Separable $K$-linear categories

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Abstract: We define and investigate separable $K$-linear categories. We show that such a category $C$ is locally finite and that every left $C$-module is projective. We apply our main results to characterize separable linear categories that are spanned by groupoids or delta categories.

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Introduction

Linear categories are important generalizations of ordinary associative algebras, that play an important role in various fields of mathematics, such as representation theory of finite dimensional algebras, mathematical physics, etc. They were introduced and studied in [2], while in [3] several homological tools were adapted to this more general framework. In particular, in loc. cit. Hochschild-Mitchell cohomology was defined as a substitute of Hochschild cohomology, which is a key homological invariant of unital associative algebras.

The aim of this short note is to investigate the basic properties of the simplest linear categories from a cohomological point of view. More precisely, we give equivalent characterizations of those linear categories with the property that their Hochschild-Mitchell cohomology groups vanish in positive degrees, see Theorem 2.1. It is worthwhile to remark that for associative algebras a similar result can be found in [6], and in [1] in the more general case of algebras in an abelian monoidal category. In analogy to the case of associative algebras, we call these linear categories separable. We also show that a separable linear category $C$ is locally finite, i.e. $\dim_K \text{Hom}_C(x, y) < \infty$, for any objects $x$ and $y$ in $C$. The later result may be seen as a generalization of Zelinsky Theorem. As applications of our main results we give necessary and sufficient conditions such that the $K$-linear categories spanned by groupoids and delta categories to be separable.

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1. Preliminaries

Throughout this paper \(C\) denotes a small category. The set of objects in \(C\) is denoted by \(C_0\) and, for simplicity, we write \(C(x, y)\) for \(\text{Hom}_C(x, y)\).

Claim 1.1.

**K-linear categories.** Let \(K\) be a field. A category \(C\) is said to be \(K\)-linear if \(C(x, y)\) is a \(K\)-vector space, for any \(x, y \in C_0\), and the composition maps in \(C\) are bilinear. Note that the composition in \(C\) can be seen as linear maps

\[
C(y, z) \otimes C(x, y) \to C(x, z), \quad g \otimes f \mapsto g \circ f.
\]

For the properties of linear categories the reader is refered to [3, 5] and the references therein.

Let \(C\) and \(D\) be two \(K\)-linear categories. A functor \(F : C \to D\) is said to be \(K\)-linear if \(F(-) : C(x, y) \to D(F(x), F(y))\) is a \(K\)-linear map, for all \(x, y \in C_0\).

Claim 1.2.

**Modules and bimodules over \(K\)-linear categories.** A left module over \(C\) is a \(K\)-linear functor \(M : C \to K\text{-Mod}\). Note that \(M\) is defined by a family of vector spaces \((M)_x \in C_0\) and \(K\)-linear maps

\[
\triangleright : C(y, x) \otimes M_x \to M_y
\]

satisfying appropriate identities, resembling the definition of modules over associative algebras. A module morphism \(f : M \to N\) is a natural transformation between the functors \(M\) and \(N\). It is given by a family \((f)_x \in C_0\) of \(K\)-linear maps

\[
\triangleright : M_x \to N_y,
\]

which are also linear with respect to the maps that define the module structures on \(M\) and \(N\). Right \(C\)-modules are defined analogously. We obtain two categories \(C\text{-Mod}\) and \(\text{Mod}-C\).

To define \(C\)-bimodules one defines a new linear category \(C \otimes_K C^{op}\), see [5] for details. By definition, a \(C\)-bimodule is a left \(C \otimes_K C^{op}\)-module, that is a family \((M_y)_{y \in C_0}\), together with left and right actions

\[
\triangleright : C(y, x) \otimes M_x \to M_y,
\]

\[
\triangleleft : M_x \otimes C(y, x) \to M_y,
\]

such that, for all \(x_0\) and \(y_0\) in \(C_0\), the pairs \((M_{y_0})_{y \in C_0}, \triangleright\) and \((M_{x_0})_{x \in C_0}, \triangleleft\) are a left and a right \(C\)-module, respectively, and these structures are compatible in the obvious sense. We shall denote these modules by \(_x M_{y_0}\) and \(_{x_0} M_x\), respectively. The category of \(C\)-bimodules is denoted by \(C\text{-Mod}-C\). Note that \(C\) can be seen in a canonical way as an object in \(C\text{-Mod}-C\). Another example of \(C\)-bimodule is \(C \otimes C\), whose components are

\[
_\triangleright (C \otimes C)_y = \bigoplus z C(x, z) \otimes C(y, z).
\]

Both left and right module structures of \(C \otimes C\) are induced by the composition in \(C\). The composition in \(C\) also defines a morphism of \(C\)-bimodules \(\text{comp} : C \otimes C \to C\).

