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1 Introduction

In the present paper, we solve the problem of reducing to the simplest and convenient for our purposes, “canonical” form for an arbitrary pair of compatible nonlocal Poisson brackets of hydrodynamic type generated by metrics of constant Riemannian curvature (compatible Mokhov–Ferapontov brackets [1]) in order to get an effective construction of the integrable hierarchies related to all these compatible Poisson brackets. As was shown in [2], [3] (see also [4]), compatible Mokhov–Ferapontov brackets are described by a consistent nonlinear system of equations integrable by the method of inverse scattering problem (the case of flat metrics see in [5]–[7]). But the problem of an effective construction of the corresponding integrable hierarchies in this case, what is the main purpose of the present paper, requires a different approach to the description of these compatible brackets. In this paper, for an arbitrary solution of the integrable system of equations describing the compatible brackets under consideration, that is, for an arbitrary pair of these compatible brackets, integrable bi-Hamiltonian systems of hydrodynamic type possessing this pair of compatible nonlocal Poisson brackets of hydrodynamic type are constructed in an explicit form. For the case of the Dubrovin–Novikov brackets [8] (the local Poisson brackets of hydrodynamic type), this problem was considered and completely solved in the present author’s works [9], [10].

In [1] the nonlocal Poisson brackets of hydrodynamic type which have the following form (the Mokhov–Ferapontov brackets):

\[ \{ I, J \} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k(u(x)) u^k_x + Ku^i_x \left( \frac{d}{dx} \right)^{-1} u^j_x \right) \frac{\delta J}{\delta u^j(x)} dx, \]  

(1.1)

where \( I[u] \) and \( J[u] \) are arbitrary functionals on the space of functions (fields) \( u^i(x) \), \( 1 \leq i \leq N \), of single independent variable \( x \), \( u = (u^1, ..., u^N) \) are local coordinates on a certain given smooth \( N \)-dimensional manifold \( M \), the coefficients \( g^{ij}(u) \) and \( b^{ij}_k(u) \) of the bracket (1.1) are smooth functions of local coordinates, \( K \) is an arbitrary constant, were introduced and studied.

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The form of bracket (1.1) is invariant with respect to local changes of coordinates. A bracket of the form (1.1) is called \textit{nondegenerate} if \( \det(g^{ij}(u)) \neq 0 \). If \( \det(g^{ij}(u)) \neq 0 \), then bracket (1.1) is a Poisson bracket if and only if \( g^{ij}(u) \) is an arbitrary pseudo-Riemannian contravariant metric of constant Riemannian curvature \( K \), \( b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u) \), where \( \Gamma^j_{sk}(u) \) is the Riemannian connection generated by the metric \( g^{ij}(u) \) (the Levi–Civita connection) \( \text{(note that the coefficients } g^{ij}(u) \text{ and } b^{ij}_k(u) \text{ of bracket (1.1) are transformed as corresponding differential-geometric objects under local changes of coordinates: a contravariant metric } g^{ij}(u) \text{ and a contravariant connection } b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u) \text{ respectively, } K \text{ is an invariant). For } K = 0 \text{ we have the local Poisson brackets of hydrodynamic type (the Dubrovin–Novikov brackets \( \text{(3)} \)).}

Recall that Poisson brackets are called \textit{compatible} if their arbitrary linear combination is also a Poisson bracket (Magri, \( \text{(1)} \)).

\section{Compatible nonlocal Poisson brackets of hydrodynamic type}

\textbf{Lemma 2.1} In the classification problem for an arbitrary pair of compatible nonlocal Poisson brackets of the form \( \{I, J\}_0 \), one can always consider one of the Poisson brackets as local without loss of generality.

Actually, if two compatible nonlocal Poisson brackets of the form \( \{I, J\}_0 \) (with a corresponding constant \( K_0 \) in the nonlocal term) and \( \{I, J\}_1 \) (with a constant \( K_1 \)) are linear independent, then in the pencil of these Poisson brackets, that is, among the Poisson brackets

\[
\{I, J\}_0 + \lambda \{I, J\}_1 \text{ for arbitrary constants } \lambda,
\]

there is necessarily a nonzero local Poisson bracket (for this bracket \( \lambda K_0 + \lambda_1 K_1 = 0 \)), which can be taken as one of the generators for all the considered pencil of compatible Poisson brackets.

