Abstract

In this paper, we investigate the trinomial probability distribution of the first and second kind from the $\mathcal{R}(p, q)$-quantum algebras. Moreover, we compute their $\mathcal{R}(p, q)$-factorial moments and derive the corresponding covariance. Particular cases of trinomial probability distribution are deduced from the formalism developed.

Keywords. $\mathcal{R}(p, q)$-calculus, quantum algebras, trinomial distribution, factorial moments, $\mathcal{R}(p, q)$-covariance.

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1 Introduction

Charalambos presented the $q-$ deformed Vandermonde and Cauchy formulae. Moreover, the $q-$ deformed univariate discrete probability distributions were investigated. Their properties and limiting distributions were derived [3].
Furthermore, the \( q \)-deformed multinomial coefficients were defined and their recurrence relations were deduced. Also, the \( q \)-deformed multinomial and negative \( q \)-deformed multinomial probability distributions of the first and second kind were presented [4].

The same author extended the multivariate \( q \)-deformed Vandermonde and Cauchy formulae, and the multivariate \( q \)-Pólya and inverse \( q \)-Pólya were constructed [5]. Moreover, the \( q \)-factorial moments of the bivariate discrete distributions were investigated [6].

Let \( p \) and \( q \) be two positive real numbers such that \( 0 < q < p < 1 \). We consider a meromorphic function \( \mathcal{R} \) defined on \( \mathbb{C} \times \mathbb{C} \) by [9]:

\[
\mathcal{R}(u, v) = \sum_{s,t=-l}^{\infty} r_{st} u^s v^t, \tag{1.1}
\]

with an eventual isolated singularity at the zero, where \( r_{st} \) are complex numbers, \( l \in \mathbb{N} \cup \{0\} \), \( \mathcal{R}(p^n, q^n) > 0 \), \( \forall n \in \mathbb{N} \), and \( \mathcal{R}(1, 1) = 0 \) by definition. We denote by \( \mathbb{D}_R \) the bidisk

\[
\mathbb{D}_R = \{ w = (w_1, w_2) \in \mathbb{C}^2 : |w_j| < R_j \},
\]

where \( R \) is the convergence radius of the series (1.1) defined by Hadamard formula as follows [14]:

\[
\lim_{s + t \to \infty} \sup_{s+t} \sqrt{|r_{st}| R_1^s R_2^t} = 1.
\]

We denote by \( \mathcal{O}(\mathbb{D}_R) \) the set of holomorphic functions defined on \( \mathbb{D}_R \).

The \( \mathcal{R}(p, q) \)-deformed numbers is defined by [9]:

\[
[n]_{\mathcal{R}(p,q)} := \mathcal{R}(p^n, q^n), \quad n \in \mathbb{N}, \tag{1.2}
\]

the \( \mathcal{R}(p, q) \)-deformed factorials and binomial coefficients are given as:

\[
[n]_{\mathcal{R}(p,q)}! := \begin{cases} 1 & \text{for } n = 0 \\ \mathcal{R}(p,q) \cdots \mathcal{R}(p^n, q^n) & \text{for } n \geq 1, \end{cases} \tag{1.3}
\]

and

\[
\binom{n}{m}_{\mathcal{R}(p,q)} := \frac{[n]_{\mathcal{R}(p,q)}!}{[m]_{\mathcal{R}(p,q)}! [n-m]_{\mathcal{R}(p,q)}!}, \quad (n, m) \in \mathbb{N} \cup \{0\}, \quad n \geq m. \tag{1.4}
\]

We consider the following linear operators on \( \mathcal{O}(\mathbb{D}_R) \) given by:

\[
Q : \varphi \mapsto Q \varphi(z) := \varphi(qz) \quad \text{and} \quad P : \varphi \mapsto P \varphi(z) := \varphi(pz), \tag{1.5}
\]

leading to define the \( \mathcal{R}(p, q) \)-deformed derivative:

\[
\partial_{\mathcal{R},p,q} := \partial_{p,q} \frac{p-q}{P-Q} \mathcal{R}(P, Q) = \frac{p-q}{pP - qQ} \mathcal{R}(pP, qQ) \partial_{p,q}. \tag{1.6}
\]
where $\partial_{p,q}$ is the $(p,q)$-derivative:

\begin{align}
\partial_{p,q} : \varphi \mapsto \partial_{p,q} \varphi (z) := \frac{\varphi(pz) - \varphi(qz)}{z(p - q)} .
\end{align}

The quantum algebra associated with the $\mathcal{R}(p,q)$-deformation, denoted by $\mathcal{A}_{\mathcal{R}(p,q)}$, is generated by the set of operators $\{1, A, A^\dagger, N\}$ satisfying the following commutation relations $[10]$:

\begin{align}
AA^\dagger &= [N + 1]_{R(p,q)}, & A^\dagger A &= [N]_{R(p,q)}, \\
[N, A] &= -A, & [N, A^\dagger] &= A^\dagger
\end{align}

with its realization on $\mathcal{O}(\mathbb{D}_R)$ given by:

\begin{align}
A^\dagger := z, & \quad A := \partial_{R(p,q)}, & \quad N := z \partial_z,
\end{align}

where $\partial_z := \frac{\partial}{\partial z}$ is the usual derivative on $\mathbb{C}$.

The $\mathcal{R}(p,q)$-deformed numbers (1.2) can be rewritten as follows [7]:

\begin{align}
[n]_{\mathcal{R}(p,q)} = \frac{\phi_1^n - \phi_2^n}{\phi_1 - \phi_2}, \quad \phi_1 \neq \phi_2,
\end{align}

where $\phi_i, i \in \{1,2\}$ are functions of the parameters deformations $p$ and $q$.

The following relations hold [8]:

\begin{align}
[x]_{\mathcal{R}(p^{-1},q^{-1})} &= (\phi_1 \phi_2)^{1-x} [x]_{\mathcal{R}(p,q)},
\end{align}

\begin{align}
[r]_{\mathcal{R}(p^{-1},q^{-1})!} &= (\phi_1 \phi_2)^{-\frac{r}{2}} [r]_{\mathcal{R}(p,q)}!,
\end{align}

\begin{align}
[x]_{\mathcal{R}(p^{-1},q^{-1})} &= (\phi_1 \phi_2)^{1-xr + \left(\frac{r+1}{2}\right)} [x]_{\mathcal{R}(p,q)}.
\end{align}

For $a, b \in \mathbb{N}$, the $\mathcal{R}(p,q)$-deformed shifted factorial is defined by [7]:

\begin{align}
(u \oplus v)^n_{\mathcal{R}(p,q)} := \prod_{i=1}^{n} (u \phi_1^{i-1} + v \phi_2^{i-1}), \quad \text{with} \quad (u \oplus v)^0_{\mathcal{R}(p,q)} := 1.
\end{align}

Analogously,

\begin{align}
(u \ominus v)^n_{\mathcal{R}(p,q)} := \prod_{i=1}^{n} (u \phi_1^{i-1} - b \phi_2^{i-1}), \quad \text{with} \quad (u \ominus v)^0_{\mathcal{R}(p,q)} := 1.
\end{align}

The $\mathcal{R}(p,q)$-deformed factorial of $u$ of order $r$ is defined by [8]:

\begin{align}
[u]_{r,\mathcal{R}(p,q)} = \prod_{i=1}^{r} [u - i + 1]_{\mathcal{R}(p,q)}, \quad r \in \mathbb{N},
\end{align}

Furthermore, the generalized Vandermonde, Cauchy formulae and univariate probability distributions induced from the $\mathcal{R}(p,q)$-deformed quantum algebras are investigated in [7].
Our aims is to determine the trinomial probability distribution of the first and second kind with properties associated to the $\mathcal{R}(p, q)$—deformed quantum algebras [10].

This paper is organized as follows: In section 2, we present the $\mathcal{R}(p, q)$—trinomial probability distribution of the first and second kind. Also their negative counterparts. Besides, the properties namely $(\mathcal{R}(p, q)$—factorial moments and covariance) are investigated. Section 3 is reserved to deduce the relevant particular cases corresponding to some quantum algebras known in the literature.

## 2 Generalized trinomial probability distribution

The trinomial probability distribution of the first and second kind from the $\mathcal{R}(p, q)$—deformed quantum algebras are investigated. Their factorial moments are also computed. Besides, the $\mathcal{R}(p, q)$—covariance is deduced.

### 2.1 $\mathcal{R}(p, q)$—trinomial distribution of the first kind

The probability function of the $\mathcal{R}(p, q)$—random vector $\mathcal{Y} = (Y_1, Y_2)$ of the $\mathcal{R}(p, q)$—trinomial probability distribution of the first kind, with parameters $n, \alpha = (\alpha_1, \alpha_2), p,$ and, $q$ is given by:

$$P(Y_1 = y_1, Y_2 = y_2) = \sum_{y_1, y_2} \frac{\alpha_1^{y_1} \alpha_2^{y_2}}{(1 + \alpha_1)^n_R(p,q) (1 + \alpha_2)^n_R(p,q)} \prod_{i=1}^{n} \phi_i^n_R(p,q),$$

where $y_j \in \{0, 1, \cdots, n\}, y_1 + y_2 \leq n, s_j = \sum_{i=1}^{j} y_j, 0 < \alpha_j < 1,$ and $j \in \{1, 2\}.$

**Theorem 2.1** The $\mathcal{R}(p, q)$—factorial moments of the $\mathcal{R}(p, q)$—trinomial probability distribution of the first kind, with parameters $n, \alpha = (\alpha_1, \alpha_2), p,$ and, $q$ are determined by the following relations:

$$E([Y_1]_{m_1,R(p,q)}) = \frac{\alpha_1^{m_1} \phi_1^{m_1}}{(1 + \alpha_1)^{m_1}_R(p,q)}, m_1 \in \{0, 1, \cdots, n\}, \quad (2.1)$$

$$E([Y_2]_{m_2,R(p,q)} | Y_1 = y_1) = \frac{\alpha_2^{m_2} \phi_2^{m_2}}{(1 + \alpha_2)^{m_2}_R(p,q)} [n - y_1]_{m_2,R(p,q)}, \quad (2.2)$$

with $m_2 \in \{0, 1, \cdots, n - y_1\}.$

$$E([Y_2]_{m_2,R(p,q)}) = \frac{\alpha_2^{m_2} \phi_1^{m_2} + m_2(n-m_2) \phi_2^{m_2}}{(1 + \alpha_2)^{m_2}_R(p,q) (\phi_1^{n-m_2} + \alpha_1 \phi_2^{n-m_2})^{m_2}_R(p,q)} [n]_{m_2,R(p,q)}, \quad (2.3)$$
with \( m_2 \in \{0, 1, \ldots, n\} \) and
\[
\begin{align*}
E([Y_1]_{m_1, \mathcal{R}(p, q)}|Y_2_{m_2, \mathcal{R}(p, q)}) = \frac{\binom{m_2}{n_2} \phi_1^{(m_2)_2} \phi_2^{(m_2)_2}}{(1 + \alpha_1)(\mathcal{R}(p, q))} \frac{\alpha_1^{m_1} \alpha_2^{m_2} [n]_{m_1 + m_2, \mathcal{R}(p, q)}}{(1 + \alpha_2)_{\mathcal{R}(p, q)}},
\end{align*}
\]
where \( m_1 \in \{0, 1, \ldots, n - m_2\} \) and \( m_2 \in \{0, 1, \ldots, n\} \).

