Zero-cycles on self-products of varieties: some elementary examples verifying Voisin’s conjecture

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Received: 5 August 2020 / Accepted: 10 September 2020 / Published online: 17 September 2020
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Abstract
An old conjecture of Voisin describes how zero-cycles on a variety $X$ should behave when pulled-back to the self-product $X^m$ for $m$ larger than the geometric genus of $X$. Using complete intersections of quadrics, we give examples of varieties in any dimension and with arbitrarily high geometric genus that verify Voisin’s conjecture.

Keywords Algebraic cycles · Chow groups · Motives · Voisin conjecture · Complete intersections of quadrics

Mathematics Subject Classification Primary 14C15 · 14C25 · 14C30

1 Introduction

Given a smooth projective variety $X$ over $\mathbb{C}$, let $A^i(X)\mathbb{Z} := CH^i(X)\mathbb{Z}$ denote the Chow groups of $X$ (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Z}$-coefficients, modulo rational equivalence [7]). Let $A^i_{\text{hom}}(X)\mathbb{Z}$ denote the subgroup of homologically trivial cycles.

The Bloch–Beilinson–Murre conjectures form a kind of Rosetta Stone, allowing to translate cohomological statements into conjectural Chow-theoretic statements [8–10,21,22,29]. The following particular instance of such a translation was first formulated by Voisin:

Conjecture 1.1 [28] Let $X$ be a smooth projective $n$-dimensional variety with $H^j(X, O_X) = 0$ for $0 < j < n$. Let $m$ be an integer strictly larger than the geometric genus $p_g(X) := \dim H^n(X, O_X)$. Then for any zero-cycles $a_1, \ldots, a_m \in A^n_{\text{hom}}(X)\mathbb{Z}$, there is an equality

$$\sum_{\sigma \in S_m} (\text{sgn}(\sigma))^{n+1} a_{\sigma(1)} \times \cdots \times a_{\sigma(m)} = 0 \text{ in } A^{mn}(X^m)\mathbb{Z}.$$
Here $\mathfrak{S}_m$ is the symmetric group on $m$ elements, and $\text{sgn}(\sigma)$ is the sign of the permutation $\sigma$. The notation $a_1 \times \cdots \times a_m$ is shorthand for the zero-cycle $p_1^*(a_1) \cdot p_2^*(a_2) \cdots p_m^*(a_m)$ on $X^m$, where the $p_j : X^m \rightarrow X$ are the various projections.\]

This conjecture is the translation in Chow-language of the fact that the Hodge structure $\bigwedge^m H^n(X, \mathbb{Q}) \subseteq H^{mn}(X^m, \mathbb{Q})$

has Hodge coniveau $> 0$ for $m > p_g(X)$ (see [28] or [29, Section 4.3.5.2] for more detailed motivation).

In case $p_g(X) = 0$, Conjecture 1.1 predicts that $A^n(X) \cong \mathbb{Z}$ (this is a form of Bloch’s conjecture [3]). In case $p_g(X) = 1$ (e.g., $X$ is a Calabi–Yau variety), the conjecture takes on a particularly appealing form: it predicts that any 2 degree 0 zero-cycles $a, a' \in A^n_\text{hom}(X)\mathbb{Z}$ verify

$$a \times a' = (-1)^n a' \times a \text{ in } A^{2n}(X \times X)\mathbb{Z}.$$\]

This is still open for a general K3 surface (in fact, I am not aware of a K3 surface with Picard number $< 9$ which is known to verify Voisin’s conjecture). Examples of surfaces of geometric genus 1 verifying the conjecture (or a variant conjecture) are given in [12,14,15,28,30]. Examples of other varieties verifying the conjecture are given in [2,5,13,16–20,27,28].

The main result of this note is that Voisin’s conjecture is true for certain complete intersections of quadrics:

**Theorem 1.2** Given $g, r \in \mathbb{N}$ and distinct numbers $\lambda_0, \ldots, \lambda_{2g+1} \in \mathbb{C}$, let $X \subset \mathbb{P}^{2g+1}(\mathbb{C})$ be the complete intersection defined by

$$\begin{align*}
x_0^2 + x_1^2 + \cdots + x_{2g+1}^2 &= 0, \\
\lambda_0 x_0^2 + \lambda_1 x_1^2 + \cdots + \lambda_{2g+1} x_{2g+1}^2 &= 0, \\
\lambda_0^2 x_0^2 + \lambda_1^2 x_1^2 + \cdots + \lambda_{2g+1}^2 x_{2g+1}^2 &= 0, \\
\vdots \\
\lambda_r^2 x_0^2 + \lambda_1^2 x_1^2 + \cdots + \lambda_{2g+1}^2 x_{2g+1}^2 &= 0.
\end{align*}$$

Then $X$ is a smooth projective variety, and Conjecture 1.1 is true for $X$.

