Which $n$-Venn diagrams can be drawn with convex $k$-gons?

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Abstract

We establish a new lower bound for the number of sides required for the component curves of simple Venn diagrams made from polygons. Specifically, for any $n$-Venn diagram of convex $k$-gons, we prove that $k \geq (2^n - 2 - n)/(n(n-2))$. In the process we prove that Venn diagrams of seven curves, simple or not, cannot be formed from triangles. We then give an example achieving the new lower bound of a (simple, symmetric) Venn diagram of seven quadrilaterals. Previously Grünbaum had constructed a 7-Venn diagram of non-convex 5-gons [‘‘Venn Diagrams II‘‘, Geombinatorics 2:25-31, 1992].

1 Introduction and Background

Venn diagrams and their close relatives, the Euler diagrams, form an important class of combinatorial objects which are used in set theory, logic, and many applied areas. Convex polygons are fundamental geometric objects that have been investigated since antiquity. This paper addresses the question of which convex polygons can be used to create Venn diagrams of certain numbers of curves. This question has been studied over several decades, for example [1, 2, 5, 6, 7]. See the on-line survey [8] for more information on geometric aspects of Venn diagrams.

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ be a family of $n$ simple closed curves in the plane. The curves are required to be finitely intersecting. We say that $\mathcal{C}$ is a Venn diagram (or $n$-Venn diagram) if all of the $2^n$ open regions $X_1 \cap X_2 \cap \cdots \cap X_n$ are non-empty and connected, where each set $X_i$ is either the bounded interior or the unbounded exterior of the curve $C_i$. If the connectedness condition is dropped the diagram is called an independent family. We can also think of the diagram as a plane edge-coloured graph whose vertices correspond to intersections of curves, and whose edges correspond to the

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segments of curves between intersections. Edges are coloured according to the curve to which they belong. We can overload the term and also refer to this graph as the Venn diagram. A Venn diagram or independent family is simple if at most two curves intersect at a common point, i.e. every vertex has degree exactly four.

Let the term \( k \)-gon designate a convex polygon with exactly \( k \) sides. Observe that two \( k \)-gons can (finitely) intersect with each other in at most \( 2k \) points. In this paper, we consider Venn diagrams and independent families composed of \( k \)-gons, for some \( k \). Note that the corners of the component \( k \)-gons are not vertices in the graph interpretation of the diagram (unless they intersect another curve at that point), and an edge, using the graph interpretation, may contain zero or more corners of the \( k \)-gon containing that edge. A side of a \( k \)-gon is the line segment connecting two of its adjacent corners; sides are not to be confused with edges.

It is clear that if an \( n \)-Venn diagram can be drawn with \( k \)-gons, it can also be drawn with \( j \)-gons for any \( j > k \), by adding small sides and making small perturbations where necessary.

Gr"unbaum first considered the problem of what polygons can be used to create Venn diagrams in [5], in which he gave a Venn diagram of six quadrilaterals, and a diagram of five triangles. He also provided an independent family of five squares, and in [7] conjectured that there is no symmetric Venn diagram with five squares.

We restate two lemmas first observed by Gr"unbaum [5], some of the consequences of which inspired this work. A FISC is a family of Finitely Intersecting Simple closed Curves in the plane, with the property that the intersection of the interiors of all the curves is not empty. Clearly, every Venn diagram is a FISC.

**Lemma 1.1.** In a FISC of \( n \) convex \( k \)-gons there are at most \( \binom{n}{2} 2k \) vertices.

**Proof.** A pair of convex \( k \)-gons can intersect with each other at most \( 2k \) times; there are \( \binom{n}{2} \) pairs. \( \square \)

**Lemma 1.2.** In a simple \( n \)-Venn diagram of \( k \)-gons,

\[
k \geq \left\lceil \frac{(2^{n-1} - 1)}{\binom{n}{2}} \right\rceil.
\]

**Proof.** Euler’s formula for plane graphs, combined with the fact that in a simple diagram all vertices have degree four, implies that the number of vertices in a simple Venn diagram is \( 2^n - 2 \). Combining this with Lemma 1.1, which gives an upper bound on the number of vertices, the inequality follows. \( \square \)

