Representations of Quantum Affine Algebras

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Introduction

Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra and \( \mathfrak{g} \) the corresponding affine Kač-Moody algebra. The notion of the fusion in the category \( \mathcal{O} \) of representations of affine Kač-Moody algebras \( \mathfrak{g} \) was introduced ten years ago by physicists in the framework of Conformal Field Theory. This notion was developed in a number of mathematical papers (see, for example, [TUY]), where the notion of fusion is rigorously defined and [D1] where the relation between the fusion and quantum groups in the quasiclassical region was established). This line of development was extended in [KL] to a construction on equivalence between the “fusion” category for an arbitrary negative charge and the category of representations of the corresponding quantum group (for simply-laced affine Kač-Moody algebras).

It is natural to try to define the notion of fusion for the category \( \mathcal{O} \) for affine quantum groups. One can hope that it might be useful for a deformed CFT (see [FR]). The usual construction does not work since it is based on the existence of a subalgebra \( \Gamma \subset \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \) which is a central extension of the Lie algebra \( \Gamma(\mathbb{P}^1 - \{0, 1, \infty\}, \mathfrak{g}) \). Unfortunately such subalgebra \( \Gamma \) does not have a natural generalization for the quantum case (see [D3] where the problem was stated).

In the remarkable paper [FR], I. Frenkel and N. Reshetikhin found a way to describe the fusion between finite-dimensional representations and representations from the category \( \mathcal{O} \) for affine quantum groups. In the present paper we reconsider this problem from the point of view of [KL]. Our main result is the construction of the quasi-associativity constraints (see Sect. 3 and 5). Main phenomena which should be mentioned in connection with the problem are the appearance of elliptic curves instead of genus zero curves.
and $\mathbb{Z}$-sheaves instead of bundles with flat connections. From a general point of view this reflects the fact that the categories we are considering do not carry monoidal structure. Nevertheless those categories of representations of affine quantum algebras carry some other interesting structures discussed in Sect.1 in the general situation. These structures explain the categorical meaning of the so-called quantum Knizhnik-Zamolodchikov equations introduced in [FR]. Among other results we can mention meromorphicity of the quantum $R$-matrix for any two finite-dimensional representations (see Sect. 4) which has also general categorical origin.

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1. Monoidal categories

1.1 Definitions and basic properties.

1.1.1. Definition. A monoidal category $\mathcal{C}$ is a triple $\mathcal{C} = (\tilde{\mathcal{C}}, \otimes, a)$ where $\tilde{\mathcal{C}}$ is a category, $\otimes : \tilde{\mathcal{C}} \times \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ is a functor and $a$ is the natural transformation between the functors $\otimes(\otimes \times \text{id})$ and $\otimes(\text{id} \times \otimes)$ from $\tilde{\mathcal{C}} \times \tilde{\mathcal{C}} \times \tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}$, $a = \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\}$, $X, Y, Z$ in $\tilde{\mathcal{C}}$ such that all $a_{X,Y,Z}$ are isomorphisms, the pentagon axiom is satisfied, and there exists an object $U$ in $\tilde{\mathcal{C}}$ and an isomorphism $u : U \otimes U \to U$ such that the functors $X \to X \otimes U$ and $X \to U \otimes X$ are autoequivalences of $\tilde{\mathcal{C}}$ (see [DM]). We call such a pair $(U, u)$ the identity object of $\mathcal{C}$. (It is clear that the identity object is determined uniquely up to a unique isomorphism.)

Let $\mathcal{C} = (\tilde{\mathcal{C}}, \otimes, A)$ be a monoidal category, and $(U, u)$ the identity object in $\mathcal{C}$.

Lemma. a) For any object $X$ in $\tilde{\mathcal{C}}$ there exist unique isomorphisms $r_X : X \to X \otimes U$, $\ell_X : X \to U \otimes X$ such that $a_{X,U,U} \circ (r_X \otimes \text{id}_U) = \text{id}_X \otimes u$ and $\text{id}_U \otimes \ell_X = a_{U,U,X} \circ (u \otimes \text{id}_X)$.

b) $r_U = u$.

c) For any morphism $\alpha : Y \to X$ in $\tilde{\mathcal{C}}$ the diagrams

$$
\begin{array}{ccc}
Y & \xrightarrow{\alpha} & X \\
\downarrow r_Y & & \downarrow r_X \\
Y \otimes U & \xrightarrow{\alpha \otimes 1} & X \otimes U
\end{array} \quad \quad 
\begin{array}{ccc}
Y & \xrightarrow{\alpha} & X \\
\downarrow \ell_Y & & \downarrow \ell_X \\
U \otimes Y & \xrightarrow{\alpha \otimes 1} & U \otimes X
\end{array}
$$

are commutative.

Proof: Well known. (See [DM].)

1.1.2 Definition. We say that a monoidal category $\mathcal{C} = (\tilde{\mathcal{C}}, \otimes, a)$ is strict if for any $X, Y, Z$ in $\tilde{\mathcal{C}}$ we have $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, $a_{X,Y,Z} = \text{id}$, there exists an object $\mathbb{1}$ in $\tilde{\mathcal{C}}$ such that $\mathbb{1} \otimes X = X \otimes \mathbb{1} = X$ for all $X$ in $\tilde{\mathcal{C}}$, and for any $X, Y$ in $\tilde{\mathcal{C}}$ the composition

$$X \otimes Y = (X \otimes \mathbb{1}) \otimes Y = X \otimes (\mathbb{1} \otimes Y) = X \otimes Y$$

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is equal to $\text{id}_{X \otimes Y}$.

Mac Lane’s theorem says that any monoidal category is equivalent to a strict one (see [M, Chapter 7]). To simplify formulas we will from now on assume that all monoidal categories are strict. We will also assume that the category $\tilde{\mathcal{C}}$ is abelian, contains inductive limits and that $\otimes$ is an additive exact functor. Let $F_{\mathcal{C}} \overset{\text{def}}{=} \text{End}_{\tilde{\mathcal{C}}} (\mathbb{1})$. It is easy to see that $F_{\mathcal{C}}$ is a commutative ring and for any $X, Y$ in $\tilde{\mathcal{C}}$ the group $\text{Hom}_{\tilde{\mathcal{C}}}(X, Y)$ has a natural structure of an $F_{\mathcal{C}}$-module.

From now on we will denote the category $\tilde{\mathcal{C}}$ simply as $\mathcal{C}$. We hope that this will not create any confusion.

1.1.3. Assume until the end of this section that $F_{\mathcal{C}}$ is a field. Any $F_{\mathcal{C}}$-vector space $R$ can be written as an inductive limit of finite dimensional spaces $R = \lim \to R_{i}$. We define $R_{\mathcal{C}} \overset{\text{def}}{=} \lim R_{i} \otimes \mathbb{1}$. It is clear that the object $R_{\mathcal{C}}$ in $\mathcal{C}$ does not depend on a choice of a presentation $R = \lim R_{i}$.

We denote by $\text{Vec}_{\mathcal{C}}$ the category of $F_{\mathcal{C}}$-vector spaces and for any object $X$ in $\mathcal{C}$ we denote by $L_{X}$ the functor from $\text{Vec}_{\mathcal{C}}$ to itself such that

$$L_{X}(R) \overset{\text{def}}{=} \text{Hom}_{\mathcal{C}}(X, R_{\mathcal{C}}) \quad \text{for all } R \in \text{Vec}_{\mathcal{C}}.$$ 

It is easy to see that the functor $L_{X}$ satisfies the conditions of Theorem 5.6.3 in [M]. Therefore the functor $L_{X}$ is representable.

**Definition.** We denote by $\langle X \rangle$ the object in $\text{Vec}_{\mathcal{C}}$ which represents the functor $L_{X}$.

As follows from the definition of $\langle X \rangle$, for any $X$ in $\mathcal{C}$ we have a natural isomorphism $\langle X \rangle^{\vee} \cong \text{Hom}_{\mathcal{C}}(X, \mathbb{1})$ where $L^{\vee} \overset{\text{def}}{=} \text{Hom}(L, F_{\mathcal{C}})$ for any $L$ in $\text{Vec}_{\mathcal{C}}$.

1.1.4. Let $\mathcal{C}$ be a strict monoidal category. We denote by $\overline{\mathcal{C}}$ the set of isomorphism classes of objects in $\mathcal{C}$. For any $X$ in $\tilde{\mathcal{C}}$ we denote by $[X] \in \overline{\mathcal{C}}$ the equivalence class of $X$.

**Definition.** We say that an object $X$ in $\mathcal{C}$ is rigid if there exists an object $Y$ in $\mathcal{C}$ and a pair of morphisms

$$i_{X} : \mathbb{1} \to X \otimes Y \quad e_{X} : Y \otimes X \to \mathbb{1}.$$
such that the compositions

\[ X = \mathbb{1} \otimes X \overset{i_X \otimes \text{id}}{\longrightarrow} X \otimes Y \otimes X \overset{\text{id} \otimes e_X}{\longrightarrow} X \]

\[ Y = Y \otimes \mathbb{1} \overset{\text{id} \otimes i_Y}{\longrightarrow} Y \otimes X \otimes Y \overset{e_Y \otimes \text{id}}{\longrightarrow} Y \]

are equal to \text{id}_X and \text{id}_Y correspondingly. In this case we say that \( Y \) is dual to \( X \).

It is easy to see that such a triple \((Y, i_X, e_X)\) if it exists, is unique up to a unique isomorphism. For any rigid \( X \) in \( C \) we denote by \([X]^* \in \overline{C}\) the isomorphism class of objects \( Y \) as in Definition 1.1.4.

1.1.5 Definition. We say that the category \( C \) is rigid if all its objects are rigid and for any \([Y]\) in \( \overline{C} \) there exists \( X \) in \( C \) such that \([X]^* = [Y]\).

If \( C \) is a rigid category, then there exists an equivalence \( * : C \rightarrow C^{\text{op}} \) of categories such that \( X^* \) is dual to \( X \) for all \( X \) in \( C \), \( \alpha \in \text{Hom}_C(X, Y) \) the morphism \( \alpha^* \in \text{Hom}_C(Y^*, X^*) \) is the composition

\[ Y^* \longrightarrow Y^* \otimes \mathbb{1} \overset{\text{id} \otimes i_Y}{\longrightarrow} Y^* \otimes X \otimes X^* \overset{\text{id} \otimes \alpha \otimes \text{id}}{\longrightarrow} Y^* \otimes Y \otimes X^* \overset{e_Y \otimes \text{id}}{\longrightarrow} \mathbb{1} \otimes X^* = X^*. \]

Moreover, such an equivalence is unique up to a unique isomorphism.

We will fix such an equivalence between the categories \( C \) and \( C^{\text{op}} \). Then the functor \( X \rightarrow X^{**} \) is an auto-equivalence of \( C \).

1.1.6. Example. Let \((H, m, \Delta, i, \varepsilon, S)\) be a Hopf algebra over a ring \( F \), \( C_H \) be the category of \( H \)-modules \( X = (\rho_X, X) \) and \( \otimes : C_H \times C_H \rightarrow C_H \) be the tensor product over \( F \). That is, \( X \otimes Y = ((\rho_X \otimes \rho_Y) \circ \Delta, X \otimes Y) \), and \( \mathbb{1} \overset{\text{def}}{=} (F, \varepsilon) \). Then \((C_H, \otimes, \mathbb{1})\) is a strict monoidal category. In this case, \( F_C = F \).

Let \( H_0 \subset H \) be the kernel of the counit \( \varepsilon \). For any \( X = (\rho_X, X) \) in \( C_H \) we define \( X^{H_{(0)}} \overset{\text{def}}{=} \rho_X(H_0)X \). Often we will write \( X_{(0)} \) instead of \( X^{H_{(0)}} \). In the case when \( F \) is a field, we can identify \( \langle X \rangle \) (see Definition 1.1.3) with the quotient \( \langle X \rangle \overset{\text{def}}{=} X/X_{(0)} \).

Assume that \( F \) is a field. An object \( X = (\rho_X, X) \) is rigid if \( \dim F X < \infty \). In this case, \( X^* = (\rho_X^*, X^*) \) where \( X^* \overset{\text{def}}{=} X^\vee = \text{Hom}_F(X, F) \), and \( \rho_X^*(h) = (\rho_X(S(h)))^* \), where for any \( \alpha \in \text{End } X \) we denote by \( \alpha^\vee \) the endomorphism of \( X^\vee \) dual to \( \alpha \). In this case, the morphism \( e_X : X^* \otimes X \rightarrow \mathbb{1} \) is induced by the natural pairing \( X^* \otimes X \rightarrow F \). This pairing defines a canonical isomorphism of the linear space \( X^{**} \) with \( X \) and the action \( \rho_{X^{**}} \) of \( H \) on \( X^{**} = X \) is given by the rule \( \rho_{X^{**}}(h) = \rho(S^2(h)) \).
1.1.7 Proposition. For any $X$ in $\mathcal{C}$ and $\varphi \in \text{Hom}(X, X)$ the diagrams

$$
\begin{array}{ccc}
1 \xrightarrow{i_X} X \otimes X^* & \xrightarrow{\varphi \otimes id} & X^* \otimes X \\
\downarrow i_X & & \downarrow 1 \otimes \varphi \\
X \otimes X^* & \xrightarrow{id \otimes \varphi^*} & X \otimes X^* \\
\end{array}
$$

are commutative.

Proof: We prove the commutativity of the first diagram. The proof of the commutativity of the second diagram is completely analogous.

Let

$$a \overset{\text{def}}{=} (\varphi \otimes id_{X^*}) \circ i_X, \quad b \overset{\text{def}}{=} (id_X \otimes \varphi^*) \circ i_X \in \text{Hom}_\mathcal{C}(1, X \otimes X^*).$$

By the definition of $i_X$ and $e_X$, $a$ is equal to the composition

$$1 \xrightarrow{i_X} X \otimes X^* \xrightarrow{\varphi \otimes id_{X^*}} X \otimes X^* \xrightarrow{i_X \otimes id_X \otimes X^*} X \otimes X^* \otimes X \otimes X^* \xrightarrow{id_X \otimes e_X \otimes id_{X^*}} X \otimes X^*.$$ 

But the composition $(i_X \otimes id_{X \otimes X^*}) \circ (\varphi \otimes id_{X^*})$ is equal to the composition 

$$(id_{X \otimes X^*} \otimes \varphi \otimes id_{X^*}) \circ (i_X \otimes id_{X \otimes X^*}).$$

Since the compositions

$$1 \xrightarrow{i_X} X \otimes X^* \xrightarrow{id_{X \otimes X^*} \otimes i_X} X \otimes X^* \otimes X \otimes X^* \text{ and } 1 \xrightarrow{i_X} X \otimes X^* \xrightarrow{i_X \otimes id_{X \otimes X^*}} X \otimes X^* \otimes X \otimes X^*$$

are equal (and both coincide with the composition $1 = 1 \otimes 1 \xrightarrow{i_X \otimes i_X} X \otimes X^* \otimes X \otimes X^*$)

we see that $a$ is equal to the composition

$$1 \xrightarrow{i_X} X \otimes X^* \xrightarrow{id_{X \otimes X^*} \otimes i_X} X \otimes X^* \otimes X \otimes X^* \xrightarrow{id_{X \otimes X^*} \otimes \varphi \otimes id_{X^*}} X \otimes X^* \otimes X \otimes X^* \xrightarrow{id_X \otimes e_X \otimes id_{X^*}} X \otimes X^*.$$ 

But the composition of the last three arrows is equal to $b = id_X \otimes \varphi^*$. Lemma 1.1.7 is proved.

1.1.8. For any $X, Y$ in $\mathcal{C}$ the morphisms $i, e$ defined as the compositions

$$i : 1 \xrightarrow{i_X} X \otimes X^* = X \otimes 1 \xrightarrow{\otimes i_Y \otimes id} (X \otimes Y) \otimes (Y^* \otimes X^*)$$

$$e : (Y^* \otimes X^*) \otimes (X \otimes Y) \xrightarrow{id \otimes e_X \otimes id} Y^* \otimes 1 \otimes Y = Y^* \otimes Y \xrightarrow{e_Y} 1$$

satisfy the conditions of Definition 1.1.4. Therefore they define isomorphisms of $(X \otimes Y)^*$ with $Y^* \otimes X^*$ such that $i = i_X \otimes Y$ and $e = e_X \otimes Y$. We will freely use this identification and will therefore identify the second dual $(X \otimes Y)^{**}$ with $X^{**} \otimes Y^{**}$.

Remark: As follows from ([DM], 1.17) the rigidity of $\mathcal{C}$ implies the semisimplicity of $1$. We will always assume that $1$ is simple. In this case $F_C$ is a field.
1.1.9. **Definition.** Let $\mathcal{C}$ be a monoidal category, $\mathcal{D} \subset \mathcal{C}$ a full monoidal subcategory such that any object of $\mathcal{D}$ is rigid.

For any $V, W$ in $\mathcal{C}$ and $X, Y$ in $\mathcal{D}$ we define the maps $\alpha_V^X : (V \otimes X) \to (X^{**} \otimes V)^{**}$ and

$$\varphi_{V, W}^{X, Y} : \hom_{\mathcal{C}}(V \otimes X, Y \otimes W) \to \hom_{\mathcal{C}}(Y^* \otimes V, W \otimes X^*)$$

as the compositions

$$\alpha_V^X : (V \otimes X)^{**} = \hom_{\mathcal{C}}(V \otimes X, \mathbb{1}) \to \hom_{\mathcal{C}}(X^{**} \otimes V \otimes X^*, X^{**} \otimes X^*)$$

$$\begin{align*}
&= (X^{**} \otimes V)^{**}.
\end{align*}$$

$$\varphi_{V, W}^{X, Y}(a) : Y^* \otimes V \xrightarrow{id \otimes Y} Y^* \otimes V \otimes X \otimes X^* \xrightarrow{id \otimes a \otimes id} Y^* \otimes Y \otimes W \otimes X^* \xrightarrow{ev} W \otimes X^*$$

for all $a \in \hom_{\mathcal{C}}(V \otimes X, Y \otimes W)$.

**Remark:** We will write $\varphi_{V, W}^{X, Y}$ instead of $\varphi_{V, W}^{X, Y}$.

**Lemma.** If $F_\mathcal{C}$ is a field, then there exist $F_\mathcal{C}$-linear maps $\beta_{X^{**}}^V : (X^{**} \otimes V) \to (V \otimes X)$ such that $\alpha_V^X = (\beta_{X^{**}}^V)^{**}$.

**Proof:** For any $R$ in $\text{Vec}_\mathcal{C}$ we can define a $F_\mathcal{C}$-linear map $\beta_{X^{**}}^V(R) : \hom(V \otimes X, R_\mathcal{C}) \to \hom(X^{**} \otimes V, R_\mathcal{C})$ exactly in the same way as we have defined the map $\alpha_V^X$. Then maps $\alpha_V^X(R)$ define a morphism from the functor $L_{V \otimes X}$ to the functor $L_{X^{**} \otimes V}$. The corresponding morphism between the representing objects is $\beta_{X^{**}}^V$.

1.1.10. For any $V$ in $\mathcal{C}$ we denote by $E_V$ the ring of endomorphisms of $V$ and by $E_V^{op}$ the opposite ring. If $V$ is rigid, the map $f \to f^*$ defines an isomorphism of $E_V$ with $E_V^{op}$.

**Lemma.** a) The linear maps $\alpha_V^X, \beta_{X^{**}}^V$ and $\varphi_{V, W}^{X, Y}$ are isomorphisms.

b) For any $V$ in $\mathcal{C}$, $X, Y$ in $\mathcal{D}$ the linear map $\alpha_V^{(X \otimes Y)}$ is equal to the composition $\alpha_{Y^{**} \otimes V}^X \circ \alpha_{V \otimes X}^Y$ and the map $\beta_{(X \otimes Y)^{**}}^V$ is equal to the composition $\beta_{V^{**} \otimes X}^Y \circ \beta_{X^{**} \otimes V}^Y$.

c) For any $f_X \in E_X$, $f_Y \in E_Y$, $f_W \in E_W, f_V \in E_V$ and $a \in \hom_{\mathcal{C}}(V \otimes X, Y \otimes W)$ we have

$$\varphi_{V, W}^{X, Y}((f_Y \otimes f_W) \circ a \circ (f_V \otimes f_X)) = (f_W \otimes f_X^*) \circ \varphi_{V, W}^{X, Y}(a) \circ (f_Y^* \otimes f_V).$$
PROOF: a) Consider the map from \( (X^{**} \otimes V)^{\vee} \) to \( (V \otimes X)^{\vee} \) defined as the composition

\[
(X^{**} \otimes V)^{\vee} = \text{Hom}_C(X^{**} \otimes V, \mathbb{1}) \longrightarrow \text{Hom}_C(X^* \otimes X^{**} \otimes V \otimes X, X^* \otimes X) \\
\xrightarrow{i_X \otimes \text{id}_V \otimes \text{id}_X \otimes \text{id}_X} \text{Hom}_C(V \otimes X, \mathbb{1})
\]

\[
= (V \otimes X)^{\vee}.
\]

As follows immediately from Definition 1.1.4, this map is the inverse to \( \alpha^X_V \). The construction of the inverse to \( \varphi^{X,Y}_{V,W} \) is completely analogous.

b) Follows immediately from the definitions.

c) We have to show that for any \( a \in \text{Hom}_C(V \otimes X, X \otimes V) \) the following equalities hold:

i) \( \varphi^{X,Y}_{V,W}(a \circ (f_V \otimes \text{id}_X)) = \varphi^{X,Y}_{V,W}(a)(\text{id}_Y \otimes f_V) \) for any \( f_V \in E_V \)

ii) \( \varphi^{X,Y}_{V,W}(a \circ (\text{id}_W \otimes f_X)) = (\text{id}_W \otimes f_X^*) \circ \varphi^{X,Y}_{V,W}(a) \) for all \( f_X \in E_X \).

iii) \( \varphi^{X,Y}_{V,W}((\text{id}_Y \otimes f_W) \circ a) = (f_W \otimes \text{id}_X) \circ \varphi^{X,Y}_{V,W}(a) \)

iv) \( \varphi^{X,Y}_{V,W}((f_Y \otimes \text{id}_W) \circ a) = \varphi^{X,Y}_{V,W}(a) \circ (f_Y^* \otimes \text{id}_V) \).

The proofs of i) and iii) are straightforward. We will show how to prove ii). The proof of (iv) is completely analogous.

By the definition the map \( \varphi^{X,Y}_{V,W}(a \circ (\text{id}_V \otimes f_X)) \) is equal to the composition

\[
Y^* \otimes V \xrightarrow{\llbracket \otimes i_X} Y^* \otimes V \otimes X \otimes X^* \xrightarrow{id \otimes f_X \otimes id} Y^* \otimes V \otimes X \otimes X^* \\
\xrightarrow{id \otimes a \otimes id} Y^* \otimes Y \otimes W \otimes X^* \xrightarrow{e_Y \otimes id} W \otimes X^*.
\]

As follows from Lemma 1.1.7 a) this composition is equal to the composition

\[
Y^* \otimes V \xrightarrow{\llbracket \otimes i_X} Y^* \otimes V \otimes X \otimes X^* \xrightarrow{id \otimes id_X \otimes f_X^*} Y^* \otimes V \otimes X \otimes X^* \\
\xrightarrow{id \otimes a \otimes id} Y^* \otimes Y \otimes W \otimes X^* \xrightarrow{e_Y \otimes id} W \otimes X^*.
\]

But the last composition is equal to \( (\text{id}_W \otimes f_X^*) \circ \varphi^{X,Y}_{V,W}(a) \).

Lemma 1.1.10 is proved.

1.1.11. Let \( H \) be a Hopf algebra, \( C = C_H \), \( X = (\rho_X, \underline{X}) \), \( Y = (\rho_Y, \underline{V}) \) be \( H \)-modules such that \( \dim C_H < \infty \).

Let \( P^X_V : \underline{X} \otimes \underline{V} \rightarrow \underline{V} \otimes \underline{X} \) be the permutation \( P^X_V(x \otimes v) = v \otimes x \). We can consider \( P^X_V \) as a linear map from \( \underline{X}^{**} \otimes \underline{V} \) to \( \underline{V} \otimes \underline{X} \).
**Lemma.** $P^X_V$ maps the subspace $(X^{**} \otimes V)_{(0)}$ into $(V \otimes X)_{(0)}$ and induces the linear map from $(X^{**} \otimes V)$ to $(V \otimes X)$ equal to $\beta^V_X$.

**Proof:** Let $(P^X_V)^* : (V \otimes X)^* \to (X^{**} \otimes V)^*$ be the linear map dual to $P^X_V$. It is sufficient to show that for any $\lambda \in (V \otimes X)^\vee \subset (V \otimes X)^*$ we have $(P^X_V)^*(\lambda) = (\alpha^X_V)(\lambda)$.

Consider first the case when $V = X^*$ and $\lambda = e_X$. It is easy to check that $\alpha^X_X(e_X) = e_X$ and the validity of Lemma 1.1.12 follows from definition of $e_X$.

