SEMI-WAVE, TRAVELING WAVE AND SPREADING SPEED FOR MONOSTABLE COOPERATIVE SYSTEMS WITH NONLOCAL DIFFUSION AND FREE BOUNDARIES

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Abstract. We consider a class of cooperative reaction-diffusion systems with free boundaries in one space dimension, where the diffusion terms are nonlocal, given by integral operators involving suitable kernel functions, and they are allowed not to appear in some of the equations in the system. The problem is monostable in nature, resembling the well known Fisher-KPP equation. Such a system covers various models arising from mathematical biology, with the Fisher-KPP equation as the simplest special case, where a “spreading-vanishing” dichotomy is known to govern the long time dynamical behaviour. The question of spreading speed is widely open for such systems except for the scalar case. In this paper, we develop a systematic approach to determine the spreading profile of the system, and obtain threshold conditions on the kernel functions which decide exactly when the spreading has finite speed, or infinite speed (accelerated spreading). This relies on a rather complete understanding of both the associated semi-waves and traveling waves. When the spreading speed is finite, we show that the speed is determined by a particular semi-wave, and obtain sharp estimates of the semi-wave profile and the spreading speed. For kernel functions that behave like $|x|^{-γ}$ near infinity, we are able to obtain better estimates of the spreading speed for both the finite speed case, and the infinite speed case, which appear to be the first for this kind of free boundary problems, even for the special Fisher-KPP equation.

Key words: Free boundary, nonlocal diffusion, spreading speed, semi-wave, traveling wave.

MSC2010 subject classifications: 35K20, 35R35, 35R09.

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1. Introduction

1.1 The equations and assumptions. In this paper we determine the spreading speed for cooperative systems with nonlocal diffusion and free boundaries of the following form:

$$\begin{cases} 
\partial_t u_i = d_i \mathcal{L}_i[u_i](t, x) + f_i(u_1, u_2, \cdots, u_m), & t > 0, \ x \in (g(t), h(t)), \ 1 \leq i \leq m_0, \\
\partial_t u_i = f_i(u_1, u_2, \cdots, u_m), & t > 0, \ x \in (g(t), h(t)), \ m_0 < i \leq m, \\
u_i(t, g(t)) = u_i(t, h(t)) = 0, & t > 0, \ 1 \leq i \leq m, \\
g'(t) = -\sum_{i=1}^{m_0} \mu_i \int_{g(t)}^{h(t)} J_i(x - y)u_i(t, y)dy \ dx, & t > 0, \\
h'(t) = \sum_{i=1}^{m_0} \mu_i \int_{g(t)}^{h(t)} J_i(x - y)u_i(t, x)dy \ dx, & t > 0, \\
u_i(0, x) = u_{i0}(x), & x \in [-h_0, h_0], \ 1 \leq i \leq m,
\end{cases}$$

(1.1)

where $1 \leq m_0 \leq m$, and for $i \in \{1, ..., m_0\}$,

$$\mathcal{L}_i[v](t, x) := \int_{g(t)}^{h(t)} J_i(x - y)v(t, y)dy - v(t, x),$$

$d_i > 0$ and $\mu_i \geq 0$ are constants, with $\sum_{i=1}^{m_0} \mu_i > 0$.

The initial functions satisfy

$$(1.2) \quad u_{i0} \in C([-h_0, h_0]), \ u_{i0}(-h_0) = u_{i0}(h_0) = 0, \ u_{i0}(x) > 0 \ \text{in} \ (-h_0, h_0), \ 1 \leq i \leq m.$$ 

The kernel functions $J_i(x)$ ($i = 1, \cdots, m_0$) satisfy

$$(J) \quad J_i \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ is nonnegative, even, } J_i(0) > 0, \ \int_{\mathbb{R}} J_i(x)dx = 1 \text{ for } 1 \leq i \leq m_0.$$ 

In order to describe the assumptions on the function $F = (f_1, ..., f_m) \in [C^1(\mathbb{R}_+^m)]^m$ with

$$\mathbb{R}_+^m := \{x = (x_1, ..., x_m) \in \mathbb{R}^m : x_i \geq 0 \text{ for } i = 1, ..., m\},$$

and for convenience of later discussions, we introduce some notations about vectors in $\mathbb{R}^m$, and also recall the definition of irreducible matrices and some associated properties.

Notations about vectors in $\mathbb{R}^m$:
(i) For $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, we simply write $(x_1, \ldots, x_m)$ as $(x_i)$. For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^m$,

$$x \succeq (\preceq) y \quad \text{means} \quad x_i \succeq (\preceq) y_i \text{ for } 1 \leq i \leq m,$$

$$x > (\prec) y \quad \text{means} \quad x \succeq (\preceq) y \text{ but } x \neq y,$$

$$x \succ (\ll) y \quad \text{means} \quad x_i > (\prec) y_i \text{ for } 1 \leq i \leq m.$$

(ii) If $x \preceq y$, then $[x, y] := \{z \in \mathbb{R}^m : x \preceq z \preceq y\}$.

(iii) **Hadamard product:** For $x = (x_i), y = (y_i) \in \mathbb{R}^m$,

$$x \circ y = (x_i y_i) \in \mathbb{R}^m.$$

(iv) Any $x \in \mathbb{R}^m$ is viewed as a row vector, namely a $1 \times m$ matrix, whose transpose is denoted by $x^T$.

**Irreducible matrix and principal eigenvalue:**

An $m \times m$ matrix $A = (a_{ij})$, with $m \geq 2$, is called **reducible** if the index set $\{1, \ldots, m\}$ can be split into the union of two subsets $S$ and $S'$ with $r \geq 1$ and $m - r \geq 1$ elements, respectively, such that $a_{ij} = 0$ for all $i \in S$ and $j \in S'$. $A$ is called **irreducible** if it is not reducible. If $D$ is a diagonal $m \times m$ matrix, clearly $A + D$ is irreducible if and only if $A$ is irreducible.

If $A$ is irreducible and all its off-diagonal elements are nonnegative, then for $\sigma > 0$ large $A + \sigma I_m$ is a nonnegative irreducible matrix, where $I_m$ denotes the $m \times m$ identity matrix. By the Perron-Frobenius theorem, $A + \sigma I_m$ has a largest eigenvalue $\lambda_1 = \lambda_1(\sigma)$ which is the only eigenvalue that corresponds to a positive eigenvector $v_1 \gg 0$: $(A + \sigma I_m)v_1 = \lambda_1 v_1$. Hence $Av_1 = \lambda_1 v_1$ with $\lambda_1 = \hat{\lambda}_1 - \sigma$, which is the largest eigenvalue of $A$ and is independent of $\sigma$. We will call $\lambda_1$ the principal eigenvalue of $A$.

A $1 \times 1$ matrix is irreducible if and only if its sole element is not $0$.

**Assumptions on $F$:**

(f1) (i) $F(u) = 0$ has only two roots in $\mathbb{R}^m_+$: $0 = (0, 0, \ldots, 0)$ and $u^* = (u_1^*, u_2^*, \ldots, u_m^*) \gg 0$.

(ii) $\partial_j f_i(u) \geq 0$ for $i \neq j$ and $u \in [0, \hat{u}]$, where either $\hat{u} = \infty$ meaning $[0, \hat{u}] = \mathbb{R}^m_+$, or $u^* \ll \hat{u} \in \mathbb{R}^m$; which implies that (1.1) is a cooperative system in $[0, \hat{u}]$.

(iii) The matrix $\nabla F(0)$ is irreducible with principal eigenvalue positive, where $\nabla F(0) = (a_{ij})_{m \times m}$ with $a_{ij} = \partial_j f_i(0)$.

(iv) If $m_0 < m$ then $\partial_j f_i(u) > 0$ for $1 \leq j \leq m_0 < i \leq m$ and $u \in [0, u^*]$.

(f2) $F(ku) \geq kF(u)$ for any $0 \leq k \leq 1$ and $u \in [0, \hat{u}]$.

(f3) The matrix $\nabla F(u^*)$ is invertible, $[u^*] \nabla F(u^*) \preceq 0$ and for each $i \in \{1, \ldots, m\}$, either

$$\sum_{j=1}^m \partial_j f_i(u^*) u_j^* < 0,$$

or

$$\sum_{j=1}^m \partial_j f_i(u^*) u_j^* = 0 \quad \text{and} \quad f_i(u) \text{ is linear in } [u^* - \epsilon_0 1, u^*] \text{ for some small } \epsilon_0 > 0,$$

where $1 = (1, \ldots, 1) \in \mathbb{R}^m$.

(f4) The set $[0, \hat{u}]$ is invariant for

$$U_t = D \circ \int_{\mathbb{R}} \mathbf{J}(x-y) \circ U(t, y)dy - D \circ U + F(U) \text{ for } t > 0, x \in \mathbb{R},$$

and the equilibrium $u^*$ attracts all the nontrivial solutions in $[0, \hat{u}]$; namely, $U(t, x) \in [0, \hat{u}]$ for all $t > 0, x \in \mathbb{R}$ if $U(0, x) \in [0, \hat{u}]$ for all $x \in \mathbb{R}$, and $\lim_{t \to \infty} U(t, \cdot) = u^*$ in $L^\infty_{\text{loc}}(\mathbb{R})$ if additionally $U(0, x) \neq 0$.

In (1.3) we have used the convention that $d_i = 0$ and $J_i \equiv 0$ for $m_0 < i \leq m$, and

$$D = (d_i), \quad \mathbf{J}(x) = (J_i(x)),$$

This convention will be used throughout the paper.
The above assumptions on $F$ indicate that the system is cooperative in $[0, \hat{u}]$, and of monostable type, with $u^*$ the unique stable equilibrium of \([1.3]\), which is the global attractor of all the nontrivial nonnegative solutions of \([1.3]\) in $[0, \hat{u}]$.

Problems \([1.1]\) and \([1.3]\) arise frequently in population and epidemic models. For example, if $m_0 = m = 1$, then \([1.1]\) reduces to the Fisher-KPP model with free boundary considered in \([8, 12]\). With $m_0 = m = 2$, \([1.1]\) contains the West Nile virus model in \([10]\) as a special case, and with $(m_0, m) = (1, 2)$, it covers the epidemic model in \([12]\). In these special cases, it is known that the long-time dynamical behaviour of the solution to \([1.1]\) exhibits a spreading-vanishing dichotomy. When spreading happens, one important question is to determine the spreading speed. For the Fisher-KPP model in \([8]\), it was shown recently in \([12]\) that the spreading speed may be finite or infinite, depending on whether a threshold condition is satisfied by the kernel function. For the epidemic models in \([16]\) and \([42]\), the spreading speed has not been determined.

Problem \([1.3]\) and its various special cases have been extensively studied in the literature; see, for example, \([1, 2, 4–6, 9, 11, 20, 21, 23, 24, 26, 29, 34, 38–40]\) for a very small sample.

The purpose of this paper is to provide a unified treatment of free boundary problems of the form \([1.1]\). Similar to the special cases mentioned in the last paragraph, it can be shown that \([1.1]\) with initial data satisfying \([1.2]\) and $U(0, x) \in [0, \hat{u}]$ has a unique positive solution $(U(t, x), g(t), h(t))$ defined for all $t > 0$. We say spreading happens if, as $t \to \infty$,

$$(g(t), h(t)) \to (\infty, \infty) \text{ and } U(t, \cdot) \to u^* \text{ component-wise in } L^\infty_{loc}(\mathbb{R}),$$

and we say vanishing happens if

$$(g(t), h(t)) \to (g_\infty, h_\infty) \text{ is a finite interval, and } \max_{x \in [g(t), h(t)]} |U(t, x)| \to 0.$$  

1.2. Main results. We aim to determine the spreading speed when spreading happens for \([1.1]\). We will show that the spreading speed is finite if and only if the following additional condition is satisfied by the kernel functions:

$$(J_1): \quad \int_0^\infty x J_i(x) dx < \infty \text{ for every } i \in \{1, \ldots, m_0\} \text{ such that } \mu_i > 0.$$  

If \((J_1)\) is not satisfied, then the spreading speed is infinite, namely accelerated spreading happens. Let us note that if for some $i \in \{1, \ldots, m_0\}$, $\mu_i = 0$, then no restriction on $J_i$ is imposed by \((J_1)\).

The proof of these conclusions rely on a complete understanding of the associated semi-wave problem to \([1.1]\), which consists of the following two equations \([1.4]\) and \([1.5]\) with unknowns $(c, \Phi(x))$:

\begin{equation}
\begin{aligned}
D \circ \int_{-\infty}^0 J(x-y) \circ \Phi(y) dy - D \circ \Phi + c \Phi(x) + F(\Phi(x)) = 0 \text{ for } -\infty < x < 0, \\
\Phi(-\infty) = u^*, \quad \Phi(0) = 0,
\end{aligned}
\end{equation}

and

\begin{equation}
c = \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^0 \int_0^\infty J_i(x-y) \phi_i(x) dy dx,
\end{equation}

where $D = (d_i)$, $J = (J_i)$, $\Phi = (\phi_i)$ and “$\circ$” is the Hadamard product.

If $(c, \Phi)$ solves \([1.4]\), we say that $\Phi$ is a semi-wave solution to \([1.3]\) with speed $c$. This is not to be confused with the semi-wave to \([1.1]\), for which the extra equation \([1.5]\) should be satisfied, yielding a semi-wave solution of \([1.3]\) with a desired speed $c = c_0$, which determines the spreading speed of \([1.1]\).

To fully understand the semi-wave solutions to \([1.3]\), we also need to examine the associated traveling wave problem for \([1.3]\). A differentiable function $\Psi$ is called a traveling wave solution...
of (1.3) with speed $c$ if $\Psi$ satisfies

$$
\begin{cases}
D \circ \int_{-\infty}^{\infty} J(x-y) \circ \Psi(y)dy - D \circ \Psi + c\Psi'(x) + F(\Psi(x)) = 0 \quad \text{for } x \in \mathbb{R}, \\
\Psi(-\infty) = u^*, \quad \Psi(\infty) = 0.
\end{cases}
$$

We are interested in semi-waves and traveling waves which are monotone and with positive speed. It turns out that for any fixed speed $c > 0$, either a semi-wave or traveling wave exists, but not both. More precisely, the following dichotomy holds:

**Theorem 1.1.** Suppose $(J)$ and $(f_1) - (f_4)$ hold. Then there exists $C_* \in (0, \infty)$ such that (1.4) has a monotone solution if and only if $0 < c < C_*$, and (1.6) has a monotone solution if and only if $c \geq C_*$. Therefore a monotone traveling wave with some positive speed $c$ exists if and only if $C_* < \infty$. We will show that $C_* < \infty$ if and only if the following condition is satisfied by the kernel functions:

$$(J_2): \quad \int_0^\infty e^{\lambda x} J_i(x) dx < \infty \quad \text{for some } \lambda > 0 \text{ and every } i \in \{1, ..., m_0\}.$$  

We have the following refinements of the conclusions in Theorem 1.1.

**Theorem 1.2.** Under the conditions of Theorem 1.1 the following hold:

(i) For $0 < c < C_*$, (1.4) has a unique monotone solution $\Phi^C = (\phi^C_i)$, and

$$
\lim_{c \nearrow C_*} \Phi^C(x) = 0 \quad \text{locally uniformly in } (-\infty, 0].
$$

(ii) $C_* \neq \infty$ if and only if $(J_2)$ holds.

(iii) The system (1.4), (1.5) has a solution pair $(c, \Phi)$ with $\Phi(x)$ monotone if and only if $(J_1)$ holds. And when $(J_1)$ holds, there exists a unique $c_0 \in (0, C_*)$ such that $(c, \Phi) = (c_0, \Phi^{c_0})$ solves (1.4) and (1.5).

It is easily checked that $(J_2)$ implies $(J_1)$, but if $J_i(x) = \xi_i(1 + |x|)^{-\eta_i}$ with $\xi_i > 0$, $\eta_i > 2$ for $1 \leq i \leq m_0$, then $(J_1)$ holds but $(J_2)$ does not.

We are now able to state our first result on the spreading speed of (1.1).

**Theorem 1.3.** Suppose the conditions in Theorem 1.1 are satisfied, $(U, g, h)$ is a solution of (1.1) with $U(0, x) \in [0, \hat{u}]$, and spreading happens. Then the following conclusions hold for the spreading speed:

(i) If $(J_1)$ is satisfied, then the spreading speed is finite, and is determined by

$$
\lim_{t \to \infty} \frac{g(t)}{t} = \lim_{t \to \infty} \frac{h(t)}{t} = c_0 \quad \text{with } c_0 \text{ given in Theorem 1.2 (iii)}.
$$

(ii) If $(J_1)$ is not satisfied, then accelerated spreading happens, namely

$$
\lim_{t \to \infty} \frac{g(t)}{t} = \lim_{t \to \infty} \frac{h(t)}{t} = \infty.
$$

Under further conditions on $F$ and the kernel functions, the conclusions in Theorem 1.3 can be sharpened. For $\alpha > 0$, we introduce the condition

$$(J_\alpha): \quad \int_0^\infty x^\alpha J_i(x) dx < \infty \quad \text{for every } i \in \{1, ..., m_0\}.$$  

Let us note that $(J^1)$ implies $(J_1)$, but unless $\mu_i > 0$ for every $i \in \{1, ..., m_0\}$, $(J_1)$ does not imply $(J^1)$. On the other hand, if $(J_2)$ holds, then $(J^\alpha)$ is satisfied for all $\alpha > 0$. 

Theorem 1.4. In Theorem 1.2, suppose additionally (J^{a}) holds for some $\alpha \geq 2$, $F$ is $C^2$ and $u^*[\nabla F(u^*)]^T \ll 0$. Then there exist positive constants $\theta$, $C$ and $t_0$ such that, for all $t > t_0$ and $x \in [g(t), h(t)]$,

$$|h(t) - c_0 t| + |g(t) + c_0 t| \leq C,$$

$$U(t, x) \geq [1 - \epsilon(t)] [\Phi^0(x - c_0 t + C) + \Phi^0(-x - c_0 t + C) - u^*],$$

$$U(t, x) \leq [1 + \epsilon(t)] \min \{\Phi^0(x - c_0 t - C), \Phi^0(-x - c_0 t - C)\},$$

where $\epsilon(t) := (t + \theta)^{-\alpha}$, and $(c_0, \Phi^0)$ is the unique pair solving (1.4) and (1.5) obtained in Theorem 1.2 (iii), with $\Phi^0(x)$ extended by 0 for $x > 0$.

Further estimates on $g(t)$ and $h(t)$ can be obtained if we narrow down more on the class of kernel functions $\{J_i : i = 1, ..., m_0\}$. We will write

$$\eta(t) \approx \xi(t) \text{ if } C_1 \xi(t) \leq \eta(t) \leq C_2 \xi(t)$$

for some positive constants $C_1 \leq C_2$ and all $t$ is the concerned range.

Our next two theorems are about kernel functions satisfying, for some $\gamma > 0$,

$$(\tilde{J}^{\gamma}) : J_i(x) \approx |x|^{-\gamma} \text{ for } |x| \gg 1 \text{ and } i \in \{1, ..., m_0\}.$$

Note that for kernel functions satisfying $(\tilde{J}^{\gamma})$, condition (J) is satisfied if and only if $\gamma > 1$, and $(J_1)$ is satisfied if and only if $\gamma > 2$. The next result determines the orders of accelerated spreading when $\gamma \in (1, 2]$.

Theorem 1.5. In Theorem 1.3, if additionally the kernel functions satisfy $(\tilde{J}^{\gamma})$ for some $\gamma \in (1, 2]$, then for $t \gg 1$,

$$-g(t), h(t) \approx t \ln t \quad \text{if } \gamma = 2,$$

$$-g(t), h(t) \approx t^{1/(\gamma - 1)} \quad \text{if } \gamma \in (1, 2).$$

For kernel functions satisfying $(\tilde{J}^{\gamma})$, clearly $(J^{a})$ holds if and only if $\gamma > 1 + \alpha$. Therefore the case $\gamma > 3$ is already covered by Theorem 1.4. The following theorem is concerned with the remaining case $\gamma \in (2, 3]$, which indicates that the result in Theorem 1.4 is sharp.

Theorem 1.6. In Theorem 1.3, suppose additionally the kernel functions satisfy $(\tilde{J}^{\gamma})$ for some $\gamma \in (2, 3]$, $F$ is $C^2$ and

$$(1.7) \quad F(v) - v[\nabla F(v)]^T \gg 0 \quad \text{for } 0 \ll v \leq u^*.$$

Then for $t \gg 1$,

$$c_0 t + g(t), c_0 t - h(t) \approx t \ln t \quad \text{if } \gamma = 3,$$

$$c_0 t + g(t), c_0 t - h(t) \approx t^{3 - \gamma} \quad \text{if } \gamma \in (2, 3).$$

Note that $(f_2)$ implies

$$F(v) - v[\nabla F(v)]^T \succeq 0 \quad \text{for } v \in [0, u^*].$$

Therefore (1.7) is a strengthened version of $(f_2)$. If we take $v = u^*$ in (1.7), then it yields $u^*[\nabla F(u^*)]^T \ll 0$. When $m = 1$, (1.7) reduces to $F(v) > F'(v)v$ for $0 < v \leq \hat{u}$, which is satisfied, for example, by $F(v) = av - bv^p$ with $a, b > 0$ and $p > 1$.

The proofs of Theorems 1.4 and 1.6 rely on some of the following estimates on the semi-wave solutions of (1.3), which are of independent interests.

Theorem 1.7. Suppose that $F$ satisfies $(f_1) - (f_4)$ and the kernel functions satisfy (J), and $\Phi(x) = (\phi_i(x))$ is a monotone solution of (1.4) for some $c > 0$. Then the following conclusions hold:
(i) If \((J^\alpha)\) holds for some \(\alpha > 0\), then for every \(i \in \{1, \ldots, m\}\),
\[
\int_{-\infty}^{-1} [u_i^* - \phi_i(x)] |x|^{\alpha - 1} \, dx < \infty,
\]
which implies, by the monotonicity of \(\phi_i(x)\),
\[
0 < u_i^* - \phi_i(x) \leq C|x|^{-\alpha} \text{ for some } C > 0 \text{ and all } x < 0.
\]

(ii) If \((J^\alpha)\) does not hold for some \(\alpha > 0\), then
\[
\sum_{i=1}^{m} \int_{-\infty}^{-1} [u_i^* - \phi_i(x)] |x|^{\alpha - 1} \, dx = \infty.
\]

(iii) If \((J_2)\) holds, then there exist positive constants \(C\) and \(\beta\) such that
\[
0 < u_i^* - \phi_i(x) \leq Ce^{\beta x} \text{ for all } x < 0, \quad i \in \{1, \ldots, m\}.
\]

1.3. Examples. It is clear that all the results here can be applied to the Fisher-KPP nonlocal diffusion model in [8,12]. When \(m = m_0 = 1\), our Theorem 1.3 here recovers the main result in [12], but Theorems 1.3, 1.5, 1.6 and 1.7 are new even in this special case.

Let us now examine the models in [16] and [42].

Example 1. The West Nile virus model in [16] is given by
\[
\begin{align*}
H_t &= d_1L_1[H](t,x) + a_1(e_1 - H)V - b_1H, \quad x \in (g(t), h(t)), \quad t > 0, \\
V_t &= d_2L_2[V](t,x) + a_1(e_2 - V)H - b_2V, \quad x \in (g(t), h(t)), \quad t > 0, \\
H(t,x) &= V(0,x) = 0, \quad t > 0, \quad x \in \{g(t), h(t)\}, \\
g'(t) &= -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_1(x-y)V(t,x) \, dy \, dx, \quad t > 0, \\
h'(t) &= \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_1(x-y)V(t,x) \, dy \, dx, \quad t > 0, \\
-g(0) &= h(0) = h_0, \quad H(0,x) = u_1^0(x), \quad V(0,x) = u_2^0(x), \quad x \in [-h_0, h_0],
\end{align*}
\]
where \(a_i, e_i\) and \(b_i\) \((i = 1, 2)\) are positive constants satisfying \(a_1a_2e_1e_2 > b_1b_2\) (which is necessary for spreading to happen). We thus have
\[
F(u) = F_1(u) := (a_1(e_1 - u_1)u_2 - b_1u_1, a_2(e_2 - u_2)u_1 - b_2u_2).
\]
\[
u^* = \left(\frac{a_1a_2 - e_1e_2 - b_1b_2}{a_1a_2e_2 + a_2b_1}, \frac{a_1a_2 - e_1e_2 - b_1b_2}{a_1a_2e_1 + a_1b_2}\right).
\]
It is straightforward to check that conditions \((f_1) - (f_3)\) are satisfied by \(F_1\) with \(\hat{u} = (e_1, e_2)\). Condition \((f_4)\) was shown to hold in [16]. It is also easy to see that \(F_1\) is \(C^2\) and
\[
F_1(u) = u[\nabla F_1(u)]^T = (a_1u_1u_2, a_2u_1u_2).
\]
Therefore (1.7) holds as well. Thus all our results apply to (1.8).
The epidemic model in [42] is given by

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d\mathcal{L}_1[u] - au + cv, \\
\frac{\partial v}{\partial t} &= -bv + G(u), \\
u(t, x) &= v(t, x) = 0, \\
g'(t) &= -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_1(x - y)u(t, x)dydx, \\
h'(t) &= \mu \int_{g(t)}^{h(t)} \int_{-\infty}^{\infty} J_1(x - y)u(t, x)dydx, \\
g(0) &= h(0) = h_0, \\n0 < u(0, x) = u_0(x), \\n0 < v(0, x) = v_0(x), \\n&(x \in [-h_0, h_0],
\end{align*}
\]

where \( a, b, c, d, \mu \) and \( h_0 \) are positive constants, and the function \( G \) is assumed to satisfy

(i) \( G \in C^1([0, \infty]), \ G(0) = 0, \ G'(z) > 0 \) for \( z \geq 0; \)

(ii) \( \left[ \frac{G(z)}{z} \right]' < 0 \) for \( z > 0 \) and \( \lim_{z \to +\infty} \frac{G(z)}{z} < \frac{ab}{c}; \)

(iii) \( G'(0) > \frac{ab}{c} \) (necessary for spreading to happen).

In this example,

\[
F(u) = F_2(u) := (-au + cv, G(u) - bu), \quad u^* = (K_1, K_2)
\]

where \( (K_1, K_2) \succcurlyeq 0 \) are uniquely determined by

\[
\frac{G(K_1)}{K_1} = \frac{ab}{c}, \quad K_2 = \frac{G(K_1)}{b}.
\]

One easily checks that \( F_2 \) satisfies (f_1) – (f_3) with \( \hat{u} = \infty \). In [42], it was proved that (f_4) also holds. Clearly \( F_2 \) is \( C^2 \). However, \( u^*[\nabla F_2(u^*)]^T \prec 0 \) does not hold. Therefore all our results apply to (1.9) except Theorems 1.4 and 1.6.

1.4. Related problems and comments. The random (local) diffusion version of various special cases or variations of (1.1) have been studied extensively in recent years, starting from the work [13]. In these random diffusion free boundary problems the spreading speed is always finite, and is determined by the associated semi-waves; see, for example, [3, 13–15, 17, 25, 28, 35, 41] for an incomplete sample of such results. In many situations, especially in population and epidemic models, it has been recognised that random diffusion is often not the best approximation of the spatial dispersal behaviour of the concerned species, and in order to include factors such as long-distance dispersal, in a large amount of literature, the random diffusion terms in the models are replaced by nonlocal diffusion operators as in (1.3). A striking difference of this type of nonlocal diffusion models to their random diffusion counterparts is that accelerated spreading may occur. For the scalar case of (1.3), namely for the Fisher-KPP equation with nonlocal diffusion, it follows from the theory in [36] that accelerated spreading occurs exactly when the kernel function does not satisfy (J_2) described above. On the other hand, when (J_2) is satisfied by the kernel function (thin-tailed kernel) then there exists some \( c_\ast > 0 \) such that the associated traveling wave problem has a monotone traveling wave with speed \( c \) if and only if \( c \geq c_\ast \), and \( c_\ast \) is the asymptotic spreading speed determined by the scalar nonlocal Fisher-KPP equation (1.3); see, for example, [27, 31, 33, 35, 40]. Related works on accelerated spreading can be found in [11, 16, 19, 22, 30, 38, 39] and the references therein. It is well known that the random diffusion version of (1.3) with compactly supported initial functions can only spread with finite speed, which is determined by the associated traveling waves [27, 36, 37, 43].

The works [8, 10] appear to be the first to consider a nonlocal diffusion version of the free boundary problem proposed in [13], where the free boundary conditions of the form in (1.1) were first introduced (independently). In [10] the authors considered the case that the reaction term is identically 0 and therefore completely different long-time dynamical behaviour was shown. Our
results here (as well as in [12]) indicate that \( J_1 \) is the threshold condition on the kernel functions which decides whether accelerated spreading happens for (1.1); in contrast, as mentioned above, for (1.3), \( J_2 \) is the corresponding threshold condition (at least in the scalar Fisher-KPP case).

There are two fundamental differences between the free boundary model (1.1) and the corresponding model (1.3) where no free boundary appears. Firstly (1.1) provides the exact location of the spreading front, which is the free boundary, while the location of the front is not prescribed in (1.3), and one usually uses suitable level sets of the solution to describe the front behaviour. Secondly, the long time dynamical behaviour of (1.1) is often governed by a spreading-vanishing dichotomy [8, 16, 42], but (1.3) predicts successful spreading all the time. Let us also note that since \( J_2 \) implies \( J_1 \) but not the other way round, (1.3) is more readily than (1.1) to give rise to accelerated spreading.

1.5. **Organisation of the paper.** The rest of the paper is organised as follows. In Section 2, we prove Theorems 1.1 and 1.2. Much of the arguments there are based on the perturbed semi-wave problem (2.9), which in the limit yields either a semi-wave solution or a traveling wave solution. This limiting process is also used in several other places of the paper; for example, it plays a crucial role in our proof for accelerated spreading in Theorem 1.3.

Section 3 is devoted to the proof of Theorem 1.3, which relies on careful constructions of upper and lower solutions building from the semi-wave solution with the desired speed \( c_0 \), and on a limiting argument when such a semi-wave does not exist, leading to accelerated spreading.

Section 4 gives the proof of Theorems 1.4 and 1.7. The proof of the former is built on the proof and conclusions of the latter, where subtle analysis is used to find out the relationship between the behaviour of the semi-wave solution and that of the kernel functions. The constructions of upper and lower solutions in the proof of Theorem 1.4 are much more subtle than that in Section 3.

Sections 5 and 6 are devoted to the proof of Theorems 1.5 and 1.6 for kernel functions behaving like \( |x|^{-\gamma} \) near infinity. In Section 5, we completely determine the growth orders of \( c_0 t - h(t) \) for \( \gamma \) in the range (2, 3], while in Section 6, we completely determine the accelerated spreading orders of \( h(t) \) when \( \gamma \) falls into the range (1, 2]. Note that when \( \gamma > 3 \), the spreading behaviour is already covered by the more general results in Section 4.

Although Sections 4, 5 and 6 are based on completely new ideas and techniques, part of the strategy in the approach of Sections 2 and 3 is borrowed from [12], albeit the situation here is much more general and complex, requiring solutions to new problems and use of novel methods. For example, (a) while in the scalar case the traveling wave solutions of (1.3) are completely understood in [40], here the corresponding result requires a rather nontrivial proof, (b) the perturbation problem (2.9) is treated here by a much simpler method than the one suggested by [12], which appears difficult to apply in the more general setting here, and (c) our proof of accelerated spreading in Subsection 3.3 uses a completely new approach.

2. **Semi-waves and traveling waves of (1.3)**

2.1. **Some preparatory results.** Since \( F = (f_i) \) is \( C^1 \) over \( \mathbb{R}_+^m \), for any vector \( K = (k_i) \gg 0 \) there is a constant \( L(K) > 0 \) such that for \( u, v \in [0, K] \) with \( u = (u_i) \) and \( v = (v_i) \),

\[
|f_i(u) - f_i(v)| \leq L(K) \sum_{i=1}^{m} |u_i - v_i|.
\]  

**Lemma 2.1.** If \( (f_1) \) holds, then there exist \( \lambda_1 > 0 \), small \( \epsilon > 0 \), and vectors \( \Theta = (\theta_i) \gg 0 \), \( \tilde{\Theta} = (\tilde{\theta}_i) \gg 0 \) such that

\[
\Theta \nabla F(0)^T = \lambda_1 \Theta, \quad \tilde{\Theta} \nabla F(0) = \lambda_1 \tilde{\Theta},
\]
and
\[(2.3) \quad F(\epsilon \Theta) \gg 0, \quad \sum_{i=1}^{m} \tilde{\theta}_i f_i(X) \geq \sum_{i=1}^{m} b_i x_i \text{ for } X = (x_i) \in [0, \epsilon 1],\]
where \(1 = (1, \ldots, 1) \in \mathbb{R}^m\) and \(b_i := \frac{\lambda_i \tilde{\theta}_i}{2} > 0\).

Proof. Let \(\lambda_1\) be the principal eigenvalue of \(\nabla F(0)\). By the Perron-Frobenius theorem, there exist positive eigenvectors \(\Theta\) and \(\tilde{\Theta}\) such that the identities in (2.2) hold.

Moreover, in view of \(F \in [C^1(\mathbb{R}_+^m)]^m\), for small \(\epsilon > 0\) and \(X = (x_i) \in [0, \epsilon 1]\),
\[
F(\epsilon \Theta) = \epsilon \Theta [\nabla F(0)^T + o(1) I_m] = \epsilon [\lambda_1 + o(1)] \Theta,
\]
\[
\sum_{i=1}^{m} \tilde{\theta}_i f_i(X) = \tilde{\Theta} [\nabla F(0) + o_\epsilon(X)] X^T = \tilde{\Theta} [\lambda_1 I_m + o_\epsilon(X)] X^T,
\]
with \(|o_\epsilon(X)| \to 0\) as \(\epsilon \to 0\) uniformly in \(X \in [0, \epsilon 1]\). Hence (2.3) holds provided that \(\epsilon > 0\) is small enough. \(\square\)

Lemma 2.2. Assume (J) holds, and for every \(i \in \{1, \ldots, m\}\), \(v_i \in C(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})\) satisfies
\[
\begin{cases}
    d_i \int_{\mathbb{R}} J_i(x-y) v_i(y) dy - d_i v_i(x) + p_i(x) v_i'(x) + \sum_{j=1}^{m} q_{ij}(x) v_j(x) \leq 0, & x < 0, \\
    v_i(x) \geq 0, & x \geq 0,
\end{cases}
\]
where \(d_i \in [0, \infty)\), \(p_i, q_{ij} \in L_\text{loc}^\infty(\mathbb{R})\) with \(q_{ij} \geq 0\) for \(i \neq j\). If \(v_i(x) \geq 0\) for all \(x \in \mathbb{R}\) and \(i \in \{1, \ldots, m\}\), with \(d_{i_0} > 0\) and \(v_{i_0}(x) \geq \epsilon > 0\) for some \(i_0 \in \{1, \ldots, m\}\), then \(v_{i_0}(x) > 0\) for \(x < 0\).

Proof. Assume by contradiction there is \(x_0 < 0\) such that \(v_{i_0}(x_0) = 0\). Since \(v_{i_0}(x) \neq 0\), we may also require that there exists a sequence \(\{x_n\}_{n=1}^\infty\) with \(x_n \to x_0\) such that \(v_{i_0}(x_n) > 0\). Then \(v_{i_0}'(x_0) = 0\), and from the inequality satisfied by \(v_{i_0}\) we deduce that
\[
d_{i_0} \int_{\mathbb{R}} J_{i_0}(x_0-y) v_{i_0}(y) dy \leq 0,
\]
which implies that \(v_{i_0}(x) = 0\) for all \(x\) near \(x_0\) due to \(J_{i_0}(0) > 0\). However, this contradicts to \(v_{i_0}(x_n) > 0\). \(\square\)

Lemma 2.3. Assume that (J) holds, \(T \in (0, \infty)\), and for every \(i \in \{1, \ldots, m\}\), \(v_i \in C([0, T] \times \mathbb{R}) \cup L^\infty([0, T] \times \mathbb{R})\), \(\partial_t v_i \in C([0, T] \times \mathbb{R})\), and
\[
\begin{cases}
    \partial_t v_i(t, x) \geq d_i(t, x) L_i[v_i](t, x) + \sum_{j=1}^{m} q_{ij}(t, x) v_j(t, x), & t \in (0, T], x \in \mathbb{R}, \\
    v_i(0, x) \geq 0, & x \in \mathbb{R},
\end{cases}
\]
where
\[
L_i[v_i](t, x) := \int_{\mathbb{R}} J_i(x-y) v_i(t, y) dy - v_i(t, x),
\]
and the functions \(d_i, q_{ij} \in L^\infty([0, T] \times \mathbb{R})\) satisfy \(d_i \geq 0\) and \(q_{ij} \geq 0\) for \(i \neq j\). Then
\[(2.4) \quad v_i(t, x) \geq 0 \text{ for } (t, x) \in [0, T] \times \mathbb{R}, 1 \leq i \leq m.
\]

Proof. For any given \(\epsilon > 0\), define
\[
w_i(t, x) := v_i(t, x) + \epsilon e^{At} \quad \text{for } (t, x) \in [0, T] \times \mathbb{R},
\]
Clearly, since if $A > \min_{t,x} \lambda \epsilon$ with
\[
\partial_t w_i - d_i \mathcal{L}_i[w_i] - \sum_{j=1}^{m} q_{ij} w_j = \partial_t v_i - d_i \mathcal{L}_i[v_i] - \sum_{j=1}^{m} q_{ij} v_j + \epsilon A e^{At} - \epsilon e^{At} \sum_{j=1}^{n} q_{ij}
\]  
(2.5)
\[
\geq \left( A - \sum_{j=1}^{n} q_{ij} \right) \epsilon e^{At} \geq \epsilon e^{At} \text{ for } (t,x) \in (0,T) \times \mathbb{R}.
\]

We claim that
\[
(2.6) \quad w_i(t,x) > 0 \text{ for } (t,x) \in [0,T] \times \mathbb{R}, 1 \leq i \leq m.
\]
Define
\[
T_0 = \sup \{ s \in (0,T) : w_i(t,x) > 0 \text{ for all } (t,x) \in [0,s] \times \mathbb{R}, 1 \leq i \leq m \}.
\]
Clearly, $T_0 > 0$ is well defined since $w_i(0,x) \geq \epsilon$ for $x \in \mathbb{R}$ and $\partial_t w_i$ has a finite lower bound due to (2.5). Moreover, the definition of $T_0$ implies that either (2.6) holds or
\[
(2.7) \quad \begin{cases} w_i(t,x) > 0 & \text{on } [0,T_0) \times \mathbb{R} \text{ for } 1 \leq i \leq m, \\ A := \min_{1 \leq i \leq m} \inf_{x \in \mathbb{R}} w_i(T_0, x) = 0, \end{cases}
\]

since if $A > 0$ in (2.7), then in view of the assumption that (2.7) does not hold, we must have $T_0 < T$, but then we could apply the fact that $\partial_t w_i$ is bounded below to deduce that for some small $\tilde{\epsilon} > 0$,
\[
w_i(t,x) > A/2 > 0 \quad \text{for } (t,x) \in [T_0, T_0 + \tilde{\epsilon}) \times \mathbb{R}, 1 \leq i \leq m,
\]
which contradicts the definition of $T_0$.

From $A = 0$ we can find $1 \leq i_0 \leq m$ such that $\inf_{x \in \mathbb{R}} w_{i_0}(T_0, x) = 0$. Then for $0 < \epsilon_n \ll 1$ with $\epsilon_n \to 0$ as $n \to \infty$, we can find $x_n \in \mathbb{R}$ such that
\[
w_{i_0}(T_0, x_n) < \epsilon_n.
\]
Clearly
\[
\inf_{(t,x) \in [0,T_0] \times \mathbb{R}} w_{i_0}(t,x) = 0.
\]
Making use of Ekeland’s variational principle [18], for $\lambda = \min\{ T_0/2, 1 \}$ and each $n \geq 1$, there is $(\tilde{t}_n, \tilde{x}_n) \in [0,T_0] \times \mathbb{R}$ such that
\[
\begin{cases} w_{i_0}(\tilde{t}_n, \tilde{x}_n) \leq w_{i_0}(T_0, x_n) < \epsilon_n, \\ |T_0 - \tilde{t}_n| + |x_n - \tilde{x}_n| < \lambda, \\ w_{i_0}(t, \tilde{x}_n) - w_{i_0}(t, \tilde{x}_n) < \frac{|\lambda - \epsilon_n|}{\lambda} \epsilon_n \quad \text{for any } t \in (0,T_0), 
\end{cases}
\]
It follows that (note that $\tilde{t}_n > T_0 - \lambda > T_0/2 > 0$)
\[
(2.8) \quad \partial_t w_{i_0}(\tilde{t}_n, \tilde{x}_n) \leq \frac{\epsilon_n}{\lambda} \text{ for all } n \geq 1.
\]

On the other hand, by (2.5) and $w_i \geq 0$ on $[0,T_0] \times \mathbb{R}$ we deuce
\[
\partial_t w_{i_0}(\tilde{t}_n, \tilde{x}_n) \geq d_{i_0} \mathcal{L}_{i_0}[w_{i_0}](\tilde{t}_n, \tilde{x}_n) + \sum_{j=1}^{m} q_{i_0j} w_j(\tilde{t}_n, \tilde{x}_n) + \epsilon e^{At} 
\[
\geq - d_{i_0} w_{i_0}(\tilde{t}_n, \tilde{x}_n) + q_{i_0i_0} w_{i_0}(\tilde{t}_n, \tilde{x}_n) + \epsilon e^{At} 
\geq - (\|d_{i_0}\|_{\infty} + \|q_{i_0i_0}\|_{\infty}) \epsilon_n + \epsilon e^{At} 
\geq \frac{1}{2} \epsilon e^{At} \quad \text{for all large } n,
\]
which is a contradiction to (2.8) since $\epsilon_n \to 0$ as $n \to \infty$.

