Existence of new singularities in Einstein-Aether theory

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Abstract. How do the global properties of a Lorentzian manifold change when endowed with a vector field? This interesting question is tackled in this paper within the framework of Einstein-Aether (EA) theory which has the most general diffeomorphism-invariant action involving a spacetime metric and a vector field. After classifying all the possible nine vacuum solutions with and without cosmological constant in Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology, we show that there exist three singular solutions in the EA theory which are not singular in the General Relativity (GR), all of them for $k = -1$, and another singular solution for $k = 1$ in EA theory which does not exist in GR. This result is cross-verified by showing the focusing of timelike geodesics using the Raychaudhuri equation. These new singular solutions show that GR and EA theories can be completely different, even for the FLRW solutions when we go beyond flat geometry ($k = 0$). In fact, they have different global structures. In the case where $\Lambda = 0$ ($k = \pm 1$) the vector field defining the preferred direction is the unique source of the curvature.

Keywords: gravity, modified gravity, dark energy theory

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1 Introduction

The theory of General Relativity (GR) underpins our best understanding of gravity. It has passed all the experimental tests, the first being the measurement of bending of light during the total solar eclipse of 1919 [1] and the most recent confirmation come from the direct detection of gravitational waves in 2015 [2], and the capture of black hole shadow image in 2019 [3]. Despite the experimental success, a fundamental theoretical problem of GR is the existence of curvature singularities such as the ones in the big bang and black holes. A curvature singularity is a point of intense gravity where spacetime, and the laws of physics break down. Stephan Hawking and Roger Penrose proved that the curvature singularities are not artifacts of coordinates or symmetries but are inevitable under reasonable energy and causality conditions [4]. Since then it has been clear that GR is not a final theory of gravitation and there have been several attempts to replace GR both at low and high energies [5]. The Einstein-Aether (EA) theory belongs to a class of infrared theories.

In this paper, we are interested in studying how the timelike vector field dubbed aether in EA theory affects the nature of singularities in comparison to GR. In order to have a clear view, we first classify all the possible cosmological vacuum solutions in EA theory. Then, we compute the Kretschmann scalar to see if curvature singularities exist in both GR and EA theories. We found four solutions that are non-singular (and one of them is non-existing) in GR theory but are singular in EA theory, three of them within the experimentally allowed parameter space. We reconfirm this by studying the focusing of congruence of timelike geodesics using the Raychaudhuri equation. It is noteworthy to mention that recently, in EA theory, we stumbled upon an interesting case of a new singularity in FLRW solution for a dark energy fluid with equation of state \( p = -\rho \), with constant energy density [6]. Even in
GR, with a rather exotic and unphysical matter, one can get physical singularities that are unaccompanied by geodesic incompleteness which are called “sudden” singularities [7–9]. A large class of such singularities were also found in a simple Friedmann cosmology containing only a scalar-field with a power-law self-interaction potential [10]. The paper is organized as follows. The section 2 presents the quick overview of EA field equations. In section 3, we classify the vacuum cosmological solutions and compare them with the corresponding cases in GR theory. We also prove the existence of curvature singularities by computing the Kretschmann curvature scalar invariant. And finally we end with a summary and conclusion in section 4.

2 Einstein-Aether theory

The EA theory in the current form was introduced in 2001 to study the preferred frame effects in gravitation and cosmology [11]. In this generally covariant theory, the local Lorentz Invariance is broken by a dynamical unit timelike vector field $u^a$ often referred to as aether. The general action of the EA theory is given by,

$$S = \int \sqrt{-g} \left( L_{\text{Einstein}} + L_{\text{aether}} + L_{\text{matter}} \right) d^4x,$$  

(2.1)

where,

$$L_{\text{Einstein}} = \frac{1}{16\pi G} (R - 2\Lambda),$$  

(2.2)

$$L_{\text{aether}} = \frac{1}{16\pi G} \left[ -K_{mn}^{ab} \nabla_a u^m \nabla_b u^n + \lambda (g_{ab} u^a u^b + 1) \right],$$  

(2.3)

with

$$K_{mn}^{ab} = c_1 g^{ab} g_{mn} + c_2 \delta_m^a \delta_n^b + c_3 \delta_m^a \delta_n^b - c_4 u^a u^b g_{mn},$$  

(2.4)

and the $c_i$ being dimensionless coupling constants, and $\lambda$ a Lagrange multiplier enforcing the unit timelike constraint on the aether.

