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The $\mathcal{F} -$Topology on Space of Zero Dimensional rings

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Abstract Let $R$ be a subring of a ring $T$, and let $\mathcal{F}$ be a non-principal ultrafilter on the natural numbers $IN$. We consider properties and applications of a countably compact, Hausdorff topology called the “$\mathcal{F}$-topology” defined on space of all zero-dimensional subring of $T$ that contains a fixed subring $R$. We show that the $\mathcal{F}$-topology is strictly finer than the Zariski topology. We extend results regarding distinguished spectral topologies on the space of zero-dimensional subring.

Keywords Zero-dimensional subring; Ultrafilter; $\mathcal{F}$-Topology, Countably compact.

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1 Introduction

Let $R$ be a subring of a ring $T$. We denote by $\mathcal{Z}(R,T)$, the collection of all zero-dimensional subrings of $T$ contain $R$. In the case where $R$ is the prime subring of $T$, $\mathcal{Z}(R,T)$ will be denoted by $\mathcal{Z}(T)$. The first topological approach to space $\mathcal{Z}(R,T)$ is due to Zariski, endowed with what is now called the Zariski topology (see [12]).

The purpose of this paper is twofold: Defined the $\mathcal{F}$-topology on $\mathcal{Z}(R,T)$, and to study the connection between
Spec($R$) and $Z(R,T)$ by different topology.
In Section 2, we give some basic propriety of $Z(R,T)$, we also introduce the notion of the ultrafilter and note that S. Garcia-Ferreira and L. M. Ruza-Montilla recently used ultrafilters to define this topology on the set of all prime ideals of a commutative ring, and then prove that this $\mathcal{F}$-topology is a countably compact, Hausdorff topology (see\cite{6}). In Section 3, we define a $\mathcal{F}$-topology on the space $\Spec(R)$. After that, we study the connection between $Z(R,T)$ and $\Spec(R)$, and modernize our understanding of the collection of all zero-dimensional rings via the category of spectral spaces.

As the Theorem 8 demonstrates, the tools are connected by a continuous surjection $\gamma : Z(R,T)^{\tau} \to \Spec(R)^{\tau}$. 

2 Preliminaries

We begin by giving the notations and preliminary results.
Let $R$ be a subring of a ring $T$, and let $Z(R,T)$ the set of all zero-dimensional subrings of $T$ containing $R$, we know by \cite{4, Proposition 2.2} that the set $Z(R,T)$ can be empty.

The following theorem under conditions for which $Z(R,T)$ is a nonempty.

**Theorem 1** (\cite{11, Proposition 1} and \cite{9, Theorem 1.6}) Let $R$ be a subring of a ring $T$, the following conditions are equivalent:

1. $Z(R,T) \neq \emptyset$.
2. The power of the ideal $xT$ is idempotent for each $x$ in $R$.
3. For each finitely generated ideal $I$, the set $\{\Ann_{R}(I)\}_{j=1}^{\infty}$ stabilizes for some $m \in \mathbb{N}$.

Now, assume that $Z(R,T) \neq \emptyset$, is that $Z(R,T)$ closed under arbitrary intersection?
In the following theorem, R. Gilmer answers this question.

**Theorem 2** (\cite{9, Theorem 2.1}) Let $R$ be a subring of a ring $T$ such that $Z(R,T) \neq \emptyset$. Then $Z(T)$ is closed under arbitrary intersection.

**Remark 1** Suppose $R$ is a subring of the ring $T$. If $Z(R,T) \neq \emptyset$, then Theorem 2 shows that $Z(R,T) \neq \emptyset$ has a unique minimal element. We denote this element by $R^0$, and call it the minimal zero-dimensional extension of $R$ in $T$. Then for each $x$ in $R$, assume that $x^{m+x}T$ is idempotent, and let $s_x$ be the pointwise inverse of $x^{m+x}$ in $T$. By \cite{9, Theorem 2.5} we have that $R^0 = R[\{s_x : x \in R\}]$.

