A GENERALIZED COMPLEX GINZBURG-LANDAU EQUATION: GLOBAL EXISTENCE AND STABILITY RESULTS

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Abstract. We consider the complex Ginzburg-Landau equation with two pure-power nonlinearities and a damping term. After proving a general global existence result, we focus on the existence and stability of several periodic orbits, namely the trivial equilibrium, bound-states and solutions independent of the spatial variable. In particular, we construct bound-states either explicitly in the real line or through a bifurcation argument for a double eigenvalue of the Dirichlet-Laplace operator on bounded domains.

1. Introduction and main results. The complex Ginzburg-Landau equation models various physical phenomena especially in theory of superconductivity and fluid dynamics. A particular Ginzburg-Landau equation can be written as

\[ \hat{c}_t A = (1 + i\alpha)\Delta A + (1 + i\beta)|A|^2 A + kA \quad (1.1) \]

which admits the development of singularities for certain values of parameters (see e.g. [5, 20, 22]). However, the introduction of a high-order term with a negative sign, like \(- (1 + ic)|A|^4 A\), allows to saturate the explosive instabilities. We refer e.g. [1, 11] for a more complete physical background.

We are concerned with the study of a generalized Ginzburg-Landau equation

\[
\begin{aligned}
\hat{c}_t u &= (a + i\alpha)\Delta u + (b + i\beta)|u|^\sigma u - (c + i\gamma)|u|^\sigma_2 u + ku, \\
B u(t, x) &= 0, \quad x \in \partial\Omega, \quad t \geq 0 \\
u(0, x) &= u_0(x)
\end{aligned} 
\]

(gCGL)

where \(B\) is the identity operator (Dirichlet condition) or \(B = \frac{\partial}{\partial n}\) (Neumann condition). We assume \(a > 0, \alpha, b, \beta, k \in \mathbb{R}, \sigma_1, \sigma_2 > 0\) and \(\Omega\) as a domain in \(\mathbb{R}^N\) of class \(C^2\) with \(\partial\Omega\) bounded. If \(c = \gamma = 0\), (gCGL) is reduced to the complex Ginzburg-Landau equation (CGL), equation widely studied under several assumptions on the

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parameters since the seminal paper [19]; see also [14, 15, 21] and the references therein.

In this paper, we extend some results of global existence of solutions and their stability and also the existence of standing wave solutions in one dimension, previously exposed for the complex Ginzburg-Landau equation in [8], where only one nonlinear term was present. Moreover, we prove the existence of standing waves in bounded domains through a bifurcation argument applied to double eigenvalues of the Dirichlet-Laplace operator, which is new even in the context of (CGL). As mentioned, the main interest in adding a higher-order term is the need for more precise physical descriptions.

Define the linear operators $-A_D = (a + i\alpha)\Delta$, $a > 0$, with domain $D(A_D) = H^2(\Omega) \cap H_0^1(\Omega)$ (Dirichlet condition) and $-A_N = (a + i\alpha)\Delta$, with domain $D(A_N) = \{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega) \}$

$\{-\int_{\Omega} \Delta u v = -\int_{\Omega} \nabla u \cdot \nabla v \, dx, \forall v \in H^1(\Omega)\}$

(Neumann condition). It is well known that these operators generate an analytic semi-group (see [13]). Denoting by $A$ any of these two operators $A_D$ or $A_N$, let us introduce the following definition:

**Definition 1.1.** A function $u(\cdot) \in C([0,T); L^2(\Omega))$, $T > 0$, is called a strong solution of (gCGL) if $u(t) \in D(A)$, $\frac{du}{dt}(t)$ exists for $t \in (0,T)$, $u(0) = u_0$ and the differential equation in (gCGL) is satisfied in $L^2(\Omega)$ for all $t \in (0,T)$.

Since $f(u) = (b + i\beta)|u|^{\sigma_1}u - (c + i\gamma)|u|^{\sigma_2}u + ku$ is locally Lipschitz in $H^1(\Omega)$ with values in $L^2(\Omega)$, for $\sigma_1, \sigma_2 \leq 2/(N-2)^+$ (with the convention $2/(N-2)^+ = +\infty$ if $N = 1, 2$), then there exists $T = T(u_0) > 0$ such that the problem (gCGL) has a unique solution on $[0,T_0)$, and this solution depends continuously of the initial data (see [18], pag. 54 and 62). We begin with a global existence result:

**Theorem 1.2.** Let $\Omega$ be a domain in $\mathbb{R}^N$ of class $C^2$ with $\partial \Omega$ bounded. Assume $0 < \sigma_j \leq 2/(N-2)^+$. Then, for any $u_0 \in H_0^1(\Omega)$ (resp. $u \in H^1(\Omega)$), there exists $T = T(u_0) > 0$ such that (gCGL) with $A = A_D$ (resp. $A = A_N$) has a unique strong solution on $[0,T)$ and this solution depends continuously of the initial data. Moreover, if $0 < \sigma_1 < \sigma_2$, $c > 0$, $\alpha \neq 0$ and $\gamma/\alpha \geq 0$, the solution is global.

As expected, the lower order nonlinear term does not influence the global existence result. This proves in particular that the addition of a higher-order term with a specific sign prevents any possible blow-up mechanisms.

The existence of standing waves for the complex Ginzburg-Landau equation remains a largely open problem. Before we proceed, we rewrite the generalized complex Ginzburg-Landau equation in its trigonometric form, following the notations of [6, 8]:

$u_t = e^{i\theta} \Delta u + e^{i\gamma_1}|u|^{\sigma_1}u + \chi e^{i\gamma_2}|u|^{\sigma_2}u + ku,$ \hspace{1cm} (gCGL*)

where $-\pi/2 < \theta < \pi/2$, $-\pi < \gamma_1, \gamma_2 \leq \pi$, $k \in \mathbb{R}$, $\chi = \pm 1$. Then one may look for solutions of (gCGL*) in the form $u = e^{i\omega t}\phi(x)$, where $\phi \in H^1(\Omega)$ is a solution of the elliptic equation

$i\omega \phi = e^{i\theta} \Delta \phi + e^{i\gamma_1}|\phi|^{\sigma_1}\phi + \chi e^{i\gamma_2}|\phi|^{\sigma_2}\phi + k\phi$. \hspace{1cm} (B-S)

For $k = 0$, the existence of standing wave solutions is already known in some particular cases : $\theta = \pm \gamma = \pm \pi/2$, which corresponds to the nonlinear Schrödinger
equation or \( \omega = 0 \) (stationary solutions). Outside of these cases, we refer [6, 8, 9], where the implicit function theorem is used to obtain the existence of standing waves of (B-S) for \( \chi = 0 \) and several constraints on the remaining parameters. In [6] it is proven that, for \( \Omega \) bounded, the equation (B-S) (with \( k = 0 \)) has a solution \((\omega, u) \in \mathbb{R} \times H_0^1(\Omega)\) bifurcating from \( u = 0 \) if \( \sigma \) is sufficiently small and \( \cos \theta \cos \gamma > 0 \). A similar result is obtained in [8] where the aim was to trade the freedom in \( k \) for the freedom in \( \sigma \). The reference [9] focuses on a bifurcation argument starting from the ground-state solution of the nonlinear Schrödinger equation, for both \( \Omega \) bounded and the whole space (under some radial assumptions).

Our first result concerns an explicit bound-state in the real line.

**Theorem 1.3.** Suppose \( \Omega = \mathbb{R} \). Fix \(-\pi/2 < \theta < \pi/2\) and \( \omega, k \in \mathbb{R} \) such that \( \omega \cos \theta + k \sin \theta \neq 0 \). Define

\[
d = \frac{k \cos \theta - \omega \sin \theta + \sqrt{\omega^2 + k^2}}{\omega \cos \theta + k \sin \theta}
\]

and let \( \gamma_j \in (-\pi, \pi) \), \((j = 1, 2)\) be the unique solutions of

\[
\tan(\gamma_j - \theta) = \frac{d(\sigma_j + 4)}{\sigma_j + 2 - 2d^2}; \quad d \sin(\gamma_j - \theta) + \cos(\gamma_j - \theta) > 0.
\]

Then a) If \( \chi = 1 \) the generalized complex Ginzburg-Landau equation admits a bound-state of the form

\[
\phi = \psi \exp(id\ln \psi),
\]

where \( \psi \) is the bound-state for the nonlinear Schrödinger equation:

\[
\psi'' = \epsilon \psi - \eta_1 \psi^{\sigma_1+1} - \chi \eta_2 \psi^{\sigma_2+1}
\]

with

\[
\epsilon = \frac{\sqrt{\omega^2 + k^2}}{1 + d^2}, \quad \eta_j = \frac{d \sin(\gamma_j - \theta) + \cos(\gamma_j - \theta)}{1 + d^2}.
\]

b) If \( \chi = -1 \), there exists a small enough \( \delta > 0 \) such that for \( 0 < \sigma_2 < \delta \) the generalized Ginzburg-Landau equation admits a bound-state of the form (1.4) with \( \psi \) the bound-state for the nonlinear Schrödinger equation satisfying (1.5).

**Remark 1.** We observe that the conditions \( \omega^2 + k^2 \neq 0 \) and \( \arg(k - i\omega) \neq \theta \) imply \( \omega \cos \theta + k \sin \theta \neq 0 \).

