Variational source condition for the reconstruction of distributed fluxes

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Abstract

This paper is devoted to the inverse problem of recovering the unknown distributed flux on an inaccessible part of boundary using measurement data on the accessible part. We establish and verify a variational source condition for this inverse problem, leading to a logarithmic-type convergence rate for the corresponding Tikhonov regularization method under a low Sobolev regularity assumption on the distributed flux. Our proof is based on the conditional stability and Carleman estimates together with the complex interpolation theory on a proper Gelfand triple.

Keywords: Inverse problem, Tikhonov regularization, variational source condition, convergence rates, stability estimate, distributed flux reconstruction.

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1 Introduction

In this paper, we analyze an inverse problem of reconstructing the distributed flux on the inaccessible part of the boundary from measurement data on the accessible part of

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the boundary. More precisely, we consider the following elliptic diffusion system:

\[
\begin{align*}
-\nabla \cdot (\alpha(x) \nabla u(x)) &= f(x) & \text{in } \Omega, \\
-\alpha(x) \frac{\partial u}{\partial n}(x) &= k(x)(u(x) - u_a(x)) & \text{on } \Gamma_a, \\
-\alpha(x) \frac{\partial u}{\partial n}(x) &= q(x) & \text{on } \Gamma_i,
\end{align*}
\]

where the given data include the source term \( f \), the ambient concentration or temperature \( u_a \), the concentration or heat transfer coefficient \( k \) and the diffusivity coefficient \( \alpha \). The boundary is given by \( \partial \Omega = \Gamma_a \cup \Gamma_i \), where \( \Gamma_a \) denotes the accessible part, and \( \Gamma_i \) is the inaccessible one. The Neumann boundary term \( -\alpha(x) \frac{\partial u}{\partial n} = q(x) \) on \( \Gamma_i \), referred to as the distributed flux, is the main concern of this work:

(IP) Given noisy data \( u^\delta \) of the exact solution \( u^\dagger \) on the accessible part \( \Gamma_a \), we aim at recovering the distributed flux \( q^\dagger \) on the inaccessible part \( \Gamma_i \).

This inverse problem finds numerous important applications in diffusive, thermal and heat transfer problems, including the real-time monitoring in steel industry [1] and the visualization by liquid crystal thermography [9]. Since it is difficult to obtain an accurate measurement on the inaccessible boundary, such as the interior boundary of nuclear reactors and steel furnaces, engineers attempt to reconstruct the flux from the measurements on the accessible boundary. This leads to the inverse problem (IP), which is a severely ill-posed Cauchy problem in Hadamard’s sense [18]: If we replace the exact data \( u^\dagger \) on the accessible part \( \Gamma_a \) by a noisy pattern \( u^\delta \) satisfying

\[\|u^\dagger - u^\delta\|_{\Gamma_a} \leq \delta, \tag{1.2}\]

where \( \delta > 0 \) represents the noisy level, then there may not exist a solution to (IP). Even if a solution exists and \( \delta > 0 \) is very small, it may be far away from the exact one \( q^\dagger \). We refer to [25, 45, 46] and the references therein for theoretical and numerical results related to (IP).

To deal with the ill-posedness, we shall consider the Tikhonov regularization technique by solving a least-squares minimization problem. Our main goal is to examine the convergence rate of the regularized solution under an appropriate choice of the noise level \( \delta > 0 \) and the Tikhonov regularization parameter. It is well-known that a smoothness assumption on the true solution (source condition), is required to obtain the convergence rate. In general, a (classical) source condition requires the Fréchet differentiability of the forward operator and further properties on the adjoint of the Fréchet-derivative (cf. [24, 28]). Our present work focuses on the so-called variational source condition (VSC). The concept of VSC was originally introduced by Hofmann et al. [22] based on the use of a linear index function. Convergence rates for a more general index function were proven independently in [5, 11, 15]. The paper [12] contains a modified proof of the convergence rate result by [15]. Compared to the classical source condition, VSC does
not require any differentiability assumption on the forward operator. More importantly, convergence rates for the regularized solutions follow immediately from VSC under an appropriate parameter choice rule (see [23]).

To the best of our knowledge, there are only very few contributions towards VSC for inverse problems governed by partial differential equations. Hohage and Weidling [19, 20] derived VSC for the Tikhonov regularization of inverse scattering problems, leading to the strong convergence with logarithmic-type rates for the corresponding regularized solutions. VSC for ill-posed backward Maxwell’s equations was analyzed in [8] (cf. also [43, 44] concerning the optimal control of nonlinear Maxwell’s equations). For abstract linear operators, we refer to [2, 4, 7, 21] and the references therein. A general criterion for the verification of VSC for linear inverse problem was established in [21]. For more details between VSC and classical source conditions, we refer the reader to [19, 20, 21]. See also [7] concerning recent results on VSC for elastic-net regularizations.

The main purpose of this paper is to establish and verify VSC for the Tikhonov regularization of the inverse problem (IP). To this end, we will first establish a sufficient condition on VSC for a general ill-posed problem (Lemma 3.1), in terms of a sequence of approximating orthogonal projectors, which is an extension to the one introduced in [21]. The proposed sufficient condition consists of two separate conditions characterizing the smoothness of the exact solution and the ill-posedness of the inverse of the forward problem, respectively. In order to apply the sufficient condition to the inverse problem (IP), we shall derive a conditional stability estimate (Theorem 2.2) for every function \( u \in H^2(\Omega) \) satisfying a specific second-order elliptic equation (2.10). In particular, the proposed estimate reveals the dependence of \( \|u\|_{H^1(\Omega)} \) on \( \|u\|_{L^2(\Gamma_a)} \) under an a priori bound on the \( H^2(\Omega) \)-norm. The main tools to prove the conditional stability estimate are the developed techniques by [3, 35] and a specialized Carleman estimate (Lemma 4.1). Eventually, our result extends [3], as the derived estimate makes use of the \( L^2(\Gamma_a) \)-norm of the Dirichlet data (Proposition 4.2 and Theorem 4.1) instead of the \( H^1(\Gamma_a) \)-norm as in [3] Propositions 2.2-2.4 & Theorems 2.2-2.3]. Finally, based on the developed conditional stability estimate in combination with the complex interpolation theory and the Gelfand triple \( H^{1/2}(\Gamma_i) \subset L^2(\Gamma_i) \subset H^{-1/2}(\Gamma_i) \), we are able to establish a sequence of approximating orthogonal projectors and prove our main result on VSC (Theorem 2.1). In particular, it leads to a logarithmic-type convergence rate for the Tikhonov regularization under a low Sobolev regularity assumption on the exact distributed flux \( q^\dagger \in H^s(\Gamma_i) \) with \( s \in (0, 1/2] \).

This paper is organized as follows. Section 2 provides the precise mathematical formulation of the inverse problem (IP) and states our main theoretical findings (Theorems 2.1 and 2.2). In Section 3 we present a sufficient condition on VSC for a general ill-posed problem. The proof of the conditional stability estimate (Theorem 2.2) is provided in Section 4. The final section is devoted to the derivation of Theorem 2.1 on VSC for (IP).
2 Mathematical formulation and main results

We begin this section by recalling some terminologies and notations used in the sequel. Given a linear operator $T : X \to X$ on a complex Hilbert space $X$, the notations $D(T)$, $\rho(T)$ and $\sigma(T)$ stand for the domain, resolvent and spectrum of $T$, respectively. A linear operator $T : D(T) \subset X \to X$ is called closed, if its graph $\{(x, Tx), \ x \in D(T)\}$ is closed in $X \times X$. Furthermore, the adjoint of a densely defined operator $T : D(T) \subset X \to X$ is denoted by $T^\ast : D(T^\ast) \subset X \to X$. We call $T : D(T) \subset X \to X$ symmetric, if $Tx = T^\ast x$ holds true for all $x \in D(T)$, i.e., $(Tx, y)_X = (x, Ty)_X$ for all $x, y \in D(T)$. If a symmetric operator $T$ satisfies that $D(T) = D(T^\ast)$, then $T$ is said to be self-adjoint.

For $1 \leq i, j \leq d$ and a function $u$ defined on $\mathbb{R}^d$, we write $\partial_i u := \partial u / \partial x_i$, $\partial_{i,j} u := \partial^2 u / \partial x_i \partial x_j$, and $\nabla u = (\partial_1 u, \ldots, \partial_d u)$. Given the Hessian matrix $u''$ of a function $u$, we write $u''(x, y) := \sum_{i,j=1}^d \partial_{i,j} u \cdot x^i \cdot y^j$ with the vectors $x = (x^1, \ldots, x^d), y = (y^1, \ldots, y^d) \in \mathbb{C}^d$. In addition, we will often use the notation $C$ to denote generic positive constant independent of the parameter or functions involved. Also, we use the expression $A \lesssim B$ to indicate that $A \leq CB$ for a positive constant $C$ that is independent of $A$ and $B$. For two Banach spaces $X$ and $Y$ that are continuously embedded in the same Hausdorff topological vector space, we denote by $[X, Y]_\theta$ ($0 \leq \theta \leq 1$) the complex interpolation space between $X$ and $Y$.

