Maximal Order of an NG-group

Faraj. A. Abdunabi

Department mathematical, Faculty science, University, Ajdabyia, Libya
Correspondence author: Faraj.a.abdunabi@uoa.edu.ly

Abstract

This study was aimed to consider the NG-group that consisting of transformations on a nonempty set $A$ has no bijection as its element. In addition, it tried to find the maximal order of these groups. It found the order of NG-group not greater than $n$. Our results proved by showing that any kind of NG-group in the theorem be isomorphic to a permutation group on a quotient set of $A$ with respect to an equivalence relation on $A$.

Keywords: NG-group, permutation group, equivalence relation, $\chi$-subgroup.

Introduction

This study considered the problem that the maximal order of a group consisting of transformations on a nonempty set $A$ and the group has no bijection as its element. Recall a permutation group on $A$ is a group consisting of bijections from $A$ to $A$ with respect to compositions of mappings. It is well known that any permutation group on a set $A$ with cardinality $n$ has an order not greater than $n!$.

In previous studies, there are some authors (1, 2), problem 1.4 in (3) considering groups which consist of non-bijective transformations on $A$ where the binary operation is the composition of mappings. Our first result is on the orders of such groups.

Theorem 1.1. let $A$ be a set with cardinality $n$. Suppose $NG$ be groups consisting of non-bijective transformations on $A$, where the binary operation
on $NG$ is the composition of transformation. Then the order of $NG$ is not greater than $(n-1)!$ and there are such groups having order $(n-1)!$.

Then it was proven Theorem 1.1 by showing that any kind of group in the theorem be isomorphic to a permutation group on a quotient set of $A$ with respect to an equivalence relation on $A$.

**Definition 1.1.** A class of group $\chi$ is called an SHP-class if it is closed under taking subgroups, homomorphic images, and products of normal subgroups. The latter condition means that if $U$ and $V$ are normal in $G$ and both $U$ and $V$ lie in $\chi$, then $UV \in \chi$. If a group $G$ belongs to $\chi$, we will say $G$ is an $\chi$-group.

**Remark 1.2.** If $\chi$ is an SHP-class and $U,V \triangleleft G$ are such that $G/U$ and $G/V$ are $\chi$-groups, then $G/(U \cap V)$ is isomorphic to a subgroup of the $\chi$-group $(U/G) \times (G/V)$, and thus $G/(U \cap V)$ is an $\chi$-group. It follows that given a finite group $G$, there exists a unique smallest normal subgroup $N$ such that $G/N \in \chi$, and we write $N= G\chi$.

The following theorem was found by the author; see also lemma 2.32 in (4).

**Theorem 1.2.** Let $\chi$ be an SHP-class, and suppose $G=UV$, where $U$ and $V$ are subnormal in $G$. Then $G\chi = U\chi V\chi$.

It could take the SHP-class to the class of $p$-groups, the class of nilpotent groups, etc. Theorem 1.2 will imply Lemma 9.15, problem 9B.5, Corollary 9.27, problem 9C.2, as corollaries.

**Remark 1.3.** It was noted in sec.4 of (5) that if it replaces the condition that $\chi$ is an SHP-class by some weaker condition that the class $\chi$ is such that whose composition factors all lie in some given set of simple groups then theorem 1.2 will fail in this case.

**Definition 1.2.** Let $\chi$ be an SHP-class and $G$ be a finite group. The result was denoted the maximal normal $\chi$-subgroup of $G$ by $G\chi$.

Then was considered the question that if $G=UV$ with $U,V$ subnormal in $G$ then it holds that $G\chi = U\chi V\chi$ or not. If $p$ is a prime and take the SHP-class $\chi$ to
be the class of all finite \( p \)-group, then for any finite group \( G \) will be \( O_p(G) \) and results have the following theorem.

**Theorem 1.3.** Let \( p \) and \( q \) be two primes such that \( q \equiv 1 \pmod{p} \). Let \( N = G_q \) be a cyclic group of order \( q \) and \( H = \langle x \rangle \times \langle y \rangle \) an elementary abelian group of order \( p^2 \). Let \( \langle x \rangle \) act on \( N \) faithfully and \( \langle y \rangle \) act on \( N \) trivially. Set \( G = N \rtimes H \) to the semidirect product of \( N \) and \( H \). Let \( U = N \langle x \rangle \) and \( V = \langle xy \rangle \). Then

1. \( U, V \) are both subnormal in \( G \).
2. \( O_p(G) = \langle y \rangle \) and \( O_p(U) = O_p(V) = 1 \). In particular, \( O_p(G) \neq O_p(U)O_p(V) \).

