On the Quantum Creation of Matter in the Expanding Universe

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where \( g \equiv 3\pi/2G, \text{ } G \) is the Newton gravitational constant. The canonical form of the action \([2]\) is

\[
I = \int_0^C dc \left( p_a \dot{a} + p_\phi \dot{\phi} - H \right),
\]

where

\[
H = -\frac{1}{2} \left( \frac{gp_a^2}{a} + \frac{a}{g} \right) + \frac{1}{2} \left( \frac{p_a^2}{2\pi^2a^3} + 2\pi^2a^3m^2\phi^2 \right)
\]

is the Hamiltonian constraint, which in fact regulates the balance of matter and gravitational field energies at the classical level. It must be equal zero as a condition of stationarity with respect to the Lagrangian multiplier \([3]\). In equation \((4)\) we introduced a time parameter \( a, a=0 \) to \( C \), which is related to the ordinary time as follows, \( dc = N dt \) \([6]\). In this case the upper limit is a free dynamical variable.

Let us turn to quantum theory. Introducing the operators of momenta,

\[
\hat{p}_a = \frac{\hbar}{i} \frac{\partial}{\partial a}, \quad \hat{p}_\phi = \frac{\hbar}{i} \frac{\partial}{\partial \phi},
\]

we define the Hamiltonian operator as follows:

\[
\hat{H} = -\hat{H}_a + \hat{H}_\phi
\]

\[
\hat{H}_a = \frac{1}{2} \left( -\hbar^2g \frac{1}{a^3} \frac{\partial}{\partial a} a^{-2}\frac{\partial}{\partial a} + \frac{a}{g} \right)
\]

\[
\hat{H}_\phi = \frac{1}{2} \left( -\frac{\hbar^2}{2\pi^2a^3} \frac{\partial^2}{\partial \phi^2} + 2\pi^2a^3m^2\phi^2 \right)
\]

Here we chose a certain ordering of non-commuting operator multipliers in \([7]\), assuming the integration measure on the minisuperspace \((a, \phi)\) to be

\[
a^q d a d \phi.
\]

We interpret the scale factor \( a, a \in [0, \infty) \) as a ”radial” coordinate in the superspace, and take for definiteness \( q = 3 \). Quantum dynamics of the universe in our theory is described by a wave function \( \psi(c, a, \phi) \), which is a solution of the Schrödinger equation

\[
\imath\hbar \frac{\partial \psi}{\partial c} = \hat{H} \psi.
\]

with corresponding initial data. Let us mentione that \( c \in [0, C] \), and \( C \) is up to now arbitrary.

The dynamical parameter \( C \) is not observable and must be excluded in the framework of QAP \([3]\). The role of observable time in our theory will play the scale factor \( a \). But one can use the description of the quantum dynamics in terms of the formal time parameter \( c \) in the Schrödinger equation \([10]\) for regularization of the theory near the singularity \( a = 0 \). In order to make this regularization, let us slightly ”improve” the theory by introducing a signature function in the Hamiltonian \([3]\):

\[
\hat{H}(c) = f(c) \hat{H}_a + \hat{H}_\phi,
\]

where, for example,

\[
f(c) = 2 \exp \left( -\frac{c}{C_{pl}} \right) - 1.
\]

Near the singularity, \( c \to 0 \), the ”improved” Hamiltonian \([11]\) is a positive definite operator. It will be used in the next section for determining the initial quantum state of the universe at the moment \( c = 0 \). After the Plankian epoc, \( c >> C_{pl} \), the Hamiltonian \([11]\) restores its Lorentzian signature.

### III. INITIAL QUANTUM STATE OF THE UNIVERSE

Let the initial state of the universe \( \psi_0(a, \phi) \) be an eigenfunction of the Hamiltonian \([11]\) taken at the moment \( c = 0 \):

\[
\hat{H}(0) \psi_0 = E \psi_0
\]

which corresponds to its minimal eigenvalue \( E \). In analogy with ordinary quantum theory of the hydrogen atom, we expect that the ground state \( \psi_0(a, \phi) \) of the universe is non-singular. In order to find the ground state one can use the ordinary variational principle for the functional

\[
F[\psi] = \frac{\langle \psi, \hat{H}(0) \psi \rangle}{\langle \psi, \psi \rangle}.
\]

An approximation to the exact ground state is given by the following two-parametric Gauss function:

\[
\tilde{\psi}_0(a, \phi) = \exp \left( -\frac{\alpha^2 a^2}{2} - \frac{\beta \phi^2}{2} \right),
\]

for which the functional \([14]\) equals

\[
F = \frac{3}{8g} \sqrt{\frac{\pi}{\alpha}} (h^2g^2\alpha^2 + 1) + \frac{\hbar^2}{8\pi\sqrt{\pi}} \alpha^{3/2} \beta + \frac{15 \pi^{5/2} m^2}{16 \alpha^{3/2} \beta^2}.
\]

