Geometric phase corrected by initial system-environment correlations

Sharoon Austin,1 Sheraz Zahid,1 and Adam Zaman Chaudhry1,∗

1School of Science & Engineering, Lahore University of Management Sciences (LUMS), Opposite Sector U, D.H.A, Lahore 54792, Pakistan

We find the geometric phase of a two-level system undergoing pure dephasing via interaction with an arbitrary environment, taking into account the effect of the initial system-environment correlations. We use our formalism to calculate the geometric phase for the two-level system in the presence of both harmonic oscillator and spin environments, and we consider the initial state of the two-level system to be prepared by a projective measurement or a unitary operation. The geometric phase is evaluated for a variety of parameters such as the system-environment coupling strength to show that the initial correlations can affect the geometric phase very significantly even for weak and moderate system-environment coupling strengths. Moreover, the correction to the geometric phase due to the system-environment coupling generally becomes smaller (and can even be zero) if initial system-environment correlations are taken into account, thus implying that the system-environment correlations can increase the robustness of the geometric phase.

PACS numbers: 03.65.-w, 03.65.Yz, 05.30.-d

I. INTRODUCTION

The geometric phase is the phase information acquired by a system due to its cyclic evolution in a curved parameter space [1,2]. This phenomenon was first studied by Pancharatnam in optics [3] and by Longuet-Higgins [4] and Stone [5] in quantum chemistry. Berry’s finding that the geometric phase arises generally in the study of closed quantum systems undergoing cyclic adiabatic evolutions ignited interest in the subject [6]. Aharonov and Anand thereafter generalized the geometric phase to non-adiabatic evolutions, showing that the phase depends on the geometry of the path followed by the system in the projective Hilbert space [7], while Uhlmann considered the geometric phase for mixed quantum states [8] which was further generalized by Sjoqvist et al. [9]. On the experimental front, the geometric phase has been observed in nuclear magnetic resonance [10], superconducting [11], and optical setups [12], amongst others.

Besides its theoretical importance, the geometric phase has practical applications as well. In particular, due to its geometric nature, the geometric phase may have intrinsic resistance to external noise, which makes it an attractive tool for robust quantum information processing [13–15]. It is then important to extend the study of the geometric phase to open quantum systems where the effect of the environment on the geometric phase can be investigated. Different approaches have been used to investigate the effect of the environment on the geometric phase [16–30]. In particular, emphasis has been on a single two-level system undergoing pure dephasing, that is, it is assumed that dephasing plays a much more dominant role compared to relaxation effects. In this case, starting from a product state of the two-level system and the environment in thermal equilibrium, the density matrix of the two-level system can be computed as a function of time, and the geometric phase can then be obtained. Of particular importance to us is Ref. [21] where the effect of non-Markovianity on the geometric phase is studied. Given that memory effects can play a role, it is then natural to consider the effect of initial system-environment correlations on the geometric phase as well [31–61]. The effect of the initial correlations is expected to be especially significant if the system-environment coupling is not weak, since in this case, the initial state can no longer be assumed to be a product state of the system and the environment thermal equilibrium state. However, to date, to the best of our knowledge, the effect of the initial system-environment correlations on the geometric phase has not been studied. In this work, we aim to study the geometric phase for the pure dephasing model, taking the initial system-environment correlations into account.

We start by deriving general expressions for the geometric phase of a two level system undergoing pure dephasing for both initially pure and mixed states. Our expressions are general in the sense that we do not make any assumptions regarding the form of the environment or the system-environment coupling, and they take the initial system-environment correlations into account. We then apply these expressions to two concrete well-known system-environment models: a two-level system undergoing dephasing via interaction with a harmonic oscillator environment, and a two-level system undergoing pure dephasing due to a spin environment. Both of these models are exactly solvable for arbitrary system-environment coupling strengths even if initial system-environment correlations are taken into account. The initial state of the two-level system is prepared either by performing a projective measurement on the system only (the initial state of the system is pure in this case), or by performing a unitary operation on the system (the initial state is now, in general, mixed). Using the exact solutions, we investigate the effect of the initial system-environment correlations on the geometric phase as various physical parameters...
such as the system-environment coupling strength and the temperature are varied. We find that, in general, the initial correlations can affect the geometric phase very significantly, even for weak and moderate system-environment coupling strengths. Interestingly, the initial correlations can make the geometric phase more robust; in fact, the correction to the geometric phase due to the environment can become zero for specific values of system-environment parameters if the initial correlations are taken into account.

This paper is organized as follows. In Sec. II, we derive expressions for the geometric phase of a two-level system undergoing pure dephasing for both initially pure and mixed system states. In Sec. III, we compute the geometric phase for a two-level system interacting with an environment of harmonic oscillators both with and without initial system-environment correlations. A similar task is performed for a spin environment in Sec. IV. Finally, we summarize our results in Sec. V. Details regarding the exact solutions of the system-environment models employed are presented in the appendices.

II. THE FORMALISM

A. Pure initial system state

Consider a two-level system with Hamiltonian \( H_S \) interacting with an arbitrary environment whose Hamiltonian is \( H_B \). The system-environment interaction is \( H_{SB} \). The total system-environment Hamiltonian is then

\[
H = H_S + H_B + H_{SB}.
\]

For a pure dephasing model, \([H_S, H_{SB}] = 0\), which means that the the eigenbasis of \( H_S \), the diagonal elements of the density matrix of the two-level system do not change. In this basis, the initial state of the two-level system (assumed to be pure) can be written as

\[
\rho(0) = \begin{bmatrix}
\cos^2 \left( \frac{\theta_0}{2} \right) & \frac{1}{2}\sin \theta_0 e^{-i\phi_0} \\
\frac{1}{2}\sin \theta_0 e^{i\phi_0} & \sin^2 \left( \frac{\theta_0}{2} \right)
\end{bmatrix}.
\]

Here \( 0 \leq \theta_0 \leq \pi, 0 \leq \phi_0 < 2\pi \) are the usual Bloch angles characterizing the initial system state. Since we are considering only pure dephasing, time evolution leads to a density matrix of the form

\[
\rho(t) = \begin{bmatrix}
\cos^2 \left( \frac{\theta_0}{2} \right)e^{-i\Omega(t)} & \frac{1}{2}\sin \theta_0 e^{-i\Omega(t)}e^{-\Gamma(t)} \\
\frac{1}{2}\sin \theta_0 e^{i\Omega(t)}e^{-\Gamma(t)} & \sin^2 \left( \frac{\theta_0}{2} \right)e^{-i\Omega(t)}e^{-\Gamma(t)}
\end{bmatrix}.
\]

It is important to note that the density matrix \( \rho(t) \) will have this form even in the presence of initial system-environment correlations - only the form of \( \Omega(t) \) and \( \Gamma(t) \) can be different. Now, in the Bloch vector representation, we can write \( \rho(t) \) as

\[
\rho(t) = \frac{1}{2} \left[ 1 + n_x \sigma_x + n_y \sigma_y + n_z \sigma_z \right],
\]

where \( n_x = \sin \theta_0 e^{-\Gamma(t)} \cos |\Omega(t)| \), \( n_y = \sin \theta_0 e^{-\Gamma(t)} \sin |\Omega(t)| \), and \( n_z = \cos \theta_0 \). Given the density matrix \( \rho(0) \), we can compute the geometric phase \( \Phi_G \) via [21]

\[
\Phi_G = \arg \left( \sum_{k=1}^{2} \sqrt{\varepsilon_k(0)\varepsilon_k(\tau)} \langle \varepsilon_k(0) | \varepsilon_k(\tau) \rangle e^{-\int_0^\tau dt \langle \varepsilon_k | \frac{\partial}{\partial t} | \varepsilon_k \rangle} \right).
\]

