Surgery on Foliations

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Abstract

In this paper, we set up two surgery theories and two kinds of Whitehead torsion for foliations. First, we construct a bounded surgery theory and bounded Whitehead torsion for foliations, which correspond to the Connes’ foliation algebra in the K-theory of operator algebras, in the sense that there is an analogy between surgery theory and index theory, and a Novikov Conjecture for bounded surgery on foliations in analogy with the foliated Novikov conjecture of P. Baum and A. Connes in operator theory. This surgery theory classifies the leaves topologically. Secondly, we construct a bounded geometry surgery for foliations, which is a generalization of blocked surgery, and a bounded geometry Whitehead torsion. The classifications in this surgery theory include the specification of the Riemannian metrics of the leaves up to quasi-isometry. We state Borel conjectures for foliations, which solves a problem posed by S. Weinberger [20], and verify these in some cases of geometrical interest.

1 Introduction

In 1980 M. Pimsner and D. Voiculescu [15] proved that the K-theory of the $C^*$-algebra of the Kronecker foliation, the irrational rotation algebra, is isomorphic to the group $\mathbb{Z} \oplus \mathbb{Z}$ in both dimension 0 and 1. Pimsner and Voiculescu used their six-term exact sequence to prove this result. In this paper we develop a general surgery theory for foliations and we prove the analogous result for this theory. We have two surgery theories: one a bounded geometry foliated surgery theory, the classifications of which include the specification of the Riemannian metrics of the leaves up to quasi-isometry, and the other a boundedly controlled foliated surgery theory where only the topology of the leaves is classified. We prove first that $M \times F$, where $F$ is the Kronecker foliation and $M$ is a manifold of dimension at least 5, has structure set $H_*(T^2; S_*(M))$ where $T^2$ is the 2-torus, and $S_*(M)$ the structure set of $M$, using codimension 1 splitting and the first author’s results with S. Hurder [6]. We then introduce foliated Whitehead torsion, a foliated s-cobordism theorem and surgery groups, analogous to Connes’ transverse index theory [10], for foliations of arbitrary codimension, with the constraint that the codimension is bounded below by the dimension of the manifold so that the leaves are dimension $\geq 5$. We prove a surgery exact sequence and state Novikov and Borel conjectures for foliations. P. Baum
and A. Connes have stated a Novikov conjecture for foliations [7] using operator K-theory of the foliation algebra which is analogous to our surgery theoretic version of the Novikov Conjecture in the bounded case but is different for the surgery theoretic case case where leaves are considered as manifolds of bounded geometry. This surgery theory is a generalization of the blocked surgery of [16]. The two surgery theories are related by the fact that one gets the bounded surgery theory by forgetting the Cheeger’s finiteness condition (that there are a finite number of types in a ball of fixed radius $r$, for any $r$) along the leaves. The theory of Baum and Connes using K-theory of operator algebras corresponds to this bounded surgery theory. The Novikov Conjecture of Baum and Connes involves the fundamental groupoid of the foliation, whereas Novikov conjecture for boundedly controlled foliated surgery involves the holonomy groupoid and the locally finite $L$-homology of the leaves. These two conjectures agree when the holonomy groupoid is the same as the fundamental groupoid of the leaves of the foliation. We use bounded surgery theory to prove this conjecture in the case where $(M, F)$ is ultra-spherical. S. Hurder [13] has proven the foliated Novikov conjecture for a large class of foliations using his exotic index theory.

The bounded geometry surgery theory works is analogous to blocked surgery, and we use this blocked surgery theory to study the problem of when a manifold is a leaf of a foliation. We note that, in the case of the Kronecker foliation, although there are a continuously infinite number of quasi-isometry types of a leaf, recurrence of leaves makes the number of types of a foliation countable [6]. We see how this observation agrees with bounded geometry surgery of the Kronecker foliation.

2 Pimsner-Voiculescu for Structure Sets

Throughout the paper we will work in the PL category. Our constructions can be smoothed as well (See [2]). In addition we will assume that all foliations have leaves of dimension $\geq 5$.

**Definition 2.1.** A foliated map is a map

$$f : (M, F) \to (N, G)$$

so that if $x$ and $y$ are on the same leaf, $f(x)$ and $f(y)$ are on the same leaf, and $f$ is a bg map when restricted to each leaf. A foliated homotopy is a foliated map which is a homotopy $f_t$ for which for each $t$, $f_0(x)$ is on the same leaf as $f_t(x)$. A foliated homotopy equivalence is one for which $f \circ g$ is foliated homotopic to the identity and $g \circ f$ is foliated homotopic to the identity, where

$$g : (N, G) \to (M, F)$$

is a foliated map.

**Definition 2.2.** A foliated simple homotopy equivalence

$$f : (M, F) \to (N, G)$$
between two codimension one foliations is a foliated map which is a bg simple homotopy equivalence [2] along the leaves and along the transverse foliation. A foliated simple homotopy equivalence

\[ f : (M, \mathcal{F}) \to (N, \mathcal{G}) \]

between two foliations of codimension \( \geq 2 \) is a foliated map which is a bg simple homotopy equivalence [2] along the leaves and along the transversals of the foliation.

**Definition 2.3.** A foliated s-cobordism is a foliated cobordism \((W, \mathcal{H})\) between two foliations \((M, \mathcal{F})\) and \((N, \mathcal{G})\) so that \((W, \mathcal{H})\) is foliated simple homotopy equivalent to \((M, \mathcal{F})\) and to \((N, \mathcal{G})\). A foliated h-cobordism \((W, \mathcal{H})\) between two foliations \((M, \mathcal{F})\) and \((N, \mathcal{G})\) so that \((W, \mathcal{H})\) is foliated homotopy equivalent to either end.

**Definition 2.4.** Let \((M, \mathcal{F})\) be a foliation. The structure set of \((M, \mathcal{F})\) is the set of foliated simple homotopy equivalences \(f : (M, \mathcal{F}) \to (N, \mathcal{G})\) with two foliated simple homotopy equivalences being the same if there is a foliated s-cobordism \(W\) between \(M\) and \(N\).

