Bounds on transport from univalence and pole-skipping

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Bounds on transport represent a way of understanding allowable regimes of quantum and classical dynamics. Numerous such bounds have been proposed, either for classes of theories, or, by using general arguments, universally for all theories. Few are exact and inviolable. I present a new set of methods for deriving exact, rigorous and sharp bounds on all coefficients of hydrodynamic dispersion relations, including diffusivity and the speed of sound. These general techniques combine analytic properties of hydrodynamics and the theory of univalent (complex holomorphic and injective) functions. Particular attention is devoted to bounds relating transport to quantum chaos, which can be established through pole-skipping in quantum field theories with classical holographic duals. Examples of such bounds are shown along with holographic theories that can demonstrate the validity of the necessary assumptions involved. I also discuss examples of bounds without relation to chaos.

Introduction.—The existence of bounds on properties of transport, such as diffusion, has persistently enthralled physicists concerned with time-dependent collective dynamics. Numerous bounds that improved our understanding of quantum and classical dynamics have been proposed. Among them is Sachdev’s relaxation time bound [1], the Mott-Ioffe-Regel limit of metallic conductivity [2, 3], lower bounds on diffusion and viscosity [4–9], upper bounds on diffusion [9, 10] and a bound on the speed of sound [11]. These bounds are heuristic and rely on basic physical principles such as the uncertainty principle and causality. Exact inequalities, even for restricted classes of theories are rare. An example is Prosen’s bound on diffusion [12]. Holographic methods to bound conductivities in disordered theories were developed in [13, 14]. Holographic advances in quantum chaos then led to the exact Maldacena-Shenker-Stanford bound on quantum Lyapunov exponents that follows from arguments of analyticity and complex analysis [15]. Another bound on the growth of (weak) quantum chaos was derived in [16].

Microscopic bounds, such as bounds on quantum chaos, should imply sharp bounds on collective transport. The purpose of this work is to introduce a new set of mathematical techniques from the theory of univalent functions, which allows for a rigorous derivation of exact inequalities of that type on diffusivity, the speed of sound and all higher-order coefficients of hydrodynamic dispersion relations. As is discussed below, because of their generality, univalence methods can also be applied to derive bounds without any reference to chaos.

Univalent functions.—A univalent (or schlicht) function \( f(z) \) is a complex holomorphic injective function. The condition of injectivity demands that \( f(z_1) \neq f(z_2) \) for all \( z_1 \neq z_2 \). Henceforth, all considered \( f(z) \) will be univalent in some simply connected region \( U \subset \mathbb{C} \). By the Riemann mapping theorem, it is then possible to map \( U \) to an open unit disk \( \mathbb{D} = \{ z \mid |z| < 1 \} \) in the complex \( \zeta \)-plane by a holomorphic invertible conformal map \( \varphi : \zeta = \varphi(z) \) and \( z = \varphi^{-1}(\zeta) \). As is conventional, we will use the normalisation \( f(\zeta = 0) = 0 \) and \( f'(\zeta = 0) = 1 \) for functions in the \( \zeta \)-plane. All such functions admit a power series representation of the following form:

\[
    f(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n. \tag{1}
\]

The series is guaranteed to converge for all \( |\zeta| < 1 \).

Locally, \( f(z) \) is univalent if \( f'(z) \neq 0 \). However, proving local univalence at every \( z \in U \) does not guarantee global univalence. Instead, one of numerous sufficient conditions for univalence must be employed [17, 18]. Once univalence is established and we have mapped \( U \to \mathbb{D} \), then we can resort to theorems bounding univalent functions on \( \zeta \in \mathbb{D} \), such as the growth theorem:

\[
    \frac{|\zeta|}{(1 + |\zeta|)^2} \leq |f(\zeta)| \leq \frac{|\zeta|}{(1 - |\zeta|)^2}, \tag{2}
\]

and the celebrated de Branges’s theorem (the Bieberbach conjecture) [19] constraining each coefficients of (1):

\[
    |b_n| \leq n, \quad \text{for all } n \geq 2. \tag{3}
\]

The inequalities (3) and the growth theorem (2) are saturated by the Koebe function (and its rotations in \( \zeta \)),

\[
    f_K(\zeta) = \frac{\zeta}{(1 - |\zeta|)^2} = \sum_{n=1}^{\infty} n \zeta^n, \tag{4}
\]

that conformally maps \( \mathbb{D} \to \mathbb{C} \setminus (-\infty, -1/4] \).

