Dynamics analysis and numerical implementations of a diffusive predator–prey model with herd behaviour and Allee effect

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ABSTRACT
This paper is concerned with a prey–predator system involving herd behaviour and strong Allee effect. First, the local stability analysis of all the possible equilibrium points is investigated. Second, the existence of Hopf bifurcation around the unique positive equilibrium are carried out. Finally, this paper is ended with a numerical simulation to understand the dynamics of the system. As it turns out that the ‘Allee effect constant’ plays a very important role in biological invasion mechanism, such as controlling and preventing biological invasion or invading successfully.

1. Introduction
It is well known that the dynamic relationship between prey and predator population is very essential in both mathematical ecology due to its universal existence and importance. For the predator–prey ecosystems, the introduction of predator facilitates the coexistence of competing prey species (Wallach et al., 2015), which could be characterized as ‘Biological invasions’, and it has also been shown that the effects of predator introduction on the coexistence of prey species can be very complicated (Noonburg & Abrams, 2005; Schmitz, 2019). In fact, the predator may interact with the prey along the outer corridor of the herd of prey (i.e. the so-called herd behaviour Bentout et al., 2022; Braza, 2012; Mezouaghi et al., 2021; Souna et al., 2021), which is more appropriate to model the response functions of prey that exhibit herd behaviour in terms of the square root of the prey population. Based on this moment, Ajraldi et al. (2011) have proposed a predator–prey model given by the following model:

\[
\begin{align*}
\frac{du}{dt} &= u(1-u) - \sqrt{uv}, \quad t > 0, \\
\frac{dv}{dt} &= \gamma v(-\beta + \sqrt{u}), \quad t > 0,
\end{align*}
\]

where \(u(t)\) and \(v(t)\) represent the prey and predator densities, respectively, at time \(t\). \(\gamma\) is the death rate of the predator in the absence of prey, \(\gamma\) is the conversion or consumption rate of prey to predator and the parameters \(\gamma\) and \(\beta\) are assumed to be positive constants. It has been shown that the sustained limit cycles are possible and the solution behaviour near the origin is more subtle and interesting than the classical predator–prey models. Given the above facts, there appears some new works about herd behaviour, such as Yuan et al. (2013), Tang and Song (2015a, 2015b, 2015c), Kooi and Venturino (2016), Tang et al. (2016), and Salman et al. (2016).

Now we consider that the prey population is subjected to the Allee effect. The Allee effect mainly signifies a positive relationship between the size of the population and average fitness of the individuals (Berec et al., 2007; Vishwakarma & Sen, 2021; Wang & Kot, 2001). In fact, the reduction of the per capita growth rate of a population of a biological species at densities smaller than a critical value is known as the Allee effect (Hadjavigousti & Ichtiaroglou, 2008). The main cause of the Allee effect is the difficulty in finding mates between the individuals of a species at low population densities. Other causes may be reduced defense against predators, special trends of social behaviour, etc. In the present work, the model (1) involving the Allee effect given by Wang and Kot (2001) is

\[
\begin{align*}
\frac{du}{dt} &= u(1-u)(u/\alpha - 1) - \sqrt{uv}, \quad t > 0, \\
\frac{dv}{dt} &= \gamma v(-\beta + \sqrt{u}), \quad t > 0,
\end{align*}
\]

where \((u/\alpha - 1)\) is the term for the Allee effect and \(\alpha \in (0, 1)\) can be called the ‘Allee effect constant’. The larger \(\alpha\) is, the stronger the Allee effect will be, and the slower the per capita growth rate of the prey population: by
introducing the Allee effect into the model (1), the per capita growth rate of the prey population is reduced from $u(1-u)$ to $u(1-u)(u/a - 1)$.

The paper is organized as follows. In Sections 2 and 3, we analyse the nonnegative constant equilibrium solutions and their stabilities, respectively, to system (1) and (2). Some numerical simulations and some remarks are shown to illustrate and supplement the analytical conclusions in Section 4.

