Semi-Lagrangian shallow water equations solver on the cubed-sphere grid as a prototype of new-generation global atmospheric model

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Abstract. Next generation weather prediction atmospheric models will have horizontal resolution of about 3-5 km. The problem dimension will be $10^4 - 10^5$. One will need to use efficiently $10^4 - 10^5$ computational cores to make a practical operational forecast. This leads to the need for the deep revision of numerical methods and algorithms used in atmospheric models. One of the problems to be solved is the horizontal discretization of atmospheric dynamics equations on the quasi-uniform spherical grids. This problem can be investigated using shallow water model that is much computationally cheaper than the use of full atmosphere model. We are developing an atmospheric dynamics solver for the next generation numerical weather prediction model at the Institute of Numerical Mathematics and Hydrometeorological center of Russia. Within this work, the solver for the shallow water equations using gnomonic cubed-sphere grid has been developed. The solver is verified using standard shallow water test cases. The accuracy of the presented solver is analysed. The good agreement to the reference solutions is achieved, when 4-th order spatial approximations are used.

1. Introduction

The progress of computational systems will allow to advance the horizontal resolution of atmospheric models to 3-5 km in the numerical weather prediction and 10-20 km in the climate studies in the nearest future. Such models have to use efficiently $O(10^4) - O(10^6)$ computational cores to deliver simulation result in the reasonable time. This means that regular latitude-longitude grid that was used in atmospheric modelling for years has to be abandoned. The meridians convergence makes the longitudinal grid spacing too small at polar regions. That forces one either to use very small and impractical time-step or apply special numerical techniques like the Fourier polar filtering that degrade parallel efficiency. It is believed that regular latitude-longitude grid cannot be used with the horizontal resolution finer than 10 km [1].

Quasi-uniform spherical meshes are needed for the high-resolution atmospheric simulations. However, each of grids known to date has one or more of the following drawbacks: non-orthogonality, overset or joining regions, complex geometry, non-quadrilateral cells, etc [1]. Considering the horizontal discretization of atmospheric dynamics equations, each problem of...
the grid can lead to the loss of accuracy. One needs to find a reasonable compromise between pros and cons of the grid taking in mind the specifics of the future model development and application. The most popular options of the quasi-uniform spherical meshes are icosahedral, cubed sphere and Yin-Yang grids [2].

The new generation atmospheric model is under development at the G.I. Marchuk Institute of Numerical Mathematics and Hydrometeorological center of Russia. We chose the equiangular cubed sphere grid for the model [3]. This grid is obtained by the central projection of the grid at the faces of cube to the inscribed sphere. The main advantages of the cubed sphere grid are high uniformity, quadrilateral cells and rectangular logical structure. The drawbacks are the lack of orthogonality and the breaks of coordinate lines on the images of cube edges.

While developing new horizontal approximations for a grid with complex geometry, it is desirable to have computationally cheap and informative test for it. In the context of atmospheric modelling, such tests are provided by the shallow water model on the rotating sphere:

\[
\frac{d\vec{v}}{dt} = -fk \times \vec{v} - f\nabla h,
\]

\[
\frac{dh}{dt} = -h\nabla \cdot \vec{v},
\]

where \(d/dt\) is the Lagrangian (material) derivative, \(\vec{v}\) is the horizontal wind vector, \(f\) is the Coriolis parameter, \(k\) is the vertical unit vector, \(h\) is the fluid layer height.

This quasi-2D model is derived from the full atmospheric hydrothermodynamics equations system using non-compressibility assumption and hydrostatic approximation in vertical [4]. Although being very idealized, this model retains analogues of basic atmospheric phenomena such as inertia-gravity and Rossby waves, advection, instability development. Besides, shallow water equations system is a stiff problem from computational point of view and therefore is a good test for temporal approximations.

We have developed finite-difference semi-implicit semi-Lagrangian (see [5]) shallow water prototype for the full atmosphere model on the cubed sphere grid. This solver is described in the current paper. The paper is organized as follows: the basic information about cubed sphere grid is given in section 2, the summary of numerical methods used is presented in section 3, numerical tests results are given in section 4.

