Online Mincut: Advice, Randomization and More

Avah Banerjee∗
ibanerjee@lsu.edu
Louisiana State University

Guoli Ding
ding@math.lsu.edu
Louisiana State University

Abstract

In this paper we study the mincut problem on connected graphs in the online setting. We consider the vertex arrival model; whenever a new vertex arrives it’s adjacency to previously revealed vertices are given. An online algorithm must make an irrevocable decision to determine the side of the cut that the vertex must belong to in order to minimize the size of the cut.

Various models are considered. 1) For classical and advice models we give tight bounds on the competitive ratio of deterministic algorithms. 2) Next we consider few semi-adversarial inputs: random order of arrival with adversarially generated and sparse graphs. 3) Lastly we introduce a new model, which we call the friendly sequence model. We look at several online optimization problems: mincut, maxcut and submodular maximization and show that there are input ordering where a greedy strategy can produce an optimal answer.

1 Introduction

Let $\Gamma$ be a (possibly infinite) graph. An online graph $G = (V, E)$ is a finite subgraph of $\Gamma$ and there is a total order $\pi$ on $V(G)$ or (and) $E(G)$. We assume $|V(G)| = n$ and $|E(G)| = m$. Sometimes $\Gamma$ is not mentioned when we describe a problem because $G$ is allowed to be any finite graph and $\Gamma$ is the disjoint union of all finite graphs (and thus there is no need to mention $\Gamma$). In the vertex arrival model, vertices of $G$ are revealed one at a time according to $\pi$, along with its neighbors in the current set of revealed vertices. In the edge arrival model, vertices of $G$ are known and edges arrive one at a time according to $\pi$. We do not explicitly consider the edge arrival model in this paper. However some of our results in the vertex arrival model can be extended to the latter setting without much effort.

1.1 Computational Models

We introduce some standard notions in online computation [5]. However we frame our discussions in terms of online graph problems. Let $P$ be some optimization problem on a graph and let $opt_P(G)$ be the optimal value of $P$ when the input is $G$. It is also known as the offline optimal. Let $A_P(G, \pi)$ be the output computed by some online algorithm $A$ given the ordering $\pi$. We use competitive analysis to measure the relative performance of $A$ with respect to the offline minimum. Specifically, we say that $A$ is $c$-competitive if for every $G$,

$$\max_\pi A_P(G, \pi) \leq c \cdot opt_P(G) + d$$

for some constant $d \geq 0$. If $d = 0$ then the algorithm is said to be strictly $c$-competitive. The smallest $c$ for which an algorithm is (strictly) $c$-competitive is known as the (strict) competitive

∗Formerly Indranil Banerjee
An algorithm is said to be competitive if it is 1-competitive. For maximization problems the competitive ratio is defined in a similar manner. It is important to note that the constant $d$ must be independent of $G$ but may depend on $A$ and $P$. We omit the subscript $P$ whenever the context is clear.

In the adversary model the input $G$ and its order of arrival (be it vertex or edge arrival) are determined by an adversary. At each step the algorithm makes an irrevocable decision based on the part of the input seen so far. No other knowledge about the input is known to the algorithm in advance (not even the length). The model is used for both deterministic and randomized algorithms \[^{5}\]. In the deterministic setting, the adversary knows the algorithm (also referred to as the online player) in advance. For every input sequence the adversary knows the sequence of actions performed by the algorithm. Hence it is often assumed that adversary creates the entire input then feeds the online algorithm one piece at a time. However, in the case of randomized algorithms the notion of an adversary is a bit more complex. Due to usage of random bits, the behaviour of an randomized online algorithm may differ in each run even with the same input sequence. Informally, the power of an adversary depends on whether they are allowed to look at the current state of the online algorithm before deciding the next input.

Many different models can be considered semi-adversarial or non-adversarial. They are often characterized in terms of an weak adversary. In some models the online algorithm is supplied with additional information by a benevolent oracle. Interested readers can refer to \[^{15, 11, 17, 9, 19}\] for a more detail overview of these models. Some of the more well known models are random-order model (for the matroid secretary problem), diffuse adversary (for paging), Markov process (for paging) etc. Resource augmentation based models, where the adversary is made weak by giving more “resources” to the online algorithm can also be thought of as semi-adversarial. A good example is the $(h, k)$-server problem $(h \leq k) \[^{11}\]$. Here $h$ is the number of server the adversary is allowed to use to process the requests they generate. These models are an important alternative to the adversarial models as they strive to represent real world situations more accurately. We consider two such models: the first one is a randomized/ restricted input model and the latter is a beyond-worst-case measure. We discuss these next.

1.1.1 Semi-adversarial Inputs

In the context of online graph problems, we look at a relevant semi-adversarial model. The arrival order of vertices are chosen uniformly at random. In this setting we consider two situation: (1) The graph $G$ is adversarially generated (2) The graph $G$ comes from a particular family of graphs which is known to the algorithm in advance. In particular we look at sparse graphs.

In the random order model we want to determine the the competitive ratio in terms of the expected value of the solution determined by the algorithm. That is,

$$\mathbb{E}[A_P(G, \pi)] = \frac{1}{n!} \sum_\pi A_P(G, \pi) \leq c \text{ opt}_P(G) + d$$

The above expectation is over the random permutation $\pi$ and possibly over the random choices made by $A_P$. Since $G$ is not random the optimal value is not a random variable.

1.1.2 Friendly Sequence Model

For many online problems the classical worst case model yields pessimistic results. A review of some well known alternatives can be found here \[^{15, 11}\]. However there are online problems, particularly in the minimization setting, where these model fail to distinguish the hardness of
these problems. Graph problems, such as finding the mincut, min-degree, minimum spanning
tree, minimum dominating set (discussed later) are good examples of online problems which are
considered hard even with many beyond worst case measures.

With this in mind we look at the following measure to evaluate the hardness of some online
problems on graphs. An extension of this idea can also be used to compare the “robustness” of
different online algorithm even if their worst case performance are indistinguishable. At a high level
we classify problems based on whether there is a “good ordering” of the inputs for every possible
input graphs such that we can always find the optimal output using a fixed (necessarily greedy)
strategy. Let $A$ be an online algorithm for a graph minimization problem $P$ under the vertex arrival
model. The input of $A$ is a permutation $\pi(V(G)) = v_1, v_2, \ldots, v_n$ of a graph $G$. To measure the
performance of $A$, we usually consider $\max A(G, \pi)$ over all permutations $\pi$ in the adversary model.
Similarly for the random order model with adversarially generated input we are interested in the
average $\sum A(G, \pi)/n!$. Along this line, one natural question we may ask is, what is $\min A(G, \pi)$,
over all $\pi$?

For many problems $P$, it is easy to construct $A$ such that $\min_\pi A(G, \pi) = \text{opt}(G)$ holds for all
graphs $G$. For instance, if $P = \text{MinCov}$ is the minimum vertex-cover problem, then it is clear that
the following $A$ satisfies the requirement: placing $v_i, v_{i+1}, \ldots, v_n$ in the cover, where $i$ is the largest
index such that $v_1, v_2, \ldots, v_{i-1}$ is an independent set. To see that $\min_\pi A(G, \pi) = \text{opt}(G)$ we only
need to take a maximum independent set $I$ and define $\pi$ to be a permutation that first lists all
vertices of $I$ and then vertices of $V \setminus I$.

However, there are also problems for which no matter what $A$ is, $\min_\pi A(G, \pi)$ is different
from $\text{opt}(G)$ for at least one graph $G$. For instance, consider the minimum domination problem
$P = \text{MinDom}$: find a smallest set $D$ of vertices of $G$ such that every vertex outside $D$ is adjacent
to at least one vertex inside $D$. Then $P$ is such a problem. Suppose otherwise that $A$ satisfies
$\min_\pi A(G, \pi) = \text{opt}(G)$ for all $G$. Then

1. $A$ must place $v_1$ in $D$ because $G$ might have only one vertex. In general, if $v_1, \ldots, v_k$ is
   independent then $A$ has to place all of them in $D$.