Consider the problem of compatibility for a pair of nonlocal and local Poisson brackets of hydrodynamic type

\[
\{I, J\}_1 = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_{k}(u(x)) u^k_x + K_1 u^i_x \right) \frac{\delta J}{\delta u^j(x)} dx
\]

and

\[
\{I, J\}_2 = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_{k}(u(x)) u^k_x \right) \frac{\delta J}{\delta u^j(x)} dx,
\]

that is, the condition that for any constant \( \lambda \) the bracket

\[
\{I, J\} = \{I, J\}_1 + \lambda \{I, J\}_2
\]

is a Poisson bracket (a pencil of Poisson brackets).

We assume further that the local bracket \( \{I, J\}_2 \) is nondegenerate, that is, \( \det g^{ij}_2(u) \neq 0 \), but we do not impose any additional conditions on the bracket \( \{I, J\}_1 \), that is, generally
speaking, it may be degenerate. The bracket (2.3) can be degenerate, therefore we need general relations for coefficients of an arbitrary bracket of the form (1.1) which are equivalent to the condition that the bracket (1.1) is a Poisson bracket. These general relations (without the assumption of nondegeneracy) were obtained in the present author’s work [12] (see also [13]):

$$g_{ij}(u) = g_{ji}(u),$$  \(2.4\)

$$\frac{\partial g_{ij}}{\partial u^k} = b_{k}^{ij}(u) + b_{k}^{ji}(u),$$  \(2.5\)

$$g^{is}(u)b_{s}^{jr}(u) = g^{js}(u)b_{s}^{ir}(u),$$  \(2.6\)

$$g^{is}(u)\left(\frac{\partial b_{s}^{jr}}{\partial u^k} - \frac{\partial b_{s}^{jr}}{\partial u^s}\right) + b_{s}^{ij}(u)b_{s}^{kr}(u) - b_{s}^{ir}(u)b_{s}^{kj}(u) = K(g^{ir}(u)\delta_{k}^{j} - g^{ij}(u)\delta_{k}^{r}),$$  \(2.7\)

$$\sum_{(i,j,r)} [b_{p}^{si}(u)\left(\frac{\partial b_{s}^{jr}}{\partial u^k} - \frac{\partial b_{s}^{jr}}{\partial u^p}\right) + b_{s}^{ij}(u)b_{s}^{kr}(u) - b_{s}^{ir}(u)b_{s}^{kj}(u)]\delta_{r}^{p} + K(b_{p}^{ij}(u) - b_{p}^{ji}(u))\delta_{r}^{p} = 0,$$  \(2.8\)

where \(\sum_{(i,j,r)}\) means summation over all cyclic permutations of the indices \(i, j, r\).

3 Canonical form for compatible pairs of brackets

According to the Dubrovin–Novikov theorem [8], for any nondegenerate local Poisson bracket of hydrodynamic type \(\{I, J\}_2\), there always exist local coordinates \(u^1, ..., u^N\) (flat coordinates of the metric \(g_{ij}^2(u)\)) in which this bracket is constant, that is, \(g_{ij}^2(u) = \eta_{ij} = \text{const}\), \(b_{2,i}^{jk}(u) = \Gamma_{2,i}^{jk}(u) = 0\). Thus we can choose flat coordinates of the metric \(g_{ij}^2(u)\) and further on consider that the Poisson bracket \(\{I, J\}_2\) is constant and has the form

$$\{I, J\}_2 = \int \delta I \frac{\delta J}{\delta u^i(x)} \eta^{ij} \frac{d}{dx} \delta u^j(x) dx, \quad (3.1)$$

where \(\eta^{ij} = \eta^{ji} = \text{const}\), \(\delta I = 0, \delta J = \delta^{ij}\). In the sequel, in the considered flat coordinates, we shall also use the covariant metric \(\eta_{ij}\) which is inverse to the contravariant metric \(\eta^{ij}\).

**Theorem 3.1** An arbitrary nonlocal Poisson bracket \(\{I, J\}_1\) of the form \((2.4)\) (may be degenerate) is compatible with the constant Poisson bracket \((3.1)\) if and only if it has the...
\[ \{ I, J \}_1 = \int \frac{\delta I}{\delta u^i(x)} \left( \eta^{is} \frac{\partial H^j}{\partial u^s} + \eta^{js} \frac{\partial H^i}{\partial u^s} - K_1 u^i u^j \right) \frac{d}{dx} + \right. \\
\left. \left[ \eta^{is} \frac{\partial^2 H^j}{\partial u^s \partial u^k} - K_1 \delta^i_k u^j \right] u_x^k + K_1 u_x^i \left( \frac{d}{dx} \right)^{-1} u_x^j \right) \frac{\delta J}{\delta u^j(x)} dx, \] (3.2)

where \( H^i(u), 1 \leq i \leq N, \) are smooth functions defined in a certain domain of local coordinates.