**Proof 2.2.** The \( \mathcal{R}(p, q) \)- random variable \( Y_1 \) follows the \( \mathcal{R}(p, q) \)- binomial probability distribution of the first kind as:
\[
P(Y_1 = y_1) = \binom{n}{y_1} \frac{\alpha_1^{y_1} \phi_1^{(n - y_1)_2} \phi_2^{(y_1)_2}}{(1 + \alpha_1)_{\mathcal{R}(p, q)}}, \quad y_1 \in \{0, 1, \ldots, n\}.
\]

From [7], the \( \mathcal{R}(p, q) \)- factorial moments of \( Y_1 \) are given by (2.1). Moreover, the conditional probability distribution of \( Y_2 \), given that \( Y_1 = y_1 \), is the \( \mathcal{R}(p, q) \)- binomial probability distribution of the first kind, with mass function:
\[
P(Y_2 = y_2|Y_1 = y_1) = \binom{n - y_1}{y_2} \frac{\alpha_2^{y_2} \phi_1^{(n - y_2)_2} \phi_2^{(y_2)_2}}{(1 + \alpha_2)_{\mathcal{R}(p, q)}}, \quad y_2 \in \{0, 1, \ldots, n - y_1\}.
\]
Using also [7], the conditional \( \mathcal{R}(p, q) \)- factorial moments of \( Y_2 \), given that \( Y_1 = y_1 \), are furnished by (2.2). Besides, we determine the \( \mathcal{R}(p, q) \)- factorial moments of \( Y_2 \) according to the formula:
\[
E([Y_2]_{m_2, \mathcal{R}(p, q)}) = E\left(E([Y_2]_{m_2, \mathcal{R}(p, q)}|Y_1)\right) = \frac{\alpha_2^{m_2} \phi_2^{(m_2)_2}}{(1 + \alpha_2)_{\mathcal{R}(p, q)}} E\left([n - Y_1]_{m_2, \mathcal{R}(p, q)}\right).
\]

Obviously,
\[
E([n - Y_1]_{m_2, \mathcal{R}(p, q)}) = \sum_{y_1 = 0}^{n - m_2} [n - y_1]_{m_2, \mathcal{R}(p, q)} \binom{n}{y_1} \frac{\alpha_1^{y_1} \phi_1^{(n - y_1)_2} \phi_2^{(y_1)_2}}{(1 + \alpha_1)_{\mathcal{R}(p, q)}}.
\]

From the relation
\[
[n - y_1]_{m_2, \mathcal{R}(p, q)} \binom{n}{y_1} = [n]_{m_2, \mathcal{R}(p, q)} \binom{n - m_2}{y_1}
\]
and the \( \mathcal{R}(p, q) \)-binomial formula [8]:
\[
(1 + t)_{\mathcal{R}(p, q)} = \sum_{k=0}^{n} \binom{n}{k} \phi_1^{(n-k)_2} \phi_2^{(k)_2} t^k, \quad t \in \mathbb{R},
\]
5
we have:

\[
E\left( [n - Y_1]_{m_2, R(p,q)} \right) = \sum_{y_1=0}^{n-m_2} \left[ n \right]_{m_2, R(p,q)} \frac{[n-m_2]_{y_1}}{y_1_{R(p,q)}} \alpha_1^{y_1} \phi_1^{\left(n-y_1\right)} \phi_2^{\left(y_1\right)} \left(1 + \alpha_1\right)^n_{R(p,q)}
\]

\[
= \frac{\phi_1^{\left(m_2\right)} + m_2(n-m_2)}{\left(1 + \alpha_1\right)^n_{R(p,q)}} \left[ n \right]_{m_2, R(p,q)} \frac{[n-m_2]_{y_1}}{y_1_{R(p,q)}} \left(1 + \alpha_1\right)^n_{R(p,q)}
\]

\[
= \frac{\phi_1^{\left(m_2\right)} + m_2(n-m_2)}{\left(\phi_1^{n-m_2} + \alpha_1 \phi_2^{n-m_2}\right)_{R(p,q)}}
\]

Thus,

\[
E\left( [Y_2]_{m_2, R(p,q)} \right) = \frac{\alpha_2^{m_2} \phi_2^{\left(m_2\right)}}{\left(1 + \alpha_2\right)^n_{R(p,q)}} \left[ n \right]_{m_2, R(p,q)} \frac{[n-m_2]_{y_1}}{y_1_{R(p,q)}} \left(1 + \alpha_1\right)^n_{R(p,q)}
\]

So, the joint \( R(p, q) \)-factorial moments \( E\left( [Y_1]_{m_1, R(p,q)} [Y_2]_{m_2, R(p,q)} \right), m_2 \in \{0, 1, \ldots, n-m_1\}, m_1 \in \{0, 1, \ldots, n\} \) can be calculated by applying the relation:

\[
E\left( [Y_1]_{m_1, R(p,q)} [Y_2]_{m_2, R(p,q)} \right) = E\left( E\left( [Y_1]_{m_1, R(p,q)} [Y_2]_{m_2, R(p,q)} \right) | Y_1 \right)
\]

\[
= \frac{\alpha_2^{m_2} \phi_2^{\left(m_2\right)}}{\left(1 + \alpha_2\right)^n_{R(p,q)}} E\left( [Y_1]_{m_1, R(p,q)} [n - Y_1]_{m_2, R(p,q)} \right).
\]

Since

\[
E\left( [Y_1]_{m_1, R(p,q)} [n - Y_1]_{m_2, R(p,q)} \right) = \sum_{y_1=m_1}^{n-m_2} \left[ y_1 \right]_{m_1, R(p,q)} \frac{[n-y_1]_{m_2, R(p,q)}}{y_1_{R(p,q)}} \alpha_1^{y_1} \phi_1^{\left(n-y_1\right)} \phi_2^{\left(y_1\right)} \left(1 + \alpha_1\right)^n_{R(p,q)}
\]

Using the relation

\[
[y_1]_{m_1, R(p,q)} [n - y_1]_{m_2, R(p,q)} \left[ \frac{n}{y_1} \right]_{R(p,q)} = \left[ n \right]_{m_1 + m_2, R(p,q)} \left[ n - m_1 - m_2 \right]_{y_1 - m_1, R(p,q)}
\]

and the \( R(p, q) \)-binomial formula, we obtain:

\[
E\left( [Y_1]_{m_1, R(p,q)} [n - Y_1]_{m_2, R(p,q)} \right) = \left[ n \right]_{m_1 + m_2, R(p,q)} \alpha_1^{m_1} \phi_1^{\left(m_1\right)} \phi_2^{\left(m_1\right)}
\]

\[
\times \sum_{y_1=m_1}^{n-m_2-m_1} \left[ \frac{n - m_1 - m_2}{y_1 - m_1} \right]_{R(p,q)} \frac{\alpha_1^{m_1} \phi_2^{\left(m_1\right)} \phi_1^{\left(n-y_1-m_1\right)} \left(1 + \alpha_1\right)^n_{R(p,q)}}{\left(1 + \alpha_1\right)^n_{R(p,q)}}
\]

\[
= \left[ n \right]_{m_1 + m_2, R(p,q)} \alpha_1^{m_1} \phi_1^{\left(m_1\right)} \phi_2^{\left(m_1\right)} \left(\phi_1^{m_1} \phi_2^{m_1} + \alpha_1 \phi_2^{n-m_1-m_2}\right)_{R(p,q)}
\]

\[
= \left[ n \right]_{m_1 + m_2, R(p,q)} \alpha_1^{m_1} \phi_1^{\left(m_1\right)} \phi_2^{\left(m_1\right)} \left(1 + \alpha_1\right)^n_{R(p,q)}
\]
Thus,

\[
E([Y_1]_{m_1,R(p,q)}[Y_2]_{m_2,R(p,q)}) = \frac{[n]_{m_1+m_2,R(p,q)} \alpha_1^m \alpha_2^n \phi_{2}^{m_2} \phi_{1}^{m_1} + \alpha_1 \phi_{2}^{m_1} \phi_{1}^{m_2}}{(1+\alpha_2)^{m_2} \alpha_1^n \phi_{1}^{m_1} \phi_{2}^{m_2}}.
\]

**Corollary 2.3** The \(R(p,q)\)-covariance of the \(R(p,q)\)-trinomial probability distribution of the first kind is presented by:

\[
Cov([Y_1]_{R(p,q)}), [Y_2]_{R(p,q)}) = \frac{\phi_{1}^{n-1} \alpha_1 \alpha_2 [n]_{R(p,q)} (\alpha_1^{n-1} + \alpha_2^{n-1})}{(1+\alpha_1)(1+\alpha_2)(\phi_{1}^{n-1} + \alpha_2^{n-1})}.
\]

**Proof 2.4** By definition, the covariance of \([Y_1]_{R(p,q)}\) and \([Y_2]_{R(p,q)}\) is given by

\[
Cov([Y_1]_{R(p,q)}, [Y_2]_{R(p,q)}) = E([Y_1]_{R(p,q)}[Y_2]_{R(p,q)}) - E([Y_1]_{R(p,q)})E([Y_2]_{R(p,q)}).
\]

Taking \(m_1 = m_2 = 1\), in the relations (2.1), (2.2), and (2.3), we get

\[
E([Y_1]_{R(p,q)})E([Y_2]_{R(p,q)} = \frac{\alpha_1 [n]_{R(p,q)}^{n-1} \alpha_2 [n]_{R(p,q)}^{n-1}}{(1+\alpha_1)(1+\alpha_2)(\phi_{1}^{n-1} + \alpha_2^{n-1})},
\]

and

\[
E([Y_1]_{R(p,q)}[Y_2]_{R(p,q)}) = \frac{\phi_{1}^{n-1} \alpha_1 \alpha_2 [n]_{R(p,q)}^{2}}{(1+\alpha_1)(1+\alpha_2)(\phi_{1}^{n-1} + \alpha_2^{n-1})}.
\]

After computation, the result follows.

### 2.2 Negative \(R(p,q)\)-trinomial distribution of the first kind

Let \(U_n\) be the number of successes until the occurrence of the \(n^{th}\) failure, in a sequence of independent Bernoulli trials, with probability of success at the \(i^{th}\) trial given as:

\[
p_i = \frac{\alpha \phi_{2}^{i-1}}{\phi_{1}^{i-1} + \alpha \phi_{2}^{i-1}}, \quad i \in \mathbb{N} \cup \{0\}.
\]

**Lemma 2.5** The \(R(p,q)\)-random variable \(U_n\) obeys the negative \(R(p,q)\)-binomial probability distribution of the first kind:

\[
P(U = u) = \binom{n + u - 1}{u} \frac{\alpha^u \phi_{1}^{n-u} \phi_{2}^{u}}{(1+\alpha)^{n+u}} R(p,q), \quad u \in \mathbb{N} \cup \{0\}. \tag{2.6}
\]

Moreover, the \(R(p^{-1}, q^{-1})\)-factorial moments are presented by the relation

\[
E([U_n]_{m,R(p^{-1}, q^{-1})}) = \binom{n + m - 1}{m} \frac{\alpha^m}{(1+\alpha)^{m+n}} R(p^{-1}, q^{-1}), \quad m \in \mathbb{N} \cup \{0\}. \tag{2.7}
\]
We denote by $W_j$ the number of successes of the $j^{th}$ kind until the occurrence of the $n$th failure of the second kind, in a sequence of Bernoulli trials with chain-composite failures. Then, the $\mathcal{R}(p, q)$-random vector $\mathbf{W} = (W_1, W_2)$ follows the negative $\mathcal{R}(p, q)$-trinomial probability distribution of the first kind, with parameters $n$, $\alpha = (\alpha_1, \alpha_2)$, $p$, and $q$. Its mass function is presented by:

$$P(W = w) = \binom{n + w_1 + w_2 - 1}{w_1, w_2} \frac{\alpha_1^{w_1} \alpha_2^{w_2} \phi_1^{\binom{n-w_1}{2}} \phi_2^{\binom{n-w_2}{2}} \phi_3^{w_1 + w_2}}{(1 + \alpha_1)^{n+w_1+w_2} (1 + \alpha_2)^{n+w_2}}, \quad (2.8)$$

where $w_j \in \mathbb{N} \cup \{0\}$, $0 < \alpha_j < 1$, and $j \in \{1, 2\}$.

