Spectral asymptotics of harmonic oscillator perturbed by bounded potential

Markus Klein * and Evgeni Korotyaev † and Alexis Pokrovski ‡

March 28, 2022

Abstract

Consider the operator $T = -\frac{d^2}{dx^2} + x^2 + q(x)$ in $L^2(\mathbb{R})$, where real functions $q$, $q'$ and $\int_0^\pi q(s) \, ds$ are bounded. In particular, $q$ is periodic or almost periodic. The spectrum of $T$ is purely discrete and consists of the simple eigenvalues $\{\mu_n\}_{n=0}^\infty$, $\mu_n < \mu_{n+1}$. We determine their asymptotics $\mu_n = (2n + 1) + (2\pi)^{-1} \int_{\pi/2}^{\pi} q(\sqrt{2n + 1} \sin \theta) \, d\theta + O(n^{-1}/3)$.

1 Introduction and main results

Consider the quantum-mechanical harmonic oscillator $T^0 = -\frac{d^2}{dx^2} + x^2$ on $L^2(\mathbb{R})$. It is well known that the spectrum of $T^0$ is purely discrete and consists of the simple eigenvalues $\mu_n^0 = 2n + 1$, $n \geq 0$ with corresponding orthonormed eigenfunctions $\psi_n^0$. Define the perturbed operator $T y = -y'' + x^2 y + q(x)y$ in $L^2(\mathbb{R})$, where $q$ belongs to the complex Banach space $\mathcal{B}$ given by

$$\mathcal{B} = \left\{ q, q', q_1 \in L^\infty(\mathbb{R}) : \|q\|_B = \sup_{x \in \mathbb{R}} \left( |q(x)| + |q'(x)| + |q_1(x)| \right) < \infty \right\}, \quad q_1(x) \equiv \int_0^x q(t) \, dt. \quad (1.1)$$

In particular, this class includes periodic, almost-periodic $q$. If $q \in \mathcal{B}$ is real, then $T$ is self-adjoint, its spectrum is purely discrete, $\sigma(T) = \{\mu_n\}_{n=0}^\infty$ and $\mu_n = \mu_n^0 + O(1)$, $n \to \infty$.

Our goal is to determine the asymptotics of $\mu_n - \mu_n^0$ as $n \to \infty$.

For decaying perturbations (e.g. $q', xq \in L^2(\mathbb{R})$) a complete inverse spectral theory is obtained in [2], [3]. At the same time we did not find in the literature any results concerning periodic and almost-periodic $q$. The existing methods (e.g. in [2]) cannot be used for $q \in \mathcal{B}$.

Let us number the points of the spectrum so that $|\mu_n| \leq |\mu_{n+1}|$ counting multiplicity.

---

*Institut für Mathematik, Universität Potsdam, Am Neuen Palais 10, 14469 Potsdam, Germany
†Institut für Mathematik, Humboldt Universität zu Berlin, Rudower Chaussee 25, 12489, Berlin, Germany
‡Institut für Mathematik, Universität Potsdam, and Laboratory of Quantum Networks, Institute for Physics, St.-Petersburg State University, Ulyanovskaya 1, 198504 St.-Petersburg, Russia.
Theorem 1.1. For any $q \in \mathcal{B}$ the spectrum of operator $T$ is discrete and the following asymptotic is fulfilled:

$$
\mu_n = \mu_n^0 + \mu_n^1 + \|q\|_BO(n^{-\frac{3}{4}}), \quad \mu_n^1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} q(\sqrt{\mu_n^0} \sin \vartheta) d\vartheta = \|q\|_BO(n^{-\frac{3}{4}}). \tag{1.2}
$$

Remark. For finite smooth $q$ the formula (1.2) gives $\mu_n^1 = \frac{1}{\pi} \int_{-\infty}^{\infty} q(s) ds$, which is the well-known leading term of asymptotics in this case [2].

Proposition 1.2. Let $q \in \mathcal{B}$ and $q(x) = \int_{\mathbb{R}} e^{ixt} d\nu(t)$ for some Borel measure $d\nu$ on $\mathbb{R}$ which satisfies the condition $C_q = \int_{\mathbb{R}} (1 + |t|^{-p}) d\nu(t) < \infty$ for some $p > \frac{3}{2}$. Then

$$
\mu_n^1 = \int_{\mathbb{R}} J_0(t \sqrt{\lambda}) d\nu(t) = \frac{\sigma(\sqrt{\mu_n^0})}{(\mu_n^0)^{\frac{1}{4}}} + C_qO(n^{-\frac{3}{4}}), \tag{1.3}
$$

where $J_0$ is the Bessel function and

$$
\sigma(s) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}} \frac{\cos(|t| s - \frac{\pi}{4})}{|t|^\frac{3}{2}} d\nu(t). \tag{1.4}
$$

Moreover, if $q$ has the form $q(x) = \sum_{k \in \mathbb{Z}} q_k e^{ikt}$, then

$$
\sigma(s) = \sqrt{\frac{2}{\pi}} \sum_{k \in \mathbb{Z}} \frac{q_k}{\sqrt{|t_k|}} \cos(s |t_k| + \frac{\pi}{4}). \tag{1.5}
$$

In Section 2 we introduce the quasiclassical change of variable. In this variable we write the integral equation for the fundamental solutions. In Sections 3 and 4 we prove convergence of the iteration series for the fundamental solutions in the sub-barrier ($x \gtrsim \sqrt{|\lambda|}$) and over-barrier ($0 < x \lesssim \sqrt{|\lambda|}$) regions, respectively. In Section 5 using these series we derive the asymptotics of the Wronskian. In Section 6 using this asymptotics we prove Theorem 1.1. We prove auxiliary properties of the quasiclassical change of variables in the Appendix.

2 Preliminaries: changes of variables

Consider the differential equation

$$
-y'' + (x^2 + q(x))y = \lambda y, \quad (x, \lambda) \in \mathbb{R} \times \mathbb{C}. \tag{2.1}
$$

We shall show that there exist fundamental solutions $\psi_{\pm}$ which satisfy the asymptotics

$$
\psi_{\pm}(x, \lambda) = (\pm \sqrt{2x})^{\frac{\lambda - 1}{2}} e^{\frac{-x^2}{2}} (1 + o(1)), \quad \psi_{\pm}'(x, \lambda) = -x(\pm \sqrt{2x})^{\frac{\lambda - 1}{2}} e^{\frac{-x^2}{2}} (1 + o(1)) \tag{2.2}
$$

as $x \to \pm \infty$ and locally uniformly in $\lambda$. If $q \equiv 0$ then these solutions have the form $\psi_0 \pm(x, \lambda) = D_{\lambda - 1}(\pm \sqrt{2x})$, where $D_r$ is the Weber (parabolic cylinder) functions (see [1]). We introduce the Wronskian $\{f, g\} = fg' - f'g$. 

2
Theorem 2.1. Let \( q, q_1 \in L^\infty(\mathbb{R}) \). Then

i) For any \( \lambda \in \mathbb{C} \) there exist unique solutions \( \psi_\pm(x, \lambda) \) of (2.1) with the asymptotics (2.2). Moreover, for each \( x \in \mathbb{R} \) the functions \( \psi_\pm(x, \cdot), \psi_\pm'(x, \cdot) \) and \( w = \{ \psi_-, \psi_+ \} \) are entire. If \( q \) is real, then the operator \( T = -\frac{d^2}{dx^2} + x^2 + q(x) \) has only simple eigenvalues.

Proof. Consider the function \( \psi_+ \), the proof for \( \psi_- \) is similar. In order to prove that \( \psi_+ \) is entire function of \( \lambda \) it is sufficient to show that it is analytic in each disc \( D(\mu) = \{ \lambda \in \mathbb{C} : |\lambda - \mu| < 1 \} \), \( \mu \in \mathbb{C} \). For \( \lambda \in \mathbb{D}(\mu) \) we have (see [1]) the uniform asymptotics

\[
\psi_0^0(x, \lambda) = g(x) \left( 1 + O(x^{-2}) \right), \quad \psi_0'(x, \lambda) = -xg(x) \left( 1 + O(x^{-2}) \right), \quad x \to +\infty, \tag{2.3}
\]

where \( g(x) = (\sqrt{2x})^{\lambda+1/2} e^{-x^2/2} \). Let \( h(x, \lambda) = \frac{1}{2\sqrt{\pi}} \Gamma(\frac{1-\lambda}{2})(\psi_0^0(x, \lambda) - \sin \frac{\lambda}{2} \psi_0'(x, \lambda)); \) note that

\[
h(x, \lambda) = \frac{1}{2xg(x)}(1 + O(x^{-1})), \quad h'(x, \lambda) = \frac{1}{2g(x)}(1 + O(x^{-1})), \quad x \to +\infty, \tag{2.4}
\]

(see [1]) uniformly for \( \lambda \in \mathbb{D}(\mu) \), so that \( \{ \psi_0^0, h \} = 1 \). Define the entire function \( M(x, y) = h(x, \lambda)\psi_0^0(y, \lambda) - \psi_0'(x, \lambda)h(y, \lambda) \). Then a solution of

\[
\psi(x, \lambda) = \psi_0^0(x, \lambda) + \lim_{t \to \infty} \int_x^t M(x, y)q(y)\psi(y, \lambda) \, dy \tag{2.5}
\]
solves (2.1). We rewrite (2.5) in the form

\[
p(x, \lambda) = p_0(x, \lambda) + \lim_{t \to \infty} \int_x^t K(x, y)q(y)p(y, \lambda) \, dy, \quad x > 1, \tag{2.6}
\]

where

\[
p = \frac{\psi(x, \lambda)}{g(x)}, \quad p_0 = \frac{\psi_0^0(x, \lambda)}{g(x)}, \quad K(x, y) = \frac{M(x, y)g(y)}{g(x)}. \tag{2.7}
\]

Let \( h_0 = 2xg(x)h(x, \lambda) \) and

\[
K = U - V, \quad U(x, y) = \frac{h_0(x)p_0(y) g^2(y)}{2x g^2(x)}, \quad V(x, y) = \frac{p_0(x)h_0(y)}{2y}. \tag{2.8}
\]

In order to study Eq. (2.6) introduce the spaces of functions

\[
\mathcal{F}_\alpha = \{ f \in C([1, \infty)) : \| f \|_\alpha \equiv \sup_{x \in [1, \infty)} |x^\alpha f(x)| < \infty \}, \quad \alpha \in \mathbb{R},
\]

and \( \mathcal{F}_{\alpha, \beta} = \{ f \in \mathcal{F}_\alpha : f' \in \mathcal{F}_\beta \} \) with the norm \( \| f \|_{\alpha, \beta} = \| f \|_\alpha + \| f' \|_\beta \), \( \beta \in \mathbb{R} \). By (2.3), for \( \lambda \in \mathbb{D}(\mu) \) we have the estimate

\[
\| p_0 \|_{0,1} \leq c < \infty. \tag{2.9}
\]

Let \( f \in \mathcal{F}_\alpha \) for some \( \alpha \in \mathbb{R} \) and \( u = Uqf \). Then we have

\[
u(x) = \int_x^\infty h_0(x)p_0(y) g^2(y) q(y)f(y) \, dy = \frac{h_0(x)}{2x^\lambda} \int_x^\infty p_0(y) e^{x^2-y^2} y^{\lambda-1} q(y) f(y) \, dy. \tag{2.10}
\]
Using (2.3), (2.4) and the estimate \( \int_x^\infty e^{-y^2} y^\gamma dy \leq C e^{-x^2} x^{\gamma-1} \) for \( x \geq 1 \) and \( \gamma \in \mathbb{R} \) we obtain

\[
\|u\|_{\alpha+2} \leq C\|q\|_\infty \|f\|_\alpha, \quad \|u'\|_{\alpha+1} \leq C\|q\|_\infty \|f\|_\alpha
\]  
(2.11)

uniformly in \( \lambda \in \mathbb{D}(\mu) \). Here and below \( C \) is some absolute constant.

Let \( f \in \mathcal{F}_{\alpha,\beta} \) for some \( \alpha \in \mathbb{R} \) and \( \beta > 0 \). Set \( v = Vqf \). Then integration by parts gives

\[
v(x) = \lim_{t \to \infty} \int_x^t p_0(y) h_0(y) \frac{1}{2y} q(y) f(y) \, dy = p_0(x) \int_x^\infty \left[ q_1(x) - q_1(y) \right] \left( \frac{h_0(y) f(y)}{2y} \right) \, dy,
\]

where the last integral converges absolutely. Using (2.3–2.4) we obtain

\[
|v(x)| \leq CC_q(\|f\|_\alpha x^{-\alpha} + \|f'\|_\beta x^{-\beta}), \quad |v'(x)| \leq CC_q(\|f\|_\alpha x^{-\alpha-2} + \|f'\|_\beta x^{-\beta-1})
\]  
(2.12)

for \( x \geq 1 \), uniformly in \( \lambda \in \mathbb{D}(\mu) \), where \( C_q = \|q\|_\infty + \|q_1\|_\infty \). Thus

\[
K : \mathcal{F}_{\alpha,\beta} \to \mathcal{F}_{\alpha',\beta'}, \quad \alpha' = \min\{\alpha + 1, \beta\}, \quad \beta' = \min\{\alpha + 1, \beta + 1\}.
\]  
(2.13)

Consider the iterations \( p_{n+1} = Kqp_n, n \geq 0 \). By (2.9), we have \( p_0 \in \mathcal{F}_{0,1} \); using (2.8), (2.11), (2.12) and (2.13), we conclude that

\[
\|p_{n+1}\|_{\alpha_{n+1},\beta_{n+1}} \leq CC_q\|p_n\|_{\alpha_n,\beta_n},
\]  
(2.14)

where \( \alpha_0 = 0, \beta_0 = 1, \)

\[
\alpha_{2n} = \alpha_0 + n, \quad \beta_{2n} = \alpha_0 + 1 + n, \quad \alpha_{2n+1} = \alpha_0 + 1 + n, \quad \beta_{2n+1} = \alpha_0 + n.
\]  
(2.15)

Using (2.9) we obtain

\[
|p_{2n}(x)| \leq (CC_q)^{2n} c x^{-n}, \quad |p'_{2n}(x)| \leq (CC_q)^{2n} c x^{-n-1},
\]

\[
|p_{2n+1}(x)| \leq (CC_q)^{2n+1} c x^{-n-1}, \quad |p'_{2n+1}(x)| \leq (CC_q)^{2n+1} c x^{-n-1}.
\]

Hence for \( x \geq x_0 = (2CC_q)^2 \) the series \( p(x) = \sum_{n \geq 0} p_n(x) \) and \( p'(x) = \sum_{n \geq 0} p'_n(x) \) converge absolutely and uniformly in \( \lambda \in \mathbb{D}(\mu) \); \( p(x) \) gives the solution of Eq. (2.6). Moreover,

\[
p(x) = 1 + O(x^{-1}), \quad p'(x) = O(x^{-1}), \quad x \to \infty,
\]  
(2.16)

uniformly for \( \lambda \in \mathbb{D}(\mu) \). Therefore \( \psi = gp \) is a solution of (2.5). By (2.7) and (2.16), \( \psi \) satisfies (2.2). For each \( n \geq 0 \) and fixed \( x \geq x_0 \) the iterations \( p_n(x, \cdot), p'_n(x, \cdot) \) are analytic in \( \mathbb{D}(\mu) \). Hence \( p(x, \cdot), p'(x, \cdot) \) are analytic in \( \mathbb{D}(\mu) \) for each fixed \( x \geq x_0 \). By (2.7), \( \psi_+(x, \lambda) \) and \( \psi'_+(x, \lambda) \) are analytic in \( \mathbb{D}(\mu) \) for each fixed \( x \geq x_0 \). Hence, the solution is also analytic in \( \lambda \) for any fixed \( x \) (see this simple fact e.g. in [4]). Thus \( \psi_+(x, \lambda) \) and \( \psi'_+(x, \lambda) \) are entire functions of \( \lambda \) for any \( x \in \mathbb{R} \).

Suppose that \( \psi_+(x, \lambda) \) is not unique and denote by \( \psi_+ \) another solution of (2.1) satisfying (2.2). Then \( \{\psi_+, \psi_+\} = 0 \) and therefore these solutions are linearly dependent. Since they have the same asymptotics (2.2), \( \psi_+ = \psi_+ \). Thus \( \psi_+ \) is unique.
ii) Let \( \lambda \) be an eigenvalue, which is not simple. Then there exist solutions \( \psi_+ \) and \( \psi \) of (2.1) in \( L^2(\mathbb{R}) \). Note that \( \psi \) and \( \psi' \) are in \( L^\infty(\mathbb{R}) \). The asymptotics (2.2) yields \( \{ \psi_+, \psi \} = 0 \). Therefore \( \psi_+ \) and \( \psi \) are linearly dependent, which gives contradiction. ■

Theorem 2.1 gives no information on high-energy asymptotics \( \lambda \to \infty \). To derive those, we introduce some notations and auxiliary functions.

Throughout the paper we use the following agreements:

- The functions \( \log z \) and \( z^\alpha = e^{\alpha \log z} \) for \( \alpha \in \mathbb{R} \) take their principal values on \( \mathbb{C} \setminus \mathbb{R}_- \).
- For \( \lambda \in \mathbb{C}_+ \setminus \{0\} \) we set \( \lambda = |\lambda|e^{2i\vartheta} \), \( \vartheta \in [0, \frac{\pi}{4}] \).

Define the function \( \langle z \rangle = (1 + |z|^2)^{\frac{1}{2}}, z \in \mathbb{C} \). For any interval \( I \subset \mathbb{R} \) we introduced a sector \( S(I) = \{ z \in \mathbb{C} : \arg z \in I \} \). Define the function

\[
\xi(t) = \int_{1}^{t} \sqrt{s^2 - 1} \, ds = \frac{1}{2} \left( t\sqrt{t^2 - 1} - \log(t + \sqrt{t^2 - 1}) \right), \quad t \in S(-\frac{\pi}{2}, 0),
\]

(2.17)

where \( \xi(t) > 0 \) for \( t > 1 \). The function \( \xi \) is a conformal mapping from \( S(-\frac{\pi}{2}, 0) \) onto \( \Xi = \mathbb{C}_- \cup \{ \text{Re} \xi < 0, \text{Im} \xi \in [0, \frac{\pi}{4}] \} \). The following uniform asymptotics is fulfilled:

\[
\xi(t) = \frac{1}{2} \left( t^2 - \frac{1}{2} - \log 2t + O(|t|^{-2}) \right), \quad |t| \to \infty, \quad t \in S[-\frac{\pi}{2}, 0].
\]

(2.18)

We introduce the function

\[
k(t) = \left( \frac{3}{2} \xi(t) \right)^{\frac{2}{3}}, \quad t \in S(-\frac{\pi}{2}, 0), \quad k(0) = - \left( \frac{3\pi}{8} \right)^{\frac{2}{3}} < 0.
\]

(2.19)

Note that \( k(t) \) is a conformal mapping from \( S(-\frac{\pi}{2}, 0) \) onto the domain \( \mathcal{K} \) given by

\[
\mathcal{K} = S[-\frac{2\pi}{3}, 0) \cup \left\{ k \in S(-\pi, -\frac{2\pi}{3}) : |k| \sin \frac{\pi}{3} \arg k \frac{3}{2} < |k(0)| \right\}.
\]

(2.20)

By (2.18), the following asymptotics and estimates are fulfilled:

\[
k(t) = \left( \frac{3}{4} \right)^{\frac{2}{3}} t^{\frac{2}{3}} \left( 1 + O(t^{-1}) \right), \quad t \in S[-\frac{\pi}{2}, 0], \quad t(k) = \left( \frac{4}{3} \right)^{\frac{2}{3}} k^{\frac{2}{3}} \left( 1 + O(k^{-\frac{2}{3}}) \right), \quad k \in \mathcal{K}, \quad k(0) = \left( \frac{3\pi}{8} \right)^{\frac{2}{3}} \leq 0,
\]

(2.21)

\[
|t'(k)| \leq C(k)^{-\frac{2}{5}}, \quad |t''(k)| \leq C(k)^{-\frac{4}{5}}, \quad k \in \mathcal{K},
\]

(2.22)

where \( t(k) \) is the inverse function for \( k(t) \). Here and below \( C \) is an absolute constant.

Consider the change of variable \( x \to k = k(x) \); it maps \( \mathbb{R}_+ \) onto the curve \( \tilde{\gamma}_\lambda = k(e^{i\vartheta} \mathbb{R}_+) \). The domain \( \mathcal{K} \) and the curve \( \tilde{\gamma}_\lambda \) are presented on Fig. 11. By (2.21), the function \( y_1(k, \lambda) = \frac{y_0(\sqrt{\lambda}(k))}{\sqrt{t'(k)}} \) solves

\[
y'_1(k, \lambda) - \lambda^2 k y_1(k, \lambda) = v_0(k)y_1(k, \lambda) + v_q(k, \lambda)y_1(k, \lambda), \quad k \in \tilde{\gamma}_\lambda,
\]

(2.23)
Figure 1: The domain $K = k(S(-\pi/2, 0))$ and the curve $\tilde{\gamma}_\lambda = k(e^{i\theta}R_+)$.

where

$$v_0(k) = t'(k)^{3/2} \left( \frac{d^2}{dt^2} \frac{1}{k'(t)} \right) \Big|_{t=t(k)}^1, \quad v_q(k, \lambda) = \lambda t'(k)q(\lambda^{3/4}t(k)). \quad (2.24)$$

Using (2.21) and (2.22) we obtain

$$|v_0(k)| \leq C(1 + |k|)^{-2}, \quad k \in K. \quad (2.25)$$

For each $\lambda \in \mathbb{C}_+ \setminus \{0\}$ we define the basic variable $z = \lambda^{3/4}k$. We have the function

$$z(x, \lambda) = \lambda^{3/4}k(x^{3/4}), \quad \lambda \in \mathbb{C}_+ \setminus \{0\}, \quad x \geq 0. \quad (2.26)$$

Each mapping $z(\cdot, \lambda) : \mathbb{R}_+ \to \Gamma_\lambda = z(\mathbb{R}_+, \lambda)$ is a real analytic isomorphism (see Fig. 2) and see Lemma 7.5 about $\Gamma_\lambda$. If $\lambda > 0$, then $\Gamma_\lambda = [-\lambda^{3/4}(3\pi/8)^{3/4}, \infty)$ is a half-line. Moreover,

$$z_0 \equiv z(0, \lambda) = -\lambda^{3/4}(3\pi/8)^{3/4}, \quad \lambda \in \mathbb{C}_+ \setminus \{0\}. \quad (2.27)$$

For any $\lambda \in \mathbb{C}_+ \setminus \{0\}$ and $0 \leq x_1 < x_2$ set $z_n = z(x_n, \lambda)$, $n = 1, 2$. We define the curves

$$\Gamma_\lambda(z_1, z_2) = \{z : z = z(x, \lambda), x \in [x_1, x_2]\}, \quad \Gamma_\lambda(z_1) \equiv \Gamma_\lambda(z_1, \infty) = \{z : z = z(x, \lambda), x \geq x_1\}.$$

By (2.28), the function $u(z, \lambda) = y_1(z\lambda^{-3/4}, \lambda)$ solves

$$\partial_z^2 u(z, \lambda) - zu(z, \lambda) = V(z, \lambda)u(z, \lambda), \quad V = V_0 + V_q, \quad z \in \Gamma_\lambda, \quad (2.28)$$

here and below $\partial_z = \frac{\partial}{\partial z}$ and $V_q$ (linear in $q$) and $V_0$ (not include $q$) are given by

$$V_0(z, \lambda) = \frac{v_0(z\lambda^{-\frac{3}{4}})}{\lambda^{3/4}}, \quad V_q(z, \lambda) = \frac{\rho^2(z, \lambda)q(\sqrt{\lambda}t(z\lambda^{-\frac{3}{4}}))}{\lambda^{3/4}}, \quad \rho(z, \lambda) \equiv t'(z\lambda^{-\frac{3}{4}}). \quad (2.29)$$
Using (2.22), we obtain for $\lambda \in \mathbb{C}_+ \setminus \{0\}$ and $z \in S[-\pi, \frac{\pi}{2}]$ the estimates

$$|\rho(z, \lambda)| \leq C|z\lambda^{-\frac{3}{2}}|^{-\frac{3}{2}}, \quad |\partial_z \rho(z, \lambda)| \leq |\lambda|^{-\frac{3}{2}}|z\lambda^{-\frac{3}{2}}|^{-\frac{3}{2}}, \quad |V_0(z, \lambda)| \leq \frac{C}{|\lambda|^{\frac{3}{4}} + |z|^2}, \quad (2.30)$$

where $C$ does not depend on $\lambda$ and $z$.