Now, we are interested in the topological structure on $S(R,T)$. Let $R$ be a subring of a ring $T$, the set $S(R,T)$ endowed with a topological structure defined by taking, as a basis for the open sets, the subsets:

$$B_S := \{F \in S(R,T) \mid S \subseteq F\}$$
For $S$ varying in $B_{fin}(T)$. This topology is called the Zariski's topology on $S(R, T)$. If $S := \{x_1, x_2, \ldots, x_n\}$ with $x_j \in T$ for each $j \in \{1, \ldots, n\}$, then

$$B_S := S(R[x_1, x_2, \ldots, x_n], T).$$

Therefore the collection of subsets $B := \{S(R[x], T) : x \in T\}$ is a base for the Zariski topology on $S(R, T)$. It is easily seen that $S(R, T)$ is a Kolmogorov topological space ($T_0$ space). Indeed, if $R_1 \neq R_2 \in S(R, T)$, we can assume, without loss of generality, that there is an element $x \in R_1 \setminus R_2$, then the open set $S(R[x], T)$ contains $R_1$ and does not contain $R_2$.

**Corollary 1** Let $R$ be a subring of a ring $T$. If $\mathcal{Z}(R, T) \neq \emptyset$, then $\mathcal{Z}(R, T)$ is a Kolmogorov topological space.

We will work in at least ZFC, that is, Zermelo-Frankel set theory with the axiom of choice. If $I$ is a set, we recall that $\mathcal{F}$ is a filter on $I$ if it is a subset of the power set of $I$ that satisfies the following conditions:

1. $\emptyset \notin \mathcal{F}$ and $I \in \mathcal{F}$;
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. If $A \in \mathcal{F}$ and $A \subset A' \subset I$, then $A' \in \mathcal{F}$.

A filter $\mathcal{F}$ on $I$ is called an ultrafilter if $\mathcal{F}$ is maximal with respect to being a filter, or equivalently, if whenever $A \subset I$, then either $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$. An ultrafilter $\mathcal{F}$ is called principal if there exists an element $i_0 \in I$ such that $\mathcal{F}$ consist of all subsets of $I$ that contain $i_0$. In this case, we denoted $\mathcal{F}$ by $\mathcal{F}_{\{i_0\}}$. Other ultrafilters are called non-principal. We denote the collection of all ultrafilters on a set $I$ by $\beta(I)$. Note that the notation $\beta(I)$ is the same for the Stone-Čech compactification of $I$, this comes naturally, as en can identify both, for more details see the recent book of Comfort and Negrepontis [1].

Now, let $R$ be a commutative ring, and let $\text{Spec}(R)$ denote the set of all prime ideals of $R$. On $\text{Spec}(R)$, we can consider the Zariski topology by taking as open sets the collection of all sets $D(a) := \{P \in \text{Spec}(R) : a \notin P\}$ for all $a \in R$, then the family $\{D_a : a \in R\}$ is a basis for the open sets of $\text{Spec}(R)^{zar}$. Zariski's topology has several attractive properties. For example is quasi-compact, Kolmogorov, but almost never compact. More precisely, $\text{Spec}(R)^{zar}$ is Hausdorff $\iff$ $\text{Spec}(R)^{zar}$ is compact $\iff$ $\dim(R) = 0$ (see [7, Theorem 3.6]).

Recently, S. Garcia-Ferreira and L.M. Ruza-Montilla in [6] have considered another topology on $\text{Spec}(R)$ by using the notion of an ultrafilter. Indeed, let $(P_n)_{n \in IN}$ be a sequence of $\text{Spec}(R)$, and let $\mathcal{F}$ be an ultrafilter on $IN$, set

$$\mathcal{F} - \lim_{n \in IN} P_n := \{a \in R : \{n \in IN : a \in P_n\} \in \mathcal{F}\}$$

It can be easily shown $\mathcal{F} - \lim_{n \in IN} P_n$ is a prime ideal. Indeed, let $ab \in \mathcal{F} - \lim_{n \in IN} P_n$, then $A = \{n \in IN : ab \in P_n\} \in \mathcal{F}$. Then $A = \{n \in IN : a \in P_n\} \cup \{n \in IN : b \in P_n\}$ and, since $\mathcal{F}$ is an ultrafilter, we have $a \in \mathcal{F} - \lim_{n \in IN} P_n$ or $b \in \mathcal{F} - \lim_{n \in IN} P_n$, then $\mathcal{F} - \lim_{n \in IN} P_n$ is a prime
ideal. This notion of $\mathcal{F}$–limit of collections of prime ideals has been used with central effect in the construction of the $\mathcal{F}$-topology on $\text{Spec}(R)$. If $(P_n)_{n \in \mathbb{N}}$ be a sequence of $C \subset \text{Spec}(R)$ and, if $\mathcal{F}$ be a principal ultrafilter on $\mathbb{N}$, we have that $\mathcal{F} - \lim_{n \in \mathbb{N}} P_n = P_k$, for some $P_k \in C$ [6, Section 2]. On the other hand, if $\mathcal{F}$ is nonprincipal, then it is not at all clear that the prime ideal $\mathcal{F} - \lim_{n \in \mathbb{N}} P_n$ should lie in $C$. That motivates the following definition.

