ABSTRACT. In this note, we study A. Beilinson’s gluing for perverse sheaves in the case of the diagonal arrangement and its relation to the Grothendieck-Teichmüller group. We also explain a relation to the Etingof-Kazhdan quantisation.

0.1. Introduction. Let $M^B(\mathbb{A}^n, S_\emptyset)$ be a category of unipotent perverse sheaves on a complex $n$-affine space which are lisse with respect to a diagonal stratification $S_\emptyset$. We propose the following:

**Hypothesis 1.**

(i) For every binary $n$-labelled tree $T$ there exist a fiber functor:

$$\omega_T : M^B(\mathbb{A}^n, S_\emptyset) \to \text{Vect}_Q$$

(ii) A collection $\mathcal{Loc}_n := \{M^B(\mathbb{A}^n, S_\emptyset), \omega_T\}_{T \in \text{Tree}(n)}$ naturally assembles into a fibered category over a category $\Pi^B_1(FM^n(\mathbb{A}))$, where $\Pi^B_1(FM^n(\mathbb{A}))$ is a Betti (=pro-unipotent) fundamental groupoid of the Fulton-MacPherson space of $n$-points in $\mathbb{A}$.

(iii) The corresponding category of cartesian sections gives a $\Sigma_n$-equivariant equivalence:

$$\Gamma_{cart}(\mathcal{Loc}_n) \cong M^B(\mathbb{A}^n, S_\emptyset),$$

where a symmetric group $\Sigma_n$ acts on $\mathbb{A}^n$ by permuting coordinates.

Note that a collection $\{\Pi^B_1(FM^n(\mathbb{A}))\}_{n \geq 1}$ has a natural structure of an operad $[\text{Kon99}]$. The equivalences from Hypothesis 1 are compatible with operadic compositions. Denote by $\text{GT}^\text{un}$ the pro-unipotent Grothendieck-Teichmüller group $[\text{Dri91}]$. By $M^B(\text{Ran}(\mathbb{A}), S_\emptyset)$ we denote a category of unipotent perverse sheaves on a Ran space of $\mathbb{A}^1$ $[\text{BD04}]$ $[\text{Kal19}]$. These lead to the following:

**Hypothesis 2.** There exists a morphism:

$$\text{GT}^\text{un} \to \text{Aut}(M^B(\text{Ran}(\mathbb{A}), S_\emptyset))$$

Under the equivalence between factorizable objects in $M^B(\text{Ran}(\mathbb{A}), S_\emptyset)$ and conilpotent Hopf algebras $[\text{KS20}]$ $[\text{Kal19}]$ Hypothesis 2 corresponds to Theorem 11.1.7 from $[\text{Fre17}]$ (there is a natural equivalence between 2-algebras (DG-algebras over and operad of little 2-disks) and DG-sheaves on a Ran space $[\text{Lur}]$. Following V. Schechtman $[\text{Sch93}]$ we consider examples of above statements in the case of affine spaces $\mathbb{A}^2$ and $\mathbb{A}^3$ and discuss a relation to the Etingof-Kazhdan quantisation $[\text{EK96}]$. A more detailed account shall appear in $[\text{Kal}]$.

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0.3. Notation. For an integer \( n \) we denote by \([n]\) a set of elements \([n] := \{1, 2, \ldots, n\}\). We denote by \( \Sigma_n \) the symmetric group on \( n \) letters. Let \( \mathcal{C} \) be an abelian category by a fiber functor \( \omega: \mathcal{A} \to \text{Vect}_Q \) we understand an exact and faithful functor [Del90], where \( \text{Vect}_Q \) is a category of finite dimensional \( \mathbb{Q} \)-vector spaces. By \( \mathcal{C}^\text{A} \) we denote the corresponding Tannakian dual coalgebra. By \( \text{Cat} \) we denote a 2-category of all small categories.

