On the correlation energies for two interacting electrons in a parabolic quantum dot

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Abstract

The correlation energies for two interacting electrons in a parabolic quantum dot are studied via a pseudo-perturbation recipe. It is shown that the central spike term, \((m^2 - 1/4)/r^2\), plays a distinctive role in determining the spectral properties of the above problem. The study is carried out for a wide range of the Coulomb coupling strength \(\lambda\) relative to the confinement.
Advances in semiconductor technology have made it possible to fabricate ultrasmall structures that confine electrons on a scale comparable to their de Broglie wavelength. Structures that restrict the motion of electrons in all directions are called quantum dots (QDs). The simplest QD consists of an insulator, e.g. AlGaAs, and a semiconductor, e.g. GaAs. Between is a potential difference that confines injected electrons to a thin layer at the interface, in which electronic motion in the direction perpendicular to the sandwich is essentially frozen out. Effectively, QDs are considered two-dimensional systems demonstrating typical quantum effects such as discrete energy levels and interference. As a realistic and computationally convenient approximation, a harmonic shape of the laterally confined potential (i.e., a two-dimensional oscillator effective in the plane of the dot) is often used [1].

Several experimental [2-6] and theoretical [7-23] methods were invested for the study of spectroscopic structure of interacting electrons in a harmonic QD. For example, electron correlation has been investigated by the many-particle Schrödinger equation [20], by perturbation [21], shifted 1/N expansion [17], and by WKB treatments and exact numerical solutions [16]. QD helium ground state in a magnetic field is obtained by a Hartree, Hartree-Fock, and exact treatments [22]. Magnetic field dependence of electron energies in QD is studied by the decoupled approximation [23], screening of an ionic potential in QD [24], · · · etc.

In this paper we consider the simplest nontrivial problem of two interacting electrons, with effective mass \( m^* \), in a harmonic quantum dot. The Hamiltonian of which is known to decouple into an exactly solvable center-of-mass Hamiltonian

\[
H_R = \frac{P_R^2}{2M} + \frac{1}{2} M \omega_o^2 R^2, \tag{1}
\]

and a non-exactly (or, at best, conditionally-exactly) solvable Hamiltonian

\[
H_r = \frac{P_r^2}{2\mu} + \frac{1}{2} \mu \omega_o^2 r^2 + \frac{e^2}{\epsilon r}. \tag{2}
\]

Where, \( \omega_o \) is the characteristic frequency of the parabolic confinement, \( \vec{R} = (\vec{r}_1 + \vec{r}_2)/2 \), \( \vec{P}_R = \vec{p}_1 + \vec{p}_2 \), \( M = 2m^* \), \( \vec{r} = \vec{r}_1 - \vec{r}_2 \), \( \vec{P}_r = (\vec{p}_1 - \vec{p}_2)/2 \), and \( \mu = m^*/2 \). Moreover, when a magnetic field \( \vec{B} \) is applied perpendicular to the plane of the dot one would simply amend the above Hamiltonians and replace \( \omega_o \) by the effective frequency \( \tilde{\omega} = \sqrt{\omega_o^2 + \omega_c^2}/4 \), where \( \omega_c = eB/m^*c \) is the cyclotron frequency, and the spin energy term \( E_s = g^* \mu_B BS_z \) (with \( S_z = [1 - (-1)^m]/2 \), \( g^* \) is the Landé factor, and \( \mu_B \) is the Bohr magneton) could be added to the total energy of the dot. However, upon the substitution \( r = \sqrt{2l_o q} \), Schrödinger equation for the relative motion Hamiltonian (2) eventually reads

\[
\left[ -\frac{d^2}{dq^2} + \frac{m^2 - 1/4}{q^2} + q^2 + \frac{\lambda}{q} \right] \Psi_{k,m}(q) = \Xi_{k,m} \Psi_{k,m}(q). \tag{3}
\]

Where \( \Xi_{k,m} = 2E_{k,m}/(h\omega_o) \), \( E_{k,m} \) is the eigenvalue of \( H_r \) in (2), \( \lambda = \sqrt{2l_o}/a^* \), \( a^* = \)
\(\hbar^2 \epsilon/(m^*e^2)\) is the effective Bohr radius, and \(l_o = \sqrt{\hbar/(m^*\omega_o)}\) is the characteristic length of the harmonic confinement.