These complete intersections are admittedly very special (and unfortunately the argument proving Theorem 1.2 does not apply to general complete intersections of quadrics), but at least they provide examples of any dimension, and with $p_g$ arbitrarily high, verifying Voisin’s conjecture. The complete intersections of Theorem 1.2 are rather similar to the Calabi–Yau varieties of [20]: both are related to products of curves, and hence to abelian varieties. Thus, to prove Theorem 1.2 we can reduce to a problem concerning zero-cycles on abelian varieties; this last problem can be solved thanks to recent work of Vial’s [27].

As a consequence of Theorem 1.2, certain instances of the generalized Hodge conjecture are verified:

**Corollary 1.3** Let $X$ be as in Theorem 1.2, and $m > p_g(X)$. The sub-Hodge structure

$$\bigwedge^m H^n(X, \mathbb{Q}) \subseteq H^{mn}(X^m, \mathbb{Q})$$

is supported on a divisor.
Conventions In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

Unless indicated otherwise, all Chow groups will be with rational coefficients: we will denote by $A_j(X)$ (resp. $A_j(X)_{\mathbb{Z}}$) the Chow group of $j$-dimensional cycles on $X$ with $\mathbb{Q}$-coefficients (resp. $\mathbb{Z}$-coefficients); for $X$ smooth of dimension $n$ the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably. The notation $A^j_{\text{hom}}(X)$ will be used to indicate the subgroups of homologically trivial cycles.

We will write $M_{\text{rat}}$ (and $M_{\text{hom}}$) for the contravariant category of pure motives with respect to rational equivalence (resp. homological equivalence), as in [22,24].

2 The proof

This section contains the proof of the main result:

Proof of Theorem 1.2 This is based on results of Terasoma, who has made an in-depth study of this kind of complete intersection [25, Sections 2.4, 2.5 and 2.6] (NB: in loc. cit., the variety $X$ is denoted $X_{g-r}$, which is (by definition) the same as $X_{r,2g+2,2}$).

First, the non-singularity of $X$ is readily checked using the Vandermonde determinant (cf. [25, Proposition 2.4.1]).

Next, to verify Conjecture 1.1 for $X$, we need to understand the Chow group of zero-cycles $A^n(X)_{\mathbb{Z}}$ (where we write $n := 2g - r$ for the dimension of $X$). Thanks to a theorem of Roitman [23], it suffices to prove that Conjecture 1.1 is true for $A^n(X)$, the Chow group with $\mathbb{Q}$-coefficients. From Terasoma’s work [25, Corollary 2.5.3], we know there exist a finite number of hyperelliptic curves $C_\chi$ and a correspondence $/\Gamma_1$ inducing an isomorphism in cohomology

$$/\Gamma_1 : \bigoplus_\chi H^1_{\text{prim}}(X,\mathbb{Q}) \xrightarrow{\cong} \bigoplus_\chi \wedge^n H^1(C_\chi,\mathbb{Q}).$$

(1)

Because curves and complete intersections verify the Lefschetz standard conjecture, we know that the inverse is also induced by a correspondence. (This is well-known, cf. for instance [26, Proof of Proposition 1.1]), where I first learned this. In a nutshell, the argument is as follows: the Lefschetz standard conjecture for $X$ implies that there is a correspondence $s \in A^n(X \times X)$ inducing a polarization on $H^1_{\text{prim}}(X,\mathbb{Q})$. A Hodge-theoretical argument [26, Lemma 1.6] then gives that $\alpha := /\Gamma_1 \circ s \circ /\Gamma_1$ induces an automorphism of $\bigoplus_\chi \wedge^n H^1(C_\chi,\mathbb{Q})$. By the Cayley–Hamilton theorem, the inverse to $\alpha_*$ is given by $P(\alpha)_*$ for some rational polynomial $P$. Then $/\Gamma_1 \circ s \circ /\Gamma_1$ acts as the identity on $\bigoplus_\chi \wedge^n H^1(C_\chi,\mathbb{Q})$, and so the correspondence $s \circ /\Gamma_1 \circ P(\alpha)$ gives an inverse to $\alpha_*$.)

