Model-Independent Bounds on $R(J/\psi)$ via Dispersive Relations

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Model-independent bounds on $R(J/\psi) \equiv \frac{BR(B_c^+ \rightarrow J/\psi\tau^+\nu_\tau)}{BR(B_c^+ \rightarrow J/\psi\mu^+\nu_\mu)}$ are obtained through a combination of dispersive relations, heavy-quark relations at zero-recoil, and the limited existing form factor determinations from lattice QCD. The resulting 95% confidence-level bound, $0.20 \leq R(J/\psi) \leq 0.39$, agrees with the recent LHCb result at $1.3\sigma$, and removes the dominant model-dependent uncertainty from theory predictions. Using the same techniques, a prediction of $R(\eta_c) = 0.29(5)$ is obtained.

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1 Introduction

The ratios of semileptonic heavy-meson decay branching fractions to different flavors represent sensitive tests of lepton universality and new physics because the matrix element can be factorized at leading order into hadronic and leptonic terms:

$$|M_{\bar{b} \to c \ell^+ \nu}|^2 = \frac{L_{\mu \nu} H_{\mu \nu}^c}{q^2 - M_W^2} + O(\alpha, G_F).$$

(1)

This expression implies that the ratios of semileptonic heavy-meson decay branching fractions can differ from unity at this level of precision only due to kinematic factors. A tension between theory and experiment exist in $R(D(D^*))$ for heavy-light meson decays $B \to D(D^*) \ell \nu$ with $\ell = \tau$ to those with $\ell = \mu$ or $e$. In light of this tension, the LHCb Collaboration has measured the rates for the heavy-heavy semileptonic meson decays $B^+_{c} \to J/\psi \ell^+ \nu_\ell$ in the $\ell = \tau, \mu$ channels, finding $R(J/\psi) = 0.71(17)(18)$ [1].

Prior to [2], only model-dependent calculations of $R(J/\psi)$ existed and although most models’ central values cluster in LHCb’s quoted theory range of $0.25 - 0.28$, this range is overly optimistic, even taking only the model’s own assessed uncertainty. A more reasonable estimate of the model predictions is $0 < R(J/\psi) < 0.48$, found by forming the union of the 95% confidence levels (CL) using only the reported theoretical uncertainties [2]. Without a clear understanding of the systematic uncertainties these assumptions introduce, even this range is suspect.

In the Standard Model, the factorization of Eq. (1) into a leptonic and a hadronic tensor reduces the problem of calculating $R(J/\psi)$ to the computation of the hadronic matrix element $\langle J/\psi |(V - A)\mu|B_c^+ \rangle$. Using this factorization, the hadronic matrix element can be written in terms of four transition form factors via [3, 4]:

$$\langle J/\psi(p, \epsilon)|(V - A)^\mu|B_c^+(P)\rangle = \frac{2i\epsilon^{\mu\nu\rho\sigma}}{M + m} \epsilon^*_\nu p_\rho P_\sigma V(q^2) - (M + m)\epsilon^*\mu A_1(q^2)$$

$$+ \frac{\epsilon^* \cdot q}{M + m} (P + p)^\mu A_2(q^2) + 2m \frac{\epsilon^* \cdot q}{q^2} q^\mu A_3(q^2)$$

$$- 2m \frac{\epsilon^* \cdot q}{q^2} q^\mu A_0(q^2),$$

(2)

where $M \equiv M_{B_c^+}$ and $m \equiv M_{J/\psi}$, the momenta $P^\mu$ and $p^\mu$, polarization $\epsilon^\mu$ of the $J/\psi$, and $q^\mu \equiv (P - p)^\mu$. While five form factors are shown, only four are independent. In the physical set, $A_0(q^2)$ is defined as the form factor that couples to timelike virtual $W$ polarizations ($\propto q^\mu$), while $A_3(q^2)$ is simply a convenient shorthand for

$$A_3(q^2) = \frac{M + m}{2m} A_1(q^2) - \frac{M - m}{2m} A_2(q^2).$$

(3)

