SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY
\((s, m)\)-CONVEX FUNCTIONS IN SECOND SENSE

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Abstract. Authors introduce the concept of harmonically \((s, m)\)-convex functions in second sense in \cite{4}. In this article, we establish some Hermite-Hadamard type inequalities of this class of functions.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function and \( a, b \in I \) with \( a < b \). Then following inequality

\[
 f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}
\]

holds. This double inequality is known in the literature as the Hermite-Hadamard integral inequality for convex function. Note that some of the classical inequalities for mean can be derived from (1.1) for the appropriate particular selection of mapping \( f \). Both inequalities in (1.1) hold in the reverse direction if \( f \) is concave.

In \cite{4}, İmdat Iscan introduced the concept of harmonically convex function, and established a variant ond Hermite-Hadamard type inequalities which holds for these classes of functions as follows:

Definition 1.1. Let \( I \subset \mathbb{R}/\{0\} \) be a real interval. A function \( f : I \to \mathbb{R} \) is said to be harmonically convex, if

\[
 f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)
\]

for all \( x, y \in I \) and \( t \in [0, 1] \). If inequality in (1.2) is reversed, then \( f \) is said to be harmonically concave.

Theorem 1.2. (see \cite{4}) Let \( f : I \subset \mathbb{R}/\{0\} \to \mathbb{R} \) be harmonically convex and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \), then following inequalities hold

\[
 f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

Theorem 1.3. (see \cite{4}). Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable on \( I^c, a, b \in I \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \( [a, b] \) for \( q \geq 1 \), then

\[
 \left|\frac{f(a) + f(b)}{2} - \frac{a b}{b-a} \int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{a b (b-a)}{2} \lambda_1^{\frac{q-1}{q}} (\lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q)^{\frac{1}{q}},
\]

where

\[
 \lambda_1 = \frac{1}{a b} - \frac{2}{(b-a)^2} \ln \left(\frac{(a+b)^2}{4 a b}\right),
\]

\[
 \lambda_2 = -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left(\frac{(a+b)^2}{4 a b}\right),
\]

\[
 \lambda_3 = \frac{1}{a(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left(\frac{(a+b)^2}{4 a b}\right).
\]
Theorem 1.4. (see [4]). Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I$, $a, b \in I$ with $a < b$ and $f' \in [a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q > 1$, then

$$
(1.5) \quad \left( f(a) + f(b) \right) - \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right)^\frac{1}{p+1} \left( \mu_1|f'(a)|^q + \mu_2|f'(b)|^q \right)^\frac{1}{p+1},
$$

where

$$
\mu_1 = \frac{[a^{2-2q} + b^{1-2q}][(b-a)(1-2q) - a]}{2(b-a)^2(1-q)(1-2q)},
$$
$$
\mu_2 = \frac{[a^{2-2q} - a^{1-2q}][(b-a)(1-2q) + b]}{2(b-a)^2(1-q)(1-2q)}.
$$

In [5], Indat Iscan introduced the concept of harmonically $s$-convex function in second sense as follow:

Definition 1.5. A function $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically $s$-convex in second sense, if

$$
(1.6) \quad f \left( \frac{xy}{tx + (1-t)y} \right) \leq t^s f(y) + (1-t)^s f(x)
$$

for all $x, y \in I$ and $t \in [0, 1]$. If inequality in (1.6) is reversed, then $f$ is said to be harmonically $s$-concave.

Remark 1.6. Note that for $s = 1$, harmonically $s$-convexity reduces to ordinary harmonically convexity.

In [3], Feixiang Chen and Shanhe Wu generalized Hermite-Hadamard type inequalities given in [4] which hold for harmonically $s$-convex functions in second sense.

