Randomized Empirical Processes and Confidence Bands via Virtual Resampling *

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In Memoriam Manny Parzen

Abstract

Let $X, X_1, X_2, \cdots$ be independent real valued random variables with a common distribution function $F$, and consider \{X_1, \cdots, X_N\}, possibly a big concrete data set, or an imaginary random sample of size $N \geq 1$ on $X$. In the latter case, or when a concrete data set in hand is too big to be entirely processed, then the sample distribution function $F_N$ and the the population distribution function $F$ are both to be estimated. This, in this paper, is achieved via viewing \{X_1, \cdots, X_N\} as above, as a finite population of real valued random variables with $N$ labeled units, and sampling its indices \{1, \cdots, N\} with replacement $m_N := \sum_{i=1}^{N} w_i^{(N)}$ times so that for each $1 \leq i \leq N$, $w_i^{(N)}$ is the count of number of times the index $i$ of $X_i$ is chosen in this virtual resampling process. This exposition extends the Doob-Donsker classical theory of weak convergence of empirical processes to that of the thus created randomly weighted empirical processes when $N, m_N \to \infty$ so that $m_N = o(N^2)$.

Keywords: Virtual resampling, big data sets, imaginary random samples, finite populations, infinite super-populations, randomized empirical processes, weak convergence, confidence bands for empirical and theoretical distributions, goodness-of-fit tests, Brownian bridge, randomized central limit theorems, confidence intervals

*Research supported by a Natural Sciences and Engineering Research Council of Canada Discovery Grant of M. Csörgő.
1 Introduction

Let \( X, X_1, X_2, \ldots \) be independent real valued random variables with a common distribution function \( F \). Consider \( \{X_1, \ldots, X_N\} \), possibly a big data set of a concrete or imaginary random sample of size \( N \geq 1 \) on \( X \) of a hypothetical infinite super-population, and define their empirical distribution function

\[
F_N(x) := \sum_{i=1}^{N} \mathbb{I}(X_i \leq x)/N, \quad x \in \mathbb{R},
\]

and the corresponding empirical process in this setting

\[
\beta_N(x) := \frac{1}{N^{1/2}} \sum_{i=1}^{N} (\mathbb{I}(X_i \leq x) - F(x)) = N^{1/2}(F_N(x) - F(x)), \quad x \in \mathbb{R},
\]

where \( \mathbb{I}(\cdot) \) is the indicator function.

In case of an imaginary random sample \( \{X_1, \ldots, X_N\} \) from an infinite super-population, or when a data set is too big to be entirely processed, then the sample distribution function \( F_N \) and the population distribution function \( F \) are both to be estimated via taking sub-samples from an imaginary random sample, or a big data set “in hand”. Naturally, the same holds true for the other sample and population parameters as well, like, for example, the sample and population means and percentiles, etc. (cf. Csörgő and Nasari (2015) and section 6 in this exposition).

To begin with, we view a concrete or imaginary random sample \( \{X_1, \ldots, X_N\} \) as a finite population of real valued random variables with \( N \) labeled units, \( N \geq 1 \), and sample its set of indices \( \{1, \ldots, N\} \) with replacement \( m_N \) times so that for each \( 1 \leq i \leq N \), \( w_i^{(N)} \) is the count of the number of times the index \( i \) of \( X_i \) is chosen in this re-sampling procedure.

In view of the definition of \( w_i^{(N)} \), \( 1 \leq i \leq N \), in this virtual re-sampling procedure, they form a row-wise independent triangular array of random variables with \( m_N := \sum_{i=1}^{N} w_i^{(N)} \) and, for each \( N \geq 1 \),

\[
\left( w_1^{(N)}, \ldots, w_N^{(N)} \right) \overset{d}{=} \text{Multinomial}(m_N, 1/N, \ldots, 1/N),
\]

i.e., the vector of the weights has a multinomial distribution of size \( m_N \) with respective probabilities \( 1/N \). Clearly, for each \( N \geq 1 \), the multinomial weights \( (w_1^{(N)}, \ldots, w_N^{(N)}) \), by definition, are independent from the finite population of the \( N \) labeled units \( \{X_1, \ldots, X_N\} \).

Notations for use throughout. Let \( (\Omega_X, \mathcal{F}_X, P_X) \) denote the probability space of the i.i.d. random variables \( X, X_1, \ldots, \) and \( (\Omega_w, \mathcal{F}_w, P_w) \) be the probability space on which

\[
(w_1^{(1)}, w_1^{(2)}, w_2^{(2)}, \ldots, (w_1^{(N)}, \ldots, w_N^{(N)}), \ldots),
\]
are defined. In view of the independence of these two sets of random variables, jointly they live on the direct product probability space \((\Omega_X \times \Omega_w, \mathcal{F}_X \otimes \mathcal{F}_w, P_{X,w} = P_X \times P_w)\). For each \(N \geq 1\), we also let \(P_{|w}()\) and \(P_{|X}()\) stand for the conditional probabilities given \(\mathcal{F}_w^{(N)} := \sigma(w_1^{(N)}, \ldots, w_N^{(N)})\) and \(\mathcal{F}_X^{(N)} := \sigma(X_1, \ldots, X_N)\), respectively, with corresponding conditional expected values \(E_{|w}()\) and \(E_{|X}()\). Also, \(E_{X,w}()\), \(E_{X}()\) and \(E_{w}()\) will stand for corresponding expected values in terms of \(P_{X,w}()\), \(P_X()\) and \(P_w()\) respectively.

We note in passing that in terms of the above notations, the distribution function \(F\) of the random variable \(X\) is \(F(x) := P_X(X \leq x), \, x \in \mathbb{R}\).

Randomizing, via the multinomial weights as in (1.3), define the randomly weighted empirical process

\[
\beta_{mN,N}^{(1)}(x) := \sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{m_N} - \frac{1}{N} \right) \mathbb{I}(X_i \leq x)
\]

\[
\beta_{mN,N}^{(1)}(x) = \frac{F_{mN,N}(x) - F_N(x)}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}}, \quad x \in \mathbb{R},
\]

(1.4)

where

\[
F_{mN,N}(x) := \sum_{i=1}^{N} \frac{w_i^{(N)}}{m_N} \mathbb{I}(X_i \leq x), \quad x \in \mathbb{R},
\]

(1.5)

is the randomly weighted sample distribution function, and define as well

\[
\beta_{mN,N}^{(2)}(x) := \frac{\sum_{i=1}^{N} \frac{w_i^{(N)}}{m_N} \left( \mathbb{I}(X_i \leq x) - F(x) \right)}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}},
\]

\[
\beta_{mN,N}^{(2)}(x) = \frac{F_{mN,N}(x) - F(x)}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2 + \frac{1}{N}}}, \quad x \in \mathbb{R}.
\]

(1.6)

Further to these two randomly weighted empirical processes, we introduce also
\( \beta_{mN,N}(x, \theta) \) := \( \sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{mN} - \frac{\theta}{N} \right) \left( \mathbb{I}(X_i \leq x) - F(x) \right) \)

\begin{align*}
&= \frac{\left( F_{mN,N}(x) - F(x) \right) - \theta \left( F_N(x) - F(x) \right)}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{mN} - \frac{\theta}{N} \right)^2}} \\
&= \frac{\left( F_{mN,N}(x) - F(x) \right) - \theta \left( F_N(x) - F(x) \right)}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{mN} - \frac{1}{N} \right)^2 + (1-\theta)^2 N}} \quad \text{, } x \in \mathbb{R}, \quad (1.7)
\end{align*}

where \( \theta \) is a real valued constant.

**Remark 1.1.** On letting \( \theta = 1 \) in (1.7), it reduces to (1.4), while letting \( \theta = 0 \) in (1.7) yields (1.6). Hence, instead of establishing the asymptotic behavior of the respective randomly weighted empirical processes of (1.4) and (1.6) individually on their own, it will suffice to conclude that of \( \beta_{mN,N}(., \theta) \) as in (1.7) to start with.

As to the weak convergence of the randomly weighted empirical process \( \beta_{mN,N}(., \theta) \) as in (1.7), it will be established via conditioning on the weights \( (w_1^{(N)}, \ldots, w_N^{(N)}) \) as in (1.3) and on assuming that \( N, mN \to \infty \) in such a way that \( mN = o(N^2) \) (cf. (2.3) of Theorem 1). This, in turn, will identify the respective weak convergence of the pivotal processes \( \beta_{mN,N}(.) \) and \( \beta_{mN,N}(., 0) \) as that of \( \beta_{mN,N}(., 1) \) under the same conditions to a Brownian bridge \( B(.) \) that, in turn, yields Corollary 2.1 as in Section 2.

Further to (1.7), defined now the randomized empirical process \( \tilde{\beta}_{mN,N}(., \theta) \) as

\begin{align*}
\tilde{\beta}_{mN,N}(x, \theta) := \sqrt{\frac{NmN}{N + mN(1-\theta)^2}} \left( F_{mN,N}(x) - F(x) \right) - \theta \left( F_N(x) - F(x) \right), \quad x \in \mathbb{R}, \quad (1.8)
\end{align*}

where \( \theta \) is a real valued constant. On letting \( \theta = 1 \) in (1.8), define also \( \tilde{\beta}_{mN,N}(.) \) as

\begin{align*}
\tilde{\beta}_{mN,N}(x) := \tilde{\beta}_{mN,N}(x, 1) = \sqrt{mN} \left( F_{mN,N}(x) - F_N(x) \right), \quad x \in \mathbb{R}, \quad (1.9)
\end{align*}
and, on letting $\theta = 0$, define $\tilde{\beta}^{(2)}_{m,N}(.)$ as

$$
\tilde{\beta}^{(2)}_{m,N}(x) := \beta^{(3)}_{m,N}(x,0) = \sqrt{\frac{N m_N}{N + m_N}} \left( F_{m,N}(x) - F(x) \right), \quad x \in \mathbb{R}. \quad (1.10)
$$

Conditioning again on the weights as in (1.3) and on assuming, as before, that $N, m_N \to \infty$ so that $m_N = o(N^2)$, it will be seen that the weak convergence of the virtually resampled empirical process $\tilde{\beta}^{(3)}_{m,N}(.\), $\theta)$ coincides with that of $\beta^{(3)}_{m,N}(.\), $\theta)$ (cf. (2.4) of Theorem 1) and, consequently, the respective weak convergence of the pivotal processes $\tilde{\beta}^{(1)}_{m,N}(.)$ and $\tilde{\beta}^{(2)}_{m,N}(.)$ as in (1.9) and (1.10) coincide with that of $\beta^{(1)}_{m,N}(.)$ and $\beta^{(2)}_{m,N}(.)$ to a Brownian bridge $B(.)$. This, in turn, yields Corollary 2.2 as in Section 2.

**Remark 1.2.** Given the sample values of an i.i.d. sample $X_1, \ldots, X_N$ on $F$ and redrawing $m_N$ bootstrap values, define the corresponding bootstrapped empirical process à la $\tilde{\beta}^{(1)}_{m_N}$. In this context the latter process is studied in Section 3.6.1 of van der Vaart and Wellner (1996) (cf. $\hat{G}_{n,k}$ therein, with $k$ bootstrap values out of $n$ sample values of an i.i.d. sample of size $n$), where it is concluded that, in our terminology, conditioning on $X_1, \ldots, X_N$, the “sequence” $\tilde{\beta}^{(1)}_{m_N}$ converges in distribution to a Brownian bridge in $P_X$ for every possible manner in which $m_N, N \to \infty$.

We note that, for $N \geq 1$,

$$
E_{X|w}(F_{m,N}(x)) = F(x), \quad E_{w|x}(F_{m,N}(x)) = F_N(x) \quad (1.11)
$$

and

$$
E_{X,w}(F_{m,N}(x)) = F(x), \quad \text{for all } x \in \mathbb{R}, \quad (1.12)
$$

i.e., when conditioning on the observations $(X_1, \ldots, X_N)$ on $X$, the randomly weighted sample distribution function $F_{m,N}(\cdot)$ is an unbiased estimator of the sample distribution function $F_N(\cdot)$ and, when conditioning on the weights $(w_1^{(N)}, \ldots, w_N^{(N)})$, it is an unbiased estimator of the theoretical distribution function $F(\cdot)$.