Here \( \varepsilon_k(t) \) are the eigenvalues of the density matrix \( \rho(t) \), \( |\varepsilon_k(t)\rangle \) are the eigenvectors, and \( \tau \) is the time after which the system completes a cyclic evolution. For our case, the eigenvalues of \( \rho(t) \) are

\[
\varepsilon_{\pm}(t) = \frac{1}{2} \left( 1 \pm \sqrt{1 + \sin^2 \theta_0 [e^{-2\Gamma(t)} - 1]} \right).
\]

Notice that the eigenvalues are independent of \( \Omega(t) \). Moreover, since \( \varepsilon_- (0) = 0 \), as is expected for a pure initial system state, our calculation of the geometric phase greatly simplifies. The corresponding eigenvectors of \( \rho(t) \) are

\[
|\varepsilon_+(t)\rangle = \cos \left( \frac{\theta}{2} \right) |0\rangle + e^{i\Omega(t)} \sin \left( \frac{\theta}{2} \right) |1\rangle,
\]

\[
|\varepsilon_-(t)\rangle = \sin \left( \frac{\theta}{2} \right) |0\rangle - e^{i\Omega(t)} \cos \left( \frac{\theta}{2} \right) |1\rangle,
\]

where

\[
\sin \theta = F(t)^{-1} \sin \theta_0 e^{-\Gamma(t)},
\]

\[
\cos \theta = F(t)^{-1} \cos \theta_0,
\]

\[
F(t) = \sqrt{1 + \sin^2 \theta_0 (e^{-2\Gamma(t)} - 1)},
\]

and \( |0\rangle \) and \( |1\rangle \) are the eigenstates of \( H_S \). Since \( \varepsilon_- (0) = 0 \),

\[
\Phi_G = \arg \left( \langle \varepsilon_+(0) | \varepsilon_+(\tau) \rangle \langle \psi_+(0) | \psi_+(\tau) \rangle e^{-\int_0^\tau dt \langle \varepsilon_+ | \frac{\partial}{\partial t} | \varepsilon_+ \rangle} \right).
\]

This further simplifies to

\[
\Phi_G = \arg \left( \langle \varepsilon_+(0) | \varepsilon_+(\tau) \rangle e^{-\int_0^\tau dt \langle \varepsilon_+ | \frac{\partial}{\partial t} | \varepsilon_+ \rangle} \right),
\]

since \( \sqrt{\varepsilon_+(0)\varepsilon_+(\tau)} \) is real. We also find that \( \langle \varepsilon_+ | \frac{\partial}{\partial t} | \varepsilon_+ \rangle = i\Omega \sin^2 \left( \frac{\theta}{2} \right) \), where the dot denotes the time derivative. Moreover,

\[
\langle \varepsilon_+(0) | \varepsilon_+(\tau) \rangle = \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta_0}{2} \right)
\]

\[\]

\[
+ \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta_0}{2} \right) \cos \left( \frac{\theta_0}{2} \right) \sin \left( \frac{\theta_0}{2} \right) \sin \left( \frac{\theta_0}{2} \right).
\]

The geometric phase can then be written as

\[
\Phi_G = \Phi_1 + \Phi_2,
\]

(8)
with $\Phi_{1} = -\int_{0}^{\tau} dt \Omega \sin^{2} \left( \frac{\theta}{2} \right)$, and $\Phi_{2} = \arg \left[ 1 + e^{i\Omega(t)}e^{-i\omega_{0}\tau} \sin \left( \frac{\theta}{2} \right) \tan \left( \frac{\theta}{2} \right) \tan \left( \frac{\theta}{2} \right) \right]$. To evaluate each of these one by one, we first note that $H_{S}$ has a characteristic frequency $\omega_{0}$ such that $\omega_{0}\tau = 2\pi$. Then, $\Omega(t)$ can be written as $\Omega(t) = \omega_{0} + \omega_{0}t + \chi(t)$, where $\chi(t)$ takes into account part of the effect of the system-environment coupling. It follows that

$$
\Phi_{1} = -\int_{0}^{\tau} dt \left( \omega_{0} + \chi(t) \right) \sin^{2} \left( \frac{\theta}{2} \right),
$$

which can be simplified to

$$
\Phi_{1} = -\pi - \frac{\chi(t)}{2} + \frac{\cos \theta_{0}}{2} I(\tau),
$$

with

$$
I(\tau) = \int_{0}^{\tau} \mathcal{F}(t)^{-1}[\omega_{0} + \chi(t)] dt.
$$

As for $\Phi_{2}$, we can write

$$
\Phi_{2} = \arg \left( 1 + e^{i\chi(t)} \tan \left[ \frac{\theta(0)}{2} \right] \tan \left[ \frac{\theta(\tau)}{2} \right] \right).
$$

Since $\tan \left( \frac{\theta}{2} \right) = \frac{\sin \theta}{1 + \cos \theta}$ and $\tan \theta = (\tan \theta_{0})e^{-\Gamma(t)}$, this further simplifies to

$$
\Phi_{2} = \arg \left( 1 + e^{i\chi(t)}e^{-\Gamma(t)} \frac{1 - \cos \theta_{0}}{\mathcal{F}(\tau) + \cos \theta_{0}} \right).
$$

With $\Phi_{1}$ and $\Phi_{2}$ found, we can thereby calculate $\Phi_{G}$. It should be noted that if the system-environment interaction strength is zero, we find that $\Phi_{1} = -\pi + \pi \cos \theta_{0}$ while $\Phi_{2} = 0$, thereby leading to the usual result $\Phi_{G} = -\pi + \pi \cos \theta_{0}$. Moreover, for $\theta_{0} = \pi/2$, $\Phi_{1} = -\pi - \frac{\chi(\tau)}{2}$ and $\Phi_{2} = \frac{\chi(\tau)}{2}$, meaning that $\Phi_{G} = -\pi$. Thus the geometric phase is robust for the states with $\theta_{0} = \pi/2$ even if initial correlations are taken into account. Consequently, we will consider $\theta_{0} \neq \pi/2$ to investigate the effect of the initial correlations on the geometric phase. Before doing so for concrete system-environment models, we generalize our results to the case where the initial state is mixed.