To analyze (2.28) we need well-known properties \cite{1} of the Airy functions $\text{Ai}$ and $\text{Bi}$:

$$\text{Ai}(z) = \frac{e^{-\frac{2}{3}z^\frac{3}{2}}}{2\sqrt{z}} (1 + O(z^{-\frac{1}{2}})), \quad |z| \to \infty, \quad |\arg z| < \pi - \varepsilon, \quad \forall \varepsilon > 0, \quad (2.31)$$

$$\{\text{Ai}(z), \text{Bi}(z)\} = 1, \quad \text{Bi}(z) = i \left(2e^{-i\frac{\pi}{3}}\text{Ai}(\omega z) - \text{Ai}(z)\right), \quad \omega = e^{2\pi i}, \quad (2.32)$$

$$\text{Ai}(z) = e^{-i\frac{\pi}{3}}\text{Ai}(z\omega) + e^{i\frac{\pi}{3}}\text{Ai}(z\bar{\omega}), \quad \text{Bi}(z) = ie^{-i\frac{\pi}{3}}\text{Ai}(z\omega) - ie^{i\frac{\pi}{3}}\text{Ai}(z\bar{\omega}). \quad (2.33)$$

Let $\Gamma \subset \mathbb{C}$ be a smooth curve. For any continuous function $f$ on $\Gamma$ we denote by $\int_{\Gamma} f(s) \, ds$ the usual complex line integral. We denote by $\int_{\Gamma} f(s) |ds|$ the line integral of $f$ along $\Gamma$ with respect to the arc length $|ds| = \sqrt{(dx)^2 + (dy)^2}$. For integration along the infinite curve $\Gamma_\lambda$, defined above, we use the standard notation p.v.$\int_{\Gamma_\lambda(z)} f(s) \, ds = \lim_{w \to \infty} \int_{\Gamma_\lambda(z, w)} f(s) \, ds$ as $w \to \infty, w \in \Gamma_\lambda^+$ whenever it exist.

We will study the formal integral equation

$$u_+(z, \lambda) = u_0(z) + \text{p.v.} \int_{\Gamma_\lambda(z)} J_0(z, s)V(s, \lambda)u_+(s, \lambda) \, ds, \quad z \in \Gamma_\lambda, \quad (2.34)$$

$$u_0(z) = \text{Ai}(z), \quad J_0(z, s) = \text{Ai}(s)\text{Bi}(z) - \text{Ai}(z)\text{Bi}(s), \quad z, s \in \mathbb{C}. \quad (2.35)$$

We rewrite (2.34) in the form

$$v_+(z) = a(z) + \text{p.v.} \int_{\Gamma_\lambda(z)} J(z, s)V(s, \lambda)v_+(s) \, ds, \quad a(z) \equiv \text{Ai}(z)e^{\frac{2}{3}z^\frac{3}{2}}, \quad z \in \Gamma_\lambda, \quad (2.36)$$

$$u_+(z) = v_+(z)e^{-\frac{2}{3}z^\frac{3}{2}}, \quad J(z, s) = J_0(z, s)e^{\frac{2}{3}(e^{\frac{2}{3}} - s^{\frac{2}{3}})}). \quad (2.37)$$

If $z < 0$ and $\lambda \in \mathbb{C}_+$, then $z^{\frac{3}{2}}$ takes its values on the lower side of the cut. This agreement provides continuity as $\arg \lambda \downarrow 0$, since for $\lambda \in \mathbb{C}_+$ the curve $\Gamma_\lambda$ lies in the lower half-plane. By (2.31) and (2.33), the following estimates are fulfilled:

$$|a(z)| \leq C|z|^{-\frac{1}{2}}, \quad \forall z \in \mathbb{C}, \quad (2.38)$$

$$|a'(z)| \leq C|z|^{-\frac{3}{2}}, \quad |\arg z| \leq \pi - \varepsilon, \quad \forall \varepsilon > 0. \quad (2.39)$$

We write (2.36) in the form

$$v_+ = a + Jv_+, \quad (2.40)$$
where the integral operator $J$ is given by

$$[Jf](z) = \text{p.v.} \int_{\Gamma(z)} J(z,s)f(s)ds. \quad (2.41)$$

The next lemma (proved in the Appendix) gives a splitting of $\Gamma_\lambda$. Here and below we fix

$$\delta \in (0,\frac{4}{3}\arccos 2^{-\frac{1}{3}}). \quad (2.42)$$

**Lemma 2.2.** For any $\lambda \in \mathbb{C}_+ \setminus \{0\}$ there exists a unique point $z_* \equiv z_*(\lambda) \in \Gamma_\lambda$ such that

1. if $0 \leq \arg \lambda < \delta$, then $|z_*| = \min_{z \in \Gamma_\lambda} |z|$,

2. if $\delta \leq \arg \lambda \leq \pi$, then $z_* = \Gamma_\lambda \cap \{z : \arg z = -\frac{\pi}{3}\}$.

For any $\lambda \in \mathbb{C}_+ \setminus \{0\}$ here and below we use $z_*$, defined by Lemma 2.2. We define the point $x_*$ by $z_* = z(x_*, \lambda)$ and let $t_* = \frac{z_*}{\sqrt{\lambda}}$ and set

$$\Gamma_\lambda^- = \Gamma_\lambda(z(0, \lambda), z_*), \quad \Gamma_\lambda^+ = \Gamma_\lambda(z_*, \infty), \quad \Gamma_\lambda = \Gamma_\lambda^- \cup \Gamma_\lambda^+. \quad (2.43)$$

![Figure 2: The point $z_*$ divides the curve $\Gamma_\lambda = \Gamma_\lambda^- \cup \Gamma_\lambda^+$ (for $\lambda = |\lambda|e^{2i\vartheta} \in \mathbb{C}_+ \setminus \{0\}$).](image)

**Lemma 2.3.** Let $\lambda \in \mathbb{C}_+ \setminus \{0\}$. Let $h(x, \lambda) = |\exp(\frac{2}{\sqrt{\lambda}}z(x, \lambda))|$ for $x \geq 0$. Then

1. if $\lambda > 0$, then $h(\cdot, \lambda)$ is strictly increasing on $[x_*, \infty)$ and $h(\cdot, \lambda) \equiv 1$ on $[0, x_*]$. 

8
2. if $0 < \arg \lambda \leq \pi$, then $h(\cdot, \lambda)$ is strictly increasing on $[0, \infty)$.

**Lemma 2.4.** Let $\lambda \in \mathbb{C}_+ \setminus \{0\}$ and $|\lambda| \geq 1$. Assume a) $z \in \Gamma_{\lambda}$, $\delta \leq \arg \lambda \leq \pi$ or b) $z \in \Gamma_{\lambda}^\circ$. Then the following estimates are fulfilled:

$$
\int_{\Gamma_{\lambda}(z)} |e^{-\frac{1}{2} s \lambda^2} |s|^{-\alpha} |ds| \leq C |e^{-\frac{1}{2} s \lambda^2} |z|^{-\alpha - \frac{1}{2}}, \quad \alpha \in \mathbb{R},
$$

(2.44)

$$
\int_{\Gamma_{\lambda}(z)} \langle s \rangle^{-\alpha} |ds| \leq C |z|^{-\alpha - 1}, \quad \alpha \geq 1,
$$

(2.45)

where $C$ is independent of $\lambda$ and $z$.

**Lemma 2.5.** Let $\lambda \in \mathbb{C}_+$ and $|\lambda| \geq 1$. Then the following estimates are fulfilled:

$$
\int_{\Gamma_{\lambda}} \langle s \rangle^{-\alpha} |ds| \leq \begin{cases} C(1 - \alpha)^{-1}|\lambda|^\frac{1}{2}(1 - \alpha) & \text{for } 0 \leq \alpha < 1, \\ C \log(|\lambda| + 1) & \text{for } \alpha = 1, \\ C(\alpha - 1)^{-1} & \text{for } \alpha > 1, \end{cases}
$$

(2.46)

$$
\int_{\Gamma_{\lambda}} \frac{|ds|}{|\lambda|^\frac{1}{2} + |s|^2} \leq \frac{C}{|\lambda|^\frac{2}{3}},
$$

(2.47)

where $C$ is independent of $\lambda$ and $z$.

### 3 Analysis of the integral equation

In this section we consider the integral equation $v_+ = a + J V v_+$ for large $|\lambda|$ in the following cases: i) $0 \leq \arg \lambda \leq \pi$, $z \in \Gamma_{\lambda}^\circ$ and ii) $\delta \leq \arg \lambda \leq \pi$, $z \in \Gamma_{\lambda}^\circ$. Moreover, we give the complete analysis for these cases. The case $0 \leq \arg \lambda \leq \pi$, $z \in \Gamma_{\lambda}^\circ$ is treated in the next section.

For $\lambda \in \mathbb{C}_+ \setminus \{0\}$ and $\alpha, \beta > 0$ define the Banach spaces of functions on $\Gamma_{\lambda}$:

$$
F_{\alpha}^\lambda = \left\{ f \in C(\Gamma_{\lambda}^\circ) : \|f\|_{\alpha} \equiv \sup_{z \in \Gamma_{\lambda}^\circ} \langle z \rangle^{\alpha} |f(z)| < \infty \right\} \quad \text{for } |\arg \lambda| < \delta,
$$

(3.1)

$$
F_{\alpha} = \left\{ f \in C(\Gamma_{\lambda}) : \|f\|_{\alpha} \equiv \sup_{z \in \Gamma_{\lambda}} \langle z \rangle^{\alpha} |f(z)| < \infty \right\} \quad \text{for } \delta \leq |\arg \lambda| \leq \pi,
$$

(3.2)

$$
F_{\alpha,\beta}^\lambda = \left\{ f \in F_{\alpha}^\lambda : f' \in F_{\beta}^\lambda \right\}, \quad \|f\|_{\alpha,\beta} = \|f\|_{\alpha} + \|f'\|_{\beta}.
$$

(3.3)

Evidently $F_{\alpha}^\lambda \subset F_{\alpha'}^\lambda$ and $F_{\alpha,\beta}^\lambda \subset F_{\alpha',\beta'}^\lambda$ for $\alpha < \alpha'$ and $\beta < \beta'$. Now we formulate the main result of this section; its proof is given in the end of the section.
Theorem 3.1. Let \( q, q_1 \in L^\infty(\mathbb{R}) \), and \( \lambda \in \overline{\mathbb{C}}_+ \). Then the equation \( v_+ = v_0 + JVv_+ \), \( v_0(z) = a(z) \), has a unique solution \( v_+ \in \mathcal{F}_+^\lambda \) for \( |\lambda|^\frac{2}{3} > 2c_0 (\|q\|_\infty + \|q_1\|_\infty + 1) \), where \( c_0 > 1 \) is an absolute constant. Moreover, the solution satisfies

\[
|v_+(z)| \leq C(z)^{-\frac{1}{4}}, \quad |v_+'(z)| \leq C(z)^{-1},
\]

\[
|v_+(z) - v_0(z)| \leq \varepsilon C(z)^{-\frac{1}{4}}, \quad |v_+'(z) - v_0'(z)| \leq \varepsilon C(z)^{-1},
\]

where \( \varepsilon = c_0|\lambda|^{-\frac{1}{6}} (\|q\|_\infty + \|q_1\|_\infty + 1) \).

We have the identity

\[
Ai(z\omega) = e^{\frac{2}{3}z^\frac{2}{3}} a(z\omega), \quad \omega = e^{\frac{2\pi i}{3}}, \quad -\pi \leq \arg z < \frac{\pi}{3}.
\]

Using (2.32), (2.33) and (3.6) we write the kernel \( J(z, s) \), given by (2.37), in terms of \( a(z) \) and \( a(\omega z) \). As a result we obtain

\[
J(z, s) = -2ie^{-i\frac{\pi}{3}} \left( a(z)a(s\omega) - e^{\frac{2}{3}(\omega - s)^\frac{2}{3}} a(z\omega)a(s) \right), \quad z, s \in S(-\pi, \frac{\pi}{3}).
\]

Note that by Lemma 7.5.1, in both cases i) and ii) we have \( \Gamma(z) \subset S[-\pi + \frac{2}{3}\delta, 0] \), so (3.7) holds on \( \Gamma(z) \).

Following (3.7), we represent \( JV_q \) as the sum of two operators. In the next two Lemmas 8.2 and 8.3 we estimate these two operators in suitable functional spaces. In Lemma 3.4 we estimate \( JV_0 \) (which is asymptotically small in comparison with \( JV_q \)). These estimates, combined in Lemma 3.6 give an a priori estimate for \( JV \). In Theorem 3.1 we prove convergence of the iterations series for the equation \( v_+ = v_0 + JVv_+ \). This gives the estimates for \( v_+ \) necessary for further analysis.

For \( v_0(z) = a(z) \) given by (2.36), due to (2.38) and Lemma 7.5.1 we have

\[
\|v_0\|_{\frac{1}{4} - \frac{2}{3}} \leq C.
\]

uniformly in \( \lambda \in \overline{\mathbb{C}} \setminus \{0\} \). For any fixed \( \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\} \) and \( z_1, z_2 \in \Gamma_\lambda \) such that \( z_j = z(x_j, \lambda), j = 1, 2 \) and \( 0 \leq x_1 \leq x_2 \) we define the function

\[
Q(z_2, z_1) \equiv \lambda^{-\frac{1}{4}} \int_{z_1}^{z_2} \dot{q}(z, \lambda) \rho(z, \lambda) dz,
\]

where \( \rho \) is given by (2.23). We have the identity \( Q(z_2, z_1) = \int_{z_2}^{z_1} q(x) dx \). Since \( q_1(x) = \int_0^x q(t) dt \in L^\infty(\mathbb{R}) \), we have

\[
|Q(z_2, z_1)| \leq 2\|q_1\|_\infty \quad \text{for any} \quad z_1, z_2 \in \Gamma_\lambda.
\]

By Lemma 7.5.2 and Lemma 7.5.3, we have

\[
C_1|\lambda|^\frac{2}{3} \leq |z| \leq C_2|\lambda|^\frac{2}{3} \quad \text{for} \quad z \in \Gamma_\lambda, \quad \delta \leq \arg \lambda \leq \pi,
\]

where \( C_1 \) and \( C_2 \) are independent of \( z \) and \( \lambda \).

Next we estimate the term of \( JV_q \), corresponding to the first term in decomposition (3.7).
Lemma 3.2. Let \( q, q_1 \in L^\infty(\mathbb{R}) \). Assume \( \lambda \in \overline{C}_+ \setminus \{0\} \), \( |\lambda| \geq 1 \) and \( f \in \mathcal{F}_{\alpha, \beta}^\lambda \) for \( \alpha > 0, \beta > \frac{3}{4} \) (that is, \( f \) is defined on \( \Gamma_\lambda \) for \( \delta \leq \arg \lambda \leq \pi \) and on \( \Gamma_\lambda^\alpha \) for \( 0 \leq \arg \lambda \leq \delta \)). Then \( g(z, \lambda) = \text{p.v.} \int_{\Gamma_\lambda(z)} a(s\omega)V_q(s)f(s)ds \in \mathcal{F}_{\alpha, \beta}^\lambda \) and satisfies

\[
|a(z)g(z, \lambda)| \leq C|\lambda|^{-\frac{1}{2}}\|q_1\|_\infty \left\{ \|f\|_\alpha(\lambda)^{-\alpha - \frac{1}{2}} + \|f'\|_\beta(\lambda)^{-\beta + \frac{1}{2}} \right\},
\]

\[
|a'(z)g(z, \lambda)| \leq C|\lambda|^{-\frac{1}{2}}\|q_1\|_\infty \left\{ \|f\|_\alpha(\lambda)^{-\alpha - \frac{3}{2}} + \|f'\|_\beta(\lambda)^{-\beta - \frac{1}{2}} \right\}.
\]

Proof. Consider the case \( 0 \leq \arg \lambda , z \in \Gamma_\lambda^\alpha \). By Lemma 7.5.1, we have \( \Gamma_\lambda(z) \subset S[-\pi + \frac{2}{3}\delta, 0] \), so the uniform estimates (2.38) and (2.39) hold on \( \Gamma_\lambda(z) \) for both \( a(z) \) and \( a(z\omega) \). Writing \( F(z) = \lambda^{-\frac{1}{4}}a(z\omega)\rho(z, \lambda) \) ( \( \rho \) is given by (2.29)), integration by parts yields

\[
g(z, \lambda) = \text{p.v.} \int_{\Gamma_\lambda(z)} (\partial_\lambda Q(s, z)) F(s)f(s)ds = -\text{p.v.} \int_{\Gamma_\lambda(z)} Q(s, z)(F(s)f(s))'ds,
\]

where we used \( Q(z, z) = 0 \) and \( \lim_{r_\lambda^{\alpha, \beta} \to \infty} Q(w, z)F(w)f(w) = 0 \) (this holds by (2.38), (2.39) and (3.10)). Thus using (2.39), (2.38), (2.39) and (3.10) we have

\[
|g(z, \lambda)| \leq C\frac{\|q_1\|_\infty}{|\lambda|^{\frac{1}{4}}} \left\{ \int_{\Gamma_\lambda(z)} \|f\|_\alpha(\lambda)^{-\alpha - \frac{1}{2}}|ds| + \int_{\Gamma_\lambda(z)} \|f'\|_\beta(\lambda)^{-\beta - \frac{1}{2}}|ds| \right\}.
\]

Due to Lemma 7.5.1 we have \( |z| = \inf_{s \in \Gamma_\lambda(z)} |s| \). Using also (2.43) we obtain

\[
|g(z, \lambda)| \leq C\frac{\|q_1\|_\infty}{|\lambda|^{\frac{1}{4}}} \left\{ \|f\|_\alpha(\lambda)^{-\alpha} \int_{\Gamma_\lambda(z)} \langle \lambda \rangle^{-\frac{1}{2}}|ds| + \int_{\Gamma_\lambda(z)} \|f'\|_\beta(\lambda)^{-\beta - \frac{1}{2}}|ds| \right\}.
\]

which together with (2.38) and (2.39) proves (3.12) and (3.13), respectively.

Consider the case \( 0 \leq \arg \lambda , z \in \Gamma_\lambda^{-} \). By Lemma 7.5.1, we have \( \Gamma_\lambda(z) \subset S[-\pi + \frac{2}{3}\delta, 0] \), so the uniform estimates (2.38) and (2.39) hold on \( \Gamma_\lambda(z) \) for both \( a(z) \) and \( a(z\omega) \). We have \( g(z, \lambda) = g_-(z, \lambda) + g_+(\lambda) \), where

\[
g_+(\lambda) = \text{p.v.} \int_{\Gamma_\lambda(z)} a(s\omega)V_q(s)f(s)ds, \quad g_-(z, \lambda) = \int_{\Gamma_\lambda(z, z)} a(s\omega)V_q(s)f(s)ds.
\]

Using (3.11) and (3.15) for \( z = z_* \) we have

\[
|g_+(\lambda)| \leq C\frac{\|q_1\|_\infty}{|\lambda|^{\frac{1}{4}}} \left\{ \|f\|_\alpha(\lambda)^{-\alpha + \frac{1}{4}} + \|f'\|_\beta(\lambda)^{-\beta - \frac{1}{4}} \right\} \leq C\frac{\|q_1\|_\infty}{|\lambda|^{\frac{1}{4}}} \left\{ \|f\|_\alpha(\lambda)^{-\alpha + \frac{1}{4}} + \|f'\|_\beta(\lambda)^{-\beta - \frac{1}{4}} \right\}.
\]

11
In order to estimate $g_-$ we integrate by parts

$$g_-(z, \lambda) = Q(z_*, z) F(z) f(z) - \int_{\Gamma_\lambda(z_*)} Q(z_* s) (F(s) f(s))' \, ds, \quad F(z) = \lambda^{-\frac{1}{\alpha}} a(z \omega) \rho(z, \lambda),$$

(3.17)

since $Q(z_*, z_*) = 0$ and $\dot{q}(s) \rho(s) = -\partial_s Q(z_*, s)$. Using (2.30), (2.38), (2.39) and (3.10) we obtain

$$|g_-(z, \lambda)| \leq C \frac{\|q_1\|_{\infty}}{|\lambda|^\frac{1}{\alpha}} \left\{ \frac{\|f\|_{\alpha}}{|z|^{\alpha - \frac{1}{\alpha}}} + \int_{\Gamma_\lambda} \left( \frac{\|f\|_{\alpha}}{|s|^{\alpha + \frac{1}{\alpha}}} + \frac{\|f\|_{\alpha}}{|\lambda|^{\frac{1}{\alpha}}} \right) |ds| + \int_{\Gamma_\lambda} \frac{\|f'\|_{\beta}}{|s|^{\beta + \frac{1}{\alpha}}} |ds| \right\}.$$  

By (2.46) and (3.11), we have

$$|g_-(z, \lambda)| \leq C \|q_1\|_{\infty} |\lambda|^{-\frac{1}{\alpha}} \left\{ \|f\|_{\alpha} |z|^{-\alpha - \frac{1}{\alpha}} + \|f'\|_{\beta} |z|^{-\beta + \frac{1}{\alpha}} \right\}.$$  

(3.18)

Combining (3.16) and (3.18) with (2.38) gives (3.12). The estimate (3.13) follows from (3.16), (3.18) and (2.39).