0.4. A. Beilinson's gluing. Let \((X, \mathcal{O}_X)\) be a complex variety space equipped with a Whitney stratification \( \mathcal{S} \). By \( \mathcal{M}(X, \mathcal{S}) \) we denote a category of perverse sheaves smooth with respect to \( \mathcal{S} \). With every regular function \( f: X \to \mathbb{A}^1 \) we can associate the following diagram of algebraic varieties \( i: D \to X \leftarrow U: j \). Here by \( D \) we have denoted the principal divisor defined by \( D := f^{-1}(0) \) and by \( U := f^{-1}(\mathbb{C}^\times) \) the corresponding open complement. Following P. Deligne and O. Gabber [BBD83] we have a functor of nearby cycles \( \Psi_f: \mathcal{M}(X, \mathcal{S}) \to \mathcal{M}(Z, \mathcal{S}) \) and a functor of vanishing cycles \( \Phi_f: \mathcal{M}(X, \mathcal{S}) \to \mathcal{M}(Z, \mathcal{S}) \). We have a natural transformation \( T_{\Psi}: \Psi_f \to \Psi_f \) (resp. \( T_{\Phi}: \Phi_f \to \Phi_f \)) called the monodromy transformation of nearby cycles (resp. monodromy transformation of vanishing cycles.) We also have canonical and variations morphism, which are natural transformation of functors: \( \text{can}: \Psi_f \leftarrow \Phi_f: \text{var} \). We also denote by \( \Psi_f^\text{\textvar} \) (resp. \( \Phi_f^\text{\textvar} \)) a part of nearby (resp. vanishing) cycles where a monodromy operator act unipotently.

Following A. Beilinson [Bei87] with a regular function \( f \) on \( X \) we associate a gluing category \( \text{Glue}_f(U, Z) \). This is a category with following objects \( \{\mathcal{E}_U, \mathcal{E}_Z, u, v\} \), where \( u: \Psi_f^\text{\textvar} \mathcal{E}_U \to \mathcal{E}_Z, \quad v: \Phi_f^\text{\textvar} \mathcal{E}_U \leftarrow \mathcal{E}_Z \), where \( \mathcal{E}_U \in \mathcal{M}(U, \mathcal{S}) \) and \( \mathcal{E}_Z \in \mathcal{M}(Z, \mathcal{S}) \), such that \( vu = T_{\Phi} - 1 \). We have the following:

**Theorem 0.4.1** (A. Beilinson). For every \( f \in \mathcal{O}_X \) we have a functor \( F_f: \mathcal{M}(X, \mathcal{S}) \to \text{Glue}_f(U, Z) \) defined by the rule:

\[
F_f: \mathcal{E} \mapsto \{j^* \mathcal{E}, \Phi_f^\text{\textvar} \mathcal{E}, \text{can}, \text{var}\}
\]

This functor extends to an equivalence between categories \( \mathcal{M}(X, \mathcal{S}) \) and \( \text{Glue}_f(U, Z) \).

0.5. Fiber functors and trees. For a natural number \( n \in \mathbb{N}_{\geq 1} \) we consider the corresponding complex affine space \( \mathbb{A}^n \) with coordinates \( (z_i)_{i=1,\ldots,n} \). We equip \( \mathbb{A}^n \) with a diagonal stratification \( \mathcal{S}_0 = \{\Delta_j\} \), where \( \Delta_j = z_i - z_j \). The unique minimal closed stratum of \( \mathcal{S}_0 \) will be denoted by \( \Delta \) and the unique maximal open stratum will be denoted by \( U^n \). We denote by \( \mathcal{M}(\mathbb{A}^n, \mathcal{S}_0) \) a category of perverse sheaves which are smooth with respect to the diagonal stratification \( \mathcal{S}_0 \) and every perverse sheaf is an extension of direct sums of perverse sheaves supported on closed strata of \( \mathbb{A}^n \) [Kho95]. Denote by \( \text{Tree}(n) \) a set (groupoid) of binary rooted trees with leaves labelled by a finite set \([n]\). We are going to define fiber functors associated with a tree \( T \in \text{Tree}(n) \):