Hence (2.6) always holds true. Letting $\epsilon \to 0$, we immediately get (2.4). \qed
Define an operator $P$

Lemma 2.1. Then the problem Lemma 2.6. Suppose Lemma 2.3 can be relaxed. If for each $(t, x)$, both the one-sided partial derivatives $\partial_t v_i(t - 0, x)$ and $\partial_x v_i(t - 0, x)$ exist, and the differential inequalities are satisfied when $\partial_t v_i(t, x)$ is replaced by both one-sided partial derivatives, then the conclusion of Lemma 2.3 remains valid. This also applies to the comparison lemmas in the rest of this paper, though we will not remark again after them.

Making use of (f$_1$) and Lemma 2.3, we have the following result.

Lemma 2.5. Assume that (J) and (f$_1$) hold, $T > 0$ and $d_i \geq 0$ are constants, and $U = (u_i)$ with $u_i \in C([0, T] \times \mathbb{R})$, $\partial_i u_i \in C([0, T] \times \mathbb{R})$ $(i = 1, ..., m)$ solves (1.3) for $t \in [0, T]$. If for every $i \in \{1, ..., m\}$, $v_i \in C([0, T] \times \mathbb{R}) \cup L^\infty([0, T] \times \mathbb{R})$ and $\partial_t v_i \in C([0, T] \times \mathbb{R})$ satisfy

$$\begin{aligned} &\partial_t v_i \geq d_i \mathcal{L}_i[v_i] + f(v_1, v_2, ..., v_m), \quad (t, x) \in (0, T] \times \mathbb{R}, \\
&v_i(0, x) \geq u_i(0, x), \quad x \in \mathbb{R},
\end{aligned}$$

where

$$\mathcal{L}_i[v_i](t, x) := \int_{\mathbb{R}} J_i(x - y)v_i(t, y)dy - v_i(t, x),$$

then

$$v_i(t, x) \geq u_i(t, x) \quad \text{for} \quad (t, x) \in [0, T] \times \mathbb{R}.$$  

2.2. A perturbed semi-wave problem. For $\delta \gg 0$, we consider the auxiliary problem

(2.9) \quad \begin{cases} 
D \circ \int_{-\infty}^{\infty} J(x - y) \Phi(y)dy - D \circ \Phi + c\Phi'(x) + F(\Phi(x)) = 0, & -\infty < x < 0, \\
\Phi(-\infty) = u^*, \quad \Phi(x) = \delta, & 0 \leq x < \infty. 
\end{cases}

If $\delta = 0$ then (2.9) is reduced to the semi-wave problem (1.4); therefore (2.9) can be viewed as a perturbed semi-wave problem. As we will see below, the semi-wave solutions and traveling wave solutions of (1.3) can be obtained as the limit of the solution of (2.9) when $\delta \to 0$, subject to suitable translations in $x$.

Define

(2.10) \quad \tilde{\sigma}(v) := F(v) + cMv - D \circ v = (f_1(v) + (cM - d_1)v_1) \quad \text{for} \quad v = (v_i) \in \mathbb{R}^m_+,

where $M > 0$ is a constant. Then the first equation in (2.9) is equivalent to

(2.11) \quad -c(e^{-Mx} \Phi)' = e^{Mx} \left( D \circ \int_{-\infty}^{\infty} J(x - y) \Phi(y)dy + \tilde{\sigma}(\Phi(x)) \right).

Since $F$ is $C^1$, we could choose $M$ large enough such that $\tilde{\sigma}(v)$ is increasing for $v \in [0, u^* + 1]$, namely

$$\tilde{\sigma}(v) \geq \tilde{\sigma}(u) \quad \text{if} \quad u, v \in [0, u^* + 1] \quad \text{and} \quad v \geq u.$$

Lemma 2.6. Suppose (J) and (f$_1$) hold. Let $\delta = \epsilon \Theta$ for small $\epsilon > 0$, where $\Theta$ is given by Lemma 2.7. Then the problem (2.9) has a solution $\Phi(x) = (\phi_i(x))$ which is nonincreasing in $x$, and can be obtained by an iteration process to be specified in the proof.

Proof. Let

$$\Omega := \{\Gamma \in [C(\mathbb{R})]^m : 0 \leq \Gamma \leq u^*\}.$$ 

Define an operator $P = (P_i) : \Omega \to [C(\mathbb{R})]^m$ by

$$P[\Gamma](x) = \begin{cases} 
e^{Mx}_- \delta + \frac{e^{Mx}}{e} \int_x^0 e^{-M \xi} \left[ D \circ \int_{-\infty}^{\infty} J(\xi - y) \Gamma(y)dy + \tilde{\sigma}(\Gamma(\xi)) \right] d\xi, & x < 0, \\
\delta, & x \geq 0. 
\end{cases}$$
Using (2.11) we easily see that (2.9) is equivalent to

\[
\begin{align*}
\Phi(x) &= P[\Phi](x) \text{ for } x \in \mathbb{R}, \\
\Phi(-\infty) &= u^*.
\end{align*}
\]

We next solve (2.12) in three steps.

**Step 1** We show that \(P\) has a fixed point in \(\Omega\).

Firstly we prove that \(P[\delta](x) \geq \delta\) with \(\delta\) regarded as a constant function. By the definition of \(P\), we have \(P[\delta](x) = \delta\) for \(x \geq 0\). For \(x < 0\),

\[
P[\delta](x) = e^{Mx} \delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [D \circ \delta + \tilde{\sigma}(\delta)] \, d\xi
\]

\[
= e^{Mx} \delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [cM \delta + F(\delta)] \, d\xi
\]

\[
\approx e^{Mx} \delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [cM \delta] \, d\xi
\]

\[
= e^{Mx} \delta - e^{Mx} \delta + \delta = \delta
\]

since \(F(\delta) \geq 0\) by Lemma 2.1.

Secondly we show \(P[u^*](x) \not\leq u^*\). Since \(\epsilon > 0\) is small, \(P[u^*](x) = \delta = \epsilon \Theta \not\leq u^*\) for \(x \geq 0\).

For \(x < 0\), we have

\[
P[u^*](x) = e^{Mx} \delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [D \circ u^* + \tilde{\sigma}(u^*)] \, d\xi
\]

\[
= e^{Mx} \delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [cM u^* + F(u^*)] \, d\xi
\]

\[
= e^{Mx} \delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [cM u^*] \, d\xi
\]

\[
\ll e^{Mx} u^* + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [cM u^*] \, d\xi
\]

\[
= e^{Mx} u^* - e^{Mx} u^* + u^* = u^*.
\]

Next we define inductively

\[
\Gamma_0(x) := \delta, \quad \Gamma_{n+1}(x) := P[\Gamma_n](x) = P^n[\Gamma_0](x) \quad \text{for} \quad n = 0, 1, 2, \ldots, x \in \mathbb{R}.
\]

Then

\[
\Gamma_0 \leq \Gamma_n \leq \Gamma_{n+1} \ll u^*
\]

due to the monotonicity of \(P\) which is a simple consequence of the fact that \(\tilde{F}(v)\) is increasing in \(v \in [0, u^*]\).

Define

\[
\widehat{\Gamma}(x) := \lim_{n \to \infty} \Gamma_n(x) \in [0, u^*].
\]

It is clear that \(\widehat{\Gamma}(x) = \delta\) for \(x \geq 0\). Making use of the Lebesgue dominated convergence theorem and \(\Gamma_{n+1}(x) = P[\Gamma_n](x)\), for \(x < 0\) we deduce

\[
\widehat{\Gamma}(x) = P[\widehat{\Gamma}](x),
\]

which also implies that \(\widehat{\Gamma}'(x)\) exists and is continuous for \(x < 0\). Hence \(\widehat{\Gamma}\) is a fixed point of \(P\) in \(\Omega\).

**Step 2.** We show that \(\widehat{\Gamma}'(x) \leq 0\) for \(x < 0\).

It suffices to prove that \(\Gamma_n'(x) \leq 0\) for \(x < 0\) and each \(n = 0, 1, 2, \ldots\), since this would imply each \(\Gamma_n\) is nonincreasing and hence \(\widehat{\Gamma}(x)\) is nonincreasing for \(x < 0\).
It is clear that $\Gamma_0(x) = \delta$ is nonincreasing. Assume $\Gamma'_n(x) \leq 0$ for $x < 0$. We show that $\Gamma'_n(x) \leq 0$ for $x < 0$.

By the definition, for $x < 0$,

$$\Gamma_{n+1}(x) = e^{Mx} \delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} g_n(\xi) d\xi,$$

where

$$g_n(\xi) = g(\xi; \Gamma_n) := D \circ \int_{-\infty}^\infty J(\xi - y) \circ \Gamma_n(y) dy + \tilde{\sigma}(\Gamma_n(\xi))$$

$$= D \circ \int_{-\infty}^\infty J(\xi + \xi) \circ \Gamma_n(y + \xi) dy + \tilde{\sigma}(\Gamma_n(\xi))$$

Let us note that $\Gamma'_n(z) \leq 0$ for $z \neq 0$, and every element of the matrix function $\nabla \tilde{\sigma}(z)$ is nonnegative for $z \in [\mathbb{R}^+]^m$. It follows that $g_n(\xi)$ is differentiable for all $\xi \in \mathbb{R}$, and $g'_n(\xi) \leq 0$ for $\xi \in \mathbb{R}$. Moreover,

$$g_n(0) = g(0; \Gamma_n) \geq 0 \geq g(0; \Gamma_0) = D \circ \delta + \tilde{\sigma}(\delta) = cM\delta + F(\delta) \geq cM\delta,$$

since $\Gamma_n \geq \Gamma_0 = \delta$, $F(\delta) \gg 0$, and $g(0; \Gamma_n)$ is nondecreasing with respect to $\Gamma_n$. Therefore, for $x < 0$,

$$(\Gamma_{n+1})'(x) = \delta M e^{Mx} + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} g_n(\xi) d\xi - \frac{1}{c} g_n(x)$$

$$= \delta M e^{Mx} + \frac{e^{Mx}}{c} \left[-\frac{e^{-M\xi}}{M} g_n(\xi) \bigg|_0^0 + \int_x^0 e^{-M\xi} g_n'(\xi) d\xi\right] - \frac{1}{c} g_n(x)$$

$$\leq \delta M e^{Mx} + \frac{e^{Mx}}{c} \left[-g_n(0) + \frac{e^{-Mx}}{M} g_n(x)\right] - \frac{1}{c} g_n(x)$$

$$= \delta M e^{Mx} - \frac{g_n(0) e^{Mx}}{c} \leq \delta M e^{Mx} - cM e^{Mx} = 0.$$

By the principle of mathematical induction, we have $\Gamma'_n(x) \leq 0$ for $x < 0$ and all $n \geq 1$.

**Step 3.** We verify $\hat{\Gamma}(-\infty) = u^*$. By step 2, $\lim_{x \to -\infty} \hat{\Gamma}(x) = K$ exists, and $0 \ll K \ll u^*$. We claim that

$$\lim_{x \to -\infty} \int_{-\infty}^\infty J(x - y) \circ \hat{\Gamma}(y) dy = K. \quad (2.13)$$

Indeed, since $\hat{\Gamma}$ is nonincreasing and $\lim_{x \to -\infty} \hat{\Gamma}(x) = K$, we have

$$\int_{-\infty}^\infty J(x - y) \circ \hat{\Gamma}(y) dy = \int_{-\infty}^\infty J(y) \circ \hat{\Gamma}(y + x) dy$$

$$\geq \int_{-\infty}^{-x/2} J(y) \circ \hat{\Gamma}(y + x) dy \geq \hat{\Gamma}(x/2) \circ \int_{-\infty}^{-x/2} J(y) dy \to K$$

as $x \to -\infty$, and on the other hand

$$\int_{-\infty}^\infty J(x - y) \circ \hat{\Gamma}(y) dy = \int_{-\infty}^\infty J(y) \circ \hat{\Gamma}(y + x) dy$$

$$\leq \int_{-\infty}^\infty J(y) \circ K dy = K,$$

which gives (2.13).

If $K \neq u^*$, then by (f1), we have $F(K) = (f_1(K)) \neq 0$. Note that $\hat{\Gamma}$ satisfies

$$D \circ \int_{-\infty}^\infty J(x - y) \circ \hat{\Gamma}(y) dy - D \circ \hat{\Gamma} + c\hat{\Gamma}'(x) + F(\hat{\Gamma}(x)) = 0, \quad -\infty < x < 0.$$
Letting $x \to -\infty$ and making use of (2.13), we deduce
\[
\lim_{t \to -\infty} \hat{\Gamma}'(x) = \lim_{t \to -\infty} F(\hat{\Gamma}(x)) = F(K) \neq 0,
\]
which contradicts the fact that $\hat{\Gamma}$ is nonincreasing and bounded. Thus, $\hat{\Gamma}(-\infty) = u^*.$

Combining Steps 1-3, we see that (2.9) admits a nonincreasing solution $\hat{\Gamma}$, which is the limit of $\Gamma_n$ obtained from an iteration process. \hfill $\square$

The following result describes the monotonic dependence on $c$ and $\delta = \epsilon \Theta$ of the solution $\Phi$ to (2.9) obtained in the above lemma. To stress these dependences, we will write $\Phi = \Phi_c^\epsilon$.

**Lemma 2.7.** Suppose (J) and (f1) hold. Let $\Phi_c^\epsilon$ be the solution of (2.9) obtained through the iteration process in Lemma 2.6, with $c > 0$ and $\delta = \epsilon \Theta$. Then
\[
\Phi_{c_1}^\epsilon \leq \Phi_{c_2}^\epsilon \quad \text{if} \quad 0 < c_1 \leq c_2 \leq 1,
\]
\[
\Phi_{c_1}^\epsilon \geq \Phi_{c_2}^\epsilon \quad \text{if} \quad 0 < c_1 \leq c_2.
\]

**Proof.** To verify the first inequality in (2.14) for fixed $c > 0$, we adopt the definition of $P$ and $\Phi_n$ in Lemma 2.6 but in order to distinguish them between $\delta = \epsilon_1 \Theta$ and $\delta = \epsilon_2 \Theta$, we write $P = P_i$ and $\Phi_n = \Phi_{i,n}$ for $\delta = \epsilon_i \Theta$, $i = 1, 2$. Thus we have
\[
\Phi_{c_1}^\epsilon(x) = \lim_{n \to \infty} \Phi_{i,n}(x).
\]

Since $P[\Phi](x)$ is nondecreasing with respect to $\delta$ and $\Phi$, respectively, we have
\[
\Phi_{1,n+1}(x) = P_1[\Phi_{1,n}](x) \leq P_1[\Phi_{2,n}](x) \leq P_2[\Phi_{2,n}](x) = \Phi_{2,n+1}(x)
\]
provided that
\[
\Phi_{1,n}(x) \leq \Phi_{2,n}(x).
\]

Since $\Phi_{1,0}(x) \equiv \epsilon_1 \Theta \leq \epsilon_2 \Theta \equiv \Phi_{2,0}(x)$, the above conclusion combined with the induction method gives $\Phi_{1,n}(x) \leq \Phi_{2,n}(x)$ for all $n = 0, 1, 2, \cdots$, which implies $\Phi_{c_1}^\epsilon(x) \leq \Phi_{c_2}^\epsilon(x)$, as desired.

We now show the second inequality in (2.14) for fixed $\delta = \epsilon \Theta$. To stress the reliance on $c_i$, we use the notions $P_i^1$ and $\Phi_{i,n}^1$ respectively, for $P$ and $\Phi$ when $c = c_i$, $i = 1, 2$. From Lemma 2.6 we have for $i = 1, 2$,
\[
\Phi_{c_i}^\epsilon(x) = \lim_{n \to \infty} \Phi_{i,n}^1(x) = \lim_{n \to \infty} P_i^1[\Phi_{i,n}^1](x).
\]

Due to $c_1 \leq c_2$ and (2.9), we have
\[
D \circ \int_{-\infty}^{\infty} J(x-y) \circ \Phi_{c_1}^\epsilon(y)dy - D \circ \Phi_{c_1} + c_2(\Phi_{c_1}^\epsilon)'(x) + F(\Phi_{c_1}^\epsilon(x)) \geq 0,
\]
which implies that
\[
\Phi_{c_1}^\epsilon(x) \geq P_2[\Phi_{c_1}^\epsilon](x).
\]

Since $P[\Phi](x)$ is increasing with respect to $\Phi$, it follows that
\[
\Phi_{c_1}^\epsilon(x) \geq P_2[\Phi_{c_1}^\epsilon](x) \geq P_2[\Phi_{n+1}^\epsilon](x) = \Phi_{n+1}^\epsilon(\cdot)
\]
provided that
\[
\Phi_{c_1}^\epsilon(x) \geq \Phi_{n}^\epsilon(\cdot).
\]

Recall that $\Phi_{c_1}^\epsilon(x) \geq \delta \equiv \Phi_{2}^\epsilon(\cdot)$. By induction, we obtain that $\Phi_{c_1}^\epsilon(x) \geq \Phi_{n}^\epsilon(\cdot)$ for all $n = 0, 1, 2, \cdots$, and so $\Phi_{c_1}^\epsilon(x) \geq \Phi_{n}^\epsilon(\cdot)$.
In later analysis of the paper, we will need the following variant of (2.9):

\[ (2.15) \begin{cases} D \circ \int_{-\infty}^{\infty} K(x-y) \circ \Psi(y) dy - D \circ \Psi + c\Psi'(x) + F(\Psi(x)) = 0, & -\infty < x < 0, \\
\Psi(-\infty) = \hat{u}^*, \quad \Psi(x) = \delta, & 0 \leq x < \infty, \end{cases} \]

where \( \delta = \epsilon \Theta \) as in (2.9), \( K(x) = (K_i(x)) \) satisfies (J) except that \( \int_{\mathbb{R}} K_i(x) dx = 1 \) \( (i = 1, \ldots, m) \) is now replaced by

\[ 1 - \epsilon \leq \int_{\mathbb{R}} K_i(x) dx \leq 1, \text{ i.e., } \|K_i\|_{L^\infty(\mathbb{R})} \in [1 - \epsilon, 1], \]

and \( \hat{u}^* \) is the positive constant equilibrium of the first equation in (2.15).

**Lemma 2.8.** Suppose that \((f_1)\) and \((f_3)\) hold. Then for all small \( \epsilon > 0 \), problem (2.15) has a solution \( \Psi(x) = (\psi_i(x)) \) which is nonincreasing in \( x \), and can be obtained by an interaction process. Moreover, if \( K(x) \leq J(x) \) for all \( x \in \mathbb{R} \), then

\[ \hat{u}^* \leq u^* \text{ and } \Psi(x) \leq \Phi(x) \text{ for } x \in \mathbb{R}, \]

where \( \Phi \) is the solution of (2.9) obtained in Lemma 2.6.

**Proof.** Define

\[ \hat{D} = (\hat{d}_i) \text{ with } \hat{d}_i := d_i\|K_i\|_{L^1(\mathbb{R})}, \quad \hat{K} = (\hat{K}_i) \text{ with } \hat{K}_i := \frac{1}{\|K_i\|_{L^1(\mathbb{R})}} K_i, \]

\[ \hat{P}(u) := F(u) - (\hat{D} - D) \circ u \text{ for } u \in \mathbb{R}^m. \]

Then clearly \( \hat{K} \) satisfies all the conditions in (J), and (2.15) can be rewritten as

\[ (2.16) \begin{cases} \hat{D} \circ \int_{-\infty}^{\infty} \hat{K}(x-y) \circ \Psi(y) dy - \hat{D} \circ \Psi + c\Psi'(x) + \hat{P}(\Psi(x)) = 0, & -\infty < x < 0, \\
\Psi(-\infty) = \hat{u}^*, \quad \Psi(x) = \delta, & 0 \leq x < \infty. \end{cases} \]

From \((f_1)\) and \((f_3)\) we easily see that for all sufficiently small \( \epsilon > 0 \), \( \hat{u}^* \) is the unique solution of \( \hat{P}(u) = 0 \) in a small neighbourhood of \( u^* \). Hence \( \hat{P} \) satisfies \((f_1)\) with \( \hat{u}^* \) in place of \( u^* \). The existence of \( \Psi \) now follows from the proof of Lemma 2.6 applied to (2.16). More precisely, let

\[ \hat{\Omega} := \{ \Gamma \in [C(\mathbb{R})]^m : 0 \leq \Gamma \leq \hat{u}^* \}, \]

and define \( \hat{P} = (\hat{P}_i) : \hat{\Omega} \to [C(\mathbb{R})]^m \) by

\[ \hat{P}[\Gamma](x) = \begin{cases} e^{Mx} \delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} \left[ D \circ \int_{-\infty}^{\infty} K(y) \circ \Gamma(y) dy + \tilde{\sigma}(\Gamma(\xi)) \right] d\xi, & x < 0, \\
\delta, & x \geq 0, \end{cases} \]

or equivalently,

\[ \hat{P}[\Gamma](x) = \begin{cases} e^{Mx} \delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} \left[ \hat{D} \circ \int_{-\infty}^{\infty} \hat{K}(y) \circ \Gamma(y) dy + \tilde{\sigma}(\Gamma(\xi)) \right] d\xi, & x < 0, \\
\delta, & x \geq 0, \end{cases} \]

where \( \tilde{\sigma} \) is given by

\[ \tilde{\sigma}(v) := F(v) + cMv - D \circ v = \hat{P}(v) + cMv - \hat{D} \circ v \text{ for } v \in \mathbb{R}^m, \]

with \( M > 0 \) large enough such that \( \tilde{\sigma}(v) \) is increasing in \( v \in [0, \hat{u}^* + 1] \). Then

\[ \Psi(x) = \lim_{n \to \infty} \hat{P}^n[\delta](x). \]

Now suppose additionally \( K(x) \leq J(x) \) for \( x \in \mathbb{R} \). Then

\[ \hat{P}(u) \leq P(u) \text{ for all } u \geq 0, \]
which together with the monotonicity of $\hat{P}(u)$ and $P(u)$ in $u$ yields

$$\hat{P}^n[\delta] \leq P^n[\delta], \quad n = 0, 1, 2, \ldots.$$ 

Thus, $\Psi \preceq \Phi$, which implies $\hat{u}^* \preceq u^*$.

## 2.3. A dichotomy between semi-waves and traveling waves.

**Theorem 2.9.** Suppose (J), (f₁), (f₂) and (f₄) hold. Then for each $c > 0$, (1.3) has either a monotone semi-wave solution with speed $c$ or a monotone traveling wave solution with speed $c$, but not both. Moreover, one of the following holds:

1. For every $c > 0$, (1.3) has a monotone semi-wave solution with speed $c$.
2. There exists $C_* \in (0, \infty)$ such that (1.3) has a monotone semi-wave solution with speed $c$ for every $c \in (0, C_*)$, and has a monotone traveling wave solution with speed $c$ for every $c \geq C*$.

We prove Theorem 2.9 by the following two lemmas.

**Lemma 2.10.** Suppose (J), (f₁) and (f₂) hold. Then for each $c > 0$, (1.3) has either a monotone semi-wave solution with speed $c$ or a monotone traveling wave solution with speed $c$, but not both.

**Proof.** Let $\Phi^c_n = (\phi^c_{n,i})$ be the solution of (2.9) defined in Lemma 2.6 with $\delta = \delta_n := \epsilon_n \Theta$, $\epsilon_n \downarrow 0$ as $n \to \infty$. Then

$$x^c_n := \max \{ x : \phi^c_{n,1}(x) = u^*_{c} / 2 \}$$

is well defined, and

$$\phi^c_{n,1}(x^c_n) = u^*_{c} / 2, \quad \phi^c_{n,1}(x) < u^*_{c} / 2 \quad \text{for } x > x^c_n.$$ 

Moreover, making use of Lemma 2.7 we have

$$0 > x^c_n \geq x^c_{n,m} \quad \text{if } n \leq m,$$

$$0 > x^c_{n1} \geq x^c_{n2} \quad \text{if } c_1 \leq c_2.$$ 

Define

$$\Phi^c_n(x) := \Phi^c_n(x + x^c_n), \quad x \in R.$$ 

Then $\Phi^c_n$ satisfies, for $x < -x^c_n$,

$$D \circ \int_{-\infty}^{\infty} J(x-y) \circ \Phi^c_n(y) \, dy - D \circ \Phi^c_n(x) + c(\Phi^c_n)'(x) + F(\Phi^c_n(x)) = 0,$$

and for $x \geq -x^c_n$, $\Phi^c_n(x) = \delta_n$. Moreover,

$$\Phi^c_{n,1}(0) = u^*_{c} / 2,$$

where $\Phi^c_{n,1}$ is the first element of the vector function $\Phi^c_n$. Since $x^c_n$ is nonincreasing in $n$,

$$x^c := -\lim_{n \to \infty} x^c_n \in (0, \infty]$$

always exists, and there are two possible cases

- Case 1. $x^c = \infty$
- Case 2. $x^c \in (0, \infty)$.

Clearly, for fixed $c > 0$, $\Phi^c_n$ and by the equation subsequently $(\Phi^c_n)'$ (for $x \neq -x^c_n$) are uniformly bounded in $n$. Then by the Arzela-Ascoli theorem and a standard argument involving a diagonal process of choosing subsequences, we see that $\{\Phi^c_n\}_{n \geq 1}$ has a subsequence, still denoted by itself for simplicity of notation, which converges to some $\Phi^c = (\phi^c) \in C(\mathbb{R})$ locally uniformly in $\mathbb{R}$. Moreover, $\Phi^c(x)$ is nonincreasing in $x$ with $\phi^c_1(0) = u^*_{c} / 2$. 

If Case 1 happens, we easily see that $\Phi^c$ satisfies
\[(2.19) \quad D \circ \int_{-\infty}^{\infty} \mathbf{J}(x-y) \circ \Phi^c(y) dy - D \circ \Phi^c(x) + c(\Phi^c)'(x) + F(\Phi^c(x)) = 0 \quad \text{for} \quad x \in \mathbb{R}.
\]

In fact, from (2.18), for $x \in \mathbb{R}$ and all large $n$ satisfying $x < -x_n^c$, we have
\[c\Phi_n^c(x) - c\Phi_n^c(0) = -D \circ \int_0^x \left[ \int_{-\infty}^{\infty} \mathbf{J}(\xi - y) \circ \Phi_n^c(y) dy - D \circ \Phi_n^c(\xi) + F(\Phi_n^c(\xi)) \right] d\xi.
\]

It then follows from the dominated convergence theorem that, for $x \in \mathbb{R}$,
\[c\Phi^e(x) - c\Phi^e(0) = -D \circ \int_0^x \left[ \int_{-\infty}^{\infty} \mathbf{J}(\xi - y) \circ \Phi^e(y) dy - D \circ \Phi^e(\xi) + F(\Phi^e(\xi)) \right] d\xi,
\]

and (2.19) thus follows by differentiating this equation. Due to the monotonicity and boundedness of $\Phi^e(x)$, the arguments in step 3 of the proof of Lemma 2.6 can be repeated to give
\[
\lim_{x \to -\infty} \left[ D \circ \int_{-\infty}^{\infty} \mathbf{J}(x-y) \circ \Phi^e(y) dy - D \circ \Phi^e(x) \right] = 0,
\]
and so
\[
\lim_{x \to -\infty} [c(\Phi^e)'(x) + F(\Phi^e(x))] = 0.
\]

Denote $K := \lim_{x \to -\infty} \Phi^e(x) \in \mathbb{R}^m_+$. Then we must have
\[F(K) = \lim_{x \to -\infty} F(\Phi^e(x)) = -\lim_{x \to -\infty} c(\Phi^e)'(x).
\]

This is possible only if $F(K) = 0$. By (f1) either $K = 0$ or $K = u^*$. Since $\Phi^e(x)$ is nonincreasing in $x$ with $\tilde{\Phi}^e_1(0) = u^*_1/2 > 0$, we have $K > 0$ and hence we must have $K = u^*$. An analogous analysis can be applied to show $\lim_{x \to -\infty} \tilde{\Phi}^e(x) = 0$. Therefore, $\Phi^e(x)$ is a monotone traveling wave of (1.3) with speed $c$.

If Case 2 happens, analogously for fixed $x < x^c$,
\[c\Phi^e(x) - c\Phi^e(0) = -D \circ \int_0^x \left[ \int_{-\infty}^{\infty} \mathbf{J}(\xi - y) \circ \Phi^e(y) dy - D \circ \Phi^e(\xi) + F(\Phi^e(\xi)) \right] d\xi,
\]
and $\Phi^e(x) = 0$ for $x \geq x^c$, which yields
\[
\begin{cases}
D \circ \int_{-\infty}^{x^c} \mathbf{J}(x-y) \circ \Phi^e(y) dy - D \circ \Phi^e(x) + c(\Phi^e)'(x) + F(\Phi^e(x)) = 0 \quad \text{for} \quad x < x^c, \\
\Phi^e(x^c) = 0.
\end{cases}
\]

Let $\Phi^c(x) := \Phi^e(x + x^c)$ for $x \leq 0$, then $\Phi^c(x)$ satisfies
\[
\begin{cases}
D \circ \int_{-\infty}^{0} \mathbf{J}(x-y) \circ \Phi^e(y) dy - D \circ \Phi^e(x) + c(\Phi^e)'(x) + F(\Phi^e(x)) = 0 \quad \text{for} \quad x < 0, \\
\Phi^c(0) = 0.
\end{cases}
\]

Moreover, as in Case 1, we can show $\lim_{x \to -\infty} \Phi^c(x) = u^*$. Therefore, $\Phi^c(x)$ is a monotone semi-wave solution of (1.3) with speed $c$.

We have thus proved that for any $c > 0$, (1.3) has either a monotone traveling wave solution with speed $c$ or a monotone semi-wave solution with speed $c$. We show next that for any given $c > 0$, (1.3) cannot have both.

Suppose, on the contrary, there is $c_0 > 0$ such that (1.3) admits a monotone traveling wave solution $\Psi = (\psi_i)$ with speed $c_0$ and also a monotone semi-wave solution $\Phi = (\phi_i)$ with speed $c_0$. We are going to drive a contradiction.
Let $\tilde{\Phi}(x) := k\Phi(x)$ for some fixed $k \in (0, 1)$. Then by (f2), $\tilde{\Phi} = (\tilde{\phi}_i)$ satisfies
\[
\begin{cases}
D \circ \int_{-\infty}^{\infty} J(x-y) \circ \tilde{\Phi}(y)dy - D \circ \tilde{\Phi}(x) + c \tilde{\Phi}(x) + F(k\Phi(x)) \geq 0, & x < 0, \\
\tilde{\Phi}(-\infty) = ku^*, \quad \tilde{\Phi}(x) = 0, & x \geq 0.
\end{cases}
\]
For $\beta \in \mathbb{R}$, define
\[
\Psi^\beta(x) := \Psi(x + \beta), \quad W^\beta(x) = (w^\beta_i(x)) := \Psi^\beta(x) - \tilde{\Phi}(x), \quad x \in \mathbb{R}.
\]
For fixed $x \leq 0$ and $i \in \{1, ..., m\}$,
\[
w^\beta_i(x) \geq \psi_i(\beta) - k\phi_i(x) \geq \psi_i(\beta) - ku^* \rightarrow (1-k)u^* > 0 \quad \text{as} \quad \beta \rightarrow -\infty.
\]
Therefore there exists $\beta_i \ll 0$ independent of $x$ such that $w^\beta_i(x) > 0 \quad \text{for} \quad x \leq 0, \quad \beta \leq \beta_i$.

On the other hand,
\[
w^\beta_i(-1) = \psi_i(\beta - 1) - k\phi_i(-1) \rightarrow -k\phi_i(-1) < 0 \quad \text{as} \quad \beta \rightarrow -\infty.
\]
Therefore we can find $\beta^*_i \in \mathbb{R}$ such that $h_i(\beta) := \inf_{x \leq 0} w^\beta_i(x) > 0 \quad \text{for} \quad \beta < \beta^*_i, \quad h_i(\beta^*_i) = 0$.

Clearly $w^\beta_i(-\infty) = (1-k)u^* > 0$ and $w^\beta_i(0) = \psi_i^{\beta_*}(0) > 0$. Therefore due to the continuity of $w^\beta_i(x)$ there exists $x^0_i \in (-\infty, 0)$ such that $w^\beta_i(x^0_i) = 0$. We can thus conclude that $w^\beta_i(x) \geq 0 \quad \text{for} \quad x \leq 0, \quad \beta \leq \beta^*_i, \quad \text{and} \quad w^\beta_i(x^0_i) = 0$.

Let $\beta^* = \min\{\beta^*_i, ..., \beta^*_{m}\}$. Then there exists $i_0 \in \{1, ..., m\}$ such that $\beta^* = \beta^*_i$, and hence we have
\[
W^{\beta^*}(x) \geq 0 \quad \text{for} \quad x \leq 0, \quad w^{\beta^*}_{i_0}(x^0_{i_0}) = 0.
\]

By the definition of $\Psi$ and $\Phi$, we see that $W^{\beta^*}$ satisfies
\[
\begin{cases}
D \circ \int_{-\infty}^{\infty} J(x-y) \circ W^{\beta^*}(y)dy - D \circ W^{\beta^*}(x) + cW^{\beta^*}(x) \\
\quad + \left(F(\Psi^{\beta^*}(x)) - F(k\Phi(x))\right) \leq 0, & x < 0, \\
W^{\beta^*}(-\infty) = (1-k)u^* \gg 0, \quad W^{\beta^*}(x) \geq 0, & x \in \mathbb{R}.
\end{cases}
\]
We have
\[
F(\Psi^{\beta^*}(x)) - F(k\Phi(x)) = [W^{\beta^*}(x)]E(x)
\]
with $E(x) = (e_{ij}(x))$ an $m \times m$ matrix given by
\[
e_{ij}(x) = \int_{-\infty}^{0} \partial_i f_j(k\Phi(x) + tW^{\beta^*}(x))dt.
\]
By (f1) we have $e_{ij} \geq 0$ for $i \neq j$. This allows us to use Lemma 2.2 to conclude that $W^{\beta^*}(x) \gg 0$ for $x < 0$, which contradicts the second part of (2.20). This completes the proof. \hfill \Box

**Lemma 2.11.** In Lemma 2.10, if further (f4) is satisfied, then one of the following holds:

(i) For every $c > 0$, \textbf{[1.3]} has a monotone semi-wave solution with speed $c$.

(ii) There exists $C_s \in (0, \infty)$ such that \textbf{[1.3]} has a monotone semi-wave solution with speed $c$ for every $c \in (0, C_s)$, and has a monotone traveling wave solution with speed $c$ for every $c \geq C_s$.

**Proof.** From [2.17] we see that $x^c$ is nonincreasing in $c$ and hence there are three possible cases:

1. For any $c > 0$, $x^c < \infty$. 


(2) There is $C_*>0$ such that $x^c < \infty$ for any $c \in (0, C_*)$, and $x^c = \infty$ for any $c > C_*$. 

(3) For any $c > 0$, $x^c = \infty$.

From the proof of Lemma 2.10, we know that in case (1), (1.3) has a monotone semi-wave with speed $c$ for any $c > 0$, and in case (2), it has a monotone semi-wave with speed $c$ for each $c \in (0, C_*)$, has a monotone traveling wave with speed $c$ for each $c > C_*$. Therefore to complete the proof it suffices to show that case (3) does not happen, and in case (2), (1.3) has a monotone traveling wave solution with speed $c = C_*$. 

We prove the latter first. Let $Ψ^c = (ψ^c_1)$ be a monotone traveling wave solution of (1.3) with speed $c > C_*$. By a suitable translation we may assume $ψ^c_1(0) = u^*_1/2$. Since $Ψ^c$ is uniformly bounded, by the equation satisfied by $Ψ^c$ we see that $(Ψ^c)'$ is also uniformly bounded in $c$ for $c > C_*$. Then by the Arzela-Ascoli theorem and a standard argument involving a diagonal process of choosing subsequences, for any sequence $c_n \searrow C_*$, \{Ψ$^{c_n}$\}$^\infty_{n=1}$ has a subsequence, still denoted by itself, which converges to some $Ψ = (ψ_1) \in [C(\mathbb{R})]^m$ locally uniformly in $\mathbb{R}$ as $n \to \infty$. Similar to the proof of Lemma 2.10, we can check at once that $Ψ$ satisfies 

\[
\begin{cases}
D \circ \int_{-\infty}^{\infty} J(x-y) \circ Ψ(y)dy - D \circ Ψ(x) + C_* Ψ'(x) + F(Ψ(x)) = 0, & x \in \mathbb{R}, \\
ψ_1(0) = u^*_1/2.
\end{cases}
\]

Making use of the monotonicity of $Ψ(x)$ inherited from $Ψ^c_1(x)$, we can use the method in Step 3 of the proof of Lemma 2.6 to show that $Ψ(-\infty) = u^*$, $Ψ(\infty) = 0$, which implies that $Ψ$ is a monotone traveling wave solution of (1.3) with speed $c = C_*$. 

We finally show that case (3) does not happen. Again we argue by contradiction. Suppose $x^c = \infty$ for all $c > 0$. Then from our earlier argument (1.3) has a monotone traveling wave solution $Ψ^c = (ψ^c_1)$ with speed $c$ for any $c > 0$. By a suitable translation, we may assume $ψ^c_1(0) = u^*_1/2$. We choose a sequence \{c_n\} such that $c_n \searrow 0$ as $n \to \infty$ and consider the sequence of monotone functions \{Ψ$^{c_n}$\}. By Helly’s selection theorem and a standard diagonal process of choosing subsequences, \{Ψ$^{c_n}$\} has a subsequence, still denoted by itself, which converges to a nonincreasing function $Ψ = (ψ_1)$ pointwisely in $\mathbb{R}$. Thus $ψ_1(0) = u^*_1/2$ and

\[x(ψ_1(x) - u^*_1/2) \leq 0 \text{ for } x \in \mathbb{R}.
\]

Since $ψ_1$ is nonincreasing, it is continuous in $\mathbb{R}$ except at some countable set (possibly empty) where jumping discontinuities may occur. Therefore we can find an interval $[a, b]$ with $a < b < 0$ such that $ψ_1$ is continuous on $[a, b]$. Then from [10] Lemma 7 we see that $ψ^{c_n}_1(x)$ converges to $ψ^1(x)$ uniformly for $x \in [a, b]$.

We now choose a function $γ_1 \in C(\mathbb{R})$ satisfying $0 \leq γ_1(x) \leq u^*_1/4$ in $\mathbb{R}$ and whose support is exactly $[a, b]$. Then by the above properties of $ψ_1$ we have

\[γ_1(x) \leq \frac{1}{2} ψ_1(x) \text{ for } x \in \mathbb{R},
\]

and hence, due to the uniform convergence of $ψ^{c_n}_1(x)$ to $ψ_1(x)$ over $[a, b]$, there is large $N > 0$ such that

\[γ_1(x) \leq ψ^{c_n}_1(x) \text{ for } x \in \mathbb{R}, \ n > N.
\]

Define $U_0 = (v_i) \in [C(\mathbb{R})]^m$ with $v_1(x) = γ_1(x)$ and $v_i(x) ≡ 0$ for $i \neq 1$. Let $U(t, x) = (u_i(t, x))$ be the solution of (1.3) with $U(0, x) = U_0(x)$. Then we have

\[(2.21) \quad U(0, x) = U_0(x) ≤ Ψ^{c_n}_1(x) \text{ for } x \in \mathbb{R}, \ n > N,
\]

and by (f$_4$),

\[\lim_{t \to \infty} U(t, x) = u^* \text{ for } x \in \mathbb{R}.
\]
It follows that
\begin{equation}
(2.22) \quad u_1(t, x_1) \geq \frac{2u_1^*}{3} \text{ for all large } t, \text{ say } t \geq T_1 > 0,
\end{equation}
where \( x_1 = \frac{a+b}{2} \in [a, b] \).