Lemma 1.2 gives us a bound, for each \( n \), of the minimum \( k \) required to form a simple \( n \)-Venn diagram of \( k \)-gons. Diagrams are well-known that achieve the bounds for \( n \leq 5 \); see [3] for examples. For \( n = 6 \), the Lemma implies \( k \geq 3 \), and Carroll [3] achieved the
Figure 1: A Venn diagram of six triangles.
lower bound by giving examples of 6-Venn diagrams formed of triangles; his diagrams are all simple. Figure 1 shows one of Carroll’s Venn diagrams of six triangles.

For \( n = 7 \), Lemma 1.2 implies that \( k \geq 3 \); however until now the diagram with the smallest known \( k \) was a 7-Venn diagram of 5-gons by Grünbaum in [7].

The contributions of this paper are, first, to prove a tighter lower bound than Lemma 1.2 for the minimum \( k \) required to draw a simple Venn diagram of \( k \)-gons; second, to show that no 7-Venn diagram of triangles (simple or not) can exist, and third, to achieve the new lower bound for \( n = 7 \) by exhibiting a Venn diagram of seven quadrilaterals.

In [5], Theorem 3 contains bounds on \( k^*(n) \), which is defined as the minimal \( k \) such that there exists a Venn diagram of \( n \) \( k \)-gons. Carroll’s results prove that \( k^*(6) = 3 \), and our results prove that \( k^*(7) = 4 \), and provide a lower bound on \( k^*(n) \) for \( n > 7 \) when considering simple diagrams.

## 2 Venn Diagrams of \( k \)-Gons

We now prove a tighter lower bound than that given by Lemma 1.2 for simple Venn diagrams. Recall that a side of a \( k \)-gon is the line connecting two of its adjacent corners, whereas an edge is the contiguous boundary of a \( k \)-gon between two adjacent vertices. Vertices are points of intersection of two \( k \)-gons, as opposed to corners, where two sides of a given \( k \)-gon meet. An edge may contain zero or more corners of the \( k \)-gon containing it.

**Observation 2.1.** In a Venn diagram, each curve has exactly one edge on the outer face.

**Proof.** This is a special case of Lemma 4.6 from [4], which states that no two edges in a face in a Venn diagram belong the same curves. \( \square \)

We now introduce some notation before proving the main theorems of this section. In a Venn diagram of \( k \)-gons, consider any two \( k \)-gons \( C_i \) and \( C_j \), \( 1 \leq i < j \leq n \). The corners of \( C_i \) may be labelled according to whether they are external (\( E \)) to \( C_j \) or internal (\( I \)) to \( C_j \) (we can assume that curves do not intersect at corners; if so, we can incrementally perturb one of the relevant curves to eliminate this situation). In a clockwise walk around \( C_i \) we obtain a circular sequence of \( k \) occurrences of \( E \) or \( I \). Let \( EI_{ij} \) denote the number of occurrences of an \( E \) label followed by an \( I \) label; the notations \( II_{ij} \) and \( IE_{ij} \) are defined in an analogous manner. We distinguish two cases when an \( E \) follows an \( E \): either \( C_i \) is intersected twice on the side between the two \( E \) corners, or it is not intersected. The notation \( EE_{ij} \) is for the case where no intersection with \( C_j \) occurs and \( EE'_{ij} \) for the case where two intersections occur. By convexity, \( C_i \) can only be intersected at most twice in a side by \( C_j \). Since these cases cover all types of corners,

\[
EI_{ij} + IE_{ij} + II_{ij} + EE_{ij} + EE'_{ij} = k.
\]
Also note that $EI_{ij} = IE_{ij}$ since there must be an even number of crossings between the curves.

**Theorem 2.2.** In a Venn diagram of $k$-gons,

$$V \leq 2k \binom{n}{2} - n(k - 1).$$

**Proof.**

Given the notation above, consider the entire collection of curves. Label corners on the outer face $\epsilon$ and the others $\iota$. Define $E_i$ to be the number of corners of $C_i$ labelled $\epsilon$ and $I_i$ to be the number labelled $\iota$. Clearly $I_i + E_i = k$.