It is well known, on the other hand, that results from perturbation theory are limited to the case where \(\lambda \ll 1\) \([10,21]\). Moreover, the shifted 1/N expansion technique (SLNT) \([17]\) and WKB treatments \([16]\) lead to dubious accuracies in connection with level ordering and energy crossings \([16,25]\). However, results from exactly solvable potentials can be used in perturbation and pseudo-perturbation theories, or they can be combined with numerical calculations. Nevertheless, in the simplest case, analytical and semianalytical calculations can aid numerical studies in areas where numerical techniques might not be safely controlled \([26-31]\).

In this work we shall use a pseudo-perturbation theory (PSLET) to study the spectroscopic structure of the relative motion Hamiltonian of two interacting electrons in a parabolic quantum dot, eq.(3). Where, in the numerous methodical predecessors of a subset of papers \([28-31]\), an alternative possibility has been sought in the power-law asymptotic expansions using some other small parameter. It has been noticed that the presence of the central spike, e.g. \((m^2 - 1/4)/q^2\) in (3), just copies the effect of the centrifugal and/or centripetal force and immediately inspires the use of small shifted inverse angular momentum quantum number. An exhaustive description of the necessary formulae of PSLET accompanied by the persuasive verifications of their numerical usefulness (by immediate comparisons of its results with available brute force numerical data) could be found in ref.s \([28-31]\).

PSLET recipe starts with the augmentation of \(m^2 - 1/4\), of the central spike in (3), by \(l_D(l_D + 1)\). Where \(l_D = l + (D - 3)/2\), \(D\) is the dimensionality, \(l\) is angular momentum when \(D = 3\) and \(l = |m|\) when \(D = 2\) (the dimensionality under consideration). Then, we simply use \(1/l\) as a pseudo-perturbation parameter, with \(\bar{l} = l_D - \beta\) and \(\beta\) is a vital shift as it removes the poles that would emerge at \(l = 0\) for \(D = 3\). Equation (3), with \(V(q) = (q^2 + \lambda/q)/2\), therefore reads

\[
\left\{ -\frac{1}{2} \frac{d^2}{dq^2} + \frac{\bar{l}^2}{q} + \frac{2\beta + 1}{2q^2} + \frac{\bar{l}^2}{Q} V(q) \right\} \Psi_{k,l}(q) = \bar{E}_{k,l} \Psi_{k,l}(q),
\]

where \(\bar{E}_{k,l} = \Xi_{k,l}/2\) and \(Q\) is a constant that scales the potential \(V(q)\) at large \(-l_D\) limit and is set equal to \(\bar{l}^2\) at the end of the calculations, for any specific choice of \(l_D\) and nodal zeros \(k\). Next, we shift the origin of the coordinate system through \(x = \bar{l}^{1/2}(q - \bar{q}_o)/\bar{q}_o\), where \(\bar{q}_o\) is currently an arbitrary point to be determined below. Expansions about this point, \(x = 0\) (i.e. \(q = \bar{q}_o\)) would lead to

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\bar{q}_o^2}{l} \bar{V}(x(q)) \right] \Psi_{k,l}(x) = \frac{\bar{q}_o^2}{l} \bar{E}_{k,l} \Psi_{k,l}(x),
\]

with

\[
\bar{q}_o^2 \bar{V}(x(q)) = \bar{q}_o^2 \left[ \frac{1}{2\bar{q}_o^2} + \frac{V(q_o)}{Q} \right] + \bar{l}^{1/2} B_1 x + \sum_{n=0}^\infty c^{(n)}(x) \bar{l}^{-n/2},
\]
where

\[ v^{(0)}(x) = B_2 x^2 + \frac{2\beta + 1}{2}, \quad (7) \]

\[ v^{(n)}(x) = B_{n+2} x^{n+2} + (-1)^n (2\beta + 1) \frac{(n + 1)}{2} x^n + (-1)^n \frac{\beta(\beta + 1)}{2} (n - 1) x^{(n-2)} \quad n \geq 1, \quad (8) \]

\[ B_n = (-1)^n \frac{(n + 1)}{2} + \left( \frac{d^n V(q_o)}{dq_o^n} \right) \frac{q_o^{n+2}}{n! Q}. \quad (9) \]

It is then convenient to expand \( \bar{E}_{k,l} \) as

\[ \bar{E}_{k,l} = \sum_{n=-2}^{\infty} E_{k,l}^{(n)} \bar{l}^{-n}. \quad (10) \]