The last term, $L_{\text{matter}}$ is the matter Lagrangian.

In the weak-field, slow-motion limit EA theory reduces to Newtonian gravity with a value of Newton’s constant $G_N$ related to the parameter $G$ in the action (2.1) by [12],

$$G = G_N \left( 1 - \frac{c_1 + c_4}{2} \right).$$  

(2.5)

The coupling constant $G$ of EA theory is equal to the usual Newtonian gravitational constant $G_N$ for $c_1 = -c_4$ and not necessarily $c_1 = c_4 = 0$. The Newtonian limit is recovered only for $c_1 + c_4 < 2$. If $c_1 + c_4 > 2$ gravity is repulsive, while for $c_1 + c_4 = 2$ the coupling constant $G$ is zero, which means that there is no coupling between gravity and matter in this theory [13].

The field equations, obtained by extremizing the action with respect to the independent variables $\lambda$, $u^a$ and $g_{mn}$ are given by [12],

$$g_{ab} u^a u^b = -1,$$  

(2.6)

$$\nabla_a (K_{mn}^{ab} \nabla_m u^n) + c_4 u^m \nabla_m u_n \nabla_b u^n + \lambda u_b = 0,$$  

(2.7)

$$G_{ab}^{\text{Einstein}} = T_{ab}^{\text{aether}} + 8\pi G T_{ab}^{\text{matter}},$$  

(2.8)
with
\[ C_{ab}^{\text{Einstein}} = R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab}, \]  
\[ T_{ab}^{\text{aether}} = \nabla_c [ J^c_{\ (a} u_b) + u^c J_{(ab)} - J_{(a}^c u_b) ] - \frac{1}{2} g_{ab} \nabla_c u^d + \lambda u_a u_b 
+ c_1 [ \nabla_a u_c \nabla_b u^c - c_4 u_a \nabla_c u_b ] + c_4 u^m \nabla_m u_a \nabla_n u_a, \]  
\[ T_{\text{matter}}^{ab} = -2 \frac{\delta (\sqrt{-g} L_{\text{matter}})}{\delta g_{ab}}. \]

By demanding the ghost-free condition of the tensor, vector and scalar parts of the linearly perturbed AE action, we get the following respective constraints on the free parameters \( c_i \) \[ 14 \].

\[ c_1 + c_3 < 1 \]
\[ c_1 + c_4 > 0 \]
\[ c_2 > -1 \]

Thus theoretical consistency requires \( \beta = c_1 + 3c_2 + c_3 > -2 \), which we will use later. These free parameters have been severely constrained using many observational/experimental tests such as the primordial nucleosynthesis \[ 15 \], ultra-high energy cosmic rays \[ 16 \], the solar system tests \[ 17, 18 \], binary pulsars \[ 19, 20 \], and more recently gravitational waves \[ 21, 22 \].

3 Classification of cosmological models

The most general isotropic and homogeneous universe is described by a FLRW metric,
\[ ds^2 = -dt^2 + B(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \]  
where, \( B(t) \) is the scale factor and \( k \) is a Gaussian curvature the space at a given time. According to observations by WMAP and Planck experiments, this metric is a good description of our universe as it is spatially homogeneous and isotropic when averaged over large scales. This leaves us with a choice of \( u^a = (1, 0, 0, 0) \).

The standard definitions of the Hubble parameter \( H(t) \), the deceleration parameter \( q(t) \) and the redshift are given by, respectively,
\[ H(t) = \frac{\dot{B}(t)}{B(t)}, \]  
\[ q(t) = -\frac{\ddot{B}(t)B(t)}{B(t)^2}, \]
where the symbol dot denotes the differentiation with respect to the time coordinate.

The Friedmann-Lemaître equations are given by,
\[ \left( 1 + \frac{\beta}{2} \right) \left( \frac{\dot{B}(t)}{B(t)} \right)^2 = \Lambda \frac{3}{B(t)^2}, \]  
\[ \left( 1 + \frac{\beta}{2} \right) \frac{\ddot{B}(t)}{B(t)} = \Lambda \frac{3}{3}, \]
where Λ is the cosmological constant. For recent literature on EA cosmology, the reader may consult the references [23, 24] and [25].