2. Qualitative analysis of system (1)

It is easy to check that system (1) has two boundary equilibria $E_0 = (0, 0)$ and $E_1 = (1, 0)$, and a unique positive equilibrium $E_2 = (u^*, v^*)$ if and only if $0 < \beta < 1$, where

$$u^* = \beta^2, \quad v^* = \beta(1 - \beta^2).$$

At the equilibrium $E = (u, v)$, the linearized operator of system (1) can be expressed by

$$L = \begin{pmatrix}
1 - 2u - \frac{v}{2\sqrt{u}} & -\sqrt{u} \\
\frac{\gamma v}{2\sqrt{u}} & -\beta \gamma + \lambda \sqrt{u}
\end{pmatrix}.$$  (3)

Letting $E = E_1$ and $E = E_2$, the linearized operators $L$ can be calculated as

$$L_1 = \begin{pmatrix}
-1 & -1 \\
0 & \gamma(1 - \beta)
\end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix}
\frac{1}{2}(1 - 3\beta^2) & -\beta \\
0 & \frac{1}{2}\gamma(1 - \beta^2)
\end{pmatrix},$$  (4)

respectively.

Theorem 2.1:

(1) For the boundary equilibrium $E_1$ : If $\beta > 1$ (0 < $\beta < 1$), then $E_1$ is a stable node (unstable saddle).

(2) For the boundary equilibrium $E_2$ : If $\frac{\sqrt{3}}{3} < \beta < 1$ (0 < $\beta < \frac{\sqrt{3}}{3}$), then $E_2$ is locally asymptotically stable (unstable). Moreover, if $0 < \gamma < \frac{(1-3\beta^2)^2}{8\beta(1-\beta^2)}$, then $E_2$ is a stable (unstable) node point; if $\gamma = \frac{(1-3\beta^2)^2}{8\beta(1-\beta^2)}$, then $E_2$ is a degenerate stable (unstable) node point; if $\gamma > \frac{(1-3\beta^2)^2}{8\beta(1-\beta^2)}$, then $E_2$ is a stable (unstable) spiral point.

Proof: (1) The trace and determinant of the matrix $L_1$ in (4) are

$$T_1 = -1 - \beta \gamma + \gamma \quad \text{and} \quad D_1 = \beta \gamma - \gamma.$$  (5)

If $\beta > 1$, then $D_1 > 0, T_1 < 0$. So $E_1$ is locally asymptotic stable. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of the matrix $L_1$.

(1) If $\beta > 1$, then $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_1 \neq \lambda_2$, which yields that $E_1$ is a stable node;

(2) If $0 < \beta < 1$, then $\lambda_1 < 0$ and $\lambda_2 > 0$, i.e. $E_1$ is an unstable saddle.

(2) The trace and determinant of the matrix $L_2$ in (4) are

$$T_2 = \frac{1}{2}(1 - 3\beta^2) \quad \text{and} \quad D_2 = \frac{1}{2} \beta \gamma(1 - \beta^2),$$

where $0 < \beta < 1$.  (6)

Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of the matrix $L_2$, and

$$\Delta = T_2^2 - 4D_2 = \frac{1}{4}(1 - 3\beta^2)^2 - 2\beta \gamma(1 - \beta^2).$$

- If $\frac{\sqrt{3}}{3} < \beta < 1$, then $D_2 > 0, T_2 < 0$ and then $E_2$ is locally asymptotic stable.

