Variations on stability

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Abstract

We explore the effects of non-abelian dynamics of D-branes on their stability and introduce Hitchin-like modifications to previously-known stability conditions. The relation to brane-antibrane systems is used in order to rewrite the equations in terms of superconnections and arrive at deformed vortex equations.
1 Introduction and Discussion

The non-abelian nature of D-branes is one of the key features of their modern understanding. The connection with non-abelian gauge theories has served as a natural basis for many explorations. In this paper we concentrate on the effects of non-abelian dynamics of D-branes on their BPS stability.

It is well-known that stability plays an essential role in the study of moduli spaces of BPS states. To see this, it is already sufficient to look at the limit in which the volume of the ambient manifold is large; in that limit, branes are described by specifying a (non-trivially) embedded manifold in the spacetime and a gauge connection on it, or to put shortly supersymmetric cycle. Typically, in the study of BPS conditions, one first gets various types of conditions on the cycles, such as being holomorphically embedded or being a special lagrangian submanifold. In addition to that, one gets equations which involve the curvature $F$ of the gauge connection $A$ on the brane. These equations, which in the holomorphic case are the Hermitian-Yang-Mills equations, do not always admit a solution. However one typically is able to formulate a condition for having solutions on the connection $A$ in purely algebraic-geometric terms, involving some inequality on subbundles. This condition is called stability; it is nice that, in a sense, the mathematical definition has anticipated the physical meaning.

Away from the large volume situation, this situation will get corrections; the equations will be deformed, and in fact the whole geometrical interpretation of branes will lose its validity. Steps towards understanding these corrections have been done in two approaches. The first one is to redo the BPS analysis starting from the complete Dirac-Born-Infeld plus Chern-Simons action \cite{1}, and gives a deformation of the Hermitian-Yang-Mills equations:

\[ \text{Im}(e^{i\theta} e^{F+\omega})_{\text{top}} = 0, \]  

(1.1)

where $\omega$ is the Kähler form, and $\theta$ is a phase. The second, more abstract, approach, which avoids the difficulty of finding correct equations, directly introduces the notion of a deformed stability condition \cite{2}.

We recall now that in earlier days of D-branes considerable attention was paid to yet another type of modifications to Yang-Mills equations involving scalars (see, e.g., \cite{3, 4} for context closely related to ours). These equations – the Hitchin equations, or rather generalized Hitchin equations as they do not have to be considered only in two dimensions – also lead to a certain notion of stability. The recent progress in understanding the non-abelian dynamics of D-branes \cite{5} warrants revisiting stability conditions with intention of incorporating the scalars. The resulting modification of stability while rather mild for the whole BPS moduli space, may turn out to be rather dramatic for certain individual cycles. In particular we will see that some previously discarded cycles may get rehabilitated under the modified stability conditions.

The connection between transverse scalars and tachyons (superconnections) is natural and intuitive – both serve the purpose of “localization”; more precisely, we will argue that the tachyon can be expressed in terms of the characteristic polynomial of the transverse scalars (see eq. (5.6) below). This connection is a key to a better conceptual understanding of the generalizations of (1.1). These can be compactly summarized as deformed vortex equations

\[ \text{Im} \left( \text{Sym} \{ \exp \left( \begin{array}{cc} i\theta_1 \text{Id}_{N_1} & 0 \\ 0 & i\theta_2 \text{Id}_{N_2} \end{array} \right) \right\} [e^{[\bar{D},\bar{D}]} e^{\omega+J}]_{\text{top}} \right) = 0 \]

in terms of a sort of holomorphic “contracting” superconnection $\bar{D}$ ($\bar{D}$), and of a real two-form $J$ introduced for bookkeeping, as we will explain in section \cite{5}.

The deformed equations we are studying here would seem to suggest, along the lines of \cite{3, 4}, the presence of an integrable system on the cotangent bundle to the moduli space of Π-stable branes, giving
thus a deformation to the so-called Hitchin integrable system [7]; this could have some contact with [8], but we will not try address this question here.

A brief outline of the paper is the following. Section 2 is a more technical introduction, containing a brief review of the deformed Hermitian Yang-Mills equation and Hitchin equations. A generalization of the former in the spirit of the latter (or vice versa) is presented in section 3. The interpretation of the new equations and their implications for stability are discussed in section 4. Finally, in section 5 we explore the relation with brane-antibrane systems and with superconnections.

2 Preliminaries

Let us start with a simple example. Consider super Yang-Mills in \( d = 4 \) with \( \mathcal{N} = 4 \). In this case, one can easily verify that the condition for preserving half of supersymmetry, if the transverse scalars \( X \) are set to their vacuum values, is the instanton equation \( F = \ast F \). In complex coordinates \( z_1, z_2 \), this reads

\[
F_{1\bar{1}} + F_{2\bar{2}} = c \text{ Id}, \quad F_{12} = 0
\]  

(2.1)

where \( F_{ij} \equiv F_{z_i,\bar{z}_j}, F_{i\bar{j}} \equiv F_{z_i,\bar{z}_j} \), and another equation \( F_{1\bar{2}} = 0 \) follows from \( A \) being antihermitian. Let us now consider the T-duals of these equations. The general procedure is the same as that of dimensional reduction, and reads \( D_i \to X_i \): a connection becomes an endomorphism (a matrix). More precisely, this rule, for T-duality, means that we are but rewriting the covariant derivatives as infinite dimensional matrices; then, the expression obtained in this way will be true also in the case in which matrices are finite dimensional. As mentioned, the outcome of this reasoning is the same as that of dimensional reduction. If T-duality is in the \( z_2, \bar{z}_2 \) directions, the result is

\[
F_{1\bar{1}} + [X, X^\dagger] = c \text{ Id}, \quad D_1 X = 0
\]

(2.2)

where we have called \( X_2 \equiv X \). These equations can be obtained also by considering super Yang-Mills in \( d = 2 \) and with \( \mathcal{N} = (8,8) \), which is indeed a dimensional reduction, or T-dual, of \( d = 4, \mathcal{N} = 4 \), and looking for solutions which preserve half supersymmetry with one complex scalar turned on. For these reasons they have already been argued to be relevant in several physical situations [3, 8, 10, 11, 12, 13]; on the mathematical side, they have been studied by Hitchin [6].

One can look for more general BPS conditions in super Yang-Mills theories starting instead from the \( d = 10, \mathcal{N} = 1 \) case, keeping all the complex scalars on:

\[
F_{11} + \ldots + F_{55} = c \text{ Id}, \quad F_{ij} = 0.
\]

(2.3)

Again, reducing these to lower dimensions gives equations involving complex scalars in the adjoint: for instance in 4 dimensions one gets

\[
F_{11} + F_{22} + [X_1, X_1^\dagger] + [X_2, X_2^\dagger] + [X_3, X_3^\dagger] = c \text{ Id}; \quad F_{12} = 0; \\
D_i X = 0, \quad i = 1, 2; \quad [X_a, X_b] = 0, \quad a, b = 1, 2, 3.
\]

(2.4)

These are a modification of the usual self-duality conditions (2.1) in 4 dimensions.

These super Yang-Mills theories are known to describe the dynamics of flat branes in \( \mathbb{R}^{10} \) in zero slope limit. One can wonder how does the situation change away from this limit. Let us first start from the abelian case. There, one knows that the effective action becomes the Dirac-Born-Infeld; putting this together with the Chern-Simons term, describing coupling to RR fields, one obtains an action whose
BPS conditions can be studied [1]. To describe the result, let us first covariantize the equations we have obtained so far in the case without scalars. All of them can be written in the form

\[ F \wedge \frac{\omega^{n-1}}{(n-1)!} = c \frac{\omega^n}{n!} \text{Id}, \quad F^{(2,0)} = 0, \]  

(2.5)

where \( n = d/2 \) is the complex dimension of the brane we are considering, and \( c \) is a constant. In this form, they are known as Hermitian-Yang-Mills (HYM), or Donaldson-Uhlenbeck-Yau [14], equations. The BPS conditions gotten from DBI + CS can be instead written in the compact form

\[ \text{Im}(e^{i\theta} e^{F+\omega})]_{\text{top}} = 0, \quad F^{(2,0)} = 0. \]  

(2.6)

The subscript \( \text{top} \) indicates that we only have to take the top-form part of the expansion \( e^{F+\omega} \). As one can see the second equation, which is the holomorphicity condition, is not changed. This agrees with the decoupling conjecture of [15] stating that for B-type branes the superpotential does not depend on Kähler moduli. The F-flatness condition \( F^{(2,0)} = 0 \) is almost trivial to reduce, and often we will not write it or its lower-dimensional descendants explicitly.