**Theorem 2.1** (Pimsner-Voiculescu for Structure Sets). The structure set of the product of the Kronecker foliation \((T^2, \mathcal{F})\) with a compact manifold \(M^n, n \geq 5\) is

\[ \mathcal{S}_{\text{foliated}}(M \times \mathcal{F}) = H_\ast(T^2; \mathcal{S}_\ast(M)) \]

**Proof:** We recall the result of Attie-Hurder [6] that leaves of a codimension one foliation of the form \(M \times \mathbb{R}\) are end-periodic. We wish to show that leaves of a foliation of the foliated simple homotopy type of the foliation \(M \times \mathcal{F}\) are of the form \(M \times \mathbb{R}\). We observe that since the leaves are bg simple homotopy equivalent to \(M \times \mathbb{R}\) they are boundedly controlled simple homotopy equivalent to \(M \times \mathbb{R}\). We can then apply boundedly controlled surgery and find, by applying the boundedly controlled Borel conjecture [11] for \(\mathbb{R}\) that the boundedly controlled structure set \(\mathcal{S}_{\text{bdd}}(M \times \mathbb{R})\) is isomorphic to \(\mathcal{S}(M)\), which proves that the leaf is boundedly controlled homeomorphic to \(M' \times \mathbb{R}\) for some structure \(M'\) on \(M\). Hence the leaves are end-periodic. To study the transverse direction we apply codimension 1 splitting to the transverse foliation, since we have a bg simple homotopy equivalence and the splitting obstruction vanishes and observe that by Attie-Hurder [6] we have end-periodicity in this direction as well. The result follows.

### 3 Surgery on Foliations

**Definition 3.1.** [12] \(T\) is a manifold of dimension equal to the codimension of the leaves, equipped with an immersion \(j : T \to X\) transverse to the leaves and where the image meets each leaf at least once. The elements of \(\Gamma_T\) are represented by the triples \((x, c, y)\) where \(c\) is a path connected \(j(y)\) to \(j(x)\) in a leaf, two paths
being equivalent if they determine the same holonomy. In other words $\Gamma_T$ is the subspace of $T \times G \times T$ formed from triples $(x, y, g)$ so that $j(x) = \beta(g)$ and $j(y) = \alpha(g)$.

The foliation $\mathcal{F}$ can be defined by an open covering $\{U_i\}$ and submersions $f_i : U_i \rightarrow T_i$, where $T_i$ is a manifold of dimension equal to the codimension of the leaves, the $f_i$ being surjective with connected fibers. The $f_i$ should satisfy the following compatibility condition: for every pair $(i, j)$, there exists a homeomorphism $g_{ij} : f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j)$ so that $f_i = g_{ij} \cdot f_j$ on $U_i \cap U_j$. Let $T$ be the disjoint of $T_i$. The local homeomorphisms $g_{ij}$ generate a pseudogroup of $\mathcal{F}$ associated to the cocycle of definition $(U_i, f_i, g_{ij})$. We denote by $\Gamma_T$ the topological groupoid of germs of its elements.

It is clear that we have a bijection between leaves of $\mathcal{F}$ and orbits of the groupoid $\Gamma_T$. To the transverse holonomy groupoid $\Gamma_T$ of a foliation $\mathcal{F}$ on a manifold $X$ is associated, its classifying space $B\Gamma_T$ and a continuous map $X \rightarrow B\Gamma_T$.

**Theorem 3.1** (s-cobordism theorem). A foliated s-cobordism $W$ where the leaves are of dimension $\geq 6$ is foliated PL homeomorphic to a product.

**Proof:** Using the bg s-cobordism theorem [2], we see that the leaves are products. To deal with the transversals we use the transversal groupoid $\Gamma_T$ in [12], to give the transversal groupoid control on the whole cobordism, via the product structure on the leaves. We have that the foliation is defined by a map $W \rightarrow B\Gamma_T$ rel one end and since the leaves are products, the transversals are products by the fact that the map is unique up to homotopy.

**Definition 3.2.** The foliated Whitehead group of $(M, \mathcal{F})$ is the group of foliated $h$-cobordisms of $(M, \mathcal{F})$ and is denoted $Wh_{\mathcal{F}}(M)$.

The following definition follows the definition of a foliated space in [9].

**Definition 3.3.** A foliated Poincare duality complex is a metric space $X$ which is a foliated space whose leaves are bg Poincare duality complexes in the following sense. The foliation $\mathcal{F}$ is an equivalence relation on $X$, equivalence classes being connected, embedded by Poincare duality spaces all of the same dimension $k$. A foliated chart in $X$ is a pair $(U, \phi)$ where $U \subseteq X$ is open and $\phi : U \rightarrow T \times Z$ is a homeomorphism where $Z$ is an open neighborhood in a fixed metric space $Z_1$ and $T$ is an open ball in a given bg Poincare duality space of finite radius. The set $P_y = \phi^{-1}(T \times y)$ where $y \in Z$ is called a plaque. If $P$ and $Q$ are plaques, then $P \cap Q$ is open in a given bg Poincare duality space which is the same for $P$ and $Q$. The union of all plaques which are open in a given bg Poincare duality space is that space itself.

The following definitions are based on [19] Chapters 9 and 10.

**Definition 3.4.** Let $(M, \mathcal{F})$ be a foliation. The foliated surgery group of $(M, \mathcal{F})$ is the group of foliated cobordism classes of unrestricted objects where an unrestricted object is:
A foliated Poincare pair $(Y, X)$ over $(M, F)$.

A foliated map $\phi : (W, G, (\partial W, H)) \to (Y, X)$ of pairs of degree 1, where $(W, G)$ is a foliation and $\phi | \partial W : \partial W \to X$ is a foliated simple homotopy equivalence.

A bg stable framing $F$ of $\tau W \oplus \phi^* (\tau)$, where $\tau$ is the Spivak normal fibration of $(Y, X)$.

A map $\omega : Y \to K$, where $K$ is a foliated complex so that the pullback of the double cover of $K$ to $Y$ is orientation preserving.

We write $\theta \sim 0$ to denote that we can construct the triples:

- a simple foliated Poincare triad $(Z; Y, Y_+)$ with $Y \cap Y_+ = X$ and a bundle $\mu$ over $Z$ extending $\nu$.
- a compact foliated manifold triad $(P; N, N_+)$ with $N \cap N_+ = M$.
- a map $\psi : (P; N, N_+) \to (Z; Y, Y_+)$ of degree 1 extending $\phi$ and inducing a foliated simple homotopy equivalence $N_+ \to Y_+$.

The set of quadruples under the equivalence relation $\sim$ is the foliated surgery group. We denote the foliated surgery group by $L^F_s(M)$.