For the analysis of the part of $J V_q$ corresponding to the second term in (3.7) we also integrate by parts. Let us introduce an analogue of $Q(z_1, z_2)$:

$$P(z) = \frac{1}{\lambda^\frac{\alpha}{2}} \int_{\Gamma_\lambda(z, \infty)} e^{-\frac{4}{3} s^2} \dot{q}(s, \lambda) \rho(s) \, ds = \int_{x}^{\infty} e^{-2 \lambda s^2 / \sqrt{\lambda}} q(s) \, ds, \quad z \in \Gamma_\lambda^+,$$

(3.19)

where $x = \sqrt{\lambda} t \left( \frac{z}{\lambda^\frac{1}{2}} \right)$. Using (2.30) and (2.41) for $q \in L^\infty(\mathbb{R})$ gives

$$|P(z)| \leq \|q\|_{\infty} C \int_{\Gamma_\lambda(z)} \left| \frac{e^{-\frac{4}{3} s^2}}{|z|^{\frac{1}{\alpha}} + |s|^{\frac{1}{\alpha}}} \right| |ds| \leq C \|q\|_{\infty} e^{-\frac{4}{3} z^2} |\langle z \rangle|^{-\frac{1}{2}}, \quad z \in \Gamma_\lambda^+.$$  

(3.20)

Using $|\rho(z)| \leq C$ and (2.41), we obtain another estimate

$$|P(z)| \leq \|q\|_{\infty} C \left\{ \frac{e^{-\frac{4}{3} z^2}}{|\lambda|^{\frac{1}{\alpha}}} \int_{\Gamma_\lambda(z)} |e^{-\frac{4}{3} s^2}| |ds| \right\} \leq C \|q\|_{\infty} e^{-\frac{4}{3} z^2} |\langle z \rangle|^{-\frac{1}{2}}, \quad z \in \Gamma_\lambda^+.$$  

(3.21)

We estimate the part of $J V_q$, corresponding to the second term in the decomposition (3.7).

**Lemma 3.3.** Let $q \in L^\infty(\mathbb{R})$. Assume $\lambda \in \mathbb{C}_+, |\lambda| \geq 1$, and $f \in \mathcal{F}_{\alpha, \beta}^\lambda$ for $\alpha > 0, \beta > 0$ (that is, $f$ is defined on $\Gamma_\lambda$ for $\delta \leq \arg \lambda \leq \pi$ and on $\Gamma_\lambda^+$ for $0 \leq \arg \lambda \leq \delta$). Then

$$g(z, \lambda) = \int_{\Gamma_\lambda(z)} a(s) e^{-\frac{4}{3} s^2} V_q(s) f(s) \, ds \in \mathcal{F}_{\alpha, \beta}^\lambda$$

and satisfies

$$|e^{\frac{4}{3} z^2} a(z) g(z, \lambda)| \leq C \|q\|_{\infty} |\lambda|^{-\frac{1}{\alpha}} \left\{ \|f\|_{\alpha} |z|^{-\alpha - \frac{1}{\alpha}} + \|f'\|_{\beta} |z|^{-\beta + \frac{1}{\alpha}} \right\},$$

(3.22)

$$\left| g(z, \lambda) \frac{d}{dz} (a(z \omega) e^{\frac{4}{3} z^2}) \right| \leq C \frac{\|q\|_{\infty}}{|\lambda|^{\frac{1}{\alpha}}} \left\{ \frac{\|f\|_{\alpha}}{|\langle z \rangle|^{\alpha + \frac{1}{\alpha}}} + \frac{\|f'\|_{\beta}}{|\langle z \rangle|^{\beta + \frac{1}{\alpha}}} \right\}.$$  

(3.23)
Proof. Assume $0 \leq \arg \lambda \leq \pi$ and $z \in \Gamma_\lambda^+$. By Lemma 7.6, we have $\Gamma_\lambda(z) \subset S[-\pi + \frac{\pi}{3}, 0]$, so the uniform estimates (2.38) and (2.39) hold on $\Gamma_\lambda(z)$ for both $a(z)$ and $a(z\omega)$. Let $F(z) = \lambda^{-\frac{\beta}{2}} a(\zeta) \rho(z)$, where $\rho$ is given by (2.29).

\[ g - PFf = I_1 + I_2, \quad I_1 = \int_{\Gamma_\lambda(z)} P(s)F'(s)f(s)ds, \quad I_2 = \int_{\Gamma_\lambda(z)} P(s)F(s)f'(s)ds. \]

Using (2.30), (2.38) and (3.20) we have

\[ |P(z)F(z)f(z)| \leq C|\lambda|^{-\frac{\beta}{2}} \|q\|_\infty |e^{-\frac{g}{2}}| \|f\|_\alpha(z)^{-\alpha - 1}. \] (3.24)

In order to estimate $I_1$ and $I_2$ we use (2.30), (2.38), (2.39), (2.44) and (3.21). This gives

\[ |I_1| \leq C \|q\|_\infty \int_{\Gamma_\lambda(z)} \left( \frac{s}{\lambda^2} \right)^{-\frac{\beta}{2}} \|f\|_\alpha |e^{-\frac{g}{2}}| \|\lambda\|^{-\frac{\beta}{2}} \|f\|_\alpha \|e^{-\frac{g}{2}}\|_{\alpha} \right) |ds| \]

\[ \leq C \|q\|_\infty |\lambda|^{-\frac{\beta}{2}} |e^{-\frac{g}{2}}\|_{\alpha} \|f\|_\alpha(z)^{-\alpha - 2}, \]

\[ |I_2| \leq C \|q\|_\infty \int_{\Gamma_\lambda(z)} \|f''\|_\beta |e^{-\frac{\beta}{2}}\|_{\beta} \right) \left( \frac{s}{\lambda^2} \right)^{-\frac{\beta}{2}} \|f''\|_\beta \|\lambda\|^{-\frac{\beta}{2}} \right) |ds| \leq C \|q\|_\infty \|e^{-\frac{g}{2}}\|_{\alpha} \|f''\|_\beta \|\lambda\|^{-\frac{\beta}{2}} \|f''\|_\beta \|\lambda\|^{-\frac{\beta}{2}}. \]

The above estimates for $I_1$, $I_2$ and (3.24) give

\[ |g(z, \lambda)| \leq C|\lambda|^{-\frac{\beta}{2}} \|q\|_\infty |e^{-\frac{g}{2}}\|_{\alpha} \left\{ \|f\|_\alpha(z)^{-\alpha - 1} + \|f''\|_\beta \|\lambda\|^{-\frac{\beta}{2}} \right\}. \] (3.25)

The last estimate together with (2.38) and (2.39) implies (3.22) and (3.23).

Assume $\delta \leq \arg \lambda \leq \pi$ and $z \in \Gamma_\lambda$. Using $\Gamma_\lambda = \Gamma_\lambda^- \cup \Gamma_\lambda^+$ we have $g = g_- + g_+$, where

\[ g_-(z, \lambda) = \int_{\Gamma_\lambda(z, z_\lambda)} a(s)e^{-\frac{\beta}{2}s^2} V_q(s)f(s)ds, \quad g_+(\lambda) = \int_{\Gamma_\lambda^+} a(s)e^{-\frac{s^2}{2}} V_q(s)f(s)ds. \]

Using (3.11) and (3.25) for $z = z_\lambda$ we obtain

\[ |g_+(\lambda)| \leq C \|q\|_\infty |e^{-\frac{\beta}{2}s^2}\| \left\{ \|f\|_\alpha |\lambda|^{-\frac{\beta}{2}} + \|f''\|_\beta |\lambda|^{-\frac{\beta}{2}} \right\} \leq C \|q\|_\infty |e^{-\frac{\beta}{2}s^2}\| \left\{ \|f\|_\alpha |\lambda|^{-\frac{\beta}{2}} + \|f''\|_\beta |\lambda|^{-\frac{\beta}{2}} \right\}. \] (3.26)

Using (2.38), (2.29), (2.30), (2.44) and (3.11) results in

\[ |g_-(z, \lambda)| \leq C \|q\|_\infty \int_{\Gamma_\lambda(z, z_\lambda)} |e^{-\frac{\beta}{2}s^2}| \|f\|_\alpha |\lambda|^{-\frac{\beta}{2}} |ds| \leq C \|q\|_\infty |e^{-\frac{\beta}{2}s^2}\| \|f\|_\alpha |\lambda|^{-\frac{\beta}{2}} |ds| \leq C \|q\|_\infty \|e^{-\frac{\beta}{2}s^2}\| |\lambda|^{-\frac{\beta}{2}}. \] (3.27)

Now (3.22) and (3.23) follow from (3.26) and (3.27) taking into account (2.38), (2.39) and the fact that, by Lemma 7.3, $\exp\left(\frac{2}{\beta}(x, \zeta, \lambda)^{\frac{1}{2}}\right)$ is strictly increasing.

In the following Lemma we estimate the operator $JV_0$. We show that as $\lambda \to \infty$ it is asymptotically small in comparison with $JV_0$.  

13
Lemma 3.4. Let $\lambda \in \overline{\mathbb{C}}_+$, $|\lambda| \geq 1$ and $f \in \mathcal{F}^\lambda_\alpha$ for some $\alpha > 0$ (that is, $f$ is defined on $\Gamma_\lambda$ for $\delta \leq \arg \lambda \leq \pi$ and on $\Gamma^+_\lambda$ for $0 \leq \arg \lambda \leq \delta$). Then $\mathbf{J}V_0f \in \mathcal{F}^\lambda_\alpha$ and
\[
|(\mathbf{J}V_0f)(z,\lambda)| \leq \frac{C}{|\lambda|^\frac{3}{2}} \frac{\|f\|_\alpha}{(z)^{\alpha+\frac{3}{2}}}, \quad \left| \frac{\partial}{\partial z} (\mathbf{J}V_0f)(z,\lambda) \right| \leq \frac{C}{|\lambda|^\frac{3}{2}} \frac{\|f\|_\alpha}{(z)^{\alpha+\frac{3}{2}}}. 
\] (3.28)

Proof. By Lemma 7.5, we have $\Gamma_\lambda(z) \subset S[-\pi + \frac{2}{3}\delta, 0]$, so the decomposition (3.7) holds on $\Gamma_\lambda(z)$. We estimate the part of $\mathbf{J}V_0$, corresponding to the first term in decomposition (3.7). Using Lemma 7.5 (2.30), (2.38), (2.44) and the inequality $(z)|\lambda|^\frac{3}{2} \leq 2(|\lambda|^\frac{1}{2} + |z|^2)$. As a result we have
\[
\left| \int_{\Gamma_\lambda(z,\infty)} a(s\omega)V_0(s)f(s)ds \right| \leq C \int_{\Gamma_\lambda(z,\infty)} \frac{\|f\|_\alpha}{(s)^{\alpha+\frac{3}{2}}} \frac{|ds|}{|\lambda|^\frac{1}{2} + |s|^2} 
\]
\[
\leq \frac{C}{|\lambda|^\frac{3}{2}} \int_{\Gamma_\lambda(z,\infty)} \frac{\|f\|_\alpha}{(s)^{\alpha+\frac{3}{2}}} \frac{|ds|}{|\lambda|^\frac{3}{2} (z)^{\alpha+\frac{3}{2}}}. 
\] (3.29)

In order to estimate the part of $\mathbf{J}V_0$, corresponding to the second term in decomposition (3.7), we use (2.30), (2.38), (2.44) and the inequality $(z)|\lambda|^\frac{3}{2} \leq 2(|\lambda|^\frac{1}{2} + |z|^2)$. This gives
\[
\left| \int_{\Gamma_\lambda(z,\infty)} a(s)e^{-\frac{4s^3}{3}}V_0(s)f(s)ds \right| \leq C \int_{\Gamma_\lambda(z,\infty)} \frac{\|f\|_\alpha}{|\lambda|^\frac{3}{2} + |s|^2} \frac{|e^{-\frac{4s^3}{3}}|}{(s)^{\alpha+\frac{3}{2}}} |ds| 
\]
\[
\leq C \frac{\|f\|_\alpha}{|\lambda|^\frac{3}{2}} \int_{\Gamma_\lambda(z,\infty)} \frac{|e^{-\frac{4s^3}{3}}|}{(s)^{\alpha+\frac{3}{2}}} |ds| \leq \frac{C}{|\lambda|^\frac{3}{2}} \frac{|e^{-\frac{4s^3}{3}}|}{(z)^{\alpha+\frac{3}{2}}}. 
\] (3.30)

The first estimate in (3.28) follows from (3.29) and (3.30) taking into account (2.38) and (3.7).

In order to estimate $\partial_z g(z,\lambda)$ we note that $\partial_z g(z,\lambda) = \int_{\Gamma_\lambda(z)} \partial_z J(z, s)V_0(s)f(s)ds$. Therefore the second estimate in (3.28) follows from (3.29) and (3.30) taking into account (2.39) and (3.7).

Now in order to estimate the operator $\mathbf{J}V = \mathbf{J}V_q + \mathbf{J}V_0$ we combine the results of the three previous Lemmas.

Lemma 3.5. Let $q, q_1 \in L^\infty(\mathbb{R})$. Assume $\lambda \in \overline{\mathbb{C}}_+$, $|\lambda| \geq 1$, and $f \in \mathcal{F}^\lambda_{\alpha,\beta}$ for $\alpha > 0, \beta > \frac{3}{4}$ (that is, $f$ is defined on $\Gamma_\lambda$ for $\delta \leq \arg \lambda \leq \pi$ and on $\Gamma^+_\lambda$ for $0 \leq \arg \lambda \leq \delta$). Then $\mathbf{J}Vf \in \mathcal{F}^\lambda_{\alpha,\beta}$ and
\[
|(\mathbf{J}Vf)(z,\lambda)| \leq \frac{C \|q\|_\infty + \|q_1\|_\infty}{|\lambda|^\frac{3}{2}} \left\{ \frac{\|f\|_\alpha}{(z)^{\alpha+\frac{3}{2}}} + \frac{\|f'\|_\beta}{(z)^{\frac{3}{2}+\beta-\frac{3}{4}}} \right\} + \frac{C}{|\lambda|^\frac{3}{2}} \frac{\|f\|_\alpha}{(z)^{\alpha+\frac{3}{2}}}. 
\] (3.31)

\[
|\partial_z (\mathbf{J}Vf)(z,\lambda)| \leq \frac{C \|q\|_\infty + \|q_1\|_\infty}{|\lambda|^\frac{3}{2}} \left\{ \frac{\|f\|_\alpha}{(z)^{\alpha+\frac{3}{2}}} + \frac{\|f'\|_\beta}{(z)^{\frac{3}{2}+\beta+\frac{3}{4}}} \right\} + \frac{C}{|\lambda|^\frac{3}{2}} \frac{\|f\|_\alpha}{(z)^{\alpha+\frac{3}{2}}}. 
\] (3.32)
Proof. Recall that $JV = JV_0 + JV_1$. By Lemma 3.31, we have $\Gamma_\lambda(z) \subset S[-\pi + \frac{2}{3}\delta, 0]$, so the decomposition (3.7) holds on $\Gamma_\lambda(z)$. Taking into account this decomposition, we deduce that the combination of Lemmas 3.2 and 3.3 gives the estimate for $JV_\lambda$ (the corresponding terms in (3.31) and (3.32) contain curved brackets). Together with the estimate (3.28) for $JV_0$ this proves (3.31) and (3.32). ■

Proof of Theorem 3.1 We present the proof for $\lambda \in \mathbb{C}_+$, for $\lambda \in \mathbb{C}_-$ it is analogous. Let $v_{n+1} = JVv_n$, $n \geq 0$. Substituting $v_n$, $g = v_{n+1}$ in (3.31), (3.32) and taking into account (3.3) for $v_0$ we obtain

$$|v_{n+1}(z)| \leq \varepsilon \|v_n\|_{\alpha_0} + \varepsilon \|v'_n\|_{\beta_0}, \quad |v'_{n+1}(z)| \leq \varepsilon \|v_n\|_{\alpha_0} + \varepsilon \|v'_n\|_{\beta_0},$$

where

$$\alpha_0 = \frac{1}{4}, \beta_0 = \frac{5}{4}, \quad \alpha_{n+1} = \min \{\alpha_n + \frac{1}{2}, \beta_n - \frac{1}{2}\}, \quad \beta_{n+1} = \min \{\alpha_n + \frac{3}{4}, \beta_n + \frac{1}{2}\}.$$ (3.33)

Therefore

$$|v_{2n}(z)| \leq \varepsilon^{2n}\|v_0\|_{\frac{1}{4} + \frac{5}{4}}(z)^{-\frac{\alpha_n}{2} - \frac{1}{4}}, \quad |v_{2n+1}(z)| \leq \varepsilon^{2n+1}\|v_0\|_{\frac{1}{4} + \frac{5}{4}}(z)^{-\frac{\beta_n}{2} - \frac{1}{4}},$$ (3.34)

$$|v'_{2n}(z)| \leq \varepsilon^{2n}\|v_0\|_{\frac{1}{4} + \frac{5}{4}}(z)^{-\frac{\alpha_n}{2} - \frac{1}{4}}, \quad |v'_{2n+1}(z)| \leq \varepsilon^{2n+1}\|v_0\|_{\frac{1}{4} + \frac{5}{4}}(z)^{-\frac{\beta_n}{2} - 1},$$ (3.35)

and for $\varepsilon < 1$ the series $v_+(z) = \sum_{n=0}^{\infty} v_n(z)$ converges absolutely and is a solution of $v_+ = a + JV v_+$. For $\varepsilon < \frac{1}{2}$ we obtain from (3.3), (3.34) and (3.35) the estimates (3.4), (3.5).

We prove the uniqueness. Suppose that there exists another solution $v_+^{(1)} \in F_{\frac{1}{4}, 1}$. We have $y = JV y$ and therefore $y = (JV)^n y$ for any integer $n \geq 1$; applying Lemma 3.3 we obtain $|y(z)| \leq C \varepsilon^n \|y\|_{\frac{1}{4}, 1}$. Taking the limit $n \to \infty$ for $\varepsilon < 1$ we obtain $y = 0$. ■

4 Uniform asymptotics

In this section we consider the equation $v_+ = a + JV_+ |\lambda|$, $|\arg \lambda| \leq \delta$ and $z \in \Gamma_\lambda^-$. The case $z \in \Gamma_\lambda^+$ was treated in the previous section.

For $|\arg \lambda| \leq \delta$ denote by $F_{\lambda}$ the class of functions $f$ on $\Gamma_\lambda^-$ such that $f, f' \in L^\infty(\Gamma_\lambda^-)$. We also set $F_{\lambda} = F_{\lambda}^+ \oplus F_{\lambda}^-$ (for the definition of $F_{\alpha, \beta}^\lambda$ see (3.3)). Our main result is

Theorem 4.1. Let $q \in B$ and $|\arg \lambda| \leq \delta$. Then the equation $v_+ = v_0 + JV v_+, v_0 \equiv a$, has a unique solution $v_+ \in F_{\lambda}$ for $|\lambda|^{\frac{1}{2}} \geq 2c_0(\|q\|_B + 1)$, where $c_0 > 1$ is an absolute constant. If, in addition, $(z, \lambda) \in \Gamma_\lambda^- \times S[-\delta, \delta]$, then the following estimates are fulfilled:

$$|v_+(z)| \leq C(z)^{-\frac{1}{2}}, \quad |v'_+(z)| \leq C(z)^{\frac{1}{2}},$$ (4.1)
\[ |v_1(z)| \leq C \varepsilon(z)^{-\frac{3}{4}}, \quad |v'_1(z)| \leq C(z)^{\frac{1}{4}} \varepsilon, \quad v_1 = JVv_0, \quad \varepsilon = c_0|\lambda|^{-\frac{1}{2}}(\|q\|_B + 1), \quad (4.2) \]

\[ |v_+(z) - v_0(z) - v_1(z)| \leq C \varepsilon^2(z)^{-\frac{1}{4}}, \quad |v'_+(z) - v'_0(z) - v'_1(z)| \leq \varepsilon^2 C(z)^{\frac{1}{4}}. \quad (4.3) \]

**Corollary 4.2.** Let \( q \in B \). Then the equation \( u_+ = \text{Ai} + Jv u_+ \) has a unique solution \( u_+(z, \lambda) \) such that \( u_+(z, \lambda) = \text{Ai}(z)(1 + o(1)), \partial_2 u_+(z, \lambda) + \sqrt{2} u_+(z, \lambda) = \text{Ai}'(z)O(z^{-\frac{5}{4}}) \) as \( \Gamma \lambda \ni \Re z \to \infty \) for \( |\lambda|^{\frac{1}{4}} \geq 2c_0(\|q\|_B + 1) \), where \( c_0 > 1 \) is an absolute constant. Moreover, \( u_+(z, \lambda) = e^{-\frac{3}{2}z^2} v_+(z, \lambda) \). The following estimates for \( u_+ \) and \( u_1(z, \lambda) = e^{-\frac{3}{2}z^2} v_1(z, \lambda) \) are fulfilled uniformly in \( z \in \Gamma \lambda \):

If \( |\arg \lambda| \leq \delta \), then

\[ |u_+(z, \lambda)| \leq C \varepsilon \frac{e^{\frac{3}{4}|\Re z|}}{(z)^{\frac{1}{4}}}, \quad |\partial_2 u_+(z, \lambda)| \leq C(z)^{\frac{1}{4}} \varepsilon e^{\frac{3}{4}|\Re z|}, \quad \varepsilon = c_0 \frac{\|q\|_B + 1}{|\lambda|^{\frac{1}{4}}}, \quad (4.4) \]

\[ |u_1(z, \lambda)| \leq C \varepsilon e^{\frac{3}{4}|\Re z|} (z)^{-\frac{1}{4}}, \quad |\partial_2 u_1(z, \lambda)| \leq C(z)^{\frac{1}{4}} \varepsilon e^{\frac{3}{4}|\Re z|}, \quad \varepsilon = c_0 \frac{\|q\|_B + 1}{|\lambda|^{\frac{1}{4}}}, \quad (4.5) \]

\[ |u_+(z, \lambda) - \text{Ai}(z) - u_1(z, \lambda)| \leq C \varepsilon^2 e^{\frac{3}{4}|\Re z|} (z)^{-\frac{1}{4}}, \quad (4.6) \]

\[ |\partial_2 u_+(z, \lambda) - \text{Ai}'(z) - \partial_2 u_1(z, \lambda)| \leq C \varepsilon^2 e^{\frac{3}{4}|\Re z|} (z)^{-\frac{1}{4}}, \quad (4.7) \]

if \( \delta \leq |\arg \lambda| \leq \pi, \) then

\[ |u_+(z, \lambda) - \text{Ai}(z)| \leq C \varepsilon e^{\frac{3}{4}|\Re z|} (z)^{-\frac{1}{4}}, \quad |\partial_2 u_+(z, \lambda) - \text{Ai}'(z)| \leq C \varepsilon \frac{e^{\frac{3}{4}|\Re z|}}{(z)^{\frac{1}{4}}}. \quad (4.8) \]

**Proof.** Set \( u_+(z, \lambda) = e^{-\frac{3}{2}z^2} v_+(z, \lambda) \), where \( v_+ \) is given by Theorem 3.1 (for \( \delta < |\arg \lambda| \leq \pi \)) and Theorem 4.1 (for \( \arg \lambda \leq \delta \)). By (2.34), (2.37) and (3.4), \( u_+ \) is a solution of (2.34) with the required asymptotics.

