ITERATED COMMUTATORS UNDER A JOINT CONDITION ON THE TUPLE OF MULTIPLYING FUNCTIONS

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ABSTRACT. We present a pair of joint conditions on the two functions $b_1, b_2$ strictly weaker than $b_1, b_2 \in \text{BMO}$ that almost characterize the $L^2$ boundedness of the iterated commutator $[b_2, [b_1, T]]$ of these functions and a Calderón–Zygmund operator $T$. Namely, we sandwich this boundedness between two bisublinear mean oscillation conditions of which one is a slightly bumped up version of the other.

1. INTRODUCTION

The study of commutators of Calderón-Zygmund operators with pointwise multiplication has been a long standing interest in the field of harmonic analysis; for example, in the fundamental paper of Coifman, Rochberg, Weiss [2] a characterization of the space $\text{BMO}(\mathbb{R}^d)$ is given with respect to the commutator taken with the Riesz transforms:

$$[b, R_i] : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

boundedly for all $i = 1, \ldots, d$

if and only if $b \in \text{BMO}(\mathbb{R}^d)$. Here $[b, T] = bT - T(b \cdot)$. Already in [2] it was shown that $b \in \text{BMO}$ is a sufficient condition for the boundedness of the iterated commutator $[b, [b, \ldots, [b, T]]]$ and the same argument extends to the case of commutators $[b_k, [b_{k-1}, \ldots, [b_1, T]]]$ with different functions, all in BMO separately.

Our object is to make the first systematic study of the iterated commutator $[b_2, [b_1, T]]$ in the case of two different functions $b_1, b_2$. In particular, we want to identify a joint condition on the pair $(b_1, b_2)$ that is weaker than the individual conditions $b_1, b_2 \in \text{BMO}$, that is as close to optimal as possible, and which still guarantees the boundedness of the commutator. This is, in some sense, similar in spirit to the case of bilinear weighted theory, where $w_1, w_2 \in A_4$ is not the optimal condition for the boundedness of bilinear singular integrals from $L^4(w_1) \times L^4(w_2)$ to $L^2(w_1^{1/2} w_2^{1/2})$, but rather there is a genuinely bilinear joint condition $(w_1, w_2) \in A_{(4,4)}$ introduced by Lerner, Ombrosi, Pérez, Torres and Trujillo-González [11]. In the weighted case the identification of this genuinely bilinear condition has been highly impactful.

We study two-sided estimates for the $L^2 \to L^2$ norm of the commutator $[b_2, [b_1, T]]$. While the upper bounds will be valid for all bounded singular integrals, the lower bounds require some suitable non-degeneracy, and here we work with the Riesz transforms

$$R_j f(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) \, dy, \quad f \in L^2(\mathbb{R}^d), \quad j = 1, \ldots, d.$$  

We show that

$$C_T(S_2(b_1, b_2) + T_2(b_1, b_2)) \leq \|[b_2, [b_1, T]]\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_T(S_2(b_2, b_2) + T_2(b_1, b_2)), \quad (1.1)$$

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where the constant $C_{\varepsilon}$ tends to infinity as $\varepsilon$ tends to zero and the joint conditions $S_p, T_p$, with $0 < p < \infty$, imposed on the complex valued functions $b_1, b_2$ are defined by

$$S_p(b_1, b_2) = \sup_Q \left( \frac{1}{|Q|} \int_Q |b_1 - \langle b_1 \rangle_Q|^p \right)^{1/p},$$

and

$$T_p(b_1, b_2) = \sup_Q \left( \frac{1}{|Q|} \int_Q |b_2 - \langle b_2 \rangle_Q|^p \right)^{1/p},$$

for all $b_1, b_2 \in L^p_{\text{loc}}(\mathbb{R}^d)$.

Here the supremums are taken over all cubes. Whenever it is well understood which functions $b_1, b_2$ are in question, we refer to these conditions shortly as $S_p$ and $T_p$.

We show by example that the lower bound in (1.1) does not improve to $S_{2+\varepsilon}(b_1, b_2) + T_{2+\varepsilon}(b_1, b_2)$ for any $\varepsilon > 0$ — that is, the obtained upper bound is not necessary. This leads us to consider joint conditions involving Young functions that can be made strictly weaker than $S_{2+\varepsilon} + T_{2+\varepsilon}$ for all $\varepsilon > 0$. Hence, we prove the commutator upper bound with these updated conditions with a version of the sparse domination principle introduced in Lerner and Ombrosi [10].

1.1. Basic notation. We denote $A \lesssim B$, if $A \leq CB$ for some constant $C > 0$ depending only on the dimension of the underlying space, on integration exponents and on other concurrently unimportant absolute constants appearing in the assumptions. Then naturally $A \sim B$, if $A \lesssim B$ and $B \lesssim A$. Subscripts on constants ($C_{a,b,c,\ldots}$) signify their dependence on those subscripts.

We also denote the space $L^p(\mathbb{R}^d)$ with $L^p$.

Integral average is by dash or brackets: $\frac{1}{|Q|} \int_Q f = \|f\|_Q = \langle f \rangle_Q$.

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2. THE CONDITIONS $S_p, T_p$ AND $S_{A,B}, T_C$

Before proceeding any further, let us precisely define the commutator $[b_2, [b_1, T]]$.

Definition 2.1. Let $b_i, i = 1, 2$, be such that $b_1, b_2, b_1b_2 \in L^p_{\text{loc}}(\mathbb{R}^d)$ and denote $b = (b_1, b_2)$. With $T$ being an operator on $L^2(\mathbb{R}^d)$, the commutator $C_b T$ on $L^\infty(\mathbb{R}^d)$ is defined as

$$C_b T = [b_2, [b_1, T]],$$

where $[A, B] = AB - BA$ for any two operations $A, B$, and $b_i f(\cdot) = b_i(\cdot) f(\cdot)$.

We deal with the second order commutator $[b_2, [b_1, T]]$ but our results concerning sufficient conditions could just as well be formulated in the higher order cases.

It follows by the John-Nirenberg inequality that if $b_1, b_2 \in \text{BMO}$, then the conditions $S_p, T_q$ hold for all $p, q \geq 1$. Hence, a natural question is immediate: Are $S_p$, and respectively $T_p$, equivalent for all $1 \leq p < \infty$. Or even in a weaker sense: if both of the conditions $S_p, T_q$ hold simultaneously, could we deduce $S_q$ or $T_q$ for some $q > p$? By Theorem 2.4, the answer is no.

The next proposition will clarify the situation and point out how the counterexample in Theorem 2.4 can be constructed.

Recall, that a function $\omega : \mathbb{R}^d \to (0, \infty)$ is said to be in the class of $A_p$ weights, $1 < p < \infty$, if

$$[w]_{A_p} = \sup_Q (w/\langle w \rangle_Q)^{p'} < \infty, \quad p' = \frac{p}{p-1},$$

where the supremum is taken over all cubes.

Proposition 2.2. Given $1 < p < q < \infty$, there exists functions $\phi, \psi \in L^p_{\text{loc}}$ satisfying the conditions $S_p, T_p$ and failing the condition $S_q$. 