Theorem 1.7. (see [3]). Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable on $I$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically $s$-convex in second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $q \geq 1$, then

$$
(1.7) \quad \left( f(a) + f(b) \right) - \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \leq \frac{ab(b-a)}{2} C_1^{1-s}(a, b)[C_2(s; a, b)|f'(a)|^q + C_3(s; a, b)|f'(b)|^q]^\frac{1}{s},
$$

where

$$
C_1(a, b) = b^{-2} \left( 2F_1(2, 2; 3, 1 - a \quad b) - 2F_1(2, 1; 2, 1 - a \quad b) + \frac{1}{2}2F_1(2, 1; 3, \frac{1}{2}(1 - a \quad b)) \right)
$$
$$
C_2(s, a, b) = b^{-2} \left( \frac{2}{s + 2}2F_1(2, s + 2; s + 3, 1 - a \quad b) - \frac{1}{s + 1}2F_1(2, s + 1; s + 2, 1 - a \quad b) \right)
$$
$$
+ \frac{1}{2}2F_1(2, s + 1; s + 3, \frac{1}{2}(1 - a \quad b))
$$
$$
C_3(s, a, b) = b^{-2} \left( \frac{2}{(s + 1)(s + 2)}2F_1(2, 2; s + 3, 1 - a \quad b) - \frac{1}{s + 1}2F_1(2, 1; s + 2, 1 - a \quad b) \right)
$$
$$
\cdot \frac{1}{2}2F_1(2, 1; 3, \frac{1}{2}(1 - a \quad b))
$$

Remark 1.8. Note that for $s = 1$, $C_1(a, b) = \lambda_1$, $C_2(1, a, b) = \lambda_2$ and $C_3(1, a, b) = \lambda_3$. Hence, Theorem 1.7 is particular case of theorem 1.5 for $s = 1$.

Theorem 1.9. (see [3]) Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable on $I$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically $s$-convex in second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $q \geq 1$, then

$$
(1.9) \quad \left( f(a) + f(b) \right) - \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \leq \frac{ab(b-a)}{2b} \left( \frac{1}{p+1} \right)^\frac{p}{p+1} \left[ \frac{1}{s + 1}[2F_1(2q, s + 1; s + 2, 1 - a \quad b)|f'(b)|^q + 2F_1(2q, 1; s + 2, 1 - a \quad b)|f'(a)|^q] \right]^\frac{1}{p+1},
$$
Remark 1.10. Note that for $s = 1$

$$\mu_1 = \frac{1}{2b^{2q}}F_1(2q, 2, 3, 1 - \frac{a}{b}),$$

and

$$\mu_2 = \frac{1}{2b^{2q}}F_1(2q, 1, 3, 1 - \frac{a}{b}).$$

Hence, Theorem 1.4 is particular case of theorem 1.9 for $s = 1$.

In ([7]), Jaekeun Park considered the class of $(s, m)$-convex functions in second sense. This class of function is defined as follow

Definition 1.11. For some fixed $s \in (0, 1]$ and $m \in [0, 1]$ a mapping $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be $(s, m)$-convex in the second sense on $I$ if

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y)$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

In ([6]), İmdat Iscan introduced the concept of harmonically $(α, m)$-convex functions and established some Hermite-Hadamard type inequalities for this class of function. This class of functions is defined as follow

Definition 1.12. The function $f : (0, \infty) \to \mathbb{R}$ is said to be harmonically $(α, m)$-convex, where $α \in [0, 1]$ and $m \in (0, 1]$, if

$$(1.7) \quad \frac{mxy}{my + (1-t)x} = f\left(\frac{t}{x} + \frac{1-t}{my}\right)^{-1} \leq t^α f(x) + m(1-t)^α f(y)$$

for all $x, y \in (0, \infty)$ and $t \in [0, 1]$. If the inequality in (1.7) is reversed, then $f$ is said to be harmonically $(α, m)$-concave.

In [11], authors introduce the concept of Harmonically $(s, m)$-convex functions in second sense which generalize the notion of Harmonically convex and Harmonically $s$-convex functions in second sense introduced by İmdat Iscan in [10][12].