In the flat case of compatible Dubrovin–Novikov brackets (\( K_1 = 0 \)), the corresponding, necessary for our purposes, statement was formulated by the present author in [14], [15] (see also the conditions on flat pencils of metrics in [16]).

**Proof.** It follows from relations (2.4)–(2.8) that, in the considered local coordinates, the conditions of compatibility for the Poisson brackets \( \{ I, J \}_1 \) and \( \{ I, J \}_2 \) or, in other words, the conditions that the bracket

\[ \{ I, J \}_1 = \int \frac{\delta I}{\delta u^i(x)} \left( (q^{ij}(u(x)) + \lambda \eta^{ij}) \frac{d}{dx} + b_{1,k}^{ij}(u(x)) u_x^k + K_1 u_x^i \left( \frac{d}{dx} \right)^{-1} u_x^j \right) \frac{\delta J}{\delta u^j(x)} dx \] (3.3)

is a Poisson bracket for all values of the parameter \( \lambda \) have the form

\[ \eta^{is} b_{1,s}^{jr}(u) = \eta^{js} b_{1,s}^{jr}(u), \] (3.4)

\[ \frac{\partial b_{1,s}^{jr}}{\partial u^k} - \frac{\partial b_{1,k}^{jr}}{\partial u^s} = K_1 (\delta^r_s \delta^j_k - \delta^r_k \delta^j_s). \] (3.5)

Let us define the functions \( A_{k}^{ij}(u) \) by the relations

\[ A_{k}^{ij}(u) = b_{1,k}^{ij}(u) - K_1 \delta^j_k u^i. \] (3.6)

It follows from formula (3.3) that

\[ \frac{\partial A_{k}^{jr}}{\partial u^k} - \frac{\partial A_{k}^{jr}}{\partial u^s} = 0, \]

that is, by the Poincaré lemma, there locally exist functions \( P^{ij}(u) \) such that

\[ A_{k}^{ij}(u) = \frac{\partial P^{ij}}{\partial u^k}, \]
and we derive from (3.6) a necessary expression for the coefficient $b_{i,k}^j(u)$:

$$b_{i,k}^j(u) = \frac{\partial P^i}{\partial u^k} + K_1 \delta_k^i u^j.$$  

Let us find the corresponding expression for the metric $g_{ij}^1(u)$. From relation (2.5), for the Poisson bracket $\{I, J\}_1$, we have

$$\frac{\partial g_{ij}^1}{\partial u^k} = \frac{\partial P^i}{\partial u^k} + \frac{\partial P^j}{\partial u^k} + K_1 \delta_j^i + K_1 \delta_k^j u^i$$

and, consequently, taking into account relation (2.4), we get

$$g_{ij}^1(u) = P_{ij}(u) + K_1 u^i u^j + c_{ij}, \quad c_{ij} = \text{const}, \quad c^{ij} = c_{ji}.$$  

Thus it is proved that the coefficients of the Poisson bracket $\{I, J\}_1$ have the so-called Liouville form (see about the important Liouville property more in detail below):

$$g_{ij}^1(u) = R_{ij}(u) + R_{ji}(u) + K_1 u^i u^j,$$

where

$$R_{ij}(u) = P_{ij}(u) + \frac{1}{2} c_{ij}.$$  

Moreover, from relation (3.4) we get additionally

$$\eta_{ps} b_{i,j}^{tr}(u) = \eta_i \delta_{1,p}^{tr}(u),$$

that is,

$$\frac{\partial (\eta_{ps} R_{sr}(u))}{\partial u^i} + K_1 \eta_{ps} u^s \delta_s^r = \frac{\partial (\eta_{is} R_{ir}(u))}{\partial u^p} + K_1 \eta_{is} u^s \delta_s^r.$$  