We will use the condition whereby if \( \text{Re} \ f'(z) > 0 \) in any convex \( U \subset \mathbb{C} \), then \( f(z) \) is univalent in \( U \) [20, 21]. If, moreover, after \( \varphi : U \to \mathbb{D} \), \( \text{Re} \ f'(\zeta) > 0 \), then \( f(\zeta) \) satisfies stronger versions of the theorems (2) and (3) [22]:

\[
    -|\zeta| + 2 \ln (1 + |\zeta|) \leq |f(\zeta)| \leq -|\zeta| - 2 \ln (1 - |\zeta|), \tag{5}
\]

\[
    |b_n| \leq \frac{2}{n}, \quad \text{for all } n \geq 2. \tag{6}
\]

Hydrodynamics.—Hydrodynamics is an effective theory of collective late-time and long-range excitations...
in fluids governed by conserved quantities such as energy, momentum and charges [23–34]. Linearised hydrodynamics predicts the structure of dispersion relations \( \omega(q^2) \), where \( \omega \) is the frequency and \( q^2 \) the momentum (squared) of a collective mode: diffusion or sound. In theories preserving spatial rotations, classical \( \omega(q^2) \) are infinite Puiseux series in the complex argument \( z \) [37, 38]:

\[
\omega_{\text{diff}}(z \equiv q^2) = -i \sum_{n=1}^{\infty} a_n z^n,
\]

(7)

\[
\omega_{\text{sound}}^\pm(z \equiv \sqrt{q^2}) = -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} z^n,
\]

(8)

where all \( a_n \), \( c_n \in \mathbb{R} \). We have \( c_1 = D \) (diffusivity) and \( a_1 = v_s \) (the speed of sound). Each series converges for \( |z| < R \equiv |z_s| \) with \( z = z_s \) being the first critical point of the associated complex curve [37, 38]. Each fully analytically continued function \( \omega(z) \) is holomorphic in the region \( z \in H \subset \mathbb{C} \), where \( H \) contains \( |z| < R \).

Different concepts of wave propagation speeds beyond \( v_s \) exist, such as the phase velocity \( v_{ph}(q) \equiv \omega/q \), the front velocity and the group velocity \( v_g(q) \equiv \partial \omega/\partial q \), where \( q \equiv \sqrt{q^2} \). Causality, for example, imposes certain conditions on these speeds (see Ref. [43]). In an analogous spirit, we will sometimes use properties of \( v_g \) to define the univalence region of hydrodynamics \( U \).

**General bounds.**—A hydrodynamic dispersion relation \( \omega(z) \) is by Puiseux’s theorem invertible at \( z = 0 \) and thus locally univalent at \( z = 0 \) [37, 38]. Beyond including \( z = 0 \) in all univalent regions \( U \subset H \), we assume that \( U \) also contains a point \( z = z_0 \) where \( \omega \equiv \omega(z_0) \) is known. \( U \) need not be maximal. A convenient way to choose \( U \) is through the sufficient condition \( \text{Re} f'(z) > 0 \), where \( f_{\text{diff}}(z) = i \omega_{\text{diff}}(z) \) and \( f_{\text{sound}}(z) = \omega_{\text{sound}}(z) \). This implies univalence for \( U = \{ z : |z| < \min(|z_g|, |R|) \} \), where

\[
\text{diff} : z_g = \frac{q_g^2}{2} \equiv \min q^2 \quad \text{Re} v_g \text{Im} |q_g| = \text{Im} v_g \text{Re} q_g \quad (9)
\]

\[
\text{sound} : z_g = q_g \equiv \min q \quad |\text{Re} v_g| = 0, \quad (10)
\]

expressed through the properties of the group velocity. If \( v_g \) vanishes at \( |z_g| \) smaller than those in (9) and (10), then univalence is lost locally due to \( f'(z_g) = 0 \). We have

\[
q_g \equiv \min q \quad v_g = 0. \quad (11)
\]

Using a conformal map \( \varphi : U \to \mathbb{D} \) with \( \varphi(z) = \zeta \) that preserves the origin, i.e. \( \varphi(0) = 0 \), we then define

\[
f_{\text{diff}}(\zeta) \equiv \frac{i \omega_{\text{diff}}(\varphi^{-1}(\zeta))}{D \partial \zeta \varphi^{-1}(\zeta)} = \zeta + \sum_{n=2}^{\infty} b_n^{\text{diff}} \zeta^n, \quad (12)
\]

\[
f_{\text{sound}}(\zeta) \equiv \frac{\omega_{\text{sound}}^+(\varphi^{-1}(\zeta))}{v_s \partial \zeta \varphi^{-1}(\zeta)} = \zeta + \sum_{n=2}^{\infty} b_n^{\text{sound}} \zeta^n. \quad (13)
\]