Also, for use in Section 6 we recall that (cf. Csörgö and Nasari (2015)), with $N$ fixed and $m = m_N \to \infty$,

$$
F_{m,N}(x) \to F_N(x) \quad \text{in probability } P_{X,w}, \quad \text{pointwise in } x \in \mathbb{R}, \quad (1.13)
$$

and, with $N, m_N \to \infty$,

$$
\left( F_{m,N}(x) - F_N(x) \right) \to 0 \quad \text{in probability } P_{X,w}, \quad \text{pointwise in } x \in \mathbb{R}. \quad (1.14)
$$
Our approach in this exposition to the formulation of the problems in hand was inspired by Hartley, H.O. and Sielken Jr., R.L. (1975). A “Super-Population Viewpoint” for Finite Population Sampling, *Biometrics* **31**, 411-422. We quote from second paragraph of 1. INTRODUCTION of this paper:

“By contrast, the super-population outlook regards the finite population of interest as a sample of size \( N \) from an infinite population and regards the stochastic procedure generating the surveyor’s sample of \( n \) units as the following two-step procedure:

Step 1. Draw a “large sample” of size \( N \) from an infinite super-population.

Step 2. Draw a sample of size \( n < N \) from the large sample of size \( N \) obtained in Step 1.

Actually, Step 1 is an imaginary step, and it is usually assumed that the resulting sample elements are independent and identically distributed.”

The material in this paper is organized as follows. Section 2 spells out weak convergence conclusions for the randomly weighted empirical processes introduced in Section 1. Section 3 is devoted to constructing asymptotically correct size confidence bands for \( F_N \) and continuous \( F \), both in terms of \( P_{X|w} \) and \( P_{X,w} \), via virtual resampling big concrete or imaginary random sample data sets that are viewed as samples from infinite super-populations. Section 4 and 5 Appendix deal with constructing confidence bands for continuous \( F \) via virtual resampling concrete large enough, or moderately small samples, that are to be compared to the classical Kolmogorov-Smirnov bands. Section 6 concludes randomized central limit theorems and, consequently, confidence intervals for \( F_N(x) \) and not necessarily continuous \( F(x) \) at fixed points \( x \in \mathbb{R} \) via virtual resampling. All the proofs are given in Section 7.

2 Weak convergence of randomly weighted empirical processes

In view of \( \{X_1, \ldots, X_N\} \) possibly being a big data set, or an imaginary random sample of size \( N \geq 1 \) on \( X \) of a hypothetical infinite super-population, the weak convergence of the randomized empirical processes \( \{\beta_{m_N,N}(x), \beta^{(3)}_{m_N,N}(x, \theta) \ ; x \in \mathbb{R}\} \), \( i = 1, 2, \ \theta \in \mathbb{R} \), \( \{\tilde{\beta}_{m_N,N}(x), \tilde{\beta}^{(3)}_{m_N,N}(x, \theta) \ ; x \in \mathbb{R}\} \), \( i = 1, 2, \ \theta \in \mathbb{R} \), is studied mainly for the sake of forming asymptotically exact size confidence bands for \( F_N \) and \( F \), based on \( \{\beta_{m_N,N}(x), x \in \mathbb{R}\} \), \( i = 1, 2 \), and \( \{\tilde{\beta}_{m_N,N}(x), x \in \mathbb{R}\} \), \( i = 1, 2 \), with sub-samples of size \( m_N < N \), (cf. Section 3).
For the right continuously defined distribution function $F(x) := P_X(X \leq x)$, we define its left continuous inverse (quantile function) by

$$F^{-1}(t) := \inf \{x \in \mathbb{R} : F(x) \geq t\}, \quad 0 < t \leq 1, \ F^{-1}(0) = F^{-1}(0+).$$

Thus, in case of a continuous distribution function $F$, we have

$$F^{-1}(t) = \inf \{x \in \mathbb{R} : F(x) = t\} \quad \text{and} \quad F(F^{-1}(t)) = t \in [0, 1].$$

Consequently, the random variable $F(X)$ is uniformly distributed on the unit interval $[0,1]$:

$$P_X(F(X) \geq t) = P_X(X \geq F^{-1}(t)) = 1 - P_X(X \leq F^{-1}(t)) = 1 - F(F^{-1}(t)) = 1 - t, \ 0 \leq t \leq 1.$$

Hence, if $F$ is continuous, then the classical empirical process (cf.(1.2)) in this setting becomes

$$\beta_N(F^{-1}(t)) = \frac{1}{N^{1/2}} \sum_{i=1}^{N} \left( \mathbb{1}(X_i \leq F^{-1}(t)) - F(F^{-1}(t)) \right)$$

$$= \frac{1}{N^{1/2}} \sum_{i=1}^{N} \left( \mathbb{1}(F(X_i) \leq t) - t \right), \ 0 \leq t \leq 1, \quad (2.1)$$

the uniform empirical process of the independent uniformly distributed random variables $F(X_1), \ldots, F(X_N), N \geq 1$.

Accordingly, when $F$ is continuous, the weak convergence of $\beta_N(x), x \in \mathbb{R}$, as in (1.2) can be established via that of $\beta_N(F^{-1}(t)), 0 \leq t \leq 1$, as in (2.1), and, as $N \to \infty$, we have (Doob (1949), Donsker (1952))

$$\beta_N(F^{-1}(\cdot)) \xrightarrow{\text{Law}} B(\cdot) \quad \text{on} \quad (D, \mathcal{D}, ||||), \quad (2.2)$$

with notations as in our forthcoming Theorem [1] where $B(\cdot)$ is a Brownian bridge on $[0,1]$, a Gaussian process with covariance function $EB(s)B(t) = s \wedge t - st$, that in terms of a standard Brownian motion $\{W(t), 0 \leq t < \infty\}$ can be defined as

$$\{B(t), 0 \leq t \leq 1\} = \{W(t) - tW(1), 0 \leq t \leq 1\}.$$

Mutatis mutandis, when $F$ is continuous, similar conclusions hold true for obtaining the weak convergence of the randomly weighted empirical processes as in (1.4), (1.6), (1.7), (1.8), (1.9) and (1.10), respectively, via establishing that of their corresponding uniform versions that, for the sake of theorem proving, are defined in terms of the independent uniformly distributed random variables $F(X_1), \ldots, F(X_N), N \geq 1$.  

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**Theorem 1.** Let $X, X_1, \ldots$ be real valued i.i.d. random variables on $(\Omega_X, \mathcal{F}_X, P_X)$. Assume that $F(x) = P_X(X \leq x)$, $x \in \mathbb{R}$, is a continuous distribution function. Then, relative to the conditional distribution $P_{X|w}$, if $N, m_N \to \infty$ so that $m_N = o(N^2)$, via (1.7), with $\theta \in \mathbb{R}$, we have

$$
\beta_{mN,N}^{(3)} \left( F^{-1}(\cdot), \theta \right) \xrightarrow{\text{Law}} B(\cdot) \text{ on } (D, \mathcal{D}, ||||), \quad \text{in probability } P_w, \tag{2.3}
$$

and, via (1.8), with $\theta \in \mathbb{R}$, we get

$$
\tilde{\beta}_{mN,N}^{(3)} \left( F^{-1}(\cdot), \theta \right) \xrightarrow{\text{Law}} B(\cdot) \text{ on } (D, \mathcal{D}, ||||), \quad \text{in probability } P_w, \tag{2.4}
$$

where $B(\cdot)$ is a Brownian bridge on $[0,1]$, $\mathcal{D}$ denotes the $\sigma$-field generated by the finite dimensional subsets of $D = D[0,1]$, and $||||$ stands for the uniform metric for real valued functions on $[0,1]$, i.e., in both cases we have weak convergence on $(D, \mathcal{D}, ||||)$ in probability $P_w$ in terms of $P_{X|w}$.

**Remark 2.1.** We note in passing that, suitably stated, the conclusions of Theorem 1 continue to hold true in terms of an arbitrary distribution function $F(.)$ as well. Namely, in the latter case, let $F(.)$ be defined to be right continuous, and define its left continuous inverse $F^{-1}(\cdot)$ as before. Then, relative to the conditional distribution $P_{X|w}$, if $N, m_N \to \infty$ so that $m_N = o(N^2)$, in lieu of (2.3), in probability $P_w$ we have

$$
\beta_{mN,N}^{(3)} \left( F^{-1}(\cdot), \theta \right) \xrightarrow{\text{Law}} B(F(F^{-1}(\cdot))) \text{ on } (D, \mathcal{D}, ||||),
$$

and, in lieu of (2.4), in probability $P_w$ we have

$$
\tilde{\beta}_{mN,N}^{(3)} \left( F^{-1}(\cdot), \theta \right) \xrightarrow{\text{Law}} B(F(F^{-1}(\cdot))) \text{ on } (D, \mathcal{D}, ||||).
$$

Consequently, limiting functional laws will depend on $F$, unless it is continuous.

**Corollary 2.1.** As $N, m_N \to \infty$ so that $m_N = o(N^2)$, via (2.3) with $\theta \in \mathbb{R}$ we conclude in probability $P_w$

$$
P_{X|w}(h(\beta_{mN,N}^{(3)}(F^{-1}(\cdot), \theta) \leq y)) \to P(h(B(\cdot)) \leq y) =: G_{h(B(\cdot))}(y), \quad y \in \mathbb{R}, \tag{2.5}
$$

on letting $\theta = 1$ in (2.3) and recalling Remark 1.1.

$$
P_{X|w}(h(\beta_{mN,N}^{(1)}(F^{-1}(\cdot)) \leq y)) \to P(h(B(\cdot)) \leq y) =: G_{h(B(\cdot))}(y), \quad y \in \mathbb{R}, \tag{2.6}
$$
and, on letting $\theta = 0$ in \(2.5\) and recalling Remark 1.1,

\[
P_{X|w}(h(\beta_{m,N}^{(2)}(F^{-1}(|\cdot|) \leq y)) \rightarrow P(h(B(|\cdot|) \leq y) =: G_{h(B(|\cdot|))}(y), \ y \in \mathbb{R}, \quad (2.7)
\]

at all points of continuity of the distribution function $G_{h(B(|\cdot|))}(|\cdot|)$ for all functionals $h : D \rightarrow \mathbb{R}$ that are $(D, \mathcal{D})$ measurable and \( \| \cdot \| \)-continuous, or \( \| \cdot \| \)-continuous, except at points forming a set of measure zero on $(D, \mathcal{D})$ with respect to the measure generated by $\{B(t), 0 \leq t \leq 1\}$.

Consequently, by a bounded convergence theorem as spelled out in Lemma 1.2 of S. Csörgö and Rosalsky (2003), as $N, m_N \rightarrow \infty$ so that $m_N = o(N^2)$, respectively via (2.5) and (2.6) we conclude also

\[
P_{X,w}(h(\beta_{m,N}^{(3)}(F^{-1}(|\cdot|, \theta)) \leq y) \rightarrow G_{h(B(|\cdot|))}(y), \ y \in \mathbb{R}, \quad (2.8)
\]

and

\[
P_{X,w}(h(\beta_{m,N}^{(1)}(F^{-1}(|\cdot|) \leq y) \rightarrow G_{h(B(|\cdot|))}(y), \ y \in \mathbb{R}, \quad (2.9)
\]

a remarkable extension of (1.14), and, via (2.7) we arrive at having as well

\[
P_{X,w}(h(\beta_{m,N}^{(2)}(F^{-1}(|\cdot|) \leq y) \rightarrow G_{h(B(|\cdot|))}(y), \ y \in \mathbb{R}, \quad (2.10)
\]

at all points of continuity of the distribution function $G_{h(B(|\cdot|))}(|\cdot|)$ for all functionals $h : D \rightarrow \mathbb{R}$ as spelled out right after (2.7) above.

In the sequel, we will make use of taking $h$ to be the sup functional on $D[0, 1]$.

