Quantum motion equation and Poincaré translation invariance of noncommutative field theory

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Abstract

We study the Moyal commutators and their expectation values between vacuum states and non-vacuum states for noncommutative scalar field theory. For noncommutative $\varphi^{*4}$ scalar field theory, we derive its energy-momentum tensor from translation transformation and Lagrange field equation. We generalize the Heisenberg and quantum motion equations to the form of Moyal star-products for noncommutative $\varphi^{*4}$ scalar field theory for the case $\theta^{0i} = 0$ of spacetime noncommutativity. Then we demonstrate the Poincaré translation invariance for noncommutative $\varphi^{*4}$ scalar field theory for the case $\theta^{0i} = 0$ of spacetime noncommutativity.

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1 Introduction

The researches of gravitation theory and superstring theories in recent years reveal that spacetime coordinates may be noncommutative under certain microscopic scales such as the Planck scale [1-4]. This has confirmed the concept of spacetime quantization proposed by Snyder many years ago [5]. For quantum field theories, they are set up in ordinary commutative spacetime. Because spacetime may be noncommutative, people need to study quantum field theories on noncommutative spacetime. A lot of research works have been carried out on noncommutative field theories in recent years. However for noncommutative field theories, there are still many problems need to be studied further.

Poincaré translation invariance and Lorentz rotation invariance are the basic spacetime symmetries for quantum field theories on ordinary commutative spacetime. For quantum field theories on noncommutative spacetime, we need to study whether these properties are still satisfied. According to superstring theories, people usually take $\theta^{\mu\nu}$ to be constant $c$-number matrix that not changed under reference system transformations. This will destroy the Lorentz rotation invariance generally for noncommutative field theory, except for certain
special forms of $\theta^{\mu\nu}$, the invariance under a subgroup $SO(1, 1) \times SO(2)$ of the usual Lorentz group is reserved [6]. In Refs. [7,8] the authors put forward the twisted Poincaré algebra explanation for the relativistic invariance of noncommutative field theories. However the quantum realization of such a proposition is not clear yet. In Ref. [9] the authors take $\theta^{\mu\nu}$ to be a $c$-number second-order antisymmetric tensor and then demonstrated the Poincaré translation invariance and Lorentz rotation invariance for noncommutative field theories from their classical field equation approach.

In this paper, we study the quantum motion equations and Poincaré translation invariance of noncommutative field theory. For simplicity, we only study the noncommutative scalar field theory. In Sec. II, we study the Moyal commutators and their expectation values between vacuum states and non-vacuum states for noncommutative scalar field theory. In Sec. III, we derive the energy-momentum tensor for noncommutative $\varphi^4$ scalar field theory from translation transformation and Lagrange field equation. In Sec. IV, we generalize the Heisenberg and quantum motion equations to the form of Moyal star-products for noncommutative $\varphi^4$ scalar field theory and then demonstrate its Poincaré translation invariance from the Heisenberg and quantum motion equation approach. However we need the condition $\theta^{0i} = 0$ for the spacetime noncommutativity. For the case $\theta^{0i} \neq 0$ of spacetime noncommutativity, it seems that Poincaré translation invariance for noncommutative scalar field theory cannot be set up from Heisenberg and quantum motion equation approach. In Sec. V, we discuss some of the problems.

## 2 Commutators of the Moyal star-products and their expectation values

According to the researches of gravitation and superstring theories [1,2], we know that spacetime may have noncommutative structures under the Planck scale. The noncommutativity of spacetime can be realized through introducing the commutation relation

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

for the spacetime coordinates, where $\theta^{\mu\nu}$ is a real antisymmetric matrix that parameterizes the noncommutativity of the spacetime. $\theta^{\mu\nu}$ has the dimension of square of length. For quantum field theories on noncommutative spacetime, they can be realized through introducing the Moyal star-product, i.e., all of the products between field functions or field operators that depend on spacetime coordinates are replaced by the Moyal star-products. The Moyal star-product of two fields $f(x)$ and $g(x)$ is defined to be

$$f(x) \star g(x) = \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1 \nu_1} \cdots \theta^{\mu_n \nu_n} \partial_{\mu_1} \cdots \partial_{\mu_n} f(x) \partial_{\nu_1} \cdots \partial_{\nu_n} g(x).$$

(2.2)
In noncommutative spacetime, the Moyal star-product can be regarded as the fundamental product operation. Therefore it is necessary to study the commutators of Moyal star-products for quantum fields on noncommutative spacetime. The Lagrangian for the noncommutative $\varphi^4$ scalar field theory is given by

$$L = \frac{1}{2}\partial^\mu \varphi \star \partial_\mu \varphi - \frac{1}{2}m^2 \varphi \star \varphi - \frac{1}{4!}\lambda\varphi \star \varphi \star \varphi \star \varphi .$$

(2.3)

The Fourier expansion of the free scalar field is given by

$$\varphi(x, t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} [a(k)e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t} + a^\dagger(k)e^{-i\mathbf{k} \cdot \mathbf{x} + i\omega t}] .$$

(2.4)

Here we adopt the usual Lorentz invariant form for the Fourier expansion, hence it is just the same as the case of the commutative spacetime, except that in Eq. (2.4) we take the spacetime coordinates to be noncommutative which satisfy the commutation relation (2.1).

It is reasonable to take the commutation relations for the creation and annihilation operators to be the same as the commutative spacetime:

$$[a(k), a^\dagger(k')] = \delta^3(\mathbf{k} - \mathbf{k}') ,$$

$$[a(k), a(k)] = 0 ,$$

$$[a^\dagger(k), a^\dagger(k')] = 0 .$$

(2.5)

The Moyal star-product of two field functions of Eq. (2.2) is defined at the same spacetime point. We can generalize Eq. (2.2) to define the Moyal star-product of two field functions on different spacetime points [4]:

$$f(x_1) \star g(x_2) = e^{i\sum_{\alpha, \beta} \theta^{\alpha \beta} \partial_\alpha \partial_\beta} f(x_1 + \alpha)g(x_2 + \beta)|_{\alpha = \beta = 0}$$

$$= f(x_1)g(x_2) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1 \nu_1} \ldots \theta^{\mu_n \nu_n} \partial_{\mu_1} \ldots \partial_{\mu_n} f(x_1)\partial_{\nu_1} \ldots \partial_{\nu_n} g(x_2) ,$$

(2.6)

which means that the commutation relation of spacetime coordinates (2.1) is generalized to arbitrary two different spacetime points:

$$[x_1^\mu, x_2^\nu] = i\theta^{\mu \nu} .$$

(2.7)

We define the commutator of two scalar fields of Moyal star-product to be

$$[\varphi(x), \varphi(y)]_\star = \varphi(x) \star \varphi(y) - \varphi(y) \star \varphi(x) .$$

(2.8)

We can call Eq. (2.8) the Moyal commutator for convenience. From the Fourier expansion of Eq. (2.4) for the scalar field, we can calculate the Moyal commutator for two scalar fields.
It is given by
\[
[\varphi(x), \varphi(y)]_\ast = \int \frac{d^3k d^3k'}{(2\pi)^3\sqrt{2\omega_k 2\omega_k'}} \left[ a(k)e^{-ikx} + a^\dagger(k)e^{ikx}, a(k')e^{-ik'y} + a^\dagger(k')e^{ik'y} \right]_\ast
\]
\[
= \int \frac{d^3k d^3k'}{(2\pi)^3\sqrt{2\omega_k 2\omega_k'}} \left\{ [a(k)e^{-ikx}, a(k')e^{-ik'y}]_\ast + [a(k)e^{-ikx}, a^\dagger(k')e^{ik'y}]_\ast + [a^\dagger(k)e^{ikx}, a(k')e^{-ik'y}]_\ast + [a^\dagger(k)e^{ikx}, a^\dagger(k')e^{ik'y}]_\ast \right\} .
\]
(2.9)

For the reason that there exist two kinds of noncommutative objects, i.e., field operators and spacetime coordinates, the spacetime coordinates now is noncommutative, we cannot apply the commutation relations for the creation and annihilation operators of Eq. (2.5) directly to obtain a \(c\)-number result for the Moyal commutator.