On the other hand, clearly \( U^n(t, x) := \Psi^n(x - c_n t) \) is the solution of \( (1.3) \) with initial function \( \Psi^c(x) \). Since \( U(0, x) \leq U^n(0, x) \) for \( n > N \) by \( (2.21) \), by Lemma 2.5 we have
\[
U(t, x) \leq U^n(t, x) = \Psi^n(x - c_n t) \quad \text{for } t \geq 0, \ x \in \mathbb{R}, \ n > N.
\]
In particular,
\[
u_1(T_1, x_1) \leq \psi_1^c(x_1 - c_n T_1) \quad \text{for } n > N.
\]
Letting \( n \to \infty \), we obtain
\[
u_1(T_1, x_1) \leq \psi_1(x_1) \leq \psi_1(0) = u_1^*/2,
\]
which contradicts (2.22). The proof is now complete. \( \square \)

2.4. Sharp criteria for the dichotomy in Theorem 2.9. From Theorem 2.9 we see that \( (1.3) \) can have a monotone traveling wave solution with some positive speed \( c \) if and only if alternative (ii) happens, and in that case, \( (1.3) \) has a monotone traveling wave solution with speed \( c \) if and only if \( c \geq C^* \). In this subsection, we prove the following sharp criteria for alternative (ii) to happen.

**Theorem 2.12.** In Theorem 2.9 suppose additionally \( (f_3) \) holds. Then alternative (ii) happens if and only if the kernel functions satisfy \( (J_2) \).

We note that under the assumptions of Theorem 2.9, it is already known in the literature that alternative (ii) happens when condition \((J_2)\) holds; see, for example, Theorems 3.6 and 3.7 of [23]. Therefore, we only need to show that if \((J_2)\) does not hold, then \((1.3)\) does not have a monotone traveling wave solution with some positive speed \( c \). To this end, we first apply the method in [9] to obtain an estimate for any monotone traveling wave solution of \( (1.3) \) when the kernel functions satisfy the following additional condition:
\begin{equation}
(2.23) \quad \int_{\mathbb{R}} |x| J_i(x) dx < \infty \quad \text{for every } i \in \{1, ..., n\}.
\end{equation}

**Lemma 2.13.** Suppose \((J_2)\), \( (2.23) \), and \( (f_1) \) are satisfied. If \((1.3)\) admits a monotone traveling wave solution \( \Psi = (\psi_i) \) with some positive speed \( c \), then there is \( \alpha > 0 \) such that \( \psi_i(x) = O(e^{-\alpha x}) \) as \( x \to \infty \), for every \( i \in \{1, ..., m\} \).

**Proof.** We first show that \( \psi_i(x) \) is integrable on \( [0, \infty) \) for every \( i \in \{1, ..., m\} \).

By \((2.23)\),
\[
\tilde{\psi}(x) := \sum_{i=1}^{m} \tilde{\theta}_i \psi_i(x)
\]
satisfies, for \( y > x \gg 1 \),
\[
c(\tilde{\psi}(x) - \tilde{\psi}(y)) = \sum_{i=1}^{m} \int_{x}^{y} \left[ \tilde{\theta}_i d_i J_i * \psi_i(z) - \tilde{\theta}_i d_i \psi_i(z) + \tilde{\theta}_i f_i(\Psi(z)) \right] dz \geq \sum_{i=1}^{m} \int_{x}^{y} \left[ \tilde{\theta}_i d_i J_i * \psi_i(z) - \tilde{\theta}_i d_i \psi_i(z) + b_i \psi_i(z) \right] dz,
\]
By \((2.23)\) and a direct calculation, we have
\[
\left| \int_{x}^{y} \left( \int_{\mathbb{R}} J_i(z - w) \psi_i(w) dw - \psi_i(z) \right) dz \right| = \int_{x}^{y} \int_{\mathbb{R}} J_i(w)(\psi_i(z + w) - \psi(z)) dw dz
\]
\[
\psi(y) - \psi(x) + \int_{x}^{y} J_i(w) \int_{0}^{1} w \psi_i'(z + sw) ds dw dz \leq u_i^* \int_{R} |y| J_i(y) dy =: M_i < \infty \text{ for } i \in \{1, \ldots, n\}.
\]

Thus, for all \( y > x \gg 1 \),
\[
\sum_{i=1}^{m} \int_{x}^{y} b_i \psi_i(z) dz \leq c(\psi(x) - \psi(y)) + \sum_{i=1}^{n} (\hat{\theta}_i d_i M_i) \leq c\psi(x) + \sum_{i=1}^{n} (\hat{\theta}_i d_i M_i),
\]
which indicates that \( \int_{x}^{\infty} \psi_i(z) dz < \infty \) and hence \( \int_{0}^{\infty} \psi_i(z) dz < \infty \) for \( i \in \{1, \ldots, m\} \). This completes the proof of the desired conclusion.

Moreover, by letting \( y \to \infty \), \( \psi \) satisfies
\[
c\psi(x) \geq \sum_{i=1}^{m} \int_{x}^{\infty} \left[ \hat{\theta}_i d_i J_i + \hat{\theta}_i d_i \psi_i(z) + b_i \psi_i(z) \right] dz,
\]
which, together with the integrability of \( \psi_i \) proved above, implies that \( J_i \psi_i(z) \) is integrable on \( (x, \infty) \) and hence on \( [0, \infty) \).

We also have
\[
\int_{x}^{\infty} [J_i \psi_i(z) - \psi_i(z)] dz = \int_{x}^{\infty} J_i(w) \int_{0}^{1} w \psi_i'(z + sw) ds dw dz
\]
\[
= - \int_{R} w J_i(w) \int_{0}^{1} \psi_i(x + sw) ds dw = - \int_{R} J_i(w) \int_{0}^{w} \psi_i(x + s) ds dw
\]
\[
= \int_{0}^{\infty} w J_i(w) \int_{0}^{w} \left[ \psi_i(x - s) - \psi_i(x + s) \right] ds dw \geq 0
\]
since \( \psi_i(x) \) is nonincreasing in \( x \). It follows that, for \( b_* := \min_{1 \leq i \leq m} b_i / \theta_i > 0, x \gg 1 \) and \( r > 0 \),
\[
c\psi(x) \geq \sum_{i=1}^{m} \int_{x}^{\infty} b_i \psi_i(z) dz \geq b_* \int_{x}^{\infty} \psi(z) dz \geq b_* \int_{x}^{x+r} \psi(z) dz \geq b_* r \psi(x + r).
\]
Let \( \phi(x) := \psi(x)e^{\alpha x} \) with \( \alpha = 1/r \ln(b_* r) / c \) for some large \( r \) such that \( \alpha > 0 \). Then for all large \( x \), say \( x \geq X \gg 1 \),
\[
\phi(x + r) = \psi(x + r)e^{\alpha(x+r)} \leq c b_* r \psi(x)e^{\alpha x} = \psi(x)e^{\alpha x} = \phi(x).
\]
Hence
\[
M^* := \sup_{x \geq X} \phi(x) = \max_{x \in [X, X+r]} \phi(x) < \infty.
\]
Thus \( \psi(x) \leq M^*e^{-\alpha x} \) for \( x \geq X \) which implies \( \psi_i(x) = O(e^{-\alpha x}) \) as \( x \to \infty \) for every \( i \in \{1, \ldots, m\} \).

\[\square\]

**Proof of Theorem 2.12** As mentioned earlier, we only need to show that if \( (J_2) \) does not hold, then \( (1.3) \) does not have a monotone traveling wave solution with some positive speed \( c \).

We argue indirectly. Suppose that \( (J_2) \) does not hold, but \( (1.3) \) has a monotone traveling wave solution \( \Phi = (\phi_i) \) with some positive speed \( c \). We aim to derive a contradiction.

For fixed small \( \epsilon > 0 \), let \( M \) be a large constant such that
\[
\int_{|x| \leq M} J_i(x) dx \geq 1 - \epsilon \text{ for every } i \in \{1, \ldots, n\}.
\]

Define
\[
K_i(x) := \begin{cases}
J_i(x) & \text{for } |x| \leq M, \\
\frac{M}{|x|} J_i(x) & \text{for } |x| \geq M,
\end{cases}
\]
and
\[
\bar{K}_i(x) := \frac{1}{\|K_i\|_{L^1(R)}} K_i(x).
\]
Then
\[(2.24) \quad K_i(x) \leq J_i(x) \quad \text{for} \quad x \in \mathbb{R}, \ 1 \leq i \leq n,\]
and \(\tilde{K}_i (i = 1, \ldots, n)\) satisfy (J) and (2.23).

Since (J2) is not satisfied, for any \(\lambda > 0\),
\[(2.25) \quad \sum_{i=1}^{n} \int_{\mathbb{R}} e^{\lambda x} K_i(x) dx \geq \sum_{i=1}^{n} \int_{M}^{\infty} J_i(x) e^{\lambda x} \frac{dx}{x} \]
provided that \(\tilde{M} > M\) is so large that \(x < e^{\frac{\tilde{M}}{2}} \) for \(\geq \tilde{M}\).

We now show that there is a monotone traveling wave solution to (1.13) with \((J_i)\) replaced by \((K_i)\). To this end, we first consider the perturbed problem (2.9) with kernel functions \(J = (J_i)\) and \(\delta = \epsilon_n \Theta\) as in the proof of Lemma \(2.10\). From that proof and our assumption that (1.3) has a monotone traveling wave solution with speed \(c\), we obtain two sequences \(\{\Phi^c_n\}\) and \(\{x^c_n\}\), with the properties
\[-x^c_n \to \infty \quad \text{and} \quad \Phi^c_n(x) \to \Phi(x) \text{ locally uniformly for } x \in \mathbb{R}\]
as \(n \to \infty\). where \(x^c_n\) is determined by the kernel functions \(J_i\) and a decreasing sequence \({\delta_n}\) of \(n\).

Next we consider the perturbed problem (2.9) with \((J_i)\) replaced by \((K_i)\) and with \(\delta = \epsilon_n \Theta\). By (2.24), Lemmas \(2.3\) and \(2.10\), we can similarly obtain two sequences \(\tilde{\Phi}^c_n\) and \(\{\tilde{x}^c_n\}\) of \(n\), and we have \(\tilde{\Phi}^c_n(x) \leq \Phi^c_n(x)\). Then by the definitions of \(x^c_n\) and \(\tilde{x}^c_n\), we deduce \(\tilde{x}^c_n \leq x^c_n\), and so \(-\tilde{x}^c_n \to \infty\) as \(n \to \infty\). We may now repeat the argument in the proof of Lemma \(2.9\) to see that, by passing to a subsequence, \(\tilde{\Phi}^c_n(x)\) converges locally uniformly to some \(\tilde{\Phi}(x) = (\tilde{\phi}_i)\) in \(\mathbb{R}\), and \(\tilde{\Phi}\) is a monotone traveling wave solution of (1.13) with speed \(c\) but with the kernel functions \((K_i)\) in place of \((J_i)\), as we wanted.

We are now ready to derive a contradiction by making use of Lemma \(2.13\), which implies that \(\tilde{\phi}_i(x) = O(e^{-ax})\) as \(x \to \infty\) for every \(i \in \{1, \ldots, m\}\) and some \(\alpha > 0\). By (2.25), there exists \(j \in \{1, \ldots, n\}\) such that
\[(2.26) \quad \int_{\mathbb{R}} e^{\lambda x} K_j(x) dy = \infty.\]
From (2.24) and the exponential decay estimates for \(\tilde{\phi}_i(x) (i = 1, \ldots, m)\) near \(x = +\infty\) we obtain, for \(\lambda \in (0, \alpha)\),
\[
\left| \int_{\mathbb{R}} e^{\lambda x} f_j(\tilde{\Phi}(x)) dx \right| \leq L \int_{\mathbb{R}} e^{\lambda x} \sum_{i=1}^{m} \tilde{\phi}_i(x) dx < \infty,
\]
and hence by the equation satisfied by \(\tilde{\phi}_j\),
\[
d_j \int_{\mathbb{R}} e^{\lambda x} K_j(x) \tilde{\phi}_j(x) dx = \int_{\mathbb{R}} e^{\lambda x} [d_j \tilde{\phi}_j(x) - c \tilde{\phi}_j'(x) - f_j(\tilde{\Phi}(x))] dx
\]
\[
= (d_j + c\lambda) \int_{\mathbb{R}} e^{\lambda x} \tilde{\phi}_j(x) dx - \int_{\mathbb{R}} e^{\lambda x} f_j(\tilde{\Phi}(x)) dx < \infty.
\]
By the Fubini-Tonelli theorem, for any finite numbers \(L_1 < L_2\) we have
\[
\int_{\mathbb{R}} e^{\lambda x} K_j(x) \tilde{\phi}_j(x) dx = \int_{\mathbb{R}} e^{\lambda x} \int_{\mathbb{R}} K_j(y) \tilde{\phi}_j(x-y) dy dx
\]
\[
\geq \int_{\mathbb{R}} e^{\lambda x} \int_{L_1}^{L_2} K_j(y) \tilde{\phi}_j(x-y) dy dx
\]
It follows that
\[
\int_{L_1}^{L_2} K_j(y)e^{\lambda y}dy \leq M_0 := \int_{\mathbb{R}} e^{\lambda x} K_j * \tilde{\phi}_j(x)dx / \int_{\mathbb{R}} \tilde{\phi}_j(x)e^{\lambda x}dx < \infty,
\]
which contradicts (2.26) since \(L_1 < L_2\) are arbitrary and we can take \(\lambda = \alpha/2\). \(\square\)

2.5. **Uniqueness and strict monotonicity of semi-wave solutions to (1.3).**

**Theorem 2.14.** Suppose that (J), (f_1) and (f_2) hold. Then for any \(c > 0\), (1.3) has at most one monotone semi-wave solution \(\Phi = \Phi_c\) with speed \(c\), and when exists, \(\Phi_c(x)\) is strictly decreasing in \(x\) for \(x \in (-\infty, 0]\). Moreover, if \(\Phi_{c_1}\) and \(\Phi_{c_2}\) both exist and \(0 < c_1 < c_2\), then \(\Phi_{c_1}(x) \nsucc \Phi_{c_2}(x)\) for fixed \(x < 0\).

**Proof.** Assume that \(\Phi^{(1)} = (\phi_i^{(1)})\) and \(\Phi^{(2)} = (\phi_i^{(2)})\) are monotone semi-wave solutions of (1.3) with speed \(c > 0\). We want to show that \(\Phi^1 \equiv \Phi^2\).

**Claim 1.** \(\phi_i^{(k)}(x) > 0\) for \(x < 0\), \(k = 1, 2\) and \(i \in \{1, ..., m_0\}\).

By (f_1), we can write, for \(k = 1, 2\),
\[
F(\Phi^{(k)}(x)) = \Phi^{(k)}(x)E^{(k)}(x)
\]
where \(E^{(k)}(x) = (e_{ij}^{(k)}(x))\) is a matrix function with \(e_{ij} \geq 0\) for \(i \neq j\). Due to 0 \(\leq \Phi^{(k)}(x) \leq u^*\) and \(\Phi^{(k)}(-\infty) = u^*\), we can apply Lemma 2.2 to conclude that \(\phi_i^{(k)}(x) < 0\) for \(x < 0\) and \(i \in \{1, ..., m_0\}\).

**Claim 2.** \((\phi_i^{(k)})'(0^-) < 0\) for \(k = 1, 2\) and \(i \in \{1, ..., m_0\}\); \((\phi_i^{(k)})'(0^-) = 0 < (\phi_i^{(k)})''(0^-)\) for \(k = 1, 2\) and \(i \in \{m_0 + 1, ..., m\}\).

From the equation satisfied by \(\Phi^{(k)}\), we deduce, for \(k = 1, 2\),
\[
\left(\Phi^{(k)}\right)'(0^-) = \lim_{x \to 0^-} \frac{\Phi^{(k)}(x)}{x} = \lim_{x \to 0^-} \frac{1}{cx} \int_0^x \left[-D \circ \int_{-\infty}^y J(z - y) \circ \Phi^{(k)}(y)dy + D \circ \Phi^{(k)}(y) - F(\Phi^{(k)}(y))\right] dy.
\]
\[
(2.27) = \lim_{x \to 0^-} \frac{1}{cx} \int_0^x \left[-D \circ \int_{-\infty}^y J(y) \circ \Phi^{(k)}(y)dy\right] dy.
\]
Hence \((\phi_i^{(k)})'(0^-) < 0\) for \(k = 1, 2\) and \(i \in \{1, ..., m_0\}\); \((\phi_i^{(k)})'(0^-) = 0\) for \(k = 1, 2\) and \(i \in \{m_0 + 1, ..., m\}\). Moreover, for \(k = 1, 2\) and \(i \in \{m_0 + 1, ..., m\}\), from
\[
c(\phi_i^{(k)})'(x) = f_i(\Phi^{(k)}(x))
\]
we deduce
\[
c(\phi_i^{(k)})''(0^-) = -\sum_{j=1}^m \partial_j f_i(0)(\phi_j^{(k)})'(0^-) > 0.
\]

Now with the help of Claim 1 and Claim 2, we are ready to define
\[
p := \inf\{\rho \geq 1 : \rho\Phi^{(1)}(x) \geq \Phi^{(2)}(x) \text{ for } x \leq 0\}.
\]
Since \(\Phi^{(k)}(-\infty) = u^* \nsucc 0\) for \(k = 1, 2\), and for each \(i \in \{1, ..., m\}\), \(\phi_i^{(1)}(x)\) is uniformly bounded for \(x\) in a small left neighbourhood of 0 by Claim 2, we see that \(p \in [1, \infty)\) is well-defined, and \(p\Phi^{(1)}(x) \geq \Phi^{(2)}(x)\) for \(x \leq 0\).
In Case 1, from (2.27) we obtain
\[ \lim_{n \to \infty} \frac{\phi_j^{(2)}(x_n)}{\phi_j^{(1)}(x_n)} = p > 1. \]

From \( \Phi^{(k)}(-\infty) = u^* \Rightarrow 0 \) for \( k = 1, 2 \) we see that \( \{x_n\} \) must be a bounded sequence, and hence by passing to a subsequence, we may assume that \( x_n \to x_* \in (-\infty, 0] \) as \( n \to \infty \). Define
\[ V(x) = (v_i(x)) := p\Phi^{(1)}(x) - \Phi^{(2)}(x). \]
Clearly \( V(x) \geq 0 \) for \( x \leq 0 \). Our discussion below is organised according to the following two possibilities:

- Case 1. \( V(x) \gg 0 \) for all \( x < 0 \).
- Case 2. There exist \( i_0 \in \{1, ..., m\} \) and \( x_0 < 0 \) such that \( v_{i_0}(x_0) = 0 \).

In Case 1, from (2.27) we obtain
\[ V'(0-) = -\frac{1}{c} D \circ \int_{-\infty}^{0} J(0-y) \circ V(y)dy, \]
which implies \( v_i'(0-) < 0 \) for \( i \in \{1, ..., m_0\} \) and \( v_i'(0-) = 0 \) for \( i \in \{m_0 + 1, ..., m\} \). Moreover, for \( i \in \{m_0 + 1, ..., m\} \),
\[ cv_i''(0-) = -\sum_{j=1}^{m} \partial_j f_i(0)v_j'(0-) > 0. \]

Let us examine the sequence \( \{x_n\} \) in (2.28). We have \( x_n \to x_* \in (-\infty, 0] \). If \( x_* < 0 \) then we deduce \( v_j(x_*) = 0 \) which is a contradiction to \( V(x) = (v_i(x)) \gg 0 \) for \( x < 0 \). Therefore we must have \( x_* = 0 \) and so \( x_n \to 0 \) as \( n \to \infty \). It then follows that
\[ \lim_{n \to \infty} \frac{\phi_j^{(2)}(x_n)}{\phi_j^{(1)}(x_n)} = \frac{(\phi_j^{(2)}(0-))'}{(\phi_j^{(1)}(0-))'} < p \text{ if } j \in \{1, ..., m_0\} \]
due to \( v_j'(0-) < 0 \) and \( (\phi_j^{(k)}(0-))' < 0 \) for \( k = 1, 2 \), and
\[ \lim_{n \to \infty} \frac{\phi_j^{(2)}(x_n)}{\phi_j^{(1)}(x_n)} = \frac{(\phi_j^{(2)}(0-))''}{(\phi_j^{(1)}(0-))''} < p \text{ if } j \in \{m_0 + 1, ..., m\} \]
due to \( v_j''(0-) > 0 \) and \( (\phi_j^{(k)}(0-))'' = 0 < (\phi_j^{(k)}(0-))'' \) for \( k = 1, 2 \). Thus we always arrive at a contradiction to (2.28) in Case 1.

In Case 2, from the assumptions (f1) and (f2), we see that \( W(x) := \Phi^{(1)}(x) - p^{-1}\Phi^{(2)}(x) \) satisfies, for \( x \leq 0 \),
\[ 0 = D \circ \int_{-\infty}^{0} J(x-y) \circ W(y)dy - D \circ W(x) + cW'(x) + F(\Phi^{(1)}(x)) - p^{-1}F(\Phi^{(2)}(x)) \]
\[ \geq D \circ \int_{-\infty}^{0} J(x-y) \circ W(y)dy - D \circ W(x) + cW'(x) + F(\Phi^{(1)}(x)) - F(p^{-1}\Phi^{(2)}(x)) \]
\[ = D \circ \int_{-\infty}^{0} J(x-y) \circ W(y)dy - D \circ W(x) + cW'(x) + W(x)E(x), \]
where \( E = (e_{ij}) \) is a matrix function with \( e_{ij} \geq 0 \) for \( i \neq j \). In view of \( W(x) \geq 0 \) for \( x \leq 0 \), and \( W(-\infty) \gg 0 \), we can apply Lemma 2.2 to conclude that
\[ w_i(x) > 0 \text{ for } x < 0, \ i = 1, ..., m_0. \]
This is already a contradiction if \( i_0 \in \{1, \ldots, m_0 \} \). If \( i_0 \in \{m_0 + 1, \ldots, m \} \), then

\[
 cw_i'(x) = \sum_{j=1}^{m} e_{ji_0}(x)w_j(x).
\]

By (f) (iv) we see that \( e_{ji_0}(x) > 0 \) for \( j \in \{1, \ldots, m_0 \} \), and hence

\[
 \sum_{j=1}^{m} e_{ji_0}(x)w_j(x) > e_{i_0i_0}(x)w_{i_0}(x) \geq -Mw_{i_0}(x) \quad \text{for } x < 0 \quad \text{and some constant } M.
\]

It follows that

\[
 cw_i'(x) > -Mw_{i_0}(x) \quad \text{for } x < 0.
\]

This combined with \( w_{i_0}(0) = 0 \) yields \( w_{i_0}(x) > 0 \) for \( x < 0 \), so we arrive at a contradiction to \( w_{i_0}(x_0) = 0 \).

We have thus proved \( p = 1 \), and so \( \Phi^{(1)}(x) \geq \Phi^{(2)}(x) \) for \( x \leq 0 \). By swapping \( \Phi^{(1)}(x) \) with \( \Phi^{(2)}(x) \) we also have \( \Phi^{(2)}(x) \geq \Phi^{(1)}(x) \) for \( x \leq 0 \). This completes our proof for uniqueness of the semi-wave solution.

Next we prove the strict monotonicity properties stated in the theorem. Let \( \Phi^c \) be a monotone semi-wave solution of \( (1.3) \) with speed \( c > 0 \). To show the strict monotonicity of \( \Phi^c(x) \) with respect to \( x \leq 0 \), it suffices to verify that \( V(x) := \Phi^c(x - \delta) - \Phi^c(x) \gg 0 \) for \( \delta > 0 \) and \( x \leq 0 \).

From (f) (ii),

\[
 F(\Phi^c(x - \delta)) - F(\Phi^c(x)) = V(x)E(x)
\]

where \( E(x) = (e_{ij}(x)) \) is a matrix function with \( e_{ij} \geq 0 \) for \( i \neq j \). Note that \( V(x) \geq 0 \) by Lemma 2.10 and \( V(0) = \Phi^c(-\delta) \gg 0 \) by the conclusions in Claim 2. Making use of Lemma 2.2, we deduce \( v_i(x) > 0 \) for \( x \leq 0 \) and \( i \in \{1, \ldots, m_0 \} \). Using this and (f) (iv), we can further show, as in the argument near the end of the proof of Claim 3, that \( v_i(x) > 0 \) for \( i \in \{m_0 + 1, \ldots, m \} \) and \( x \leq 0 \).

Now we show that for fixed \( x < 0 \), \( \Phi^c(x) \) is strictly decreasing with respect to \( c > 0 \), namely, \( \Phi^{c_1}(x) = (\phi^{c_1}(x)) \gg \Phi^{c_2}(x) = (\phi^{c_2}(x)) \) for \( c_2 > c_1 > 0 \). Denote \( W(x) := \Phi^{c_1}(x) - \Phi^{c_2}(x) \). By Lemma 2.7 and the proof of Lemma 2.10 without shifting \( \Phi^c \), we see that \( W(x) \geq 0 \) for \( x \leq 0 \).

By (f) (ii),

\[
 F(\Phi^{c_1}(x)) - F(\Phi^{c_2}(x)) = W(x)E(x)
\]

where \( E(x) = (e_{ij}(x)) \) is a matrix function with \( e_{ij} \geq 0 \) for \( i \neq j \). This, combined with \( c_1(\Phi^{c_1})' - c_2(\Phi^{c_2})' > c_1W'(x) \), allows us to apply Lemma 2.2 to conclude that \( w_i(x) > 0 \) for \( x < 0 \) and \( i \in \{1, \ldots, m_0 \} \). We may then use this and (f) (iv) to deduce, as before, \( w_i(x) > 0 \) for \( x < 0 \) and \( i \in \{m_0 + 1, \ldots, m \} \).

\[
 \Box
\]

2.6. Semi-wave solution with the desired speed.

**Theorem 2.15.** Suppose that (J), (f), (f) hold, and \( \Phi^c(x) \) is the unique monotone semi-wave solution of \( (1.3) \) with speed \( c \in (0, C_*) \), where \( C_* \in (0, \infty) \) is given by Theorem 2.4. Then

\[
 \lim_{c \uparrow C_*} \Phi^c(x) = 0 \quad \text{locally uniformly in } (-\infty, 0].
\]

Moreover, (1.4) and (1.5) have a solution pair \((c, \Phi)\) with \( \Phi(x) \) monotone if and only if (J) holds. And when (J) holds, there exists a unique \( c_0 \in (0, C_*) \) such that \((c, \Phi) = (c_0, \Phi^{c_0})\) solves (1.4) and (1.5).

**Proof.** We first prove (2.29). Since \( \Phi^c = (\phi^c) \) is decreasing with respect to \( c \), \( \Phi(x) := \lim_{c \uparrow C_*} \Phi^c(x) \) is well-defined, and \( \Phi(x) \in [0, u^*] \) for \( x \leq 0 \). Moreover, by the uniform boundedness of \((\Phi^c)'(x)\)
Therefore, \( \Phi^c(x) \) is locally uniform in \((-\infty, 0]\). If \( C_* = \infty \), then from
\[
\Phi^c(x) = \frac{1}{c} \int_{0}^{x} \int_{-\infty}^{0} J(z-y) \Phi^c(y) \, dy + D \int_{-\infty}^{0} \Phi^c(z) - F(\Phi^c(z)) \, dz
\]
we immediately obtain \( \Phi(x) \equiv 0 \). If \( C_* < \infty \) then \( \Phi \) satisfies
\[
\begin{align*}
D \int_{-\infty}^{\infty} J(x-y) \Phi(y) \, dy - D \Phi(x) + C_\ast \Phi'(x) + F(\Phi(x)) &= 0, & x &< 0, \\
\Phi(0) &= 0.
\end{align*}
\]
Note that \( \Phi(x) \) is nonincreasing since \( \Phi^c(x) \) is. As in Step 3 of the proof of Lemma 2.6, we can show that \( \Phi(-\infty) = u^* \) or \( 0 \). By Theorem 2.9 (1.3) admits no monotone semi-wave solution for \( c = C_* \), and hence necessarily \( \Phi(-\infty) = 0 \). Thus we also have \( \Phi \equiv 0 \), and (2.29) is proved.

Next we show that if \((J_1)\) holds, then \((1.4)-(1.5)\) have a unique solution pair \((c_0, \Phi^{c_0})\). It suffices to prove that
\[
P(c) := c - M(c), \text{ with } M(c) := \sum_{i=1}^{n} \mu_i \int_{-\infty}^{0} \int_{0}^{\infty} J_i(x-y) \phi_1^c(x) \, dy \, dx,
\]
has a unique root in \((0, C_*)\). Let us observe that when \((J_1)\) holds, \( M(c) \) is well-defined and strictly decreasing in \( c \) by Theorem 2.14. Indeed, an elementary calculation yields
\[
\int_{-\infty}^{0} \int_{0}^{\infty} J_i(x-y) \, dy \, dx = \int_{0}^{\infty} \int_{0}^{\infty} J_i(x+y) \, dy \, dx = \int_{0}^{\infty} J_i(y) \, dy,
\]
which implies that \( M(c) \) is well-defined.

Using the uniqueness of \( \Phi^c \), we can apply a similar convergence argument as used above to prove (2.29) to show that \( \Phi^{c_n} \to \Phi^c \) as \( c_n \to c \in (0, C_*) \), which yields the continuity of \( \Phi^c(x) \) in \( c \in (0, C_*) \) uniformly for \( x \) over any bounded interval of \((-\infty, 0]\). Note that we can easily see that \( \Phi(x) := \lim_{c_n \to c} \Phi^{c_n}(x) \) satisfies \( \Phi(-\infty) = u^* \) by comparing \( \Phi^{c_n} \) to some \( \Phi^c \) with \( \hat{c} \in (c, C_*) \) and using the monotonicity of \( \Phi^c \) in \( c \).

Hence \( P(c) \) is increasing and continuous in \( c \). For \( c \in (0, C_*/2) \) close to 0, we have \( P(c) \leq c - M(C_*/2) < 0 \), and for all \( c \) close to \( C_* \), \( M(c) \) is small and hence \( P(c) > 0 \). Thus there is a unique \( c_0 \in (0, C_*) \) such that \( P(c_0) = 0 \).

Finally we verify that \((J_1)\) holds if \((1.4)-(1.5)\) have a solution pair \((c_0, \Phi^{c_0})\). Since
\[
c_0 = \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_{0}^{\infty} J_i(x-y) \phi^c_i(x) \, dy \, dx,
\]
for every \( i_0 \in \{1, \ldots, m_0\} \) such that \( \mu_{i_0} > 0 \) we have
\[
\int_{-\infty}^{0} \int_{0}^{\infty} J_{i_0}(x-y) \phi^c_{i_0}(x) \, dy \, dx < \infty.
\]
By Theorem 2.14 \( \Phi^{c_0}(x) \) is decreasing in \( x \). Hence,
\[
\int_{-\infty}^{0} \int_{0}^{\infty} J_{i_0}(x-y) \phi^c_{i_0}(x) \, dy \, dx \geq \phi^c_{i_0}(-1) \int_{-\infty}^{0} \int_{0}^{\infty} J_{i_0}(x-y) \, dy \, dx.
\]
and so
\[
\int_{-\infty}^{0} \int_{0}^{\infty} J_{i_0}(x-y) \, dy \, dx = \int_{-1}^{0} \int_{0}^{\infty} J_{i_0}(x-y) \, dy \, dx + \int_{-\infty}^{-1} \int_{0}^{\infty} J_{i_0}(x-y) \, dy \, dx + \int_{-\infty}^{-\infty} \int_{0}^{\infty} J_{i_0}(x-y) \, dy \, dx \\
\leq 1 + \int_{-\infty}^{-1} \int_{0}^{\infty} J_{i_0}(x-y) \, dy \, dx < \infty.
\]
Therefore, \((J_1)\) holds. \( \Box \)
3. Spreading speed

3.1. Further comparison results. The following maximum principle is a direct consequence of [16] Lemma 3.1, where a more general case is considered.

Lemma 3.1. Suppose that $T$ and $d_i$ for $i \in \{1, ..., m_0\}$ are positive constants, $g, h \in C([0, T])$ satisfy $g(0) < 0 < h(0)$, and both $-g(t)$ and $h(t)$ are nondecreasing for $t \in [0, T]$, $\Omega_T := \{(t, x) : t \in (0, T], x \in (g(t), h(t))\}$, $m \geq m_0 \geq 1$, and for $i, j \in \{1, 2, ..., m\}$, $\phi_i, \partial_t \phi_i \in C(\Omega_T)$, $c_{ij} \in L^{\infty}(\Omega_T)$ and

$$
\begin{align*}
\partial_t \phi_i & \geq d_i L_i[\phi_i] + \sum_{j=1}^{m} c_{ij} \phi_j, & (t, x) \in \Omega_T, & 1 \leq i \leq m_0, \\
\partial_t \phi_i & \geq \sum_{j=1}^{m} c_{ij} \phi_j, & (t, x) \in \Omega_T, & m_0 + 1 \leq i \leq m, \\
\phi_i(t, g(t)) & \geq 0, & t \in (0, T), & 1 \leq i \leq m, \\
\phi_i(0, x) & \geq 0, & x \in [g(0), h(0)], & 1 \leq i \leq m,
\end{align*}
$$

where

$$L_i[v](t, x) := \int_{g(t)}^{h(t)} J_i(x - y)v(t, y)dy - v(t, x),$$

with every $J_i$ ($i = 1, ..., m_0$) satisfying (J). Then the following conclusions hold:

(i) If $c_{ij} \geq 0$ on $\Omega_T$ for $i, j \in \{1, ..., m\}$ and $i \neq j$, then $\phi_i \geq 0$ on $\Omega_T$ for $i \in \{1, ..., m\}$.

(ii) If in addition $\phi_{i_0}(0, x) \neq 0$ in $[-h_0, h_0]$ for some $i_0 \in \{1, ..., m_0\}$, then $\phi_{i_0} > 0$ in $\Omega_T$.

Lemma 3.2. Assume (J), (f1) hold, and $(U, g, h)$ with $U = (u_i)$ is the solution of (1.1) with initial function $U(0, x)$ satisfying (1.2) and $U(t, x) \in [0, \tilde{U}]$ for $t \geq 0, x \in [g(t), h(t)]$. Suppose $\tilde{g}, \tilde{h} \in C([0, T])$, $\tilde{U}$ is continuous in $\Omega_T$, and

$$L[\tilde{U}](t, x) := \int_{\tilde{g}(t)}^{\tilde{h}(t)} J(x - y)\tilde{U}(t, y)dy - \tilde{U}(t, x),$$

where $T \in (0, \infty)$ and

$$\Omega_T := \{(t, x) : t \in (0, T], x \in (\tilde{g}(t), \tilde{h}(t))\}.$$

Then the following conclusions hold.

(i) If $\tilde{g}, \tilde{h}$ and the continuous vector function $\tilde{U} = (\tilde{u}_i)$ satisfy

$$
\begin{align*}
\tilde{g}(t) & \equiv g(t), & \tilde{h}(0) & \geq h(0), & t \in [0, T], \\
\tilde{U}(0, x) & \geq 0, & x \in [\tilde{g}(0), \tilde{h}(0)], \\
\tilde{U}(0, x) & \geq U(0, x), & x \in [-h_0, h_0],
\end{align*}
$$

and

$$
\begin{align*}
\partial_t \tilde{U} & \geq D \circ L[\tilde{U}](t, x) + F(\tilde{U}), & \tilde{U}(t, x) & \in [0, \tilde{U}], & t \in (0, T], x \in (g(t), \tilde{h}(t)), \\
\tilde{U}(t, g(t)) & \geq 0, & \tilde{U}(t, \tilde{h}(t)) & \geq 0, & t \in (0, T], \\
\tilde{h}'(t) & \geq \sum_{i=1}^{m_0} \mu_i \int_{g(t)}^{\tilde{h}(t)} \int_{h(t)}^{\tilde{h}(t)} J_i(x - y)\tilde{u}_i(t, x)dydx, & t \in (0, T),
\end{align*}
$$

then

$$\tilde{h}(t) \geq h(t), \quad \tilde{U}(t, x) \geq U(t, x) \quad \text{for } t \in (0, T], \ x \in [g(t), \tilde{h}(t)].$$
(ii) If \( \tilde{g}, \tilde{h} \) and the continuous vector function \( \tilde{U} = (\tilde{u}_i) \) satisfy
\[
\begin{align*}
\partial_t \tilde{U} &\leq D \circ \mathcal{L}[\tilde{U}](t, x) + F(\tilde{U}), \quad \tilde{U}(t, x) \leq \tilde{u}, \quad t \in (0, T), \ x \in (\tilde{g}(t), \tilde{h}(t)), \\
\tilde{U}(t, \tilde{g}(t)) &\leq 0, \quad \tilde{U}(t, \tilde{h}(t)) \leq 0, \quad t \in (0, T), \\
\tilde{h}'(t) &\leq \sum_{i=1}^{m_0} \mu_i \int_{\tilde{g}(t)}^{\tilde{h}(t)} J_i(x - y)\tilde{u}_i(t, x)dydx, \quad t \in (0, T), \\
\tilde{g}'(t) &\geq -\sum_{i=1}^{m_0} \mu_i \int_{\tilde{g}(t)}^{\tilde{h}(t)} J_i(x - y)\tilde{u}_i(t, x)dydx, \quad t \in (0, T),
\end{align*}
\]
and
\[
g(0) \leq \tilde{g}(0) < 0 < \tilde{h}(0) \leq h(0),
\]
\[
\tilde{U}(0, x) \leq U(0, x), \quad x \in [\tilde{g}_0, \tilde{h}_0],
\]
then
\[
g(t) \leq \tilde{g}(t), \quad \tilde{h}(t) \leq h(t), \quad t \in (0, T),
\]
\[
\tilde{U}(t, x) \leq U(t, x), \quad t \in (0, T), \ x \in [\tilde{g}(t), \tilde{h}(t)].
\]

Here and in what follows, we use the notation \( u_+ := (\max\{u_i, 0\}) \) for \( u \in \mathbb{R}^m \).

**Proof.** Lemma 3.2 follows from Lemma 3.1 by an argument similar to the proof of [8, Theorem 3.1]; since the required changes are rather obvious, we omit the details. \( \square \)

### 3.2. Finite spreading speed: Proof of Theorem 1.3 (i).

**Lemma 3.3.** Let \((J), (J_1)\) and \((f_i)\) with \(i = 1, 2, 3, 4\) be satisfied, \((U, g, h)\) be a solution of \((1.1)\) with \(U(0, x) \in [0, \tilde{u}]\) for \(x \in [-h_0, h_0]\). If spreading happens, then

\[
lm c_0 > 0 \text{ is given by Theorem 1.2}
\]

**Proof.** Let \((c_0, \Phi^{c_0})\) be the unique solution pair of \((1.4)-(1.5)\). To simplify notations, we write \( \Phi = \Phi^{c_0} = (\phi_i) \). For fixed small \( \epsilon > 0 \), we define \( \delta := 2\epsilon c_0 \) and

\[
\begin{align*}
\hat{h}(t) &:= (c_0 + \delta)t + K, \quad t \geq 0, \\
\overline{U}(t, x) &:= (1 + \epsilon)\Phi(x - \underline{h}(t)), \quad t \geq 0, \ x \leq \underline{h}(t),
\end{align*}
\]

where \( K > 0 \) is a large constant to be determined.

By \((f_4)\), the unique solution \( v(t) \) of the ODE system
\[
v' = F(v), \quad v(0) = (\max_{x} u_i(0, x)) \in \{0, \tilde{u}\},
\]
which also solves \((1.3)\), satisfies \( v(t) \to u^* \) as \( t \to \infty \). By Lemma 3.1 and \((f_1)\), we easily see that \( v(t) \geq U(t, x) \) for \( t \geq 0, x \in [g(t), h(t)] \), and hence there is a large constant \( t_0 > 0 \) such that

\[
U(t + t_0, x) \leq (1 + \epsilon/2)u^* \text{ for } t \geq 0, \ x \in [g(t + t_0), h(t + t_0)].
\]

Due to \( \Phi(-\infty) = u^* \), we may choose sufficient large \( K > 0 \) such that \( \underline{h}(0) = K > h(t_0), \ -\underline{h}(0) = -K < g(t_0), \) and also

\[
\overline{U}(0, x) = (1 + \epsilon)\Phi(-K) \geq (1 + \epsilon/2)u^* \geq U(t_0, x) \text{ for } x \in [g(t_0), h(t_0)].
\]

Next we verify that, with \( \overline{U} = (\tilde{u}_i) \),

\[
\hat{h}'(t) \geq \sum_{i=1}^{m_0} \mu_i \int_{-\hat{h}(t)}^{\hat{h}(t)} J_i(x - y)\tilde{u}_i(t, x)dydx \text{ for } t > 0,
\]

\[
\overline{U}(t, x) \geq U(t, x), \quad t \in (0, T), \ x \in [\hat{g}(t), \hat{h}(t)].
\]
and for \(t > 0, x \in (g(t + t_0), h(t))\),

\[
(3.4) \quad \nabla t(t, x) \geq D \circ \int_{g(t + t_0)}^{h(t)} (J)(x - y) \circ \nabla t(y, y) dy - D \circ \nabla t(x, x) + F(\nabla t(t, x)).
\]

A direct calculation gives, for \(t > 0\),

\[
\sum_{i=1}^{m_0} \mu_i \int_{g(t + t_0)}^{h(t)} \int_{-\infty}^{\infty} J_i(x - y) \bar{u}_i(t, x) dy dx \leq \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{h(t)} \int_{-\infty}^{\infty} J_i(x - y) \bar{u}_i(t, x) dy dx
\]

\[
= \sum_{i=1}^{m_0} \mu_i (1 + \epsilon) \int_{0}^{\infty} \int_{-\infty}^{\infty} J_i(x - y) \bar{\phi}_i(x) dy dx = (1 + \epsilon) c_0 < c_0 + \delta = \bar{h}'(t),
\]

which yields (3.3).

