Fluid/gravity correspondence for general non-rotating black holes

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Abstract

In this paper, we investigate the fluid/gravity correspondence in spacetime with general non-rotating weakly isolated horizon. With the help of a Petrov-like boundary condition and large mean curvature limit, we show that the dual hydrodynamical system is described by a generalized forced incompressible Navier–Stokes equation. Specially, for stationary black holes or those spacetime with some asymptotically stationary conditions, such a system reduces to a standard forced Navier–Stokes system.

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1. Introduction

The correspondence between gravity and fluid was found in the 1970s of the last century. Damour [1] firstly noticed that there was a formal correspondence between the Raychauduri equations and fluid equations. He suggested that the black-hole horizon could be viewed as a membrane of fluid. Such idea had been studied by other researchers [2] and this method was called ‘membrane paradigm of black hole’ [3]. These pieces of work imply that there should be some relations between the gravitational perturbation of the black-hole horizon and the dynamics of a fluid membrane.

More than 20 years later, such a topic has been reconsidered in the framework of the AdS/CFT correspondence. Based on the AdS/CFT correspondence, the gravity in bulk should be dual to some quantum field theory on the boundary of spacetime. Under certain conditions,
any quantum field theory can be effectively described by hydrodynamics at the long wavelength limit. So there should be a natural correspondence between the dynamics of long wavelength gravitational perturbation and hydrodynamics. Such correspondence was firstly studied by Policastro et al [4]. They considered the hydrodynamics of super-symmetric gauge theory on the conformal boundary of spacetime and found the similar properties as found by Damour on the black-hole horizon. Following this idea, many important pieces of work have been done and there is a nice review on this topic [5]. This so-called fluid/gravity duality was further deepened by Bhattacharyya et al to a full correspondence involving the nonlinear fluid dynamics [6]. The similarity between the fluid/gravity duality in the AdS/CFT framework and the black-hole membrane indicates that there should be some relations between them. Indeed, this kind of universality is considered in [7], and is later interpreted as the Wilson renormalization group flow in the AdS/CFT framework [8]. Some further developments following this approach can be found in [9–11].

The membrane paradigm has been generalized to some asymptotic AdS black holes as well. Kovtun et al first found that long-time, long-distance fluctuations of plane-symmetric horizons exhibit universal hydrodynamic behavior [12]. Later, Eling, Oz and their colleagues (EO) found [13], with the non-relativistic limit and long wavelength limit, that the gravitational perturbation of black brane satisfies the Navier–Stokes equation. The EO method is a quite nice way to establish the membrane paradigm of horizon because it is a local method which only requires the geometric information near horizon but does not need the knowledge of the asymptotic region. But this approach contains some drawbacks. As emphasized by them, the long wavelength limit plays a crucial role in this method; thus, it can only be applicable to the spatially non-compact case, typically in which the horizon is plane-symmetric. Some improvement has been made recently. In [14], Lysov and Strominger (LS) studied the fluid/gravity correspondence with the help of the Petrov-like boundary condition in a flat spacetime. They found that, with the help of the Petrov-like boundary condition and large mean curvature limit (which is equivalent to the near horizon limit), the correspondence between the Einstein equation and the Navier–Stokes equation can be established. The main idea of their work is as follows. First, they consider the perturbations of the extrinsic curvature $K_{ab}$ of a hypersurface $\Sigma_c$ and impose a Petrov-like boundary condition for Weyl curvature on $\Sigma_c$. Since the Weyl tensor is traceless, the Petrov-like boundary condition provides $\frac{p(p+1)}{2} - 1$ constraints on the $(p+1)(p+2)/2$ components of $K_{ab}$. The Gaussian equation also gives another constraint equation for the perturbation of $K_{ab}$, so the remaining $p+1$ independent components of $K_{ab}$ may be interpreted as the velocity field $v^i$ and the pressure $P$ of fluid living on this hypersurface. The Gaussian equation on $\Sigma_c$ can be viewed as an equation of state for this fluid and the Codazzi equations on $\Sigma_c$ will give the evolution equation of this fluid. Then, taking a suitable non-relativistic limit and a large mean curvature limit (near the horizon limit), the Codazzi equations have the form of incompressible Navier–Stokes equations. In [9], Strominger and his colleague have shown that the long wave perturbation solution of gravity satisfies the Petrov-like boundary condition. This implies that we can use the Petrov-like boundary condition instead of the long-wave condition to evade the difficulty of the compactness of the hypersurface. Like EO’s method, LS’s method is also a local method in linking the Einstein equation and the Navier–Stokes equation. Since in this approach one only imposes the Petrov-type boundary condition and the near horizon limit rather than the long wavelength limit, the requirement of the spatial uncompactness for the hypersurface becomes unnecessary. Some generalizations of Strominger’s work have been considered. Reference [15, 16] generalized this framework to some curved cases and the cases with a cosmological constant, respectively. Reference [17] further considered the situation of spacetime with an electromagnetic field.
It is worthwhile to point out that up to now the fluid/gravity correspondence in this route has only been investigated in some concrete background such as static black-hole solutions in the literature. However, based on the accumulated experience in the study of AdS/CFT, one has reasons to believe that such a correspondence should be a quite general notion. At least, it should be applicable to general stationary black holes, so a general proof for such correspondence in a more general setting is needed. This paper will focus on this topic. We will show that for any spacetime which contains a non-rotating weakly isolated horizon (WIH), LS’s realization of the fluid/gravity correspondence can always be established.

This paper is organized as follows. In section 2, the definition of the WIH is introduced and the near horizon geometry is studied. In section 3, we introduce the Petrov-like condition for gravity and analyze its detailed behavior under the near horizon limit. All results are summarized into two lemmas. The equivalence between Strominger’s large mean curvature limit and near horizon limit has been addressed and the Gaussian equation has also been considered in this section. Because the proof of these two lemmas are little longer, we put them into appendices to make the paper more clear. In section 4, we derive the incompressible Navier–Stokes equation from the Codazzi equation for gravity. Section 5 contains some discussions of our work.

2. The geometry near non-rotating isolated horizon

In order to consider the existence of fluid/gravity correspondence for a general black hole, a general definition of the black-hole horizon is needed. The WIH advocated by Ashtekar et al is a nice choice [18]. This sort of horizon preserves many important properties of the traditional black-hole horizon, but is applicable to more general cases. It has been shown that the stationary black-hole horizon is all WIH [18, 19]. Roughly speaking, the WIH is a non-expansion light cone with almost stationary inner geometry. A nice review of the WIH can be found in [18]. In [19], Ashtekar’s definition has been generalized to higher dimensional spacetime. The definition of WIH in \((p + 2)\)-dimensional spacetime is given as follows [19].

**Definition 1.** (WIH in \((p + 2)\)-dimensional spacetime) Let \((M, g)\) be a \((p + 2)\)-dimensional Einstein manifold with or without a cosmological constant. \(\mathcal{H}\) is a \((p + 1)\)-dimensional null hypersurface in \(M\) and \(l\) is the null normal of \(\mathcal{H}\). \(\mathcal{H}\) is called a WIH in \(M\) if

1. there exists an embedding \( : S \times [0, 1] \to M\), \(\mathcal{H}\) is the image of this map, \(S\) is a \(p\)-dimensional compact, connected manifold and for every maximal null curve in \(\mathcal{H}\) there exists \(x \in S\) such that the curve is the image of \(x \times [0, 1]\);
2. the expansion of \(l\) vanishes everywhere on \(\mathcal{H}\);
3. \(\mathcal{R}_{ab}l^a l^b|\mathcal{H}\) = 0;
4. let \(\mathcal{D}\) denote the induced connection on \(\mathcal{H}\), \((\mathcal{L}_l, \mathcal{D})|l = 0\) holds on \(\mathcal{H}\).