**Remark 2.** In [8], the uniqueness and stability of bound-states defined on \( \mathbb{R} \) was studied. The same arguments may be applied in our framework without any extra difficulty.

For \( \Omega \) bounded, following the spirit of [6, 8] for (CGL), we wish to construct solutions of (B-S) through a bifurcation argument applied to the trivial solution \( u \equiv 0 \). Therein, a bifurcation from simple eigenvalues of the Laplacian is built directly as an application of the Implicit Function Theorem. In the context of the (B-S), a similar procedure should be applicable. Instead, we turn our focus to the bifurcation problem for eigenvalues of multiplicity two, inspired in the methodology presented in [3]. We remark that, even in the special case \( \chi = 0 \), this is an open problem. A classic example where one has double eigenvalues is the case of the square \( \Omega = (-1, 1)^2 \); we refer to [12] for a bifurcation result in this specific case. Our main result is the following:
Theorem 1.4. Given $\Omega \subset \mathbb{R}^N$ bounded and $2 \leq \sigma_1 + 1 \leq \sigma_2 < 4/(N-2)^+$, suppose that $\lambda_0$ is a double eigenvalue of the Dirichlet-Laplace operator with $L^2$-orthogonal real-valued eigenfunctions $u_1, u_2$. Suppose that the equation

$$ P(\alpha) = \int_{\Omega} |u_1 + \alpha u_2|^\sigma \frac{1}{\sigma} (u_1 + \alpha u_2)(\alpha u_1 - u_2) = 0 $$

has a solution $\alpha_0 \in \mathbb{C}$ satisfying $P'(\alpha_0) \neq 0$. Then there exist $\delta > 0$ and a Lipschitz mapping

$$ \epsilon \in [0, \delta) \rightarrow (y, \lambda, \alpha) \in H^1_0(\Omega) \times \mathbb{C} \times \mathbb{C}, $$

with $(y(0), \lambda(0), \alpha(0)) = (0, e^{i\theta}\lambda_0, \alpha_0)$ and $(y, u_1)_{L^2} = (y, u_2)_{L^2} = 0$, such that

$$ u = y(\epsilon) + \epsilon u_1 + \alpha(\epsilon) u_2 $$

is a solution to (B-S) for $k - i\omega = \lambda(\epsilon)$. Moreover,

$$ \lambda(\epsilon) = e^{i\theta}\lambda_0 - e^{i\gamma_1}\epsilon^{\sigma_1} \int_{\Omega} |u_1 + \alpha_0 u_2|^\sigma (u_1 + \alpha_0 u_2) u_1 dx + o(e^{\sigma_1}). \quad (1.6) $$

Remark 3. As a consequence of the above result, if $P$ does not have multiple roots, the number of branches bifurcating at $e^{i\theta}\lambda_0$ is equal to the number of simple roots of $P$ (counting permutations of $u_1$ and $u_2$, see Example 1).

We now focus on the stability of the equilibrium solution $u \equiv 0$, the asymptotic decay of the global solutions of (gCGL) depending on the parameters and the stability of some particular time periodic solutions. To be more precise, we give the following definition:

Definition 1.5. We say that the equilibrium point $u \equiv 0$ is $E$-stable if for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$ u_0 \in E, \quad \|u_0\|_E < \varepsilon \Rightarrow \sup_{t \geq 0} \|u(t)\|_E < \delta. $$

In addition, we say that it is asymptotically stable if there exists $\eta > 0$ such that

$$ \lim_{t \to \infty} \|u(t)\|_E = 0 \quad \text{for all } u_0 \in H^1_0(\Omega), \quad \|u_0\|_E < \eta. $$

First, we have the following result:

Theorem 1.6. Concerning the Dirichlet problem, assume the hypothesis of Theorem 1.2 and $0 < \sigma_1 < \sigma_2$.

1. $L^p$ stability: If

$$ k \leq 0, \quad |\alpha| \frac{p-2}{2} \leq a, \quad b \frac{\sigma_1}{\sigma_2} \leq c \quad \text{and} \quad b \frac{\sigma_2 - \sigma_1}{\sigma_2} \leq |k|, $$

the equilibrium point $0$ is $L^p$-stable for $2 \leq p < \frac{2N}{N-2}$, if $N > 2$, $2 \leq p < \infty$ if $N = 1, 2$.

In addition, if $k < 0$ and $b \frac{\sigma_2 - \sigma_1}{\sigma_2} < |k|$ we have the asymptotic stability and

$$ \|u(t, x)\|_{L^p} \to 0 \quad \text{as} \quad t \to \infty, \quad \text{for all} \quad u_0 \in H^1_0(\Omega). $$

In the particular case $p = 2$, if $\Omega$ is a bounded domain,

$$ k > 0, \quad b \frac{\sigma_1}{\sigma_2} \leq c \quad \text{and} \quad b + \frac{\sigma_2 - \sigma_1}{\sigma_2} + k < a \left( \frac{1}{\Omega} \right)^{-2/N}, $$

where $b^+ = \max\{0, b\}$, $\omega_N$ represents the volume of the unit ball in $\mathbb{R}^N$ and $|\Omega|$ the volume of $\Omega$, then $\|u(t)\|_{L^2} \to 0$ as $t \to \infty$, for all $u_0 \in H^1_0(\Omega)$. 
2. \(H^1\) stability:

Assume \(\alpha/a = \beta/b = \gamma/c\). Then, the equilibrium point 0 is asymptotically stable in \(H^1\) if

1. \(k < 0\) and

\[
\frac{b}{\sigma_1 + 2 \sigma_2} \leq \frac{c}{\sigma_2 + 2}, \quad \frac{b \sigma_2 - \sigma_1}{\sigma_2} \leq \frac{|k|}{2}, \quad b(\sigma_1 + 1) < \min\{c, |k|\}.
\]

2. \(k = 0\), \(\Omega\) is a bounded domain and

\[
\frac{b}{\sigma_1 + 2 \sigma_2} \leq \frac{c}{\sigma_2 + 2}, \quad b(\sigma_1 + 1) < \min\left\{c, \left(\frac{1}{\omega_N|\Omega|}\right)^{-2/N}\right\}.
\]

In both cases,

\[
\|u(t)\|_{H^1} \to 0 \text{ as } t \to \infty, \quad \text{for all } u_0 \in H^1_0(\Omega).
\]

Remark 4. If \(b = 0\), one may easily prove the asymptotic stability in \(H^1\) with the additional condition

\[
0 < k \leq \frac{a}{2} \left(\frac{1}{\omega_N|\Omega|}\right)^{-2/N}.
\]

Remark 5. The results stated in the theorem extended trivially, with a slight modification, to the (gCGL) equation with a Neumann condition, in the case \(k < 0\).

Finally, we study the stability of some particular time periodic solutions of the generalized complex Ginzburg-Landau equation. Consider the (gCGL) equation on a bounded domain \(\Omega\) with the Neumann condition on the boundary and assume \(0 < \sigma_1 < \sigma_2\). Take the associated ordinary differential equation,

\[
\dot{u} = (b + i\beta)|u|^\sigma_1 u - (c + i\gamma)|u|^\sigma_2 u + ku
\]

and look for periodic solutions. If we assume that there exists \(r_0 > 0\) such that

\[
br_0^{\sigma_1} - cr_0^{\sigma_2} + k = 0
\]

we obtain the explicit periodic solution

\[
u(t) = r_0 \exp(it(\beta r_0^{\sigma_1} - \gamma r_0^{\sigma_2})).
\]

We consider now the two following cases

1. \(c = 0\) and \(bk < 0\); the condition (1.8) can be verified and the equation (1.7) allows a \(T_1\)-periodic solution (1.9) which we denote by \(p(t)\), \(T_1 > 0\).

2. \(k = 0\) and \(bc > 0\); we obtain a \(T_2\)-periodic solution (1.9) which we denote by \(q(t)\), \(T_2 > 0\).

It is clear that the (gCGL) equation with the Neumann condition on the boundary allows the time periodic solutions \(P(x, t) \equiv p(t), Q(x, t) \equiv q(t)\) for all \(x \in \Omega\).

**Theorem 1.7.** Let \(\Omega \subset \mathbb{R}^N\) a bounded domain and consider the (gCGL) equation with a Neumann condition on the boundary. Suppose the conditions of Theorem 1.2 are verified.

1. Assume \(c = 0\).

   If \(b < 0\) and \(k > 0\), the \(T_1\)-periodic solution \(P(x, t)\) is orbitally asymptotically stable, i.e. there exists \(\delta > 0\) and \(\zeta > 0\) such that, if

\[
\min_{0 \leq t \leq T_1} \|u_0 - P(t)\|_{H^1} < \delta
\]
the solution \( u(t) \) of (gCGL) with initial data \( u(0) = u_0 \) exists on \( 0 \leq t < \infty \) and there exists a real \( \omega \) and \( c > 0 \) such that

\[
\| u(t) - P(t - \omega) \|_{H^1} \leq c e^{-\epsilon t}.
\]

If \( b > 0 \) and \( k < 0 \), \( P(x, t) \) is strongly unstable: for \( u_n^0 = r_0 + 1/n \), the solution \( u_n \) with initial condition \( u_n(0) = u_n^0 \) blows-up in finite time.