The categories \(C\text{-Mod}, \text{Mod}-C\) and \(C\text{-Mod}-C\) are abelian and have enough projective and injective objects, cf. [5]. Thus we may consider \(\text{Ext}\) functors in these categories.

Claim 1.3.

**Hochschild–Mitchell cohomology.** Let \(C\) be a \(K\)-linear category, and let \(M\) be a \(C\)-bimodule. Hochschild–Mitchell cohomology of \(C\) with coefficients in \(M\) is defined by

\[
H^*(C, M) := \text{Ext}^*_C(C, M),
\]

where \(\text{Ext}^*_C(\cdot, \cdot)\) denote the Ext functors in the category \(C\text{-Mod}-C\).
Many of the properties of Hochschild-Mitchell cohomology follow immediately from the fact that this cohomology theory is defined using derived functors in an abelian category. The most important ones for our work are the following.
First, if \( n \in \mathbb{N}^+ \) and \( M \) is an injective \( \mathcal{C} \)-bimodule, then \( H^n(\mathcal{C}, M) = 0 \). This equality also holds if \( \mathcal{C} \) is a projective as a \( \mathcal{C} \)-bimodule.
Second, if \( 0 \to M \to N \to P \to 0 \) is an exact sequence of \( \mathcal{C} \)-bimodules, then the exact sequence of the Ext functor, applied to the above short exact sequence, yields the following long exact sequence:

\[
0 \to H^0(\mathcal{C}, M) \to H^0(\mathcal{C}, N) \to H^0(\mathcal{C}, P) \to H^1(\mathcal{C}, M) \to H^1(\mathcal{C}, N) \to H^1(\mathcal{C}, P) \to \ldots \tag{1}
\]

2. Separable linear categories

We are going to study the \( K \)-linear categories that are simple from a cohomological point of view. More exactly, we are going to study the properties of a \( K \)-linear category \( \mathcal{C} \) such that its Hochschild-Mitchell cohomology in positive degrees is trivial.

**Definition 2.1.**
A \( K \)-linear category \( \mathcal{C} \) is separable if \( H^1(\mathcal{C}, M) = 0 \), for any \( \mathcal{C} \)-bimodule \( M \).

**Lemma 2.1.**
The bimodule \( \mathcal{C} \otimes \mathcal{C} \) is projective (i.e. it is a projective object in \( \mathcal{C}\text{-Mod-}\mathcal{C} \)).

**Proof.** Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{C} & \xrightarrow{\psi} & \mathcal{C} \\
\downarrow & & \downarrow
\end{array}
\]

where \( \varphi \) and \( \pi \) are arbitrary bimodule morphisms with \( \pi \) epimorphism. By definition we have \( \psi(\mathcal{C} \otimes \mathcal{C})_g = \bigoplus_y \mathcal{C}(z, x) \otimes \mathcal{C}(y, z) \). Since \( \pi \) is an epimorphism, for every \( z \in \mathcal{C}_0 \), there is \( z, m_z \in \mathcal{C}_M \) such that \( z, \pi((z, m_z) = z, \varphi(1_z \otimes 1_z) \). We define

\[
\psi_y : \mathcal{C}(z, x) \otimes \mathcal{C}(y, z) \to \mathcal{C}_M,
\]

\[
\psi_y(f \otimes g) = f \triangleright z, m_z \triangleleft g.
\]

Let \( \psi_y : \mathcal{C}(z, x) \to \mathcal{C}_M \) be the \( K \)-linear map induced by the family \( \{\psi_y\}_{z \in \mathcal{C}_0} \). It is easy to see that the family \( \{\psi_y\}_{z \in \mathcal{C}_0} \) is a morphism of bimodules such that \( \pi \circ \psi = \varphi \). Thus \( \mathcal{C} \otimes \mathcal{C} \) is projective.