For instance, in 4 dimensions, first equation in (2.6) can be written as

\[ iF \wedge \omega = \tan(\theta) \left( \frac{1}{2} F^2 + \frac{1}{2} \omega^2 \right); \]  

(2.7)

here we have used the fact that \( F \) is anti-hermitian and so, in the abelian case, purely imaginary, and that \( \omega \) is real. The \((1,1)\) part of the equations (2.3) in 4 dimensions (the self duality equations) can be seen as a linearized in \( F \) version of (2.7), with \( c = -i/2 \tan \theta \).

Strictly speaking, the equations were derived only in the abelian case. In the non-abelian case, the action is not completely known. Nevertheless, (2.6) admit a very natural non-abelianization by treating \( F \) as a matrix, and putting an identity in the \( \omega^n \) part. This will clearly be well-defined, thanks to the transformation of the curvature under gauge transformation \( F \rightarrow UFU^\dagger \). In fact, we can write this non-abelianized version again in a compact form as

\[ \text{Im}(e^{i\theta} e^{F+\omega}\text{Id})]_{\text{top}} = 0; \]  

(2.8)

this has now to be read with a little caveat, namely that \( \text{Im}(\cdot) \) is now \( (\cdot) - (\cdot)^\dagger)/2i \) (this is because now \( F \) is not purely imaginary, but anti-hermitian). Practically, this means that for instance in 4 dimensions we get again the equations (2.7). Notice that the equations come automatically with a symmetrizer, due to the fact that they are written in terms of forms.

So far we have seen that:

- the BPS conditions in the zero slope limit, in which there is a super Yang-Mills description, are given by dimensional reductions of the 10 dimensional equations (2.3), which can be rewritten as \( F \omega^4 = c \omega^5 \) (from now on we will omit \( \wedge \) and Id if there is no danger of confusion).

- away from the zero slope limit, the deformation to these equations are known in all dimensions, and in particular in 10 dimensions.

It is natural, then, to assume that dimensional reduction of the deformed equations (2.8) gives the correct BPS conditions in lower dimensions with transverse complex scalars turned on. This will give a deformation of the Hitchin-like equations we have shown before to arise from super Yang-Mills, for instance (2.2) and (2.4).
Before we go on and find the general deformed equations with scalars, let us write also first equations in (2.2) and (2.4) in a covariant form. For (2.2) it is easier: a simple way is to write it is
\[ F - i[X, X^\dagger]\omega = c\omega. \] (2.9)

We could have as well made \( X \) a one form; this second choice is the version studied by Hitchin, and we will have more to say about this in section 4.1. As to the second, (2.4), there are more scalars in this case. Let us then introduce a transverse form \( \omega_\perp \), with which the indices of the \( X^a \) and of the \( \bar{X}^a \equiv X^{\dagger a} \), which are scalars in the spacetime but vectors in the transverse directions, can contract. This is a very natural thing to do, in light of the 10 dimensional origin of the equations. Then we can write (2.4) in the form
\[ F\omega + \frac{\omega^2}{2} (i_X + i_{\bar{X}^\dagger})^2\omega_\perp = c\omega^2, \] (2.10)

where \( i_X \equiv X^a i_{e_a}, i_{\bar{X}^\dagger} \equiv X^{\dagger a} i_{\bar{e}_a} \) are contractions with the holomorphic and antiholomorphic parts \( X, X^\dagger \) of the transverse scalars. Notice that \( i_X^2, i_{\bar{X}^\dagger}^2 \) are zero because they reproduce the commutators \( [X_a, X_b] \) and \( [X_a, \bar{X}_b] \); \( i_X i_{\bar{X}^\dagger} + i_{\bar{X}^\dagger} i_X \) then gives the desired combination of commutators. Notice that, in order to keep \( \omega_\perp \) real, we have chosen it to be of the form \( i\sum_a dx^a d\bar{x}^a \).

We confine ourselves to a holomorphic setup, and thus always consider the case when both the original equations and the reductions are even-dimensional. In principle we could have as well reduced an odd number of covariant derivatives. Even though we will not do this here let us notice that such reductions could also lead to interesting equations. For instance, the odd-codimension reductions of self-duality equations yields two important equations: Nahm equations in 0+1 dimension, and the Bogomol’nyi equations for monopoles in 2+1 dimensions. Another interesting situation involving an odd-dimensional reduction is the gauge theory on the coassociative cycles in manifolds of \( G_2 \) holonomy.

### 3 The equations

Let us now turn to the reduction of the equations (2.8). For illustrative purposes, we will first do the reduction from 4 dimensions to 2, which can be viewed as reducing from 10 to 2 but with only one complex scalar on. Then we will tackle the case with all the complex scalars on and reduce from 10 dimensions to 4. After that we will cast the result in a dimension-independent form analogous to that of (2.8). The most convenient four-dimensional starting point is in the form (2.7). Reducing this in 2 dimensions one gets
\[ \frac{1}{2} \{ F(-i)[X, X^\dagger] - i[X, X^\dagger]F - i(DX^\dagger\bar{D}X - \bar{D}X DX^\dagger) \} + \omega = i\tan(\theta)(F - i[X, X^\dagger]\omega); \] (3.1)

this is, as expected, a deformation (by the first three terms) of (2.4); these two equations are the same in linear order.

Coming now to the more laborious task of reducing from 10 to 4, we will not actually perform the reduction of the equations, but reduce instead the expression
\[ [e^{F+\omega\text{Id}}]_{\text{top}} = \frac{1}{5!}F^5 + \frac{1}{4!}F^4\omega + \frac{1}{3!2!}F^3\omega^2 + \frac{1}{2!3!}F^2\omega^3 + \frac{1}{4!}F\omega^4 + \frac{1}{5!}\omega^5 \]

and then restore the phase \( \theta \) at the end. To write down the result in 4 dimensions, we introduce some little more piece of notation. Let \( d_A \equiv D + \bar{D} \equiv dz^i D_i + d\bar{z}^j D_{\bar{j}}, \) where \( D_i, D_{\bar{j}} \) are the covariant derivatives. With the help of this we will write the result in a form halfway from an explicit and a contracted one, and then explain how to go on in either direction:
At first sight, this formula may appear a little strange since a non-homogeneous object is exponentiated. However, the present computation, apart from clarifying the meaning of such a formal expression, constitutes a non-trivial check since in the process of reduction, some forms become scalars. For instance, let us note that (3.2) reduces, in the case without scalars, to\[ e^{F + \omega} \text{Id} \] top = \(\frac{1}{2!} F^2 + F \omega + 1/2! \omega^2\), as it should. In fact the whole formula (3.2) can be rewritten in a nicer and more suggestive form:

\[
[e^{F + \omega \text{Id}}]_{\text{top}} = \frac{1}{2!} F^2 + F \omega + 1/2! \omega^2,
\]

or, alternatively:

\[
[e^{F + [D,i_x] + [i_x,\hat{D}]}]_{\text{top}} = \frac{1}{2!} F^2 + F \omega + 1/2! \omega^2.
\]

At first sight, this formula may appear a little strange since a non-homogeneous object is exponentiated. Note that formally this expression coincides with one obtained by reduction of \(F\) directly in the exponent of (2.8). However, the present computation, apart from clarifying the meaning of such a formal expression, constitutes a non-trivial check since in the process of reduction, some forms become scalars. For instance, from a similar formal argument (reduction of the covariant derivative) one can write the exponent of (3.3) in an even more compact expression, turning its two forms into respectively:

\[
[e^{\frac{1}{2}[D+\hat{D},i_x + i_{X_1}]}]_{\text{top}} = \frac{1}{2!} F^2 + F \omega + 1/2! \omega^2.
\]
if one understands the commutator in a super-sense: $i_X$ is treated as “fermionic” and the usual super-Lie bracket, which is an anticommutator on two fermions, is used. (Moreover, one should not forget the extra signs coming from the usual grading of forms: for instance $[A_1, A_2] = A_1A_2 + A_2A_1$.) This reminds one very much of the formalism and of the expressions appearing in the computation of Chern characters with superconnections [14], as already noted in [17] in the context of the modified D-brane Chern-Simons couplings [5]. Actually, at this point we just observe a similarity; tachyons and $X$ scalars, although related by tachyon condensation, are not quite the same object, of course. We will, nevertheless, come back to this later, arguing for the proper place of these similarities.

