Three Remarks on Carleson Measures for Dirichlet Space

GUOZHENG CHENG, XIANG FANG, ZIPENG WANG, AND JIAYANG YU

Abstract

In this paper we prove that all doubling measures on the unit disk $\mathbb{D}$ are Carleson measures for the standard Dirichlet space $\mathcal{D}$. The proof has three new ingredients. The first one is a characterization of Carleson measures which holds true for general reproducing kernel Hilbert spaces. The second one is another new equivalent condition for Carleson measures, which holds true only for the standard Dirichlet space. The third one is an application of dyadic method to our setting.

1 Introduction and the Main Result

In this paper we study Carleson measures on the standard Dirichlet space $\mathcal{D}$, which consists of analytic functions over the unit disk $\mathbb{D} \subset \mathbb{C}$ under the norm

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$ 

A positive measure $\mu$ on $\mathbb{D}$ is called a $\mathcal{D}$-Carleson measure if there exists a constant $C$ such that

$$\int_{\mathbb{D}} |f|^2 d\mu \leq C \|f\|_{\mathcal{D}}^2, \quad f \in \mathcal{D}.$$ 

This is certainly a well studied topic in operator theory with a long history. Carleson measures were originally introduced by L. Carleson in his 1962 solution of the corona problem, and have become one of the most cherished tools
in analysis. They have proved to be powerful in a great variety of problems. In case of the Hardy space or the Bergman space on the unit disk, Carleson measures are characterized by beautiful geometric conditions [16]. In contrast, $\mathcal{D}$-Carleson measures remain somehow elusive after several decades of study and their characterization is capacitary [6], [11], [14], hence hard to check. This makes the search for sufficient conditions meaningful. In particular, the recent work of El-Fallah-Kellay-Mashreghi-Ransford [7] contains an elegant sufficient condition for $\mathcal{D}$-Carleson measures, which is especially effective if armed with growth estimates near the boundary. This is a recurrent theme for $\mathcal{D}$-Carleson measures in the past, both for characterizations and for sufficient conditions. The purpose of this paper is to show that a large class of commonly used measures are all $\mathcal{D}$-Carleson.

Nowadays there exists an extensive literature on generalized Dirichlet spaces with generalized Carleson measures. Such aspects will not be considered in this paper. But we do keep this group of readers in mind and make things convenient for them when it is at not much extra cost. (See the remark before Lemma 9.)

A measure $\mu$ on the unit disk $\mathbb{D}$ is called a doubling measure if there exists a constant $C$ such that

$$\mu(B(z, 2r)) \leq C \mu(B(z, r))$$

for all $z \in \mathbb{D}$ and $r > 0$, where $B(z, r) = \{ w \in \mathbb{D} : |z - w| < r \}$. Doubling measures arise naturally in various ways in analysis and geometry and there is an industry building on them [12]. We restrict our attention to those which are absolutely continuous with respect to the Lebesgue measure and we call them weights.

**Theorem 1.** If $\sigma$ is a finite reverse doubling measure that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{D}$, then it is a $\mathcal{D}$-Carleson measure.

Then [9] (Lemma 6) implies the claimed result for doubling measures. For the definition of reverse doubling, see Definition 8. Our arguments can actually prove that a larger class of measures are $\mathcal{D}$-Carleson. But stating the result in its full strength will involve contrive technicality, hence undesirable, although our proof might attract some to probe. Our approach to $\mathcal{D}$-Carleson measures is indeed quite different from the past.
2 The Proof

First we give a general characterization of Carleson measures on a reproducing kernel Hilbert space in terms of the reproducing kernel. This is a small but necessary extension of a result of [3].

Lemma 2. Let \( \mu \) be a positive measure on \( \mathbb{D} \). Suppose that \( H \) is a Hilbert space of analytic functions over \( \mathbb{D} \) with a reproducing kernel \( k(z,w) \), such that for any \( z \in \mathbb{D} \), the function \( k(z,\cdot) \) is continuous on \( \overline{\mathbb{D}} \). Let

\[
T_{k,\mu}f(z) = \int_{\mathbb{D}} f(w)k(z,w)d\mu(w), \quad f \in L^2(\mathbb{D},\mu),
\]

and

\[
T_{\text{Re}(k),\mu}f(z) = \int_{\mathbb{D}} f(w)\text{Re}(k)(z,w)d\mu(w), \quad f \in L^2(\mathbb{D},\mu),
\]

where \( \text{Re}(k) \) denotes the real part of \( k \). Then the following conditions are equivalent:

(a) \( T_{\text{Re}(k),\mu} : L^2(\mathbb{D},\mu) \to L^2(\mathbb{D},\mu) \) is bounded.

(b) \( T_{k,\mu} : L^2(\mathbb{D},\mu) \to L^2(\mathbb{D},\mu) \) is bounded.