For every $-\infty < s < \infty$, we define fractional Sobolev space

$$H^s(\mathbb{R}^d) := \{ u \in \mathcal{S}(\mathbb{R}^d)' \mid \| u \|_{H^s(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 d\xi < +\infty \},$$

where $\mathcal{F} : \mathcal{S}(\mathbb{R}^d)' \to \mathcal{S}(\mathbb{R}^d)'$ is the Fourier transform and $\mathcal{S}(\mathbb{R}^d)'$ denotes the tempered distribution space (see, e.g., [33, 41, 42]). For a bounded domain $U \subset \mathbb{R}^d$ with a Lipschitz boundary $\partial U$, the space $H^s(U)$ with a possibly non-integer exponent $s \geq 0$ is defined as the space of all complex-valued functions $v \in L^2(U)$ satisfying $V|_U = v$ for some $V \in H^s(\mathbb{R}^n)$, endowed with the norm

$$\| v \|_{s,U} := \inf_{V|_U = v} \| V \|_{H^s(\mathbb{R}^n)}.$$

When no confusion may be caused, we simply drop $U$ in the subscription of $\| \cdot \|_{s,U}$. For every $s \in [0, \infty)$, we denote by $[s] \in [0, s]$ the largest integer less or equal to $s$. In the case of $s \in (0, \infty)$ with $s = [s] + \sigma$ and $0 < \sigma < 1$, the norm $\| \cdot \|_{s,U}$ is equivalent to (cf. [42])

$$\left( \sum_{|\alpha| \leq [s]} \| D^\alpha u \|^2_{L^2(U)} + \sum_{|\alpha| \leq [s]} \int_{x \in U} \int_{y \in U} |D^\alpha u(x) - D^\alpha u(y)|^2 \frac{dx \, dy}{|x - y|^{n + 2\sigma}} \right)^{1/2}.$$

If $s$ is a non-negative integer, then $H^s(U)$ coincides with the classical Sobolev space. For a compact, $d$-dimensional $C^{k,\kappa}$-manifold $M$ with an integer $k \geq 0$ and $\kappa \in \{0, 1\}$, we can define the Sobolev space $H^s(M)$ on $M$ for all $0 \leq s \leq k + \kappa$ via partitions of unity and
Sobolev spaces $H^s(\mathbb{R}^d)$ (see, e.g., [11]). More precisely, let $M$ be a $d$-dimensional $C^{k,\kappa}$ compact manifold with an admissible $C^{k,\kappa}$-atlas $\{(U_i, \alpha_i)\}_{i \in I}$ for $M$ and a subordinate partition of unity $\beta_i$. Since $M$ is compact, the indexing set $I$ can be chosen to be finite, and possibly by shrinking $U_i$, the mappings $\alpha_i \circ \alpha_j^{-1}$ are $C^{k,\kappa}$-diffeomorphisms (cf. [11 Proposition 4.1]). Then, for $0 \leq s \leq k + \kappa$, $H^s(M)$ is specified by the space of all $u \in L^2(M)$ such that for all $i \in I$ the functions

$$(u \cdot \beta_i) \circ \alpha_i^{-1} : \alpha_i(U_i) \subset \mathbb{R}^d \to \mathbb{C}$$

belong to $H^s(\mathbb{R}^d)$. This forms a Hilbert space equipped with

$$\langle u, v \rangle_{H^s(M)} = \sum_{i \in I} (u \cdot \beta_i) \circ \alpha_i^{-1}, (v \cdot \beta_i) \circ \alpha_i^{-1} \rangle_{H^s(\mathbb{R}^d)} \quad \text{and} \quad \|u\|_{s,M} := \sqrt{\langle u, u \rangle_{H^s(M)}}.$$

One can show further that this definition is independent of the particular choice of atlas and partition of unity. In particular, for a bounded domain $U$ of class $C^{k,\kappa}$, its boundary $\partial U$ is a compact $C^{k,\kappa}$ manifold and then the Sobolev space $H^s(\partial U)$ is defined as above. By $H^s(M)$ we denote the dual of $H^s(M)$ with respect to the inner product in $L^2(M)$. We also write $H^0(M) = L^2(M)$ and denote its norm and scalar product by $\| \cdot \|_M$ and $\langle \cdot, \cdot \rangle_M$. By a standard argument (as used in [33, Theorem 7.7]), for every $-(k + \kappa) \leq s_1 < s_2 \leq k + \kappa$, one has for all $\theta \in [0,1]$,

$$[H^{s_1}(M), H^{s_2}(M)]_{\theta} = H^{s_1(1-\theta)+s_2\theta}(M), \quad (2.1)$$

with equivalent norms.

Let us now formulate the general assumptions on the domain $\Omega$ and the coefficients involved in (1.1).

(H1) The domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is connected, bounded and of class $C^{1,1}$. There exist $(d-1)$-dimensional compact $C^{1,1}$-manifolds $\Gamma_a, \Gamma_i \subset \mathbb{R}^d$ such that $\Gamma_a \cap \Gamma_i = \emptyset$ and $\partial \Omega = \Gamma_i \cup \Gamma_a$.

(H2) In addition, $\alpha \in C^{1,1}(\overline{\Omega}), k \in C^1(\Gamma_a)$ with

$$\alpha_{\min} := \min_{x \in \overline{\Omega}} \alpha(x) > 0, \quad k_{\min} := \min_{x \in \Gamma_a} k(x) > 0, \quad (2.2)$$

and $f \in L^2(\Omega)$ and $u_a \in H^{1/2}(\Gamma_a)$.

In particular, by (H1), the trace operator $tr : H^1(\Omega) \to H^{1/2}(\partial \Omega)$ and the (outward) normal derivative trace operator $\frac{\partial}{\partial n} : H^2(\Omega) \to H^{1/2}(\partial \Omega)$ are well-defined as linear and bounded operators (see e.g. [17] [11]). For simplicity, we also use the abbreviation $\frac{\partial}{\partial n} = \partial_n$.

We now recall the basic variational approach for a general inverse problem:

$$F(q) = g, \quad (2.3)$$
where $F : D(F) \subset X \to Y$, with domain $D(F)$, is an operator between two Hilbert spaces $X$ and $Y$ (see e.g. [12, 18, 39]). In general, $F^{-1}$ is unbounded so that the operator equation (2.3) is ill-posed. To tackle the ill-posedness, we consider the Tikhonov regularization method: For a given perturbation $g^\delta$ of the exact data $F(q^\dagger)$ and a regularization parameter $\alpha > 0$, we look for the minimizer $q^\delta_\alpha \in D(F)$ of

$$
\min_{q \in D(F)} \left[ \frac{1}{\alpha} \|F(q) - g^\delta\|^2_Y + \frac{1}{2} \|q\|^2_X \right]. 
$$

In general, the convergence rate for $\|q^\dagger - q^\delta_{\alpha}\|$ as $\delta, \alpha \to 0$ may be arbitrary slow (see [10]). To achieve a convergence rate of for the regularized solutions $\{q^\delta_{\alpha}\}$, a source condition on the true solution $q^\dagger$ is required (see [12, 18, 39]). As pointed out in the introduction, we focus on the variational source condition (VSC) of the form

$$
\beta \frac{1}{2} \|q - q^\dagger\|^2_X \leq \frac{1}{2} \|q\|^2_X - \frac{1}{2} \|q^\dagger\|^2_X + \Psi(\|F(q^\dagger) - F(q)\|_Y) \quad \text{for all } q \in D(F),
$$

where $\beta \in (0, 1]$ and $\Psi$ is an index function, that is a continuous and strictly increasing function $\Psi : (0, \infty) \to (0, \infty)$ satisfying $\lim_{t \to 0^+} \Psi(t) = 0$. If we are able to prove the existence of an index function $\Psi : (0, \infty) \to (0, \infty)$ and $\beta \in (0, 1]$ satisfying (2.5), then the following convergence rate

$$
\|q^\delta_{\alpha(\delta)} - q^\dagger\|^2_X = \Psi(\delta) \to 0 \text{ as } \delta \to 0^+
$$

is obtained under an appropriate parameter choice on $\delta$ and $\alpha(\delta)$ (see e.g. [7]).

Let us now discuss the Tikhonov regularization method for (IP). To this aim, we first introduce the solution operator

$$
S : L^2(\Gamma_i) \to H^1(\Omega),
$$
that assigns to every element \( q \in L^2(\Gamma_i) \) the unique solution \( u \in H^1(\Omega) \) of the weak formulation to (1.1):
\[
\int_{\Omega} \alpha \nabla u \cdot \nabla v dx + \int_{\Gamma_a} k u \nabla v \, dS = \int_{\Omega} f v dx - \int_{\Gamma_i} q v \, dS - \int_{\Gamma_a} k u_a \nabla v \, dS \quad \forall v \in H^1(\Omega),
\]
where for simplicity we set \( v = \text{tr}(v) \) on \( \partial \Omega \) to express boundary values of a Sobolev function \( v \in H^1(\Omega) \). Furthermore, we introduce
\[
A : L^2(\Gamma_i) \to L^2(\Gamma_a), \quad A(q) = \text{tr}(Sq) |_{\Gamma_a}.
\]
Then, the inverse problem (IP) is equivalent to solve the operator equation
\[
A(q) = u^\dagger. \tag{2.6}
\]
The operator \( A : L^2(\Gamma_i) \to L^2(\Gamma_a) \) is compact due to the compactness of the embedding \( H^{1/2}(\Gamma_a) \hookrightarrow L^2(\Gamma_a) \). Therefore, by a well-known argument, the inverse operator equation (2.6) is ill-posed (see also [15, Theorem 2.2] for the parabolic cases). To tackle with the ill-posedness, we consider the Tikhonov regularization method:
\[
\min_{q \in U_{q^\dagger}} \left[ \frac{1}{\alpha} \| A(q) - u^\dagger \|^2_{\Gamma_a} + \frac{1}{2} \| q \|^2_{\Gamma_i} \right], \tag{2.7}
\]
where \( U_{q^\dagger} \) is a non-empty, convex and closed subset of \( L^2(\Gamma_i) \) and \( u^\delta \) is a noisy pattern of the exact data \( u^\dagger = A(q^\dagger) \), i.e., \( \| u^\delta - A(q^\dagger) \|_{0, \Gamma_i} \leq \delta \). By classical arguments, this quadratic minimization problem admits a unique solution \( q^\delta \in U_{q^\dagger} \). In this paper, the admissible set \( U_{q^\dagger} \) is specified as
\[
U_{q^\dagger} := \{ q \in L^2(\Gamma_i) \mid \| q - q^\dagger \|_{1/2, \Gamma_i} \leq M_0 \};
\]
with a given positive constant \( M_0 > 0 \).

