FLUCTUATIONS OF THE FREE ENERGY OF THE MIXED
\textit{p}-SPIN MEAN FIELD SPIN GLASS MODEL

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Abstract. We prove the convergence in distribution of the fluctuations of the free energy of the mixed \textit{p}-spin Sherrington-Kirkpatrick model with non-vanishing 2-spin component at high enough temperature. The limit is Gaussian, and the fluctuations are seen to arise from weighted cycle counts in the complete graph on the spin indices weighted by the 2-spin interaction matrix.

1. Introduction

The Sherrington-Kirkpatrick [SK75] model and its variants [Der80; GM84; Tal00] are important models of disordered magnetic systems and paradigmatic examples in the theory of complex systems [MPV87; Tal10; Tal11; Pan13b].

An important step in the solution of the model is the computation of the free energy. The Parisi formula [Par80; Par79; Gue03; Tal06; Pan13a] gives the typical value of the free energy in the form of a law of large numbers. This article studies the fluctuations of free energy around its typical value. We prove that the distribution of the fluctuations for mixed \textit{p}-spin Sherrington-Kirkpatrick models without external field at high enough temperature are asymptotically Gaussian, provided a non-vanishing 2-spin component is present. We achieve this by proving an estimate for the free energy which is sharp to subleading order, where the subleading term arises from certain cycle counts in the complete weighted graph defined by the disorder matrix of the 2-spin component. Thus the fluctuations of the free energy can be understood as arising from the fluctuations of these cycle counts.

To formally state our results let

$$\xi(x) = \sum_{p \geq 2} \alpha_p x^p,$$

be a non-zero power series with radius of convergence greater than one with $\alpha_p \geq 0$ for all $p$, called the mixture. Let $H_N(\sigma)$ be a centered Gaussian process on the sphere $\{\sigma \in \mathbb{R}^n : |\sigma| = \sqrt{N}\}$ with covariance

$$E_N [H_N(\sigma) H_N(\sigma')] = N \xi(\sigma \cdot \sigma' N),$$

called a \textit{mixed \textit{p}-spin Hamiltonian} with mixture $\xi$. The left-hand side is a well-defined covariance function on the sphere for any $\xi$ by Schonenberg’s theorem [Sch42], and can be explicitly constructed as a polynomial with Gaussian random coefficients, see e.g. [Pan13b, (1.12)-(1.15)].

Let $E$ be the uniform measure on $\{\pm 1\}^N$ and let the free energy be given by

$$F_N = \log Z_N, \text{ where } Z_N = E [\exp (\beta H_N(\sigma))],$$

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for an inverse temperature $\beta \geq 0$. Let $\mathcal{N}(\mu, \sigma^2)$ denote the normal distribution with mean $\mu$ and variance $\sigma^2$ and write $\xrightarrow{D}$ for convergence in distribution. We have the following result for the fluctuation of the free energy:

**Theorem 1.1.** For any $\xi$ with $\alpha_2 > 0$ there exists a $\beta_\xi \in (0, \frac{1}{\sqrt{2\alpha_2}}]$ such that if $0 < \beta < \beta_\xi$ then letting

$$s^2 = -\frac{1}{2} \log \left(1 - 2\alpha_2 \beta^2\right),$$

we have

$$F_N - N\frac{\beta^2}{2} \xi(1) \xrightarrow{D} N\left(-\frac{1}{2}s^2, s^2\right),$$

as $N \to \infty$. If $\xi(x) = \alpha_2 x^2$ then $\beta_\xi = \frac{1}{\sqrt{2\alpha_2}}$.

The leading order term in (3) comes from the annealed partition function

$$E_N[Z_N] = \exp \left(N\frac{\beta^2}{2} \xi(1)\right).$$

Theorem 1.1 will follow from a precise estimate of $\log Z_N$ to subleading order. To state it we construct the 2-spin component $H^2_N$ by setting

$$H^2_N(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i,j} J_{ij} \sigma_i \sigma_j,$$

for $J = (J_{ij})_{i,j=1,\ldots,N}$ a GOE random matrix, i.e.

$$J_{ii} \sim \mathcal{N}(0, 2) \text{ for all } i, J_{ij} \sim \mathcal{N}(0, 1) \text{ for } i \neq j, J_{ij} \text{ independent for } i \leq j,$$

defined under a probability $\mathbb{P}_N$. Note that then $E_N[H^2_N(\sigma)H^2_N(\sigma')] = N(\sigma \cdot \sigma'/N)^2$.

We construct $H_N(\sigma)$ by letting

$$H_N(\sigma) = \sqrt{\alpha_2} H^2_N(\sigma) + \tilde{H}_N(\sigma),$$

where $\tilde{H}_N(\sigma)$ is a centered Gaussian process in $\mathbb{R}^N$, independent of $H^2_N$ and also defined under $\mathbb{P}_N$, with covariance $E_N[\tilde{H}_N(\sigma)\tilde{H}_N(\sigma')] = N\tilde{\xi} \left(\frac{\sigma \cdot \sigma'}{N}\right)$ for $\tilde{\xi}(x) = \xi(x) - \alpha_2 x^2$. The process $H_N(\sigma)$ then satisfies (1).

Having constructed the 2-spin component $H^2_N$, consider now its centered weighted cycle counts

$$C_{N,k} = \frac{1}{N^2} \sum_{i_0, \ldots, i_{k-1} \text{ distinct}} J_{i_0 i_1} \ldots J_{i_{k-1} i_0} - (N-1)1_{k=2}, \quad k = 1, 2, \ldots .$$

The sum in (7) has mean zero if $k \neq 2$ since each summand is a product of distinct independent centered Gaussians, and if $k = 2$ it equals $\sum_{i_0 \neq 1} J_{i_0 i_1}^2$ which has mean $N(N-1)$. Note that $C_{N,k} = 0$ for $k > N$.

Finally letting

$$\mu_k = \left(\sqrt{2\alpha_2 \beta}\right)^k, \quad k = 1, 2, \ldots ,$$

and writing $\xrightarrow{P}$ for convergence in probability we have the following precise estimate for the free energy:
Theorem 1.2. For any $\xi$ with $\alpha_2 > 0$ there exists a $\beta_\xi \in (0, \frac{1}{\sqrt{2\alpha_2}})$ such that if $0 < \beta < \beta_\xi$ then
\[ F_N - N \frac{\beta^2}{2} \xi(1) - \sum_{k=1}^\infty \frac{2\mu_k C_{N,k} - \mu_k^2}{4k} \to 0, \]
as $N \to \infty$. If $\xi(x) = \alpha_2 x^2$ then $\beta_\xi = \frac{1}{\sqrt{2\alpha_2}}$.

Theorem 1.2 identifies the origin of the fluctuations of the free energy as fluctuations of the cycle counts arising from the 2-spin component.

1.1. Previous work. To the best of our knowledge the first work on fluctuations of the free energy was [ALR87], which obtained the fluctuations in the standard Sherrington-Kirkpatrick model (that is the case $\xi(x) = x^2$) up to the critical temperature. It derived both the leading order of the free energy and the fluctuations through a graphical analysis, in which cycle counts give the main contribution. In the present work we rely on the second moment method for the leading order, and find a different way to use graphical analysis to study the fluctuations (see Section 1.2 below for more details).

The work [BKL+02] obtained the law of the fluctuations for pure $p$-spin Hamiltonians (that is for the case $\xi(x) = x^p$ for some $p \geq 2$) for small enough $\beta$ using a martingale method. Furthermore [CDP17] did the same for mixed $p$-spin Hamiltonians without odd $p$-terms and with non-zero external field at all temperatures using a combination of interpolation, the Chen-Stein method and the Parisi formula. For the spherical SK model the fluctuations of the free energy at high temperature has been obtained in [BL16] using the random matrix techniques that are applicable to that special case. Similar techniques are used in the related works [Lan20; LS20; LS19].

1.2. Outline of proof. We now give a high-level sketch of the proof of the estimate Theorem 1.2 of $\log Z_N$ in terms of cycle counts. The method has previously been used by the first author to study fluctuations in a stochastic block model [Ban+18], in a hypothesis testing problem for spiked random matrices [BM18] and in the SK model with Curie-Weiss interaction [Ban20].

It is known since [ALR87] that the subleading fluctuations of the pure 2-spin model are determined by the cycle counts $C_{N,k}$. In the approach of [ALR87] the cycle counts appear as the leading contributions in a graphical cluster expansion. Cycle counts are also relevant in the study of fluctuations in the stochastic block model [MNS13] and random regular graphs [Jan95]. In this approach the cycle counts enter the analysis via certain Radon-Nikodym derivatives. Our method is inspired by the latter approach.

Once one suspects that the cycle counts determine the fluctuations of $Z_N$ one can guess the form of the fluctuations as follows: One views $\tilde{Z}_N = Z_N/E_N[Z_N]$ as a Radon-Nikodym derivative $\frac{dP_N}{dQ_N}$ and considers the law of the sequence
\[ C_{N,1}, C_{N,2}, C_{N,3}, \ldots, \]
of cycle counts under the measures $Q_N$. Under the measure $P_N$ one can show using the moment method that the cycle counts are asymptotically Gaussian:
\[ C_{N,k} \overset{D}{\to} \mathcal{N}(0, 2k) \text{ under } P_N, \]
jointly for finitely many \( k \), with the \( C_{N,k} \) becoming independent in the limit. For more details on this important computation see the next subsection and Proposition 2.1. It turns out that \( \tilde{Z}_N \) as a Radon-Nikodym derivative changes the law of the \( J_{ij} \) in a simple way: it gives them a random non-zero mean, but otherwise the law stays the same (Lemma 3.1 Corollary 3.2). Using this one can show that under \( Q_N \) the cycle counts are also asymptotically Gaussian but with a non-zero mean, namely

\[
C_{N,k} \rightarrow \mathcal{N}(\mu_k, 2k) \quad \text{under} \quad Q_N,
\]

jointly for finitely many \( k \), still independent in the limit (see Proposition 3.3).

