A note on the eigenvalues of double band matrices

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Abstract

We consider matrices containing two diagonal bands of positive entries. We show that all eigenvalues of such matrices are of the form \( r \zeta \), where \( r \) is a nonnegative real number and \( \zeta \) is a \( p \)th root of unity, where \( p \) is the period of the matrix, which is computed from the distance between the bands.

KEYWORDS: Eigenvalues, Nonnegative matrices, Irreducible matrices

1 Introduction

Let \( b \) and \( k \) be positive integers and consider the double band matrix

\[
A = \begin{pmatrix}
0 & 0 & \cdots & a_{1,k+1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & a_{2,k+2} & \cdots & 0 \\
\vdots & & & & & \ddots & \\
a_{b+1,1} & 0 & \cdots & & & & \\
0 & a_{b+2,2} & \cdots & & & & \\
\vdots & & & & & & \\
\end{pmatrix},
\]

where the diagonals \( a_{b+j,j}, a_{i,k+i} \) are positive real numbers and the remaining entries are zero. Such matrices arise, for example, in second order differential equations. See [2] for an application to the Lamé equation. The matrices as in (1) have a period \( p \), which is the greatest common divisor of the lengths of cycles in the directed graph \( \Gamma(A) \). Thus it is given by

\[
p = \frac{b + k}{\gcd(b, k)}.
\]

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†The period is also sometimes referred to as the index, or index of imprimitivity. A primitive matrix is one in which \( p = 1 \).
If \( b \) and \( k \) are relatively prime then \( A \) is irreducible. The Perron-Frobenius theorems \([1]\) tells us that the spectrum of a nonnegative and irreducible matrix with period \( p \) is invariant under rotations by \( 2\pi/p \) and that there are exactly \( p \) eigenvalues with maximal modulus \( \rho(A) \) given by 
\[
\rho(A) \exp(i2\pi j/p), \, j = 0, \ldots, p - 1.
\]

The main result of this paper is to show that for matrices with the special form \([1]\), in addition to this, the eigenvalues all lie on the lines through the \( p \)th roots of unity in the complex plane.

**Theorem 1.1.** Let \( A \) be an \( n \times n \) matrix as in \([1]\). Let \( n = mp + q \) for \( 0 \leq q < p \), and let \( g = \gcd(b, k) \). If \( g = 1 \) then \( A \) has a zero eigenvalue of multiplicity \( q \), and \( mp \) eigenvalues
\[
r_s e^{i2\pi j/p}, \, j = 0, \ldots, p - 1, \, s = 1, \ldots, m,
\]
where \( r_s, s = 1, \ldots, m, \) are distinct, real and positive.

More generally, the nonzero eigenvalues of \( A \) are given by
\[
r_{s,t} e^{i2\pi j/p}, \, j = 0, \ldots, p - 1, \, t = 1, \ldots, g, \, s = 1, \ldots,
\]
where for each fixed \( t \), the \( r_{s,t} \)'s are distinct, real and positive. The zero eigenvalue of \( A \) has multiplicity
\[
g \left( \left\lfloor \frac{n}{g} \right\rfloor \mod p \right) + (n \mod g) \left( \left\lfloor \frac{n}{g} \right\rfloor + 1 \right) \mod p - \left\lfloor \frac{n}{g} \right\rfloor \mod p \]

Before proceeding to the proof we make a few remarks. First, if \( b = k \), i.e. the bands are an equal distance from the diagonal, then \( p = 2 \), so all the eigenvalues are real, and come in positive and negative pairs. The case \( b = 1, k = 2 \) was considered in \([2]\), where it was shown that all eigenvalues are of the form \( r \exp(i2\pi j/3) \). Theorem 1.1 is a generalization of the result in that paper. The following corollary is an immediate consequence of the construction of the proof of the above theorem, and generalizes to the case when the two bands contain nonnegative and complex entries.