Using (1.4) and (f_2), we deduce

\[
\nabla t(t, x) = -(1 + \epsilon)(c_0 + \delta) \Phi'(x - \bar{h}(t)) \geq -(1 + \epsilon) c_0 \Phi'(x - \bar{h}(t))
\]

\[
= (1 + \epsilon) \left[ D \circ \int_{-\infty}^{\bar{h}(t)} J(x - y) \circ \Phi(y - \bar{h}(t)) dy - D \circ \Phi(x - \bar{h}(t)) + F(\Phi(x - \bar{h}(t))) \right]
\]

\[
\geq D \circ \int_{-\infty}^{\bar{h}(t)} J(x - y) \circ \nabla t(y, x) dy - D \circ \nabla t(x, x) + (1 + \epsilon) F(\Phi(x - \bar{h}(t)))
\]

\[
\geq D \circ \int_{g(t + t_0)}^{h(t)} J(x - y) \circ \nabla t(y, x) dy - D \circ \nabla t(x, x) + F(\nabla t(t, x)),
\]

which proves (3.4).

Moreover, we have

\[
\nabla t(t, g(t + t_0)) > 0, \quad \nabla t(h(t)) = (1 + \epsilon) \Phi(\bar{h}(t) - \bar{h}(t)) = 0 \quad \text{for} \quad t \geq 0.
\]

This combined with (3.2), (3.3) and (3.4) allows us to use Lemma 3.2 (i) to conclude that

\[
h(t + t_0) \leq \bar{h}(t), \quad t \geq 0,
\]

\[
U(t + t_0, x) \leq \nabla t(t, x), \quad t \geq 0, \quad x \in [g(t + t_0), \bar{h}(t)].
\]

Hence

\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq \limsup_{t \to \infty} \frac{\bar{h}(t - t_0)}{t} = c_0 + \delta,
\]

which yields (3.1) by letting \(\delta \to 0\). \(\square\)

**Lemma 3.4.** Under the conditions of Lemma 3.3, we have

\[
(3.5) \quad \liminf_{t \to \infty} \frac{h(t)}{t} \geq c_0.
\]

**Proof.** With \(\Phi = (\phi_i)\) as in the proof of Lemma 3.3 we define, for small \(\epsilon > 0, \delta := 2c_0\epsilon\) and

\[
\begin{aligned}
\bar{h}(t) &:= (c_0 - \delta) t + K, \quad t \geq 0, \\
\nabla t(t, x) &:= (1 - \epsilon) [\Phi(x - \bar{h}(t)) + \Phi(-x - \bar{h}(t)) - u^*], \quad t \geq 0, \quad x \in [-\bar{h}(t), \bar{h}(t)],
\end{aligned}
\]

for some large \(K > 0\) to be determined.

For later analysis involving \(\bar{h}\) and \(\nabla\), we first prepare some simple estimates. Due to \(\Phi(-\infty) = u^*\) there is \(K_0 > 0\) such that

\[
\Phi(-K_0) > (1 - \epsilon) u^*,
\]

which implies

\[
(3.6) \quad \Phi(x - \bar{h}(t)), \Phi(-x - \bar{h}(t)) \in [(1 - \epsilon) u^*, u^*] \quad \text{for} \quad x \in [-\bar{h}(t) + K_0, \bar{h}(t) - K_0], \quad t \geq 0
\]
provided that $\theta(0) = K > K_0$. Define

$$\epsilon_1 := \min_{1 \leq i \leq m} \inf_{x \in [-K_0,0]} |\phi_i'(x)| > 0.$$  

Then for every $i \in \{1, \ldots, m\}$,

$$\begin{cases} 
\phi_i'(x - \theta(t)) \leq -\epsilon_1 & \text{for } x \in [\theta(t) - K_0, \theta(t)], \ t \geq 0, \\
\phi_i'(-x - \theta(t)) \leq -\epsilon_1 & \text{for } x \in [-\theta(t), -\theta(t) + K_0], \ t \geq 0.
\end{cases}$$  

Define

$$\epsilon_2 := \frac{\epsilon_1 \delta}{2mL} \quad \text{with } L = L(u^*) \text{ is given by (2.1).}$$

**Step 1.** We prove, with $U = (\theta_i)$,

$$\begin{align*}
\frac{d}{dt} \theta_i(t) & \leq \sum_{i=1}^{m_0} \mu_i \int_{\theta(t)}^{\theta(t)} J_i(x-y)u_i(t,x)dydx \quad \text{for } t > 0, \\
\frac{d}{dt} \theta_i(t) & \geq -\sum_{i=1}^{m_0} \mu_i \int_{\theta(t)}^{-\theta(t)} J_i(x-y)u_i(t,x)dydx \quad \text{for } t > 0.
\end{align*}$$

Since $U(t, x) = U(t, -x)$ and $J(x) = J(-x)$, we just need to verify (3.8).

A simple calculation yields

$$\begin{align*}
\sum_{i=1}^{m_0} \mu_i \int_{\theta(t)}^{\theta(t)} \int_{\theta(t)}^{\infty} J_i(x-y)u_i(t,x)dydx &= (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-2\theta(t)}^{0} \int_{0}^{\infty} J_i(x-y)\phi_i(x)dy \\
& \quad + (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{0}^{2\theta(t)} \int_{0}^{\infty} J_i(x-y)[\phi_i(-x-2\theta(t)) - u_i^*]dydx \\
& = (1 - \epsilon) c_0 - (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{-2\theta(t)} \int_{0}^{\infty} J_i(x-y)\phi_i(x)dydx \\
& \quad - (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-2\theta(t)}^{0} \int_{0}^{\infty} J_i(x-y)[u_i^* - \phi_i(-x-2\theta(t))]dydx.
\end{align*}$$

In view of $J_1$, there is a large constant $K_1 > 0$ such that for $K > K_1$,

$$\begin{align*}
(1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{-2\theta(t)} \int_{0}^{\infty} J_i(x-y)\phi_i(x)dydx & \leq (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i u_i^* \int_{-\infty}^{-2K_1} \int_{0}^{\infty} J_i(x-y)dydx < \frac{c_0 \epsilon}{4}.
\end{align*}$$

Moreover,

$$\begin{align*}
0 & \leq (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{0}^{2\theta(t)} \int_{0}^{\infty} J_i(x-y)[u_i^* - \phi_i(-x-2\theta(t))]dydx \\
& \leq (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_{0}^{\infty} J_i(x-y)[u_i^* - \phi_i(-x-2\theta(t))]dydx \\
& \leq (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i u_i^* \int_{-\infty}^{-2K_1} \int_{0}^{\infty} J_i(x-y)dydx
\end{align*}$$
\[ + (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-2K_i}^0 \int_0^\infty J_i(x-y)[u^*_i - \phi_i(-x - 2h(t))] dy dx \]
\[ < \frac{c_0 \epsilon}{4} + (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i [u^*_i - \phi_i(2K_1 - 2K)] \int_{-2K_i}^0 \int_0^\infty J_i(x-y) dy dx \]
\[ < \frac{c_0 \epsilon}{2} \text{ for } t \geq 0 \text{ and } K > K_2 \gg K_1. \]

Hence, for \( K > K_2 \) and all \( t \geq 0 \),
\[ \sum_{i=1}^{m_0} \mu_i \int_{-h(t)}^{h(t)} \int_{-h(t)}^{h(t)} J_i(x-y) u_i(t,x) dy \geq (1 - \epsilon)c_0 - \frac{3c_0 \epsilon}{4} \geq c_0 - 2c_0 \epsilon > h'(t), \]
which finishes the proof of (3.8).

**Step 2.** We show that for \( t > 0 \) and \( x \in (-h(t), h(t)), \)
\[ (3.9) \quad \underline{u}_i(t, x) \leq D \circ \int_{-h(t)}^{h(t)} J(x-y) \circ \underline{u}(t, y) dy - D \circ \underline{u}(t, x) + F(\underline{u}(t, x)_+). \]

By (1.4), for \( t > 0 \) and \( x \in (-h(t), h(t)), \)
\[-c_0 \Phi'(x - h(t)) = D \circ \int_{-h(t)}^{h(t)} J(x-y) \circ \Phi(y - h(t)) dy - D \circ \Phi(x - h(t)) + F(\Phi(x - h(t))) \]
and
\[-c_0 \Phi'(-x - h(t)) = D \circ \int_{-h(t)}^{h(t)} J(x-y) \circ \Phi(-y - h(t)) dy - D \circ \Phi(-x - h(t)) + F(\Phi(-x - h(t))). \]

Thus, for such \( t \) and \( x, \)
\[ \underline{u}_i(t, x) = -(1 - \epsilon)(c_0 - \delta)[\Phi'(x - h(t)) + \Phi'(-x - h(t))] \]
\[ = (1 - \epsilon)\delta \left[ \Phi'(x - h(t)) + \Phi'(-x - h(t)) \right] \]
\[ + (1 - \epsilon) \left[ D \circ \int_{-h(t)}^{h(t)} J(x-y) \circ \Phi(y - h(t)) dy - D \circ \Phi(x - h(t)) \right] \]
\[ + D \circ \int_{-h(t)}^{h(t)} J(-x-y) \circ \Phi(-y - h(t)) dy - D \circ \Phi(-x - h(t)) \]
\[ + (1 - \epsilon) \left[ F(\Phi(x - h(t))) + F(\Phi(-x - h(t))) \right] \]
\[ = (1 - \epsilon)\delta \left[ \Phi'(x - h(t)) + \Phi'(-x - h(t)) \right] + D \circ \int_{-h(t)}^{h(t)} J(x-y) \circ \underline{u}(t, y) dy - D \circ \underline{u}(t, x) \]
\[ + (1 - \epsilon) \left[ D \circ \int_{-h(t)}^{h(t)} J(x-y) \circ [\Phi(y - h(t)) - u^*_i] dy \right] \]
\[ + D \circ \int_{-h(t)}^{h(t)} J(-x-y) \circ [\Phi(-y - h(t)) dy - u^*_i] dy \]
\[ + (1 - \epsilon) \left[ F(\Phi(x - h(t))) + F(\Phi(-x - h(t))) \right] \]
\[ \leq D \circ \int_{-h(t)}^{h(t)} J(x-y) \circ \underline{u}(t, y) dy - D \circ \underline{u}(t, x) + F(\underline{u}(t, x)_+) + \Delta(t, x), \]
with
\[ \Delta(t, x) := (1 - \epsilon)\delta \left[ \Phi'(x - h(t)) + \Phi'(-x - h(t)) \right] \]
\[ + (1 - \epsilon) \left[ F(\Phi(x - h(t))) + F(\Phi(-x - h(t))) \right] - F(\underline{u}(t, x)_+). \]
To complete the proof of Step 2, it remains to verify that 
\[ \Delta(t, x) = (\Delta_i(t, x)) \leq 0 \] for \( t > 0, \ x \in (-h(t), h(t)) \).

Next we estimate \( \Delta(t, x) \) separately for \( x \) in the following three intervals:

\[ I_1(t) := [h(t) - K_0, h(t)], \ I_2(t) := [-h(t), -h(t) + K_0], \ I_3(t) := [-h(t) + K_0, h(t) - K_0]. \]

For \( x \in I_1(t) \), we have

\[ 0 \Rightarrow \Phi(-x - h(t)) - u^* \geq \Phi(K_0 - 2h(t)) - u^* \geq \Phi(K_0 - 2K) - u^* \geq -\epsilon_2 \mathbf{1} \]

provided \( K > K_2 \) for some \( K_2 \gg K_0 \). Then by (3.21) and (f2),

\[ F(\Phi(-x - h(t))) = F(\Phi(-x - h(t))) - F(u^*) \leq m\epsilon_2 \mathbf{1} \]

and

\[ F(U(t, x)_+) \geq (1 - \epsilon) \bigg[ F\big( \Phi(x - h(t)) + \Phi(-x - h(t)) - u^* \big) \bigg] \]

\[ \geq (1 - \epsilon) \bigg[ F(\Phi(x - h(t))) - m\epsilon_2 \mathbf{1} \bigg]. \]

Thus from (3.7) and the definition of \( \epsilon_2 \), we deduce

\[ \Delta(t, x) \leq (1 - \epsilon) \bigg( \delta \big[ \Phi'(x - h(t)) + \Phi'(-x - h(t)) \big] + 2m\epsilon_2 \mathbf{1} \bigg) \]

\[ \leq (1 - \epsilon) \big( \delta \epsilon_1 + 2m\epsilon_2 \mathbf{1} \big) = 0. \]

For \( x \in I_2(t) \), since \( \Delta(t, x) \) is even in \( x \), the above inequality is also valid.

Finally we consider \( x \in I_3(t) \). By (f3), for each \( i \in \{1, \ldots, m\} \), either

(i) \[ \sum_{j=1}^{m} \partial_j f_i(u^*)u_j^* < 0, \] or

(ii) \[ \sum_{j=1}^{m} \partial_j f_i(u^*)u_j^* = 0 \] and \( f_i(u) \) is linear in \([u^* - \epsilon_0 \mathbf{1}, u^*] \) for some small \( \epsilon_0 > 0 \).

Note that since \( f_i(u^*) = 0 \), in case (ii) we have, for \( u \in [u^* - \epsilon_0 \mathbf{1}, u^*] \),

\[ f_i(u) = \sum_{j=1}^{m} a_j (u_j - u_j^*) \] for some constants \( a_j, j = 1, \ldots, m \).

For \( x \in I_3(t) \), (3.9) holds and hence

\[ -(3 - 2x) \epsilon u^* \leq U(t, x) - u^* \leq -\epsilon u^*. \]

Therefore, by choosing \( \epsilon > 0 \) sufficiently small, we may assume that for \( t > 0 \) and \( x \in I_3(t) \),

\[ \Phi(x - h(t)), \Phi(-x - h(t)), (1 - \epsilon)^{-1}U(t, x) \in [u^* - \epsilon_0 \mathbf{1}, u^*] \]

Thus in case (ii) we have, in view of (f2), for such \( t > 0 \) and \( x \),

\[ \Delta_i(t, x) \leq (1 - \epsilon) \bigg[ f_i(\Phi(x - h(t))) + f_i(\Phi(-x - h(t))) \bigg] - f_i(U(t, x)) \]

\[ \leq (1 - \epsilon) \bigg[ f_i(\Phi(x - h(t))) + f_i(\Phi(-x - h(t))) \bigg] - f_i((1 - \epsilon)^{-1}U(t, x)) \]

\[ = (1 - \epsilon) \sum_{j=1}^{m} a_j \big[ \phi_j(x - h(t)) - u_j^* + \phi_j(-x - h(t)) - u_j^* - (1 - \epsilon)^{-1}w_j(t, x) + u_j^* \big] \]

\[ = 0. \]

In case (i), we have, for such \( t \) and \( x \),

\[ \Delta_i(t, x) \leq (1 - \epsilon) \bigg[ f_i(\Phi(x - h(t))) + f_i(\Phi(-x - h(t))) \bigg] - f_i(U(t, x)) \]

\[ = (1 - \epsilon) \sum_{j=1}^{m} \partial_j f_i(u^*) \big[ \phi_j(x - h(t)) - u_j^* \big] + (1 - \epsilon) \sum_{j=1}^{m} \partial_j f_i(u^*) \big[ \phi_j(-x - h(t)) - u_j^* \big] \]
By the above lemmas, we have
\[
- \sum_{j=1}^{m} \partial_j f_i^*(u^*_j) [u_j(x) - u_j^*] + o(\epsilon)
= \epsilon \sum_{j=1}^{m} \partial_j f_i^* u_j^* + o(\epsilon) < 0,
\]
provided that \( \epsilon > 0 \) is small enough. The proof of Step 2 is finished.

**Step 3.** We prove (3.5) by using Lemma 3.2.

It is clear that
\[
U(t, \pm h(t)) = (1 - \epsilon)|\Phi(-2h(t)) - u^*| \leq 0 \quad \text{for} \quad t \geq 0.
\]

Since spreading happens for \((U, g, h)\), for fixed \( K > \max\{K_0, K_2, K_2\} \) there exists a large constant \( t_0 > 0 \) such that
\[
g(t_0) < -K = -h(0) \quad \text{and} \quad h(0) = K < h(t_0),
\]
\[
U(t_0, x) \geq 1 - \epsilon \geq U(0, x) \quad \text{for} \quad x \in [-h(0), h(0)],
\]
which combined with the conclusions proved in Steps 1 and 2 allows us to apply Lemma 3.2 to conclude that
\[
g(t + t_0) \leq -h(t), \quad h(t + t_0) \geq h(t), \quad t \geq 0,
U(t + t_0, x) \geq U(t, x), \quad t \geq 0, \quad x \in [-h(t), h(t)].
\]

Therefore
\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq \liminf_{t \to \infty} \frac{h(t - t_0)}{t} = c_0 - \delta,
\]
and (3.5) follows by letting \( \delta \to 0 \).

**Proof of Theorem 1.3 (i):** By the above lemmas, we have
\[
\lim_{t \to \infty} \frac{h(t)}{t} = c_0.
\]

It remains to show
\[
\lim_{t \to \infty} \frac{g(t)}{t} = -c_0.
\]

Let \( \tilde{U}(t, x) := U(t, -x), \tilde{h}(t) := -g(t) \) and \( \tilde{g}(t) := -h(t) \). Then \((\tilde{U}, \tilde{g}, \tilde{h})\) satisfies (1.1) with initial function \( \tilde{U}(0, x) := U(0, -x) \), and spreading happens. Hence we can apply Lemmas 3.3 and 3.4 to conclude that
\[
\lim_{t \to \infty} \frac{-g(t)}{t} = \lim_{t \to \infty} \frac{\tilde{h}(t)}{t} = c_0.
\]

**3.3. Accelerated spreading: Proof of Theorem 1.3 (ii).** Throughout this subsection, we assume that the conditions in Theorem 1.3 (ii) are satisfied.

For each positive integer \( n \) and \( i \in \{1, \ldots, m_0\} \), we define
\[
J_i^n(x) := \begin{cases} J_i(x) & \text{for} \ |x| \leq n, \\ \frac{n}{|x|} J_i(x) & \text{for} \ |x| \geq n. \end{cases}
\]

Then clearly \( J_i^n(x) \leq J_i(x) \), \( |x| J_i^n(x) \leq n J_i(x) \), and for each \( \alpha > 0 \), there exists \( c_{n, \alpha} > 0 \) depending on \( n \) and \( \alpha \) such that
\[
e^{\alpha |x|} J_i^n(x) \geq c_{n, \alpha} e^{\frac{\alpha}{2} |x|} J_i(x).
\]

Therefore
\[
\int_0^\infty x J_i^n(x) dx \leq \frac{n}{2}, \quad \int_0^\infty e^{\alpha x} J_i^n(x) dx \geq c_{n, \alpha} \int_0^\infty e^{\frac{\alpha}{2} x} J_i(x) dx.
\]

(3.10)
Next we define \( \tilde{J}_i^n = (\tilde{J}_i^n) \) with

\[
\tilde{J}_i^n(x) := \frac{J_i^n(x)}{\|J_i^n\|_{L^1(\mathbb{R})}} \quad \text{for } i \in \{1, ..., m_0\}, \quad \tilde{J}_i^n(x) \equiv 0 \quad \text{for } i \in \{m_0 + 1, ..., m\}.
\]

Then by \((3.10)\) we see that \( \tilde{J}^n \) satisfies \((J)\) and \((J_1)\), but not \((J_2)\). Moreover, \( J^n \) is nondecreasing in \( n \), and

\[
\lim_{n \to \infty} J^n(x) = \lim_{n \to \infty} \tilde{J}^n(x) = \{J(x) \text{ locally uniformly for } x \in \mathbb{R}.}
\]

We now consider the following variation of \((1.1)\):

\[
\begin{aligned}
&\tilde{U}_t = D \circ \left[ \int_{\tilde{g}(t)}^{\tilde{h}(t)} J^n(x-y) \circ \tilde{U}(t,y)dy - \tilde{U}(t,x) \right] + F(\tilde{U}), \quad t > 0, \ x \in (\tilde{g}(t), \tilde{h}(t)), \\
&\tilde{U}(t, \tilde{g}(t)) = \tilde{U}(t, \tilde{h}(t)) = 0, \quad t > 0, \\
&\tilde{g}'(t) = -\sum_{i=1}^{m_0} \mu_i \int_{\tilde{g}(t)}^{\tilde{h}(t)} J_i^n(x-y)\tilde{u}_i(t,x)dydx, \quad t > 0, \\
&\tilde{h}'(t) = \sum_{i=1}^{m_0} \mu_i \int_{\tilde{g}(t)}^{\tilde{h}(t)} J_i^n(x-y)\tilde{u}_i(t,x)dydx, \quad t > 0, \\
&\tilde{g}(0) = g(T), \ \tilde{h}(0) = h(T), \ \tilde{U}(0, x) = U(T, x), \quad x \in [g(T), h(T)],
\end{aligned}
\]

where \( T > 0, \tilde{U} = (\tilde{u}_i) \), and \((U, g, h)\) with \( U = (u_i)\) is the solution of \((1.1)\) with \( U(0, x) \in [0, \bar{u}] \) and spreading happens.

**Lemma 3.5.** For all large \( n \), problem \((3.12)\) admits a unique positive solution \((\tilde{U}, \tilde{g}, \tilde{h}) = (U^n, g^n, h^n)\) with \( U^n = (u^n_i)\), and

\[
\begin{aligned}
&[g^n(t), h^n(t)] \subset [g(t+T), h(t+T)], \quad t \geq 0, \\
&U^n(t, x) \leq U(t+T, x), \quad t \geq 0, \ x \in [g^n(t), h^n(t)].
\end{aligned}
\]

**Proof.** The first equation in \((3.12)\) can be rewritten as

\[
\tilde{U}_t = D^n \circ \left[ \int_{\tilde{g}(t)}^{\tilde{h}(t)} \tilde{J}_i^n(x-y) \circ \tilde{U}(t,y)dy - \tilde{U}(t,x) \right] + F^n(\tilde{U}),
\]

where \( D^n = (d^n_i) \) with \( d^n_i := d_i \| J_i^n \|_{L^1(\mathbb{R})} \) and

\[
F^n(w) = F(w) + (D^n - D) \circ w, \quad w \in \mathbb{R}^{m_0}.
\]

Since \( F \) satisfies \((f_1)\) for \( i \in \{1, 2, 3\} \), it is easily seen that for all large \( n \), \( F^n \) also satisfies \((f_1)\) for \( i \in \{1, 2, 3\} \) (subject to some obvious modifications). One modification is to replace \( u^* \) by some \( u^n_* \) with \( u^n_* \to u^* \) as \( n \to \infty \), and \( F^n(u^n_*) = 0 \). The existence and uniqueness of the solution to \((3.12)\) now follow by similar arguments to those in \([16, \text{Theorem 4.1}]\). Moreover, due to \( J^n \preceq J \) we can apply Lemma \((3.2)\) to obtain \((3.13)\).

**Lemma 3.6.** For all large \( n \), the system

\[
\begin{aligned}
&D \circ \int_{-\infty}^{0} J^n(x-y) \circ \Phi(y)dy - D \circ \Phi(x) + c\Phi'(x) + F(\Phi(x)) = 0, \quad -\infty < x < 0, \\
&\Phi(-\infty) = u^n_*, \quad \Phi(0) = \Phi^0,
\end{aligned}
\]

and

\[
c = \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_{0}^{\infty} J_i^n(x-y)\phi_i(x)dydx,
\]

has a solution pair \((c, \Phi) = (c^n, \Phi^n)\) with \( \Phi^n(x) = (\phi^n_i(x)) \) monotone in \( x \).
Proof. We already see that for all large $n$, $F^n$ satisfies (f1) for $i \in \{1, 2, 3\}$ (subject to some obvious modifications). In the first equation of (3.15), we can replace $(D, J^n, F)$ by $(D^n, J^n, F^n)$ as in the proof of Lemma 3.5. Moreover, since $\tilde{h}$ does not satisfy (j2), we can argue as in the proof of Theorem 2.12 to show that the corresponding traveling wave problem of (3.15) with any $c > 0$ has no monotone solution. Therefore we can apply Lemma 2.10 to conclude that for any $c > 0$, (3.15) has a monotone solution $\Phi = \Phi_c$. Furthermore, since $\tilde{J}^n$ satisfies (j1), by Theorem 2.14 and the proof of Theorem 2.15 we see that there exists a unique $c^n > 0$ such that (3.16) is satisfied with $c = c^n$ and $(\phi_1) = \Phi^n := \Phi_{c^n}$.

Lemma 3.7. For all large $n$, we have

$$\liminf_{t \to \infty} \frac{-g(t)}{t} \geq c^n, \quad \liminf_{t \to \infty} \frac{h(t)}{t} \geq c^n,$$

where $c^n$ is given in Lemma 3.6.

Proof. Similarly to the proof of Lemma 3.4, we construct a lower solution to (3.12) by defining, for small $\epsilon > 0$, $\delta := 2c^n\epsilon$ and

$$\frac{h^n(t) := (c^n - \delta) + K, \quad t \geq 0,}{U^n(t, x) := (1 - \epsilon)[\Phi^n(x - h^n(t)) + \Phi^n(-x - h^n(t)) - u^n_n]], \quad t \geq 0, \quad x \in [-h^n(t), h^n(t)],}$$

where $K > 0$ is to be determined. Denote $U^n := (u^n_n)$. As in Steps 1 and 2 of the proof of Lemma 3.4, one can choose a large enough constant $K$ such that

$$\begin{aligned}
&\partial_t U^n \leq D \int_{-h^n(t)}^{h^n(t)} J^n(x - y) \circ U^n(y)dy - D \circ U^n + F(U^n), \quad t > 0, \quad x \in (-h^n(t), h^n(t)), \\
&\frac{-1}{h^n(t)} \int_{-h^n(t)}^{h^n(t)} J^n(x - y) \circ U^n(y)dy \leq \frac{1}{h^n(t)} \int_{-h^n(t)}^{h^n(t)} J^n(x - y) \circ U^n(y)dy, \quad t > 0, \\
&\frac{1}{h^n(t)} \int_{-h^n(t)}^{h^n(t)} J^n(x - y) \circ U^n(y)dy \leq \frac{1}{h^n(t)} \int_{-h^n(t)}^{h^n(t)} J^n(x - y) \circ U^n(y)dy, \quad t > 0.
\end{aligned}$$

Clearly, $U^n(t, \pm h^n(t)) = (1 - \epsilon)[\Phi^n(-2h^n(t)) - u^n_n] \to 0$ for $t \geq 0$, and

$$U^n(0, x) = (1 - \epsilon)[\Phi^n(x - K) + \Phi^n(-x - K) - u^n_n] \to (1 - \epsilon)u^n_n \to (1 - \epsilon)u^*$$
as $n \to \infty$. Since spreading happens for $(U, g, h)$, there exists a large constant $T^n > 0$ such that

$$[\limsup_{n \to \infty} (0, x) \leq U^n(T^n, x) \text{ for } x \in [-K, K].$$

Taking $T = T^n$ in (3.12), we have $g_n(0) = g(T^n), h_n(0) = h(T^n)$ and $U_n(0, x) = U(T^n, x)$, and hence we can apply Lemma 3.2 to obtain

$$[-h^n(t), h^n(t)] \subset [g^n(t), h^n(t)], \quad t \geq 0,$$

$$U^n(t, x) \leq U^n(t, x), \quad t \geq 0, \quad x \in [-h^n(t), h^n(t)].$$

It follows that

$$\liminf_{t \to \infty} \frac{h^n(t)}{t} \geq \liminf_{t \to \infty} \frac{h^n(t)}{t} = c^n - \delta,$$

Letting $\delta \to 0$ we obtain $\liminf_{t \to \infty} \frac{h^n(t)}{t} \geq c^n$. We may now use (3.13) to deduce

$$\liminf_{t \to \infty} \frac{h(t)}{t} \geq \liminf_{t \to \infty} \frac{h(t) - T}{t} \geq c^n.$$
Proof. Suppose on the contrary that there is a subsequence of \( \{c^n\}_{n=1}^{\infty} \), still denoted by itself, converging to a finite number \( \hat{c} \). Arguing as in the proof of Lemma 2.10 we see that one of the following two statements must be true:

(i) Problem (1.3) with \( c = \hat{c} \) has a monotone solution \( \Phi = (\phi_i) \), and along some sequence \( n_k \to \infty \),
\[
\Phi^{n_k}(x) \to \Phi(x) \quad \text{locally uniformly for } x \in (-\infty, 0].
\]

(ii) Problem (1.3) has a monotone traveling wave solution with speed \( \hat{c} \).

Since \( (J_2) \) does not hold, it follows from Theorem 2.12 that there is no monotone traveling wave solution for (1.3). Therefore (ii) cannot happen, and necessarily (i) holds.

From (3.16) we obtain
\[
c^n \geq \sum_{i=1}^{m_0} \mu_i \int_{-L_2}^{-L_1} \int_0^\infty J_i^n(x-y)\phi_i^n(x)dydx
\]
for any \( L_2 > L_1 > 0 \). Take \( n = n_k \) and let \( k \to \infty \). By the dominated convergence theorem we obtain
\[
\hat{c} \geq \sum_{i=1}^{m_0} \mu_i \int_{-L_2}^{-L_1} \int_0^\infty J_i(x-y)\phi_i(x)dydx.
\]
Since \( \Phi(-\infty) = u^* \gg 0 \), we can fix \( L_1 > 0 \) large enough such that \( \phi_i(x) \geq u_i^*/2 \) for \( x \leq -L_1 \), and hence
\[
\hat{c} \geq \sum_{i=1}^{m_0} \mu_i \frac{u_i^*}{2} \int_{-L_2}^{-L_1} \int_0^\infty J_i(x-y)dydx \text{ for all } L_2 > L_1.
\]
This implies that, for any \( i \in \{1, \ldots, m_0\} \) with \( \mu_i > 0 \),
\[
\int_0^\infty \int_0^\infty J_i(x-y)dydx = \int_0^\infty xJ_i(x)dx < \infty,
\]
which is a contradiction to the assumption that \( (J_1) \) is not satisfied. Therefore, \( \lim_{n \to \infty} c^n = \infty \).

Theorem 1.3 (ii) clearly follows directly from Lemmas 3.7 and 3.8.

\[\square\]

4. Sharper estimates for the semi-wave and spreading speed

4.1. Asymptotic behaviour of semi-wave solutions to (1.3). The purpose of this subsection is to prove the following three theorems, which imply Theorem 1.7.

**Theorem 4.1.** Suppose that \( F \) satisfies \((f_1) - (f_4)\) and the kernel functions satisfy \((J)\) and \((J^\alpha)\) for some \( \alpha > 0 \). If \( \Phi(x) = (\phi_i(x)) \) is a monotone solution of (1.4) for some \( c > 0 \), then for every \( i \in \{1, \ldots, m\} \),
\[
\int_{-\infty}^{-1} [u_i^* - \phi_i(x)]|x|^\alpha dx < \infty,
\]
which implies, by the monotonicity of \( \Phi(x) \),
\[
0 < |x|^\alpha [u_i^* - \phi_i(x)] \leq C \text{ for some } C > 0 \text{ and } x < 0, \ i \in \{1, \ldots, m\}.
\]

Under the condition \((J)\), if the kernel functions satisfy \((J^\alpha)\) for some \( \alpha = \alpha_0 \geq 1 \), then it is easily seen that \((J^\alpha)\) is satisfied for all \( \alpha \in [0, \alpha_0) \). Therefore if \((J^\alpha)\) is satisfied for some but not for all \( \alpha \geq 1 \), then there exists \( \alpha^* \in [1, \infty) \) such that the kernel functions satisfy \((J^\alpha)\) if and only if \( \alpha \in I^{\alpha^*} = [0, \alpha^*) \) or \([0, \alpha^*) \) (depending on whether or not \((J^{\alpha^*})\) is satisfied), namely
For fixed $\sigma$, then for any $\alpha > 0$, (4.4) Lemma 4.4.

Suppose that $\beta$, $\Phi(\sigma) = (\phi_i(x))$ is a monotone solution of (4.2) for some $c > 0$, then we have

\begin{align*}
\sum_{i=1}^{m} \int_{-\infty}^{0} |u_i - \phi_i(x)||x|^{\alpha-1}dx < \infty \quad \text{for every } \alpha \in I^\alpha. \\
\end{align*}

Therefore, by Theorem 4.1 we have

\begin{align*}
\sum_{i=1}^{m} \int_{-\infty}^{0} |u_i - \phi_i(x)||x|^{\alpha-1}dx = \infty \quad \text{for } \alpha \in (0, \infty) \setminus I^\alpha. \\
\end{align*}

The next result shows that this estimate is sharp.

**Theorem 4.2.** Suppose that $F$ satisfies $(f_1) - (f_4)$ and the kernel functions satisfy $(J)$. If $(J^\alpha)$ is not satisfied for some $\alpha > 0$, and $\Phi(x) = (\phi_i(x))$ is a monotone solution of (4.4) for some $c > 0$, then

\begin{align*}
\sum_{i=1}^{m} \int_{-\infty}^{0} |u_i - \phi_i(x)||x|^{\alpha-1}dx = \infty.
\end{align*}

**Theorem 4.3.** Suppose that $F$ satisfies $(f_1) - (f_4)$ and the kernel functions satisfy $(J)$. If $(J_2)$ holds, and $\Phi(x) = (\phi_i(x))$ is a monotone solution of (4.4) for some $c > 0$, then there exist positive constants $\beta$ and $C$ such that

\begin{align*}
0 < u_i - \phi_i(x) \leq Ce^{\beta x} \quad \text{for all } x < 0, \quad i \in \{1, \ldots, m\}.
\end{align*}

The following lemmas play a crucial role in the proof of Theorem 4.1

**Lemma 4.4.** Suppose that $J(x)$ has the properties described in $(J)$ and satisfies $(J^\alpha)$ for some $\alpha \geq 1$. If $\psi \in L^1((-\infty, 0])$ is nonnegative, continuous and nondecreasing in $(-\infty, 0]$, and

\begin{align*}
\int_{-\infty}^{0} |x|^\beta \psi(x)dx < \infty \quad \text{for some } \beta \geq 0,
\end{align*}

then for any $\sigma \in (0, \min\{\beta + 1, \alpha\}]$, there exists $C > 0$ such that

\begin{align*}
I = I_M := \int_{-M}^{0} |x|^\sigma \left[ \int_{-\infty}^{0} J(x-y)\psi(y)dy - \psi(x) \right]dx \in [-C, C] \quad \text{for all } M > 0.
\end{align*}

**Proof.** For fixed $M > 0$ we have

\begin{align*}
\int_{-M}^{0} \int_{-\infty}^{0} |x|^\sigma J(x-y)\psi(y)dydx &= \int_{0}^{M} \int_{-\infty}^{x} x^\sigma J(y)\psi(y-x)dydx \\
&= \int_{0}^{M} \int_{-\infty}^{x} x^\sigma J(y)\psi(y-x)dydx + \int_{0}^{M} \int_{0}^{x} x^\sigma J(y)\psi(y-x)dydx \\
&= \int_{-\infty}^{0} \int_{0}^{M} x^\sigma J(y)\psi(y-x)dydx + \int_{0}^{M} \int_{0}^{y} x^\sigma J(y)\psi(y-x)dxdy \\
&= \int_{-\infty}^{0} \int_{-y}^{M-y} (x+y)^\sigma J(y)\psi(-x)dxdy + \int_{0}^{M} \int_{0}^{M-y} (x+y)^\sigma J(y)\psi(-x)dxdy,
\end{align*}

and

\begin{align*}
\int_{-M}^{0} |x|^\sigma \psi(x)dx &= \int_{\mathbb{R}} \int_{0}^{M} x^\sigma J(y)\psi(-x)dxdy.
\end{align*}
Therefore we can write

\[ I = \sum_{j=1}^{3} I_j \]

with

\[
I_1 := \int_{-\infty}^{0} \int_{-y}^{M-y} [(x+y)\sigma - x^\sigma] J(y)\psi(-x) dx dy
\]

\[ + \int_{0}^{M} \int_{-y}^{M-y} [(x+y)\sigma - x^\sigma] J(y)\psi(-x) dx dy, \]

\[
I_2 := \int_{-\infty}^{0} \int_{M}^{M-y} x^\sigma J(y)\psi(-x) dx dy - \int_{-\infty}^{0} \int_{0}^{M-y} x^\sigma J(y)\psi(-x) dx dy,
\]

\[
I_3 := - \int_{0}^{M} \int_{M-y}^{M} x^\sigma J(y)\psi(-x) dx dy - \int_{M}^{\infty} \int_{0}^{M} x^\sigma J(y)\psi(-x) dx dy.
\]

To estimate \( I_1 \) we will make use of some elementary inequalities. If \( s, t > 0 \) and \( \sigma \in (0, 1] \), then it is easily checked that

\[
(4.5) \quad (s + t)^\sigma - s^\sigma \leq t^\sigma.
\]

If \( \sigma = n + \theta \) with \( n \geq 1 \) an integer, and \( \theta \in (0, 1] \), then by the mean value theorem

\[
(s + t)^\sigma - s^\sigma = \sigma(s + \zeta t)^{\sigma - 1} t \leq \sigma t(s + t)^{\sigma - 1} = \sigma t s^{\sigma - 1} + \sigma t \left[(s + t)^{\sigma - 1} - s^{\sigma - 1}\right]
\]

\[
\leq \sum_{k=1}^{\infty} \left[\Pi_{j=0}^{k-1} (\sigma - j) t^{\sigma - k}\right] + \Pi_{j=0}^{n-1} (\sigma - j) t^n \left[(s^\theta + t^\theta) - s^\theta\right]
\]

\[
\leq \sum_{k=1}^{\infty} \left[\Pi_{j=0}^{k-1} (\sigma - j) t^{\sigma - k}\right] + \Pi_{j=0}^{n-1} (\sigma - j) t^{n+\theta}
\]

\[
= \sum_{k=1}^{n} c_k t^k s^{\sigma - k} + c_{n+1} t^\sigma
\]

where \( \zeta \in [0, 1] \), and \( c_k = c_k(\sigma) > 0 \) for \( k \in \{1, ..., n+1\} \).

Applying this inequality to \((x+y)^\sigma - x^\sigma\) with \( x + y > 0 \) and \( x > 0 \), we obtain, for the case \( \sigma > 1 \),

\[
|(x+y)^\sigma - x^\sigma| \leq \sum_{k=1}^{n} c_k |y|^k x^{\sigma - k} + c_{n+1} |y|^\sigma
\]

with \( \sigma - n = \theta \in (0, 1] \) and \( n \geq 1 \) an integer, \( c_k = c_k(\sigma) > 0 \) for \( k \in \{1, ..., n+1\} \).