In order to obtain a \(c\)-number result for the Moyal commutator, we can calculate its vacuum state expectation value. We have
\[
\langle 0|[\varphi(x), \varphi(y)]_\ast |0\rangle
\]
\[
= \langle 0| \int \frac{d^3k d^3k'}{(2\pi)^3\sqrt{2\omega_k 2\omega_k'}} \left( a(k)a^\dagger(k')e^{-ikx} \ast e^{ik'y} - a(k')a^\dagger(k)e^{-ik'y} \ast e^{ikx} \right) |0\rangle
\]
\[
= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left[ e^{-ikx} \ast e^{iky} - e^{-iky} \ast e^{ikx} \right]
\]
\[
= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left[ \exp(\frac{i}{2}k \times k)e^{-i(x-y)} - \exp(\frac{i}{2}k \times k)e^{i(x-y)} \right] ,
\]
(2.10)
where we have applied Eq. (2.6) and we note
\[
k \times p = k_\mu \theta^{\mu\nu} p_\nu .
\]
(2.11)
Because \(\theta^{\mu\nu}\) is antisymmetric, \(k \times k = 0\), we obtain
\[
\langle 0|[\varphi(x), \varphi(y)]_\ast |0\rangle = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left[ e^{-ik(x-y)} - e^{ik(x-y)} \right]
\]
\[
= -i \int \frac{d^3k}{(2\pi)^3\omega_k} e^{ik(x-y)} \sin \omega_k(x_0 - y_0) = i\Delta(x - y) .
\]
(2.12)

We can see that the result of Eq. (2.12) is just equal to the commutator of two scalar fields in ordinary commutative spacetime:
\[
[\varphi(x), \varphi(y)] = -\frac{i}{(2\pi)^3} \int \frac{d^3k}{\omega_k} e^{ik(x-y)} \sin \omega_k(x_0 - y_0) = i\Delta(x - y) .
\]
(2.13)
It is obvious to see that this equality relies on the antisymmetry of $\theta^{\mu\nu}$.

The vacuum state expectation value of the equal-time Moyal commutator of two scalar fields can be obtained from Eq. (2.12):

$$\langle 0 | [\varphi(x, t), \varphi(y, t)] \star | 0 \rangle = \Delta(x - y, 0) = 0 .$$  \hfill (2.14)

The time derivative of the function $\Delta$ at the origin of the coordinates is given by

$$\left. \frac{\partial \Delta(x - y)}{\partial x_0} \right|_{x_0 = y_0} = -\delta^3(x - y) .$$  \hfill (2.15)

The conjugate momentum for the scalar field is defined to be

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi} .$$  \hfill (2.16)

From Eqs. (2.12) and (2.16) we can obtain the following vacuum state expectation value for the equal-time Moyal commutators:

$$\langle 0 | [\pi(x, t), \pi(y, t)] \star | 0 \rangle = 0 ,$$  \hfill (2.17)

$$\langle 0 | [\pi(x, t), \varphi(y, t)] \star | 0 \rangle = -i\delta^3(x - y) .$$  \hfill (2.18)

We can also calculate the expectation values between non-vacuum states for the Moyal commutators. Let $|\Psi\rangle$ represent a normalized non-vacuum physical state which is in the occupation eigenstate:

$$|\Psi\rangle = |N_{k_1}N_{k_2}\cdots N_{k_i}\cdots, 0 \rangle ,$$  \hfill (2.19)

where $N_{k_i}$ represents the occupation number of the momentum $k_i$. We can suppose that the occupation numbers are nonzero only on some separate momentums $k_i$. For all other momentums, the occupation numbers are zero. We use 0 to represent that the occupation numbers are zero on all the other momentums in Eq. (2.19). The state vector $|\Psi\rangle$ has the following properties:

$$\langle N_{k_1}N_{k_2}\cdots N_{k_i}\cdots | N_{k_1}N_{k_2}\cdots N_{k_i}\cdots \rangle = 1 ,$$  \hfill (2.20)

$$\sum_{N_{k_1}N_{k_2}\cdots} |N_{k_1}N_{k_2}\cdots N_{k_i}\cdots \rangle \langle N_{k_1}N_{k_2}\cdots N_{k_i}\cdots | = 1 ,$$  \hfill (2.21)

$$a(k_i)|N_{k_1}N_{k_2}\cdots N_{k_i}\cdots \rangle = \sqrt{N_{k_i}}|N_{k_1}N_{k_2}\cdots (N_{k_i} - 1)\cdots \rangle ,$$  \hfill (2.22)

$$a^\dagger(k_i)|N_{k_1}N_{k_2}\cdots N_{k_i}\cdots \rangle = \sqrt{N_{k_i} + 1}|N_{k_1}N_{k_2}\cdots (N_{k_i} + 1)\cdots \rangle .$$  \hfill (2.23)

Equation (2.21) is the completeness expression for the state vector $|\Psi\rangle$. Therefore Eq. (2.19) can represent an arbitrary scalar field quantum system. Then from Eq. (2.9) we can obtain
the expectation value between any non-vacuum physical states for the Moyal commutator (2.8) to be

\[
\langle \Psi | [\varphi(x), \varphi(y)]_{\ast} | \Psi \rangle = \langle \Psi | \int \frac{d^3k d^3k'}{(2\pi)^3 \sqrt{2\omega_k 2\omega_{k'}}} \left\{ [a(k)e^{-ikx}, a'(k')e^{ik'y}]_{\ast} + [a'(k)e^{ikx}, a(k')e^{-ik'y}]_{\ast} \right\} | \Psi \rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ e^{-ikx} \ast e^{iky} - e^{-iky} \ast e^{ikx} \right] = i\Delta(x - y),
\]

(2.24)

which is just equal to the vacuum state expectation value of Eq. (2.12). For the equal-time commutators, we can also obtain

\[
\langle \Psi | [\varphi(x, t), \varphi(y, t)]_{\ast} | \Psi \rangle = \Delta(x - y, 0) = 0,
\]

(2.25)

\[
\langle \Psi | [\pi(x, t), \pi(y, t)]_{\ast} | \Psi \rangle = 0,
\]

(2.26)

\[
\langle \Psi | [\pi(x, t), \varphi(y, t)]_{\ast} | \Psi \rangle = -i\delta^3(x - y).
\]

(2.27)

The properties of the commutation relations for the creation and annihilation operators of Eq. (2.5) are still reflected in the above evaluations for the non-vacuum state expectation values for the Moyal commutators.

Because \(\Delta(x - y)\) is a Lorentz invariant singular function, it has the property

\[
\Delta(x - y) = 0 \quad \text{for} \quad (x - y)^2 < 0.
\]

(2.28)

This means

\[
\langle 0 | [\varphi(x), \varphi(y)]_{\ast} | 0 \rangle = 0 \quad \text{for} \quad (x - y)^2 < 0,
\]

(2.29)

and similarly

\[
\langle \Psi | [\varphi(x), \varphi(y)]_{\ast} | \Psi \rangle = 0 \quad \text{for} \quad (x - y)^2 < 0.
\]

(2.30)

For quantum field theories on ordinary spacetime, the commutators satisfy the microscopic causality. For scalar field theory we have

\[
[\varphi(x), \varphi(y)] = 0 \quad \text{for} \quad (x - y)^2 < 0.
\]

(2.31)

This means that any two fields as physical observables commute with each other when they are separated by a spacelike interval. Thus any two fields as physical observables at different spacetime points can be measured precisely and independently of each other only if they cannot be connected by a light signal or any other physical information. For scalar field theory on noncommutative spacetime, although we cannot conclude that \( [\varphi(x), \varphi(y)]_{\ast} = 0 \) for \((x - y)^2 < 0\) from Eqs. (2.29) and (2.30), Eqs. (2.29) and (2.30) still represent the satisfying of microscopic causality for scalar field theory on noncommutative spacetime. This is because any physical measurement is taken under certain physical state. What the observer measures are all expectation values in fact.
However in the above we have only analyzed the microcausality property for the linear operator $\varphi(x)$. For the quadratic operators of free scalar field on noncommutative spacetime such as $\varphi(x) \star \varphi(x)$, their microcausality properties need to be studied further. Some of their results have been obtained in Refs. [10,11]. In addition, the microcausality problem discussed here is only restricted to free fields. Because for noncommutative field theories, there exist the UV/IR mixing problems [12,13]. The infrared singularities that come from non-planar diagrams may need one to invoke certain nonlocal terms in the renormalization of noncommutative field theories. These nonlocal terms may destroy the microcausality for quantum field theories on noncommutative spacetime.