This means that there is an isomorphism of homological motives

$$[\Gamma] : H^n_{\text{prim}}(X,\mathbb{Q}) \xrightarrow{\cong} \bigoplus_\chi \text{Sym}^n h^1(C_\chi) \quad \text{in } M_{\text{hom}}.$$

(2)

Let $h^n_{\text{prim}}(X) \in M_{\text{rat}}$ be the Chow motive such that

$$h(X) = h^n_{\text{prim}}(X) \oplus \bigoplus I(*) \quad \text{in } M_{\text{rat}}$$

(that is, $h^n_{\text{prim}}(X)$ is defined by the projector $\pi_{h,n}^{\text{prim}} := \Delta_X - \frac{1}{2\pi i} \sum_j h^i \times h^{n-j}$, where $h \in A^1(X)$ denotes a hyperplane section). The variety $X$ is isomorphic to a quotient $D^{2g-r}/G$.
where $D$ is a smooth curve [25, Theorem 2.4.2], and so $X$ is Kimura finite-dimensional [10]. We recall Kimura’s fundamental nilpotence theorem [10], which implies that the functor from finite-dimensional Chow motives to homological motives is conservative, i.e. detects isomorphisms. The isomorphism (2) can thus be upgraded to an isomorphism of Chow motives
\[ h^n_{prim}(X) \xrightarrow{\cong} \bigoplus_{\chi} \text{Sym}^n h^1(C_{\chi}) \quad \text{in } M_{\text{rat}}. \] (3)

Let $J_{\chi} := \text{Jac}(C_{\chi})$ denote the Jacobian of the curve $C_{\chi}$. There are isomorphisms $h^1(C_{\chi}) \cong h^1(J_{\chi})$. Moreover, $\text{Sym}^n h^1(B) \cong h^n(B)$ for any abelian variety $B$ (here $h^n(B)$ refers to the Deninger–Murre decomposition of abelian schemes [6]), and so the isomorphism (3) induces an isomorphism
\[ h^n_{prim}(X) \xrightarrow{\cong} \bigoplus_{\chi} h^n(J_{\chi}) \quad \text{in } M_{\text{rat}}. \]

The following lemma (which we state separately for possible future reference) now closes the proof of the theorem:

**Lemma 2.1** Let $X$ be a smooth projective variety of dimension $n$, and assume that
\[ h(X) \cong \bigoplus_{\chi} h^n(B_{\chi}) \oplus \bigoplus \mathbb{1}(*) \quad \text{in } M_{\text{rat}}, \] (4)
where $B_{\chi}$ are abelian varieties, and $h^n(B_{\chi})$ refers to the Deninger–Murre decomposition of the motive of abelian schemes [6]. Then Conjecture 1.1 is true for $X$.

It remains to prove the lemma. (NB: In proving the lemma, we use some results concerning the Chow motive of an abelian variety $B$ and the Beauville decomposition $A^*_{(n)}(B)$ of the Chow ring. For a quick survey containing all the results we use, one could look at [24, Section 5].)

Taking Chow groups on both sides of (4), we obtain an isomorphism
\[ A^n_{\text{hom}}(X) \xrightarrow{\cong} \bigoplus_{\chi} A^n(h^n(B_{\chi})). \] (5)

Writing $g_{\chi} := \dim B_{\chi}$, let us proceed to analyze the various summands in (5):

- In case $n > g_{\chi}$, the summand $A^n(h^n(B_{\chi}))$ is zero for dimension reasons. The piece $A^{g_{\chi}}(B_{\chi})$ of the Beauville decomposition is also zero.
- In case $n = g_{\chi}$, we have that $A^n(h^n(B_{\chi})) = A^{g_{\chi}}(B_{\chi})$, since the Deninger–Murre decomposition $h^n(B)$ induces the Beauville decomposition $A^*_{(n)}(B)$ in the sense that $A^j_{(s)}(B) = A^j(h^{2j-s}(B))$ for all integers $j$ and $s$ and any abelian variety $B$.
- Finally, let us assume $n < g_{\chi}$, say $n = g_{\chi} - s$ where $s > 0$. In this case, the work of Künemann [11] provides a “hard Lefschetz” isomorphism of motives
\[ h^n(B_{\chi}) \xrightarrow{\cong} h^{n+2s}(B_{\chi})(s) \quad \text{in } M_{\text{rat}}. \]

