Finite groups with an automorphism of large order

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Abstract

Let $G$ be a finite group, and assume that $G$ has an automorphism of order at least $\rho|G|$, with $\rho \in (0, 1)$. Generalizing recent analogous results of the author on finite groups with a large automorphism cycle length, we prove that if $\rho > 1/2$, then $G$ is abelian, and if $\rho > 1/10$, then $G$ is solvable, whereas in general, the assumption implies $[G : \text{Rad}(G)] \leq \rho^{-1.78}$, where $\text{Rad}(G)$ denotes the solvable radical of $G$. Furthermore, we generalize an example of Horoševskiǐ to show that in finite groups, the quotient of the maximum automorphism order by the maximum automorphism cycle length may be arbitrarily large.

1 Introduction

1.1 Motivation and main results

The purpose of this paper is to study finite groups that may be viewed as “extreme” with respect to their maximum automorphism order. More generally, many authors have studied finite groups satisfying “extreme” quantitative conditions of various kinds. We mention the following examples: A variety of papers deals with finite groups in which some automorphism raises some minimum fraction of elements to the $e$-th power for $e = -1, 2, 3$, see [15, 16, 11, 12, 13, 14, 17, 4, 19, 7]. Wall classified the finite groups $G$ having more than $\frac{1}{2}|G| - 1$ involutions [18], and this was extended to a classification of those $G$ with more than $\frac{1}{5}|G| - 1$ subgroups of prime order by Burness and Scott [3].

In [1] and [2], we studied finite groups having an automorphism with a cycle of length at least $\rho|G|$, for some fixed $\rho \in (0, 1)$. We proved that if $\rho > \frac{1}{2}$, then $G$ is abelian [1, Theorem 1.1.7], and if $\rho > \frac{1}{10}$, then $G$ is solvable [2, Corollary 1.1.2(1)].

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Furthermore, we showed that for any fixed value of $\rho$, the index of the solvable radical $\text{Rad}(G)$ in $G$ is bounded from above in terms of $\rho$ if $G$ has such a long automorphism cycle [2, Theorem 1.1.1(1)].

In this paper, we strengthen these results, replacing automorphism cycle lengths by automorphism orders:

**Theorem 1.1.1.** Let $G$ be a finite group.

1. If $G$ has an automorphism of order greater than $\frac{1}{2}|G|$, then $G$ is abelian.
2. If $G$ has an automorphism of order greater than $\frac{1}{10}|G|$, then $G$ is solvable.
3. For any $\rho \in (0, 1)$, if $G$ has an automorphism of order at least $\rho|G|$, then $[G : \text{Rad}(G)] \leq \rho^{E_1}$, where $E_1 = (\log_{60}(6) - 1)^{-1} = -1.7781\ldots$.

A few comments relating this to the results on cycle lengths. Automorphisms $\alpha$ of finite groups having a cycle of length $\text{ord}(\alpha)$ (following the terminology in [6], such cycles will be referred to as regular) have been extensively studied by Horoševskiǐ in [9]. He gave examples of automorphisms of finite groups without a regular cycle (i.e., whose order is larger than the largest cycle length). In Section 3, we will generalize one of Horoševskiǐ’s examples to show that in finite groups, the quotient of the maximum automorphism order by the maximum automorphism cycle length may be arbitrarily large.

On the other hand, Horoševskiǐ also gave conditions on finite groups $G$ assuring that every automorphism of $G$ has a regular cycle, namely if $G$ is either semisimple (i.e., has no nontrivial solvable normal subgroups) [9, Theorem 1] or nilpotent [9, Corollary 1]. Note that in view of the latter, Theorem 1.1.1 implies the following: *Every automorphism $\alpha$ of a finite group $G$ such that $l := \text{ord}(\alpha) > \frac{1}{2}|G|$ has a cycle of length $l$. In particular, for $\rho > \frac{1}{2}$, the conditions “$G$ has an automorphism with a cycle of length $\rho|G|$,” and “$G$ has an automorphism of order $\rho|G|$,” are equivalent, and by [1, Corollary 1.1.8], we obtain a complete classification of the pairs $(G, \alpha)$ where $G$ is a finite group and $\alpha$ an automorphism of $G$ such that $\text{ord}(\alpha) > \frac{1}{2}|G|$.*