We can now conclude by putting back $\theta$, and writing the equations in the form

$$\text{Im} \left( e^{i\theta} \left[ e^{F + [D, i \chi]} + [i \chi, D] + (i \chi + i \chi^\dagger)^2 \epsilon \omega + \omega_\perp \right]_\text{top} \right) = 0. \quad (3.5)$$

Note that formally an expansion of (3.5) in $n$ dimensions contains terms of degree $n = n_\parallel + n_\perp$ in self-explanatory notation. Here we keep only the purely-longitudinal forms $n = n_\parallel$ ($n_\perp = 0$). The need of defining extra rules and the presence of $\omega_\perp$ in the equations is not particularly nice, however we find this form to be the most convenient for the analysis of the next section. In section 5 we will reinterpret this equation and write it in a more conceptual form.

4 Geometrical considerations

We have, so far, dealt with flat space and branes; but we have tried to write our formulas in the way most independent from this situation. So we can try to extrapolate the equations to more interesting geometrical cases; let us think of an ambient manifold $M$ which is a factor of the space in which string theory lives (we will mostly have in mind a Calabi-Yau threefold); the rest will be flat Minkowski space.

What we know is that, when one wraps several branes on a submanifold $B$ of the given ambient manifold $M$, the scalars defining transverse fluctuations of the brane – which, in the abelian case, would be sections of the normal bundle $N(B, M)$ – become also matrices; that is, they are now sections of $N(B, M) \otimes \text{End}(E)$, where $E$ is the gauge bundle [3]. To derive the equations for this more general case, one has to start from the analogue of (2.8) for a brane wrapping the whole Calabi-Yau. For the case of CY threefolds, this can be read in eq.(3.27)(a) in [1]: its imaginary part is still identical to the covariantization of its flat space counterpart, (2.8), that we have already seen. Then, when the Calabi-Yau is fibred in tori over the cycle, the same logic of T-duality as in flat space applies, and we get again our equation with scalars [3,5]. Once more we have to argue as in the flat space case: T-duality gives in principle only a rewriting of the equations, in which $D$’s are written as infinite dimensional $X$’s; but then the expression obtained in this way is the same for finite dimensional $X$. Moreover, in this case, we are supposing that the equations obtained in the case in which there is a fibration in tori will be correct even in the general case. The procedure is similar to that of [3, 4].

In principle one may imagine another method to get the equations with scalars, namely to try to directly non-abelianize relevant equations in [4]. However, this would be correct only if we had the complete $\kappa$-symmetric non-abelian action, which is not the case. So, this second method will not give the full equations; in particular, one cannot get terms with commutators $[X, X^\dagger]$.

We may also recall that the equations (2.4) were related in [4] to noncommutative Hermitian Yang-Mills equations via Seiberg-Witten map. The latter strictly speaking applies to abelian fields only. With a progress in finding a map applicable to more general situations it would be interesting so see if (3.3) can lead to a simple form of noncommutative Hitchin equations.

In principle the branes can be extended or not extended in the extra flat directions; in the latter case we can have also scalars describing fluctuations in these directions, but we will ignore these issues altogether in this section.
One could wonder what the piece with $\omega_\perp$ appearing in (3.3) is supposed to mean in this more general case. As we mentioned at the end of the section 3 due to the appearance in the equations of the combination $\omega + \omega_\perp$, we can substitute it more generally directly with (the pull-back of) $\omega_M$, the Kähler form on the ambient manifold $M$.

A point which deserves emphasizing is that now all the covariant derivatives we have been writing in the flat space case should not only be covariant with respect to the usual gauge part, but also contain a connection $a$ coming from the normal bundle $[18]$. This is simply because, as we noted, the $X$ are now sections of $\text{End}(E) \otimes N(B, M)$.

4.1 Twisting

The equations we have been writing so far were all covariantized keeping $X$ as scalars, as required by physics. However, as we noticed after (2.9), mathematically there would have been in principle another possibility, that of making $X$ a form. We will see here that this corresponds in fact to the possibility of twisting the supersymmetric brane theory. The basic idea comes from [3]: consider the case in which the ambient manifold $M$ is a K3, and $B$ is a divisor in it. As we have said, $X$ are sections of the normal bundle tensor the matrix part, $N(B, M) \otimes \text{End}(E)$; but in this case, due to adjunction formula, we have $N(B, M) = K_B$, the canonical bundle, which is in this case nothing but the cotangent $\Omega_B$. So the $X$ gets substituted in this case by a antiholomorphic one form (that is, a $(0, 1)$ form annihilated by the holomorphic covariant derivative: $D\phi = 0$) with values in $\text{End}(E)$, and the equation (2.2) can be written in the Hitchin form $F + [\phi, \phi^\dagger] = c\omega$.

Now, it is clear that we can do the same trick when $M_{n+1}$ is a higher dimensional Calabi-Yau and $B_n$ is again a divisor in it. In that case, the only change is that the canonical bundle is now not be the cotangent: $\phi$ is a top antiholomorphic form on $B$. In this way, for instance, equation (2.4) with one scalar can be covariantized in the form

$$F\omega + i[\phi, \phi^\dagger] = \frac{e_{\text{S}}^2}{2}.$$ 

It is now very natural to wonder whether this twisting can be applied also to our deformed equations (3.3). Let us stick to the cases that we have been considering so far, in which $B$ is a divisor in a Calabi-Yau $n + 1$-fold $M$. In this class of cases, there is only one complex scalar on. Thus, we can consider the easier equations obtained reducing from $n + 1$ dimensions to $n$. These equations can again be summarized by (3.3), but now remembering that the transverse space has dimension 2.

First of all, for terms like $[X, X^\dagger](e^{F+\omega})_{\text{top}}$ it is rather easy to see how we can manage the twisting. Since $X$ is now replaced by the two form $\phi$, the commutator is now $[\phi, \phi^\dagger]$, which is a top form. But there is really no great difference between a scalar and a top form, thanks to Hodge duality; so we can simply, for example, contract this top form with the rest and get a scalar equation $[\phi, \phi^\dagger] \cdot (e^{F+\omega})_{\text{top}}$. It is harder to understand the twisting of terms involving $D\phi$. The guiding principle in doing this is again Hodge duality, and a local (anti)holomorphic Hodge duality (see previous footnote). The result is that now, wherever $DX$ appeared, now we have to use the $(0, n - 1)$ form $D^\dagger \phi$, where $D^\dagger$ is the adjoint of the $D$ operator. In this way, we have that $D^\dagger \phi D^\dagger \phi^\dagger$ is a $(n - 1, n - 1)$, which can be contracted with the rest again to give a scalar equation. The whole formula becomes thus

$$\text{Sym}\{(e^{F+\omega})_{2n} - i[X, X^\dagger](e^{F+\omega})_{2n} + \frac{i}{2} \left(DXDX^\dagger - DX^\dagger DX\right)(e^{F+\omega})_{2n-2}\}$$

\*\*This is due to the fact that, reducing equations like $F_{ij} = 0$, we obtain actually antiholomorphic scalars $D_iX_{\bar{n}}$; see for instance (3.3).

\*\*If the divisor were a Calabi-Yau itself, this would be not really a replacing, but rather a rearranging of the equations using a holomorphic Hodge dual [19].

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\[
\rightarrow \text{Sym}\{s(e^{F+\omega})_{2n} + i^{n-1}[\phi, \phi^\dagger] \cdot (e^{F+\omega})_{2n} + \frac{i^{n-1}}{2}(\bar{D}^\dagger \phi D^\dagger \phi^\dagger + (-)^n D^\dagger \phi^\dagger \bar{D}^\dagger \phi) \cdot (e^{F+\omega})_{2n-2}\}.
\]

4.2 Stability

The existence of solutions to the type of equations we have been considering so far is usually equivalent to a mathematical concept of stability. The precise definition of this stability depends on the equation which one considers, but roughly it is always an inequality involving subbundles of the bundle on which the connection is defined. It is very easy to understand at least why such a condition is necessary. Let us start from the simple equation \( F_A = c_1 \omega \) in 2 dimensions (along with \( F^{(2,0)} \), as usual) where we have explicitly indicated that \( F_A \) is the curvature of a connection \( A \) on a bundle \( E \). Taking trace and integrating, we get \( c = (2\pi i)c_1(E)/(rk(E)Vol) \equiv (2\pi i)\mu(E)/Vol \). To avoid use of induction, let us consider the case in which \( rk(E) = 2 \). Suppose now there is a holomorphic subbundle (in this case it can only be a line bundle) \( L \hookrightarrow E \), with a connection \( A' \) on it (on a line bundle we can choose it to have constant curvature, like \( A \) has). The embedding \( s \) is a section of \( \text{End}(L, E) \); \( A' \) and \( A \) induce on this bundle a connection \( B \), and the condition that the subbundle be holomorphic can be explicitly expressed as \( \bar{D}Bs = 0 \) (here, as above, \( D \) denotes the holomorphic part of the covariant derivative). Finally, let us put a hermitian metric \((\cdot, \cdot)\) on the bundle \( \text{End}(L, E) \). We can now consider the equalities

\[
0 = \int \partial(DBs, s) = \int (\bar{D}Bs, s) - \int (Ds, DBs).
\]

Since, in the form notation we have been using so far, \( DB\bar{D}B + D\bar{D}DB = [DB, DB] = F_B^{(1,1)} = F_B \), and \( \bar{D}Bs = 0 \), we have

\[
\int (Ds, DBs) = \int (F Bs, s) = (2\pi i)\frac{\int (s, s)\omega}{Vol}(\mu(E) - \mu(L));
\]

but, since the (imaginary part of the) lhs is non negative, we have that if on \( E \) there is a holomorphic connection that satisfies \( F = c\omega \), \( E \) satisfies the following property, called \( \mu \)-semistability: for any holomorphic subbundle \( L \hookrightarrow E \), one has \( \mu(E) \geq \mu(L) \). Sufficiency of this property, in this and in the other cases discussed in the mathematical literature, can also be shown using moment maps in the infinite-dimensional space of connections. A similar, slightly stronger notion is \( \mu \)-stability, for which we simply substitute \( \geq \) with \( > \).