**Definition 3.5.** Let $X, F$ be a foliated Poincare duality complex. The group of normal invariants of $X$, $NI_F(X)$ is the bordism group of quadruples $(M, \phi, \nu, F)$, where $M$ is a foliation, $\phi : M \to X$ of degree 1, $\nu$ a vector bundle over $X$, and trivialization $F$ of $\tau_M \oplus \phi^* \nu$.

The formulation of the following conjectures is a problem in [20].

**Definition 3.6** (Borel Conjecture for Foliations). Let $(M, F)$ be a foliation. If the leaves of $F$ are aspherical, then there is an isomorphism $\theta$

$$NI_F(M) \to L^F_s(M)$$

given by taking the surgery obstruction of a normal map in $NI_F(M)$.

**Definition 3.7** (Novikov Conjecture for Foliations). Let $(M, F)$ be a foliation. If the leaves of $F$ are aspherical, then the above map $\theta$ is injective.

**Definition 3.8** (Surgery Exact Sequence for foliations). The following sequence is exact:

$$\ldots \to L^F_{s+1}(M) \to S_F(M) \to NI_F \to L^F_s(M) \to \ldots$$
4 Algebraic K-Theory

Definition 4.1. Let \((M, \mathcal{F})\) be a manifold equipped with a foliation. The holonomy groupoid \(H = \text{Hol}(M, \mathcal{F})\) is a smooth groupoid with \(H_0 = M\) as the space of objects. If \(x, y \in M\) are two points on different leaves there are no arrows between \(x\) and \(y\) in \(H\). If \(x\) and \(y\) lie on the same leaf \(L\), an arrow \(h : x \rightarrow y\) in \(H\) is an equivalence class \(h = [\alpha]\) of smooth paths \(\alpha : [0, 1] \rightarrow L\) with \(\alpha(0) = x\) and \(\alpha(1) = y\). To explain the equivalence relation, let \(T_x\) and \(T_y\) be small \(q\)-disks through \(x\) and \(y\) transverse to the leaves of the foliation. If \(x' \in T_x\) is a point sufficiently close to \(x\) on a leaf \(L'\), then \(\alpha\) can be “copied” inside \(L'\) to give a path \(\alpha'\) near \(\alpha\) with endpoint \(y' \in T_y\). In this way one obtains the germ of a diffeomorphism from \(T_x\) to \(T_y\) sending \(x\) to \(y\) and \(x'\) to \(y'\). This germ is called the holonomy of \(\alpha\) and denoted \(\text{hol}(\alpha)\). Two paths \(\alpha\) and \(\beta\) from \(x\) to \(y\) in \(L\) are equivalent, i.e. define the same arrow \(x \rightarrow y\), if and only if \(\text{hol}(\alpha) = \text{hol}(\beta)\). We obtain a well defined smooth groupoid \(H = \text{Hol}(M, \mathcal{F})\). This groupoid is a foliation groupoid, and the (discrete) isotropy group \(H_x\) at \(x\) is called the holonomy group of the leaf through \(x\).

Definition 4.2. \([\text{I}]\) Let \(X_1\) and \(X_2\) be spaces equipped with continuous maps \(p_1, p_2\) to a metric space \(Z\). Then a map \(f : X_1 \rightarrow X_2\) is boundedly controlled if there exists an integer \(m \geq 0\) so that for all \(z \in Z\) of \(r \geq 0\), \(p_1^{-1}(B_r(z)) \subseteq f(p_2^{-1}(B_{r+m}(z)))\), where \(B_r(z)\) denotes the metric ball in \(Z\) of radius \(r\) about \(z\). Or, equivalently, there is a constant \(m \geq 0\) so that

\[
\text{dist}_Z(p_2 \circ f(x), p_1(x)) < m
\]

for all \(x \in X_1\).

Definition 4.3. (See \([\text{I}]\)). Let \((X_1, \mathcal{F}_1)\) and \((X_2, \mathcal{F}_2)\) be foliated spaces, and let \((Z, \mathcal{G})\) be a foliated space such that there are foliated maps \(p_1 : (X_1, \mathcal{F}_1) \rightarrow (Z, \mathcal{G})\) and \(p_2 : (X_2, \mathcal{F}_2) \rightarrow (Z, \mathcal{G})\). Then a foliated map \(f : (X_1, \mathcal{F}_1) \rightarrow (X_2, \mathcal{F}_2)\) is boundedly controlled over the foliated space \((Z, \mathcal{G})\) such that for any leaf \(L\) and for any transversal \(T\) in \(X_1\) there is a constant \(m \geq 0\) so that

\[
\text{dist}_Z(p_2 \circ f(x), p_1(x)) < m
\]

for \(x \in L\) or \(x \in T\).

Definition 4.4. (See \([\text{I}]\)). Let \((X, \mathcal{F})\) be a foliated space controlled over a metric space \(Z\) by a map \(p\). Denote by \(\mathcal{P}\) the category of metric balls in \(Z\) with morphisms given by inclusions. Define \(\mathcal{P}\text{Hol}(X, \mathcal{F})\) to be the category whose objects are pairs \((x, K)\) where \(K \in \mathcal{P}\) is an object of \(\mathcal{P}\) and if \(x\) and \(y\) are on the same leaf, a morphism \((x, K) \rightarrow (y, L)\) is a pair \((\omega, i)\) where \(i \in \mathcal{P}(K, L)\) is a morphism in \(\mathcal{P}\) from \(K\) to \(L\) and \(\omega\) is a holonomy class of paths in \(p^{-1}L\) from \(y\) to \(p^{-1}(i(x))\).

Definition 4.5. (See \([\text{I}]\)). Let \((S, \sigma)\) be a controlled basis. A controlled free \(\mathcal{P}\text{Hol}(X, \mathcal{F})\)-module is defined so that given a controlled basis \((S, \sigma),\)

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i) For any metric ball $B$, $F(\sigma)(B)$ is the free $R$-module on the basis

$$\{(\beta, s) \mid s \in S, \beta \text{ a path from } \sigma(s) \text{ to } b\}$$

ii) The controlled holonomy group $\mathcal{PH}o(\mathcal{M}, \mathcal{F})$ acts by composition of paths.

The following bounded category corresponds to S.Hurder’s exotic index theory for foliations $^{[13]}$.