We note that both in [1] and in [2], we did not only study automorphisms, but a larger class of permutations on finite groups, so-called bijective affine maps:

**Definition 1.1.2.** Let $G$ be a finite group.

1. For an element $x \in G$ and an endomorphism $\varphi$ of $G$, the (left-)-affine map of $G$ with respect to $x$ and $\varphi$ is the map $A_{x,\varphi} : G \to G, g \mapsto x\varphi(g)$.
2. The group of bijective affine maps of $G$ (which are just those $A_{x,\varphi}$ where $\varphi$ is an automorphism of $G$) is denoted by $\text{Aff}(G)$.

We also had results for such maps, namely that a finite group $G$ having a bijective affine map with a cycle of length greater than $\frac{1}{2}|G|$ is solvable [2, Theorem 1.1.1(2)] and that $[G : \text{Rad}(G)]$ is bounded from above in terms of $\rho$ in a finite group $G$ having a bijective affine map cycle of length at least $\rho|G|$. This can be strengthened in the same way as the results on automorphisms:

**Theorem 1.1.3.** (1) Let $G$ be a finite group such that some $A \in \text{Aff}(G)$ has order greater than $\frac{1}{2}|G|$. Then $G$ is solvable.

2. Let $\rho \in (0, 1)$ and let $G$ be a finite group such that some $A \in \text{Aff}(G)$ has order at least $\rho|G|$. Then $[G : \text{Rad}(G)] \leq \rho^{E_2}$, where $E_2 = (\log_{60}(30) - 1)^{-1} = -5.9068\ldots$.
Finally, we remark that, just as for the results on cycle lengths, the constants $\frac{1}{2}$, $\frac{1}{m}$ and $\frac{1}{4}$ in Theorems [1,1.1,1.2] and [1,1.3,1] respectively cannot be lowered further, as follows by considering maximum automorphism orders in finite dihedral groups (for Theorem [1,1.1,1]) and in the alternating group $A_5$.

1.2 Notation

By $\mathbb{N}^+$, we denote the set of positive integers. For a function $f$ and a set $M$, we denote by $f[M]$ the element-wise image of $M$ under $f$, and by $f|_M$ the restriction of $f$ to $M$. The order of a group element $g$ is denoted by $\text{ord}(g)$. We write $\text{char} G$ for “$N$ is a characteristic subgroup of $G$”. The finite field with $q$ elements is denoted by $\mathbb{F}_q$, and the logarithm with respect to a base $c > 1$ by $\log_c$. We also use the following notation, most of which was already introduced in [1] and [2]:

**Notation.** (1) Let $X$ be a finite set, $\psi$ a permutation on $X$. We denote by $\Lambda(\psi)$ the largest cycle length of $\psi$ and set $\lambda(\psi) := \frac{1}{|X|} \Lambda(\psi)$.

(2) For a finite group $G$, we define $\Lambda(G) := \max_{\alpha \in \text{Aut}(G)} \Lambda(\alpha)$, $\lambda(G) := \frac{1}{|G|} \Lambda(G)$, $\Lambda_{\text{aff}}(G) := \max_{\alpha \in \text{Aff}(G)} \Lambda(\alpha)$ and $\lambda_{\text{aff}}(G) := \frac{1}{|G|} \Lambda_{\text{aff}}(G)$.

(3) For a finite group $G$, we denote by $\text{meo}(G)$ the maximum element order of $G$ and set $\text{mao}(G) := \text{meo}(\text{Aut}(G))$ and $\text{maff}(G) := \text{meo}(\text{Aff}(G))$.

2 Finite dynamical systems and finite dynamical groups

This section gives a quick overview on some basic concepts and facts which we will need.

**Definition 2.1.** (1) A **finite dynamical system** (FDS) is a finite set $S$ together with a function $f : S \to S$, a so-called self-transformation of $S$. It is called **periodic** if and only if $f$ is bijective.