Equation (5), along with (6-9), is evidently the one-dimensional Schrödinger equation for a harmonic oscillator \( \Omega^2 x^2/2 \), with \( \Omega^2 = 2B_2 \), and the remaining terms in Eq.(5) are considered as an infinite power series perturbation to the harmonic oscillator. One would then imply that

\[ E_{k,l}^{(-2)} = \frac{1}{2q_o^2} + \frac{V(q_o)}{Q}, \quad (11) \]

\[ E_{k,l}^{(-1)} = \frac{1}{q_o^2} \left[ \frac{2\beta + 1}{2} + (k + \frac{1}{2}) \Omega \right]. \quad (12) \]

Where \( q_o \) is chosen to minimize \( E_{k,l}^{(-2)} \), i.e.

\[ \frac{dE_{k,l}^{(-2)}}{dq_o} = 0 \quad \text{and} \quad \frac{d^2E_{k,l}^{(-2)}}{dq_o^2} > 0. \quad (13) \]

Equation (13) in turn gives, with \( \bar{l} = \sqrt{Q} \),

\[ \]
\[ l_D - \beta = \sqrt{q_o^3 V''(q_o)}. \] (14)

The shifting parameter \( \beta \) is determined by choosing \( \bar{E}_{k,l}^{(-1)} = 0 \). Hence

\[ \beta = -\left[ \frac{1}{2} + (k + \frac{1}{2})\Omega \right], \quad \Omega = \sqrt{3 + \frac{q_o V''(q_o)}{V'(q_o)}} \] (15)

where primes of \( V(q_o) \) denote derivatives with respect to \( q_o \). Then equation (5) reduces to

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \sum_{n=0}^{\infty} v^{(n)}(x) \tilde{l}^{-n/2} \right] \Psi_{k,l}(x) = \left[ \sum_{n=1}^{\infty} q_o^2 E_{k,l}^{(n-1)} \tilde{l}^{-n} \right] \Psi_{k,l}(x). \] (16)

Setting the wave functions with any number of nodes \( k \) as

\[ \Psi_{k,l}(x(q)) = F_{k,l}(x) \exp(U_{k,l}(x)), \] (17)

equation (16) readily transforms into the following Riccati equation:

\[
F_{k,l}(x) \left[ -\frac{1}{2} \left( U_{k,l}'(x) + U_{k,l}(x) U_{k,l}'(x) \right) + \sum_{n=0}^{\infty} v^{(n)}(x) \tilde{l}^{-n/2} \right. 
- \sum_{n=1}^{\infty} q_o^2 E_{k,l}^{(n-1)} \tilde{l}^{-n} \right] - F_{k,l}'(x) U_{k,l}'(x) - \frac{1}{2} F_{k,l}''(x) = 0, \] (18)

where the primes denote derivatives with respect to \( x \). It is evident that this equation admits solution of the form

\[ U_{k,l}'(x) = \sum_{n=0}^{\infty} U_k^{(n)}(x) \tilde{l}^{-n/2} + \sum_{n=0}^{\infty} G_k^{(n)}(x) \tilde{l}^{-(n+1)/2}, \] (19)

\[ F_{k,l}(x) = x^k + \sum_{n=0}^{k-1} d_{n,k}^{(n)} x^n \tilde{l}^{-n/2}, \] (20)

\[ U_k^{(n)}(x) = \sum_{m=0}^{n+1} D_{m,n,k} x^{2m-1}; \quad D_{0,n,k} = 0, \] (21)
\[ G_k^{(n)}(x) = \sum_{m=0}^{n+1} C_{m,n,k} x^{2m}. \]  

(22)

Substituting equations (19) - (22) into equation (16) implies

\[
F_{k,l}(x) \left[ -\frac{1}{2} \sum_{n=0}^{\infty} \left( U_k^{(n)} l^{-n/2} + G_k^{(n)} l^{-(n+1)/2} \right) 
- \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( U_k^{(m)} U_k^{(n-m)} l^{-n/2} + G_k^{(m)} G_k^{(n-m)} l^{-(n+2)/2} \right) 
+ 2U_k^{(m)} G_k^{(n-m)} l^{-(n+1)/2} \right] + \sum_{n=0}^{\infty} \nu(n) l^{-n/2} - \sum_{n=1}^{\infty} q_0^2 E_{k,l}^{(n-1)} l^{-n} \right] 
- \frac{1}{2} F'_{k,l}(x) = 0 \]  

(23)

The solution of equation (22) follows from the uniqueness of power series representation. Therefore, for a given \( k \) we equate the coefficients of the same powers of \( \bar{l} \) and \( x \), respectively.