**Theorem 2.1.**
Let \( \mathcal{C} \) be a \( K \)-linear category. The following statements are equivalent:

1. \( \mathcal{C} \) is separable.
2. \( H^n(\mathcal{C}, M) = 0 \) for all \( n > 0 \) and \( M \in \mathcal{C}\text{-Mod-}\mathcal{C} \).
3. \( \mathcal{C} \) is projective as a bimodule.
4. The canonical morphism \( \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \) splits in the category \( \mathcal{C}\text{-Mod-}\mathcal{C} \).
5. There is a family \( \{a^2_x\}_{x \in \mathcal{C}_0} \) with the following properties:
   
   \( \begin{enumerate} 
   \item The element \( a^2_x \in \mathcal{C}(y, x) \otimes \mathcal{C}(x, y) \), for all \( x, y \in \mathcal{C}_0 \).
   \item For any \( x \in \mathcal{C}_0 \), the family \( \{a^2_x\}_{y \in \mathcal{C}_0} \) is of finite support.
   \end{enumerate} \)
(c) For every object \( x \), we have \( \sum_{y \in C_0} \text{comp}(a^x_y) = 1_x \), the comp denotes the map induced by the composition in the category \( C \).

(d) If \( I \in C(x, z) \), then \( I \triangleright a^x_y = a^x_y \triangleleft I \).

**Proof.** (1) \( \Leftrightarrow \) (2) Recall that, by definition, \( C \) is separable if and only if \( H^1(C, M) = 0 \), for every bimodule \( M \). Therefore, (2) \( \Rightarrow \) (1) is trivial. The other implication can be proved by induction as follows. We assume that \( H^n(C, Q) = 0 \), for all \( n > 0 \) and \( Q \in \text{C-Mod-C} \). As in the category of \( C \)-bimodules there are enough injective objects, there exists a monomorphism \( i : M \to I \) in \( \text{C-Mod-C} \). Let \( Q \) be the cokernel of \( i \), so the following sequence is exact in \( \text{C-Mod-C} \):

\[
0 \longrightarrow M \longrightarrow I \longrightarrow Q \longrightarrow 0.
\]

From (1) we get the following exact sequence:

\[
H^n(C, Q) \rightarrow H^{n+1}(C, M) \rightarrow H^{n+1}(C, I)
\]

By induction hypothesis, \( H^n(C, Q) = 0 \). On the other hand \( H^{n+1}(C, I) = 0 \), as \( I \) is injective as a \( C \)-bimodule. Thus \( H^{n+1}(C, M) = 0 \), too.

(2) \( \Leftrightarrow \) (3) Hochschild-Mitchell cohomology of \( C \) with coefficients in \( M \) is defined by \( H^n(C, M) = \text{Ext}^n_{C-\text{Mod}}(C, M) \). Furthermore, an object \( X \) in an abelian category \( A \) is projective if, and only if, \( \text{Ext}^n_X(Y, Y) = 0 \), for all \( n > 0 \) and \( Y \in \text{Ob}(A) \). Thus, \( C \) is projective as a \( C \)-bimodule if and only if \( H^n(C, M) = 0 \), for all \( n > 0 \) and \( M \in \text{C-Mod-C} \).

(3) \( \Rightarrow \) (4) Consider the following diagram in \( \text{C-Mod-C} \):

\[
\begin{array}{ccc}
C & \xrightarrow{C} & C \\
\downarrow & & \downarrow \\
C \otimes C & \xrightarrow{\text{comp}} & C
\end{array}
\]

If \( C \) is projective, then there exists a morphism of \( C \)-bimodules \( \varphi : C \rightarrow C \otimes C \) such that \( \text{comp} \circ \varphi = 1_C \). This proves that \( \varphi \) is a section of \( \text{comp} \), that is \( \varphi \) splits in \( \text{C-Mod-C} \).

(4) \( \Rightarrow \) (3) Suppose that the canonical morphism \( \text{comp} : C \otimes C \rightarrow C \) has a section in \( \text{C-Mod-C} \). It results that \( M \), the kernel of \( \text{comp} \), is a complement of \( C \) in \( C \otimes C \). From the above lemma \( C \otimes C \) is projective, so \( C \) is a projective \( C \)-bimodule, since it is a direct summand in a projective bimodule.