**Definition 1** The set $C$ is $\mathcal{F}$–closed in $\text{Spec}(R)$ if it contains all of its $\mathcal{F}$–limit.

It is not hard to see that the $\mathcal{F}$–closed subsets of $\text{Spec}(R)$ define a topology on the set $\text{Spec}(R)$, called $\mathcal{F}$–topology on $\text{Spec}(R)$ (see [6, Theorem 4.2]). We denote by $\text{Spec}(R)^\mathcal{F}$ the set $\text{Spec}(R)$ endowed with the $\mathcal{F}$–topology. One of the main results of a recent article by Garcia-Ferreira and Ruza-Montilla is the following.

**Theorem 3** ([6, Theorem 4.4]) Let $R$ be a commutative ring. If $\mathcal{F}$ is a nonprincipal ultrafilter on $\mathbb{N}$, then $\tau_\mathcal{F}$ is countably compact.

3 The $\mathcal{F}$–topology on $\mathcal{Z}(R, T)$.

Let $R$ be a subring of a ring $T$. Taking as starting point the situation on the prime spectrum of a ring, and under the condition, the next goal is to study the $\mathcal{F}$–topology on space $\mathcal{Z}(R, T)$.

We begin by recalling a very useful definition.

**Definition 2** Let $T$ be a ring, and let $I$ an infinite set. If $R_i \in S(T)$, for each $i \in I$, and $\mathcal{F}$ is an ultrafilter on $I$, we define the $\mathcal{F}$–lim of a sequence of subring $R_i$’s as:

$$
\mathcal{F} - \lim_{i \in I} R_i := \{a \in T : \{i \in I : a \in R_i\} \in \mathcal{F}\}.
$$

We note that $\mathcal{F} - \lim_{i \in I} R_i$ is also a subring of $T$, and we have that:

$$
\mathcal{F} - \lim_{i \in I} R_i = \bigcup_{X \in \mathcal{F}} \left( \bigcap_{i \in X} R_i \right)
$$

Now, we state without proof some easy and well-known properties (For proof see [6]).

**Proposition 1** Let $A$ be a set, $I$ an infinite set, $\mathcal{F}$ an ultrafilter on $I$ and $\{S_i : i \in I\} \subseteq S(A)$, then:

1. $\mathcal{F}_{(k)} - \lim_{i \in I} S_i = S_k$, for each $k \in I$;
2. If $J \in \mathcal{F}$, then $\mathcal{F} - \lim_{i \in I} S_i = \mathcal{F} |_{J} - \lim_{i \in J} S_i$;
3. Let \( \Gamma \) be an infinite set, and let \( \sigma : \Delta \to \Gamma \) be a surjective function. For each \( j \in \Gamma \) put \( T_j = S_i \) if \( \sigma(i) = j \), then
\[
\mathcal{F} - \lim_{i \in \Delta} S_i = C - \lim_{j \in \mathcal{F}} T_j.
\]
where \( \sigma(\mathcal{F}) = \{ \sigma[F] : F \in \mathcal{F} \} = C \).

The interest in studying the topology on \( Z(R, T) \) comes from the following theorem.

**Theorem 4** Let \( R \) be a subring of a ring \( T \), and let \( Z(R, T) \neq \emptyset \). If \( R_i \in Z(R, T) \), for each \( i \in I \), and \( \mathcal{F} \) is an ultrafilter on \( I \), then \( \mathcal{F} - \lim_{i \in I} R_i \) is also a zero-dimensional.

**Proof** According to [12, Proposition 4.2], \( \mathcal{F} - \lim_{i \in I} R_i \) is a direct union of zero-dimensional ring. Then the conclusion follows immediately by [11, Introduction].

**Proposition 2** Let \( R \) be a subring of a ring \( T \). If \( X \subseteq S(R, T) \), \( I \) an infinite set, \( \mathcal{F} \) an ultrafilter on \( I \) and \( \{ R_i : i \in I \} \subseteq X \), let the map
\[
\pi : \beta(I) \to X, \ \mathcal{F} \mapsto \mathcal{F} - \lim_{i \in I} R_i.
\]
Then :

1. \( \{ R_i : i \in I \} \subseteq \text{Im}(\pi) \)
2. \( \pi \) is a surjection if and only if for each \( \mathcal{F} \in \beta(I) \), \( X \) stable by the \( \mathcal{F} - \lim \).

