U(1) Gauge Theory with Villain Action on Spherical Lattices

C. B. Lang* and P. Petreczky†

Institut für Theoretische Physik, Universität Graz, A-8010 Graz, AUSTRIA

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Abstract

We have studied the U(1) gauge field theory with Villain (periodic Gaussian) action on spherelike lattices. The effective size of the systems studied ranges from 6 to 16. We do not observe any 2-state signal in the distribution function of the plaquette expectation value at the deconfining phase transition. The observed finite-size scaling behavior is consistent with a second order phase transition. The obtained value of the critical exponent is $\nu = 0.366(12)$ and thus neither Gaussian ($\nu = 0.5$) nor discontinuous ($\nu = 0.25$) type, indicating a nontrivial continuum limit.

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*e-mail: cbl@kfunigraz.ac.at
†on leave from L. Kossuth University, Debrecen, Hungary
1 Introduction

In this work we study the Villain formulation \cite{1} of the U(1) pure gauge theory on 4D lattices with spherelike topology. We were motivated by recent results obtained in the theory with Wilson action (with and without a charge two coupling $\gamma$) on spherelike lattices \cite{2, 3, 4}. It turned out that on such lattices there are no 2-state signals even on the Wilson line ($\gamma = 0$) \cite{3}. For hypercubic lattices with periodic boundary conditions (the usual torus geometry) one finds such signals \cite{5, 6, 7, 8} even at sizable negative values of $\gamma$ \cite{6}; under certain assumption a tricritical point (TCP) was predicted at $\gamma = -0.11$ \cite{6}. It has been argued \cite{2} that the disturbing effects may be due to the interplay of periodic boundary condition with the important rôle of the topological monopole excitations in the transition. This issue is still unsettled, however. In case the two-state signal persists in an infinite-volume limit, the transition is first order; if the signal is spurious, it prevents a careful finite-size scaling (FSS) analysis of the critical transition. Up to date taking a continuum limit has not been possible due to this problem.

We hope that in this situation simulations with the Villain action help to clarify some of the problems mentioned above. This formulation is of certain interest because it lends itself to theoretical analysis. In this form of the theory the partition function can be decomposed into Gaussian fluctuations and monopole excitations \cite{9}. There are further relationships to the noncompact U(1) Higgs model in the limit of large negative squared bare mass (frozen 4D superconductor) \cite{10, 11} and to an effective string theory equivalent to that model \cite{12, 13}. The action obeys reflection positivity.

The leading terms of a character expansion of the Villain action shows that the phase transition (PT) is near $\beta W = 1.16$ (we denote by $\beta W$ the coupling in the Wilson action) and $\gamma = -0.22$. Although this value of $\gamma$ is below the value for a conjectured TCP \cite{6} metastability signals were observed (for torus b.c.) here, too \cite{14, 15}. The PT lies in a hypersurface of PTs for the other actions mentioned. It is therefore of interest, whether the two-state signals vanish for spherelike lattice for the Villain action as well, and, if this is the case, whether the critical behavior is in the same universality class.

We proceed in the following way: In section 2 we discuss the details and implementation of the Villain action, in section 3 the lattices with spherelike topology are introduced and the general strategy of the simulation and the FSS analysis are presented. Section 4 summarizes our conclusions.
2 The Villain action

The Villain or heat-kernel action is defined through the Boltzmann factor per plaquette

$$\exp (-S_P(\beta, U_P)) \equiv K(1/\beta, U_P),$$

and is the solution of the diffusion equation in group space

$$\Delta K(t, U) = -\frac{d}{dt} K(t, U),$$

with the boundary conditions

$$K(0, U) = \delta(1, U), \quad K(\infty, U) = 1.$$

It connects the constant distribution at strong coupling ($t = \infty$) with a distribution peaked at the group unit element in the weak coupling limit ($t = 0$).

The heat equation is implemented by introducing the metric tensor and the Laplace-Beltrami operator on the group manifold. In the case of the $U(1)$ gauge group the solution can be written as

$$\exp(-S_P) = \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{\beta}{2} (\theta_P - 2n\pi)^2 \right]$$

where $\theta_P$ is the angle of the plaquette variable $U_P = \cos \theta_P$ and given as a sum over all link angles $\theta_{x,\mu} \in [-\pi, \pi]$

$$\theta_P = \theta_{x,\mu} + \theta_{x+\mu,\nu} - \theta_{x+\nu,\mu} - \theta_{x,\nu}$$

In the simulation $S_P$ has been calculated by interpolating pretabulated values.

