Generation of Electro and Magneto Static Solutions of the Scalar-Tensor Theories of Gravity

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Abstract

The field equations of the scalar-tensor theories of gravitation are presented in different representations, related to each other by conformal transformations of the metric. One of the representations resembles the Jordan-Brans-Dicke theory, and is the starting point for the generation of exact electrostatic and magnetostatic exterior solutions. The corresponding solutions for each specific theory can be obtained by transforming back to the original canonical representation, and the conversions are given for the theories of Jordan-Brans-Dicke, Barker, Schwinger, and conformally invariant coupling. The electrostatic solutions represent the exterior metrics and fields of configurations where the gravitational and electric equipotential surfaces have the same symmetry. A particular family of electrostatic solutions is developed, which includes as special case the spherically symmetric solutions of the scalar-tensor theories. As expected, they reduce to the well-known Reissner-Nordström metric when the scalar field is set equal to a constant. The analysis of the Jordan-Brans-Dicke metric yields an upper bound for the mass-radius ratio of static stars, for a class of interior structures.

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I Introduction and Summary

The scalar-tensor theories of gravitation\textsuperscript{1–4} differ from Einstein’s general relativity since they admit the existence of a long range scalar field coupled to matter in such a way that the theories satisfy the weak principle of equivalence, and therefore are viable alternatives for the explanation of gravitational phenomena. However, the strong principle of equivalence is not valid, since the scalar field connects the local physics with the mass-energy of the rest of the universe. In fact, the Jordan-Brans-Dicke[JBD] theory, which has the simplest scalar coupling, was formulated as a relativistic model allowing a better representation of Mach’s principle and Dirac’s ideas concerning the connection between the Newtonian gravitational constant and the age of the universe\textsuperscript{5}. The roots of the scalar-tensor theories go back to the original Kaluza-Klein five dimensional unification of the gravitational and electromagnetic forces, and the modifications by Einstein-Mayer\textsuperscript{6}, Jordan\textsuperscript{7}, and Thiry\textsuperscript{8}. The five dimensional metric of these theories has five additional degrees of freedom, where four of them serve to represent the electromagnetic vector potential while the other behaves as a scalar field in a four-dimensional geometry. Thus we have that a scalar field appears as a natural outcome of the intriguing Kaluza-Klein unification and, furthermore, is the simplest long range field by means of which the matter distribution of the universe can affect local physics. The scalar-tensor formalism is able to incorporate this hypothetical but interesting interaction consistently and in agreement with experiments, and hence opens a broader theoretical framework to approach the study of gravitation.

Aside from the physical and philosophical motivations for considering the existence of
scalar fields, they could also provide heuristic representations of matter fields which, due to their mathematical simplicity, help us gain physical insights. For instance, it is well known that an isentropic fluid with pressure equal to energy density can be represented by a scalar field. Moreover, they could also be interpreted as large perturbations, serving in the study of the stability of general relativity backgrounds.

There are several alternative representations of the scalar-tensor theories which are related to the original canonical representation, Eqs. (2.1) and (2.2), by a conformal transformation of the metric, where the conformal factor is a function of the scalar field. For example, in the Einstein-scalar representation the new metric, $\bar{g}_{\mu\nu}$, and the original metric, $g_{\mu\nu}$, are related by

$$\bar{g}_{\mu\nu} = \phi g_{\mu\nu}.$$  

The field equations for $\bar{g}_{\mu\nu}$, given by Eqs. (2.6) and (2.7), are as in Einstein’s theory but with an added energy-momentum tensor for the scalar field. A more general transformation is

$$\tilde{g}_{\mu\nu} = \frac{\phi}{\tilde{\phi}} g_{\mu\nu},$$

where the new scalar field $\tilde{\phi}$ is an arbitrary function of $\phi$, and we arrived at the field equations (2.12) and (2.13) which are remarkably similar to the originals. Alternatively, we can reexpress the Einstein-scalar representation using now $\tilde{\phi}$ instead of $\phi$ as the scalar field to obtain Eqs. (2.17) and (2.18). A particular choice of the function $\tilde{\phi}(\phi)$ makes the theory look similar to the JBD theory in the sense that the second order equations for the metric, Eqs. (2.12) and (2.17), take the same form as the corresponding one in the JBD theory, with a constant.
parameter, $\omega_0$, playing the role of the scalar field coupling constant. When $\omega_0$ is chosen to be zero the theory can be written as a five dimensional Einstein’s field equations in which the geometry is independent of the fifth dimension (Eqs. (2.22) - (2.28)).

The representation $\bar{g}_{\mu\nu}, \tilde{\phi}$, is suitable for the mathematical analysis, and therefore is the starting point in Section III for the generation of exact scalar-tensor electrostatic solutions, starting from given static vacuum metrics in general relativity. This mapping gives the exterior fields of configurations of arbitrary shape, for which the surfaces of constant electrostatic and gravitational potential coincide. They can be converted into new solutions of the original canonical representation, $\phi, g_{\mu\nu}$, for each particular theory, and the necessary transformations are then given for the specific theories of JBD, Barker, Schwinger, and for conformally invariant coupling. Solutions representing the exterior gravitational, electric, and scalar fields of oblates or prolates configurations are presented explicitly in Section IV. These solutions reduce to the Reissner-Nordström metric when the scalar field is set equal to a constant and the configuration is taken to be spherically symmetric.

The spherically symmetric solutions are study in Section V, where it is found that, in contrast to general relativity, in the scalar-tensor theories the analysis of the exterior solution alone is not sufficient to establish and upper bound for the mass-radius ratio of the static configurations, and it is necessary to consider the interior configurations as well. This is done in detail for the JBD theory, where an upper bound is determined for equations of state where the volume integral of the trace of the energy-momentum tensor is positive.

In the last section, exact scalar-tensor magnetostatic solutions are given in terms of
vacuum metrics in general relativity. Then an explicit family of solutions is worked out, that represents prolates or oblates configurations immersed in magnetic and scalar fields. These solutions are scalar-tensor theories generalizations of Melvin’s magnetized metric of general relativity.