Although the energy series, equation (23), could appear divergent, or, at best, asymptotic for small \( \bar{l} \), one can still calculate the eigenenergies to a very good accuracy by forming the sophisticated Padé approximation

\[ P_N^M(1/\bar{l}) = (P_0 + P_1/\bar{l} + \cdots + P_M/\bar{l}^M)/(1 + q_1/\bar{l} + \cdots + q_N/\bar{l}^N) \]  

(24)

to the energy series (23). The energy series is calculated up to \( E_{k,l}^{(11)}/\bar{l}^{11} \) by

\[ E_{k,l} = \bar{l}^2 E_{k,l}^{(-2)} + E_{k,l}^{(0)} + \cdots + E_{k,l}^{(11)}/\bar{l}^{11} + O(1/\bar{l}^{12}), \]  

(25)

and with the \( P_5^5(1/\bar{l}) \) Padé approximant it becomes

\[ E_{k,l}[10, 9] = \bar{l}^2 E_{k,l}^{(-2)} + P_5^5(1/\bar{l}). \]  

(26)

Following the above procedure, PSLET results are compared, in table 1, with the exact numerical ones (obtained by direct numerical integrations, DNI) [16] for \( \lambda = 1 \) and \( \lambda = 10 \). To avoid exhaustive numbers of tables we do not list Garcia-Castelan et. al’s results [16] from WKB, WKB single-parabola (WKB-SP), and WKB double-parabola (WKB-DP). In contrast with the WKB, WKB-SP, WKB-DP [16], and SLNT [13] results the comparison between PSLET and DNI results implies excellent agreement.
In order to make remediabale analysis on the effect of $\lambda$, hence of the characteristic length $l_o (\lambda \sim l_o)$, we list ( in tables II-IV) PSLET results for $k = 0, 1, 2$ and $\lambda = 0, 1, 2, 4, 6, 8, 10, 12$ at different values of $|m|$. They are also plotted in figure 1.

Figure 1 (along with tables II-IV) shows that the degeneracies associated with the harmonic oscillator confinement at $\lambda = 0$ are only partially lifted as $\lambda$ increases from zero ( of course, such degeneracies would completely be lifted when a magnetic field is applied perpendicular to the plane of the dot). It also shows that the equidistance form of the energy levels at $\lambda = 0$ changes in the following manners; (i) for a given $k$, the spacing between two successive $|m|$ states decreases as $\lambda$ increases, and increases as $|m|$ increases for a given $\lambda$, whilst (ii) for a given $\lambda$, the spacing increases as the nodal quantum number $k$ increases. One should nevertheless notice that (iii) s-states (with $m = 0$) shift up more rapidly than states with $|m| \geq 1$, and for $|m| \geq 1$ states with lower $|m|$ shift up faster than states with higher $|m|$ as $\lambda$ increases from zero.

The above mentioned features (i)-(iii), in fact, build up the sought after scenario for the change in level ordering, that manifests energy crossings and spin-singlet ($S_z = 0$) spin-triplet ($S_z = 1$) oscillations, and inspires the vital role of the central spike term in (3). More specifically, the twofold nature of the central spike term in the effective potential, of eq.(3),

$$V_{\text{eff}}(q) = \frac{m^2 - 1/4}{q^2} + q^2 + \frac{\lambda}{q}$$

explains the energy crossings as follows; (a) for $m = 0$ it represents an attractive core that strengthens the confinement $q^2$, whereas (b) for $|m| \geq 1$ it represents a repulsive core which renders, along with the Coulomb repulsion, the potential less potent. This is why, for a given $k$, the energy of a lower $|m|$ state increases much faster ( more rapidly for $m = 0$) than that of a higher $|m|$, as $\lambda$ increases, and catches up with it ( hence energy crossings and singlet-triplet spin oscillations occur, or, at most, energy levels clustering is manifested). On the physical sides, the two electrons are farther apart for higher $|m|$. Moreover, for a given $k$ energy crossings are not feasible between the corresponding states with different $|m|$. Whereas, states with a given $k$ and $|m|$ cross with states at lower $k$ and higher $|m|$. Therefore, the lowest three states $(0,0)$, $(0,1)$, and $(0,2)$ never cross any other state ( i.e., they can never be depressed into a lower $k$-state).

