Network Criticality Analysis for Finite Sized Wireless Sensor Networks

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Abstract—The topology of a sensor network changes very frequently due to node failures because of power constraints or physical destruction. Robustness to topology changes is one of the important design factors of wireless sensor networks which makes them suitable to military, communications, health and surveillance applications. Network criticality is a measure which capture the properties of network robustness to environmental changes. In this work, we derived the analytical formulas for network criticality, normalized network criticality and studied the effect of number of neighbors and network dimension on the network criticality and normalized network criticality.

Index Terms—Wireless sensor networks, Regular Graphs, Network Robustness, Network Criticality, Spectral graph theory

I. INTRODUCTION

Sensor network composed of large number of low power sensor nodes which are densely deployed in inaccessible or remote geographic places. Due to lack of power or physical damages or environmental changes, nodes fail to operate which leads to frequent changes in topology. Robustness to topology changes is one of the important design factors [1], which can be studied by the network criticality metric. Network criticality studies the adaptability of a network to the topology changes which can be further classified into link criticality and node criticality. In [2], robustness of different graph topologies has been studied and established the network robustness conditions. Ali Tizghadam et. al proposed a method in [3] for capacity assignment to optimize the network robustness and they also proposed a robust routing framework in [4] for core networks. In our work we derived the network criticality and normalized network criticality expressions for \( r \)-nearest neighbor networks. The motivation behind the using finite sized networks is most of the practical wireless sensor networks are finite sized. The \( r \)-nearest neighbor networks [6] with varying number of neighbors represents the notion of geographical proximity in the wireless sensor networks. Here, the nearest neighbors \( r \) captures the overhead or nodes’ transmission radius. The advantage of this kind of analysis and theoretical results play a critical role in the design of wireless networks, and also easier to perform than real experiments and thousands of simulation trails. This work provides the understanding of network criticality in terms of number of nodes, nearest neighbors and network dimension and gives the important insights for designing robust wireless networks. The rest of the paper is organized as follows. In Section II, we have given a brief overview about network criticality. In Section III, we have derived the generalized expressions for eigen values of the Laplacian matrix for \( r \)-nearest neighbor networks. In Section IV, network criticality and normalized network criticality expressions for \( r \)-nearest neighbor cycle, \( r \)-nearest neighbor torus and \( m \)-dimensional \( r \)-nearest neighbor networks have been derived. In Section V, we compared the simulation results with analytical results obtained in the Section IV.

II. NETWORK CRITICALITY

Let \( G = (V, E) \), be an undirected graph with node set \( V = \{1, 2, \ldots, n\} \) and an edge set \( E \subseteq V \times V \). Further, let \( A \) be \( n \times n \) symmetric adjacency matrix of the graph \( G \), each entry of adjacency matrix is represented by \( a_{ij} \), which is 1 if node \( i \) is connected to node \( j \), else it is 0.

The degree matrix \( D \) is defined as the diagonal matrix with entries \( d_i = \deg(v_i) \). \( L \) is the Laplacian matrix which describes the topology of the network. The Laplacian matrix of the graph \( G \) is the \( n \times n \) symmetric matrix \( L = D - A \), whose entries are

\[
l_{ij} = l_{ji} = \begin{cases} 
\deg(v_i) & \text{if } j = i \\
-a_{ij} & \text{if } j \neq i
\end{cases}
\]

\[ (1) \]

Definition 1: Let \( \eta_l \) be the node criticality of node \( l \), then it can be defined as the random walk betweenness of a node over the sum of the weights of its incident links. Let \( \eta_{lj} \) be the link criticality of link \( l = (i,j) \), then it can be defined as the betweenness of a link over its weight.

Lemma 1: Network criticality \((\tau)\) [4] evaluates the network robustness to topology changes and it can be expressed as

\[
\tau = \sum_{s} \sum_{d} \tau_{sd}
\]

\[ (2) \]

where \( \tau_{sd} = l_{ss}^+ + l_{dd}^- - 2l_{sd}^+ \). \( L^+ = [l_{ij}^+] \) is the Moore-Penrose inverse of graph Laplacian matrix.
Lemma 2: Normalized network criticality ($\hat{\tau}$) studies the effect of topology and community of interest via betweenness of a node or link and it can be expressed as

$$\hat{\tau} = \frac{\tau}{n(n-1)}$$  \hspace{1cm} (3)

where $n$ is the number of nodes.

