Testing perfection is hard

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Abstract

A graph property \( P \) is strongly testable if for every fixed \( \epsilon > 0 \) there is a one-sided \( \epsilon \)-tester for \( P \) whose query complexity is bounded by a function of \( \epsilon \). In classifying the strongly testable graph properties, the first author and Shapira showed that any hereditary graph property (such as \( P \) the family of perfect graphs) is strongly testable. A property is easily testable if it is strongly testable with query complexity bounded by a polynomial function of \( \epsilon^{-1} \), and otherwise it is hard. One of our main results shows that testing perfectness is at least as hard as testing triangle-freeness, which is hard. On the other hand, we show that induced \( P_3 \)-freeness is easily testable. This settles one of the two exceptional graphs, the other being \( C_4 \) (and its complement), left open in the characterization by the first author and Shapira of graphs \( H \) for which induced \( H \)-freeness is easily testable.

1 Introduction

Property testing is an active area of computer science where one wishes to quickly distinguish between objects that satisfy a property from objects that are far from satisfying that property. The study of this notion was initiated by Rubinfeld and Sudan [22], and subsequently Goldreich, Goldwasser, and Ron [14] started the investigation of property testers for combinatorial objects. Graph property testing in particular has attracted a great deal of attention. A property \( P \) is a family of (undirected) graphs closed under isomorphism. A graph \( G \) with \( n \) vertices is \( \epsilon \)-far from satisfying \( P \) if one must add or delete at least \( \epsilon n^2 \) edges in order to turn \( G \) into a graph satisfying \( P \).

An \( \epsilon \)-tester for \( P \) is a randomized algorithm, which given \( n \) and the ability to check whether there is an edge between a given pair of vertices, distinguishes with probability at least \( 2/3 \) between the cases \( G \) satisfies \( P \) and \( G \) is \( \epsilon \)-far from satisfying \( P \). Such an \( \epsilon \)-tester is one-sided if, whenever \( G \) satisfies \( P \), the \( \epsilon \)-tester determines this with probability 1. A property \( P \) is strongly-testable if for every fixed \( \epsilon > 0 \) there exists a one-sided \( \epsilon \)-tester for \( P \) whose query complexity is bounded only by a function of \( \epsilon \), which is independent of the size of the input graph.

We call a property \( P \) easily testable if it is strongly testable with a one-sided \( \epsilon \)-tester whose query complexity is polynomial in \( \epsilon^{-1} \), and otherwise \( P \) is hard. This is analogous to classical complexity

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theory, where an algorithm whose running time is polynomial in the input size is considered fast, and otherwise slow. Call a hereditary graph property extendable if for all but finitely many graphs in the family, there is a larger graph in the family containing it as an induced subgraph. Most of the well-known hereditary graph properties are extendable. As mentioned briefly in [15] and proved in detail in [3], there is a universal one-sided \( \epsilon \)-tester for extendable hereditary graph properties which has query complexity at most quadratic in the minimum possible query complexity of an optimal one-sided \( \epsilon \)-tester. Indeed, it samples \( d \) random vertices (for some \( d \)), and if the subgraph they induce is in \( \mathcal{P} \), it accepts, and otherwise it rejects. The query complexity of this tester is \( (d^2) \), and it is at least as accurate as any tester with query complexity at most \( d/2 \). The query complexity is a lower bound for the running time of an \( \epsilon \)-tester, and, if there is a polynomial time recognition algorithm for membership in \( \mathcal{P} \), the running time is polynomial in the query complexity. So while query complexity and running time are different notions, they are often of comparable order.