**Example 0.5.1.** We start with the simplest (nontrivial) case \( \mathbb{A}^2 \) with coordinates \((z_1, z_2)\). There are two binary 2-labelled trees:

\[
T_1 = \begin{array}{c}
1 \\
\end{array} \\
2
\]

\[
T_2 = \begin{array}{c}
1 \\
2
\end{array}
\]

\[\text{Here we assume the middle perversity function in the sense of [BBD83].}\]

\[\text{We shift cycles by } [-1] \text{ in order to make them } t\text{-exact.}\]
We define functors $\omega_T: M^B(\mathbb{A}^2, S_0) \to \text{Vect}_\mathbb{Q}$ $i=1,2$ by the rule:

$$\omega_T: = \Gamma(\mathbb{A}, \Psi_*^{u_{z_1-z_2}} \oplus \Phi_*^{u_{z_2-z_3}})[-1], \quad \omega_T: = \Gamma(\mathbb{A}, \Psi_*^{u_{z_2-z_1}} \oplus \Phi_*^{u_{z_2-z_1}})[-1].$$

By the construction these functors are exact and moreover by Theorem 0.4.1 implies that this functor is faithful and hence it is a fiber functor. Indeed we get the classical $(\Psi, \Phi)$-description of the category of perverse sheaves: we have a morphism $f_T: \mathbb{A}^2 \to \mathbb{A}$ (resp. $f_T: \mathbb{A}^2 \to \mathbb{A}$) defined by the rule $(z_1, z_3) \mapsto (z_1 - z_2)$ (resp. $(z_1, z_2) \mapsto (z_1 - z_2)$). The shifted pushforward defines an equivalence between $M^B(\mathbb{A}^2, S_0)$ and category of perverse sheaves on $\mathbb{A}$, which are smooth with respect to a stratification $\{0\} \subset \mathbb{A}$. The corresponding Tannakian dual coalgebra is a quiver coalgebra of the quiver $\bullet \to \bullet$.

**Example 0.5.2.** Consider a case of $\mathbb{A}^3$ with a coordinate $(z_1, z_2, z_3)$. For example we take the following tree:

![Tree diagram](image)

Let $\mathbb{A}^2$ with a coordinate $(t_1, t_2)$. Consider the morphism $f_T: \mathbb{A}^3 \to \mathbb{A}^2$, defined by the rule $f_T: (z_1, z_2, z_3) \mapsto (z_3 - z_2, z_1 - z_2)$. Denote by $S$ stratification on $\mathbb{A}^2$ associated with hyperplanes $t_1 = t_2, t_1 = 0, t_2 = 0$. The morphism $f_T$ respect these stratification and defines an equivalence of abelian categories $f_{T*}[-1]: M^B(\mathbb{A}^3, S_0) \to M^B(\mathbb{A}^2, S)$. Consider the following quiver (we assume that $u$-morphisms go up and $v$-morphisms go down):

![Quiver diagram](image)

where $V_{ij}$ and $V_{012}$ and $V$ are vector spaces. Building on Theorem 0.4.1 in [Sch93] (see also [Sch92a, Sch92b]) it was proved that the datum of the quiver together with some relations (see ibid.) determines a perverse sheaf in $M^B(\mathbb{A}^2, S)$ and vivre versa. Applying the equivalence above (here we use an interaction property of nearby and vanishing cycles for a pushforward along a proper morphism i.e. $\Psi_f g_* \cong g_* \Psi_g f$) one defines a fiber functor $\omega_T: M^B(\mathbb{A}^3, S_0) \to \text{Vect}_\mathbb{Q}$ by the rule:

$$\omega_T := \Gamma(\mathbb{A}, \Psi_*^{u_{z_1-z_2}} \Psi_*^{u_{z_3-z_2}} \oplus \Phi_*^{u_{z_1-z_2}} \Psi_*^{u_{z_3-z_2}} \oplus \Psi_*^{u_{z_2-z_3}} \Phi_*^{u_{z_2-z_3}} \oplus \Phi_*^{u_{z_2-z_3}} \Phi_*^{u_{z_3-z_2}})[-1].$$

Analogously one defines a fiber functor for any tree $T \in \text{Tree}(3)$.

