Level compressibility for the Anderson model on regular random graphs and the absence of non-ergodic extended eigenfunctions

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We calculate the level compressibility \(\chi(W, L)\) of the energy levels inside \([-L/2, L/2]\) for the Anderson model on infinitely large regular random graphs with on-site potentials distributed uniformly in \([-W/2, W/2]\). We show that \(\chi(W, L)\) approaches the limit \(\lim_{L\to\infty} \chi(W, L) = 0\) for a broad interval of the disorder strength \(W\) within the extended phase, including the region of \(W\) close to the critical point for the Anderson transition. These results strongly suggest that the energy levels follow the Wigner-Dyson statistics in the extended phase, which implies on the absence of non-ergodic extended wavefunctions. This picture is consistent with earlier analytical predictions derived using the supersymmetric method. Our results are obtained from the accurate numerical solution of an exact set of equations for the level-compressibility of infinitely large regular random graphs.

Introduction. Despite more than fifty years since the seminal work of Anderson [1], the localization of single particle wavefunctions in disordered quantum systems continues to attract a significant interest [2]. Exactly solvable models naturally have played a crucial role in the understanding of the transition between localized and extended wavefunctions. One of the most important models consists of a single particle hopping among the nodes of an infinitely large Cayley tree with on-site disorder [3]. In contrast to its counterpart in finite dimensions, this mean-field version of the Anderson model is exactly solvable thanks to the absence of loops.

The statistics of energy levels and eigenfunctions of the Anderson model on locally tree-like random graphs have lately re-emerged as a central problem in condensed matter theory. This is due to the connection between this class of models and localization in interacting many-body systems. Essentially, the structure of localized wavefunctions in the Fock space of many-body quantum systems can be mapped on the localization problem of a single particle hopping on a tree-like graph with quenched disorder [4–6]. The phenomena of many-body localization and ergodicity breaking in isolated quantum systems prevent them to equilibrate, which has serious consequences for the foundations of equilibrium statistical mechanics [7, 8].

The prototypical model to inspect the statistics of energy levels and eigenfunctions in the Anderson model is realized on a regular random graph (RRG). Regular random graphs have a local treelike structure, but loops containing typically \(O(ln N)\) sites are present. Another difference of a RRG with respect to a Cayley tree (both with finite \(N\)) is the absence of boundary nodes in the former case. The majority of sites of a Cayley tree lies on its boundary, which pathologically influences the eigenfunctions within the delocalized phase [9, 10]. Although the complete characterization of the phase diagram of the Anderson model on a RRG is still a work in progress [11–13], it is well established that the extended phase appears at the center of the band as long as the disorder strength \(W\) is smaller than a critical value \(W_c\) [3, 14].

Recently there has been an intense debate concerning the ergodicity of the eigenfunctions within the extended phase of the Anderson model on RRGs and two main pictures have emerged. At one side, it has been put forward that, for a certain interval \(W_E < W < W_c\), there exists an intermediate phase where the eigenfunctions are multifractal [14–17] and the inverse participation ratio scales as \(N^{-\tau(W)}\) \((N \gg 1)\), with \(0 < \tau(W) < 1\) [16, 17]. The results supporting this picture are mostly based on a numerical diagonalization study of the fractal exponents [15–17], combined with a semi-analytical approach to solve the self-consistent equations derived in [3]. The transition between ergodic and non-ergodic extended eigenstates at \(W_E\) is discontinuous [16, 17], analogous somehow to the one-step replica symmetry breaking transition observed in some spin-glass systems [18, 19].

According to the other side, the inverse participation ratio scales as \(1/N\) \((N \gg 1)\) and the energy-levels follow the Wigner-Dyson statistics within the whole extended phase [10, 20, 21]. The results supporting the ergodicity of the extended eigenstates are mainly based on numerical diagonalization [10, 20]. The statistical properties of the energy levels and eigenfunctions display a non-monotonic behavior for increasing \(N\), reducing essentially the non-ergodicity of the eigenfunctions to a finite size effect. This picture is consistent with earlier analytical predictions for the problem of a single quantum particle hopping on an Erdős-Renyi random graph [22, 23], a treelike model closely related to the Anderson model on a RRG.