The effect of correlation, between two interacting electrons in a harmonic QD, is therefore clear in the full energy spectrum for $\lambda > 0$ with all $(k,|m|)$-states for the relative motion as shown in figure 1 and documented in tables I-IV. However, it should be noted that the level ordering reported by Garcia-Castelan et. al [15] is now changed, namely for the $(0,4)$ and $(1,1)$ states. Moreover, the $(1,4)$ and $(2,1)$ states seem to change order as $\lambda$ increases form 12.

To sum up, we have used a pseudo-perturbation recipe (PSLET) to study the characteristic length effect on the correlation energies for two interacting electrons in a parabolic QD. We have proved PSLET persuasive numerical reliability in comparison with direct numerical integration method ( in table 1) for $\lambda=1$, and 10. Next, we have obtained the correlation energies for $\lambda = 2, 4, 6, 8, 12$ and $k = 0, 1, 2$, and documented ( through figure I) that the level ordering reported by Garcia-Castelan et. al [15] is not absolute but continually bound to change as $\lambda$ increases from zero. Finally, the almost forgotten twofold effect of the central
spike term, in the effective two-dimensional potential (27), is now clarified to inherit a major responsibility for energy crossings.
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TABLE I. Comparison of PSLET energies (in $\hbar \omega_o/2$ units) and the exact ones from direct numerical integration [16] for $\lambda=1$ and 10.

| $(k, |m|)$ | Exact | PSLET | $(k, |m|)$ | Exact | PSLET |
|-----------|-------|-------|-----------|-------|-------|
| (1,7)     | 20.3587 | 20.3587 | (1,7)     | 23.5040 | 23.5040 |
| (0,9)     | 20.3280 | 20.3280 | (0,9)     | 23.2188 | 23.2188 |
| (3,2)     | 18.5351 | 18.5351 | (3,2)     | 23.0339 | 23.0339 |
| (2,4)     | 18.4388 | 18.4388 | (2,4)     | 22.2217 | 22.2217 |
| (1,6)     | 18.3843 | 18.3843 | (3,1)     | 21.8721 | 21.8715 |
| (0,8)     | 18.3472 | 18.3472 | (1,6)     | 21.7355 | 21.7355 |
| (3,1)     | 16.6498 | 16.6498 | (0,8)     | 21.3954 | 21.3954 |
| (2,3)     | 16.4895 | 16.4895 | (0,8)     | 21.3954 | 21.3954 |
| (1,5)     | 16.4163 | 16.4163 | (0,8)     | 21.3954 | 21.3954 |
| (0,7)     | 16.3701 | 16.3701 | (0,8)     | 21.3954 | 21.3954 |
| (3,0)     | 14.9850 | 14.9881 | (0,7)     | 21.3954 | 21.3954 |
| (2,2)     | 14.5646 | 14.5646 | (0,7)     | 21.3954 | 21.3954 |
| (1,4)     | 14.4579 | 14.4579 | (0,7)     | 21.3954 | 21.3954 |
| (0,6)     | 14.3983 | 14.3983 | (0,7)     | 21.3954 | 21.3954 |
| (2,1)     | 12.6961 | 12.6961 | (0,7)     | 21.3954 | 21.3954 |
| (1,3)     | 12.5154 | 12.5154 | (0,7)     | 21.3954 | 21.3954 |
| (0,5)     | 12.4340 | 12.4340 | (0,7)     | 21.3954 | 21.3954 |
| (2,0)     | 11.0848 | 11.0883 | (0,7)     | 21.3954 | 21.3954 |
| (1,2)     | 10.6024 | 10.6024 | (0,7)     | 21.3954 | 21.3954 |
| (0,4)     | 10.4814 | 10.4814 | (0,7)     | 21.3954 | 21.3954 |
| (1,1)     | 8.7594  | 8.7594  | (0,7)     | 21.3954 | 21.3954 |
| (0,3)     | 8.5485  | 8.5485  | (0,7)     | 21.3954 | 21.3954 |
| (1,0)     | 7.2340  | 7.2362  | (0,7)     | 21.3954 | 21.3954 |
| (0,2)     | 6.6538  | 6.6538  | (0,7)     | 21.3954 | 21.3954 |
| (0,1)     | 4.8553  | 4.8553  | (0,7)     | 21.3954 | 21.3954 |
| (0,0)     | 3.4952  | 3.4968  | (0,7)     | 21.3954 | 21.3954 |
TABLE II. PSLET correlation energies (in $\hbar \omega / 2$ units) for $k=0$, $|m|=0,1,2,3,4,5$ and $\lambda=0,1,2,4,6,8,10,12$

