Rescaled Lotka-Volterra Models Converge to Super Stable Processes

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Abstract

Recently, it has been shown that stochastic spatial Lotka-Volterra models when suitably rescaled can converge to a super Brownian motion. We show that the limit process could be a super stable process if the kernel of the underlying motion is in the domain of attraction of a stable law. The corresponding results in Brownian setting were proved by Cox and Perkins (2005, 2008). As applications of the convergence theorems, some new results on the asymptotics of the voter model started from single 1 at the origin are obtained which improve the results by Bramson and Griffeath (1980).

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Abbreviated Title: Lotka-Volterra model

1 Introduction

1.1 Motivation

Originally, super Brownian motion arises as the limit of branching random walks; see [10, 11, 18]. Recently, it has been shown that many interacting particle systems with very different dynamics, when suitably rescaled, all converge to super Brownian motion. Such examples include the voter model, the contact process, interacting diffusion process and the spatial Lotka-Volterra model; see [4, 11, 5, 7, 9]. Donsker’s invariance principle is deeply involved in those results; see [22] for an excellent nontechnical introduction. So if we assume that the kernel of the underlying motion has finite variance, super Brownian motion is obtained as the limit process. On the other hand, the general class of stable distribution was introduced and given this name by the famous French mathematician Paul Lévy. The inspiration for Lévy was the desire to generalize the Central Limit Theorem which is the foundation of

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Donsker’s principle. Thus we can expect that if we let the kernel of the underlying motion be in the domain of attraction of a stable law, the limit process could be a super stable process.

A motivation for proving those limit theorems is to actually use it in the study of complicated approximating systems. For example, the Lotka-Volterra invariance principle established in [7] was used to study the coexistence and survival problem of the Lotka-Volterra model; see [8]. Cox and Perkins [6] used the voter invariance principle to give a probabilistic proof of the asymptotics for the voter model obtained in [3]. In this paper, we will show that rescaled stochastic spatial Lotka-Volterra models can converge to super stable processes and also use those limit theorems to get some new results on the asymptotics for the voter model. Coexistence and survival for the Lotka-Volterra model will be discussed in a future work.

1.2 Our model

A stochastic spatial version of the Lotka-Volterra model was first introduced and studied by Neuhauser and Pacala [17]. In this paper, we follow the construction of the model suggested by [7] but we assume that the kernel of the model is in the domain of attraction of a symmetric stable law. We first briefly describe the model. Let \( \{p(x, y)\} \) be a random walk kernel on \( \mathbb{Z}^d \) (the \( d \)-dimensional integer lattice). Suppose at each site of \( \mathbb{Z}^d \) there is a plant of one of two type. We label the two types 0 and 1. At random times plants die and are replaced by new plants. The times and the types depend on the configuration of surrounding plants. We denote by \( \xi_t \), an element of \( \{0, 1\}^\mathbb{Z}^d \), the state of the system at time \( t \) and \( \xi_t(x) \) gives the type of the plant at \( x \) at time \( t \). To describe the evolution of the system, for \( \xi \in \{0, 1\}^\mathbb{Z}^d \), define

\[
    f_i(x, \xi) = \sum_{y \in \mathbb{Z}^d} p(x, y) 1_{\{\xi(y) = i\}}, \quad i = 0, 1.
\]

Let \( \alpha_0, \alpha_1 \) be nonnegative parameters. Define the Lotka-Volterra rate function \( c(x, \xi) \) by

\[
    c(x, \xi) = \begin{cases} 
    f_1(f_0 + \alpha_0 f_1) & \text{if } \xi(x) = 0, \\
    f_0(f_1 + \alpha_1 f_0) & \text{if } \xi(x) = 1.
    \end{cases}
\]

The Lotka-Volterra process \( \xi_t \) is the unique \( \xi \in \{0, 1\}^\mathbb{Z}^d \)-valued Feller process with rate function \( c(x, \xi) \), meaning that the generator of \( \xi_t \) is the closure of the operator \( \Omega \)

\[
\Omega \phi(\xi) = \sum_x c(x, \xi)(\phi(\xi^x) - \phi(\xi))
\]

on the set of function \( \phi : \xi \in \{0, 1\}^\mathbb{Z}^d \to \mathbb{R} \) depending on only finitely many coordinates, where \( \xi^x(y) = \xi(y) \) for \( y \neq x \) and \( \xi^x(x) = 1 - \xi(x) \).

Note that \( f_0 + f_1 = 1 \). The dynamics of \( \xi_t \) can now be described as follows: at site \( x \) in configuration \( \xi \), the coordinate \( \xi(x) \) makes transitions

\[
\begin{align*}
    0 &\to 1 \quad \text{at rate} \quad f_1(f_0 + \alpha_0 f_1) = f_1 + (\alpha_0 - 1)f_1^2, \\
    1 &\to 0 \quad \text{at rate} \quad f_0(f_1 + \alpha_1 f_0) = f_0 + (\alpha_1 - 1)f_0^2.
\end{align*}
\]
These rates are interpreted in [17] as follows. A plant of type \( i \) site \( x \) dies at rate \( f_i + \alpha_i f_{1-i} \), and is replaced by a plant of type \( \xi \) where \( \xi \) is chosen with probability \( p(x, y) \). \( \alpha_i \) measures the strength of interspecific competition of type \( i \) and we set the self-competition parameter equal to one.

In [4] an invariance principle was proved for the voter model. That is appropriately rescaled voter models converge to super-Brownian motion. Thus we can expect that when the parameters \( \alpha_i \) are close to one a similar result holds for the Lotka-Volterra model. The results in [7] and [9] say that it is true. The intuition of the voter invariance principle is that when appropriately rescaled, the dependence on the local density of particles gets washed out and the rescaled voter models should behave like the rescaled branching random walk. The asymptotics behavior of the latter is well known: it approaches super-Brownian motion. On the other hand, if the kernel of the underlying motion is in the domain of attraction of a stable law, appropriately rescaled branching random walk could approach a super stable process. Our main results in this paper will show that it is the case.

Let \( M(\mathbb{R}^d) \) denote the space of finite measures on \( \mathbb{R}^d \), endowed with the topology of weak convergence of measures. Let \( \Omega_D = D([0, \infty), M(\mathbb{R}^d)) \) be the Skorohod space of càdlàg paths taking values in \( M(\mathbb{R}^d) \). Let \( \Omega_C \) be the space of continuous \( M(\mathbb{R}^d) \)-valued paths with the topology of uniform convergence on compact set. We denote by \( X_t(\omega) = \omega \) the coordinate function. We write \( \mu(\phi) \) for \( \int \phi d\mu \). For \( 1 \leq n \leq \infty \) let \( C^n_b(\mathbb{R}^d) \) be the space of bounded continuous function whose partial derivatives of order less than \( n + 1 \) are also bounded and continuous, and let \( C^n_0(\mathbb{R}^d) \) be the space of those functions in \( C^n_b(\mathbb{R}^d) \) with compact support.

A \( \mathbb{R}^d \)-valued Lévy process \( Y_t \) is said to be a symmetric \( \alpha \)-stable process with index \( \alpha \in (0, 2] \) and diffusion speed \( \sigma^2 > 0 \) if

\[
\Psi(\eta) := E(e^{i\eta \cdot Y_t}) = e^{-\sigma^2|\eta|^\alpha},
\]

where \( |\eta| \) is the Euclidean norm of \( \eta \). The distribution of \( Y_t \) will be called \((\sigma^2, \alpha)\)-stable law. When \( \alpha = 2 \), \( Y_t \in \mathbb{R}^d \) is a \( d \)-dimensional \( \sigma^2 \)-Brownian motion whose generator is \( \mathcal{A} \phi = \frac{\sigma^2 \Delta \phi}{2} \) for \( \phi \in \mathcal{C}_b^2(\mathbb{R}^d) \). When \( 0 < \alpha < 2 \), the generator of \( Y_t \) is given by

\[
\mathcal{A} \phi(x) = \frac{\sigma^2 \Delta \phi(x)}{2} = \sigma^2 \int \left[ \phi(x + y) - \phi(x) - \frac{1}{1 + |y|^2} \sum_{i=1}^d y_i D_j \phi(x) \right] \nu(dy)
\]

for \( \phi \in \mathcal{C}_b^2(\mathbb{R}^d) \) and \( D_j = \frac{\partial}{\partial x_j} \), where

\[
\nu(dy) = c|y|^{-d-\alpha}1_{\{|y| \neq 0\}}(dy)
\]

for an appropriate \( c > 0 \); see [20] for details. In both cases, \( C^\infty_b(\mathbb{R}^d) \) is a core for \( \mathcal{A} \) in that the \( \text{bp}-\)closure of \( \{ (\phi, \mathcal{A} \phi) : \phi \in C^\infty_b \} \) contains \( \{ (\phi, \mathcal{A} \phi) : \phi \in \mathcal{D}(\mathcal{A}) \} \), where \( \mathcal{D}(\mathcal{A}) \) denotes the domain of the weak generator for the process \( Y \); see [18].

An adapted a.s.-continuous \( M(\mathbb{R}^d) \)-valued process \( \{ X_t : t \geq 0 \} \) on a complete filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \) is said to to a super symmetric \( \alpha \)-stable process with branching rate \( b \geq 0 \), drift \( \theta \in \mathbb{R} \) and diffusion coefficient \( \sigma^2 > 0 \) starting at \( X_0 \in M(\mathbb{R}^d) \) if it solves the following martingale problem:
For all \( \phi \in C_b^{\infty}(\mathbb{R}^d) \),

\[
M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s \left( \frac{\sigma^2 \Delta^{\alpha/2}(\phi(x))}{2} \right) ds - \theta \int_0^t X_s(\phi) ds
\]  

(1.3)
is a continuous \((\mathcal{F}_t)\)-martingale, with \( M_0(\phi) = 0 \) and predictable square function

\[
\langle M(\phi) \rangle_t = \int_0^t X_s(b\phi^2) ds.
\]

(1.4)
The existence and uniqueness in law of a solution to this martingale problem is well known; see Theorem II.5.1 and Remark II.5.13 of [18]. Let \( P_{b,\theta,\sigma^2,\alpha} \) denote the law of the solution on \( \Omega_C \). So \( b \) and \( \theta \) can be regarded as branching parameters and parameters \( \sigma \) and \( \alpha \) determine the underlying motion.

Let \( \{Z_n : n \geq 1\} \) be a discrete time random walk on \( \mathbb{Z}^d \),

\[
Z_n = z_0 + \sum_{i=1}^{n} U_i,
\]

where \( z_0 \in \mathbb{Z}^d \) and the random variables \( (U_i : i \geq 1) \) are independent identically distributed on \( \mathbb{Z}^d \). Let \( \{p(x, y)\} \) be a random walk kernel. In the following of this paper we assume that

(A1): \( p(x, y) = p(x - y) \) is an irreducible, symmetric, random walk kernel on \( \mathbb{Z}^d \) and \( p(0) = 0 \). For \( \alpha \in (0, 2] \) and \( \sigma^2 > 0 \), \( \{p(x)\} \) is in the domain of attraction of a symmetric \((\sigma^2, \alpha)\)-stable law; i.e.,

\[
P(U_1 = x) = p(x)
\]

and there exists a function \( b(n) \) of regular variation of index \( 1/\alpha \) such that

\[
b(n)^{-1} \sum_{i=1}^{n} U_i \xrightarrow{(d)} Y_1 \quad \text{as} \quad n \to \infty,
\]

(1.5)

where \( Y_1 \) is determined by \([1.2]\) and the symbol \( \xrightarrow{(d)} \) means convergence in distribution.

We will call a random walk (discrete time or continuous time) with kernel satisfying assumption (A1) a stable random walk. In the following of this paper, we always assume that

(A1) holds for some \( \sigma > 0 \) and \( \alpha \in (0, 2] \).

**Remark 1.1** Without loss of generality, we may and will assume that function \( b \) is continuous and monotonically increasing from \( \mathbb{R}^+ \) onto \( \mathbb{R}^+ \) and \( b(0) = 0 \); see [13] or [15]. We also have that

\[
b(x) = x^{1/\alpha} s(x), \quad x > 0,
\]

where \( s : (0, \infty) \to (0, \infty) \) is a slowly varying function, meaning that for any \( c > 0 \),

\[
\lim_{x \to \infty} \frac{s(cx)}{s(x)} = 1
\]

where the convergence holds uniformly when \( c \) varies over the interval \([\epsilon, 1/\epsilon]\) for any \( \epsilon > 0 \); see Lemma 2 of VIII.8 of [13].
Remark 1.2 According to Proposition 2.5 of [15] and its proof, we have that under (A1), random walk \( \{Z_n\} \) is transient if and only if

\[
\sum_{k=1}^{\infty} b(k)^{-d} < \infty.
\]

By Lemma 2 in Section VIII.8 of [13], the random walk is always transient when \( d > \alpha \).

Typically, when \( d = \alpha = 1 \), the random walk is recurrent if only if

\[
\sum_{k=1}^{\infty} \frac{1}{ks(k)} = \infty.
\]

Now, we are ready to define our rescaled Lotka-Volterra models. For \( N = 1, 2, \cdots \), let

\[
S_N = \mathbb{Z}^d / b(N).
\]

Define the kernel \( p_N \) on \( S_N \) by

\[
p_N(x) = p(xb(N)), \quad x \in S_N.
\]

For \( \xi \in \{0,1\}^{S_N} \), define the densities \( f_i^N = f_i^N(\xi) = f_i^N(x, \xi) \) by

\[
f_i^N(x, \xi) = \sum_{y \in S_N} p_N(y - x)1\{\xi(y) = i\}, \quad i = 0, 1.
\]

Let \( \alpha_i = \alpha_i^N \) depend on \( N \) and let \( \xi^i_N \) be the process taking values in \( \{0,1\}^{S_N} \) determined by the rates: at site \( x \) in configuration \( \xi \), the coordinate \( \xi(x) \) makes transitions

\[
0 \to 1 \quad \text{at rate} \quad N f_1^N(f_0^N + \alpha_0^N f_1^N),
\]

\[
1 \to 0 \quad \text{at rate} \quad N f_0^N(f_1^N + \alpha_1^N f_0^N).
\]

That is \( \xi^i_N \) is rate-\( N \) Lotka-Volterra process determined by the parameters \( \alpha_i^N \) and the kernel \( p_N \). More precisely, if set

\[
c_N(x, \xi) = \begin{cases} 
N f_1^N(f_0^N + \alpha_0^N f_1^N) & \text{if } \xi(x) = 0, \\
N f_0^N(f_1^N + \alpha_1^N f_0^N) & \text{if } \xi(x) = 1,
\end{cases}
\]

\( \xi^N_t \) is the unique Feller process taking values in \( \{0,1\}^{S_N} \) whose generator is the closure of the operator

\[
\Omega_N \phi(\xi) = \sum_{x \in S_N} c_N(x, \xi)(\phi(\xi^x) - \phi(\xi))
\]

on the set of function \( \phi : \xi \in \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R} \) depending on only finitely many coordinates. Here \( \xi^x(y) = \xi(y) \) for \( y \neq x \) and \( \xi^x(x) = 1 - \xi(x) \).

Remark 1.3 If we assume \( \sum_{x \in \mathbb{Z}^d} x^i x^j p(x) = \delta_{ij} \sigma^2 < \infty \), then \( p(x) \) is in the domain of attraction of a normal law. That is the case of \( \alpha = 2 \). So we recover the fixed kernel models in [7]. For critical case, since there are significant differences between the case of \( d = \alpha = 1 \) and the case of \( d = \alpha = 2 \), we only consider the case of \( d = \alpha = 1 \). For \( d = \alpha = 2 \), please see the work in [9].
Define
\[ g(x) = \int_1^x b(s)^{-1}ds \]
for \( d = \alpha = 1 \) and \( x \geq 0 \). According to Remark 1.2, the one-dimensional random walk \( Z \) is recurrent if and only if \( \lim_{x \to \infty} g(x) = \infty \).

Set
\[ N' = \begin{cases} 
N, & \text{if } d > \alpha, \\
N, & \text{if } d = \alpha = 1 \text{ and } \lim_{x \to \infty} g(x) < \infty, \\
N/g(N), & \text{if } d = \alpha = 1 \text{ and } \lim_{x \to \infty} g(x) = \infty.
\end{cases} \]

That is when the stable random walk is transient \( N' = N \) and \( N' = N/g(N) \) if the stable random walk is recurrent.