**Theorem 2.6** For $m_1 \in \mathbb{N} \cup \{0\}$ and $m_2 \in \mathbb{N} \cup \{0\}$, the $\mathcal{R}(p^{-1}, q^{-1})$-factorial moments of the negative $\mathcal{R}(p, q)$-trinomial probability distribution of the first kind, with parameters $n$, $\alpha = (\alpha_1, \alpha_2)$, $p$, and $q$ are given as follows:

$$E(W_1^{m_1} | W_1 = w_1) = \alpha_1^{m_1} [n + w_1 + 1]_{m_1, \mathcal{R}(p,q)}, \quad (2.9)$$

$$E(W_2^{m_2} | W_2 = w_2) = \alpha_1^{m_1} [n + w_2 + 1]_{m_2, \mathcal{R}(p,q)}, \quad (2.10)$$

and

$$E\left(\frac{W_1^{m_1} W_2^{m_2}}{(\phi_1^{n+w_1} + \alpha_2 \phi_2^{n+w_2})^{m_1}}\right) = \frac{[n + m_1 + m_2 - 1]_{m_1+m_2, \mathcal{R}(p,q)}}{\alpha_1^{-m_1} \alpha_2^{-m_2}}, \quad (2.11)$$

$$E\left(\frac{W_1^{m_1} W_2^{m_2}}{(\phi_1^{n+w_1} + \alpha_2 \phi_2^{n+w_2})^{m_1}}\right) = \frac{[n + m_1 + m_2 - 1]_{m_1+m_2, \mathcal{R}(p,q)}}{\alpha_1^{-m_1} \alpha_2^{-m_2}}, \quad (2.12)$$

**Proof 2.7** According to the relations (2.6) and (2.7), we derive the $\mathcal{R}(p^{-1}, q^{-1})$-factorial moments of $W_2$ given by (2.9). Moreover, the conditional distribution of the $\mathcal{R}(p, q)$-random variable $W_1$, given that $W_2 = w_2$, is a negative $\mathcal{R}(p, q)$-trinomial probability distribution of the first kind, with mass function:

$$P(W_1 = w_1 | W_2 = w_2) = \binom{n + w_1 + w_2 - 1}{w_1} \frac{\alpha_1^{w_1} \phi_1^{\binom{n-w_1}{2}} \phi_2^{w_2}}{(1 + \alpha_1)^{n+w_1+w_2}, \ m_1 \in \mathbb{N} \cup \{0\}}.$$ 

Using, once the relations (2.6) and (2.7), the conditional $\mathcal{R}(p^{-1}, q^{-1})$-factorial moments of $W_1$, given that $W_2 = w_2$, are given by (2.10).

Furthermore, the expected value of the $\mathcal{R}(p^{-1}, q^{-1})$-function of $\mathbf{W} = (W_1, W_2)$

$$\Gamma := \frac{[W_1^{m_1} W_2^{m_2}]_{\mathcal{R}(p^{-1}, q^{-1})}}{(\phi_1^{n+w_1} + \alpha_2 \phi_2^{n+w_2})^{m_1}}, \ m_1 \in \mathbb{N} \cup \{0\}$$

can be calculated according to the relation

$$E(\Gamma) = E(E(\Gamma | W_2)) = \alpha_1^{m_1} E\left(\frac{[n + W_2 + m_1 - 1]_{m_1, \mathcal{R}(p,q)}}{(\phi_1^{n+w_2} + \alpha_2 \phi_2^{n+w_2})^{m_1}}\right).$$
Since
\[ E \left( \frac{n + W_2 + m_1 - 1}{\phi_1^{n+W_2} \oplus \alpha_2 \phi_2^{n+W_2}}_{\mathcal{R}(p,q)}^{m_1} \right) = \sum_{w_2}^{\infty} \left[ n + w_2 + m_1 - 1 \right]_{\mathcal{R}(p,q)} \left[ n + w_2 - 1 \right]_{\mathcal{R}(p,q)} \times \left[ \frac{\alpha_2 w_2 \phi_1^{(n-w_2)} \phi_2^{(w_2)}}{(1 + \alpha_2)^{n+m_1+w_2}} \right]. \]

From the relation
\[ [n+w_2+m_1-1]_{\mathcal{R}(p,q)} \left[ n + w_2 - 1 \right]_{\mathcal{R}(p,q)} = [n+m_1-1]_{\mathcal{R}(p,q)} \left[ n + m_1 + w_2 - 1 \right]_{\mathcal{R}(p,q)} \]
and the negative \( \mathcal{R}(p,q) \)-binomial formula [8], we have:
\[ E \left( \frac{[n + W_2 + m_1 - 1]}{\phi_1^{n+W_2} \oplus \alpha_2 \phi_2^{n+W_2}}_{\mathcal{R}(p,q)}^{m_1} \right) = \sum_{w_2=0}^{\infty} \left[ n + m_1 + w_2 - 1 \right]_{\mathcal{R}(p,q)} \times \alpha_2 w_2 \phi_1^{(n-w_2)} \phi_2^{(w_2)}{(1 + \alpha_2)^{n+m_1+w_2}} \]
\[ = [n + m_1 - 1]_{\mathcal{R}(p,q)}. \]

Thus, the relation (2.11) holds. Analogously, the expected value of the \( \mathcal{R}(p^{-1}, q^{-1}) \)-function of \( W = (W_1, W_2) \)
\[ \Lambda := \frac{[W_1]_{\mathcal{R}(p^{-1}, q^{-1})} [W_2]_{\mathcal{R}(p^{-1}, q^{-1})}}{\left( \phi_1^{n+W_2} \oplus \alpha_2 \phi_2^{n+W_2} \right)_{\mathcal{R}(p,q)}^{m_1}}, m_1 \in \mathbb{N} \cup \{0\}, m_2 \in \mathbb{N} \cup \{0\} \]
may be computed using the relation
\[ E(\Lambda) = E(E(\Lambda|W_2)) = \alpha_1^{m_1} E \left( \frac{[W_2]_{\mathcal{R}(p^{-1}, q^{-1})}[n + W_2 + m_1 - 1]_{\mathcal{R}(p,q)}}{\left( \phi_1^{n+W_2} \oplus \alpha_2 \phi_2^{n+W_2} \right)_{\mathcal{R}(p,q)}^{m_1}} \right). \]

Since
\[ E \left( \frac{[W_2]_{\mathcal{R}(p^{-1}, q^{-1})}[n + W_2 + m_1 - 1]_{\mathcal{R}(p,q)}}{\left( \phi_1^{n+W_2} \oplus \alpha_2 \phi_2^{n+W_2} \right)_{\mathcal{R}(p,q)}^{m_1}} \right) = \sum_{w_2=m_2}^{\infty} \left[ w_2 \right]_{\mathcal{R}(p^{-1}, q^{-1})} \left[ n + w_2 + m_1 - 1 \right]_{\mathcal{R}(p,q)} \times \left[ n + w_2 - 1 \right]_{\mathcal{R}(p,q)} \frac{\alpha_2 w_2 \phi_1^{(n-w_2)} \phi_2^{(w_2)}}{(1 + \alpha_2)^{n+w_2}}. \]

Using the relations
\[ [w_2]_{\mathcal{R}(p^{-1}, q^{-1})} = \phi_1^{w_2} \phi_2^{(w_2+1)} \left[ w_2 \right]_{\mathcal{R}(p,q)}, \]
\[
\binom{w_2}{2} + \binom{m_2 + 1}{2} - w_2 m_2 = \binom{w_2 - m_2}{2}
\]

and

\[
[w_2]_{m_2, R(p, q)} [n + w_2 + m_1 - 1]_{m_1, R(p, q)} \left[ \begin{array}{c} n + w_2 - 1 \\ w_2 \end{array} \right]_{R(p, q)} = [n + m_1 + m_2 - 1]_{m_1 + m_2, R(p, q)} \\
\times \left[ \begin{array}{c} n + m_1 + w_2 - 1 \\ w_2 - m_2 \end{array} \right]_{R(p, q)},
\]

together with the negative \( R(p, q) \)-binomial formula (8), we get

\[
\mathbb{E} \left( \frac{[W_2]_{m_2, R(p^{-1}, q^{-1})} [n + W_2 + m_1 - 1]_{m_1, R(p, q)}}{(\phi_1^{n+W_2} \oplus \alpha_2 \phi_2^{n+W_2})_{R(p, q)}} \right) = [n + m_2 + m_1 - 1]_{m_1 + m_2, R(p, q)} \alpha_2^{m_2}
\]

and the proof is achieved.

**Corollary 2.8** The \( R(p, q) \)-covariance of the \( R(p, q) \)-random variables \( \hat{W} := (\phi_1^{n+W_2} \oplus \alpha_2 \phi_2^{n+W_2})^{-1} [W_1]_{m_1, R(p^{-1}, q^{-1})} \) and \( \overline{W} := [W_2]_{m_2, R(p^{-1}, q^{-1})} \) is determined by:

\[
\text{Cov}(\hat{W}, \overline{W}) = [n]_{R(p, q)} \alpha_1 \alpha_2 \left( [n + 1]_{R(p, q)} - [n]_{R(p, q)} \right),
\]

where the \( R(p, q) \)-random vector \( \overline{W} = (W_1, W_2) \) follows the negative \( R(p, q) \)-trinomial probability distribution of the first kind, with parameters \( n, \alpha = (\alpha_1, \alpha_2), p, \) and, \( q \).

**Proof 2.9** By definition, the \( R(p, q) \)-covariance of \( \hat{W} \) and \( \overline{W} \) is given by

\[
\text{Cov}(\hat{W}, \overline{W}) = \mathbb{E} \left( \hat{W} \overline{W} \right) - \mathbb{E}(\hat{W}) \mathbb{E}(\overline{W}).
\]

Taking \( m_1 = m_2 = 1 \), in the relations (2.9), (2.11), and (2.12), the result follows.

**2.3 \( R(p, q) \)-trinomial distribution of the second kind**

The probability function of the \( R(p, q) \)-random vector \( \underline{X} = (X_1, X_2) \) of the \( R(p, q) \)-trinomial probability distribution of the second kind, with parameters \( n, \beta = (\beta_1, \beta_2), p, \) and, \( q \) is given by:

\[
P(\underline{X} = \underline{x}) = \binom{n}{x_1, x_2}_{R(p, q)} \beta_1^{x_1} \beta_2^{x_2} (1 \ominus \beta_1)^{n-x_1} (1 \ominus \beta_2)^{n-x_1-x_2},
\]

(2.13)

where \( x_j \in \{0, 1, \cdots, n\}, x_1 + x_2 \leq n, s_j = \sum_{i=1}^{j} x_i, 0 < \beta_j < 1, \) and \( j \in \{1, 2\} \).
Theorem 2.10  The $R(p, q)$-factorial moments of the $R(p, q)$-trinomial probability distribution of the second kind, with parameters $n, \beta = (\beta_1, \beta_2), p, \text{and}, q,$ are given by:

$$E([X_1]_{m_1, R(p,q)}) = \beta_1^{m_1} [n]_{m_1, R(p,q)}, \quad m_1 \in \{0, 1, \ldots, n\},$$  \hspace{2.5cm} (2.14)

$$E([X_2]_{m_2, R(p,q)}|X_1 = x_1) = \beta_2^{m_2} [n - x_1]_{m_2, R(p,q)},$$  \hspace{2.5cm} (2.15)

where $m_2 \in \{0, 1, \ldots, n - x_1\}$,

$$E \left( \frac{[X_2]_{m_2, R(p,q)}[X_2]_{m_2, R(p,q)}}{(\phi_1^{n-m_2-X_1} \ominus \beta_1 \phi_2^{n-m_2-X_1})_{R(p,q)}^{m_2}} \right) = \beta_2^{m_2} [n]_{m_2, R(p,q)},$$  \hspace{2.5cm} (2.16)

with $m_2 \in \{0, 1, \ldots, n\}$, and

$$E \left( \frac{[X_1]_{m_1, R(p,q)}[X_2]_{m_2, R(p,q)}}{(\phi_1^{n-m_2-X_1} \ominus \beta_1 \phi_2^{n-m_2-X_1})_{R(p,q)}^{m_2}} \right) = \beta_1^{m_1} \beta_2^{m_2} [n]_{m_1+m_2, R(p,q)},$$  \hspace{2.5cm} (2.17)

where $m_1 \in \{0, 1, \ldots, n - m_2\}$ and $m_2 \in \{0, 1, \ldots, n\}$.