**Proof** 1. For each \( R_k \), taking the principal ultrafilter \( \mathcal{F}_{\{k}\} \), according to Proposition 1, we have \( \mathcal{F}_{\{k}\} - \lim_{i \in I} R_i = R_k \).
2. Noted that, if \( X = S(R, T) \), then for each \( \{ R_i : i \in I \} \) collection of \( X \), if \( \mathcal{F} \) is an ultrafilter on \( I \), we have \( \mathcal{F} - \lim_{i \in I} R_i \in X \). According to Proposition 2, \( \pi \) be a surjection (the surjection comes from the fact that \( X \) is stable by \( \mathcal{F} - \lim \)), so to generalize the result, we take \( X \) stable by \( \mathcal{F} - \lim \).

**Example 1** With the notation of the previous Proposition 2, and by Theorem 4, if we let \( X = Z(R, T) \). Then \( \pi \) is a surjection.

The previous Proposition leads naturally to the following crucial definition of this section.

**Definition 3** Let \( R \) be a subring of a ring \( T \), and let \( \mathcal{F} \) be an ultrafilter on \( \text{IN} \). We say that \( C \subseteq S(R, T) \) is \( \mathcal{F} - \text{closed} \) if for each sequence \( \{ R_n \}_{n \in \text{IN}} \) in \( C \), we have that \( \mathcal{F} - \lim_{n \in \text{IN}} R_n \in C \).

We shall define a new topology on the set of subring of a commutative ring.

**Theorem 5** Let \( R \) be a subring of a ring \( T \), and let \( \mathcal{F} \) be an ultrafilter on \( \text{IN} \). Then, the collection of all \( \mathcal{F} - \text{closed} \) subsets of \( S(R, T) \) is the family of closed sets for a topology on \( S(R, T) \) called the \( \mathcal{F} - \text{topology} \) and denoted by \( \tau_{\mathcal{F}} \).
Proof  

The empty set and $S(R, T)$ are clearly $\mathcal{F}$–closed subsets. Now, consider two $\mathcal{F}$–closed subsets $C_1, C_2$ of $S(R, T)$, set $C := C_1 \cup C_2$, and let $(R_n)_{n \in \mathbb{N}}$ be a sequence in $C$. By an argument similar to that used in [6, Theorem 4.2], we have $\mathcal{F} - \lim_{n \in \mathbb{N}} R_n \subseteq C$. It is evident that the intersection of $\mathcal{F}$–closed subsets is a $\mathcal{F}$–closed subset.

Remark 2  

Note that, by Proposition 2 we have $\mathcal{F} - \lim_{n \in \mathbb{N}} R_n \cap R = R_k$ for each sequence $(R_n)_{n \in \mathbb{N}}$ of $S(R, T)$, then we deduce that $\tau_\mathcal{F}$ is the discrete topology.

Lemma 1  

Let $R$ be a subring of a ring $T$, and let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$. If $\{R_n : n \in \mathbb{N}\} \subseteq S(R, T)$ is an infinite set, then $\mathcal{F} - \lim_{n \in \mathbb{N}} R_n$ is an accumulation point of $\{R_n : n \in \mathbb{N}\}$ inside the topology $\tau_\mathcal{F}$.

As we have introduced, we are interested in the spaces $\mathcal{Z}(R, T)$, $\mathcal{A}(R, T)$ and $\mathcal{D}(R, T)$, is like $\mathcal{Z}(R, T)$ the largest closed for the topology $\tau_\mathcal{F}$ which contains $\mathcal{A}(R, T)$ and $\mathcal{D}(R, T)$ (see Theorem 4), we restrict in the following to $\mathcal{Z}(R, T)$.

Naturally we start by comparing this topology with the usual topology on $\mathcal{Z}(R, T)$.

Theorem 6  

Let $R$ be a subring of a ring $T$, and let $\mathcal{Z}(R, T) \neq \emptyset$.

1. The $\mathcal{F}$–topology is finer than the Zariski topology on $\mathcal{Z}(R, T)$;
2. The $\mathcal{F}$–topology is hausdorff topology on $\mathcal{Z}(R, T)$;
3. The $\mathcal{F}$–topology is countably compact.

