Chaos after Two-Field Inflation

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We show that two-field inflation can be followed by an era in which the field dynamics become chaotic, and discuss the possible consequences of this for two-field inflationary models.

1 Introduction

The simplest realizations of the inflationary paradigm are based on a single scalar field, minimally coupled to Einstein gravity. However, there are many incentives for considering models based on two (or more) interacting fields, including a reduced need for fine-tuning, and a more diverse range of phenomenological predictions.\(^1\)

It is hardly surprising that more complicated models admit more complicated behavior. However, the purpose of this paper is to highlight the possible role of chaotic dynamics in two-field inflation. In the case of single field, \(\phi\), the equations of motion can be expressed entirely in terms of \(\phi\) and \(\dot{\phi}\), and therefore constitute a two dimensional, autonomous system which does not contain enough degrees of freedom to exhibit chaos. However, with two scalar fields, the evolution equations form a four dimensional system, which can be chaotic. Even comparatively simple chaotic systems may exhibit extremely complex dynamics, and chaotic behavior differs both qualitatively and quantitatively from non-chaotic behavior. Consequently, the possible existence of chaos in two-field inflationary models raises the possibility of new, and previously unsuspected, phenomenology which may easily be overlooked by studies based on approximate techniques.

A full description of this work is given in a preprint\(^3\) written with my collaborator, Kei-ichi Maeda.

2 Chaos and Two-Field Inflation

The equations of motion for inflationary models driven by two scalar fields with a combined potential \(V(\phi, \psi)\) are

\[
H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi m_{\text{pl}}^2}{3} \left[\dot{\phi}^2 + \dot{\psi}^2 + V(\phi, \psi)\right],
\]

\[
\ddot{\phi} = -3H\dot{\phi} - \frac{\partial V}{\partial \phi}, \quad \ddot{\psi} = -3H\dot{\psi} - \frac{\partial V}{\partial \psi},
\]
where \( a \) is the scale factor of the Robertson-Walker spacetime metric, \( m_{\text{pl}} \) is the Planck mass, and dots denote differentiation with respect to time. The specific results presented here are obtained for the potential

\[
V(\phi, \psi) = \left( M^2 - \frac{\sqrt{\lambda}}{2} \psi^2 \right)^2 + \frac{m^2}{2} \phi^2 + \frac{\gamma}{2} \phi^2 \psi^2, \quad (3)
\]

but the key ingredient needed for chaos is simply that the fields are coupled by a term like \( \frac{\gamma}{2} \phi^2 \psi^2 \), and the precise form of the potential is not important.

Several workers have examined the possible role of chaotic dynamics in inflationary models. However, previous studies focussed on the overall evolution of the universe, and it is not clear that the chaos they found would lead to consequences detectable by a realistic observer, whose perspective is confined to the interior of a single universe.

In contrast, we consider a chaotic era which occurs during the oscillatory phase after the end of inflation, when the damping terms proportional to \( H \) no longer dominate the fields’ equations of motion. The post-inflationary dynamics are then accurately approximated by the Hamiltonian system that is derived by dropping the friction term from the equations of the motion. It is straightforward to show that the frictionless system is chaotic for a broad range of parameter choices. The next step is to demonstrate that the chaotic properties persist once the dissipative effects caused by the expansion of the universe are reinstated. There is typically a minimum energy (density) below...
which chaos cannot occur, so we immediately deduce that the chaotic era in
the post-inflationary universe is transient, since the expansion of the universe
guarantees that the energy density is strictly decreasing.

As a specific example of the consequences of a chaotic era occurring after
the end of inflation, Figure 1 shows how the growth of the universe during the
oscillatory era depends on the precise values of the fields and their velocities
at the end of inflation. We compared a model with $\gamma = 0$, where there is no
chance of chaotic evolution, to one with $\gamma = 0.5$, which is initially chaotic. The
qualitative difference between the chaotic and non-chaotic cases is clear. In the
latter case, the amount of growth experienced by the universe after the end of
inflation varies discontinuously with (small) changes in initial conditions, and
the variation is a large fraction of the average amount of expansion. Conversely,
in the non-chaotic case, the amount of expansion varies smoothly with the
changing initial conditions, and the total amount of variation is much smaller.

3 Discussion

We have restricted our attention to ensembles of homogeneous universes, rather
than a single inhomogeneous universe. Since variations on spatial scales significa-
tantly larger than the post-inflationary horizon volume do not directly affect
the dynamics, we are therefore effectively considering an ensemble of homoge-
neous horizon volumes. However, the rapid growth of small initial differences
is the hallmark of chaotic dynamics, so the initial differences in the field val-
ues in different volumes will grow rapidly during the chaotic era. We expect
that the spatial gradient terms in the fields will become large, and they must
be included if we are to make quantitative predictions about the role of
chaos in the post-inflationary universe. Work on the inhomogeneous problem
is currently in progress.

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