Proof 2.11  The $R(p, q)$-random variable $X_1$ obey the $R(p, q)$-binomial probability distribution of the second kind, with mass function:

$$P(X_1 = x_1) = \binom{n}{x_1} \beta_1^{x_1} (1 \ominus \beta_1)^{n-x_1}, \quad x_1 \in \{0, 1, \ldots, n\}.$$  \hspace{2.5cm}

Thus, using [7], the relation (2.14) follows. Furthermore, the conditional distribution of the $R(p, q)$-random variable $X_2$, given that $X_1 = x_1$, is a $R(p, q)$-binomial probability distribution of the second kind, with mass density

$$P(X_2 = x_2|X_1 = x_1) = \binom{n - x_1}{x_2} \beta_2^{x_2} (1 \ominus \beta_2)^{n-x_1-x_2}.$$  \hspace{2.5cm}

According again to [7], the conditional $R(p, q)$-factorial moments of $X_2$, given that $X_1 = x_1$, are furnished by (2.15). The expected value of the $R(p, q)$-function of $X = (X_1, X_2)$

$$\Delta := \frac{[X_2]_{m_2, R(p,q)}}{(\phi_1^{n-m_2-X_1} \ominus \beta_1 \phi_2^{n-m_2-X_1})_{R(p,q)}^{m_2}}, \quad m_2 \in \{0, 1, \ldots, n\}$$  \hspace{2.5cm}

can be computed according to the relation

$$E(\Delta) = E[E(\Delta|X_1)] = \beta_2^{m_2} E \left[ \frac{[n - X_1]_{m_2, R(p,q)}}{(\phi_1^{n-m_2-X_1} \ominus \beta_1 \phi_2^{n-m_2-X_1})_{R(p,q)}^{m_2}} \right].$$  \hspace{2.5cm}
Thus,

\[
E\left[\frac{[n - X_1]_{m_2, R(p, q)}}{(\phi_1^{n-m_2-X_1} \oplus \beta_1 \phi_2^{n-m_2-X_1})_{R(p, q)}}\right] = \sum_{x_1=0}^{n-m_2} [n - x_1]_{m_2, R(p, q)} \left[\frac{n}{x_1}ight]_{R(p, q)} \times \frac{\beta_1^{x_1} (1 \oplus \beta_1)^{n-x_1}}{(\phi_1^{n-m_2-x_1} \oplus \beta_1 \phi_2^{n-m_2-x_1})_{R(p, q)}}.
\]

Using the relations:

\[
[n - x_1]_{m_2, R(p, q)} \left[\frac{n}{x_1}\right]_{R(p, q)} = [n]_{m_2, R(p, q)} \left[\frac{n - m_2}{x_1}\right]_{R(p, q)},
\]

\[
(1 \oplus \beta_1)^{n-x_1} = (1 \oplus \beta_1)^{n-m_2-x_1} (\phi_1^{n-m_2-x_1} \oplus \beta_1 \phi_2^{n-m_2-x_1})_{R(p, q)},
\]

and the \( R(p, q) \)-binomial formula \[8\], we have:

\[
E\left[\frac{[n - X_1]_{m_2, R(p, q)}}{(\phi_1^{n-m_2-X_1} \oplus \beta_1 \phi_2^{n-m_2-X_1})_{R(p, q)}}\right] = [n]_{m_2, R(p, q)} \sum_{x_1=0}^{n-m_2} \left[\frac{n - m_2}{x_1}\right]_{R(p, q)} \times \beta_1^{m_1} (1 \oplus \beta_1)^{n-m_2-x_1}.
\]

Thus,

\[
E\left(\frac{[X_2]_{m_2, R(p, q)}}{\left(\phi_1^{n-m_2-X_1} \oplus \beta_1 \phi_2^{n-m_2-X_1}\right)_{R(p, q)}}\right) = \beta_2^{m_2} [n]_{m_2, R(p, q)}.
\]

Similarly, the expected value of the \( R(p, q) \)-function of \( X = (X_1, X_2) \)

\[
\nabla := [X_1]_{m_1, R(p, q)} [X_2]_{m_2, R(p, q)} \left(\phi_1^{n-m_2-X_1} \oplus \beta_1 \phi_2^{n-m_2-X_1}\right)_{R(p, q)}
\]

may be calculated using the relation:

\[
E(\nabla) = E\left[E(\nabla | X_1)\right] = \beta_2^{m_2} E\left[\frac{[X_1]_{m_1, R(p, q)} [n - X_1]_{m_2, R(p, q)}}{\left(\phi_1^{n-m_2-X_1} \oplus \beta_1 \phi_2^{n-m_2-X_1}\right)_{R(p, q)}}\right].
\]

Since

\[
E\left[\frac{[X_1]_{m_1, R(p, q)} [n - X_1]_{m_2, R(p, q)}}{\left(\phi_1^{n-m_2-X_1} \oplus \beta_1 \phi_2^{n-m_2-X_1}\right)_{R(p, q)}}\right] = \sum_{x_1=0}^{n-m_2} [x_1]_{m_1, R(p, q)} [n - x_1]_{m_2, R(p, q)} \left[\frac{n}{x_1}\right]_{R(p, q)} \times \frac{\beta_1^{x_1} (1 \oplus \beta_1)^{n-x_1}}{(\phi_1^{n-m_2-x_1} \oplus \beta_1 \phi_2^{n-m_2-x_1})_{R(p, q)}}.
\]
From the $\mathcal{R}(p,q)$-binomial formula \([8]\), the relations (2.18), and

\[
[x_1]_{m_1,\mathcal{R}(p,q)}[n-x_1]_{m_2,\mathcal{R}(p,q)} = [n]_{m_1+m_2,\mathcal{R}(p,q)}\left[\begin{array}{c} n-m_1-m_2 \\ x_1-m_1 \end{array}\right]_{\mathcal{R}(p,q)},
\]

we obtain:

\[
E\left[\left[\frac{[X_1]_{m_1,\mathcal{R}(p,q)}[n-X_1]_{m_2,\mathcal{R}(p,q)}}{(\phi_1^{n-m_2-X_1} \ominus \beta_1 \phi_2^{n-m_2-X_1})_{\mathcal{R}(p,q)}}\right]\right] = [n]_{m_1+m_2,\mathcal{R}(p,q)}\beta_1^{m_1}
\]

and the realtion (2.17) follows.

**Corollary 2.12** The $\mathcal{R}(p,q)$- covariance of the functions $\hat{X} := [X_1]_{\mathcal{R}(p,q)}$ and $\overline{X} := (\phi_1^{n-1-x_1} - \beta_1 \phi_2^{n-1-x_1})^{-1}[X_2]_{\mathcal{R}(p,q)}$ is given by:

\[
\text{Cov}(\hat{X}, \overline{X}) = [n]_{\mathcal{R}(p,q)}\beta_1 \beta_2 ([n-1]_{\mathcal{R}(p,q)} - [n]_{\mathcal{R}(p,q)}),
\]

with $X = (X_1, X_2)$ a $\mathcal{R}(p,q)$-random vector satisfying the $\mathcal{R}(p,q)$- trinomial probability distribution of the second kind, with parameters $n$, $\underline{\beta} = (\beta_1, \beta_2)$, $p$, and, $q$.

**Proof 2.13** Taking $m_1 = m_2 = 1$ in the relations (2.14), (2.16), and (2.17), we compute $E(\hat{X}\overline{X})$ and $E(\hat{X}) E(\overline{X})$. Therefore, the proof is achieved.

### 2.4 Negative $\mathcal{R}(p,q)$-trinomial distribution of the second kind

Let $T_n$ be the number of failures until the occurrence of the $n^{th}$ success, in a sequence of independent geometric sequences of trials. Then, the distribution of the $\mathcal{R}(p,q)$- random variable $T_n$ is called negative $\mathcal{R}(p,q)$-binomial distribution of the second kind, with parameters $n$, $\underline{\beta} = (\beta_1, \beta_2)$, $p$, and, $q$.

**Lemma 2.14** The probability function of the negative $\mathcal{R}(p,q)$- binomial probability distribution of the second kind, with parameters $n$, $\underline{\beta}$, $p$, and, $q$, is given by:

\[
P(T = t) = \left[\begin{array}{c} n + t - 1 \\ t \end{array}\right]_{\mathcal{R}(p,q)} \beta^t (1 \ominus \beta)^n_{\mathcal{R}(p,q)}, \quad t \in \mathbb{N} \cup \{0\}.
\] (2.19)

Furthermore, its $\mathcal{R}(p,q)$-factorial moments are presented as follows:

\[
E([T]_{m,\mathcal{R}(p,q)}) = \left[\begin{array}{c} n + m - 1 \\ m \end{array}\right]_{\mathcal{R}(p,q)} \beta^m_{\mathcal{R}(p,q)}, \quad m \in \mathbb{N}.
\] (2.20)

Let $\overline{V} = (V_1, V_2)$ be a $\mathcal{R}(p,q)$-random vector obeying the negative $\mathcal{R}(p,q)$- trinomial probability distribution of the second kind, with parameters $n$, $\underline{\beta} = (\beta_1, \beta_2)$, $p$, and, $q$. Then, its mass function is given by:

\[
P(\overline{V} = v) = \left[\begin{array}{c} n + v_1 + v_2 - 1 \\ v_1, v_2 \end{array}\right]_{\mathcal{R}(p,q)} \beta_1^{v_1} \beta_2^{v_2} (1 \ominus \beta_1)^{n+v_1} (1 \ominus \beta_2)^n_{\mathcal{R}(p,q)}, \quad v_j \in \mathbb{N} \cup \{0\},
\]

where $0 < \beta_j < 1$, and $j \in \{1, 2\}$. 

13
Theorem 2.15 For \( m_1 \in \mathbb{N} \cup \{0\} \) and \( m_2 \in \mathbb{N} \cup \{0\} \), the \( \mathcal{R}(p, q) \)-factorial moments of the negative \( \mathcal{R}(p, q) \)-trinomial probability distribution of the second kind, with parameters \( n, \beta = (\beta_1, \beta_2) \), \( p \), and \( q \) are presented as follows:

\[
E([V_2]_{m_2, \mathcal{R}(p, q)}) = \frac{\beta_2^{m_2} [n + m_2 - 1]_{m_2, \mathcal{R}(p, q)}}{\left( \phi_1^n \odot \beta_2 \phi_2^n \right)_{\mathcal{R}(p, q)}}, \tag{2.21}
\]

\[
E([V_1]_{m_1, \mathcal{R}(p, q)}|V_2 = v_2) = \frac{\beta_1^{m_1} [n + v_2 + m_1 - 1]_{m_1, \mathcal{R}(p, q)}}{\left( \phi_1^n \odot \beta_1 \phi_2^n \right)_{\mathcal{R}(p, q)}}, \tag{2.22}
\]

\[
E\left( \frac{[V_1]_{m_1, \mathcal{R}(p, q)} [V_2]_{m_2, \mathcal{R}(p, q)}}{\left( \phi_1^n \odot \beta_1 \phi_2^n \right)_{\mathcal{R}(p, q)}}^{-m_1} \right) = \frac{\beta_1^{m_1} [n + m_1 - 1]_{m_1, \mathcal{R}(p, q)}}{\left( \phi_1^n \odot \beta_2 \phi_2^n \right)_{\mathcal{R}(p, q)}}, \tag{2.23}
\]

and

\[
E\left( \frac{[V_1]_{m_1, \mathcal{R}(p, q)} [V_2]_{m_2, \mathcal{R}(p, q)}}{\left( \phi_1^n \odot \beta_1 \phi_2^n \right)_{\mathcal{R}(p, q)}}^{-m_1} \right) = \frac{\beta_1^{m_1} [n + m_1 + m_2 - 1]_{m_1 + m_2, \mathcal{R}(p, q)}}{\left( \phi_1^n \odot \beta_2 \phi_2^n \right)_{\mathcal{R}(p, q)}}. \tag{2.24}
\]

Proof 2.16 The relation (2.21) comes from (2.19) and (2.20). Furthermore, the conditional distribution of the \( \mathcal{R}(p, q) \)-random variable \( V_1 \), given that \( V_2 = v_2 \), is a negative \( \mathcal{R}(p, q) \)-binomial distribution of the second kind, with probability function:

\[
P(V_1 = v_1|V_2 = v_2) = \frac{[V_1]_{m_1, \mathcal{R}(p, q)} [V_2]_{m_2, \mathcal{R}(p, q)}}{\left( \phi_1^n \odot \beta_1 \phi_2^n \right)_{\mathcal{R}(p, q)}}, \quad v \in \mathbb{N} \cup \{0\}.
\]

