Nonlinear Spinor Fields in LRS Bianchi type-I spacetime: Theory and observation

Bijan Saha and Victor S. Rikhvitsky

Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Moscow region, Russia

Within the scope of a LRS Bianchi type-I cosmological model we study the role of the nonlinear spinor field in the evolution of the Universe. In doing so we consider a polynomial type of nonlinearity that describes different stages of the evolution. Finally we also use the observational data to fix the problem parameters that match best with the real picture of the evolution. The assessment of the age of the Universe in case of the soft beginning of expansion (initial speed of expansion in a point of singularity is equal to zero) the age was found 15 billion years, whereas in case of the hard beginning (nontrivial initial speed) it was found that the Universe is 13.7 billion years old.

PACS numbers: 98.80.Cq
Keywords: Spinor field, dark energy, anisotropic cosmological models, isotropization
I. INTRODUCTION

The discovery and further confirmation of the existence of the accelerated mode of expansion of the present day Universe [1, 2] lead cosmologists to construct new theories able to explain this new cosmological findings. Though cosmological constant, quintessence, Chaplygin gas etc. are the prime candidates, thanks to a number of remarkable works [3–17], recently many authors considered the spinor fields as a possible alternative to these models. And it is because of the spinor fields’ ability to simulate different type of source fields from ekpyrotic matter to phantom matter and Chaplygin gas [18–22]. Moreover it was found that a nonlinear spinor field can also (i) generate singularity-free Universe [5–7, 9, 10]; (ii) accelerate the isotropization process of the initially anisotropic spacetime [7, 9, 12] and (iii) give rise to a late time accelerated mode of expansion [11, 13–15].

Some recent studies show that the non-diagonal components of the energy-momentum tensor of the spinor field can play significant role on the geometry of spacetime as well as on the components of the spinor field itself, namely on the spinor field nonlinearity [23–30]. In those papers it was shown that depending on the specificity of metric in some cases the spinor field nonlinearity and the mass terms vanish all together, whereas in other cases both the mass term and nonlinear term do not only disappear but also play a crucial role in the evolution of the Universe. In a recent paper [30] it was found that within the scope of a LRS Bianchi type-I spacetime the spinor field nonlinearity depending on the sign of self-coupling constant allows either an expansion with acceleration or an oscillatory mode of evolution of the Universe. In this note we develop that theoretical work further and fix the values of the problem parameters using the recent observational data.

II. BASIC EQUATIONS

The LRS Bianchi type-I (BI) model is the ordinary Bianchi type-I model with two of the three metric functions being equal to each other and can be given by

\[ ds^2 = dt^2 - a_1^2 [dx^2 + dy^2] - a_3^2 dz^2, \]

with \( a_1 \) and \( a_3 \) being the functions of time only.

Keeping this in mind the symmetry between \( \psi \) and \( \bar{\psi} \) we choose the symmetrized Lagrangian [31] for the spinor field as [7]:

\[ L = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - m_{sp} \bar{\psi} \psi - F, \]

where the nonlinear term \( F \) describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field. We consider \( F = F(K) \), with \( K = \{ I, J, I + J, I - J \} \). It can be shown that such a choice describes the nonlinearity in its most general form.

Varying (2.2) with respect to \( \bar{\psi} (\psi) \) one finds the spinor field equations:

\[ i \gamma^\mu \nabla_\mu \psi - m_{sp} \bar{\psi} - 2F_K (SK_I + iPK_J \gamma^5) \psi = 0, \quad (2.3a) \]

\[ i \nabla_\mu \bar{\psi} \gamma^\mu + m_{sp} \bar{\psi} + 2F_K \bar{\psi} (SK_I + iPK_J \gamma^5) = 0. \quad (2.3b) \]

Here we denote \( F_K = dF / dK, K_I = dK / dI \) and \( K_J = dK / dJ \).