3 The energy-momentum tensor of noncommutative $\varphi^{*4}$ scalar field theory

In ordinary commutative field theories, Poincaré translation invariance results the existence of locally conserved energy-momentum tensors. In this section we will derive the energy-momentum tensor for noncommutative $\varphi^{*4}$ scalar field theory from translation transformation and Lagrange field equation. The Lagrangian for noncommutative $\varphi^{*4}$ scalar field theory is given by Eq. (2.3). To consider an infinitesimal displacements of spacetime coordinates

$$x'^\mu = x^\mu + \epsilon^\mu ,$$

we can see in Eq. (2.1), because $\theta^{\mu\nu}$ does not depend on the coordinates, the spacetime noncommutative relations (2.1) and (2.7) are translation invariant. The Lagrangian of Eq. (2.3) does not depend on the coordinates explicitly, hence it is translation invariant. For the Lagrangian (2.3) in the form of Moyal star-products, it only contains the fields and their first order derivative, therefore we have

$$\mathcal{L} = \mathcal{L}(\varphi, \partial \varphi / \partial x_\mu) .$$

The field equation can still be obtained from the variation principle formally. It is given by

$$\frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial (\partial \varphi / \partial x_\mu)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 .$$

To substitute Eq. (2.3) into Eq. (3.3), we obtain

$$\Box \varphi + \frac{1}{3!}\lambda \varphi \star \varphi \star \varphi = 0 .$$

Under the infinitesimal displacements of Eq. (3.1), the Lagrangian will have a little displacement:

$$\delta \mathcal{L} = \mathcal{L}(x') - \mathcal{L}(x) = \epsilon_\mu \frac{\partial \mathcal{L}}{\partial x_\mu} .$$

On the other hand, from Eq. (3.2) we have

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \star \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial \varphi / \partial x_\mu)} \star \delta \left( \frac{\partial \varphi}{\partial x_\mu} \right) ,$$
where
\[ \delta \varphi = \varphi(x + \epsilon) - \varphi(x) = \epsilon_\mu \frac{\partial \varphi(x)}{\partial x_\mu}. \] (3.7)

From Eqs. (3.5), (3.6), and (3.3), we obtain

\[ \epsilon_\mu \frac{\partial \mathcal{L}}{\partial x_\mu} = \frac{\partial}{\partial x_\mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial \varphi/\partial x_\mu)} \star \epsilon_\nu \frac{\partial \varphi}{\partial x_\nu} \right). \] (3.8)

Because Eq. (3.8) is satisfied for arbitrary \( \epsilon_\mu \), we obtain

\[ \frac{\partial \mathcal{T}_{\mu\nu}}{\partial x_\mu} = 0, \] (3.9)

where \( \mathcal{T}_{\mu\nu} \) is the energy-momentum tensor defined to be

\[ \mathcal{T}_{\mu\nu} = -g_{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial (\partial \varphi/\partial x_\mu)} \star \frac{\partial \varphi}{\partial x_\nu}. \] (3.10)

To notice that the Moyal star-products are not invariant generally under the commutation of the orders of two functions, we need to write Eq. (3.10) in a symmetrized form for the Moyal star-products:

\[ \mathcal{T}_{\mu\nu} = -g_{\mu\nu} \mathcal{L} + \frac{1}{2} \left( \frac{\partial \mathcal{L}}{\partial (\partial \varphi/\partial x_\mu)} \star \frac{\partial \varphi}{\partial x_\nu} + \frac{\partial \varphi}{\partial x_\nu} \star \frac{\partial \mathcal{L}}{\partial (\partial \varphi/\partial x_\mu)} \right). \] (3.11)

In fact this can be resulted through symmetrizing the orders of the Moyal star-products between two functions properly in the above derivation. To substitute Eq. (2.3) into Eq. (3.11) we obtain

\[ \mathcal{T}_{\mu\nu} = \frac{1}{2} (\partial_\mu \varphi \star \partial_\nu \varphi + \partial_\nu \varphi \star \partial_\mu \varphi) - g_{\mu\nu} \mathcal{L}. \] (3.12)

In Refs. [14,15], Eq. (3.12) has been obtained from the translation invariance of the action. Here we derive it from a different method.

However the divergence of the energy-momentum tensor obtained above is not zero. In fact the divergence of the energy-momentum tensor of Eq. (3.12) is given by [14,15]

\[ \partial^\mu \mathcal{T}_{\mu\nu} = \lambda \left[ [\varphi, \partial_\nu \varphi]_{\star} \varphi^2 \right]_{\star}. \] (3.13)

Therefore the energy-momentum tensor of Eq. (3.12) is not locally conserved. From the integral property of the Moyal star-product, \( \mathcal{T}_{\mu\nu} \) is conserved to the integral over the whole spacetime:

\[ \partial^\mu \int d^4x \mathcal{T}_{\mu\nu} = \int d^4x \partial^\mu \mathcal{T}_{\mu\nu} = 0. \] (3.14)

Equation (3.14) can also be resulted from the Gauss theorem. We can analyze why we cannot obtain a locally conserved energy-momentum tensor from the above derivation. The reason is that: the Lagrangian (2.3) contains higher order derivative terms when it is expanded.
according to the Moyal star-product (2.2). If we expand the Lagrangian completely, it will contain higher order derivative terms up till infinite order. We may write it in the form

$$\mathcal{L} = \mathcal{L}(\phi, \partial\phi/\partial x_\mu, \partial^2\phi/\partial x_\mu\partial x_\nu, \cdots), \quad (3.15)$$

where the star-products are canceled with the ordinary products left. In such a case, the Lagrange field equation is not Eq. (3.3) in fact, and the variation of the Lagrangian under infinitesimal displacements of Eq. (3.1) is not Eq. (3.6). Therefore the above derivation of the energy-momentum tensor for the Lagrangian (2.3) given in the form of Moyal star-products is only formal. Thus we cannot obtain a locally conserved energy-momentum tensor from such a method. Perhaps one can obtain a locally conserved energy-momentum tensor from the completely expanded Lagrangian of the Moyal star-products. However because it will contain infinite terms, it is not realizable actually. In Ref. [16] the action for noncommutative gauge field theory is expanded to the first order of the parameter $\theta^{\mu\nu}$ and the energy-momentum tensor to the first order of $\theta^{\mu\nu}$ has been derived. However the energy-momentum tensor obtained there is not locally conserved, except to define a modified energy-momentum tensor. We consider that the reason is related with the fact the total Lagrangian of a noncommutative field theory contains infinite terms when it is expanded according to the Moyal star-product. In Ref. [17] the action for noncommutative gauge field theory is expanded to the second order of the parameter $\theta^{\mu\nu}$.

Although the energy-momentum tensor of Eq. (3.12) is not locally conserved, we can still define such a four-momentum $P_\mu$ formally:

$$P_\mu = \int d^3x T_{0\mu}. \quad (3.16)$$

From Eq. (3.12) we have

$$H = \int d^3x T_{00} = \int d^3x (\pi \star \pi - \mathcal{L}) = \int d^3x (\pi \star \dot{\phi} - \mathcal{L}), \quad (3.17)$$

$$P_i = \int d^3x T_{0i} = \int d^3x \frac{1}{2} (\pi \star \partial_i \phi + \partial_i \phi \star \pi). \quad (3.18)$$

We use the Bjorken metric throughout this paper [18]. Hence the three-dimensional momentum is $P = (-P_1, -P_2, -P_3)$. From Eq. (3.17), we can see that $H$ is equal to the definition of the total energy of noncommutative scalar field theory in the form of Moyal star-products. Because

$$\int d^3x \partial^\mu T_{\mu\nu} = \partial^0 \int d^3x T_{0\nu} + \int d^3x \partial^\nu T_{i\nu} = \partial^0 \int d^3x T_{0\nu}, \quad (3.19)$$

we obtain

$$\partial^0 P_\mu = \int d^3x \frac{\lambda}{4!} \left[ [\phi, \partial_\mu \phi]_\star, \phi^4 \right]_\star. \quad (3.20)$$

