An Interface/Boundary-Unfitted EXtended HDG Method for Linear Elasticity Problems

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Abstract
An interface/boundary-unfitted eXtended hybridizable discontinuous Galerkin (X-HDG) method of arbitrary order is proposed for linear elasticity interface problems on unfitted meshes with respect to the interface and domain boundary. The method uses piecewise polynomials of degrees \( k \) (\( \geq 1 \)) and \( k - 1 \) respectively for the displacement and stress approximations in the interior of elements inside the subdomains separated by the interface, and piecewise polynomials of degree \( k \) for the numerical traces of the displacement on the inter-element boundaries inside the subdomains and on the interface/boundary of the domain. Optimal error estimates in \( L^2 \)-norm for the stress and displacement are derived. Finally, numerical experiments confirm the theoretical results and show that the method also applies to the case of crack-tip domain.

Keywords EXtended HDG method · Linear elasticity · Interface/boundary-unfitted · Error estimate · Crack-tip domain

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain with piecewise smooth boundary $\partial \Omega = \partial \Omega_D \cup \partial \Omega_N$, where $\text{meas}(\partial \Omega_D) > 0$ and $\partial \Omega_D \cap \partial \Omega_N = \emptyset$. The domain $\Omega$ is divided into two subdomains, $\Omega_i$ ($i = 1, 2$), by a piecewise smooth interface $\Gamma$ (cf. Fig. 1 for an example). Consider the linear elasticity interface problem

\[ A \sigma - \epsilon(u) = 0 \quad \text{in } \Omega_1 \cup \Omega_2, \]
\[ \nabla \cdot \sigma = f \quad \text{in } \Omega_1 \cup \Omega_2, \]
\[ u = g_D \quad \text{on } \partial \Omega_D, \]
\[ \sigma n = g_N \quad \text{on } \partial \Omega_N, \]
\[ [u] = 0, \quad [\sigma n] = g_\Gamma \quad \text{on } \Gamma, \]

where $\sigma : \Omega \to \mathbb{R}^{d \times d}_{\text{sym}}$ denotes the symmetric $d \times d$ stress tensor field, $u : \Omega \to \mathbb{R}^d$ the displacement field, $\epsilon(u) = (\nabla u + (\nabla u)^T)/2$ the strain tensor, and $A \in \mathbb{R}^{d \times d}_{\text{sym}}$ the compliance tensor with

\[ A \sigma = \frac{1}{2\mu} \left( \sigma - \frac{\lambda}{2\mu + d\lambda} \text{tr}(\sigma) I \right). \]

Here $\text{tr}(\sigma)$ denotes the trace of $\sigma$, $I$ the $d \times d$ identity matrix, and $\lambda$ and $\mu$ the Lamé coefficients with $\lambda|_{\Omega_i} = \lambda_i > 0$ and $\mu|_{\Omega_i} = \mu_i > 0$. $f$ is the body force, $g_D$ and $g_N$ are respectively the surface displacement on $\partial \Omega_D$ and the surface traction on $\partial \Omega_N$, and $n$ in (1.1d) and (1.1e) denotes respectively the unit outer normal vector along $\partial \Omega_N$ and the unit normal vector along $\Gamma$ pointing to $\Omega_2$. The jump of a function $w$ across the interface $\Gamma$ is defined by $[w] = (w|_{\Omega_1})|_\Gamma - (w|_{\Omega_2})|_\Gamma$. Elasticity interface problems are usually used to describe complicated elasticity structure characterized by discontinuous or even singular material properties, and have many applications in materials science and continuum mechanics [29, 30, 39, 42, 55, 62, 66].

![Fig. 1](image) The geometry of domain with circle interface or fold line interface
For elliptic interface problems, the global regularity of the solutions is generally very low, which may lead to reduced accuracy of finite element discretizations [2, 71]. To tackle this situation there are mainly two types of methods in the literature: interface-fitted methods and interface-unfitted methods. The fitted methods use interface-fitted meshes to dominate the approximation error caused by the low regularity of solutions [6, 10, 13, 14, 18, 38, 48, 56]; see Fig. 2 for an example. However, it is usually expensive to generate interface-fitted meshes, especially when the interface is of complicated geometry or moving with time or iteration.

The unfitted methods, based on meshes independent of the interface, employ certain types of modification in the finite element discretization for approximating functions around the interface so as to avoid the loss of numerical accuracy. One representative unfitted method is the eXtended/generalized Finite Element Method [3, 4, 8, 36, 41, 54, 65, 67, 68], where additional basis functions characterizing the solution singularity around the interface are adopted for the corresponding approximation function space. For elasticity interface problems, we refer to [37] for an XFEM of displacement-type, and to [7, 37] for a mixed XFEM based on a displacement-pressure formulation, both of which use the cut linear polynomials around the interface as additional basis functions to enrich the standard linear element displacement spaces. We also refer to [9, 28, 47, 60, 61] for some applications of XFEMs in the simulation of crack propagation in fracture mechanics and [11, 12, 32] for curved domains.

The immersed finite element method (IFEM) is another type of interface-unfitted methods, where special finite element basis functions are constructed to satisfy the interface jump conditions [1, 31, 43, 49–51, 57, 72]. We refer to [52, 53] for linear/bilinear immersed finite elements and a nonconforming immersed rectangular element for planar elasticity interface problems.

The hybridizable discontinuous Galerkin (HDG) framework [19] provides a unifying strategy for hybridization of finite element methods. In this framework, a trace variable defined on the mesh skeleton is introduced, as a Lagrange multiplier, so as to relax the continuity constraint of the approximation solution on the inter-element boundaries. Thus, the HDG method allows for local elimination of unknowns defined in the interior of elements and leads to a reduction of the number of degrees of freedom in the final discrete system. We refer to [16, 17, 20, 21, 23, 24, 44–46, 58, 63] for some developments of the HDG method and [22, 25, 26, 59, 64] for HDG method to deal with domain with curved boundary. In [27] an unfitted HDG method was developed for two-dimensional Poisson interface problems by constructing a novel ansatz function in the vicinity of the interface. Based on the XFEM philosophy.
and a level set description of interface, an equal order eXtended HDG (X-HDG) method was proposed in [34] for diffusion problems with voids and later applied to heat bimaterial problems [33]. In [35] two arbitrary order X-HDG methods with optimal convergence rates were presented and analyzed for diffusion interface problems in two and three dimensions.

This paper aims to develop an interface/boundary-unfitted X-HDG method of arbitrary order for the linear elasticity interface problem (1.1). The main features of our X-HDG method are as follows.

- The method uses piecewise polynomials of degrees $k \geq 1$ and $k - 1$ respectively for the displacement and stress approximations in the interior of elements inside the subdomains separated by the interface, and piecewise polynomials of degree $k$ for the numerical traces of the displacement on the inter-element boundaries inside the subdomains and on the interface/boundary of the domain. We note that the unfitted methods in [7, 37, 52, 53] are low order ones, and that the methods in [15, 58] for linear elasticity problems (without interface) use piecewise polynomials of degrees $k + 1 (k \geq 1)$, $k$ and $k$ respectively for the displacement, stress approximations in the interior of elements and the numerical traces of displacement on the inter-element boundaries.
- The method inherits the following advantages of X-FEM and HDG: does not require the used meshes to fit the interface or boundary; allows for local elimination; and does not require the stabilization parameters to be “sufficiently large”.
- The derived error estimates for the displacement and stress approximations are optimal.
- The method applies to any piecewise $C^2$ smooth interface and any crack-tip domain.

The rest of the paper is organized as follows. Section 2 introduces the X-HDG scheme for the elasticity problem on interface-unfitted meshes and boundary-unfitted meshes, respectively. Section 3 is devoted to the a priori error estimation for the X-HDG method. Section 4 provides several numerical examples to verify the theoretical results. Finally, Sect. 5 gives some concluding remarks.

## 2 X-HDG Scheme

### 2.1 Notation

For any bounded polygonal/polyhedral domain $\Lambda \subset \mathbb{R}^s$ ($s = d, d - 1$) and nonnegative integer $m$, let $H^m(\Lambda)$ and $H^m_0(\Lambda)$ be the usual $m$-th order Sobolev spaces on $\Lambda$, with norm $\| \cdot \|_{m, \Lambda}$ and semi-norm $| \cdot |_{m, \Lambda}$. In particular, $L^2(\Lambda) := H^0(\Lambda)$ is the space of square integrable functions, with the inner product $(\cdot, \cdot)_{\Lambda}$. When $\Lambda \subset \mathbb{R}^{d-1}$, we use $(\cdot, \cdot)_\Lambda$ to replace $(\cdot, \cdot)_\Lambda$. We set

$$
H^m(\Omega_1 \cup \Omega_2) := \{ v \in L^2(\Omega) : v|_{\Omega_1} \in H^m(\Omega_1), \text{ and } v|_{\Omega_2} \in H^m(\Omega_2) \},$
$$
$$
\| \cdot \|_m := \| \cdot \|_{m, \Omega_1 \cup \Omega_2} = \sum_{i=1}^2 \| \cdot \|_{m, \Omega_i}, \quad | \cdot |_m := | \cdot |_{m, \Omega_1 \cup \Omega_2} = \sum_{i=1}^2 | \cdot |_{m, \Omega_i}.
$$

For integer $k \geq 0$, $P_k(\Lambda)$ denotes the set of all polynomials on $\Lambda$ with degree no more than $k$. We note that bold face fonts will be used for vector (or tensor) analogues of the Sobolev spaces along with vector-valued (or tensor-valued) functions. In particular, for the tensor case we set

$$
L^2(\Omega, S) := \{ w \in [L^2(\Omega)]^{d \times d} : w \text{ is symmetric} \},
$$

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Let \( T_h = \bigcup \{ K \} \) be a shape-regular triangulation of the domain \( \Omega \) consisting of open triangles/tetrahedrons, which is unfitted with the interface. We define the set of all elements intersected by the interface \( \Gamma \) as

\[
T_h^\Gamma := \{ K \in T_h : K \cap \Gamma \neq \emptyset \}.
\]

For any \( K \in T_h^\Gamma \) which is called an interface element, let \( \Gamma_K := K \cap \Gamma \) be the part of \( \Gamma \) in \( K \), \( K_i = K \cap \Omega_i \) be the part of \( K \) in \( \Omega_i \) (\( i = 1, 2 \)), and \( \Gamma_{K,h} \) be the straight line/plane segment connecting the intersection between \( \Gamma_K \) and \( \partial K \). To ensure that \( \Gamma \) is reasonably resolved by \( T_h \), we make the following standard assumptions on \( T_h \) and interface \( \Gamma \):

- **(A1)**. For \( K \in T_h^\Gamma \) and any edge/face \( F \subset \partial K \) which intersects \( \Gamma \), \( F_\Gamma := \Gamma \cap F \) is simply connected with either \( F_\Gamma = F \) or \( \text{meas}(F_\Gamma) = 0 \).
- **(A2)**. For \( K \in T_h^\Gamma \), there is a smooth function \( \psi \) which maps \( \Gamma_{K,h} \) onto \( \Gamma_K \).
- **(A3)**. For any two different points \( x, y \in \Gamma_K \), the unit normal vectors \( n(x) \) and \( n(y) \), pointing to \( \Omega_2 \), at \( x \) and \( y \) satisfy

\[
|n(x) - n(y)| \leq \gamma h_K,
\]

with \( \gamma \geq 0 \) (cf. [18, 71]). Note that \( \gamma = 0 \) when \( \Gamma_K \) is a straight line/plane segment.

