Einstein Metrics, Harmonic Forms, and Symplectic Four-Manifolds

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Abstract

Let $M$ be the underlying 4-manifold of a Del Pezzo surface. We show that a specific open region in the space of Riemannian metrics on $M$ contains all the known Einstein metrics on $M$, but no others; consequently, this region contributes exactly one connected component to the moduli space of Einstein metrics on $M$. Our methods also yield new results concerning the geometry of almost-Kähler 4-manifolds.

1 Introduction

Given a smooth compact 4-manifold $M$, one would like to completely understand the moduli space $\mathcal{E}(M)$ of the Einstein metrics it carries. Here, of course, an Einstein metric [7] means a Riemannian metric $h$ which has constant Ricci curvature, in the sense that it solves the Einstein equation

$$r = \lambda h$$

where $r$ is the Ricci tensor of $h$ and $\lambda$ is a real number, called the Einstein constant of $h$. The moduli space $\mathcal{E}(M)$ of Einstein metrics on $M$ is by definition the quotient of the set of Einstein metrics by the action of the group $\text{Diff}(M) \times \mathbb{R}^\times$ of self-diffeomorphisms and constant rescalings; for simplicity, we may give $\mathcal{E}(M)$ the quotient topology induced by the $C^\infty$-topology on the space of smooth metric tensors, but it is worth noting that, for reasons of elliptic regularity [16], this coincides [1] with the metric topology induced by the Gromov-Hausdorff distance between unit-volume Einstein metrics.

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While our understanding of this problem is rather limited for general $M$, there are specific cases where our knowledge is quite complete. In particular, if $M$ is the 4-torus $[6]$, or $K3$ $[21,26]$, or a compact quotient $\mathcal{H}^4/\Gamma$ of hyperbolic 4-space $[8]$, or a compact quotient $\mathbb{C}H_2/\Gamma$ of the complex-hyperbolic plane $[29]$, the moduli space is completely understood, and in particular is known to be connected. This is dramatically unlike the unruly state of affairs that predominates in higher dimensions, where moduli spaces of Einstein metrics are typically disconnected $[9,10]$, and indeed often have infinitely many connected components $[56,57]$.

The moduli problem described above encompasses both existence and uniqueness questions. To enquire whether the Einstein moduli space of $M$ is non-empty is obviously the same as asking whether $M$ admits an Einstein metric. In contrast to the situation in higher dimensions, there are known obstructions to the existence of Einstein metrics on 4-manifolds $[18,19,21,22,27,31,33,43,45,52]$, so there are certainly many choices of $M^4$ for which the moduli space $\mathcal{E}(M)$ can be shown to be empty. But the technology needed to prove existence of Einstein metrics is still in its infancy, and, as a result, we remain unable to determine whether $\mathcal{E}(M) \neq \emptyset$ for many common-garden 4-manifolds $M$. However, the situation becomes noticeably better if we restrict our attention to 4-manifolds which admit symplectic structures. In particular, the following classification result was proved in $[34]$, building on results from $[12]$:

**Theorem.** Let $M$ be a smooth compact 4-manifold which admits a symplectic structure $\omega$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if it is diffeomorphic to one of the following compact complex surfaces: (i) a Del Pezzo surface, (ii) a $K3$ surface, (iii) an Enriques surface, (iv) an Abelian surface, or (v) a hyper-elliptic complex surface.

Here the metric $g$ and the symplectic form $\omega$ are not assumed to be related in any way, beyond residing on the same manifold. But the “if” and “only if” parts of the proof rely on entirely different toolkits. In the “if” direction, the listed manifolds can in fact all be shown $[12,15,41,49,54]$ to admit Einstein metrics which are *conformally Kähler*. In the “only if” direction, one instead uses the Hitchin-Thorpe inequality $[7,21,52]$ and Seiberg-Witten theory $[37,51]$ to exclude all the symplectic manifolds that don’t appear on the list.

The complex surfaces listed in items (ii)–(v) of the above Theorem all satisfy $c_1^2 = 0$, and the boundary case of the Hitchin-Thorpe inequality $[21]$ therefore implies that every Einstein metric on any of these manifolds must
be Ricci-flat Kähler; the Calabi-Yau theorem [59] thus allows us to thoroughly understand the associated Einstein moduli spaces $\mathcal{E}(M)$ for these $M$, and in particular to prove [5] [48] that these moduli spaces are all connected. Unfortunately, however, no currently available result seems to provide comparable information about the moduli space $\mathcal{E}(M)$ in case (i), when $M$ is a Del Pezzo surface. While we do know that each Del Pezzo surface admits at least one Einstein metric, we are unable to determine whether they also carry other, “exotic” Einstein metrics that are entirely unlike the ones now familiar to us. In this article, we will venture into this difficult terrain, and make some modest progress towards an answer. Indeed, on each Del Pezzo surface, we will show that a specific open region in the set of Riemannian metrics contains all the known Einstein metrics, but no others.