II The Field Equations

The scalar-tensor theories field equations, without a cosmological term, can be written in the following canonical form:

\[ \phi G_{\mu\nu} = 8\pi GT_{\mu\nu} + \frac{\omega}{\phi} \left[ \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi_{,\alpha} \right] + \phi_{;\mu\nu} - g_{\mu\nu} \phi_{;\alpha} \phi_{,\alpha}, \]

\[ (2\omega + 3) \phi_{,\alpha} = \frac{2\omega + 3}{\sqrt{-g}} \left[ \sqrt{-g} \phi_{,\alpha} \right]_{,\alpha} = 8\pi GT_{\alpha}^{\alpha} - \phi_{,\alpha} \phi_{,\sigma} \frac{d\omega}{d\phi}, \]

where, as usual, commas and semicolons mean partial and covariant derivatives, respectively, \( G_{\mu\nu} \) is the Einstein tensor, and \( T_{\mu\nu} \) is the energy momentum tensor for all fields excluding the scalar, \( \phi \), and the metric, \( g_{\mu\nu} \), fields. We are essentially adopting the notation convention of Misner-Thorne and Wheeler\(^9\). It follows from Eqs. (2.1), (2.2) and the Bianchi identities that we still have, as it is in general relativity, the energy-momentum conservation:

\[ T^{\mu\nu}_{;\nu} = 0. \]

Thus, in this representation, the scalar field does not enter the equation of motion, and consequently free falling test particles move in geodesics of the metric \( g_{\mu\nu} \). With the simplest choice of coupling \( \omega = \text{const} \), we get the well known JBD field equations. Other examples
of interesting theories can be defined by
\[ \omega = \frac{1}{2} \left[ \frac{k_1}{k_2 \phi + k_3} - 3 \right], \quad (2.4) \]
where \( k_1, k_2 \) and \( k_3 \) are constants. For instance, if \( k_1 = k_2 = -k_3 = 1 \), we have the theory due to Barker\(^{10}\), while the case \( k_3 = 0 \) was motivated by Schwinger\(^{11}\). Another attractive model is the conformally invariant curvature coupling\(^1\), in which \( k_1 = 3, k_2 = -1, \) and \( k_3 = 1 \).

An alternative representation of these theories is provided by the conformal transformation\(^{12}\)
\[ \bar{g}_{\mu \nu} = \phi g_{\mu \nu}, \quad (2.5) \]
which gives rise to the Einstein’s scalar form of the field equations:
\[ \bar{G}_{\mu \nu} = 8\pi \bar{T}_{\mu \nu} + \left( \omega + \frac{3}{2} \right) \left[ \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} \bar{g}_{\mu \nu} \Phi_{,\alpha} \Phi_{,\alpha} \right], \quad (2.6) \]
\[ \frac{2\omega + 3}{\sqrt{-g}} \left[ \sqrt{-g} \Phi_{,\alpha} \right]_{,\alpha} = 8\pi \bar{T}_{\alpha}^{\alpha} - \Phi_{,\alpha} \Phi_{,\alpha} \frac{d\omega}{d\Phi}. \quad (2.7) \]
where
\[ \Phi \equiv \ell n\phi, \quad (2.8) \]
and
\[ \bar{T}_{\mu \nu} = \frac{T_{\mu \nu}}{\phi}. \quad (2.9) \]
The bar over \( G_{\mu \nu} \) means that \( \bar{g}_{\mu \nu} \) is replacing \( g_{\mu \nu} \), and now indices are raised with \( \bar{g}_{\mu \nu} \).

A more general transformation is generated by letting
\[ \tilde{g}_{\mu \nu} = \frac{\phi g_{\mu \nu}}{\phi} = \frac{\bar{g}_{\mu \nu}}{\phi}, \quad (2.10) \]
\[ \tilde{\phi} = \tilde{\phi}(\phi), \quad (2.11) \]
where $\tilde{\phi}$ is an arbitrary function of $\phi$. The new field equations can be written in the form

$$\tilde{\phi} \tilde{G}_{\mu\nu} = 8\pi G \tilde{T}_{\mu\nu} + \frac{d\tilde{\omega}}{\tilde{\phi}} \left[ \tilde{\phi}_{,\mu} \tilde{\phi}_{,\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\phi}^{,\alpha} \tilde{\phi}_{,\alpha} \right] + \tilde{\phi}_{,\mu\nu} - \tilde{g}_{\mu\nu} \tilde{\phi}^{,\alpha}_{,\alpha},$$

(2.12)

$$\frac{2\tilde{\omega} + 3}{\sqrt{-\tilde{g}}} \left[ \sqrt{-\tilde{g}} \tilde{\phi}^{,\alpha} \right]_{,\alpha} = 8\pi \frac{G \tilde{T}^{\alpha}_{\alpha}}{\sigma} - \tilde{\phi}^{,\alpha} \tilde{\phi}_{,\alpha} \frac{d\tilde{\omega}}{d\tilde{\phi}},$$

(2.13)

where

$$\tilde{T}_{\mu\nu} = \frac{T_{\mu\nu} \tilde{\phi}}{\phi},$$

(2.14)

$$\sigma \equiv \frac{d\ln \tilde{\phi}}{d\ln \phi},$$

(2.15)

$$2\tilde{\omega} + 3 \equiv \frac{2\omega + 3}{\sigma^2}.$$

(2.16)

Yet, if instead of $\tilde{g}_{\mu\nu}$, the Einstein-scalar metric $\bar{g}_{\mu\nu}$ is used, we will obtain

$$\bar{G}_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu} + \left( \frac{\tilde{\omega}}{2} + \frac{3}{2} \right) \left[ \bar{\phi}_{,\mu} \bar{\phi}_{,\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{\phi}^{,\alpha} \bar{\phi}_{,\alpha} \right]$$

(2.17)

$$\frac{2\bar{\omega} + 3}{\sqrt{-\bar{g}}} \left[ \sqrt{-\bar{g}} \bar{\phi}^{,\alpha} \right]_{,\alpha} = 8\pi \frac{G \bar{T}^{\alpha}_{\alpha}}{\sigma} - \bar{\phi}^{,\alpha} \bar{\phi}_{,\alpha} \frac{d\bar{\omega}}{d\bar{\phi}},$$

(2.18)

where

$$\bar{\Phi} \equiv \ell n \bar{\phi}.$$

(2.19)

An interesting choice for the new scalar field is

$$\tilde{\Phi} = \frac{1}{\sqrt{2\omega_0 + 3}} \int \sqrt{2\omega + 3d\Phi} + \text{Const.}$$

(2.20)

$$\omega_0 = \text{Const};$$

(2.21)

since it then can be shown from Eqs. (2.15) and (2.16) that $\tilde{\omega} = \omega_0$, and therefore the field equations take the JBD form, except for the factor $\frac{1}{\sigma}$ multiplying $\tilde{T}^{\alpha}_{\alpha}$ and $\bar{T}^{\alpha}_{\alpha}$ in Eqs. (2.13)
and (2.18). Nevertheless, here $\omega_0$ is an arbitrary parameter, which, of course, for the JBD theory can be set equal to $\omega$.