To generalize fluid/gravity correspondence to the WIH case, the geometry near horizon is needed. So in this section, we will get some control on the behavior of metric near horizon. Suppose \(\mathcal{H}\) is a WIH in a \((p + 2)\)-dimensional spacetime, based on the definition of [18, 19]. To control geometry near a null hypersurface, a Bondi-like coordinate system is always a convenient choice [20]. Let \(l\) be the tangent vector of a null generator of \(\mathcal{H}\) with a parameter \(t\). The level set of \(t\) in \(\mathcal{H}\) gives a space-like foliation \(\{S_t\}\) of \(\mathcal{H}\). On section \(t = 0\), there are coordinates \(\{x^i\}\). The generator of horizon will bring these coordinates to the whole horizon. On each section \(S_t\), let \(\{E_i\}\) be a set of orthonormal basis for the tangent space of \(S_t\). Under some suitable rotation, \(\{E_j\}\) can always be chosen such that \((\mathcal{L}_l E_j)|\mathcal{H} \propto E_j\). Therefore,
we have a tetrad \((l, E_i)\) at each point of \(\mathcal{H}\). Then choose a past-pointed null vector field \(n\) on \(\mathcal{H}\) such that \(\langle l, n \rangle = 1\) and \(\langle n, E_i \rangle = 0\). It is easy to see that \(n\) is unique at each point of \(\mathcal{H}\). For any \(p \in \mathcal{H}\), there exists a unique null geodesic \(\gamma\) with respect to \(n\). The affine parameter of \(\gamma\) is \(r\) and \(\gamma(0) = p\). One can extend coordinates \((t, r, x')\) into spacetime by Lie dragging them along \(\gamma\), and \((t, r, x')\) is a coordinates’ system near horizon. Similarly, one can also extend the null tetrad \((l, n, E_i)\) into the spacetime by requiring \(\nabla_n (l, n, E_i) = 0\). We call coordinates \((t, r, x')\) as the Bondi-like coordinates near horizon \(\mathcal{H}\) and the above chosen tetrad as the Bondi-like tetrad near horizon. With this choice of Bondi-like coordinates, the Bondi-like tetrad can be expressed as

\[
\begin{align*}
n &= \partial_r, \\
l &= \partial_t + U \partial_r + X^i \partial_i, \\
E_i &= W_i \partial_r + e^i_j \partial_j,
\end{align*}
\]

where \((U, X^i, W_i, e^i_j)\) are the functions of \((t, r, x')\) and satisfy \(U \equiv X^i \equiv W_i \equiv 0\). (Note. Here we follow the notation of [18]. * \(\equiv\) * means equality holds only on horizon \(\mathcal{H}\). In the following, we also use \(f\) to denote the value of function \(f\) on horizon.) Using the null tetrad, the metric can be expressed as \(g^{ab} = \pi^n_0 (n^b + n^a t^b + E_i^b e^j_i)\); then, components of the metric are

\[
(g^{\mu \nu}) = \begin{pmatrix}
0 & 1 & 0 \\
1 & 2U + W_i W_i & X^i + W_i e^i_j \\
0 & X^i + W_i e^i_j & e^i_j e^j_i
\end{pmatrix}.
\]

With this Bondi gauge, it is also easy to see that the following relation holds:

\[
\alpha_I := -\langle l, \nabla n \rangle, \quad \pi_I := \langle E_i, \nabla n \rangle
\]

\[
\alpha_I + \pi_I = 0.
\]

Based on the discussion in [18, 19], \(\pi_j\) is related with the angular momentum of horizon. In this paper, we consider non-rotating black holes, so we require the non-rotating condition : \(\pi_j \equiv 0\).

In order to obtain the behavior of metric near horizon, let us consider the Cartan structure equations,

\[
[n, E_l] = \frac{\partial W_l}{\partial r} \partial_r + \frac{\partial e^i_j}{\partial r} \partial_i \\
= \alpha_l \partial_r - \theta^j_l (W_j \partial_r + e^j_i \partial_i),
\]

\[
[n, l] = \frac{\partial U}{\partial r} \partial_r + \frac{\partial X^i}{\partial r} \partial_i \\
= \varepsilon \partial_r - \pi_I (W_I \partial_r + e^i_j \partial_j).
\]

Above equations imply

\[
\frac{\partial W_l}{\partial r} = \alpha_l - \theta^j_l W_j, \quad \frac{\partial e_j^i}{\partial r} = -\theta^j_l e^i_j, \quad \frac{\partial U}{\partial r} = \varepsilon - \pi_I W_I, \quad \frac{\partial X^i}{\partial r} = -\pi_I e^i_j.
\]

where \(\theta^j_l := \langle E_l, \nabla n \rangle, \varepsilon := \langle n, \nabla l \rangle\). Based on the discussion of [18, 19], \(\varepsilon |_{\mathcal{H}}\) is a constant and just the surface gravity of horizon \(\mathcal{H}\). Combining with equation (2), we obtain the first-order derivative of the metric.

To control the order of the metric more accurately, the second derivation of metric is needed. Taking the \(r\)-derivative on both sides of equations in (5),
\[
\frac{\partial^2 U}{\partial r^2} = \frac{\partial \varepsilon}{\partial r} - \frac{\partial \pi_I}{\partial r} W_I - \frac{\partial W_I}{\partial r}, \\
\frac{\partial^2 W_I}{\partial r^2} = \frac{\partial \alpha_I}{\partial r} - \frac{\partial \theta_{II}}{\partial r} W_J - \frac{\partial W_J}{\partial r}, \\
\frac{\partial^2 X^i}{\partial r^2} = -\frac{\partial \pi_I}{\partial r} e_I^i - \frac{\partial e_I^i}{\partial r} \pi_I, \\
\frac{\partial \varepsilon}{\partial r} = \nabla_n \varepsilon = \nabla_n [n, \nabla l] = R_{nln} - \alpha_l \pi_l. 
\]

(6)

so the values of $\partial_r \varepsilon$, $\partial_r \pi_I$, $\partial_r \alpha_I$ and $\partial_r \theta_{JI}$ are needed. With the help of second Cartan structure equations,

\[
\partial_r \varepsilon = \nabla_n \varepsilon = \nabla_n [n, \nabla l] = R_{nln} - \alpha_l \pi_l. 
\]

(7)

Similarly, we have

\[
\partial_r \pi_I = R_{nln} - \theta_{IJ} \pi_J, \\
\partial_r \alpha_I = R_{nln} - \alpha_l \theta_{JI}, \\
\partial_r \theta_{IJ} = R_{nln} - \theta_{JK} \theta_{IJ}. 
\]

(8)

Plugging these equations into (6), we have

\[
\frac{\partial^2 U}{\partial r^2} = R_{nln} - R_{nln} W_I - 2 \alpha_l \pi_I + 2 \theta_{IJ} \pi_I W_I, \\
\frac{\partial^2 W_I}{\partial r^2} = R_{nln} - R_{nln} W_J - \alpha_I (\theta_{II} + \theta_{II}) + W_J \theta_{IJ} (\theta_{II} + \theta_{II}), \\
\frac{\partial^2 X^i}{\partial r^2} = -R_{nln} e_I^i + 2 \theta_{IJ} \pi_I e_I^i. 
\]

(9)

Using the non-rotating condition $\pi_I = 0$, all unknown functions near horizon are

\[
U = \hat{\varepsilon} r + \frac{1}{2} \hat{R}_{nln} r^2 + O(r^3), \\
W_I = \frac{1}{2} \hat{R}_{nln} e_I^2 + O(r^3), \\
X^i = \frac{1}{2} \hat{R}_{nln} e_I^2 + O(r^3), \\
e_I^i = \hat{e}_I^i - \theta_{IJ} \hat{e}_J^i + O(1). 
\]

(10)

Finally, the asymptotic extensions of the metric near horizon are

\[
g^{rr} = 1, \quad g^{ij} = 0, \\
g^{rr} = 2U + \sum W_I^2 = 2 \hat{\varepsilon} r + \hat{R}_{nln} r^2 + O(r^3), \\
g^{ij} = X^i + \sum W_I e_I^i = \hat{R}_{nln} e_I^2 + O(r^3), \\
g^{ij} = \sum e_I^i e_I^j + O(1). 
\]

(11)

and

\[
g_{rt} = -g^{rr} + g^{ij} g_{ij} e_I^2 = -2 \hat{\varepsilon} r - \hat{R}_{nln} r^2 + O(r^3), \\
g_{ti} = -g_{ij} e_I^j = -\hat{R}_{nln} \hat{e}_j e_I^2 + O(r^3), \\
g_{rt} = 1, \quad g_{ri} = 0, \quad g_{ij} \sim O(1). 
\]

(12)
3. The Petrov-like condition for non-rotating weakly isolated horizon

As firstly pointed out by Strominger et al., the Petrov-like boundary condition plays an essential role in the construction of the gravity/fluid correspondence. The main reason is that imposing such a condition can guarantee that there is no out-going gravitational radiation across the boundary [14]. Originally, the Petrov condition is proposed to classify the geometry of the whole spacetime. But here such conditions are only specified on the cutoff surface; thus, one may call it the Petrov-like boundary condition.