For any $\lambda \in (V \otimes X)^\vee = \text{Hom}_C(V \otimes X, \mathbb{1})$ we have $\lambda = e_X \circ (\tilde{\lambda} \otimes \text{id}_X)$ where $\tilde{\lambda} \in \text{Hom}_C(V, X^*)$ and Lemma 1.1.12 follows from the functoriality of $\alpha^X_V$ and $P^X_V$. Lemma 1.1.12 is proved.

**Definition.** A strict endomorphism $\mathcal{T}$ of $\mathcal{C}$ is a functor from the category $\mathcal{C}$ to itself such that

- a) $\mathcal{T}(\mathbb{1}) = \mathbb{1}$,
- b) $\mathcal{T}(X \otimes Y) = \mathcal{T}(X) \otimes \mathcal{T}(Y)$ for all $X, Y$ in $\mathcal{C}$ and those identifications are compatible with morphisms in $\mathcal{C}$, and
- c) for any rigid object $X$ in $\mathcal{C}$ we have $\mathcal{T}(X^*) = (\mathcal{T}(X))^*$,
- d) $\mathcal{T}(i_X) = i_{\mathcal{T}(X)}, \mathcal{T}(e_X) = e_{\mathcal{T}(X)}$.

We say that $\mathcal{T}$ is a strict automorphism if it is an equivalence of categories.

If $\mathcal{T}$ is a strict automorphism of $\mathcal{C}$, then for any $X$ in $\mathcal{C}$, $\mathcal{T}$ defines an isomorphism from $\text{Hom}_C(X, \mathbb{1}) = \langle X \rangle^\vee$ to $\text{Hom}_C(\mathcal{T}(X), \mathbb{1}) = \langle \mathcal{T}(X) \rangle^\vee$. It is easy to see that this isomorphism $\langle X \rangle^\vee \sim \langle \mathcal{T}(X) \rangle^\vee$ comes from an isomorphism $\langle X \rangle \sim \langle \mathcal{T}(X) \rangle$.

**Example:** For any rigid monoidal category $\mathcal{D}$ the functor $X \to X^{**}$ is a strict automorphism. We denote the inverse to this automorphism by $\mathcal{T}$. It is clear that any strict automorphism $\mathcal{T}$ of $\mathcal{D}$ commutes with $\mathcal{T}$.

**Remark:** We can interpret the linear map $\beta^V_X$ defined in Lemma 1.1.9 as the linear map from $\langle X \otimes V \rangle$ to $\langle V \otimes \mathcal{T}(X) \rangle$.

**Let $\mathcal{C}$ be a monoidal category, $\mathcal{D}$ and $\mathcal{O}$ full subcategories of $\mathcal{C}$ such that $\mathcal{D}$ is rigid and $\mathcal{T}$ a strict automorphism of $\mathcal{D}$.**
**Definition.** A right $\mathcal{T}$-braiding of $\mathcal{D}$ on $\mathcal{O}$ is a functorial isomorphism

$$s_{V,X} : V \otimes X \to \mathcal{T}(X) \otimes V,$$

defined for all $V$ in $\mathcal{O}$ and $X$ in $\mathcal{D}$ such that $s_{V,\mathbb{I}} = id_V$ and for all $X,Y$ in $\mathcal{D}$ we have

$$s_{V,X \otimes Y} = (id_{\mathcal{T}(X)} \otimes s_{V,Y}) \circ (s_{V,X} \otimes id_Y).$$

**1.1.17 Proposition.** For any $X$ in $\mathcal{D}$ and $V$ in $\mathcal{O}$ the diagram

$$\begin{array}{ccc}
\mathcal{T}(X)^* \otimes V \otimes X & \xrightarrow{id_{\mathcal{T}(X)} \otimes s_{V,X}} & \mathcal{T}(X)^* \otimes \mathcal{T}(X) \otimes V \\
\downarrow s_{V,X \otimes X}^{-1} \otimes id_X & & \downarrow e_{\mathcal{T}(X)} \\
V \otimes X^* \otimes X & \xrightarrow{e_X} & V
\end{array}$$

is commutative.

**Proof:** As follows from the functoriality of $s_{V,X}$ and the equality $s_{V,\mathbb{I}} = id$, the diagram

$$\begin{array}{ccc}
V \otimes (X^* \otimes X) & \xrightarrow{s_{V,X \otimes X}} & (X^* \otimes X) \otimes V \\
\downarrow e_X & & \downarrow e_X \\
V & = & V
\end{array}$$

is commutative. On the other hand, we know that

$$s_{V,X^* \otimes X} = (id_{\mathcal{T}(X^*)} \otimes s_{V,X}) \circ (s_{V,X^*} \otimes id_X).$$

Therefore the diagram

$$\begin{array}{ccc}
\mathcal{T}(X)^* \otimes V \otimes X & \xrightarrow{id_{\mathcal{T}(X^*)} \otimes s_{V,X}} & \mathcal{T}(X)^* \otimes \mathcal{T}(X) \otimes V \\
\downarrow s_{V,X^*} \otimes id_X & & \downarrow e_{\mathcal{T}(X)} \\
V \otimes X^* \otimes X & \xrightarrow{e_X} & V
\end{array}$$

is commutative. Proposition 1.1.17 is proved.
1.1.18 Proposition. For any $V$ in $O$ and any $X$ in $D$

$$\varphi^X_{V,T(X)}(s_{V,X}) = s^{-1}_{V,X}.$$ 

Proof: By the definition the map $\varphi^X_{V,T(X)}(s_{V,X}) \in Hom_C(T(X)^* \otimes V, V \otimes X^*)$ is defined as the composition

$$T(X)^* \otimes V \xrightarrow{id \otimes i_X} T(X)^* \otimes V \otimes X^* \xrightarrow{s_{V,X}} T(X) \otimes V \otimes X^* \xrightarrow{id \otimes e_{T(X)} \otimes id} V \otimes X^*.$$ 

As follows from Lemma 1.1.17, this composition is equal to the composition

$$T(X)^* \otimes V \xrightarrow{id \otimes i_X} T(X)^* \otimes V \otimes X^* \otimes X^* \xrightarrow{s_{V,X}^{-1} \otimes id} V \otimes X^*.$$ 

As follows from the definition of rigidity, the composition of the last two morphisms is equal to $s_{V,X}^{-1}$. Proposition 1.1.18 is proved.

1.1.19 Definition. Let $C$ be a strict monoidal category. A braiding $s$ on $C$ is a functorial system of isomorphisms $s_{X,Y} \in Isom(X \otimes Y, Y \otimes X)$ for $X,Y$ in $C$ such that

a) for any $X,Y,Z$ in $C$ we have

$$s_{X \otimes Y,Z} = (s_{X,Z} \otimes id_Y) \circ (id_X \otimes s_{Y,Z}),$$ 
$$s_{X,Y \otimes Z} = (id_X \otimes s_{X,Z}) \circ (s_{X,Y} \otimes id_Z);$$

b) $s_{X,\mathbb{1}} = s_{\mathbb{1},X} = id_X$ for all $X$ in $C$.

1.2 KZ-data.

1.2.1 Definition. Let $D$ be a rigid monoidal category. A weak braiding $b = (D^{(2)}, s)$ on $D$ is

1) A choice of a subset $\overline{D}^{(2)}$ of $\overline{D} \times \overline{D}$ such that

a) For any $X,Y,Z$ in $D$ such that $([X],[Y]),([X],[Z])$ are in $\overline{D}^{(2)}$ the pair $([X],[Y \otimes Z])$ is in $\overline{D}^{(2)}$.

b) For any $X,Y,Z$ in $D$ such that $([X],[Z])$ and $([Y],[Z])$ in $\overline{D}^{(2)}$ the pair $([X \otimes Y],[Z])$ is in $\overline{D}^{(2)}$. 

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c) \((\overline{X}, 11) \in \overline{D}^{(2)}\) and \((11, X) \in \overline{D}^{(2)}\), for all \(\overline{X} \in \overline{D}\).

2) A functorial isomorphism \(s_{X,Y}\) between the restrictions of functors

\((X, Y) \to X \otimes Y, (X, Y) \to (Y \otimes X)\) on \(D^{(2)}\) such that

a) \(s_{11, X} = s_{X, 11} = \text{id}\) for all \(X \in D\)

b) \(s_{X,Y \otimes Z} = (1_Y \otimes s_{X,Z}) \circ (s_{X,Y} \otimes 1_Z)\) for all \(X, Y, Z \in D\) such that

\((X, Y), (X, Z) \in D^{(2)}\)

c) \(s_{X \otimes Y, Z} = (s_{X,Z} \otimes 1_Y) \circ (1_X \otimes s_{Y,Z})\) for all \(X, Y, Z \in D\) such that

\((X, Z), (Y, Z) \in D^{(2)}\),

where we denote by \(D^{(2)}\) the full subcategory of \(D \times D\) of pairs \((X, Y)\) such that

\(([X], [Y]) \in \overline{D}^{(2)}\).

1.2.2 Definition. A \(KZ\)-data consists of a monoidal category \(C\), its full subcategories \(D, C^{\pm}\), strict automorphisms \(T^{\pm}\) of \(D\), right \(T^{\pm}\) braiding \(s^{\pm}\) of \(D\) on \(C^{\pm}\), and a weak braiding \((D^{(2)}, s)\) compatible with the automorphisms \(T^{\pm}\).

1.2.3. We say that \(KZ\)-data \((C, D, C^{\pm}, s^{\pm}, T^{\pm}, D^{(2)}, s)\) is rigid if \(D\) is rigid. In this case we denote by \(T\) the automorphism of \(D\) which is the composition \(T = T_{\pm} T_{-} T_{+}\).

1.2.4 Definition. Let \(K = (C, D, C^{\pm}, s^{\pm}, T^{\pm}, D^{(2)}, s)\) be a rigid \(KZ\)-data.

a) We say that a pair \((X, Y) \in D \times D\) is \(T\)-generic (or simply generic) if for any \(r \in \mathbb{Z}\),

\((T^r(X), Y) \in D^{(2)}\).

b) For any \(n \in \mathbb{Z}\) we denote by \(S_n = S_n^T\) the set of \(n\)-tuples \((X_1, \ldots, X_n) \in D^n\) such that

all the pairs \((X_i, X_j), 1 \leq i \neq j \leq n\) are generic.

c) For any \(i, 1 \leq i \leq n - 1\) we denote by \(p_i : S_n^T \to S_{n-1}^T\) the map

\[ (X_1, \ldots, X_n) \to (X_1, \ldots, X_{i-1}, X_i \otimes X_{i+1}, X_{i+2}, \ldots, X_n). \]

d) We define an action of the group \(\mathbb{Z}^n\) on \(S_n^T\) by the rule

\[ E_i(X_1, \ldots, X_n) = (X_1, \ldots, X_{i-1}, T(X_i), X_{i+1}, \ldots, X_n), \]

where \(\{E_i\}, 1 \leq i \leq n\), is the standard set of generators of the group \(\mathbb{Z}^n\).
e) For any pair $V \in C^+, W \in C^-$ and a point $x = (X_1, \ldots, X_n) \in S_n$ we define a vector space $F_{V,W}^{(n)}(x) \overset{\text{def}}{=} \langle V \otimes X_1 \otimes \cdots \otimes X_n \otimes W \rangle$. We will consider $F_{V,W}^{(n)}$ as a set-theoretical vector bundle over $S_n$.

It is clear that for any $i$, $1 \leq i \leq n - 1$ we have a canonical isomorphism

$$F_{V,W}^{(n)} \sim p_i^*(F_{V,W}^{(n-1)}).$$

1.2.5 Definition. We denote by $\theta_n : S_n \to S_n$ an automorphism given by the rule

$$\theta_n(X_1, \ldots, X_n) = (X_2, \ldots, X_n, T(X_1))$$

and by $\hat{\theta}_n$ its lifting $\hat{\theta}_n : F_{V,W}^{(n)} \to \theta_n^*(F_{V,W}^{(n)})$ to the bundle defined as the composition

$$\hat{\theta}_n : F_{V,W}^{(n)}(x) = \langle V \otimes X_1 \otimes \cdots \otimes X_n \otimes W \rangle \to \langle V \otimes X_1 \otimes Y \otimes W \rangle \to \langle V \otimes Y \otimes W \otimes T_\ast T_+(X_1) \rangle \to \langle V \otimes Y \otimes T(X_1) \otimes W \rangle = F_{V,W}^{(n)}(\theta_n(x))$$

for all $x = (X_1, \ldots, X_n) \in S_n$ where $Y = X_2 \otimes \cdots \otimes X_n$ and the isomorphism $\beta_Z^U : \langle Z \otimes U \rangle \sim \langle U \otimes T_\ast(Z) \rangle$ is defined in 1.1.15.

It is clear that $p_{n-1} \circ \theta_n^2 = \theta_{n-1} \circ p_{n-1}$. Since $F_{V,W}^{(n)} = p_{n-1}^*(F_{V,W}^{(n-1)})$ we can consider $\hat{\theta}_n^2$ and $p_{n-1}^*(\hat{\theta}_{n-1})$ as automorphisms of $F_{V,W}^{(n)}$ lifting the transformation $\theta_n^2$ of $S_n$. Analogously for any $i$, $1 \leq i < n - 1$ we can consider $p_i^*(\hat{\theta}_{n-1})$ as an automorphism of $F_{V,W}^{(n)}$ over $\theta_n$.

1.2.6 Proposition.

a) $\hat{\theta}_n$ commutes with endomorphisms of $Y = X_1 \otimes \cdots \otimes X_{n-1}$ and for any endomorphism $a$ of $X_n$ we have $(\theta_n \circ a) = T(a) \circ \theta_n$.

b) $\hat{\theta}_n^2 = p_{n-1}^*(\hat{\theta}_{n-1})$.

c) $\hat{\theta}_n = p_i^*(\hat{\theta}_{n-1})$ for all $i$, $1 \leq i < n - 1$.

Proof: Follows immediately from the definition and the properties of $s_\pm$ and $s$. 

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1.2.7 Definition. For any \( i, 1 \leq i \leq n \) we denote by \( \delta_i^{(n)} : \mathcal{F}_{V,W}^{(n)}(x) \to \mathcal{F}_{V,W}^{(n)}(E_i(x)) \) a lifting of \( E_i \) to \( \mathcal{F}_{V,W}^{(n)} \) defined by a composition

\[
\delta_i^{(n)} : \mathcal{F}_{V,W}^{(n)}(x) = \langle V \otimes X^+ \otimes X_i \otimes X^- \otimes W \rangle \rightarrow \langle V \otimes X_i \otimes X^+ \otimes X^- \otimes W \rangle \xrightarrow{s_{X_i}^{-1}} \langle V \otimes X^+ \otimes X^- \otimes T(X_i) \otimes W \rangle \xrightarrow{s_{X^- \tau(X_i)}} \langle V \otimes X^+ \otimes T(X_i) \otimes X^- \otimes W \rangle = \mathcal{F}_{V,W}^{(n)}(E_i(x))
\]

for all \( x = (X_1, \ldots, X_n) \in S_n \) where \( X^+ = X_1 \otimes \cdots \otimes X_{i-1}, X^- = X_{i+1} \otimes \cdots \otimes X_n \).

1.2.8 Proposition.

a) \( \delta_{i+1}^{(n)} \delta_i^{(n)} = p_i^*(\delta_i^{(n-1)}) \)

b) \( \delta_i^{(n)} \delta_{i+1}^{(n)} = p_i^*(\delta_i^{(n-1)}) \)

c) If \( j < i \), then \( \delta_i^{(n)} = p_j^*(\delta_{i-1}^{(n-1)}) \)

d) If \( j > i + 1 \), then \( \delta_i^{(n)} = p_j^*(\delta_{i-1}^{(n-1)}) \).

Proof: a) For simplicity we consider the case \( i = 1 \). The proof in the general case is completely analogous. Let \( x = (V_1, \ldots, V_n) \in S_n \). We define \( Y = X_3 \otimes \cdots \otimes X_n \). To prove

\[
\delta_2^{(n)} \delta_1^{(n)}(x) : \langle V \otimes X_1 \otimes X_2 \otimes Y \otimes W \rangle \to \langle V \otimes Y \otimes T(X_1) \otimes T(X_2) \otimes Y \otimes W \rangle.
\]

Using the equality \( s_{X_2 \otimes Y,X_1} = (s_{X_2,X_1} \otimes id_Y) \circ (id_{X_2} \otimes s_{Y,X_1}) \) we can write this map as the composition

\[
\langle V \otimes X_1 \otimes X_2 \otimes Y \otimes W \rangle \xrightarrow{\tilde{\theta}_n} \langle V \otimes X_2 \otimes Y \otimes T(X_1) \otimes W \rangle \xrightarrow{s_{Y,T(X_1)}} \langle V \otimes X_2 \otimes T(X_1) \otimes Y \otimes W \rangle \xrightarrow{\tilde{\theta}_n} \langle V \otimes T(X_1) \otimes X_2 \otimes Y \otimes W \rangle \xrightarrow{s_{Y,T(X_2)}} \langle V \otimes T(X_1) \otimes T(X_2) \otimes Y \otimes W \rangle.
\]

By Proposition 1.2.6 a) this composition is equal to the composition

\[
\langle V \otimes X_1 \otimes X_2 \otimes Y \otimes Y \otimes W \rangle \xrightarrow{\tilde{\theta}_n} \langle V \otimes X_2 \otimes Y \otimes T(X_1) \otimes W \rangle \xrightarrow{\tilde{\theta}_n} \langle V \otimes Y \otimes T(X_1) \otimes T(X_2) \otimes W \rangle \xrightarrow{s_{Y,T(X_1)}} \langle V \otimes T(X_1) \otimes Y \otimes T(X_2) \otimes W \rangle \xrightarrow{s_{Y,T(X_2)}} \langle V \otimes T(X_1) \otimes T(X_2) \otimes Y \otimes W \rangle
\]

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Part a) follows now from Proposition 1.2.6 b) and the definition of weak braiding.

To prove part b) we consider the composition

\[
\delta_i^{(n)} \delta_j^{(n)}(x) : \langle V \otimes X_1 \otimes X_2 \otimes Y \otimes W \rangle \xrightarrow{s_{X_2,X_1}} \langle V \otimes X_2 \otimes X_1 \otimes Y \otimes W \rangle \xrightarrow{\delta_n} \\
\rightarrow \langle V \otimes X_1 \otimes Y \otimes T(X_2) \otimes W \rangle \xrightarrow{s_{Y,T}(X_2)} \langle V \otimes X_1 \otimes T(X_2) \otimes Y \otimes W \rangle \xrightarrow{\delta_n} \\
\langle V \otimes T(X_2) \otimes Y \otimes T(X_1) \otimes W \rangle \xrightarrow{s_{Y,T}(X_1)} \langle V \otimes T(X_2) \otimes T(X_1) \otimes Y \otimes W \rangle \rightarrow \\
\xrightarrow{s_{\tau(X_2),T}(X_1)} \langle V \otimes T(X_2 \otimes X_1) \otimes Y \otimes W \rangle \xrightarrow{T(s_{X_2,X_1})} \langle V \otimes T(X_1 \otimes X_2) \otimes Y \otimes W \rangle.
\]

It follows now from Proposition 1.2.6 a) that \( \delta_i^{(n)} \delta_j^{(n)}(x) = p_1^*(\delta_i^{(n-1)}) \).

**1.2.9 Proposition.** The isomorphisms \( \delta_i^{(n)} : F_{V,W}^{(n)} \rightarrow E_i^{(n)}(F_{V,W}^{(n)}) \), \( 1 \leq i \leq n \) commute.

**Proof:** It follows from Proposition 1.2.8 that \( \delta_i^{(n)} \) and \( \delta_i^{(n)} \) commute. To prove that \( \delta_i^{(n)} \) and \( \delta_i^{(n)} \) commute we observe that (by Proposition 1.2.8) \( \delta_i^{(n-2)} \) and \( \delta_i^{(n-1)} \) commute. Therefore

\[
p_{n-2}(\delta_i^{(n-1)}) p_{n-2}(\delta_i^{(n-1)}) = p_{n-2}(\delta_i^{(n-1)}) p_{n-2}(\delta_i^{(n-1)}).
\]

By Lemma 1.2.8 we can rewrite this equality in the form

\[
\delta_i^{(n)} \delta_i^{(n)} = \delta_i^{(n)} \delta_i^{(n)}.
\]

Since the transformation \( \delta_i^{(n)} \) is invertible and, by the same Proposition 1.2.8 it commutes with both \( \delta_i^{(n)} \) and \( \delta_i^{(n)} \) we see that \( \delta_i^{(n)} \) and \( \delta_i^{(n)} \) commute. Analogous arguments show that \( \delta_i^{(n)} \), \( \delta_j^{(n)} \) commute for all \( i, j, 1 \leq i, j \leq n \). Proposition 1.2.9 is proved.

**1.3. A useful formula.**

**1.3.1.** Let \( (H, 1, \varepsilon, m, \Delta) \) be a Hopf algebra. \( H_0 = \ker \varepsilon \subset H \) and \( (H \otimes H)_0 \subset H \otimes H \) be the subgroup of linear combinations of elements of the form

\[
\Delta(x_0)a, \quad x_0 \in H_0, \quad a \in H \otimes H.
\]

Let \( S \) be the antipode of \( H \).
1.3.2 Lemma. For any $x \in H$ such that $\Delta(x) = \sum_{r=0}^{n} x'_r \otimes x''_r$ we have

$$x \otimes 1 = \sum_{r=0}^{n} \Delta(x'_r)(1 \otimes S(x''_r)).$$

Proof: The right side is equal to $(id \otimes m)(id \otimes id \otimes S)(\Delta \otimes 1)\Delta(x) = (id \otimes m)(id \otimes id \otimes S)(1 \otimes \Delta)\Delta(x)$. By the definition of the antipode this expression is equal to $(id \otimes \epsilon)\Delta(x) = x \otimes 1$. Lemma 1.3.2 is proved.

1.3.3 Lemma. For any $x \in H$ we have $x \otimes 1 - 1 \otimes S(x) \in (H \otimes H)_0$.

Proof: Since $(\epsilon \otimes 1)\Delta(x) = 1 \otimes x$ we can write $\Delta(x)$ in the form $\Delta(x) = 1 \otimes x + \sum_{r=1}^{n} x'_r \otimes x''_r$ where $x'_r \in H_0, x''_r \in H, 1 \leq r \leq n$. Then it follows from Lemma 1.3.2 that

$$x \otimes 1 = \sum_{r=1}^{n} \Delta(x'_r)(1 \otimes S(x''_r)) + 1 \otimes S(x).$$

Lemma 1.3.3 is proved.
§2. Quantum affine algebras

In this section we will use freely notations from [L].

2.1 Basic definitions.

2.1.1. Let \((I, \cdot)\) be an affine irreducible Cartan datum and \((\Lambda^\vee, \Lambda, \langle \cdot, \cdot \rangle)\) be a simply connected root datum of type \((I, \cdot)\) (see [L] 2.1 and 2.2) which is an affinization of a finite root datum. Any such datum is either a symmetric root datum (see [L] 2.1) or is obtained as a quotient of a symmetric one by a finite group of automorphisms which preserve some special vertex (see [L] 14.1.5). We denote by \(i_0 \in I\) this special vertex. As follows from [L] 14.1.4 such a vertex is defined uniquely up to an automorphism of the root datum and \((i_0 \cdot i_0) = 2\). We define \(\mathcal{T} \overset{\text{def}}{=} I - \{i_0\}\). (In the terminology of [K] we consider the non-twisted affine case).

Let \(Z[I] \to \Lambda\) be the group homomorphism such that \(i \mapsto i'\) for all \(i \in I\) (see [L] 2.2.1). This homomorphism is not injective, it has a kernel isomorphic to \(\mathbb{Z}\). As follows from [K] 6.2 there exists a generator \(\sum_{i \in I} n_i \cdot i\) of this kernel such that \(n_{i_0} = 1\). We define the dual Coxeter number \(h^\vee\) as the sum \(h^\vee \overset{\text{def}}{=} \sum_{i \in I} n_i \frac{(i \cdot i)}{2}\) and denote by \(\Lambda_0^\vee \subset \Lambda^\vee\) the subgroup generated by an element \(\sum n_i \frac{(i \cdot i)}{2}\).

Let \(\bar{\Lambda}^\vee \subset \Lambda^\vee\) and \(\overline{\Lambda} \subset \Lambda\) be the subgroups generated by elements \(i\) and \(i'\) correspondingly for \(i \in \mathcal{T}\). Since \(n_{i_0} = 1\) we have a direct sum decomposition \(\Lambda^\vee = \bar{\Lambda}^\vee \oplus \Lambda_0^\vee\) which induces a direct sum decomposition \(\Lambda = \bar{\Lambda} \oplus \Lambda_0\), where \(\Lambda_0 \overset{\text{def}}{=} \{\lambda \in \Lambda | \langle i, \lambda \rangle = 0 \ \forall i \in \mathcal{T}\}\) and \(\bar{\Lambda} = \bar{\Lambda} \otimes \mathbb{Z}Q \cap \Lambda\). Then \(\bar{\Lambda}\) is a subgroup of finite index \(d\) in \(\bar{\Lambda}^\vee\). As follows from the definition of a root datum we have \(\langle i \cdot i' \rangle = 2(i \cdot \lambda)\) for all \(i \in I, \lambda \in \Lambda^\vee\). The map \(\lambda \mapsto \lambda'\) defines an imbedding \(\bar{\Lambda}^\vee \hookrightarrow \bar{\Lambda}\) and there exists unique symmetric bilinear form \([,] : \bar{\Lambda}^\times \bar{\Lambda} \to 1/d\mathbb{Z}\) such that \([\lambda', \mu'] = (\lambda \cdot \mu)\) for all \(\lambda, \mu \in \bar{\Lambda}^\vee\).