Because the right hand side of Eq. (3.20) is not zero generally, we cannot define a conserved four-momentum $P_\mu$ for noncommutative $\varphi^4$ scalar field theory.
For the case $\theta^0_i = 0$ of spacetime noncommutativity, a totally conserved energy-momentum four-vector can be defined for noncommutative $\varphi^4$ scalar field theory [14,15]. This is because the integral of Moyal star-products on three-dimensional space satisfies the cyclic property
\[
\int d^3x (\varphi_1 \star \varphi_2 \star \cdots \star \varphi_n)(x) = \int d^3x (\varphi_n \star \varphi_1 \star \cdots \star \varphi_{n-1})(x) \tag{3.21}
\]
when $\theta^0_i = 0$. Therefore we have
\[
\partial^0 P^\mu = \int d^3x \frac{\lambda}{4!} \left[ [\varphi, \partial^\mu \varphi]_\star, \varphi^2 \right]_\star = 0 \tag{3.22}
\]
for the case $\theta^0_i = 0$. In Sec. IV, we will demonstrate the Poincaré translation invariance for noncommutative $\varphi^4$ scalar field theory from the Heisenberg and quantum motion equation approach. However this will need the condition that the energy-momentum $P^\mu$ are conserved quantities. Because for the case $\theta^0_i = 0$ of spacetime noncommutativity, the energy-momentum $P^\mu$ are conserved quantities, this make us possible to establish the Poincaré translation invariance for noncommutative $\varphi^4$ scalar field theory from Heisenberg and quantum motion equation approach for the case $\theta^0_i = 0$ of spacetime noncommutativity. From the above analysis we can see that for noncommutative field theories, although the Lagrangians and actions may be translation invariant, it may not necessarily result the existence of locally conserved energy-momentum tensors.

4 Quantum motion equation and Poincaré translation invariance for noncommutative scalar field theory

4.1 Heisenberg equation on noncommutative spacetime

The fundamental evolution equation for quantum fields in ordinary commutative spacetime is Heisenberg equation. The Heisenberg equations for the scalar field and its conjugate momentum are
\[
\begin{align*}
i[H, \varphi(x)] &= \frac{\partial \varphi(x)}{\partial t}, \tag{4.1} \\
i[H, \pi(x)] &= \frac{\partial \pi(x)}{\partial t}.	ag{4.2}
\end{align*}
\]
The solutions of Eqs. (4.1) and (4.2) can be given by the following formal integrals:
\[
\begin{align*}
\varphi(\mathbf{x}, t) &= e^{iHt} \varphi(\mathbf{x}, 0) e^{-iHt}, \tag{4.3} \\
\pi(\mathbf{x}, t) &= e^{iHt} \pi(\mathbf{x}, 0) e^{-iHt} \tag{4.4}.
\end{align*}
\]
For quantum field theories on noncommutative spacetime, such as noncommutative scalar field theory, its Hamiltonian can be defined by
\[
H = \int d^3 x \mathcal{H} = \int d^3 x (\pi \star \dot{\varphi} - \mathcal{L}), \tag{4.5}
\]
where $L$ is the Lagrangian on noncommutative spacetime. For noncommutative $\varphi^4$ scalar field theory, its Lagrangian is given by Eq. (2.3). Furthermore to consider that the Moyal star-products are not invariant generally under the commutation of the orders of two functions, we need to write Eq. (4.5) in a symmetrized form for fields and their conjugate momentums such as

$$H = \int d^3x \mathcal{H} = \int d^3x \left( \frac{1}{2} \pi \ast \dot{\varphi} + \frac{1}{2} \dot{\varphi} \ast \pi - L \right). \tag{4.6}$$

Because in Eqs. (4.1) and (4.2), the commutators are the usual commutators in commutative spacetime, for quantum field theories on noncommutative spacetime, in order to make Eqs. (4.1) and (4.2) as the fundamental evolution equations for fields and their conjugate momentums, we need to expand the Hamiltonian according to the Moyal star-product of Eq. (2.2) in principle. Supposing that we have expanded the Hamiltonian of Eq. (4.6) according to the Moyal star-product (2.2), then we can still regard Eqs. (4.1) and (4.2) as the fundamental evolution equations for fields and their conjugate momentums on noncommutative spacetime.

We can find that the formal solutions of Eqs. (4.1) and (4.2) can also be given by the following integrals:

$$\varphi(x, t) = e^{iHt} \ast \varphi(x, 0) \ast e^{-iHt}, \tag{4.7}$$

$$\pi(x, t) = e^{iHt} \ast \pi(x, 0) \ast e^{-iHt}, \tag{4.8}$$

i.e., in Eqs. (4.3) and (4.4), we replace the ordinary products by the Moyal star-products. This can be verified through substituting Eqs. (4.7) and (4.8) into Eqs. (4.1) and (4.2). On the other hand, we can see that Heisenberg equations (4.1) and (4.2) can be rewritten in the form

$$i[H, \varphi(x)]_* = \frac{\partial \varphi(x)}{\partial t}, \tag{4.9}$$

$$i[H, \pi(x)]_* = \frac{\partial \pi(x)}{\partial t}. \tag{4.10}$$

This is because we can consider that the integral of Hamiltonian density over the whole space in Eq. (4.6) has been carried out first, thus $H$ does not rely on the space variables. On the other hand, because $H$ is the total energy of the system, we can think that it does not rely on the time variable. This makes

$$H \cdot \varphi(x) = H \ast \varphi(x) \quad \text{and} \quad \varphi(x) \cdot H = \varphi(x) \ast H,$$

$$H \cdot \pi(x) = H \ast \pi(x) \quad \text{and} \quad \pi(x) \cdot H = \pi(x) \ast H \tag{4.11}$$

according to the definition of the Moyal star-product (2.2). Thus we have

$$[H, \varphi(x)] = [H, \varphi(x)]_* \quad \text{and} \quad [H, \pi(x)] = [H, \pi(x)]_* \tag{4.12}.$$

This makes the establishment of Eqs. (4.9) and (4.10). In fact we can see that Eqs. (4.7) and (4.8) are also the formal integrals of Eqs. (4.9) and (4.10), under the condition that $H$ does not rely on the spacetime variables. This means that Heisenberg equations (4.1) and
(4.2) can be generalized to the form of noncommutative spacetime directly. However we have seen in Sec. III that for noncommutative field theories, the total energy of the system may not be a conserved quantity generally. This means that the Hamiltonian $H$ may depend on the time variable. For the case that $H$ depends on the time variable, the satisfying of Eqs. (4.11) and (4.12) need the condition $\theta^0 = 0$ for the spacetime noncommutativity. Under such a condition, the formal integrals of Eqs. (4.9) and (4.10) can still be given by Eqs. (4.7) and (4.8). Therefore for noncommutative field theories of general case, the generalization of Heisenberg equations (4.1) and (4.2) to the form of Eqs. (4.9) and (4.10) need the condition $\theta^0 = 0$ of the spacetime noncommutativity.

4.2 Heisenberg relations on noncommutative spacetime

Quantum field theories in ordinary spacetime satisfy the Poincaré translation invariance. To consider an infinitesimal displacement of the spacetime coordinates

$$x'^\mu = x^\mu + \epsilon^\mu,$$  

(4.13)

whose generation operator is given by

$$U(\epsilon) = \exp(i\epsilon_\mu P^\mu) \approx 1 + i\epsilon_\mu P^\mu,$$  

(4.14)

where $P^\mu$ are the generators of the displacement transformation. Under the displacement transformation (4.13), the scalar field satisfies

$$U(\epsilon)\varphi(x)U(\epsilon)^{-1} = \varphi(x + \epsilon).$$  

(4.15)

From Eq. (4.15) we obtain

$$i[P^\mu, \varphi(x)] = \frac{\partial \varphi(x)}{\partial x_\mu},$$  

(4.16)

which is the Heisenberg relations that should be satisfied for the Poincaré translation invariance of quantum field theory. In Eq. (4.16), $P^\mu$ are constructed from the energy-momentum tensor. Because $P^\mu$ are the generators of the displacement transformation, they should be conserved quantities of motion. For quantum field theories on ordinary spacetime, Eq. (4.16) is verified through the equal-time commutation relations and Lagrange field equations [18].