2. If $v_1$ is adjacent to $v_2$ then $A$ must place $v_2$ outside $D$ because $G$ might be $K_2$. In general, if
   $v_1$ is adjacent to $v_2, \ldots, v_k$ and $\{v_2, \ldots, v_k\}$ is independent then $A$ must place $v_2, \ldots, v_k$ outside
   $D$ since $G$ might be $K_{1,k-1}$.

Now let $H$ be the tree with five edges 13, 23, 34, 45 and 46. Then $H$ has a unique minimum
dominating set $\{3, 4\}$. If $\pi$ is a permutation so that $A(H, \pi) = \{v_3, v_4\}$, by (1) above we may
assume $v_1 = 3$. Then (2) implies a contradiction. So, no matter what $A$ is, $A(H, \pi) \neq \text{opt}(H)$ for
all $\pi$.

The above two examples show the two extremes concerning $\min_\pi A(G, \pi)$. In this paper we
establish that if $P$ is the mincut problem then there exists an algorithm with $\min_\pi A(G, \pi) = \text{opt}(G)$
holds for all $G$. We extend our results to other graph optimization problems such as online maxcut
and sub-modular function maximization \cite{20}.

1.1.3 Advice Model

Advice in the context of online computation is a model where some information about the future
inputs are available to the algorithm. Its inception is somewhat recent \cite{6 \cite{10}. The informal idea
is as follows. The online algorithm is given access to a friendly oracle which knows the input in
advance. The oracle is assumed to have unlimited computational power. The algorithm is allowed
to ask arbitrary questions to this oracle at any stage of the computation. We do not care about the
nature of information received rather than the amount, in terms of the number of bits. This quantity is known as the advice complexity of the algorithm. Given some online problem $P$ we want to determine the lower (upper) bound of the amount of advice needed by any (some) algorithm to achieve a certain competitive ratio. This model have been shown to be useful in proving certain lower bounds for online problems.

There are various flavors of advice models, which are more or less equivalent. The model we use here is a variant of the tape model [15]. Let $P$ be some online graph minimization problem. Let $A_P^{adv}$ be an algorithm solving $P$ which has access to an advice string $adv$. We say $A_P^{adv}$ is $c$-competitive with advice complexity $b$ for $P$ if there is an advice string $adv$ of size at most $b$ such that:

\[ \max \ A_P^{adv}(G, \pi) \leq c \ \text{opt}_P(G) + d \]

Where $d$ is some constant independent of the size of $G$. The advice complexity $b$ can be a function of the size of $G$, however it is not dependent on $G$ itself. In the above definition we implicitly assume the length of the advice string is known to the algorithm. Otherwise we may assume advice strings are self delimiting adding to a $O(\log b)$ overhead.

1.2 Problem Definition and Notations

Let $G = (V, E)$ be a graph. For any disjoint subsets $X, Y \subseteq V$, we denote by $E(X, Y)$ the set of all edges of $G$ that are between $X$ and $Y$. A partition of $V$ is a pair $(X, Y)$ of disjoint subsets of $V$ with $X \cup Y = V$. A cut of $G$ is a set $C \subseteq E$ that can be expressed as $E(X, Y)$ for a partition $(X, Y)$ of $V$ with $X \neq \emptyset \neq Y$. Note that every graph with two or more vertices must have at least one cut.

The minimum cut problem (MinCut) is to minimize $|C|$ over all cuts $C$ of $G$. Note that the minimum is finite for all $G$ with two or more vertices, and the minimum is $\infty$ if $|V(G)| = 1$ since we are minimizing over the empty set. For a graph with a positive edge weights $w : E \rightarrow \mathbb{R}^+$ the problem (MinCut$^+$) is to minimize $w(C)$, where $w(C) = \sum_{e \in C} w(e)$.

Let $\mathcal{G}$ be a class of graphs. All graphs considered in the paper are simple. By $\text{MinCut}[\mathcal{G}]$ we denote the problem MinCut with its input limited to graphs in $\mathcal{G}$. According to our definition (in section 1.1), an online algorithm $A$ for $\text{MinCut}[\mathcal{G}]$ is called $c$-competitive if there exists a constant $d$ (which may depend on $\mathcal{G}$) such that

\[ \max_{G \in \mathcal{G}} \max_{\pi} A(G, \pi) \leq c \cdot \text{opt}(G) + d. \]

For any integer $k \geq 0$, let $\mathcal{G}_k$ denote the class of $k$-edge-connected graphs. Equivalently, $\mathcal{G}_k$ consists of all graphs $G$ with $\text{opt}(G) \geq k$. In addition, every graph in $\mathcal{G}_k$ has at least $k + 1$ vertices. We use $\mathcal{G}_{(n)}$ to denote an infinite collection of graphs. The collection contains graphs of size $n$ whenever $n$ is sufficiently large.

We consider the minimum cut problem in the advice model as follows. The input, which is generated by the adversary, is a graph $G$ together with a total order $\pi$ on its vertices. We denote the vertices, under $\pi$, by $v_1, v_2, ..., v_n$ throughout our discussion. A partial input sequence $(v_1, \ldots, v_i)$ is termed as a prefix sequence. By symmetry we assume $v_1 \in X$. The algorithm may choose to ask questions even before $v_1$ is revealed. Since the placement of $v_1$ is fixed, it does not matter if these questions are asked before or after $v_1$ is revealed. To be consistent with all other steps, we assume that $A$ does not ask anything before $v_1$ is revealed. So the process goes as follows: Step 1: $v_1$ is revealed and is placed in $X$.

Step 2: $v_2$ is revealed, then $A$ asks a question and gets an answer, then $v_2$ is placed in $X$ or $Y$.  

4
Step 3: $v_3$ is revealed, then $A$ asks a question and gets ... and so on.

At the $i$th ($i > 1$) step of the computation, $A$ has placed $v_1, ..., v_{i-1}$ in $X$ or $Y$ already, $v_i$ is just revealed, and $A$ needs to decide where to place $v_i$. At this point, $A$ will ask a question about $(G, \pi)$, with the knowledge of $G[v_1, ..., v_i]$ (the subgraph of $G$ induced on $v_1, ..., v_i$) and possibly other information about $(G, \pi)$ that was obtained by $A$ from the previous inquires. We define $\Gamma_i$ as the collection of potential inputs $G$ after seeing the first $i$ vertices. A partition $(X_i, Y_i)$ of $\{v_1, ..., v_i\}$ is called extendable if it can be extended into an optimal solution.

1.3 Related Work

To the best of our knowledge MinCut and its other siblings (like min-bisection) have not been studied in the online setting. In contrast there have been few results related to MaxCut. The folklore randomized 2-approximation for the offline MaxCut also works in the online setting. In [2] authors gave a almost tight bound of $3\sqrt{3}/2$ for the competitive ratio of the maximum directed cut problem under the vertex arrival model.

Few other studies have been made for online minimization problems on graphs. Two important problems in this area are online minimum spanning tree and coloring[13, 12, 3]. For the minimum spanning tree problem generally the edge arrival model is used. In [18] authors study this problem when the edge weights are selected uniformly at random from $[0, 1]$. More recently this problem has been studied in the advice setting [4].

1.4 Our Results

This section serves to summarize our main results and present some selected proofs. The proofs that are omitted can be found in the appendix.

1.4.1 Adversarial Input and Advice Complexity

Theorem 1. (i) Let $A$ be an online algorithm for MinCut$[G_k]$, where $A$ knows $n$ in advance. Then the following hold.

(a) If $k = 0$ then $A$ is not $c$-competitive for any $c$.
(b) If $k \geq 1$ and $A$ is $c$-competitive then $c \geq \frac{n-p}{k}$ for some $p \geq 1$.

(ii) Suppose $k \geq 1$. Then there exists an online algorithm $A$ for MinCut$[G_k]$, where $A$ does not know $n$ in advance, such that $A$ is $\frac{n-p}{k}$-competitive for all $p \geq 1$.