(c) \( \mu \) is an \( H \)-Carleson measure. That is, there exists a constant \( c \) such that

\[
\|f\|_{L^2(\mathbb{D},\mu)} \leq c\|f\|_H \quad \text{for all} \quad f \in H.
\]

Moreover, in the above cases, we have

\[
\|T_{\text{Re}(k),\mu}\|_{L^2(\mathbb{D},\mu)\to L^2(\mathbb{D},\mu)} \leq \|T_{k,\mu}\|_{L^2(\mathbb{D},\mu)\to L^2(\mathbb{D},\mu)} \leq 2\|T_{\text{Re}(k),\mu}\|_{L^2(\mathbb{D},\mu)\to L^2(\mathbb{D},\mu)}.
\]

Proof. The equivalence between (a) and (c) was proved in [3]. The equivalence between (a) and (b) follows from the three lemmas below.

Lemma 3. Let \( \mu \) be a positive measure over \( \mathbb{D} \). Suppose that \( T \) is a linear operator (not necessarily bounded) from \( L^2(\mathbb{D},\mu) \) to some Hilbert space \( H \). If there exists a constant \( c \) such that

\[
\|Tf\|_H \leq c\|f\|_{L^2(\mathbb{D},\mu)}
\]

for all real-valued functions \( f \) in \( L^2(\mathbb{D},\mu) \), then \( T \) is bounded. Moreover,

\[
\|T\|_{L^2(\mathbb{D},\mu)\to H} \leq \sqrt{2}c.
\]
Proof. The proof is elementary and we include it for completeness. For \( f \in L^2(\mathbb{D}, d\mu) \), decompose it into real and imaginary parts as \( f = f_1 + if_2 \). Then

\[
|\langle T f, T f \rangle_H| \leq |\langle T f_1, T f_1 \rangle_H| + |\langle T f_2, T f_2 \rangle_H| + 2 |\langle T f_1, T f_2 \rangle_H|
\]

\[
\leq c^2 \| f_1 \|^2_{L^2(\mathbb{D}, \mu)} + c^2 \| f_2 \|^2_{L^2(\mathbb{D}, \mu)} + 2c^2 \| f_1 \|_{L^2(\mathbb{D}, \mu)} \| f_2 \|_{L^2(\mathbb{D}, \mu)}
\]

\[
\leq 2c^2 \| f \|^2_{L^2(\mathbb{D}, \mu)}.
\]

\[\square\]

Lemma 4. Let \( \mu \) be a positive measure over \( \mathbb{D} \) and \( k : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \) be a measurable function such that for any \( z \in \mathbb{D} \) the function \( k(z, \cdot) \) is continuous over \( \mathbb{D} \). If \( T_{k,\mu} \) is positive, not necessarily bounded, on \( L^2(\mathbb{D}, \mu) \), then \( T_{k,\mu} \) is bounded on \( L^2(\mathbb{D}, \mu) \) if and only if \( T_{Re(k),\mu} \) is bounded. When \( T_{Re(k),\mu} \) is bounded, we have

\[
\| T_{Re(k),\mu} \|_{L^2(\mathbb{D}, \mu) \to L^2(\mathbb{D}, \mu)} \leq \| T_{k,\mu} \|_{L^2(\mathbb{D}, \mu) \to L^2(\mathbb{D}, \mu)} \leq 2 \| T_{Re(k),\mu} \|_{L^2(\mathbb{D}, \mu) \to L^2(\mathbb{D}, \mu)}.
\]

Proof. Assume that \( T_{k,\mu} \) is bounded on \( L^2(\mathbb{D}, \mu) \). Let

\[
f \in L^2(\mathbb{D}, \mu)
\]

and

\[
g \in L^2(\mathbb{D}, \mu),
\]

we have

\[
\langle T_{Re(k),\mu} f, g \rangle_{L^2(\mathbb{D}, \mu)} = \frac{1}{2} \langle T_{k,\mu} f, g \rangle_{L^2(\mathbb{D}, \mu)} + \frac{1}{2} \overline{\langle T_{k,\mu} \overline{f}, \overline{g} \rangle_{L^2(\mathbb{D}, \mu)}}.
\]

Thus

\[
\| T_{Re(k),\mu} \|_{L^2(\mathbb{D}, \mu) \to L^2(\mathbb{D}, \mu)} \leq \| T_{k,\mu} \|_{L^2(\mathbb{D}, \mu) \to L^2(\mathbb{D}, \mu)}.
\]

Let

\[
f \in L^2(\mathbb{D}, \mu)
\]

be a real-valued function. Then

\[
\langle T_{Re(k),\mu} f, f \rangle_{L^2(\mathbb{D}, \mu)} = \langle T_{k,\mu} f, f \rangle_{L^2(\mathbb{D}, \mu)}.
\]

If \( T_{Re(k),\mu} \) is bounded on \( L^2(\mathbb{D}, \mu) \), then there is a constant \( c \) such that for any real-valued function

\[
f \in L^2(\mathbb{D}, \mu),
\]

4
we have
\[ \|T_{k,\mu} f\|_{L^2(\mathbb{D},\mu)} \leq c \|f\|_{L^2(\mathbb{D},\mu)}. \]

