THE K-THEORY OF TOEPLITZ C*-ALGEBRAS OF RIGHT-ANGLED ARTIN GROUPS

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Abstract. Toeplitz C*-algebras of right-angled Artin groups were studied by Crisp and Laca. They are a special case of the Toeplitz C*-algebras T(G, P) associated with quasi-lattice ordered groups (G, P) introduced by Nica. Crisp and Laca proved that the so called ”boundary quotients” C_Q^*(Γ) of C^*(Γ) are simple and purely infinite. For a certain class of finite graphs Γ we show that C_Q^*(Γ) can be represented as a full corner of a crossed product of an appropriate C* -subalgebra of C_Q^*(Γ) built by using C^*(Γ'), where Γ' is a subgraph of Γ with one less vertex, by the group \( \mathbb{Z} \). Using induction on the number of the vertices of Γ we show that C_Q^*(Γ) are nuclear and belong to the small bootstrap class. We also use the Pimsner-Voiculescu exact sequence to find their K-theory. Finally we use the Kirchberg-Phillips classification theorem to show that those C*-algebras are isomorphic to tensor products of \( \mathcal{O}_n \) with \( 1 \leq n \leq \infty \).

1. Introduction

Toeplitz C*-Algebras of right-angled Artin Groups generalize both the Toeplitz algebra and the Cuntz algebras. Coburn showed in [4] that the C*-algebra, generated by a single nonunitary isometry is unique, i.e. every two C*-algebras, each generated by a single nonunitary isometry are \( \ast \)-isomorphic. Similar uniqueness theorems about C*-algebras generated by isometries were proved by Cuntz [7], Douglas [10], Murphy [13], and others. Laca and Raeburn in [12] and Crisp and Laca in [5] proved such uniqueness theorems for a large class of C*-algebras, corresponding to quasi-lattice ordered groups (G, P). One of the key point they use was to project onto the ”diagonal” C* -algebra generated by the range projections of those isometries, an idea originating from [10].

These C*-algebras can be viewed as crossed products of commutative C*-algebras (the C*-algebras generated by the range projections of the isometries) by semigroups of endomorphisms. Crisp and Laca used techniques from [11] about such crossed products together with the uniqueness theorems mentioned above to prove a structure theorem for the universal C*-algebra C*(G, P) (which by the uniqueness theorems is isomorphic to the ”reduced one” T(G, P)) for a large class of quasi-lattice ordered groups (G, P). We will now state [6, Corollary 8.5] and [6, Theorem 6.7] and use them throughout this note. A graph will always mean a simple graph with countable set of vertices.

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Theorem 1.1 ([6], Theorem 6.7). Suppose that $\Gamma$ is a graph with a set of vertices $S$ (finite or infinite) such that $\Gamma^\text{opp}$ has no isolated vertices. Then the universal $C^*$-algebra with generators $\{V_s|s \in S\}$ subject to the relations:

1. $V_s^*V_s = I$ for each $s \in S$;
2. $V_sV_t = V_tV_s$ and $V_s^*V_t = V_tV_s^*$ if $s$ and $t$ are adjacent in $\Gamma$;
3. $V_s^*V_t = 0$ if $s$ and $t$ are distinct and not adjacent in $\Gamma$;
4. $\prod_{s \in S_\lambda} (I - V_sV_s^*) = 0$ for each $S_\lambda \subset S$ spanning a finite connected component of $\Gamma^\text{opp}$.

is purely infinite and simple.

We will denote the $C^*$-algebra from this theorem by $C^*_Q(\Gamma)$.

Theorem 1.2 ([6], Corollary 8.5). Suppose that $\Gamma$ is a graph with a set of vertices $S$ (finite or infinite) such that $\Gamma^\text{opp}$ has no isolated vertices. Let $C^*(\Gamma)$ denote the universal $C^*$-algebra with generators $\{V_s|s \in S\}$ subject to the relations:

1. $V_s^*V_s = I$ for each $s \in S$;
2. $V_sV_t = V_tV_s$ and $V_s^*V_t = V_tV_s^*$ if $s$ and $t$ are adjacent in $\Gamma$;
3. $V_s^*V_t = 0$ if $s$ and $t$ are distinct and not adjacent in $\Gamma$;
4. $\prod_{s \in S_\lambda} (I - V_sV_s^*) = 0$, where each $S_\lambda \subset S$ spans a finite union of finite connected components of $\Gamma^\text{opp}$.

Then each quotient of $C^*(\Gamma)$ is obtained by imposing a further collection of relations of the form

(R) $\prod_{s \in S_\lambda} (I - V_sV_s^*) = 0$, where each $S_\lambda \subset S$ spans a finite connected component of $\Gamma^\text{opp}$.

We remind that by definition the opposite graph of the graph $\Gamma$ is

\[ \Gamma^\text{opp} = \{(v,w) | v, w \in S, (v, w) \notin \Gamma \}. \]

$\Gamma^\text{opp}$ is also called the complement or the inverse of the graph $\Gamma$.

Let $\Gamma$ be a finite graph with set of vertices $S$ such that the opposite graph $\Gamma^\text{opp}$ is connected and has more than 1 vertex. Then $C^*_Q(\Gamma)$ is the quotient of $C^*(\Gamma)$ by the ideal generated by $\prod_{s \in S} (I - V_sV_s^*)$. Let $I_\Gamma : \langle \prod_{s \in S} (I - V_sV_s^*) \rangle_{C^*(\Gamma)} \to C^*(\Gamma)$ be the inclusion map of this ideal, and $Q_\Gamma : C^*(\Gamma) \to C^*_Q(\Gamma)$ be the quotient map. Theorem 1.2 implicitly contains the uniqueness theorem ([5, Theorem 24]). In particular we have the following faithful representation $\pi_\Gamma : C^*(\Gamma) \to B(H_\Gamma)$ which corresponds to $T(A_\Gamma, A_\Gamma^+)$, where $A_\Gamma = \{S|ss' = s's \text{ if } (s, s') \in \Gamma\}$: Let $H_\Gamma$ be the Hilbert space with an orthonormal basis

\[ \{E[s_1, s_2, \ldots, s_n] | n \in \mathbb{N}_0, s_1, \ldots, s_n \in S\} / \sim, \]

where the relation $\sim$ means $E[s_1, s_2, \ldots, s_n] \sim E[s'_1, s'_2, \ldots, s'_m]$ if and only if $V_{s_1} \cdots V_{s_n} = V_{s'_1} \cdots V_{s'_m}$ subject to commutation relation (2) from Theorem 1.2. Let $\pi_\Gamma$ be given on a generating family of operators and vectors by

\[ \pi_\Gamma(V_s)(E[0]) = E[s], \]

\[ \pi_\Gamma(V_s)(E[s_1, s_2, \ldots, s_n]) = E[s, s_1, s_2, \ldots, s_n]. \]

For this representation it is true that the ideal $\langle \pi_\Gamma(\prod_{s \in S} (I - V_sV_s^*)) \rangle_{\pi_\Gamma(C^*(\Gamma))}$ coincides with $\mathcal{K}(H_\Gamma)$ - the compact operators on $H_\Gamma$. 
In [7] Cuntz introduced a certain type of $C^*$-algebras $O_n$, $n = 2, 3, \ldots, \infty$ generated by a set of isometries with mutually orthogonal ranges. He was able to represent $K \otimes O_n$ as a crossed product of an AF-algebra by $\mathbb{Z}$ ($K$ stands for the $C^*$-algebra of the compact operators on a separable Hilbert space). There have been generalizations of these algebras that depend on the "crossed product by $\mathbb{Z}$" idea, for example Cuntz-Krieger algebras [9], Cuntz-Pimsner algebras [17] and others.

In our note for a fixed finite graph with at least three vertices $\Gamma$ with $\Gamma^\text{opp}$ connected we choose a subgraph $\Gamma'$ one less vertex such that $(\Gamma')^\text{opp}$ is connected. Then we represent $C^*_Q(\Gamma)$ as a full corner of a crossed product of a $C^*$-algebra, built by using $C^*_Q(\Gamma')$, by the group $\mathbb{Z}$. After doing so we can use some results about $C^*$-algebras which are crossed products by $\mathbb{Z}$. Most importantly we use the Pimsener-Voiculescu exact sequence for the $K$-theory ([18]). Using induction on the number of the vertices of the graph we conclude that $C^*_Q(\Gamma)$ is nuclear and belong to the small bootstrap class (see [2, IV.3.1], [1, §22]) and thus the classification result for purely infinite simple $C^*$-algebras of Kirchberg-Phillips [16] applies. From this we conclude that $C^*_Q(\Gamma)$ is isomorphic to $O_{1+|\chi(\Gamma)|}$, where $\chi(\Gamma)$ is an analogue of Euler characteristic, introduced in [6]. Then we extend this result to the case when $\Gamma$ is an infinite graph with countably many vertices and such that $\Gamma^\text{opp}$ is connected, since this graph can be represented as an increasing sequence of finite subgraphs. The general case is a graph $\Gamma$ with at least two and at most countably many vertices which is such that $\Gamma^\text{opp}$ has no isolated vertex. It can be treated easily using Theorem 1.1 and the special cases described above. The conclusion is that $C^*_Q(\Gamma)$ is isomorphic to tensor products of $O_n$ for $1 \leq n \leq \infty$, where we define $O_1$ to be the unital Kirchberg algebra with $K_0(O_1) = \mathbb{Z}[1_{O_1}]$ and $K_1(O_1) = \mathbb{Z}$. A Kirchberg algebra is by definition a separable, nuclear, simple, purely infinite $C^*$-algebra that satisfies the Universal Coefficient Theorem.

2. Some $C^*$-Subalgebras of $C^*_Q(\Gamma)$ and the Crossed Product Construction

If $\Gamma$ has two vertices and no edges, then from the construction of $C^*(\Gamma)$ is clear that $C^*(\Gamma)$ is generated by isometries $V_1$ and $V_2$ with orthogonal ranges and such that $V_1V_1^* + V_2V_2^* < I$. This is the $C^*$-algebra $\mathcal{E}_2$ from [8] which is an extension of $O_2$ by the compacts. Thus $C^*_Q(\Gamma) \cong O_2$.

Suppose now that $\Gamma$ has a set of vertices $S$ such that $2 < \text{card}(S) < \infty$ and suppose that $\Gamma^\text{opp}$ is connected. Since $\Gamma^\text{opp}$ is connected if it is not a tree we can remove an arbitrary edge from its arbitrary cycle and the graph obtained in this way (let’s denote it by $\Gamma^\text{opp}_1$) will remain connected. Continuing in this fashion in finitely many (say $l$) steps we will arrive at $\Gamma^\text{opp}_l$ which will be a tree. Let $s \in S$ be a "leaf" for $\Gamma^\text{opp}_l$. Removing $s$ and the edge that comes out of $s$ from $\Gamma^\text{opp}_l$ will not alter the connectedness. All this shows that if $\Gamma'$ is the graph, obtained from $\Gamma$ by removing the vertex $s$ and all the edges that come out of $s$, then its opposite graph $(\Gamma')^\text{opp}$ will be connected.
Let $S' \subseteq S$ be the set of edges of $\Gamma'$. We can suppose that $S = \{1, \ldots, n, n + 1\}$ and that $S' = \{1, \ldots, n\}$ for some $n \geq 2$. We want to describe the words in letters $\{V_1, \ldots, V_n, V_{n+1}, V_1^*, \ldots, V_n^*, V_{n+1}^*\}$.

**Lemma 2.1.** Every word in letters $\{V_1, \ldots, V_n, V_{n+1}, V_1^*, \ldots, V_n^*, V_{n+1}^*\}$ can be written in the form $w_1w_2^*$, where $w_1, w_2$ are words in letters $\{V_1, \ldots, V_n, V_{n+1}\}$.

**Proof.** We will use induction on the length of the words. The words of length one are $V_i$ and $V_i^*$ and they are of such form. Suppose that the statement of the lemma is true for all words of length $m > 1$ and less. Take a word $w$ of length $m + 1$. We have two cases for $w$:

1) $w = w'V_i^*$ and 2) $w = w'V_i$ for some $1 \leq i \leq n + 1$ and some word $w'$ of length $m$.

