Classification of continuously transitive circle groups

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Let $G$ be a closed transitive subgroup of $\text{Homeo}(S^1)$ which contains a non-constant continuous path $f : [0, 1] \to G$. We show that up to conjugation $G$ is one of the following groups: $\text{SO}(2, \mathbb{R})$, $\text{PSL}(2, \mathbb{R})$, $\text{PSL}_d(2, \mathbb{R})$, $\text{Homeo}_d(S^1)$, $\text{Homeo}(S^1)$. This verifies the classification suggested by Ghys in [5]. As a corollary we show that the group $\text{PSL}(2, \mathbb{R})$ is a maximal closed subgroup of $\text{Homeo}(S^1)$ (we understand this is a conjecture of de la Harpe). We also show that if such a group $G \subset \text{Homeo}(S^1)$ acts continuously transitively on $k$–tuples of points, $k > 3$, then the closure of $G$ is $\text{Homeo}(S^1)$ (cf [1]).

37E10; 22A05, 54H11

1 Introduction

Let $\text{Homeo}(S^1)$ denote the group of orientation preserving homeomorphisms of $S^1$ which we endow with the uniform topology. Let $G$ be a subgroup of $\text{Homeo}(S^1)$ with the topology induced from $\text{Homeo}(S^1)$. We say that $G$ is transitive if for every two points $x, y \in S^1$, there exists a map $f \in G$, such that $f(x) = y$. We say that a group $G$ is closed if it is closed in the topology of $\text{Homeo}(S^1)$. A continuous path in $G$ is a continuous map $f : [0, 1] \to G$.

Let $\text{SO}(2, \mathbb{R})$ denote the group of rotations of $S^1$ and $\text{PSL}(2, \mathbb{R})$ the group of Möbius transformations. The first main result we prove describes transitive subgroups of $\text{Homeo}(S^1)$ that contain a non constant continuous path.

**Theorem 1.1** Let $G$ be a transitive subgroup of $\text{Homeo}(S^1)$ which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:

1. $G$ is conjugate to $\text{SO}(2, \mathbb{R})$ in $\text{Homeo}(S^1)$.
2. $G$ is conjugate to $\text{PSL}(2, \mathbb{R})$ in $\text{Homeo}(S^1)$.
3. For every $f \in \text{Homeo}(S^1)$ and each finite set of points $x_1, \ldots, x_n \in S^1$ there exists $g \in G$ such that $g(x_i) = f(x_i)$ for each $i$. 

Published: 18 September 2006 DOI: 10.2140/gt.2006.10.1319
(4) $G$ is a cyclic cover of a conjugate of $\text{PSL}(2, \mathbb{R})$ in $\text{Homeo}(S^1)$ and hence conjugate to $\text{PSL}_k(2, \mathbb{R})$ for some $k > 1$.

(5) $G$ is a cyclic cover of a group satisfying condition 3 above.

Here we write $\text{PSL}_k(2, \mathbb{R})$ and $\text{Homeo}_k(S^1)$ to denote the cyclic covers of the groups $\text{PSL}(2, \mathbb{R})$ and $\text{Homeo}(S^1)$ respectively, for some $k \in \mathbb{N}$.

The proof begins by showing that the assumptions of the theorem imply that $G$ is continuously 1–transitive. This means that if we vary points $x, y \in S^1$ in a continuous fashion, then we can choose corresponding elements of $G$ which map $x$ to $y$ that also vary in a continuous fashion. In Theorems 3.8 and 3.10 we show that this leads us to two possibilities, either $G$ is conjugate to $\text{SO}(2, \mathbb{R})$, or $G$ is a cyclic cover of a group which is continuously 2–transitive.

We then analyse groups which are continuously 2–transitive and show that they are in fact all continuously 3–transitive. Furthermore, if such a group is not continuously 4–transitive, we show that it is a convergence group and hence conjugate to $\text{PSL}(2, \mathbb{R})$. On the other hand if it is continuously 4–transitive, then we use an induction argument to show that it is continuously $n$–transitive for all $n \geq 4$. This implies that for every $f \in \text{Homeo}(S^1)$ and each finite set of points $x_1, \ldots, x_n \in S^1$ there exists a group element $g$ such that $g(x_i) = f(x_i)$ for each $i$.

The remaining possibilities, namely cases 2 and 3, arise when the aforementioned cyclic cover is trivial.

In the case where the group $G$ is also closed we can use Theorem 1.1 to make the following classification.

**Theorem 1.2** Let $G$ be a closed transitive subgroup of $\text{Homeo}(S^1)$ which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:

1. $G$ is conjugate to $\text{SO}(2, \mathbb{R})$ in $\text{Homeo}(S^1)$.
2. $G$ is conjugate to $\text{PSL}_k(2, \mathbb{R})$ in $\text{Homeo}(S^1)$ for some $k \geq 1$.
3. $G$ is conjugate to $\text{Homeo}_k(S^1)$ in $\text{Homeo}(S^1)$ for some $k \geq 1$.

The above theorem provides the classification of closed, transitive subgroups of $\text{Homeo}(S^1)$ that contain a non-trivial continuous path. This classification was suggested by Ghys for all transitive and closed subgroups of $\text{Homeo}(S^1)$ (See [5]).

One well known problem in the theory of circle groups is to prove that the group of Möbius transformations is a maximal closed subgroup of $\text{Homeo}(S^1)$. We understand
that this is a conjecture of de la Harpe (see [1]). The following theorem follows directly from our work and answers this question.

**Theorem 1.3** \( \text{PSL}(2, \mathbb{R}) \) is a maximal closed subgroup of \( \text{Homeo}(\mathbb{S}^1) \).

In the following five sections we develop the techniques needed to prove our results. Here we prove the results about the transitivity on \( k \)-tuples of points. In Section 7 we give the proofs of all the main results stated above.

## 2 Continuous Transitivity

Let \( G < \text{Homeo}(\mathbb{S}^1) \) be a transitive group of orientation preserving homeomorphisms of \( \mathbb{S}^1 \). We begin with some definitions which generalize the notion of transitivity.

Set,
\[
P_n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{S}^1, x_i = x_j \iff i = j \}
\]
to be the set of distinct \( n \)-tuples of points in \( \mathbb{S}^1 \). Two \( n \)-tuples
\[
(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in P_n
\]
have matching orientations if there exists \( f \in \text{Homeo}(\mathbb{S}^1) \) such that \( f(x_i) = y_i \) for each \( i \).

**Definition 2.1** \( G \) is \( n \)-transitive if for every pair \( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in P_n \) with matching orientations there exists \( g \in G \) such that \( g(x_i) = y_i \) for each \( i \).

**Definition 2.2** \( G \) is uniquely \( n \)-transitive if it is \( n \)-transitive and for each pair \( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in P_n \) with matching orientations there is exactly one element \( g \in G \) such that \( g(x_i) = y_i \). Equivalently, the only element of \( G \) fixing \( n \) distinct points is the identity.

Endow \( \mathbb{S}^1 \) with the standard topology and \( P_n \) with the topology it inherits as a subspace of the \( n \)-fold Cartesian product \( \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \). These are metric topologies. With the topology on \( P_n \) being induced by the distance function
\[
d_{P_n}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{d_{\mathbb{S}^1}(x_i, y_i) : i = 1, \ldots, n\},
\]
where \( d_{\mathbb{S}^1} \) is the standard Euclidean distance function on \( \mathbb{S}^1 \).
We have the following lemma. We will call a pair of paths $\gamma: [0, 1] \to X$. If $\gamma: [0, 1] \to P_n$ is a path in $P_n$, we will write $x_i(t) = π_i \circ γ(t)$, where $π_i$ is projection onto the $i$-th component of $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, so that we can write $γ(t) = (x_1(t), \ldots, x_n(t))$. We will call a pair of paths $\gamma, \delta: [0, 1] \to P_n$ compatible if there exists a path $h: [0, 1] \to \text{Homeo}(\mathbb{S}^1)$ with $h(t)(x_i(t)) = y_i(t)$ for each $i$ and $t$.

**Definition 2.3** $G$ is continuously $n$–transitive if for every compatible pair of paths $\gamma, \delta: [0, 1] \to P_n$ there exists a path $g: [0, 1] \to G$ with the property that $g(t)(x_i(t)) = y_i(t)$ for each $i$ and $t$.

**Definition 2.4** A continuous deformation of the identity in $G$ is a non constant path of homeomorphisms $f_t \in G$ for $t \in [0, 1]$ with $f_0 = \text{id}$.

We have the following lemma.

**Lemma 2.5** For $n \geq 2$ the following are equivalent:

1. $G$ is continuously $n$–transitive.
2. $G$ is continuously $n - 1$–transitive and the following holds. For every $n - 1$–tuple $(a_1, \ldots, a_{n-1}) \in P_{n-1}$ and $x \in \mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$ there exists a continuous map $F_x: I \to G$ satisfying the following conditions,
   - (a) $F_x(y)$ fixes $a_1, \ldots, a_{n-1}$ for all $y \in I_x$
   - (b) $(F_x(y))(x) = y$ for all $y \in I_x$
   - (c) $F_x(x) = \text{id}$
   where $I_x$ is the component of $\mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$ containing $x$.
3. $G$ is continuously $n - 1$–transitive and there exists $(a_1, \ldots, a_{n-1}) \in P_{n-1}$ with the following property. There is a component $I$ of $\mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$, a point $\tilde{x} \in I$ and a continuous map $F_{\tilde{x}}: I \to G$ satisfying the following conditions,
   - (a) $F_{\tilde{x}}(y)$ fixes $a_1, \ldots, a_{n-1}$ for all $y \in I$
   - (b) $(F_{\tilde{x}}(y))(\tilde{x}) = y$ for all $y \in I$
   - (c) $F_{\tilde{x}}(\tilde{x}) = \text{id}$.
(4) \( G \) is continuously \( n-1 \)-transitive and there exists \( (a_1, \ldots, a_n) \in P_{n-1} \) with the following property. There is a component \( I \) of \( S^1 \setminus \{a_1, \ldots, a_n\} \), such that for each \( x \in I \) there exists a continuous deformation of the identity \( f_t \), satisfying \( f_t(a_i) = a_i \) for each \( t \) and \( i \) and \( f_t(x) \neq x \) for some \( t \).

Proof We start by showing \([1 \Rightarrow 4]\). As \( G \) is continuously \( n \)-transitive, it will automatically be continuously \( n-1 \) transitive. Take \( (a_1, \ldots, a_n) \in P_{n-1} \) and \( x \in S^1 \setminus \{a_1, \ldots, a_n\} \). Let \( I_x \) be the component of \( S^1 \setminus \{a_1, \ldots, a_n\} \) which contains \( x \). Take \( y \in I_x \setminus \{x\} \) and let \( x_t \) be an injective path in \( I_x \) with \( x_0 = x \) and \( x_1 = y \).