For quantum field theories on noncommutative spacetime, if they satisfy the Poincaré translation invariance, it need to be guaranteed same by the Heisenberg relations (4.16). However the four-momentum $P^\mu$ should be conserved quantities of motion. But we have seen in Sec. III that it is difficult to define a locally conserved energy-momentum tensor and hence a conserved four-momentum for noncommutative field theories generally. If $P^\mu$ are not the conserved quantities of motion, the satisfying of Heisenberg relations has no meaning. However we have seen in Sec. III that for noncommutative $\varphi^4$ scalar field theory, for the case $\theta^{0i} = 0$ of spacetime noncommutativity, the four-momentum $P^\mu$ defined through Eqs. (3.16) to (3.18) are conserved quantities of motion. Therefore for such a case, we are possible to verify the Poincaré translation invariance from the Heisenberg relations (4.16).
As in Eqs. (4.1) and (4.2), the commutators in Eq. (4.16) are commutators of ordinary products. Thus in order to verify the Poincaré translation invariance from Eq. (4.16), we need to expand the four-momentum $P^\mu$ according to the Moyal star-product. However there will be infinite expansion terms and we cannot carry out to obtain them completely in fact. Therefore we hope to keep the products between the field operators to be the Moyal star-products in $P^\mu$ and to apply the Moyal commutators to verify Eq. (4.16) for quantum field theories on noncommutative spacetime.

In Eq. (4.16), if we consider that the integrals of the four-momentum densities over the whole space have been carried out first, then $P^\mu$ do not rely on the space variables. At the same time, if $P^\mu$ are conserved quantities of motion, then $P^\mu$ do not rely on the time variable. Therefore the products between $P^\mu$ and $\varphi(x)$ have the property

$$P^\mu \cdot \varphi(x) = P^\mu \star \varphi(x), \quad \varphi(x) \cdot P^\mu = \varphi(x) \star P^\mu$$

(4.17)

according to the definition of the Moyal star-product (2.2) if we consider that the space integral operations have been performed first. However for Eq. (4.17), we can see that we need the condition $\theta^{0i} = 0$ if the four-momentum $P^\mu$ are not the conserved quantities which depend on time. But we know that for noncommutative field theories, the four-momentum $P^\mu$ that obtained and defined from the method of Sec. III are not conserved quantities of motion generally. Therefore we need the condition $\theta^{0i} = 0$ of spacetime noncommutativity for the relation (4.17) for noncommutative field theories generally. Under such a condition, equation (4.16) can be rewritten in the form

$$i[P^\mu, \varphi(x)]_\star = \frac{\partial \varphi(x)}{\partial x_\mu}$$

(4.18)

equivalently according to the relation (4.17). This means that it is possible for us to verify the Poincaré translation invariance for noncommutative field theories from the Moyal commutators, i.e., to keep the products between field operators to be the Moyal star-products in $P^\mu$ and not to expand them according to Eq. (2.2). And we know that under the condition $\theta^{0i} = 0$, the four-momentum $P^\mu$ defined through Eqs. (3.16) to (3.18) are conserved quantities of motion for noncommutative $\varphi^4$ scalar field theory.

### 4.3 Moyal commutators for coupling fields

For quantum field theories on ordinary spacetime, Heisenberg relations (4.16) are verified through the equal-time commutation relations. For scalar field theory these equal-time commutation relations are

$$[\varphi(x, t), \varphi(y, t)] = 0,$$

(4.19)

$$[\pi(x, t), \pi(y, t)] = 0,$$

(4.20)

$$[\pi(x, t), \varphi(y, t)] = -i\delta^3(x - y).$$

(4.21)

These equal-time commutation relations are first established for free fields, then through Heisenberg equations they are proved to be true also for coupling fields [18]. For the commutators of Moyal star-products, as seen in Sec. II, they are not c-number results. In order
to obtain the $c$-number results for the Moyal commutators, we need to evaluate their vacuum state or non-vacuum state expectation values. For scalar field, we have obtained in Sec. II the expectation values for the Moyal commutators. For the sake of convenience, we write down them here again:

$$\langle \Psi | [\varphi(x,t) , \varphi(y,t)] | \Psi \rangle = 0 \ , \quad (4.22)$$

$$\langle \Psi | [\pi(x,t) , \pi(y,t)] | \Psi \rangle = 0 \ , \quad (4.23)$$

$$\langle \Psi | [\pi(x,t) , \varphi(y,t)] | \Psi \rangle = -i \delta^3(x - y) \ . \quad (4.24)$$

Here we use $|\Psi\rangle$ to represent the vacuum state and non-vacuum state together. As shown in Sec. II, these relations are established for free fields at this time.

We hope to verify that these relations are also satisfied for coupling fields. Like that for quantum field theories on ordinary commutative spacetime, we can suppose that at time $t=0$, the interactions have not participated, the interactions participate adiabatically at time $t > 0$. Then at time $t=0$ the field operators $\varphi(x,0)$ and $\pi(x,0)$ can be expanded in the form of free fields:

$$\varphi(x,0) = \int \frac{d^3k}{\sqrt{(2\pi)^32\omega_k}} [a(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x}] \ , \quad (4.25)$$

$$\pi(x,0) = \varphi^\dagger(x,0) = \int \frac{d^3k}{\sqrt{(2\pi)^32\omega_k}} (-i\omega)[a(k)e^{ik\cdot x} - a^\dagger(k)e^{-ik\cdot x}] \ . \quad (4.26)$$

To suppose that the expansion coefficients $a^\dagger(k)$ and $a(k)$ satisfy the same commutation relations (2.5) of free fields, this will make the coupling fields satisfy the relations (4.22) to (4.24) at time $t=0$:

$$\langle \Psi | [\varphi(x,0) , \varphi(y,0)] | \Psi \rangle = 0 \ , \quad (4.27)$$

$$\langle \Psi | [\pi(x,0) , \pi(y,0)] | \Psi \rangle = 0 \ , \quad (4.28)$$

$$\langle \Psi | [\pi(x,0) , \varphi(y,0)] | \Psi \rangle = -i \delta^3(x - y) \ . \quad (4.29)$$

These relations can be resulted through repeating the procedure of Sec. II. Or equivalently we can regard that the fields are in the free propagating state from time $t = -\infty$ to $t = 0$. Then the coupling fields satisfy the relations (4.27) to (4.29) at time $t=0$ as a corollary of the results of Sec. II.

Now we need to verify that the relations (4.27) to (4.29) are also satisfied for coupling fields at an arbitrary time. To take Eq. (4.29) for example, we product it with $e^{iEt}$ from the left hand side and product it with $e^{-iEt}$ from the right hand side:

$$e^{iEt} \langle \Psi | [\pi(x,0) , \varphi(y,0)] | \Psi \rangle e^{-iEt} = e^{iEt}[-i \delta^3(x - y)]e^{-iEt} = -i \delta^3(x - y) \ , \quad (4.30)$$

where $E$ is the total energy of the state $|\Psi\rangle$. As postulated in Sec. II, the state vector $|\Psi\rangle$ is in the occupation eigenstate, we can suppose that it is also in the energy eigenstate at the same time. Hence we have

$$\langle \Psi | e^{iHt} = \langle \Psi | e^{iEt} \quad \text{and} \quad e^{-iHt}|\Psi\rangle = e^{-iEt}|\Psi\rangle \ . \quad (4.31)$$
To apply Eq. (4.31) we have

\[
\begin{align*}
e^{iEt} \langle \Psi | [\pi(x,0), \varphi(y,0)]_\star | \Psi \rangle e^{-iEt} \\
= & \langle \Psi | e^{iHt} [\pi(x,0), \varphi(y,0)] e^{-iHt} | \Psi \rangle \\
= & \langle \Psi | [e^{iHt} \pi(x,0) e^{-iHt}, e^{iHt} \varphi(y,0) e^{-iHt}]_\star | \Psi \rangle ,
\end{align*}
\]

(4.32)

where in the third line, we have taken \( H \) as being integrated therefore it does not depend on the space coordinates. Then from Eqs. (4.3) and (4.4), we obtain

\[
\langle \Psi | [\pi(x,t), \varphi(y,t)]_\star | \Psi \rangle = -i\delta^3(x - y)
\]

(4.33)

for the coupling fields at an arbitrary time. For Eqs. (4.27) and (4.28), we can also obtain

\[
\langle \Psi | [\varphi(x,t), \varphi(y,t)]_\star | \Psi \rangle = 0
\]

(4.34)

and

\[
\langle \Psi | [\pi(x,t), \pi(y,t)]_\star | \Psi \rangle = 0
\]

(4.35)

for the coupling fields at an arbitrary time. To mention here that in the above derivation, the state vector \( | \Psi \rangle \) is an arbitrary state vector for the scalar field system including the vacuum state, while not necessarily to be the state of the interacting and evolitional quantum field system at the time \( t \). Of course the relations (4.33) to (4.35) are also satisfied for the vacuum state for coupling fields at an arbitrary time:

\[
\begin{align*}
\langle 0 | [\pi(x,t), \varphi(y,t)]_\star | 0 \rangle = -i\delta^3(x - y) , \\
\langle 0 | [\varphi(x,t), \varphi(y,t)]_\star | 0 \rangle = 0 , \\
\langle 0 | [\pi(x,t), \pi(y,t)]_\star | 0 \rangle = 0 .
\end{align*}
\]

(4.36) (4.37) (4.38)

However in the following demonstration, what we need are relations (4.36) to (4.38), while not (4.33) to (4.35).