Proof. Let
\[ ψ(x) = x^{-\frac{2}{p+q}}1_{(0,1)}, \quad φ = ψ^{-1}1_{(0,1)}. \]
We check that the conditions \( S_2, T_q \) hold. Let \( [a, b] \) be an arbitrary interval such that \( [a, b] \cap [0, 1) = [c, d] \neq \emptyset \) (if the intersection is empty, then the claim is trivial). First,
\[ \frac{1}{b - a} \int_a^b |ψ - ⟨ψ⟩| dα \frac{1}{b - a} \int_a^b |φ - ⟨φ⟩| dα \leq \frac{1}{d - c} 2^{p+1} \int_c^d φ^p \frac{1}{d - c} 2^{p+1} \int_c^d φ^p. \]
Then, by the fact (see Grafakos [1]) that \( |x|^{-\frac{2}{p+q}} \in A_2 \), we have
\[ \frac{1}{d - c} \int_c^d φ^p \frac{1}{d - c} \int_c^d φ^p \leq |x|^{-\frac{2}{p+q}}A_2. \]
It follows that \( S_2(ψ, φ) \leq 1. \)
By the above estimates and \( φψ \leq 1 \), it follows for an arbitrary interval \( I \) that
\[ \frac{1}{|I|} \int_I |ψ - ⟨ψ⟩|^p |φ - ⟨φ⟩|^p \leq 4^p \frac{1}{|I|} \left( \int_I φ^p φ^p + \int_I φ^p ⟨φ⟩^p + \int_I ⟨φ⟩^p ⟨φ⟩^p \right) \leq 1. \]
Hence \( T_q(b_1, b_2) < ∞. \)
On the other hand by \(-2q/(p+q) < -1,\) the singularity in \( \int_0^1 |ψ - ⟨ψ⟩| \) is not integrable, and by \( j_0^1 |φ - ⟨φ⟩| > 0, \) we have \( S_2(ψ, φ) = ∞. \)

**Remark 2.3.** If one wishes to have \( ψ, φ ∈ L^∞_{loc}, \) say to have the joint conditions well-defined, Proposition 2.2 can be modified by considering multiple copies of the situation spread out through \( R \) and introducing the singularities in \( ψ \)'s only gradually as is done in the next theorem.

**Theorem 2.4.** There exist functions \( ψ, φ \) in \( L^∞_{loc} \) failing the condition \( S_{2+} \) for all \( ε > 0, \) such that \( [φ, [ψ, H]] : L^2 → L^2 \) boundedly, where \( H \) is the Hilbert transform, i.e. the 1-dimensional Riesz transform.

**Remark 2.5.** By Theorem 3.3, the \( L^2 \) boundedness implies that \( ψ, φ \) satisfy the conditions \( T_2, S_2. \)

**Proof of Theorem 2.4.** Let
\[ ψ_0^k(x) = c_k x^{-\frac{k}{2} - 1 + \frac{k}{2} - 1}, \quad η_k = \frac{1}{k + 1}, \quad φ_0(x) = x^{1/2}1_{(1,0)}(x), \]
where \( c_k \) depends on \( k \) and will be determined later. Let \( τ_k f(x) = f(x - h). \) Then set \( φ_k = τ_k φ_0 \) and \( ψ_k = τ_k ψ_0. \) Finally we define
\[ φ = \sum_{k ∈ 2Z} φ_k, \quad ψ = \sum_{k ∈ 2Z} ψ_k. \]
Let \( k ∈ 4N + 2 \) be fixed. We first show that the pair \( (ψ_k, φ) \) satisfies \( S_2, T_q \) for \( q = q_k = 2 + k^{-1}/2. \) Since \( τ_k φ = φ \) it suffices to prove that \( (ψ_0^k, φ) \) satisfies \( S_2, T_q. \) Again, for any interval \( I, \) we have
\[ \frac{1}{|I|} \int_I |ψ_k^k - ⟨ψ_k⟩|^p |φ - ⟨φ⟩|^p \leq 4^{q+1}(|ψ_0^k|^p |φ|^q |φ|^q)_I. \]
We first consider the case when \( ℓ(I) ≤ 1 \) and we may further assume that \( I ⊂ (0, 1). \) Since \( qη_k < 1 \) we know that \( |x|^{-qη_k} ∈ A_2 \) and hence, by \( I ⊂ (0, 1), \)
\[ 4^{q+1}(|ψ_0^k|^p |φ|^q |φ|^q)_I ≤ 4^{q+1}2^{5/2} |x|^{-qη_k} A_2. \]
It remains to consider the case when \( ℓ(I) > 1. \) Since certainly \( (0, 1) ∩ I \neq ∅ \) (as otherwise there is nothing to prove) we know that \( (0, 1) ⊂ 3I. \) Then due to that \( φ \) is a periodic function we have
\[ 4^{q+1}(|ψ_0^k|^p |φ|^q |φ|^q)_I ≤ 4^{q+1}(|ψ_0^k|^p |φ|^q |φ|^q)_I ≤ 4^{q+1}2^{5/2}/ |x|^{-qη_k} A_2. \]
Therefore, we conclude that
\[ S_q ≤ 4^{q+1/2} c_k 2^{5/2} |x|^{-qη_k} A_2. \]
On the other hand, since \( ψ_0^k φ_0 ≤ c_k, \) then by similar arguments as in Proposition 2.2 we have
\[ T_q ≤ 4^{q+1/2} c_k 2^{5/2} |x|^{-qη_k} A_2. \]
Hence by Theorem 4.6 (see below) we know that the commutator $[\phi, [\psi_k, H]]$ is bounded on $L^2$ with norm $\sim c_k^{5/2} \| |x|^{-\eta_k} A_k \|_{L^2} M_{q'}^{2} \| \sigma_k \|_{L^2 \rightarrow L^2}^2$. Thus, we may further demand the constant $c_k$ to be so small that $\| [\phi, [\psi_k, H]] \|_{L^2 \rightarrow L^2} \leq 2^{-k}$. Then $[\phi, [\psi, H]]$ also is bounded on $L^2$:

$$\| [\phi, [\psi, H]] \|_{L^2 \rightarrow L^2} = \| [\phi, [ \sum_{k \in 4n+2} \psi_k, H]] \|_{L^2 \rightarrow L^2} \leq \sum_{k=1}^{\infty} \| [\phi, [\psi_k, H]] \|_{L^2 \rightarrow L^2} \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$ 

It remains to check that the pair $(\psi, \phi)$ is precisely what we need. It is obvious that $\psi, \phi \in L^\infty_{loc}$. It remains to verify that $(\psi, \phi)$ fails $S_{2+\epsilon}$ for any $\epsilon > 0$. By Hölder’s inequality we can assume $0 < \epsilon < 1$. Find $\ell \in \mathbb{N}$ such that with $k := 4l + 2$ it holds that $(2 + \epsilon)\eta_k > 1 + (2k)^{-1}$. Hence, with $I = (k, k + 1)$ we get

$$\int_I |\psi - \langle \psi \rangle_I|^{2+\epsilon} = \int_0^1 |\psi_0 - \langle \psi_0 \rangle_{(0,1)}|^{2+\epsilon} \geq c_k^{2+\epsilon} \int_{\epsilon_k^{-1} e^{-100k^2}}^{\epsilon_k^{-1} e^{-100k^2}} x^{-\frac{1}{5+k}} dx \geq c_k^{2+\epsilon} \int_{\epsilon_k^{-1} e^{-100k^2}}^{\epsilon_k^{-1} e^{-100k^2}} x^{-\frac{1}{5+k}} dx \geq c_k^{-1} e^{50k}. $$

On the other hand,

$$\int_I |\phi - \langle \phi \rangle_I|^{2+\epsilon} = \int_0^1 |\phi_0 - \langle \phi_0 \rangle_{(0,1)}|^{2+\epsilon} \sim 1.$$ 

We conclude the proof by letting $\ell \to \infty$. □

**Theorem 2.2** leads us to consider weaker joint conditions involving Young functions that can be made strictly weaker than $S_{2+\epsilon}, T_{2+\epsilon}$ for all $\epsilon > 0$.