We formally prove this result in appendix A.1. There is a trivial algorithm which creates a partition $G$ with one part having a solitary vertex. We show that this is the best one can do without any additional information about the input except for its length. This result stands in contrast to that for the online maxcut problem. In the case of online minimization problems like mincut making a single mistake can prove to be costly. Can advice help? There are two interesting cases to consider. One where we want to find the optimal cut and the other where an approximate value would suffice. As it turns out the advice complexity of these two problems are more or less the same. This is a bit surprising as there are AOC-complete online problems for which this is not the case. Here AOC stands for asymmetric online cover which was introduce in [7]. For problems in this class a $c$-competitive algorithm requires $\Omega(n/c)$-bits of advice and these bounds are tight. But for the minimum cut the results are pessimistic. The following two theorems gives the advice complexity for optimality.

Theorem 2. There is a competitive algorithm that finds a minimum cut with $n-1$ bits of advice.
This is proved formally in appendix A.2. To achieve this we simply ask the oracle if putting \( v_i \) in \( X \) makes the current solution extendable. If not we put \( v_i \) in \( Y \). Note that in this case answer to each question is a 1-bit answer. Hence there are no overhead due to self-delimiting strings. The algorithm correctly determines a minimum cut even if the graph \( G \) is disconnected. Unfortunately as theorem 3 shows this naive strategy is almost optimal.

**Theorem 3.** There is a collection \( \mathcal{G}_{(n)} \) of graphs such that any competitive algorithm solving MinCut[\( \mathcal{G}_{(n)} \)] requires at least \( n - 5 \) bits of advice.

**Proof.** For every \( n \geq 6 \) we present a graph for which a competitive algorithm requires at least \( n - 5 \) bits of advice. Each graph \( G \) in the collection has path a \( P = (x_1, x_2, x_3, x_4) \) of length 4 (see Figure 1). Additionally, all other vertices of \( G \) are divided into two parts \( S \) and \( T \). Each vertex in \( S \) is adjacent to both \( x_1, x_2 \) and each vertex in \( T \) is adjacent to both \( x_3, x_4 \). There are no other edges in \( G \). Suppose the adversary first reveals the vertices in \( S \cup T \). The induced subgraph \( G[S \cup T] \) forms an independent set. Let \( \Gamma_{n-4} \) be the set of potential graphs remain after processing the set \( \{v_1, v_2, \ldots, v_{n-4}\} \). First we show \( |\Gamma_{n-4}| = 2^{n-4} \). This follows from the fact that the set \( \{v_1, v_2, \ldots, v_{n-4}\} \) can be partitioned in \( 2^{n-4} \) different ways depending on which vertices (if any) are adjacent to \( \{x_1, x_2\} \). Since the labels \( S \) and \( T \) are interchangeable there are exactly \( |\Gamma_{n-4}|/2 = 2^{n-5} \) pairwise distinct optimal solutions in \( \Gamma_{n-4} \).

An optimal algorithm, without advice, must be able distinguish between these pairwise distinct solutions before the path \( P \) is revealed. By the standard information theoretic argument we see that \( \geq n - 5 \) advice bits are necessary to solve MinCut[\( \mathcal{G}_{(n)} \)] optimally.

Next we ask : how much advice is necessary and sufficient to approximate the value of the mincut value. Theorem 1 gives a \( O(n/k) \)-competitive algorithm even without advice whenever \( k \geq 1 \). However, with only \( O(\log n + \log \log n) \) bits of advice we can achieve a \( \frac{\delta(G)}{k} \)-competitive algorithm. Here \( \delta(G) \) is the minimum degree of \( G \). At the beginning we ask the oracle the position of a vertex with the minimum degree, which requires \( O(\log n + \log \log n) \) bits. The log \log n term correspond to the extra bits used to make the advice string self-delimiting. The algorithm puts this vertex in one part and all other vertices into the other part resulting in a cut of size \( \delta(G) \). Unfortunately, if \( \delta(G) = O(n) \) then it is no better than the algorithm without advice. In the next theorem we show that this is essentially the best one can do.

**Theorem 4.** Let \( A^{adv} \) be a \( c \)-competitive algorithm for MinCut[\( \mathcal{G}_k \)] where \( 1 \leq k \leq \left\lfloor \frac{n-4}{2} \right\rfloor \). For every \( \frac{k+1}{n} < \epsilon < \frac{1}{2}(1 - \frac{1}{n}) \), if \( A^{adv} \) uses \( b < n - 2[en] - 1 \) bits of advice then \( c \geq \frac{\epsilon n - 1}{k} \).

The proof uses a counting argument similar to the one we used above. The main technical contribution is devising a hard instance for the problem. We used the graph from Figure 2 for this
purpose. This graph has a unique minimum cut of size $k$. For any other partition of $G$ the size of the resulting cut is large. The technical details are given in appendix 2.1.

In some respect, the minimum cut problem shows a limitation of the advice model. Unlike AOC-complete problems the advice complexity for mincut has a sharp phase transition. Either we have sufficient amount of advice to produce an optimal solution or a sub-linear competitive ratio cannot be guaranteed.

### 1.4.2 On Semi-adversarial Models

In the previous section we showed that there is $O(n/k)$-competitive algorithm when both the input graph and the order of arrival is determined by an adversary. This upper bound also holds when the order of arrival is determined by a random permutation. Unfortunately, it turns out this is the best we can do without any restriction on the input graph. We show this next. We complement this with an $O(1)$ upper bound for sparse connected graphs.

**Theorem 5.** For any deterministic algorithm $A$ for $\text{MinCut}[G_k]$ under the random-vertex order model there exists a class of graphs $G_{(n)}$ for infinitely many values of $n$ for which,

$$\mathbb{E}[A(G)] \geq \frac{n}{64k} \cdot \text{opt}(G).$$

Here the expectation is taken over the random order.

![Figure 2: A graph $G \in G_{(n)}$ used in the proof of theorem 4](image)

**Proof.** We use the class of graphs $G_{(n)}$ from theorem 4 (Figure 2). Here we take $|C| = |D| = \epsilon n$ and $|A| = |B| = (1/2 - \epsilon)n$. We note that $\lambda(G) = k$, same as before. An optimal offline algorithm returns this value. Consider any online algorithm $A$. Without loss of generality we may assume $v_1$ is assigned to the partition $X$. Let $V_i = \{v_1, \ldots, v_i\}$ be the set of vertices to arrive so far. Let $\mathbb{E}[A(G, V_i)]$ be the expected value of the mincut computed by the online algorithm after processing the vertices $v_1$ through $v_i$. Let $A_i$ is the following algorithm which has two phases: online and offline. In the online phase it processes the first $i$ vertices same as $A$ creating a partial solution. Then it is allowed to read the rest of the input just like an offline algorithm. This is the offline phase. It outputs a final partition that minimizes the cut value while respecting the decisions made during its online phase. Let $\mathbb{E}[A_i(G)]$ be the expected value of the minimum cut computed
by $A_i$. It is clear that $\mathbb{E}[A(G)] = \mathbb{E}[A(G, V_n)] = \mathbb{E}[A_n(G)]$. Further, the function $\mathbb{E}[A_i(G)]$ is monotonically increasing in $i$. Hence we have,
\[ \mathbb{E}[A(G, V_n)] \geq \mathbb{E}[A_2(G)] \]

We give a lower bound for $\mathbb{E}[A_2(G)]$ as claimed in the theorem. Let $\lambda(G, X_i, Y_i)$ be the minimum cut achievable after assigning the first $i$ vertices by $A_i$, where $(X_i, Y_i)$ is the resulting partition. There are two cases as follows.