In view of (2.6) and (2.9), as $N, m_N \rightarrow \infty$ so that $m_N = o(N^2)$, with $\overset{d}{\rightarrow}$ standing for convergence in distribution, we have

\[
\sup_{x \in \mathbb{R}} \left| \beta_{m,N}^{(1)}(x) \right| = \frac{1}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(a)}}{m_N} - \frac{1}{N} \right)^2}} \sup_{x \in \mathbb{R}} \left| F_{m,N}(x) - F_N(x) \right| = \sup_{0 \leq t \leq 1} \left| \beta_{m,N}^{(1)}(F^{-1}(t)) \right| \overset{d}{\rightarrow} \sup_{0 \leq t \leq 1} |B(t)| \quad (2.11)
\]

both in terms of $P_{X|w}$ and $P_{X,w}$, while in view of (2.7) and (2.10), we conclude

\[
\sup_{x \in \mathbb{R}} \left| \beta_{m,N}^{(2)}(x) \right| = \frac{1}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(a)}}{m_N} - \frac{1}{N} \right)^2 + \frac{1}{N}}} \sup_{x \in \mathbb{R}} \left| F_{m,N}(x) - F(x) \right| = \sup_{0 \leq t \leq 1} \left| \beta_{m,N}^{(2)}(F^{-1}(t)) \right| \overset{d}{\rightarrow} \sup_{0 \leq t \leq 1} |B(t)| \quad (2.12)
\]

both in terms of $P_{X|w}$ and $P_{X,w}$.
As a consequence of [2.5] and [2.8], as \( N, m_n \to \infty \) so that \( m_N = o(N^2) \), for \( \beta_{m,N}^{(\cdot)}(x, \theta) \) as in (1.7) with \( F(.) \) continues, we conclude

\[
\sup_{x \in \mathbb{R}} \left| \beta_{m,N}^{(3)}(x, \theta) \right| = \sup_{x \in \mathbb{R}} \left| \left( F_{m,N}(x) - \theta F_N(x) \right) - (1 - \theta) F(x) \right| \leq \sqrt{\sum_{j=1}^{N} \left( \frac{w_{m}}{m} \right)^2 + \frac{(1-\theta)^2}{N}}
\]

\[
= \sup_{0 \leq t \leq 1} \left| \beta_{m,N}^{(3)}(F^{-1}(t), \theta) \right| \to \sup_{0 \leq t \leq 1} |B(t)| \tag{2.13}
\]

both in terms of \( P_{X|w} \) and \( P_{X,w} \).

**Remark 2.2.** We note in passing that on letting \( \theta = 1 \), the statement of (2.13) reduces to that of (2.11), while on letting \( \theta = 0 \), we arrive at that of (2.12). With \( \{ \theta \in \mathbb{R} | \theta \neq 1 \} \) in (2.13), we have \( F_{m,N}(x) - \theta F_N(x) \) estimating \( (1 - \theta) F(x) \) uniformly in \( x \in \mathbb{R} \), a companion to \( F_{m,N}(x) \) also estimating \( F(x) \) uniformly in \( x \in \mathbb{R} \) as in (2.12), as well as to \( F_N(x) \) alone estimating \( F(x) \) uniformly in \( x \in \mathbb{R} \) as right below in (2.14).

Naturally, for the classical empirical process \( \beta_N(\cdot) \) (cf. (1.2) and (2.1) with \( F(x) = P_X(X \leq x) \) continuous), as \( N \to \infty \), via (2.2) we arrive at

\[
\sup_{x \in \mathbb{R}} |\beta_N(x)| = \sup_{0 \leq t \leq 1} |\beta_N(F^{-1}(t))| \to \sup_{0 \leq t \leq 1} |B(t)| \tag{2.14}
\]

in terms of \( P_X \).

**Corollary 2.2.** As \( N, m_n \to \infty \) so that \( m_N = o(N^2) \), via (2.4) and (1.8) with \( \theta \in \mathbb{R} \), we conclude in probability \( P_w \)

\[
P_{X|w} \left( h\left( \beta_{m,N}^{(3)}(F^{-1}(\cdot), \theta) \right) \leq y \right) \to P \left( h\left( B(.) \right) \leq y \right) = G_{h(B(.))}(y), \ y \in \mathbb{R}, \tag{2.15}
\]

on letting \( \theta = 1 \) in (1.8) and recalling (1.9)

\[
P_{X|w} \left( h\left( \beta_{m,N}^{(1)}(F^{-1}(\cdot)) \right) \leq y \right) \to P \left( h\left( B(.) \right) \leq y \right) = G_{h(B(.))}(y), \ y \in \mathbb{R}, \tag{2.16}
\]

and on letting \( \theta = 0 \), in (1.8) and recalling (1.10)

\[
P_{X|w} \left( h\left( \beta_{m,N}^{(2)}(F^{-1}(\cdot)) \right) \leq y \right) \to P \left( h\left( B(.) \right) \leq y \right) = G_{h(B(.))}(y), \ y \in \mathbb{R}, \tag{2.17}
\]

at all points of continuity of the distribution function \( G_{h(B(.))}(\cdot) \) for all functionals \( h : D \to \mathbb{R} \) that are \((D, \mathcal{D})\) measurable and || -continuous, or || -continuous, except at points forming a set of measure zero on \((D, \mathcal{D})\) with respect to the measure generated by \( \{B(t), 0 \leq t \leq 1\} \).
Consequently, again in view of Lemma 1.2 of S. Csörgő and Rosalsky (2003), as $N, m_N \to \infty$ so that $m_N = o(N^2)$, respectively via (2.15), (2.16) and (2.17), we conclude also

\[
P_{X,w} \left( h\left( \tilde{\beta}^{(3)}_{m_N,N}(F^{-1}(\cdot), \theta) \right) \leq y \right) \to G_{h(B(\cdot))}(y), \ y \in \mathbb{R},
\]

(2.18)

\[
P_{X,w} \left( h\left( \tilde{\beta}^{(1)}_{m_N,N}(F^{-1}(\cdot)) \right) \leq y \right) \to G_{h(B(\cdot))}(y), \ y \in \mathbb{R},
\]

and

(2.19)

\[
P_{X,w} \left( h\left( \tilde{\beta}^{(2)}_{m_N,N}(F^{-1}(\cdot)) \right) \leq y \right) \to G_{h(B(\cdot))}(y), \ y \in \mathbb{R},
\]

(2.20)

at all points of continuity of the distribution function $G_{h(B(\cdot))}(\cdot)$ for all functionals $h : D \to \mathbb{R}$ as spelled out right after (2.17) above.

On taking $h$ to be the sup functional on $D[0,1]$ in (2.15) - (2.20), as $N, m_N \to \infty$ so that $m_N = o(N^2)$, we conclude, both in terms of $P_{X|w}$ and $P_{X,w}$,

\[
\sup_{x \in \mathbb{R}} \left| \tilde{\beta}^{(3)}_{m_N,N}(x, \theta) \right| = \sqrt{\frac{Nm_N}{N + m_N(1 - \theta)^2}} \sup_{x \in \mathbb{R}} \left| F_{m_N,N}(x) - \theta F_N(x) \right| - (1 - \theta)F(x)
\]

(2.21)

\[
\sup_{x \in \mathbb{R}} \left| \tilde{\beta}^{(1)}_{m_N,N}(x) \right| = \sqrt{\frac{NmN}{N + m_N}} \sup_{x \in \mathbb{R}} \left| F_{m_N,N}(x) - F_N(x) \right|
\]

(2.22)

\[
\sup_{x \in \mathbb{R}} \left| \tilde{\beta}^{(2)}_{m_N,N}(x) \right| = \sqrt{\frac{Nm_N}{N + m_N}} \sup_{x \in \mathbb{R}} \left| F_{m_N,N}(x) - F_N(x) \right|
\]

(2.23)

\textbf{Remark 2.3.} \textit{Mutatis mutandis, the conclusions of Remark 2.2 continue to be valid concerning (2.21) - (2.23) and (2.14).}
3 Confidence bands for empirical and theoretical distributions via virtual resampling big concrete or imaginary random sample data sets that are viewed as samples from infinite super-populations

Let \( \{X_1, \ldots, X_N\} \) be a big i.i.d. data set on \( X \), or an imaginary random sample (a finite population) of size \( N \geq 1 \) on \( X \), of a hypothetical infinite super-population with distribution function \( F(x) = P_X(X \leq x), \; x \in \mathbb{R} \). As in Section 2 we assume throughout this section that \( F(.) \) is a continuous distribution function.

Big data sets in this section refer to having too many data in one random sample as above that in some cases need to be stored on several machines, on occasions even on thousands of machines. In some cases processing samples of this size may be virtually impossible or, simply, even undesirable to make use of the whole sample.

To deal with this problem, as well as with that of sampling from a finite population, we explore and exploit the virtual resampling method of taking sub-samples from the original big data set, or finite population, of size \( N \) so that only the reduced number of the picked elements \( m_N \) of the original sample are to be used to infer about the parameters of interest of the big data set, or of a finite population, as well as those of their super-population. This can be done by generating a realization of multinomial random variables \( \left(w_1^{(N)}, \ldots, w_N^{(N)}\right) \) of size \( m_N = \sum_{i=1}^{N} w_i^{(N)} \), independently from the data (cf. (1.3)) so that \( m_N \ll N \). The generated weights are to be put in a one-to-one correspondence with the indices of the members of the original concrete or imaginary data set. Then, only those data in the set in hand are to be observed whose corresponding weights are not zero.

We note in passing that in the case when the sub-sample size \( m_N \) and the sample size \( N \) are so that \( m_N/N \leq 0.05 \), then for each \( i, 1 \leq i \leq N, \; w_i^{(N)} \) is either zero or one almost all the time. Thus, in this case, our virtual resampling with replacement method as in (1.3), practically reduces to reduction via virtual resampling without replacement.

In the present context of viewing big data sets and finite populations as random samples and imaginary random samples respectively from an infinite super-population, we are to construct asymptotically exact size confidence bands for both \( F_N(.) \) and \( F(.) \), via our randomized empirical processes. To achieve this goal, we may use the respective conclusions of (2.11) and (2.12), or, asymptotically equivalently (cf. Lemma 7.2), those of (2.22) and (2.23). In view of the more familiar appearance of the respective norming sequences in the conclusions of (2.22) and (2.23), in this section we are to make use of the latter two that are also more convenient for doing calculations.

Consider \( \tilde{\beta}^{(1)}_{m_N,N}(x) \), as in (1.9) and define the event, a confidence set for \( F_N(.) \),
\[\mathcal{A}_{m_N,N}^{(1)}(c_\alpha) := \{ L_{m_N,N}^{(1)}(x) \leq F_N(x) \leq U_{m_N,N}^{(1)}(x), \forall x \in \mathbb{R} \}, \quad (3.1)\]

where

\[L_{m_N,N}^{(1)}(x) := F_{m_N,N}(x) - c_\alpha/\sqrt{m_N}, \quad (3.2)\]

\[U_{m_N,N}^{(1)}(x) := F_{m_N,N}(x) + c_\alpha/\sqrt{m_N}, \quad (3.3)\]

and, given \(\alpha \in (0, 1)\), \(c_\alpha\) is the \((1 - \alpha)\)th quantile of the distribution function of the random variable \(\sup_{0 \leq t \leq 1} |B(t)|\), i.e.,

\[K(c_\alpha) := P\left( \sup_{0 \leq t \leq 1} |B(t)| \leq c_\alpha \right) = 1 - \alpha. \quad (3.4)\]

Then, as \(N, m_N \to \infty\) so that \(m_N = o(N^2)\), by (2.22) we obtain an asymptotically correct \((1 - \alpha)\) size confidence set for \(F_N(\cdot)\), both in terms of \(P_{X|w}\) and \(P_{X,w}\), that via \(P_{X,w}\) reads as follows

\[P_{X,w}(\mathcal{A}_{m_N,N}^{(1)}(c_\alpha)) = P_{X,w}\left( \sup_{x \in \mathbb{R}} |\tilde{\beta}_{m_N,N}(x)| \leq c_\alpha \right) \longrightarrow 1 - \alpha. \quad \]

Consequently, since \(F_N(x), x \in \mathbb{R}\), takes on its values in \([0, 1]\), it follows that, under the same conditions,

\[\left\{ \max \left( 0, L_{m_N,N}^{(1)}(x) \right) \leq F_N(x) \leq \min \left( 1, U_{m_N,N}^{(1)}(x) \right), \forall x \in \mathbb{R} \right\} \quad (3.5)\]

is an asymptotically correct \((1 - \alpha)\) size confidence band for \(F_N(\cdot)\), both in terms of \(P_{X|w}\) and \(P_{X,w}\).