Since we are interested in the curvature singularities, an important quantity to be computed is the Kretschmann scalar. For FLRW metric, it is given by

\[
K = \frac{12}{B^4} \left[ k^2 + 2k\dot{B}(t)^2 + \dot{B}(t)^4 + \ddot{B}(t)^2B(t)^2 \right].
\]

(3.7)

A singularity always implies focusing of geodesics, although focusing alone cannot imply a singularity as pointed out by Landau [26]. Having already established the presence of singularities, we now use the focusing of timelike geodesics to reinforce our results. For a similar analysis, see [27]. The expansion rate of the congruence of geodesics as seen by comoving observers is given by the Raychaudhuri equation [28–31],

\[
\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma_{\mu\nu}\sigma^{\mu\nu} + w_{\mu\nu}w^{\mu\nu} - R_{\mu\nu}k^{\mu}k^{\nu},
\]

(3.8)

where \(\theta, \sigma^{\mu\nu}, w^{\mu\nu}\) are respectively the expansion, shear and twist of the congruence of geodesics, and \(\tau\) is the proper time along a geodesic with a tangent vector field, \(k^\mu = dx^\mu/d\tau\). The Raychaudhuri equation has geometrical meaning and has no connection \textit{a priori} to the gravitational theory which only enters through the term \(-R_{\mu\nu}\xi^{\mu}\xi^{\nu}\). For the FLRW metric, \((\tau = t)\), assuming the vector \(k^\mu = \delta_{ti}\) as the four-velocity, both shear \(\sigma^{\mu\nu}\) and twist \(w^{\mu\nu}\) are zero, while the expansion and curvature terms for timelike geodesics are given by,

\[
\theta = 3\frac{\dot{B}(t)}{2B(t)} \quad \text{and} \quad -R_{tt} = -R_{tt},
\]

(3.9)

(3.10)

where \(R_{tt}\) is the component \(tt\) of the Ricci tensor. Thus, the expansion rate of congruence of timelike geodesics is given by

\[
\frac{d\theta}{dt} = -3 \left[ \frac{\dot{B}(t)}{B(t)} \right]^2 + 3 \frac{\ddot{B}(t)}{B(t)} = -3H^2 \left( q + \frac{1}{4} \right).
\]

(3.11)

In the next sections we will calculate the Kretschmann scalar and the expansion rate for all the obtained solutions (for both \(B_1(t)\) and \(B_2(t)\), except when they are static or imaginary). Only for calculating the focussing of geodesics, we have used the solutions before normalizations for the sake of simplicity of the analysis.

3.1 Solutions for \(\Lambda > 0\)

3.1.1 \(k = 1\)

The Friedmann-Lemaître equations yield the following two solutions for \(\Lambda > 0, k = 1\),

\[
B_1(t) = \frac{1}{2\sqrt{\Lambda(\beta + 2)}} \left[ e^{\sqrt{\frac{2\beta}{\beta + 2}}(t_0 - t)} + 3(\beta + 2) e^{-\sqrt{\frac{2\beta}{\beta + 2}}(t_0 - t)} \right],
\]

(3.12)

\[
B_2(t) = \frac{1}{2\sqrt{\Lambda(\beta + 2)}} \left[ e^{\sqrt{\frac{2\beta}{\beta + 2}}(t - t_0)} + 3(\beta + 2) e^{-\sqrt{\frac{2\beta}{\beta + 2}}(t - t_0)} \right].
\]

(3.13)
Note that there is an integration constant $t_0$ which must be chosen in a such way that we have the Sitter solutions \cite{32} at the limit of GR theory. Thus,

$$ t_0 = \frac{\ln 6}{2} \sqrt[3]{\frac{3}{\Lambda}}. \quad (3.14) $$

As usual the $t_0$ may be the current time where the scale factor is normalized $B(t_0) = 1$. Thus,

$$ B_1(t) = \frac{1}{2\sqrt{\Lambda}} \left[ \frac{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t_0-t)}}{\sqrt{\Lambda + \epsilon \sqrt{\Lambda - 3}}} + \frac{\sqrt{\Lambda + \epsilon \sqrt{\Lambda - 3}}}{e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t_0-t)}} \right], \quad (3.15) $$

$$ B_2(t) = \frac{1}{2\sqrt{\Lambda}} \left[ \frac{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t-t_0)}}{\sqrt{\Lambda + \epsilon \sqrt{\Lambda - 3}}} + \frac{\sqrt{\Lambda + \epsilon \sqrt{\Lambda - 3}}}{e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t-t_0)}} \right], \quad (3.16) $$

where $\epsilon = \pm 1$. The solution only exists for $\Lambda - 3 > 0$ and $\beta + 2 > 0$, including $\beta = 0$.