We define the corresponding measure-valued process \( X_t^N \) by
\[ X_t^N = \frac{1}{N'} \sum_{x \in S_N} \xi_t^N(x) \delta_x. \tag{1.6} \]

As in [7] and [9], we make the following assumptions:
\begin{enumerate}
\item \( \sum_{x \in S_N} \xi_0^N(x) < \infty \).
\item \( X_0^N \to X_0 \) in \( M(\mathbb{R}^d) \) as \( N \to \infty \). \hfill (A2)
\item \( \theta_i^N = N'(\alpha_i^N - 1) \to \theta_i \in \mathbb{R} \) as \( N \to \infty \), \( i = 0, 1 \).
\end{enumerate}

Now, we are ready to describe our main results.

### 1.3 Main results

To describe the limit process, we introduce a coalescing random walk systems \( \{\hat{B}_x^z, x \in \mathbb{Z}^d\} \). Each \( \hat{B}_x^z \) is a rate 1 random walk on \( \mathbb{Z}^d \) with kernel \( p \), with \( \hat{B}_0^x = x \). The walks move independently until they collide, and then move together after that. For finite \( A \subset \mathbb{Z}^d \), let
\[ \hat{\tau}(A) = \inf\{s : |\{\hat{B}_x^z, x \in A\}| = 1\} \]
be the time at which the particles starting from \( A \) coalesce into a single particle, and write \( \hat{\tau}(a, b, \cdots) \) when \( A = \{a, b, \cdots\} \). Note that when the stable random walk is transient, we can define the “escape” probability by
\[ \gamma_e = \sum_{e \in \mathbb{Z}^d} p(e)P(\hat{\tau}(0, e) = \infty). \]

We also define
\begin{align*}
\beta &= \sum_{e,e' \in \mathbb{Z}^d} p(e)p(e')P(\hat{\tau}(e, e') < \infty, \hat{\tau}(0, e) = \hat{\tau}(0, e') = \infty), \\
\delta &= \sum_{e,e' \in \mathbb{Z}^d} p(e)p(e')P(\hat{\tau}(0, e) = \hat{\tau}(0, e') = \infty).
\end{align*}
We also need a collection of independent (noncoalescing) rate-1 continuous time random walks with step function $p$, which we will denote $\{B^x_t : x \in \mathbb{Z}^d\}$, such that $B^x_0 = x$. Define the collision times

$$
\tau(x,y) = \inf\{t \geq 0 : B^x_t = B^y_t\}, \quad x,y \in \mathbb{Z}^d.
$$

Let $P_N$ denote the law of $X^N$. Our first result is following.

**Theorem 1.1** Assume (A1), (A2) and $d \geq \alpha$. If the stable random walk is transient, then

$$
P_N^{(d)} \xrightarrow{d} P^{2\gamma,\theta,\sigma^2,\alpha}_{X_0}
$$

as $N \to \infty$, where $\theta = \theta_0 \beta - \theta_1 \delta$.

Note that if we assume $\sum_{x \in \mathbb{Z}^d} x^i x^j p(x) = \delta_{ij} \sigma^2 < \infty$, then $\{p(x)\}$ is in the domain of attraction of a normal law with $b(N) = \sqrt{N}$. So Theorem 1.1 generalizes Theorem 1.2 in [7].

Next, we consider the recurrent case. And for some technical reasons we need to assume that the $\{p(x)\}$ is in the domain of normal attraction of $(\sigma^2, 1)$-stable law; see Remark 4.5 below. To state our result, we introduce the one-dimensional potential kernel $a(x)$,

$$
a(x) = \int_0^\infty \left[P(B^0_t = 0) - P(B^x_t = 0)\right] dt.
$$

(1.7)

We will discuss the existence of $a(x)$ later. Note that $a(x) \geq 0$. Let $\{p_t(x) : t \geq 0, x \in \mathbb{R}\}$ denote the transition density of $\{Y_t\}$. Now we define

$$
\gamma^* = (p_1(0))^{-1} \int_0^\infty \sum_{x,y,e,e'} p(e)p(e')P(\tau(0,e) \wedge \tau(0,e') > \tau(e,e') \in du, B^0_u = x, B^e_u = y) a(y-x).
$$

(1.8)

Our critical Lotka-Volterra invariance principle is

**Theorem 1.2** Assume (A2), $d = \alpha = 1$, (A1) holds with $b(t) = t$ and $N' = N/\log N$. Then

$$
P_N^{(d)} \xrightarrow{d} P^{2\gamma,\theta,\sigma^2,1}_{X_0}
$$

as $N \to \infty$, where $\theta = \gamma^*(\theta_0 - \theta_1)$ and $\hat{p} = (p_1(0))^{-1}$.

**Remark 1.4** According to Remark 1.2, the assumption that (A1) holds with $b(t) = t$ implies that the stable random walk is recurrent.

Now, we consider the applications of the convergence theorems. One can see from the rate function form that if we set $\alpha_0 = \alpha_1 = 1$, $\xi_t$ is just the well known voter model. Identify $\xi_t$ with the set $\{x : \xi_t(x) = 1\}$ and let $\xi^A_t$ denote the voter model starting from 1’s exactly on $A$, $\xi^A_0 = A$. Write $\xi^x_t$ for $\xi^\{x\}_t$. The usual additive construction of the voter models yields

$$
\xi^A_t = \bigcup_{x \in A} \xi^x_t.
$$
The fact that $|\xi_t^0| = \sum_x \xi_t^0(x)$ is martingale tells us $|\xi_t^0|$ hits 0 eventually with probability 1. Letting $p_t = P(|\xi_t^0| > 0)$, it follows that $p_t \rightarrow 0$ as $t \rightarrow \infty$. People always want to determine the rate at which $p_t \rightarrow 0$. By using a result in [21], Bramson and Griffeath [3] were able to obtain precise asymptotics under the assumption that the underlying motion is a simple random walk. By making the voter model invariance principle, Cox and Perkins [6] reproved the main result in [3] under a weaker assumption that the jump kernel has finite variance.

In this paper as applications of the convergence theorems above we want to determine the rate at which $p_t \rightarrow 0$ under the assumption (A1). With notation $f(t) \sim g(t)$ as $t \rightarrow \infty$ we mean $\lim_{n \rightarrow \infty} f(t)/g(t) = 1$. Our result is following theorem.

**Theorem 1.3** Assume $d \geq \alpha$ and (A1) holds with $b(t) = t^{1/\alpha}$; i.e., $\{p(x)\}$ is in the domain of normal attraction of the $(\sigma, \alpha)$-stable law. Let $\gamma_1 = p_1(0)^{-1}$ for $d = \alpha$. Then as $t \rightarrow \infty$

$$p_t \sim \frac{\log t}{\gamma_1 t} \quad d = \alpha,$$

$$\sim (\gamma t)^{-1} \quad d > \alpha.$$

Moreover,

$$P(p_t|\xi_t^0| > u|\xi_t^0| > 0) \xrightarrow{t \rightarrow \infty} e^{-u}, \quad u > 0.$$

At last, we introduce some notations which will play important roles in our proofs of the main results. First, according to [13], for $0 < \alpha \leq \alpha$, we can define

$$|p|_\alpha := \sum_{x \in \mathbb{Z}^d} |x|^\alpha p(x) < \infty.$$

And by (A2), define

$$\bar{\theta} = 1 \vee \sup_{N,i} N^i \alpha_i - 1 < \infty.$$

For $D \subset \mathbb{R}^d$ and $\phi : D \rightarrow \mathbb{R}$, define

$$||\phi||_{\text{Lip}} = ||\phi||_{\infty} + \sup_{x \neq y} \frac{\phi(x) - \phi(y)}{|x - y|}.$$

For $0 < \alpha \leq 1$, let

$$||\phi||_\alpha = \begin{cases} \sup_{x \neq y, |x - y| \leq 1} \frac{\phi(x) - \phi(y)}{|x - y|^\alpha} \vee 2||\phi||_{\infty}, & \phi \equiv c \text{ for some constant } c \in \mathbb{R} \\ \phi \neq c \text{ and for } \alpha > 1 \text{ let} \end{cases}$$

$$||\phi||_\alpha^2 = 2||\phi||_{\text{Lip}}.$$

Note that for $\alpha \leq 1$,

$$\sup_{x \neq y, |x - y| \leq 1} \frac{\phi(x) - \phi(y)}{|x - y|^\alpha} \leq \sup_{x \neq y} \frac{\phi(x) - \phi(y)}{|x - y|}.$$

Thus for any $\alpha > 0$

$$||\phi||_\alpha \leq 2||\phi||_{\text{Lip}} \quad \text{and} \quad |\phi(x) - \phi(y)| \leq ||\phi||_\alpha |x - y|^\alpha. \quad (1.9)$$
Remark 1.5 Since $p(\cdot)$ in this paper may not have bounded moment of the first order, we cannot use Lipschitz norm to do estimates. Thus a ‘Hölder’ norm is introduced.

The remaining of this paper is organized as follows. In Section 2 we first give some random walk estimates and then deduce the semimartingale decompositions for the approximating processes. Finally, we prove a key result, uniform convergence of random walk generators to the generator of the symmetric stable process. In Section 3 and Section 4, we follow the strategy in [7] and [9] to prove our convergence theorems, Theorem 1.1 and Theorem 1.2. Our proofs will be deeply involved due to the lack of high moments. We will carry out in detail only the part that differs. Theorem 1.3 will be proved in Section 5.

2 Preliminaries

2.1 Random walk estimate

Recall that $\{B_t^x, x \in \mathbb{Z}^d\}$ is a collection of rate-one independent stable random walks with $B_0^x = x$. Let $p_t(x, y) = P(B_t^x = y)$ denote the transition function of $\{B_t^x\}$. We denote by $l$ the inverse of $b$. Define the characteristic function of the step function $p(\cdot)$ by

$$
\psi(\eta) = \sum_x p(x)e^{-iy\cdot\eta} \quad \text{for} \quad \eta \in T^d := (-\pi, \pi]^d.
$$

Since $p$ is symmetric, $\psi(\eta)$ is real. So

$$
p_t(0, x) \leq p_t(0, 0).
$$

The following proposition is taken from [15].

Proposition 2.1 The following are equivalent:

1. $p(\cdot)$ is in the domain of attraction of $(\sigma^2, \alpha)$-stable law.
2. $\psi(\eta) = 1 - \frac{\sigma^2}{l(1/|\eta|)} + o\left(\frac{1}{l(1/|\eta|)}\right)$ as $|\eta|$ tends to 0.
3. $\psi\left(\frac{\eta}{\psi(\eta)}\right)^n \xrightarrow{n \to \infty} \Psi(\eta), \quad \eta \in \mathbb{R}^d.$

We also have that $l$ is of regular variation of index $\alpha$ and

$$
l(x) = x^\alpha t(x),
$$

where

$$
t(x) = s(l(x))^{-\alpha}.
$$

By Lemma 2.1 in [15], for any $\epsilon > 0$, we have that there exist two positive constants $C_\epsilon, C'_\epsilon$ such that, for any $1 \leq y \leq z$,

$$
C_\epsilon y^{\alpha-\epsilon} \leq l(y) \leq C'_\epsilon y^{\alpha+\epsilon} \quad \text{and} \quad C_\epsilon \left(\frac{z}{y}\right)^{\alpha-\epsilon} \leq \frac{l(z)}{l(y)} \leq C'_\epsilon \left(\frac{z}{y}\right)^{\alpha+\epsilon}.
$$

(2.2)
A similar result also holds for $b$, with $\alpha$ replaced by $1/\alpha$. Since $p(\cdot)$ is symmetric and irreducible, $\psi$ is real and $\psi(\eta) = 1$ if and only if $\eta = 0$; see [23]. According to Proposition 2.1 we may assume that there exists a constant $C > 0$ such that

$$\frac{C}{l(1/|\eta|)} \leq 1 - \psi(\eta) \leq 1$$

for every $\eta \in T^d$. [22] tells us that for $b(t) \geq d\pi$, and $0 \leq \epsilon \leq \alpha$,

$$t(1 - \psi(\frac{\eta}{b(t)})) \geq \frac{Cl(b(t))}{l(b(t)/|\eta|)} \geq (C_\epsilon \vee C_\epsilon')(\alpha + |\eta|^{\alpha-\epsilon}). \quad (2.3)$$

Recall that $\{p_t(x) : t \geq 0, x \in \mathbb{R}\}$ denote the transition density of $\{Y_t\}$. The local limit theorem for the stable random walk which plays an important role in our proofs of main results will be given in the following proposition.

**Proposition 2.2** If (A1) holds,

$$\lim_{t \to \infty} \sup_{x \in \mathbb{Z}^d} \left| b(t)^d p_t(0, x) - p_1 \left( \frac{x}{b(t)} \right) \right| = 0 \quad (2.4)$$

and there exists a constant $C$ depending on $p(\cdot)$ such that for every $t \geq 0$ and $x \in \mathbb{R}^d$,

$$p_t(0, x) \leq C b(t)^{-d}. \quad (2.5)$$

Moreover, if $b(t) = t$ and $d = 1$,

$$\sup_{x \in \mathbb{Z}} P(B^0_t = x) \leq C(t + 1)^{-1}. \quad (2.6)$$

**Proof.** Since $l$ is a function of regular variation, by Proposition 2.1, for each $|\eta| > 0$,

$$\lim_{t \to \infty} t \left( 1 - \psi \left( \frac{\eta}{b(t)} \right) \right) = \lim_{t \to \infty} \frac{l(b(t))}{l(b(t)/|\eta|)} (\sigma^2 + o(1)) = \sigma^2 |\eta|^\alpha. \quad (2.7)$$

Then

$$\left| b(t)^d p_t(0, x) - p_1 \left( \frac{x}{b(t)} \right) \right|$$

$$\leq (2\pi)^{-d} \left| \int_{b(t)T^d} e^{-ix \cdot (\eta/b(t))} \exp \left\{-t \left( 1 - \psi \left( \frac{\eta}{b(t)} \right) \right) \right\} d\eta - \int_{b(t)T^d} e^{-ix \cdot \eta} \Psi(\eta) d\eta \right| + (2\pi)^{-d} \int_{\mathbb{R}^d \setminus b(t)T^d} \exp \left\{-\sigma^2 |\eta|^\alpha \right\} d\eta$$

$$\leq (2\pi)^{-d} \left| \int_{b(t)T^d} \exp \left\{-t \left( 1 - \psi \left( \frac{\eta}{b(t)} \right) \right) \right\} - \exp \left\{-\sigma^2 |\eta|^\alpha \right\} d\eta \right| + (2\pi)^{-d} \int_{\mathbb{R}^d \setminus b(t)T^d} \exp \left\{-\sigma^2 |\eta|^\alpha \right\} d\eta.$$
Then the Dominated Convergence Theorem with (2.3) yields (2.4). For (2.5), when \( b(t) \geq d \pi \),

\[
p_t(0, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \eta} \exp \{-t (1 - \psi(\eta))\} \, d\eta
\]

\[
\leq (2\pi)^{-d} b(t)^{-d} \int_{b(t)\mathbb{R}^d} \exp \{-t \left(1 - \psi \left(\frac{\eta}{b(t)}\right)\right)\} \, d\eta
\]

\[
= (2\pi)^{-d} b(t)^{-d} \int_{\mathbb{R}^d} \exp \{-C_p \sqrt{C_p} |\eta|^{\alpha + \epsilon} + |\eta|^{\alpha - \epsilon}\} \, d\eta
\]

\[
\leq C b(t)^{-d},
\]

where the second inequality follows from (2.3). Then (2.5) holds for every \( t \geq 0 \). We complete the proof. \( \square \)

The following two propositions consider the growth of the stable random walk.

