Special reduction of density matrixes and entanglement between two bunches of particles

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By using of a special reduction way of density matrices, in this Letter we find the entanglement between two bunches of particles, its measure can be represented by the entanglement of formation. PACC numbers: 03.65Ud, 03.67-a, 03.65Bz, 03.67Hk.

In this Letter, we discuss a special way of reduction of density matrices, and prove that in an arbitrary multipartite qubit state there is the new kind of entanglement, i.e. the entanglement between two bunches of particles, independent what happens to the remaining particles, and its measure can be represented by the entanglement of formation\cite{1} \(E_f\). Some examples are discussed.

It is known that the problems of the measure of the entanglement of bipartite qubit (pure- and mixed-)states are solved, e.g. see \cite{1,2}. However, the description of entanglement of multipartite qubit states is a formidable task as yet. Recently many new results have been obtained, one way in which is to describe the properties of entanglement of multipartite qubit states by using of the bipartite reduced density matrices and of the entanglement of formation \(E_f\). For instance, in \cite{3} the entanglement between two particles, independent what happens to the remaining particles, is described by a bipartite reduced operator, and its measure is represented by \(E_f\). Some more new results concerned, see \cite{4,5}.

For the spin particle \(M_k (k = 1, 2, \cdots, N, N \geq 3)\), we simply write \(\uparrow M_k \equiv | 0 >_k \) and \(\downarrow M_k \equiv | 1 >_k\), or in unior by \(| i >_k (i = 0 \text{ and } 1)\). \(| i >_k\) spans the Hilbert space \(H_k\). The main points of the above ways are as follows: If \(\rho\) is a density matrix acting upon \(H_1 \otimes H_2 \otimes \cdots \otimes H_N\), let the \(\rho_{kl}\) be the bipartite reduced density matrix defined by

\[
\rho_{kl} \equiv tr_{1\cdots k\cdots l\cdots N} (\rho), \quad (1 \leq k < l \leq N)
\]

where the \(tr\) denotes the trace of a matrix, and symbol \(\Box\) denotes the deletion operator. Since \(\rho_{kl}\) is a bipartite density matrix, we can use the entanglement of formation \(E_f\). Therefore once we use \(\rho_{kl}\) and \(E_f [\rho_{kl}]\) to describe the entanglement status between two particles \(| i >_k\) and \(| i >_l\), independent what happens to the remaining particles.

About the above kind of ways by using of reduced density matrices \(\rho_{kl}\) and \(E_f [\rho_{kl}]\), we need to consider at least the following two problems: First, why only consider we two particles, but not more particles? In fact, for the case of entanglement among more particles the problem backs again to the original status, i.e. we need to handle other multipartite qubit state, the above way will runs up against difficulties. For instance, when \(N \geq 5\), and we need to consider how to describe the entanglement among four particles \(| i >_1, | i >_2, | i >_3\) and \(| i >_4\), independent what happens to the remaining particles \(| i >_5, \cdots | i >_N\), then we use the reduced matrix \(\rho_{1234} \equiv tr_{5\cdots N} (\rho)\), however \(\rho_{1234}\) is not a bipartite qubit state, we cannot use \(E_f\). Although we still can write the set \(\{E_f [\rho_{12}], E_f [\rho_{13}], \cdots, E_f [\rho_{14}]\}\), it cannot show more contents of above entanglement among the particles \(| i >_1, | i >_2, | i >_3\) and \(| i >_4\), but the latter must contain other more contents. How are we to surmount this difficulty? Secondly, for some important multipartite qubit entangled states, say \(\text{GHZ}_N \equiv \frac{1}{\sqrt{2}} \left( | 000 \cdots 0 > + | 111 \cdots 1 > \right)\), \(\rho_{\text{GHZ}_N} \equiv \text{GHZ}_N \equiv <\text{GHZ}_N |\), all \(\rho_{kl}\) are disentangled, i.e. all \(E_f [\rho_{kl}] = 0\), this is somewhat making one puzzled: Is there some possible entanglement shared between two parts of system \(\text{GHZ}_N\) with non-zero \(E_f\)? In this Letter, we suggest a way that ones should consider some special ways of reduction of density matrices, then we prove that there is a new kind of entanglement in multipartite qubit systems, i.e. the entanglement between two bunches of particles, its measure can be represented by \(E_f\). These results bring to light the more properties of multipartite qubit entangled states.