The peculiar value $\omega_0 = 0$ yields a representation that can be cast in the following five-dimensional language:

\[
\begin{align*}
\tilde{g}_{\mu\nu} &= \tilde{g}_{\mu\nu}; \quad \mu, \nu = \{1-4\}, \\
\tilde{g}_{\mu 5} &= 0 \quad (2.23) \\
\tilde{g}_{55} &= \tilde{g}^2 \\
\tilde{G}_{\mu\nu} &= \frac{\tilde{G}_{\mu\nu}}{\phi}; \quad \tilde{G} = \text{const.}, \\
\tilde{G}_{\mu 5} &= 0 \\
\tilde{G}_{55} &= \frac{1}{2} \tilde{\phi} \left[ 1 - \frac{1}{\sigma} \right] \tilde{G}_{\alpha\alpha}
\end{align*}
\]

and hence the scalar tensor field equations become

\[
\begin{align*}
\tilde{G}_{AB} &= 8\pi \tilde{G}_{T\mu AB}; \quad A, B = \{1-5\},
\end{align*}
\]

where $\tilde{G}_{AB}$ is the five-dimensional Einstein’s tensor for the metric $\tilde{g}_{AB} (X^\mu)$.

We can readily integrate Eq. (2.20) for the $\omega$ given by Eq. (2.4), and we obtain:

(i) Barker’s Theory

\[
\ell n \tilde{\phi} = \frac{2}{\ell} Tan^{-1} \sqrt{\phi - 1} + \text{Const.} ; \quad \ell \equiv \sqrt{2\omega_0 + 3},
\]

\[
\phi = Tan^2 \frac{\ell}{2} [\ell n \tilde{\phi} - \text{Const.}] + 1.
\]

(ii) Schwinger’s Theory

\[
\ell n \tilde{\phi} = \frac{-2}{\ell \sqrt{\alpha \tilde{\phi}}} + \text{Const.} ; \quad \alpha \equiv \frac{k_2}{k_1}
\]
\[
\phi = \frac{4}{\alpha \ell^2} [\text{Const.} - \ell n\tilde{\phi}]^{-2}.
\] (2.32)

(iii) **Conformally Invariant Curvature Coupling**

\[
\ell n\tilde{\phi} = \frac{\sqrt{3}}{\ell} \ell n \left[ \frac{1 - \tau}{1 + \tau} \right] + \text{Const.}, \quad \tau \equiv \sqrt{1 - \phi}
\] (2.33)

\[
\phi = \frac{4\text{Const.} \ell^2/\sqrt{3}}{[\tilde{\phi}^2/\sqrt{3} + \text{Const.}]^2}
\] (2.34)

## III Generation of Electrostatic Solutions

Let’s consider the field equations (2.17) and (2.18) in the representation where \( \tilde{\omega} = \omega_0 \) [JBD like representation], and assume that we have the energy-momentum tensor of an electromagnetic field:

\[
8\pi G T_{\mu\nu} = 2 \left[ F_{\mu \alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu} F^2 \right]
\] (3.1)

\[
F_{\mu\nu} = \frac{\partial A_\nu}{\partial X^\mu} - \frac{\partial A_\mu}{\partial X^\nu}.
\] (3.2)

Furthermore, let’s assume that the space-time is static, and the vector potential only has the component \( A_4 \). Hence, we are dealing with three potentials; the electrostatic potential \( A_4 \), and the gravitational potentials which are represented by \( \tilde{g}_{44} \) and \( \tilde{\phi} \). We know that when the configurations are spherically symmetric these potentials depend only on the radius variable that characterize the equipotential surfaces. Moreover, even if the configurations are not spherically symmetric, we could still expect that the gravitational and electric equipotentials are surfaces of the same shape, and, consequently, these potentials once more would be identified by a single function, say \( u \). Thus, in searching for new solutions, it is quite natural
to start with the following ansatz:

\[ A_4 = A_4(u), \quad (3.3) \]
\[ \tilde{\phi} = \tilde{\phi}(u), \quad (3.4) \]
\[ \bar{g}_{44} = \bar{g}_{44}(u). \quad (3.5) \]

Then one finds (Appendix A) that the electrostatic solutions of the JBD-like field equations are expressed in terms of solutions, \( g'_{\mu\nu} \), of Einstein’s static vacuum field equations, \( G'_{\mu\nu} = 0 \), in the following way:

\[ -\bar{g}_{44} = \frac{4ab(-g'_{44})^c}{[b - (-g'_{44})^c]^2}, \quad (3.6) \]
\[ \bar{g}_{ij} = \left[ b - (-g'_{44})^c \right]^2 (g'_{44})^{1-c} g'_{ij}; i, j = 1, 2, 3, \quad (3.7) \]
\[ A_4 = \pm \sqrt{a [b + (-g'_{44})^c]} \quad (3.8) \]
\[ \Phi = e \ln(-g'_{44}) + \text{Const.}, \quad (3.9) \]
\[ e \equiv \pm \sqrt{\frac{2 - c^2}{2\omega_0 + 3}} \quad \text{for } c^2 < 1, \quad (3.10) \]

where \( a, b, c, \) and \( d \) are constants.

For asymptotically Minkowski space-time where

\[ -g'_{44} \approx 1 - \frac{2GM'}{r}; \quad r = \sqrt{X_iX^i} \to \infty, \quad (3.11) \]

we have that, with the choice

\[ a = \frac{(b - 1)^2}{4b}, \quad (3.12) \]
\[ d = \pm \sqrt{a \frac{[1 + b]}{[1 - b]}}, \quad (3.13) \]
the asymptotic behavior of $g_{\mu\nu}, \tilde{\phi}$ and $A_4$ are

$$-g_{44} \approx 1 - \frac{2GM}{r}, \quad (3.14)$$

$$g_{ij} \approx g'_{ij} \left[ 1 + \frac{2GM'}{r} \left( c - 1 + \frac{2c}{b-1} \right) \right], \quad (3.15)$$

$$\tilde{\phi} \approx \text{const.} \left[ 1 - \frac{2GeM'}{r} \right], \quad (3.16)$$

$$A_4 \approx -\frac{kQ}{r}, \quad (3.17)$$

where $k$ is a constant, and

$$M = cM' \left( \frac{b+1}{b-1} \right), \quad (3.18)$$

$$kQ = \pm \frac{2c\sqrt{bGM'}}{b-1}, \quad (3.19)$$

$$GM^2 = (GcM')^2 + (kQ)^2 \quad (3.20)$$

Note that if $b \to \infty$, then $Q \to 0$, and $M = cM'$. Thus for $M$ and $M'$ to be positive we must assume $c > 0$. From Eqs. (3.14) and (3.17) we see that $M$ and $Q$ can be identified with the total inertial mass and electric charge, respectively.