By the induction hypothesis $w'$ can be represented as $w' = w'_1(w'_2)^*$, where $w'_1$ and $w'_2$ are words in letters $\{V_1, \ldots, V_n, V_{n+1}\}$. In case 1) $w = w'_1(w'_2)^*V_i^*$, so setting $w_1 = w'_1$ and $w_2 = V_iw'_2$ shows that $w$ can be written in the desired form. For case 2) if the word $w'_2$ is empty then setting $w_1 = w'_1V_i$ and $w_2 = I$ shows that $w$ has the desired form. If $w'_2 = V_jw'_2'$ with $w'_2'$ a word in letters $\{V_1, \ldots, V_{n+1}\}$ then

$$w = w'_1(w'_2')^*V_j^*V_i = \begin{cases} 0, & \text{if } (i, j) \notin \Gamma \\ w'_1(w'_2')^*, & \text{if } i = j \\ w'_1(w'_2')^*V_j^*V_i, & \text{if } (i, j) \in \Gamma. \end{cases}$$

The first and the second case in the above equation are words of the desired form. In the third case we have that $w'_1(w'_2')^*V_i$ is a word of length $m$ so it can be represented as $\omega_1\omega_2^*$. Then $w'_1(w'_2')^*V_iV_j^* = \omega_1\omega_2^*V_j^*$ is of the desired form. This concludes the induction and proves the lemma. \qed

Let’s denote by $V$ the isometry $V_{n+1} \in C^\circ_Q(\Gamma)$ and suppose without loss of generality that $V^*V_i = 0$ for $k < i \leq n$ (notice that since $\Gamma^\circ$ is connected, $k < n$). If $k > 0$ then also $V$ commutes and $\ast$-commutes with $V_1, \ldots, V_k$.

Let $T_0 = C^\ast(V_1, \ldots, V_n)$. Then from Theorem 1.2 it is easy to see that $T_0 \cong C^\ast(\Gamma')$. Define by induction $T_m$ to be the closed linear span of elements of $C^\circ_Q(\Gamma)$ of the form $wVt_{m-1}V^*(w')^*$, where $w, w'$ are words in letters $\{V_1, \ldots, V_n\}$ and $t_{m-1} \in T_{m-1}$. The following lemma characterizes the sets $T_m$.

**Lemma 2.2.** $T_m$ is a $\ast$-subalgebra of $C^\ast(\Gamma)$, isomorphic to $K^\circ_{m} \otimes T_0 \cong K \otimes C^\ast(\Gamma')$.

**Proof.** Let us denote by $\Omega$ the set of all words $\omega$ in letters $\{V_1, \ldots, V_n\}$ such that the letters of the word $\omega V$ cannot be commuted pass $V$, i.e. $\omega V = \omega_1V\omega_2$ for some words $\omega_1, \omega_2$ in letters $\{V_1, \ldots, V_n\}$ implies $\omega_2 = I$. It is easy to see that from the connectedness of $\Gamma^\circ$ follows that $\Omega$ is an infinite countable set therefore we can enumerate its elements: $\Omega = \{\omega_0, \omega_1, \omega_2, \ldots\}$, setting $\omega_0 = I$. We assume that the words in $\Omega$ don’t repeat, i.e. $\omega_p \neq \omega_q$ for $p \neq q$ after using the commutation relation. Suppose by induction that $T_{m-1} \cong K^\circ_{(m-1)} \otimes T_0$ for some $m \geq 1$. We want to show that $T_m \cong K \otimes T_{m-1}$. Clearly $\{\omega_pVt_{m-1}V^*\omega_q^*|p, q \in \mathbb{N}_0\}$ is a $\ast$-closed set. It is easy to see that each element $wVt_{m-1}V^*w^*$ of $T_m$ after applying the commutation relations (2) from Theorem 1.2 can be written in the form $\omega_pVt_{m-1}^pV^*\omega_q^*$ for some $p, q \in \mathbb{N}_0$. 

and some $t'_{m-1} \in T_{m-1}$. Therefore $\{\omega_p V_{t_{m-1}} V^* \omega_q^* | p, q \in \mathbb{N}_0, t_{m-1} \in T_{m-1}\}$ spans a dense subset of $T_m$. We conclude that $T_m$ is *-closed.

We want to show now that $V^* \omega_q^* \omega_p V = \delta_{p,q} I$. Write $\omega_p = V_{j_1} \cdots V_{j_s}$ and $\omega_q = V_{i_1} \cdots V_{i_t}$. Then $V^* \omega_q^* \omega_p V = V^* V_{i_t}^* \cdots V_{i_1}^* V_{j_1} V_{j_2} \cdots V_{j_s} V$. There are three cases:

1) If $V_{j_1}$ commutes with $V_{i_1}^*, \ldots, V_{i_r}^* (1 \leq r < t)$ and $i_{r+1} = j_1$ then $V_{i_r}^*$ will commute with $V_{i_1}^*, \ldots, V_{i_r}^*$, so the word $\omega_q$ can be written in the form $\omega_q = V_{i_1} V_{i_2} \cdots V_{i_t}$, with $i_1 = j_1$. Then we can write $V^* \omega_q^* \omega_p V = V^* V_{i_t} \cdots V_{i_2} V_{j_2} \cdots V_{j_s} V$ and continue the argument with this word.

2) If $V_{j_1}$ commutes with $V_{i_1}^*, \ldots, V_{i_r}^* (1 \leq r < t)$ and $(j_1, i_{r+1}) \notin \Gamma$, then $V^* \omega_q^* \omega_p V = 0$.

Also if $j_1 > k$ and $V_{j_1}$ commutes with $V_{i_1}^*, \ldots, V_{i_t}^*$ we also have $V^* \omega_q^* \omega_p V = 0$.

3) If $V_{j_2}$ commutes with $V_{i_1}, \ldots, V_{i_t}$ and $V$ clearly then $j_1 \leq k$ and from the definition of $\Omega$ follows that $V_{j_1}$ doesn’t commute with all $V_{j_2}, \ldots, V_{j_s}$. Suppose that $V_{j_1}$ doesn’t commute with $V_{j_r}$ $(2 \leq r \leq s)$ and if $r > 2 V_{j_2}$ commutes with $V_{j_3}, \ldots, V_{j_{r-1}}$. Notice that $j_r \notin \{i_1, \ldots, i_t\}$ since $V_{j_2}$ commutes with $V_{i_1}^*, \ldots, V_{i_t}^*$ and not with $V_{j_r}$.

Suppose that $V^* \omega_q^* \omega_p V \neq 0$. Then suppose that $V_{j_1}, \ldots, V_{j_r}$ can be dealt with by using repeatedly case 1). If $r_1 = s = t$ then $V^* \omega_q^* \omega_p V = \delta_{p,q} I$ is proven. If $r_1 = s < t$ then $V^* \omega_q^* \omega_p V$ reduces to $V^* V_{i_t}^* \cdots V_{i_s}^* V$. If $i_t \leq k$ then $V_{i_t}^*$ would commute with $V^*$ contradicting the fact that $\omega_q \in \Omega$. $i_t > k$ implies immediately $V^* V_{i_t}^* \cdots V_{i_s}^* V = 0$ because $V$ does not commute with all of $V_{i_1}^*, \ldots, V_{i_{s+1}}^*$ so it has an orthogonal range with some of them. The case $r_1 = t < s$ is similar. If $r_1 < s$ and $r_1 < t$ then suppose that for $V_{j_1+1}$ case 3) applies. We will obtain a contradiction with the fact that $\omega_p \in \Omega$. By case 3) we can find $r_2 > r_1 + 1$ such that $V_{j_1+1}$ doesn’t commute with $V_{j_{r_2}}$ and if $r_2 > r_1 + 2$ then $V_{j_{r_1+1}}$ commutes with $V_{j_{r_2}+1}, \ldots, V_{j_{r_2-1}}$. Also $j_{r_2} \notin \{i_{r_1+1}, \ldots, i_t\}$ ($V_{j_{r_1+1}}$ commutes with $V_{i_{r_1+1}}, \ldots, V_{i_t}$ and not with $V_{j_{r_2}}$) and so case 1) cannot be applied to $V_{j_{r_2}}$. We can repeat this process finitely many times until we reach the isometry $V_{j_s}$ for which case 3) must apply since case 1) cannot be applied as we saw above and case 2) cannot be applied by assumption. But then $j_s \leq k$ and $V_{j_s}$ commutes with $V$ which contradicts $\omega_p \in \Omega$. This proves $V^* \omega_q^* \omega_p V = \delta_{p,q} I$.

It follows that $\omega_p V_{t_{m-1}} V^* \omega_q^* \omega_p V_{t_{m-1}} V^* \omega_q^* = \delta_{p,q} V_{t_{m-1}} V^* \omega_q^* \omega_p V_{t_{m-1}} V^* \omega_q^*$ and thus $T_m$ is a $C^*$-algebra. The equation $V^* \omega_q^* \omega_p V = \delta_{p,q} I$ implies that $C^*(\{\omega_p V V^* \omega_q^* | 0 \leq p, q \leq l - 1\}) \cong M_l(\mathbb{C})$. It is clear that $V T_{m-1} V^*$ is a $C^*$-algebra, isomorphic to $T_{m-1}$. Therefore

$$C^*(\{\omega_p V_{t_{m-1}} V^* \omega_q^* | 0 \leq p, q \leq l - 1, t_{m-1} \in T_{m-1}\}) \cong$$

$$C^*(\{\sum_{i=0}^{l-1} (\omega_i V_{t_{m-1}} V^* \omega_i^*) | t_{m-1} \in T_{m-1}\}) \otimes C^*(\{\omega_p V V^* \omega_q^* | 0 \leq p, q \leq l - 1\})$$

$$\cong T_{m-1} \otimes M_l(\mathbb{C}) = M_l(T_{m-1}),$$

since $\sum_{i=0}^{l-1} (\omega_i V_{t_{m-1}} V^* \omega_i^*)$ commutes with $\omega_p V V^* \omega_q^*$ for each $0 \leq p, q \leq l - 1$ and each $t_{m-1} \in T_{m-1}$. Taking limit $l \to \infty$ concludes the proof of the lemma.

From the proof of this lemma easily follows that $T_m$ is the closed linear span of

$$\{\omega_{p_1} V \cdots V \omega_{p_m} V_{t_0} V^* \omega_{q_1} V^* \cdots V^* \omega_{q_m}^* | \omega_{p_1}, \ldots, \omega_{p_m}, \omega_{q_1}, \ldots, \omega_{q_m} \in \Omega, t_0 \in T_0\}.$$
This implies that \( T_m \cdot T_l \subseteq T_m \) and \( T_l \cdot T_m \subseteq T_l \) for each \( m \geq l \geq 0 \).

Now we introduce the following \( C^* \)-subalgebras of \( C^*_Q(\Gamma) \). Define \( B_0 = T_0 \) and \( B_m = C^*(B_m-1 \cup T_m) = C^*(T_0 \cup \cdots \cup T_m) \). From what we said above is clear that \( T_m \) is an ideal of \( B_m \). Therefore we have an extension
\[
0 \rightarrow T_m \xrightarrow{i_m} B_m \xrightarrow{p_m} B_m/T_m \rightarrow 0,
\]
where \( i_m : T_m \rightarrow B_m \) is the inclusion map and \( p_m : B_m \rightarrow B_m/T_m \) is the quotient map.

From [14, Theorem 3.1.7] (or [2] Corollary II.5.1.3] follows that \( B_m = B_{m-1} + T_m \) as a linear space. From [14, Remark 3.1.3] follows that the map \( \pi_m : B_{m-1}/(B_{m-1} \cap T_m) \rightarrow B_m/T_m \) given by \( b_{m-1} + B_{m-1} \cap T_m \mapsto b_{m-1} + T_m \) is an isomorphism \( (b_{m-1} \in B_{m-1}) \).

Define \( \mathcal{I}_m \overset{def}{=} \langle V^n \prod_{i=1}^n (I - V_i V_i^*) \rangle \mathcal{I}_m \). Since \( T_0 \cong C^*(\Gamma') \) from Theorem 1.2 follows that \( \mathcal{I}_0 \) is the unique nontrivial ideal of \( T_0 \) and it is isomorphic to \( \mathcal{K} \). Then from Lemma 2.2 follows that \( \mathcal{I}_m \) is the unique nontrivial ideal of \( T_m \) and it is isomorphic to \( \mathcal{K}^\otimes m \otimes \mathcal{K} \). The ideal \( \mathcal{I}_m \) can be described as the closed linear span of
\[
\{ \omega_{p_1} V \cdots V \omega_{q_1} V_{l_0} V^* \omega_{q_2} V^* \cdots V^* \omega_{q_m} | \omega_{p_1}, \ldots, \omega_{p_m}, \omega_{q_1}, \ldots, \omega_{q_m} \in \Omega, l_0 \in \mathcal{I}_0 \}.
\]
Therefore it is easy to see that \( V^n \prod_{i=1}^n (I - V_i V_i^*) = V^n \mathcal{I}_0 (V^*)^m = V^n \mathcal{I}_0 (V^*)^m \).