(4) \( \Rightarrow \) (5) Let \( \varphi : C \rightarrow C \otimes C \) be a \( C \)-bimodule morphism such that \( \text{comp} \circ \varphi = 1_C \). We fix \( (x, y) \in C_0 \times C_0 \). Let \( x \varphi_y : C(x, y) \rightarrow \bigoplus_{y \in C_0} C(x, y) \otimes C(y, z) \) be the corresponding component of \( \varphi \). We have

\[
x \varphi_y(1_x) \in \bigoplus_{y \in C_0} C(y, x) \otimes C(x, y),
\]

so \( x \varphi_y(1_x) = (a^x_y)_{y \in C_0} \), where \( a^x_y \in C(y, x) \otimes C(x, y) \). Hence the family \( (a^x_y)_{x, y \in C_0} \) satisfies the first property in (5). It also satisfies the second property as \( x \varphi_y(1_x) \) is an element in \( \bigoplus_{y \in C_0} C(y, x) \otimes C(x, y) \), so the family \( (a^x_y)_{y \in C_0} \) has finite support. Moreover, \( \text{comp}(x \varphi_y(1_x)) = 1_x \), since \( \varphi \) is a section of \( \text{comp} \). Thus, for all \( x \in C_0 \)

\[
\sum_{y \in C_0} \text{comp}(a^x_y) = 1_x,
\]

i.e. \( (a^x_y)_{y \in C_0} \) satisfies the third property. Let us now show that \( (a^x_y)_{x, y \in C_0} \) satisfies the last property. Let \( I \in C(x, z) \). Since \( a^x_y \in C(y, x) \otimes C(x, y) \), we can write this element as a sum

\[
a^x_y = \sum_{j=1}^{n_{x,y}} f^i_{y,x} \otimes g^j_{x,y},
\]
where \( f_{y,x}^i \in \mathcal{C}(y,x) \) and \( g_{x,y}^i \in \mathcal{C}(x,y) \). Then we get
\[
f \triangleright a_y^x = \sum_{i=1}^{n_{x,y}} f \circ f_{y,x}^i \otimes g_{x,y}^i \quad \text{and} \quad a_y^x \prec f = \sum_{i=1}^{n_{x,y}} f_{y,x}^i \otimes g_{x,y}^i \circ f.
\]

On the other hand
\[
x \varphi_y(f) = x \varphi_y(f \triangleright 1_x) = f \triangleright x \varphi_y(1_x) = f \triangleright (a_y^x)_{y \in \mathcal{C}_0} = (f \triangleright a_y^x)_{y \in \mathcal{C}_0}
\]
and, similarly,
\[
x \varphi_y(f) = x \varphi_y(1_x \prec f) = a_y^x(1_x \prec f) = (a_y^x)_{y \in \mathcal{C}_0} \prec f = (a_y^x \prec f)_{y \in \mathcal{C}_0}.
\]
So \( f \triangleright a_y^x = (a_y^x \prec f)_{y \in \mathcal{C}_0} \). We deduce that \( f \triangleright a_y^x = a_y^x \prec f \), for all \( y \in \mathcal{C}_0 \).

(5) \( \Rightarrow \) (4) Let \( (a_y^x)_{x,y \in \mathcal{C}_0} \) a family which satisfies (a)-(d). We define:
\[
x \varphi_y : \mathcal{C}(y,x) \to \bigoplus_{x \in \mathcal{C}_0} \mathcal{C}(x,x) \otimes \mathcal{C}(y,x), \quad x \varphi_y(f) = (f \triangleright a_y^x)_{x \in \mathcal{C}_0}.
\]

The map \( x \varphi_y \) is well-defined because \( (a_y^x)_{x \in \mathcal{C}_0} \) is of finite support. Obviously \( \varphi = (x \varphi_y)_{x,y \in \mathcal{C}_0} \) is a morphism of left \( \mathcal{C} \)-modules. The family \( \varphi \) defines a morphism of right \( \mathcal{C} \)-modules because \( (a_y^x)_{x,y \in \mathcal{C}_0} \) satisfies (d). Finally, taking into account (c), \( \varphi : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \) is a section for \( \text{comp} : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \) in \( \mathcal{C} \text{-Mod} \). \( \square \)

Using the equivalent characterization of separable linear categories in Theorem 2.1 we shall now prove a generalization of Zelinsky Theorem, which states that a separable algebra is finite dimensional (as a vector space over the base field).