A Del Pezzo surface is by definition a compact complex surface $(M^4, J)$ whose first Chern class is a Kähler class. In other words, the Del Pezzo surfaces are exactly the Fano manifolds of complex dimension 2. Exactly ten differentiable 4-manifolds $M$ arise as Del Pezzo surfaces: namely, $S^2 \times S^2$ and the connected sums $\mathbb{CP}_2 \# m \overline{\mathbb{CP}}_2$ for $m = 0, 1, \ldots, 8$. As complex manifolds, the Del Pezzo surfaces are exactly $\mathbb{CP}_1 \times \mathbb{CP}_1$ and the blow-ups of $\mathbb{CP}_2$ at $m$ distinct points, $0 \leq m \leq 8$, such that no three points are on a line, no six are on a conic, and no eight are on a nodal cubic with one of the given points as its node [14] [38]. When $b_2 \leq 5$, there is consequently, up to biholomorphism, only one Del Pezzo complex structure for each diffeotype, since we can simultaneously move up to four generically located points in the projective plane to standard positions via a suitable projective linear transformation. For larger values of $b_2$, the choice of complex structure instead essentially depends on $2b_2 - 10$ complex parameters; however, the various possibilities still form a single connected family, since the set of prohibited configurations of $m = b_2 - 1$ points in $\mathbb{CP}_2$ is a union of complex hypersurfaces, and so has real codimension 2.

Given a Del Pezzo surface $(M, J)$ with fixed complex structure, there is always a $\lambda > 0$ Einstein metric of the form $h = s^{-2} g$, where $g$ is a $J$-compatible Kähler metric of scalar curvature $s > 0$. Indeed, in most cases, we can simply take $h = g$, so that $h$ is a Kähler-Einstein metric. By a celebrated result of Tian [40] [53], one can in fact find such a Kähler-Einstein metric iff $(M, J)$ has reductive automorphism group, and it is easy to see that this excludes only two Del Pezzo surfaces $(M, J)$. The two oddballs are the ones diffeomorphic to $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ and $\mathbb{CP}_2 \# 2 \overline{\mathbb{CP}}_2$; and because the automorphism groups of these Del Pezzo surfaces are not reductive, a result of Matsushima implies that neither, in fact, admits a Kähler-Einstein metric. Nonetheless, Page [41] had been able to construct an explicit cohomogeneity-
one Einstein metric on $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$, and Derdziński [15] later discovered that this metric is actually conformal to an extremal Kähler metric on the one-point blow-up of the complex projective plane. Eventually, Chen, Weber, and the present author proved [12] that $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ also admits an Einstein metric which is conformally Kähler. For a simplified proof, see [36].

Later, the present author proved [35] that if an Einstein metric on a compact complex surface is merely Hermitian with the respect to the given complex structure, then it is either Kähler-Einstein, or is isometric to a constant multiple of the Page metric or Chen-LeBrun-Weber metric. In particular, up to complex automorphisms and constant rescalings, there is exactly one conformally Kähler, Einstein metric on each Del Pezzo surface. In most cases, this uniqueness primarily follows from the work of Bando and Mabuchi [4]; but for the toric Del Pezzos, the proof actually requires a more general uniqueness result [13] for extremal Kähler metrics, as well as a detailed study of the Calabi energy of extremal Kähler metrics on these manifolds.

Let us now venture into a broader world by considering Riemannian metrics $h$ on a smooth compact oriented 4-manifold $M$ with $b_+(M) \neq 0$. For any $h$ we pick, there is exactly a $b_+(M)$-dimensional space of self-dual harmonic 2-forms $\omega$ associated with $h$; and since the fact that these forms are self-dual and harmonic is unaffected by multiplying $h$ by an arbitrary positive function, we may choose to think of them as being associated with the conformal class $[h]$ rather than with the any particular metric in it. We will henceforth say that the conformal class $[h]$ is of symplectic type if there is a harmonic self-dual 2-form $\omega$ on $(M, h)$ such that $\omega \neq 0$ everywhere on $M$. When this happens, $\omega$ is a closed, non-degenerate 2-form, and $(M, \omega)$ is consequently a symplectic 4-manifold. The key point for us is that this is an open condition on $[h]$, in the sense that the set of conformal classes of symplectic type is automatically open [32] in the $C^2$ topology.

