Surfaces with constant extrinsically Gaussian curvature in the Heisenberg group

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Abstract

In this work we study constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group which are invariant under the 1-parameter groups of isometries.

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1. Introduction

In 1982, W. P. Thurston formulated a geometric conjecture for three dimensional manifolds, namely every compact orientable 3-manifold admits a canonical decomposition into pieces, each of them having a canonical geometric structure from the following eight maximal and simply connected homogenous Riemannian spaces: \( \mathbb{E}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, SL(2, \mathbb{R}), \mathbb{H}_3 \) and \( Sol_3 \). See e.g. [34].

During the recent years, there has been a rapidly growing interest in the geometry of surfaces in three homogenous spaces focusing on flat and constant Gaussian curvature surfaces. Many works are studying the geometry of surfaces in homogenous 3-manifolds. See for example [2–4, 9, 12, 14–16, 21, 22, 24, 36].

The concept of translation surfaces in \( \mathbb{R}^3 \) can be generalized the surfaces in the three dimensional Lie group, in particular, homogeneous manifolds. In Euclidean
3-space, every cylinder is flat. Conversely, complete flat surfaces in $\mathbb{E}^3$ are cylinders over complete curves. See [20]. López and Munteanu [17] studied invariant surfaces with constant mean curvature and constant Gaussian curvature in $Sol_3$ space. Yoon and Lee [37] studied translation surfaces in Heisenberg group $\mathbb{H}_3$ whose position vector $x$ satisfies the equation $\Delta x = Ax$, where $\Delta$ is the Laplacian operator of the surface and $A$ is a $3 \times 3$-real matrix.

Flat $G_4$-invariant surfaces are nothing but surfaces invariant under $SO(2)$-action, i.e. rotational surfaces. Flat rotational surfaces are classified by Caddeo, Piu and Ratto in [8].

In [14], J. I. Inoguchi give a classification of intrinsically flat $G_1$-invariant translation surfaces in Heisenberg group $\mathbb{H}_3$. Let $M$ be a surface invariant under $G_3$, then $M$ is locally expressed as

$$X(u, v) = (0, 0, v).((x(u), y(u), 0) = (x(u), y(u), v), \ u \in I, \ v \in \mathbb{R}.$$ 

Here $I$ is an open interval and $u$ is the arclength parameter. Note that $(x, y, 0)$ and $(0, 0, v)$ commute. Then the sectional curvature $K(X_x \wedge X_y) = \frac{1}{4}$ and the extrinsically Gaussian curvature $K_{ext} = -\frac{1}{4}$. Direct computation show that $M$ is flat. (cf. [12–14, 28]).

The paper is divided according the type of surfaces invariant under 1-parameter subgroups of isometries $\{G_i\}_{i=1,2,3,4}$. So, in section 3 we classify $G_1$-invariant surfaces of the Heisenberg group $\mathbb{H}_3$ with constant extrinsically Gaussian curvature $K_{ext}$, including extrinsically flat $G_1$-invariant surfaces.

In section 4 we classify $G_2$-invariant surfaces of the Heisenberg group $\mathbb{H}_3$ with constant extrinsically Gaussian curvature $K_{ext}$, including extrinsically flat $G_2$-invariant surfaces.

2. Preliminaries

The 3-dimensional Heisenberg group $\mathbb{H}_3$ is the simply connected and connected 2-step nilpotent Lie group. Which has the following standard representation in $GL(3, \mathbb{R})$

$$\begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$$

with $r, s, t \in \mathbb{R}$. The Lie algebra $\mathfrak{h}_3$ of $\mathbb{H}_3$ is given by the matrices

$$A = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$
with \(x, y, z \in \mathbb{R}\). The exponential map \(\exp : \mathfrak{h}_3 \to \mathbb{H}_3\) is a global diffeomorphism, and is given by
\[
\exp(A) = I + A + \frac{A^2}{2} = \begin{pmatrix}
1 & x & z + \frac{xy}{2} \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}.
\]

The Heisenberg group \(\mathbb{H}_3\) is represented as the cartesian 3-space \(\mathbb{R}^3(x, y, z)\) with group structure:
\[
(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}x_1y_2 - \frac{1}{2}x_2y_1).
\]