By the definition of \( C^*_Q(\Gamma) \) we have \( (I - V V^*) \prod_{i=1}^n (I - V_i V_i^*) = 0 \) or \( \prod_{i=1}^n (I - V_i V_i^*) = V V^* \prod_{i=1}^n (I - V_i V_i^*) \). Therefore using relations (2) and (3) from Theorem 1.1 we get
\[
\prod_{i=1}^n (I - V_i V_i^*) = V V^* \prod_{i=1}^n (I - V_i V_i^*) = V \prod_{i=1}^k (I - V_i V_i^*) \prod_{i=k+1}^n (I - V_i V_i^*) = V \prod_{i=1}^k (I - V_i V_i^*) V^* \in T_1.
\]

It follows also that \( V^n \prod_{i=1}^k (I - V_i V_i^*) V^* (V^*)^m = V^m T_1 (V^*)^m \subset T_{m+1} \). It is easy to see that \( T_{m+1} \cdot B_m \subset T_{m+1} \) and \( B_m \cdot T_m \subset T_m \). This implies that \( T_m \cap T_{m+1} \) is an ideal of \( T_m \) and that \( T_m \cap T_{m+1} \) is an ideal of \( B_m \). From this we can conclude that \( \mathcal{I}_m \subset (T_m \cap T_{m+1}) \) for each \( m \in \mathbb{N} \). The reverse inclusion is also true:

**Lemma 2.3.** \( B_m \cap T_{m+1} = \mathcal{I}_m \) for each \( m \in \mathbb{N}_0 \).

**Proof.** Since \( \mathcal{I}_0 \) is the unique nontrivial ideal of \( T_0 \) and since \( T_0 \cap T_1 \) is an ideal of \( T_0 \), then if we assume that \( \mathcal{I}_0 \subset T_0 \cap T_1 \) it will follow that \( T_0 = T_0 \cap T_1 \). Then \( I = 1_{\mathcal{I}_0} = 1_{C^*_Q(\Gamma)} \in T_0 \subset T_1 \). This will imply that \( T_1 \cong \mathcal{K} \otimes T_0 \) is a unital \( C^* \)-algebra which is a contradiction. Therefore \( \mathcal{I}_0 = T_0 \cap T_1 \).

It is easy to see that for each \( m \in \mathbb{N} \) we have \( V^n (V^*)^m T_m V^n (V^*)^m = V^m T_0 (V^*)^m \cong T_0 \) and that \( V^n (V^*)^m T_{m+1} V^n (V^*)^m = V^m T_1 (V^*)^m \cong T_1 \). Thus if we assume that \( T_m = T_m \cap T_{m+1} \) it will follow that \( V^n T_0 (V^*)^m \subset V^m T_1 (V^*)^m \) and therefore that
This is a contradiction with what we proved in the last paragraph. Therefore \( T_m \cap T_{m+1} \subsetneq T_m \) and thus \( T_m \cap T_{m+1} = \mathcal{I}_m \).

To conclude the proof of the lemma we have to show that \( T_{m+1} \cap T_j = 0 \) for each \( 0 \leq j < m \). In this case we have once again that \( T_{m+1} \cap T_j \) is an ideal of \( T_j \). Therefore the assumption \( T_{m+1} \cap T_j \neq 0 \) implies that \( T_{m+1} \) contains the minimal nonzero ideal of \( T_j \), \( \mathcal{I}_j \). In particular \( V_j^n (I - V_i V_i^*) (V_i^*)^j = V_j^{j+1} \prod_{i=1}^k (I - V_i V_i^*) (V_i^*)^{j+1} \in T_{m+1} \).

This implies

\[
V_j^{j+1} \prod_{i=1}^k (I - V_i V_i^*) (V_i^*)^{j+1} = V_j^{j+1} (V_i^*)^{j+1} \prod_{i=1}^k (I - V_i V_i^*) (V_i^*)^{j+1} V_j^{j+1} (V_i^*)^{j+1} \\
\in V_j^{j+1} (V_i^*)^{j+1} T_j V_j^{j+1} (V_i^*)^{j+1} = V_j^{j+1} T_{j-1} (V_i^*)^{j+1}.
\]

Therefore \( \prod_{i=1}^k (I - V_i V_i^*) \in T_{m-j} \). Since also \( \prod_{l=1}^k (I - V_i V_i^*) \in T_0 \), then the ideal \( T_0 \cap T_{m-j} \) of \( T_0 \) contains \( \prod_{i=1}^k (I - V_i V_i^*) \). We will show that \( \prod_{i=1}^k (I - V_i V_i^*) \notin \mathcal{I}_0 \) this will imply that \( T_0 \subset T_{m-j} \) for \( m-j > 0 \) and therefore obtaining a contradiction with the fact that \( T_{m-j} \) is not unital for \( m-j > 0 \).

Suppose that \( \prod_{i=1}^k (I - V_i V_i^*) \in \mathcal{I}_0 \). Then since \( T_0 = C^*(\Gamma') \) we have \( Q_{\Gamma'}(\prod_{i=1}^k (I - V_i V_i^*)) = 0 \). From the connectedness of \((\Gamma')^{opp}\) follows that we can find \( j, 1 \leq j \leq k \) and \( l, k < l \leq n \) with \((j, l) \notin \Gamma'\). Then

\[
0 = Q_{\Gamma'}(V_i^*) Q_{\Gamma'}(\prod_{i=1}^k (I - V_i V_i^*)) Q_{\Gamma'}(V_i) = Q_{\Gamma'}(V_i^*) Q_{\Gamma'}(\prod_{(i,j) \in \Gamma'}^{1 \leq i \leq k} (I - V_i V_i^*)) Q_{\Gamma'}(V_i) = Q_{\Gamma'}(V_i^* V_i) Q_{\Gamma'}(\prod_{(i,j) \in \Gamma'}^{1 \leq i \leq k} (I - V_i V_i^*)) = Q_{\Gamma'}(\prod_{(i,j) \in \Gamma'}^{1 \leq i \leq k} (I - V_i V_i^*)).
\]

By repeating this argument finitely many times we will arrive at the equality \( Q_{\Gamma'}(I) = 0 \) which is a contradiction. Therefore \( \prod_{i=1}^k (I - V_i V_i^*) \notin \mathcal{I}_0 \). This completes the proof of the lemma. \( \square \)

This lemma shows that we have an extension

\[
(2) \quad 0 \to \mathcal{I}_{m-1} \overset{i_m'}{\to} B_{m-1} \overset{p_m'}{\to} B_{m-1}/\mathcal{I}_{m-1} \to 0,
\]

where \( i_m' : \mathcal{I}_{m-1} \to B_{m-1} \) is the inclusion map and \( p'_m : B_{m-1} \to B_{m-1}/\mathcal{I}_{m-1} \) is the quotient map.
From equations (1) and (2) we have the commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{I}_{m-1} & \xrightarrow{i_m} & B_{m-1} & \xrightarrow{p_m} & B_{m-1}/\mathcal{I}_{m-1} & \longrightarrow & 0 \\
& & t_m \downarrow & & t_m \downarrow & & \cong \downarrow \pi_m & & \\
0 & \longrightarrow & T_m & \xrightarrow{i_m} & B_m & \xrightarrow{p_m} & B_m/T_m & \longrightarrow & 0,
\end{array}
\]

where \( I'_m : \mathcal{I}_{m-1} \rightarrow T_m \) and \( I_m : B_{m-1} \rightarrow B_m \) are the inclusion maps.

Define \( \tilde{B} \overset{\text{def}}{=} \bigcup_{i=0}^{\infty} B_i \subset C^*(\Gamma) \) or in other words \( \tilde{B} \overset{\text{def}}{=} \lim_{m \to \infty}(B_m, I_m) \). Notice that if \( t_m \in T_m \) then \( V_{t_m} \in T_{m+1} \). Thus we have a well defined injective endomorphism \( \beta : B \to B \) given by \( b \mapsto VbV^* \).

Similarly to the Cuntz' construction from [7] we define \( \tilde{B} \overset{\text{def}}{=} \lim_{m \to \infty}(B_m, \alpha_m) \) as the limit of the sequence (which is also a commutative diagram)

\[
\begin{align*}
\ldots & \xrightarrow{\alpha_{m-1}} B^{-m} \xrightarrow{\alpha_m} \ldots \xrightarrow{\alpha_1} B^0 \xrightarrow{\alpha_0} B^1 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_m} B^m \xrightarrow{\alpha_m} \ldots \\
& \xrightarrow{j_m \cong} \ldots \xrightarrow{j_1 \cong} \ldots \xrightarrow{j_0 \cong} \ldots \xrightarrow{j_m \cong} \ldots \\
\ldots & \xrightarrow{\beta} B \xrightarrow{\beta} \ldots \xrightarrow{\beta} B \xrightarrow{\beta} \ldots \xrightarrow{\beta} B \xrightarrow{\beta} \ldots ,
\end{align*}
\]

where \( j_m : B^m \to B \) are \(*\)-isomorphisms. Since \( \tilde{B} \) is a limit \( C^*\)-algebra we have \(*\)-homomorphisms \( \alpha_m : B^m \to \tilde{B} \), s.t. \( \alpha_m = \alpha^{m+1} \circ \alpha_m \) for all \( m \in \mathbb{Z} \).

Now we define a \(*\)-homomorphism \( \Phi \) of \( \tilde{B} \) to itself, which is induced by "shift to the left" on \( \mathbb{I} \). In other words if we have a stabilizing sequence \( (b^m)_{m=-\infty}^{+\infty} \), where \( b^m \in B^m \) for each \( m \), then \( \Phi((b^m)_{m=-\infty}^{+\infty}) = (j_{m-1} \circ j_{m+1}(b^m))_{m=-\infty}^{+\infty} \). In particular for \( b \in B \) the element \( \alpha_m \circ j_m(b) \) can be represented as the sequence \( (0, \ldots, 0, j_{m-1}(b), \alpha_m \circ j_m(b), \alpha_{m+1} \circ \alpha_m \circ j_m(b), \ldots) = (0, \ldots, 0, j_{m-1}(b), j_m \circ \beta(b), j_{m+1} \circ \beta(b), \ldots) \).

Therefore \( \Phi(\alpha_m \circ j_m(b)) \) can be represented as the sequence \( (0, \ldots, 0, j_{m-1}(b), j_m \circ \beta(b), j_{m+1} \circ \beta(b), \ldots) \). This shows that \( \Phi(\alpha_m \circ j_m(b)) = \alpha_m \circ j^{-1}_m \circ \beta(b) \).

The extension of this map to the whole of \( \tilde{B} \) (we call it \( \Phi \) also) is a \(*\)-isomorphism, because \( \Phi \) is isometric on the dense set of all stabilizing sequences (since \( j_m \) are all isomorphisms).

Now let \( \tilde{A} \) be the crossed product of \( \tilde{B} \) by the automorphism \( \Phi \). We represent \( \tilde{A} \) faithfully on a Hilbert space \( \mathfrak{H} \) so that \( \Phi \) is implemented by a unitary \( U \) on \( \mathfrak{H} \): \( \Phi(b) = UbU^* \) for \( b \in \tilde{B} \). Then \( \tilde{A} = C^*(\tilde{B} \cup \{U\}) \).

Every element of \( \tilde{A} \) is a limit of elements of the form \( \tilde{a} = \sum_{i=-N}^{N} b_i U^i = U^{i} b_i + b_0 + \sum_{i=1}^{N} b_i U^i \), with \( b_i \in \tilde{B} \), where \( \tilde{a} = U^{-i}b_i U^i \in \tilde{B} \) for \( i = -N, \ldots, -1 \). Therefore the set of the elements of \( \tilde{A} \) of the above form is dense in \( \tilde{A} \).

Set \( \tilde{P}_m \overset{\text{def}}{=} \alpha_m(1_{B^m}) \in \tilde{B} \) for each \( m \in \mathbb{Z} \). Notice that \( \alpha_m(1_{B^m}) = \alpha_m \circ j_m^{-1}(I) = \alpha^{m+1} \circ \alpha_m \circ j_m^{-1}(I) = \alpha^{m+1} \circ j_{m+1}^{-1}(\beta(I)) \).