2. Cubed sphere grid
The equiangular gnomonic cubed-sphere grid is obtained via central projection of the grid on the cube faces onto inscribed sphere. The construction procedure, basic differential operators and other information about this grid is given in [3]. The grid on the cube faces is set using angular coordinates. If cube edges are aligned with coordinate axis, cube center is in the origin and edge length is 2, the grid nodes on the face with outward normal \((0, 0, 1)\) are \((\tan \alpha_i, \tan \beta_j, 1)\). Here \(\alpha_i = -\pi/4 + (i - 0.5)\Delta, \beta_j = -\pi/4 + (j - 0.5)\Delta, i, j \in [1, N_c], \Delta = \pi/(2N_c), N_c\) is the grid dimension. The coordinates of corresponding points on the inscribed sphere of unit radius are \((\tan \alpha_i, \tan \beta_j, 1)/\sqrt{1 + \tan^2 \alpha_i, \tan^2 \beta_j}\). The cubed-sphere grid with \(N_c = 16\) is shown in figure 1a.

We use the non-staggered grid (Arakawa [6] type-A) where \(h\) and \(\vec{v}\) components are stored at the same grid points. Numerical methods on A-type grid fail to correctly represent short inertia-gravity waves, but are generally faster than staggered grid methods. Given our further choice of methods described in section 3 the short-wave part of spectrum is dumped, therefore using A-type grid is acceptable.
3. Numerical methods

3.1. Semi-implicit semi-Lagrangian time integration

The material derivative $d\psi/dt$, where $\psi$ is an arbitrary quantity, is the rate of change of $\psi$ with time along the trajectory of some fluid parcel. At each step, semi-Lagrangian (SL) methods consider the family of parcels that arrive to the computational grid nodes by the end of this time-step.

An arbitrary equation $d\psi/dt = A(\psi)$ can be integrated in time giving:

$$\psi^{n+1} = \psi^*_n + \int_{t^n}^{t^{n+1}} A(\psi)d\psi,$$

(3)

where $\psi^*_n$ is $\psi$ at time $t^n$ at the departure position of considered fluid parcel. The $\psi^*_n$ is found by interpolation of $\psi^n$ grid point values.

SL methods allow to hide advective terms ($-\vec{v}\cdot\nabla \psi$) inside the material derivative and thus circumvent CFL time-step restriction for the wind transport process. The stability of SL methods is limited by less restrictive Lipschitz condition that means non-crossing of individual parcels trajectories. SL methods can usually be used with CFL=3-5.

We use SETTLS SL scheme [7] to find the departure positions of fluid parcels. Vector SL formulation [8] is used to account for Earth curvature in the vector equations. The values at departure positions are calculated using Lagrangian interpolation with 4-th order polynomials.

Semi-implicit time-integration methodology [9] consists in splitting model operator into linear and non-linear parts. Linear part is integrated implicitly and explicit method is used for non-linear part. When linear operator describes the fastest waves in the model (that is usual for atmospheric modelling), semi-implicit method allows to increase the step size several times as compared to fully explicit method at the cost of solving one system of linear equations per time step.

We select linear and non-linear parts of the operator by splitting the right-hand side of (2) as

$$-h\nabla\cdot\vec{v} = -h'\nabla\cdot\vec{v} - H_0\nabla\cdot\vec{v},$$

where $H_0 = const \gg h'$. Then, the combination of Crank-Nicolson and SETTLS schemes is used to approximate integral in (3):

$$\psi^{n+1} = \psi^*_n + \frac{\Delta t}{2} \left[ L\psi^{n+1} + (L\psi^n)_s + N(\psi^n) + \left(N(\psi^{n+1})_s\right)_s\right] + O(\Delta t^3),$$

(4)
where $L$ and $N$ are the linear and non-linear parts of operator correspondingly, $(..)_*$ denotes the interpolation to departure point location.

### 3.2. Finite-difference approximation

We use the following 4-th order accurate formula to approximate partial derivatives with respect to $\alpha$ and $\beta$ in both divergence and gradient operators:

$$
\left(\frac{\partial \psi}{\partial x}\right)_i = -\psi_{i+2} + 8\psi_{i+1} - 8\psi_{i-1} + \psi_{i-2} + \frac{1}{12\Delta} + O(\Delta^4),
$$

where $x$ is $\alpha$ or $\beta$.

