KOSTKA NUMBERS AND FOURIER DUALITY

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To Yuri Ivanovich Manin on his 85th birthday with admiration

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ABSTRACT. We relate the Fourier transform of perverse sheaves smooth along the coordinate hyperplane configuration in a complex vector space to the Deligne-Lusztig duality of unipotent representations of a general linear group over a finite field. A similar relation is established for arbitrary finite Coxeter groups.

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1. Introduction

1.1. As soon as perverse sheaves were discovered about 40 years ago, there appeared a problem of concrete linear algebraic description of abelian categories of perverse sheaves smooth along some particular stratifications ("How to glue perverse sheaves"). One important case is a complex affine space stratified by a hyperplane arrangement. In case such an arrangement is real, a solution of this problem was suggested in [KS] (hyperbolic sheaves).

In the present note we study a most trivial example of a hyperplane arrangement: namely the one of coordinate hyperplanes. In this case another description of the category of perverse sheaves was proposed in [GGM] (iterated vanishing/nearby cycles). It turns out that already in this simplest example, the interplay between the two different descriptions of the same abelian category leads to some nontrivial combinatorial consequences.

Namely, we go even further along the route of simplification and consider semisimple perverse sheaves on \( \mathbb{C}^n \) with trivial monodromy, however with coefficients not in the category of vector spaces, but in the category \( \mathcal{C} \) of representations of the symmetric group \( S_{n+1} \) (over a field \( k \) of characteristic 0). We consider a perverse sheaf \( \mathcal{F}_\mu \) whose generic stalk is the irreducible \( S_{n+1} \)-module
V_\mu (for a partition \mu of n + 1), and whose hyperbolic stalks are equal to induction of V_\mu to S_{n+1} from appropriate parabolic subgroups. When \mu = (n + 1), so that V_{(n+1)} is trivial, we call the multiplicities of simple perverse sheaves in F_{(n+1)} small Kostka numbers \kappa_{\lambda,I}. According to Theorem 3.5.1, \kappa_{\lambda,I} is equal to the number of standard Young tableaux of shape \lambda with descent set I. We also discuss an extension of (small) Kostka numbers to an arbitrary finite Coxeter group W related to [So2].

The hyperbolic calculus developed in [FKS] allows to compute the Fourier-Sato transform of F_\mu. Namely, we have \text{FT} F_\mu \cong F_\mu^t (transposed partition). Also, the hyperbolic calculus provides a master complex M_\mu computing the vanishing cycles \Phi_0 F_\mu of F_\mu at the origin. It turns out that \Phi_0 F_\mu \cong V_\mu^t, and the master complex goes back at least to [So1].

For a prime power q, there is a Benson-Curtis isomorphism k[S_{n+1}] \rightarrow \mathcal{H}_{n+1} with the Iwahori-Hecke algebra. Thus \mathcal{C} is equivalent to the category \mathcal{C}_q of \mathcal{H}_{n+1} modules. Under this equivalence, the master complex M_\mu corresponds to a complex q\text{Mas}_\mu* going back at least to [Kat]. According to loc.cit., a similar complex exists for the Hecke algebra of an arbitrary finite Coxeter group.

Furthermore, \mathcal{C}_q is equivalent to the category of unipotent representations of GL(n + 1, \mathbb{F}_q), and under this equivalence, q\text{Mas}_\mu* corresponds to the complex of [DL] defining the Alvis-Curtis-Deligne-Lusztig-Kawanaka duality [A, C, Kaw, DL].

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2. Hyperbolic sheaves over coordinate arrangements

If (X, \leq) is a poset, we call a bisheaf on X with values in a category \mathcal{C} a couple B = (\gamma, \delta) of functors

\gamma: X \rightarrow \mathcal{C}, \delta: X^{op} \rightarrow \mathcal{C}

such that for all x \in X, \gamma(x) = \delta(x) =: B(x). Thus, for all x \leq y we have two arrows

\gamma_{x,y}: B(x) \xrightarrow{\sim} B(y) : \delta_{y,x}.

2.1. Real coordinate arrangement. Let V = \mathbb{R}^n with coordinates x_1, \ldots, x_n;

\mathcal{H} = \{H_1, \ldots, H_n\},

where H_i \subset V is given by the equation x_i = 0.

S is the corresponding stratification of V.
The strata of $S$ are in bijection with sequences
\[ s = (s_1, \ldots, s_n) \in \{-1, 0, 1\}^n, \]
where $C_s$ is given by $n$ equalities and inequalities: $x_i = 0$ if $s_i = 0$, and $s_i x_i > 0$ if $s_i = \pm 1$. So all the strata are cells, and in what follows we will refer to the strata as cells.

To put it differently, the cells are numbered by couples $(I, \epsilon)$ where $I$ is a subset of $[n] := \{1, \ldots, n\}$, and $\epsilon = (\epsilon_j) \in \{1, -1\}^{[n] \setminus I}$ is a collection of signs on the complement. We denote the corresponding cell by $C_{(I, \epsilon)}$.

We have $C_{(I, \epsilon)} \leq C_{(I', \epsilon')}$ (i.e. $C_{(I, \epsilon)}$ lies in the closure of $C_{(I', \epsilon')}$) if and only if

(a) $I \supset I'$, whence $I := [n] \setminus I \subset I'$,

and

(b) $\epsilon_j = \epsilon'_j$ for all $j \in I$.

In other words, $(I, \epsilon)$ is obtained from $(I', \epsilon')$ by replacing some elements $\epsilon'_j$ in the sequence $\epsilon'$ by 0's.

Consider two cells $C_s$ and $C' = C_{s'}$ where the sequence $s'$ is obtained from $s$ by replacing some element $s_i = \pm 1$ by $s'_i = -s_i$. We call such two cells neighbours. Let $C'' = C_{s''}$ where $s''$ is obtained from $s$ by replacing the element $s_i$ by 0. This $C''$ is called the wall between $C$ and $C''$.

Following Tits, let us call a gallery a sequence $C_0, \ldots, C_r$ where all $C_i, C_i+1$ are neighbours. We call $C_0$ and $C_r$ faraway neighbours.

Each cell $C$ of dimension $p$ has $2^p$ faraway neighbours (including itself), whose union is dense in $L(C)$: the flat of $C$, the $\mathbb{R}$-vector space spanned by $C$.

We denote by $C_I$ the positive cell $C_{(I, \epsilon)}$ with all $\epsilon_j = 1$. So, there are $2^n$ positive cells.

2.2. Complex coordinate arrangement. Let
\[ \mathcal{H}_C = \{H_1^C, \ldots, H_n^C\} \]
be the complex coordinate arrangement in $V_C = \mathbb{C}^n$, and let $S_C$ be the corresponding stratification of $V_C$.

Its strata are $S_I, I \subset [n] := \{1, \ldots, n\}$, where
\[ S_I = \{x \in V_C | x_i = 0 \text{ for } i \in I, x_i \neq 0 \text{ for } i \notin I\}. \]

We write $S \leq S'$ iff $S \subset \overline{S'}$.

In the complex case $S_I \leq S_J$ iff $J \subset I$.

A complex stratum $S_I$ with $|I| = p$ is of real codimension $2p$, and contains $2^{n-p}$ real cells $C_{(I, \epsilon)}$. 

2.3. **GGM sheaves.** We have two linear algebra descriptions of the category $\mathcal{M}(\mathcal{C}, \mathcal{S}_C) = \text{Perv}(\mathcal{C}, \mathcal{S}_C)$.

The first one is due to [GGM]. Namely, $\mathcal{M}(\mathcal{C}, \mathcal{S}_C)$ is equivalent to a category $\mathcal{GGM}_n$ whose objects are certain bisheaves $\Phi$ on the poset $\mathcal{S}_C$ with values in $\text{Vect}$, i.e.

- collections of $2^n$ spaces
  \[ \Phi(I) = \Phi(S_I) \in \text{Vect}, \quad I \subset [n], \]
- and **canonical** maps
  \[ u_{IJ}: \Phi(I) \to \Phi(J), \quad I \subset J, \]
and **variation** maps
  \[ v_{JI}: \Phi(J) \to \Phi(I), \quad I \subset J, \]
satisfying the transitivity for triples $I \subset J \subset K$.

They must satisfy the following further conditions.

For all $I, i \in I$ denote

\[ v_i = v_{I, I \setminus \{i\}}: \Phi(I) \xrightarrow{v} \Phi(I \setminus \{i\}) : u_{I \setminus \{i\}, I} = u_i. \]

Then the **mixed commutativity** condition: $uv = vu$ must hold, that is, for any $i, j \in I, i \neq j$, the square

\[ \begin{array}{ccc}
\Phi(I \setminus \{i\}) & \xrightarrow{v_j} & \Phi(I \setminus \{i, j\}) \\
u_i & u_i & \\
\Phi(I) & \xrightarrow{v_j} & \Phi(I \setminus \{j\})
\end{array} \]

must commute.

Also, the **invertibility** condition of $1 + vu$ must hold, that is for all $i \in I \subset [n]$, the **monodromy** operator

\[ T_i := 1 + v_i u_i: \Phi_I \to \Phi_I \]

must be invertible.

Note that $1 + v_i u_i$ is invertible if and only if $1 + u_i v_i$ is invertible.

This definition makes sense if we replace the coefficient category $\text{Vect}$ by an arbitrary **additive** category $\mathcal{C}$; let us denote the corresponding GGM category $GGM_n(\mathcal{C})$.

2.4. **Fourier-Sato transform.** The *Fourier-Sato transform* $\text{Perv}(\mathcal{C}, \mathcal{S}_C) \xrightarrow{\sim} \text{Perv}(\mathcal{C}, \mathcal{S}_C)$ induces the same named auto-equivalence $GGM_n \xrightarrow{\sim} GGM_n$. More generally, for an arbitrary additive category $\mathcal{C}$, we have an auto-equivalence

\[ \text{FT}: \text{GGM}_n(\mathcal{C}) \xrightarrow{\sim} \text{GGM}_n(\mathcal{C}) \]
that sends an object $\Phi = (\Phi(I), u, v)$ to $FT(\Phi) = (\Phi'(I) := \Phi(I), u', v')$. Here we set for $i \notin I \subset [n]$,
$$
u'_i := -\nu_i : \Phi(\bar{I}) \to \Phi(\bar{I} \setminus \{i\}), \quad \text{and} \quad v'_i := u_i(1 + v_i u_i)^{-1} : \Phi(\bar{I} \setminus \{i\}) \to \Phi(\bar{I}),$$
cf. [BK, Proposition 4.5] and [Mal, (II.2.1), (II.3.2)].

2.5. Hyperbolic sheaves. The second, “Möbius dual”, linear algebraic description of $M(V_C, S_C)$ is a version of [KS]. The category $M(V_C, S_C)$ is equivalent to a category $\text{Hyp}_n$ of hyperbolic sheaves. An object of $\text{Hyp}_n$ is a bisheaf $E$ with values in Vect on the poset of real strata $S$, i.e.

