Route to Lambda in conformally coupled phantom cosmology

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In this letter we investigate acceleration in the flat cosmological model with a conformally coupled phantom field and we show that acceleration is its generic feature. We reduce the dynamics of the model to a 3-dimensional dynamical system and analyze it on an invariant 2-dimensional submanifold. Then the concordance FRW model with the cosmological constant Λ is a global attractor situated on a 2-dimensional invariant space. We also study the behaviour near this attractor, which can be approximated by the dynamics of the linearized part of the system. We demonstrate that trajectories of the conformally coupled phantom scalar field with a simple quadratic potential crosses the cosmological constant barrier infinitely many times in the phase space. The universal behaviour of the scalar field and its potential is also calculated. We conclude that the phantom scalar field conformally coupled to gravity gives a natural dynamical mechanism of concentration of the equation of state coefficient around the magical value \( w_{\text{eff}} = -1 \). We demonstrate route to Lambda through the infinite times crossing the \( w_{\text{eff}} = -1 \) phantom divide.

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At present the scalar fields play a crucial role in modern cosmology. In an inflationary scenario they generate an exponential rate of evolution of the universe as well as density fluctuations due to vacuum energy. The Lagrangian for a phantom scalar field on the background of the Friedmann-Robertson-Walker (FRW) universe is assumed in the form

\[
\mathcal{L}_\psi = \frac{1}{2} [g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + \xi R \psi^2 - 2U(\psi)],
\]

where \( g^{\mu\nu} \) is the metric of the spacetime manifold, \( \psi = \psi(t) \), \( t \) is the cosmological time, \( R = R(g) \) is the Ricci scalar for the spacetime metric, \( \xi \) is a coupling constant which assumes zero for a scalar field minimally coupled to gravity and 1/6 for a conformally coupled scalar field, \( U(\psi) \) is a potential of the scalar field.

The minimally coupled slowly evolving scalar fields with a potential function \( U(\psi) \) are good candidates for a description of dark energy. In this model, called quintessence [1,2], the energy density and pressure from the scalar field are \( \rho_\psi = -1/2 \dot{\psi}^2 + U(\psi) \), \( p_\psi = -1/2 \dot{\psi}^2 - U(\psi) \). From recent studies of observational constraints we obtain that \( w_\psi = \rho_\psi/p_\psi < -0.55 \) [3]. This model has been also extended to the case of a complex scalar field [4,5].

Observations of distant supernovae support the cosmological constant term which corresponds to the case \( \dot{\psi} \sim 0 \). Then we obtain that \( w_\psi = -1 \). But there emerge two problems in this context. Namely, the fine tuning and the cosmic coincidence problems. The first problem comes from the quantum field theory where the vacuum expectation value is of 123 orders of magnitude larger than the observed value of \( 10^{-47} \text{GeV} \). The lack of a fundamental mechanism which sets the cosmological constant almost zero is called the cosmological constant problem. The second problem called “cosmic conundrum” is a question why the energy densities of both dark energy and dark matter are nearly equal at the present epoch.

One of the solutions to this problem offers the idea of quintessence, which is a version of the time varying cosmological constant conception. Quintessence solves the first problem through the decaying Λ term from the beginning of the Universe to a small value observed at the present epoch. Also the ratio of energy density of this field to the matter density increases slowly during the expansion of the Universe because the specific feature of this model is the variation of the coefficient of the equation of state with respect to time. The quintessence models [2,6] describe the dark energy with the time varying equation of state for which \( w_X > -1 \), but recently quintessence models have been extended to the phantom quintessence models with \( w_X < -1 \). In this class of models the weak energy condition is violated and
such a theoretical possibility is realized by a scalar field with a switched sign in the kinetic term \( \dot{\psi}^2 \rightarrow -\dot{\psi}^2 \). From theoretical point of view it is necessary to explore different evolutionary scenarios for dark energy which provide a simple and natural transition to \( w_X = -1 \). The methods of dynamical systems with notion of attractor (a limit set with an open inset) offers the possibility of description of transition trajectories to the regime with \( w_X = -1 \). Moreover they demonstrate whether this mechanism is generic.