In this paper, we establish some results connected with the right side of new inequality similar to (1.1) for this class of functions such that results given by İmdat Iscan [4], Feixiang Chen and Shanhe Wu [3] are obtained for the particular values of $s, m$.

Definition 1.13. The function $f : I \subset (0, \infty) \to \mathbb{R}$ is said to be harmonically $(s, m)$-convex in second sense, where $s \in (0, 1]$ and $m \in (0, 1]$ if

$$f\left(\frac{mxy}{my + (1-t)x}\right) = f\left(\frac{t}{x} + \frac{1-t}{my}\right)^{-1} \leq t^s f(x) + m(1-t)^s f(y)$$

$\forall x, y \in I$ and $t \in [0, 1]$.

Remark 1.14. Note that for $s = 1$, $(s, m)$-convexity reduces to harmonically $m$-convexity and for $m = 1$, harmonically $(s, m)$-convexity reduces to harmonically $s$-convexity in second sense (see [3]) and for $s, m = 1$, harmonically $(s, m)$-convexity reduces to ordinary harmonically convexity (see [4]).

Proposition 1.15. Let $f : (0, \infty) \to \mathbb{R}$ be a function

a) if $f$ is $(s, m)$-convex function in second sense and non-decreasing, then $f$ is harmonically $(s, m)$-convex function in second sense.

b) if $f$ is harmonically $(s, m)$-convex function in second sense and non-increasing, then $f$ is $(s, m)$-convex function in second sense.

Proof. For all $t \in [0, 1]$, $m \in (0, 1)$ and $x, y \in I$, we have

$$t(1-t)(x-ym)^2 \geq 0$$
Hence, the following inequality holds
\[(1.8) \quad \frac{mxy}{mty + (1-t)x} \leq tx + m(1-t)y\]
By the inequality \((1.8)\), the proof is completed. \(\square\)

**Remark 1.16.** According to proposition 1.15, every non-decreasing \((s, m)\)-convex function in second sense is also harmonically \((s, m)\)-convex function in second sense.

**Example 1.17.** (see\[2\]) Let \(0 < s < 1\) and \(a, b, c \in \mathbb{R}\), then function \(f : (0, \infty) \rightarrow \mathbb{R}\) defined by
\[
f(x) = \begin{cases} 
a, & x = 0 
bx^s + c, & x > 0 
\end{cases}
\]
is non-decreasing \(s\)-convex function in second sense for \(b \geq 0\) and \(0 \leq c \leq a\). Hence, by proposition 1.16 \(f\) is harmonically \((s, 1)\)-convex function.

**Proposition 1.18.** Let \(s \in [0, 1]\), \(m \in (0, 1]\), \(f : [a, mb] \subset (0, \infty) \rightarrow \mathbb{R}\), be an increasing function and \(g : [a, mb] \rightarrow [a, mb]\), \(g(x) = \frac{mab}{a+mb-x}\), \(a < mb\). Then \(f\) is harmonically \((s, m)\)-convex in second sense on \([a, mb]\) if and only if \(fog\) is \((s, m)\)-convex in second sense on \([a, mb]\).

**Proof.** Since
\[(1.9) \quad (fog)(ta + m(1-t)b) = f\left(\frac{mab}{mbt + (1-t)a}\right)\]
for all \(t \in [0, 1]\), \(m \in (0, 1]\). The proof is obvious from equality \((1.9)\). \(\square\)

The following result of the Hermite-Hadamard type holds.