**Proposition 2.3** (a) If \( z_T \in \mathbb{Z}^d \) and \( t_T > 0 \) satisfy

\[
\lim_{T \to \infty} \frac{z_T}{b(T)} = z \text{ and } \lim_{T \to \infty} \frac{t_T}{T} = s > 0
\]

then

\[
\lim_{T \to \infty} b(T)^d P(B_T^0 = z_T) = \frac{p_1(z/s)}{s^d}.
\]

(b) For each \( K > 0 \), there is a constant \( C_{2.10} > 0 \) such that

\[
\liminf_{T \to \infty} \inf_{|x| \leq Kb(T)} b(T)^d P(B_T^0 = x) \geq C_{2.10}(K).
\]

**Proof.** By (2.8) and Remark 1.1, we have \( \lim_{T \to \infty} \frac{b(t_T)}{b(T)} = s \). Then (2.9) follows from (2.4). For (b), when \( \alpha = 2 \), by (2.4), the desired result is immediate. When \( 0 < \alpha < 2 \), recall that \( \{p_t(x) : t \geq 0, x \in \mathbb{R}^d\} \) is the transition density of a symmetric \( \alpha \)-stable process. By the arguments after Remark 5.3 of [1], there exists two positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \leq p_t(x) \leq c_2 \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right).
\]

By above bounds and (2.4),

\[
\liminf_{T \to \infty} \inf_{|x| \leq Kb(T)} b(T)^d P(B_T^0 = x) = \liminf_{T \to \infty} \inf_{|x| \leq Kb(T)} p_1(x/b(T)) \\
\geq c \left(1 \wedge K^{d+\alpha}\right).
\]

The desired result follows readily. \( \square \)

**Proposition 2.4** Assume \( d = 1 \). If \( g_1 \) and \( g_2 \) are two positive functions on \( \mathbb{R}^+ \) such that \( g_1(x) \to +\infty, \ g_2(x) \to +\infty \) as \( x \to +\infty \), then there is exists a constant \( C_{2.12} \) which only depends on \( p \) such that

\[
P\left(|B_T^0_{g_1(N)}| \geq g_2(N)\right) \leq \frac{C_{2.12} p_1(N)}{l(g_2(N))}.
\]
Proof. First,
\[
P \left( |B_0^0(N)| \geq g_2(N) \right) \leq P \left( \max_{u \leq g_1(N)} |B_u^0| \geq g_2(N) \right).
\]
Note that \( \{B_u^0 : u \geq 0\} \) is a compound Poisson process whose Lévy measure is given by
\[
\nu_0(dz) := \sum_{y \in \mathbb{Z}^d} p(y) \delta_y(dz),
\]
which is a symmetric measure. According to the arguments in Section 3 of \[19\],
\[
P \left( \max_{u \leq g_1(N)} |B_u^0| \geq g_2(N) \right) \leq C g_1(N) \left( \nu_0(z : |z| > g_2(N)) + g_2(N)^{-2} \int_{|z| \leq g_2(N)} z^2 \nu_0(dz) \right),
\]
where \( C \) is a positive constant; see (3.2) of \[19\]. Since \( p(\cdot) \) is in the domain of attraction of \((\sigma, \alpha)\)-stable law, we have
\[
\frac{x^2[\nu_0(z : |z| > x)]}{\int_{|z| \leq x} z^2 \nu_0(dz)} \rightarrow \frac{2 - \alpha}{\alpha}
\]
and
\[
\frac{x \int_{|z| \leq b(x)} z^2 \nu_0(dz)}{b(x)^2} \rightarrow C_0
\]
as \( x \to \infty \) for some constant \( C_0 > 0 \); see (5.16) and (5.23) in Chapter XVII of \[13\]. By (2.13) there exists a constant \( C_1 \) independent of \( N \) such that
\[
\nu_0(z : |z| > g_2(N)) \leq C_1 g_2(N)^{-2} \int_{|z| \leq g_2(N)} z^2 \nu_0(dz).
\]
According to (2.14), there exists another constant \( C_2 \) independent of \( N \) such that
\[
g_2(N)^{-2} \int_{|z| \leq g_2(N)} z^2 \nu_0(dz) \leq \frac{C_2}{l(g_2(N))},
\]
(Recall that \( l \) is the inverse function of \( b \).) Thus
\[
P \left( \max_{u \leq g_1(N)} |B_u^0| \geq g_2(N) \right) \leq CC_2(C_1 + 1) \frac{g_1(N)}{l(g_2(N))}
\]
which yields the desired result. \( \square \)

2.2 Semimartingale decompositions

Some results in this subsection are exactly the same with those in Section 3 of \[9\]. For complement, we list them here. Let \( \xi_t^N \) be the rescaled Lotka-Volterra model we have constructed in Section 1.2. As in \[9\], we introduce the following notation. If \( \phi = \phi_s(x), \ \dot{\phi}_s(x) \equiv \frac{\partial}{\partial s} \phi(s, x) \in C_b([0, T] \times \mathbb{S}_N), \)
and \( s \leq T \), define

\[
\mathcal{A}_N(\phi_s)(x) = \sum_{y \in \mathcal{S}_N} N p_N(y - x)(\phi_s(y) - \phi_s(x))
\]

(2.15)

\[
D_t^{N,1}(\phi) = \int_0^t X_s^N(\mathcal{A}_N \phi_s + \dot{\phi}_s)ds
\]

(2.16)

\[
D_t^{N,2}(\phi) = \frac{N(\alpha_0^N - 1)}{N'} \int_0^t \sum_{x \in \mathcal{S}_N} \phi_s(x)1_{\xi_i(x) = 0}(f_1^N(x, \xi_s^N))^2ds
\]

(2.17)

\[
D_t^{N,3}(\phi) = \frac{N(\alpha_1^N - 1)}{N'} \int_0^t \sum_{x \in \mathcal{S}_N} \phi_s(x)1_{\xi_i(x) = 1}(f_1^N(x, \xi_s^N))^2ds
\]

(2.18)

\[
\langle M^N(\phi) \rangle_{1, t} = \frac{N}{(N')^2} \int_0^t \sum_{x \in \mathcal{S}_N} \phi_s^2(x) \sum_{y \in \mathcal{S}_N} p_N(y - x)(\xi_s^N(y) - \xi_s^N(x))^2ds
\]

(2.19)

\[
\langle M^N(\phi) \rangle_{2, t} = \frac{1}{(N')^2} \int_0^t \sum_{x \in \mathcal{S}_N} \phi_s^2(x) [(\alpha_0^N - 1)1_{\xi_i(x) = 0}(f_1^N(x, \xi_s^N))^2
\]

\[+(\alpha_1^N - 1)1_{\xi_i(x) = 1}(f_1^N(x, \xi_s^N))^2]ds
\]

(2.20)

If \( X \) is a process let \((\mathcal{F}_t^N, t \geq 0)\) be the right-continuous filtration generated by \( X \). The following proposition is a version of Proposition 3.1 of [9]. For its proof, please go to Section 2 of [7].

**Proposition 2.5** For \( \phi, \dot{\phi} \in C_b([0, T] \times \mathcal{S}_N) \) and \( t \in [0, T] \),

\[
X_t^N(\phi_t) = X_0^N(\phi_0) + D_t^N(\phi) + M_t^N(\phi),
\]

(2.21)

where

\[
D_t^N(\phi) = D_t^{N,1}(\phi) + D_t^{N,2}(\phi) - D_t^{N,3}(\phi)
\]

(2.22)

and \( M_t^N(\phi) \) is an \( \mathcal{F}_t^N \)-square-integrable martingale with predictable square function

\[
\langle M^N(\phi) \rangle_t = \langle M^N(\phi) \rangle_{1, t} + \langle M^N(\phi) \rangle_{2, t}.
\]

(2.23)

The following lemma is a generalization of Lemma 3.5 of [7] and Lemma 4.8 of [9].

**Lemma 2.1** There is a constant \( C \) such that if \( \phi : [0, T] \times \mathcal{S}_N \to \mathbb{R} \) is a bounded measurable function, then

(a) \( \langle M^N(\phi) \rangle_{2, t} = \int_0^t m_{2, s}^N(\phi)ds \), where

\[
|m_{2, s}^N(\phi)| \leq C \frac{||\dot{\phi}_s||^2}{(N')^2} X_s^N(1).
\]

(2.24)

(b) For \( \alpha < 1 \land \alpha \),

\[
\langle M^N(\phi) \rangle_{1, t} = 2 \int_0^t X_s^N((N/N')^{\alpha}f_{\alpha}^N(\xi_s^N))ds + \int_0^t m_{1, s}^N(\phi_s)ds,
\]

(2.25)
where
\[ |m_{1,s}^N(\phi)| \leq \left[ X_s^N (1) \frac{2N||\phi||_2^2 |p|_\alpha}{N' b(N')^2} \right] \wedge \left[ \frac{2N||\phi||_\infty^2 X_s^N (1)}{N'} \right]. \] (2.26)

(c) For \( i = 2, 3 \), \( D_t^{N,i}(\phi) = \int_0^t d_s^{N,i}(\phi) ds \) for \( t \leq T \), where for all \( N, s \leq T \),
\[ |d_s^{N,i}(\phi)| \leq C||\phi||_\infty X_s^N \left( (N/N') f_0^N (\xi_s^N) \right). \]

**Remark 2.1** Note that when \( N' = N \), since \( f_0^N \leq 1 \),
\[ |d_s^{N,i}(\phi)| \leq C||\phi||_\infty X_s^N (1), \quad i = 2, 3. \]

**Proof.** (a) In the following of this proof, with \( C \) we denote a positive constant which may change from line to line. Since \( f_0^N \leq 1, f_1^N \leq 1 \) and \( 1_{\{\xi_s^N(1) = 1\}} = \xi_s^N(x) \), the definition of \( \langle M^N(\phi) \rangle_{2,t} \) and the fact that \( f_0^N + f_1^N = 1 \) imply
\[ |m_{2,s}^N(\phi)| \leq \frac{||\phi||_\infty^2 \sup_{N'} \frac{N'^3}{(N')^3} (f_1^N(x, \xi_s^N)) \sum_{x \in S_N} (f_1^N(x, \xi_s^N)) - 1|}{N'^2} \sum_{x \in S_N} (f_1^N(x, \xi_s^N)) (1_{\{\xi_s^N(x) = 1\}}) \sum_{y \in S_N} (f_1^N(y, \xi_s^N)) + \frac{C||\phi||_\infty^2}{(N')^2} X_s^N (1), \]
where the second inequality follows from (A2). For (b), note that
\[ \langle M^N(\phi) \rangle_{2,t} = \frac{1}{(N')^2} \int_0^t \sum_{x \in S_N} \phi_s^2(x) \sum_{y \in S_N} Np_N(y - x)(\xi_s^N(y) - \xi_s^N(x))^2 ds \]
\[ = \frac{1}{(N')^2} \int_0^t \sum_{x \in S_N} \phi_s^2(x) \sum_{y \in S_N} Np_N(y - x) (2\xi_s^N(x)(1 - \xi_s^N(y))) ds \]
\[ + \frac{1}{(N')^2} \int_0^t \sum_{x \in S_N} \phi_s^2(x) \sum_{y \in S_N} Np_N(y - x) (\xi_s^N(y) - \xi_s^N(x)) ds. \]

Thus (2.26) holds with
\[ m_{1,s}^N(\phi) = \frac{N}{(N')^2} \sum_{x \in S_N} \phi_s^2(x) \sum_{y \in S_N} p_N(y - x)(\xi_s^N(y) - \xi_s^N(x)) \]
\[ = \frac{N}{(N')^2} \sum_{x \in S_N} \phi_s^2(x) \sum_{y \in S_N} p_N(y - x)(\xi_s^N(y) - \xi_s^N(x)) \sum_{x,y \in S_N} p_N(y - x)(\xi_s^N(y) - \xi_s^N(x)) \]
\[ \leq 2N||\phi||^2_\infty X_s^N (1). \]
On the other hand, 
\[ |\phi_s^2(x) - \phi_s^2(y)| \leq 2||\phi||_2^2 |x - y|^{\alpha} \]
for \( \alpha < 1 \land \alpha \). Thus
\[
m^{N}_{1,s}(\phi) \leq 2(N/N')||\phi||_2^2 \frac{1}{N'} \sum_y \xi_s^N(y) \sum_x |y - x|^\alpha p_N(y - x) \]
\[
\leq X^N_s(1) \frac{2N||\phi||_2^2 \alpha}{N'b(N)^\alpha}. \]

We complete the proof of (b). For (c), according to (A2), the fact that both \( f_0^N \) and \( f_1^N \) are less than 1 yields
\[
|d^{N,i}_s(\phi)| \leq \frac{N}{N'} \sup_N N'|\alpha_{2}^N - 1| ||\phi_s||_\infty \frac{1}{N'} \sum_x \sum_y p_N(y - x) \xi_s^N(x)(1 - \xi_s^N(y)) \]
\[
\leq C||\phi_s||_\infty X^N_s((N/N')f_0^N(\xi_s^N)). \]

We are done. \( \square \)

2.3 Convergence of Generators

In this subsection we consider the uniform convergence of \( \mathcal{A}_N \). Recall the definition of generators of symmetric stable processes and the stable random walk \( Z_n \) defined in section 1.2. For each \( N > 1 \), let \( \{P^N_t: t \geq 0\} \) be a rate-\( N \) Poisson process which is independent of \( \{U_i: i \geq 1\} \). Then
\[
\hat{Z}^N_t = b(N)^{-1} \sum_{i=1}^{P^N_t} U_i
\]
is a compound Poisson process on \( \mathbb{R}^d \) whose Lévy measure is given by
\[
\nu_N(dy) := \sum_{z \in S_N} Np_N(z)\delta_z(dy);
\]
see [20]. Note that both the law of \( \hat{Z}^N_1 \) and the \((\sigma^2, \alpha)\)-stable law are infinitely divisible distributions. We also have that
\[
\mathbf{E} \left( e^{-i\hat{Z}^N_1 \cdot \eta} \right) = \exp \left\{ -N \left( \psi \left( \frac{\eta}{b(N)} \right) - 1 \right) \right\}.
\]
By [2.7],
\[
\hat{Z}^N_1 \xrightarrow{(d)} Y_1 \quad \text{as} \quad N \to \infty.
\]
According to Theorem 8.7 of [20] and its proof, we see
\[
\rho_N(dy) := \frac{|y|^2}{1 + |y|^2} \nu_N(dy) \to \rho(dy) := \frac{\sigma^2 |y|^2}{1 + |y|^2} \nu(dy) \quad \text{in} \ M(\mathbb{R}^d).
\]
For \( f \in C_b(\mathbb{R}^d) \), define
\[
||f||_{BL} = \sup_x |f(x)| \vee \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.
\]

Let \( P, Q \) be two probability measures on \( \mathbb{R}^d \). Set
\[
||P - Q||_{BL} := \sup_{||f||_{BL} = 1} \left| \int f dP - \int f dQ \right|.
\]

It is easy to see that
\[
||P - Q||_{BL} = \sup_{||f||_{BL} < \infty} \left| \int f dP - \int f dQ \right|. 
\] (2.27)

By Problem 3.11.2 of [12],
\[
||P - Q||_{BL} \leq 3 \mathcal{M}(P, Q),
\] (2.28)
where \( \mathcal{M} \) denotes the Prohorov metric; see Chapter 3 of [12].

**Lemma 2.2** For \( \phi \in C^{1,3}_b([0, T] \times \mathbb{R}^d) \),
\[
\lim_{N \to \infty} \sup_{s \leq T} \left\| A_N\phi_s - \frac{\sigma^2 \Delta^{\alpha/2}\phi_s}{2} \right\|_{\infty} = 0.
\]

Moreover, for each \( R < \infty \), the rate of convergence is uniform on
\[
H_R := \left\{ \phi \in C^{1,3}_b([0, T] \times \mathbb{R}^d) : \sup_{s,i,j,k} \left( ||\phi||_{\infty} + ||(\phi)_i||_{\infty} + ||(\phi)_ij||_{\infty} + ||(\phi)_{ijk}||_{\infty} \right) < R \right\},
\]
where the subscripts \( i, j, k \) indicate partial derivatives with respect to the spatial variable.