For a graph \( H \), let \( \mathcal{P}_H \) denote the property of being \( H \)-free, i.e., it is the family of graphs which do not contain \( H \) as a subgraph. The triangle removal lemma of Ruzsa and Szemerédi [23] is one of the most influential applications of Szemerédi’s regularity lemma. It states that for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that any graph on \( n \) vertices with at most \( \delta n^3 \) triangles can be made triangle-free by removing at most \( \epsilon n^2 \) edges. The triangle removal lemma is equivalent to the fact that \( \mathcal{P}_{K_3} \) is strongly testable. Indeed, the algorithm samples \( t = 2\delta^{-1} \) triples of vertices uniformly at random, where \( \delta \) is picked according to the triangle removal lemma, and accepts if none of them form a triangle, and otherwise rejects. Any triangle-free graph is clearly accepted. If a graph is \( \epsilon \)-far from being triangle-free, then it contains at least \( \delta n^3 \) triangles, and the probability that none of the sampled triples forms a triangle at most \( (1 - \delta)^t < 1/3 \). Notice that the query complexity depends on the bound in the triangle removal lemma. As observed by Ruzsa and Szemerédi, the triangle removal lemma gives a simple proof of Roth’s theorem [21] that every dense subset of the integers contains a 3-term arithmetic progression. From Behrend’s construction [7], which gives a large subset of the first \( n \) positive integers without a 3-term arithmetic progression, it follows that \( \delta \leq e^{c \log \epsilon} \) in the triangle removal lemma. This implies that testing triangle-freeness is hard. Indeed, in the universal algorithm described earlier, in a random sample of \( d \) vertices, the expected number of triangles is at most \( \delta d^3 \), and hence in the universal one-sided \( \epsilon \)-tester for triangle-freeness, \( 1/3 \leq \delta d^3 \), or equivalently, \( d \geq (3\delta)^{-1/3} \). As discussed earlier, the query complexity of any one-sided \( \epsilon \)-tester for triangle-freeness is at least \( d/2 \).

The triangle removal lemma was extended in [3] (see also [2]) to the graph removal lemma. It says that for each \( \epsilon > 0 \) and graph \( H \) on \( h \) vertices there is \( \delta = \delta(\epsilon, H) > 0 \) such that every graph on \( n \) vertices with at most \( \delta n^h \) copies of \( H \) can be made \( H \)-free by removing at most \( \epsilon n^2 \) edges. The graph removal lemma similarly implies that testing \( H \)-freeness is strongly testable. The proof, which uses Szemerédi’s regularity lemma, gives a bound on the query complexity which is a tower of height a power of \( \epsilon^{-1} \). This was somewhat improved recently by the second author [12] to a tower of height logarithmic in \( \epsilon^{-1} \). The first author [1] showed that \( H \)-freeness is easily testable if and only if \( H \) is bipartite.

For a graph \( H \), let \( \mathcal{P}_H^{\ast} \) denote the property of being induced \( H \)-free, i.e., it is the family of graphs which do not contain \( H \) as an induced subgraph. The graph removal lemma was extended by the first author, Fischer, Krivelevich, Szegedy [3] to the induced graph removal lemma, which states that
for every $\epsilon > 0$ and graph $H$ on $h$ vertices there is $\delta > 0$ such that any graph on $n$ vertices with at most $\delta n^h$ induced copies of $H$ can be made induced $H$-free by adding or removing at most $\epsilon n^2$ edges. The induced graph removal lemma is equivalent to the fact that, for any graph $H$, the property $\mathcal{P}_H$ is strongly testable. The proof, which uses a strengthening of Szemerédi’s regularity lemma, gives a bound on the query complexity which is wowzer of height a power of $\epsilon^{-1}$, which is one higher in the Ackermann hierarchy than the tower function. This has recently been improved by Conlon and the second author [10] to the tower function.

The length of a path is the number of edges it contains, and we let $P_k$ denote the path of length $k$. The first author and Shapira [4] showed that for any graph $H$ other than the paths of length at most 3, a cycle of length 4, and their complements, testing induced $H$-freeness is hard. For $H$ a path of length at most 2 or their complements, induced $H$-freeness is easily testable. They left open the cases that $H$ is a path of length 3 or a cycle of length 4 (and equivalently its complement). Here we settle one of the two remaining cases.

**Theorem 1.1** Induced $P_3$-freeness is easily testable.

A well-known result of Seinsche [24] gives a simple structure theorem for induced $P_3$-free graphs. These graphs, also known as cographs, are generated from the single vertex graph by complementation and disjoint union. This is equivalent to the statement that every induced $P_3$-free graph or its complement is not connected.

A quite general result of the first author and Shapira [5] states that every hereditary family $\mathcal{P}$ of graphs is strongly testable. They further asked which hereditary graph properties are easily testable, and, in particular, for a few of the well-known hereditary families of graphs, including perfect graphs and comparability graphs.