**Theorem 1.19.** Let \(f : I \subset (0, \infty) \rightarrow \mathbb{R}\) be a harmonically \((s, m)\)-convex function in second sense with \(s \in [0, 1]\) and \(m \in (0, 1]\). If \(0 < a < b < \infty\) and \(f \in L[a, b]\), then one has following inequality
\[
\frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \min\left[\frac{f(a) + mf\left(\frac{b}{m}\right)}{s+1}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{s+1}\right]
\]
**Proof.** Since, \(f : I \subset (0, \infty) \rightarrow \mathbb{R}\) is a harmonically \((s, m)\)-convex function in second sense. We have, for \(x, y \in I \subset (0, \infty)\)
\[
f\left(\frac{xy}{ty + (1-t)x}\right) = f\left(\frac{mxy}{mty + (1-t)x}\right) \leq t^s f(x) + m(1-t)^s f\left(\frac{y}{m}\right)
\]
which gives
\[
f\left(\frac{ab}{tb + (1-t)a}\right) \leq t^s f(a) + m(1-t)^s f\left(\frac{b}{m}\right)
\]
and
\[
f\left(\frac{ab}{ta + (1-t)b}\right) \leq t^s f(b) + m(1-t)^s f\left(\frac{a}{m}\right)
\]
for all \(t \in [0, 1]\). Integrating on \([0, 1]\) w.r.t ‘\(t\)’, we obtain
\[
\int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) dt \leq \frac{f(a) + mf\left(\frac{b}{m}\right)}{s+1}
\]
and
\[
\int_0^1 f\left(\frac{ab}{ta + (1-t)b}\right) dt \leq \frac{f(b) + mf\left(\frac{a}{m}\right)}{s+1}
\]
However,
\[
\int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) dt = \int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) dt = \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx
\]
Hence, required inequality is established. \(\square\)
Corollary 1.20. If we take \( m = 1 \) in theorem 1.19 then we get
\[
\frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \leq \frac{f(a) + f(b)}{s+1}
\]

Corollary 1.21. If we take \( s = 1 \) in theorem 1.19 then we get
\[
\frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \leq \min \left[ \frac{f(a) + m f(\frac{a+b}{m})}{2}, \frac{f(b) + m f(\frac{a+b}{m})}{2} \right]
\]

2. Main Results

For finding some new inequalities of Hermite-Hadamard type for the functions whose derivatives are harmonically \((s,m)\)-convex in second sense, we need the following lemma

Lemma 2.1. Let \( f : I \subset \mathbb{R}/\{0\} \to \mathbb{R} \) be a differentiable function on \( I^\circ \) with \( a < b \). If \( f \in L[a,b] \), then
\[
\frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx = \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{(tb+(1-t)a)^m} f'\left(\frac{ab}{tb+(1-t)a}\right) dt
\]

Theorem 2.2. Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^\circ \), \( a, \frac{b}{m} \in I^\circ \) with \( a < b \), \( m \in (0,1] \) and \( f' \in L[a,b] \). If \( |f'|^q \) is harmonically \((s,m)\)-convex in second sense on \([a, \frac{b}{m}]\) for \( q \geq 1 \) with \( s \in [0,1] \), then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2^{q-\frac{s}{n}}} \left[ \rho_1(s,q;a,b) |f'(a)|^q + m \rho_2(s,q;a,b) |f'(\frac{b}{m})|^q \right]^\frac{1}{q}
\]

where
\[
\rho_1(s,q;a,b) = \frac{\beta(1,s+2)}{b^{2q}} \text{B} \left( 2q, 1; s + 3; 1 - \frac{a}{b} \right) - \frac{\beta(2,s+1)}{b^{2q}} \text{B} \left( 2q, 2; s + 3; 1 - \frac{a}{b} \right) + \frac{2^{2q-s} \beta(2,s+1)}{(a+b)^{2q}} \text{B} \left( 2q, 2; s + 3; 1 - \frac{2a}{a+b} \right)
\]

\[
\rho_2(s,q;a,b) = \frac{\beta(s+1,2)}{b^{2q}} \text{B} \left( 2q, s+1; s + 3; 1 - \frac{a}{b} \right) - \frac{\beta(s+1,2)}{b^{2q}} \text{B} \left( 2q, s+1; s + 3; 1 - \frac{a}{b} \right) + \frac{\beta(s+2,1)}{b^{2q}} \text{B} \left( 2q, s+2; s + 3; 1 - \frac{a}{b} \right)
\]