Let \( \varepsilon_h \) be the set of all edges (faces) of all elements in \( T_h \) and \( \varepsilon_h^\Gamma \) be the partition of \( \Gamma \) with respect to \( T_h \), i.e.

\[
\varepsilon_h^\Gamma := \{ F : F = \Gamma_K \text{ or } F = \Gamma \cap \partial K \text{ if } \Gamma \cap \partial K \text{ is an edge/face of } K, \forall K \in T_h \},
\]

and set \( \varepsilon_h^* := \varepsilon_h \setminus \varepsilon_h^\Gamma \). For any \( K \in T_h \) and \( F \in \varepsilon_h^* \cup \varepsilon_h^\Gamma \), \( h_K \) and \( h_F \) denote respectively the diameters of \( K \) and \( F \), and \( n_K \) denotes the unit outward normal vector along \( \partial K \). We denote by \( h := \max_{K \in T_h} h_K \) the mesh size of \( T_h \), and by \( \nabla h \), \( \nabla h^r \), and \( \epsilon_h \) the piecewise-defined gradient, divergence and strain operators with respect to \( T_h \), respectively.

Throughout the paper, we use \( a \leq b(a \geq b) \) to denote \( a \leq C b (a \geq C b) \), where \( C \) is a generic positive constant independent of mesh parameters \( h, h_K, h_e \), and the location of the interface relative to the meshes, and may be different at its each occurrence.

### 2.2 X-HDG Scheme on Interface-Unfitted Meshes

For \( i = 1, 2 \), let \( \chi_i \) be the characteristic function on \( \Omega_i \), and for integer \( r \geq 0 \), let \( \Pi_r^b : L^2(\Lambda) \to P_r(\Lambda) \) be the standard \( L^2 \) orthogonal projection operator for any bounded domain \( \Lambda \). Vector or tensor analogues of \( \Pi_r^b \) are denoted by \( \Pi_r^b \), respectively. Set

\[
\oplus \chi_i P_r(K) := \chi_1 P_r(K) + \chi_2 P_r(K), \quad L^2(\Omega, S) := \{ w \in [L^2(\Omega)]^{d \times d} : w^T = w \}.
\]

Let us introduce the following X-HDG finite element spaces:

\[
W_h = \{ w \in L^2(\Omega, S) : \forall K \in T_h, w|_K \in P_{k-1}(K) \text{ if } K \cap \Gamma = \emptyset ; w|_K \in \oplus \chi_i P_{k-1}(K) \text{ if } K \cap \Gamma \neq \emptyset \},
\]

\[
V_h = \{ v \in L^2(\Omega) : \forall K \in T_h, v|_K \in P_k(K) \text{ if } K \cap \Gamma = \emptyset ; v|_K \in \oplus \chi_i P_k(K) \text{ if } K \cap \Gamma \neq \emptyset \},
\]

\[
M_h = \{ \tilde{\mu} \in L^2(\varepsilon_h^*) : \forall F \in \varepsilon_h^*, \tilde{\mu}|_F \in P_k(F) \text{ if } F \cap \Gamma = \emptyset ; \tilde{\mu}|_F \in \oplus \chi_i P_k(F) \text{ if } F \cap \Gamma \neq \emptyset \},
\]

\[
\hat{M}_h = \{ \tilde{\mu} \in L^2(\varepsilon_h^\Gamma) : \tilde{\mu}|_F \in P_k(F)_F, \forall F \in \varepsilon_h^\Gamma, \text{ with } F = \tilde{K} \cap \Gamma \text{ for } K \in T_h \},
\]
\[ M_h(g_D) = \{ \tilde{\mu} \in M_h : \tilde{\mu}|_{F \cap \Omega_i} = \Pi^b_k(g_D|_{F \cap \Omega_i}), \forall F \in \varepsilon_h^* \text{ with } F \subset \partial \Omega_D \} \]

Then the X-HDG method is given as follows: seek \((\sigma_h, u_h, \hat{u}_h, \tilde{u}_h) \in W_h \times V_h \times M_h(g_D) \times \tilde{M}_h\) such that

\[
\begin{align*}
(A \sigma_h, w)_{T_h} + (u_h, \nabla_h \cdot w)_{T_h} - (\hat{u}_h, w)_{\partial T_h \setminus \Gamma_h^\varepsilon} - (\tilde{u}_h, w)_{\Omega_i, \Gamma} &= 0, \\
(\nabla_h \cdot \sigma_h, v)_{T_h} - (\tau(u_h - \hat{u}_h), v)_{\partial T_h \setminus \Gamma_h^\varepsilon} - (\eta(u_h - \tilde{u}_h), v)_{\Omega_i, \Gamma} &= (f, v), \\
(\sigma_h n, \tilde{\mu})_{\partial T_h \setminus \Gamma_h^\varepsilon} - (\tau(u_h - \hat{u}_h), \tilde{\mu})_{\partial T_h \setminus \Gamma_h^\varepsilon} &= (g_N, \tilde{\mu})_{\partial T_h \setminus \Gamma_h^\varepsilon}, \\
(\sigma_h n, \mu)_{\Omega_i, \Gamma} - (\eta(u_h - \tilde{u}_h), \mu)_{\Omega_i, \Gamma} &= \left( \frac{1}{2} g_N^\varepsilon, \mu \right)_{\Omega_i, \Gamma}
\end{align*}
\]

for all \((w, v, \tilde{\mu}, \mu) \in W_h \times V_h \times M_h(0) \times \tilde{M}_h\). Here

\[
(\cdot, \cdot)_{T_h} := \sum_{K \in T_h} (\cdot, \cdot)_K, \quad (\cdot, \cdot)_{\partial T_h \setminus \Gamma_h^\varepsilon} := \sum_{K \in T_h} (\cdot, \cdot)_{\partial K \setminus \Gamma_h^\varepsilon},
\]

and, for vectors \(\mu, v\) and tensor \(w\) with \(\mu_i = \mu|_{F \cap \Omega_i}, v_i = v|_{F \cap \Omega_i}\) and \(w_i = w|_{F \cap \Omega_i}\),

\[
\begin{align*}
(\mu, v)_{\Omega_i, \Gamma} &= \sum_{F \in \varepsilon_h^\Gamma} \int_F (\mu_1 \cdot v_1 + \mu_2 \cdot v_2) ds, \\
(wn, v)_{\Omega_i, \Gamma} &= \sum_{F \in \varepsilon_h^\Gamma} \int_F ((\omega_1 n_1) \cdot v_1 + (\omega_2 n_2) \cdot v_2) ds,
\end{align*}
\]

where \(n_i\) denotes the unit normal vector along \(\Gamma\) pointing from \(\Omega_i\) to \(\Omega_j\) with \(i, j = 1, 2\) and \(i \neq j\). The stabilization functions \(\tau\) and \(\eta\) are defined as below: for \(F \in \varepsilon_h, K \in T_h\) and \(i = 1, 2,\)

\[
\begin{align*}
\tau|_{F \cap \Omega_i} &= 2\mu_i h_K^{-1}, \quad \text{if } F \subset \partial K \setminus \varepsilon_h^\Gamma \text{ and } F \cap \Omega_i \neq \emptyset, \\
\eta|_{F \cap \Omega_i} &= 2\mu_i h_K^{-1}, \quad \text{if } F = \Gamma_K \text{ or } F \subset \partial (K \cap \Omega_i).
\end{align*}
\]

\begin{remark}
In fact, if the homogeneous interface condition \(\underline{u} = 0\) in the model problem (1.1) is generalized as

\[
\underline{u} = g_D^\varepsilon \neq 0
\]

and the interface \(\Gamma\) is a piecewise straight segment/polygon, then we can introduce the space

\[
\tilde{M}_h(g_D^\varepsilon) = \{ \tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2) : \tilde{\mu}_i|_{F} \in P_k(F), \ \| \tilde{\mu} \|_F = \tilde{\mu}_1|_{F} - \tilde{\mu}_2|_{F} = \Pi^b_k(g_D^\varepsilon|_{F}), \ \forall F \in \varepsilon_h^\Gamma \}
\]

so as to obtain the corresponding X-HDG scheme: seek \((\sigma_h, u_h, \hat{u}_h, \tilde{u}_h) \in W_h \times V_h \times M_h(g_D^\varepsilon) \times \tilde{M}_h(g_D^\varepsilon)\) such that the equations (2.2) hold for \((w, v, \tilde{\mu}, \mu) \in W_h \times V_h \times M_h(0) \times \tilde{M}_h(0)\). We note that all the analyses hereafter also apply to this case.

\end{remark}

\begin{theorem}
For \(k \geq 1\), the X-HDG scheme (2.2) admits a unique solution \((\sigma_h, u_h, \hat{u}_h, \tilde{u}_h) \in W_h \times V_h \times M_h(g_D^\varepsilon) \times \tilde{M}_h\).
\end{theorem}

\begin{proof}
Since the (2.2) is a linear square system, it suffices to show that if all of the given data vanish, i.e. \(f = g_D = g_N = g_N^\varepsilon = 0\), then we get the zero solution. Taking \((w, v, \tilde{\mu}) = (\sigma_h, u_h, \hat{u}_h, \tilde{u}_h)\) in (2.2) and adding these equations together, we have

\[
(A \sigma_h, \sigma_h)_{T_h} + (\tau(u_h - \hat{u}_h), u_h - \hat{u}_h)_{\partial T_h \setminus \Gamma_h^\varepsilon} + (\eta(u_h - \tilde{u}_h), u_h - \tilde{u}_h)_{\Omega_i, \Gamma} = 0,
\]

\end{proof}
which, together with the relation
\[
(Aw, w)_{T_h} = \left( \frac{1}{2\mu} \left( w - \frac{1}{d} \text{tr}(w) I \right), w - \frac{1}{d} \text{tr}(w) I \right)_{T_h} + \left( \frac{1}{d(d\lambda + 2\mu)} \text{tr}(w), \text{tr}(w) \right)_{T_h}, \forall w \in W_h,
\]
shows that
\[
\sigma_h = 0, \quad \text{in } T_h,
\]
\[
u_h - \hat{u}_h = 0, \quad \text{on } \partial T_h \setminus \varepsilon_h,
\]
\[
\{(u_h - \hat{u}_h)^2 \} = 0, \quad \text{on } \Gamma.
\]
Here \(\{ \cdot \}\) is defined by \(\{ v \} = \frac{1}{2}(v_1 + v_2)\) with \(v_i = v|_{\Gamma \cap \tilde{\Omega}_i}\) for \(i = 1, 2\). These relations, plus (2.2a) and integration by parts, yield
\[
(A\sigma_h, w)_{T_h} + \langle \epsilon_h(u_h), w \rangle_{T_h} - \langle u_h - \hat{u}_h, \partial_n \rangle_{\partial T_h \setminus \varepsilon_h} = \langle u_h - \hat{u}_h, \partial n \rangle_{\partial \Omega} = 0.
\]
Taking \(w = \epsilon_h(u_h)\) in this relation leads to \(\epsilon_h(u_h) = 0\). On the other hand, from the facts
\[
(u_h - \hat{u}_h)|_{\partial T_h \setminus \varepsilon_h} = 0, \quad \hat{u}_h|_{\partial \Omega} = 0, \quad \{(u_h - \hat{u}_h)^2\}|_{\Gamma} = 0, \quad \text{and } \hat{u}_h|_{\Gamma} = 0,
\]
it follows that \(u_h \in C^0(\Omega)\) and \(u_h|_{\partial \Omega} = 0\). Hence, we have \(u_h = 0\). The other two conclusions \(\hat{u}_h = 0\) and \(\tilde{u}_h = 0\) also follow. This completes the proof. \(\square\)