Recall that if of \( C \sim \mathcal{N}(0, \sigma^2) \) then the Radon-Nikodym derivative \( \exp\left(\frac{2\mu C - \mu^2}{2\sigma^2}\right) \) changes the law of \( C \) to \( \mathcal{N}(\mu, \sigma^2) \). Now if under a measure \( P \) the sequence (10) is exactly independent Gaussian with mean 0 and variance of \( k \)-th variable given by \( 2k \), and \( Q \) is the measure where they have the same distribution but with the mean of the \( k \)-th variable given by \( \mu_k \) instead of 0, then we necessarily have

\[
\frac{dP}{dQ} = \exp\left(\sum_{k=1}^{\infty} \frac{2\mu_k C_{N,k} - \mu_k^2}{4k}\right).
\]

Thus a possible approximation of \( \log \tilde{Z}_N \) is as log of the right-hand side, which is precisely the sum that appears in (9). We prove Theorem 1.2 by making this approximation rigorous and using (4).

To achieve this we roughly speaking use a second moment estimate for the difference between the normalized partition function \( \tilde{Z}_N \) and the the RHS of (13), together with the Chebyshev inequality. Since for finite \( N \) the cycle counts have tails that decay too slowly for the RHS of (13) to have a finite second moment, we carry out this argument on a limiting probability space where after taking \( N \rightarrow \infty \) the cycle counts become exactly Gaussian and thus do have finite exponential moments. The second moment argument depends on the second moment \( E_N[Z_N^2] \) of the partition function being asymptotic to \( c_\xi E_N[Z_N^2] \), where \( c_\xi \) is a constant depending on \( \xi \), and therefore works precisely when the vanilla second moment method proves that the leading order free energy is given by its annealed value.

Theorem 1.1 is a simple consequence of Theorem 1.2 and the asymptotic normality and independence of the cycle counts.

### 1.3. Cycle counts

As mentioned above a crucial step in both the proof of Theorem 1.2 and the derivation of Theorem 1.1 is the asymptotic normality (11) of the cycle counts, which we prove in Proposition 2.1.

The cycle counts are related to traces of a power of a GOE random matrix studied in random matrix theory; indeed if the sum in (7) is taken over all \( i_0, \ldots, i_{k-1} \) without the requirement that they be distinct then this sum is precisely \( N^{-k/2} \text{Tr}(J^k) \). It is well-known that the traces satisfy a CLT with the same normalization \( N^{-k/2} \) but different recentering (see e.g. [SS98; AZ06; AGZ10, Chapter 1]). The traces do not become asymptotically independent, and the variance of \( N^{-k/2} \text{Tr}(J^k) \) is different from that of the corresponding cycle count. To prove (11) we adapt in Section 2 the random matrix method to study traces, namely the moment method together with a graphical computation of the moments. The latter computation turns out to be simpler for cycle counts than for traces, since the restriction to distinct \( i_0, \ldots, i_{k-1} \) leads to a simpler collection of graphs, namely only cycles.
1.4. Discussion. It is natural to ask for how large $\beta$ the claims of Theorems 1.1-1.2 remain true. Let $\beta_*$ denote the supremum of all such $\beta$.

One may note that the recentering and variance in (3) explodes as $\beta$ approaches $1/\sqrt{2\alpha}$, so that certainly $\beta_* \leq 1/\sqrt{2\alpha}$. We must also have $\beta_* \leq \beta_c$, where $\beta_c$ is the critical inverse temperature of the static phase transition for the Hamiltonian, since for $\beta > \beta_c$ even the leading order term in (3) is incorrect.

As alluded to above, the $\beta_\xi$ in our theorems is the largest inverse temperature for which the vanilla second moment proves a lower bound for the free energy, and thus $\beta_\xi = \beta_* = \beta_c$ only for the pure 2-spin model, and otherwise we expect that $\beta_\xi < \beta_* \leq \beta_c$.

An interesting question is whether (3) holds all the way up to $\beta_c$, i.e. if $\beta_* = \beta_c$, as is the case for the 2-spin model, or if there is a second regime with different fluctuations at high temperature, as is the case for the REM model [BKL+02].

2. Asymptotic normality of cycle counts using method of moments and Wick’s formula

Recall that $\mathbb{P}_N$ is the probability of the probability space on which $J$ and $\tilde{H}_N$ are defined. In this section we will prove the following result on the convergence in distribution of the weighted cycle counts $C_{N,k}$ from (7) under the measure $\mathbb{P}_N$.

Proposition 2.1 (Limiting law of centered cycle counts under $\mathbb{P}_N$). For any $k \geq 1$

$$\mathbb{P}_N\text{-law of } (C_{N,1}, C_{N,2}, \ldots, C_{N,k}) \overset{D}{\rightarrow} (C_{\infty,1}, C_{\infty,2}, \ldots, C_{\infty,k}),$$

where $(C_{\infty,1}, \ldots, C_{\infty,k})$ is a centered independent Gaussian vector where $C_{\infty,k}$ has variance $2k$.

The proof will use the method of moments, Wick’s formula and the combinatorial framework from [AZ06] and [AGZ10, Chapter 1]. We first state some elements of the latter framework.

Definition 2.2 (Word). For a given $N \geq 1$, a letter is an element of $\{1, \ldots, N\}$. A word $w$ is a finite sequence of letters $s_1 \ldots s_n$, at least one letter long. A word $w$ is closed if its first and last letters are the same.

For any word $w = (w_0 \ldots w_{k-1})$, we use $l(w) = k$ to denote the length of $w$ and supp$(w)$ to denote the support of $w$, i.e. the set of letters appearing in $w$. To any word $w$ we may associate a graph as follows.

Definition 2.3 (Graph associated with a word). Given a word $w = (w_0, \ldots, w_{k-1})$, we let $G_w = (V_w, E_w)$ be the graph with vertex set $V_w = \text{supp}(w)$ and edge set $E_w = \{\{w_i, w_{i+1}\}; i = 0, \ldots, k-2\}$.

Note that $G_w$ is an undirected simple graph permitting loops. The word $w$ defines a walk on the graph $G_w$ which further starts and terminates at the same vertex if the word is closed. For $e \in E_w$, we use $N^w_e$ to denote the number of times this walk traverses the edge $e$ (in any direction).

In this paper we shall mainly deal with a special class of words, namely cyclic words.

Definition 2.4 (Set $\mathcal{M}_1$ of cyclic words). We call a word $w$ cyclic if $l(w) = 2, 3$ and the word is closed, or if $l(w) \geq 4$ and the word is closed, the graph $G_w$ is a cycle.
and \( N^w_e = 1 \) for each edge \( e \) in \( G_w \). We write \( \mathcal{W}_l \) for the set of all such words of length \( l \).

Note that for \( k \geq 1 \) we have \( w \in \mathcal{W}_{k+1} \) iff \( w = (i_0, \ldots, i_{k-1}, i_0) \) for \( i_0, \ldots, i_{k-1} \) distinct. Thus

\[
|\mathcal{W}_{k+1}| = \begin{cases} N(N-1) \cdots (N-k+1) & \text{for } 0 \leq k \leq N, \\ 0 & \text{for } k > N. \end{cases}
\]

Also we see that the sum in (7) is exactly a sum over all \((i_0, \ldots, i_{k-1}, i_0) \in \mathcal{W}_{k+1}\), so that

\[
C_{N,k} = \frac{1}{N^{\frac{k}{2}}} \sum_{w \in \mathcal{W}_{k+1}} J_w - (N-1) \mathbf{1}_{\{k=2\}} \quad \text{for } k \geq 1,
\]

where we define the weight

\[
J_w = \prod_{i=0}^{l(w)-1} J_{w_i w_{i+1}},
\]

of a cyclic word \( w \). We define also the centered weight

\[
\hat{J}_w = J_w - \mathbb{E}_N[J_w] = \begin{cases} J_w & \text{if } l(w) \neq 3, \\ J_{w_0 w_1} - 1 & \text{if } l(w) = 3, \end{cases}
\]

of a cyclic word \( w \) (note that the words \( w \in \mathcal{W}_{k+1} \) for \( k = 2 \) are special since they satisfy \( \mathbb{E}_N[J_w] = \mathbb{E}_N[J_{w_0 w_1}^2] = 1 \), otherwise \( \mathbb{E}_N[J_w] = 0 \)). Using this and for the case \( k = 2 \) that \(|\mathcal{W}_3| = N(N-1)\) we obtain from (16) the formula

\[
C_{N,k} = \frac{1}{N^{\frac{k}{2}}} \sum_{w \in \mathcal{W}_{k+1}} \hat{J}_w, \quad k \geq 1.
\]

To make use of this formula we will use the following properties of centered word weights.

\[\textbf{Lemma 2.5} \text{ (Properties of centered word weights).} \quad \text{For all cyclic words } w \]

\[
\mathbb{E}_N[\hat{J}_w] = 0.
\]

Furthermore for all cyclic words \( w, v \)

\[
\mathbb{E}_N[\hat{J}_w \hat{J}_v] = \begin{cases} 0 & \text{if } G_w \neq G_v, \\ a_k & \text{if } G_w = G_v \text{ and } w, v \in \mathcal{W}_{k+1} \text{ for } k \geq 1, \end{cases}
\]

where \( a_1 = a_2 = 2, a_k = 1, k \geq 3 \).