Note that the chromatic number of a graph is at least its clique number as the vertices of any clique must receive different colors in a proper coloring. A graph is perfect if every induced subgraph of it satisfies that its clique number and chromatic number are equal. The study of perfect graphs was started by Berge, partly motivated by the study of the Shannon capacity in information theory, which lies between the clique number and chromatic number of a graph. Perfect graphs form a relatively large class of graphs for which several fundamental algorithmic problems which are known to be NP-hard for general graphs, such as the graph coloring problem, the maximum clique problem, and the maximum independent set problem, can all be solved in polynomial time (see [16]). Also, it has significant connections with the study of linear and integer programming (see, e.g., [20]).

A famous conjecture of Berge, which was proved a few years ago by Chudnovsky, Robertson, Seymour and Thomas [9], states that a graph is perfect if and only if it contains no induced odd cycle of length at least five or the complement of one. The proof in fact establishes a stronger structural theorem for perfect graphs which was conjectured by Conforti, Cornuèjols, and Vušković. It says that every perfect graph falls into one of a few basic classes, or admits one of a few kinds of special decompositions. Shortly afterwards, a proof that perfect graphs can be recognized in polynomial time (as a function of the number of vertices of the graph) was discovered by Chudnovsky, Cornuèjols, Liu, Seymour, and Vušković [8].
Another well-studied hereditary family of graphs are comparability graphs. A comparability graph is a graph that connects pairs of elements that are comparable to each other in a partial order. Gallai classified these graphs by forbidden induced subgraphs, and Dilworth’s theorem is equivalent to the statement that the complement of comparability graphs are perfect. Further, comparability graphs can be recognized in polynomial time (see McConnell and Spinrad). Every cograph is a comparability graph, and every comparability graph is a perfect graph. It is natural to suspect that the structure theorem could hint at a polynomial in $\epsilon$ tester for perfectness similar to testing cographs. However, we show that testing perfectness essentially requires as much query complexity (or time) as testing triangle-freeness, which is hard.

**Theorem 1.2** Testing perfectness is hard.

Indeed, Theorem 3.1 shows that from a graph on $n$ vertices which is $14\epsilon$-far from being triangle-free but a random sample of $d$ vertices is with probability at least $1/2$ triangle-free, we can construct a graph on $5n$ vertices which is $\epsilon/25$-far from being induced $C_5$-free but a random sample of $d$ vertices in it is a comparability graph with probability at least $1/2$. Since every comparability graph is perfect, every perfect graph is induced $C_5$-free, and testing triangle-freeness is hard, this implies the above theorem that testing perfectness is hard, and further that testing for comparability graphs is hard.

**Theorem 1.3** Testing for comparability graphs is hard.

In the next section, we show that induced $P_3$-freeness is easily testable. In Section 3 we show that testing perfectness is at least as hard as testing triangle-freeness, which is hard. We finish with some concluding remarks. Throughout the paper, we systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs.

## 2 Induced $P_3$-freeness is easily testable

A cut for a graph $G = (V, E)$ is a partition $V = V_1 \cup V_2$ into nonempty subsets such that there are no edges between $V_1$ and $V_2$ or $V_1$ is complete to $V_2$. The following definition is a natural relaxation of a cut. For $\beta > 0$, define a $\beta$-cut for a graph $G = (V, E)$ as a partition $V = V_1 \cup V_2$ into nonempty subsets such that $e(V_1, V_2) \leq \beta |V_1||V_2|$ or $e(V_1, V_2) \geq (1 - \beta)|V_1||V_2|$. For a graph $G$ and vertex subset $S$, let $G[S]$ denote the induced subgraph of $G$ with vertex set $S$. Let $c(\beta, n)$ be the least $\delta$ for which there is a graph $G = (V, E)$ on $n$ vertices which has no $\beta$-cut and has $\delta n^4$ induced copies of $P_3$.

**Theorem 2.1** We have $c(\beta, n) \geq (\beta/100)^{12}$.

**Proof:** Suppose for contradiction that there is a graph $G$ on $n$ vertices which does not have a $\beta$-cut and has less than $\delta n^4$ induced copies of $P_3$, where $\delta = (\beta/100)^{12}$. Since $G$ has no $\beta$-cut, then $G$ contains an induced $P_3$. Hence, $1 \leq \delta n^4$ and $n \geq \delta^{-1/4} \geq (100/\beta)^3$. 

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Since $G$ has at most $\delta n^4$ induced copies of $P_3$, a random sample of $r = (8\delta)^{-1/4} \geq 10^5 \beta^{-3}$ vertices has in expectation at most $\delta r^4 = 1/8$ induced copies of $P_3$. Hence, with probability at least $7/8$, a random sample of $r$ vertices contains no induced $P_3$.