Inflation and quintessence with non-minimal coupling constant are studied in the context of formulation of necessary conditions for the acceleration of the universe \( 10 \) (see also \( 11, 12 \)). We can find two important arguments which favour the choice of conformal coupling over \( \xi \neq 1/6 \). The first, equation for the massless scalar field is conformally invariant \( 13, 14 \). The second argument is that if the scalar field satisfy Klein-Gordon equation in the curved space then \( \psi \) does not violate the equivalence principle, and \( \xi \) is forced to assume the value \( 1/6 \) \( 15 \).

While recent astronomical observations give support that the equation of state parameter for dark energy is close to constant value \(-1\) they do not give a “corridor” around this value. Moreover, Alam et al. \( 16 \) pointed out that evolving state parameter is favoured over constant \( w_X = -1 \). The first step in the direction of description of the dynamics of the dark energy seems to be investigation of the system with evolving dark energy in the close neighbourhood of the value \( w_X = -1 \). For this aim we linearize dynamical system at this critical point and then describe the system in a good approximation (following the Hartman-Grobman theorem \( 17 \)) by its linearized part.

Other dark energy models like the Chaplygin gas model \( 18, 19, 20 \), \( 9, \) and references therein\) and the model with tachyonic matter can also be interpreted in terms of a scalar field with some form of a potential function.

Recent applications of the Bayesian framework \( 21, 22, 23, 24, 25 \) of model selection to the broad class of cosmological models with acceleration indicate that a posteriori probability for the ΛCDM model is 96%. Therefore the explanation why the current universe is such close to the ΛCDM model seems to be a major challenge for modern theoretical cosmology.

In this letter we present the simplest mechanism of concentration around \( w_X = -1 \) basing on the influence of a single scalar field conformally coupled to gravity acting in the radiation epoch. Phantom cosmology non-minimally coupled to the Ricci scalar was explored in the context of superquintessence \((w_X < -1)\) by Faraoni \( 24, 27 \) and there was pointed out that the superacceleration regime can be achieved by the conformally coupled scalar field in contrast to the minimally coupled scalar field.

Let us consider the flat FRW model which contains a negative kinetic scalar field conformally coupled to gravity \((\xi = 1/6)\) (phantom) with the potential function \( U(\psi) \). For the simplicity of presentation we assume \( U(\psi) \propto \psi^2 \). In this model the phantom scalar field is coupled to gravity via the term \( \xi R \psi^2 \). We consider massive scalar fields (for recent discussion of cosmological implications of massive and massless scalar fields see \( 28 \)). The dynamics of a non-minimally coupled scalar field for some self-interacting potential \( U(\psi) \) and for an arbitrary \( \xi \) is equivalent to the action of the phantom scalar field (which behaves like a perfect fluid) with energy density \( \rho_\psi \) and pressure \( p_\psi \) \( 29 \)

\[
\rho_\psi = -\frac{1}{2} \dot{\psi}^2 + U(\psi) - 3\xi H^2 \psi^2 - 3\xi H (\psi^2),
\]

\[
p_\psi = -\frac{1}{2} \dot{\psi}^2 - U(\psi) + \xi \left[ 2H(\psi^2) + (\dot{\psi}^2) \right] + \xi \left[ 2H + 3H^2 \right] \psi^2,
\]

where the conservation condition \( \dot{\rho}_\psi = -3H(\rho_\psi + p_\psi) \) gives rise to the equation of motion for the field

\[
\ddot{\psi} + 3H \dot{\psi} + \xi R \psi^2 - U'(\psi) = 0,
\]

where \( R = 6(H^2 + 2H^2) \) is the Ricci scalar.