Let \( \lambda : [0, 1] \to P_n \) be the constant path defined by \( \lambda(t) = (a_1, \ldots, a_n, x_0) \) and let \( \gamma : [0, 1] \to P_n \) be the path defined by \( \gamma(t) = (a_1, \ldots, a_n, x_t) \). Then since \( x_t \in I_y \) for every time \( t \) these form an compatible pair of paths. Consequently, there exists a path \( g_t \in G \) which fixes each \( a_i \) and such that \( g_t(x) = (x_t) \). Defining \( f_t = g_t \circ (g_0)^{-1} \) gives us the required continuous deformation of the identity.

We now show that \([4 \Rightarrow 3]\). For \( \tilde{x} \in I \) set \( K_{\tilde{x}} \) to be the set of points \( x \in I \) for which there is a path of homeomorphisms \( f_t \in G \) satisfying,

1. \( f_0 = \text{id} \)
2. \( f_t(a_i) = a_i \) for each \( i \) and \( t \)
3. \( f_t(\tilde{x}) = x \).

Obviously, \( K_{\tilde{x}} \) will be a connected subset of \( I \) and hence an interval for each \( \tilde{x} \in I \).

Choose \( \tilde{x} \in I \) and take \( x \in K_{\tilde{x}} \). Let \( f_t \) and \( g_t \) be continuous deformations of the identity which fix the \( a_i \) for all \( t \) and such that \( f_{t_0}(x) \neq x \) for some \( t_0 \in (0, 1] \) and \( g_1(\tilde{x}) = x \). \( f_t \) exists by the assumptions of condition 4, and \( g_t \) exists because \( x \in K_{\tilde{x}} \). The following paths show that the interval between \( f_{t_0}(x) \) and \( (f_{t_0})^{-1}(x) \) is contained in \( K_{\tilde{x}} \):

\[
    h_1(t) = \begin{cases} 
        g_{2t} & t \in [0, 1/2] \\
        f_{t_0(2t-1)} \circ g_1 & t \in [1/2, 1] 
    \end{cases}
\]

\[
    h_2(t) = \begin{cases} 
        g_{2t} & t \in [0, 1/2] \\
        (f_{t_0(2t-1)})^{-1} \circ g_1 & t \in [1/2, 1] 
    \end{cases}
\]

As \( x \) is contained in this interval and cannot be equal to either of its endpoints we see that \( K_{\tilde{x}} \) is open for every \( \tilde{x} \in I \). On the other hand, \( \tilde{x} \in K_{\tilde{x}} \) for each \( \tilde{x} \in I \) and if \( x_1 \in K_{\tilde{x}_2} \) then \( K_{\tilde{x}_1} = K_{\tilde{x}_2} \). Consequently, the sets \( \{K_{\tilde{x}} : \tilde{x} \in I\} \) form a partition of \( I \) and hence \( K_{\tilde{x}} = I \) for every \( \tilde{x} \in I \).
We now construct the map $F_{\bar{x}}$. To do this, take a nested sequence of intervals $[x_n, y_n]$ containing $\bar{x}$ for each $n$ and such that $x_n, y_n$ converge to the endpoints of $I$ as $n \to \infty$. We define $F_{\bar{x}}$ inductively on these intervals. Since $K_{\bar{x}} = I$ we can find a path of homeomorphisms $f_t \in G$ satisfying,

1. $f_0 = \text{id}$
2. $f_t(a_i) = a_i$ for each $i$ and $t$
3. $f_1(\bar{x}) = x_1$.

We now show that there exists a path $\tilde{f}_t \in G$, which also satisfies the above, but with the additional condition that the path $\tilde{f}_t(\bar{x})$ is simple.

To see this, let $[x^*, \bar{x}]$ be the largest subinterval of $[x_1, \bar{x}]$ for which there exists a path $\tilde{f}_t \in G$ which satisfies,

1. $\tilde{f}_0 = \text{id}$
2. $\tilde{f}_t(a_i) = a_i$ for each $i$ and $t$
3. $\tilde{f}_1(\bar{x}) = x^*$
4. $\tilde{f}_t(\bar{x})$ is simple.

We want to show that $x^* = x_1$. Assume for contradiction that $x^* \neq x_1$. Then since $x^* \in [x_1, \bar{x}]$ there exists $s \in [0, 1]$ such that $f_s(\bar{x}) = x^*$ and for small $\epsilon > 0$, we have that $f_{s+\epsilon}(\bar{x}) \notin [x^*, \bar{x}]$. Then if we concatenate the path $\tilde{f}_t$ with $f_{s+\epsilon} \circ f_{s-1} \circ \tilde{f}_1$ for small $\epsilon$ we can construct a simple path satisfying the same conditions as $\tilde{f}_t$ but on an interval strictly bigger than $[x^*, \bar{x}]$, this contradicts the maximality of $x^*$ and we deduce that $x^* = x_1$.

We can use the path $\tilde{f}_t$ to define a map $F_{x^*}^1: [x_1, y_1] \to G$ satisfying,

1. $F_{x^*}^1(y)$ fixes each $a_i$ for each $y \in I$
2. $(F_{x^*}^1(y))(\bar{x}) = y$ for all $y \in I$
3. $F_{x^*}^1(\bar{x}) = \text{id}$.

by taking paths of homeomorphisms that move $\bar{x}$ to $x_1$ and $y_1$ along simple paths in $S^1$.

Now assume we have defined a map $F_{x^*}^k: [x_k, y_k] \to G$ satisfying,

1. $F_{x^*}^k(y)$ fixes each $a_i$ for each $y \in I$
2. $(F_{x^*}^k(y))(\bar{x}) = y$ for all $y \in I$
3. $F_{x^*}^k(\bar{x}) = \text{id}$.
We can use the same argument used to produce $F^1_\bar{x}$ to show that there exists a map $\bar{\xi}_x: [x_{k+1}, x_k] \to G$ such that \( \bar{\xi}_x(x) \) fixes the $a_i$ for each $x$, $\bar{\xi}_x(x_k) = \text{id}$ and $(\bar{\xi}_x(x))(x_k) = x$. Similarly there exists a map $\bar{\xi}_y: [y_k, y_{k+1}] \to G$ such that $\bar{\xi}_y(x)$ fixes the $a_i$ for each $x$, $\bar{\xi}_y(y_k) = \text{id}$ and $(\bar{\xi}_y(x))(y_k) = x$.

This allows us to define, $F^{k+1}_x: [x_{k+1}, y_{k+1}] \to G$ by:

$$
F^{k+1}_x(x) = \begin{cases} 
F^k_{\bar{x}}(x) & x \in [x_k, y_k] \\
(\bar{\xi}_x(x)) \circ F^k_{\bar{x}}(x_k) & x \in [x_{k+1}, x_k] \\
(\bar{\xi}_y(x)) \circ F^k_{\bar{x}}(y_k) & x \in [y_k, y_{k+1}] 
\end{cases}
$$

Inductively, we can now define the full map $F_x: I \to G$.

We now show that $[3 \Rightarrow 2]$. So take $x' \in I$ with $x' \neq \bar{x}$ and define $F_{x'}: I \to G$ by

(1) $F_{x'}(y) = F_{\bar{x}}(y) \circ (F_{\bar{x}}(x'))^{-1}$

Then $F_{x'}$ satisfies,

(1) $F_{x'}(y)$ fixes $a_1, \ldots, a_{n-1}$ for all $y \in I$

(2) $(F_{x'}(y))(x') = y$ for all $y \in I$

(3) $F_{x'}(x') = \text{id}$.

Moreover, we can use (1) to define a map $F: I \times I \to G$ which is continuous in each variable and satisfies,

(1) $F(x, y)$ fixes $a_1, \ldots, a_{n-1}$ for all $x, y \in I$

(2) $(F(x, y))(x) = y$ for all $x, y \in I$

(3) $F(x, x) = \text{id}$ for all $x \in I$.

Now take $x'$ to be a point in $S^1 \setminus I \cup \{a_1, \ldots, a_{n-1}\}$ and let $I'$ be the component of $S^1 \setminus \{a_1, \ldots, a_{n-1}\}$ which contains $x'$. Then since $G$ is continuously $n - 1$–transitive there exists $g \in G$ which permutes the $a_i$ so that $g(I) = I'$. Define $F_{x'}: I' \to G$ by

$$
F_{x'}(y) = g \circ F_{g^{-1}(x')}(g^{-1}(y)) \circ g^{-1}
$$

for $y \in I'$. Then $F_{x'}$ satisfies,

(1) $F_{x'}(y)$ fixes $a_1, \ldots, a_{n-1}$ for all $y \in I$

(2) $(F_{x'}(y))(x') = y$ for all $y \in I'$

(3) $F_{x'}(x') = \text{id}$. 

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Now let \((b_1, \ldots, b_{n-1}) \in P_{n-1}\) have the same orientation as \((a_1, \ldots, a_{n-1})\) then since \(G\) is continuously \(n-1\)-transitive there exists \(g \in G\) so that \(g(a_i) = b_i\) for each \(i\). Let \(x' \in S^1 \setminus \{b_1, \ldots, b_{n-1}\}\) and let \(I'\) be the component of \(S^1 \setminus \{b_1, \ldots, b_{n-1}\}\) in which it lies. Define \(F_{x'} : I' \to G\) by
\[
F_{x'}(y) = g \circ F_{g^{-1}(x')}(g^{-1}(y)) \circ g^{-1}
\]
for \(y \in I'\). Then \(F_{x'}\) satisfies,

1. \(F_{x'}(y)\) fixes \(b_1, \ldots, b_{n-1}\) for all \(y \in I\)
2. \((F_{x'}(y))(x') = y\) for all \(y \in I'\)
3. \(F_{x'}(x') = \text{id}\)

and we have that \([3 \Rightarrow 2]\)

Finally we have to show that \([2 \Rightarrow 1]\). Let \(\mathcal{X}, \mathcal{Y} : [0, 1] \to P_n\) be an compatible pair of paths. We define \(\mathcal{X}' : [0, 1] \to P_{n-1}\) by
\[
\mathcal{X}'(t) = (x_1(t), \ldots, x_{n-1}(t))
\]
and \(\mathcal{Y}' : [0, 1] \to P_{n-1}\) by
\[
\mathcal{Y}'(t) = (y_1(t), \ldots, y_{n-1}(t)).
\]
Notice that \(\mathcal{X}'\) and \(\mathcal{Y}'\) will also be a compatible pair of paths. Furthermore, as \(G\) is continuously \(n-1\)-transitive there will exist a path \(g' : [0, 1] \to G\) such that \(g'(t)(x_i(t)) = y_i(t)\) for \(1 \leq i \leq n-1\).