Both (12) and (13) have the form of (1). The growth theorem (2) applied at \( \zeta_0 \equiv \varphi(z_0) \) now yields lower and upper bounds on diffusivity and the speed of sound:

\[
\frac{\omega_0 |1 - |\zeta_0| \rangle^2}{|\zeta_0| |\partial \zeta \varphi^{-1}(0)|} \leq (D \lor v_s) \leq \frac{\omega_0 |1 + |\zeta_0| \rangle^2}{|\zeta_0| |\partial \zeta \varphi^{-1}(0)|}, \quad (14)
\]

where \((D \lor v_s)\) means either \( D \) or \( v_s \) depending on whether we used (12) or (13). If, in addition to univalence, \( \text{Re} f'(\zeta) > 0 \) for \( |\zeta| < 1 \), then (5) gives

\[
\frac{|\omega_0|}{|\partial \zeta \varphi^{-1}(0)|} \ln \frac{e^{-|\zeta_0|}}{(1 - |\zeta_0|)^2} \leq (D \lor v_s) \leq \frac{|\omega_0|}{|\partial \zeta \varphi^{-1}(0)|} \ln e^{-|\zeta_0|} (1 + |\zeta_0|)^2. \quad (15)
\]

To bound higher-order coefficients, we use the de Branges’s theorem (3) on each term of the series (12) or (13). This establishes a chain of inequalities on \( c_n \) or \( a_n \) in terms of all \( c_m \) or \( a_m \) with \( m < n \). For a diffusive dispersion relation (7), we first use \( |b_2| \leq 2 \) to bound \( c_2 \):

\[
c_2 + \frac{D}{2} \frac{\partial^2 \varphi^{-1}(0)}{|\partial \zeta \varphi^{-1}(0)|^2} \leq \frac{2D}{|\partial \zeta \varphi^{-1}(0)|}, \quad (16)
\]

further eliminating \( D \) through (14). Next, \( |b_2| \leq 3 \) is used to bound \( c_3 \) and so on for all \( c_n \geq 4 \). If \( \text{Re} f'(\zeta) > 0 \), then the bound (16) has another factor of \( 1/2 \) on the right-hand-side due to \( |b_2| \leq 1 \) in Eq. (6). An analogous procedure can be used for bounding \( a_n \) by \( v_s \) and \( \varphi \). All bounds are determined purely in terms of a single known \( \omega_0(z_0) \) and the chosen original region of univalence \( U \) through the conformal map \( \varphi : U \to \mathbb{D} \).

**Quantum chaos and pole-skipping.**—Of particular interest are bounds that stem from the underlying microscopic quantum chaos. While the general relation between transport and chaos is unknown, precise connection has been established through the phenomenon of pole-skipping in quantum field theories with a large number of local degrees of freedom (large-\( N \) theories) that possess a classical gravitational holographic dual [44–47].

Pole-skipping is an indeterminacy of two-point functions associated with dispersion relations (7)–(8). In the longitudinal channel of energy-momentum fluctuations (e.g. sound or energy diffusion), pole-skipping implies

\[
\omega_0(q_0^2) = i \lambda_L, \quad q_0^2 = -\lambda_L^2/v_B^2. \quad (17)
\]

Hence, for such modes, we have \( q_0 = i \lambda_L/v_B \). Here, \( \lambda_L \) is the maximal Lyapunov exponent \( \lambda_L = 2\pi T \), \( T \) is the temperature, and \( v_B \) the butterfly velocity characterising the exponential growth of the out-of-time-ordered correlator used to probe chaos, \( e^{\lambda_L(t-|x|)/v_B} \) [15, 48].

In neutral theories, a related expression exists also for transverse fluctuations (e.g. momentum diffusion) [38, 49]:

\[
\omega_0(q_0^2) = -i \lambda_L, \quad q_0^2 = \lambda_L^2/v_B^2. \quad (18)
\]
In charged theories, pole-skipping (18) at \( \omega_0 = -i\lambda_L \) generically exhibits a more complicated \( q_0^2 \). Since the pole-skipping points can be easily computed from dual gravity, and they relate chaos to transport, we will use them as \( \omega_0(z_0) \) in most bounds below.

**Diffusion I: Maximal univalence.**—In our first, simple and very special example, assume that a diffusive dispersion relation \( \omega(z) = \omega_{\text{diff}}(z) \) (cf. Eq. (7)) is maximally univalent \((U = H)\) and holomorphic on the entire \( z \in \mathbb{C} \) except at a branch point \( z_* \) and at \( z = \infty \). We define \( \omega_* \equiv \omega(z_*) \). Under \( \text{Im} \ z \to -\text{Im} \ z \), \( \text{Re} \ \omega \) is odd and \( \text{Im} \ \omega \) is even. To have a single \( z_* \), we need \( \text{Re} \omega_* = 0 \), hence \( z_* \in \mathbb{R} \). For concreteness, we take \( z_* > 0 \) and choose the branch cut so that \( U = \mathbb{C} \setminus [z_*, \infty) \). \( R = z_* \) is the radius of convergence of the hydrodynamic series (7).