2. Assume \( k = 0 \).

If \( b > 0 \) and \( c > 0 \), the \( T_2 \)-periodic solution \( Q(x, t) \) is orbitally asymptotically stable.

If \( b < 0 \) and \( c < 0 \), \( Q(x, t) \) is strongly unstable.

Remark 6. In the Neumann case, the solutions of (1.7) automatically embed in the flow for (gCGL). The above theorem says that, concerning the stability of \( P \) and \( Q \), both flows have precisely the same dynamic behavior.

The paper is organized as follows: in Section 2, we prove the global existence result (Theorem 1.2). In Section 3, we focus on the construction of bound-states on the real line and on bounded domains. In Section 4, we study the stability of the trivial solution. Finally, Section 5 is devoted to the stability of periodic solutions.

2. Proof of the Theorem 1.2.

Proof. To prove the global existence of a solution, multiply (gCGL) equation by \( \tilde{\pi}, -\Delta \tilde{\pi} \) and \( |u|^2 \tilde{\pi} \), integrate on \( \Omega \) and take the real part. One obtains

\[
\frac{1}{2} \frac{d}{dt} \| u \|_{L^2}^2 = -a \| \nabla u \|_{L^2}^2 + b \| u \|^2_{L^2} + c \| u \|^2_{L^2} + k \| u \|_{L^2}, \quad (2.1)
\]

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 = -a \| \Delta u \|_{L^2}^2 - b \Re \left\{ \int_{\Omega} \Delta \bar{\pi} |u|^2 u \, dx + \beta \Im \left\{ \int_{\Omega} \Delta \bar{\pi} |u|^1 u \, dx \right\} \right\} + c \Re \left\{ \int_{\Omega} \Delta \bar{\pi} |u|^2 u \, dx - \gamma \Im \left\{ \int_{\Omega} \Delta \bar{\pi} |u|^1 u \, dx \right\} \right\}, \quad (2.2)
\]

\[
\frac{1}{\sigma_2 + 2} \frac{d}{dt} \| u \|_{L^{\sigma_2+2}}^2 = a \Re \left\{ \int_{\Omega} \Delta u |u|^2 \pi \, dx - \alpha \Im \left\{ \int_{\Omega} \Delta u |u|^2 \pi \, dx \right\} \right\} + b \| u \|_{L^{\sigma_1+\sigma_2+2}}^2 + c \| u \|_{L^{2\sigma_2+2}}^2 + k \| u \|_{L^{2\sigma_2+2}}^2. \quad (2.3)
\]

Next, if we multiply (2.3) by \( \gamma/\alpha \) (with \( \alpha \neq 0 \)) and add to (2.1) + (2.2), one obtains:

\[
\frac{d}{dt} \left[ \frac{1}{2} \| u \|_{H^1}^2 + \frac{\gamma}{\alpha(\sigma_2 + 2)} \| u \|_{L^{\sigma_2+2}}^2 \right] = k \left[ \| u \|_{H^1}^2 + \frac{\gamma}{\alpha} \| u \|_{L^{\sigma_2+2}}^2 \right] - a \| \Delta u \|_{L^2}^2 - a \| \nabla u \|_{L^2}^2 - b \| u \|_{L^{\sigma_1+2}}^2 + c \| u \|_{L^{2\sigma_2+2}}^2 + \frac{\gamma b}{\alpha} \| u \|_{L^{\sigma_1+\sigma_2+2}}^2 + \frac{\gamma c}{\alpha} \| u \|_{L^{2\sigma_2+2}}^2 - \beta \Re \left\{ \int_{\Omega} \Delta |u|^2 \pi \, dx \right\} + \beta \Im \left\{ \int_{\Omega} \Delta \bar{\pi} |u|^1 u \, dx \right\} - b \Re \int_{\Omega} \Delta \bar{\pi} |u|^1 u \, dx + \beta \Im \left\{ \int_{\Omega} \Delta \bar{\pi} |u|^1 u \, dx \right\} . \quad (2.4)
\]

By interpolation we have

\[
\| u \|_{L^{\sigma_1+2}}^2 \leq \| u \|_{L^{\sigma_2+2}}^{2\sigma_1/(\sigma_1+\sigma_2)} \| u \|_{L^{\sigma_2+2}}^{\sigma_2/(\sigma_1+\sigma_2)}
\]

and by the well-known Young inequality

\[
abla \leq \varepsilon a^p + \frac{p-1}{p} \varepsilon^{-\frac{1}{p'}} b^{p'}, \quad p > 1, \, \varepsilon > 0,
\]
with $p = \sigma_2/\sigma_1$, we obtain
\[ \|u\|_{L^{\sigma_1+2}}^{\sigma_1+2} \leq \varepsilon \|u\|_{L^{\sigma_2+2}}^{\sigma_2+2} + \frac{p - 1}{p'} \varepsilon \|u\|_{L^2}^2 \]
and we choose $\varepsilon$ such that $b \varepsilon = c$ (if $b > 0$). It follows that
\[ b \|u\|_{L^{\sigma_1+2}}^{\sigma_1+2} - c \|u\|_{L^{\sigma_2+2}}^{\sigma_2+2} < c_1 \|u\|_{L^2}^2 \]
with $c_1 = |b| \frac{p - 1}{p'} \varepsilon \frac{1}{p'}$. Similarly
\[ \|u\|_{L^{\sigma_1+\sigma_2+2}}^{\sigma_1+\sigma_2+2} \leq \delta \|u\|_{L^{\sigma_2+\sigma_2+2}}^{\sigma_2+\sigma_2+2} + C(\delta) \|u\|_{L^2} \]
and if we choose $\delta$ such that $\delta b < c/2$ (if $b > 0$), we get
\[ \frac{\gamma_b}{\alpha} \|u\|_{L^{\sigma_1+\sigma_2+2}}^{\sigma_1+\sigma_2+2} - \frac{\gamma_c}{\alpha} \|u\|_{L^{\sigma_2+\sigma_2+2}}^{\sigma_2+\sigma_2+2} < -\frac{\gamma_c}{2\alpha} \|u\|_{L^{\sigma_2+\sigma_2+2}}^{\sigma_2+\sigma_2+2} + c_2 \|u\|_{L^2}^2 \]
with $c_2 = c_2(\delta)$. Next, we estimate
\[ \langle |b| + |\beta| \rangle \int_\Omega \Delta \pi |u|^{\sigma_1} |u| \leq (|b| + |\beta|) \|\Delta u\|_{L^2} \|u\|_{L^{2\sigma_1+2}}^{\sigma_1+1} \]
\[ \leq \eta(|b| + |\beta|) \|\Delta u\|_{L^2}^2 + \frac{|b| + |\beta|}{\eta} \|u\|_{L^{2\sigma_1+2}}^{2\sigma_1+2} \]
and we take $\eta$ such that $\eta(|b| + |\beta|) \leq a$. By interpolation,
\[ \|u\|_{2\sigma_1+2} \leq \|u\|_{2\sigma_2+2} \|u\|_{L^2} \]
and using the Young inequality (with $p = \frac{\sigma_2}{\sigma_2-\sigma_1}$, $p' = \frac{\sigma_2}{\sigma_1}$) we get
\[ \|u\|_{L^{2\sigma_2+2}}^{2\sigma_2+2} \leq \rho \|u\|_{2\sigma_2+2}^{2\sigma_2+2} + C(\rho) \|u\|_{L^2}^2 \]
and we choose $\rho$ such that $\frac{|b| + |\beta|}{\eta} \leq \frac{\sigma_2}{\sigma_1}$. Finally, notice that $\Re \int_\Omega \Delta \pi |u|^{\sigma_2} u \, dx \leq 0$. By (2.5), (2.6), (2.7), (2.8) and (2.4) we obtain the conclusion by the Gronwall inequality.

**Remark 7.** The complex Ginzburg-Landau equation on $\Omega \subset \mathbb{R}^N$ with the Dirichlet condition,
\[ u_t = e^{i\theta} (\Delta u + |u|^\sigma u) + ku, \quad k \geq 0, \quad -\pi/2 < \theta < \pi/2 \]
allows explosive solutions in a finite time, $u(t)$, under the condition that the energy
\[ \frac{1}{2} \int_\Omega |\nabla u_0|^2 \, dx - \frac{1}{\sigma + 2} \int_\Omega |u_0|^{\sigma + 2} \, dx < 0 \]
(see [4, 5]). This result remains true (with essentially the same proof) for the generalized Ginzburg-Landau equation:
\[ u_t = e^{i\theta} [\Delta u + |u|^{\sigma_1} u - \nu |u|^{\sigma_2} u] + ku, \quad k \geq 0, \quad \nu \in \mathbb{R}. \]
More precisely, we have

**Proposition 1.** Assume $-\pi/2 < \theta < \pi/2$, $\sigma_1, \sigma_2 > 0$, $k \geq 0$ and $\nu \leq 0$ or $\nu > 0$ and $\sigma_2 \leq \sigma_1$. Let $u_0 \in H_0^1(\Omega)$ and $u(t)$ the corresponding maximal solution of (2.9). If $E(u_0) < 0$ with
\[ E(u_0) = \int_\Omega \left( \frac{1}{2} |\nabla u_0|^2 - \frac{1}{\sigma_1 + 2} |u_0|^{\sigma_1+2} + \nu \frac{1}{\sigma_2 + 2} |u_0|^{\sigma_2+2} \right) \, dx < 0 \]
then $u$ blows up in a finite time.
3. Existence of bound-states of \((gCGL^*)\).