The solutions (3.6) - (3.9) can be converted into the corresponding solutions of the original scalar-tensor field equations (2.1) and (2.2) by using the transformation (2.10):

$$g_{\mu\nu} = \frac{\overline{g}_{\mu\nu}}{\bar{\phi}}, \quad (3.21)$$

and then $\phi(\tilde{\phi})$ could be determined by the inversion of Eq. (2.20), with $\tilde{\phi}$ given, by Eq. (3.9), explicitly as a function of $g'_{44}$. For instance, for the JBD theory we can have $\phi = \tilde{\phi}$, while for Barker’s Schwinger’s and conformally invariant curvature coupling theories, the necessary inversions are found in Eqs. (2.30), (2.32), and (2.34), respectively.
IV  Weyl Type Solutions

An interesting family of solutions are generated with the transformation (3.6)-(3.9) when we take as seed metric, $g'_{\mu\nu}$, the following well known Weyl solution:

\begin{align}
  -g'_{44} &= \left[ 1 - \frac{2\beta}{r} \right]^\delta \\
  g'_{rr} &= \frac{\left[ 1 - \frac{2\beta}{r} \right]^{\delta^2 - \delta - 1}}{\left[ 1 - \frac{2\beta}{r} + \frac{\beta^2}{r^2} \sin^2 \theta \right]^{\delta^2 - 1}} \\
  g'_{\theta\theta} &= \frac{r^2 \left[ 1 - \frac{2\beta}{r} \right]^{\delta^2 - \delta}}{\left[ 1 - \frac{2\beta}{r} + \frac{\beta^2}{r^2} \sin^2 \theta \right]^{\delta^2 - 1}} \\
  g'_{\varphi\varphi} &= r^2 \left[ 1 - \frac{2\beta}{r} \right]^{1 - \delta} \sin^2 \theta
\end{align}

where $\beta$ and $\delta$ are constants. Clearly, the above solutions reduce to the Schwarzschild metric when $\delta = 1$. Otherwise, they represent the gravitational field exterior to oblates ($\delta > 1$) or prolates ($\delta < 1$) ellipsoidal configurations\(^{13}\).

The new solutions generated by Eq. (3.6)-(3.9) are:

\begin{align}
  -\mathcal{G}_{44} &= \left( \frac{b - 1}{b - \left[ 1 - \frac{2\beta}{r} \right]^{\delta c}} \right)^2 \left[ 1 - \frac{2\beta}{r} \right]^{\delta c} \equiv \psi \left[ 1 - \frac{2\beta}{r} \right]^{\delta c} \\
  \mathcal{G}_{rr} &= \frac{\left[ 1 - \frac{2\beta}{r} \right]^{\delta^2 - \delta - 1}}{\psi \left[ 1 - \frac{2\beta}{r} + \frac{\beta^2}{r^2} \sin^2 \theta \right]^{\delta^2 - 1}} \\
  \mathcal{G}_{\theta\theta} &= \frac{r^2 \left[ 1 - \frac{2\beta}{r} \right]^{\delta^2 - \delta c}}{\psi \left[ 1 - \frac{2\beta}{r} + \frac{\beta^2}{r^2} \sin^2 \theta \right]^{\delta^2 - 1}} \\
  \mathcal{G}_{\varphi\varphi} &= \frac{r^2 \left[ 1 - \frac{2\beta}{r} \right]^{1 - \delta c}}{\psi \sin^2 \theta} \\
  A_4 &= \pm \left[ \frac{b - 1}{2\sqrt{b}} \left( \frac{b + 1 - \frac{2\beta}{r}^{\delta c}}{b - \left[ 1 - \frac{2\beta}{r} \right]^{\delta c}} \right) - \frac{b + 1}{2\sqrt{b}} \right]
\end{align}
\begin{equation}
\Phi = \delta e \ln \left[1 - \frac{2\beta}{r}\right] + \text{Const.} \tag{4.10}
\end{equation}

where

\begin{equation}
\psi \equiv \left(\frac{b - 1}{b - [1 - \frac{2\beta}{r}]c}\right)^2 \tag{4.11}
\end{equation}

The solutions (4.5)-(4.10) represent static gravitational, electric, and scalar fields with oblate or prolate equipotential surfaces (Appendix B). When \( \delta = 1 \) the solutions are spherically symmetric, and the metric takes this simple form:

\begin{equation}
\overline{g}_{\theta\theta} = \frac{g_{\varphi\varphi}}{\sin^2 \theta} = \left(\frac{b - [1 - \frac{2\beta}{r}]c}{b - 1}\right)^2 \left[1 - \frac{2\beta}{r}\right]^{1-c} \overline{R}^2 \tag{4.12}
\end{equation}

\begin{equation}
-g_{44} = \overline{g}_{rr}^{-1} = \frac{(b - 1)^2 [1 - \frac{2\beta}{r}] c}{(b - [1 - \frac{2\beta}{r}]c)^2} \tag{4.13}
\end{equation}

The above solution is a scalar-tensor generalization of the Reissner-Nordström metric, to which it reduces when \( c = 1 \) (Appendix C).

If \( c \neq 1 \), and \( r = 2\beta = 2GM' \), we see from Eq. (4.13) that \( g'_{44} = 0 \). However, in contrast to general relativity where \( r = 2GM' \) represents an event horizon, here we have, from Eq. (4.12), that

\begin{equation}
\overline{R} \rightarrow 0,
\end{equation}

\begin{equation}
\lim_{r \rightarrow 2\beta}
\end{equation}

and, since the proper areas of constant radius of the new solutions are equal to \( 4\pi \overline{R}^2 \), then we see that the Schwarzschild horizon is mapped into the origin of the JBD like representation. Thus, we have that the static metric (4.12), (4.13) and scalar field (4.10) describe the whole space \( 0 < \overline{R} < \infty \), for \( 2GM' < r < \infty \), without a horizon. Furthermore, this exterior metric
does not imply a lower bound for $\overline{R}$, analogous to $r > 2GM'$ appearing in general relativity. Nevertheless, we have to be cautious with the physical interpretation of $\overline{R}$, since now we are working in the JBD like representation where the physical meaning of the metric and scalar field $\tilde{\phi}$ are not transparent, for we have that free falling test particles do not move in geodesics of the metric $\tilde{g}_{\mu\nu}$. Instead, we should look at the behavior of the radius, $R(r)$, in the original canonical representation, $g_{\mu\nu}, \phi$, which, by virtue of Eq. (2.5), is related to $\overline{R}$ in the following way

$$R = \sqrt{g_{\theta\theta}} = \sqrt{\overline{g}_{\theta\theta}} = \frac{\overline{R}}{\sqrt{\phi}}.$$  \hspace{1cm} (4.15)

In the next section we will take on this analysis for the JBD theory.