### B. Mixed initial system state

We now derive expressions for the geometric phase for initially mixed states. Our approach will be to write the state for the two-level system in a form similar to that in Eqs. (3) and (4) so that we obtain an expression for the geometric phase similar to that in Eq. (5). As such, we start by noting that the initial density matrix, even for a mixed state, can be written as

$$
\rho(0) = \begin{pmatrix}
\cos^{2} \left( \frac{\theta_{0}}{2} \right) & \frac{1}{2} e^{-\Gamma_{0}} \sin \theta_{0} e^{-i\phi_{0}} \\
\frac{1}{2} e^{-\Gamma_{0}} \sin \theta_{0} e^{i\phi_{0}} & \sin^{2} \left( \frac{\theta_{0}}{2} \right)
\end{pmatrix}.
$$

Note that $\tilde{\theta}_{0}$ is not a Bloch angle here. $\Gamma_{0} > 0$ takes into account that the initial state is mixed. It follows that

$$
\rho(t) = \begin{pmatrix}
\cos^{2} \left( \frac{\tilde{\theta}_{0}}{2} \right) & \frac{1}{2} \sin \tilde{\theta}_{0} e^{-i\Omega(t) - \Gamma_{0} - \Gamma(t)} \\
\frac{1}{2} \sin \tilde{\theta}_{0} e^{i\Omega(t) - \Gamma_{0} - \Gamma(t)} & \sin^{2} \left( \frac{\tilde{\theta}_{0}}{2} \right)
\end{pmatrix},
$$

with $\Omega(t) = \omega_{0}t + \chi(t) + \phi_{0}$ as before. The eigenvalues of the density matrix $\rho(t)$ are now a simple extension of Eq. (6), that is,

$$
\varepsilon_{\pm} = \frac{1}{2} \left( 1 \pm \tilde{\mathcal{F}}(t) \right),
$$

with

$$
\tilde{\mathcal{F}}(t) = \sqrt{1 + \sin^{2} \tilde{\theta}_{0}(e^{-2t\Gamma}e^{-2t\Gamma(t)} - 1)}.
$$

The corresponding eigenvectors are similarly

$$
|\varepsilon_{+}\rangle = \cos \left( \frac{\tilde{\theta}_{0}}{2} \right) |0\rangle + e^{i\Omega(t)} \sin \left( \frac{\tilde{\theta}_{0}}{2} \right) |1\rangle,
$$

$$
|\varepsilon_{-}\rangle = \sin \left( \frac{\tilde{\theta}_{0}}{2} \right) |0\rangle - e^{i\Omega(t)} \cos \left( \frac{\tilde{\theta}_{0}}{2} \right) |1\rangle,
$$

where $\sin \tilde{\theta} = \sin \tilde{\theta}_{0} e^{-\Gamma_{0}} e^{-\Gamma(t)} \tilde{\mathcal{F}}(t)^{-1}$ and $\cos \tilde{\theta} = \cos \tilde{\theta}_{0} \tilde{\mathcal{F}}(t)^{-1}$. With the density matrix $\rho(t)$ found, the geometric phase $\Phi_{G}$ can be written as

$$
\Phi_{G} = \Phi_{1} + \Phi_{2} + \Phi_{3},
$$

with

$$
\Phi_{1} = \arg \left( e^{-\int_{0}^{\tau} dt \langle \varepsilon_{+} | \mathcal{H} | \varepsilon_{+} \rangle} \right),
$$

$$
\Phi_{2} = \arg \langle \varepsilon_{+}(0) | \varepsilon_{+}(\tau) \rangle,
$$

$$
\Phi_{3} = \arg \left( 1 + \sqrt{\frac{\varepsilon_{-}(0) e^{\varepsilon_{-}(-\tau)} \langle \varepsilon_{-}(0) | \varepsilon_{-}(-\tau) \rangle}{\varepsilon_{+}(0) e^{\varepsilon_{+}(-\tau)} \langle \varepsilon_{+}(0) | \varepsilon_{+}(-\tau) \rangle} \times e^{\int_{0}^{\tau} dt \langle \varepsilon_{-} | \mathcal{H} | \varepsilon_{-} \rangle - \langle \varepsilon_{+} | \mathcal{H} | \varepsilon_{+} \rangle} \right).
$$

The calculations for $\Phi_{1}$ and $\Phi_{2}$ can be performed as done before to obtain

$$
\Phi_{1} = -\pi - \frac{\chi(\tau)}{2} + \frac{1}{2} \cos \tilde{\theta}_{0} \tilde{I}(\tau),
$$

where

$$
\tilde{I}(\tau) = \int_{0}^{\tau} dt \frac{\omega_{0} + \chi(t)}{\sqrt{1 + \sin^{2} \tilde{\theta}_{0}(e^{-2t\Gamma}e^{-2t\Gamma(t)} - 1)}},
$$

and

$$
\Phi_{2} = \arg \left( 1 + e^{i\chi(\tau)} \tan \left[ \frac{\theta(0)}{2} \right] \tan \left[ \frac{\theta(\tau)}{2} \right] \right).
$$
Finally, we compute \( \Phi_3 \) and find that

\[
\Phi_3 = \arg \left(1 + a(\tau)b(\tau) e^{-i \cos \theta_0 I(\tau)}\right),
\]

(16)

where

\[
a(\tau) = \sqrt{\frac{\xi_-(0) \xi_-(\tau)}{\xi_+(0) \xi_+(\tau)}},
\]

and

\[
b(\tau) = \frac{\tan \left(\frac{\theta(0)}{2}\right) \tan \left(\frac{\theta(\tau)}{2}\right) + e^{i \chi(\tau)}}{1 + e^{i \chi(\tau)} \tan \left(\frac{\theta(0)}{2}\right) \tan \left(\frac{\theta(\tau)}{2}\right)}.
\]

Finding the geometric phase now is simply a matter of finding the parameters \( \theta_0, \phi_0, \) and \( \Gamma_0 \) characterizing the initial state as well as the functions \( \Gamma(t) \) and \( \chi(t) \) that go into the time evolution of the system density matrix. It is important to realize that if the system-environment interaction is zero, the geometric phase is, in general, no longer \(-\pi + \pi \cos \theta_0\) since the initial state is mixed. However, for \( \theta_0 = \pi/2 \), we again obtain \( \Phi_3 = -\pi \).

We will now use the expressions for the geometric phase to perform calculations with concrete system-environment models, both with and without initial system-environment correlations.

### III. TWO-LEVEL SYSTEM INTERACTING WITH AN ENVIRONMENT OF HARMONIC OSCILLATORS

We first apply our formalism to the paradigmatic example of a single two-level system undergoing pure dephasing via interaction with a collection of harmonic oscillators [62]. The total system-environment Hamiltonian is \( H = H_S + H_B + H_{SB} \), where (we set \( \hbar = 1 \) throughout)

\[
H_S = \frac{\omega_0}{2} \sigma_z, \quad H_B = \sum_k \omega_k b_k^\dagger b_k, \\
H_{SB} = \sigma_z \sum_k (g_k^* b_k + g_k b_k^\dagger),
\]

and \( \sigma_z \) is the usual Pauli matrix, \( \omega_0 \) is the energy bias, and \( b_k \) (\( b_k^\dagger \)) are the annihilation (creation) operators for the harmonic oscillator modes. Since \( \{H_S, H_{SB}\} = 0 \), \( \langle \sigma_z \rangle \) does not change with time, and only dephasing takes place. Assuming that the initial system-environment state is a product state with the environment in a thermal equilibrium state \( \rho_B = e^{-\beta H_B}/Z_B \), where \( Z_B = \text{Tr}_B[e^{-\beta H_B}] \), the evolution of the off-diagonal elements of the density matrix is given by [62]

\[
\langle \sigma_{\pm}(t) \rangle = \langle \sigma_{\pm} \rangle e^{+ i \omega_0 t - \Gamma_{uc}(t)},
\]

(17)

where

\[
\Gamma_{uc}(t) = \sum_k 4 |g_k|^2 \coth(\beta \omega_k/2) \frac{1 - \cos \omega_k t}{\omega_k^2}.
\]

For completeness, the derivation of this result is presented in Appendix A. On the other hand, if the system and the environment have interacted for a long time beforehand, the initial state of the environment is not the thermal equilibrium state \( e^{-\beta H_B}/Z_B \). Instead, the system and the environment together are in a thermal equilibrium state, that is, \( e^{-\beta H}/Z \), where \( Z = \text{Tr}_{S,B}[e^{-\beta H}] \) [63]. Then, at time \( t = 0 \), we can perform either a projective measurement or a unitary operation on the system to prepare the desired initial system state. We now analyze these scenarios one by one.