With the previous result, for a cutoff surface $\Sigma_c := \{ p \in M| r(p) = r_c \}$, the induced line element on $\Sigma_c$ could be written as

$$ds^2_{\Sigma_c} = g_{tt} dt^2 + 2 g_{ti} dt dx^i + g_{ij} dx^i dx^j, \quad (i, j = 1, ..., p).$$

(13)

The Petrov-like condition for gravity is defined by [14]

$$C_{\mu \nu \rho \sigma} l^\mu E^\nu i l^\rho E^\sigma |_{\Sigma_c} = 0,$$

(14)

where $C_{\mu \nu \rho \sigma}$ is the Weyl tensor of spacetime, $l^\mu$ and $E^\mu$ are the null tetrad as introduced in the previous section.

It has been shown in the last section that $l^\mu$ has the form $l^\mu = U \partial_r + \partial_t + X^i \partial_i$. From the normal covector $N^a = (dr)_a/\sqrt{g_{rr}}$ of $\Sigma_c$, one obtains

$$(\partial_r)^a = \frac{1}{\sqrt{g_{rr}}} N^a - \frac{1}{g_{rr}} (\partial_t)^a - \frac{g_{ri}}{g_{rr}} (\partial_i)^a.$$

Thus,

$$l^a = \frac{U}{\sqrt{g_{rr}}} N^a + \left(1 - \frac{U}{g_{rr}}\right) (\partial_t)^a + \left(X^i - \frac{U g_{gi}}{g_{rr}}\right) (\partial_i)^a.$$

(15)

The Petrov-like condition could be expressed as

$$0 = \left[ \frac{U^2}{g_{rr}} C_{NlNj} + \left(1 - \frac{U}{g_{rr}}\right)^2 C_{ij} + \frac{U}{\sqrt{g_{rr}}} \left(1 - \frac{U}{g_{rr}}\right) \left(C_{Nlj} + C_{Nji}\right) 
+ \frac{U}{\sqrt{g_{rr}}} \left(X^k - \frac{U g_{k2}}{g_{rr}}\right) \left(C_{Nkj} + C_{Njk}\right) + \left(X^k - \frac{U g_{k2}}{g_{rr}}\right) \left(X^m - \frac{U g_{m2}}{g_{rr}}\right) C_{lkmj} 
+ \left(1 - \frac{U}{g_{rr}}\right) \left(X^k - \frac{U g_{k2}}{g_{rr}}\right) \left(C_{Nkj} + C_{Njk}\right) \right]_{\Sigma_c}.$$

(16)

It has been mentioned in the introduction that the intrinsic metric of the cutoff surface is fixed, only the extrinsic curvature is perturbed. So the relation between the spacetime Weyl tensor and the curvature of the cutoff surface is needed. In the absence of matter field, the spacetime Weyl tensor may be decomposed by the curvature of the cutoff surface:

$$C_{abcd} = \bar{R}_{abcd} - K_{ab} K_{cd} + K_{ad} K_{bc},$$

$$C_{abcN} = D_a K_{bc} - D_b K_{ac},$$

$$C_{abNlN} = -\bar{R}_{ab} + KK_{ab} - K_{ab} K^b,$$

(17)

Here, $D$ is the induced connection on $\Sigma_c$ and $\bar{R}_{abcd}$ is the associated Riemann curvature. Thus, equation (16) can be expressed in terms of the intrinsic and extrinsic curvatures of the cutoff surface.

In the original method proposed by Strominger [14], to obtain the fluid/gravity correspondence, one should consider the near horizon limit and non-relativistic limit. Two such limits can be implemented in the following way: introduce a rescaling parameter $\lambda$, define new time coordinate $\tau = 2\delta \lambda^2 r$ and choose $r_c = 2\delta \lambda^2$. Consider the limit $\lambda \to 0$,
where $\hat{\kappa}$ is the surface gravity which has been introduced in last section. In the following parts, we will consider the Petrov-like condition in such limits.

As a result of equation (10),
\[
X^k - \frac{U g^k}{g^\tau} \Rightarrow X^k - \frac{UX^k + UW_i e^k_j}{2U + W_j W_i} \sim O(\tau^3).
\]

Equation (16) has a simpler expression,
\[
0 = \frac{U^2}{g^\tau} \left( - \tilde{R}_{ij} + KK_{ij} - K_c K^c_{ij} \right) + \left( 1 - \frac{U}{g^\tau} \right)^2 \left( \tilde{R}_{ij} - K_{i\tau} K_{j\tau} + K_{ij} K_{\tau\tau} \right) \\
+ \frac{U}{\sqrt{g^\tau}} \left( 1 - \frac{U}{g^\tau} \right) \left( 2D_{ij} K_{ij\nu} - 2D_{\nu} K_{ij} \right) + O(\lambda^5).
\]

The induced metric of $\Sigma_c$ in the new coordinate is
\[
\text{d}s^2_{p+1} = \frac{g_{\tau\nu}}{4\tilde{\kappa}^2\lambda^4} \text{d}\tau^2 + 2 \frac{g_{\tau\nu}}{2\tilde{\kappa}^2\lambda^2} \text{d}\tau' + g_{ij} \text{d}x^i \text{d}x^j.
\]

In the coordinate $(\tau, r, x')$, the Petrov-like condition (19) becomes
\[
0 = \frac{U^2}{g^\tau} \left( KK_{ij} - K_c K^c_{ij} - \tilde{R}_{ij} h^{hi} \right) + 4\tilde{\kappa}^2\lambda^4 \left( 1 - \frac{U}{g^\tau} \right)^2 \left( \tilde{R}_{ij} h^{hi} - K_{i\tau} K^c_{j\nu} + K_{ij} K^c_{\tau\nu} \right) \\
+ \frac{U}{\sqrt{g^\tau}} \left( 1 - \frac{U}{g^\tau} \right) 2\tilde{\kappa}^2\lambda^4 \left( 2D_{ij} K_{ij\nu} - 2D_{\nu} K_{ij} \right) + O(\lambda^5),
\]

where $D_{ij}$ is the induced connection on $\Sigma_c$.

According to the fluid/gravity correspondence, the stress tensor of a gravity system corresponds to the energy–momentum tensor of a fluid. It is well known that for gravity such a stress tensor can be described by the Brown–York tensor $t_{ab}$
\[
t_{ab} = K h_{ab} - K_{ab}.
\]

We remark that in the standard AdS/CFT framework, there should be a counter term in addition to the bare Brown–York tensor. However, in our approach the counter term will not affect the final result. A short argument is given in the section 5. Here we just ignore it. Plugging into equation (21), we obtain the Petrov-like conditions in terms of the stress tensor,
\[
0 = -\frac{U^2}{g^\tau} \tilde{R}_{ij} h^{hi} + 4\tilde{\kappa}^2\lambda^4 \left( 1 - \frac{U}{g^\tau} \right)^2 \tilde{R}_{i\tau} h^{hi} + \frac{U^2}{g^\tau} \left( -t^i_j t^j_i + \frac{\tau}{p} t^i_j + \frac{\tau}{\rho} t^j_i - t^j_i \right) \\
- 4\tilde{\kappa}^2\lambda^4 \left( 1 - \frac{U}{g^\tau} \right)^2 \left[ h_{ij} \left( \frac{t^2_i}{p^2} \delta^j_i - t^i_j \frac{\tau}{p} \delta^j_i - \frac{\tau}{\rho} t^i_j + \frac{\tau}{\rho} t^j_i - t^j_i \right) \\
+ h_{ij} \left( \frac{\tau}{p} t^i_j \delta^j_i - \frac{\tau}{\rho} t^i_j + \frac{\tau}{\rho} t^j_i - t^j_i \right) \right] \\
+ \frac{U}{\sqrt{g^\tau}} \left( 1 - \frac{U}{g^\tau} \right) 2\tilde{\kappa}^2\lambda^4 \left[ -2h_{i\tau} D_{ij} t^j_i + 2h_{ijm} \left( D_{ij} \frac{t}{p} \delta^m_i - D_{ij} t^m_i \right) \\
+ 2h_{i\tau} D_{ij} t^j_i - 2h_{i\tau} D_{ij} \left( \frac{t}{p} \delta^m_i - t^m_i \right) \right] + O(\lambda^5).
\]

Now we have obtained the expression of the Petrov-like boundary condition in terms of the intrinsic geometry and extrinsic curvature of $\Sigma_c$. This boundary condition plays an extremely important role in LS’s realization of fluid/gravity duality. From the view point of an initial-boundary value problem of an Einstein system [23], such boundary condition means there is no out-going gravitational radiation through the boundary. Such boundary condition will provide some restriction between the intrinsic geometry and extrinsic curvature of $\Sigma_c$.  