In a Venn diagram each of the $n$ $k$-gons has one outer edge, by Observation 2.1, and so all corners of $C_i$ labelled $\epsilon$ must appear contiguously; thus

$$\sum_{i \neq j} EE_{ij} \geq \sum_i (E_i - 1)$$

since the left-hand term will also count corners external to some curve but internal to others.

Since any corner labelled $\iota$ is internal to some curve,

$$\sum_{i \neq j} (II_{ij} + IE_{ij}) \geq \sum_i I_i$$

since the left-hand term will double count any corner on $C_i$ internal to more than 1 curve.

Since each $EI$ and $IE$ accounts for one intersection and $EE'$ for two intersections,

$$2V = \sum_{i \neq j} (EI_{ij} + IE_{ij} + 2EE'_{ij})$$

$$= \sum_{i \neq j} (2k - 2II_{ij} - 2EE_{ij} - EI_{ij} - IE_{ij})$$

$$= 4k \binom{n}{2} - \sum_{i \neq j} (2II_{ij} + 2EE_{ij} + 2EI_{ij})$$

$$\leq 4k \binom{n}{2} - 2 \sum_i (E_i - 1) - 2 \sum_i I_i$$

by [2] and [4].

$$= 4k \binom{n}{2} - 2 \sum_i (E_i - 1 + I_i)$$

$$= 4k \binom{n}{2} - 2 \sum_i (k - 1)$$

$$\leq 4k \binom{n}{2} - 2n(k - 1).$$

Dividing by 2 gives $V \leq 2k\binom{n}{2} - n(k - 1)$, as desired.

**Theorem 2.3.** In any simple $n$-Venn diagram of $k$-gons,

$$k \geq \left\lceil \frac{2^n - 2 - n}{n(n - 2)} \right\rceil.$$

**Proof.** For simple Venn diagrams, we have that the number of vertices is $2^n - 2$. Combined with Theorem 2.2, we have

$$2^n - 2 \leq 2k\binom{n}{2} - n(k - 1) = n(n - 1)k - nk + n.$$

Thus

$$2^n - 2 - n \leq k(n(n - 1) - n)$$

or

$$\left\lceil \frac{2^n - 2 - n}{n(n - 2)} \right\rceil \leq k,$$

as desired.

Theorem 2.3 gives a lower bound of the minimum $k$ required to construct a simple $n$-Venn diagram of $k$-gons. Table 1 shows the bound for small values of $n$.

For an upper bound on $k$, note that there are many general constructions for Venn diagrams that produce diagrams of $k$-gons where the value $k$ is a function of $n$ (for examples, see [8] or [5]). In Grünbaum’s convex construction in [6], the $n$th curve is a convex $2^{n-2}$-gon; this gives the upper bounds in Table 1 for $n > 7$. Including this paper’s contributions, diagrams are known for $n \leq 7$, thus solving these cases.

| $n$ | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|
| $k \geq$ | 1  | 2  | 2  | 3  | 4  | 4  | 6  | 8  | 13 | 21 | 35 | 58 |
| $k \leq$ | 1  | 2  | 2  | 3  | 4  | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |

Table 1: Minimum $k$ required to construct a simple $n$-Venn diagram of $k$-gons.

### 3 7-Venn Diagrams of Triangles

In this section we prove that there is no 7-Venn diagram, simple or not, composed of triangles. The bound in Theorem 2.3 for $n = 7$, gives $k \geq 4$, which proves the simple case.
In a non-simple diagram, there must exist at least one vertex where at least three curves intersect. This vertex can be reduced in degree by the operation of incrementally translating one of the intersecting curves in a direction orthogonal to itself; this operation will create a new face. We call this operation *splitting* the vertex. See Figure 2 for an example; first the heavy edge is translated, and then the dashed edge, reducing a vertex of degree eight to six degree-four vertices, and three new faces are created.

Note that splitting a vertex can never remove a face and must add at least one face to the resulting diagram. Thus, after splitting any large degree vertex, the resulting diagram will no longer be a Venn diagram as some face must be duplicated, but it will still be an independent family.