**Definition 2.2.**
We say that a \( K \)-linear category \( \mathcal{C} \) is **locally finite dimensional** if \( \dim_K \mathcal{C}(x,y) < \infty \) for all \( x, y \in \mathcal{C}_0 \).

**Theorem 2.2.**
A \( K \)-linear separable category \( \mathcal{C} \) is locally finite.

**Proof.** Since \( \mathcal{C} \) is separable, there is a family \( (a_y^x)_{x,y \in \mathcal{C}_0} \) that satisfies the properties (a)-(d) in Theorem 2.1 (5). We write each \( a_y^x \) as a sum
\[
a_y^x = \sum_{i=1}^{n_{x,y}} f_{y,x}^i \otimes g_{x,y}^i,
\]
with \( f_{y,x} \in \mathcal{C}(y,x) \) and \( g_{x,y} \in \mathcal{C}(x,y) \). From all representations of \( a_y^x \) as in (2) we choose one such that the number \( n_{x,y} \) is minimal and the set \( \{ g_{x,y}^i \mid i = 1, \ldots, n_{x,y} \} \) is linearly independent. Thus, for every \( i = 1, \ldots, n_{x,y} \), there is a \( K \)-linear application \( a_y^x : \mathcal{C}(x,y) \to K \) such that \( a_y^x(g_{x,y}^i) = \delta_{ij} \), for any \( 1 \leq j \leq n_{x,y} \). If \( f \in \mathcal{C}(x,z) \) and \( y \in \mathcal{C}_0 \) then \( f \triangleright a_y^x = a_y^x \preceq f \). Equivalently, we have the following identity in \( \mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \)
\[
\sum_{p=1}^{n_{x,y}} f \circ f_{y,x}^p \otimes g_{x,y}^p = \sum_{q=1}^{n_{x,y}} f_{y,x}^q \otimes g_{x,y}^q \preceq f.
\]

Let \( \mathcal{V}_{y,x} \) denote the vector space generated by \( f_{y,x}^i \), where \( i = 1, \ldots, n_{x,y} \). By construction, we have \( \dim_K \mathcal{V}_{y,x} < \infty \). For a given \( i \), let us apply \( 1_{\mathcal{C}(y,x)} \otimes a_y^x \) to the left and right sides of (3). For \( x, y, z \in \mathcal{C}_0 \) and \( f \in \mathcal{C}(x,z) \), one obtains
\[
f \circ f_{y,x}^i = \sum_{q=1}^{n_{x,y}} a_y^x(g_{x,y}^q \preceq f) f_{y,x}^q.
\]
This relation shows that $f \circ f_{y,x}^i \in \mathcal{V}_{y,z}$. Furthermore, the composition in $\mathcal{C}$ induces an application

$$\varphi_{x,y,z} : C(x, z) \to \text{Hom}_K(\mathcal{V}_{y,x}, \mathcal{V}_{y,z}), \quad \varphi_{x,y,z}(f)(g) = f \circ g.$$  

We fix $(x, z) \in \mathcal{C}_0 \times \mathcal{C}_0$. The family $\{a_{y}^{ij}\}_{y \in \mathcal{C}_0}$ is of finite support, so there exist $y_1, \ldots, y_p \in \mathcal{C}_0$ such that $a_{y}^{ij} = 0$, for every $y$ which does not belong to $\{y_1, \ldots, y_p\}$. We define

$$\varphi_{x,z} : C(x, z) \to \bigoplus_{j=1}^{p} \text{Hom}_K(\mathcal{V}_{y_j,x}, \mathcal{V}_{y_j,z}), \quad \varphi_{x,z}(f) = \{\varphi_{x,y_j,z}(f)\}_{1 \leq j \leq p}.$$  