Randomly sample a set $R = S \cup T$ of $r = s + t$ vertices from $V$, where $s = t = r/2$. Let $E_0$ be the event that $G[R]$ is induced $P_3$-free, so the probability of event $E_0$ is at least $7/8$.

Since $G$ does not have a $\beta$-cut, each vertex has more than $\beta(n-1)$ neighbors and less than $(1-\beta)(n-1)$ neighbors. Let $\alpha = \beta/2$. Hoeffding (see Section 6 of [18]) proved that the hypergeometric distribution is at least as concentrated as the corresponding binomial distribution. Thus, by the Azuma-Hoeffding inequality (see, e.g., [19]), and the fact that each vertex $v \in S$ has more than $\beta(n-1)$ neighbors, the probability that a particular $v \in S$ has less than $\alpha(s-1)$ neighbors in $S$ is a most

$$e^{-((\beta-\alpha)(s-1))^2/(2(s-1))} = e^{-(\beta-\alpha)^2(s-1)/2} \leq e^{-\beta^2s/16} \leq \frac{1}{168}.$$  

Similarly, the probability that $v$ has more than $(1-\alpha)(s-1)$ neighbors in $S$ is at most $\frac{1}{168}$. Let $E_1$ be the event that every vertex in $S$ has at least $\alpha(s-1)$ and at most $(1-\alpha)(s-1)$ neighbors in $S$, i.e., the induced subgraph $G[S]$ has minimum degree at least $\alpha(s-1)$ and maximum degree at most $(1-\alpha)(s-1)$. By the union bound, the probability of event $E_1$ is at least $1 - 2s \cdot \frac{1}{168} = 7/8$.

Let $U$ be the set of vertices $v \in V \setminus S$ which are complete or empty to $S$. As the degree of each vertex of $G$ is at least $\beta(n-1)$ and at most $(1-\beta)(n-1)$, the probability that for a given vertex $v$, a random subset of $s$ vertices of $V \setminus \{v\}$ are all neighbors of $v$ or all nonneighbors of $v$ is at most $2(1-\beta)^s$. Hence, a given vertex has probability at most $2(1-\beta)^s$ of being in $U$. By linearity of expectation, the expected size of $U$ is at most $2(1-\beta)^s n$. Let $E_2$ be the event that $|U| \leq 16(1-\beta)^s n \leq 16e^{-\beta^2 n} \leq \frac{\delta}{8} n$. By Markov’s inequality, the probability of $E_2$ is at least $1 - 1/8 = 7/8$.

Let $E_3$ be the event that $T$ contains no vertex from $U$. By linearity of expectation, $\mathbb{E}[|U \cap T|] = \mathbb{E}[|U|] t/n \leq 2(1-\beta)^s t \leq 2e^{-\beta^2 t} \leq \frac{1}{8}$. Therefore, event $E_3$ occurs with probability at least $7/8$.

The probability that events $E_0$ and $E_1$ both occur is at least $7/8 - 1/8 = 3/4$. If both of these events occur, then $G[S]$ has at least one and at most $2^{a-1}$ cuts. Consider such a cut $S = S_1 \cup S_2$ of $G[S]$, and suppose $S_1$ is complete to $S_2$ (the case $S_1$ is empty to $S_2$ can be treated similarly). For each such cut, consider the partition $V \setminus S = U \cup V_0 \cup V_1 \cup V_2$ of vertices, where $v \in V \setminus S$ satisfies $v \in V_0$ if $v \notin U$ and it is not complete to $S_1$ and not complete to $S_2$, $v \in V_1$ if it is complete to $S_2$ but not complete to $S_1$, and $v \in V_2$ if it is complete to $S_1$ but not to $S_2$.

Note that if $T$ contains a vertex from $V_0$, then the cut $S = S_1 \cup S_2$ of $G[S]$ does not extend to a cut of $G[R]$. If events $E_i$ for $i = 0, 1, 2, 3$ occur, which happens with probability at least $1/2$, then $G[R]$ is induced $P_3$-free, so it has a cut, and no vertex in $T$ is complete or empty to $S$. In this case one of the cuts of $G[S]$ extends to a cut of $G[R]$, and hence, for at least one cut of $G[S]$, no vertex of $T$ is in the corresponding $V_0$.