The energy-momentum tensor of the spinor field is given by

\[ T_\mu^\nu = \frac{i}{4} g^{\rho \nu} \left( \bar{\psi} \gamma_\rho \nabla_\nu \psi + \psi \bar{\psi} \gamma_\rho \nabla_\nu \psi - \nabla_\rho \bar{\psi} \gamma_\nu \psi - \nabla_\rho \psi \bar{\psi} \gamma_\nu \psi \right) - \delta_\mu^\nu L_{sp} \]
where $L_{sp}$ in view of (2.3) can be rewritten as

$$L_{sp} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - m_{sp} \bar{\psi} \psi - F(K),$$

$$= \frac{i}{2} \bar{\psi} [\gamma^\mu \nabla_\mu - m_{sp} \bar{\psi}] - \frac{i}{2} [\nabla_\mu \bar{\psi} \gamma^\mu + m_{sp} \bar{\psi}] \psi - F(K),$$

$$= 2F_K (IK + JK) - F = 2KF_K - F(K). \quad (2.5)$$

It can be shown that the spinor field in this case possesses nontrivial non-diagonal components of the energy-momentum tensor. On account of that the system of Einstein equations can be written as [30]

$$\ddot{a}_3 a_3 + \ddot{a}_1 a_1 + \dot{a}_3 \dot{a}_1 = \kappa (F(K) - 2KF_K), \quad (2.6a)$$

$$2 \ddot{a}_1 a_1 + \dot{a}_1^2 = \kappa (F(K) - 2KF_K), \quad (2.6b)$$

$$\dot{a}_3^2 + 2 \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} = \kappa (m_{sp}S + F(K)), \quad (2.6c)$$

$$0 = \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} \right) A^2, \quad (2.6d)$$

$$0 = \left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right) A^1. \quad (2.6e)$$

where $A^\mu = \bar{\psi} \gamma^5 \gamma^\mu \psi$ are the components of the pseudovector.

### III. SOLUTION TO THE FIELD EQUATIONS

In this section we solve the equations obtained in the previous section. The off-diagonal components of the Einstein equations (2.6a) and (2.6c) impose the following restrictions either on the components of the spinor field or on the metric functions:

$$A^2 = 0, \quad A^1 = 0, \quad \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} \right) = 0. \quad (3.1a)$$

$$\left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right) = 0. \quad (3.1b)$$

From (3.1b) we duly find $a_3 = q_0 a_1$ with $q_0$ being some constant. In this case the system can be described by a FRW model from the very beginning. Hence we don’t consider this case here. We will do it in some of our forthcoming papers on FRW model.

So we consider (3.1a) when the restriction is imposed on the components of the spinor field. Subtraction of (2.6b) from (2.6a) gives

$$\frac{\ddot{a}_3}{a_3} - \dot{a}_3 a_1 - \frac{\dot{a}_1}{a_1} \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} \right) = 0, \quad (3.2)$$

that leads to [7]

$$a_1 = D_1 V^{1/3} \exp \left( X_1 \int \frac{dt}{\sqrt{V}} \right), \quad a_3 = (1/D_1^3) V^{1/3} \exp \left( -2X_1 \int \frac{dt}{\sqrt{V}} \right). \quad (3.3)$$
with \( D_i \) and \( X_i \) being the integration constants. Thus we see that the metric functions can be expressed in terms of \( V \).

Our next step will be to define \( V \). Combining the diagonal Einstein equations (2.6a), (2.6b) and (2.6c) in a certain way for \( V \) we find [7]

\[
\dot{V} = \frac{3\kappa}{2} m_{sp} S \left( m_{sp} V + 2 \left( F(K) - KF_K \right) V \right).
\]

(3.4)

Now order to solve (3.4) we have to know the relation between the spinor and the gravitational fields. Using the equations

\[
\begin{align*}
\dot{S}_0 + \mathcal{H} A_0^0 &= 0, \\
\dot{P}_0 - \Phi A_0^0 &= 0,
\end{align*}
\]

(3.5a)

(3.5b)

where we denote \( S_0 = SV, P_0 = PV \) it can be show that

\[
K = \frac{V_0^2}{V^2}, \quad \mathcal{K} = \{1, J, I + J, I - J\}.
\]

(3.6)

The relation (3.6) holds for \( K = \{J, I + J, I - J\} \) only for massless spinor field, while for \( K = I \) it holds both for massless and massive spinor field. In case of \( K = I + J \) one can write \( S = \sin(V_0/V) \) and \( P = \cos(V_0/V) \), whereas for \( K = I - J \) one can write \( S = \cosh(V_0/V) \) and \( P = \sinh(V_0/V) \).