(2) For FDSs $(S,f)$ and $(T,g)$, an (FDS) **homomorphism** between $(S,f)$ and $(T,g)$ is a function $\eta : S \to T$ such that $\eta \circ f = g \circ \eta$. The **image** of $\eta$, denoted by $\text{im}(\eta)$, is the FDS $(\eta[S],g|_{\eta[S]})$. An (FDS) **isomorphism** is a bijective FDS homomorphism.

(3) If $(S_1,f_1), \ldots, (S_r,f_r)$ are FDSs, their (FDS) **product** is defined as the FDS $(S_1 \times \cdots \times S_r, f_1 \times \cdots \times f_r)$, where $f_1 \times \cdots \times f_r$ maps $(s_1, \ldots, s_r)$ to $(f_1(s_1), \ldots, f_r(s_r))$.

[8] Sections 1–3 provides an introduction to the theory of FDSs. We will only need the following proposition summarizing some elementary facts on cycle lengths in FDS products:

**Proposition 2.2.** Let $(S_1,f_1), \ldots, (S_r,f_r)$ be periodic FDSs.

(1) The cycle length of $(s_1, \ldots, s_r) \in S_1 \times \cdots \times S_r$ under $f_1 \times \cdots \times f_r$ is the least common multiple of the cycle lengths of the $s_i$ under $f_i$.

(2) If each $f_i$ has a regular cycle, then so does $f_1 \times \cdots \times f_r$. $\square$
Corollary 2.4. Let \((G, \alpha)\) be a periodic FDG such that \(\lambda(\alpha) > \frac{1}{2}\), and let \((Q, \beta)\) be a homomorphic image of it. If \(\text{ord}(\beta) = 1\), then the group \(Q\) is trivial.

**Proof.** Note that \(\text{ord}(\beta) = 1\) implies that \(g_1^{-1}g_2 \in \ker \eta\) whenever \(g_1, g_2 \in G\) lie on the same cycle of \(\alpha\). Since \(\alpha\) has a cycle of length greater than \(\frac{1}{2}|G|\) by assumption, it follows that \(|\ker \eta| > \frac{1}{2}|G|\), whence \(\ker \eta = G\) by Lagrange's theorem, and we are done. \(\square\)

### 3 On the quotient \(\text{mao}(G)/\Lambda(G)\)

The sole purpose of this section is to prove the following:

**Proposition 3.1.** \(\sup_G \text{mao}(G)/\Lambda(G) = \infty\), where \(G\) ranges over finite groups.

**Proof.** Fix \(C \in \mathbb{N}^+\). Denote by \(p_1, \ldots, p_{3C}\) the first \(3C\) odd primes in increasing order. For \(i = 1, \ldots, 3c\), set \(B_i := \mathbb{Z}/p_i\mathbb{Z}\), and let \(B := \prod_{i=1}^{3C} B_i\). Observe that those automorphisms of \(B\) that act by inversion on precisely one of the \(B_i\) and identically on the others generate an elementary abelian \(2\)-subgroup \(E \leq \text{Aut}(B)\) of \(\mathbb{F}_2\)-dimension \(3C\). Let \(\psi : \mathbb{F}_2^{3C} \to E\) denote the \(\mathbb{F}_2\)-isomorphism mapping a vector \(v\) to the automorphism \(\psi(v) =: \alpha_v\) of \(B\) acting identically on \(B_i\) if the \(i\)-th component of \(v\) is \(0\), and otherwise by inversion.

Consider the 2-dimensional subspace \(U\) of \(\mathbb{F}_2^{3}\) spanned by the vectors \((0, 1, 1)^t\) and \((1, 0, 1)^t\) together with the inclusion map \(i : U \hookrightarrow \mathbb{F}_2^3\). Form an external direct sum \(U_C\) of \(C\) copies of \(U\). The product of \(C\) copies of \(i\) (in the sense of Definition 3.4) is an embedding \(i' : U_C \hookrightarrow \oplus_{i=1}^C \mathbb{F}_2^3\). Furthermore, there is an isomorphism \(\sigma : \oplus_{i=1}^C \mathbb{F}_2^3 \to \mathbb{F}_2^{3C}\) mapping the \(i\)-th standard basis vector, \(i = 1, 2, 3\), of the \(j\)-th summand, \(j = 1, \ldots, C\), of the source to the \((3(j - 1) + i)\)-th standard basis vector of \(\mathbb{F}_2^{3C}\).