**Remark 0.5.3.** (i) One extends the definition above to an arbitrary dimension.

Let $T \in \text{Tree}(n)$, we define $\omega_T: M^B(\mathbb{A}^n, S_0) \to \text{Vect}_\mathbb{Q}$ by the rule:

$$\omega_T := \bigoplus_{\Lambda=\Psi^*, \Phi^*} \Gamma(\mathbb{A}, \Lambda_{z_{i_1}-z_{i_2}} \oplus \cdots \oplus \Lambda_{z_{i_1}-z_{i_2}})[-1]$$

where $(i_1, i_2)$ is pair of leaves which collide in the tree $T$ first (we orient a tree towards a root) as the second pair we take leaves $(i_3, i_2)$ which collide...
next (here we assume that $i_3$ is the closest leave to $i_2$. Note that if two pairs collide at the same we do not distinguish the order in the composition, indeed by Lemma 10.2 [BFS98] two compositions are identically equal. Hence (2) is well defined.

(ii) Recall that Theorem 0.4.1 holds for $\mathcal{D}$-modules and more generally mixed Hodge modules [Sai90]. Hence fiber functors (2) can be defined in the mixed Hodge (more generally motivic) setting. It would be very interesting to define and study $\mathcal{M}_{\mathcal{D}}(\mathbb{A}^n, S_0)$ as an object of the category of mixed Hodge structures. Note that this coalgebra is closely related to universal enveloping algebra of P. Deligne’s motivic fundamental group [Del89] at tangential base points.

0.6. Local systems of categories. Further we assume that $n = 1, 2, 3$. Recall that a Fulton–MacPherson compactification $F^n_M(\mathbb{A}^1)$ is defined as a real blowup of the space of $n$ distinct complex points [FM94]. This space is naturally a manifold with corners such that its interior can be identified with $U^n$ (modulo affine transformation). We consider a Betti fundamental groupoid (pro-unipotent completion of the Poincaré groupoid) $\Pi^n_B(\mathcal{F}_M^n(\mathbb{A}^1))$. We have the following:

**Proposition 0.6.1.** We have a $\Sigma$-equivariant equivalence of categories:

$$\Gamma_{\text{cart}}(\text{Loc}_n) \cong M^B(\mathbb{A}^2, S_0), \quad \Gamma_{\text{cart}}(\text{Loc}_3) \cong M^B(\mathbb{A}^3, S_0).$$

**Proof.** We leave it to the reader, however see [KST16] (Subsection 9A) for $n = 2$.

Denote by $\mathcal{G}_n$ a group of automorphisms of $\Pi^n_B(\mathcal{F}_M^n(\mathbb{A}^1))$ which are identical on objects. From Proposition 0.6.1 one gets the following:

**Corollary 0.6.2.** For $n = 1, 2, 3$ we have a canonical action of a group $\mathcal{G}_n$ on $M^B(\mathbb{A}^n, S_0)$.

**Remark 0.6.3.**

(i) Note that $\{\Pi^n_B(\mathcal{F}_M^n(\mathbb{A}^1))\}_{n \geq 1}$ is naturally an operad in the category of groupoids. One shows that $\{\text{Loc}\}$ is naturally a local system on the operad $\{\Pi^n_B(\mathcal{F}_M^n(\mathbb{A}^1))\}_{n \geq 1}$ in the sense of [KG94] and the category of section is equivalent to the category of perverse sheaves on the Ran space.