#### A. System state preparation by projective measurement

If the initial system state \( |\psi\rangle \) is prepared by a projective measurement, described by the projector \( P_\psi = |\psi\rangle \langle \psi | \), then the initial system-environment state is \( \rho(0) = \frac{1}{2} P_\psi e^{-\beta H} P_\psi \) with \( Z = \text{Tr}_{S,B}[P_\psi e^{-\beta H}] \). With
For completeness, the derivation of these results is sketched in Appendix A. Note that the effect of the initial correlations is to modify the decoherence rate as well as to introduce a phase shift. Moreover, for zero temperature, these expressions further simplify to $\Gamma_{\text{corr}}(t) = 0$ and $\chi(t) = \Phi(t)$. In this case, the decoherence rate is not modified and the effect of the initial correlations is a simple phase shift given by $\Phi(t)$.

With the system density matrix found, the geometric phase can then be evaluated. To calculate the sum over the environment modes, the sum is converted to an integral via the spectral density $J(\omega)$, which allows us to write $\sum_{k} |g_{k}|^{2}(\ldots) \rightarrow \int_{0}^{\infty} d\omega J(\omega)(\ldots)$. We consider the spectral density to be of the form $\int_{0}^{\infty} d\omega J(\omega)(\ldots)$, where $\lambda$ is a dimensionless constant characterizing the system-environment interaction strength, $s$ is the so-called Ohmicity parameter, and $\omega_{c}$ is the cutoff frequency [62]. In Figs. 3(a) and (b), we have plotted the behavior of the correction to the geometric phase $|\delta \Phi_{G}|$, as the Ohmicity parameter is varied, for weak system-environment coupling strength. It is clear from these figures that for weak system-environment coupling strength, the effect of the initial correlations on the geometric phase is generally negligible since the dashed blue line largely overlaps with the solid black curve. Nevertheless, for sub-Ohmic environments (that is, $s < 1$), the initial correlations can still play a role. Interestingly, taking the initial correlations into account generally makes the correction to the geometric phase smaller. In fact, for a particular value of the Ohmicity parameter, the correction to the geometric phase is zero. Proceeding along these lines, in Figs. 4(a) and (b) we have shown the correction to the geometric phase at zero temperature for stronger system-environment coupling strengths. Three points are evident from these figures. First, for a range of values of $s$, the initial correlations have a very small effect on the geometric phase. Second, for sub-Ohmic environments as well as for very super-Ohmic environments, the contribution of the initial correlations to the geometric phase is very significant. Third, the initial correlations generally reduce the correction to the geometric phase, thereby implying that the initial correlations increase the robustness of the geometric phase. As before, for a particular value of the Ohmicity parameter, the correction to the geometric phase becomes zero. We have also found that, as expected, as the temperature is increased, the effect of the initial correlations decreases [see Figs. 4(a) and (b)].

It is also interesting to analyze the correction to the geometric phase as the system-environment coupling strength is varied. The results are illustrated in Fig. 5(a) and (b). For the sub-Ohmic environment considered in Fig. 5(a), the initial correlations greatly reduce the correction to the geometric phase. Surprisingly, as the system-environment coupling strength is increased, the correction to the geometric phase, in the case where the initial correlations are taken into account, can decrease. In fact, for particular non-zero values of the system-environment interaction strength, the correction to the geometric phase becomes zero. This is not the case for an Ohmic environment [see Fig. 5(b)].
B. System state preparation by unitary operation

We now analyze the effect of the initial correlations if a unitary operation, instead of a projective measurement, is used to prepare the initial system state. The initial system-environment state in this case is \( \rho(0) = \frac{1}{2} \Omega e^{-\beta H} \Omega^\dagger \), where \( \Omega \) is a unitary operation performed on the system. The off-diagonal elements of the system density matrix are given by \(49, 51\)

\[
\langle \sigma_{\pm}(t) \rangle = \langle \sigma_{\pm}(0) \rangle e^{\pm i \omega_t \lambda + \chi(t)} e^{-\Gamma(t)},
\]

with

\[
\Gamma(t) = \Gamma_{uc}(t) + \Gamma_{corr}(t),
\]

where

\[
\Gamma_{corr}(t) = -\ln \left\{ \abs{ e^{-\beta \omega_0/2} \langle 0| \Omega^\dagger \sigma_+ \Omega |0 \rangle e^{-i \Phi(t)} + e^{\beta \omega_0/2} \langle 1| \Omega^\dagger \sigma_+ \Omega |1 \rangle e^{+i \Phi(t)} } \right\}, \]

\[
\chi(t) = \arg \left[ \cos(\Phi(t)) + i \sin(\Phi(t)) \right] \left\{ e^{\beta \omega_0/2} \langle 0| \Omega^\dagger \sigma_+ \Omega |1 \rangle e^{-\beta \omega_0/2} - (0 \langle 0| \Omega^\dagger \sigma_+ \Omega |0 \rangle e^{-\beta \omega_0/2}) \right\}.
\]

IV. TWO-LEVEL SYSTEM INTERACTING WITH SPIN ENVIRONMENT

We now consider the central two-level system to be interacting with a collection of \( N \) two-level systems \(64, 65\). The system Hamiltonian \( H_S \) is still \( \frac{\omega}{2} \sigma_z \), while the environment Hamiltonian is now \( \sum_i \omega_i \sigma_i^z \), and the system-environment interaction is described by \( \sigma_z \sum_i \lambda_i \sigma_i^z \). Since \( [H_S, H_{SB}] = 0 \), this is also a pure dephasing model. If the initial system-environment state is a product state of the form \( \rho(0) = \rho_S(0) \otimes e^{-\beta H_B}/Z_B \), then the evolution of the off-diagonal elements is given by \(65, 67\)

\[
\langle \sigma_{\pm}(t) \rangle = \langle \sigma_{\pm}(0) \rangle e^{\pm i \omega t} e^{-\Gamma_{uc}(t)},
\]

where

\[
\Gamma_{uc}(t) = -\sum_j \ln \left\{ 1 - \frac{2 \lambda_j^2}{\lambda_j^2 + \omega_j^2} \sin^2 \left( \sqrt{\lambda_j^2 + \omega_j^2} t \right) \right\},
\]

and the sum is over the environment spins. The derivation of this result is reproduced in Appendix \(13\). However, as emphasized before, this result may questionable since the initial system-environment correlations are disregarded. To investigate the effect of these correlations, we consider the system state to be prepared by a projective measurement as well as by a unitary operation starting from the total system-environment equilibrium state \( e^{-\beta H}/Z \). We note that, to the best of our knowledge, this model has not been solved taking initial correlations into account before.

