Finite-type invariants for curves on surfaces

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Abstract: In this note, we define a notion of finite-type for invariants of curves on surfaces as an analogue of the notion of finite-type for invariants of knots and 3-manifolds (Section 3). We also present a systematic construction for a large family of finite-type invariants \( SCI_n \) for curves on surfaces (Section 5). Arnold’s invariants of plane isotopy classes of plane curves occur as invariants of order 1. Our theory of finite-type invariants of curves on surfaces is developed using the topological theory of words.

Key words: Finite-type invariants; immersed curves; topological theory of words; Arnold’s invariants.

1. Introduction. V. A. Vassiliev developed a method for knot classification by applying singularity theory to knots [10]. This method attempts to classify knots by using finite-type invariants (Vassiliev invariants). It remains unknown whether finite-type invariants can classify knots (the Vassiliev conjecture) [3, 10]. Every \( C \)-valued invariant \( v \) of oriented knots is extended inductively to singular knots, knots which may have double points, by resolving the double points using the formula:

\[
(1) \quad v \left( \begin{array}{c}
\vline \\
\vline \\
\vline \\
\end{array} \right) = v \left( \begin{array}{c}
\vline \\
\vline \\
\vline \\
\end{array} \right) - v \left( \begin{array}{c}
\vline \\
\vline \\
\vline \\
\end{array} \right).
\]

We say that \( v \) is a finite-type invariant of order less than or equal to \( n \) if \( v \) vanishes on every singular knot with at least \( n + 1 \) double points, where \( v \) is extended by (1). It is known that there are many finite-type invariants of knots, though it is not easy to construct them.

V. I. Arnold introduced invariants of generic plane curves using a theory similar to that used by Vassiliev [1, 2]. M. Polyak and O. Viro gave a concrete construction of second and third order Vassiliev invariants by Gauss diagram formulas [5]: a Gauss diagram formula is a formula given by a sum over subdiagrams of a given diagram. Polyak also reconstructed Arnold’s invariants by Gauss diagram formulas in a similar manner and combinatorially defined finite-type invariants of plane curves [4]. On the other hand, V. Turaev suggested that words be considered as generalisations of curves and knots and demonstrated that it is possible to classify words in the same manner as knots [7–9].

In this note, we define a notion of finite-type for invariants of curves on surfaces by replacing “double points” in the definition of Vassiliev invariants of knots with “self-tangency points and triple points” of curves on surfaces (Definition 1). Arnold’s invariants of plane curves are finite-type invariants of first order in this sense. We also give a systematic construction of a large family of finite-type invariants \( SCI_n \) for curves on surfaces using the topological theory of words (Main Theorem). An idea of our construction of finite-type invariants is to consider a sum over subwords of a given word, like a Gauss diagram formula.

In this paper, we present the main results and the ideas of the proofs. The details and generalisations will be presented elsewhere.

2. Curves. A curve is a smooth immersion of an oriented circle into an oriented surface. A curve is generic if it has only transversal double points of self-intersection. A curve is singular if it has only transversal double points, self-tangency points, and triple points of self-intersection. A pointed curve is a generic curve with a base point on the curve distinct from the self-intersections. Two curves are stably homeomorphic if there is a homeomorphism of their regular neighbourhoods in the ambient surfaces that maps the first curve onto the second one preserving the orientations of the curve and the surface. Simi-
larly, two pointed curves are said to be stably homeomorphic if there is a homeomorphism as above preserving the base point.

3. Definition of finite-type invariants. In this note, every local self-crossing is either a transversal double point or a point looking like the left hand side of the formulae (2)–(4) and the mirror image of the picture on the left hand side of (3). Note that each of these self-crossings is uniquely determined up to homeomorphism preserving orientation. Self-tangency points or triple points are called singular points. In particular, a self-tangency point is a direct self-tangency point if the two tangent branches are oriented in the same direction; otherwise, it is called an inverse self-tangency point. The direction of the resolution of a self-tangency point is positive if the two tangent branches are oriented in the opposite direction; otherwise, it is negative. The direction of the resolution of the triple point is positive if the two tangent branches are oriented in the same direction; otherwise, it is negative.