**Corollary 1.1.** Let \( A \) be an \( n \times n \) complex matrix with \( a_{ij} = 0 \) for all \( i - j \neq b \) or \( -k \). Then the spectrum of \( A \) is invariant under rotations by \( 2\pi/p \). If, in addition, \( a_{ij} \) is real and nonnegative, then the eigenvalues of \( A \) are all of the form \( r e^{i2\pi j/p} \), where \( r \) is a nonnegative real number.

## 2 Proof of Theorem 1.1

We first prove the following lemma that allows us to reduce the problem to the case when \( b \) and \( k \) are relatively prime. If \( g > 1 \) we can move the diagonals “inward” at the expense of rearranging
them and inserting zeros. In the proofs that follow we will employ many of the results from the theory of nonnegative matrices, which can be found in [3] and [1].

**Lemma 2.1.** A is cogredient to a direct sum of $g$ matrices of the form (1) where the $b$ and $k$ are relatively prime. That is, there is a permutation matrix $P$ such that

$$B = PAP^T = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & B_g \end{pmatrix}$$

where $B_i$ is an $\left(\left\lfloor \frac{n - i}{g} \right\rfloor + 1\right) \times \left(\left\lfloor \frac{n - i}{g} \right\rfloor + 1\right)$ square matrix with non-zero entries at the positions $(n_b + j, j)$ and $(i, n_k + i)$ and zeros everywhere else, where $n_b = b/g$ and $n_k = k/g$.

**Proof.** We construct $P$ explicitly. Let $\sigma$ be the permutation of the integers $1, \ldots, n$ defined as follows. For $i = 1, \ldots, g$, define

$$n_i = \left\lfloor \frac{n - i}{g} \right\rfloor + 1$$

and the partial sums $N_1 = 0, N_i = n_1 + \cdots + n_{i-1}$ for $i = 2, \ldots g$. A simple calculation shows that $n_1 + n_2 + \cdots + n_g = n$. Now, for each $i = 1, \ldots, g$ define

$$\sigma_{N_i + j} = i + (j - 1)g, \quad j = 1, 2, \ldots, n_i$$

Then $\sigma$ is a permutation of $(1, \ldots, n)$, and we have

$$\sigma_{n_b + j} - \sigma_j = b,$$

$$\sigma_{n_k + i} - \sigma_i = k,$$

for all pairs $N_i < n_b + j, j \leq N_i + n_i$ for some $i$, and likewise for $n_k + i, i$. Now we define the permutation matrix $P$ by setting $p_{j, \sigma_j} = 1$ for $j = 1, \ldots, n$. Then, define $B = PAP^T$, so that the $(i, j)$th entry of $B$ is $b_{ij} = a_{\sigma_i, \sigma_j}$. Since $a_{ij} \neq 0$ if and only if $i - j = b$ or $i - j = -k$, as long as $N_i < n_b + j, j \leq N_i + n_i$ for some $i$ and $N_l < n_k + i, i \leq N_l + n_l$ for some $l$, we have

$$b_{n_b + j, j} = a_{\sigma_{n_b + j}, \sigma_j} = a_{s + b, s},$$

$$b_{n_k + i, i} = a_{\sigma_i, \sigma_{n_k + i}} = a_{t, t + k},$$

for some $s$ and $t$.  

\footnote{A matrix $A$ is cogredient to $B$ if there is a permutation matrix $P$ such that $A = PBP^T$.}
Notice that each square block \( N_l < i, j \leq N_l + n_l \) on the diagonal of \( B \) contains \( n_l - n_b \) nonzero elements at the positions \((n_b + j, j)\) and \( n_l - n_k \) nonzero elements in the positions \((i, n_k + i)\), for a total of \( n - gn_b = n - b \) elements in the positions \((n_b + j, j)\) and \( n - gn_k = n - k \) elements in the positions \((i, n_k + i)\). Thus, since \( P A P^T \) moves each element of \( A \) to a unique position, we see that \( B \) has the form (6), as required.