The last formula is equivalent to the relation

$$\frac{\partial (\eta_{ps} R_{sr}(u) + K_1 \eta_{ps} u^s u^r)}{\partial u^i} = \frac{\partial (\eta_{is} R_{ir}(u) + K_1 \eta_{is} u^s u^r)}{\partial u^p}.$$  

Consequently, by the Poincaré lemma, there locally exist functions $H^r(u)$ such that

$$\eta_{ps} R_{sr}(u) + K_1 \eta_{ps} u^s u^r = \frac{\partial H^r}{\partial u^p}.$$  

Thus, it is proved that

$$R_{sr}(u) = \eta^{sp} \frac{\partial H^r}{\partial u^p} - K_1 u^s u^r,$$  

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and the coefficients of the Poisson bracket \( \{ I, J \} \) have the form:

\[
 g^{ij}(u) = \eta^{is} \frac{\partial H^j}{\partial u^s} + \eta^{js} \frac{\partial H^i}{\partial u^s} - K_1 u^i u^j, \tag{3.7}
\]

\[
 b^{ij}_{1,k}(u) = \eta^{is} \frac{\partial^2 H^j}{\partial u^s \partial u^k} - K_1 \delta_k^i u^j. \tag{3.8}
\]

Since in this case, as is easy to check, all the relations of compatibility (3.4) and (3.5) are satisfied, theorem 3.1 is proved.

4 Integrable equations for canonical compatible pairs of brackets

**Theorem 4.1** An arbitrary nonlocal bracket of the form (3.2) (may be degenerate) is a Poisson bracket if and only if the following equations are satisfied:

\[
 \frac{\partial^2 H^i}{\partial u^k \partial u^s} \eta^{sp} \frac{\partial^2 H^j}{\partial u^p \partial u^r} = \frac{\partial^2 H^j}{\partial u^k \partial u^s} \eta^{sp} \frac{\partial^2 H^i}{\partial u^p \partial u^r}, \tag{4.1}
\]

\[
 \left( \eta^{ir} \frac{\partial H^s}{\partial u^r} + \eta^{sr} \frac{\partial H^i}{\partial u^r} - K_1 u^i u^s \right) \eta^{jp} \frac{\partial^2 H^k}{\partial u^p \partial u^s} =
\]

\[
 \left( \eta^{ir} \frac{\partial H^s}{\partial u^r} + \eta^{sr} \frac{\partial H^j}{\partial u^r} - K_1 u^j u^s \right) \eta^{ip} \frac{\partial^2 H^k}{\partial u^p \partial u^s}. \tag{4.2}
\]

In the flat case \( (K_1 = 0) \), the corresponding theorem was stated by the present author in [15], where was also stated the conjecture on the integrability of the corresponding system (4.1), (4.2) (see also the conditions on the flat pencils of metrics in [16]).

For every nonlocal bracket of the form (3.2) relations (2.4), (2.5), (2.8) are satisfied identically. Relation (2.7) takes the form

\[
 b^{ij}_{1,s}(u) b^{sr}_{1,k}(u) = b^{ij}_{1,s}(u) b^{sr}_{1,k}(u) \tag{4.3}
\]

that gives the equations

\[
 \left( \eta^{ip} \frac{\partial^2 H^j}{\partial u^p \partial u^s} - K_1 \delta^j_k u^s \right) \left( \eta^{sl} \frac{\partial^2 H^r}{\partial u^l \partial u^k} - K_1 \delta^r_k u^s \right) =
\]

\[
 \left( \eta^{ip} \frac{\partial^2 H^r}{\partial u^p \partial u^s} - K_1 \delta^r_k u^s \right) \left( \eta^{sl} \frac{\partial^2 H^j}{\partial u^l \partial u^k} - K_1 \delta^j_k u^s \right). \tag{4.4}
\]

equivalent to the equations (4.1).
Relation (2.6) gives the equations

\[
\left( \eta_{ij} \frac{\partial H^s}{\partial u^p} + \eta_{sp} \frac{\partial H^i}{\partial u^p} - K_1 u^i u^s \right) \left( \eta_{jl} \frac{\partial^2 H^r}{\partial u^l \partial u^s} - K_1 \delta^r_j u^r \right) = \left( \eta_{jp} \frac{\partial H^s}{\partial u^p} + \eta_{sp} \frac{\partial H^j}{\partial u^p} - K_1 u^j u^s \right) \left( \eta_{il} \frac{\partial^2 H^r}{\partial u^l \partial u^s} - K_1 \delta^r_i u^r \right)
\]

(4.5)
equivalent to the equations (4.2).