We now condition on the occurrence of events $E_i$ for $i = 0, 1, 2, 3$. Note that since the probability that this happens is at least $1/2$, for any other event $E$, the conditional probability that $E$ occurs given that $E_i$ occur for $i = 0, 1, 2, 3$ is at most twice the probability of $E$ without any conditioning. To complete the proof we claim that with positive probability $E_0, E_1, E_2, E_3$ occur and yet the induced subgraph on $S \cup T$ contains an induced $P_3$, contradicting $E_0$. To do so we apply the union bound over
all cuts in $G[S]$ to show that with positive probability, for each such cut, either $T$ contains a vertex of $V_0$ (and hence the cut cannot be extended to one in $G[R]$) or $T$ contains a vertex $v_1$ in $V_1$ and a vertex $v_2$ in $V_2$, which are nonadjacent, providing an induced $P_3$ in $G[S \cup T]$ on the vertices $v_1, v_2$ together with a vertex $s_1 \in S_1$ not adjacent to $v_1$ and a vertex $s_2 \in S_2$ not adjacent to $v_2$.

We proceed with the proof of this claim. Conditioning on $E_i$ for $i = 0, 1, 2, 3$, fix a cut $(S_1, S_2)$ in $G[S]$ and let $V_0, V_1, V_2$ be as above. Consider two possible cases.

Case 1: $|V_0| \geq \frac{2}{4t} n$.

In this case, the probability that $T$ contains no vertex of $V_0$ is at most

$$ (1 - \frac{2}{4t})^t \leq e^{-2/\alpha} < 2^{-\alpha^{-1}-1}, $$

showing that even after our conditioning the probability of this event is smaller than $2^{-\alpha^{-1}}$.

Case 2: $|V_0| < \frac{2}{4t} n \leq \frac{3}{8} n$.

Let $x = |U| + |V_0|$, $y = |S_1| + |V_1|$, and $z = |S_2| + |V_2|$, so $x + y + z = n$. Assume without loss of generality that $y \leq z$. Since the partition $V = (S_1 \cup V_1) \cup (S_2 \cup V_2 \cup U \cup V_0)$ is not a $\beta$-cut, there are at least $\beta y(z + x)$ missing edges between these two sets. Since, in addition, $S_1$ is complete to $S_2$, $S_1$ is complete to $V_2$, and $V_1$ is complete to $S_2$, then these missing edges go between $V_1$ and $V_2$ and between $S_1 \cup V_1$ and $U \cup V_0$. Thus

$$ \frac{\beta}{2} yn \leq \beta y(z + x) \leq |V_1||V_2| - e(V_1, V_2) + yx. $$

If events $E_i$ for $i = 0, 1, 2, 3$ occur, then $x \leq \frac{3}{4} n$, and hence there are at least $\frac{\beta}{4} yn$ missing edges between $V_1$ and $V_2$. In this case, every vertex of $S_1$ is complete to $S_2 \cup V_2$, and hence

$$ (1 - \beta)(n - 1) \geq z = n - x - y \geq n - \frac{\beta}{4} n - y $$

and

$$ y \geq \frac{3\beta}{4} n - 1 \geq \frac{\beta}{2} n. $$

Thus, the number of missing pairs between $V_1$ and $V_2$ in the case events $E_i$ for $i = 0, 1, 2, 3$ occur is at least $\frac{\beta}{4} yn \geq \frac{\beta^2}{8} n^2$.

Let $E_4$ be the event that $T$ contains the two vertices of at least one of the nonedges between $V_1$ and $V_2$. Given that there are at least $\frac{\beta^2}{8} n^2$ edges missing between $V_1$ and $V_2$, the probability that event $E_4$ occurs is at least the probability that at least one of $t/2$ random pairs of vertices of $G$ contains one of the nonedges between $V_1$ and $V_2$. The probability that this does not happen is at most

$$ \left(1 - \frac{\beta^2 n^2 / 8}{\binom{n}{2}}\right)^{t/2} \leq e^{-\beta^2 t/8} = e^{-\beta^2 10^5/(8 \cdot 2 \beta^2)} = e^{-10^5/(32 \alpha)} < 2^{-\alpha^{-1}-1}, $$

and hence even after our conditioning the probability of this event is smaller than $2^{-\alpha^{-1}}$. 


By the union bound it now follows that with positive probability $E_i$ for $i = 0, 1, 2, 3$ occur and yet $G[S \cup T]$ contains an induced $P_3$. This is a contradiction, completing the proof.