In the first place, we discuss the simplest case, i.e. the tripartite qubit entangled states. The general form of a \(\Psi\) in \(H_a \otimes H_b \otimes H_c\) is as \(\Psi = \sum_{i,j,k=0,1} c_{ijk} \ | i >_a \otimes j >_b \otimes k >_c ( c_{ijk} \in \mathbb{C})\). We define \(H_{a/bc}, H_{a/b\bar{c}}, \) respectively by

\[
H_{a/bc} \equiv \left\{ \Psi : \text{The form of } \Psi \text{ is as } \Psi_{a/bc} = \sum_{i,j,k=0,1} c_{ikk} \ | i >_a \otimes j >_b \otimes k >_c \right\}
\]

\[
H_{a/b\bar{c}} \equiv \left\{ \Psi : \text{The form of } \Psi \text{ is as } \Psi_{a/b\bar{c}} = \sum_{i,j,k=0,1} c_{ik(1-k)} \ | i >_a \otimes j >_b \otimes 1-k >_c \right\}
\]

\(H_{a/bc}, H_{a/b\bar{c}}\) are two 4-dimensional subspaces orthogonal to each other, we have the direct sum decomposition

\[
H_a \otimes H_b \otimes H_c = H_{a/bc} \oplus H_{a/b\bar{c}}
\]
Now we make two formal bases $| i >_x$ and $| i >_y$, since $H_{a/bc}$ and $H_{a/bc}$ both are 4-dimensional spaces, we take the 1-1 correspondences as

$$| i >_a \otimes | k >_b \otimes | c > \equiv | i >_a \otimes | k >_x \quad \text{and} \quad | i >_a \otimes | k >_b \otimes | 1 - k >_c \equiv | i >_a \otimes | k >_y$$  \hspace{1cm} (4)

then we have the following isomorphisms

$$H_{a/bc} \cong H_{a} \otimes H_{x}, \quad H_{a/bc} \cong H_{a} \otimes H_{y}, \quad H_{a} \otimes H_{b} \otimes H_{c} \cong H_{a} \otimes H_{x} \otimes H_{a} \otimes H_{y}$$  \hspace{1cm} (5)

From Eq.(5), any quantum state $\Psi$ can be expressed in only one form as a sum of two orthogonal states $\Psi_{a/bc}$ and $\Psi_{a/bc}$, especially, we can explain $\Psi_{a/bc}$ and $\Psi_{a/bc}$ as follows: $\Psi_{a/bc}$ is the ‘wave function describing two bunches ($| i >_a$) and ($| k >_b \otimes | c >$), where in bunch ($| k >_b \otimes | c >$) the spin-directions of particles $b$ and $c$ always are the same, hence their behavior of spin can be regarded, as a whole, like to a single spin particle. $\Psi_{a/bc}$ is the ‘wave function describing two bunches ($| i >_a$) and ($| k >_b \otimes | 1 - k >_c$), where in bunch ($| k >_b \otimes | 1 - k >_c$) the spin directions of particles $b$ and $c$ always are contrary, hence their behavior also can be regarded, as a whole, like to other single spin particle.

The projection from $H_{a} \otimes H_{b} \otimes H_{c}$ to $H_{a} \otimes H_{x}$ and $H_{a} \otimes H_{y}$, respectively, are two left multiplication operators as

$$P_{a/bc} \equiv \sum_{i,k=0,1} | i >_a \otimes | k >_b \otimes | c > \equiv \sum_{i,k=0,1} | i >_a \otimes | k >_x$$  \hspace{1cm} (6)

$$P_{a/bc} \equiv \sum_{i,k=0,1} | i >_a \otimes | k >_b \otimes | 1 - k >_c \equiv \sum_{i,k=0,1} | i >_a \otimes | k >_y$$

And the interior mappings $I_{ax} : H_{a} \otimes H_{x} \rightarrow H_{a} \otimes H_{b} \otimes H_{c}$ and $I_{ay} : H_{a} \otimes H_{y} \rightarrow H_{a} \otimes H_{b} \otimes H_{c}$ are two left multiplication operators as

$$I_{ax} \equiv \sum_{i,k=0,1} | i >_a \otimes | k >_b \otimes | c > \equiv \sum_{i,k=0,1} | i >_a \otimes | k >_x$$  \hspace{1cm} (7)

$$I_{ay} \equiv \sum_{i,k=0,1} | i >_a \otimes | k >_b \otimes | 1 - k >_c \equiv \sum_{i,k=0,1} | i >_a \otimes | k >_y$$

Obviously, $P_{a/bc} \circ I_{ax}$ and $P_{a/bc} \circ I_{ay}$, respectively, are the identical mappings upon $H_{x}$ and $H_{y}$.

