Alcuin’s Propositiones de Civitatibus: 
the Earliest Packing Problems

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Abstract

We consider three problems about cities from Alcuin’s Propositiones ad acuendos juvenes. These problems can be considered as the earliest packing problems difficult also for modern state-of-the-art packing algorithms. We discuss the Alcuin’s solutions and give the known (to the author) best solutions to these problems.

1 Introduction

The manuscript Propositiones ad acuendos juvenes (Problems to Sharpen the Young) attributed to Alcuin of York is considered as the earliest collection of mathematical problems in Latin [5].

In Alcuin’s Opera Omnia published by Frobenius Forster [1] and revised and republished by J.-P. Migne [7] there is a version of Propositiones containing 53 problems. Another version of Propositiones with three additional problems was published in Venerable Bede’s Opera Omnia [6]; it was revised and republished by J.-P. Migne [8]. The attribution of this version to Bede is considered to be spurious. A modern edition of Propositiones was made by Menso Folkerts [2]. He found 12 manuscripts, the oldest one is dating from the late 9th century. It is incomplete, but already contains the three additional problems of Bede’s text.

There are two translations of Propositiones into English. One of them was made by John Hadley with commentary by David Sigmaster and John Hadley [5]. A literal translation with commentary was made by Peter J. Burkholder [1]. Also, there is a German translation by Helmhuth Gericke with commentary by Menso Folkerts and Helmhuth Gericke [3].

According to Singmaster, Propositiones contain “first River Crossing Problems (3 types); first Explorer’s Problem; first Division of Casks; first Apple-sellers’ Problem; first Collecting Stones; unusual solution of Posthumous Twins Problem; first Three Odds Make an Even; first Strange Families” [9].

On the other hand, some geometric problems from Propositiones with probably Roman origin use rather crude approximations (for example, some of them require $\pi = 4$). Nevertheless certain geometric problems are of particular interest.

In Propositiones there are 11 geometric problems, IX, X, XXI–XXV, XXVII–XXI, dealing with the area of figures (triangles, rectangles, circles and quadrangles). Moreover, problems XXVII–XXVIII are formulated as packing problems. So they can be considered as the earliest packing problems. They are all about cities (of different shapes) in which it is required to put rectangular houses of equal known sizes. Problem XXVIII dealing with triangular city is easier than XXVII and XXVIIII about quadrangular and round cities, respectively. And the formers are difficult even for modern state-of-the-art packing algorithms. We give the original problem formulations and solutions of the problems with the translation by Burkholder [1] and give the best known solutions.

2 Propositiones de civitatibus (problems about cities)

2.1 Quadrangular city problem

XXVII. Propositio de civitate quadrangula.

Est civitas quadrangula quae habet in uno latere pedes mille centum; et in alio latere pedes mille; et in fronte pedes $d_c$, et in alio pedes $d_c$. Volo ibidem tecta $600$ feet, et in altera pedes $d_c$. Dicat, qui velit, quot casas capere debet?

Solutio. Si fuerunt duas hujus civitatis longitudines junctae, facient $2100$ feet. Similiter duae, si fuerunt longitudines junctae, facient $2100$ feet.

27. Proposition concerning the quadrangular city. There is a quadrangular city which has one side of 1100 feet, another side of 1000 feet, a front of 600 feet, and a final side of 600 feet. I want to put some houses there so that each house is 40 feet long and 30 feet wide. Let him say, he who wishes, How many houses ought the city to contain?

Solution. If the two lengths of this city were joined together, they would measure 2100 [feet]. Likewise, if
latitudines junctae, faciunt $1200 \text{ cc}$. Ergo duc mediam de $1200 \text{ cc}$, faciunt $600 \text{ cc}$, rursusque duc mediam de $2100 \text{ cc}$, faciunt $1050 \text{ cc}$. Et quia unaquaque domus habet in longitude $40 \times 30 = 1200 \text{ feet}$, take a fourtieth part of $1200$, making $26$. Therefore, take half of $1200$, i.e. $600$, and half of $2100$, i.e. $1050$. Because each house is $40$ feet long and $30$ feet wide, take a fourtieth part of $1050$, making $26$.

Then, take a thirtieth of $600$, which is $20$. $20$ times $26$ is $520$, which is the number of houses to be contained in the city.

Alcuin uses the Egyptian (or Roman) formula for calculating the area of a quadrilateral

$$S = \frac{a + c}{2} \cdot \frac{b + d}{2},$$

where $a$, $b$, $c$, $d$ are length of four sides of the quadrilateral (the product of the half sum of its opposite sides). According it, $S = 1050 \times 600 = 630,000 \text{ sq ft}$, that is equal to $525$ house areas. If the shape of the city is an isosceles trapezium then its area is $627,808.689 \ldots \text{ sq ft}$, that is equal to $523.174 \ldots \text{ house areas}$.

Seemingly Alcuin is not interested in whether the city really could contain the number of houses indicated by him, though discarding the fractional part in dividing $1050$ by $40$ could purpose to take into account this.

Singmaster [5] notes that he can get $516$ houses in (if the shape of the city is an isosceles trapezium); but “by turning the houses the other way” he can get $517$ in. “By having some houses either way” he can get $519$ in. I don’t know these solutions and I can fit only $510$ houses in; see Fig. 1.

2.2 Triangular city problem

XXVIII. Propositio de civitate triangula. Est civitas triangula quae in uno habet latere pedes $c$, et in alio latere pedes $c$, et in fronte pedes $x$, volo enim ibidem aedificia domorum construere [Bede: Volo ut fiat ibi domorum constuctio . . .], sic tamen, ut unaquaque domus habeat in longitudine pedes $x$, et in latitudine pedes $x$. Dicat, qui potest, quot domus capi debent?