To illustrate the reduction of the number of data that is needed for covering \(F_N(x)\) for all \(x \in \mathbb{R}\) in case of a big data set or a finite population, say of size \(N = 10^4\), on taking \(m_N = \sqrt{N}\), we have

\[m_{10^4} = \sum_{i=1}^{10^4} w_i^{(10^4)} = (10^4)^{1/2} = 100, \]

where the random multinomially distributed weights \((w_1^{(10^4)}, \ldots, w_{10^4}^{(10^4)})\) are generated independently from the data \(\{X_1, \ldots, X_{10^4}\}\) with respective probabilities \(1/10^4\), i.e.,

\[(w_1^{(10^4)}, \ldots, w_{10^4}^{(10^4)}) \overset{d}{=} \text{Multinomial}\left(100; 1/10^4, \ldots, 1/10^4\right). \]

These multinomial weights, in turn, are used to construct an asymptotically correct \((1 - \alpha)\) size confidence set \(\text{a la} (3.1)\) and (3.5), covering the unobserved sample distribution function \(F_N(x)\) uniformly in \(x \in \mathbb{R}\).

Consider now \(\tilde{\beta}_{m_N,N}^{(2)}\), as in (1.10), and define the event, a confidence set for \(F(\cdot)\),
\[ A^{(2)}_{m,N}(c_\alpha) := \{ L^{(2)}_{m,N}(x) \leq F(x) \leq U^{(2)}_{m,N}(x), \forall x \in \mathbb{R} \}, \] (3.6)

where
\[
L^{(2)}_{m,N}(x) := F_{m,N}(x) - c_\alpha \sqrt{1/m_N + 1/N}, \\
= F_{m,N}(x) - c_\alpha \frac{1 + m_N/N}{m_N}, \quad (3.7)
\]
\[
U^{(2)}_{m,N}(x) := F_{m,N}(x) + c_\alpha \sqrt{1/m_N + 1/N}, \\
= F_{m,N}(x) + c_\alpha \frac{1 + m_N/N}{m_N}, \quad (3.8)
\]

and, given \( \alpha \in (0, 1) \), \( c_\alpha \) is again as in \((3.4)\).

Thus, as \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), by \((2.23)\), we now obtain an asymptotically correct \((1 - \alpha)\) size confidence set for \( F(.) \), both in terms of \( P_{X|w} \) and \( P_{X,w} \), that via \( P_{X,w} \) reads as follows
\[
P_{X,w}(A^{(2)}_{m,N}(c_\alpha)) = P_{X,w}\left( \sup_{x \in \mathbb{R}} |\tilde{\beta}_{m,N}(x)| \leq c_\alpha \right) \longrightarrow 1 - \alpha. \quad (3.9)
\]

Consequently, since \( F(x), x \in \mathbb{R} \), takes on its values in \([0,1]\), it follows that, under the same conditions,
\[
\{ \max (0, L^{(2)}_{m,N}(x)) \leq F(x) \leq \min (0, U^{(2)}_{m,N}(x)), \forall x \in \mathbb{R} \} \quad (3.10)
\]
is an asymptotically correct \((1 - \alpha)\) size confidence band for \( F(.) \), both in terms of \( P_{X|w} \) and \( P_{X,w} \).

As to the sub-samples of size \( m_N, N \geq 1 \), that are to be used for covering \( F_N(.) \) and \( F(.) \), for all \( x \in \mathbb{R} \), respectively as in \((3.5)\) and \((3.10)\), we have \( N, m_N \to \infty \) so that \( m_N = o(N^2) \) in both cases. Thus we may for example consider having \( N, m_N \to \infty \) so that \( m_N = o(N) \), say \( m_N = N^\varepsilon \) with \( 0 < \varepsilon < 1 \). Then the respective lower and upper bounds in \((3.1)\) and \((3.6)\) for covering \( F_N(.) \) and \( F(.) \) respectively via \((3.5)\) and \((3.10)\) will eventually coincide, and the asymptotically correct \((1 - \alpha)\) confidence bands therein will practically be of equal width. For instance, using the above illustrative example right after \((3.5)\), where \( N = 10^4 \) and \( m_N = m_{10^4} = 100 \), in the event \( A^{(1)}_{m,N}(c_\alpha) \), as in \((3.1)\), in this case \( c_\alpha \) is multiplied by \( \sqrt{1/100} \), while in the event \( A^{(2)}_{m,N}(c_\alpha) \) as in \((3.6)\), \( c_\alpha \) is multiplied by \( \sqrt{1/100 + 1/10^4} \).

In view of having \( m_N, N \to \infty \) so that \( m_N = o(N^2) \) in both \((3.5)\) and \((3.10)\), we may also consider the case \( m_N = O(N) \) as \( N \to \infty \), and can thus, e.g., also have \( m_N = cN \) with a small constant \( 0 < c << 1 \), in the context of this section.

**Remark 3.1.** We can also make use of the functional \( \sup_{x \in \mathbb{R}} |\tilde{\beta}_{m,N}(x)| \) as in \((2.23)\) for goodness of fit tests for \( F \) against general alternatives in our present context.
of virtual resampling big data sets and finite populations when they are viewed as samples from infinite super-populations with continuous $F$. Namely, for testing the null hypothesis $H_0: F = F_0$, where $F_0$ is a given continuous distribution function, we let $F = F_0$ in (2.23), and reject $H_0$ in favor of the alternative $H_1: F \neq F_0$, for large values of the thus obtained statistic at significance level $\alpha \in (0, 1)$ as $N, m_N \to \infty$ so that $m_N = o(N^2)$. Thus, in view of (2.23), as $N, m_N \to \infty$ so that $m_N = o(N^2)$, an asymptotic size $\alpha \in (0, 1)$ Kolmogorov type test for $H_0$ versus $H_1$ has the rejection region, both in terms of $P_{X|W}$ and $P_{X,W}$,

$$\sqrt{\frac{Nm_N}{N + m_N}} \sup_{x \in \mathbb{R}} |F_{m_N,N}(x) - F_0(x)| \geq c_\alpha,$$

where $c_\alpha$ is as in (3.4).

**Remark 3.2.** In view of the conclusions of (2.17) and (2.20), that are asymptotically equivalent to those of (2.6) and (2.10), we may also consider other functionals $h(.)$ for goodness of fit tests for $F$ against general alternatives in our present context of virtual resampling big data sets and finite populations as right above in Remark 3.1. For example, based on (2.17) and (2.20), as $N, m_N \to \infty$ so that $m_N = o(N^2)$, we have, both in terms of $P_{X|W}$ and $P_{X,W}$,

$$\frac{Nm_N}{N + m_N} \int_{-\infty}^{+\infty} \left( F_{m_N,N}(x) - F(x) \right)^2 dF(x) = \int_0^1 \left( \tilde{\beta}^{(2)}_{m_N,N}(F^{-1}(t)) \right)^2 dt \to \int_0^1 B^2(t) dt. \tag{3.12}$$

Thus, in view of (3.12), as $N, m_N \to \infty$ so that $m_N = o(N^2)$, an asymptotic size $\alpha \in (0, 1)$ Cramér-von Mises-Smirnov type test for testing $H_0$ versus $H_1$ as in Remark 3.1 has the rejection region

$$\omega^2_{N,m_N} := \frac{Nm_N}{N + m_N} \int_{-\infty}^{+\infty} \left( F_{m_N,N}(x) - F_0(x) \right)^2 dF_0(x) \geq \nu_\alpha \tag{3.13}$$

where $\nu_\alpha$ is the $(1 - \alpha)$th quantile of the distribution function of the random variable $\omega^2 = \int_0^1 B^2(t) dt$, i.e.,

$$V(\nu_\alpha) := P(\omega^2 \leq \nu_\alpha) = 1 - \alpha. \tag{3.14}$$

### 4 Confidence bands for theoretical distributions via virtual resampling from large enough, or moderately small, samples

When all $N$ observables of large enough, or moderately small, samples are available to be processed, then (2.14) yields the asymptotically exact $(1 - \alpha)$ size classical
Kolmogorov confidence band for a continuous distribution function \( F(x) \). Namely, as \( N \to \infty \), with probability \( (1 - \alpha) \), \( \alpha \in (0, 1) \), we have

\[
\left\{ \max \left( 0, F_N(x) - c_\alpha / \sqrt{N} \right) \leq F(x) \leq \min \left( 1, F_N(x) + c_\alpha / \sqrt{N} \right), \forall x \in \mathbb{R} \right\} \to 1 - \alpha,
\]

where \( c_\alpha \) is as in (3.4).

Next, in the present context, in the asymptotically correct \((1 - \alpha)\) size confidence band for \( F(.) \) as in (3.10) that obtains as \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), and is valid both in terms of \( P_{X,w} \) and \( P_{X,w} \), we may for example let \( m_N = N \) in case of having large enough samples, and \( m_N = N^\varepsilon \) with \( 1 < \varepsilon < 2 \), when having moderately small samples, and compare the thus obtained bands in (3.10) to that of the classical one in (4.1).

Also, the conclusion of (3.5) provides an asymptotically correct \((1 - \alpha)\) size confidence band for the observable \( F_N(.) \) itself in a similar way, via the also observable \( F_{m_N,N}(.) \), say as above, with \( m_N = N \) in case of moderately large samples and, with \( m_N = N^\varepsilon \), \( 1 < \varepsilon < 2 \), when having only moderately small samples.

The goodness of fit tests as in (3.11) and (3.13) can also be used for example with \( m_N \) as above, when all observables of large enough, or moderately small, samples are available to be processed.

Furthermore, as noted in Remarks 2.2 and 2.3 respectively, with \( \{ \theta \in \mathbb{R} | \theta \neq 1 \} \), in (2.12) and, asymptotically equivalently, in (2.21), the linear combination of the respective empirical distributions as in (1.1) and (1.5), \( \{ F_{m_N,N}(x) - \theta F_N(x), x \in \mathbb{R} \} \), estimates \((1 - \alpha)F(x)\) uniformly in \( x\mathbb{R} \). For an elaboration on this topic, we refer to Appendix.

5 Numerical illustrations

In this section we provide a brief numerical illustration of conclusions in Sections 3 and 4 on capturing the empirical and theoretical distributions via the sup-norm functional.
Figure 1: Illustration of the subsampled confidence band (3.10) for $F$ in the context of Section 4. The lighter shaded subsampled confidence band (3.10) overlays the classical one (the darker shaded band) for $F$ as in (4.1), for $F = \chi^2$ with df $= 6$. The solid curve is the cdf of $F$ from which a sample of size $N = 100$ was simulated. The left and right panels are 95% confidence bands for $F$, with $c_\alpha = 1.358$, based on subsamples of respective sizes $m_N = N = 100$ and $m_N = \lceil (100)^{1.9} \rceil = 6310$. The subsampled band for $F$ as in (3.10) on the right panel is just about the same as that of the classical one as in (4.1).

Figure 2: Illustration of the subsampled confidence bands (3.5) and (3.10) respectively for $F_N$ and $F$ in the context of Section 3. The left panel is a 95% subsample confidence band for $F_N$ via (3.5) with $c_\alpha = 1.358$ based on a subsample of size $m_N = (36000)^{1/2} = 6000$ from a random sample of size $N = 36000$ from $F = \chi^2_6$. The right panel is a 95% subsample confidence band for $F$ via (3.10) with $c_\alpha = 1.358$ based on the same subsample of size $m_N = 6000$ from the same random sample of size $N = 36000$ from $F = \chi^2_6$. 
Remark 5.1. As $N, m_N \to \infty$ so that $m_N = o(N^2)$, the thus obtained asymptotically correct band of (3.10) for $F$ is a bit wider than that of (3.5) for $F_N$. It, i.e., (3.10), correctly views and parameterizes the problem of estimating $F$ via the randomly weighted sample distribution function $F_{m_N,N}$ (cf. (1.5)) as a two sample problem in terms of the respective subsample and sample sizes $m_N$ and $N$. On the other hand, (3.5) quite naturally treats and parameterizes the problem of estimating $F_N$ via $F_{m_N,N}$ as a one sample problem in terms of having $m_N$ observations out of $N$ available for performing this inference for $F_N$. On choosing $m_N$ large so that $m_N = o(N)$ when $N$ is large, (3.10) and (3.5) tend to coincide, as, e.g., in Figure 2. However, to begin with, for the sake of establishing exact asymptotic correctness, one should use (3.5) for estimating $F_N$ and, respectively, (3.10) for estimating $F$ and, depending on how big a size $N$ one has for a concrete data set or for an imaginary random sample, one should explore selecting smaller values for $m_N = o(N^2)$ accordingly. For example, having a not very large sample size $N$, say $N = 10000$, one may like to consider taking $m_N = O(N)$, say $m_N = N/2$.