By Lemma 3, \( T_{k,\mu} \) is bounded on \( L^2(\mathbb{D},\mu) \). So, for the rest of the proof, it is sufficient to show that
\[ \|T_{k,\mu}\|_{L^2(\mathbb{D},\mu)\rightarrow L^2(\mathbb{D},\mu)} \leq 2\|T_{\text{Re}(k),\mu}\|_{L^2(\mathbb{D},\mu)\rightarrow L^2(\mathbb{D},\mu)}. \]

Since \( T_{k,\mu} \) is bounded and positive, we have
\[
\|T_{k,\mu}\|_{L^2(\mathbb{D},\mu)\rightarrow L^2(\mathbb{D},\mu)} = \sup \left\{ \|T_{\text{Re}(k),\mu} f\|^2_{L^2(\mathbb{D},\mu)} : \|f\|_{L^2(\mathbb{D},\mu)} \leq 1 \right\} 
\leq 2 \sup \left\{ \|T_{\text{Re}(k),\mu} f\|^2_{L^2(\mathbb{D},\mu)} : f \text{ is real-valued and } \|f\|_{L^2(\mathbb{D},\mu)} \leq 1 \right\} 
= 2 \sup \left\{ \langle T_{\text{Re}(k),\mu} f, f \rangle_{L^2(\mathbb{D},\mu)} : f \text{ is real-valued and } \|f\|_{L^2(\mathbb{D},\mu)} \leq 1 \right\} 
\leq 2\|T_{\text{Re}(k),\mu}\|_{L^2(\mathbb{D},\mu)\rightarrow L^2(\mathbb{D},\mu)}. \]

\[ \square \]

**Lemma 5.** If \( k(z, w) \) is a reproducing kernel for some Hilbert space \( H \) of functions over \( \mathbb{D} \) such that for any \( z \in \mathbb{D} \) the function \( k(z, \cdot) \) is continuous over \( \mathbb{D} \) and \( T_{k,\mu} \) is defined as in Lemma 2 on \( L^2(\mathbb{D},\mu) \), then \( T_{k,\mu} \) is positive.

**Proof.** For
\[ f \in L^2(\mathbb{D},\mu), \]
we have
\[
\langle T_{k,\mu} f , f \rangle_{L^2(\mathbb{D},\mu)} = \int_{\mathbb{D}} \int_{\mathbb{D}} k(z, w) f(w) d\mu(w) \overline{f(z)} d\mu(z)
= \int_{\mathbb{D}} \int_{\mathbb{D}} \left\langle k(\cdot, w), k(\cdot, z) \right\rangle_H f(w) d\mu(w) \overline{f(z)} d\mu(z)
= \left\langle \int_{\mathbb{D}} k(u, w) f(w) d\mu(w), \int_{\mathbb{D}} k(u, z) f(z) d\mu(z) \right\rangle_H
= \langle T_{k,\mu} f , T_{k,\mu} f \rangle_H
\geq 0. \]

\[ \square \]
The rest of the proof of Lemma 2 follows from [3]. For completeness, we sketch a short, slightly different proof. We overlook a needed density argument to simplify the presentation.

Proof of Part \((b) \Rightarrow \text{Part } (c)\) in Lemma 2. Let \(I\) be the identity map from \(H \rightarrow L^2(\mathbb{D}, \mu)\), i.e.,

\[
I(f) = f, \quad f \in H.
\]

Since \(H\) is spanned by functions of the form \(k(\cdot, w), (w \in \mathbb{D})\), \(I\) is densely defined. Moreover, \(I\) is clearly closed, so the adjoint \(I^*\) is also a densely defined, closed operator. Formally, \(I^*\) is given by

\[
I^*f(z) = \int_{\mathbb{D}} f(w)k(z, w)d\mu(w).
\]

Indeed, for \(f \in \text{Dom}(I)\) with \(I^*f \in H\), we have, by the reproducing property,

\[
I^*f(z) = \langle I^*f, k_z \rangle_H = \langle f, k_z \rangle_{L^2(\mathbb{D}, \mu)} = \int_{\mathbb{D}} f(w)k(z, w)d\mu(w).
\]

Then

\[
\|I^*f\|_H^2 = \langle I^*f, I^*f \rangle_H = \left\langle \int_{\mathbb{D}} f(w)k(z, w)d\mu(w), \int_{\mathbb{D}} f(u)k(z, u)d\mu(w) \right\rangle_H = \int_{\mathbb{D}} \int_{\mathbb{D}} k(u,w)f(w)d\mu(w)f(u)d\mu(u) = \langle T_{k,\mu}f, f \rangle_{L^2(\mathbb{D}, \mu)}.
\]

(1)

This implies \((b) \Rightarrow (c)\).

Proof of Part \((c) \Rightarrow \text{Part } (a)\) in Lemma 2. It suffices to consider the restriction of \(T_{\text{Re}(k), \mu}\) to real-valued functions in \(L^2(\mathbb{D}, \mu)\) by Lemma 3. Since

\[
k(z, w) = k(w, z),
\]
$T_{Re(k),\mu}$ is a symmetric operator on $L^2(\mathbb{D}, \mu)$. Condition (c) means that $I^*$ is bounded, it follows from (1) that

$$\text{Dom}(T_{Re(k),\mu}) = L^2(\mathbb{D}, \mu).$$

Therefore, $T_{Re(k),\mu}$ is bounded.