Now let us introduce gravitational perturbation. Based on the main picture of the AdS/CFT correspondence, the intrinsic geometry of the boundary is fixed and the extrinsic curvature of the boundary is perturbed. Obviously, perturbing the extrinsic curvature is equivalent to perturbing the Brown–York tensor; we follow LS’s way [14] to introduce the gravitational perturbation as

\[ t^{a}_{b} = \bar{t}^{a}_{b} + \sum_{k=1} \tilde{t}^{a(k)}_{b} \lambda^{k} , \]  

(24)

where \( \bar{t}^{a}_{b} \) is the Brown–York tensor of back ground spacetime. In the perturbed case, the Petrov-like boundary condition will give some restrictions between the perturbed Brown–York tensor \( \{ \tilde{t}^{a(k)}_{b} \} \) which will help us to establish the fluid/gravity correspondence. In order to obtain the concrete form of such restrictions, we need to consider the \( \lambda \)-extension of the right-hand side of equation (23) in the perturbed case. This is a quite long calculation. To make the paper more clear, we summarize all results in following lemmas and put the proofs into appendices.

**Lemma 1.** *(Near horizon behavior of extrinsic curvature of \( \Sigma_{c} \))*

\[ K^{j} = \sqrt{2\hat{\varepsilon}} \xi^{j}_{i} \lambda + O(\lambda^{3}) \sim O(\lambda). \]  

(25)

\[ K^{r} = 2\hat{\varepsilon} \lambda^{2} K^{i}_{i} = \hat{\varepsilon} \lambda^{2} (h^{i}_{i} K^{i}_{i} + h^{i}_{i} K^{i}_{i}) \sim O(\lambda^{4}) \]  

(26)

\[ K^{r} = \frac{1}{2\lambda} + \beta \lambda + O(\lambda^{3}) \sim O\left( \frac{1}{\lambda} \right). \]  

(27)

\[ K = \frac{1}{2\lambda} + \sqrt{2\hat{\varepsilon}} (\beta + \xi) \lambda + O(\lambda^{3}) \sim O\left( \frac{1}{\lambda} \right). \]  

(28)

Here, \( \xi^{j}_{i} = \hat{\varepsilon}^{k}[\sqrt{2\hat{\varepsilon}} \hat{\theta}^{j}_{j} + (2\hat{\varepsilon})^{-1/2} \delta_{ij} \hat{\theta}^{j}_{j}] \) is the coefficient of the leading order of \( K^{j} \), while \( \beta \) is the coefficient of the order \( O(\lambda) \) in \( K^{r} \) and \( \xi = \xi^{j}_{i} \).

We remark that equation (28) shows that the mean curvature of \( \Sigma_{c} \) satisfies \( K^{-1} \sim 2\lambda + O(\lambda^{3}) \) for general non-rotational isolated black hole, which is similar to the large mean curvature expansion obtained by LS in the Rindler case [14]. This means up to the order of \( \lambda \) we will use, LS’s large mean curvature limit is equivalent to our near horizon limit.

**Lemma 2.** *(The Petrov-like boundary condition (23) implies the following relation)*

\[ t^{(1)}_{j} = 2h^{i}_{k} t^{(1)}_{j} - 2h^{i}_{k} \tilde{\nabla}^{(1)}_{(j} t^{(1)}_{k)} + \frac{t^{(1)}_{j}}{\rho} \delta^{j}_{j} + \xi^{j}_{j} \sqrt{2\hat{\varepsilon}} - \bar{K}^{j}. \]  

(29)

(Note: in above equation, the meaning of \( \tilde{\nabla}^{(1)}_{j} \) is that we see \( t^{(1)}_{j} \) as a 1-form on \( S_{t} \) and take the covariant derivative on that vector inside \( S_{t} \). This relation comes from the \( O(\lambda^{3}) \) coefficient of \( \lambda \)-extension of (23). Clearly, the Petrov-like boundary condition gives restriction for the perturbed Brown–York tensor, i.e. \( t^{(1)}_{j} \) are functions of \( t^{(1)}_{j} \) and other geometric quantities on horizon. This restriction is similar to what has been obtained by previous works for a concrete spacetime background [14–17].)

Besides the Petrov-like boundary condition, there is another constraint equation on \( \Sigma_{c} \), i.e. the Gaussian equation. Under the \( \lambda \)-extension, the first non-trivial relation from Gaussian equation is

\[ t^{(1)}_{r} = \bar{R} - 2h^{i}_{j} t^{(1)}_{j} t^{(1)}_{j} - \sqrt{\frac{\hat{\varepsilon}}{2}} \frac{\xi}{\rho}. \]  

(30)
where ξ is a parameter introduced in equation (28). Similar to previous works, the Gaussian equation helps us to fix the perturbation $t^\tau_\tau$ in terms of $t^\tau_\tau$ and horizon geometry. Based on the identification of fluid/gravity correspondence [14], $t^\tau_\tau$ is the energy density of the dual fluid and $t^\tau_\tau$ relates to the velocity of the dual fluid. This equation could be viewed as the equation of state for dual fluid.

4. Navier–Stokes equation

In the previous section, we have considered the large mean curvature extension of the Petrov-like boundary condition for general non-rotating black holes. Like the discussion by Lysov and Strominger [14], the Petrov-like condition reduces the degree of freedom of the gravity to those of a fluid. The remaining $p + 2$ variables may be interpreted as the hydrodynamic variables of a fluid living on the boundary. The Gaussian equation for gravity then can be viewed as the equation of state. In this section, we consider the Codazzi equation on $\Sigma$ and show that it will lead to the Navier–Stokes equation at the large mean curvature limit.

The Codazzi equation in terms of the Brown–York tensor is

$$0 = D_a \sum_{k=1}^{\lambda^k}. (31)$$

Because the background spacetime satisfies the Codazzi equation automatically, we only need to consider the equation for perturbation:

$$0 = D_a \sum_{k=1}^{\lambda^k} t^\tau_\tau = 0, (32)$$

With the help of equation (A.3), it is easy to see that

$$t^\tau_\tau = h^\tau_\tau t^\tau_\tau + h^\tau_\tau t^\tau_\tau = O(\lambda^2),$$

$$t^\tau_\tau = h^\tau_\tau t^\tau_\tau + h^\tau_\tau t^\tau_\tau = h^\tau_\tau t^\tau_\tau + O(\lambda^4) \sim O(\lambda^0),$$

$$t^\tau_\tau = h^\tau_\tau t^\tau_\tau + h^\tau_\tau t^\tau_\tau = h^\tau_\tau t^\tau_\tau + O(\lambda^4) \sim O(\lambda^0).$$

Using equations (A.3) and (B.4),

$$0 = D_a \sum_{k=1}^{\lambda^k} t^\tau_\tau = 0, (34)$$

Based on the basic identification [14],

$$2t^\tau_\tau \leftrightarrow v_j, \quad 2t^\tau_\tau / p \leftrightarrow P, (35)$$

under the large mean curvature limit, the leading order of equation (34) implies $\tilde{\nabla}^i v_i = 0$, i.e. the incompressible condition.
Similarly,

\[ 0 = D_a \sum_{k=1}^{\infty} t^{a(j(k)} \lambda^k \]

\[
= \partial_t \sum_{k=1}^{\infty} t^{a(j(k)} \lambda^k + \delta_i \sum_{k=1}^{\infty} t^{i(j(k)} \lambda^k + \dot{\Gamma}_{\tau \tau} \sum_{k=1}^{\infty} t^{\tau(j(k)} \lambda^k + \dot{\Gamma}_r^i \sum_{k=1}^{\infty} t^{r(j(k)} \lambda^k + \dot{\Gamma}_m^i \sum_{k=1}^{\infty} t^{m(j(k)} \lambda^k + \dot{\Gamma}_{\tau m}^i \sum_{k=1}^{\infty} t^{\tau m(j(k)} \lambda^k + \dot{\Gamma}_{r \tau} \sum_{k=1}^{\infty} t^{r \tau(j(k)} \lambda^k + 2 \dot{\Gamma}_{\tau r} \sum_{k=1}^{\infty} t^{\tau r(j(k)} \lambda^k + \dot{\Gamma}_{r m}^i \sum_{k=1}^{\infty} t^{r m(j(k)} \lambda^k + \dot{\Gamma}_{\tau m} \sum_{k=1}^{\infty} t^{\tau m(j(k)} \lambda^k + \dot{\Gamma}_{r \tau} \sum_{k=1}^{\infty} t^{r \tau(j(k)} \lambda^k + \dot{\Gamma}_{r m} \sum_{k=1}^{\infty} t^{r m(j(k)} \lambda^k + O(\lambda^2). \]