Using once the relations (2.19) and (2.20), the conditional \( \mathcal{R}(p, q) \)-factorial moments of \( V_1 \), given that \( V_2 = v_2 \), are given by (2.22). Besides, the expected value of the \( \mathcal{R}(p, q) \)-function of \( V = (V_1, V_2) \)

\[
\hat{\Delta} := \frac{[V_1]_{m_1, \mathcal{R}(p, q)} [V_2]_{m_2, \mathcal{R}(p, q)}}{\left( \phi_1^n \odot \beta_1 \phi_2^n \right)_{\mathcal{R}(p, q)}}, \quad m_1 \in \mathbb{N} \cup \{0\}
\]

may be evaluated according to the relation:

\[
E(\hat{\Delta}) = E[E(\hat{\Delta}|V_2)] = \beta_1^{m_1} E\left( [n + V_2 + m_1 - 1]_{m_1, \mathcal{R}(p, q)} \right).
\]

Since

\[
E\left( [n + V_2 + m_1 - 1]_{m_1, \mathcal{R}(p, q)} \right) = \sum_{v_2=0}^{\infty} [n + v_2 + m_1 - 1]_{m_1, \mathcal{R}(p, q)} \beta_2^{m_2} (1 \odot \beta_2)^n_{\mathcal{R}(p, q)}.
\]

\[
\sum_{v_2=0}^{\infty} [n + v_2 - 1]_{\mathcal{R}(p, q)} \beta_2^{m_2} (1 \odot \beta_2)^n_{\mathcal{R}(p, q)}.
\]

14
From the relation
\[
[n + v_2 + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \left[ \begin{array}{c} n + v_2 - 1 \\ v_2 \end{array} \right]_{\mathcal{R}(p,q)} = [n + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \left[ \begin{array}{c} n + m_1 + v_2 - 1 \\ v_2 \end{array} \right]_{\mathcal{R}(p,q)},
\]
and the negative $\mathcal{R}(p,q)$-binomial formula \[8\], we have:
\[
E\left( [n + V_2 + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \right) = [n + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \sum_{v_2=0}^{\infty} \left[ \begin{array}{c} n + v_2 - 1 \\ v_2 \end{array} \right]_{\mathcal{R}(p,q)} \beta_2^{v_2} m_1 = [n + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \left( \phi_1^n + \beta_2 \phi_2^n \right)^m_{\mathcal{R}(p,q)}.
\]
Thus, the relation \[2.23\] holds.
So, the expected value of the $\mathcal{R}(p,q)$-function of $V = (V_1, V_2)$
\[
\hat{\nabla} := \frac{[V_1]_{m_1, \mathcal{R}(p,q)}[V_2]_{m_2, \mathcal{R}(p,q)}}{\left( \phi_1^n + \beta_1 \phi_2^n \right)^{m_1}_{\mathcal{R}(p,q)}} ,
\]
can be calculated using the relation:
\[
E(\hat{\nabla}) = E\left[ E(\hat{\nabla} | V_2) \right] = \beta_1^{m_1} E\left[ [V_2]_{m_2, \mathcal{R}(p,q)} [n + V_2 + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \right].
\]
Since
\[
E\left[ [V_2]_{m_2, \mathcal{R}(p,q)} [n + V_2 + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \right] = \sum_{v_2=m_2=0}^{\infty} [v_2]_{m_2, \mathcal{R}(p,q)} [n + v_2 + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \left[ \begin{array}{c} n + v_2 - 1 \\ v_2 \end{array} \right]_{\mathcal{R}(p,q)} \beta_2^{v_2} (1 \oplus \beta_2)^{n}_{\mathcal{R}(p,q)}.
\]
From the $\mathcal{R}(p,q)$-binomial formula \[8\], the relations \[2.18\], and
\[
[v_2]_{m_2, \mathcal{R}(p,q)} [n + v_2 + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \left[ \begin{array}{c} n + v_2 - 1 \\ v_2 \end{array} \right]_{\mathcal{R}(p,q)} = [n + m_1 + m_2 - 1]_{m_1 + m_2, \mathcal{R}(p,q)} \left[ \begin{array}{c} n + m_1 + v_2 - 1 \\ v_2 - m_2 \end{array} \right]_{\mathcal{R}(p,q)},
\]
we obtain:
\[
E\left[ [V_2]_{m_2, \mathcal{R}(p,q)} [n + V_2 + m_1 - 1]_{m_1, \mathcal{R}(p,q)} \right] = \frac{[n + m_1 + m_2 - 1]_{m_1 + m_2, \mathcal{R}(p,q)} \beta_2^{m_2} \left( \phi_1^n \oplus \beta_2 \phi_2^n \right)^{m_1+m_2}_{\mathcal{R}(p,q)}}{\left( \phi_1^n \oplus \beta_2 \phi_2^n \right)^{m_1}_{\mathcal{R}(p,q)}}
\]
and the relation \[2.24\] follows.
Corollary 2.17 The $\mathcal{R}(p, q)$-covariance of the functions and $\widehat{V} := (\phi_1^{n+V_2} \ominus \beta_2 \phi_2^{n+V_2})[V_1]_{m_1, \mathcal{R}(p, q)}$ and $\overline{V} := [V_2]_{m_2, \mathcal{R}(p, q)}$ is given by:

$$
\text{Cov}(\widehat{V}, \overline{V}) = \frac{[n]_{\mathcal{R}(p, q)} \beta_1 \beta_2}{(\phi_1^n - \beta_2 \phi_2^n)} \left( \frac{[n+1]_{\mathcal{R}(p, q)}}{(\phi_1^{n+1} - \beta_2 \phi_2^{n+1})} - \frac{[n]_{\mathcal{R}(p, q)}}{(\phi_1^n - \beta_2 \phi_2^n)} \right),
$$

where $\overline{V} = (V_1, V_2)$ is a $\mathcal{R}(p, q)$-random vector satisfying the negative $\mathcal{R}(p, q)$-trinomial probability distribution of the second kind, with parameters $n$, $\beta = (\beta_1, \beta_2)$, $p$, and $q$.

Proof 2.18 Putting $m_1 = m_2 = 1$, in the relations (2.21), (2.23), and (2.24), we obtain:

$$
E(\overline{V}) = \frac{[n+1]_{\mathcal{R}(p, q)} [n]_{\mathcal{R}(p, q)} \beta_1 \beta_2}{(\phi_1^n - \beta_2 \phi_2^n)(\phi_1^{n+1} - \beta_2 \phi_2^{n+1})} \quad \text{and} \quad E(\widehat{V})E(\overline{V}) = \frac{[n]_{\mathcal{R}(p, q)} [n]_{\mathcal{R}(p, q)} \beta_2 \beta_1}{(\phi_1^n - \beta_2 \phi_2^n)^2}.
$$

After computation, the result follows.

3 Trinomial distributions and particular quantum algebras

In this section, we deduce the $\mathcal{R}(p, q)$-trinomial probability distribution of the first and second kind, their negative and properties from some quantum algebra existing in the literature.

3.1 Trinomial distribution and Biedenharn-Macfarlane algebra [13]

Taking $\mathcal{R}(x) = \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{q - q^{-1}}$, we obtain:

(i) The $q$-trinomial distribution of the first kind is given by

$$
P(Y_1 = y_1, Y_2 = y_2) = \binom{n}{y_1, y_2} \frac{\alpha_1^{y_1} \alpha_2^{y_2} q^{(n-y_1)(n-y_2) - (n-y_2)^2}}{(1 + \alpha_1)n_q(1 + \alpha_2)n^{-y_1}_q} \quad (3.1)
$$

and the $q$-factorial moments are:

$$
E([Y_1]_{m_1,q}) = \frac{[n]_{m_1,q} \alpha_1^{m_1} q^{-\left(\frac{m_1}{2}\right)}}{(1 + \alpha_1)_q^{m_1}}, \quad m_1 \in \{0, 1, \ldots, n\},
$$

$$
E([Y_2]_{m_2,q} | Y_1 = y_1) = \frac{[n-y_1]_{m_2,q} \alpha_2^{m_2} q^{-\left(\frac{m_2}{2}\right)}}{(1 + \alpha_2)_q^{m_2}}, \quad m_2 \in \{0, 1, \ldots, n-y_1\},
$$

$$
E([Y_2]_{m_2,p,q} | Y_1 = y_1) = \frac{[n]_{m_2,p,q} \alpha_2^{m_2} p^{\left(\frac{m_2}{2}\right)} q^{(n-m_2)p_2 + m_2(n-m_2)}}{(1 + \alpha_2)_p^{m_2}(1 + \alpha_1)_q^{m_2}}, \quad m_2 \in \{0, 1, \ldots, n\},
$$

and

$$
E([Y_1]_{m_1,q} | [Y_2]_{m_2,q}) = \frac{[n]_{m_1,m_2,q} \alpha_1^{m_1} \alpha_2^{m_2} q^{\left(\frac{m_1}{2}\right) + m_2(n-m_2)} q^{-\left(\frac{m_2}{2}\right) - (n-m_2)}}{(1 + \alpha_1)_q^{m_1}(1 + \alpha_2)_q^{m_2}(1 + \alpha_1)_q^{m_2}}.
$$
where \( m_1 \in \{0, 1, \ldots, n - m_2\} \) and \( m_2 \in \{0, 1, \ldots, n\} \). Moreover, the \( q\)-covariance is derived as follows:

\[
\text{Cov}([Y_1]_q, [Y_2]_q) = \frac{q^{n_1-n_2} \alpha_1 \alpha_2 [n]_q (\lceil n - 1 \rceil_q - [n]_q)}{(1 + \alpha_1)(1 + \alpha_2)(q^{n-1} + \alpha_1 q^{n+1})}.
\]

(ii) The negative \( q\)-trinomial distribution of the first kind, with parameters \( n, \alpha = (\alpha_1, \alpha_2) \), and \( q \) is presented by:

\[
P(W = w) = \binom{n + w_1 + w_2 - 1}{w_1, w_2} \frac{\alpha_1^{w_1} \alpha_2^{w_2} q^{\binom{n-w_1}{2} + \binom{n-w_2}{2} q^{-\binom{w_1}{2} - \binom{w_2}{2}}}}{(1 + \alpha_1)^q q^{\binom{n+w_1+w_2}{2} (1 + \alpha_2)^q}}
\]

where \( w_j \in \mathbb{N} \cup \{0\}, 0 < \beta_j < 1, \) and \( j \in \{1, 2\} \). Besides, for \( m_1 \in \mathbb{N} \cup \{0\} \) and \( m_2 \in \mathbb{N} \cup \{0\} \),
its \( q^{-1}\)-factorial moments are given as follows:

\[
E([W_2]_{m_2,q^{-1}}) = [n + m_2 - 1]_{m_2,q} \alpha_2^{m_2}, m_2 \in \mathbb{N} \cup \{0\},
\]

\[
E([W_1]_{m_1,q^{-1}}; W_2 = w_2) = [n + w_2 + m_1 - 1]_{m_1,q} \alpha_1^{m_1}, m_1 \in \mathbb{N} \cup \{0\},
\]

\[
E\left(\frac{[W_1]_{m_1,q^{-1}}}{p^{n+W_2} \oplus \alpha_2 q^{n-W_2}}\right)_{m_1,q} = [n + m_1 - 1]_{m_1,q} \alpha_1^{m_1}, m_1 \in \mathbb{N} \cup \{0\},
\]

and

\[
E\left(\frac{[W_1]_{m_1,q^{-1}} [W_2]_{m_2,q^{-1}}}{p^{n+W_2} \oplus \alpha_2 q^{n+W_2}}\right)_{m_1,m_2,q} = [n + m_1 + m_2 - 1]_{m_1+m_2,q} \alpha_1^{m_1} \alpha_2^{m_2},
\]