**Theorem 2.1.** Let (H1) – (H2). Suppose that \( q^\dagger \in H^s(\Gamma_i) \) for some \( s \in (0, 1/2] \). Then, for every \( \kappa \in (0, 1) \), there exists a concave index function \( \Psi : (0, \infty) \to (0, \infty) \) satisfying the variational source condition
\[
\frac{1}{4} \| q - q^\dagger \|^2_{\Gamma_i} \leq \frac{1}{2} \| q \|^2_{\Gamma_i} - \frac{1}{2} \| q^\dagger \|^2_{\Gamma_i} + \Psi(\| A(q^\dagger) - A(q) \|_{\Gamma_a}) \quad \forall q \in U_{q^\dagger}, \tag{2.8}
\]
and the decay rate condition
\[
\Psi(\delta) \lesssim \frac{1}{(\log \frac{4\delta}{\alpha})^{1+2s}} \quad \text{as} \quad \delta \to 0^+, \tag{2.9}
\]
for some positive constant \( C > 0 \).
Remark 2.1. As mentioned above, this result implies that the regularized solution \( \{q_\delta^\alpha\} \) of (2.7) possess the following convergence rate
\[
\|q_\delta^\alpha - q^\dagger\|_{L^2}^2 \lesssim \frac{1}{(\log \frac{1}{\delta})^{\frac{4s}{1+2s}}} \delta \to 0^+,
\]
when the parameter \( \alpha = \alpha(\delta, g^\delta) \) is chosen appropriately (see [7, 23] for more details).

The key tool to prove Theorem 2.1 is a conditional stability estimate for every function \( u \in H^2(\Omega) \) satisfying
\[
\begin{align*}
\nabla \cdot (\alpha(x) \nabla u(x)) &= 0 \quad \text{in } \Omega, \\
-\alpha(x) \frac{\partial u(x)}{\partial n} &= k(x) u(x) \quad \text{on } \Gamma_a.
\end{align*}
\]
(2.10)

We shall prove an estimate for \( \|u\|_{1,\Omega} \) depending on \( \|u\|_{\Gamma_a} \) under the a priori bounded set:
\[
\mathcal{M}_M := \{ u \in H^2(\Omega) : \|u\|_{2,\Omega} \leq M \}, \tag{2.11}
\]
where the constant \( M > 0 \) is a prescribed constant.

Theorem 2.2. Assume that (H1) – (H2) hold. Let \( u \in H^2(\Omega) \) be a function satisfying (2.10) within \( \mathcal{M}_M \) defined by (2.11). Then, for every \( \kappa \in (0,1) \), there exist two positive constants \( C, C_0 > 0 \), independent of \( u \) and \( M \), such that
\[
\|u\|_{1,\Omega} \leq \frac{C M \log \left( \frac{C_0 M}{\|u\|_{\Gamma_a}} \right)^{\kappa}}{\|u\|_{\Gamma_a}}. \tag{2.12}
\]
In particular, if \( \|u\|_{\Gamma_a} = 0 \), then \( u \) vanishes.

3 Sufficient condition for VSC

In this section, we present a sufficient condition to verify VSC for (2.3), which is an extension of [21] Theorem 2.1. Evidently, VSC of the form (2.5) is equivalent to the form below:
\[
\Re(q^\dagger, q^\dagger - q)_X \leq \frac{1-\beta}{2} \|q^\dagger - q\|_X^2 + \Psi(\|F(q^\dagger) - F(q)\|_Y) \quad \forall \ q \in D(F) \tag{3.1}
\]
Thus, we shall verify (3.1) instead of (2.5) directly. The result below yields a sufficient condition for (3.1). The proof follows the lines of [21] Theorem 2.1 and [4] Theorem 2.5]. However, since our result considers the concavity of index functions as well as the complex settings, we could not find a precise reference covering our situation. In addition, significant modifications to the original idea are necessary. Therefore, we include a proof for the sake of completeness.
Lemma 3.1. Let $0 \neq q^\dagger \in D(F)$, $\lambda_0 \geq 0$, \{P_\lambda\}_{\lambda \geq \lambda_0}$ be a family of orthogonal projectors from $X$ to $X$, and let $f, g : [\lambda_0, \infty) \to \mathbb{R}^+$ be continuous functions satisfying

1. $f$ is strictly decreasing and fullfils $\lim_{\delta \to \infty} f(\delta) = 0$
2. $g$ is strictly increasing and fullfils $\lim_{\delta \to \infty} g(\delta) = +\infty$.

Furthermore, suppose that there exist a concave index function $\Psi_0$ and two constants $\tilde{C} \geq 0$ and $\beta \in (0, 1)$ such that

$$\|q^\dagger - P_\lambda q^\dagger\|_X \leq f(\lambda),$$

(3.2)

$$\mathcal{R}(q^\dagger, P_\lambda(q^\dagger - q))_X \leq g(\lambda)\Psi_0(\|F(q^\dagger) - F(q)\|_Y) + \tilde{C} f(\lambda)\|q^\dagger - q\|_X$$

(3.3)

$$\forall q \in D(F) \text{ with } \|q^\dagger - q\|_X < \frac{2}{1 - \beta}\|q^\dagger\|_X$$

holds for all $\lambda \geq \lambda_0$. Then, (3.1) holds true with the following concave index function:

$$\Psi : (0, \infty) \to (0, \infty), \quad \Psi(t) := \inf_{\lambda \geq \lambda_0} \left( g(\lambda)\Psi_0(t) + \frac{(\tilde{C} + 1)^2}{2(1 - \beta)} f(\lambda)^2 \right).$$

(3.4)

This index function satisfies the decay estimate

$$\Psi(\delta) \lesssim (f^2 \circ \Theta^{-1})(\Psi_0(\delta)) \text{ as } \delta \to +0,$$

(3.5)

where $\Theta^{-1}$ is the inverse of $\Theta : [\lambda_0, \infty) \to (0, \infty)$, $\lambda \mapsto \frac{(\tilde{C} + 1)^2}{2(1 - \beta)} g(\lambda)^2$, which obviously satisfies $\lim_{\lambda \to +\infty} \Theta(\lambda) = 0$.

Proof. For each $q$ satisfying $\|q^\dagger - q\|_X < \frac{2}{1 - \beta}\|q^\dagger\|_X$, the Cauchy-Schwarz inequality implies

$$\mathcal{R}(q^\dagger, q^\dagger - q)_X \leq \frac{1 - \beta}{2}\|q^\dagger - q\|_X^2.$$

Therefore, we only need to show (3.1) for $\|q^\dagger - q\|_X < \frac{2}{1 - \beta}\|q^\dagger\|_X$. For this case, using the orthogonal projection \{P_\lambda\}_{\lambda \geq \lambda_0} and (3.2)-(3.3), it follows that

$$\mathcal{R}(q^\dagger, q^\dagger - q)_X = \mathcal{R}(P_\lambda q^\dagger, q^\dagger - q)_X + \mathcal{R}((I - P_\lambda)q^\dagger, q^\dagger - q)_X$$

$$\leq \mathcal{R}(q^\dagger, P_\lambda(q^\dagger - q))_X + f(\lambda)\|q^\dagger - q\|_X$$

$$\leq g(\lambda)\Psi_0(\|F(q^\dagger) - F(q)\|_Y) + \frac{1 - \beta}{2}\|q^\dagger - q\|_X^2 + \frac{(\tilde{C} + 1)^2}{f(\lambda)^2},$$

where we have used Young’s inequality for the last inequality. In conclusion, for every $\lambda \geq \lambda_0$, the inequality

$$\mathcal{R}(q^\dagger, q^\dagger - q)_X \leq \frac{1 - \beta}{2}\|q^\dagger - q\|_X^2 + g(\lambda)\Psi_0(\|F(q^\dagger) - F(q)\|_Y) + \frac{(\tilde{C} + 1)^2}{2(1 - \beta)} f(\lambda)^2.$$
holds for all $q \in D(F)$. Thus, defining $\Psi : (0, \infty) \to (0, \infty)$ as in (3.4), we see that (3.1) is satisfied. It remains to show that $\Psi : (0, \infty) \to (0, \infty)$ is a concave index function. First, since $\Psi : (0, \infty) \to (0, \infty)$ is an infimum of concave functions, we obtain that $\Psi : (0, \infty) \to (0, \infty)$ is concave. Since for $t \in (0, \infty)$, $-\Psi(t) > -\infty$ and $-\Psi$ is convex over $(0, \infty)$, we infer that $\Psi : (0, \infty) \to (0, \infty)$ is continuous [47, Corollary 47.6].

Finally, we prove the decay estimate (3.5), which also implies the continuity of $\Psi$ at $0$. Since $\lim_{\delta \to 0^+} \Psi_0(\delta) = 0$, if $\delta$ is sufficiently small, there exists a unique $\lambda$ such that

$$
\Psi_0(\delta) = \frac{\hat{C} + 1}{1 - \beta} f(\lambda^2),
$$

as the function $\lambda \mapsto \frac{(\hat{C} + 1)^2 f(\lambda)^2}{2(1 - \beta) g(\lambda)}$ is continuous, strictly decreasing and convergent to 0 as $\lambda \to \infty$. If we set $\Theta(\xi) := \frac{(\hat{C} + 1)^2 f(\lambda)^2}{2(1 - \beta) g(\lambda)}$, then we obtain $\lambda = \Theta^{-1}(\Psi_0(\delta))$, which yields

$$
\Psi(\delta) \leq \frac{(\hat{C} + 1)^2}{1 - \beta} f(\Theta^{-1}(\Psi_0(\delta)))^2 \quad \text{as} \quad \delta \to +0,
$$

and consequently the decay estimate (3.5) is obtained.

To show that $\Psi$ is strictly increasing, we choose $t_1, t_2$ with $0 < t_1 < t_2$. For $t_2 > 0$, it holds that

$$
g(\lambda) \Psi_0(t_2) + \frac{(\hat{C} + 1)^2}{2(1 - \beta)} f(\lambda)^2 \to \infty, \quad \text{as} \quad \lambda \to +\infty.
$$

Thus, according to the definition (3.4), there exist $\lambda_*$ such that $\Psi(t_2) = g(\lambda_*) \Psi_0(t_2) + \frac{(\hat{C} + 1)^2}{2(1 - \beta)} f(\lambda_*^2)$. Then, as $\Psi_0$ is strictly increasing, it follows that

$$
\Psi(t_1) \leq \Psi(t_2),
$$

as $\delta \to 0^+$ (3.6).