In this work we add an important contribution to this heated debate following a path which we believe is conceptually simpler. We probe the ergodicity of the wavefunctions by solving an exact set of equations for the level compressibility \(\chi\) of
the number of energy levels inside the interval $[-L/2, L/2]$. By considering the limit $L \to 0^+$ (see below), this quantity allows to distinguish among the three conventional statistical behaviors of the energy levels found in Anderson models: Wigner-Dyson statistics (extended and ergodic states) [24], Poisson statistics (localized and non-ergodic states) [24] and sub-Poissonian statistics (multifractal and non-ergodic states) [25–28]. We calculate $\chi$ as a function of $L \ll 1$ within the extended phase, including some values of $W < W_c$ in an interval of disorder strength close to the critical point. This is the relevant regime of $W$ where one would expect the presence of multifractal eigenstates, according to recent numerical results [16]. Our results consistently show that $\chi$ approaches zero in the limit $L \to 0^+$ for all values of $W < W_c$ considered here, which strongly supports the absence of multifractal states for $W < W_c$.

The level-compressibility has a non-monotonic behavior as a function of $L$, which resembles the system size dependency discussed in previous works [16, 20]. Our approach is based on the numerical solution of an exact set of equations derived previously through the replica-symmetric method and valid in the limit $N \to \infty$ [29]. We discuss the possible role of replica symmetry breaking in $\chi$.

The Anderson model on a regular random graph. The tight-binding Hamiltonian for a spinless particle moving on a random potential is given by

$$\mathcal{H} = -\sum_{ij=1}^{N} t_{ij} \left( c_i^\dagger c_j + c_j^\dagger c_i \right) \sum_{i=1}^{N} \epsilon_i c_i^\dagger c_i , \quad \text{(1)}$$

where $t_{ij}$ is the energy for the hopping between nodes $i$ and $j$, while $\epsilon_1, \ldots, \epsilon_N$ are the on-site random potentials drawn from the uniform distribution $P_\delta(\epsilon) = (1/W)\delta(W/2 - |\epsilon|)$, with $W \geq 0$. The hopping coefficients $\{t_{ij}\}_{i,j=1,\ldots,N}$ correspond to the entries of the adjacency matrix of a regular random graph (RRG) with connectivity $k + 1$ [30, 31]. A matrix element $t_{ij}$ is equal to one if there is a link between nodes $i$ and $j$, and $t_{ij} = 0$ otherwise. The ensemble of random graphs can be defined through the full distribution of the adjacency matrix elements $\{t_{ij}\}_{i,j=1,\ldots,N}$. For the explicit form of this distribution, we refer the reader to [29]. For $k = 2$, where each node is connected precisely to three neighbors, all eigenfunctions become localized provided $W > W_c \approx 17.4$ [3, 11]. The value of $W_c$ is the same for the infinitely large Cayley tree and the RRG.

The level-compressibility. Let $\mathcal{I}_L^{(N)}$ denotes the number of energy levels inside $[-L/2, L/2]$

$$\mathcal{I}_L^{(N)} = N \int_{-L/2}^{L/2} dE \rho_N(E) , \quad \text{(2)}$$

where $\rho_N(E) = (1/N) \sum_{i=1}^{N} \delta(E - E_i)$ is the density of energy levels $E_1, \ldots, E_N$ between $E$ and $E + dE$. We define the $N \to \infty$ limit of the level-compressibility as follows [24, 28]

$$\chi(L, W) = \lim_{N \to \infty} \frac{n^{(N)}(L)}{\mathcal{I}_L^{(N)}}, \quad \text{(3)}$$

with the number variance

$$n^{(N)}(L) = \left< \left( \mathcal{I}_L^{(N)} \right)^2 \right> - \left< \mathcal{I}_L^{(N)} \right>^2 \quad \text{(4)}$$

characterizing the fluctuation of the energy levels within $[-L/2, L/2]$. The symbol $\langle \ldots \rangle$ represents the ensemble average with respect to the graph distribution and the distribution of the on-site potentials.