In order to prove uniqueness, we suppose that there exists another solution \( u^{(1)}_+ \) with the same asymptotics as \( \Gamma \lambda \ni \Re z \to \infty \). Then \( v^{(1)}_+(z, \lambda) = e^{-\frac{3}{2}z^2} u^{(1)}_+(z, \lambda) \) is in \( \mathcal{F}_\lambda \) (for \( |\arg \lambda| \leq \delta \)) or in \( \mathcal{F}_{\lambda,1}(\delta < |\arg \lambda| \leq \pi) \) and solves \( v_+ = v_0 + JVv_+ \). Hence, by Theorems 3.1 and 4.1, \( v^{(1)}_+ = v_+ \), implying \( u^{(1)}_+ = u_+ \).

The estimates (4.4–4.7) follow from (3.1–3.3). The estimate (4.8) follows from (3.5). \( \Box \)

Below we consider only the case \( 0 \leq |\arg \lambda| \leq \delta \), the case \(-\delta \leq |\arg \lambda| \leq 0 \) is analogous.

By Lemma 4.1, \( \Gamma \lambda \) is arbitrarily close to \( \mathbb{R}_- \) as \( \arg \lambda \to 0 \). Thus we cannot use (2.39) in order to estimate the terms in the decomposition (3.7) for \( J(z, s) \). So we use another representation. We have

\[ \text{Ai}(z) = e^{-\frac{3}{2}z^2} a(z) + \omega = e^{\frac{3}{2}z^2}, \quad \omega = e^{\frac{3}{2}z^2}. \quad (4.9) \]
Using (2.32), (2.33) and (3.6), we obtain from (2.35) and (2.37)
\[ J(z, s) = -2i \left( a(z\omega) a(s\omega) - e^{\frac{4}{3}z^\frac{3}{2}} - \frac{4}{3} z^\frac{3}{2} a(z\omega) \operatorname{Ai}(s\omega) \right), \quad z \in \Gamma^-_\lambda, \quad s \in \Gamma_\lambda, \quad (4.10) \]
where, by Lemma 7.3.1, \( \Gamma^-_\lambda \subset S[-\pi, -\frac{\pi}{3} + \frac{5}{12}\delta] \subset S[-\pi, -\frac{\pi}{3}] \) and \( \Gamma^+_\lambda \subset S[-\frac{\pi}{2} - \frac{\delta}{4}, 0] \subset S(-\pi, \pi). \)

As it follows from the decomposition (4.10), the formal operator \( J^\pi \) can be presented as the sum \( a(z\omega) \times \text{(integral operator)} + e^{\frac{4}{3}z^\frac{3}{2}} a(z\omega) \times \text{(integral operator)}. \) Thus in order to estimate \( J^\pi \) we estimate these integral operators, introducing for functions on \( \Gamma^-_\lambda \) a decomposition in the spirit of (4.10).

For \( f \in \mathcal{F}^\lambda \) and a fixed decomposition
\[ f(z) = a(z\omega) f_p(z) + e^{\frac{4}{3}z^\frac{3}{2}} a(z\omega) f_e(z), \quad z \in \Gamma^-_\lambda, \quad \lambda \in S[0, \delta] \bigcap \{ \lambda \in \mathbb{C} : |\lambda| \geq 1 \}, \quad (4.11) \]
we introduce the functionals
\[ p_0(f, \lambda) = \sup_{z \in \Gamma^-_\lambda} \left( |f_p(z)| + |e^{\frac{4}{3}z^\frac{3}{2}} f_e(z)| \right), \quad p_1(f, \lambda) = \sup_{z \in \Gamma^-_\lambda} \left( |f'_p(z)| + |e^{\frac{4}{3}z^\frac{3}{2}} f'_e(z)| \right). \quad (4.12) \]
If \( f \in \mathcal{F}^\lambda \) has a decomposition (4.11), then we have
\[ |f(z)| \leq C p_0(f, \lambda) \lambda^{-\frac{1}{4}}, \quad |f'(z)| \lambda^{-\frac{1}{5}} \leq C \left( p_0(f, \lambda) + p_1(f, \lambda) \lambda^{-\frac{1}{4}} \right), \quad z \in \Gamma^-_\lambda. \quad (4.13) \]
If \( f \in \mathcal{F}^\lambda \) has a decomposition (4.11), then we have
\[ |f_p(z)| \leq p_0(f, \lambda), \quad |f_e(z)| \leq |e^{\frac{4}{3}z^\frac{3}{2}}| p_0(f, \lambda), \quad (4.14) \]
\[ |f'_p(z)| \leq p_1(f, \lambda) \lambda^{-\frac{1}{2}}, \quad |f'_e(z)| \leq |e^{\frac{4}{3}z^\frac{3}{2}}| \lambda^{-\frac{1}{4}} p_1(f, \lambda). \quad (4.15) \]
Using (3.6), (4.9) and the identity (2.33) we obtain for \( 0 \leq \arg \lambda \leq \delta \)
\[ v_0(z) = a(z\omega) = e^{i\frac{\pi}{3}} a(z\omega) + e^{-i\frac{\pi}{3}} e^{\frac{4}{3}z^\frac{3}{2}} a(z\omega), \quad z \in \Gamma^-_\lambda \subset S[-\pi, -\frac{\pi}{3}). \quad (4.16) \]
For this decomposition using (2.38) and (2.39) we obtain
\[ p_0(v_0, \lambda) \leq 2, \quad p_1(v_0, \lambda) = 0, \quad 0 \leq \arg \lambda \leq \delta. \quad (4.17) \]

Below we estimate \( \left. (J^\pi f) \right|_{\Gamma^-_\lambda} \) in terms of \( p_0 \) and \( p_1 \) assuming that \( f \in \mathcal{F}^\lambda = \mathcal{F}^-_\lambda \oplus \mathcal{F}^\lambda_4 \).

Using (4.10) for some function \( f \) on \( \Gamma^-_\lambda \) we obtain the following formal decomposition:
\[ (J^\pi f)(z) = a(z\omega) g_p(z) + e^{\frac{4}{3}z^\frac{3}{2}} a(z\omega) g_e(z), \quad g_p = J_p V f, \quad g_e = J_e V f, \quad z \in \Gamma^-_\lambda. \quad (4.18) \]
where \( \lambda \in S[0, \delta] \cap \{ \lambda \in \mathbb{C} : |\lambda| \geq 1 \} \) and the operators \( J_p \) and \( J_e \) are given by

\[
(J_p f)(z) = -2i \int_{\Gamma_\lambda(z)} a(s\omega) f(s) \, ds, \quad (J_e f)(z) = 2i \int_{\Gamma_\lambda(z)} e^{-\frac{2}{3} s^2} \text{Ai}(s\omega) f(s) \, ds. \tag{4.19}
\]

In the applications below the integral along the infinite curve \( \Gamma^+_\lambda \) exists in the principal value sense. In this section we will always define \( p_0(JV f, \lambda) \) and \( p_1(JV f, \lambda) \) using the decomposition (4.18) and (4.19).

In order to estimate \( J_p V \) and \( J_e V \) we decompose the integrals in (4.19) corresponding to the splitting \( \Gamma_\lambda(z) = \Gamma^-_\lambda(z, z_+) \cup \Gamma^+_\lambda \). Thus we have

\[
(J_p f)(z) = (j_p f)(z) + h_p(f), \quad z \in \Gamma^- \subset S[-\pi, -\frac{\pi}{3}), \tag{4.20}
\]

where, by definition

\[
(j_p f)(z) = -2i \int_{\Gamma_\lambda(z, z_+)} a(s\omega) f(s) \, ds, \quad h_p(f) = -2i \int_{\Gamma^+_\lambda} a(s\omega) f(s) \, ds. \tag{4.21}
\]

A similar decomposition of \( J_e \) gives

\[
(J_e f)(z) = (j_e f)(z) + h_e(f), \quad z \in \Gamma^- \subset S[-\pi, -\frac{\pi}{3}), \tag{4.22}
\]

where, by definition

\[
(j_e f)(z) = 2i \int_{\Gamma_\lambda(z, z_+)} e^{-\frac{2}{3} s^2} \text{Ai}(s\omega) f(s) \, ds, \quad h_e(f) = 2i \int_{\Gamma^+_\lambda} e^{-\frac{2}{3} s^2} \text{Ai}(s\omega) f(s) \, ds. \tag{4.23}
\]

Note that the standard asymptotics (2.31) for \( \text{Ai}(s\omega) \) fails in the neighborhood of \( \arg s = -\frac{\pi}{3} \). Taking into account (4.19) we obtain

\[
(j_e f)(z) = 2i \int_{\Gamma_\lambda(z, z_+)} e^{-\frac{4}{3} s^2} a(s\omega) f(s) \, ds. \tag{4.24}
\]

By (2.33), (2.36), (3.6) and (4.9), for \( s \in \Gamma^+_\lambda \subset S(-\pi, \frac{\pi}{3}) \) we have \( e^{-\frac{2}{3} s^2} \text{Ai}(s\omega) = e^{i\pi} a(s\omega) - e^{-\frac{4}{3} s^2} \omega a(s) \). This gives

\[
h_e(f) = 2i \int_{\Gamma^+_\lambda} \left( e^{i\pi} a(s\omega) - e^{-\frac{4}{3} s^2} \omega a(s) \right) f(s) \, ds. \tag{4.25}
\]

Using (4.19–4.25), for \( z \in \Gamma^- \subset S[-\pi, -\frac{\pi}{3}) \) we have the decomposition

\[
(J f)(z) = a(z\omega) ((j_p f)(z) + h_p(f)) + e^{\frac{4}{3} s^2} a(z\omega)((j_e f)(z) + h_e(f)). \tag{4.26}
\]

For \( z_j = z(x_j, \lambda) \in \Gamma^- \), \( j = 1, 2 \) and \( x_1 < x_2 \) we define

\[
P_{z}(z_1, z_2) = \lambda^{\frac{1}{6}} \int_{\Gamma_{\lambda}(z_1, z_2)} e^{\pm \frac{4}{3} s^2} \rho^{-1}(s)V_q(s) \, ds. \tag{4.27}
\]
Lemma 4.3. Let $q \in \mathcal{B}$. Assume $0 \leq \arg \lambda \leq \delta$, $|\lambda| \geq 1$. Assume that $f|_{\Gamma_\lambda^{-}} \in \mathcal{F}_\lambda^-$ has a decomposition (4.11) and $f|_{\Gamma_\lambda^{+}} = 0$. Then $(\mathcal{J}V_q f)(z) = a(z\overline{\omega})(j_p V_q f)(z) + e^{\frac{i}{3}z^3}a(z\omega)(j_e V_q f)(z) \in \mathcal{F}_\lambda^-$ and for this decomposition the following estimates are fulfilled:

$$p_0(\mathcal{J}V_q f, \lambda) \leq C\|q\||\lambda|^{-\frac{1}{6}}\left(p_0(f, \lambda) + p_1(f, \lambda) \log(|\lambda| + 1)\right), \quad (4.28)$$

$$p_1(\mathcal{J}V_q f, \lambda) \leq C\|q\||\lambda|^{-\frac{1}{3}}p_0(f, \lambda). \quad (4.29)$$

Proof. By Lemma 7.3.1, $\Gamma_\lambda^- \subset S[-\pi, -\frac{\pi}{3}]$, so (4.10) holds. Using (4.11), (4.18), (4.20), (4.21), (4.22), (4.23) and (4.24) we obtain

$$(j_p V_q f)(z) = -2i(I_{pp}(z) + I_{pe}(z)), \quad (j_e V_q f)(z) = 2i(I_{ep}(z) + I_{ee}(z)), \quad (4.30)$$

where

$$I_{pp}(z) = \int_{\Gamma_\lambda(z, z)} a(s\omega)a(s\overline{\omega})V_q(s)f_p(s) ds, \quad I_{pe}(z) = \int_{\Gamma_\lambda(z, z)} a(s\omega)^2 e^{\frac{i}{3}s^3} V_q(s)f_e(s) ds,$$

$$I_{ep}(z) = \int_{\Gamma_\lambda(z, z)} a(s\overline{\omega})^2 e^{-\frac{i}{3}s^3} V_q(s)f_p(s) ds, \quad I_{ee}(z) = \int_{\Gamma_\lambda(z, z)} a(s\omega)a(s\overline{\omega})V_q(s)f_e(s) ds.$$

We estimate $I_{pp}$. Define the functions $F_j(s) = \lambda^{-\frac{1}{3}}a(s\omega)a(s\overline{\omega})f_j(s)\rho(s)$ for $j = e, p$. Integration by parts yields

$$I_{pp}(z) = -Q(z, z)F_p(z) + \int_{\Gamma_\lambda(z, z)} Q(z, s)F'_p(s)ds, \quad (4.31)$$

where $Q$ is given by (4.9). Using (2.29), (2.30), (2.38), (2.39) and (3.10), we obtain

$$|I_{pp}(z)| \leq C\|q_1\|\|\lambda\|^{-\frac{1}{6}}\left(p_0(f, \lambda) + p_1(f, \lambda) \log(|\lambda| + 1)\right), \quad z \in \Gamma_\lambda^-. \quad (4.32)$$

Due to (2.46), (4.14) and (4.15) we have

$$|I_{pp}(z)| \leq C\|q_1\|\|\lambda\|^{-\frac{1}{6}}\left(p_0(f, \lambda) + p_1(f, \lambda) \log(|\lambda| + 1)\right), \quad z \in \Gamma_\lambda^-. \quad (4.32)$$

We estimate $I'_{pp}(z) = -a(z\omega)a(z\overline{\omega})V_q(z)f_p(z)$. Using (2.29), (2.30) and (2.38) we have

$$|I'_{pp}(z)| \leq C\|q_1\|\|f_p(z)\|\|\lambda\|^{-\frac{1}{3}}z^{-\frac{1}{2}}, \quad z \in \Gamma_\lambda^-. \quad (4.33)$$

The estimate of $I_{ee}$ is similar. Using $F_e$ and $Q$ we integrate by parts. Using (2.29), (2.30), (2.38), (2.39) and (3.10), we obtain

$$|I_{ee}(z)| \leq C\|q_1\|\|\lambda\|^{-\frac{1}{6}}\left(|f_e(z)| + \int_{\Gamma_\lambda} \left\{|f'_e(s)| + \left|\rho(s)\right|^{-\frac{1}{2}} + \left|\lambda\right|^{-\frac{1}{3}}|f_e(s)|\right\} ds\right).$$
We have \(|e^{-\frac{4}{3}z^\frac{3}{2}}| \geq 1\) for \(z \in \Gamma^{-}_{\lambda} \subset S[-\pi, -\frac{\pi}{3}]\). Using (2.46), (2.34) and (4.15) we have

\[
|I_{ee}(z)| \leq C \frac{\|q\|_\infty}{|\lambda|^\frac{3}{2}} e^{-\frac{4}{3}z^\frac{3}{2}} \left( p_0(f, \lambda) + p_1(f, \lambda) \log(|\lambda| + 1) \right), \quad z \in \Gamma^{-}_{\lambda} \subset S[-\pi, -\frac{\pi}{3}]. \tag{4.34}
\]

We estimate \(I'_{ee}(z) = -a(z\omega) a(z\overline{\omega}) V_q(z) f_e(z)\). Using (2.29), (2.30) and (2.38) we obtain

\[
|I'_{ee}(z)| \leq C \|q\|_\infty |f_e(z)| |\lambda|^{-\frac{3}{2}} |z|^{-\frac{1}{2}}, \quad z \in \Gamma^{-}_{\lambda} \subset S[-\pi, -\frac{\pi}{3}]. \tag{4.35}
\]

In order to estimate \(I_{pe}\) we use \(P_+(s, z)\), given by (4.27). Integrating by parts we have

\[
I_{pe}(z) = -P_+(z_*, z) F(z_*) + \int_{\Gamma_\lambda(z_*, z_*)} P_+(s, z) F'(s) \, ds, \quad F(z) = \lambda^{-\frac{3}{4}} a(z\omega)^2 f_e(z) \rho(z).
\]

Using (2.27), (2.38), (2.39), (2.29) and (2.30) we obtain

\[
|I_{pe}(z)| \leq C \frac{\|q\|_B |\lambda|^{-\frac{3}{2}}}{|\lambda|^{\frac{3}{2}}} \left[ |f_e(z_*)| |e^{\frac{4}{3}z^\frac{3}{2}}| + \int_{\Gamma_\lambda} |e^{\frac{4}{3}z^\frac{3}{2}}| \left\{ \left| \frac{f'_p(s)}{\langle s \rangle^{\frac{3}{2}}} \right| + |f_p(s)| \left( \langle s \rangle^{-\frac{3}{2}} + \frac{|\lambda|^{-\frac{3}{2}}}{\langle s \rangle^{\frac{3}{2}}} \right) \right\} |ds| \right].
\]

We have \(|e^{\frac{4}{3}z^\frac{3}{2}}| \leq 1\) for \(s \in \Gamma^{-}_{\lambda} \subset S[-\pi, -\frac{\pi}{3}]\). Using (2.46) and (4.14) (4.15) we obtain

\[
|I_{pe}(z)| \leq C \|q\|_B |\lambda|^{-\frac{3}{2}} \left( p_0(f, \lambda) + p_1(f, \lambda) \log(|\lambda| + 1) \right), \quad z \in \Gamma^{-}_{\lambda}. \tag{4.36}
\]

We estimate \(I'_{pe}(z) = -a(z\omega)^2 e^{\frac{4}{3}z^\frac{3}{2}} V_q(z) f_e(z)\). Using Lemma 2.3 (2.29), (2.30) and (2.38) we get

\[
|I'_{pe}(z)| \leq C \|q\|_\infty |e^{\frac{4}{3}z^\frac{3}{2}} f_e(z)| |\lambda|^{-\frac{3}{2}} |z|^{-\frac{1}{2}}, \quad z \in \Gamma^{-}_{\lambda}. \tag{4.37}
\]

In order to estimate \(I_{ep}\) we use \(P_-(s, z)\), given by (4.27). Integrating by parts we have

\[
I_{ep}(z) = -P_-(z_*, z) F(z_*) + \int_{\Gamma_\lambda(z_*, z_*)} P_-(s, z) F'(s) \, ds, \quad F(z) = \lambda^{-\frac{3}{4}} a(z\overline{\omega})^2 f_p(z) \rho(z).
\]

Using (2.27), (2.38), (2.39), (2.29) and (2.30) we obtain

\[
|I_{ep}(z)| \leq C \frac{\|q\|_B |\lambda|^{-\frac{3}{2}}}{|\lambda|^{\frac{3}{2}}} \left[ \left| f_p(z_*) e^{-\frac{4}{3}z^\frac{3}{2}} \right| + \int_{\Gamma_\lambda(z_*, z_*)} \left| e^{-\frac{4}{3}z^\frac{3}{2}} \right| \left\{ \left| \frac{f'_p(s)}{\langle s \rangle^{\frac{3}{2}}} \right| + |f_p(s)| \left( \langle s \rangle^{-\frac{3}{2}} + \frac{|\lambda|^{-\frac{3}{2}}}{\langle s \rangle^{\frac{3}{2}}} \right) \right\} |ds| \right].
\]

By Lemma 2.3 we have \(\max_{\Gamma_\lambda(z_*, z_*)} |e^{-\frac{4}{3}z^\frac{3}{2}}| = |e^{-\frac{4}{3}z^\frac{3}{2}}|\). Using (2.46), (4.14) and (4.15) we obtain

\[
|I_{ep}(z)| \leq C \|q\|_B |\lambda|^{-\frac{3}{2}} |e^{-\frac{4}{3}z^\frac{3}{2}}| \left( p_0(f, \lambda) + p_1(f, \lambda) \log(|\lambda| + 1) \right), \quad z \in \Gamma^{-}_{\lambda}. \tag{4.38}
\]
We estimate $I'_{cp}(z) = -a(z)\overline{\alpha} e^{-\frac{\alpha}{2} z^2} V_q(z) f_p(z)$. Using Lemma 2.3, 2.29, 2.30 and 2.38 we obtain

$$|I'_{cp}(z)| \leq C\|q\|_\infty |\omega|^{-\frac{1}{2}} |\omega|^\frac{3}{2} f_p(z) |\lambda|^{-\frac{1}{2}} (z)^{-\frac{1}{2}}, \quad z \in \Gamma_0^\lambda.$$  