4 Derivation of Theorem 2.2

The goal of this section is to prove Theorem 2.2, which is a critical auxiliary result for the proof of Theorem 2.1. The importance of Theorem 2.2 lies in the fact that it establishes the continuous dependence of $q^t - q$ on $(u(q^t) - u(q))_{\Gamma_a}$ under a priori bound on $u(q^t) - u(q)$. After proving Theorem 2.2, we shall construct suitable orthogonal projections $\{P_{\lambda}\}_{\lambda \geq \lambda_0}$ such that $P_{\lambda}(q^t - q)$ can be estimated by $(u(q^t) - u(q))_{\Gamma_i}$. To prove Theorem 2.2, we follow and modify the techniques by [3]. To be more precise, we establish a Carleman estimate and derive a series of local conditional estimates results. Combining all these theoretical findings yields the desired global result. Let us underline that our results extend [3]: We study a class of second-order elliptic operators of divergence
form, while [3] focuses on the Laplace operator. Furthermore, our estimates make use of
the $L^2(\Gamma_\alpha)$-norm of the Dirichlet data (Proposition 4.2 and Theorem 11) instead of the
$H^1(\Gamma_\alpha)$-norm as considered in [3, Propositions 2.2-2.4 & Theorems 2.2-2.3].

Let us first derive a global Carleman estimate for the following second-order elliptic
operator:

$$(L_\alpha u)(x) := -\nabla \cdot (\alpha(x)\nabla u(x)). \quad (4.1)$$

Carleman estimate was initially introduced to study the quantification of unique con-
tinuation, which goes back to the early work of T. Carleman himself. In recent years,
Carleman’s estimate has been applied to the fields of control theory and inverse problem
for PDEs (see e.g. [37, 38]). Our derivation here follows from a computa-
tional method

**Lemma 4.1.** Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\psi \in C^{1,1}(\overline{U})$ be a real-valued
function such that

$$|\nabla \psi(x)| \geq c \text{ for all } x \in \overline{U} \text{ and } D^2\psi \in L^\infty(U)^{d \times d}$$

with some positive constant $c > 0$. Furthermore, let $\varphi = e^{\gamma \psi}$ with $\gamma > 0$. Then, there
exist $\gamma_* \geq 1$ and $\tau_* \geq 1$ such that

$$\int_U (\tau^3 \gamma \varphi^3 e^{2\tau \varphi} |u|^2 + \tau \gamma^2 e^{2\tau \varphi} |\nabla u|^2) dx \lesssim \int_U e^{2\tau \varphi} |L_\alpha u|^2 dx + \int_{\partial U} (\tau^3 \gamma \varphi^3 e^{2\tau \varphi} |u|^2 + \tau \gamma^2 e^{2\tau \varphi} |\nabla u|^2) dS \quad (4.2)$$

for all $\tau \geq \tau_*, \gamma \geq \gamma_*$ and $u \in H^2(U)$.

**Remark 4.1.** Obviously, $\psi(x) = e^{x_1}$ satisfies the assumption of Lemma 4.1. For our
applications, we only make use of a simpler version of Lemma 4.1. More precisely, by
fixing a $\gamma > \gamma_*$, we may drop $\gamma^2$ out of (4.2) and conclude that the estimate

$$\int_U e^{2\tau \varphi} (\tau^3 |u|^2 + \tau |\nabla u|^2) dx \lesssim \int_{\partial U} (\tau^{3} |u|^2 + \tau^3 |u|^2) dS + \int_U e^{2\tau \varphi} |L_\alpha u|^2 dx$$

holds true for all $\tau \geq \tau_*$ and $u \in H^2(U)$. This estimate now is of the same form as the
ones from [26].

**Proof.** Let $u \in H^2(U)$, $f := L_\alpha u$ and $v := e^{\tau \varphi} u$. We define the conjugated operator

$$W_\varphi v := e^{\tau \varphi} L_\alpha (e^{-\tau \varphi} v) = -e^{\tau \varphi} \nabla \cdot (\alpha \nabla (e^{-\tau \varphi} v)). \quad (4.3)$$

From the expansion of $W_\varphi$, we further introduce

$$W_2 v := -\nabla \cdot (\alpha \nabla v) - \tau^2 \gamma^2 \alpha \varphi^2 |\nabla \psi|^2 v,$$

$$W_1 v := 2\tau \gamma \alpha \varphi \nabla \psi \cdot \nabla v + 2\tau \alpha \gamma^2 \varphi |\nabla \psi|^2 v.$$
Straightforward computations yield
\[ e^{\tau \varphi} f = W_\varphi v = W_2 v + W_1 v - \tau \gamma \varphi \nabla \alpha \cdot \nabla \psi v - \tau \alpha \gamma^2 \varphi |\nabla \psi|^2 v + \gamma \tau \alpha \varphi \Delta v, \quad (4.4) \]
where we have used \( \varphi = e^{\gamma \psi} \) and the identity \( \Delta \varphi = \gamma^2 \varphi |\nabla \psi|^2 + \gamma \varphi \Delta \psi \). By (4.3),
\[ \| \frac{1}{\sqrt{\alpha}} W_2 v \|^2 + \| \frac{1}{\sqrt{\alpha}} W_1 v \|^2 + 2 \Re \langle W_2 v, \alpha^{-1} W_1 v \rangle_U = \| g \|^2_U, \]
with
\[ g := \frac{1}{\sqrt{\alpha}} (e^{\tau \varphi} f + \tau \gamma \varphi \nabla \alpha \cdot \nabla \psi v + \tau \alpha \gamma^2 \varphi |\nabla \psi|^2 v - \gamma \tau \alpha \varphi \Delta v). \quad (4.5) \]
Then, by the definition of \( \varphi \), it follows that
\[ 2 \Re \langle W_2 v, \alpha^{-1} W_1 v \rangle_U \leq \| g \|^2_U. \quad (4.6) \]
Our goal now is to establish a proper lower estimate for the left-hand side of (4.6). To this end, let us denote by \( I_{ij} \) the real part of the scalar product between the \( i \)-th term of \( W_2 \) and the \( j \)-th term of \( \alpha^{-1} W_1 \).

**Term \( I_{11} \):** Integration by parts yields
\[ I_{11} := -2 \tau \gamma \Re \langle \nabla \cdot (\alpha \nabla v), \varphi \nabla \psi \cdot \nabla v \rangle_U \]
\[ = 2 \tau \gamma \Re \int_U \alpha \nabla v \cdot \nabla (\varphi \nabla \psi \cdot \nabla \psi) dx - 2 \tau \gamma \Re \int_{\partial U} \alpha \varphi \partial_n v (\nabla \psi \cdot \nabla \psi) dS \]
\[ = 2 \tau \gamma \int_U \alpha \varphi \psi'' (\nabla v, \nabla \psi) dx + 2 \tau \gamma^2 \int_U \alpha \varphi |\nabla v \cdot \nabla \psi|^2 dx + \tau \gamma \int_U \alpha \varphi \nabla \psi \cdot \nabla |\nabla v|^2 dx \]
\[ - 2 \tau \gamma \Re \int_{\partial U} \alpha \varphi \partial_n v (\nabla \psi \cdot \nabla \psi) dS, \]
\[ = 2 \tau \gamma \int_U \alpha \varphi \psi'' (\nabla v, \nabla \psi) dx + 2 \tau \gamma^2 \int_U \alpha \varphi |\nabla v \cdot \nabla \psi|^2 dx - \tau \gamma \int_U \nabla \cdot (\alpha \varphi \nabla \psi) |\nabla v|^2 dx \]
\[ + \tau \gamma \int_{\partial U} \alpha \varphi \partial_n \psi |\nabla v|^2 dx - 2 \tau \gamma \Re \int_{\partial U} \alpha \varphi \partial_n v (\nabla \psi \cdot \nabla \psi) dS, \]
where \( \psi''(\nabla v, \nabla \psi) \) := \( \sum_{1 \leq i,j \leq d} \partial_{i,j} \psi \partial_i v \partial_j \overline{v} \), which is obviously real-valued. As the third term of the above sum can be rewritten as
\[ -\tau \gamma \int_U \nabla \cdot (\alpha \varphi \nabla \psi) |\nabla v|^2 dx = -\tau \gamma \int_U (\nabla \alpha \cdot \nabla \psi + \alpha \gamma |\nabla \psi|^2 + \alpha \Delta \psi) \varphi |\nabla v|^2 dx, \]
we obtain that
\[ I_{11} = 2 \tau \gamma \int_U \alpha \varphi \psi'' (\nabla v, \nabla \psi) dx + 2 \tau \gamma^2 \int_U \alpha \varphi |\nabla v \cdot \nabla \psi|^2 dx \]
\[ - \tau \gamma \int_U (\nabla \alpha \cdot \nabla \psi + \alpha \gamma |\nabla \psi|^2 + \alpha \Delta \psi) \varphi |\nabla v|^2 dx \]
\[ + \tau \gamma \int_{\partial U} \alpha \varphi \partial_n \psi |\nabla v|^2 dx - 2 \tau \gamma \Re \int_{\partial U} \alpha \varphi \partial_n v (\nabla \psi \cdot \nabla \psi) dS. \quad (4.7) \]
Term $I_{12}$: Similarly,

\[
I_{12} = -2\tau \gamma \Re(\nabla \cdot (\alpha \nabla v), \varphi |\nabla \psi|^2 v) \bigg|_U
= 2\tau \gamma \Re \int_U \alpha \nabla v \cdot \nabla (\varphi |\nabla \psi|^2 v) dx - 2\tau \gamma \Re \int_{\partial U} \alpha \varphi |\nabla \psi|^2 \partial_n v \overline{v} dS
\]

\[
= 2\tau \gamma^2 \int_U \alpha \varphi |\nabla \psi|^2 |\nabla v|^2 dx + 2\tau \gamma \Re \int_U \alpha (\nabla (\varphi |\nabla \psi|^2) \cdot \nabla v) \overline{v} dx
- 2\tau \gamma^2 \Re \int_{\partial U} \alpha \varphi |\nabla \psi|^2 \partial_n v \overline{v} dS.
\]