Lemma 3: Network criticality ($\tau$) can be rewritten as $2nTr(L^+)$ and normalized network criticality ($\hat{\tau}$) can be rewritten as $\hat{\tau} = \frac{2Tr(L^+)}{n(n-1)}$. Where $Tr(L^+)$ represents trace of the Moore-Penrose inverse of Laplacian matrix.

Proof:
From Lemma 1, we can write the Network criticality as,

$$\tau = \sum_s \sum_d \tau_{sd}$$

$$= \sum_s \sum_d l_{ss}^+ + \sum_s \sum_d l_{dd}^+ - 2 \sum_s \sum_d l_{sd}^+$$

$$= 2n \sum_i l_{ii}^+$$

$$= 2nTr(L^+)$$

So normalized network criticality can be written as

$$\hat{\tau} = \frac{2Tr(L^+)}{n(n-1)}$$  \hspace{1cm} (5)

III. LAPLACIAN SPECTRA OF $r$-NEAREST NEIGHBOR NETWORKS

In $r$-nearest neighbor networks, an edge will be existed between every pair of neighbors that are $r$ hops away. The eigen value expressions for $r$-nearest neighbor cycle, $r$-nearest neighbor torus and $m$ dimensional $r$-nearest neighbor torus networks have been derived in this section.

A. $r$-nearest neighbor cycle

The $r$-nearest neighbor cycle $C^n_r$ can be represented by a circulant matrix $[5]$. A circulant matrix is defined as

$$\begin{bmatrix}
    a_1 & a_2 & \ldots & a_{n-1} & a_n \\
    a_n & a_1 & \ldots & a_{n-2} & a_{n-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_3 & a_4 & \ldots & a_1 & a_2 \\
    a_2 & a_3 & \ldots & a_{n-1} & a_1 \\
\end{bmatrix}$$  \hspace{1cm} (6)

and $j$-th eigen value of a circulant matrix can be expressed as

$$\lambda_j = a_1 + a_2 \omega^j + \ldots + a_n \omega^{(n-1)j}$$  \hspace{1cm} (7)

where $\omega$ be the $n$-th root of $1$. Then $\omega$ is the complex number:

$$\omega = \cos \left( \frac{2\pi}{n} \right) + i \sin \left( \frac{2\pi}{n} \right) = e^{\frac{2\pi i}{n}}$$  \hspace{1cm} (8)

The 1-nearest cycle and 2-nearest cycle are shown in Fig.1 and Fig.2 respectively. Let the adjacency matrix $A$ and the degree matrix $D$ of 1-nearest cycle, then they can be written as

$$A = \begin{bmatrix}
    0 & 1 & 0 & \ldots & \ldots & 0 \\
    1 & 0 & 1 & \ldots & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \ddots & 0 & 0 \\
    0 & 0 & \ldots & 0 & 0 & 1 \\
    1 & 0 & 0 & \ldots & \ldots & 1 \\
\end{bmatrix}$$  \hspace{1cm} (9)

$$D = \begin{bmatrix}
    2 & 0 & 0 & \ldots & 0 & 0 \\
    0 & 2 & 0 & \ldots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \ddots & 0 & 0 \\
    0 & 0 & \ldots & 0 & 2 & 0 \\
    0 & 0 & \ldots & 0 & 0 & 2 \\
\end{bmatrix}$$  \hspace{1cm} (10)

Theorem 1: The generalized expression for eigenvalues of Laplacian matrix $L$ for $r$-nearest neighbor cycle $C^n_r$ can be expressed as,

$$\lambda_j(L(C^n_r)) = 2r + \frac{\sin \left( \frac{(2r+1)\pi j}{n} \right)}{\sin \frac{\pi j}{n}}$$  \hspace{1cm} (11)

where $j = 0, 1, \ldots, (n-1)$.

Proof: From [7], we can observe that, the first row is enough to obtain the eigen values of any circulant matrix.