Let \(W_C\) denote the image of \(U_C\) under the embedding \(\sigma \circ i'\) into \(\mathbb{F}_2^{3C}\), and let \(V_C \subseteq E\) denote the image of \(W_C\) under \(\psi\). For \(i = 1, \ldots, 3C\), denote by \(\pi_i : \mathbb{F}_2^{3C} \to \mathbb{F}_2\) the projection onto the \(i\)-th component. Observe that \(W_C\) (resp. \(V_C\)) has the following two properties:

(i) For each \(i = 1, \ldots, 3C\), there exists \(v \in W_C\) such that \(\pi_i(v) = 1\). Hence for each \(i = 1, \ldots, 3C\), there exists \(\alpha_i \in V_C\) acting by inversion on \(B_i\).

(ii) For each \(v \in W_C\), \(\pi_i(v) = 0\) for at least \(C\) values of \(i \in \{1, \ldots, 3C\}\). Thus each \(\alpha_i \in V_C\) acts identically on at least \(C\) of the \(B_i\). Let \(G_C\) be the subgroup of Hol\((B) = B \rtimes \text{Aut}(B)\) generated by \(B\) and \(V_C\); then \(G_C = B \times V_C\). We will be done once we have showed that \(\text{mao}(G_C)/\Lambda(G_C) \geq 2^{C-1}\).
Let $\xi$ be an automorphism of $G_C$. Since the only elements of order $p_i$ in $G_C$ are the nontrivial elements of $B_i$, each $B_i$ (and hence $B$) is $\xi$-invariant; fixing a nontrivial element $b_i \in B_i$, we can write $\xi(b_i) = b_i^{k_i}$ with $k_i \in (\mathbb{Z}/p_i\mathbb{Z})^\ast$. Furthermore, since elements from different cosets of $B$ in $G_C$ act identically on different collections of the $B_i$, $\xi$ restricts to a permutation on each coset of $B$. Hence for studying the dynamics of $\xi$, we can partition $G_C$ into the cosets of $B$ and study the dynamics on each coset.

Let $\alpha_v \in \mathcal{V}_C$. Observe that if $b$ is any element of $B$ having nontrivial $B_i$-component, where $i$ is such that $\pi_i(v) = 0$, then $b\alpha_v$ does not have order 2. Hence we can write $\xi(\alpha_v) = \prod_{i=1}^{3C} b_i^{l_i} \alpha_v$ with $l_i \in \mathbb{Z}/p_i\mathbb{Z}$ and $l_i = 0$ if $\pi_i(v) = 0$. It is not difficult to verify that the map $B\alpha_v \rightarrow B, b\alpha_v \rightarrow b$, is an isomorphism between the FDSs $(B\alpha_v, \xi|_{B\alpha_v})$ and the FDS given by $B = \prod_{i=1}^{3C} \mathbb{Z}/p_i\mathbb{Z}$ together with the product of the affine self-transformations $A_{i, k_i}$ of the $\mathbb{Z}/p_i\mathbb{Z}$ given by $x \mapsto k_ix + l_i$. Each $A_{i, k_i}$ has a regular cycle (so that $\xi|_{B\alpha_v}$ has a regular cycle by Proposition 2.2.2), and the order of $A_{i, k_i}$ equals the order of $k_i \in (\mathbb{Z}/p_i\mathbb{Z})^\ast$ (which is a divisor of the even number $p_i - 1$) if $k_i \neq 1$, and otherwise, it equals the order of $l_i \in \mathbb{Z}/p_i\mathbb{Z}$ (which is a divisor of $p_i$). Hence we always have $\text{ord}(A_{i, k_i}) \leq p_i$, and for those $i$ where $\pi_i(v) = 0$, the order of $A_{i, k_i}$ is a divisor of $p_i - 1$. Since there are at least $C$ such $i$ by property (ii) above, this implies that the order (or largest cycle length) of $\xi|_{B\alpha_v}$ is bounded from above by $\frac{1}{2C} \prod_{i, \pi_i(v) = 0} (p_i - 1) \prod_{i, \pi_i(v) = 1} p_i \leq \frac{1}{2C} \prod_{i=1}^{3C} p_i$.