**Definition 4.6.** The category of controlled free $\mathbb{Z}PHo(X, \mathcal{F})$-modules is defined to be the category of controlled modules over the leaves of $\mathcal{F}$, with morphisms equal to the controlled morphisms. We denote this category by $\mathbb{Z}PHo(X, \mathcal{F})^{bg}$.

The $bg$ modules are new and would correspond to a foliated version of Roe’s uniform index theorem.

**Definition 4.7.** The category of controlled free by $\mathbb{Z}PHo(X, \mathcal{F})$-modules is defined to be the category of controlled modules so that in any ball of fixed radius of a leaf or transverse leaf the modules fall into a finite number of types. Morphisms are defined to be morphisms of the modules so that if the control space is partitioned into neighborhoods of a fixed radius, the restrictions fall into a finite number of equivalence classes. We will denote this category by $\mathbb{Z}PHo(X, \mathcal{F})^{bdd}$.

**Definition 4.8.** We define $K_1(\mathbb{Z}PHo(X, \mathcal{F})^{bg})$ to be the abelian group generated by $[F, \alpha]$ where $F$ is a controlled free by module and $\alpha$ is an automorphism of $F$ so that

i) $[F, \alpha] = [F', \alpha']$ if there is an isomorphism $\phi : F \to F'$ so that $\phi \alpha = \alpha' \phi$.

ii) $[F \oplus F', \alpha \oplus \alpha'] = [F, \alpha] + [F', \alpha']$

iii) $[F, \alpha \beta] = [F, \alpha] + [F, \beta]$.

**Definition 4.9.** We define $K_1(\mathbb{Z}PHo(X, \mathcal{F})^{bdd})$ to be the abelian group generated by $[F, \alpha]$ where $F$ is a controlled free module and $\alpha$ is an automorphism of $F$ so that

i) $[F, \alpha] = [F', \alpha']$ if there is an isomorphism $\phi : F \to F'$ so that $\phi \alpha = \alpha' \phi$.

ii) $[F \oplus F', \alpha \oplus \alpha'] = [F, \alpha] + [F', \alpha']$

iii) $[F, \alpha \beta] = [F, \alpha] + [F, \beta]$.

**Definition 4.10.** $Wh_{\mathcal{F}, bg}(\mathcal{PH}o(\mathcal{X}))$ is defined to be the quotient of $K_1(\mathbb{Z}PHo(X)^{bg})$ by the subgroup of elements of the form

$$[F(\sigma), u_{F(\sigma)}] \text{ and } [F(\sigma), F(\alpha, \nu)]$$

where $(S, \alpha)$ is any basis over $\mathcal{PH}o(\mathcal{X}, \mathcal{F})$, $u_{F(\sigma)}$ is multiplication by a unit and $F(\alpha, \nu)$ is an automorphism of bases.

**Definition 4.11.** $Wh_{\mathcal{F}, bdd}(\mathcal{PH}o(\mathcal{X}))$ is defined to be the quotient of $K_1(\mathbb{Z}PHo(X)^{bdd})$ by the subgroup of elements of the form

$$[F(\sigma), u_{F(\sigma)}] \text{ and } [F(\sigma), F(\alpha, \nu)]$$

where $(S, \alpha)$ and any $bdd$ basis over $\mathcal{PH}o(\mathcal{X}, \mathcal{F})$, $u_{F(\sigma)}$ is multiplication by a unit and $F(\alpha, \nu)$ is an automorphism of bases.
Definition 4.12. Let \( X, F \) be a foliated space. Let \( DR^b_G(X) \) be the collection of all pairs \( (Y, X) \), where \( Y, G \) is a foliated space, for which \( X \) is a foliated by strong deformation retract of \( Y \).

Theorem 4.1. Every element of \( Wh_F(X) \) has a representative of the form

\[
(Y, q) = (X, p) \cup_{\phi_r} (S_r \times I^r, q_r) \cup_{\phi_{r+1}} (S_{r+1} \times I^{r+1}, q_{r+1}),
\]

for controlled \( r \)-cell \( (S_r \times I^r, q_r) \) and controlled \( r+1 \)-cell \( (S_{r+1} \times I^{r+1}, q_{r+1}) \)
and attaching maps \( \phi_r \) and \( \phi_{r+1} \).

Proof: The proof is the same as in [2] Theorem 3.11.

Theorem 4.2. The group \( Wh^b_F(X) \) is isomorphic to the group \( Wh_F(PHol(X)^b) \). The group \( Wh^{bd} F(X) \) is isomorphic to the group \( Wh_F(PHol(X)) \).

Proof: Observe that \( H^*_c \) of the universal cover of a leaf \( L \) of \( F \) in \( DR^b G(L) \) is a \( \mathbb{Z}P G_1(L) \) module. We let the torsion of a chain complex of such modules which is acyclic be the class of the boundary map in \( Wh_F(PHol(X)^b) \). More specifically, define for \( bg \) bases \( b_i = (\{ S_i, \sigma_i \}) \) the element \( [b_2/b_1] = [F(\sigma_2), F(\alpha, \nu)\phi_1^{-1}\phi_2] \), where \( (\alpha, \nu) \) is an isomorphism. Now define the torsion of a \( bg \) foliated chain complex: by noting that \( B_i = Im \partial_i \) is stably free, so using the basis \( b_i \),

\[
\tau(C_s) = \sum_i (-1)^i [b_i h_i b_{i-1} / c_i].
\]

This is clearly invariant under leafwise uniform subdivision of \( C_s \). Now define the isomorphism \( \tau : Wh_F(M) \to Wh_F(PHol(M)^b) \) by \( \tau([X, Y, q]) = \tau([C_s, \text{cell}](Y, X)) \). This is an isomorphism.

The same argument works in the bounded case.

5 Algebraic L-theory

Definition 5.1. Let \((M, F)\) be a manifold \(M\) with foliation \(F\). Let \(A\) be an additive category, \(\pi\) a group. Then the category \(C^F_M(A[\pi])\) is defined to the one whose objects are formal direct sums

\[
M = \sum_{x \in BG} \sum_{y \in L} M(y)
\]

of objects \(M(y)\) in \(A[\pi]\), where \(L\) is the leaf corresponding to the point \(x\) in the classifying space of the holonomy group \(BG\) which fall into a fixed finite number of types inside of each ball of fixed radius in \(L\) and transverse to \(L\). Here \(A[\pi]\) is the category with the one object \(M[\pi]\) for each object \(M\) in \(A\), and with morphisms linear combinations of morphisms \(f_g : M \to N\) in \(A\) finite.