Another relation for the equal-time Moyal commutators that we need to use in the verifying of the Poincaré translation invariance for noncommutative scalar field theory is

\[
\langle 0 | [\partial_i \varphi(x,t), \varphi(y,t)]_\star | 0 \rangle = 0 .
\]

(4.39)

The corresponding relation of Eq. (4.39) in ordinary commutative spacetime is

\[
[\partial_i \varphi(x,t), \varphi(y,t)] = 0 .
\]

(4.40)

For the case of the free field, Eq. (4.39) can be calculated like that for Eq. (2.10). We need not to write down the calculation completely. We only write down its last expression here:

\[
\begin{align*}
\langle 0 | [\partial_i \varphi(x,t), \varphi(y,t)]_\star | 0 \rangle &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} (ik_i) \left[ e^{ik(x-y)} + e^{-ik(x-y)} \right] ,
\end{align*}
\]

(4.41)

where we note \( k = (k_1, k_2, k_3) \). In Eq. (4.41) because the integrand now is an odd function, this makes the whole integral to be zero. Then through the argument of this subsection, we can obtain that Eq. (4.41) is also held for coupling fields.
4.4 Poincaré translation invariance

Now we can verify the Poincaré translation invariance for noncommutative $\varphi^4$ scalar field theory from the Heisenberg relations (4.18) on noncommutative spacetime. From Eqs. (3.17) and (3.18), we have

$$P^0 = H = \int d^3 x \mathcal{H}(\pi, \varphi),$$

$$\mathcal{H}(\pi, \varphi) = \frac{1}{2} [\pi(x, t) * \pi(x, t) + \partial_i \varphi(x, t) * \partial_i \varphi(x, t)$$

$$+ m^2 \varphi(x, t) * \varphi(x, t) + \frac{2}{4!} \lambda \varphi(x, t) * \varphi(x, t) * \varphi(x, t) * \varphi(x, t)],$$

and

$$P^i = - \int d^3 x \frac{1}{2} [\pi(x, t) * \partial_i \varphi(x, t) + \partial_i \varphi(x, t) * \pi(x, t)].$$

We first verify the space components of Eq. (4.18). For the three space components of Eq. (4.18), they are in the equal positions. Thus we need to check only one of them. We take the $x^1$-component for example. We have

$$i[P^1, \varphi(x', t)]_* = - \frac{1}{2} \int d^3 x [\pi(x, t) * \partial_1 \varphi(x, t) + \partial_1 \varphi(x, t) * \pi(x, t), \varphi(x', t)]_*.$$

To use the formula

$$[a * b, c]_* = a * [b, c]_* + [a, c]_* * b,$$

we have

$$[\pi(x, t) * \partial_1 \varphi(x, t), \varphi(x', t)]_* = \pi(x, t) * [\partial_1 \varphi(x, t), \varphi(x', t)]_* + [\pi(x, t), \varphi(x', t)]_* * \partial_1 \varphi(x, t),$$

$$[\partial_1 \varphi(x, t) * \pi(x, t), \varphi(x', t)]_* = \partial_1 \varphi(x, t) * [\pi(x, t), \varphi(x', t)]_* + [\partial_1 \varphi(x, t), \varphi(x', t)]_* * \pi(x, t).$$

For the reason that the Moyal commutators are not $c$-number functions, we can evaluate the vacuum expectation values for Eqs. (4.44) and (4.46). We have

$$\langle 0 | [\pi(x, t) * \partial_1 \varphi(x, t), \varphi(x', t)]_* | 0 \rangle$$

$$= \langle 0 | \pi(x, t) * [\partial_1 \varphi(x, t), \varphi(x', t)]_* | 0 \rangle + \langle 0 | [\pi(x, t), \varphi(x', t)]_* * \partial_1 \varphi(x, t) | 0 \rangle,$$

$$\langle 0 | [\partial_1 \varphi(x, t) * \pi(x, t), \varphi(x', t)]_* | 0 \rangle$$

$$= \langle 0 | \partial_1 \varphi(x, t) * [\pi(x, t), \varphi(x', t)]_* | 0 \rangle + \langle 0 | [\partial_1 \varphi(x, t), \varphi(x', t)]_* * \pi(x, t) | 0 \rangle.$$

(4.47)
In order to use the previous result for the Moyal commutators, we insert the unit operator $|0\rangle \langle 0| = I$ inside Eq. (4.47). Thus we obtain

$$\langle 0 | [\pi(x, t) \star \partial_i \varphi(x, t), \varphi(x', t)]_\star |0\rangle = \langle 0 |[\partial_i \varphi(x, t), \varphi(x', t)]_\star |0\rangle + \langle 0 |[\pi(x, t), \varphi(x', t)]_\star |0\rangle \star \langle 0 | \partial_i \varphi(x, t)|0\rangle ,$$

(4.48)

To apply Eqs. (4.36) and (4.39), we obtain

$$\langle 0 | [\pi(x, t) \star \partial_i \varphi(x, t), \varphi(x', t)]_\star |0\rangle = -i \delta^3(x - x') \star \langle 0 | \partial_i \varphi(x, t)|0\rangle ,$$

$$\langle 0 | [\partial_i \varphi(x, t) \star \pi(x, t), \varphi(x', t)]_\star |0\rangle = -i \langle 0 | \partial_i \varphi(x, t)|0\rangle \star \delta^3(x - x') .$$

(4.49)

To substitute Eq. (4.49) into Eq. (4.44) we have

$$\langle 0 | i [P^1, \varphi(x', t)]_\star |0\rangle = -\frac{1}{2} \int d^3 x \delta^3(x - x') \star \langle 0 | \partial_i \varphi(x, t)|0\rangle + \langle 0 | \partial_i \varphi(x, t)|0\rangle \star \delta^3(x - x')]$$

$$= \frac{1}{2} \langle 0 | \int d^3 x [\delta^3(x - x') \star \partial^1 \varphi(x, t) + \partial^1 \varphi(x, t) \star \delta^3(x - x')] |0\rangle .$$

(4.50)

To compare the two sides of Eq. (4.50), we can move the vacuum states away. Thus we obtain

$$i [P^1, \varphi(x', t)]_\star = \frac{1}{2} \int d^3 x [\delta^3(x - x') \star \partial^1 \varphi(x, t) + \partial^1 \varphi(x, t) \star \delta^3(x - x')] .$$

(4.51)

---

1Here, we can suppose the state vector $|0\rangle$ is in the momentum representation, therefore it does not rely on the coordinates, we have $|0\rangle \langle 0| = |0\rangle \star |0\rangle$. To be more reasonable, we may insert the complete set of the state vectors such as that of Eq. (2.21). We rewrite Eq. (2.21) as

$$\sum_{N_k_1 N_k_2 \cdots} |N_k_1 N_k_2 \cdots \rangle \langle N_k_1 N_k_2 \cdots | = |0\rangle \langle 0| + \sum_{M=1}^{\infty} |M\rangle \langle M| = |0\rangle \star |0\rangle + \sum_{M=1}^{\infty} |M\rangle \star \langle M| .$$