Due to (4.30), (4.32), (4.34), (4.36) and (4.38) we have

$$p_0(J_q f, \lambda) = |(j_q f)| (z) + |e^{\frac{3}{2} z} (j_e q f)(z)| \leq C \frac{\|q\|_B}{|\lambda|^2} \left( p_0(f, \lambda) + p_1(f, \lambda) \log(|\lambda| + 1) \right),$$

where $z \in \Gamma_0^\lambda$. This proves (4.28). Due to (4.30), (4.33), (4.35), (4.37) and (4.39) we have

$$p_1(J_q f, \lambda) = |\partial_z (j_q f)(z)| + \left| e^{\frac{3}{2} z} \partial_z (j_e q f)(z) \right| \leq C \frac{\|q\|_\infty}{|\lambda|^2} (|f_p(z)| + |e^{\frac{3}{2} z} f_e(z)|),$$

where $z \in \Gamma_0^\lambda$. This proves (4.29).}

**Lemma 4.4.** Let $q \in B$ and $0 \leq \arg \lambda \leq \delta$, $|\lambda| \geq 1$. Assume $f = f^+ + f^-$, where $f^+ = f|_{\Gamma_0^\lambda} \in \mathcal{F}_{\alpha, \beta}^\lambda$ for $\alpha > 0$, $\beta > \frac{3}{4}$ and $f^- = f|_{\Gamma_0^\lambda} \in \mathcal{F}_{-\alpha, -\beta}^\lambda$. Then $g = (J_q f)|_{\Gamma_0^\lambda} \in \mathcal{F}_{\alpha, \beta}^\lambda$ and for the decomposition $g(z) = a(z) (j_q q f)(z) + e^{\frac{3}{2} z} a(z \omega) (j_e q f)(z)$ the following estimates are fulfilled:

$$p_0(g, \lambda) \leq C \frac{\|q\|_B}{|\lambda|^2} \left( p_0(f^-, \lambda) + p_1(f^-, \lambda) \log(|\lambda| + 1) + \|f^+\|_{\alpha, \beta} \right) + \frac{C}{|\lambda|^2} p_0(f^-, \lambda),$$

$$p_1(g, \lambda) \leq C |\lambda|^{-\frac{1}{2}} (\|q\|_\infty + 1) p_0(f^-, \lambda).$$

**Proof.** By (4.26), we have $g = g_0 + g_+ + g_+$, where

$$g_+ = a(z) h_p(q f^+) + e^{\frac{3}{2} z} a(z \omega) h_e(q f^+),$$

$$g_+ = a(z) (j_q q f^+)(z) + e^{\frac{3}{2} z} a(z \omega) (j_e q f^+)(z),$$

$$g_0 = a(z) (j_q q f)(z) + e^{\frac{3}{2} z} a(z \omega) (j_e q f)(z).$$

Firstly, we estimate $g_+$. From (3.15) and (3.25) we have

$$|h_p(q f)| \leq C \|q\|_\infty |\lambda|^\frac{1}{2} |f^+|_{\alpha, \beta}, \quad |h_e(q f)| \leq C (\|q\|_\infty |\omega|^{-\frac{1}{2}} + \|q\|_\infty) |\lambda|^\frac{1}{2} |f^+|_{\alpha, \beta}.$$  

By Lemma 2.3 we have $|e^{\frac{3}{2} z} a(z \omega)| \leq 1$ for $z \in \Gamma_0^\lambda$. Thus

$$p_0(g_+, \lambda) \leq C (\|q\|_\infty + \|q\|_\infty) |\lambda|^\frac{1}{2} |f^+|_{\alpha, \beta}, \quad p_1(g_+, \lambda) = 0.$$  

Secondly, we estimate $g_0$. Using (4.21), (4.24), (2.29), (2.30), (2.38), (2.47) and Lemma 2.3 gives

$$|(j_q q f)| (z) \leq C \int_{\Gamma_0^\lambda} \frac{p_0(f^-, \lambda) |ds|}{|\lambda|^\frac{1}{2} + |s|^2 s} \frac{ds|}{|\lambda|^\frac{1}{2}} \leq C \frac{p_0(f^-, \lambda)}{|\lambda|^\frac{1}{2}} , \quad z \in \Gamma_0^\lambda.$$  

21
By Lemma 2.3, we have

\[ |(j_e V_0 f)(z)| \leq C|e^{-\frac{\mu}{2}z^2}| \int_{\Gamma^*} \frac{p_0(f^-, \lambda)}{|\lambda|^\frac{\mu}{2} + |s|^2 \langle s \rangle^\frac{\mu}{2}} \leq C|e^{-\frac{\mu}{2}z^2}| \frac{p_0(f^-, \lambda)}{|\lambda|^\frac{\mu}{2}} \quad z \in \Gamma^* . \quad (4.45) \]

Using Lemma 2.3, 3.29 and 3.30 we have

\[ |h_p(0V_0 f)| \leq C|\lambda|^{-\frac{\mu}{2}} \|f^+\|_{\alpha}, \quad |h_e(0V_0 f)| \leq C|\lambda|^{-\frac{\mu}{2}} |f^+|_{\alpha} . \quad (4.46) \]

By Lemma 2.3 we have \(|e^{\frac{1}{2}|z|^2 - \frac{\mu}{2}z}| \leq 1\) for \(z \in \Gamma^*\). Thus substituting (4.44), (4.45) and (4.46) in (1.26) we obtain

\[ p_0(g_0, \lambda) \leq C(p_0(f^-, \lambda) + \|f^+\|_{\alpha})|\lambda|^{-\frac{\mu}{2}} . \quad (4.47) \]

Using (4.21), (4.22), (2.38), (2.29), (2.30) and (4.13) we obtain

\[ |\partial_z (j_e V_0 f)(z)| \leq C \frac{p_0(f^-, \lambda)}{|\lambda|^\frac{\mu}{2} \langle z \rangle^\frac{\mu}{2}}, \quad |\partial_z (j_e V_0 f)(z)| \leq C \frac{e^{-\frac{\mu}{2}z^2}}{|\lambda|^\frac{\mu}{2}} \frac{p_0(f^-, \lambda)}{\langle z \rangle^\frac{\mu}{2}} , \]

which proves

\[ p_1(g_0, \lambda) \leq C p_0(f^-, \lambda)|\lambda|^{-\frac{\mu}{2}} . \quad (4.48) \]

Finally, apply (4.28) to \(g_-\); together with (4.43) and (4.47) this gives (4.40). Applying (1.29) to \(g_-\) together with (4.43) and (4.48) gives (4.41). \(\square\)

**Proof of Theorem 4.1.** We consider the case \(0 \leq \arg \lambda \leq \delta\), the proof for \(-\delta \leq \arg \lambda \leq 0\) is similar. Let \(v_{n+1} = JV_n, n \geq 0\), where \(v_0 \equiv a\). Introduce \(v_n^\pm = v_n|_{\Gamma^\pm}\). By (3.31, 3.32), for some absolute constant \(c_0 > 0\) we have

\[ \|v_n^+\|_{\alpha_n, \beta_n, \pm} \leq \varepsilon \|v_n^+\|_{\alpha_n, \beta_n, \pm} . \]

where \(\alpha_n\) and \(\beta_n\) are given by (3.33). We estimate \(v_n^-\) in terms of \(p_0\) and \(p_1\), using the decomposition (4.16) for \(v_0^-\) and (4.18), (4.19) for \(v_n^-\), \(n \geq 1\). Substituting \(f = v_n\) and \(g = v_{n+1}\) in (4.41), (4.44) and choosing \(c_0\) sufficiently large, we obtain

\[ p_0(v_{n+1}^-, \lambda) \leq \varepsilon \left( p_0(v_n^-, \lambda) + p_1(v_n^-, \lambda) \log(|\lambda| + 1) + \|v_n^+\|_{\alpha_n, \beta_n} \right), \quad p_1(v_{n+1}^-) \leq \varepsilon \frac{p_0(v_n^-, \lambda)}{|\lambda|^\frac{\mu}{2}} . \]

Using (3.8) and (4.17) (in particular, \(p_1(v_0, \lambda) = 0\)) we obtain for \(v_1\) and \(v_2\)

\[ p_0(v_1^-, \lambda) \leq \varepsilon L, \quad p_1(v_1^-, \lambda) \leq \varepsilon |\lambda|^{-\frac{\mu}{2}} L, \quad L = 2 + \|v_0\|_{\alpha} , \]

\[ p_0(v_2^-, \lambda) \leq \varepsilon \left( \varepsilon L + \varepsilon L|\lambda|^{-\frac{\mu}{2}} \log(|\lambda| + 1) + \|v_1^+\|_{\alpha_1, \beta_1} \right) \leq \varepsilon^2 \left( 2 + |\lambda|^{-\frac{\mu}{2}} \log(|\lambda| + 1) \right) L , \]

\[ p_1(v_2^-) \leq \varepsilon^2 |\lambda|^{-\frac{\mu}{2}} L . \]
Increasing the constant $\epsilon_0$ and using the induction principle we obtain for each integer $n \geq 0$

$$p_0(v_n^-, \lambda) \leq \epsilon^n L, \quad p_1(v_n^-, \lambda) \leq \frac{1}{|\lambda|^{\frac{1}{4}}} \epsilon^n L, \quad \|v_n^+\|_{\alpha_n, \beta_n} \leq \epsilon^n \|v_0\|_{\frac{1}{4}, \frac{1}{2}}. \quad (4.49)$$

By Theorem 3.1 for $\epsilon < 1$ the series $v_n^+ = \sum_{n=0}^{\infty} v_n^+$ converges in $F_{\frac{1}{4}, 1}$-norm and gives a solution of $v_+ = v_0 + Jv_+$ on $\Gamma_\lambda^+$. By (4.49), for $\epsilon < 1$ the series $\sum_{n=0}^{\infty} p_0(v_n^-, \lambda)$ and $\sum_{n=0}^{\infty} p_1(v_n^-, \lambda)$ converge; for $\epsilon \leq \frac{1}{2}$ the following estimates are fulfilled:

$$p_0(v_+|_{\Gamma_\lambda^-}, \lambda) \leq 2L, \quad p_1(v_+|_{\Gamma_\lambda^-}, \lambda) \leq \frac{2L}{|\lambda|^{\frac{1}{4}}}, \quad p_0(v_1^-, \lambda) \leq 2\epsilon L, \quad p_1(v_1^-, \lambda) \leq 2\epsilon L, \quad (4.50)$$

$$p_0\left(v_+|_{\Gamma_\lambda^-} - v_0^- - v_1^-, \lambda\right) \leq 2\epsilon^2 L, \quad p_1\left(v_+|_{\Gamma_\lambda^-} - v_0^- - v_1^-, \lambda\right) \leq 2|\lambda|^{-\frac{1}{4}}\epsilon^2 L. \quad (4.51)$$

By (4.13), convergence of $\sum_{n=0}^{\infty} p_0(v_n^-, \lambda)$ and $\sum_{n=0}^{\infty} p_1(v_n^-, \lambda)$ implies convergence of $\sum_{n=0}^{\infty} v_n^-$ in $C^1$-norm on $\Gamma_\lambda^-$. Thus for $\epsilon < 1$ the series $v_+ = \sum_{n=0}^{\infty} v_n^-$ converges and solves the equation $v_+ = v_0 + Jv_+$ on $\Gamma_\lambda^-$. Using (4.13) and (4.49) we obtain (4.1) and (4.2) from (4.50); similarly we obtain (4.3) from (4.51).

We prove uniqueness. Suppose that there exists another solution $v_+^{(1)} \in F_\lambda = F_\lambda^- + F_{\frac{1}{4}, 1}^\lambda$. By Theorem 3.1 we have $v_+^{(1)}|_{\Gamma_\lambda^+} = v_+|_{\Gamma_\lambda^+}$. Consider the difference $y = (v_+ - v_+^{(1)})|_{\Gamma_\lambda^-} \in F_\lambda^-$. We have $y = (Jv)^n y$ for any $n > 1$; applying Lemma 4.13 we obtain $p_0(y, \lambda) \leq C\epsilon^n (p_0(y, \lambda) + p_1(y, \lambda))$. Taking the limit $n \to \infty$ for $\epsilon < 1$ gives $y = 0$. \hfill \blacksquare

## 5 Asymptotics of the Wronskian

In this section we shall determine the asymptotics of $w(\lambda) = \{\psi_-, \psi_+\}$ as $\lambda \to \infty$. To this end we find the asymptotics of $u_1(z, \lambda) = e^{-\frac{i}{2} \frac{\psi_-}{4}} (Jv_0)(z)$ (given by Corollary 4.2) as $\lambda \to \infty$ in the sector $|\arg \lambda| \leq \delta$. Introduce an auxiliary function

$$E_+(\lambda) = 2 \int_{\Gamma_\lambda^-} \text{Ai}(s\omega) \text{Ai}(s\bar{\omega}) V_\delta(s) ds, \quad \omega = e^{\frac{2\pi i}{3}}. \quad (5.1)$$

By (2.27), we have

$$\frac{2}{3} z_0^{\frac{3}{4}} = \pm i \frac{\pi}{4} \lambda, \quad 0 \leq \pm \arg \lambda \leq \pi, \quad z_0 = z(0, \lambda) = -\lambda^\frac{2}{3} \left(\frac{3\pi}{8}\right)^{\frac{2}{3}}. \quad (5.2)$$

**Lemma 5.1.** 1. Let $q \in B$ and $|\arg \lambda| \leq \delta, |\lambda| \to \infty$. Then

$$u_+(z_0, \lambda) = z_0^{-\frac{1}{4}} \left[\sin \frac{\pi}{4} (\lambda + 1) \left(1 + O(\lambda^{-\frac{1}{6}})\right) + E_+(\lambda) \cos \frac{\pi}{4} (\lambda + 1)\right] + O \left(\frac{e^{\frac{2\pi i}{3} \text{Im} \lambda}}{\lambda^\frac{2}{3}}\right), \quad (5.3)$$
Using (4.16), (2.32), (2.33), (3.6), (4.9) and (4.20–4.25) we obtain

\[
\partial_\bar{z} u_+(z_0, \lambda) = z_0^{\frac{1}{2}} \left[ -\cos \frac{\pi}{4}(\lambda + 1) \left( 1 + O(\lambda^{-\frac{1}{2}}) \right) + E_+(\lambda) \sin \frac{\pi}{4}(\lambda + 1) \right] + O\left( \frac{e^{\frac{\pi}{4}|\text{Im}\lambda|}}{\lambda^{\frac{1}{2}}} \right),
\]

(5.4)

2. Let \( q \in \mathcal{B} \) and \( \delta \leq \pm \arg \lambda \leq \pi, |\lambda| \to \infty \). Then

\[
u_+(z_0, \lambda) = \frac{e^{2 \pi i (\lambda+1)}}{2z_0^{\frac{1}{2}}} + O\left( \frac{e^{\frac{\pi}{4}|\text{Im}\lambda|}}{\lambda^{\frac{1}{2}}} \right), \quad \partial_\bar{z} u_+(z_0, \lambda) = -\frac{z_0^{\frac{1}{2}}}{2} e^{2 \pi i (\lambda+1)} + O\left( \frac{e^{\frac{\pi}{4}|\text{Im}\lambda|}}{\lambda^{\frac{1}{2}}} \right).
\]

(5.5)

**Proof.** We present the proof only for \( \text{Im} \lambda \geq 0 \), for \( \text{Im} \lambda \leq 0 \) it is analogous.

1. Let \( 0 \leq \arg \lambda \leq \delta \). We have

\[
u_1(z, \lambda) = e^{-\frac{\pi}{2}z^{-\frac{3}{2}}} v_1(z) = e^{-\frac{\pi}{2}z^{-\frac{3}{2}}} (J V_0)(z) = \text{Ai}(z\omega)(J_p V_0)(z) + \text{Ai}(\omega)(J_{\omega} V_0)(z).
\]

By (4.47), \((J V_0)(z) = O(\lambda^{-\frac{1}{2}})\). Since \( V = V_q + V_0 \), we have as \( \lambda \to \infty \)

\[
u_1(z, \lambda) = e^{-\frac{\pi}{2}z^{-\frac{3}{2}}} v_1(z) = e^{-\frac{\pi}{2}z^{-\frac{3}{2}}} (J V_0)(z) + O\left( |\lambda|^{-\frac{3}{2}} \right) - e^{\frac{\pi}{4}|\omega| e^{-\frac{3}{4}|\text{Re} z^{-\frac{1}{2}}|}}.
\]

(5.6)

Using (4.16), (2.32), (2.33), (3.6), (4.9) and (4.20–4.25) we obtain

\[
\begin{aligned}
e^{-\frac{\pi}{4}z^{-\frac{3}{2}}} (J V_q)(z) &= \text{Ai}(z\omega)(J_p V_q)(z) + \text{Ai}(\omega)(J_{\omega} V_q)(z) \\
&= 2i\omega \text{Ai}(z) \int_{\Gamma_{\lambda}^{+}} a(s\omega) a(s) V_q(s) \, ds - 2i\omega \text{Ai}(\omega) \int_{\Gamma_{\lambda}^{+}} e^{-\frac{\pi}{4}s^{-\frac{3}{2}}} a^2(s) V_q(s) \, ds \\
&+ 2\text{Bi}(z) \int_{\Gamma_{\lambda}^{+}} a(s\omega) a(s\omega) V_q(s) \, ds - 2i\omega \text{Ai}(z\omega) \int_{\Gamma_{\lambda}^{+}} e^{\frac{\pi}{4}s^{-\frac{3}{2}}} a^2(s\omega) V_q(s) \, ds \\
&+ 2ie^{\frac{\pi}{4}z^{-\frac{1}{2}}} \text{Ai}(\omega) \int_{\Gamma_{\lambda}^{+}} e^{-\frac{\pi}{4}s^{-\frac{3}{2}}} a^2(s\omega) V_q(s) \, ds,
\end{aligned}
\]

where \( \omega = e^{\frac{\pi}{2}i}. \) Now we set \( z = z_0 \) and write the asymptotics of \( u_1 \) in terms of \( E_+ \). By Lemma 2.32, 3.6, 4.9 and 2.38, we have

\[
u_1(z_0, \lambda) = 2i\omega \text{Ai}(z_0) \int_{\Gamma_{\lambda}^{+}} a(s\omega) a(s) V_q(s) \, ds + \text{Bi}(z_0) E_+(\lambda) + O\left( \lambda^{-\frac{3}{2}} z_0^{-\frac{3}{4}} |e^{-\frac{\pi}{4}z^{-\frac{3}{2}}}| \right).
\]

(5.7)

Using (2.38) we obtain for the derivative

\[
\partial_\bar{z} u_1(z_0, \lambda) = 2i\omega \text{Ai}'(z_0) \int_{\Gamma_{\lambda}^{+}} a(s\omega) a(s) V_q(s) \, ds + \text{Bi}'(z_0) E_+(\lambda) + O\left( \lambda^{-\frac{3}{2}} z_0^{-\frac{3}{4}} |e^{-\frac{\pi}{4}z^{-\frac{3}{2}}}| \right).
\]

(5.8)

From Lemma 3.2, 4.6, 2.38 and 2.39 we deduce that

\[
\int_{\Gamma_{\lambda}^{+}} a(s\omega) a(s) V_q(s) \, ds = O(\lambda^{-\frac{1}{2}}). \]

Therefore using (4.16), (4.7), (2.32), (3.6), (5.7) and (5.8) we obtain

\[
u_+(z_0, \lambda) = \text{Ai}(z_0)(1 + O(\lambda^{-\frac{1}{2}})) + \text{Bi}(z_0) E_+(\lambda) + O\left( \lambda^{-\frac{1}{2}} e^{\frac{\pi}{4}|\text{Im}\lambda|} \right),
\]

(5.9)
\[ \partial_x u_+(z_0, \lambda) = \text{Ai}'(z_0)(1 + O(\lambda^{-\frac{4}{3}})) + \text{Bi}'(z_0)E_+(\lambda) + O\left(\lambda^{-\frac{4}{3}}e^{\frac{2}{3}\text{Im}\lambda}\right), \]

where \( z_0 \) is given by (2.27). Recall the standard uniform asymptotics of Airy functions [1] in the sector \(|\arg z| < \frac{2}{3} - \varepsilon, \varepsilon > 0\), as \( z \to \infty \):

\[ \text{Ai}(-z) = z^{-\frac{1}{3}}(\sin \eta + O(F(z))), \quad \text{Ai}'(-z) = -z^\frac{1}{3}(\cos \eta + O(F(z))), \]

\[ \text{Bi}(z) = z^{-\frac{1}{3}}(\cos \eta + O(F(z))), \quad \text{Bi}'(-z) = z^\frac{1}{3}(\sin \eta + O(F(z))), \]

where \( \eta = \frac{2}{3}z^\frac{2}{3} + \frac{\pi}{4} \) and \( F(z) = z^{-\frac{2}{3}}e^{\frac{2}{3}\text{Im}z^{\frac{2}{3}}} \). Note that for \( 0 \leq \arg \lambda \leq \delta \) we have \(-\pi \leq \arg z_0 \leq -\pi + \frac{\pi}{3} \). Thus we substitute (5.11) and (5.12) into (5.9 - 5.10). This gives (5.3) - (5.4).

2. Let \( \delta \leq \arg \lambda \leq \pi \). By (4.8), we have

\[ u_+(z_0, \lambda) = \text{Ai}(z_0) + O\left(\lambda^{-\frac{2}{3}}e^{\frac{2}{3}\text{Im}\lambda}\right), \quad u_+'(z_0, \lambda) = \text{Ai}'(z_0) + O\left(\lambda^{-\frac{2}{3}}e^{\frac{2}{3}\text{Im}\lambda}\right). \]

Note that for \( \delta \leq \arg \lambda \leq \pi \) we have \(-\pi + \frac{\pi}{3} \leq \arg z_0 \leq 0\). Thus we apply (2.31) and (5.2) to (5.13), which gives (5.5).]

Introduce the function

\[ \phi(\lambda) = 2^\frac{3}{4} \left(\frac{\lambda}{2e}\right)^{\lambda/4}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_-, \quad \phi(\lambda) > 0 \quad \text{for} \quad \lambda > 0. \]

In the next Lemma we write \( \psi_+ \), defined by (2.2), in terms of \( u_+ \).