![FIG. 5. (Color online) Correction to the geometric phase \( \delta \Phi_G = \Phi_G - \Phi_0 \) (where \( \Phi_0 \) is the geometric phase for the two-level system if the system-environment interaction strength is zero) with a harmonic oscillator environment as a function of the Ohmicity parameter \( s \) if the initial state is prepared via a unitary operation. The solid, black curve shows the geometric phase when the initial state is prepared via the unitary operation \( \Omega = e^{i \pi \sigma_z/3} \), while the dashed, blue curve is for an uncorrelated initial state. In (a), we have \( \beta = 3 \) while in (b), we have \( \beta = 1 \). Once again, we have set \( \omega_0 = 1 \), and we have used \( \omega_c = 5 \) and \( \lambda = 0.5 \).](attachment:image.png)
A. System state preparation by projective measurement

If the initial state is \( \rho(0) = P_\psi e^{-\beta H} P_\psi / Z \), then the off-diagonal elements of the density matrix are given by

\[
\langle \sigma_\pm(t) \rangle = \langle \sigma_\pm \rangle e^{\pm i \omega_0 t + \chi(t)} e^{-\Gamma(t)},
\]

where, similar to the form obtained for the harmonic oscillator environment,

\[
\tan[\chi(t)] = \frac{\sinh(\beta \omega_0/2) - \cos \theta_0 \cosh(\beta \omega_0/2)}{\cosh(\beta \omega_0/2) - \cos \theta_0 \sinh(\beta \omega_0/2)} \tan[\Phi(t)],
\]

with \( \theta_0 \) the Bloch angle characterizing the initial state. We now have

\[
\Phi(t) = \sum_j \arg[A_j(t) + i B_j(t)],
\]

where \( A_j(t) = 1 - 2 \frac{\lambda_j^2}{\alpha_j^2} \sin^2(\alpha_j t) \) and \( B_j(t) = \frac{\lambda_j^2}{\alpha_j^2} \tanh(\beta \alpha_j) \sin(2 \alpha_j t) \) with \( \alpha_j = \sqrt{\lambda_j^2 + \omega_j^2} \). Also,

\[
\Gamma(t) = \Gamma_{uc}(t) + \Gamma_{corr}(t), \quad \text{where} \quad \Gamma_{corr}(t) = \Gamma_{corr}^{(1)}(t) + \Gamma_{corr}^{(2)}(t),
\]

Interestingly, in this case, even if the temperature is zero, the initial correlations change the decay rate of the off-diagonal elements since \( \Gamma_{corr}^{(1)}(t) \neq 0 \) at zero temperature while \( \Gamma_{corr}^{(2)}(t) = 0 \). On the other hand, at zero temperature, \( \chi(t) \) is once again equal to \( \Phi(t) \).

With the system density matrix found, we compute the correction to the geometric phase \( \delta \Phi_G = \Phi_G - \Phi_U \).

The behavior of the correction \( \delta \Phi_G \) as a function of the two-level system-environment coupling strength is shown in Figs. 6(a) and (b). The effect of the initial correlations is again very significant; in particular, the initial correlations can make the geometric phase more robust.

For particular values of the system-environment interaction strength \( \lambda \), the correction to the geometric phase becomes zero.

B. System state preparation by unitary operation

We now prepare the initial system state via a unitary operation. We find that for the initial system-environment state \( \rho(0) = \frac{1}{Z} \Omega e^{-\beta H} \Omega^\dagger \), the off-diagonal

\[
\begin{array}{c}
\text{(a)} \\
\text{(b)}
\end{array}
\]
elements of the density matrix are, as for the harmonic
oscillator environment,
\[ \langle \sigma_\pm(t) \rangle = \langle \sigma_\pm(0) \rangle e^{\pm i \omega_0 t + \chi(t)} e^{-\Gamma(t)}, \quad (27) \]

\[ \Gamma^{(2)}_{corr}(t) = -\ln \left\{ \text{abs} \left[ \frac{e^{-\beta \omega_0/2} \langle 0 | \Omega \sigma \Omega | 0 \rangle e^{-i \Phi(t)} + e^{\beta \omega_0/2} \langle 1 | \Omega \sigma \Omega | 1 \rangle e^{i \Phi(t)} - e^{-\beta \omega_0/2} \langle 0 | \Omega \sigma \Omega | 0 \rangle + e^{\beta \omega_0/2} \langle 1 | \Omega \sigma \Omega | 1 \rangle} \right] \right\}. \]

Also, \( \chi(t) \) is of the same form as in Eq. (22), but with \( \Phi(t) \) now given by Eq. (24). Details can be found in Appendix B. Once again, for zero temperature, we find that the dynamics are the same as the case where the initial state is prepared by a projective measurement. However, as illustrated in Figs. 7(a) and (b), even for non-zero temperatures, the contribution to the geometric phase due to the initial correlations can be very significant. Once again, if we increase the temperature, the effect of the initial correlations decreases as expected.

\[ \text{V. CONCLUSION} \]

In summary, we have presented exact expressions for the geometric phase of a two-level system undergoing pure dephasing to investigate the effect of the initial system-environment correlations on the geometric phase. As concrete examples, we have applied these expressions to two different environments: a collection of harmonic oscillators, and a collection of spins. Our results illustrate that the effect of the initial correlations on the geometric phase can be very significant, with a non-trivial dependence on the system-environment parameters. For instance, increasing the system-environment coupling strength may not always increase the correction to the geometric phase; in fact, for certain values of the coupling strength, the correction becomes zero, implying that the initial correlations can increase the robustness of the geometric phase. Our work on the geometric phase should be important not only for studies of the geometric phase itself as well as its practical implementations, but also for investigating the role of system-environment correlations in open quantum systems.

\[ \text{ACKNOWLEDGEMENTS} \]

The authors acknowledge support from the LUMS FIF Grant FIF-413. A. Z. C. is also grateful for support from HEC under grant No 5917/Punjab/NRPU/R&D/HEC/2016. Support from the National Center for Nanoscience and Nanotechnology is also acknowledged.

\[ \text{where } \Gamma(t) = \Gamma_{uc}(t) + \Gamma^{(1)}_{corr}(t) + \Gamma^{(2)}_{corr}(t) \text{ with } \Gamma^{(1)}_{corr}(t) \text{ the same as before [see Eq. (25)], while } \Gamma^{(2)}_{corr}(t) \text{ is given by} \]

\[ \text{Appendix A: Solution for harmonic oscillator environment} \]

For completeness, we sketch how to solve for the system dynamics for the total system-environment Hamiltonian \( H = H_S + H_B + H_{SB} \), where \( 49, 51 \)

\[ H_S = \frac{\omega_0}{2} \sigma_z, \quad H_B = \sum_k \omega_k b_k^\dagger b_k, \]

\[ H_{SB} = \sigma_z \sum_k (g_k^* b_k + g_k b_k^\dagger). \]

First, we transform to the interaction picture to obtain
\[ H_I(t) = e^{i(H_S + H_B)t} H_{SBE^{-i(H_S + H_B)t}}, \]
\[ = \sigma_z \sum_k (g_k^* b_k e^{-i \omega_k t} + g_k b_k^\dagger e^{i \omega_k t}). \quad (A1) \]

We next find the time evolution operator \( U_I(t) \) corresponding to \( H_I(t) \) using the Magnus expansion as
\[ U_I(t) = \exp\{\sigma_z \sum_k [b_k^\dagger \alpha_k(t) - b_k \alpha_k^*(t)]/2\}, \quad (A2) \]

and the total unitary time-evolution operator is \( U(t) = e^{-i \omega_0 \sigma_z t/2} U_I(t) \). We now define
\[ [\rho_S(t)]_{10} = \text{Tr}_{SB} [U(t) \rho(0) U^\dagger(t) | 0 \rangle \langle 1 |]. \]

Defining \( P_{01}(t) = U^\dagger(t) | 0 \rangle \langle 1 | U(t) \), this can be written as
\[ [\rho_S(t)]_{10} = \text{Tr}_{SB} [\rho(0) P_{01}(t)]. \]