Proof of Theorem 1.3. We look for solutions \(\phi \in H^1(\mathbb{R})\) of the elliptic equation (B-S), or in an equivalent form,

\[
\phi'' = \omega e^{i\theta} \phi - e^{i\gamma_1} |\phi|^\sigma_1 \phi + \chi e^{i\gamma_2} |\phi|^\sigma_2 \phi + ike^{i\theta} \phi
\]

with \(\tilde{\theta} = \pi/2 - \theta, \tilde{\gamma}_1 = \gamma_1 - \theta, \tilde{\gamma}_2 = \gamma_2 - \theta\).

1. First consider the case \(\chi = 1\). Let us search for a solution \(\phi \in H^1(\mathbb{R})\) of the equation (3.1) of the form

\[
\phi = \psi \exp(id \ln \psi)
\]

where \(d \in \mathbb{R}\) and \(\psi > 0\) is the unique solution (up to translations of the origin) of the stationary Schrödinger equation

\[
\psi'' = \varepsilon \psi - \eta |\psi|^\sigma_1 + 1 - \zeta |\psi|^\sigma_2 + 1 =: -f(\psi), \quad \varepsilon, \eta, \zeta > 0.
\]

Note that the existence of the solution \(\psi\) follows from the fact that

\[
x_0 = \inf\{x > 0 : F(x) = 0\} > 0 \quad \text{with} \quad F(z) = \int_0^z f(s)ds
\]

and \(f(x_0) > 0\) (see [2], Th.5).

First, one has

\[
\phi''(x) = \left[\psi''(x)(1 + id) + id(1 + id) \frac{\psi'(x)^2}{\psi(x)}\right] \exp(id \ln \psi(x))
\]

and we note that if \(\psi\) is a solution of (3.2), then a direct integration of the equation yields

\[
\frac{(\psi')^2}{\psi} = \epsilon \psi - \frac{2\eta}{\sigma_1 + 2} |\psi|^\sigma_1 + 1 - \frac{2\zeta}{\sigma_2 + 2} |\psi|^\sigma_2 + 1.
\]

It follows from (3.1) that

\[
\psi'' - d^2 \frac{(\psi')^2}{\psi} = \omega \cos \tilde{\theta} \psi - k \sin \tilde{\theta} \psi - \cos \tilde{\gamma}_1 |\psi|^\sigma_1 + 1 - \cos \tilde{\gamma}_2 |\psi|^\sigma_2 + 1,
\]

\[
d\psi'' + d \frac{(\psi')^2}{\psi} = \omega \sin \tilde{\theta} \psi + k \cos \tilde{\theta} \psi - \sin \tilde{\gamma}_1 |\psi|^\sigma_1 + 1 - \sin \tilde{\gamma}_2 |\psi|^\sigma_2 + 1
\]

and so

\[
(1 + d^2) \psi'' = \left[\omega (d \sin \tilde{\theta} + \cos \tilde{\theta}) + k(d \cos \tilde{\theta} - \sin \tilde{\theta})\right] \psi
\]

\[
- (d \sin \tilde{\gamma}_1 + \cos \tilde{\gamma}_1) |\psi|^\sigma_1 + 1 - (d \sin \tilde{\gamma}_2 + \cos \tilde{\gamma}_2) |\psi|^\sigma_2 + 1,
\]

\[
(1 + d^2) \frac{(\psi')^2}{\psi} = \left[\omega \left(\frac{\sin \tilde{\theta}}{d} - \cos \tilde{\theta}\right) \psi + k \left(\frac{\cos \tilde{\theta}}{d} + \sin \tilde{\theta}\right) \psi
\]

\[
- \left(\frac{\sin \tilde{\gamma}_1}{d} - \cos \tilde{\gamma}_1\right) |\psi|^\sigma_1 + 1 - \left(\frac{\sin \tilde{\gamma}_2}{d} - \cos \tilde{\gamma}_2\right) |\psi|^\sigma_2 + 1\right].
\]

Hence, writing

\[
\epsilon = \frac{\omega (d \sin \tilde{\theta} + \cos \tilde{\theta}) + k(d \cos \tilde{\theta} - \sin \tilde{\theta})}{1 + d^2},
\]

\[
\eta = \frac{d \sin \tilde{\gamma}_1 + \cos \tilde{\gamma}_1}{1 + d^2}
\]

and

\[
\zeta = \frac{d \sin \tilde{\gamma}_2 + \cos \tilde{\gamma}_2}{1 + d^2},
\]
we require that
\[ \omega \left( \frac{\sin \theta}{d} - \cos \hat{\theta} \right) + k \left( \frac{\cos \hat{\theta}}{d} + \sin \hat{\theta} \right) = \omega (d \sin \hat{\theta} \cos \hat{\theta}) + k (d \cos \hat{\theta} - \sin \hat{\theta}), \tag{3.7} \]
\[ \frac{\sin \gamma_1}{d} - \cos \gamma_1 = \frac{2}{\sigma_1 + 2} (d \sin \gamma_1 + \cos \gamma_1) \tag{3.8} \]
and
\[ \frac{\sin \gamma_2}{d} - \cos \gamma_2 = \frac{2}{\sigma_2 + 2} (d \sin \gamma_2 + \cos \gamma_2). \tag{3.9} \]
From (3.7) we derive
\[ d = \frac{k \sin \hat{\theta} - \omega \cos \hat{\theta} \pm \sqrt{\omega^2 + k^2}}{\omega \sin \hat{\theta} + k \cos \hat{\theta}} =: d_\pm \tag{3.10} \]
and so
\[ \epsilon = \pm \sqrt{\omega^2 + k^2}. \]
Since \( \epsilon > 0 \) (see [2]), we must have \( d = d_+ \). Finally, the conditions (3.8), (3.9) and \( \eta, \zeta > 0 \) are equivalent to (1.3).

2. Now we consider the case \( \chi = -1 \). Keeping the same notation, we obtain again the conclusions (3.7), (3.8), and (3.9) assuming the existence of the solution of the stationary Schrödinger equation
\[ \psi'' = \epsilon \psi - \eta \psi^{\sigma_1 + 1} + \zeta \psi^{\sigma_2 + 1} \]
with \( \epsilon, \eta, \zeta > 0 \). Set \( f(z) = -\epsilon z + \eta z^{\sigma_1 + 1} - \zeta z^{\sigma_2 + 1} \) and take the primitive
\[ F(z) = \int_0^z f(s) ds = z^2 \left[ -\epsilon + \frac{2 \eta}{\sigma_1 + 2} z^{\sigma_1} - \frac{2 \zeta}{\sigma_2 + 2} z^{\sigma_2} \right] \]
It is clear that \( z_0 := \inf \{ z > 0 : F(z) = 0 \} > 0 \) and
\[ f(z_0) = z_0 \left[ -\epsilon + \eta z_0^{\sigma_1} - \zeta z_0^{\sigma_2} \right] > 0 \]
since
\[ z_0^{\sigma_2} \left[ \frac{\sigma_1 \eta}{\sigma_1 + 2} z_0^{\sigma_1 - \sigma_2} - \frac{\sigma_2 \zeta}{\sigma_2 + 2} \right] > 0 \]
which is verified for \( \sigma_2 \) small enough.

