Short chaotic strings and their behaviour in the scaling region

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Coupled map lattices are a paradigm of higher-dimensional dynamical systems exhibiting spatio-temporal chaos. A special case of non-hyperbolic maps are one-dimensional map lattices of coupled Chebyshev maps with periodic boundary conditions, called chaotic strings. In this short note we show that the fine structure of the self energy of this chaotic string in the scaling region (i.e. for very small coupling) is retained if we reduce the length of the string to three lattice points.

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I. INTRODUCTION

Coupled map lattices (CMLs) as introduced by Kaneko and Kapral [1,2] are a paradigm of higher-dimensional dynamical systems exhibiting spatio-temporal chaotic behaviour. There is a variety of applications for CMLs to model hydrodynamical flows, turbulence, chemical reactions, biological systems, and quantum field theories (see e.g. reviews in [3,4]). For hyperbolic maps with slope of the local maps always larger than one a variety of analytical results exists [5,6,7,8]. However, promising results have been achieved also for the physically more interesting non-hyperbolic maps [9,10,11,12,13,14,15,16]. Local non-hyperbolic maps given by N-th order Chebyshev polynomials promise to answer a long-standing problem of elementary particle physics, namely the mass spectrum of elementary particles. With surprising evidence it could be shown that the self-energy of the chaotic string, i.e. a one-dimensional lattice based on diffusely coupled Chebyshev maps, shows a fine structure with minima that can be related to the fundamental mass parameters of the standard model of elementary particle physics [4,13]. It was recently shown that this fine structure disappears if we use a lattice of more than one dimension [17]. In addition the chaotic string can serve as a useful model for vacuum fluctuations and dark energy in the universe [18].

It is interesting to know whether under special circumstances a long one-dimensional lattice with periodic boundary conditions can be replaced by a short one. Dettmann et al. studied a string of length \( L = 2 \), i.e. the case of two coupled Chebyshev maps [19] using periodic orbit theory. In this note we examine short strings with periodic boundary conditions (closed strings) as well as strings with open ends (open strings) and fixed ends (fixed strings). Our central result is that closed strings of length \( L \geq 3 \) as well as other short string configurations lead to the same self energy fine structure in the scaling region (i.e. for the coupling close to zero) as for the case of maps with rather long lengths (\( L = 10000 \)). This fact supports hopes for understanding the reason for the fine structure in the scaling region.

II. APPROACHING THE SCALING REGION

In this note we consider the CML given by the coupled Chebyshev map system

\[
\phi_i^{n+1} = (1-a)T_2(\phi_i^n) + \frac{a}{2}(T_2(\phi_{i+1}^n) + T_2(\phi_{i-1}^n))
\]

which is denoted as 2A-string in Ref. [4]. The index \( n \) counts the iterations while \( i \) names the position on the one-dimensional string. We are calling these numbers the temporal and the spatial position. Periodic boundary conditions close the string by implying \( \phi_{L+1}^n = \phi_1^n \) where \( L \) is the string length. \( a \) is the coupling and \( T_2(\phi) \) is the second order Chebyshev polynomial

\[
T_2(\phi) = 2\phi^2 - 1.
\]

It can be shown that in the scaling region (\( a \) small) the 2B-string (with the last two Chebyshev polynomials in Eq. (1) replaced by identities) has the same behaviour.

The self energy of the chaotic string is the temporal and spatial average

\[
\langle V(\phi) \rangle_a = \langle \phi - \frac{2}{3}\phi^3 \rangle.
\]

It is obtained by performing the map (1) iteratively and taking the average over all values \( \phi_i^n \) on the string. Obviously, the precision of the result is growing with the product of the number of iterations and the length of the (closed) string. The self energy depends on the coupling \( a \) in a non-obvious way. In Fig. 1 we show the self energy for a closed string of length \( L = 10000 \), a closed string of length \( L = 3 \), and an open string of length \( L = 4 \).
The latter is given by the map \( I \) for \( i = 2, 3 \) combined with \( \phi_{i+1}^n = T_2(\phi_{i}^n) \) and \( \phi_{i}^{n+1} = T_2(\phi_{i}^n) \). The boundary values are not influenced but themselves influence the central value. Because of the ergodicity of the Chebyshev map they are distributed according to the invariant probability density \( \rho_0(\phi) = \frac{1}{\pi \sqrt{1 - \phi^2}} \) \[ (4) \]