In what follows, we will consider the case for \( K = I \), setting \( F = \sum_k \lambda_k I_n^k = \sum_k \lambda_k S_{2n_k} \), as in this case further setting spinor mass \( m_{sp} = 0 \) we can revive the results for other cases.

Then inserting \( F = \sum_k \lambda_k I_n^k = \sum_k \lambda_k S_{2n_k} \) into (3.4) and taking into account that in this case \( S = V_0/V \) we find

\[
\dot{V} = \frac{3\kappa}{2} \left[ m_{sp} V_0 + 2 \sum_k \lambda_k (1 - n_k) V_0^{2n_k} V^{1 - 2n_k} \right],
\]

(3.7)

with the solution in quadrature

\[
\int \frac{dV}{\sqrt{3\kappa \left[ m_{sp} V_0 V + \sum_k \lambda_k V_0^{2n_k} V^{2(1 - n_k) + \tilde{C}} \right]}} = t + t_0,
\]

(3.8)

with \( \tilde{C} \) and \( t_0 \) being some arbitrary constants.

In what follows we solve the equations for \( V \), i.e., (3.7) numerically. But before doing that we write the solution to the spinor field equations explicitly.

The solutions to the spinor field equations (2.3) in this case can be presented as

\[
\psi_{1,2}(t) = \frac{C_{1,2}}{\sqrt{V}} \exp \left( -i \int \Phi dt \right), \quad \psi_{3,4}(t) = \frac{C_{3,4}}{\sqrt{V}} \exp \left( i \int \Phi dt \right),
\]

(3.9)

with \( C_1, C_2, C_3, C_4 \) being the integration constants and related to \( V_0 \) as

\[
C_1^* C_1 + C_2^* C_2 - C_3^* C_3 - C_4^* C_4 = V_0.
\]

Thus we see that the metric functions, the components of spinor field as well as the invariants constructed from metric functions and spinor fields are some inverse functions of \( V \) of some degree. Hence at any spacetime point where \( V = 0 \) it is a singular point. So we consider the initial value of \( V(0) \) is small but non-zero. As a result for the nonlinear term to prevail in (3.7) we should have \( 1 - 2n_k < 0 \), i.e., \( n_k > 1/2 \), whereas for an expanding Universe when \( V \to \infty \) as \( t \to \infty \) one should have \( 1 - 2n_k > 0 \), i.e., \( n_k < 1/2 \). As is seen from (3.7), \( n_k = 1/2 \) leads to a term that can
be added to the mass term. So without losing the generality we can consider \( n_0 = 1/2, n_1 = 0 \) and \( n_2 = 2 \).

In this case we obtain

\[
\ddot{V} = \Phi_1(V), \quad \Phi_1(V) = \frac{3\kappa}{2} \left[ (m_{sp} + \lambda_0) V_0 + 2\lambda_1 V - 2\lambda_2 V_0^4 V^{-3} \right]. \tag{3.10}
\]

Equation (3.10) allows the first integral

\[
\dot{V} = \Phi_2(V), \quad \Phi_2(V) = \sqrt{3\kappa \left[ (m_{sp} + \lambda_0) V_0 V + \lambda_1 V^2 + \lambda_2 V_0^4 V^{-2} + \bar{C} \right]}. \tag{3.11}
\]

The solution to the equation (3.10) can be written in quadrature as follows

\[
\int \frac{dV}{\Phi_2(V)} = t + t_0. \tag{3.12}
\]