V Analysis Of $R(r)$.

For the JBD theory we can have $\omega = \omega_0, \phi = \tilde{\phi}$, and hence from Eq. (4.10) we find that

$$\phi = \left[ \frac{1 - 2GM'}{r} \right]^e, \delta = 1,$$ \hspace{1cm} (5.1)

where the constant of integration has been chosen such that $\phi \to 1$ as $r \to \infty$. Using Eqs. (4.12), (4.15) and (5.1), we get

$$R = r \left[ b - \frac{(1 - \frac{2GM'}{r})^c}{b - 1} \right] \left[ 1 - \frac{2GM'}{r} \right]^{\frac{1-e}{2}} ,$$  \hspace{1cm} (5.2)

and depending on the sign of $S \equiv \frac{1-e-c}{2}$ we will have as $r \to 2GM'$ that either $R \to 0 (S > 0)$, or $R \to \infty (S < 0)$ after reaching a minimum value for some $r > 2GM'$. On the other hand, the sign of $S$ is determined by the value of $c$, which, in turn, depends on the interior structure of the star, as described below.
The field equations (2.17) and (2.18) with $\tilde{\omega} = \omega_0$ imply for a static system that

\[
\left[\sqrt{-g}(\ell n - g_{44})^i\right]_i = -16\pi G \left[ T_4^i - \frac{1}{2} T^i_\alpha \right] \sqrt{-g},
\]

\[
\left[\sqrt{-g}\tilde{\Phi}^i\right]_i = \frac{8\pi G T^i_\alpha}{\sigma(2\omega_0 + 3)} \sqrt{-g},
\]

and consequently

\[
\int \sqrt{-g}(\ell n - g_{44})^i d^2 S_i = -16\pi G \int \left[ T_4^i - \frac{1}{2} T^i_\alpha \right] \sqrt{-g} d^3 X.
\]

\[
\int \sqrt{-g}\tilde{\Phi}^i d^2 S_i = 8\pi G \int \frac{T^i_\alpha \sqrt{-g} d^3 X}{\sigma(2\omega_0 + 3)}.
\]

Thus, if the two dimensional surface of integration, $d^2 S_i$, is chosen outside the boundary of the star, we can then substitute, in the left hand side of Eqs. (5.5) and (5.6), for $g_{44}$ and $\tilde{\Phi}$ the expressions given in Eqs. (3.6) and (3.9). At this point we simplified the problem by assuming that the configuration is neutral ($Q = 0$), and consequently Eqs. (5.5) and (5.6) become

\[
c \int \sqrt{-g}(\ell n - g_{44})^i d^2 S_i = -16\pi G \int \left[ T_4^i - \frac{1}{2} T^i_\alpha \right] \sqrt{-g} d^3 X,
\]

\[
e \int \sqrt{-g}(\ell n - g_{44})^i d^2 S_i = 8\pi G \int \frac{T^i_\alpha \sqrt{-g} d^3 X}{\sigma(2\omega_0 + 3)}.
\]

From which it follows that

\[
\frac{e}{c} = \frac{\int \frac{T^i_\alpha \sqrt{-g} d^3 X}{\sigma(2\omega_0 + 3)}}{-2 \int \left[ T_4^i - \frac{1}{2} T^i_\alpha \right] \sqrt{-g} d^3 X} = \frac{\int \frac{T^i_\alpha \sqrt{-g} d^3 X}{\sigma(2\omega_0 + 3)}}{-2 \int \left[ T_4^i - \frac{1}{2} T^i_\alpha \right] \sqrt{-g} d^3 X},
\]

where we used Eqs. (2.5) and (2.9). For a perfect fluid we obtain

\[
\frac{e}{c} = \frac{\int \frac{(3p - \rho) \sqrt{-g} d^3 X}{\sigma(2\omega_0 + 3)}}{-\int (3p + \rho) \sqrt{-g} d^3 X}.
\]
In particular, if the pressure, $p$, is proportional to the energy density, $p = \epsilon \rho$, we find

$$\frac{e}{c} = \frac{(3\epsilon - 1)}{(3\epsilon + 1)\sqrt{2\omega_0 + 3}(\sqrt{2\omega + 3})},$$

where we use Eq. (2.16) with $\bar{\omega} = \omega_0$, and

$$\left(\frac{1}{\sqrt{2\omega + 3}}\right) \equiv \frac{\int \rho \sqrt{-g} d^3X}{\int \rho \sqrt{-g} d^3\bar{x}}.$$ \hspace{1cm} (5.12)

Hence for the JBD theory we get, with $\omega_0 = \omega$,

$$\frac{e}{c} = \frac{3\epsilon - 1}{(2\omega + 3)(3\epsilon + 1)},$$

from which, using Eq. (3.10),

$$c = \sqrt{\frac{2\omega + 3}{2\omega + 3 + \left(\frac{3\epsilon - 1}{3\epsilon + 1}\right)^2}}.$$ \hspace{1cm} (5.14)

If $\epsilon$ is not a constant, or we do not have a perfect fluid we can still use Eqs. (5.13) and (5.14) by replacing $\epsilon$ in them by

$$\bar{\epsilon} \equiv -\frac{1}{3} \left[ \frac{\int T^0_4 \sqrt{-g} d^3X}{\int T^4_4 \sqrt{-g} d^3X} \right] = \frac{1}{3} \left[ \frac{-\int T^0_\alpha \sqrt{-g} d^3X}{\int T^\alpha_4 \sqrt{-g} d^3X} + 1 \right].$$ \hspace{1cm} (5.15)

Using Eq. (5.13) and (5.14), we can express $S$ as a function of $\omega$ and $\bar{\epsilon}$:

$$S = \frac{1}{2}(1 - c - e) = \frac{1}{2} \left( 1 - \frac{2\omega + 3}{\sqrt{2\omega + 3 + \left(\frac{3\epsilon - 1}{3\epsilon + 1}\right)^2}} \left[ 1 + \frac{3\bar{\epsilon} - 1}{(2\omega + 3)(3\bar{\epsilon} + 1)} \right] \right).$$ \hspace{1cm} (5.16)

Thus, assuming that $2\omega + 3 > 0$, if $0 \leq \bar{\epsilon} < \frac{1}{3}$, then $S > 0$, and when $\bar{\epsilon} > \frac{1}{3}$, one finds that $S < 0$. The case $\bar{\epsilon} = \frac{1}{3}$ yields $c = 1$, which implies that $\phi = const$, and we are back to the Schwarzschild metric of general relativity. According to Eq. (5.15), for a positive energy density, $-T^4_4 > 0$, we have $\bar{\epsilon} > \frac{1}{3}$ if $\int T^\alpha_\alpha \sqrt{-g} d^3x > 0(\leq 0)$. 