Let us consider the behavior of $\chi(L, W)$ when $L = s/N$, i.e., the interval $[-L/2, L/2]$ is measured in units of the mean level spacing $1/N$. For ergodic eigenfunctions, the corresponding energy levels strongly repel each other and the number variance scales as $n^{(N)}(L) \propto \ln \left( \mathcal{I}_L^{(N)} \right)$ ($s \gg 1$), yielding $\chi(L, W) = 0$ [24]. In the case of localized eigenfunctions, the energy levels are uncorrelated random variables with a Poisson distribution, the number variance is given by $n^{(N)}(L) = \left( \mathcal{I}_L^{(N)} \right)$ ($s \gg 1$) and, consequently, we have $\chi(L, W) = 1$ [24]. Finally, if the eigenstates are multifractal, the number variance scales linearly with $\mathcal{I}_L^{(N)}$ ($s \gg 1$), but $0 < \chi(L, W) < 1$ [25–28]. Thus, the level-compressibility is a suitable quantity to distinguish among ergodic, localized and multifractal eigenstates.

The first $\kappa_1^{(N)}$ and second $\kappa_2^{(N)}$ cumulants of the random variable $\mathcal{I}_L^{(N)}$ read

$$\kappa_1^{(N)}(L, W) = \frac{\partial^2 \mathcal{I}_L^{(N)}(y)}{\partial y^2} \big|_{y=0} = \frac{\left< \mathcal{I}_L^{(N)} \right>}{N} , \quad \text{(5)}$$

$$\kappa_2^{(N)}(L, W) = -\frac{\partial^2 \mathcal{I}_L^{(N)}(y)}{\partial y^2} \big|_{y=0} = \frac{n^{(N)}(L)}{N} , \quad \text{(6)}$$

where the cumulant generating function $\mathcal{F}_L^{(N)}(y)$ for the statistics of $\mathcal{I}_L^{(N)}$ is given by

$$\mathcal{F}_L^{(N)}(y) = -\frac{1}{N} \ln \left< e^{-y \mathcal{I}_L^{(N)}} \right> . \quad \text{(7)}$$

Substituting eqs. (5) and (6) in eq. (3), we see that the level-compressibility can be written in terms of the cumulants

$$\chi(L, W) = \frac{\kappa_2^{(N)}(L, W)}{\kappa_1^{(N)}(L, W)} , \quad \text{(8)}$$

with $\kappa_{1,2}(L, W) \equiv \lim_{N \to \infty} \kappa_{1,2}^{(N)}(L, W)$. Thus, the calculation of $\chi(L, W)$ reduces to evaluate $\mathcal{F}_L^{(N)}(y)$ in the limit $N \to \infty$, from which the first and second cumulants are readily obtained.

Here we briefly recall the analytical approach to calculate $\lim_{N \to \infty} \mathcal{F}_L^{(N)}(y)$ and then we present the final equations for the first and second cumulants. A detailed account of this
computation is presented in [29]. Our first task consists in expressing $\mathcal{F}_L^{(N)}(y)$ in terms of the disordered Hamiltonian $\mathcal{H}$, such that we are able to compute analytically the ensemble average $\langle \ldots \rangle$ in eq. (7). This is achieved by representing the Heaviside step function $\Theta(x)$ ($x \in \mathbb{R}$) in terms of the discontinuity of the complex logarithm along the negative real axis, i.e., $\Theta(-x) = \frac{1}{2\pi i} \lim_{\eta \to 0^+} [\ln(x + i\eta) - \ln(x - i\eta)]$. Using this prescription in eq. (2), we derive

$$\mathcal{I}_L^{(N)} = \frac{1}{\pi i} \lim_{\eta \to 0^+} \ln \left[ \frac{Z(L/2 - i\eta)Z(-L/2 + i\eta)}{Z(L/2 + i\eta)Z(-L/2 - i\eta)} \right],$$  \tag{9}$$

where $Z(z) = [\det(\mathcal{H} - z\mathbb{I})]^{-1/2}$ ($z \in \mathbb{C}$), with $\mathbb{I}$ the $N \times N$ identity matrix, $(\cdots)^*$ the complex conjugation, and $\eta > 0$ a regularizer. Combining eqs. (9) and (7), one can write

$$\mathcal{F}_L^{(N)}(y) = -\lim_{\eta \to 0^+} \frac{1}{N} \ln Q_L^{(N)}(y),$$  \tag{10}$$

with

$$Q_L^{(N)}(y) = \left\langle \frac{Z^{2\eta} (L/2 + i\eta)Z^{2\eta} (-L/2 - i\eta)}{Z^{2\eta} (L/2 - i\eta)Z^{2\eta} (-L/2 + i\eta)} \right\rangle.$$  \tag{11}$$