We equip \(\mathbb{H}_3\) with the following left invariant Riemannian metric
\[
g := dx^2 + dy^2 + \left( dz + \frac{1}{2}(ydx - xdy) \right)^2.
\]

The identity component \(I^\circ(\mathbb{H}_3)\) of the full isometry group of \((\mathbb{H}_3, g)\) is the semi-direct product \(SO(2) \ltimes \mathbb{H}_3\). The action of \(SO(2) \ltimes \mathbb{H}_3\) is given explicitly by
\[
A = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2}(a \sin \theta - b \cos \theta) \\
\frac{1}{2}(a \cos \theta + b \sin \theta)
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
+ \begin{bmatrix}
a \\
b \\
c
\end{bmatrix}.
\]

In particular, rotational around the \(z\)-axis and translations:
\[
(x, y, z) \to (x, y, z + a), a \in \mathbb{R}
\]
along the \(z\)-axis are isometries of \(\mathbb{H}_3\).

The Lie algebra \(\mathfrak{h}_3\) of \(I^\circ(\mathbb{H}_3)\) is generated by the following Killing vector fields:
\[
F_1 = \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial z}, \quad F_2 = \frac{\partial}{\partial y} - \frac{x}{2} \frac{\partial}{\partial z},
\]
\[
F_3 = \frac{\partial}{\partial z}, \quad F_4 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.
\]

One can check that \(F_1, F_2, F_3\) are infinitesimal transformations of the 1-parameter groups of isometries defined by
\[
G_1 = \{(t, 0, 0) | t \in \mathbb{R}\}, \quad G_2 = \{(0, t, 0) | t \in \mathbb{R}\}, \quad G_3 = \{(0, 0, t) | t \in \mathbb{R}\},
\]
respectively. Here this groups acts on \(\mathbb{H}_3\) by the left translation. The vector field \(F_4\) generates the group of rotations around the \(z\)-axis. Thus \(G_4\) is identified with \(SO(2)\).
Definition 2.1. A surface $\Sigma$ in the Heisenberg space $\mathbb{H}_3$ is said to be invariant surface if it is invariant under the action of the 1-parameter subgroups of isometries $\{G_i\}$, with $i \in \{1, 2, 3, 4\}$.

The Lie algebra $\mathfrak{h}_3$ of $\mathbb{H}_3$ has an orthonormal basis $\{E_1, E_2, E_3\}$ defined by

\[
E_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.
\]

The Levi-Civita connection $\nabla$ of $g$, in terms of the basis $\{E_i\}_{i=1,2,3}$ is explicitly given as follows

\[
\begin{aligned}
\nabla_{E_1} E_1 &= 0, \quad \nabla_{E_1} E_2 = \frac{1}{2} E_3, \quad \nabla_{E_1} E_3 = -\frac{1}{2} E_2 \\
\nabla_{E_2} E_1 &= -\frac{1}{2} E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = \frac{1}{2} E_1 \\
\nabla_{E_3} E_1 &= -\frac{1}{2} E_2, \quad \nabla_{E_3} E_2 = \frac{1}{2} E_1, \quad \nabla_{E_3} E_3 = 0
\end{aligned}
\]

The Riemannian curvature tensor $R$ is a tensor field on $\mathbb{H}_3$ defined by

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

The components $\{R^l_{ijk}\}$ are computed as

\[
R^1_{212} = -\frac{3}{4}, \quad R^1_{313} = \frac{1}{4}, \quad R^2_{323} = \frac{1}{4}.
\]

Let us denote $K_{ij} = K(E_i, E_j)$ the sectional curvature of the plane spanned by $E_i$ and $E_j$. Then we get easily the following:

\[
K_{12} = -\frac{3}{4}, \quad K_{13} = -\frac{1}{4}, \quad K_{23} = -\frac{1}{4}.
\]

The Ricci curvature $Ric$ is defined by

\[
Ric(X,Y) = \text{trace}\{Z \to R(Z,X)Y\}.
\]

The components $\{R_{ij}\}$ of $Ric$ are defined by

\[
Ric(E_i, E_j) = R_{ij} = \sum_{k=1}^{3} \langle R(E_i, E_k)E_k, E_j \rangle.
\]