- a collection of $3^n$ spaces $E(C) = E(I, \epsilon) \in \text{Vect}, C \in S$,
- maps $\gamma_{CC'} : E(C) \to E(C')$, $\delta_{C'C} : E(C) \to E(C')$, $C \leq C'$.

These maps must satisfy the following conditions:

*Idempotence:* For $C \leq C'$,
$$
\gamma_{CC'} \delta_{C'C} = \text{Id}_{E(C')},
$$

Let $C, C'$ be any pair of cells. There exists a cell $C'' \leq C$ and $C'' \leq C'$, for example $C'' = \{0\}$. Define a map
$$
\phi_{CC'} = \gamma_{C''C} \delta_{CC''}.
$$
Due to the idempotence axiom this map does not depend on a choice of $C''$.

*Invertibility.* If $\dim C = \dim C' = p$, and $C, C'$ belong to the same linear subspace $L \subset V$ of dimension $p$ then $\phi_{CC'}$ is invertible.

The invertibility axiom is equivalent to a seemingly weaker requirement:

*Weak invertibility.* The map $\phi_{CC'} = \gamma_{C''C} \delta_{CC''}$ is an isomorphism for all neighbours.

*Mixed commutativity:* $\gamma \delta = \delta \gamma$. That is, for $C \leq D$, $C' \leq D'$, $C \leq C'$, $D \leq D'$, the square
$$
\begin{array}{ccc}
E(C') & \xrightarrow{\delta} & E(D') \\
\gamma \downarrow & & \gamma \downarrow \\
E(C) & \xrightarrow{\delta} & E(D)
\end{array}
$$
must commute.

This definition makes sense if we replace the coefficient category Vect by an arbitrary category $\mathcal{C}$; let us denote the corresponding category $\text{Hyp}_n(\mathcal{C})$.

2.6. Morita equivalence. We have equivalences of categories $L_h : M(V_C, S_C) \xrightarrow{\sim} \text{Hyp}_n$ (“hyperbolic, or big localization”) and $L_g : M(V_C, S_C) \xrightarrow{\sim} \text{GGM}_n$ (“vanishing cycles, or small localization”), so there should exist equivalences of quiver
Below we will construct equivalences

\[(2.6.1) \quad P: \text{Hyp}_n \xrightarrow{\sim} \text{GGM}_n, \quad Q: \text{GGM}_n \xrightarrow{\sim} \text{Hyp}_n.\]

for an arbitrary additive category \(\mathcal{C}\).

2.7. **Example** \(n = 1\). The construction below is a version of [KS, 9.A].

An object of Hyp\(_1\)(\(\mathcal{C}\)) is a triple \(E = (E_0, E_{\pm})\) of objects of \(\mathcal{C}\), and 4 maps

\[\gamma_{\pm}: E_0 \to E_{\pm}, \quad \delta_{\pm}: E_{\pm} \to E_0,\]

such that \(\gamma_{\pm}\delta_{\mp}\) are isomorphisms.

An object of GGM\(_1\)(\(\mathcal{C}\)) is a couple \(\Phi, \Psi \in (\mathcal{C})\) and 2 maps

\[v: \Phi \leftrightarrow \Psi : u\]

such that \(t := 1 + vu\) is an isomorphism.

Let \(E \in \text{Hyp}_1(\mathcal{C})\); we set \(\Psi := E_+, \quad \Phi := \text{Ker}\gamma_-\), and \(v\) is the composition

\[\Phi \hookrightarrow E_0 \xrightarrow{\gamma_+} \Psi,\]

while \(u\) is the composition

\[\Psi \xrightarrow{\delta_+} E_0 \xrightarrow{1 + \delta_- \gamma_-} \Phi.\]

This way we get an object

\[P(E) = (\Phi, \Psi, v, u) \in \text{GGM}_1(\mathcal{C}).\]

In the opposite direction, given \(G = (\Phi, \Psi, v, u) \in \text{GGM}_1(\mathcal{C})\) with \(t := 1 + vu\), we define

\[Q(G) = (E_+, E_\pm, \gamma_\pm, \delta_\pm) \in \text{Hyp}_1(\mathcal{C})\]

by

\[E_+ = E_- = \Psi; \quad E_0 = \Phi \oplus \Psi;\]

where \(\delta_-\) (resp. \(\gamma_-\)) is given by the obvious inclusion (resp., projection), whereas

\[\gamma_+(x, y) = -v(x) + y,\]

and

\[\delta_+(y) = (u(y), t(y)).\]
2.8. General $n$. We note that

$$GGM_n(\mathcal{C}) \cong GGM_1(GGM_{n-1}(\mathcal{C})), \quad \text{and} \quad Hyp_n(\mathcal{C}) \cong Hyp_1(Hyp_{n-1}(\mathcal{C})).$$

Whence we define $P_n, Q_n$ by induction: $P_n$ being given as the composition

$$\text{(2.8.1) } Hyp_n(\mathcal{C}) \cong Hyp_1(Hyp_{n-1}(\mathcal{C})) \xrightarrow{P_1} GGM_1(Hyp_{n-1}(\mathcal{C}))$$
$$P_n \rightarrow GGM_1(GGM_{n-1}(\mathcal{C})) \cong GGM_n(\mathcal{C}),$$

and similarly for $Q_n$.

2.9. Vanishing cycles at the origin: the Master complex. We will give an explicit description of the equivalence $P : Hyp_n \rightarrow GGM_n$ via vanishing cycles.

Given $E = (E(C), \gamma, \delta) \in Hyp_n$, we are going to describe the corresponding GGM-sheaf $P(E) = (\Phi(I), u, v)$.

First let us describe its vanishing cycles at the origin, $\Phi_0 := \Phi([n])$. Geometrically, let $M \in M(V_C, S_C)$ be the perverse sheaf corresponding to $E$; the sheaf of vanishing cycles $\Phi_f(M)$ corresponding to a function $f(x) = \sum_{i=1}^n x_i$ is a skyscraper concentrated at 0, and $\Phi_0$ is its stalk at 0:

$$\Phi_0 = \Phi_f(M)_0.$$

Algebraically the procedure of computing $\Phi_f(M)$ is described in [FKS], for an arbitrary real hyperplane arrangement. For the coordinate arrangement it reduces to the following.

We consider the positive cells $C_I$, and denote $E(I) := E(C_I)$. Let $S^+ \subset S$ denote the subset of positive cells. Let

$$\text{Sub}_n^i = \{ I \subset [n] : |I| = n - i \} \cong S_i^+ := \{ C \in S^+ : \dim C = i \}.$$

We consider a complex

$$\text{(2.9.1) } \Phi^*(E) : 0 \rightarrow E([n]) \rightarrow \bigoplus_{I \in S^+_i} E(I) \rightarrow \ldots \rightarrow E(\emptyset) \rightarrow 0,$$

concentrated in degrees 0, $\ldots, n$.

The differentials are $\pm \gamma$. More precisely, if $J = I \sqcup j$, then the $JI$ matrix coefficient of the differential is $(-1)^s \gamma_{C(J)C(I)}$, where $s = |\{ i \in I : i < j \}|$.

**Proposition 2.9.1.** The complex $\Phi^*(E)$ is acyclic in positive degrees, and

$$\Phi_0 = H^0(\Phi^*(E)).$$

This is a particular case of [FKS, Theorem 2.3]. This complex $\Phi^*(E)$ will be called the Master complex of vanishing cycles for a hyperbolic sheaf $E \in Hyp_n$. 
2.10. **Explicit description of $P$.** Similarly, for an arbitrary $I \subset [n]$, $|I| = p$, let $f_I(x) = \sum_{i \in I} x_i$. The sheaf of vanishing cycles 

$$\Phi_{f_I}(M)$$

is supported on the closure of the stratum $S(I)$, and $\Phi(I)$ is the stalk of $\Phi_{f_I}(M)$ at $C_I$. It is computed via an exact sequence

$$0 \to \Phi(I) \to E(I) \to \oplus_{I' \subset I, |I'| = p-1} E(I') \to \ldots \to E(\emptyset) \to 0,$$

where the differentials are $\pm$.

In particular, $\Phi(I) \subset E(I)$, $\Phi(I \setminus \{i\}) \subset E(I \setminus \{i\})$, and the variation map $v_i: \Phi(I) \to \Phi(I \setminus \{i\})$ (notation of §2.3) is induced by

$$\gamma_{C_I,C_I \setminus \{i\}}: E(I) \to E(I \setminus \{i\}).$$

Finally, the canonical map $u_i: \Phi(I \setminus \{i\}) \to \Phi(I)$ is induced by the composition

$$E(I \setminus \{i\}) \xrightarrow{\delta_{C_I \setminus \{i\}} -\delta_{C_I}} E(I) \xrightarrow{1 + \delta - \gamma} E(I),$$

where $E(I) = E(C_I) = E(C_{I,i})$ $\xrightarrow{\gamma} E(C_{I \setminus \{i\}},i')$ $\xrightarrow{\delta - \gamma} E(C_{I,i}) = E(C_I) = E_I$, and $i'$ consists of all positive signs except for one negative sign of $i$.

**Remark 2.10.1.** We will not give an explicit description of the equivalence $Q: \text{GGM}_n \cong \text{Hyp}_n$; we just mention that for $\mathcal{G} = (\Phi(I), u, v)$, the **positive** hyperbolic stalks of the corresponding $E = Q(\mathcal{G}) = (E(C), \gamma, \delta) \in \text{Hyp}_n$ are given by

$$E(I) = \oplus_{J \subset I} \Phi(J)$$

(“Takeuchi formula”, cf. [KS, 4.C.(1)] and [T]).

2.11. **Amonodromic semisimple sheaves.** Fix a semisimple abelian category $\mathcal{C}$. The category of perverse sheaves $\mathcal{M}_{\mathcal{C}}(V, S_{\mathcal{C}}) = \text{Perv}_{\mathcal{C}}(V, S_{\mathcal{C}})$ with coefficients in $\mathcal{C}$ contains a full semisimple subcategory $\text{Perv}_{\mathcal{C}}^{\text{ass}}(V, S_{\mathcal{C}})$ formed by the direct sums of constant sheaves along the strata closures. Under the equivalences $\text{Perv}_{\mathcal{C}}(V, S_{\mathcal{C}}) \cong \text{Hyp}_n(\mathcal{C})$ and $\text{Perv}_{\mathcal{C}}(V, S_{\mathcal{C}}) \cong \text{GGM}_n(\mathcal{C})$ the subcategory $\text{Perv}_{\mathcal{C}}^{\text{ass}}(V, S_{\mathcal{C}}) \subset \text{Perv}_{\mathcal{C}}(V, S_{\mathcal{C}})$ corresponds to the full semisimple subcategories $\text{Hyp}_n^{\text{ass}}(\mathcal{C}) \subset \text{Hyp}_n(\mathcal{C})$ and $\text{GGM}_n^{\text{ass}}(\mathcal{C}) \subset \text{GGM}_n(\mathcal{C})$.