**Definition 2.1.** A binary relation \( \sim \) in \( A \) is called an equivalence relation on \( A \). If it satisfies the following three conditions:

(i) \( a \sim a \) for any \( a \in A \);
(ii) for any \( a, b \in A \), if \( a \sim b \) then \( b \sim a \);
(iii) for any \( a, b, c \in A \), if \( a \sim b \) and \( b \sim c \) then \( a \sim c \).

For the set \( A \), where use \( A^A \) to dented the set of all its transforms, for any \( f \in A^A \), we use \( \text{Im}(f) \) to denote the image of \( f \). Also, \( Z \) and \( Z_+ \) will respective dented the set of integers and positive integers.

**Definition 2.2.** Let \( \sim \) be an equivalence relation on \( A \), for an element \( a \in A \), it is call \( \{ x \in A | x \sim \} \) the equivalence class of a determined by \( \sim \), which is denoted by \( [a]_\sim \). And \( A/\sim = \{ [a]_\sim | a \in A \} \) is called the quotient set of \( S \) relative to the equivalence relation \( \sim \).

**Lemma 2.3.** Theorem 1 (1). for any \( f \in G \) and the \( e \) the identity element of \( G \), \( \sim_e = \sim_f \)

*Proof.* for any \( a \in A \), so result goal is to show that \( [a]_e = [a]_f \).
On one hand, if \( x \in [a]_f \), i.e. \( f(x) = f(a) \). Since \( G \) is a group with identity element \( e \), there is a transformation \( f' \in G \) such that \( ff = ef'f \). Therefore,
\[
e(x) = f'(f(x)) = f'(f(a)) = e(a),
\]
Which yields that \( x \in [a]_e \).

On the other hand, if \( y \in [a]_e \) i.e. \( e(a)e(y) \). Hence,
\[
f(a) = (fe)(a) = f(e(y)) = fe(y) = f(y),
\]
Which implies \( y \in [a]_f \). It follows that \( [a]_e = [a]_f \) for any \( a \in A \), as wanted.

**Remark 2.1.** For Lemma 2.4, the current result see that \( \sim_f \sim_g \) for any element \( f, g \in G \). The following Theorem is the revised version of Theorem 2, (I).

**Theorem 2.5.** Let \( f \) be an element in \( A^A \) and \( \hat{f} \) be the induced transformation of \( f \) on \( A/\sim_f \), i.e.
\[
\hat{f} : A/\sim_f \rightarrow A/\sim_f \quad [x]_f \mapsto [f(x)]_f.
\]
Then the following hold:

(i) The exists a groups \( G \subseteq A^A \) containing \( f \) as the identity element iff \( \hat{f} \) is bijective on \( A/\sim_f \).

(ii) There is a groups \( G \subseteq A^A \) containing \( f \) as the identity element iff \( \hat{f} \) is bijective on \( A/\sim_f \).

The following two corollaries are from (I), and we make some corrections to the original proofs. Actually, this adopt the restriction of finiteness on \( A \) in the first corollary from the original one. And then used the finiteness on \( A \) in the second corollary; the original one did not use it.

**Corollary 2.6.** Let \( f \) be an element in \( A^A \). Then \( f^2 = f \) iff the induced mapping \( \hat{f} \) on \( A/\sim_f \) is the identity element.

**Proof.** On one hand, suppose that \( f^2 = f \). Then for any \( [x]_f \in A/\sim_f \), as \( f(x) = f(f(x)) \), then see that \( [x]_f = [f(x)]_f \). It follows that
\[
\hat{f}([x]_f) = [f(x)]_f = [x]_f;
\]
This implies that \( \hat{f} \) is the identity mapping on \( A/\sim_f \).

On the other hand, assume that \( \hat{f} \) is the identity mapping on \( A/\sim_f \). Then for any \( [x]_f \in A/\sim_f \), the condition that \( \hat{f}([x]_f) = [x]_f \) will imply that \( [f(x)]_f = [x]_f \) and hence \( f(f(x)) = f(x) \). It follows that \( f^2 = f \) as required.