Lastly for any sets \( A, B \) of cyclic words

\[
\left( \hat{J}_w \right)_{w \in A} \text{ is independent of } \left( \hat{J}_v \right)_{v \in B} \quad \text{if } \left( \bigcup_{w \in A} E(G_w) \right) \cap \left( \bigcup_{v \in B} E(G_v) \right) = \emptyset.
\]

\[\text{Proof.} \quad \text{The claim } (20) \text{ is immediate from the definition } (18) \text{ of } \hat{J}_w.\]

Turning to (21), recall first (6). Note that if \( G_w \neq G_v \) then there is an edge \( e = \{i, j\} \) that is in only one of \( G_w \) and \( G_v \), so that \( J_{ij} = J_{ji} \) appears in the product \( \hat{J}_w \hat{J}_v \) exactly once, as either a factor \( J_{ij} \) or a factor \( J_{ij}^2 - 1 \) for \( i \neq j \), both of which have mean zero, so that the independence of the \( J_{ij} \) implies that
$E_N[\hat{J}_w, \hat{J}_v] = 0$. Now consider the case $G_w = G_v$, which can only occur if $w, v \in W_{k+1}$ for $k \geq 1$. If $k \geq 3$ then $\hat{J}_w \hat{J}_v = \prod_{(i,j) \in E(G_w)} J_{ij}^2$ which has mean 1, and if $k = 2$ then $\hat{J}_w \hat{J}_v = (J_{wv}^2 - 1)^2$ which has mean 2 and finally if $k = 1$ then $\hat{J}_w \hat{J}_v = J_{wv}^2$ which has mean 2. This proves (21).

The claim (22) follows because if the words in $A$ do not share any edges with the words in $B$ then there is no random variable $J_{ij}$ that appears in both in $\hat{J}_w$ for some $w \in A$ and in $\hat{J}_v$ for some $v \in B$. $\square$

We now compute the mean and the variance $C_{N,k}$ using (19) and the previous lemma.

Lemma 2.6 (Mean and variance of $C_{N,k}$). For all $N$ it holds under $\mathbb{P}_N$ that

(23) \[ E_N[C_{N,k}] = 0 \text{ for all } k \geq 1, \]

and

(24) \[ E_N[C_{N,k}C_{N,l}] = 0 \text{ for all } k \neq l, \]

and

(25) \[ \text{Var}[C_{N,k}] = \begin{cases} 2k N^{(N-1)\ldots(N-k+1)} / N^k & \text{for } k \leq N, \\ 0 & \text{for } k > N. \end{cases} \]

For fixed $k \geq 1$

(26) \[ \text{Var}[C_{N,k}] \to 2k \text{ as } N \to \infty. \]

Proof. The claim (23) follows from (19) and (20). To compute the (co-)variances note that for all $k, l$

(27) \[ E_N[C_{N,k}C_{N,l}] = \frac{1}{N^{k+l}} \sum_{w \in W_{k+1}, v \in W_{l+1}} E_N[\hat{J}_w \hat{J}_v]. \]

Recalling (21) we note that since $G_w \neq G_v$ if $w \in W_{k+1}, v \in W_{l+1}$ for $k \neq l$ the claim (24) follows. Setting $w = v$ we get for $k \geq 1$

(28) \[ E_N[C_{N,k}^2] = \frac{a_k}{N^k} \sum_{w \in W_{k+1}} |\{v \in W_{k+1} : G_v = G_w\}|, \]

for $a_k$ as in (21). If $G_w = G_v$ then the sequence $v$ must be a walk of the graph $G_w$ of length $k + 1$ that visits all $k$ vertices of $G_w$ and ends at the vertex where it started.

For $k = 1$ there is one such walk, so $|\{v \in W_{k+1} : G_v = G_w\}| = 1$ which together with $|W_2| = N$ and $a_1 = 2$ gives $\text{Var}[C_{N,1}] = 2$ and proves (25) for $k = 1$.

If $k = 2$ there is one such walk for each of the two possible starting vertices, so $|\{v \in W_{k+1} : G_v = G_w\}| = 2$ which together with $|W_3| = N(N-1)$ and $a_k = 2$ gives $\text{Var}[C_{N,2}] = 4(N-1)/N$ and proves (25) for $k = 2$.

If $k \geq 3$ then all such walks can be enumerated by picking one of $k$ starting vertices, and then picking one of two directions to traverse the cycle. Therefore $|\{v \in W_{k+1} : G_v = G_w\}| = 2k$, so that with $a_k = 1$ we get

(29) \[ \text{Var}[C_{N,k}] = \frac{2k |W_{k+1}|}{N^k}, \]

which implies (25) for $k \geq 3$ by (15). Finally (26) is a simple consequence of (25). $\square$
This shows that the mean and covariance of the vector on the LHS of (14) converge to the those of the vector on the RHS. To prove the convergence in law we will verify the convergence of higher moments and use the method of moments in the form we now state.

**Lemma 2.7** (Method of moments). Let \((Y_{N,1}, \ldots, Y_{N,l}), N \geq 1\), be a sequence of random vectors of dimension \(l\). Assume that:

1) (Mixed moments converge) For any fixed \(m\) and \(i_1, \ldots, i_m \in \{1, \ldots, l\}\) the limit
\[
\lim_{N \to \infty} \mathbb{E}[Y_{N,i_1} \ldots Y_{N,i_m}]
\]
exists.

2) (Carleman’s Condition; [Car26]) It holds
\[
\sum_{h=1}^{\infty} \left( \lim_{N \to \infty} \mathbb{E}[Y_{N,i}^{2h}] \right)^{-\frac{1}{2h}} = \infty
\]
for all \(1 \leq i \leq l\).

Then the vector \((Y_{N,1}, \ldots, Y_{N,l})\) converges in distribution to some random vector \((Z_1, \ldots, Z_l)\). Further, if \(m, i_1, \ldots, i_m\) are as in 1) then the limit in (30) equals \(\mathbb{E}[Z_{i_1} \ldots Z_{i_m}]\).

Next we recall Wick’s formula.

**Lemma 2.8** (Wick’s formula; [Wic50]). Let \((Y_1, \ldots, Y_l)\) be a centered random vector of dimension \(l\) with covariance matrix \(\Sigma\) (possibly singular). Then \((Y_1, \ldots, Y_l)\) is jointly Gaussian if and only if for any integer \(m\) and \(i_1, \ldots, i_m \in \{1, \ldots, l\}\) we have with \(X_r = Y_{i_r}\) that
\[
\mathbb{E}[X_1 \ldots X_m] = \begin{cases} 
\sum_{\eta} \prod_{(i,j) \in \eta} \mathbb{E}[X_i X_j] & \text{for } m \text{ even} \\
0 & \text{for } m \text{ odd},
\end{cases}
\]
where the sum is over pairings \(\eta\) of \(\{1, \ldots, m\}\) (that is of partitions of this set into \(\frac{m}{2}\) sets containing exactly 2 elements).

To compute the higher and mixed moments of the \(C_{N,k}\) we will use further combinatorial concepts from [AZ06].

**Definition 2.9** (Sentences and corresponding graphs). A sentence \(a = [w_i]_{i=1}^n = [(\alpha_{i,j})_{j=1}^{l(w_i)}]_{i=1}^n\) is an ordered collection of \(n\) words \(w_1, \ldots, w_n\) of length \((l(w_1), \ldots, l(w_n))\).

We define the graph \(G_a = (V_a, E_a)\) to be the graph with
\[
V_a = \text{supp}(a), \quad E_a = \{\alpha_{i,j}, \alpha_{i,j+1} \mid i = 1, \ldots, n; j = 1, \ldots, l(w_i) - 1\}.
\]

We define an equivalence relation on sentences by saying that sentences \(a\) and \(b\) are equivalent if there is a permutation of \(\{1, \ldots, N\}\) which turns \(a\) into \(b\) when applied to each letter of each word of \(a\).

From (19) one sees that a mixed moment of \(C_{N,l}\)-s can be written as a sum over sentences:
\[
\mathbb{E}_N[C_{N,l_1} \ldots C_{N,l_m}] = \sum_{a \in \mathcal{W}_{l_1+1} \times \cdots \times \mathcal{W}_{l_m+1}} \mathbb{E}_N \left[ \hat{j}_a \right],
\]
where
\[
\hat{j}_a = \prod_{i=1}^m \hat{j}_{a_i},
\]
is the weight of a sentence \(a\) of length \(m\).
Definition 2.10 (Weak CLT sentences). A sentence $a = [w_i]_{i=1}^n$ is called a weak CLT sentence if the following conditions are true:

1) All the words $w_i$ are closed.
2) Jointly the words $w_i$ visit each edge of $G_a$ at least twice.
3) For each $i \in \{1, \ldots, n\}$, there is another $j \neq i \in \{1, \ldots, n\}$ such that $G_{w_i}$ and $G_{w_j}$ have at least one edge in common.

We have that

$$ (34) \quad \mathbb{E}_N [\hat{J}_a] = 0 \text{ if } a \text{ is a sentence of cyclic words and not a weak CLT sentence,} $$

since a sentence of cyclic words that is not a weak CLT sentence must violate either 2) or 3); if 2) is violated then the product $\hat{J}_a$ contains some $J_{ij}, i \leq j$ exactly once, so that $\mathbb{E}_N [\hat{J}_a] = 0$ by the independence of the $J_{ij}$, and if 3) is violated then there is an $i$ such that $G_{a_i}$ is disjoint from $\bigcup_{j \neq i} G_{a_j}$ so that $\mathbb{E}_N [\hat{J}_a] = 0$ by (22) and (20).

By (34) only weak CLT sentences can give a non-zero contribution to the mixed moment in (32).