\( \beta \) is Euler Beta function defined by
\[
\beta(x,y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \ x, y > 0
\]

and \( _2F_1 \) is hypergeometric function defined by
\[
_2F_1(a,b;c,z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \ c > b > 0, \ |z| < 1
\]

Proof. From above Lemma and using power mean inequality, we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2} \int_0^1 \left| \frac{1-2t}{(tb+(1-t)a)^m} \right|^q |f'(\frac{ab}{tb+(1-t)a})| dt
\]
\[
\leq \frac{ab(b-a)}{2} \left( \int_0^1 |1-2t|^{\frac{1}{q}} dt \right)^{1-\frac{s}{q}} \times \left( \int_0^1 |1-2t| \left| f'(\frac{ab}{tb+(1-t)a}) \right|^q dt \right)^{\frac{1}{q}}
\]
Since, $|f'|^q$ is harmonically $(s,m)$-convex function in second sense, we have

$$
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
\leq \frac{ab(b-a)}{2} \left( \frac{1}{2} \right)^{1-\frac{q}{s}} \left( \int_0^1 \frac{1 - 2t^s}{(tb + (1-t)a)^{2q}} \, dt \right)^{1-\frac{q}{s}}
$$

$$
= \frac{ab(b-a)}{2} \left( \frac{1}{2} \right)^{1-\frac{q}{s}} \left[ |f'(a)|^q \int_0^1 \frac{1 - 2t^s}{(tb + (1-t)a)^{2q}} \, dt + m|f'(b)|^q \int_0^1 \frac{1 - 2t^s}{(tb + (1-t)a)^{2q}} \, dt \right]^{1-\frac{q}{s}}
$$

$$
= \frac{ab(b-a)}{2} \left( \frac{1}{2} \right)^{1-\frac{q}{s}} \left[ \rho_1(s; q; a, b)|f'(a)|^q + m\rho_2(s; q; a, b)|f'(b)|^q \right]^{1-\frac{q}{s}}
$$

It is easy to check that

$$
\int_0^1 \frac{1 - 2t^s}{(tb + (1-t)a)^{2q}} \, dt = \int_0^{\frac{1}{2}} \frac{1 - 2t^s}{(tb + (1-t)a)^{2q}} \, dt + \int_0^1 \frac{1 - 2t^s}{(tb + (1-t)a)^{2q}} \, dt
$$

$$
= \frac{2^{2q-\beta}(2, s + 1)}{(a+b)^{2q}} \cdot 2F_1(2q, 2; s + 3, 1 - \frac{2a}{a+b}) - \frac{\beta(2, s + 1)}{b^{2q}} \cdot 2F_1(2q, 2; s + 3, 1 - \frac{a}{b}) + \frac{\beta(1, s + 2)}{b^{2q}} \cdot 2F_1(2q, 1; s + 3, 1 - \frac{a}{b})
$$

$$
= \rho_1(s; q; a, b)
$$

and

$$
\int_0^1 \frac{1 - 2t^s}{(tb + (1-t)a)^{2q}} \, dt = \frac{\beta(1, s + 2)}{b^{2q}} \cdot 2F_1(2q, 2; s + 3, 1 - \frac{a}{b}) + \frac{\beta(2, s + 1)}{b^{2q}} \cdot 2F_1(2q, s + 1; s + 3, 1 - \frac{a}{b}) + \frac{\beta(2, s + 1)}{b^{2q}} \cdot 2F_1(2q, s + 2; s + 3, 1 - \frac{a}{b})
$$

$$
= \rho_2(s; q; a, b)
$$

This completes the proof. \(\square\)

If we take \(s = m = 1\) in Theorem 2.2, we get the following

**Corollary 2.3.** Let \(f : I \subset (0, \infty) \to \mathbb{R}\) be differentiable function on \(I\), \(a, b \in I\) with \(a < b\) and \(f' \in L[a, b]\). If \(|f'|^q\) is \((1,1)\)-harmonically convex in second sense or harmonically convex function on \([a, b]\) for \(q \geq 1\), then

$$
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2} \left[ \rho_1(1; q; a, b)|f'(a)|^q + \rho_2(1; q; a, b)|f'(b)|^q \right]^{1-\frac{q}{s}}
$$