We first use a rescaling M"obius transformation to map \( z_* \to -1/4 \), keeping \( z = \infty \) at \( \infty \). The branch cut is now chosen to lie along \((-\infty, -1/4] \). Next, we use an inverse of the Koebe function (4) to map \( \mathbb{C} \setminus (-\infty, -1/4] \to \mathbb{D} \). The full conformal map \( \varphi : U \to \mathbb{D} \) is thus

\[
\zeta = \varphi(z) = \frac{z - 2z_* + 2\sqrt{z_*^2 - z_z}}{z},
\]

\[
z = \varphi^{-1}(\zeta) = -4z_* f_K(\zeta) = -\frac{4z_* + (1 - \zeta)^2}{(1 + \zeta)^2},
\]

with \( \partial^b \varphi^{-1}(0) = -4n^2(n-1)!R \). Using the pole-skipping relations (17) or (18), the diffusivity bounds (14) become

\[
z_0 = -\frac{\lambda^2}{v_B^2} < 0 : \quad \frac{v_B^2}{\lambda_L} \leq D \leq \frac{v_B^2}{\lambda_L} + \frac{\lambda_L}{R},
\]

\[
0 < z_0 = \frac{\lambda^2}{v_B^2} < R : \quad \frac{v_B^2}{\lambda_L} - \frac{\lambda_L}{R} \leq D \leq \frac{v_B^2}{\lambda_L}.
\]

Since \([z_*, \infty) \notin U\), we do not consider \( z_0 \geq R \). Eqs. (21) and (22) correspond to the longitudinal (energy diffusion) and, assuming (18), the transverse (momentum diffusion) channels, respectively. The inequalities are fixed by pole-skipping and the radius of convergence. The lower bound in (21) and the upper bound in (22) have the form of the relation between \( D \) and \( v_B^2/\lambda_L \) first noticed by Blake [6]. Moreover, our results imply that if a univalent diffusive \( \omega(z) \) is entire (holomorphic everywhere except at infinity, so that \( R \to \infty \)), then

\[
D = \frac{v_B^2}{\lambda_L}.
\]

Using Eq. (16) for general \( R \), we can now find bounds on \( c_2 \) (a third-order hydrodynamic coefficient [50, 51]):

\[
0 \leq c_2 \leq \frac{D}{R}.
\]

The upper bound from either (21) or (22) eliminates \( D \) from (24). Simple algebraic manipulations give further bounds on \( c_3 \), \( c_4 \) and so on. If we can take \( R \to \infty \), then \( c_2 = 0 \). Moreover, all \( c_{n>2} = 0 \) in this limit. Hence, for entire univalent \( \omega_{\text{diff}}(z) \), the dispersion relation truncates at first order for all \( q_0^2 \), with \( D \) fixed by pole-skipping:

\[
\omega_{\text{diff}}(q_0^2) = -iDq_0^2 = -i\frac{v_B^2}{\lambda_L} q_0^2.
\]

A theory that exhibits diffusive properties discussed here is a holographic model with broken translational invariance and energy diffusion [52]. At a special point in the parameter space of the background fields, symmetry enhancement allows to analytically find the exact diffusive \( \omega(z) \) [53]. Pole-skipping and hydrodynamic convergence in this theory were studied in [38, 46], finding \( z_0 = -8\pi^2 T^2 \), \( v_B^2 = 1/2 \) and \( z_* = R = \pi^2 T^2 \). The bounds from (21) and (24), and also for \( c_3 \), are then

\[
\frac{v_B^2}{\lambda_L} = \frac{1}{4\pi T} \leq D \leq \frac{9v_B^2}{4\pi T} = \frac{9}{4\pi T},
\]

\[
0 \leq c_2 \leq \frac{D}{\pi^2 T^2} \leq \frac{9}{4\pi^3 T^3},
\]

\[
-\frac{27}{32\pi^5 T^5} \leq -3\frac{D}{8\pi^4 T^4} \leq c_3 \leq \frac{D}{\pi^2 T^2} \leq \frac{9}{4\pi^5 T^5}.
\]

The actual values of \( D = 1/2\pi T \), \( c_2 = 1/8\pi^3 T^3 \) and \( c_3 = 1/16\pi^5 T^5 \) all satisfy the inequalities.