Term $I_{21}$: Also, Integration by parts results in

\[
I_{21} = -2\tau \Re(\alpha |\nabla \psi|^2 \varphi^2 v, \varphi \nabla \psi \cdot \nabla v) \bigg|_U = -2\tau \gamma^3 \int_U \alpha |\nabla \psi|^2 \varphi^3 \nabla \psi \cdot \nabla |v|^2 dx
= 2\tau \gamma^3 \int_U \nabla \cdot (\alpha |\nabla \psi|^2 \varphi^3 \nabla \psi) |v|^2 dx - 2\tau \gamma^3 \int_{\partial U} \alpha |\nabla \psi|^2 \varphi^3 \partial_n v |v|^2 dx.
\]

Term $I_{22}$: A direct computation implies

\[
I_{22} = -2\tau \gamma^4 \int_U \alpha |\nabla \psi|^4 \varphi^3 |v|^2 dx.
\]

Regrouping the terms of all expansions of $I_{ij}$ and as the second term in the right-hand side of (4.7) is nonnegative, we have

\[
2\Re(W_2 v, \alpha^{-1} W_1 v) \geq \int_U \tau^3 \gamma^4 \alpha_0 |v|^2 dx + \int_U \tau \gamma^2 \alpha_1 |\nabla v|^2 dx + T_1 + T_2,
\]

with

\[
\alpha_0 := \frac{2}{\gamma} \nabla \cdot (\alpha |\nabla \psi|^2 \varphi^3 \nabla \psi) - 2\alpha |\nabla \psi|^4 \varphi^3, \quad \alpha_1 := (\alpha |\nabla \psi|^2 - \frac{1}{\gamma} \nabla \alpha \cdot \nabla \psi - \frac{\alpha}{\gamma} \Delta \psi) \varphi,
\]

\[
T_1 := 2\tau \gamma \int_U \alpha \varphi \partial_x \nabla (v, \nabla \psi) dx + 2\tau \gamma \Re \int_U \alpha (\nabla (\varphi |\nabla \psi|^2) \cdot \nabla v) \overline{v} dx,
\]

\[
T_2 := \tau \gamma \int_{\partial U} \alpha \varphi \partial_n v \nabla |v|^2 dx - 2\tau \gamma \Re \int_{\partial U} \alpha \varphi \partial_n v (\nabla \psi \cdot \nabla \overline{v}) dS - 2\tau \gamma \Re \int_{\partial U} \alpha \varphi |\nabla \psi|^2 \partial_n v \overline{v} dS
- 2\tau \gamma^3 \int_{\partial U} \alpha |\nabla \psi|^2 \varphi^3 \partial_n v |v|^2 dx.
\]

Let us now verify the following estimates:

\[
\varphi^3 \lesssim \alpha_0 \text{ a.e. in } U \quad \text{and} \quad \varphi \lesssim \alpha_1 \text{ a.e. in } U,
\]

for all sufficiently large $\gamma > 0$. Indeed, since $\alpha(x) \geq \alpha_{\text{min}} > 0$ and $|\nabla \psi(x)| \geq c > 0$ hold true for all $x \in \overline{U}$, and $D^2 \psi \in L^\infty(U)^{d \times d}$, we find that

\[
\alpha_1 \geq (\alpha_{\text{min}} c^2 - \frac{1}{\gamma} \nabla \alpha \cdot \nabla \psi - \frac{\alpha}{\gamma} \Delta \psi) \varphi \geq \frac{\alpha_{\text{min}} c^2}{2} \varphi \text{ a.e. in } U.
\]
holds for all sufficiently large $\gamma > 0$. Furthermore, by virtue of $\varphi = e^{\gamma \psi}$, straightforward computations imply that

$$\alpha_0 = 4\alpha|\nabla \psi|^4 \varphi^3 - \frac{2}{\gamma} \nabla \cdot (\alpha|\nabla \psi|^2 \nabla \psi) \varphi^3 \geq \alpha_{\min} c^4 \varphi^3 \text{ a.e. in } U$$

holds for all sufficiently large $\gamma > 0$. In conclusion, (4.9) is valid. Let us now derive upper estimates for $|T_1|$ and $|T_2|$. Again, by the definition $\varphi = e^{\gamma \psi}$,

$$\nabla (\varphi |\nabla \psi|^2) = \nabla \varphi |\nabla \psi|^2 + \varphi \nabla |\nabla \psi|^2 = \varphi (\gamma \nabla \psi |\nabla \psi|^2 + \nabla |\nabla \psi|^2).$$

Thus, for all $\gamma \geq 1$, Young’s inequality implies for every $\varepsilon > 0$ that

$$|T_1| \lesssim \tau \gamma \int_U \varphi |\nabla v|^2 dx + \tau \gamma^3 \int_U \varphi |\nabla v||v|dx \lesssim \tau \gamma \int_U \varphi |\nabla v|^2 dx + \varepsilon \tau \gamma^2 \int_U \varphi |\nabla v|^2 dx + \frac{\tau \gamma^4}{\varepsilon} \int_U \varphi |v|^2 dx. \quad (4.10)$$

On the other hand, in view of $\psi \in C^{1,1}(\overline{U})$, we infer that

$$|T_2| \lesssim (\tau \gamma \int_{\partial U} \varphi |\nabla v|^2 dS + \tau \gamma^3 \int_{\partial U} \varphi^3 |v|^2 dS) + \tau \gamma \int_{\partial U} |\nabla v||v|\varphi dS.$$

Then, since $\varphi \geq 1$ on $\partial U$, Young’s inequality implies for all $\tau \geq 1$ that

$$|T_2| \lesssim \tau \gamma \int_{\partial U} \varphi |\nabla v|^2 dS + \tau \gamma^3 \int_{\partial U} \varphi^3 |v|^2 dS. \quad (4.11)$$

Now, according to (4.5),

$$||g||_U^2 \lesssim \int_U e^{2\tau \varphi} |f|^2 dx + \int_U (\tau^2 \gamma^2 + \tau^2 \gamma^4) \varphi^2 |v|^2 dx. \quad (4.12)$$

It follows therefore from (4.6), (4.8), (4.9), (4.10) with a sufficiently small $\varepsilon > 0$ and (4.12) that

$$\int_U (\tau^3 \gamma^4 \varphi^3 |v|^2 + \tau \gamma^2 \varphi |\nabla v|^2) dx \lesssim \int_U e^{2\tau \varphi} |f|^2 dx + \int_{\partial U} (\tau \gamma (\varphi |\nabla v|^2 + \tau \gamma^3 \varphi^3 |v|^2) dS$$

holds true for all sufficiently large $\gamma, \tau \geq 1$. Substituting the identities

$$v = e^{\tau \varphi} u \quad \text{and} \quad \nabla v = e^{\tau \varphi} \nabla u + \tau \gamma \varphi e^{\tau \varphi} u \nabla \psi \quad (4.13)$$

into the inequality above, we obtain that

$$\int_U (\tau^3 \gamma^4 \varphi^3 |v|^2 + \tau \gamma^2 \varphi |\nabla v|^2) dx \lesssim \int_U e^{2\tau \varphi} |f|^2 dx + \int_{\partial U} (\tau \gamma \varphi e^{2\tau \varphi} |\nabla u|^2 + \tau \gamma^3 \varphi^3 e^{2\tau \varphi} |u|^2) dS,$$

for all sufficiently large $\gamma, \tau \geq 1$. In addition, (4.13) also yields

$$\int_U (\tau^3 \gamma^4 \varphi^3 e^{2\tau \varphi} |u|^2 + \tau \gamma^2 \varphi e^{2\tau \varphi} |\nabla u|^2) dx \lesssim \int_U (\tau^3 \gamma^4 \varphi^3 e^{2\tau \varphi} |v|^2 + \tau \gamma^2 \varphi |\nabla v|^2) dx.$$

Combining these two inequalities, we finally come to the conclusion that (4.12) is valid. □
By virtue of Lemma 4.1, the same arguments as in [3, Propositions 2.2-2.4] yield the following result:

**Proposition 4.1.**

(a) Let \( \omega_1, \omega_2 \) be two domains such that \( \omega_1 \subseteq \Omega \) and \( \omega_2 \subseteq \Omega \). Then, there exist \( s, c, \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0] \) and \( u \in \mathcal{H}^2(\Omega) \),

\[
\|u\|_{1, \omega_1} \leq \frac{c}{\varepsilon} (\|L_\alpha u\|_\Omega + \|u\|_{1, \omega_2}) + \varepsilon^s \|u\|_{1, \Omega}. \tag{4.14}
\]

(b) Let \( x_0 \in \Gamma_a \). Then, there exist a neighborhood \( \omega_0 \) of \( x_0 \), and positive constants \( s, c, \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0] \) and \( u \in \mathcal{H}^2(\Omega) \),

\[
\|u\|_{1, \Omega \cap \omega_0} \leq \frac{c}{\varepsilon} (\|L_\alpha u\|_\Omega + \|u\|_{1, \Gamma_a} + \|\partial_n u\|_{\Gamma_a}) + \varepsilon^s \|u\|_{1, \Omega}. \tag{4.15}
\]

(c) Let \( x^* \in \partial \Omega \). Then, there exist a neighborhood \( \omega \) of \( x^* \) and an open domain \( \omega_1 \subseteq \Omega \) such that for each \( \kappa \in (0,1) \), there exist \( c, \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0] \) and \( u \in \mathcal{H}^2(\Omega) \),

\[
\|u\|_{1, \Omega \cap \omega} \leq e^{c/\varepsilon} (\|L_\alpha u\|_\Omega + \|u\|_{1, \omega_1}) + \varepsilon^\kappa \|u\|_{2, \Omega}. \tag{4.16}
\]