(ii) Recall that the group of automorphisms of the operad $\{\Pi^n_B(\mathcal{F}_M^n(\mathbb{A}^1))\}_{n \geq 1}$ is the Grothendieck–Teichmüller group $\Gamma_{\text{un}}$ [Fre17]. Hence one proves Hypothesis [2]. It would be very interesting to consider a "derived" version of this picture in particular to relate M. Kontsevich’s graph complex to deformation of the category of $\mathcal{L}$-sheaves on a Ran space of $\mathbb{A}^1$.

0.7. Quantisations. In [Kal19] the problem of quantisation of Lie bialgebras was transformed to the problem of constructing isomorphisms between certain fiber functors. Consider a fiber functor $\omega^H$ from *ibid*. This functor is defined as the
zero cohomology of the smallest real diagonal with coefficients in sections with real support (hyperbolic stalk) see [KS16]. We will discuss the space of isomorphisms between functors $\omega^B$ and $\omega_T$.\footnote{In [FPS22] the same problem was studied in the case of a normal crossing arrangement $z_1 \ldots z_n = 0$.}

Let $\mathcal{S}_{\emptyset, \mathbb{R}}$ be a diagonal stratification of a real $n$-affine space $\mathbb{A}^n_{\mathbb{R}}$, we assume that $(x_1, \ldots, x_n)$ is a coordinates of the real affine space such that $\mathcal{R}(z_i) = x_i$. According to \textit{ibid.} a perverse sheaf is completely determined by a so-called hyperbolic sheaf i.e. a collection of vector spaces $E_C$ where $C \in \mathcal{S}_{\emptyset, \mathbb{R}}$ is a face and operators: $\gamma^C_f : E_f \to E_C$, $\delta^C_f : E_C \to E_f$ when $C \subseteq \overline{C}$ together with some relations (see \textit{ibid.}). The hyperbolic stalks are defined the rule $E_C := \Gamma(C, \mathbb{R}^\ast \mathcal{A}^n_{\mathbb{R}})$. We usually denote chambers $C \in \mathcal{S}_{\emptyset, \mathbb{R}}$ as totally ordered real numbers i.e. $x_1 < x_2 < x_3$, we also sometimes denote by $\Delta^\mathbb{R}$ the minimal diagonal. The following real-analytic interpretation of nearby and vanishing cycles will be important to us:

**Lemma 0.7.1** (M. Kashiwara and P. Schapira [KS90]). For every regular function $f \in \mathcal{O}_X$ we have the following isomorphism of functors:

$$\Phi_f \xrightarrow{i} i^! \mathbb{R}^\ast \Gamma_{\mathcal{R}(f) > 0}[1]$$

where $i : f^{-1}(0) := D \hookrightarrow X$.

Let $X = \mathbb{A}^f$ we denote by $\Phi_f^{\text{fake}}$ (resp. $\Psi_f^{\text{fake}}$) the following functor $i^! \mathbb{R}^\ast \Gamma_{\mathcal{R}(f) > 0}$ (resp. $i^! \mathbb{R}^\ast \Gamma_{\mathcal{R}(f) < 0}$) and called it a fake vanishing (resp. nearby) cycles functor. From the standard Gysin triangle we have the distinguished triangle:

$$(3) \quad \Phi_f[-1]^{\text{fake}} \to i^! \mathbb{R}^\ast \Gamma_{\mathcal{R}(f) > 0} \to \Psi_f^{\text{fake}} \to \Phi_f^{\text{fake}}$$

These functor are equipped with a natural transformations $\Psi_f^{\text{fake}} \to \Psi_f$ and $\Phi_f^{\text{fake}} \to \Phi_f$ (Lemma 0.7.1) which induce equivalences on the sections with support on a real locus and hence $\mathbb{R}^\ast \Gamma(D, \Psi_f^{\text{fake}}) \cong \mathbb{R}^\ast \Gamma(D, \Psi_f)$ and $\mathbb{R}^\ast \Gamma(D, \Phi_f^{\text{fake}}) \cong \mathbb{R}^\ast \Gamma(D, \Phi_f)$ [FKS21].

