbers), and we can calculate bulk commutators of quasilocal quantities by using simple ADM expressions. More detailed description of Brown-York energy in Ashtekar variables can be found in the paper of Lau \[21\].

5 Einstein’s equations as a local conservation law

It is interesting to notice that the torsion of the connection \(33\) with the spacetime tetrad

\[T^A_A' = \nabla^- \wedge e^A_A'\]  

(47)

projection of which on a boundary defines the ADM energy is a locally conserved quantity in a covariant sense. Indeed the covariant divergence of \(T^A_A'\) vanishes due to Einstein’s equations:

\[\nabla \wedge T^A_A' = \nabla \wedge \nabla^- \wedge e^A_A' = F^{-B}_A \wedge e^A_B = 0.\]  

(48)

Here \(F^{-B}_A\) is the curvature of the self-dual connection. The quantities \(C^A_B\) from \(28\) which define quasilocal angular momentum when projected on the boundary are also locally conserved due to the total connection being torsion-free:

\[\nabla \wedge C^A_B = \nabla \wedge \left(\frac{1}{\Lambda} F^A_B + \sigma^A_C \wedge \sigma^C_B\right) = \nabla \wedge \left(e^A_A' \wedge e^A_B'\right) = 0.\]  

(49)

From \(48,49\) it follows that the complete set of equations of GR is simply equivalent to the condition of conservation of \(T^A_A'\) and \(C^A_B\). Therefore all the Einstein’s equations can be put in a form of a local conservation law.

One of the conserved quantities, \(T^A_A'\), is not a tensor (it does not transform covariantly with respect to right-handed gauge transformations). This is like the divergence-freeness condition of stress-energy tensor can be reexpressed as a genuine conservation law for some pseudotensor. In the present situation we can however rewrite all the equation as a conservation of a covariant quantities. This can be done on an arbitrary slice of spacetime. To each such slice one can associate a triad \(e^A_A\) which is the projection of the tetrad \(e^A_A\) on it. The torsion of \(\sigma^A_B\), \(H^A_B = \nabla \wedge \sigma^A_B\), is a covariant quantity. In particular, if the slice is timelike this is a constraint \(23\) of the boundary theory \(23\). The covariant divergence of \(H^A_B\) is equivalent to a subset of Einstein’s equations:

\[\nabla \wedge H^A_B = \nabla \wedge \nabla \wedge \sigma^A_B = F^A_C \wedge \sigma^C_B = 0\]  

(50)

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These equations are not necessarily defined on a single slice. One can consider one parameter family of slices or foliation of the whole spacetime to define it everywhere. One foliation is however not enough to recover the whole set of Einstein's equations (equations with components normal to the foliation are still missing). At least two foliations which are different everywhere are required.

The equations (48,49) pulled back on a spacial slice $\Sigma$ form the complete set of constraints of GR. Formally all the constraints are now Gaussian – they say that the electric field $C^A_B$ and some other field $T^A_A'$ are divergence-free. Quantum-mechanically this form of constraints is very difficult to treat because of complicated dependance of $H^A_B$ and even $T^A_A'$ on the basic canonical variables. However it provides us with some intuition about what generic solutions of the bulk theory must look like. If we introduce a perturbation on the boundary which has certain energy and momentum it has to be continued into the bulk as a certain flux of energy-momentum. Constraint equations simply mean that the lines of such fluxes must be continuous.

6 Discussion: Towards the definition of excitations around the vacuum

In this section we are making a proposal on how to define excitations around the vacuum which will be developed in subsequent papers.

Any physical excitation must have nonzero energy. As the energy is defined on a boundary of spacetime, any excitation must be seen from the boundary. Energy together with other boundary observables, such as momentum and angular momentum, provide exactly enough information to specify any excitation. Therefore to introduce an excitation in the vacuum one just has to change somehow the value of the boundary observables.

A natural (and in fact the simplest) way to give a nonzero value to the boundary observables is to put up a topological puncture on the boundary. Because the symmetry of the boundary theory is SO(3,1), containing rotations and boosts parallel to the boundary, and translations tangential to it, to each puncture one can associate a representation of SO(3,1) group (a charge of SO(3,1) gauge field) labeled by casimirs such as mass and spin, and a vector in this representation. This vector specifies the momentum and the polarization of a particle moving along the boundary.

The constraints of general relativity must then continue the excitation introduced on the boundary into the bulk. Because puncture on the boundary is an SO(3,1) charge it must induce a flux of SO(3,1) gauge field. As it was discussed in section 5 the constraints simply mean that the lines of such fluxes must be continuous.

The continuous lines of fluxes of a gauge field may be expressed as Wilson lines of this field. Now the gauge group is SO(3,1) and the Wilson line is

$$W(a) = Pe^{\int a_a dx^a},$$

where the connection $a$ is that from (26).

The loops (51) are very difficult to treat quantum-mechanically, because the connection $a$ entering them is non-commutative with respect to the bulk simplectic form and
the expression for a commutator between different components of this connection is very involved. It is therefore difficult to say if the loops (51) indeed satisfy all the constraints of general relativity or not. Yet (although in a different context) a formalism treating non-commutative loops has been developed [23] and some definite physical predictions have been extracted from it.

The bulk equations of motion can in turn put some constraints on the boundary observables, in particular they can relate masses and spins of particles propagating along the boundary. A glimpse of such a constraint can be found in the work of Thiemann [24] on ADM energy of spin-network states. The loops with commutative connection considered there are special cases of loops (51) where the representation is labeled by only one casimir which is the spin $j$. In [24] it was shown that in a simple case in which the ADM energy can be diagonalized its eigenvalues scale like

$$ E_{ADM} \sim \sqrt{j(j+1)}. $$

(52)

Similar expression for quasilocal energy was obtained in [25] for a completely different regularization. So the relation (52) seems to be generic. When momentum is zero the energy is proportional to the mass of the particle and as a consequence for large spins we have the following relation

$$ m^2 \sim j. $$

(53)

This is Regge type of relation for string oscillation modes. This may indicate that some sort of string theory may arise here. Of course this doesn’t have to be any kind of critical strings so far known.

The problem with (52) is that the energy spectrum is discrete and therefore cannot describe propagating particles (they must have continuous kinetic energy). This can be solved by including the more general type of loops (51) for SO(3,1) group for which the spectrum must be continuous. The last statement is somehow supported by some of the results of [23].

All the above will be developed in subsequent papers.

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