Simplifying \( P_{01}(t) \) using the unitary time-evolution operator \( U(t) \), we find that
\[ P_{01}(t) = e^{i \omega_0 t} e^{-R_{01}(t)} P_{01}, \quad (A3) \]

where
\[ R_{01}(t) = \sum_k [b_k^\dagger \alpha_k(t) - b_k \alpha_k^*(t)], \quad (A4) \]

with
\[ \alpha_k(t) = \frac{2 g_k (1 - e^{i \omega_k t})}{\omega_k}. \]

Consequently,
\[ [\rho_S(t)]_{10} = e^{i \omega_0 t} \text{Tr}_{SB} [e^{-R_{01}(t)} P_{01} \rho(0)]. \quad (A5) \]
This is a general result because it applies to an arbitrary initial density $ρ(0)$. Now, if $ρ(0) = ρ_S(0) ⊗ ρ_B$, where $ρ_B = e^{−β_H B} Z_B^{-1}$ with $Z_B = Tr_B[e^{−β_H B}]$, then

$$[ρ_S(t)]_{10} = [ρ_S(0)]_{10} e^{iω_0 t} Tr_B[e^{−R_{01}(t) ρ_B}]. \quad (A6)$$

The trace over the environment computes to

$$Tr_B[e^{−R_{01}(t) ρ_B}] = \exp \left[ -\sum_k 4|g_k|^2 \frac{[1−\cos(ω_k t)]}{ω_k} \coth \left( \frac{βω_k}{2} \right) \right], \quad (A7)$$

thereby yielding

$$[ρ_S(t)]_{10} = [ρ_S(0)]_{10} e^{iω_0 t} e^{−Γ_{uc}(t)}, \quad (A8)$$

with

$$Γ_{uc}(t) = \sum_k 4|g_k|^2 \frac{[1−\cos(ω_k t)]}{ω_k^2} \coth \left( \frac{βω_k}{2} \right). \quad (A9)$$

We now consider what happens if the initial state is of the form $ρ(0) = \frac{1}{2} Ω e^{−β_H Ω^†}$, with $Z$ the normalization factor. Currently, the $Ω$ operator can be a projection operator or a unitary operator. To first simplify $Z$, we use the completeness relation $\sum_i |i⟩⟨i| = I$, where $|z⟩|l⟩ = (-1)^l |l⟩$. Then,

$$Z = \sum_i e^{−βω_0(-1)^l/2}⟨l|Ω|l⟩ Tr_B[e^{−β_H B}], \quad (A10)$$

with

$$H_B^{(i)} = H_B + (-1)^l \sum_k (g_k^† b_k + g_k b_k^†). \quad (A11)$$

To simplify further, we introduce the displaced harmonic oscillator modes

$$B_{k,l} = b_k + (-1)^l g_k, \quad (A12)$$

$$B_{k,l}^† = b_k^† + (-1)^l g_k^†, \quad (A13)$$

allowing us to write

$$Z = \sum_i e^{−βω_0(-1)^l/2}⟨l|Ω|l⟩ e^{β Σ_k |g_k|²} Z_B, \quad (A14)$$

where $Z_B = Tr_B[e^{−β Σ_k ω_k B_{k,l}^† B_{k,l}}]$. With $Z$ found, we then substitute our initial state in Eq. (A3) and introduce $\sum_i |l⟩⟨l|$ to simplify the resulting $Tr_B[e^{−R_{01}(t) e^{−β_H B} }]$. Using the displaced harmonic oscillator modes as before, we find that

$$R_{01}(t) = \sum_k [α_k(t) B_{k,l}^† − α_k^†(t) B_{k,l}] + i(-1)^l Φ(t), \quad (A15)$$

where

$$Φ(t) = \sum_k 4|g_k|^2 \sin(ω_k t). \quad (A16)$$

We then find that

$$Tr_B[e^{−R_{01}(t) e^{−β_H B}}] = e^{−i(1)^l Φ(t)} Z_B e^{β Σ_k |g_k|^2} e^{−Γ_{uc}(t)}. \quad (A17)$$

Putting this all together, and rearranging, we obtain

$$[ρ_S(t)]_{10} = [ρ_S(0)]_{10} e^{iω_0 t} e^{−Γ_{uc}(t)} X(t), \quad (A18)$$

with

$$X(t) = \sum_i ⟨i|Ω|l⟩ P_i Ω|l⟩ e^{−i(1)^l Φ(t)} e^{βω_0(1)^l/2} \sum_j ⟨i|Ω|l⟩ P_j Ω|l⟩ e^{−βω_0(1)^l/2}.$$  

Now assuming that $Ω$ is a projection operator, that is, $Ω = |ψ⟩⟨ψ|$, we can further simplify and write $X(t)$ in polar form to obtain Eq. (19). On the other hand, if $Ω$ is taken to be a unitary operator, we obtain Eq. (19).

**Appendix B: Dynamics with a spin environment**

We now consider the total system-environment Hamiltonian $H = H_S + H_B + H_{SB}$, where

$$H_S = \frac{ω_0}{2} σ_z, \quad H_B = \sum_i ω_i σ_x, \quad H_{SB} = σ_z \sum_i λ_i σ_i^†.$$  

Once again, since $[H_S, H_{SB}] = 0$, this is a pure dephasing model. Our aim is to then calculate $⟨σ_±(t)⟩$. We note that $e^{it(H_B + H_{SB})} |l⟩ = e^{it(H_B + (-1)^l V)} |l⟩$, where

$$V = \sum_i λ_i σ_i^†. \quad (B1)$$

Using the completeness relation $\sum_s |l⟩⟨s| = I$, we can simplify $σ_±(t) = e^{iH t} σ_± e^{-iH t}$ to find

$$σ_±(t) = e^{±iω_0 t} e^{ιH_B V} e^{−it(H_B + V)} σ_±. \quad (B2)$$

We now consider initial states of the form

$$ρ(0) = ρ_S(0) ⊗ ρ_B, \quad ρ_B = e^{−β_H B} / Z_B. \quad (B3)$$

For simplicity, we only show the calculation for $⟨σ_+(t)⟩$. Using Eq. (B2), we obtain

$$⟨σ_+(t)⟩ = Tr[σ_+(t) ρ(0)]$$

$$= \frac{e^{iω_0 t}}{Z_B} ⟨σ_+(0)⟩ Tr_B[R(t) e^{−β_H B}], \quad (B4)$$

where $R(t) = e^{it(H_B + V)} e^{−it(H_B − V)}$. Our remaining task is to compute $Tr_B[R(t) e^{−β_H B}]$. To this end, we first write $R(t)$ as $e^{it Σ_j α_j (n_j^† σ_j)} e^{−it Σ_j α_j (n_j σ_j)}$, where $n_j^† = \frac{1}{2}(ω_j, 0, λ_j)$, $n_j = \frac{1}{2}(ω_j, 0, −λ_j)$ and $α_j = \sqrt{ω_j^2 + λ_j^2}$. The exponentials can then be combined and the resulting expression is further simplified to obtain

$$Tr_B[R(t) e^{−β_H B}] = 2N \prod_j \cos e_j \cos(iβω_j). \quad (B5)$$
where \( \cos c \beta = 1 - 2 \left( \frac{\Delta}{\alpha} \right)^2 \sin^2(\alpha_j t) \), and \( Z_B = 2^{N+1} \cos(\beta \omega_j) \). Putting it all together, we finally have that

\[
(\sigma_+ (t)) = (\sigma_+ (0)) e^{i \omega_0 t} \prod_j \{ 1 - 2 \left( \frac{\alpha_j}{\alpha} \right)^2 \sin^2(\alpha_j t) \}.
\]  

(B6)

We now consider initially correlated states of the form

\[
\rho(0) = \frac{1}{Z} \Omega e^{-\beta H \Omega^\dagger}
\]

As before, we find that \( \langle \sigma_+ (t) \rangle = \text{Tr}_{S,B} [e^{i \omega_0 t} R(t) \sigma_+ (0)] \).