**Corollary 4.1** Any linear function

\[ H^i = c^i_k u^k + c^i = \text{const} \]

is a trivial solution of the nonlinear system of equations (4.1), (4.2). Thus the bracket

\[
\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left( \eta^{is} c^j_s + \eta^{js} c^i_s - K_1 u^i u^j \right) \frac{d u^j}{d x} - K_1 \delta^i_k u^j u^k + K_1 \delta^i_k u^j \left( \frac{d}{d x} \right)^{-1} \frac{\delta J}{\delta u^j(x)} d x
\]

(4.6)
is a Poisson bracket for any constants \( c^i_k \). In particular, for any symmetric constant matrix \( (\mu^{ij}) \), \( \mu^{ij} = \mu^{ji} \), \( \mu^{ij} = \text{const} \), \( \mu^{ij} - K u^i u^j \) is always a metric of constant Riemannian curvature \( K \) if it is nondegenerate (it is obvious that under the condition of nondegeneracy for the matrix \( (\mu^{ij}) \) this metric is nondegenerate at least for small \( u^i \)), and in addition the Levi-Civita connection generated by this metric is defined by the relation

\[ \Gamma^i_{jk}(u) = \mu^{is} - K u^i u^s \Gamma^i_{sk}(u) = K \delta^i_k u^j. \]

Consider the contravariant metric of the form

\[
g^{ij}(u) = a^i \delta^{ij} - K u^i u^j,
\]

(4.7)
where \( a^i \), \( 1 \leq i \leq N \), are arbitrary nonzero constants, so that the metric \( g^{ij}(u) \) is nondegenerate. It is easy to prove that

\[
\det(g^{ij}(u)) = a^1 \cdot \cdot \cdot a^N \left( 1 - K \sum_{s=1}^{N} (u^s)^2 \right).
\]

(4.8)
The covariant metric \( g_{ij}(u) \), which is inverse to the contravariant metric \( g^{ij}(u) \), \( g^{is}(u) g_{sj}(u) = \delta^i_j \), has the form

\[
g_{ij}(u) = \frac{1}{a^i} \delta_{ij} + \frac{K u^i u^j}{a^i a^j \left( 1 - K \sum_{s=1}^{N} (u^s)^2 \right)}
\]

(4.9)
and is a metric of constant Riemannian curvature \( K \) for all values of the nonzero constants \( a^i \). The considered local coordinates are geodesic at the point with the coordinates \( (0, \ldots, 0) \) \((u^i = 0)\), but they are not the normal Birkhoff coordinates and even not Riemannian coordinates.
Note that the metric $g^{ij}(u)$ is nondegenerate if and only if not more than one of $N + 1$ constants: $a^i, 1 \leq i \leq N$, and $K$, equals to zero. All the formulae are easily adapted for the case when one of these constants equals to zero. If $a^m = 0$, then
\[
\det(g^{ij}(u)) = -K \left( \prod_{s \neq m} a^s \right) (u^m)^2.
\] (4.10)
In this case, the components of the covariant metric $g_{ij}(u)$ of constant Riemannian curvature $K$ have the form
\[
g_{mm}(u) = -\frac{1}{K(u^m)^2} \left( 1 - K \sum_{s \neq m} \frac{(a^s)^2}{a^s} \right),
\] (4.11)
\[
g_{im}(u) = g_{mi}(u) = -\frac{1}{a^i u^m}, \quad i \neq m,
\] (4.12)
\[
g_{ii}(u) = 1, \quad i \neq m,
\] (4.13)
\[
g_{ij}(u) = 0, \quad i \neq j, \quad i \neq m, \quad j \neq m.
\] (4.14)
All these models of the spaces of constant curvature play an important role in the Hamiltonian theory of systems of hydrodynamic type. The Mokhov–Ferapontov brackets generated by the metrics of constant Riemannian curvature (4.7) for $a^i = \varepsilon^i = \pm 1$ are called in [17] canonical. The canonical brackets arose naturally also in applications in [18].

**Theorem 4.2** ([2], [3]) The system of nonlinear equations (4.1), (4.2) is integrated by the method of inverse scattering problem.

Note that in [2], [3] the system of nonlinear equations describing the compatible nonlocal Poisson brackets of the form (1.1) in other local coordinates, more convenient for the integration (in these coordinates, the metrics of both compatible brackets are diagonal), was derived and integrated.