Let $f(\epsilon, n)$ be the least $\delta$ for which there is a graph $G = (V, E)$ on $n$ vertices which is $\epsilon$-far from being induced $P_3$-free and has $\delta n^4$ induced copies of $P_3$.

**Theorem 2.2** There is $n_0 \geq c n$ such that $f(\epsilon, n) \geq c(\epsilon, n_0) \epsilon^4 \geq (\epsilon/100)^{16}$.

**Proof:** Let $G = (V, E)$ be a graph on $n$ vertices which is $\epsilon$-far from being induced $P_3$-free. Partition $V$ into two parts along an $\epsilon$-cut, and continue refining parts along $\epsilon$-cuts of the subgraphs induced by the parts until no part has an $\epsilon$-cut, and let $V = V_1 \cup \ldots \cup V_k$ be the resulting partition. We modify edges along these $\epsilon$-cuts to turn them into cuts, letting $G'$ be the resulting graph. The total fraction of pairs of vertices changed in making $G'$ from $G$ is at most $\epsilon n^2 - \epsilon(n^2/2) \geq \epsilon n^2/2$ edges must be changed from the resulting graph $G'$ to make it induced $P_3$-free. We can modify edges in each $V_i$ to make it induced $P_3$-free, and the resulting graph on $V$ is induced $P_3$-free. If $|V_i| \leq \epsilon n$ for $1 \leq i \leq k$, then the number of edge modifications made to $G'$ to obtain an induced $P_3$-free graph is at most

$$\sum_{i=1}^{k} \left(\frac{|V_i|}{2}\right) \leq \frac{n}{2} \max_{1 \leq i \leq k} (|V_i| - 1) < \frac{\epsilon n^2}{2},$$

a contradiction. Thus, one of the parts $V_i$, call it $V_0$, has $n_0 > \epsilon n$ vertices, and $G[V_0]$ has no $\beta$-cut. Therefore, the induced subgraph $G[V_0]$, and hence $G$, has at least

$$c(\epsilon, n_0) n_0^4 \geq c(\epsilon, n_0) \epsilon^4 n^4 \geq (\epsilon/100)^{12} \epsilon^4 n^4 \geq (\epsilon/100)^{16} n^4$$

induced copies of $P_3$, completing the proof. □

Consider the following one-sided $\epsilon$-tester for induced $P_3$-freeness. Let $\delta = (\epsilon/100)^{16}$. The algorithm samples $t = 2\delta^{-1}$ quadruples of vertices uniformly at random, and accepts if none of them form an induced $P_3$, and otherwise rejects. Any induced $P_3$-free graph is clearly accepted. If a graph is $\epsilon$-far from being induced $P_3$-free, then it contains at least $\delta n^4$ induced $P_3$ by Theorem 2.2, and the probability that none of the sampled quadruples forms an induced $P_3$ is at most $(1 - \delta)^t < 1/3$. Note that the query complexity for this algorithm depends linearly on $\delta^{-1}$, and hence polynomially on $\epsilon^{-1}$, completing the proof of Theorem 1.1. □

### 3 Testing perfectness

We first observe a couple of equivalent versions of the triangle removal lemma. The **triangle edge cover number** $\nu(G)$ of a graph $G$ is the minimum number of edges of $G$ that cover all triangles in $G$, i.e., it is the minimum number of edges of $G$ whose deletion makes $G$ triangle-free. The triangle removal lemma thus says that for each $\epsilon > 0$ there is $\delta > 0$ such that every graph on $n$ vertices with at most $\delta n^3$ triangles satisfies $\nu(G) \leq \epsilon n^2$. 

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The triangle packing number $\tau(G)$ of a graph $G$ is the maximum number of edge disjoint triangles in $G$. The following simple bounds hold for all graphs:
\[ \tau(G) \leq \nu(G) \leq 3\tau(G). \]

Indeed, at least one edge from each of the edge-disjoint triangles is needed in any edge cover of the triangles in $G$, and deleting the $3\tau(G)$ edges from a maximum collection of edge-disjoint triangles leaves a triangle-free graph. We remark that a well known conjecture of Tuza states that the upper bound can be improved to $\nu(G) \leq 2\tau(G)$. Haxell [17] improved the upper bound factor to $3 - \frac{3}{27}$.