We define the orientation of a resolution of a triple point following [1]. An arbitrary triple point gives rise to the newborn triangle which exists just after a resolution of a triple point as shown in Fig. 1. For the newborn triangle, we define the sign of a triangle. By the definition, each singular curve is an immersion of an oriented circle into an oriented surface. Every triple point of a curve has three preimages on the circle and their cyclic order is well defined. Then we obtain a well-defined orientation of the newborn triangle given by the order in which the immersed curve visits its sides. Each side of the immersed curve also has its own orientation. The orientation may coincide with the orientation defined by the cyclic order of the sides of the triangle or may be opposite to it. Let \( q \) be the number of the sides of the newborn triangle whose orientations coincide with that given by the cyclic order. The sign of the triangle is defined as \((-1)^q\) (Fig. 1).

The direction of the resolution of the triple point is positive if the sign of the newborn triangle is 1 after the resolution. The direction of the resolution of the singular point is negative if the direction is non-positive. It is possible to resolve singular points away from the base point.

Let \( G \) be some Abelian group. Every \( G \)-valued invariant \( \varphi \) of generic curves is extended inductively to singular curves by resolving the singular points using (2), (3), and (4):

\[
\varphi \left( \begin{array}{c} \circ \end{array} \right) = \varphi \left( \begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \right) - \varphi \left( \begin{array}{c} \circ \end{array} \right),
\]

\[
\varphi \left( \begin{array}{c} \circ \end{array} \right) = \varphi \left( \begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \right) - \varphi \left( \begin{array}{c} \circ \end{array} \right),
\]

\[
\varphi \left( \begin{array}{c} \circ \end{array} \right) = \varphi \left( \begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \right) - \varphi \left( \begin{array}{c} \circ \end{array} \right).
\]

**Definition 1.** We say that \( \varphi \) is a finite-type invariant of order less than or equal to \( n \) if \( \varphi \) vanishes on every singular curve with at least \( n + 1 \) singular points (self-tangency points or triple points), where \( \varphi \) is extended by (2), (3) and (4).

Examples of finite-type invariants will be given later in this paper (Sect. 5).

4. Signed words. In this section, we introduce a signed word. To define signed words, we review the definition of nanowords following [7–9].

An alphabet is a set and letters are its elements. A word of length \( n \geq 1 \) in an alphabet \( \mathcal{A} \) is a mapping \( w : \mathbb{Z} \to \mathcal{A} \) where \( \mathbb{Z} = \{ i \in \mathbb{Z} | 1 \leq i \leq n \} \). Such a word is encoded by the sequence \( w(1)w(2)\cdots w(n) \).

A word \( w : \mathbb{Z} \to \mathcal{A} \) is a Gauss word if the inverse image of each element of \( \mathcal{A} \) consists of precisely two elements of \( \mathbb{Z} \).

For a set \( \alpha \), an \( \alpha \)-alphabet is a set \( \mathcal{A} \) endowed with a mapping \( | : \mathcal{A} \ni A \mapsto |A| \in \alpha \). A nanoword \( (\mathcal{A}, w) \) over \( \alpha \) is a pair (an \( \alpha \)-alphabet \( \mathcal{A} \), a Gauss word in the alphabet \( \mathcal{A} \)). By definition, there is a unique empty nanoword \( \emptyset \) of length 0.