We claim that $\varphi_{x,z}$ is injective. If $f \in \ker \varphi_{x,z}$, then $\varphi_{x,y_j,z}(f)(g) = 0$ for any $1 \leq j \leq p$. Thus $\varphi_{x,y_j,z}(f)(g) = 0$, for all $g \in \mathcal{V}_{y_j,x}$. In particular, by taking $g := f_{y_j,x}^i$, we get $f \circ f_{y_j,x}^i = 0$ for any $1 \leq i \leq n_{x,y_j}$. It results that

$$f \triangleright a_{y}^{ij} = \sum_{i=1}^{n_{x,y_j}} f \circ f_{y_j,x}^i \otimes g_{y_j}^{i} = 0,$$

for all $1 \leq j \leq p$. On the other hand, if $y \notin \{y_1, \ldots, y_p\}$ then $a_{y}^{ij} = 0$. We deduce that

$$f = f \triangleright 1_x = f \triangleright \text{comp}\left(\sum_{j=1}^{p} a_{y}^{ij}\right) = \sum_{j=1}^{p} \text{comp}(f \triangleright a_{y}^{ij}) = 0.$$

In conclusion, $\varphi_{x,z}$ is injective, as we claimed. Therefore, $C(x, z)$ can be embedded in the vector space $\mathcal{V} = \bigoplus_{j=1}^{p} \text{Hom}_K(\mathcal{V}_{y_j,x}, \mathcal{V}_{y_j,z})$. Note that $\mathcal{V}$ is a finite dimensional vector space, being a finite direct sum of finite dimensional vector spaces. Thus, $C(x, z)$ is obviously finite dimensional, for every $x, z \in \mathcal{C}_0$.  

Let $A$ be a not necessarily linear category. The $K$-linearization of $A$ is the $K$-linear category $K[A]$ that has the same objects as $A$, but

$$K[A]x,y := \{f \mid f \in A(x,y)\}_K.$$  

Therefore, by definition, $K[A]x,y$ is the $K$-vector space having $A(x,y)$ as a basis. The composition in $K[A]$ is the unique bilinear extension of the composition in $A$. Recall that a category $\mathcal{G}$ is a groupoid if, and only if, all morphisms in $\mathcal{G}$ are invertible. We can now prove the following corollary, that generalizes Maschke Theorem from group algebras.

**Corollary 2.1.**  

Let $\mathcal{G}$ be a small groupoid. Then $K[\mathcal{G}]$ is separable if, and only if, $\mathcal{G}(x,y)$ is a finite set and $|\mathcal{G}(x,y)|$ is nonzero in $K$, for all $x,y \in \mathcal{G}_0$.

**Proof.** Let us first assume that $K[\mathcal{G}]$ is separable. Since any separable linear category is locally finite it follows that $\mathcal{G}(x,y)$ is a finite set, for any $x,y \in \mathcal{G}_0$. Let $\{a_{x, y}^{ij}\}_{x,y \in \mathcal{G}_0}$ be a family which satisfies relations (a)-(d) in Theorem 2.1,(5). We fix $x$ and $y$ in $\mathcal{G}_0$. Hence $a_{x, y}^{ij} \in K[\mathcal{G}]\{y, x\} \otimes K[\mathcal{G}]\{x, y\}$. Note that $\{g \otimes h \mid g \in \mathcal{G}(y, x) \text{ and } h \in \mathcal{G}(x, y)\}$ is a basis on $K[\mathcal{G}]\{y, x\} \otimes K[\mathcal{G}]\{x, y\}$. Thus

$$a_{x, y}^{ij} = \sum_{g \in \mathcal{G}(y, x)} \sum_{h \in \mathcal{G}(x, y)} a_{g,h} g \otimes h,$$

where $a_{g,h}$ is a certain element in $K$, for every $g \in \mathcal{G}(y, x)$ and $h \in \mathcal{G}(x, y)$. Taking into account that $\{a_{x, y}^{ij}\}_{x,y \in \mathcal{G}_0}$ satisfies (c), it follows easily that

$$\sum_{g \in \mathcal{G}(y, x)} a_{g,g^{-1}} = 1. \quad (4)$$
On the other hand, since \( \{a_{i,y}^x\}_{x,y \in \mathcal{G}} \) satisfies (d), for every \( f \in \mathcal{G}(x,y) \) we have \( f \triangleright a_{i,y}^x = a_{i,y}^x \triangleleft f \). It follows

\[
\sum_{g \in \mathcal{G}(y,z)} \sum_{h \in \mathcal{G}(z,x)} a_{g,h} (f \circ g) \otimes h = \sum_{g \in \mathcal{G}(y,z)} a_{g,h} g \otimes (h \circ f).
\]