Furthermore, the finiteness of Eq. (2) as $q^2 \to 0$ requires $A_3(0) = A_0(0)$, which is useful in constructing bounds. For notational simplicity, $t \equiv q^2$, and two important kinematic points are defined via $t_\pm = (M \pm m)^2$. 
The state-of-the-art lattice QCD calculations for $B_c^+ \to J/\psi$ are limited to preliminary results from the HPQCD Collaboration for $V(q^2)$ at two $q^2$ values and $A_1(q^2)$ at three $q^2$ values \[4\] and are reproduced in Fig. 1. At present, there are no lattice results for $A_0(q^2)$ or $A_2(q^2)$.

While this decomposition is useful for lattice QCD, it is not the best decomposition for the dispersive analysis. The second convention we use is the helicity basis, which exchanges the form factors $V, A_i$ for $g, f, F_1, F_2$ via the relations

\[ g = \frac{2}{M + m} V, \quad f = (M + m) A_1, \]
\[ \mathcal{F}_1 = \frac{1}{m} \left[ -\frac{2k^2 t}{M + m} A_2 - \frac{1}{2}(t - M^2 + m^2)(M + m) A_1 \right], \quad \mathcal{F}_2 = 2 A_0, \]

where $k = \sqrt{\frac{(t_+ - t)(t_- - t)}{4t}}$ The differential cross section for the semileptonic decay is

\[
\frac{d\Gamma}{dt} = \frac{G_F^2 |V_{cb}|^2 k}{192\pi^3 M^3 t^{5/2}} (t - m^2_\ell)^2 \left\{ (2t + m^2_\ell) \left[ 2t |f|^2 + |\mathcal{F}_1|^2 + 2k^2 t^2 |g|^2 \right] + 3m^2_\ell k^2 t |\mathcal{F}_2|^2 \right\}. \tag{5}
\]

### 2 Heavy-Quark Spin Symmetry

In the decay $B_c^+ \to J/\psi$, relations between the form factors at zero-recoil can be derived analogous to the Isgur-Wise function \[5, 6\]. In the heavy-heavy systems, unlike the heavy-light, the difference between the heavy-quark kinetic energy operators produces energies no longer negligible compared to those of the spectator $c$, and this effect spoils the flavor symmetry in heavy-heavy systems. Furthermore, the spectator $c$ receives a momentum transfer from the decay of $b \to c$ of the same order as the momentum imparted to the $c$, so one cannot justify a normalization of the form factors at the zero-recoil point based purely upon symmetry.

While the heavy-flavor symmetry is lost, the separate spin symmetries of $b$ and $\bar{c}$ quarks remain, with an additional spin symmetry from the heavy spectator $c$. Together, the spin symmetries imply that the four form factors are related to a single, universal function $h (\Delta$ in Ref. \[7\]), but only at the zero-recoil point, and no symmetry-based normalization for $h$ can be derived \[7\].

In \[7, 8\], the relative normalization between the four $Qq \to \bar{Q}'q$ form factors near the zero-recoil point [where the spatial momentum transfer to $q$ is $O(m_q)$]. The relations are:

\[
g(w = 1) = \frac{2\rho + (1 + \rho)\sigma}{4M^2 r \rho} f(w = 1),
\]
\[
\mathcal{F}_1(w = 1) = M(1 - r)f(w = 1),
\]
\[
\mathcal{F}_2(w = 1) = \frac{2(1 + r)\rho + (1 - r)(1 - \rho)\sigma}{4Mr \rho} f(w = 1), \tag{6}
\]
where \( r \equiv m/M \), \( \rho \equiv m_Q/m_Q \), and \( \sigma \equiv m_q/m_Q \). These relations reproduce the standard Isgur-Wise result \( [5, 6, 9] \) when \( \sigma = 0 \). The relation between \( F_1(w = 1) \) and \( f(w = 1) \) follows directly from the definition of Eq. (4), independent of heavy-quark symmetries. Terms that break these relations should be \( O(m_c/m_b, \Lambda_{QCD}/m_c) \approx 30\% \), and in the fits is allowed to be as large as 50%.

