Monopole-Monopole solutions of Einstein-Yang-Mills-Higgs Theory

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New static regular axially symmetric solutions of SU(2) Yang-Mills-Higgs theory are constructed. They are asymptotically flat and represent gravitating monopole-monopole pairs. The solutions form two branches linked to the second Bartnik-McKinnon solution on upper mass branch and to the monopole-monopole configuration in flat space on the lower branch, respectively.

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I. INTRODUCTION

The nontrivial vacuum structure of SU(2) Yang-Mills-Higgs (YMH) theory allows for the existence of regular non-perturbative finite energy solutions, such as spherically symmetric monopoles [1], various axially symmetric multimonopoles [2, 3, 4, 5] and monopole-antimonopole systems [6, 7]. In the Bogomol’nyi-Prasad-Sommerfield (BPS) limit of vanishing scalar potential, axially symmetric multimonopole configurations are known analytically [4]. In these solutions the nodes of the Higgs field are superimposed at a single point. Since in the BPS limit the repulsive and attractive forces between monopoles exactly compensate, the BPS monopoles experience no net interaction [9]. Thus, the BPS configuration with multiple node at the origin can be continuously deformed into a system of individual monopoles with unit topological charge (see, e.g., [10, 11]). However, as scalar field becomes massive, the fine balance of forces between the monopoles is broken and there is only repulsion between non-BPS multimonopole solutions [5].

On the other hand, there are axially symmetric saddlepoint solutions of the YMH theory, where the Higgs field vanishes at several isolated points on the symmetry axis [6, 7]. Simplest such a solution represent a monopole-antimonopole (MA) pair, a magnetic dipole [6, 7].

When gravity is coupled to YMH theory, branches of gravitating solutions arise [12, 13, 14]. The lower branch emerges from the flat space configurations as coupling constant $\alpha$ entering the Einstein-Yang-Mills-Higgs (EYMH) equations, is increases from zero. However, there is a difference between behavior of the gravitating solutions with a single or superimposed nodes of the scalar field [12, 13] and gravitating monopole-antimonopole chains and vortex rings [14]. While in the former case the lower branch ends at a critical value, beyond which the core of the configuration would be smaller than the Schwarzschild radius [12, 13], for the gravitating monopole-antimonopole chains and vortex rings the second branch emerges which is extended back to $\alpha = 0$ [14]. In this limit the configurations shrink to zero size and the scaled solutions approach corresponding solutions of pure Einstein-Yang-Mills (EYM) theory [15, 16, 17].

It was mentioned that the additional attraction in the YMH system due to the coupling to gravity also allows for bound monopoles with superimposed zeros of the scalar field which are not present in flat space [13]. However, one can conjecture if the EYMH theory allows for further static axially-symmetric solutions representing, for example, separated monopole-monopole (MM) pair. Evidently, beyond BPS limit these solutions do not possess counterparts in flat space but gravitational attraction may reinforce the effect of the scalar interaction and a bound state of the gravitating monopoles can exist.

In this letter we report the existence of one such new solution representing MM pair. On the upper branch it is related to the second Bartnik-McKinnon solution with two zeros [13]. The properties of the gravitating monopole pair are compared with those of MA pair, which on the upper branch is linked to the first Bartnik-McKinnon solution with one zero, and with solution for the gravitating charge 2 axially symmetric monopole, linked to the extremal Reissner-Nordstr"om black hole solution.

In section II we present the Lagrangian of the EYMH theory, the axially symmetric ansatz and the boundary conditions. In section III we discuss the properties of the gravitating MM pair.

II. EINSTEIN-YANG-MILLS-HIGGS MODEL AND AXIALLY SYMMETRIC ANSATZ

We consider the SU(2) Einstein-Yang-Mills-Higgs theory with action

$$S = \int \left\{ \frac{R}{16\pi G} - \frac{1}{2} \text{Tr} \left( F_{\mu\nu} F^{\mu\nu} \right) - \frac{1}{4} \text{Tr} \left( D_\mu \Phi D^\mu \Phi \right) - \frac{\lambda}{8} \text{Tr} \left( \Phi^2 - \eta^2 \right)^2 \right\} \sqrt{-g} d^4x$$

(1)

Here $G$ and $\lambda$ denote the gravitational and scalar coupling constants, respectively, $\eta$ is the vacuum expectation value of the Higgs field and $R$ is Ricci scalar.
In isotropic coordinates the static axially symmetric metric reads

\[ ds^2 = -f dt^2 + \frac{m}{f} dr^2 + \frac{m^2}{f} d\theta^2 + \frac{l r^2 \sin^2 \theta}{f} d\varphi^2 , \]

(2)

where the metric functions \( f, m \) and \( l \) are functions of the coordinates \( r \) and \( \theta \), only. The \( z \)-axis \((\theta = 0, \pi)\) represents the symmetry axis. Regularity on this axis requires \( m = l \) there.