Any large-degree vertex can be reduced to a set of vertices of degree at most four by repeated splitting, as in the example above. We use this operation to prove the following:

**Lemma 3.1.** There is no non-simple Venn diagram of seven triangles.

**Proof.** Assume such a diagram exists; call it $D_0$. Since $D_0$ is non-simple, some vertices have degree greater than four. Let $D_1$ be the simple independent family formed by splitting all of the high-degree vertices in $D_0$. It is clear that this can be performed while still retaining the fact that $D_1$ is composed of triangles, by incrementally perturbing the corners of the component triangles. Let $F_i$, $E_i$, and $V_i$ be the number of faces, edges, and vertices in $D_i$, for $i \in \{0, 1\}$.

Since $D_0$ is a Venn diagram, $F_0 = 128$, and $F_1 > F_0$ since some new faces must have been created by splitting vertices to form $D_1$. Since $D_1$ has all degree-four vertices, summing the vertex degrees gives us $E_1 = 2V_1$. Using Euler’s formula, $V_1 + F_1 - E_1 = 2$, and substituting for $E_1$ gives $V_1 = F_1 - 2 > F_0 - 2 = 126$, and so $V_1 > 126$.

However, $D_1$ is composed of triangles, two of which can only intersect at most six times, and thus $V_1 \leq 6 \binom{7}{2} = 126$, which provides a contradiction. \( \square \)
**Theorem 3.2.** There is no Venn diagram of seven triangles.

**Proof.** By Theorem 2.3 and Lemma 3.1.

4 7-Venn Diagrams of Quadrilaterals

The proof of the previous section raises the question of how close we can get: is there a Venn diagram constructed of seven 4-gons? In this section we answer this question in the affirmative with a simple diagram; this shows that the bound in Theorem 2.3 is tight for $n \leq 7$.

Figure 3 shows a simple 7-Venn diagram of quadrilaterals. The diagram is also symmetric: it possesses a rotational symmetry about a centre point, and the seven quadrilaterals are congruent as each maps onto the next by a rotation of $2\pi/7$. The figure is in fact isomorphic as a plane graph (i.e., can be transformed by continuous deformation in the plane) to the 7-Venn diagram “Victoria” discovered by Frank Ruskey [9, “Symmetric Venn Diagrams”].

Table 2 gives the coordinates for the four corners of one of the component quadrilaterals: the other six quadrilaterals may be constructed by rotating the given coordinates around the origin by an angle of $2\pi i/n$, for $1 \leq i \leq 6$.

| $(x, y)$       |
|---------------|
| $(-0.446, \ 0.000)$ |
| $(-0.123, -0.433)$ |
| $(\ 0.699, \ 0.061)$ |
| $(-0.081, \ 0.451)$ |

Table 2: Coordinates of corners of a quadrilateral in Figure 3

This diagram was discovered using a simple software tool to manipulate polygons in the plane and compute intersections between them.

5 Open Problems

It is not known whether the bound in Theorem 2.3 is tight for $n > 7$. Note that the non-simple result in Lemma 3.1 works because of the fortuitous fact that $2^n - 2 = \binom{n}{2} 2k$ for $n = 7$ and $k = 3$, which is not true for $n \geq 8$. Thus this technique will not work for establishing a non-simple lower bound for $n \geq 8$. A nice open problem is thus to find a tight lower bound on $k$ for the existence of simple and non-simple $n$-Venn diagrams of $k$-gons in general. It appears difficult to generalize Theorem 2.3 to the non-simple case; nevertheless we offer:
Figure 3: A (symmetric, simple) Venn diagram of seven quadrilaterals.
Conjecture 5.1. The bound in Theorem 2.3 also holds for non-simple diagrams.

Not all 7-Venn diagrams can be drawn with quadrilaterals; for example, in the diagram M4 from [9, “Symmetric Venn Diagrams”], each curve has another curve intersect with it 10 times, implying that at least 5-gons are required to draw the figure with $k$-gons. Thus, what is the maximum over all $n$-Venn diagrams of the minimum $k$ required to draw each diagram as a collection of $k$-gons?

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