Definition 5.2. The algebraic surgery group \(L^{F,bg}_*(M)\) is defined as

\[
L^{F,bg}_*(M) = L_*(\mathbb{Z}PHol(M)^b)
\]
Definition 5.3. The algebraic surgery group $L_*^{bd}(M)$ is defined as

$$L_*^{bd}(M) = L_*(ZPmol(M)^{bd})$$

Definition 5.4. [17] A chain duality $(T, e)$ on an additive category $\mathcal{A}$ is a contravariant functor $T : \mathcal{A} \to \mathcal{B}(\mathcal{A})$ together with a natural transformation $e : T^2 \to 1 : \mathcal{A} \to \mathcal{B}(\mathcal{A})$ such that for each object $A$ in $\mathcal{A}$

1. $e(T(A)) \cdot T(e(A)) = 1 : T(A) \to T^3(A) \to T(A)$.

2. $e(A) : T^2(A) \to A$ is a chain equivalence.

Definition 5.5. [17] Use the standard free $\mathbb{Z}[\mathbb{Z}_2]$-modules resolution of $\mathbb{Z}$ to define for any finite chain complex $C$ in $\mathcal{A}$ the $\mathbb{Z}$-module chain complex $W_\% C = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_\mathcal{A} C)$

Definition 5.6. [17] An $n$-dimensional quadratic Poincaré complex $(C, \psi)$ is a finite chain complex $C$ in $\mathcal{A}$ together with an $n$-cycle $\psi \in (W_\%)_n$ such that the chain map

$$(1 + T)\psi_0 : C^n \to C$$

is a chain equivalence in $\mathcal{A}$.

Definition 5.7. [17] Let $f : C \to D$ be a chain map. The algebraic mapping cone of $f$, $C(f)$ is defined by

$$d_{C(f)} = \begin{pmatrix} d_D & (-1)^r f \\ 0 & d_C \end{pmatrix}$$

$$(1 + T)f_0 : C_{n-*} \to C$$

Definition 5.8. [17] A chain map $f : C \to D$ of finite chain complexes in $\mathcal{A}$ induces $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$f \otimes f : C \otimes_\mathcal{A} C \to D \otimes_\mathcal{A} D$$

and hence a $\mathbb{Z}$-module chain map

$$f_\% : W_\% C \to W_\% D$$

Definition 5.9. [17] Given an additive category $\mathcal{A}$ and a simplicial complex $K$ are combined to define an additive category $\mathcal{A}^*(K)$ of $K$-based objects in $\mathcal{A}$ which depends contravariantly on $K$. We define $\mathcal{A}_*(K)$ of $K$-based objects in $\mathcal{A}$ which depends covariantly on $K$. An object $M$ in an additive category $\mathcal{A}$ is $K$-based if it is expressed as a direct sum

$$M = \sum_{\sigma \in K} M(\sigma)$$
of objects $M(\sigma)$ in $A$, such that $\{\sigma \in K \mid M(\sigma) \neq 0\}$ is finite. A morphism $f : M \to N$ of $K$-based objects is a collection of morphisms in $A$

$$f = \{f(\tau, \sigma) : M(\sigma) \to N(\tau) \mid \sigma, \tau \in K\}$$

Let $A^*(K)$ be the additively category of $K$-based objects $M$ in $A$, with morphisms $f : M \to N$ such that $f(\tau, \sigma) : M(\sigma) \to N(\tau)$ is 0 unless $\tau \leq \sigma$, so that

$$f(M(\sigma)) \subseteq \sum_{\tau \leq \sigma} N(\tau)$$

Let $A_*(K)$ be the additive category of $K$-based objects $M$ with morphisms $f : M \to N$ so that $f(\tau, \sigma) : M(\sigma) \to N(\tau)$ is 0 unless $\tau \geq \sigma$ so that

$$f(M(\sigma)) \subseteq \sum_{\tau \geq \sigma} N(\tau)$$

Forgetting the $K$-based structure defines the covariant assembly functor

$$A^*(K) \to A; M \to M^*(K) = \sum_{\sigma \in K} M(\sigma)$$

$$A_*(K) \to A; M \to M_*(K) = \sum_{\sigma \in K} M(\sigma)$$

Definition 5.10. [17] Let $A^*[K]$ be the additive category with objects the covariant functors $M : K \to A; \sigma \to M[\sigma]$ such that $\{\sigma \in K \mid M[\sigma] \neq 0\}$ is finite. The morphisms are the natural transformations of such functors. Let $A_*[K]$ be the additive category with objects the contravariant functors as above.

Definition 5.11. [17] Let $A(R)$ be the category of finitely generated free $R$-modules, $R$ a ring. The additive category $A(R)_*[K]$ will be denoted by $A[R, K]$ and the additive category $A(R)_*(K)$ will be denoted $A(R, K)$. Their objects will be called $[R, K]$-modules and $(R, K)$-modules.

Definition 5.12. [17] For any additive category with chain duality $A$ there is an algebraic bordism category

$$\Lambda(A) = (A, B(A), C(A))$$

with $B(A)$ the category of finite chain complexes in $A$, and $C(A) \subseteq B(A)$ the subcategory of contractible complexes.

Definition 5.13. [17] A subcategory $C \subseteq B(A)$ is closed if it is a full additive subcategory which is invariant under $T$, such that the algebraic mapping cone $C(f)$ of any chain map $f : C \to D$ in $C$ is an object in $C$. 


Definition 5.14. [17] A chain complex $C$ in $\mathcal{A}$ is $\mathcal{C}$-contractible if it belongs to $\mathcal{C}$. A chain map $f : C \to D$ is a $\mathcal{C}$-equivalence if the algebraic mapping cone $C(f)$ is $\mathcal{C}$-contractible.

Definition 5.15. [17] An $n$-dimensional quadratic complex $(C, \psi)$ in $\mathcal{A}$ is $\mathcal{C}$-Poincaré if the chain complex

$$\partial C = S^{-1}C((1 + T)\psi_0 : C^{n-*} \to C)$$

is $\mathcal{C}$-contractible.

Definition 5.16. [17] Let $\Lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ be an algebraic bordism category. An $n$-dimensional quadratic complex $(C, \psi)$ in $\mathcal{A}$ which is $\mathcal{B}$-contractible and $\mathcal{C}$-Poincaré. The quadratic $L$-group $L_n(\Lambda)$ is the cobordism group of $n$-dimensional quadratic complexes in $\Lambda$.