Now we come to a simple yet pivotal point in the proof. The next lemma is easily proved by direct calculation, but its discovery is a fortunate coincidence since it requires us to explicitly factor a given kernel into the convolution of two. This is usually impossible. We can make it work only for the standard Dirichlet space, and generalizing it to other spaces may require different ideas.

**Lemma 6.** Suppose that $\mu$ is a positive measure on $\mathbb{D}$. It is a $\mathcal{D}$-Carleson measure if and only if $K_1 : L^2(\mathbb{D}) \to L^2(\mathbb{D}, \mu)$ is bounded, where

$$K_1 f(z) = \int_{\mathbb{D}} \frac{f(w)}{1 - \overline{z}w} dA(w).$$

Before the proof we observe a result to reduce $L^p(\mathbb{D})$, the domain of $K_1$, to $L^p(\mathbb{D})$ via the Bergman projection $P$ for any $p > 1$. The proof is skipped.

**Lemma 7.** If $1 < p < \infty$, then $K_1 f = K_1(P f)$ for all $f \in L^p(\mathbb{D})$.

**Proof of Lemma 6.** Let $\tilde{K}_1$ be the restriction of $K_1$ onto $L^2_0(\mathbb{D})$. By Lemma 7,

$$K_1 : L^2(\mathbb{D}) \to L^2(\mathbb{D}, \mu)$$

is bounded if and only if

$$\tilde{K}_1 : L^2_0(\mathbb{D}) \to L^2(\mathbb{D}, \mu)$$

is bounded if and only if

$$\tilde{K}_1 \tilde{K}_1^* : L^2(\mathbb{D}, \mu) \to L^2(\mathbb{D}, \mu)$$

is bounded. With direct calculations,

$$\tilde{K}_1 \tilde{K}_1^* f(z) = \int_{\mathbb{D}} f(w) k_D(z, w) d\mu(w),$$

7
where
\[
k_D(z, w) = \frac{1}{zw} \log \frac{1}{1-z\overline{w}}.
\]

So Lemma 2 completes the proof.

Now the hard work begins. Our target is

\[K_1 : L^2(\mathbb{D}) \to L^2(\mathbb{D}, \sigma),\]

which is a special case of the notorious two weight problem, whose full resolution seems out of reach. But we manage to get what is sufficient to resolve our problem. Indeed our arguments can prove much more, although we state our result only for the sleekest case (Theorem 1). On the other hand, although the techniques we use are not new in harmonic analysis, the way they are modified (from [13]) and applied to operator theory in this paper should be applicable to some other problems in operator theory.

**Definition 8.** [9] A measure \( \sigma \) on the unit disk has the reverse doubling property if there is a constant \( \delta < 1 \) such that

\[
\frac{|B_I|_\sigma}{|Q_I|_\sigma} < \delta,
\]

for any interval \( I \subset \mathbb{T} \). Here

\[Q_I = \{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I \}\]

and

\[B_I = \{ z \in \mathbb{D} : 1 - \frac{|I|}{2} < |z| < 1, \frac{z}{|z|} \in I \}.
\]

Starting now we present several lemmas with general parameters \((p, q)\) instead of mere \((2, 2)\). There are two reasons to do this. First of all, this should appeal to those interested in the (rather large) literature of generalized Dirichlet spaces and it is indeed at almost no extra cost for us. Second, some—including us—might find \((p, q)\) arguments to be more illustrative and they won’t really complicate the reading of any serious reader.
Lemma 9. Let $\alpha > 0$ and $1 < p \leq q < \infty$. Let $\mu$ and $\nu$ be weights on $\mathbb{D}$ such that $\mu^{1-p'}$ and $\nu$ have the reverse doubling property. If

$$
\sup_{I \subset \mathbb{T}} \frac{|Q_I|^{\frac{1}{p}}|Q_I|^{\frac{p}{\mu_{1-p'}}}}{|Q_I|^{\frac{1}{q}}} < \infty,
$$

then there exists a constant $c$ such that for $f \in L^p(\mathbb{D}, \mu)$,

$$
\left( \int_{\mathbb{D}} |K_\alpha(f)|^q \nu(z) dA(z) \right)^{\frac{1}{q}} \leq c \left( \int_{\mathbb{D}} |f(z)|^p \mu(z) dA(z) \right)^{\frac{1}{p}}.
$$

Here $p'$ is the conjugate index, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$
K_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^\alpha} dA(w).
$$

Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Consider the following two dyadic grids on $\mathbb{T}$,

$$
\mathcal{D}^0 = \left\{ \left[ \frac{2\pi m}{2^j}, \frac{2\pi (m+1)}{2^j} \right) : m \in \mathbb{Z}_+, j \in \mathbb{Z}_+, 0 \leq m < 2^j \right\}
$$

and

$$
\mathcal{D}^\frac{1}{3} = \left\{ \left[ \frac{2\pi m}{2^j} + \frac{2\pi}{3}, \frac{2\pi (m+1)}{2^j} + \frac{2\pi}{3} \right) : m \in \mathbb{Z}_+, j \in \mathbb{Z}_+, 0 \leq m < 2^j \right\}.
$$

For each $\beta \in \{0, \frac{1}{3}\}$, let $Q^{\beta}$ denote the collection of Carleson boxes $Q_I$ with $I \in \mathcal{D}^\beta$ and we call $Q^{\beta}$ a Carleson box system over $\mathbb{D}$.