Table 1: Illustration of the performance of the randomized bands (3.10) for the theoretical distribution $F$ to the classical band as in (4.1) based on 1000 replications of the specified distributions. For each simulated sample of size $N$ simultaneous subsamples of size $m_N = \lceil N^a \rceil$, $a = 0.5, 1, 1.9$, were drawn from it to construct the randomized confidence bands (3.10). Nominal coverage probability for the bands is 95%.

| Distribution | $N$ | $m_N$ | empiric coverage of (3.10) | empiric coverage of (4.1) |
|--------------|-----|-------|--------------------------|--------------------------|
| $\chi^2_1$  | 50  | $\lceil N^{0.5} \rceil$ | 0.985                    | 0.970                    |
|              |     | $N$   | 0.976                    |                          |
|              |     | $\lceil N^{1.9} \rceil$ | 0.966                    |                          |
|              | 200 | $\lceil N^{0.5} \rceil$ | 0.978                    | 0.953                    |
|              |     | $N$   | 0.971                    |                          |
|              |     | $\lceil N^{1.9} \rceil$ | 0.954                    |                          |
| $t_{15}$     | 50  | $\lceil N^{0.5} \rceil$ | 0.985                    | 0.969                    |
|              |     | $N$   | 0.977                    |                          |
|              |     | $\lceil N^{1.9} \rceil$ | 0.97                      |                          |
|              | 200 | $\lceil N^{0.5} \rceil$ | 0.981                    |                          |
|              |     | $N$   | 0.964                    |                          |
|              |     | $\lceil N^{1.9} \rceil$ | 0.96                      |                          |
Table 2: Illustration of the performance of the randomized bands (3.5) for the empirical distribution $F_N$ based on 1000 replications. For each simulated sample of size $N$ simultaneous subsamples of size $m_N = \lceil Na\rceil$, $a = 0.5$, 1, 1.9, were drawn from it to construct the randomized confidence bands (3.5). The nominal coverage probability for the bands is 95%.

| Distribution | $N$ | $m_N$ | empiric coverage of (3.5) |
|--------------|-----|-------|--------------------------|
| $\chi^2_1$   | 50  | $[N^{0.5}]$ | 0.985                    |
|              |     | $N$     | 0.971                    |
|              |     | $[N^{1.9}]$ | 0.957                    |
|              | 200 | $[N^{0.5}]$ | 0.978                    |
|              |     | $N$     | 0.966                    |
|              |     | $[N^{1.9}]$ | 0.955                    |
| $t_{15}$     | 50  | $[N^{0.5}]$ | 0.981                    |
|              |     | $N$     | 0.973                    |
|              |     | $[N^{1.9}]$ | 0.967                    |
|              | 200 | $[N^{0.5}]$ | 0.977                    |
|              |     | $N$     | 0.97                      |
|              |     | $[N^{1.9}]$ | 0.954                    |

6 Randomized central limit theorems and confidence intervals for empirical and theoretical distributions at fixed points via virtual resampling

In case of a continuous distribution function $F(.)$, our discussion of confidence bands as in Sections 3 and 4 can naturally be reformulated in terms of confidence intervals, pointwise for any fixed $x \in \mathbb{R}$. In the latter context, i.e., pointwise, we can, however, do better in general, namely also when having random samples $X_1, \ldots, X_N$, $N \geq 1$, on $X$ with an arbitrary distribution function $F(.)$.

As before, let $F_N(.)$ be the empirical distribution function as in (1.1). Define the standardized empirical process

$$\alpha_N(x) := \frac{N^{-1/2} \sum_{i=1}^{N} \left( \mathbb{1}(X_i \leq x) - F(x) \right)}{\sqrt{F(x)(1 - F(x))}}, \quad x \in \mathbb{R},$$

(6.1)
and the sample variance of the indicator random variables \( \mathbb{I}(X_i \leq x) \), \( i = 1, \ldots, N \), as

\[
S_N^2(x) := \frac{1}{N} \sum_{i=1}^{N} \left( \mathbb{I}(X_i \leq x) - F_N(x) \right)^2
= F_N(x)(1 - F_N(x)), \quad x \in \mathbb{R},
\]

(6.2)

where \( F_N(.) \) is the empirical distribution function.

Define also the Studentized empirical process

\[
\hat{\alpha}_N(x) := \frac{N^{1/2}}{\sqrt{F_N(x)(1 - F_N(x))}} \left( F_N(x) - F(x) \right), \quad x \in \mathbb{R}.
\]

(6.3)

As a consequence of the classical central limit theorem (CLT) for Bernoulli random variables, as \( N \to \infty \), we have for any fixed \( x \in \mathbb{R} \)

\[
\alpha_N(x) \xrightarrow{d} Z,
\]

(6.4)

where \( Z \) here, and also throughout, stands for a standard normal random variable.

On combining the latter conclusion with the Glivenko-Cantelli theorem, as \( N \to \infty \), we also have

\[
\hat{\alpha}_N(x) \xrightarrow{d} Z
\]

(6.5)

for any fixed \( x \in \mathbb{R} \).

With \( m_N = \sum_{i=1}^{N} w_i^{(N)} \) and multinomial weights \( (w_1^{(N)}, \ldots, w_N^{(N)}) \) as in (1.3) that are independent from the random sample \( X_1, \ldots, X_N \), define the randomized standardized empirical process \( \alpha_{m_N,N}^{(1)}(x) \), a standardized version of \( \beta_{m_N,N}^{(1)}(x) \) as in (1.4), as follows

\[
\alpha_{m_N,N}^{(1)}(x) := \frac{\sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{m_N} - \frac{1}{N} \right) \mathbb{I}(X_i \leq x)}{\sqrt{F(x)(1 - F(x))} \sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}}
= \frac{F_{m_N,N}(x) - F_N(x)}{\sqrt{F(x)(1 - F(x))} \sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}}, \quad x \in \mathbb{R},
\]

(6.6)

where \( F_{m_N,N}(.) \) is the randomly weighted sample distribution function as in (1.5).

Define also the randomized subsample variance of the indicator random variables \( \mathbb{I}(X_i \leq x) \), \( i = 1, \ldots, N \), as
\[ S^2_{m,N,N} := \frac{1}{m_N} \sum_{i=1}^{N} w_i^{(N)} \left( \mathbb{1}(X_i \leq x) - F_{m_N,N} \right)^2 \]
\[ = F_{m_N,N}(x)(1 - F_{m_N,N}(x)), \quad x \in \mathbb{R}. \quad (6.7) \]

With \( N \) fixed and \( m = m_N \to \infty \), via (1.13), pointwise in \( x \in \mathbb{R} \), we arrive at
\[ S^2_{m,N,N}(x) \to F_N(x)(1 - F_N(x)) = S^2_N(x) \text{ in probability } P_{X,w}, \quad (6.8) \]

and, as a consequence of (1.14), as \( N,m_N \to \infty \), pointwise in \( x \in \mathbb{R} \), we also have
\[ \left( S^2_{m,N,N}(x) - S^2_N(x) \right) \to 0 \text{ in probability } P_{X,w} \quad (6.9) \]

with \( S^2_{m,N,N}(x) \) and \( S^2_N(x) \) respectively as in (6.7) and (6.2).

Further to the randomized standardized empirical process \( \alpha^{(1)}_{m,N,N}(x) \), we now define
\[ \alpha^{(2)}_{m,N,N}(x) := \frac{\sum_{i=1}^{N} \frac{w_i^{(N)}}{m_N} \left( \mathbb{1}(X_i \leq x) - F(x) \right)}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} \right)^2}}, \quad x \in \mathbb{R}, \quad (6.10) \]

a standardized version of \( \beta^{(2)}_{m,N,N}(x) \) as in (1.6) with an arbitrary distribution function \( F(.) \).

Along the above lines, we also define the standardized version of \( \beta^{(3)}_{m,N,N}(x,\theta) \) of (1.7), with an arbitrary distribution function \( F(.) \), namely
\[ \alpha^{(3)}_{m,N,N}(x,\theta) := \frac{\sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{m_N} - \frac{\theta}{N} \right) \left( \mathbb{1}(X_i \leq x) - F(x) \right)}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{\theta}{N} \right)^2}}, \quad x \in \mathbb{R}, \quad (6.11) \]

where \( \theta \) is a real valued constant.
Remark 6.1. On letting $\theta = 1$ in (6.11), it reduces to $\alpha _{m_N,N}^{(1)}(x)$ of (6.6), while letting $\theta = 0$ in (6.11) yields $\alpha _{m_N,N}^{(2)}(x)$ of (6.10).

For the proof of the results of our next proposition, we refer to Remark 7.2.

Proposition 6.1. As $N,m \to \infty$ so that $m = o(N^2)$, for $\alpha _{m_N,N}(\cdot,\theta)$ as in (6.11), we have the following central limit theorem (CLT)

$$P_{X,w} \left( \alpha _{m_N,N}(x,\theta) \leq t \right) \to \Phi(t) \text{ in probability } P_w$$

(6.12)

for all $x,t \in \mathbb{R}$, as well as the unconditional CLT

$$P_{X,w} \left( \alpha _{m_N,N}(x,\theta) \leq t \right) \to \Phi(t) \text{ for all } x,t \in \mathbb{R},$$

(6.13)

where $\Phi(.)$ is the unit normal distribution function.

Corollary 6.1. In view of Remark 6.1 and Proposition 6.1, as $N,m \to \infty$ so that $m = o(N^2)$, we have

$$\alpha _{m_N,N}(x) \xrightarrow{d} Z \text{ for all } x \in \mathbb{R},$$

(6.14)

with $s = 1$ (cf. (6.6)) and also for $s = 2$ (cf. (6.10)), both CLTs in terms of both $P_{X,w}$ (cf. (6.12)) and $P_{X,w}$ (cf. (6.13)).

Remark 6.2. On combining the two respective conclusions of Proposition 6.1 with the Glivenko-Cantelli theorem, the latter two continue to hold true with $\sqrt{F_N(x)(1-F_N(x))}$ replacing $\sqrt{F(x)(1-F(x))}$ in the definition of $\alpha _{m_N,N}(x,\theta)$ as in (6.11), as well as with $\sqrt{F_{m_N,N}(x)(1-F_{m_N,N}(x))}$ replacing $\sqrt{F(x)(1-F(x))}$ therein, on account of the conclusion of (6.9). Consequently, similar respective versions of (6.14) of Corollary 6.1 also hold true. We are now to spell out these conclusions in our next three corollaries.

Corollary 6.2. As $N,m \to \infty$ so that $m = o(N^2)$, we have in terms of both $P_{X,w}$ and $P_{X,w}$

$$\hat{\alpha }_{m_N,N}^{(3)}(x,\theta) := \frac{\left( F_{m_N,N}(x) - \theta F_N(x) \right) - (1-\theta) F(x)}{\sqrt{F_N(x)(1-F_N(x))} \sqrt{\sum_{j=1}^{N} \frac{w_j(N)}{m_N} - \frac{1}{N}}^2 + \frac{(1-\theta)^2}{N}}$$

$$\xrightarrow{d} Z \text{ for all } x \in \mathbb{R},$$

(6.15)

and
\[ \hat{\alpha}^{(1)}_{m_N,N}(x, \theta) := \frac{(F_{m_N,N}(x) - \theta F_N(x)) - (1 - \theta) F(x)}{\sqrt{F_{m_N,N}(x)(1 - F_{m_N,N}(x))}\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2 + \frac{(1 - \theta)^2}{N}}} \]

\[ \xrightarrow{d} Z \text{ for all } x \in \mathbb{R}, \quad (6.16) \]

with any constant \( \theta \in \mathbb{R} \) in both cases.

On taking \( \theta = 1 \), respectively \( \theta = 0 \), in Corollary 6.2, we arrive at the following two corollaries.