Namely, $\text{GGM}_n^{\text{ass}}(\mathcal{C}) \subset \text{GGM}_n(\mathcal{C})$ consists of all GM sheaves $(\Phi(I), u, v)$ as in §2.3 with all $u_{IJ} = v_{JI} = 0$.

The description of $\text{Hyp}_n^{\text{ass}}(\mathcal{C}) \subset \text{Hyp}_n(\mathcal{C})$ is a little bit more elaborate. Given two cells $C_{(I, \epsilon)} \leq C_{(I', \epsilon')}$ as in §2.1, we define the **reflected cell** $C_{(I', \epsilon')}$ as follows. Recall from §2.1 that $(I, \epsilon)$ is obtained from $(I', \epsilon')$ by replacing some elements $\epsilon'_j$ in the sequence $\epsilon'$ by $0$’s. Now to obtain $(I'', \epsilon'')$ we replace these elements by the opposite ones (i.e. change their signs). In particular, $I'' = I'$, and $\dim C_{(I', \epsilon')} = \dim C_{(I'', \epsilon'')}$. 


Finally, a hyperbolic sheaf \((E(C), \gamma, \delta)\) lies in \(\text{Hyp}^\text{ass}_n(C)\) if for any cells \(C_{(I,\epsilon)} \leq C_{(I',\epsilon')}\) and the corresponding reflected cell \(C_{(I',\epsilon')}\) we have
\[
(2.11.1) \quad \ker\gamma_{C_{(I,\epsilon)}} = \ker\gamma_{C_{(I',\epsilon')}} \quad \text{and} \quad \im\delta_{C_{(I',\epsilon')}} = \im\delta_{C_{(I,\epsilon)}}.
\]

The equivalence \(P: \text{Hyp}^\text{ass}_n(C) \to \text{GGM}^\text{ass}_n(C)\) takes \((E(C), \gamma, \delta)\) to \((\Phi(I), 0, 0)\), where \(\Phi(I) := \bigcap_{J \subseteq I} \ker\gamma_{C_{(I,\epsilon)}}\).

The inverse equivalence \(Q: \text{GGM}^\text{ass}_n(C) \to \text{Hyp}^\text{ass}_n(C)\) takes \((\Phi(I), 0, 0)\) to \((E(C), \gamma, \delta)\), where \(E(C_{(I,\epsilon)}) := \bigoplus_{J \subseteq I} \Phi(J)\), while \(\gamma\) are given by the natural embeddings of direct summands, and \(\delta\) are given by the natural projections onto direct summands.

Remark 2.11.1. The autoequivalence of §2.4
\[
\text{FT}: \text{GGM}_n(C) \to \text{GGM}_n(C)
\]
induces an involutive autoequivalence
\[
(2.11.2) \quad \text{FT}: \text{GGM}^\text{ass}_n(C) \to \text{GGM}^\text{ass}_n(C), \quad (\Phi(I), 0, 0) \mapsto (\Phi(I), 0, 0).
\]
The corresponding involutive autoequivalence
\[
(2.11.3) \quad \text{FT}: \text{Hyp}^\text{ass}_n(C) \to \text{Hyp}^\text{ass}_n(C)
\]
is a particular case of [FKS, Theorem 4.15].

3. Kostka numbers

In the next section we will study the categories of amonodromic semisimple perverse sheaves \(\text{Hyp}^\text{ass}_n(C)\), \(\text{GGM}^\text{ass}_n(C)\) for the category \(\mathcal{C} = \text{Rep}(S_{n+1})\) of representations of the symmetric group \(S_{n+1}\) on \(n+1\) letters over a field \(k\) of characteristic 0.

In this section we prepare some basic material about Young tableaux, Kostka numbers, and \(\text{Rep}(S_{n+1})\). We also discuss a generalization of Kostka numbers to a finite Coxeter group \(W\).

3.1. Partitions and compositions. A partition \(\lambda\) of \(m\) is a sequence of integers
\[
\lambda = (\lambda_1, \ldots, \lambda_k), \quad \lambda_1 \geq \ldots \geq \lambda_k > 0, \quad \sum \lambda_i = m.
\]
The Young diagram of a partition \(\lambda\) is the set \(\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq k, \ 1 \leq j \leq \lambda_i\}\). The elements \((i, j)\) of a Young diagram, called boxes, are arranged on the plane and indexed as elements of a matrix, i.e., \(i\) is the row index of box \((i, j)\) and \(j\) is its column index. Slightly abusing notation, we identify a partition \(\lambda\) with its Young diagram and denote the latter by the same letter \(\lambda\). For example, we have
\[
\lambda = (3, 3, 1) = \begin{array}{ccc}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & & \\
\end{array}
\]
Let $\mathcal{P}_m$ be the set of partitions of $m$, equivalently, the set of Young diagrams with $m$ boxes, and let $\mathcal{P} := \bigcup_{m \geq 0} \mathcal{P}_m$.

A composition $\beta$ of $m$ is a sequence positive integers $\beta = (\beta_1, \ldots, \beta_k)$ with $\sum \beta_i = m$.

Let $\mathfrak{Q}_m$ be the set of all compositions of $m$. We have an obvious inclusion

$$(3.1.1) \quad \iota: \mathcal{P}_m \hookrightarrow \mathfrak{Q}_m$$

with a left inverse $\mathfrak{Q}_m \to \mathcal{P}_m$, that associates to a composition its weakly decreasing permutation.

### 3.2. Young tableaux.
For a Young diagram $\lambda \in \mathfrak{P}$, a semi-standard Young tableau $T$ of shape $\lambda$ is a filling of boxes of $\lambda$ by positive integers such that the entries are weakly increasing along each row and strictly increasing along each column of $T$. The weight of $T$ is a nonnegative integer sequence $\beta = (\beta_1, \ldots, \beta_k)$, where $\beta_i$ is the number of $i$'s in $T$. (Here we identify weights $\beta$ obtained from each other by adding 0’s at the end: $\beta = (\beta_1, \ldots, \beta_k) = (\beta_1, \ldots, \beta_k, 0, \ldots, 0)$.)

A standard Young tableau $T$ a semi-standard Young tableau of weight $\beta = (1, \ldots, 1)$. We denote by $SSYT(\lambda, \beta)$ the set of all semi-standard Young tableaux of shape $\lambda$ and weight $\beta$. Also $SYT(\lambda)$ denotes the set of all standard Young tableaux of shape $\lambda$. For example, here is a semi-standard Young tableau $T \in SSYT((4,3,1),(2,2,3,1))$ and a standard Young tableau $T' \in SYT((4,3,1))$:

\[
T = \begin{array}{cccc} 
1 & 1 & 2 & 3 \\
2 & 3 & 3 \\
4 & & & \\
\end{array} \quad T' = \begin{array}{cccc} 
1 & 2 & 4 & 7 \\
3 & 5 & 6 \\
8 & & & \\
\end{array}
\]

Clearly, $\# SSYT(\lambda, \beta) = \# SSYT(\lambda, \beta')$, where $\beta'$ is the composition obtained from weight $\beta$ by removing all 0’s.

### 3.3. Kostka numbers.
For a partition $\lambda$ and a composition $\beta$, the Kostka number $K_{\lambda, \beta}$ is defined as the number of semi-standard Young tableaux of shape $\lambda$ and weight $\beta$:

$$K_{\lambda, \beta} := \# SSYT(\lambda, \beta).$$

The Kostka numbers $K_{\lambda, \beta}$ are known to be invariant under permutations of parts of $\beta$. Thus, without loss of generality, the Kostka numbers can be labelled by a pair of partitions $\lambda, \mu \in \mathfrak{P}$, $K_{\lambda \mu} = K_{\lambda, \iota(\mu)}$, where $\iota$ denotes the inclusion $(3.1.1)$. If $\mu = \lambda$, the only semi-standard Young tableau of the same shape and weight $\lambda$ is obtained by filling the $i$th row of $\lambda$ all all $i$'s, for $i = 1, 2, \ldots$. So $K_{\lambda \lambda} = 1.$
3.4. Small Kostka numbers. Let us fix a nonnegative integer \( n \). Let \([n] := \{1, \ldots, n\}\), and let \(\text{Sub}_n := 2^{[n]}\) denote the set of subsets \( I \subset [n] \). The set \(\text{Sub}_n\) is in bijection with the set \(\mathfrak{Q}_{n+1}\) of compositions of \( n + 1 \).

Lemma 3.4.1. The following two maps \( \varrho = \varrho_n : \text{Sub}_n \sim \mathfrak{Q}_{n+1} \) and \( \varrho^{-1} : \mathfrak{Q}_{n+1} \sim \text{Sub}_n \) are bijective and inverse to each other:

\[
\varrho : I = \{i_1 < i_2 < \ldots < i_r\} \mapsto \beta = (i_1, i_2 - i_1, \ldots, i_r - i_{r-1}, n + 1 - i_r),
\]

\[
\varrho^{-1} : \beta = (\beta_1, \ldots, \beta_k) \mapsto I = \{\beta_1, \beta_1 + \beta_2, \ldots, \beta_1 + \ldots + \beta_{k-1}\}.
\]

For example, \( \varrho([n]) = (1, 1, \ldots, 1) \) and \( \varrho(\emptyset) = (n + 1) \).

Definition 3.4.2. For \( \lambda \in \mathfrak{P}_{n+1} \) and \( I \in \text{Sub}_n \), the small Kostka number \( \kappa_{\lambda,I} \) is defined by

\[
\kappa_{\lambda,I} := \sum_{J \subset I} (-1)^{|J| - |I|} K_{\lambda,\varrho(J)}.
\]

Proposition 3.4.3. For \( \lambda \in \mathfrak{P}_{n+1} \) and \( I \in \text{Sub}_n \), we have

\[
K_{\lambda,\varrho(I)} = \sum_{J \subset I} \kappa_{\lambda,J},
\]

or, equivalently, \( K_{\lambda,\beta} = \sum_{J \subset \varrho^{-1}(\beta)} \kappa_{\lambda,J} \). This formula defines the numbers \( \kappa_{\lambda,I} \) uniquely.

Proof. The claim follows from the Möbius inversion formula. \(\square\)

Example 3.4.4. Let \( n = 2 \) and \( \lambda = (2, 1) \in \mathfrak{P}_3 \). We have \( K_{\lambda,\varrho(\emptyset)} = K_{\lambda,(3)} = 0 \), \( K_{\lambda,\varrho(\{1\})} = K_{\lambda,(1,2)} = 1 \), \( K_{\lambda,\varrho(\{2\})} = K_{\lambda,(2,1)} = 1 \), \( K_{\lambda,\varrho(\{1,2\})} = K_{\lambda,(1,1,1)} = 2 \). Thus the small Kostka numbers are \( \kappa_{\lambda,\emptyset} = 0 \), \( \kappa_{\lambda,\{1\}} = 1 - 0 = 1 \), \( \kappa_{\lambda,\{2\}} = 1 - 0 = 1 \), \( \kappa_{\lambda,\{1,2\}} = 2 - 1 - 1 + 0 = 0 \).