The first appearance of shifted dyadic grids in print is probably on page 30 of [5]. A quick way to appreciate why shifted dyadic grids are powerful is to look at [1], [8] and [10]. In particular, [1] contains a nice application to Sarason’s problem on Toeplitz products.

**Lemma 10.** [10] Let $J \subset \mathbb{T}$ be an interval. Then there exists an interval

$$
L \in \mathcal{D}^0 \cup \mathcal{D}^\frac{1}{3}
$$

such that

$$
J \subset L \text{ and } |L| \leq 6|J|.
$$
The proof of the next lemma will be skipped.

**Lemma 11.** There is a positive constant $c$ such that for any $z, w \in \mathbb{D}$, there exists a Carleson box $Q_I$ such that $z, w \in Q_I$ and

$$\frac{1}{c}|Q_I|^\frac{3}{2} \leq |1 - zw| \leq c|Q_I|^\frac{3}{2}.$$

By Lemma 10 and Lemma 11, there is a constant $c$ such that for any $z, w \in \mathbb{D}$, we can find an $L \in \mathcal{D}^0 \cup \mathcal{D}^\frac{1}{3}$ such that

$$\frac{1}{|1 - zw|^{\alpha}} \leq \frac{c\chi_{Q_L}(z)\chi_{Q_L}(w)}{|Q_L|^\frac{3}{2}}.$$  \(2\)

For each $\beta \in \{0, \frac{1}{3}\}$, we define

$$K_\alpha^\beta f(z) = \sum_{I \in \mathcal{D}^\beta} \int_{\mathbb{D}} \frac{f(w)\chi_{Q_I}(w)}{|Q_I|^\frac{3}{2}}dA(w)\chi_{Q_I}(z).$$

By (2), for any positive function $f$ on $\mathbb{D}$, we have

$$|K_\alpha f(z)| \leq c(K_\alpha^0 f(z) + K_\alpha^\frac{1}{3} f(z))$$

for any $z \in \mathbb{D}$. It easily follows that

**Lemma 12.** Let $\sigma$ and $\omega$ be two weights on $\mathbb{D}$ and $1 \leq p, q \leq \infty$. If both $K_\alpha^0$ and $K_\alpha^{\frac{1}{3}}$ are bounded from $L^p(\mathbb{D}, \sigma)$ into $L^q(\mathbb{D}, \omega)$, then

$$K_\alpha : L^p(\mathbb{D}, \sigma) \rightarrow L^q(\mathbb{D}, \omega)$$

is bounded.

Now we come to another key technical point. Namely, we prove an off-diagonal Carleson embedding theorem over the unit disk. A good entry point to this area of techniques is [13] which contains various Carleson embedding theorems of diagonal type over the Euclidean space $\mathbb{R}^n$. For operator
theorists, it is probably desirable in general to see how to adopt those (rich) techniques on $\mathbb{R}^n$ to the study of analytic function spaces on the unit disk $\mathbb{D}$.

Let $\sigma$ be a weight on $\mathbb{D}$ and $\beta \in \{0, \frac{1}{3}\}$. Let $\mathcal{Q}^\beta$ be a dyadic Carleson system over $\mathbb{D}$. For any $Q_I \in \mathcal{Q}^\beta$, let

$$E_{Q_I}^\sigma f = \frac{1}{|Q_I|_\sigma} \int_{Q_I} f(z)\sigma(z)dA(z).$$

Then a tree mapping on the dyadic Carleson system $\mathcal{Q}^\beta$ is given by

$$\Lambda_\beta : f \mapsto E_{Q_I}^\sigma f$$

for $f \in L^1(\mathbb{D}, \sigma)$.

Next, we endow $\mathcal{Q}^\beta, \beta \in \{0, \frac{1}{3}\}$, with a measurable space structure. For any $t \in \mathbb{R}$ and $Q_I \in \mathcal{Q}^\beta$, let

$$a_t(Q_I) = |Q_I|_\sigma^t.$$

Then $\{\mathcal{Q}^\beta, a_t(\cdot)\}$ is a measurable space. Furthermore, for $1 \leq p < \infty$,

$$f \in L^p(\mathcal{Q}^\beta, a_t(\cdot))$$

means

$$\|f\|_{L^p(\mathcal{Q}^\beta, a_t(\cdot))} = \left( \sum_{Q_I \in \mathcal{Q}^\beta} a_t(Q_I) |f(Q_I)|^p \right)^{\frac{1}{p}} < \infty.$$