Now, given a conformal class of symplectic type and a nowhere-zero harmonic self-dual 2-form $\omega$, we may also consider the function $W^+(\omega, \omega)$ obtained by contracting the self-dual Weyl tensor with two copies of the symplectic form $\omega$. If we replace $h$ with $uh$ for some positive function $u$, then $W_+(\omega, \omega)$ is replaced with $u^{-3}W_+(\omega, \omega)$, so that, in particular, the sign of $W_+(\omega, \omega)$ at any given point is unaltered by conformal rescaling. We may thus declare the conformal class $[h]$ to be of positive symplectic type if, for some choice of $h$-compatible self-dual harmonic 2-form $\omega$,

$$W^+(\omega, \omega) > 0$$

everywhere on $M$. Once again, this condition is open in the $C^2$ topology.
If $g$ is a Kähler metric of positive scalar curvature on a compact complex surface $M$, its conformal class $[g]$ is automatically of positive symplectic type, and this observation now tells us that there is a non-empty, and indeed large and interesting, open set of conformal classes of positive symplectic type on any rational or ruled complex surface. Moreover, on any Del Pezzo surface, this region includes the conformal classes of all the known Einstein metrics, as described above. The purpose of this article is to use this property, conversely, to characterize the known Einstein metrics:

**Theorem A.** Let $(M, h)$ be a 4-dimensional Einstein manifold. If the conformal class $[h]$ is of positive symplectic type, then $h$ is isometric to

- a Kähler-Einstein metric on a Del Pezzo surface; or
- a constant multiple of the Page metric on $\mathbb{CP}_2 \# \mathbb{CP}_2$; or
- a constant multiple of the Chen-LeBrun-Weber metric on $\mathbb{CP}_2 \# 2 \mathbb{CP}_2$.

The proof of this result, which may be found at the end of §2, proceeds by showing that any such $h$ must be conformally Kähler. We emphasize that the classification offered by Theorem A is actually sharp, since the conformal class of each listed Einstein metric is indeed of positive symplectic type.

Here is a simple but interesting consequence:

**Corollary 1.** Let $M$ be the underlying smooth compact 4-manifold of a Del Pezzo surface. Let $\mathcal{E}(M)$ denote the the moduli space of Einstein metrics $h$ on $M$, and let $\mathcal{E}_\omega^+(M) \subset \mathcal{E}(M)$ be the open subset arising from Einstein metrics $h$ for which the corresponding conformal classes $[h]$ are of positive symplectic type. Then $\mathcal{E}_\omega^+(M)$ is connected. Moreover, if $b_2(M) \leq 5$, then $\mathcal{E}_\omega^+(M)$ exactly consists of a single point.

In particular, this gives us a simple proof of the following fact:

**Corollary 2.** For any Del Pezzo surface $M$, $\mathcal{E}_\omega^+(M)$ is exactly a connected component of $\mathcal{E}(M)$.

Indeed, it suffices to prove that the path-connected space $\mathcal{E}_\omega^+(M)$ is both open and closed in $\mathcal{E}(M)$. Since $\mathcal{E}(M)$ has the quotient topology, the fact that it is open follows from the fact that the set of metrics with positive symplectic conformal class is open and invariant under the action of $\text{Diff}(M) \times \mathbb{R}^\times$. On the other hand, it is also closed, because, except in cases where $\mathcal{E}_\omega^+(M)$ is now known to be a single point, the Einstein metrics in question are all Kähler, and requiring that a Riemannian metric carry a parallel almost-complex structure is a closed condition.
2 Harmonic Self-Dual Weyl Curvature

Recall that we say that a conformal class \([h]\) on an a compact oriented 4-manifold \(M\) is of \textit{symplectic type} if there is a harmonic self-dual 2-form \(\omega\) on \((M, h)\) such that \(\omega \neq 0\) everywhere on \(M\). This is indeed a conformally invariant condition, because the Hodge star operator is conformally invariant; moreover, it is an open condition \[^{32}\]\ with respect to the \(C^2\) topology. Since any self-dual 2-form \(\omega\) satisfies
\[
\omega \wedge \omega = \omega \wedge * \omega = \|\omega\|^2_h \, d\mu_h,
\]
it follows that an appropriate \(\omega\) is actually a symplectic form on \(M\) if \([h]\) is of symplectic type. Assuming this, the conformally related metric \(g \in [h]\) given by \(g = 2^{-1/2} |\omega|_h h\) is then an \textit{almost-Kähler metric}, in the sense that \(g\) is related to the symplectic form \(\omega\) via \(g = \omega(\cdot, J\cdot)\) for a unique almost-complex structure \(J\) on \(M\). For our purposes, the important point is that, in dimension 4, the almost-Kähler condition is equivalent to saying that \(\omega\) is harmonic and self-dual with respect to \(g\), and that \(|\omega|^2_g \equiv 2\).