Definition 5.17. Let $\mathcal{A}_F^\mathcal{F}(K)$ be the additive category of $K$-based objects in $\mathcal{A}_F^\mathcal{F}$ with morphisms $f : M \to N$ such that $f(\tau, \sigma) = 0 : M(\sigma) \to N(\tau)$ unless $\tau \geq \sigma$ so that $f(M(\sigma)) \subseteq \sum_{\tau \geq \sigma} N(\tau)$.

Definition 5.18. The quadratic foliated structure groups of $(R, K)$ are the cobordism groups

$$S_n^\mathcal{F}(R, K) = L_{n-1}(\mathcal{A}_F^\mathcal{F}(R, K), \mathcal{C}_F^R(R, K), \mathcal{C}_F^F(R)^*(K))$$

Definition 5.19. Define the local, uniformly finite, finitely generated free $(R, K)$-modules

$$\Lambda(R)_F^\mathcal{F}(K) = (\mathcal{A}_F^\mathcal{F}(R, K), \mathcal{B}_F^\mathcal{F}(R, K), \mathcal{C}_F^\mathcal{F}(R)^*(K))$$

where $\mathcal{B}_F^\mathcal{F}(R, K)$ is the category of finite chain complexes of f.g. free foliated $(R, K)$-modules. An object in $\mathcal{C}_F^\mathcal{F}(R)^*(K)$ is finite f.g. free foliated $(R, K)$-module chain complex $C$ such that each $[C][\sigma](\sigma \in K)$ is a contractible f.g. free $R$-module chain complex.

Definition 5.20. Let $M$ be a compact manifold with foliation $\mathcal{F}$. Define the foliated normal invariants for $(M, \mathcal{F})$ to be:

$$H_n(BG; \mathcal{L}) = L_n(\Lambda(Z)^\mathcal{F}(M))$$

where $BG$ is the classifying space of the holonomy groupoid of $M$, and $\mathcal{L}$ is the cosheaf assigning to each point $x$ of $BG$ the $L$-homology of the leaf $S_x$ through $x$.

Theorem 5.1 (Surgery Exact Sequence). Let $(M, \mathcal{F})$ be a manifold $M$ with a foliation $\mathcal{F}$. Then we have an algebraic surgery exact sequence

$$\ldots \to H_n(BG; \mathcal{L}) \to L_n^{\mathcal{F}, bg}(M) \to S_n^{\mathcal{F}, bg}(R, M) \to H_{n-1}(BG; \mathcal{L}) \to \ldots$$

where $BG$ is the classifying space of the holonomy groupoid and $\mathcal{L}$ is the cosheaf of the uniformly finite homology with coefficients in the the $L$-spectrum of the leaf $S_x$ through $x \in BG$. 

Theorem 5.2 (Haefliger Cor.3.2.4). Let $F$ be a foliation on a manifold $X$ so that the holonomy coverings of the leaves are $(k-1)$-connected. Then the holonomy groupoid $G$ of $F$ considered as a $G$-principal bundle with base $X$ by the end projection is $k$-universal. Thus the space $X$ itself is $k$-classifying and the map $i$ of $X$ to $BG$ is $k$-connected.

Theorem 5.3 (Haefliger Cor.3.1.5). Suppose the target projection of the holonomy groupoid $G$ of the foliation $F$ on a manifold $X$ is a locally trivial fibration, whose fiber is the common holonomy covering $L$ of all the leaves. Then the map $i : X \to BG$ is homotopy equivalent to a locally trivial fibration with base $BG$ and fiber $L$.

The hypothesis of this theorem are satisfied in the following cases:

i. $X$ is compact and $F$ possesses a transverse Riemannian metric.

ii. The leaves of $F$ are transverse to the fibers of a compact fibration and the foliation is analytic.

iii. The leaves of $F$ are the trajectories of a flow without closed orbits or the holonomy group of each closed orbit is infinite.

Because of this theorem of Haefliger, foliated surgery has as a special case blocked surgery [16], and in analogy with the blocked surgery diagram at the end of [16] we have the following exact sequence:

Theorem 5.4 (Blocked surgery exact sequence).

$$
\ldots \to H^n(BG; \mathcal{L}) \to H^n(BG; \mathcal{L}^b(S_x)) \to H^n(BG; S^b(S_x)) \to \ldots
$$

where $\mathcal{L}$ is cosheaf of the uniformly finite homology with coefficients in the $L$ spectrum of the leaf $S_x$ through $x$ in $BG$, $\mathcal{L}^b$ is the cosheaf of the bg $L$-theory of the leaf $S_x$ through $x$ in $BG$. $S^b$ is the bg structure set of $S_x$ the leaf through $x$ in $BG$.

In addition there are exact sequences

Theorem 5.5 (Leafwise Assembly).

$$
\ldots \to \text{Fiber}^N_n \to H_n(BG; \mathcal{L}) \xrightarrow{A^N} H_n(M; \mathcal{L}) \to \text{Fiber}^N_{n-1} \to \ldots
$$

$$
\ldots \to \text{Fiber}^L_n \to H_n(BG; \mathcal{L}^b) \xrightarrow{A^L} L_n(\pi_1(M)) \to \text{Fiber}^L_{n-1} \to \ldots
$$

$$
\ldots \to \text{Fiber}^S_n \to H_n(BG; S^b(S_x)) \xrightarrow{A^S} S_n(M) \to \text{Fiber}^S_{n-1} \to \ldots
$$

where $\text{Fiber}^N_n, \text{Fiber}^L_n, \text{Fiber}^S_n$ are the fibers of the leafwise assembly maps, and classify the non-leaves, and the maps $A^N : H_n(BG; \mathcal{L}) \to H_n(M; \mathcal{L})$, $A^L : H_n(BG; \mathcal{L}^b) \to L_n(\pi_1(M))$ and $A^S : H_n(BG; S^b(S_x)) \to S_n(M)$ are the leafwise assembly maps.