Suppose that $T$ is an linear operator,

$$T : H_{a} \otimes H_{b} \otimes H_{c} \rightarrow H_{a} \otimes H_{b} \otimes H_{c}, \ \ \ P = T (\Psi)$$  \hspace{1cm} (8)

then we can obtain the induced mappings $T_{ax}$ and $T_{ay}$ from $T$,

$$T_{ax} : H_{a} \otimes H_{x} \rightarrow H_{a} \otimes H_{b}, \quad T_{ax} : H_{a} \otimes H_{x}, \quad \Psi_{ax} \rightarrow \Psi_{ax} \equiv T_{ax} \Psi_{ax} = P_{a/bc} \circ T \circ I_{ax} \Psi_{ax}$$

$$T_{ay} : H_{a} \otimes H_{y} \rightarrow H_{a} \otimes H_{b}, \quad T_{ay} : H_{a} \otimes H_{y}, \quad \Psi_{ay} \rightarrow \Psi_{ay} \equiv T_{ay} \Psi_{ay} = P_{a/bc} \circ T \circ I_{ay} \Psi_{ay}$$  \hspace{1cm} (9)

We take especially an interest in the case of that $T$ is a ( pure or mixed) density operator $\rho$ on $H_{a} \otimes H_{b} \otimes H_{c}$, in this case the results obtained are two bipartite operators $\Upsilon_{axd}$ and $\Upsilon_{ayd}$ as

$$\Upsilon_{ax} : \Psi_{axd} \rightarrow \Psi_{axd} \equiv \Upsilon_{ax} \Psi_{axd}, \quad \Upsilon_{ax} \equiv P_{a/bc} \circ \rho \circ I_{ax}$$

$$\Upsilon_{ay} : \Psi_{ayd} \rightarrow \Psi_{ayd} \equiv \Upsilon_{ay} \Psi_{ayd}, \quad \Upsilon_{ay} \equiv P_{a/bc} \circ \rho \circ I_{ay}$$  \hspace{1cm} (10)

By using of Eqs.(6), (7) and (9), we find the entries of $\Upsilon_{ax}$ and $\Upsilon_{ay}$, respectively, are

$$[\Upsilon_{ax}]_{ij,k} = [\rho]_{ij,k}, \quad [\Upsilon_{ay}]_{ij,k} = [\rho]_{ij(1-j),kl(1-l)}, \quad (i, j, k, l = 0, 1)$$  \hspace{1cm} (11)

where $[\rho]_{ijm,kn}$ are the entries of density matrix $\rho$. By normalization, we can write

$$\Upsilon_{ax} = \eta_{ax} \rho_{ax}, \quad \eta_{ax} = \sum_{r,s=0,1} | \rho_{rss},rss |, \quad \rho_{ax} = \frac{1}{\eta_{ax}} \Upsilon_{ax}, \quad \Upsilon_{ay} = \eta_{ay} \rho_{ay}, \quad \eta_{ay} = \sum_{r,s=0,1} | \rho_{r(1-s),s(1-s)} |, \quad \rho_{ay} = \frac{1}{\eta_{ay}} \Upsilon_{ay}$$  \hspace{1cm} (12)
Since $\rho$ is a density matrix, from Eqs.(10),(11) and (12) we can directly verify that $\rho_{ax}$ and $\rho_{ay}$ both are bipartite density operators. Obviously, $\rho_{ax}$ describes the status of entanglement between bunches ($|i >_a$) and ($|k >_b$), and $\rho_{ay}$ describes the status of entanglement between bunches ($|i >_a$) and ($|k >_b \otimes 1 - k >_c$). In addition, there is the relation $\eta_{ax} + \eta_{ay} = 1$. This means that we can consider the operator $\rho_{(a,b,c)}$ defined by