Equation (36)

Using lemma 2, above equation becomes

\[
0 = \tilde{\nu}^i \dot{\partial}_t \tau^{i(1)} + \tilde{\nu}^i \tau^{i(1)} + \dot{\Gamma}_{\tau \tau} \tau^{i(1)} + \dot{\Gamma}_r^i \tau^{i(1)} + 2 \dot{\Gamma}_{\tau r} \tau^{i(1)} + O(\lambda^2) \]

Then with equations (34) and (35), we obtain the leading order of above equation as

\[
0 = \partial_t v^j + v^j \nabla_j v^i - \tilde{\nu} v^j - \tilde{R}^j v^i + \tilde{\nu} P + \tilde{f}^j + \tilde{\Gamma}_{\tau \tau} v^j + \tilde{\Gamma}_r^i v^j + 2 \tilde{\Gamma}_{\tau r} v^j, \]

\[
(37) \]

where \( f^j = \tilde{\nabla} (\tilde{\nu} \sqrt{\tilde{\nu}} - \tilde{R}^i) \) only depends on the horizon geometry, so it can be seen as an external force caused by the curved space.

**Remark.** In equation (37), we obtain a generalized Navier–Stokes equation. The first line of equation (37) is in the form of the standard Navier–Stokes equation with an external force. The problem is the extra terms in the second line. These terms are related with the Christoffel symbol on horizon. From equation (B.4), it is easy to see that

\[
\begin{align*}
\dot{\Gamma}_{\tau \tau} & \equiv \frac{1}{2 \tilde{\nu}} \tilde{\partial}_t \tilde{R}_{\text{total}}, \\
\dot{\Gamma}_r^i & \equiv \frac{1}{2} \tilde{\nu}_{\beta \gamma} \tilde{\partial}_f (\tilde{\partial}_\beta \tilde{\partial}_\gamma) \tilde{\partial}_f, \\
\dot{\Gamma}_m^i & \equiv \frac{1}{2} \tilde{\partial}_f (\tilde{\partial}_\beta \tilde{\partial}_\gamma) \tilde{\partial}_f. 
\end{align*}
\]

(38)

If the black hole is stationary, then \( \tilde{\partial}_t \) is a Killing vector. Obviously, all these additional terms vanish, so we obtain the standard Navier–Stokes equation. This means that fluid/gravity correspondence exists for a general non-rotating stationary black hole. Furthermore, simple calculation shows that these additional terms will be zero if \( \tilde{\partial}_t \) satisfies \( \mathcal{L}_{\tilde{g}} \tilde{\partial}_t \sim O(r^2) \), so such correspondence also holds for asymptotic stationary black holes (Note: based on equation (A.5), \( \mathcal{L}_{\tilde{g}} \tilde{\partial}_t \sim O(r) \) for general WIH, so our asymptotic stationary condition \( \mathcal{L}_{\tilde{g}} \tilde{\partial}_t \sim O(r^2) \) is not very restricted.). This result tells us that the fluid/gravity correspondence
can also exist in some dynamical cases. Another remark is about the external force term. In previous works [14, 9, 15, 16], all cases satisfy $\xi_i = 0$ and $\tilde{\nabla}_i \tilde{R}_j = 0$, so there are no external force terms and our result agrees with previous results. For the general horizon case, the inhomogeneity of the section of horizon will cause an external force term in the dual Navier–Stokes equation.

5. Summary and discussion

In this paper, we have proved that Lysov and Strominger’s realization of the fluid/gravity correspondence exists for any non-rotating stationary isolated black-hole horizon. We have also found that such correspondence can be generalized to some non-stationary black holes. As discussed in the introduction, the Petrov-like boundary condition helps us to reduce the degrees of freedom of gravity to that of fluid. Such condition also help us to deal with the compact horizon section cases without the use of the method of long wavelength expansion. From the viewpoint of the initial-boundary value problem of Einstein equations, the Petrov-like boundary condition is also a quite natural choice. Based on the previous work [23], $C_{i j}$ is just the free boundary data of Einstein equations. This observation should be helpful for us to generalize the Petrov-like boundary condition to non-vacuum cases such as in [17], where some generalization for the Einstein–Maxwell case has been discussed.

As discussed in [17], when we take the Brown–York tensor, there should be a counter term which depends only on the induced geometry of the boundary [21, 22] in addition to the extrinsic curvature terms. In our approach, the Navier–Stokes equation emerges from the conservation law on the boundary only at the perturbative level, while our boundary condition requires that the induced geometry should be fixed [14]. Therefore, the counter term has no effect on the perturbations of the dual fluid on the boundary and we can ignore this term in the procedure.

In previous works [16, 17], there is an artificial coefficient $\alpha$ in the near horizon limit $r_c = \alpha^2 \lambda^2$ as $\lambda \to 0$. The value of $\alpha$ has been chosen artificially in order to obtain the Navier–Stokes equation. In this paper, we have clarified that this coefficient is related to the surface gravity $\hat{\epsilon}$ of the black hole as $\alpha^2 = 2\hat{\epsilon}$.

Because of equation (17), the cases we considered is spacetime without a cosmological constant. A little longer but quite similar calculation can show that the same result also holds for non-rotating weakly isolated horizon with a cosmological constant. The cosmological constant gives no contribution for the gravity/fluid correspondence. This also agrees with the previous work [16].

Non-vacuum cases, especially Einstein–Maxwell cases as well as Einstein–Maxwell–Scalar cases, are very interesting topic that will be considered in future works. Recently, such setups have been used to consider holographic magnetofluid [17] and holographic superfluid [24].

The non-rotating condition is crucial for our work. If we give up this condition, then the dual equation will become very complex. For instance, the incompressible condition will be lost. How to understand the gravity/fluid correspondence for rotating black holes is also an interesting open problem.

Another interesting problem is about the higher order gravity perturbation. In [25], Compère et al found, in Minkowski case, that there was a systematic way to construct such correspondence to any order gravity perturbation and to extract the physical properties of the dual fluid. With the help of long wavelength expansion, they found that the dual fluid system satisfies a modified Navier–Stokes equation. This dual system is no longer incompressible and more complex. Interactions between shear, twist and expansion of fluid will appear in
the modified Navier–Stokes equations. Some recent work [26] has considered the Petrov-like boundary condition for higher order perturbation in the Minkowski case. The higher order LS’s realization of the fluid/gravity correspondence in general spacetime will be considered in future work.

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Appendix

In these appendices, we will give the detail proof of lemmas 1 and 2 in section 3. Appendix A contains the proof of lemma 1 and the proof of lemma 2 is in appendix B. In appendix C, we will analyze the \( \lambda \)-extension of the Gaussian equation and show that it will give a non-relativistic energy relation.

Appendix A. Proof of lemma 1

First of all, we will calculate the near horizon behavior of the extrinsic curvature. By definition,

\[
K_{ab} = \frac{1}{2} \tilde{h}_{ab} \Gamma^c_{ab} = \frac{1}{\sqrt{g}} \sqrt{g} \Gamma^c_{ab}, \quad a, b = t, i,
\]

(A.1)

where \( N^a \) and \( h_{ab} \) are the normal vector and the induced metric of \( \Sigma_c \), respectively. The components of \( N^a \) are

\[
N^a = \sqrt{g}, \quad N^t = \frac{1}{\sqrt{g}}, \quad N^i = \frac{g^i}{\sqrt{g}}.
\]

(A.2)

With the help of equations (11) and (12), the near horizon behaviour of \( N^a \) is clear:

\[
\sqrt{g} = \sqrt{2\hat{\varepsilon}} \left( 1 + \hat{R}_{\text{nlnl}} r + O(r^2) \right) = (2\hat{\varepsilon} + 4\hat{\varepsilon}^2 b \lambda^3) + O(\lambda^5),
\]

\[
g^{ij} = \hat{R}_{\text{nlnl}} + O(r^3) = 4\hat{\varepsilon}^2 c \lambda^4 + O(\lambda^6),
\]

where \( b = \hat{R}_{\text{nlnl}} / 4\hat{\varepsilon}, c = \hat{R}_{\text{nlnl}} \).