Note that the assembly map $A^N : H_n(BG; \mathcal{L}) \to H_n(M; \mathcal{L})$ was used in [1] to prove Bott Integrability in the simply-connected case. It was generalized in [4] and [5] to prove more general versions of Bott Integrability.
Theorem 5.6 (Embeddability of Leaves). The image of $\text{Fiber}_n^S \to H_n(BG; S^{bg}(S_x))$ is the kernel of $A^S : H_n(BG; S^{bg}(S_x)) \to S_n(M)$ and therefore we take $H_n(BG; S^{bg})/\text{Ker}A^S$ to be the foliations on $M$ with the leaves of a given bg homotopy type. This answers to a certain extent the question of when an open manifold embeds as a leaf of a foliation in a given closed manifold $M$.

For example, take $N \times F$, where $F$ is the Kronecker foliation on $T^2$, a foliation on $N \times T^2$. $BG = T^2$, the leaves are bg homotopy equivalent to $N \times R$, $S^{bg}(N \times R) = H_{uf}^\ast(R; S^\ast(N))$ and $\text{Ker}A^S$ is the set of non-end-periodic leaves [6]. This gives us the Pimsner-Voiculescu theorem from Section 2, as the set of end-periodic leaves corresponds to $Z \subset H_0^uf(R)$ as in [6] and therefore the structure set is $H_\ast(BG; S(N)) = H_\ast(T^2; S(N))$.

Theorem 5.7 (Bounded Surgery Exact Sequence). Let $(M, F)$ be a manifold $M$ with a foliation $F$. Then we have an algebraic surgery exact sequence

$$\cdots \to H_n(BG; \mathcal{L}) \to L^{F,bdd}_n(M) \to S^{F,bdd}_n(R, M) \to H_{n-1}(BG; \mathcal{L}) \to \cdots$$

where $BG$ is the classifying space of the holonomy groupoid and $\mathcal{L}$ is the cosheaf of the locally finite homology of the L-spectrum of the leaf $S_x$ through $x \in BG$.

We have two Novikov and Borel Conjectures for foliations. The first, ones for bounded surgery, correspond to the Baum-Connes Conjecture for foliations and the Novikov Conjecture for foliations due to Baum and Connes [7].

Conjecture 5.1 (Bounded Novikov Conjecture for Foliations). The map

$$H_\ast(BG; \mathcal{L}^I) \to L^{F,bdd}_\ast(M)$$

where $\mathcal{L}^I$ is the cosheaf assigning the locally finite L-homology of the leaf through $x \in BG$ to the point $x \in BG$, is injective, provided the leaves of $(M, F)$ are uniformly contractible.

Conjecture 5.2 (Bounded Borel Conjecture for Foliations). If the leaves of $(M, F)$ are uniformly contractible, then the map

$$H_\ast(BG; \mathcal{L}^I) \to L^{F,bdd}_\ast(M)$$

where $\mathcal{L}^I$ is the cosheaf assigning the locally finite L-homology of the leaf through $x \in BG$ to the point $x \in BG$ is an isomorphism.

The second two conjectures are for bounded geometry surgery:

Conjecture 5.3 (Bounded Geometry Novikov Conjecture for Foliations). If the leaves of $(M, F)$ are uniformly contractible, then

$$H_\ast(BG; \mathcal{L}^{uf}(S_x)) \to L^{F,bg}_\ast(M)$$

is injective, where $S_x$ is the leaf of $F$ through $x \in BG$ and $\mathcal{L}^{uf}$ is the uniformly finite L-homology cosheaf of the leaves over $BG$. 
Conjecture 5.4 (Bounded Geometry Borel Conjecture for Foliations). If the leaves of \((M,F)\) are uniformly contractible, then
\[
H_\ast(BG; \mathcal{L}^{uf}(S_x)) \to L^{F, bg}_\ast(M)
\]
is an isomorphism, where \(S_x\) is the leaf of \(F\) through \(x \in BG\) and \(\mathcal{L}^{uf}\) is the uniformly finite \(L\)-homology cosheaf of the leaves over \(BG\).

6 Examples of Foliations Satisfying the Novikov and Borel Conjectures

We recall the cases where the Baum-Connes conjecture has been verified. We will prove the bounded Borel conjecture for these foliations.

1. Fibrations \(F \to M \to B\). In this case the leaf space is identified with the base space of the fibration \(B\).
2. Foliations induced by free actions of \(\mathbb{R}^n\), and free actions of a solvable simply connected Lie group \(\Gamma\).
3. The Reeb foliations on \(T^2\) and \(S^3\).
4. Foliations without holonomy. In this case \(BG\) is a torus \(T^n\) provided with a linear foliation, and we apply 2.
5. Foliations almost without holonomy. Applying graphs of groups, this is reduced to 4.
6. Nontrivial examples: the Sacksteder foliation, the Hirsch foliation, \(\mathbb{Z}\)-periodic foliations.

Our Pimsner-Voiculescu theorem is a case of the Borel Conjecture for foliations, the case of the Kronecker foliation. We will verify the Borel Conjecture in all cases where the Baum-Connes Conjecture is known.

Theorem 6.1. The Borel Conjecture is true for fibrations

\[
F \to B \to M
\]
where \(F\) is compact.

Proof: In this case the leaf space is \(B\). We see that

\[
H_n(BG; H^{lf}_{m}(F; L)) = H_n(B; H^{lf}_{m}(F; L)) = H_{n+m}(M; L)
\]

Where \(H^{lf}_{m}(F; L)\) is the constant cosheaf. The surgery group of the foliation is isomorphic to this.

Theorem 6.2. The Borel Conjecture is true for the Reeb foliation on \(T^2\) and \(S^3\)

Proof: This can be seen easily by codimension 1 splitting as for the case of Pimsner-Voiculescu.
Theorem 6.3. The Borel Conjecture is true for free actions of $\mathbb{R}^n$ and free actions of a solvmanifold $\Gamma$.

Proof: We again use codimension 1 splitting along the leaves inductively down to $\mathbb{R}^{n-1}$ etc.

Theorem 6.4. The Borel Conjecture is true for foliations without holonomy.

Proof: Codimension 1 splitting applies where $BG$ is $T^n$ and the leaves are linear.

Theorem 6.5. The Borel Conjecture is true for foliations almost without holonomy.

Proof: Apply the previous using graphs of groups.

Theorem 6.6. The Borel Conjecture is true for the Hirsch foliation.

Proof: Apply codimension 1 splitting to the leaves, which are in the form of an $n$-partite graph $[8]$.

Similarly, the Borel conjecture can be verified for the Sacksteder foliation and $\mathbb{Z}$-periodic foliations from codimension 1 splitting.