By induction

\[
\tilde{P}_m = \alpha^{m+i} \circ j_{m+i}^{-1}(\beta^i(I)), \quad m \in \mathbb{Z}, \quad i \in \mathbb{N}.
\]
Therefore we can write

\[ \hat{P}_m = \Phi^{-m}(\hat{P}_0), \quad m \in \mathbb{Z}. \]

Consider the $C^*$-algebra $\hat{P}_0 \tilde{A} \hat{P}_0$. Clearly $\hat{P}_0 \tilde{B} \hat{P}_0 \subset \hat{P}_0 \tilde{A} \hat{P}_0$. Since elements of the form $\tilde{a} = \sum_{i=-N}^{1} U^i b_i + b_0 + \sum_{i=1}^{N} b_i U^i (b_i \in \tilde{B})$ are dense in $\tilde{A}$, then elements of the form

\[ \hat{P}_0 \tilde{a} \hat{P}_0 = \sum_{i=-N}^{-1} \hat{P}_0 U^i b_i \hat{P}_0 + \hat{P}_0 b_0 \hat{P}_0 + \hat{P}_0 \sum_{i=1}^{N} b_i U^i \hat{P}_0 \]

are dense in $\hat{P}_0 \tilde{A} \hat{P}_0$. It is easy to see that $U \hat{P}_0 U^* = \Phi(\hat{P}_0) < \hat{P}_0$, so the range of $U \hat{P}_0$ is contained in $\hat{P}_0$ and therefore $\hat{P}_0 U \hat{P}_0 = U \hat{P}_0$. Then

\[ \hat{P}_0 \tilde{a} \hat{P}_0 = \sum_{i=-N}^{-1} \hat{P}_0 U^i b_i \hat{P}_0 + \hat{P}_0 b_0 \hat{P}_0 + \hat{P}_0 \sum_{i=1}^{N} b_i U^i \hat{P}_0 = \]

\[ = \sum_{i=-N}^{-1} (\hat{P}_0 U^i)(\hat{P}_0 b_i \hat{P}_0) + \hat{P}_0 b_0 \hat{P}_0 + \hat{P}_0 \sum_{i=1}^{N} (\hat{P}_0 b_i \hat{P}_0)(U^i \hat{P}_0). \]

This shows that if we set $S \overset{def}{=} U \hat{P}_0$ then $\hat{P}_0 \tilde{A} \hat{P}_0 = C^*(\hat{P}_0 \tilde{B} \hat{P}_0 \cup \{S\})$. Let us also set $S_i \overset{def}{=} \alpha^0(j_0^{-1}(V_i)), \quad i = 1, \ldots, n$.

It is easy to see that $\text{Span}(\bigcup_{l=0}^{\infty} T_l)$ is dense in $B$. Then it follows that $\text{Span}(\bigcup_{l=0}^{\infty} \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l))$ is dense in $\tilde{B}$. Therefore $\hat{P}_0 \text{Span}(\bigcup_{l=0}^{\infty} \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l)) \hat{P}_0 = \text{Span}(\hat{P}_0 \bigcup_{l=0}^{\infty} \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l))$ is dense in $\hat{P}_0 \tilde{B} \hat{P}_0$. For each $i \in \mathbb{N}$ we have

\[ \hat{P}_0 \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l) \hat{P}_0 = \alpha^i \circ j_i^{-1}(\beta^i(I)) \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l) \alpha^i \circ j_i^{-1}(\beta^i(I)) = \]

\[ = \alpha^i \circ j_i^{-1}(\beta^i(I)) (\bigcup_{l=0}^{\infty} T_l) \beta^i(I)) = \alpha^i \circ j_i^{-1}(V^i(V^*)^i) (\bigcup_{l=0}^{\infty} T_l) (V^i(V^*)^i) = \]

\[ = \alpha^i \circ j_i^{-1}((V^i(V^*)^i)^2 (\bigcup_{l=0}^{\infty} T_l)(V^i(V^*)^i)^2) \subset \alpha^i \circ j_i^{-1}((V^i(V^*)^i T_l)(\bigcup_{l=0}^{\infty} T_l)(V^i(V^*)^i)) \subset \]

\[ \subset \alpha_i \circ j_i^{-1}(V^i(V^*)^i) (\bigcup_{l=0}^{\infty} T_l) (V^i(V^*)^i) = \alpha^i \circ j_i^{-1}(V^i(\bigcup_{l=0}^{\infty} T_l))(V^i(V^*)^i) = \alpha^i \circ j_i^{-1}(\beta^i(\bigcup_{l=0}^{\infty} T_l)) = \]

\[ = \alpha^i \circ \alpha_{i-1} \circ \alpha_{i-2} \circ \cdots \circ \alpha_1 \circ \alpha_0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty} T_l) = \alpha^0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty} T_l). \]

From this it follows that $\alpha_0 \circ j_0^{-1}(\text{Span}(\bigcup_{l=0}^{\infty} T_l))$ is dense in $\hat{P}_0 \tilde{B} \hat{P}_0$ and therefore also that $\alpha^0(B^0) = \hat{P}_0 \tilde{B} \hat{P}_0$. This shows that $\hat{P}_0 \tilde{A} \hat{P}_0 = C^*(\alpha_0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty} T_l) \cup \{S\})$.\]
Observe that

\[(6) \quad S\alpha^0 \circ j_0^{-1}(b)S^* = U\tilde{P}_0\alpha^0 \circ j_0^{-1}(b)\tilde{P}_0U^* = U\alpha^0 \circ j_0^{-1}(b)U^* = \Phi(\alpha^0 \circ j_0^{-1}(b)) =
\]

\[= \alpha^0 \circ j_0^{-1} \circ \beta(b) = \alpha^0 \circ j_0^{-1}(VbV^*).
\]

Since for every \(m > 0\) \(T_m\) can be constructed from \(T_0\) and "Ad(V)" equation \(\text{(6)}\) shows that \(\tilde{P}_0\tilde{\Delta}\tilde{P}_0 = C^*(\alpha_0 \circ j_0^{-1}(\bigcup_{i=0}^{\infty} T_i) \cup \{S\}) = C^*(\alpha_0 \circ j_0^{-1}(T_0) \cup \{S\}) = C^*(\{S_1, \ldots, S_n, S\})\).

We want to apply now Theorem 1.1 to the C*-algebra \(A \overset{\text{def}}{=} \tilde{P}_0\tilde{\Delta}\tilde{P}_0\). \(S_i = \alpha^0 \circ j_0^{-1}(V_i)\) are clearly isometries \((i = 1, \ldots, n)\). \(S^*S = \tilde{P}_0U^*U\tilde{P}_0 = \tilde{P}_0\) and therefore \(S\) is also an isometry. Thus condition (1) holds. It is clear from \(\text{(6)}\) that \(SS^* = \alpha^0 \circ j_0^{-1}(VV^*)\). Therefore

\[0 = \alpha^0 \circ j_0^{-1}(0) = \alpha^0 \circ j_0^{-1}((I - VV^*) \prod_{i=1}^{n} (I - V_iV_i^*)) =
\]

\[(\tilde{P}_0 - \alpha^0 \circ j_0^{-1}(VV^*)) \prod_{i=1}^{n} (\tilde{P}_0 - \alpha^0 \circ j_0^{-1}(V_iV_i^*)) = (\tilde{P}_0 - SS^*) \prod_{i=1}^{n} (\tilde{P}_0 - S_iS_i^*).\]

This proves that condition (4) holds. Conditions (2) and (3) obviously hold for all pairs of isometries from \(\{S_1, \ldots, S_n\}\). If \(n \geq i > k\) then \(S_iS_i^*SS^* = \alpha^0 \circ j_0^{-1}(V_iV_i^*VV^*) = 0\), so condition (3) holds also for all pairs \((S_i, S)\) with \(k < i \leq n\). For \(1 \leq i \leq k\) one has

\[SS_i = S\alpha^0 \circ j_0^{-1}(V_i) = S\alpha^0 \circ j_0^{-1}(V_i)S^*S = \Phi(\alpha^0 \circ j_0^{-1}(V_i))S = \alpha^0 \circ j_0^{-1}(VV_iV^*)S =
\]

\[= \alpha^0 \circ j_0^{-1}(V_iVV^*)S = \alpha^0 \circ j_0^{-1}(V_i)\alpha^0 \circ j_0^{-1}(VV^*)S = S_iSS^*S = S_iS.
\]

This shows that \(SS_i = S_iS\). In the same way one can show that \(SS_i^* = S_i^*S\). Therefore condition (4) holds for all pairs \((S_i, S)\) with \(1 \leq i \leq k\). Applying Theorem 1.1 we get \(A \cong C_Q^*(\Gamma)\). Obviously we also have \(C_Q^*(\Gamma) \cong \tilde{P}_m\tilde{\Delta}\tilde{P}_m\) for each \(m \in \mathbb{Z}\).

We reming here (see [2] IV.3.1, [11, §22]) that each C*-algebra in the small bootstrap class \(\mathfrak{M}\) satisfies the Universal Coefficient Theorem. The small bootstrap class \(\mathfrak{M}\) is the smallest class of C*-algebras that satisfy:

(i) \(\mathbb{C} \in \mathfrak{M}\).

(ii) \(\mathfrak{M}\) is closed under stable isomorphism.

(iii) \(\mathfrak{M}\) is closed under inductive limits.

(iv) \(\mathfrak{M}\) is closed under crossed-products by \(\mathbb{Z}\).

(v) If \(0 \rightarrow \mathcal{J} \rightarrow \mathfrak{A} \rightarrow \mathfrak{A}/\mathcal{J} \rightarrow 0\) is an exact sequence, and two of \(\mathcal{J}, \mathfrak{A}, \mathfrak{A}/\mathcal{J}\) are in \(\mathfrak{M}\), so is the third.

The C*-algebras in this class are all nuclear.

The following proposition holds:

**Proposition 2.4.** In the above settings: \(\tilde{A} \cong \tilde{B} \times_{\varphi} \mathbb{Z}\) and \(A \cong C_Q^*(\Gamma)\) is Morita equivalent to \(\tilde{A}\). Both of the C*-algebras \(A\) and \(\tilde{A}\) are simple, belong to \(\mathfrak{M}\) and
$K_*(\tilde{A}) = K_*(A)$. Also if we suppose that $[\tilde{P}_0]_0$ generates $K_0(\tilde{A})$ then it follows that $[\tilde{P}_0]_0$ generates $K_0(A)$.

Proof. We showed above that $\tilde{P}_m\tilde{A}\tilde{P}_m \cong C^*_Q(\Gamma)$ for each $m \in \mathbb{Z}$. It is easy to see that $\tilde{A} = \bigcup_{m=0}^{\infty} \tilde{P}_m\tilde{A}\tilde{P}_m$ and since each $\tilde{P}_m\tilde{A}\tilde{P}_m$ is simple from this follows that $\tilde{A}$ is simple too. Therefore every projection in $\tilde{A}$ is full. In particular $\tilde{P}_0$ is a full projection and therefore $A = \tilde{P}_0\tilde{A}\tilde{P}_0$ is a full corner of $\tilde{A}$ and is therefore Morita equivalent to $\tilde{A}$. It follows that $A$ and $\tilde{A}$ are stably isomorphic (by Brown’s Theorem [3]) and therefore $K_*(A) = K_*(\tilde{A})$.

If $\tilde{A}$ belongs to $\mathfrak{N}$ then from the definition follows that $A$ also does since it is stably isomorphic to $\tilde{A}$.

To conclude the proof of the lemma it remains to show that starting from any finite graph $G$ with $G^\text{opp}$ connected and going through the above construction the $C^*$-algebra (let us denote it by $\tilde{A}_G$ - the analogue of $\tilde{A}$ for $G$) belongs to $\mathfrak{N}$. We will do this by using induction on the number of the vertices of $G$. If $G$ has only two vertices and no edges then $C^*_Q(G) \cong O_2$ and $C^*_G(G) \cong E_2$ so the statement for this graph is true. Suppose that the statement is true for any graph $G$ with at most $n \geq 2$ vertices such that its opposite graph $G^\text{opp}$ is connected. In particular $C^*_Q(\Gamma')$ (and therefore also $C^*(\Gamma')$) belong to $\mathfrak{N}$. Then $T_0 \cong C^*(\Gamma')$ as constructed above also does. Since the bootstrap category is closed under stabilization, extensions, inductive limits and crossed products by $\mathbb{Z}$ we conclude using induction that the $C^*$-algebra $\tilde{A}$ is also nuclear and belong to the small bootstrap class (we use diagram (3) together with Lemma 2.2 and the fact that $\pi_m$ is an isomorphism for all $m \in \mathbb{N}$). Finally as we showed in the last paragraph this implies that $A$ belongs to $\mathfrak{N}$. This concludes the inductive step because $A \cong C^*_Q(\Gamma)$ and $\Gamma$ is an arbitrary graph with $n+1$ vertices such that $\Gamma^\text{opp}$ is connected.

The final statement of the proposition is obvious.

The proposition is proved. \qed

3. The Computation of the $K$-Theory

For a finite graph $G$ with $G^\text{opp}$ connected Crisp and Laca conjectured in [6] that the order of $[1_{C^*_Q(G)}]_0$ in $K_0(C^*_Q(G))$ is $|\chi(G)|$, where $\chi(G)$ is the Euler characteristics of $G$. $\chi(G)$ is defined as

$$\chi(G) = 1 - \sum_{j=1}^{\infty} (-1)^{j-1} \times \{ \text{number of complete subgraphs of } G \text{ on } j \text{ vertices} \}.$$  

We will use the settings from the previous section. Denote $P_m \overset{\text{def}}{=} V^m(V^*)^m$, $m \in \mathbb{N}_0$. Denote also $Q \overset{\text{def}}{=} \prod_{i=1}^{k} (I - V_iV_i^*)$. Let $\Gamma_k = \{(i,j) | 1 \leq i, j \leq k, (i,j) \in \Gamma'\} \subset \Gamma'$. 