This simple equation is the lowest form of life in the zoo of equations we have been considering so far. We have now to consider several generalizations of this. First of all, we can go up in the dimension. The equation becomes now \( \text{Sym}^2(e^{F+\omega})_{2n} \), and the analogue of what we have seen above in 2 dimensions works perfectly; \( \mu \) is now \( \text{deg} / rk \), where \( \text{deg} = c_1 \cdot [\omega]^{n-1} \) depends now on the Kähler class \( [\omega] \).

Since in general the definition of \( \mu \)-(semi)stability involves checks of the inequality for all coherent subsheaves \( E' \) of rank 0 \(< rk(E') < rk(E) \), the argument above may seem to have a caveat. Fortunately it is sufficient to carry out the check for only so-called reflexive sheaves \( \mathcal{E} \), and the only such sheaves of rank one are line bundles. More generally, it is shown in \( \text{[21]} \) that HYM implies stability even in the stronger sense (that is, including subsheaves). One however needs stability with subsheaves in order to show that stability implies the existence of a HYM \( \mathcal{E} \). Low dimensions (two and four) are exception \( \text{[21]} \), and the definition of the stability used above is adequate. Finally, there are other more refined notions of stability; we have in mind Gieseker stability in particular. It is related \( \text{[22]} \) to an asymptotically large-volume analysis of deformed equation \( \text{[23]} \). One of the differences between Gieseker and \( \mu \) stability is that now subsheaves are not even required to have smaller rank; see for example \( \text{[21]} \) for a more detailed discussion. Similar considerations to the ones in this paragraph should be taken in account whenever we speak of subbundles.
Deformations

Now we turn to the deformations (2.8). These are expected to be related, modulo some corrections, with II-stability [2, 24]. The latter is a stability for branes very similar in spirit to the stabilities considered by mathematicians (the similarity can indeed be summarized using the powerful mathematical language of categories): for each subbrane $B'$ of a given brane $B$, the relation $\phi(B) \geq \phi(B')$ should be satisfied, where $\phi$ is, modulo some subtleties, the argument of the central charge of the brane. This stability interpolates between different known stabilities in various limiting points of the moduli space; in particular, in the limit in which branes can be considered as geometric objects (holomorphic cycles with bundles on them), the inequality with $\phi$ reduces to the inequality with $\mu$, and we recover the usual stability for bundles. This is clearly similar to the way in which (2.8) reduces to (2.5) in the limit of large $\omega$ (or small $F$). To be more precise, let us underline once again the reason for which $\mu$ is appearing in the usual stability. The reason is that $\mu$ is proportional to the constant $c$ appearing in HYM (2.5). Now, we have a constant in (2.8) as well: it is $\theta$. As in the HYM case, the way to uncover the relation of this with the topological constants is to take the trace of the equation and integrate. Here we get

$$\int \text{Im}(e^{i\theta} e^{F+\omega})_{\text{top}} =$$

$$\sin(\theta) \int (\omega^n + \omega^{n-2} \text{Tr} F^2 + \ldots) + \cos(\theta) \int (\omega^{n-1} \text{Tr}(-iF) + \omega^{n-3} \text{Tr}(-iF^3) + \ldots) =$$

$$\sin(\theta) \Re \int (\omega^n + \omega^{n-1} \text{Tr} F + \ldots) + \cos(\theta) \Im \int (\omega^n + \omega^{n-1} \text{Tr} F + \ldots) =$$

$$\text{Im} \left( e^{i\theta} \int e^{F+\omega} \right)$$

The equation (2.8) sets this to zero; thus we find that $\theta = \arg(\int e^{F+\omega})(+\pi ki)$. Since in the trivial geometries, this expression is the same as the central charge, this suggests that the corrections [25] to the equation in a nontrivial geometrical setting amount to introduction of an additional factor $e^{-F_a/2} \sqrt{A(R)/A(F_a)}$, where $a$ is the connection on the normal bundle, $F_a$ its curvature and $R$ the Ricci curvature on the brane. In this way we get $\theta = \arg \left( \int e^{F+\omega} e^{-F_a/2} \sqrt{A(R)/A(F_a)} \right) = \arg(Z) = \phi$. Unfortunately, it is now harder to reproduce the proof we saw for HYM due to the nonlinearity of equation (2.8). One of the central points of that proof was the appearance of $F$ from the anticommutator of two covariant derivatives; the analogue of this is not totally clear. The lack of supersymmetric completion for CS terms involving gravitational corrections is another obstacle in this direction. One could try a different approach by attempting to generalize the usual moment map method (see e.g. [26]) but this is beyond the scope of our work.

Transverse scalars

Let us come now to the main theme of this paper, the introduction of the transverse scalars. Foreseeing problems due to non-linearity of the equations, one can first introduce scalars in the “linearized” case, the HYM, and perform the analysis there. This should serve as a base for extending the modifications to the non-linear equations. To be specific, let us concentrate on the Hitchin equations in four dimensions

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In general, II-stability is actually defined in terms of distinguished triangles in the derived category [24]; it is however tempting to examine the possibility that also in other areas of the moduli space the analysis can be reduced to one which only involves a simpler abelian category, that is, one in which it still makes sense to talk about subobjects. These subobjects are the things that we will call subbranes in this paper.
and find a stability condition for this equation. We will denote by $\langle \cdot, \cdot \rangle$ the inner product $\int vol(\cdot, \cdot)$, with $vol$ the volume form, and make use of the Weitzenböck equalities [21]

$$\frac{1}{2} D_A^\dagger D_A = \partial_A^\dagger \partial_A - i \omega \cdot F_A .$$

As in the easier case of two dimensions and no scalars, discussed in the beginning of this subsection, we will choose for simplicity a rank 2 bundle $E$, and consider a sub-line-bundle $L$ on which, without loss of generality, we can put a connection $a$ with constant curvature. Applying a Hodge star (and changing $c$) we rewrite (2.10) in the form

$$i \omega \cdot F + \sum_{a=1}^{3} [X_a, X_a^\dagger] = c .$$

(4.1)

On the bundle $\text{Hom}(L, E)$ we can now consider again the connection $B$ induced by the connections $a$ and $A$ on $L$ and $E$. We can apply this covariant derivative again to the holomorphic section $s$ of $\text{Hom}(L, E)$ expressing the subbundle relation (the embedding):

$$\frac{1}{2} \langle D_B^\dagger D_B s, s \rangle = \langle \partial_B^\dagger \partial_B s, s \rangle - i \langle \omega \cdot F_B s, s \rangle .$$

Using adjunction (that is, integrating by parts), holomorphy of $s$ and (4.1) we have

$$2 \pi \frac{\langle s, s \rangle}{Vol} (\mu(E) - \mu(L)) = \langle D_B s, D_B s \rangle + \sum \langle [X_a, X_a^\dagger] s, s \rangle .$$

(4.2)

Suppose now that $X s = \lambda s$; this equation can be seen in a sense as looking for an “eigenvalue” $\lambda$ which is actually a section of $N(B, M)$. With a little abuse of language we will be calling $s$ an “eigenvector”. Then we have for the last term

$$\langle [X, X^\dagger] s, s \rangle = \langle X^\dagger s, X^\dagger s \rangle - |\lambda|^2 \langle s, s \rangle = \langle (X^\dagger - \bar{\lambda})s, (X^\dagger - \bar{\lambda})s \rangle .$$

Since this is non-negative, from (4.3) we get the condition we wanted: for any subbundle $L \hookrightarrow E$ which is $X$-invariant (namely, the embedding $s$ is eigenvector of all the $X_a$’s) we have the relation $\mu(E) \geq \mu(L)$. This is the four-dimensional analogue of Hitchin stability. We have here only shown that it is necessary for solution of (2.10) to exist; a proof of sufficiency is more difficult, but essentially standard along the lines of other linear examples, using orbits of the complexified gauge group and analytic arguments, as for example in [14, 1].