(1) If $\frac{\sqrt{3}}{3} < \beta < 1$ and $0 < \gamma < \frac{(1-3\beta^2)^2}{8\beta(1-\beta^2)}$, then $(\Delta > 0)$ $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_1 \neq \lambda_2$, which implies that $E_2$ is stable node;

(2) If $\frac{\sqrt{3}}{3} < \beta < 1$ and $\gamma = \frac{(1-3\beta^2)^2}{8\beta(1-\beta^2)}$, then $(\Delta = 0)$ $\lambda_1 = \lambda_2 < 0$, i.e. $E_2$ is a degenerate stable node;

(3) If $\frac{\sqrt{3}}{3} < \beta < 1$ and $\gamma > \frac{(1-3\beta^2)^2}{8\beta(1-\beta^2)}$, then $(\Delta < 0) \lambda_{1,2}$ is the form of $aa \pm ibb$, where $aa < 0, bb > 0$. So, $E_2$ is a stable spiral.

- If $0 < \beta < \frac{\sqrt{3}}{3}$, then $T_2 > 0, D_2 > 0$ and then $E_2$ is unstable. Similar to the above statements, we have the instability result about $E_2$:

(1) If $0 < \beta < \frac{\sqrt{3}}{3}$ and $0 < \gamma < \frac{(1-3\beta^2)^2}{8\beta(1-\beta^2)}$, then $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_1 \neq \lambda_2$, which yields that $E_2$ unstable node;

(2) If $0 < \beta < \frac{\sqrt{3}}{3}$ and $\gamma = \frac{(1-3\beta^2)^2}{8\beta(1-\beta^2)}$, then $\lambda_1 = \lambda_2 > 0$, i.e. $E_2$ is a degenerate unstable node;

(3) If $0 < \beta < \frac{\sqrt{3}}{3}$ and $\gamma > \frac{(1-3\beta^2)^2}{8\beta(1-\beta^2)}$, then $\lambda_{1,2}$ is the form of $aa \pm ibb$, where $aa, bb > 0$. So, $E_2$ is an unstable focus.

\[\square\]

Remark 2.1:

(1) Theorem 2.1(1) shows that $E_1 = (1, 0)$ is stable if $\beta > 1$, which implies that if the death rate $\beta$ of the predator is large enough and the initial value conditions $u(0)$ and $v(0)$ of the prey and predator are near to $E_1$, then the predator population will fail to invade, the prey population will survive eventually.

(2) Theorem 2.1(2) shows that $E_2 = (u^*, v^*)$ is stable if $\frac{\sqrt{3}}{3} < \beta < 1$, which implies that if the death rate $\beta$ of the predator is in some range and the initial value conditions $u(0)$ and $v(0)$ of the prey and predator are near to $E_2$, then the predator population will invade successfully, and the prey and predator populations will coexist eventually.
Remark 2.2 (Hopf bifurcation): By (6) and Theorem 2.1, $\beta_H = \sqrt{3}$ is Hopf bifurcation point, and $E_2$ is locally asymptotically stable (unstable) if $\beta_H < \beta < 1$ ($0 < \beta < \beta_H$). In other words, system (1) will undergo a continuous Hopf bifurcation around the unique positive equilibrium $E_2 = (\beta^2, \beta(1 - \beta^2))$.

3. Qualitative analysis of system (2)

It is easy to check that system (2) has two boundary equilibria $E_0 = (0, 0), E_1 = (1, 0)$ and $E_2 = (\alpha, 0)$, and a unique positive equilibrium $E_3 = (u^*, v^*)$ if and only if $\sqrt{\alpha} < \beta < 1$, where

$$u^* = \beta^2, \quad v^* = \beta(1 - \beta^2)\left(\frac{\beta^2}{\alpha} - 1\right).$$

At the equilibrium $E = (u, v)$, the linearized operator of system (1) can be expressed by

$$L = \begin{pmatrix}
-\frac{1}{\alpha} [3u^2 - 2(1 + \alpha)u + \alpha] - \frac{v}{2\sqrt{u}} & -\sqrt{u} \\
\frac{v}{2\sqrt{u}} & -\beta \gamma + \gamma \sqrt{u}
\end{pmatrix}
\tag{7}
$$

Theorem 3.1:

(1) For the boundary equilibrium $E_1$: if $\beta > 1$, then $E_1$ is a stable node. Moreover, if $\gamma = \frac{1-\alpha}{\alpha(1-\beta)}$, then $E_1$ is a degenerate stable node point.