Since the vertex \( n + 1 \) of \( \Gamma \) is connected with each of the vertices \( 1, \ldots, k \) and none of the others we have

\[
\chi(\Gamma) = 1 - \sum_{j=1}^{n} (-1)^{j-1} \times \{ \text{number of complete subgraphs of } \Gamma' \text{ on } j \text{ vertices} \} - \\
(1 - \sum_{j=1}^{k} (-1)^{j-1} \times \{ \text{number of complete subgraphs of } \Gamma_k \text{ on } j \text{ vertices} \}).
\]

Therefore

\[
(7) \quad \chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k).
\]

The following lemma is based on the "Euler characteristics idea" and is essentially due to Crisp and Laca:

**Lemma 3.1.** If \( E \) is a \( C^* \)-subalgebra of \( B \) that contains \( T_m \) (for \( m \in \mathbb{N}_0 \)) we have

\[
(8) \quad \chi(\Gamma')[P_m]_0 = [P_{m+1}Q]_0 \text{ (in } K_0(E)).
\]

If \( E \) is a \( C^* \)-subalgebra of \( B \) that contains \( T_m \) and \( T_{m+1} \) (for \( m \in \mathbb{N}_0 \)) we have

\[
(9) \quad \chi(\Gamma')[P_m]_0 = \chi(\Gamma_k)[P_{m+1}]_0 \text{ (in } K_0(E)).
\]

If \( E \) is a \( C^* \)-subalgebra of \( B \) that contains \( T_{m+1} \) (for \( m \in \mathbb{N}_0 \)) we have

\[
(10) \quad [P_{m+1}Q]_0 = \chi(\Gamma_k)[P_{m+1}]_0 \text{ (in } K_0(E)).
\]

**Proof.** In the last section we showed that

\[
(11) \quad \prod_{i=1}^{n}(I - V_iV_i^*) = V \prod_{i=1}^{k}(I - V_iV_i^*)V^*.
\]

Since \( V^m \prod_{i=1}^{n}(I - V_iV_i^*)(V^*)^m = \prod_{i=1}^{n}(V^m(V^*)^m - V^mV_iV_i^*(V^*)^m) \) then by multiplying equation (11) by \( V^m \) on the left and by \( (V^*)^m \) on the right we get

\[
\prod_{i=1}^{n}(V^m(V^*)^m - V^mV_iV_i^*(V^*)^m) = V^{m+1}\prod_{i=1}^{k}(I - V_iV_i^*)(V^*)^{m+1} = \\
= V^{m+1}(V^*)^{m+1}Q.
\]

This equation is actually three equations which hold in certain \( C^* \)-subalgebras of \( B \). We record them here:

If \( E \) is a \( C^* \)-subalgebra of \( B \) that contains \( T_m \) (for \( m \in \mathbb{N}_0 \)) we have

\[
(12) \quad \prod_{i=1}^{n}(V^m(V^*)^m - V^mV_iV_i^*(V^*)^m) = V^{m+1}(V^*)^{m+1}Q.
\]

If \( E \) is a \( C^* \)-subalgebra of \( B \) that contains \( T_m \) and \( T_{m+1} \) (for \( m \in \mathbb{N}_0 \)) we have

\[
(13) \quad \prod_{i=1}^{n}(V^m(V^*)^m - V^mV_iV_i^*(V^*)^m) = V^{m+1}\prod_{i=1}^{k}(I - V_iV_i^*)(V^*)^{m+1}.
\]
If $E$ is a $C^*$-subalgebra of $B$ that contains $T_{m+1}$ (for $m \in \mathbb{N}_0$) we have

\begin{equation}
V^{m+1} \prod_{i=1}^{k} (I - V_i V_i^*) (V^*)^{m+1} = V^{m+1} (V^*)^{m+1} Q.
\end{equation}

Note that if $E$ is an appropriate $C^*$-subalgebra of $B$ then for each projection $P$ that commutes with $V_1 V_1^*$ we have $[V^m P (V^*)^m - V^m P V_1 V_1^* (V^*)^m]_0 = [V^m P (V^*)^m]_0 - [V^m P V_1 V_1^* (V^*)^m]_0$. Suppose by induction that for some $n > l \geq 1$ if $P$ is a projection that commutes with $V_1 V_1^*, \ldots, V_l V_l^*$ we have

\begin{equation}
[V^m \prod_{i=1}^{l} (P - P V_i V_i^*) (V^*)^m]_0 = [V^m P (V^*)^m]_0 - \sum_{i=1}^{l} [V^m P V_i V_i^* (V^*)^m]_0 + \\
+ \sum_{j=2}^{l} (-1)^j \left( \sum_{1 \leq i_1 < \cdots < i_j \leq l \atop (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j} [V^m P V_{i_1} \cdots V_{i_{j-1}} V_{i_j} V_{i_j}^* \cdots V_{i_1}^* (V^*)^m]_0 \right).
\end{equation}

We know that $V_{l+1} V_{l+1}^*$ commutes with each of $V_1 V_1^*, \ldots, V_l V_l^*$. If $P$ commutes with $V_1 V_1^*, \ldots, V_{l+1} V_{l+1}^*$ then we can apply \([15]\) to the family $V_1 V_1^*, \ldots, V_l V_l^*$ and the projection $P V_{l+1} V_{l+1}^*$ to obtain the following equation:

\begin{align*}
[V^m \prod_{i=1}^{l} (P - P V_i V_i^*) (V^*)^m]_0 &= [V^m \prod_{i=1}^{l} (P V_{l+1} V_{l+1}^* - P V_{l+1} V_{l+1}^* V_i V_i^*) (V^*)^m]_0 = \\
&= [V^m P V_{l+1} V_{l+1}^* (V^*)^m]_0 - \sum_{i=1}^{l} [V^m P V_{l+1} V_{l+1}^* V_i V_i^* (V^*)^m]_0 + \\
+ \sum_{j=2}^{l} (-1)^j \left( \sum_{1 \leq i_1 < \cdots < i_j \leq l \atop (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j} [V^m P V_{l+1} V_{l+1}^* V_{i_1} \cdots V_{i_{j-1}} V_{i_j} V_{i_j}^* \cdots V_{i_1}^* (V^*)^m]_0 \right).
\end{align*}
Now since $V^m V_{l+1} V_{l+1}^* \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m < V^m \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m$ it is easy to see that we have

$$[V^m (P - PV_{l+1} V_{l+1}^*) \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m]_0 =$$

$$= [V^m \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m - V^m V_{l+1} V_{l+1}^* \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m]_0 =$$

$$= [V^m \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m]_0 - [V^m V_{l+1} V_{l+1}^* \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m]_0 =$$

$$= [V^m P (V^*)^m]_0 - \sum_{i=1}^l [V^m PV_i V_i^* (V^*)^m]_0 +$$

$$+ \sum_{j=2}^l (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq l, (i_s, i_l) \in \Gamma^*, 1 \leq s \leq t \leq j} [V^m PV_{i_1} \cdots V_{i_j} V_{i_1}^* \cdots V_{i_j}^* (V^*)^m]_0 - [V^m PV_{l+1} V_{l+1}^* (V^*)^m]_0 =$$

$$- \sum_{j=2}^l (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq l, (i_s, i_l) \in \Gamma^*, 1 \leq s \leq t \leq j} [V^m PV_{i_1} \cdots V_{i_j} V_{i_1}^* \cdots V_{i_j}^* (V^*)^m]_0 =$$

$$= [V^m P (V^*)^m]_0 - \sum_{i=1}^{l+1} [V^m PV_i V_i^* (V^*)^m]_0 +$$

$$+ \sum_{j=2}^{l+1} (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq l+1, (i_s, i_l) \in \Gamma^*, 1 \leq s \leq t \leq j} [V^m PV_{i_1} \cdots V_{i_j} V_{i_1}^* \cdots V_{i_j}^* (V^*)^m]_0.$$

Then by induction follows that for $l = k$ or $l = n$ we get

$$[\prod_{i=1}^l (V^m (V^*)^m - V^m V_i V_i^* (V^*)^m)]_0 =$$

$$= [l]_0 - \sum_{i=1}^l [V^m V_i V_i^* (V^*)^m]_0 + \sum_{j=2}^l (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq l, (i_s, i_l) \in \Gamma^*, 1 \leq s \leq t \leq j} [V^m V_{i_1} \cdots V_{i_j} V_{i_1}^* \cdots V_{i_j}^* (V^*)^m]_0.$$

Combining the last equation with equations (12), (13) and (14) we obtain the following equations:
If $E$ is a $C^*$-subalgebra of $B$ that contains $T_m$ (for $m \in \mathbb{N}_0$) we have

\begin{equation}
[V^m(V^*)^m]_0 - \sum_{i=1}^{n} [V^m V_i V_i^* (V^*)^m]_0 + \sum_{j=2}^{n} (-1)^j \left( \sum_{1 \leq i_1 < \ldots < i_j \leq n} [V^m V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^*(V^*)^m]_0 \right) = [V^{m+1}(V^*)^{m+1}Q]_0.
\end{equation}

If $E$ is a $C^*$-subalgebra of $B$ that contains $T_m$ and $T_{m+1}$ (for $m \in \mathbb{N}_0$) we have

\begin{equation}
[V^m(V^*)^m]_0 - \sum_{i=1}^{n} [V^m V_i V_i^* (V^*)^m]_0 + \sum_{j=2}^{n} (-1)^j \left( \sum_{1 \leq i_1 < \ldots < i_j \leq n} [V^m V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^*(V^*)^m]_0 \right) = [V^{m+1}(V^*)^{m+1}]_0 - \sum_{i=1}^{k} [V^{m+1} V_i V_i^* (V^*)^{m+1}]_0 + \sum_{j=2}^{k} (-1)^j \left( \sum_{1 \leq i_1 < \ldots < i_j \leq k} [V^{m+1} V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^*(V^*)^{m+1}]_0 \right).
\end{equation}

If $E$ is a $C^*$-subalgebra of $B$ that contains $T_{m+1}$ (for $m \in \mathbb{N}_0$) we have

\begin{equation}
[V^{m+1}(V^*)^{m+1}]_0 - \sum_{i=1}^{k} [V^{m+1} V_i V_i^* (V^*)^{m+1}]_0 + \sum_{j=2}^{k} (-1)^j \left( \sum_{1 \leq i_1 < \ldots < i_j \leq k} [V^{m+1} V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^*(V^*)^{m+1}]_0 \right) = [V^{m+1}(V^*)^{m+1}Q]_0.
\end{equation}

It is easy to see that in each $C^*$-subalgebra of $B$ that contains $T_m$ the projection $V^m V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^*(V^*)^m$ is Murray - von Neumann equivalent to $V^m(V^*)^m$ via the partial isometry $V^m V_{i_1} \cdots V_{i_j} (V^*)^m \in T_m$, where $\{i_1, \ldots, i_j\} \subset \{1, \ldots, n\}$.