We denote by \(\rho' \in \bar{\Lambda}\) the unique element such that \(\langle i, \rho' \rangle = 1\) for all \(i \in \mathcal{T}\). Then \([i', \rho'] = \frac{(i' \cdot i)}{2}\) for all \(i \in \mathcal{T}\).

Remark: In the terminology of [K], \(\Lambda\) is the weight lattice, \(\Lambda^\vee\) is the coroot lattice, \(\bar{\Lambda}\) is the weight lattice of finite-dimensional root datum \((\mathcal{T}, \cdot)\).
2.1.2. We fix a number \( \tilde{q} \in \mathbb{C}^* \) such that \( |\tilde{q}| < 1 \) and define \( q \overset{\text{def}}{=} \tilde{q}^{dh^\vee}, q_i \overset{\text{def}}{=} q^{(i-i)/2} \) for all \( i \in I \).

For any \( n \in \mathbb{N}, i \in I \) we define \( [n]_i \overset{\text{def}}{=} \frac{q_i^n-q_i^{-n}}{q_i-q_i^{-1}} \) and \( [n]_i! \overset{\text{def}}{=} \prod_{s=1}^n [s]_i. \)

2.1.3. We denote by \( \mathring{U} \) the \( \mathbb{C} \)-algebra generated by elements \( E_i, F_i, K_\mu, i \in I, \mu \in \Lambda^\vee \) and relations (a)-(e) below.

(a) \( K_0 = 1, K_\mu K_{\mu'} = K_{\mu+\mu'}, \mu, \mu' \in \Lambda^\vee. \)
(b) \( K_\mu E_i = q^{(\mu,i')} E_i K_\mu, i \in I, \mu \in \Lambda^\vee. \)
(c) \( K_\mu F_i = q^{-\langle \mu,i \rangle} F_i K_\mu, i \in I, \mu \in \Lambda^\vee. \)
(d) \( E_i F_j - F_j E_i = \delta_{i,j} \frac{\tilde{K}_i - \tilde{K}_{-i}}{q_i - q_i^{-1}} \) for \( i, j \in I \), where
\[
\tilde{K}_{\pm i} = K_{\pm (\mu,i)}
\]
(e) \( \sum_{p+p'=1-2i,j/i} (-1)^{p'} E_i^{(p)} F_j E_i^{(p')} = 0 \) for all \( i \neq j \in I \), where \( E_i^{(p)} \overset{\text{def}}{=} E_i^p/[p]_i! \)

2.1.4. Let \( \tilde{Z} \overset{\text{def}}{=} \prod_{i \in I} K_i^{-n_i} \). Then \( \tilde{Z} \) lies in the center of \( \mathring{U} \) we define \( U = \mathring{U}[Z] \) where \( Z \) is defined as a central element such that \( Z^{dh^\vee} = \tilde{Z} \).

2.1.5. Let \( \Lambda_{C^*} \) be the tensor product \( \Lambda_{C^*} \overset{\text{def}}{=} \Lambda \otimes \mathbb{C}^* \) and \( \Lambda \hookrightarrow \Lambda_{C^*} \) be the imbedding induced by the imbedding \( \mathbb{Z} \hookrightarrow \mathbb{C}^* : n \mapsto q^n \). We extend an imbedding \( \mathbb{Z} \hookrightarrow \mathbb{C}^* \) to an imbedding \( \frac{1}{d} \mathbb{Z} \hookrightarrow \mathbb{C}^* \) in such a way that \( \frac{1}{d} \mapsto \tilde{q}^{h^\vee} \). This imbedding defines an imbedding \( \frac{1}{d} \Lambda \hookrightarrow \Lambda_{C^*} \).

We denote the group structure on \( \Lambda_{C^*} \) as \( + \). The pairing \( \langle , \rangle : \Lambda^\vee \times \Lambda \rightarrow \mathbb{Z}, (\mu, \lambda) \mapsto \langle \mu, \lambda \rangle \) defines the pairing \( \frac{1}{d} \Lambda^\vee \times \Lambda_{C^*} \rightarrow \mathbb{C}^* \) which we will also denote by \( (\mu, \lambda) \mapsto \langle \mu, \lambda \rangle \).

For any representation \( (\rho, V) \) of \( U \) and \( \lambda \in \Lambda_{C^*} \) we define
\[
V_\lambda = \{ v \in V | K_\mu v = \langle \mu, \lambda \rangle v \} \text{ for all } \mu \in \Lambda^\vee.
\]

**Definition.** We denote by \( C \) the category of representations \( (\rho, V) \) of \( U \) such that \( V = \bigoplus_{\lambda \in \Lambda_{C^*}} V_\lambda \). For any commutative ring \( B \) containing \( \mathbb{C} \) we keep the notation \( C \) for the category of \( \mathbb{U}_B \)-modules (where \( \mathbb{U}_B = U \otimes_{\mathbb{C}} B \)) obtained from \( C \) by changing the scalars.

For any complex-valued function \( F \) on \( \Lambda_{C^*} \) and any \( V = (\rho, V) \) in \( C \) we denote by \( \mathring{F} \) the linear endomorphism of \( V \) which preserves the direct sum decomposition \( V = \bigoplus_{\lambda \in \Lambda_{C^*}} V_\lambda \) and such that
\[ F|_{V_\lambda} = F(\lambda)Id_{V_\lambda} \text{ for all } \lambda \in \Lambda_{C^*}. \]

For any \( V = (\rho, V) \) in \( C \) we define \( V^* \) as the direct sum \( V^* \defeq \bigoplus_{\lambda \in \Lambda_{C^*}} \text{Hom}(V_\lambda, C). \) Then \( V^* \) has a natural structure of a \( U \)-module. We denote this \( U \)-module as \( V^* \). By the definition \( V^* \) belongs to \( C \).

In this paper by an expression “a \( U \)-module” we will always understand “a \( U \)-module from the category \( C \)”.

2.1.6. For any \( z \in \mathbb{C}^* \) we denote by \( \mathcal{C}_z \) the full subcategory of \( C \) of representations \( (\rho, V) \) such that \( \rho(Z) = zId_V \). The full subcategory of \( \mathcal{C}_1 \) consisting of finite-dimensional representations is denoted by \( \mathcal{D} \).

2.1.7. Let \( \Lambda_{C^*} \defeq \Lambda \otimes \mathbb{C}^* \) and \( r : \Lambda_{C^*} \to \Lambda_{C^*} \) be the projection induced by the direct sum decomposition \( \Lambda = \Lambda \oplus 0 \). By the definition \( \langle i, \lambda \rangle = 0 \) for all \( i \in \mathbb{T}, \lambda \in \ker r \). Therefore we obtain a pairing \( \Lambda^* \times \Lambda_{C^*} \to \mathbb{C}^* \) which we also denote as \( \langle , \rangle \). It is clear that this pairing coincides with the restriction of the pairing \( \langle , \rangle : \frac{1}{d}\Lambda^* \times \Lambda_{C^*} \to \mathbb{C}^* \) to \( \Lambda^* \times \Lambda_{C^*} \).

The imbedding \( \Lambda \hookrightarrow \Lambda_{C^*} \) induces the imbedding \( \Lambda \hookrightarrow \Lambda_{C^*} \). We denote by \( \mathcal{C} \) the torus \( \Lambda_{C^*}/\Lambda \) and by \( \Pi : \Lambda_{C^*} \to \mathcal{C} \) the natural projection of \( \Lambda_{C^*} \) to \( \mathcal{C} \).

**DEFINITION.** a) For any \( s \in \mathcal{C} \) we denote by \( ^s\mathcal{C} \) the full subcategory of \( U \)-modules \( (\rho, V) \) such that \( V_\lambda = \{0\} \) for all \( \lambda \in \Lambda_{C^*} \) such that \( \Pi(r(\lambda)) \neq s \).

b) For any \( s \in \mathcal{C}, z \in \mathbb{C}^* \) we define \( ^s\mathcal{C}_z \) as the intersection of \( ^s\mathcal{C} \) and \( \mathcal{C}_z \).

**REMARK:** In this paper, we will almost always assume that our \( U \)-modules are in the subcategory

\[ [0]\mathcal{C} \defeq \bigcup_{s \in \Pi(\Lambda)} ^s\mathcal{C}, \]

The reason to consider the more general category \( \mathcal{C} \) is to have a possibility to prove results for \( ^s\mathcal{C} \), where \( s \) is “generic” first and then to “deform” \([0]\mathcal{C}\) to \( ^{-s}\mathcal{C} \) for “generic” \( s \) (see, for example, the proof of Theorem 3.2.2).

2.1.8. Let \( U_0 \subset U \) be the subalgebra generated by \( Z \) and \( K_\mu, \mu \in \Lambda^* \) and \( U_0^f \subset U_0 \) its subalgebra generated by \( K_\mu, \mu \in \Lambda^* \). For any \( \nu \in \mathbb{N}[I] \) we denote by \( U_{\nu^*}^+ \) the subspace of \( U \) spanned by elements of the form \( x^+ \) where \( x \in \mathfrak{f}_\nu \) (see [L] 3.1.1). We define \( U^+_\mu \) as a
direct sum

\[ U_+^n \overset{\text{def}}{=} \bigoplus_{\nu \in \mathbb{N}[I]} U_+^\nu \]

and denote by \( U_+^{>n} \) the direct sum \( \oplus_{\nu \geq n} U_+^{\nu} \). We denote \( U_+^{>1} \) as \( U_+^> \) and define \( U_+ \subset U \) to be the subalgebra generated by \( U_0 \) and \( U_+^> \) and \( U_+^{f} \subset U_+ \) as the subalgebra generated by \( U_0^f \) and \( U_+^{>f} \). Then \( U_+ = U_+^f[Z, Z^{-1}] \).

2.1.9. We denote by \( U \subset U \) the subalgebra generated by \( K_\mu, \mu \in \Lambda^\vee \) and \( E_i, F_i \) for \( i \in I \). Then \( U \) is the quantum algebra corresponding to the root datum \((\Lambda^\vee, \Delta)\).

2.1.10. Let \( \Delta : U \rightarrow U \otimes U \) be the comultiplication such that

\[
\begin{align*}
\Delta(E_i) &= E_i \otimes 1 + \tilde{K}_i \otimes E_i, & i \in I \\
\Delta(F_i) &= F_i \otimes \tilde{K}_{-i} + 1 \otimes F_i, & i \in I \\
\Delta(K_\mu) &= K_\mu \otimes K_\mu, & \mu \in \Lambda^\vee \\
\Delta(Z) &= Z \otimes Z
\end{align*}
\]

(see [L] 23.1.5).

We define a co-unit \( \epsilon \) on \( U \) as the unique homomorphism \( \epsilon : U \rightarrow \mathbb{C} \) such that

\[ \epsilon(E_i) = \epsilon(F_i) = 0 \quad i \in I, \]
\[ \epsilon(K_\mu) = 1, \mu \in \Lambda^\vee \quad \text{and} \quad \epsilon(Z) = 1. \]

Then \( U \) is a Hopf algebra where the antipode \( S \) is given by the formulas \( S(E_i) = -\tilde{K}_{-i}E_i \), \( S(F_i) = -F_i\tilde{K}_i \), \( S(K_\mu) = K_{-\mu} \) and \( S(Z) = Z^{-1} \).

This comultiplication defines a strict monoidal structure on the category \( \mathcal{C} \).

2.1.11. Let \( \omega \) be the involution of the algebra \( U \) such that

\[ \omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad i \in I, \quad \omega(K_\mu) = K_{-\mu}, \quad \mu \in \Lambda^\vee, \quad \omega(Z) = Z^{-1} \]

(see [L] 3.1.3).

For any \( U \)-module \((\rho, M)\) we denote by \( \omega M \) the representation \((\tilde{\rho}, \tilde{M})\) of \( U \) on the same space \( M \) such that \( \tilde{\rho}(x) = \rho(\omega(x)) \).

2.1.12. Let \( F \) be a field containing \( \mathbb{C} \), \( U_F = U \otimes_{\mathbb{C}} F \). For any central invertible \( u \in U_F \) we define two automorphisms \( \varphi \) and \( \psi \) of the Hopf algebra \( U_F \) where
\[ \varphi_u(E_{i_0}) = u^{d i i} E_{i_0}, \quad \varphi_u(F_{i_0}) = u^{-d i i} F_{i_0}, \quad \varphi_u(E_i) = E_i, \quad \varphi_u(F_i) = F_i, \quad i \in \mathcal{T}, \]
\[ \varphi_u(K_\mu) = K_\mu, \quad \mu \in \Lambda^\vee, \quad \varphi_u(Z) = Z, \]
\[ \psi_u(E_i) = u^{d(i,i)} E_i, \quad i \in I, \quad \psi_u(F_i) = u^{-d(i,i)} F_i, \quad i \in I, \quad \psi_u(K_\mu) = K_\mu, \quad \mu \in \Lambda^\vee, \quad \psi_u(Z) = Z. \]

**Remark:** The automorphisms \( \varphi_u \) and \( \psi_u \) of \( \mathbf{U}_F \) define strict automorphisms of the category \( \mathcal{C} \) which we denote as \( \mathcal{T}_u^\varphi \) and \( \mathcal{T}_u \) respectively.

2.1.13. Fix a point \( s \in \mathcal{S} \) and an element \( a \in \Pi^{-1}(s) \subset \mathbf{\Lambda}_{\mathcal{C}^\star} \). For any central invertible \( u \in \mathbf{U}_F \) we denote by \( \mathcal{L}_u^a \) the function on \( \Pi^{-1}(s) \) with values in \( \mathbf{U}_F \) such that \( \mathcal{L}_u^a(a) = 1 \) and
\[
\mathcal{L}_u^a(\lambda + i') = u^{d(i,i)/2, \lambda} \mathcal{L}_u^a(\lambda) \quad \text{for any} \quad \lambda \in \Pi^{-1}(s), \quad i \in \mathcal{T}.
\]

We extend \( \mathcal{L}_u^a \) to \( \Lambda_{\mathcal{C}^\star} \) in such a way that \( \mathcal{L}_u^a(\lambda) = 0 \) if \( \lambda \notin \Pi^{-1}(s), \lambda \in \mathbf{\Lambda}_{\mathcal{C}^\star} \) and \( \mathcal{L}_u^a \) is constant on the fibers of \( r \).

**Definition.** For any \( V = (\rho, \underline{V}) \in \overset{\star}{\mathcal{C}} \) we denote by \( \mathcal{L}_u^a \) a linear endomorphism of \( \underline{V} \) corresponding to the function \( \mathcal{L}_u^a \) (see 2.1.5).

**Remark:** For any two \( a', a'' \in \Pi^{-1}(s) \) the operators \( \mathcal{L}_u^{a'} \) and \( \mathcal{L}_u^{a''} \) are proportional and the coefficient of proportionality does not depend on a choice of \( V \) in \( \overset{\star}{\mathcal{C}} \). Therefore we can consider \( \mathcal{L}_u^a \) as a section of a line bundle \( \hat{\mathcal{L}}_u \) on \( \mathcal{S} \).

2.1.14. **Proposition.** For any \( V = (\rho, \underline{V}) \) in \( \overset{\star}{\mathcal{C}} \) the map \( \mathcal{L}_u^a \) defines an isomorphism of the \( \mathbf{U}_F \)-module \( \mathcal{T}_u^\varphi(V) \) with the \( \mathbf{U}_F \)-module \( \mathcal{T}_u(V) \).

**Proof:** We have to show that for any \( x \in \mathbf{U}_F \), we have \( \mathcal{L}_u^a \varphi_u(x) = \psi_u(x) \mathcal{L}_u^a \). This equality is obvious if \( x = K_\mu, \mu \in \Lambda^\vee \) or \( x = Z \). In the case when \( x = E_i \) or \( F_i, \ i \in \mathcal{T} \) the claim follows immediately from the definition of \( \mathcal{L}_u^a \) and the equalities \( x^+ \underline{V}_\lambda \subset \underline{V}_{\lambda + \nu}, \]
\[ x^- \underline{V}_\lambda \subset \underline{V}_{\lambda - \nu} \] for \( x \in \mathfrak{f}_\nu \). In the case when \( x = E_{i_0} \) or \( F_{i_0} \) the claim follows from the definition of the dual Coxeter number \( h^\vee \). Proposition 2.1.14 is proved.

2.1.15 **Proposition.** For any finite-dimensional \( X = (\rho_X, \underline{X}) \in \mathcal{C} \) the identity map \( \underline{X} \to \underline{X} \) defines an isomorphism \( X^{**} \to \mathcal{T}_{q - 2h^\vee}(X) \) of \( \mathbf{U}_F \)-modules.

**Proof:** By the definition \( X^{**} = (\rho_{X^{**}}, \underline{X}) \) where \( \rho_{X^{**}}(a) = \rho_X(S^2(a)), \ a \in \mathbf{U}_F \) and \( S : \mathbf{U}_F \to \mathbf{U}_F \) is the antipode. The claim follows now from the formulas for the antipode \( S \) in 2.1.10.
2.2 Sugawara operators.

2.2.1. For any \( V = (\rho, V) \) in \( \mathcal{C} \) and \( n \in \mathbb{N} \) we define a subspace \( V(n) \) of \( V \)

\[
V(n) \overset{\text{def}}{=} \big\{ v \in V \mid av = 0 \ \forall a \in U^n_+ \big\}.
\]

It is clear that \( V(1) \subset V(2) \subset \cdots \subset V(n) \subset \cdots \). We define a subspace \( V(\infty) \) of \( V \) as the union of all \( V(n), n > 0 \).

**Proposition.** For any \( i \in I, n \in \mathbb{N} \) we have

\[
(E_i) V(n) \subset V(n) \quad \text{and} \quad (F_i) V(n) \subset V(n + 1).
\]

**Proof:** The first part follows immediately from the definitions and the second follows from [L] 3.1.6.

**Corollary.** \( V(\infty) \) is a \( U \)-invariant subspace of \( V \).

We denote the corresponding \( U \)-module by \( V(\infty) \). Let \( \mathcal{C}^+ \) be the full subcategory of \( \mathcal{C} \) of modules \( V \) such that \( V(\infty) = V \) and let \( \mathcal{C}^- \) be the subcategory of \( \mathcal{C} \) of \( U \)-modules \( W \) such that \( uW \in \mathcal{C}^+ \). For any \( W \) in \( \mathcal{C}^- \) we define \( D(W) = W^*(\infty) \) where \( W^* \) is as in 2.1.5. We define \( {}^s\mathcal{C}^\pm \) as \( {}^s\mathcal{C} \cap \mathcal{C}^\pm \) and \( \mathcal{C}_z^\pm \) as \( \mathcal{C}^\pm \cap \mathcal{C}_z \).

**Remark:** For any \( V \) in \( \mathcal{C} \) the submodule \( V(\infty) \subset V \) is the maximal subobject of \( V \) contained in \( \mathcal{C}^+ \).

2.2.2. For any \( V \) in \( \mathcal{C}^+ \) and \( W \) in \( \mathcal{C}^- \) we define a linear map \( \text{Hom}_U(V, D(W)) \rightarrow \text{Hom}_U(V \otimes W, \mathbb{C}) \) as in [KL] 2.30. As follows from 1.1.6 we can identify the linear space \( \text{Hom}_U(V \otimes W, \mathbb{C}) \) with \( \langle V \otimes W \rangle^\vee \).

**Lemma.** The map \( \text{Hom}_U(V, D(W)) \rightarrow \langle V \otimes W \rangle^\vee \) is an isomorphism.

**Proof:** Analogous to the proof of Lemma 2.3.1 in [KL].

2.2.3. Let \( \Omega_{<p} \) be elements defined in [L] 6.1.1. Then for any \( M = (\rho, M) \) in \( \mathcal{C}^+ \) there exists an operator \( \Omega \) on \( M \) such that \( \Omega(m) = \Omega_{<p}(m) \) for any \( m \in M \) and all sufficiently big \( p \). We have \( \tilde{K}_{-i}E_i\Omega = \tilde{K}_i\Omega E_i, \Omega F_i = F_i\tilde{K}_i\Omega \tilde{K}_i \) and \( \Omega K_\mu = K_\mu\Omega, i \in I, \mu \in \Lambda^\vee \).
2.2.4. Fix a point \( s \in \mathfrak{S} \) and an element \( a \in \Pi^{-1}(s) \subset \Lambda_{\mathbb{C}^+} \). Let \( G^a : \Pi^{-1}(s) \to \mathbb{C}^* \) be the function such that \( G^a(a) = 1 \) and for any \( \lambda \in \Pi^{-1}(s), i \in I \) we have

\[
G^a(\lambda)G^a(\lambda - i')^{-1} = (i, \lambda)^{(i:i)}.
\]

We continue \( G^a \) to \( \Lambda_{\mathbb{C}^+} \) in the same way as we did with \( L^a_u \) in 2.1.13. For any \( V = (\rho, \underline{\nu}) \) in \( {}^*\mathbb{C} \) we denote by \( G^a \) the corresponding endomorphism of \( \underline{\nu} \) (see (2.1.5)).

**Remark:** As in the case of \( L^a_u \) we can consider \( G^a \) as a section of a line bundle on \( \mathfrak{S} \).

**Definition.** For any \( s \in \mathfrak{S}, V = (\rho, \underline{\nu}) \in {}^*\mathbb{C}^+ \) and \( a \in \Pi^{-1}(s) \) we define linear endomorphisms \( T^a_{\nu} \) (or simply \( T^a \)) of \( \underline{\nu} \) as composition \( T^a \overset{\text{def}}{=} \underline{\nu}_{(Z^h\tilde{\nu})^{-2}} \Omega G^a \). We will call all of them Sugawara operators.

**Proposition.**

\[
T^a \in \text{Hom}_U(V, T_{(Z^h\tilde{\nu})^{-2}}(V)).
\]

**Proof:** We have to check that for any \( x \in U \) we have \( T^a x = \psi_{(Z^h\tilde{\nu})^{-2}}(x)T^a \). Let \( T^a \overset{\text{def}}{=} \Omega \cdot G^a \). It is sufficient to prove that

1. \( T^a K_\mu = K_\mu T^a, \quad T^a Z = Z T^a \)
2. \( T^a E_i = E_i T^a, \quad T^a F_i = F_i T^a \) for \( i \in I \)
3. \( T^a E_{i_0} = (Z^h\tilde{\nu})^{-2} \Omega E_{i_0} T^a, \quad T^a F_{i_0} = (Z^h\tilde{\nu})^{2} \Omega F_{i_0} T^a \)

The equalities 1) are obviously true. The proof of equalities 2) is completely analogous to the proof of Proposition 6.1.7 in [L]. So we give only the proof of equalities 3). Actually we only give a proof of the equality \( T^a E_{i_0} = (Z^h\tilde{\nu})^{2} \Omega E_{i_0} T^a \). The proof of the second part of 3) is completely analogous.

If \( v \in \underline{V}_\lambda \) then \( E_{i_0} v \in \underline{V}_{\lambda+i_0} \). Let \( \nu \overset{\text{def}}{=} \sum_{i \in I} n_i i \in \Lambda \). Since \( \lambda + i_0 = \lambda - \nu' \) we have

\[
T^a E_{i_0} v = G^a(\lambda - \nu') \Omega E_{i_0} v = G^a(\lambda - \nu') K_{i_0}^{-2} E_{i_0} \Omega v =
\]

\[
= G^a(\lambda - \nu') K_{i_0}^{-2} E_{i_0} \Omega v = G^a(\lambda - \nu') \bar{Z}^{-2} \prod_{i \in I} K_i^{2n_i} E_{i_0} \Omega v =
\]

\[
= Z^{-2d} G^a(\lambda - \nu') G^a(\lambda)^{-1} \prod_{i \in I} K_i^{2n_i} E_{i_0} T^a v = (Z^h\tilde{\nu})^{2} E_{i_0} T^a v.
\]

Proposition 2.2.4 is proved.
2.2.5. In the case when \( V = (\rho_V, \underline{V}) \) lies in \([0]\mathcal{C}\) we can give a more explicit formula for the operators \( \mathcal{L}_u \) and \( \mathcal{G} \) on \( \underline{V} \). Let \( \mathcal{L}_u \) be the function on \( \mathcal{A} \) with values in the center of \( U \) and let \( \mathcal{G} \) be the complex-valued function on \( \mathcal{A} \) such that

\[
\mathcal{L}_u(\lambda) = u^{[\lambda, \rho']} \quad \text{and} \quad \mathcal{G}(\lambda) = q^{([\lambda + \rho', \lambda + \rho'] - [\rho', \rho'])}
\]

for \( \lambda \in \mathcal{A} \) where the bilinear form \([ , ]\) is as 2.1.1. They define \( \mathcal{L}_u \) and \( \mathcal{G} \) in the obvious way.