Proof  

1. Since $\mathcal{B} := \{\mathcal{Z}(R[x], T) : x \in T\}$ is a base for the open sets on $\mathcal{Z}(R, T)$ endowed with the zariski topology , it is enough to prove that $C := \mathcal{Z}(R, T) \setminus \mathcal{Z}(R[x], T)$ is $\mathcal{F}$–closed for every $x \in T$. Assume, by contradiction, that there exists an ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $\mathcal{F} - \lim_{n \in \mathbb{N}} R_n \notin C$ for each sequence $(R_n)_{n \in \mathbb{N}}$ in $C$. It follows that $x \in \mathcal{F} - \lim_{n \in \mathbb{N}} R_n$, then $\{n \in \mathbb{N} : x \in R_n\} \in \mathcal{F}$, by the definition of $C$, and the fact that $\emptyset \notin \mathcal{F}$, we have a contradiction.

2. According to (1), the basic open sets of the Zariski topology on $\mathcal{Z}(R, T)$ are both open and closed in the $\mathcal{F}$–topology. Then the $\mathcal{F}$–topology is finer than some topology defined as the coarsest topology for which the set $\mathcal{Z}(R[x], T)$ is both open and closed, for every $x \in T$. On the other hand, this topology is a Hausdorff. Indeed, let $V_1$ and $V_2$ be two distinct elements of $\mathcal{Z}(R, T)$, and, without loss of generality, we can take an element $y \in V_1 \setminus V_2$. By assumption, the sets $\mathcal{Z}(R, T) \setminus \mathcal{Z}(R[y], T)$ and $\mathcal{Z}(R[y], T)$ are disjoint open neighborhoods of $V_1$ and $V_2$. Hence the confirmation comes from the fact that the $\mathcal{F}$–topology is finer than this topology.

3. In general, a Hausdorff space $X$ is called countably compact if every infinite subset of $X$ has an accumulation point. Then by (2) and Lemma 1, $\mathcal{F}$–topology is countably compact.

Remark 3  

The space $\mathcal{Z}(R, T)^{\mathcal{F}}$ is countably compact but not compact in general.
Theorem 7 Let $R$ be a subring of a ring $T$ such that $\mathcal{Z}(R, T) \neq \emptyset$, and let $\mathcal{F}_r$ the Frechet ultrafilter on $IN$. Then, the following conditions are equivalent:

1. $\mathcal{Z}(R, T)$ is countably closed;
2. $\mathcal{Z}(R, T)$ is spectral space;
3. $\mathcal{F}_r - \lim_{n \in IN} R_n$ exist, for any $\{R_n : n \in IN\} \subseteq \mathcal{Z}(R, T)$, such that $\{R_n : n \in IN\} \models \emptyset_0$.

Proof 1) $\Rightarrow$ 2). According to [2, Theorem 3.10.3], every countably compact, countable Hausdorff space is compact, then $\mathcal{Z}(R, T)^{\tau_{zar}}$ is compact. Inspired by the idea given in as in [6], by [12, Lemma 4.4] the $\mathcal{F}$--topology and ultrafilter topology are the same, then by [3, Corallary 3.3] $\mathcal{Z}(R, T)$ is a spectral space.

2) $\Rightarrow$ 3). Let $\mathcal{Z}(R, T) \cong \text{Spec}(S)$, for some ring $S$ and let $\varphi : \mathcal{Z}(R, T) \to \text{Spec}(S)$. If $\{P_n : n \in IN\} \subseteq \text{Spec}(S)$ such that $\{P_n : n \in IN\} \models \emptyset_0$, then according to [2, Theorem 3.10.3 (v)] $\mathcal{F} - \lim_{n \in IN} P_n$ exist for each nonprincipal ultrafilter on $IN$. On the other hand $\varphi^{-1}(\mathcal{F} - \lim_{n \in IN} P_n) = \mathcal{F} - \lim_{n \in IN} \varphi^{-1}(P_n)$. Thus, it suffices to choose $R_n := \varphi^{-1}(P_n)$, since $\mathcal{F} - \lim_{n \in IN} R_n$ exist for every nonprincipal ultrafilter $\mathcal{F}$ on $IN$. Then, $\mathcal{F}_r - \lim_{n \in IN} R_n$ exist by applying [5, Definition 1.1].