While our primary aim here is to learn something about Einstein metrics, we will more generally focus on oriented Riemannian 4-manifolds \((M, h)\) with \textit{harmonic self-dual Weyl curvature}, in the sense that \(\delta W^+ = -\nabla \cdot W^+ = 0\). When \(h\) is Einstein, this property holds, as a consequence of the second Bianchi identity. However, we will see in due course that \(\delta W^+ = 0\) is in general much weaker than the Einstein condition.

When \([h]\) is of symplectic type, it will prove profitable to study this equation from the point of view of the conformally related almost-Kähler metric \(g\). This is quite tractable, because the divergence-free condition on a section of \(\mathcal{C}^2_0 \Lambda^+\) is conformally invariant, albeit \[^{12}\] with an unexpected \textit{conformal weight}. In practice, this means that if \(h = f^2 g\) has the property that \(\delta W^+ = 0\), then \(g\) will instead have the property that \(\delta(f W^+)=0\). For us, the important point is that this then implies a Weitzenböck formula
\[
0 = \nabla^* \nabla (f W^+) + \frac{s}{2} f W^+ - 6 f W^+ \circ W^+ + 2 f |W^+|^2 I
\]
for \(f W^+\), considered as a section of \(\text{End}(\Lambda^+)\); cf. \[^{15, 20, 42}\].

To exploit this effectively, we will need the following identity:

\textbf{Lemma 1.} Any 4-dimensional almost-Kähler manifold satisfies
\[
\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle = [W^+ (\omega, \omega)]^2 + 4 |W^+ (\omega)|^2 - s W^+ (\omega, \omega)
\]
at every point.
Proof. First notice that the oriented Riemannian 4-manifold \((M, g)\) satisfies
\[
\Lambda^+ \otimes \mathbb{C} = \mathbb{C} \omega \oplus K \oplus \overline{K},
\]
where \(K = \Lambda^2_{J,0}\) is the canonical line bundle of the almost-complex manifold \((M, J)\). Locally choosing a unit section \(\varphi\) of \(K\), we thus have
\[
\nabla \omega = \alpha \otimes \varphi + \bar{\alpha} \otimes \bar{\varphi}
\]
for a unique 1-form \(\alpha \in \Lambda^1_{J,0}\), since \(\nabla [\alpha \omega_{bc}] = 0\) and \(\omega^{bc} \nabla_a \omega_{bc} = 0\). If \(\otimes : \Lambda^+ \times \Lambda^+ \to \otimes^2_0 \Lambda^+\) denotes the symmetric trace-free product, we therefore have
\[
(\nabla \omega) \otimes (\nabla^c \omega) = 2|\alpha|^2 \varphi \otimes \bar{\varphi} = -\frac{1}{4}|\nabla \omega|^2 \omega \otimes \omega
\]
and we thus deduce that
\[
\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle = 2W^+(\omega, \nabla^* \nabla \omega) - 2W^+(\nabla \omega, \nabla^c \omega)
\]
\[
= 2W^+(\omega, \nabla^* \nabla \omega) + \frac{1}{2} |\nabla \omega|^2 W^+(\omega, \omega)
\]
\[
= 2W^+(\omega, 2W^+(\omega) - \frac{8}{3} \omega) + \left[ W^+(\omega, \omega) - \frac{8}{3} \right] W^+(\omega, \omega)
\]
\[
= -\frac{2}{3} s W^+(\omega, \omega) + 4 |W^+(\omega)|^2 + \left[ W^+(\omega, \omega) - \frac{8}{3} \right] W^+(\omega, \omega)
\]
\[
= |W^+(\omega, \omega)|^2 + 4 |W^+(\omega)|^2 - s W^+(\omega, \omega)
\]
where we have used the Weitzenböck formula
\[
0 = \nabla^* \nabla \omega = 2W^+(\omega) + \frac{8}{3} \omega
\]
for the harmonic self-dual 2-form \(\omega\), as well as the associated key identity
\[
\frac{1}{2} |\nabla \omega|^2 = W^+(\omega, \omega) - \frac{8}{3}
\]
resulting from the fact that \(|\omega|^2 \equiv 2\).