The ensemble average in eq. (11) can be calculated analytically using the replica approach of spin-glass theory [29, 32]. The limit $N \to \infty$ of $\mathcal{F}_L^{(N)}(y)$ follows from the solution of a saddle-point integral, in which we have restricted our analysis to those saddle-points that are replica symmetric, i.e., invariant with respect to the permutation of two or more replicas. For all details involved in the calculation of $\lim_{N \to \infty} \mathcal{F}_L^{(N)}(y)$, we refer the reader to the supplemental material of [29].

Following this approach we find the expressions for the first two cumulants:

$$\kappa_1(L, W) = \frac{1}{\pi} \lim_{\eta \to 0^+} \left[ \frac{(k + 1)}{2} \langle F \rangle_{\nu} - \langle R \rangle_{\mu} - (k + 1) \langle R \rangle_{\nu} \right],$$  \tag{12}$$

with the functions $R = R(u, v)$ and $F = F(u, v; u', v')$

$$R(u, v) = \frac{i}{2} \ln \left[ \frac{uv}{(uv)^*} \right],$$  \tag{14}$$

$$F(u, v; u', v') = R(u, v) + R(u', v') + \varphi(u, u') + \varphi(v, v'),$$  \tag{15}$$

and

$$\varphi(u, u') = -\frac{i}{2} \ln \left[ \frac{1 + uu'}{(1 + uu')^*} \right].$$  \tag{16}$$

The average $\langle \ldots \rangle_{\mathcal{P}}$ of integer powers of $R(u, v)$ and $F(u, v; u', v')$ with an arbitrary distribution $\mathcal{P}$ is defined by the general formula

$$\langle R^m F^n \rangle_{\mathcal{P}} = \int du dv dv' du' \mathcal{P}(u, v) \mathcal{P}(u', v') R^m(u, v) F^n(u, v; u', v'),$$  \tag{17}$$

where $m \geq 0$ and $n \geq 0$. The distributions $\mu(u, v)$ and $\nu(u, v)$, which enter in the averages appearing in eqs. (12) and (13), are determined from the self-consistent equations
where $\langle \ldots \rangle_n$ is the average over the on-site random potentials.

Equations (12-19) are exact for $N \to \infty$ and $L = \mathcal{O}(1)$ fixed, independently of the system size $N$, and the level-compressibility is evaluated with high accuracy using the population dynamics algorithm [29]. However, as we discussed above, we should calculate $\chi(L, W)$ at the scale $L = \mathcal{O}(1/N)$ in order to probe the presence of non-ergodic extended states. The reason is twofold: (i) by considering the regime $L = \mathcal{O}(1/N)$, we are inspecting the fluctuations of low-lying energies of $O(1/N)$ (or long time scales of the order $O(N)$, much larger than the typical size $\ln N$ of the loops); (ii) the average density of states $\langle \rho(E) \rangle$ is uniform along an interval of size $L = \mathcal{O}(1/N)$, and we avoid the spurious influence on $\chi(L, W)$ of significant variations of $\langle \rho(E) \rangle$ [29].

In principle, one should employ the formalism of [29] and determine the cumulants when $L = s/N$ ($s \gg 1$). However, one immediately concludes that the terms arising from rescaling $L \to s/N$ and $\eta \to \eta/N$ in the formal development of [29] enter only in an eventual calculation involving finite size corrections, i.e., when one considers $N$ large but finite [33, 34]. Thus, the leading behavior of the level-compressibility $\lim_{N \to \infty} \kappa_2^{(N)} / \kappa_1^{(N)}$ in the scaling regime $L = \mathcal{O}(1/N)$ should already be given by eqs. (12-13) in the limit $L \to 0^+$ and $\eta \to 0^+$. The central idea here is to explore numerically this limit using population dynamics. Note that the imaginary part of the energy $\eta$ is also going to zero and the interesting limit is $L \to 0^+$ and $\eta \to 0^+$, keeping the ratio $L/\eta$ large. Essentially, $\eta$ plays the role of the level spacing in a regularized density of states and we want to have many levels within $[-L/2, L/2]$.