Solutio. Duo igitur hujus civitatis latera juncta fiunt $cc$, atque duc medium de $cc$, fiunt $c$. Sed quia in fronte habet pedes $x$, duc median de $xc$, fiunt $xv$. Et quia longitudo uniuscujusque domus habet pedes $xx$, et latitudo ipsarum pedes $x$, duc $xx$ partem in [Bede: de] $c$, fiunt $v$. Et pars decima quadragerannii $iv$ sunt. Duc itaque quinquies $iii$, fiunt $xx$. Tot domos hujusmodi captura [Bede: capienda] est civitas.

28. Proposition concerning the triangular city. There is a triangular city which has one side of $100$ feet, another side of $100$ feet, and a third of $90$ feet. Inside of this, I want to build a structure of houses, however, in such a way that each house is $20$ feet in length, $10$ feet in width. Let him say, he who can, How many houses should be contained [within this structure]?

Solution. Two sides of the city joined together make $200$; taking half of $200$ makes $100$. But because the front is $90$ feet, take half of $90$, making $45$. And since the length of each house is $20$ feet while the width is $10$, take $20$ into $100$, making five. A tenth part of $40$ is four; thus, take four five times, making $20$. The city is to contain this many houses in this way.
Figure 2: Problem XXVIII. One can put 16 houses in the triangular city.

For calculation the area Alcuin uses the Egyptian formula that for triangles has the form

$$S = \frac{a + b}{2} \cdot \frac{c}{2}.$$

According it, the area of the city must be $100 \times 45 = 4500$ sq ft, i.e. $22\frac{1}{2}$ house areas. The correct value of the area is $4018\frac{6}{7}$, i.e. $20\frac{0}{9}$ house areas. As Singmaster notes [5], “The conversion of 45 to 40 may be an attempt to compensate for the inaccuracy of the formula, or to allow for the difficulty of fitting rectangular houses into the triangular town.” Singmaster can get 15 houses in. Hadley can get 18 houses in if “the walls can be bent slightly” [5]. Here we give a solution with 16 houses (and with straight walls). It is shown in Fig. 2.

Note that there is a small gap between houses 7 and 15. Also, there is a gap between houses 8 and 16 or/and between houses 14 and 16.

2.3 Round city problem

XXVIII. Propositio de civitate rotunda. Est civitas rotunda quae habet in circuitu pedum VIII millia. Dicit, qui potest, quot domos capere debet, ita ut unaquaeque habeat in longitudine pedes xxx, et in latitudine pedes xx?

Solutio. In hujus civitatis ambitu VIII millia pedum numerantur, qui sesquialtera proportione dividuntur in XXXX dccc, et in IIII CC. In illis autem longitudine domorum, in istis latitudine versatur. Subtrahe itaque de utraque summa medietatem, et remanent de majori Pi CCC: de minore vero I DC. Hos igitur I DC divide in vicenos et invenies octagies viginti, rursusque majora summa, id est Pi CCCC, in XXX partitii, octagies triginta diunumerantur. Duc octagies LXXX, et fiunt VI millia CCCC. Tot in hujusmodi civitate domus, secundum propositionem supra scriptam, construi [Bede: constitui] possunt.

As Burkholder notes [1], Alcuin replaces the original round city by a rectangular city with the same circumference 8000 feet, whose sizes are proportional to the house sizes. The rectangle obtained has sizes $2400 \times 1600$ and contains $6400$ houses. The shortcoming here is that closed curves of equal lengths can enclose different areas.

Folkerts [2] gives a second solution:

Ambitus huius civitatis viii complectitur pedum. Duc ergo quartam de viii partem, fiunt Pi. Rursusque duc tertiam de viii partem, fiunt Pi dclxv. Duc vero medium de duobus milibus, fiunt I, atque iterum de

29. Proposition concerning the round city. There is a city which is 8000 feet in circumference. Let him say, he who is able, How many houses should the city contain, such that each [house] is 30 feet long, and 20 feet wide?

Solution. This city measures 8000 feet around, which is divided into proportions of one-and-a-half to one, i.e. 4800 and 3200. The length and width of the houses are to be of these [dimensions]. Thus, take half of each of the above [measurements], and from the larger number there shall remain 2400, while from the smaller, 1600. Then, divide 1600 into twenty [parts] and you will obtain 80 times 20. In a similar fashion, [divide] the larger number, i.e. 2400, into 30 pieces, deriving 80 times 30. Take 80 times 80, making 6400. This many houses can be built in the city, following the above-written proposal.
Figure 3: Problem XXVIII. One can put 8349 houses in the round city.
duobus milibus dclxvi medium assume partem, fiunt 1 cccxxxvi. Deinde duce partem tricesimam de 1 cccxxxvi, fiunt 2 cccxxxvi, rursusque duce partem vigesimam de 1, fiunt 3 cccxxxvi. Duc vero quinquagesam cccxxxvi, total number of houses.

Here the following formula for the area of a circle is used,

\[ S = \frac{\ell^2}{12}, \]

where \( \ell \) is the circumference of the circle. The formula requires \( \pi = 3 \). In our case the area could be 5,333,333.33\ldots sq ft, i.e. 8888.889\ldots house areas. The correct area of the circle is

\[ S = \frac{\ell^2}{4\pi} = 5,092,958.179\ldots \text{ sq ft}, \]

i.e. 8488.264\ldots house areas.

Singmaster has fitted in 8307 houses [5] but he notes that “it is probably possible to fit more in”. I found a solution with 8349 houses; see Fig. 3 There are 8342 houses put “horizontally” plus 7 houses rotated.

References

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