The Hubble parameter $H(t)$ and the deceleration parameter $q(t)$ corresponding to this metric are given by

$$ H_1(t) = \sqrt{\frac{2\Lambda}{3(\beta+2)}} \left[ \frac{-3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t_0-t)}}{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t_0-t)}} + \left( \frac{\sqrt{\Lambda + \epsilon \sqrt{\Lambda - 3}}}{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t_0-t)}} \right) \right] \quad (3.17) $$

$$ H_2(t) = -\sqrt{\frac{2\Lambda}{3(\beta+2)}} \left[ \frac{-3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t-t_0)}}{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t-t_0)}} + \left( \frac{\sqrt{\Lambda + \epsilon \sqrt{\Lambda - 3}}}{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t-t_0)}} \right) \right] \quad (3.18) $$

$$ q_1(t) = -\left[ \frac{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t_0-t)}}{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t_0-t)}} + \left( \frac{\sqrt{\Lambda + \epsilon \sqrt{\Lambda - 3}}}{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t_0-t)}} \right) \right]^2 \quad (3.19) $$

$$ q_2(t) = -\left[ \frac{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t-t_0)}}{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t-t_0)}} + \left( \frac{\sqrt{\Lambda + \epsilon \sqrt{\Lambda - 3}}}{3 e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}} (t-t_0)}} \right) \right]^2 \quad (3.20) $$

This metric is non-singular for all values of $\beta + 2 > 0$.

The Raychaudhuri equation is given by

$$ \frac{d\theta}{d\tau}(B_1) = \frac{3\Lambda}{2(\beta+2)} \left[ 3(\beta+2) + \frac{e^\beta \sqrt{\frac{2\Lambda}{3(\beta+2)}} (-t+C1)}{2} \right]^2 $$

$$ \times \left[ \frac{4 e \sqrt{\frac{2\Lambda}{3(\beta+2)}} (-t+C1) + 10(\beta+2) e \sqrt{\frac{2\Lambda}{3(\beta+2)}} (-t+C1) + 9(\beta+2)^2}{4} \right], \quad (3.21) $$

$$ \frac{d\theta}{d\tau}(B_2) = \frac{3\Lambda}{2(\beta+2)} \left[ 3(\beta+2) + \frac{e^\beta \sqrt{\frac{2\Lambda}{3(\beta+2)}} (-t+C2)}{2} \right]^2 $$

$$ \times \left[ e^{-\frac{4}{3} \sqrt{\frac{2\Lambda}{3(\beta+2)}} (-t+C2) + 10(\beta+2) e^{-\frac{4}{3} \sqrt{\frac{2\Lambda}{3(\beta+2)}} (-t+C2) + 9(\beta+2)^2}} \right], \quad (3.22) $$

where $C1$ and $C2$ are, hereinafter, integration constants.
3.1.2 \( k = 0 \)

The Friedmann-Lemaître equations yield the following two solutions for \( \Lambda > 0, k = 0, \)

\[
B_1(t) = e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)}, \quad (3.23)
\]
\[
B_2(t) = e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t - t_0)}. \quad (3.24)
\]

Note again that the integration constant \( t_0, \) given by equation (3.14), must be chosen in such a way that we have the Sitter solutions [32] at the limit of GR theory. Thus, In the usual normalization \( B(t_0) = 1 \) gives the same solutions.

The solution only exists for \( \beta + 2 > 0. \)

The Hubble parameter \( H(t) \) and the deceleration parameter \( q(t) \) corresponding to the cases of this metric are given by

\[
H_1(t) = -\sqrt{\frac{2\Lambda}{3(\beta + 2)}}, \quad (3.25)
\]
\[
H_2(t) = \sqrt{\frac{2\Lambda}{3(\beta + 2)}}, \quad (3.26)
\]
\[
q_1(t) = -1, \quad (3.27)
\]
\[
q_2(t) = -1. \quad (3.28)
\]

This metric is non-singular for all values of \( \beta + 2 > 0 \) since the Kretschmann scalar is a constant \( (K = \frac{32}{3(\beta + 2))}. \)

The Raychaudhuri equation is given by

\[
\frac{d\theta}{d\tau}(B_1) = \frac{3}{2} \frac{\Lambda}{\beta + 2}, \quad (3.29)
\]
\[
\frac{d\theta}{d\tau}(B_2) = \frac{3}{2} \frac{\Lambda}{\beta + 2}. \quad (3.30)
\]