Moreover, $f \in L^{p, \infty}(\mathcal{Q}^\beta, a_t(\cdot))$ means

$$\|f\|_{L^{p, \infty}(\mathcal{Q}^\beta, a_t(\cdot))} = \sup_{\lambda > 0} \left\{ \lambda \left[ \sum_{Q_I \in \mathcal{Q}_\lambda} a_t(Q_I) \right]^{\frac{1}{p}} \right\} < \infty,$$

where

$$\mathcal{Q}_\lambda = \{Q_I \in \mathcal{Q}^\beta : |f(Q_I)| > \lambda\}.$$
Lemma 13. Let $1 < p \leq q < \infty$, $t = \frac{q}{p}$ and $\sigma$ be a weight on $\mathbb{D}$. Let $Q^\beta$ be a dyadic Carleson system on $\mathbb{D}$, $\beta \in \{0, \frac{1}{3}\}$. Let
\[
\Lambda_\beta : f \mapsto E^\sigma_Q f
\]
be the tree mapping on $Q^\beta$. If there is a constant $c_1$ such that for any $K \in D^\beta$,
\[
\sum_{Q_I \in Q^\beta : Q_I \subset Q_K} |Q_I|_\sigma^t \leq c_1 |Q_K|_\sigma^t,
\]
then the tree mapping $\Lambda_\beta$ is bounded from $L^p(\mathbb{D}, \sigma)$ into $L^q(Q^\beta, a_t(\cdot))$. That is, there is a constant $c_2$ such that
\[
\left( \sum_{Q_I \in Q^\beta} |Q_I|_\sigma^t (E^\sigma_Q f)^q \right)^\frac{1}{q} \leq c_2 \left( \int_{\mathbb{D}} |f(z)|^p \sigma(z) dA(z) \right)^\frac{1}{p}, \quad f \in L^p(\mathbb{D}, \sigma).
\]

In order to prove Lemma 13, we need the Marcinkiewicz interpolation theorem which we recall for the convenience of the readers.

Lemma 14. ([4], Page 6) Let $0 < p_0, q_0, p_1, q_1 \leq \infty$ and $p_0 \neq p_1$. Let $(X, \sigma)$ and $(Y, \omega)$ be measurable spaces. Let $T$ be a linear operator such that
(a) $T : L^{p_0}(X, \sigma) \to L^{q_0, \infty}(Y, \omega)$ is bounded with norm $c_3$;
(b) $T : L^{p_1}(X, \sigma) \to L^{q_1, \infty}(Y, \omega)$ is bounded with norm $c_4$.

Set
\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}
\]
and
\[
\frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}
\]
for some $0 \leq \theta \leq 1$. If $p \leq q$, then
\[
T : L^p(X, \sigma) \to L^q(Y, \omega)
\]
is bounded with norm $c_5 = c(c_3, c_4, \theta)$. 

12
Proof of Lemma 13. Let 
\[ \beta \in \{0, \frac{1}{3}\} \]
be fixed. First, the tree mapping
\[ \Lambda_\beta : f \to E_{Q_I}^\sigma f = \frac{1}{|Q_I|_\sigma} \int_{Q_I} f(z)\sigma(z)dA(z) \]
is bounded from \( L^\infty(\mathbb{D}, \sigma) \) into \( L^\infty(Q^\beta, a_t(\cdot)) \) with norm \( c_3 \leq 1 \).

Claim: \( \Lambda_\beta : L^1(\mathbb{D}, \sigma) \to L^{t,\infty}(Q^\beta, a_t(\cdot)) \) is bounded.

If we can verify this claim, then let
\[ \theta = \frac{1}{p} \]
and it follows that
\[ \Lambda_\beta : L^p(\mathbb{D}, \sigma) \to L^q(Q^\beta, a_t(\cdot)) \]
is bounded which will complete the proof of Lemma 13.

Proof of Claim: Let \( f > 0 \) and \( \lambda > 0 \). Let
\[ Q_\lambda = \{Q_I \in Q^\beta : |E_{Q_I}^\sigma f| > \lambda\}. \]

Let
\[ \{Q_{I_j} : j \in \Gamma\} \]
be the collection of maximal (with respect to inclusion) Carleson boxes in \( Q_\lambda \) for some index set \( \Gamma \). Then
\[ \sum_{Q_I \in Q_\lambda} a_t(Q_I) = \sum_{j \in \Gamma} \sum_{Q_I \subset Q_{I_j}} |Q_I|_\sigma^t \]
\[ \leq c_1 \sum_{j \in \Gamma} |Q_{I_j}|_\sigma^t \]
\[ \leq c_1 \sum_{j \in \Gamma} |Q_{I_j}|_\sigma^t. \]
Now the proof is complete by observing
\[
\sum_{j \in \Gamma} |Q_I|_\sigma \leq \frac{1}{\lambda} \sum_{j \in \Gamma} \int_{Q_I} f(z) \sigma(z) dA(z) \\
\leq \frac{1}{\lambda} \int_D f(z) \sigma(z) dA(z).
\]

Next, we need another result on the reverse doubling property from \cite{9} (Lemma 7).