The paths \(\mathcal{X}', \mathcal{Y}' : [0, 1] \to P_{n-1}\) will also be compatible with the constant paths,
\[
\mathcal{X}'_0 : [0, 1] \to P_{n-1}
\]
\[
\mathcal{X}'_0(t) = \mathcal{X}'(0)
\]
and
\[
\mathcal{Y}'_0 : [0, 1] \to P_{n-1}
\]
\[
\mathcal{Y}'_0(t) = \mathcal{Y}'(0)
\]
respectively. So that there exist paths \(g'_x, g'_y : [0, 1] \to G\) with \(g'_x(x_i(0)) = x_i(t)\) and \(g'_y(y_i(0)) = y_i(t)\) for \(1 \leq i \leq n-1\). Furthermore, by pre composing with \((g'_x(0))^{-1}\) and \((g'_y(0))^{-1}\) if necessary, we can assume that \(g'_x(0) = g'_y(0) = \text{id}\).

We now construct a path \(g_x : [0, 1] \to G\) which satisfies,
\[
g_x(t)(x_i(0)) = x_i(t)
\]
for \(1 \leq i \leq n\). To do this let \(I\) be the component of \(S^1 \setminus \{x_1(0), \ldots, x_{n-1}(0)\}\) containing \(x_n(0)\). By assumption we have a continuous map \(F_{x_n(0)} : I \to G\) satisfying

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(1) \( F_{x_n(0)}(y) \) fixes \( x_1(0), \ldots, x_{n-1}(0) \) for all \( y \in I \)

(2) \( (F_{x_n(0)}(y))(x) = y \) for all \( y \in I \)

(3) \( F_{x_n(0)}(x) = \text{id.} \)

Define \( g_x : [0, 1] \to G \) by

\[
g_x(t) = g_y'(t) \circ (F_{x_n(0)}((g_x'(t))^{-1}(x_n(t))))^{-1}.\]

Then \( g_x(t)(x_i(0)) = x_i(t) \) for \( 1 \leq i \leq n \). We can repeat this process with \( g_y' \) to construct a path \( g_y : [0, 1] \to G \) satisfying \( g_y(t)(y_i(0)) = y_i(t) \) for \( 1 \leq i \leq n \).

The map \( g'(0) \) which we defined earlier will map \( x_i(0) \) to \( y_i(0) \) for \( 1 \leq i \leq n - 1 \). Moreover, \( g'(0)(x_n(0)) \) will lie in the same component of \( S^1 \setminus \{y_1(0), \ldots, y_{n-1}(0)\} \) as \( y_n(0) \). So we have a map \( F_{g'(0)(x_n(0))}(y_n(0)) \) which maps \( g'(0)(x_n(0)) \) to \( y_n(0) \) and fixes the other \( y_i(0) \). Putting all of this together allows us to define \( g : [0, 1] \to G \) by

\[
g(t) = g_y(t) \circ F_{g'(0)(x_n(0))}(y_n(0)) \circ g'(0) \circ (g_x(t))^{-1}.
\]

This is a path in \( G \) which satisfies \( g_x(t)(x_i(t)) = y_i(t) \) for each \( i \) and \( t \). Since we can do this for any two compatible paths, \( G \) is continuously \( n \)-transitive and we have shown that \( 2 \Rightarrow 1 \).

**Proposition 2.6** If \( G \) is \( 1 \)-transitive and there exists a continuous deformation of the identity \( f_t : [0, 1] \to G \) in \( G \), then \( G \) is continuously \( 1 \)-transitive.

**Proof** Let \( x_0 \in S^1 \) be such that \( f_{t_0}(x_0) \neq x_0 \) for some \( t_0 \in [0, 1] \). Take \( x \in S^1 \) then there exists \( g \in G \) such that \( g(x) = x_0 \). Consequently, \( g^{-1} \circ f_t \circ g \) is a continuous deformation of the identity which doesn’t fix \( x \) for some \( t \). Since these deformations exist for each \( x \in S^1 \) the proof follows in exactly the same way as \( 4 \Rightarrow 1 \) from the proof of Lemma 2.5.

From now on we will assume that \( G \) contains a continuous deformation of the identity, and hence is continuously \( 1 \)-transitive.

### 3 The set \( J_x \)

**Definition 3.1** For \( x \in S^1 \) we define \( J_x \) to be the set of points \( y \in S^1 \) which satisfy the following condition. There exists a continuous deformation of the identity \( f_t \in G \) which fixes \( x \) for all \( t \) and such that \( f_{t_0}(y) \neq y \) for some \( t_0 \in [0, 1] \).
It follows directly from this definition that \( x \not\in J_x \).

**Lemma 3.2** \( J_{f(x)} = f(J_x) \) for every \( f \in G \) and \( x \in \mathbb{S}^1 \).

**Proof** Let \( y \in J_{f(x)} \) and let \( f_t \) be the corresponding continuous deformation of the identity with \( f_{t_0}(y) \neq y \). Then \( f^{-1} \circ f_t \circ f \) is also a continuous deformation of the identity which now fixes \( x \), and for which \( f_{t_0}(f^{-1}(y)) \neq f^{-1}(y) \). This means that \( f^{-1}(y) \in J_x \) and hence \( y \in f(J_x) \) so that \( J_{f(x)} \subset f(J_x) \). The other inclusion is an identical argument. \( \square \)

**Lemma 3.3** \( J_x \) is open for every \( x \in \mathbb{S}^1 \).

**Proof** Let \( y \in J_x \) and take \( f_t \) to be the corresponding continuous deformation of the identity with \( f_{t_0}(y) \neq y \) for some \( t_0 \in [0, 1] \). Then since \( f_{t_0} \) is continuous there exists a neighborhood \( U \) of \( y \) such that \( f_{t_0}(z) \neq z \) for all \( z \in U \). This implies that \( U \subset J_x \) and hence that \( J_x \) is open. \( \square \)

**Lemma 3.4** \( J_x = \emptyset \) for every \( x \in \mathbb{S}^1 \) or \( J_x \) has a finite complement for every \( x \in \mathbb{S}^1 \).

To prove this lemma we will use the Hausdorff maximality Theorem which we now recall.

**Definition 3.5** A set \( P \) is partially ordered by a binary relation \( \leq \) if,

1. \( a \leq b \) and \( b \leq c \) implies \( a \leq c \)
2. \( a \leq a \) for every \( a \in P \)
3. \( a \leq b \) and \( b \leq a \) implies that \( a = b \).

**Definition 3.6** A subset \( Q \) of a partially ordered set \( P \) is totally ordered if for every pair \( a, b \in Q \) either \( a \leq b \) or \( b \leq a \). A totally ordered subset \( Q \subset P \) is maximal if for any member \( a \in P \setminus Q \), \( Q \cup \{a\} \) is not totally ordered.

**Theorem 3.7** (Hausdorff Maximality Theorem) Every nonempty partially ordered set contains a maximal totally ordered subset.

We now prove Lemma 3.4.
**Proof** Assume that there exists \( x \in S^1 \) for which \( J_x = \emptyset \). Then for every \( y \in S^1 \) there exists a map \( g \in G \) such that \( g(x) = y \). Consequently, 

\[
J_y = J_{g(x)} = g(J_x) = g(\emptyset) = \emptyset
\]

for every \( y \in S^1 \).

Assume that \( J_x \neq \emptyset \) for every \( x \in S^1 \) and let \( S_x = S^1 \setminus J_x \) denote the complement of \( J_x \). This means that \( S_x \) consists of the points \( y \in S^1 \) such every continuous deformation of the identity which fixes \( x \) also fixes \( y \). The set \( P = \{ S_x : x \in S^1 \} \) is partially ordered by inclusion so that by Theorem 3.7 there exists a maximal totally ordered subset, \( Q = \{ S_x : x \in A \} \), where \( A \) is the appropriate subset of \( S^1 \).

If we set \( S = \bigcap_{x \in A} S_x \) then we have the following:

1. \( S \neq \emptyset \)
2. if \( x \in S \) then \( S_x = S \).

1 follows from the fact that \( S \) is the intersection of a descending family of compact sets, and hence is nonempty.

To see that 2 is also true, fix \( x \in S \). Then from the definition of \( S \), we will have \( x \in S_a \) for each \( a \in A \). In other words, if we take \( a \in A \), then every continuous deformation of the identity which fixes \( a \) will also fix \( x \). Furthermore, if \( y \in S_x \) then every continuous deformation of the identity which fixes \( a \) not only fixes \( x \) but \( y \) too, so that \( S_x \subset S_a \). This is true for every \( a \in A \) so that \( S_x \subset S \). On the other hand, by the maximality of \( Q \), it must contain \( S_x \). Consequently, if \( x \in S \) then \( S_x = S \).

Fix \( x_0 \in S \) and assume for contradiction that \( S_{x_0} \) is infinite. Take a sequence \( x_n \in S_{x_0} \) and let \( x_{n_k} \) be a convergent subsequence with limit \( x' \). This limit will also be in \( S_{x_0} \) as it is closed. As \( J_{x_0} \) is a nonempty open subset of \( S^1 \) it will contain an interval \((a, b)\) with \( a, b \in S_{x_0} \). Take maps \( g_a, g_b \in G \) so that \( g_a(x') = a \) and \( g_b(x) = b \). Since \( x', a \in S_{x_0} \) we have that,

\[
g_a(S_{x_0}) = g_a(S_{x_0}) = S_{g_a(x')} = S_a = S_{x_0}
\]

and similarly for \( g_b \). As a result \( g_a(x_n), g_b(x_n) \in S_{x_0} \) for each \( n \), but \( g_a, g_b \) are orientation preserving homeomorphisms so that at least one of these points will lie in \((a, b)\), a contradiction.

We have shown that \( S_{x_0} \) is finite. If we now take any other point \( x \in S^1 \) then there exists a map \( g \in G \) such that \( g(x_0) = x \). This means that the set \( S_x = S_{g(x_0)} = g(S_{x_0}) \) will also be finite and we are done.

**Theorem 3.8** If \( J_x = \emptyset \) for all \( x \in S^1 \) then \( G \) is conjugate in \( \text{Homeo}(S^1) \) to the group of rotations \( \text{SO}(2, \mathbb{R}) \).

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We require the following lemma for the proof of this Theorem.

**Lemma 3.9** If \( f : \mathbb{R} \to \mathbb{R} \) is a homeomorphism which conjugates translations to translations, then it is an affine map.

**Proof** Let \( f \) be a homeomorphism which conjugates translations to translations and set \( f_1 = T \circ f \) where \( T \) is the translation that sends \( f(0) \) to 0. Then \( f_1 \) fixes 0 and also conjugates translations to translations. In particular there exists \( \alpha \) such that \( f_1 \) conjugates \( x \mapsto x + 1 \) to the map \( x \mapsto x + \alpha \). Notice that \( \alpha \neq 0 \) since the identity is only conjugate to itself.