A similar case to this one, though with a different “twisting”, has been extensively studied in the mathematical literature under the name of Higgs bundles [27].

Moving now finally to our equations (3.3), it is very natural to propose a Hitchin-like modification of stability, in a more general abelian category (see previous footnote: what we need here is a category in which it makes sense to speak about exact sequences): for each $X$-invariant holomorphic subbrane $B'$ of a given brane $B$, one has $\phi(B) \geq \phi(B')$. The only step left is to specify what “$X$-invariant” is supposed to mean, since in general (away from geometrical limits in the moduli space) branes are not bundles on cycles (that is, we are in a different abelian category). For this, we notice that one can reformulate this notion in a completely abstract way by completing the embedding $B' \hookrightarrow B$ to the exact sequence

$$0 \to B' \xrightarrow{i} B \xrightarrow{\pi} B'' \to 0 .$$

Using the maps $i$ and $\pi$, one can see that being $X$-invariant means that $\pi \circ X \circ i = 0$ as an element of $\text{Hom}(B, B'')$, where $\circ$ is composition. We emphasize again that the status of the abelian category
in question is rather uncertain; all we know is that its objects should involve in some sense the cycles and their embeddings in $M$, or equivalently the normal bundles. The morphisms in this category should also contain these data, in order to match the known large-volume limit, in which the category is the one of sheaves and the morphism $X$ is a section of $\text{End}(E) \otimes N(B, M)$. Though this proposal is the natural melting of II-stability and of Hitchin’s one, let us stress that again, as for the proposed connection between II stability and $(2.8)$, a complete proof is very difficult to find due to the non-linearity of the equations.

We would like to underline that these modifications à la Hitchin to the usual stabilities is substantial. Once one fixes a cycle $B$, a bundle $E$ on it and an endomorphism $X$, it can happen in fact that even if $E$ was unstable with respect to the usual definition, it is stable with respect to the modified one. Let us analyze for example $\mu$-stability. Suppose that there exists only one holomorphic subbundle $E'$ which destabilizes $E$, $\mu(E') > \mu(E)$. Then, if this subbundle is not $X$-invariant, $E$ is still $\mu$-Hitchin-stable. (Or more accurately, the couple $(E, X)$ is $\mu$-Hitchin-stable.)

**Quivers and $\theta$-stability**

A logical consequence of this is also that the $\theta$-stability that one finds going to the Gepner point [2, 28] should be modified in a similar fashion. Let us give a quick look at this here. First of all, recall that, near the Gepner point of the moduli space, branes are described to some extent by the same approach describing branes on an orbifold singularity [29]. Thus one has a supersymmetric gauge theory whose gauge content is summarized by a quiver; here, however, we will not need this explicitly, and keep all of the chiral multiplets in a set of total chiral fields $\Phi$, whose blocks are the matrices which represent the quiver. Then, we can write the D-term and F-term equations which describe the moduli space of the theory as

$$\sum_a [\Phi_a, \Phi_a^\dagger] = \Theta \text{Id}; \quad \frac{\partial W}{\partial \Phi_a} = 0 \quad (4.3)$$

As in the geometrical limit, the F-term equation (the $(2, 0)$ part) is holomorphic and does not get modified by Kähler moduli. The equation that leads to a stability condition is again the single real equation for $\theta$-stability that one finds going to the Gepner point [2, 28]. The block-diagonal matrix $\Theta$ contains FI terms.

The condition for existence of solutions for these equations, again implies a stability condition. Indeed, consider a subrepresentation of the quiver. From the point of view of the total fields $\Phi_a$ (of rank, say, $k$), this means there is an injection $i$ such that smaller matrices $\Phi_a'$ of rank $k' < k$, satisfying $\Phi_a \circ i = i \circ \Phi_a'$ can be found (we will omit $\circ$ from now on). It is useful to introduce as well a matrix $i^\dagger$ such that $i^\dagger i = \text{Id}_{k'}$; then $ii^\dagger$ is a rank $k'$ projector $p$, and we can rewrite $\Phi_a' = i^\dagger \Phi_a i$. Having introduced such a notation, let us see what happens each time we have a subrepresentation $i$. Then we start from a quantity manifestly non negative and expand it:

$$0 \leq \sum_a \text{tr} \left( (p\Phi_a(1-p))(p\Phi_a(1-p))^\dagger \right) = \sum_a \text{tr} (p\Phi_a(1-p)\Phi_a^\dagger) = \sum_a \left( \text{tr} (p\Phi_a \Phi_a^\dagger) - \text{tr} (p\Phi_a p\Phi_a^\dagger) \right); \quad (4.4)$$

but then, being

$$\text{tr} (p\Phi_a p\Phi_a^\dagger) = \text{tr} (i\Phi_a i\Phi_a^\dagger) = \text{tr} (i\Phi_a i\Phi_a^\dagger i) = \text{tr} (i\Phi_a \Phi_a^\dagger i) = \text{tr} (i\Phi_a \Phi_a^\dagger i);$$

we can reexpress $(4.4)$ as

$$\sum_a \text{tr} (p[\Phi_a, \Phi_a^\dagger]) = \text{tr} (p\Theta).$$

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So, each time we have a subrepresentation of the quiver, the relation $\text{tr}(\rho \Theta) \geq 0$ should hold. This is called $\theta$-stability. This direct and explicit proof of necessity of stability for solving the D-term equations is exactly along the lines of the one we gave at the beginning of this long section. Now, it is not evident that the modification à la Hitchin of the stability that we have introduced will survive till this point of the moduli space; after all it is not clear to what the endomorphism which was called $X$ in the geometrical limit will correspond in the quiver limit. In particular it could correspond to the zero endomorphism, thus giving no modification at all. But, at least mathematically, we can give an example of an equation whose solutions would imply a $\theta$-Hitchin-stability.

Thanks to this analysis, this is now almost trivial. First, one must set the condition that the quiver representation is $X$-invariant – the equation $[X, \Phi_a] = 0$. Then, one finds the appropriate modification to the D-term equation in the form

$$\sum_a [\Phi_a + \alpha_a X, \Phi_a^\dagger + \alpha_a^* X^\dagger] = \Theta \text{Id}.$$  (4.6)

Indeed, for a subrepresentation to be $X$-invariant, the injection $i$ should satisfy the additional condition (similar to the above with $\Phi$, $\Phi'$) that a smaller matrix $X'$ should exist, such that $Xi = iX'$. If a subrepresentation satisfies this, there is no problem in carrying out the steps (4.5) in exactly the same way, with $\Phi \rightarrow \Phi + \alpha X$; otherwise this is not possible. Thus, the result is that we have the inequality $\text{tr}(\rho \Theta) \geq 0$ only for $X$-invariant representations.

Thus we obtain a modification to $\theta$-stability very similar to ones in geometric phase, which again may in particular rehabilitate representations previously discarded as unstable. One can view this as the survival of Hitchin-like modification all over the moduli space. We will see, however, that from the point of view of the moduli space of all the BPS states, this modification is not so dramatic as one could think, thanks to the fact that branes can be lifted to coverings, as we will now explain.

**Coverings**

The starting point for this technique (which has been used already for instance in [12, 30]) is the standard interpretation for non-abelian transverse scalars. Their virtue is that they allow to describe, in different vacua, configurations with several branes wrapped on the same locus or the same number of branes, but each wrapped in a different locus. For instance, the vacuum $X = 0$ represents $N$ (the rank of the matrix) branes wrapped on the submanifold $B$ on which the non-abelian brane theory is defined; whereas the vacuum $X = \text{diag}(x^{(1)}, \ldots, x^{(N)})$ with $x^{(1)} \neq \ldots \neq x^{(N)}(\neq 0)$ describes again $N$ branes, but displaced from one another and from the “initial locus” $B$. Usually the configurations for the transverse scalars are taken constant, because one wants to consider the vacua. Here we will, however, consider general solutions to our equations.