We can now proceed to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Suppose, first of all, that \( b \) and \( k \) are relatively prime. Then \( p = b + k \). Also, we may assume that \( b \leq k \) without loss of generality. Since \( A \) is irreducible with period \( p \) it is cogredient to a matrix in superdiagonal block form. That is, there is a permutation matrix \( P \) such that

\[
C = P A P^T = \begin{pmatrix}
0 & C_1 & 0 & \cdots & 0 \\
0 & 0 & C_2 & \cdots & 0 \\
\vdots & & \ddots & & \\
0 & 0 & 0 & \cdots & C_{p-1} \\
C_p & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

where the diagonal zero blocks are square, but the blocks \( C_j \) may be rectangular. The key to the proof is that we can find a \( P \) such that the matrices \( C_j \) are bidiagonal. We construct \( P \) explicitly. We first construct a permutation \( \sigma \) of the integers from 1 to \( n \). For each \( i = 0, \ldots, p - 1 \) define \( \gamma_i, z_i, \) and \( n_i \) by

\[
\gamma_i = (ik + 1) \mod p = (ik + 1) + z_i, \quad \quad (14)
\]

\[
n_i = \left\lfloor \frac{n - \gamma_i}{p} \right\rfloor + 1, \quad \quad (15)
\]

and \( n_{-1} = 0 \). An elementary calculation shows that \( n_0 + \cdots + n_{p-1} = n \). To simplify notation we also define the partial sums \( N_i = n_0 + \cdots + n_i \). Now, for each \( i = 0, \ldots, p - 1 \), define

\[
\sigma_{N_{i-1}+j} = \gamma_i + (j - 1)p, \quad j = 1, \ldots, n_i.
\]

Thus \( \sigma = (\sigma_1, \ldots, \sigma_n) \) is a permutation of the numbers \( (1, \ldots, n) \).

Now we define the permutation matrix \( p_{j, \sigma(j)} = 1 \). Then the \((i, j)\) entry of \( C = P A P^T \) is \( c_{ij} = a_{\sigma_i, \sigma_j} \). For ordered subsets \( \alpha, \beta \) of \( \{1, \ldots, n\} \), let \( A(\alpha, \beta) \) be the submatrix with rows in \( \alpha \) and columns in \( \beta \). We define the submatrices \( C_i \) as follows:

\[
C_i = C(\{N_{i-2} + 1, \ldots, N_{i-2} + n_{i-1}\}, \{N_{i-1} + 1, \ldots, N_{i-1} + n_i\}), \quad i = 1, \ldots, p - 1, \quad (17)
\]

\[
C_p = C(\{N_{p-2} + 1, \ldots, n\}, \{1, \ldots, n_0\})
\]

(18)
Note that since \( b \) and \( k \) are relatively prime, and \( b \leq k \), \( z_i - z_{i+1} \) is either 0 or 1. We now use this fact to show that each \( C_i \) is bidiagonal.

Consider the main, lower and upper diagonals of \( C_i \) for \( i = 1, \ldots, p - 1 \). Note that

\[
\sigma_{N_{i-1}+j} - \sigma_{N_i+j} = \gamma_i + (j - 1)p - (\gamma_{i+1} + (j - 1)p)
\]
\[
= -k + (z_i - z_{i+1}) p
\]
\[
= \begin{cases} 
-k & \text{if } z_i - z_{i+1} = 0 \\
 b & \text{if } z_i - z_{i+1} = 1
\end{cases} \quad (19)
\]

\[
\sigma_{N_{i-1}+j+1} - \sigma_{N_i+j} = \gamma_i + jp - (\gamma_{i+1} + (j - 1)p)
\]
\[
= b + (z_i - z_{i+1}) p
\]
\[
= \begin{cases} 
b & \text{if } z_i - z_{i+1} = 0 \\
b + p & \text{if } z_i - z_{i+1} = 1
\end{cases} \quad (20)
\]

\[
\sigma_{N_{i-1}+j} - \sigma_{N_i+j+1} = \gamma_i + (j - 1)p - (\gamma_{i+1} + jp)
\]
\[
= -k + (z_i - z_{i+1}) p - p
\]
\[
= \begin{cases} 
-k - p & \text{if } z_i - z_{i+1} = 0 \\
-k & \text{if } z_i - z_{i+1} = 1
\end{cases} \quad (21)
\]