(2) For the boundary equilibrium $E_2$: if $\beta > 1$, then $E_2$ is an unstable node. Moreover, if $\beta > \sqrt{\alpha}$ and $0 < \gamma < \frac{1-\alpha}{\alpha(1-\beta)}$, then $E_1$ is an (degenerate) unstable node.

Proof: Letting $E = E_1$ and $E = E_2$, the linearized operators $L$ can be calculated as

$$L_1 = \begin{pmatrix}
\frac{\alpha - 1}{\alpha} & -1 \\
0 & -\beta \gamma + \gamma
\end{pmatrix}
$$

and

$$L_2 = \begin{pmatrix}
1 - \frac{\alpha}{\alpha} & -\sqrt{\alpha} \\
0 & \gamma(\sqrt{\alpha} - \beta)
\end{pmatrix}
\tag{8}
$$

(1) The trace and determinant of the matrix $L_1$ in (8) are

$$T_1 = -\frac{1-\alpha}{\alpha} - \beta \gamma + \gamma \quad \text{and} \quad D_1 = \frac{1-\alpha}{\alpha}(\beta \gamma - \gamma)
\text{with} \quad \alpha \in (0, 1).
\tag{9}
$$

Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of the matrix $L_1$.

- If $\beta > 1$, then $T_1 < 0$ and $D_1 > 0$. So, $E_1$ is locally asymptotically stable.
  (1) If $0 < \beta < 1$ and $0 < \gamma \neq \frac{1-\alpha}{\alpha(1-\beta)}$, then $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_1 \neq \lambda_2$, which implies that $E_1$ is a stable node point;
  (2) If $\beta > 1$ and $\gamma = \frac{1-\alpha}{\alpha(1-\beta)}$, then $\lambda_1 = \lambda_2 < 0$, which implies that $E_1$ is a degenerate stable node point (Figure 1(a)).

- If $0 < \beta < 1$, then $D_1 < 0$. So, $E_1$ is unstable.
  (1) If $0 < \beta < 1$ and $0 < \gamma \neq \frac{1-\alpha}{\alpha(1-\beta)}$, then $(T_1 > 0)$ $\lambda_1 > 0, \lambda_2 < 0$ and $\lambda_1 > |\lambda_2|$, which yields that $E_1$ is an unstable saddle point;
  (2) If $0 < \beta < 1$ and $\gamma = \frac{1-\alpha}{\alpha(1-\beta)}$, then $(T_1 = 0)$ $\lambda_1 = -\lambda_2 \neq 0$, which yields that $E_1$ is an unstable saddle point;
  (3) If $0 < \beta < 1$ and $\gamma > \frac{1-\alpha}{\alpha(1-\beta)}$, then $(T_1 < 0)$ $\lambda_1 > 0, \lambda_2 < 0$ and $\lambda_1 < |\lambda_2|$, which yields that $E_1$ is an unstable saddle point.

Figure 1. The phase portraits of the temporal local system (2) with fixed $\alpha = 0.5, \gamma = 2$. (a) $\beta = 1.5$; (b) $\beta = 0.9$. 
The trace and determinant of the matrix $L_2$ in (8) are
\[ T_2 = 1 - \alpha + \gamma(\sqrt{\alpha} - \beta) \quad \text{and} \quad D_2 = \gamma(1 - \alpha)(\sqrt{\alpha} - \beta), \quad \alpha \in (0, 1). \quad (10) \]

Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of the matrix $L_2$.
If $\beta > \sqrt{\alpha}$, then $D_2 < 0$; if $\beta < \sqrt{\alpha}$, then $T_2 > 0$. So, $E_2$ is always unstable. Similar to the above statements, we have the instability result about $E_2$.