The remainder of this section is dedicated to the proof of Theorem 1.4. Throughout the proof, \( (\cdot, \cdot) \) will denote the complex \( L^2 \)-inner product
\[ (u, v) = \int_{\Omega} u(x) \overline{v(x)} dx, \quad u, v \in L^2(\Omega). \]
Denote by \( \lambda_0 \) a double eigenvalue of the Laplace-Dirichlet operator \( -\Delta \) in \( L^2(\Omega) \) and let \( u_1, u_2 \in H^{2}(\Omega) \cap H^1_{0}(\Omega) \) be two \( L^2 \)-orthonormal eigenfunctions, spanning the eigenspace \( V \). Furthermore, define the orthogonal projection \( P : L^2(\Omega) \rightarrow V^\perp \). As a consequence, for all \( \lambda \) near \( \lambda_0 \), one has
\[ \| (\lambda + P \Delta)^{-1} f \|_{H^1_{0}(\Omega)} \lesssim \| f \|_{H^{-1}(\Omega)}, \quad \text{for all} \; f \in H^{-1}(\Omega). \]
To simplify notations, set
\[ L := -\epsilon i^\theta \Delta, \quad M_1 u = e^{i\gamma_1} |u|^{\sigma_1} u, \quad M_2 u = \chi e^{i\gamma_2} |u|^{\sigma_2} u, \quad M u = M_1 u + M_2 u. \]
Then equation (B-S) can be rewritten as
\[ \lambda u - Lu + Mu = 0, \quad \lambda = k - i\omega \in \mathbb{C}. \] (3.12)
Applying the Lyapunov-Schmidt reduction, equation (3.12) is equivalent to the system
\[ P(\lambda u - Lu + Mu) = 0 \] (3.13)
\[ (\lambda u - Lu + Mu, u_j) = 0, \quad j = 1, 2. \] (3.14)
We write \[ u = y + \epsilon_1 u_1 + \epsilon_2 u_2, \ y \in V^\perp. \] Since (3.12) enjoys a gauge symmetry, we may assume, without loss of generality, that \( \epsilon_1 > 0. \) By (3.13),
\[ y = (\lambda - PL)^{-1}[-PM(y + \epsilon_1 u_1 + \epsilon_2 u_2)] \] (3.15)
On the other hand, equation (3.14) reduces to
\[ \epsilon_j(\lambda - e^{i\theta} \lambda_0) = -(M(y + \epsilon_1 u_1 + \epsilon_2 u_2), u_j), \quad j = 1, 2. \] (3.16)
Setting \( \alpha = \epsilon_2/\epsilon_1, \) it follows from (3.16) that
\[ \lambda - e^{i\theta} \lambda_0 = -\frac{1}{\epsilon_1}(M(y + \epsilon_1 u_1 + \epsilon_2 u_2, u_1) \]
\[ = -\epsilon_1 \lambda (M_1(u_1 + \alpha u_2), u_1) + Q_1(y, \epsilon_1, \alpha), \] (3.17)
where
\[ Q_1(y, \epsilon_1, \alpha) = -\frac{1}{\epsilon_1}(M(y + \epsilon_1 u_1 + \epsilon_2 u_2) - M_1(\epsilon_1 u_1 + \epsilon_2 u_2, u_1). \]
On the other hand, again by (3.16),
\[ \alpha M(y + \epsilon_1 u_1 + \epsilon_2 u_2, u_1) = -\alpha \epsilon_1(\lambda - e^{i\theta} \lambda_0) = -\epsilon_2(\lambda - e^{i\theta} \lambda_0) \]
\[ = (M(y + \epsilon_1 u_1 + \epsilon_2 u_2, u_2). \] (3.18)
Setting
\[ Q_2(y, \epsilon_1, \alpha) = \frac{1}{\epsilon_1}(M(y + \epsilon_1 u_1 + \epsilon_2 u_2) - M_1(\epsilon_1 u_1 + \epsilon_2 u_2, u_1 - u_2), \]
equation (3.18) becomes
\[ (M_1(u_1 + \alpha u_2), u_1 - u_2) + Q_2(y, \epsilon_1, \alpha) = 0. \] (3.19)
The proof of Theorem 1.4 will follow from the following steps: first, we show that, for each \( \epsilon_1, \alpha, \lambda \) fixed, \( y \) may be found through a fixed-point argument applied to (3.15). Afterwards, we apply a Lipschitz version of the Implicit Function Theorem to solve (3.17) and (3.19).
Since \( \lambda_0 \) is not an eigenvalue of \( PD, \) there exists \( r > 0 \) small such that
\[ V_r := \{ \lambda \in \mathbb{C} : |\lambda - e^{i\theta} \lambda_0| \leq r \} \subset \rho(PL) \] (3.20)
Notice that
\[ H^1_0(\Omega) \hookrightarrow L^{\sigma_j+2}(\Omega), \quad j = 1, 2, \]
and, by duality,
\[ L^{\sigma_j+2}_{\sigma_j+2}(\Omega) \hookrightarrow H^{-1}(\Omega), \quad j = 1, 2. \]
Lemma 3.1. Let $\lambda \in V_\epsilon$. Then, for all $\delta > 0$ small enough and $|\epsilon_1|, |\epsilon_2| \leq \delta$, there exists a solution $y = y(\epsilon_1, \epsilon_2, \lambda) \in H_0^1(\Omega)$ of (3.15). Moreover, for some universal constants $C > 0, K > 0$,
\begin{equation}
\|y(\epsilon_1, \epsilon_2, \lambda)\|_{H^1_0} \leq C\delta^{\sigma_1+1}
\end{equation}
and
\begin{equation}
\|y(\epsilon_1, \epsilon_2, \lambda) - y(\tilde{\epsilon}_1, \tilde{\epsilon}_2, \tilde{\lambda})\|_{H^1_0} \leq K\delta^{\sigma_1+1}|(\epsilon_1, \epsilon_2, \lambda) - (\tilde{\epsilon}_1, \tilde{\epsilon}_2, \tilde{\lambda})|
\end{equation}
for all $\lambda_1, \tilde{\lambda}_1 \in V_\epsilon$ and $|\epsilon_j|, |\tilde{\epsilon}_j| < \delta, j = 1, 2$.

Proof. Denote by $S_\lambda = S(\epsilon_1, \epsilon_2, \lambda)y$ the right-hand side of (3.15) and by $R(\lambda)$ the resolvent $(\lambda - PL)^{-1}$, which is a bounded operator from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$, uniformly in $\lambda \in V_\epsilon$. Then, for $\|y_1\|_{H^1_0}, \|y_2\|_{H^1_0} \leq \delta$ and $r_j = (\sigma_j + 2)/(|\sigma_j + 1|, j = 1, 2$,
\begin{equation}
\|S_y\|_{H^1_0} \leq \|M(y_1 + \epsilon_1 u_1 + \epsilon_2 u_2)\|_{H^{-1}}
\end{equation}
\begin{equation}
\leq \sum_{j=1,2} M_j(y_1 + \epsilon_1 u_1 + \epsilon_2 u_2)\|_{L^{r_j}}
\end{equation}
\begin{equation}
\leq \sum_{j=1,2} \left(\|y_1\|_{L^{r_j+2}} + |\epsilon_1|\|u_1\|_{L^{r_j+2}} + |\epsilon_2|\|u_2\|_{L^{r_j+2}}\right)^{\sigma_j+1} \leq (\delta^{\sigma_1+1} + \delta^{\sigma_2+1})
\end{equation}
and
\begin{equation}
\|S_y - S_{y_2}\|_{H^1_0} \leq \|M(y_1 + \epsilon_1 u_1 + \epsilon_2 u_2) - M(y_2 + \epsilon_1 u_1 + \epsilon_2 u_2)\|_{H^{-1}}
\end{equation}
\begin{equation}
\leq \sum_{j=1,2} M_j(y_1 + \epsilon_1 u_1 + \epsilon_2 u_2) - M_j(y_2 + \epsilon_1 u_1 + \epsilon_2 u_2)\|_{L^{r_j}}
\end{equation}
\begin{equation}
\leq \sum_{j=1,2} \left(\|y_1\|_{L^{r_j+2}} + |\epsilon_1|\|u_1\|_{L^{r_j+2}} + |\epsilon_2|\|u_2\|_{L^{r_j+2}}\right)^{\sigma_j} \|y_1 - y_2\|_{H^1_0}
\end{equation}
\begin{equation}
\leq (\delta^{\sigma_1} + \delta^{\sigma_2})\|y_1 - y_2\|_{H^1_0}.
\end{equation}
Thus, for all $\delta > 0$ small, it follows from the Banach fixed point theorem that there exists a unique $y = y(\epsilon_1, \epsilon_2, \lambda)$ solution to (3.15) with $\|y(\epsilon_1, \epsilon_2, \lambda)\|_{H^1_0} \leq \delta$. By (3.24), this estimate can be improved to (3.21).