For the upcoming results, we shall also make use of the following auxiliary lemma:

**Lemma 4.2 ([3, Lemma 2.3 and Corollary 2.1]).**

(i) Let \( s, \beta, A \text{ and } B \) denote four non-negative real numbers such that \( \beta \leq B \). If there exist \( c, \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0] \), it holds

\[
\beta \leq \frac{c}{\varepsilon} A + \varepsilon^s B,
\]

then there exists \( C \), only depending on \( s \) and \( c \), such that

\[
\beta \leq CA^{s/\varepsilon} B^{1/\varepsilon^s}.
\]

(ii) Let \( \beta, \delta, M \) denote three non-negative numbers such that \( \beta \leq M \) and \( \delta \leq C_0 M \) with some constant \( C_0 > 0 \). If there exist \( c, \varepsilon_0, \kappa > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0] \), it holds

\[
\beta \leq e^{c/\varepsilon} \delta + \varepsilon^\kappa M,
\]

then there exists \( C \), only depending on \( c, \varepsilon_0, \) and \( \kappa \), such that

\[
\beta \leq C \frac{M}{\log(\delta_0 M)^\kappa}.
\]
By virtue of Lemma 4.2, the estimate (4.14) in Proposition 4.1 (a) implies
\[ \|u\|_{1,\omega_1} \lesssim (\|L_\alpha u\|_\Omega + \|u\|_{1,\omega_2}) \frac{1}{\epsilon} \frac{1}{\|u\|_{1,\omega_1}} \quad \forall u \in H^2(\Omega). \] (4.17)
Furthermore, the estimate (4.16) in Proposition 4.1 (c) yields
\[ \|u\|_{1,\Omega - \omega} \lesssim \frac{\|u\|_{2,\Omega}}{\log(\frac{C}{\|u\|_{2,\Omega}})} \quad \forall u \in H^2(\Omega), \] (4.18)
for some \( C > 0. \)

To obtain our final result, we need to reduce the \( H^1 \)-regularity term of Dirichlet data in Proposition 4.1 (b) to a term of \( L^2 \)-regularity by possibly enlarging the term \( \epsilon^s \|u\|_{1,\Omega} \).

**Proposition 4.2.** Let \( x_0 \in \Gamma_\alpha. \) Then, there exist a neighborhood \( \omega_0 \) of \( x_0 \), and positive constants \( s, c, \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0) \) and \( u \in H^2(\Omega) \),
\[ \|u\|_{1,\Omega - \omega_0} \leq \frac{c}{\epsilon}(\|L_\alpha u\|_\Omega + \|u\|_{\Gamma_\alpha} + \|\partial_\alpha u\|_{\Gamma_\alpha}) + \epsilon^s \|u\|_{2,\Omega}. \] (4.19)

**Proof.** Since \( \Gamma_\alpha \) itself is a compact \( C^{1,1} \)-manifold, we have the following well-known interpolation result
\[ \|u\|_{1,\Gamma_\alpha} \lesssim \|u\|_{3,2,\Gamma_\alpha}^{2/3} \|u\|_{0,\Gamma_\alpha}^{1/3} \quad \forall u \in H^{3/2}(\Gamma_\alpha) \]
(see (2.1) and e.g. [33, Theorem 7.7] ). Then, Young’s inequality yields for all \( \epsilon > 0 \) that
\[ \|u\|_{1,\Gamma_\alpha} \lesssim \epsilon^3/2 \|u\|_{3/2,\Gamma_\alpha} + \epsilon^{-3} \|u\|_{\Gamma_\alpha}. \]
Applying this inequality to Proposition 4.1 (b), we find constants \( s', c', \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0) \) and \( u \in H^2(\Omega) \), it holds
\[ \|u\|_{1,\Omega - \omega_0} \leq c' \frac{\epsilon}{\epsilon^3}(\|L_\alpha u\|_\Omega + \|\partial_\alpha u\|_{\Gamma_\alpha}) + \epsilon^{s'} \|u\|_{2,\Omega} \]
Setting \( s'' := \min\{1/2, s'\} \) and by the embedding result \( \|u\|_{3/2,\Gamma_\alpha} \lesssim \|u\|_{2,\Omega} \), there exists a further constant \( c'' > 0 \) such that
\[ \|u\|_{1,\Omega - \omega_0} \leq \frac{c''}{\epsilon^4}(\|L_\alpha u\|_\Omega + \|\partial_\alpha u\|_{\Gamma_\alpha} + \|u\|_{\Gamma_\alpha}) + c'' \epsilon^{-s''} \|u\|_{2,\Omega}, \]
for all \( 0 < \epsilon \leq \min\{1, \epsilon_0\} \). By a reparametrization of \( \epsilon \), i.e., replacing \( \epsilon \) by \( \hat{c} \epsilon^\frac{4}{s''} \) for a sufficiently large \( \hat{c} > 0 \), we may find positive constants \( c, s, \epsilon_0 > 0 \) such that (4.16) is valid.

Employing the developed local results, we are now in the position to prove global estimates. Roughly speaking, Proposition 4.2 enables us to “transfer” Cauchy data on \( \Gamma_\alpha \) to a neighborhood \( \omega_0 \) of every point \( x_0 \in \Gamma_\alpha \), in particular to a subdomain \( \omega_2 \) such that \( \omega_2 \subset \omega_0 \). Proposition 4.1 (a) allows us to “transfer” data from this open domain \( \omega_2 \) to another small domain \( \omega_1 \subset \Omega \), in particular, to the one “near” to a point \( x_0 \in \Gamma_\alpha \). Lastly, Proposition 4.1 (c) “transfer” data on an open domain \( \omega_1 \subset \Omega \) to a neighborhood \( \omega \) of \( x_\alpha \in \Gamma_\alpha \) (See Fig. 2 below). In the sequel we explain what exactly does “transfer” mean.
Theorem 4.1. For every \( \kappa \in (0, 1) \), there exist \( c > 0 \) and \( \epsilon_0 \) such that

\[
\|u\|_{1, \Omega} \leq e^{c/\epsilon}(\|L_\alpha u\|_\Omega + \|u\|_{\Gamma_a} + \|\partial_n u\|_{\Gamma_a}) + \epsilon \|u\|_{2, \Omega}
\]

(4.20)

holds true for all \( \epsilon \in (0, \epsilon_0] \) and \( u \in H^2(\Omega) \).

Proof. Let \( \kappa \in (0, 1) \). We first study the local boundary estimate over \( \Gamma_i \). Let \( x^* \in \Gamma_i \) be arbitrarily fixed. Proposition 4.1 (c) implies the existence of a neighborhood \( \omega \) of \( x^* \), an open domain \( \omega_1 \supseteq \Omega \) and positive constants \( c', \epsilon'_0 > 0 \) such that

\[
\|u\|_{1, \Omega \cap \omega} \leq e^{c/\epsilon}(\|L_\alpha u\|_\Omega + \|u\|_{1, \omega_1}) + \epsilon \|u\|_{2, \Omega}
\]

(4.21)

holds for all \( \epsilon \in (0, \epsilon'_0] \) and \( u \in H^2(\Omega) \). Next, let \( x_0 \in \Gamma_a \) be arbitrarily fixed. Proposition 4.2 and Lemma 4.2 (i) imply the existence of a neighborhood \( \omega_0 \) of \( x_0 \) and a constant \( \theta' \in (0, 1) \) such that

\[
\|u\|_{1, \Omega \cap \omega_0} \lesssim (\|L_\alpha u\|_\Omega + \|u\|_{1, \omega_1})^{\theta'} \|u\|_{1, \Omega}^{1-\theta'} \quad \forall \ u \in H^2(\Omega).
\]

(4.22)

Our goal now is to prove that the estimation (4.21) remains true with \( \|u\|_{1, \omega_1} \) replaced by the Dirichlet data on \( \Gamma_a \). To this aim, let us select a domain \( \omega_2 \) such that \( \omega_2 \in \omega_0 \cap \Omega \) (see Fig. 2). By Proposition 4.1 (a) and Lemma 4.2 (i), it holds for some \( \theta' \in (0, 1) \) that

\[
\|u\|_{1, \Omega \cap \omega_2} \lesssim (\|L_\alpha u\|_\Omega + \|u\|_{1, \omega_2})^{\theta'} \|u\|_{1, \Omega}^{1-\theta'} \quad \forall \ u \in H^2(\Omega).
\]

(4.23)