**Lemma 15.** Let $\beta \in \{0, \frac{1}{3}\}$. Let $D^\beta$ be a dyadic grid of $T$, and $Q^\beta$ be a dyadic Carleson system over $D$. If a weight $\sigma$ has the reverse doubling property, then there is a constant $c$ such that for any $K \in D^\beta$
\[
\sum_{Q_I \in Q^\beta : Q_I \subset Q_K} |Q_I|_\sigma \leq c |Q_K|_\sigma.
\tag{3}
\]

(3) is known as the Carleson embedding condition. Combining with the result in \cite{2}, one may conjecture that this condition a necessary and sufficient condition for Carleson measures on $D$.

Combining Lemma 13 and Lemma 15, we have

**Corollary 16.** Let $1 < p \leq q < \infty$, $t = \frac{q}{p}$ and $\sigma$ be a weight with the reverse doubling property. Then there is a constant $c$ such that
\[
\left( \sum_{Q_I \in Q^\beta} |Q_I|_\sigma \left( \mathcal{E}_{Q_I} f \right)^q \right)^{\frac{1}{q}} \leq c \left( \int_D |f(z)|^p \sigma(z) dA(z) \right)^{\frac{1}{p}}.
\]

Next, we prove an inequality for $K^\alpha_{\beta}$ whose formulation is of independent interests, although the proof is more or less standard now.

**Lemma 17.** Let $1 < p \leq q < \infty$. Let $\mu$ and $\nu$ be weights on $D$ such that $\mu^{1-p'}$ and $\nu$ have the reverse doubling property. If
\[
\sup_{I \subset \Gamma} \frac{|Q_I|_{\nu}^{\frac{1}{p}} |Q_I|_{\mu^{1-p'}}^{\frac{1}{p'}} < \infty,
\]

\]

14
then there exists a constant $c_1$ such that for $f \in L^p(\mathbb{D}, \mu)$,
\[
\left( \int_{\mathbb{D}} |K_\alpha^\beta(f)|^q \nu(z) dA(z) \right)^{\frac{1}{q}} \leq c_1 \left( \int_{\mathbb{D}} |f(z)|^p \mu(z) dA(z) \right)^{\frac{1}{p}}.
\]

Proof. It is sufficient to show that there is a constant $c_1$ such that
\[
\left( \int_{\mathbb{D}} |K_\alpha^\beta(f\mu^{1-p'})|^q \nu(z) dA(z) \right)^{\frac{1}{q}} \leq c_1 \left( \int_{\mathbb{D}} |f(z)|^p \mu(z) dA(z) \right)^{\frac{1}{p}}.
\] (4)

Then Lemma 17 holds by replacing $f$ in (4) with $f\mu^{p'-1}$.

Let
\[
f \in L^p(\mathbb{D}, \mu^{1-p'})
\]
and
\[
g \in L^{q'}(\mathbb{D}, \nu).
\]

Let
\[
c_2 = \sup_{I \in \mathcal{D}} \frac{|Q_I|^\frac{1}{p'} |Q_I|^{\frac{1}{p}}}{|Q_I|^\frac{1}{2}}.
\]

Then
\[
|\langle K_\alpha^\beta(f\mu^{1-p'}), g \rangle_{L^2(\nu)}| = \left| \sum_{Q_I \in Q^\delta} \frac{1}{|Q_I|^\frac{1}{2}} \int_{Q_I} f(z) \mu^{1-p'}(z) dA(z) \int_{Q_I} g(z) \nu(z) dA(z) \right|
\]
\[
\leq c_2 \sum_{Q_I \in Q^\delta} \frac{1}{|Q_I|^\frac{1}{2}} \frac{1}{|Q_I|^{\frac{1}{p}}} \int_{Q_I} |f(z)| \mu^{1-p'}(z) dA(z)
\]
\[
\times \int_{Q_I} |g(z)| \nu(z) dA(z)
\]
\[
= c_2 \sum_{Q_I \in Q^\delta} |Q_I|^{\frac{1}{p} - \frac{1}{p'}} \frac{1}{|Q_I|^{\frac{1}{p}}} \int_{Q_I} |f(z)| \mu^{1-p'}(z) dA(z)
\]
\[
\times |Q_I|^{\frac{1}{p'}} \frac{1}{|Q_I|^{\frac{1}{p'}}} \int_{Q_I} |g(z)| \nu(z) dA(z)
\]
\[
\leq c_2 \left( \sum_{Q_I \in Q^\delta} |Q_I|^{\frac{2}{p} - \frac{1}{p'}} \left( \frac{1}{|Q_I|^{\frac{1}{p}}} \int_{Q_I} |f(z)| \mu^{1-p'}(z) dA(z) \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}
\]
\[
\times \left( \sum_{Q_I \in Q^\delta} |Q_I|^{\frac{1}{p'}} \left( \frac{1}{|Q_I|^{\frac{1}{p'}}} \int_{Q_I} |g(z)| \nu(z) dA(z) \right)^{\frac{1}{q}} \right)^{\frac{1}{q'}}.
\]
Since $\mu^{1-p'}$ and $\nu$ have the reverse doubling property, by the $p$-$q$ Carleson embedding (Corollary 16),