We now prove the Lipschitz estimate (3.22) in $\lambda$, as the estimate in the remaining variables is straightforward. For $|\epsilon_1|, |\epsilon_2| < \delta$ fixed, take $\lambda_1, \lambda_2 \in V_\epsilon$ and consider $y_j = y(\epsilon_1, \epsilon_2, \lambda_j), j = 1, 2$. Then
\begin{equation}
y_1 - y_2 = (R(\lambda_1) - R(\lambda_2))[PM(y_1 + \epsilon_1 u_1 + \epsilon_2 u_2)]
\end{equation}
\begin{equation}
+ R(\lambda_2) [PM(y_1 + \epsilon_1 u_1 + \epsilon_2 u_2) - PM(y_2 + \epsilon_1 u_1 + \epsilon_2 u_2)]
\end{equation}
\begin{equation}
= (\lambda_2 - \lambda_1)[R(\lambda_1) - R(\lambda_2)] [PM(y_1 + \epsilon_1 u_1 + \epsilon_2 u_2)]
\end{equation}
\begin{equation}
+ R(\lambda_2) [PM(y_1 + \epsilon_1 u_1 + \epsilon_2 u_2) - PM(y_2 + \epsilon_1 u_1 + \epsilon_2 u_2)].
\end{equation}
Therefore, proceeding as in (3.24),
\begin{equation}
\|y_1 - y_2\|_{H^1_0} \leq \sum_{j=1,2} |\lambda_1 - \lambda_2| \left(\|y_1\|_{H^1_0} + |\epsilon_1 + |\epsilon_2|\right)^{\sigma_j+1}
\end{equation}
\begin{equation}
+ \left(\|y_1\|_{H^1_0} + \|y_2\|_{H^1_0} + |\epsilon_1 + |\epsilon_2|\right)^{\sigma_j} \|y_1 - y_2\|_{H^1_0}
\end{equation}
\begin{equation}
\leq (\delta^{\sigma_1+1} + \delta^{\sigma_2+1})|\lambda_1 - \lambda_2| + (\delta^{\sigma_1} + \delta^{\sigma_2})|y_1 - y_2|_{H^1_0}.
\end{equation}
The estimate follows for $\delta$ small enough.
Proof of Theorem 1.4. We wish to solve system (3.17)-(3.19). First, when one drops
the remainder terms $R$ and $Q$, the system reduces to
\[
\begin{aligned}
F_1(\epsilon, \lambda, \alpha) &= \lambda - e^{i\theta} \lambda_0 + \epsilon_1^2 (M_1(u_1 + \alpha u_2), u_1) = 0 \\
F_2(\epsilon, \lambda, \alpha) &= P(\alpha) = 0
\end{aligned}
\]
which, by assumption, satisfies the conditions of the Implicit Function Theorem at
$(\epsilon, \lambda, \alpha) = (0, e^{i\theta} \lambda_0, \alpha_0)$. Now observe that, due to (3.21) and (3.22), $Q_1$ and $Q_2$
are Lipschitz continuous in $\epsilon$, $\lambda$ and $\alpha$, with constant proportional to $\epsilon_1^{\gamma_1-1}$, $\epsilon_1^{\gamma_1}$
and $\epsilon_1^{\gamma_1}$, respectively. We exemplify by proving the Lipschitz estimate for $Q_2$ with
respect to $\lambda$: for $\epsilon_1$ small and $\alpha$ fixed, using Hölder inequality,
\[
\begin{aligned}
|Q_2(y(\lambda_1)) - Q_2(y(\lambda_2))| &\leq \frac{1}{\epsilon_1^{\gamma_1+1}} \| (M(y(\lambda_1) + \epsilon_1 u_1 + \epsilon_2 u_2) - M(y(\lambda_2) + \epsilon_1 u_1 + \epsilon_2 u_2), u_1) \\
&\leq \frac{1}{\epsilon_1^{\gamma_1+1}} \sum_{j=1,2} (\|y(\lambda_1)\|_{L^{\gamma_j+2}} + \|y(\lambda_2)\|_{L^{\gamma_j+2}} + \epsilon_1 + \epsilon_2)^{\sigma_j+1} \|y(\lambda_1) - y(\lambda_2)\|_{L^{\gamma_j+2}} \\
&\leq \frac{1}{\epsilon_1^{\gamma_1+1}} (\epsilon_1^{\gamma_1} + \epsilon_2^{\gamma_2}) \|y(\lambda_1) - y(\lambda_2)\|_{H_0^1} \leq \epsilon_1^{\gamma_1} |\lambda_1 - \lambda_2|.
\end{aligned}
\]
Therefore (3.17)-(3.19) is a Lipschitz perturbation, small in $\alpha$ and $\lambda$, of (3.25). The
conclusion now follows from [7, Section 7.1].

Remark 8. The above proof can be easily applied to the case of simple eigenvalues.
Indeed, the Lyapunov-Schmidt reduction yields the system
\[
\begin{aligned}
y &= (\lambda - PL)^{-1}[-PM(y + \epsilon_1 u_1)] \\
\lambda - e^{i\theta} \lambda_0 &= -\frac{1}{\epsilon_1} (M(y + \epsilon_1 u_1), u_1)
\end{aligned}
\]
The first equation can be solved through a fixed point argument, while the second is
in the conditions of the Implicit Function Theorem (in the Lipschitz formulation).

Example 1. Let $\Omega = (-1,1)^2$ and $\sigma_1 = 2$. As it is well-known, the second eigen-
value of the Laplacian $\lambda_2 = 5\pi^2/4$ is double, with associated eigenfunctions
\[
v_1(x,y) = \cos \left( \frac{\pi x}{2} \right) \sin (\pi y), \quad v_2(x,y) = \cos \left( \frac{\pi y}{2} \right) \sin (\pi x).
\]
If we choose $u_1 = v_1$ and $u_2 = v_2$, the function $P$ takes the form
\[
P(\alpha) = \frac{3}{16} (\alpha^3 - \alpha),
\]
and we find three bifurcation branches with $\alpha_0 = 0, \pm 1$. The permutation $u_1 = v_2,
 u_2 = v_1$ provides yet another branch (formally identifiable with $\alpha_0 = \infty$). In
conclusion, we recover the results of [12], which are specific for $\Omega = (-1,1)^2$, $\sigma_1 = 2$
and $\theta, \gamma_1, \gamma_2 = 0$.

4. Stability of the trivial equilibrium. In this section, we study the stability
of the equilibrium solution $u \equiv 0$ and the asymptotic decay of global solutions of
(gCGL) depending on the parameters and the coefficient for the driving term $k$. Let
 denote by $S(t)$ the dynamical system associated to (gCGL): $S(t)u_0 = u(t; u_0), \ t \geq 0$. 
Definition 4.1. We say that \( u_0 \in H^1_0(\Omega) \) is stable if for any \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that
\[
v \in H^1_0(\Omega), \| u_0 - v \|_{H^1} < \varepsilon \Rightarrow \sup_{t \geq 0} \| S(t)u_0 - S(t)v \|_{H^1} < \delta.
\]
In addition, we say that \( u_0 \) is asymptotically stable if \( u_0 \) is stable and there exists \( \eta > 0 \) such that \( \lim_{t \to \infty} \| S(t)u_0 - y \|_{H^1} = 0 \) for all \( y \in H^1_0(\Omega) \), \( \| u_0 - y \|_{H^1} < \eta \).

More generally if \( S(t) \) denote a dynamical system on a Banach space \( H \) we recall that a Lyapunov function is a continuous function \( W : H \to \mathbb{R} \) such that
\[
\dot{W}(u) := \lim_{t \to 0^+} \frac{1}{t} [W(S(t)u) - W(u)] \leq 0
\]
for all \( u \in H \). The next lemma is mainly proved in [16].

Lemma 4.2. Let \( S(t) \) be a dynamical system on a Banach space \((D, \|\cdot\|)\). Let \( E \) a normed space such that \( D \hookrightarrow E \) and \( W \) a Lyapunov function on \( D \) such that
\[
W(u_0) \geq k_1\|u_0\|_E, \quad k_1 > 0, \quad u_0 \in D.
\]
Then, the equilibrium point \( 0 \) is \( \|\cdot\|_E \)-stable in the sense that
\[
u_0 \in D, \|u_0\| \to 0 \Rightarrow \|S(t)u_0\|_E \to 0,
\]
uniformly in \( t \geq 0 \).
Assume in addition that
\[
\dot{W}(u_0) \leq -k_2\|u_0\|_E, \quad k_2 > 0, \quad u_0 \in D.
\]
Then, \( \lim_{t \to \infty} \|S(t)u_0\|_E = 0 \) for any \( u_0 \in D \).

Proof of Theorem 1.6. 1. Let us denote by \( S(t)u_0 \equiv u(t, u_0) \) the unique global solution of \((gCGL)\) under the hypothesis of the Theorem (1.2) and define
\[
W_p(u) = \int_{\Omega} |u(x)|^p dx,
\]
with \( 2 \leq p \leq \frac{2N}{N-2} \) if \( N > 2 \), \( 2 \leq p < \infty \) if \( N=1,2 \) and \( u = u(t, u_0) \). It is clear that \( W_p : H^1_0(\Omega) \to \mathbb{R} \), is a continuous functional and, from \( \dot{W}_p(u) = \nabla W(u) \cdot \nabla u(t) \), we get
\[
\dot{W}_p(u) = p \Re \int_{\Omega} \left| u \right|^p \overline{\left((a + i\alpha) \Delta u + (b + i\beta) |u|^{q_1} u -(c + i\gamma)|u|^{q_2} u + ku \right)} dx
\]
\[
\leq pk \int_{\Omega} |u|^p dx - ap \int_{\Omega} |u|^{p-2} \nabla u |^2 dx + pb \int_{\Omega} |u|^p |u|^{q_1} dx
\]
\[
- pc \int_{\Omega} |u|^p |u|^{q_2} dx + p\alpha \Re \int_{\Omega} \nabla \left( |u|^{p-2} \overline{\nabla u} \right) dx.
\]
Since
\[
\nabla \left( |u|^{p-2} \overline{\nabla u} \right) = \frac{p-2}{2} |u|^{p-4} (u \nabla \overline{u} + \overline{u} \nabla u) \overline{u},
\]
we obtain
\[
\left| p\alpha \Re \int_{\Omega} \nabla \left( |u|^{p-2} \overline{\nabla u} \right) dx \right| \leq p|\alpha| \frac{p-2}{2} \int_{\Omega} |u|^{p-2} |\nabla u|^2 dx.
\]
Furthermore, by interpolation, one has
\[
\|u\|_{L^{p+\sigma_1}_\lambda} \leq \|u\|_{L^{p+\sigma_2}_\lambda} \|u\|_{L^p}^{p(\sigma_2 - \sigma_1)}
\]
and by the Young inequality
\[ \|u\|_{L^{p+\sigma_1}}^{p+\sigma_1} \leq \frac{\sigma_2}{\sigma_1} \|u\|_{L^{p+\sigma_2}}^{p+\sigma_2} + \frac{\sigma_2 - \sigma_1}{\sigma_2} \|u\|_{L^p}^p. \]
Hence, if
\[ k \leq 0, \quad \frac{p - 2}{2} \leq a, \quad b \frac{\sigma_1}{\sigma_2} \leq c \quad \text{and} \quad b \frac{\sigma_2 - \sigma_1}{\sigma_2} \leq |k|, \]
we derive that \( \tilde{W}_p(u) \leq pk \|u\|_{L^p}^p \) and the conclusion follows from the Lemma 4.2. If \( p = 2 \) and \( \Omega \) is bounded, by the Poincaré inequality, we obtain the same conclusion under the conditions
\[ k > 0, \quad b \frac{\sigma_1}{\sigma_2} \leq c \quad \text{and} \quad b + \frac{\sigma_2 - \sigma_1}{\sigma_2} + k < a \left( \frac{1}{\omega_N} |\Omega| \right)^{2/N}, \]
with \( b^+ = \max\{0, b\} \).