**Case 1:** $[v_1$ and $v_2$ are not adjacent]. $A_2$ either puts (1) of them in $X$ or (2) puts $v_2$ in $Y$. Suppose $A_2$ chooses (1). Then,
\[ \mathbb{E}[A_2(G) | v_1, v_2 \text{ are not adjacent}] = \mathbb{P}[v_1, v_2 \in C \text{ or } v_1, v_2 \in D] \cdot k \]
\[ + 2\mathbb{P}[v_1 \in A \text{ and } v_2 \in B] \cdot \alpha_1 \]
\[ + 2\mathbb{P}[v_1 \in C \text{ and } v_2 \in D] \cdot \alpha_2 \]
\[ \geq 2\mathbb{P}[v_1 \in C \text{ and } v_2 \in D] \cdot \alpha_2 \]

Here $\alpha_2$ is a lower bound on the minimum cut found by $A_2$ when $v_1$ and $v_2$ are in different stable sets $C$ and $D$. Clearly $\alpha_2 \geq |A| = (1/2 - \epsilon)n$. Since $v_1, v_2$ are picked from a random order,
\[ \mathbb{P}[v_1 \in C \text{ and } v_2 \in D] = \epsilon^2 \]

From Equation 1 we get:
\[ \mathbb{E}[A_2(G) | v_1, v_2 \text{ are not adjacent}] \geq 2\epsilon^2(1/2 - \epsilon)n \]

Now suppose $A_2$ puts $v_2$ in $Y$. A similar argument to the one above can be made to show that,
\[ \mathbb{E}[A_2(G) | v_1, v_2 \text{ are not adjacent}] \geq 2\epsilon^2(1/2 - \epsilon)n \]

**Case 2:** $[v_1$ and $v_2$ are adjacent.] Again we have two possibilities. (1) $A_2$ puts $v_2$ in $X$ and (2) $A_2$ puts $v_2$ in $Y$. For the first case we have,
\[ \mathbb{E}[A_2(G) | v_1, v_2 \text{ are adjacent}] = \mathbb{P}[v_1, v_2 \in A \text{ or } v_1, v_2 \in B] \cdot k \]
\[ + 2\mathbb{P}[v_1 \in A \text{ and } v_2 \in B] \cdot (1/2 - \epsilon)n \]
\[ \geq (1/2 - \epsilon)^3n \]

In a similar manner we find that if $A_2$ puts $v_2$ in $Y$ then,
\[ \mathbb{E}[A_2(G) | v_1, v_2 \text{ are adjacent}] \geq \epsilon^2(1/2 - \epsilon)n \]

In all of the of the above cases regardless of what $A_2$ chooses do with $v_2$ we have ,
\[ \mathbb{E}[A_2(G)] \geq \min(\epsilon^2(1/2 - \epsilon)n, (1/2 - \epsilon)^3n) \]

The right hand side of the above expression is maximized when $\epsilon = 1/4$ and we get $\mathbb{E}[A_2(G)] \geq n/64$.

For sparse connected graphs with $O(n)$-edges we can do significantly better in the random order model. The proof the following is given in appendix B.

**Theorem 6.** In the random order model there is $O(1)$-competitive algorithm in expectation for sparse connected graphs with $O(n)$ edges.
1.4.3 Friendly Sequence and Greedy Order

As we have discussed in section 1.1.2, the performance of $\min_{\pi} A(G, \pi)$ could serve as a measure on the complexity of an online problem $P$. In this section we will study $\min_{\pi} A(G, \pi)$ for online mincut and maxcut problems. In both cases, we establish that there exists $A$ satisfying $\min_{\pi} A(G, \pi) = \text{opt}(G)$ for all $G$. In addition, we obtain an analogous result for maximizing a submodular function and we refute the existence of such a result for minimizing a submodular function.

We begin by clarifying our Terminology. Let $G = (V, E)$ be a graph. For the current discussion, we allow parallel edges but not loops in $G$. This is the same as allowing a nonnegative weight $w$ on edges and measuring the size of a cut $C$ by the total weight $\sum\{w(e) : e \in C\}$. In mincut and maxcut problems, a cut $C \subseteq E$ is represented by a partition $(X, Y)$ of $V$ such that $C$ is the set of edges between $X$ and $Y$. We insist that neither $X$ nor $Y$ could be empty and we assume that $G$ has at least two vertices.

For any disjoint $X, Y \subseteq V$, let $|X, Y|$ denote the number of edges of $G$ between $X$ and $Y$. We will write $|x, Y|$ or $|X, y|$ for $|X, Y|$ if $X = \{x\}$ or $Y = \{y\}$, respectively. If $E(X, Y)$ is a minimum or maximum cut of $G$ for a partition $(X, Y)$ of $V$ then we may simply call $(X, Y)$ is a minimum, maximum cut of $G$, respectively. If $U \subseteq V$ then we use $G[U]$ to denote the subgraph of $G$ induced on $U$.

We will consider a greedy type algorithm $A$. Let $\pi = v_1, v_2, \ldots, v_n$ be a permutation of $V$. Let $X, Y$ be the partition determined by $A$ during the process. Since $n$ is unknown to the algorithm, $A$ has to place $v_1 \in X$ and $v_2 \in Y$, because $A$ needs to ensure $X \neq \emptyset \neq Y$ even when $n = 2$. In the $i$th iteration ($i \geq 3$), vertex $v_i$ is revealed and $A$ need to decide if $v_i$ should go to $X$ or $Y$. A simple greedy strategy is to make the choice depending on $f_X$ and $f_Y$, which are the number of edges from $v_i$ to $X$ and $Y$, respectively. In the mincut problem, $v_i$ goes to $X$ if $f_X > f_Y$, while in the maxcut problem, $v_i$ goes to $X$ if $f_X < f_Y$. When $f_X = f_Y$, $A$ needs to have a tie breaking rule to decide where $v_i$ should go.

Such a greedy strategy is a common sense approach. The difficulty in studying such an algorithm is to come up with a simple tie breaking rule. It turns out that letting $v_i$ go with $v_{i-1}$ will make things work. To be more specific, in case $f_X = f_Y$, then $v_i$ goes to $X$ if $v_{i-1}$ went to $X$, and $v_i$ goes to $Y$ if $v_{i-1}$ went to $Y$. Let $A_{\min}$ and $A_{\max}$ be our greedy algorithms with this tie breaking rule for mincut and maxcut problems, respectively. For every graph $G$ with two or more vertices, we constructed two permutations $\pi_*$ and $\pi^*$ such that $A_{\min}(G, \pi_*)$ is a minimum cut of $G$, and $A_{\max}(G, \pi^*)$ is a maximum cut of $G$. To achieve this, we need the following graph theoretical result.

**Theorem 7.** Every loopless graph $G = (V, E)$ has a minimum cut $(X, Y)$ for which there exists a permutation $v_1, \ldots, v_n$ of $V$ such that the following conditions are satisfied. For each $i \geq 1$, let $X_i = X \cap \{v_1, \ldots, v_i\}$ and $Y_i = Y \cap \{v_1, \ldots, v_i\}$.

(i) $v_1 \in X$ and $v_2 \in Y$.

(ii) For every $i \geq 3$, if $v_i \in X$ then $|v_i, X_{i-1}| \geq |v_i, Y_{i-1}|$ and if $v_i \in Y$ then $|v_i, Y_{i-1}| \geq |v_i, X_{i-1}|$.

(iii) If $i \geq 3$ is minimum with $v_i \in X$ then $i = |Y| + 2$ and $|v_i, X_{i-1}| > |v_i, Y_{i-1}|$.

This theorem suggests the tie-breaking rule (R) we mentioned above:

(R) If $f_X = f_Y$ then $v_i$ goes to where $v_{i-1}$ went.

This is equivalent to the following rule.

(R) If $f_X = f_Y$ and $|X| = 1$ then $Y = Y \cup \{v_i\}$; if $f_X = f_Y$ and $|X| > 1$ then $X = X \cup \{v_i\}$.

Now we can formally describe our Greedy Algorithm.