16
Let us first consider the case $\tau > \frac{1}{3}$, $S < 0$, where it can be shown that $R$ has a minimum value at
\[ R = R_{\text{min}} \equiv 2GM'(1 + U)^{1+U} \frac{1}{U';UU}; \quad U \equiv -S > 0. \quad (5.17) \]
Consequently, the solution does not cover the totality of space, since it leaves out the region $0 < R < R_{\text{min}}$. Then, in order to be able to match an interior static solution to this exterior solution, it must be required that the radius of the interior configuration is larger than $R_{\text{min}}$, thus implying that
\[ R > R_{\text{min}} = \frac{2GM(1 + U)^{1+U}}{c U';UU} > 2GM, \quad (5.18) \]
where we used $M = cM'$. The value of $U$ can be calculated as a function of $\omega$ for a given interior model using Eqs. (5.15) and (5.16).

We now turn to the case $\tau < \frac{1}{3}$, $S > 0$, which implies that
\[ R \geq 0; \quad r \geq 2GM', \quad (5.19) \]
and therefore no lower bound appears for $R$ from the above considerations of the exterior space-time alone, and further interior analysis is necessary.

Besides their mathematical interest and heuristic value, the solutions (3.6)-(3.9) provide a necessary tool for confronting with reality the predictions of a very large class of viable theories, since with the more refined observations of strong fields it is increasingly important to use exact results. In particular, the solutions of this paper serve to investigate the trajectories of photons and particles in the space exterior to static systems. For instance, if the configurations are spherically symmetric, then, from
\[ U'^\alpha U_\alpha = U'^\phi U_\phi + U'^r U_r + U'^4 U_4 = -1, \quad (5.20) \]
we derived the following expressions for the orbits:

\[
\varphi(r) - \varphi(r_0) = \pm \int_{r_0}^{r} \frac{L \, dr}{R \left[1 + \overline{g}_{44} \left(\frac{1}{U_4^2} + \frac{L_P^2}{R^2}\right)\right]^{1/2}},
\]

\[
t(r) - t(r_0) = \pm \int_{r_0}^{r} \frac{dr}{\overline{g}_{44} \left[1 + \overline{g}_{44} \left(\frac{1}{U_4^2} + \frac{L_P^2}{R^2}\right)\right]^{1/2}},
\]

where, without loss of generality, we have assumed that the motions occur in the plane \( \theta = \pi/2 \), and

\[
L \equiv |\frac{\overline{U}_\varphi}{U_4}|,
\]

is the so-called angular momentum per mass or impact parameter. Since the field are independent of \( t \) and \( \varphi \), then the energy-momentum conservation, \( T^{\mu\nu} ; \nu = 0 \), yields

\[
P_\varphi = mU_\varphi = \frac{m\overline{U}_\varphi}{\phi^2} = \text{const.}; m = \text{const.},
\]

\[
P_4 = mU_4 + eA_4 = \frac{m\overline{U}_4}{\phi^2} + eA_4 = \text{const.},
\]

from which we obtain that

\[
L = \frac{\bar{T}}{1 - \frac{eA_4}{P_\varphi}}; \bar{T} = \text{const.},
\]

and thus, for a neutral particle \( L = \bar{T} \), and \( \overline{U}_4 = \text{const.}\phi^2 \). Therefore, with the metric (4.12), (4.13) and the expressions (5.22), (5.23), we are in a position to extract the values of \( \varphi(r) \) and \( t(r) \), from computer tabulations, and hence to obtain precise predictions for relativistic celestial mechanics effects, like time delay, light deflection, perihelium shift, etc.
VI Magnetostatic Solutions

Let us now assume that the metric and fields are static and independent of a coordinate $X^a$. Then we can generate magnetostatic JBD like solutions starting from an Einstein’s vacuum static metric, $g'_{\mu\nu}$, in the following way (Appendix A):

\[-g_{44} = -\frac{(-g'_{44})^c}{4a_mb_m} \left[ b_m + g'_{aa}(-g'_{44})^{1-c} \right]^2 \equiv \frac{(-g'_{44})^c}{\psi_m}, \quad (6.1)\]

\[\Psi_{aa} = \psi_m(-g'_{44})^{1-c}g'_{aa}, \quad (6.2)\]

\[\Psi_{ij} = \frac{(-g'_{44})^{1-c}}{\psi_m} g'_{ij}; \ i, j \neq 4, a, \quad (6.3)\]

\[A_a \equiv \pm \sqrt{a_m} \frac{[g'_{aa}(-g'_{44})^{1-c} - b_m]}{g'_{aa}(-g'_{44})^{1-c} + b_m} + d_m, \quad (6.4)\]

\[\Phi = e^\ell n(-g'_{44}) + \text{const.}, \quad (6.5)\]

where

\[\psi_m \equiv \frac{4a_mb_m}{[b_m + g'_{aa}(-g'_{44})^{1-c}]^2}. \quad (6.6)\]

In order for $g_{\mu\nu}$ to reduce to $g'_{\mu\nu}$ when $c = 1$ and $A_a = 0$, the constants $a_m, b_m$ and $d_m$ should be related as follows:

\[d_m = \pm \sqrt{a_m}, \quad (6.7)\]

\[b_m = 4a_m. \quad (6.8)\]

