Study of new chaotic flows on a family of 3-dimensional systems with quadratic nonlinearities

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Abstract. Based on a wider systematic search on a family of 3-dimensional systems with quadratic nonlinearities, three new simple chaotic systems were found. One of them has the unusual feature of having a stable equilibrium point, and it is the simplest one of other chaotic flows with this property. The others have some interesting special properties.

1. Introduction
The search of chaotic flows began two decades ago, e.g., in [1] a systematic search of 3-dimensional autonomous systems with quadratic nonlinearities, allowed them to find 19 distinct simple chaotic flows. It is well-known that mathematically simple dynamical systems can present chaotic flows [2], [3]. Recent works have investigated simple chaotic flows with quadratic nonlinearities, and were found with either a stable equilibrium point or a line equilibrium, see [4], [5] and references there. This class of systems have hidden attractors; a hidden attractor has a basin of attraction that does not intersect with small neighbourhoods of any equilibrium points. In particular any dissipative chaotic flow with no equilibrium or with only stable equilibrium must have a hidden strange attractor [4]. Hidden attractors are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or an airplane wing [4]. Another class of interesting chaotic flows are those with only an equilibrium point [6]. The goal of this work is to present 3 new chaotic flows, one of them with a stable equilibrium point which is, to our best knowledge, the simplest mathematical system in the literature today with this property. The last is one of the few known chaotic systems in the literature with only one equilibrium point. These systems are found via a systematic computer search for chaos in 3-dimensional autonomous systems with quadratic nonlinearities.

2. The new chaotic flows
The 3-dimensional family of systems with quadratic nonlinearities considered in this work can be expressed as

\[ \dot{X} = A_x + M \times X + N \times R + Q \times S \]  

(1)
where

\[ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; \quad R = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}; \quad S = \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} \] (2)

and parameters as

\[ A_0 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}; \quad M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}; \quad N = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}; \quad Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \] (3)

The parameters are searched in a space determined as follows:

- \( A_0 \) parameters were set to zero.
- \( M \) parameters were set within the interval \([-50, 50]\) with 2.5 steps and including \{-1, 1\}.
- \( N \) and \( Q \) parameters were set only to \{-1, 0, 1\}.

We use Monte Carlo methods to generate random numbers and assign values to the corresponding parameters. Our intention is to have few parameters in \( N \) and \( Q \). Each generated new set of equations are solved using a Runge-Kutta non-stiff fourth-order method with fixed step size, and stored if it does not diverge. We calculate Lyapunov exponents using [7] for each stored system and keep it if it’s largest Lyapunov exponent is greater than 0.5.

We present 3 new systems with chaotic flows founded with this method. The criteria to determine whether the system shows a chaotic behavior or not is based on the bifurcation diagram, the Lyapunov exponents, and the state space diagram.

### 2.1 First chaotic flow

The first system is given by

\[
\begin{align*}
\frac{dx}{dt} &= y(z - 50) \\
\frac{dy}{dt} &= 15(z - y) \\
\frac{dz}{dt} &= x(25 - y)
\end{align*}
\] (4)

Where the Lyapunov exponents are \{2.2, 0.014, -17.2\}. In consequence the system is dissipative and with the following two equilibrium points: \{0, 0, 0\} and \{0, 50, 50\}. The eigenvalues for \{0, 0, 0\} are \{-32.6, 8.8 - 22.3i, 8.8 + 22.3i\} and the eigenvalues for \{0, 50, 50\} are \{25\sqrt{2}i, -25\sqrt{2}i, -15\}. Note that the last equilibrium point is stable. This system has a hidden attractor although the other equilibrium point is not stable. In the Figure 1 the bifurcation diagram and in Figure 2 the state space diagram are shown where the attractor is presented. This system is interesting particularly because is very simple, it is more simple than the 23 systems in [4], since this system has only 6 terms and none of them are pure quadratic. Therefore this
system is, to our best knowledge, the simplest mathematical system in the literature today with a stable equilibrium point.

![Figure 1. Bifurcation diagram for equations (4). Varying parameter –50 of first equation.](image1)

2.2 Second chaotic flow

The second system is given by

\[
\begin{align*}
\frac{dx}{dt} &= -35x + yz + 32.5x \\
\frac{dy}{dt} &= -2.5y + 50z + z^2 \\
\frac{dz}{dt} &= -xy - 50y + 15z
\end{align*}
\] (5)

The Lyapunov exponents are \(8.1, 0.009, -19.4\). Also, the system is dissipative and with the following 3 equilibrium points: \([-98.14, 15.82, -50.77]\), \([-18.32, -23.11, -48.81]\) and \([0, 0, 0]\). The corresponding eigenvalues for \([-98.14, 15.82, -50.77]\) are \(8.93 + 57.96i, 8.93 - 57.96i, -30.36\), for \([-18.32, -23.11, -48.81]\) the eigenvalues are \([55.34, -33.92 + 22.99i, -33.92 - 22.99i]\) and the eigenvalues for \([0, 0, 0]\) are \([6.25 + 49.22i, 6.25 - 49.22i, -25]\). In Figure 3 the bifurcation diagram and in Figure 4 the state space diagram are shown. This system shows a special mismatch over the trajectories as can be seen from the state space diagram. Also, this system has only 9 terms and three are quadratic, clearly it is a simple system with three hyperbolic equilibrium points.

![Figure 3. Bifurcation diagram for equations (5). Varying parameter –25 of first equation.](image2)

![Figure 4. State space diagram for equations (5)](image3)
2.3 Third chaotic flow

The third system is given by
\[
\begin{align*}
\frac{dx}{dt} &= -30y \\
\frac{dy}{dt} &= 25x - 47.5z \\
\frac{dz}{dt} &= -xy + 27.5y - y^2 - yz - 37.5z
\end{align*}
\] (6)

The Lyapunov exponents are given by \{1.43, 0.006, -31.0\}. Again, the system is dissipative and with only one equilibrium point in \{0, 0, 0\}. The eigenvalues are \{9.38 + 20.29i, 9.38 - 20.29i, -56.26\}. In Figure 5 the bifurcation diagram and Figure 6 the state space diagram are shown.

This system is linear in the first and second equations and quadratic in the third equation with 5 terms. It is not common to find chaotic flows with only one equilibrium point, for example in [7] a chaotic system with only one stable equilibrium point was recently reported. In this case we have one hyperbolic equilibrium point.

![Figure 5. Bifurcation diagram for equations (6). Varying parameter -30 of first equation.](image1)

![Figure 6. State space diagram for equations (6)](image2)

3. Conclusions

We present three new simple chaotic systems with the following properties: the first is a very simple system with a stable equilibrium point and a hyperbolic equilibrium point. The second shows a special mismatch over the trajectories. The third is a chaotic flow with only one hyperbolic equilibrium point. Based on the number of searches performed, we agree with [4] and [5] in the sense that these classes of systems seem to be more common than usually thought.

References

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