On the other hand, considering the inner automorphism $\xi$ of $G_C$ with respect to the element $b_1 \cdots b_{3C}$, $\xi$ fixes each $b_i$ (so that $k_i = 1$ for all $i$ in the above notation), and $\xi(\alpha_v) = \prod_{i, \pi_i(v) = 1} b_i^{l_i} \alpha_v$ for $v \in \mathcal{V}_C$. In view of the above observations, this implies that every cycle length of $\xi$ is a product of some of the $p_i$, and by property (i) above, each $p_i$ occurs as a divisor of some cycle length. Hence $\text{ord}(\xi) = \prod_{i=1}^{3C} p_i$. It follows that $\text{mao}(G_C)/\Lambda(G_C) \geq \prod_{i=1}^{3C} p_i/\left(\frac{1}{2C} \prod_{i=1}^{3C} p_i\right) = 2^{C-1}$.

Remark 3.2. As mentioned earlier, the groups $G_C$ described in the proof of Proposition 3.1 are generalizations of an example given by Horoševskii, see [4], remarks after Corollary 1; Horoševskii’s example is our group $G_1$.

4 On the functions $\text{mao}_\text{rel}$ and $\text{mafo}_\text{rel}$

In this section, we study the functions assigning to each finite group the quotient of its maximum automorphism (resp. bijective affine map) order by the group order.

We start with a very simple general lemma:

**Lemma 4.1.** Let $f$ be a function from the class $G_{\text{fin}}$ of finite groups to the interval $(0, \infty)$ such that $f(G_1) = f(G_2)$ whenever $G_1 \cong G_2$ and $f(G/\text{Rad}(G)) \geq f(G)$ for all finite groups $G$. Furthermore, assume that for finite semisimple groups $H$, $f(H) \rightarrow 0$ as $|H| \rightarrow \infty$; more explicitly, fix a function $g : (0, \infty) \rightarrow (0, \infty)$ such that for any $\rho \in (0, \infty)$, $f(H) < \rho$ whenever $H$ is a finite semisimple group with $|H| > g(\rho)$.

Then for any $\rho \in (0, \infty)$, if $G$ is a finite group such that $f(G) \geq \rho$, then $|G : \text{Rad}(G)| \leq g(\rho)$.

**Proof.** By assumption, we have $f(G/\text{Rad}(G)) \geq f(G) \geq \rho$. Since $G/\text{Rad}(G)$ is semisimple, this implies $|G : \text{Rad}(G)| = |G/\text{Rad}(G)| \leq g(\rho)$ by choice of $g$. □
Lemma 4.1 summarizes our original approach to prove the weaker versions of Theorems 1.1.1(3) and 1.1.3(2) with cycle lengths instead of orders. Indeed, on the one hand, we observed that it follows from [1, Lemma 2.1.4] that \( \lambda_{\text{aff}}(G/N) \geq \lambda_{\text{aff}}(G) \) for any finite group \( G \) and \( N \) char \( G \) (implying the first assumption \( f(G/\text{Rad}(G)) \geq f(G) \) for these two \( f \)). On the other hand, assume that for some function \( f : \mathcal{G}^{\text{fin}} \to (0, \infty) \), we have \( |H| \cdot f(H) \leq |H|^{e} \) for some \( e \in (0, 1) \) and all finite semisimple groups \( H \). Then clearly, if \( p \in (0, 1) \) and \( H \) is a finite semisimple group such that \( |H| > p^{1/(e-1)} \), then \( f(H) < p \), whence \( g(\rho) \) from Lemma 4.1 can be chosen as \( p^{1/(e-1)} \). By [2, Lemma 3.4], we have \( \Lambda(H) \leq |H|^{\log_{60}(6)} \) and \( \Lambda_{\text{aff}}(H) \leq |H|^{\log_{60}(50)} \) for all finite semisimple groups \( H \), thus explaining the exponents in [2, Theorem 1.1.1].