Let us assume that both the homogeneous scalar field \( \psi(t) \) and the potential \( U(\psi) \) depend on time through the scale factor, i.e.

\[
\psi(t) = \psi(a(t)), \quad U(\psi) = U(\psi(a));
\]

then due to this simplified assumption the coefficient of the equation of state \( w_\psi \) is parameterized by the scale factor only

\[
w_\psi = w_\psi(a), \quad p_\psi = w_\psi(a) \rho_\psi(a),
\]

and

\[
w_\psi = \frac{-\frac{1}{2} \dot{\psi}^2 H^2 a^2 - U(\psi) + \xi \left[ 2(\psi^2)H^2 a + (\dot{\psi}^2) \right] + \xi \left[ \dot{H} + 3H^2 \right] \psi^2}{-\frac{1}{2} \dot{\psi}^2 H^2 a^2 + U(\psi) - 3\xi H^2 \psi^2 - 3\xi (\psi^2)H^2 a}
\]

\( \text{Eq. (7)} \)
where prime denotes the differentiation with respect to the scale factor.

We assume the flat model with the FRW geometry, i.e., the line element has the form

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d	heta^2 + \sin^2 \theta d\varphi^2)],$$  \(8\)

where \(0 \leq \varphi \leq 2\pi, 0 \leq \theta \leq \pi\) and \(0 \leq r \leq \infty\) are comoving coordinates, \(t\) stands for the cosmological time. It is also assumed that a source of gravity is the phantom scalar field \(\psi\) with the conformal coupling to gravity \(\xi = 1/6\). The dynamics is governed by the action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ m_p^2 R + (g^\mu\nu \partial_\mu \psi \partial_\nu \psi + \frac{1}{6} R \psi^2 - 2U(\psi)) \right]$$  \(9\)

where \(m_p^2 = (8\pi G)^{-1}\); for simplicity and without lost of generality we assume \(4\pi G/3 = 1\) and \(U(\psi)\) is the scalar field potential

$$U(\psi) = \frac{1}{2} m^2 \psi^2.$$  \(10\)

After dropping the full derivatives with respect to time, rescaling phantom field \(\psi \to \phi = \psi a\) and the time variable to the conformal time \(dt = a d\eta\) we obtain the energy conservation condition

$$E = \frac{1}{2} a'^2 + \frac{1}{2} a^2 \phi'^2 - \frac{1}{2} m^2 a^2 \phi^2 = \rho_{r,0}$$  \(11\)

where \(\rho_{r,0}\) is constant corresponding to the radiation in the model. The equations of motion are

$$\begin{cases}
a'' = m^2 a \phi^2, \\
\phi'' = m^2 a^2 \phi
\end{cases}$$  \(12\)

where a prime denotes the differentiation with respect to the conformal time \(dt = a d\eta\) and \(m^2 > 0\).

From the energy conservation condition we have

$$\frac{1}{2} a'^2 = \frac{\rho_{r,0} + \frac{1}{2} m^2 a^2 \phi^2}{1 + \phi^2}$$  \(13\)

and now from the equations of motion \(12\) we receive

$$(\rho_{r,0} + \frac{1}{2} m^2 a^2 \phi^2) \dot{\phi} + \frac{1}{2} m^2 a \phi (1 + \phi^2) (\phi \dot{\phi} - a) = 0.$$  \(14\)

The effective equation of state parameter is

$$w_{\text{eff}} = \frac{p_\phi + \frac{4}{3} \rho_r}{\rho_\phi + \rho_r},$$  \(15\)

for our model this parameter reduces to

$$w_{\text{eff}} = -\frac{1}{3} \left[ \frac{1}{2} \phi'^2 + \frac{1}{2} m^2 a^2 \phi^2 - \rho_{r,0} \right]$$  \(16\)

where a prime denotes the differentiation with respect to the conformal time and finally taking into account equation \(13\) we have

$$w_{\text{eff}} = -\frac{1}{3} \left[ \frac{1}{2} \phi'^2 + \frac{1}{2} m^2 a^2 \phi^2 - \rho_{r,0} \right] (1 + \phi^2).$$  \(17\)