Remark 3.4.5. For \( n \leq 4 \), all small Kostka numbers \( \kappa_{\lambda,I} \) are either 0 or 1. For \( n \geq 5 \), the numbers \( \kappa_{\lambda,I} \) can be greater than 1. For example, for \( n = 3k - 1 \), we have \( \kappa_{\lambda,(\{k\})} = k \).

For \( \lambda \in \mathfrak{P}_{n+1} \), let \( \lambda^t \) denote the conjugate partition of \( \lambda \), i.e., the partition whose Young diagram is obtained from \( \lambda \) by transposition. For \( I \in \text{Sub}_n \), let \( I^t := [n] \setminus I \).

Theorem 3.4.6. (1) The small Kostka numbers \( \kappa_{\lambda,I} \) are non-negative integers.

(2) The small Kostka numbers have the symmetry \( \kappa_{\lambda,I} = \kappa_{\lambda^t,I^t} \).

This theorem follows from a combinatorial and a representation-theoretical interpretations of the small Kostka numbers \( \kappa_{\lambda,I} \) given below, see Theorem 3.5.1 and Corollary 3.8.3.
3.5. Descents in Young tableaux. For a standard Young tableau $T \in \text{SYT}(\lambda)$ of shape $\lambda \in \mathcal{P}_{n+1}$, we say that $i \in [n]$ is a descent of $T$ if the entry $i+1$ is located in $T$ below the entry $i$, i.e., the row containing $i+1$ is below the row containing $i$. The descent set $\operatorname{Des}(T) := \{i \in [n] \mid i \text{ is a descent of } T\} \in \text{Sub}_n$, the set of all descents of $T$.

Let $T^t$ denote the standard Young tableau obtained by transposing $T$. Clearly, the descent set of $T^t$ is the complement to the descent set of $T$: $\operatorname{Des}(T^t) = [n] \setminus \operatorname{Des}(T)$. For example, for $T = \begin{array}{cccc} 1 & 2 & 4 & 8 \\ 3 & 5 & 7 & \\ 6 & \end{array}$ and $T^t = \begin{array}{ccc} 1 & 3 & 6 \\ 2 & 5 & \\ 4 & 7 & \\ 8 & \end{array}$, we have $\operatorname{Des}(T) = \{2, 4, 5\}$ and $\operatorname{Des}(T^t) = [7] \setminus \{2, 4, 5\} = \{1, 3, 6, 7\}$.

Theorem 3.4.6 follows immediately from the following claim.

**Theorem 3.5.1.** The small Kostka number $\kappa_{\lambda, I}$ equals the number of standard Young tableaux of shape $\lambda$ with descent set $\operatorname{Des}(T) = I$.

**Proof.** Define the standardization map

$$\text{std} : \text{SSYT}(\lambda, \beta) \to \text{SYT}(\lambda)$$

as follows. For a semi-standard Young tableau $T$ of weight $\beta = (\beta_1, \ldots, \beta_k)$, the standardization $\text{std}(T)$ of $T$ is the standard Young tableau obtained from $T$ by replacing its $\beta_i$ entries $i$ by the integers in the interval $[\beta_1 + \cdots + \beta_{i-1} + 1, \beta_1 + \cdots + \beta_i]$ arranged in the increasing order from left to right, for $i = 1, \ldots, k$. For example, we have

$$T = \begin{array}{ccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 \\ 4 & \end{array} \quad \text{and} \quad T^t = \begin{array}{ccc} 1 & 2 & 4 & 7 \\ 3 & 5 & 6 \\ 8 & \end{array}$$

The standardization map gives a bijection between the set $\text{SSYT}(\lambda, \beta)$ of all semi-standard Young tableaux $T$ of shape $\lambda$ and weight $\beta$ and the subset of standard Young tableaux $T' \in \text{SYT}(\lambda)$ that have no descents in the union of intervals

$$[1, \beta_1 - 1] \cup [\beta_1 + 1, \beta_1 + \beta_2 - 1] \cup \cdots \cup [\beta_1 + \cdots + \beta_{k-1} + 1, \beta_1 + \cdots + \beta_k - 1],$$

or, equivalently, $\operatorname{Des}(T') \subset \varrho^{-1}(\beta) = \{\beta_1, \beta_1 + \beta_2, \ldots, \beta_1 + \beta_2 + \cdots + \beta_{k-1}\}$:

$$\text{std} : \text{SSYT}(\lambda, \beta) \sim \{T' \in \text{SYT}(\lambda) \mid \operatorname{Des}(T') \subset \varrho^{-1}(\beta)\}.$$
which is exactly equation (3.4.1) that defines the small Kostka numbers. Thus 
\( \kappa_{\lambda,I} = \bar{\kappa}_{\lambda,I} \), as needed. □

### 3.6. Skew Young diagrams.

For two Young diagrams \( \lambda, \mu \in \mathcal{P} \) such that \( \lambda \supseteq \mu \), i.e., \( \lambda \) contains \( \mu \) as a subset, the skew Young diagram \( \lambda/\mu \) is the set-theoretic difference of the Young diagrams \( \lambda \) and \( \mu \). For example, for \( \lambda = (3,3,1) \) and \( \mu = (2,1) \), we have

\[
\begin{align*}
\lambda/\mu &= \\
&= 
\end{align*}
\]

Usual Young diagrams can be considered as a special case of skew Young diagrams \( \lambda/\mu \) with \( \mu = \emptyset \).

A connected component of a skew Young diagram \( \lambda/\mu \) is a connected component of the graph on the set of boxes \( (i,j) \in \lambda/\mu \) with the edge set \( \{((i,j), (i',j')) \mid |i - i'| + |j - j'| = 1\} \). For example, the above skew Young diagram has two connected components.

We consider two special types of skew Young diagrams: horizontal strips and ribbons.\(^1\) A skew Young diagram \( \lambda/\mu \) is a horizontal strip if each connected component of \( \lambda/\mu \) consists of a single row of boxes. A skew Young diagram is a ribbon if it contains no \( 2 \times 2 \) square and has exactly one connected component.

For a composition \( \beta = (\beta_1, \ldots, \beta_k) \), let \( \text{hstrip}(\beta) \) be the horizontal strip whose \( i \)th row contains \( \beta_i \) boxes, for \( i = 1, \ldots, k \) (with rows touching each other at the corners). Also let \( \text{ribbon}(\beta) \) be the ribbon whose \( i \)th row contains \( \beta_i \) boxes, for \( i = 1, \ldots, k \). For example,

\[
\begin{align*}
\text{hstrip}((2,3,1,2)) &= \\
\text{ribbon}((2,3,1,2)) &= 
\end{align*}
\]

### 3.7. Young symmetrizers.

Let \( k[S_{n+1}] \) be the group algebra of the symmetric group \( S_{n+1} \). Let \( \lambda/\mu \) be a skew Young diagram with \( n+1 \) boxes. Let \( T \) be any filling of boxes of \( \lambda/\mu \) by \( 1, \ldots, n+1 \) (without repetitions), i.e., \( T \) is any bijection \( [n+1] \to \{\text{boxes of } \lambda/\mu\} \). (Here \( T \) is not necessarily a standard Young tableau.) Let \( R(T) \) (resp., \( C(T) \)) denote the subgroup of \( S_{n+1} \) consisting of all permutations preserving the rows (resp., the columns) of \( T \).

The Young symmetrizer \( y_T \in k[S_{n+1}] \) is defined as

\[
\begin{align*}
\alpha_T &= \sum_{w \in R(T)} w, \\
\beta_T &= \sum_{w \in C(T)} \epsilon(w) w, \quad \text{and} \quad y_T = \beta_T \alpha_T.
\end{align*}
\]

\(^1\)Ribbons appear in the literature under many different names. They are MacMahon’s zigzag diagrams, see [Mac]. They also called rim hooks, border strips, etc.
Here \( \varepsilon(w) \in \{1, -1\} \) denotes the sign of permutation \( w \in S_{n+1} \).

The left ideal

\[
L_T := k[S_{n+1}] \cdot y_T \subset k[S_{n+1}]
\]

is a representation of \( S_{n+1} \).

If \( T \) and \( T' \) have the same shape \( \lambda/\mu \), then \( y_T = y_{T'} \cdot u \), for some \( u \in S_{n+1} \). Thus \( L_T \cong L_{T'} \) are isomorphic \( S_{n+1} \)-modules.

The Specht module \( V_{\lambda/\mu} \) is defined (up to an isomorphism) as

\[
V_{\lambda/\mu} := L_T,
\]

for any filling \( T \) of the shape \( \lambda/\mu \). A choice of \( T \) gives an embedding of the Specht module \( V_{\lambda/\mu} \) as a left submodule \( L_T \) of the group algebra \( k[S_{n+1}] \).

For \( \mu = \emptyset \), the Specht modules \( V_{\lambda}, \lambda \in \mathfrak{P}_{n+1} \), are exactly (the isomorphism classes of) all irreducible representations of the symmetric group \( S_{n+1} \). We have the following decomposition of the group algebra \( k[S_{n+1}] \) into a direct sum of left submodules:

\[
k[S_{n+1}] = \bigoplus_{T \in \text{SYT}_{n+1}} L_T = \bigoplus_{\lambda \in \mathfrak{P}_{n+1}} \left( \bigoplus_{T \in \text{SYT}(\lambda)} L_T \right),
\]

where \( \text{SYT}_{n+1} = \bigcup_{\lambda \in \mathfrak{P}_{n+1}} \text{SYT}(\lambda) \), see [We, Theorem 1.3.6].

### 3.8. Induced modules and ribbon modules

The semitic filling of a skew Young diagram \( \lambda/\mu \) is the filling \( T \) of boxes of \( \lambda/\mu \) by 1, 2, 3, ... reading the boxes by rows right-to-left top-to-bottom. For example, here are the semitic fillings of the horizontal strip \( hstrip((2, 3, 1, 2)) \) and the ribbon \( \text{ribbon}((2, 3, 1, 2)) \):

\[
\begin{array}{cccc}
2 & 1 & 5 & 4 \\
6 & 8 & 7 & 3
\end{array}
\quad
\begin{array}{cccc}
2 & 1 & 5 & 4 \\
6 & 8 & 7 & 3
\end{array}
\]

For a composition \( \beta = (\beta_1, \ldots, \beta_k) \) of \( n+1 \), define the induced module \( M_\beta := L_T \), where \( T \) is the semitic filling of the horizontal strip \( hstrip(\beta) \). Equivalently,

\[
M_\beta := k[S_{n+1}] \cdot \left( \sum_{w \in S_{\beta_1} \times \cdots \times S_{\beta_k}} w \right) \cong \text{Ind}^{S_{n+1}}_{S_{\beta_1} \times \cdots \times S_{\beta_k}} k,
\]

where the direct product of groups \( S_{\beta_1} \times \cdots \times S_{\beta_k} \) is standardly embedded as a subgroup of \( S_{n+1} \). For instance, \( M_{(1,\ldots,1)} = k[S_{n+1}] \) is the regular representation, and \( M_{(n+1)} = k \) is the trivial representation of \( S_{n+1} \).