Looking at Fig. 1 we see that close to \( a = 0 \) the differences between the three results seem to vanish. For larger values of \( a \) the self energy of the closed string of length \( L = 3 \) disagrees to positive values, the self energy of the open string to negative values. The noticeable disagreement between the closed strings for length \( L = 10000 \) and \( L = 3 \) close to \( a = 0.04 \) is due to the formation of stable orbits on the short string. A detailed analysis of the interval \( a \in [0.039, 0.043] \) shows that stable orbits of different periods interchange with chaotic behaviour. However, we do not dwell on this point here because the value \( a = 0.04 \) is still considered to be outside of the scaling region.

### III. FINE STRUCTURE OF THE SELF ENERGY

The name “scaling region” for the region close to \( a = 0 \) is due to the fact that we observe the scaling behaviour

\[
(V(\phi))_a - (V(\phi))_0 = f^{(N)}(\ln a) \sqrt{a} \tag{5}
\]

where \( f^{(N)} \) is a periodic function of \( \ln a \) with period \( \ln N^2 \) \[ 4]. Using perturbative methods, this scaling behaviour could be shown in Refs. \[ 15, 16 \]. Looking at a single period for the 2A-string (i.e. \( N = 2 \)) we observe a fine structure of the self energy which is not yet fully understood. However, the local minima of the self energy could be shown to be related to Yukawa and gravitational couplings for all quark and lepton flavours modulo \( N^2 = 4 \). Therefore, the fine structure of the self energy as shown in Fig. 2 is of central interest for us.

### IV. CLOSED STRING OF LENGTH \( L = 2 \)

If we reduce the string length to two points as in Ref. \[ 19 \] by keeping the periodic boundary conditions we observe a quite different shape of the self energy (Fig. 3). A fine structure is still visible but is different from the one for the long string. Relative minima are reallocated and, therefore, are not related to physical parameter values. The self-similar structure is much more evident as in the other cases. The enhancement close to \( a \approx 0.000577 \) (similar to the one for the closed string of length \( L = 3 \) near \( a = 0.04 \)) is caused by stable orbits interchanged with chaotic behaviour. At \( a = 0.0005768 \) for instance we find an orbit of period 6 which reads

\[
+0.7070 \quad +0.9954 \\
+0.0003 \quad +0.9811 \\
-0.9989 \quad +0.9238 \\
+0.9954 \quad +0.7070 \\
+0.9811 \quad +0.0003 \\
+0.9238 \quad -0.9989 
\tag{6}
\]

The deviation between the fine structures in Figs. 2 and 3 can be measured by the mean value of the differences of the self energy value mediated over \( a \in [0.00015, 0.00060] \) which is given by \((-1.2 \pm 10.1) \times 10^{-3} \). The distribution function of this difference measured on the interval \( a \in [0.00015, 0.00060] \) differs very much from a Gaussian distribution function (cf. Fig. 4). While the large standard deviation is due to the orbit region, the shape of the distribution function and the large value of the order of \( 10^{-3} \) for the mean value indicates the visible deviation of the two fine structures.

### V. CLOSED STRING OF LENGTH \( L = 3 \)

If we are looking at the closed string of length \( L = 3 \), we observe that the fine structure is recovered (Fig. 5). The mean value of the differences \((0.2 \pm 4.4) \times 10^{-3} \) is
FIG. 3: Self energy of the 2A-string of length \( L = 2 \) with periodic boundary conditions in the range \( a \in [0.00015, 0.0006] \), the number of iterations being \( 3 \times 10^7 \).