Moreover, we know by [2, Theorem 2.2.3] that \( \text{maffo}(H) = \Lambda(H) \) and \( \text{maffo}(H) = \Lambda_{\text{aff}}(H) \) for all finite semisimple groups \( H \). Hence by Lemma 4.1 and the remarks from the last paragraph, Theorems 1.1.1(3) and 1.1.3(2) are clear once we have proved the following:

**Lemma 4.2.** Define functions \( \text{mao}_{\text{rel}}, \text{maffo}_{\text{rel}} : \mathcal{G}^{\text{fin}} \to (0, \infty) \) by \( \text{mao}_{\text{rel}}(G) := \frac{1}{\log |G|} \text{mao}(G) \) and \( \text{maffo}_{\text{rel}}(G) := \frac{1}{\log |G|} \text{maffo}(G) \). Then \( \text{mao}_{\text{rel}}(G/\text{Rad}(G)) \geq \text{mao}_{\text{rel}}(G) \) and \( \text{maffo}_{\text{rel}}(G/\text{Rad}(G)) \geq \text{maffo}_{\text{rel}}(G) \) for all finite groups \( G \).

Before proving Lemma 4.2, we show:

**Lemma 4.3.** Let \( B \) be a finite elementary abelian group, and fix \( \beta \in \text{Aut}(B) \). Then \( \text{lcm}_{x \in B} \text{ord}(A_{x, \beta}) \leq |B| \).

*Proof.* For \( x \in B \), let \( \text{sh}_{\beta}(x) := x \beta(x) \cdots \beta^{\text{ord}(\beta)-1}(x) \in B \). We observed in [2] that \( \text{ord}(A_{x, \beta}) = \text{ord}(\beta) \cdot \text{ord}(\text{sh}_{\beta}(x)) \). Hence the least common multiple in question is either equal to \( \text{ord}(\beta) \), which is bounded from above by \( |B| \) by [3, Theorem 2], or to \( p \cdot \text{ord}(\beta) \), where \( p \) is the prime base of \( |B| \). Hence assume, for a contradiction, that some \( \text{sh}_{\beta}(x) \) is nontrivial and that \( \text{ord}(\beta) > \frac{1}{p} |B| \). Considering the primary rational canonical form of \( \beta \) as an \( \mathbb{F}_p \)-automorphism (corresponding to a decomposition of \( B \) into a maximal number of subspaces that are cyclic for \( \beta \)), we may assume by induction that \( \beta \) can be represented by the companion matrix of \( P(X)^k \) for some irreducible \( P(X) \in \mathbb{F}_p[X] \). Note that all \( \text{sh}_{\beta}(x) \) are fixed points of \( \beta \), and so \( \beta \) has a nontrivial fixed point by assumption. This implies that for some nonzero \( Q(X) \in \mathbb{F}_p[X] \) of degree less than \( \deg(P(X)^k) \), we have \( X \cdot Q(X) \equiv Q(X) \pmod{P(X)^k} \), or equivalently \( P(X)^k \mid Q(X) \cdot (X - 1) \). Since \( P(X)^k \mid Q(X) \), it follows that \( P(X) \mid X - 1 \), and thus \( P(X) = X - 1 \) by irreducibility. In view of the formula for the order of the companion matrix of \( P(X)^k \) (first proved by Elspas [3, Appendix II, 9], see also [10, Theorem 3.11] and [3, Theorem 5 and remarks afterward]), it follows that \( \text{ord}(\beta) = p^{\lceil \log_p(k) \rceil} \leq p^{k-1} = \frac{1}{p} |B| \), a contradiction. \( \square \)