Plugging this into our Weitzenböck formula (1) and integrating by parts, we thus see that whenever a compact almost-Kähler 4-manifold \((M, g, \omega)\)
satisfies $\delta(fW^+) = 0$, we then automatically have

$$0 = \left\langle \nabla^* \nabla fW^+ + \frac{s}{2} fW^+ - 6 fW^+ \circ W^+ + 2 |fW^+|^2 I, \omega \otimes \omega \right\rangle d\mu$$

$$= \int_M \left[ (W^+, \nabla^* \nabla (\omega \otimes \omega)) + \frac{s}{2} W^+(\omega, \omega) - 6 |W^+(\omega)|^2 + 2 |W^+|^2 |\omega|^2 \right] f d\mu$$

$$= \int_M \left[ (|W^+(\omega, \omega)|^2 + 4 |W^+(\omega)|^2 - sW^+(\omega, \omega)) + \frac{s}{2} W^+(\omega, \omega) - 6 |W^+(\omega)|^2 + 4 |W^+|^2 \right] f d\mu$$

$$= \int_M \left[ |W^+(\omega, \omega)|^2 - \frac{s}{2} W^+(\omega, \omega) - 2 |W^+(\omega)|^2 + 4 |W^+|^2 \right] f d\mu .$$

In other words, letting $W^+(\omega)^\perp$ denote the component of $W^+(\omega)$ perpendicular to $\omega$, any compact almost-Kähler manifold $(M, g, \omega)$ with $\delta(fW^+) = 0$ satisfies the identity

$$\int_M sW^+(\omega, \omega) f d\mu = 8 \int_M \left( |W^+|^2 - \frac{1}{2} |W^+(\omega)^\perp|^2 \right) f d\mu . \quad (2)$$

To proceed further, we will now need another algebraic observation:

**Lemma 2.** Any 4-dimensional almost-Kähler manifold satisfies

$$|W^+|^2 - \frac{1}{2} |W^+(\omega)^\perp|^2 \geq \frac{3}{8} [W^+(\omega, \omega)]^2$$

at every point, and equality can only hold at points where $W^+(\omega)^\perp = 0$.

**Proof.** If $A = [A_{jk}]$ is any symmetric trace-free $3 \times 3$ matrix, the fact that $A_{33} = -(A_{11} + A_{22})$ implies that

$$\sum_{jk} A_{jk}^2 \geq 2A_{21}^2 + A_{11}^2 + A_{22}^2 + A_{33}^2 = 2A_{21}^2 + \frac{3}{2} A_{11}^2 + 2(A_{11}^2 + A_{22}^2)$$

and we therefore conclude that

$$|A|^2 \geq 2A_{21}^2 + \frac{3}{2} A_{11}^2 .$$

If we now let $A$ represent $W^+: \Lambda^+ \to \Lambda^+$ with respect to an orthogonal basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$ for $\Lambda^+$ such that $\omega = \sqrt{2}\varepsilon_1$ and $W^+(\omega)^\perp \propto \varepsilon_2$, this inequality becomes

$$|W^+|^2 \geq |W^+(\omega)^\perp|^2 + \frac{3}{8} [W^+(\omega, \omega)]^2$$

which not only proves the desired inequality, but shows that it is actually strict whenever $\omega$ is not an eigenvector of $W^+$. \(\square\)
Combining (2) with Lemma 2 now yields the global inequality
\[\int_M s W^+(\omega, \omega) f \, d\mu \geq 3 \int_M [W^+(\omega, \omega)]^2 f \, d\mu,\]  
with equality only if \(W^+(\omega) \equiv 0\). It thus follows that
\[0 \geq \int_M W^+(\omega, \omega) \left( W^+(\omega, \omega) - \frac{s}{3} \right) f \, d\mu.\]

However, since \(\frac{1}{2} |\nabla \omega|^2 = W^+(\omega, \omega) - \frac{s}{3}\) for any almost-Kähler 4-manifold, this proves the following:

**Proposition 1.** Let \((M^4, g, \omega)\) be a compact almost-Kähler manifold, and suppose that, for some positive function \(f\), the conformally related metric \(h = f^2 g\) has harmonic self-dual Weyl curvature. Then \((M, g, \omega)\) satisfies the inequality
\[0 \geq \int_M W^+(\omega, \omega)|\nabla \omega|^2 f \, d\mu.\]

This has an interesting immediate consequence:

**Proposition 2.** Let \((M^4, g, \omega)\) be a compact connected almost-Kähler manifold with \(W^+(\omega, \omega) \geq 0\), and suppose that the conformally related metric \(h = f^2 g\) satisfies \(\delta W^+ = 0\). Then either \(g\) is a Kähler metric with scalar curvature \(s = c/f\) for some constant \(c > 0\), or else \(g\) satisfies \(W^+ \equiv 0\), and so is an anti-self-dual metric.