Let \( (M \otimes N)_0 \subset M \otimes N \) be the subspace defined as in 1.1.6.

2.2.6. For any \( V = (\rho_V, \underline{V}) \) in \( ^*\mathcal{C}^- \) we define an endomorphism \( \tilde{T}^a \) of \( \underline{V} \) by the rule

\[
\tilde{T}^a \overset{\text{def}}{=} (T^a_{\omega V})^{-1},
\]

where the module \( ^aV \subset ^*\mathcal{C}^+ \) is defined as in 2.1.11.

**Remark:** We can consider \( T^a \) and \( \tilde{T}^a \) as elements in appropriate completions of \( U \). Then \( \tilde{T}^a = \omega(T^a) \).

2.2.7 **Proposition.** For any \( i \in I \) we have

a) \( T^a E_i = (Zq^h)^{-((i-i)^d} E_i T^a \),

b) \( \tilde{T}^a E_i = (Zq^{-h})^{-((i-i)^d} E_i \tilde{T}^a \),

c) \( T^a F_i = (Zq^h)^{(i-i)^d} F_i T^a \),

d) \( \tilde{T}^a F_i = (Zq^{-h})^{-(i-i)^d} F_i \tilde{T}^a \).

**Proof:** Parts a) and c) are corollaries of Proposition 2.2.4. We prove b). The proof of d) is completely analogous. We have

\[
\tilde{T}^a E_i = \omega[((T^a)^{-1} F_i)] = \omega[(Zq^h)^{-((i-i)^d} F_i (T^a)^{-1}] = (Zq^{-h})^{(i-i)^d} E_i \tilde{T}^a,
\]

where the second equality is a restatements of c). Proposition 2.2 is proved.

2.2.8. For any \( V = (\rho_V, \underline{V}) \) in \( ^*\mathcal{C}^+ \) and \( W = (\rho_W, \underline{W}) \) in \( ^*\mathcal{C}^- \) the linear map \( T^a \otimes \tilde{T}^a \) of \( \underline{V} \otimes \underline{W} \) does not depend on a choice of \( a \in \Pi^{-1}(s) \) and we denote this operator as \( T \otimes \tilde{T} \).
Proposition. \((T \otimes \tilde{T})\) preserves the subspace \((M \otimes N)_0\) of \(\underline{M} \otimes \underline{N}\).

Proof: It is sufficient to show that for any \(m \in M\), \(n \in N\), \(i \in I\) and any \(\mu \in \Lambda^\vee\), we have

a) \((T \otimes \tilde{T})(E_i)(m \otimes n) \in (M \otimes N)_0\),

b) \((T \otimes \tilde{T})(F_i)(m \otimes n) \in (M \otimes N)_0\),

c) \((T \otimes \tilde{T})[(K_{\mu})(m \otimes n) - (m \otimes n)] \in (M \otimes N)_0\).

Proof: of a): By the definition of the action of \(U\) on \(M \otimes N\) we have

\[
(T \otimes \tilde{T})E_i(m \otimes n) = TE_i m \otimes \tilde{T}n + T\tilde{K}_i m \otimes \tilde{T}E_i n.
\]

Therefore it follows from Proposition 2.2.7 that

\[
(T \otimes \tilde{T})E_i(m \otimes n) = (Z\tilde{q}^{-h^\vee})^{-d(i-i)} E_i Tm \otimes \tilde{T}n + \tilde{K}_i Tm \otimes (Z\tilde{q}^{-h^\vee})^{d(i-i)} E_i \tilde{T}n
\]

\[
= \tilde{q}^{-d(i-i)h^\vee} \Delta(E_i)(Z^{-d(i-i)} Tm \otimes \tilde{T}n) - (Z^{-d(i-i)} A \otimes B - A \otimes Z^{d(i-i)} B)
\]

\[
= \tilde{q}^{-d(i-i)h^\vee} \Delta(E_i)(Z^{-d(i-i)} Tm \otimes \tilde{T}n) - (\Delta(Z^{-d(i-i)}))
\]

\[
- \epsilon(Z^{-d(i-i)})(A \otimes Z^{d(i-i)} B),
\]

where \(A = \tilde{K}_i Tm\) and \(B = \tilde{T}n\).

The inclusion a) is proved. The proof of the inclusion is completely analogous and the proof of c) is obvious.

Proposition 2.2.7 is proved.

Remark: We will show later (see 3.2) that \(T \otimes \tilde{T}\) induces the identity map on the quotient \(<M, N> \defeq \underline{M} \otimes \underline{N}/(M \otimes N)_0\).

We extend \(L_u\) and \(G\) to functions on \(\Lambda_{C^*}\) which are constant on fibers of \(r\) and are zero outside of \(r^{-1}(\hat{\Lambda})\) and denote by \(\tilde{L_u}\) and \(\tilde{G}\) the corresponding automorphism of \(\tilde{V}\) for any \(V = (\rho, \underline{V})\) in \([0]C\). Then \(\tilde{L_u}\) and \(\tilde{G}\) are sections of the line bundles \(\tilde{L_u}\) and \(\tilde{G}\) over the finite set \(\Pi(\hat{\Lambda}) \subset \mathcal{G}\). For any \(V = (\rho_V, \underline{V})\) in \([0]C^+\) (\(\defeq C^+ \cap [0]C\)) and \(W = (\rho_W, \underline{W})\) in \([0]C^-\) (\(\defeq C^- \cap [0]C\)) we denote by \(T \in \text{End } \underline{V}\) and \(\tilde{T} \in \text{End } \underline{W}\) the Sugawara operators corresponding to the functions \(\tilde{L_u}\) and \(\tilde{G}\).
2.2.9 Proposition. Let \( M = (\rho_M, M) \in [^0]C^+_z \) be a \( U \)-module, \( \lambda \in \frac{1}{d} \Lambda \) and let \( m \in M_\lambda \), be such that \( E_i m = 0 \) for all \( i \in I \). Then

\[
T_m = q[[\lambda+\rho',\lambda+\rho']] z^{-2d[\lambda,\rho']} q^{-2[\lambda,\rho']}_m = q^\lambda \lambda z^{-2d[\lambda,\rho']} m.
\]

Proof: Follows immediately from the definition of \( T \).

2.2.10. Let \( M = (\rho_M, M) \) be a \( U \)-module in \([^0]C\). For any \( n \in \mathbb{N} \) we denote by \( M^{(n)} \subset M \) the subspace generated by vectors of the norm \( xm, x \in U \geq n, m \in M \). For any \( N = (\rho_N, N) \) in \( C^- \) we define \( N^{(n)} = (\omega N)(n) \subset N \) where we identify the spaces \( N \) and \( \omega N \). We define \( M_n \overset{\text{def}}{=} M/M^{(n)}, N_n \overset{\text{def}}{=} N/N^{(n)} \) and denote by \( \pi \) the natural projections \( M \rightarrow M_n \) and \( N \rightarrow N_n \).

2.2.11. For any \( p \in \mathbb{N} \), we define

\[
T(p) = \frac{L}{(Z_{\rho,h}) - \Omega \leq p \Omega}.
\]

where \( \Omega \leq p \in U \) is as in [L] 6.1.1.

Proposition. a) For any \( M \) in \([^0]C\) and any pair \( m > n \in \mathbb{N} \) we have \( T(m) M^{(n)} \subset M^{(n)} \).

b) The induced operator \( T_n \) on \( M_n \) does not depend on a choice of \( m > n \).

c) The system \( \{ T_n \in \text{End}M_n \} \) is compatible with the natural projections \( M_n \rightarrow M_{n-1} \).

Proof: a) and b) follow from [L] 6.1.1, and c) is obvious.

2.2.12. For any \( M \) in \( C \) we define \( \widehat{M} \overset{\text{def}}{=} \lim M_n \) and denote by \( \widehat{\pi}_n : \widehat{M} \rightarrow M_n \) the natural projection. Let \( \widehat{M}(\infty) \subset \widehat{M} \) be the submodule as in 2.2.1. Since \( \widehat{M}(\infty) \in C^+ \) we can define a linear transformation \( T_{\widehat{M}} \in \text{End} \widehat{M} \).

Proposition. \( \widehat{\pi}_n \circ T_{\widehat{M}} = T_n \circ \widehat{\pi}_n \) for all \( n \in \mathbb{N} \).

Proof: Fix \( \widehat{m} \in \widehat{M}(\infty) \) and \( n \in \mathbb{N} \) such that \( U_\infty \geq n \widehat{m} = 0 \). We may assume that \( p > n \). Let \( m \in M \) be such that \( \pi_n(m) = \pi_n(\widehat{m}) \). We have \( \widehat{\pi}_n(T_{\widehat{M}} \widehat{m}) = \widehat{\pi}_n(T(p) \widehat{m}) = \pi_n(T(p)m) = T_n \pi_n(m) = T_n \widehat{\pi}_n(\widehat{m}) \). Proposition 2.2.12 is proved.
2.3. The \( R \)-matrix.

2.3.1. Let \( \Xi \) be the complex-valued function on \( \Lambda \times \Lambda \) such that \( \Xi(\lambda, \lambda') = q^{-[\lambda, \lambda']}. \)

For any \( U \)-modules \( V' = (\rho', V'), V'' = (\rho'', V'') \) in \([\lambda] \) we denote by \( \Xi \) the endomorphism of \( V' \otimes V'' \) which preserves subspaces \( V'_\lambda \otimes V''_{\lambda'} \) for all \( \lambda, \lambda' \in \Lambda_{\mathbb{C}} \) and such that

\[
\Xi_{V'_\lambda \otimes V''_{\lambda'}} = \Xi(\lambda, \lambda') Id,
\]

where \( \lambda' \overset{\text{def}}{=} r(\lambda') \) and \( \lambda'' \overset{\text{def}}{=} r(\lambda''). \)

2.3.2 DEFINITION. We say that a pair \( V', V'' \), \( V' = (\rho', V'), V'' = (\rho'', V'') \in [\lambda] \),

is admissible if for any \( \nu' \in V', \nu'' \in V'' \) we have \( \Theta_{\nu'}(\nu'' \otimes \nu') = 0 \) for almost all \( \nu \in \mathbb{Z}[I] \)

where \( \Theta_{\nu} \in U^{(2)} \overset{\text{def}}{=} U \otimes U \) are as in \([L], 4.1.1. \)

If \( \nu' = (\rho', V'), \nu'' = (\rho'', V'') \) is an admissible pair we denote by \( \Theta \) an endomorphism of \( V'' \otimes V' \) such that

\[
\Theta(\nu'' \otimes \nu') = \sum_{\nu \in \mathbb{Z}[I]} \Theta(\nu'' \otimes \nu') \quad \text{for all} \quad \nu' \in V', \nu'' \in V''.
\]

REMARK: If \( V', V'' \in \mathcal{C} \) are such that either \( V' \in \mathcal{C}^+ \) or \( V'' \in \mathcal{C}^- \) then the pair \((V', V'')\) is admissible.

2.3.3. If \( V', V'' \) is an admissible pair we define linear map \( 's_{V', V''} \) and \( s_{V', V''} \) from \( V' \otimes V'' \)

to \( V'' \otimes V' \) as compositions:

\[
's_{V', V''} \overset{\text{def}}{=} \Theta \circ \Xi \circ \sigma, \quad s_{V', V''} \overset{\text{def}}{=} 's_{V', V''} \circ (L_{Z-1} \otimes L_{Z-1})
\]

where \( \sigma : V' \otimes V'' \rightarrow V'' \otimes V' \) is the isomorphism such that \( \sigma(\nu' \otimes \nu'') = \nu'' \otimes \nu' \) for all \( \nu' \in V', \nu'' \in V'. \)

2.3.4. Let \( \varphi^{(2)} \) be the automorphism of \( U \otimes U \), such that

\[
\varphi^{(2)}(x \otimes y) = \begin{cases} 
    x \otimes y & \text{if } x \otimes y \in \{K_{\lambda} \otimes K_{\mu}, E_{i} \otimes 1, 1 \otimes E_{i}, F_{i} \otimes 1, 1 \otimes F_{i} \} \\
    for \lambda, \mu \in \Lambda_{\mathbb{C}}, i \in I; \\
    \tilde{Z}^{-1} \otimes E_{i_0} & \text{if } x \otimes y = 1 \otimes E_{i_0}; \\
    E_{i_0} \otimes \tilde{Z}^{-1} & \text{if } x \otimes y = E_{i_0} \otimes 1; \\
    \tilde{Z} \otimes F_{i_0} & \text{if } x \otimes y = 1 \otimes F_{i_0}; \\
    F_{i_0} \otimes \tilde{Z} & \text{if } x \otimes y = F_{i_0} \otimes 1.
\end{cases}
\]

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Proposition. If $V' = (\rho', \nu')$, $V'' = (\rho'', \nu'') \in \mathcal{C}$ is an admissible pair, then

\[ 's_{V', V''} \circ \varphi^{(2)}(\Delta(x)) = \Delta(x) \circ 's_{V', V''} \quad \text{for all} \quad x \in U. \]

2.3.5 Proof: As in [L] 32.1, we have to show that $\Theta \cdot \Xi \varphi^{(2)}(t \Delta(x)) = \Delta(x) \Theta \Xi$ when $x$ runs through a system of generators of $U$.

Let $\alpha$ be an automorphism of $U(2)$ as in [L] 32.1. Then we have $\Delta(x) \Theta = \Theta \alpha^t \Delta(x)$ for all $x \in U$ (see [L] 32.1). We have to show that

\[ \varphi^{(2)}(t \Delta(x)) = \Xi^{-1} \alpha(\Delta(x)) \Xi \]

for a system of generators $x$ in $U$.

For any $\lambda \in \Lambda$ we define

\[ V'_\lambda \overset{\text{def}}{=} \bigoplus_{\lambda \in r^{-1}(\lambda)} V'_\lambda, \quad V''_\lambda \overset{\text{def}}{=} \bigoplus_{\lambda \in r^{-1}(\lambda)} V''_\lambda. \]

Then it is sufficient to prove the equality

\[ \varphi^{(2)}(t \Delta(x))(v'' \otimes v') = \Xi^{-1} \alpha(\Delta(x)) \Xi(v'' \otimes v') \]

for all $v' \in V'_\lambda$, $v'' \in V''_\lambda$, $\lambda', \lambda'' \in \Lambda_C^*$ and a system of generators $x$ of $U$. If $x \in U_0$, then the validity of (*) is obvious. If $x = E_i$ or $F_i$ for $i \in T$ then the validity of (*) is proven in [L] 32.1. So it is sufficient to prove (*) in the case when $x = E_{i_0}$ and $x = F_{i_0}$. We prove (*) in the case when $x = E_{i_0}$. The case $x = F_{i_0}$ is completely analogous.

2.3.6. If we compute the right side of (*) for $x = E_{i_0}$ we find that it is equal to $q[i_0, \lambda']E_{i_0} v'' \otimes v' + \tilde{Z}^{-1} v'' \otimes E_{i_0} v'$. On the other hand,

\[ \varphi^{(2)}(t \Delta(E_0)) = \varphi^{(2)}(E_{i_0} \otimes K_{i_0} + 1 \otimes E_{i_0}) = E_{i_0} \otimes \tilde{Z}^{-1} K_{i_0} + \tilde{Z}^{-1} \otimes E_{i_0} = E_{i_0} \otimes K_{\nu}^{-1} + \tilde{Z}^{-1} \otimes E_{i_0}. \]

where $\nu = \sum_{i \in T} n_i (i \cdot i) - i$. and we see that the left side of (*) is equal to the right side.

Proposition 2.3.4 is proved.
2.3.7 Corollary. Let $V', V''$ be an admissible pair of $U$-modules such that $V' \in C_{z'}, V'' \in C_{z''}$, $z', z'' \in \mathbb{C}^*$. Then $s_{V', V''} : V' \otimes V'' \to V'' \otimes V'$ is an $U$-module isomorphism between $U$-modules $\mathcal{T}_{z'_-1}(V') \otimes \mathcal{T}_{z''_-1}(V'')$ and $V'' \otimes V'$.

2.3.8. Let $V', V''$ be as in 2.3.7.

Corollary. The linear map $s_{V', V''}$ defines a $U$-module isomorphism between $U$-modules $\mathcal{T}_{z''_-1}(V') \otimes \mathcal{T}_{z'_-1}(V'')$ and $V'' \otimes V'$.

2.3.9 Lemma. Let $V', V''$, $V \in [0]^C$ be such that the pairs $(V, V')$, $(V, V'')$ and $(V, V' \otimes V'')$ are admissible. Then

$$s_{V, V' \otimes V''} = (Id_{V'} \otimes s_{V, V''}) \circ (s_{V, V'} \otimes Id_{V''}).$$

Proof: Completely analogous to the proof of Proposition 32.2.4 in [L].

2.4. Quantum algebras over $\mathbb{C}[t]$.

2.4.1. Let $A = \mathbb{C}[t], F = \mathbb{C}(t), A_n = A/t^n A, n \in \mathbb{N}$. For any $U$-module $M$ and $n \in \mathbb{N}$ we define $nM \overset{\text{def}}{=} M \otimes_A A_n$. Let $U$ be as in 2.1.4,

$$U_A \overset{\text{def}}{=} U \otimes \mathbb{C} A, \quad U_F \overset{\text{def}}{=} U \otimes \mathbb{C} F, \quad nU = U \otimes \mathbb{C} A_n,$$

$$U_A^{(2)} \overset{\text{def}}{=} U_A \otimes_A U_A \quad \text{and} \quad nU^{(2)} = U_A^{(2)} \otimes_A A_n.$$  

We will consider $U$ as a subalgebra in $U_A$ and $U_n$. The comultiplications $\Delta : U \to U \otimes U$ defines $A$-linear comultiplications $U_A \to U_A^{(2)}$ and $nU \to nU^{(2)}$ which we also denote by $\Delta$.

2.4.2. Let $\psi \overset{\text{def}}{=} \psi_t$ be the automorphism of $U_F$ as in 2.1.12 and

$$\Gamma = \{ x \in U_A | (1 \otimes \psi)(\Delta(x)) \in U_A^{(2)} \}.\]$$

As follows from [L], Proposition 3.2.4, we have a direct sum decomposition $U = \bigoplus_{\nu} U_{\nu}$, where $U_{\nu} \overset{\text{def}}{=} U_+(U_{\nu}^-)$. We consider the $\mathbb{C}$-linear map $\eta : U \to U_A$ such that $\eta(u) = t^{\nu |d| u}$ for $u \in U_{\nu}$ and extend it to an $A$-linear map $\eta_A : U \otimes \mathbb{C} A \to U_A$. Here $|\nu| = \sum_{p=1}^n \nu_p \frac{(i_{\nu_p} + i_{\nu_p})}{2}$ if $\nu = (\nu_1, \cdots, \nu_n)$. This is called a degree of $\nu$.  

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Proposition. a) $\Gamma$ is a subalgebra of $U_A$,

b) $\text{Im } \eta \subset \Gamma$

c) The map $\eta_A : U \otimes_C A \to \Gamma$ is an isomorphism of $A$-modules.

d) The algebra $\Gamma$ is generated by $U_+$ and $t_i^d F_i$, $i \in I$, where $t_i = t^{(i,i)/2}$.

e) $\Delta(\Gamma) \subset \Gamma \otimes \Gamma$.

2.4.3 Proof: Part a) follows immediately from the definition of $\Gamma$ since $(1 \otimes \psi) \circ \Delta$ is an algebra homomorphism from $U_F$ to $U_F \otimes U_F$. Part b) follows from a) and the inclusions $(1 \otimes \psi) \circ \Delta(U_+) \subset (1 \otimes \psi)(U_+ \otimes U_+) \subset U_+ \otimes U_+ \subset U_A^{(2)}$ and

$$(1 \otimes \psi) \circ \Delta(t_i^d F_i) = (1 \otimes \psi)(t_i^d F_i \otimes \bar{K}_{-i}) + t_i^d(1 \otimes F_i)) = t_i^d(F_i \otimes \bar{K}_{-i}) + 1 \otimes F_i \in U_A^{(2)}.$$

2.4.4. To prove part c) we choose a basis $B$ in $f$ consisting of homogeneous elements and containing 1 (see [L] 3.2.4). For any $b \in B$ we denote by $|b|$ its degree, i.e., $|b| = |\nu|$ if $b \in f_\nu$. As follows from Proposition 3.2.4 in [L] we can write $x \in \Gamma$ as a sum $x = \sum_{b',\mu,b} c_{b',\mu,b} b'^+ K_\mu b^-$ with $c_{b',\mu,b} \in A$. We have to show that $c_{b',\mu,b} \in t^{(b|)A}$.

Since the map $\Delta : U \to U \otimes U$ is a monomorphism, we see that the inclusion $(1 \otimes \psi) \circ \Delta(x) \in U_A \otimes U_A$ implies the inclusion $c_{b',\mu,b}(1 \otimes \psi)\Delta(b'^+ K_\mu b^-) \in U_A \otimes U_A$ for all $b, b' \in B, \mu \in \Lambda^\vee$. Therefore we may assume that $x = ab'^+ K_\mu b^-$ for $b', b \in B, \mu \in \Lambda^\vee$ and $a \in A$.

2.4.5. Assume that $a \notin t^{(b|)A}$. We want to show that $(1 \otimes \psi)(\Delta(x)) \notin U_A^{(2)}$. Choose $n \in \mathbb{N}$ such that $a \in t^n A - t^{n+1} A$ and define $\bar{x} \overset{\text{def}}{=} t^{(b|)-n} x$. Let $- : U_A^{(2)} \to U \otimes U$ be the natural projection (= reduction mod $t$). As follows from part b), $(1 \otimes \psi)\Delta(\bar{x}) \in U_A^{(2)}$. Let $\bar{x}$ be the image of this element in $U \otimes U$. Since $|b| - n > 0$ it is sufficient to show that $\bar{x} \neq 0$.

We have

$$(1 \otimes \psi) \circ \Delta(\bar{x}) = t^{(b|)-n} a(1 \otimes \psi)\Delta(b'^+)(\Delta(K_\mu))(1 \otimes \psi) \cdot (\Delta(b^-)).$$

Therefore it follows from formulas in 2.1.10 and the definition of $\psi$ that

$$\bar{x} = \bar{a}b'^+ K_\mu \otimes K_\mu b^-$$

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where $\overline{a} \in \mathbb{C}$ is the reduction of $at^{-n} \in A \mod t$. By the definition of $n$, we have $\overline{a} \neq 0$. Therefore, $\overline{a} \neq 0$. Part c) of Proposition 2.4.2 is proved.

Part d) follows immediately from c) and part e) follows from d) and explicit formulas for $\Delta$ (see 2.1.12). Proposition 2.4.2 is proved.

**2.4.6 Corollary.** $\Gamma = \{ x \in U_A | (\psi \otimes 1)(\Delta(x)) \in U_A^{(2)} \}$. 

**Proof:** Let $\Gamma' = \{ x \in U_A | (\psi \otimes 1)\Delta(x) \in U_A^{(2)} \}$. It follows from Proposition 2.4.2 that $\Gamma \subset \Gamma'$. An analogous argument shows that $\Gamma' \subset \Gamma$. So $\Gamma = \Gamma'$.

**2.4.7.** The map $\Delta : U_0 \to U_0 \otimes U_0 \subset U \otimes U$ defines an imbedding of $U_0$ in $U \otimes U$. Since $U_0 \subset \Gamma$ and $(1 \otimes \psi) \circ \Delta|_{U_0} = \Delta|_{U_0}$ we can consider $U_0$ as a subalgebra of $\Gamma$. Therefore the imbeddings $(1 \otimes \psi) \circ \Delta : \Gamma \hookrightarrow U_A^{(2)}$ and $U_- \otimes U_+ \hookrightarrow U_A^{(2)}$ define an $A$-linear map $\alpha : \tilde{U}_A^{(2)} \to U_A^{(2)}$, where $\tilde{U}_A^{(2)} \overset{\text{def}}{=} \Gamma \otimes U_0 (U_- \otimes U_+)$. Analogously for any $n \in \mathbb{N}$ we can define an $A_n$-linear map $\alpha_n : \tilde{n}U^{(2)}(2) \to nU^{(2)}(2)$, where $nU^{(2)}(2) \overset{\text{def}}{=} \tilde{U}_A^{(2)}/t^n \tilde{U}_A^{(2)}$.

**Theorem.** The map $\alpha$ is an isomorphism.

**2.4.8 Proof:** We start the proof with the following general result.

**Lemma.** Let $M, N$ be free $A_n$-modules, let $\alpha : M \to N$ be a morphism such that the induced map $\overline{\alpha} : M/tM \to N/tN$ is an isomorphism. Then $\alpha$ is an isomorphism.