To solve the (3.10) we should choose the problem parameters \( V_0, m_{sp}, \kappa, \bar{C}, \lambda_k \) as well as the initial value of \( V(0) \) in such a way that does not leads to

\[
(m_{sp} + \lambda_0) V_0 V + \lambda_1 V^2 + \lambda_2 V_0^4 V^{-2} + \bar{C} < 0.
\]

In a recent paper [30] we considered the case with \( \lambda_0 = \lambda_1 = \lambda_2 = \lambda \). It was shown that in case of positive \( \lambda \) we have an accelerated mode of expansion of the Universe, while for negative \( \lambda \) we have oscillatory solution.

In this paper we do not perform numerical analysis to obtain different type of solutions for different values of problem parameters. In what follows we study the equation (3.10) numerically to find the problem parameters that fit best with the observational data.

**IV. COMPARISON WITH OBSERVATIONS**

In what follows we numerically solve the equation (5.7). Let us rewrite this equation as follows:

\[
\dot{V} = A + \sum_{k=1}^{3} B_k (1 - n_k) V^{1 - 2n_k}. \tag{4.1}
\]

Here \( A = \frac{3\kappa}{2} (m_{sp} + \lambda_0) V_0 \), i.e., the constant term, whereas \( B_k = 2\lambda_k V_0^{2n_k} \).

Defining Hubble parameter and red-shift as follows

\[
H = \frac{1}{3} \frac{\dot{V}}{V} = \frac{\dot{a}}{a}, \tag{4.2a}
\]

\[
z + 1 = \left( \frac{1}{V} \right)^{1/3} = \frac{1}{a}, \tag{4.2b}
\]

we rewrite (4.1) in the form

\[
\frac{\partial}{\partial z} \left( \frac{H}{(z+1)^4} \right)^2 = -\frac{2}{3} \frac{A}{(z+1)^4} - \frac{2}{3} \sum_{k=1}^{3} B_k (1 - n_k) (z+1)^{6n_k-7},
\]

\[
\frac{\partial}{\partial z} t = -\frac{1}{(z+1)^4 \sqrt{H}}
\]

with initial values \( t(0) = 0, H(0) = H_0 \).
The foregoing equation allows the following solution

\[ H(z) = \sqrt{\frac{2}{9}A(z+1)^3 + \frac{1}{9} \sum_{k=1}^{3} B_k (z+1)^{6n_k} + C_H(z+1)^6}, \]  

where \( C_H \) is constant of integration.

The numerical values of the parameters such as spinor mass, power of nonlinearity etc. are determined by comparing the solutions to astrophysical observations exploiting the maximum likelihood method by minimizing the functional

\[ \chi^2 = \sum \left( \frac{H(z_i) - H_i}{\sigma_{H_i}} \right)^2, \]  

where \( z_i, H_i \) and \( \sigma_{H_i} \) were taken from the tables given in [32–34].

From (4.2b) it follows that \( z \to \infty \) leads to \( a(t) \to 0 \), i.e., there occurs a space-time singularity. For an expanding Universe it means \( z \to \infty \) at the time of Big Bang. Let us assume that \( \dot{a}|_{z = \infty} = 0 \) (soft origin). From \( \dot{a} = \frac{H}{z+1} \to 0 \) we see that it may happen it \( z \) increases faster than \( H \) (not necessarily \( H = 0 \)). Then from (4.4) we can conclude that \( A = 0, C_H = 0 \) and \( n_k < 1/3 \). The equality \( A = 0 \) states that in this case the spinor mass is either zero or compensated by some constant. Analogical results were found in the papers where dark energy was simulated by spinor field [27].

The value of \( n_k \) is searched using the method of random walk with the selection of successful steps. At the beginning \( n_k(0) = 1 \) and \( n_k(0) = n_k(0) \). At each \( m \)-th step we replace \( n_k^{(m+1)} = n_k^{(m)} + 0.01 \xi \) using standard normal distribution of random variable \( \xi \in N(0,1) \), but under the condition \( \chi^2(n_k^{(m+1)}) < \chi^2(n_k^{(m)}) \).