Now, since \( a_{i,j} \neq 0 \) if and only if \( i - j = b \) or \( i - j = -k \), we see that each \( C_i \), \( i = 1, \ldots, p - 1 \), is upper bidiagonal if \( z_{i-1} - z_i = 1 \) and lower bidiagonal if \( z_{i-1} - z_i = 0 \), and that the entries in the two nonzero diagonals are all positive.

Next, consider \( C_p \). We have

\[
\sigma_{N_{p-2}+j} - \sigma_j = \gamma_{p-1} + (j - 1)p - (1 + (j - 1)p)
\]
\[
= b \quad (22)
\]

\[
\sigma_{N_{p-2}+j} - \sigma_{j+1} = \gamma_{p-1} + (j - 1)p - (1 + jp)
\]
\[
= -k \quad (23)
\]

Thus \( C_p \) is an upper bidiagonal \( n_{p-1} \times n_0 \) matrix.

Now we have \( C \) in the form (13). Thus

\[
C^p = \begin{pmatrix}
D_1 & 0 & \cdots & 0 \\
0 & D_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & D_p
\end{pmatrix}
\]

(24)
where
\[ D_1 = C_1 C_2 \cdots C_p, \quad D_2 = C_2 C_3 \cdots C_p C_1, \quad \ldots, \quad D_p = C_p C_1 \cdots C_{p-1} \] (25)

It follows that the non-zero spectra of each of the products \( D_j \) are all equal and that the non-zero spectra of \( C \) consists of the \( p \)th roots of the eigenvalues of \( D_j \). Notice that each of the \( D_j \) are square \( n_j \times n_j \) matrices. Furthermore, for each \( j \), \( n_j = m = \lfloor n/p \rfloor \) or \( n_j = m + 1 \), and for at least one \( j \), \( n_j = m \). Thus, if \( m = 0 \) all the eigenvalues are zero. So we assume that \( m > 0 \).

So let \( j \) be such that \( n_{j-1} = m \), so that \( D_j \) is \( m \times m \). We will show that the eigenvalues of \( D_j \) are all real, positive and distinct. For this it is sufficient to show that \( D_j \) is oscillatory. First, we recall a few definitions. A matrix is totally nonnegative (positive) if all minors of all sizes are nonnegative (positive). A matrix \( X \) is oscillatory if there exists a positive integer \( s \) such that \( X^s \) is totally positive. Oscillatory matrices have the remarkable property that their eigenvalues are distinct, real and positive \(^3\) cf. Ch. 6, Theorem 3.2]. Moreover, a matrix \( X \) is oscillatory if and only if it is (a) totally nonnegative, (b) nonsingular, and (c) satisfies \( x_{i,i+1}, x_{i+1,i} > 0 \) for all \( i \). We will verify these three conditions for \( D_j \).

Condition (c) follows from the fact that \( z_0 = z_1 = 0 \), so \( C_1 \) is lower bidiagonal. And since \( C_p \) is upper bidiagonal, \( D_j \) is the product of upper and lower bidiagonal matrices. Thus, the super and sub-diagonal entries of \( D_j \) are all positive. Conditions (a) and (b) follow from the Cauchy-Binet identity. For any pair of ordered subsets \( \alpha, \beta \subset \{ 1, 2, \ldots, m \} \) of the same cardinality, the determinant of \( D_j(\alpha, \beta) \) is