### 2.3 X-HDG Scheme on Boundary-Unfitted Meshes

In this subsection, we shall extend the X-HDG method in Sect. 2.1 to the case using boundary-unfitted meshes (cf. Fig. 3 for an example). For simplicity, we consider the following linear elasticity problem:
\[
A\sigma - \epsilon_h(u) = 0, \quad \text{in }\Omega, \quad (2.5a)
\]
\[
\nabla \cdot \sigma = f, \quad \text{in }\Omega, \quad (2.5b)
\]
\[
u = g_D, \quad \text{on }\partial \Omega, \quad (2.5c)
\]
\[
\sigma n = g_N, \quad \text{on }\partial \Omega_N. \quad (2.5d)
\]

Let \(\mathbb{B} \supset \Omega\) be a simpler domain than \(\Omega\) (Fig. 3), denote \(\Omega^c := \mathbb{B} \setminus \tilde{\Omega}\), and introduce a vector function \(\tilde{f}\) defined on \(\mathbb{B}\) with \(\tilde{f}|_{\Omega} = f\) and \(\tilde{f}|_{\Omega^c} = 0\). Then we can rewrite problem (2.5) as an ‘interface’ problem:
\[
A\sigma - \epsilon(u) = 0, \quad \text{in }\Omega \cup \Omega^c, \quad (2.6a)
\]
\[
\nabla \cdot \sigma = \tilde{f}, \quad \text{in }\Omega \cup \Omega^c, \quad (2.6b)
\]
\[
u = 0, \quad \sigma = 0, \quad \text{in }\Omega^c, \quad (2.6c)
\]
\[
[u] = g_D, \quad \text{on }\partial \Omega, \quad (2.6d)
\]
\[
[\sigma n] = g_N, \quad \text{on }\partial \Omega_N. \quad (2.6e)
\]

Here \([u] := (u|_{\Omega})|_{\partial \Omega_D} - (u|_{\Omega^c})|_{\partial \Omega_D}\) and \([\sigma n] := (\sigma n|_{\Omega})|_{\partial \Omega_N} - (\sigma n|_{\Omega^c})|_{\partial \Omega_N}\). We note that the problem (2.6) can be viewed as a special interface problem with \(\partial \Omega\) being the interface, for which we only need to approximate the solution in \(\Omega\), since the solution in \(\Omega^c\) is zero.
The X-HDG scheme for problem (2.6) reads as follows: seek \( \sigma_h, u_h, \hat{u}_h, \tilde{u}_h \) in \( W_h \times V_h \times M_h^i \times M_h^\alpha (g_D) \) such that

\[
\begin{align*}
(A\sigma_h, w)_{T_h} + (u_h, \nabla_h \cdot w)_{T_h} - \langle \tilde{u}_h, w \rangle_{\partial T_h \setminus \Gamma_h} - \langle \hat{u}_h, w \rangle_{\partial T_h \setminus \Gamma_h} &= 0, \\
- (\nabla_h \cdot \sigma_h, v)_{T_h} + \langle \tau (u_h - \hat{u}_h), v \rangle_{\partial T_h \setminus \Gamma_h} + \langle \eta (u_h - \tilde{u}_h), v \rangle_{\partial T_h \setminus \Gamma_h} &= \langle f, v \rangle, \\
\langle \sigma_h n, \tilde{\mu} \rangle_{\partial T_h \setminus \Gamma_h} - \langle \tau (u_h - \hat{u}_h), \tilde{\mu} \rangle_{\partial T_h \setminus \Gamma_h} &= 0, \\
\langle \sigma_h n, \tilde{\mu} \rangle_{\partial T_h \setminus \Gamma_h} - \langle \eta (u_h - \tilde{u}_h), \tilde{\mu} \rangle_{\partial T_h \setminus \Gamma_h} &= \langle g_N, \tilde{\mu} \rangle_{\partial T_h \setminus \Gamma_h}
\end{align*}
\]

Let \( T_h = \bigcup \{ K \} \) be a shape-regular triangulation of the domain \( \mathbb{B} \) consisting of open triangles/tetrahedrons. Define the following sets of elements and edges/ faces:

\[
\begin{align*}
T_h^i : &= \{ K \in T_h : K \cap \Omega = K \}, \\
T_h^\Gamma : &= \{ K^\Gamma : K^\Gamma = K \cap \Omega, K \in T_h \text{ and } K \cap \partial \Omega \neq \emptyset \}, \\
T_h^\varepsilon : &= T_h^i \cup T_h^\Gamma, \\
\varepsilon_h^i : &= \{ F : F \text{ is a edge/face of element in } T_h^i \text{ and } F \cap \partial \Omega = \emptyset \}, \\
\varepsilon_h^\Gamma : &= \{ F : F \text{ is a edge/face of element in } T_h^\Gamma \text{ and } F \cap \partial \Omega = \emptyset \}, \\
\varepsilon_h^\alpha : &= \{ F : F = K \cap \partial \Omega, \forall K \in T_h^\Gamma \text{ or } F \text{ is a edge/face of } K, \forall K \in T_h^\Gamma \text{ and } \bar{K} \cap \partial \Omega \neq \emptyset \}, \\
\varepsilon_h : &= \varepsilon_h^i \cup \varepsilon_h^\Gamma \cup \varepsilon_h^\alpha.
\end{align*}
\]

Introduce the following X-HDG finite element spaces:

\[
\begin{align*}
W_h : &= \{ w \in L^2(\Omega, S) : w|_K \in P_{k-1}(K) \text{ if } K \in T_h^i \}, \\
V_h : &= \{ v \in L^2(\Omega) : v|_K \in P_k(K) \text{ if } K \in T_h^i \}, \\
M_h^i : &= \{ \mu \in L^2(\varepsilon_h \setminus \varepsilon_h^\alpha) : \mu|_F \in P_k(F) \text{ if } F \in \varepsilon_h \setminus \varepsilon_h^\alpha \}, \\
M_h^\alpha : &= \{ \tilde{\mu} \in L^2(\varepsilon_h^\alpha) : \tilde{\mu}|_F \in P_k(K)|_F, \forall F \in \varepsilon_h^\alpha \text{ with } F = \bar{K} \cap \partial \Omega \text{ for } K \in T_h^\Gamma \}, \\
M_h^\alpha (g_D) : &= \{ \tilde{\mu} \in L^2(\varepsilon_h^\alpha) : \langle \tilde{\mu}, \mu^* \rangle_F = \langle g_D, \mu^* \rangle_F, \forall F \in \varepsilon_h^\alpha \cap \partial \Omega_D, \text{ and } \tilde{\mu}|_F, \mu^*|_F \in P_k(K)|_F \text{ for some } K \in T_h^\Gamma \}.
\end{align*}
\]
for all \((w, v, \mu, \tilde{\mu}) \in W_h \times V_h \times M^i_h \times M^\beta_h(0)\), and the stabilization coefficient is given by
\[
\tau|_F = \eta|_F = 2\mu h^{-1}, \quad \forall F \in \partial K \text{ or } F = K \cap \Omega.
\] (2.8)

**Remark 2.2** By following the same routine as in the proof of Theorem 2.1, we can easily know that the scheme (2.7) admits a unique solution for \(k \geq 1\).

### 3 A Priori Error Estimation

This section is devoted to the error analysis of X-HDG scheme (2.2) for the linear elasticity interface problem (1.1). To this end, we make the following assumption on the interface \(\Gamma\) and the triangulation \(T_h\).

\((A4)\). \(\Gamma \cap \partial \Omega_D\) contains at most some vertexes (2D) or edges (3D) of elements in \(T_h\).

#### 3.1 Some Basic Results

The following lemma from \([69, 70]\) will be used to carry out error estimation of projections around the interface \(\Gamma\) (cf. Lemma 3.2).

**Lemma 3.1** There exists a positive constant \(h_0\) depending only on the interface \(\Gamma\), the shape regularity of the mesh \(T_h\), and \(\gamma\) in (2.1), such that for any \(h \in (0, h_0]\) and \(K \in T_h^\Gamma\), the following estimates hold for either \(i = 1\) or \(i = 2\):
\[
\|v\|_{0, \Gamma_K} \leq h^{-1/2}_{K} \|v\|_{0, K \cap \Omega_i} + \|\nabla v\|_{0, K \cap \Omega_i}^{1/2}, \quad \forall v \in H^1(K \cap \Omega_i),
\] (3.1)
\[
\|v_h\|_{0, \Gamma_K} \leq h^{-1/2}_{K} \|v_h\|_{0, K \cap \Omega_i}, \quad \forall v_h \in P_r(K).
\] (3.2)

**Remark 3.1** We note that the condition \(h \in (0, h_0]\) in this lemma is not required when \(\Gamma_K\) is a straight line/plane segment, and this condition is easy to satisfy when \(\Gamma_K\) is a curved line/surface segment.

**Remark 3.2** The inverse trace inequality (3.2) is essential in the subsequent error estimation. Notice that the inverse trace inequality (3.2) may not be valid in one of the sub-elements \(K \cap \Omega_1\) and \(K \cap \Omega_2\). In fact, when \(\Gamma_K\) is close to an edge or a node of \(K\), the hidden constant may blow up in the small sub-element. Therefore, we need to introduce extension operators so as to avoid using (3.2) on the small sub-element in the error analysis.

For \(i = 1, 2\), set \(\tilde{\Omega}_i := \{\cup \tilde{K} : K \cap \Omega_i \neq \emptyset, \ K \in T_h\}\), and introduce extension operators \(E_i : H^s(\Omega_i) \rightarrow H^s(\tilde{\Omega}_i)\), with integer \(s \geq 1\), such that
\[
(E_i w)|_{\Omega_i} = w \quad \text{and} \quad \|E_i w\|_{H^s(\tilde{\Omega}_i)} \leq \|w\|_{H^s(\Omega_i)}, \quad \forall w \in H^s(\Omega_i).
\]

For any \(v \in H^1(\Omega_1 \cup \Omega_2)\) and \(i = 1, 2\), set \(\tilde{v}_i := E_i(v|_{\Omega_i})\), and define \(Q_r v\) and \(Q^\beta_r v\), with integer \(r \geq 0\), by
\[
(Q_r v)|_K := \chi_1 \Pi_r(\tilde{v}_1|_K) + \chi_2 \Pi_r(\tilde{v}_2|_K), \quad \forall K \in T_h,
\] (3.3)
\[
(Q^\beta_r v)|_F := \chi_1 \Pi^\beta_r(\tilde{v}_1|_F) + \chi_2 \Pi^\beta_r(\tilde{v}_2|_F), \quad F \in e^F_h.
\] (3.4)
Here $\Pi_r : L^2(K) \to P_r(K)$ denotes the standard $L^2$ orthogonal projection operator, and we recall that $\Pi_r^b$ is the $L^2$ orthogonal projection operator from $L^2(F)$ onto $P_r(F)$. Notice that if $K \cap \Omega_i = K$ and $F \cap \Omega_i = F$, then

$$
(Q_r v)|_K := \Pi_r(v|_K), \quad (Q_r^b v)|_F := \Pi_r^b(v|_F).
$$

Vector or tensor analogues of $Q_r$ and $Q_r^b$ are denoted by $Q_r$ and $Q_r^b$, respectively.

Based on Lemma 3.1 and standard properties of the $L^2$ projection operators, we have the following conclusion.