**Example 0.7.2.** Consider the case $\mathbb{A}^2$ and a fiber functor $\omega_{T_1}$. Following [KS16] Subsection 9A we have:

$$\Gamma(\mathbb{A}, (\Phi_{z_1 - z_2} \oplus \Phi_{z_1 - z_2}) \cong \Gamma(\mathbb{A}, i^! \mathbb{R}^\ast \Gamma_{x_1 < x_2} \oplus i^! \mathbb{R}^\ast \Gamma_{x_1 > x_2})[1]$$

Applying (3) we get a morphism from a functor $\omega^B$ to a functor $\omega_{T_1}$. One shows (see \textit{ibid.}) that this is an equivalence.

**Example 0.7.3.** Consider $\mathbb{A}^3$ with a binary tree $T$ from Example 0.5.2 Applying base change one easily computes that $V := E_{x_1 < x_3 < x_2}$. Namely denote by $\mathcal{R}(z_3) < \mathcal{R}(z_2)$ the locus $\mathbb{A}^3$ which consists of real numbers $(x_1, x_2, x_3)$ such that $x_3 < x_2$. Consider the following diagram:

$\begin{array}{ccc}
\mathcal{R}(z_3) < \mathcal{R}(z_2) & \xrightarrow{\nu} & \mathbb{A}^3 \\
\downarrow q & & \downarrow i_1 \\
\mathcal{R}(z_3) = \mathcal{R}(z_2) > \mathcal{R}(z_1) & \xrightarrow{r} & \mathbb{A}^2 \\
\downarrow l & & \downarrow i_2 \\
\mathbb{A}^2 & \xrightarrow{p} & \mathbb{A}_{z_2 = z_2} \\
\downarrow k & & \downarrow i_2 \\
\mathbb{A}^1 & \\
\end{array}$
\( \Psi^{fake}_{z_1 - z_2} := R^* i^* h_* v_* v^! h^! \) and \( \Phi^{fake}_{z_1 - z_2} := R^* p_* q^* v_* v^! h^!\). Hence
\[\Psi^{fake}_{z_1 - z_2} \Phi^{fake}_{z_3 - z_2} = R^* k_* l^* r_* v^* v^! h^! .\]
Note that \( h^! \) is an exact functor (see [KST16]) (it takes perverse sheaves to combinatorial sheaves on the real affine space \( A^3_R \)). Hence it is enough to compute the \( *\)-extension of the corresponding combinatorial sheaves: let \( K = v^* h^! E, E \in M^B(A^3, \mathcal{S}_0) \), be a combinatorial sheaf on \( \mathcal{R}(z_3) < \mathcal{R}(z_2) \) (we denote the corresponding combinatorial data (section over faces) by \( E_c \), where \( C \in \mathcal{S}_0 \)). We are interested in sections of \( R^* v_* K \) over faces which have a non empty intersection with an image of \( q \). These chambers are \( \{x_1 < x_3 = x_2 \} \) and \( \{x_1 > x_3 = x_2 \} \). Moreover since further we take a pullback along \( r \) it is enough to consider \( \{x_1 < x_3 = x_2 \} \). To compute \( \Gamma(\{x_1 < x_3 = x_2 \}, R^* v_* K) \) we need to take sections over chambers whose closure contains \( \{x_1 < x_3 = x_2 \} \) and have a non empty intersection with \( \mathcal{R}(z_3) < \mathcal{R}(z_2) \). This chamber is \( \{x_1 < x_3 < x_2 \} \), hence \( V = E_{x_1 < x_3 < x_2} \).