**Lemma 5.2.** Let \( q \in \mathcal{B} \). Let \( \psi_+ \) be the solution of Eq. (2.4), satisfying (2.2); let \( u_+ \) be the solution of Eq. (2.34), given by Corollary 4.2. Then for \( |\lambda|^\frac{3}{2} \geq c_0 (\|q\|_s + 1) \), where \( c_0 \geq 1 \) is some absolute constant, the following identity holds:

\[ \psi_+(x, \lambda) = \phi_+(x, \lambda), \quad \text{where} \quad \phi_+(x, \lambda) = \phi(\lambda)\frac{u_+(z(x, \lambda), \lambda)}{\sqrt{z'(x, \lambda)}}, \quad x \geq 0. \]

**Proof.** The function \( \phi_+ \), given by (5.15), solves Eq. (2.1) by changing variables according to (2.26). In order to prove (5.15) it is sufficient to demonstrate that \( \phi_+ \) has the asymptotics (2.2). Using (2.21), (2.26), (2.31), Corollary 4.2 and (5.2), we have for \( x \to \infty \)

\[ \phi_+(x, \lambda) = e^{-x^2/2} (\sqrt{2\pi})^{1/2} \left(1 + O\left(\frac{1}{x^3}\right)\right), \quad \frac{\partial \phi_+(x, \lambda)}{\partial x} = -\frac{e^{-x^2/2} (\sqrt{2\pi})^{1/2}}{\sqrt{2}} \left(1 + O\left(\frac{1}{x^3}\right)\right), \]

which proves (5.13). ]

In order to obtain the asymptotics of the Wronskian \( w(\lambda) = \{\psi_-, \psi_+\} \) we need also the asymptotics of the fundamental solution \( \psi_- \) (see Theorem 2.1). We use our results for \( x > 0 \) and consider the reflected potential \( q_-(x) = q(-x) \) for \( x \in \mathbb{R}_+ \). We define

\[ V_q^-(z) = \lambda^{-\frac{4}{3}}\rho^2(z, \lambda)\tilde{q}_-(z), \quad \tilde{q}_-(z, \lambda) \equiv q_-(\sqrt{\lambda}(z\lambda^{-\frac{4}{3}})), \]

25
where $\rho$ is given by (2.29). Let $u_-(z, \lambda) = u_+(z, \lambda; V_q^-)$, where $u_+$ is given by Corollary 4.2 (with $V_q$ replaced by $V_q^-$). By Lemma 5.2, the fundamental solution $\psi_-$ is related to $u_-$ by
\[
\psi_-(x, \lambda) = \phi(\lambda) \frac{u_-(x, \lambda, \lambda)}{\sqrt{z'(x, \lambda)}}, \quad x \geq 0.
\]
(5.16)

Introduce $E_-(\lambda)$ by
\[
E_-(\lambda) = 2 \int_{\Gamma_\lambda^{-}} \operatorname{Ai}(s \omega) \operatorname{Ai}(s \overline{\omega}) V_q^-(s) \, ds, \quad \omega = e^{\frac{2\pi i}{3}}.
\]
(5.17)

By Lemma 5.1 for $|\arg \lambda| \geq \delta$ the solution $u_-(z_0, \lambda)$ has the asymptotics (5.5); for $|\arg \lambda| \leq \delta$ it has the asymptotics (5.3), (5.4) with $E_+$ replaced by $E_-:
\[
\begin{align*}
\psi_-(z, \lambda) &= z_0^{-\frac{3}{2}} \left[ \sin \frac{\pi}{4} (\lambda + 1) \left( 1 + O(\lambda^{-\frac{5}{8}}) \right) \right] + O \left( \lambda^{-\frac{5}{2}} e^{\frac{3}{4} |\text{Im}\lambda|} \right), \\
\psi'_-(z, \lambda) &= z_0^\frac{1}{2} \left[ -\cos \frac{\pi}{4} (\lambda + 1) \left( 1 + O(\lambda^{-\frac{5}{8}}) \right) \right] + O \left( \lambda^{-\frac{3}{2}} e^{\frac{3}{4} |\text{Im}\lambda|} \right).
\end{align*}
\]
(5.18)

Below we use the function $E = E_- + E_+$.

**Lemma 5.3.** Let $q \in \mathcal{B}$. Then for $|\lambda| \to \infty$ the following uniform asymptotics hold:
\[
w(\lambda) = \frac{\phi^2(\lambda)}{2} \left[ e^{\frac{3}{8} i \pi} + O \left( \frac{e^{\frac{3}{4} |\text{Im}\lambda|}}{\lambda^{\frac{1}{2}}} \right) \right] = \phi^{-2}(\lambda) \left( 4 + O(\lambda^{-\frac{5}{8}}) \right), \quad \delta \leq \pm |\arg \lambda| \leq \pi.
\]
(5.20)

**Proof.** Using (2.26), (5.15) and (5.16) we obtain
\[
w(\lambda) = -\phi^2(\lambda) \left( u_+(z_0, \lambda) u'_-(z_0, \lambda) + u'_+(z_0, \lambda) u_-(z_0, \lambda) \right) - \phi^2(\lambda) \frac{u_+(z_0, \lambda) u_-(z_0, \lambda)}{16 \lambda k^2(0)}.
\]
Substituting (5.3), (5.4) and (5.18), (5.19) in the last identity, we obtain (5.20); substituting (5.3) and the same formulae for $u_-$, we obtain (5.21). $\blacksquare$

### 6 The proof of Theorem 1.1

Using (5.14) we write the following uniform asymptotics of the unperturbed Wronskian $w_0(\lambda) = \{\psi^0_0, \psi^0_+\} = -\frac{2\sqrt{\pi}}{1 + \frac{1}{2}}$ (see, for example, [1]):
\[
w_0(\lambda) = \phi^2(\lambda) \cos \frac{\pi \lambda}{2} \cdot (1 + O(\lambda^{-1})), \quad |\arg \lambda| \leq \delta
\]
(6.1)
\[
w_0(\lambda) = \phi^{-2}(-\lambda) \left( 4 + O(\lambda^{-1}) \right), \quad |\arg \lambda| \geq \delta.
\]
(6.2)
Lemma 6.1. Let $q \in \mathcal{B}$. Then there is $N_0 \in \mathbb{Z}$ such that for each integer $N > N_0$ the operator $T$ has exactly $N$ simple eigenvalues in the disc $\{ z : |z| < 2N \}$ and for each $n > N$, exactly one simple eigenvalue in the disc $\{ z : |z - \mu_n^0| < n^{-\frac{1}{\delta}} \}$. There are no other eigenvalues.

Proof. Consider the contours $|\lambda| = 2n$, $|\lambda - \mu_n^0| = \delta_n$, $n > N$, $n \in \mathbb{N}$, where $\delta_n = n^{-\frac{1}{\delta}} < \frac{\log 2}{n}$. Then $|\cos \frac{\pi}{2} \lambda| \geq 4e^{\frac{\pi}{2} |\text{Im} \lambda|}$ for $|\lambda| = 2n$ and $\frac{1}{\pi} \cos \frac{\pi}{2} |\lambda| \geq 4e^{\frac{\pi}{2} |\text{Im} \lambda|}$ for $|\lambda - \mu_n^0| = \delta$, $\delta < \frac{\log 2}{n}$. By the asymptotics (5.21), (5.22), (5.24) and (6.1), there exist integer $N_0 > \left( \frac{\pi}{\log 2} \right)^6$ such that for any integer $N > N_0$ on these contours $|w(\lambda) - w_0(\lambda)| \leq \frac{1}{2} |w_0(\lambda)|$. It follows that $w(\lambda)$ does not vanish on these contours. Hence, by Rouche’s theorem, $w(\lambda)$ has as many roots, counted with multiplicities, as $w_0(\lambda)$ in each of the bounded regions and in the remaining unbounded region. Since $w_0(\lambda)$ has only the simple roots $\{ \mu_n^0 \}_{n=0}^\infty$, the Lemma is proved.

Lemma 6.2. Let $q \in \mathcal{B}$. Then for $|\lambda| \to \infty$ the following asymptotics are fulfilled:

$$E(\lambda + \varepsilon) - E(\lambda) = O \left( \lambda^{-\frac{1}{\delta}} \right), \quad \varepsilon = O \left( \lambda^{-\frac{1}{\delta}} \right), \quad -\frac{\delta}{2} \leq \arg \lambda \leq \frac{\delta}{2},$$

(6.3)

$$E(\lambda) = -\frac{1}{2} \int_{-1}^{1} \frac{q(t \sqrt{\lambda}) dt}{\sqrt{1 - t^2}} + O \left( \|q\|_{\mathcal{B}} \lambda^{-\frac{1}{\delta}} \right), \quad E(\lambda) = O \left( \|q\|_{\mathcal{B}} \lambda^{-\frac{1}{\delta}} \right), \quad \lambda > 0.$$  

(6.4)

Proof. Consider the case $0 \leq \arg \lambda \leq \delta$; for $-\delta \leq \arg \lambda \leq 0$ the proof is analogous. Consider $E_+$, given by (5.1). Recall that $t_* = \frac{\sqrt{\lambda}}{\lambda}$, where $z_*$ is defined by $z_* = z(x_*, \lambda)$ (see Lemma 222 and below). We have $\lambda = |\lambda| e^{\pi i \delta}$. For some $c \in [0, 1]$ we set $t_1 = t_* - \frac{c}{\sqrt{\lambda}} e^{-i \delta}$, $x_1 = \frac{t_1}{\sqrt{\lambda}}$ and $z_1 = z_1(x_1, \lambda)$. The length of $\Gamma_{\lambda}(z_1, z_*)$ is $|\Gamma_{\lambda}(z_1, z_*)| = |\lambda|^{\frac{1}{\delta}} \int_{t_1, t_*} |k(t)| dt$

By Lemma 7.34, $|t_*|$ is bounded uniformly in $\lambda$. Thus using (2.21) we conclude that $|\Gamma_{\lambda}(z_1, z_*)| \leq C$. Therefore using (2.29), (2.30), (2.36), (3.0) and (1.3) gives

$$\left| \int_{\Gamma_{\lambda}(z_1, z_*)} \text{Ai}(s \bar{\omega}) \text{Ai}(s \omega) V_q(s) ds \right| \leq C \cdot \|q\|_{\mathcal{B}} \frac{1}{|\lambda|^\frac{1}{\delta}}.$$  

(6.5)

Next, using the asymptotics (2.31) and Lemma 7.36, we have uniformly in $|\arg \lambda| \leq \delta$

$$\left| \text{Ai}(z \omega) \text{Ai}(z \bar{\omega}) + \frac{i}{4 \pi z^{\frac{3}{2}}} \right| \leq \frac{C}{|z|^{\frac{3}{2}} (1 + |z|)^{\frac{1}{2}}}, \quad z \in \Gamma_{\lambda}^{-} \subset S[\pi, -\frac{\pi}{3}].$$

(6.6)

Substituting (6.3) and (6.6) in (5.1), we have

$$E_+(\lambda) = 2 \pi \int_{\Gamma_{\lambda}(z_0, z_1)} \text{Ai}(s \omega) \text{Ai}(s \bar{\omega}) V_q(s) ds + O \left( \|q\|_{\mathcal{B}} \frac{1}{|\lambda|^\frac{1}{\delta}} \right) = -\frac{i}{2} \int_{\Gamma_{\lambda}(z_0, z_1)} V_q(s) s^{-\frac{1}{\delta}} ds + O \left( \|q\|_{\mathcal{B}} \frac{1}{|\lambda|^\frac{1}{\delta}} \right).$$

(6.7)

Making the substitution $s = k(t) \cdot \lambda^{\frac{3}{2}}$, $ds = \lambda^{\frac{3}{2}} k'(t) dt$ and using (2.29) we obtain

$$E_+(\lambda) = -\frac{i}{2} \int_{e^{-\imath \theta_0 t_1}} \frac{q(\sqrt{\lambda} t)}{\sqrt{k(t) k'(t)}} dt + O \left( \|q\|_{\mathcal{B}} \frac{1}{|\lambda|^\frac{1}{\delta}} \right).$$

(6.8)
Using (2.17) and (2.19) we have \( k'(t)\sqrt{k(t)} = \sqrt{t^2 - 1} = i\sqrt{1 - t^2} \), where the last root is positive for \(-1 < t < 1\). Hence

\[
E_+(\lambda) = -\frac{1}{2} \int_0^d \frac{q(t|\lambda|^{1/2})}{\sqrt{e^{2i\theta} - t^2}} dt + O\left(\frac{\|q\|}{|\lambda|^{1/2}}\right), \quad d = 1 - \frac{c}{|\lambda|^{1/2}}.
\]

Using similar arguments for \( E_- \), we obtain

\[
E(\lambda) = E_c(\lambda) + O\left(\frac{\|q\|}{|\lambda|^{1/2}}\right), \quad E_c(\lambda) = -\frac{1}{2} \int_{-d}^d \frac{q(t\sqrt{|\lambda|})}{\sqrt{e^{2i\theta} - t^2}} dt. \tag{6.9}
\]

In order to prove (6.3) we set \( c > 0 \). We estimate the partial derivatives of \( E_c(\lambda) \) with respect to real and complex parts of \( \lambda = \mu + i\nu = |\lambda|e^{2i\theta} \). This gives \( \|\partial_\mu E_c(\lambda)\| \leq C \|\nu\| |\lambda|^{-1/2} \), \( \|\partial_\nu E_c(\lambda)\| \leq C \|\mu\| |\lambda|^{-1/2} \). Therefore as \( \lambda \to \infty \)

\[
|E_c(\lambda + \varepsilon) - E_c(\lambda)| \leq \varepsilon \cdot \sup_{|\lambda' - \lambda| \leq \varepsilon} |\nabla E_c(\lambda')| \leq C \|q\| |\lambda|^{-5/6}, \quad \varepsilon = O\left(|\lambda|^{-5/6}\right),
\]

which together with (6.9) proves (6.3).

Setting \( c = 0 \) and \( \lambda > 0 \) in (6.9) proves the first asymptotics in (6.4). In order to prove the second one we use the decomposition \( E_0(\lambda) = E_1(\lambda) + I_2(\lambda), \lambda > 0 \), where

\[
I_2(\lambda) = -\frac{1}{2} \int_{1-\lambda^{-1/2}}^{\lambda^{1/2}} \frac{q(t\sqrt{\lambda}) + q(-t\sqrt{\lambda})}{\sqrt{1 - t^2}} dt.
\]

Using \( \int_{\lambda^{1/2}}^{\lambda^{-1/2}} q(x) dx = q_1(\lambda^{1/2} - 1) - q_1(\lambda^{-1/2} t) \) we integrate the expression for \( E_1 \) by parts. This gives \( |E_1(\lambda)| \leq C \|q_1\| |\lambda|^{-1/2} \). Direct estimate gives \( |I_2(\lambda)| \leq \|q_1\| \lambda^{-1/2} \). Combining these estimates proves the second relation in (6.4).

**Proof of Theorem 1.1** By Lemma 6.1 in each disc \( D_n = \{ \lambda : |\lambda - \mu_0^n| \leq \frac{1}{n^2} \} \) there exists exactly one simple eigenvalue \( \mu_n \) for \( n \) sufficiently large; now we improve this estimate. Using the asymptotics (5.20) and \( \sin(\pi/2) \lambda | \geq \frac{1}{2} \) for \( \lambda \in D_n \), we obtain

\[
\cot(\pi/2 \mu_n) + E(\mu_n) + O\left(n^{-1/2}\right) = 0, \quad n \to \infty. \tag{6.10}
\]

Using (6.4), we write

\[
\mu_n - \mu_0^n = -\frac{2}{\pi} E(\mu_0^n) + \varepsilon_n, \quad \varepsilon_n = O\left(n^{-1/6}\right), \quad n \to \infty. \tag{6.11}
\]

Substituting (6.11) into (6.10) we have

\[
\frac{\pi}{2} \varepsilon_n + E(\mu_n) - E(\mu_0^n) = O\left(n^{-1/3}\right). \tag{6.12}
\]
Similarly, we have

\[ E(\mu_n) - E(\mu_n^0) = O(n^{-\frac{1}{4}}). \]

Thus in (6.11), the error term \( \varepsilon_n \) is \( O(n^{-\frac{1}{4}}) \), which yields \( \mu_n = \mu_n^0 - \frac{1}{2} E(\mu_n^0) + O(n^{-\frac{1}{4}}) \). The change of variable \( t = \sin \theta \) gives \( \mu_n^1 = -\frac{1}{2} E(\mu_n^0) = (2\pi)^{-1} \int_{-\pi}^{\pi} q(\mu_n^0 \sin \theta) \, d\theta \). This proves (1.2).

**Proof of Proposition 1.2.** Substituting \( q(x) = \int_{\mathbb{R}} e^{ixt} \, dv(t) \) into (6.9) and using the identity for Bessel function \( J_0(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{iz \sin \varphi} \, d\varphi \) (see [1]) we have

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} q(\sqrt{\lambda} \sin \varphi) \, d\varphi = \int_{\mathbb{R}} J_0(t\sqrt{\lambda}) \, dv(t) = I_1 + I_2, \quad \lambda > 0, \tag{6.13}
\]

where

\[
I_1 = \int_{|t| < \varepsilon} J_0(t\sqrt{\lambda}) \, dv(t), \quad I_2 = \int_{|t| > \varepsilon} J_0(t\sqrt{\lambda}) \, dv(t), \quad \varepsilon = \lambda^{-\frac{3}{4}}, \quad \frac{3}{4p} < \frac{1}{2}. \tag{6.14}
\]

Using \( J_0(-z) = J_0(z) \) and the asymptotics (see [1])

\[
J_0(z) = \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{\pi}{4} \right) + O \left( \frac{e^{\text{Im} z}}{z^{\frac{3}{2}}} \right), \quad |\text{arg } z| \leq \frac{\pi}{2},
\]

we obtain

\[
I_2 = \frac{1}{\lambda^{\frac{3}{4}}} \int_{|t| > \varepsilon} \sqrt{\frac{2}{\pi |t|}} \cos \left( \sqrt{\lambda} |t| - \frac{\pi}{4} \right) \, dv(t) + \frac{O(1)}{\lambda^{\frac{3}{4}}} \int_{|t| > \varepsilon} \frac{dv(t)}{t^{\frac{3}{2}}}, \tag{6.15}
\]

where the last term is \( O(\lambda^{-\frac{3}{4}}) \). Next,

\[
|I_1| \leq C \int_{|t| < \varepsilon} dv(t) \leq C\varepsilon^p \int_{|t| < \varepsilon} \frac{dv(t)}{|t|^p} = C\gamma \varepsilon^p = O(\lambda^{-\frac{3}{4}}). \tag{6.16}
\]

Similarly we have

\[
\left| \lambda^{-\frac{1}{4}} \int_{|t| < \varepsilon} \sqrt{\frac{2}{\pi t}} \cos \left( \frac{|t|\sqrt{\lambda} - \frac{\pi}{4}}{4} \right) \, dv(t) \right| \leq C\lambda^{-\frac{1}{4}} \int_{|t| < \varepsilon} \frac{dv(t)}{|t|^{\frac{3}{2}}} \leq C\lambda^{-\frac{1}{4}} \varepsilon^{p-\frac{3}{2}} = O(\lambda^{-\frac{3}{4}}). \tag{6.17}
\]

Using (6.13) (6.17) and setting \( \lambda = \mu_n^0 \) gives \( \mu_n^1 = \sigma(\mu_n^0) + O(n^{-\frac{3}{4}}) \) which implies (1.3).

Moreover, substituting \( dv(t) = \sum_{k \in \mathbb{Z}} \delta(t-t_k)q_k dt \) into (1.4) we obtain (1.5). ■

**7 Appendix**

For fixed \( \text{arg } \lambda = 2\vartheta \) we have \( t = \frac{x}{\sqrt{\lambda}} \in e^{-i\vartheta} \mathbb{R}_+ \). We rewrite \( t \in S[-\frac{\pi}{2}, 0] \) and \( \xi \) in the form

\[
t = re^{-i\vartheta} = 1 + \eta e^{-i\varphi}, \quad \varphi \in [0, \pi], \quad \eta \geq 0, \tag{A.1}
\]

\[
\xi(t) = e^{-i\frac{2\vartheta}{\pi}} \int_0^\eta \sqrt{2 + se^{-i\varphi}} \cdot s^{\frac{3}{2}} ds, \quad t \in S[-\frac{\pi}{2}, 0]. \tag{A.2}
\]
Lemma 7.1. Let \( t \in S[-\frac{\pi}{2}, 0] \), \( R(t) = |\xi(t)| \), \( \Phi(t) = \arg \xi(t) \). Then

1. \( t = re^{-i\vartheta} = 1 + ne^{-i\varphi} \), where \( r, n \geq 0 \), \( \vartheta \in [0, \frac{\pi}{2}] \), \( \varphi \in [0, \pi] \),

2. \( \Phi(t) + \frac{3\varphi}{2} \in [-\frac{\varphi}{2}, 0] \),

3. if \( n \leq 1 \), then \( \frac{2}{3}|t - 1|^\frac{3}{2} \leq R(t) \leq 2|t - 1|^\frac{3}{2} \),

   \( \frac{2}{3}|t - 1|^\frac{3}{2} \leq R(t) \leq 2|t - 1|^\frac{3}{2} \),

4. if \( \vartheta \in (0, \frac{\pi}{2}) \), then \( \cup_{r \in \mathbb{R}^+} \Phi(t) = [-\frac{3\varphi}{2}, -2\vartheta] \),

5. if \( -\pi - \vartheta \leq \Phi(t) \), then \( \arg(e^{2i\varphi} \partial_t \xi(t)) \in [-\frac{\pi}{2} + \vartheta, \frac{\varphi}{2}] \).

Proof. The proof of 1 is simple. 2 Consider the integrand in formula (A.2). For \( s \in [0, n] \) we have \( \arg(\sqrt{2 + re^{-i\varphi}}) \in [-\frac{\varphi}{2}, 0] \), hence \( \Phi(t) + \frac{3\varphi}{2} = \arg\left(\int_0^n \sqrt{2 + re^{-i\varphi}} \cdot s^{\frac{3}{2}} ds \right) \in [-\frac{\varphi}{2}, 0] \).

3 Consider the integrand in (A.2). For \( s \in [0, n] \) we have \( 1 < \text{Re} \sqrt{2 + re^{-i\varphi}} \) and \( |\sqrt{2 + re^{-i\varphi}}| < 3 \). Using \( R(t) = \left|\int_0^n \sqrt{2 + re^{-i\varphi}} \cdot s^{\frac{3}{2}} ds \right| \) gives

\[
\frac{2}{3}n^{\frac{3}{2}} \leq R(t) < 3 \cdot \frac{2}{3}n^{\frac{3}{2}}, \quad n = |t - 1|.
\]

The relation \( \min_{t \in \left(-\vartheta, \vartheta\right]} |t - 1| = \sin \vartheta \) for \( \vartheta \in (0, \frac{\pi}{2}) \) finishes the proof.

4 Fix \( \vartheta \in (0, \frac{\pi}{2}) \). By direct calculation, \( \xi(S[-\frac{\pi}{2}, 0]) \subset S[-\frac{3\pi}{2}, 0] \), so \( \cup_{r \in \mathbb{R}^+} \Phi(t) \subset S[-\frac{3\pi}{2}, 0] \).

Using (2.18) we have \( \arg(e^{2i\varphi} \xi(t)) \rightarrow 0 \) as \( r \rightarrow \infty \). Also we have \( \xi(0) = i\varphi \). Since \( \xi(t) \) is continuous, this gives \( \cup_{r \in \mathbb{R}^+} \Phi(t) \supset [-\frac{3\varphi}{2}, -2\vartheta] \). Thus we need only prove that \( \arg(e^{2i\varphi} \xi(t)) < 0 \).