### 3.3. Halo-zones

We need the values of quantities at the grid points located beyond their face (that means $|\alpha| > \pi/4$ or $|\beta| > \pi/4$, see figure 1b) to perform numerical differentiation via equation (6) or interpolation (see section 3.1) near edges of the cube without loss of accuracy. We call such points virtual (contrary to the real points located inside the face). The field values at virtual points are found by interpolation from real points of adjacent faces. The interesting feature of gnomonic cubed sphere grid is that virtual points lay on the same great circle arc as real points of adjacent face. Therefore, we need only 1D interpolation in most of cases. Cubic Lagrangian interpolation is used.

### 4. Results of Numerical experiments

![Figure 2](image-url)

**Figure 2.** Solution convergence as a function of grid resolution. Panel a) – the root mean squared error (RMSE) of the fluid height field at day 10 in steady state geostrophic balance test. Panel b) – RMSE of the wind at day 6 in barotropic instability test case with respect to $513 \times 1024$ regular lat-lon grid solution (green curve) and the solution of the same model with $N_c = 512$ (orange curve). Upper and lower black lines on both panels indicate 3rd and 2nd order convergence correspondingly.
Figure 3. Day 6 vorticity field in the barotropic instability test. Numerical solutions using model [10] at 513×1024 regular latitude-longitude grid and the presented model at cubed sphere grids of various resolution.

The generally accepted practice is to test numerical shallow water models with tests from suite [11] extended by the case [12]. We carried out all of these tests and see the good agreement with results published in literature, the two most demonstrative cases are analysed below.

The first case is steady state non-linear geostrophically balanced flow from [11]. This problem tests the accuracy of model in reproduction of the triple balance between pressure gradient force, Coriolis force and curvature terms. Also, the accuracy of divergence operator approximation is tested. We run the test using $\Delta t = 2400$ s with $N_c = 96$ and keep $N_c \Delta t$ the same at finer grids. Such choice of time step gives inertia-gravity waves and wind CFL of about 3.5 and 0.8 correspondingly. Figure 2a shows the root mean square error (RMSE) of fluid height field after 10 days of integration. The solution convergence is between 2nd and 3rd order.

The second presented test is barotropic instability development [12]. The initial conditions describe narrow jet in barotropically unstable geostrophic balance with superimposed small
height field perturbation. Unstable modes grow exponentially till day 5 of experiment when non-linear effects become important and waves break. We use the same grids and time-steps configurations as in the previous test, that give CFL of about 8 and 2.5 for inertia gravity waves and wind respectively.

Figure 2b shows the RMSE of wind field at day 6 with respect to the same model high-resolution solution with $N_c = 512$ and to the solution of the shallow water model at $513 \times 1024$ regular latitude-longitude grid [10]. Self-convergence of the presented model is between 3-rd and 2-nd order, that coincides with the results of the steady-state test. The presented model shows 3-rd order convergence to the regular lat-lon grid reference solution at coarse resolution. The convergence almost stagnates at finer grids. This is due to the solution uncertainty that arises because of inherent instability, unequal representation of initial conditions on different grids, and various model aspects like numerical dissipation that cannot be made similar in different models.

Figure 3 shows relative vorticity field at day 6. One can note that already at $N_c = 96$ grid physical signal dominates over the spurious generation of waves number 4 and 8 (that is a common drawback of low-resolution cubed-sphere solutions). The vorticity field becomes more and more developed as resolution increases. At $N_c = 256$ grid, the solution is visually indistinguishable from the regular lat-lon grid solution. Also, good agreement can be found with the high-resolution results from the high-order Eulerian shallow water models at cubed sphere grid [13], [14].

5. Conclusion
We developed the semi-implicit semi-Lagrangian shallow water model on the cubed-sphere grid. The accuracy of the presented model in the standard test cases is nearly similar to its Eulerian analogues [13], [14]. At the same time, the stability properties of semi-implicit semi-Lagrangian approach allow to use large time-steps (CFL of about 5-10) that results in better computational efficiency as compared to Eulerian models. These results encourage us to extend the used methods to the case of full 3D atmosphere hydrothermodynamics equations system.

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