$$
\rho_{(a,b,c)} = \rho_{ax} + \rho_{ay} = \eta_{ax}\rho_{ax} + \eta_{ay}\rho_{ay}
$$

$$
= \begin{bmatrix}
|\rho|_{000,000} + |\rho|_{000,001} |\rho|_{000,010} + |\rho|_{001,010} |\rho|_{000,100} + |\rho|_{001,100} |\rho|_{000,111} + |\rho|_{001,110} \\
|\rho|_{011,000} + |\rho|_{010,100} |\rho|_{011,010} + |\rho|_{010,110} |\rho|_{011,100} + |\rho|_{010,110} |\rho|_{011,111} + |\rho|_{010,110}
\end{bmatrix}
$$

(13)

then $\rho_{(a,b,c)}$ can be taken as a bipartite qubit mixed-state, which describes the status of entanglement between two bunches of particles $(a)$ and $(b,c)$. In addition, the procedure in accordance with the rules in Eqs.(11), (12) and (13), in fact, is a special reduction of density matrices.

Similarly, we take

$$
H_{b/c} = \sum_{i,k=0,1} c_{ik} |k >_a \otimes |i >_b \otimes |k >_c
$$

(14)

$$
H_{b/c} \approx H_b \otimes H_x \Rightarrow H_{b/c} \approx H_b \otimes H_y
$$

and similarly construct the projections $P_{b/c}$, $P_{b/c}$, the interior mappings $I_{b/c}$, $I_{b/c}$ and the induced mappings $\mathcal{Y}_{bx}, \mathcal{Y}_{by}, \cdots$ etc. They lead to

$$
\rho_{(b,ca)} = \mathcal{Y}_{bx} + \mathcal{Y}_{by} \cdot \rho_{(b,ca)} \cdot \mathcal{Y}_{bx} = \rho_{ij,kl} = |\rho|_{ij,kl} + |\rho|_{(1-j),i,k,l}
$$

(15)

$$
\rho_{(b,ca)} = \begin{bmatrix}
\rho_{000,000} + \rho_{000,100} \rho_{000,010} + \rho_{001,010} \rho_{000,100} + \rho_{001,010} \rho_{000,111} + \rho_{001,110} \\
\rho_{011,000} + \rho_{010,100} \rho_{011,010} + \rho_{010,110} \rho_{011,100} + \rho_{010,110} \rho_{011,111} + \rho_{010,110}
\end{bmatrix}
$$

(16)

Notice that although we can yet write $\rho_{(ab,c)}, \cdots$, there are repeats, e.g. $\rho_{(ab,c)} = \rho_{(c,ab)}, \cdots$, etc., there only are three independent $\rho_{(a,\bullet,\bullet)}$, i.e. $\rho_{(a,bc)}, \rho_{(b,ca)}$ and $\rho_{(c,ab)}$.

Since $\rho_{(a,bc)}, \rho_{(b,ca)}$ and $\rho_{(c,ab)}$ all are bipartite density matrix, we naturally use $E_f$ to represent their entanglement measure. For a given $\rho$ this measure $E_f$ can be concretely calculated by using of the so-called ‘concurrence’ [6,7]. For instance, for $\rho_{(a,bc)}$ defined as in Eq.(13)

$$
E_f [\rho_{(a,bc)}] = h \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - C^2} \right)
$$

(17)

where $h$ is the binary entropy function $h(x) \equiv -x \log_2 x - (1 - x) \log_2 (1 - x)$, the concurrence $C$ is determined by

$$
C = \max \{0, -\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4\}
$$

(18)

where $\lambda_i$ are the eigenvalues, in decreasing order, of the Hermitian matrix $R \equiv \sqrt{\rho_{(a,bc)}} \overline{\rho_{(a,bc)}} \sqrt{\rho_{(a,bc)}} \overline{\rho_{(a,bc)}} \overline{\rho_{(a,bc)}} \cdot \sigma_2 \otimes \sigma_2 \overline{\rho_{(a,bc)}} \overline{\sigma_2 \otimes \sigma_2}, \sigma_2$ is the Pauli matrix $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, the star is the complex conjugation. Similarly, for $\rho_{(b,ca)}$ and $\rho_{(c,ab)}$. Therefore, we can obtain the complete set of measures of entanglement shared between every pair
of two bunches, i.e. \( \{ E_f [\rho_{(a,b,c)}], E_f [\rho_{(b,c,a)}], E_f [\rho_{(c,a,b)}] \} \), it describes some character of the entanglement status among three particles \( a, b, \) and \( c \).