Definition 2.11 (CLT sentence). Let $a = [w_i]_{i=1}^m$ be a weak CLT sentence consisting of $m$ words with length $l_1+1, \ldots, l_m+1$ respectively. Then $a$ is called a CLT sentence if $|V(G_a)| = \sum_{r=1}^m l_r$.

We will see that CLT sentences give the main contribution to mixed moment (32), since the entropy of any sentence $a$ with smaller $|V(G_a)|$ will be of lower order in $N$.

The following proposition will be used to show that the sum in (32) restricted to CLT sentences factors in way that makes Wick’s formula hold in the limit $N \to \infty$.

Proposition 2.12. (Proposition 4.9 in [AZ06]) Let $a = [w_i]_{i=1}^m$ be a weak CLT sentence consisting of $m$ words with length $l_1+1, \ldots, l_m+1$ respectively. Then we have $|V(G_a)| \leq \sum_{r=1}^m l_r$. Suppose equality occurs (i.e. $a$ is a CLT sentence), then the words $w_i$ of the sentence $a$ are perfectly paired in the sense that for all $i$ there exists a unique $j$ distinct from $i$ such that $w_i$ and $w_j$ have a letter in common. In particular, $m$ is even.

Using this proposition and (22) we have for any CLT sentence $a$ with pairing $\eta(a)$ that

$$ (35) \quad \mathbb{E}_N [\hat{J}_a] = \mathbb{E}_N \left[ \prod_{(i,j) \in \eta(a)} \hat{J}_{a_i} \hat{J}_{a_j} \right] = \prod_{(i,j) \in \eta(a)} \mathbb{E}_N [\hat{J}_{a_i} \hat{J}_{a_j}]. $$

We have now stated all the ingredients necessary to prove Proposition 2.1.

Proof of Proposition 2.1. Consider the vector $(C_{N,1}, \ldots, C_{N,k})$. We will show that the mixed moments of this vector converge and that the limit is a sum over pairings, to be able to apply Lemmas 2.7 and 2.8.

To this end consider for any $m \geq 1, l_1, \ldots, l_m \in \{1, \ldots, k\}$ the mixed moment

$$ \mathbb{E}_N [X_{N,1} \ldots X_{N,m}], $$
where $X_{N,i} = C_{N,i}$. By \((19)\) this equals
\[
\frac{1}{N^{\frac{1+\ldots+l_m}{2}}} \sum_{a \in \mathfrak{M}_{l_1+1} \times \ldots \times \mathfrak{M}_{l_m+1}} \mathbb{E}_N \left[ \hat{j}_{w_1} \ldots \hat{j}_{w_m} \right]
\]
which by \((31)\) is the same as
\[
\frac{1}{N^{\frac{1+\ldots+l_m}{2}}} \sum_{a \in \mathfrak{M}_{l_1+1} \times \ldots \times \mathfrak{M}_{l_m+1}} \mathbb{E}_N \left[ \hat{j}_{a} \right].
\]

Consider the magnitude of the contribution of weak CLT sentences whose corresponding graph has a less than maximal number of vertices, i.e.
\[
(36) \quad \mathbb{E}_N \left[ \hat{j}_{a} \right] \quad \text{with } a \in \mathfrak{M}_{l_1+1} \times \ldots \times \mathfrak{M}_{l_m+1}, \text{weak CLT sequence } |V(G_a)| < \frac{1+\ldots+l_m}{2}
\]

Note that the moment $\mathbb{E}_N \left[ \hat{j}_{a} \right]$ depends only on the equivalence class of $a$. Furthermore the set of equivalence classes is a function only of $m, l_1, \ldots, l_m$ and not of $N$. Thus we obtain that
\[
\sup_{a \in \mathfrak{M}_{l_1+1} \times \ldots \times \mathfrak{M}_{l_m+1}} \mathbb{E}_N \left[ \hat{j}_{a} \right] \leq M,
\]
where $M < \infty$ does not depend on $N$, and \((36)\) is bounded above by
\[
(37) \quad \frac{M}{N^{\frac{1+\ldots+l_m}{2}}} \sum_{a \in \mathfrak{M}_{l_1+1} \times \ldots \times \mathfrak{M}_{l_m+1}, \text{weak CLT sequence, } |V(G_a)| < \frac{1+\ldots+l_m}{2}} 1.
\]

Noting that the number of vertices of $G_a$ depends only on the equivalence class of $a$, and that the number $K$ of equivalence classes of sentences such that $|V(G_a)| < \frac{1+\ldots+l_m}{2}$ does not depend on $N$, we can crudely bound the sum in \((37)\) by $K \times N^{-\frac{1+\ldots+l_m}{2}}$. Thus \((36)\) is at most $cN^{-1}$ for a constant $c$ that depends only on $m, l_1, \ldots, l_m$, and so (recalling Definition \(2.11\))
\[
\left| \mathbb{E}_N \left[ X_{N,1} \ldots X_{N,m} \right] - \frac{1}{N^{\frac{1+\ldots+l_m}{2}}} \sum_{a \in \mathfrak{M}_{l_1+1} \times \ldots \times \mathfrak{M}_{l_m+1}} \mathbb{E}_N \left[ \hat{j}_{a} \right] \right| \to 0.
\]

Now by Proposition \(2.12\) all CLT sentences $a$ have length $m$ that is even, which in particular implies that $\mathbb{E}_N \left[ X_{N,1} \ldots X_{N,m} \right] \to 0$ if $m$ is odd. Furthermore recalling \((35)\) we have
\[
\sum_{a \in \mathfrak{M}_{l_1+1} \times \ldots \times \mathfrak{M}_{l_m+1}} \mathbb{E}_N \left[ \hat{j}_{a} \right] = \sum_{\eta} \sum_{a \in C_{\eta}} \prod_{i,j \in \eta} \mathbb{E}_N \left[ \hat{j}_{a_i} \hat{j}_{a_j} \right],
\]
where the sum over $\eta$ is over all pairings of $\{1, \ldots, m\}$ and with $C_{\eta}$ denoting the set of all $a \in \mathfrak{M}_{l_1+1} \times \ldots \times \mathfrak{M}_{l_m+1}$ that are CLT sentences whose pairing $\eta(a)$ satisfies...
\( \eta(a) = \eta. \) Now note that for any fixed \( \eta \) we have using (24)

\[
\frac{1}{N^{1 + \frac{1}{2} + l_m}} \left| \sum_{a \in \mathcal{C}_n} \prod_{\{i,j\} \in \eta} E_N \left[ \hat{J}_{a_i} \hat{J}_{a_j} \right] - \sum_{a \in \mathfrak{M}_{1+} \times \mathfrak{M}_{m+1}} \prod_{\{i,j\} \in \eta} E_N \left[ \hat{J}_{a_i} \hat{J}_{a_j} \right] \right| \leq \frac{cM}{N^{1 + \frac{1}{2} + l_m}} \left| \sum_{a \in \mathcal{C}} \left\{ G_{a_i} = G_{a_j} \text{ for all } \{i,j\} \in \eta \right\} - \sum_{a \in \mathfrak{M}_{1+} \times \mathfrak{M}_{m+1}} \left\{ G_{a_i} = G_{a_j} \text{ for all } \{i,j\} \in \eta \right\} \right|.
\]

All sentences \( a \) that are CLT sentences appear in both sums on the bottom line, while the only sentences that appears only in one are those for which \( G_{a_i} = G_{a_j} \) for all \( \{i,j\} \in \eta \) but \( G_{a_i} \) shares a vertex with \( G_{a_j} \) for some \( \{i,j\} \notin \eta \). If so \( G_{a_i} \) must necessarily have less than \( \frac{l_1 + \ldots + l_m}{2} \) vertices. Therefore the difference in the bottom line of (38) is bounded above by \( KN \frac{1}{1 + \frac{1}{2} + l_m} \), for the constant \( K \) from before, so that (38) goes to zero as \( N \to \infty \). Since also the number of pairings \( \eta \) does not depend on \( N \) we have

\[
\left| E_N \left[ X_{N,1} \ldots X_{N,m} \right] - \sum_{\eta} \frac{1}{N^{1 + \frac{1}{2} + l_m}} \sum_{a \in \mathfrak{M}_{1+} \times \mathfrak{M}_{m+1}} \prod_{\{i,j\} \in \eta} E_N \left[ \hat{J}_{a_i} \hat{J}_{a_j} \right] \right| \to 0.
\]

Finally the last sum over \( a \) factors as

\[
\frac{1}{N^{1 + \frac{1}{2} + l_m}} \sum_{a \in \mathfrak{M}_{1+} \times \mathfrak{M}_{m+1}} \prod_{\{i,j\} \in \eta} E_N \left[ \hat{J}_{a_i} \hat{J}_{a_j} \right] = \prod_{\{i,j\} \in \eta} \frac{1}{N^{\frac{l_1}{2}}} \sum_{w \in \mathfrak{M}_{l_1+1} \times \mathfrak{M}_{l_j+1}} E_N \left[ \hat{J}_w \right] \sum_{v \in \mathfrak{M}_{l_j+1} \times \mathfrak{M}_{l_j+1}} \hat{J}_v \right] = \prod_{\{i,j\} \in \eta} E_N \left[ C_{N,i}C_{N,j} \right].
\]