To determine \( A \) and \( B_k \) at each \( m \)-th step for fixed \( n_k \) the functional (4.5) is approximately substituted by

\[ \tilde{\chi}^2 = \sum \left( \frac{H(z_i)^2 - H_i^2}{2H_i\sigma_{H_i}} \right)^2, \]  

what is quadratic to coefficients to be determined and the quest of the minimum is reduced to the solution of linear equations. In Fig. 1 we have plotted the dynamics of the process of finding the minimum \( \chi \).

**FIG. 1.** Dynamics of the process of finding the minimum \( \chi^2 \).
The adjusted values of parameters are
\[ n_1 = 0.1961075852, \quad B_1 = 8.033381 \cdot 10^6 = 3\kappa\lambda_1 V_0^{2n_1}, \]
\[ n_2 = 0.1922624717, \quad B_2 = -1.073812 \cdot 10^6 = 3\kappa\lambda_2 V_0^{2n_2}, \] 
\[ n_3 = 0.1805648222, \quad B_3 = 2.709600 \cdot 10^6 = 3\kappa\lambda_3 V_0^{2n_3}. \] (4.7)

At present epoch model time \( t = 0 \), red-shift \( z = 0 \) and Hubble parameter \( H_0 = 73.5 \). It is known that \( z = Ht \) for small \( z \), so comparing at red-shift \( z = 0.001 \) the astronomical time \( t^a(0.001) = 0.001/H_0 \) with model time \( t(0.001) \) we can obtain the age of Universe \( T = [t(\infty)/t(0.001)]t^a(0.001) = 15.0 \cdot 10^9 \) years. The confidence level is found to be \( CL = 0.92 \).

In case of \( A \neq 0 \) (hard origin), the adjusted values of parameters are
\[ A = -39093.17887 = \frac{3\kappa}{2}(m_{sp} + \lambda_0) V_0, \]
\[ n_1 = 0.3318171453, \quad B_1 = 10.70355153 \cdot 10^6 = 3\kappa\lambda_1 V_0^{2n_1}, \]
\[ n_2 = 0.3122973942, \quad B_2 = -21.81174544 \cdot 10^6 = 3\kappa\lambda_2 V_0^{2n_2}, \]
\[ n_3 = 0.2939425415, \quad B_3 = 11.23402881 \cdot 10^6 = 3\kappa\lambda_3 V_0^{2n_3}. \] (4.8)

At present epoch model time \( t = 0 \), red-shift \( z = 0 \) and Hubble parameter \( H_0 = 72.8 \), we can obtain the age of Universe \( T = 13.7 \cdot 10^9 \) years. The confidence level is found to be \( CL = 0.93 \).

In Figs. 2 and 3 we have plotted the \( H(z)/(1+z) \) data (32 points) and model prediction (line for best-fit model) as a function of red-shift (logarithmic scale) for hard and soft origins, respectively. In Figs. 4 and 5 \( H(z)/(1+z) \) model prediction as a function of time \( t \) (years) has been drawn for hard and soft origins, respectively. Figs. 6 and 7 we have demonstrated the evolution of volume scale \( V \) (logarithmic scale) model prediction as a function of time \( t \) (years) for hard and soft origins, respectively.
Within the scope of LRS Bianchi type-I model we have studied the role of spinor field in the evolution of the Universe. Since the expression for metric functions, components of the spinor field and invariants constructed from these quantities are in functional dependence on volume scale $V$ it is important to find and solve the equation for $V$. In this report we have solved the equation.

V. CONCLUSION
in question and in doing so we have exploited the astronomical data available to fix the problem parameters such as coupling constants etc. that provides the best correspondence with observation. The assessment of the age of the Universe in case of the soft beginning of expansion (initial speed of expansion in a point of singularity is equal to zero) the age was found 15 billion years, whereas in case of the hard beginning (nontrivial initial speed) it was found that the Universe is 13.7 billion years old.