This observation together with equations (16), (17) and (18) give:

If $E$ is a $C^*$-subalgebra of $B$ that contains $T_m$ we have

\begin{equation}
[P_m]_0 - \sum_{i=1}^{n} [P_m]_0 + \sum_{j=2}^{n} (-1)^j \left( \sum_{1 \leq i_1 < \ldots < i_j \leq n} [P_m]_0 \right) = [P_{m+1}Q]_0.
\end{equation}
If $E$ is a $C^*$-subalgebra of $B$ that contains $T_m$ and $T_{m+1}$ then we have

$$ (20) \quad [P_m]_0 - \sum_{i=1}^{n} [P_m]_0 + \sum_{j=2}^{n} (-1)^j \left( \sum_{1 \leq i_1 < \cdots < i_j \leq n} [P_m]_0 \right) = $$

$$ = [P_{m+1}]_0 - \sum_{i=1}^{k} [P_{m+1}]_0 + \sum_{j=2}^{k} (-1)^j \left( \sum_{1 \leq i_1 < \cdots < i_j \leq k} [P_{m+1}]_0 \right). $$

If $E$ is a $C^*$-subalgebra of $B$ that contains $T_{m+1}$ we have

$$ (21) \quad [P_{m+1}]_0 - \sum_{i=1}^{k} [P_{m+1}]_0 + \sum_{j=2}^{k} (-1)^j \left( \sum_{1 \leq i_1 < \cdots < i_j \leq k} [P_{m+1}]_0 \right) = [P_{m+1}Q]_0. $$

The last three equations are what we had to prove. \qed

**Remark 3.2.** It also follows from this lemma that if we denote the isometries that generate $C^*(\Gamma)$ by $\tilde{V}, \tilde{V}_1, \ldots, \tilde{V}_n$, then

$$ [(I - \tilde{V}V^*) \prod_{i=1}^{n} (I - \tilde{V}_i \tilde{V}_i^*)]_0 = \chi(\Gamma)[I]_0 \text{ (in } K_0(C^*(\Gamma))) \text{.} $$

Therefore in the extension

$$ (22) \quad 0 \to \langle (I - \tilde{V}V^*) \prod_{i=1}^{n} (I - \tilde{V}_i \tilde{V}_i^*) \rangle \xrightarrow{\text{tr}} C^*(\Gamma) \xrightarrow{Q} C_Q^*(\Gamma) \to 0 $$

the map $I_{\Gamma^*}$ on $K_0$ is given by

$$ [(I - \tilde{V}V^*) \prod_{i=1}^{n} (I - \tilde{V}_i \tilde{V}_i^*)]_0 \mapsto \chi(\Gamma)[I]_0. $$

Now we can state and prove the following

**Proposition 3.3.** Suppose that $G$ is a finite graph with at least two vertices and suppose that $G^{\text{opp}}$ is connected. Then

$$ (23) \quad K_0(C_Q^*(G)) = \begin{cases} \mathbb{Z}_{\chi(G)}, & \text{if } \chi(G) \neq 0, \\ \mathbb{Z}, & \text{if } \chi(G) = 0, \end{cases} \quad K_1(C_Q^*(G)) = \begin{cases} 0, & \text{if } \chi(G) \neq 0, \\ \mathbb{Z}, & \text{if } \chi(G) = 0, \end{cases} $$

and $[1_{C_Q^*(G)}]_0$ generates $K_0(C_Q^*(G))$ in all cases.

Moreover $K_0(C^*(G)) = \mathbb{Z}$, $K_1(C^*(G)) = 0$ and $[1_{C^*(G)}]_0$ generates $K_0(C^*(G))$ in all cases.

**Proof.** We will use induction on the number of vertices of $G$. If $G$ has two vertices (and no edges) then $C_Q^*(G) = \mathcal{O}_2$ and $C^*(G) = \mathcal{E}_2$ and in this case certainly the statement is true. Suppose that the statement is true for all graphs $G$ with at most $n \geq 2$ vertices and with $G^{\text{opp}}$ connected. The graph $\Gamma$ considered above was a randomly
chosen graph with $n + 1$ vertices and with the property that $\Gamma^{\text{opp}}$ is connected. If we show that the statement holds for $\Gamma$ than this will prove the statement by induction.

We note that from Lemma 2.2 and the assumption follows that $K_0(T_m) = \mathbb{Z}[P_m]_0$ and $K_1(T_m) = 0$ for all $m \in \mathbb{N}_0$. Also since $I_m \cong K$ we have $K_0(I_m) = \mathbb{Z}[P_mQ]_0$ and $K_1(I_m) = 0$ for all $m \in \mathbb{N}_0$. Finally we remind that $\pi_m$ is an isomorphism for all $m \in \mathbb{N}_0$.

From the $K$-theory six term exact sequences for the two exact rows of (3) we have the following commutative diagram:

\[
\begin{array}{cccc}
K_0(I_{m-1}) & \xrightarrow{i_{m*}} & K_0(B_{m-1}) & \xrightarrow{p_{m*}} & K_0(B_{m-1}/I_{m-1}) \\
\downarrow I'_{m*} & & \downarrow I_{m*} & & \cong \downarrow \pi_{m*} \\
K_0(T_m) & \xrightarrow{i_{m*}} & K_0(B_m) & \xrightarrow{p_{m*}} & K_0(B_m/I_m) \\
\uparrow \gamma_{m\text{ind}} & & \uparrow \delta_{m\text{ind}} & & \downarrow \\
K_1(B_m/I_m) & \xrightarrow{p_{m*}} & K_1(B_m) & \leftarrow & 0 \\
\cong \uparrow \pi_{m*} & & \uparrow I_{m*} & & \end{array}
\]

where $\gamma_{m\text{ind}}$ and $\delta_{m\text{ind}}$ are the index maps for the corresponding six term exact sequences.

Since $I_{m-1}$ is generated by $P_mQ$ from Lemma 3.1 follows that the map $i_{m*} : K_0(I_{m-1}) \rightarrow K_0(B_{m-1})$ is induced by $[P_mQ]_{K_0(I_{m-1})} \mapsto \chi(\Gamma')[P_{m-1}]_{K_0(B_{m-1})}$. Also the map $I'_{m*} : K_0(I_{m-1}) \rightarrow K_0(T_m)$ is induced by $[P_mQ]_{K_0(I_{m-1})} \mapsto \chi(\Gamma_k)[P_m]_{K_0(T_m)}$.

When we “apply” $\beta$ to equations (1) and (2) we obtain the following commutative diagrams with exact rows:

\[
\begin{array}{ccc}
0 & \rightarrow & I_{m-1} \xrightarrow{i_{m*}} B_{m-1} \xrightarrow{p_{m*}} B_{m-1}/I_{m-1} \rightarrow 0 \\
\beta & \downarrow & \beta & \downarrow \beta \\
0 & \rightarrow & I_m \xrightarrow{i_{m+1}} B_m \xrightarrow{p_{m+1}} B_m/I_m \rightarrow 0
\end{array}
\]

and

\[
\begin{array}{ccc}
0 & \rightarrow & T_m \xrightarrow{i_m} B_m \xrightarrow{p_m} B_m/T_m \rightarrow 0 \\
\beta & \downarrow & \beta & \downarrow \beta \\
0 & \rightarrow & T_{m+1} \xrightarrow{i_{m+1}} B_{m+1} \xrightarrow{p_{m+1}} B_{m+1}/T_{m+1} \rightarrow 0
\end{array}
\]
where $\bar{\beta}$ and $\bar{\beta}'$ are induced by $\beta$ on the above quotients.

We can now start examining the five different cases depending on $\chi(\Gamma')$ and $\chi(\Gamma_k)$:

(case I): $\chi(\Gamma') = 0$ and $\chi(\Gamma_k) = 0$.

By assumption $i_m^* = 0 = I_m^*$. From (24) is easy to see that $\delta_m^\text{ind} = 0$. Therefore (24) splits into two:

$$
\cdots \xrightarrow{i_m^*=0} K_0(B_{m-1}) \xrightarrow{p_m^* = \cong} K_0(B_{m-1}/T_{m-1}) \longrightarrow 0
$$

(27)

$$
\xrightarrow{\delta_m^\text{ind}=0} \quad K_0(T_m) \xrightarrow{i_m^*} K_0(B_m) \xrightarrow{p_m^*} K_0(B_m/T_m) \longrightarrow 0,
$$

$$
0 \longrightarrow K_1(B_{m-1}) \xrightarrow{p_m^*} K_1(B_{m-1}/T_{m-1}) \xrightarrow{\delta_m^\text{ind}=0} K_0(T_{m-1}) \xrightarrow{i_m^*=0} \cdots
$$

(28)

$$
0 \longrightarrow K_1(B_m) \xrightarrow{p_m^*} K_1(B_m/T_m) \xrightarrow{\delta_m^\text{ind}=0} \cdots .
$$

Suppose by induction that $K_0(B_{m-1}) = \mathbb{Z}[P_0] \oplus \cdots \oplus \mathbb{Z}[P_{m-1}]$. Notice that for $m = 1$ we have $K_0(B_0) = \mathbb{Z}[P_0]$. Then from (27) follows that $K_0(B_m) = I_m^*(K_0(B_{m-1})) \oplus i_m^*(K_0(T_m))$ since all extensions of free abelian groups are trivial. Noting that $K_0(T_m) = \mathbb{Z}[P_m]$ concludes the induction. Therefore $K_0(B_m) = \mathbb{Z}[P_0] \oplus \cdots \oplus \mathbb{Z}[P_m]$ for each $m \in \mathbb{N}$. Notice that we can write $K_0(B_m) = \mathbb{Z}[P_0] \oplus \cdots \oplus \mathbb{Z}[\beta_m^\text{ind}([P_0])]$

Suppose by induction that

$$
K_1(B_{m-1}) = \mathbb{Z}(p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma_1^\text{ind})^{-1}([P_1Q]_0) \oplus \cdots
$$

$$
\cdots \oplus \mathbb{Z}(p_{m-1*})^{-1} \circ \pi_{m-1*} \circ (\gamma_m^\text{ind})^{-1}([P_{m-1}Q]_0).
$$

This is trivially true for $m = 1$. From (28) we see that $K_1(B_m) = (p_m^*)^{-1} \circ \pi_m^* (K_1(B_{m-1}/T_{m-1}))$. Since all groups are free abelian all the extensions are trivial and therefore $K_1(B_m) = I_m^*(K(B_{m-1})) \oplus \mathbb{Z}(p_m^*)^{-1} \circ \pi_m^* \circ (\gamma_m^\text{ind})^{-1}([P_mQ]_0)$. This concludes the induction. From the functoriality of the index map and from equations (25) and (26) follows that $(p_m^*)^{-1} \circ \pi_m^* \circ (\gamma_m^\text{ind})^{-1}([P_mQ]_0) = (p_m^*)^{-1} \circ \pi_m^* \circ (\gamma_m^\text{ind})^{-1} \circ \beta^*(P_{m-1}Q_0) = \beta \circ (p_{m-1})^{-1} \circ \pi_{m-1*} \circ (\gamma_{m-1})^{-1}([P_{m-1}Q]_0)$. Therefore we can write

$$
K_1(B_m) = \mathbb{Z}(p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma_1^\text{ind})^{-1}([P_1Q]_0) \oplus \cdots
$$

$$
\cdots \oplus \mathbb{Z}(p_{m-1*})^{-1} \circ \pi_{m-1*} \circ (\gamma_{m-1})^{-1}([P_{m-1}Q]_0).
$$

If $u \in B_1$ is a unitary with $[u]_1 = (p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma_1^\text{ind})^{-1}([P_1Q]_0)$ then we can write

$$
K_1(B_m) = \mathbb{Z}[u]_1 \oplus \cdots \oplus \mathbb{Z}[\beta_{m-1}^\text{ind}([u]_1)].
$$
From this we get $K_0(B) = \bigoplus_{i=0}^{\infty} \mathbb{Z} \beta_+^i([I_0])$ and $K_1(B) = \bigoplus_{i=1}^{\infty} \mathbb{Z} \beta_+^{i-1}([u_1])$.

Let $\tilde{u} = \alpha^0 \circ j_0(u) \in \tilde{B}$. Then it is easy to see that $K_0(\tilde{B}) = \bigoplus_{i=\infty}^{\infty} \mathbb{Z} \Phi^i([\tilde{P}_0])$ and $K_1(\tilde{B}) = \bigoplus_{i=\infty}^{\infty} \mathbb{Z} \Phi^i([-\tilde{u}_1])$.

The Pimsner-Voiculescu gives

$$
\begin{array}{cccc}
\bigoplus_{i=\infty}^{\infty} \mathbb{Z} \Phi^i([\tilde{P}_0]) & \xrightarrow{id_{\mathbb{Z}}-\Phi_*} & \bigoplus_{i=\infty}^{\infty} \mathbb{Z} \Phi^i([\tilde{P}_0]) & \longrightarrow & K_0(\tilde{A}) \\
\uparrow & & \downarrow & & \\
K_1(\tilde{A}) & \xleftarrow{id_{\mathbb{Z}}-\Phi_*} & \bigoplus_{i=\infty}^{\infty} \mathbb{Z} \Phi^i([-\tilde{u}_1]) & \xrightarrow{id_{\mathbb{Z}}-\Phi_*} & \bigoplus_{i=\infty}^{\infty} \mathbb{Z} \Phi^i([-\tilde{u}_1]).
\end{array}
$$

From this we can conclude that $K_0(\tilde{A}) = \mathbb{Z}[\tilde{P}_0], \ K_1(\tilde{A}) = \mathbb{Z}$. From Proposition 2.4 follows that $K_0(\tilde{A}) \cong K_0(C^*_Q(\Gamma)) = \mathbb{Z}$ and $K_1(\tilde{A}) \cong K_1(C^*_Q(\Gamma)) = \mathbb{Z}$ and that $[1_{C_2(\Gamma)}_0]$ generates $K_0(C^*_Q(\Gamma))$.

From Remark 3.2 follows that in the extension (22) the map $I_{\Gamma_0}$ on $K_0$ is zero. This shows that $K_0(C^*_r(\Gamma)) = \mathbb{Z}[1_{C^*_r(\Gamma)}_0]$ and $K_1(C^*_r(\Gamma)) = 0$.