1. If $\beta > \sqrt{\alpha}$, then $E_2$ is an unstable saddle;
2. If $\beta < \sqrt{\alpha}$ and $0 < \gamma \neq \frac{1 - \alpha}{\sqrt{\alpha} - \beta}$ ($\gamma = \frac{1 - \alpha}{\sqrt{\alpha} - \beta}$), then $E_2$ is an (degenerate) unstable node.

**Theorem 3.2:** Let $\beta_c = \frac{3(1+\alpha) + \sqrt{9\alpha^2 - 2\alpha + 9}}{10}$. Then $\beta_c < 1$ and $E_3$ is locally asymptotically stable (unstable) if $\beta > \beta_c$ ($\sqrt{\alpha} < \beta < \beta_c$).

**Proof:** Note that $E_3 = (u^*, v^*)$ where $u^* = \beta^2$, $v^* = \beta(1 - \beta^2)(\frac{\beta^2}{\alpha} - 1)$ and $\sqrt{\alpha} < \beta < 1$. Letting $E = E_3$, the linearized operators $L$ can be calculated as
\[ L_3 = \begin{pmatrix} -1 + \frac{1}{\alpha}[3\beta^4 - 2(1 + \alpha)\beta^2 + \alpha] - \frac{1}{2}(1 - \beta^2)(\frac{\beta^2}{\alpha} - 1) & -\beta \\ \frac{\gamma}{2}(1 - \beta^2)(\frac{\beta^2}{\alpha} - 1) & 0 \end{pmatrix}, \quad (11) \]
which could be simplified to obtain that
\[ L_3 = \begin{pmatrix} -1 + \frac{1}{\alpha}[3\beta^4 - 2(1 + \alpha)\beta^2 + \alpha] - \frac{1}{2}(1 - \beta^2)(\frac{\beta^2}{\alpha} - 1) & -\beta \\ \frac{\gamma}{2}(1 - \beta^2)(\frac{\beta^2}{\alpha} - 1) & 0 \end{pmatrix}. \quad (12) \]

**Figure 2.** The phase portraits of the temporal local system (2) with fixed $\alpha = 0.5$, $\gamma = 2$ and $\beta = 0.877 < \beta_H = 0.8776$.

**Figure 4.** Asymptotic stability of $E_2$ to system (2), where $\beta_1 = \sqrt{\alpha}$ and $\beta_2 = \frac{3(1+\alpha) + \sqrt{9\alpha^2 - 2\alpha + 9}}{10}$ ($< 1$).
Let $T_3$ and $D_3$ be the trace and determinant of the matrix $L_3$ in (12), where

\[
T_3 = \frac{1}{2a} [5\beta^4 - 3(1 + \alpha)\beta^2 + \alpha] \quad \text{and} \\
D_3 = \frac{\beta \gamma}{2} (1 - \beta^2) \left( \frac{\beta^2}{\alpha} - 1 \right). \tag{13}
\]

Since $\sqrt{\alpha} < \beta < 1$, we have $D_3 > 0$. Let

\[
h(z) = 5z^2 - 3(1 + \alpha)z + \alpha, \quad \text{where} \quad \alpha \leq z \leq 1.
\]

Then $h(\alpha) < 0$ and $h(1) < 0$. Let \( \Delta = 9(1 + \alpha)^2 - 20\alpha = 9\alpha^2 - 2\alpha + 9 > 0 \) and $z_0 = \frac{3(1 + \alpha) + \sqrt{\Delta}}{10} (< 1)$.

**Figure 5.** Numerical simulation of solutions for the temporal local system (2) with fixed $\gamma = 2, \beta = 0.877 < \beta_H = 0.8776$ and variable ‘Allee effect constant’ $\alpha$. 
If $\alpha < z < z_0$, then $h(z) = 5z^2 - 3(1 + \alpha)z + \alpha < 0$. So we obtain that if $\beta > \sqrt{z_0}$ ($\sqrt{\alpha} < \beta < \sqrt{z_0}$), then $T_3 < 0$ ($T_3 > 0$) and then $E_3$ is stable (unstable) (Figure 1(b)).