Applying also Lemma 2.6 we have showed that

\[
E_N \left[ X_{N,1} \ldots X_{N,m} \right] \to \sum_{\eta} \prod_{\{i,j\} \in \eta} E \left[ C_{\infty,i}C_{\infty,j} \right]
\]

i.e. that all the mixed moments converge. Applying this with \( m = 2h \) and \( X_i = C_{N,i} \), and (31) with \( X_i = C_{\infty,i} \), one sees that \( \lim_{N \to \infty} E_N \left[ C_{N,i}^{2h} \right] = E \left[ C_{\infty,i}^{2h} \right] \) for all positive integers \( h \), so that Carleman’s condition is easily verified. Therefore by Lemma 2.7 the vector \( (C_{N,1}, \ldots, C_{N,k}) \) converges in distribution to a random vector \( (Z_1, \ldots, Z_k) \). By the same lemma we have \( E_N \left[ C_{N,i}C_{N,j} \right] \to E \left[ Z_iZ_j \right] \), so that in fact \( E \left[ Z_iZ_j \right] = E \left[ C_{\infty,i}C_{\infty,j} \right] \) for all \( i, j \), which means that for all \( m \geq 1, 1 \leq l_1, \ldots, l_m \leq k \) we have with \( X_j = Z_{l_j} \) that

\[
E \left[ X_{1} \ldots X_{m} \right] = \begin{cases} \sum_{\eta} E \left[ X_{1}X_{j} \right] & \text{if } m \text{ is even} \\ 0 & \text{otherwise.} \end{cases}
\]

Thus \( (Z_1, \ldots, Z_k) \) satisfies Wicks formula (31) so by Lemma 2.8 the vector \( (Z_1, \ldots, Z_k) \) is Gaussian. Since its covariance matches that of \( (C_{\infty,1}, \ldots, C_{\infty,k}) \) also its law does. \( \square \)
3. Fluctuations of partition function determined by cycle counts; Proof of Theorem 1.2

In this section we will prove Theorem 1.2. Denote the normalized partition function by
\[ \hat{Z}_N = \frac{Z_N}{E_N[Z_N]}, \]
so that
\[ E_N[\hat{Z}_N] = 1. \]
Since also \( \hat{Z}_N \geq 0 \) we can use it to define a tilted measure \( Q_N \) via
\[ dQ_N = \hat{Z}_N dP_N. \]
Let \( Q_{N,\sigma} \) be a measure under which the \( J_{ij} \) are Gaussian with the same covariance as under \( P_N \) (see (6)), but where \( J_{ij} \) has mean \( \frac{1}{\sqrt{N}} \beta \alpha_2 \sigma_i \sigma_j \). Note that
\[ dQ_{N,\sigma} = \exp \left( \sum_i \left\{ \frac{\beta \sqrt{\alpha_2} J_{ii}^2}{2N} - \frac{\beta^2 \alpha_2}{N} \right\} + \sum_{i<j} \left\{ \frac{\beta \sqrt{2 \alpha_2} \sigma_i \sigma_j J_{ij}}{\sqrt{N}} - \frac{\beta^2 \alpha_2}{2N} \right\} \right) dQ_N. \]
We then have the following, which can be interpreted as saying that under \( Q_N \) the \( J_{ij} \) have the law of a mixture of Gaussians. Namely conditionally on \( \tilde{H}_N \) one samples \( \sigma \) according to the Gibbs measure of \( \tilde{H}_N \), and then samples \( J_{ij} \) as Gaussians with mean \( \frac{1}{\sqrt{N}} \beta \alpha_2 \sigma_i \sigma_j \) and the covariance of \( J_{ij} \) under \( P_N \).

Lemma 3.1 (Radon-Nikodym derivative identity). It holds that
\[ \frac{dQ_N}{dP_N} = \sum_{\sigma \in \{-1,1\}^N} \frac{1}{2^N} \frac{\exp \left( \beta \tilde{H}_N (\sigma) \right)}{E_N[\exp (\beta \tilde{H}_N (\sigma))]}. \]

Proof. Since \( H_N(\sigma) = \sqrt{\alpha_2} H_N^2(\sigma) + \tilde{H}_N(\sigma) \) and \( H_N^2 \) and \( \tilde{H}_N \) are independent we obtain that
\[ \frac{dQ_N}{dP_N} = \frac{1}{E_N[\exp (\beta H_N(\sigma))]} \frac{\exp \left( \beta \tilde{H}_N (\sigma) \right)}{E_N[\exp (\beta \tilde{H}_N (\sigma))]} = \sum_{\sigma \in \{-1,1\}^N} \frac{1}{2^N} \frac{\exp \left( \beta \sqrt{\alpha_2} H_N^2 (\sigma) \right)}{E_N[\exp (\beta \sqrt{\alpha_2} H_N^2 (\sigma))]} \frac{\exp \left( \beta \tilde{H}_N (\sigma) \right)}{E_N[\exp (\beta \tilde{H}_N (\sigma))]} \]
From the definition (5) of \( H_N^2 \) and that \( \text{Var}[H_N^2(\sigma)] = 1 \) it follows that
\[ \frac{\exp \left( \beta \sqrt{\alpha_2} H_N^2 (\sigma) \right)}{E_N[\exp (\beta \sqrt{\alpha_2} H_N^2 (\sigma))]} = \frac{\exp \left( \frac{\beta \sqrt{\alpha_2}}{2N} \sum_i J_{ii} + \frac{\beta \sqrt{\alpha_2}}{\sqrt{N}} \sum_{i<j} J_{ij} \sigma_i \sigma_j \right)}{\exp \left( \frac{\beta^2 \alpha_2}{2N} \right)}, \]
which equals \( \frac{dQ_{N,\sigma}}{dP_N} \) by (41).

The following is an easy consequence of the lemma and the independence of \( \tilde{H}_N \) and \( \frac{dQ_{N,\sigma}}{dP_N} \).

Corollary 3.2. We have
\[ Q_N [A] = E[Q_{N,\sigma} [A]] \text{ for any event } A, \]
measurable with respect to the \( J_{ij} \).
We will need a variant of Proposition 2.1 for the law of the cycle counts under the measure $Q_N$, namely the following.

**Proposition 3.3** (Limiting law of centered cycle counts under $Q_N$). Suppose that $\beta \sqrt{2\alpha_2} \leq 1$. For any $k \geq 1$ it holds that

$$Q_N\text{-law of } (C_{N,1}, C_{N,2}, \ldots, C_{N,k}) \xrightarrow{D} (\mu_1 + C_{\infty,1}, \ldots, \mu_k + C_{\infty,k}),$$

for $(C_{\infty,1}, \ldots, C_{\infty,k})$ as in Proposition 2.1 and $\mu_k$ as in (8).

**Proof.** By (16) we have for all $l \geq 1$

$$C_{N,l} = \frac{1}{N^2} \sum_{w \in \mathcal{M}_{l+1}} \prod_{j=0}^{l-1} J_{w_{j+1}} - (N - 1)1_{l=2}$$

(46)

$$= \frac{1}{N^2} \sum_{w \in \mathcal{M}_{l+1}} \prod_{j=0}^{l-1} \left( J_{w_{j+1}} - \sigma_{w_{j+1}} + \sigma_{w_{j+1}} \right) - (N - 1)1_{l=2},$$

where we use the shorthand

$$\sigma_{ij} = \frac{\beta \sqrt{2\alpha_2}}{\sqrt{N}} \sigma_i \sigma_j.$$ 

Recall from above (41) that under $Q_{N,\sigma}$ the $J_{ij} - \sigma_{ij}$ have the same law as the $J_{ij}$ under $P_N$. Now letting $B_{ij}$, $i \leq j$ have this same law under an auxiliary probability $P$ and letting $\sigma_i$ be IID Rademacher random variables under $P$, independent also of the $B_{ij}$, we have from (44) that

$$Q_N \text{-law of } C_{N,l} = P - \text{law of } \frac{1}{N^2} \sum_{w \in \mathcal{M}_{l+1}} \prod_{j=0}^{l-1} \left( B_{w_{j+1}} + \sigma_{w_{j+1}} \right) - (N - 1)1_{l=2}. 

(47)$$

If $l = 2$ then the RHS can be written as

$$\frac{1}{N^2} \sum_{w \in \mathcal{M}_{l+1}} \prod_{j=0}^{l-1} B_{w_{j+1}} - (N - 1) + \frac{2}{N} \sum_{i \neq j} B_{ij} \sigma_{ij} + \frac{1}{N^2} \sum_{w \in \mathcal{M}_{l+1}} \prod_{j=0}^{l-1} \sigma_{w_{j+1}}. 

(48)$$

If $l \neq 2$ the product $\prod_{j=0}^{l-1} \left( B_{w_{j+1}} + \sigma_{w_{j+1}} \right)$ can be written as a sum over subgraphs of $G_w$, namely

$$\sum_{H \subseteq G_w} \left( \prod_{e \in E(G_w \setminus H)} B_e \right) \left( \prod_{\sigma \in E(H)} \sigma_e \right),$$

where $G_w \setminus H$ denotes the graph on $\{1, 2, \ldots, N\}$ with vertex set $V(G_w)$ and edge set $E(G_w) \setminus E(H)$. Thus the quantity on the right-hand side of (47) equals

$$\frac{1}{N^2} \sum_{w \in \mathcal{M}_{l+1}} \prod_{j=0}^{l-1} B_{w_{j+1}} + \sum_{w \in \mathcal{M}_{l+1}} V_{N,l,w} + \frac{1}{N^2} \sum_{w \in \mathcal{M}_{l+1}} \prod_{j=0}^{l-1} \sigma_{w_{j+1}},$$

for

$$V_{N,l,w} = \frac{1}{N^2} \sum_{\emptyset \neq H \subseteq G_w} \left( \prod_{e \in E(G_w \setminus H)} B_e \right) \left( \prod_{\sigma \in E(H)} \sigma_e \right).$$