The induced metric is easy to work out:

\[
h_{\tau\tau} = \frac{g_{\tau\tau}}{4\hat{\varepsilon}^2 \lambda^2} = -\frac{1}{\lambda^2} - \hat{R}_{\text{nlnl}} + O(\lambda^2), \quad h^{\tau\tau} = -\frac{\lambda^4}{g^{\tau\tau}} = -\lambda^2 + O(\lambda^4);
\]

\[
h_{\tau i} = \frac{g_{\tau i}}{2\hat{\varepsilon} \lambda^2} \sim O(\lambda^2), \quad h^{\tau i} = -2\hat{\varepsilon} \lambda^2 \frac{g^{\tau j}}{g^{\tau\tau}} \sim O(\lambda^4);
\]

\[
h_{ij} = g_{ij} \sim O(\lambda^6), \quad h^{ij} = g^{ij} - \frac{g^{\tau j} g^{\tau i}}{g^{\tau\tau}} = g^{ij} + O(\lambda^6).
\]

(A.3)

For late use, \( \delta_i h_{ij} \) need to be calculated out explicitly. From equations (5) and (11),

\[
g^{ij} = \hat{g}^{ij} + 2\hat{\varepsilon} \left( \delta_i \delta_j \right) r + O(r^2)
\]

\[
= \hat{g}^{ij} - 2\hat{\varepsilon} \delta_i \delta_j r + O(r^2),
\]

\[
g_{ij} = \hat{g}_{ij} + 2\hat{\varepsilon} \delta_i \delta_j \hat{g}_{ij} r + O(r^2).
\]

(A.4)
The weakly isolated horizon (WIH) condition implies \( \partial_t \tilde{g}_{ij} = 0 \). Thus, 
\[
\partial_t g_{ij} = \left[ 2 \tilde{g}_{ik} \partial_t e_i^j \tilde{e}^k_j \tilde{g}_{mj} + \tilde{g}_{ik} \partial_t \tilde{e}^i_j \tilde{e}^k_j \tilde{g}_{mj} \right] r + O(r^2). \tag{A.5}
\]

We need the values of \( \partial_t e_i^j \) and \( \partial_t \theta_{ij} \) on horizon. With the help of facts \( X^i = U = W_i = 0 \), the commutator between \( l \) and \( E_l \) gives 
\[
[l, E_l] = (\partial_t e_i^j) \partial_i. \tag{A.6}
\]

On the other hand, 
\[
[l, E_l] = \nabla_l E_l - \nabla l = -\pi_l + \epsilon_{jl} E_j - \alpha_i l - \theta_{jl} E_j.
\]

Here, we use abbreviations \( \epsilon_{jl} = \langle E_j, \nabla_l E_l \rangle, \theta_{jl} = \langle E_j, \nabla l \rangle \). The fact that on horizon \( l \) is geodesic leads to \( \langle l, \nabla E_l \rangle = -\langle E_l, \nabla l \rangle \equiv 0 \). The Bondi gauge of the null tetrad and the fact that \( l \) is normal direction of horizon implies \( \epsilon_{lj} \equiv 0 \). For the WIH condition \( tr \theta = 0 \), it implies that \( \theta_{lj} \equiv 0 \) by the Raychaudhuri equation. Thus, we obtain \( [l, E_l] \equiv 0 \) for a non-rotating black hole. This implies that \( \partial_t e_i^j \equiv 0 \).

As for \( \partial_l \theta_{lj} \), consider the derivative 
\[
\nabla_l \theta_{lj} = \nabla_l (E_j, \nabla_m E_l) \equiv R_{lijm} - \epsilon_{lj} \theta_{lm}. \tag{A.7}
\]

where the fact \( \nabla_l n_l \equiv 0 \), \( \pi_j \equiv \alpha_l \equiv \theta_{lj} \theta^l_k \equiv 0 \) and \( U \equiv X^i \equiv 0 \) are used. On the other hand, because of equation (1) 
\[
\nabla_l \theta_{lj} \equiv \partial_l \theta_{lj}. \tag{A.9}
\]

Thus, we have 
\[
\partial_l \theta_{lj} \equiv R_{lijm} - \epsilon_{lj} \theta_{lm}. \tag{A.10}
\]

Plugging into equation (A.5), we may deduce that 
\[
\partial_t g_{ij} = 2 \tilde{g}_{ik} \partial_t \tilde{e}^k_j \tilde{e}^i_j \tilde{g}_{mj} 2 \tilde{\xi} \lambda^2 + O(\lambda^4) \sim O(\lambda^2). \tag{A.11}
\]

We also need the value of \( \partial_t g_{ij} \). Because \( g_{kl} \delta^{ij} = \delta_i^j \) and \( g^{ij} = \sum \delta_i^j \), it is easy to see 
\[
\partial_t g_{ij} = -g_{mj} (\partial_t e_i^m) \tilde{e}^k_j \tilde{g}_{kj} - g_{mj} (\partial_t e_j^m) \tilde{e}^k_i \tilde{g}_{ki} = 2 g_{mj} (\partial_t e_i^m) \tilde{e}^k_j \tilde{g}_{kj} \sim O(\lambda^0). \tag{A.12}
\]

In the last step, equation (5) is used.

Based on these preparation, the near horizon extensions of the extrinsic curvature are 
\[
K_{ij}^q = h^{qi} K_{qj} = h^{qi} K_{ij} + h^{qi} \lambda K_{ij}
\]

\[
= \frac{1}{2} \tilde{g}^{ij} \left( \sqrt{\tilde{g}^{jl}} \partial_l g_{ij} + \frac{1}{\sqrt{\tilde{g}^{jl}}} \partial_l g_{ij} \right) + O(\lambda^3)
\]

\[
= \sqrt{2} \tilde{e}^i_l \lambda + O(\lambda^3) \sim O(\lambda^3). \tag{A.13}
\]

\[
K_{ij}^e = \lambda^2 K_{ij} = \sqrt{2} \tilde{e}^i_l \lambda^2 (h^{qi} K_{qj} + h^{qi} \lambda K_{qj}) \sim O(\lambda^4). \tag{A.14}
\]

\[
K_{ij}^c = K_{ij} = h^{qi} K_{qj} + h^{qi} \lambda K_{qj}
\]

\[
= \frac{1}{2} \tilde{g}^{ij} \left( \sqrt{\tilde{g}^{jl}} \partial_l g_{ij} + \frac{1}{\sqrt{\tilde{g}^{jl}}} \partial_l g_{ij} + 2 \tilde{g}_{ijkl} \partial_i \tilde{g}_{kl} + \sqrt{\tilde{g}^{jl}} \right) + O(\lambda^3)
\]

\[
= \frac{1}{2 \lambda} + \beta \lambda + O(\lambda^3) \sim O(\lambda^{1-.}). \tag{A.15}
\]
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\[ K = K'_i + K''_i \]
\[ = \frac{1}{2\lambda} + \sqrt{2\hat{\xi}}(\beta + \xi)\lambda + O(\lambda^3) \sim O\left(\frac{1}{\lambda}\right). \] (A.16)

Here, \( \hat{\xi}' \) is the coefficient of the leading order of \( K'_i \), while \( \beta \) is the coefficient of the order \( O(\lambda) \) in \( K'_i \) and \( \xi = \hat{\xi}' \). These are the results in lemma 1.