**Proof.** Recall that \( D_j = \frac{\partial}{\partial x_j} \). Define
\[
g_s(x, y) = \left[ \phi_s(x + y) - \phi_s(x) - \frac{1}{1 + |y|^2} \sum_{i=1}^d y_j D_j \phi_s(x) \right] \cdot \frac{1 + |y|^2}{|y|^2}.
\]

Since \( p_N \) is symmetric, we may rewrite
\[
A_N\phi_s(x) = \int g_s(x, y) \rho_N(dy)
\]
and we also have that
\[
\frac{\sigma^2 \Delta^{\alpha/2}\phi_s(x)}{2} = \int g_s(x, y) \rho(dy).
\]

Let \( h : \mathbb{R}^d \to [0, 1] \) be a \( C^\infty \) function such that
\[
B(0, 1) \subset \{ x : h(x) = 0 \} \subset \{ x : h(x) < 1 \} \subset B(0, 2)
\]
and
\[
B(0, 2)^c \subset \{ x : h(x) = 1 \}.
\]
Define $h_k(x) = h(kx)$ for $k \geq 1$. Let

$$g_k(s, x, y) := h_k(y)g_s(x, y).$$

Then $g_k(s, x, y) = g_s(x, y)$ for $|y| > 2/k$. One can check that

$$\sup_{k} \sup_{\phi \in H_R} \sup_s \sup_x (||g_k(s, x, \cdot)||_\infty + ||g_s(x, \cdot)||_\infty) < C_d R$$

and for each $k \geq 1$

$$\sup_{\phi \in H_R} \sup_s \sup_x \frac{\sum_{j=1}^d \partial g_k(s, x, y)}{\partial y_j} \|g_k(s, x, y)\|_\infty < kC_d R,$$

where $C_d$ is a constant which only depend on $d$. Typically, for each $k \geq 1$,

$$\sup_{\phi \in H_R} \sup_s \sup_x \|g_k(s, x, \cdot)\|_{BL} < (k + 1)C_d R.$$

By (2.27) and (2.28), we obtain

$$\left| \frac{\int g_k(s, x, y)\rho_N(dy) - \int g_k(s, x, y)\rho(dy)}{\rho_N(\mathbb{R}^d) - \rho(\mathbb{R}^d)} \right|$$

$$\leq (k + 1)C_d R \cdot 3M \left( \frac{\rho_N}{\rho_N(\mathbb{R}^d)}, \frac{\rho}{\rho(\mathbb{R}^d)} \right)$$

$$\rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

By triangle inequality,

$$\sup_{\phi \in H_R} \sup_s \sup_x \left| \int g_k(s, x, y)\rho_N(dy) - \int g_k(s, x, y)\rho(dy) \right|$$

$$\leq C_d R \left| \rho_N(\mathbb{R}^d) - \rho(\mathbb{R}^d) \right|$$

$$+ \rho(\mathbb{R}^d) \sup_{\phi \in H_R} \sup_s \sup_x \left| \int g_k(s, x, y)\rho_N(dy) - \int g_k(s, x, y)\rho(dy) \right|$$

$$\rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Using triangle inequality again,

$$\sup_{\phi \in H_R} \sup_s \sup_x \left| A_N \phi_s - \frac{\sigma^2 \Delta^{n/2} \phi_s}{2} \right|_\infty$$

$$\leq \sup_{\phi \in H_R} \sup_s \sup_x \left| \int g_s(x, y)\rho_N(dy) - \int g_k(s, x, y)\rho_N(dy) \right|$$

$$+ \sup_{\phi \in H_R} \sup_s \sup_x \left| \int g_k(s, x, y)\rho_N(dy) - \int g_k(s, x, y)\rho(dy) \right|$$

$$+ \sup_{\phi \in H_R} \sup_s \sup_x \left| \int g_k(s, x, y)\rho(dy) - \int g_s(x, y)\rho(dy) \right|$$

$$\leq C_d R \rho_N\{y : |y| \leq 2/k\} + C_d R \rho\{y : |y| \leq 2/k\}$$

$$+ \sup_{\phi \in H_R} \sup_s \sup_x \left| \int g_k(s, x, y)\rho_N(dy) - \int g_k(s, x, y)\rho(dy) \right|$$
Note that $\rho(dy)$ is absolutely continuous with respect to the Lebesgue measure. Letting $N$ go to infinity above yields

$$\lim_{N \to \infty} \sup_{\phi \in H_R} \sup_{s \leq T} ||A_N \phi_s - \frac{\sigma^2 \Delta^{\alpha/2} \phi_s}{2}||_\infty \leq 2C_d R \rho(\{y : |y| \leq 2/k\}).$$

Then since $\rho(\{0\}) = 0$ the desired result follows readily if we let $k \to \infty$. $\square$

3 Proof of Theorem 1.1

In this section, we assume the stable random walk $Z$ is transient, which is equivalent to

$$\int_1^\infty \frac{dx}{b(x)^d} < \infty.$$ 

When $d = \alpha = 1$, above condition implies that $s(x) \to \infty$ as $x \to \infty$. The strategy of the proof is the same with that used in [7]. In [7] the authors worked with a more general class of particle systems they called voter perturbations. As a result we will specialize the setting there for the reader’s convenience. Let $\{\hat{B}^{N,x}_N : x \in S_N\}$ denote a rate-$N$ continuous time coalescing random walk system on $S_N$ with step function $p_N$ such that $\hat{B}^{N,x}_0 = x$. For a finite set $A \subset S_N$, let

$$\hat{\tau}^N(A) = \inf\{t \geq 0 : |\{\hat{B}^{N,x}_N, x \in A\}| = 1\}.$$ 

We also need a collection of independent (noncoalescing) rate-$N$ continuous time random walks on $S_N$ with step function $p_N$, which we will denote $\{B^{N,x}_N : x \in S_N\}$, such that $B^{N,x}_0 = x$. For any finite subset $A$ of $Z^d$, let $\hat{\tau}^N(A) = \hat{\tau}(A/b(N))$. We first check the kernel assumptions in Section 1.2 of [7].

Lemma 3.1 There exists a positive sequence $\{\epsilon^*_N\}$ with $\epsilon^*_N \to 0$ and $N \epsilon^*_N \to \infty$. such that the following hold:

$$\lim_{N \to \infty} N P(B^{N,0}_N) = 0 \quad \text{(3.1)}$$

$$\lim_{N \to \infty} \sum_{e \in S_N} p_N(e) P(\hat{\tau}^N(\{0,e\}) \in (\epsilon^*_N, t]) = 0 \quad \text{for all} \quad t > 0,$$

$$\lim_{N \to \infty} \sum_{e \in S_N} p_N(e) P(\hat{\tau}^N(\{0,e\}) > \epsilon^*_N) = \gamma_e. \quad \text{(3.2)}$$

and if we define $\sigma_N(A) = P(\hat{\tau}^N(A) \leq \epsilon^*_N)$ for any finite subset $A$ of $Z^d$, then

$$\lim_{N \to \infty} \sigma_N(A) = \sigma(A) \quad \text{exists.} \quad \text{(3.3)}$$

Proof. First, consider the case $d > \alpha$. We may assume $\epsilon^*_N = N^{-\epsilon^*}$ for some $0 < \epsilon^* < 1$. We need to find a suitable condition on $\epsilon^*$. Recall that $b$ is a function of regular variation with index $1/\alpha$. Given $\epsilon < 1/2$, there exist two positive constants $C_\epsilon$, $C'_\epsilon$ such that for $y \geq 1$,

$$C_\epsilon y^{1/\alpha - \epsilon} \leq b(y) \leq C'_\epsilon y^{1/\alpha + \epsilon}.$$
By (2.5), we see

\[ NP(B_{\epsilon_N}^{N,0} = 0) = NP(B_{N\epsilon_N}^{N,0} = 0) \leq C N (N\epsilon_N^*)^{-d} \leq \frac{C N (N\epsilon_N^*)^d}{\epsilon_N^{d/\alpha}}. \]

A simple calculation shows that given \( \epsilon < 1/2 \), we can set

\[ \epsilon_N^* = N^{-\epsilon} \quad \text{for} \quad \epsilon^* < 1 - \frac{\alpha}{d - \alpha d\epsilon} < 1. \]

Then \( NP(B_{\epsilon_N}^{N,0} = 0) \to 0 \) as \( N \to \infty \). When \( d = \alpha = 1 \), since \( s(x) \to \infty \) as \( x \to \infty \), we can set \( x(0) = 0 \) and \( \forall k \geq 1 \), there exists \( x(k) > x(k-1) \), such that if \( x > x(k) \), \( s(x) > k \). Then \( x(k) \to \infty \) as \( k \to \infty \). Define function \( s' \) on \( \mathbb{R}^+ \) such that \( s'(x) = 1 \) for \( 0 \leq x \leq x(1) \) and

\[ s'(x) = k, \quad \text{for} \quad x(k) < x \leq x(k+1) \quad \text{and} \quad k \geq 1. \]

It is easy to see that \( s'(x) \uparrow \infty \) as \( x \to \infty \) and \( \forall x > x(1), \ s'(x) < s(x) \). Define

\[ \epsilon_N^* := \left( (\log N) \wedge \frac{1}{s'(N/\log N)} \right)^{-1}. \]

Then \( N\epsilon_N^* \geq N/\log N \) and \( N\epsilon_N^* \to \infty \) as \( N \to \infty \). Thus when \( N \) is large enough \((N\epsilon_N^* > x(1))\),

\[ \epsilon_N^* s(N\epsilon_N^*) \geq s'(N\epsilon_N^*)/\sqrt{s'(N/\log N)} \geq \sqrt{s'(N/\log N)} \xrightarrow{N \to \infty} \infty. \]

We have that

\[ NP(B_{\epsilon_N}^{N,0} = 0) \leq C N b(N\epsilon_N^*)^{-1} = \frac{1}{\epsilon_N s(N\epsilon_N^*)} \to 0 \]

as \( N \to \infty \). Next,

\[
\sum_{e \in S_N} p_N(e) P(\hat{\tau}^N(\{0,e\}) > \epsilon_N^*) = \sum_{e \in \mathbb{Z}^d} p(e) P(\hat{\tau}(0,e) > N\epsilon_N^*) \\
\quad \to \sum_{e \in \mathbb{Z}^d} p(e) P(\hat{\tau}(0,e) = \infty) = \gamma_e.
\]

Note that

\[ P(\hat{\tau}^N(\{0,e\}) \in (\epsilon_N^*, t]) = P(\hat{\tau}^N(\{0,e\}) > \epsilon_N^*) - P(\hat{\tau}^N(\{0,e\}) > t). \]

Then the second limit also holds. For any finite set \( A \subset \mathbb{Z}^d \),

\[ \sigma_N(A) = P(\hat{\tau}^N(A) \leq \epsilon_N^*) = P(\hat{\tau}(A) \leq N\epsilon_N^*) \to P(\hat{\tau}(A) < \infty) = \sigma(A). \]

We are done. \( \square \)

Next, we consider the ‘perturbation’ term. As in [7], let \( P_F \) denote the set of finite subsets of \( \mathbb{Z}^d \). For \( A \in P_F, \ x \in S_N, \ \xi \in \{0,1\}^{S_N} \), define

\[
\chi_N(A, x, \xi) = \prod_{e \in A/b(N)} \xi(x + e).
\]
We also define

\[
\beta_N(A) = \begin{cases} 
\theta^N_0(p(e))^2, & A = \{e\}, \\
2\theta^N_0 p(e)p(e'), & A = \{e, e'\}, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\delta_N(A) = \begin{cases} 
\theta^N_1, & A = \emptyset, \\
\theta^N_1[(p(e))^2 - 2p(e)], & A = \{e\}, \\
2\theta^N_0 p(e)p(e'), & A = \{e, e'\}, \\
0, & \text{otherwise}.
\end{cases}
\]

**Remark 3.1** According to the arguments in Section 1.2 of [7], the ‘Perturbation assumptions’ (P1) to (P5) there are satisfied by the above coefficients with \(l_N = b(N)\).

The following proposition is exactly the same with Proposition 3.3 of [7]. The Proposition 3.3 of [7] was proved in Section 4 there in which the proof of the results did not use any of the kernel assumptions. Thus we can state the following proposition without proof.

**Proposition 3.1** For \(K, T > 0\), there exists a finite constant \(C_1(K, T)\) such that if \(\sup_N X^N_0(1) \leq K\), then

\[
\sup_N E \left( \sup_{t \leq T} X^N_t(1)^2 \right) \leq C_1(K, T).
\]

This bound allows us to employ the \(L^2\) arguments of [7]. Next, we consider another technical result, a version of Proposition 3.4 of [7]. For \(A \in P_F\), \(\phi : [0, T] \times S_N \rightarrow \mathbb{R}\) bounded and measurable, \(K > 0\) and \(t \in [0, T]\), define

\[
\mathcal{E}_N(A, \phi, K, t) \defeq \sup_{X^N \in \mathcal{A}} E \left( \left( \int_0^t \left[ \sum_{x} \phi(s,x) \chi_N(A, x, \xi_s^N) - \sigma_N(A) X^N_s(\phi_s) \right] ds \right)^2 \right).
\]

Set \(c_\beta = \sup_N |\theta^N_0|^p \sum_{e,e' \in \mathbb{Z}^d} p(e)p(e')\) and \(\bar{c} = c_\beta + k_\delta\), where \(k_\delta = \sup_N |\theta^N_1|\). The following proposition is a version of Proposition 3.4 of [7].

**Proposition 3.2** There is a positive sequence \(\epsilon_N \rightarrow 0\) as \(N \rightarrow \infty\), and for any \(K, T > 0\), a constant \(C_2(K, T) > 0\), \(\alpha \leq 1 \wedge \alpha\), such that for any \(\phi \in C_b([0, T] \times S_N)\) satisfying \(\sup_{s \leq T} \|\phi_s\|_{\text{Lip}} \leq K\), nonempty \(A \in P_F\), \(\bar{a} \in A\), \(J \geq 1\) and \(0 \leq t \leq T\),

\[
\mathcal{E}_N(A, \phi, K, t) \leq C_4(K, T) \left[ \epsilon_N^{2} e^{\delta_N} + J^{-2} \right. \left. + J^{2} \left( \epsilon_N |A| + (\sigma_N(A) \wedge (\epsilon_N + \left| \frac{\bar{a}}{b(N)} \right|) \right) \right].
\]

In particular, \(\lim_{N \rightarrow \infty} \sup_{t \leq T} \mathcal{E}_N(A, \phi, K, t) = 0\).
Proof. We can follow the arguments in Section 5 and Section 6 of [7]. In fact, only a small trick is needed. For $\alpha \in (0, 2]$ and $d > \alpha$, we may find an $\alpha < \alpha$ which is close enough to $\alpha$ so that

$$E(|B_{N}^{\epsilon_N}|^\alpha) = \frac{N\epsilon_N^\alpha |p|}{b(N)^\alpha} \to 0 \quad \text{as } N \to \infty. \quad (3.5)$$

(Note that $b$ is a function of regular variation with index $1/\alpha$ and recall the choice of $\epsilon_N^*$ in Lemma 3.1 when $d > \alpha$). Fix this $\alpha$. For $||\phi||_{\text{Lip}} \leq K$, (1.9) implies

$$E \left( \left| \phi \left( y - \frac{\bar{a}}{b(N)} + B_{N}^{\epsilon_N} \right) - \phi(y) \right| \right) \leq 2KE \left| B_{N}^{\epsilon_N} - \frac{\bar{a}}{b(N)} \right|^\alpha \leq 2KE \left| B_{N}^{\epsilon_N} \right|^\alpha + 2K \frac{\bar{a}}{b(N)} |^\alpha.$$

When $\alpha > 1$, we may assume $\alpha \wedge 1 = 1 \leq \alpha$. (3.5) suggests

$$E \left( |B_{N}^{\epsilon_N}|^\alpha \right) \to 0 \quad \text{as } N \to \infty.$$

When $d = \alpha = 1$, for any $\alpha < 1$, by (1.9),

$$E \left( \left| \phi \left( y - \frac{\bar{a}}{b(N)} + B_{N}^{\epsilon_N} \right) - \phi(y) \right| \right) \leq 2KE \left( |B_{N}^{\epsilon_N} - \frac{\bar{a}}{b(N)}|^{\alpha} \right) < s(N)^{-1} + 2||\phi||_{\infty}P \left( \left| B_{N}^{\epsilon_N} \right| \geq s(N)^{-1} \right) \leq \frac{2K}{s(N)^{\alpha}} + 2K \frac{\bar{a}}{b(N)} |^\alpha + 2KP \left( \left| B_{N}^{\epsilon_N} \right| \geq s(N)^{-1} \right).$$

We want to estimate the last term above for $s = \epsilon_N^*$. First,

$$P \left( \left| B_{N}^{\epsilon_N} \right| \geq s(N)^{-1} \right) = P \left( \left| B_{N}^{0} \right| \geq N \right).$$

By Proposition 2.1.2 and (2.2), $P(|B_{N}^{\epsilon_N}| \geq s(N)^{-1})$ is bounded by

$$C_{2.12} \frac{N\epsilon_N^*}{l(N)} = C_{2.12} \frac{l(N\epsilon_N^*s(N\epsilon_N^*))}{l(N)} \leq C_{2.12} \frac{s(N\epsilon_N^*)}{C_s(N\epsilon_N^*)^{1-\epsilon}}.$$

Recall the choice of $\epsilon_N^*$ in the Lemma 3.1 when $d = \alpha = 1$. The last term above goes to zero when $N \to \infty$. Set

$$\epsilon_N = 2KE(|B_{N}^{\epsilon_N}|^\alpha) \quad \text{for } d > \alpha$$

and

$$\epsilon_N = \frac{2K}{s(N)^{\alpha}} + 2KP \left( \left| B_{N}^{\epsilon_N} \right| \geq s(N)^{-1} \right) \quad \text{for } d = \alpha = 1.$$

Then $\epsilon_N \to 0$ as $N \to \infty$ and

$$E \left( \left| \phi \left( y - \frac{\bar{a}}{b(N)} + B_{N}^{\epsilon_N} \right) - \phi(y) \right| \right) \leq \epsilon_N + 2K \frac{\bar{a}}{b(N)} |^\alpha. \quad (3.6)$$

With (3.6) in mind, the reader may go back to [7] for the proof of this proposition. In fact, as in [7], we first define $\eta_N$ as (5.1) of [7] and decompose it into four error terms.
$\eta_i^N$, $i = 1, 2, 3, 4$. And decompose $\eta_3^N$ into two terms, $\eta_{3,1}^N$ and $\eta_{3,2}^N$, as in (5.15) and (5.16) of [7] respectively. (3.6) will be used when we estimate $\eta_{3,2}^N(s)$ as on p.944 of [7]. Only a part of the proof at the end of Section 5 of [7] is needed to be modified. When estimate $\eta_{3,1}^N$, we also need (2.1).