**Proof.** Recall that \(f > 0\) by convention, and that \(W^+(\omega, \omega) \geq 0\) by assumption. Thus (4) implies that
\[\int_M W^+(\omega, \omega)|\nabla \omega|^2 f \, d\mu = 0,\]
so that \(\nabla \omega = 0\) wherever \(W^+(\omega, \omega) \neq 0\). If \(U \subset M\) is the open subset where \(W^+(\omega, \omega) \neq 0\), the restriction of \(g\) to \(U\) is therefore Kähler. On the other hand, by hypothesis, \(g\) satisfies \(\delta(fW^+) = 0\). However, for any Kähler manifold of real dimension 4, \(W^+\) is the trace-free part of \((s/4)\omega \otimes \omega\), where the scalar curvature \(s\) satisfies \(s = 3W^+(\omega, \omega)\). It follows that \(d[fW^+(\omega, \omega)] = 0\) on \(U\). By continuity, we therefore have \(d[fW^+(\omega, \omega)] = 0\) on the closure \(\overline{U}\) of \(U\), too. On the other hand, \(fW^+(\omega, \omega) \equiv 0\) on \(M - \overline{U}\), so we also have \(d[fW^+(\omega, \omega)] = 0\) on the open set \(M - \overline{U}\). Hence \(d[fW^+(\omega, \omega)] = 0\) on all of \(M\). Since \(M\) is connected, it follows that \(fW^+(\omega, \omega) = c/3\) for
some non-negative constant $c \geq 0$. If $c > 0$, $M = U$, and $(M, g)$ is a Kähler manifold, with $s = 3W^+(\omega, \omega) = c/f$. On the other hand, if $c = 0$, we have $W^+(\omega, \omega) \equiv 0$, and therefore have equality in (3). However, this implies that $W^+(\omega) \perp \equiv 0$, and (2) therefore implies that $W^+ \equiv 0$, as claimed.

We now recast this in a narrower and more useful form.

**Theorem 1.** Let $(M, h)$ be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$. If the conformal class $[h]$ is of positive symplectic type, then $h = s^{-2}g$ for a unique Kähler metric $g$ of scalar curvature $s > 0$. Conversely, if $g$ is any Kähler metric of positive scalar curvature, the conformally related metric $h = s^{-2}g$ satisfies $\delta W^+ = 0$.

**Proof.** To say that $[h]$ is of positive symplectic type means that there is a self-dual harmonic 2-form $\omega$ on $(M, h)$ such that $W^+(\omega, \omega) > 0$ at every point of $M$. Rescaling $h$ to make $\omega$ have constant norm $\sqrt{2}$ results in an almost-Kähler metric $\hat{g}$ such that $h = c^{-1}\hat{g}$ for some positive function $c$. If $h$ satisfies $\delta W^+ = 0$, the almost-Kähler metric $\hat{g}$ then satisfies with $W^+(\omega, \omega) > 0$ and $\delta(\hat{f}W^+) = 0$, so Proposition 2 then tells us that $\hat{g}$ is actually Kähler, with scalar curvature $\hat{s} = c/\hat{f}$ for some positive constant $c$. In particular, $M$ admits a Kähler metric with positive scalar curvature, and Yau’s vanishing theorem 55 for the geometric genus therefore implies that $b_+(M) = 1$. Thus the choice of $\omega$ is in fact unique up to an overall multiplicative constant, and the choice of $\hat{g}$ is therefore determined up to constant rescalings. But if, for a positive constant $a$, we replace $\hat{g}$ with $g = s^{-2}g$, we must also replace $\hat{f}$ with $f = a^{-1}\hat{f}$; and note that the scalar curvature of $g$ is then $s = a^{-2}\hat{s}$. Since $\hat{f} = c\hat{s}^{-1}$, we then have $h = f^2\hat{g} = c^2s^{-2}\hat{g} = c^2(a^2s)^{-2}(a^{-2}g) = (ca^{-3})^2s^{-2}g$. This shows that setting $a = \sqrt[3]{c}$ results in a Kähler metric $g$ such that $h = s^{-2}g$, and moreover shows that this choice yields the only Kähler metric with this property.