To simplify \( \rho(0) \), we use the fact that \( e^{-\beta H} |s\rangle = e^{(-1)^{j+1} \beta \omega_0/2} e^{-\beta (H_B + (-1)^j V)} |s\rangle \). We then have

\[
(\sigma_+ (t)) = \frac{e^{i \omega_0 t}}{Z} \left[ \langle 0 | \Omega^\dagger \sigma_+ (0) | e^{-\beta \omega_0/2} \text{Tr}_{B} [R(t) e^{-\beta (H_B + V)}] \right.
\]

\[
+ \langle 1 | \Omega^\dagger \sigma_+ (0) | e^{i \omega_0 t/2} \text{Tr}_{B} [R(t) e^{-\beta (H_B - V)}] \right\}.
\]

(B7)

We now sketch the calculation for \( \text{Tr}_{B} [R(t) e^{-\beta (H_B + V)}] \) as the calculation for \( \text{Tr}_{B} [R(t) e^{-\beta (H_B - V)}] \) is very similar. The trick is to write \( \text{Tr}_{B} [R(t) e^{-\beta (H_B + V)}] \) as \( \text{Tr}_{B} [e^{-i t (H_B - V)} e^{i \gamma (H_B + V)}] \) we have defined \( \gamma = t + i \beta \). The exponentials can then be manipulated as before to obtain

\[
\text{Tr}_{B} [R(t) e^{-\beta (H_B + V)}] = C_0 \Pi_j (A_j - i B_j),
\]

where \( A_j = 1 - 2 \left( \frac{\alpha_j}{\alpha} \right)^2 \sin^2(\alpha_j t) \), \( B_j = 2 \left( \frac{\alpha_j}{\alpha} \right)^2 \tan(\beta \omega_j) \sin(\alpha_j t) \), and \( C_0 = \Pi_j 2 \cosh(\beta \alpha_j) \). We can then further simplify to

\[
(\sigma_+ (t)) = (\sigma_+ (0)) e^{i \omega_0 t} e^{-\Gamma(t)}
\]

\[
\times \sum_j (\langle \Omega^\dagger \sigma_+ (0) | e^{-\beta \omega_0/2} e^{-i(1)^j \Phi(t)} + \langle 1 | \Omega^\dagger \sigma_+ (0) | e^{i \omega_0 t/2} e^{-\beta \omega_0/2} e^{-i(1)^j \Phi(t)} \right),
\]

where

\[
\Gamma(t) = \sum_j \Gamma_j(t), \quad \Phi(t) = \sum_j \Phi_j(t), \quad (B8)
\]

and \( F_j(t) = A_j(t) + i B_j(t) = e^{-\Gamma_j(t)} e^{i \Phi_j(t)} \). It is then a simple matter of specifying that \( \Omega \) is a projection operator or a unitary operator to work out the dynamics.

[1] E. Sjöqvist, Geometric phases in quantum information, Int. J. Quantum Chem. 115, 1311 (2015).
[2] E. Cohen, H. Larocque, F. Bouchard, F. Nejadsat-tari, Y. Gfen, and E. Karimi, Geometric phase from aharonovbohlm to pancharatnamberry and beyond, Nat. Rev. Phys. 1, 437 (2019).
[3] S. Pancharatnam, Generalized theory of interference and its applications, Proc. Indian Acad. Sci. A 44, 398 (1956).
[4] H. Longuet-Higgins, The intersection of potential energy surfaces in polyatomic molecules, Proc. R. Soc. London A 344, 147 (1975).
[5] A. J. Stone, Spin-orbit coupling and the intersection of potential energy surfaces in polyatomic molecules, Proc. R. Soc. London A 351, 141 (1976).
[6] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. R. Soc. London A 392, 45 (1984).
[7] Y. Aharonov and J. Anandan, Phase change during a cyclic quantum evolution, Phys. Rev. Lett. 58, 1993 (1987).
[8] A. Uhmann, On berry phases along mixtures of states, Ann. Phys. 501, 63 (1989).
[9] E. Sjöqvist, A. K. Pati, A. Ekert, J. S. Anandan, M. Ericsson, D. K. L. Oi, and V. Vedral, Geometric phases for mixed states in interferometry, Phys. Rev. Lett. 85, 2845 (2000).
[10] D. Suter, G. Chingas, R. Harris, and A. Pines, Berry’s phase in magnetic resonance, Mol. Phys. 61, 1327 (1987).
[11] P. J. Leek, J. M. Fink, A. Blais, R. Bianchetti, M. Göppi, J. M. Gambetta, D. I. Schuster, L. Frunzio, R. J. Schoelkopf, and A. Wallraff, Observation of berry’s phase in a solid-state quibt. Science 318, 1889 (2007).
[12] R. Simon, H. J. Kimble, and E. C. G. Sudarshan, Evolving geometric phase and its dynamical manifestation as a frequency shift: An optical experiment, Phys. Rev. Lett. 61, 19 (1988).
[13] P. Zanardi and M. Rasetti, Holonomic quantum computation, Phys. Lett. A 264, 94 (1999).
[14] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Geometric quantum computation using nuclear magnetic resonance, Nature 403, 869 (2000).
[15] G. Falci, R. Fazio, G. Massimo Palma, J. Siewert, and V. Vedral, Detection of geometric phases in superconducting nanocircuits, Nature 407, 355 (2000).
[16] L.-M. Duan, J. I. Cirac, and P. Zoller, Geometric manipulation of trapped ions for quantum computation, Science 292, 1695 (2001).
[17] W. Xiang-Bin and M. Keiji, Nonadiabatic conditional geometric phase shift with nmr, Phys. Rev. Lett. 87, 097901 (2001).
[18] D. Liebfried, B. DeMarco, V. Meyer, D. Lucas, M. Barrett, J. Britton, W. M. Itano, B. Jelenkovic, C. Langer, T. Rosenband, and D. J. Wineland, Experimental demonstration of a robust, high-fidelity geometric two ion-quantum phase gate, Nature 422, 412 (2000).
[19] M. Ericsson, E. Sjöqvist, J. Bräullund, D. K. L. Oi, and A. K. Pati, Generalization of the geometric phase to completely positive maps, Phys. Rev. A 67, 020101 (2003).
[20] A. Carollo, I. Fuentes-Guridi, M. F. m. c. Santos, and V. Vedral, Geometric phase in open systems, Phys. Rev. Lett. 90, 160402 (2003).
[21] D. M. Tong, E. Sjöqvist, L. C. Kwek, and C. H. Oh, Kinematic approach to the mixed state geometric phase in nonunitary evolution, Phys. Rev. Lett. 93, 080405 (2004).

[22] R. S. Whitney, Y. Makhlin, A. Shnirman, and Y. Gefen, Geometric nature of the environment-induced berry phase and geometric dephasing, Phys. Rev. Lett. 94, 070407 (2005).