3.1.3 \( k = -1 \)

The Friedmann-Lemaître equations yield the following two solutions for \( \Lambda > 0, k = -1. \)

\[
B_1(t) = \frac{1}{2\sqrt{\Lambda(\beta + 2)}} \left[ e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)} - 3(\beta + 2) e^{-\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)} \right], \quad (3.31)
\]
\[
B_2(t) = \frac{1}{2\sqrt{\Lambda(\beta + 2)}} \left[ e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t - t_0)} - 3(\beta + 2) e^{-\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t - t_0)} \right]. \quad (3.32)
\]

As before the integration constant \( t_0 \) given by equation (3.14), it is chosen in such a way that we have the Sitter solutions [32] at the limit of GR theory. Thus, as usual the \( t_0 \) may be the current time where the scale factor is normalized \( B(t_0) = 1. \) Thus,

\[
B_1(t) = \frac{1}{2\sqrt{\Lambda}} \left[ \frac{3e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)}}{e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)}} + \frac{e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)}}{e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)}} \right], \quad (3.33)
\]
\[
B_2(t) = \frac{1}{2\sqrt{\Lambda}} \left[ \frac{3e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)}}{e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)}} + \frac{e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)}}{e^{\sqrt{\frac{2\Lambda}{3(\beta + 2)}}(t_0 - t)}} \right]. \quad (3.34)
\]

The solution only exists for \( \beta + 2 > 0. \)
The solution only exists for \( \beta \). The Friedmann-Lemaître equations yield the following two solutions for \( \Lambda = 0 \)
\[ k_{3.2.1} \]

The Raychaudhuri equation is given by
\[ \beta \]
The metric is not singular for \( \beta \). The Kretschmann scalar for the metric is singular at
\[ q_{2(t-t_0)} = \frac{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t-t_0)} - (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2}{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t-t_0)} + (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2} \]  
(3.38)

The Hubble parameter \( H(t) \) and the deceleration parameter \( q(t) \) corresponding to this metric are given by
\[ H_1(t) = -\sqrt{\frac{2\Lambda}{3(\beta + 2)}} \left[ \frac{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t_0-t)} + (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2}{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t_0-t)} - (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2} \right], \]  
(3.35)
\[ H_2(t) = \sqrt{\frac{2\Lambda}{3(\beta + 2)}} \left[ \frac{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t-t_0)} + (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2}{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t-t_0)} - (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2} \right], \]  
(3.36)
\[ q_1(t) = -\frac{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t_0-t)} - (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2}{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t_0-t)} + (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2}, \]  
(3.37)
\[ q_2(t) = -\frac{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t-t_0)} - (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2}{3 e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t-t_0)} + (\epsilon \sqrt{\Lambda + 3 + \sqrt{\Lambda}})^2}. \]  
(3.38)

The Kretschmann scalar for the metric is singular at
\[ t_{\text{sing}}(B_1) = t_0 - \sqrt{\frac{3(\beta + 2)}{8\Lambda}} \ln \left[ \frac{2}{3} \Lambda + \frac{2}{3} \epsilon \sqrt{\Lambda(\Lambda + 3) + 1} \right], \]  
(3.39)
\[ t_{\text{sing}}(B_2) = t_0 + \sqrt{\frac{3(\beta + 2)}{8\Lambda}} \ln \left[ \frac{2}{3} \Lambda + \frac{2}{3} \epsilon \sqrt{\Lambda(\Lambda + 3) + 1} \right]. \]  
(3.40)

The metric is not singular for \( \beta = 0 \) since in this time the curvature invariant being \( \frac{8\Lambda^2}{\pi^2} \). But, the metric is singular for \( \beta + 2 > 0 \) with \( \beta \neq 0 \). This means it is a new singularity.

The Raychaudhuri equation is given by
\[ \frac{d\theta}{d\tau}(B_1) = \frac{3\Lambda}{2(\beta + 2)} \left[ \frac{1}{-3(\beta + 2) + e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t+C1)} \sqrt{6 \Lambda}} \right] \times \left[ e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t+C1)} - 10(\beta + 2)e^{\frac{3\Lambda}{\beta + 2}(t+C1)} + 9(\beta + 2)^2 \right], \]  
(3.41)
\[ \frac{d\theta}{d\tau}(B_2) = \frac{3\Lambda}{2(\beta + 2)} \left[ \frac{1}{-(3\beta + 2) + e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t+C2)} \sqrt{6 \Lambda}} \right] \times \left[ -e^{\frac{\sqrt{6 \Lambda}}{\beta + 2}(t+C2)} - 10(\beta + 2)e^{\frac{3\Lambda}{\beta + 2}(t+C2)} + 9(\beta + 2)^2 \right]. \]  
(3.42)