Here we use $|1\rangle$ to represent the state vectors that the occupation numbers on each momentum are no more than one, however at least one of the occupation number is one. We use $|2\rangle$ to represent the state vectors that the occupation numbers on each momentum are no more than two, however at least one of the occupation number is two. Similarly for $M \geq 3$. We have supposed that the state vectors $|M\rangle$ are in the momentum representation, therefore they do not rely on the coordinates, and this makes $|M\rangle \langle M| = |M\rangle \star \langle M|$. In the above expression, we have omitted a summation index for each $|M\rangle$. We can obtain that in Eq. (4.48), the contributions that come from the state vectors $|1\rangle$, $|2\rangle$, $\ldots$ are all zero. For example, we can obtain $\langle 1 | [\partial_i \varphi(x, t), \varphi(x', t)]_\star |0\rangle = 0$, $\langle 0 | \pi(x, t)|2\rangle = 0$, $\langle 3 | [\partial_i \varphi(x, t), \varphi(x', t)]_\star |0\rangle = 0$, $\langle 0 | \pi(x, t)|3\rangle = 0$, etc. For free fields it is so. For coupling fields, it is also as such. We omit to write down the detailed analysis here. Therefore we can only insert the expression $|0\rangle \langle 0| = I$ equivalently.
Under the condition $\theta^{0i} = 0$ of the spacetime noncommutativity, we have the formula
\[
\int d^3x \delta^3(x - x') \ast f(x, t) = \int d^3x f(x, t) \ast \delta^3(x - x') = f(x', t) .
\] (4.52)

A derivation for Eq. (4.52) is given in the Appendix. To apply Eq. (4.52) in Eq. (4.51), we obtain
\[
i[P^1, \varphi(x', t)]_\ast = \partial^1 \varphi(x', t) .
\] (4.53)

Or equivalently, we write Eq. (4.53) as
\[
i[P^1, \varphi(x, t)]_\ast = \frac{\partial \varphi(x, t)}{\partial x_{\mu}} .
\] (4.54)

Thus we have finished the proof for the $x$-component of the Heisenberg relations (4.18). For the $y$- and $z$-component of the Heisenberg relations (4.18), because they are the same as the $x$-component, we need not to repeat it.

Now we need to check the time component of the Heisenberg relations (4.18). The method is the same as for the $x$-component. To use Eq. (4.45) and to insert the unit operator $|0\rangle \ast \langle 0| = I$, \footnote{The same reason at that of Eq. (4.48), we need to insert the complete set of the state vectors. However the contribution that come from non-vacuum states are all zero in Eq. (4.55). Hence we can only insert the expression $|0\rangle \langle 0| = I$ equivalently. Because the state vector $|0\rangle$ is in the momentum representation, it does not rely on the coordinates, we have $|0\rangle \langle 0| = |0\rangle \ast \langle 0|.$} then to use Eqs. (4.37) and (4.39), we have
\[
\langle 0| [\partial_t \phi(x, t) \ast \partial_t \phi(x, t), \varphi(x', t)]_\ast |0\rangle = 0 ,
\]
\[
\langle 0| [\varphi(x, t) \ast \varphi(x, t), \varphi(x', t)]_\ast |0\rangle = 0 ,
\]
\[
\langle 0| [\varphi(x, t) \ast \varphi(x, t) \ast \varphi(x, t) \ast \varphi(x, t), \varphi(x', t)]_\ast |0\rangle = 0
\]
\[
+ \langle 0| [\varphi(x, t) \ast \varphi(x, t), \varphi(x', t)]_\ast \ast \varphi(x, t) \ast \varphi(x, t) |0\rangle = 0 .
\] (4.55)

To use Eq. (4.45) we obtain
\[
\langle 0| i[H, \varphi(x', t)]_\ast |0\rangle
\]
\[
= \frac{i}{2} \langle 0| \int d^3x [\pi(x, t) \ast \pi(x, t), \varphi(x', t)]_\ast |0\rangle
\]
\[
= \frac{i}{2} \int d^3x \left( \langle 0| [\pi(x, t), \varphi(x', t)]_\ast \ast \pi(x, t) |0\rangle + \langle 0| [\pi(x, t), \varphi(x', t)]_\ast |0\rangle \right) .
\] (4.56)

To insert the unit operator $|0\rangle \ast \langle 0| = I$, the same reason as that of Eqs. (4.48) and (4.55), then to use Eqs. (4.36) and (4.52), we obtain
\[
\langle 0| i[H, \varphi(x', t)]_\ast |0\rangle = \langle 0| [\pi(x', t)]_\ast |0\rangle = \langle 0| \phi(x', t) |0\rangle .
\] (4.57)
To move away the vacuum states in both sides of Eq. (4.57) and to replace $x'$ by $x$, we obtain

$$i[H, \varphi(x, t)]_* = \frac{\partial \varphi(x, t)}{\partial t} . \quad (4.58)$$

Thus we have proved the time component of the Heisenberg relations (4.18).

All together we have proved that for noncommutative $\varphi^4$ scalar field theory, under the condition $\theta^{0i} = 0$ of the spacetime noncommutativity, $\varphi(x)$ satisfies the equations

$$i[P^\mu, \varphi(x)]_* = \frac{\partial \varphi(x)}{\partial x_\mu} , \quad (4.59)$$

which is the Heisenberg relations in the form of Moyal star-products. From Eq. (4.17), we know that Eq. (4.59) (i.e., Eq. (4.18)) is equivalent to Eq. (4.16) for the case $\theta^{0i} = 0$ of the spacetime noncommutativity. This means that Heisenberg relations (4.16) in the form of the commutation relations of ordinary products are satisfied for noncommutative $\varphi^4$ scalar field theory. Thus we have proved that Poincaré translation invariance is satisfied for noncommutative $\varphi^4$ scalar field theory from the Heisenberg and quantum motion equation approach.

However we have seen that for our proof, we need the condition $\theta^{0i} = 0$ of the spacetime noncommutativity, i.e., we need the spacetime noncommutativity to be spacelike. For the four-momentum $P^\mu$ to be conserved quantities of motion for noncommutative $\varphi^4$ scalar field theory, the equivalence between Eq. (4.59) and (4.16), and the satisfying of Eq. (4.52), they all need $\theta^{0i}$ to be zero. This means that for noncommutative $\varphi^4$ scalar field theory, its Poincaré translation invariance can only be held for spacelike noncommutativity of the spacetime. On the other hand, it was shown in Refs. [19-21] that the unitarity of noncommutative field theories will also lose when $\theta^{0i} \neq 0$. For the light-like noncommutativity, i.e., for the case $\theta^{0i} = -\theta^{1i}$ with all other components of $\theta^{\mu\nu}$ to be zero, it was argued in Ref. [22] that the unitarity of noncommutative field theories are still held. However according to the study of this paper, the Poincaré translation invariance for noncommutative field theories may not hold for light-like noncommutativity of the spacetime either.

Because the demonstration of the Poincaré translation invariance for noncommutative $\varphi^4$ scalar field theory in this paper depends on the definition of the energy-momentum tensor of Eq. (3.12), while we have seen in Eq. (3.13) that the energy-momentum tensor of Eq. (3.12) is not locally conserved, this makes the demonstration for the Poincaré translation invariance of noncommutative $\varphi^4$ scalar field theory of this paper not very ideal. A remedy for this problem is to exert a constraint equation for the energy-momentum tensor:

$$\frac{\lambda}{4!} [\varphi, \partial_\mu \varphi]_* \cdot \varphi^2_* = 0 . \quad (4.60)$$

This will make the energy-momentum tensor of Eq. (3.12) to be locally conserved necessarily. However this will bring more complications for the classical and quantum motions for the noncommutative $\varphi^4$ scalar field theory.