Now define \( f_2 = f_1 \circ M_\alpha \) where \( M_\alpha(x) = \alpha x \). A simple calculation shows that \( f_2 \) conjugates \( x \mapsto x + 1 \) to itself and conjugates translations to translations. Since \( f_2 \) fixes 0, we see that \( f_2 \) must fix all the integer points.

Now, for \( n \in \mathbb{N} \) let \( \gamma \in \mathbb{R} \) be such that \((f_2)^{-1} \circ T_{1/n} \circ f_2 = T_\gamma \) where \( T_\alpha(x) = x + \alpha \). It follows that,

\[
T_1 = (f_2)^{-1} \circ (T_{1/n})^n \circ f_2 = ((f_2)^{-1} \circ T_{1/n} \circ f_2)^n = (T_\gamma)^n
\]

so that \( \gamma = 1/n \) and \((f_2)^{-1} \circ T_{1/n} \circ f_2 = T_{1/n} \) for every \( n \in \mathbb{N} \). Combining this with the fact that \( f_2 \) fixes 0, we deduce that \( f_1 \) and hence \( f \) are affine. \( \square \)

We can now prove Theorem 3.8.

**Proof** Let \( \hat{G} < G \) denote the path component of the identity in \( G \). We are going to show that \( \hat{G} \) is a compact group. Proposition 4.1 in [5] will then imply that it is conjugate in \( \text{Homeo}(S^1) \) to a subgroup of \( \text{SO}(2, \mathbb{R}) \). Moreover, as \( \hat{G} \) is 1–transitive it will be equal to the whole of \( \text{SO}(2, \mathbb{R}) \).

For \( x \in S^1 \) let \( \pi_x : \mathbb{R} \to S^1 \) be the usual projection map which sends each integer to \( x \) and for each integer translation \( T : \mathbb{R} \to \mathbb{R} \) satisfies \( \pi_x \circ T = \pi_x \).

If we fix \( x \in S^1 \) then since \( G \) is continuously 1–transitive we can choose a continuous path \( g : [0, 1] \to G \) such that \( g(t)(x) = \pi_x(t) \) and \( g(0) = \text{id} \). Notice that this path is contained in \( \hat{G} \) and \( g(1) \) is not necessarily the identity even though it fixes \( x \).

For \( x \in S^1 \) we define a continuous map \( F_x : \mathbb{R} \to \hat{G} \) by

\[
F_x(t) = g(t - \lfloor t \rfloor) \circ g(1)^{\lfloor t \rfloor}
\]

where \( \lfloor t \rfloor \) is the greatest integer less than or equal to \( t \). Set \( f = F_x(1) \). Note that \( F_x(n) = f^n \) for every \( n \in \mathbb{Z} \).

We claim that \( F_x \) has the following properties,
Assume now that the sequence \( s_n \) exists a unique \( x \) such that
\[
s_n \xrightarrow{m \to \infty} x.
\]

(1) \( F(x)(t) = \pi_s(t) \) for every \( t \in \mathbb{R} \)
(2) \( F_x(0) = \text{id} \)
(3) The map \( F_x \) is a surjection, that is \( F_x(\mathbb{R}) = \hat{G} \)
(4) If the map \( f = F_x(1) \) is not equal to the identity map then \( F_x \) is a bijection

The first two properties follow directly from the definition. To see that the third property holds, let \( h_s \) be a path in \( \hat{G} \), \( s \geq 0 \), \( h_0 = \text{id} \). Let \( \alpha(s) = h_s(x) \). We have that \( \alpha \) is a continuous map from the non-negative reals \( \mathbb{R}^+ \) into the circle. Since the set \( \mathbb{R}^+ \) is contractible we can lift the map \( \alpha \) into the universal cover of the circle. That is, there is a map \( \beta: \mathbb{R}^+ \to \mathbb{R} \) such that \( \pi \circ \beta = \alpha \). We have \( F_x(\beta(s))(x) = h_s(x) \).

Then \( h_s^{-1} \circ F_x(\beta(s))(x) = x \). It follows from the assumption of the theorem that \( F_x(\beta(s)) = h_s \) and \( F_x \) is surjective. The map \( F_x \) is injective for \( 0 \leq t < 1 \), because \( F_x(t)(x) = \pi_s(t) \). If \( F_x(1) \) is not the identity, and since \( F_x(1)(x) = x \) we have that \( F_x(m) = F_x(n) \) if and only if \( m = n \), for every two integers \( m, n \). This implies the fourth property.

It follows from (\( \ast \)), and the surjectivity of \( F_x \), that \( \hat{G} \) is a compact group if and only if the cyclic group generated by \( F_x(1) = f \) is a compact group. We will prove that \( f = \text{id} \).

Assume that \( f \) is not the identity map. Since \( F_x \) is a bijection for each \( t \in \mathbb{R} \), there exists a unique \( s_n(t) \in \mathbb{R} \) such that,
\[
f^n \circ F_x(t) \circ f^{-n} = F_x(s_n(t)). \tag{**}
\]

This defines a function \( s_n: \mathbb{R} \to \mathbb{R} \) which we claim is continuous for each \( n \). To see this, fix \( n \) and let \( t_m \in \mathbb{R} \) be a convergent sequence with limit \( t' \). Since \( F_x \) is continuous,
\[
f^n \circ F_x(t_m) \circ f^{-n} \xrightarrow{m \to \infty} f^n \circ F_x(t') \circ f^{-n}
\]
and so \( F_x(s_n(t_m)) \to F_x(s_n(t')) \) as \( m \to \infty \).

Now, if \( s_n(t_m) \) is a convergent subsequence, with limit \( t_0 \), then using continuity \( F_x(s_n(t_m)) \) will converge to \( F_x(t_0) \). Since \( F_x \) is a bijection this gives us that \( t_0 = s_n(t') \). Consequently, if the sequence \( s_n(t_m) \) were bounded, then it would converge to \( t' \).

Assume now that the sequence \( s_n(t_m) \) is unbounded and take a divergent subsequence \( s_n(t_{m_k}) \). Consider the corresponding sequence,
\[
F_x(s_n(t_{m_k})) = g(s_n(t_{m_k}) - [s_n(t_{m_k})]) \circ f[s_n(t_{m_k})].
\]

Since \( s_n(t_{m_k}) - [s_n(t_{m_k})] \in [0, 1) \) for each \( m \), there exists a subsequence \( t_{m_{k_i}} \) of \( t_{m_k} \) such that \( s_n(t_{m_{k_i}}) - [s_n(t_{m_{k_i}})] \) converges to some \( t_0 \in [0, 1] \). Now since \( g \) is continuous and the sequence \( F_x(s_n(t_{m_k})) \) converges to a homeomorphism \( F_x(s_n(t')) \) we have that
$f^{[s_n(t_m)]}$ converges to a homeomorphism as $l \to \infty$. However, as $s_n(t_m)$ is divergent $[s_n(t_m)]$ will be divergent too.

Let $S_f$ denote the set of fixed points of $f$. Note that $x \in S_f$. Since we assume that $f$ is not the identity we have that $S^1 \setminus S_f$ is non-empty. Let $J$ be a component of $S^1 \setminus S_f$ and let $a, b \in S^1$ be its endpoints. Since $f$ fixes $J$, and has no fixed points inside $J$ we deduce that on compact subsets of $J$ the sequence $f^{[s_n(t_m)]}$ converges to one of the endpoints and consequently, can not converge to a homeomorphism. This is a contradiction, so $s_n(t_m)$ can not be unbounded and $s_n$ is continuous.

Notice that $s_n(0) = 0$ and if $t \in \mathbb{Z}$ then $F_x(t)$ will commute with the $f^n$ so we have $s_n(m) = m$ for all $m \in \mathbb{Z}$. This yields that $s_n([0, 1]) = [0, 1]$ for every $n \in \mathbb{Z}$.

Let $U_f \subset S^1$ be the set defined as follows. We say that $y \in U_f$ if there exists an open interval $I$, $y \in I$, such that $|f^n(I)| \to 0$, $n \to \infty$. Here $|f^n(I)|$ denotes the length of the corresponding interval. The set $U_f$ is open. We show that $U_f$ is non-empty and not equal to $S^1$. As before, let $J$ be a component of $S^1 \setminus S_f$ and let $a, b \in S^1$ be its endpoints. Since $f$ fixes $J$, and has no fixed points inside $J$ we deduce that on compact subsets of $J$ the sequence $f^n$ converges to one of the endpoints, say $a$. This shows that $J \subset U_f$. Also, this shows that the point $b$ does not belong to $U_f$.

Let $y \in U_f$, and let $I$ be the corresponding open interval so that $y \in I$ and $|f^n(I)| \to 0$, $n \to \infty$. Set $f^n(I) = I_n$. Consider the interval $F_x(s_n(t))(I_n)$, $t \in [0, 1]$. Since $s_n([0, 1]) = [0, 1]$ we have that $F_x(s_n([0, 1]))$ is a compact family of homeomorphisms. This allows us to conclude that $|F_x(s_n(t))(I_n)| \to 0$, $n \to \infty$, uniformly in $n$ and $t \in [0, 1]$. Set $J_t = F_x(t)(I)$. From (**) we have that $|f^n(J_t)| \to 0$, $n \to \infty$, for a fixed $t \in [0, 1]$. This implies that the point $F_x(t)(y)$ belongs to the set $U_f$ for every $t \in [0, 1]$.

Let $J$ be a component of $U_f$, and let $a, b$ be its endpoints. Note that the points $a, b$ do not belong to $U_f$. Since $F_x(t)$ is a continuous path and $F_x(0) = id$, for small enough $t$ we have that $F_x(t)(J) \cap J \neq \emptyset$. Since $F_x(t)(J) \subset U_f$, and since $a, b$ are not in $U_f$ we have that $F_x(t)(J) = J$. By continuity this extends to hold for every $t \in [0, 1]$. But this means that $F_x(t)(a) = a$ for every $t \in [0, 1]$. However, for appropriately chosen inverse $t_0 = \pi^{-1}_x(a)$, we have that $F_x(t_0)(x) = a$, which contradicts the fact that $F_x(t_0)$ is a homeomorphism. This shows that $f = id$, and therefore we have proved that $\hat{G}$ is a compact group.