### 2.6. Young functions, their basic properties and the conditions $S_{A,B}, T, C$

We may also define joint conditions involving Young functions. A function $A : [0, \infty) \to [0, \infty)$ is called a Young function if it is continuous, convex, strictly increasing and satisfies

$$A(0) = 0, \quad \lim_{t \to \infty} A(t)/t = \infty.$$ 

Given a Young function $A$, the complementary Young function $\tilde{A}$ is defined by

$$\tilde{A}(t) = \sup_{s > 0} \{ st - A(s) \}, \quad t > 0.$$ 

We also have the maximal function associated with a Young function $A$:

$$M_A f (x) = \sup_{Q \ni x} \langle |f| \rangle_{A, Q},$$ 

where the Luxemburg norm is defined by

$$\langle |f| \rangle_{A, Q} = \inf \{ \lambda > 0 : \frac{1}{|Q|} \int_Q A(|f|/\lambda) \leq 1 \}. $$

We say that $f \in L^A_{loc}$ if $\langle |f| \rangle_{A, Q} < \infty$ for all cubes $Q$. The relative sizes of Young functions $A, B$ are compared with the symbol $\preceq$; we say that $B \succeq A$, if there exist constants $C, t_0 > 0$ such that $A(t) \leq C B(t)$, when $t > t_0$. Finally, we define the $B_p$ class: a Young function $A \in B_p$ for $p > 1$ if

$$\int_1^\infty \frac{A(t) dt}{t^{p-1}} < \infty.$$ 

We record the following properties, which can be found at least in [3] Chapter 5 [see also [15]]:

**Proposition 2.7.** Given a Young function $A$, it holds that

i) for any $t \geq 0$, $t \leq A^{-1}(t)A^{-1}(t) \leq 2t$,
for any cube $Q$,
\[ \langle |fg| \rangle_Q \leq 2 \langle |f| \rangle_{A,Q} \langle |g| \rangle_{A,Q}. \]  \hfill (2.1)

More generally, if $A$, $B$, and $C$ are Young functions such that for all $t \geq t_0 > 0$,
\[ B^{-1}(t)C^{-1}(t) \leq cA^{-1}(t), \]
then
\[ \langle |fg| \rangle_{A,Q} \lesssim \langle |f| \rangle_{B,Q} \langle |g| \rangle_{C,Q}. \]

ii) if $B \geq A$, then $\langle |f| \rangle_{A,Q} \lesssim \langle |f| \rangle_{B,Q}$ and $M_A \lesssim M_B$.

Proposition 2.8. Let $M_A : L^p \to L^p$ boundedly if and only if $A \in B_p$.

Now we are ready to give the following definition:

Definition 2.9. Given Young functions $A, B, C$ such that $B, C \in B_p$, $\tilde{A} \in B_p$ and a pair of complex valued functions $b_1 \in L^A_{loc}(\mathbb{R}^d), b_2 \in L^B_{loc}(\mathbb{R}^d)$, we say that the joint condition $S_{A,B}$ holds if
\[ S_{A,B}(b_1, b_2) := \sup_Q \langle |b_1 - \langle b_1 \rangle_Q| \rangle_{A,Q} \langle |b_2 - \langle b_2 \rangle_Q| \rangle_{B,Q} < \infty, \]
and for $b_1^\prime, b_2^\prime \in L^C_{loc}(\mathbb{R}^d)$, we say that the joint condition $T_C$ holds if
\[ T_C(b_1, b_2) := \sup_Q \langle |b_1 - \langle b_1 \rangle_Q| |b_2 - \langle b_2 \rangle_Q| \rangle_{C,Q} < \infty. \]

We remark immediately, that in Theorem 5.1 we find a commutator that is unbounded on $L^2$ and that satisfies the conditions $S_2 + T_2$ but fails the conditions $S_{A,B} + T_C$ for all Young functions $A, B, C \in B_2$.

3. Necessary Conditions

We move on to derive the lower bounds $T_2$ and $S_2$ for the iterated commutator taken with the Riesz transforms. Our proof separates into two cases, to odd and even dimensions. Later we see that the conditions $T_2$ and $S_2$ are not strong enough to imply the $L^2$ boundedness of the commutator, however.

Lemma 3.1. Let $R_j$ be the $j$th Riesz transform on $\mathbb{R}^d$, $j = 1, \ldots, d$, $f_1, f_2 \in L^\infty_c$ and $b_1, b_2, b_1 b_2 \in L^1_{loc}$.

Under these assumptions, for all cubes $Q$, we have that
\[ \left| \int_Q \int_Q \prod_{i=1}^2 (b_i(x) - b_i(y)) f_2(y) f_1(x) \, dy \, dx \right| \leq C_d \sum_{i=1}^k \| C_b R_i \|_{L^p \to L^p} \left( \int_Q |f_1|^p \right)^{1/p} \left( \int_Q |f_2|^p \right)^{1/p}. \]

Proof. Case 1, $d$ is odd: Let $d = 2k + 1$ for some $k \in \mathbb{N}$. By composing back and forth with the translation $x \mapsto x - cQ$, we may assume that the cube $Q$ is centred at the origin. We begin with introducing 1 as
\[ 1 = \sum_{i=1}^d \frac{(x_i - y_i)^2}{|x - y|^d} = \sum_{i=1}^d \frac{(x_i - y_i)(x_i - y_i)|x - y|^{d-1}}, \]

denote $b(x, y) = \prod_{i=1}^d (b_i(x) - b_i(y))$, and then proceed with:
\[ \left| \int_Q \int_Q b(x, y) f_2(y) f_1(x) \, dy \, dx \right| = \left| \int_Q \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} b(x, y) f_2(y) 1_{Q}(y) f_1(x) \, dy \, dx \right| \]
\[ = \left| \int_Q \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} b(x, y) \sum_{i=1}^d \frac{(x_i - y_i)^2}{|x - y|^d} f_2(y) 1_{Q}(y) f_1(x) \, dy \, dx \right| \]
\[ = \left| \int_Q \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} b(x, y) \sum_{i=1}^d \frac{Y_i(x - y)^2}{|x - y|^{2d}} f_2(y) 1_{Q}(y) f_1(x) \, dy \, dx \right|, \]
where $Y_i(x) = x_i |x|^{d-1}$. We momentarily force the expression into this form in order to contrast it with the similar argument employing spherical harmonics given in [2].

For a given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$, let $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$. We continue with

$$
\left| \int_{Q} \lim_{\varepsilon \to 0} \int_{|x-y| < \varepsilon} b(x,y) \sum_{i=1}^d \frac{Y_i(x-y)^2}{|x-y|^{2d}} f_2(y) 1_Q(y) f_1(x) \, dy \, dx \right|
= \left| \int_{Q} \lim_{\varepsilon \to 0} \int_{|x-y| < \varepsilon} b(x,y) \sum_{i=1}^d \frac{x_i - y_i}{|x-y|^{d+1}} \left( \sum_{j=1}^d (x_j - y_j)^2 \right)^k f_2(y) 1_Q(y) f_1(x) \, dy \, dx \right|
= \left| \int_{Q} \lim_{\varepsilon \to 0} \int_{|x-y| < \varepsilon} \sum_{i=1}^d \sum_{\alpha, \beta = \alpha} a_{\alpha, \beta} \frac{x_i - y_i}{|x-y|^{d+1}} \phi_i f_1(x) x^\alpha y^\beta f_2(y) 1_Q(y) f_1(x) \, dy \, dx \right|
\leq \sum_{i=1}^d \sum_{\alpha, \beta = \alpha} |a_{\alpha, \beta}| \left\| f_1 \right\|_{L^p(Q)} \left\| C_b R_i (\cdot)^\beta f_2 1_Q \right\|_{L^p(Q)}
\leq \sum_{i=1}^d \sum_{\alpha, \beta = \alpha} |a_{\alpha, \beta}| \left\| f_1 \right\|_{L^p(Q)} \left\| C_b R_i f_2 \right\|_{L^p(Q)}
\leq C_d \sum_{i=1}^d \sum_{\alpha, \beta = \alpha} |a_{\alpha, \beta}| \left\| C_b R_i \right\|_{L^p \to L^p} \left\| f_1 \right\|_{L^p(Q)} \left\| f_2 \right\|_{L^p(Q)}
\leq C_d |Q| \sum_{i=1}^d \left\| C_b R_i \right\|_{L^p \to L^p} \left\| f_1 \right\|_{L^p(Q)} \left\| f_2 \right\|_{L^p(Q)}.
$$