For the gauge and the Higgs field we employ the known ansatz \[ 8 \]

\[ A_\mu dx^\mu = \left( \frac{K_1}{r} dr + (1 - K_2) d\theta \right) \tau_2^{(n)} \frac{r^2}{2e} - n \sin \theta \left( \frac{K_3 \tau_r^{(n,m)}}{2e} + (1 - K_4) \tau_\theta^{(n,m)} \right) d\varphi , \]

(3)

\[ \Phi = \eta \left( \Phi_1 \tau_r^{(n,m)} + \Phi_2 \tau_\theta^{(n,m)} \right) . \]

(4)

where the \( su(2) \) matrices \( \tau_r^{(n,m)}, \tau_\theta^{(n,m)}, \) and \( \tau_\varphi^{(n)} \) are defined as products of the spatial unit vectors

\[ \begin{align*}
\hat{e}_r^{(n,m)} &= (\sin(m \theta) \cos(n \varphi), \sin(m \theta) \sin(n \varphi), \cos(m \theta)) , \\
\hat{e}_\theta^{(n,m)} &= (\cos(m \theta) \cos(n \varphi), \cos(m \theta) \sin(n \varphi), -\sin(m \theta)) , \\
\hat{e}_\varphi^{(n)} &= (-\sin(n \varphi), \cos(n \varphi), 0) ,
\end{align*} \]

(5)

with the Pauli matrices \( \tau^a \).

For \( m = 2, n = 1 \) the ansatz corresponds to the one for the MA pair solutions \[ 6, 7, 12 \], while for \( m = 1, n > 1 \) it corresponds to the ansatz for axially symmetric multimonopoles \[ 8, 18 \]. In particular, for \( m = 1, n = 2 \) we have non-BPS extension of the charge 2 monopole solution \[ 8 \].

The four gauge field functions \( K_i \) and two Higgs field functions \( \Phi_i \) depend on the coordinates \( r \) and \( \theta \), only. To construct regular solutions we have to fix the gauge condition \( r \partial_r K_1 - \partial_\theta K_2 = 0 \). Further, we introduce the dimensionless coordinate \( x = e r \eta \) and rescale the Higgs field as \( \Phi \to \phi / \eta \). Then the dimensionless coupling \( \alpha = 4\pi G \eta^2 \) enters the equations.

To obtain asymptotically flat solutions which are regular and corresponds to the gravitating MM pair, we need to impose the boundary conditions. Regularity of the solutions at the origin \( (r = 0) \) requires for the metric functions the boundary conditions \( \partial_r f(r, \theta)|_{r=0} = \partial_r m(r, \theta)|_{r=0} = \partial_r l(r, \theta)|_{r=0} = 0 \), whereas the gauge field functions \( K_i \) satisfy \( K_1(0, \theta) = K_3(0, \theta) = 0, \ K_2(0, \theta) = K_4(0, \theta) = 1 \), and the Higgs field functions \( \Phi_i \) satisfy

\[ \begin{align*}
\sin \theta \ \Phi_1(0, \theta) + \cos \theta \ \Phi_2(0, \theta) &= 0 , \\
\theta_r [\cos \theta \ \Phi_1(r, \theta) - \sin \theta \ \Phi_2(r, \theta)]|_{r=0} &= 0 .
\end{align*} \]

(6)

(7)

These conditions are the same both for MA-pair and MM-pair. For the charge 2 monopole both \( \Phi_1 \) and \( \Phi_2 \) must vanish at the origin.

The boundary conditions at infinity shall provide correct asymptotic behavior of the fields depending on the configuration. Evidently, asymptotical flatness requires \( f \to 1, \ m \to 1, \ l \to 1 \) for any solution. Also the Higgs field at infinity have to approach the ‘hedgehog’ shape, i.e., \( \Phi_1 \to 1, \ \Phi_2 \to 0 \). But the boundary conditions on the gauge functions can be different.