An isomorphism of \( \alpha \)-alphabets \( \mathcal{A}_1, \mathcal{A}_2 \) is a bijection \( f : \mathcal{A}_1 \to \mathcal{A}_2 \) such that \(|A| = |f(A)|\) for all \( A \in \mathcal{A}_1 \). Two nanowords \( (\mathcal{A}_1, w_1) \) and \( (\mathcal{A}_2, w_2) \) over \( \alpha \) are isomorphic if there is an isomorphism of \( \alpha \)-alphabets \( f : \mathcal{A}_1 \to \mathcal{A}_2 \) such that \( w_2 = f w_1 \). For an arbitrary nanoword \( (\mathcal{A}, w) \) over \( \alpha \), a subnanoword of \( (\mathcal{A}, w) \) is obtained by deleting a certain set of letters from both the \( \alpha \)-alphabet \( \mathcal{A} \) and Gauss word \( w \).

**Definition 2.** Let \( \alpha_0 = \{-1, 1\} \). A signed word of length \( 2n \) is a nanoword \( (\mathcal{A}, w) \) over \( \alpha_0 \) where the length of \( w \) is \( 2n \).

We consider a simple presentation for signed words. For every letter \( A \in \mathcal{A} \) of a signed word \( (\mathcal{A}, w) \), replace \( A \) with \( \overline{A} \) if \(|A| = -1 \) and leave \( A \)
unchanged if $|A| = 1$. This allows us to encode a signed word by a sequence of letters $A$ or $\overline{A}$ for $A \in \mathcal{A}$. The isomorphism of two signed words $w$ and $w'$ is written as $w \simeq w'$. For example, $\overline{A}BB\overline{A} \simeq \overline{B}C\overline{C}B$. However, $\overline{A}BB\overline{A}$ is not isomorphic to $\overline{A}\overline{B}B\overline{A}$.

For an arbitrary signed word $w$, we define a subword of $w$ as a sub-nanoword of $(A, w)$ over $\alpha_0$. For an arbitrary signed word $w$, $u \prec w$ means that $u$ is a subword of $w$. For example, all the subwords of the signed word $\overline{A}B\overline{A}C\overline{B}\overline{C}B\overline{C}$ are $\emptyset, A, A\overline{A}, \overline{A}A, \overline{A}B, A\overline{A}B, AA\overline{A}C, \overline{A}A\overline{A}C, \overline{A}B\overline{A}C, AC\overline{B}\overline{C}, \overline{A}B\overline{A}C, \overline{A}B\overline{A}C\overline{B}$.

5. Construction of the invariant. For two arbitrary signed words $u$ and $w$, define $\langle \cdot, \cdot \rangle$ by

$$\langle u, w \rangle = \sum_{t \in \mathcal{A}} (u, v),$$

where $(u, v)$ is 1 if $u \simeq v$ and is 0 otherwise. Let $k$ be a field, $W$ the $k$-linear space generated by all the isomorphism classes of the signed words, and $W_n$ the $k$-linear space generated by the isomorphism classes of signed words of length $2n$. We denote by $U$ the dual space of a given $k$-linear space $U$. We extend $\langle \cdot, \cdot \rangle$ linearly to $\langle \cdot, \cdot \rangle : W \times W \rightarrow k$.

We associate with an arbitrary pointed curve $\Gamma$ a nanoword $w(\Gamma)$ over $\alpha_0$ following [8]. Let us label the double points of $\Gamma$ by distinct letters $A_1, A_2, \ldots, A_m$ where $m$ is the number of double points. Starting at the base point of $\Gamma$ and following along $\Gamma$ in the positive direction, we write down the labels of all double points until we return to the base point. Since every double point is traversed twice, this gives a Gauss word, $w(\Gamma)$, on the alphabet $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. Let $t^1_i$ (resp., $t^2_i$) be the tangent vector to $\Gamma$ at the double point labeled $A_i$ appearing at the first (resp., second) time we pass through this double point. Set $|A_i| = -1$ if the pair $(t^1_i, t^2_i)$ is positively oriented and $|A_i| = 1$ otherwise. This makes $\mathcal{A}$ into an $\alpha_0$-alphabet and makes $w(\Gamma)$ into a nanoword over $\alpha_0$. This nanoword is well defined up to isomorphism. This nanoword yields a signed word. For an arbitrary generic curve $\Gamma$ with $m$ double points, we denote by $w_\Gamma$ a signed word: $\overline{2m} \rightarrow A$ that is determined by selecting an arbitrary base point as above.