The components $\{R_{ij}\}$ are computed as

\[
R_{11} = -\frac{1}{2}, \quad R_{12} = R_{13} = R_{23} = 0, \quad R_{22} = -\frac{1}{2}, \quad R_{33} = \frac{1}{2}.
\]

The scalar curvature $S$ of $\mathbb{H}_3$ is constant and we have

\[
S = trRic = \sum_{i=1}^{3} Ric(E_i, E_i) = -\frac{1}{2}.
\]
3. Constant extrinsically Gaussian curvature

$G_1$-invariant translation surfaces in Heisenberg group $\mathbb{H}_3$

3.1.

In this subsection we study complete extrinsically flat translation surfaces $\Sigma$ in Heisenberg group $\mathbb{H}_3$ which are invariant under the one parameter subgroup $G_1$. Clearly, such a surface is generated by a curve $\gamma$ in the totally geodesic plane $\{x = 0\}$. Discarding the trivial case of a vertical plane $\{y = y_0\}$. Thus $\gamma$ is given by $\gamma(y) = (0, y, v(y))$. Therefore the generated surface is parameterized by

$$X(x, y) = (x, 0, 0)(0, y, v(y)) = (x, y, v(y) + \frac{x y}{2}), \ (x, y) \in \mathbb{R}^2.$$ 

We have an orthogonal pair of vector fields on $(\Sigma)$, namely,

$$e_1 := X_x = (1, 0, \frac{y}{2}) = E_1 + yE_3.$$ 

and

$$e_2 := X_y = (0, 1, v' + \frac{x}{2}) = E_2 + v'E_3.$$ 

The coefficients of the first fundamental form are:

$$E = \langle e_1, e_1 \rangle = 1 + y^2, \ F = \langle e_1, e_2 \rangle = yv', \ G = \langle e_2, e \rangle = 1 + v'^2.$$ 

As a unit normal field we can take

$$N = \frac{-y}{\sqrt{1 + y^2 + v'^2}} E_1 - \frac{v'}{\sqrt{1 + y^2 + v'^2}} E_2 + \frac{1}{\sqrt{1 + y^2 + v'^2}} E_3$$ 

The covariant derivatives are

$$\tilde{\nabla}_{e_1} e_1 = -yE_2$$ 

$$\tilde{\nabla}_{e_1} e_2 = \frac{y}{2} E_1 - \frac{v'}{2} E_2 + \frac{1}{2} E_3$$ 

$$\tilde{\nabla}_{e_2} e_2 = v'E_1 + v''E_3.$$ 

The coefficients of the second fundamental form are

$$l = \langle \tilde{\nabla}_{e_1} e_1, N \rangle = \frac{yv'}{\sqrt{1 + y^2 + v'^2}}$$ 

$$m = \langle \tilde{\nabla}_{e_1} e_2, N \rangle = \frac{-y^2 + v'^2 + 1}{\sqrt{1 + y^2 + v'^2}}$$ 

$$n = \langle \tilde{\nabla}_{e_2} e_2, N \rangle = \frac{-yv' + v''}{\sqrt{1 + y^2 + v'^2}}$$
Let $K_{ext}$ be the extrinsic Gauss curvature of $\Sigma$,

$$K_{ext} = \frac{\ln - m^2}{EG - F^2} = \frac{-y^2 v'^2 + yv'v'' - \left( -\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2} \right)^2}{(1 + y^2 + v'^2)^2}.$$ 

Thus $\Sigma$ is extrinsically flat invariant surface in Heisenberg group $H_3$ if and only if

$$K_{ext} = 0,$$

that is, if and only if

$$-y^2 v'^2 + yv'v'' - \left( -\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2} \right)^2 = 0 \quad (3.1)$$

to classify extrinsically flat invariant surfaces must solve the equation (3.1). We can writes equation (3.1) as

$$y^2 + yv'v'' - \left( \frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2} \right)^2 = 0 \quad (3.2)$$

we assume that $z = \frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}$. Then

$$\begin{cases} z' = y + v'v'' \\ v'v'' = z' - y \\ v'^2 = 2z - y^2 - 1. \end{cases} \quad (3.3)$$

Therefore equation (3.2) becomes

$$yz' - z^2 = 0. \quad (3.4)$$

equation (3.4) implies that

$$-\frac{z'}{z^2} = -\frac{1}{y}, \quad (3.5)$$

and equation (3.5) implies that

$$z = \frac{1}{-\ln(y) + \alpha}. \quad (3.6)$$

where $\alpha \in \mathbb{R}$, and if $y \neq e^\alpha$.