\[
\det (D_j(\alpha, \beta)) = \det ((C_j C_{j+1} \cdots C_{[j+p-1]}) (\alpha, \beta))
\]

\[
= \sum_{\theta_j, \ldots, \theta_{j+p-2}} \prod_{i=j}^{(j+p-1)} \det (C_{[i]} (\theta_{i-1}, \theta_i))
\] (26)

where \([i] = i \bmod p\), \( \theta_{j-1} = \alpha, \theta_{j+p-1} = \beta \), and the sum is taken over all ordered subsets \( \theta_j, \ldots, \theta_{j+p-2} \) where the submatrices are defined. A simple exercise shows that all minors of bidiagonal matrices with positive entries on the two diagonals are nonnegative. It immediately follows from (26) that \( D_j \) is totally nonnegative. Thus, condition (a) is satisfied.

It remains to be shown that \( D_j \) is nonsingular. For this we use (26) again, with \( \alpha = \beta = \{ 1, 2, \ldots, m \} \). We have

\[
\det (D_j) = \prod_{i=j}^{(j+p-1)} \det (C_{[i]} (\alpha, \alpha))
\]

\[
+ \sum_{\theta_j, \ldots, \theta_{j+p-2} \neq \alpha} \prod_{i=j}^{(j+p-1)} \det (C_{[i]} (\theta_{i-1}, \theta_i))
\] (27)
\[ \theta_{j-1} = \theta_{j+p-1} = \alpha. \] The leading principal submatrices \( C_i(\alpha, \alpha) \) are bidiagonal with positive entries on their diagonals, so the first term in (27) is positive, and the remaining terms are nonnegative. Hence \( \det(D_j) > 0 \) and condition (b) is satisfied.

Since \( D_j \) is oscillatory it has \( m \) distinct, positive eigenvalues \( \omega_1, \ldots, \omega_m \), and thus the spectrum of \( A \) consists of \( q \) zeros together with the numbers

\[
\omega_s^{1/p} e^{i2\pi j/p}, \quad j = 1, \ldots, p, \quad s = 1, \ldots, m. \tag{28}
\]

This completes the proof in the case when \( b \) and \( k \) are relatively prime.

In the case when \( g = \gcd(b, k) > 1 \), we first form \( B \) as the direct sum of matrices as in (13). The spectrum of \( A \) is the union of the spectra of the \( B_i \)'s, so we apply the previous result to each of the matrices \( B_1, \ldots, B_g \). Note that for \( i = 1, \ldots, n \mod g \), \( B_i \) is an \( ([n/g] + 1) \times ([n/g] + 1) \) square matrix and for \( i = n \mod g + 1, \ldots, g \), \( B_i \) is an \( [n/g] \times [n/g] \) square matrix. The formulas (11) and (13) are obtained by applying the result for relatively prime \( b \) and \( k \) to each of the matrices \( B_i \) and then counting. In the general case, although all of the nonzero eigenvalues are of the form \( r e^{i2\pi j/p} \), there is no guarantee that the \( r \)'s are all distinct since it may happen that the spectra of two or more of the \( B_i \)'s overlap.

In the preceding proof the form of \( C = PAP^T \) in (13) and (24) depended only on the zero pattern of \( A \). So, as long as \( a_{ij} = 0 \) outside of the two diagonal bands \( (b + j, j) \) and \( (i, k + i) \), the eigenvalues of \( A \) are the \( p \)th roots of the eigenvalues of \( D_j \), and hence the spectrum is invariant under rotations by \( 2\pi/p \). Moreover, if we impose the weaker condition that \( a_{b+j,j}, a_{i,k+i} \geq 0 \), then all minors of the bidiagonal matrices \( C_i \) are still nonnegative, so the \( D_j \)'s in (24) are all totally nonnegative. The eigenvalues of a totally nonnegative matrix are real and nonnegative, but not necessarily distinct and positive [3, cf. Ch. 6, Theorem 2.5]. Thus, we have proved Corollary 1.1.

References

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