### 3.2 Solutions for \( \Lambda = 0 \)

#### 3.2.1 \( k = 1 \)

The Friedmann-Lemaître equations yield the following two solutions for \( \Lambda = 0, k = 1 \),
\[ B_1(t) = 1 + \sqrt{\frac{-2}{2 + \beta}}(t_0 - t), \]  
(3.43)
\[ B_2(t) = 1 + \sqrt{\frac{-2}{2 + \beta}}(t - t_0). \]  
(3.44)

The solution only exists for \( \beta + 2 < 0 \).
The Hubble parameter $H(t)$ and the deceleration parameter $q(t)$ corresponding to this metric are given by

$$H_1(t) = -\left[\sqrt{\frac{2+\beta}{-2}} + (t_0 - t)\right]^{-1}, \quad (3.45)$$

$$H_2(t) = \left[\sqrt{\frac{2+\beta}{-2}} + (t - t_0)\right]^{-1}, \quad (3.46)$$

$$q_1(t) = 0, \quad (3.47)$$

$$q_2(t) = 0. \quad (3.48)$$

The Kretschmann scalar for the metric is singular at

$$t_{\text{sing}}(B_1) = t_0 + \sqrt{\frac{2+\beta}{-2}}, \quad (3.49)$$

$$t_{\text{sing}}(B_2) = t_0 - \sqrt{\frac{2+\beta}{-2}}. \quad (3.50)$$

For $\beta = 0$, the solution does not exist. That means it is a new singularity but exists for EA theory.

The Raychaudhuri equation is given by

$$\frac{d\theta}{d\tau}(B_1) = -\frac{3}{\left[-2t + C_1 \sqrt{-2(\beta + 2)}\right]^2}, \quad (3.51)$$

$$\frac{d\theta}{d\tau}(B_2) = -\frac{3}{\left[-2t + C_2 \sqrt{-2(\beta + 2)}\right]^2}. \quad (3.52)$$

### 3.2.2 $k = 0$

The Friedmann-Lemaître equations yield the following solution for $\Lambda = 0, k = 0$,

$$B(t) = 1. \quad (3.53)$$

The Hubble parameter $H(t)$ and the deceleration parameter $q(t)$ corresponding to this metric are given by

$$H(t) = 0,$$

$$q(t) = 0. \quad (3.54)$$

This metric is never singular and independent of the value of $\beta$.

### 3.2.3 $k = -1$

The Friedmann-Lemaître equations yield the following two solutions for $\Lambda = 0, k = -1$,

$$B_1(t) = 1 + \sqrt{\frac{2}{2+\beta}}(t_0 - t) \quad (3.55)$$

$$B_2(t) = 1 + \sqrt{\frac{2}{2+\beta}}(t - t_0) \quad (3.56)$$

The solution only exists for $\beta + 2 > 0$. 
The Hubble parameter $H(t)$ and the deceleration parameter $q(t)$ corresponding to this metric are given by

$$H_1(t) = -\left[\sqrt{\frac{2+\beta}{2}} + (t_0 - t)\right]^{-1}, \quad (3.57)$$

$$H_2(t) = \left[\sqrt{\frac{2+\beta}{2}} + (t - t_0)\right]^{-1}, \quad (3.58)$$

$$q_1(t) = 0, \quad (3.59)$$

$$q_2(t) = 0. \quad (3.60)$$

The Kretschmann scalar for the metric is singular at

$$t_{\text{sing}}(B_1) = t_0 + \sqrt{\frac{2+\beta}{2}}, \quad (3.61)$$

$$t_{\text{sing}}(B_2) = t_0 - \sqrt{\frac{2+\beta}{2}}. \quad (3.62)$$

For $\beta = 0$ which corresponds to GR, the solution exists and is never singular with the curvature invariant being null. But, the metric is singular for $\beta + 2 > 0$ such that $\beta \neq 0$. This means it is a new singularity.