From (3.3) and (3.6), we have

$$v'^2 = 2z - y^2 - 1$$

$$= \frac{2}{-\ln(y) + \alpha} - y^2 - 1.$$ 

Thus

$$v' = \sqrt{\frac{2}{-\ln(y) + \alpha} - y^2 - 1}.$$ 

As conclusion, we have
Theorem 3.1. • The only non-extendable extrinsically flat translation surfaces in the 3-dimensional Heisenberg group $H_3$ invariant under the 1-parameter subgroup $G_1 = \{(t, 0, 0) \in H_3 / t \in \mathbb{R}\}$, are the surfaces whose parametrization is $X(x, y) = (x, y, v(y) + \frac{xy}{2})$ where $y$ and $v$ satisfy

$$v(y) = \int \sqrt{\frac{2}{-\ln(y) + \alpha - y^2}} - 1 \, dy,$$

where $\alpha \in \mathbb{R}$, and $y \neq e^{\alpha}$.

• There are no complete extrinsically flat translation surfaces in the 3-dimensional Heisenberg group $H_3$ invariant under the 1-parameter subgroup $G_1 = \{(t, 0, 0) \in H_3 / t \in \mathbb{R}\}$.

Remark 3.2. Let $\Sigma$ be a $G_1$-invariant translation surfaces in the 3-dimensional Heisenberg space. Then $\Sigma$ is locally expressed as

$$X(x, y) = (0, y, v(y)). (x, 0, 0) = \left(x, y, v(y) - \frac{xy}{2}\right).$$

Then the extrinsically Gaussian curvature $K_{ext}$ of $\Sigma$ is computed as

$$K_{ext} = \frac{(v' - x)^2 - 1}{4 (1 + (v' - x)^2)^2}.$$ 

Thus $\Sigma$ can not be of constant extrinsically Gaussian curvature.

3.2.

In this subsection we study complete constant extrinsically Gaussian curvature translation surfaces $\Sigma$ in Heisenberg group $H_3$ which are invariant under the one parameter subgroup $G_1$. Clearly, such a surface is generated by a curve $\gamma$ in the totally geodesic plane $\{x = 0\}$. Discarding the trivial case of a vertical plane $\{y = y_0\}$. Thus $\gamma$ is given by $\gamma(y) = (0, y, v(y))$. Therefore the generated surface is parameterized by

$$X(x, y) = (x, 0, 0). (0, y, v(y)) = (x, y, v(y) + \frac{xy}{2}), (x, y) \in \mathbb{R}^2.$$ 

Theorem 3.3. • The $G_1$-invariant constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group $H_3$, are:

1. $K_{ext} = -\frac{1}{4}$.

   The surfaces of equation

   $$z = v(y) + \frac{xy}{2} = \frac{xy}{2} + \frac{1}{2} y \sqrt{2\beta - y^2} + \arctan \left(\frac{y}{\sqrt{\beta - y^2}}\right),$$

   where $\beta \in \mathbb{R}$. 
2. \( K_{\text{ext}} \neq -\frac{1}{4} \).

Then \( y \) and \( v \) satisfy

\[
v(y) = \int \frac{\sqrt{1 - 2(K_{\text{ext}} + \frac{1}{4}) \ln(y) + \gamma}}{2(K_{\text{ext}} + \frac{1}{4})} - y^2 - 1 \, dy.
\]

where \( \gamma \in \mathbb{R} \), and \( y \neq e^{\frac{\gamma}{2(K_{\text{ext}} + \frac{1}{4})}} \).

- There are no complete constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group \( \mathbb{H}_3 \) invariant under the 1-parameter subgroup \( G_1 \).