Hence \( a_{x,y}^{-1} \circ u_v \circ a_{x,y}^{-1} \), for any \( u \in \mathcal{G}(y,z) \) and \( v \in \mathcal{G}(z,x) \). We fix \( g_0 \in \mathcal{G}(y,z) \). Thus, by taking \( f := u_0 \circ u^{-1} \) in the above identity, we get

\[
a_{u,v}^{-1} = a_{u_0,0}^{-1} = a_{u_0,0}^{-1} \circ u_0 u^{-1} = a_{u_0,0}^{-1}.
\]

In conclusion the element \( a_{u,v}^{-1} \) does not depend on \( u \in \mathcal{G}(y,z) \). Using (4) one gets \( \mathcal{G}(y,z) \) is nonzero in \( K \). It is easy to check that the elements

\[
a_i^y := \frac{1}{|\mathcal{G}(x,y)|} \sum_{g \in \mathcal{G}(x,y)} g \otimes g^{-1}
\]

define a family which satisfies the properties (a)-(d) in Theorem 2.1.(5), so \( K[\mathcal{G}] \) is separable.

Recall that a category \( \mathcal{A} \) is said to be skeletal if its only isomorphisms are automorphisms. A skeletal category \( \mathcal{A} \) is called a delta category if the only endomorphisms in \( \mathcal{A} \) are the identities, cf. [3, p. 83]. If \( \mathcal{A} \) is a delta category then there is a partial order relation \( \leq \) on \( \mathcal{A}_0 \) such that \( \mathcal{A}(x,y) \neq \emptyset \) if, and only if, \( x \leq y \). Note that any poset (regarded as a category) is a delta category. A category \( \mathcal{A} \) is discrete if, for every homomorphism \( f \in \mathcal{A} \), there is \( x \in \mathcal{A}_0 \) such that \( f = 1_x \). Discrete categories are, of course, examples of posets (with respect to the trivial order relation).

**Corollary 2.2.**

Let \( \mathcal{A} \) be a delta category. Then \( K[\mathcal{A}] \) is separable if, and only if \( \mathcal{A} \) is a discrete category.

**Proof.** Clearly, for a discrete category \( \mathcal{A} \), the \( K \)-liniarization \( K[\mathcal{A}] \) is separable. Indeed, the elements

\[
a_i^y := \begin{cases} 
0, & x \neq y; \\
1_x \otimes 1_y, & x = y;
\end{cases}
\]

define a family \( \{a_i^y\}_{y \in \mathcal{A}_0} \) which satisfies the properties (a)-(d) in Theorem 2.1.(5). Conversely, let us assume that \( K[\mathcal{A}] \) is separable. We have to prove that \( \mathcal{A}(x,z) = \emptyset \), for any \( x < z \). Let \( \{a_i^y\}_{y \in \mathcal{A}_0} \) which satisfies the properties (a)-(d). Since \( \mathcal{A} \) is a delta category it follows that either \( K[\mathcal{A}](x,y) = 0 \) or \( K[\mathcal{A}](y,x) = 0 \), provided that \( x \neq y \). Therefore \( a_i^y = 0 \), for any \( x \) and \( y \) such that \( x \neq y \). Let us take \( f : x \rightarrow z \) in \( K[\mathcal{A}] \), where \( x < z \). Since \( f \triangleright a_i^x = a_i^x \triangleleft f = 0 \triangleleft f \), we deduce that \( f \triangleright a_i^x = 0 \) for any \( x, y \in \mathcal{A}_0 \). Hence

\[
f = f \triangleright 1_x = \sum_{y \in \mathcal{A}_0} \text{comp}(f \triangleright a_i^y) = 0.
\]

It follows that \( K[\mathcal{A}](x,z) = 0 \), for all \( x < z \). Therefore \( \mathcal{A}(x,z) = \emptyset \). \( \Box \)

**Remark 2.1.**

For a different proof of the above corollary see [3, Proposition 33.1].