Therefore, in the case \( \sigma > 1 \),

\[
|I_1| \leq \int_{-\infty}^{0} \int_{-y}^{M-y} \left[\sum_{k=1}^{n} c_k |y|^k x^{\sigma - k} + c_{n+1} |y|^\sigma\right] J(y)\psi(-x) dx dy
\]

\[ + \int_{0}^{M} \int_{-y}^{M-y} \left[\sum_{k=1}^{n} c_k |y|^k x^{\sigma - k} + c_{n+1} |y|^\sigma\right] J(y)\psi(-x) dx dy,
\]

\[
\leq 2 \sum_{k=1}^{n} c_k \int_{0}^{\infty} x^{\sigma - k} \psi(-x) dx \int_{0}^{\infty} y^k J(y) dy + 2c_{n+1} \int_{0}^{\infty} \psi(-x) dx \int_{0}^{\infty} y^\sigma J(y) dy
\]

\[
:= C_1.
\]

Since \( 1 \leq k \leq n < \sigma \leq \min\{\beta + 1, \alpha\} \), by the assumptions on \( J \) and \( \psi \) we see that \( C_1 \) is a finite number.
If \( \sigma \in (0, 1] \), then
\[
|I_1| \leq \int_{-\infty}^{0} \int_{-y}^{M-y} |y|^\sigma J(y) \psi(-x) \, dx \, dy + \int_{0}^{M} \int_{-y}^{M-y} |y|^\sigma J(y) \psi(-x) \, dx \, dy
\]
\[
\leq 2 \int_{0}^{\infty} \psi(-x) \, dx \int_{0}^{\infty} y^\sigma J(y) \, dy := \tilde{C}_1 < \infty.
\]
Since \( \psi(x) \) is nondecreasing, from (4.4) we easily deduce
\[
\psi(-x) \leq \frac{M_1}{x^\sigma} \text{ for some } M_1 > 0 \text{ and all } x > 0.
\]
Similarly, using \((J^\alpha)\) we obtain
\[
M \int_{M}^{\infty} J(y) \, dy \leq M^{1-\alpha} \int_{M}^{\infty} y^\alpha J(y) \, dy \leq \int_{1}^{\infty} y^\alpha J(y) \, dy := M_2 \text{ for } M \geq 1,
\]
and hence
\[
M \int_{M}^{\infty} J(y) \, dy \leq \min \left\{ \int_{0}^{\infty} J(y), M_2 \right\} := M_3 < \infty \text{ for all } M > 0.
\]
Therefore
\[
|I_2| \leq \int_{-\infty}^{0} \int_{M}^{M-y} M_1 J(y) \, dx \, dy + \int_{-\infty}^{0} \int_{0}^{y} M_1 J(y) \, dx \, dy
\]
\[
= 2M_1 \int_{0}^{\infty} yJ(y) \, dy := C_2 < \infty,
\]
and
\[
|I_3| \leq \int_{0}^{M} M_1 yJ(y) \, dy + \int_{M}^{\infty} M_1 M J(y) \, dy
\]
\[
\leq M_1 \int_{0}^{\infty} yJ(y) \, dy + M_1 M_3 := C_3 < \infty.
\]
We thus have
\[
|I| \leq C_1 + \tilde{C}_1 + C_2 + C_3 := C < \infty \text{ for all } M > 0.
\]
The proof is complete. \( \square \)

**Lemma 4.5.** Suppose that \( J(x) \) has the properties described in \((J)\) and satisfies \((J^\alpha)\) for some \( \alpha \in (0, 1) \). Let \( \psi \) be nonnegative, continuous and nondecreasing in \(( -\infty, 0 ] \). Then there exists \( C > 0 \) such that
\[
S = S_M := \int_{-M}^{0} |x|^{\sigma-1} \int_{-\infty}^{0} J(x-y) \psi(y) dy \, dx - \psi(x) \, dx \leq C \text{ for all } M > 0.
\]

**Proof.** As in the proof of Lemma 4.6, we deduce for fixed \( M > 0 \) and \( \sigma > -1 \),
\[
\int_{-M}^{0} \int_{-\infty}^{0} |x|^\sigma J(x-y) \psi(y) \, dy \, dx
\]
\[
= \int_{-\infty}^{0} \int_{-y}^{M-y} (x+y)^\sigma J(y) \psi(-x) \, dx \, dy + \int_{0}^{M} \int_{0}^{M-y} (x+y)^\sigma J(y) \psi(-x) \, dx \, dy.
\]
and
\[
\int_{-M}^{0} |x|^\sigma \psi(x) \, dx = \int_{\mathbb{R}} \int_{0}^{M} |x|^\sigma J(y) \psi(-x) \, dx \, dy.
\]
Hence
\[
S = \sum_{i=1}^{3} \tilde{I}_i
\]
Therefore we can use the same reasoning as in the proof of Lemma 2.1 to find two vectors $u$ such that, for $x > 0$ and $y > 0$,

$$(x + y)^a - x^a < 0$$

and hence, by \((J^a)\) and $\sigma + 1 = \alpha \in (0, 1)$,

$$I_1 := \int_{-\infty}^{0} \int_{-\infty}^{M-y} [(x + y)^a - x^a] J(y)\psi(-x) dxdy$$

$$\leq \psi(0) \int_{-\infty}^{0} \int_{-\infty}^{M-y} [(x + y)^a - x^a] J(y) dxdy$$

$$= \frac{\psi(0)}{\sigma + 1} \int_{-\infty}^{0} \int_{-\infty}^{M-y} [(M - y)^{\sigma+1} - (M - y)^{\sigma+1}] J(y) dxdy$$

$$\leq \frac{\psi(0)}{\sigma + 1} \int_{-\infty}^{0} \int_{-\infty}^{M-y} (M - y)^{\sigma+1} J(y) dxdy$$

Moreover, by \((J^a)\), $\sigma + 1 = \alpha \in (0, 1)$ and \((4.5)\),

$$I_2 := \int_{-\infty}^{0} \int_{M}^{M-y} x^a J(y)\psi(-x) dxdy$$

$$\leq \psi(0) \int_{-\infty}^{0} \int_{M}^{M-y} x^a J(y) dxdy$$

$$= \frac{\psi(0)}{\sigma + 1} \int_{-\infty}^{0} \int_{M}^{M-y} [(M - y)^{\sigma+1} - (M - y)^{\sigma+1}] J(y) dxdy$$

$$\leq \frac{\psi(0)}{\sigma + 1} \int_{0}^{\infty} y^{\sigma+1} J(y) dy := C_2 < \infty.$$
for some \( b > 0 \).

Since \( \Psi(-\infty) = 0 \) and \( \Psi(x) = (\psi_i(x)) \Rightarrow 0 \) for \( x < 0 \), we have \( 0 < \psi_i(x) < \epsilon \) for \( x \ll -1 \), and so

\[
(4.7) \quad \sum_{i=1}^{m} \tilde{a}_i g_i(\Psi(x)) \leq -b\tilde{\psi}(x) \quad \text{for} \quad x \ll -1, \quad \text{with}
\]

\[
(4.8) \quad \tilde{\psi}(x) := \sum_{j=1}^{m} \tilde{a}_j \psi_j(x).
\]

**Lemma 4.6.** Suppose \((J)\) and \((f_1) - (f_4)\) are satisfied. If \((J^0)\) holds for some \( \alpha \geq 1 \), then

\[
\int_{-\infty}^{0} \tilde{\psi}(x)dx < \infty.
\]

**Proof.** A simple calculation gives

\[
D \circ \int_{-\infty}^{0} J(x-y) \circ \Psi(y)dy - D \circ \Psi + D \circ u^* \circ \int_{0}^{\infty} J(x-y)dy
\]

\[
= -D \circ \int_{-\infty}^{0} J(x-y) \circ \Phi(y)dy + D \circ \Phi.
\]

Integrating the equation satisfied by \( \tilde{\psi} \) over the interval \((x, y)\) with \( x < y \ll -1 \), and making use of (4.7), we obtain

\[
c(\tilde{\psi}(y) - \tilde{\psi}(x)) + \sum_{i=1}^{m} y \int_{x}^{y} a_i d_i \left[ \int_{-\infty}^{0} J_i(z-w)\psi_i(w)dw - \psi_i(z) \right] dz
\]

\[
+ \sum_{i=1}^{m} \int_{x}^{y} a_i d_i u_i^* \int_{0}^{\infty} J_i(z-w)dw dz
\]

\[
= c(\tilde{\psi}(y) - \tilde{\psi}(x)) - \sum_{i=1}^{m} y \int_{x}^{y} a_i d_i \left[ \int_{-\infty}^{0} J_i(z-w)\psi_i(w)dw - \psi_i(z) \right] dz
\]

\[
= -\int_{x}^{y} \sum_{i=1}^{m} \tilde{a}_i g_i(\Psi(z))dz \geq b \int_{x}^{y} \tilde{\psi}(z)dz.
\]

We extend \( \Phi \) to \( \mathbb{R} \) by define \( \phi_i(x) = 0 \) for \( x > 0 \). Then the new function \( \Phi \) is differentiable on \( \mathbb{R} \) except at \( x = 0 \). Due to \((J^{\alpha})\), we have, for \( i \in \{1, ..., m_0\}, \)

\[
\left| \int_{x}^{y} \left( \int_{-\infty}^{0} J_i(z-w)\psi_i(w)dw - \psi_i(z) \right) dz \right| = \left| \int_{x}^{y} \left( \int_{\mathbb{R}} J_i(z-w)\phi_i(w)dw - \phi_i(z) \right) dz \right|
\]

\[
= \left| \int_{\mathbb{R}} J_i(w) \phi_i(z+w) - \phi_i(z) dw dz \right| = \left| \int_{x}^{y} \int_{\mathbb{R}} J_i(w) \int_{0}^{1} w\phi'_i(z+sw)ds dw dz \right|
\]

\[
\leq a_i^* \int_{\mathbb{R}} |y| |J_i(y)|dy =: M_i < \infty.
\]

Thus, for \( x < y \ll -1, \)

\[
b \int_{x}^{y} \tilde{\psi}(z)dz \leq c(\tilde{\psi}(y) - \tilde{\psi}(x)) + \sum_{i=1}^{m} \tilde{a}_i d_i M_i \leq \sum_{i=1}^{m} \tilde{a}_i (cu_i^* + d_i M_i),
\]

which implies \( \int_{-\infty}^{0} \tilde{\psi}(z)dz < \infty. \) \( \square \)
Proof of Theorem 4.1  Case 1. \( \alpha > 1 \).

With \( \psi = \sum_{i=1}^{m} \tilde{a}_i \psi_i \) given by (4.8), it suffices to show
\[
\int_{-\infty}^{0} \tilde{\psi}(x)|x|^{\alpha-1} \, dx < \infty.
\]

By Lemma 4.6 we have
\[
\int_{-\infty}^{0} \tilde{\psi}(x) \, dx < \infty \quad \text{and hence} \quad \int_{-\infty}^{0} \psi_i(x) \, dx < \infty \quad \text{for} \quad i \in \{1, \ldots, m\}.
\]

So there is nothing to prove if \( \alpha = 1 \), and we only need to consider the case \( \alpha > 1 \).

Suppose \( \alpha > 1 \) and
\[
\int_{-\infty}^{0} |x|^\gamma \tilde{\psi}(x) \, dx < \infty \quad \text{for some} \quad \gamma \geq 0.
\]

Then by Lemma 4.4 for any \( \beta \) satisfying \( 0 < \beta \leq \min\{\gamma + 1, \alpha\} \), and \( i \in \{1, \ldots, m_0\} \),
\[
\int_{-M}^{0} \left[ \int_{-\infty}^{0} J_i(x-y)\psi_i(y) \, dy - \psi_i(x) \right] |x|^{\beta} \, dx \leq C \quad \text{for some} \quad C > 0 \quad \text{and all} \quad M < 0.
\]

Moreover, if we fix \( M_0 > 1 \) so that (4.7) holds for \( x \leq -M_0 \), then for \( M < M_0 \) and \( \beta \) as above, we have
\[
\begin{align*}
&\hat{b} \int_{-M}^{-M_0} \tilde{\psi}(x) |x|^\beta \, dx \\
&\leq - \sum_{i=1}^{m} \int_{-M}^{-M_0} \tilde{a}_i g_i(\Psi(x)) |x|^\beta \, dx \\
&= c \int_{-M}^{-M_0} \tilde{\psi}'(x) |x|^\beta \, dx + \sum_{i=1}^{m} \tilde{a}_i d_i \int_{-M}^{-M_0} \left[ \int_{-\infty}^{0} J_i(x-y)\psi_i(y) \, dy - \psi_i(x) \right] |x|^\beta \, dx \\
&\quad + \sum_{i=1}^{m} \tilde{a}_i d_i u_i^* \int_{-M}^{-M_0} \int_{0}^{\infty} |x|^{\beta} J_i(x-y) \, dy \, dx.
\end{align*}
\]

By (4.10),
\[
\begin{align*}
&\sum_{i=1}^{m_0} \tilde{a}_i d_i \int_{-M}^{-M_0} \left[ \int_{-\infty}^{0} J_i(x-y)\psi_i(y) \, dy - \psi_i(x) \right] |x|^\beta \, dx \\
&\leq C \sum_{i=1}^{m_0} \tilde{a}_i d_i \left( - \sum_{i=1}^{m_0} \tilde{a}_i d_i \int_{-M}^{-M_0} \int_{-\infty}^{0} J_i(x-y)\psi_i(y) \, dy - \psi_i(x) \right) |x|^\beta \, dx \\
&=: C_1 < \infty \quad \text{for all} \quad M > M_0.
\end{align*}
\]

Moreover, if we assume additionally that \( \beta \leq \alpha - 1 \), then we have, for \( i \in \{1, \ldots, m_0\} \),
\[
\begin{align*}
&\int_{-M}^{-M_0} \int_{0}^{\infty} |x|^{\beta} J_i(x-y) \, dy \, dx \\
&\leq \int_{0}^{M} \int_{0}^{\infty} x^{\beta} J_i(x+y) \, dy \, dx = \int_{0}^{M} \int_{x}^{\infty} x^{\beta} J_i(y) \, dy \, dx \\
&\leq \int_{0}^{\infty} \int_{x}^{\infty} x^{\beta} J_i(y) \, dy = \frac{1}{\beta + 1} \int_{0}^{\infty} y^{\beta+1} J_i(y) \, dy := C_2 < \infty.
\end{align*}
\]

Therefore, for \( \beta \in (0, \min\{\gamma + 1, \alpha - 1\}] \) and \( M > M_0 \),
\[
\hat{b} \int_{-M}^{-M_0} \tilde{\psi}(x) |x|^\beta \, dx \leq c \int_{-M}^{-M_0} \tilde{\psi}'(x) |x|^\beta \, dx + C_1 + \sum_{i=1}^{m} \tilde{a}_i d_i u_i^* C_2
\]
\[
\int_{-1}^{\infty} \beta(x)|x|^\beta dx < \infty.
\]

It follows that

\[(4.11)\]

\[
\int_{-\infty}^{0} \beta(x)|x|^\beta dx < \infty.
\]

Thus we have proved that \[(4.9)\] implies \[(4.11)\] for any \(\beta \in (0, \min\{\gamma + 1, \alpha - 1\})\).

If we write \(\alpha - 1 = n + \theta\) with \(n \geq 0\) an integer and \(\theta \in (0, 1]\). Then by the above conclusion and an induction argument we see that \[(4.11)\] holds with \(\beta = n\). Thus \[(4.9)\] holds for \(\gamma = n\).

So applying the above conclusion once more we see that \[(4.11)\] holds for every \(\beta \in (0, \min\{n + 1, \alpha - 1\}) = (0, \alpha - 1]\), as desired.

**Case 2.** \(\alpha \in (0, 1]\).

Let \(\beta = \alpha - 1\). As in Case 1, for \(M > M_0\),

\[
\int_{-M}^{-M_0} \bar{\psi}(x)|x|^\beta dx \\
\leq c \int_{-M}^{-M_0} \bar{\psi}'(x)|x|^\beta dx + c \int_{-M}^{-M_0} \bar{\psi}'(-x)|x|^\beta dx \\
\leq c \int_{-M}^{-M_0} \bar{\psi}'(x)|x|^\beta dx + c \int_{-M}^{-M_0} \bar{\psi}'(-x)|x|^\beta dx \\
\leq c \int_{-M}^{-M_0} \bar{\psi}'(x)|x|^\beta dx + c \int_{-M}^{-M_0} \bar{\psi}'(-x)|x|^\beta dx.
\]

where \(\bar{C}_1 > 0\) is obtained by making use of Lemma \(4.5\). By \((J_\alpha)\) and \(\beta + 1 = \alpha\),

\[
\int_{-M}^{-M_0} \int_{0}^{\infty} |x|^\beta J_\alpha(x-y) dy dx \\
\leq \int_{0}^{\infty} x^{\beta} J_\alpha(y) dy dx \\
= \frac{1}{\alpha} \int_{0}^{\infty} y^{\alpha} J_\alpha(y) dy := \bar{C}_2 < \infty.
\]

Due to \(\beta < 0\), we have

\[
\int_{-M}^{-M_0} \bar{\psi}'(x)|x|^\beta dx = \int_{M_0}^{M} \bar{\psi}'(-x)|x|^\beta dx \\
= \bar{\psi}(-M_0)M_0^\beta - \bar{\psi}(-M)M^\beta + \beta \int_{M_0}^{M} \bar{\psi}(-x)x^{\beta-1} dx \\
\leq \bar{\psi}(-M_0)M_0^\beta := \bar{C}_3 < \infty.
\]

Hence

\[
\bar{b} \int_{-M}^{-M_0} \bar{\psi}(x)|x|^\beta dx \leq \bar{C}_1 + \bar{C}_2 \sum_{i=1}^{m_0} \bar{a}_i d_i \bar{u}_i + c\bar{C}_3 < \infty
\]

for all \(M > M_0\), which implies

\[
\int_{-\infty}^{-1} \bar{\psi}(x)|x|^\alpha dx < \infty.
\]

The proof is completed.
Proof of Theorem 4.2 We have
\[ |g_i(\Psi(x))| \leq L \sum_{j=1}^{m} \psi_j(x) := L \hat{\psi}(x) \] for some \( L > 0 \) and all \( x < 0 \), \( i \in \{1, \ldots, m\} \).

Now for \( M > 1 \) and \( \beta = \alpha - 1 \),
\[
L \int_{-M}^{-1} \hat{\psi}(x)|x|^\beta \, dx \geq - \sum_{i=1}^{m} \int_{-M}^{-1} g_i(\Psi(x))|x|^\beta \, dx \\
= c \int_{-M}^{-1} \hat{\psi}'(x)|x|^\beta \, dx + \sum_{i=1}^{m_0} \int_{-M}^{0} \left[ \int_{-\infty}^{0} J_i(x-y) \psi_i(y) \, dy - \psi_i(x) \right] |x|^\beta \, dx \\
+ \sum_{i=1}^{m_0} d_i u_i^* \int_{-M}^{-1} \int_{0}^{\infty} |x|^\beta J_i(x-y) \, dy \, dx \\
\geq - \sum_{i=1}^{m_0} \int_{-M}^{-1} \psi_i(x)|x|^\beta \, dx + \sum_{i=1}^{m_0} d_i u_i^* \int_{-M}^{0} \int_{0}^{\infty} |x|^\beta J_i(x-y) \, dy \, dx
\]
Therefore, with \( \bar{L} := L + \sum_{i=1}^{m_0} d_i \), we have
\[
\bar{L} \int_{-M}^{-1} \hat{\psi}(x)|x|^\beta \, dx \geq \sum_{i=1}^{m_0} d_i u_i^* \int_{-M}^{-1} \int_{0}^{\infty} |x|^\beta J_i(x-y) \, dy \, dx \\
= \sum_{i=1}^{m_0} d_i u_i^* \int_{1}^{M} \int_{0}^{\infty} x^\beta J_i(y) \, dy \, dx \\
= \sum_{i=1}^{m_0} d_i u_i^* \left[ \int_{1}^{M} \int_{0}^{\infty} \int_{1}^{M} x^\beta J_i(y) \, dy \, dx \right] \\
= \sum_{i=1}^{m_0} \frac{d_i u_i^*}{\beta + 1} \left[ \int_{1}^{M} y^{\beta+1} J_i(y) \, dy - \int_{1}^{\infty} J_i(y) \, dy \right] \to \infty \text{ as } M \to \infty,
\]
since \( \beta + 1 = \alpha \). Therefore, (4.2) holds, as we wanted.

To prove Theorem 4.3 we need the following lemma.

Lemma 4.7. Let the assumptions in Theorem 4.3 be satisfied and \( \Psi(x) = (\psi_i(x)) =: u^* - \Phi(x) \). Then for every small \( \epsilon > 0 \), there exist \( \beta = \beta(\epsilon) \in (0, \lambda) \) and \( C = C(\epsilon) > 0 \) such that for all \( M > 0 \) and \( i \in \{1, \ldots, m\} \),

\[
(4.12) \quad Q^{(i)} = Q^{(i)}_M := \int_{-M}^{0} e^{-\beta x} \int_{-\infty}^{0} J_i(x-y) \psi_i(y) \, dy \, dx \leq (1 + \epsilon) \int_{-M}^{0} e^{-\beta x} \psi_i(x) \, dx + C.
\]

Proof. By a change of variables, we deduce
\[
Q^{(i)} = \int_{-M}^{0} e^{-\beta x} \int_{-\infty}^{x} J_i(y) \psi_i(x+y) \, dy \, dx = \int_{0}^{M} \int_{-\infty}^{x} e^{\beta x} J_i(y) \psi_i(y-x) \, dy \, dx \\
= \int_{0}^{M} \left( \int_{-\infty}^{0} + \int_{0}^{x} \right) e^{\beta x} J_i(y) \psi_i(y-x) \, dy \, dx \\
= \int_{-\infty}^{0} \int_{0}^{M} e^{\beta x} J_i(y) \psi_i(y-x) \, dx \, dy + \int_{0}^{M} \int_{y}^{M} e^{\beta x} J_i(y) \psi_i(y-x) \, dx \, dy
\]
We have
$$
I = \int_{-M}^{0} e^{\beta y} J_i(y) \int_{-y}^{M-y} e^{\beta x} \psi_i(-x) dx dy + \int_{-\infty}^{-M} e^{\beta y} J_i(y) \int_{-y}^{M-y} e^{\beta x} \psi_i(-x) dx dy
$$
$$
= \int_{-M}^{0} e^{\beta y} J_i(y) \left( \int_{-y}^{M} + \int_{M}^{M-y} \right) e^{\beta x} \psi_i(-x) dx dy + \int_{-\infty}^{-M} e^{\beta y} J_i(y) \int_{-y}^{M-y} e^{\beta x} \psi_i(-x) dx dy
$$
$$
= \int_{-M}^{0} e^{\beta y} J_i(y) \int_{-y}^{M-y} e^{\beta x} \psi_i(-x) dx dy + \int_{-M}^{0} e^{\beta y} J_i(y) \int_{M}^{M-y} e^{\beta x} \psi_i(-x) dx dy
$$
$$
+ \int_{-\infty}^{-M} e^{\beta y} J_i(y) \int_{-y}^{M-y} e^{\beta x} \psi_i(-x) dx dy
$$
$$
:= B_1^{(i)} + A_1^{(i)} + A_2^{(i)},
$$
and
$$
II = \int_{0}^{M} e^{\beta y} J_i(y) \int_{0}^{M} e^{\beta x} \psi_i(-x) dx dy - \int_{0}^{M} e^{\beta y} J_i(y) \int_{M}^{M-y} e^{\beta x} \psi_i(-x) dx dy
$$
$$
:= B_2^{(i)} + A_3^{(i)}.
$$
Hence,
$$
Q^{(i)} = I + II = (B_1^{(i)} + B_2^{(i)}) + (A_1^{(i)} + A_2^{(i)} + A_3^{(i)})
$$
$$
\leq \int_{-M}^{0} e^{\beta y} J_i(y) \int_{-y}^{M} e^{\beta x} \psi_i(-x) dx dy + \int_{0}^{M} e^{\beta y} J_i(y) \int_{0}^{M} e^{\beta x} \psi_i(-x) dx dy
$$
$$
+ (A_1^{(i)} + A_2^{(i)} + A_3^{(i)})
$$
$$
= \int_{-M}^{0} e^{\beta y} J_i(y) dy \int_{0}^{M} e^{\beta x} \psi_i(-x) dx + (A_1^{(i)} + A_2^{(i)} + A_3^{(i)}).
$$
Set
$$
P(\gamma) := \int_{\mathbb{R}} e^{\gamma y} J_i(y) dy = \int_{0}^{\infty} [e^{\gamma y} + e^{-\gamma y}] J_i(y) dy.
$$
Clearly $P(\gamma)$ is increasing and continuous in $\gamma \in [0, \alpha]$, with $P(0) = 1$. Hence there exists small $\beta_* = \beta_*(\epsilon) \in (0, \lambda]$ such that for all $0 < \beta \leq \beta_*(\epsilon)$,
$$
P(\beta) = \int_{\mathbb{R}} e^{\beta y} J_i(y) dy \leq 1 + \epsilon.
$$
Thus, for such $\beta$,
$$
Q^{(i)} \leq (1 + \epsilon) \int_{0}^{M} e^{\beta x} \psi_i(-x) dx + (A_1^{(i)} + A_2^{(i)} + A_3^{(i)}).
$$
It remains to verify that $A_1^{(i)} + A_2^{(i)} + A_3^{(i)}$ has an upper bound which is independent of $M \in (0, \infty)$. Using the monotonicity of $\psi_i$, we deduce
$$
A_1^{(i)} + A_3^{(i)} = \int_{-M}^{0} e^{\beta y} J_i(y) \int_{-y}^{M-y} e^{\beta x} \psi_i(-x) dx dy - \int_{0}^{M} e^{\beta y} J_i(y) \int_{M}^{M-y} e^{\beta x} \psi_i(-x) dx dy
$$
$$
\leq \psi_i(-M) \int_{-M}^{0} e^{\beta y} J_i(y) \int_{-y}^{M-y} e^{\beta x} dx dy - \psi_i(-M) \int_{0}^{M} e^{\beta y} J_i(y) \int_{M}^{M-y} e^{\beta x} dx dy
$$
By Lemma 4.7, there exist \( \beta_0 \) and \( M_0 \),

\[
\int_0^M e^{\beta y} J_i(y) \left[ e^{\beta M} - e^{\beta(M-y)} \right] dy = \psi_i(-M) \frac{e^{\beta M}}{\beta} \int_{-M}^0 J_i(y) \left[ 1 - e^{\beta y} \right] dy
\]

This together with (4.12) implies

\[
\int_0^M J_i(y) \left[ 2 - e^{-\beta y} - e^{-\beta y} \right] dy \leq 0,
\]

and

\[
A_2^{(i)} = \int_{-\infty}^{-M} e^{\beta y} J_i(y) \int_{-y}^{M-y} e^{\beta x} \psi_i(-x) dx dy \leq \psi_i(0) \int_{-\infty}^{-M} e^{\beta y} J_i(y) \int_{-y}^{M-y} e^{\beta x} dx dy
\]

\[
= \frac{u_i^*(e^{\beta M} - 1)}{\beta} \int_{-\infty}^{\infty} e^{\beta y} J_i(y) dy \leq \frac{u_i^*(e^{\beta M} - 1)}{\beta} \int_0^{\infty} e^{\beta y} J_i(y) dy := C < \infty,
\]

since \( \beta \leq \lambda \). Hence (4.12) holds.

Proof of Theorem 4.3: With \( \tilde{\psi} = \sum_{i=1}^m \tilde{a}_i \psi_i \) given by (4.8), it suffices to show that there exists \( \beta \in (0, \lambda] \) such that

\[
\tilde{\psi}(x) = O(e^{\beta x}) \text{ for large negative } x.
\]

By Lemma 4.7 there exist \( \epsilon > 0 \) and \( \beta \in (0, \lambda] \) small such that (4.12) holds and \( \hat{b} \geq \sum_{i=1}^m \tilde{a}_i d_i \epsilon + c \beta \). Multiplying \( e^{-\beta x} \) on both sides of the equation satisfied by \( \tilde{\psi} \) and then integrating the resulting equation over the interval \([-M, 0]\) with an arbitrary \( M > 0 \), we obtain

\[
\sum_{i=1}^m \tilde{a}_i d_i \int_{-M}^{0} \tilde{a}_i g_i(\Psi(x)) e^{-\beta x} dx - \int_{-M}^{0} c \tilde{\psi}'(x)(-x)^\beta dx
\]

\[
(4.13)
\]

\[
= \sum_{i=1}^m \tilde{a}_i d_i \int_{-M}^{0} \left[ \int_{-\infty}^{0} J_i(x-y) \psi_i(y) dy - \psi_i(x) \right] e^{-\beta x} dx
\]

\[
+ \sum_{i=1}^m \tilde{a}_i d_i u_i^* \int_{-M}^{0} e^{-\beta x} \int_{0}^{\infty} J_i(x-y) dy dx =: S_1(M) + S_2(M).
\]

In view of (J2) and \( \beta \in (0, \lambda] \), we have

\[
S_2(M) = \sum_{i=1}^m \tilde{a}_i d_i u_i^* \int_{-M}^{0} e^{-\beta x} \int_{-\infty}^{\infty} J_i(y) dy dx \leq \sum_{i=1}^m \tilde{a}_i d_i u_i^* \int_{-\infty}^{0} e^{-\beta x} \int_{-\infty}^{\infty} J_i(y) dy dx
\]

\[
= \sum_{i=1}^m \tilde{a}_i d_i u_i^* \int_{-\infty}^{0} \int_{-\infty}^{\infty} e^{-\beta x} J_i(y) dy dx = \sum_{i=1}^m \tilde{a}_i d_i u_i^* \frac{\beta}{\beta} \int_{0}^{\infty} [e^{\beta y} - 1] J_i(y) dy < \infty.
\]

This together with (4.12) implies

\[
(4.14)
S_1(M) + S_2(M) \leq \sum_{i=1}^m \tilde{a}_i d_i \epsilon \int_{-M}^{0} e^{-\beta x} \psi_i(x) dx + C_1
\]

for some \( C_1 > 0 \) independent of \( M \).

On the other hand, by (4.7) and \( \hat{b} \geq \sum_{i=1}^m \tilde{a}_i d_i \epsilon + c \beta \) we obtain, for \( M > M_0 \gg 1 \),

\[
- \sum_{i=1}^m \int_{-M}^{0} \tilde{a}_i g_i(\Psi(x)) e^{-\beta x} dx - \int_{-M}^{0} c \tilde{\psi}'(x) e^{-\beta x} dx
\]

\[
\geq \hat{b} \int_{-M}^{0} \tilde{\psi}(x) e^{-\beta x} dx - \int_{-M}^{0} c \tilde{\psi}'(x) e^{-\beta x} dx
\]
\[-\sum_{i=1}^{m} \int_{-M_0}^{0} \tilde{a}_i g_i(\Psi(x)) e^{-\beta x} \, dx \]
\[= \hat{b} \int_{-M}^{0} \tilde{\psi}(x) e^{-\beta x} \, dx - \int_{-M}^{0} c \tilde{\psi}'(x) e^{-\beta x} \, dx + C_2 \]
\[\geq \sum_{i=1}^{m} \tilde{a}_i d_i \int_{-M}^{0} \tilde{\psi}(x) e^{-\beta x} \, dx + c \beta \int_{-M}^{0} \tilde{\psi}(x) e^{-\beta x} \, dx - \int_{-M}^{0} c \tilde{\psi}'(x) e^{-\beta x} \, dx + C_2 \]
\[= \sum_{i=1}^{m} \tilde{a}_i d_i \int_{-M}^{0} \tilde{\psi}(x) e^{-\beta x} \, dx - \int_{-M}^{0} \tilde{\psi}(x) e^{-\beta x} \, dx \]
\[= \sum_{i=1}^{m} \tilde{a}_i d_i \int_{-M}^{0} \tilde{\psi}(x) e^{-\beta x} \, dx - \int_{-M}^{0} \tilde{\psi}(x) e^{-\beta x} \, dx - c\tilde{\psi}(0) + \tilde{\psi}(-M) e^{\beta M} + C_2, \]

where
\[C_2 := - \sum_{i=1}^{m} \int_{-M_0}^{0} \tilde{a}_i g_i(\Psi(x)) e^{-\beta x} \, dx - \int_{-M_0}^{0} \tilde{\psi}(x) e^{-\beta x} \, dx.\]

Therefore, by (4.13) and (4.14),
\[c\tilde{\psi}(-M) e^{\beta M} \leq c\tilde{\psi}(0) + C_1 - C_2 \text{ for all } M > M_0,\]
which implies \(\tilde{\psi}(x) = O(e^{\beta x})\) for \(x \ll -1\). The proof is completed. \(\square\)

4.2. Bounds for \(c_0 t - h(t), c_0 t + g(t)\) and \(U(t, x)\). Let us first observe that using the reasoning in the proof of Theorem 1.3 (i), it suffices to estimate \(h(t) - c_0 t\), since that for \(g(t) + c_0 t\) follows by a simple change of the initial function.

Theorem 4.3 will follow easily from the following two lemmas and their proofs, where more general and stronger conclusions are proved.

**Lemma 4.8.** In Theorem 1.3 if additionally (J) holds for some \(\alpha \geq 1\), \(F \) is \(C^2\) and \(u^* \nabla F(u^*) \leq 0\), then there exists \(C > 0\) such that for \(t \geq 0\),
\[h(t) - c_0 t \geq -C \left[1 + \int_{0}^{t} (1 + x)^{-\alpha} \, dx + \int_{0}^{\infty} x^2 \tilde{J}(x) \, dx + t \int_{\frac{t^2}{4}}^{\infty} x \tilde{J}(x) \, dx \right],\]
where \(c_0 > 0\) is given in Theorem 1.2 and \(\tilde{J}(x) := \sum_{i=1}^{m_0} \mu_i J_i(x)\).

**Proof.** Let \((c_0, \Phi^\alpha)\) be the unique solution pair of (1.3)-(1.5) obtained in Theorem 1.2. To simplify notations we write \(\Phi^\alpha(x) = \Phi(x) = (\phi_i(x))\). By Theorem 4.1 there is \(C > 0\) such that
\[(4.15) \quad \sum_{i=1}^{m_0} \int_{0}^{\infty} J_i(y) y^\alpha \, dy \leq C, \quad 0 < u^*_i - \phi_i(x) \leq \frac{C}{x^\alpha} \text{ for } x < 0, i \in \{1, ..., m\}.\]

Define
\[
\begin{cases}
\begin{aligned}
\tilde{h}(t) := c_0 t + \delta(t), & t \geq 0, \\
L(t, x) := (1 - \epsilon(t)) [\Phi(x - \tilde{h}(t)) + \Phi(-x - \hat{h}(t)) - u^*], & t \geq 0, x \in [-\hat{h}(t), \hat{h}(t)],
\end{aligned}
\end{cases}
\]
where \(\epsilon(t) := (t + \theta)^{-\alpha}\) and
\[\delta(t) := K_1 - K_2 \int_{0}^{t} \epsilon(\tau) d\tau - 2 \sum_{i=1}^{m_0} \mu_i u^*_i \int_{0}^{t} \int_{-\infty}^{-\frac{\hat{h}(\tau + \theta)}{2}} J_i(x-y) \, dy \, dx \, d\tau,
\]
with \(\theta, K_1\) and \(K_2\) large positive constants to be determined.
For any $M > 0$ and $i \in \{1, \ldots, m_0\}$,
\[
\int_{-\infty}^{-M} \int_0^\infty J_i(x-y)dydx = \int_{-\infty}^y \int_x^\infty J_i(y)dydx = \int_{M}^\infty \int_0^y J_i(y)dydM = \int_{M}^\infty (y-k)J_i(y)dy \leq \int_{M}^\infty yJ_i(y)dy.
\]

Hence, due to $\int_0^\infty yJ_i(y)dy < \infty$, we have
\[
2\sum_{i=1}^{m_0} \mu_i u_i^* \int_0^t \int_{-\infty}^{-\frac{c}{2}(\tau+\theta)} \int_0^\infty J_i(x-y)dydx\tau \leq 2\sum_{i=1}^{m_0} \mu_i u_i^* \int_0^t \int_{-\infty}^{-\frac{c}{2}\theta} \int_0^\infty J_i(x-y)dydx\tau \leq \left[ 2\sum_{i=1}^{m_0} \mu_i u_i^* \int_0^\infty yJ_i(y)dy \right] t \leq \frac{c_0}{4} t
\]
provided that $\theta > 0$ is large enough, say $\theta \geq \theta_0$.

For any given small $\epsilon_0 > 0$, due to $\Phi(-\infty) = u^*$ there is $K_0 = K_0(\epsilon_0) > 0$ such that
\[
(1 - \epsilon_0)u^* \leq \Phi(-K_0),
\]
which implies that
\[
(4.16) \quad \Phi(x - h(t)), \Phi(x - \bar{h}(t)) \in [(1 - \epsilon_0)u^*, u^*] \quad \text{for} \quad x \in [-h(t) + K_0, \bar{h}(t) - K_0],
\]
where we have assumed $\bar{h}(0) = K_1 > K_0$.

Clearly
\[
K_2 \int_0^{t} (\tau + \theta)^{-\alpha} d\tau \leq K_2 \theta^{-\alpha} t \leq \frac{c_0}{4} t
\]
provided $\theta \geq (4K_2/c_0)^{1/\alpha}$. Therefore
\[
(4.17) \quad \bar{h}(t) \geq \frac{c_0}{2} t + K_1 \geq \frac{c_0}{2} (t + \theta) > K_0 \quad \text{for all} \quad t \geq 0 \quad \text{provided that}
\]
\[
(4.18) \quad K_1 \geq \frac{c_0}{2} \theta \quad \text{and} \quad \theta \geq \max \left\{ (4K_2/c_0)^{1/\alpha}, \theta_0, 2K_0/c_0 \right\}.
\]

Define
\[
\epsilon_1 := \inf_{1 \leq i \leq m} \inf_{x \in [-K_0,0]} |\phi_i'(x)| > 0.
\]

Then
\[
(4.19) \quad \begin{cases}
\Phi(x - h(t)) < -\epsilon_1 1 \quad \text{for} \quad x \in [h(t) - K_0, h(t)], \\
\Phi(-x - h(t)) < -\epsilon_1 1 \quad \text{for} \quad x \in [-h(t), -h(t) + K_0].
\end{cases}
\]

**Claim 1:** With $U = (u_i)$, and suitably chosen $\theta$, $K_1$, $K_2$, we have
\[
(4.20) \quad h'(t) \leq \sum_{i=1}^{m} \mu_i \int_{-h(t)}^{h(t)} J_i(x-y)u_i(t,x)dy, \quad t > 0
\]
and
\[
-\bar{h}'(t) \geq -\sum_{i=1}^{m} \mu_i \int_{-\bar{h}(t)}^{\bar{h}(t)} J_i(x-y)u_i(t,x)dy, \quad t > 0.
\]
Due to $U(t, x) = U(t, -x)$ and $J(x) = J(-x)$, we just need to verify (4.20). We calculate
\[
\sum_{i=1}^{m_0} \mu_i \int_{-h(t)}^{h(t)} \int \infty J_i(x - y) u_i(t, x) dy dx 
= (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_0^{2h(t)} \int_0^{\infty} J_i(x - y) \phi_i(x) dy dx 
+ (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-2h(t)}^{2h(t)} \int_0^{\infty} J_i(x - y) [\phi_i(-x - 2h(t)) - u_i^*(x)] dy dx 
= (1 - \epsilon) c_0 - (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{\infty} \int_0^{\infty} J_i(x - y) [\phi_i(-x - 2h(t))] dy dx.
\]
From (4.17), for $t \geq 0$,
\[
(1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_0^{\infty} J_i(x - y) \phi_i(x) dy dx 
+ (1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_0^{\infty} J_i(x - y) [u_i^* - \phi_i(-x - 2h(t))] dy dx 
\leq 2 \sum_{i=1}^{m_0} \mu_i u_i^* \int_{-\infty}^{\infty} \int_0^{\infty} J_i(x - y) dy dx 
\leq 2 \sum_{i=1}^{m_0} \mu_i u_i^* \int_{-\infty}^{\infty} \int_0^{\infty} J_i(x - y) dy dx.
\]
And by (4.15), we have, for $t > 0$,
\[
(1 - \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-h(t)}^{0} \int_0^{\infty} J_i(x - y) [u_i^* - \phi_i(-x - 2h(t))] dy dx 
\leq \sum_{i=1}^{m_0} \mu_i [u_i^* - \phi_i(-h(t))] \int_{-h(t)}^{0} \int_0^{\infty} J_i(x - y) dy dx 
\leq \sum_{i=1}^{m_0} \mu_i \frac{C}{h(t)\alpha} \int_{-\infty}^{0} \int_0^{\infty} J_i(x - y) dy dx 
= \sum_{i=1}^{m_0} \mu_i \frac{C}{h(t)\alpha} \int_0^{\infty} \int_0^{\infty} J_i(x - y) dy dx 
\leq \sum_{i=1}^{m_0} \mu_i \frac{C}{h(t)\alpha} \int_0^{\infty} \int_0^{\infty} J_i(y) dy \leq \sum_{i=1}^{m_0} \mu_i \frac{C^2}{(c_0/2)^\alpha (t + \theta)\alpha} \frac{K_2 - c_0}{(t + \theta)\alpha}
\]
if
\[
(4.21) \quad K_2 \geq c_0 + \frac{C^2}{(c_0/2)^\alpha \sum_{i=1}^{m} \mu_i}.
\]
Hence, when $\theta, K_1$ and $K_2$ are chosen such that (4.18) and (4.21) hold, then
\[
\sum_{i=1}^{m} \mu_i \int_{-\infty}^{\infty} \int_0^{\infty} J_i(x - y) u_i(t, x) dy dx 
\geq (1 - \epsilon) c_0 - 2 \sum_{i=1}^{m_0} \mu_i u_i^* \int_{-\infty}^{\infty} \int_0^{\infty} J_i(x - y) \phi_i(x) dy dx \frac{K_2 - c_0}{(t + \theta)\alpha}.
\]
which finishes the proof of (4.20).