**Proof.** From (4.1) and (3.2) we have

\[
K_{\text{ext}} = \frac{\ln - m^2}{EG - F^2} = \frac{y^2 + yv'v'' - \frac{1}{4}(1 + y^2 + v'^2)^2}{(1 + y^2 + v'^2)^2}. \tag{3.7}
\]

1. If \( K_{\text{ext}} = -\frac{1}{4} \). Then equation (3.7) becomes

\[
y^2 + yv'v'' = 0 \tag{3.8}
\]

We note that \( y \) equal zero is solution of the equation (3.8).

If \( y \) is different to zero \((y \neq 0)\), equation (3.8) becomes

\[
v'v'' = -y.
\]

Integration gives us

\[
v(y) = \frac{1}{2}y\sqrt{2\beta - y^2} + \arctan \left( \frac{y}{\sqrt{\beta - y^2}} \right),
\]

where \( \beta \in \mathbb{R} \).

2. If \( K_{\text{ext}} \neq -\frac{1}{4} \). Then equation (3.7) becomes

\[
y^2 + yv'v'' = (K_{\text{ext}} + \frac{1}{4})(1 + y^2 + v'^2)^2.
\]

In fact, put \( z = 1 + y^2 + v'^2 \). Then \( z \) satisfies

\[
\frac{1}{2}yz' = (K_{\text{ext}} + \frac{1}{4})z^2.
\]

Hence we have

\[
z = \frac{1}{-2(K_{\text{ext}} + \frac{1}{4})y + \gamma},
\]

where \( \gamma \in \mathbb{R} \), and \( y \neq e^{\frac{\gamma}{2(K_{\text{ext}} + \frac{1}{4})}} \). Using the equation \( z = 1 + y^2 + v'^2 \), we get

\[
v'^2 = \frac{1}{-2(K_{\text{ext}} + \frac{1}{4})y + \gamma} - y^2 - 1. \quad \square
\]
4. Constant extrinsically Gaussian curvature

$G_2$-invariant translation surfaces in Heisenberg group $\mathbb{H}_3$

In this section we study constant complete extrinsically flat translation surfaces $\Sigma$ in Heisenberg group $\mathbb{H}_3$ which are invariant under the one parameter subgroup $G_2$. Clearly, such a surface is generated by a curve $\gamma$ in the totally geodesic plane \{\(y = 0\)\}. Discarding the trivial case of a vertical plane \{\(x = x_0\)\}. Thus $\gamma$ is given by $\gamma(x) = (x, 0, f(x))$. Therefore the generated surface is parameterized by

$$X(x, y) = (0, y, 0). (x, 0, f(x)) = (x, y, f(x) - \frac{xy}{2}), \ (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on $\Sigma$, namely,

$$e_1 := X_x = (1, 0, f' - \frac{y}{2}) = E_1 + f'E_3.$$

and

$$e_2 := X_y = (0, 1, -\frac{x}{2}) = E_2 - xE_3.$$

The coefficients of the first fundamental form are:

$$E = \langle e_1, e_1 \rangle = 1 + f'^2, \ F = \langle e_1, e_2 \rangle = -xf', \ G = \langle e_2, e \rangle = 1 + x^2.$$

As a unit normal field we can take

$$N = \frac{-f'}{\sqrt{1 + x^2 + f'^2}}E_1 + \frac{x}{\sqrt{1 + x^2 + f'^2}}E_2 + \frac{1}{\sqrt{1 + x^2 + f'^2}}E_3.$$

The covariant derivatives are

$$\tilde{\nabla}_{e_1} e_1 = -f'E_2 + f''E_3 - \frac{y}{\sqrt{1 + y^2 + v'^2}}E_3.$$

$$\tilde{\nabla}_{e_1} e_2 = \frac{f'}{2}E_1 + \frac{x}{2}E_2 - \frac{1}{2}E_3.$$

$$\tilde{\nabla}_{e_2} e_2 = -xE_1.$$

The coefficients of the second fundamental form are

$$l = \langle \tilde{\nabla}_{e_1} e_1, N \rangle = \frac{-xf' + f''}{\sqrt{1 + x^2 + f'^2}}$$