On the other hand, if $g$ is a Kähler metric with $s > 0$, $s^{-1}W^+$ is parallel, so that, in particular, we have $\delta(s^{-1}W^+) = 0$. Thus $h = s^{-2}g$ satisfies $\delta W^+ = 0$, as promised.

Theorem A is now a straightforward consequence. Indeed, since the second Bianchi identity implies that any Einstein metric on an oriented 4-manifold satisfies $\delta W^+ = 0$, Theorem 2 tells us that every Einstein metric $h$ with conformal class $[h]$ of positive symplectic type must be conformally Kähler. Moreover, since the conformal class $[h]$ contains a representative $g$ with $s > 0$, the constant scalar curvature $4\lambda$ of $h$ must 55 be positive, too. Theorem A therefore follows from the known classification 35 of conformally Kähler, Einstein metrics on compact 4-manifolds.
Almost-Kähler Manifolds Revisited

The results of §2 also have interesting applications in the narrower context of almost-Kähler geometry. Our main result in this direction is the following:

**Theorem 2.** Let \((M, g, \omega)\) be a compact almost-Kähler 4-manifold with non-negative scalar curvature and harmonic self-dual Weyl tensor:

\[ s \geq 0, \quad \delta W^+ = 0. \]

Then \((M, g, \omega)\) is a constant-scalar-curvature Kähler manifold.

**Proof.** For any almost-Kähler manifold,

\[ W^+(\omega, \omega) = \frac{s}{3} + \frac{1}{2} |\nabla \omega|^2 \]

so that the hypothesis \(s \geq 0\) implies \(W^+(\omega, \omega) \geq 0\). Proposition 2 with \(f = 1\), therefore tells us that \((M, g)\) is Kähler, with scalar curvature \(s = c/f = c\) for some positive constant \(c\), or else that \(W^+ \equiv 0\). In the latter case, we then have \(0 = 3W^+(\omega, \omega) \geq s \geq 0\), so \(s \equiv 0\), and hence \(|\nabla \omega|^2 = 2W^+(\omega, \omega) - 2s/3 = 0\). Thus \((M, g)\) is constant-scalar-curvature Kähler, even in the exceptional case. \(\square\)

Conversely, any constant-scalar-curvature Kähler manifold of real dimension 4 satisfies \(\delta W^+ = 0\), independent of the sign of \(s\). While the study of “cscK” (constant-scalar-curvature Kähler) metrics on compact complex surfaces is an active area of ongoing research, many existence results are already available [3, 17, 21, 28, 44, 47]. However, we should emphasize that the non-negativity of the scalar curvature plays a crucial role in Theorem 2. For example, there exist many compact almost-Kähler manifolds with \(W^+ \equiv 0\) which are not Kähler. Indeed, such examples can be obtained [23] by deforming scalar-flat Kähler metrics through anti-self-dual conformal classes, and then conformally rescaling to make \(|\omega| \equiv \sqrt{2}\). Examples of this type automatically have \(s \leq 0\), with \(s < 0\) on an open dense subset.

Since any Einstein 4-manifold satisfies \(\delta W^+ = 0\), Theorem 2 provides a new proof of Sekigawa’s breakthrough result [46] on the Goldberg conjecture:

**Corollary 3 (Sekigawa).** Every compact almost-Kähler Einstein 4-manifold with non-negative Einstein constant is Kähler-Einstein.
This result of course helped inspire the present piece of research.

The proof of Theorem 2 still works if we merely impose the ostensibly weaker hypothesis that \( s + tW^+(\omega, \omega) \geq 0 \) for some constant \( t \geq 0 \), since any such hypothesis will imply that \( W^+(\omega, \omega) \geq 0 \), with \( s = 0 \) if equality holds. In particular, one reaches exactly the same conclusion if we assume that the so-called star-scalar curvature

\[
s^* = s + |\nabla \omega|^2 = \frac{s}{3} + 2W^+(\omega, \omega)
\]

is non-negative:

**Proposition 3.** Let \( (M, g, \omega) \) be a compact almost-Kähler 4-manifold with non-negative star-scalar curvature and harmonic self-dual Weyl tensor:

\[
s^* \geq 0, \quad \delta W^+ = 0.
\]

Then \( (M, g, \omega) \) is a constant-scalar-curvature Kähler manifold.