**Diffusion II: M"obius transformations.**—A general diffusive dispersion relation has multiple branch points and branch cuts. Generalising the scenario in which \( U \) is determined by the group velocity conditions (9)–(11), let \( U \) of \( \omega_{\text{diff}}(z) \) be a disk with a centre at \( z = z_c \) and two boundary points at \( z = z_c \pm z_b \) (on its closure), containing \( z = 0 \) and \( z = z_0 \), and with \( z_c \in \mathbb{C} \) and \( z_0 \in \mathbb{R}_+ \). \( U \) can be mapped to \( \mathbb{D} \) by the M"obius transformation \( \zeta = \varphi(z) \), which we choose to be

\[
\varphi(z) = \frac{z_b z - z_c z^2 + z_c^2}{z_b z^2 + z_c^2},
\]

\[
\varphi^{-1}(\zeta) = \frac{\zeta^2 + z_c^2}{z_b^2 + z_c^2},
\]

satisfying \( \varphi(0) = 0 \) and mapping \( z_c \pm iz_b \to \pm i \). We have \( \partial^b \varphi^{-1}(0) = n!(-z_c)^{n-1}(z_b^2 + z_c^2)/z_b^n \). All of the above bounds can now be easily constructed given specific \( z_0 \), \( z_b \) and \( z_c \). For example, Eq. (14) becomes

\[
\frac{v_B^2}{\lambda_L} \left| 1 - \frac{z_c z_0}{z_b^2 + z_c^2} \right| C_- \leq D \leq \frac{v_B^2}{\lambda_L} \left| 1 - \frac{z_c z_0}{z_b^2 + z_c^2} \right| C_+, \tag{30}
\]

where \( z_0 = \pm \lambda^2/v_B^2 \), depending on whether we use (17) or (18). \( C_+ \) are defined as

\[
C_+ \equiv (1 + |\zeta_0|^2)^2, \quad |\zeta_0| = \frac{\lambda^2}{v_B^2} \left| \frac{z_b}{-z_c z_0 + z_b^2 + z_c^2} \right|. \tag{31}
\]

Of particular interest are cases with \( z_c = 0 \) so that \( \varphi \) rescales a disk of radius \( z_c = \min(|z_c|, R) \) to \( \mathbb{D} \). The
only non-zero \( \partial_c^2 \varphi^{-1}(0) \) is then \( \partial_c \varphi^{-1}(0) = z_b \), and \( b_n = z_b^{n-1} c_n / D \) for \( n \geq 2 \). The bounds on (7) follow:

\[
\frac{v_B^2}{\lambda_L} \left( 1 - \frac{\lambda_t^2}{v_B} \right)^2 \leq D \leq \frac{v_B^2}{\lambda_L} \left( 1 + \frac{\lambda_t^2}{v_B} \right)^2,
\]

\[
-\frac{nD}{z_b} \leq c_n \leq \frac{nD}{z_b}.
\]

As required, in the \( z_b \to \infty \) limit, we again recover the exact dispersion relation (25). If univalence of \( f(\zeta) \) is ensured by \( \text{Re } f'(\zeta) > 0 \), then (32) and (33) are improved:

\[
\frac{\lambda_L / z_b}{\ln \left( \frac{1 - \lambda_t^2/z_b v_B}{1 - \lambda_t^2} \right)} \leq D \leq \frac{\lambda_L / z_b}{\ln \left( \frac{1 - \lambda_t^2/z_b v_B}{1 + \lambda_t^2/z_b v_B} \right)}.
\]

\[
-\frac{2D}{n z_b} \leq c_n \leq \frac{2D}{n z_b}.
\]

If \( z_b \to \infty \), \( \omega_{\text{diff}}(q^2) \) still reduces to the form in Eq. (25).

To demonstrate the existence of such theories, we consider momentum diffusion in two strongly coupled, large-\( N \) theories at finite temperature: 3d worldvolume theory of M2 branes and 4d \( N = 4 \) supersymmetric Yang-Mills (SYM) theory. Diffusive \( \omega_{\text{diff}}(z) \) is determined by dual transverse metric fluctuations in 4d [54] and 5d [55] Einstein-Hilbert theories with a negative cosmological constant and Anti-de Sitter-Schwarzschild black brane backgrounds. We check numerically that in both theories, \( \text{Re } f'(z) > 0 \) on the entire disks of hydrodynamic convergence, establishing univalence for \( |z| < z_b = R \). For the \( N = 4 \) SYM diffusion, we depict this in Fig. 1. The 3d M2 brane case qualitatively matches the plot in Fig. 1, with \( R \approx 69.423 T^2, \lambda_L = 2\pi T \) and \( v_B = \sqrt{3}/2 \).

In 4d \( N = 4 \) SYM theory, \( R \approx 87.800 T^2, \lambda_L = 2\pi T \) and \( v_B = \sqrt{2}/3 \) [37, 38]. Given these values, we can numerically verify the validity of the bounds (34)–(35). For example, (34) evaluates to \( 0.046 / T < D < 1/4 \pi T < 0.080 / T \leq 0.201 / T \). Moreover, the bounds become extremely tight as \( n \) grows. Assuming that \( c_{n-\infty} \) become of the order of the bounds is consistent with the ratio test for convergence then giving \( \lim_{n \to \infty} |c_n / c_{n+1}| = z_b \), which is the radius of convergence of (7).