**Lemma 3.2** Let $s$ be an integer with $1 \leq s \leq r + 1$. For any $K \in \mathcal{T}_h$, $h \in (0, h_0]$ and $v \in H^1(\Omega_1 \cup \Omega_2)$, we have

$$
\sum_{K \in \mathcal{T}_h} \| v - Q_r v \|_{0,K}^2 + h^2 \sum_{K \in \mathcal{T}_h} \| v - Q_r v \|_{1,K}^2 \leq h^{2s} \| v \|_{s,\Omega_1 \cup \Omega_2}^2, \quad (3.5)
$$

$$
\sum_{K \in \mathcal{T}_h} \| v - Q_r v \|_{0,\partial K}^2 + \sum_{K \in \mathcal{T}_h} \| v - Q_r v \|_{0,\Gamma_K}^2 \leq h^{2s-1} \| v \|_{s,\Omega_1 \cup \Omega_2}^2, \quad (3.6)
$$

$$
\sum_{K \in \mathcal{T}_h} \| v - Q_r^b v \|_{0,\partial K}^2 \leq h^{2s-1} \| v \|_{s,\Omega_1 \cup \Omega_2}^2, \quad (3.7)
$$

where the notations $\| \cdot \|_s$ and $\| \cdot \|_{0,\partial K}$ for $K \in \mathcal{T}_h$ are understood respectively as $\| \cdot \|_s = \sum_{i=1}^2 \| \cdot \|^2_{s,K \cap \Omega_i}$ and $\| \cdot \|_{0,\partial K} = \sum_{i=1}^2 \| \cdot \|^2_{0,\partial K \cap \Omega_i}$.

**Proof** From the definition of $Q_r$ and the properties of the extensive operator $E_i$ and the projection $\Pi_r$ it follows

$$
\sum_{K \in \mathcal{T}_h} \| v - Q_r v \|_{0,K}^2 + h^2 \sum_{K \in \mathcal{T}_h} \| v - Q_r v \|_{1,K}^2 \leq \sum_{i=1}^2 \sum_{K \in \mathcal{T}_h, K \cap \bar{\Omega}_i \neq \emptyset} \left( \| E_i v - \Pi_r E_i v \|_{0,K}^2 + h^2 \| E_i v - \Pi_r E_i v \|_{1,K}^2 \right)
$$

$$
\leq \sum_{i=1}^2 \sum_{K \in \mathcal{T}_h, K \cap \bar{\Omega}_i \neq \emptyset} h^{2s} \| E_i v \|_{s,K}^2
$$

$$
\leq h^{2s} \| v \|_{s,\Omega_1 \cup \Omega_2}^2,
$$

which yields (3.5). Similarly, by the trace inequality and Lemma 3.1 we get

$$
\sum_{K \in \mathcal{T}_h} \| v - Q_r v \|_{0,\partial K}^2 + \sum_{K \in \mathcal{T}_h} \| v - Q_r v \|_{0,\Gamma_K}^2 \leq h^{-1} \sum_{i=1}^2 \sum_{K \in \mathcal{T}_h, K \cap \bar{\Omega}_i \neq \emptyset} \left( \| E_i v - \Pi_r E_i v \|_{0,K}^2 + h \| E_i v - \Pi_r E_i v \|_{1,K}^2 \right)
$$

$$
\leq h^{2s-1} \| v \|_{s,\Omega_1 \cup \Omega_2}^2
$$

and

$$
\sum_{K \in \mathcal{T}_h} \| v - Q_r^b v \|_{0,\partial K}^2 \leq \sum_{i=1}^2 \sum_{K \in \mathcal{T}_h, K \cap \bar{\Omega}_i \neq \emptyset} \| E_i v - \Pi_r^b E_i v \|_{0,\partial K}^2
$$

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Here, for any $\text{Lemma 3.3}$

\[ \sum_{i=1}^{2} \sum_{K \in T_h, K \cap \Omega_i \neq \emptyset} \| E_i v \|^2_{s,K} \leq h^{2s-1} \| v \|^2_{s,\Omega_1 \cup \Omega_2}. \]

This completes the proof. \(\square\)

**Remark 3.3** In fact, for $K \in T_h$ with $K \cap \Omega_i = K$ ($i = 1$ or 2) and $v \in H^s(K)$, it is easy to see that

\[ \| v - Q_r v \|_{0,K} + h \| v - Q_r v \|_{1,K} \leq h^s \| v \|_{s,K}. \]

\[ \| v - Q_r v \|_{0,\partial K} + \| v - Q_r^b v \|_{0,\partial K} \leq h^{s-1/2} \| v \|_{s,K}. \]

### 3.2 Error Estimation for Stress and Displacement Approximations

Let $(\sigma, u)$ be the solution of (1.1). For simplicity of presentation, we define

\[ e_h^\sigma := Q_{k-1}\sigma - \sigma_h, \quad e_h^u := Q_k u - u_h, \quad e_h^\sigma := Q_k^\Gamma u - \tilde{u}_h, \quad e_h^u := Q_k^\Gamma u - \tilde{u}_h. \quad (3.8) \]

Here, for any $F \in \Gamma_h^\Gamma$,

\[ (Q_k^\Gamma u)|_F := \begin{cases} \Pi_b^k(u|_F), & \text{if } F \text{ is a straight line/plane segment;} \\ \frac{1}{2}(\Pi_k(\bar{u}_1|K)|_F + \Pi_k(\bar{u}_2|K)|_F), & \text{if } F = F_K = F \cap K \text{ is not a straight line/plane segment for } K \in T_h^\Gamma, \end{cases} \quad (3.9) \]

where $\bar{u}_i := E_i(u|_{\Omega_i}), \ i = 1, 2$.

We have the following lemma on error equations.

**Lemma 3.3** For all $(w, v, \mu, \tilde{\mu}) \in W_h \times V_h \times M_h(0) \times \tilde{M}_h$, it holds

\[ (Ae_h^\sigma, w)_{T_h} + (e_h^u, \nabla_h \cdot w)_{T_h} - (\tilde{e}_h^\sigma, wn)_{\partial T_h \setminus \Gamma_h} - (e_h^u, wn)_{s,\Gamma} = L_1(w), \quad (3.10a) \]

\[ - (\nabla_h \cdot e_h^\sigma, v)_{T_h} + \langle \tau(e_h^u - e_h^\sigma), v \rangle_{\partial T_h \setminus \Gamma_h} + \langle \eta(e_h^u - e_h^\sigma), v \rangle_{s,\Gamma} = L_2(v) + L_3(v) + L_4(v), \quad (3.10b) \]

\[ \langle e_h^\sigma n, \tilde{\mu} \rangle_{\partial T_h \setminus \Gamma_h} - \langle \tau(e_h^u - e_h^\sigma), \tilde{\mu} \rangle_{\partial T_h \setminus \Gamma_h} = -L_2(\tilde{\mu}), \quad (3.10c) \]

\[ \langle e_h^u n, \tilde{\mu} \rangle_{s,\Gamma} - \langle \eta(e_h^u - e_h^\sigma), \tilde{\mu} \rangle_{s,\Gamma} = -L_3(\tilde{\mu}), \quad (3.10d) \]

where

\[ L_1(w) := -\langle u - Q_k^\Gamma u, wn \rangle_{s,\Gamma} - \langle u - Q_k u, \nabla_h \cdot w \rangle_{T_h^\Gamma} - (A(\sigma - Q_{k-1}\sigma), w)_{T_h^\Gamma} \]

\[ - -\langle u - Q_k^b u, wn \rangle_{\partial T_h \setminus \Gamma_h}, \]

\[ L_2(v) := \langle (\sigma - Q_{k-1}\sigma) n, v \rangle_{\partial T_h} + \langle \tau(Q_k u - Q_k^b u), v \rangle_{\partial T_h \setminus \Gamma_h}, \]

\[ L_3(v) := \langle (\sigma - Q_{k-1}\sigma) n, v \rangle_{s,\Gamma} + \langle \eta(Q_k u - Q_k^\Gamma u), v \rangle_{s,\Gamma}, \]

\[ L_4(v) := -(\sigma - Q_{k-1}\sigma, \epsilon_h(v))_{T_h^\Gamma}, \]

with $(\cdot, \cdot)_{T_h^\Gamma} := \sum_{K \in T_h^\Gamma} (\cdot, \cdot)_K$ and $(\cdot, \cdot)_{\partial T_h \setminus \Gamma_h} := \sum_{K \in T_h^\Gamma} (\cdot, \cdot)_{\partial K \setminus \Gamma_h}^\Gamma$. 

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Proof From the definitions of the operators $Q_{k-1}$, $Q_k$, $Q_k^b$ and $Q_k^\Gamma$, we obtain for any $(w, v) \in W_h \times V_h$ that

\[
(\langle A Q_{k-1} \sigma, w \rangle_{T_h} + (Q_k u, \nabla_h \cdot w)_{T_h} - \langle Q_k^b u, w n \rangle_{\partial T_h \setminus \Omega_h^\Gamma} - \langle Q_k^\Gamma u, w n \rangle_{\partial \Omega_h})_{T_h^\Gamma} + (\langle u - Q_k^\Gamma u, w n \rangle_{\partial T_h \setminus \Omega_h^\Gamma} + (A(\sigma - Q_k \sigma), w)_{T_h^\Gamma} \\
+ (\langle u - Q_k^b u, w n \rangle_{\partial \Omega_h})_{T_h^\Gamma} \quad \text{and (2.2d) and the relations (3.10c) and (3.10d) follow from (2.2c), (2.2d) and the relations (3.12).}
\]

Subtracting (2.2a) and (2.2b) from the above two equations respectively yields (3.10a) and (3.10b). And (3.10b) follows from (2.2c), (2.2d) and the relations

\[
\langle \sigma n, \hat{\mu} \rangle_{T_h^\Gamma} = \left\{ \frac{1}{2} g_N^\Gamma, \hat{\mu} \right\}_{T_h^\Gamma}, \quad \langle \sigma n, \hat{\mu} \rangle_{\partial T_h \setminus \Omega_h^\Gamma} = \left\{ g_N, \hat{\mu} \right\}_{\partial T_h \setminus \Omega_h^\Gamma}.
\]

Let us define

\[
\| \cdot \|_{A, T_h} := (\langle A \cdot, \cdot \rangle_{T_h}^{\frac{1}{2}}, \quad \| \cdot \|_{0, T_h} := (\langle \cdot, \cdot \rangle_{T_h}^{\frac{1}{2}}, \quad \| \cdot \|_{0, \partial T_h \setminus \Omega_h} := (\langle \cdot, \cdot \rangle_{\partial T_h \setminus \Omega_h}^{\frac{1}{2}}, \quad \| \cdot \|_{0, \partial T_h \setminus \Omega_h}^{\Gamma} := (\langle \cdot, \cdot \rangle_{\partial T_h \setminus \Omega_h}^{\Gamma}^{\frac{1}{2}}),
\]

and introduce a semi-norm $\| (w, v, \mu, \hat{\mu}) \| := \left( \| w \|_{A, T_h}^2 + \| v - \hat{\mu} \|_{0, \partial T_h \setminus \Omega_h}^{\Gamma} \right)^{\frac{1}{2}}$. (3.11)

Lemma 3.4 Let $(\sigma, u) \in H^k(\Omega_1 \cup \Omega_2, S) \times H^{k+1}(\Omega_1 \cup \Omega_2)$ and $(\sigma_h, u_h, \hat{u}_h, \tilde{u}_h) \in W_h \times V_h \times M_h(g_D) \times \tilde{M}_h$ be the solutions of the problem (1.1) and the X-HDG scheme (2.2), respectively. For any $h \in (0, h_0]$, it holds

\[
\| (e_h^\sigma, e_h, e_h^\mu, e_h) \|^2 = \sum_{i=1}^4 E_i, \quad (3.12)
\]

\[
\| \sqrt{\mu} e_h^\sigma \|_{0, T_h} \leq \| (e_h^\sigma, e_h^\mu, e_h^\mu, e_h^\mu) \| + \| \frac{1}{\sqrt{\mu}} e_h^\sigma \|_{0, T_h} + h^k \left( \| \sqrt{\mu} u \|_{k+1, \Omega_1 \cup \Omega_2} \right.
\]

\[
\left. + \| \frac{1}{\sqrt{\mu}} \| \sigma \|_{k, \Omega_1 \cup \Omega_2} \right), \quad (3.13)
\]

where

\[
E_1 := -\langle (u - Q_k^\Gamma u, e_h^\sigma)_{T_h^\Gamma} - (A(\sigma - Q_k \sigma), e_h^\sigma)_{T_h^\Gamma} \rangle + (\langle u - Q_k^b u, e_h^\sigma n \rangle_{\partial T_h \setminus \Omega_h} + (\langle u - Q_k^b u, e_h^\sigma n \rangle_{\partial T_h \setminus \Omega_h})_{T_h^\Gamma}.
\]

\[
E_2 := \langle (\sigma - Q_k \sigma) n, e_h^\mu - e_h^\mu \rangle_{\partial T_h \setminus \Omega_h} + \langle (\sigma - Q_k \sigma) n, e_h^\mu - e_h^\mu \rangle_{T_h^\Gamma}.
\]

\[
E_3 := \langle (Q_k u - Q_k^b u), e_h^\mu - e_h^\mu \rangle_{\partial T_h \setminus \Omega_h} + \langle (Q_k u - Q_k^b u), e_h^\mu - e_h^\mu \rangle_{T_h^\Gamma}.
\]

\[
E_4 := -\langle (\sigma - Q_k \sigma), e_h^\mu (e_h^\mu) \rangle_{T_h^\Gamma}.
\]
Proof Taking \((w, v, \mu, \tilde{\mu}) = (e_h^\sigma, e_h^\mu, \tilde{e}_h^\mu, \tilde{e}_h^\sigma)\) in the four equations in Lemma 3.10 and adding up them yield (3.12). Then it remains to show (3.13). We only consider the case that the interface is not a piecewise line/plane segment, since the other case is easier.