The following technical lemma will be used in checking the Compact Containment Condition.

**Lemma 3.2** Let $P_t^N$ denote the semigroup associated with generator $A_N$. We have

$$X_0^N(P_s^N(1_{B(0,n)\cap})) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly in $N$ and $s \leq t$.

**Proof.** Since

$$X_0^N(P_s^N(1_{B(0,n)\cap})) \leq X_0^N(B(0,n/2)\cap) + X_0^N(1)P(|B_s^N| > n/2),$$

and (A2) holds, it suffices to show $P(|B_s^N| > n/2)$ goes to 0 uniformly as $n \rightarrow \infty$. For $0 < c < 1$, note that

$$P(|B_s^N| > cn) = P(|B_s^N| > cnb(N)). \quad (3.7)$$

When $\alpha = 2$, the desired result follows from Chebyshev’s inequality. We only need to consider the case of $\alpha < 2$. Clearly, we can deal separately with the different coordinates of $B_s^N$, and the distribution of each coordinate of $Y_1$ is a dimension-one $(\sigma^2, \alpha)$-stable distribution. (A1) implies that each coordinate of $p(\cdot)$ is in the domain of attraction of the dimension-one $(\sigma^2, \alpha)$-stable distribution. Thus, for this proof only, we can assume $d = 1$ (Here we drop the assumption $d \geq \alpha$). By Proposition 2.4 and (2.2), the right hand side of (3.7) is bounded by

$$C_{2.12}^{Ns} l(b(N)s) \leq C_{2.12}^{Ns/C^c(cn)\alpha/\epsilon},$$

where the inequality holds for $cn > 1$. The desired result is then immediate. □

**Proof of Theorem 1.1.** Now, we are in position to prove Theorem 1.1. First, we check the compact containment condition. Let $h_n : \mathbb{R}^d \rightarrow [0, 1]$ be a $C^\infty$ function such that

$$B(0,n) \subset \{ x : h_n(x) = 0 \} \subset \{ x : h_n(x) < 1 \} \subset B(0,n+1)$$

and

$$\sup_n \sum_{i,j,k \leq d} \|(h_n)_i\|_{\infty} + \|(h_n)_{ij}\|_{\infty} + \|(h_n)_{ijk}\|_{\infty} \equiv C_h < \infty.$$
establish the Compact Containment Condition, we may follow the proof of Proposition 3.9 of [7]. In fact, the argument above and Lemma 3.2 show that
\[
\lim_{(N,n) \to \infty} E \left( \int_0^t X_s^N (|A_N h_n|) ds \right) = 0.
\]

Then the following argument for the compact containment condition are exactly the same with that in [7]. Next, with Lemma 2.2, Proposition 3.1 and Lemma 2.1 in hand, the proof of C-tightness is analogous to that of Proposition 3.7 of [7]. By Proposition 3.1, we see that the $L^2$-method in [7] is available. Thus, we may use the arguments in the proof of Proposition 3.2 in [7] with some trivial modifications to obtain the desired convergence theorem, Theorem 1.1.

\[ \Box \]

4 Proof of Theorem 1.2

In this section we assume that
\[ d = \alpha = 1 \quad \text{and} \quad b(t) = t. \]

With we always mean a constant which is strictly less than 1. We can adopt some of the arguments of [9] to prove some analogous results to those in [9] without using the fact that $p(\cdot)$ is in the domain of attraction of a stable law. We will refer the reader to these results as we use them.

4.1 Characterization of $\gamma^*$

Recall the definitions of $\hat{\tau}$ and $\tau$ in Section 1.3. For $e, e' \in \mathbb{Z}$ define the event $\Gamma_T(e, e') = \{ \hat{\tau}(e, e') < T, \hat{\tau}(0, e) \wedge \hat{\tau}(0, e') > T \}$, and let
\[
q_T = \sum_{e, e'} p(e) p(e') P(\Gamma_T(e, e')). \tag{4.1}
\]

We have the following characterization of $\gamma^*$.

**Proposition 4.1**

\[
\gamma^* = \lim_{T \to \infty} (\log T) q_T < \infty. \tag{4.2}
\]

To prove Proposition 4.1, we follow the arguments in Section 2 of [9]. Let $\tau_x = \inf\{t \geq 0 : B_t^0 = x\}$, and write $P^x$ to indicate the law of the walk $B^x$. Let $\tilde{P}(\cdot) = \sum_e p(e) P^e(\cdot)$, and define
\[
H(t) = \tilde{P}(\tau_0 > t). \tag{4.3}
\]

The following proposition is a version of Proposition 2.2 of [9].

**Proposition 4.2**

\[
\lim_{t \to \infty} H(t) \log t = p_1(0)^{-1}. \tag{4.4}
\]
\[
\frac{P_x(\tau_0 > t)}{H(t)} \leq 2a(x) \quad \text{for all} \quad x \in \mathbb{Z}, \ t > 0. \tag{4.5}
\]
\[
\lim_{t \to \infty} \frac{P_x(\tau_0 > t)}{H(t)} = a(x) \quad \text{for all} \quad x \in \mathbb{Z}. \tag{4.6}
\]
\[
a(x)/|x|, \ x \neq 0 \text{ is bounded on } \mathbb{Z}. \tag{4.7}
\]

**Proof.** For (4.4), let \( G(t) = \int_0^t p_s(0,0)ds \). Proposition 2.2 implies \( G(t) \sim p_1(0) \log t \) as \( t \to \infty \) in \( d = 1 \). Then one can follow the arguments in the proof of Lemma A.3 in [4] by using the last exit time decomposition of Lemma A.2 there and with (A.7) replaced by (2.5) to obtain that \( G(t)H(t) \to 1 \) as \( t \to \infty \); see the arguments after (A.8) of [4]. Then (4.4) holds.

Recall that \( \{Z_n : n = 0, 1, 2, \cdots \} \) denote the discrete time stable random walk defined in Section 1.2. With abuse of notation, let \( P_x \) denote the law of the walk starting at \( Z_0 = x \). Let \( \sigma_x = \inf\{n \geq 1 : Z_n = x\} \). By T29.1 of [23],
\[
a(x) = \lim_{n \to \infty} \sum_{k=0}^n [P^0(Z_k = 0) - P^0(Z_k = x)] < \infty \quad \text{exists for all } x \in \mathbb{Z}.
\]

Note that P11.1, P11.2 and P11.3 in Chapter III of [23] are available for one-dimensional recurrent random walk; see arguments before P28.1 of [23]. Meanwhile, according to T29.1 and P30.1 of [23], (i)’ and (ii)’ on page 116 in Chapter III of [23] also hold for one-dimensional random walk. Then we can check that both P11.4 and P11.5 in Chapter III of [23] are also available. Thus we have
\[
P^0(\sigma_x < \sigma_0) = 1/2a(x).
\]
Since the sequences of states visited by the walk \( B_t^0 \) is equal in law to the sequences visited by the walk \( Y_n \) (with \( Y_0 = 0 \)), we have \( \tilde{P}(\tau_x < \tau_0) = 1/2a(x) \). The strong Markov property implies that
\[
H(t) \geq \sum_e p(e)P^e(\tau_x < \tau_0, \tau_0 > t) \geq \sum_e P^e(\tau_x < \tau_0)P^x(\tau_0 > t)
\]
and then (4.5) follows.

For (4.6), by T32.1 of [23],
\[
\lim_{n \to \infty} \frac{P_x(\sigma_0 > n)}{P^0(\sigma_0 > n)} = a(x). \tag{4.8}
\]
Define
\[
h(n) = \sum_{0 \leq k \leq n} P^0(Y_k = 0).
\]
Then
\[
h(n) \sim p_1(0) \sum_{k=1}^n \frac{1}{k} \quad \text{as } n \to \infty; \tag{4.9}
\]
see Page 696 of [15]. We also have that
\[
P^0(\sigma_0 > n) = \frac{1}{h(n)} + o\left(\frac{1}{h(n)^2}\right);
\]
see the proof of Theorem 6.9 of \[15\]. Thus
\[ P^0(\sigma_0 > n) \log n \to p_1(0)^{-1}. \] (4.10)

According to a standard large deviations estimate for a rate-1 Poisson process, say \( S(t) \),

\[ e^{Ct} P(S(t) \notin [t/2, 2t]) \to 0 \] as \( n \to \infty \) for some constant \( C > 0 \). Then the fact that \( Y_{S(t)} \)
is a realization of \( B^0 \) yields
\[(1 - o(e^{-Ct})) P^x(\sigma_0 > 2t) \leq P^x(\sigma_0 > t) \leq o(e^{-Ct}) + P^x(\sigma_0 > t/2).\]
The inequalities above, together with (4.8) and (4.10), imply
\[
\lim_{t \to \infty} \frac{P_x^x(\sigma_0 > t)}{P_x^x(\sigma_0 > t/2)} = 1.
\] (4.11)

By (4.4) we see \( H(t)/P^0(\sigma_0 > t) \to 1 \) as \( t \to \infty \). Then (4.8) and (4.11) tell us (4.6) holds readily. Finally, (4.7) follows from the fact that
\[
\lim_{|x| \to \infty} \frac{a(x)}{|x|} = 0;
\]
see P29.3 of \[23\] and elsewhere. We have completed the proof. \( \Box \)

The proof of Proposition 4.1 is now exactly as that of Proposition 2.1 in Section 2 of \[9\]. We omit it here.

### 4.2 Voter and Biased Voter Estimates

In this subsection, we consider voter, biased voter bounds. We follow the arguments in Section 5 of \[9\] step by step. For \( b, \nu \geq 0 \), the 1-biased voter model \( \bar{\xi}_t \) is the Feller process taking values in \( \{0, 1\}^\mathbb{Z} \), with rate function
\[
\bar{c}(x, \xi) = \begin{cases} 
(\nu + b)f_1(x, \xi) & \text{if } \xi(x) = 0, \\
\nu f_0(x, \xi) & \text{if } \xi(x) = 1,
\end{cases}
\] (4.12)

where \( f_i(x, \xi) \) is as in (1.1). The 0-biased voter model is the Feller process \( \hat{\xi}_t \) taking values in \( \{0, 1\}^\mathbb{Z} \) with rate function
\[
\check{c}(x, \xi) = \begin{cases} 
\nu f_1(x, \xi) & \text{if } \xi(x) = 0, \\
(\nu + b)f_0(x, \xi) & \text{if } \xi(x) = 1.
\end{cases}
\] (4.13)

The voter model \( \hat{\xi}_t \) is the 1-biased voter model with bias \( b = 0 \). Then by Theorem III.1.5 of \[16\], assuming \( \zeta_0 = \xi_0 = \hat{\xi}_0 \), we may define \( \zeta_t, \hat{\xi}_t \) and \( \bar{\xi}_t \) on a common probability space so that
\[
\zeta_t \leq \hat{\xi}_t \leq \bar{\xi}_t \text{ for all } t \geq 0.
\] (4.14)

For \( \xi, \zeta \in \{0, 1\}^\mathbb{Z} \), \( \xi \preceq \zeta \) means \( \xi(x) \preceq \zeta(x) \) for all \( x \in \mathbb{Z} \).
Let us recall the voter model duality; see [16]. Recall also the coalescing random walk system \( \{ \hat{B}_t^x : x \in \mathbb{Z} \} \) defined in Subsection 1.3. The duality equation for the rate-1 \((\nu = 1)\) voter model is: for finite \( A \subset \mathbb{Z} \),

\[
P(\hat{\xi}_t(x) = 1 \forall x \in A) = P(\hat{\xi}_0(\hat{B}_t^x) = 1 \forall x \in A).
\] (4.15)

Define the mean range of the random walk \( B_t^0 \) by

\[
R(t) = E \left( \sum_x 1_{\{B_{s^t} = x \text{ for some } s \leq t\}} \right).
\]

By a result for the range of the discrete time stable random walk in [15],

\[
\lim_{t \to \infty} \frac{R(t)}{t / \log t} = p_1(0)^{-1};
\] (4.16)

see (1.e) of [15] and recall (4.9) for the asymptotic behavior of \( h(n) \).

First, we consider the voter estimates. Let \( P_t, t \geq 0 \) be the semigroup of a rate-1 random walk with step distribution \( p(\cdot) \). Recall the definition of \( |p|_\alpha \) in Section 3. For \( \phi : \mathbb{Z} \to \mathbb{R}^+ \) and \( \xi \in \{0,1\}^\mathbb{Z} \), let

\[
\xi(\phi) = \sum_x \phi(x)\xi(x).
\]

**Lemma 4.1** Let \( \hat{\xi}_t \) denote the rate-\( \nu \) voter model. Then for all bounded \( \phi : \mathbb{Z} \to \mathbb{R}^+ \), \( 0 < \alpha < 1 \) and \( t \geq 0 \),

\[
E(\hat{\xi}(\phi f_0(\hat{\xi}_t))) \leq (\nu t |p|_\alpha H(2\nu t))^{1/2} ||\phi||_{\alpha/2} |\bar{\xi}_0| + H(2\nu t)\hat{\xi}_0(\phi).
\] (4.17)

**Remark 4.1** (4.17) is just a version of (5.8) in Lemma 5.1 of [9]. We slightly abuse our notation and we can prove that the other statements in Lemma 5.1 of [9] ((5.6), (5.7) and (5.9) there) hold without modifying any arguments of their proofs.

**Remark 4.2** Recall the definition of \( ||\phi||_\alpha \) in Section 3. We see for \( \phi = 1 \), the right side of (4.17) is just \( H(2\nu t)|\hat{\xi}_0| \).

**Proof.** It suffices to consider \( \nu = 1 \). Using the voter duality equation (4.13) and following the arguments in the proof of (5.8) of [9], we have

\[
E(\hat{\xi}(\phi f_0(\hat{\xi}_t))) \leq \sum_{e,z} \hat{\xi}_0(z) p(e) E \left( \phi(z + B_t^0) 1_{\{\tau(0,e) > t\}} \right).
\]

For any \( z \) and \( 0 < \alpha < 1 \),

\[
\sum_e p(e) E \left( \phi(z + B_t^0) 1_{\{\tau(0,e) > t\}} \right) \\
\leq \sum_e p(e) E \left( (||\phi||_{\alpha/2} |B_t|^\alpha/2 + \phi(z)) 1_{\{\tau(0,e) > t\}} \right)
\]
\begin{align*}
\leq ||\phi||_2/2 \left( E(|B_t^0|) \sum_e p(e)P(\tau(0, e) > t) \right)^{1/2} \\
+ \phi(z) \sum_e p(e)P(\tau(0, e) > t).
\end{align*}

Since $E(|B_t^0|) \leq t|p|_{\infty}$, this proves (4.17). \hfill \Box

Next, we give some biased voter model bounds. Let $\xi_t$ be the 1-biased voter model with rate function (4.12). By the same arguments in Section 4 of [7], we can prove the following inequalities without using any of kernel assumptions.

\begin{align*}
E(|\xi_t|) &\leq e^{bt} |\xi_0|, \quad (4.18) \\
E(|\xi_t|^2) &\leq e^{2bt} \left( |\xi_0|^2 + \frac{2\nu + b}{b} (1 - e^{-bt}) |\xi_0| \right) \quad (4.19) \\
&\leq e^{2bt} \left( |\xi_0|^2 + (2\nu + b)t|\xi_0| \right) \quad (4.20)
\end{align*}

In the subsection 4.3 below, we will compare the Lotka-Volterra model $\xi_t^N$ with the biased voter models $\bar{\xi}_t^N, \bar{\xi}_t^N$ on $S_N$. In order to construct coupling $\xi_t^N \leq \bar{\xi}_t^N \leq \bar{\xi}_t^N$ we assume that the voting and bias rates $\nu_N$ and $b_N$ are

$$
\nu = \nu_N = N - \bar{\theta} \log N \quad \text{and} \quad b = b_N = 2\bar{\theta} \log N. \quad (4.21)
$$

As in [9], we need improved versions of (4.18) and (4.19). For $p \geq 2$ and $0 < \alpha < 1$ define

\begin{align*}
\kappa_p &= \kappa_p(b, \nu) = 3(bH(2\nu/b^p) + e^2) \quad \text{and} \quad \kappa = \kappa_3, \\
A &= A(b, \nu) = bR(2\nu/b^3) + 3e^2(1 + 2\nu/b), \\
B_p &= B_p(b, \nu, \alpha) = (|p|_\alpha \nu b^2 - pH(2\nu/b^p))^{1/2} + bH(2\nu/b^p)(|p|_\alpha(\nu/b^p + 1))^{1/2}
\end{align*}

and

\begin{align*}
h_1(b, \nu)(t) &= e^{2t} - 1/3 + 2Ke^{2+2\nu t}, \\
h_2(b, \nu)(t) &= e^{2t} - 1/3 (1 + 2\nu/b) + 5Ke^{1+3\nu t}.
\end{align*}

Put $P\phi(x) = \sum_y p(y-x)\phi(y)$ and define the operators

$$
\bar{A}\phi = \nu(P\phi - \phi) \quad \text{and} \quad \bar{A}^* = (1+b/\nu)\bar{A}
$$

(4.22)

and denote the associated semigroups by $\bar{P}_t$ and $\bar{P}^*_t$ respectively.