This concludes the proof of (case I).

(case II): $\chi(\Gamma') \neq 0$ and $\chi(\Gamma_k) = 0$.

By assumption $K_0(B_0) = \mathbb{Z}[P_0], \ K_0(B_0/I_0) = \mathbb{Z}1_{\chi(\Gamma')}p_1([P_0]), \ K_1(B_0) = 0$ and $K_1(B_0/I_0) = 0$.

Suppose by induction that

$$K_0(B_{m-1}) = \mathbb{Z}[P_{m-1}]_{0} \oplus \mathbb{Z}1_{\chi(\Gamma')}[P_{m-2}]_{0} \oplus \cdots \oplus \mathbb{Z}1_{\chi(\Gamma')}[P_0],$$

$$K_0(B_{m-1}/I_{m-1}) = \mathbb{Z}1_{\chi(\Gamma')}p_{m*}([P_{m-1}]_{0}) \oplus \mathbb{Z}1_{\chi(\Gamma')}p_{m*}([P_{m-2}]_{0}) \oplus \cdots \oplus \mathbb{Z}1_{\chi(\Gamma')}p_{m*}([P_0]),$$

$$K_1(B_{m-1}) = 0 \text{ and } K_1(B_{m-1}/I_{m-1}) = 0.$$  

Then from diagram (24) immediately follows that $K_1(B_m) = 0$ and $K_1(B_m/I_{m}) = 0$.

Then (24) reduces to the following commutative diagram with exact rows:

$$
\begin{array}{cccc}
0 & \longrightarrow & K_0(I_{m-1}) & \xrightarrow{i_{m*}} & K_0(B_{m-1}) & \xrightarrow{p_{m*}} & K_0(B_{m-1}/I_{m-1}) & \longrightarrow & 0 \\
0 & \xrightarrow{I_{m*}} & K_0(T_{m}) & \xrightarrow{i_{m*}} & K_0(B_{m}) & \xrightarrow{p_{m*}} & K_0(B_{m}/T_{m}) & \longrightarrow & 0.
\end{array}
$$

(29)

From Lemma 3.3 we have that $\chi(\Gamma')1_{P_0} = 0$ in $K_0(B_m)$ for $l = 0, \ldots, m - 1$. Since $\pi_{m*}$ is an isomorphism then by the induction hypothesis and (29) it is easy to see that $p_{m*}$ restricted to $\mathcal{G} = \langle [P_0], \ldots, [P_{m-1}]_0 \rangle$ is an isomorphism. This fact also implies that there are no relations between $[P_m]_0$ and $\mathcal{G}$ (since the bottom row of (29) is exact). Since $i_{m*}$ is injective then $[P_m]_0$ in of infinite order in $K_0(B_m)$. Clearly $K_0(B_m)$ is generated by $[P_m]_0$ and $\mathcal{G}$. Therefore

$$K_0(B_m) = \mathbb{Z}[P_m]_0 \oplus \mathbb{Z}1_{\chi(\Gamma')}[P_{m-1}]_{0} \oplus \cdots \oplus \mathbb{Z}1_{\chi(\Gamma')}[P_0].$$
From the following six term exact sequence
\[
\begin{array}{c}
K_0(I_m) \xrightarrow{i_{m*}} K_0(B_m) \xrightarrow{p_{m*}} K_0(B_m/I_m) \\
\uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
We conclude that $\delta_m^{\text{ind}}$ is "multiplication by $\chi(\Gamma_k)"$. Thus $[P_m]_0$ in $K_0(B_m)$ is of order $\chi(\Gamma_k)$ (as should be by Lemma 3.1). Therefore

$$K_0(B_m) = \mathbb{Z}[P_0]_0 \oplus \mathbb{Z}_{[\chi(\Gamma_k)]}[P_1]_0 \oplus \cdots \oplus \mathbb{Z}_{[\chi(\Gamma_k)]}[P_m]_0.$$  

We also showed that $\delta_m^{\text{ind}}$ is injective and therefore $K_1(B_m) = 0$.

Now we easily get $K_0(B) = \mathbb{Z}[P_0]_0 \oplus \mathbb{Z}_{[\chi(\Gamma_k)]}[P_1]_0$ and $K_1(B) = 0$. From this follows that $K_0(\tilde{B}) = \oplus_{i=1}^{\infty} \mathbb{Z}_{[\chi(\Gamma_k)]}[\tilde{P}_0]_0$ and $K_1(\tilde{B}) = 0$.

The Pimsner-Voiculescu exact sequence gives

$$\begin{array}{c}
\mathbb{Z}_{[\chi(\Gamma_k)]}[\tilde{P}_0]_0 \\ \mathbb{Z}_{[\chi(\Gamma_k)]}[\tilde{P}_0]_0 \oplus \mathbb{Z}_{[\chi(\Gamma_k)]}[\tilde{P}_0]_0 \\
\mathbb{Z}_{[\chi(\Gamma_k)]}[\tilde{P}_0]_0 \oplus \mathbb{Z}_{[\chi(\Gamma_k)]}[\tilde{P}_0]_0 \\
\end{array}$$

From Proposition 2.3 we get $K_0(C^*_\Gamma(\Gamma)) = \mathbb{Z}_{[\chi(\Gamma)]}[1_{C^*_\Gamma(\Gamma)}]_0$ and $K_1(C^*_\Gamma(\Gamma)) = 0$ (notice that $\chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k) = 0 - \chi(\Gamma_k)$).

From Remark 3.2 follows that $I_{\Gamma_*}$ is "multiplication by $\chi(\Gamma)"$, so $K_0(C^*_\Gamma(\Gamma)) = \mathbb{Z}_{[1_{C^*_\Gamma(\Gamma)}]}_0$ and $K_1(C^*_\Gamma(\Gamma)) = 0$.

This concludes the proof of (Case III).

(cases IV and V): $\chi(\Gamma') \neq 0, \chi(\Gamma_k) \neq 0$.

Let's denote $x = \chi(\Gamma'), y = \chi(\Gamma_k)$ and let $\text{GCD}(x, y) = d > 0$ be the greatest common divisor of $x$ and $y$. Then by the Bézout’s identity there exist $a, b \in \mathbb{Z}$ such that $ax + by = d$. Denote also $x' = x/d$ and $y' = y/d$. Then $ax' + by' = 1$.

By assumption we have that $K_0(B_0) = \mathbb{Z}[P_0]_0$, $K_1(B_0/\mathcal{I}_0) = 0$, $K_1(B_0) = 0$ and $K_0(B_0/\mathcal{I}_0) = \mathbb{Z}[x]_{P_1}[P_0]_0$.

Then $0 = K_1(B_0/\mathcal{I}_0) \cong K_1(B_1/T_1)$ which implies $K_1(B_1) = 0$. Diagram (21) for $m = 1$ reduces to

$$\begin{array}{c}
0 \rightarrow K_0(\mathcal{I}_0) \rightarrow K_0(B_0) \rightarrow K_0(\mathcal{I}_0/T_0) \rightarrow 0 \\
0 \rightarrow K_0(T_1) \rightarrow K_0(B_1) \rightarrow K_0(\mathcal{I}_0/B_1) \rightarrow 0. \\
\end{array}$$

Clearly in $K_0(B_1)$ we have $x[P_0]_0 - y[P_1]_0 = 0$. Consider $g = b[P_0]_0 + a[P_1]_0$, $g' = x'[P_0]_0 - y'[P_1]_0 \in K_0(B_1)$. Since $ag' + y'g = (ax' + y'b)[P_0]_0 = [P_0]_0$ and $x'g - bg' = (x'a + y'b)[P_1]_0 = [P_1]_0$ it follows that $g$ and $g'$ generate $K_0(B_1)$. Since $i_{1*}$ is injective on $K_0$ it follows that $K_0(B_1)$ is an infinite group. Clearly $dg' = dx'[P_0]_0 - dy'[P_1]_0 = 0$. Therefore $g$ is of infinite order in $K_0(B_1)$ and moreover $g$ and $g'$ are not related (or otherwise $g$ would be of finite order). If we suppose that $0 < d' \mid d$ and $d'g' = 0$ then it will follow that $d'x'[P_0]_0 = d'y'[P_1]_0 \in \ker(p_{1*})$. But the order of $p_{1*}([P_0]_0)$ in $K_0(B_1/T_1)$ is $|x|$, so $d'x' \geq x$ or $d' \geq d$. Therefore $d' = d$ and $K_0(B_1) = \mathbb{Z}(b[P_0]_0 + a[P_1]_0) \oplus \mathbb{Z}[x'[P_0]_0 - y'[P_1]_0]$.
Therefore we showed that \(K_0(B_1) = \{\mathbb{Z}[P_0]_0 \oplus \mathbb{Z}[P_1]_0 | x[P_0]_0 - y[P_1]_0 = 0\}\). Suppose by induction that for \(m \geq 2\), \(K_1(B_{m-1}) = 0\) and that 
\[K_0(B_{m-1}) = \{\mathbb{Z}[P_0]_0 \oplus \cdots \oplus \mathbb{Z}[P_{m-1}]_0 | x[P_0]_0 - y[P_1]_0 = 0, \ldots, x[P_{m-2}]_0 - y[P_{m-1}]_0 = 0\}\).  

From the induction hypothesis follows that \([P_{m-1}]_0\) is of infinite order in \(K_0(B_{m-1})\) and therefore that \(i_{m*} : K_0(\mathcal{I}_{m-1}) \rightarrow K_0(B_{m-1})\) is injective and therefore \(0 = K_1(B_{m-1}/I_{m-1}) \cong K_1(B_m/T_m)\). This shows that \(K_1(B_m) = 0\) and that (24) reduces to 
\[
\begin{align*}
0 & \longrightarrow K_0(I_{m-1}) \xrightarrow{i_{m*}} K_0(B_{m-1}) \xrightarrow{i_{m*}} K_0(B_{m-1}/I_{m-1}) \longrightarrow 0 \\
& \begin{array}{c}
0 \longrightarrow K_0(T_m) \xrightarrow{i_{m*}} K_0(B_m) \xrightarrow{p_{m*}} K_0(B_m/T_m) \longrightarrow 0.
\end{array}
\end{align*}
\]

It is easy to see that 
\[K_0(B_{m-1}/I_{m-1}) = \{\mathbb{Z}P'_{m*}([P_{m-1}]_0) \oplus \cdots \oplus \mathbb{Z}P'_{m*}([P_0]_0) | xP'_{m*}([P_{m-1}]_0) = 0, \ldots, xP'_{m*}([P_0]_0) = 0\}\]. Since \(i'_{m*}\) is “multiplication by \(\chi(\Gamma_k)\)” and therefore injective then by the Five Lemma follows that \(I_{m*}\) is also injective. Therefore if we denote \(G = I_{m*}(K_0(B_{m-1}))\) then \(K_0(B_m) = ([P_0]_0, G)\). One obvious relation in \(K_0(B_m)\) beside the relations that come from \(K_0(B_{m-1})\) is \(x[P_{m-1}]_0 - y[P_0]_0 = 0\) and this relation follows from Lemma 3.1. Therefore \(K_0(B_m)\) is a quotient of the group 
\[F = \{\mathbb{Z}\rho_0 \oplus \cdots \oplus \mathbb{Z}\rho_m | x\rho_{m-1} - y\rho_m = 0, \ldots, x\rho_0 - y\rho_1 = 0\}\], where the quotient map \(f : F \rightarrow K_0(B_m)\) is defined on the generators as \(\rho_l \mapsto [P_l]_0\), \(l = 0, \ldots, m\). Then if \(F' = \mathbb{Z}\rho_m\) the quotient \(F' = F/F'\) is isomorphic to 
\[F_q = \{\mathbb{Z}\rho_0 \oplus \cdots \oplus \mathbb{Z}\rho_m | x\rho_{m-1} - y\rho_m = 0, \ldots, x\rho_0 - y\rho_1 = 0, \rho_m = 0\} = \{\mathbb{Z}\rho_0 \oplus \cdots \oplus \mathbb{Z}\rho_{m-1} | x\rho_{m-1} = 0, x\rho_{m-2} - y\rho_{m-1} = 0, \ldots, x\rho_0 - y\rho_1 = 0\}\].  

Obviously we have the commutative diagram of abelian groups with exact rows 
\[
\begin{array}{c}
0 \longrightarrow F' \xrightarrow{f'} \longrightarrow F \xrightarrow{f} \longrightarrow F_q \longrightarrow 0 \\
0 \longrightarrow K_0(T_m) \xrightarrow{i_{m*}} K_0(B_m) \xrightarrow{p_{m*}} K_0(B_m/T_m) \longrightarrow 0,
\end{array}
\]

where \(f_q\) is the homomorphism induced by \(f\) and \(f'\) is the restriction of \(f\) to \(F'\). Then obviously \(f'\) and \(f_q\) are isomorphisms (since \(\pi_{m*}\) is an isomorphism). Therefore by the Five Lemma follows that \(f\) is also an isomorphism. 