Appendix B. Proof of lemma 2

As emphasized in section 3, the intrinsic geometry of \( \Sigma_c \) is fixed and Brown–York tensor is perturbed as equation (24). With the results of lemma 1, the Brown–York tensor satisfy

\[ t^i_j = -K^i_j + \sum_{k=1} t^i_{jk} \lambda^k \]
\[ = \lambda t^{(1)} + O(\lambda^2), \]
\[ t^i_j = K - K^i_j + \sum_{k=1} t^i_{jk} \lambda^k \]
\[ = (\sqrt{2\hat{\xi}} + f^{(1)}_i)\lambda + O(\lambda^2), \]
\[ t^i_j = K\delta^i_j - K^i_j + \sum_{k=1} t^i_{jk} \lambda^k \]
\[ = \frac{1}{2\lambda} \delta^i_j + \left[ \sqrt{2\hat{\xi} \left[ (\beta + \xi)\delta^i_j - \xi^i_j \right] + t^{(1)}_i \right] \lambda + O(\lambda^2), \]
\[ t^i_j = h^i_j h^j_i t^i_j = h^i_j h_{ij} t^i_j + h^i_j h_{ij} t^i_j + h^i_j h_{ij} t^i_j + h^i_j h_{ij} t^i_j = h^i_j h_{ij} t^i_j + O(\lambda), \]
\[ t = pK + \sum_{k=1} t^i_{jk} \alpha^k \lambda^k \]
\[ = \frac{p}{2\lambda} + (p\sqrt{2\hat{\xi} (\beta + \xi)+ t^{(1)}_i} \lambda + O(\lambda^2). \] (B.1)

Substituting these results into the Petrov-like condition (23), we want to show that \( t^{(1)}_j \) could be expressed in terms of \( t^{(1)}_i \). The coefficients of equation (23) are

\[ \frac{U^2}{g''} = \hat{\xi}^2 \lambda^2 + O(\lambda^4), \]
\[ = \left( 1 - \frac{U}{g''} \right)^2 4\hat{\xi}^2 \lambda^4 = -\hat{\xi}^2 \lambda^2 + O(\lambda^6), \] (B.2)
\[ \frac{U}{\sqrt{g'}} \left( 1 - \frac{U}{g''} \right) 2\hat{\xi} \lambda^2 = \hat{\xi}^2 \lambda^3 + O(\lambda^5). \]

Using equations (A.3) and (B.1), the terms in parentheses of equation (23) which are related to the Brown–York tensor could be calculated straightly,

\[ -\frac{t^i_j t^i_j}{p} + \frac{t^i_j t^i_j}{p} - \frac{t^i_j t^i_j}{p} = h^i_j h^j_i t^i_j + \frac{1}{2} \left( -t^{(1)}_j - \frac{t^{(1)}_j}{p} \delta^i_j \right) + \sqrt{\hat{\xi} / 2} \xi^i_j + O(\lambda), \]
\[ \frac{t^2_j}{p^2} \delta^i_j - \frac{t^2_j}{p} \delta^i_j - \frac{t^2_j}{p} t^i_j t^i_j - \frac{t^2_j}{p} t^i_j t^i_j \]
\[ h_{kk}' \left( \frac{4}{p} g_{k}^j h_{j}^k - t_{k}^{(1)} + \frac{4}{p} \epsilon \xi_{j}^k \epsilon \xi_{j}^k - \xi_{j}^k \xi_{j}^k \right) \sim O(1). \]  

Then, we consider the derivations \( D_{\mu} \xi_{(i)}^k \). In order to do that, we need the connection coefficients \( \Gamma_{\mu}^{\nu}_{\rho} \) on \( \Sigma \). Since the induced metric on \( \Sigma \) is fixed, by definition, \( \Gamma_{\mu}^{\nu}_{\rho} = \frac{1}{2} h^{\alpha\beta} (h_{\alpha\beta} + h_{\beta\alpha} - h_{\alpha\beta}) \). Straightforward calculation shows

\[ \Gamma_{\mu}^{\nu}_{\rho} = \frac{1}{2} g_{\alpha\beta} \partial_{\nu} R_{\alpha\beta \mu} + O(\lambda^2) \sim O(\lambda^0), \]

where \( \Gamma_{\mu}^{\nu}_{\rho} = \frac{1}{2} g_{\alpha\beta} (\frac{\partial}{\partial \nu} g_{\alpha\beta} + g_{\alpha\beta} \partial_{\nu} g_{\alpha\beta}) \) is the Christoffel symbol on the section \( S_i \) of horizon. Based on these preparations, we obtain

\[ -D_{\mu} \xi_{(i)}^k = -2 \epsilon \xi_{(i)}^k h_{\mu}^k + O(\lambda^2) \sim O(\lambda^0), \]

\[ = \frac{1}{2} \left( \frac{4}{p} \epsilon \xi_{j}^k - \xi_{j}^k \right)^2 \sim O(\lambda^0). \]  

Now most parts of equation (23) have been written out. Only two terms which are related to the intrinsic curvature remain, i.e. the two terms in the first line of equation (23). Each of them contains Riemannian curvature of the boundary \( \Sigma \). Now we consider these two terms

\[ -\frac{U^2}{g_{\nu\tau}} \bar{R}_{\nu\lambda} h_{\lambda\tau} + 4 \epsilon^2 \lambda^4 \left( 1 - \frac{U}{g_{\nu\tau}} \right)^2 \bar{R}_{\nu\lambda} h_{\lambda\tau}. \]  

By definition,

\[ \bar{R}_{\nu\lambda} = h_{\nu\lambda} \bar{R}_{\nu\lambda}^\mu = h_{\nu\lambda} \bar{R}_{\nu\lambda}^\mu + h_{\nu\lambda} \bar{R}_{\nu\lambda}^\mu, \]  

\[ \bar{R}_{\nu\lambda} = \Gamma_{\nu\lambda}^\tau - \Gamma_{\nu\tau}^\mu - \Gamma_{\lambda\tau}^\mu \sim O(\lambda^0). \]
\[\tilde{R}_{tkr} = \tilde{\Gamma}_{rk}^{i} - \tilde{\Gamma}_{tk}^{i} - \tilde{\Gamma}_{rk}^{\mu} \tilde{\Gamma}_{tk}^{\nu} - \tilde{\Gamma}_{tk}^{\mu} \tilde{\Gamma}_{rk}^{\nu} \]
\[= - \tilde{\Gamma}_{tk}^{i} + O(\lambda^2) \sim O(\lambda^{-2}), \quad (B.9)\]

with equation (B.4),
\[4\tilde{e}^2 \lambda^4 \left(1 - \frac{U}{g^r} \right)^2 \tilde{R}_{tir} h^{ki} \sim O(\lambda^2). \quad (B.10)\]

For the Ricci tensor of \(\Sigma_r\),
\[\tilde{R}_{kj} = \tilde{R}_{kj}^{\mu} = \tilde{R}_{tj}^{r} + \tilde{R}_{kj}^{i}, \quad (B.11)\]

Straight calculation shows
\[\tilde{R}_{tkr}^{r} = \tilde{\Gamma}_{rk}^{i} - \tilde{\Gamma}_{tk}^{i} + \tilde{\Gamma}_{rk}^{\mu} \tilde{\Gamma}_{tk}^{\nu} - \tilde{\Gamma}_{tk}^{\mu} \tilde{\Gamma}_{rk}^{\nu} = \tilde{\Gamma}_{tkr}^{i} + O(\lambda^2)\]
\[= \frac{1}{2\tilde{e}} (\partial_i \partial_j \tilde{e}^l_k) \tilde{e}^l_k + O(\lambda^2) \sim O(\lambda^0), \quad (B.12)\]

and
\[\tilde{R}_{kj}^{i} = \tilde{\Gamma}_{kj}^{i} - \tilde{\Gamma}_{ij}^{i} + \tilde{\Gamma}_{kj}^{\mu} \tilde{\Gamma}_{ij}^{\nu} - \tilde{\Gamma}_{ij}^{\mu} \tilde{\Gamma}_{kj}^{\nu} \]
\[= \tilde{\Gamma}_{ij}^{i} - \tilde{\Gamma}_{ij}^{k} + \tilde{\Gamma}_{ij}^{\mu} \tilde{\Gamma}_{ik}^{\nu} - \tilde{\Gamma}_{ik}^{\mu} \tilde{\Gamma}_{ij}^{\nu} + O(\lambda^2)\]
\[= \tilde{R}_{kj} + O(\lambda^0) \sim O(\lambda^0). \quad (B.13)\]

where \(\tilde{R}_{i}^{j}\) is the Riemann curvature of \(S_r\). It is obvious that \(\tilde{R}_{kj} \sim O(\lambda^0)\). Then, it is easy to see that
\[- \frac{U^2}{g^r} \tilde{R}_{kj} h^{ki} = -[\tilde{e}^2 \lambda^2 + \tilde{R}_{mot} \tilde{e}^2 \lambda^2 + O(\lambda^6)] \tilde{R}_{kj} h^{ki}\]
\[= -[\tilde{e}^2 \lambda^2 + \tilde{R}_{mot} \tilde{e}^2 \lambda^2 + O(\lambda^6)] \times \left[ \frac{1}{2\tilde{e}} (\partial_i \partial_j \tilde{e}^l_k) \tilde{e}^l_k + \tilde{R}_{kj} + O(\lambda^2) \right] h^{ki} \]
\[= -\frac{\tilde{e}}{2} (\partial_i \partial_j \tilde{e}^l_k) \tilde{e}^l_k \tilde{R}_{kj} + O(\lambda^2)\]
\[= -\tilde{R}_{kj} + \tilde{R}_{kj} \tilde{R}_{kj}^{i} + O(\lambda^4). \quad (B.14)\]