**Remark 4.3** Comparing the constants and functions defined above with those defined in (5.16) and (5.17) of [9], we see that only $B_p$ is different. We replaced $2\sigma^2$ by $|p|_\infty$.

**Remark 4.4** For the parameters $\nu = \nu_N$, $b = b_N$ in (4.21), (4.4) and (4.16) imply that $\kappa_p = O(1)$, $A = O(N/\log N)$ and $B_p = O(N^{1/2}(\log N)^{(1-p)/2})$ as $N \to \infty$.

**Remark 4.5** The estimates in Remark 4.4 will play important roles in the following proofs. That is why we are forced to assume that $\{p(x)\}$ is in the domain of normal attraction of a stable law. Or we need to replace $\log N$ by $\int_1^N b(s)^{-1}ds$. Then the estimates in Remark 4.4 will not be available.
The following proposition is a version of Proposition 5.4 of \cite{9}.

**Proposition 4.3** Assume \( b \geq 1 \) and \( p \geq 2 \). For all \( t \geq 0 \),

\[
E(|\xi_t|) \leq e^{bl^{-p} + \kappa_p t} |\xi_0|, \tag{4.23}
\]
\[
E(|\xi_t|^2) \leq e^{2 + 2\kappa t} |\xi_0|^2 + 4Ae^{1 + 3\kappa t} |\xi_0|, \tag{4.24}
\]
\[
bE(\xi_t(f_0(\xi_t))) \leq h_1(t)|\xi_0|, \tag{4.25}
\]
\[
bE(\xi_t^2) \leq h_1(t)|\xi_0|^2 + h_2(t)|\xi_0|. \tag{4.26}
\]

For all bounded \( \phi : \mathbb{Z} \to [0, \infty) \), \( p \geq 3 \) and \( 0 < \alpha < 1 \),

\[
E(\xi_t(\phi)) \leq e^{bl^{-p} + (1 + \kappa_p)t} \left( \tilde{\xi}_0(P^*_t(\phi)) + [\kappa_pb^{2-p}||\phi||_\infty + B_p||\phi||_{H/2}] |\xi_0| \right). \tag{4.27}
\]

**Remark 4.6** Proposition 5.4 of \cite{9} was proved with the help of Lemma 5.1, Lemma 5.5 and Lemma 5.6 there. We can adopt the arguments in \cite{9} to obtain similar results in Lemma 5.5 and Lemma 5.6 of \cite{9}. With abuse of notation, in the following we assume that those two lemmas are available for us.

**Remark 4.7** The only difference between Proposition 4.3 and Proposition 5.4 of \cite{9} is that inequality (4.27) is different from inequality (5.23) there. In fact, the key reason is that when prove the inequality (4.27), we will use estimate (4.17) in Lemma 4.4 of this paper replacing the estimate (5.8) of Lemma 5.1 of \cite{9}.

**Proof.** According to Remark 4.1, Remark 4.6 and the coupling (4.14), we can follow the arguments in \cite{9} to obtain that (5.36), (5.37) and (5.38) there are available which will be used in the following proof. Put \( \epsilon = b^{-p} \) and assume \( \phi \geq 0 \). We also have that

\[
E(|\xi_t(b\phi f_0(\xi_t))|) \leq 2b||\phi||_\infty E(|\xi_t| - |\xi_t|) \leq 2b(e^{be} - 1)||\phi||_\infty |\xi_0| \tag{4.28}
\]

which is just a version of (5.39) of \cite{9} (In fact, they are the same). The voter model estimate (4.17) tells us

\[
E(\tilde{\xi}_s(b\phi f_0(\tilde{\xi}_s))) \leq 2eb^2e||\phi||_\infty |\tilde{\xi}_0| + b(|p|\alpha
ueH(2\nu e))^{1/2}||\phi||_{H/2}|\tilde{\xi}_0| + bH(\nu e)|\tilde{\xi}_0(\phi). \tag{4.29}
\]

By using Markov property, we see for \( s \geq \epsilon \),

\[
E(\tilde{\xi}_s(b\phi f_0(\tilde{\xi}_s))|\mathcal{F}_{s-\epsilon}) \leq (2eb^2e||\phi||_\infty + b(|p|\alpha
ueH(2\nu e))^{1/2}||\phi||_{H/2}) |\tilde{\xi}_{s-\epsilon}| + bH(\nu e)|\tilde{\xi}_{s-\epsilon}(\phi). \tag{4.30}
\]

Take expectations in (4.30) for \( \phi = 1 \) and recall the definition \( ||\phi||_\Delta \) in Section 3. We have for \( s \geq \epsilon \)

\[
E(\tilde{\xi}_s(b\phi f_0(\tilde{\xi}_s))) \leq \kappa_p E(|\tilde{\xi}_{s-\epsilon}|). \tag{4.31}
\]

Using this inequality in (5.36) of \cite{9} yields for \( s \geq \epsilon \),

\[
E(|\xi_t|) \leq E(|\xi_\epsilon|) + \kappa_p \int_\epsilon^t E(|\xi_{s-\epsilon}|) ds \leq e^{be} + \kappa_p \int_0^t E(|\xi_s|) ds,
\]
where the second inequality follows from (5.38) of [9]. This bound also holds for $t \leq \epsilon$. Then Gronwall’s inequality implies that (4.23) holds.

Again using (5.38) of [9] gives that for $\psi : \mathbb{Z} \to \mathbb{R}^+$,

$$|E(\bar{\xi}_{t}(\psi)) - \bar{\xi}_{0}(\psi)| \leq (e^{b\epsilon} - 1)\bar{\xi}_{0}(P^*_t\psi) + |\bar{\xi}_{0}(P^*_t\psi) - \bar{\xi}_{0}(\psi)|.$$ 

Note that

$$|P^*_t\psi(x) - \psi(x)| \leq ||\psi||_{\mathfrak{A}/2}E(|B^0_{\nu e(1+b/\nu)}|_{\mathfrak{A}/2}) \leq ||\psi||_{\mathfrak{A}/2}(\epsilon(\nu + b)|p|_{\mathfrak{A}})^{1/2}.$$ 

Thus

$$|E(\bar{\xi}_{s-\epsilon}(\psi)) - \bar{\xi}_{0}(\psi)| \leq (e\epsilon ||\psi||_{\infty} + ||\psi||_{\mathfrak{A}/2}(\epsilon(\nu + b)|p|_{\mathfrak{A}})^{1/2})|\bar{\xi}_{0}|.$$ 

Then by using Markov property, for $s \geq \epsilon$,

$$E(\bar{\xi}_{s-\epsilon}(\psi)) \leq E(\bar{\xi}_{s}(\psi)) + (e\epsilon ||\psi||_{\infty} + ||\psi||_{\mathfrak{A}/2}(\epsilon(\nu + b)|p|_{\mathfrak{A}})^{1/2})E(|\bar{\xi}_{s-\epsilon}|).$$ 

Since $||P^*_t\phi||_{\mathfrak{A}/2} \leq ||\phi||_{\mathfrak{A}/2}$, using above inequality in (4.30) with $\psi = P^*_t\phi$ replacing $\phi$, we have for $s \geq \epsilon$,

$$E(\bar{\xi}_{s}(bP^*_t\phi|_{\mathfrak{A}/2})) \leq (\kappa_p b^2\epsilon ||\phi||_{\infty} + B_p||\phi||_{\mathfrak{A}/2})E(|\bar{\xi}_{s-\epsilon}|) + \kappa_p E(\bar{\xi}_{s}(P^*_t\phi)), \quad (4.32)$$

which is a version of (5.43) of [9]. Then the following arguments for proving (4.27) are very similar to those after (5.43) in [9]. We have proved (4.23) and (4.27). The other statements in the proposition can be proved in a similar way to that used to prove their counterparts in [9] (recall Remark 4.1, Remark 4.6). We omit it here. \hfill \Box

**Remark 4.8** We have followed the arguments in Section 5 of [9] to obtain some voter and biased voter estimates. In fact, we only replaced (5.8) and (5.23) in Section 5 of [9] by (4.17) and (4.27) respectively and modified the arguments in the proof of (5.19) and (5.23) of [9]; please compare (4.29)-(4.32) with their counterparts (5.40)-(5.43) in Section 5 of [9]. We can also adopt the arguments there to obtain similar results to all other statements in Section 5 of [9] without using the fact the $p(\cdot)$ is in the domain of attraction of a stable law. In the next subsection, we will directly refer to them.

### 4.3 Four Key Results

In this subsection, we will give analogous results to Propositions 4.3, 4.4, 4.5 and 4.7 of [9]. We first list those results and will give their proofs later. Let

$$g(s) = C_{4.33}^{-1/3}e^{C_{4.33}}, \quad (4.33)$$

where $C_{4.33}$ will be chosen later.

**Proposition 4.4** (a) For $T > 0$ there is a constant $C_{4.34}(T)$ such that for all $N \in \mathbb{N}$,

$$\sup_{t \leq T} E(X^N_t(1)) \leq C_{4.34}(T)X^N_0(1), \quad (4.34)$$

$$E\left(\sup_{t \leq T} X^N_t(1)^2\right) \leq C_{4.34}(T)(X^N_0(1)^2 + X^N_0(1)). \quad (4.35)$$
(b) For all \( s > 0 \) and \( N \in \mathbb{N} \),

\[
(\log N) E(X^N_s(f^N_0(\cdot, \xi^N_s))) \leq g(s)X^N_0(1),
\]

(4.36)

\[
(\log N) E(X^N_s(1)X^N_s(f^N_0(\cdot, \xi^N_s))) \leq g(s)(X^N_0(1)^2 + X^N_s(1)).
\]

(4.37)

Let \( A_N(\psi) = \frac{1}{N}(N + \bar{\theta}) \log N)A_N(\psi) \) with semigroup \( P^N_t \).

**Proposition 4.5** For \( p \geq 3 \) there is a constant \( C_{\text{4.38}}(p) \) such that for any \( t \geq 0 \) and \( \phi : \mathbb{R} \to \mathbb{R}^+ \),

\[
E(X^N_t(\phi)) \leq e^{(\log N)^{1-p}p} C_{\text{4.38}}(p) X^N_0(P^N_t, \phi)
\]

\[
+ C_{\text{4.38}}(p) ||\phi||_{1/2}(\log N)^{(1-p)/2} X^N_0(1).
\]

(4.38)

**Proposition 4.6** For \( p \geq 3 \) there is a constant \( C_{\text{4.39}}(p) \) such that for all \( \phi : \mathbb{R} \to \mathbb{R}^+ \), if \( \epsilon = (\log N)^{-p} \), then

\[
E(X^N_{\epsilon}(\log N\phi f^N_0(\cdot, \xi^N_\epsilon))) \leq C_{\text{4.39}}(p) X^N_0(1)||\phi||_{1/2}(\log N)^{(1-p)/2} + C_{\text{4.39}}(p) X^N_0(\phi).
\]

(4.39)

Let \( \sup_{K,T} \) indicate a supremum over all \( X^N_0 \in M(\mathcal{S}_N) \), \( \phi : \mathbb{R} \to \mathbb{R} \) and \( t \geq 0 \) satisfying \( X^N_0(1) \leq K \), \( ||\phi||_{\text{Lip}} \leq K \) and \( t \leq T \).

**Remark 4.9** Note that if \( ||\phi||_{\text{Lip}} \leq K \), then \( ||\phi||_{\text{Lip}} \leq 2K \) for any \( 0 \leq \alpha \leq 1 \).

**Proposition 4.7** For every \( K, T > 0 \) and \( 0 < p < 2 \),

\[
\lim_{N \to \infty} \sup_{K,T} E\left( \left| \int_0^t X^N_s(\log N\phi^2 f^N_0(\cdot, \xi^N_s)) - p_1(0)^{-1}X^N_0(\phi^2) \right|^p \right) = 0
\]

(4.40)

and for \( i = 2, 3 \),

\[
\lim_{N \to \infty} \sup_{K,T} E\left( \left| D^N_{t,i} - \int_0^t \theta^{-2} \gamma^* X^N_s(\phi) ds \right|^p \right) = 0.
\]

(4.41)

Recall the rescaled Lotka-Volterra models in Section 1.2 and assume (A2) holds. Also recall the 1-biased voter model and 0-biased voter model with rates \( \nu = \nu_N \) and \( b = \bar{b}_N \) defined in the last subsection. Set \( \xi^N_\ell(x) = \xi_\ell(Nx) \) and \( \xi^N_\omega(x) = \xi_\omega(Nx) \) for \( x \in \mathcal{S}_N \). Thus the rate function of \( \xi^N_\ell \) is given by

\[
\tilde{c}(x, \xi) = \begin{cases} 
(\nu_N + b_N)f^N_1(x, \xi) & \text{if } \xi(x) = 0, \\
\nu_N f^N_0(x, \xi) & \text{if } \xi(x) = 1,
\end{cases}
\]

and the rate function of \( \xi^N_\omega \) is given by

\[
\zeta(x, \xi) = \begin{cases} 
(\nu_N f^N_1(x, \xi) & \text{if } \xi(x) = 0, \\
(\nu_N + b_N)f^N_0(x, \xi) & \text{if } \xi(x) = 1.
\end{cases}
\]
Assume \( N \) is large enough (\( N \geq N_0 \)) so that \( \nu_N > 0 \) and \( b_N > 1 \). As in the last subsection, we may construct the three processes on one probability space so that \( \xi_0^N = \xi_0^N = \xi_0^N \) and

\[
\xi_t^N \leq \hat{\xi}_t^N \leq \check{\xi}_t^N \quad \text{for all } t \geq 0. \tag{4.42}
\]

Define

\[
\hat{X}_t^N = \frac{1}{N} \sum_{x \in \mathbb{S}_N} \hat{\xi}_t^N(x) \delta_x \quad \text{and} \quad \check{X}_t^N = \frac{1}{N} \sum_{x \in \mathbb{S}_N} \check{\xi}_t^N(x) \delta_x.
\]

It follows that

\[
\check{X}_t^N \leq X_t^N \leq \hat{X}_t^N \quad \text{for all } t \geq 0. \tag{4.43}
\]

Keep Remark 4.4 in mind. Applying Proposition 4.3 gives that there are constants \( C_{4.44} \) and \( C_{4.45} \) such that for all \( N \geq N_0 \) and \( t \geq 0 \),

\[
E(\hat{X}_t^N(1)) \leq C_{4.44} \hat{X}_0^N(1), \\
E(\check{X}_t^N(1)^2) \leq C_{4.44} (\check{X}_0^N(1)^2 + \check{X}_0^N(1))
\]

and if \( g \) is as in (4.33), then

\[
(\log N) E(\check{X}_t^N(f_0^N(\cdot, \check{\xi}_t^N))) \leq g(t) \check{X}_t^N(1), \\
(\log N) E(\check{X}_t^N(1)\check{X}_t^N(f_0^N(\cdot, \check{\xi}_t^N))) \leq g(t) (\check{X}_0^N(1)^2 + \check{X}_0^N(1)). \tag{4.47}
\]

Typically, we have there exists a constant \( C_{4.48} \) such that

\[
E(\check{X}_t^N(1)) - E(\check{X}_t^N(1)) \leq C_{4.48} (\log N)^{-2} + t \check{X}_0^N(1), \quad 0 \leq t \leq 1 \tag{4.48}
\]

whose counterpart in [9] is (6.7). We first prove Proposition 4.4. In fact, we only give an outline.