3 Dispersive Relations

The derive constraints on the form factors of \( B_c^+ \rightarrow J/\psi \) are obtained using analyticity and unitarity constraints on a particular two-point Green’s function and a conformal parameterization in the manner implemented by Boyd, Grinstein, and Lebed (BGL) \( [10] \). A slightly different set of free parameters was used to simplify the computation for the \( B_c^+ \) decays, and is laid out in detail in \( [2, 11] \).

Mapping the complex \( t \) plane to the unit disk in a variable \( z \) (with the two sides of the branch cut forming the unit circle \( C \)) can be achieved using the conformal variable transformation

\[
z(t; t_0) \equiv \frac{\sqrt{t_* - t} - \sqrt{t_* - t_0}}{\sqrt{t_* - t} + \sqrt{t_* - t_0}},
\]

where \( t_* \) is the branch point around which one deforms the contour, and \( t_0 \) is a free parameter used to improve the convergence of functions at small \( z \). In this mapping, \( z \) is real for \( t \leq t_* \) and a pure phase for \( t \geq t_* \).

To avoid issues with nonanalyticities within the unit circle, \( t_* = (M_B^{(*)} - M_D)^2 \), which is the lightest two particle state with the correct quantum numbers for the dispersive relations. This is possible because for semileptonic decays \( m_\ell^2 \leq t \leq t_- \) which is always less than \( t_* \). This choice ensures that the only nonanalytic features within the unit circle \( |z| = 1 \) are simple poles corresponding to single particles \( B_c^{(*)+} \), which can be removed by Blaschke factors \( [12, 13] \). Using this formalism, each form factor \( F_i(t) \) can be written as a non-analytic prefactor and an expansion in \( z \) corresponding to an analytic function. In this way, once can show that the sum of the squares of the coefficients of \( z \)–expansion are bounded by one.

4 Results

The above constraints are summarized as:

- The coefficients \( a_n \) of each form factor’s \( z \)–expansion are constrained by \( \sum_n a_n^2 \leq 1 \).
- Using Eq. (6), the values \( g(t_-) \) and \( F_2(t_-) \) are related to the value of \( f(t_-) \), which in turn is computed from lattice QCD, to within 50%.
• All form factors are assumed maximal at the zero-recoil point \( t = t_- \) since the universal form factor \( h \) represents an overlap matrix element between initial and final states.

• The relation \( \mathcal{F}_1(t_-) = M(1 - r)f(t_-) \), which is related to the \( q^2 = 0 \) limit of the form factors, is exact.

• \( \mathcal{F}_1(0) = \frac{1}{2}M^2(1 - r^2)\mathcal{F}_2(0) \) follows from the condition \( A_3(0) = A_0(0) \).

Imposing these, the fit is performed in two steps, reflecting the difference between the two lattice-computed form factors \((V, A_1)\), and the two \((A_0, A_2)\) without.

In the first step, random Gaussian-distributed points are sampled for the form factors \( g \) and \( f \) whose mean gives the HPQCD results and with an uncertainty dominated by an added systematic uncertainty, \( f_{\text{lat}} \) (expressed as a percentage of the form-factor point value) that estimates the uncomputed lattice uncertainties. The resulting bands of allowed form factors are shown for \( f_{\text{lat}} = 20\% \) in Fig. 1 alongside the HPQCD results.

In the second step, the unknown \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are constrained. The form factors that yield the numerical maximum and minimum \( R(J/\psi) \) values, subject to the computed \( f, g \) values and the constraints listed above, are obtained. In this way, the only uncertainties are derived from \( f_{\text{lat}} \) and the violations of the heavy-quark spin-symmetry relations. The resulting bands of allowed form factors that produce the minimum and maximum values of \( R(J/\psi) \) are plotted in Fig. 1. The 95\% CL ranges for \( R(J/\psi) \) as
TABLE I. 95% CL upper and lower bounds on $R_{J/\psi}$ as a function of the truncation power $n$ of coefficients and the systematic lattice uncertainty $f_{\text{lat}}$.

| $f_{\text{lat}}$ | $n = 1$       | $n = 2$       |
|------------------|---------------|---------------|
| 1                | [0.21, 0.33]  | [0.20, 0.35]  |
| 5                | [0.20, 0.33]  | [0.20, 0.35]  |
| 20               | [0.20, 0.36]  | [0.20, 0.39]  |

a function of the truncation power $n = 1, 2$ in the dispersive analysis coefficients and $f_{\text{lat}}$ are shown in Table I.