To construct MA-pair, which is a deformation of the topologically trivial sector, the gauge field at infinity required to tend to a pure gauge \( A_\mu \to i \partial_\mu U U^\dagger \), where \( U = \exp(-i \theta r_\varphi^{(n)}) \). In terms of the functions \( K_i \) these boundary conditions read:

\[ K_1 \to 0 , \quad K_2 \to -1 , \quad K_3 \to 2 \sin \theta , \quad K_4 \to 1 - 2 \cos \theta , \]

(8)

whereas for the charge 2 monopole we required

\[ K_1 \to 0 , \quad K_2 \to 0 , \quad K_3 \to 0 , \quad K_4 \to 0 . \]

(9)

To construct monopole-monopole pair configuration on the same axially symmetric ansatz \[ 8, 18 \], let us note that the multimonopoles can be nicely described in terms of the effective electromagnetic quantities \[ 8, 18 \], like magnetic charge

\[ g = \frac{1}{4\pi} \int \frac{1}{2} \text{Tr} (F_{ij} D_k \Phi) \varepsilon_{ijk} d^3 r \]

(10)
and the dimensionless magnetic dipole moment $\mu$, which is associated with asymptotic behavior of the function $K_3$ as $K_3 \to (1 - \cos \theta)/\sin \theta + \mu \sin \theta/r$ [8]. For a MA pair, for example, the integrated magnetic charge is zero, however the charge density distribution $g(x) = \frac{1}{2} \text{Tr} (F_{ij} D_k \Phi) \varepsilon_{ijk}$ is not trivial, it has a maximum associated with node of the Higgs field on positive $z$-axis and symmetrically located minimum associated with the node on negative $z$-axis. The MA pair has non-vanishing magnetic dipole moment which can be relatively good evaluated by consideration of the magnetic charges as point charges located at the nodes [8, 18]. On the other hand, the charge 2 monopole possess zero dipole moment since both nodes coincide.

Therefore, we can conjecture that the MM pair possesses zero dipole moment. This condition implies that the boundary conditions on the functions, which enter the component $A_\phi$ of the gauge potential, have to be modified and we impose

$$K_1 \to 0 \, , \quad K_2 \to -1 \, , \quad r^2 \partial_\theta K_3 \to 0 \, , \quad r^2 \partial_\theta K_4 \to 0 \, ,$$

We find that this modification yields new branch of gravitating MM solutions.

Finally, regularity on the $z$-axis requires $\partial_\theta f = \partial_\theta m = \partial_\theta l = 0$, whereas the matter field functions satisfy $K_1 = K_4 = \Phi_2 = 0$, $\partial_\theta K_2 = \partial_\theta K_4 = \partial_\theta \Phi_1 = 0$ for all these configurations.

### III. NUMERICAL RESULTS

Subject to the above boundary conditions [11], we solve the system of 9 coupled non-linear partial differential equation numerically in compact radial coordinate $x = r/(1 + r) \in [0 : 1]$. The numerical calculations are based on the Newton-Raphson iterative procedure [19].

For a small non-vanishing values of $\lambda$ and $\alpha$ we obtain the new static solution with two nodes of the Higgs field on the $z$-axis, which smoothly evolves as coupling constants begin to vary. Furthermore, there is a limiting branch of gravitating monopoles as $\lambda = 0$. These solutions are quite different from the known MA configurations (see Fig. 1), in particular, the metric function $f$ does not possess a minimum at the origin.

To confirm that these solutions can be interpreted as the monopole-monopole pair, let us consider the magnetic charge density defined by integrand in (10). For the configuration which we are analysing, the charge density remains positive everywhere. It has a shape of two tori those maxima form two rings in planes parallel to the $xy$-plane, intersecting the symmetry axis close to the nodes of the scalar field (see Fig. 2 left). The energy density of the configuration possesses two maxima on the $z$-axis associated with nodes of the scalar field (Fig. 2 right).