We have \( \Phi^{fake}_{z_1 - z_2} := R^* i^* j^* j^! = R^* p^* q^* j^! \) and hence \( \Psi^{fake}_{z_1 - z_2} \Phi^{fake}_{z_3 - z_2} = R^* k_* l^* r_* q^* j^! \).
Let us compute section of \( q^* j^! \) over a chamber \( \{x_3 = x_2 > x_1 \} \). Let \( K \) be a combinatorial sheaf on \( A^3_R \) (which is a \( !\)-restriction of a perverse sheaf) we need to compute its sections with support \( \mathcal{R}(z_3) \geq \mathcal{R}(z_2) \) over faces which have a non empty intersection with \( \mathcal{R}(z_3) \geq \mathcal{R}(z_2) \). Hence the cohomology of the following complex
\[ C^* := \{E_{x_3 = x_2 > x_1} \oplus E_{x_3 > x_2 > x_1} \overset{\gamma + \gamma + id}{\longrightarrow} E_{x_3 > x_2 > x_1} \oplus E_{x_3 > x_2 > x_1} \}. \]
Since we are working with a perverse sheaf this complex has only cohomology in degree zero \( H^0(C) = Ker(E_{x_2 = x_3 > x_1} \rightarrow E_{x_2 > x_3 > x_1}) \). We set \( V_{01} := Ker(E_{x_2 = x_3 > x_1} \rightarrow E_{x_2 > x_3 > x_1}) \).

Consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{R}(z_3) \geq \mathcal{R}(z_2) & \xrightarrow{j} & A^3 \\
\downarrow{q} & & \downarrow{i_1} \\
\mathcal{R}(z_3) = \mathcal{R}(z_1) \geq \mathcal{R}(z_2) & \xrightarrow{r} & A^2_R \xrightarrow{i} A^2_{z_3 = z_2} \\
\downarrow{q} & & \downarrow{i_2} \\
A_R & \xrightarrow{k} & A^1 \\
\end{array}
\]

By previous computations we have \( \Phi^{fake}_{z_1 - z_2} \Phi^{fake}_{z_3 - z_2} := R^* k_* s^* r^* q^* j^! \). Let \( K \) be a combinatorial sheaf (again a \( !\)-restriction of a perverse sheaf) on \( A^3_R \) we need to compute its sections with support \( \mathcal{R}(z_3) \geq \mathcal{R}(z_2) \) (we restrict ourselves to chamber \( \Delta^R \)).
To compute sections over $\Delta^R$ we perform computation analogous to the previous case and find that it is equal:

$$A := \text{Ker} \left( \begin{array}{c} E_{x_1=x_2=x_3} \\ \oplus \gamma \end{array} \right) \bigoplus_{C \in A_3^3 \setminus \mathfrak{R}(z_3) \geq \mathfrak{R}(z_2)} \text{dim } C = 2 \bigoplus E_C .$$

Hence we set $V_{012} := A$.

Consider the following diagram:

```
\begin{tikzpicture}
  \node (A1) at (0,0) {$A_2^3$};
  \node (A2) at (2,0) {$A_1^3$};
  \node (A3) at (0,-1) {$A_2^3$};
  \node (A4) at (2,-1) {$A_1^3$};
  \node (A5) at (0,-2) {$A_2^3$};
  \node (A6) at (2,-2) {$A_1^3$};
  \node (A7) at (1,-2.5) {$H^0(C)$};
  \node (A8) at (1,-3) {$A$};

  \draw[->] (A1) -- (A2) node[midway,above] {$q$};
  \draw[->] (A3) -- (A4) node[midway,above] {$i_1$};
  \draw[->] (A5) -- (A6) node[midway,above] {$i_2$};
  \draw[->] (A1) -- (A5) node[midway,left] {$s$};
  \draw[->] (A2) -- (A6) node[midway,left] {$r$};
  \draw[->] (A3) -- (A4) node[midway,left] {$j$};
  \draw[->] (A1) -- (A7) node[midway,above] {$d$};
  \draw[->] (A2) -- (A7) node[midway,above] {$i$};
  \draw[->] (A3) -- (A7) node[midway,below] {$k$};
  \draw[->] (A4) -- (A7) node[midway,below] {$l$};
  \draw[->] (A5) -- (A7) node[midway,below] {$m$};
  \draw[->] (A6) -- (A7) node[midway,below] {$n$};
\end{tikzpicture}
```