where at * we used the fact that the limits exist separately as $R_i (\cdot)^\beta f_2 b(x, \cdot)(x)$, and where in the second to last estimate we used the assumption that $Q$ is centered at the origin. Dividing by $|Q|^2$ gives the claim.

**Case 2, $d \geq 2$:** In the previous estimate we saw that the key issue with the lower bound for $C_b R_i$'s is the following: We introduce 1 as $\sum_{i=1}^d (x_i - y_i)^2 |x-y|^{-2}$, and would like to view this as $(x_i - y_i)|x-y|^{d-1}$ times functions that depend only on $x$ and only on $y$. As we saw:

$$
1 = \sum_{i=1}^d \frac{(x_i - y_i)^2}{|x-y|^2} = \sum_{i=1}^d \frac{(x_i - y_i)^2}{|x-y|^{d+1}} (x_i - y_i)|x-y|^{d-1},
$$

and the problem becomes about expanding $|x-y|^{d-1}$ when $d$ is even, hence $d$ was odd.

Consider the function $z_1 |z|^{d-1}$ of $z \in \mathbb{R}^d$. By induction, we check that $\partial^\alpha (z_1 |z|^{d-1})$ is a linear combination of terms of the form $z^\beta |z|^{d-|\alpha|-|\beta|}$, where $|\beta| \leq |\alpha| + 1$. In particular, when $|\alpha| = d + 1$, then $\partial^\alpha (z_1 |z|^{d-1})$ is a linear combination of terms of the form $z^\beta |z|^{d-1-|\beta|}$. In particular, $|\partial^\alpha (z_1 |z|^{d-1})| \lesssim |z|^{-1} \in L^1_{\text{loc}}(\mathbb{R}^d)$ for $d \geq 2$.

Now let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a smooth bump such that $\phi \equiv 1$ in $Q(0, \frac{1}{2})$, and $\phi$ is supported in $Q(0, \frac{1}{2})$ (cube of centre 0 and “radius” $\frac{1}{2}$, hence sidelength 1). We consider the function $\phi_1(z) = \phi(z) z_1 |z|^{d-1}$. By the previous computation and product rule, this satisfies

$$
|\partial^\alpha \phi_1| \lesssim |z|^{-1} 1_{Q(0, \frac{1}{2})} \in L^1(\mathbb{R}^d)
$$

for $|\alpha| = d + 1$ and $d \geq 2$.

Thus the Fourier transform of $\phi_1$ satisfies for all $|\alpha| = d + 1$,

$$
|k^\alpha \hat{\phi}_1(k)| \sim \left| \int \partial^\alpha \phi_1(z) e^{-i2\pi k \cdot z} \, dz \right| \leq \int |\partial^\alpha \phi_1(z)| \, dz < \infty,
$$
Lemma 3.2. Let $Q$ be a cube and $b_i \in L^3_{loc}$, $i = 1, 2$, be such that $\int_Q b_i = 0$. Then

$$\left| \int_Q \int_Q (b_1(x) - b_1(y))(b_2(x) - b_2(y))\overline{b_1(x)b_2(y)} \, dy \, dx \right| \geq \int_Q |b_1|^2 \int_Q |b_2|^2$$  \hspace{1cm} (3.4)

and

$$\int_Q \int_Q (b_1(x) - b_1(y))(b_2(x) - b_2(y))\overline{b_1(x)b_2(x)} \, dy \, dx = \int_Q |b_1b_2|^2 + \int_Q |b_1b_2|^2,$$  \hspace{1cm} (3.5)

where we have replaced the latter occurrence of $b_2(y)$ with $b_2(x)$.

Proof. Multiplying out shows that

$$\int_Q \int_Q (b_1(x) - b_1(y))(b_2(x) - b_2(y))\overline{b_1(x)b_2(y)} \, dy \, dx$$

$$= \int_Q b_1(x)b_2(x)\overline{b_1(x)} \, dx \int_Q b_2(y) \, dy - \int_Q b_1(x)b_1(x) \, dx \int_Q b_2(y)b_2(y) \, dy$$

$$- \int_Q b_2(x)\overline{b_1(x)} \, dx \int_Q b_1(y)b_2(y) \, dy + \int_Q b_1(x) \, dx \int_Q b_1(y)b_2(y) \, dy$$

$$= -\int_Q |b_1(x)|^2 \, dx \int_Q |b_2(y)|^2 \, dy - \int_Q |b_1(x)b_2(x)|^2 \, dx \int_Q |b_2(y)|^2 \, dy,$$

whence

$$\left| \int_Q \int_Q (b_1(x) - b_1(y))(b_2(x) - b_2(y))\overline{b_1(x)b_2(y)} \, dy \, dx \right|$$

$$= \int_Q |b_1(x)|^2 \, dx \int_Q |b_2(y)|^2 \, dy + \int_Q |b_1(x)b_2(x)|^2 \, dx \int_Q |b_2(y)|^2 \, dy \geq \int_Q |b_1(x)|^2 \, dx \int_Q |b_2(y)|^2 \, dy.$$
As for (5.5) we compute:
\[
\int_Q \int_Q (b_1(x) - b_1(y)) (b_2(x) - b_2(y)) \overline{b_1(x) b_2(x)} \, dy \, dx
\]
\[
= \int_Q |b_1 b_2|^2 - \int_Q b_2 \int_Q |b_1|^2 \overline{b_1 b_2} - \int_Q b_1 \int_Q |b_2|^2 + \int_Q b_1 b_2 \int_Q |b_1 b_2|
\]
\[
= \int_Q |b_1 b_2|^2 + \int_Q b_1 b_2 \bigg| b_1 b_2 \bigg|^2.
\]
□

The lower bounds now follow by combining lemmas 3.1 and 3.2.