*Proof of Lemma 4.2.* We only prove that \( \text{maffo}_{\text{rel}}(G/\text{Rad}(G)) \geq \text{maffo}_{\text{rel}}(G) \), as the argument for \( \text{mao}_{\text{rel}}(G) \) is similar. The proof is by induction on \( |\text{Rad}(G)| \). For the induction step, fix \( A = A_{x, \alpha} \in \text{Aff}(G) \) such that \( \text{ord}(A) = \text{maffo}(G) \). Following the argument in [9, proof of Theorem 2], we may fix a minimal \( \alpha \)-invariant elementary abelian normal subgroup \( B \) of \( G \). By the induction hypothesis, it is sufficient to show
that \( \text{maffo}_{\text{rel}}(G/B) \geq \text{maffo}_{\text{rel}}(G) \). Denoting by \( \tilde{A} = A_{\pi(x)_{\alpha}} \) (where \( \pi : G \to G/B \) is the canonical projection and \( \tilde{\alpha} \) the induced automorphism on \( G/B \)) the induced affine map of \( G/B \), we find that by \cite[Lemma 2.1.4]{1}, every cycle length of \( A \) is a product of some cycle length of \( \tilde{A} \) with some cycle length of a bijective affine map of \( B \) of the form \( A_{b,\alpha_{\beta}}. \) Hence the order of \( A \) divides the product of \( \text{ord}(\tilde{A}) \) with \( \text{lcm}_{B\in B}(A_{b,\alpha_{\beta}}) \). In particular, by \cite[Lemma 1.1.1(3)]{1} \( \text{ord}(A) \leq \text{ord}(\tilde{A}) \cdot |B| \). It follows that \( \text{maffo}_{\text{rel}}(G) = \frac{1}{|B|} \text{ord}(A) \leq \frac{1}{|G/B|} \text{ord}(\tilde{A}) \leq \text{maffo}_{\text{rel}}(G/B). \)

5 Proof of the main results

As explained in Section 4, Theorems 1.1.1(3) and 1.1.3(2) follow from Lemmata 4.1 and 4.2 as well as the remarks between them, and deriving Theorem 1.1.1(2) (resp. 1.1.3(1)) from Theorem 1.1.1(3) (resp. 1.1.3(2)) is like in \cite[proof of Corollary 1.1.2, Section 3]{2}. Hence it only remains to prove Theorem 1.1.1(1).

Fix an automorphism \( \alpha \) of \( G \) such that \( \text{ord}(\alpha) > \frac{1}{2}|G| \). We prove that \( G \) is abelian by induction on \( |G| \). For the induction step, observe that \( G \) cannot be semisimple, since otherwise, by \cite[Theorem 1]{1}, \( \alpha \) would have a regular cycle and hence \( G \) would be abelian by \cite[Theorem 1.1.7]{1}, contradicting its semisimplicity.

Like in the proof of Lemma 4.2, following the argument in \cite[proof of Theorem 2]{5}, we fix a minimal \( \alpha \)-invariant elementary abelian normal subgroup \( B \) of \( G \). We may of course assume that \( B \) is proper in \( G \). Denote by \( \tilde{\alpha} \) the induced automorphism of \( \tilde{G}/B \), set \( m := \text{ord}(\tilde{\alpha}) \), \( n := \text{ord}(\alpha_{B}) \) and denote by \( C \) the set of fixed points of \( \alpha^{m} \) in \( B \). Horoševskiĭ proceeded to show that either \( C = \{1\} \) or \( C = B \) (by minimality of \( B \)) and to derive upper bounds for \( \text{ord}(\alpha) \) in both cases, which imply that \( \text{ord}(\alpha) \leq m \cdot |G/B| \) in any case and thus \( m \geq \text{ord}(\alpha)/|G/B| = |B| \cdot \text{ord}(\alpha)/|G| > \frac{1}{2}|B| \), whence \( G/B \) is abelian by the induction hypothesis.

In particular, we have \( G' \leq B \) and \( \lambda(\tilde{\alpha}) > \frac{1}{2} \) by \cite[Corollary 1]{5}. Consider the homomorphism \( \varphi : G \to \text{Aut}(B) \) corresponding to the conjugation action of \( G \) on \( B \). Since \( B \) is abelian, we have \( B \leq \ker(\varphi) \), and so there is a homomorphism \( \overline{\varphi} : G/B \to \text{Aut}(B) \) such that \( \overline{\varphi} \circ \pi_{B} = \varphi \), where \( \pi_{B} : G \to G/B \) is the canonical projection.