The total action is defined through

$$\exp (-S(\beta)) = \prod_P \exp (-S_P(\beta, U_P))$$

and the observables (quantum averages) are given as expectation values with regard to the Gibbs measure $\prod_{x,\mu} dU_{x,\mu} \exp (-S(\beta))$. Useful observables are the internal energy

$$E_{HK} = -\langle \sum_P \frac{\partial S_P}{\partial \beta} \rangle$$
and the specific-heat

\[ C_{HK} = \frac{\partial E_{HK}}{\partial \beta} = \langle \left( \sum_p \frac{\partial S_P}{\partial \beta} \right)^2 \rangle - \langle \sum_p \frac{\partial S_P}{\partial \beta} \rangle^2 - \langle \sum_p \frac{\partial^2 S_P}{\partial \beta^2} \rangle. \] (8)

This form differs from the usual definition due to the non-linear dependence of \( S_P \) on \( \beta \). Note, that \( \beta \) is different from the coupling \( \beta_W \) used in the Wilson-action.

In our calculations we measure the expectation value of the Wilson plaquette variable

\[ E \equiv \langle \sum_p w_P \text{Re} U_P \rangle \] (9)

and its distributions (the weight factors are due to correction factor for the spherical lattice shape and are discussed below). For the investigation of the PT both sets of observables are equally suited; we find the later set more useful, however. The histogram analysis along the lines of Ferrenberg and Swendsen [16] is implemented in that variable \( E \) and the analytic continuation therefore is in the conjugate coupling variable \( \beta_W \). From the histograms in \( E \) corresponding higher order cumulants are determined,

\[ c_V(\beta, L) = \frac{1}{6V} \langle (E - \langle E \rangle)^2 \rangle, \] (10)

\[ V_{CLB}(\beta, L) = -\frac{1}{3} \frac{\langle (E^2 - \langle E \rangle^2)^2 \rangle}{\langle E^2 \rangle^2}, \] (11)

\[ U_4(\beta, L) = \frac{\langle (E - \langle E \rangle)^4 \rangle}{\langle (E - \langle E \rangle)^2 \rangle^2}. \] (12)

(For simplicity we call \( c_V \) specific-heat, too.) The positions and values of their respective extrema are used for the FSS analysis.

### 3 Numerical simulation and results

We have performed simulation on the following lattices with spherelike topology:

- SH[N], the surface of 5D hypercubes of size \( N^5 \);
- S[N], with the topology of SH[N] but introducing weight factors for the plaquette action: \( S = \sum P w_P S_P \), correcting for spherical shape. i.e.
distributing the curvature over the lattice (details about the geometry and these factors are given in [4]).

Most of our simulations were done on $S[N]$ lattices with $N = 4, 6, 8, 10$; these lattices have roughly the same number of variables as $6^4$, $9^4$, $12^4$, $16^4$ hypercubic lattices. In average we have performed about half a million sweeps for each lattice size, using a 3-hit Metropolis algorithm. We have measured the values and distribution functions (histograms) for $E_{HK}$ and $E$ and found no indication of double peaks. Unfortunately a multihistogram analysis combining various histograms determined at different values of $\beta$ is not applicable here because of the specific form of the action: $S_P$ does not depend linearly on $\beta$. However, we are able to do the analysis based on single histograms in $E$ in the vicinity of the PT and extrapolate into the “Wilson direction” $\beta_W$ based on the individual histogram,

$$\langle f(E) \rangle = \frac{1}{A} \sum_E h(E, \beta) f(E) e^{-\beta_W E}, \quad A \equiv \sum_E h(E, \beta) e^{-\beta_W E}, \quad (13)$$

where $h(E, \beta)$ denotes the histogram entry at $E$. The number of bins was taken sufficiently large (typically $> 2000$) to exclude systematic errors.

For each lattice size we try to simulate as close as possible to the peak position of the specific-heat. For this aim we first determined the cumulants as functions of $\beta_W$ from histograms at various values of $\beta$. This preparatory analysis was done at the statistics of 100K sweeps. From this we infer a relationship between $\beta_W(\text{peak})$ and $\beta$ and determine the presumed value of the corresponding pseudocritical coupling $\beta_0$. These are given in Table 1. Then we performed long runs (about 500K sweeps) at these couplings. The final results are histograms determined almost on top of the specific-heat peak. The individual histogram analysis allows us to extrapolate away from this point in coupling space into the direction $\beta_W$ and the table also shows this distance to the actual peak position of the specific-heat. This provides further (small) corrections.

For our analysis we use the values of the cumulants at their (extrapolated) maxima or minima, which agree within the errors with the values measured directly at $\beta_0$.

The final histograms are similar to those determined for the Wilson action [4] and show no 2-state signal. (Note that these are individual histograms and not the result of a combination.) We proceed to study the FSS of the extrema value of the specific-heat (maximum) and Binder cumulants $V_{CLB}$ and $U_4$.
Table 1: Couplings $\beta_0$ where the long runs were performed; we also give the extrapolation distance to the peak position of the specific-heat in direction of $\beta_W$ (as determined a posteriori from the histogram analysis).