Then the following is an immediate consequence of Theorem 7.
Algorithm 1: A Greedy proto-Algorithm MinCut

1: **Input:** A graph $G$.
2: **Output:** A cut of $G$.
3: Initialize: $X \leftarrow \{v_1\}$, $Y \leftarrow \{v_2\}$ and $i \leftarrow 0$.
4: **while** $2 < i \leq n$ **do**
5: $f_X = |v_i, X|$ and $f_Y = |v_i, Y|$ 
6: **if** $f_X > f_Y$ **then**
7: $X \leftarrow X \cup \{v_i\}$
8: **else if** $f_X < f_Y$ **then**
9: $Y \leftarrow Y \cup \{v_i\}$
10: **else**
11: Using tie breaker (R) to decide if $X \leftarrow X \cup \{v_i\}$ or $Y \leftarrow Y \cup \{v_i\}$.
   This decision is based on $G[\{v_1, \ldots, v_i\}]$
12: **end if**
13: **end while**
14: $i \leftarrow i + 1$

**Theorem 8.** For the online mincut problem there exists a greedy algorithm $A$ with the following property. For every loopless graph $G$ there exists a permutation of $V(G)$ such that when taking this permutation as its input $A$ produces a minimum cut.

To establish a similar result for the online MaxCut problem we need the following theorem.

**Theorem 9.** Every loopless graph $G = (V, E)$ has a maximum cut $(X, Y)$ for which there exists a permutation $v_1, \ldots, v_n$ of $V$ such that the following conditions are satisfied. For each $i \geq 1$, let $X_i = X \cap \{v_1, \ldots, v_i\}$ and $Y_i = Y \cap \{v_1, \ldots, v_i\}$.
(i) $v_1 \in X$ and $v_2 \in Y$.
(ii) For every $i \geq 3$, if $v_i \in X$ then $|v_i, X_{i-1}| \leq |v_i, Y_{i-1}|$ and if $v_i \in Y$ then $|v_i, Y_{i-1}| \leq |v_i, X_{i-1}|$.
(iii) If $i \geq 3$ and $|v_i, Y_{i-1}| = |v_i, X_{i-1}|$ then either $\{v_{i-1}, v_i\} \subseteq X$ or $\{v_{i-1}, v_i\} \subseteq Y$.

This theorem leads to the following.

**Theorem 10.** For online MaxCut there exists a greedy algorithm $A$ satisfying the following property. For every loopless graph $G$ there exists a permutation of $V(G)$ such that when taking this permutation as its input $A$ produces a maximum cut.

**Proof.** We consider the greedy algorithm $A$ given in Algorithm 2. To see that $A$ satisfies the theorem, for any loopless graph $G$, let partition $(X, Y)$ and permutation $v_1, \ldots, v_n$ be determined as in Theorem 9 Then $A$ produces exactly partition $(X, Y)$, which is a maximum cut, as required. 

Therefore, for MinCut and MaxCut problems, we established the existence of an algorithm $A$ with $\min_\pi A(G, \pi) = \text{opt}(G)$ for all $G$.

There are two related problems. Let $\Omega$ be a finite set and let $f$ be a function defined on all subsets of $\Omega$. If $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$ holds for all $X, Y \subseteq \Omega$ then $f$ is called a submodular function. Suppose $G = (V, E)$ is a graph and $f : 2^V \to \mathbb{R}$ such that $f(X)$ is the number of edges between $X$ and $V \setminus X$. Then it is not difficult to verify that $f$ is a submodular function. So minimizing and maximizing a submodular function can be considered as a generalization of mincut and maxcut. However, for the corresponding online problems there is a subtle difference. For the online submodular problem, if $\Omega'$ is the set of currently revealed elements, then the algorithm can
Algorithm 2 A Greedy Algorithm for MaxCut

1: **Input:** A graph $G$.
2: **Output:** A cut of $G$.
3: Initialize: $X \leftarrow \{v_1\}$, $Y \leftarrow \{v_2\}$ and $i \leftarrow 0$.
4: **while** $2 < i \leq n$ **do**
5: $f_X = |v_i, X|$ and $f_Y = |v_i, Y|$  
6: **if** $f_X < f_Y$ **then**
7: $X \leftarrow X \cup \{v_i\}$
8: **else if** $f_X > f_Y$ **then**
9: $Y \leftarrow Y \cup \{v_i\}$
10: **else**
11: Put $v_i$ where $v_{i-1}$ went.
12: **end if**
13: **end while**
14: $i \leftarrow i + 1$

access to $f(X)$ for all $X$ contained in $\Omega$. In contrast, if $V'$ is the set of currently revealed vertices and if $X \subseteq V'$, the algorithm cannot access to $f(X)$, it can only compute the number of edges between $X$ and $V' \setminus X$.

Nevertheless, we developed a greedy type algorithm $A^*$, which behaves very similar to $A_{\text{min}}$ and $A_{\text{max}}$. In particular, for every submodular function $f$, we constructed a permutation $\pi$ of $\Omega$ such that $A^*(f, \pi)$ is a subset of $\Omega$ that maximizes $f$. In other words, we establish that $\min_{\pi} A^*(f, \pi) = \text{opt}(f)$ holds for all submodular functions $f$. Finally, remark that no such $A^*$ exists for minimizing a submodular function. An example is presented in appendix C.4.

References

[1] Nikhil Bansal, Marek Eliš, Lukasz Jež, and Grigoris Koumoutsos. The (h, k)-server problem on bounded depth trees. *ACM Transactions on Algorithms (TALG)*, 15(2):1–26, 2019.

[2] Amotz Bar-Noy and Michael Lampis. Online maximum directed cut. *Journal of combinatorial optimization*, 24(1):52–64, 2012.

[3] Yair Bartal, Amos Fiat, and Stefano Leonardi. Lower bounds for on-line graph problems with application to on-line circuit and optical routing. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 531–540, 1996.

[4] Maria Paola Bianchi, Hans-Joachim Böckenhauer, Tatjana Brülisauer, Dennis Komm, and Beatrice Palano. Online minimum spanning tree with advice. *International Journal of Foundations of Computer Science*, 29(04):505–527, 2018.

[5] Allan Borodin and Ran El-Yaniv. *Online computation and competitive analysis*. cambridge university press, 2005.

[6] Joan Boyar, Lene M Fanhvoldt, Christian Kudahl, Kim S Larsen, and Jesper W Mikkelsen. Online algorithms with advice: a survey. *Acm Sigact News*, 47(3):93–129, 2016.
[7] Joan Boyar, Lene M Favrelholdt, Christian Kudahl, and Jesper W Mikkelsen. Advice complexity for a class of online problems. In 32nd International Symposium on Theoretical Aspects of Computer Science (STACS 2015). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2015.

[8] Chandra Chekuri, Shalmoli Gupta, and Kent Quanrud. Streaming algorithms for submodular function maximization. In International Colloquium on Automata, Languages, and Programming, pages 318–330. Springer, 2015.

[9] Sina Dehghani, Soheil Ehsani, MohammadTaghi Hajiaghayi, Vahid Liaghat, and Saeed Seddighin. Stochastic k-server: How should uber work? arXiv preprint arXiv:1705.05755, 2017.

[10] Stefan Dobrev, Rastislav Královíč, and Dana Pardubská. Measuring the problem-relevant information in input. RAIRO-Theoretical Informatics and Applications, 43(3):585–613, 2009.

[11] Reza Dorrigiv. Alternative measures for the analysis of online algorithms. 2010.

[12] Michal Forišek, Lucia Keller, and Monika Steinová. Advice complexity of online coloring for paths. In International Conference on Language and Automata Theory and Applications, pages 228–239. Springer, 2012.

[13] Magnús M Halldórsson and Mario Szegedy. Lower bounds for on-line graph coloring. Theoretical Computer Science, 130(1):163–174, 1994.

[14] Elad Hazan and Satyen Kale. Online submodular minimization. Journal of Machine Learning Research, 13(Oct):2903–2922, 2012.

[15] Benjamin Hiller and Tjark Vredeveld. Probabilistic alternatives for competitive analysis. Computer Science-Research and Development, 27(3):189–196, 2012.

[16] Juraj Hromkovič, Rastislav Královíč, and Richard Královíč. Information complexity of online problems. In International Symposium on Mathematical Foundations of Computer Science, pages 24–36. Springer, 2010.

[17] Elias Koutsoupias and Christos H Papadimitriou. Beyond competitive analysis. SIAM Journal on Computing, 30(1):300–317, 2000.

[18] Jan Remy, Alexander Souza, and Angelika Steger. On an online spanning tree problem in randomly weighted graphs. Combinatorics, Probability and Computing, 16(1):127–144, 2007.