The Raychaudhuri equation is given by

$$\frac{d\theta}{d\tau}(B_1) = -\frac{3}{-2t + C_1\sqrt{2(\beta + 2)}} \quad (3.63)$$

$$\frac{d\theta}{d\tau}(B_2) = -\frac{3}{-2t + C_2\sqrt{2(\beta + 2)}} \quad (3.64)$$

### 3.3 Solutions for $\Lambda < 0$

#### 3.3.1 $k = 1$

The Friedmann-Lemaître equations yield the following two solutions for $\Lambda < 0, k = 1$,

$$B_1(t) = \sqrt{-\frac{3}{|\Lambda|}} \sin \left[\sqrt{\frac{2|\Lambda|}{3(\beta + 2)}}(t_0 - t)\right], \quad (3.65)$$

$$B_2(t) = \sqrt{-\frac{3}{|\Lambda|}} \sin \left[\sqrt{\frac{2|\Lambda|}{3(\beta + 2)}}(t - t_0)\right]. \quad (3.66)$$

These solutions are imaginaries and substituting into the Friedmann equations we can show the solutions only exist for $\Lambda = 3$, thus there are no solutions for this case, since $\Lambda < 0$.

#### 3.3.2 $k = 0$

The Friedmann-Lemaître equations yield the following two solutions for $\Lambda < 0, k = 0$,

$$B_1(t) = e^{\sqrt{\frac{-2|\Lambda|}{3(\beta + 2)}}(t-t_0)} \quad (3.67)$$

$$B_2(t) = e^{\sqrt{\frac{2|\Lambda|}{3(\beta + 2)}}(t_0-t)} \quad (3.68)$$

The solution only exists for $\beta + 2 < 0$. 
The Hubble parameter \( H(t) \) and the deceleration parameter \( q(t) \) corresponding to this metric are given by

\[
H_1(t) = \sqrt{-\frac{2|\Lambda|}{3(\beta + 2)}}, \\
H_2(t) = -\sqrt{-\frac{2|\Lambda|}{3(\beta + 2)}},
\]

(3.69)

\[
q_1(t) = -1, \\
q_2(t) = -1.
\]

(3.71)

For \( \beta = 0 \), the solution does not exist. The solution exists and is non-singular only for EA theory.

The Raychaudhuri equation is given by

\[
\frac{d\theta}{d\tau}(B_1) = -\frac{3}{2} \frac{|\Lambda|}{\beta + 2},
\]

(3.73)

\[
\frac{d\theta}{d\tau}(B_2) = -\frac{3}{2} \frac{|\Lambda|}{\beta + 2}.
\]

(3.74)

### 3.3.3 \( k = -1 \)

The Friedmann-Lemaître equations yield the following two solutions for \( \Lambda < 0, k = -1, \)

\[
B_1(t) = \sqrt{\frac{3}{|\Lambda|}} \sin \left( \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (t - t_0) \right)
\]

\[
B_2(t) = \sqrt{\frac{3}{|\Lambda|}} \sin \left( \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (t_0 - t) \right)
\]

(3.75)

In the usual normalization \( B(t_0) = 1 \) we have

\[
B_1(t) = \sqrt{\frac{3}{|\Lambda|}} \sin \left( \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (t - t_0) + \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \right)
\]

(3.76)

\[
B_2(t) = \sqrt{\frac{3}{|\Lambda|}} \sin \left( \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (t_0 - t) + \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \right)
\]

(3.77)

The solution only exists for \( \beta + 2 > 0 \).

The Hubble parameter \( H(t) \) and the deceleration parameter \( q(t) \) corresponding to this metric are given by

\[
H_1(t) = \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} \cot \left( \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (t - t_0) + \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \right)
\]

(3.78)

\[
H_2(t) = \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} \cot \left( \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (t - t_0) - \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \right)
\]

(3.79)

\[
q_1(t) = \tan \left( \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (t - t_0) + \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \right)^2
\]

(3.80)

\[
q_2(t) = \tan \left( \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (t - t_0) - \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \right)^2
\]

(3.81)
The Kretschmann scalar for the metric is singular at

\[ t_{\text{sing}}(B_1) = t_0 - \sqrt{\frac{3(\beta + 2)}{2|\Lambda|}} \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \]  
(3.82)

\[ t_{\text{sing}}(B_2) = t_0 + \sqrt{\frac{3(\beta + 2)}{2|\Lambda|}} \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \]  
(3.83)

For \( \beta = 0 \) the solution exists and is never singular with the curvature invariant being \( \frac{8|\Lambda|^2}{3} \). But, the metric is singular for \( \beta + 2 > 0 \) with \( \beta \neq 0 \). This means it is a new singularity.