**Corollary 2.7.** Suppose that \( A \) is a finite set and \( f \) is an element in \( A^A \). Then there is a group \( G \subseteq A^A \) containing \( f \) as an element iff \( \text{Im}(f) = \text{Im}(f^2) \).
Proof. On one hand, suppose that there is a group \( G \subseteq A^A \) containing \( f \) as an element. Let \( e \) be the identity element of \( G \). Then by Theorem 2.5, the induced mapping \( \hat{f} \) is a bijection on \( A/\sim_f \). In particular, \( \hat{f} \) is surjective and thus for any \( x \in A \), there is a \([y]_f \in A/\sim_f\) such that \( \hat{f}([y]_f) = [x]_f = [f(y)]_f \); Which yields that \( f(x) = f(f(y)) = (\hat{f}^2)(y) \). As a result, \( \text{Im}(\hat{f}) \subseteq \text{Im}(\hat{f}^2) \) and thus \( \text{Im}(\hat{f}) = \text{Im}(\hat{f}^2) \).

On the other hand, suppose that \( \text{Im}(\hat{f}) = \text{Im}(\hat{f}^2) \). Then for any \( f(x) \in \text{Im}(\hat{f}) \) there is a \( y \in A \) such that \( f(x) = f(f(y)) \) and hence \( \hat{f}([y]_f) = [x]_f \); which implies that \( \hat{f} \) is surjective on \( A/\sim_f \). Note that results are assuming that \( A \) is finite and so is \( A/\sim_f \). This study has that the induced mapping \( \hat{f} \) is bijective. By Theorem 2.5, the assertion follows.

Remark 2.2. Let \( G \subseteq A^A \) be a group. That has seen, in Remark 2.1, that \( \sim = \sim_g \) for any elements in \( G \) and we will denote the common equivalence relation by \( \sim \). Also, by Theorem 2.5, each element \( f \in G \) will induce a bijection \( \hat{f} \) on \( A/\sim \).

The following theorem is crucial since it turns a group \( G \subseteq A^A \) into a permutation group.

**Theorem 2.8.** Let \( G \subseteq A^A \) be a group. Set 
\( \hat{G} = \{ \hat{f} \mid f \in G \} \); then \( \hat{G} \) is a permutation group on \( A/\sim \) and \( \rho : G \to \hat{G} \), \( f \to \hat{f} \), is an isomorphism.

*Proof.* For any \( f, g \in G \) and any \([a] \in A/\sim\), results have \( \rho(fg)([a]) = ([fg](a)) = [f[g(a)]] = \rho(f)([g(a)]) = (\rho(f) \rho(g))(a) \); which implies that \( \rho(fg) = \rho(f) \rho(g) \) and thus \( \rho \) is a homomorphism. By the definition of \( \hat{G} \), it is obvious that \( \rho \) is surjective.

Now suppose that \( \rho(f) = \rho(g) \) for two elements \( f, g \in G \), i.e.\([f(a)] = [g(a)]\), \( \forall a \in A \). Let \( e \) be the identity element of \( G \), then we have \([f(a)]_e = [g(a)]_e \); \( \forall a \in A \). It follows that \( e(f(a)) = e(g(a)) \); \( \forall a \in A \). Hence \( f(a) = (ef)(a) = (\hat{e}f)(a) = \hat{e}(g(a)) = g(a) \), \( \forall a \in A \), and therefore \( f = g \). So it conclude that \( \rho \) is injective. As a consequence, \( \rho \) is an isomorphism.

**Definition 2.3.** A subgroup \( H \) of a group \( G \) is called characteristic in \( G \), denoted \( H \text{ char } G \), if every automorphism of \( G \) maps \( H \) to itself, that is \( \rho(H) = H \) for all \( \rho \in \text{Aut}(G) \).

**Remark 2.3.** If \( H \) is characteristic in \( G \) in \( K \) and \( K \) is characteristic in \( G \), then \( H \) is characteristic in \( G \).
Let $G$ be a finite group. It has the following two lemmas. They are from Section 2 of (5).

**Lemma 2.9.** Suppose that $\chi$ is an SHP-class.

(a) Let $\leq G$ be a subgroup. Then $H^\chi \leq G^\chi$.

(b) Let $N \triangleleft G$ be a normal subgroup of $G$ and write $\overline{G}=G/N$, then $\overline{G}^\chi = \overline{G^\chi}$.

(c) $G^\chi$ is characteristic in $G$.

(d) $O^\chi(G)$ is characteristic in $G$.

The following lemma is a generalization of Problem 2A.1 in (8).

**Lemma 2.10.** Let $A$ and $B$ be two subnormal $\chi$-subgroups of $G$. Then the subgroup $\langle A,B \rangle$ generated by $A$ and $B$ are $\chi$-subgroup of $G$.