Furthermore, the covariance of \( \tilde{W} := (q^{n+W_2} \oplus \alpha_2 q^{n-W_2})^{-1}[W_1]_{m_1,q^{-1}} \) and \( \overline{W} := [W_2]_{m_2,q^{-1}} \) is determined by:

\[
\text{Cov}(\tilde{W}, \overline{W}) = [n]_q \alpha_1 \alpha_2 ([n + 1]_q - [n]_q).
\]

(iii) The \( q\)-trinomial distribution of the second kind, with parameters \( n, \beta = (\beta_1, \beta_2) \), and \( q \) is given by:

\[
P(X_1 = x_1, X_2 = x_2) = \binom{n}{x_1, x_2} \beta_1^{x_1} \beta_2^{x_2} (1 \oplus \beta_1)^{n-x_1} (1 \oplus \beta_2)^{n-x_1-x_2},
\]

where \( x_j \in \{0, 1, \ldots, n\}, x_1 + x_2 \leq n, s_j = \sum_{i=1}^{j} x_j, 0 < \beta_j < 1, \) and \( j \in \{1, 2\} \). Besides, its \( q\)-factorial moments are given by:

\[
E([X_1]_{m_1,q}) = [n]_{m_1,q} \beta_1^{m_1}, m_1 \in \{0, 1, \ldots, n\},
\]

\[
E([X_2]_{m_2,q}|X_1 = x_1) = [n - x_1]_{m_2,q} \beta_2^{m_2}, m_2 \in \{0, 1, \ldots, n - x_1\},
\]
\[
E\left(\frac{[X_2]_{m_2,q}}{(p^{n-m_2-X_1} \ominus \beta_1 q^{n-m_2-X_1})_{m_2}^q}\right) = [n]_{m_2,q} \beta_2^{m_2}, \quad m_2 \in \{0, 1, \ldots, n\},
\]
and
\[
E\left(\frac{[X_1]_{m_1,q}[X_2]_{m_2,q}}{(p^{n-m_2-X_1} \ominus \beta_1 q^{n-m_2-X_1})_{m_2}^q}\right) = [n]_{m_1+m_2,q} \beta_1^{m_1} \beta_2^{m_2},
\]
where \(m_1 \in \{0, 1, \ldots, n - m_2\}\) and \(m_2 \in \{0, 1, \ldots, n\}\). Moreover, the \(q\)-covariance of the functions \((q^{n-1-X_1} - \beta_1 q^{-n+1+X_1})^{-1}[X_2]_q\) and \([X_1]_q\) is given by:
\[
Cov\left([X_1]_q, (q^{n-1-X_1} - \beta_1 q^{-n+1+X_1})^{-1}[X_2]_q\right) = [n]_q \beta_1 \beta_2 ([n - 1]_q - [n]_q).
\]

(iii) The negative \(q\)-trinomial probability distribution of the second kind, with parameters \(n, \beta = (\beta_1, \beta_2)\), and, \(q\) is given by:
\[
P(V = v) = \left[\frac{n + v_1 + v_2 - 1}{v_1, v_2}\right] \beta_1^{v_1} \beta_2^{v_2} (1 \ominus \beta_1)_q^{n+v_1} (1 \ominus \beta_2)_q^{n+v_2}, \quad v_j \in \mathbb{N} \cup \{0\},
\]
where \(0 < \beta_j < 1\), and \(j \in \{1, 2\}\). Moreover, its \(q\)-factorial moments of are presented as follows:
\[
E([V_2]_{m_2,q}) = \left[\frac{n + m_2 - 1}{p^n \ominus \beta_2 q^n} \beta_2^{m_2}\right]_{m_2}^q,
\]
\[
E([V_1]_{m_1,q} [V_2]_{m_2,q} | V_2 = v_2) = \left[\frac{n + v_2 + m_1 - 1}{p^{n+v_2} \ominus \beta_1 q^{n+v_2}} \beta_1^{m_1}\right]_{m_1}^q,
\]
\[
E\left(\frac{[V_1]_{m_1,q}}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})_{m_1}^q}\right) = \left[\frac{n + m_1 - 1}{p^n \ominus \beta_2 q^n} \beta_1^{m_1}\right]_{m_1}^q,
\]
and
\[
E\left(\frac{[V_1]_{m_1,q} [V_2]_{m_2,q}}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})_{m_2}^q}\right) = \left[\frac{n + m_1 + m_2 - 1}{p^n \ominus \beta_2 q^n} \beta_1^{m_1} \beta_2^{m_2}\right]_{m_1+m_2}^q,
\]
where \(m_1 \in \mathbb{N} \cup \{0\}\) and \(m_2 \in \mathbb{N} \cup \{0\}\). Furthermore, the \(q\)-covariance of the functions and \(\widehat{V} := (q^{n+v_2} - \beta_2 q^{-n-v_2}) [V_1]_{m_2,q}\) and \(\overline{V} := [V_2]_{m_1,q}\) is given by:
\[
Cov(\widehat{V}, \overline{V}) = \frac{[n]_q \beta_1 \beta_2}{(q^n - \beta_2 q^{-n})} \left[\frac{n + 1}{q^{n+1} - \beta_2 q^{-n+1}} - \frac{[n]_q}{q^n - \beta_2 q^{-n}}\right].
\]
3.2 Trinomial distribution and Jagannathan-Srinivasa algebra [12]

For illustration, we investigate by taking \( R(x, y) = (p - q)^{-1}(x - y) \):

(i) The \((p, q)\)-trinomial distribution of the first kind is given by

\[
P(Y_1 = y_1, Y_2 = y_2) = \binom{n}{y_1, y_2}_{p,q} \frac{\alpha_{1}^{y_1} \alpha_{2}^{y_2} p^{(n-y_1)} q^{(n-y_2)}}{(1 + \alpha_1)^n_{p,q}(1 + \alpha_2)^{n-y_1}_{p,q}}
\]

and the \((p, q)\)-factorial moments are:

\[
E([Y_1]_{m_1,p,q}) = \binom{n}{m_1}_{p,q} \frac{\alpha_{1}^{m_1} q^{(n-m_1)}}{(1 + \alpha_1)^{m_1}_{p,q}}, \quad m_1 \in \{0, 1, \ldots, n\},
\]

\[
E([Y_2]_{m_2,p,q}|Y_1 = y_1) = \frac{[n - y_1]_{m_2,p,q} \alpha_{2}^{m_2} q^{(m_2)}}{(1 + \alpha_2)^{m_2}_{p,q}}, \quad m_2 \in \{0, 1, \ldots, n - y_1\},
\]

\[
E([Y_2]_{m_2,p,q}) = \frac{[n]_{m_2,p,q} \alpha_{2}^{m_2} q^{(m_2)}}{(1 + \alpha_2)^{m_2}_{p,q}(1 + \alpha_1)^{n-m_2}_{p,q}}, \quad m_2 \in \{0, 1, \ldots, n\},
\]

and

\[
E([Y_1]_{m_1,p,q}[Y_2]_{m_2,p,q}) = \frac{[n]_{m_1,m_2,p,q} \alpha_{1}^{m_1} \alpha_{2}^{m_2} p^{(m_2)} q^{(m_1)}}{(1 + \alpha_1)^{m_1}_{p,q}(1 + \alpha_2)^{m_2}_{p,q}(1 + \alpha_1)^{n-m_2}_{p,q}},
\]

where \( m_1 \in \{0, 1, \ldots, n - m_2\} \) and \( m_2 \in \{0, 1, \ldots, n\} \). Moreover, the \((p, q)\)-covariance is derived as follows:

\[
Cov([Y_1]_{p,q}, [Y_2]_{R(p,q)}) = \frac{p^{n-1} \alpha_1 \alpha_2 [n]_{p,q} - [n]_{p,q}}{(1 + \alpha_1)(1 + \alpha_2)(p^{n-1} + \alpha_1 q^{n-1})}
\]

(ii) The negative \((p, q)\)-trinomial distribution of the first kind, with parameters \( n, \alpha = (\alpha_1, \alpha_2), p, \) and, \( q \) is presented by:

\[
P(W = w) = \binom{n + w_1 + w_2 - 1}{w_1, w_2}_{p,q} \frac{\alpha_{1}^{w_1} \alpha_{2}^{w_2} p^{(n-w_1)} q^{(n-w_2)}}{(1 + \alpha_1)^{n+w_1+w_2}_{p,q}(1 + \alpha_2)^{n+w_2}_{p,q}},
\]

where \( w_j \in \mathbb{N} \cup \{0\}, 0 < \alpha_j < 1, \) and \( j \in \{1, 2\} \). Besides, for \( m_1 \in \mathbb{N} \cup \{0\} \) and \( m_2 \in \mathbb{N} \cup \{0\} \), its \((p^{-1}, q^{-1})\)-factorial moments are given as follows:

\[
E([W_2]_{m_2,p^{-1},q^{-1}}) = [n + m_2 - 1]_{m_2,p,q} \alpha_{2}^{m_2}, \quad m_2 \in \mathbb{N} \cup \{0\},
\]

\[
E([W_1]_{m_1,p^{-1},q^{-1}}|W_2 = w_2) = [n + w_2 + m_1 - 1]_{m_1,p,q} \alpha_{1}^{m_1}, \quad m_1 \in \mathbb{N} \cup \{0\},
\]

\[
E\left(\frac{[W_1]_{m_1,p^{-1},q^{-1}}}{(p^{n+W_2} + \alpha_2 q^{n+W_2})_{p,q}}\right) = [n + m_1 - 1]_{m_1,p,q} \alpha_{1}^{m_1}, \quad m_1 \in \mathbb{N} \cup \{0\},
\]

19
and
\[
E \left( \frac{[W_1]_{m_1, p^{-1}, q^{-1}} [W_2]_{m_2, p^{-1}, q^{-1}}}{(p^{n+W_2} \oplus \alpha_2 q^{n+W_2})_{p,q}^{m_1}} \right) = [n + m_1 + m_2 - 1]_{m_1 + m_2, p,q} \alpha_1^{m_1} \alpha_2^{m_2},
\]

Furthermore, the covariance of \( \widehat{W} := (p^{n+W_2} \oplus \alpha_2 q^{n+W_2})^{-1} [W_1]_{m_1, p^{-1}, q^{-1}} \) and \( \overline{W} := [W_2]_{m_2, p^{-1}, q^{-1}} \) is determined by:
\[
Cov(\widehat{W}, \overline{W}) = [n]_{p,q} \alpha_1 \alpha_2 \left( \left[ n + 1 \right]_{p,q} - \left[ n \right]_{p,q} \right).
\]

(iii) The \((p, q)\)-trinomial distribution of the second kind, with parameters \(n, \beta = (\beta_1, \beta_2), p, \) and, \( q \) is given by:
\[
P(X_1 = x_1, X_2 = x_2) = \left[ \begin{array}{c} n \\ x_1, x_2 \end{array} \right]_{p,q} \beta_1^{x_1} \beta_2^{x_2} (1 \ominus \beta_1)_{p,q}^{n-x_1} (1 \ominus \beta_2)_{p,q}^{n-x_1-x_2},
\]
where \(x_j \in \{0, 1, \ldots, n\}, x_1 + x_2 \leq n, s_j = \sum_{i=1}^{j} x_j, 0 < \beta_j < 1, \) and \( j \in \{1, 2\} \).