Taking \(w = 2\mu \epsilon_h(e_h^n)\) in (3.10a) and applying integration by parts, we obtain

\[
\langle A e_h^n, 2\mu \epsilon_h(e_h^n) \rangle_{\Gamma_h} - \langle e_h^n(\epsilon_h^n - 2\mu \epsilon_h(e_h^n)) \rangle_{\Gamma_h} + \langle e_h^n - e_h^\mu, 2\mu \epsilon_h(e_h^n) \rangle_{\partial \Omega \setminus \Gamma_h} + \langle e_h^n - e_h^\mu, 2\mu \epsilon_h(e_h^n) \rangle_{\partial \Omega \setminus \Gamma_h} = L_1(2\mu \epsilon_h(e_h^n)),
\]

which, together with the Cauchy–Schwarz inequality, the properties of the extensive operator \(E_i\) and the trace inequality, implies

\[
\left\| \sqrt{2\mu} \epsilon_h(e_h^n) \right\|_{0, \Gamma_h}^2 \\
= (A e_h^n, 2\mu \epsilon_h(e_h^n))_{\Gamma_h} + \langle e_h^n - e_h^\mu, 2\mu \epsilon_h(e_h^n) \rangle_{\partial \Omega \setminus \Gamma_h} + \langle e_h^n - e_h^\mu, 2\mu \epsilon_h(e_h^n) \rangle_{\partial \Omega \setminus \Gamma_h} - L_1(2\mu \epsilon_h(e_h^n)) \\
\leq \sum_{i=1}^2 \sum_{k \in \mathcal{K}_{i, i}} \left( \left\| \sqrt{2\mu} E_i \epsilon_h(e_h^n) \right\|_{0, k} \right) \left( \left\| \sqrt{2\mu} A e_h^n \right\|_{0, \Gamma_h} + \left\| \tau^\frac{1}{2} (e_h^n - e_h^\mu) \right\|_{0, \partial \Omega \setminus \Gamma_h} \right)
\]

Under the assumption (A4) it holds \(Q^i_k(u)|_F = \Pi^i_k(u|_F)\) on \(F \subset \partial \Omega D\). Thus, By (1.2) and Lemma 3.2 we further get

\[
\left\| \sqrt{2\mu} \epsilon_h(e_h^n) \right\|_{0, \Gamma_h} \\
\leq \left\| \sqrt{2\mu} \epsilon_h^n \right\|_{0, \Gamma_h} + \left\| \tau^\frac{1}{2} (e_h^n - e_h^\mu) \right\|_{0, \partial \Omega \setminus \Gamma_h} + \left\| \eta^\frac{1}{2} (e_h^n - e_h^\mu) \right\|_{0, \partial \Omega \setminus \Gamma_h} \\
+ h^k \left\| \sqrt{2\mu} u \right\|_{k+1, \Omega_1 \cup \Omega_2} + h^k \left\| \sqrt{2\mu} \sigma \right\|_{k, \Omega_1 \cup \Omega_2}.
\]
which yields the desired estimate (3.13). □

Based on Lemma 3.4, we can obtain the following result.

**Lemma 3.5** Let \((\sigma, u) \in H^k(\Omega_1 \cup \Omega_2, S) \times H^{k+1}(\Omega_1 \cup \Omega_2)\) and \((\sigma_h, u_h, \tilde{u}_h, \hat{u}_h) \in W_h \times V_h \times M_h(g_D) \times \bar{M}_h\) be the solutions of the problem (1.1) and the X-HDG scheme (2.2), respectively. Then it holds

\[
\|e_h^\sigma\|_{0, T_h} \lesssim \| (e_h^\sigma, e_h^u, e_h^{\hat{u}}, e_h^{\tilde{u}}) \| .
\]

(3.14)

Further more, for any \(h \in (0, h_0]\),

\[
\| (e_h^\sigma, e_h^u, e_h^{\hat{u}}, e_h^{\tilde{u}}) \| \lesssim h^k \left( \| \sqrt{\mu} u \|_{k+1, \Omega_1 \cup \Omega_2} + \| \frac{1}{\sqrt{\mu}} \|_{k, \Omega_1 \cup \Omega_2} \right).
\]

(3.15)

**Proof** For any \(w \in \mathbb{R}^{d \times d}\), let \(w_* := w - \frac{1}{d} tr(w) I\) denote its deviatoric tensor. Then we can easily have, for all \(w, \tau \in [L^2(\Omega)]^{d \times d}\),

\[
(Aw, \tau)_{T_h} = \left( \frac{1}{2\mu} w_* , \tau_* \right)_{T_h} + \left( -\frac{1}{d(d\lambda + 2\mu)} tr(w), tr(\tau) \right)_{T_h},
\]

\[
\| \tau \|_{0, T_h}^2 = \| \tau_* \|_{0, T_h}^2 + \frac{1}{d} \| tr(\tau) \|_{0, T_h}^2,
\]

which, together with the definition of \(\| \cdot \|\), indicate

\[
\|e_h^\sigma\|_{0, T_h} \leq \tilde{C} \sqrt{\lambda_{\max} + \mu_{\max}} \| (e_h^\sigma, e_h^u, e_h^{\hat{u}}, e_h^{\tilde{u}}) \|,
\]

i.e. (3.14) holds. Here and in what follows \(\tilde{C}\) denotes a generic positive constant independent of \(h, \lambda_{\max} = \max_{i=1,2} \lambda_i, \mu_{\max} = \max_{i=1,2} \mu_i\) and \(\mu_{\min} = \min_{i=1,2} \mu_i\).

From (3.12), the Cauchy–Schwarz inequality, the properties of the extensible operators, the inverse inequality and Lemma 3.2 it follows

\[
\| (e_h^\sigma, e_h^u, e_h^{\hat{u}}, e_h^{\tilde{u}}) \|^2 \leq \| u - Q_k^\Gamma u \|_{* \Gamma} \| e_h^\sigma \|_{* \Gamma} + \| u - Q_k^\Gamma u \|_{0, T_h \Gamma} \| \nabla_h \cdot e_h^\sigma \|_{0, T_h \Gamma}
\]

\[
+ \left( \| \frac{1}{\sqrt{2\mu}} (\sigma - Q_{k-1} \sigma) \|_{0, T_h \Gamma} \right)^2 + \left( \| \frac{1}{\sqrt{2\mu}} e_h^\sigma \|_{0, T_h \Gamma} \right)^2
\]

\[
+ \| u - Q_k^\Gamma u \|_{0, \partial T_h \setminus \Gamma} \| e_{h}^{\hat{u}} \|_{0, \partial T_h \setminus \Gamma}
\]

\[
+ \left( \| \frac{1}{\sqrt{2\mu}} (\sigma - Q_{k-1} \sigma) \|_{0, \partial T_h \setminus \Gamma} \right)^2 + \left( \| \frac{1}{\sqrt{2\mu}} e_{h}^{\hat{u}} \|_{0, \partial T_h \setminus \Gamma} \right)^2
\]

\[
+ \left( \| \tau \|_{\frac{1}{2} (\sigma - Q_{k-1} \sigma)} \|_{0, \partial T_h \setminus \Gamma} \right)^2 + \left( \| \eta \|_{\frac{1}{2} (\sigma - Q_{k-1} \sigma)} \|_{\frac{1}{2} (\sigma - Q_{k-1} \sigma)} \right)^2
\]

\[
+ \| u - Q_k^\Gamma u \|_{0, \partial T_h \setminus \Gamma}
\]

\[
\leq \tilde{C} h^k \left( \| \sqrt{2\mu} u \|_{k+1, \Omega_1 \cup \Omega_2} + \| \frac{1}{\sqrt{2\mu}} \|_{k, \Omega_1 \cup \Omega_2} \right)
\]

\[
\left( \| \tau \|_{\frac{1}{2} (\sigma - Q_{k-1} \sigma)} \|_{0, \partial T_h \setminus \Gamma} + \| \eta \|_{\frac{1}{2} (\sigma - Q_{k-1} \sigma)} \|_{0, \partial T_h \setminus \Gamma} \right)^2.
\]
In this situation, we can follow the same line as in the proof of [15, Theorem 3.4] to get the estimate (3.15) holds. i.e. the estimate (3.15) holds.

Remark 3.4 Note that the hidden constant factors in the upper bounds of (3.14) and (3.15) depend on the Lamé coefficient \( \lambda \) (cf. (3.16) and (3.17) for the explicit dependence). This is due to the use of extension operators \( E_i \) in the interface elements in \( T_h^i \). In fact, if the estimates in Lemma 3.1 hold simultaneously for \( i = 1, 2 \), then in the analysis we can avoid to introduce \( E_i \) just by defining the operators \( Q_r, Q_b \) as

\[
(Q_r \psi)|_K := \chi_1 \Pi_r(\psi|_{K \cap \Omega_1}) + \chi_2 \Pi_r(\psi|_{K \cap \Omega_2}), \quad (Q_b \psi)|_F := \chi_1 \Pi^b_r(\psi|_{F \cap \Omega_1}) + \chi_2 \Pi^b_r(\psi|_{F \cap \Omega_2}).
\]