## 5 Liouville and special Liouville coordinates

Local coordinates $u = (u^1, ..., u^N)$ are called Liouville for an arbitrary Poisson bracket $\{ I, J \}$ if the functions (the fields) $u^i(x)$ are densities of integrals in involution with respect to this bracket, that is,
\[
\{ U^i, U^j \} = 0, \quad 1 \leq i, j \leq N,
\] (5.1)
where $U^i = \int u^i(x) dx$, $1 \leq i \leq N$. In this case the Poisson bracket is also called Liouville in these coordinates. Liouville coordinates naturally arise and play an essential role in the Dubrovin–Novikov procedure of averaging of Hamiltonian equations [3]. The physical
coordinates derived by averaging of the densities of the participating in the Dubrovin–Novikov procedure \( N \) involutive local integrals of an initial Hamiltonian system are always Liouville for the corresponding averaged bracket. This property was a motivation for the definition of Liouville coordinates for local Poisson brackets of hydrodynamic type in [8]. For general nonlocal Poisson brackets of hydrodynamic type (the Ferapontov brackets [19, 20]), Liouville coordinates were introduced in [17].

A nonlocal Poisson bracket of hydrodynamic type \((1.1)\) is Liouville in the local coordinates \( u = (u^1, ..., u^N) \) if and only if there exists a matrix function \( \Phi^{ij}(u) \) such that

\[
b^{ij}_k (u) = \frac{\partial \Phi^{ij}}{\partial u^k} - K \delta^i_k u^j.
\]

(5.2)

In this case, by virtue of relations (2.4), (2.5), the metric \( g^{ij}(u) \) must have the form

\[
g^{ij}(u) = \Phi^{ij}(u) + \Phi^{ji}(u) - Ku^i u^j.
\]

(5.3)

(here the function \( \Phi^{ij}(u) \) can be corrected by a constant matrix function \( c^{ij} = \text{const} \)). The matrix function \( \Phi^{ij}(u) \) is called a Liouville function.

Thus a nonlocal Poisson bracket \((1.1)\) is Liouville if it has the form

\[
\{I, J\} = \int \frac{\delta F}{\delta u^i(x)} \left( (\Phi^{ij}(u) + \Phi^{ji}(u) - Ku^i u^j) \frac{d}{dx} + \left( \frac{\partial \Phi^{ij}}{\partial u^k} - K \delta^i_k u^j \right) u^k_x + Ku^j_x \left( \frac{d}{dx} \right)^{-1} u^i_x \right) \frac{\delta J}{\delta u^j(x)} dx.
\]

(5.4)

From theorem 3.1 it follows

**Theorem 5.1** Flat coordinates of an arbitrary nondegenerate local Poisson bracket of hydrodynamic type \( \{I, J\}_2 \) are always Liouville for any nonlocal Poisson bracket \( \{I, J\}_1 \) of the form \((1.1)\) compatible with \( \{I, J\}_2 \). Moreover, in addition the corresponding Liouville function \( \Phi^{ij}(u) \) always has the special form

\[
\Phi^{ij}(u) = \eta_{is} \frac{\partial H^j}{\partial u^s}.
\]

(5.5)

Local coordinates \( u = (u^1, ..., u^N) \) are called special Liouville coordinates [15], [21] for an arbitrary Poisson bracket \( \{I, J\} \) if there exists a nonzero constant symmetric matrix \( (\eta_{ij}) \) such that the functions (the fields) \( u^i(x), 1 \leq i \leq N, \) and \( \eta_{ij} u^i(x) u^j(x) \) are densities of integrals in involution with respect to this bracket, that is,

\[
\{U^i, U^j\} = 0, \quad 1 \leq i, j \leq N + 1,
\]

(5.6)

where \( U^i = \int u^i(x) dx, 1 \leq i \leq N, U^{N+1} = \int \eta_{ij} u^i(x) u^j(x) dx \). In this case the Poisson bracket is also called special Liouville in these coordinates. The special Liouville coordinates were introduced in [15], [21]. The most important case is the case of nondegenerate matrix \( \eta_{ij} \).
Theorem 5.2 An arbitrary nonlocal Poisson bracket of the form \([1.4]\) is special Liouville in the local coordinates \(u = (u^1, \ldots, u^N)\) if and only if it is Liouville with a special Liouville function \(\Phi^{ij}(u)\) such that
\[
\eta_{ks} \Phi^{ij}(u) = \frac{\partial H^j}{\partial u^k}. \tag{5.7}
\]

In this case, for a nondegenerate matrix \((\eta_{ij})\), we get exactly our bracket \([3.2]\) from the canonical compatible pair.