$$m = \langle \tilde{\nabla}_{e_1} e_2, N \rangle = \frac{-\frac{f'^2}{2} + \frac{x^2}{2} - \frac{1}{2}}{\sqrt{1 + x^2 + f'^2}}$$

$$n = \langle \tilde{\nabla}_{e_2} e_2, N \rangle = \frac{-yv' + v''}{\sqrt{1 + y^2 + v'^2}}.$$
Let \( K_{\text{ext}} \) be the extrinsic Gauss curvature of \( \Sigma \),
\[
K_{\text{ext}} = \frac{\ln - m^2}{\mathcal{E} \mathcal{G} - F^2} = \frac{x^2 + xf'f'' - \frac{1}{4}(x^2 + f'^2 + 1)^2}{(1 + x^2 + f'^2)^2}.
\]  
(4.1)

Thus \( \Sigma \) is extrinsically flat invariant surface in Heisenberg group \( \mathbb{H}_3 \) if and only if
\[
K_{\text{ext}} = 0,
\]
that is, if and only if
\[
x^2 + xf'f'' - \frac{1}{4}(x^2 + f'^2 + 1)^2 = 0.
\]  
(4.2)

To classify extrinsically flat invariant surfaces must solve the equation (4.2).

We remark that the equation (4.2) is similarly to the equation (3.1), It is sufficient to change \( y \) by \( x \) and \( v \) by \( f \).

As conclusion, we have

**Theorem 4.1.** • The only non-extendable extrinsically flat translation surfaces in the 3-dimensional Heisenberg group \( \mathbb{H}_3 \) invariant under the 2-parameter subgroup \( G_2 = \{ (0, t, 0) \in \mathbb{H}_3 / t \in \mathbb{R} \} \), are the surfaces whose parametrization is \( X(x, y) = \left( x, y, f(x) - \frac{xy}{2} \right) \) where \( x \) and \( f \) satisfy
\[
f(x) = \int \sqrt{-\ln(x) + \alpha - x^2 - 1} dy.
\]

where \( \alpha \in \mathbb{R} \), and \( x \neq e^\alpha \).

• There are no complete extrinsically flat translation surfaces in the 3-dimensional Heisenberg group \( \mathbb{H}_3 \) invariant under the 1-parameter subgroup \( G_2 = \{ (0, t, 0) \in \mathbb{H}_3 / t \in \mathbb{R} \} \).

**Remark 4.2.** Let \( \Sigma \) be a \( G_2 \)-invariant translation surfaces in the 3-dimensional Heisenberg space. Then \( \Sigma \) is locally expressed as
\[
X(x, y) = (x, 0, f(x)) \cdot (0, y, 0) = \left( x, y, f(x) + \frac{xy}{2} \right).
\]

Then the extrinsically Gaussian curvature \( K_{\text{ext}} \) of \( \Sigma \) is computed as
\[
K_{\text{ext}} = -\frac{(f' + y)^2 - 1)^2}{4(1 + (v' - x)^2)^2}.
\]

Thus \( \Sigma \) can not be of constant extrinsically Gaussian curvature.

**Theorem 4.3.** • The \( G_2 \)-invariant constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group \( \mathbb{H}_3 \), are:
1. \( K_{\text{ext}} = -\frac{1}{4} \).

The surfaces of equation
\[
z = f(x) - \frac{xy}{2} = -\frac{xy}{2} + \frac{1}{2}x\sqrt{2\beta - x^2} + \arctan\left(\frac{x}{\sqrt{2\beta - x^2}}\right),
\]
where \( \beta \in \mathbb{R} \).

2. \( K_{\text{ext}} \neq -\frac{1}{4} \).

Then \( x \) and \( f \) satisfy
\[
f(x) = \int \sqrt{-2}(K_{\text{ext}} + \frac{1}{4})\ln(x) + x^2 - 1dy.
\]
where \( \gamma \in \mathbb{R} \), and \( x \neq e^{\gamma(K_{\text{ext}} + \frac{1}{4})} \).

- There are no complete constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group \( \mathbb{H}_3 \) invariant under the 1-parameter subgroup \( G_2 \).

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