For \(a, \phi \gg \rho_{r,0}\) this equation reduces to

$$w_{\text{eff}} = -\frac{1}{3} (2 \dot{\phi}^2 + 1),$$  \(18\)

and it is clear that for any value of \(\dot{\phi}\) \(w_{\text{eff}}\) is always negative.
To analyze equation (14) we reintroduce the original phantom field variable $\psi = \frac{\dot{a}}{a}$ and $da/a = d \ln a$. Now equation (14) reads

$$\left( \rho_r \frac{\dot{a}}{a^2} + \frac{1}{2} m^2 \psi^2 \right) (\psi'' + \psi') + \frac{1}{2} m^2 \psi (1 + (\psi' + \psi)^2) (\psi (\psi' + \psi) - 1) = 0$$

where a prime now denotes the differentiation with respect to a natural logarithm of the scale factor. Introducing new variables $y = \psi'$ and $\rho_r = \rho_r \dot{a} a^{-4}$ we can represent this equation as an autonomous dynamical system

$$\psi' = y$$
$$y' = -y - \frac{1}{2} m^2 \psi (\psi (y + \psi) - 1)(1 + (y + \psi)^2)$$
$$\rho_r' = -4 \rho_r.$$

There are the two critical points in the phase space $(\psi, y, \rho_r)$, namely $\psi = \pm 1$, $y = 0$, $\rho_r = 0$. The linearization matrix reads

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -2(1 + \psi^2) & -1 - (1 + \psi^2) & 2 m^2 \psi (\psi^2 - 1)(1 + \psi^2) \\ 0 & 0 & -4 \end{bmatrix}.$$  

The eigenvalues for this matrix are $\lambda_{1,2} = \frac{1}{2} (-3 \pm i \sqrt{7})$ and $\lambda_3 = -4$.

To find a global phase portrait it is necessary to study the system in the neighbourhood of the critical points which correspond, from the physical point of view, stationary states (or asymptotic solutions). Then the Hartman-Grobman theorem guaranties us that the linearized system at this point is a well approximation of the nonlinear system. First, we must note that $\rho_r = 0$ is in the invariant submanifold of the 3-dimensional nonlinear system. It is also useful to calculate the eigenvectors for any eigenvalue. We obtain following eigenvectors

$$v_{1,2} = \begin{bmatrix} -\frac{3 \sqrt{7}}{8} \\ 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} \frac{7}{8} \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$  

They are helpful in construction of the exact solution of the linearized system

$$\vec{x}(t) = \vec{x}(0) \exp t \begin{bmatrix} 0 & 1 & 0 \\ -4 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3 \sqrt{7}}{8} & x_0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-4t} \\ e^{-\frac{3}{2}t} \cos \frac{\sqrt{7}}{2} t \\ e^{-\frac{3}{2}t} \sin \frac{\sqrt{7}}{2} t \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{3}{2}t} \cos \frac{\sqrt{7}}{2} t \\ e^{-\frac{3}{2}t} \sin \frac{\sqrt{7}}{2} t \end{bmatrix} \begin{bmatrix} x_0 \\ x_0 \\ x_0 \end{bmatrix},$$

where $x = \psi - \psi_0$, $y = \psi' - \psi_0'$, $z = \rho_r - \rho_0$ and $x_0, y_0, z_0$ are initial conditions and we have substituted $\ln a = t$.

If we consider linearized system on the invariant stable submanifold $z = 0$, it is easy to find the exact solution. If we return to the original variables $\psi, \psi'$, then $\psi(\ln a)$ is the solution of the linear equation

$$(\psi - \psi_0)'' + 3(\psi - \psi_0)' + 4(\psi - \psi_0) = 0,$$

i. e.,

$$\psi(\psi_0) = C_1 \exp \left( -\frac{3}{2} \ln a \right) \cos \left( \frac{\sqrt{7}}{2} \ln a \right) + C_2 \exp \left( -\frac{3}{2} \ln a \right) \sin \left( \frac{\sqrt{7}}{2} \ln a \right)$$

or

$$\phi(\phi_0) = C_1 a^{-\frac{3}{2}} \cos \left( \frac{\sqrt{7}}{2} \ln a \right) + C_2 a^{-\frac{3}{2}} \sin \left( \frac{\sqrt{7}}{2} \ln a \right).$$