**Theorem 3.3.** Let \( R_j, j = 1, \ldots, d, \) be the Riesz transforms, \( b_1 b_2 \in L^2_{\text{loc}}, b_1, b_2 \in L^3_{\text{loc}}. \) Then

\[
S_2(b_1, b_2) + T_2(b_1, b_2) \leq C_d \sum_{j=1}^{d} \| C_b R_j \|_{L^2 \to L^2}.
\]

**Proof.** Denote \( \psi_i = b_i - (b_i)|_Q, i = 1, 2. \) Then \( \int_Q \psi_i = 0 \) and the assumptions of Lemma 3.2 are satisfied by which by (5.3) and lemma 5.1 we get the necessary condition \( S_2 \)

\[
\int_Q |\psi_1(x)|^2 \, dx \int_Q |\psi_2(y)|^2 \, dy \leq \left| \int_Q \int_Q (\psi_1(x) - \psi_1(y))(\psi_2(x) - \psi_2(y)) \overline{\psi_1(x) \psi_2(y)} \, dx \, dy \right|
\]
\[
\leq C_d \sum_{i=1}^{k} \| C_b R_i \|_{L^2 \to L^2} \left( \int_Q |\psi_1|^2 \right)^{1/2} \left( \int_Q |\psi_2|^2 \right)^{1/2}.
\]

For the condition \( T_2, \) we apply lemma 5.1 with \( f_2 = 1_Q, f_1 = \overline{\psi_1 \psi_2} \) and lemma 5.2 by (5.5) with \( b_1 = \psi_1 \) to attain

\[
\int_Q |\psi_1 \psi_2|^2 \leq \left| \int_Q \int_Q (b_1(x) - b_1(y))(b_2(x) - b_2(y)) \overline{\psi_1(x) \psi_2(x)} 1_Q(y) \, dy \, dx \right|
\]
\[
\leq C_d \sum_{i=1}^{k} \| C_b R_i \|_{L^p \to L^p} \left( \int_Q |\psi_1|^2 \right)^{1/2} \left( \int_Q |f_2|^2 \right)^{1/2} = C_d \sum_{i=1}^{k} \| C_b R_i \|_{L^p \to L^p} \left( \int_Q |\psi_1|^2 \right)^{1/2}.
\]

Dividing out equal factors and summing gives the claim. □

4. SUFFICIENT CONDITIONS

Next, our focus will be on Calderón-Zygmund operators satisfying the Dini condition. We begin with partially recalling, with only minor modifications, a sparse domination from Lerner (9) (also see Lerner, Ombrosi (10). See also Ibáñez-Firnkorn - Rivera-Ríos (7).

**Definition 4.1.** A \( d \)-dimensional Calderón-Zygmund operator \( T \) with an \( \omega \)-Dini -kernel is a \( L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) bounded operator with the representation

\[
Tf(x) = \int \overline{K(x, y)} f(y) \, dy, \quad x \notin \text{spt}(f),
\]

with the kernel \( K : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \to \mathbb{C} \) satisfying the size condition \( |K(x, y)| \leq C|x - y|^{-d} \) and the regularity condition

\[
|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{C'}{|x - y|^d} \omega \left( \frac{|x - x'|}{|x - y|} \right),
\]

whenever \( |x - x'| \leq \frac{1}{2}|x - y| \), with the modulus of continuity \( \omega : [0, 1] \to \mathbb{R}_+ \) that is continuous, increasing, subadditive, satisfies \( \omega(0) = 0 \) and \( \| \omega \|_{\text{Dini}} := \int_0^1 \omega(t) \frac{dt}{t} < \infty. \)

**Definition 4.2.** Given a \( \gamma \in (0, 1), \) we say that a collection of sets \( \mathcal{F} \) is \( \gamma \)-sparse, if for all distinct elements \( S, R \in \mathcal{F}, \) there exist sets \( E_S \subset S, E_R \subset R \) such that \( E_S \cap E_R = \emptyset \) and \( |E_S| > \gamma |S|. \)
Definition 4.3. Let $T$ be as in Definition 3.1. We have the following maximal operators on $L^2(\mathbb{R}^d)$:

i) the maximal operator $T, f(x) = \sup_{\varepsilon > 0} |Tf|_{1B(x,\varepsilon r)}(x)$,

ii) the grand maximal operator $\mathcal{M}T(f)(x) = \sup_{Q \ni x} \text{ess sup}_{y \in Q} |T(f)_{1_{B_y}}(\xi)|$,

iii) and its localized version $\mathcal{M}T, Q(f)(x) = \sup_{Q \supset P \ni x} \text{ess sup}_{\xi \in P} |T(f)_{1_{B_\xi}}(\xi)|$, where $Q, P$ are cubes.

The control over the grand maximal operator is given by

Lemma 4.4. [9 Lemma 3.2] Let $f \in L^2_{\text{loc}}$. The following pointwise estimates hold:

i) for a.e. $x \in Q$ we have $|T(|f|_Q(x))| \leq C_d \|T\|_{L^1 \to L^{1,\infty}} |f(x)| + \mathcal{M}T, Q(f)(x)$;

ii) for all $x \in \mathbb{R}^d$ we have $\mathcal{M}T(f)(x) \leq C_d (\omega)_{\text{Dini}} + C_T \mathcal{M}f(x) + T, f(x)$.

For a more refined argument for the sparse domination in Theorem 4.5 without Lemma 4.4, see the latest version of the sparse domination principle in Lerner, Ombrosi [10].

Theorem 4.5. Let $T$ be a $d$-dimensional Calderón-Zygmund operator with a Dini kernel and denote $b(x, y) = (b_1(x) - b_1(y))(b_2(x) - b_2(y))$. We assume that $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, and further to make everything well-defined that $b_1, b_2, b_1b_2, b_1b_2f, b_2f, b_2f, f, b_2f \in L^1_{\text{loc}}$.

From these assumptions it follows that there exists a sparse collection $S$ of cubes on $\mathbb{R}^d$ such that

$$\left| C_b T f(x) \right| \leq C_{T,d} \sum_{i=1}^4 S_i f(x),$$

where

$$S_1 f = \sum_{Q \in S} |b_1 - (b_1)_Q| \|b_2 - (b_2)_Q\| \langle |f|_Q \rangle_1 Q,$$

$$S_2 f = \sum_{Q \in S} |b_2 - (b_2)_Q| \|b_1 - (b_1)_Q\| \langle |f|_Q \rangle_1 Q,$$

$$S_3 f = \sum_{Q \in S} |b_1 - (b_1)_Q| \|b_2 - (b_2)_Q\| \langle |f|_Q \rangle_1 Q,$$

$$S_4 f = \sum_{Q \in S} |b_2 - (b_2)_Q| \|b_1 - (b_1)_Q\| \langle |f|_Q \rangle_1 Q,$$

and the sparse constant denoted with $\gamma$ depends only on the dimension $d$.

Proof. We recall only the part of the proof where the exceptional set is defined and control over the appearing terms is established. In addition, a comment is made about the rest of the proof, the details for which we refer the reader to the proof of Theorem 1.1 in [12] or [9].

For an arbitrary integrable function $\psi \neq 0$ on $Q$ define

$$E_1(\psi) = \{ x \in Q : |\psi(x)| > \alpha(|\psi|_{3Q}) \}, \quad E_2(\psi) = \{ x \in Q : \mathcal{M}T, Q(\psi(x)) > \alpha(|\psi|_{3Q}) \}$$

and let the exceptional set be

$$E = \bigcup_{i=1,2} E_i(f) \cup E_i(b_1f) \cup E_i(b_2f) \cup E_i(b_1b_2f).$$

Since the localized version of the grand maximal operator is controlled with the non-localized by

$$\mathcal{M}T, Q f \leq \mathcal{M}T f_{1_{3Q}},$$

and by the well-known facts that $\mathcal{M}, T : L^1 \to L^{1,\infty}$ boundedly, it follows from the weak (1,1) bounds implied by ii) of Lemma 4.4 in conjunction with the local integrability of all functions in question that we may choose some $\alpha > 0$ independent of the cube $Q$ so that $|E| \leq 2^{-(d+2)}|Q|$.