Remark 3.1:

1) Theorem 3.1(1) shows that $E_1 = (1, 0)$ is stable if $\beta > 1$, which implies that if the death rate $\beta$ of the predator and the initial value conditions $u(0)$ and $v(0)$ of the prey and predator are near to $E_1$, then the predator population will fail to invade, the prey population will survive eventually.

2) Theorem 3.1(2) shows that $E_2 = (\alpha, 0)$ is unstable, which implies if the initial value conditions $u(0)$ and $v(0)$ of the prey and predator are near to $E_2$, then the predator population maybe invade successfully.

3) Theorem 3.2 shows that $E_3 = (u^*, v^*)$ is stable if $\beta > \frac{3(1 + \alpha) + \sqrt{9\alpha^2 - 2\alpha + 9}}{10}$, which implies if the death rate $\beta$ of the predator is sufficiently large and the initial value conditions $u(0)$ and $v(0)$ of the prey and predator are near to $E_3$, then the predator population will invade successfully.

![Figure 6. The phase portraits of the temporal local system (1) with fixed $\gamma = 2$.](image)

(a) $\beta = 1.5$; (b) $\beta = 0.6$.

![Figure 7. Asymptotic stability of $E_2$ to system (1) as $\beta$ vary.](image)
Remark 3.2 (Hopf bifurcation): By (13) and Theorem 3.2,

\[ \beta_H = \frac{3(1 + \alpha) + \sqrt{9\alpha^2 - 2\alpha + 9}}{10} \quad (\ll 1) \]

is Hopf bifurcation point, and \( E_3 \) is locally asymptotically stable (unstable) if \( \beta > \beta_H \left( \sqrt{3} < \beta < \beta_H \right) \). In other words, system (2) will undergo a continuous Hopf bifurcation around the unique positive equilibrium \( E_3 = (\beta^2, \beta(1 - \beta^2)(\frac{\beta^2}{\alpha} - 1)) \) (Figures 2, 3 and 4).

4. Numerical simulations and some remarks

In this section, we have presented some simulation results of our proposed model by taking some hypothetical values of the parameters.

4.1. Numerical simulation results about system (1)

For the system (1), letting \( \gamma = 2 \), we consider the dynamic behaviour of system (1). By Section 2, we know that (1) If \( \beta > 1 \), then \( E_1 = (1, 0) \) is linearly stable (Figure 5); (2) If \( \sqrt{3}/3 < \beta < 1 \), then \( E_2 = (\beta^2, \beta(1 - \beta^2)) \) exists and it is stable (Figure 6(b)); (3) For the equilibrium point \( E_2, \beta = \)

![Figure 8. The phase portraits of the temporal local system (1) with fixed \( \gamma = 2 \) and \( \beta = 0.577 < \beta_H = \frac{\sqrt{3}}{3} \approx 0.5774 \).](image)

![Figure 9. Periodic solutions and associated limit cycle for the temporal local system (1) with fixed \( \gamma = 2 \) and \( \beta = 0.577 < \beta_H = \frac{\sqrt{3}}{3} \approx 0.5774 \).](image)
$\beta_H = \sqrt{3}/3$ is a Hopf bifurcation point (Figure 7), i.e. there will exist a periodic solution (limit cycle) surrounding $E_2$ (Figures 8 and 9).

### 4.2. Numerical simulation results about system (2)

For the system (2), letting $\alpha = 2$ and $\gamma = 2$, we consider the dynamic behaviour of system (2). By Section 3, we
know that (1) If $\beta > 1$, then $E_1 = (1,0)$ is linearly stable (Figure 1(a)); (2) If $\frac{3(1+\alpha)+\sqrt{9\alpha^2-2\alpha+9}}{10} < \beta < 1$, then $E_3 = (\beta^2, \beta(1-\beta^2)(\frac{\beta^2}{2\alpha}-1))$ exists and it is stable (Figure 1(b)); (3) For the equilibrium point $E_3$, $\beta = \beta_H = \frac{3(1+\alpha)+\sqrt{9\alpha^2-2\alpha+9}}{10}$ is a Hopf bifurcation point (Figure 4), i.e., there will exist a periodic solution (limit cycle) surrounding $E_3$ (Figures 2 and 3).