To finish the argument, it remains to show that $G = \hat{G}$. Let $\Phi \in \text{Homeo}(S^1)$ be a map which conjugates $\hat{G}$ to $\text{SO}(2, \mathbb{R})$ and take $g \in G \setminus \hat{G}$. Since $\hat{G}$ is a normal subgroup of $G$, $\Phi \circ g \circ \Phi^{-1}$ conjugates rotations to rotations. Lifting to the universal cover we get that every lift of $\Phi \circ g \circ \Phi^{-1}$ conjugates translations to translations. If we choose one...
then by Lemma 3.9 it will be affine. On the other hand, it must be periodic, and hence is a rotation. So that $\Phi \circ g \circ \Phi^{-1}$ is itself a rotation and we are done. 

\begin{theorem}
If $J_x \neq \emptyset$ then one of the following is true:

1. $J_x = S^1 \setminus \{x\}$ in which case $G$ is continuously 2–transitive.
2. There exists $R \in \text{Homeo}(S^1)$ which is conjugate to a finite order rotation and satisfies $R \circ g = g \circ R$ for every $g \in G$. Moreover, $G$ is a cyclic cover of a group $G_1$ which is continuously 2–transitive, where the covering transformations are the cyclic group generated by $R$.

\end{theorem}

\textbf{Proof}  If $J_x = S^1 \setminus \{x\}$ then we are in case 4 of Lemma 2.5 with $n = 2$. In this situation we know that $G$ will be continuously 2–transitive.

We already know that $S_x = S^1 \setminus J_x$ must contain $x$ and by Lemma 3.4 must be finite. Moreover, as $f(J_x) = J_{f(x)}$ the sets $S_x$ contain the same number of points for each $x \in S^1$. Define $R : S^1 \rightarrow S^1$ by taking $R(x)$ to be the first point of $S_x$ you come to as you travel anticlockwise around $S^1$. Now take $g \in G$ and $x \in S^1$, then since $J_{g(x)} = g(J_x)$ and $g$ is orientation preserving $R \circ g(x) = g \circ R(x)$ for all $x \in S^1$.

We now show that $R$ is a homeomorphism. To see this take any continuous path $x_t \in S^1$, we will show that $R(x_t) \rightarrow R(x_0)$ as $t \rightarrow 0$. Since $G$ is continuously 1–transitive, there exists a continuous path $g_t \in G$ satisfying $g_t(x_0) = x_0$, so that,

$$
\lim_{t \rightarrow 0} R(x_t) = \lim_{t \rightarrow 0} (g_t)^{-1}(R(g_t(x_t))) = \lim_{t \rightarrow 0} (g_t)^{-1}(R(x_0)) = R(x_0),
$$

where the first equality follows from the fact that $R \circ g(x) = g \circ R(x)$ for all $x \in S^1$. This shows that $R$ is continuous. If we take $y \notin J_x$ then $J_x \subset J_y$, and hence $S_x \supset S_y$ but in this case since $S_x$ and $S_y$ contain the same number of points they will be equal. Consequently, $R$ has an inverse defined by taking $R^{-1}(x)$ to be the first point of $S_x$ you come to by traveling clockwise around $S^1$ and this inverse is continuous by the same argument as for $R$. Consequently, $R \in \text{Homeo}(S^1)$. Furthermore, $R$ is of finite order equal to the number of points in $S_x$ and hence conjugate to a rotation.

Let $\Gamma$ denote the cyclic subgroup of $\text{Homeo}(S^1)$ generated by $R$. Define $\pi : S^1 \rightarrow S^1/\Gamma \cong S^1$, in the usual way with $\pi(x)$ being the orbit of $x$ under $\Gamma$. Since $R \circ g(x) = g \circ R(x)$ for all $x \in S^1$, each $g \in G$ defines a well defined homeomorphism of the quotient space $S^1/\Gamma$ which we call $g_\Gamma$. This gives us a homomorphism $\pi_\Gamma : G \rightarrow \text{Homeo}(S^1)$, defined by $\pi_\Gamma(g) = g_\Gamma$. Let $G_\Gamma$ denote the image of $G$ under $\pi_\Gamma$, then $G$ is a cyclic cover of $G_\Gamma$. 

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It remains to see that $G\Gamma$ is continuously 2–transitive. This follows from the fact that if we take $x_0 \in S^1$ then $J_{\pi(x_0)} = \pi(J_{x_0})$, where $J_{\pi(x_0)}$ is the set of points that can be moved by continuous deformations of the identity in $G\Gamma$ which fix $\pi(x_0)$. Consequently, $J_{\pi(x_0)} = S^1 \setminus \{x_0\}$ so that $G\Gamma$ is continuously 2–transitive by the first part of this proposition.

4 Implications of continuous 2–transitivity

We now know that if $G$ is transitive and contains a continuous deformation of the identity then it is either conjugate to the group of rotations $SO(2, R)$, is continuously 2–transitive, or is a cyclic cover of a group which is continuously 2–transitive. For the rest of the paper we assume that $G$ is continuously 2–transitive and examine which possibilities arise.

For $n \geq 2$ and $(x_1 \ldots x_n) \in P_n$ we define $J_{x_1 \ldots x_n}$ to be the subset of $S^1$ containing the points $x \in S^1$ which satisfy the following condition. There exists a continuous deformation of the identity $f_i \in G$, with $f_i(x_i) = x_i$ for each $i$ and $t$ and such that there exists $t_0 \in [0, 1]$ with $f_{x_0}(x) \neq x$. This generalizes the earlier definition of $J_x$ and we get the following analogous results.

**Lemma 4.1** $J_{f(x_1) \ldots f(x_n)} = f(J_{x_1 \ldots x_n})$ for every $f \in G$.

**Lemma 4.2** $J_{x_1 \ldots x_n}$ is open.

We also have the following.

**Lemma 4.3** If $J_{x_1 \ldots x_n}$ is nonempty and $G$ is continuously $n$–transitive, then it is equal to $S^1 \setminus \{x_1 \ldots x_n\}$.

**Proof** Assume that $J_{x_1 \ldots x_n} \subset S^1 \setminus \{x_1, \ldots, x_n\}$ is nonempty. By Lemma 4.2 it is also open and hence is a countable union of open intervals. Pick one of these, and call its endpoints $b_1$ and $b_2$. Assume for contradiction that at least one of $b_1$ and $b_2$ is not one of the $x_i$. Interchanging $b_1$ and $b_2$ if necessary we can assume that this point is $b_1$. Since $G$ is continuously $n$–transitive there exist elements of $G$ which cyclically permute the $x_i$. Using these elements and the fact that $J_{f(x_1) \ldots f(x_n)} = f(J_{x_1 \ldots x_n})$ for every $f \in G$, we can assume without loss of generality that $b_1$ and hence the whole interval lies in the component of $S^1 \setminus \{x_1, \ldots, x_n\}$ whose endpoints are $x_1$ and $x_2$. 

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We now claim that \( J_{b_1, b_2, x_3, \ldots, x_n} \supset J_{x_1, x_2} \). To see this, take \( x \in J_{x_1, x_2} \), then there exists a continuous deformation of the identity \( f_t \) which fixes \( x_1, \ldots, x_n \) and for which there exists \( t_0 \) such that \( f_{t_0}(x) \neq x \). Now since \( b_1, b_2 \not\in J_{x_1, x_2} \), \( f_t \) must also fix \( b_1 \) and \( b_2 \) for all \( t \), consequently we can use \( f_t \) to show that \( x \in J_{b_1, b_2, x_3, \ldots, x_n} \). In particular, this means that \( J_{b_1, b_2, x_3, \ldots, x_n} \) contains the whole interval between \( b_1 \) and \( b_2 \).

Take \( g \in G \) which maps \( \{ b_1, b_2 \} \) to \( \{ x_1, x_2 \} \) and fixes the other \( x_i \), such an element exists as \( G \) is continuously \( n \)–transitive. Then,

\[
J_{x_1, x_2, x_3, \ldots, x_n} = J(g(b_1), g(b_2), g(x_1), \ldots, g(x_n)) = g(J_{b_1, b_2, x_3, \ldots, x_n})
\]

so that \( J_{x_1, x_2, x_3, \ldots, x_n} \) must contain the whole interval between \( x_1 \) and \( x_2 \). This is a contradiction, since \( b_1 \) lies between \( x_1 \) and \( x_2 \) but is not in \( J_{x_1, x_2} \).

**Proposition 4.4** Let \( G \) be continuously \( n \)–transitive for some \( n \geq 2 \) and suppose there exist \( n \) distinct points \( a_1, \ldots, a_n \in S^1 \) and a continuous deformation of the identity \( g_t \in G \), which fixes each \( a_t \) for all \( t \). Then \( G \) is continuously \( n + 1 \)–transitive.

**Proof** \( J_{a_1, \ldots, a_n} \neq \emptyset \) so by Lemma 4.3 \( J_{a_1, \ldots, a_n} = S^1 \setminus \{ a_1, \ldots, a_n \} \). We can now apply Lemma 2.5 to see that \( G \) is continuously \( n + 1 \)–transitive.

**Corollary 4.5** If \( G \) is continuously 2–transitive and there exists \( g \in G \setminus \{ \text{id} \} \) with an open interval \( I \subset S^1 \) such that the restriction of \( g \) to \( I \) is the identity, then \( G \) is continuously \( n \)–transitive for every \( n \geq 2 \).

**Proof** Let \( I \subset S^1 \) be a maximal interval on which \( g \) acts as the identity, so that if \( I' \supset I \) is another interval containing \( I \) then \( g \) doesn’t act as the identity on \( I' \). Let \( a \) and \( b \) be the endpoints of \( I \) and let \( a_t \) and \( b_t \) be continuous injective paths with \( a_0 = a, b_0 = b \) and \( a_t, b_t \not\in I \) for each \( t \neq 0 \). This is possible because \( g \neq \text{id} \) so that \( S^1 \setminus I \) will be a closed interval containing more than one point. Let \( g_t \) be a continuous path in \( G \) so that \( g_0 = \text{id}, g_t(a) = a_t \) and \( g_t(b) = b_t \), such a path exists as \( G \) is continuously 2–transitive. Consider the path \( h_t = g_t^{-1} \circ g_t \circ g_t^{-1} \) since \( g_0 = \text{id} \) we get \( h_0 = \text{id} \). Now \( g_t \circ g_t^{-1} \) acts as the identity on the interval between \( a_t \) and \( b_t \) and by maximality of \( I \), \( g_t^{-1} \) will not act as the identity for \( t \neq 0 \). Consequently, \( h_t \) is a continuous deformation of the identity which acts as the identity on \( I \). So if \( G \) is continuously \( k \)–transitive for \( k \geq 2 \), by taking \( k \)–points in \( I \) and using Proposition 4.4 we get that \( G \) is \( k + 1 \)–transitive. As a result, since \( G \) is continuously 2–transitive it will be \( n \)–transitive for every \( n \geq 2 \).

\( \text{SO}(2, \mathbb{R}) \) is an example of a subgroup of \( \text{Homeo}(S^1) \) which is continuously 1–transitive but not continuously 2–transitive. However, as the next result shows, there are no subgroups of \( \text{Homeo}(S^1) \) which are continuously 2–transitive but not continuously 3–transitive.
Proposition 4.6  If $G$ is continuously 2–transitive, then it is continuously 3–transitive.