3) $\Rightarrow$ 1) Let $\mathcal{F}_r - \lim_{n \in IN} R_n$ exist, then $\mathcal{F} - \lim_{n \in IN} R_n$ for every nonprincipal ultrafilter $\mathcal{F}$ on $IN$, since $\mathcal{Z}(R, T)^{\tau_{zar}}$ is $\mathcal{F}$-compact for each $\mathcal{F}$, according to [13, Theorem 1.6] $\mathcal{Z}(R, T)^{\tau_{zar}}$ is a compact space. Then the conclusion follows immediately by [2, Theorem 3.10.3].

Proposition 3 Let $R$ be a subring of a ring $T$ such that $\mathcal{Z}(R, T) \neq \emptyset$. Then $\gamma : \mathcal{Z}(R, T)^{\tau_{zar}} \to \text{Spec}(R)^{\tau_{zar}}$ is a continuous map.

Proof According to [4, Remark], we can construct a map $\gamma : \mathcal{Z}(R, T) \to \text{Spec}(R)$ sending a zero dimensional ring $S \in \mathcal{Z}(R, T)$, with a prime ideal $Q$, to the prime ideal $Q \cap R$ of $R$. If $R$ is a zero dimensional, then $\gamma$ is a surjection map.

Next, if we consider $\mathcal{Z}(R, T)^{\tau_{zar}}$ and $\text{Spec}(R)^{\tau_{zar}}$ as topological spaces both endowed with the Zariski topology, we check that $\gamma$ is continuous, it is enough to show that $\gamma^{-1}(D_x)$ is open, where $D_x = \{P \in \text{Spec}(R) : x \in P\}$ is a basic Zariski-open subset of $\text{Spec}(R)$. According to [9, Theorem 2.5], for each $x \in R$ exist $s_x$ be the pointwise inverse of $x^{m(x)}$ in $T$ such that $R[s_x]$, $x \in R$ is minimal zero-dimensional extension of $R$ in $T$, then $\gamma^{-1}(D_x) = \mathcal{Z}(R, R[s_x])$, forme where $\gamma$ is a continuous map.

Corollary 2 The map $\gamma : \mathcal{Z}(R, T)^{\tau_{zar}} \to \text{Spec}(R)^{\tau_{zar}}$ is a homeomorphism if and only if $\gamma$ is injective.

The next goal is to study the map $\gamma$ when $\mathcal{Z}(R, T)$ and $\text{Spec}(T)$ are both equipped with the $\mathcal{F}$--topology.

Theorem 8 Let $T$ be a ring and $R$ a subring of $T$ such that $\mathcal{Z}(R, T) \neq \emptyset$. Then, the surjective map $\gamma : \mathcal{Z}(R, T)^{\tau_{zar}} \to \text{Spec}(R)^{\tau_{zar}}$ is continuous and closed.
Proof. According to Theorem 6, $\mathcal{Z}(R,T)$ is Hausdorff space, by straightforward topological arguments, it is enough to show that is continuous. Let $C$ be a $\mathcal{F}$–closed subset of $\text{Spec}(R)\tau$, and let $\{S_n : n \in \mathbb{N}\} \subseteq \gamma^{-1}(C)$, $\mathcal{F}$ ultrafilter on $\mathbb{N}$. Then, it suffices to show that $\mathcal{F} - \lim_{n \in \mathbb{N}} S_n \in \gamma^{-1}(C)$. According to Theorem 4, we have $\mathcal{F} - \lim_{n \in \mathbb{N}} S_n \in \mathcal{Z}(R,T)$, then $R \subseteq \mathcal{F} - \lim_{n \in \mathbb{N}} S_n$. On the other hand, for each $(P_n) \in C$ we can also consider the ideal:

$$\mathcal{F} - \lim_{n \in \mathbb{N}} P_n = \{a \in R \mid \{n \in \mathbb{N} : a \in P_n\} \in \mathcal{F}\}$$

which is a prime ideal of $R$. By [4], there exists a prime ideal $Q$ of $\mathcal{F} - \lim_{n \in \mathbb{N}} S_n$ such that, $Q \cap R = \mathcal{F} - \lim_{n \in \mathbb{N}} P_n$. Since, by [6], $C$ is a $\mathcal{F}$–closed subset, we have $\gamma(\mathcal{F} - \lim_{n \in \mathbb{N}} S_n) = \mathcal{F} - \lim_{n \in \mathbb{N}} P_n \in C$, and so $\mathcal{F} - \lim_{n \in \mathbb{N}} S_n \in \gamma^{-1}(C)$. Therefore, we deduce that $\gamma^{-1}(C)$ is closed $\mathcal{Z}(R,T)\tau$, hence the conclusion.

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