Thus, up to a constant factor change in $\epsilon$, the triangle removal lemma is the same as saying that a graph $G$ on $n$ vertices with at least $\epsilon n^2$ edge disjoint triangles contains at least $\delta n^3$ triangles. We can further suppose, up to a constant factor change in $\epsilon$, that $G$ is tripartite. Indeed, every graph has a tripartite subgraph which contains at least $2/9$ of the total, and there is a tripartition for which the number of edge-disjoint triangles is at least the expected number. We may thus assume $G$ is tripartite.

**Theorem 3.1** Let $T$ be a graph on $n$ vertices which is $14\epsilon$-far from being triangle-free such that a random sample of $d$ vertices of $T$ is triangle-free with probability at least $1/2$. Then there is a graph $G$ on $5n$ vertices which is $\epsilon/25$-far from being induced $C_5$-free, such that a random sample of $d$ vertices of $G$ is a comparability graph with probability at least $1/2$.

**Proof:** By the remarks above, $T$ contains a tripartite subgraph $F$ which contains at least $\frac{1}{3} \cdot \frac{2}{9} \cdot 14\epsilon n^2 > \epsilon n^2 = (\epsilon/25)(5n)^2$ edge-disjoint triangles. Denote the three parts of $F$ by $V_2, V_3, V_5$.

Let $G = (V, E)$ be the graph on $5n$ vertices with partition $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$, where $V_1$ and $V_4$ are of size $2n$ each, and $V_2, V_3, V_5$ are the parts of $F$. We next specify the edges between the various parts of $G$. Each part $V_i$, $1 \leq i \leq 5$, is an independent set. There are no edges between $V_1$ and $V_2$, between $V_1$ and $V_3$, between $V_3$ and $V_4$, and between $V_4$ and $V_5$. There is a complete bipartite graph between $V_1$ and $V_4$, between $V_2$ and $V_4$, between $V_3$ and $V_5$, and between $V_2$ and $V_5$. The edges of $G$ are precisely the edges of $F$ between $V_2$ and $V_3$, and between $V_3$ and $V_5$. Finally, between $V_2$ and $V_5$, the edges of $G$ are precisely the nonedges of $F$.

Arbitrarily order $T_1, \ldots, T_t$ a maximum collection of $t = \tau(F) \geq \epsilon n^2$ edge-disjoint triangles in $F$. As $F$ is a tripartite graph on $n$ vertices, $t = \tau(F)$ is at most the product of the two smallest parts, which is at most $n^2/9$. For every triangle in $F$, the same three vertices in $G$ with a vertex in $V_1$ and a vertex in $V_4$ form an induced $C_5$. We next show that this implies that there are $t$ induced copies of $C_5$ in $G$, labeled $L_1, \ldots, L_t$, such that each pair intersects in at most one vertex. In fact, we greedily construct $L_1, \ldots, L_t$ so that they further satisfy that the vertex set of each $L_i$ consists of the vertices of $T_i$ together with a vertex in $V_1$ and a vertex in $V_4$.

Suppose we have already constructed $L_j$ for $j < i$ satisfying the desired properties. We next show how to construct $L_i$ with the desired properties. Note that in a tripartite graph, the number of edge-disjoint triangles containing a given vertex $v$ is at most the minimum order of the two parts not containing $v$. [8]
It follows that $T_1$ has nonempty intersection with at most $n$ of the $t$ triangles $T_1, \ldots, T_t$. Hence, for $h = 1, 4$, at most $n$ vertices in $V_h$ are in at least one $L_j$ with $j < i$ for which $T_j$ and $T_i$ share a vertex in common. For $h = 1, 4$, delete these vertices from $V_h$, and denote the resulting subset of $V_h$ as $V'_h$, so $|V'_h| \geq |V_h| - n = n$. As $i - 1 < t < n^2 \leq |V'_h| |V'_h|$, there is a pair $(v_1, v_4) \in V'_1 \times V'_4$ that is not in any $L_j$ with $j < i$. We pick $L_i$ to be the induced $C_5$ in $G$ with vertices $v_1, v_4$ and the vertices of $T_i$. It is clear from this construction that $L_i$ intersects each $L_j$ with $j < i$ in at most one vertex. We therefore can greedily construct the desired $t$ induced copies of $C_5$, and conclude that $G$ is $\epsilon/25$-far from being induced $C_5$-free.