FIG. 4: Distribution function for the difference values of the self energies of the closed strings for \( L = 2 \) and \( L = 10000 \) mediated over the intervall \( a \in [0.00015, 0.0006] \). The standard deviation is due to statistical errors which could be reduced by calculating the two fine structures to higher accuracy. Finally, the distribution function is very much of the shape of a Gaussian curve (Fig. 6). The same is true for other closed strings of short length, for instance for a string consisting of \( L = 4 \) or \( L = 6 \) points (mean difference values are \((0.3 \pm 6.6) \times 10^{-5}\) and \((0.2 \pm 5.7) \times 10^{-5}\), resp.).

VI. SHORT OPEN STRINGS

If we replace the periodic boundary conditions by the open string boundary conditions (map (1) for \( i \neq 1, L \) together with \( \phi_{n+1}^{i} = T_2(\phi_{n}^{i}) \) and \( \phi_{n+1}^{L} = T_2(\phi_{n}^{L}) \)), we obtain a different picture (cf. Fig. 7). The open string of length \( L = 3 \) corresponds to the perturbative settings given in Refs. [15, 16]. A fine structure is visible but is again different from the one for the closed string. This fact is indicated also by the mean difference \((1.6 \pm 6.2) \times 10^{-5}\) and the distribution function which is divided up into two maxima. However, if we use an open string of length \( L = 4 \) (Fig. 8), the original pattern of Fig. 2 is recovered again (mean difference \((1.6 \pm 6.2) \times 10^{-5}\)). In both cases we observe some scattered points which deviate from the main distribution. An analysis shows that unlike the case of closed strings, these points do not represent stable orbits. Instead, their occurrence depends on the starting conditions and would vanish if we sum over a sufficient large set of random starting values.

VII. SHORT FIXED STRINGS

A final analysis is devoted to a one-dimensional map lattice with border values set to \( \phi_{n}^{1} = \phi_{n}^{L} = 0 \) (fixed string). The first recurrence of the characteristic fine structure is obtained for the fixed string of length \( L = 5 \) for the central value of this string (\( i = 3 \)). The neighboured points (\( i = 2, 4 \)) also show a fine structure but more of the kind we were used to from the closed string of length \( L = 2 \). Both fine structures are shown in Fig. 9.
FIG. 7: Self energy of the 2A-string of length $L = 3$ with open ends in the range $a \in [0.00015, 0.0006]$ after $6 \times 10^7$ iterations

\[ \langle \Phi \rangle = \Phi - 2\Phi^3/3 \]

FIG. 8: Self energy of the 2A-string of length $L = 4$ with open ends in the range $a \in [0.00015, 0.0006]$ after $6 \times 10^7$ iterations

\[ \langle \Phi \rangle = \Phi - 2\Phi^3/3 \]

FIG. 9: Self energy of the 2A-string of length $L = 5$ with fixed ends in the range $a \in [0.00015, 0.0006]$ after $3 \times 10^7$ iterations for the central point $i = 3$ and the neighboured points ($i = 2, 4$)

\[ \langle \Phi \rangle = \Phi - 2\Phi^3/3 \]

VIII. CONCLUSIONS

We have compared the self energy of the 2A-string of length $L = 10\,000$ and periodic boundary conditions (closed string) with the same closed string with short lattice sizes. It turns out that for $L \geq 3$ the fine structure of the self energy in the scaling region close to $a = 0$ can be reproduced. Using an open string of short length it figures out that the fine structure is reproduced for a string of length $L \geq 4$. Finally, the central points of a string of length $L \geq 5$ with fixed ends shows again the same fine structure of the self energy. We conclude that the closed string of length $L = 3$, the open string of length $L = 4$, or the fixed string of length $L = 5$ are minimal requirements for understanding the origin of the fine structure of the self energy dependence on the coupling $a$ in the scaling region.

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[20] Note that a possible alternative to this prescription for the open string is to replace the deterministic evolution given by the Chebyshev map by a stochastic variable without time correlation, distributed according to $\rho_0(\phi)$. 

\[ \langle \Phi \rangle = \Phi - 2\Phi^3/3 \]