**Proof of Proposition 4.4** With inequalities (4.44), (4.45) and the coupling (4.43) in hand, part (a) follows from the strong \( L^2 \) inequality for non-negative submartingales and the fact that \( \hat{X}_t^N(1)^2 \) is a submartingale; see Remark 4.8 and (5.29) of [9]. For part (b), if we have similar results to those in Proposition 6.1 of [9], then part (b) follows from Remark 4.4. But the proof of Proposition 6.1 of [9] works here if we replace (5.40) there by (4.29) in the last subsection; see Remark 4.8. \( \square \)

**Proof of Proposition 4.3** Recall that \( \check{\xi}_t \) is the biased voter model with rates \( \nu = N - \theta \log N \) and \( b = 2\theta \log N \), and \( \check{\xi}_t^N(x) = \check{\xi}_t(Nx), \ x \in \mathbb{S}_N \). For \( \psi : \mathbb{R} \to \mathbb{R}^+ \), define \( \phi : \mathbb{Z} \to \mathbb{R}^+ \) by \( \phi(x) = \psi(x/N) \). Then \( ||\phi||_\infty = ||\psi||_\infty \) and for \( 0 < \alpha < 1 \),

\[
\sup_{x \neq y, |x - y| \leq 1} \frac{|\phi(x) - \phi(y)|}{|x - y|^{\alpha/2}} \leq \sup_{x \neq y, |x - y| \leq 1} \frac{|\phi(x) - \phi(y)|}{|x - y|^{1/2}} \leq N^{-1/2} \sup_{x \neq y, |x - y| \leq 1/N} \frac{|\psi(x) - \psi(y)|}{|x - y|^{1/2}}.
\]

Thus \( ||\phi||_{\alpha/2} \leq N^{-1/2} ||\psi||_{1/2} \). Note that \( \mathcal{A}_N^* \psi(x) = (N + \theta \log N) \sum_{y \in \mathbb{S}_N} p_N(y - x) \psi(y) \) with semigroup \( P_t^{N,*} \) and \( \mathcal{A}^* \phi(x) = (N + \theta \log N) \sum_{y} p(y - x) \phi(y) \) with semigroup \( P_t^* \); see (4.22).
for the definition of $\mathcal{A}^*$. We have that $P_t^* \phi(x) = P_t^{N,*} \psi(x/N)$ and $\tilde{\xi}_t^N(\psi) = \tilde{\xi}_t(\phi)$. According to (1.27), we obtain

$$E(\tilde{\xi}_t^N(\psi)) \leq e^{b_1 - p + (1 + \kappa_p) t} \left( \tilde{\xi}_t^N(P_t^{N,*}(\psi)) + \left[ \kappa_p b_2^{-2-p} \|\psi\|_\infty + B_p N^{-1/2} \|\psi\|_{1/2} \|\tilde{\xi}_t^N\| \right] \right).$$

Since $p \geq 3$, Remark 4.4 implies $\kappa_p b_2^{-p} + B_p N^{-1/2} = O((\log N)^{(1-p)/2})$ as $N \to \infty$. Then the fact that $\theta \geq 1$ implies $b \geq \log N$ and the coupling (4.33) yield the desired inequality (1.38).

Proof of Proposition 4.6. Let $\epsilon = b^{-p}$. According to Remark 4.8, we may use (5.32) of \cite{[9]} to obtain that

$$E(X_\epsilon^N(b \phi f_0(\xi^N_\epsilon))) \leq E(\tilde{X}_\epsilon^N(b \phi f_0(\xi^N_\epsilon))) + 2b \|\phi\|_\infty (E(\tilde{X}_\epsilon^N(1) - X_\epsilon^N(1))).$$

Applying (5.62) of \cite{[9]} and (4.29) gives

$$E(X_\epsilon^N(b \phi f_0(\xi^N_\epsilon))) \leq (6eb^{-p} \|\phi\|_\infty + B_p N^{-1/2} \|\phi\|_{1/2}) X_0^N(1) + \kappa_p X_0^N(\phi).$$

Then Remark 4.4 yields (1.39).

We will give the proof of Proposition 4.7 in the final subsection. In the next subsection with the help of the four propositions in this subsection we prove Theorem 1.2.

### 4.4 Convergence Theorem

In this subsection, we follow the strategy in the Section 4 of \cite{[9]} to obtain Theorem 1.2. First, we check the compact containment condition.

Proposition 4.8 For all $\epsilon > 0$ there is an $n \in \mathbb{N}$, so that

$$\sup_N P \left( \sup_{t \leq \epsilon^{-1}} X_t^N(B(0, n)_\epsilon) > \epsilon \right) < \epsilon.$$

Proof. The proof is similar to that for Proposition 4.12 of \cite{[9]}. We only give an outline here. Recall that $b(N) = N$. Let $h_n : \mathbb{R}^d \to [0, 1]$ be a $C^\infty$ function such that

$$1_{\{|x| > n + 1\}} \leq h_n(x) \leq 1_{\{|x| > n\}}$$

and

$$\sup_n \sum_{i,j,k \leq d} \| (h_n)_i \|_\infty + \| (h_n)_{ij} \|_\infty + \| (h_n)_{ijk} \|_\infty \equiv C_h < \infty.$$

By the semimartingale decomposition

$$\sup_{t \leq T} X_t^N(h_n) \leq X_0^N(h_n) + \sum_{i=1}^3 \sup_{t \leq T} |D_t^{N,i}(h_n)| + \sup_{t \leq T} |M_t^N(h_n)|.$$

We need to check the right hand side tends to zero as $N, n \to \infty$. Let

$$\eta_N := \sup_n \| A_N(h_n) - \frac{\sigma^2 \Delta^{1/2} \theta_n}{2} \|_\infty.$$
Then $\lim_{N \to \infty} \eta_N = 0$ by Lemma 2.2. Note that
\[
\frac{1}{N'} \sum_{x,y} |h_n(x) - h_n(y)|p_N(x - y)\xi_N^y(y) \leq \|h_n\|_\infty N' \sum_y |x - y|p_N(x - y)\xi_N^y(y)
\leq \frac{C_h}{N^\alpha} \xi_N^y(1).
\]
Set $\eta'_N(T) = C_{4.34}(T)(\eta_N + \tilde{\theta} C_h \log N |p|_{\infty}/N^\alpha T)$. We have, as in the deviation of (4.17) in [9]
\[
E \left( \sup_{t \leq T} X_t^N(h_n) \right) \leq X_0^N(h_n) + 2(\langle M^N(h_n) \rangle_T)^{1/2} + \eta'_N X_0^N(1)
+C_h \int_0^T E(X_s^N(h_n)) ds + 2\tilde{\theta} \int_0^T E(X_s^N(h_n \log N f_0^N(\xi_s^N))) ds.
(4.49)
\]
Applying Proposition 4.4 and (4.34), we obtain the last integral above is bounded by
\[
\eta''_N(T)X_0^N(1) + C_{4.39} \int_0^T E(X_s^N(h_n)) ds,
(4.50)
\]
where $\eta''_N(T) = C_{4.34}(T)[\log N]^{-2} + C_{4.34} C_4 T / \log N$. By Lemma 2.1 and (4.34) there is a constant $C_{4.51}(T)$ such that if $\phi = \psi$, then for any $\alpha < 1$ and $0 \leq s \leq T$,
\[
E(|m_{1,s}^N| + |m_{2,s}^N|) \leq C_{4.51}(T)||\phi||^2_{\infty}(\log N / N^\alpha) X_0^N(1).
(4.51)
\]
Then the above inequality, (4.50) and Lemma 2.1 gives (recall $N'/N' = \log N$)
\[
E(\langle M^N(h_n) \rangle_T) \leq \eta''_N(T)X_0^N(1) + 2C_{4.39} \int_0^T E(X_s^N(h_n)) ds,
(4.52)
\]
where $\eta''_N(T) = 2\eta''_N(T) + C_{4.51}(T)TC_h^2 \log N / N^\alpha$. Finally, let $B_{t,s}^{N,*}$ be the continuous random walk with semigroup $P_t^{N,*}$ defined before Proposition 4.5. $B_0^{N,*} = 0$. Note that
\[
P \left( |B_{t,s}^{N,*}| \geq \frac{n - 1}{2} \right) = P \left( |B_{(N+\tilde{\theta} \log N)s}^0 \geq \frac{N(n - 1)}{2} \right).
\]
Since $b(t) = l(t) = t$, Proposition 2.2 yields that the left hand side above goes to 0 uniformly in $N \in \mathbb{N}$ and $0 \leq s \leq T$ as $n \to \infty$. Thus with the help of Proposition 4.5 and the inequalities (4.49), (4.50), (4.52) we can conclude: for any $T, \epsilon > 0$ there is an $N_0$ such that
\[
\text{for } N \geq N_0, n \geq N_0, E(\sup_{t \leq T} X_t^N(h_n)) < \epsilon.
\]
The desired result is immediate. \[\square\]

**Proof of Theorem 1.2** In fact, we have already completed all tasks. First, with (4.36) and (4.37) in hand, by the same arguments as those in the proof of Lemma 4.10 of [9], we have there exists a constant $C_{4.53}(T)$ such that for all $0 \leq s \leq t \leq T$,
\[
E \left( \left[ \int_s^t X_r^N(\log N f_0^N(\xi_r^N)) dr \right]^2 \right) \leq C_{4.53}(T)(t - s)^{4/3}(X_0^N(1)^2 + X_0^N(1)).
(4.53)
\]
Now, recall the decomposition of $X_\nu^N(\phi_i)$ in Section 2.2. With the help of Lemma 2.1 and (4.33), by the the same arguments as those in the proof of Proposition 4.11 of [9], for each $\phi \in C^1_b(\mathbb{R}_+ \times \mathbb{R})$, each of families $\{X^N(\phi), N \in \mathbb{N}\}$, $\{D^N, N \in \mathbb{N}\}$, $\{(M^N(\phi)), N \in \mathbb{N}\}$, and $\{M^N(\phi), N \in \mathbb{N}\}$ is C-tight in $D([0, \infty), \mathbb{R})$. The C-tightness of $\{P_N, N \in \mathbb{N}\}$ is now immediate from Proposition 4.8 and Theorem II.4.1 of [18]. Then to check any limit point of $\{P_N\}$ is the law claimed in the Theorem, one can follow the same arguments as those in the proof of proposition 4.2 of [9], using Proposition 4.7 above.

\section{Proof of Proposition 4.7}

For $N$ fixed, let $\hat{\xi}_t$ be the rate $\nu_N = N - \theta \log N$ voter model on $\mathbb{Z}$ with rate as in (4.12) for $b = 0$ and $\nu = \nu_N$. Define $\hat{\xi}_t^N(x) = \hat{\xi}_t(xN)$, $x \in S_N$, the rate $\nu_N$ voter model on $S_N$. Recall the independent and coalescing random walks system $\{N_i, N \in \mathbb{N}\}$ is now immediate from Proposition 4.8 and Theorem II.4.1 of [18]. Then to check any limit point of $\{P_N\}$ is the law claimed in the Theorem, one can follow the same arguments as those in the proof of proposition 4.2 of [9], using Proposition 4.7 above.

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Lemma 4.2 There is a constant $C_{4.57}$ such that
\[
\frac{\log N}{N'} \sum_{x,e} p_N(e) P\left(\hat{\xi}_0^N(B_{tN}^{N,x}) = \hat{\xi}_0^N(B_{tN}^{N,x+e}) = 1, \tau^N(x, x + e) > t_N\right) \leq C_{4.57} \epsilon'_N \int \int_{|w-z| \leq \delta_N} d\hat{X}_0^N(w) d\hat{X}_0^N(z) + C_{4.57} \hat{X}_0^N(1)^2. \tag{4.57}
\]

Proof. By translation invariance and symmetry, the left side of (4.57) is
\[
(N')^{-2} \sum_{w,z} \hat{\xi}_0^N(w) \hat{\xi}_0^N(z) \sum_e p_N(e) \times \left[ \sum_x N P(\hat{B}_{tN}^{N,0} = w - x, \hat{B}_{tN}^{N,e} = z - x, \tau^N(0, e) > t_N) \right] = (N')^{-2} \sum_{w,z} \hat{\xi}_0^N(w) \hat{\xi}_0^N(z) \sum_e p_N(e) N P(\hat{B}_{2tN}^{N,e} = z - w, \tau_0^{N,e} > 2t_N) \equiv \Sigma_d^N + \Sigma_c^N, \tag{4.58}
\]
where $\tau_0^{N,e} = \inf\{s : B_{s}^{N,e} = 0\}$, and $\Sigma_d^N$, respectively, $\Sigma_c^N$, denotes the contribution to (4.58) from $w, z$ satisfying $|w - z| \leq K_N t_N$, respectively, $|w - z| > K_N t_N$. Let
\[
\hat{P}((B_{tN}^N, \tau_0^N) \in \cdot) = \sum_e p_N(e) P((B_{tN}^{N,e}, \tau_0^{N,e}) \in \cdot).
\]
For $\Sigma_d^N$, use (2.5) and the Markov property at time $t_N$ to see that
\[
N \hat{P}^N(\hat{B}_{2tN}^N = z - w, \tau_0^N > 2t_N) \leq \nu \hat{P}^N(\hat{B}_{tN}^{N,0} = z - w, B_{2tN}^N(w) ; \tau^N_0 > t_N) \leq CN \hat{P}(\tau^N_0 > t_N) (\nu_{tN})^{-1} \leq C_N \hat{P}(\tau^N_0 > t_N) \nu_{tN},
\]
By (4.4), there is a constant $C_{4.59}$ such that
\[
\Sigma_d^N \leq C_{4.59} \epsilon'_N \int \int_{|w-z| \leq K_N t_N} d\hat{X}_0^N(w) d\hat{X}_0^N(z). \tag{4.59}
\]
It is more complicated to bound $\Sigma_c^N$. Using the Markov property at time $\hat{\eta}_{NtN}$ gives
\[
\hat{P}^N(\hat{B}_{2tN}^N = w - z, \tau_0^N > 2t_N) \leq \hat{P}\left(\tau_0^N > \hat{\eta}_{NtN} | B_{\hat{\eta}_{NtN}}^N \right) \sup_{x'} P\left(B_{(2-\hat{\eta}_{NtN})tN}^{N,0} = x' \right) + \hat{P}\left(P\left(B_{(2-\hat{\eta}_{NtN})tN}^{N,0} = w - z - B_{(2-\hat{\eta}_{NtN})tN}^N \right) ; \tau_0^N > \hat{\eta}_{NtN} | B_{\hat{\eta}_{NtN}}^N \right) \leq \frac{K_N t_N}{2}
\]
\[
= \Sigma_{1c}^N + \Sigma_{2c}^N, \text{ say.}
\]
Note that
\[
\hat{P}\left(|B_{\hat{\eta}_{NtN}}^N| > \frac{K_N t_N}{2} \right) = \sum_e p_N(e) P\left(|B_{\hat{\eta}_{NtN}}^N + e| > \frac{N K_N t_N}{2} \right).
\]
which is bounded by
\[
\frac{2|p|_{1/2}}{(NK_N^2 t_N)^{1/2}} + P\left(|B_{N\hat{\eta}_N t_N}^0| > \frac{NK_N t_N}{4}\right).
\]
By Proposition 2.4,
\[
P\left(|B_{N\hat{\eta}_N t_N}^0| > \frac{NK_N t_N}{4}\right) \leq \frac{4C(2,12)N\hat{\eta}_N t_N}{NK_N t_N} = 4C(2,12)\hat{\eta}_N/K_N.
\]
(Note that \(l(t) = b(t) = t\).) Thus by (2.0)
\[
\Sigma_{1c}^N \leq C \left(\frac{\hat{\eta}_N/K_N + 1/(NK_N t_N)^{1/2}}{\nu_N(2 - \eta_N)t_N}\right),
\]
(4.60)
Let us consider \(\Sigma_{2c}^N\). By the definition of \(\varepsilon(t)\) and (2.11) (recall \(d = \alpha = 1\),
\[
p_t(0, x) \leq \frac{\varepsilon(t)}{t} + \frac{p_1(x/t)}{t}
\]
\[
\leq \frac{1}{t} \left(\varepsilon(t) + c_2 \left(1 \wedge \left|\frac{t}{x}\right|^2\right)\right).
\]
(4.61)
Note that for \(|w - z| > K_N t_N\), on \(\{|B_{\hat{\eta}_N t_N}^N| \leq \frac{K_N t_N}{2}\}\),
\[
|w - z - B_{\hat{\eta}_N t_N}^N|^{-1} \leq \frac{2}{K_N t_N}.
\]
Thus by inequality (4.61), \(\Sigma_{2c}^N\) is less than
\[
\left(\varepsilon(\nu_N(2 - \hat{\eta}_N)t_N) + c_2 \left(1 \wedge \left(\frac{2\nu_N(2 - \hat{\eta}_N)^2}{NK_N}\right)\right)\right) \frac{H(\nu_N\hat{\eta}_N t_N)}{\nu_N(2 - \hat{\eta}_N)t_N}.
\]
Thus by \(a_N\epsilon_N' = \nu_N(2 - \hat{\eta}_N)t_N\) and (4.3),
\[
\Sigma_{2c}^N \leq C \left(\varepsilon(\nu_N(2 - \hat{\eta}_N) + 1/K_N^2) \frac{\log N}{\nu_N\epsilon_N' \log(\nu_N\hat{\eta}_N t_N)}\right)
\]
\[
\leq C \left(\varepsilon(\nu_N(2 - \hat{\eta}_N)/(N\epsilon_N') + (N \log N\epsilon_N')^{-1})\right)
\]
\[
\leq C \left(\varepsilon(\nu_N(2 - \hat{\eta}_N)/(N\epsilon_N) + \log \log N/N \log N)\right)
\]
\[
= C\epsilon_N/N,
\]
(4.62)
where \(C\) may change its values from line to line and the second inequality follows from
\[
\log(\nu_N\hat{\eta}_N t_N) = \log(\epsilon_N') + \log(\nu_N) - \log \log N - \sqrt{\log N
\]
and \(\lim_{N \to \infty} \frac{N}{\nu_N} = 1\). With (4.59), (4.60) and (4.62) in hand, (4.58) yields the desired result, (4.57).