By previous computations we have $\Phi_{f a k e} := \mathbb{R}^* k_* s^* r^* q^* d_* j^!$. In order to compute the iterated cycles we need to find sections of a sheaf $d_* d^* j^! \mathcal{E}$ over chambers $\{x_3 = x_1 > x_2\}$, $\{x_3 = x_1 < x_2\}$, and $\{x_1 = x_2 = x_3\}$. Over the first chamber are trivial since there are no chambers in $\mathfrak{R}(z_3) < \mathfrak{R}(z_2)$ such that their closure contains this chamber. Sections over the second chamber are given by the vector space $E_{x_3<x_1<x_2}$, since this chamber lie in the space $\mathfrak{R}(z_3) < \mathfrak{R}(z_2)$. Lets compute section over the minima chamber: we need to calculate the $*$-extension of the sheaf $d^* j^! \mathcal{E}$, applying standard methods one get the following vector spaces:

$$E_{x_3<x_1<x_2}, \quad E_{x_3=x_1<x_2}, \quad E_{x_3=x_1=x_2} .$$

Acting like before we finally set:

$$V_{12} \oplus V_{02} := E_{x_3<x_1<x_2} \oplus E_{x_3<x_1=x_2} .$$

Hence we have the following quiver:

```
\begin{tikzpicture}
  \node (A1) at (0,0) {$E_{x_1<x_3<x_2}$};
  \node (A2) at (0,-1) {$E_{x_3=x_1<x_2}$};
  \node (A3) at (0,-2) {$E_{x_3<x_1=x_2}$};
  \node (A4) at (2,0) {$E_{x_3<x_1<x_2}$};
  \node (A5) at (2,-1) {$E_{x_3=x_1<x_2}$};
  \node (A6) at (2,-2) {$E_{x_3<x_1=x_2}$};
  \node (A7) at (1,-2.5) {$H^0(C)$};
  \node (A8) at (1,-3) {$A$};

  \draw[->] (A1) -- (A2) node[midway,above] {$u_{12}$};
  \draw[->] (A2) -- (A3) node[midway,above] {$u_{12}$};
  \draw[->] (A3) -- (A4) node[midway,above] {$u_{12}$};
  \draw[->] (A4) -- (A5) node[midway,above] {$u_{12}$};
  \draw[->] (A5) -- (A6) node[midway,above] {$u_{12}$};
  \draw[->] (A6) -- (A1) node[midway,above] {$u_{12}$};
  \draw[->] (A7) -- (A1) node[midway,above] {$w_{12}$};
  \draw[->] (A7) -- (A2) node[midway,above] {$w_{12}$};
  \draw[->] (A7) -- (A3) node[midway,above] {$w_{12}$};
  \draw[->] (A7) -- (A4) node[midway,above] {$w_{12}$};
  \draw[->] (A7) -- (A5) node[midway,above] {$w_{12}$};
  \draw[->] (A7) -- (A6) node[midway,above] {$w_{12}$};
\end{tikzpicture}
```

We leave it to the reader to determine canonical and variation operators.

**Remark 0.7.4.** Recall that in [Kal19] we consider 'de Rham' fiber functor $\omega^{dR}$ for $\mathcal{D}$-modules. One can also construct an isomorphism between fiber functor $\omega^{dR}$ and the de Rham version of a functor $\omega_T$. One can summarise by saying that there is a 'canonical' Betti functor $\omega^B$ and a 'canonical' de Rham functor $\omega^{dR}$. A functor $\omega^B$ (resp. $\omega^{dR}$) is responsible for associative (resp. Lie) bialgebras and a problem of quantization [Dr92] transfers to establishing an isomorphism between these two functors. The latter construction passes through 'non-canonical' fiber functors $\omega_T$. 
These functors have an advantage being "motivic" in contrast to functors $\omega^{dR}$ and $\omega^B$.

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