In this situation, we can follow the same line as in the proof of [15, Theorem 3.4] to get the coercivity inequality

\[
\|\mathbf{w}\|_{0,T_h} \leq \tilde{C} \sqrt{\mu_{\text{max}}} \left( \|\mathbf{w}\|_{A,T_h} + \left\| \frac{h}{2\mu} (\mathbf{w}n - \hat{\mathbf{w}}n) \right\|_{0,\partial T_h \setminus \Gamma^h} + \left\| \frac{h}{2\mu} (\mathbf{w}n - \tilde{\mathbf{w}}n) \right\|_{*,\Gamma^h} \right)
\]

for all \((\mathbf{w}, \hat{\mathbf{w}}, \tilde{\mathbf{w}}) \in W_h \times L^2(\mathbf{e}^h_h) \times L^2(\mathbf{e}^h_b)\) satisfying

\[
\langle \mathbf{w}, \mathbf{e}_h(\mathbf{Q}_k \psi) \rangle - \langle \hat{\mathbf{w}}n, \mathbf{Q}_k \psi \rangle_{\partial T_h \setminus \Gamma^h} - \langle \tilde{\mathbf{w}}n, \mathbf{Q}_k \psi \rangle_{*,\Gamma^h} = 0, \quad \forall \psi \in H^1(\Omega) \text{ and } \psi|_{\partial \Omega_D} = 0,
\]

\[
\langle \hat{\mathbf{w}}n, \hat{\mathbf{\mu}} \rangle_{\partial T_h \setminus \Gamma^h} = 0, \quad \forall \hat{\mathbf{\mu}} \in \mathbf{M}_h(0),
\]

\[
\langle \tilde{\mathbf{w}}n, \tilde{\mathbf{\mu}} \rangle_{*,\Gamma^h} = 0, \quad \forall \tilde{\mathbf{\mu}} \in \tilde{\mathbf{M}}_h,
\]

\[
tr(\mathbf{w}) \in L^2_0(\Omega), \text{ if } \Gamma_N = \emptyset.
\]

By this uniform coercivity, we can further obtain

\[
\|\mathbf{e}^h_\sigma\|_{0,T_h} \leq \tilde{C} \left( \|\mathbf{e}^h_\sigma\|, \mathbf{e}^h_\sigma, \mathbf{e}^h_\sigma, \mathbf{e}^h_\sigma \right) \leq \tilde{C} h^k (\|\mathbf{\sqrt{\mu}}u\|_{k+1,\Omega_1 \cup \Omega_2} + \|\frac{1}{\sqrt{\mu}} \mathbf{\sigma}\|_{k,\Omega_1 \cup \Omega_2}),
\]

which finally leads to error estimates which are uniform in \( \lambda \) (cf. Remark 3.5).

In light of Lemmas 3.2, 3.4, 3.5 and the triangle inequality, we can easily derive the following optimal error estimates for the stress and displacement approximations.

**Theorem 3.1** Let \((\mathbf{\sigma}, \mathbf{u}) \in H^k(\Omega_1 \cup \Omega_2) \times H^{k+1}(\Omega_1 \cup \Omega_2)\) and \((\mathbf{\sigma}_h, \mathbf{u}_h, \hat{\mathbf{u}}_h, \tilde{\mathbf{u}}_h) \in W_h \times V_h \times \mathbf{M}_h(\mathbf{Q}) \times \tilde{\mathbf{M}}_h\) be the solutions of the problem (1.1) and the X-HDG scheme (2.2), respectively. Then for any \( h \in (0, h_0] \) it holds

\[
\|\mathbf{\sigma} - \mathbf{\sigma}_h\|_{0,T_h} \leq h^k (\|\mathbf{u}\|_{k+1,\Omega_1 \cup \Omega_2} + \|\mathbf{\sigma}\|_{k,\Omega_1 \cup \Omega_2}),
\]

\[
\|\mathbf{\epsilon}(\mathbf{u}) - \mathbf{\epsilon}(\mathbf{u}_h)\|_{0,T_h} \leq h^k (\|\mathbf{u}\|_{k+1,\Omega_1 \cup \Omega_2} + \|\mathbf{\sigma}\|_{k,\Omega_1 \cup \Omega_2}).
\]
3.3 $L^2$ Estimation for Displacement Approximation

To derive an $L^2$ error estimate for the displacement approximation by the Aubin-Nitsche’s technique of duality argument, we need to introduce an auxiliary problem:

$$A \Phi - e_h(\phi) = 0, \quad \text{in } \Omega_1 \cup \Omega_2$$  \hspace{1cm} (3.23a)

$$\nabla : \Phi = e_h^\mu, \quad \text{in } \Omega_1 \cup \Omega_2$$  \hspace{1cm} (3.23b)

$$\phi = 0, \quad \text{on } \partial \Omega_D$$  \hspace{1cm} (3.23c)

$$\Phi n = 0, \quad \text{on } \partial \Omega_N$$  \hspace{1cm} (3.23d)

$$\| \Phi \|_{H^1(\Omega_1 \cup \Omega_2)} + \mu \| \Phi \|_{H^2(\Omega_1 \cup \Omega_2)} \leq \| e_h^\mu \|_{L^2(\Omega_1 \cup \Omega_2)}. \quad (3.24)$$

We note that this regularity result holds when $\Omega$ is convex and $\Gamma$ is smooth with $\Gamma \cap \partial \Omega = \emptyset$ (cf. [18, 40]).

**Theorem 3.2** Let $(\sigma, u) \in H^k(\Omega_1 \cup \Omega_2, S) \times H^{k+1}(\Omega_1 \cup \Omega_2)$ and $(\sigma_h, u_h, \hat{u}_h, \tilde{u}_h) \in W_h \times V_h \times M_h(\mathbf{g}_D) \times \tilde{M}_h$ be the solutions of the problem (1.1) and the X-HDG scheme (2.2), respectively. Then for any $h \in (0, h_0)$ it holds the error estimate

$$\| u - u_h \|_{0, \Gamma} \leq h^{k+1}(\| u \|_{k+1, \Omega_1 \cup \Omega_2} + \| \sigma \|_{k, \Omega_1 \cup \Omega_2}). \quad (3.25)$$

**Proof** Testing the equations (3.23b) by $e_h^\mu$ and using the projection properties and integration by parts, we have

$$\| e_h^\mu \|_{0, \Gamma}^2 = (\nabla \cdot e_h^\mu, e_h^\mu)_\Gamma = (\nabla \cdot Q_{k-1} \Phi, e_h^\mu)_\Gamma + \langle (\Phi - Q_{k-1} \Phi)n, e_h^\mu \rangle_{\partial \Gamma} + \langle \nabla \cdot e_h^\mu, e_h^\mu \rangle_{\partial \Omega_2 \setminus \Gamma}$$

$$+ \langle \Phi - Q_{k-1} \Phi, \nabla e_h^\mu \rangle_{\partial \Gamma} + \langle \Phi - Q_{k-1} \Phi, e_h^\mu \rangle_{\partial \Omega_1}. \quad (3.26)$$

Taking $(v, \hat{\mu}, \tilde{\mu}) = (Q_k \phi, Q_k \phi, Q_k \phi)$ in (3.10b)-(3.10d) yields that

$$- (\nabla \cdot e_h^\mu, Q_k \phi)_\Gamma + \langle \tau(e_h^\mu - \hat{\mu}_h), Q_k \phi \rangle_{\partial \Omega_2 \setminus \Gamma} + \langle \eta(e_h^\mu - \hat{\mu}_h), Q_k \phi \rangle_{\partial \Omega_1 \setminus \Gamma} \quad \sum_{i=2}^{4} L_i(Q_k \phi),$$

$$\langle e_h^\mu, \hat{\mu}_h \rangle_{\partial \Omega_2 \setminus \Gamma} - \langle e_h^\mu, \tilde{\mu}_h \rangle_{\partial \Omega_1 \setminus \Gamma} = -L_2(Q_k \phi),$$

$$\langle e_h^\mu, Q_k \phi \rangle_{\partial \Omega_1 \setminus \Gamma} + \langle \eta(e_h^\mu - \hat{\mu}_h), Q_k \phi \rangle_{\partial \Omega_1 \setminus \Gamma} = -L_3(Q_k \phi).$$

These relations plus (3.23e) lead to

$$\langle A \Phi, e_h^\mu \rangle_{\Gamma} = -\langle \phi, \nabla \cdot e_h^\mu \rangle_{\Gamma} + \langle \Phi, e_h^\mu n \rangle_{\partial \Omega_2 \setminus \Gamma} + \langle \phi, e_h^\mu n \rangle_{\partial \Omega_1 \setminus \Gamma}$$

$$= -(Q_k \phi, \nabla \cdot e_h^\mu)_\Gamma + \langle Q_k \phi, e_h^\mu n \rangle_{\partial \Omega_2 \setminus \Gamma} + \langle Q_k \phi, e_h^\mu n \rangle_{\partial \Omega_1 \setminus \Gamma} + \langle \phi - Q_k \phi, e_h^\mu n \rangle_{\partial \Omega_1 \setminus \Gamma}.$$
Similarly, we can obtain

\[ \langle \tau (e_h^u - e_h^\tilde{u}), Q_h^\Gamma \phi - Q_h^\Gamma \Phi \rangle_{\partial \Omega_h \setminus e_h^\Gamma} + \langle \eta (e_h^u - e_h^\tilde{u}), Q_h^\Gamma \phi - Q_h^\Gamma \Phi \rangle_{*, \Gamma} + \langle \phi - Q_h^\Gamma \phi, e_h^\nu n \rangle_{*, \Gamma}. \]

By (3.26) we further get

\[ \| e_h^u \|^2_{0,T_h} = \sum_{j=1}^{4} I_j + \langle (Q_{k-1} \Phi - \Phi, A e_h^\nu)_{T_h} + (\Phi - Q_{k-1} \Phi, \nabla_h e_h^u)_{T_h \Gamma}, \]

where

\[ I_1 = \langle (Q_{k-1} \Phi - \Phi) n, e_h^u - e_h^\tilde{u} \rangle_{\partial \Omega_h \setminus e_h^\Gamma} + \langle (Q_{k-1} \Phi - \Phi) n, e_h^u - e_h^\tilde{u} \rangle_{*, \Gamma}, \]

\[ I_2 = L_2 (Q_k \phi - Q_k^\Gamma \phi) + L_3 (Q_k \phi - Q_k^\Gamma \phi), \]

\[ I_3 = \langle \tau (e_h^u - e_h^\tilde{u}), Q_h^\Gamma \phi - Q_h^\Gamma \Phi \rangle_{\partial \Omega_h \setminus e_h^\Gamma} + \langle \eta (e_h^u - e_h^\tilde{u}), Q_h^\Gamma \phi - Q_h^\Gamma \Phi \rangle_{*, \Gamma}, \]

\[ I_4 = \langle \phi - Q_h^\Gamma \phi, e_h^\nu n \rangle_{*, \Gamma} + L_1 (Q_{k-1} \Phi) + L_4 (Q_k \phi). \]

In light of the Cauchy–Schwarz inequality and Lemma 3.2, we obtain

\[ (Q_{k-1} \Phi - \Phi, A e_h^\nu)_{T_h} + (\Phi - Q_{k-1} \Phi, e_h^u (e_h^u))_{T_h \Gamma} \leq (A e_h^\nu \|0,T_h \]

\[ + \| e_h (e_h^u) \|_{0,T_h} \| Q_{k-1} \Phi - \Phi \|_{0,T_h} \]

\[ \leq h \| \mu^{-1} e_h^u \|_{0,T_h} \]

\[ + \| e_h (e_h^u) \|_{0,T_h} \| \Phi \|_{1, \Omega_1 \Omega_2}. \]

From the definition of \( \| \cdot \| \) it follows

\[ I_1 \leq \| (Q_{k-1} \Phi - \Phi) \|_{\partial \Omega_h \setminus e_h^\Gamma} \| \tau^{-\frac{1}{2}} \tau^{-\frac{1}{2}} (e_h^u - e_h^\tilde{u}) \|_{\partial \Omega_h \setminus e_h^\Gamma} \]

\[ + \| (Q_{k-1} \Phi - \Phi) \|_{*, \Gamma} \| \eta^{-\frac{1}{2}} \eta^{-\frac{1}{2}} (e_h^u - e_h^\tilde{u}) \|_{*, \Gamma} \]

\[ \leq h \| \Phi \|_{1, \Omega_1 \Omega_2} \| \mu^{-\frac{1}{2}} (e_h^\nu, e_h^u, e_h^\tilde{u}) \| . \]