2. We now define the new functional:
\[ V(u) := \frac{a}{2} \int_\Omega |\nabla u|^2 dx - \frac{b}{\sigma_1 + 2} \int_\Omega |u|^\sigma_1^2 dx + \frac{c}{\sigma_2 + 2} \int_\Omega |u|^\sigma_2^2 dx - \frac{k}{2} \int_\Omega |u|^2 dx. \]
It is clear that \( V \) is a continuous real function on \( H_0^1(\Omega) \). By interpolation and the Young inequality, we have
\[ \|u\|_{L^{\sigma_1+2}} \leq \|u\|_{L^{\sigma_1+2}}^{\sigma_2^2/(\sigma_1^2+2)} \|u\|_{L^{\sigma_2^2}}^{\sigma_1^2/(\sigma_2^2+2)} \leq \frac{\sigma_1}{\sigma_2} \|u\|_{L^{\sigma_1+2}} + \frac{\sigma_2 - \sigma_1}{\sigma_2} \|u\|_{L^2}^2. \]
Then we have \( V(u) \geq M \|u\|_{H^1}^2, \quad M > 0 \), if
\[ k < 0, \quad \frac{b}{\sigma_1 + 2} \frac{\sigma_1}{\sigma_2} \leq \frac{c}{\sigma_2 + 2} \quad \text{and} \quad \frac{b}{\sigma_1 + 2} \frac{\sigma_2 - \sigma_1}{\sigma_2} \leq \frac{|k|}{2} \] (4.3)
or
\[ k = 0, \quad \frac{b}{\sigma_1 + 2} \frac{\sigma_1}{\sigma_2} \leq \frac{c}{\sigma_2 + 2} \quad \text{and} \quad \frac{b}{\sigma_1 + 2} \frac{(\sigma_2 - \sigma_1)}{(\sigma_2 + 2)\sigma_2} \leq \frac{a}{2} \left( \frac{1}{\omega_N} |\Omega| \right)^{-2/N} \] (4.4)
and \( \Omega \) is a bounded domain.

In addition, for any \( u \in H_0^1(\Omega) \cap H^2(\Omega) \) and \( h \in H_0^1(\Omega) \), we have \( V(u + h) = V(u) + L \cdot h + o(\|h\|_{H^1}) \), where
\[ L : h = -\Re \int_\Omega [a\Delta u - b|u|^\gamma u + k\pi] h \; dx. \]
Therefore, for all \( u = u(t) \in H_0^1(\Omega) \cap H^2(\Omega) \),
\[ \dot{V}(u) = -\int_\Omega [a\Delta u + b|u|^{\sigma_1} u - c|u|^{\sigma_2} u + ku|^2 \; dx \]
\[ - \Re \int_\Omega (a\Delta u + b|u|^{\sigma_1} u - c|u|^{\sigma_2} u) i(\alpha\Delta u + \beta|u|^{\sigma_1} u - \gamma|u|^{\sigma_2} u) \; dx \]
\[ - \Re \int_\Omega i k\pi \alpha \Delta u + \beta|u|^{\sigma_1} u - \gamma|u|^{\sigma_2} u \; dx \]
and, for \( \frac{\alpha}{\sigma_1} = \frac{\beta}{\sigma_2} = \frac{\gamma}{\sigma_2} \), we obtain
\[ \dot{V}(u(t)) = -\int_{\Omega_t} [a\Delta u + b|u|^{\sigma_1} u - c|u|^{\sigma_2} u + ku|^2 \; dx \leq 0, \quad t > 0. \]
(4.5)
Note that
\[ \frac{1}{t} [V(S(t)u_0) - V(u_0)] = \dot{V}(S(t^*)u_0) \]
for some \(0 < t^* < t\) and so (4.5) is true for all \(t \geq 0\). Hence, the functional \(V\) is a Lyapunov function and, under the conditions (4.3), (4.4), we have the stability in \(H_0^1(\Omega)\) of the equilibrium solution \(u \equiv 0\).

We prove now the asymptotic stability. We have

\[
-\dot{V}(u) = a^2 \int_{\Omega} |\Delta u|^2 dx + b^2 \int_{\Omega} |u|^{2\sigma_1 + 2} dx \\
+ c^2 \int_{\Omega} |u|^{2\sigma_2 + 2} dx + k^2 \int_{\Omega} |u|^2 dx + 2ab \Re \int_{\Omega} \Delta u |u|^\sigma \overline{\pi} dx \\
- 2ac \Re \int_{\Omega} \Delta u |u|^\sigma \overline{\pi} dx + 2ak \Re \int_{\Omega} \Delta u \overline{\pi} dx + 2bk \int_{\Omega} |u|^\sigma_1 + 2 dx \\
- 2ck \int_{\Omega} |u|^\sigma_2 + 2 dx - 2bc \int_{\Omega} |u|^\sigma_1 + \sigma_2 + 2 dx. \tag{4.6}
\]

Next one has the following estimate:

\[
\Re \int_{\Omega} \Delta u |u|^\sigma \overline{\pi} dx \\
= - \int_{\Omega} \nabla u^2 |u|^\sigma_2 dx - \frac{\sigma_2}{2} \Re \int_{\Omega} |u|^\sigma_2 - 2 \nabla u \cdot (\nabla u \overline{\pi} + u \nabla \overline{\pi}) \overline{\pi} dx \\
= - \int_{\Omega} \nabla u^2 |u|^\sigma_2 dx - \frac{\sigma_2}{2} \int_{\Omega} |\nabla u|^2 |u|^\sigma_2 dx - \frac{\sigma_2}{2} \Re \int_{\Omega} |u|^\sigma_2 - 2 (\nabla u \cdot \nabla u) \overline{\pi}^2 dx \\
\leq - \int_{\Omega} |\nabla u|^2 |u|^\sigma dx
\]

and so

\[
- 2ac \Re \int_{\Omega} \Delta u |u|^\sigma \overline{\pi} dx \geq 2ac \int_{\Omega} |\nabla u|^2 |u|^\sigma_2 dx. \tag{4.7}
\]

Also

\[
2ak \Re \int_{\Omega} \Delta u \overline{\pi} dx = -2ak \int_{\Omega} |\nabla u|^2 dx. \tag{4.8}
\]

Since

\[
\int_{\Omega} |u|^{\sigma_1 + \sigma_2 + 2} dx \leq \|u\|_{L^{\sigma_1 + 2}} \|u\|_{L^{\sigma_2 + 2}},
\]

we obtain

\[
- 2bc \int_{\Omega} |u|^{\sigma_1 + \sigma_2 + 2} dx \geq -b^2 \|u\|_{L^{\sigma_1 + 2}}^{\sigma_2 + 2} - c^2 \|u\|_{L^{\sigma_2 + 2}}^{\sigma_2 + 2} \tag{4.9}
\]

and, if \(b_1/\sigma_2 < c\) and \(b(\sigma_2 - \sigma_1)/\sigma_2 < |k|/2\), it follows from (4.2)

\[
2bk \|u\|_{L^{\sigma_1 + 2}}^{\sigma_2 + 2} < 2c|k| \|u\|_{L^{\sigma_2 + 2}}^{\sigma_2 + 2} + k^2 \|u\|_{L^2}^2. \tag{4.10}
\]

Finally we remark that

\[
\left| \int_{\Omega} \Delta u |u|^\sigma \overline{\pi} dx \right| \leq (\sigma_1 + 1) \int_{\Omega} |\nabla u|^2 |u|^\sigma_1 dx
\]

and so, if we assume \(|b| (\sigma_1 + 1) < \min\{c, |k|\}\), we get

\[
\left| 2ab \int_{\Omega} \Delta u |u|^\sigma \overline{\pi} dx \right| \\
\leq 2ab |b| (\sigma_1 + 1) \int_{\Omega} |\nabla u|^2 dx + 2a|b|(\sigma_1 + 1) \int_{\Omega} |\nabla u|^2 |u|^\sigma_2 dx \\
< 2ac \int_{\Omega} |\nabla u|^2 |u|^\sigma_2 dx + 2a|k| \int_{\Omega} |\nabla u|^2 dx. \tag{4.11}
\]
If $k < 0$, is now clear that the asymptotic stability of $u \equiv 0$ follows from (4.6) and (4.7), (4.8), (4.9), (4.10), (4.11). With $k = 0$ and $\Omega$ a bounded domain, we estimate
\[
\int_\Omega |\nabla u|^2 dx \leq \left( \int_\Omega |\Delta u|^2 dx \right)^{1/2} \left( \int_\Omega |u|^2 dx \right)^{1/2}
\]
and by the Poincaré inequality,
\[
\left( \int_\Omega |\Delta u|^2 dx \right)^{1/2} \geq \left( \frac{1}{\omega_N} |\Omega| \right)^{-1/N} \left( \int_\Omega |\nabla u|^2 dx \right)^{1/2}.
\]
Hence
\[
a^2 \int_\Omega |\Delta u|^2 dx \geq a^2 \left( \frac{1}{\omega_N} |\Omega| \right)^{-2/N} \int_\Omega |\nabla u|^2 dx.
\] (4.12)
Since $k = 0$, it is sufficient to estimate the fifth term in the r.h.s of (4.6) with $b > 0$. From the estimations (4.7), (4.11) and (4.12) we must require
\[
b(\sigma_1 + 1) \leq c, \quad \text{and} \quad b(\sigma_1 + 1) < \frac{a}{2} \left( \frac{1}{\omega_N} |\Omega| \right)^{-2/N}
\]
and we note that this second condition imply the last stability condition in (4.4). The proof is now complete. \hfill $\square$