Since \( \omega_2 \in \Omega \cap \omega_0 \), it follows from (4.22)-(4.23) and \( \|L_\alpha u\|_\Omega \lesssim \|u\|_{2, \Omega} \) that

\[
\|u\|_{1, \omega_1} \lesssim (\|L_\alpha u\|_\Omega + (\|L_\alpha u\|_\Omega + \|u\|_{\Gamma_a} + \|\partial_n u\|_{\Gamma_a})^{\theta'} \|u\|_{1, \Omega}^{1-\theta}) \|u\|_{1, \Omega}^{1-\theta} + \|L_\alpha u\|_{\Omega}^{\theta'} \|u\|_{2, \Omega}^{1-\theta'} \|u\|_{1, \Omega}^{1-\theta} \quad \forall \ u \in H^2(\Omega).
\]
Then, by an elementary inequality $x^\theta + y^\theta \leq 2^{1-\theta}(x + y)^\theta$ for all $x, y \geq 0$, we obtain
\[\|u\|_{1,\omega_1} \lessapprox (\|L_{\alpha}u\|_\Omega + \|u\|_{\Gamma_a} + \|\partial_{n}u\|_{\Gamma_a})^{\theta_\theta}\|u\|_{2,\Omega}^{(1-\theta_\theta)}\|u\|_1^{1-\theta_\theta} \quad \forall u \in H^2(\Omega),\]
and hence
\[\|u\|_{1,\omega_1} \lessapprox (\|L_{\alpha}u\|_\Omega + \|u\|_{\Gamma_a} + \|\partial_{n}u\|_{\Gamma_a})^{\theta_\theta}\|u\|_{2,\Omega}^{1-\theta_\theta} \quad \forall u \in H^2(\Omega). \tag{4.24}\]
Combining (4.24) and (4.21) together, we obtain for all $\epsilon \in (0, \epsilon_0]$ and $u \in H^2(\Omega)$ that
\[\|u\|_{1,\Omega \cap \omega} \leq \epsilon^{\epsilon/\epsilon}(\|L_{\alpha}u\|_\Omega + (\|L_{\alpha}u\|_\Omega + \|u\|_{\Gamma_a} + \|\partial_{n}u\|_{\Gamma_a})^s\|u\|_{2,\Omega}^{1-s}) + \epsilon^\epsilon\|u\|_{2,\Omega},\]
with $s = \theta_\theta$, from which it follows that
\[\|u\|_{1,\Omega \cap \omega} \lessapprox \epsilon^{\epsilon/\epsilon}(\|L_{\alpha}u\|_\Omega + \|u\|_{\Gamma_a} + \|\partial_{n}u\|_{\Gamma_a})^s\|u\|_{2,\Omega}^{1-s} + \epsilon^\epsilon\|u\|_{2,\Omega}. \tag{4.25}\]
Furthermore, Young’s inequality implies
\[\epsilon^{\epsilon/\epsilon}(\|L_{\alpha}u\|_\Omega + \|u\|_{\Gamma_a} + \|\partial_{n}u\|_{\Gamma_a})^s\|u\|_{2,\Omega}^{1-s} \leq \frac{\epsilon^{\epsilon/\epsilon}}{\epsilon^{\epsilon/\epsilon}}(\|L_{\alpha}u\|_\Omega + \|u\|_{\Gamma_a} + \|\partial_{n}u\|_{\Gamma_a}) + \epsilon^\epsilon\|u\|_{2,\Omega}.\]
Applying this inequality to (4.25) and choosing a sufficiently large $c > 0$ and a sufficiently small $\epsilon_0 > 0$, we obtain the desired estimate
\[\|u\|_{1,\Omega \cap \omega} \leq \epsilon^{\epsilon/\epsilon}(\|L_{\alpha}u\|_{L^2} + \|u\|_{\Gamma_a} + \|\partial_{n}u\|_{\Gamma_a}) + \epsilon^\epsilon\|u\|_{2,\Omega}, \tag{4.26}\]
for all $\epsilon \in (0, \epsilon_0]$ and all $u \in H^2(\Omega)$.

For the case $x_0$ being a point on $\Gamma_a$, we can readily prove a better estimate. Indeed, using (4.22) and Young’s inequality again, we know that there exist some $c > 0$ such that for all $u \in H^2(\Omega)$,
\[\|u\|_{1,\Omega \cap \omega_0} \lessapprox \frac{c}{\epsilon^{\epsilon/\epsilon}}(\|L_{\alpha}u\|_{L^2} + \|u\|_{\Gamma_a} + \|\partial_{n}u\|_{\Gamma_a}) + \epsilon^\epsilon\|u\|_{2,\Omega}.\]
This estimate is still true if we replace the term $\frac{c}{\epsilon^{\epsilon/\epsilon}}$ by $\epsilon^{\epsilon}$, provided $\epsilon$ is sufficiently small enough. Therefore, the estimate (4.26) holds true with $\|u\|_{1,\Omega \cap \omega}$ replaced by $\|u\|_{1,\Omega \cap \omega_0}$. Of course, we may need to choose another constants $c$ and $\epsilon_0$ if necessarily. In the same manner, for every $\omega' \in \Omega$, one can show that estimate (4.26) is still valid with $\|u\|_{1,\Omega \cap \omega}$ replaced by $\|u\|_{1,\omega'}$. Patching together all local estimates, we conclude from the compactness of $\overline{\Omega}$ that (4.20) is valid. \hfill \Box

We close this section by proving Theorem 2.2.

**Proof of Theorem 2.2.** Let $u \in H^2(\Omega)$ be a solution of (2.10) within $\mathcal{M}_M$ and $\kappa \in (0,1)$. Since $L_{\alpha}u = 0$, Theorem 4.1 and Lemma 4.2 (ii) imply
\[\|u\|_{1,\Omega} \leq \frac{C'}{\log(C_0\|u\|_{2,\Omega})}\cdot \frac{\|u\|_{2,\Omega}}{\log(C_0\|u\|_{2,\Omega})^{\kappa}}.\]
for some positive constants \( C \) and \( C_0 \), independent of \( u \). Since the mapping \( y \mapsto \frac{y}{(\ln(y/y_0))^\kappa} \) is increasing over \( (y_0, +\infty) \) and \( \|u\|_{2,\Omega} \leq M \), it holds that

\[
\|u\|_{H^1(\Omega)} \leq C \frac{M}{\ln \left( C_0 \|u\|_{\Gamma_a} + \|\partial_n u\|_{\Gamma_a} \right)^\kappa} \tag{4.27}
\]

Since \( \|u\|_{\Gamma_a} \lesssim \|\partial_n u\|_{\Gamma_a} \), we conclude (by changing \( C \) and \( C_0 \), if necessary) that Theorem \ref{thm:2.2} is valid.

\[ \square \]

5 Proof of Theorem \ref{thm:2.1}

To prove Theorem \ref{thm:2.1} we apply Lemma \ref{lem:3.1} by constructing suitable orthogonal projections on \( L^2(\Gamma_i) \). Our proof is based on the conditional stability estimate (Theorem \ref{thm:2.2}) along with the complex interpolation theory and the following Gelfand triple:

(G1) \( H^{1/2}(\Gamma_i) \subset L^2(\Gamma_i) \subset H^{-1/2}(\Gamma_i) \) with dense and continuous embeddings;

(G2) \( \{H^{1/2}(\Gamma_i), H^{-1/2}(\Gamma_i)\} \) forms an adjoint pair with the duality product \( \langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma_i), H^{1/2}(\Gamma_i)} \);

(G3) the duality pairing \( \langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma_i), H^{1/2}(\Gamma_i)} : H^{-1/2}(\Gamma_i) \times H^{1/2}(\Gamma_i) \to \mathbb{C} \) satisfies

\[
\langle v, u \rangle_{H^{-1/2}(\Gamma_i), H^{1/2}(\Gamma_i)} = \langle v, u \rangle_{L^2(\Gamma_i)} \quad \forall u \in H^{1/2}(\Gamma_i), \ v \in L^2(\Gamma_i).
\]

Since the inner-product \( \langle \cdot, \cdot \rangle_{H^{1/2}(\Gamma_i)} \) is a symmetric sesquilinear form over \( H^{1/2}(\Gamma_i) \), the operator \( \mathcal{B} : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i) \) defined by

\[
\langle \mathcal{B}u, v \rangle_{H^{-1/2}(\Gamma_i), H^{1/2}(\Gamma_i)} := \langle u, v \rangle_{H^{1/2}(\Gamma_i)} \quad \forall u, v \in H^{1/2}(\Gamma_i)
\]

is linear and bounded. We can then define an unbounded operator \( \mathcal{A} : D(\mathcal{A}) \subset L^2(\Gamma_i) \to L^2(\Gamma_i) \) as follows:

\[
\mathcal{A}u := \mathcal{B}u \quad \forall u \in D(\mathcal{A}),
\]

with the domain

\[
D(\mathcal{A}) = \{ u \in H^{1/2}(\Gamma_i) \mid \mathcal{B}u \in L^2(\Gamma_i) \}.
\]

One can infer that \( \mathcal{A} : D(\mathcal{A}) \subset L^2(\Gamma_i) \to L^2(\Gamma_i) \) is a densely defined and closed operator (cf. \cite[Theorem 1.25]{22}). Further properties of this operator is summarized in the following lemma:

Lemma 5.1 \cite[Chapter 1, Section 8]{22}. The operator \( \mathcal{A} : D(\mathcal{A}) \subset L^2(\Gamma_i) \to L^2(\Gamma_i) \) is densely defined, closed, self-adjoint and \( m \)-accretive. Furthermore, it satisfies

\[
\langle \mathcal{A}u, v \rangle_{\Gamma_i} = \langle u, v \rangle_{H^{1/2}(\Gamma_i)} \quad \forall u, v \in D(\mathcal{A}). \tag{5.1}
\]
By (5.1), we obtain that
\[(Au, u)_{G_i} = \|u\|_{1/2, G_i}^2 \geq \|u\|_{0, G_i}^2, \quad \forall u \in D(A).\]

Then, in view of the compactness of the embedding \(D(A) \subset L^2(G_i)\), we infer that there exists a complete orthonormal basis \(\{e_n\}_{n=1}^{\infty} \subset L^2(G_i)\) such that
\[(Au, u)_{G_i} = \sum_{n=1}^{\infty} \lambda_n |(u, e_n)_{G_i}|^2 \quad \forall u \in D(A), \quad (5.2)\]

where \(1 \leq \lambda_1 \leq \lambda_2 \leq \cdots\), \(\lim_{n \to \infty} \lambda_n = +\infty\), and, for every \(n \in \mathbb{N}^+\), \(e_n\) is the eigenfunction of \(A\) for the eigenvalue of \(\lambda_n\), i.e.,
\[Ae_n = \lambda_n e_n \quad \forall n \in \mathbb{N}.\]

For every \(s \in \mathbb{R}\), the fractional power \(A^s\) of \(A\) can be defined as
\[A^s u := \sum_{n=1}^{\infty} \lambda_n^s |(u, e_n)_{G_i}|^2 \quad \forall u \in D(A^s), \quad (5.3)\]

where the domain \(D(A^s)\) is given by
\[D(A^s) = \{u \in L^2(G_i) \mid \sum_{n=1}^{\infty} \lambda_n^{2s} |(u, e_n)_{G_i}|^2 < \infty\}. \quad (5.4)\]

Then, for each \(s \geq 0\), \(A^s : D(A^s) \subset L^2(G_i) \to L^2(G_i)\) is also self-adjoint, and \(D(A^s)\) is a Banach space equipped with the norm
\[\|u\|_{D(A^s)} := \|A^s u\|_{L^2(G_i)} = \left( \sum_{n=1}^{\infty} \lambda_n^{2s} |(u, e_n)_{G_i}|^2 \right)^{1/2} \quad \forall u \in D(A^s), \quad (5.5)\]

which is also equivalent to the corresponding graph norm of \((A^s, D(A^s))\) (for more details, we refer to [36, 41]). Let us mention that for all \(\theta \in [0, 1/2]\), it holds that
\[D(A^\theta) = [L^2(G_i), H^{1/2}(G_i)]_{2\theta} = H^\theta(G_i) \quad (5.6)\]
with norm equivalence. The first identity is from [42, Corollary 2.4], while the second one is due to [22, 11].