[19] José A Soto. Matroid secretary problem in the random-assignment model. SIAM Journal on Computing, 42(1):178–211, 2013.

[20] Matthew Streeter and Daniel Golovin. An online algorithm for maximizing submodular functions. In Advances in Neural Information Processing Systems, pages 1577–1584, 2009.

Appendix

A Adversarial Input and Advice Complexity

A.1 Classical Model

Theorem 1. (i) Let A be an online algorithm for MinCut[Gk], where A knows n in advance. Then the following hold.
(a) If $k = 0$ then $A$ is not $c$-competitive for any $c$.

(b) If $k \geq 1$ and $A$ is $c$-competitive then $c \geq \frac{n-p}{k}$ for some $p \geq 1$.

(ii) Suppose $k \geq 1$. Then there exists an online algorithm $A$ for MinCut[$G_k$], where $A$ does not know $n$ in advance, such that $A$ is $\frac{n-p}{k}$-competitive for all $p \geq 1$.

Proof. (a) Let $G$ be obtained from $K_{n-1} \backslash e$ (where $n \geq 4$ and $e = xy$) by adding an isolated vertex $z$. The adversary first reveal two nonadjacent vertices $v_1, v_2$. If $A$ places $v_1, v_2$ in the same part of the partition, then the adversary can declare $v_1 = x$ and $v_2 = z$. In this case $A(G, \pi) \geq n - 3$.

If $A$ places $v_1, v_2$ in different parts of the partition then the adversary can declare $v_1 = x$ and $v_2 = y$. In this case $A(G, \pi) \geq n - 3$ holds again. If $A$ is $c$-competitive, then there exists a number $d$ independent of $G$ and $\pi$ such that $A(G, \pi) \leq c \cdot \text{opt}(G) + d$ holds for all our $G$ and $\pi$. It follows that $n - 3 \leq c \cdot 0 + d$ holds for all $n \geq 4$. This is impossible and thus $A$ is not $c$-competitive for any $c$.

(b) Since $A$ is $c$-competitive, there exists a constant $d$ satisfying $A(G, \pi) \leq c \cdot \text{opt}(G) + d$ for all $G \in G_k$ and all $\pi$ on $G$. Without loss of generality, we assume $d \geq 0$. Let $G$ be obtained from $K_{n-1} (n > k)$ by adding a new vertex $z$ and joining it to $k$ vertices of $K_{n-1}$. Then $\text{opt}(G) = k$ and thus $G$ belongs to $G_k$. The adversary first reveal two adjacent vertices $v_1, v_2$. If $A$ places $v_1, v_2$ in the same part of the partition, then the adversary can declare $v_1 = z$. In this case $A(G, \pi) \geq n - 2$.

If $A$ places $v_1, v_2$ in different parts of the partition then the adversary can declare that neither $v_1$ nor $v_2$ is $z$. In this case $A(G, \pi) \geq n - 2$ holds again. Let $p = d + 2$. Then $p \geq 1$. In addition, $n - 2 \leq c \cdot k + d$, implying $c \geq \frac{n-p}{k}$, as required.

(ii) Let $A$ be the following simple online algorithm for MinCut[$G_k$]: placing the first revealed vertex in the first part of the partition and all other vertices in the second part of the partition. Note that $A$ does not need to to know $|G|$ in advance. We now prove that $A$ is $\frac{n-p}{k}$-competitive for all $p \geq 1$. To do so, we choose $d = (p - 1) + \frac{(p-1)^2}{k}$ and we show that $A(G, \pi) \leq \frac{n-p}{k} \cdot \text{opt}(G) + d$ holds for all $G \in G_k$ and all $\pi$ on $G$, which will prove (ii). We consider two cases.

If $n \leq p$ then $A(G, \pi) \leq n - 1 \leq p - 1 \leq (p - 1) + \frac{(p-1)^2}{k} + \frac{n-p}{k} \cdot (n - 1) = \frac{n-p}{k} \cdot (n - 1) + d \leq \frac{n-p}{k} \cdot \text{opt}(G) + d$.

If $n > p$ then $A(G, \pi) \leq n - 1 = \frac{n-p}{k} \cdot k + (p - 1) \leq \frac{n-p}{k} \cdot \text{opt}(G) + (p - 1) \leq \frac{n-p}{k} \cdot \text{opt}(G) + d$.

Thus (ii) is verified.

\[\Box\]

A.2 Advice Complexity of MinCut

In theorem \[\text{2}\] and theorem \[\text{3}\] we give upper and lower bounds for the advice complexity of competitive algorithms.

Theorem 2. There is an competitive algorithm that finds a minimum cut with $n - 1$ bits of advice.

Proof. Let $A^{adv}$ define $X_1 = \{v_1\}$ and $Y_1 = \emptyset$ when it receives $v_1$. For each $i = 2, ..., n$, suppose $X_{i-1}$ and $Y_{i-1}$ have been constructed. When $v_i$ is revealed $A^{adv}$ asks: is $(X_{i-1} \cup \{v_i\}, Y_{i-1})$ extendable?

If the answer is yes then set $X_i = X_{i-1} \cup \{v_i\}$ and $Y_i = Y_{i-1}$; if the answer is no then set $X_i = X_{i-1}$ and $Y_i = Y_{i-1} \cup \{v_i\}$. At the end, $A^{adv}$ finds an optimal solution with $n - 1$ bits of advice.

\[\Box\]

Theorem 4. Let $A^{adv}$ be a $c$-competitive algorithm for MinCut[$G_k$] where $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. For every $k+1 \leq n < \frac{1}{2} (1 - \frac{1}{n})$, if $A^{adv}$ uses $b < n - 2[\epsilon n] - 1$ bits of advice then $c \geq \frac{cn-1}{k}$.

Proof. We will show that there is an infinite family $G(n)$ of graphs for which the theorem holds. Consider a graph $G \in G(n)$ as shown in Figure\[\text{2}\]. The induced subgraphs $G[A]$ and $G[B]$ are both cliques of size $p > k + 1$. We connect $A$ and $B$ via the sets $A'$ and $B'$. Since the minimum cut is
The total number of edges is \( O \). The induced subgraphs \( G[C] \) and \( G[D] \) are both independent sets and \( |C| + |D| \geq 2 \). Each vertex in \( C \) (resp. \( D \)) is adjacent to all vertices in \( A \) (resp. \( B \)). The adversary sends the vertices in the set \( C \cup D \) before sending any of the vertices in \( A \cup B \). Let \( \Gamma_{C \cup D} \) be the set of potential graphs after \( G[C \cup D] \) has been revealed. Depending on how the vertices in \( C \cup D \) are connected to \( A \cup B \) there are \( 2^{|C|+|D|}-1 = 2^{n-2p-1} \) pairwise different optimal solutions with a minimum cut of \( k \) corresponding to the set \( \Gamma_{C \cup D} \). This is essentially the same argument we used when proving theorem 3. With \( b \) bits of advice there are only \( 2^b \) possible advice strings. Hence there exists some advice string \( \phi \) which is read by \( A^{adv} \) for at least \( 2^{n-2p-1}/2^b \) pairwise different optimal solutions. Let this set be \( S \). If \( |S| > 1 \) then the adversary can fool \( A^{adv} \) in choosing a non-optimal solution. Suppose after reading \( \phi \), \( A^{adv} \) chooses a partition of \( C \cup D \) according to a solution \( (X', Y') \) (aka a partition of \( G \)) in \( S \). Then adversary sends the rest of \( G \) (aka the vertices in \( A \cup B \)) according to some other partition \( (X'', Y'') \in S \). Since \( A^{adv} \) has no means of distinguishing these to case based on the advice string \( \phi \) it will fail to optimally partition \( G \). It is easy to see that for any non-optimal partition \( (X', Y') \neq (X, Y) \) of \( G \) we have \( \lambda(X', Y') \geq p - 1 = \frac{\epsilon n}{2} \text{opt}(G) \). Thus we must have \( |S| \leq 1 \), which implies \( n - 2p - 1 - b \leq 0 \). Taking \( p = \lceil \epsilon n \rceil \) we see \( b \geq n - 2\lceil \epsilon n \rceil - 1 \) if \( A^{adv} \) to be less than \( \epsilon \)-competitive.