(49)
Observe that \( B_{N,l} := \frac{1}{N^2} \sum_{w \in \mathcal{W}_{l+1}} \prod_{j=1}^{l-1} B_{w_jw_{j+1}} - (N - 1)1_{\{l=2\}} \) have exactly same joint distribution under \( \mathbb{P} \) as the \( C_{N,l} \) do under \( \mathbb{P}_N \). Hence from Proposition 2.1 we have that

\[
\mathbb{P}-\text{law of } (B_{N,1}, B_{N,2}, B_{N,3} \ldots, B_{N,k}) \xrightarrow{D} (C_{\infty,1}, \ldots, C_{\infty,k}).
\]

On the other hand we have that

\[
\frac{1}{N^2} \sum_{w \in \mathcal{W}_{l+1}} \prod_{j=0}^{l-1} \sigma_{w_jw_{j+1}} = \frac{1}{N^2} \sum_{w \in \mathcal{W}_{l+1}} \left( \beta \frac{\sqrt{2\alpha_2}}{\sqrt{N}} \right)^l \leq (1 + o(1)) \mu_l.
\]

A variance calculation shows that

\[
\frac{1}{N} \sum_{i \neq j} B_{ij} \sigma_{ij} = \frac{\beta \sqrt{2\alpha_2}}{N^2} \sum_{i \neq j} B_{ij} \sigma_{ij} \xrightarrow{P} 0 \text{ under } \mathbb{P}.
\]

In the remainder we will show that

\[
\sum_{w \in \mathcal{W}_{l+1}} V_{N,l,w} \xrightarrow{P} 0 \text{ under } \mathbb{P} \text{ for } 3 \leq l \leq k.
\]

Recalling (47), (48) and (49) the claim (45) follows from (50)-(53) (note that \( V_{N,1,w} = 0 \) and Slutsky’s theorem).

It thus only remains to prove (53). To this end we compute

\[
\mathbb{E} \left[ \left( \sum_{w \in \mathcal{W}_{l+1}} V_{N,l,w} \right)^2 \right] = \sum_{w,v \in \mathcal{W}_{l+1}} \mathbb{E} [V_{N,l,w} V_{N,l,v}].
\]

This second moment equals

\[
\frac{1}{N^l} \sum_{w,v \in \mathcal{W}_{l+1}} 0 \neq H \subseteq G_v, \emptyset \neq H' \subseteq G_v \left( \prod_{e \in E(H) \cup E(H')} \sigma_e \right) \mathbb{E} \left[ \left( \prod_{e \in E(G_v \setminus H)} B_e \right) \left( \prod_{e \in E(G_v \setminus H')} B_e \right) \right].
\]

We have

\[
\left( \prod_{e \in E(H) \cup E(H')} \sigma_e \right) \leq \left( \frac{\beta \sqrt{2\alpha_2}}{\sqrt{N}} \right)^{|E(H)| + |E(H')|} \leq N^{-\frac{|E(H)| + |E(H')|}{2}},
\]

(since \( \beta \sqrt{2\alpha_2} \leq 1 \)) and since the \( B_e \) are independent and have mean zero

\[
\mathbb{E} \left[ \left( \prod_{e \in E(G_v \setminus H)} B_e \right) \left( \prod_{e \in E(G_v \setminus H')} B_e \right) \right] = \begin{cases} 1 & \text{ if } G_v \setminus H = G_v \setminus H', \\ 0 & \text{ else}. \end{cases}
\]

Thus

\[
\mathbb{E} \left[ \left( \sum_{w \in \mathcal{W}_{l+1}} V_{N,l,w} \right)^2 \right] \leq \frac{1}{N^l} \sum_{w,v \in \mathcal{W}_{l+1}} 0 \neq H \subseteq G_v, \emptyset \neq H' \subseteq G_v 1_{\{G_v \setminus H = G_v \setminus H'\}} N^{-\frac{|E(H)| + |E(H')|}{2}}
\]

Since \( G_v \setminus H = G_v \setminus H' \) implies \( |E(H)| = |E(H')| \), and \( 1_{\{G_v \setminus H = G_v \setminus H'\}} \leq 1_{\{G_v \setminus H \subseteq G_v\}} \) and there are most \( 2^l \) ways to choose \( H' \), this is at most

\[
\frac{2^l}{N^l} \sum_{w \in \mathcal{W}_{l+1}} 0 \neq H \subseteq G_w \sum_{v \in \mathcal{W}_{l+1}} 1_{\{G_v \setminus H \subseteq G_v\}} N^{-|E(H)|}
\]

Since \( G_w \setminus H = G_v \setminus H' \) implies \( |E(H)| = |E(H')| \), and \( 1_{\{G_v \setminus H = G_v \setminus H'\}} \leq 1_{\{G_v \setminus H \subseteq G_v\}} \) and there are most \( 2^l \) ways to choose \( H' \), this is at most
We now bound the sum over $v$ combinatorially. For a pair $(w, H)$ such that $w = (w_0, w_1, \ldots, w_l) \in \mathcal{W}_{l+1}$ and $H \subset G_w$ we encode $H$ as a vector $h(w, H) \in \{0, 1\}^l$ by setting $h_i(w, H) = 1$ if $\{w_i, w_{i+1}\} \in E(H)$. Next define an equivalence relation for triples $(w, h, v) \in \mathcal{W}_{l+1} \times \{0, 1\}^l \times \mathcal{W}_{l+1}$ under which $(w, h, v)$ is equivalent to $(w', h', v')$ if $h = h'$ and $(w, v) \sim (w', v')$ in the equivalence relation for sentences. Let $[(w, h, v)]$ denote the equivalence class of $(w, h, v)$ and let $\mathcal{E}$ denote the set of all equivalence classes.

The indicator $1_{[(G_w \setminus H) \leq G_v]}$ is a function only of the equivalence class $e = [(w, h (w, H), v)]$. Denote this function by $f(e)$. Using these constructions one can write

$$\sum_{v \in \mathcal{W}_{l+1}} 1_{[(G_w \setminus H) \leq G_v]} = \sum_{e \in \mathcal{E}} \left| \left\{ v \in \mathcal{W}_{l+1} : [(w, h (w, H), v)] = e \right\} \right| f(e).$$

Now note the following:

- The number of vertices in $G_v \setminus (G_w \setminus H)$ depends only on the equivalence class $e = [(w, h (w, H), v)]$. Denote this number of vertices by $A(e)$.
- For given $w, H$ and $e \in \mathcal{E}$, to construct a $v$ such that $[(w, h (w, H), v)] = e$ one must pick indices in $\{1, \ldots, N\}$ for $A(e)$ vertices (the indices of the other vertices are fixed by $w$), so that

$$\left| \left\{ v \in \mathcal{W}_{l+1} : [(w, h (w, H), v)] = e \right\} \right| \leq N^{A(e)}.$$

- For all $w, H, v$ it holds that $A([(w, h (w, H), v)]) \leq |E(G_v)| - |E(G_w \setminus H)| - 1 = |E(G_w)| - |E(G_w \setminus H)| - 1 = |E(H)| - 1$, where we have equality in the bound if $H = \emptyset$, $E(G_w)$ is a line, and one uses that $H \neq \emptyset, E(G_w)$.
- The number of equivalence classes $|\mathcal{E}|$ is finite and independent of $N$.

With these facts we get that

$$\sum_{v \in \mathcal{W}_{l+1}} 1_{[(G_w \setminus H) \leq G_v]} \leq |\mathcal{E}| N^{|E(H)| - 1}.$$

Thus

$$\mathbb{E} \left[ \left( \sum_{w \in \mathcal{W}_{l+1}} V_{N,l,w} \right)^2 \right] \leq \frac{2^l |\mathcal{E}|}{N^l} \sum_{w \in \mathcal{W}_{l+1}} \sum_{\emptyset \neq H \leq G_w} \frac{1}{N} \leq \frac{4^l |\mathcal{E}|}{N^{l+1}} |\mathcal{W}_{l+1}| \leq \frac{4^l |\mathcal{E}|}{N},$$

since $|\mathcal{W}_{l+1}| \leq N^l$ by (15). This proves (53) and thus concludes the proof of the proposition. \(\square\)

We will also need a precise asymptotic for the second moment of $\tilde{Z}_N$ under $\mathbb{P}_N$.

**Lemma 3.4** (Asymptotic for second moment of partition function). For every $\xi$ there is a $\beta_\xi \in (0, \frac{1}{\sqrt{2\alpha_2}})$ such that if $0 \leq \beta < \beta_\xi$ then

$$\mathbb{E}_N \left[ \tilde{Z}_N^2 \right] \to \frac{1}{\sqrt{1 - 2\alpha_2\beta^2}} \quad \text{as } N \to \infty.$$

If $\xi(x) = \alpha_2 x^2$ then $\beta_\xi = \frac{1}{\sqrt{2\alpha_2}}$. 
Proof. We have
\[
\mathbb{E}_N \left[ \tilde{Z}_N \right] = \mathbb{E}_N \left[ \frac{1}{Z_N^2} \mathbb{E}_N [Z_N^2] \right] = \mathbb{E}_N \left[ \frac{1}{Z_N^2} E^{\otimes 2} [\mathbb{E}_N [\exp (\beta H_N (\sigma) + \beta H_N (\sigma'))]] \right]
\]
(54)
\[
\mathbb{E}_N \left[ \frac{1}{Z_N^2} E^{\otimes 2} [\exp (\beta^2 \xi (1) N + \beta^2 N \xi (\frac{\sigma'}{\sqrt{N}}))] \right]
\]
\[
E \left[ \exp \left( \beta^2 N \xi \left( \sum_{i=1}^N \frac{\sigma_i}{\sqrt{N}} \right) \right) \right] = E \left[ \exp \left( \beta^2 N \xi \left( \sum_{i=1}^N \frac{\sigma_i}{\sqrt{N}} \right) \right) \right] \text{1}\left\{ \left| \sum_{i=1}^N \frac{\sigma_i}{\sqrt{N}} \right| \leq \varepsilon \right\}
\]
\[
+ E \left[ \exp \left( \beta^2 N \xi \left( \sum_{i=1}^N \frac{\sigma_i}{\sqrt{N}} \right) \right) \right] \text{1}\left\{ \varepsilon \leq \left| \sum_{i=1}^N \frac{\sigma_i}{\sqrt{N}} \right| \leq \sqrt{N} \right\}
\]
\[
+ E \left[ \exp \left( \beta^2 N \xi \left( \sum_{i=1}^N \frac{\sigma_i}{\sqrt{N}} \right) \right) \right] \text{1}\left\{ \left| \sum_{i=1}^N \frac{\sigma_i}{\sqrt{N}} \right| \geq \varepsilon \right\}
\]