This time, the most relevant feature we will need, that we have not exploited so far, is the fact that the $(2,0)$ part of the equations does not get deformed. This part is (compare with (2.4))

$$F^{(2,0)} = 0, \quad [X_a, X_b] = 0, \quad DX_a = 0 \quad (4.7)$$

We will for most of the time analyze the case with only one transverse scalar $X$, and come later to more general cases. Let us begin from the consequences of last equation in (4.7), which is now $DX = 0$; and let us first pretend we are in flat space. We have recalled the standard interpretation that the eigenvalues of $X$ give the classical position of the branes. These eigenvalues are given by the roots of the characteristic polynomial $p_X(x) = \det(X - x)$; last equation in (4.7) implies $\partial p_X(x) = 0$. Then this characteristic polynomial can be viewed as an equation $p(z_1, \ldots, z_n, x) = 0$ in $B \times \mathbb{C} = \mathbb{C}^n \times \mathbb{C}$, and thus cuts out a complex $n$-dimensional locus. This is the same dimensionality of the base, as it should, and in fact this
locus is a $N$-fold covering of $B$: over a generic point of $B$, there are $N$ counterimages, and so locally this looks the same as the usual picture of several branes, wrapped on different loci, described by the eigenvalues of $X$ (our previous example $X = \text{diag}(x^{(1)}, \ldots, x^{(N)})$). But this time, apart from the most trivial case in which $p$ is constant in the $z_i$’s, the covering will be branched; that is, the eigenvalues will come to coincide somewhere, and thus the resulting covering brane will be one. This is the so-called spectral manifold [7, 31].

Summarizing: in generic cases, the eigenvalues of $X$ describe a single brane $\tilde{B}$ which is geometrically a branched covering of the base $B$. We have described this for the flat space case for simplicity; but this so-called spectral manifold can be defined also in the most general case, in which the base manifold $B$ on which the non-abelian brane theory is defined is an arbitrary manifold, and $X$ is a section of $N(B, M) \otimes \text{End}(E)$. The equation $p_X(x) = 0$ becomes now an equation in the total space of the line bundle $N(B, M)$ (or $K_B$, which is the same, if the ambient manifold $M$ is a Calabi-Yau). Now $DX = 0$ means, as we have underlined before the beginning of subsection [4.1], $dX + [A, X] + aX = 0$, where $A$ is the part of the connection in $\text{End}(E)$ and $a$ is a connection on the bundle $N(B, M)$. Thus we have now $(\partial + a)p_X(x) = 0$; this can be considered as giving a holomorphic structure to the submanifold $p_X(x) = 0$ in the total space of the bundle $N(B, M)$. With these modifications, the geometrical interpretation is intuitively the same.

Actually, what we want is not quite a submanifold defined on the total space of the normal bundle $N(B, M)$; this is only a local (around the initial brane $B$) description. Then we can make $\tilde{B}$ a submanifold of the ambient manifold via the map $N(B, M) \to M$, as it is done in K-theory to find Gysin map from the Thom isomorphism.

Finally, as noted many times to specify a submanifold is not enough for giving a brane, we have to provide the bundle on it as well. Since one expects the covering brane $\tilde{B}$ to be a single object, pulling back the non-abelian gauge bundle from the base $B$ would not be a good choice: we want a line bundle. The right construction is remarkably simple [7, 31]. The line bundle on the covering brane $\tilde{B}$ can be constructed recalling the very definition of $\tilde{B}$: over each point of the base $B$, we consider the $N$ eigenvalues of $X$ as values of the extra transverse coordinate. The line bundle is defined by considering, on each point of the covering (which is an eigenvalue), the corresponding eigenspace. This line bundle $L$ is natural in the sense that its push-forward from $\tilde{B}$ to the base $B$ (this operation can be defined by resorting to sheaves) is indeed the gauge bundle on the base: $\pi_*L = E$, where $\pi : \tilde{B} \to B$ is the covering map.

Coverings, stability and the BPS states

We have now a non-abelian brane theory defined on a submanifold $B$ of a given ambient manifold $M$. We have argued that stability gets modified à la Hitchin, with the condition on subbundles (or subbranes) replaced by the same condition on $X$-invariant subbundles.

The first case is the one in which $X = 0$. In this situation the branes are all wrapped on the base manifold $B$. But in this case, the modification we have proposed does not affect anything, because being $X$-invariant is an automatically satisfied condition.

If, on the other hand, $X$ is a non trivial configuration, then we have seen that this describes branes on a covering $\tilde{B}$ of the initial brane $B$. In particular, we have seen how we can lift the gauge bundle from $B$ to $\tilde{B}$. This is important for the following reason. If one considers BPS states on a fixed base $B$, there are substantial contributions coming from the presence of the scalars $X$, as we have seen. But, if one considers the whole moduli space of BPS states, namely the union of the former moduli space for all the base branes $B$, the covering technique actually shows that most of the new BPS states one obtains
for one base brane $B$ are actually copies of other BPS states pertaining to another brane $\tilde{B}$.

The discussion we had so far has some limitations. First, there are subtleties concerning the covering mechanism. Indeed, we have supposed so far that the eigenvalues are all distinct. If the characteristic polynomial is irreducible (which means that the covering is connect; this we can assume with no loss of generality) the only way we have to duplicate eigenvalues is to duplicate all of them, taking thus a characteristic polynomial which is a power: $p_X(x) = (p_X(x))^k$. In this case, lifting yields a vector bundle $\tilde{E}$ of rank $\leq k$ over the covering brane $\tilde{B}$; a priori this is not guaranteed to be in the usual moduli space of BPS states pertaining to the cycle $\tilde{B}$, since $\tilde{E}$ could be not stable. But the modification of the stability à la Hitchin that we have proposed means now that the subbundles $E'$ of the gauge bundle $E$ on the base $B$ should be lifttable too. This way, the stability condition will get translated into a stability for the vector bundle $\tilde{E}$ over the lifted brane $\tilde{B}$.

More importantly, so far we have analyzed the covering mechanism in the case in which there is only one transverse scalar. When there are more, although there is an equation $[X_a, X_b] = 0$ from the undeformed $(2,0)$ part of the equations, still the commutators like $[X_a, X_b^\dagger]$ do not vanish: they appear in the deformed $(1,1)$ equations. This means that in general we are entering the realm of noncommutative – perhaps better, fuzzy – BPS states; we will not analyze this, but for the small note that follows.

More transverse scalars

The analysis of this case allows us to appreciate the importance of the part of the connection corresponding to the normal bundle $N(B, M)$. Let us first of all tackle the flat space situation in which the brane is at least 2 dimensional but there are more than one complex scalar.

In this case our equations have a strange feature. If we consider more than one transverse scalars, again because of the equation $DX_a = 0$, all of them will have an antiholomorphic characteristic polynomial, $\partial \det (X_a - x_a) = 0$. But the second equation of (4.7) implies, in the generic case in which the eigenvalues are not coincident, that the matrices are simultaneously diagonalizable. This in turn implies that the monodromies of the eigenvalues are the same, and thus that the characteristic polynomial are the same\footnote{We may present another somewhat indirect argument in favor of this assertion by recalling the definition of $\phi(E)$. As it was argued in \cite{13} taking into account the non-abelian dynamics of D-branes amounts to replacing the RR coupling $C \wedge Y$ by Clifford multiplication of $Y$ by RR fields while leaving intact the form of the D-brane charge $Y$, and thus the element in $K(M)$. We should note however that transforming CS data into equations of motion is not straightforward in view of difference in “natural” conventions for the kinetic and CS parts. We will meet another similar clash of conventions in section \ref{section:another clash}}. In this situation the $X_a$ turn out to be multiples of each other. In general, we can relax the assumption that the eigenvalues are distinct, and find situations in which the matrices are not multiples of each other; so it still makes sense to keep (3.3) in its form with several scalars. But the characteristic polynomials remain the same, and so in a way for the covering considerations we have done before only one of the scalars is sufficient.

Let us now see what differences come in when we are in a geometrically less trivial situations; let $B$ be a brane wrapped on a 2-cycle (or higher) on the ambient manifold $M$. (In the main example we have had in mind, that of $M$ being a Calabi-Yau threefold, the only such case is the one in which $B$ is a 2-cycle.) Then, as we have seen in the case with one scalar only, the equation $DX = 0$ does not imply any longer that $\partial \det (X_a - x_a) = 0$, but that $\partial_{N(B,M)} \det (X_a - x_a) = 0$, where $\partial_{N(B,M)}$ is a connection on $N(B, M)$. This prevents the arguments we expounded in the paragraph above to be applied in this more general case. Finally, we can consider instead the case in which $B$ is a 0-cycle. Then the above considerations do not apply in flat space. The equation $DX_a = 0$ becomes now one of the already present equations $[X_a, X_b] = 0$, and so all of its consequences that we have explored in the preceding paragraph are not there any longer. In this case we are thus completely in the realm of fuzzy solutions; but we will
5 Deformed tachyon equations

In this section, we use again dimensional reduction, this time in a different way: we will obtain equations that we will argue to be relevant for the tachyon of the Dp–Dp system.