**Proof.** Let $A$ be a subnormal $\chi$-subgroup of $G$. The resulting use induction on the subnormal depth $r$, $A \subseteq O^\chi_r$ of $A$ in $G$ to show that if $r=1$, then $A\triangleleft$ and thus $A \subseteq O^\chi(G)$ since $O^\chi(G)$ is the largest normal $\chi$-subgroup of $G$.

Suppose $r > 1$ and the containment holds for $r-1$. Let $A_1 = A \triangleleft \cdots \triangleleft H_{r-1} \triangleleft H_r = G$ be a subnormal series from $A$ to $G$: Then $A \subseteq O^\chi_r(G)$ by inductive hypothesis. Since $O^\chi_r(G)$ char $H_{r-1}$ and $H_r \triangleleft G$; $O^\chi_r(G) \triangleleft G$ and then $O^\chi_r(G) (H_{r-1}) \subseteq O^\chi(G)$.

It was concluded that $A \subseteq O^\chi_r(G)$.

In general, for any two subnormal $\chi$-subgroups $A$ and $B$, $A,B \subseteq O^\chi_r(G)$ and thus $\langle A,B \rangle \subseteq O^\chi_r(G)$ as wanted.

**Proofs of Main Results**

Now let $A$ be a set having $n$ letters written as $\{1, 2, \ldots, n\}$. The results have the following theorem, which is Theorem 1.1.

**Theorem 3.1.** Let $A$ be a set with cardinality $n$ with $n \geq 3$. Suppose $NG$ is a group consisting of non-bijective transformations on $A$, where the binary operation on $NG$ is the composition of transformations. Then the order of $NG$ is not greater than $(n-1)!$ and there are such groups having order $(n-1)!$.

**Proof.** Let $NG$ be a group consisting of non-bijective transformations on $A$. By Remark 2.1, it is known that $\sim = \sim_\chi$ for any element $f,g \in NG$ and it denote the common equivalence relation by $\sim$. Note that $NG$ is a group consisting of non-bijective transformations, then we see that the equivalence relation is not the equality relation $=on A$. Thus, these results have that the quotient set $A/\sim$ has an order less than $n-1$. 

*Libyan Journal of Basic Sciences, Vol: 13, No: 1, P: 1-, 38- 47, April. 2021*
Additionally, $NG$ is isomorphic to a permutation group on $A/\simeq$ by Theorem 2.8. It follows that the order of $NG$ is less than $(n-1)!$ as any permutation group on $A/\simeq$ has order less than $(n-1)!$.

Note that in defining a permutation $s$ on the set $\{1, 3, \ldots, n\}$, there are $n-1$ choices for $\rho(1)$, $n-2$ choices of $\rho(3) \neq \rho(1)$, etc., i.e. totally $(n-1)(n-2)! = (n-1)!$.

**Theorem 3.2.** Let $\chi$ be an SHP-class, and suppose $G = UV$; where $U$ and $V$ are subnormal in $G$. Then $G^\chi = U^\chi V^\chi$.

**Proof.** This work use induction of the subnormal depth of $U$ in $G$ to prove the result.

First, if the subnormal depth of $U$ in $G$ is one, i.e. $U \trianglelefteq G$. Since $U^\chi$ is characteristic in $U$ and $U$ is normal in $G$ we see that $U^\chi$ is normal in $G$.

Let $\tilde{G} = G/U^\chi$. By the hypothesis, $\tilde{G} = \bar{U} \bar{V}$ where $\bar{U} = U/U^\chi$, $\bar{V} = V/U^\chi$. Therefore, $\bar{U}$ is a normal $\chi$-group of $\tilde{G}$ and $\bar{V}$ is subnormal in $\tilde{G}$. By Lemma 2.10, we have $\tilde{G}^\chi = \bar{V}^\chi$. By Lemma 2.9 (b), $G^\chi = \bar{G}^\chi$, $\bar{V}^\chi = \bar{V} = \bar{U} \bar{\chi} \bar{V} \bar{\chi}$.

By correspondence theorem, it has $G^\chi = U^\chi V^\chi$; as required.