Kirchberg [25] has elsewhere investigated almost-Kähler 4-manifolds with harmonic Weyl tensor and positive star-scalar curvature. Since the hypothesis \( \delta W = 0 \) is equivalent to

\[
\delta W^+ = \delta W^- = 0,
\]

and is therefore stronger than the hypothesis \( \delta W^+ = 0 \) of Proposition 3, we can recover several of Kirchberg’s results from our own. In particular, we can deduce the following clarification of [25, Corollary 3.13]:

**Proposition 4.** Let \( (M, g, \omega) \) be a compact almost-Kähler 4-manifold with non-negative scalar curvature and harmonic Weyl tensor:

\[
s \geq 0, \quad \delta W^+ = \delta W^- = 0.
\]

Then \( (M^4, g, J) \) is either a Kähler-Einstein manifold with \( \lambda \geq 0 \), or else is locally symmetric, with universal cover \((\tilde{M}, \tilde{g})\) isometric to the Riemannian product of two constant-curvature surfaces. In the latter case, moreover, the complex manifolds \((M^4, J)\) admitting these non-Einstein Kähler metrics are precisely the geometrically ruled surfaces arising as projectivizations of polystable rank-2 holomorphic vector bundles over compact complex curves.
Proof. By Theorem [2] we know that \((M, g, J)\) is a Kähler manifold of constant scalar curvature. But since the entire Weyl tensor is also assumed to be harmonic, the second Bianchi identity also tells us that
\[
\nabla_{[c^r d]} h = \nabla_a W^{abcd} + \frac{1}{6} g_{[c} \nabla_d] s = 0.
\]
The covariant derivative \(\nabla r\) of the Ricci tensor is therefore completely symmetric. Decomposing \(\otimes^3 \Lambda^1_C\) into \(\otimes^3 (\Lambda^{1,0} \oplus \Lambda^{0,1})\), we thus have
\[
\nabla_n r_{\mu \bar{\nu}} = \nabla_{\bar{\nu}} r_{\mu \bar{n}} = 0 \quad \text{and} \quad \nabla_n r_{\mu \bar{\nu}} = \nabla_{\mu} r_{\bar{\nu} \bar{n}} = 0.
\]
This shows that the Ricci tensor of our Kähler manifold is parallel; and, consequently, the primitive part \(\hat{\rho} \in \Lambda^-\) of its Ricci form must be parallel, too. If \(\hat{\rho} = 0\), \((M, g, J)\) is Kähler-Einstein. Otherwise, \(\omega\) and \(\sqrt{2|\hat{\rho}| / |\hat{\rho}|}\) are the Kähler forms for two \(g\)-compatible, oppositely-oriented parallel complex structures, and the holonomy of \((M, g)\) is therefore contained in \(U(1) \times U(1) \subset U(2) \subset SO(4)\). By the deRham splitting theorem [7], the universal cover \((\tilde{M}, \tilde{g})\) of \((M, g)\) is therefore a Riemannian product \((M_1, g_1) \times (M_2, g_2)\) of two complete, simply connected Riemannian 2-manifolds. Moreover, since the constant scalar curvature of \(g\) can be expressed in terms of the Gauss curvatures of \(g_1\) and \(g_2\) as \(s = 2K_1 + 2K_2\), it follows that \(K_1\) and \(K_2\) are both constant.

Since \(s \geq 0\), and since the Ricci tensor of \(g\) is given by \(r = K_1 g_1 + K_2 g_2\), one of the surfaces, say \((M_1, g_1)\), must have constant positive curvature unless \((M, g)\) is Ricci-flat. Each of the resulting submanifolds \(\mathbb{CP}^1 \times \{pt\}\) of \(\tilde{M}\) moreover covers a leaf of the foliation of \(M\) tangent to the \(K_1\)-eigenvectors of \(r\), and since each such leaf is oriented by \(J\), each such \(\mathbb{CP}^1\) necessarily embeds in \(M\). Since these leaves are all simply connected, the induced flat connection on the normal bundle of each leaf has trivial holonomy, and the space of leaves is therefore a manifold \(\Sigma\). The projection \(\tilde{M} \to \Sigma\) now makes \((M, J)\) into a holomorphic \(\mathbb{CP}^1\)-bundle, and the transverse foliation corresponding to the \(M_2\) factor gives this bundle a flat connection with structure group \(SO(3) = PSU(2)\). The Narasimhan-Seshadri theorem [39] thus asserts that \(M\) is the projectivization of a polystable holomorphic vector bundle \(E \to \Sigma\), and conversely shows that any ruled surface arising from a polystable bundle conversely carries a flat projective unitary connection, and so admits Kähler metrics of constant positive scalar curvature that are uniformized by the round 2-sphere times a 2-dimensional space of constant curvature.

For related results concerning constant-scalar-curvature Kähler metrics on geometrically ruled surfaces, see [2, 11, 30].
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