On the other hand, the only triples $a < b < c$ of vertices in a linear ordering which puts the vertices in $V_1$ before $V_j$ if $i < j$ with $a$ adjacent to $b$, $b$ adjacent to $c$, and $a$ not adjacent to $c$ are with $a \in V_2$, $b \in V_3$, and $c \in V_5$ the vertices of a triangle in $F$. Thus, by sampling $d$ vertices uniformly at random from $G$, we sample at most $d$ vertices uniformly at random from $F$. These at most $d$ vertices are triangle-free in $F$ with probability at least $1/2$, and hence the $d$ random vertices in $G$ form a comparability graph with probability at least $1/2$. This completes the proof.

As discussed toward the end of the introduction, Theorem 3.1 implies Theorem 1.2 that testing perfectness is hard, and Theorem 1.3 that testing for comparability graphs is hard.

A partially ordered set (poset) is a directed graph on a vertex set $P$ which

- has no loops, i.e., no pair $(x, x)$ is an edge,
- has no antiparallel edges, i.e., if $(x, y)$ is an edge, then $(y, x)$ is not an edge,
- is transitive, i.e., if $(x, y)$ is an edge and $(y, z)$ is an edge, then $(x, z)$ is also an edge.

The fact that testing for posets is hard (at least as hard as testing for triangle-freeness) follows from Theorem 3.1 by adding directions. However, we next sketch a simpler proof. Let $T$ be a tripartite graph on $n$ vertices with parts $V_1, V_2, V_3$ which is $\epsilon$-far from being triangle-free. Consider the directed graph $G$ on the same vertex set as $T$ with $(v_1, v_2) \in V_1 \times V_2$ an edge of $G$ if it is an edge of $T$, $(v_2, v_3) \in V_2 \times V_3$ an edge of $G$ if it is an edge of $T$, $(v_1, v_3) \in V_1 \times V_3$ an edge of $G$ if it is not an edge of $T$, and there are no other edges. At least one pair in every triangle of $T$ must be modified to turn $G$ into a poset, so $G$ is $\epsilon$-far from being a poset. Also, any subset of vertices which is triangle-free in $T$ induces a poset in $G$. This implies that testing for posets is at least as hard as testing for triangle-freeness.

4 Concluding Remarks

We believe that comparing the number of queries needed to test various properties, as done in this paper comparing testing perfectness and triangle-freeness, could be an interesting direction for further research. This is the analogue in property testing to the powerful technique of hardness reductions in complexity theory. One general class of hard graph properties for testing for which to compare with is (not necessarily induced) $H$-freeness for $H$ a fixed odd cycle.
We showed that testing perfectness is hard. This is equivalent to showing that there is a graph which is $\epsilon$-far from being perfect such that a random set of vertices of size polynomial in $\epsilon^{-1}$ is perfect with probability at least 1/2. This still leaves the possibility of getting a small witness if the graph is far from being perfect. That is, does every graph which is $\epsilon$-far from being perfect contain an induced odd cycle or its complement of size at least 5 and at most a polynomial in $\epsilon^{-1}$?

We showed that testing induced $P_3$-freeness is easy, which is a step toward completing the classification of graphs $H$ for which induced $H$-free testing is easy. It remains to determine whether or not induced $C_4$-freeness is easy.

Finally, it will be very interesting to characterize all easily testable graph properties. As all these properties have to be strongly testable, it follows from the main result of [5] that if we restrict ourselves only to natural properties, in the sense of [5], then these properties have to be essentially hereditary. Among the hereditary properties, properties that are known to be easily testable include the property of being $k$-colorable for any fixed $k$, as shown in [14], as well as a natural extension of it, as proved in [15]. As mentioned in the introduction, additional easily testable (hereditary) properties are $H$-freeness for any bipartite $H$, and induced $H$-freeness for any path $H$ on at most 4 vertices or its complement (where the case of 4 vertices is proved in Section 2).

Hereditary properties which are not easily testable are $H$-freeness for nonbipartite $H$, induced $H$-freeness for all graphs besides the paths on at most 4 vertices and their complements, as well as possibly the cycle of length 4 and its complement, perfectness and comparability. Our techniques here can be applied to provide several additional examples of easily testable and of non-easily testable hereditary properties, but most of these are somewhat artificial and not familiar graph properties. Does the above list of known results suggest a (conjectured) characterization of all easily testable hereditary graph properties? At the moment we are unable to formulate such a conjecture.

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