**Proof of Theorem 2.1.** If \(q^* \neq 0\), then (2.8) holds true for every concave index function \(\Psi\).

Let therefore \(0 \neq q \in H^s(G_i)\) for some \(s \in (0, 1/2]\). Let us introduce a family of \(\{P_\lambda\}_{\lambda \geq \lambda_1}\) in \(L^2(G_i)\), where \(P_\lambda : L^2(G_i) \to L^2(G_i)\) is define by
\[P_\lambda q = \sum_{\lambda_n \leq \lambda} (q, e_n)_{G_i} e_n.\]
Since $q^\dagger \in H^s(\Gamma_i)$, it follows from \((5.6)\) that $q^\dagger \in D(\mathcal{A}^*)$. Then, using \((5.5)\), we have
\[
\|(I - P_\lambda)q^\dagger\|^2_{0,\Gamma_i} = \sum_{\lambda_n > \lambda} |(q^\dagger, e_n)_{\Gamma_i}|^2 \\
\leq \sum_{\lambda_n > \lambda} \frac{\lambda_n^{2s}}{\lambda_n^{2s}} |(q^\dagger, e_n)_{\Gamma_i}|^2 \leq \frac{1}{\lambda_n^{2s}} \|q^\dagger\|^2_{D(\mathcal{A}^*)}. \tag{5.7}
\]
Thus, it remain to estimate the inner product $(q^\dagger, q^\dagger - q)_{0,\Gamma_i}$ for every $q \in \mathcal{U}_{q^\dagger}$. To this aim, let $q \in \mathcal{U}_{q^\dagger}$. As $q^\dagger \in D(\mathcal{A}^*)$, there exits $\omega := \mathcal{A}^*q^\dagger \in L^2(\Gamma)$ such $q^\dagger = \mathcal{A}^{-s}\omega$. From the definition of $\mathcal{A}^*$, it follows that
\[
\Re(q^\dagger, P_\lambda(q^\dagger - q))_{\Gamma_i} = \Re(\omega, \mathcal{A}^{-s}P_\lambda(q^\dagger - q))_{\Gamma_i}. \tag{5.8}
\]
Since $\|q^\dagger - q\|_{\Gamma_i} = \sum_{n=1}^\infty |(q^\dagger - q, e_n)_{\Gamma_i}|^2$, we have
\[
\|\mathcal{A}^{-s}P_\lambda(q^\dagger - q)\|^2_{0,\Gamma_i} = \sum_{\lambda_n \leq \lambda} \lambda_n^{-2s} |(q^\dagger - q, e_n)_{\Gamma_i}|^2 \tag{5.9}
\]
Since $H^{1/2}(\Gamma_i)$ is dense in $L^2(\Gamma_i)$ and by \((G3)\), we have
\[
|(q^\dagger - q, e_n)_{\Gamma_i}| \leq \|q^\dagger - q\| - \frac{1}{2}\Gamma_i\|e_n\|_{\frac{1}{2}\Gamma_i} \lesssim \sqrt{\lambda_n} \|q^\dagger - q\| - \frac{1}{2}\Gamma_i, \tag{5.10}
\]
where we have uses \((5.5)\). Combining \((5.9)\) and \((5.10)\) yields
\[
\|\mathcal{A}^{-s}P_\lambda(q^\dagger - q)\|^2_{0,\Gamma_i} \lesssim \lambda_n^{1-2s} \|q^\dagger - q\|^2_{-\frac{1}{2}\Gamma_i}. \tag{5.11}
\]
According to the definition of $\mathcal{U}_{q^\dagger}$, it holds that $q - q^\dagger \in H^{1/2}(\Gamma_i)$ with $\|q - q^\dagger\|_{1/2,\Gamma_i} \leq M_0$. Thus, by a classical elliptic regularity result (cf. [17], [33]), there exists a constant $M > 0$ such that $u := S(q) - S(q^\dagger) \in \mathcal{M}_M$; see \([2, 11]\) for the definition of $\mathcal{M}_M$. We note that the constant $M$ may depend on $M_0$, but not on $q \in \mathcal{U}_{q^\dagger}$. Furthermore, by definition, $u = S(q) - S(q^\dagger) \in H^2(\Omega)$ satisfies
\[
\langle q^\dagger - q, \text{tr}(v) \rangle_{H^{-1/2}(\Gamma_i), H^{1/2}(\Gamma_i)} = \int_{\Gamma_i} (q^\dagger - q) \text{tr}(\nabla) dS = \int_{\Omega} \alpha \nabla u \cdot \nabla \nu dx + \int_{\Gamma_a} k \text{tr}(u) \text{tr}(\nu) dS \\
= \int_{\Gamma_i} \text{tr}(\nabla u \cdot n) \text{tr}(\nu) dS \quad \forall v \in H^1(\Omega) \text{ satisfying } \text{tr}(v) = 0 \text{ on } \Gamma_a
\]
Thus, as $\text{tr} : H^1(\Omega) \to H^{1/2}(\partial \Omega)$ is surjective, we deduce from $\Gamma_a \cap \Gamma_i = \emptyset$, the boundedness of $H^1(\Omega) \ni u \mapsto \partial_n u \in H^{-1/2}(\Gamma_i)$ and the estimate
\[
\|\alpha v\|_{\frac{1}{2},\Gamma_i} \lesssim \|\alpha\|_{C^1(\Gamma_i)} \|\partial_n u\|_{\frac{1}{2},\Gamma_i} \lesssim C_n \|\alpha\|_{C^1(\Gamma_i)} \|u\|_{1,\Omega}. 
\]
(see e.g. [42, Page 49]) that
\[
\|q^\dagger - q\|_{-\frac{1}{2},\Gamma_i} \lesssim \|\alpha\|_{C^1(\Gamma_i)} \|\partial_n u\|_{-\frac{1}{2},\Gamma_i} \leq C_n \|\alpha\|_{C^1(\Gamma_i)} \|u\|_{1,\Omega}.
\]
with $C_n > 0$, independent of $u$, $q$ and $q^\dagger$. By this inequality and since $u \in M_M$ satisfies (2.10), Theorem 2.2 and (5.11) imply the existence of positive constants $C, C_0 > 0$, independent of $q$ and $q^\dagger$ such that

$$\|A^{-s} P_\lambda (q^\dagger - q)\|_0, \Gamma_i \leq \lambda^{1/2-s} \|\alpha\|_{C^1(\Gamma_i)} C_n \frac{CM}{\log(C_0 M \|A(q^\dagger) - A(q)\|_0, \Gamma_a)^\kappa)},$$

which, together with (5.8), implies that

$$\Re(q^\dagger P_\lambda (q^\dagger - q)_{\Gamma_i} \leq \lambda^{1/2-s} \|\omega\|_{\Gamma_i} \|\alpha\|_{C^1(\Gamma_i)} C_n \frac{CM}{\log(C_0 M \|A(q^\dagger) - A(q)\|_0, \Gamma_a)^\kappa}). \quad (5.12)$$

To apply Lemma 3.1 we first notice that the continuity of the trace operator yields $\|A(q^\dagger) - A(q)\|_0, \Gamma_a \leq c'M$ with a constant $c' > 0$, independent of $q^\dagger$ and $q$. Without loss of generality, we assume that $C_0$ is large enough so that $c' M > e^{\kappa+1}$. Otherwise, we can enlarge $C_0$ and $C$ at the same time since the mapping $M \mapsto \frac{M \log(M\delta)}{\log(M\delta)^\kappa}$ is increasing over $[\delta, +\infty)$. It is readily checked that the function

$$G_0(\delta) = \|\omega\|_{\Gamma_i} \|\alpha\|_{C^1(\Gamma_i)} \frac{CM}{\log(C_0 M \|A(q^\dagger) - A(q)\|_0, \Gamma_a)^\kappa)} \forall \delta \in (0, c'M],$$

is concave, continuous and strictly increasing over $(0, c'M]$. In conclusion, the mapping

$$\Psi_0 : (0, \infty) \rightarrow (0, \infty), \quad \Psi_0(\delta) := \begin{cases} (d_\delta G)(c'M)(x - c'M) + G(c'M) & \text{if } \delta \in (c'M, \infty) \\ G_0(\delta) & \text{if } \delta \in (0, c'M] \end{cases}$$

is a concave index function satisfying

$$\Re(q^\dagger P_\lambda (q^\dagger - q)_{\Gamma_i} \leq \lambda^{1/2-s} \Psi_0(\|A(q^\dagger) - A(q)\|_0, \Gamma_a) \forall q \in U_{q^\dagger}. \quad (5.13)$$

In view of (5.7) and (5.13), if $0 < s < 1/2$, then Lemma 3.1 is applicable and yields the assertion. If $s = 1/2$, then (5.13) implies

$$\Re(q^\dagger P_\lambda (q^\dagger - q)_{\Gamma_i} \leq \Psi_0(\|A(q^\dagger) - A(q)\|_0, \Gamma_a) \forall q \in U_{q^\dagger}.$$ 

By passing $\lambda \rightarrow \infty$, we see that the claim is true for $\Psi = \Psi_0$.

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