Consider \( e^{2i\varphi} \xi(t) \), \( t = re^{-i\varphi} \), as a function of real parameter \( r \in \mathbb{R} \). Note that by Lemma 7.2, \( \text{Im}(e^{2i\varphi} \xi(t)) \) strictly decreases in \( r \). Since \( \xi(S[-\frac{\pi}{2}, 0]) \subset S[-\frac{3\pi}{2}, 0] \) and \( \xi(0) = i\varphi \), as \( r \) increases \( e^{2i\varphi} \xi(t) \) hits only the negative half of the imaginary axis. Therefore, as soon as \( \arg(e^{2i\varphi} \xi(t)) > -\frac{\pi}{2} \), we have \( \text{Im}(e^{2i\varphi} \xi(t)) < 0 \). Hence \( \arg(e^{2i\varphi} \xi(t)) < 0 \), which finishes the proof.

5 By (A.2), we have \( e^{2i\varphi} \partial_t \xi(t) = e^{-i\vartheta} e^{i\varphi} \sqrt{2}\eta \sqrt{1 + \frac{r e^{-i\varphi}}{2}} \), where \( \sqrt{1 + \frac{r e^{-i\varphi}}{2}} \in S[-\frac{\varphi}{2}, 0] \). Therefore

\[
\arg(e^{2i\varphi} \partial_t \xi(t)) \in [-\frac{\varphi}{2} + \vartheta, -\frac{\varphi}{2} + \vartheta]. \tag{A.3}
\]

Next, by hypothesis, \( \Phi \in [-\pi - \vartheta, -2\vartheta] \). Thus using 2 we obtain \( -\varphi \in [-\frac{\pi}{2}(\pi + \vartheta), -\vartheta] \). Substituting this into (A.3) proves 5. \( \square \)

Lemma 7.2. Let \( t = re^{-i\vartheta} \in S[-\frac{\pi}{2}, 0] \), \( R(t) = |\xi(t)| \), \( \Phi(t) = \arg \xi(t) \). Then

1. \( \arg \partial_t \xi(t) \in [-\frac{\pi}{2} - \vartheta, -2\vartheta] \),

30
2. if \( \vartheta \in (0, \frac{\pi}{2}] \) and \( r \geq 0 \) then \( \operatorname{Im}(e^{2i\vartheta}\partial_r \xi(t)) < 0 \) and \( \operatorname{Re}(e^{2i\vartheta}\partial_r \xi(t)) > 0 \),
    if \( \vartheta = 0 \) and \( r \in [0, 1) \), then \( \operatorname{Im}(e^{2i\vartheta}\partial_r \xi(t)) < 0 \) and \( \operatorname{Re} \xi(t) = 0 \),
    if \( \vartheta = 0 \) and \( r \in (1, \infty) \), then \( \operatorname{Im} \xi(t) = 0 \) and \( \operatorname{Re}(e^{2i\vartheta}\partial_r \xi(t)) > 0 \),
3. if \( \Phi(t) \in (-\pi - 2\vartheta, -\frac{\pi}{2} - \vartheta) \), then \( \partial_r \Phi(t) > 0 \),
4. if \( \Phi(t) \in (-\frac{3\pi}{2} - 2\vartheta, -\pi - \vartheta) \), then \( \partial_r R(t) < 0 \),
5. if \( \Phi(t) \in (-\frac{\pi}{2} - 2\vartheta, -\vartheta) \), then \( \partial_r R(t) > 0 \).

Proof. 1. By direct calculation, \( \partial_r \xi(t) = e^{-i\vartheta} \xi'(t) = e^{-i\vartheta} \sqrt{t^2 - 1} \). We have \( (t^2 - 1) \in S[-\pi, -2\vartheta] \), so that \( \arg \sqrt{t^2 e^{-2i\vartheta} - 1} \in [-\frac{\pi}{2}, -\vartheta) \).
2. For \( \vartheta \in (0, \frac{\pi}{2}) \) the result follows from 1. For \( \vartheta = 0 \), the result follows from (2.17) by direct calculation.
3. We have \( \partial_r \Phi(t) = \frac{\xi'(t)}{\xi(t)} \sin \{ \arg \partial_r \xi(t) - \Phi(t) \} \). By 1 \( \partial_r \Phi(t) \) is strictly positive for \( \Phi(t) \in (-\pi - 2\vartheta, -\frac{\pi}{2} - \vartheta) \).
4. and 5. We have \( \partial_r R(t)^2 = 2|\xi'(t)||\xi(t)| \cos \{ \arg \partial_r \xi(t) - \Phi(t) \} \). By 1 \( \partial_r R(t) \) is positive for \( \Phi(t) \in (-\frac{\pi}{2} - 2\vartheta, -\pi - \vartheta) \) and negative for \( \Phi(t) \in (-\frac{3\pi}{2} - 2\vartheta, -\pi - \vartheta) \). ■

For \( \lambda = |\lambda| e^{2i\vartheta} \in \mathbb{C}_+ \setminus \{0\} \) we have
\[
\frac{2}{3} z(x, \lambda)^{\frac{2}{3}} = \lambda \xi(t), \quad t = \frac{x}{\sqrt{\lambda}} = re^{-i\vartheta} \in S[-\frac{\pi}{2}, 0]. \quad (A.4)
\]

Lemma 7.3. For each \( \lambda = |\lambda| e^{2i\vartheta} \in S[0, \delta) \setminus \{0\} \) there exist a unique \( z_* \in \Gamma_\lambda \) such that \( |z_*| = \min_{z \in \Gamma_\lambda} |z| \). Moreover, the following relations are valid (\( x_* \) defined by \( z_* = z(x_*, \lambda) \)):

1. if \( \arg \lambda = 0 \), then \( z_* = 0 \) and \( x_* = \sqrt{\lambda} \),
   if \( \arg \lambda \in (0, \delta) \), then \( z_* \in S[-\frac{\pi}{2} - \frac{\vartheta}{2}, -\frac{\pi}{2} + \frac{\vartheta}{2} + \frac{\vartheta}{6}] \).
2. \( |z(\cdot, \lambda)| \) is strictly decreasing on \([0, x_*)\) and strictly increasing on \((x_*, \infty)\),
3. if \( t = \frac{x}{\sqrt{\lambda}} \) for \( x \in [0, x_*) \), then \( \arg(e^{2i\vartheta} \partial_r \xi(t)) \in [-\frac{\pi}{2}, -\frac{\pi}{2} + \frac{\vartheta}{4}] \),
4. if \( t_* = \frac{x_*}{\sqrt{\lambda}} \), then \( |t_*| \leq \frac{1}{\sin(\frac{\vartheta}{4})} \).

Proof. Let us prove uniqueness of \( z_* \). By (A.4), it is sufficient to show that for each \( \lambda \) there exists a unique solution \( t_* \) of \( \partial_r \xi(t) = 0 \) for \( t = re^{-i\vartheta}, r \in \mathbb{R}_+ \). For \( \arg \lambda = 0 \) direct calculation gives \( z_* = 0 \) and \( x_* = \sqrt{\lambda} \). Next we show the existence of \( t_* \) in the case \( \lambda = |\lambda| e^{2i\vartheta} \in S(0, \delta) \setminus \{0\} \).

Since \( \xi(t) \neq 0 \), we have \( 2\partial_r |\xi(t)| = |\xi(t)|^{-1} \operatorname{Re} \left( \xi(t) \frac{\partial \xi(t)}{\partial r} \right) \). Let us use the representation (A.1): \( t = re^{-i\vartheta} = 1 + \eta e^{-i\varphi} \). For any \( t \in S[-\frac{\pi}{2}, 0) \) Lemma 7.12 implies \( \arg \xi(t) \in \)
there exist a unique solution $\vartheta$, similarly Lemma 7.2.1 implies $\arg \frac{\partial}{\partial r} \xi(t) \in [-\frac{\vartheta}{2}, -\frac{3\vartheta}{2}, -\vartheta]$. Thus $\arg \frac{\partial}{\partial r} |\xi(t)|^2 \in [-\varphi + \frac{\vartheta}{2}, -\varphi + \frac{3\vartheta}{2}]$ and we have

$$\partial_r |\xi(t)| > 0 \quad \text{for } -\varphi \in \left(-\frac{\pi}{2} - \frac{\vartheta}{2}, -\frac{\pi}{2} - \frac{3\vartheta}{2}\right),$$

(A.5)

$$\partial_r |\xi(t)| < 0 \quad \text{for } -\varphi \in \left(-\frac{3\pi}{2} - \frac{\vartheta}{2}, -\frac{\pi}{2} - \vartheta\right).$$

(A.6)

By (A.5) and (A.6), there exist at least one solution of $\partial_r |\xi(t)| = 0$. Moreover, any solution satisfies

$$-\varphi \in \left[-\frac{\pi}{2} - \frac{3\vartheta}{2}, -\frac{\vartheta}{2}ight].$$

(A.7)

Now we show the uniqueness of $t_*$. We need only verify that for any $t$, satisfying (A.7), we have

$$\frac{1}{2} \left| \xi(t) \right|^2 = \left| \frac{\partial \xi(t)}{\partial r} \right|^2 + \Re \left\{ \xi(t) \cdot \frac{\partial^2 \xi(t)}{\partial r^2} \right\} > 0.$$ 

This is true if

$$\left| \partial_r \xi(t) \right|^2 > |\xi(t)| \left| \partial_r^2 \xi(t) \right|.$$ 

(A.8)

We have $\frac{\partial \xi(t)}{\partial r} = e^{-i\varphi} \sqrt{t^2 - 1}$, $\frac{\partial^2 \xi(t)}{\partial r^2} = e^{-3i\varphi} r \sqrt{t^2 - 1}$. Therefore (A.8) is equivalent to

$$\left| (t^2 - 1)^{\frac{3}{2}} / t \right| > |\xi(t)|.$$ 

(A.9)

Due to (A.7), $|t| \leq 1$. Therefore we have for $t = 1 + \eta e^{-i\varphi}$

$$\frac{|t^2 - 1|^{\frac{3}{2}}}{|t|} \geq \frac{|t - 1|^{\frac{3}{2}} |t + 1|^{\frac{1}{2}}}{1} = \eta^{\frac{3}{2}} (2 + \eta e^{-i\varphi})^{\frac{3}{2}}, \quad |2 + \eta e^{-i\varphi}| \geq 2 \cos \frac{3\vartheta}{2}.$$ 

(A.10)

Thus using $2\vartheta \in (0, \varphi) \subset [0, \frac{\pi}{3}, \arccos \frac{1}{2}]$ we obtain

$$\frac{|t^2 - 1|^{\frac{3}{2}}}{|t|} \geq \eta^{\frac{3}{2}} \left(2 \cos \frac{3\vartheta}{2}\right)^{\frac{3}{2}} > 2\eta^{\frac{3}{2}}.$$ 

(A.11)

Since $\vartheta \in [0, \varphi) \subset [0, \frac{\pi}{3}]$, (A.7) gives $\eta \leq 1$. Thus we apply Lemma 7.1.3, which gives $2\eta^{\frac{3}{2}} \geq |\xi(t)|$. Substituting the last estimate in (A.11) yields (A.9) and (A.8). Therefore there exist a unique solution $t_*$ of $|\partial_r \xi(t)| = 0$. Thus $z_\ast = z(x_\ast, \lambda)$ for $x_\ast = t_\ast \sqrt{\lambda}$.

1 and 2 Using (A.7) and Lemma 7.1.2 we obtain

$$\arg \xi(t_\ast) \in \left[-\frac{3\pi}{4}, -\frac{11\vartheta}{4}, \frac{3\pi}{4}, -\frac{3\vartheta}{4}\right].$$ 

(A.12)

By (A.4), this proves 1 Uniqueness of $t_\ast$ and the relation (A.4) prove 2.

3 By (A.5), for $r < |t_\ast|$ we have $-\pi \leq -\varphi \leq -\frac{\pi}{2} - \frac{\vartheta}{2}$. Using Lemma 7.1.2 we obtain

$$-\frac{\pi}{2} \leq \arg \left(e^{-i\varphi} \partial_r \xi(t)\right) = \arg \left(e^{i\varphi} e^{-i\varphi} \sqrt{1 - \frac{1}{2} e^{-i\varphi}}\right) \leq -\varphi - \frac{3\vartheta}{2} \leq -\frac{\pi}{4} - \frac{3\vartheta}{4}.$$ 

4 Geometric considerations show that in terms of (A.1), $r = \frac{\sin \varphi}{\sin(\varphi - \theta)}$. By (A.7) and Lemma 7.1.2, for $t_\ast = 1 + \eta_\ast e^{-i\varphi}$, we have $\frac{2\pi}{3} - \frac{\vartheta}{3} \geq \varphi - \vartheta \geq \frac{\pi}{3}$, so that $r_\ast = \frac{\sin \varphi}{\sin(\varphi - \theta)} \leq \frac{1}{\sin \frac{\pi}{3}}$.
Lemma 7.4. For each \( \lambda = |\lambda|e^{2i\vartheta} \in \mathbb{C}_+ \setminus \{0\} \) there exist a unique \( z_* \in \Gamma_{\lambda} \) such that \( \arg z_* = -\frac{\pi}{3} \). Moreover, the following relations are valid (here \( x_* \) defined by \( z_* = z(x_*,\lambda) \)):

1. if \( x \in [0, x_*) \), then \( \arg z(x,\lambda) < -\frac{\pi}{3} \),
   if \( x \in (x_*, \infty) \), then \( \arg z(x,\lambda) \rangle \frac{\pi}{3} \).
2. \(|z(\cdot, \lambda)| \) is strictly increasing on \((x_*, \infty)\),
3. if \( t_* = \frac{x_*}{\sqrt{3}} \), then \(|t_*| \leq \frac{1}{\sin^{\frac{\pi}{\lambda}}} \) uniformly in \( \lambda \).

Proof. By Lemma 7.1.1 and (A.1), \( \Gamma_{\lambda} \) intersects the ray \( \{z : \arg z = -\frac{\pi}{3}\} \) at least once. By (A.2), the sector in \( z \)-plane \( S[-\frac{\pi}{3} - \varepsilon, -\frac{\pi}{3} + \varepsilon] \) for small \( \varepsilon > 0 \) corresponds to \( S[-\frac{\pi}{3}, -2\vartheta - \frac{3\pi}{2}, -\frac{\pi}{3} - 2\vartheta - \frac{3\pi}{2}] \) in \( \xi \)-plane. By Lemma 7.2.1, in this sector \( \partial, \Phi(t) \geq 0 \). Therefore, by (A.1), \( \frac{\partial}{\partial x} \arg z(x,\lambda) > 0 \) for \( z \in S[-\frac{\pi}{3}, -\varepsilon, -\frac{\pi}{3} + \varepsilon] \). Hence \( z_* \) is unique and holds.

2. By (A.2), for \( x \in (x_*, \infty) \) and \( t = \frac{x}{\sqrt{3}} \) the hypothesis of Lemma 7.2.6 is fulfilled. Therefore \( \frac{\partial}{\partial x} R(t) > 0 \) and \(|z(x,\lambda)| \) is strictly increasing in \( x \).

3. By (A.1), we have \( \arg \xi(t_*) = -\frac{\pi}{6} - 2\vartheta \). By Lemma 7.1.2, for \( t_* = 1 + \eta_* e^{-i\varphi_*} \) we have \( \varphi_* - \vartheta \in [2\pi/3, -2\pi/3] \). Therefore using the relation \( r = \frac{\sin \varphi \sin (\varphi - \vartheta)}{\sin \varphi - \sin \varphi} \) in terms of (A.1), we obtain \(|t_*| \leq \frac{1}{\sin^{\frac{\pi}{\lambda}}} \).

In the next Lemma we analyze the curves \( \Gamma_{\lambda}^\pm \), defined in (2.43).

Lemma 7.5. Let \( \lambda = |\lambda|e^{2i\vartheta} \in \mathbb{C}_+ \setminus \{0\} \). Then

1. if \( \vartheta = 0 \), then \( \Gamma_{\lambda}^{\pm} = [-\lambda^{\frac{5}{6}}(\frac{3\vartheta}{\lambda})^{\frac{2}{3}}, \infty) \),
   if \( 0 < \vartheta \leq \frac{\pi}{2} \), then \( \Gamma_{\lambda}^{\pm} \subset \mathcal{S}_{-\pi + \frac{2\vartheta}{3}, -\frac{2\vartheta}{3}} \), \( \Gamma_{\lambda}^{\pm} \subset \mathcal{S}_{-\frac{2\vartheta}{3}, 0} \),
2. \( \inf_{z \in \Gamma_{\lambda}^{\pm}} |z| \geq |\lambda|^{\frac{5}{6}} \sin \vartheta \),
3. \( \Gamma_{\lambda}^{-} \subset \{z : |z| \leq C|\lambda|^{\frac{5}{6}}\} \), the length of \( \Gamma_{\lambda}^{-} \) satisfies \(|\Gamma_{\lambda}^{-}| \leq C|\lambda|^{\frac{5}{6}} \),
4. the function \(|z(\cdot, \lambda)| \) is strictly increasing on \([x_*, \infty)\).

Proof. 1. Direct calculation yields the result for \( \vartheta = 0 \). For \( 0 < \vartheta \leq \frac{\pi}{2} \) the assertion on \( \Gamma_{\lambda}^{\pm} \) follows from Lemma 7.1.1 and \( z = \lambda^{\frac{5}{6}}(\frac{3\vartheta}{\lambda})^{\frac{2}{3}} \). The assertion on \( \Gamma_{\lambda}^{\pm} \) for \( 0 \leq \arg \lambda \leq \delta \) follows from 11 of the present Lemma and Lemma 7.3.1. For \( \delta \leq \arg \lambda \leq \pi \) it follows from Lemma 7.4.1.

2 follows from Lemma 7.2.8 and \( z = \lambda^{\frac{5}{6}}(\frac{3\vartheta}{\lambda})^{\frac{2}{3}} \).

3. By Lemma 7.3.1 and Lemma 7.4.8, \(|t_*| \leq \frac{1}{\sin^{\frac{\pi}{\lambda}}} \) uniformly in \( \lambda \in \mathbb{C}_+ \setminus \{0\} \). Using the definition (2.19) we conclude that \( k(t) \) and \( k'(t) \) for \( t \in [0, t_*] \) are also uniformly bounded. Now the relations \(|\Gamma_{\lambda}^{-}| = |\lambda|^{\frac{5}{6}} \int_{[0, t_*]} |k'(t)| dt | \) and \(|z(x,\lambda)| = |\lambda|^{\frac{5}{6}} k(t) | \), \( t = \frac{x}{\sqrt{3}} \) complete the proof.

4. We deduce from (A.4) that \( \frac{\partial}{\partial x} |z(x,\lambda)| > 0 \) if \( \partial, \xi(t) > 0 \) for \( t = re^{-i\vartheta} = \frac{x}{\sqrt{3}} \). Thus for \( \delta \leq \arg \lambda \leq \pi \) the result follows from Lemma 7.4.2. For \( 0 \leq \arg \lambda \leq \delta \) it follows from Lemma 7.3.2. ■
Proof of Lemma 2.2. Case 1 (or 2) follows from Lemma 7.3 (or Lemma 7.4).

Proof of Lemma 2.3. Fix $\lambda = |\lambda|e^{2i\vartheta}$. By (A.4), we have $h(x) = e^{Re\lambda(t)}$ for $t = \frac{\pi}{\sqrt[3]{\lambda}} = re^{i\vartheta}$.

By Lemma 7.2.2, if either $0 < \arg \lambda \leq \pi$, $r \geq 0$ or $\arg \lambda = 0$, $r > 1$, then $\partial_r$ Re $(\lambda \xi(t)) > 0$. It remains to consider the case $\vartheta = 0$ and $x \in [0, x_*)$; by Lemma 7.2.2, Re $\xi(t) = 0$ and therefore $h = 1$.

In order to prove Lemma 2.4 for a fixed $\lambda = |\lambda|e^{2i\vartheta} \in \mathbb{C} \setminus \{0\}$ define the function $\Phi(z) = \frac{2z^2 + 2}{3|\lambda|^2}$. $\Phi$ maps $\Gamma_\lambda(z_1, z_2)$ onto $\gamma_\lambda(u_1, u_2)$, $u_j = \Phi(z_j)$, $j = 1, 2$. Similarly, we set $\gamma_\lambda(u) = \Phi(\Gamma_\lambda(z))$ for $u = \Phi(z)$ and $\gamma_\lambda^+ = \Phi(\Gamma_\lambda^+)$.

Consider the two cases a') and b').

Let us show that in both cases a') and b') the following estimate holds:

\[
\int_{\gamma_\lambda(u)} |f(v)||dv| \leq C \int_{Re u}^{\infty} |f(\varpi + i \Im v(\varpi))| \left| \frac{dv}{d\varpi} \right| d\varpi.
\]

Let us estimate $\left| \frac{dv}{d\varpi} \right|$. In terms of the parametrization $v(r) = e^{2i\vartheta} \xi(r e^{-i\vartheta})$ we have

\[
\frac{d \Im v(\varpi)}{d\varpi} = \frac{\partial_r \Im \left(e^{2i\vartheta} \xi(t)\right)}{\partial_r \Re \left(e^{2i\vartheta} \xi(t)\right)} = \tan \arg \partial_r e^{2i\vartheta} \xi(t).
\]

Consider the two cases a') and b').

a') Let $\delta \leq \arg \lambda \leq \pi$ and $u \in \gamma_\lambda$. For each point $v(r) \in \gamma_\lambda$ consider $t = re^{-i\vartheta}$ such that $v(r) = e^{2i\vartheta} \xi(t)$. By Lemma 7.2.1, $\partial_r \xi(t) \in [-\frac{\pi}{2} - \vartheta, -2\vartheta]$. Hence $\arg \partial_r e^{2i\vartheta} \xi(t) \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta, 0]$ and

\[
\left| \frac{d \Im v(\varpi)}{d\varpi} \right| \leq \frac{1}{\sin \frac{\pi}{2}}
\]

uniformly for $\delta \leq \arg \lambda \leq \pi$. Therefore $\left| \frac{dv}{d\varpi} \right|$ is also uniformly bounded.

b') Let $0 \leq \arg \lambda \leq \delta$ and $u \in \gamma_\lambda^+ = \Phi(\Gamma_\lambda^+)$. For each point $v(r) \in \gamma_\lambda^+$ consider $t = re^{-i\vartheta}$ such that $v(r) = e^{2i\vartheta} \xi(t)$. Since $0 \leq \vartheta \leq \frac{\pi}{2}$, (A.12) (equivalent to Lemma 7.3.1) implies that $-\pi - \vartheta \leq \arg \xi(t)$. Therefore by Lemma 7.1.5, for $v \in \gamma_\lambda^+$ we have $\arg e^{2i\vartheta} \xi(t) \in [-\frac{\pi}{2} + \frac{\pi}{2}, \frac{\pi}{2}]$. Hence

\[
\left| \frac{d \Im v(\varpi)}{d\varpi} \right| \leq \frac{1}{\sin \frac{\pi}{2}}
\]

and $\left| \frac{dv}{d\varpi} \right|$ is uniformly bounded.