**Corollary 2.4.** If we take \(m = 1\) in Theorem 2.2, then we get

$$
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2} \left[ \rho_1(s; q; a, b)|f'(a)|^q + \rho_2(s; q; a, b)|f'(b)|^q \right]^{1-\frac{q}{s}}
$$

**Corollary 2.5.** If we take \(s = 1\) in Theorem 2.2, we get

$$
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2} \left[ \rho_1(1; q; a, b)|f'(a)|^q + \rho_2(1; q; a, b)|f'(b)|^q \right]^{1-\frac{q}{s}}
$$

**Theorem 2.6.** Let \(f : I \subset (0, \infty) \to \mathbb{R}\) be a differentiable function on \(I\), \(ma, b \in I\) with \(a < b\), \(m \in (0,1]\) and \(f' \in L[a, b]\). If \(|f'|^q\) is harmonically \((s,m)\)-convex in second sense on \([a, b]\) for \(q \geq 1\) with \(s \in (0,1]\), then

$$
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2} \rho_1^{1-\frac{q}{s}}(0, q; a, b)|f'(a)|^q + m\rho_2(0, q; a, b)|f'(m)|^q \right]^{1-\frac{q}{s}}
$$
Proof. From Lemma, Power mean inequality and harmonically \((s, m)\)-convexity in second sense of \(|f'|^q\) on \([a, b/m]\), we have
\[
\left|\frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx\right| \leq \frac{ab(b - a)}{2} \int_0^1 \left|1 - 2t\right| \left|f'(\frac{ab}{tb + (1 - t)a})\right| \, dt
\]
\[
\leq \frac{ab(b - a)}{2} \left(\int_0^1 \left|1 - 2t\right| \left(1 - \frac{tb}{(1 - t)a}\right)^2 \, dt\right)^{1/2} \left(\int_0^1 \left|1 - 2t\right| \left|\frac{f'(\frac{ab}{tb + (1 - t)a})}{(1 - \frac{tb}{(1 - t)a})^q}\right|^q \, dt\right)^{1/2}
\]
\[
\leq \frac{ab(b - a)}{2} \rho_1 \left[\frac{1}{2} + \frac{1}{q}\right] (0, a, b) \left[\rho_1 (s, q; a, b) |f'(a)|^q + \rho_2 (s, q; a, b) |f'(b)|^q\right]^{1/4}
\]
which is \textbf{Theorem 1.7} proved by Feixiang Chen and Shanhe Wu in \cite{4}.

\textbf{Corollary 2.7.} If we take \(m = 1\) in \textbf{Theorem 2.6}, then we get
\[
\left|\frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx\right| \leq \frac{ab(b - a)}{2} \rho_1 \left[\frac{1}{2} + \frac{1}{q}\right] (0, a, b) \left[\rho_1 (1, q; a, b) |f'(a)|^q + \rho_2 (1, q; a, b) |f'(b)|^q\right]^{1/4}
\]
This is \textbf{Theorem 1.7} proved by Feixiang Chen and Shanhe Wu in \cite{4}.

\textbf{Corollary 2.8.} If we take \(s = 1\) in \textbf{Theorem 2.6}, we get
\[
\left|\frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx\right| \leq \frac{ab(b - a)}{2} \rho_1 \left[\frac{1}{2} + \frac{1}{q}\right] (0, a, b) \left[\rho_1 (1, q; a, b) |f'(a)|^q + \rho_2 (1, q; a, b) |f'(b)|^q\right]^{1/4}
\]

\textbf{Corollary 2.9.} If we take \(s = m = 1\) in \textbf{Theorem 2.6}, we get
\[
\left|\frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx\right| \leq \frac{ab(b - a)}{2} \rho_1 \left[\frac{1}{2} + \frac{1}{q}\right] (0, a, b) \left[\rho_1 (1, q; a, b) |f'(a)|^q + \rho_2 (1, q; a, b) |f'(b)|^q\right]^{1/4}
\]
which is \textbf{Theorem 1.7} proved by Imdat Iscan in \cite{2}.