Now the kernel of \( \overline{\varphi} \) consists by definition of those \( \pi_{B}(g) \in G/B \) such that \( gB \subseteq C_{G}(B) \). Clearly, since \( B \) is \( \alpha \)-invariant, so ist \( C_{G}(B) \), and thus \( \ker(\overline{\varphi}) \) is \( \tilde{\alpha} \)-invariant. It follows that there exists an automorphism \( \overline{\alpha} \) on the image \( \overline{\varphi}(G/B) \leq \text{Aut}(B) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
G/B & \xrightarrow{\overline{\alpha}} & G/B \\
\overline{\varphi} \downarrow & & \downarrow \overline{\varphi} \\
\overline{\varphi}(G/B) & \xrightarrow{\overline{\alpha}} & \overline{\varphi}(G/B)
\end{array}
\]

In other words, the FDG \((\overline{\varphi}(G/B), \overline{\alpha})\) is the image of the FDG \((G/B, \tilde{\alpha})\) under the FDG homomorphism \( \overline{\varphi} : (G/B, \tilde{\alpha}) \to (\overline{\varphi}(G/B), \overline{\alpha}) \). By this definition of \( \overline{\alpha} \), it is
clear that $\text{ord}(\overline{\alpha}) | \text{ord}(\tilde{\alpha}) = m$.

We give an alternative definition of $\overline{\alpha}$. The element $\overline{\alpha}(gB) \in \overline{\alpha}(G/B)$, which is by definition the restriction of conjugation by $g$ to $B$, is mapped by $\overline{\alpha}$ to $\overline{\alpha}(\overline{\alpha}(gB)) = \overline{\alpha}(\alpha(g)B)$, which is the restriction of conjugation by $\alpha(g)$ to $B$. But this implies that $\overline{\alpha}$ is the restriction of conjugation by $\alpha|_B$ in $\text{Aut}(B)$ to its subgroup $\overline{\alpha}(G/B)$. In particular, $\text{ord}(\overline{\alpha}) | \text{ord}(\alpha|_B) = n$.

We now distinguish two cases. First, assume that $B$ is cyclic. Then $\text{Aut}(B)$ is abelian, and so by the second definition of $\overline{\alpha}$, it is clear that $\overline{\alpha} = \text{id}_{\overline{\alpha}(G/B)}$. By Corollary 2.3, this implies that $\overline{\alpha}$ is the trivial homomorphism $G/B \to \text{Aut}(B)$, and by definition of $\overline{\alpha}$, this just means that $B \leq \zeta G$. In particular, we have $G' \leq \zeta G$, whence $G$ is nilpotent of class 2. By [3, Corollary 1], this implies that $\lambda(\alpha) = \text{ord}(\alpha) > \frac{1}{2}|G|$, and so $G$ is abelian by [1, Theorem 1.1.7].

Now assume that $B \cong (\mathbb{Z}/p\mathbb{Z})^n$ for some prime $p$ and $n \geq 2$. By the argument in [9, proof of Theorem 2], if $C = B$, we have $\text{ord}(\alpha) \leq m \cdot p \leq |G/B| \cdot \frac{1}{p}|B| = \frac{1}{p}|G|$, a contradiction. Hence $C = \{1\}$, whence by [3, Lemma 3a], we have $\text{ord}(\alpha) = \text{lcm}(m,n)$. If $\gcd(m,n) > 1$, it follows that $\text{ord}(\alpha) \leq \frac{1}{2} \cdot m \cdot n \leq \frac{1}{2} \cdot |G/B| \cdot |B| \leq \frac{1}{2}|G|$, a contradiction. Therefore, $\gcd(m,n) = 1$, which implies that $\text{ord}(\overline{\alpha}) = 1$. Now repeat the argument from the first case to conclude the proof.

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