**Sound.**—By extending our holographic analysis to sound in the \( N = 4 \) SYM theory, we find that \( \text{Re } f'(z) \neq 0 \) on the hydrodynamic convergence disk \( |z| < R \), where \( R = 2\sqrt{2/\pi} T \approx 8.886 T \) [37, 38]. Instead, \( \text{Re } f'(z) > 0 \) for \( |z| < |z_g| < R \), with \( z_g = q_g \) determined by the local condition (11). We depict the univalence condition in Fig. 2.

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4 Convergence of hydrodynamics in the holographic M2 brane theory is analysed by using the methods from Refs. [37, 38]. The transverse channel pole-skipping (18) follows from the methods of [38, 46, 49]. We prove analytically that \( \omega_{\text{diff}}(z) \) passes through an infinite sequence of pole-skipping points: \( \omega_{\text{diff}}(z_n) = -2\pi T \ln \left( -q_n^{\text{tr}} = 16\pi^2 T^2 \sqrt{3} \right) \) for all \( n \in Z_+ \cup \{0\} \).

**FIG. 1.** The univalence condition \( \text{Re } f'(\zeta) \), with \( \zeta = |\zeta| e^{i \phi} \), plotted as a function of \( \phi \) for momentum diffusion in \( N = 4 \) SYM theory. The colour gradient indicates different \( |\zeta| \), from \( |\zeta| = 0 \) (red) to \( |\zeta| = 0.92 \) (blue). We find that \( \text{Re } f'(\zeta) > 0 \) for all \( |\zeta| < 1 \), with \( |\zeta| = 1 \) mapped by \( \varphi \) from \( |z| = z_b = \|z_g\| \), where \( v_B(z_g) = 0 \).

**FIG. 2.** \( \text{Re } f'(\zeta) \), with \( \zeta = |\zeta| e^{i \phi} \), plotted for sound in \( N = 4 \) SYM theory. The colour gradient runs from \( |\zeta| = 0 \) (red) to \( |\zeta| = 1 \) (blue), with \( |\zeta| = 1 \) mapped by the \( z_c = 0 \) Möbius transformation \( \varphi \) from \( |z| = z_b = \|z_g\| \), where \( v_B(z_g) = 0 \).

2. Numerically, we find that \( z_g \approx -3.79 i T \). Since \( z_g \) lies within the hydrodynamic radius of convergence, its value can be crudely approximated by conformal first-order hydrodynamics: \( z_g \approx -3i v_s / AD = -5.441 i T \) with \( v_s = 1/\sqrt{3} \) and \( D = 1/4 \pi T \).

A crucial difference between this case and diffusion above is that the pole-skipping \( z_0 = i \lambda_L / v_B \) (cf. Eq. (17)) is no longer in the \( |z| < |z_g| \) disk of univalence \( U \) (i.e., \( |z_g| < |z_0| = \lambda_L / v_B \approx 7.695 T \)). However, it can be checked numerically that another univalent disk \( z \in U \) can be chosen with \( z_c \approx 2.548 i T \) and \( z_b \approx 6.338 T \) (cf. Eq. (29)). The bounds on \( \omega_{\text{sound}}(z) \) then follow from Eqs. (14) and (3) (not (15) and (6)) and \( \text{Re } f'(\zeta) \neq 0 \) for all \( |\zeta| < 1 \) after \( \varphi : U \to \mathbb{D} \), with \( |\zeta| = \lambda_L \) and \( \zeta_0 \) and the derivatives of \( \varphi^{-1}(0) \) computable from (29).

The maximally univalent sound analogue of (25) is recovered when \( z_c = 0 \) and \( z_b \to \infty \). Then, we find an exact truncated dispersion relation \( \omega_{\text{sound}}(q) = \pm v_B q \).
Bounds without pole-skipping.—In the absence of pole-skipping considerations, we can derive bounds on transport purely in terms of the wave propagation speeds. For $U = \{ z \mid |z| < \min||z_g|, R \}$, with $z_g$ given by the group velocity conditions (9)–(10) or (11), it follows that if the limit $|\zeta_0| \to 1$ exists, then Eq. (14) implies the following bounds expressed in terms of the phase velocities and momentum $\hat{q}$, such that $|\hat{q}| = \min||q_g|, |q_s||$

$$0 \leq D \leq 4 |v_{ph}(q^2)|/\hat{q}, \quad 0 \leq v_s \leq 4 |v_{ph}(\hat{q})|. \quad (36)$$

If we can use the inequalities from Eq. (15), then 4 in the upper bounds is improved to $1/(2\ln 2 - 1)$. Higher-order coefficients are bounded either by Eq. (3) or (6).