Claim 2: With \( \theta \), \( K_1 \), \( K_2 \) chosen such that (4.18) and (4.21) hold, and \( K_2 \) suitably further enlarged (see (4.22) below), \( \theta_0 \gg 1 \) and \( 0 < \epsilon_0 \ll 1 \), we have, for all \( t > 0 \) and \( x \in (-h(t), h(t)) \),

\[
\frac{U(t, x)}{J(x - y) \circ U(t, y) \, dy} - D \circ U(t, x) + F(U(t, x)).
\]

A simple calculation gives

\[
\frac{U(t, x)}{J(x - y) \circ U(t, y) \, dy} - D \circ U(t, x) + F(U(t, x))
\]

and using the equation satisfied by \( \Phi \) we deduce

\[
= (1 - \epsilon) \left[ D \circ \int_{-h(t)}^{h(t)} J(x - y) \circ \Phi(y - \bar{h}(t)) \, dy - D \circ \Phi(x - \bar{h}(t)) \right]
\]

\[
+ (1 - \epsilon) \left[ F(\Phi(x - \bar{h}(t))) + F(\Phi(-x + \bar{h}(t))) \right]
\]

\[
= D \circ \left[ \int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y) \, dy - U(t, x) \right]
\]

\[
+ (1 - \epsilon) \left[ D \circ \int_{-\infty}^{-h(t)} J(x - y) \circ [\Phi(y - \bar{h}(t)) - \Phi^*(y)] \, dy
\]

\[
+ D \circ \int_{h(t)}^{\infty} J(-x - y) \circ [\Phi(-y + \bar{h}(t)) - \Phi^*(y)] \, dy
\]

\[
+ (1 - \epsilon) \left[ F(\Phi(x - \bar{h}(t))) + F(\Phi(-x + \bar{h}(t))) \right]
\]

\[
\leq D \circ \left[ \int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y) \, dy - U(t, x) \right]
\]

\[
+ (1 - \epsilon) \left[ F(\Phi(x - \bar{h}(t))) + F(\Phi(-x + \bar{h}(t))) \right].
\]

Hence

\[
\frac{U(t, x)}{J(x - y) \circ U(t, y) \, dy} - D \circ U(t, x) + F(U(t, x)) + A_1(t, x) + A_2(t, x),
\]

where

\[
A_1(t, x) := (1 - \epsilon) \left[ \Phi(x - \bar{h}(t)) + \Phi(-x + \bar{h}(t)) - \Phi^*(y) \right],
\]

\[
A_2(t, x) := - (1 - \epsilon) \delta'(t) \left[ \Phi(x - \bar{h}(t)) + \Phi(-x + \bar{h}(t)) \right]
\]

\[
+ (1 - \epsilon) \left[ F(\Phi(x - \bar{h}(t))) + F(\Phi(-x + \bar{h}(t))) \right] - F(U(t, x)).
\]
To finish the proof of Claim 2, it remains to check that
\[ A_1(t, x) + A_2(t, x) \leq 0 \quad \text{for} \quad t > 0, \ x \in (-\bar{h}(t), \bar{h}(t)). \]

We next prove this inequality for \( x \) in the following three intervals, separately:
\[ I_1(t) := [\bar{h}(t) - K_0, \bar{h}(t)], \ I_2(t) := [-\bar{h}(t), -\bar{h}(t) + K_0], \ I_3(t) := [-\bar{h}(t) + K_0, \bar{h}(t) - K_0]. \]

For \( x \in I_1(t) \), by (4.15),
\[ 0 > \Phi(-x - \bar{h}(t)) - u^* \geq \Phi(K_0 - 2\bar{h}(t)) - u^* \geq \Phi(-\bar{h}(t)) - u^* \geq \frac{-C}{h(t)\alpha}1 \]

Then by (2.1) and (f2),
\[ F(\Phi(-x - \bar{h}(t))) = F(\Phi(-x - \bar{h}(t))) - F(u^*) \leq L\frac{C}{h(t)^\alpha}1 \]

and
\[ F(U(t, x)) \geq (1 - \epsilon)F\left(\Phi(x - \bar{h}(t)) + \Phi(-x - \bar{h}(t)) - u^*\right) \]
\[ \geq (1 - \epsilon)\left[F(\Phi(x - \bar{h}(t))) - L\frac{C}{h(t)^\alpha}1\right]. \]

Thus from the definition of \( \delta(t) \), (4.17) and (4.19), we deduce
\[ A_2(t, x) \leq (1 - \epsilon)\left[\delta'(t)\left(\Phi'(x - \bar{h}(t)) + \Phi'(-x - \bar{h}(t))\right) + F(\Phi(x - \bar{h}(t))) \right. \]
\[ + \left. F(\Phi(-x - \bar{h}(t))) - F\left(\Phi(x - \bar{h}(t)) + \Phi(-x - \bar{h}(t)) - u^*\right)\right] \]
\[ \leq (1 - \epsilon)\left[-\delta'(t)\epsilon_1 + 2L\frac{C}{h(t)^\alpha}\right]1 \leq (1 - \epsilon)\left[-K_2(t + \theta)^{-\alpha}\epsilon_1 + \frac{2LC}{h(t)^\alpha}\right]1 \]
\[ \leq (1 - \epsilon)(t + \theta)^{-\alpha}\left[-K_2\epsilon_1 + 2LC(2/c_0)^\alpha\right]1. \]

Moreover,
\[ A_1(t, x) \leq \alpha(t + \theta)^{-\alpha - 1}u^* \leq 2|u^*|(1 - \epsilon)\alpha(t + \theta)^{-\alpha - 1}1, \]

where \( |u^*| := \max_{1 \leq i \leq m} u_i^* \) and by enlarging \( \theta_0 \) we have assumed that \( \epsilon(t) \leq \theta_0^{-\alpha} < 1/2 \). Hence
\[ A_1(t, x) + A_2(t, x) \leq (1 - \epsilon)(t + \theta)^{-\alpha}\left[-K_2\epsilon_1 + 2LC(2/c_0)^\alpha + 2|u^*|\alpha\theta_0^{-1}\right]1 \leq 0 \]

if additionally
\[ (4.22) \quad K_2 \geq \epsilon_1^{-1}\left[2LC(2/c_0)^\alpha + 2|u^*|\alpha\theta_0^{-1}\right]. \]

This proves the desired inequality for \( x \in I_1(t) \).

Since \( A_1(t, x) + A_2(t, x) \) is even in \( x \), the desired inequality is also valid for \( x \in I_2(t) = -I_1(t) \).

It remains to prove the desired inequality for \( x \in I_3(t) \).

The case \( x \in I_3(t) \) requires some preparations. Define
\[ G(u, v) = (g_i(u, v)) := (1 - \epsilon)[F(u) + F(v)] - F((1 - \epsilon)(u + v - u^*)), \ u, v \in \mathbb{R}^m. \]

For \( u \in [0, u^*] \) and \( v \in [0, u^*] \), and each \( i \in \{1, ..., m\} \), we may apply the mean value theorem to the function
\[ \xi_i(t) := g_i(u^* + t(u - u^*), u^* + t(v - u^*)) \]

to obtain
\[ \xi_i(1) = \xi_i(0) + \xi_i'(\zeta_i) \text{ for some } \zeta_i \in [0, 1]. \]

Denote
\[ \tilde{u} = \tilde{u}^i := u^* + \zeta_i(u - u^*), \ \tilde{v} = \tilde{v}^i := u^* + \zeta_i(v - u^*). \]
Thus where

\[ g_i(u, v) = g_i(u^*, u^*) + \nabla_u g_i(\tilde{u}, \tilde{v}) \cdot (u - u^*) + \nabla_v g_i(\tilde{u}, \tilde{v}) \cdot (v - u^*) \]

\[ = - f_i((1 - \epsilon)u^*) + (1 - \epsilon)\nabla f_i(\tilde{u}) \cdot (u - u^*) + (1 - \epsilon)\nabla f_i(\tilde{v}) \cdot (v - u^*) \]

\[ - (1 - \epsilon)\nabla f_i((1 - \epsilon)(\tilde{u} + \tilde{v} - u^*)) \cdot (u - u^*) \]

\[ - (1 - \epsilon)\nabla f_i((1 - \epsilon)(\tilde{u} + \tilde{v} - u^*)) \cdot (v - u^*). \]

Let us note that \( \tilde{u} \in [u, u^*] \) and \( \tilde{v} \in [v, u^*] \). Since \( F \in C^2 \), there is \( C_1 \) such that

\[ |\partial_{jk} f_i(u)| \leq C_1 \text{ for } u \in [0, \hat{u}], i, j, k \in \{1, ..., m\}. \]

A simple calculation gives

\[ (1 - \epsilon)\nabla f_i(\tilde{u})(u - u^*) - (1 - \epsilon)\nabla f_i((1 - \epsilon)(\tilde{u} + \tilde{v} - u^*)) \cdot (u - u^*) \]

\[ = (1 - \epsilon) \left[ \nabla f_i(\tilde{u}) - \nabla f_i((1 - \epsilon)(\tilde{u} + \tilde{v} - u^*)) \right] \cdot (u - u^*) \]

\[ \leq (1 - \epsilon) b_1 \sum_{j=1}^{m} (u_j^* - u_j), \]

where

\[ b_1 := C_1 |\tilde{u} - (1 - \epsilon)(\tilde{u} + \tilde{v} - u^*)| \]

\[ = C_1 |e\tilde{u} - (1 - \epsilon)(\tilde{v} - u^*)| \leq C_1 \sum_{j=1}^{m} |e\tilde{u}_j + (1 - \epsilon)(u_j^* - \tilde{v}_j)| \]

\[ \leq C_2 \epsilon + C_1 \sum_{j=1}^{m} (u_j^* - v_j) \text{ with } C_2 := C_1 \sum_{j=1}^{m} u_j^*. \]

Similarly,

\[ (1 - \epsilon)\nabla f_i(\tilde{v}) \cdot (v - u^*) - (1 - \epsilon)\nabla f_i((1 - \epsilon)(\tilde{u} + \tilde{v} - u^*)) \cdot (v - u^*) \]

\[ \leq (1 - \epsilon) b_2 \sum_{j=1}^{m} (u_j^* - v_j), \]

where

\[ b_2 := C_1 |e\tilde{u} - (1 - \epsilon)(\tilde{u} - u^*)| \leq C_2 \epsilon + C_1 \sum_{j=1}^{m} (u_j^* - u_j). \]

Thus

\[ g_i(u, v) \leq - f_i((1 - \epsilon)u^*) + (1 - \epsilon)b_1 \sum_{j=1}^{m} (u_j^* - v_j) + (1 - \epsilon)b_2 \sum_{j=1}^{m} (u_j^* - u_j) \]

\[ \leq - f_i((1 - \epsilon)u^*) + \left[ C_2 \epsilon + C_1 \sum_{j=1}^{m} (u_j^* - v_j) \right] \sum_{k=1}^{m} (u_k^* - u_k) \]

\[ + \left[ C_2 \epsilon + C_1 \sum_{j=1}^{m} (u_j^* - u_j) \right] \sum_{k=1}^{m} (u_k^* - v_k) \]

\[ = - f_i((1 - \epsilon)u^*) + C_2 \epsilon \sum_{k=1}^{m} (u_k^* - u_k) + (u_k^* - v_k) \]
provided that we thus obtain

$$C = \epsilon \nabla f_i(u^*) \cdot u^* + o(\epsilon) + C_2 \epsilon \sum_{k=1}^{m} (u_k^* - u_k) + C_1 \sum_{j,k=1}^{m} (u_j^* - u_j)(u_k^* - u_k)$$

$$+ 2C_1 \sum_{j,k=1}^{m} (u_j^* - u_j)(u_k^* - u_k),$$

where $o(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$. Since $\epsilon(t) \leq \theta_0^{-1}$, we see that

$$o(\epsilon) = o(1) \epsilon \text{ with } o(1) \to 0 \text{ as } \theta_0 \to \infty.$$

For our discussions below, it is convenient to introduce the notations

$$P(t, x) = (p_i(t, x)) := u^* - \Phi(x - h(t)), \quad Q(t, x) = (q_i(t, x)) := u^* - \Phi(-x - h(t)).$$

Then by (4.16) we have

$$(4.23) \quad P(t, x), \quad Q(t, x) \in [0, \epsilon_0 u^*] \text{ for } x \in I_3(t), t > 0.$$ \hspace{1cm} (*)

Moreover, since $\min\{x - h(t), -x - h(t)\} \leq -h(t)$ always holds, by (4.15) and (4.17), if we denote $C_3 := C(\epsilon_0/2)^{-\alpha}$, then

$$(4.24) \quad p_j(t, x)q_k(t, x) \leq \frac{\epsilon_0}{h(t)^2} \leq C_3 \epsilon_0\epsilon(t) \text{ for } x \in I_3(t), t > 0, \quad j, k \in \{1, ..., m\}.$$ \hspace{1cm} (**)

Let $A^i_2$ denote the $i$-th component of $A_2$. Now due to $\delta'(t) < 0$ and $\Phi' \not\equiv 0$, we have, by (4.23) and (4.24),

$$A^i_2(t, x) \leq g_i(u^* - P, u^* - Q)$$

$$\leq \epsilon \nabla f_i(u^*) \cdot u^* + o(\epsilon) + C_2 \epsilon \sum_{k=1}^{m} (p_k + q_k) + 2C_1 \sum_{j,k=1}^{m} p_j q_k$$

$$\leq \epsilon \nabla f_i(u^*) \cdot u^* + o(\epsilon) + C_4 \epsilon_0 \epsilon$$

$$= \epsilon \left[ u^* \cdot \nabla f_i(u^*) + o(1) + C_4 \epsilon_0 \epsilon \right] \text{ for } x \in I_3(t), t > 0, \quad i \in \{1, ..., m\},$$

with $C_4 := 2m(C_2 + C_1 C_3)$. Since

$$A^i_1(t, x) \leq \alpha(t + \theta)^{-\alpha-1} u^*_i \leq \alpha |u^*_i| \theta_0^{-1} \epsilon(t)$$

and

$$u^* [\nabla F(u^*)]^T \not\equiv 0,$$

we thus obtain

$$A^i_1 + A^i_2 \leq \epsilon \left[ u^* \cdot \nabla f_i(u^*) + \left[ o(1) + C_4 \epsilon_0 + \alpha u^*_i \theta_0^{-1} \right] \right] < 0 \text{ for } x \in I_3(t), t > 0, \quad i \in \{1, ..., m\}$$

provided that $\theta_0$ is sufficiently large and $\epsilon_0$ is sufficiently small. The proof of Claim 2 is now complete.

**Claim 3:** There exists $t_0 > 0$ such that

$$(4.25) \quad \begin{cases} g(t + t_0) \leq -h(t), \quad h(t + t_0) \geq h(t) \text{ for } t \geq 0, \\ U(t + t_0, x) \geq U(t, x) \text{ for } t \geq 0, \quad x \in [-h(t), h(t)]. \end{cases}$$

It is clear that

$$\underline{U}(t, \pm h(t)) = (1 - \epsilon(t))[\Phi(-2h(t)) - u^*] \not\equiv 0 \text{ for } t \geq 0.$$ \hspace{1cm} ($\ast$)

Since spreading happens for $(U, g, h)$, there exists a large constant $t_0 > 0$ such that

$$g(t_0) < -K_1 = -\underline{h}(0) \text{ and } \underline{h}(0) = K_1 < h(t_0),$$

for
Due to \( \Phi(\alpha) \) conclude that (4.25) is valid.

Claim 4: There exists \( C > 0 \) such that

\[
\delta(t) \geq -C \left[ 1 + \int_0^t (1 + x)^{-\alpha} dx + \int_0^\infty x^2 J(x) dx + t \int_0^\infty x J(x) dx \right].
\]

Clearly

\[
\int_0^t \epsilon(\tau) d\tau = \int_0^t (x + \theta)^{-\alpha} dx < \int_0^t (x + 1)^{-\alpha} dx.
\]

By changing order of integrations we have

\[
\int_0^t \int_{-\infty}^{\Phi(t+\theta)} \int_{\Phi(t)}^{\Phi(t)+\infty} J_i(x - y) dy dx d\tau \leq \int_0^t \int_{-\infty}^\infty \int_{\Phi(t)}^{\Phi(t)+\infty} J_i(x - y) dy dx d\tau
\]

\[
= \int_0^t \int_{\Phi(t)}^{\Phi(t)+\infty} \left[ y - \frac{c_0}{2} \right] J_i(y) dy d\tau \leq \int_0^t \int_{\Phi(t)}^{\Phi(t)+\infty} y J_i(y) dy d\tau
\]

\[
= \frac{c_0}{2} \int_0^\infty y^2 J_i(y) dy + t \int_{\Phi(t)}^{\Phi(t)+\infty} y J_i(y) dy.
\]

The desired inequality now follows directly from the definition of \( \delta(t) \). \( \square \)

Next we prove an upper bound for \( h(t) - c_0 t \). Let us note that we do not need the condition \( (J^\alpha)^T \) in the following result.

Lemma 4.9. Under the assumptions of Theorem 1.3 (i), if \( (J_1) \) holds, and additionally \( F \) is \( C^2 \) and \( u^* [\nabla F(u^*)]^T \neq 0 \), then there exits \( C > 0 \) such that

\[
(4.26) \quad h(t) - c_0 t \leq C \quad \text{for all} \quad t > 0.
\]

Proof. As in the proof of Lemma 4.8 \( (c_0, \Phi(c_0)) \) denotes the unique solution pair of (1.4)-(1.5) obtained in Theorem 1.2 and to simplify notations we write \( \Phi(c_0) = \Phi(x) = (\Phi_1(x)) \).

For fixed \( \beta > 1 \), and some large constants \( \theta > 0 \) and \( K_1 > 0 \) to be determined, define

\[
\begin{align*}
\bar{h}(t) &:= c_0 + \delta(t), \quad t \geq 0, \\
\bar{U}(t, x) &:= (1 + \epsilon(t)) \Phi(x - \bar{h}(t)), \quad t \geq 0, \quad x \leq \bar{h}(t),
\end{align*}
\]

where \( \epsilon(t) := (t + \theta)^{-\beta} \)

\[
\delta(t) := K_1 + \frac{c_0}{1 - \beta} [(t + \theta)^{1-\beta} - \theta^{1-\beta}].
\]

Clearly, there is a large constant \( t_0 > 0 \) such that

\[
U(t + t_0, x) \geq (1 + \frac{1}{2} \epsilon(0)) u^* \quad \text{for} \quad t \geq 0, \quad x \in [g(t), h(t)].
\]

Due to \( \Phi(-\infty) = u^* \), we may choose sufficient large \( K_1 > 0 \) such that \( \bar{h}(0) = K_1 > 2g(t_0), \)

\[
\bar{h}(0) = -K_1 < 2g(t_0), \quad \text{and also}
\]

\[
(4.27) \quad \bar{U}(0, x) = (1 + \epsilon(0)) \Phi(-K_1/2) > (1 + \frac{1}{2} \epsilon(0)) u^* \geq U(t_0, x) \quad \text{for} \quad x \in [g(t_0), h(t_0)].
\]

Claim 1: We have, with \( \bar{U} = (\bar{u}_i) \),

\[
\bar{h}'(t) \geq \sum_{i=1}^{m} \mu_i \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_i(x - y) \bar{u}_i(t, x) dy \quad \text{for} \quad t > 0.
\]
A direct calculation shows
\[
\sum_{i=1}^{m} \mu_i \int_{g(t+t_0)}^{h(t_0)} J_i(x-y) \tilde{u}_i(t,x) \, dy \\
\leq \sum_{i=1}^{m} \mu_i \int_{-\infty}^{h(t_0)} J_i(x-y) \tilde{u}_i(t,x) \, dy \\
= (1 + \epsilon) \sum_{i=1}^{m} \mu_i \int_{0}^{h(t_0)} J_i(x-y) \phi_i(x) \, dy \\
= (1 + \epsilon)c_0 = \tilde{h}(t),
\]
as desired.

**Claim 2:** If \( \theta > 0 \) is sufficiently large, then for \( t > 0 \) and \( x \in (g(t+t_0), h(t)) \), we have
\[
(4.28) \quad \overline{U}_i(t,x) \geq D \circ \int_{g(t+t_0)}^{h(t)} J(x-y) \circ \overline{U}(t,y) \, dy - D \circ \overline{U}(t,x) + F(\overline{U}(t,x)).
\]

By (1.4), we have
\[
\overline{U}_i(t,x) = - (1 + \epsilon)[c_0 + \delta(t)] \Phi'(x-h(t)) + \epsilon'(t) \Phi(x-h(t)) \\
= - (1 + \epsilon)c_0 \Phi'(x-h(t)) - (1 + \epsilon)\delta(t) \Phi(x-h(t)) - \beta(t+\theta)^{-\beta-1} \Phi(x-h(t)) \\
\geq D \circ \int_{h(t_0)}^{h(t)} J(x-y) \circ \overline{U}(t,y) \, dy - D \circ \overline{U}(t,x) + F(\overline{U}(t,x)) + A(t,x)
\]
with
\[
A(t,x) := (1 + \epsilon) F(\Phi(x-h(t))) - F((1 + \epsilon) \Phi(x-h(t))) \\
- (1 + \epsilon)\delta(t) \Phi'(x-h(t)) - \beta(t+\theta)^{-\beta-1} \Phi(x-h(t)).
\]

To prove the claim, we need to show
\[
A(t,x) \geq 0 \quad \text{for} \quad x \in [g(t_0 + t), h(t)] \quad \text{and} \quad t > 0.
\]

Let \( \epsilon_0, \epsilon_1 \) and \( K_0 \) be given as in the proof of Lemma 4.8. For \( x \in [\tilde{h}(t) - K_0, \tilde{h}(t)] \) and \( t > 0 \), by (4.19), we have
\[
A(t,x) \geq (1 + \epsilon) \beta(t+\theta)^{-\beta-1} \Phi(x-h(t)) \\
\geq c_0(t+\theta)^{-\beta-1} [c_0\epsilon_1 \mathbf{1} - \beta\mathbf{u}^*] \geq 0,
\]
provided \( \theta \) is large enough.

We next estimate \( A(t,x) \) for \( x \in [g(t+t_0), \tilde{h}(t) - K_0] \). Define
\[
G'(u) = (g_i(u)) := (1 + \epsilon) F(u) - F((1 + \epsilon)u), \quad u, v \in \mathbb{R}^m.
\]

Then for \( u, v \in [0, \mathbf{u}^*] \) and \( i \in \{1, \ldots, m\} \),
\[
g_i(u) = g_i(\tilde{u}) + \nabla g_i(\tilde{u}) \cdot (u - \mathbf{u}^*) \\
= - f_i((1 + \epsilon)\mathbf{u}^*) + (1 + \epsilon) \nabla f_i((1 + \epsilon)\tilde{u}) \cdot (u - \mathbf{u}^*) - (1 + \epsilon) \nabla f_i((1 + \epsilon)\tilde{u}) \cdot (u - \mathbf{u}^*) \\
= - f_i((1 + \epsilon)\mathbf{u}^*) + (1 + \epsilon) \nabla f_i((1 + \epsilon)\tilde{u}) \cdot (u - \mathbf{u}^*)
\]
for some \( \tilde{u} = \tilde{u}^i \in [u, \mathbf{u}^*] \). Since \( F \in C^2 \), there exists \( C_1 > 0 \) such that
\[
|\partial_{jk} f_i(u)| \leq C_1 \quad \text{for} \quad u \in [0, \tilde{u}], \quad i, j, k \in \{1, \ldots, m\}.
\]
Therefore
\[ g_t(u) \geq -f_t((1 + \epsilon)u^*) - (1 + \epsilon)b_1 \sum_{j=1}^{m} (u_j^* - u_j) \]
with
\[ b_1 := C_1|\epsilon\bar{u}| \leq C_1|u^*| := C_2\epsilon. \]
Thus
\[ g_t(u) \geq -\epsilon \nabla f_t(u^*) \cdot u^* + o(\epsilon) - 2C_2\epsilon \sum_{j=1}^{m} (u_j^* - u_j). \]

By (4.16) we have
\[ -\epsilon_0 u^* \leq \Phi(x - \bar{h}(t)) - u^* \ll 0 \quad \text{for} \quad x \in [g(t_0 + t), \bar{h}(t) - K_0], \quad t > 0. \]

Using (4.16), \( \delta' > 0, \Phi' \ll 0 \) and \( \epsilon = (t + \theta)^{-\beta} \leq \theta^{-\beta}, \) we obtain
\begin{align*}
A^i(t, x) &\geq (1 + \epsilon)f_t(\Phi(x - \bar{h}(t))) - f_t((1 + \epsilon)\Phi(x - \bar{h}(t))) - \beta(t + \theta)^{-\beta-1}\phi_t(x - \bar{h}(t)) \\
&= g_t(\Phi(x - \bar{h}(t)) - \beta(t + \theta)^{-\beta-1}\phi_t(x - \bar{h}(t)) \\
&\geq \epsilon \left[-u^* \cdot \nabla f_t(u^*) + o(1) - 2\epsilon_0 C_2 \sum_{j=1}^{m} u_j^* - \beta\theta^{-\beta-1}u_i^* \right] \\
&> 0 \quad \text{for} \quad x \in [g(t_0 + t), \bar{h}(t) - K_0], \quad t > 0, \quad i \in \{1, \ldots, m\},
\end{align*}
provided \( \theta \) is large enough and \( \epsilon_0 > 0 \) is small enough, since \( u^*[\nabla F(u^*)]^T \ll 0. \) We have now proved (4.28).

Due to the inequalities proved in Claims 1 and 2, (4.27) and
\[ \bar{U}(t, g(t + t_0)) > 0, \quad \bar{U}(t, \bar{h}(t)) = (1 + \epsilon)\Phi(\bar{h}(t) - \bar{h}(t)) = 0 \quad \text{for} \quad t \geq 0, \]
we are now able to apply Lemma 3.2 to conclude that
\begin{align*}
h(t + t_0) &\leq \bar{h}(t), \quad t \geq 0, \\
U(t + t_0, x) &\leq \bar{U}(t, x), \quad t \geq 0, \quad x \in [g(t + t_0), \bar{h}(t)].
\end{align*}
The desired inequality (4.26) follows directly from \( \delta(t) \leq K_1 + \frac{c_0}{\beta-1}\theta^{1-\beta} \) and \( h(t + t_0) \leq \bar{h}(t). \)
The proof is complete. \( \square \)

Proof of Theorem 1.3. Since \( \alpha \geq 2, \) from the proof of Lemmas 4.8 and 4.9 it is easily seen that
\[ C_0 := \sup_{t>0} \left[ |\bar{h}(t) - c_0 t| + |\bar{h}(t) - c_0 t| \right] < \infty. \]
Hence for large fixed \( \theta > 0 \) and all large \( t, \) say \( t \geq t_0, \)
\[ [g(t), h(t)] \supset [\bar{h}(t - t_0), \bar{h}(t - t_0)] \supset [-c_0 t + C, c_0 t - C] \quad \text{with} \quad C := C_0 + c_0 t_0, \]
and
\[ U(t, x) \geq \bar{U}(t, x) \geq (1 - \epsilon(t)) \left[ \Phi^0(x - c_0 t + C) + \Phi^c(x - c_0 t + C) - u^* \right] \]
for \( x \in [-c_0 t + C, c_0 t - C], \) where \( \epsilon(t) = (t + \theta)^{-\alpha}. \) This inequality for \( U(t, x) \) also holds for \( x \in [g(t), h(t)] \) if we assume that \( \Phi^0(x) = 0 \) for \( x > 0, \) since when \( x \) lies outside of \( [-c_0 t + C, c_0 t - C] \) the right side is \( 0. \)

Using the reasoning in the proof of Theorem 1.3, from the proof of Lemma 4.9 we see that the following analogous inequalities hold:
\[ g(t) \geq -\bar{h}(t - t_0), \quad U(t, x) \leq (1 + \epsilon(t))\Phi^0(x - \bar{h}(t - t_0)) \]
for \( t > t_0 \) and \( x \in [g(t), h(t)]. \) We thus have
\[ [g(t), h(t)] \subset [-\bar{h}(t - t_0), \bar{h}(t - t_0)] \subset [-c_0 t - C, c_0 t + C], \]
and
\[ U(t,x) \leq U(t,x) \leq (1 - \epsilon(t)) \min \left\{ \Phi^\circ(x - c_0 t - C), \Phi^\circ(-x - c_0 t - C) \right\} \]
for \( t > t_0 \) and \( x \in [g(t), h(t)] \). The proof is complete. \( \square \)

5. The growth orders of \( c_0 t - h(t) \) and \( c_0 t + g(t) \)

Recall that \((U(t,x), g(t), h(t))\) is the unique positive solution of (1.1), and we assume that spreading happens. Under the assumptions of Theorem 1.3, we have
\[ - \lim_{t \to \infty} \frac{g(t)}{t} = \lim_{t \to \infty} \frac{h(t)}{t} = c_0 > 0. \]

In this section, we determine the growth order of \( c_0 t - h(t) \) and \( c_0 t + g(t) \) when the kernel functions satisfy, for some \( \gamma \in (2,3) \), \( \omega \in (\gamma - 1, \gamma) \), \( C > 0 \) and all \( |x| \geq 1 \),
\begin{align}
J_i(x) \approx |x|^{-\gamma} & \quad \text{if } i \in \{1, \ldots, m_0\} \text{ and } \mu_i \neq 0, \\
J_i(x) \leq C|x|^{-\omega} & \quad \text{if } i \in \{1, \ldots, m_0\} \text{ and } \mu_i = 0.
\end{align}

Clearly, \((\tilde{J})\) implies \((5.1)\).

The main result of this section is the following theorem.

**Theorem 5.1.** In Theorem 1.3, if additionally \((J^1)\), \((5.1)\) and \((1.7)\) hold, then for \( t \gg 1 \),
\begin{align}
\begin{cases}
\quad c_0 t + g(t), \\
c_0 t - h(t) \approx t^{3-\gamma} & \text{if } \gamma \in (2,3), \\
c_0 t - h(t) \approx \ln t & \text{if } \gamma = 3.
\end{cases}
\end{align}

It is clear that the conclusion of Theorem 1.6 follows directly from Theorem 5.1. Note that if \( \omega > 2 \) in \((5.1)\), then \((J^1)\) automatically holds.

By \((f_1)\) and the Perron-Frobenius theorem, we know that the matrix \( \nabla F(0) - \tilde{D} \) with \( \tilde{D} = \text{diag}(d_1, \ldots, d_m) \) has a principal eigenvalue \( \tilde{\lambda}_1 \) with a corresponding eigenvector \( V^* = (v_1^*, \ldots, v_m^*) \gg 0 \), namely
\[ V^*(\nabla F(0)^T - \tilde{D}) = \tilde{\lambda}_1 V^*. \]

To prove Theorem 5.1, the difficult part is to find the lower bound for \( c_0 t - h(t) \), which will be established according to the following two cases: (i) \( \tilde{\lambda}_1 < 0 \), (ii) \( \tilde{\lambda}_1 \geq 0 \).

As before, we will only estimate \( c_0 t - h(t) \), since the estimate for \( c_0 t + g(t) \) follows by making the variable change \( x \to -x \) in the initial functions.

5.1. The case \( \tilde{\lambda}_1 < 0 \).

**Lemma 5.2.** Suppose that the assumptions in Theorem 5.1 are satisfied. If \( \tilde{\lambda}_1 < 0 \), then there exists \( \sigma = \sigma(\gamma) > 0 \) such that for all large \( t > 0 \),
\begin{align}
\begin{cases}
c_0 t - h(t) \geq \sigma t^{3-\gamma} & \text{if } \gamma \in (2,3), \\
c_0 t - h(t) \geq \sigma \ln t & \text{if } \gamma = 3.
\end{cases}
\end{align}

**Proof.** Let \( \beta := \gamma - 2 \in (0,1] \), and \((c_0, \Phi)\) be the solution of (1.4)-(1.5). Define
\[ \epsilon(t) := K_1(t + \theta)^{-\beta}, \quad \delta(t) := K_2 - K_3 \int_0^t \epsilon(\tau)d\tau \]
and
\[ \begin{cases}
\tilde{h}(t) := c_0 t + \delta(t), \\
\tilde{U}(t,x) := (1 + \epsilon(t))\Phi(x - \tilde{h}(t)) + \rho(t,x),
\end{cases} \quad t \geq 0, \quad x \leq \tilde{h}(t), \]
where
\[ \rho(t,x) := K_4 \xi(x - \tilde{h}(t))\epsilon(t)V^*, \]
with $\xi \in C^2(\mathbb{R})$ satisfying
\begin{equation}
0 \leq \xi(x) \leq 1, \quad \xi(x) = 1 \text{ for } |x| < \bar{\epsilon}, \quad \xi(x) = 0 \text{ for } |x| > 2\bar{\epsilon},
\end{equation}
and the positive constants $\theta, K_1, K_2, K_3, K_4, \bar{\epsilon}$ are to be determined.

We are going to show that, it is possible to choose these constants and some $t_0 > 0$ such that
\begin{equation}
\vartheta_j(t) = \vartheta_{\jmath,0} + \int_0^t J(x-y, t) \vartheta_j(y, t) dy
\end{equation}
for $t > 0, \ x \in (g(t+\theta), \bar{\theta}(t))$, and the desired inequality for $\sigma$
with $\bar{\theta}$ such that
\begin{equation}
\vartheta_j(t) \geq \vartheta_{\jmath,0} + \int_0^t J(x-y, t) \nu_j(t, x) dy
\end{equation}
for $t > 0$.

Therefore, to complete the proof, it suffices to prove the above inequalities. We divide the arguments below into several steps.

Firstly, by Theorem 1.3 there is $C_1 > 1$ such that
\begin{equation}
eg g(t), h(t) \leq (c_0 + 1)t + C_1 \quad \text{for } t \geq 0.
\end{equation}

Let us also note that (5.7) holds trivially.

**Step 1.** Choose $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (5.8) holds.

For later analysis, we need to find $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (5.8) holds and at the same time they have less than linear growth in $\theta$.

Let $W^* \gg 0$ be an eigenvector corresponding to the maximal eigenvalue $\lambda$ of $\nabla F(u^*)$. By our assumptions on $F$, we have $\lambda < 0$. Hence there exists small $\epsilon > 0$ such that for any $k \in (0, \epsilon)$,
\begin{align*}
F(u^* + kW^*) &= kW^* \left( [\nabla F(u^*)]^T + o(1) I_m \right) \\
&\leq \frac{k}{2} \lambda W^* \ll 0,
\end{align*}
\begin{align*}
F(u^* - kW^*) &= -kW^* \left( [\nabla F(u^*)]^T + o(1) I_m \right) \\
&\geq -\frac{k}{2} \lambda W^* \gg 0.
\end{align*}

It follows that, for $\bar{\sigma} = \lambda/2$,
\begin{align*}
\bar{W}(t) &= u^* + \epsilon_e e^{\bar{\sigma}t} W^*, \quad \bar{W}(t) = u^* - \epsilon_e e^{\bar{\sigma}t} W^*
\end{align*}
are a pair of upper and lower solution of the ODE system $W' = F(W)$ with initial data $W(0) \in [u^* - \epsilon, W^*, u^* + \epsilon W^*]$.

By (f4), the unique solution of the ODE system
\begin{equation}
W' = F(W), \quad W(0) = (\|u_{t1}\|_\infty, \cdots, \|u_{m0}\|_\infty)
\end{equation}
satisfies $\lim_{t \to \infty} W(t) = u^*$. Hence there exists $t_0 > 0$ such that
\begin{equation}
W(t_0) \in [u^* - \epsilon W^*, u^* + \epsilon W^*].
\end{equation}

Using the above defined upper solution $\bar{W}(t)$ we obtain
\begin{equation}
W(t + t_0) \leq u^* + \epsilon e^{\bar{\sigma}t} W^* \leq (1 + \epsilon e^{\bar{\sigma}t}) u^* \quad \text{for } t \geq 0,
\end{equation}
where $\epsilon > 0$ is chosen such that $\epsilon W^* \leq \epsilon u^*$. By the comparison principle we deduce
\begin{equation}
U(t + t_0, x) \leq \bar{W}(t_0) \leq (1 + \epsilon e^{\bar{\sigma}t}) u^* \quad \text{for } t \geq 0, \ x \in [g(t + t_0), h(t + t_0)].
\end{equation}
Hence
\[ U(t_0, x) \leq (1 + \frac{\epsilon(0)}{2})u^* \text{ for } x \in [g(t_0), h(t_0)] \]
provided that
\[ t_0 = t_0(\theta) := \frac{\beta}{|\sigma|} \ln \theta + \frac{\ln(2\bar{\epsilon}/K_1)}{|\sigma|} + t_* . \]

By (5.1), for any fixed \( \omega_\ast \in (\beta, \omega - 1) \), we have
\[ \int_{\mathbb{R}} J(x)|x|^{\omega_\ast} dx < \infty . \]
Then by Theorem 1.7, there is \( C_2 \) such that
\[ u^* - \Phi(x) \leq \frac{C_2}{|x|^\omega} u^* \text{ for } x \leq -1 . \]
Hence, for \( K > 1 \) we have
\[
(1 + \epsilon(0))\Phi(-K) - (1 + \epsilon(0)/2)u^* \\
\geq (1 + \epsilon(0))[1 - C_2K^{-\omega_\ast}]u^* - (1 + \epsilon(0)/2)u^* \\
= [K_1\theta^{-\beta}/2 - C_2K^{-\omega_\ast}(1 + K_1\theta^{-\beta})]u^* \\
\geq 0
\]
provided that
\[ K^{\omega_\ast} \geq 2C_2 + \frac{2C_2^2}{K_1} . \]
Therefore, for all \( K_1 \in (0, 1], \theta \geq 1 \) and \( K \geq (4C_2/K_1)^{1/\omega_\ast}\theta^{\beta/\omega_\ast} \), we have
\[ (1 + \epsilon(0))\Phi(-K) - (1 + \epsilon(0)/2)u^* \geq 0 . \]

Now define
\[ K_2(\theta) := 2 \max \left\{ (4C_2/K_1)^{1/\omega_\ast}\theta^{\beta/\omega_\ast}, (c_0 + 1)t_0(\theta) + C_1 \right\} . \]
Then for \( K_2 = K_2(\theta) \) we have
\[ \bar{h}(0) = K_2 > K_2/2 \geq (c_0 + 1)t_0 + C_1 \geq h(t_0) , \]
and for \( x \in [g(t_0), h(t_0)] \),
\[ \bar{U}(0, x) = (1 + \epsilon(0))\Phi(x - K_2) \geq (1 + \epsilon(0))\Phi(-K_2/2) \geq (1 + \epsilon(0)/2)u^* . \]
Thus (5.8) holds if \( t_0 \) and \( K_2 \) are chosen as above, for any \( \theta \geq 1 \), \( K_1 \in (0, 1] \).