Definition 3. The linear mapping $\nu$ on $W_n$ is defined by $\nu(AxAy) = x\overline{A}y\overline{A}$ and $\nu(A\overline{A}x\overline{A}y) = xA\overline{A}A$ where $x$ and $y$ are words and $A$ is a letter. For two arbitrary signed words $w$ and $w'$, the cyclic equivalence $\sim$ is defined as $w \sim w'$ if and only if there exists $l \in \mathbb{N}$ such that $\nu^l(w) = w'$. Let the subspace $W_n^w$ of $W_n$ be the linear space generated by $\{w \in W_n | \nu(w) = w\}$. For a signed word $v$, $[v]$ denotes the sum of all the signed words which are cyclic equivalent to $v$. The formula $v \mapsto [v]$ extends to an endomorphism of $W_n$ also denoted by the square brackets.

For example, for $v = A\overline{A}B\overline{B}$, the cyclic equivalence class containing $v$ is $\{A\overline{A}B\overline{B}, \overline{A}A\overline{A}B, B\overline{A}A, BA\overline{B}\overline{A}, B\overline{A}BA, BA\overline{B}\overline{A}A\}$, and $[v]$ is $A\overline{A}B\overline{B} + \overline{A}A\overline{A}B + B\overline{A}A + BA\overline{B}\overline{A}A + BA\overline{B}\overline{A}A + B\overline{A}BA$ because $A\overline{A}B\overline{B} \simeq B\overline{A}BA$.

We denote the linear space generated by all stable homeomorphism classes of curves on oriented closed surfaces by $C$.

Remark 1. We denote the linear space generated by all stable homeomorphism classes of curves with $n$ double points on oriented closed surfaces by $C_n$. There exists a bijective mapping from $C_n$ to $W_n^w$; this has been proved by V. Turaev [8]. Specifically, every signed word determines a regular neighborhood of a curve $\Gamma$ on a surface $S$, where $\Gamma$ gives the CW-decomposition of $S$ (Fig. 2) [6]. In the rest of this paper, we identify $C_n$ with $W_n^w$ and $C$ with $W^w$ by $\Gamma \mapsto [w_\Gamma]$, where $W^w$ is the linear space generated by $\{w \in W \mid \nu(w) = w\}$.

For an arbitrary natural number $n$, we define a signed curve invariant of order $n$, $SCI_n : W^w \rightarrow W_n^w$, as follows: For a generic curve $\Gamma \in W^w$, we define $SCI_n(\Gamma) : W_n^w \rightarrow k$ by

$$SCI_n(\Gamma)(v) = \langle [v], w_\Gamma \rangle \quad (v \in W_n),$$

where $w_\Gamma$ is a signed word determined by $\Gamma$. We can verify, from the definitions of $[\cdot]$ and $\langle \cdot, \cdot \rangle$, that $SCI_n(\Gamma)$ is defined independently of the choice of base point of $\Gamma$.

Main theorem. For any generic curve $\Gamma$ on an oriented surface, $SCI_n(\Gamma)$ is a finite-type invariant of generic curves of order less than or equal to $n$.