**Proof of Lemma:** Follows from Nakayama’s lemma.

**2.4.9 Proposition.** The maps $\alpha_n : n \in \mathbb{N}$ are isomorphisms.

**Proof:** Consider first the case $n = 1$. We have $^1U^{(2)} = U \otimes U$, $^1\tilde{U}^{(2)} = \Gamma_1(U_- \otimes U_+)$ where the subalgebra $\Gamma_1 \subset U \otimes U$ is generated by elements $x^+ \otimes 1, 1 \otimes x'^-$ and $K_\mu \otimes K_\mu$ for $x, x' \in \mathfrak{f}$ and $\mu \in \Lambda^\vee$. Therefore in the case when $n = 1$ Proposition 2.4.9 follows from the triangular decomposition for $U$ (see [L] 3.2). The general case follows now from Lemma 2.4.8. Proposition 2.4.9 is proved.

**2.4.10 Corollary.** The map $\alpha$ is a monomorphism.

**Proof:** Let $x \in \tilde{U}_A^{(2)}$ be such that $\alpha(x) = 0$. It follows from Proposition 2.4.9 that the image of $x$ in $t^n \tilde{U}_A^{(2)}$ is equal to zero for all $n \in \mathbb{N}$. Therefore $x$ is divisible in $\tilde{U}_A^{(2)}$ by $t^n$ for all $n \in \mathbb{N}$. But the $A$-module $\tilde{U}_A^{(2)}$ is free. Therefore $x = 0$. The Corollary is proved.
2.4.11. For any \( m \in \mathbb{N} \) we denote by \( U^{(2)}(m) \) the subspace of \( U \otimes U \) spanned by elements of the forms \( x^+ K_\mu x'^- \otimes y^- K_\mu y'^+ \), where \( \mu, \mu' \in \Lambda^\vee \) and \( x, x', y, y' \) are homogeneous elements such that \( |x| + |y| \leq m \) and we define \( U^{(2)}_A(m) = U^{(2)}(m) \otimes \mathbb{C} A \subset U^{(2)}_A \).

**Lemma.** For all \( m \in \mathbb{N} \) we have \( U^{(2)}_A(m) \subseteq \text{Im}(\alpha) \).

2.4.12 **Proof:** We will prove Lemma 2.4.11 by the induction in \( m \). If \( m = 0 \), then the result follows from the inclusion \( U^{(2)}(0) \subseteq U_- \otimes U_+ \). Assume that the lemma is true for \( m - 1 \). It is sufficient to show that for any \( \mu, \mu', x, x', y, y' \) as in 2.4.11 there exists \( \gamma \in \Gamma \) and \( \bar{u} \in U_- \otimes U_+ \) such that \( x^+ K_\mu x'^- \otimes y^- K_\mu y'^+ - (1 \otimes \psi) \Delta(\gamma) \bar{u} \in U^{(2)}_A(m - 1) \). But we can take \( \gamma = \ell^{[y]} x^+ y^- \) and \( \bar{u} = K_{-\delta} K_\mu x'^- \otimes K_{-\delta'} K_\mu y'^+ \), where \( \Delta(x^+) = x^+ \otimes K_{\delta'} + \cdots \), \( \Delta(y^-) = K_{\delta} \otimes y^- + \cdots \). Lemma 2.4.12 is proved.

2.4.13. Now we can finish the proof of Theorem 2.4.7. It follows from the triangular decomposition that \( U^{(2)}_A = \bigcup_m U^{(2)}_A(m) \). Therefore Lemma 2.4.11 implies the surjectivity of \( \alpha \). On the other hand, the injectivity of \( \alpha \) follows from Corollary 2.4.10. Theorem 2.4.7 is proved.

2.4.14. We will use a following version of Theorem 2.4.7. As follows from Corollary 2.4.6 we have \( (\psi \otimes 1) \Delta(\gamma) \in U^{(2)}_A \) for all \( \gamma \in \Gamma \). Let \( \beta : \Gamma \otimes U_0 (U_+ \otimes U_) \to U^{(2)}_A \) be the morphism such that \( \beta(\gamma \otimes u) = (\psi \otimes 1) \Delta(\gamma) u \) for all \( \gamma \in \Gamma \) and \( u \in U_+ \otimes U_- \).

**Theorem.** \( \beta \) is an isomorphism of \( A \)-modules.

2.4.15 **Corollary.** Let \( \beta^{(1)} \) be the linear map from \( U \otimes U_+ \otimes U_- \) to \( U \otimes U \) such that \( \beta^{(1)}(x \otimes u) = \Delta(x) u \). Then \( \beta^{(1)} \) is an isomorphism of linear spaces.

**Proof:** Let \( ev^1 : A \to \mathbb{C} \) be the morphism of evaluation at \( t = 1 \). Since \( \beta^{(1)} = \beta \otimes_A \mathbb{C} \), where \( A \) acts on \( \mathbb{C} \) by \( ev^1 \). Theorem 2.4.14 implies the validity of the corollary.

2.4.16. As follows from part e) of Proposition 2.4.2 \( \Gamma \) has a natural structure of a Hopf \( A \)-algebra. Let \( \Gamma_0 \subseteq \Gamma \) be the kernel of the counit \( \epsilon_A : U_A \to A \) to \( \Gamma \). We denote by \( U^{(2)}_0 \subseteq U^{(2)}_A \) the span of elements of the form \( (\psi \otimes 1)(\Delta(\gamma))(x) \), \( \gamma \in \Gamma_0 \), \( x \in U^{(2)}_A \).

**Proposition.** For any \( x \in f_\nu \) there exists \( y, z \in U^{(2)}_0 \) such that \( (1 \otimes x^+) - y \in t^{[\nu]} U^{(2)}_A \) and \( (x^- \otimes 1) - z \in t^{[\nu]} U^{(2)}_A \).
For any \( \psi \)

On the other hand, the inclusion \( x \) is completely analogous.

Proposition 2.4.20

\[
\psi(S^{-1}(x^+)) \otimes 1 - 1 \otimes x^+ = \sum_{1 \leq r \leq R} (\psi \otimes 1) \circ \Delta(a_r) \cdot (\psi \otimes 1)b_r.
\]

Since \( \psi(U_+) \subset U_+ \otimes A \), we see that \( a_r \in \Gamma \) and \( (\psi \otimes 1)b_r \in U_A^{(2)} \) for all \( r, 1 \leq r \leq R \).

Proposition 2.4.16 is proved.

2.4.18. For any \( U \)-module \( M = (\rho, M) \) the \( A \)-module \( M_A \) has a natural \( U \)-module structure. Therefore two imbeddings \( id \) and \( \psi \) of \( \Gamma \) into \( U_A \) define two \( \Gamma \)-module structures on \( M_A \).

2.4.19. For any \( \nu \in \mathbb{N}[I] \) we define \( \Theta_\psi \) where \( \Theta_\nu \in U_A^{(2)} \) are as in 2.3.2. Then \( \Theta_\psi \in \mathbb{I}^{d[\nu]}U_A^{(2)} \) for all \( \nu \in \mathbb{N}[I] \). Therefore for any \( M' = (\rho_{M'}, M'), M'' = (\rho_{M''}, M'') \) in \([0]C \) and any \( n \in \mathbb{N} \) we have \( n(\rho_{M''} \otimes \rho_{M'})\Theta_\psi = 0 \) for almost all \( \nu \in \mathbb{N}[I] \) and we can define an \( A_n \)-linear map \( n s_{M'[t], M''} \) from \( M' \otimes_{C} M'' \otimes_{C} A_n \) to \( M'' \otimes_{C} M' \otimes_{C} A_n \) as a finite sum

\[
n s_{M'[t], M''} \overset{\text{def}}{=} \sum_{\nu \in \mathbb{N}[I]} n(\rho_{M''} \otimes \rho_{M'})\Theta_\psi \Xi_\sigma \circ (L_{Z-1} \otimes L_{Z-1})
\]

where \( \Xi, \sigma \) and \( L_{Z-1} \) are as in 2.3.3.

2.4.20 Proposition. For any \( M', M'' \in [0]C \) such that \( M \in C_z, M'' \in C_{z''}, z', z'' \in \mathbb{C}^* \) and any \( n \in \mathbb{N} \) the map \( n s_{M'[t], M''} \) is a \( \Gamma \)-module morphism from \( n(T_{z''-1}(M)[t] \otimes T_{z'-1}(M'')) \) to \( n(M'' \otimes M'[t]) \).

Proof: Clear.
2.4.21 **Proposition.** For any $M', M'' , M$ in $^{[0]}\mathcal{C}$ and $n \in \mathbb{N}$ we have

$$n^{s_{M[t],M'} \otimes M''} = (Id_{M'} \otimes n^{s_{M[t],M''}}) \circ (n^{s_{M[t],M'}} \otimes Id_{M''}).$$

**Proof:** Analogous to the proof of Proposition 32.2.4 in [L].

2.4.22. For any $M, N$ in $\mathcal{C}$ and $u \in C^*$ we denote by $\tilde{u}$ the automorphism of $A$ such that $\tilde{u}(t) = ut$ and by $\hat{u} = \hat{u}_{M,N}$ the $\mathbb{C}$-linear endomorphism of the space $M \otimes \mathbb{C} N \otimes A$ such that $\hat{u}(m \otimes n \otimes a) = m \otimes n \otimes \tilde{u}(a)$ for all $m \in M, n \in N, a \in A$. It is clear that $\hat{u}$ is a $\tilde{u}$-linear automorphism of $M \otimes \mathbb{C} N \otimes A$.

**Proposition.** For any $M, N$ in $\mathcal{C}$ and $u \in C^*$ we have $\hat{u}(M[t] \otimes N)(0) \subset (T_u(M)[t] \otimes N)(0)$, where the subspaces $(M[t] \otimes N)(0)$ and $(T_u(M)[t] \otimes N)(0)$ of $M \otimes N \otimes A$ are as in 1.1.6.

**Proof:** Follows immediately from the definitions.

**Corollary.** The $\tilde{u}$-morphism $\hat{u}$ defines a $\tilde{u}$-linear isomorphism between $A$-modules

$$\langle M[t] \otimes N \rangle \sim \langle T_u(M)[t] \otimes N \rangle,$$

where $\langle V \rangle$ denote $\langle V \rangle_\Gamma$.

**Proof:** Follows from Proposition 2.4.22.

We denote the induced isomorphisms from $\langle M[t] \otimes N \rangle$ to $\langle T_u(M)[t] \otimes N \rangle$ also by $\hat{u}$. 

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2.5 Γ-coinvariants.

2.5.1. For any Γ-module \( M \) we denote by \( \langle M \rangle_\Gamma \) (or simply \( \langle M \rangle \)) the \( A \)-module of Γ-coinvariants (see 1.1.6).

2.5.2. Let \( \mathcal{M} = (\rho_\mathcal{M}, \mathcal{M}) \) be a \( U_+ \)-module, \( \mathcal{N} = (\rho_\mathcal{N}, \mathcal{N}) \) be a \( U_- \)-module and \( X = (\rho_X, X) \) a \( U \)-module. Then \( \mathcal{M} \otimes X \otimes \mathcal{N} \) has a natural structure of a \( U_0 \)-module and we denote by \( \langle \mathcal{M} \otimes X \otimes \mathcal{N} \rangle_{U_0} \) the space of \( U_0 \)-coinvariants of \( \mathcal{M} \otimes X \otimes \mathcal{N} \).

Let \( M \overset{\text{def}}{=} U \otimes_{U_+} \mathcal{M} \) and \( N \overset{\text{def}}{=} U \otimes_{U_-} \mathcal{N} \) be the induced \( U \)-modules and let \( j : (\mathcal{M} \otimes \mathcal{N}) \to M \otimes N \subset M[t] \otimes N \) be the natural imbedding, that is, \( j(m \otimes n) = (1 \otimes m)(1 \otimes n) \) for \( m \in \mathcal{M} \), \( n \in \mathcal{N} \). Then \( j \) is a morphism of \( U_0 \)-modules and it induces a morphism of \( A \)-modules

\[
j : \langle \mathcal{M} \otimes \mathcal{N} \rangle_{U_0} \otimes A \to \langle M[t] \otimes N \rangle.
\]

**Proposition.** The map \( j \) is an isomorphism.

**Proof:** Follows from Theorem 2.4.15.

**Corollary.** For any \( n \in \mathbb{N} \) the natural map

\[
\langle (M[t] \otimes N) \otimes A_n \rangle \to \langle^n (M[t] \otimes N) \rangle
\]

is an isomorphism.

**Proof:** Clear.

2.5.3 Lemma. The map \( \langle \mathcal{M} \otimes \mathcal{N} \rangle_{U_0} \to \langle M \otimes N \rangle_U \) induced by the imbedding \( j \) is an isomorphism.

**Proof:** Using the same arguments as in the proof of Corollary 2.4.15 we deduce Lemma 2.5.3 from Proposition 2.5.2.

2.5.4. Let \( V, W \) be \( U \)-modules such that there exist exact sequences \( M_1 \to M_0 \to V \to 0 \) and \( N_1 \to N_0 \to W \to 0 \) of \( U \)-modules such that \( M_i = U \otimes_{U_+} \mathcal{M}_i \), \( N_i = U \otimes_{U_-} \mathcal{N}_i \), \( i = 0, 1 \), where \( \mathcal{M}_i \) (resp. \( \mathcal{N}_i \)) are \( U_+ \) (resp. \( U_- \)) modules.
Proposition. For any \( n \in \mathbb{N} \) the natural \( A_n \)-morphism
\[
\langle V[t] \otimes W \rangle \otimes_{A} A_n \longrightarrow \langle^n (V[t] \otimes W) \rangle
\]
is an isomorphism.

Proof: The functor \( \otimes_{\mathbb{C}} \) is exact and therefore the sequence of
\[
(M_1[t] \otimes N_0) \oplus (M_0[t] \otimes N_1) \rightarrow M_0[t] \otimes N_0 \rightarrow V[t] \otimes W \rightarrow 0
\]
of \( \Gamma \)-modules is exact. Since \( \langle \ , \, \rangle \) is a right exact functor we see that the sequence
\[
\langle M_1[t] \otimes N_0 \rangle \oplus \langle M_0[t] \otimes N_1 \rangle \rightarrow \langle M_0[t] \otimes N_0 \rangle \rightarrow \langle V[t] \otimes W \rangle \rightarrow 0
\]
is exact. Analogously one shows that for any \( n \in \mathbb{N} \) the sequence
\[
\langle^n (M_1[t] \otimes N_0) \rangle \oplus \langle^n (M_0[t] \otimes N_1) \rangle \rightarrow \langle^n (M_0[t] \otimes N_0) \rangle \rightarrow \langle^n (V[t] \otimes W) \rangle \rightarrow 0
\]
is also exact. Therefore Proposition 2.5.4 follows from Corollary 2.5.2 and the five-homomorphism lemma.

2.5.5. Assume that \( V, W \) are \( U \)-modules as in 2.5.4 and \( \dim \mathcal{M}_k < \infty, \dim \mathcal{N}_k < \infty \) for \( k = 0, 1 \).

Lemma. \( \langle V[t] \otimes W \rangle \) is a finitely generated \( A \)-module.

Proof: Follows from the exactness of the sequence (*) and Proposition 2.5.2.

2.5.6. Let \( D \) be the rigid category of finite-dimensional \( U \)-modules \( X \) such that \( Z \) acts trivially on \( X \). Then \( D \) is a subcategory of \( [0] \mathbb{C} \).

Given \( z \in C^* \), \( V \in [0]C_z^+, X \in D \) and \( W \in [0]C_{z-1} \) we denote by \( n\delta_{V,X,W} \) the \( A_n \)-linear map from \( \langle^n ((V \otimes X)[t] \otimes W) \rangle \) to \( \langle^n ((V \otimes T_{z \sim q^{2h} \sqrt{\lambda}} X)[t] \otimes W) \rangle \) defined as the composition
\[
\langle^n ((V \otimes X)[t] \otimes W) \rangle \xrightarrow{s_{V,X} \otimes id_W} \langle^n ((T_z(X) \otimes V)[t] \otimes W) \rangle = \langle^n ((T_z(X)) \otimes V[t] \otimes A_n^n W_A) \rangle \xrightarrow{\beta} \langle^n (T_z(X)[t]) \otimes A_n^n W_A \otimes A_n^n (T_{z \sim q^{2h} \sqrt{\lambda}} (X)[t]) \rangle \xrightarrow{id_{V[t]} \otimes s_{T_{z \sim q^{2h} \sqrt{\lambda}} (X), W}^{-1}} \langle^n ([V[t] \otimes A_n^n (T(X)[t]) \otimes A_n^n W_A) = \langle^n ((V \otimes T(X))[t] \otimes W)) \rangle,
\]
where \( \beta \overset{\text{def}}{=} \beta_n^{V[t] \otimes A_n^n W_A} \) is as in 1.1.15 and \( T \overset{\text{def}}{=} T_{(z \sim q^{2h} \sqrt{\lambda})^2} \).

Remark: The map \( n\delta_{V,X,W} \) is an analog of the map \( \delta_{n}^{(3)} \) in 1.2.7.
2.5.7. We have assumed that $W$ lies in $[0]C_{z-1}^-$. Then the composition
\[(T^{-1} \otimes id \otimes \hat{T}^{-1}) \circ \delta_{V,X,W} \] defines a $\Gamma$-module morphism $\varphi_{V,X,W}$ from $\langle n(V \otimes X)[t] \otimes W \rangle$ to
\[\langle n((T(V) \otimes T(X))[t] \otimes T(W)) \rangle = \langle T(n((V \otimes X)[t] \otimes W)) \rangle.\]

As follows from 1.1.3 we can identify the $A_n$-module $\langle T(n((V \otimes X)[t] \otimes W)) \rangle$ with $\langle (V \otimes X)[t] \otimes W \rangle$. Therefore we can consider $\varphi_{V,X,W}$ as an endomorphism of the $A_n$-module $\langle (V \otimes X)[t] \otimes W \rangle$.

**Theorem.** $\varphi_{V,X,W} = id$.

We will prove this theorem at the end of section 3.2.
§3. The category of smooth representation

3.1 The category \( \mathcal{O}_z^+ \).

3.1.1 Definition. A finite dimensional \( \mathbf{U}_f^+ \)-module \( \mathcal{N} \) is called a nil-module if there exists \( n \in \mathbb{N} \) such that the \( \mathbf{U}_+^2n \mathcal{N} = 0 \) and \( \mathcal{N} = \oplus_{\lambda \in \Lambda_\mathbb{C}} \mathcal{N}_\lambda \) (see 2.1.5).

For any nil-module \( \mathcal{N} \) and a number \( z \in \mathbb{C}^* \) we extend the action of \( \mathbf{U}_f^+ \) on \( \mathcal{N} \) to an action of \( \mathbf{U}_+ \) on \( \mathcal{N} \) in such a way that \( Z \) acts as \( z \text{Id} \) and define

\[ \mathcal{N}^z \overset{\text{def}}{=} \mathbf{U} \otimes_{\mathbf{U}_+} \mathcal{N}. \]

The map \( n \to 1 \otimes n \) defines a \( \mathbf{U}_0 \)-covariant imbedding \( \mathcal{N} \hookrightarrow \mathcal{N}^z \) and we will always consider \( \mathcal{N} \) as a \( \mathbf{U}_0 \)-submodule of \( \mathcal{N}^z \).

3.1.2 Definition. \( \mathcal{N}^z \) is called a generalized Verma module.

Remark: \( \mathcal{N}^z \) lies in \( \mathcal{C}_z^+ \).

For any \( a \in \overline{\Lambda}_\mathbb{C}^* \) we denote by \( \mathcal{V}_a \) the one-dimensional representation of \( \mathbf{U}_f^+ \) such that \( K_\mu \) acts as a multiplication by \( \langle \mu, a \rangle \) for all \( \mu \in \overline{\Lambda}^\vee \) and \( \mathbf{U}_+^2 \) acting as zero. In this case the \( \mathbf{U} \)-module \( \mathcal{N}^z \) is denoted by \( \mathcal{V}_a^z \); it is called a Verma module.

3.1.3 Proposition. a) For any generalized Verma module \( \mathcal{N}^z \) there exists a finite filtration by submodules such that the successive quotients are Verma modules.

b) Let \( V \) be an object in \( \mathcal{C}_z^+ \). Then \( V \) is a quotient of a generalized Verma module if and only if there exists \( n \geq 1 \) such that \( \dim V(n) < \infty \) and \( V(n) \) generates \( V \) as a \( \mathbf{U} \)-module.

c) Any object \( V \) in \( \mathcal{C}_+^* \) is a union of subobjects which are isomorphic to quotients of generalized Verma modules.

Proof: Analogous to the proof of Proposition 2.5 in [KL].

3.1.4 The following result concerns the action of the Sugarawa operator \( T^a \) on a Verma module \( \mathcal{V}_a^z \) (see 2.2.4). For any \( \ell \in \mathbb{C}^* \) we set \( \ell \mathcal{V}_a^z = \{ v \in \mathcal{V}_a^z | T^a(v) = \ell v \} \).
**Proposition.**

a) \( V^z_a = \bigoplus_{\ell \in \mathbb{C}^*} \ell V^z_{\ell} \).

b) If \( |\tilde{q}^{h \vee} z| \neq 1 \), then \( \ell V^z_{\ell} \) is a finite-dimensional vector space, for all \( \ell \in \mathbb{C} \).

c) If \( \ell V^z_{\ell} \neq 0 \), then \( \ell = (z^d q)^{2n} \) for some \( n \in \mathbb{N} \).

d) \( 1 V^z_a = V_a \).

e) \( V^z_a = V^z_a(\infty) \).

f) The \( U \)-module \( V^z_a \) has unique irreducible quotient \( L^z_a \).

g) If \( |\tilde{q}^{h \vee} z| > 1 \) then \( V^z_a \) has finite length for any \( a \in \Lambda_{\mathbb{C}^*} \).

**Proof:** The proof of a) - f) is analogous to the proofs of Propositions 2.7 - 2.9 in [KL] when one uses Proposition 2.2.7. In order to prove g) one should observe that if \( V^z_{a'} \to V^z_a \) is a non-trivial homomorphism of \( U \)-modules then \( L^a_{(z^d q)^{-2n}} G^a(a') = (z^d q)^{2nd} \) for such \( n \in \mathbb{N} \) that \( a - a' = \sum_{p=1}^{n} i_p \). Under our assumptions the LHS is bounded and the RHS increases when \( n \to \infty \). This implies that there are finitely many such \( a' \) for a given \( a \). Then one can finish the proof along the lines of [KL], 2.22.

**Definition.** a) Complex numbers \( z_1, \cdots, z_n \in \mathbb{C}^* \) are multiplicatively independent if for any \( \vec{r} = (r_1, \cdots, r_n) \in \mathbb{Z}^n \), \( \vec{r} \neq 0 \) we have \( z_1^{r_1} \cdots z_n^{r_n} \neq 1 \).

b) A pair \( (s, z) \), \( s \in \mathfrak{g} \), \( z \in \mathbb{C}^* \) is generic if for any \( a \in \Pi^{-1}(s) \) the complex numbers \( a(i) \overset{\text{def}}{=} \langle i, a \rangle \), \( i \in \mathfrak{t} \), \( z \) and \( q \) are multiplicatively independent.

c) A nil-module \( N \) is \( z \)-generic if for any \( \lambda \in \Lambda_{\mathbb{C}^*} \) such that \( N_\lambda \neq \{0\} \) the pair \( (\lambda, z) \) is generic.

**3.1.6 Proposition.** a) If a pair \( (s, z) \in \mathfrak{g} \times \mathbb{C}^* \) is generic, then the Verma module \( V^z_a \) is irreducible for \( a \in \Pi^{-1}(s) \).

b) If a nil-module \( N \) is \( z \)-generic then the corresponding generalized Verma module \( N^z \) is a direct sum of Verma modules.

**Proof:** Follows from the same arguments as Propositions 9.9 and 9.10 in [K].

**3.1.7 Definition.** We denote by \( \Omega \geq 0 \subset \mathbb{C}^* \) the set of numbers such that \( |z| \leq 1 \).
In the remainder of this paper we assume \( z \in \mathbb{C}^* \) is such that \( \tilde{q}^{h_V} z \notin \mathfrak{Q}_{\geq 0} \).

3.1.8 Proposition. Let \( V \) be an object in \( \mathcal{C}_z^+ \). The following conditions are equivalent:

a) There exists a finite composition series of \( V \) with subquotients of the form \( L_a^z \) for various \( a \in \mathbb{N}^I \).

b) \( V \) is a quotient of a generalized Verma module.

c) There exists \( n \geq 1 \) such that \( V(n) \) generates \( V \) as a \( \mathbb{U} \)-module and \( \dim V(n) < \infty \).

d) \( \dim V(1) < \infty \).

Proof: Analogous to the proof of Theorems 2.22 and 3.2 in [KL].

3.1.9 Definition. \( \mathcal{O}_z^+ \) is the full subcategory of \( \mathcal{C}_z^+ \) consisting of modules satisfying the conditions of Proposition 3.1.8.

3.1.10 Corollary. Any object in \( \mathcal{C}_z^+ \) can be represented as an inductive limit of objects from \( \mathcal{O}_z^+ \).