Taking a Calderón-Zygmund decomposition of the function $1_E$ at the height $2^{-(d+1)}$ yields a collection $\mathcal{F}$ of cubes satisfying:

$$\sum_{P \in \mathcal{F}} |P| \leq \frac{1}{2}|Q|, \quad |E \setminus \bigcup_{P \in \mathcal{F}} P| = 0 \quad \text{and} \quad P \cap E^c \neq \emptyset \quad \forall P \in \mathcal{F}.$$

Then one decomposes

$$(C_b T f_{1_{3Q}}) 1_Q = (C_b T f_{1_{3Q}}) 1_{Q \setminus P} + \sum_{P \in \mathcal{F}} (C_b T f_{1_{3Q}}) 1_{P} + \sum_{P \in \mathcal{F}} (C_b T f_{1_{3P}}) 1_{P}$$
and uses the properties of the collection $\mathcal{F}$, Lemma 2.4 and that the commutator is unchanged modulo constants in the functions $b_1, b_2$ to derive
\[
C_Tf13Q 1_Q \leq C_{T,d} \left( |b_2 - \langle b_2 \rangle_{3Q}|b_1 - \langle b_1 \rangle_{3Q}|f|_{3Q} \right.
\]
\[
+ |b_2 - \langle b_2 \rangle_{3Q}|(\langle b_1 \rangle_{3Q} - |f|_{3Q})_{3Q}
\]
\[
+ |b_1 - \langle b_1 \rangle_{3Q}|(\langle b_2 \rangle_{3Q} - |f|_{3Q})_{3Q}
\]
\[
+ \langle |b_1 - \langle b_1 \rangle_{3Q}|b_2 - \langle b_2 \rangle_{3Q}|f|_{3Q} \rangle_{1_Q} + \sum_{P \in \mathcal{F}} C_Tf13P 1_P.
\]

From this situation one first iterates the above estimate with the last term and then transfers the limit construction from the local to the global.

\[\square\]

**Theorem 4.6.** Assume that a pair of functions $b_1 \in L^A_{loc}(\mathbb{R}^d)$ and $b_2 \in L^B_{loc}(\mathbb{R}^d)$ with $b_1^3, b_2^3 \in L^C_{loc}(\mathbb{R}^d)$ satisfy the conditions $T_C$ and $S_{A,B}$ for some Young functions $A, B$, $C$ with $\tilde{A}, \tilde{B}, \tilde{C} \in B_2$, then
\[S_i : L^2(\mathbb{R}^d) \cap L^2_{loc}(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad i = 1, 2, 3, 4\]
bondedly.

Especially, it follows with a standard density argument by Theorem 4.5 that
\[C_T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)\]
bondedly when notation and assumptions are retained.

**Proof.** The pairs of terms $S_1, S_3$ and $S_2, S_4$ are symmetric with respect to dual pairings. Hence, we show the estimate in the two distinct cases of $S_1$ and $S_3$. By duality it is enough to estimate the pairings $\langle S_i(f), \psi \rangle$.

First, for the term $S_1$ we only use the assumptions involving the functions $A, B$. By sparseness we get
\[
\left| \langle S_1(f), \psi \rangle \right| \leq \sum_{Q \in S} \int_Q |b_1 - \langle b_1 \rangle_Q||\psi||b_2 - \langle b_2 \rangle_Q||f|_Q
\]
\[
\leq \sum_{Q \in S} |Q||\langle b_1 - \langle b_1 \rangle \rangle_A||\psi||b_2 - \langle b_2 \rangle_Q||f|_B.Q
\]
\[
\leq S_{A,B}(b_1, b_2) \sum_{Q \in S} |Q||\psi||A,Q||f|_B.Q \leq \gamma^{-1} S_{A,B}(b_1, b_2) \sum_{Q \in E_1} |\psi||A,Q||f|_B.Q
\]
\[
\leq \gamma^{-1} S_{A,B}(b_1, b_2) \sum_{Q \in S} |\psi||A,Q||f|_B.Q \leq \sum_{Q \in S} |\psi||A,Q||f|_B.Q
\]
\[
\leq S_{A,B}(b_1, b_2) \sum_{Q \in S} |\psi||A,Q||f|_B.Q
\]
\[
\leq S_{A,B}(b_1, b_2)||\psi||L^2||f||L^2
\]
where we have used Proposition 2.3 in the last step.

Next we use the condition $T_C$ to control the term $S_3$:
\[
\left| \langle S_3(f), \psi \rangle \right| \leq \sum_{Q \in S} |Q||\psi||Q||b_1 - \langle b_1 \rangle_Q||b_2 - \langle b_2 \rangle_Q||f|_Q
\]
\[
\leq \sum_{Q \in S} |Q||\psi||Q||b_1 - \langle b_1 \rangle_Q||b_2 - \langle b_2 \rangle_Q||c,Q
\]
\[
\leq T_C(b_1, b_2) \sum_{Q \in S} |Q||\psi||Q||c,Q \leq \gamma^{-1} T_C(b_1, b_2) ||M \psi||_{L^2}||M_c f||_{L^2}
\]
\[
\leq T_C(b_1, b_2) ||\psi||_{L^2}||f||_{L^2}.
\]
\[\square\]

Since with $A(t) = t^p, \tilde{A} \in B_2$, for $p > 2$, we immediately get:
We prove the result via the following example. Let $C_\lambda T: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ boundedly.

We close this section with some remarks.

Remark 4.8. For Theorem 4.6 the difference in the case $p \neq 2$ is that we need to introduce 3 more Young functions to manage the now non-symmetric dual pairings from the terms $S_2, S_4$. According to Definition 2.9 the existing Young functions are replaced with ones satisfying

$$\sigma \in B_{p'}, \quad \bar{B}, \bar{C} \in B_p$$

and are supplemented with Young functions $D, E, F$ satisfying

$$\bar{D}, \bar{F} \in B_p, \quad \bar{E} \in B_{p'}$$

and $S_{D,E}(b_1, b_2) + T_F(b_1, b_2) < \infty$.

Remark 4.9. Given a $q \in (2, \infty)$, adapting the proof of Theorem 4.6 shows that if $b_1, b_2$ satisfy the conditions $S_{q+\varepsilon}, T_{q+\varepsilon}$ for any $\varepsilon > 0$, then $C_\lambda T: L^q \rightarrow L^q$ boundedly.

On the other hand, for $q \in (1, 2)$, the conditions $S_p, T_p$ with $p \in (q, 2)$ are not strong enough to conclude that $C_\lambda T: L^q \rightarrow L^q$ boundedly. Indeed, if they were, then by duality and interpolation $C_\lambda T: L^2 \rightarrow L^2$ boundedly and Theorem 5.3 would imply the condition $S_2$. This gives a contradiction since by Proposition 2.2 there exists functions $\phi, \psi$ such that $S_p, T_p$ are satisfied and $S_2$ is not.

5. Conjecture and related examples

In the last section we continue discussing the conditions $S_{A,B}, T_C,$ and their interdependence with the boundedness properties of the commutator on different $L^p$ spaces.

Theorem 5.1. There exists $b_1, b_2 \in L^\infty_{\text{loc}}(\mathbb{R})$ such that $S_2(b_1, b_2) + T_2(b_1, b_2) < \infty$, but $S_{A,B}(b_1, b_2) = \infty$ and $T_C(b_1, b_2) = \infty$ for arbitrary Young functions $A, B, C$ with $A, B, C \in B_2$. Moreover, $C_\lambda H : L^2 \not\rightarrow L^2$.