The first row of adjacency matrix $A$, degree matrix $D$ and Laplacian matrix $L$ can be written as follows,

$$A_{1n} = \begin{bmatrix}
    0 & 1 & 1 & \ldots & \ldots & 0 & 1 & 1 \\
    1 & 0 & 1 & \ldots & \ldots & 1 & 0 & 1 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \ddots & 0 & 1 & 1 & 0 \\
    0 & 0 & \ldots & 0 & 0 & 0 & 1 & 1 \\
    1 & 0 & 0 & \ldots & \ldots & 0 & 1 & 1 \\
\end{bmatrix}$$  \hspace{1cm} (12)

$$D_{1n} = \begin{bmatrix}
    2r & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
    0 & 2r & 0 & \ldots & 0 & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 & 0 \\
    0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$  \hspace{1cm} (13)

$$L_{1n} = \begin{bmatrix}
    2r & -1 & -1 & \ldots & \ldots & -1 & -1 & -1 \\
    -1 & 2r & -1 & \ldots & \ldots & -1 & -1 & -1 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \ddots & 0 & -1 & -1 & -1 \\
    -1 & -1 & \ldots & 0 & 0 & -1 & -1 & -1 \\
    -1 & -1 & \ldots & -1 & -1 & 0 & 0 & -1 \\
\end{bmatrix}$$  \hspace{1cm} (14)

By using (14) and (7), we can write the

$$\lambda_j(L(C^n_r)) = 2r - 2 \sum_{i=1}^{r} \cos \left( \frac{2\pi ji}{n} \right)$$  \hspace{1cm} (15)

Lemma 1: Trigonometric identity of Dirichlet kernel [7]

$$1 + 2 \sum_{j=1}^{r} \cos(jx) = \frac{\sin \left( r + \frac{1}{2} \right) x}{\sin \left( \frac{x}{2} \right)}$$  \hspace{1cm} (16)

Hence, from the Lemma 1, (15) can be rewritten as,

$$\lambda_j(L(C^n_r)) = 2r + 1 - \frac{\sin \left( \frac{(2r+1)\pi j}{n} \right)}{\sin \frac{\pi j}{n}}$$  \hspace{1cm} (17)
B. r-nearest neighbor torus

A torus can be seen in Fig. 3 and it can be represented by the $n \times n$ block circulant matrix $A$ as

\[
A = \begin{bmatrix}
A_0 & A_1 & \ldots & A_{n-1} & A_{n-1}
\end{bmatrix}
\]

where the number of nodes $n = n_1^2$, then each block $A_i$, for $i = 0, 1, \ldots (n_1 - 1)$ represents $n_1 \times n_1$ circulant matrices.

**Lemma 2:** Let $G$ be the cartesian product of two graphs $G'$ and $G''$ with vertex sets $V'$ and $V''$ and edge sets $E'$ and $E''$. Let the eigen values of $G'$ are $\lambda_1(G'), \ldots, \lambda_p(G')$ and $G''$ are $\lambda_1(G''), \ldots, \lambda_q(G'')$, where $p = |V'|$ and $q = |V''|$. Let the vertex set of $G$ is $r = |V|$, which can be expressed as $V = \sqrt{|V'|} \times \sqrt{|V''|}$. Then, the eigen values of $G$ can be expressed as

\[
\lambda_k(G) = \lambda_i(G') + \lambda_j(G'')
\]

where $i \in \{1, 2, ..., p\}$, $j \in \{1, 2, ..., q\}$ and $k \in \{1, 2, ..., r\}$.

**Remark 1:** [19] also holds for eigen values of the Laplacians $L'$ and $L''$ of graphs of $G'$ and $G''$ respectively [8].

**Theorem 2:** The generalized expression for eigenvalues of Laplacian matrix $L$ for r-nearest neighbor torus $T_n^r$ can be expressed as

\[
\lambda_{j_1,j_2} \left(L(T_{k_1,k_2}^r)\right) = 4r + 2 - \frac{\sin \left(\frac{(2r+1)\pi j_1}{k_1}\right)}{\sin \frac{\pi j_1}{k_1}} - \frac{\sin \left(\frac{(2r+1)\pi j_2}{k_2}\right)}{\sin \frac{\pi j_2}{k_2}}
\]

where $j_1 = 0, 1, 2, \ldots (k_1 - 1)$, $j_2 = 0, 1, 2, \ldots (k_2 - 1)$.

**Proof:** $T_n^r$ can be represented by Cartesian product of two $r$-nearest neighbor cycles. So from the Lemma 2, we can write the $\lambda_{j_1,j_2} \left(L(T_{k_1,k_2}^r)\right)$ as

\[
\lambda_{j_1,j_2} \left(L(T_{k_1,k_2}^r)\right) = \lambda_{j_1} \left(L(C_{k_1}^r)\right) + \lambda_{j_2} \left(L(C_{k_2}^r)\right)
\]

From (11), we can write the expressions for $\lambda_{j_1} \left(L(C_{k_1}^r)\right)$ and $\lambda_{j_2} \left(L(C_{k_2}^r)\right)$, substituting them in (21) proves the theorem.