Now suppose that the subnormal depth of $U$ in $G$ is $r$ with $r > 1$: Let $U_1 = U \trianglelefteq \ldots \trianglelefteq U_r \trianglelefteq G$ be a subnormal series from $U$ to $G$ with length $r$. By Dedekind’s lemma, $U_r = U(V \cap U_r)$. As both $U$ and $V \cap U_r$ are subnormal in $U_r$ and $U$ has subnormal depth $r-1$ in $U_r$, then obtain that 

$$(U_r)^\chi = U^\chi (V \cap U_r)^\chi$$

by inductive hypothesis. Also, $G = U_r V$ with $U_r$ normal in $G$ and $V$ subnormal in $G$, and hence 

$G^\chi = (U_r)^\chi V^\chi$ by the first paragraph of the proof. It follows that 

$G^\chi = (U_r)^\chi V^\chi = U^\chi (V \cap U_r)^\chi V^\chi = U^\chi V^\chi$,

because $(V \cap U_r)^\chi \subseteq V^\chi$ by Lemma 2.9 (a).

**Theorem 3.3.** Let $p$ and $q$ be two primes such that $q \equiv 1 (modp)$. Let $N = C_q$ be a cyclic group of order $q$ and $H = <x>$ an elementary abelian group of order $p^2$. Let $<x>$ act on $N$ faithfully and $<y>$ act on $N$ trivially. Set $G = N \rtimes H$ to be the semidirect product of $N$ and $H$. Let $U = N <x>$ and $V = N <xy>$. Then

(i) $U, V$ are both subnormal in $G$ and $G = UV$.

(ii) $O_p(G) = <y>$ and $O_p(U) = O_p(V ) = 1$. In particular, $O_p(G) \neq O_p(U)O_p(V )$. 

---

Libyan Journal of Basic Sciences, Vol: 13, No: 1, P: 1-, 38-47, April. 2021
Proof. Since $N$ is normal in $G$ and the quotient group $G=N/H$ is abelian, it deduced that the derived subgroup $G'$ is contained in $N$. It follows that both $U$ and $V$ contain $G'$ as a subgroup, which implies that $U$ and $V$ are normal in $G$. Obviously, $G = UV$. Assertion (i) holds.

Note that the Sylow $p$-subgroup of $G$ is not normal since $\langle x \rangle$ act on $N$ faithfully and hence $O_p(G)$ has an order less than $p^2$. However, as $\langle y \rangle$ act on $N$ trivially, $N$ normalizes $\langle y \rangle$ which yields that $\langle y \rangle$ is a normal $p$-subgroup of $G$. It is easy to see that $O_p(G) = \langle y \rangle$. Both $\langle x \rangle$ and $\langle xy \rangle$ act faithfully on $N$, which yields that $O_p(U) = O_p(V) = 1$; as wanted.

References:

1. Y. Wu, X. Wei, Condition of the groups generated by nonbijective transformations on a set. *Journal of Hubei University (Natural Science)* 27, 1 (2005).
2. Y. Zhu, J. Hai, Groups Generated by Non-bijective Transformations on a Set and Their Representations in the Matrix Algebra. *Journal of Capital Normal University (Natural Science)* 24, 5 (2003).
3. I. M. Isaacs, *Algebra (A Graduate Course)*. (Wadsworth, Inc, 2003).
4. C. E. Finch, L. Jones, A curious connection between Fermat numbers and finite groups. *The American mathematical monthly* 109, 517 (2002).
5. I. Isaacs, U. Meierfrankenfeld, Repeated and final commutators in group actions. *Proceedings of the American Mathematical Society* 140, 3777 (2012).
6. O. Bogopolski, Introduction to group theory, European Math. Soc., Zurich, (2008).
7. H. Kurzweil, B. Stellmacher, *The Theory of Finite Groups: An Introduction*. Universitext Series. (Springer-Verlag New York, Inc, 2004).
8. I. M. Isaacs, *Finite group theory*. (American Mathematical Soc., 2008), vol. 92.
تبية ترتيب لزمر
فرج أرخيص عبدالنبي
قسم الرياضيات، كلية العلوم، جامعة أجدابيا

الملخص العربي

هدفت هذه الدراسة إلى اعتبار زمر 
التي تتكون من تحولات غير تقابليه على مجموعة
NG غير فارغة A والتي تكون زمر غير جزئية من الزمر التبديلية ونطلق على هذه الزمر ب:
NG-Transformation. 
وستحصل على انه لايمكن ان يكون درجة اعلى زمرة أكبر من (ن - 1).! بالإضافة إلى ذلك، سوف نثبت نتيجتنا من خلال إظهار أن أي نوع من هذه
الزمر في النظرية المتحصل عليها سيكون متماثلاا لزمرة التباديل المعروفة على مجموعة
A بالنسبة لعلاقة التكافؤ على. A

الكلمات المفتاحية: مجموعة NG، زمرة تبديلية، علاقة تكافؤ، المجموعة الفرعية.