(minima, cf. 2 3 for more details on the definitions and FSS properties) to check for consistent critical (i.e. second order ) FSS behavior, which in leading order may be parameterized

$$c_{V,max}(L) \approx a_c + b_c L^{\alpha/\nu},$$

$$V_{CLB,min}(L) \approx (a_u + b_u L^{\alpha/\nu})L^{-4},$$

$$U_{4,min}(L) \approx a_u + b_u L^{-\alpha/\nu}.$$  

In our analysis we assume validity of the hyperscaling relation $\alpha = 2 - D\nu$ ($D = 4$ in our case). The length scale $L \equiv V^{\frac{1}{4}}$, where the volume $V = \frac{1}{6} \sum_P w_P$, roughly proportional to the number of sites (in fact, this definition gives the number of sites in the situation of the usual hypercubic lattice with torus boundary conditions).

A log-log plot of the peak value of the specific-heat as a function of $L$ is shown in Fig. 1a. From the fit we obtain the value $\alpha/\nu = 1.479(135)$ and $\nu = 0.365(9)$ ($\chi^2/d.f. = 1.3$), which is far from what one expects in the case of a first order PT, where $\alpha/\nu = 4$ and $\nu = 1/D = 0.25$. This value is also compatible with the values obtained in 3 4. The cumulant $V_{CLB,min}$ is shown in Fig 1b; its FSS behavior is in good agreement with that of the specific-heat. The fit due to [14] leads to $\nu = 0.359(10)$ ($\chi^2/d.f. = 0.8$). The cumulant $U_{4,min}$ is compatible with the expected FSS; it approaches a constant $\simeq 2.75(3)$ but has too large errors to determine the nonleading scaling term, although it is consistent with the values of $\nu$ determined from the other quantities.

Let us discuss briefly the FSS of the pseudocritical couplings (as summarized in Table 1). Generally the FSS of pseudocritical couplings may be
written in the form
\[
\beta(V) = \beta(\infty) + cL^{-\lambda},
\]
where \(\lambda\) is the so-called shift-exponent. For many models \(\lambda = 1/\nu\), but this is not necessarily so in general (see the discussion in [17]; a recent study of the 2D Ising model on lattices with spherelike topology yielded \(\lambda \neq 1/\nu\) [18]). Therefore the FSS behavior of the pseudocritical coupling is not necessarily suitable for an independent determination of \(\nu\). A fit to the values in Table 1 gives \(\lambda = 2.71(26)\) – consistent with \(\lambda = 1/\nu\) within the large error bars – and \(\beta(\infty) = 0.6496(3)\). Note, that due to the geometry correction factors

Figure 1: The logarithm of the extrema values of (a) the specific-heat and (b) the Challa-Landau-Binder \(V_{CLB}\) cumulant vs \(\ln L\); the fits represent the leading FSS behavior given in (14).
this latter value does not have to agree with the corresponding value for torus topology. A more detailed discussion in a similar context can be found in [4].

Figure 2: The imaginary part of the closest Fisher zero as the function of the size $L$.

Another efficient way to determine the exponent $\nu$ lies in the FSS of the imaginary part of the (Fisher) zeroes $z_0$ of the partition function (in the complex coupling plane) closest to the real axis. The corresponding plot is shown on Fig.2; the expected scaling is $\text{Im} z_0 \propto L^{-1/\nu}$. From the fit one obtains $\nu = 0.366(12)$, using all lattice sizes ($\chi^2/\text{d.f.} = 0.5$). This is in excellent agreement with the values obtained from the cumulants.

We have also performed simulations on SH[N] lattices with $L = 4, 6, 8$. Since here the statistics was only 20% of that for the S[N] lattices we do not quote errors and did not analyze these data in more detail. The main result is that also for SH[N] we do not see any 2-state signal. The pseudocritical couplings $\beta_0$ are 0.64, 0.64375, 0.645 for corresponding lattice sizes. The peak values of the specific-heat are 1.48, 2.77, 3.32 correspondingly. All these results are close to those obtained on S[N] lattices.
4 Conclusion

We have studied the compact U(1) gauge theory with the Villain action on 4D lattices with spherelike topology. We do not observe any metastability (i.e. 2-state signal) in the distribution function of the internal energy. The FSS for various quantities is consistent with critical behavior. The obtained value of the critical exponent \( \nu \) is compatible with neither the Gaussian value 0.5 nor the “discontinuity” value 0.25. In our determination for the Villain action we get values in the range 0.36–0.37, a smaller range of values, but in perfect agreement with those obtained in [4] with mixed action on spherical lattices. Note that these actions are at quite separate points in the space of plaquette actions given by a character expansion.

These results indicate that below the Wilson line the deconfining PT is of second order and both the mixed action and the Villain action belong to one universality class of models with a nontrivial continuum limit.

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