Proof  Let $a, b \in S^1$ be distinct points. Construct two injective paths $a(t), b(t)$ in $S^1$ with disjoint images, such that $a(0) = a$, $b(0) = b$ and such that $a(t)$ and $b(t)$ lie in the same component of $S^1 \setminus \{a, b\}$ for $t \in (0, 1]$. We label this component $I$ and the other $I'$.

Since $G$ is continuously 2–transitive, there exists a path $g(t) \in G$ such that $g(0) = \text{id}$, $g(t)(a) = a(t)$ and $g(t)(b) = b(t)$ for every $t$. Now for every $t$ the restriction of $g(t)$ to the closure of $I$, is a continuous map of a closed interval into itself, and hence must have a fixed point, $c(t)$. This point will normally not be unique, but since $g(t)$ is continuous, for a small enough time interval we can choose it to depend continuously on $t$. Likewise for the restriction of $g(t)^{-1}$ to the closure of $I'$, for a small enough time interval we can choose a path of fixed points $d(t)$, which must therefore also be fixed points for $g(t)$.

Now pick points $c \in I$ and $d \in I'$. Using continuous 2–transitivity of $G$ construct a path $h(t) \in G$ such that $h(t)(c) = c(t)$ and $h(t)(d) = d(t)$. Then $h(t)^{-1} \circ g(t) \circ h(t)$ is only the identity when $t = 0$ because the same is true of $g(t)$ and we have constructed a continuous deformation of the identity which fixes $c$ and $d$ for all $t$. Consequently we can use Proposition 4.4 to show that $G$ is continuously 3–transitive.  \hfill \Box

5  Convergence Groups

Definition 5.1  A subgroup $G$ of Homeo($S^1$)is a convergence group if for every sequence of distinct elements $g_n \in G$, there exists a subsequence $g_{n_k}$ satisfying one of the following two properties:

1. There exists $g \in G$ such that,
   $$\lim_{k \to \infty} g_{n_k} = g \quad \text{and} \quad \lim_{k \to \infty} g_{n_k}^{-1} = g^{-1}$$
   uniformly in $S^1$.

2. There exist points $x_0, y_0 \in S^1$ such that,
   $$\lim_{k \to \infty} g_{n_k} = x_0 \quad \text{and} \quad \lim_{k \to \infty} g_{n_k}^{-1} = y_0$$
   uniformly on compact subsets of $S^1 \setminus \{y_0\}$ and $S^1 \setminus \{x_0\}$ respectively.

The notion of convergence groups was introduced by Gehring and Martin [4] and they have proceeded to play a central role in geometric group theory. The following theorem has been one of the most important and we shall make frequent use of it.

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Theorem 5.2  \( G \) is a convergence group if and only if it is conjugate in \( \text{Homeo}(\mathbb{S}^1) \) to a subgroup of \( \text{PSL}(2, \mathbb{R}) \).

This Theorem was proved by Gabai in [3]. Prior to that, Tukia [7] proved this result in many cases and Hinkkanen [6] proved it for non discrete groups. Casson and Jungreis proved it independently using different methods [2]. See [2], [3], [7] for references to other papers in this subject.

For the rest of this section we shall assume that \( G \) is continuously \( n \)–transitive, but not continuously \( n + 1 \)–transitive for some \( n \geq 3 \).

Take \((x_1, \ldots, x_{n-1}) \in \mathbb{P}_{n-1}\) and define 
\[
G_0 = \{ g \in G : g(x_i) = x_i \quad i = 1, \ldots, n - 1 \}.
\]

Choose a component \( I \) of \( \mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\} \) and denote its closure by \( \bar{I} \). We construct a homomorphism \( \Phi : G_0 \to \text{Homeo}(\mathbb{S}^1) \) as follows. Take \( g \in G_0 \), then since \( g \) fixes the endpoints of \( I \) and is orientation preserving, we can restrict it to a homeomorphism \( g' \) of \( \bar{I} \). By identifying the endpoints of \( \bar{I} \) we get a copy of \( \mathbb{S}^1 \) and we define \( \Phi(g) \) to be the homeomorphism of \( \mathbb{S}^1 \) that \( g' \) descends to under this identification. We label the identification point \( \bar{x} \) and set \( \hat{G}_0 = \Phi(G_0) \) to be the image of \( G_0 \) under \( \Phi \).

In this situation Lemma 2.5 implies the following. For every \( x \in I \), there exists a continuous map \( F_x : \mathbb{S}^1 \setminus \bar{x} \to \hat{G}_0 \) satisfying the properties,

\[
\begin{align*}
(1) & \quad (F_x(y))(x) = y \quad \forall \ y \in \mathbb{S}^1 \setminus \bar{x} \\
(2) & \quad F_x(x) = \text{id}.
\end{align*}
\]

Proposition 5.3  \( \Phi : G_0 \to \hat{G}_0 \) is an isomorphism.

Proof  Surjectivity is trivial. If we assume that \( \Phi \) is not injective then there will exist \( g \in G_0 \) which is non-trivial and acts as the identity on \( I \). Then by Corollary 4.5 \( G \) will be \( n + 1 \)–transitive, a contradiction.

Let \( \hat{G}_0 \) denote the path component of the identity in \( G_0 \), we now analyze the group \( \hat{G}_0 = \Phi(G_0) \).

Proposition 5.4  \( \hat{G}_0 \) is a convergence group.

Proof  Choose \( x \in I \) then we know there exists a continuous map \( F_x : \mathbb{S}^1 \setminus \bar{x} \to \hat{G}_0 \) satisfying the properties,
(1) \((F_{x}(y))(x) = y \; \forall \; y \in \mathbb{S}^1 \setminus \bar{x}\)

(2) \(F_{x}(x) = \text{id}\).

Now since \(F_{x}(x) = \text{id}\) and \(F_{x}\) is continuous, the image of \(F_{x}\) will lie entirely in \(\hat{G}_0\).

In fact, \(F_{x}\) gives a bijection between \(\mathbb{S}^1 \setminus \bar{x}\) and \(\hat{G}_0\). To see this we first observe that injectivity follows directly from condition 1. To see that it is also surjective, take \(g \in \hat{G}_0\). Then there exists a path \(g_{t} \in \hat{G}_0\) for \(t \in [0, 1]\) with \(g_{0} = \text{id}\) and \(g_{1} = g\). So that \(g_{t}(x)\) is a path in \(\mathbb{S}^1 \setminus \bar{x}\) from \(x\) to \(g(x)\). Consider the path \((F_{x}(g_{t}(x)))^{-1} \circ g_{t} \in \hat{G}_0\), it fixes \(x\) for every \(t\), and so must be the identity for each \(t\). Otherwise, by Proposition 4.4, \(G\) would be continuously \(n + 1\)–transitive, which would contradict our assumptions. As a result \(g = F_{x}(g(x))\) so \(F_{x}\) is a bijection, with inverse given by evaluation at \(x\).

Fix \(x_{0} \in \mathbb{S}^1 \setminus \bar{x}\), let \(g_{n}\) be a sequence of elements of \(\hat{G}_0\) and consider the sequence of points \(g_{n}(x_{0})\), since \(\mathbb{S}^1\) is compact \(g_{n}(x_{0})\) has a convergent subsequence \(g_{n_{k}}(x_{0})\) converging to some point \(x'\). If \(x' \neq \bar{x}\) then by continuity of \(F_{x_{0}}\), \(g_{n_{k}}\) will converge to \(F_{x_{0}}(x')\). Now if there does not exist a subsequence of \(g_{n}(x_{0})\) converging to some \(x' \neq \bar{x}\), then take a subsequence \(g_{n_{k}}\) such that \(g_{n_{k}}(x_{0})\) converges to \(\bar{x}\). If we can show that \(g_{n_{k}}(x)\) converges to \(\bar{x}\) for every \(x \in \mathbb{S}^1 \setminus \bar{x}\) then we shall be done.

Suppose for contradiction that there exists \(x \in \mathbb{S}^1 \setminus \bar{x}\) such that \(g_{n_{k}}(x)\) does not converge to \(\bar{x}\). Then there exists a subsequence of \(g_{n}(x)\) which converges to \(x' \neq \bar{x}\), but then by the previous argument the corresponding subsequence of \(g_{n_{k}}\) will converge to the homeomorphism \(F_{x}(x')\). This is a contradiction since \(F_{x}(x')(x_{0})\) would have to equal \(\bar{x}\).

\begin{proof}
Let \(g\) be an element of \(\hat{G}_0\). If \(g\) fixes a point in \(\mathbb{S}^1 \setminus \bar{x}\) then it is the identity.
\end{proof}

\begin{corollary}
Let \(g\) be an element of \(\hat{G}_0\). If \(g\) fixes a point in \(\mathbb{S}^1 \setminus \bar{x}\) then it is the identity.
\end{corollary}

\begin{proof}
Let \(x \in \mathbb{S}^1 \setminus \bar{x}\) be a fixed point of \(g\). From the previous proof we know that \(F_{x}: I \to \hat{G}_0\) is a bijection. So that \(F_{x}(g(x)) = g\), but \(g\) fixes \(x\) so that \(g = F_{x}(x) = \text{id}\).
\end{proof}

\begin{corollary}
The restriction of the action of \(\hat{G}_0\) to \(\mathbb{S}^1 \setminus \bar{x}\) is conjugate to the action of \(\mathbb{R}\) on itself by translation.
\end{corollary}

\begin{proof}
By Theorem 5.2 and Proposition 5.4 \(\hat{G}_0\) is conjugate in \(\text{Homeo}(\mathbb{S}^1)\) to a subgroup of \(\text{PSL}(2, \mathbb{R})\) which fixes the point \(\bar{x}\). Moreover, from Corollary 5.5 this is the only point fixed by a non trivial element. By identifying \(\mathbb{S}^1\) with \(\mathbb{R} \cup \{\infty\}\) so that \(\bar{x}\) is identified with \(\{\infty\}\) in the usual way, we see that \(\hat{G}_0\) is conjugate to a subgroup of the

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Möbius group acting on $\mathbb{R} \cup \{\infty\}$. Since every element will fix $\{\infty\}$, their restriction to $\mathbb{R}$ will be an element of $\text{Aff}(\mathbb{R})$ acting without fixed points, so can only be a translation. On the other hand the group must act transitively on $\mathbb{R}$ and so must be the full group of translations. This gives the result.

**Proposition 5.7** The restriction of the action of $G_0$ to $I$ is conjugate to the action of a subgroup of the affine group $\text{Aff}(\mathbb{R})$ on $\mathbb{R}$. In particular, each non trivial element of $G_0$ can act on $I$ with at most one fixed point.