Proof. We prove the result via the following example. Let $I_0 = [-1, 1]$ and

$$\sigma = 1_{I_0}, \quad w = M(\sigma)^{-1},$$

notice that both $\sigma$ and $w$ are even functions. It is immediate to see that

$$\sup_I \langle \sigma\rangle_I \langle w\rangle_I \leq \sup_I \inf_{x \in I} M(\sigma)(x) \langle w\rangle_I \leq \sup_I \langle M(\sigma) w\rangle_I = 1. \tag{5.1}$$

Now define

$$b_1(x) := \text{sgn}(x)\sigma(x), \quad b_2(x) := \text{sgn}(x)w^{1/2}(x),$$

and notice that immediately $b_1, b_2 \in L^\infty_{\text{loc}}$. By (5.1) we see that

$$S_2(b_1, b_2)^2 = \sup_I \langle |b_1 - \langle b_1\rangle_I|^2 \rangle_I \langle |b_2 - \langle b_2\rangle_I|^2 \rangle_I \leq 16 \sup_I \langle |b_1|^2 \rangle_I \langle |b_2|^2 \rangle_I \leq 16 < \infty. \tag{5.2}$$

We also have

$$|b_1 - \langle b_1\rangle_I|^2|b_2 - \langle b_2\rangle_I|^2 \leq 4(|b_1|^2 + |\langle b_1\rangle_I|^2)(|b_2|^2 + |\langle b_2\rangle_I|^2),$$

and by $|b_1 b_2| \leq 1$, direct calculations give us

$$T_2(b_1, b_2)^2 = \sup_I \langle |b_1 - \langle b_1\rangle_I|^2|b_2 - \langle b_2\rangle_I|^2 \rangle_I \leq 4 + 12\langle |b_1|^2 \rangle_I \langle |b_2|^2 \rangle_I \leq 16 < \infty. \tag{5.3}$$

However, for $J_k = (-k, k), k \geq 2$, since $b_1$ and $b_2$ are odd functions,

$$S_{A,B}(b_1, b_2) \geq \lim_{k \to \infty} \langle |b_1|\rangle_{A,J_k} \langle |b_2|\rangle_{B,J_k} \geq \lim_{k \to \infty} \langle |b_1|\rangle_{A,J_k} \langle |b_2|^2 \rangle_{J_k}^{1/4}$$
\[
\sim \lim_{k \to \infty} \frac{k^{\frac{1}{2}}}{A^{-1}(k)} \sim \lim_{k \to \infty} \frac{\tilde{A}(k)}{k^{\frac{1}{2}}} = \lim_{k \to \infty} \left( \frac{(\tilde{A}(k))^{2}}{A(A^{-1}(k))} \right)^{\frac{1}{2}},
\]

where we have used the fact that \(M(1_{I_0})(x) \sim (1 + |x|)^{-1}\). To conclude that immediately by definition \(\lim_{t \to \infty} \tilde{A}(t) = \infty\) and

\[
\lim_{t \to \infty} \frac{\tilde{A}(t)}{t^{2}} \leq \frac{4}{\log 2} \lim_{t \to \infty} \int_{t}^{2t} \frac{\tilde{A}(s)}{s^{2}} \, ds = 0.
\]

On the other hand with \(I_k = (0, k), k \geq 100\), we have

\[
T_{C}(b_1, b_2) \geq \lim_{k \to \infty} \left\langle |b_1 - \langle b_1 \rangle_{I_k}|b_2 - \langle b_2 \rangle_{I_k} \right\rangle_{C,I_k} \geq \lim_{k \to \infty} \left\langle |b_1 - \langle b_1 \rangle_{I_k}|b_2 - \langle b_2 \rangle_{I_k}|1_{(0,1)} \right\rangle_{C,I_k}
\]

\[
= \lim_{k \to \infty} (1 - k^{-1}) \left\langle |b_2 - \langle b_2 \rangle_{I_k}|1_{(0,1)} \right\rangle_{C,I_k}.
\]

Since for \(x > 1, b_2(x) = \left(\frac{\sqrt{2}}{3k}\right)^{\frac{1}{2}}\) and for \(0 < x \leq 1, b_2(x) = 1\), another direct calculation shows that

\[
\langle b_2 \rangle_{I_k} = \frac{\sqrt{2}}{3k} \left[ (k + 1)^{\frac{1}{2}} - 2\sqrt{2} \right]
\]

by which and the assumption \(k \geq 100\) we see that

\[
|b_2 - \langle b_2 \rangle_{I_k}|1_{(0,1)} \geq c k^{\frac{1}{2}}.
\]

Hence

\[
T_{C,2}(b_1, b_2) \geq \lim_{k \to \infty} c(1 - k^{-1}) k^{\frac{1}{2}}(1_{(0,1)})_{C,I_k} \geq \lim_{k \to \infty} c(1 - k^{-1}) \frac{k^{\frac{1}{2}}}{C^{-1}(k)} = \infty.
\]

Next, we show that \(C_{b}H : L^2 \not\to L^2\). To see this, let

\[
f(x) = x^{-\frac{1}{2}}(\log x)^{-1}1_{[100,\infty)}(x) \in L^2(\mathbb{R}).
\]

We claim that \(|C_{b}H f(x)| = \infty\) for all \(x \in I_0\), and in showing this, hence conclude the unboundedness of \(C_{b}H\) on \(L^2\). Indeed, since for \(y \in [100, \infty)\)

\[
b_2(y) = (M(1_{I_0}))^{-\frac{1}{2}} = \sqrt{\frac{y + 1}{2}},
\]

and \(b_1(x) = b_2(x) = \text{sgn}(x), \) for any \(x \in I_0\), we have

\[
|C_{b}H f(x)| = \left| \int_{100}^{\infty} (b_1(x) - b_1(y))(b_2(x) - b_2(y)) \frac{f(y)}{x - y} \, dy \right| = \left| \int_{100}^{\infty} (\text{sgn}(x) - \sqrt{\frac{y + 1}{2}}) \frac{f(y)}{x - y} \, dy \right|
\]

\[
\sim \int_{100}^{\infty} \sqrt{\frac{y + 1}{2} \frac{f(y)}{x - y}} \, dy \sim \int_{100}^{\infty} \frac{f(y)}{\sqrt{y}} \, dy = \infty.
\]

If we take \(A(t) = B(t) = C(t) = t^{2+\varepsilon}\), where \(\varepsilon > 0\), we immediately have the following

**Corollary 5.2.** The conditions \(S_2, T_2\) holding simultaneously does not improve to \(S_{2+\varepsilon}\) or \(T_{2+\varepsilon}\) for any \(\varepsilon > 0\).

For our next example, we note that functions \(\Phi : [0, \infty) \to [0, \infty)\) of the form

\[
\Phi(t) = t^{p} \log(e + t)^{p - 1 + \delta}, \quad p \in (1, \infty), \quad \delta \in (0, \infty)
\]

are called log -bumps. These are Young functions, and we recall some facts from [3] Chapter 5:

- If \(\Phi(t) = t^{p} \log(e + t)^{p - 1 + \delta}\), then

  \[
  \Phi^{-1}(t) \sim t^{1/p} \log(e + t)^{-\delta/2} \quad \text{and} \quad \Phi(t) \sim t^{p \left[ \log(e + t) \right]^{-1 - (p - 1)\delta}} \in B_{p'}.
  \]
If $\Phi(t) = t^p \log(e + t)^{p-1} \log \log(e + t)^{p-1+\delta}(\text{which is referred to as a loglog-bump})$, then

$$\Phi^{-1}(t) \sim t^{1/p} \log(e + t)^{-1/p} \log \log(e + t)^{-1/p} \cdot \frac{1}{t},$$
and

$$\Phi(t) \sim t^{p'} \log(e + t)^{-1} \log \log(e + t)^{1-(p' - 1)\delta} \in B_{p'}. $$

**Theorem 5.3.** There exist functions $b_1, b_2 \in L^p_{\infty}$ such that $C_b H : L^2 \to L^2$ boundedly, but $S_{A,B}(b_1, b_2) = \infty$ and $T_{C}(b_1, b_2) = \infty$, for all log-bumps $A, B, C$ with $A, B, C \in B_2$.