Let us consider a pair of gauge bundles, \( E_1 \) and \( E_2 \), describing the gauge theory on the brane and on the antibrane, and a morphism of bundles \( T \) connecting them, that can be thought of as a section of \( \text{End}(E_1, E_2) \). In [32], a set of equations has been described that implies the equations of motion of this \( \text{Dp–Dp} \) system, much in the same way in which the instanton equations imply the equations of motion for Yang-Mills, and more generally in the same way in which BPS conditions imply equations of motion for a supersymmetric action. These equations read

\[
F_1 \cdot \omega - iT^\dagger = \lambda_1 \text{Id}_{N_1}, \quad F_2 \cdot \omega + iT^\dagger T = \lambda_2 \text{Id}_{N_2}, \quad \partial T + A_1^{(1,0)} T - TA_2^{(1,0)} = 0, \tag{5.1}
\]

where \( N_i = rk(E_i) \); along with the usual \( F^{(2,0)} \). As for the HYM equations and their deformation we have been studying so far, solution to them is equivalent to a stability condition on the triple \( (E_1, E_2, T) \).

Now, the point interesting for us is that these equations come naturally again from dimensional reduction of HYM, although in a more formal way. Namely, we reduce from complex dimension \( n + 1 \) to \( n \), in such a way that only one complex scalar will appear; this we will call \( W \), as it is not one of the scalars parameterizing transverse fluctuations (for instance, the procedure we are following will give us the tachyon equations also in the maximal dimension \( n = 5 \), that is for the D9–D9 system, in which case there are no transverse fluctuations at all). Another modification to the reduction we have done before is that we have now to take a slightly more general non-abelianization of them, considering a direct sum bundle; thus, in the rhs of (2.5), we can be more general and write, instead of \( \lambda \text{Id} \), \( \text{diag}(\lambda_1 \text{Id}_{N_1}, \lambda_2 \text{Id}_{N_2}) \) (\( \text{diag} \) is here in a block sense). With this specifications, let us now do dimensional reduction of HYM (2.3) taking the particular choice

\[
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \tag{5.2}
\]

(remember that \( W \) has the formal role that \( X \) had in previous sections, appearing in Hitchin equations in its stead). In this way one obtains precisely the tachyon equations (5.1): first two come from the \( (1,1) \) part, the third comes from the \( (2,0) \) part.

We have obtained these tachyon equations from reduction of HYM; but we have seen that the latter get deformed by stringy corrections to (2.3). Thus we expect that, if we now reduce (2.8) in the same way, we will obtain the right deformation of the tachyon equations. This is by now an easy task: first of all, as usual, the \( (2,0) \) part does not change, so we will keep on getting \( DT \equiv \partial T + A_1^{(1,0)} T - TA_2^{(1,0)} = 0 \). Then, let us start from our expression (3.3) and first of all specialize it to the case in which there is only one complex scalar. We get

\[
\text{Sym}\{ (e^{F+\omega \text{Id}})_{\text{top}} (1 - i[W, W^\dagger]) + \frac{i}{2} \left( [D, W^\dagger][D, W] - [\bar{D}, W][D, W^\dagger] \right) (e^{F+\omega \text{Id}})_{\text{top} - 2} \}. \tag{5.3}
\]

Once more, let us now extend this to the case in which the bundle is a direct sum; in this case this means to substitute \( e^{i\theta} \) in (3.3) with \( \exp\{\text{diag}(i\theta_1 \text{Id}_{N_1}, i\theta_2 \text{Id}_{N_2})\} \). Then, putting in this expression the

\[\text{This choice can be better motivated [33] if one takes the fibre to be } \mathbb{P}^1, \text{ and chooses then the bundle on the whole fibration to be } p_1^* E_1 \oplus p_2^* E_1 \otimes p_2^* \mathcal{O}(2). \text{ Here we are doing a formal reduction as in sections 2 and 3.}\]
particular choice (5.2), one obtains our “deformed vortex equations”

\[
\text{Im} \left( e^{i\theta_1} \text{Sym}\{(e^{F_1 + \omega \text{Id}_{N_1}})_{\text{top}}(1 - iT\bar{T}^\dagger) - \frac{i}{2} DT\bar{D}T^\dagger (e^{F_1 + \omega \text{Id}_{N_1}})_{\text{top}-2}\} \right) = 0
\]

\[
\text{Im} \left( e^{i\theta_2} \text{Sym}\{(e^{F_2 + \omega \text{Id}_{N_2}})_{\text{top}}(1 + iT^\dagger T) + \frac{i}{2} DT\bar{D}T (e^{F_2 + \omega \text{Id}_{N_2}})_{\text{top}-2}\} \right) = 0,
\]  

(5.4)

where \(\bar{D}T \equiv \partial T + A_1^{(0,1)} - T A_2^{(0,1)}\) is the antiholomorphic covariant derivative of the tachyon. Notice that these equations are less decoupled than usual: not only does the tachyon appear in both, but also, in the deformation term with \(\bar{D}T\), both \(A_1\) and \(A_2\) appear.

It is natural to speculate that (5.4) can be expressed in terms of superconnections. Let us try to make this expectation more precise. One would like to exploit the considerations made after (3.3) to write an expression which contains a superconnection in the exponent. To do so, the procedure is just like the one we have done above: i) replacing all the \(\text{Id}\) with \(\text{diag}(\text{Id}_{N_1}, \text{Id}_{N_2})\), ii) taking \(A\) and \(T\) of the particular form (5.2). But this time, one wants to start not from the more explicit form of the equation (5.3), but from one of the more imaginative forms (3.3) or (3.4). Explicitly, one gets

\[
\text{Im} \left( \text{Sym}\{\exp \left( \begin{array}{cc} i\theta_1 \text{Id}_{N_1} & 0 \\ 0 & i\theta_2 \text{Id}_{N_2} \end{array} \right) \right) \left[ e^F e^{\omega + J}_{\text{top}} \right] \right) = 0,
\]  

(5.5)

where

\[
\mathcal{F} \equiv \left( \begin{array}{ccc} F_1 + iT_i T^i & iT_i \bar{D}_2 - \bar{D}_1 & iT_i \\ D_2 i_T^i & i \bar{T} & iD_1 \\ F_2 + i_T^i & D_1 & iT_i \end{array} \right) = [D, \bar{D}]; \quad D = \left( \begin{array}{cc} D_1 & iT_i \\ 0 & D_2 \end{array} \right).
\]

These equations need of course some comments. First, we remind the reader that we are using here a super-Lie bracket instead of the usual one, as explained after (3.4); this explains the sign of \(iT_i iT_i\) in \(\mathcal{F}\).

Second, we introduced in (5.5) a symbol \(J\) which is a two-form \(idw\bar{w}\) in the formal transverse space spanned by \(W, W^\dagger\). Once again, this is not the physical transverse space, and this \(J\), though it has the same formal role of \(\omega_{\perp}\) in (3.3) and similar equations, is a different object, introduced here only as a convenient device for bookkeeping. Given that there is only one holomorphic object \(T\) here, this choice could appear rather baroque; and one could wonder if, instead, it would not have been better to start from an expression halfway between (3.3) and (5.5); namely, an expression compact as (3.3) is, with an exponential structure, but with \(iW\) replaced in some way by \(W\). In fact, such an equation does not exist. To replace \(iT_i W\) with \(W\) (or \(iT_i T_i\) with \(T_i\)) before expanding the exponential would be wrong, as one can convince oneself by noting the minus sign in the expression \([D, W^\dagger][\bar{D}, W] - [\bar{D}, W][D, W^\dagger]\) in (5.3). Moreover, the expression one could obtain this way will necessarily have a piece proportional to \(\text{diag}(TT^\dagger, -T^\dagger T)\), which comes from a commutator \([W, W^\dagger]\) (compare with (5.1)). Thus, this expression would be in any case different from the superconnection arising from the CS system of the \(D - \bar{D}\) system. It is hardly expected that the two superconnections are the same. Since we are using the equations of motion, there is a contribution from the DBI part. However, it is interesting to notice that the latter do not spoil completely the presence of the \(Z_2\)-graded structures.

Finally, let us notice that it could be that (5.4), alias (5.5), is the right deformation to solve some of the problems of interpretation for stability of triples raised in (3.3).

**Tachyons and transverse fluctuations**

By expanding on the observation in section 3 on the similarity of equations involving the scalars to those appearing in the computation of Chern character with superconnections, we conclude by a comment on relation between tachyons \(T\) and transverse scalars \(X\) on D-branes. So far this is only a formal correspondence, as \(T\) and \(X\) are quite different objects. One may furthermore note that the
reduction performed in this section for the tachyon equations is formal, unlike that for $X$ which was given by T-duality. For instance, there is no reason for which the tachyon equations we propose should not be valid in the important case of D9-$\overline{\text{D9}}$ system (in which case there are no transverse scalars).