**B Semi-adversarial Inputs**

**B.1 Upper Bounds For Sparse**

In this section we present a result on sparse connected graphs. Sparseness here is defined to mean that the graph has linear number of edges. Many important families of graphs falls in this category such as planer graphs, degree bounded expanders etc.

**Algorithm 3** An algorithm for sparse graphs

1. **Input:** A sparse connected graph \( G \) with an vertex arrival order chosen uniformly at random.

2. **Output:** A cut of \( G \).

3. Initialize: \( X \leftarrow \emptyset \), \( Y \leftarrow \emptyset \), and \( i \leftarrow 1 \).

4. **while** \( i \leq n \) **do**

5.  **if** \( i == 1 \) **then**

6.  \( Y \leftarrow \{v_i\} \)

7.  **else**

8.  \( X \leftarrow X \cup \{v_i\} \)

9.  **end if**

10. \( i \leftarrow i + 1 \)

11. **end while**

**Theorem 6.** In the random order model there is \( O(1) \)-competitive algorithm in expectation for sparse connected graphs with \( O(n) \) edges.

**Proof.** We show algorithm 3 is \( O(1) \)-competitive. Suppose the graph \( G \) has \( O(n) \) edges and is connected. Algorithm 3 essentially puts a random vertex in \( Y \) and rest in \( X \). Since \( G \) is connected \( \lambda(G) \geq 1 \). Let \( v^* \) be the vertex chosen to be in \( Y \) and \( \text{E}[d(v^*)] \) be its expected degree. Let \( d_1 > \ldots > d_n \) be the degree sequence of \( G \). The number of vertices of degree \( d_i \) is at most \( n_i \). Since the total number of edges is \( O(n) \) we have \( \sum n_i d_i = 2|E| = O(n) \).
We have,
\[ \mathbb{E}[d(v^*)] = \sum_i \mathbb{P}[d(v^*) = d_i] d_i = \frac{1}{n} \sum_i n_i d_i = \frac{1}{n} O(n) = O(1) \]

Hence the competitive ratio is bounded.

Corollary 11. For a class of connected graphs with \( m \)-edges there is an \( O(m/n) \)-competitive algorithm.

Proof. Immediately follows from theorem 6.

C Greedy algorithms and Friendly Sequence Model

C.1 Online MinCut

Theorem 7. Every loopless graph \( G = (V, E) \) has a minimum cut \((X,Y)\) for which there exists a permutation \( v_1, ..., v_n \) of \( V \) such that the following conditions are satisfied. For each \( i \geq 1 \), let \( X_i = X \cap \{v_1, ..., v_i\} \) and \( Y_i = Y \cap \{v_1, ..., v_i\} \).

(i) \( v_1 \in X \) and \( v_2 \in Y \).

(ii) For every \( i \geq 3 \), if \( v_i \in X \) then \( |v_i, X_{i-1}| \geq |v_i, Y_{i-1}| \) and if \( v_i \in Y \) then \( |v_i, Y_{i-1}| \geq |v_i, X_{i-1}| \).

(iii) If \( i \geq 3 \) is minimum with \( v_i \in X \) then \( i = |Y| + 2 \) and \( |v_i, X_{i-1}| > |v_i, Y_{i-1}| \).

Proof. Let us choose a minimum cut \((X,Y)\) with \(|X|\) as small as possible. We prove that, with respect to this partition \((X,Y)\), there exists a permutation satisfying (i-iii).

Claim 1. If \(|X| = 1\) then the desired permutation exists.

Let \( v_1 \) be the unique member of \( X \) and let \( v_2 \) be an arbitrary vertex of \( Y \). We prove that there is a desired permutation starting with the two specified terms \( v_1, v_2 \). Note that no matter how the permutation \( v_3, ..., v_n \) is determined, conditions (i) and (iii) are always satisfied. So when we define \( v_3, ..., v_n \) we only need to ensure condition (ii), which is equivalent to: for each \( i \geq 3 \), \(|v_i, \{v_2, ..., v_{i-1}\}| \geq |v_i, v_1|\) holds.

We define permutation \( v_3, ..., v_n \) inductively. Suppose terms \( v_2, ..., v_{i-1} \) have been selected, where \( 3 \leq i \leq n \). Let \( Y' = \{v_2, ..., v_{i-1}\} \). We prove that there exists a vertex in \( Y \setminus Y' \), which we call \( v_i \), such that \(|v_i, Y'| \geq |v_i, v_1| \). Suppose otherwise that \(|y, Y'| < |y, v_1| \) holds for all \( y \in Y \setminus Y' \). Then \(|Y \setminus Y', Y'| < |Y \setminus Y', v_1| \), implying \(|Y', V \setminus Y'| = |Y', v_1| + |Y', Y' \setminus Y'| < |Y', v_1| + |Y' \setminus Y', v_1| = |X, Y| \), a contradiction. Thus \( v_i \) can be selected, and this proves Claim 1.

Claim 2. If \(|X| > 1\) then there exist distinct \( x, x' \in X \) with \(|x, x'| > |x', Y|\).

Suppose this is not the case. Then \(|x, x| \leq |x', Y| \) holds for all distinct \( x, x' \in X \). Consequently, for any fixed \( x \in X \), we have \(|X \setminus x, x| \leq |X \setminus x, Y| \), which implies \(|x, V \setminus x| = |x, Y| + |x, X \setminus x| \leq |x, Y| + |X \setminus x, Y| = |X, Y| \). This contradicts the minimality of \(|X|\) and thus Claim 2 is proved.

Now we are ready to construct the required permutation for the case \(|X| > 1\). Let \( x, x' \) be chosen as in Claim 2. Let \( v_1 = x \) and \( v_{|Y|+2} = x' \). We will make vertices of \( Y \) (specified below) \( v_2, ..., v_{|Y|+1} \) and vertices of \( X \) (also specified below) \( v_1, v_{|Y|+2}, ..., v_{|V|} \). Note that under this arrangement, conditions (i) and (iii) are satisfied.

To determine a permutation of \( Y \) we consider \( G' \) obtained from \( G \) by contracting \( X \) into a single vertex, which we denote by \( x^* \). Then \((\{x^*\}, Y)\) is a minimum cut of \( G' \). By Claim 1, vertices of \( Y \) can be permuted to satisfy (ii). Note that satisfying (ii) in \( G' \) and satisfying (ii) in \( G \) are the same thing for vertices of \( Y \). So we have obtained a required permutation for \( Y \). Similarly, to determine a permutation of \( X' = X \setminus \{x, x'\} \) we consider \( G'' \) obtained from \( G \) by contracting \( Y \) into a single vertex \( y^* \) and also contracting \( \{x, x'\} \) into a single vertex \( x^* \). Again, \((X' \cup \{x^*\}, \{y^*\})\) is a minimum
cut of $G''$. By Claim 1, vertices of $G''$ can be permuted to satisfy (ii), where if $u_1, ..., u_{|X|}$ is the permutation of $V(G'')$ then $u_1 = y^*$ and $u_2 = x^*$ (as shown in the proof of Claim 1). It follows that setting $v_{|Y| + i} = u_i$ ($3 \leq i \leq |X|$) results in a permutation of $V$ that satisfies (ii).

\[ \square \]

C.2 Online Undirected MaxCut

**Theorem 9.** Every loopless graph $G = (V, E)$ has a maximum cut $(X, Y)$ for which there exists a permutation $v_1, ..., v_n$ of $V$ such that the following conditions are satisfied. For each $i \geq 1$, let $X_i = X \cap \{v_1, ..., v_i\}$ and $Y_i = Y \cap \{v_1, ..., v_i\}$.

(i) $v_1 \in X$ and $v_2 \in Y$.

(ii) For every $i \geq 3$, if $v_i \in X$ then $|v_i, X_{i-1}| \leq |v_i, Y_{i-1}|$ and if $v_i \in Y$ then $|v_i, Y_{i-1}| \leq |v_i, X_{i-1}|$.