Acknowledgments
This work is supported in part by a joint Romanian-LIT, JINR, Dubna Research Project, theme no. 05-6-1060-2005/2013.

[1] A.G. Riess et al., *Astron. J.* **116**, 1009 (1998)
[2] S Perlmutter et al., *Astrophys. J.* **517**, 565 (1999)
[3] M. Henneaux *Phys. Rev. D* **21**, 857 (1980)
[4] U. Ochs and M. Sorg *Int. J. Theor. Phys.* **32**, 1531 (1993)
[5] B. Saha and G.N. Shikin *Gen. Relat. Grav.* **29**, 1099 (1997)
[6] B. Saha and G.N. Shikin *J Math. Phys.* **38**, 5305 (1997)
[7] B. Saha *Phys. Rev. D* **64**, 123501 (2001)
[8] C. Armendáriz-Picón and P.B. Greene *Gen. Relat. Grav.* **35**, 1637 (2003)
[9] B. Saha and T. Boyadjiev *Phys. Rev. D* **69**, 124010 (2004)
[10] B. Saha *Phys. Rev. D* **69**, 124006 (2004)
[11] M.O. Ribas, F.P. Devecchi, and G.M. Kremer *Phys. Rev. D* **72**, 123502 (2005)
[12] B. Saha *Phys. Particle. Nuclei.* **37**. Suppl. 1, S13 (2006)
[13] B. Saha *Phys. Rev. D* **74**, 124030 (2006)
[14] B. Saha *Grav. & Cosmol.* **12**(2-3)(46-47), 215 (2006)
[15] B. Saha *Romanian Rep. Phys.* **59**, 649 (2007).
[16] R.C de Souza and G.M. Kremer *Class. Quantum Grav.* **25**, 225006 (2008)
[17] G.M. Kremer and R.C de Souza arXiv:1301.5163v1 [gr-qc]
[18] V.G.Krechet, M.L. Fel’chenkov, and G.N. Shikin *Grav. & Cosmol.* **14** No 3(55), 292 (2008)
[19] B. Saha *Cent. Euro. J. Phys.* **8**, 920 (2010a)
[20] B. Saha *Romanian Rep. Phys.* **62**, 209 (2010b)
[21] B. Saha *Astrophys. Space Sci.* **331**, 243 (2011)
[22] B. Saha *Int. J. Theor. Phys.* **51**, 1812 (2012)
[23] L. Fabbri *Int. J. Theor. Phys.* **52** 634 (2013)
[24] L. Fabbri *Phys. Rev. D* **85** 0475024 (2012)
[25] S. Vignolo, L. Fabbri, and R. Cianci *J. Math. Phys.* **52** 112502 (2011)
[26] B. Saha *Int. J. Theor. Phys.* **53** 1109 (2014)
[27] B. Saha *Astrophys. Space Sci.* **357** 28 (2015)
[28] B. Saha arXiv: 1504.03883v1 [gr-qc] (2015)
[29] B. Saha arXiv: 1507.03847v1 [gr-qc] (2015)
[30] B. Saha arXiv: 1507.06236 [gr-qc] (2015)
[31] T.W.B. Kibble *J. Math. Phys.* **2**, 212 (1961)
[32] O.Farooq, D.Mania, and B.Ratra. *Observational constraints on non-flat dynamical dark energy cosmological models* (Table 1) arXiv:1308.0834v2 [astro-ph.CO] (2014)
[33] O.Farooq. *An abstract of dissertation* (Table D.2) arXiv:1309.3710 [astro-ph.CO] (2013)
[34] Yun Chen, Chao-Qiang Geng, Shuo Cao, Yu-Mei Huang, Zong-Hong Zhu. *Constraints on a $\phi CDM$ model from strong gravitational lensing and updated Hubble parameter measurements* (Table 1) arXiv:1312.1443v2 [astro-ph.CO] (2014)