For the final example, assume that a class of theories has univalence properties of sound whereby $|\partial_k \varphi^{-1}(0)| = 4|\omega_0(z_0)|\sqrt{d-1}$, with $d$ the number of spacetime dimensions. Moreover, assume that $\zeta_0 = \varphi(z_0)$ is infinitesimally close to the boundary of $\mathbb{D}$ and that $|\zeta_0| \to 1$ again exists. The growth theorem (14) then implies the following conformal upper bound on the speed of sound: $0 \leq v_s \leq \sqrt{1/(d-1)} \quad (11)$. It would be intriguing to find physical examples of theories, such as quantum chromodynamics [56], that satisfy the necessary univalence property, particularly in relation to their equations of state.

Discussion.—To use the above construction of bounds, one must first establish univalence in $U$. Generally, as stated in Eqs. (9)–(11), hydrodynamic dispersion relation will be univalent up to at least the physically-motivated group velocity conditions in complicated momentum space. In holographic theories, this can be checked explicitly by numerical calculations. Finding more efficient methods, possibly by directly using the associated bulk differential equations remains an open problem. Another open problem is to explore univalence and the emergent bounds in weakly coupled theories and kinetic theory, where, as with considerations of the radius of convergence [37], we expect the regions of univalence to become smaller and bounds less tight.

While pole-skipping was chosen in most examples due to our interest in relating bounds on transport to quantum chaos, as well as for convenience, any known value of $\omega_0(z_0)$ in $U$ could also have been chosen. Two such examples were provided in the last section. Further simple examples can arise from the pole-skipping points without a clear connection to chaos (see Footnotes 3 and 4). In fact, such choices may lead to more restrictive bounds. This naturally opens a general problem to find the tightest possible bounds within the scope of univalence techniques. As the univalence methods help pave the way towards more precise analytic explorations of transport, these and other questions will be addressed in the future.

Acknowledgements.—I am grateful to Mike Blake, Richard Davison, Pavel Kovtun, Hong Liu, Andrei Starinets and Petar Tadić for useful and stimulating discussions on related topics. This work was supported by the U.S. DOE grant DE-SC0011090.

[1] S. Sachdev, Quantum Phase Transitions, 2nd ed. (Cambridge University Press, 2011).
[2] A. Ioffe and A. Regel, Prog. Semicond 4, 237 (1960).
[3] N. Mott, Philosophical Magazine 26, 1015 (1972).
[4] P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. 94, 111601 (2005), arXiv:hep-th/0405231.
[5] S. A. Hartnoll, Nature Phys. 11, 54 (2015), arXiv:1405.3651 [cond-mat.str-el].
[6] M. Blake, Phys. Rev. Lett. 117, 091601 (2016), arXiv:1603.08510 [hep-th].
[7] J. Zaanen, SciPost Phys. 6, 061 (2019), arXiv:1807.10951 [cond-mat.str-el].
[8] K. Trachenko and V. Brazhkin, Science Advances 6, eaba3747 (2020).
[9] M. Baggioli and W.-J. Li, (2020), arXiv:2005.06482 [hep-th].
[10] T. Hartman, S. A. Hartnoll, and R. Mahajan, Phys. Rev. Lett. 119, 141601 (2017), arXiv:1706.00019 [hep-th].
[11] A. Cherman, T. D. Cohen, and A. Nellore, Phys. Rev. D 80, 066003 (2009), arXiv:0905.0903 [hep-th].
[12] T. Prosen, Physical Review E 89 (2014), 10.1103/physrev.e89.012142.
[13] S. Grozdanov, A. Lucas, S. Sachdev, and K. Schalm, Phys. Rev. Lett. 115, 221601 (2015), arXiv:1507.00003 [hep-th].
[14] S. Grozdanov, A. Lucas, and K. Schalm, Phys. Rev. D 93, 061901 (2016), arXiv:1511.05970 [hep-th].
[15] J. Maldaecna, S. H. Shenker, and D. Stanford, JHEP 08, 106 (2016), arXiv:1503.01409 [hep-th].
[16] I. Kukuljan, S. Grozdanov, and T. Prosen, Phys. Rev. B96, 060301 (2017), arXiv:1701.09147 [cond-mat.stat-mech].
[17] P. Duren, Univalent Functions, Grundliche der mathematischen Wissenschaften (Springer New York, 2010).
[18] O. Lehto, Univalent Functions and Teichmüller Spaces, Graduate Texts in Mathematics (Springer New York, 2011).
[19] L. Branges, Acta Math. 154, 137 (1985).
[20] K. Noshiro, Journal of the Faculty of Science Hokkaido Imperial University. Ser. 1 Mathematics 2, 129 (1934).
[21] S. E. Warschawski, Transactions of the American Mathematical Society 38, 310 (1935).
[22] T. H. Macgregor, Transactions of the American Mathematical Society 104, 532 (1962).
[23] L. Landau and E. Lifshits, Fluid Mechanics (Pergamon Press, New York, 1987).
[24] P. Kovtun, INT Summer School on Applications of String Theory Seattle, Washington, USA, July 18-29, 2011, J. Phys. A45, 473001 (2012), arXiv:1205.5040 [hep-th].
[25] S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, Phys. Rev. D85, 085029 (2012), arXiv:1107.0731 [hep-th].
[26] S. Grozdanov and J. Polonyi, Phys. Rev. D91, 105031 (2015), arXiv:1305.3670 [hep-th].
[27] M. Crossley, P. Glorioso, and H. Liu, JHEP 09, 095 (2017), arXiv:1511.03647 [hep-th].
[28] P. Glorioso, M. Crossley, and H. Liu, JHEP 09, 096 (2017), arXiv:1701.07817 [hep-th].
[29] F. M. Haehl, R. Loganayagam, and M. Rangamani, JHEP 01, 184 (2016), arXiv:1510.02494 [hep-th].
[30] F. M. Haehl, R. Loganayagam, and M. Rangamani, JHEP 04, 039 (2016), arXiv:1511.07809 [hep-th].
[31] K. Jensen, N. Pinzani-Fokeeva, and A. Yarom, JHEP 09, 127 (2018), arXiv:1701.07436 [hep-th].