Because of the lack of alternatives to the mysterious cosmological constant [21, 22] we allow that energy might vary in time following assumed a priori parameterization of $\psi(z)$. In the popular parameterization [20, 31, 32, 33] appears a free function in most scenarios which is a source of difficulties in constraining parameters by observations. However most parameterizations of the dark energy equation of state cannot reflect real dynamics of cosmological models with
dark energy. The assumed form of \( w(z) \) can be incompatible with the \( w(z) \) obtained from the underlying dynamics of the cosmological model. For example some of parameters can be determined from the dynamics which can be crucial in testing and selection of cosmological models \[21\]. Our point of view is to obtain the form of \( w(z) \) specific for given class of cosmological models from dynamics of this models and apply it in further analysis both theoretical and empirical. In practice we put the cosmological model in the form of the dynamical system and linearize it around the neighbourhood of the present epoch to find the exact formula of \( w(z) \). For the phantom scalar field model this incompatibility manifests by the presence of a focus type critical point (therefore damping oscillations) in the phase space rather than a stable node (Fig. 1 and its 3D version Fig. 2).

The properties of the minimally coupled phantom field in the FRW cosmology using the phase portrait have been investigated by Singh et al. \[33\] (see also \[32\] for more recent studies). Authors showed the existence of the deSitter attractor and damped oscillations (the ratio of the kinetic to the potential energy \([T/U]\) to oscillate to zero).

We can also express \( w_{\text{eff}} \) in these new variables

\[
\begin{align*}
\dot{w}_{\text{eff}} &= \frac{1}{3} \left\{ (\psi + \psi')^2 - \frac{\rho_r - \frac{1}{2} m^2 \psi^2}{\rho_r + \frac{1}{2} m^2 \psi^2} [1 + (\psi + \psi')^2] \right\} \\

\ddot{w}_{\text{eff}} &= -\frac{2}{3} \left\{ \frac{m^2 \psi^2}{\rho_r + \frac{1}{2} m^2 \psi^2} (\psi + \psi')(\psi' + \psi'') + \rho_\gamma m^2 \psi^2 \frac{2 + \psi'}{\rho_r + \frac{1}{2} m^2 \psi^2} [1 + (\psi + \psi')^2] \right\}. 
\end{align*}
\]

Recently Caldwell and Linder \[36\] have discussed dynamics of quintessence models of dark energy in terms of \( w - w' \) phase variables, where \( w' \) was the differentiation with respect to the logarithm of the scale factor. These methods were extended to the phantom and quintom models of dark energy \[37, 38\]. Guo et al. \[38\] examined the two-field quintom models as the illustration of the simplest model of transition across the \( w_X = -1 \) barrier. The interesting mechanism of acceleration with a periodic crossing of the \( w = -1 \) barrier have been recently discussed in the context of the cubic superstring field theory \[39\]. In the model under consideration we obtain this effect but trajectories cross the barrier infinitely many times. The main advantage of the discovered road to \( \Lambda \) is that it takes place in the simple flat FRW model with the quadratic potential of the scalar field.

It is easy to check that at the critical points \( \dot{w}_{\text{eff}} = -1 \) and \( \frac{\ddot{w}_{\text{eff}}}{d \ln a} = 0 \). Since these points are sinks there is infinite many crossings of \( w_{\text{eff}} = -1 \) during the evolution.

The methods of the Lyapunov function are useful in discussion of stability of the critical point of the non-linear system. The stability of any hyperbolic critical point of dynamical system is determined by the signs of the real parts of the eigenvalues \( \lambda_i \) of the Jacobi matrix. A hyperbolic critical point is asymptotically stable iff real \( \lambda_i < 0 \) \( \forall i \), if \( x_0 \) is a sink. The hyperbolic critical point is unstable iff it is either a source or a saddle. The method of the Lyapunov function is especially useful in deciding the stability of a non-hyperbolic critical points \[17, p.129\]. The construction of the Lyapunov function was used by \[40\] for demonstration that periodic behaviour of a single scalar field is not possible for minimally coupled phantom scalar field (see also \[41\]).