\textbf{Theorem 2.10.} Let \(f : I \subset (0, \infty) \to \mathbb{R}\) be a differentiable function on \(I^o\), \(m, a, b \in I^o\) with \(a < b\), \(m \in (0, 1]\) and \(f' \in L[a, b]\). If \(|f'|^q\) is harmonically \((s, m)\)-convex in second sense on \([a, b/m]\) for \(q > 1\), \(\frac{1}{p} + \frac{1}{q} = 1\) with \(s \in [0, 1]\), then
\[
\left|\frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx\right| \leq \frac{ab(b - a)}{2} \left(\frac{1}{p} + 1\right)^{1/4} \left[\nu_1 (s, q; a, b) |f'(a)|^q + \nu_2 (s, q; a, b) |f'(b)|^q\right]^{1/4}
\]
where
\[
\nu_1 (s, q; a, b) = \frac{\beta(1, s + 1/b^2q)}{b^2q} \cdot 2F_1(2q, 1; s + 2, 1 - a/b)
\]
and
\[
\nu_2 (s, q; a, b) = \frac{\beta(s + 1, 1)}{b^2q} \cdot 2F_1(2q, s + 1; s + 2, 1 - a/b)
\]
Proof. From Lemma, Hölder’s inequality and harmonically \((s, m)\)-convexity of \(|f'|^q\) on \([a, \frac{b}{m}]\), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| 
\leq \frac{ab(b-a)}{2} \left( \int_0^1 |1 - 2t|^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 \frac{1}{(tb + (1-t)a)^2} |f'(\frac{ab}{tb + (1-t)a})| \, dt \right)^{\frac{1}{q}}
\]

\[
\leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ |f'(a)|^q \nu_1(s, q; a, b) + m|f'(b)|^q \nu_2(s, q; a, b) \right]^{\frac{1}{q}}
\]

where an easy calculation gives

\[
\int_0^1 |1 - 2t|^p \, dt = \frac{1}{p+1}
\]

\[
\int_0^1 \frac{t^s}{(tb + (1-t)a)^{2q}} \, dt = \frac{\beta(1, s+1)}{b^{2q}} \cdot 2F1(2q, 1; s + 2, 1 - \frac{a}{b}) := \nu_1(s, q; a, b)
\]

and

\[
\int_0^1 \frac{(1-t)^s}{(tb + (1-t)a)^{2q}} \, dt = \frac{\beta(s+1, 1)}{b^{2q}} \cdot 2F1(2q, s = 1; s + 2, 1 - \frac{a}{b}) := \nu_2(s, q; a, b)
\]

This completes the proof. \(\square\)

Corollary 2.11. If we take \(m = 1\) in Theorem 2.11, then we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| 
\leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ \nu_1(s, q; a, b)|f'(a)|^q + \nu_2(s, q; a, b)|f'(b)|^q \right]^{\frac{1}{q}}
\]

this is Theorem 1.4 proved by Feixiang Chen and Shanhe Wu in [3].

Corollary 2.12. If we take \(s = 1\) in above Theorem, then we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| 
\leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ \nu_1(1, q; a, b)|f'(a)|^q + m\nu_2(1, q; a, b)|f'(b)|^q \right]^{\frac{1}{q}}
\]

Corollary 2.13. If we take \(s = m = 1\) in above Theorem, then we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| 
\leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ \nu_1(1, q; a, b)|f'(a)|^q + \nu_2(1, q; a, b)|f'(b)|^q \right]^{\frac{1}{q}}
\]

This is Theorem 1.4 proved by İmdat Iscan in [4].
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