**Corollary 6.3.** As \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), we have in terms of both \( P_{X|w} \) and \( P_{X,w} \)

\[ \hat{\alpha}^{(1)}_{m_N,N}(x, 1) = \frac{F_{m_N,N}(x) - F_N(x)}{\sqrt{F_N(x)(1 - F_N(x))}\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}} \]

\[ \xrightarrow{d} Z \text{ for all } x \in \mathbb{R}, \quad (6.17) \]

and

\[ \hat{\alpha}^{(1)}_{m_N,N}(x) := \hat{\alpha}^{(3)}_{m_N,N}(x, 1) = \frac{F_{m_N,N}(x) - F_N(x)}{\sqrt{F_{m_N,N}(x)(1 - F_{m_N,N}(x))}\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2 + \frac{1}{N}}} \]

\[ \xrightarrow{d} Z \text{ for all } x \in \mathbb{R}. \quad (6.18) \]

**Corollary 6.4.** As \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), we have in terms of both \( P_{X|w} \) and \( P_{X,w} \)

\[ \hat{\alpha}^{(2)}_{m_N,N}(x) := \hat{\alpha}^{(3)}_{m_N,N}(x, 0) = \frac{F_{m_N,N}(x) - F(x)}{\sqrt{F_N(x)(1 - F_N(x))}\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2 + \frac{1}{N}}} \]

\[ \xrightarrow{d} Z \text{ for all } x \in \mathbb{R}, \quad (6.19) \]

and

\[ \hat{\alpha}^{(2)}_{m_N,N}(x) := \hat{\alpha}^{(3)}_{m_N,N}(x, 0) = \frac{F_{m_N,N}(x) - F(x)}{\sqrt{F_{m_N,N}(x)(1 - F_{m_N,N}(x))}\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2 + \frac{1}{N}}} \]

\[ \xrightarrow{d} Z \text{ for all } x \in \mathbb{R}. \quad (6.20) \]
In view of Lemma 7.2, Corollaries 6.2, 6.3 and 6.4 have the following asymptotically equivalent respective forms with more familiar looking norming sequences that are also more convenient for doing calculations.

**Corollary 6.5.** As \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), we have in terms of both \( P_{X|w} \) and \( P_{X,w} \)

\[
\tilde{\alpha}_{m,N}(x, \theta) := \sqrt{\frac{Nm_N}{N + m_N(1-\theta)^2}} \frac{(F_{m,N,N}(x) - \theta F_N(x) - (1-\theta) F(x)}{\sqrt{F_N(x)(1 - F_N(x))}} \\
\xrightarrow{d} Z \text{ for all } x \in \mathbb{R}, \tag{6.21}
\]

and

\[
\hat{\alpha}_{m,N}(x, \theta) := \sqrt{\frac{Nm_N}{N + m_N(1-\theta)^2}} \frac{(F_{m,N,N}(x) - \theta F_N(x) - (1-\theta) F(x)}{\sqrt{F_{m,N}(x)(1 - F_{m,N}(x))}} \\
\xrightarrow{d} Z \text{ for all } x \in \mathbb{R}, \tag{6.22}
\]

with any constant \( \theta \in \mathbb{R} \) in both cases.

On taking \( \theta = 1 \), respectively \( \theta = 0 \), in Corollary 6.5, we arrive at the following two corollaries, respective companions of Corollaries 6.3 and 6.4.

**Corollary 6.6.** As \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), we have in terms of both \( P_{X|w} \) and \( P_{X,w} \)

\[
\tilde{\alpha}_{m,N}(x) := \tilde{\alpha}_{m,N}(x, 1) = \frac{\sqrt{m_N} \left( F_{m,N,N}(x) - F_N(x) \right)}{\sqrt{F_N(x)(1 - F_N(x)))}} \xrightarrow{d} Z \text{ for all } x \in \mathbb{R}, \tag{6.23}
\]

and

\[
\hat{\alpha}_{m,N}(x) := \hat{\alpha}_{m,N}(x, 1) = \frac{\sqrt{m_N} \left( F_{m,N,N}(x) - F_N(x) \right)}{\sqrt{F_{m,N,N}(x)(1 - F_{m,N,N}(x))}} \xrightarrow{d} Z \text{ for all } x \in \mathbb{R}. \tag{6.24}
\]

**Corollary 6.7.** As \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), we have in terms of both \( P_{X|w} \) and \( P_{X,w} \)
\[ \tilde{\alpha}_{m,N}^{(2)}(x) := \tilde{\alpha}_{m,N}(x,0) = \sqrt{N m_N} \frac{F_{m,N}(x) - F(x)}{N + m_N \sqrt{F_N(x)(1 - F_N(x))}} \xrightarrow{d} Z \text{ for all } x \in \mathbb{R}, \]

(6.25)

and

\[ \tilde{\alpha}_{m,N}^{(3)}(x) := \tilde{\alpha}_{m,N}(x,0) = \sqrt{N m_N} \frac{F_{m,N}(x) - F(x)}{N + m_N \sqrt{F_{m,N}(x)(1 - F_{m,N}(x))}} \xrightarrow{d} Z \text{ for all } x \in \mathbb{R}. \]

(6.26)

In the context of viewing big data sets and finite populations as random samples respectively from an infinite super-population, it is of interest to estimate both \( F_N(x) \) and \( F(x) \) also pointwise in \( x \in \mathbb{R} \). The asymptotically equivalent CLT's for \( \hat{\alpha}_{m,N}^{(1)}(x) \) and \( \hat{\alpha}_{m,N}(x) \) as in (6.18) and (6.24) can be used to construct asymptotically exact \( (1 - \alpha) \) size confidence sets for any \( \alpha \in (0, 1) \) and pointwise in \( x \in \mathbb{R} \) for the empirical distribution function \( F_N(x) \) in terms of both \( P_{X|w} \) and \( P_{X,w} \). We note in passing that these two CLT's are essential extensions of the pointwise in \( x \in \mathbb{R} \) estimation of \( F_N(x) \) by \( F_{m,N}(x) \) in (1.14), when \( N, m_N \to \infty \) so that \( m_N = o(N^2) \) is assumed.

In the same context, the asymptotically equivalent CLT's for \( \tilde{\alpha}_{m,N}^{(2)}(x) \) and \( \tilde{\alpha}_{m,N}(x) \) as in (6.20) and (6.26) can, in turn, be used to construct asymptotically exact \( (1 - \alpha) \) size confidence intervals for any \( \alpha \in (0, 1) \) and pointwise in \( x \in \mathbb{R} \) for the arbitrary distribution function \( F(x) \), in terms of both \( P_{X|w} \) and \( P_{X,w} \). We remark as well that these two CLT's constitute significant extensions of the pointwise in \( x \in \mathbb{R} \) estimation of \( F(x) \) by \( F_{m,N}(x) \) in view of (1.14), when \( N, m_N \to \infty \) so that \( m_N = o(N^2) \) is assumed.

As to how to go about constructing these confidence intervals in hand, and to illustrate the kind of reduction of the number of data we can work with in context of a big data set or a finite population, say of of size \( N = 10^4 \), mutatis mutandis, we refer back to the illustrative example right after (3.5), where we outline taking virtual sub-samples of sizes \( m_{10^4} = \sqrt{10^4} = 100. \)

The CLT's for \( \hat{\alpha}_{m,N}(.) \) and \( \hat{\alpha}_{m,N}(.) \) in Corollary 6.3 were already concluded on their own in Csörgő and Nasari (2015) (cf. the respective conclusions of (62) and (63) with \( s = 1 \) and that of (64) in Section 6 therein), where the use of \( \hat{\alpha}_{m,N}(.) \) for constructing pointwise confidence sets for \( F_N(.) \) and pointwise confidence intervals for \( F(.) \) is also detailed.

When all \( N \) observables are available and desirable to be processed, then it is inviting to study and compare the asymptotic confidence intervals that are respectively provided for the arbitrary distribution function \( F(x) \) pointwise in \( x \in \mathbb{R} \) by the classical Studentized process as in (6.3), via (6.5), and the ones we can construct...
using the randomized Studentized empirical processes, say via the CLT’s as in (6.25) and (6.26) respectively, both with \( m_N = N \), and \( N \to \infty \). Also, more generally, when indexed by \( \{ \theta \in \mathbb{R} \mid \theta \neq 1 \} \), the CLT’s in (6.21) and (6.22) yield a family of confidence intervals for an arbitrary distribution function \( F(x) \) pointwise in \( x \in \mathbb{R} \) that, as \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), can be studied along the lines of Section 4 and Appendix.

7 Proofs

Our Theorem 1 for the randomly weighted empirical processes, respectively as in (1.7) and (1.8) that, with \( F(x) \) assumed to be continuous, read as

\[
\{ \beta_{m_N,N}^{(3)}(x, \theta), x \in \mathbb{R} \} = \{ \beta_{m_N,N}^{(3)}(F^{-1}(t), \theta), 0 \leq t \leq 1 \}
\]

and

\[
\{ \tilde{\beta}_{m_N,N}^{(3)}(x, \theta), x \in \mathbb{R} \} = \{ \tilde{\beta}_{m_N,N}^{(3)}(F^{-1}(t), \theta), 0 \leq t \leq 1 \},
\]

is based on well known results on the weak convergence of scalar-weighted empirical processes (cf., e.g., Section 3.3 in Shorack and Wellner (1986) and Section 2.2 in Koul (2002)). The original papers dealing with such empirical processes date back to Koul (1970) and Koul and Staudte (1972) (cf. Shorack (1979) for further references in this regard). In our context, we make use of Corollary 2.2.2 in Koul (2002), and spell it out as follows.

**Theorem 2.** Let \( X, X_1, \ldots \) be real valued i.i.d. random variables on \((\Omega_X, \mathcal{F}_X, P_X)\) and assume that the distribution function \( F(x) = P_X(X \leq x) \) is continuous. Let \( \{d_{i,N}\}_{i=1}^N \) be a triangular array of real numbers, and define the weighted empirical process

\[
\beta_{d,N}(x) := \sum_{i=1}^N d_{i,N}(\mathbb{1}(X_i \leq x) - F(x)), \ x \in \mathbb{R},
\]

\[
= \sum_{i=1}^N d_{i,N}(\mathbb{1}(F(X_i) \leq t) - t)
\]

\[
=: \beta_{d,N}(F^{-1}(t)), \ 0 \leq t \leq 1.
\]

Assume that, as \( N \to \infty \),

\[
\mathcal{H}_N := \max_{1 \leq i \leq N} d_{i,N}^2 \to 0, \quad (7.1)
\]

and

\[
\sum_{i=1}^N d_{i,N}^2 = 1 \quad \text{for each} \quad N \geq 1. \quad (7.2)
\]
Then, as $N \to \infty$,

$$\beta_{d,N}(F^{-1}(t)) \overset{\text{Law}}{\to} B(t) \quad \text{on} \quad (D, \mathcal{D}, \|\|),$$

(7.3)

where $B(.)$ is a Brownian bridge on $[0, 1]$, $\mathcal{D}$ denotes the $\sigma$-field generated by the finite dimensional subsets of $D = D[0,1]$, and $\|\|$ stands for the uniform metric for real valued functions on $[0,1]$.

In order to “translate” Theorem 2 to the first conclusion (2.3) of our Theorem 1, it will suffice to conclude the following maximal negligibility of the weights in probability $P_w$.

**Lemma 7.1.** Let $N, m_N \to \infty$ so that $m_N = o(N^2)$. Then

$$\mathcal{H}_N = \mathcal{H}_N(\theta) := \max_{1 \leq i \leq N} \left( \frac{w_i^{(N)}}{m_N} - \frac{\theta}{N} \right)^2 \to 0 \quad \text{in probability} \quad P_w$$

(7.4)

with any arbitrary real constant $\theta$.

**Remark 7.1.** With $\theta = 1$, the conclusion (7.4) of Lemma 7.1 was established in Csörgő, Martsynyuk and Nasari (2014) in their Lemma 5.2.

The second conclusion (2.4) of Theorem 1 follows from that of its (2.3) via the following asymptotic equivalence of the weights in probability $P_w$.