Taking Chow groups, this induces an isomorphism
\[ A^n(h^n(B_{\chi})) \xrightarrow{\cong} A^{n+s}(h^{n+2s}(B_{\chi})) = A^{n+s}_{(n)}(B_{\chi}) = A^{g_{\chi}}_{(n)}(B_{\chi}). \]
In view of this analysis, we can rewrite (5) in order to have zero-cycles on both sides:

\[ A^n_{\text{hom}}(X) \xrightarrow{\cong} \bigoplus_{\chi} A^{g_{\chi}}_{(n)}(B_{\chi}) \]  

(6)

We now invoke a result of Vial:

**Theorem 2.2** (Vial [27]) Let \( B \) be an abelian variety of dimension \( g \), and let \( \pi^i_B \) denote the Deninger–Murre projectors [6].

(i) Assume \( n \in \mathbb{N} \) is even. Then

\[ (\wedge^m \pi^g_{B} \cdot \otimes \chi_{g_{\chi}} A_0(B^m) = 0 \ \forall \ m > \binom{g}{n} . \]

(ii) Assume \( n \in \mathbb{N} \) is odd. Then

\[ (\text{Sym}^m \pi^g_{B} \cdot \otimes \chi_{g_{\chi}} A_0(B^m) = 0 \ \forall \ m > \binom{g}{n} . \]

(This is [27, Theorem 4.1]. Vial’s result generalizes a result of Voisin [29, Example 4.40], which was the case \( n = g \) of Theorem 2.2.)

Armed with Vial’s result, we are in position to prove the lemma. Let us treat in detail the case \( n \) even (the case \( n \) odd is similar and will be left to the zealous reader). Let us write \( \pi_{\text{prim}}^X \) for the projector such that

\[ (X, \pi_{\text{prim}}^X, 0) \sim \bigoplus_{\chi} h^n(B_{\chi}). \]

Given an integer \( m > p^g_{\chi}(X) \), using (6) we find

\[ (\wedge^m \pi_{\text{prim}}^X \cdot \otimes \chi_{g_{\chi}} A_0{(X^m)} \cong \bigoplus_{m_\chi=m} \bigotimes_{\chi}(\wedge^m \pi^g_{B_{\chi}} \cdot \otimes \chi_{g_{\chi}} A_0{(B_{m_\chi}^\chi)} . \]  

(7)

We note that (4) gives us an equality

\[ p^g_{\chi}(X) = \sum_{\chi} \binom{g_{\chi}}{n} . \]

Since (by assumption) \( m > p^g_{\chi}(X) \), it follows that in each partition \( m = \sum m_\chi \) there exists some \( \chi \), say \( \chi_0 \), such that

\[ m_{\chi_0} > \binom{g_{\chi_0}}{n} . \]

Theorem 2.2(i) applied to \( B_{\chi_0} \) guarantees that in the sum (7) each summand

\[ \bigotimes_{\chi}(\wedge^m \pi^g_{B_{\chi}} \cdot \otimes \chi_{g_{\chi}} A_0(B_{m_\chi}^\chi)) \]

vanishes. It follows that the whole sum (7) vanishes, i.e.

\[ (\wedge^m \pi_{\text{prim}}^X \cdot \otimes \chi_{g_{\chi}} A_0{(X^m)} = 0 . \]

Since \( (\pi_{\text{prim}}^X \cdot A_0(X) = A^n_{\text{hom}}(X) \), this proves Conjecture 1.1 for \( X \), in case \( n \) is even.

The argument for \( n \) odd is similar, the difference being that one considers \( \text{Sym}^m \pi_{\text{prim}}^X = \text{Sym}^m \pi^n_{\chi} \) instead of \( \wedge^m \pi_{\text{prim}}^X \), and one relies on part (ii) of Theorem 2.2 instead of part (i). 

\( \square \)
Proof of Corollary 1.3  As Voisin had already remarked [28, Corollary 3.5.1], this is implied by the truth of Conjecture 1.1 for $X$ (the implication can be seen using the Bloch–Srinivas argument [4]; this is explained in detail in [18, Corollary 2.7]).

Acknowledgements  Thanks to Kai for enjoyable bike trips in the Alsace countryside. Thanks to the referee for many constructive comments that helped to improve the presentation.

Compliance with ethical standards

Conflict of interest  The author states that there is no conflict of interest.

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