Now, we consider the influence of the impact of the Allee effect on the predator interaction using numerical simulation. Note that system (2) has a unique positive equilibrium $E_3 = (u^*, v^*)$ if and only if $\sqrt{\alpha} < \beta < 1$, where

$$u^* = \beta^2, \quad v^* = \beta(1-\beta^2)\left(\frac{\beta^2}{\alpha} - 1\right).$$

Let $\beta = 0.877$. First, we consider the impact on the visibility of the equilibria $(u^*, v^*)$. Obviously, $u^* = \beta^2 = 0.7691$, and the graph of $v^*$ as a function of $\alpha$ can be characterized by Figure 10. It is obvious that $v^* \to +\infty$ as $\alpha \to 0^+$ and $v^* \to 0$ as $\alpha \to 0^+$ by Figure 10.

Second, we give the influence of the impact of the Allee effect on the unique positive equilibrium $E_3 = (u^*, v^*)$ in Figure 5, from which we conclude that by increasing the control parameter ‘Allee effect constant’ $\alpha$ of the predator (0.15 $\to$ 0.45 $\to$ 0.5 $\to$ 0.55 $\to$ 0.75), the solution sequence ‘stable homogeneous stationary solution’ $E_3$ (cf. Figure 5(a–d)) $\to$ ‘periodic solution around $E_3$ (cf. Figure 5(e,f))’ $\to$ ‘stable homogeneous stationary solution (0,0) (cf. Figure 5(g–h))’ reveals. In other words, there will exist a critical value $\alpha_c$ such that the predator population will invade successfully if $\alpha < \alpha_c$ and the prey and predator populations will coexist or coexist periodically eventually.

4.3. Some remarks about comparison of dynamic behaviour between (1) and (2)

Note that biological invasions are one of the most important ecological disturbances that threaten native biodiversity (Petrovskii et al., 2005). Here, we understand the population $u$ and $v$ as the native species and the invasive species, and focus on the comparison between the dynamic behaviour between (1) and (2), due to the Allee effect constant $\alpha$.

(1) Allee effect on the boundary equilibrium $(1,0)$: we know that if $0 < \beta < 1$, then $(1,0)$ is unstable for system (1) and (2) simultaneously, which implies that the introduction of ‘Allee effect’ cannot prevent biological invasion.

(2) For the boundary equilibrium $(\alpha,0)$, by Theorem 3.1, it is always unstable if $0 < \alpha < 1$. Now, we recall the
linearized operators $\mathcal{I}_2$ in (8):

$$
\mathcal{I}_2 = \begin{pmatrix}
1 - \alpha & -\sqrt{\alpha} \\
0 & \gamma(\sqrt{\alpha} - \beta)
\end{pmatrix}.
$$

In fact, besides $\alpha \in (0, 1)$, we could also take $\alpha \in (1, \infty)$. Similar to the qualitative analysis in Theorem 3.1, the new boundary equilibrium state $(\alpha, 0)$, which is relative to system (1), is linearly stable if $\beta > \sqrt{\alpha}$ (Figure 11), which implies that the introduction of ‘Allee effect’ cannot prevent biological invasion. At this moment, it could be argued that, by formulating the relevant effective countermeasures and making the ‘Allee effect constant’ greater than 1, we could control and prevent biological invasion.

(3) For the unique positive equilibrium, by Figures 12 and 13, we find that the introduction of ‘Allee effect’ will make the period of corresponding periodic oscillations bigger.

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**Data availability statement**

Some or all data, models or code generated or used during the study are available from the corresponding author by request. (W. Yang).

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