It is well-known, cf. [Kos] and [FH, Corollary 4.39], that the Kostka number \( K_{\lambda,\beta} \) is exactly the multiplicity of the irreducible \( S_{n+1} \)-module \( V_{\lambda} \) in the induced
module \( M_\beta \):

\[
(3.8.2) \quad K_{\lambda,\beta} = \dim \text{Hom}_{S_{n+1}}(M_\beta, V_\lambda), \quad \text{equivalently,} \quad M_\beta \cong \bigoplus_{\lambda \in \mathcal{P}_{n+1}} V_\lambda^{\oplus K_{\lambda,\beta}}.
\]

Define the ribbon module \( R_\beta \) as

\[
(3.8.3) \quad R_\beta := L_{T'},
\]

where \( T' \) is the semitic filling of \( \text{ribbon}(\beta) \).

Recall the bijection \( \varrho : I \mapsto \beta \) between subsets \( I \in \text{Sub}_n \) and compositions \( \beta \in \mathcal{Q}_{n+1} \), see Lemma 3.4.1. We can also label induced modules and ribbon modules by subsets \( I \in \text{Sub}_n \):

\[
(3.8.4) \quad M_I := M_{\varrho(I)} \quad \text{and} \quad R_I := R_{\varrho(I)}.
\]

Solomon [So2, Theorem 2] constructed a decomposition of the group algebra of a finite Coxeter group, see Theorem 3.10.1 below. For the symmetric group \( S_{n+1} \), this decomposition can be described, as follows.

**Theorem 3.8.1.** [So2, Theorem 2] The group algebra \( k[S_{n+1}] \) decomposes into a direct sum of ribbon modules:

\[
(3.8.5) \quad k[S_{n+1}] = \bigoplus_{I \in \text{Sub}_n} R_I.
\]

More generally, any induced module \( M_I \) decomposes into a direct sum of ribbon modules.

**Theorem 3.8.2.** For \( I \in \text{Sub}_n \), the induced \( S_{n+1} \)-module \( M_I \) decomposes into a directed sum of ribbon modules:

\[
(3.8.6) \quad M_I = \bigoplus_{J \subseteq I} R_J,
\]

This theorem is a special case of Corollary 3.11.3 below. Note that here we treat all modules as concrete left submodules of the group algebra \( k[S_{n+1}] \) (and not as equivalence classes of representations of \( S_{n+1} \)).

**Corollary 3.8.3.** For a partition \( \lambda \in \mathcal{P}_{n+1} \) and a subset \( I \in \text{Sub}_n \), the small Kostka number \( \kappa_{\lambda,I} \) is the multiplicity of the irreducible \( S_{n+1} \)-module \( V_\lambda \) in the ribbon module \( R_I \):

\[
\kappa_{\lambda,I} = \dim \text{Hom}_{S_{n+1}}(R_I, V_\lambda), \quad \text{equivalently,} \quad R_I \cong \bigoplus_{\lambda \in \mathcal{P}_{n+1}} V_\lambda^{\oplus \kappa_{\lambda,I}}.
\]

**Proof.** Let \( \tilde{\kappa}_{\lambda,I} \) be the multiplicity of \( V_\lambda \) in \( R_I \). Taking multiplicities of \( V_\lambda \) in all terms of the decomposition (3.8.6), we get \( K_{\lambda,\varrho(I)} = \sum_{J \subseteq I} \tilde{\kappa}_{\lambda,J} \), which is exactly equation (3.4.1) that defines the small Kostka numbers. Thus \( \kappa_{\lambda,I} = \tilde{\kappa}_{\lambda,I} \), as needed. \( \Box \)
Remark 3.8.4. For any skew Young diagram $\lambda/\mu$, the multiplicities of irreducible representations $V_\nu$ of the symmetric group in the Specht module $V_{\lambda/\mu}$ are called the Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$:

$$V_{\lambda/\mu} \cong \bigoplus_\nu V_\nu \otimes c_{\lambda\mu}^{\nu}.$$

The Littlewood-Richardson rule is a combinatorial rule for the $c_{\lambda\mu}^{\nu}$. The combinatorial interpretation of the small Kostka numbers (Theorem 3.5.1) can be shown to be equivalent to the special case of the Littlewood-Richardson rule when $\lambda/\mu$ is a ribbon.

Theorem 3.5.1, decomposition (3.7.2), and Corollary 3.8.3 imply the following claim. For $I \in \text{Sub}_n$, let $\text{SYT}(\lambda, I)$ be the set of all standard-Young tableaux $T$ of shape $\lambda$ with $n+1$ boxes with descent set $\text{Des}(T) = I$. Also let $\text{SYT}_I := \bigcup_{\lambda \in \Psi_{n+1}} \text{SYT}(\lambda, I)$. According to Theorem 3.5.1, $\kappa_{\lambda, I} = \# \text{SYT}(\lambda, I)$.

**Corollary 3.8.5.** Let $I \in \text{Sub}_n$. The ribbon module $R_I$ is isomorphic to the direct sum of irreducible submodules of $k[S_{n+1}]$:

$$R_I \cong \bigoplus_{T \in \text{SYT}_I} L_T.$$

The induced module $M_I$ is isomorphic to the direct sum of irreducible submodules of $k[S_{n+1}]$:

$$M_I \cong \bigoplus_{J \subseteq I} \left( \bigoplus_{T \in \text{SYT}_J} L_T \right).$$

3.9. **Kostka numbers for Coxeter groups.** The symmetric group $S_{n+1}$ is a Coxeter group. Some of the above constructions can be extended to an arbitrary finite Coxeter group $W$.

Let $W$ be a finite Coxeter group of rank $n$ with Coxeter generators $s_1, \ldots, s_n$. Let $k[W]$ be the group algebra of $W$ over a field $k$ of characteristic 0.

For a subset $I \subseteq [n] := \{1, \ldots, n\}$, let $W_I$ denote the parabolic subgroup of $W$ generated by $s_i$, for $i \in I$. In particular, $W_{\emptyset} = W$ and $W_{\emptyset} = \{1\}$.

For $I \subset [n]$, define the **induced module** $M_I$ as the following left $W$-submodule of the group algebra $k[W]$

$$(3.9.1) \quad M_I := k[W] \left( \sum_{w \in \overline{I}} w \right) \cong \text{Ind}^{W}_{W_I} k,$$

where, as usual, $\overline{I} := [n] \setminus I$.

If $W = S_{n+1}$, then the induced module $M_I$ is exactly the module $M_\beta \cong \text{Ind}_{S_{\beta_1} \times \cdots \times S_{\beta_k}}^{S_{n+1}} k$, where $\beta = \varrho(I)$, from §3.8.
By analogy with (3.8.2) and Definition 3.4.2, we define the $W$-Kostka numbers and the small $W$-Kostka numbers, as follows.

**Definition 3.9.1.** Let $V$ be any irreducible representation of $W$, and let $I \in \text{Sub}_n$ be any subset of $[n]$. The $W$-Kostka number $K_{V,I}$ is the multiplicity of $V$ in the induced module $M_I$:

(3.9.2) \[ K_{V,I} := \dim \text{Hom}_{k[W]}(M_I, V). \]

The small $W$-Kostka number $\kappa_{V,I}$ is the alternating sum of $K$-Kostka numbers:

(3.9.3) \[ \kappa_{V,I} := \sum_{J \subset I} (-1)^{|J|-|I|} K_{V,J}. \]

Using Möbius inversion, we can express the $K$-Kostka numbers in terms of the small $W$-Kostka numbers as

(3.9.4) \[ K_{V,I} = \sum_{J \subset I} \kappa_{V,J}. \]

Let $k_\epsilon$ be the *sign representation* of $W$, which is the 1-dimensional $k[W]$-module, where each Coxeter generator as $s_i : x \mapsto -x$. For a $W$-module $V$, let

\[ V^\epsilon := V \otimes k_\epsilon. \]

The following claim generalizes Theorem 3.4.6.

**Theorem 3.9.2.** (1) The small $W$-Kostka numbers $\kappa_{V,I}$ are non-negative integers.

(2) The small Kostka numbers have the symmetry $\kappa_{V,I} = \kappa_{V^\epsilon,I^\epsilon}$.

This theorem follows from Corollary 3.11.4 and Lemma 3.11.5 below. It would be nice to find a simple combinatorial rule like Theorem 3.5.1 for calculation of the small $W$-Kostka numbers (e.g. when $W$ is a Weyl group of type $B$ or $D$).

3.10. **Solomon’s decomposition of $k[W]$**. Define the following elements of the group algebra $k[W]$

\[ a_I := \sum_{w \in W_I} w \quad \text{and} \quad b_I := \sum_{w \in W_I} \epsilon(w) w, \]

where $\epsilon(w) = (-1)^{\ell(w)}$ is the sign of $w \in W$ and $\ell(w) := \min\{l \mid w = s_{i_1} \cdots s_{i_l}\}$ is the length of $w$.

Notice the induced module $M_I$ defined by (3.9.1) is $M_I = k[W] a_\emptyset$.

Define the ribbon $W$-module $R_I$ as

(3.10.1) \[ R_I := k[W] b_I a_\emptyset. \]

If $W = S_{n+1}$, then the ribbon $W$-module $R_I$ is exactly the ribbon module (3.8.3) from §3.8.
**Theorem 3.10.1.** [So2, Theorem 2] The group algebra $k[W]$ decomposes into a direct sum of $2^n$ ribbon modules:

$$k[W] = \bigoplus_{I \in \text{Sub}_n} R_I.$$

3.11. **Descent basis.** Let us give a $k$-linear basis of $k[W]$ which agrees with Solomon’s decomposition.

The (right) descent set $\text{Des}(w)$ of an element $w \in W$ is

$$\text{Des}(w) := \{i \in I \mid \ell(ws_i) < \ell(w)\}.$$

For $w \in W$, define the following element of the group algebra

$$(3.11.1) \quad d_w := wb_{\text{Des}(w)}a_{[n] \setminus \text{Des}(w)} \in k[W].$$

**Example 3.11.1.** For $W = S_3 = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$, we have

$$d_1 = 1 + s_1 + s_2 + s_1s_2 + s_2s_1 + s_1s_2s_1,$$

$$d_s = s_1(s_2 - s_1)(s_1 + s_2) = -1 + s_1 - s_2 + s_1s_2,$$

$$d_s = s_2(1 - s_2)(1 + s_1) = -1 - s_1 + s_2 + s_2s_1,$$

$$d_{s_1s_2} = s_1s_2(1 - s_2)(1 + s_1) = -1 - s_1 + s_1s_2 + s_1s_2s_1,$$

$$d_{s_2s_1} = s_2s_1(1 - s_1)(1 + s_2) = -1 - s_2 + s_2s_1 + s_1s_2s_1,$$

$$d_{s_1s_2s_1} = -1 + s_1 + s_2 - s_1s_2 - s_2s_1 + s_1s_2s_1.$$

**Theorem 3.11.2.** (1) The set of elements $\{d_w \mid w \in W\}$ is a $k$-linear basis of the group algebra $k[W]$.

(2) For any $I \in \text{Sub}_n$, the set of elements $\{d_w \mid w \in W \text{ such that } \text{Des}(w) = I\}$ is a $k$-linear basis of the ribbon module $R_I$. In particular,

$$\dim R_I = \#\{w \in W \mid \text{Des}(w) = I\}.$$

(3) For any $I \in \text{Sub}_n$, the set of elements $\{d_w \mid w \in W \text{ such that } \text{Des}(w) \subset I\}$ is a $k$-linear basis of the induced module $M_I$. In particular,

$$\dim M_I = \#\{w \in W \mid \text{Des}(w) \subset I\}.$$

A proof of this theorem is given in §3.12 below. It is immediate from the definitions that $d_w \in R_I$ if $\text{Des}(w) = I$, and that $d_w \in M_I$ if $\text{Des}(w) \subset I$.