**Proof** The restriction of $\widehat{G}_0$ to $S^1 \setminus \bar{x}$ is isomorphic to the restriction of $\widehat{G}_0$ to $I$. So that by Corollary 5.6 there exists a homeomorphism $\phi : I \to \mathbb{R}$ which conjugates the restriction of $\widehat{G}_0$ to $I$, to the action of $\mathbb{R}$ on itself by translation. Take $h \in G_0 \setminus \widehat{G}_0$ then $h' = \phi \circ h \circ \phi^{-1}$ is a self-homeomorphism of $\mathbb{R}$. Since $\widehat{G}_0$ is a normal subgroup of $G_0$, $h'$ conjugates every translation to another one and so by Lemma 3.9 is itself an affine map and the proof is complete.

Let $g$ be a nontrivial element of $G_0$, then $g \in \widehat{G}_0$ if and only if it acts on each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ as a conjugate of a non trivial translation. Furthermore, if $g \not\in \widehat{G}_0$ then it acts on each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ as a conjugate of a affine map which is not a translation, each of which must have a fixed point. This situation cannot actually arise as the next proposition will show.

**Proposition 5.8** $G_0 = \widehat{G}_0$

**Proof** Let $g \in G_0 \setminus \widehat{G}_0$, then $g$ acts on each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ as a conjugate of a affine map which is not a translation. Consequently, $g$ will have a fixed point in each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$. Label the fixed points of $g$ in the components of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ whose boundaries both contain $x_1$ as $y_1$ and $y_2$. Since $G$ is $n$–transitive, there exists a map $g'$ which sends $y_1$ to $x_1$ and fixes all the other $x_i$. Then $g' \circ g \circ (g')^{-1}$ fixes all the $x_i$ and hence is an element of $G_0$. On the other hand, $g' \circ g \circ (g')^{-1}$ also fixes $g'(x_1)$ and $g'(y_2)$ which lie in the same component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$, this is impossible since every non-trivial element of $G_0$ can only have one fixed point in each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$.

**Corollary 5.9** The restriction of the action of $G_0$ to $I$ is conjugate to the action of $\mathbb{R}$ on itself by translation. In particular the action is free.
We finish this section by comparing the directions that a non-trivial element of $G_0$ moves points in different components of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$. So endow $\mathbb{S}^1$ with the anti-clockwise orientation, this gives us an ordering on any interval $I \subset \mathbb{S}^1$, where for distinct points $x, y \in I$, $x \prec y$ if one travels in an anti-clockwise direction to get from $x$ to $y$ in $I$. Let $g \in G_0 \setminus \{\text{id}\}$ if $I$ is a component of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$ then we shall say that $g$ acts positively on $I$ if $x \prec g(x)$ and negatively if $x \succ g(x)$ for one and hence every $x \in I$.

Let $I$ and $I'$ be the two components of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$ whose boundaries contain $x_i$. Labeled so that in the order on the closure of $I$, $x \prec x_i$ for each $x \in I$, whereas in the order on the closure of $I'$, $x_i \prec x$ for each $x \in I'$. Then we have the following.

**Proposition 5.10** Let $g$ be a non-trivial element of $G_0$, if $g$ acts positively on $I$ then it acts negatively on $I'$ and if $g$ acts negatively on $I$ then it acts positively on $I'$.

**Proof** Let $x, x' \in I$ and $y, y' \in I'$ be points such that $x \prec x'$ and $y \succ y'$. There exists $g \in G$ fixing $x_1, \ldots, x_{i-1}$ and $x_{i+1}, \ldots, x_{n-1}$ and sending $x$ to $x'$ and $y$ to $y'$. This map will have a fixed point $\tilde{x}$ between $x'$ and $y'$, since it maps the interval between them into itself.

Let $g' \in G$ fix $x_1, \ldots, x_{i-1}$ and $x_{i+1}, \ldots, x_{n-1}$ and send $\tilde{x}$ to $x_i$. Then $g_0 = g' \circ g \circ (g')^{-1}$ will fix $x_1, \ldots, x_{n-1}$ and hence lie in $G_0$. Moreover, $g_0$ acts positively on $I$ and negatively on $I'$.

Now let $g_1 \in G_0$ be any non-trivial element which acts positively on $I$. Then there exists a path $g_t$ in $G_0$ from $g_0 = g' \circ g \circ (g')^{-1}$ to $g_1$, so that $g_t \neq \text{id}$ for any $t$. Since $g_t$ is never the identity and $g_0$ acts negatively on $I'$, $g_1$ must also act negatively on $I'$.

If $h \in G_0$ is a non-trivial element which acts negatively on $I$, then $h^{-1}$ will act positively on $I$. So that, by the above argument, $h^{-1}$ will act negatively on $I'$. This means that $h$ will act positively on $I'$ as required. 

**Corollary 5.11** If $G$ is $n$–transitive but not $n + 1$–transitive for $n \geq 3$ then $n$ is odd.

**Proof** Let $g$ be a non-trivial element of $G_0$ which acts positively on some component $I$ of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$. Then by Proposition 5.10 as we travel around $\mathbb{S}^1$ in an anti-clockwise direction the manner in which it acts on each component will alternate between negative and positive. Consequently, if $n$ was even, when we return to $I$ we would require that $g$ acted negatively on $I$, a contradiction, so $n$ is odd.
6 Continuous 3–transitivity and beyond

We begin this section by analyzing the case where $G$ is continuously 3–transitive but not continuously 4–transitive. We shall show that such a group is a convergence group and consequently conjugate to a subgroup of $\text{PSL}(2, \mathbb{R})$.

Fix distinct points $x_0, y_0 \in S^1$ and define

$$G_0 = \{ g \in G : g(x_0) = x_0, g(y_0) = y_0 \}$$

$$\bar{G} = \{ g \in G : g(x_0) = x_0 \}$$

then we have the following propositions.

**Proposition 6.1** $G_0$ is a convergence group.

**Proof** From Corollary 5.9, we know that the restriction of $G_0$ to each of the components of $S^1 \setminus \{x_0, y_0\}$ is conjugate to the action of $\mathbb{R}$ on itself by translation. Let $g_n$ be a sequence of distinct elements of $G_0$ and take a point $x \in S^1 \setminus \{x_0, y_0\}$. Then the sequence of points $g_n(x)$ will have a convergent subsequence $g_{n_k}(x)$. If this sequence converges to $x_0$ or $y_0$, then from Proposition 5.10 so will the sequences $g_{n_k}(y)$ for all $y \in S^1 \setminus \{y_0\}$ or $S^1 \setminus \{x_0\}$ respectively.

Let $I_x$ be the component of $S^1 \setminus \{x_0, y_0\}$ containing $x$. Assume that the sequence of points $g_{n_k}(x)$ converges to a point $x' \in I_x$. Now let $y$ be a point in the other component, $I_y$ of $S^1 \setminus \{x_0, y_0\}$, and consider the sequence of points $g_{n_k}(y)$ in $I_y$. If it had a subsequence which converged to $x_0$ or $y_0$ then the sequence $g_{n_k}(x)$ would have to as well. This is impossible so $g_{n_k}(y)$ must stay within a compact subset of $I_y$ and hence $g_{n_k}$ has a subsequence, $g_{n_{k_l}}$ for which $g_{n_{k_l}}(y)$ converges to some point $y' \in I_y$.

By Corollary 5.9 there exist self homeomorphisms of $I_x$ and $I_y$ to which the sequence $g_{n_{k_l}}$ converges uniformly on $I_x$ and $I_y$ respectively. Gluing these together at $x_0$ and $y_0$ gives us an element of Homeo$(S^1)$ which $g_{n_k}$ converges to uniformly. Consequently, $G_0$ is a convergence group.

**Proposition 6.2** $\bar{G}$ is a convergence group.

**Proof** Let $f_n$ be a sequence of elements of $\bar{G}$. If for every $y \in S^1 \setminus \{x_0\}$ every convergent subsequence of $f_n(y)$ converges to $x_0$ then we would be done. So assume that this is not the case, take $y \in S^1 \setminus \{x_0\}$ such that the sequence of points $f_n(y)$ has a convergent subsequence $f_{n_{k_l}}(y)$ converging to some point $\tilde{y} \neq x_0$. Let $I$ be a small open interval around $\tilde{y}$, not containing $x_0$ then since $G$ is continuously 3–transitive, there exists a map $F_{\tilde{y}} : I \to \bar{G}$ satisfying the following,
(1) \( F_\gamma(x)(\gamma) = x \) for all \( x \in I \)
(2) \( F_\gamma(\gamma) \) is the identity.

Let \( g_1, g_2 \in \tilde{G} \) satisfy \( g_1(\gamma) = y_0 \) and \( g_2(y_0) = \gamma \) consider the sequence,
\[
h_k = g_1 \circ F_\gamma(f_n(y))^{-1} \circ f_n \circ g_2
\]
of elements of \( \tilde{G} \). They all fix \( y_0 \), and since \( g_1 \circ F_\gamma(f_n(y))^{-1} \) converges to \( g_1 \) as \( k \to \infty \) we have the following.

(1) If \( h_k \) contains a subsequence \( h_{k_l} \) such that there exists a homeomorphism \( h \) with,
\[
\lim_{l \to \infty} h_{k_l} = h \quad \text{and} \quad \lim_{l \to \infty} (h_{k_l})^{-1} = h^{-1}
\]
then so does \( f_{n_k} \).

(2) Furthermore, if there exist points \( x', y' \in S^1 \) and a subsequence \( h_{k_l} \) of \( h_k \) such that,
\[
\lim_{l \to \infty} h_{k_l} = x' \quad \text{and} \quad \lim_{l \to \infty} (h_{k_l})^{-1} = y'
\]
uniformly on compact subsets of \( S^1 \setminus \{y'\} \) and \( S^1 \setminus \{x'\} \) respectively, then so does \( f_{n_k} \) \( (x' \) and \( y' \) will be replaced by \( g_1^{-1}(x') \) and \( g_1^{-1}(y') \)).

Now, since \( G_0 \) is a convergence group, one of the above situations must occur. Consequently, \( \bar{G} = \{ g \in G : g(x_0) = x_0 \} \) is a convergence group. \( \square \)

**Proposition 6.3** If \( G \) is a subgroup of \( \text{Homeo}(S^1) \) which is continuously 3–transitive but not continuously 4–transitive then \( G \) is a convergence group.

**Proof** This proof is almost identical to the previous one but we write it out in full for clarity.

Choose \( x_0 \in S^1 \) and let \( f_n \) be a sequence of elements of \( G \). Then since \( S^1 \) is compact, the sequence of points \( f_n(x_0) \) will have a convergent subsequence, \( f_{n_k}(x_0) \), converging to some point \( \tilde{x} \). Let \( I \) be a small open interval around \( \tilde{x} \), then since \( G \) is continuously 3–transitive, there exists a map \( F_\tilde{x} : I \to G \) satisfying the following,

(1) \( F_\tilde{x}(x)(\tilde{x}) = x \) for all \( x \in I \)
(2) \( F_\tilde{x}(\tilde{x}) \) is the identity.