The proof of the main theorem is given in Sect. 7.
Let $m_w$ be the number of representative elements of the cyclic equivalence class that contains a signed word $w$. Let $\overline{w} = \frac{1}{m_w}[w] \in W^*$. Then, $SCI_n(\Gamma) = \langle [\cdot], \overline{\Gamma} \rangle$. For two arbitrary signed words $v$ and $u$, $v'$ denotes a linear mapping such that $v'(u) = (v, u)$. We extend $*$ linearly to a map $*: W \to W$ by $(au + bv)' = au' + bv'$ for $a, b \in k$. $SCI_n$ restricted to $W^*$ gives an isomorphism between $W_n^*$ and $(W_n^*)^*$, sending $\overline{\Gamma}$ to $[w_1]^*$. We denote this isomorphism by $i_n$. Let us consider some examples of $SCI_n$. Note that $SCI_n(\Gamma)(v) = \sum_u[u]^*(v)$ where $u < w$ and $u \in W_n$. If $\Gamma$ is a given curve and $w_1 = ABCACB$, then $SCI_1(\Gamma) = [A\overline{A}]^* + [B\overline{B}]^* + [CC]^*$. $SCI_2(\Gamma) = [A\overline{A}B\overline{B}]^* + [ACAC]^* + [BCC\overline{B}]^*$. $SCI_3(\Gamma) = [A\overline{B}CA\overline{B}]^*$. $SCI_n(\Gamma) = 0$ (n $\geq$ 4). $SCI_1(\Gamma)(XX) = [A\overline{A}]^*(XX) + [B\overline{B}]^*(XX) + [CC]^*(XX) = 3$.

**Theorem 5.1.** For arbitrary $k$, $l$ such that $1 \leq k \leq l \leq n$, the following holds:

$$ (7) \quad \left( \begin{array}{c} n-l \\ k-l \end{array} \right) SCI_1 = SCI_1|_{W^*} \circ i_{k-l} \circ SCI_k. $$

*Proof.* The equality $(n - k + 1) \cdot (\binom{n-1}{k-1}) = \binom{n}{k}$ implies $SCI_{n-k} = SCI_{n-k}|_{W^*} \circ i_{k-1} \circ SCI_k$ for every $k$ such that $2 \leq k \leq n$. By using (8), we see that $(n-l)!SCI_1 = (n-k)!SCI_1|_{W^*} \circ i_{k-1} \circ SCI_{n-k}|_{W^*} \circ i_{k-1} \circ SCI_k = (n-k)!SI_1|_{W^*} \circ i_{k-1} \circ SCI_k$. \hfill \Box

6. Arnold's invariants and $SCI_n$. In Sect. 6, $\Gamma$ is used to denote an arbitrary generic plane curve. Let $w_1$ be a signed word determined by selecting an arbitrary base point, and by using the definition in Sect. 5. Arnold's basic invariants $(J^+, J^-, St)$ and $SCI_n (n \in \mathbb{N})$ are invariants for plane isotopy classes of curves on a plane. The definitions of $J^+$, $J^-$ and $St$ are provided in [1, 2].

Let $i(\Gamma)$ be the rotation number of a generic plane curve $\Gamma$. Then, $i(\Gamma)$ is a finite-type invariant of order 0 because the left-hand sides of (2), (3) and (4) should be equal to 0. We obtain the following equations:

$$ (9) \quad J^+(\Gamma) - J^-(\Gamma) = SCI_1(\Gamma)(AA), $$

$$ (10) \quad J^-(\Gamma) + 6St(\Gamma) = -2SCI_2(\Gamma)(AABB) $$

$$ - A\overline{A}B\overline{B} + A\overline{B}A\overline{B} + i^2(\Gamma) - 1.$$
Fig. 3. An example of \( w_{\Gamma_0} \), obtained by a negative resolution at a direct self-tangency point \( P_i \).

\[
\begin{align*}
\Gamma_{\sigma_0} = xAByABz \\
w_{\Gamma_0} &= xAByABz \\
w_{\Gamma_0} &= xyz
\end{align*}
\]

Fig. 4. An example of \( w_{\Gamma_1} \), obtained by a negative resolution at an inverse self-tangency point \( P_i \).

\[
\begin{align*}
\Gamma_{\sigma_0} = xAByBCzCT \\
w_{\Gamma_0} &= xAByBCzCT \\
w_{\Gamma_0} &= xBCACzBCT
\end{align*}
\]