3.1.11. In this subsection \( \mathcal{A} \) denotes the ring of regular functions on \( \mathfrak{X}_C \times \mathbb{C}^* \), \( \mathcal{A} \mathbb{U} \stackrel{\text{def}}{=} \mathbb{U} \otimes_\mathbb{C} \mathcal{A} \), \( \mathcal{A} \mathbb{U}_+ \stackrel{\text{def}}{=} \mathbb{U}_+ \otimes_\mathbb{C} \mathcal{A} \), etc. We denote by \( \hat{\mathcal{A}} \) the \( \mathcal{A} \mathbb{U}_+ \)-module which is isomorphic to \( \mathcal{A} \) as an \( \mathcal{A} \)-module and such that \( \mathcal{A} \mathbb{U}_+ \) acts trivially, \( K_\mu \) acts as a multiplication by the function \( \langle \mu, \overline{\lambda} \rangle \) if \( \mu \in \Lambda_0^+ \) and \( Z \) acts as a multiplication by \( z \), where \((\overline{\lambda}, z)\) are natural coordinates on \( \mathfrak{X}_C \times \mathbb{C}^* \). We denote by \( \mathcal{V} \) the induced \( \mathcal{A} \mathbb{U} \)-module \( \mathcal{V} = \mathcal{A} \mathbb{U} \otimes_\mathcal{A} \mathbb{U}_+ \hat{\mathcal{A}} \).

3.1.12. For any \( z_0 \in \mathbb{C}^* \) we define by \( ev_{z_0} : \mathcal{A} \to \mathbb{C}^* \) the homomorphism of the evaluation at the point \((0, z_0) \in \mathfrak{X}_C \times \mathbb{C}^* \). This homomorphism defines an algebra homomorphism \( ev_{z_0} : \mathcal{A} \mathbb{U} \to \mathbb{U} \). Given a \( \mathbb{U}_+ \)-nil-module \( \mathcal{N} \) and \( z_0 \in \mathbb{C}^* \) we can use \( ev_{z_0} \) to define a structure of \( \mathcal{A} \mathbb{U}_+ \)-module on \( \mathcal{N} \). We denote by \( \mathcal{A} \mathcal{N} \) the \( \mathcal{A} \mathbb{U}_+ \)-module which is defined as a tensor product \( \mathcal{A} \mathcal{N} = \mathcal{N} \otimes_\mathbb{C} \mathcal{V} \). Let \( \mathcal{A} \mathcal{N} \) be the induced module \( \mathcal{A} \mathcal{N} \stackrel{\text{def}}{=} \mathcal{A} \mathbb{U} \otimes_\mathcal{A} \mathbb{U}_+ \mathcal{A} \mathcal{N} \).

For any point \((\overline{\lambda}, z) \in \mathfrak{X}_C \times \mathbb{C}^* \) we denote by \( m_{\overline{\lambda}, z} \subset \mathcal{A} \) the maximal ideal of functions equal to zero at \((\overline{\lambda}, z)\). Define \( (\overline{\lambda}, z)^{\mathcal{N}} \stackrel{\text{def}}{=} \mathcal{A} \mathcal{N} / m_{\overline{\lambda}, z} \mathcal{N} \). Then \((\overline{\lambda}, z)^{\mathcal{N}} \) has a natural structure of a \( \mathbb{U} \)-module.
3.1.13 Proposition. a) $\mathcal{A}\tilde{N}$ is a free $\mathcal{A}$-module.

b) For all $(\lambda, z) \in \mathcal{T}_{c^*} \times C^*$ the $U$-module $(\lambda, z)\tilde{N}$ is a generalized Verma module obtained from the nil-module $(\lambda, z)N \overset{\text{def}}{=} \mathcal{A}N/m_{\lambda, z}\mathcal{A}N$.

c) For almost all $(\lambda, z) \in \mathcal{T}_{c^*} \times C^*$ the module $(\lambda, z)\tilde{N}$ is a direct sum of Verma modules.

Proof: a) and b) follow from definitions and c) follows from Proposition 3.1.6 b).

3.1.14 Definition. We denote by $O^-_z \subset C^-_{z-1}$ the category of $U$-modules $M$ such that $\omega M$ lies in $O^+_z$.

For any nil-module $N$ and $z \in C^*$ we define $N^-_z \overset{\text{def}}{=} \omega(N^z)$. Then $N^-_z$ lies in $C^-$ and, as before, we have a nature imbedding $\omega N \hookrightarrow N^-_z$ of $U_0$-modules where we identify $\omega N$ with $N$ as a vector space and the action of $U_0$ on $\omega N$ is given by the map $(x, n) \rightarrow \omega(x)n$, $x \in U_0, n \in N$.

3.1.15. For any Hopf algebra $H$ and two $H$-modules $M = (\rho_M, M)$ and $N = (\rho_N, N)$ we define $\langle M, N \rangle_H \overset{\text{def}}{=} \langle M \otimes N \rangle$ (see 1.1.6) and denote by $pr_{M, N}$ (or simply $pr$) the natural projection $pr : M \otimes N \rightarrow \langle M, N \rangle_H$.

3.1.16. Let $M, N$ be nil-modules $z \in C^*$ and $M \otimes \omega N \overset{i}{\hookrightarrow} M^z \otimes N^-_z$ be the natural imbedding. Then we have a linear map $pr \circ i : \langle M \otimes \omega N \rangle \rightarrow \langle M^z, N^-_z \rangle_U$ which factorizes through the map $\tilde{\iota} : \langle M, \omega N \rangle_U \rightarrow \langle M^z, N^-_z \rangle_U$.

Proposition. The map $\tilde{\iota}$ is an isomorphism.

Proof: Using Theorem 2.4.15 one can immediately apply the arguments of the proof of Proposition 9.15 in [KL].

3.2 The action of Sugawara operators on coinvariants.

3.2.1. In this section we prove Theorem 2.5.7. We start with the special case when $X = h$. So $V = (\rho_V, V)$, $W = (\rho_W, W)$ be $U$-modules such that $V \in C^+_z$ and $W \in C^-_{z-1}$. Let $T \otimes \tilde{T} \in \text{End}(V \otimes W)$ be the linear map as in 2.2.8. As follows from Proposition 2.2.8 the $T \otimes \tilde{T}$ induces an endomorphism of the space $\langle V, W \rangle$ which we denote as $\varphi_{V, W}$. 41
3.2.2 Theorem. \( \varphi_{V,W} = id \).

Proof of Theorem 3.2.2: As follows from Proposition 3.1.3 c) it is sufficient to consider the case when \( V \) and \( W \) are generalized Verma modules, \( V = \mathcal{M}^z, W = \mathcal{N}^w \) where \( \mathcal{M} \) and \( \mathcal{N} \) are nil-modules. It is clear that \( \langle V, W \rangle = \{0\} \) if \( z \neq w \).

We start with the following special case.

3.2.3 Proposition. Theorem 3.2.2 is true in the case when \( V \) and \( W \) are Verma modules.

Proof of Proposition 3.2.3: Let \( V = V_a^z, W = \omega(V_b^z) \). As follows from Proposition 3.1.16 the map \( \tilde{t} : \langle V_a \otimes \omega(V_b) \rangle U_a \rightarrow \langle V, W \rangle U \) is an isomorphism. On the other hand, it follows from Proposition 2.2.9 that the operator \( T \otimes \hat{T} = T^a \otimes \hat{T}^a \) preserves the subspace \( \mathbb{C}_a \otimes \omega(\mathbb{C}_b) \subset \mathcal{M} \otimes \mathcal{N} \) and acts trivially on this subspace. Proposition 3.2.3 is proved.

3.2.4. Consider now the case when \( V \) and \( W \) are arbitrary generalized Verma modules. Let \( \mathcal{M}, \mathcal{N} \) be nil-modules, \( z_0 \in \mathbb{C}^* \) and \( \mathcal{M}, \mathcal{N} \) be modules as in 3.1.12. As follows from Proposition 2.2.8, the operator \( T \otimes \hat{T} \) defines an endomorphism of the \( \mathcal{A} \)-module \( \langle \mathcal{M}, \mathcal{N}^- \rangle \) \( \varepsilon \langle \mathcal{M}, \mathcal{N} \rangle \). We denote this endomorphism of \( \langle \mathcal{M}, \mathcal{N} \rangle \) by \( \Phi \). For any \( (\lambda, z) \in \Lambda_{\mathbb{C}^*} \times \mathbb{C}^* \) we denote by \( \Phi_{\lambda,z} \) the natural morphism \( \Phi_{\lambda,z} : \langle \mathcal{M}, \mathcal{N}^- \rangle \rightarrow \langle \lambda, z \rangle \mathcal{M}, \langle \lambda, z \rangle \mathcal{N}^- \).

3.2.5 Lemma. a) The \( \mathcal{A} \)-module \( \langle \mathcal{M}, \mathcal{N}^- \rangle \) is free as an \( \mathcal{A} \)-module,

b) \( \Phi_{\lambda,z} \) is surjective for all \( \lambda \in \Lambda_{\mathbb{C}^*}, z \in \mathbb{C}^* \),

c) the kernel of \( \Phi_{\lambda,z} \) is equal to \( m_{\lambda,z} \langle \mathcal{M}, \mathcal{N}^- \rangle \).

Proof: Follows from Propositions 3.1.13 and 3.1.16.

Now we can finish the proof of Theorem 3.2.2 in the case when \( V \) and \( W \) are generalized Verma modules. Really since the morphism \( \Phi_{0,1} \) is surjective it is sufficient to prove that \( \Phi = Id \). On the other hand, since the \( \mathcal{A} \)-module \( \langle \mathcal{M}, \mathcal{N}^- \rangle \) is free, it is sufficient to show that the induced endomorphism \( \Phi_{\lambda,z} \) on \( \langle \lambda, z \rangle \mathcal{M}, \langle \lambda, z \rangle \mathcal{N} \) is equal to \( Id \) for generic \( (\lambda, z) \).

But this follows from Propositions 3.1.13 c) and 3.2.3.

Theorem 3.2.2 is proved.

3.2.6 Proposition. Theorem 2.5.7 is true in the case when \( V \) and \( W \) are Verma modules.
Proof: We have $V = V_a^z, W = \omega(V_b^z)$ for some $a, b \in \Lambda$. As follows from Theorem 2.4.7 the natural imbedding $V_a \otimes X \otimes \omega(V_b) \hookrightarrow (V \otimes X)[t] \otimes W$ defines an isomorphism $\langle V_a \otimes X \otimes \omega(V_b) \rangle_{U_0 \otimes \mathbb{C}A_n} \cong \langle ((V \otimes X)[t] \otimes W) \rangle$ (cf. 2.5.2). So it is sufficient to show that for any $x \in X_{b-a}$ we have $\varphi_{V,X,W}(1_a \otimes x \otimes 1_b) = 1_a \otimes x \otimes 1_b$, where $1_a, 1_b$ are generators of 1-dimensional spaces $V_a$ and $V_b$ and we identify $(V_a \otimes X \otimes \omega(V_b))_{U_0}$ with its image in $\langle V_a \otimes X \otimes \omega(V_b) \rangle$. But this follows immediately from the definitions. Proposition 3.2.6 is proved.

3.2.7. The same arguments as in 3.2.4-3.2.5 show that Theorem 2.5.7 follows from Proposition 3.2.6. Theorem 2.5.7 is proved.

3.3 Completions.

3.3.1. We assume until the end of the section all our infinite-dimensional modules are in $[0]^\mathbb{C}$. It is easy to see that for any $N$ in $[0]^\mathbb{C}$ and any finite dimensional $U$-module $V$ the tensor product $N \otimes V$ lies in $[0]^\mathbb{C}$.

3.3.2. We fix until the end of this section a number $z$ such that $zq^{h^+} \in \mathbb{C}^* - \Omega_{\geq 0}$. For any $M = (\rho, M)$ in $\mathcal{C}_z$ we define the spaces $M_n$ and $M_{(n)}$ as in 2.2.10.

3.3.3 Proposition. If $N^z$ is a generalized Verma module, $V$ is a finite-dimensional representation of $U$ and $M = N^z \otimes V$ then the natural morphism $N \otimes V \longrightarrow M_1$ is an isomorphism.

Proof: Follows from 2.4.15.

3.3.4. We denote by $\mathcal{E}_z \subset \mathcal{C}_z$ be the full subcategory of modules $M$ such that $\dim M_1 < \infty$.

3.3.5 Proposition. a) For any $M$ in $\mathcal{E}_z$ and any $n \in \mathbb{N}$ we have $\dim M_n < \infty$.

b) For any $N$ in $\mathcal{O}_z^+$ and any $V$ in $\mathcal{D}$ the tensor product $N \otimes V$ lies in $\mathcal{E}_z$.

Proof: a) The proof is completely analogous to one in §§7.6-7.7 of [KL].

b) Follows from Proposition 3.3.3.
3.3.6. For \( n \in \mathbb{N} \) we denote by \( L_n \subset \mathbb{C}^* \) the set of eigenvalues of \( T_n \) on \( \overline{M}_{(n-1)} \), where \( \overline{M}_{(n-1)} \subset M_n \) is the image \( M_{(n-1)} \).

**Proposition.** For any \( M \in \mathcal{E}_z \) and \( n \in \mathbb{N} \) we have

\[
L_{n+1} \subset \bigcup_{n \leq k \leq 3n} q_1^k L_1,
\]

where \( q_1 \overset{\text{def}}{=} (zq^h)^2 d = (z d q)^2 \).

3.3.7 Proof: We prove the result by induction in \( n \). If \( n = 0 \) then there is nothing to prove. Assume that we know the Proposition for \( n = n_0 \) and we prove it for \( n = n_0 + 1 \). Any element in \( M_{(n-1)} \) is a linear combination of elements of the form \( F_i m, i \in I, m \in M_{(n-2)} \). Let \( \overline{m} \) be the image of \( m \) in \( \overline{M}_{(n-1)} \). We may assume that \( \overline{m} \) is a generalized eigenvector of \( T_{n-1} \) on \( \overline{M}_{(n-2)} \) with an eigenvalue equal to \( \lambda \). It follows then from Proposition 2.2.7 that the image of \( F_i m \) in \( \overline{M}_{(n-1)} \) is a generalized eigenvector for \( T_n \) with the eigenvalue \( \lambda \cdot q_1^{(ii)/2} \). The inclusion \( \lambda \cdot q_1^{(ii)/2} \in \bigcup_{n \leq k \leq 3n} q_1^k L_1 \) follows now from the inductive assumption and the inclusion \( (ii)/2 \in \{1, 2, 3\} \). Proposition 3.3.7 is proved.

3.3.8. For any \( M \in \mathcal{E}_z \) we define \( \widehat{M} \overset{\text{def}}{=} \lim_{\leftarrow} M_n, \widehat{T} \overset{\text{def}}{=} \lim_{\leftarrow} T_n \subset \text{End}(\overline{M}) \) and denote by \( \hat{\pi}_n : \widehat{M} \to M_n \) the natural projection (see 2.2.10). Given \( M \in \mathcal{E}_z \) we define (as in [KL], §29) for any \( \ell \in \mathbb{C} \), and \( n \in \mathbb{N} \) the number \( d_n(\ell) \) to be the dimension of the space \( \iota M_n \), where \( \iota M_n \overset{\text{def}}{=} \cup_m \ker(T_n - \ell)^m \). It is clear that for any \( \ell \in \mathbb{C} \), we have

\[
d_1(\ell) \leq d_2(\ell) \leq \cdots \leq d_n(\ell) \leq \cdots.
\]

We define \( d(\ell) = \lim_{n \to \infty} d_n(\ell) \).

3.3.9 Proposition. For any \( \ell \in \mathbb{C} \), we have \( d(\ell) < \infty \).

Proof: Since all operators \( T_n \) are invertible we may assume that \( \ell \in \mathbb{C}^* \). It follows from Proposition 3.3.7 that there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), \( \ell \) is not an eigenvalue of the restriction of \( T_n \) on \( \overline{M}_{(n-1)} \). Therefore \( d_n(\ell) = d_{n_0}(\ell) \) for all \( n \geq n_0 \). Proposition 3.3.9 is proved.
Remark: We can rephrase the statement of Proposition 3.3.9 by saying that $\hat{T}$ induces an admissible automorphism of $\hat{M}$ (see §29 in [KL]).

3.3.10. For any $M$ in $E_z$ we define a submodule $\hat{M}(\infty) \subset \hat{M}$ as in 2.2.1 and define a subspace $\hat{M}^\infty \overset{\text{def}}{=} \bigoplus_{\ell \in \mathbb{C}^*} \ell \hat{M} \subset \hat{M}$ as in Proposition 29.5 of [KL]. Since $\hat{M}(\infty)$ lies in $[0]^+\mathbb{C}$ one can define the Sugawara operator $T \in \text{End} \hat{M}(\infty)$.

**Proposition.**

a) $\hat{M}^\infty \subset \hat{M}(\infty)$.

b) For any $n \in \mathbb{N}$ we have $\pi_n(\hat{M}^\infty) = M_n$.

**Proof:** Part a) follows immediately from Propositions 3.3.9, 3.3.6. Part b) follows from Proposition 29.1 in [KL].

3.3.11. For any $M$ in $E_z$ the projective system $\pi_n : M_n \to M_{n-1}$ defines an inductive system $\pi_n^* : M_{n-1}^* \to M_n^*$, where $M_n^* \overset{\text{def}}{=} \bigoplus_{\ell \in \Lambda C^*} \text{Hom}(M_n, \mathbb{C})$ (see 2.1.5). We denote by $M^*$ the inductive limit $M^* = \lim_{\to} M_n^*$. It is easy to see that $M^*$ has a natural structure of $U$-module. Moreover, $M^*$ lies in $[0]^-\mathbb{C}$ and therefore we can define the Sugawara operator $\hat{T} \in \text{End}(M^*)$.

3.3.12 **Proposition.** For any $\zeta \in M_n^* \subset M^*$ and $m \in M$ we have $\zeta(T_n(\pi_n(m))) = (\hat{T}^{-1}\zeta)(m)$ where we consider $\zeta$ as a linear functional on $M_n$ and $\hat{T}^{-1}\zeta \in M^*$ as a linear functional on $M$.

**Proof:** Follows from Theorem 3.2.1.

3.3.13 **Proposition.** Let $M$ be an object in $E_z$.

a) For any $m \in \hat{M}(\infty)$ and any $n \in \mathbb{N}$ we have $\pi_n(Tm) = T_n\pi_n(m)$.

b) $\hat{M}(\infty) = \hat{M}^\infty$.

c) The $U$-module $\hat{M}(\infty)$ belongs to $O_z^+$.

d) The natural map $(\hat{M}(\infty))_n \to (\hat{M})_n = M_n$ is an isomorphism for all $n \in \mathbb{N}$.

e) $\hat{M}$ belongs to $E_z$.

**Proof:** a) Follows from 2.2.12.
To prove b) we observe that the inclusion $\hat{M}^\infty \subset \hat{M}(\infty)$ follows from Proposition 3.3.6 and the inclusion $\hat{M}(\infty) \subset \hat{M}^\infty$ follows from Proposition 3.1.8 (see the proof of Lemma 26.4 in [KL]).

Part c) follows from Proposition 3.1.8, part d) follows from Proposition 3.3.10 b), and part e) is clear. Proposition 3.3.13 is proved.

**3.3.14 COROLLARY.** a) For any module $M$ in $\mathcal{E}_z$ the morphisms

$$M \rightarrow \hat{M} \leftarrow \hat{M}^\infty$$

induce isomorphisms

$$M^* \leftarrow (\hat{M})^* \rightarrow (\hat{M}^\infty)^*.$$

b) For any $W$ in $\mathcal{O}_z^-$ and any $M$ in $\mathcal{E}_z$ the morphisms

$$\text{Hom}_U(W, M^\vee) \leftarrow \text{Hom}_U(W, (\hat{M})^\vee) \rightarrow \text{Hom}_U(W, (\hat{M}^\infty)^\vee)$$

are isomorphisms, where as always $M^\vee \overset{\text{def}}{=} \text{Hom}(M, \mathbb{C}).$

**PROOF:** Part a) is equivalent to part d) of Theorem 3.3.13. To prove b) we observe that it follows from the definitions the natural map $\text{Hom}(W, M^*) \rightarrow \text{Hom}(W, M^\vee)$ is an isomorphism. Therefore part b) is a restatement of part a).

**3.3.15.** In 3.3.15 - 3.3.18 $\langle M \rangle$ denotes coinvariants of $M$ with respect to $U$. For any two $U$-modules $W$ and $M$ in $\mathcal{C}$ we define the map

$$\langle M \otimes W \rangle^\vee \rightarrow \text{Hom}_U(W, M^\vee), \ r \rightarrow r^\vee,$$

where for any $r \in \langle M \otimes W \rangle^\vee \subset \text{Hom}(M \otimes W, \mathbb{C})$ and any $w \in W$ we define $r^\vee(w) \in M^\vee$ by the rule $r^\vee(w)(m) \overset{\text{def}}{=} r(m \otimes w)$.

**PROPOSITION.** The map $\langle M \otimes W \rangle \rightarrow \text{Hom}_U(W, M^\vee)$ is an isomorphism.

**PROOF:** Clear.

**3.3.16 COROLLARY.** For any $M$ in $\mathcal{E}_z$ and $W$ in $\mathcal{O}_z^-$ the morphisms

$$\langle M \otimes W \rangle \rightarrow \langle \hat{M} \otimes W \rangle \leftarrow \langle \hat{M}^\infty \otimes W \rangle$$
are isomorphisms.

3.3.17. For any finite dimensional \( U \)-module \( X \) we denote by \( \hat{\otimes}X \) the functor from \( \mathcal{O}^+_z \) to itself such that

\[
V \hat{\otimes}X \overset{\text{def}}{=} (V \hat{\otimes}X)^\infty (= (V \hat{\otimes}X)(\infty))
\]

for all \( V \) in \( \mathcal{O}^+_z \).

For any finite dimensional \( U \)-module \( Y \) and \( W \) in \( \mathcal{O}^-_z \) we define \( Y \hat{\otimes}W \overset{\text{def}}{=} \omega(Y \otimes \omega W) \) and \( Y \check{\otimes}W \overset{\text{def}}{=} \omega(Y \otimes \omega W) \).

3.3.18 Proposition. For any \( V \) in \( \mathcal{O}^+_z \), \( W \) in \( \mathcal{O}^-_z \) and a finite dimensional \( U \)-module \( X \) the maps

\[
\langle V \otimes X \otimes W \rangle \leftarrow \langle (V \hat{\otimes}X) \otimes W \rangle \rightarrow \langle (V \check{\otimes}X) \otimes W \rangle
\]

are isomorphisms.

Proof: Follows from Corollary 3.3.16 in the case when \( M = V \otimes X \).

3.3.19. We say that a \( U \)-module \( V \) is locally \( \overline{U} \)-finite if for any \( v \in V \), \( \overline{U}v \) is a finite-dimensional subspace of \( V \).

Proposition. Let \( M \) be a \( U \)-module in \( \mathcal{E}_z \) which is locally \( \overline{U} \)-finite. Then the module \( \hat{M}^\infty \) is also locally \( \overline{U} \)-finite.

3.3.20 Proof: The proposition is an immediate consequence of the following general and easy result.

Claim: Let \( V = (\rho, V) \) be a locally finite representation of \( U \), \( V^{\vee\vee} \overset{\text{def}}{=} (\rho, V^{\vee\vee}) \) be the full second dual to \( V \) and \( V^{**} \subset V^{\vee\vee} \) be the subspace of vectors which are \( \overline{U}^+ \)-finite. Then \( V^{**} \) is a locally finite representation of \( U \).

We will not give a proof of this claim since we will never use Proposition 3.3.19.

3.3.21 Corollary. Let \( \mathcal{O}^0_z \subset \mathcal{O}^+_z \) be the subcategory of locally \( \overline{U} \)-finite modules. For any \( V \) in \( \mathcal{O}^0_z \) and any finite-dimensional representation \( X \) of \( U \) the module \( V \hat{\otimes}X \) lies in \( \mathcal{O}^0_z \).

3.3.22. As in [KL] §27 we denote by \( \mathcal{A}_z \subset \mathcal{O}^+_z \) the full subcategory of objects \( V \) which admit a filtration \( 0 = V_0 \subset V_1 \subset \cdots \subset V_N = V \) such that each quotient \( V_n \overset{\text{def}}{=} V_n/V_{n-1} \),
1 \leq n \leq N$, is isomorphic to a Verma module $V_{a_i}^z$, $a_i \in \Lambda_{C^*}$. We denote by $[V]$ the element of the group ring $C[\Lambda_{C^*}]$ defined as a sum $[V] \overset{\text{def}}{=} \sum_{n=1}^{N} a_i$. As follows from the Jordan-Hölder theorem, the element $[V]$ does not depend on a choice of filtration.

3.3.23. For any finite-dimensional representation $X = (\rho, \mathcal{X})$ of $U$ we define

$$\{X\} \overset{\text{def}}{=} \sum_{\lambda \in \Lambda_{C^*}} \dim(X_\lambda) \cdot \lambda \in C[\Lambda_{C^*}],$$

where the subspace $X_\lambda$ of $X$ is defined as in 2.1.5.