Nevertheless, there is a relation between the two objects, provided by tachyon condensation. First of all, let us begin with a couple of considerations on the general meaning of what we are going to analyze. Consider a $\text{D}p-\overline{\text{D}p}$ system on some manifold $M$. This defines, through relative K-theory, some element of the $K(M)$. Superconnections were initially introduced $[14]$ as a method to compute the Chern character of this element. If we consider instead a non-abelian brane theory defined on some submanifold $B$, by the covering mechanism this describes a brane wrapped on some other submanifold $\tilde{B}$ of $M$. This is again an element of $K(M)$. Tachyon condensation connects these two ways of obtaining a K-theory element; the result, as we will see shortly, is that $T$ is the characteristic polynomial of $X$.

To be more specific, consider a D9-$\overline{\text{D9}}$ system on $C \times M$, where $M$ is now a 8 dimensional manifold. Let us first consider the case in which the bundles on the brane and the antibrane are chosen in such a way that the tachyon is $T = x$, where $x$ is the coordinate on $C$. Thus the system condenses to a D7 wrapped on $M$, described by the locus $x = 0$. What we want to emphasize is that both $T$ for the D9-$\overline{\text{D9}}$ and $X$ for the D7, the vanishing locus is the same, although from a different perspective: in the first case the tachyon is a 10 dimensional field whose zeroes indicate where the resulting lower-dimensional brane will be; in the second case $X$ it is a 8 dimensional, whose zeroes indicate in which position of the transverse direction that it parameterizes the vacuum is located. Let us go ahead with this and consider now a less trivial case: let the tachyon be $T(x, z_i)$, where $z_i$ are coordinates on $M$; let us suppose it is holomorphic, and moreover that is a polynomial in $x$ of degree $N$. Now, the D7 resulting after condensation can be obtained as a classical configuration in the non-abelian brane theory with base $B = M$. This is accomplished by the covering mechanism we have described above: it is sufficient to take a configuration $X$ with characteristic polynomial $p_X(x, z_i) = T(x, z_i)$. Thus, more generally we can say that $T$ is the characteristic polynomial of $X$.

Let us give a geometrical twist to this. We can describe this $C \times M$ as a trivial line bundle over $M$; more generally we can consider a different line bundle $L$ on $M$, and take the 10 dimensional space as the total space $Q$ of this line bundle. In this situation the locus is again described by the zeroes of the tachyon $T(x, z_i)$; but the tachyon itself now is a section of the $N$-th power of the bundle which is the pull-back of the bundle $L$ to its own total space $Q$. This bundle has an obvious so-called tautological section, and $x$ is to be understood now as this section.

Finally, it is now an easy generalization to consider the case in which the result of the condensation has codimension higher than 2: locally around the lower-dimensional brane, the tachyon will be in that case

$$T = \sigma_a \det(X_a - x_a),$$

where $\gamma_a = \left( \begin{array}{cc} 0 & \sigma_a \\ 0 & 0 \end{array} \right)$ is the $\gamma$ matrix relative to the holomorphic coordinate $x^a$. This formula makes more explicit in general the relationship between tachyon field and transverse scalars we have been using implicitly in deriving (5.4) and (5.5).

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References

[1] M. Marino, R. Minasian, G. Moore and A. Strominger, “Nonlinear instantons from supersymmetric p-branes,” JHEP0001 (2000) 005 [hep-th/9911206].

[2] M. R. Douglas, B. Fiol and C. Romelsberger, “Stability and BPS branes,” hep-th/0002037.

[3] M. Bershadsky, C. Vafa and V. Sadov, “D-Branes and Topological Field Theories,” Nucl. Phys. B 463 (1996) 420 [hep-th/9511222].

[4] J. A. Harvey and G. Moore, “On the algebras of BPS states,” Commun. Math. Phys. 197, 489 (1998) [hep-th/9609017].

[5] R. C. Myers, “Dielectric-branes,” JHEP9912 (1999) 022 [hep-th/9910053].

[6] N. J. Hitchin, “The Self-duality Equations On A Riemann Surface,” Proc. Lond. Math. Soc. 55 (1987) 59.

[7] N. J. Hitchin, “Stable bundles and integrable systems,” Duke Math. J. 54, 1, (1987), 91.

[8] R. Donagi, L. Ein and R. Lazarsfeld, “A non-linear deformation of the Hitchin dynamical system,” alg-geom/9504017.

[9] M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, “Topological reduction of 4-d SYM to 2-d sigma models,” Nucl. Phys. B 448 (1995) 166 [hep-th/9501096].

[10] C. Vafa and E. Witten, “A Strong coupling test of S duality,” Nucl. Phys. B 431 (1994) 3 [hep-th/9408074].

[11] S. B. Giddings, F. Hacquebord and H. Verlinde, “High energy scattering and D-pair creation in matrix string theory,” Nucl. Phys. B 537 (1999) 260 [hep-th/9804121].

[12] G. Bonelli, L. Bonora and F. Nesti, “Matrix string theory, 2D SYM instantons and affine Toda systems,” Phys. Lett. B 435 (1998) 303 [hep-th/9805071]; G. Bonelli, L. Bonora and F. Nesti, “String interactions from matrix string theory,” Nucl. Phys. B 538 (1999) 100 [hep-th/9807232]; G. Bonelli, L. Bonora, F. Nesti and A. Tomasiello, “Matrix string theory and its moduli space,” Nucl. Phys. B 554 (1999) 103 [hep-th/9901093].

[13] G. Bonelli, L. Bonora, S. Terna and A. Tomasiello, “Instantons and scattering in N = 4 SYM in 4D,” hepth/9912227.

[14] K. Uhlenbeck & S. T. Yau, On the Existence of Hermitian-Yang-Mills Connections in Stable Vector Bundles, Comm. in Pure and Appl. Math., 39, S257-S293 (1986)

[15] I. Brunner, M. R. Douglas, A. E. Lawrence and C. Romelsberger, “D-branes on the quintic,” JHEP0008, 015 (2000) [hep-th/9906200].

[16] D. Quillen, “Superconnections and the Chern character,” Topology 24 (1985), 89-95.

[17] V. Periwal, “Deformation quantization as the origin of D-brane non-Abelian degrees of freedom,” JHEP0008 (2000) 021 [hep-th/0008046].

[18] S. F. Hassan and R. Minasian, “D-brane couplings, RR fields and Clifford multiplication,” hepth/0008149.
[19] S. K. Donaldson and R. P. Thomas, “Gauge theory in higher dimensions,” in The Geometric Universe: Science, Geometry, And The Work Of Roger Penrose, Oxford University Press, 1998.

[20] M. Lübke, “Stability of Einstein-Hermitian vector bundles,” Man. Math. 42 (1983) 245-257.

[21] S. K. Donaldson & P. B. Kronheimer, The Geometry of Four-Manifolds, Oxford 1990.

[22] N. C. Leung, “Einstein type metrics and stability on vector bundles,” J. Diff. Geom. 45 (1997) 514-546.

[23] D. Huybrechts and M. Lehn, The geometry of moduli spaces of sheaves, Viehweg, 1997.

[24] M. R. Douglas, “D-branes, categories and N = 1 supersymmetry,” hep-th/0011017.

[25] M. R. Douglas, “D-Branes on Calabi-Yau Manifolds,” math.ag/0009209.

[26] R. Thomas, “D-branes: Bundles, derived categories, and Lagrangians,” to appear in the proceedings of the 2000 Clay Mathematics Institute school on Mirror Symmetry.

[27] See for instance C. T. Simpson, “Higgs bundles and Local systems,” Publ. Math. IHES 75 (1992), 5-95.

[28] A. King, “Moduli of representations of finite dimensional algebras,” Quarterly J. Math. Oxford 45 (1994), 515-530.

[29] D. Diaconescu and M. R. Douglas, “D-branes on stringy Calabi-Yau manifolds,” hep-th/0006224.

[30] G. Bonelli, “The geometry of the M5-branes and TQFTs,” hep-th/0012075.

[31] N. J. Hitchin, “Riemann surfaces and integrable systems,” in Integrable Systems: Twistors, Loop Groups, and Riemann Surfaces, Oxford Graduate Texts in Mathematics, Clarendon Press (1999).

[32] Y. Oz, T. Pantev and D. Waldram, “Brane-antibrane systems on Calabi-Yau spaces,” JHEP0102 (2001) 045 [hep-th/0009112].

[33] S. Bradlow and O. García-Prada, “Stable triples, equivariant bundles and dimensional reduction,” Math. Ann. 304 (1996), 225-252, alg-geom/9401008.