**Proposition 2.1.**

If \( \mathcal{C} \) is separable then any left \( \mathcal{C} \)-module \( M \) is projective.
Proof. Since \( C \) is separable there is a family \( (a^y_i)_{y \in C_0} \) as in Theorem 2.1.(5). It is sufficient to prove that the canonical morphism of left \( C \)-modules \( \varphi : C \otimes M \to M \) has a section in \( C \)-Mod. Let

\[
\varphi_x : \bigoplus_{y \in C_0} C(y, x) \otimes_y M \to \mathcal{C}(x, M)
\]

be the corresponding component of degree \( x \). We define

\[
\psi^x : \mathcal{C}(x, M) \to C(y, x) \otimes_y M, \quad \psi^x (m) = \sum_{i=1}^{n_{x,y}} f_{y,x}^i \otimes (g_{x,y}^i \triangleright m)
\]

where the elements \( f_{y,x}^i \) and \( g_{x,y}^i \) define \( a^y_i \in C(y, x) \otimes C(x, y) \) as in relation (2). Since the family \( (a^y_i)_{y \in C_0} \) is of finite support, it follows that the family \( (\psi^x(m))_{y \in C_0} \) is also of finite support. Thus it makes sense to define \( \psi_x : \mathcal{C}(x, M) \to \bigoplus_{y \in C_0} (C(y, x) \otimes_y M) \) by \( \psi_x (m) = (\psi^x(m))_{y \in C_0} \). For \( m \in \mathcal{C}(x, M) \) we get

\[
(\varphi_x \circ \psi_x) (m) = \sum_{y \in C_0} \sum_{i=1}^{n_{x,y}} f_{y,x}^i \otimes (g_{x,y}^i \triangleright m) = \sum_{y \in C_0} \sum_{i=1}^{n_{x,y}} (f_{y,x}^i \circ g_{x,y}^i) \triangleright m = \left( \sum_{y \in C_0} \text{comp}(a^y_i) \right) \triangleright m = 1_x \triangleright m = m.
\]

Note that the first equality follows by the definition of the maps \( \varphi_x \) and \( \psi_x \). The second and the last equalities are consequences of the definition of \( C \)-modules, while for the third relation we used the fact that the family \( (a^y_i)_{y \in C_0} \) satisfies property (c) in Theorem 2.1.(5). Summarizing, we have proved that \( \psi := (\psi_x)_{x \in C_0} \) is a section of \( \varphi \). It remains to show that \( \psi \) is a morphism of left \( C \)-modules. Let \( f \in C(z, x) \) and \( m \in \mathcal{C}(x, M) \), where \( x \) and \( z \) are given objects in \( C_0 \). We have

\[
\psi_x (f \triangleright m) = (\psi^x (f \triangleright m))_{y \in C_0} = \left( \sum_{i=1}^{n_{y,z}} f_{y,z}^i \otimes (g_{y,z}^i \triangleright m) \right)_{y \in C_0} = (f \triangleright \psi^y (m))_{y \in C} = f \triangleright (\psi^y (m))_{y \in C_0} = f \triangleright \psi_y (m).
\]

The first and the second identities follow by the definition of \( \psi_x \) and \( \psi^y_x \), respectively. For the third equality one uses property (d) in Theorem 2.1.(5), while the fourth one is obtained from the definition of the action on \( C \otimes M \). Finally, the fifth and the last relations are consequences of the definition of \( \psi^y_x \) and \( \psi_y \), while for the sixth identity one uses the definition of the \( C \)-module structure on \( M := \bigoplus_{y \in C_0} y \mathcal{C}(x, M) \).

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References

[1] Ardizzoni A., Menini C., Ştefan D., Hochschild cohomology and “smoothness” in monoidal categories, J. Pure Appl. Algebra, 2007, 208, 297–330
[2] Mitchell B., Rings with several objects, Adv. Math., 1972, 8, 1–161
[3] Mitchell B., Theory of categories, Academic Press Inc., New York, 1965
[4] McCarthy R., The cyclic homology of an exact category, J. Pure Appl. Algebra, 1994, 93, 251–296
[5] Herscovich E., Solotar A., Hochschild-Mitchell cohomology and Galois extensions, J. Pure Appl. Algebra, 2007, 209, 37–55
[6] Weibel C.A., An introduction to homological algebra, Cambridge University Press, Cambridge, 1995