**Lemma 7.2.** Let $N, m_N \to \infty$ so that $m_N = o(N^2)$. Then

$$\left| \sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{m_N} - \frac{1}{N} \right)^2 - \frac{1}{m_N} \right| = o_{P_w}(1).$$

(7.5)

Based on Lemmas 7.1 and 7.2, we are to first conclude Theorem 1 via Theorem 2 and then proceed to prove these two lemmas in hand.

**Proof of Theorem 1**

Put

$$d_{i,N} = d_{i,N}(\theta) := \frac{w_i^{(n)}/m_N - \theta/N}{\sqrt{(w_i^{(n)}/m_N - \theta/N)^2}}, \quad 1 \leq i \leq N.$$
\[ \sum_{i=1}^{N} d_{i,N}^2 = 1, \text{ for each } N \geq 1, \quad (7.6) \]

and, in view of Lemma 7.1 as \( N, m_N \to \infty \) so that \( m_N = o(N^2) \),

\[ \mathcal{H}_N := \max_{1 \leq i \leq N} d_{i,N}^2 \to \text{in probability } P_w. \quad (7.7) \]

Consequently, every subsequence \( \{N_k\}_k \) of \( N, N \geq 1 \), contains a further subsequence \( \{N_{k_l}\}_l \) such that as \( l \to \infty \), then

\[ \mathcal{H}_{N_{k_l}} := \max_{1 \leq i \leq N_{k_l}} d_{i,N_{k_l}}^2 \to \text{almost surely in } P_w. \quad (7.8) \]

In view of the characterization of convergence in probability in terms of almost sure convergence of subsequences, the respective conclusions of (7.7) and (7.8) are equivalent to each other, and the latter holds true for a set of weights \( \hat{\Omega}_w \in \hat{\mathcal{F}}_w \) with \( P(\hat{\Omega}_w) = 1 \).

Thus, in our present context, condition (7.1) of Theorem 2 is satisfied almost surely in \( P_w \) as in (7.8), and that via (7.7) leads to concluding the statement of (2.3) of Theorem 1. Also, the latter in combination with Lemma 7.2 leads to concluding the second statement (2.4) of Theorem 1 as well. \( \square \)

**Remark 7.2.** Similarly to the above proof of (2.3) of Theorem 2 via the Lindeberg-Feller CLT based Lemma 5.1 of Csörgő et al. (2014) in combination with our Lemma 7.1, we arrive at concluding Proposition 6.1. The latter, in turn, combined with Lemma 7.2 results in having also Corollary 6.5. Thus, the content of Section 6 becomes self-contained.

Using on occasions Lemma 7.2 as well, we now proceed with proving Lemma 7.1. In turn, we then prove Lemma 7.2.

**Proof of Lemma 7.1**

First we show that as \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), then

\[ \frac{\max_{1 \leq i \leq N} \left( \frac{w_{i,N}}{m_N} - \frac{\theta}{N} \right)^2}{\sum_{j=1}^{N} \left( \frac{w_{j,N}}{m_N} - \frac{1}{N} \right)^2} \to 0 \text{ in } P_w \quad (7.9) \]

with any arbitrary real constant \( \theta \).

We have
\[
\left( \frac{w_i^{(N)}}{m_N} - \frac{\theta}{N} \right)^2 = \left( \frac{w_i^{(N)}}{m_N} - \frac{1}{N} + \frac{1}{N} - \frac{\theta}{N} \right)^2
\]

\[
= \left( \frac{w_i^{(N)}}{m_N} - \frac{1}{N} \right)^2 + 2 \left( \frac{w_i^{(N)}}{m_N} - \frac{1}{N} \right) \left( \frac{1}{N} \right) - \left( \frac{1}{N} \right)^2, \quad 1 \leq i \leq N.
\]

Consequently,

\[
\max_{1 \leq i \leq N} \left( \frac{w_i^{(N)}}{m_N} - \frac{\theta}{N} \right)^2 \leq \mathcal{H}_N(1) + I_2(N) + I_3(N), \quad (7.10)
\]

where \( \mathcal{H}_N(1) = \mathcal{H}_N(\theta) \) with \( \theta = 1 \), as in (7.4), i.e., as \( N, m_N \to \infty \) so that \( m_N = o(N^2) \), via Remark 7.1,

\[
\mathcal{H}_N(1) = o_{P_w}(1), \quad (7.11)
\]

\[
I_2(N) := 2|1 - \theta| \left( \frac{\max_{1 \leq i \leq N} \left| \frac{w_i^{(N)}}{m_N} - \frac{1}{N} \right|}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}} \right) \left( \frac{1}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}} \right)
\]

\[
= o_{P_w}(1) \left( \frac{1}{\sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}} \right)
\]

\[
= o_{P_w}(1) \left( \frac{1}{\sqrt{o_{P_w}(1) + 1/m_N}} \right)
\]

\[
= o_{P_w}(1) \left( \frac{1}{\sqrt{m_N/1} \left( \frac{1}{\sqrt{m_N o_{P_w}(1) + 1}} \right)} \right)
\]

\[
= o_{P_w}(1) o(1) \left( \frac{1}{\sqrt{m_N o_{P_w}(1) + 1}} \right)
\]

\[
= o_{P_w}(1), \quad (7.12)
\]

in view of (7.11), Lemma 7.2 and \( m_N = o(N^2) \), and

\[
I_3(N) := \frac{(1 - \theta)^2}{N^2 \sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2} = \frac{(1 - \theta)^2}{N^2 (o_{P_w}(1) + 1/m_N)}
\]

\[
= \frac{(m_N/1) \left( \frac{1}{\sqrt{m_N o_{P_w}(1) + 1}} \right)}{o_{P_w}(1) o(1) \left( \frac{1}{\sqrt{m_N o_{P_w}(1) + 1}} \right)}
\]

\[
= o_{P_w}(1), \quad (7.13)
\]
in view of Lemma 7.2 and $m_N = o(N^2)$.

Combining now (7.11), (7.12) and (7.13), via (7.10) we arrive at (7.9).

Next we write $H_N = H_N(\theta)$ of (7.3) as

$$H_N(\theta) = \left( \max_{1 \leq i \leq N} \left( \frac{w_i(N)}{m_N} - \frac{\theta}{N} \right)^2 \right) \left( \sum_{j=1}^{N} \left( \frac{w_j(N)}{m_N} - \frac{1}{N} \right)^2 \right)$$

$$= o_P(1) I_4(N), \quad (7.14)$$

on account of (7.9) as $N, m_N \to \infty$ so that $m_N = o(N^2)$.

As to the denominator of the term $I_4(N)$, we have

$$\sum_{j=1}^{N} \left( \frac{w_j(N)}{m_N} - \frac{\theta}{N} \right)^2 = \sum_{j=1}^{N} \left( \left( \frac{w_j(N)}{m_N} - \frac{1}{N} \right)^2 + \left( \frac{1}{N} - \frac{\theta}{N} \right)^2 \right)$$

$$= \sum_{j=1}^{N} \left( \frac{w_j(N)}{m_N} - \frac{1}{N} \right)^2 + (1 - \theta)^2 / N$$

$$= \sum_{j=1}^{N} \left( \frac{w_j(N)}{m_N} - \frac{1}{N} \right)^2 \left( 1 + \frac{(1 - \theta)^2}{N(\sum_{j=1}^{N} \left( \frac{w_j(N)}{m_N} - \frac{1}{N} \right)^2 - 1/m_N) + N/m_N} \right)$$

$$= \sum_{j=1}^{N} \left( \frac{w_j(N)}{m_N} - \frac{1}{N} \right)^2 \left( 1 + \frac{(1 - \theta)^2}{N(o_P(1) + 1/m_N)} \right),$$

on account of Lemma 7.2 as $N, m_N \to \infty$ so that $m_N = o(N^2)$.

Consequently, asymptotically accordingly, for $I_4(N)$ as in (7.14), we have

$$I_N(4) = \frac{1}{1 + (1/N)(o_P(1) + 1/m_N)}, \quad (7.15)$$

and thus, via (7.14) and (7.15), we conclude that $H_N(\theta) = o_P(1)$, as claimed by Lemma 7.1. □

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Proof of Lemma 7.2

To prove this lemma we note that $E_w\left(\sum_{i=1}^{N} \left(\frac{w_i^{(N)}}{m_N} - \frac{1}{N}\right)^2 \right) = \frac{(1-\frac{1}{N})}{m_N}$, and apply Chebyshev’s inequality, with arbitrary positive $\varepsilon$. Accordingly, we have

$$P\left(|\sum_{i=1}^{N} \left(\frac{w_i^{(N)}}{m_N} - \frac{1}{N}\right)^2 - \frac{1}{m_N}| > \varepsilon\right) \leq \frac{m_n^2}{\varepsilon^2(1-\frac{1}{N})^2} E_w\left(\sum_{i=1}^{N} \left(\frac{w_i^{(N)}}{m_N} - \frac{1}{N}\right)^2 - \frac{1}{m_N}\right)^2$$

$$= \frac{m_n^2}{\varepsilon^2(1-\frac{1}{N})^2} E_w\left\{\left(\sum_{i=1}^{N} \left(\frac{w_i^{(N)}}{m_N} - \frac{1}{N}\right)^2 - \frac{1}{m_N}\right)^2\right\}$$

$$= \frac{m_n^2}{\varepsilon^2(1-\frac{1}{N})^2} \left\{N E_w\left(\frac{w_1^{(N)}}{m_N} - \frac{1}{N}\right)^4\right\}$$

$$+ N(N-1) E_w\left[\left(\frac{w_1^{(N)}}{m_N} - \frac{1}{N}\right)^2 \left(\frac{w_2^{(N)}}{m_N} - \frac{1}{N}\right)^2 - \frac{1}{m_N}\right] \right\}$$

(7.16)

Considering that the random weights $w_i^{(N)}$ are multinomially distributed, after some algebra one obtains the following upper bound for the right hand side of (7.16):

$$\frac{m_n^2}{\varepsilon^2(1-\frac{1}{N})^2} \left\{\frac{1}{N^3 m_N^3} + \frac{(1-\frac{1}{N})^4}{m_N^3} + \frac{(m_N - 1)(1-\frac{1}{N})^2}{N m_N^3} + \frac{4(N-1)}{N^2 m_N^3} + \frac{1}{m_N^2}\right\}$$

$$- \frac{1}{N m_N^2} + \frac{N - 1}{N^3 m_N^3} + \frac{4(N-1)}{N^2 m_N^3} - \frac{(1-\frac{1}{N})^2}{m_N^2}\right\}.$$}

The preceding term vanishes as $N, m_N \to \infty$ so that $m_N = o(N^2)$. This observation concludes the proof of Lemma 7.2 □
Appendix: Confidence bands for theoretical distributions in combination with virtual resampling from large enough, or moderately small, samples

When all \( N \) observables of large enough, or moderately small, samples are available to be processed, then (2.14) yields the asymptotically exact \((1 - \alpha)\) size classical Kolmogorov confidence band for a continuous distribution function \( F(x) \). Namely, as \( N \to \infty \), with probability \((1 - \alpha)\), \( \alpha \in (0, 1) \), we have

\[
\{ \max \left( 0, \frac{F_N(x) - c_\alpha}{\sqrt{N}} \right) \leq F(x) \leq \min \left( 1, \frac{F_N(x) + c_\alpha}{\sqrt{N}} \right), \forall x \in \mathbb{R} \} \to 1 - \alpha, \tag{8.1}
\]

where \( c_\alpha \) is as in (3.4).