We can get explicit magnetostatic solutions by choosing as background the axially symmetric Weyl metric (4.1)-(4.4). In this case, $X^a$ can be identified with the azymuthal angle $\varphi$, and we obtain

\[\varphi_{44} = -\Lambda^2 \left[ 1 - \frac{2\beta}{r} \right]^{\delta e}, \quad (6.9)\]
\[
\bar{g}_{rr} = \frac{\Lambda^2[1 - \frac{2\beta}{r}]^{\delta c - \delta c - 1}}{[1 - \frac{2\beta}{r} + \frac{\beta^2}{r^2} \sin^2 \theta]^{\delta c - 1}}.
\]
\[(6.10)\]
\[
\bar{g}_{\theta \theta} = \frac{\Lambda^2 r^2 [1 - \frac{2\beta}{r}]^{\delta c - \delta c}}{[1 - \frac{2\beta}{r} + \frac{\beta^2}{r^2} \sin^2 \theta]^{\delta c - 1}}.
\]
\[(6.11)\]
\[
\bar{g}_{\phi \phi} = \frac{r^2 [1 - \frac{2\beta}{r}]^{1 - \delta c}}{\Lambda^2} \sin^2 \theta,
\]
\[(6.12)\]
\[
A_\phi = \frac{Br^2 [1 - \frac{2\beta}{r}]^{1 - \delta c}}{\Lambda} \sin^2 \theta,
\]
\[(6.13)\]
\[
\tilde{\Phi} = \delta c \left[ \ln \left( 1 - \frac{2\beta}{r} \right) \right],
\]
\[(6.14)\]

where
\[
\Lambda = B^2 r^2 \left[ 1 - \frac{2\beta}{r} \right]^{1 - \delta c} \sin^2 \theta + 1,
\]
\[(6.15)\]
\[
B \equiv \frac{\pm 1}{2 \sqrt{a_m}}.
\]
\[(6.16)\]

When \( \delta = 1, c = 1, \) and \( \beta = 0, \) the above solutions reduce to the well known magnetized Melvin solutions\(^{14}\), which represents a cylindrically symmetric bundle of pure magnetic flux. If \( \beta \neq 0 \) we have a Schwarzschild black hole embedded in the magnetic flux\(^{15}\). If, furthermore, \( \delta \neq 1 \) and \( c \neq 1 \), the configuration is not spherically symmetric, and a scalar field is also present.

\section{VII Acknowledgements}

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Appendix A

Suppose that the JBD like field equations are independent of the coordinate $X^a$, and that the metric and electromagnetic vector have the form

$$\overline{g}_{\mu\nu} = \begin{pmatrix} \overline{g}_{aa} & 0 \\ 0 & \overline{g}_{ij} \end{pmatrix},$$  \hspace{1cm} (A.1)

$$A_{\mu} = A_a \delta_{\mu a}. \hspace{1cm} (A.2)$$

Then, with the following transformation of the metric

$$\overline{g}_{aa} = \overline{g}'_{aa} \psi \Omega, \hspace{1cm} (A.3)$$

$$\overline{g}_{ij} = \frac{\overline{g}'_{ij}}{\psi \Omega}, \hspace{1cm} (A.4)$$

we obtain, from the JBD like equations,

$$\overline{G}_{ij} - 8\pi G \overline{T}_{ij} - (\omega_0 + \frac{3}{2})[\overline{\Phi}_{i} \overline{\Phi}_{j} - \frac{1}{2} \overline{g}_{ij} \overline{\Phi}^k \overline{\Phi}_{k}] = \overline{G}'_{ij} - \frac{C_{ijik}}{2} [A^l_k + B^l_k] = 0, \hspace{1cm} (A.5)$$

where

$$C_{ijik} \equiv g'_{il} g'_{jk} - \frac{1}{2} g'_{lj} g'_{ik}, \hspace{1cm} (A.6)$$

$$A^l_k \equiv (\ell \ln g''_{aa})^l (\ell \ln \psi)^k + (\ell \ln g'_{aa})^l (\ell \ln \psi)^k + (\ell \ln \psi)^l (\ell \ln \psi)^k + \frac{4A'_a A'_a}{g''_{aa} \psi}, \hspace{1cm} (A.7)$$

$$B^l_k \equiv (\ell \ln g'_{aa})^l (\ell \ln \Omega)^k + (\ell \ln g'_{aa})^k (\ell \ln \Omega)^l + (\ell \ln \Omega)^l (\ell \ln \Omega)^k + (2\omega_0 + 3) \overline{\Phi}^l \overline{\Phi}^k, \hspace{1cm} (A.8)$$

$$g''_{aa} \equiv g'_{aa} \Omega. \hspace{1cm} (A.9)$$

Furthermore,

$$\overline{R}'_{a} = \psi \Omega \overline{R}'_{a} - \frac{1}{2} \psi \Omega [\sqrt{-g'} (\ell \ln \psi)^k]_{k} = \frac{A^k_a A_{a,k}}{g''_{aa}}, \hspace{1cm} (A.10)$$
\begin{align*}
F^{\alpha\nu}_{\mu} &= -\psi\Omega \left[ \frac{\sqrt{-g} A_a}{g''_{aa}\psi} \right]^k = 0, \\
\tilde{\Phi}^{\alpha\nu}_{;\alpha} &= \psi\Omega \left[ \frac{\sqrt{-g} \tilde{\Phi}}{g''_{aa}\psi} \right]^k = 0.
\end{align*}

Note that in the right hand side of the equations the indexes are raised with $g^{ij}$.

We are looking for functions $\psi$ and $\Omega$ such that $A^{ik} = 0, B^{lk} = 0$, and, thereby, $G'_{ij} = 0$.

The inspection of the expressions (A.7) and (A.8) suggest that we work with the following ansatz:

\begin{align*}
A_a &= A_a(g_{aa}'), \\
\psi &= \psi(g_{aa}'), \\
\tilde{\Phi} &= \tilde{\Phi}(g_{aa}'), \\
\Omega &= \Omega(g_{aa}').
\end{align*}

The hypothesis (A.13) and (A.14) are equivalent to the assumptions (3.3)-(3.5), and hence these are solutions where the surfaces $u \equiv g_{aa}' = \text{const.}$ are equipotentials of $A_a, \tilde{\Phi}_{aa}$ and $\tilde{\phi}$. Thus, for instance, for static systems, $a = 4$, the surfaces of constant gravitational and electric potentials coincide, which is to be expected in astrophysical systems in equilibrium.