**Step 2.** We verify that (5.6) holds if \( \theta, K_1, K_3 \) and \( K_4 \) are chosen suitably. Denote
\[ C_3 := \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_{0}^{+\infty} J_1(x - y)dydx = \sum_{i=1}^{m_0} \mu_i \int_{0}^{+\infty} J_1(y)ydy . \]

With \( \rho = (\rho_i) \), a direct calculation shows
\[
\sum_{i=1}^{m_0} \mu_i \int_{g(t)^{t+\theta}}^{+\infty} J_1(x - y)\bar{u}_i(t, x)dydx \\
= \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{h(t)} \int_{h(t)^{t+\theta}}^{+\infty} J_1(x - y)\bar{u}_i(t, x)dydx - \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{g(t)^{t+\theta}} \int_{h(t)}^{+\infty} J_1(x - y)\bar{u}_i(t, x)dydx \\
= \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_{0}^{+\infty} J_1(x - y)(1 + \epsilon)\Phi(x + \rho_i(t, x + \bar{h}(t)))dydx
\]
By elementary calculus, for any $k > 1$,

\[
\int_{-\infty}^{-k} \int_{-\infty}^{\infty} \frac{1}{|x-y|^{2+\beta}} dy \, dx = \int_{-k}^{\infty} \int_{-\infty}^{\infty} \frac{1}{y^{2+\beta}} dy \, dx = \frac{1}{(1+\beta)k^{-\beta}}.
\]

From (5.1) and (5.9), there exists $C_4 > 0$ such that

\[
\sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J_i(x-y)(1+\epsilon)\phi_i(x) \, dy \, dx \\
\geq (1+\epsilon)c_0 + C_3 K_4 \epsilon |V^*| - \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J_i(x-y) \phi_i(x) \, dy \, dx \\
\geq (1+\epsilon)c_0 + C_3 K_4 \epsilon |V^*| - \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J_i(x-y) \phi_i(x) \, dy \, dx,
\]

for all large $\theta > 0$ so that

\[
\theta > \frac{(c_0 + 1) t_0 + C_3 + K_2}{(2c_0 + 1)},
\]

which is possible since $t_0(\theta)$ and $K_2(\theta)$ grow slower than linearly in $\theta$, we have

\[
\sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} J_i(x-y) u_i(x,y) \, dy \, dx \\
\leq (1+\epsilon) c_0 + C_4 K_4 \epsilon |V^*| - \frac{\phi_* C_4}{\beta(1+\beta)(2c_0 + 1)^\beta} (t + \theta)^{-\beta} \\
= c_0 + \epsilon(t) \left[ c_0 + C_4 K_4 |V^*| - K_3 \phi_* C_4 \right] \\
\leq c_0 - K_3 \epsilon(t) = h'(t)
\]

provided that $K_1, K_3$ and $K_4$ are small enough so that

\[
K_1 (c_0 + C_4 K_4 |V^*| + K_3) \leq \frac{\phi_* C_4}{\beta(1+\beta)(2c_0 + 1)^\beta}.
\]

Therefore (5.6) holds if we first fix $K_1, K_3, K_4$ small so that (5.15) holds, and then choose $\theta$ large such that (5.14) is satisfied.
Step 3. We show that \((5.10)\) holds when \(K_3\) and \(K_4\) are chosen suitably small and \(\theta\) is large. From \((1.4)\), we deduce
\[
\begin{align*}
\mathcal{U}_t(t, x) &= -(1 + \epsilon)[c_0 + \delta'(t)]\Phi'(x - \bar{h}(t)) + \epsilon'(t)\Phi(x - \bar{h}(t)) + \rho_t(t, x),
\end{align*}
\]
and
\[
-(1 + \epsilon)c_0 \Phi'(x - \bar{h}(t)) = (1 + \epsilon) \left[ D \circ \int_{-\infty}^{\bar{h}(t)} J(x - y) \circ \Phi(y - \bar{h}(t)) dy - D \circ \Phi(x - \bar{h}(t)) + F(\Phi(x - \bar{h}(t))) \right]
\]
\[
= D \circ \int_{-\infty}^{\bar{h}(t)} J(x - y) \circ \mathcal{U}(t, y) - \rho(t, y) \circ \Phi(y - \bar{h}(t)) dy - D \circ \mathcal{U}(t, x) - \rho(t, x) + (1 + \epsilon)F(\Phi(x - \bar{h}(t)))
\]
\[
= D \circ \int_{\bar{h}(t)}^{g(t + t_0)} J(x - y) \circ \mathcal{U}(t, y) dy - D \circ \mathcal{U}(t, x) + F(\mathcal{U}(t, x))
\]
\[
+ D \left[ \rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x - y) \circ \rho(t, y) dy \right] + (1 + \epsilon)F(\Phi(x - \bar{h}(t))) - F(\mathcal{U}(t, x)).
\]
Hence
\[
\mathcal{U}_t(t, x) = D \circ \int_{\bar{h}(t)}^{g(t + t_0)} J(x - y) \circ \mathcal{U}(t, y) dy - D \circ \mathcal{U}(t, x) + F(\mathcal{U}(t, x))
\]
with
\[
A(t, x) := D \circ \left[ \rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x - y) \circ \rho(t, y) dy \right] + (1 + \epsilon)F(\Phi(x - \bar{h}(t))) - F(\mathcal{U}(t, x))
\]
\[
-(1 + \epsilon)\delta'(t)\Phi'(x - \bar{h}(t)) + \epsilon'(t)\Phi(x - \bar{h}(t)) + \rho_t(t, x).
\]
Therefore to complete this step, it suffices to show that we can choose \(K_3, K_4\) and \(\theta\) such that \(A(t, x) \geq 0\). We will do that for \(x \in [\bar{h}(t) - \bar{\epsilon}, \bar{h}(t)]\) and for \(x \in [g(t_0 + t), \bar{h}(t) - \bar{\epsilon}]\) separately.

Claim 1. If \(\bar{\epsilon} > 0\) in \((5.4)\) is sufficiently small and \(\theta\) is sufficiently large, then
\[
D \circ \left[ \rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x - y) \circ \rho(t, y) dy \right] + (1 + \epsilon)F(\Phi(x - \bar{h}(t))) - F(\mathcal{U}(t, x)) \geq \frac{1}{4} |\lambda_1| \rho(t, x) > 0 \quad \text{for} \quad x \in [\bar{h}(t) - \bar{\epsilon}, \bar{h}(t)].
\]
Since \(\lambda_1 < 0\) and \(D \circ V* = V* \bar{D}\), using \((5.2)\) we deduce, for \(x \in [\bar{h}(t) - \bar{\epsilon}, \bar{h}(t)]\),
\[
D \circ \left[ \rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x - y) \circ \rho(t, y) dy \right]
\]
\[
= K_4 \epsilon(t) \left[ D \circ V* - D \circ \int_{-\infty}^{0} J(x - \bar{h}(t) - y) \circ \xi(y)V* dy \right]
\]
\[
\geq K_4 \epsilon(t) \left[ D \circ V* - D \circ \int_{-2\bar{\epsilon}}^{0} J(x - \bar{h}(t) - y) \circ V* dy \right]
\]
\[
= K_4 \epsilon(t) \left[ V* \nabla F(0) - \lambda_1 V* - D \circ \int_{\bar{h}(t) - 2\bar{\epsilon}}^{\bar{h}(t) - x} J(y) \circ V* dy \right]
\]
\[
\geq K_4 \epsilon(t) \left[ V* \nabla F(0) - \lambda_1 V* - D \circ \int_{-2\bar{\epsilon}}^{-\bar{\epsilon}} J(y) \circ V* dy \right]
\]
\[ \geq K_4 \epsilon(t) \left[ V^* \nabla F(0) - \frac{\lambda_1}{2} V^* \right] = \rho(t, x) \nabla F(0) - \frac{\lambda_1}{2} \rho(t, x), \]

provided \( \tilde{\epsilon} \in (0, \epsilon_1] \) for some small \( \epsilon_1 > 0 \).

On the other hand, for \( x \in [\tilde{h}(t) - \tilde{\epsilon}, \tilde{h}(t)] \), by (f2) we obtain
\[
(1 + \epsilon) F(\Phi(x - \tilde{h}(t))) - F(\bar{U}(t, x)) \\
\geq F(1 + \epsilon) \Phi(x - \tilde{h}(t))) - F(\bar{U}(t, x)) \\
= F(\bar{U}(t, x) - \rho(t, x)) - F(\bar{U}(t, x),
\]

and
\[
0 \leq \bar{U}(t, x) \leq (1 + \epsilon) \Phi(\tilde{\epsilon}) + K_4 \epsilon V^* \leq 2 \Phi(\tilde{\epsilon}) + \theta^{-1} V^*,
\]

So the components of \( \bar{U}(t, x) \) and \( \rho(t, x) \) are small for small \( \tilde{\epsilon} \) and large \( \theta \). It follows that
\[
F(\bar{U}(t, x) - \rho(t, x)) - F(\bar{U}(t, x)) = -\rho(t, x)[\nabla F(\bar{U}(t, x)) + o(1) I_m] \\
= -\rho(t, x)[\nabla F(0) + o(1) I_m] \geq -\rho(t, x) \nabla F(0) + \frac{\lambda_1}{4} \rho(t, x)
\]

for \( x \in [\tilde{h}(t) - \tilde{\epsilon}, \tilde{h}(t)] \), provided that \( \tilde{\epsilon} \) is small and \( \theta \) is large. Hence, (5.16) holds.

Denote
\[ M := \max_{1 \leq i \leq m, x \leq 0} |\phi_i'(x)|. \]

For \( x \in [\tilde{h} - \tilde{\epsilon}, \tilde{h}] \), by (5.16) we have
\[
A(t, x) \geq \frac{\lambda_1}{4} \rho(t, x) - (1 + \epsilon) \delta'(t) \Phi'(x - \tilde{h}(t)) + \epsilon'(t) \Phi(x - \tilde{h}(t)) + \rho(1, x) \\
\geq \epsilon(t) \left[ \frac{\lambda_1}{4} K_4 V^* - 2 K_3 \epsilon M - \beta(t + \theta)^{-1} u^* - K_4 \beta(t + \theta)^{-1} V^* \right] \\
\geq \epsilon(t) \left[ \frac{\lambda_1}{4} K_4 V^* - 2 K_3 \epsilon M - \theta^{-1} \beta \left( u^* + K_4 V^* \right) \right] \\
\geq 0
\]

provided that we first fix \( K_3 \) and \( K_4 \) so that (5.15) holds and at the same time
\[ (5.17) \]
\[ \frac{\lambda_1}{4} K_4 V^* - 2 K_3 \epsilon M \gg 0, \]

and then choose \( \theta \) sufficiently large.

Next, for fixed small \( \tilde{\epsilon} > 0 \), we estimate \( A(t, x) \) for \( x \in [g(t + t_0), \tilde{h}(t) - \tilde{\epsilon}] \).

Claim 2. For any given \( 1 \gg \eta > 0 \), there is \( c_1 = c_1(\eta) \) such that
\[ (5.18) \]
\[ (1 + \epsilon) F(v) - F(1 + \epsilon v) \geq c_1(\eta) v \] for \( v \in [\eta 1, u^*] \) and \( 0 < \epsilon \ll 1 \).

Indeed, by (1.7) there exists \( c_1 > 0 \) depending on \( \eta \) such that
\[ F(v) - v [\nabla F(v)]^T \geq 2 c_1 1 \text{ for } v \in [\eta 1, u^*]. \]

Since
\[ \lim_{\epsilon \to 0} \frac{(1 + \epsilon) F(v) - F((1 + \epsilon) v)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon F(v) - [F(v + \epsilon v) - F(v)]}{\epsilon} \]
\[ = F(v) - v [\nabla F(v)]^T \geq 2 c_1 1 \]

uniformly for \( v \in [\eta 1, u^*] \), there exists \( \epsilon_0 > 0 \) small so that
\[ \frac{(1 + \epsilon) F(v) - F((1 + \epsilon) v)}{\epsilon} \geq c_1 1 \]
for $v \in [\eta_1, u^*]$ and $\epsilon \in (0, \epsilon_0]$. This proves Claim 2.

By Claim 2 and the Lipschitz continuity of $F$, there exist positive constants $C_I$ and $C_f$ such that, for $v = \Phi(x - \tilde{h}(t)) \in [\Phi(-\epsilon), u^*],$

$$(1 + \epsilon)F(v) - F((1 + \epsilon)v + \rho)$$

$$= (1 + \epsilon)F(v) - F((1 + \epsilon)v) + F((1 + \epsilon)v) - F((1 + \epsilon)v + \rho)$$

$$\geq C_I \epsilon 1 - C_f K_4 \epsilon 1$$

when $\epsilon = \epsilon(t)$ is small.

We also have

$$D \circ \left[ \rho(t, x) - \int_{-\infty}^{h(t)} J(x - y) \circ \rho(t, x) dy \right] \geq -D \circ \int_{-\infty}^{\tilde{h}(t)} J(x - y) \circ \rho(t, x) dy$$

$$\geq -K_4 \epsilon(t) D \circ V^* \geq -C_d K_4 \epsilon(t) 1$$

for some $C_d > 0$, and

$$\rho_t(t, x) = -\xi_t \tilde{h}' K_4 \epsilon(t) V^* + \xi_4 K_4 \epsilon(t) V^*$$

$$\geq -\xi_4 K_4 \epsilon(t) V^* - K_4 \beta(t + \theta)^{-1} \epsilon(t) V^*$$

$$\geq - (\xi_* + \beta \theta^{-1}) K_4 \epsilon(t) V^*,$$

with $\xi_* := c_0 \max_{x \in \mathbb{R}} |\xi'(x)|$.

Using these we obtain, for $x \in [g(t_0 + t), \tilde{h}(t) - \epsilon]$,

$$A(t, x) \geq - C_I \epsilon(t) 1 + (1 + \epsilon)F(\Phi(x - \tilde{h}(t))) - F(\tilde{U}(t, x)) + 2M \delta'(t) 1 + \epsilon(t) u^* + \rho_t(t, x)$$

$$\geq C_I \epsilon(t) 1 - (C_f + C_d) K_4 \epsilon(t) 1 - 2M K_3 \epsilon(t) 1 - \beta(t + \theta)^{-1} \epsilon(t) u^* - (\xi_* + \beta \theta^{-1}) K_4 \epsilon(t) V^*$$

$$= \epsilon(t) \left[ C_I 1 - K_4 (C_f + C_d) 1 - 2M K_3 1 - \beta(t + \theta)^{-1} u^* - (\xi_* + \beta \theta^{-1}) K_4 V^* \right]$$

$$\geq \epsilon(t) \left[ C_I 1 - K_4 (C_f + C_d) 1 - 2M K_3 1 - \xi_* K_4 V^* - \beta \theta^{-1} (u^* + K_4 V^*) \right]$$

$$\geq 0$$

provided that we first choose $K_3$ and $K_4$ small such that

$$C_I 1 - K_4 (C_f + C_d) 1 - 2M K_3 1 - \xi_* K_4 V^* \gg 0$$

while keeping both (5.15) and (5.17) hold, and then choose $\theta > 0$ sufficiently large.

Therefore, (5.5) holds when $K_3, K_4$ and $\theta$ are chosen as above. The proof of the lemma is now complete. \[\square\]

5.2. The case $\tilde{\lambda}_1 \geq 0$.

**Lemma 5.3.** Suppose that the assumptions in Theorem 5.1 are satisfied. If $\tilde{\lambda}_1 \geq 0$, then (5.3) still holds.

**Proof.** This is a modification of the proof of Lemma 5.2. We will use similar notations. Let $\beta = \gamma - 2 \in (0, 1]$, and $(c_0, \Phi)$ be the solution of (1.4)-(1.5). For fixed $\epsilon > 0$, let $\xi \in C^2(\mathbb{R})$ satisfy

$$0 \leq \xi(x) \leq 1, \quad \xi(x) = 1 \text{ for } |x| < \epsilon, \quad \xi(x) = 0 \text{ for } |x| > 2\epsilon.$$

Define

$$\tilde{h}(t) := c_0 t + \delta(t),$$

$$\tilde{U}(t, x) := (1 + \epsilon(t)) \Phi(x - \tilde{h}(t) - \lambda(t)) - \rho(t, x), \quad t \geq 0, \quad x \leq \tilde{h}(t),$$

$$\tilde{\lambda}(t) := \lambda(t) + \epsilon(t).$$
where

\[ \epsilon(t) := K_1(t + \theta)^{-\beta}, \quad \delta(t) := K_2 - K_3 \int_0^t \epsilon(\tau) d\tau, \]

\[ \rho(t, x) := K_4 \xi(x - \bar{h}(t)) \epsilon(t) V^*, \quad \lambda(t) := K_5 \epsilon(t), \]

and the positive constants \( \theta, \bar{\epsilon} \) and \( K_1, K_2, K_3, K_4, K_5 \) are to be determined.

Let

\[ C_\epsilon := \min_{1 \leq i \leq m} \min_{x \in [-2\bar{\epsilon}, 0]} |\phi'_i(x)|. \]

Then for \( x \in [\bar{h}(t) - 2\bar{\epsilon}, \bar{h}(t)] \) and \( i \in \{1, \ldots, m\} \),

\[ \bar{u}_i(t, x) \geq \phi_i( - \lambda(t)) - \rho_i(t, x) \geq C_\epsilon \lambda(t) - K_4 \epsilon(t) v^*_i \]

\[ \geq \epsilon(t)(C_\epsilon K_5 - K_4 v^*_i) > 0 \]

if

\[ (5.19) \quad K_4 = C_\epsilon K_5/(2 \max_{1 \leq i \leq m} v^*_i), \]

which combined with \( \xi(x) = 0 \) for \( |x| \geq 2\bar{\epsilon} \) implies

\[ (5.20) \quad \mathcal{U}(t, x) \geq 0 \quad \text{for} \quad t \geq 0, \quad x \leq \bar{h}(t). \]

Let \( t_0 = t_0(\theta) \) and \( K_2 = K_2(\theta) \) be given by Step 1 in the proof of Lemma 5.22. Then \([g(t_0), \bar{h}(t_0)] \subset (-\infty, K_2/2)\), and due to \( \rho(0, x) = 0 \) for \( x \leq \bar{h}(t_0) < K_2/2 < \bar{h}(0) \), we have

\[ (5.21) \quad \mathcal{U}(0, x) = (1 + \epsilon(0)) \Phi(x - K_2 - \lambda) \geq (1 + \epsilon(0)) \Phi(-K_2/2) \]

\[ \geq (1 + \epsilon(0)/2) u^* \geq U(t_0, x) \quad \text{for} \quad x \in [g(t_0), \bar{h}(t_0)]. \]

**Step 1.** We verify that by choosing \( K_1, K_3 \) and \( K_5 \) suitably small,

\[ (5.22) \quad \bar{h}'(t) \geq \sum_{i=1}^{m_0} \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{-\bar{h}(t)}^{+\infty} J_i(x - y) \bar{u}_i(t, x) dy dx \quad \text{for all} \quad t > 0. \]

By direct calculations we have

\[
\sum_{i=1}^{m_0} \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x - y) \bar{u}_i(t, x) dy dx \\
\leq \sum_{i=1}^{m_0} \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x - y)(1 + \epsilon) \Phi_i(x - \bar{h}(t) - \lambda(t)) dy dx \\
= (1 + \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_{0}^{+\infty} J_i(x - y) \Phi_i(x - \lambda(t)) dy dx \\
- (1 + \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{g(t+t_0) - \bar{h}(t)} \int_{0}^{+\infty} J_i(x - y) \Phi_i(x - \lambda(t)) dy dx \\
\leq (1 + \epsilon) c_0 + (1 + \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_{0}^{+\infty} J_i(x - y) [\Phi_i(x - \lambda) - \Phi_i(x)] dy dx \\
- (1 + \epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{g(t+t_0) - \bar{h}(t)} \int_{0}^{+\infty} J_i(x - y) \Phi_i(x) dy dx
\]
Let $M_1 := \max_{1 \leq i \leq m} \sup_{x \leq 0} |\phi'_i(x)|$ and $C_3$ be given by (5.11). Then

$$(1 + \epsilon) \sum_{i=1}^{m} \phi_i(x) - \phi_i(x - \lambda(t)) - \phi_i(x) \leq 2C_3M_1\lambda(t).$$

By (5.13),

$$\sum_{i=1}^{m} \phi_i(x) - \phi_i(x - \lambda(t)) \leq \frac{\phi_4C_3}{\beta(1 + \beta)(2c_0 + 1)^\beta} (t + (c_0 + 1)t_0 + C_1 + K_2)^{-\beta}.$$

Therefore, as in the proof of Lemma 5.2, for sufficiently large $\theta$ so that

$$\theta > \frac{(c_0 + 1)t_0 + C_1 + K_2}{(2c_0 + 1)}$$

holds, we have

$$\sum_{i=1}^{m} \phi_i(x) - \phi_i(x - \lambda(t)) \leq \frac{\phi_4C_3}{\beta(1 + \beta)(2c_0 + 1)^\beta} (t + (c_0 + 1)t_0 + C_1 + K_2)^{-\beta}.$$

provided that $K_1, K_3$ and $K_5$ are suitably small so that

$$K_1(c_0 + 2C_3M_1K_5 + K_3) \leq \frac{\phi_4C_3}{\beta(1 + \beta)(2c_0 + 1)^\beta}.$$

**Step 2.** We show that by choosing $K_3, K_5$ suitably small and $\theta$ sufficiently large, for $t > 0$, $x \in [g(t + t_0), \tilde{h}(t)]$,

$$(5.25) \quad \tilde{U}(t, x) \geq D \circ \int_{g(t + t_0)}^{\tilde{h}(t)} J(x - y) \circ \tilde{U}(t, y)dy - \tilde{U}(t, x) + F(\tilde{U}(t, x)).$$

Using the definition of $\tilde{U}$, we have

$$\tilde{U}(t, x) = - (1 + \epsilon)(\tilde{h} + \lambda')\Phi'(x - \tilde{h} - \lambda) + \epsilon\Phi(x - \tilde{h} - \lambda) - \rho_t$$

and from (1.4), we obtain

$$= (1 + \epsilon) \left[ D \circ \int_{-\infty}^{\tilde{h} + \lambda} J(x - y) \circ \Phi(y - \tilde{h} - \lambda)dy - D \circ \Phi(x - \tilde{h} - \lambda) + F(\Phi(x - \tilde{h} - \lambda)) \right]$$

$$\geq (1 + \epsilon) \left[ D \circ \int_{-\infty}^{\tilde{h}} J(x - y) \circ \Phi(y - \tilde{h} - \lambda)dy - D \circ \Phi(x - \tilde{h} - \lambda) + F(\Phi(x - \tilde{h} - \lambda)) \right]$$

$$= D \circ \int_{-\infty}^{\tilde{h}} J(x - y) \circ [\tilde{U}(t, y) + \rho]dy - D \circ [\tilde{U}(t, x) + \rho] + (1 + \epsilon)F(\Phi(x - \tilde{h} - \lambda))$$

$$= D \circ \int_{-\infty}^{\tilde{h}(t)} J(x - y) \circ \tilde{U}(t, y)dy - D \circ \tilde{U}(t, x).$$
On the other hand, for 
\[ x \] with 
\[ K \]
To show (5.25), it remains to choose suitable 
\[ \tilde{x}_0 \], such that 
\[ \tilde{\eta} \in [\bar{B}(t,x)] \].

\[
B(t,x) := - D \circ \left[ \rho(t,x) - \int_{-\infty}^{\tilde{h}} J(x-y) \circ \rho(t,y) dy \right] + (1 + \epsilon) F(\Phi(x - \tilde{h} - \lambda)) - F(\mathcal{U})
\]
\[
- (1 + \epsilon)(\delta + \lambda')\Phi'(x - \tilde{h} - \lambda) + \epsilon' \Phi(x - \tilde{h} - \lambda) - \rho_t.
\]

To show (5.25), it remains to choose suitable \( K_3, K_5 \) and \( \theta \) such that \( B(t,x) \geq 0 \) for \( t > 0 \) and 
\[ x \in [g(t + t_0), \tilde{h}(t)]. \]

**Claim:** There exist small \( \bar{\epsilon}_0 \in (0, \bar{\epsilon}/2) \) and some \( \tilde{J}_0 > 0 \) depending on \( \bar{\epsilon} \) but independent of \( \bar{\epsilon}_0 \), such that

\[
(5.26) \quad - D \circ \left[ \rho(t,x) - \int_{-\infty}^{\tilde{h}} J(x-y) \circ \rho(t,y) dy \right] + (1 + \epsilon) F(\Phi(x - \tilde{h} - \lambda)) - F(\mathcal{U}(t,x)) \geq \tilde{J}_0 \rho(t,x) \quad \text{for} \quad x \in [\tilde{h}(t) - \bar{\epsilon}_0, \tilde{h}(t)].
\]

Indeed, for \( x \in [\tilde{h}(t) - \bar{\epsilon}_0, \tilde{h}(t)] \),

\[
D \circ \left[ \rho(t,x) - \int_{-\infty}^{\tilde{h}} J(x-y) \circ \rho(t,y) dy \right] = K_2 \epsilon(t) \left[ D \circ V^* - D \circ \int_{-\infty}^{\tilde{h}} J(x-y) \circ \xi(y - \tilde{h}(t)) V^* dy \right] \leq K_2 \epsilon(t) \left[ D \circ V^* - D \circ \int_{\tilde{h}(t) - \bar{\epsilon}}^{\tilde{h}} J(x-y) \circ V^* dy \right] \leq K_2 \epsilon(t) \left[ D \circ V^* - D \circ \int_{\tilde{h}(t) - \bar{\epsilon}}^{\tilde{h}(t) - x} J(x-y) \circ V^* dy \right] \leq D \circ \left[ 1 - \int_{-\bar{\epsilon} + \bar{\epsilon}_0}^{0} J(y) dy \right] \leq D \circ \left[ 1 - \int_{-\bar{\epsilon}/2}^{0} J(y) dy \right].
\]

On the other hand, for \( x \in [\tilde{h}(t) - \bar{\epsilon}_0, \tilde{h}(t)] \), we have

\[
(1 + \epsilon) F(\Phi(x - \tilde{h} - \lambda)) - F(\mathcal{U}) \geq F((1 + \epsilon) \Phi(x - \tilde{h} - \lambda)) - F(\mathcal{U}) = F(\mathcal{U} + \rho) - F(\mathcal{U}) = \rho \left( |\nabla F(\mathcal{U})|^T + o(1) I_m \right) = K_4 \epsilon(t) V^* \left( |\nabla F(0)|^T + o(1) I_m \right).
\]
\[= K_4 \epsilon(t)[V^* \bar{D} + \tilde{\lambda}_1 V^* + o(1)V^*] = K_4 \epsilon(t)[D \circ V^* + \tilde{\lambda}_1 V^* + o(1)V^*] = D \circ \rho + \tilde{\lambda}_1 \rho + o(1)\rho.\]

since both \(\bar{U}(t, x)\) and \(\rho(t, x)\) are close to \(0\) for \(x \in [\bar{h}(t) - \tilde{\epsilon}_0, \bar{h}(t)]\) with \(\tilde{\epsilon}_0\) small.

Hence, for such \(x\) and \(\tilde{\epsilon}_0\), since \(\tilde{\lambda}_1 \geq 0\),
\[
-D \circ \left[ \rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)dy \right] + (1 + \epsilon) F(\Phi(x - \bar{h}(t))) - F(\bar{U}(t, x)) \\
\geq D \circ \rho \left[ -1 + \int_{-\tilde{\epsilon}/2}^{0} J(y)dy \right] + D \circ \rho + \tilde{\lambda}_1 \rho + o(1)\rho \\
\geq \tilde{J}_0 \rho(t, x), \quad \text{with} \quad \tilde{J}_0 := \frac{1}{2} \min_{1 \leq i \leq m} d_i \int_{-\tilde{\epsilon}/2}^{0} J(y)dy \text{ if } m_0 = m.
\]

This proves (5.20) when \(m_0 = m\).

If \(m_0 < m\), we need to modify \(V^*\) in the definition of \(\rho\) slightly. In this case, for \(\tilde{\delta} > 0\) small we define
\[\tilde{V}^* := V^* + \tilde{\delta} D = (v_i^* + \tilde{\delta} d_i).\]

Since \(d_i = 0\) for \(i = m_0 + 1, \ldots, m\) and \(d_i > 0\) for \(i = 1, \ldots, m_0\), by (f_1) (iv) we see that
\[W = (w_i) := D[\nabla F(0)]^T \]
satisfies \(w_i > 0\) for \(i = m_0 + 1, \ldots, m\). Let us write
\[W = W^1 + W^2 = (w_i^1) + (w_i^2) \text{ with } \begin{cases} w_i^1 = 0 \text{ for } i = m_0 + 1, \ldots, m, \\ w_i^2 = 0 \text{ for } i = 1, \ldots, m_0. \end{cases} \]

Then
\[\tilde{V}^* \left( [\nabla F(0)]^T - D \right) = \tilde{\lambda}_1 V^* + \tilde{\delta} \tilde{W}^1 + \tilde{\delta} W^2 \text{ with } \tilde{W}^1 := W^1 - D \bar{D}. \]

It is important to observe that the vector \(\tilde{W}^1 = (\tilde{w}_i^1)\) has its last \(m - m_0\) components 0, namely \(\tilde{w}_i^1 = 0\) for \(i = m_0 + 1, \ldots, m\).

Replacing \(V^*\) by \(\tilde{V}^*\) in the definition of \(\rho\), we see that the analysis above is not affected, except that, for \(\tilde{\epsilon}_0 > 0\) small and \(x \in [\bar{h}(t) - \tilde{\epsilon}_0, \bar{h}(t)]\),
\[
(1 + \epsilon) F(\Phi(x - \bar{h} - \lambda)) - F(\bar{U}) \\
\geq K_4 \epsilon(t) \tilde{V}^* \left( [\nabla F(0)]^T + o(1)I_m \right) \\
= K_4 \epsilon(t) \left( [\tilde{V}^* \bar{D} + \tilde{\lambda}_1 V^* + o(1)V^*] + \tilde{\delta} \tilde{W}^1 + \tilde{\delta} W^2 \right) \\
= K_4 \epsilon(t) \left( D \circ \tilde{V}^* + \tilde{\lambda}_1 V^* + o(1)V^* + \tilde{\delta} \tilde{W}^1 + \tilde{\delta} W^2 \right) \\
\geq D \circ \rho + K_4 \epsilon(t) \left( o(1)V^* + \tilde{\delta} \tilde{W}^1 + \tilde{\delta} W^2 \right).
\]

Hence, for such \(x\) and \(\tilde{\epsilon}_0\), we now have
\[
-D \circ \left[ \rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)dy \right] + (1 + \epsilon) F(\Phi(x - \bar{h}(t))) - F(\bar{U}(t, x)) \\
\geq D \circ \rho \left[ -1 + \int_{-\tilde{\epsilon}/2}^{0} J(y)dy \right] + D \circ \rho + K_4 \epsilon(t) \left( o(1)V^* + \tilde{\delta} \tilde{W}^1 + \tilde{\delta} W^2 \right) \\
\geq K_4 \epsilon(t) \left( \min_{1 \leq i \leq m_0} v_i^* \int_{-\tilde{\epsilon}/2}^{0} J(y)dyD + o(1)V^* + \tilde{\delta} \tilde{W}^1 + \tilde{\delta} W^2 \right).\]
We now fix $\tilde{\delta} > 0$ small enough such that
\[
\tilde{\delta}W^1 \lessapprox \frac{1}{2} \min_{1 \leq i \leq m_0} v_i^* \int_{-\bar{\epsilon}/2}^{0} J(y)dy D,
\]
and notice that
\[
\hat{W} := \frac{1}{2} \min_{1 \leq i \leq m_0} v_i^* \int_{-\bar{\epsilon}/2}^{0} J(y)dy D + \tilde{\delta}W^2 \gg 0.
\]
Therefore there exists $\tilde{J}_0 > 0$ such that
\[
\frac{1}{2} \hat{W} \geq \tilde{J}_0 \hat{V}^*.
\]
Then
\[
K_4 \epsilon(t) \left( \min_{1 \leq i \leq m_0} v_i^* \int_{-\bar{\epsilon}/2}^{0} J(y)dy D + o(1)V^* + \tilde{\delta}W^1 + \tilde{\delta}W^2 \right)
\geq K_4 \epsilon(t) \left( \hat{W} + o(1)V^* \right) \geq K_4 \epsilon(t) \frac{1}{2} \hat{W} \geq K_4 \epsilon(t)\tilde{J}_0 \hat{V}^* = \tilde{J}_0 \rho,
\]
provided that $\tilde{\epsilon}_0 > 0$ is chosen sufficiently small.
Therefore for $\tilde{\epsilon}_0 > 0$ small and $x \in [\bar{h}(t) - \tilde{\epsilon}_0, \bar{h}(t)]$, we finally have
\[
-D \circ \rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t, y)dy + (1 + \epsilon)F(\Phi(x - \bar{h}(t))) - F(\Omega(t, x))
\geq \tilde{J}_0 \rho(t, x), \quad \text{as desired.}
\]
With $\tilde{\delta} > 0$ chosen as above, we will from now on denote
\[
\hat{V}^* := \begin{cases} V^* & \text{if } m_0 = m, \\
\tilde{V}^* & \text{if } m_0 < m,
\end{cases}
\]
but keep the notation for $\rho$ unchanged.
Clearly
\[
-\rho(t, x) = \beta K_4 K_1 (t + \theta)^{-\beta-1} \hat{V}^* \geq 0.
\]
Recalling $M_1 := \max_{1 \leq i \leq m} \sup_{x \leq 0} |\phi'(x)|$, we obtain, for $x \in [\bar{h}(t) - \tilde{\epsilon}_0, \bar{h}(t)]$ and small $\tilde{\epsilon}_0,$
\[
B(t, x) \geq \tilde{J}_0 K_2 \epsilon(t) \hat{V}^* + 2(\beta'(t) + \lambda'(t))M_1 1 + \epsilon'(t)u^*
= \tilde{J}_0 K_2 \epsilon(t) \hat{V}^* + 2(\beta - K_3 - K_5 \beta(t + \theta)^{-1})M_1 1 - \beta(t + \theta)^{-1} \epsilon(t)u^*
\geq \epsilon(t) \left[ \tilde{J}_0 K_2 \hat{V}^* - 2(K_3 + K_5 \beta \theta^{-1})M_1 1 - \beta \theta^{-1} u^* \right]
= \epsilon(t) \left[ \tilde{J}_0 K_2 \hat{V}^* - 2K_3 M_1 1 - \theta^{-1} \left( K_5 \beta M_1 1 + \beta u^* \right) \right]
\geq 0
\]
provided that $K_3$ is chosen small so that (5.24) holds,
\[
(5.27) \quad \tilde{J}_0 K_2 \hat{V}^* - 2K_3 M_1 1 \gg 0,
\]
and $\theta$ is chosen sufficiently large\[1\]

\[1\] In fact, by the choice of $K_3 = K_2(\theta)$ in (5.10), for fixed $K_3$, (5.27) always holds for large enough $\theta.$
We next estimate $B(t, x)$ for $x \in [g(t + t_0), h(t) - \varepsilon_0]$. From Claim 2 in the proof of Lemma 5.2 and the Lipschitz continuity of $F$, there exist positive constants $C_l = C_l(\bar{\varepsilon}_0)$ and $C_f$ such that, for $v = \Phi(x - \bar{h}(t - \lambda(t))) \in [\Phi(\varepsilon_0), u^*]$,\[ (1 + \epsilon)F(v) - F((1 + \epsilon)v - \rho) \geq (1 + \epsilon)F(v) - F((1 + \epsilon)v) + F((1 + \epsilon)v) - F((1 + \epsilon)v - \rho) \geq C_l f_1 - C_f \rho \geq C_l f_1 - C_f K_4 \varepsilon \hat{V}^* \]
when $\epsilon = \epsilon(t)$ is small. Hence
\[ (1 + \epsilon)F(\Phi(x - \bar{h} - \lambda)) - F(\bar{U}) \geq C_l f_1 - C_f K_4 \varepsilon \hat{V}^* \text{ for } x \in [g(t + t_0), \bar{h}(t) - \varepsilon_0], \ 0 < \varepsilon_0 \ll 1. \]
Clearly,
\[ -D \circ \left[ \rho(t, x) - \int_{-\infty}^{\bar{h}(t)} \mathcal{J}(x - y) \circ \rho(t, x) dy \right] \geq -K_4 \varepsilon(t)D \circ \hat{V}^*, \]
and
\[ \rho(t, x) = -K_4 \varepsilon(t)\hat{h}^\prime(t)\varepsilon \hat{V}^* + K_4 \varepsilon(t)\hat{V}^* \preceq \varepsilon \hat{V}^* \]
with $\varepsilon : = c_0 \max_{x \in \mathbb{R}} |\xi'(x)|$.

We thus obtain, for $x \in [g(t + t_0), \bar{h}(t) - \varepsilon_0]$ and $0 < \varepsilon_0 \ll 1$,
\[ B(t, x) \geq -K_4 \varepsilon(t)D \circ \hat{V}^* + (1 + \epsilon)F(\phi(x - \bar{h})) - F(\bar{U}) + 2M_1(\delta^\prime + \lambda^\prime)1 + \epsilon u^* - \rho_t \]
\[ \geq C_l f_1 - K_4 \varepsilon(t)(D \circ \hat{V}^* + C_f \hat{V}^* + \varepsilon \hat{V}^*) + 2M_1(-K_3 \varepsilon(t) + K_5 \varepsilon(t))1 + \epsilon(t)u^* \]
\[ \geq \epsilon(t) \left[ C_l 1 - K_4(\varepsilon(t)D \circ \hat{V}^* + C_f \hat{V}^* + \varepsilon \hat{V}^*) - 2M_1(K_3 + K_5 \beta(t + \theta)^{-1})1 - \beta(t + \theta)^{-1}u^* \right] \]
\[ \geq 0 \]
if we choose $K_3$ and $K_5$ small so that $(5.24)$ and $(5.27)$ hold and at the same time, due to $(5.19)$
\[ C_l f_1 - K_4(\varepsilon(t)D \circ \hat{V}^* + C_f \hat{V}^* + \varepsilon \hat{V}^*) - 2M_1K_31 \geq 0, \]
and then choose $\theta$ sufficiently large. Hence, $(5.25)$ is satisfied if $K_3$ and $K_5$ are chosen small as above, and $\theta$ is sufficiently large.

From $(5.20)$, we have
\[ \bar{U}(t, g(t + t_0)) \succeq 0, \ \bar{U}(t, \bar{h}(t)) \succeq 0 \text{ for } t \geq 0. \]
Together with $(5.21)$, $(5.22)$ and $(5.25)$, this enables us to use the comparison principle to conclude that
\[ h(t + t_0) \leq \bar{h}(t), \ U(t + t_0, x) \preceq \bar{U}(t, x) \text{ for } t \geq 0, \ x \in [g(t + t_0), \bar{h}(t)], \]
which implies $(5.3)$. The proof of the lemma is now complete. \hfill \Box

5.3. **Proof of Theorem 5.1.** Since $(J^1)$ holds, by Lemma 4.8 and then by $(5.1)$, there exists $C_0 > 0$ such that
\[ h(t) - c_0 t \geq -C \left[ 1 + \int_0^t (1 + x)^{-1}dx + \int_0^{\frac{q_0}{t}} x^2 \hat{J}(x) dx + t \int_{\frac{q_0}{2t}}^{\infty} x \hat{J}(x) dx \right] \]
\[ \geq -C \left[ 1 + \int_0^t \hat{J}(x) dx + \ln(t + 1) + C_0 \int_1^{\frac{q_0}{2t}} x^{2-\gamma} dx + C_0 t \int_{\frac{q_0}{2t}}^{\infty} x^{1-\gamma} dx \right]. \]
Therefore when $\gamma \in (2, 3)$ we have, for $t \geq 1$,
\[
h(t) - c_0 t \geq - C \left[ \hat{C} + \ln(t + 1) + \hat{C}_1 t^{3-\gamma} \right] \geq -\hat{C}_1 t^{3-\gamma}
\]
for some $\hat{C}_1, \hat{C}, \hat{C}_1 > 0$, and when $\gamma = 3$, for $t \geq 1$,
\[
h(t) - c_0 t \geq - \hat{C}_2 \ln t
\]
for some $\hat{C}_2 > 0$. This combined with Lemmas 5.2 and 5.3 gives the desired conclusion of Theorem 5.1. The proof is completed. $\square$

6. The growth orders of infinite spreading speed

Let $(U, g, h)$ be the unique positive solution of (1.1), and assume that spreading happens. Under the assumptions of Theorem 1.3, we have
\[
- \lim_{t \to \infty} \frac{g(t)}{t} = \lim_{t \to \infty} \frac{h(t)}{t} = \infty.
\]

Suppose $(\hat{J})$ holds for some $\gamma \in (1, 2]$, namely, for $|x| \gg 1$ we have
\[
J_i(x) \approx |x|^{-\gamma} \quad \text{for } i \in \{1, ..., m_0\} \quad \text{and some } \gamma \in (1, 2].
\]

Then
\[
\int_R J_i(x)dx < \infty, \quad \int_R |x|J_i(x)dx = \infty \quad \text{for } i \in \{1, ..., m_0\}.
\]

So (J1) is not satisfied.

The purpose of this section is to prove Theorem 1.5, which we restate as

**Theorem 6.1.** Assume that (J) and (f1) – (f4) are satisfied. If spreading happens, and additionally (6.1) holds, then for large $t > 0$,

\[
\begin{cases}
-g(t), \quad h(t) \approx t^{1/(\gamma-1)} & \text{if } \gamma \in (1, 2), \\
-g(t), \quad h(t) \approx t \ln t & \text{if } \gamma = 2.
\end{cases}
\]

We will only prove the estimate for $h(t)$, since that for $g(t)$ follows by the change of variable $x \to -x$. Theorem 6.1 will follow directly from the lemmas in Subsections 6.1 and 6.2 below.

6.1. Upper bound. To prove the upper bound a slightly weaker condition than (6.1) is enough. We assume that there exist positive constants $C_1$ and $C_2$ such that
\[
\frac{C_1}{|x|^{\gamma} + 1} \leq \sum_{i=1}^{m_0} \mu_i J_i(x) \leq \frac{C_2}{|x|^{\gamma} + 1} \quad \text{for } x \in \mathbb{R} \quad \text{and some } \gamma \in (1, 2].
\]

Obviously, (6.2) has no restriction for the kernel function $J_{i_0}$ whenever $\mu_{i_0} = 0$, and (6.1) implies (6.2) for the same $\gamma$.