We can evaluate the charge by straightforward substitution of the numerical solutions into the definitions above, the calculation gives $g = 2$. Thus, the solution represent gravitationally bounded pair of monopoles. In the limiting case $\alpha \to 0$ the solution approaches flat space limit where the bound state ceases to exist and monopoles set to be free. Separation between the monopoles in this limit depends on the value of the Higgs self-coupling $\lambda$, it is minimal for BPS monopoles: $d_{min} = 6.53$ in scaled units. As $\alpha$ increases the monopoles approach the flat space limit on a larger separation, e.g., $d_{min} = 7.31$ for $\lambda = 0.1$. 

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Figure 1: The gauge field function $K_4$ (left) and the metric function of the lower branch monopole-monopole pair solution are shown at $\lambda = 0.5$, $\alpha = 0.05$.
MM configuration: charge density at $\lambda = 0.1$, $\alpha = 0.03$

MM configuration: energy density at $\lambda = 0.1$, $\alpha = 0.03$

Figure 2: The charge density (left) and the energy density (right) distributions of the monopole-monopole pair solution are shown as functions of the coordinates $z$, $\rho = r \sin \theta$ at $\lambda = 0.1$, $\alpha = 0.03$.

Figure 3: The nodes of the Higgs field are shown as functions of the coupling constant $\alpha$ for the MM pair and the MA pair solutions at $\lambda = 0$ (left). Also the scaled masses of the MM pair, the MA pair and the charge 2 monopole solution are shown as functions of the coupling constant $\alpha$ at $\lambda = 0$.

This branch of gravitating monopoles extends up to a maximal value $\alpha_{cr}$, where it merges with a second branch similar to the case of the gravitating MA pair [14]. As scalar coupling increases the $\alpha_{cr}$ decreases, e.g., for $\lambda = 0$ we have $\alpha_{cr} = 0.838$ while for $\lambda = 0.1$ $\alpha_{cr} = 0.616$ and for $\lambda = 0.5$ $\alpha_{cr} = 0.562$. The second branch extends back to $\alpha = 0$, as seen in Fig 3. In this limit the mass diverges and the configuration shrink to zero size. However, rescaling the coordinate $x \rightarrow \alpha x$ and the scalar field $\Phi \rightarrow \Phi / \alpha$ leads to a limiting solution with finite size and finite scaled mass [7]. The scaled mass of the MM pair and the nodes of the Higgs field as functions of the coupling constant $\alpha$ are exhibited in Fig 3. Note that there is a principal difference between the metric function $f$ of the gravitating MM pair and that of the MA pair: the minimum of the former function is clearly associated with position of nodes of the scalar field while the minimum of the latter function remains at the origin [7].

Considering the limit $\alpha \rightarrow 0$ on the upper branch in scaled coordinates, we observe that the solution may be thought of as composed of a scaled charge 2 monopole solution in the inner region and the second Bartnik-McKinnon solution with two nodes of the gauge field function in the outer region, as seen in Fig 4. The inner region shrinks to zero size as $\alpha \rightarrow 0$.

Comparing the solutions with the gravitating charge 2 monopole [13] we observe that on the lower mass branch, the mass of the monopole-monopole pair for the same value of $\lambda = 0$ is lower (see Fig 3). Since both configurations are in the same topological sector with charge 2, one may expect that gravitating 2-monopole is unstable. However, these solutions evolve differently, as $\alpha$ increases, the branch of gravitating 2-monopole approaches the extremal Reissner-Nordström black hole with magnetic charge 2 [12] while both gravitating MM pair and gravitating MA pair evolve...
back on the upper branch and they are linked to the Bartnik-McKinnon solutions of pure EYM theory.

In is instructive to compare the limiting behavior of these three different configurations, all possessing two nodes of the Higgs field. In Fig 5 we exhibit the corresponding gauge field functions $K_2$ and the metric functions $f$ for the upper branch MM pair solution, the upper branch MA pair solution and charge 2 monopole.

Concluding, we have found new static axially symmetric solutions of SU(2) EYMH theory, which represent gravitating monopole-monopole pairs. This results holds both for zero and for finite Higgs self-coupling [20]. We expect that there are also gravitating multiply magnetically charged solutions which are counterparts of the vortex rings discussed in [8]. Since we observe that for the MM pair the minimum of the metric function $f$ coincides with location of the monopoles, some interesting features of the black hole solutions related to these configuration may be observed. Study of these and other solutions of the new class, only simplest of those was discussed in this note, will be presented elsewhere [20].

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