Thus $\left| \frac{dv}{d\varpi} \right|$ is uniformly bounded in both cases a') and b'), implying (A.13).

Proof of Lemma 2.4. Firstly we prove (2.44) for the case $\delta \leq \arg \lambda \leq \pi$. By Lemma 7.3.2, we have $\text{dist}(\Gamma_\lambda, \{0\}) > \text{const}$ uniformly in $\lambda$. Thus we replace $\langle \cdot \rangle$ by $| \cdot |$. The change of variables $v = \Phi(s)$ in (2.44) results in the equivalent relation

\[
\int_{\gamma_\lambda(u)} \left| \frac{e^{-2|\lambda|Re u}}{\|v\|^3|\alpha + \frac{2}{3}\|} \right| dv \leq C \left| \frac{e^{-2|\lambda|Re u}}{|\lambda| |u|^3|\alpha + \frac{2}{3}|} \right|, \quad u = \frac{2z^2}{3|\lambda|} \in \gamma_\lambda.
\]

(A.14)
By (A.13), we have the auxiliary estimate
\[ \int_{\gamma_\lambda(u)} e^{-2|\lambda| \text{Re} v} |dv| \leq C e^{-2|\lambda| \text{Re} u} |\lambda|, \quad u \in \gamma_\lambda. \tag{A.15} \]

We show that (A.14) follows from (A.15). If \( u \in \gamma_\lambda^+ \), then Lemma 7.3.4 yields \(|u| = \min_{v \in \gamma_\lambda(u)} |v|\). Therefore (A.14) follows from (A.15). If \( u \in \gamma_\lambda^- \), then \( \text{dist}(\gamma_\lambda, \{0\}) > \text{const} \) and we have
\[ \int_{\gamma_\lambda(u)} e^{-2|\lambda| \text{Re} v} |dv| \leq C \int_{\gamma_\lambda(u)} e^{-2|\lambda| \text{Re} u} |dv|, \quad u \in \gamma_\lambda^- \tag{A.16} \]

By Lemma 7.3.3, \( \gamma_\lambda^- \) is uniformly bounded. Therefore, \(|u| \) is bounded on \( \gamma_\lambda^- \), so (A.16) and (A.15) imply (A.14).

b) We prove (2.41) for the case \( z \in \Gamma_\lambda^+ \). It suffices to show that for any \( z \in \Gamma_\lambda^+ \) we have
\[ \int_{\Gamma_\lambda(z)} |e^{-\frac{4+s_2}{3}}| \, |ds| \leq C |e^{-\frac{4+s_2}{3}}|, \quad |z| \leq 1; \quad \int_{\Gamma_\lambda(z)} \frac{|e^{-\frac{4+s_2}{3}}|}{|s|^{\frac{\alpha}{2}}} \, |ds| \leq C \frac{|e^{-\frac{4+s_2}{3}}|}{|z|^{\alpha+1}}, \quad |z| > 1. \tag{A.17} \]

By the change of variable \( u = \Psi(z) \) the first estimate in (A.17) follows from (A.15). For the proof of the second estimate in (A.17), we observe that by Lemma 7.3.4, for \( z \in \Gamma_\lambda^+ \) we have \(|z| = \min_{v \in \Gamma_\lambda(z)} |v|\). Hence, (A.17) follows from (A.15). This proves (2.41).

Next we shall prove (2.45). By the change of variable \( u = \Psi(z) \) we have
\[ \int_{\Gamma_\lambda(z)} \frac{|ds|}{\langle s \rangle^{\alpha}} = \left( \frac{2}{3} \right)^{\frac{\alpha}{2}} \frac{I}{|\lambda|^\frac{\alpha}{2}(\alpha-1)}, \quad I = \int_{\gamma_\lambda(u)} \frac{|dv|}{|v|^{\frac{\alpha}{2}}(\varepsilon + |v|^{\frac{1}{2}})} \, u = \frac{2 \varepsilon^\frac{1}{2}}{3 |\lambda|^\frac{\alpha}{2}} \tag{A.18} \]

where \( \varepsilon = \left( \frac{3}{2} |\lambda| \right)^{-\frac{2}{3}} \leq C \).

Assume \( z \in \Gamma_\lambda^+, \) \( 0 < \arg \lambda \leq \pi \). Set \( a = \text{Re} u \) and \( b = \text{Im} u \). By Lemma 7.3.1, Lemma 7.3.3, Lemma 7.3.5, and definition of \( z^* \), \( b \leq 0 \). By Lemma 7.2.2, \( \text{Im} v \) is strictly decreasing on \( \gamma_\lambda^+ \), so that \(|b| = \min_{v \in \gamma_\lambda(u)} |\text{Im} v| \). Thus using (A.13) we obtain
\[ I \leq CJ, \quad J = \int_a^\infty \frac{d\zeta}{|\zeta|^\frac{1}{2}(\varepsilon + |b|^{\frac{1}{2}} + |\zeta|^{\frac{1}{2}})^{\alpha}}. \]

Consider two cases: \( a \geq 0 \) and \( a < 0 \). Firstly, let \( a \geq 0 \). Then we have
\[ J = \frac{3}{2} \int_a^\infty \frac{dx}{(\varepsilon + |b|^{\frac{1}{2}} + x)^{\alpha}} = \frac{3}{2} \frac{1}{\alpha-1} \frac{1}{(\varepsilon + |b|^{\frac{1}{2}} + a^{\frac{1}{2}})^{\alpha-1}} \leq \frac{C}{(\varepsilon + |u|^{\frac{1}{2}})^{\alpha-1}} \]

Therefore
\[ \frac{I}{|\lambda|^\frac{\alpha}{2}(\alpha-1)} \leq \frac{C}{|\lambda|^\frac{\alpha}{2}(\varepsilon + |u|^{\frac{1}{2}})^{\alpha-1}} = \frac{C}{\langle z \rangle^{\alpha}}. \]
which together with (A.18) proves (2.45). Secondly, let $a < 0$. Again, by $x = z^\frac{2}{3}$, we obtain

$$J \leq C \int_0^\infty \frac{d\chi}{\chi^\frac{1}{2}(\varepsilon + |b|^\frac{2}{3} + z^\frac{2}{3})^\alpha} \leq \frac{C}{(\varepsilon + |b|^\frac{2}{3})^{\alpha-1}}.$$ 

Due to Lemma 7.3.1 and Lemma 7.4.1, for $u = a + ib \in \gamma_+^+$ and $a < 0$ we have $|a| \leq C|b|$ uniformly in $0 \leq \arg \lambda \leq \pi$. Therefore $|u| \leq Cb$ and

$$\frac{1}{(\varepsilon + |b|^\frac{2}{3})^{\alpha-1}} \leq \frac{C}{(\varepsilon + |u|^\frac{2}{3})^{\alpha-1}}.$$ 

Thus

$$\frac{I}{|\lambda|^\frac{4}{3}(\alpha-1)} \leq \frac{C}{|\lambda|^\frac{4}{3}(\alpha-1)(\varepsilon + |u|^\frac{2}{3})^\alpha} = \frac{C}{\langle z \rangle^\alpha},$$ 

which together with (A.18) proves (2.45).

Assume $z \in \Gamma^-_\lambda$ and $\delta \leq \arg \lambda \leq \pi$. Recall that $u_* = \frac{2}{3} \frac{z}{|\lambda|}$. We have $I = I_- + I_+$, where

$$I_- = \int_{\gamma_+(u,u_*)} |dv| |v|^\frac{1}{2}(\varepsilon + |v|^\frac{2}{3})^\alpha, \quad I_+ = \int_{\gamma_+(u_*)} |dv| |v|^\frac{1}{2}(\varepsilon + |v|^\frac{2}{3})^\alpha, \quad u = \frac{2}{3} \frac{z}{|\lambda|}.$$ 

By Lemma 7.6.2, we have $|z| \geq \sin \frac{\delta}{2}$ for $z \in \Gamma_\lambda$, so $|u| \geq \frac{2}{3} (\sin \frac{\delta}{2})^\frac{2}{3}$ for $u \in \gamma_\lambda$. By Lemma 7.4.3, the length of the curve $|\gamma_\lambda^-| < C$. Therefore $I_- \leq C$. By (A.19) for $u = u_*$, we have $I_+ \leq C$ so that $I \leq C$. Next, by Lemma 7.3.3, $\gamma^-_\lambda$ is uniformly bounded, implying $C \leq (\varepsilon + |u|^\frac{2}{3})^{-1\alpha}$ and

$$\frac{I}{|\lambda|^\frac{4}{3}(\alpha-1)} \leq \frac{C}{|\lambda|^\frac{4}{3}(\alpha-1)(\varepsilon + |u|^\frac{2}{3})^\alpha} = \frac{C}{\langle z \rangle^\alpha}.$$

which together with (A.18) proves (2.45).

**Proof of Lemma 2.5.** First we prove (2.46). By the change of variable $u = \Psi(z)$, we have

$$\int_{\Gamma^-_\lambda} \frac{|ds|}{\langle s \rangle^\alpha} = \left(\frac{2}{3}\right)^\frac{3}{2}(\alpha+\frac{1}{2}) \frac{I}{|\lambda|^{\frac{4}{3}(\alpha-1)}}, \quad I = \int_{\gamma^-_\lambda} |dv| |v|^\frac{1}{2}(\varepsilon + |v|^\frac{2}{3})^\alpha,$$ 

where $\varepsilon = (\frac{3}{2} |\lambda|)^{-\frac{2}{3}} \leq C$. By Lemma 7.2.2, Im $v$ is strictly decreasing on $\gamma^-_\lambda$. Thus we parameterize the last integral by $\chi = \text{Im} v$, so that $v(\chi) = \text{Re} v(\chi) + i\chi$. By Lemma 7.5.3, $\gamma^-_\lambda \subset \{ u \in \mathbb{C} : |u| < c \}$ for some $c > 0$ independent of $\lambda$. Therefore

$$I \leq 2 \int_0^c \frac{|dv|}{|d\chi|} \frac{d\chi}{\chi^{\frac{3}{2}(\varepsilon + \chi^\frac{2}{3})^\alpha}},$$

where $\left| \frac{d\chi}{d\chi} \right| = \sqrt{1 + \left( \frac{d\text{Re} v(\chi)}{d\chi} \right)^2}$. Recall that $\lambda = |\lambda|e^{2i\theta}$ and $t = re^{-i\theta}$. In order to estimate $\left| \frac{d\text{Re} v(\chi)}{d\chi} \right|$ we make use of the parametrization
\[ \gamma_\lambda(u) = \left\{ v \in \mathbb{C} : v = v(r) \text{ for } r \in [r_u, \infty], \text{ where } v(r) = e^{2i\theta} \xi \left( r \sqrt{|\lambda|}, \lambda \right) \right\}. \]

Thus we have
\[
\frac{d \text{Re} v(\chi)}{d\chi} = \partial_r \text{Re} \left( e^{2i\theta} \xi(t) \right) = \text{cot} \left( e^{2i\theta} \partial_r \xi(t) \right).
\]

By Lemma 7.3.1, Lemma 7.4.1, (2.42) and Lemma 7.1.3, \( e^{2i\theta} \partial_r \xi(t) \) is in a sector isolated from the real axis uniformly in \( \lambda \). Therefore \( \left| \frac{d \text{Re} v(\chi)}{d\chi} \right| \leq C \) and \( \left| \frac{dv}{dx} \right| \leq C \). Substituting this estimate in (A.21) and making change of variable \( x = \varepsilon^{-\frac{2}{3}} \chi \) we obtain
\[
I \leq C \int_0^{C_1\varepsilon^{-\frac{2}{3}}} \frac{dx}{x^\frac{1}{3}(1 + x^\frac{4}{3})^{\alpha}}, \quad \varepsilon = \left( \frac{3}{2} |\lambda| \right)^{-\frac{2}{3}} \leq C, \tag{A.23}
\]

Recall that \( |\lambda| \geq R > 0 \). Then the proof of (2.46) follows from (A.21) and (A.23).

Now we prove (2.47). By the change of variable \( u = \Psi(z) \), we have
\[
\int_{\Gamma_\lambda} \frac{|ds|}{|\lambda|^\frac{1}{3} + |s|^2} = \frac{I_- + I_+}{|\lambda|^\frac{1}{3}}, \quad I_\pm = \int_{\gamma_\lambda^\pm} \frac{|dv|}{\left( \frac{3}{2} |v| \right)^\frac{1}{3}(1 + \left( \frac{3}{2} |v| \right)^\frac{1}{3})}. \tag{A.24}
\]

For \( I_+ \) we use (A.13), which gives
\[
I_+ \leq C \int_0^\infty \frac{dx}{x^\frac{1}{3}(1 + x^\frac{4}{3})} \leq C. \tag{A.25}
\]

In order to estimate \( I_- \) we use the parametrization of \( \gamma_\lambda^- \) by \( \chi = \text{Im} v \). Repeating the arguments used above for the proof of (2.46), we conclude that \( \left| \frac{dv}{dx} \right| \leq C \) and
\[
I_- \leq C \int_0^\infty \frac{d\chi}{\chi^\frac{1}{3}(1 + \chi^\frac{4}{3})} \leq C. \tag{A.26}
\]

Now (A.25) and (A.26) give \( I \leq C \), which by (A.24) proves (2.47). \( \blacksquare \)

**Lemma 7.6.** Let \( \lambda = |\lambda|e^{2i\theta} \in S[0, \delta] \) and \( q, q' \in L^\infty(\mathbb{R}) \). Then for some absolute constant \( C \) the following estimates are fulfilled:
\[
|P_-(z, z_*)| \leq C \left( |q'\|_\infty + \|q\|_\infty |\lambda|^{-\frac{2}{3}} \right), \quad z \in \Gamma_\lambda^- \tag{A.27}
\]
\[
|P_+(z_1, z_2)| \leq C \left( |q'\|_\infty + \|q\|_\infty |\lambda|^{-\frac{2}{3}} \right). \tag{A.28}
\]

**Proof.** Let \( x_1 \leq x_2 \leq x_* \), \( t_{1,2} = \frac{x_{1,2}}{\sqrt{\lambda}} \) and \( z_{1,2} = z(x_{1,2}, \lambda) \in \Gamma_\lambda^- \). Let \( \xi_{1,2} = \xi(t_{1,2}) \). Then, Lemma 7.3.2 yields \( |\xi_1 - \xi_2| \leq 2|\xi_1| \). Lemma 2.3 gives \( |e^{2\lambda(\xi_1 - \xi_2)} - 1| \leq 2|\lambda||\xi_1 - \xi_2| \) and \( |e^{2\lambda(\xi_1 - \xi_2)} - 1| \leq C \). Therefore
\[
|\left( e^{2\lambda(\xi_1 - \xi_2)} - 1 \right)/\lambda | \leq C(1 + |\lambda||\xi_1|)^{-1}. \tag{A.29}
\]
Using $z_{1,2} = \lambda^{\frac{2}{3}}(\sqrt[3]{2}z_{1,2})^{\frac{2}{3}}$, we obtain the following estimate uniformly in $0 \leq \arg \lambda \leq \delta$:

$$|(e^{\frac{4}{3}z^{\frac{3}{2}} - s^{\frac{3}{2}}}) - 1)z^{-\frac{3}{2}}| \leq C(z_1)^{\frac{3}{2}}. \quad (A.30)$$

Introduce the functions $R_\pm(s, z) = e^{i\frac{4}{3}z^{\frac{3}{2}}^s} - e^{i\frac{4}{3}z^{\frac{3}{2}}^s}$. Evidently $\partial_s R_\pm(s, z) = \pm 2\sqrt{3}e^{i\frac{4}{3}z^{\frac{3}{2}}^s}$. Integrating by parts, we have

$$P_-(z_*, z) = R_-(z, z_*)f(z) + \int_{\Gamma_\lambda(z, z_*)} R_-(s, z_*)f'(s)\, ds, \quad f(s) = \rho(s)\hat{q}(s)/(2\sqrt{3}\lambda^{\frac{1}{2}}). \quad (A.31)$$

Therefore using (2.29), (2.30), (A.30) and $|\frac{d}{ds}\hat{q}(s)| \leq C||q'||\infty|\lambda|^{-\frac{1}{6}}$ we obtain

$$|P(z, z_*)| \leq C \left| e^{\frac{4}{3}z^{\frac{3}{2}} - s^{\frac{3}{2}}} \right| \left[ \|q\|\infty \left\frac{||q\|\infty}{\langle s \rangle^{\frac{1}{2}}} \langle z \rangle^{\frac{3}{2}} + \int_{\Gamma_\lambda(z, z_*)} |ds| \left( |\lambda|^{-\frac{1}{6}} \|q\|\infty \left\frac{||q\|\infty}{\langle s \rangle^{\frac{1}{2}}} + |\lambda|^{-\frac{2}{3}} \|q\|\infty \left\frac{||q\|\infty}{\langle s \rangle^{\frac{1}{2}}} \right) \right) \right], \quad (A.32)$$

which together with (2.46), (2.46) gives (A.27). Similarly we have

$$P_+(z_1, z_2) = -R_+(z_1, z_2)f(z_1) - \int_{\Gamma_\lambda(z_1, z_2)} R_+(s, z_2)f'(s)\, ds. \quad (A.33)$$

Again using (2.29), (2.30), (A.30) and $|\frac{d}{ds}\hat{q}(s)| \leq C||q'||\infty|\lambda|^{-\frac{1}{6}}$ we have

$$|P(z_1, z_2)| \leq C \left| e^{\frac{4}{3}(z_1^{\frac{3}{2}} - s^{\frac{3}{2}})} \right| \left[ \|q\|\infty \left\frac{||q\|\infty}{(1 + |z_2|)^{\frac{3}{2}}} + \int_{\Gamma_\lambda(z_1, z_2)} |ds| \left( |\lambda|^{-\frac{1}{6}} \|q\|\infty \left\frac{||q\|\infty}{\langle s \rangle^{\frac{1}{2}}} + |\lambda|^{-\frac{2}{3}} \|q\|\infty \left\frac{||q\|\infty}{\langle s \rangle^{\frac{1}{2}}} \right) \right) \right],$$

which together with (2.46), (2.46) gives (A.28). ■

**Lemma 7.7.** Let $q \in \mathcal{B}$. Define $F(z) = a(z)e^{\pm \frac{4}{3}z^{\frac{3}{2}}})V_q(z)$. Then uniformly in $z \in \Gamma_\lambda$ and $\lambda \in S[-\delta, \delta]$ the following estimates hold for sufficiently large $|\lambda|$:

$$\left| \int_{\Gamma_\lambda(z)} e^{i\frac{4}{3}(z^{\frac{3}{2}} - s^{\frac{3}{2}})} F(s)\, ds \right| \leq C \frac{||q\|\infty}{|\lambda|^{\frac{1}{2}}}, \quad \left| \int_{\Gamma_\lambda(z, z_*)} e^{i\frac{4}{3}(z^{\frac{3}{2}} - s^{\frac{3}{2}})} F(s)\, ds \right| \leq C \frac{||q\|\infty}{|\lambda|^{\frac{1}{2}}}. \quad (A.34)$$

**Proof.** We show the first estimate in (A.34). Using (2.29), (2.30), (2.38) and (2.44) we obtain

$$\left| \int_{\Gamma_\lambda} e^{i\frac{4}{3}(z^{\frac{3}{2}} - s^{\frac{3}{2}})} F(s)\, ds \right| \leq C \frac{||q\|\infty}{|\lambda|^{\frac{1}{2}}}. \quad (A.35)$$

Using $R_-(s, z_*) = e^{-i\frac{4}{3}s^{\frac{3}{2}}} - e^{i\frac{4}{3}s^{\frac{3}{2}}}$ we integrate by parts the integral over $\Gamma_\lambda(z, z_*)$. We have

$$I = \int_{\Gamma_\lambda(z, z_*)} e^{i\frac{4}{3}(z^{\frac{3}{2}} - s^{\frac{3}{2}})} F(s)\, ds = \frac{e^{\frac{4}{3}z^{\frac{3}{2}}}}{2} \left[ R_-(z, z_*) \frac{F(z)}{\sqrt{z}} + \int_{\Gamma_\lambda(z, z_*)} R_-(s, z_*) \left( \frac{F(s)}{\sqrt{s}} \right)'\, ds \right]. \quad (A.36)$$
Therefore, using Lemma 2.3, \( |\frac{d}{ds} \hat{q}(s)| \leq C|\lambda|^{-\frac{1}{3}}\|q\|_{\infty}, (2.29), (2.30), (2.38), (2.39) \) and (A.30), we obtain from (A.36)

\[
|I| \leq C \left[ \frac{\|q\|_{\infty}}{|\lambda|^{\frac{1}{3}} \langle s \rangle^{\frac{1}{2}}} + \int_{\lambda}^{\lambda'} \left( \frac{|\lambda|^{-\frac{1}{3}}\|q'\|_{\infty}}{\langle s \rangle^{\frac{1}{2} + \frac{1}{2}}} + \frac{|\lambda|^{-\frac{2}{3}}\|q\|_{\infty}}{\langle s \rangle} \right) ds \right].
\] (A.37)

By Lemma 2.5, for sufficiently large \(|\lambda|\) this implies \(|I| \leq C|\lambda|^{-\frac{1}{3}}\|q\|_{B} \left( 1 + |\lambda|^{-\frac{1}{3}} \log(|\lambda| + 1) \right)\). Together with (A.35) this proves the first estimate in (A.34). In order to prove the second estimate in (A.34) we use similar arguments with \(R_+(s, z_s) = e^{\frac{4}{3}s^\frac{3}{2}} - e^{\frac{4}{3}z_s^\frac{3}{2}}\). □

References

[1] Abramowitz, M. and Stegun, A., eds.: *Handbook of Mathematical Functions.* N.Y.: Dover Publications Inc.

[2] Chelkak, D., Kargaev, P., Korotyaev, E.: An Inverse Problem for an Harmonic Oscillator Perturbed by Potential: Uniqueness. Lett. Math. Phys. 64(1), 7-21 (2003)

[3] Chelkak D., Kargaev P., Korotyaev E.: Inverse problem for harmonic oscillator perturbed by potential, characterization. Comm. Math. Phys. in press

[4] Pöschel, J., Trubowitz, E.: *Inverse Spectral Theory.* Boston: Academic Press, 1987