Note that
\[
\xi (x) = \alpha_2 x^2 + O_\varepsilon (x^3).
\]
Using this and Bernstein’s inequality and (56), the second line on the right-hand side is bounded by
\[
\sum_{M \leq m \leq \varepsilon \sqrt{N}} \exp \left( \beta^2 \alpha_2 m^2 + c \frac{m^3}{\sqrt{N}} \right) \exp \left( -\frac{m^2}{2(1+m)} \right)
\]
\[
\leq \sum_{M \leq m \leq \varepsilon \sqrt{N}} \exp \left( -m^2 \left( \frac{1}{2(1+\varepsilon)} - c \varepsilon + \beta^2 \alpha_2 \right) \right) = o_M (1),
\]
provided \(\varepsilon\) is chosen small enough depending on \(\xi, \beta, \alpha_2\). By a large deviation bound the last line of (55) is bounded by
\[
\sum_{d \leq \xi \leq c \varepsilon} \exp \left( N \left( \beta^2 \xi \left( \frac{1}{N} \right) - I \left( \frac{1}{N} \right) + o (1) \right) \right)
\]
\[
\leq N \exp \left( N \sup_{|\alpha| \geq \varepsilon} \left\{ \beta^2 \xi \left( \frac{1}{N} \right) - I \left( \frac{1}{N} \right) + o (N) \right\} \right) \rightarrow 0, \text{ as } N \rightarrow \infty,
\]
since the supremum is negative for all positive \(\varepsilon > 0\) when \(\beta < \beta_\varepsilon\). Finally the CLT implies that \(\sum_{i=1}^N \frac{\sigma_i}{\sqrt{N}} \rightarrow N (0, 1)\) under \(E\), so that taking first the limit \(N \rightarrow \infty\) and
then the limit $M \to \infty$ the first line the RHS of (55), namely

$$
E \left[ \exp \left( \beta^2 N \xi \left( \frac{1}{N} \sum_{i=1}^{N} \sigma_i \right) \right) 1 \left\{ \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sigma_i \right\| \leq M \right\} \right] \\
= E \left[ \exp \left( \beta^2 \alpha_2 \left( \frac{1}{N} \sum_{i=1}^{N} \sigma_i \right)^2 + O \left( \frac{\alpha_2}{\sqrt{N}} \right) \right) 1 \left\{ \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sigma_i \right\| \leq M \right\} \right],
$$

converges to

$$
\frac{1}{\sqrt{2\pi}} \int e^{\beta^2 \alpha_2 x^2 - \frac{x^2}{2}} dx = \frac{1}{\sqrt{1 - 2\alpha_2 \beta^2}},
$$

which is then also the limit of $\mathbb{E}_N[\tilde{Z}_N^2]$.

The next lemma gives a bound on the tail of the sum appearing in Theorem 1.2, and will be used to truncate this sum in the proofs of Theorems 1.1, 1.2.

**Lemma 3.5** (Bound on tail of cycle count sum). If $\beta < \frac{1}{\sqrt{2\alpha_2}}$ it holds or any $x$ and $K$ that

(57) $\sup_{N \geq 1} \mathbb{P}_N \left( \left\{ \sum_{k=K+1}^{\infty} \left( C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right) \right\} \geq x \right) \leq \frac{2}{1 - 2\alpha_2 \beta^2} \frac{(2\alpha_2 \beta^2)^{K+1}}{x^2}.$

**Proof.** Using Lemma 2.6 we have that

(58)

$$
\mathbb{E}_N \left[ \left( \sum_{k=K+1}^{\infty} \left( C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right) \right)^2 \right] = \mathbb{E}_N \left[ \left( \sum_{k=K+1}^{\infty} C_{N,k} \frac{\mu_k}{2k} \right)^2 \right] + \left( \sum_{k=K+1}^{\infty} \frac{\mu_k^2}{4k} \right)^2 \\
= \sum_{k=K+1}^{\infty} \frac{\mu_k^2}{4k^2} \mathbb{E}_N [C_{N,k}^2] + \left( \sum_{k=K+1}^{\infty} \frac{\mu_k^2}{4k} \right)^2 \\
\leq \sum_{k=K+1}^{\infty} \frac{\mu_k^2}{2k} + \left( \sum_{k=K+1}^{\infty} \frac{\mu_k^2}{4k} \right)^2,
$$

where the last inequality follows since $\mathbb{E}_N [C_{N,k}^2] \leq 2k$ for all $k$ by (25). Now observe that $\sum_{k=K+1}^{\infty} \frac{\mu_k^2}{2k} \leq \frac{1}{1 - 2\alpha_2 \beta^2} (2\alpha_2 \beta^2)^{K+1}$ by (8). Then (57) follows from an application of Chebyshev’s inequality.

We have now prepared all the tools needed to prove Theorem 1.2. Before giving the formal proof we give a more detailed heuristic sketch. The Radon-Nikodym derivative $\tilde{Z}_N = \frac{d\mathbb{P}_N}{d\mathbb{P}_N^{(\mu)}}$ changes the law of the sequence

$$
C_{N,1}, C_{N,2}, \ldots
$$

from approximately independent Gaussian such that $C_{N,k} \sim \mathcal{N}(0, 2k)$ (Proposition 2.1) to approximately independent Gaussian such that $C_{N,k} \sim \mathcal{N}(\mu_k, 2k)$ (Proposition 3.3). As mentioned in the introduction, a Radon-Nikodym derivative that changes the law of a sequence $C_{N,1}, C_{N,2}, \ldots$ from exactly independent
with $C_{N,k} \sim \mathcal{N}(0, 2k)$ to exactly independent with $C_{N,k} \sim \mathcal{N}(\mu_k, 2k)$ is necessarily equal to

$$\exp \left( \sum_{k=1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right).$$

We seek to prove that $\hat{Z}_N$ is approximately equal to this expression, using that the $C_{N,k}$ are approximately Gaussian under $\mathbb{P}_N$ and $\mathbb{Q}_N$.

A naive attempt would be to use Chebyshev’s inequality

$$\mathbb{P}_N \left( \left| \hat{Z}_N - \exp \left( \sum_{k=1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right) \right| \geq \varepsilon \right) \leq \frac{\mathbb{E}_N \left( \left| \hat{Z}_N - \exp \left( \sum_{k=1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right) \right| \right)^2}{\varepsilon^2}.$$

However $\sum_{k=1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\}$ has no well-defined exponential moments (in fact $C_{N,3}$ does not since a product of three independent Gaussians does not have any finite exponential moments), and therefore $\exp \left( \sum_{k=1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right)$ does not have a second moment. Thus we instead use an argument involving subsequential limits to define limiting random variables $\hat{Z}_{\infty}, C_{\infty,k}$ where the $C_{\infty,k}$ are exactly Gaussian and independent, so that $\sum_{k=1}^{\infty} \left\{ C_{\infty,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\}$ has well-defined exponential moments, and apply the Chebyshev inequality argument in the limit.

**Proof of Theorem 7.2** Let $\varepsilon > 0$ be arbitrary. Let $\delta > 0$ and assume for contradiction that

$$\limsup_{N \to \infty} \mathbb{P}_N \left( \left| \log \hat{Z}_N - \sum_{k=1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right| \geq \varepsilon \right) \geq \delta.$$

To truncate the sum (which must be done to make the above sketch precise) note that because of the inequality

$$e^{-b} |a - b| \leq |e^a - e^b| \text{ for all } a, b \in \mathbb{R},$$

with $b = \hat{Z}_N$ we have for any $K$ and $R$ that

$$\mathbb{P}_N \left( \left| \log \hat{Z}_N - \sum_{k=1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right| \geq \varepsilon \right) \
\leq \mathbb{P}_N \left( \left| \sum_{k=K+1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right| \geq \varepsilon \right) + \mathbb{P}_N \left( \hat{Z}_N \geq R \right) \
+ \mathbb{P}_N \left( \left| \hat{Z}_N - \exp \left( \sum_{k=1}^{K} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right) \right| \geq e^{-R} \frac{\varepsilon}{2} \right),$$

Pick $R$ large enough so that

$$\mathbb{P}_N \left( \hat{Z}_N \geq R \right) \leq \frac{\mathbb{E}_N \left( \hat{Z}_N \right)}{R} \leq \frac{1}{R} \leq \frac{\delta}{3}.$$