**Theorem 3:** The generalized expression for eigenvalues of Laplacian matrix $L$ for $m$-dimensional $r$-nearest neighbor torus can be expressed as

\[
\lambda_{j_1,j_2,\ldots,j_m} \left(L(T_{k_1,k_2,\ldots,k_m}^r)\right) = (2r + 1)n - \sum_{i=1}^{m} \left(\frac{\sin \left(\frac{(2r+1)\pi j_i}{k_i}\right)}{\sin \frac{\pi j_i}{k_i}}\right)
\]

**Proof:** r-nearest neighbor $m$-dimensional torus can be represented by Cartesian product of $m$ number of r-nearest neighbor cycles. So from the Lemma 2, we can write the $\lambda_{j_1,j_2,\ldots,j_m} \left(L(T_{k_1,k_2,\ldots,k_m}^r)\right)$ as,

\[
\lambda_{j_1,j_2,\ldots,j_m} \left(L(T_{k_1,k_2,\ldots,k_m}^r)\right) = \lambda_{j_1} \left(L(C_{k_1}^r)\right) + \lambda_{j_2} \left(L(C_{k_2}^r)\right) + \ldots + \lambda_m \left(L(C_{k_m}^r)\right)
\]

From (11), we can substitute the expressions for $\lambda_{j_1} \left(L(C_{k_1}^r)\right)$, $\lambda_{j_2} \left(L(C_{k_2}^r)\right)$ and $\lambda_m \left(L(C_{k_m}^r)\right)$ in (23), which proves the theorem.

IV. NETWORK CRITICALITY ANALYSIS FOR r-NEAREST NEIGHBOR NETWORKS

**Theorem 5:** The network criticality of r-nearest neighbor cycle $C_n^r$ between every arbitrary pair of nodes is
\[ \tau(C_n^r) = \sum_{j=2}^{n} \left( \frac{2n}{(2r+1) \sin \left( \frac{(2r+1)\pi j}{n} \right)} \right) \]  

(24)

Proof: From (11), we can write

\[ Tr(L^+) = \sum_{j=2}^{n} \frac{1}{x_j^r(L_j)} \]

(25)

Substituting the (25) in (4) proves the theorem.

**Theorem 6:** The network criticality of \( r \)-nearest neighbor torus \( T_{k_1,k_2} \) between every arbitrary pair of nodes is

\[ \tau(T_{k_1,k_2}) = \sum_{j=1}^{k_1-1} \sum_{j_2=0}^{k_2-1} \left( \frac{2(k_1+k_2)}{\sin \left( \frac{(2r+1)\pi j}{n} \right) \sin \left( \frac{(2r+1)\pi j_2}{n} \right)} \right) \]

(26)

Proof: From (20), we can write

\[ Tr(L^+) = \sum_{j=1}^{k_1-1} \sum_{j_2=0}^{k_2-1} \frac{1}{x_j^r(L_j)} \]

(27)

Substituting the (27) in (4) proves the theorem.

**Theorem 7:** The network criticality of \( r \)-nearest neighbor torus \( T_{k_1,k_2,...,k_m} \) between every arbitrary pair of nodes is

\[ \tau(T_{k_1,k_2,...,k_m}) = \sum_{j=1}^{k_1-1} \sum_{j_2=0}^{k_2-1} \cdots \sum_{j_m=0}^{k_m-1} \left( \frac{2}{\sin \left( \frac{(2r+1)\pi j}{n} \right) \sin \left( \frac{(2r+1)\pi j_2}{n} \right) \cdots \sin \left( \frac{(2r+1)\pi j_m}{n} \right)} \right) \]

(28)

Proof: From (22), we can write

\[ Tr(L^+) = \sum_{j_1=1}^{k_1-1} \sum_{j_2=0}^{k_2-1} \cdots \sum_{j_m=0}^{k_m-1} \left( \frac{1}{x_{j_1,j_2,...,j_m}} \right) \]

(29)

Substituting the (29) in (3) proves the theorem.