Besides, its \((p, q)\)-factorial moments are given by:
\[
E([X_1]_{m_1, p,q}) = [n]_{m_1, p,q} \beta_1^{m_1}, m_1 \in \{0, 1, \ldots, n\},
\]
\[
E([X_2]_{m_2, p,q}, X_1 = x_1) = [n - x_1]_{m_2, p,q} \beta_2^{m_2}, m_2 \in \{0, 1, \ldots, n - x_1\},
\]
\[
E \left( \frac{[X_2]_{m_2, p,q}}{(p^{n-m_2-X_1} \oplus \beta_1 q^{n-m_2-X_1})_{p,q}^{m_2}} \right) = [n]_{m_2, p,q} \beta_2^{m_2}, m_2 \in \{0, 1, \ldots, n\},
\]
and
\[
E \left( \frac{[X_1]_{m_1, p,q}, [X_2]_{m_2, p,q}}{(p^{n-m_2-X_1} \oplus \beta_1 q^{n-m_2-X_1})_{p,q}^{m_2}} \right) = [n]_{m_1 + m_2, p,q} \beta_1^{m_1} \beta_2^{m_2},
\]
where \(m_1 \in \{0, 1, \ldots, n - m_2\}\) and \(m_2 \in \{0, 1, \ldots, n\}\). Moreover, the \((p, q)\)-covariance of the functions \((p^{n-X_1} - \beta_1 q^{n-X_1})^{-1} [X_2]_{p,q}\) and \([X_1]_{p,q}\) is given by:
\[
Cov\left([X_1]_{p,q}, (p^{n-X_1} - \beta_1 q^{n-X_1})^{-1} [X_2]_{p,q}\right) = [n]_{p,q} \beta_1 \beta_2 \left( \left[ n - 1 \right]_{p,q} - \left[ n \right]_{p,q} \right).
\]

(iii) The negative \((p, q)\)-trinomial probability distribution of the second kind, with parameters \(n, \widehat{\beta} = (\beta_1, \beta_2), p, \) and, \( q \) is given by:
\[
P(V = v) = \left[ \begin{array}{c} n + v_1 + v_2 - 1 \\ v_1, v_2 \end{array} \right]_{p,q} \beta_1^{v_1} \beta_2^{v_2} (1 \ominus \beta_1)_{p,q}^{n+v_2} (1 \ominus \beta_2)_{p,q}^{n}, v_j \in \mathbb{N} \cup \{0\},
\]

20
where $0 < \beta_j < 1$, and $j \in \{1, 2\}$. Moreover, its $(p, q)$-factorial moments of are presented as follows:

$$E([V_2]_{m_2, p, q}) = \frac{[n + m_2 - 1]_{m_2, p, q} \beta_2^{m_2}}{(p^n \ominus \beta_2 q^n)_{p, q}^{m_2}},$$

$$E([V_1]_{m, p, q}|V_2 = v_2) = \frac{[n + v_2 + m_1 - 1]_{m_1, p, q} \beta_1^{m_1}}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})_{p, q}^{m_1}},$$

$$E\left(\frac{[V_1]_{m_1, p, q}[V_2]_{m_2, p, q}}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})_{p, q}^{-m_2}}\right) = \frac{[n + m_1 + m_2 - 1]_{m_1 + m_2, p, q} \beta_1^{m_1} \beta_2^{m_2}}{(p^n \ominus \beta_2 q^n)_{p, q}^{m_1 + m_2}},$$

and

$$E\left(\frac{[V_1]_{m_1, p, q}[V_2]_{m_2, p, q}}{(p^n \ominus \beta_2 q^n)^{-m_2}}\right) = \frac{[n]_{p, q} \beta_1 \beta_2}{(p^n - \beta_2 q^n)} \left(\frac{[n + 1]_{p, q}}{(p^{n+1} - \beta_2 q^{n+1})} - \frac{[n]_{p, q}}{(p^n - \beta_2 q^n)}\right).$$

### 3.3 Trinomial distribution and Chakrabarty and Jagannathan algebra [2]

Putting $\mathcal{R}(x, y) = \frac{1-xy}{(p-\alpha)q}$, we obtain:

(i) The $(p^{-1}, q)$-trinomial distribution of the first kind is given by

$$P(Y_1 = y_1, Y_2 = y_2) = \binom{n}{y_1, y_2}_{p^{-1}, q} \alpha_1^{y_1} \alpha_2^{y_2} p^{-y_1 - y_2} q^{y_1 y_2},$$

and the $(p^{-1}, q)$-factorial moments are:

$$E([Y_1]_{m_1, p^{-1}, q}) = \frac{[n]_{m_1, p^{-1}, q} \alpha_1^{m_1} q^{(m_1)_2}}{(1 + \alpha_1)_{p^{-1}, q}^{m_1}}, \quad m_1 \in \{0, 1, \ldots, n\},$$

$$E([Y_2]_{m_2, p^{-1}, q}|Y_1 = y_1) = \frac{[n - y_1]_{m_2, p^{-1}, q} \alpha_2^{m_2} q^{(m_2)_2}}{(1 + \alpha_2)_{p^{-1}, q}^{m_2}}, \quad m_2 \in \{0, 1, \ldots, n - y_1\},$$

$$E([Y_2]_{m_2, p^{-1}, q}) = \frac{[n]_{m_2, p^{-1}, q} \alpha_2^{m_2} q^{(m_2)_2} p^{-m_2 - m_2(n-m_2)}}{(1 + \alpha_2)_{p^{-1}, q}^{m_2} (p^{n-m_2} + \alpha_1 q^{n-m_2})_{p^{-1}, q}^{m_2}}, \quad m_2 \in \{0, 1, \ldots, n\},$$
and

\[
E([Y_1]_{m_1,p^{-1},q}[Y_2]_{m_2,p^{-1},q}) = \frac{[n]_{m_1+m_2,p^{-1},q} \alpha_1^{m_1} \alpha_2^{m_2} p^{-(m_2)} - m_2(n-m_2) q^{(m_2)}}{(1 + \alpha_1)^{m_1} (1 + \alpha_2)^{m_2} p^{-1,q} (p^{n-m_2} + \alpha_1 q^{n-m_2})^{m_2}},
\]

where \(m_1 \in \{0, 1, \ldots, n - m_2\}\) and \(m_2 \in \{0, 1, \ldots, n\}\). Moreover, the \((p^{-1}, q)\)-covariance is derived as follows:

\[
\text{Cov}([Y_1]_{p^{-1},q}, [Y_2]_{p^{-1},q}) = \frac{p^{-n+1} \alpha_1 \alpha_2 [n]_{p^{-1},q} ([n-1]_{p^{-1},q} - [n]_{p^{-1},q})}{(1 + \alpha_1)(1 + \alpha_2)(p^{-n+1} + \alpha_1 q^{n-1})},
\]

(ii) The negative \((p^{-1}, q)\)-trinomial distribution of the first kind, with parameters \(n, \underline{\alpha} = (\alpha_1, \alpha_2), p, \) and, \(q\) is presented by:

\[
P(W = w) = \left[n + w_1 + w_2 - 1\right]_{w_1,w_2} \frac{\alpha_1^{w_1} \alpha_2^{w_2} p^{-(w_1)} - (n-w_2) q^{(w_1)+w_2}}{(1 + \alpha_1)^{n+w_1+w_2} (1 + \alpha_2)^{n+w_2}},
\]

where \(w_j \in \mathbb{N} \cup \{0\}, 0 < \alpha_j < 1, \) and \(j \in \{1, 2\}\). Besides, for \(m_1 \in \mathbb{N} \cup \{0\}\) and \(m_2 \in \mathbb{N} \cup \{0\}\),

its \((p, q^{-1})\)-factorial moments are given as follows:

\[
E([W_2]_{m_2,p,q^{-1}}) = [n + m_2 - 1]_{m_2,p^{-1},q} \alpha_2^{m_2}, \quad m_2 \in \mathbb{N} \cup \{0\},
\]

\[
E([W_1]_{m_1,p,q^{-1}}[W_2 = w_2]) = [n + w_2 + m_1 - 1]_{m_1,p^{-1},q} \alpha_1^{m_1}, \quad m_1 \in \mathbb{N} \cup \{0\},
\]

\[
E\left(\frac{[W_1]_{m_1,p,q^{-1}}}{(p^{n+w_2} + \alpha_2 q^{n+w_2})_{m_1}}\right)_{p^{-1},q} = [n + m_1 - 1]_{m_1,p^{-1},q} \alpha_1^{m_1}, \quad m_1 \in \mathbb{N} \cup \{0\},
\]

and

\[
E\left(\frac{[W_1]_{m_1,p,q^{-1}}[W_2]_{m_2,p,q^{-1}}}{(p^{n+w_2} + \alpha_2 q^{n+w_2})^{m_1}_{m_2}}\right)_{p^{-1},q} = [n + m_1 + m_2 - 1]_{m_1+m_2,p^{-1},q} \alpha_1^{m_1} \alpha_2^{m_2}.
\]

Furthermore, the covariance of \(\widehat{W} := (p^{-n-W_2} - \alpha_2 q^{n+W_2})^{-1}[W_1]_{m_1,p,q^{-1}}\) and \(\overline{W} := [W_2]_{m_2,p,q^{-1}}\) is determined by:

\[
\text{Cov}(\widehat{W}, \overline{W}) = [n]_{p^{-1},q} \alpha_1 \alpha_2 ([n+1]_{p^{-1},q} - [n]_{p^{-1},q}).
\]

(iii) The \((p^{-1}, q)\)-trinomial distribution of the second kind, with parameters \(n, \underline{\beta} = (\beta_1, \beta_2), p, \) and, \(q\) is given by:

\[
P(X_1 = x_1, X_2 = x_2) = \left[x_1,x_2\right]_{p^{-1},q} \beta_1^{x_1} \beta_2^{x_2} (1 \oplus \beta_1)^{n-x_1} (1 \oplus \beta_2)^{n-x_1-x_2},
\]

where \(x_j \in \{0, 1, \ldots, n\}, x_1 + x_2 \leq n, s_j = \sum_{i=1}^{j} x_j, 0 < \beta_j < 1, \) and \(j \in \{1, 2\}\). Besides, its \((p^{-1}, q)\)-factorial moments are given by:

\[
E([X_1]_{m_1,p^{-1},q}) = [n]_{m_1,p^{-1},q} \beta_1^{m_1}, \quad m_1 \in \{0, 1, \ldots, n\},
\]

22
The negative probability distribution presented as follows:

\[ E([X_2]_{m_2, p^{-1}, q}|X_1 = x_1) = [n - x_1]_{m_2, p^{-1}, q} \beta_2^{m_2}, \ m_2 \in \{0, 1, \ldots, n - x_1\}, \]

\[ E \left( \frac{[X_2]_{m_2, p^{-1}, q}}{(p^{n-m_2-X_1} \ominus \beta_1 q^{n-m_2-X_1})^{m_2}_{p^{-1}, q}} \right) = [n]_{m_2, p^{-1}, q} \beta_2^{m_2}, \ m_2 \in \{0, 1, \ldots, n\}, \]

and

\[ E \left( \frac{[X_1]_{m_1, p^{-1}, q} [X_2]_{m_2, p^{-1}, q}}{(p^{n-m_2-X_1} \ominus \beta_1 q^{n-m_2-X_1})^{m_2}_{p^{-1}, q}} \right) = [n]_{m_1 + m_2, p^{-1}, q} \beta_1^{m_1} \beta_2^{m_2}, \]

where \( m_1 \in \{0, 1, \ldots, n - m_2\} \) and \( m_2 \in \{0, 1, \ldots, n\} \). Moreover, the \((p^{-1}, q)\)-covariance of the functions \((p^{-n+1+X_1} - \beta_1 q^{n-1-X_1})^{-1}[X_2]_{p^{-1}, q}\) and \([X_1]_{p^{-1}, q}\) is given by:

\[ \text{Cov} \left( [X_1]_{p^{-1}, q}, (p^{-n+1+X_1} - \beta_1 q^{n-1-X_1})^{-1}[X_2]_{p^{-1}, q} \right) = [n]_{p^{-1}, q} \beta_1 \beta_2 ([n - 1]_{p^{-1}, q} - [n]_{p^{-1}, q}). \]