For \(\phi : \mathbb{R}^2 \to \mathbb{R}, \zeta \in \{0, 1\}^\mathcal{S}\) and \(X(\phi) = (1/N') \sum_x \phi(x)\zeta(x)\), define
\[
\Delta_1^{N,+}(\phi, \zeta) = X(\log N \phi^2 f_0^N(\cdot, \zeta))
\]
\[
\Delta_2^{N,+}(\phi, \zeta) = \frac{1}{N'} \sum_x (1 - \zeta(x))\phi(x) \log N f_1^N(x, \zeta)^2
\]
\[
\Delta_3^{N,+}(\phi, \zeta) = X(\log N \phi f_0^N(\cdot, \zeta)^2)
\]
and

\[ \Delta^N_j(\phi, \zeta) = \Delta^{N,+}_j(\phi, \zeta)\gamma_j X(\phi), \quad j = 1, 2, 3, \]

where \( \gamma_1 = p_1(0)^{-1} \) and \( \gamma_2 = \gamma_3 = \gamma^* \). Define

\[ m(1) = 2 \quad \text{and} \quad m(2) = m(3) = 1. \]

The following proposition is a version of Proposition 7.5 of [9].

**Proposition 4.9** There is a constant \( C_{4.63} \) and a sequence \( \eta(N) \downarrow 0 \) such that for \( j = 1, 2, 3 \), if \( \phi : \mathbb{R}^2 \to \mathbb{R} \), then for any \( 0 < \alpha < 1 \)

\[
|E(\Delta^N_j(\phi, \hat{\xi}_N))| \leq \eta_{4.63}(N) \left( \hat{X}_0^N(1) + \hat{X}_0^N(1)^2 \right) \| \phi \|_{\infty}^{m(j)}
+ C_{4.63} \| \phi \|_{\infty}^{m(j)} \frac{\epsilon'}{\epsilon_N} \int \int_{|w-z| \leq \epsilon_N} \hat{X}_0^N(w)d\hat{X}_0^N(z).
\]  

(4.63)

**Proof.** To prove the proposition, we can define \( \Sigma^N_j, i = 1, 2 \) for \( j = 1 \) and \( i = 1, 2, 3 \) for \( j = 2, 3 \) as in (7.20), (7.21) and (7.22) of [9] and decompose each \( E(\Delta^{N,+}_j) \) into a sum of those terms. We omit the definitions and decompositions here, since they are the same. By Lemma 4.2 we can show that

\[
\Sigma^2,N \leq C_{4.57} \| \phi \|_{\infty}^{m(j)} \left( \epsilon'_{N} \right)^{-1} \int \int_{|w-z| \leq \delta_N} \hat{X}_0^N(w)d\hat{X}_0^N(z) + \epsilon_N \hat{X}_0^N(1)^2.
\]

(4.64)

For \( \Sigma^3,N, j = 2, 3 \), with Proposition 4.2 in hand, one can check that a similar conclusion to that in Lemma 2.5 of [9] is available. Following the proof of Proposition 7.5 of [9], we have

\[
\Sigma^3,N + \Sigma^3,N \leq C_{4.65} \| \phi \|_{\infty} \hat{X}_0^N(1)(\log N)^{-1/2}.
\]

(4.65)

Now, we need to establish that there is a sequence \( \eta(N) \rightarrow 0 \) such that for \( j = 1, 2, 3 \),

\[
|\Sigma^N_j - \gamma_j \hat{X}_0^N(\phi)| \leq \eta(N) \| \phi \|_{\infty}^{m(j)} \hat{X}_0^N(1).
\]

(4.66)

Let \( \epsilon \) denote independent random variable with law \( p(\cdot) \). First,

\[
P \left( B_{t_N}^{N,e} > \sqrt{\epsilon_N} \right) = P \left( |B_{\nu_{Nt}}^0 + \epsilon| > N \sqrt{\epsilon_N} \right).
\]

We also have

\[
P \left( |B_{\nu_{Nt}}^0 + \epsilon| > N \sqrt{\epsilon_N} \right) \leq \frac{2 |p|_{\alpha}}{N \sqrt{\epsilon_N}} + P \left( |B_{\nu_{Nt}}^0| > N \sqrt{\epsilon_N}/2 \right)
\]

\[
\leq \frac{2 |p|_{\alpha}}{N \sqrt{\epsilon_N}} + \frac{C_{2.12} \mu_{Nt}}{N \sqrt{\epsilon_N}},
\]

(4.67)

where the second inequality follows from Proposition 2.4. Typically, we have

\[
P \left( B_{t_N}^{N,0} > \sqrt{\epsilon_N} \right) \leq \frac{C_{2.12} \mu_{Nt}}{N \sqrt{\epsilon_N}} = \frac{C_{2.12} \mu \sqrt{\epsilon_N}}{N \log N}.
\]

(4.68)
Now, we consider the case of $j = 2$. By the same arguments as in \([9]\), we can show
\[
|\Sigma_2^{1,N} - \gamma^* \hat{X}_0^N(\phi)|
\leq \frac{1}{N'} \sum_w \xi_0^N(w) \log N E \left( |\phi(w - \hat{B}_{t_N}^{N,e}) - \phi(w)|; \hat{\tau}^N(0,e) > t_N, \right)
\leq \frac{1}{N'} \sum_w \xi_0^N(w) \log N E \left( |\phi(w - \hat{B}_{t_N}^{N,e}) - \phi(w)|; \hat{\tau}^N(0,e) > t_N \right)
\leq ||\phi||_\infty \hat{X}_0^N(1) \log N \left( \sqrt{\epsilon_N} \right) + 2 ||\phi||_\infty \hat{X}_0^N(1) \log N \left( \sqrt{\epsilon_N} \right)
\leq ||\phi||_\infty \hat{X}_0^N(1) \left( \sqrt{\epsilon_N} \right) + 2 \log N \left( \sqrt{\epsilon_N} \right)
\leq ||\phi||_\infty \hat{X}_0^N(1) ||\phi||_\infty \hat{X}_0^N(1)
\leq \eta_{4.69}(N) ||\phi||_\infty \hat{X}_0^N(1).
\]
where the second inequality follows from Cauchy-Schwarz inequality and considering the cases $|\hat{B}_{t_N}^{N,e}| > \sqrt{\epsilon_N}$ and $|\hat{B}_{t_N}^{N,e}| \leq \sqrt{\epsilon_N}$. Thus by \((4.41), (4.67)\) and Proposition \(4.1\) there exists a sequence \(\eta_{4.69}(N)\) which goes to 0 as $N \to \infty$ such that
\[
|\Sigma_2^{1,N} - \gamma^* \hat{X}_0^N(\phi)| \leq \eta_{4.69}(N) ||\phi||_\infty \hat{X}_0^N(1).
\]
By replacing $\hat{B}_{t_N}^{N,e}, \hat{B}_{t_N}^{N,0}$ with $\hat{B}_{t_N}^{N,0}, \hat{B}_{t_N}^{N,0}$ respectively, the same argument as that above gives the same bound for $|\Sigma_3^{1,N} - \gamma^* \hat{X}_0^N(\phi)|$. Typically, inequality \((4.67)\) could be simplified. Next, we turn to $\Sigma_1^{1,N}$. Following the strategy of the proof for term on $\Sigma_2^{1,N}$, we have that
\[
|\Sigma_1^{1,N} - p_1(0)^{-1} \hat{X}_0^N(\phi^2)|
= \left| \frac{1}{N'} \sum_w \xi_0^N(w) \left( \log N E \left( \phi^2(w - \hat{B}_{t_N}^{N,0}); \tau^N(0,e) > t_N \right) - p_1(0)^{-1} \phi^2(w) \right) \right|
\leq \left| \frac{1}{N'} \sum_w \xi_0^N(w) \left( \log N E \left( \phi^2(w - \hat{B}_{t_N}^{N,0}); \tau^N(0,e) > t_N \right) \right) \right|
+ \left| \frac{1}{N'} \sum_w \xi_0^N(w) \phi^2(w) \log N \left( \nu_N t_N \right) - p_1(0)^{-1} \phi^2(w) \right|
\leq \left( 2 ||\phi||_\infty \log N \left( \nu_N t_N \right) + 2 ||\phi||_\infty \log N \left( \nu_N t_N \right) \right) ||\phi||_\infty \hat{X}_0^N(1).
\]
According to \((4.67)\) and \((4.4)\), we can conclude
\[
|\Sigma_1^{1,N} - p_1(0)^{-1} \hat{X}_0^N(\phi^2)| \leq \eta_{4.70}(N) ||\phi||^2_\infty \hat{X}_0^N(1)
\]
where $\eta_{4.70}(N) \to 0$ as $N \to \infty$. Thus we get the \((4.69)\). By decompositions in (7.18) of \([9]\), we obtain the desired result.

With Proposition 7.1 in hand, Proposition 4.7 follows from the following two propositions which are analogous to Proposition 7.1 and Proposition 7.2 in \([9]\) and a similar argument to that in Section 8 of \([9]\).
Proposition 4.10 There is a constant $C_{4.71} (K)$ and sequence $\eta_{4.71} (N) \downarrow 0$ such that for all $\phi : \mathbb{R} \to [0, \infty)$ satisfying $||\phi||_{Lip} \vee X_0^N (1) \leq K$ and $j = 1, 2, 3,$

$$|E(\Delta_j (\phi, \xi^N_{1N}))| \leq C_{4.71} (K) \left( \eta_{4.71} (N) (X_0^N (1) + X_0^N (1)^2) \right) + (\epsilon'_N)^{-1} \int_{|w-z| \leq \delta_N} dX_0^N (w) dX_0^N (z).$$

(4.71)

Proof. First, we can obtain follow the strategy in the proof of Lemma 7.8 in [9] to obtain an analogous result to that in Lemma 7.8 of [9]. Then with our coupling, (4.48) and Proposition 4.9 in hand, following the argument in [9], one can get the desired result. \(\square\)

Proposition 4.11 There is a constant $C_{4.72}$ such that for all $0 \leq t \leq T$,

$$E \left( \int \int_{|w-z| \leq \delta_N} dX_0^N (w) dX_0^N (z) \right) \leq C_{4.72} \left( X_0^N (1) + X_0^N (1)^2 \right) \times \delta_N \left( t^2 + \delta_N t^{-1/3} \log(1 + \frac{t}{\delta_N}) \right).$$

(4.72)

The proof of Proposition 4.11 is also exactly the same with that of Proposition 7.2 of [9]. In fact, we only need to prove the following random walk estimate which is a version of Corollary 7.9 of [9] and can be deduced directly from (2.6) and Proposition 2.3. Let $B_t^{N,*}$ be the random walk with semigroup $(P_t^{N,*}, t \geq 0)$ from Proposition 4.5, at rate $\nu_N + b_N = N + \bar{\theta} \log N$, $B_t^{N,*}$ takes steps with $p_N (-)$ and $B_0^{N,*} = 0$.

Corollary 4.1 (a) For all $x \in \mathbb{S}_N$ and $t \geq 0$,

$$P(B_t^{N,*} = x) \leq \frac{C_{2.6}}{1 + Nt}.$$  

(4.73)

(b) Assume $\delta'_N \downarrow 0$ and $N\delta'_N \to \infty$. For each $K > 0$ there is a constant $C_{4.74} (K) > 0$ such that

$$\inf_{N \geq 1, w \in \mathbb{S}_N, |w| \leq K \delta'_N} N \delta'_N P(B_t^{N,*} = w) \geq C_{4.74} (K) > 0.$$  

(4.74)

Now, one follows the argument in [9] to get Proposition 4.11. To obtain Proposition 4.7, the following arguments are similar to those in Section 8 of [9]. We omit it here.

5 Voter Model’s Asymptotics

In this section, we will prove Theorem 1.3 and we assume that assumption (A1) holds with $b(t) = t^{1/\alpha}$. Recall that $p_t = P(|\xi^0_t| > 0)$. Our first object is to prove that

$$p_t = O \left( \frac{\log t}{t} \right) \quad \text{as} \quad t \to \infty \quad d = \alpha,$$

$$p_t = O(t^{-1}) \quad \text{as} \quad t \to \infty \quad d > \alpha.$$

(5.1)
The asymptotics above are similar to the results in Theorem 1 of [3]. Note that Theorem 1 of [3] could be proved under the assumption that the underlying motion has finite variance and one only need to modify the proof of Lemma 5 of [3]; see Lemma 2 of [2]. For our purpose we also need to generalize the asymptotic results in (14) of [3].

Recall that \( \{B^x_t, x \in \mathbb{Z}^d\} \) is a collection of rate-one independent stable random walks with \( B^x_0 = x \). Let \( p_t(x, y) = P(B^x_t = y) \) denote the transition function of \( \{B^x_t\} \). Define the mean range of the stable random walk \( B^0_t \) by

\[
R(t) = E \left( \sum_{x \in Z^d} 1_{\{B^0_s = x \text{ for some } s \leq t\}} \right).
\]

By the results for the range of the discrete time stable random walk in [15], we see

\[
\lim_{t \to \infty} \frac{R(t)}{t} = \frac{\gamma_e}{d} > \alpha.
\]

With this in hand, one can generalize the asymptotics results in (14) of [3]. Now, to prove (5.1) we only need to prove some analogous results to those in Lemma 5 of [3]. Set \( G_t(x) = \int_0^t p_s(0, x)ds \) and let \( \tau(x) = \inf\{t \geq 0 : B^x_t = 0\} \), define \( H_t(x) = P(\tau(x) \leq t) \).

**Lemma 5.1** If \( x \in \mathbb{Z}^d \) with \( |x| = r \), then there is a constant \( C_{d,\alpha} > 0 \) such that

\[
H_{r^\alpha}(x) \geq C_{d,\alpha} \log r \quad d = \alpha,
\]

\[
\geq C_{d,\alpha} r^{\alpha - d} \quad d > \alpha.
\]

**Proof.** We first consider the asymptotics for the Green’s function. According to (2.4) and (2.11), when \( r \) large enough,

\[
G_{r^\alpha}(x) = \int_0^{r^\alpha} p_s(0, x)ds \geq c_1 \int_{r^\alpha/2}^{r^\alpha} \frac{s}{s^{d+\alpha}} ds - \int_{r^\alpha/2}^{r^\alpha} s^{-d/\alpha} ds.
\]

A bit of calculation show that there exist a constant \( \check{C}_{d,\alpha} > 0 \) such that

\[
G_{r^\alpha}(x) \geq \check{C}_{d,\alpha} r^{\alpha - d} \quad d > \alpha,
\]

\[
\geq \check{C}_{d,\alpha} \quad d = \alpha.
\]

By (2.5), we see that there exist constants \( C_{d,\alpha} > 0 \) such that

\[
G_{r^\alpha}(0) \leq C_{d,\alpha} \log r \quad d = \alpha.
\]

Then the desired result follows from inequality \( H_t(x) \geq G_t(x)/G_t(0) \). \( \Box \)

Now, one can follow the arguments in Section 3 of [3] to obtain (5.1) (Note that when prove an analogous result to that in Lemma 4 of [3] one may need to set \( s_t = d[(2p_t^{-1})^{1/d}]^{\alpha} \). With (5.1), Theorem 1.1 and Theorem 1.2 in hand, the following proof for Theorem 1.3 are
exactly the same with that in \[6\]. We left it to the interested readers. The intuition is that the underlying motion has nothing to do with the total mass process.

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