**Proof.** The idea is to construct a pair of functions $(b_1, b_2)$ such that it satisfies the assumption in Theorem 4.6 so that we can conclude the boundedness of $C_b H$ directly, meanwhile, the related bump function increases slower than log-bumps. Let $\Phi_0 = t^2 \log(e + t) \log \log(e + t)^{3/2}$, and define

$$b_1(x) = \text{sgn}(x)1_{I_0}(x), \quad b_2(x) = \text{sgn}(x)\Phi_0^{-1}\left((M1_{I_0}(x))^{-1}\right), \quad I_0 = [-1, 1].$$

We will show that $(b_1, b_2)$ is what we need. First of all, it is easy to check that for any cube $I$,

$$\langle |b_1| \rangle_{\Phi_0, I} \langle |b_2| \rangle_{\Phi_0, I} \leq \langle \Phi_0^{-1}\left((M1_{I_0}(x))^{-1}\right) \rangle_{\Phi_0, I} \leq 1.$$ 

Then by the triangle inequality and general Hölder’s inequality we have

$$\langle |b_1 - \langle b_1 \rangle_{I_0}| \rangle_{\Phi_0, I} \langle |b_2 - \langle b_2 \rangle_{I_0}| \rangle_{\Phi_0, I} \lesssim 1.$$ 

On the other hand, since $|b_1 b_2| \leq 1$, using triangle inequality and general Hölder’s inequality again we have

$$\langle |b_1 - \langle b_1 \rangle_{I_0}|^2 \rangle_{\Phi_0, I} \langle |b_2 - \langle b_2 \rangle_{I_0}|^2 \rangle_{\Phi_0, I} \lesssim 1.$$ 

Using Theorem 4.6 we know that $C_b H$ is bounded on $L^2$, thanks to $\Phi_0 \in B_2$.

It remains to show that $S_{A,B}(b_1, b_2) = \infty$ and $T_{C}(b_1, b_2) = \infty$, for all log-bumps $A, B, C$ with $A, B, C \in B_2$. Without loss of generality we can assume that $A(t) = t^2 \log(e + t)^{1+\alpha}$, $B(t) = t^2 \log(e + t)^{1+\beta}$ and $C(t) = t^2 \log(e + t)^{1+\gamma}$, where $\alpha, \beta, \gamma > 0$. For $S_{A,B}(b_1, b_2)$ again we test with the interval $I_k = (-k, k)$ with $k \geq 2$. Since $b_1$ and $b_2$ are odd functions, we have

$$\langle |b_1 - \langle b_1 \rangle_{I_k}| \rangle_{A, I_k} \langle |b_2 - \langle b_2 \rangle_{I_k}| \rangle_{B, I_k} = \langle |b_1| \rangle_{A, I_k} \langle |b_2| \rangle_{B, I_k} \lesssim k^{-\frac{3}{2}} \log(e + k)^{\frac{1+\alpha}{2}} k^{-\frac{3}{2}} \log(e + k)^{-\frac{1+\alpha}{2}} k^{-\frac{3}{2}} \log(e + k)^{-\frac{1+\alpha}{2}} \to \infty$$

For $T_{C}(b_1, b_2)$, we test with the cube $I_k = (0, k)$, $k \geq 100$, we have

$$T_{C}(b_1, b_2) \geq \lim_{k \to \infty} \langle |b_1 - \langle b_1 \rangle_{I_k}|^2 \rangle_{C, I_k} \geq \lim_{k \to \infty} \langle |b_1 - \langle b_1 \rangle_{I_k}|^2 \rangle_{C, I_k} \lesssim k^{-\frac{3}{2}} \log(e + k)^{\frac{1+\alpha}{2}} k^{-\frac{3}{2}} \log(e + k)^{-\frac{1+\alpha}{2}} k^{-\frac{3}{2}} \log(e + k)^{-\frac{1+\alpha}{2}} = \infty.$$ 

**Corollary 5.4.** The conditions $S_{2+\varepsilon}, T_{2+\varepsilon}$ are not precise enough to yield a characterization of $C_b H : L^2 \to L^2$.

**Proof.** The iterated commutator $C_b H$ of Theorem 5.3 is bounded on $L^2$. We show that the conditions $S_{2+\varepsilon}(b_1, b_2)$ and $T_{2+\varepsilon}(b_1, b_2)$ are not satisfied for any $\varepsilon > 0$. To see this, it is enough to notice that for all log-bumps $A, B, C$ with $A, B, C \in B_2$, we have $t^{2+\varepsilon} \leq A(t), B(t), C(t)$ by which by Proposition 1.4 and the estimates in Theorem 5.3 it follows that

$$\langle |b_1 - \langle b_1 \rangle_{I_k}|^{2+\varepsilon} \rangle_{I_k} \langle |b_2 - \langle b_2 \rangle_{I_k}|^{2+\varepsilon} \rangle_{I_k} \lesssim \langle |b_1 - \langle b_1 \rangle_{I_k}| \rangle_{A, I_k} \langle |b_2 - \langle b_2 \rangle_{I_k}| \rangle_{B, I_k} \to \infty,$$
and

$$\langle |b_1 - \langle b_1 \rangle_{I_k}|^{2+\varepsilon} |b_2 - \langle b_2 \rangle_{I_k}|^{2+\varepsilon} \rangle_{I_k} \lesssim \langle |b_1 - \langle b_1 \rangle_{I_k}|^2 \rangle_{C, I_k} \to \infty,$$
a as $k \to \infty$, showing that $S_{2+\varepsilon}(b_1, b_2) = \infty$ and $T_{2+\varepsilon}(b_1, b_2) = \infty$. 

\[\square\]
Corollary 5.5. The commutator of Theorem 5.3 is bounded on $L^2$ and unbounded on all $L^p$, $p \in (1, \infty) \setminus \{2\}$.

Proof. Let $p > 2$, $q \in (2, p)$ and $f(x) = x^{-1/q} 1_{[100, \infty)}(x) \in L^p$. For all $x \in [-1, 1]$,

$$
|C_b H f(x)| = \left| \int (b_1(x) - b_1(y)) (b_2(x) - b_2(y)) \frac{f(y)}{x-y} \, dy \right| \\
\sim \int_{100}^{\infty} \frac{y^b}{\log(e+y)^{\frac{a}{q}} \log(e+y)^{\frac{1}{q}}} \frac{y^{-\frac{1}{q}}}{y} \, dy = \infty,
$$

showing that $C_b H : L^p \not\to L^q$. It follows by duality that also $C_b H : L^q \not\to L^p$. \qed

Remark 5.6. Alternatively, we can prove Corollary 5.4 by Corollary 5.5. Indeed, if the conditions $S_{2+\varepsilon}(b_1, b_2), T_{2+\varepsilon}(b_1, b_2)$ hold for some $\varepsilon > 0$, then by Remark 4.9 we have $C_b H : L^q \to L^q$ boundedly for all $q \in (2, 2 + \varepsilon)$, a contradiction with Corollary 5.5.

The above considerations lead us to conjecture:

Conjecture 5.7. With the functions $b_1, b_2$ subject to the same assumptions as those in Theorems 4.6 and 5.3, the boundedness of $[b_1, b_2, H]_p$ on $L^2(\mathbb{R})$ is equivalent with the existence of Young functions $A, B, C$ with $A, B, C \in B_2$ such that $S_{A,B}(b_1, b_2) + T_C(b_1, b_2) < \infty$.

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