The Raychaudhuri equation is given by

\[ \frac{d\theta}{d\tau}(B_1) = -\frac{|\Lambda|}{2(\beta + 2)} \frac{3\cos^2 \left[ \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (-t + C_1) \right] - 4}{\cos^2 \left[ \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (-t + C_1) \right] - 1}, \]  
(3.84)

\[ \frac{d\theta}{d\tau}(B_2) = -\frac{|\Lambda|}{2(\beta + 2)} \frac{3\cos^2 \left[ \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (-t + C_2) \right] - 4}{\cos^2 \left[ \sqrt{\frac{2|\Lambda|}{3(\beta + 2)}} (-t + C_2) \right] - 1}. \]  
(3.85)

4 Conclusions

The initial singularity in cosmology has been the center of much research even before the Big Bang model was vindicated by the COBE results in 1990 [33]. In fact, even before the advent of the singularity theorems, the issue of breakdown of predictability in classical physics at the spacetime singularities had particularly worried John Wheeler [34]. There have been several efforts both classically and quantum mechanically to eliminate or avoid such singularities. See [35], for example.

In this paper, we investigated how the presence of timelike vector field in EA theory affects the nature of singularities in comparison to GR theory. For this, we first classified all the possible cosmological vacuum solutions in EA theory and found three cases which are non-singular in GR but are singular in EA theory. They are 3.1.3 (\( \Lambda > 0, k = -1 \)), 3.2.3 (\( \Lambda = 0, k = -1 \)) and 3.3.3 (\( \Lambda < 0, k = -1 \)) which all are within the experimentally allowed parameter space satisfying \( \beta + 2 > 0 \). Besides we found another singular solution in EA, 3.3.2 (\( \Lambda < 0, k = 0 \)), which does not have a counterpart in GR, albeit it exists in EA only for \( \beta + 2 < 0 \) which is ruled out by experiments.

Our conclusions are reinforced by studying the focusing of congruence of timelike geodesics using the Raychaudhuri equation. For all the three singular cases 3.2.3 (\( \Lambda = 0, k = -1 \)), 3.3.2 (\( \Lambda < 0, k = 0 \)) and 3.3.3 (\( \Lambda < 0, k = -1 \)), the expansion rate is \( \frac{d\theta}{d\tau} \) < 0, assuring the convergence of the congruences. However, the convergence of the congruences of the case 3.1.3 (\( \Lambda > 0, k = -1 \)) depends on the interval of time considered. For all the non-singular cases we have \( \frac{d\theta}{d\tau} > 0 \) as it should be.

The important take away is that the new singular solutions show that GR and EA theories can have different global structures, even for the FLRW solutions when we go beyond flat geometry (\( k = 0 \)). The cases 3.2.1 and 3.2.3 with \( \Lambda = 0 \) are particularly interesting since the vector field defining the preferred direction is the unique source of the curvature. This result challenges several preconceived ideas about the nature of spacetime singularities in modified theories of gravity.
Among the gravitational theories that break Lorentz invariance (LI) by construction, the most popular are Horava-Lifshitz (HL) theory and EA theory. In the case of HL theory, the breaking of LI is implemented by introducing a preferred foliation of space-time, but no additional structure. Whereas EA theory breaks LI by coupling general relativity to a dynamical unit timelike vector field. Jacobson [36, 37] showed that any hypersurface orthogonal solution to EA theory is a solution to the IR limit of a particular version of HL gravity, although the converse does not appear to be true. For the purpose of the current paper, we would like to comment that the lowest dimension terms (the infrared limit (IR)) of the BPS [38, 39] version of HL action are equivalent to those of EA theory when the aether vector is assumed to be hypersurface orthogonal. But, this does not mean that the EA theory is an IR theory of gravity. Unfortunately, there does not exist much literature on the ultraviolet (UV) behavior of EA theory. The second thing we would like to comment about is that we have confirmed the singularities purely from geometry without appealing to the energy conditions in EA theory, which are shown to be violated in some solutions [40]. The third thing we would like to comment on is the role of breaking of LI on singularities. Perhaps the roots of EA theory go back to Gasperini [41] who suggested that in the UV, the Lorentz gauge symmetry of general relativity could breakdown leading a way of evading the singularity theorems at least in a classical geometric context. Although Gasperini showed the avoidance of singularities in FLRW universe with the matter by invoking repulsive gravitational interactions associated with a non-minimal breaking of the local Lorentz symmetry, that is not the case here.

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