$$\left( \sum_{Q_I \in \mathcal{Q}^g} |Q_I|^{\frac{q}{q'}} \left( \frac{1}{|Q_I|^{\mu^{1-p'}}} \int_{Q_I} |f(z)|^{\mu^{1-p'}}(z)dA(z) \right)^{q'} \right)^{\frac{1}{q'}} \leq c_3 \left( \int_D |f(z)|^{p\mu^{1-p'}}(z)dA(z) \right)^{\frac{1}{p}}$$

and

$$\left( \sum_{Q_I \in \mathcal{Q}^g} |Q_I|^{\nu} \left( \frac{1}{|Q_I|^{\nu}} \int_{Q_I} |g(z)|^{\nu}(z)dA(z) \right)^{\frac{1}{q'}} \right)^{\frac{1}{q'}} \leq c_4 \left( \int_D |g(z)|^{q'\nu}(z)dA(z) \right)^{\frac{1}{q'}}.$$

Let

$$c_1 = c_2 c_3 c_4.$$

Then we have

$$|\langle K_\alpha^{\beta} f^{\mu^{1-p'}}, g \rangle|_{L^2(\nu)} \leq c_1 \left( \int_D |f(z)|^{p\mu^{1-p'}}(z)dA(z) \right)^{\frac{1}{p}} \left( \int_D |g(z)|^{q'\nu}(z)dA(z) \right)^{\frac{1}{q'}}.$$

The proof of Lemma 17 is complete now.

Now Lemma 9 follows from Lemma 12 and Lemma 17. Then Theorem 1 follows from Lemma 9 when applied to $K_1$ with $p = q = 2$ and $\mu$ being the Lebesgue measure.

**Acknowledgement**

G. Cheng is supported by NSFC (11471249), Zhejiang Provincial Natural Science Foundation of China (LY14A010021). X. Fang is supported by NSC of Taiwan (106-2115-M-008-001-MY2) and NSFC 11571248 during his visit to Soochow University in China. Z. Wang is supported by NSFC (11601296) and NSF of Shaanxi (2017JQ1008). J. Yu is supported by NSFC (11501384).

**References**

[1] A. Aleman, S. Pott and M. Reguera, Sarason conjecture on the Bergman space, Int. Math. Res. Notices. (IMRN), **2017**(2017), no. 14, 4320-4349.
[2] N. Arcozzi, R. Rochberg and E. Sawyer, Carleson measures for analytic Besov spaces, Rev. Mat. Iberoamericana 18 (2002), no. 2, 443-510.

[3] N. Arcozzi, R. Rochberg and E. Sawyer, Carleson measures for the Drury-Arveson Hardy space and other Besov-Sobolev spaces on complex balls, Adv. Math. 218 (2008), 1107-1180.

[4] J. Bergh and J. Lofstrom, Interpolation Spaces-An Introduction, Grundlehren der Mathematischen Wissenschaften, 223, Springer-Verlag, 1976.

[5] M. Christ, Weak Type (1, 1) Bounds for Rough Operators, Ann. Math. 128 (1988), 19-42.

[6] O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, A Primer on the Dirichlet Space, Cambridge Tracts in Mathematics, 203, Cambridge University Press, Cambridge, 2014.

[7] O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, One-box conditions for Carleson measures for the Dirichlet space, Proc. Amer. Math. Soc. 143 (2015), no. 2, 679-684.

[8] J. Garnett and P. Jones, BMO from dyadic BMO, Pacific J. Math. 2(1982), 351-371.

[9] X. Fang and Z. Wang, Two weight inequalities for the Bergman projection with doubling measures, Taiwanese J. Math. 19 (2015), no. 3, 919-926.

[10] T. Mei, BMO is the intersection of two translates of dyadic BMO, C. R. Acad. Sci. Paris, Ser. I. 336 (2003), 1003-1006.

[11] D. Stegenga, Multipliers on the Dirichlet space, Illinois J. Math. 24 (1980), no.1, 113-139.

[12] E. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Math. Ser., vol. 43, Princeton University Press, Princeton, NJ, 1993.

[13] X. Tolsa, Analytic Capacity, the Cauchy Transform, and Non-homogeneous Calderon-Zygmund Theory, Progress in Mathematics, 307, Birkhauser/Springer, 2014.
[14] Z. Wu, Carleson measures and multipliers for Dirichlet spaces, J. Funct. Anal. 169 (1999), no. 1, 148-163.

[15] Z. Wu, A new characterization for Carleson measures and some applications, Integr. Equ. Oper. Theory 71 (2011), 161-180.

[16] K. Zhu, Operator Theory in Function Spaces, Second edition, Mathematical Surveys and Monographs, 138, American Mathematical Society, Providence, RI, 2007.

G. Cheng, Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, P. R. China E-mail address: chgzh09@gmail.com

X. Fang, Department of Mathematics, National Central University, Chung-Li 32001, Taiwan; Email: xfang@math.ncu.edu.tw

Z. Wang, College of Mathematics and Information Sciences, Shaanxi Normal University, Xi’an 710062, P. R. China; Email: zipengwang@snnu.edu.cn

J. Yu, School of Mathematics, Sichuan University, Chengdu 610064, P. R. China; Email: jiayangyu@scu.edu.cn