**Proposition.** For any $V$ in $A_z$ and any $U$-module $X$ in $D$

$a)$ $V \hat{\otimes} X \in A_z$ and

$b)$ $[V \hat{\otimes} X] = [V] \cdot \{X\}$.

**Proof:** The proof of a) is completely analogous to the proof of Proposition 28.1 in [KL] and the proof of b) is completely analogous to the proof of Theorem 28.1 in [KL].

### 3.4 Comparison of coinvariants.

3.4.1. Given $V \in \mathcal{O}_z^+$, $X, Y \in \mathcal{D}$ and $W \in \mathcal{O}_z^-$ we consider $\Gamma$-modules

$P(V, X, Y, W) \overset{\text{def}}{=} (V \otimes X)[t] \otimes Y \otimes W$, \quad $\hat{P}(V, X, Y, W) \overset{\text{def}}{=} (V \hat{\otimes} X)[t] \otimes (Y \hat{\otimes} W)$ and

$Q(V, X, Y, W) \overset{\text{def}}{=} (V \hat{\otimes} X)[t] \otimes (Y \hat{\otimes} W)$. We will often write $P, \hat{P}$ and $Q$ instead of $P(V, X, Y, W)$, $\hat{P}(V, X, Y, W)$ and $Q(V, X, Y, W)$. For any $n \in \mathbb{N}$ we define $^nP \overset{\text{def}}{=} P \otimes_A A_n$, $^{\hat{P}} \overset{\text{def}}{=} \hat{P} \otimes_A A_n$, and $^Q \overset{\text{def}}{=} Q \otimes_A A_n$. We denote by $\langle ^nP \rangle$, $\langle ^{\hat{P}} \rangle$ and $\langle ^Q \rangle$ the corresponding $A_n$-modules of coinvariants.

The natural imbeddings $V \otimes X \hookrightarrow V \hat{\otimes} X \hookrightarrow V \hat{\otimes} X$ and $Y \otimes W \hookrightarrow Y \hat{\otimes} W \hookrightarrow Y \hat{\otimes} W$ induce the imbeddings $^nP \hookrightarrow ^{\hat{P}} \hookrightarrow ^Q$ and the corresponding $A_n$-morphism of $\Gamma$-coinvariants

$$\langle ^nP \rangle \xrightarrow{n^\sigma} \langle ^{\hat{P}} \rangle \xleftarrow{n^\eta} \langle ^Q \rangle.$$

**Theorem.** The morphisms $n^\sigma$ and $n^\eta$ are isomorphisms for all $V \in \mathcal{O}_z^+$, $X, Y \in \mathcal{D}$ and $W \in \mathcal{O}_z^-$.  

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3.4.2 Proof: Let \( nP_0 \subset nP \) be the \( A_n \)-submodule generated by vectors of the form \( \gamma p \), where \( \gamma \in \Gamma_0 \) and \( p \in nP \). In other words, \( nP_0 \) is the kernel of the natural projection \( nP \to \langle nP \rangle \). We start with the following result.

3.4.3 Lemma. 
\( (V \otimes X)(n) \otimes (Y \otimes W) \subset nP_0 \)
\( (V \otimes X) \otimes (Y \otimes W)(n) \subset nP_0 \).

Proof of Lemma: We prove part a). The proof of part b) is completely analogous. By the definition of the subspace \( (V \otimes X)(n) \otimes (Y \otimes W) \) it is sufficient to show that for any \( \tilde{v} \in V \otimes X \), \( \tilde{w} \in Y \otimes W \) and \( x \in f_\nu \), \( |\nu| \geq n \) we have \( x - \tilde{v} \otimes \tilde{w} \in nP_0 \). But this follows from Proposition 2.4.16.

3.4.4 Corollary. The map \( ^n\sigma : \langle nP \rangle \to \langle n\hat{P} \rangle \) is an isomorphism.

3.4.5. Consider the map
\[ ^n\zeta \overset{\text{def}}{=} (^n\sigma)^{-1} \circ ^n\eta : \langle nQ \rangle \to \langle nP \rangle \].

Lemma. The map \( ^n\zeta \) is surjective.

Proof: Follows from Proposition 3.3.13 d).

3.4.6 Proposition. Theorem 3.4.1 is true in the case when V and W are generalized Verma modules.

Proof: In this case we have \( V = M \) and \( W = N \) where \( M \) and \( N \) are nil-modules and it follows from Proposition 2.5.2 that the \( A_n \)-module \( nP \) is free of rank equal to \( \dim (M \otimes X \otimes Y \otimes \omega N)_{U_0} \). Therefore \( \dim \langle nP \rangle = n \cdot \dim (M \otimes X \otimes Y \otimes \omega N)_{U_0} \). On the other hand, it follows from Proposition 3.3.23 and the right exactness of the functor \( \langle , \rangle \) that \( \dim \langle nQ \rangle \leq n \cdot \dim (M \otimes X \otimes Y \otimes \omega N)_{U_0} \). Therefore Proposition 3.4.6 follows from Lemma 3.4.5.

3.4.7. We can now finish the proof of Theorem 3.4.1. Really for any \( V \in O^+_z \), \( W \in O^-_z \) we can find exact sequences \( M_1 \to M_0 \to V \to 0 \) and \( N_1 \to N_0 \to W \to 0 \) such that
$M_0, M_1$ are generalized Verma modules in $\mathcal{O}_z^+$ and $N_0, N_1$ are generalized Verma modules in $\mathcal{O}_z^-$. Therefore it follows from Proposition 3.4.6, the right exactness of the functor $\langle \ , \ \rangle$ and the five-homomorphisms lemma that $n\bar{\xi} : \langle^nQ \rangle \to \langle^nP \rangle$ is an isomorphism. Theorem 3.4.1 is proved.

3.4.8 Theorem. Let $\mathcal{T} = \mathcal{T}((z\bar{q}h)^2)$. The diagram

$$
\begin{array}{ccc}
\langle^nQ(V, X, Y, W)\rangle & \xrightarrow{T^{\mathcal{T}}_{V} \otimes X \otimes id} & \langle^nQ(\mathcal{T}(V), \mathcal{T}(X), Y, W)\rangle \\
\downarrow n\bar{\xi} & & \downarrow n\bar{\xi} \\
\langle^nP(V, X, Y, W)\rangle & \xrightarrow{(T^{\mathcal{T}}_{V}^{-1} \otimes id) \cdot n\delta_{X, Y} \otimes W} & \langle^nP(\mathcal{T}(V), \mathcal{T}(X), Y, W)\rangle
\end{array}
$$

is commutative.

Proof: We decompose the diagram into a sequence of simpler diagrams whose commutativity was proven already.
where the commutativity of (1) follows from Theorem 3.2.2, the commutativity of (4) from Theorem 2.5.7, the vertical map in (2), (3), (5) and (6) are isomorphisms coming from natural imbeddings (see Theorem 3.4.1) and the right vertical isomorphism in (7) comes from the natural isomorphism $\langle M \rangle \sim \langle T(M) \rangle$ as in 1.1.13. Theorem 3.4.5 is proved.
3.4.9. Fix $z$ as in 3.1.7, $V$ in $O^+_z$, $W$ in $O^-_z$, $X,Y$ in $D$ and define $u \eqdef (z\tilde{q}^h)^{-2}$. For any $n \in \mathbb{N}$ we define $^n R \eqdef \text{Hom}_{A^n}(\langle ^n Q(V,X,Y,W) \rangle, \langle ^n P(V,X,Y,W) \rangle)$ and $^n R' \eqdef \text{Hom}_{A^n}(\langle ^n Q(T(V),T(X),Y,W) \rangle, \langle ^n P(T(V),T(X),Y,W) \rangle)$ and denote by $^n \nabla'$ the $A_n$-linear map from $^n R$ to $^n R'$ such that $^n \nabla'(a) = (T_V^{-1} \otimes \text{id}) \circ ^n \delta_{V,X,Y} \otimes W \circ (a \cdot T_V^{-1} \rtimes X \otimes \text{id})$.

3.4.10. The $\tilde{u}$-linear isomorphism $\hat{u}$ which we constructed in 2.4.22 define an $\tilde{u}$-linear isomorphism $\check{u} = (\hat{u})^{-1}$ such that $\check{u} : ^n R' \to ^n R$ and we define $^n \nabla \eqdef \check{u} \circ ^n \nabla'$. Then $^n \nabla : ^n R \to ^n R$ is a $\tilde{u}$-linear automorphism. We can restate Theorem 3.4.8 as follows.

**Theorem.** $^n \nabla(\tilde{\xi}) = ^n \tilde{\xi}$. 
4. Finite-dimensional representations

4.1 The category $\mathcal{D}$.

4.1.1 As before, we denote by $\mathcal{D}$ the category of unital finite-dimensional $U$-modules $M$ such that $Z$ acts on $M$ as identity.

**Remark:** If $\tilde{U}$ is the quotient of $U$ by the ideal generated by $Z - 1$ we can identify $\mathcal{D}$ with the category of finite dimensional unital $\tilde{U}$-modules.

It is clear that $\mathcal{D}$ has a natural structure of a strict monoidal rigid category (see §1).

4.1.2. The algebra $\tilde{U}$ admits “loop-like” generators $x_{ik}^\pm, h_{ik}, i \in \mathcal{T}, k \in \mathbb{Z}$ such that $h_{i0} = K_i, x_{i0}^\pm = E_i, \quad i \in \mathcal{T}$; and the generators are subject to certain commutation relations explained in [D2] (see also [Be1]). We will use only the following relations:

\[
\begin{align*}
(\alpha) & \quad [h_{ik}, h_{j\ell}] = 0 \\
(\beta) & \quad [h_{ik}, x_{j\ell}^\pm] = \pm \frac{1}{q} [k \theta_{ij}] \cdot x_{j,k+\ell}^\pm \\
(\gamma) & \quad [x_{ik}^+, x_{j\ell}^-] = \delta_{\epsilon (\theta_{ik_{k+\ell}} - \varphi_{ik_{k+\ell}})} \frac{q - q^{-1}}{q - q^{-1}},
\end{align*}
\]

where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and the elements $\theta_{i,p}$ and $\varphi_{i,p}$, $p > 0$, are defined from the relations:

\[
\begin{align*}
\exp((q - q^{-1}) \sum_{p > 0} h_{ip} \tau^p) &= 1 + \sum_{p > 0} K_i^{-1} \theta_{ip} \tau^p \\
\exp((q - q^{-1}) \sum_{p < 0} h_{ip} \tau^p) &= 1 + \sum_{p < 0} K_i \varphi_{ip} \tau^p,
\end{align*}
\]

where we consider $\tau$ as a formal parameter.

4.1.3. Let $A^+$ (resp. $A^-$) $\subset \tilde{U}$ be the subalgebra with unity generated by $x_{ik}^\pm, h_{ik}, \quad i \in \mathcal{T}$, where $k > 0$ ($k < 0$ resp.).

**Proposition.** For any $X = (\rho, \underline{X})$ in $\mathcal{D}$ we have $\rho(A^+) = \rho(A^-) = \rho(\tilde{U})$.

**Proof:** We prove that $\rho(A^+) = \rho(\tilde{U})$. The proof of the equality $\rho(A^-) = \rho(\tilde{U})$ is completely analogous.

Let $\mathcal{H}(n) \subset \tilde{U}$ be the span of $h_{ik}, k \geq n, \quad 1 \leq i \leq r$. Put $H = \cap_{n \in \mathbb{N}} \rho(\mathcal{H}(n))$. Since $\dim \underline{\text{End}} \underline{X} < \infty$ there exists $N > 0$ such that $H = \rho(\mathcal{H}(N + p))$ for any $p \geq 0$.  

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Let us prove that for any \( \ell < 0 \), \( \rho(x_{j\ell}^+)^{\dagger} \in \rho(A^+) \). Fix a vertex \( j_0 \in T \) such that \( a_{j_0j} \neq 0 \). Then
\[
[\rho(h_{j_0N}^N), \rho(x_{j\ell}^+)] = \frac{1}{k}[ka_{j_0j}]\rho(x_{j\ell}^+),
\]
where \( m = \ell - N \).

We have \( \rho(h_{j_0N}) \in H = \rho(H(N + p)) \), for all \( p \geq 0 \) and therefore \( \rho(h_{j_0N}) \) lies in \( \rho(H(N - \ell + 1)) \). Since \( \rho(h_{j_0N}) \) lies in \( \rho(H(N - \ell + 1)) \) we can write \( \rho(h_{j_0N}) \) as a linear combination of operators of the form \( \rho(h_{is}) \), \( i \in T \), \( s > N - \ell \). Therefore \( \frac{1}{k}[ka_{j,\ell}]\rho(x_{j\ell}^+) \) is a linear combination of operators of the form
\[
[\rho(h_{is}), \rho(x_{jm}^+)] = \frac{1}{k}[ka_{ij}]x_{j,m+s}^+, \quad s > N - \ell, \ m = \ell - N.
\]
Therefore we see that \( \frac{1}{k}[ka_{j_0j}]x_{j\ell}^+ \) lies in \( \rho(A^+) \). Since the number \( q \) is not a root of 1 we have \( \frac{1}{k}[ka_{j_0j}] \neq 0 \), and therefore \( \rho(x_{j\ell}^+) \in \rho(A^+) \). In a similar way, one shows that \( \rho(x_{j\ell}^-) \in \rho(A^+) \) for any \( \ell \in \mathbb{Z} \). \( \rho(h_{ik}) \in \rho(A^+) \) for \( k \geq 0 \) and for any \( i \in T \), we have: \( \rho(\theta_{ik}) \in \rho(A^+) \) for \( k \geq 0 \). The relation \( (\gamma) \) in 4.1.2 implies that \( \rho(\varphi_{ik}) \in \rho(A^+) \) for \( k \geq 0 \) and all \( i \in T \). Since \( \varphi_{ik} = \theta_{ik} = 0 \) if \( k \leq 0 \) we see that the images of all Drinfeld’s generators of \( \overline{U} \) belong to \( \rho(A^+) \). Proposition 4.1.3 is proved.

4.2 Endomorphisms of tensor products.

4.2.1. Let \( F = \mathbb{C}[t] \) be the field of fractions of \( A(= \mathbb{C}[t]) \), \( \overline{A} = \mathbb{C}[[t]] \) be the completion of \( A \) and \( \overline{F} \) be the field of fractions of \( \overline{A} \). It is clear that \( \overline{X}(t) = X[t] \otimes_A F \) and \( X((t)) = X[t] \otimes_A \overline{F} \) carry the structures of \( U_F \) and \( U_{\overline{F}} \)-modules respectively. We will denote them by \( X(t) = (\rho_X(t), X(t)) \), \( X((t)) = (\rho_X((t)), X((t))) \).

For any \( X = (X, \rho_X), Y = (Y, \rho_Y) \) in \( D \), the \( \Gamma \)-module structure \( \rho_X[t] \otimes \rho_Y \) on \( X[t] \otimes_{\mathbb{C}} Y \) defines a \( \Gamma \)-module structure on finite dimensional, respectively, \( F \) and \( \overline{F} \) vector spaces
\[
X(t) \otimes Y \overset{\text{def}}{=} (X[t] \otimes Y) \otimes_A F \quad \text{and} \quad X((t)) \otimes Y \overset{\text{def}}{=} (X[t] \otimes Y) \otimes_A \overline{F}.
\]
We denote the corresponding \( \Gamma \)-modules by \( X(t) \otimes Y = (\rho_X(t) \otimes_Y, X(t) \otimes Y) \) and
\[ X((t)) \otimes Y = (\rho_{X((t)) \otimes Y}(U_F)) \subset \text{End}(X((t)) \otimes Y) \] the \( \overline{F} \)-space of the image of \( \rho_{X((t)) \otimes Y} \).

4.2.2. For any \( X, Y \) in \( \mathcal{D} \) we denote (as in 1.1.10) the ring of endomorphisms of \( X \) and \( Y \) to be \( E_X, E_Y \) and denote by \( E \) the tensor product \( E = E_{X,Y} \equiv E_X \otimes_\mathbb{C} E_Y \). Let

\[ \mathcal{E} \equiv \text{End}_{U_F}(X((t)) \otimes_\mathbb{C} Y), \quad \mathcal{E} \equiv \text{End}_{U_F}(X(t) \otimes_\mathbb{C} Y). \]

Then \( \mathcal{E} \) (resp. \( \mathcal{E} \)) has a natural structure of an \( \overline{F} \) (resp. \( F \)) module and we have a natural imbedding \( E \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{E} \).

4.2.3 Theorem. The natural morphism \( E \otimes_\mathbb{C} F \rightarrow \mathcal{E} \) is an isomorphism.

Proof: We start with the following obvious result.

4.2.4. Let \( X, Y \) be finite-dimensional \( \overline{F} \)-vector spaces, \( B_X \subset \text{End}_{\overline{F}}(X) \), \( B_Y \subset \text{End}_{\overline{F}}(Y) \) be \( \overline{F} \)-subalgebras containing \( \text{id}_X \) and \( \text{id}_Y \), \( E_X \) and \( E_Y \) be centralizers of \( B_X \) and \( B_Y \) in \( \text{End}_{\overline{F}}(X) \) and \( \text{End}_{\overline{F}}(Y) \) respectively.

Lemma. The centralizer of \( B_X \otimes B_Y \) in \( \text{End}_{\overline{F}}(X \otimes Y) = \text{End}_{\overline{F}}(X) \otimes_{\overline{F}} \text{End}_{\overline{F}}(Y) \) is equal to \( E_X \otimes E_Y \).

Proof: Well known.

Since \( \rho_{X((t)) \otimes Y}(U_F) \subset \rho_{X((t))}(U_F) \otimes_{\overline{F}} \rho_Y(U_F) \), Lemma 4.2.4 shows that the following result implies the validity of Theorem 4.2.3.

4.2.5 Proposition. For any \( X = (\rho_X, X), Y = (\rho_Y, Y) \) in \( \mathcal{D} \)

\[ \rho_{X((t)) \otimes Y}(\Gamma) \supset \rho_{X((t))}(U_F) \otimes_{\overline{F}} \rho_Y(U_F). \]

4.2.6 Proof of Proposition 4.2.5: Let \( B_X \equiv \rho_X(U) \subset \text{End}_\mathbb{C}(X), \)

\( B_Y \equiv \rho_Y(U) \subset \text{End}_\mathbb{C}(Y), \) \( p : \text{End}_A(\mathcal{X}_t \otimes \mathcal{Y}) \rightarrow \text{End}_\mathbb{C}(\mathcal{X} \otimes \mathcal{Y}) \) be the reduction mod \( m \), where \( m \equiv tA \subset A \) and \( \overline{p} \) be the composition

\[ \overline{p} = p \circ \rho_{X([t]) \otimes Y} : \Gamma \rightarrow \text{End}_\mathbb{C}(\mathcal{X} \otimes \mathcal{Y}) = \text{End}_\mathbb{C}(\mathcal{X}) \otimes_\mathbb{C} \text{End}_\mathbb{C}(\mathcal{Y}). \]

Nakayama's lemma shows that Theorem 4.2.3 is a consequence of the following result.
CLAIM. $\varpi(\Gamma) \supset B_X \otimes B_Y$.

4.2.7 PROOF OF CLAIM: It follows from [Be], Th. 4.7 and Prop. 5.3 that for any $k > 0$, $i \in I$ we have $h_{ik}, x_{ik} \in \Gamma$ and moreover for any $\alpha \in \{h_{ik}, x_{ik}^{\pm}\}, i \in I, k > 0$ we have

$$(id \otimes \psi)\Delta(\alpha) - \alpha \otimes 1 \in m(U_A \otimes_A U_A).$$

Here $\psi$ means the automorphism defined in 2.4.2. Therefore $\varpi(\alpha) = \rho_X(\alpha) \otimes id_Y$ and we see that $\varpi(\Gamma) \supset \rho_X(A^+) \otimes id_Y$. As follows from Proposition 4.1.3, $\varpi(\Gamma) \supset B_X \otimes id_Y$.

Analogously we see that for any $i \in I, k < 0$ and $x \in \{h_{ik}, x_{ik}^{\pm}\}$ we have $t^{-k}\alpha \in \Gamma$ and $(id \otimes \psi)\Delta(t^{-k}\alpha) - 1 \otimes \alpha \in m(U_A \otimes U_A)$. So $\varpi(t^{-k}\alpha) = 1 \otimes \alpha$ and we have $\varpi(\Gamma) \supset id_X \otimes \rho_Y(A^-)$. As follows from Proposition 4.1.3 we have $\varpi(\Gamma) \supset id_X \otimes B_Y$. This finishes the proof of the Claim. Theorem 4.2.3 is proved.

4.2.8 COROLLARY. The natural morphism $E \otimes_C F \to \mathcal{E}$ is an isomorphism.

PROOF: We can consider $\mathcal{E}$ and $\text{End}_F(X((t)) \otimes Y)$ as subspaces in $\text{End}_F(X((t)) \otimes Y)$. Then $\mathcal{E} = \mathcal{E} \cap \text{End}_F(X(t) \otimes Y)$ and the Corollary follows immediately from Theorem 4.2.3.

4.2.9. Let $G$ be the algebraic $C$-group such that $G(C)$ is the group of invertible elements in $E_X \otimes E_Y$. We can restate Corollary 4.2.7 in the following way.

COROLLARY. The natural morphisms $G(F) \to \text{Aut}(X(t) \otimes Y)$ and $G(F) \to \text{Aut}(X(t) \otimes Y)$ are isomorphisms.

4.2.10. Since the group $G$ is defined over $C$ any automorphism $\eta$ of the field $\overline{F}$ preserving the subfield $F \subset \overline{F}$ defines group automorphisms of groups $\text{Aut}(X(t) \otimes Y)$ and $\text{Aut}(X((t)) \otimes Y)$ which we denote by $\hat{\eta}$.

4.3 Intertwiners from $X(t) \otimes Y$ to $Y \otimes X(t)$.

4.3.1. For any $X, Y$ in $D$ we denote by

$$\overline{J} \subset \text{Hom}_{\overline{F}}(X((t)) \otimes Y, Y \otimes X((t))),$$

$$J \subset \text{Hom}_F(X(t) \otimes Y, Y \otimes X(t))$$

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the subsets of morphisms which are isomorphisms. The group $G(F)$ acts naturally on $\mathcal{J} : (a, f) \rightarrow af, a \in \mathcal{J}, f \in G(F)$.

Let $\text{Hom}(X(t) \otimes Y, Y \otimes X(t))$ be the $F$-linear space $\text{Hom}_U (X(t) \otimes Y, Y \otimes X(t))$ considered as an algebraic variety.

4.3.2 Lemma. a) $\mathcal{J}$ is a set of $F$-points of a Zariski open subset $\mathcal{J}$ in a linear subspace of $\text{Hom}(X(t) \otimes Y, Y \otimes X(t))$, $\mathcal{J} = \mathcal{J}(F)$ and the action of $G(F)$ on $\mathcal{J}$ comes from an algebraic action of $G$ on $\mathcal{J}$.

b) Either $\mathcal{J}$ is empty or $G$ acts simply transitively on $\mathcal{J}$.

Proof: Clear.

4.3.3 Proposition. For any $s \in \mathcal{J}$ there exists an element $f \in G(F)$ and such that $s = s_0 f$, where $s_0 \in \mathcal{J}$.

Proof: Since the algebraic $\mathbb{C}$-group $G$ is a group of invertible elements in an algebra $E$ (see 4.2.2) it is isomorphic to a semidirect product $L \rtimes N$, where $L$ is a direct product of a number of general linear groups and $N$ is a unipotent group. Let $G(F)$ be the $F$-group obtained from $G$ by the extension of scalars from $\mathbb{C}$ to $F$. Then $G(F)$ is also a semidirect product of $L^*_F$ and $N^*_F$, where $L^*_F$ is a direct product of general linear groups and $N^*_F$ is unipotent. As follows from Corollary 4.2.8 and Lemma 4.3.2 b) $\mathcal{J}$ is a principal homogeneous $G$-space. It follows from Proposition 1.33, Propositions 3.1 and 3.6 in [S] that $\mathcal{J} \neq \emptyset$ and moreover $\mathcal{J} = G(F)\mathcal{J}$. Proposition 4.3.3 is proved.

4.4 The functional equation.

4.4.1. For any $X = (\rho_X, X)$, $Y = (\rho_Y, Y)$ in $\mathcal{D}$ we denote by $s_{X[t], Y} \in \mathcal{J}$ the inverse limit of elements $^n s_{X[t], Y}$, where $^n s_{X[t], Y}$ are defined in 2.4.19. It can be treated as a linear map $X[t] \otimes Y \rightarrow Y \otimes X[t]$, where $X[t] = X \otimes \mathbb{A}$.

Lemma. There exists an element $\hat{f} \in G(F)$ such that $s_{X[t], Y} \cdot \hat{f}^{-1} \in \mathcal{J}$ and such an element is unique up to a left multiplication by an element in $G(F)$.

Proof: Follows from Proposition 4.3.3.
We denote by $L_{X,Y}$ the subset of all elements $f \in G(\overline{F})$ such that $s_{X[[t]],Y} \cdot f^{-1} \in \mathcal{J}$.