Now, we return to the problems mentioned in the start of this Letter. In the first place, if \( \rho \) is a density operator upon \( H_1 \otimes H_2 \otimes \cdots \otimes H_N \) \( (N \geq 3) \), then \( \text{tr} \left( \otimes_{j=k}^{l=n} \rho \right) \) \( (1 \leq j < k < l \leq N) \) is a tripartite density operator acting upon \( H_i \otimes H_k \otimes H_l \), therefore the status of the entanglement between two bunches of particles \( (M_j) \) and \( (M_l, M_k) \), independent what happens to the remaining particles, can be described by state
\[
\left[ \text{tr} \left( \otimes_{j=k}^{l=n} \rho \right) \right]_{(j,kl)}.
\]
In the following we simply write
\[
\rho_{(j,kl)} \equiv \left[ \text{tr} \left( \otimes_{j=k}^{l=n} \rho \right) \right]_{(j,kl)}.
\]
(19)

Here we must stress that \( \rho_{(j,kl)} \) is a special reduced matrix of \( \rho \) by two reduction procedures in succession: The first is the ordinary reduction, the second is in accordance with the special rules as in Eqs.(11), (12) and (13). The measure \( E_f [\rho_{(j,kl)}] \) can be calculated as in Eqs.(17) and (18). Similarly, for \( \rho_{(j,l,k)} \) and \( \rho_{(l,j,k)} \). For three particles \( | i > j, | i > k, | i > l \), the set \( \{ E_f [\rho_{(j,kl)}], E_f [\rho_{(k,j,l)}], E_f [\rho_{(l,j,k)}] \} \) completely describes all entanglement between every pair consisting of a bunch containing single particle and a bunch containing two particles. Obviously, this shows a character of the entanglement among three particles \( | i > j, | i > k, | i > l \) in the multipartite qubit state \( \rho \), independent what happens to the remaining particles. This cannot be obtained only by using of the ordinary reduction as in Eq.(1) and \( E_f \).

The generalization of more high dimensional bunches is straightforward, e.g. we can obtain \( \rho_{1234} \rho_{1234} \rho_{1234} \rho_{1234} \) and obtain \( \rho_{1234} \rho_{1234} \rho_{1234} \rho_{1234} \) (notice that, in fact, \( \rho_{1234} \rho_{1234} \rho_{1234} \rho_{1234} = \rho_{1234} \rho_{1234} \rho_{1234} \rho_{1234} \)). Similarly, \( \rho_{1234} \rho_{1234} \rho_{1234} \rho_{1234} \). At last, when \( N \) particles are divided into two bunches \( \{ r_1, \ldots, r_m \} \) and \( \{ s_1, \ldots, s_n \} \), where \( 1 \leq r_1 < r_2 < \cdots < r_m \leq N, 1 \leq s_1 < s_2 < \cdots < s_n \leq N \), \( \{ r_1, \ldots, r_m \} \cap \{ s_1, \ldots, s_n \} = \emptyset \) and \( \{ r_1, \ldots, r_m \} \cup \{ s_1, \ldots, s_n \} = \{ 1, 2, \ldots, N \} \), then we obtain \( \rho \) by two reduction procedures in succession: \( \{ \{ i_1 > k_1, \ldots, i_m > k_m \}, \{ j_1 > l_1, \ldots, j_n > l_n \} \} \). The set of all possible \( E_f [\rho_{\{ (i_1), (j_1) \}}] \) (notice that there are repeats in \( \{ \{ i_1 > k_1, \ldots, i_m > k_m \}, \{ j_1 > l_1, \ldots, j_n > l_n \} \} \)), is yet a description of the character of the entanglement among the \( m + n \) particles \( | i_1 > k_1, \ldots, i_m > k_m, | j_1 > l_1, \ldots, j_n > l_n \) in the \( N \)-partite qubit state \( \rho \), independent what happens to the remaining \( N - m - n \) particles.