The Novikov conjecture for foliations has been stated by P.Baum and A.Connes [7]. Let $\pi$ be the fundamental groupoid along the leaves. $B\pi$ denotes the classifying space of the topological groupoid $\pi$. Since $V$ is the units of $\pi$ there is a canonical map:

$$\lambda : V \to B\pi$$

$\pi$ is itself a principal $\pi$-bundle over $V$ and the map $\lambda$ is the classifying map of this principal $\pi$-bundle.

Conjecture 6.1. Let $(V,F)$ and $(V',F)$ be orientable $C^\infty$ foliations with $V,V'$ compact. Let $f : V' \to V$ be a leafwise homotopy equivalence. Choose orientations for $V$ and $V'$ so that $f$ is orientation preserving. Then in $H_*(B\pi; \mathbb{Q})$ there is the equality:

$$\lambda_* (L(TV) \cap [V]) = \lambda_* f_* (L(TV') \cap [V'])$$

Baum and Connes proved the Foliated Novikov Conjecture in the following special case:

Let $\mathcal{L}$ be a leaf of the foliation $(V,F)$. If for every leaf $\mathcal{L}$, $\pi_i(\mathcal{L}) = 0$ for all $i \geq 2$, then $B\pi = V$, where $B\pi$ is the classifying space of the fundamental groupoid along the leaves. If $V$ is compact and $(V,F)$ has negatively curved leaves this is the case, so the Novikov conjecture holds.

S.Hurder [13] has extended this to the class of ultra-spherical foliations, which will be defined below.
Definition 6.1. \[13\] Let \((M, \mathcal{F})\) be a foliation and \(\mathcal{G}_F\) be its holonomy groupoid. A leafwise path \(\gamma\) is a continuous map \(\gamma : [0, 1] \to M\) whose image is contained in a single leaf of \(\mathcal{F}\). There are natural continuous maps \(s, r : \mathcal{G}_F \to M\) defined by \(s(\gamma) = \gamma(0)\) and \(r(\gamma) = \gamma(1)\). For a point \(x \in M\), the pre-image \(s^{-1}(x) = L_x\) is the holonomy cover of the leaf \(L_x\) of \(\mathcal{F}\) through \(x\).

Definition 6.2. \[13\] For \(x \in M\) and a leafwise path \(\gamma : [0, 1] \to L_x\), define the plaque length function \(N_T(\gamma)\) to be the least number of plaques to cover the image of \(\gamma\). Define the plaque distance function \(D_x(\cdot, \cdot)\) on the holonomy cover \(\tilde{L}_x\) using the plaque length function: for \(y, y' \in \tilde{L}_x\)

\[
D_x(y, y') = \inf \{N_T | \gamma \text{ is a leafwise path from } y \text{ to } y'\}
\]

Definition 6.3. \[13\] Denote by \(C_u(\mathcal{G}_F) \subset C(\mathcal{G}_F)\) the subspace of uniformly continuous functions on \(\mathcal{G}_F\), the holonomy groupoid of \(\mathcal{F}\).

Definition 6.4. \[13\] For \(x \in M\) and \(r > 0\), define the fiberwise variation function

\[
V_s(x, r) : C(\tilde{L}_x) \to [0, \infty]
\]

\[
V_s(x, r)(h)(y) = \sup \{|h(y') - h(y)| \text{ such that } D_x(y, y') \leq r\}
\]

We say that \(f \in C(\mathcal{G}_F)\) has uniformly vanishing variation at infinity, where \(\mathcal{G}_F\) is the holonomy groupoid of \(\mathcal{F}\) if there exists a function \(D : (0, \infty) \to [0, \infty)\) so that if \(D_x(y, \ast x) > D(\epsilon)\) then \(V_s(x, r)((t_x^\ast f)(y)) < \epsilon\). Here \(\ast x\) denotes the constant path at \(x\). Let \(C_h(\mathcal{F}) \subset C_u(\mathcal{F})\) denote the subspace of uniformly continuous functions which have uniformly vanishing variation at infinity.

Definition 6.5. \[13\] Let \(\mathcal{F}\) be a topological foliation of a paracompact manifold \(M\) equipped with a regular foliation atlas. The corona, \(\partial_h \mathcal{F}\), of \(\mathcal{F}\) is the spectrum of the quotient \(C^*\)-algebra \(C_h(\mathcal{F})/C_0(\mathcal{F})\).

Definition 6.6. \[13\] A foliation \(\mathcal{F}\) on a connected manifold \(M\) is said to be ultra-spherical if there exists a map of coronas \(\sigma : \partial_h \mathcal{F} \to SF\) which commutes with the projections onto \(M\), and so the \(\sigma^* \Theta \in H^*(\partial_h \mathcal{F})\) is nonzero.

We next prove the foliated Novikov Conjecture using bounded surgery.

Theorem 6.7. Let \(\mathcal{F}\) be an oriented ultra-spherical foliation with uniformly contractible leaves and Hausdorff holonomy groupoid. Then the foliated Novikov Conjecture is true for \(\mathcal{F}\).

Proof: See \[13\]. Let

\[
\mu : KO(BG) \otimes \mathbb{Q} \to L_{x,bdd}^\infty(M) \otimes \mathbb{Q}
\]

be the assembly map, which we must show to be injective. There exists a boundary class \(u \in KO^*(\partial_h \mathcal{F})\). Let \(\eta \in KO(SF)\) with KO-theory boundary \(\beta[T\mathcal{F}] \in KO(T\mathcal{F})\) and set \([u] = \sigma^* \eta\). There is a continuous extension of \(\sigma\) to a map of pairs

\[
\tilde{\sigma} : (G_F, \partial_h \mathcal{F}) \to (T\mathcal{F}, SF)
\]

which commutes with the projection onto \(M\). By naturality of the boundary map, \(\partial[u] = \tilde{\sigma} \beta[T\mathcal{F}]\). The rest follows as in \[13\], Theorem 6.9.
7 Further Directions

The difference between boundedly controlled foliated surgery and bounded geometry foliated surgery should give rise to a fiber (to the forgetful map) and an exact sequence. The fiber is the set of metrics on the leaves given the boundedly controlled topological type. We have not investigated the situation where a foliation has a symmetry or group action, and the resulting equivariant surgery theory. Furthermore, the symmetric L-groups due to Mishchenko and Ranicki \[17\], and visible L-groups due to Weiss \[17\] also merit investigation.

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