Its minimum value equals

\[
F_m = \frac{3^{1/4}}{2} \sqrt{\frac{\pi}{g}} h \left( \sqrt{\frac{15\pi}{4\sqrt{2}}} \right) \frac{m^2}{h},
\]

when

\[
\alpha = \frac{1}{\sqrt{3gh}}, \quad \beta = \frac{15}{2} \frac{3^{3/4}m^{3/2}}{\sqrt{h}}
\]
The main result of our consideration is the estimation of mean value of the scale factor in the initial ground state of the universe: it has a non-zero value of the order of the Plank length,

\[ \langle a \rangle \equiv \frac{\langle \psi_0, a \psi_0 \rangle}{\langle \psi_0, \psi_0 \rangle} = \frac{3^{5/4}}{4} \sqrt{\pi} \sqrt{\hbar g} \simeq l_{pl}. \] (19)

The energy of the scalar field in the ground state (the second term in (17)) is close to the vacuum energy \((1/2) \hbar m\). Therefore, in analogy with the ordinary quantum theory of the hydrogen atom the initial ground state of the universe is non-singular. The universe will remain in the ground state up to the moment \(c = c_{pl}\), when the signature of the Hamiltonian \((14)\) will be changed. This moment may be interpreted as "birth" of the universe. Now one can go to the semi-classical description of the universe dynamics.

### IV. CREATION OF MATTER IN THE EXPANDING UNIVERSE

Let us turn to the second stage of the universe dynamics after its "birth". Let us shift the initial moment of the proper time to the moment of "birth", and consider the universe dynamics on the interval \(c \in [0, C]\) in the semi-classical approximation proposed in [3]. The semi-classical approximation in the framework of QAP is achieved by means of an "improvement" of the original classical action before quantization. We modify the first part \((7)\) of the Hamiltonian, introducing additional complex variables \(\lambda = \lambda_1 + i\lambda_2\) and \(d = d_1 + id_2\) for description of the scale factor dynamics, as follows:

\[ \tilde{H}_a = -\frac{1}{2g} |d|^2 + \lambda (d - gp_a - ia) + \lambda (d - gp_a + ia). \] (20)

It is obvious that at the classical level the modified Hamiltonian is equivalent to the original one, if we consider \(\lambda, d\) as independent dynamical variables with canonical momenta equal zero. But now the Hamiltonian becomes linear with respect to \(p_a\). The idea [3] is to quantize the theory with the arbitrary variables \(\lambda (c), d (c)\) at the condition that they are fixed at the quantum level in the framework of QAP. The corresponding modified Schrödinger equation may be written in a form:

\[ \left( i\hbar \frac{\partial}{\partial c} - 2i\hbar g \lambda_1 \frac{\partial}{\partial a} \right) \psi = \left[ -\frac{1}{2ga} (d_1^2 + d_2^2) + (\lambda_1 d_1 - \lambda_2 d_2) + 2a\lambda_2 + \tilde{H}_{\phi} \right] \psi. \] (21)

Now, one can consider the scale factor as a function of time \(a (c)\) and the expression in the round brackets in the left hand side of (21) as a full derivative of a wave function \(\psi (c, a (c), \phi)\) with respect to \(c\), if we take \(-2g\lambda_1 \equiv \tilde{a}\). The function \(a (c)\) must be defined by means of QAP, as well.

The quantum action in QAP is defined as the real phase of the transition amplitude for a given quantum transition [3]. Let us consider the transition of the quantum oscillator \(\phi\) from the (normalized) vacuum state \(|0\rangle\),

\[ |0\rangle = \pi^{-1/4} \exp \left( -\frac{1}{2} 2\pi^2 a_0^3 \frac{m}{\hbar} \phi^2 \right), \] (22)

at the moment \(c = 0\) with \(a_0 = l_{pl}\) to an excited (normalized) state \(|n\rangle\) at the moment \(c = C\),

\[ |n\rangle = \frac{\pi^{-1/4}}{\sqrt{2^m n!}} \tilde{H}_n \left( \sqrt{2\pi^2 a_1^3 \frac{m}{\hbar}} \phi \right) \exp \left( -\frac{1}{2} 2\pi^2 a_1^3 \frac{m}{\hbar} \phi^2 \right), \] (23)

where \(a_1 \equiv a (C)\) is a final value of the scale factor. It is this quantity that will play the role of an observable time parameter in the expanding universe. The corresponding transition amplitude

\[ A_{n0} \equiv \langle n | \tilde{U}_C | 0 \rangle, \] (24)

where \(\tilde{U}_C\), is the evolution operator of the modified Schrödinger equation (21) in the interval \([0, C]\), can be written in a form:

\[ A_{n0} = \exp \left( \frac{i}{\hbar} \Lambda_{Ca} \right) \langle n | \tilde{U}_C \phi | 0 \rangle, \] (25)

where \(\tilde{U}_C\phi\) is the evolution operator of the Schrödinger equation for only scalar field part on a classical homogeneous space-time background with arbitrary function \(a (c)\):

\[ i\hbar \frac{\partial \psi}{\partial c} = \tilde{H}_{\phi} \psi, \] (26)

and

\[ \Lambda_{Ca} \equiv \int_0^C dc \left[ \frac{1}{2ga} (d_1^2 + d_2^2) + \left( \frac{\tilde{a}}{2} d_1 + \lambda_2 d_2 \right) - 2a\lambda_2 \right]. \] (27)

At this stage we can find the stationary value of the phase (27) as a function of additional variables \(d_1, \lambda_2\). A resulting quantity is equal to the gravitational part of the original classical action (22).

It is the equation (26), that describes the creation of matter in the expanding universe (minisuperspace model) in the approach mentioned above [1], at the condition that the dynamics of \(a (c)\) is classical. The parametric excitation of the scalar field \(\phi\) arises due to the dependence of \(\tilde{H}_{\phi}\) on \(a (c)\) in accordance with [8]. If we only take into account this mechanism of excitation, we will lose the balance between matter and gravitation field energies, that is regulated by the equation (4) in classical theory. We shall restore this balance by taking into account the dynamics of the scale factor \(a (c)\) in the framework of QAP, and the important additional condition of stationarity of a quantum action with respect to
Therefore, in our approach we have overcame only half of the way.

Let us turn to the formulation of QAP. We define the full quantum action as a sum,
\[ \Lambda = \Lambda_{Ca} + \Lambda_{C\phi}. \] (28)
Here $\Lambda_{Ca}$ is the classical action for the scale factor $a(c)$, $\Lambda_{C\phi}$ is the real phase of the transition amplitude $\langle n|\hat{U}_{C\phi}|0\rangle$ written in the exponential form:
\[ \langle n|\hat{U}_{C\phi}|0\rangle = R_{C\phi} \exp \left( \frac{i}{\hbar} \Lambda_{C\phi} \right). \] (29)
In analogy with classical action principle, we formulate QAP as a set of the stationarity conditions of the quantum action:
\[ \frac{\delta \Lambda}{\delta a(c)} = 0, \quad \frac{\partial \Lambda}{\partial C} = 0. \] (30)
The first equation in (30) defines the (semi-classical) dynamics of the scale factor $a(c)$ with fixed boundary values $a(0) = a_0$, $a(C) = a_1$. This equation of motion takes into account the back-reaction of quantum dynamics of the scalar field $\phi$. The second equation in (30) fixes the proper time $C$. It is this condition, that restores the balance of energies in the creation of matter process. The balance is restored by tuning of the proper time $C$ to given parameters of the final state of the universe, i.e. the quantum number $n$, and the spatial size of the universe $a_1$. Let us stress that the semi-classical history of the scale factor and the length of the proper time interval are defined for a given quantum transition $|0\rangle \rightarrow |n\rangle$. We must substitute these quantities into the amplitude (29).

Calculation of probabilities in QAP needs a special consideration. The quantum evolution of matter described by the Schrödinger equation (26) is unitary. But additional conditions (30) and a posteriori manipulations with amplitude (29) destroy the unitarity. We must renormalize all amplitudes (29) ($n = 0, 1, 2, \ldots$) after taking into account the back-reaction of the creation of matter process on the dynamics of the scale factor and the energy balance. The properly normalized transition amplitude of the process $|0\rangle \rightarrow |n\rangle$ is
\[ K_n(a_1) = \frac{1}{\sqrt{Z}} \langle n|\hat{U}_{C\phi}|0\rangle, \quad Z \equiv \sum_{n=0}^{\infty} \left| \langle n|\hat{U}_{C\phi}|0\rangle \right|^2. \] (31)
This is the probability density to detect $n$ quanta of the homogeneous scalar field $\phi$ at the moment when the radius of the universe will be equal $a_1$. We expect that the probability will be maximum for states with energy equal to the gravitational energy corresponding to a macroscopic value of the scale factor $a_1$.

V. CONCLUSIONS

The balance between matter and gravitational energies in the universe, which is guaranteed by the Hamiltonian constraint equation in classical General Relativity is realized in the present work at the quantum level as a condition of the stationarity of a quantum action with respect to an internal time of the universe. The latter is not observable quantity in a quantum universe. The role of observable time in the universe plays its spatial scale factor $a$. We expect that the properly normalized transition amplitude (31), which takes into account the energy balance, will give us a sufficient rate for the quantum creation of matter in the expanding universe.

Acknowledgments

We are thanks V. A. Franke and A. V. Goltsev for useful discussions.