Secondly, as a special example we consider the GHZ state \( \rho_{\text{GHZ}_N} \equiv \text{GHZ}_N \). By using of the above \( (k_i)_m \) and \( (l_i)_n \), we have the following results
\[
\rho_{\{ (k_i), (l_i) \}} \text{ is disentangled}, \quad E_f \left[ \rho_{\{ (k_i), (l_i) \}} \right] = 0, \quad \text{for } 2 \leq m + n < N
\]
\[
\rho_{\{ (k_i), (l_i) \}} \text{ is maximally entangled}, \quad E_f \left[ \rho_{\{ (k_i), (l_i) \}} \right] = 1, \quad \text{for } m + n = N
\]
(20)
The proof only is a straightforward calculation by Eqs.(17) and (18). This result shows fully the character of \( \rho_{\text{GHZ}_N} \), i.e. only when all \( N \) particles are divided into two parts (every particle must be in one and only one of them), the entanglement between this two parts does not vanish, and it is maximal. Therefore in view of this, for \( N \geq 3 \) the result that all \( E_f [\rho_{\text{GHZ}_N}] = 0 \) is not at all surprising.

Other interesting example is \( w \) is a given integer, \( 1 \leq w < N \)
\[
\phi^+ (N,w) = \frac{1}{\sqrt{2}} (| 0 > \otimes \cdots \otimes | 0 > w \otimes | 1 > w+1 \otimes \cdots \otimes | 1 > N + | 1 > 1 \otimes \cdots \otimes | 1 > w \otimes | 0 > w+1 \otimes \cdots \otimes | 0 > N )
\]
(21)
it like to the Bell state \( \phi^+ = 1 / \sqrt{2} (| 0 > a \otimes | 1 > b + | 1 > a \otimes | 0 > b ) \). For \( B^+_{(N,w)} \equiv | \phi^+ (N,w) \rangle > \phi^+ (N,w) \rangle \), it is easily verified that
\[
\left( B^+_{(N,w)} \right)_{\{ (k_1, \ldots, k_m), \{ l_1, \ldots, l_n \} \}} \text{ is disentangled}, \quad E_f \left[ \left( B^+_{(N,w)} \right)_{\{ (k_1, \ldots, k_m), \{ l_1, \ldots, l_n \} } \right] = 0, \quad \text{for } 2 \leq m + n < N
\]
\[
\left( B^+_{(N,w)} \right)_{\{ (k_1, \ldots, k_m), \{ l_1, \ldots, l_n \} \}} \text{ is entangled}, \quad E_f \left[ \left( B^+_{(N,w)} \right)_{\{ (k_1, \ldots, k_m), \{ l_1, \ldots, l_n \} } \right] > 0, \quad \text{for } m + n = N
\]
(22)
More generally, if \( N \leq M \) and \( (r_k)_N \equiv \{r_1, \cdots, r_N\} \) is a subset containing \( N \) elements in the set \( \{1, 2, \cdots, M\} \), we can construct \( \phi^+_{(r_k)_N, w} \) and \( B^+_{(r_k)_N, w} \), according to Eq.(21) for the set \( \{|i_k > r_k\} (k=1, \cdots, N) \) of \( N \) particles. We define the state

\[
|\Psi_{(M,N,w)}\rangle \equiv |\phi^+_{(r_k)_N, w}\rangle \otimes |0\cdots0\rangle_{\text{rest}}
\]

and the mixed state

\[
B^+_{(M,N,w)} \equiv \sum_{\text{all possible } (r_k)_N \subset \{1, 2, \cdots, M\}} x_{(r_k)_N} |\Psi_{(M,N,w)}\rangle <\Psi_{(M,N,w)}| 
\]

where the real numbers \( x_{(r_k)_N} \) obey \( 0 < x_{(r_k)_N} \leq 1 \) and \( \sum_{\text{all possible } (r_k)_N \subset \{1, 2, \cdots, M\}} x_{(r_k)_N} = 1 \), then Eq.(22) still holds for \( B^+_{(M,N,w)} \). The action of \( B^+_{(M,N,w)} \) is somewhat like to an ‘entanglement molecule’[3].

Sum up, in a multipartite qubit state \( \rho \) there is a new kind of entanglement, i.e. the entanglement between two bunches of particles, independent what happens to the remaining particles, which can be described by the special bipartite reduced density operators \( \rho_{(k_i, l_j)_n} \), and the measure of entanglement can be represented by

\[
E_f \left[ \rho_{(k_i, l_j)_n} \right].
\]

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