| $|m|$ | $\lambda=0$ | $\lambda=1$ | $\lambda=2$ | $\lambda=4$ |
|-----|----------|----------|----------|----------|
| 0   | 2        | 3.4968   | 4.6391   | 6.4428   |
| 1   | 4        | 4.8553   | 5.6557   | 7.1251   |
| 2   | 6        | 6.6538   | 7.2872   | 8.4994   |
| 3   | 8        | 8.5485   | 9.0864   | 10.1331  |
| 4   | 10       | 10.4814  | 10.9564  | 11.8885  |
| 5   | 12       | 12.4340  | 12.8638  | 13.7112  |

|   | $\lambda=6$ | $\lambda=8$ | $\lambda=10$ | $\lambda=12$ |
|---|------------|-------------|--------------|--------------|
| 0 | 7.9373     | 9.2644      | 10.4816      | 11.6184      |
| 1 | 8.4599     | 9.6938      | 10.8496      | 11.9425      |
| 2 | 9.6480     | 10.7425     | 11.7903      | 12.7975      |
| 3 | 11.1440    | 12.1226     | 13.0720      | 13.9947      |
| 4 | 12.7978    | 13.6861     | 14.5547      | 15.4049      |
| 5 | 14.5429    | 15.3599     | 16.1628      | 16.9525      |
TABLE III. Same as table 2 for $k=1$.

| $|m|$ | $\lambda=0$ | $\lambda=1$ | $\lambda=2$ | $\lambda=4$ |
|-----|------------|------------|------------|------------|
| 0   | 6          | 7.2362     | 8.2945     | 10.0462    |
| 1   | 8          | 8.7594     | 9.4879     | 10.8608    |
| 2   | 10         | 10.6024    | 11.1913    | 12.3314    |
| 3   | 12         | 12.5154    | 13.0233    | 14.0173    |
| 4   | 14         | 14.4579    | 14.9110    | 15.8031    |

|      | $\lambda=6$ | $\lambda=8$ | $\lambda=10$ | $\lambda=12$ |
|------|------------|------------|------------|------------|
| 0    | 11.5189    | 12.8317    | 14.0381    | 15.1665    |
| 1    | 12.1368    | 13.3327    | 14.4622    | 15.5362    |
| 2    | 13.4252    | 14.4772    | 15.4916    | 16.4721    |
| 3    | 14.9840    | 15.9254    | 16.8431    | 17.7388    |
| 4    | 16.6773    | 17.5343    | 18.3753    | 19.2009    |
TABLE IV. Same as table 2 for $k=2$.

| $m$ | $\lambda=0$ | $\lambda=1$ | $\lambda=2$ | $\lambda=4$ |
|-----|-------------|------------|-------------|-------------|
| 0   | 10          | 11.0883    | 12.0757     | 13.7327     |
| 1   | 12          | 12.6961    | 13.3720     | 14.6650     |
| 2   | 14          | 14.5646    | 15.1195     | 16.2015     |
|     | $\lambda=6$ | $\lambda=8$ | $\lambda=10$ | $\lambda=12$ |
| 0   | 15.1801     | 16.4736    | 17.6660     | 18.7833     |
| 1   | 15.8856     | 17.0418    | 18.1420     | 19.1937     |
| 2   | 17.2480     | 18.2613    | 19.2438     | 20.1978     |
Figures captions

**Fig.1**: PSLET correlation energies for two interacting electrons in a harmonic quantum dot vs $\lambda = \sqrt{2} l_o / a^*$ (the ratio of the oscillator length $l_o$ and the effective Bohr radius $a^*$). The energies are normalized with the oscillator energy $\hbar \omega_o/2$. The full lines represent states with $k = 0$, dashed lines for $k = 1$, and dashed dotted lines for $k = 2$. 