**Theorem 9:** The normalized network criticality of \( r \)-nearest neighbor cycle \( C_n^r \) between every arbitrary pair of nodes is

\[ \hat{\tau}(C_n^r) = \sum_{j=2}^{n} \left( \frac{2}{(n-1) \sin \left( \frac{(2r+1)\pi j}{n} \right)} \right) \]

(30)

Proof: Substituting the (29) in (5) proves the theorem.

**Theorem 10:** The normalized network criticality of \( r \)-nearest neighbor torus \( T_{k_1,k_2} \) between every arbitrary pair of nodes is

\[ \hat{\tau}(T_{k_1,k_2}) = \sum_{j_1=1}^{k_1-1} \sum_{j_2=0}^{k_2-1} \cdots \sum_{j_m=0}^{k_m-1} \left( \frac{2}{\sin \left( \frac{(2r+1)\pi j}{n} \right) \sin \left( \frac{(2r+1)\pi j_2}{n} \right) \cdots \sin \left( \frac{(2r+1)\pi j_m}{n} \right)} \right) \]

(31)

Proof: Substituting the (29) in (5) proves the theorem.

**Theorem 11:** The normalized network criticality of \( r \)-nearest neighbor torus \( T_{k_1,k_2,...,k_m} \) between every arbitrary pair of nodes is

\[ \hat{\tau}(T_{k_1,k_2,...,k_m}) = \sum_{j_1=1}^{k_1-1} \sum_{j_2=0}^{k_2-1} \cdots \sum_{j_m=0}^{k_m-1} \left( \frac{2}{\sin \left( \frac{(2r+1)\pi j_1}{n} \right) \sin \left( \frac{(2r+1)\pi j_2}{n} \right) \cdots \sin \left( \frac{(2r+1)\pi j_m}{n} \right)} \right) \]

(32)

V. SIMULATION RESULTS

We have done simulations using MATLAB. As shown in Fig. 4, for \( n=300 \), we have plotted network criticality versus nearest neighbors \( r \). Network criticality decreases with the nearest neighbors. From Fig. 5, we can observe that network criticality increases with \( n \), for \( r=1 \). To know the effect of network dimension \( m \) on \( \tau \), we have plotted the network criticality versus \( k_1 \) and \( k_2 \), for \( r=1 \). To know the effect of network dimension \( m \) on \( \tau \), we have plotted the Fig. 8, for \( k_1=16, k_2=18, k_3=20 \) and \( k_4=22 \). Fig. 8 also shows the effect of \( r \) on \( \tau \) for multi-dimensional networks. Similarly, as shown in the Fig. 9, Fig. 10, Fig. 11, Fig. 12 and Fig. 13, we plotted the normalized network criticality \( \hat{\tau} \) against number of nodes, nearest neighbors and network dimension for \( r \)-nearest neighbor networks. Both network criticality \( \tau \) and normalized network criticality \( \hat{\tau} \) decreases with \( r \) and \( m \) and increases with \( n \). As shown in the figures, the simulation results agreed with analytical results derived in the section IV.

VI. CONCLUSIONS

We have derived the analytical expressions for network criticality and normalized network criticality for \( r \)-nearest neighbor cycle, \( r \)-nearest neighbor torus and \( m \)-dimensional networks.
Fig. 5. Network Criticality versus number of nodes for $r$-nearest neighbor cycle

Fig. 6. Network Criticality versus Nearest Neighbors for $r$-nearest neighbor torus

Fig. 7. Network Criticality versus $k_1$ and $k_2$ for $r$-nearest neighbor torus

Fig. 8. Network Criticality versus Dimension for $r$-nearest neighbor torus

Fig. 9. Normalized Network Criticality versus Nearest Neighbors for $r$-nearest neighbor cycle

Fig. 10. Normalized Network Criticality versus number of nodes for $r$-nearest neighbor cycle

Fig. 11. Normalized Network Criticality versus Nearest Neighbors for $r$-nearest neighbor torus

Fig. 12. Normalized Network Criticality versus $k_1$ and $k_2$ for $r$-nearest neighbor torus

Fig. 13. Normalized Network Criticality versus Dimension for $r$-nearest neighbor torus

$r$-nearest neighbor networks to study the robustness properties of finite size wireless sensor networks. Our analytical formulas derived in this work agreed with simulation results, and observed that the both network criticality and normalized network criticality decreases with nearest neighbors and network dimension and increases with number of nodes. As the variable $r$ which represents nearest neighbors captures the node overhead and transmission radius, this work can be further extended for network robustness control and optimization problems.
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