Due to the $\eta$-dependency of eqs. (12-19), it is convenient to introduce the level compressibility $\chi_0(W, L)$ for fixed $\eta$. For a given value of $L$ and $W$, we have

$$
\chi(W, L) = \lim_{\eta \to 0^+} \chi_0(W, L).
$$

Henceforth, we restrict ourselves to $k = 2$. For this connectivity, the eigenfunctions at the center of the band undergo an Anderson localization transition at $W_c = 17.4$ [3, 11].

In figure 1 we present results for $\chi_0(W, L)$ as a function of $W$ for fixed $\eta = 10^{-6}$ and different values of the size $L$ of the interval. As $L$ decreases, it is clear that $\chi_0(W, L)$ converges to $\chi_0(W, L) = 1$ or $\chi_0(W, L) = 0$ for $W > W_c$ or $W < W_c$, respectively, as long as $W$ is not too close to the critical value $W_c \approx 17.4$. Importantly, one observes a non-monotonic behavior of $\chi_0(W, L)$ as a function of $L$ for some values of $W < W_c$. This is particularly evident for $W = 10$ and $W = 12.5$. However, it is not clear from figure 1 that $\eta$ is small enough such that the limit $\eta \to 0^+$ has been reached, especially close to the critical point.

In order to have reliable data in the delocalized phase $W < W_c$, it is crucial to understand the dependence of $\chi_0(W, L)$ with respect to $\eta$. We have thus solved eqs. (12-19) for several values of $\eta$, keeping $L$ fixed [35]. For sufficiently small $\eta < \eta_0 \sim L, 1 - \chi_0(W, L)$ can be well fitted by the function $\chi_0 + a\eta^b$, where the fitting parameters $\chi_0(W, L), a(W, L)$ and $b(W, L)$ are determined with high accuracy [35]. This procedure allows to obtain $\chi(W, L) = \lim_{\eta \to 0^+} \chi_0(W, L)$ simply by reading the value of $\chi_0$. Performing this numerical computation for many values of $W$ and $L$ is highly time consuming, so we have focused on some values of $W$ within the extended phase for which the eigenfunctions would be multifractal, according to recent works [15-17].

The main outcome of this calculation is shown in figure 2, where we show $\chi(W, L)$ as a function of $L$. As we approach $W_c$ from the delocalized side, the level compressibility $\chi(W, L)$ behaves non-monotonically as a function of $L$: initially it tends to its Poisson value $\chi(W, L) = 1$, but as $L$ is further decreased, the level compressibility clearly moves towards the limit $\lim_{L \to 0^+} \chi(W, L) = 0$, character-
The issue of replica symmetry breaking could arise in the limit \( L \to 0^+ \), since we expect to approach the local scale of \( L = s/N (s \gg 1) \) characterizing the mean level spacing. From the replica calculation of the connected part of the two-level correlation function \( R^{(c)}(s) \) for the GUE ensemble \([41]\), the replica-symmetric saddle-point yields the decay \( R^{(c)}(s) \propto 1/s^2 \), which already gives the leading contribution \( \chi(W, s/N) \propto s^{-1} \ln s \xrightarrow{s \to \infty} 0 \) \([24, 28]\). The inclusion of saddle-points that break replica symmetry allow to capture the oscillatory behavior of \( R^{(c)}(s) \) \([41]\), which does not affect the leading value \( \lim_{s \to \infty} \chi(W, s/N) = 0 \), but only introduces sub-leading corrections due to finite \( s \). We expect the situation to be similar for the Anderson model on regular random graphs, i.e., replica symmetry breaking is important only when one is interested in sub-leading corrections for finite \( s \).

This is also supported by the absence of many solutions to the cavity or population dynamics eqs. \((18)\) and \((19)\), which is a common sign of replica symmetry breaking \([32]\).

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