The quantity \( w'_{\text{eff}} \) in terms of \( w_{\text{eff}} \) and \((\ln \psi)'\) reads

\[
\dot{w}_{\text{eff}} = -(1 - 3w_{\text{eff}})(1 + w_{\text{eff}} + \frac{2}{3}(\ln \psi)'),
\]

It is interesting that equation \[29\] can be solved in terms of \( \bar{w}(a) \) – the mean of the equation of the state parameter in the logarithmic scale defined by Rahvar and Movahed \[42\] as

\[
\bar{w}(a) = \frac{\int_{\ln a}^a w(a')d(\ln a')}{\int_{\ln 1}^a d(\ln a')},
\]

namely:

\[
w(a) = \frac{1}{3} - 4a^{3(1+\bar{w}(a))} \psi^2.
\]

They argued that this phenomenological parameterization removes the fine tuning of dark energy and \( \rho_X / \rho_m \propto a^{-3\bar{w}(a)} \) approaches a unity at the early universe. Note that in \( \bar{w}(a) = -1 \) that

\[
w(z) + 1 = \frac{4}{3}(1 - \psi^2),
\]
FIG. 1: The phase portrait \((w_{\text{eff}}, w'_{\text{eff}})\) of the investigated model on the submanifold \(\rho_c = 0\). This figure illustrates the evolution of the dark energy equation of the state parameter as a function of redshift for different initial conditions. In all cases trajectories cross the boundary line \(w_{\text{eff}} = -1\) infinite many times but this state also represents the global attractor.

where

\[
\psi = \psi_0 + (1 + z)^{\frac{3}{2}} \left\{ C_1 \cos(\frac{\sqrt{7}}{2} \ln(1 + z)) - C_2 \sin(\frac{\sqrt{7}}{2} \ln(1 + z)) \right\}.
\]  

(33)

In Fig. 3 we present the relation \(w(z)\) for different values of parameters \(\psi_0 = \pm 1, C_1\) and \(C_2\).

In this letter we regarded the phantom scalar field conformally coupled to gravity in the context of the problem of acceleration of the Universe. We applied the methods of dynamical systems and the Hartman-Grobman theorem to find universal behaviour at the late times – damping oscillations around \(w_{\text{eff}} = -1\). We argued that most parameterizations of the dark energy, such as linear evolution of \(w(z)\) in redshift or the scale factor, cannot reflect realistic physical models because of the presence of non-hyperbolic critical point of a focus type on the phase plane \((w, w')\). We suggested a parameterization of a type

\[
w_X(z) = -1 + (1 + z)^{\frac{3}{2}} \left\{ C_1 \cos(\ln(1 + z)) + C_2 \sin(\ln(1 + z)) \right\}
\]

(34)

which parameterizes damping oscillations around \(w_X = -1\) “phantom divide”, and finally, with the help of this formula one can simply calculate energy density for dark energy \(\rho_X\)

\[
\rho_X = \rho_{X,0} \exp \left( -B \right) \exp \left( (1 + z)^{\frac{3}{2}} [A \sin(\ln(1 + z)) + B \cos(\ln(1 + z))] \right).
\]

(35)

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FIG. 2: The phase portrait of the 3-dimensional dynamical system (20) in terms of the variables \((w_{\text{eff}}, w'_{\text{eff}}, \rho_r)\) and projections of trajectories on the submanifold \((w_{\text{eff}}, w'_{\text{eff}}, 0)\). The critical point represents the state where \(w_{\text{eff}} = -1\) – the cosmological constant. The trajectories approach this point as the scale factor goes to infinity (or the redshift \(z \to -1\)). Before this stage the weak energy condition is violated infinite number of times.

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FIG. 3: The relation \( w(z) \) for different values of the parameters: \( C_1 = 0.01, C_2 = 0 \) dash-dotted line (red el. version); \( C_1 = 0, C_2 = -0.01 \) dotted line (green el. version); \( C_1 = 0.01, C_2 = -0.01 \) solid line (blue el. version) and \( \psi_0 = \pm 1 \). For all solutions \( w(z) \) approaches the cosmological constant as \( z \to -1 \). It is obvious that this relation is true only in the vicinity of the critical point \( (w_{\text{eff}}, w_{\text{eff}}', \rho_r) = (-1, 0, 0) \).

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