5. Stability of some time periodic solutions of (gCGL). Consider the (gCGL) equation on a bounded domain $\Omega$ with the Neumann condition on the boundary. We study now the stability of some particular time periodic solutions. Let be $\vartheta(t)$ a $T$-periodic solution of the ordinary differential equation (1.7),
\[
\dot{\vartheta} = (b + i\beta)|u|^\sigma(u - (c + i\gamma)|u|^\sigma u + ku)
\]
associated to the (gCGL) equation.

Proof of Theorem 1.10. First we linearise the (gCGL) equation around the $T$-periodic solution $\Theta(x,t) \equiv \vartheta(t)$. We obtain the linear variational equation
\[
\dot{v} = A_N v + B(t)v
\] (5.1)
where $A_N = (a + i\alpha)\Delta$ denote the Neumann operator. If we set $v = v_1 + iv_2$ and $\vartheta = \vartheta_1 + i\vartheta_2$ we have
\[
\Re(B(t)v) = b|\vartheta|^\sigma v_1 - \beta|\vartheta|^{\sigma_1}v_2 + b\sigma_1|\vartheta|^{\sigma_2-1}v_1 \Re(\vartheta \overline{v})
\]
\[
- \beta \sigma_1|\vartheta|^{\sigma_1-2}v_2 \Re(\vartheta \overline{v}) - c|\vartheta|^{\sigma_2}v_1 + \gamma|\vartheta|^{\sigma_2}v_2
\]
\[
- c \sigma_2|\vartheta|^{\sigma_2-2}v_1 \Re(\vartheta \overline{v}) + \gamma \sigma_2|\vartheta|^{\sigma_2-2}v_2 \Re(\vartheta \overline{v})
\] (5.2)
\[
\Im(B(t)v) = \beta|\vartheta|^\sigma v_1 + b|\vartheta|^{\sigma_1}v_2 + b\sigma_1|\vartheta|^{\sigma_2-1}v_1 \Re(\vartheta \overline{v})
\]
\[
+ \beta \sigma_1|\vartheta|^{\sigma_1-2}v_1 \Re(\vartheta \overline{v}) - c|\vartheta|^{\sigma_2}v_2 - \gamma|\vartheta|^{\sigma_2}v_1
\]
\[
- c \sigma_2|\vartheta|^{\sigma_2-2}v_2 \Re(\vartheta \overline{v}) - \gamma \sigma_2|\vartheta|^{\sigma_2-2}v_1 \Re(\vartheta \overline{v}).
\] (5.3)
Notice that $B(t)$ is $T$-periodic.

Now, let $R(t,s)$ the evolution operator for (5.1), i.e.
\[
R(t,s)v_0 = v(t,s,v_0)
\]
is the solution of (5.1) with initial data, $v(s) = v_0$, and recall that the eigenvalues of the period map, $U_0 = R(T,0)$, are the characteristic multipliers. Since $A_N$ has compact resolvent, $U_0$ is compact and so, the spectrum $\sigma(U_0) \setminus \{0\}$ is entirely
composed by characteristic multipliers (see [18, pg. 197]). Next, we prove the following claim: the characteristic multipliers of (5.1) are the multipliers of the planar system

$$\dot{v} = -\lambda v + B(t)v$$

(5.6)

for any $\lambda$, eigenvalue of the Neumann operator $-A_N = -(a + i\alpha)\Delta$.

In fact, let $\tilde{R}(t,s)$ be the evolution operator for the planar system $\dot{v} = B(t)v$. By the Floquet representation we have

$$\tilde{U}_0 := \tilde{R}(T,0) = P(T)e^{CT}P(T)^{-1}\quad where C is a constant matrix and P(T) is an invertible matrix. Then we obtain

$$U_0 = R(T, 0) = e^{A_N T} \tilde{R}(T, 0) = e^{A_N T} P(T)e^{CT}P(T)^{-1} = P(T)e^{(A_N + C)T}P(T)^{-1}\quad and so the eigenvalues of U_0 are the eigenvalues of $e^{(A_N + C)T}$, i.e. the characteristic multipliers of (5.1) are those of (5.6).

Denote this multipliers by $\mu_j, (j = 1, 2)$. It is well known that $\mu_j$ must meet the condition (see, e.g. [10])

$$\mu_1\mu_2 = \exp \left( \int_0^T \text{Tr}(-\lambda I + B(t)) \right).$$

(5.7)

We consider now the two cases stated in the theorem:

1. In $(gCGL)$ equation let $c = 0$ and assume $bk < 0$. Take the $T_1$-periodic solution $P(x, t) \equiv p(t)$ for all $x \in \Omega$. We obtain, for each $\lambda$ eigenvalue of $-\Delta$ with the Neumann condition,

$$\mu_1\mu_2 = \exp \left( \int_0^{T_1} b\sigma_1|p(t)|^{\sigma_1} (2 + \sigma_1) + 2k - 2\lambda dt \right) = \exp \left( \int_0^{T_1} \sigma_1 b|p(t)|^{\sigma_1} - 2\lambda dt \right)$$

(5.8)

since $b|p(t)|^{\sigma_1} + k = 0$ for all $t \in [0, T_1]$ (recall that the $T_1$-periodic solution $p(t)$ has his orbit in the circle $|z| = r_1$, with $br_1^{\sigma_1} + k = 0$). If $b < 0$ it is clear that

$$\mu_1\mu_2 = \exp(-k\sigma_1T_1 - 2\lambda T_1) < \exp(-k\sigma_1T_1) < 1$$

for all $\lambda \in \sigma(-A_N)$, which implies the asymptotic stability of $P(x, t)$ (see [18], Th.8.2.3).

If $b > 0$ (and $k < 0$), easily we find the instability of the solution $p(t)$ of (and so the instability of $P(x, t)$). In fact, multiply (1.7) by $\bar{u}$ and take the real part. We obtain

$$\frac{d}{dt}|p|^2 = 2b|p|^{\sigma_1 + 2} + 2k|p|^2$$

and the solution $|p(t)|^2$ with initial data $|p(0)|^2 = r_1^2 + \varepsilon, (\varepsilon > 0)$, blow up in a finite time, since $2b\sigma_1^{\sigma_1 + 2} + 2k r_1^2 = 0$.

2. Assume now $k = 0$ and $bc > 0$. Take the $T_2$-periodic solution $Q(x, t) \equiv q(t)$ and recall that $q(t)$ has his orbit in the circle $|z| = r_2$, with $b - c\sigma_2^{\sigma_2 - \sigma_1} = 0$. We have

$$\mu_1\mu_2 = \exp \left( \int_0^{T_2} 2b|q(t)|^{\sigma_1} + b\sigma_1|q(t)|^{\sigma_1} - 2c|q(t)|^{\sigma_2} - c\sigma_2|q(t)|^{\sigma_2} - 2\lambda dt \right)$$

$$= \exp \left( \int_0^{T_2} b\sigma_1|q(t)|^{\sigma_1} - c\sigma_2|q(t)|^{\sigma_2} - 2\lambda dt \right) = \exp(-c(\sigma_2 - \sigma_1)r_2^{\sigma_2}T_2 - 2\lambda T_2).$$

We proceed just like before: if $c > 0$ (and so $b > 0$) we have $\mu_1\mu_2 = \exp(-c(\sigma_2 - \sigma_1)r_2^{\sigma_2}T_2 - 2\lambda T_2) \leq \exp(-c(\sigma_2 - \sigma_1)r_1^{\sigma_2}T_2) < 1$ for all $\lambda \in \sigma(-A_N)$ which proves the the asymptotic stability of $Q(x, t)$. 

A GENERALIZED COMPLEX GINZBURG-LANDAU EQUATION 17
If \( c < 0 \) and \( b < 0 \), from (1.7) we derive
\[
\frac{d}{dt}|q|^2 = 2b|q|^\sigma_1 + 2c|q|^\sigma_2 + 2
\]
which implies the blow-up in a finite time since \( 2b|q|^\sigma_1 + 2c|q|^\sigma_2 + 2 = 0 \). In particular we have prove, in this case, the instability of \( Q(x,t) \).

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