Thus our problem on compatible nonlocal Poisson brackets of hydrodynamic type is equivalent to the problem of classification of the special Liouville coordinates for nonlocal Poisson brackets of hydrodynamic type.

Theorem 5.3 An arbitrary nonlocal Poisson bracket of hydrodynamic type of the form \([1.4]\) is compatible with the constant Poisson bracket \([3.1]\) if and only if the functions \(u^i(x), 1 \leq i \leq N,\) and \(\eta_{ij} u^i(x) u^j(x), \eta^{ij} \delta^i_j,\) are densities of integrals in involution with respect to the Poisson bracket \([1.4]\).

Note that \(u^i(x), 1 \leq i \leq N,\) are the densities of the annihilators of the bracket \([3.1],\) and \(\frac{1}{2} \eta_{ij} u^i(x) u^j(x)\) is the density of the momentum of the bracket \([3.1].\)

Theorem 5.4 An arbitrary nonlocal Poisson bracket of hydrodynamic type of the form \([1.4]\) is compatible with an arbitrary nondegenerate local Poisson bracket of hydrodynamic type \([2.2]\) if and only if \(N\) annihilators and the momentum of the bracket \([2.2]\) are integrals in involution with respect to the Poisson bracket \([1.4]\).

6 Integrable bi-Hamiltonian systems of hydrodynamic type

Consider an arbitrary pair of compatible nonlocal Hamiltonian operators of hydrodynamic type \(P^{ij}_1\) and \(P^{ij}_2\) generated by metrics of constant Riemannian curvature. As is shown above in lemma \([2.4]\), one of these operators, let us assume \(P^{ij}_2,\) can be considered as local without loss of generality. If the local Hamiltonian operator \(P^{ij}_2\) is nondegenerate, then it follows from theorem \([3.1]\) that, by local change of coordinates, the pair of compatible Hamiltonian operators \(P^{ij}_1\) and \(P^{ij}_2\) can be reduced to the following canonical form:
\[
P^{ij}_2[v(x)] = \eta^{ij} \frac{d}{dx}, \tag{6.1}
\]
\[
P^{ij}_1[v(x)] = \left( \eta^{is} \frac{\partial h^j}{\partial v^s} + \eta^{js} \frac{\partial h^i}{\partial v^s} - K v^i v^j \right) \frac{d}{dx} + \left( \eta^{is} \frac{\partial^2 h^j}{\partial v^s \partial v^k} - K \delta^i_k v^j \right) v_k^s + K v^i \left( \frac{d}{dx} \right)^{-1} v^j, \tag{6.2}
\]
where \((\eta^{ij})\) is an arbitrary nondegenerate constant symmetric matrix: \(\det(\eta^{ij}) \neq 0, \eta^{ij} = \text{const}, \eta^{ij} = \eta^{ji}; K\) is an arbitrary constant; \(h^i(v), 1 \leq i \leq N,\) are smooth functions defined...
in a certain domain of local coordinates and such that the operator (6.2) is Hamiltonian, that is, the functions \( h^i(v) \) satisfy to the integrable equations (4.1), (4.2) (see theorems 4.1 and 4.2 above).

**Remark 6.1** It is obvious that here we can always consider that \( \eta^{ij} = \varepsilon^i \delta^{ij}, \varepsilon^i = 1 \) for \( i \leq p, \varepsilon^i = -1 \) for \( i > p \), where \( p \) is the positive index of inertia of the metric, \( 0 \leq p \leq N \), and, in addition, it is necessary to classify the Hamiltonian operators (6.2) with respect to the action of the group of motions for the corresponding \( N \)-dimensional pseudo-Euclidean space \( R^N_p \), but for our purposes it is sufficient (and more convenient) to use the indicated above representation for canonical compatible pair (“conventionally canonical” representation).