**Corollary.** $L_{X,Y}$ is a left $G(F)$ coset in $G(\overline{F})$.

**4.4.2.** Let $\eta$ be the continuous automorphism of the field $\overline{F}$ over $\mathbb{C}$ such that $\eta(t) = \overline{q^{2h}} \cdot t$. Let $\tilde{\eta}$ be the corresponding automorphisms of the groups $G(F)$ and $G(\overline{F})$ (see 4.2.10).

**Theorem.** $\tilde{\eta}(L_{X,Y}) = L_{X,Y}$.

**4.4.3.** We start the proof of Theorem 4.4.2 with the following observation. Let
$$
\mathcal{J}^* \subset \text{Hom}_{U_F}(X(t) \otimes Y^*, Y^* \otimes X(t)), \quad \mathcal{J}' \subset \text{Hom}_{U_{\ell}}(X((t)) \otimes Y^*, Y^* \otimes X((t))),
$$
$$
\mathcal{J}^{**} \subset \text{Hom}_{U_F}(X(t) \otimes Y^{**}, Y^{**} \otimes X(t)) \quad \mathcal{J}^{**'} \subset \text{Hom}_{U_{\ell}}(X((t)) \otimes Y^{**}, Y^{**} \otimes X((t))
$$
be the subsets of isomorphisms as in 4.3.1 and we denote by $\mathcal{J}^*$ the algebraic subvariety of the linear space $\text{Hom}(X(t) \otimes Y^*, Y^* \otimes X(t))$ corresponding to $\mathcal{J}^*$.

There exists an action
$$
\phi : (E_X^{op} \otimes E_Y^{op}) \times \text{Hom}_{U_F}(X(t) \otimes Y^*, Y^* \otimes X(t)) \to \text{Hom}_{U_F}(X(t) \otimes Y^*, Y^* \otimes X(t))
$$
of the algebra $E_X^{op} \otimes E_Y^{op}$ on the space $\text{Hom}_{U_F}(X(t) \otimes Y^*, Y^* \otimes X(t))$ such that
$$
\phi(a, f_X \otimes f_Y) = (id \otimes f_X) \circ a \circ (id \otimes f_Y^*)
$$
for $f_X \in E_X, f_Y \in E_Y, a \in \text{Hom}_{U_F}(X(t) \otimes Y^*, Y^* \otimes X(t))$. The action $\phi$ of the algebra $(E_X \otimes E_Y)^{op}$ on $\text{Hom}(X(t) \otimes Y^*, Y^* \otimes X(t))$ defines an algebraic action $(a, g) \mapsto a \ast g$ of $G(F)$ on $\mathcal{J}^*$ such that $a \ast g \overset{\text{def}}{=} \phi(a, g^{-1})$ for $a \in \mathcal{J}^*, g \in G(F)$.

**4.4.4.** We consider $s_{X[[t]],Y^*}$ as an element in $\mathcal{J}^*$ and denote by $\mathcal{M}_{X,Y^*} \subset G(\overline{F})$ the set of all $g \in G(\overline{F})$ such that $(s_{X[[t]],Y^*}) \ast (g^{-1}) \in \mathcal{J}^*$.

**Lemma.** $\mathcal{M}_{X,Y^*}$ is a left $G(F)$ coset in $G(\overline{F})$.

**Proof:** Clear.

**4.4.5 Proposition.** $\mathcal{M}_{X,Y^*} = L_{X,Y}$.

**Proof of Proposition:** Consider the action $\psi$ of the algebra $E_X \otimes E_Y$ on the space $\text{Hom}_{U_F}(Y^* \otimes X(t), X(t) \otimes Y^*)$ such that $\psi(b, f_X \otimes f_Y) = (id \otimes f_Y^*) \circ b \circ (id \otimes f_X)$ for $f_X \in E_X, f_Y \in E_Y, b \in \text{Hom}_{U_F}(Y^* \otimes X(t), X(t) \otimes Y^*)$.
**Lemma.** For any $a \in J^*$, $g \in G$ we have $(a \ast g)^{-1} = \psi(a^{-1}, g)$.

**Proof:** Follows from the definitions.

4.4.6. The action $\psi$ defines an algebraic action $(b, g) \mapsto b \hat{\otimes} g$ of $G(F)$ on $\text{Hom}_{U_F}(Y^* \otimes X(t), X(t) \otimes Y^*)$ such that $b \hat{\otimes} g = \psi(b, g)$ for $b \in \text{Hom}_{U_F}(Y^* \otimes X(t), X(t) \otimes Y^*)$, $g \in G(F)$. As follows from Lemma 4.4.5 we have

$$ M_{X,Y^*} = \{ g \in G(F) | s_{X,Y^*}^{-1}(a \cdot f_{X,Y}) \hat{\otimes} g^{-1} \in \text{Hom}(Y^* \otimes X(t), X(t) \otimes Y^*) \}.$$

4.4.7. Now we can prove Proposition 4.4.5. Choose any $f_{X,Y} \in L_{X,Y}$. Then we have

$$s_{X,Y} = a \cdot f_{X,Y}, \text{ where } a \in J.\text{ As follows from Proposition 1.1.18 we have } \varphi_{X[[t]]} Y^* X_{[t]}, Y^* Y^* = s_{X[[t]]}^{-1} Y^* X_{[t]}, Y^*.\text{ So } s_{X[[t]]}^{-1} Y^* X_{[t]}, Y^* = \varphi_{X[[t]]} Y^* (a \cdot f_{X,Y}).\text{ By Proposition 1.1.10 the right side is equal to } \varphi_{X[[t]]} Y^* Y^* X_{[t]}, Y^* Y^* X_{[t]}\text{ maps } \text{Hom}_{U_F}(X(t) \otimes Y, Y \otimes X(t))\text{ into } \text{Hom}_{U_F}(Y^* \otimes X(t), X(t) \otimes Y^*)\text{ we see that } f_{X,Y} \in M_{X,Y^*}.\text{ Proposition 4.4.5 is proved.}$$

4.4.8 **Proposition.** $L_{X,Y^*} = M_{X,Y}$ for all $X, Y$ in $D$.

**Proof:** Completely analogous to the proof of Proposition 4.4.5.

**Corollary.** $L_{X,Y^{**}} = L_{X,Y}$.

4.4.9. For $u \in \mathbb{C}^*$ we denote by $\theta_u$ the automorphism of the field $F$ such that $\theta_u(t) = ut$. We also denote by $\theta_u$ the continuous extension of $\theta_u$ on $F$. The automorphism $\theta_u$ define automorphisms of the groups $G(F)$ and $G(F)$ which we denote by $\hat{\theta}_u$.

**Lemma.** For any $X, Y$ in $D$ and $u \in \mathbb{C}^*$ we have

$$ L_{T_u(X),Y} = \hat{\theta}_u(L_{X,Y}) \text{ and } L_{X,T_u(Y)} = \hat{\theta}_u^{-1}(L_{X,Y}).$$

**Proof:** Follows immediately from the definitions.

4.4.10. Now we can finish the proof of Theorem 4.4.2. Really it follows from Corollary 4.4.8, Lemma 4.4.9 and Proposition 2.1.15. Theorem 4.4.2 is proved.

**Corollary.** For any $f_{X,Y} \in L_{X,Y}$ we have

$$(*) \quad \hat{\eta}(f_{X,Y}) f_{X,Y}^{-1} \in G(F).$$
Lemma. There exists $f_{X,Y} \in \mathcal{L}_{X,Y}$ such that $f_{X,Y}$ and $X[[t]] \otimes Y$ preserve $X[[t]] \otimes Y \subset X((t)) \otimes Y$ and $f_{X,Y}(0) = id$.

Proof: The standard (profinite) topology on $\mathcal{F}$ defines a topology on the group $G(\mathcal{F})$. Since the group $G$ is a semidirect product $L \rtimes N$, where $L$ is the direct product of a number of copies of $GL_n$ and $N$ is a unipotent group, the subgroup $G(\mathcal{F})$ is dense in $G(\mathcal{F})$. Lemma 4.4.11 follows now immediately from the equality $s_{X[[t]],Y} \equiv \sigma \pmod{m}$, where $\sigma : X \otimes Y \rightarrow Y \otimes X$ is permutation as in 2.3.3.

4.4.12. Let $A_{mer}$ be the subring of the power series which converges in a neighborhood of zero and define a meromorphic function in $\mathbb{C}$.

Proposition. For any $X,Y$ in $\mathcal{D}$ there exists $f_0 \in G(A_{mer})$ which satisfies the condition of Lemma 4.4.11.

Proof: Choose any $f_{X,Y} \in \mathcal{L}_{X,Y}$ and define $r = \hat{\eta}(f_{X,Y})^{-1}$. As follows from Theorem 4.4.2 we have $r \in G(\mathbb{C}(t) \cap A)$ and $r(0) = Id$. Consider the equation

$$\hat{\eta}(f)f^{-1} = r$$

on $f \in G(A)$. It is easy to see that there exists a unique solution $f_0$ of (**') in $G(A)$ such that $f_0(0) = Id$ and moreover any solution of (**') has a form $f = f_0 \cdot c$ for some $c \in G(\mathbb{C})$. One can write $f_0 \in G(A)$ as an infinite convenient product $f_0(t) = \prod_{n=0}^{\infty} r^{-1}(2^{2^nh^n}t)$. It is easy to see that this product is also convergent in $G(A_{mer})$. Proposition 4.4.12 is proved.

Corollary. The map $s_{X[[t]],Y}$ from $X \otimes_{\mathbb{C}} \overline{Y} \otimes_{\mathbb{C}} \overline{A}$ to $Y \otimes X \otimes A$ maps the subspace $X \otimes_{\mathbb{C}} \overline{Y} \otimes_{\mathbb{C}} A_{mer}$ to $Y \otimes_{\mathbb{C}} \overline{X} \otimes_{\mathbb{C}} A_{mer}$.

4.4.13. As follows from the previous Corollary we can consider $s = s_{X[[t]],Y}$ as a meromorphic function in $t$ with values in the space $\text{Hom}_\mathbb{C}(X \otimes Y, X \otimes X)$. The union of the set of poles of $s$ and the points $t \in \mathbb{C}$ such that $s(t)$ is not an isomorphism is denoted by $\Lambda_{X,Y}$.

Lemma. $\Lambda_{X,Y}$ is a discrete subset of $\mathbb{C}$.

Proof: It is clear that either $\Lambda_{X,Y} = \mathbb{C}$ or it is discrete. Since $s(0)$ is invertible,
\[ \Lambda_{X,Y} \neq \emptyset. \text{ The Lemma is proved.} \]

4.4.14. Let \( \tilde{D}^{(2)} \) be the full subcategory of the pair of objects in \((X,Y)\) in \(D\) such that 
\[ 1 \notin \Lambda_{X,Y}. \]
As follows from Lemma 4.4.13 for all pairs \((X,Y)\) in \(D\) we have 
\[ (T_u(X), Y) \in \tilde{D}^{(2)} \]
for all \(u\) in a dense open set \(\mathbb{C}^* - \Lambda_{X,Y}\). We can consider \((\tilde{D}^{(2)}, s)\) as a 
weak braiding on \(D\) (see 1.2.1).

**Proposition.** The data \((C, D, \mathcal{O}_z^\pm, s_\pm, T_\pm, \tilde{D}^{(2)}, s)\) is a rigid KZ-data for all \(z\) such that 
\[ zq^{h^\vee} \in \mathbb{C}^* - \Omega_{\geq 0}, \text{ where } T_\pm = T_{\pm 1}, \text{ } s_{\pm (V,X)} \overset{\text{def}}{=} (s_{V,X})^{\pm 1}, \text{ and } s_{V,X} \text{ is as in 2.3.3}. \]

**Proposition:** Follows immediately from the definitions and 4.4.14, 2.3.8.

4.4.15 **Remark.**: Similarly to 4.4.3 one can prove that “the square of weak braiding” 
\[ s_{X,Y}(t)s_{Y,X}(t^{-1}) \] is an elliptic function on the curve \(\mathbb{C}^*/q^{2h^\vee}\mathbb{Z}\) with values in 
\(\text{Hom}_\mathbb{C}(X \otimes Y, Y \otimes X)\).
§5. The quasi-associativity morphism.

5.1 The definition.

5.1.1. Fix $z$ such that $z \tilde{q}^h \in \mathbb{C}^* - \mathfrak{Q}_{\geq 0}$ and $U$-modules $V$ in $\mathcal{O}_z^+$, $W$ in $\mathcal{O}_z^-$, $X, Y$ in $\mathbb{D}$. We assume that $(z \tilde{q}^h)^{2m} \notin \Lambda_{X,Y}$ for $m \in \mathbb{Z}$. Let $P = P(V, X, Y, W)$ be the $\Gamma$-module as in 3.4.1 and $\langle P \rangle$ the corresponding $A$-module of coinvariants.

Proposition. $\langle P \rangle$ is a finitely generated $A$-module.

Proof: As follows from Proposition 3.1.8, there exist exact sequences $M'_1 \to M'_0 \to V \to 0$ and $N'_1 \to N'_0 \to W \to 0$ where $M'_k$ and $N'_k$ are generalized Verma modules in $\mathcal{O}_z^+$ and $\mathcal{O}_z^-$, respectively, $k = 0, 1$. Therefore Proposition 5.1.1 follows from Lemma 2.5.5 and the right exactness of the functor $\langle \rangle$.

5.1.2. Let $Q, nP, nQ$ be $\Gamma$-modules as in 3.4.1 and $\langle Q \rangle, \langle nP \rangle$ and $\langle nQ \rangle$ be the corresponding modules of coinvariants.

Proposition. a) $\langle Q \rangle$ is a finitely generated $A$-module.

b) The natural $A_n$-module morphisms $n\pi_P : \langle P \rangle \otimes_A A_n \to \langle nP \rangle$ and $n\pi_Q : \langle Q \rangle \otimes_A A_n \to \langle nQ \rangle$ are isomorphisms for all $n \in \mathbb{N}$.

Proof: a) Follows from Proposition 5.1.1 and Theorem 3.4.1 and b) follows from Proposition 2.5.4.

5.1.3. For any $n \in \mathbb{N}$ we denote by $n\xi : \langle P \rangle \otimes_A A_n \to \langle Q \rangle \otimes_A A_n$ the composition $n\xi = (n\pi_P)^{-1} \circ n\tilde{\xi} \circ n\pi_Q$ where $n\tilde{\xi}$ is the isomorphism from $\langle nQ \rangle$ to $\langle nP \rangle$ as in 3.4.5. It follows from Theorem 3.4.1 and Proposition 5.1.2 that $n\xi$ is an isomorphism for all $n \in \mathbb{N}$. It is clear that the isomorphisms $n\xi$ are compatible with the natural projections $\langle P \rangle \otimes_A A_{n+1} \to \langle P \rangle \otimes_A A_n$ and $\langle Q \rangle \otimes_A A_{n+1} \to \langle Q \rangle \otimes_A A_n$. Therefore the family $n\xi$ defines an isomorphism $\tilde{\xi} : \langle Q \rangle \to \langle P \rangle$ of $\mathcal{A}$-modules, where $\langle Q \rangle \overset{\text{def}}{=} \lim_{\leftarrow} \langle Q \rangle \otimes_A A_n$, $\langle P \rangle \overset{\text{def}}{=} \lim_{\leftarrow} \langle P \rangle \otimes_A A_n$ and $\mathcal{A} = \lim_{\leftarrow} A_n$.

Since $\langle P \rangle, \langle Q \rangle$ are finitely generated $A$-modules we can identify the $\mathcal{A}$-modules $\langle P \rangle$, $\langle Q \rangle$ with the tensor products $\langle P \rangle \otimes_A \mathcal{A}$ and $\langle Q \rangle \otimes_A \mathcal{A}$ correspondingly.

Let $A_{mer}$ be the ring as in Section 4.
5.1.4. Let \( \langle P \rangle_{\text{mer}} \overset{\text{def}}{=} \langle P \rangle \otimes_{A} A_{\text{mer}} \), \( \langle Q \rangle_{\text{mer}} \overset{\text{def}}{=} \langle Q \rangle \otimes_{A} A_{\text{mer}} \). We can identify \( \langle P \rangle \) with \( \langle P \rangle_{\text{mer}} \otimes_{A} A_{\text{mer}} \) and \( \langle Q \rangle \) with \( \langle Q \rangle_{\text{mer}} \otimes_{A} A_{\text{mer}} \).

**Theorem.** There exists an isomorphism \( \xi_{\text{mer}} : \langle Q \rangle_{\text{mer}} \to\to \langle P \rangle_{\text{mer}} \) such that \( \bar{\xi} = \xi_{\text{mer}} \otimes 1 \).

**Proof:** For any \( n \in \mathbb{N} \) we denote by \( nR \) the \( A_{n} \)-module \( \text{Hom}_{A_{n}}(\langle nQ \rangle, \langle nP \rangle) \) and by \( n\nabla \) the \( \tilde{u} \)-linear automorphism of \( nR \) as in 3.4.10.

Let \( \mathcal{R} \overset{\text{def}}{=} \lim_{\leftarrow} nR \). We can identify \( \mathcal{R} \) with the \( \overline{A} \)-module \( \text{Hom}_{\overline{A}}(\langle Q \rangle, \langle P \rangle) \). Let \( R_{\text{mer}} \overset{\text{def}}{=} \text{Hom}_{A_{\text{mer}}}(\langle Q \rangle_{\text{mer}}, \langle P \rangle_{\text{mer}}) \). One can consider \( R_{\text{mer}} \) as an \( A_{\text{mer}} \)-module of \( \mathcal{R} \) and \( \mathcal{R} = R_{\text{mer}} \otimes_{A_{\text{mer}}} \overline{A} \). We denote by \( \nabla \) the \( \tilde{u} \)-linear automorphism of the module \( \mathcal{R} \overset{\text{def}}{=} \lim_{\leftarrow} nR \) which is the projective limit of \( n\nabla \).

5.1.5 **Lemma.** \( \nabla(R_{\text{mer}}) \subset R_{\text{mer}} \).

**Proof:** Follows from Corollary 4.4.12, Proposition 2.4.22 and the definition of \( \nabla \).

5.1.6. Our proof of Theorem 5.1.4 is based on the following result.

**Proposition.** Let \( \varphi : D_{R} \to M_{n}(\mathbb{C}) \) be a holomorphic function in a disc \( |t| \subset D_{R} = \{t \in \mathbb{C} | |t| < R \} \) such that \( \det \varphi(0) \neq 0 \), \( \psi : \mathbb{C} \to GL_{n}(\mathbb{C}) \) be a polynomial function, \( p \) a complex number such that \( 0 < |p| < 1 \) and \( F(t) \in \overline{A} \otimes_{\mathbb{C}} M_{n}(\mathbb{C}) \) be a formal power series solution of the difference equation

\[
(*) \quad F(pt)\psi(t) = \varphi(t)F(t)
\]

such that \( F(0) = Id \). Then the series \( F(t) \) is convergent in the disc \( |t| < \frac{R}{\|\varphi(0)\| + 1} \). Moreover if \( \varphi(t) \) has a meromorphic continuation to \( \mathbb{C} \) then \( F(t) \) has a meromorphic continuation to \( \mathbb{C} \).

**Proof:** As follows from \( (*) \) we have \( \varphi(0) = \psi(0) \). Let us write the expansions for \( \varphi, \psi \) and \( F \)

\[
\varphi(t) = \sum_{j=0}^{\infty} \varphi_{j}t^{j}, \quad \psi(t) = \sum_{j=0}^{N} \psi_{j}t^{j}, \quad F(t) = \sum_{j=0}^{\infty} f_{j}t^{j}.
\]

Then we can rewrite \( (*) \) in the form

\[
(**) \quad p^{j}f_{j}\varphi(0) - \varphi(0)f_{j} = \sum_{k=1}^{j} \varphi_{k}f_{j-k} - \sum_{k=1}^{N} f_{j-k}\psi_{k}.
\]
Since \( \varphi \) is convergent in \( D_R \) we have \( \| \varphi_j \| \leq r^{-j} \) for all \( r < R \). Let \( K_j \) be an endomorphism of \( M_n(\mathbb{C}) \) such that \( K_j M \overset{\text{def}}{=} p^j M \varphi(0) - \varphi(0) M \) for \( M \subset M_n(\mathbb{C}) \) and \( K > 0 \) be a constant such that \( \| K_j^{-1} \| < K \) for \( j \gg 0 \). Let us prove that \( \| f_j \| < C(r/K')^{-j} \) suitable constants \( K' \) and \( C \). This will imply the validity of the first part of Proposition 5.1.6.

5.1.7. Applying the norm to both sides of (**) we get an inequality (for \( j \gg 0 \))

\[
\| f_j \| \leq K' \left( \sum_{k=1}^{j} r^{-k} \| f_{j-k} \| \right).
\]

This implies that the sequence \( \| f_j \| \) is dominated by the sequence \( g_j \) which satisfies the equalities \( g_j = K' \left( \sum_{k=1}^{j} r^{-k} g_{j-k} \right) \) for all \( j \gg 0 \). The sequence \( g_j \) satisfies the equation

\[
rg_{j+1}/K' - g_j/K' = g_j
\]

and therefore \( g_j \) is a geometric progression. So \( f_j \) is dominated by a geometric progression and therefore \( F(t) \) is convergent in a neighborhood of 0. If \( \varphi \) has a meromorphic continuation to \( \mathbb{C} \), it follows then from (*) that \( F \) has a meromorphic continuation to \( \mathbb{C} \). Proposition 5.1.6 is proved.

5.1.8 Corollary. Let \( L, M \) be finitely generated \( A \)-modules, \( \varphi \in (\text{End} \ L) \otimes_A \text{Amer} \), \( \psi \in (\text{End} \ M) \) and \( \xi \in \text{Hom}(L, M) \otimes_A \bar{A} \) be such that

\[
\varphi(0) \in \text{Aut} \ L/tL, \quad \psi(0) \in \text{Aut} \ M/tM, \quad \bar{\xi}(0) \in \text{Isom}(L/tL, M/tM)
\]

and the formal power series \( \bar{\xi} \) satisfy the equation

\[
\bar{\xi}(pt) = \varphi(t)\bar{\xi}(t)\psi(t).
\]

then \( \bar{\xi} \in \text{Hom}(L, M) \otimes_A \text{Amer} \).

5.1.9. We can now prove Theorem 5.1.4. We put \( p = (zq^{h\xi})^{-1} \). As follows from Proposition 3.1.8 it is sufficient to prove Theorem 5.1.4 in the case when \( V \) and \( W \) are generalized Verma modules. In this case it follows from Proposition 2.5.2 that \( P \) and \( Q \) are free \( A \)-modules and the result follows from Theorem 3.4.8 and Corollary 5.1.8. Theorem 5.1.4 is proved.
5.2.1. Let \( z, V, W, X \) and \( Y \) satisfy the assumptions of 5.1.1. It follows from the Theorem 5.1.4 that there exists a small disc \(|t| < \epsilon\) such that \( \xi_{\text{mer}} \) defines an isomorphism of \( \langle Q \rangle_{\text{mer}} \) and \( \langle P \rangle_{\text{mer}} \) at any point of this disc. Since \( |z q^h| > 1 \) we can find an even integer \( m < 0 \) such that \( |(z q^h)^m| < \epsilon \) so we get an isomorphism of \( \langle Q \rangle \) and \( \langle P \rangle \) at \( t_0 = (z q^h)^m \). Iterating \( \tilde{u} \)-linear isomorphism \( \nabla \) from Lemma 5.1.5 and using Corollary 2.4.22 and the condition \( (z q^h)^n \notin \Lambda_{X,Y} \), for \( n \in \mathbb{Z} \) we obtain the isomorphism of \( \langle Q \rangle \) and \( \langle P \rangle \) at \( t_1 = 1 \).

5.2.2 Proposition. We have constructed a functorial isomorphism

\[
\nabla : \langle V \hat{\otimes} X, Y \hat{\otimes} W \rangle_U \simeq \langle V, X, Y, W \rangle_U
\]

where \( \langle \cdot \rangle_U \) denotes the vector space of \( U \)-coinvariants.

5.2.3. Corollary. Let \( z, V, X \) and \( Y \) be as in 5.1.1. Then the isomorphism \( \nabla \) from 5.2.2 gives rise to the functorial quasi-associativity constraint

\[
a_{V,X,Y} : (V \hat{\otimes} X) \hat{\otimes} Y \simeq V \hat{\otimes} (X \otimes Y).
\]

It satisfies the pentagon axiom (see [M]) with respect to \( X \) and \( Y \).

Proof: Follows from 5.2.2 in the same way as the isomorphism 18.2 (b) follows from the theorem 17.29 in [KL].

5.2.4 Remark: The following is the tautological reformulation of 5.2.3. We say that \( X \) and \( Y \) in \( \mathcal{D} \) are in generic position with respect to the elliptic curve \( E = \mathbb{C}^*/(q^h z)^{2\mathbb{Z}} \) if the image of \( \Lambda_{X,Y} \) under the natural projection \( \mathbb{C}^* \to E \) does not contain the unity of the group \( E \). If \( X \) and \( Y \) are in generic position with respect to the elliptic curve \( E \) then 5.2.3 holds.
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