Also pick $K_0 = K_0(\varepsilon, \delta)$ large enough so that if $K \geq K_0$ we have

$$\mathbb{P}_N \left( \left| \sum_{k=K+1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right| \geq \frac{\varepsilon}{2} \right) \leq \frac{\delta}{3},$$

$$\mathbb{P}_N \left( \left| \hat{Z}_N - \exp \left( \sum_{k=1}^{K} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right) \right| \geq \frac{\varepsilon}{2} \right) \leq \frac{\delta}{3},$$

$$\mathbb{P}_N \left( \left| \hat{Z}_N - \exp \left( \sum_{k=1}^{\infty} \left\{ C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right\} \right) \right| \geq \frac{\varepsilon}{2} \right) \leq \frac{\delta}{3}.$$
by Lemma 3.5. Then if (59) holds we must have for such $K$ that

$$\limsup_{N \to \infty} \mathbb{P}_N \left( \left| \hat{Z}_N - \exp \left( \sum_{k=1}^K \left\{ C_{N,k} \mu_k \frac{\mu_k^2}{2k} - \frac{\mu_k^2}{4k} \right\} \right) \right| \geq e^{-R\varepsilon/2} \right) \geq \frac{\delta}{3}.$$  

If (60) holds then there is a subsequence along which this probability is always at least $\frac{\delta}{6}$. Since $\hat{Z}_N$ is in $L_1$ it is tight and there is a further subsequence $N_i$ along which the probability is at least $\frac{\delta}{6}$ and $\hat{Z}_{N_i}$ converges in distribution. As $(C_{N,1}, \ldots, C_{N,K})$ converge in distribution by Proposition 2.1 it also converges along this subsequence, so that

$$\left( \hat{Z}_{N_i}, C_{N_i,1}, \ldots, C_{N_i,K} \right) \overset{D}{\to} \left( \hat{Z}_\infty, C_{\infty,1}, \ldots, C_{\infty,K} \right),$$

where the latter random vector is defined on some auxiliary probability space with probability $\mathbb{P}_\infty$. We must have

$$\mathbb{P}_\infty \left( \left| \hat{Z}_\infty - \exp \left( \sum_{k=1}^K \left\{ C_{\infty,k} \mu_k \frac{\mu_k^2}{2k} - \frac{\mu_k^2}{4k} \right\} \right) \right| \geq e^{-R\varepsilon/2} \right) \geq \frac{\delta}{6}.$$  

Note that by Fatou’s lemma and Lemma 3.4

$$\mathbb{E}_\infty [\hat{Z}_\infty^2] \leq \frac{1}{\sqrt{1 - 2\alpha_2/\beta^2}} < \infty.$$  

Because of this this and the fact that the $C_{\infty,k}$ have finite exponential moments since they are exactly Gaussian under $\mathbb{P}_\infty$, we can use the Chebyshev inequality to bound

$$\mathbb{P}_\infty \left( \left| \hat{Z}_\infty - \exp \left( \sum_{k=1}^K \left\{ C_{\infty,k} \mu_k \frac{\mu_k^2}{2k} - \frac{\mu_k^2}{4k} \right\} \right) \right| \geq e^{-R\varepsilon/2} \right) \leq \frac{\mathbb{E}_\infty \left[ \left( \hat{Z}_\infty - \exp \left( \sum_{k=1}^K C_{\infty,k} \mu_k \frac{\mu_k^2}{2k} - \frac{\mu_k^2}{4k} \right) \right)^2 \right]}{e^{-2R\varepsilon^2/4}}.$$  

We can compute an upper bound of the second moment on the RHS (63) from the convergence in law of $\hat{Z}_{N_i}, C_{N_i,k}$ in a way that does not require the existence of a second moment of $C_{N,k}$ as follows. Define

$$M_{\infty, K} = \mathbb{E}_\infty \left[ \hat{Z}_\infty | \mathcal{F}_K \right],$$

where $\mathcal{F}_K$ is the $\sigma$-algebra generated by $C_{\infty,1}, \ldots, C_{\infty,K}$. We now show that in fact

$$M_{\infty, K} = \exp \left( \sum_{k=1}^K \left\{ C_{\infty,k} \mu_k \frac{\mu_k^2}{2k} - \frac{\mu_k^2}{4k} \right\} \right) \text{ a.s.,}$$

which follows if we show that for any continuous bounded $f$

$$\mathbb{E}_\infty \left[ f \left( C_{\infty,1}, \ldots, C_{\infty,K} \right) \hat{Z}_\infty \right] = \mathbb{E}_\infty \left[ f \left( C_{\infty,1}, \ldots, C_{\infty,K} \right) M_{\infty, K} \right].$$

This can be proven from the convergence in law of $C_{N,k}$ (without requiring that $C_{N,k}$ has a second moment) because by definition $\hat{Z}_N = \frac{d\mathbb{Q}_N}{d\mathbb{P}_N}$ implying

$$\lim_{l \to \infty} \mathbb{E}_{N_l} \left[ f \left( C_{N_l,1}, \ldots, C_{N_l,K} \right) \hat{Z}_{N_l} \right] = \lim_{l \to \infty} \mathbb{Q}_{N_l} \left[ f \left( C_{N_l,1}, \ldots, C_{N_l,K} \right) \right],$$

and then by Proposition 3.3 the right-hand side equals

$$\mathbb{E}_\infty \left[ f \left( C_{\infty,1} + \mu_1, \ldots, C_{\infty,K} + \mu_K \right) \right].$$
which in turns equals $\mathbb{E}_\infty \left[ f \left( C_{\infty,1}, \ldots, C_{\infty,K} \right)^2 M_{\infty,K} \right]$ since $M_{\infty,K}$ is precisely the Radon-Nikodym derivative which changes the mean of $C_{\infty,k}$ to $\mu_k$ while leaving all covariances fixed.

Thus the second moment on the RHS of (63) is

$$\mathbb{E}_\infty \left[ (\hat{Z}_\infty - M_{\infty,K})^2 \right] = \mathbb{E}_\infty \left[ \hat{Z}_\infty^2 \right] - \mathbb{E}_\infty \left[ M_{\infty,K}^2 \right],$$

where the equality follows since $M_{\infty,K} = \mathbb{E}_\infty \left[ \hat{Z}_\infty|\mathcal{F}_K \right]$. By explicit computation

$$\mathbb{E}_\infty \left[ M_{\infty,K}^2 \right] = \exp \left( \sum_{k=1}^{K} \frac{\left( 2\alpha_2 \beta^2 \right)^k}{2k} \right) = \frac{1}{\sqrt{1 - 2\alpha_2 \beta^2}} + o_K(1),$$

which together with (62) gives

$$\mathbb{E}_\infty \left[ \hat{Z}_\infty^2 \right] - \mathbb{E}_\infty \left[ M_{\infty,K}^2 \right] = o_K(1).$$

Thus we obtain from (63) that for any $K \geq K_0$

$$\mathbb{P}_\infty \left( |\hat{Z}_\infty - \exp \left( \sum_{k=1}^{K} \frac{\mu_k C_{\infty,k}}{2k} - \frac{\mu_k^2}{4k} \right)| \geq \frac{e^{-R \varepsilon}}{2} \right) \leq \frac{o_K(1)}{e^{-2R \varepsilon^2}}.$$ 

We may now for any $R$ and $\varepsilon > 0$ pick a $K \geq K_0$ large enough so that this contradicts (61). Thus (59) can not hold for any $\delta > 0$. Thus in fact for all $\varepsilon > 0$

$$\mathbb{P}_N \left( \left| \log \hat{Z}_N - \sum_{k=1}^{\infty} \left( C_{N,k} \frac{\mu_k}{2k} - \frac{\mu_k^2}{4k} \right) \right| \geq \varepsilon \right) \rightarrow 0,$$

which together with (39) and (4) implies (3).

\[\square\]

4. Derivation of Theorem 1.1 from Theorem 1.2

It only remains to derive Theorem 1.1 from Theorem 1.2 and the asymptotic normality of the $C_{N,k}$.

Proof of Theorem 1.1. Let $s^2_K = \sum_{k=1}^{K} \frac{\mu_k^2}{2k}$, and note that

$$s^2_K \rightarrow \sum_{k=1}^{\infty} \frac{\mu_k^2}{2k} = \sum_{k=1}^{\infty} \frac{\left( 2\alpha_2 \beta^2 \right)^k}{2k} = -\frac{1}{2} \log \left( 1 - 2\alpha_2 \beta^2 \right) = s^2.$$

Theorem 1.1 follows from from Theorem 1.2 once we have shown that

$$\sum_{k=1}^{\infty} \frac{2\mu_k C_{N,k} - \mu_k^2}{4k} \overset{D}{\rightarrow} \mathcal{N} \left( -\frac{1}{2} s^2, s^2 \right) \text{ as } N \rightarrow \infty \text{ under } \mathbb{P}_N.$$

Proposition 2.1 implies that for any $K$

$$\sum_{k=1}^{K} \frac{2\mu_k C_{N,k} - \mu_k^2}{4k} \overset{D}{\rightarrow} \mathcal{N} \left( -\frac{1}{2} s^2_K, s^2_K \right).$$
Now for any $z \in \mathbb{R}$
\[
P_N \left( \sum_{k=1}^\infty \frac{2\mu_k C_{N,k} - \mu_k^2}{4k} \leq z \right) 
\leq P_N \left( \sum_{k=1}^K \frac{2\mu_k C_{N,k} - \mu_k^2}{4k} \leq z + \varepsilon \right) + P_N \left( \left| \sum_{k=K+1}^\infty \frac{2\mu_k C_{N,k} - \mu_k^2}{4k} \right| \geq \varepsilon \right),
\]
Taking first the limit $N \to \infty$ and using (65) and Lemma 3.5 and then the limit $K \to \infty$ on the right hand side we obtain
\[
\limsup_{N \to \infty} P \left( \sum_{k=1}^\infty \frac{2\mu_k C_{N,k} - \mu_k^2}{4k} \leq z \right) \leq F(z),
\]
where $F$ is the CDF of $\mathcal{N}(−s^2/2, s^2)$. We get the corresponding lower bound similarly, proving (64) and therefore Theorem 1.1. □

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