Consider the recursion operator generated by canonical compatible Hamiltonian operators (6.1), (6.2):
\[
R_i^j[v(x)] = \left[ P_1[v(x)] (P_2[v(x)])^{-1} \right]^i_l = \left( \eta^{is} \frac{\partial h^j}{\partial v^s} + \eta^{js} \frac{\partial h^i}{\partial v^s} - K v^i v^j \right) \frac{d}{dx} + \left( \eta^{is} \frac{\partial^2 h^j}{\partial v^s \partial v^k} - K \delta^j_k v^i \right) v_x^k + K v^i x \left( \frac{d}{dx} \right)^{-1} v_x^j \eta_{jl} \left( \frac{d}{dx} \right)^{-1} \tag{6.3}
\]
(what about recursion operators generated by pairs of compatible Hamiltonian operators, see [22]–[26], [13]).

Let us apply the derived recursion operator (6.3) to the system of translations with respect to \( x \), that is, the system of hydrodynamic type
\[
v^i_t = v^i_x, \tag{6.4}
\]
which is, obviously, Hamiltonian with the Hamiltonian operator (6.1):
\[
v^i_t = v^i_x = P_2^{sj} \frac{\delta H}{\delta v^j(x)}, \quad H = \frac{1}{2} \int \eta_{jl} v^j(x) v^l(x) dx. \tag{6.5}
\]

Any system from the hierarchy
\[
v^i_{tn} = (R^n)^i_j v^j_x, \quad n \in Z, \tag{6.6}
\]
is a multi-Hamiltonian integrable system.

In particular, any system of the form
\[
v^i_t = R^j_i v^j_x, \tag{6.7}
\]
that is, the system of hydrodynamic type
\[
v^i_t = \left( \eta^{is} \frac{\partial h^j}{\partial v^s} + \eta^{js} \frac{\partial h^i}{\partial v^s} - K v^i v^j \right) \frac{d}{dx} + \left( \eta^{is} \frac{\partial^2 h^j}{\partial v^s \partial v^k} - K \delta^j_k v^i \right) v_x^k + \left( \eta^{is} \frac{\partial^2 h^j}{\partial v^s \partial v^k} - K \delta^j_k v^i \right) v_x^k + \ldots \]

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where \( h^i(v) \), \( 1 \leq i \leq N \), is an arbitrary solution of the integrable system (6.1), (6.2), is integrable.

This system of hydrodynamic type is bi-Hamiltonian with the pair of canonical compatible Hamiltonian operators (6.1), (6.2):

\[
\begin{align*}
Kv_x^i \left( \frac{d}{dx} \right)^{-1} v_x^j &\equiv \left( \eta^{is} \frac{\partial h^j}{\partial v^s} - K \frac{\partial^2 h^j}{\partial v^s \partial \eta_{vk} v^k} - K \eta_{sk} v^i v^s - \right. \\
K &\left. \frac{\delta^i_{jk} \eta_{al} v^l v^j}{2} \right) v_x^k \equiv \left( h^i(v) + \eta^{is} \frac{\partial h^j}{\partial v^s} \eta_{jl} v^j - K \frac{\eta_{sk} v^i v^s}{2} \right)_x,
\end{align*}
\]

(6.8)

The next system in the hierarchy (6.6) (for \( n = 2 \)) is the integrable system of hydrodynamic type

\[
\begin{align*}
v^i = \eta^{is} \frac{d}{dx} \left( \frac{\delta H_2}{\delta v^j(x)} \right), \quad H_2 = \int \left( \eta_{jk} h^k(v(x)) v^j(x) - K \frac{1}{8} \eta_{jk} \eta_{al} v^j v^k v^l v^j \right) dx.
\end{align*}
\]

(6.10)

The next system in the hierarchy (6.6) (for \( n = 2 \)) is the integrable system of hydrodynamic type

\[
\begin{align*}
v^i = \eta^{is} \frac{d}{dx} \left( \frac{\delta H_2}{\delta v^j(x)} \right), \quad H_2 = \int \left( \eta_{jk} h^k(v(x)) v^j(x) - K \frac{1}{8} \eta_{jk} \eta_{al} v^j v^k v^l v^j \right) dx.
\end{align*}
\]

(6.10)

Even the trivial, linear with respect to the fields \( v^i(x) \), solutions of the system (6.1), (6.2) generate nontrivial integrable systems of hydrodynamic type. All the results presented in this work are generalized directly to the important case of general nonlocal Poisson brackets of hydrodynamic type (the Ferapontov brackets [13], [20]), although the corresponding formulae become considerably more cumbersome and less effective. These results will be published in another paper.
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