(iii) The negative \((p^{-1}, q)\)-trinomial probability distribution of the second kind, with parameters \( n, \beta = (\beta_1, \beta_2), p, \) and \( q \) is given by:

\[ P(V = v) = \begin{bmatrix} n + v_1 + v_2 - 1 \\ v_1, v_2 \end{bmatrix}_{p^{-1}, q} \beta_1^{v_1} \beta_2^{v_2} (1 \ominus \beta_1)^{n+v_1+v_2} (1 \ominus \beta_2)^n q^{v_1+v_2}, \ v_j \in \mathbb{N} \cup \{0\}, \]

where \( 0 < \beta_j < 1 \) and \( j \in \{1, 2\} \). Moreover, its \((p^{-1}, q)\)-factorial moments of are presented as follows:

\[ E([V_2]_{m_2, p^{-1}, q}) = \frac{[n + m_2 - 1]_{m_2, p^{-1}, q} \beta_2^{m_2}}{(p^n \ominus \beta_2 q^n)^{m_2}_{p^{-1}, q}}, \]

\[ E([V_1]_{m_1, p^{-1}, q}|V_2 = v_2) = \frac{[n + v_2 + m_1 - 1]_{m_1, p^{-1}, q} \beta_1^{m_1}}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})^{m_1}_{p^{-1}, q}}, \]

\[ E \left( \frac{[V_1]_{m_1, p^{-1}, q}}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})^{m_1}_{p^{-1}, q}} \right) = \frac{[n + m_1 - 1]_{m_1, p^{-1}, q} \beta_1^{m_1}}{(p^n \ominus \beta_2 q^n)^{m_1}_{p^{-1}, q}}, \]

and

\[ E \left( \frac{[V_1]_{m_1, p^{-1}, q} [V_2]_{m_2, p^{-1}, q}}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})^{m_1}_{p^{-1}, q}} \right) = \frac{[n + m_1 + m_2 - 1]_{m_1 + m_2, p^{-1}, q} \beta_1^{m_1} \beta_2^{m_2}}{(p^n \ominus \beta_2 q^n)^{m_1 + m_2}_{p^{-1}, q}}, \]

where \( m_1 \in \mathbb{N} \cup \{0\} \) and \( m_2 \in \mathbb{N} \cup \{0\} \). Furthermore, the \((p^{-1}, q)\)-covariance of the functions and \( \overline{V} := (p^{-n-V_2} - \beta_2 q^{n+V_2})[V_1]_{m_2, p^{-1}, q} \) and \( \overline{V} := [V_2]_{m_1, p^{-1}, q} \) is given by:

\[ \text{Cov}(\overline{V}, \overline{V}) = \frac{[n]_{p^{-1}, q} \beta_1 \beta_2}{(p^n - \beta_2 q^n)} \left( \frac{[n + 1]_{p^{-1}, q}}{(p^{n-1} - \beta_2 q^{n+1})} - \frac{[n]_{p^{-1}, q}}{(p^n - \beta_2 q^n)} \right). \]
3.4 Trinomial distribution and Hounkonnou-Ngompe generalization of $q-$ Quesne algebra \[11\]

Setting $R(x, y) = \frac{xy}{(q-p)x+y}$, we obtain the following results:

(i) The Hounkonnou-Ngompe generalization of $q$- Quesne-trinomial distribution of the first kind is given by:

$$P(Y_1 = y_1, Y_2 = y_2) = \frac{\binom{n}{y_1, y_2} p^{(n-y_1)} q^{-(y_1)} (y_2)^2}{(1+\alpha_1)^{n+1} (1+\alpha_2)^{n+2}}$$

and their factorial moments are:

$$E([Y_1]^{Q}_{m_1,p,q}) = \frac{\left[m_1\right]^{Q}_{m_1,p,q} p^{(m_1)} q^{-(m_1)}}{(1+\alpha_1)^{m_1}}, m_1 \in \{0, 1, \ldots, n\},$$

$$E([Y_2]^{Q}_{m_2,p,q}|Y_1 = y_1) = \frac{\left[n-y_1\right]^{Q}_{m_2,p,q} p^{(m_2)} q^{-(m_2)}}{(1+\alpha_2)^{m_2}}, m_2 \in \{0, 1, \ldots, n-y_1\},$$

$$E([Y_2]^{Q}_{m_2,p,q}) = \frac{\left[n\right]^{Q}_{m_2,p,q} p^{(m_2)} q^{-(m_2)}}{(1+\alpha_2)^{m_2}}, m_2 \in \{0, 1, \ldots, n\},$$

and

$$E([Y_1]^{Q}_{m_1,p,q}|[Y_2]^{Q}_{m_2,p,q}) = \frac{\left[m_1+m_2\right]^{Q}_{m_1+m_2,p,q} p^{(m_1+m_2)} q^{-(m_1+m_2)}}{(1+\alpha_1)^{m_1+m_2}},$$

where $m_1 \in \{0, 1, \ldots, n-m_2\}$ and $m_2 \in \{0, 1, \ldots, n\}$. Moreover, the covariance is derived as follows:

$$Cov([Y_1]^{Q}_{p,q}, [Y_2]^{Q}_{p,q}) = \frac{\binom{n}{p+1} p^{(n+1)} q^{-(n+1)}}{(1+\alpha_1)(1+\alpha_2)(p^{n+1}+\alpha_1 q^{n+1})}.$$

(ii) The negative Hounkonnou-Ngompe generalization of $q-$ Quesne-trinomial distribution of the first kind, with parameters $n, \alpha = (\alpha_1, \alpha_2), p$, and $q$ is presented by:

$$P(W = w) = \binom{n+w_1+w_2-1}{w_1, w_2} p^{(w_1)} q^{-(w_1)} (w_2)^2$$

where $w_j \in \mathbb{N} \cup \{0\}$, $0 < \alpha_j < 1$, and $j \in \{1, 2\}$. Besides, its factorial moments are given as follows:

$$E([W_2]^{Q}_{m_2,p,q-1}) = \alpha_2^{m_2} [n+m_2-1]^{Q}_{m_2,p,q},$$

$$E([W_1]^{Q}_{m_1,p,q-1}|W_2 = w_2) = \alpha_1^{m_1} [n+w_2+m_1-1]^{Q}_{m_1,p,q},$$

$$E\left(\frac{[W_1]^{Q}_{m_1,p,q-1}}{(p^{n+W_2}+\alpha_2 q^{n+W_2})^{m_1}}\right) = \alpha_1^{m_1} [n+m_1-1]^{Q}_{m_1,p,q},$$

24
and
\[
E\left( \frac{[W_1]^Q_{m_1,p^{-1},q^{-1}} [W_2]^Q_{m_2,p^{-1},q^{-1}}}{(p^{n+W_1} \oplus \alpha_2 q^{n+W_2})_{p,q}^m} \right) = \alpha_1^{m_1} \alpha_2^{m_2} [n + m_1 + m_2 - 1]^Q_{m_1+m_2,p,q},
\]
where \( m_1 \in \mathbb{N} \cup \{0\} \) and \( m_2 \in \mathbb{N} \cup \{0\} \). Furthermore, the covariance of \( \tilde{W} := (p^{n+W_1} \oplus \alpha_2 q^{n+W_2})^{-1}[W_1]^Q_{m_1,p^{-1},q^{-1}} \) and \( \mathcal{W} := [W_2]^Q_{m_2,p^{-1},q^{-1}} \) is determined by:
\[
Cov(\tilde{W}, \mathcal{W}) = [n]^Q_{p,q} \alpha_1 \alpha_2 ([n + 1]^Q_{p,q} - [n]^Q_{p,q}).
\]

(iii) The Hounkonnou-Ngompe generalization of \( q \)-Quene-trinomial distribution of the second kind, with parameters \( n, \beta = (\beta_1, \beta_2) \), \( p \), and, \( q \) is given by:
\[
P(X_1 = x_1, X_2 = x_2) = \begin{bmatrix} n \\ x_1, x_2 \end{bmatrix}^Q_{p,q} \beta_1^{x_1} \beta_2^{x_2} (1 \oplus \beta_1)^{n-x_1} (1 \oplus \beta_2)^{n-x_1-x_2},
\]
where \( x_j \in \{0, 1, \cdots, n\} \), \( x_1 + x_2 \leq n \), \( s_j = \sum_{i=1}^{j} x_j \), \( 0 < \beta_j < 1 \), and \( j \in \{1, 2\} \). Besides, its factorial moments are given by:
\[
E([X_1]^Q_{m_1,p,q}) = [n]^Q_{m_1,p,q} \beta_1^{m_1}, m_1 \in \{0, 1, \cdots, n\},
\]
\[
E([X_2]^Q_{m_2,p,q}|X_1 = x_1) = [n-x_1]^Q_{m_2,p,q} \beta_2^{m_2}, m_2 \in \{0, 1, \cdots, n-x_1\},
\]
\[
E\left( \frac{[X_2]^Q_{m_2,p,q}}{(p^{n-m_2-X_1} \oplus \beta_1 q^{n-m_2-X_1})_{p,q}^{m_2}} \right) = [n]^Q_{m_2,p,q} \beta_2^{m_2}, m_2 \in \{0, 1, \cdots, n\},
\]
and
\[
E\left( \frac{[X_1]^Q_{m_1,p,q}[X_2]^Q_{m_2,p,q}}{(p^{n-m_2-X_1} \oplus \beta_1 q^{n-m_2-X_1})_{p,q}^{m_2}} \right) = [n]^Q_{m_1+m_2,p,q} \beta_1^{m_1} \beta_2^{m_2},
\]
where \( m_1 \in \{0, 1, \cdots, n-m_2\} \) and \( m_2 \in \{0, 1, \cdots, n\} \). Moreover, the covariance of the functions \((p^{n-1-X_1} - \beta_1 q^{1+X_1-n})^{-1}[X_2]^Q_{p,q} \) and \([X_1]^Q_{p,q} \) is given by:
\[
Cov([X_1]^Q_{p,q}, (p^{n-1-X_1} - \beta_1 q^{1+X_1-n})^{-1}[X_2]^Q_{p,q}) = [n]^Q_{p,q} \beta_1 \beta_2 ([n-1]^Q_{p,q} - [n]^Q_{p,q}).
\]

(iii) The negative Hounkonnou-Ngompe generalization of \( q \)-Quene-trinomial probability distribution of the second kind, with parameters \( n, \beta = (\beta_1, \beta_2) \), \( p \), and, \( q \) is given by:
\[
P(V = v) = \begin{bmatrix} n + v_1 + v_2 - 1 \\ v_1, v_2 \end{bmatrix}^Q_{v_1, v_2} \beta_1^{v_1} \beta_2^{v_2} (1 \oplus \beta_1)^{n+v_1} (1 \oplus \beta_2)^{n}, v_j \in \mathbb{N} \cup \{0\},
\]

25
where $0 < \beta_j < 1$, and $j \in \{1, 2\}$. Moreover, for $m_1 \in \mathbb{N} \cup \{0\}$ and $m_2 \in \mathbb{N} \cup \{0\}$, its factorial moments of are presented as follows:

$$E([V_2]_{m_2,p,q}^Q) = \frac{[n + m_2 - 1]_{m_2,p,q}^Q \beta_2^{m_2}}{(p^n \ominus \beta_2 q^n)^{m_2}_{p,q}},$$

$$E([V_1]_{m_1,p,q}^Q | V_2 = v_2) = \frac{[n + v_2 + m_1 - 1]_{m_1,p,q}^Q \beta_1^{m_1}}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})^{m_1}_{p,q}},$$

$$E\left(\frac{[V_1]_{m_1,p,q}^Q}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})^{m_1}_{p,q}}\right) = \frac{[n + m_1 - 1]_{m_1,p,q}^Q \beta_1^{m_1}}{(p^n \ominus \beta_2 q^n)^{m_1}_{p,q}},$$

and

$$E\left(\frac{[V_1]_{m_1,p,q}^Q [V_2]_{m_2,p,q}^Q}{(p^{n+v_2} \ominus \beta_1 q^{n+v_2})^{m_2}_{p,q}}\right) = \frac{[n + m_1 + m_2 - 1]_{m_1+m_2,p,q}^Q \beta_1^{m_1} \beta_2^{m_2}}{(p^n \ominus \beta_2 q^n)^{m_1+m_2}_{p,q}}.$$

Furthermore, the covariance of the functions $\hat{V} := (p^{n+v_2} \ominus \beta_2 q^{n+v_2}) [V_1]_{m_2,p,q}^Q$ and $\nabla := [V_2]_{m_1,p,q}^Q$ is given by:

$$\text{Cov}(\hat{V}, \nabla) = \frac{[n]_{p,q} \beta_1 \beta_2}{(p^n - \beta_2 q^n)} \left( \frac{[n + 1]_{p,q}}{(p^{n+1} - \beta_2 q^{n+1})} - \frac{[n]_{p,q}}{(p^n - \beta_2 q^n)} \right).$$

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