Besides, straight calculation also shows that
\[-\lambda^2 \tilde{R}_{kj} + \lambda^2 \tilde{R}_{tir} h^{ki} + \lambda^2 D_t \left( \frac{t}{p} \tilde{R}_{kj} - t_j \right) = -\lambda^2 \tilde{R}_{kj} + O(\lambda^3) \quad (B.15)\]

All the terms in the Petrov-like condition have been worked out now. Combining these results together, obviously the leading term of \(\lambda\)-extension of equation (23) is second order and the coefficient of such term gives
\[t_j^{(1)} = 2\tilde{h} t_k^i t_l^r t_j^{r(1)} - 2h \tilde{\nabla} \tilde{\nabla}_{ijkl}^{(1)} + \frac{t}{p} \tilde{R}_{kj} + \tilde{R}_{kj}^{i} \tilde{R}_{kj}^{i} - \tilde{R}_{kj}. \quad (B.16)\]

So we finish the proof of lemma 2.
Appendix C. Gaussian equation

In this appendix, we will consider the \( \lambda \)-extension of the Gaussian equation. The Gaussian equation for gravity takes the form

\[
\bar{R} + K^{\mu\nu} K_{\mu\nu} - K^2 = 0.
\]

In terms of the Brown–York tensor \( I_{\text{BY}} \), it becomes

\[
\bar{R} + \left( t_i^{(1)} \right)^2 - \frac{2h_{ij} t_j^{(1)}}{\lambda^2} t_i^{(1)} t_j^{(1)} + \frac{t^2}{p} + t_j^{(1)} = 0.
\]

With the results of equations (A.3), (B.12) and (B.13), expanding the scalar curvature \( \bar{R} \) in terms of parameter \( \lambda \) near the horizon, we have

\[
\bar{R} = \bar{R}_{\mu\nu} h^{\mu\nu} = \bar{R}_\tau \tau + 2\bar{R}_t h^{tt} + \bar{R}_{ij} h^{ij}
\]

\[
= -\frac{1}{2\bar{R}_t} h^{im} \partial_m h_{im} + \bar{R} + O(\lambda^2)
\]

\[
\sim O(\lambda^0)
\]

Then, zero-order component of the Gaussian equation gives

\[
t_j^{(1)} = \bar{R} - 2h_{ij} t_i^{(1)} t_j^{(1)} - \sqrt{\frac{\bar{R}}{2}} \xi,
\]

where \( \xi \) is a parameter introduced in equation (A.16). Similar to previous works, the Gaussian equation helps us to fix the perturbation \( t_j^{(1)} \) in terms of \( t_i^{(1)} \) and horizon geometry. This equation could be viewed as the equation of state for dual fluid.

References

[1] Damour T 1978 Phys. Rev. D 18 3598

[2] Damour T 1979 Quelques propriétés mécaniques, électromagnétiques, thermodynamiques et quantiques des trous noirs Thèse de Doctorat d’État Université Pierre et Marie Curie, Paris VI

[3] Damour T 1982 Surface effects in black hole physics Proceedings of the Second Marcel Grossmann Meeting on General Relativity ed R Ruffini (Amsterdam: North-Holland)

[4] Smarr L 1973 Phys. Rev. Lett. 30 71

[5] Callaway D J E 1996 Phys. Rev. E 53 3738

[6] Parikh M and Wilczek F 1998 Phys. Rev. D 58 064011

[7] Cardoso V and Dias O J C 2006 Phys. Rev. Lett. 96 181601

[8] Padmanabhan T 2011 Phys. Rev. D 83 044048

[9] Kolekar S and Padmanabhan T 2012 Phys. Rev. D 85 024004

[10] Price R H and Thorne K S 1986 Phys. Rev. D 33 915

[11] Thorne K S, Price R H and Macdonald D A 1986 Black Holes: the Membrane Paradigm (New Haven, CT: Yale University Press)

[12] Policastro G, Son D T and Starinets A O 2002 J. High Energy Phys. JHEP09(2002)043

[13] Son D T and Starinets A O 2007 Annu. Rev. Nucl. Part. Sci. 57 95

[14] Bhattacharyya S, Hubeny V E, Minwalla S and Rangamani M 2008 J. High Energy Phys. JHEP02(2008)045 (arXiv:0712.2456)

[15] Bhattacharyya S, Minwalla S and Wadia S R 2009 J. High Energy Phys. JHEP08(2009)059 (arXiv:0810.1545)

[16] Iqbal N and Liu H 2009 Phys. Rev. D 79 025023 (arXiv:0809.3808)

[17] Breidberg I, Keeler C, Lysov V and Strominger A 2011 J. High Energy Phys. JHEP03(2011)141 (arXiv:1006.1902)

[18] Breidberg I, Keeler C, Lysov V and Strominger A 2011 From Navier–Stokes to Einstein arXiv:1101.2451 [hep-th]

[19] Cai R-G, Li L and Zhang Y-L 2011 J. High Energy Phys. JHEP07(2011)027 (arXiv:1104.3281)
[11] Niu C, Tian Y, Wu X-N and Ling Y 2012 Phys. Lett. B 711 411 (arXiv:1107.1430)
[12] Kovtun P, Son D T and Starinets A O 2003 J. High Energy Phys. JHEP10(2003)064 (arXiv:hep-th/0309213)
[13] Eling C, Fouzon I and Oz Y 2009 Phys. Lett. B 680 496
   Eling C and Oz Y 2010 J. High Energy Phys. JHEP02(2010)069
   Eling C, Neiman Y and Oz Y 2010 J. High Energy Phys. JHEP10(2010)110
   Eling C and Oz Y 2011 J. High Energy Phys. JHEP06(2011)007
[14] Lysov V and Strominger A 2011 From Petrov–Einstein to Navier–Stokes arXiv:1104.5502
[15] Huang T-Z, Ling Y, Pan W-J, Tian Y and Wu X-N 2011 J. High Energy Phys. JHEP10(2011)079
[16] Huang T-Z, Ling Y, Pan W-J, Tian Y and Wu X-N 2012 Phys. Rev. D 85 123531
[17] Zhang C-Y, Ling Y, Niu C, Tian Y and Wu X-N 2012 Phys. Rev. D 86 084043
[18] Ashtekar A and Krishnan B 2004 Living Rev. Rel. 7 10
[19] Korzynski M, Lewandowski J and Pawlowski T 2005 Class. Quantum Grav. 22 2001
[20] Friedrich H 1981 Proc. Roy. Soc. Lond. A 378 169–84
   Friedrich H 1981 Proc. Roy. Soc. Lond. A 378 401–21
[21] Brown J D and York J W 1993 Phys. Rev. D 47 1407
[22] Skenderis K 2002 Class. Quantum Grav. 19 5849
[23] Friedrich H and Nagy G 1999 Commun. Math. Phys. 201 619
   Kreiss H O, Reula O, Sarbach O and Winicour J 2009 Commun. Math. Phys. 289 1099
   Sarbach O and Tiglio M 2012 Living Rev. Rel. 15 9 (www.livingreviews.org/lrr-2012-9)
[24] Adams A, Chesler P M and Liu H 2012 Holographic vortex liquids and superfluid turbulence arXiv:1212.0281
[25] Compère G, McFadden P, Skenderis K and Taylor M 2011 J. High Energy Phys. JHEP07(2011)050
[26] Cai R G, Li L, Yang Q and Zhang Y L 2013 Petrov type I condition and dual fluid dynamics arXiv:1302.2016