**Corollary 3.11.3.** For $I \in \text{Sub}_n$, we have the decomposition of the induced module $M_I$ into a direct sum of ribbon modules:

$$M_I = \bigoplus_{J \subset I} R_J.$$

**Corollary 3.11.4.** For an irreducible $W$-representation $V$ and $I \in \text{Sub}_n$, the small $W$-Kostka number $\kappa_{V, I}$ is the multiplicity of $V$ in the ribbon module $R_I$:

$$\kappa_{V, I} = \dim \text{Hom}_{k[W]}(V, R_I).$$
Proof. Let $\bar{\kappa}_{V,I}$ be the multiplicity of $V$ in $R_I$. By Corollary 3.11.3,

$$K_{V,I} := \dim \text{Hom}_k[\mathcal{W}](V, M_I) = \sum_{J \subset I} \dim \text{Hom}_k[\mathcal{W}](V, R_J) = \sum_{J \subset I} \bar{\kappa}_{V,I},$$

which is exactly equation (3.9.4) defining the small Kostka numbers. Thus $\bar{\kappa}_{V,I} = \kappa_{V,I}$. $\square$

Recall that $k_\epsilon$ is the sign representation of $W$.

Lemma 3.11.5. We have the isomorphism of $W$-modules:

$$R_I \otimes k_\epsilon \cong R_\mathcal{T}.$$

Proof. By [So2, Lemma 12], we have the isomorphism $k[\mathcal{W}] b_J a_I \cong k[\mathcal{W}] a_I b_J$. Thus $R_I \otimes k_\epsilon = k[\mathcal{W}] a_I b_\mathcal{T} \cong k[\mathcal{W}] b_\mathcal{T} a_I = R_\mathcal{T}$. $\square$

Now Theorem 3.9.2 follows from Corollary 3.11.4 and Lemma 3.11.5.

3.12. Proof of Theorem 3.11.2. For $I \subset [n]$, define the (right) symmetrization and the antisymmetrization operators $\text{Sym}_I$ and $\text{Asym}_I$ acting on $k[\mathcal{W}]$ by

$$\text{Sym}_I : f \mapsto f a_I \quad \text{and} \quad \text{Asym}_I : f \mapsto f b_I.$$

Let $\text{Im}(\text{Sym}_I), \text{Ker}(\text{Sym}_I), \text{Im}(\text{Asym}_I), \text{Ker}(\text{Asym}_I) \subset k[\mathcal{W}]$ be the images and kernels of these operators.

Lemma 3.12.1. The ribbon module $R_I$ is isomorphic to the quotient module

$$R_I \cong \text{Im}(\text{Asym}_I)/(\text{Ker}(\text{Sym}_I) \cap \text{Im}(\text{Asym}_I)).$$

More precisely, the map $\text{Sym}_\mathcal{T}$ restricted to $\text{Im}(\text{Asym}_I)$ induces the isomorphism between $\text{Im}(\text{Asym}_I)/(\text{Ker}(\text{Sym}_I) \cap \text{Im}(\text{Asym}_I))$ and $R_I$.

Proof. We have $R_I := k[\mathcal{W}] b_I a_\mathcal{T} = \{f a_\mathcal{T} \mid f \in \text{Im}(\text{Asym}_I)\} = \{\text{Sym}_\mathcal{T}(f) \mid f \in \text{Im}(\text{Asym}_I)\}$. $\square$

For $i \in [n]$, define the subspaces $S^i := \{f \in k[\mathcal{W}] \mid f s_i = f\}$ and $A^i := \{f \in k[\mathcal{W}] \mid f s_i = -f\}$ of the group algebra $k[\mathcal{W}]$. Clearly, $S^i = \text{Im}(\text{Sym}_{\{i\}}) = \text{Ker}(\text{Asym}_{\{i\}})$ and $A^i = \text{Im}(\text{Asym}_{\{i\}}) = \text{Ker}(\text{Sym}_{\{i\}})$.

Lemma 3.12.2. The image of the symmetrization operator $\text{Sym}_I$ is the intersection of subspaces:

$$\text{Im}(\text{Sym}_I) = \bigcap_{i \in I} S^i,$$

and the kernel of $\text{Sym}_I$ is the linear span of subspaces:

$$\text{Ker}(\text{Sym}_I) = \text{Span}_{i \in I} A^i.$$
Similarly, the image and the kernel of the antisymmetrization operator \( \text{Asym}_I \) are

\[
\text{Im}(\text{Asym}_I) = \bigcap_{i \in I} A^i \quad \text{and} \quad \text{Ker}(\text{Asym}_I) = \text{Span} \, S^i.
\]

Proof. An element \( f \in k[W] \) belongs to the image of \( \text{Sym}_I \) if and only if \( f w = f \), for any \( w \in W_I \). Equivalently, we should have \( f s_i = f \), for all \( i \in I \), that is, \( f \in \bigcap_{i \in I} S^i \).

An element \( g = \sum_{w \in W} g_w w \in k[W] \) belongs to the kernel of \( \text{Sym}_I \) if and only if the sum of its coefficients \( g_w \) over any right \( W_I \)-coset \( C \) in \( W \) is zero: \( \sum_{w \in C} g_w = 0 \). Such \( g \) should be a linear combination of the elements \( w - w s_i \), for \( w \in W \) and \( i \in I \). This follows from the fact that the right \( W_I \)-cosets in \( W \) are exactly the connected components of the graph on set of vertices \( W \) with the set of edges \{ \( (w, w s_i) \mid w \in W, i \in I \) \}. For fixed \( i \in I \), the elements \( w - w s_i \), for \( w \in W \), span the subspace \( A^i \). Thus \( g \) should belong to the linear span of the subspaces \( A^i \) over all \( i \in I \).

The claim about the image and the kernel of \( \text{Asym}_I \) is proved analogously. \( \square \)

For \( w \in W \), let

\[
c_w := w b_{\text{Des}(w)} \in k[W].
\]

Lemma 3.12.3. The set \( \{ c_w \mid w \in W \} \) is a \( k \)-linear basis of \( k[W] \).

Proof. Notice that \( w \) is the unique maximal by length element in the expansion of \( c_w \). Thus the set of elements \( \{ c_w \mid w \in W \} \) is related to the standard linear basis \( \{ w \mid w \in W \} \) of \( k[W] \) by a triangular matrix with 1’s on the diagonal. Thus \( \{ c_w \mid w \in W \} \) is a linear basis of \( k[W] \). \( \square \)

Lemma 3.12.4. In the linear basis \( \{ c_w \mid w \in W \} \), the subspace \( A^i \) is the coordinate subspace of \( k[W] \) spanned by the basis elements \( c_w \), for all \( w \) such that \( i \in \text{Des}(w) \):

\[
A^i = \text{Span} \, c_w.
\]

Moreover, for any \( I \subset [n] \), \( \text{Im}(\text{Asym}_I) \) and \( \text{Ker}(\text{Sym}_I) \) are the coordinate subspaces given by

\[
\text{Im}(\text{Asym}_I) = \text{Span} \, c_w \mid I \subset \text{Des}(w) \quad \text{and} \quad \text{Ker}(\text{Sym}_I) = \text{Span} \, c_w \mid I \cap \text{Des}(w) \neq \emptyset.
\]

Proof. Clearly, \( c_w \in A^i \) if \( i \in \text{Des}(w) \). Since \( \# \{ w \in W \mid i \in \text{Des}(w) \} = |W|/2 = \dim A^i \), we deduce that the basis elements \( c_w \) such that \( i \in \text{Des}(w) \) span the subspace \( A^i \). Now the claims about \( \text{Im}(\text{Asym}_I) \) and \( \text{Ker}(\text{Sym}_I) \) follow from Lemma 3.12.2. \( \square \)
Now we can prove part (2) of Theorem 3.11.2. By Lemmas 3.12.1 and 3.12.4, the map \( \text{Sym}_I \), restricted to \( \text{Span} \{ c_w \mid I \subset \text{Des}(w) \} \), induces the isomorphism

\[
\left( \text{Span} \left\{ c_w \mid I \subset \text{Des}(w) \right\} \right) / \left( \text{Span} \left\{ c_w \mid I \subset \text{Des}(w), T \cap \text{Des}(w) \neq \emptyset \right\} \right) \sim \rightarrow \mathbb{R}^I.
\]

Thus the elements \( \text{Sym}_I(c_w) \), for all \( w \in W \) such that \( \text{Des}(w) = I \), form a linear basis of \( \mathbb{R}^I \). This is exactly the claim of Theorem 3.11.2(2), because \( \text{Sym}_I(c_w) = w b_{\text{Des}(w)} a_T =: d_w \).

Part (1) of Theorem 3.11.2 follows from part (2) and Solomon’s decomposition (Theorem 3.10.1).

To prove part (3) Theorem 3.11.2, notice that \( d_w \in M_I \), for all \( w \in W \) such that \( \text{Des}(w) \subset I \). Such \( w \)'s are exactly the minimal length coset representatives for the right cosets \( W/W_I \). So we get \( |W|/|W_I| \) linearly independent elements \( d_w \) in the space \( M_I \cong \text{Ind}_{W_I}^W k \) of dimension \( |W|/|W_I| \). Thus they form a \( k \)-linear basis of \( M_I \).

4. Kostka sheaves

Let \( \mathcal{C} = \text{Rep}(S_{n+1}) = k[S_{n+1}]-\text{mod} \) denote the category of finite dimensional representations of \( S_{n+1} \) over \( k \), that is the category of finite dimensional left \( k[S_{n+1}] \)-modules. The key object of this section is a semisimple amonodromic perverse sheaf \( \mathcal{F} \) smooth along the coordinate stratification of \( \mathbb{C}^n \). Namely, it is a direct sum of constant sheaves \( \mathcal{F}_I \) on the strata closures \( \overline{S}_I, I \subset \text{Sub}_n \). Finally,

\[
\mathcal{F}_I = \bigoplus_{\lambda \in \mathcal{X}_{n+1}} V_{\lambda}^\oplus \mathbb{Z}_\lambda / [n - |I|].
\]

4.1. The sheaf \( \mathcal{F} \) in GGM realization. The corresponding amonodromic semisimple GGM sheaf with values in \( \mathcal{C} \) is

\[
\mathcal{P} = \mathcal{P}^{(n)} \in \text{GGM}_{n}^{\text{ass}}(\text{Rep}(S_{n+1})).
\]

By definition, for any \( I \in \text{Sub}_n \) we have \( \mathcal{P}(I) = R_I \), the ribbon module defined by (3.8.3) and (3.8.4).