Let \( g \in G \) send \( \tilde{x} \) to \( x_0 \) and consider the sequence,
\[
h_k = g \circ F_\tilde{x}(f_{n_k}(x_0))^{-1} \circ f_{n_k}
\]
of elements of \( G \). They all fix \( x_0 \), and since \( g \circ F_\tilde{x}(f_{n_k}(x_0))^{-1} \) converges to \( g \) as \( k \to \infty \) we have the following.
(1) If \( h_k \) contains a subsequence \( h_{k_l} \) such that there exists a homeomorphism \( h \) with,
\[
\lim_{l \to \infty} h_{k_l} = h \quad \text{and} \quad \lim_{l \to \infty} (h_{k_l})^{-1} = h^{-1}
\]
then so does \( f_{n_k} \).

(2) Furthermore, if there exist points \( x', y' \in S^1 \) and a subsequence \( h_{k_l} \) of \( h_{k_l} \) such that,
\[
\lim_{l \to \infty} h_{k_l} = x' \quad \text{and} \quad \lim_{l \to \infty} (h_{k_l})^{-1} = y'
\]
uniformly on compact subsets of \( S^1 \setminus \{ y' \} \) and \( S^1 \setminus \{ x' \} \) respectively, then so does \( f_{n_k} \) (\( x' \) and \( y' \) will be replaced by \( g^{-1}(x') \) and \( g^{-1}(y') \)).

Now, since \( \bar{G} = \{ g \in G : g(x_0) = x_0 \} \) is a convergence group \( G \) is too. \( \square \)

We now look at the case where \( G \) is continuously 4–transitive. In this case, we show that \( G \) must be \( n \)–transitive for every \( n \in \mathbb{N} \).

**Theorem 6.4** If \( G \) is continuously \( n \)–transitive for \( n \geq 4 \), then it is continuously \( n + 1 \)–transitive.

**Proof** Fix \( n \geq 4 \) and assume for contradiction that \( G \) is continuously \( n \)–transitive but not continuously \( n + 1 \)–transitive. Take \( (a_1, \ldots, a_{n-2}) \in P_{n-2} \) and define,
\[
\tilde{G} = \{ g \in G : g(a_i) = a_i \ \forall i \}
\]

Let \( I \) be a component of \( S^1 \setminus \{ a_1, \ldots, a_{n-2} \} \). Construct a homomorphism \( \Psi : \tilde{G} \to \text{Homeo}(S^1) \) in the same way as \( \Phi : G_0 \to \text{Homeo}(S^1) \) was constructed in Section 5. Explicitly, take \( g \in \tilde{G} \), restrict it to a self homeomorphism of \( \tilde{I} \) and identify the endpoints to get an element of \( \text{Homeo}(S^1) \).

Let \( \tilde{G} \) denote the image of \( \tilde{G} \) under \( \Psi \). Then as in Proposition 5.3 \( \tilde{G} \) is isomorphic to \( \bar{G} \). Using the arguments from the earlier Propositions in this section we can show that \( \tilde{G} \) is a convergence group and hence conjugate to a subgroup of \( \text{PSL}(2, \mathbb{R}) \). On the other hand, \( \tilde{G} \) is 2–transitive on \( I \) and every element fixes the identification point. This means that the action of \( \tilde{G} \) on \( I \) must be conjugate to the action of \( \text{Aff}(\mathbb{R}) \) on \( \mathbb{R} \).

Let \( I \) and \( I' \) be two components of \( S^1 \setminus \{ a_1, \ldots, a_{n-2} \} \) and let \( \phi : I \to \mathbb{R} \) be a homeomorphism which conjugates the action of \( \tilde{G} \) on \( I \) to the action of \( \text{Aff}(\mathbb{R}) \) on \( \mathbb{R} \). Let \( a_{n-1}, a'_{n-1} \) be two distinct points in \( I' \). Consider the groups
\[
G_0 = \{ g \in \tilde{G} : g(a_{n-1}) = a_{n-1} \}
\]

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and

\[ G'_0 = \{ g \in \tilde{G} : g(a'_{n-1}) = a'_{n-1} \} \]

They each act transitively on \( I \) and by Corollary 5.5 and Proposition 5.8 without fixed points. Consequently, \( \phi \) conjugates both of these actions to the action of \( \mathbb{R} \) on itself by translation. Let \( g \in G_0 \) and \( g' \in G'_0 \) be elements which are conjugated to \( x \mapsto x + 1 \) by \( \phi \). Then \( g^{-1} \circ g' \) acts on \( I \) as the identity. However, if it is equal to the identity, then \( g' = g \) fixes \( a_{n-1} \) and \( a'_{n-1} \), this is impossible as non-trivial elements of \( \tilde{G} \) can have at most one fixed point in \( I' \). So \( g^{-1} \circ g \) is a non-trivial element of \( G \) which acts as the identity on \( I \) and so by Corollary 4.5 we have that \( G \) is continuously \( n+1 \)-transitive.

\[ \square \]

7 Summary of Results

**Theorem 7.1** Let \( G \) be a transitive subgroup of \( \text{Homeo}(\mathbb{S}^1) \) which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:

1. \( G \) is conjugate to \( \text{SO}(2, \mathbb{R}) \) in \( \text{Homeo}(\mathbb{S}^1) \).
2. \( G \) is conjugate to \( \text{PSL}(2, \mathbb{R}) \) in \( \text{Homeo}(\mathbb{S}^1) \).
3. For every \( f \in \text{Homeo}(\mathbb{S}^1) \) and each finite set of points \( x_1, \ldots, x_n \in \mathbb{S}^1 \) there exists \( g \in G \) such that \( g(x_i) = f(x_i) \) for each \( i \).
4. \( G \) is a cyclic cover of a conjugate of \( \text{PSL}(2, \mathbb{R}) \) in \( \text{Homeo}(\mathbb{S}^1) \) and hence conjugate to \( \text{PSL}_k(2, \mathbb{R}) \) for some \( k > 1 \).
5. \( G \) is a cyclic cover of a group satisfying condition 3 above.

**Proof** Let \( f : [0, 1] \to G \) be a non constant continuous path. Then

\[ f(0)^{-1} \circ f : [0, 1] \to G \]

is a continuous deformation of the identity in \( G \). Consequently, Proposition 2.6 tells us that \( G \) is continuously 1–transitive.

If \( J_x = \emptyset \) for every \( x \in \mathbb{S}^1 \) then by Theorem 3.8 \( G \) is conjugate to \( \text{SO}(2, \mathbb{R}) \) in \( \text{Homeo}(\mathbb{S}^1) \). If \( J_x \neq \emptyset \) for some and hence all \( x \in \mathbb{S}^1 \) then by Theorem 3.10 \( G \) is either continuously 2–transitive or is a cyclic cover of a group \( G' \) which is continuously 2–transitive.

So assume that \( G \) is continuously 2–transitive, then by Proposition 4.6 it is continuously 3–transitive. If moreover \( G \) is not continuously 4–transitive, then by Proposition 6.3
it is a convergence group and hence conjugate to a subgroup of $\text{PSL}(2, \mathbb{R})$. On the other hand, since $G$ is continuously 3–transitive, it is 3–transitive, and hence must be conjugate to the whole of $\text{PSL}(2, \mathbb{R})$.

If we now assume that $G$ is continuously 4–transitive then by Theorem 6.4 it is continuously $n$–transitive and hence $n$–transitive for every $n \in \mathbb{N}$. So if we take $f \in \text{Homeo}(\mathbb{S}^1)$ and a finite set of points $x_1, \ldots, x_n \in \mathbb{S}^1$ there exists $g \in G$ such that $g(x_i) = f(x_i)$ and we are done.

**Theorem 7.2** Let $G$ be a closed transitive subgroup of $\text{Homeo}(\mathbb{S}^1)$ which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:

1. $G$ is conjugate to $\text{SO}(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$.
2. $G$ is conjugate to $\text{PSL}_k(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$ for some $k \geq 1$.
3. $G$ is conjugate to $\text{Homeo}_k(\mathbb{S}^1)$ in $\text{Homeo}(\mathbb{S}^1)$ for some $k \geq 1$.

**Proof** Since $G$ is a transitive subgroup of $\text{Homeo}(\mathbb{S}^1)$ which contains a non constant continuous path, Theorem 7.1 applies. It remains to show that if $G$ satisfies condition 3 in Theorem 7.1 then its closure is $\text{Homeo}(\mathbb{S}^1)$.

To see this, let $f$ be an arbitrary element of $\text{Homeo}(\mathbb{S}^1)$. If we can find a sequence of elements of $G$ which converges uniformly to $f$ then we shall be done. So let $\{a_n : n \in \mathbb{N}\}$ be a countable and dense set of points in $\mathbb{S}^1$. Choose a sequence of maps $g_n \in G$ so that $g_n(a_k) = f(a_k)$ for $1 \leq k \leq n$. Then $g_n$ will converge uniformly to $f$ so that the closure of $G$ will equal $\text{Homeo}(\mathbb{S}^1)$.

**Theorem 7.3** $\text{PSL}(2, \mathbb{R})$ is a maximal closed subgroup of $\text{Homeo}(\mathbb{S}^1)$.

**Proof** Let $G$ be a closed subgroup of $\text{Homeo}(\mathbb{S}^1)$ containing $\text{PSL}(2, \mathbb{R})$. Then $G$ is 3–transitive and by applying Theorem 7.2 we can see that $\text{Homeo}(\mathbb{S}^1)$ and $\text{PSL}(2, \mathbb{R})$ are the only possibilities for $G$. 

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References

[1] M Bestvina, Questions in geometric group theory Available at http://www.math.utah.edu/~bestvina/

[2] A Casson, D Jungreis, Convergence groups and Seifert fibered 3–manifolds, Invent. Math. 118 (1994) 441–456 MR1296353

[3] D Gabai, Convergence groups are Fuchsian groups, Ann. of Math. (2) 136 (1992) 447–510 MR1189862

[4] F W Gehring, G J Martin, Discrete quasiconformal groups I, Proc. London Math. Soc. (3) 55 (1987) 331–358 MR896224

[5] É Ghys, Groups acting on the circle, Enseign. Math. (2) 47 (2001) 329–407 MR1876932

[6] A Hinkkanen, Abelian and nondiscrete convergence groups on the circle, Trans. Amer. Math. Soc. 318 (1990) 87–121 MR1000145

[7] P Tukia, Homeomorphic conjugates of Fuchsian groups, J. Reine Angew. Math. 391 (1988) 1–54 MR961162

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Proposed: David Gabai Received: 12 December 2005
Seconded: Leonid Polterovich, Benson Farb Revised: 22 June 2006

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