(iii) If $i \geq 3$ and $|v_i, Y_{i-1}| = |v_i, X_{i-1}|$ then either $v_{i-1}, v_i \subseteq X$ or $v_{i-1}, v_i \subseteq Y$.

**Proof.** We first observe that there exists $k \in \{2, ..., n\}$ for which there exists a maximum cut $(X, Y)$ and a permutation $v_1, ..., v_n$ of $V$ such that (i) is satisfied and (ii-iii) are satisfied for all $i \in \{3, ..., k\}$. To see this we only need to take $k = 2$ and take any maximum cut $(X, Y)$, any $v_1 \in X$, any $v_2 \in Y$, and any permutation of $V$ starting with $v_1v_2$.

Let us choose $k$ as large as possible under the above requirements. To prove the theorem we only need to show that $k = n$. Suppose on the contrary that $k < n$. Without loss of generality, let us assume $v_k \in Y$. If there exists $y \in Y \setminus Y_k$ with $|y, Y_k| \leq |y, X_k|$ then setting $v_{k+1} = y$ (with the same maximum cut $(X, Y)$) would contradict the maximality of $k$. So $|y, Y_k| > |y, X_k|$ holds for all $y \in Y \setminus Y_k$. Similarly, from the maximality of $k$ we deduce that $|x, Y_k| \leq |x, X_k|$ holds for all $x \in X \setminus X_k$. Consequently, we must have $Y_k = Y$ and $|X \setminus X_k, X_k| = |X \setminus X_k, Y_k|$ because otherwise $(X_k \cup (Y \setminus Y_k), Y_k \cup (X \setminus X_k))$ would be a cut bigger than $(X, Y)$, a contradiction. But then replacing $(X, Y)$ with $(X_k, Y_k \cup (X \setminus X_k))$ and setting $v_{k+1} = x$ for any $x \in X \setminus X_k$ would contradict the maximality of $k$. Therefore, we must have $k = n$ and thus the theorem is proven.

\[ \square \]

C.3 Definitions and Preliminaries

Let $E$ be a finite set. A function $f : 2^E \to \mathbb{R}$ is called submodular if

$$ f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) $$

holds for all $X, Y \subseteq E$. This can also be equivalently defined as

$$ f(X \cup \{y\}) + f(X \cup \{z\}) \geq f(X) + f(X \cup \{y, z\}) $$

holds for all $X \subseteq E$ and all distinct $y, z \in E \setminus X$. The second definition is the same as saying that $f_e(X) := f(X \cup \{e\}) - f(X)$ is a non-increasing function on $2^{E \setminus e}$ for all $e \in E$. In other words,

$$ f(X \cup \{e\}) - f(X) \geq f(Y \cup \{e\}) - f(Y) $$

holds for all $e \in E$ and all $X \subseteq Y \subseteq E \setminus e$.

**Remark.** In some papers like in [8, 13] the following definition of a submodular function is used:

$$ f(X \cup \{e\}) - f(X) \geq f(Y \cup \{e\}) - f(Y) $$

holds for all $e \in E$ and all $X \subseteq Y \subseteq E$. It is easy to see that this condition is the same as saying $f_e$ is non-increasing on $2^E$ for all $e \in E$. Note that this definition is not what we have above since they have different domains. In fact, it is easy to see that $f$ satisfies this definition if and only if $f$ is submodular and non-decreasing (i.e. $f(X) \leq f(Y)$ holds for all $X \subseteq Y \subseteq E$).
C.4 Online Submodular Maximization

We consider the following online model for maximizing a submodular function $f$. First, we assume that $f$ is given by an oracle. That is, for any $X \subseteq E$, obtaining the value of $f(X)$ does not require extra work. The objective of the maximization problem is to find a maximizer $X$ of $f$, which means that $f(X) = \max \{ f(Y) : Y \subseteq E \}$. We assume that elements of $E$ are revealed one by one. At each step, when a new element $e$ is revealed, the algorithm has to decide if or not to place $e$ in $X$. This is an irrevocable decision. In the following we present an analogue of Theorem 10.

**Lemma 12.** Let $f$ be a submodular function on a set $E$. If $X \subseteq E$ is a maximizer of $f$ then
(i) $f(Y) \leq f(Y \cup \{ x \})$ holds for every $x \in X$ and every $Y \subseteq X \setminus x$;
(ii) if $X$ is a maximal (under inclusion) maximizer of $f$ then $f(Y \cup \{ x \}) < f(Y)$ holds for all $x \in E \setminus X$ and all $Y \subseteq E \setminus x$ with $Y \supseteq X$.

**Proof.** Suppose (i) is false. Then $f(Y) > f(Y \cup \{ x \})$ holds for some $x \in X$ and some $Y \subseteq X \setminus x$. From the submodularity of $f$ we deduce that $f(X \setminus x) + f(Y \cup \{ x \}) \geq f(X) + f(Y)$, which implies $f(X \setminus x) \geq f(X) + f(Y) - f(Y \cup \{ x \}) > f(X)$, contradicting the maximality of $f(X)$.

Suppose (ii) is false. Then $f(Y) \leq f(Y \cup \{ x \})$ holds for some $x \in E \setminus X$ and some $Y \subseteq E \setminus x$ with $Y \supseteq X$. Again, by the submodularity of $f$ we have $f(X \cup \{ x \}) + f(Y) \geq f(X) + f(Y \cup \{ x \})$, and thus $f(X \cup \{ x \}) \geq f(X) + f(Y \cup \{ x \}) - f(Y) \geq f(X)$. This implies that $X \cup \{ x \}$ is a maximizer of $f$, contradicting the choice of $X$.

**Theorem 13.** There exists an online greedy algorithm $A$ with the following property. For any submodular function $f$ defined on a finite set $E$, there exists a permutation of $E$ such that when taking this permutation as its input $A$ produces a maximizer of $f$.

**Algorithm 4** A greedy algorithm for maximizing a submodular function online

1: **Input:** A submodular function $f$.
2: **Output:** A subset of $E$.
3: Initialize: $X \leftarrow \emptyset$ and $i \leftarrow 0$.
4: while $0 < i \leq |E|$ do
5: if $f(X \cup \{ e_i \}) \geq f(X)$ then
6: $X \leftarrow X \cup \{ e_i \}$
7: end if
8: $i \leftarrow i + 1$
9: end while

**Proof.** We consider the above greedy algorithm $A$ as shown above. Let $e_1, e_2, \ldots$ be the input sequence.

To see that $A$ satisfies the requirements, for any submodular function $f$ defined on $E$, let $X \subseteq E$ be a maximal maximizer of $f$. Consider a permutation of $E$ such that its first $|X|$ elements are from $X$. Then the result follows immediately from Lemma 12.

**Remarks.** 1. We called Theorem 13 an analogue of Theorem 10 because they both deal with submodular functions and elements of the ground set are received one by one. However, we should point the main difference between them. In the MaxCut problem, if we use $f$ to denoted submodular function defined on $V(G)$, that is, $f(X) = |X, V \setminus X|$, we can see that at each iteration, we do not really now the values of $f(X)$. Instead, what we have is an approximation of it.
2. One may naturally ask for an analogue of Theorem 8. But such a result do not exist, as shown by the following example. Consider a function $f$ defined on $E = \{x, y\}$, with $f(\emptyset) = 0$, $f(x) = f(y) = 1$, and $f(E) = -1$. This function is submodular: to see it we only need to verify inequalities $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for incomparable subsets $X, Y$ of $E$. But this is clear since there is only such inequality $f(x) + f(y) = 2 \geq -1 = f(E) + f(\emptyset)$.

Observe that $E$ is the unique minimizer for $f$, and upto symmetry, there is only one permutation $xy$ of $E$. However, the values of the corresponding subsets $\emptyset, \{x\}, \{x, y\}$ are 0, 1, $-1$. Therefore, any greedy algorithm would return $\emptyset$ as the minimizer, which is not the real minimizer.