4.2. The sheaf \( \mathcal{F} \) in hyperbolic realization. The corresponding amonodromic semisimple hyperbolic sheaf with values in \( \mathcal{C} \) is

\[
\mathcal{Q} = \mathcal{Q}^{(n)} \in \text{Hyp}_{n}^{\text{ass}}(\text{Rep}(S_{n+1})).
\]

Explicitly, by §2.11 its hyperbolic stalks are

\[
\mathcal{Q}(C_{(I, \epsilon)}) = \bigoplus_{J \subseteq I} R_J.
\]

Therefore, by Theorem 3.8.2, they are isomorphic to the induced modules

\[
\mathcal{Q}(C_{(I, \epsilon)}) = \mathcal{Q}(I) \cong M_I = M_{\varphi(I)}.
\]
We recall that $\varrho$ denotes the isomorphism $\varrho: \text{Sub}_n \cong \Omega_{n+1}$ of Lemma 3.4.1, so that $\varrho(I) = \alpha$ is a composition of $n+1$, and $M_\alpha$ is the $S_{n+1}$-module induced from the trivial representation of the subgroup $S_\alpha \subset S_{n+1}$.

### 4.3. A functor $\text{Rep}(S_{n+1}) \to \text{Hyp}^\text{ass}_n(\text{Rep}(S_{n+1}))$. More generally, for any $M \in \text{Rep}(S_{n+1})$ we define a hyperbolic sheaf
\[
\mathcal{Q}_M = (M \otimes \mathcal{Q}(C_{I,\epsilon}), \text{Id}_M \otimes \gamma, \text{Id}_M \otimes \delta).
\]
This way we get a “localization” functor
\[
\mathcal{Q}: \text{Rep}(S_{n+1}) \to \text{Hyp}^\text{ass}_n(\text{Rep}(S_{n+1})).
\]

**Theorem 4.3.1.** For $M \in \text{Rep}(S_{n+1})$, we have an isomorphism $F_T \mathcal{Q}_M \cong \mathcal{Q}_M \otimes k_\epsilon$.

**Proof.** It suffices to construct an isomorphism $F_T \mathcal{Q} \cong \mathcal{Q} \otimes k_\epsilon$. Equivalently, in the GGM realization, we have to construct an isomorphism $F_T \mathcal{P} \cong \mathcal{P} \otimes k_\epsilon$. The existence of the desired isomorphism follows immediately from the isomorphism $V_\lambda \otimes k_\epsilon \cong V_{\lambda'}$, Theorem 3.4.6, and the formula for Fourier-Sato transform in §2.4. □

### 4.4. Induced from parabolics as functions on flags. Let us call a type a sequence of integers $\chi = (\chi_1, \ldots, \chi_p)$ with
\[
1 \leq \chi_1 < \ldots < \chi_p \leq n + 1.
\]
Notation: $\{\chi\} = \{\chi_1, \ldots, \chi_p\}$.

The number $p = \ell(\chi)$ will be called the length of $\chi$. The set of types of length $p$ will be denoted $\text{Typ}_p$; this set contains $\binom{n+1}{p}$ elements.

To each type corresponds a composition of $n+1$
\[
(4.4.1) \quad \alpha(\chi) = (\chi_1, \chi_2 - \chi_1, \ldots, n+1 - \chi_p) \in \Omega_{n+1}
\]

A flag of type $\chi$ is a chain $I_\bullet$ of subsets $I_1 \subset \ldots \subset I_p \subset [n+1]$ with $|I_i| = \chi_i$, and $p$ is the length of $I_\bullet$.

We denote by $F\ell_\chi$ the set of all flags of type $\chi$, and by $F\ell_p$ the set of all flags of length $p$.

The set $F\ell_\chi$ is acted upon from the left by $S_{n+1}$ and is isomorphic to $F\ell_\chi \cong S_{n+1} / S_\chi$, where a parabolic subgroup $S_\chi$ is the stabilizer of the standard flag
\[
[\chi_1] \subset \ldots \subset [\chi_p].
\]

We have
\[
S_\chi \cong S_{\alpha_1} \times S_{\alpha_2} \times \ldots \times S_{\alpha_{p+1}} = S_\alpha,
\]
where $\alpha = \alpha(\chi)$, cf. (4.4.1).

Let $M_\chi$ denote the space of maps of sets
\[
\text{Maps}_{\text{Sets}}(F\ell_\chi, k) = \{f: F\ell_\chi \to k\};
\]
it is an $S_{n+1}$-representation isomorphic to $M_\alpha$ introduced §3.8.

This realization of the modules $M_\alpha$ is convenient for describing some natural morphisms between them.

For instance, there are $(n+1)!$ flags of length $n+1$, all of them having type $\chi_0 = (1, 2, \ldots, n+1)$. The corresponding representation $M_{\chi_0}$ is the regular one. We postulate that there is a single type $\emptyset$ of length zero, and define $M_\emptyset$ to be the trivial representation $k$.

4.5. **Induction and restriction.** For two types $\chi, \theta$ we write $\theta \subset \chi$ if $\{\theta\} \subset \{\chi\}$.

We have obvious maps $\partial_{\chi\theta}: F\ell_\chi \to F\ell_\theta$, wherefrom we obtain the "restriction", or pullback, morphisms in $\text{Rep}(S_{n+1})$

$$r_{\theta\chi} = \partial_{\chi\theta}^*: M_\theta \to M_\chi.$$ 

Their conjugate are "induction", or pushout, morphisms

$$i_{\chi\theta} = \partial_{\chi\theta*}: M_\chi \to M_\theta$$

are explicitly defined as follows: for $f \in M_\chi$, $I_\bullet \in F\ell_\theta$,

$$i_{\chi\theta}(f)(I_\bullet) = \sum_{J_\bullet \in \partial_{\chi\theta}^{-1}(I_\bullet)} f(J_\bullet).$$

4.6. **Parabolic complexes.** These are two dual complexes of length $n$ in $\text{Rep}(S_{n+1})$.

The **master complex** going back at least to [Kat, (1.2)] (specialized to the case when the Weyl group $W = S_{n+1}$, the module $M$ is trivial, and $q = 1$) is

$$(4.6.1) \quad \text{Mas}^\bullet: 0 \to M_{\chi_0} \to \bigoplus_{|\chi| = n-1} M_\chi \to \cdots \to \bigoplus_{|\chi| = 1} M_\chi \to M_\emptyset \to 0$$

The differentials

$$d_p: \bigoplus_{|\chi| = p} M_\chi \to \bigoplus_{|\chi| = p-1} M_\chi$$

are induced by the maps $i_{\chi\theta}$ with appropriate signs. More precisely, their nonzero matrix elements are the maps $d_{\chi\chi'}: M_\chi \to M_{\chi'}$ with $\chi' \subset \chi$, $|\chi'| = |\chi| - 1$. Given a type $\chi$ of length $p$, there are $p$ subtypes $\chi' = \partial_i \chi$, $1 \leq i \leq p$, of length $p - 1$.

By definition, $d_{\chi, \partial_i \chi} = (-1)^i i_{\chi, \partial_i \chi}$. We consider the "master complex" (4.6.1) as concentrated in degrees $[0, n]$.

The **conjugate master complex** is a similar one with differentials induced by the restriction maps $r_{\chi', \chi}$, $|\chi'| = |\chi| - 1$ (with signs):

$$(4.6.2) \quad \text{Mas}^{\bullet\text{c}}: 0 \to M_\emptyset \to \bigoplus_{|\chi| = 1} M_\chi \to \cdots \to M_{\chi_0} \to 0$$

By [Kat, (#) at page 942] applied to the case when the Weyl group $W = S_{n+1}$, the module $M$ is trivial, and $q = 1$,

$$(4.6.3) \quad H^i(\text{Mas}^{\bullet}) = 0, \quad i > 0,$$
and the only nonzero cohomology is
(4.6.4) \[ H^0(\text{Mas}^\bullet) = k, \]
the sign representation.

**Corollary 4.6.1.** \[ H^i(\text{Mas}^\bullet) = 0, \ i > 0. \]

### 4.7. Comparison with the vanishing cycles master complex.

Recall the complex \( \Phi^\bullet(\mathcal{Q}) \) introduced in (2.9.1) for arbitrary hyperbolic sheaf \((E, \gamma, \delta)\).

**Theorem 4.7.1.** There exists an isomorphism of complexes in \( \text{Rep}(S_{n+1}) \)
(4.7.1) \[ \text{Mas}^\bullet \sim \rightarrow \Phi^\bullet(\mathcal{Q}). \]

**Proof.** We know already that the individual terms of the above complexes are isomorphic, see (4.2.1). Both of them start with the regular representation \( M_{[n]} = M_{(11\ldots1)} = k[S_{n+1}]. \) By (4.6.4), \( H^0(\text{Mas}^\bullet) = k. \)

On the other hand, \( H^0(\Phi^\bullet(\mathcal{Q})) \cong k. \) Indeed, the Euler characteristic of \( \Phi^\bullet(\mathcal{Q}) \) is \( k \), e.g. by [So1, Theorem 2], and \( H^{>0}(\Phi^\bullet(\mathcal{Q})) = 0 \) by Proposition 2.9.1.

We will apply the following elementary

**Lemma 4.7.2.** Let \( \mathcal{C} \) be a semisimple category, \( A \subset B, A' \subset B' \) objects of \( \mathcal{C} \). If \( A \cong A' \) and \( B \cong B' \) then \( A/B \cong A'/B' \).

We can reformulate the lemma as follows. Choose an isomorphism \( \phi_A : A \sim \rightarrow A' \). Since \( \mathcal{C} \) is semisimple, there are objects \( C, C' \) such that \( B \cong A \oplus C \) and \( B' \cong A' \oplus C' \). According to the lemma, \( C \cong C' \). Whence there exists an isomorphism \( \phi_B : B \sim \rightarrow B' \) making the square

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\phi_A \downarrow & & \downarrow \phi_B \\
A' & \longrightarrow & B'
\end{array}
\]

commutative.

Now we can construct the desired isomorphism \( \text{Mas}^\bullet \sim \rightarrow \Phi^\bullet(\mathcal{Q}) \) inductively from left to right, using the isomorphism \( H^0(\Phi^\bullet(\mathcal{Q}(n))) \cong k \) and the acyclicity of both complexes in positive degrees. \( \square \)

### 4.8. Relation to the permutohedron.

We present a geometric interpretation of the master complex of vanishing cycles \( \Phi^\bullet(\mathcal{Q}) \), cf. [Kat, So1, Wi].