[32] H. Liu and P. Glorioso, Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: Physics at the Fundamental Frontier (TASI 2017): Boulder, CO, USA, June 5-30, 2017, PoS TASI2017, 008 (2018), arXiv:1805.09331 [hep-th].

[33] S. Grozdanov, D. M. Hofman, and N. Iqbal, Phys. Rev. D 95, 096003 (2017), arXiv:1610.07392 [hep-th].

[34] X. Chen-Lin, L. V. Delacretaz, and S. A. Hartnoll, Phys. Rev. Lett. 122, 091602 (2019), arXiv:1811.12540 [hep-th].

[35] P. Kovtun and L. G. Yaffe, Phys. Rev. D68, 025007 (2003), arXiv:hep-th/0303010 [hep-th].

[36] L. V. Delacretaz, (2020), arXiv:2006.01139 [hep-th].

[37] S. Grozdanov, P. K. Kovtun, A. O. Starinets, and P. Tadić, Phys. Rev. Lett. 122, 251601 (2019), arXiv:1904.01018 [hep-th].

[38] S. Grozdanov, P. K. Kovtun, A. O. Starinets, and P. Tadić, Phys. Rev. Lett. 111, 097 (2019), arXiv:1904.12862 [hep-th].

[39] B. Withers, JHEP 06, 059 (2018), arXiv:1803.08058 [hep-th].

[40] M. P. Heller, A. Serantes, M. Spaliński, V. Svensson, and B. Withers, (2020), arXiv:2007.05524 [hep-th].

[41] N. Abbasi and S. Tahery, (2020), arXiv:2007.10024 [hep-th].

[42] A. Jansen and C. Pantelidou, (2020), arXiv:2007.14418 [hep-th].

[43] E. Krotscheck and W. Kundt, Communications in Mathematical Physics 60, 171 (1978).

[44] S. Grozdanov, K. Schalm, and V. Scopelliti, Phys. Rev. Lett. 120, 231601 (2018), arXiv:1710.00921 [hep-th].

[45] M. Blake, H. Lee, and H. Liu, JHEP 10, 127 (2018), arXiv:1801.00010 [hep-th].

[46] M. Blake, R. A. Davison, S. Grozdanov, and H. Liu, JHEP 10, 035 (2018), arXiv:1809.01169 [hep-th].

[47] S. Grozdanov, JHEP 01, 048 (2019), arXiv:1811.09641 [hep-th].

[48] S. H. Shenker and D. Stanford, JHEP 03, 067 (2014), arXiv:1306.0622 [hep-th].

[49] M. Blake, R. A. Davison, and D. Vegh, JHEP 01, 077 (2020), arXiv:1904.12883 [hep-th].

[50] S. Grozdanov and N. Kaplis, Phys. Rev. D93, 066012 (2016), arXiv:1507.02461 [hep-th].

[51] S. M. Diles, L. A. Mamani, A. S. Miranda, and V. T. Zanchin, JHEP 05, 019 (2020), arXiv:1909.05199 [hep-th].

[52] T. Andrade and B. Withers, JHEP 05, 101 (2014), arXiv:1311.5157 [hep-th].

[53] R. A. Davison and B. Gouteraux, JHEP 01, 039 (2015), arXiv:1411.1062 [hep-th].

[54] C. P. Herzog, JHEP 12, 026 (2002), arXiv:hep-th/0210126.

[55] G. Policastro, D. T. Son, and A. O. Starinets, JHEP 09, 043 (2002), arXiv:hep-th/0205052.

[56] E. Annala, T. Gorda, A. Kurkela, J. Nättilä, and A. Vuorinen, Nature Phys. (2020), 10.1038/s41567-020-0914-9, arXiv:1903.09121 [astro-ph.HE].