**Lemma 6.2.** Assume that (J) and (f1) – (f4) hold. If spreading happens, and (6.2) is satisfied, then there exists $C = C(\gamma) > 0$ such that
\[
\begin{cases}
h(t) \leq Ct^{1/(\gamma-1)} & \text{if } \gamma \in (1, 2), \\
h(t) \leq C t \ln t & \text{if } \gamma = 2.
\end{cases}
\]

**Proof.** Define, for $t \geq 0$,
\[
\bar{h}(t) := \begin{cases}
(Kt + \theta)^{1/(\gamma-1)} & \text{if } \gamma \in (1, 2), \\
(Kt + \theta) \ln(Kt + \theta) & \text{if } \gamma = 2,
\end{cases}
\]
and
\[
\bar{U}(t, x) := \bar{u} 1, \quad \bar{u} := \max_{1 \leq i \leq m} \{\|u_{i0}\|_{\infty}, u_i^*\}, \quad x \in [-\bar{h}(t), \bar{h}(t)],
\]

where $\hat{u}(t) := \max_{1 \leq i \leq m} \{\|u_{i0}\|_{\infty}, u_i^*\}, \quad x \in [-\bar{h}(t), \bar{h}(t)]$.
with positive constants $\theta$ and $K$ to be determined.

We start by showing

$$(6.4) \quad \tilde{h}'(t) \geq \sum_{i=1}^{m_0} \mu_i \int_{-\tilde{h}(t)}^{\tilde{h}(t)} \int_{-\tilde{h}(t)}^{\tilde{h}(t)} J_i(x - y)\tilde{u}_i(t, x)dydx \quad \text{for} \quad t > 0,$$

and

$$-\tilde{h}'(t) \leq -\sum_{i=1}^{m_0} \mu_i \int_{-\tilde{h}(t)}^{\tilde{h}(t)} \int_{-\infty}^{\tilde{h}(t)} J_i(x - y)\tilde{u}_i(t, x)dydx \quad \text{for} \quad t > 0.$$

Since $U(t, x) = U(t, -x)$ and $J_i(x) = J_i(-x)$, it suffices to prove $(6.4)$.

By simple calculations and $(6.2)$, for any $k > 0$,

$$\sum_{i=1}^{m_0} \mu_i \int_{-k}^{k} \int_{0}^{\infty} J_i(x - y)dydx = \sum_{i=1}^{m_0} \mu_i \int_{0}^{k} \int_{x}^{\infty} J_i(y)dydx$$

$$= \sum_{i=1}^{m_0} \mu_i \int_{0}^{k} J_i(y)dy + \sum_{i=1}^{m_0} \mu_i k \int_{k}^{\infty} J_i(y)dy$$

$$\leq \int_{0}^{k} \frac{C_2y}{y^\gamma + 1} dy + k \int_{k}^{\infty} \frac{C_2}{y^\gamma + 1} dy \leq \int_{0}^{1} C_2dy + \int_{1}^{k} \frac{C_2y}{y^\gamma} dy + k \int_{k}^{\infty} \frac{C_2}{y^\gamma} dy,$$

and so

$$(6.5) \begin{cases} \sum_{i=1}^{m_0} \mu_i \int_{-k}^{k} \int_{0}^{\infty} J_i(x - y)dydx \leq C_2 + \frac{C_2}{2 - \gamma} (k^{2-\gamma} - 1) + \frac{C_2k^2}{\gamma - 1} & \text{if} \quad \gamma \in (1, 2), \\ \sum_{i=1}^{m_0} \mu_i \int_{-k}^{k} \int_{0}^{\infty} J_i(x - y)dydx \leq 2C_2 + C_2 \ln k & \text{if} \quad \gamma = 2. \end{cases}$$

A direct calculation gives

$$\int_{-\tilde{h}(t)}^{\tilde{h}(t)} \int_{-\tilde{h}(t)}^{\tilde{h}(t)} J_i(x - y)\tilde{u}_i(t, x)dydx = \tilde{u} \int_{-2\tilde{h}(t)}^{0} \int_{0}^{+\infty} J_i(x - y)dydx.$$

Hence for $1 < \gamma < 2$, by $(6.5)$,

$$\sum_{i=1}^{m_0} \mu_i \int_{-\tilde{h}(t)}^{\tilde{h}(t)} \int_{-\tilde{h}(t)}^{\tilde{h}(t)} J_i(x - y)\tilde{u}_i(t, x)dydx$$

$$\leq \tilde{u} \left[ C_2 + 2^{2-\gamma} \left( \frac{C_2}{2 - \gamma} + \frac{C_2}{\gamma - 1} \right) (Kt + \theta)^{(2-\gamma)/(1-\gamma)} \right]$$

$$\leq \frac{K}{\gamma - 1} (Kt + \theta)^{(2-\gamma)/(1-\gamma)} = \tilde{h}'(t)$$

provided that $K > 0$ is large enough. And for $\gamma = 2$,

$$\sum_{i=1}^{m_0} \mu_i \int_{-\tilde{h}(t)}^{\tilde{h}(t)} \int_{-\tilde{h}(t)}^{\tilde{h}(t)} J_i(x - y)\tilde{u}_i(t, x)dydx$$

$$\leq \tilde{u} \left( 2C_2 + C_2 \ln[2(Kt + \theta)\ln(Kt + \theta)] \right)$$

$$\leq \frac{K}{\gamma} (Kt + \theta) + K = \tilde{h}'(t)$$

if $K \gg 1$. This finishes the proof of $(6.4)$. 


Since $\mathbf{U} \geq \mathbf{u}^*$ is a constant vector, we have, for $t > 0$, $x \in [-\tilde{h}(t), \tilde{h}(t)]$,
\begin{equation}
\nabla \mathbf{U}(t, x) \equiv 0 \geq D \circ \int_{-\tilde{h}(t)}^{\tilde{h}(t)} \mathbf{J}(x - y) \circ \mathbf{U}(t, y) dy - D \circ \mathbf{U}(t, x) + F(\mathbf{U}(t, x)).
\end{equation}

Moreover, $\tilde{h}(0) \geq h_0$ for large $\theta$, and obviously
\[ \mathbf{U}(t, \pm \tilde{h}(t)) \geq 0 \text{ for } t \geq 0, \]
\[ \mathbf{U}(0, x) \geq U(0, x) \text{ for } x \in [-h_0, h_0]. \]

Hence we can apply the comparison principle (Lemma 3.2) to conclude that
\[ [g(t + t_0), h(t + t_0)] \subseteq [-\tilde{h}(t), \tilde{h}(t)], \quad t \geq 0, \]
\[ U(t + t_0, x) \leq \mathbf{U}(t, x), \quad t \geq 0, \ x \in [g(t + t_0), h(t + t_0)]. \]

Thus (6.3) holds.

6.2. **Lower bound.** The lower bound is more difficult to obtain, and we will consider the cases $\gamma \in (1, 2)$ and $\gamma = 2$ separately.

6.2.1. **The case $\gamma \in (1, 2)$.** We start with a result from [16].

**Lemma 6.3.** [16] (2.11) If $\bar{J}$ satisfies (J), then for any $\epsilon > 0$, there is $L_\epsilon > 0$ such that for all $l > L_\epsilon$ and $\psi_1(x) := l - |x|$, 
\begin{equation}
\int_{-l}^{l} \bar{J}(x - y)\psi_1(y) dy \geq (1 - \epsilon)\psi_1(x) \text{ in } [-l, l].
\end{equation}

**Lemma 6.4.** Assume that the conditions in Theorem 6.1 are satisfied and $\gamma \in (1, 2)$. Then there exists $C = C(\gamma) > 0$ such that
\begin{equation}
h(t) \geq Ct^{1/(\gamma - 1)} \text{ for } t \gg 1.
\end{equation}

**Proof.** Define
\[ \tilde{h}(t) := (K_1 t + \theta)^{1/(\gamma - 1)}, \quad t \geq 0, \]
\[ \mathbf{U}(t, x) := K_2 \frac{\tilde{h}(t) - |x|}{\tilde{h}(t)} \Theta, \quad t \geq 0, \ x \in [-\tilde{h}(t), \tilde{h}(t)], \]
with positive constants $\theta$ and $K_1, K_2$ to be determined, where the vector $\Theta = (\theta_i)$ is given by Lemma 2.1.

**Step 1.** We show that, for large $K_1$, 
\begin{equation}
\tilde{h}'(t) \leq \mu_i \int_{-\tilde{h}(t)}^{\tilde{h}(t)} \int_{\tilde{h}(t)}^{+\infty} \mathbf{J}(x - y) \psi_1(t, x) dy dx \text{ for } t > 0.
\end{equation}

By simple calculations and (5.2), we obtain
\[ \sum_{i=1}^{m_0} \mu_i \int_{-\tilde{h}(t)}^{\tilde{h}(t)} \int_{\tilde{h}(t)}^{+\infty} \mathbf{J}_i(x - y) \psi_i(t, x) dy dx \]
\[ \geq \sum_{i=1}^{m_0} \mu_i K_2 \theta_i \int_{0}^{\tilde{h}(t)} \int_{\tilde{h}(t)}^{+\infty} \mathbf{J}_i(x - y) \frac{\tilde{h}(t) - x}{\tilde{h}(t)} dy dx \]
\[ = \sum_{i=1}^{m_0} \mu_i K_2 \theta_i \int_{0}^{\tilde{h}(t)} \int_{-\tilde{h}(t)}^{0} \mathbf{J}_i(x - y)(-x) dy dx \]
\[ = \sum_{i=1}^{m_0} \mu_i K_2 \theta_i \int_{0}^{\tilde{h}(t)} \int_{0}^{+\infty} \mathbf{J}_i(y) x dy dx \]
By (6.1), there exists \( \hat{\theta} \) such that

\[
\int_{h(t)}^{0} J_i(y)dy \geq \sum_{i=1}^{m_0} \mu_i \frac{K_2}{h(t)} \int_{0}^{h(t)} J_i(y)dy \geq \sum_{i=1}^{m_0} \mu_i \frac{K_2 C_1}{2h(t)} \int_{0}^{h(t)} y^2 dy \geq \sum_{i=1}^{m_0} \mu_i \frac{K_2 C_1 h(t)^{3-\gamma}}{4h(t)} \geq \sum_{i=1}^{m_0} \mu_i \frac{K_2 C_1 h(t)^{3-\gamma}}{3-\gamma} = \hat{C}_0(K_1 t + \theta)^{(2-\gamma)/(\gamma-1)} \geq \frac{K_1}{\gamma-1} (K_1 t + \theta)^{(2-\gamma)/(\gamma-1)} = h'(t)
\]

provided that \( K_1 \geq \hat{C}_0(\gamma - 1) \). This finishes the proof of Step 1.

**Step 2.** We show that, by choosing \( K_1, K_2 \) and \( \theta \) properly, for \( t > 0, x \in (-h(t), h(t)) \),

\[
U(t, x) \geq D \int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y)dy - D \circ U(t, x) + F(U(t, x)) \tag{6.10}
\]

From the definition of \( U \), for \( t > 0, x \in (-h(t), h(t)) \),

\[
U(t, x) = K_2 \Theta \frac{|x|h'(t)}{h(t)} \leq K_2 \Theta \frac{h'(t)}{h(t)} = \frac{K_1 K_2 \Theta}{\gamma - 1} h(t)^{1-\gamma}.
\]

**Claim 1.** For \( x \in [-h(t), h(t)] \), there exists a positive constant \( \hat{C}_1 \) depending only on \( \gamma \) such that

\[
\int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y)dy \geq \hat{C}_1 K_2 \Theta h(t)^{1-\gamma}.
\]

By (6.10), there exists \( \hat{C}_1 > 0 \) such that

\[
J_i(x) \geq \frac{\hat{C}_1}{|x|^\gamma + 1} \text{ for } x \in \mathbb{R}, \ i = 1, ..., m_0.
\]

Hence

\[
\int_{-h}^{h} J(x - y) \circ U(t, y)dy = \int_{-h}^{-h-x} J(y) \circ U(t, y + x)dy \geq K_2 \Theta \int_{-h}^{-h-x} \frac{\hat{C}_1}{|y|^\gamma + 1} \frac{h - |y + x|}{h} dy.
\]

Thus, for \( x \in [h/4, h] \),

\[
\int_{-h}^{h} J(x - y) \circ U(t, y)dy \geq K_2 \Theta \int_{-h/4}^{0} \frac{\hat{C}_1}{|y|^\gamma + 1} \frac{h - |y + x|}{h} dy \geq K_2 \Theta \int_{-h/4}^{0} \frac{\hat{C}_1}{|y|^\gamma + 1} \frac{h - (y + x)}{h} dy \geq K_2 \Theta \int_{-h/4}^{0} \frac{\hat{C}_1}{|y|^\gamma + 1} \frac{-y}{h} dy \geq \frac{K_2 \Theta}{h} \int_{0}^{h/4} \frac{\hat{C}_1 y}{y^\gamma + 1} dy \geq \frac{\hat{C}_1 K_2 \Theta}{2(h/4)^{2-\gamma}} \frac{h}{2h} \int_{1}^{h/4} y^{1-\gamma} dy \geq \frac{\hat{C}_1 K_2 \Theta}{2(2-\gamma)h^{2-\gamma}} = \hat{C}_1 K_2 \Theta h^{1-\gamma}.
\]

And for \( x \in [0, h/4] \),

\[
\int_{-h}^{h} J(x - y) \circ U(t, y)dy \geq K_2 \Theta \int_{0}^{h/4} \frac{\hat{C}_1}{|y|^\gamma + 1} \frac{h - |y + x|}{h} dy
\]
\[ K_2 \Theta \int_0^{\frac{h}{4}} \frac{\hat{C}_1}{y^{\gamma + 1}} \frac{y}{h} dy \geq \hat{C}_1 K_2 \Theta h^{1-\gamma} \]

by repeating the last a few steps in the previous calculations.

This proves (6.11) for \( x \in [0, \frac{h}{2}] \). (6.11) also holds for \( x \in [-\frac{h}{2}, 0] \) since both \( J(x) \) and \( U(t, x) \) are even in \( x \).

Claim 2. We can choose small \( K_2 \) and large \( \theta \) such that, for \( x \in [-\frac{h(t)}{2}, \frac{h(t)}{2}] \) and \( t \geq 0 \),

\[ D \circ \int_{-\frac{h(t)}{2}}^{\frac{h(t)}{2}} J(x-y) \circ U(t, y) dy - D \circ U(t, x) + F(U(t, x)) \geq F_\ast \int_{-\frac{h(t)}{2}}^{\frac{h(t)}{2}} J(x-y) \circ U(t, y) dy \]

for some positive constant \( F_\ast \). It is clear that \( U \leq K_2 \Theta \), and thus for small \( K_2 > 0 \) from the definition of \( \Theta \) in Lemma 2.1

\[ F(U(t, x)) = K_2 \frac{h(t) - |x|}{h(t)} \Theta \left( \nabla F(0) \right)^T + o(1)I_n \geq K_2 \frac{h(t) - |x|}{h(t)} \frac{3}{4} \lambda_1 \Theta = \frac{3}{4} \lambda_1 U(t, x), \]

where \( \lambda_1 > 0 \) is given in Lemma 2.1. Moreover, by (6.7), there is \( L_1 > 0 \) such that for \( \theta^{1/(\gamma-1)} \geq L_1 \),

\[ D \circ \int_{-\frac{h(t)}{2}}^{\frac{h(t)}{2}} J(x-y) \circ U(t, y) dy + \frac{\lambda_1}{4} U(t, x) \geq D \circ U(t, x) \text{ for } x \in [-\frac{h(t)}{2}, \frac{h(t)}{2}]. \]

Therefore Claim 2 is valid with \( F_\ast = \lambda_1/2 \).

Combining Claim 1 and Claim 2, we obtain

\[ D \circ \int_{-\frac{h(t)}{2}}^{\frac{h(t)}{2}} J(x-y) \circ U(t, y) dy - D \circ U(t, x) + F(U(t, x)) \geq F_\ast \hat{C}_1 K_2 \Theta h(t)^{1-\gamma} \geq \frac{K_1 K_2 \Theta}{\gamma - 1} h(t)^{1-\gamma} \geq U(t, x) \]

provided that \( K_1 \leq F_\ast \hat{C}_1 (\gamma - 1) \).

This proves (6.10).

Step 3. We prove (6.8) by the comparison principle.

It is clear that \( U(t, \pm h(0)) = 0 \text{ for } t \geq 0 \).

Since spreading happens for \((U, g, h)\), for fixed \( \theta \) and small \( K_1, K_2 \) as chosen above, there exists a large \( t_0 > 0 \) such that

\[ [-h(0), h(0)] \subset [g(t_0)/2, h(t_0)/2], \]

\[ U(t_0, x) \geq K_2 \Theta \geq U(0, x) \text{ for } x \in [-h(0), h(0)]. \]

Moreover, since \( J(x) \) and \( U(t, x) \) are both even in \( x \), (6.9) implies

\[ -h'(t) \geq \mu_i \int_{-\frac{h(t)}{2}}^{\frac{h(t)}{2}} \int_{-\frac{h(t)}{2}}^{\frac{h(t)}{2}} J(x-y) u_i(t, x) dy dx \text{ for } t > 0. \]

These combined with the estimates in Step 1 and Step 2 allow us to apply Lemma 3.2 to conclude that

\[ [-h(t), h(t)] \subset [g(t + t_0), h(t + t_0)], \]

\[ U(t, x) \leq U(t + t_0, x), \]

\[ t \geq 0, \quad x \in [-h(t), h(t)]. \]

Hence (6.8) holds. \( \square \)
6.2.2. The case $\gamma = 2$. The following simple result will play an important role in our analysis later.

**Lemma 6.5.** Let $l_1$ and $l_2$ with $0 < l_1 < l_2$ be two constants, and define

$$
\psi(x) = \psi(x; l_1, l_2) := \min \left\{ 1, \frac{l_2 - |x|}{l_1} \right\}, \quad x \in \mathbb{R}.
$$

If $J$ satisfies (J), then for any $\epsilon > 0$, there is $L_\epsilon > 0$ such that for all $l_1 > L_\epsilon$ and $l_2 - l_1 > L_\epsilon$,

$$
\int_{l_2}^{l_2 - l_1} J(x - y)\psi(y)dy \geq (1 - \epsilon)\psi(x) \text{ in } [-l_2, l_2].
$$

**Proof.** Since $\int_{\mathbb{R}} J(x)dx = 1$, there exists $B > 0$ such that

$$
\int_{-B}^{B} J(x)dx > 1 - \epsilon/2.
$$

In the following discussion we always assume that $l_1 \gg B$ and $l_2 - l_1 \gg B$. Clearly, for $x \in [-l_2, l_2]$, due to

$$
\psi(x) = 1 \text{ in } [-l_2, l_2],
$$

we have

$$
\int_{-l_2}^{l_2} J(x - y)\psi(y)dy \geq \int_{-l_2}^{l_2 - l_1} J(x - y)\psi(y)dy = \int_{-l_2}^{l_2 - l_1} J(x - y)dy
$$

$$
= \int_{-l_2}^{l_2 - l_1 - x} J(y)dy \geq \int_{-B}^{B} J(y)dy \geq 1 - \epsilon/2 > (1 - \epsilon)\psi(x).
$$

It remains to prove (6.13) for $x \in [-l_2, -(l_2 - l_1) + B] \cup [(l_2 - l_1) - B, l_2]$. By the symmetric property of $\psi(x)$ and $J(x)$ with respect to $x$, we just need to verify (6.13) for $x \in [(l_2 - l_1) - B, l_2]$, which will be carried out according to the following three cases:

(i) $x \in [l_2 - l_1 - B, l_2 - l_1 + B]$; (ii) $x \in [l_2 - l_1 + B, l_2 - B]$; (iii) $x \in [l_2 - B, l_2].$

(i) For $x \in [l_2 - l_1 - B, l_2 - l_1 + B]$, since $\psi(z)$ is nonincreasing for $z \geq 0$, we have

$$
\int_{-l_2}^{l_2} J(x - y)\psi(y)dy = \int_{-l_2}^{l_2 - x} J(y)\psi(y + x)dy
$$

$$
\geq \int_{-2l_2 + l_1 + B}^{B} J(y)\psi(y + x)dy \geq \int_{-B}^{B} J(y)\psi(y + x)dy
$$

$$
\geq \int_{-B}^{B} J(y)\psi(y + l_2 - l_1 + B)dy.
$$

By the definition of $\psi$,

$$
\psi(y + l_2 - l_1 + B) = \frac{l_2 - (y + l_2 - l_1 + B)}{l_1} = 1 - \frac{y + B}{l_1}, \quad y \in [-B, B].
$$

Hence,

$$
\int_{-B}^{B} J(y)\psi(y + l_2 - l_1 + B)dy = \int_{-B}^{B} J(y)dy - \int_{-B}^{B} J(y)\frac{y + B}{l_1}dy
$$

$$
\geq 1 - \epsilon/2 - \|J\|_{L^\infty(\mathbb{R})} \frac{2B^2}{l_1} \geq 1 - \epsilon \geq (1 - \epsilon)\psi(x)
$$

provided

$$
l_1 \geq \frac{4\|J\|_{L^\infty(\mathbb{R})}B^2}{\epsilon},
$$
which then gives
\[
\int_{-l_2}^{l_2} \tilde{J}(x-y)\psi(y)dy \geq (1-\epsilon)\psi(x) \quad \text{for} \quad x \in [l_2-l_1-B, l_2-l_1+B],
\]

(ii) For \(x \in [l_2-l_1+B, l_2-B]\),
\[
\int_{-l_2}^{l_2} \tilde{J}(x-y)\psi(y)dy = \int_{-l_2-x}^{l_2-x} \tilde{J}(y)\psi(y+x)dy \geq \int_{-2l_2-B+l_1}^{B} \tilde{J}(y)\psi(y+x)dy.
\]

From the definition of \(\psi\), for \(x \in [l_2-l_1+B, l_2-B]\) and \(y \in [-B, B]\),
\[
\psi(y+x) = \frac{l_2-(y+x)}{l_1} = \frac{l_2-x}{l_1} - \frac{y}{l_1} = \psi(x) - \frac{y}{l_1}.
\]
Thus, by (6.14),
\[
\int_{-l_2}^{l_2} \tilde{J}(x-y)\psi(y)dy \geq \int_{-B}^{B} \tilde{J}(y)\psi(y+x)dy = \psi(x) \int_{-B}^{B} \tilde{J}(y)dy - \int_{-B}^{B} \tilde{J}(y)\frac{y}{l_1}dy = \psi(x) \int_{-B}^{B} \tilde{J}(y)dy \geq (1-\epsilon)\psi(x).
\]

(iii) For \(x \in [l_2-B, l_2]\),
\[
\int_{-l_2}^{l_2} \tilde{J}(x-y)\psi(y)dy = \int_{-l_2-x}^{l_2-x} \tilde{J}(y)\psi(y+x)dy \geq \int_{-2l_2-B+l_1}^{B} \tilde{J}(y)\psi(y+x)dy = \int_{-B}^{B} \tilde{J}(y)\psi(y+x)dy - \int_{l_2-x}^{B} \tilde{J}(y)\psi(y+x)dy.
\]

As in (ii), we see that
\[
\int_{-B}^{B} \tilde{J}(y)\psi(y+x)dy = \psi(x) \int_{-B}^{B} \tilde{J}(y)dy \geq (1-\epsilon)\psi(x).
\]

By the definition of \(\psi\),
\[
\psi(y+x) \leq 0 \quad \text{for} \quad x \in [l_2-B, l_2], \quad y \in [l_2-x, B],
\]
which indicates
\[
\int_{-l_2}^{l_2} \tilde{J}(x-y)\psi(y)dy \geq \int_{-B}^{B} \tilde{J}(y)\psi(y+x)dy \geq (1-\epsilon)\psi(x).
\]

The proof is now complete. □

**Lemma 6.6.** If the conditions in Theorem 6.1 are satisfied and \(\gamma = 2\), then there exits \(C > 0\) such that
\[
(6.15) \quad h(t) \geq Ct \ln t \quad \text{for} \quad t \gg 1.
\]

**Proof.** For fixed \(\beta \in (0, 1)\), define
\[
\begin{aligned}
& h(t) := K_1(t+\theta) \ln(t+\theta), \quad t \geq 0, \\
& U(t, x) := K_2 \min \left\{ 1, \frac{h(t)-|x|}{(t+\theta)^\beta} \right\} \Theta, \quad t \geq 0, \quad x \in [-h(t), h(t)],
\end{aligned}
\]
for constants $\theta \gg 1$ and $1 \gg K_1 > 0, 1 > K_2 > 0$ to be determined, where $\Theta$ is given in Lemma 2.1. Obviously, for any $t > 0$, the function $\partial U(t, x)$ exists for $x \in [-h(t), h(t)]$ except when $|x| = h(t) - (t + \theta)$. However, the one-sided partial derivatives $\partial U(t \pm 0, x)$ always exist.

**Step 1.** We show that by choosing $\theta$ and $K_1, K_2$ suitably,

\begin{equation}
\hat{h}'(t) \leq \sum_{i=1}^{m_0} \mu_i \int_{-\hat{h}(t)}^{\hat{h}(t)} \int_0^{+\infty} J_i(x - y) u_i(t, x) dy dx \quad \text{for } t > 0,
\end{equation}

\begin{equation}
-\hat{h}'(t) \geq -\sum_{i=1}^{m_0} \mu_i \int_{-\hat{h}(t)}^{\hat{h}(t)} \int_{-\infty}^{\hat{h}(t)} J_i(x - y) u_i(t, x) dy dx \quad \text{for } t > 0.
\end{equation}

Since $U(t, x) = U(t, -x)$ and $J(x) = J(-x)$, we see that (6.17) follows from (6.16).

By elementary calculations and (6.2), we have

\begin{align*}
&\sum_{i=1}^{m_0} \mu_i \int_{-\hat{h}(t)}^{\hat{h}(t)} \int_{-\hat{h}(t)}^{\hat{h}(t)} J_i(x - y) u_i(t, x) dy dx \\
&\geq \sum_{i=1}^{m_0} \mu_i \int_{0}^{\hat{h}(t)} \int_{-\hat{h}(t)}^{\hat{h}(t)} J_i(x - y) u_i(t, x) dy dx \\
&= \sum_{i=1}^{m_0} \mu_i K_2 \int_{-\hat{h}(t)}^{\hat{h}(t)} \int_{0}^{+\infty} J_i(x - y) dy dx = \sum_{i=1}^{m_0} \mu_i K_2 \int_{0}^{\hat{h}(t)} \int_{x}^{+\infty} J_i(y) dy dx \\
&= \sum_{i=1}^{m_0} \mu_i K_2 \left( \int_{x}^{\hat{h}(t)} \frac{r^{\theta}}{(t + \theta)^\beta} + \int_{\hat{h}(t)}^{x} \frac{r^{\theta}}{(t + \theta)^\beta} \right) J_i(y) dy dx \\
&\geq \sum_{i=1}^{m_0} \mu_i K_2 \int_{x}^{\hat{h}(t)} \frac{r^{\theta}}{(t + \theta)^\beta} y dy \geq \sum_{i=1}^{m_0} \mu_i C_1 K_2 \int_{x}^{\hat{h}(t)} \frac{y - (t + \theta)^\beta}{y^2 + 1} dy \\
&\geq \sum_{i=1}^{m_0} \mu_i C_1 K_2 \int_{x}^{\hat{h}(t)} \frac{y - (t + \theta)^\beta}{2y^2} dy \\
&= \sum_{i=1}^{m_0} \mu_i C_1 K_2 \frac{1}{2} \left( \ln h(t) - \beta \ln(t + \theta) + \frac{(t + \theta)^\beta}{h(t)} - 1 \right) \\
&\geq \sum_{i=1}^{m_0} \mu_i C_1 K_2 \frac{1}{2} \left( \ln h(t) - \beta \ln(t + \theta) - 1 \right) \\
&= \sum_{i=1}^{m_0} \mu_i C_1 K_2 \frac{1}{2} \left( \ln K_1 + \ln(t + \theta) + \ln(\ln(t + \theta)) - \beta \ln(t + \theta) - 1 \right) \\
&\geq \sum_{i=1}^{m_0} \mu_i C_1 K_2 \frac{1}{2} \left( \ln(t + \theta) + 1 \right) \geq K_1 \ln(t + \theta) + C_1 = \hat{h}'(t)
\end{align*}

if

\begin{align*}
\begin{cases}
\ln(\ln \theta) \geq -\ln K_1 + 2 \\
K_1 \leq K_2 \sum_{i=1}^{m_0} \mu_i C_1 \theta_i (1 - \beta) / 2,
\end{cases}
\end{align*}

which then finishes the proof of Step 1.

**Step 2.** We show that by choosing $K_1, K_2$ and $\theta$ suitably, for $t > 0$ and $x \in (-h(t), h(t))$,

\begin{equation}
U(t, x) \leq D \int_{-h(t)}^{h(t)} J(x - y) U(t, y) dy - D \int_{-h(t)}^{h(t)} U(t, x) + F(U(t, x)).
\end{equation}
From the definition of $U$, for $t > 0$,

$$U_s(t, x) = K_1 K_2 \frac{(1 - \beta) \ln(t + \theta) + 1}{(t + \theta)^\beta} \Theta + \frac{K_2 \beta| x |}{(t + \theta)^{1+\beta}} \Theta, \quad h(t) - (t + \theta)^\beta < | x | \leq h(t),$$

$U_s(t, x) = 0, \quad | x | < h(t) - (t + \theta)^\beta.$

**Claim 1.** For $x \in [-h(t), -h(t) + (t + \theta)^\beta] \cup [h(t) - (t + \theta)^\beta, h(t)]$ and large $\theta$,

$$\int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y) dy \geq \frac{\tilde{C} K_2 \ln(t + \theta)}{4(t + \theta)^\beta} \Theta,$$

where $\tilde{C} > 0$ is given by (6.12).

A simple calculation yields, for $x \in [h(t) - (t + \theta)^\beta, h(t)]$,

$$\int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y) dy \geq K_2 \Theta \int_{h(t) - (t + \theta)^\beta}^{h(t)} J(x - y) \frac{h - y}{(t + \theta)^\beta} dy$$

$$= \frac{K_2 \Theta}{(t + \theta)^\beta} \int_{h(t) - (t + \theta)^\beta - x}^{h(t) - x} J(y)(h(t) - (y + x)) dy.$$ 

Hence, for $x \in [h(t) - \frac{3}{4}(t + \theta)^\beta, h(t)]$, by simple calculations and (6.12),

$$\int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y) dy \geq \frac{K_2 \Theta}{(t + \theta)^\beta} \int_{-(t + \theta)^\beta/4}^{0} J(y)(-y) dy$$

$$= \frac{\tilde{C} K_2 \Theta}{2(t + \theta)^\beta} \int_{1}^{(t + \theta)^\beta/4} y^{-1} dy = \frac{\tilde{C} K_2 \Theta}{2(t + \theta)^\beta} [\beta \ln(t + \theta) - \ln 4]$$

$$\geq \frac{\tilde{C} K_2 \ln(t + \theta)}{4(t + \theta)^\beta} \Theta$$

if

$$\frac{\beta}{2} \ln \theta \geq \ln 4.$$

And for $x \in [h(t) - (t + \theta)^\beta, h(t) - \frac{3}{4}(t + \theta)^\beta]$,

$$\int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y) dy \geq \frac{K_2 \Theta}{(t + \theta)^\beta} \int_{0}^{3(t + \theta)^\beta/4} J(y)(h(t) - (y + x)) dy$$

$$\geq \frac{K_2 \Theta}{(t + \theta)^\beta} \int_{0}^{(t + \theta)^\beta/4} J(y) dy \geq \frac{\tilde{C} K_2 \ln(t + \theta)}{4(t + \theta)^\beta} \Theta.$$ 

This proves (6.20) for $x \in [h(t) - (t + \theta)^\beta, h(t)]$. 

For $x \in [-h(t), -h(t) + (t + \theta)^\beta], \quad (6.11)$ also holds since both $J(x)$ and $U(t, x)$ are even in $x$.

Claim 1 is thus proved.

**Claim 2.** We can choose small $K_2$ and large $\theta$ such that, for $x \in [-h(t), h(t)]$,

$$D \circ \int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y) dy - D \circ U(t, x) + F(U(t, x)) \geq F_* \int_{-h(t)}^{h(t)} J(x - y) \circ U(t, y) dy$$

for some $F_* > 0$.

For small $K_2 > 0$, from $0 \leq U \leq K_2 \Theta$ and the definition of $\Theta$ in Lemma 2.1, we have

$$F(U(t, x)) = U(t, x) \left( [\nabla F(0)]^T + o(1) L_m \right)$$
Thus (6.15) holds. This completes the proof of the lemma. □

(6.24)

Hence, by (6.13), there is large

where

Therefore (6.22) holds with

Applying (6.20) and (6.22), we have, for

if apart from the earlier requirements, we further have

(6.24)

For

For

Thus (6.19) holds. (Let us stress that it is possible to find \(K_1, K_2\) and large \(\theta\) such that (6.18), (6.21), (6.23) and (6.24) hold simultaneously.)

Step 3. We finally prove (6.15).

Clearly, \(\bar{U}(t, \pm h(t)) = 0\) for \(t \geq 0\). Since spreading happens for \((U, g, h)\) and \(K_2 > 0\) is small, there is a large constant \(t_0 > 0\) such that

By Remark 2.4 and Lemma 3.2 we obtain

Thus (6.15) holds. This completes the proof of the lemma.

\(\square\)
REFERENCES

[1] M. Alfaro and J. Coville, Propagation phenomena in monostable integro-differential equations: Acceleration or not? *J. Differ. Equ.* 263 (2017), 5727-5758.

[2] F. Andreu-Vaillo, J.M. Mazón, J.D. Rossi, J. Toledo-Melero, *Nonlocal Diffusion Problems*, Mathematical Surveys and Monographs, AMS, Providence, Rhode Island, 2010.

[3] I. Ahn, S. Beak, Z. Lin, The spreading fronts of an infective environment in a man-environment-man epidemic model, *Appl. Math. Model.*, 40 (2016), 7082-7101.

[4] P. Bates, G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, *J. Math. Anal. Appl.*, 332 (2007), 428-440.

[5] H. Berestycki, J. Coville, H. Vo, Persistence criteria for populations with non-local dispersion, *J. Math. Biol.*, 72 (2016), 1693-1745.

[6] E. Bouin, J. Garnier, C. Henderson, F. Patout, Thin front limit of an integro-differential Fisher-KPP equation with fat-tailed kernels, *SIAM J. Math. Anal.* 50 (2018), 3365-3394.

[7] X. Cabré and J-M. Roquejoffre, The influence of fractional diffusion in Fisher-KPP equations, *Comm. Math. Phys.* 320 (2013), 679-722.

[8] J. Cao, Y. Du, F. Li, and W.-T. Li, The dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries, *J. Functional Anal.* 277 (2019), 2772-2814.

[9] J. Carr and A. Chmaj, Uniqueness of travelling waves for nonlocal monostable equations, *Proc. Amer. Math. Soc.* 132 (2004), 2433-2439.

[10] C. Cortázar, F. Quiros, N. Wolanski, A nonlocal diffusion problem with a sharp free boundary, *Interfaces Free Bound.*, 21 (2019), 441-462.

[11] J. Coville, J. Dávila, S. Martinez, Existence and uniqueness of solutions to a nonlocal equation with nonmonotone nonlinearity, *SIAM J. Math. Anal.*, 39(2008),1693-1709.

[12] Y. Du, F. Li and M. Zhou, Semi-wave and spreading speed of the nonlocal Fisher-KPP equation with free boundaries, preprint, 2019 (arXiv: 1909.03711).

[13] Y. Du, Z. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.* 42 (2010), 377-405.

[14] Y. Du, B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, *J. Eur. Math. Soc.* 17 (2015), 2673-2724.

[15] Y. Du, H. Matsuzawa, M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, *SIAM J. Math. Anal.* 46 (2014), 375-396.

[16] Y. Du, W. Ni, Analysis of a West Nile virus model with nonlocal diffusion and free boundaries, *Nonlinearity*, 33(2020), 4407-4448.

[17] Y. Du, M. Wang, M. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, *J. Math. Pure Appl.*, 107(2017), 253-287.

[18] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* 47 (1974), 324-353.

[19] J. Fang, G. Faye, Monotone traveling waves for delayed neural field equations, *Math. Models Methods Appl. Sci.* 26 (2016), 1919-1954.

[20] D. Finkelshtein, P. Tkachov, Accelerated nonlocal nonsymmetric dispersion for monostable equations on the real line, *Appl. Anal.*, 98 (2019), 756-780.

[21] J. Garnier, Accelerating solutions in integro-differential equations, *SIAM J. Math. Anal.*, 43 (2011), 1955-1974.

[22] F. Hamel and L. Roques, Fast propagation for KPP equations with slowly decaying initial conditions, *J. Differential Equations* 249 (2010), 1726-1745.

[23] C. Hu, Y. Kuang, B. Li, H. Liu, Spreading speeds and traveling wave solutions in cooperative integral-differential systems, *Discrete Contin. Dyn. Syst.* 20 (2015), 1663-1684.

[24] V. Hutson, S. Martinez, K.Mischakow, G. Vickers, The evolution of dispersal, *J. Math. Biol.* 47 (2003), 483-517.

[25] Y. Kaneko, H. Matsuzawa and Y. Yamada, Asymptotic profiles of solutions and propagating terrace for a free boundary problem of nonlinear diffusion equation with positive bistable nonlinearity, *SIAM J. Math. Anal.*, 52 (2020), 65-103.

[26] W.-T. Li, Y.-J. Sun, Z.-C. Wang, Entire solutions in the Fisher-KPP equation with nonlocal dispersal, *Nonlinear Anal. Real World Appl.*, 4 (2010), 2302-2313.

[27] X. Liang and X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, *J. Funct. Anal.*, 259 (2010), 857-903.

[28] Z. Lin, H. Zhu, Spatial spreading model and dynamics of West Nile virus in birds and mosquitoes with free boundary, *J. Math. Biol.*, 75 (2017), 1381-1409.

[29] W. Shen, A. Zhang, Stationary solutions and spreading speeds of nonlocal monostable equations in space periodic habitats, *Proc. Amer. Math. Soc.*, 140 (2012), no. 5, 1681-1696.
[30] P.E. Souganidis, A. Tarfulea, Front propagation for integro-differential KPP reaction-diffusion equations in periodic media, *NoDEA Nonlinear Differential Equations Appl.*, 26 (2019), no. 4, Paper No. 29, 41 pp.
[31] H.R. Thieme, Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread of populations, *J. Reine Angew. Math.*, 306 (1979), 94-121.
[32] H.R. Thieme, Density-dependent regulation of spatially distributed populations and their asymptotic speed of spread, *J. Math. Biol.* 8 (1979), 173-187.
[33] H.R. Thieme, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models, *J. Differential Equations* 195 (2003), no. 2, 430-470.
[34] J.B. Wang, W.-T. Li, J.W. Sun, Global dynamics and spreading speeds for a partially degenerate system with nonlocal dispersal in periodic habitats, *Proc. R. Soc. Edinb. A*, 148 (2018), 849-880.
[35] Z.G. Wang, H. Nie, Y. Du, Spreading speed for a West Nile virus model with free boundary, *J. Math. Biol.*, 79 (2019), no. 2, 433-466.
[36] H. Weinberger, Long-time behavior of a class of biological models, *SIAM J. Math. Anal.* 13 (1982), 353-396.
[37] S.L. Wu, Y.J. Sun, S.Y. Liu, Traveling fronts and entire solutions in partially degenerate reaction-diffusion systems with monostable nonlinearity, *Discrete Contin. Dyn. Syst.*, 33 (2013), 921-946.
[38] W.B. Xu, W.-T. Li, G. Lin, Nonlocal dispersal cooperative systems: acceleration propagation among species, *J. Differential Equations* 268 (2020), no. 3, 1081-1105.
[39] W.B. Xu, W.-T. Li, S. Ruan, Fast propagation for reaction-diffusion cooperative systems, *J. Differ. Equ.*, 265 (2018), 645-670.
[40] H. Yagisita, Existence and nonexistence of traveling waves for a nonlocal monostable equation, *Publ. Res. Inst. Math. Sci.* 45 (2009), 925-953.
[41] M. Zhao, W.-T. Li, W. Ni, Spreading speed of a degenerate and cooperative epidemic model with free boundaries, *Discrete Contin. Dyn. Syst., Ser. B*, 25 (2020), 981-999.
[42] M. Zhao, Y. Zhang, W.-T. Li and Y. Du, The dynamics of a degenerate epidemic model with nonlocal diffusion and free boundaries, *J. Diff. Eqns.* 269(2020), 3347-3386.
[43] X.-Q. Zhao, W. Wang, Fisher waves in an epidemic model, *Discrete Contin. Dyn. Syst., Ser. B*, 4 (2004), 1117-1128.