[23] X. X. Yi, D. M. Tong, L. C. Wang, L. C. Kwek, and C. H. Oh, Beyond the markov approximation and weak-coupling limit, Phys. Rev. A 73, 052103 (2006).

[24] F. C. Lombardo and P. I. Villar, Geometric phases in open systems: A model to study how they are corrected by decoherence, Phys. Rev. A 74, 042311 (2006).

[25] J. Dajka, M. Mierzejewski, and J. Luczka, Geometric phase of a qubit in dephasing environments, J. Phys. A: Math. Theor. 41, 012001 (2008).

[26] F. C. Lombardo and P. I. Villar, Environmentally induced effects on a biparticle two-level system: Geometric phase and entanglement properties, Phys. Rev. A 81, 022115 (2010).

[27] F. M. Cucchietti, J.-F. Zhang, F. C. Lombardo, P. I. Villar, and R. Lalanne, Geometric phase with nonunitary evolution in the presence of a quantum critical bath, Phys. Rev. Lett. 105, 240406 (2010).

[28] P. I. Villar and F. C. Lombardo, Geometric phases in the presence of a composite environment, Phys. Rev. A 83, 052121 (2011).

[29] F. C. Lombardo and P. I. Villar, Nonunitary geometric phases: A qubit coupled to an environment with random noise, Phys. Rev. A 87, 032338 (2013).

[30] F. C. Lombardo and P. I. Villar, Correction to the geometric phase by structured environments: The onset of non-markovian effects, Phys. Rev. A 91, 042111 (2015).

[31] V. Hakim and V. Ambegaokar, Quantum theory of a free particle interacting with a linearly dissipative environment, Phys. Rev. A 32, 423 (1985).

[32] F. Haake and R. Reibold, Strong damping and low-temperature anomalies for the harmonic oscillator, Phys. Rev. A 28, 2462 (1985).

[33] H. Grabert, P. Schramm, and G.-L. Ingold, Quantum brownian motion: The functional integral approach, Phys. Rep. 168, 115 (1988).

[34] C. M. Smith and A. O. Caldeira, Application of the generalized feynman-vernon approach to a simple system: The damped harmonic oscillator, Phys. Rev. A 41, 3103 (1990).

[35] R. Karlsson and H. Grabert, Exact time evolution and master equations for the damped harmonic oscillator, Phys. Rev. E 55, 153 (1997).

[36] L. Dávila Romero and J. Pablo Paz, Decoherence and initial correlations in quantum brownian motion, Phys. Rev. A 55, 4070 (1997).

[37] E. Lutz, Effect of initial correlations on short-time decoherence, Phys. Rev. A 67, 022109 (2003).

[38] S. Banerjee and R. Ghosh, General quantum brownian motion with initially correlated and nonlinearly coupled environment, Phys. Rev. E 67, 056120 (2003).

[39] N. G. van Kampen, A new approach to noise in quantum mechanics, J. Stat. Phys. 115, 1057 (2004).

[40] M. Ban, Quantum master equation for dephasing of a two-level system with an initial correlation, Phys. Rev. A 80, 064103 (2009).

[41] M. Campisi, P. Talkner, and P. Hänggi, Fluctuation theorem for arbitrary open quantum systems, Phys. Rev. Lett. 102, 210401 (2009).

[42] C. Uchiyama and M. Aihara, Role of initial quantum correlation in transient linear response, Phys. Rev. A 82, 041104 (2010).

[43] A. G. Dijkstra and Y. Tanmura, Non-markovian entanglement dynamics in the presence of system-bath coherence, Phys. Rev. Lett. 104, 250401 (2010).

[44] A. Smirne, H.-P. Breuer, J. Piilo, and B. Vacchini, Initial correlations in open-systems dynamics: The jamessumsings model, Phys. Rev. A 82, 062114 (2010).

[45] J. Dajka and J. Luczka, Distance growth of quantum states due to initial system-environment correlations, Phys. Rev. A 82, 012341 (2010).

[46] Y.-J. Zhang, X.-B. Zou, Y.-J. Xia, and G.-C. Guo, Different entanglement dynamical behaviors due to initial system-environment correlations, Phys. Rev. A 82, 022108 (2010).

[47] H.-T. Tan and W.-M. Zhang, Non-markovian dynamics of an open quantum system with initial system-reservoir correlations: A nanocavity coupled to a coupled-resonator optical waveguide, Phys. Rev. A 83, 032102 (2011).

[48] C. K. Lee, J. Cao, and J. Gong, Noncanonical statistics of a spin-boson model: Theory and exact monte carlo simulations, Phys. Rev. E 86, 021109 (2012).

[49] Y. G. Morozov, S. Mathey, and G. Röpke, Decoherence in an exactly solvable qubit model with initial qubit-environment correlations, Phys. Rev. A 85, 022101 (2012).

[50] V. Semin, I. Sinayskiy, and F. Petruccione, Initial correlation in a system of a spin coupled to a spin bath through an intermediate spin, Phys. Rev. A 86, 062114 (2012).

[51] A. Z. Chaudhry and J. Gong, Amplification and suppression of system-bath-correlation effects in an open many-body system, Phys. Rev. A 87, 012129 (2013).

[52] A. Z. Chaudhry and J. Gong, Role of initial system-environment correlations: A master equation approach, Phys. Rev. A 88, 052107 (2013).

[53] A. Z. Chaudhry and J. Gong, The effect of state preparation in a many-body system, Can. J. Chem. 92, 119 (2013).

[54] J. Reina, C. Susa, and F. Fanchini, Extracting information from qubit-environment correlations, Sci. Rep. 4, 7443 (2014).

[55] Y.-J. Zhang, W. Han, Y.-J. Xia, Y.-M. Yu, and H. Fan, Role of initial system-bath correlation on coherence trapping, Sci. Rep. 5, 13359 (2015).

[56] C.-C. Chen and H.-S. Goan, Effects of initial-system-environment correlations on open-quantum-system dynamics and state preparation, Phys. Rev. A 93, 032113 (2016).

[57] I. de Vega and D. Alonso, Dynamics of non-markovian open quantum systems, Rev. Mod. Phys. 89, 015001 (2017).

[58] J. C. Halimeh and I. de Vega, Weak-coupling master equation for arbitrary initial conditions, Phys. Rev. A 95, 052108 (2017).

[59] S. Kitajima, M. Ban, and F. Shibata, Expansion formulas for quantum master equations including initial correlation, J. Phys. A: Math. Theor. 50, 125303 (2017).

[60] M. Buser, J. Cerrillo, G. Schaller, and J. Cao, Initial system-environment correlations via the transfer-tensor
method. [Phys. Rev. A 96, 062122 (2017)]

[61] M. Majeed and A. Z. Chaudhry, Effect of initial system-environment correlations with spin environments, Eur. Phys. J. D 73, 16 (2019).

[62] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, Oxford, 2007).

[63] U. Weiss, Quantum dissipative systems (World Scientific, Singapore, 2008).

[64] F. Cucchietti, J. P. Paz, and W. Zurek, Decoherence from spin environments, Phys. Rev. A 72, 052113 (2005).

[65] S. Camalet and R. Chitra, Effect of random interactions in spin baths on decoherence, Phys. Rev. B 75, 094434 (2007).

[66] M. Schlosshauer, Decoherence and the quantum-to-classical transition (Springer, Berlin, 2007).

[67] P. I. Villar, Spin bath interaction effects on the geometric phase, Phys. Lett. A 373, 206 (2009).