Similarly, we can obtain

\[ I_2 \leq h^{k+1} \| \mu \Phi \|_{2, \Omega_1 \Omega_2} (\| \mu^{-1} \sigma \|_{k, \Omega_1 \Omega_2} + \| u \|_{k+1, \Omega_1 \Omega_2}), \]

\[ I_3 \leq h \| \mu \Phi \|_{2, \Omega_1 \Omega_2} \| \mu^{-\frac{1}{2}} (e_h^\nu, e_h^u, e_h^\tilde{u}) \| . \]

It remains to estimate \( I_4 \). Due to the fact that \( u, Q_h^\Gamma u \) and \( \Phi \) are all single-valued on \( F \in e_h^* \), we have

\[ I_4 = \langle \phi - Q_h^\Gamma \phi, e_h^\nu n \rangle_{*, \Gamma} + \langle u - Q_h^\Gamma u, Q_{k-1} \Phi n \rangle_{*, \Gamma} \]

\[ + (u - Q_k u, \nabla_h \cdot Q_{k-1} \Phi)_{T_h \Gamma} \]

\[ + (A(\sigma - Q_{k-1} \sigma), Q_{k-1} \Phi)_{T_h \Gamma} + \langle u - Q_h^\Gamma u, Q_{k-1} \Phi n \rangle_{\partial \Omega_h \setminus e_h^\Gamma} - (\sigma \]

\[ - Q_{k-1} \sigma, \nabla_h Q_k \Phi \rangle_{T_h \Gamma} \]

\[ = \langle \phi - Q_h^\Gamma \phi, e_h^\nu n \rangle_{*, \Gamma} + \langle u - Q_h^\Gamma u, Q_{k-1} \Phi - \Phi n \rangle_{*, \Gamma} \]

\[ + (u - Q_k u, \nabla_h \cdot Q_{k-1} \Phi)_{T_h \Gamma} \]

\[ + (\sigma - Q_{k-1} \sigma, Q_{k-1} e_h(\phi))_{T_h \Gamma} + \langle u - Q_h^\Gamma u, (Q_{k-1} \Phi - \Phi n) \rangle_{\partial \Omega_h \setminus e_h^\Gamma} - (\sigma \]

\[ \leq h \| \Phi \|_{1, \Omega_1 \Omega_2} \| \mu^{-\frac{1}{2}} (e_h^\nu, e_h^u, e_h^\tilde{u}) \| . \]

\[ \leq h \| \Phi \|_{1, \Omega_1 \Omega_2} \| \mu^{-\frac{1}{2}} (e_h^\nu, e_h^u, e_h^\tilde{u}) \| . \]
Here we recall that \( \tilde{\nu} \) note that the material tends to incompressible as \( \mu \to \infty \).

\[ \frac{1}{2} \mu \| \sigma \|_{0,T_h} \leq C h^k \sqrt{\mu_{\max}} \left( \| \sqrt{\mu} u \|_{k+1, \Omega_1 \cup \Omega_2} + \| \frac{1}{\sqrt{\mu}} \sigma \|_{k, \Omega_1 \cup \Omega_2} \right), \quad (3.27) \]

\[ \| \epsilon(u) - \epsilon(u_h) \|_{0,T_h} \leq \tilde{C} h^k \sqrt{\mu_{\max}} \left( \| \sqrt{\mu} u \|_{k+1, \Omega_1 \cup \Omega_2} + \| \frac{1}{\sqrt{\mu}} \sigma \|_{k, \Omega_1 \cup \Omega_2} \right), \quad (3.28) \]

\[ \| u - u_h \|_{0,T_h} \leq \tilde{C} h^{k+1} \sqrt{\mu_{\max}} \left( \| \sqrt{\mu} u \|_{k+1, \Omega_1 \cup \Omega_2} + \| \frac{1}{\sqrt{\mu}} \sigma \|_{k, \Omega_1 \cup \Omega_2} \right). \quad (3.29) \]

All the above estimates, together with the regularity assumption (3.24) and Theorem 3.1, imply the desired estimate (3.25). \( \square \)

**Remark 3.5** We note that the hidden constant \( C \) in the estimates of Theorems 3.1 and 3.2 depends on the Lamé coefficient \( \lambda \), although the numerical results in Sect. 4 demonstrate the uniform convergence of the X-HDG method.

In fact, as shown in Remark 3.4, if the estimates in Lemma 3.1 hold simultaneously for \( i = 1, 2 \) (e.g. when \( \Gamma \) is not close to an edge or a vertex of element), then we can use (3.18), instead of (3.3)-(3.4), to define the operators \( Q_r, Q_r^b \) in the whole error analysis. As a result, we can derive the following uniform optimal error estimates:

\[ \| \sigma - \sigma_h \|_{0,T_h} \leq \tilde{C} h^k \sqrt{\mu_{\max}} \left( \| \sqrt{\mu} u \|_{k+1, \Omega_1 \cup \Omega_2} + \| \frac{1}{\sqrt{\mu}} \sigma \|_{k, \Omega_1 \cup \Omega_2} \right), \quad (3.27) \]

\[ \| \epsilon(u) - \epsilon(u_h) \|_{0,T_h} \leq \tilde{C} h^k \sqrt{\mu_{\max}} \left( \| \sqrt{\mu} u \|_{k+1, \Omega_1 \cup \Omega_2} + \| \frac{1}{\sqrt{\mu}} \sigma \|_{k, \Omega_1 \cup \Omega_2} \right), \quad (3.28) \]

\[ \| u - u_h \|_{0,T_h} \leq \tilde{C} h^{k+1} \sqrt{\mu_{\max}} \left( \| \sqrt{\mu} u \|_{k+1, \Omega_1 \cup \Omega_2} + \| \frac{1}{\sqrt{\mu}} \sigma \|_{k, \Omega_1 \cup \Omega_2} \right). \quad (3.29) \]

Here we recall that \( \tilde{C} \) is a generic positive constant independent of \( h, \mu \) and \( \lambda \).

**4 Numerical Experiments**

In this section, we shall provide several numerical examples to verify the performance of the proposed interface/boundary-unfitted X-HDG method.

**Example 4.1** A plane strain test with a circular interface.

This example is a plane strain test. In (1.1) we set (cf. Fig. 4)

\[ \Omega = [0, 1]^2, \quad \Omega_2 = \left\{ (x, y) : \left( x - \frac{1}{2} \right)^2 + \left( y - \frac{1}{2} \right)^2 < \frac{3}{64} \right\}, \quad \text{and} \quad \Omega_1 = \Omega \setminus \Omega_2. \]

The exact solution \( (u, \sigma) \) in \( \Omega_1 \cup \Omega_2 \) is given by

\[ u(x, y) = \left( -x^2(x - 1)^2y(y - 1)(2y - 1) \right), \quad \sigma(x, y) = 2\mu \epsilon_h(u) + \lambda \text{div} u I, \]

where the Lamé coefficients \( \mu = \frac{E}{2(1+v)}, \quad \lambda = \frac{E v}{(1+v)(1-2v)} \), with the Young’s modulus \( E |_{\Omega_1 \cup \Omega_2} = 3 \), the Poisson ratio \( v |_{\Omega_1} = 0.4 \) and \( v |_{\Omega_2} = 0.4, 0.49, 0.4999, 0.499999 \). We note that the material tends to incompressible as \( v \rightarrow 0.5 \) (or \( \lambda \rightarrow \infty \)). The force term, boundary conditions and interface conditions can be derived explicitly.

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We use $N \times N$ uniform triangular meshes for the computation. Errors of displacement and stress approximations with $k = 1, 2$ are shown in Table 1. We can see that our X-HDG method (2.2) yields $(k + 1)$-th and $k$-th orders of convergence for $\|u - u_h\|_0$ and $\|\sigma - \sigma_h\|_0$, respectively, which are uniform as $\nu$ tends to 0.5. These results are conformable to Theorems 3.2 and 3.1.

**Remark 4.1** We note that in implementation of the scheme (2.2) on very refined meshes one may need some special handling of the approximation space $\tilde{M}_h = \{\tilde{\mu} \in L^2(F) : \tilde{\mu}|_F \in P_k(K)|_F, \forall F \in \varepsilon_h^\Gamma \}$. Taking the circular interface in 2-dimension as an example, when the mesh size $h$ becomes small enough, $F \in \varepsilon_h^\Gamma$ will be close to a line segment. In this situation, the coordinates $x$ and $y$ on $F$ are approximately linearly-dependent. Thus, the direct use of $\tilde{M}_h$ may lead to a very large condition number of the resultant stiffness matrix. In this situation, one can replace $\tilde{M}_h$ with

$$
\tilde{M}_h^{*1} = \{\tilde{\mu} \in L^2(F) : \tilde{\mu}|_F \in \text{span}\{1, x, \cdots, x^k\}, \forall F \in \varepsilon_h^\Gamma \}
$$

or

$$
\tilde{M}_h^{*2} = \{\tilde{\mu} \in L^2(F) : \tilde{\mu}|_F \in \text{span}\{1, y, \cdots, y^k\}, \forall F \in \varepsilon_h^\Gamma \}
$$

according to the average slope of $F$.

Numerical tests indicate that such a modification does not affect the accuracy of the scheme.

**Example 4.2** A plane strain test on a circular domain: boundary-unfitted meshes.
Table 1  History of convergence: Example 4.1

| ν/Ω₂ | Mesh   | k = 1  |                      |                      | k = 2  |                      |                      |
|------|--------|--------|----------------------|----------------------|--------|----------------------|----------------------|
|      |        | Error  | Order               | Error                | Error  | Order               | Error                |
|      |        | ∥u−uh∥₀|                     | ∥u∥₀                 | ∥σ−σₕ∥₀|                     | ∥σ∥₀                 |
| 0.4  | 8 × 8  | 1.3656E−01 | −                    | 3.0139E−01 | −         | 8.6182E−03 | −                    |
|      | 16 × 16| 3.7733E−02 | 1.86                | 1.4693E−01 | 1.04      | 1.3317E−03 | 2.69                |
|      | 32 × 32| 9.8295E−03 | 1.94                | 7.2247E−02 | 1.02      | 1.8939E−04 | 2.81                |
|      | 64 × 64| 2.4915E−03 | 1.98                | 3.5934E−02 | 1.01      | 2.5407E−05 | 2.90                |
| 0.49 | 8 × 8  | 1.3343E−01 | −                    | 3.0096E−01 | −         | 8.6171E−03 | −                    |
|      | 16 × 16| 3.6398E−02 | 1.87                | 1.4674E−01 | 1.04      | 1.3315E−03 | 2.69                |
|      | 32 × 32| 9.4020E−03 | 1.95                | 7.2150E−02 | 1.02      | 1.8936E−04 | 2.81                |
|      | 64 × 64| 2.2876E−03 | 2.04                | 3.5885E−02 | 1.01      | 2.5404E−05 | 2.90                |
| 0.499| 8 × 8  | 1.3312E−01 | −                    | 3.0093E−01 | −         | 8.6171E−03 | −                    |
|      | 16 × 16| 3.6268E−02 | 1.88                | 1.4673E−01 | 1.04      | 1.3315E−03 | 2.69                |
|      | 32 × 32| 9.3616E−03 | 1.95                | 7.2142E−02 | 1.02      | 1.8936E−04 | 2.81                |
|      | 64 × 64| 2.2709E−03 | 2.04                | 3.5881E−02 | 1.01      | 2.5404E−05 | 2.90                |
| 0.4999| 8 × 8  | 1.3311E−01 | −                    | 3.0093E−01 | −         | 8.6171E−03 | −                    |
|      | 16 × 16| 3.6266E−02 | 1.88                | 1.4673E−01 | 1.04      | 1.3315E−03 | 2.69                |
|      | 32 × 32| 9.3612E−03 | 1.95                | 7.2142E−02 | 1.02      | 1.8936E−04 | 2.81                |
|      | 64 × 64| 2.2709E−03 | 2.04                | 3.5881E−02 | 1.01      | 2.5404E−05 | 2.90                |
This example is to test the performance of the X-HDG scheme (2.7) with boundary-unfitted meshes (cf. Fig. 5). In (2.5) we set

$$\Omega = \{(x, y) : (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{3}{16}\}.$$ 

And the exact solution \((u, \sigma)\) has the same form as in Example 4.1, i.e.

$$u(x, y) = \left(\frac{y^4}{x^4}\right), \quad \sigma(x, y) = \frac{E}{1 + \nu} \varepsilon_h(u) + \lambda \text{div} u I,$$

where the Young’s modulus \(E = 3\), and the Poisson ratio \(\nu = 0.49, 0.4999, 0.499999\).