Aside from the assumptions (A.13) and (A.14) the functions $\psi$ and $\Omega$ are arbitrary and can be chosen to satisfy the following relations

\begin{align*}
\frac{d\ln \psi}{d\ln g''_{aa}} &= -1 \pm \sqrt{1 - 4g''_{aa} \psi k_1^2}, \\
\frac{d\ln \Omega}{d\ln g''_{aa}} &= -1 \pm \sqrt{1 - k_2^2},
\end{align*}

where

\begin{align*}
k_1^2 &= \left[ \frac{1}{\psi} \frac{d A_a}{d g''_{aa}} \right]^2, \\
k_2^2 &= (2\omega_0 + 3) \left( \frac{d \tilde{\Phi}}{d\ln g'_{aa}} \right)^2.
\end{align*}
which can be shown to guarantee that $A^{\ell k}$ and $B^{\ell k}$ vanish. Then (A.5) implies that $G'_{ij} = 0$, and as a consequence of the Bianchi identities, $G'^{\mu\nu} = 0$, we also have $G'_{aa} = 0$, or $G'_{\mu\nu} = 0$.

On the other hand, from Einstein’s vacuum equations, $G'_{\mu\nu} = 0$, we get

$$R'^a_a = -\frac{1}{2\sqrt{-g'}} \left[ \sqrt{-g'} (\ell n g'_{aa})^k \right]_{,k} = 0,$$

which together with Eqs. (A.11), (A.12), (A.16) and (A.17) imply, if $g'_{ij} \geq 0$, that

$$k_1 = \text{const.},$$

$$k_2 = \text{const.}$$

Then from Eqs. (A.15)-(A.18), we obtain

$$\mathcal{G}_{aa} = g''_{aa} \psi = \frac{4abg''_{aa}}{[b + g''_{aa}]^2}; k_1^2 \equiv \frac{1}{4a}, b = \text{const.},$$

$$A_a = \pm \sqrt{a[b - g''_{aa}]} + \text{const.},$$

$$\Omega = g'_a c^{-1}; c \equiv \pm \sqrt{1 - \frac{k_2^2}{b}},$$

$$\ell n \tilde{\Phi} = \pm k_2 \ell n g'_{aa} \sqrt{2\omega_0 + 3} + \text{const.}$$

We can also verify that Eq. (A.10) is satisfied by the above equations.

Equations (A.22)-(A.25) and Eq. (A.4) can be converted into Eqs. (3.6)-(3.9) by identifying ‘a’ with ‘4’. If, on the other hand, $x^a$ is a space-like coordinate, then $A_a$ represents a magnetic field.

If the fields are also independent of another coordinate $x^b$, then we can consider instead the conformal transformation

$$\mathcal{G}_{aa} = \frac{g'_{aa} \psi}{\Omega},$$
$$\bar{g}_{bb} = \frac{g'_{bb} \Omega}{\psi},$$  \hspace{1cm} (A.27)

$$\bar{g}_{ij} = \frac{g'_{ij}}{\psi \Omega}; \hspace{0.5cm} i, j \neq a, b.$$  \hspace{1cm} (A.28)

and then Eqs. (A.5) are derived with the following changes in $A^{\ell k}$ and $B^{\ell k}$: now in $A^{\ell k}$, $g''_{aa}$ means

$$g''_{aa} \equiv \frac{g'_{aa}}{\Omega};$$  \hspace{1cm} (A.29)

and in $B^{\ell k}$, $g'_{aa}$ is substituted by $g'_{bb}$. Consequently, following the same routine that led to Eqs. (A.22) - (A.25), we will again obtain Eqs. (A.22) and (A.23), but now

$$\Omega = g'_{bb} c^{-1},$$  \hspace{1cm} (A.30)

and

$$\ell n \tilde{\Phi} = \frac{\pm k_2}{\sqrt{2\omega_0 + 3}} \ell n g'_{bb} + \text{const.}$$  \hspace{1cm} (A.31)

Choosing $b = 4$ gives rise to the magnetostatic solutions of Section VI. If, on the other hand, $x_b$ is space-like, then we would get electrostatic ($a = 4$) or magnetic solutions ($a \neq 4$) that were not studied in this paper.
Appendix B

We see from Eqs. (4.5), (4.9) and (4.10) that the surfaces with \( r = \text{const.} \) are equipotentials for the gravitational, electric, and scalar fields. It is easy to calculate the proper distance on these surfaces along the circumferences \( \theta = \text{const.} \), and we get, using Eq. (4.8)

\[
C_\phi(\theta) = \int_0^{2\pi} \sqrt{g_{\phi\phi}} d\phi = \frac{2\pi r[1 - \frac{2\beta}{r}]^{\frac{1-\delta}{2}}}{\sqrt{\psi}} \sin \theta \equiv 2\pi R \sin \theta. \tag{B.1}
\]

On the other hand, the proper distance around the curves \( \varphi = \text{const.} \) are given by the integral

\[
C_\theta = 2 \int_0^\pi \sqrt{g_{\theta\theta}} d\theta = 2R \int_0^\pi \left(1 + \frac{\beta^2 \sin^2 \theta}{r^2[1 - \frac{2\beta}{r}]^{1-\delta^2}} \right) \frac{1-\delta^2}{2} d\theta \tag{B.2}
\]

where we used Eq. (4.7). We can see that since \( r > 2\beta \),

\[
C_\theta > C_\phi(90^\circ) = 2\pi R \; ; \; 0 < \delta < 1, \tag{B.3}
\]

\[
C_\theta < C_\phi(90^\circ) \; , \; \delta > 1, \tag{B.4}
\]

and therefore the equipotential surfaces have shapes like prolate (\( 0 < \delta < 1 \)) or oblate (\( \delta > 1 \)).
Appendix C. The Reissner-Nordström Solution.

If \( c = 1 \), one finds from Eq. (4.12) that

\[
    r = \frac{R}{b} - \frac{2\beta}{b - 1}, \tag{C.1}
\]

and then Eq. (4.13) becomes

\[
    -g_{44}^{-1} = g_{rr}^{-1} = 1 - \frac{2\beta}{R} \left( \frac{b+1}{b-1} \right) + \frac{4\beta^2 b}{(b-1)^2 R^2}. \tag{C.2}
\]

On the other hand, recalling Eqs. (3.18) and (3.19), and since \( c = 1, \beta = GM' \), we find that

\[
    -g_{44} = g_{rr}^{-1} = 1 - \frac{2GM}{R} + k^2 \frac{Q^2}{R^2}, \tag{C.3}
\]

which together with Eq. (4.12) is the standard form for the Reissner-Nordström metric.

Furthermore, Eq. (4.9) with \( \delta = c = 1 \) gives the familiar electrostatic potential:

\[
    A_4 = \frac{-kQ}{R}, \tag{C.4}
\]
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