Fig. 5. An example of \( w_{\Gamma_2} \), obtained by a negative resolution at a triple point \( P_i \).

\[
\Gamma_{\sigma_0} = xAByACzBCT \\
w_{\Gamma_0} &= xAByACzBCT \\
w_{\Gamma_1} &= xBCACzBCT
\]

Let \( u' \) be a subword of \( w_{\Gamma_0} \) and let \( \sigma(u') \) be the subword of \( w_{\Gamma_0} \) corresponding to \( u' \). Note that \( \sigma(u') \) may be \( \emptyset \) for a negative resolution at a self-tangency point. The subword \( v' \) of \( w_{\Gamma_1} \) is represented as \( \sigma(u') \).

By using (5) and the notion above,

\[
\sum_{\sigma \in \{-1,1\}^{m}} \text{sign} \sigma(u, w_{\Gamma_0}) = \sum_{\sigma \in \{-1,1\}^{m}} \text{sign} \sigma \sum_{u' \prec w_{\Gamma_0}} (u, u') = \sum_{\sigma \in \{-1,1\}^{m}} \sum_{u' \prec w_{\Gamma_0}} (u, \sigma(u'))
\]

(13)

Let \( J' \) be \( \{ i \in J \mid \text{a letter of } \sigma(u') \text{ is generated by a resolution at } P_i \} \). The sum over \( \sigma \) can be divided into one part with \( \sigma_i \) \( (i \in J' \setminus J) \) and another part with \( \sigma_i \) \( (i \in J \setminus J') \). Hence, we have

\[
\sum_{\sigma \in \{-1,1\}^{m}} \text{sign} \sigma(u, \sigma(u')) = \sum_{\sigma_i \in \{-1,1\}} \prod_{i \in J' \setminus J} \sigma_i \sum_{\sigma_i \in \{-1,1\}} \Pi_{i \in J} \sigma_i (u, \sigma(u')).
\]

(14)

For an arbitrary \( u \), where \( \sigma(u') \in W_n \), there exists at least one singular point \( P_i \) that is not related to \((u, \sigma(u'))\) because \( u \) and \( u' \in W_n \) and \( m \geq n + 1 \).

In other words, \( J \setminus J' \) is not empty. Then,

\[
\sum_{\sigma_i \in \{-1,1\}} \prod_{i \in J \setminus J'} \sigma_i (u, \sigma(u')) = (u, \sigma(u')) \prod_{\sigma_i \in \{-1,1\}} \prod_{i \in J \setminus J'} \sigma_i = 0.
\]

(15)

Here, (15) implies that \( I(\Gamma)(v) \) in (12) vanishes, and \( I(\Gamma) = 0 \).

8. Summary. In this note, the author defines a notion of finite-type for invariants of curves on surfaces by using Arnold’s idea related to the singularity theory among various definitions. The author is unable to determine the structure of the linear space \( V_n \) generated by finite-type invariants of order less than or equal to \( n \).

Because of (6) and (8) we have the following relations between the subspaces \( (W_n^\nu)^* \) of \( V_n \).

\[
\begin{align*}
&(C_n) \\
&\downarrow SCI_{n-1} \circ \iota_n^{-1} \\
&\downarrow SCI_{n-1} \circ \iota_{n-1}^{-1} \\
&\downarrow SCI_{k} \circ \iota_{k+1}^{-1} \\
&\downarrow SCI_{k-1} \circ \iota_k^{-1} \\
&\downarrow SCI_{1} \circ \iota_2^{-1} \\
&\downarrow SCI_{1} \circ \iota_1^{-1}
\end{align*}
\]

(16)

This diagram shows how \( SCI_k \) \((1 \leq k \leq n)\) reduce information from \( C_n \simeq (W_n^\nu)^* \) to \( (W_1^\nu)^* \).

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