In (2.7) we take \(\mathcal{B} = [0, 1]^2\), and use \(N \times N\) uniform triangular meshes. Numerical results listed in Table 3 for \(k = 1\) and \(k = 2\) show that our X-HDG method (2.7) yields \((k + 1)\)-th and \(k\)-th orders of uniform convergence for \(\|u - u_h\|_0\) and \(\|\sigma - \sigma_h\|_0\), respectively.

**Example 4.3** A test on a non-convex domain: inner-boundary-unfitted meshes.

This example is also used to test the performance of the X-HDG scheme (2.7) with boundary-unfitted meshes (cf. Fig. 6).

Set

$$\Omega = [0, 1]^2 \setminus \{(x, y) : (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{3}{64}\}.$$ 

The exact solution \((u, \sigma)\) of problem (2.5) is given by

$$u(x, y) = \left(\frac{y^4}{x^4}\right), \quad \sigma(x, y) = 2\mu \varepsilon_h(u) + \lambda \text{div} u I,$$

where the Lamé coefficients \(\mu = 1, \lambda = 1, 10^9\).

In (2.7) we take \(\mathcal{B} = [0, 1]^2\), and use \(N \times N\) uniform triangular meshes. Numerical results in Table 3 for \(k = 1\) and \(k = 2\) demonstrate that the proposed X-HDG method is of \((k + 1)\)-th and \(k\)-th orders of convergence for \(\|u - u_h\|_0\) and \(\|\sigma - \sigma_h\|_0\), respectively.
Table 2  History of convergence: Example 4.2

| $\nu$  | Mesh   | $k = 1$ |          | $k = 2$ |          |
|--------|--------|---------|----------|---------|----------|
|        |        | $\|u - u_h\|_0$ | $\|\sigma - \sigma_h\|_0$ | $\|u - u_h\|_0$ | $\|\sigma - \sigma_h\|_0$ |
|        |        | Error   | Order    | Error   | Order    | Error   | Order    | Error   | Order    |
| 0.49   | 8 × 8  | 5.3684E−02 | –        | 2.4740E−01 | –        | 4.8695E−03 | –        | 3.1598E−02 | –        |
|        | 16 × 16| 1.4048E−02 | 1.93     | 1.2945E−01 | 0.93     | 6.7638E−04 | 2.85     | 8.4985E−03 | 1.89     |
|        | 32 × 32| 3.5215E−03 | 2.00     | 6.4187E−02 | 1.01     | 9.2824E−05 | 2.87     | 2.1532E−03 | 1.98     |
|        | 64 × 64| 8.8990E−04 | 1.98     | 3.1914E−02 | 1.01     | 1.2318E−05 | 2.91     | 5.3899E−04 | 2.00     |
| 0.4999 | 8 × 8  | 5.3551E−02 | –        | 2.4783E−01 | –        | 4.8595E−03 | –        | 3.1711E−02 | –        |
|        | 16 × 16| 1.3991E−02 | 1.94     | 1.2974E−01 | 0.93     | 6.7533E−04 | 2.85     | 8.5228E−03 | 1.90     |
|        | 32 × 32| 3.5087E−03 | 2.00     | 6.4289E−02 | 1.01     | 9.2709E−05 | 2.86     | 2.1575E−03 | 1.98     |
|        | 64 × 64| 8.8752E−04 | 1.98     | 3.1944E−02 | 1.01     | 1.2305E−05 | 2.91     | 5.3966E−04 | 2.00     |
| 0.499999 | 8 × 8 | 5.3549E−02 | –        | 2.4784E−01 | –        | 4.8594E−03 | –        | 3.1713E−02 | –        |
|        | 16 × 16| 1.3991E−02 | 1.94     | 1.2975E−01 | 0.93     | 6.7532E−04 | 2.85     | 8.5231E−03 | 1.90     |
|        | 32 × 32| 3.5085E−03 | 2.00     | 6.4290E−02 | 1.01     | 9.3270E−05 | 2.86     | 2.1670E−03 | 1.98     |
|        | 64 × 64| 8.8749E−04 | 1.98     | 3.1944E−02 | 1.01     | 1.2305E−05 | 2.92     | 5.3967E−04 | 2.01     |
Table 3  History of convergence: Example 4.3

| $k$ | Mesh  | $\lambda = 1$ | $\lambda = 10^9$ |
|-----|-------|---------------|------------------|
|     |       | $\|u - u_h\|_0/\|u\|_0$ | $\|\sigma - \sigma_h\|_0/\|\sigma\|_0$ | $\|u - u_h\|_0/\|u\|_0$ | $\|\sigma - \sigma_h\|_0/\|\sigma\|_0$ |
|     |       | Error | Order | Error | Order | Error | Order | Error | Order |
| 1   | 8 x 8 | 2.3323E−02 | –     | 8.7293E−02 | –     | 1.7339E−02 | –     | 1.2011E−01 | –     |
|     | 16 x 16 | 6.2669E−03 | 1.90  | 4.1513E−02 | 1.07  | 5.1284E−03 | 1.76  | 5.3700E−02 | 1.16  |
|     | 32 x 32 | 1.6009E−03 | 1.97  | 2.0172E−02 | 1.04  | 1.4018E−03 | 1.87  | 2.2952E−02 | 1.23  |
|     | 64 x 64 | 4.0156E−04 | 2.00  | 9.9689E−03 | 1.02  | 3.6201E−04 | 1.95  | 1.0511E−02 | 1.13  |
| 2   | 8 x 8 | 1.5152E−03 | –     | 4.9384E−03 | –     | 1.4237E−03 | –     | 5.8621E−03 | –     |
|     | 16 x 16 | 2.2757E−04 | 2.74  | 1.1363E−03 | 2.12  | 2.2009E−04 | 2.69  | 1.2826E−03 | 2.19  |
|     | 32 x 32 | 3.0812E−05 | 2.88  | 2.6608E−04 | 2.09  | 3.0196E−05 | 2.87  | 2.8741E−04 | 2.16  |
|     | 64 x 64 | 3.9847E−06 | 2.95  | 6.3934E−05 | 2.06  | 3.9278E−06 | 2.94  | 6.7190E−05 | 2.10  |
Example 4.4 A plane stress test in crack-tip domain.

This example is a near crack-tip plane stress problem [5]. In (2.5) we take
\[ \Omega = [0, 1]^2 \setminus \{(x, y_c) : 0 \leq x \leq x_c\} \]
with the crack \( \partial \Omega_N = \{(x, y_c) : 0 \leq x \leq x_c\} \) (cf. Fig. 7), and \( \partial \Omega_D = \partial \Omega \setminus \partial \Omega_N \), where \( (x_c, y_c) = \left( \frac{1}{2}, \frac{1}{2} \right) \) is the crack tip. The Lamé coefficients \( \mu = \frac{E_2}{2(1+v)} \) and \( \lambda = \frac{E_\nu}{(1+v)(1-\nu)} \) with \( v = 1/3, \ E = 8/3 \). The exact solution \( (u, \sigma) \) is given by
\[
\begin{align*}
\begin{pmatrix}
K_I 
\frac{\sqrt{2\pi}}{r_2} \cos \left( \frac{\theta}{2} \right) \left( \kappa - 1 + 2\sin^2 \left( \frac{\theta}{2} \right) \right) \\
K_I 
\frac{\sqrt{2\pi}}{r_2} \sin \left( \frac{\theta}{2} \right) \left( \kappa + 1 - 2\cos^2 \left( \frac{\theta}{2} \right) \right)
\end{pmatrix},
\end{align*}
\]
\[
\begin{align*}
\begin{pmatrix}
\sigma(x, y) = \\
\begin{pmatrix}
\frac{K_I}{\sqrt{2\pi}} \cos \left( \frac{\theta}{2} \right) \left( 1 - \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{3\theta}{2} \right) \cos \left( \frac{3\theta}{2} \right) \right), \\
\frac{K_I}{\sqrt{2\pi}} \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{3\theta}{2} \right), \\
\frac{K_I}{\sqrt{2\pi}} \cos \left( \frac{\theta}{2} \right) \left( 1 + \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{3\theta}{2} \right) \right)
\end{pmatrix}
\end{pmatrix},
\end{align*}
\]
where \( r \) is the distance from the crack tip, \( \theta = \arctan2(y - y_c, x - x_c) \), \( \kappa = \frac{3-v}{1+v} \), and the stress intensity factor (SIF) \( K_I = \frac{\sqrt{\pi}}{2} \). We note that the boundary condition along the line crack is a homogeneous Neumann condition, i.e. \( \sigma_n \mid_{\partial \Omega_N} = 0 \), and that \( u \notin H^{3/2}(\Omega) \) but \( u \in H^{3/2-\epsilon}(\Omega) \) for any \( \epsilon > 0 \).

In the X-HDG scheme (2.7) we take \( \mathbb{B} = [0, 1]^2 \), and use \( N \times N \) uniform triangular meshes. Due to the low regularity of the exact solution, we only consider the lowest order case of the scheme, i.e. \( k = 1 \). From the numerical results in Table 4, we can see that the convergence rate is 0.5 for the stress error \( \|\sigma - \sigma_h\|_0 \), which is as same as that in [5], and that the convergence rate is 1 for the displacement error \( \|u - u_h\|_0 \). The second component of the displacement approximation \( u_h \) at 129 \times 129 mesh is also plotted in Fig. 8.

5 Concluding Remarks

In this paper, we have proposed and analyzed an arbitrary order interface/boundary-unfitted eXtended hybridizable discontinuous Galerkin method of optimal convergence for linear
Fig. 7  The crack domain in Example 4.4: $9 \times 9$ mesh

Table 4  History of convergence for Example 4.4

| $k$ | Mesh | $\|\sigma - \sigma_h\|_0 / \|\sigma\|_0$ Error | Order | $\|\sigma - \sigma_h\|_0 / \|\sigma\|_0$ Error | Order |
|-----|------|---------------------------------|-------|---------------------------------|-------|
| 1   | $9 \times 9$ | 3.5241E−02 | –      | 2.5316E−01 | –      |
| 17  | $17 \times 17$ | 1.9991E−02 | 0.89   | 1.8320E−01 | 0.51   |
| 33  | $33 \times 33$ | 1.0712E−02 | 0.94   | 1.3107E−01 | 0.50   |
| 65  | $65 \times 65$ | 5.5568E−03 | 0.97   | 9.3236E−02 | 0.50   |
| 129 | $129 \times 129$ | 2.8322E−03 | 0.98   | 6.6125E−02 | 0.50   |

elasticity interface problems. Numerical experiments have demonstrated the performance and robustness of the method.
Fig. 8 The second component of displacement approximation in Example 4.4: 129 × 129 mesh

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**Data Availability** Enquiries about data availability should be directed to the authors.

**Declarations**

**Conflict of interest** The authors have not disclosed any competing interests.

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