5 Discussion

In the above sections, we have studied the Moyal commutators and Poincaré translation
invariance for noncommutative scalar field theory. In noncommutative spacetime we can regard the Moyal star-product as the basic product operation. Thus we need to study the commutators of Moyal star-products for quantum fields on noncommutative spacetime. In Sec. II of this paper, we analyzed the Moyal commutators for scalar fields. We find that because of the noncommutativity of spacetime coordinates, the Moyal commutators are not $c$-number functions. In order to obtain the $c$-number results for Moyal commutators, we need to evaluate their vacuum state or non-vacuum state expectation values. We obtain that the expectation values of Moyal commutators are equal to the corresponding commutators in ordinary commutative spacetime. Then from the expectation values of the Moyal commutators for scalar fields, we analyzed the microcausality problem for noncommutative scalar field theory. We find that the expectation values of the Moyal commutators are zero if two spacetime points are separated by a spacelike interval. Therefore the microcausality is held for the linear operator $\phi(x)$ of free scalar field on noncommutative spacetime. For the quadratic operators of free scalar field on noncommutative spacetime such as $\phi(x) \star \phi(x)$, their microcausality properties need to be studied further. Some of their results have been obtained in Refs. [10,11]. In addition, the microcausality problem discussed here is only restricted to free fields. Because for noncommutative field theories, there exist the UV/IR mixing problems [12,13]. The infrared singularities that come from non-planar diagrams may need people to invoke certain nonlocal terms in the renormalization of noncommutative field theories. These nonlocal terms may destroy the microcausality for quantum field theories on noncommutative spacetime. In Ref. [6], through supposing the spectral measure to be the form of $SO(1,1) \times SO(2)$ invariance, the authors obtained that microcausality is violated for quantum fields on noncommutative spacetime generally. However, such a conclusion is a necessary result of the breakdown of the Lorentz invariance in Ref. [6].

Poincaré translation invariance is a fundamental property for quantum field theories. For quantum field theories in ordinary spacetime, Poincaré translation invariance is verified through the satisfying of Heisenberg relations (4.16). For noncommutative field theories, when the four-momentum $P^\mu$ are expanded according to the Moyal star-product (2.2), there will exist infinite expansion terms hence it is impossible for us to verify the Poincaré translation invariance for quantum field theories on noncommutative spacetime from Eq. (4.16). Under the condition that $P^\mu$ are conserved quantities of motion, or $\theta^0 = 0$ if $P^\mu$ are not conserved quantities of motion, we can rewrite the Heisenberg relations in the form of Moyal commutators. Thus we obtained Eq. (4.18). From Eq. (4.18) and to use the vacuum expectation values for the Moyal commutators, we have proved the Poincaré translation invariance for noncommutative $\varphi^4$ scalar field theory. However we need the spacetime noncommutativity to be spacelike in various occasions. This means that for noncommutative $\varphi^4$ scalar field theory, its Poincaré translation invariance can only be held for $\theta^0 = 0$ of the spacetime noncommutativity. On the other hand, it was shown in Refs. [19-21] that the unitarity of noncommutative field theories will also be lost when $\theta^0 \neq 0$. For the light-like noncommutativity, it was argued in Ref. [22] that the unitarity of noncommutative field theories are still held. However according to the demonstration of this paper, Poincaré translation invariance may not hold generally for noncommutative field theories for the light-like noncommutativity of the spacetime.
For the noncommutative parameters $\theta^{\mu \nu}$, usually people take them to be constants which are antisymmetric to the indexes $\mu$ and $\nu$. We can see that the spacetime noncommutative relations (2.1) and (2.7) are translation invariant. The Lagrangian (2.3) is also translation invariant. This makes us be able to define the energy-momentum tensor according to the method of Sec. III. However, the energy-momentum tensor obtained in Sec. III is not divergence free. This means that for noncommutative field theories, the translation invariance of the Lagrangians and actions may not necessarily result the existence of locally conserved energy-momentum tensors. From the classical field equation approach, in Ref. [9] the authors have demonstrated the Poincaré translation invariance for noncommutative field theories. In addition to take $\theta^{\mu \nu}$ to be a $c$-number second-order antisymmetric tensor, in Ref. [9] the authors have demonstrated the Lorentz rotation invariance for noncommutative field theories from their classical field equation approach. If we take $\theta^{\mu \nu}$ to be a $c$-number second-order antisymmetric tensor in this paper, we can same prove the Poincaré translation invariance for noncommutative scalar field theory. This is because $\theta^{\mu \nu}$ is invariant under a translation transformation even though it is a second-order antisymmetric tensor. The spacetime noncommutative relations (2.1) and (2.7) and the Lagrangian (2.3) are still translation invariant if $\theta^{\mu \nu}$ is a second-order antisymmetric tensor. Therefore, the demonstration for the Poincaré translation invariance of noncommutative scalar field theory in this paper is still held for $\theta^{\mu \nu}$ to be a second-order antisymmetric tensor.

We have also generalized the Heisenberg evolution equation (4.1) to the form of Eq. (4.9) of Moyal star-products. However, because the Hamiltonian defined as in Eqs. (4.5) and (4.6) for noncommutative field theories are not conserved quantities of motion generally, we need the condition $\theta^{0i} = 0$ for the spacetime noncommutativity generally to obtain Eq. (4.9) for noncommutative field theories. Under the condition $\theta^{0i} = 0$ for the spacetime noncommutativity, we can regard Eq. (4.9) as the fundamental evolution equation for quantum fields on noncommutative spacetime. We can establish the $S$-matrix from Eq. (4.9) in the interaction picture, so that to keep the field operators in the form of Moyal star-products in $H_{int}$. Then from the corresponding Wick’s theorem of field operators of Moyal star-products, we can also explain why there are non-planar diagrams in noncommutative field theories [4,23]. Many problems for the quantum equation of motion of noncommutative field theories have also been studied in Ref. [24] from the different approaches and methods. For the Poincaré translation invariance of other noncommutative field theories, we can also expect to construct the proofs using the method of this paper. It can also be generalized to the demonstration of the Lorentz rotation invariance for noncommutative field theories. However, we can expect that they will be rather complicated. A different method for the understanding of the relativistic invariance of noncommutative field theories is the twisted Poincaré algebra approach [7,8]. However, the quantum field realization of such an approach is not very clear at present.
**APPENDIX: A DERIVATION FOR EQUATION (4.52)**

In this Appendix we give a derivation for Eq. (4.52):

\[
\int d^3 x \delta^3(x - x') \star f(x, t) = \int d^3 x f(x, t) \star \delta^3(x - x') = f(x', t) . \tag{A1}
\]

The Fourier integral representation of \(\delta^3(x - x')\) is given by

\[
\delta^3(x - x') = \frac{1}{(2\pi)^3} \int e^{ik \cdot (x-x')} d^3 k . \tag{A2}
\]

Under the condition \(\theta^{0i} = 0\) of the spacetime noncommutativity, the Moyal star-products and the integral operations in Eq. (A1) do not relate with the time variable, so we can write the Fourier integral for \(f(x, t)\) as

\[
f(x, t) = \int d^3 q f(q, t) e^{i q \cdot x} . \tag{A3}
\]

To substitute Eqs. (A2) and (A3) into Eq. (A1), we obtain

\[
\begin{align*}
\int d^3 x \delta^3(x - x') & \star f(x, t) \\
& = \frac{1}{(2\pi)^3} \int d^3 x \int e^{ik \cdot (x-x')} d^3 k \star \int d^3 q f(q, t) e^{i q \cdot x} \\
& = \frac{1}{(2\pi)^3} \int d^3 q f(q, t) \int d^3 k \int d^3 x e^{i k \cdot (x-x')} \star e^{i q \cdot x} \\
& = \frac{1}{(2\pi)^3} \int d^3 q f(q, t) \int d^3 k \int d^3 x e^{-\frac{i}{2} k \times q} e^{i k \cdot (x-x')} e^{i q \cdot x} \\
& = \frac{1}{(2\pi)^3} \int d^3 q f(q, t) \int d^3 k e^{-\frac{i}{2} k \times q} e^{-i k \cdot x'} \int d^3 x e^{i (k+q) \cdot x} \\
& = \int d^3 q f(q, t) \int d^3 k \delta^3(k + q) e^{-\frac{i}{2} k \times q} e^{-i k \cdot x'} \\
& = \int d^3 q f(q, t) e^{i q \cdot x'} e^{\frac{i}{2} q \times q} = f(x', t) ,
\end{align*}
\]

(A4)

where in the above derivation \(k \times q = k_i \theta^{ij} q_j\). Similarly we have

\[
\int d^3 x f(x, t) \star \delta^3(x - x') = f(x', t) . \tag{A5}
\]

Thus we have proved Eq. (4.52) to be satisfied under the condition \(\theta^{0i} = 0\) of the spacetime noncommutativity. In the case \(\theta^{0i} \neq 0\), in the third line of Eq. (A4), we cannot move \(f(q, t)\) into the front directly because it will take part in the Moyal star-product operation, hence we cannot obtain the last result or Eq. (A1).
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