In Fig. 2 the previous model-dependent values of $R(J/\psi)$ are plotted alongside the LHCb result and our 95% CL bound of $0.20 \leq R(J/\psi) \leq 0.39$, as a function of publication date. While many of the previous model results lie within our 95% CL band, some are either partially or entirely excluded.

FIG. 2. $R(J/\psi)$ from the LHCb experiment (blue open square, 1σ uncertainty denoted by blue dashed lines), our bound (red dash-dotted lines), and models (points colored by model type). Reproduced from [2]

The most important piece of new lattice data to obtain is a value of $F_2(t_-)$, which currently controls the upper bound’s uncertainties. This zero-recoil form factor is directly related by $F_2(t_-) = 2A_0(t_-)$ to a traditional lattice form factor. Synthetic data suggests that a value for $F_2(t_-)$ could improve the bound by the same amount as reducing the current lattice results uncertainty by a factor of 20 using far less computing resources.
The model-independent bound on $R(J/\psi)$ was found to be $0.20 \leq R(J/\psi) \leq 0.39$ at the 95\% CL. The LHCb result is therefore consistent with the Standard Model at 1.3 $\sigma$. The near-term outlook for a higher-statistics LHCb measurement, coupled with new lattice results, promises to reduce the experimental and theoretical uncertainties dramatically. Using the same procedure provides a prediction of $R(\eta_c) = 0.29(5)$ \[11\] which may be obtainable in the future with LHCb.

\[1\] R. Aaij et al. (LHCb Collaboration), Phys. Rev. Lett. 120, 121801 (2018), arXiv:1711.05623 [hep-ex].
\[2\] T. D. Cohen, H. Lamm, and R. F. Lebed, \textcopyright{} (2018), arXiv:1807.02730 [hep-ph].
\[3\] M. Wirbel, B. Stech, and M. Bauer, Z. Phys. C29, 637 (1985).
\[4\] B. Colquhoun, C. Davies, J. Koponen, A. Lytle, and C. McNeile (HPQCD Collaboration), Proceedings, 34th International Symposium on Lattice Field Theory (Lattice 2016): Southampton, UK, July 24–30, 2016, PoS LATTICE 2016, 281 (2017), arXiv:1611.01987 [hep-lat].
\[5\] N. Isgur and M. B. Wise, Phys. Lett. B232, 113 (1989).
\[6\] N. Isgur and M. B. Wise, Phys. Lett. B237, 527 (1990).
\[7\] E. E. Jenkins, M. E. Luke, A. V. Manohar, and M. J. Savage, Nucl. Phys. B390, 463 (1993), arXiv:hep-ph/9204238 [hep-ph].
\[8\] V. V. Kiselev, A. K. Likhoded, and A. I. Onishchenko, Nucl. Phys. B569, 473 (2000), arXiv:hep-ph/9905359 [hep-ph].
\[9\] C. G. Boyd, B. Grinstein, and R. F. Lebed, Phys. Rev. D56, 6895 (1997), arXiv:hep-ph/9705252 [hep-ph].
\[10\] B. Grinstein and R. F. Lebed, Phys. Rev. D92, 116001 (2015), arXiv:1509.04847 [hep-ph].
\[11\] A. Berns and H. Lamm, (2018), arXiv:1808.07360 [hep-ph].
\[12\] I. Caprini, Z. Phys. C61, 651 (1994).
\[13\] I. Caprini, Phys. Lett. B339, 187 (1994), arXiv:hep-ph/9408238 [hep-ph].