**Proposition 4.8.1.** The complex \( \Phi^\bullet(\mathcal{Q}) \) is isomorphic, as a complex of \( k \)-vector spaces, to the complex \( C^\bullet(\text{Perm}_n) \) of cochains of the \( n \)-th permutohedron.

For instance, let \( n = 2 \). The complex \( \Phi^\bullet(\mathcal{Q}) = \Phi^\bullet(\mathcal{Q}(2)) \) has the form
\[
0 \rightarrow k[S_3] \rightarrow k[S'] \oplus k[S''] \rightarrow k \rightarrow 0.
\]
Here $S', S'' \cong S_2$ are the subgroups of $S_3$ generated by transpositions $s_1 = (12)$ and $s_2 = (23)$ respectively. So the dimensions of its terms are $6, 6, 1$, and it is isomorphic to the complex of cochains of the hexagon $\text{Perm}_2$.

Proof. Our complex $\Phi^\bullet (\mathcal{Q})$ has the form

$$0 \to k[W] \to \bigoplus_{i \in S} k[W/W_{\{i\}}] \to \bigoplus_{\{i,j\} \subset S} k[W/W_{\{i,j\}}] \to \ldots \to k \to 0,$$

where $W = S_{n+1}$, and for a subset $I \subset S$ of the set of Coxeter generators, $W_I \subset W$ denotes the subgroup generated by $s_i, i \in I$.

On the other hand, consider the root arrangement in $\mathbb{R}^S$; the permutohedron $P_W$ is a polygon dual to this arrangement. Thus, its vertices are in one-to-one correspondence with the chambers that form a set on which $W$ acts simply transitively, i.e. after choosing a fundamental chamber $C$ we can identify this set with $W$, i.e. with a base of $k[W]$.

Next, the edges of $P_W$ are in bijection with walls; there are $n$ walls $M_i$ on the boundary of $C$ which correspond to simple reflections $s_i$; the stabilizer of $M_i$ being the subgroup $W_{\{i\}}$. So the set of walls is in bijection with $\bigsqcup_{i \in S} W/W_{\{i\}}$, etc.

4.9. Warning. One might think that the maps $i_\chi \theta$ and $r_\theta \chi$ give rise to a hyperbolic sheaf, but this is not so: the collinearity axiom does not hold, as the example of $S_3$ already shows.

Namely, let $n = 2$. Consider the square

$$
\begin{array}{ccc}
M_{(1)} & \xrightarrow{r} & M_0 \\
\downarrow & & \downarrow \\
M_{(12)} & \xrightarrow{r} & M_{(2)}.
\end{array}
$$

Let $f \in M_{(1)}$, i.e. $f$ is a function from one-element subsets of $[3]$ to $k$. Then $r(f) = \sum_{i=1}^3 f(i)$, and $ir(f)$ is the constant function with value $r(f)$.

On the other hand, $i(f)(\{i\} \subset \{i, j\}) = f(i)$, and $ir(f)(\{i, j\}) = f(i) + f(j)$. Thus, $ir \neq ri$, the square is not commutative.

5. From $S_{n+1}$ to $\text{GL}(n+1, \mathbb{F}_q)$

We denote $G = G_{n+1} = \text{GL}_{n+1}(\mathbb{F}_q)$.

5.1. Flag spaces and induced representations. We fix a vector space $U = \mathbb{F}_q^{n+1}$. For a type $\chi = (\chi_1, \ldots, \chi_p) \in \mathcal{Y}_{yp}$ (notation of §4.4), a flag of type $\chi$ is a chain of linear subspaces

$$F_\bullet : 0 = U_0 \subset U_1 \subset \ldots \subset U_p \subset U;$$

such that $\dim U_i = \chi_i$. We will call $p$ the length of $F_\bullet$. 
We denote by $F\ell_\chi(q)$ the set of all flags of type $\chi$, and by $F\ell_p(q)$ the set of all flags of length $p$.

The set $F\ell_\chi(q)$ is acted upon from the left by $G$ and is a homogeneous set $F\ell_\chi(q) \cong G/P_\chi$, where $P_\chi$ is the standard parabolic subgroup, the stabilizer of the standard flag $\mathbb{P}_q^1 \subset \cdots \subset \mathbb{P}_q^p$.

For $\chi = (1,2,\ldots,n,n+1)$, we have $P_\chi = B$, the standard Borel.

Let $M_\chi(q)$ denote the space of maps of sets $\text{Maps}(F\ell_\chi(q), k) = \{ f : F\ell_\chi(q) \to k \}$; it is a $G$-module isomorphic to $\text{Ind}_P^G k$.

5.2. Induction and restriction. We have the natural $q$-analogues of the master complex and its conjugate of §4.6.

Let $\theta \subset \chi$ be two types. We have obvious maps $\partial_{\chi \theta} : F\ell_\chi(q) \to F\ell_\theta(q)$, wherefrom the restriction, or pullback, morphisms in $\text{Rep}(G)$ $r_{\theta \chi} = \partial_{\chi \theta}^* : M_\theta(q) \to M_\chi(q)$.

Their conjugate are induction, or pushout, morphisms $i_{\chi \theta} = \partial_{\chi \theta*} : M_\chi(q) \to M_\theta(q)$, that are explicitly defined as follows: for $f \in M_\chi(q), F_\bullet \in F\ell_\theta$,

\[
i_{\chi \theta}(f)(F_\bullet) = \sum_{F'_\bullet \in \partial_{\chi \theta}^{-1}(F_\bullet)} f(F'_\bullet).
\]

Proceeding as in §4.6 we obtain a parabolic complex

\[
\text{DL}(k)^* : 0 \to M_{\chi_0}(q) \to \bigoplus_{\chi:|\chi|=n-1} M_\chi(q) \to \cdots \to M_\theta(q) \to 0
\]

This is nothing but the Deligne-Lusztig complex [DL, (1.2)] for the case of trivial representation $E = k$ of $\text{GL}(n+1, \mathbb{F}_q)$. According to [DL, Theorem in §2],

\[
H^i(\text{DL}(k)^*) = 0, \ i > 0.
\]

The only nonzero cohomology is $H^0(\text{DL}(k)^*) = \text{St}$, the Steinberg module [St].

5.2.1. Example $n = 1$. Consider the following hyperbolic sheaf $E \in \mathcal{M}(\mathbb{C}, 0)$ with values in $\text{Rep}(\text{GL}(2, \mathbb{F}_q))$:

\[
E_0 = \text{Ind}_B^G k = \text{Maps}(\mathbb{P}^1, k), \ E_+ = E_- = k.
\]

We have

\[
\text{Ind}_B^G k \cong k \oplus \text{St}.
\]

The Fourier-Sato transform $\text{FT}(E)$ has the hyperbolic stalks

\[
E_0^\vee = E_0 = \text{Ind}_B^G k, \ E_+^\vee = E_-^\vee = \text{St}.
\]
5.2.2. Example $n = 2$. We have $4 = 2^2$ standard parabolics $G, P_1, P_2, B$ with
$$G/G = \{1\}, G/P_1 = \mathbb{P}^2, G/P_2 = \mathbb{P}^{2\nu}.$$ The real stratification $S_R$ of $\mathbb{R}^2$ has 4 2-dimensional cells $C_i^{\pm}$, $i = 1, 2$; and 4 1-dimensional cells $\ell_i^{\pm}$, $i = 1, 2$, and the 0-dimensional cell $\{0\}$.

We define a hyperbolic sheaf $E \in \mathcal{M}(\mathbb{C}^2, S)$ with hyperbolic stalks
$$E_{C_i^{\pm}} = k,$$
$$E_{\ell_i^{\pm}} = \text{Ind}_{P_i}^G k = \text{Map}(\mathbb{P}^2, k),$$
$$E_0 = \text{Ind}_B^G k = \text{Map}(G/B, k).$$

We have the decomposition into irreducibles
$$E_0 \cong k \oplus \text{St}' \otimes k^2 \oplus \text{St}.$$ We have $\dim \text{St}' = q^2 + q$, $\dim \text{St} = q^3$, and $|G/B| = q^3 + 2q^2 + 2q + 1$.

The Fourier-Sato transform $\text{FT}(E)$ has hyperbolic stalks
$$E_0' = E_{C_i^{\pm}}, E_{\ell_i^{\pm}}' = E_{\ell_i^{\pm}}, E_{\ell_i^{\pm}}' = E_{\ell_i^{\pm}}, E_{C_i^{\pm}}' = E_0.'$$

5.3. Unipotent representations of $\text{GL}(n + 1, \mathbb{F}_q)$. Let $\mathcal{H}_{n+1}$ denote the Iwahori-Hecke algebra of $S_{n+1}$ at $q$. There is an isomorphism $[BC]$ of algebras $\mathbb{k}[S_{n+1}] \cong \mathcal{H}_{n+1}$. At the price of extending $\mathbb{k}$ by $\sqrt{q}$, one can choose an explicit isomorphism of $[Lus]$. It gives rise to an equivalence of semisimple abelian categories $\mathcal{C} = \text{Rep}(S_{n+1}) \cong \text{Rep}(\mathcal{H}_{n+1}) =: \mathcal{C}_q$. Under this equivalence, the master complex $\text{Mas}^\bullet$ goes to the complex $q\text{Mas}^\bullet$ of $[Kat, (1.2)]$ (specialized to the case when the Weyl group $W = S_{n+1}$, and the representation $M$ is trivial).

Furthermore, $\mathcal{C}_q$ is canonically equivalent to the semisimple abelian category $\mathcal{C}_G := \text{Rep}_{\text{unip}}(G_{n+1})$ of unipotent representations of $\text{GL}_{n+1}(\mathbb{F}_q)$: direct sums of constituents of the natural representation of $\text{GL}_{n+1}(\mathbb{F}_q)$ in the space of functions on the flag variety. Under this equivalence, $q\text{Mas}^\bullet$ goes to the Deligne-Lusztig complex $[DL]$ $\text{DL}(k)$ of the trivial $G$-module $k$.

More generally, composing the functor $\mathcal{D}$ of §4.3 with the above equivalence $\mathcal{C}_G \cong \mathcal{C}$, we obtain a functor
$$\mathcal{D}^G : \text{Rep}_{\text{unip}}(G_{n+1}) \to \text{Hyp}^\text{ass}_n(\text{Rep}_{\text{unip}}(G_{n+1})).$$ Now given a unipotent representation $K \in \text{Rep}_{\text{unip}}(G_{n+1})$, we consider the complex of vanishing cycles $\Phi^\bullet(\mathcal{D}^G(K))$ of (2.9.1). Then it follows from Theorem 4.7.1 that $\Phi^\bullet(\mathcal{D}^G(K))$ is isomorphic to the Deligne-Lusztig complex $\text{DL}(K)$. Finally, it follows from Theorem 4.3.1 that we have an isomorphism $\text{FT} \mathcal{D}^G(K) \cong \mathcal{D}^G(K'^\vee)$, where $K'^\vee$ stands for the Deligne-Lusztig dual of $K$. 
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