This shows that 
\[K_0(B_m) = \{\mathbb{Z}[P_0]_0 \oplus \cdots \oplus \mathbb{Z}[P_{m}]_0 | x[P_0]_0 - y[P_1]_0 = 0, \ldots, x[P_{m-1}]_0 - y[P_m]_0 = 0\}\]. We also showed above that \(K_1(B_m) = 0\) and this concludes the induction. 

Now it is easy to see that \(K_1(B) = 0\) and that 
\[K_0(B) = \{\oplus_{i=0}^{\infty} \mathbb{Z}\beta^i_\tau([P_0]_0) | \chi(\Gamma')\beta^i_\tau([P_0]_0) - \chi(\Gamma_k)\beta^{i+1}_\tau([P_0]_0) = 0, i \in \mathbb{N}_0\}\].
Then $K_1(\tilde{B}) = 0$ and
\[
K_0(\tilde{B}) = \{ \sum_{i=-\infty}^{\infty} \mathbb{Z}\Phi_i^*([\tilde{P}_0]_0)|\chi(\Gamma')\Phi_i^*([\tilde{P}_0]_0) - \chi(\Gamma_k)\Phi_i^*([\tilde{P}_0]_0) = 0, \ i \in \mathbb{Z} \}.
\]

The Pimsner-Voiculescu exact sequence gives
\[
\begin{array}{cccc}
K_0(\tilde{B}) & \xrightarrow{id_* - \Phi_*} & K_0(\tilde{B}) & \xrightarrow{} K_0(\tilde{A}) \\
\uparrow & & \downarrow & \\
K_1(\tilde{A}) & \leftarrow 0 & \leftarrow 0.
\end{array}
\]

(Case IV): $\chi(\Gamma') \neq 0$, $\chi(\Gamma_k) \neq 0$ and $\chi(\Gamma') = \chi(\Gamma_k)$.

In this case
\[
K_0(\tilde{A}) = \{ \sum_{i=-\infty}^{\infty} \mathbb{Z}\Phi_i^*([\tilde{P}_0]_0)|\chi(\Gamma')\Phi_i^*([\tilde{P}_0]_0) - \chi(\Gamma_k)\Phi_i^*([\tilde{P}_0]_0) = 0,
\]
\[
\Phi_i^*([\tilde{P}_0]_0) - \Phi_i^*([\tilde{P}_0]_0) = 0, \ i \in \mathbb{Z} \} = 
\]
\[
\{ \sum_{i=-\infty}^{\infty} \mathbb{Z}\Phi_i^*([\tilde{P}_0]_0)|\Phi_i^*([\tilde{P}_0]_0) - \Phi_i^*([\tilde{P}_0]_0) = 0, \ i \in \mathbb{Z} \} = \mathbb{Z}[\tilde{P}_0]_0.
\]

To examine $\ker(id_* - \Phi_*)$ take $\omega = \sum_{i=-j}^{j} t_i\Phi_i^*([\tilde{P}_0]_0) \in \ker(id_* - \Phi_*)$, where $t_i \in \mathbb{Z}$. Then
\[
0 = (id_* - \Phi_*)(\omega) = \sum_{i=-j}^{j} t_i(id_* - \Phi_*)(\Phi_i^*([\tilde{P}_0]_0)) = \sum_{i=-j}^{j} t_i(\Phi_i^*([\tilde{P}_0]_0) - \Phi_i^*([\tilde{P}_0]_0)).
\]

Therefore $t_i = s_i|\chi(\Gamma')|$ for some integers $s_i, \ i = -j, \ldots, j$. From this easily follows that $\omega = \sum_{i=-j}^{j} s_i|\chi(\Gamma')|[\tilde{P}_0]_0$. Thus $\ker(id_* - \Phi_*) = |\chi(\Gamma')|\mathbb{Z}[\tilde{P}_0]_0$. This shows that $K_1(\tilde{A}) = \mathbb{Z}$.

From Proposition 2.4 follows that $K_0(\tilde{A}) \cong K_0(C^*_Q(\Gamma)) = \mathbb{Z}$, $K_1(\tilde{A}) \cong K_1(C^*_Q(\Gamma)) = \mathbb{Z}$ and that $[1_{C^*_Q(\Gamma)}]_0$ generates $K_0(C^*_Q(\Gamma))$.

From Remark 3.2 follows that in the extension (22) the map $I_{\Gamma_*}$ on $K_0$ is zero (since $\chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k) = 0$). Therefore $K_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $K_1(C^*(\Gamma)) = 0$.

This concludes the proof of (case IV).

(cas eV): $\chi(\Gamma') \neq 0$, $\chi(\Gamma_k) \neq 0$ and $\chi(\Gamma') \neq \chi(\Gamma_k)$.

In this case
\[
K_0(\tilde{A}) = \{ \sum_{i=-\infty}^{\infty} \mathbb{Z}\Phi_i^*([\tilde{P}_0]_0)|\chi(\Gamma')\Phi_i^*([\tilde{P}_0]_0) - \chi(\Gamma_k)\Phi_i^*([\tilde{P}_0]_0) = 0,
\]
\[
\Phi_i^*([\tilde{P}_0]_0) - \Phi_i^*([\tilde{P}_0]_0) = 0, \ i \in \mathbb{Z} \} = 
\]
\[
\{ \mathbb{Z}[\tilde{P}_0]_0|\chi(\Gamma')\tilde{P}_0 - \chi(\Gamma_k)[\tilde{P}_0]_0 = 0 \} = \mathbb{Z}[\chi(\Gamma')\tilde{P}_0]_0 = \mathbb{Z}[\chi(\Gamma)][\tilde{P}_0]_0.
\]
We only need to show that $K_1(\tilde{A}) = 0$ or that $\text{id}_* - \Phi_*$ is injective.

Take $\omega = \sum_{i=-j}^j t_j \Phi_*^i([\tilde{P}_0]_0)$, $t_i \in \mathbb{Z}$ and suppose that $(\text{id}_* - \Phi_*) (\omega) = 0$. Then

$$0 = (\text{id}_* - \Phi_*) (\omega) = \sum_{i=-j}^j t_j (\Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0)) =$$

$$= t_{-j} \Phi_*^{-j}([\tilde{P}_0]_0) + \sum_{i=-j+1}^j (t_i - t_{i-1}) \Phi_*^i([\tilde{P}_0]_0) - t_j \Phi_*^{i+1}([\tilde{P}_0]_0).$$

If $\chi(\Gamma')$ doesn’t divide $t_{-j}$ then the equality $-t_{-j} \Phi_*^{-j}([\tilde{P}_0]_0) = (t_i - t_{i-1}) \Phi_*^i([\tilde{P}_0]_0) - t_j \Phi_*^{i+1}([\tilde{P}_0]_0)$ is impossible. If $\chi(\Gamma')$ divides $t_{-j}$ then $\omega$ can be expressed in terms of $\Phi_*^{-j}([\tilde{P}_0]_0), \ldots, \Phi_*^j([\tilde{P}_0]_0)$. By induction we see that we can write $\omega = t[\tilde{P}_0]_0$ for some $t \in \mathbb{Z}$. But then clearly $(\text{id}_* - \Phi_*) (\omega) = 0$ is possible if and only if $t = 0$. This shows that $\text{id}_* - \Phi_*$ is injective and therefore that $K_1(\tilde{A}) = 0$.

From Proposition 2.4 we get $K_0(C_0^*(\Gamma)) = \mathbb{Z}[\chi(\Gamma)][1_{C_0^*(\Gamma)}]_0$ and $K_1(C_0^*(\Gamma)) = 0$.

From Remark 3.2 follows that $I_{\Gamma_*}$ is “multiplication by $\chi(\Gamma)$”, so $K_0(C_*^*(\Gamma)) = \mathbb{Z}[1_{C_*^*(\Gamma)}]_0$ and $K_1(C^*(\Gamma)) = 0$.

This concludes the proof of (Case V).

The Proposition is proved. □

Now we can apply the Kirchberg-Phillips Classification theorem ([16]) to $C_*^*(G)$ for a finite graph $G$ such that $G^\text{opp}$ is connected and with at least two vertices, using Theorem 1.1, Proposition 2.4 and Proposition 3.3. We obtain

$$C_*^*(G) \cong \mathcal{O}_{1+|\chi(G)|}.$$ (33)

For infinite graphs with connected opposite graphs we can argue similarly as in [8, Corollary 3.11] to prove the following:

**Proposition 3.4.** Let $G$ an infinite graph with countably many vertices and such that $G^\text{opp}$ is connected. Then $C^*(G) (= C_0^*(G))$ is nuclear and belongs to the small bootstrap class. Moreover $K_0(C_*^*(G)) = \mathbb{Z}[1_{C_*^*(G)}]_0$ and $K_1(C^*(G)) = 0$.

**Proof.** By induction we will find a increasing sequence $G_n$ of subgraphs of $G$ with $n$ vertices, $n \geq 2$ which are such that $G_n^{\text{opp}}$ is connected for each $n \geq 2$ and also $G_n \xrightarrow{n \rightarrow \infty} G$. Obviously we can find two vertices $v_1$ and $v_2$ that are not connected (since $G^{\text{opp}}$ is connected). Then we chose $G_2$ to be the graph with vertices $v_1$ and $v_2$ and no edges. Suppose we have defined the subgraph $G_n$ for some $n \geq 2$. Let $v_1, \ldots, v_n$ be the vertices of $G_n$. Since $G^{\text{opp}}$ is connected we can find a vertex $v_{n+1}$ of $G$ different from $v_1, \ldots, v_n$ such that $v_{n+1}$ is not connected with all of the vertices $v_1, \ldots, v_n$. Then obviously the subgraph $G_{n+1}$ of $G$ on vertices $v_1, \ldots, v_{n+1}$ and edges coming from $G$ is such that $G_{n+1}^{\text{opp}}$ is connected. This completes the induction.

From Proposition 3.3 we have $K_0(C_*^*(G_n)) = \mathbb{Z}[1_{C_*^*(G_n)}]_0$ and $K_1(C^*(G_n)) = 0$. It is easy to see that $C^*(G) = \lim_{n \rightarrow \infty} C^*(G_n)$. Therefore from Proposition 2.4 we get that $C_*^*(G)$ is nuclear and belongs to the small bootstrap category $\mathcal{N}$. Also $K_0(C^*(G)) = \lim_{n \rightarrow \infty} K_0(C^*(G_n)) = \mathbb{Z}[1_{C^*(G)}]_0$ and $K_1(C^*(G)) = \lim_{n \rightarrow \infty} K_1(C^*(G_n)) = 0$.

This proves the proposition. □
From Theorem 1.1 we know that $C^*(G) = C^*_Q(G)$ is purely infinite and simple. Again using Kirchberg-Phillips theorem we get that if $G$ is an infinite graph on countably many vertices such that $G^{op}$ is connected then $C^*(G) = C^*_Q(G) \cong O_\infty$. If we define for an infinite countable graph $G$ with $G^{op}$ connected $\chi(G) \overset{def}{=} \infty$ then we can write once again $C^*_Q(G) \cong O_{1+|\chi(G)|}$.

**Remark 3.5.** Let $G_1$ and $G_2$ be two disjoint graphs. Then by $G_1 \ast G_2$ we denote their join which is the graph obtained from $G_1$ and $G_2$ by connecting each vertex of $G_1$ with each vertex of $G_2$. Then if we start with a graph $G$ on countably many vertices which is such that $G^{op}$ doesn’t have any isolated vertices then we can find a sequence of subgraphs $G_n$, $n \in \mathbb{N}$ (some of $G_n$’s can have zero vertices) such that $G^{op}$ are all connected and such that $G = \bigast_{n=1}^\infty G_n$. For a graph $F$ with zero vertices we write $C^*_Q(F) = \mathbb{C}$.

Then from Theorem 1.1 easily follows that $C^*_Q(G) = \bigotimes_{n=1}^\infty C^*_Q(G_n)$.

Now we can record our main result:

**Theorem 3.6.** Let $G$ be a graph with at least two and at most countably many vertices such that $G^{op}$ has no isolated vertices. Write $G = \bigast_{n=1}^\infty G_n$ as in Remark 3.5 with $G_n$ being a subgraph of $G$ such that $G_n^{op}$ is connected.

Then

$$C^*_Q(G) = \bigotimes_{n=1}^\infty C^*_Q(G_n) \cong \bigotimes_{n=1}^\infty O_{1+|\chi(G_n)|}.$$  \hfill (34)

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