Moreover, as noted already in Remarks 2.2 and 2.3 respectively, with \( \{ \theta \in \mathbb{R} | \theta \neq 1 \} \), in (2.13) and, asymptotically equivalently, in (2.21), the linear combination of the respective empirical process as in (1.1) and (1.5), \( \{ F_N(x) - \theta F_N(x), x \in \mathbb{R} \} \), estimates \((1 - \theta)F(x)\) uniformly in \( x \in \mathbb{R} \) (cf. (1.11) and (1.12) as well). In particular, via the conclusion of (2.21), as \( N, m \to \infty \) so that \( m = o(N^2) \), with \( \{ \theta \in \mathbb{R} | \theta \neq 1 \} \), we arrive at having

\[
\{ L^{(3)}_{m_N,N}(x, \theta) \leq F(x) \leq U^{(3)}_{M_N,N}(x, \theta), \forall x \in \mathbb{R} \} \tag{8.2}
\]

as an asymptotically exact \((1 - \alpha)\) size confidence set for \( F(.) \), both in terms of \( P_{X|x} \) and \( P_{X,w} \), where

\[
L^{(3)}_{m_N,N}(x, \theta) := \begin{cases} 
\frac{1}{1 - \theta} \left( \theta F_N(x) - F_{m_N,N}(x) \right) - c_\alpha \sqrt{\frac{1}{(1 - \theta)^2 m_N} + \frac{1}{N}}, & \theta - 1 > 0. \\
\frac{1}{1 - \theta} \left( F_{m_N,N}(x) - \theta F_N(x) \right) - c_\alpha \sqrt{\frac{1}{(1 - \theta)^2 m_N} + \frac{1}{N}}, & 1 - \theta > 0.
\end{cases}
\tag{8.3}
\]

\[
U^{(3)}_{m_N,N}(x, \theta) := \begin{cases} 
\frac{1}{1 - \theta} \left( F_{m_N,N}(x) - \theta F_N(x) \right) + c_\alpha \sqrt{\frac{1}{(1 - \theta)^2 m_N} + \frac{1}{N}}, & 1 - \theta > 0. \\
\frac{1}{1 - \theta} \left( \theta F_N(x) - F_{m_N,N}(x) \right) + c_\alpha \sqrt{\frac{1}{(1 - \theta)^2 m_N} + \frac{1}{N}}, & \theta - 1 > 0.
\end{cases}
\tag{8.4}
\]

and, given \( \alpha \in (0, 1) \), \( c_\alpha \) is again as in (3.4).

Since \( 0 \leq F(x) \leq 1 \), \( \forall x \in \mathbb{R} \), as a consequence of (8.2) it follows that

\[
\{ \max \left( 0, L^{(3)}_{m_N,N}(x, \theta) \right) \leq F(x) \leq \min \left( 1, U^{(3)}_{M_N,N}(x, \theta) \right), \forall x \in \mathbb{R} \} \tag{8.5}
\]
is an asymptotically correct \((1 - \alpha)\) size confidence band for \(F(.)\), both in terms of \(P_{X|w}\) and \(P_{X,w}\), when \((1 - \theta) > 0\), or \((\theta - 1) > 0\), with \(\{\theta \in \mathbb{R} | \theta \neq 1\}\).

The performance of the family of confidence bands for \(F(.)\) as in (8.5), indexed by \(\{\theta \in \mathbb{R} | \theta \neq 1\}\), is to be compared to that of the classical one as in (4.1).

**Remark 8.1.** When \(\theta = 0\), the confidence band for \(F(.)\) in (8.5) coincides with that of (3.10) with \(L_{m,N}^{(2)}(.)\) and \(U_{m,N}^{(2)}(.)\) as in (3.7) and (3.8), respectively. For \(\theta = 2\), (8.3) and (8.4) yield the bounds

\[
2F_N(x) - F_{m,N}(x) \pm c_\alpha \sqrt{\frac{1}{m_N} + \frac{1}{N}},
\]

for an asymptotically exact \((1 - \alpha)\) size confidence set for \(F(.)\) that is to be used in (8.5) for \(F(.)\), whose width in this case coincides with that of (3.10), as mentioned right above, when \(\theta = 0\). Furthermore, indexed by \(\{\theta \in \mathbb{R} | 0 < (1 - \theta)^2 < 1\}\), i.e., when \(\theta \in (0, 2)\), the confidence bands for \(F(.)\) in (8.5) are wider than the ones provided by it, when taking \(\theta = 0\), or \(\theta = 2\), as above. On the other hand, the confidence bands for \(F(.)\) in (8.5) when indexed by \(\{\theta \in \mathbb{R} | (1 - \theta)^2 > 1\}\), i.e., when \(\theta < 0\) or \(\theta > 2\), are narrower than the just mentioned latter ones that are obtained via taking \(\theta = 0\), \(\theta = 2\). Thus, gaining better probability coverage via wider bands versus wanting narrower bands is completely governed by the windows provided by \(\{\theta \in \mathbb{R} | \theta \in (0, 2)\}\) versus \(\{\theta \in \mathbb{R} | \theta < 0\}\) or \(\{\theta \in \mathbb{R} | \theta > 2\}\).

As illustrative examples of wider bands, with \(\theta = 1/2\), (8.3) and (8.4) yield

\[
(2F_{m,N}(x) - F_N(x)) \pm c_\alpha \sqrt{\frac{4}{m_N} + \frac{1}{N}},
\]

and with \(\theta = 3/2\), (8.3) and (8.4) yield

\[
(3F_N(x) - 2F_{m,N}(x)) \pm c_\alpha \sqrt{\frac{4}{m_N} + \frac{1}{N}},
\]

as respective lower upper bounds to be used in (8.5) in these two cases for constructing the thus resulting respective confidence bands for \(F(.)\).

As illustrative examples of narrower bands, we mention the case of \(\theta = -1\), resulting in having (8.5) with respective lower and upper confidence bounds for \(F(.)\)

\[
\frac{1}{2}(F_{m,N}(x) + F_N(x)) \pm c_\alpha \sqrt{\frac{1}{4m_N} + \frac{1}{N}},
\]

and the case of \(\theta = 3\), yielding

\[
\frac{1}{2}(3F_N(x) - F_{m,N}(x)) \pm c_\alpha \sqrt{\frac{1}{4m_N} + \frac{1}{N}},
\]

as respective lower and upper confidence bounds for for \(F(.)\) in (8.5).
Remark 8.2. On taking $m_N = N$ in (8.5), with $\theta = 0$ we obtain (3.10) with $m_N = N$, and with $\theta \neq 1$ in general, we arrive at (8.3) and (8.4) that are to be used in (8.5) with $m_N = N$. All these bands for $F(.)$ are to be compared to that of the classical one as in (4.1). In particular, when $\theta = 0$, $F_{N,N}(.)$ estimates $F(.)$ alone, as compared to having $2F_{N}(.) - F_{N,N}(.)$ estimating $F(.)$ as in (8.6) when $\theta = 2$, with the same band width, that is a bit wider than that of the classical one in (4.1). The illustrative examples of even wider bands for $F(.)$ respectively provided by the bounds as in (8.7) and (8.8) are also to be compared to each other when $m_N = N$, as well as to that of classical case as in (4.1). Similarly, the illustrative examples of having “two-sample” narrower bands for $F(.)$ respectively provided by (8.10) and (8.11) are to be compared to each other when $m_N = N$, as well as to that of the classical case as in (4.1).

Remark 8.3. On letting $m_N = N$ in (8.3) and (8.4) the widths of the thus obtained bands for $F(.)$ in (8.5) is determined via $\pm \sqrt{\frac{1}{(1 - \theta)^2N} + \frac{1}{N}}$, indexed by $\{\theta \in \mathbb{R} \mid \theta \neq 1\}$. Thus, to begin with, given $N \geq 1$, we may take virtual subsamples of size $m_N = (1 - \theta)^2N$, indexed by $\{\theta \in \mathbb{R} \mid \theta \neq 1\}$, and choose desirable values for $\theta$ via $\{\theta \in \mathbb{R} \mid 0 < (1 - \theta)^2 < 1\}$ in case of wanting wider confidence bands, respectively via $\{\theta \in \mathbb{R} \mid (1 - \theta)^2 > 1\}$ in case of wanting narrower bands, as indicated in Remark 8.1 and illustrated by examples thereafter.

Remark 8.4. The family of functionals $\sup_{x \in \mathbb{R}} |\tilde{\beta}^{(3)}_{m_N,N}(x, \theta)|$ as in (2.21), indexed by $\{\theta \in \mathbb{R} \mid \theta \neq 1\}$, can also be used for goodness of fit tests for $F$ against general alternatives in our present context, i.e., when all $N$ observables of large enough, or moderately small, samples are available to be processed. Namely, for testing $H_0: F = F_0$, where $F_0$ is a given continuous distribution function, we let $F = F_0$ in (2.21), and reject $H_0$ in favor of the alternative hypothesis $H_1: F \neq F_0$, for large values of the thus obtained statistic with any desirable value of $\{\theta \in \mathbb{R} \mid \theta \neq 1\}$ at significance level $\alpha \in (0, 1)$ as $N,m_N \to \infty$ so that $m_N = o(N^2)$. Thus, in view of (2.21), this test with the rejection region

$$\sqrt{\frac{Nm_N}{N + m_N(1 - \theta)^2}} \sup_{x \in \mathbb{R}} \left| (F_{m_N,N}(x) - \theta F_N(x)) - (1 - \theta)F_0(x) \right| \geq c_\alpha \text{(8.11)}$$

where $c_\alpha$ is as in (3.4), is asymptotically of size $\alpha \in (0, 1)$, both in terms of $P_{X|w}$ and $P_{X,\alpha}$, with any desirable value of $\{\theta \in \mathbb{R} \mid \theta \neq 1\}$. With $\theta = 0$, the rejection region (8.11) reduces to that of (3.11) as in Remark 3.1 that can, of course, be also viewed and used in our present context. For values of $\{\theta \in \mathbb{R} \mid \theta \neq 1, \theta \neq 0\}$, the test in hand with rejection region as in (8.11) can only be used in our present context of large enough, or moderately small, samples, for then we are to compute $F_{m_N,N}(x) - \theta F_N(x)$ with some desirable value of $\theta$, like in (8.9) or (8.10), for example.
Remark 8.5. Along the lines of Remark 3.2, on taking $h(.)$ to be the Cramér-von Mises-Smirnov functional, in view of the conclusions (2.15) and (2.18), as $N, m_N \rightarrow \infty$ so that $m_N = o(N^2)$, we have, both in terms of $P_{X|w}$ and $P_{X,w}$,

$$
\frac{N m_N}{N + m_N(1-\theta)^2} \int_{-\infty}^{+\infty} \left( F_{m_N,N}(x) - \theta F_N(x) - (1-\theta)F(x) \right)^2 dF(x) = \int_0^1 \left( \tilde{\beta}_{m_N,N}(F^{-1}(t),\theta) \right)^2 dt \stackrel{d}{\rightarrow} \int_0^1 B^2(t) dt. \tag{8.12}
$$

Consequently, as $N, m_N \rightarrow \infty$ so that $m_N = o(N^2)$, an asymptotic size $\alpha \in (0,1)$ Cramér-von Mises-Smirnov test for $H_0$ versus $H_1$ as in Remark 8.4 has the rejection region

$$
\omega^2_{N,m_N}(\theta) := \frac{N m_N}{N + m_N(1-\theta)^2} \int_{-\infty}^{+\infty} \left( F_{m_N,N}(x) - \theta F_N(x) - (1-\theta)F_0(x) \right)^2 dF_0(x) \geq \nu_\alpha, \tag{8.13}
$$

both in terms of $P_{X|w}$ and $P_{X,w}$ with any desirable value of $\{\theta \in \mathbb{R} | \theta \neq 1\}$, where $\nu_\alpha$ is as in (3.14). With $\theta = 0$, the rejection region (8.13) reduces to that of (8.13), that can also be considered and used in our present context. For values of $\{\theta \in \mathbb{R} | \theta \neq 1, \theta \neq 0\}$, just like that of (8.11), the test in hand with rejection region as in (8.13) can only be used in our present context of large enough, or moderately small, samples.

As an illustration, let $m_N = N = 100$. Then,

$$m_{100} = \sum_{i=1}^{100} w_i^{(100)} = 100,$$

where the multinomially distributed random weights $(w_1^{(100)}, \ldots, w_{100}^{(100)})$ are generated independently from the data $\{X_1, \ldots, X_{100}\}$ with respective probabilities $1/100$, i.e.,

$$(w_1^{(100)}, \ldots, w_{100}^{(100)}) \overset{d}{=} \text{Multinomial}(100,1/100,\ldots,1/100).$$

These multinomial weights, in turn, are used to construct the asymptotic $(1-\alpha)$ size confidence bands for $F(.)$ as in Remark 8.2 and for $F_N(.)$ as in (3.5).

Acknowledgments

I most sincerely wish to thank my colleague Masoud M. Nasari for his interest in, and attention to, the progress of my preparing the presentation of the results of this
paper for publication in a number of years by now. Our discussions in this regard
had much helped me in the process of concluding it in its present form.

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