ON THE POLlicOTT–RUELLE RESONANCES

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ABSTRACT. The purpose of this survey is to present the recent advances about the Pollicott–Ruelle resonances.

1. INTRODUCTION

Suppose that $M$ is a smooth compact manifold and let $\varphi : M \longrightarrow M$ be a smooth flow such that $M$ is considered a $\phi_t$–invariant set for some $t > 0$. The flow $\varphi_t$ is called Anosov flow if rather is a hyperbolic set for $\varphi_t$ in the sense of $[\text{KaHa}, \text{Definition 6.4.18}]$ and is generated by some smooth vector field $V$ such that $\varphi_t := e^{tV}$. Let $f, g \in C^\infty(M)$ be smooth functions and $\mu$ a $\phi$–invariant probability measure, then the correlation function is defined as

$$\rho_{f,g}(t) = \int_M f(\varphi_{-t}(x))g(x)d\mu, \quad \text{for any } x \in M. \quad (1.1)$$

The power spectrum of (1.1) is the Fourier transform such that

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty \rho_{f,g}(t)e^{i\lambda t}dt, \quad (1.2)$$

is meromorphic for $|\text{Im } \lambda| > 0$. Thus, the asymptotic behaviour of $\rho_{f,g}$ is controlled by the poles of the extension $\hat{\rho}_{f,g}$, such poles are known as the Pollicott–Ruelle resonances. In other words, they are complex numbers, which describe fine of decay of correlations for an Anosov Flow on a smooth compact manifold, and were initially studied by M. Pollicott $[\text{Po85, Po86}]$ and D. Ruelle $[\text{Ru86, Ru87}]$. From another point of view, the Pollicott–Ruelle resonances are also the singularities of the meromorphic extension of the Ruelle zeta function, which was conjectured by S. Smale in 1967 $[\text{Sm}]$. Such conjecture, has been proved by Giulietti–Liverani-Pollicot $[\text{GiLiPo}]$ for compact manifolds. Later, Arnoldi-Faure-Weich $[\text{AFW}]$ defined resonances on open hyperbolic surfaces and Faure–Tsujii $[\text{FaTsb}]$ defined resonances for the Grassmanian bundle of an Anosov flow. Recently, Dyatlov–Guillarmou $[\text{DyGu14}]$ were able to define Pollicott–Ruelle resonances for open hyperbolic systems on a more general way compared to $[\text{AFW, FaTsb}]$ via a microlocal approach of Faure–Sjöstrand $[\text{FaSj}]$ and Dyatlov–Zworski $[\text{DyZw13}]$, holding the results of $[\text{Po86, §7}]$ and as a consequence, they were able to show that the Ruelle zeta function extends meromorphically to the entire complex plane.
2. On the compact case

2.1. On the functional analysis proof. In 2012, Giulietti–Liverani–Pollicott [GiLiPo] showed the existence of Pollicott–Ruelle resonances for compact manifolds, such proof was given thought the Ruelle zeta function for $C^r$ Anosov flows for $r > 2$ on a compact smooth orientable manifold, where they proved that for $C^\infty$ flows the zeta function is meromorphic on the entire complex plane. Based on the statement that from $C^r$ flows, we can obtain a strip in which $\zeta_{\text{Ruelle}}(z)$ is meromorphic of width unboundedly increasing with $r$ [Fr], such work was an expansion of [GoLi, BuLi, LiTs, Lia, Lib, BaLi].

2.1.1. Definitions. In this subsection, we work with the following assumptions:

(B1) $M$ is a $d$–dimensional connected, compact and orientable $C^\infty$ Riemannian manifold for some $d \in \mathbb{Z}_+$, $V$ is a $C^\infty$ nonvanishing vector field on $M$, and $\varphi_t = e^{tV}$ is the corresponding flow;

(B2) for each $x \in M$, there is a splitting

$$ T_xM = E_s(x) \oplus E_0(x) \oplus E_u(x), $$

where $E_0$ is the one–dimensional subspace tangent to the flow, such that for some constants $C, \gamma > 0$

$$ |d\varphi_t(x) \cdot v| \leq Ce^{-\gamma|t|}|v| \quad \text{if } t \geq 0, v \in E_s; $$
$$ |d\varphi_t(x) \cdot v| \leq Ce^{-\gamma|t|}|v| \quad \text{if } t \leq 0, v \in E_u; $$
$$ C^{-1}|v| \leq |d\varphi_t(v)| \leq C|v| \quad \text{if } t \in \mathbb{R}, v \in E_0. $$

We denote $d_s = \dim(E_s)$ and $d_u = \dim(E_u)$ to distinguish the dimension of the stable and unstable subspaces, respectively.

In the context of the assumptions (B1)–(B2), we define the Ruelle zeta function as related to the Riemann zeta function (i.e., $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$), replacing $p$ by primitive closed orbits. Thus,

$$ \zeta_{\text{Ruelle}}(z) = \prod_{\tau \in T_p} (1 - e^{-z\lambda(\tau)})^{-1}, \quad z \in \mathbb{C}, $$

where $T_p$ denotes the set of prime orbits and $\lambda(\tau)$ denotes the periodic of the closed orbit $\tau$. According to [PaPo, §10], for weak mixing Anosov flows, the $\zeta_{\text{Ruelle}}(z)$ is analytic and nonzero for $\Re(z) \geq h_{\text{top}}(\varphi_1)$ apart for a single pole at $z = h_{\text{top}}(\varphi_1)$. In order to understand the whole case, some relevant definitions will be given below, however, the assertions of some concepts will be cited and won’t be part of the main proofs.
As part of the main definitions, Ruelle [Ru76] related the transfer operator with the dynamical Fredholm determinant and defined $D_\ell$ as the dynamical determinants, which are functions defined from weighted periodic orbit data of a differentiable dynamical system.

**Definition 2.1.** The dynamical (also known as Fredholm–Ruelle) determinant $D_\ell(z)$ are defined as

$$D_\ell(z) = \exp \left( -\sum_{\tau \in T} \frac{\text{tr}(\wedge^\ell(D_{\text{hyp} \varphi - \lambda(\tau)}) e^{-z \lambda(\tau)})}{\mu(\tau) \epsilon(\tau) |\det(1 - D_{\text{hyp} \varphi - \lambda(\tau)})|} \right),$$  

where $\epsilon(\tau)$ is 1 if the flow preserves the orientation of $E_s$ along $\tau$ and -1 otherwise. More precisely, $\epsilon(\tau) = \text{sign}(\det(D_{\varphi - \lambda(\tau)}|_{E_s}))$.

The symbol $D_{\text{hyp} \varphi - t}$ in Definition 2.1, indicates the derivative of the map induced by the local transverse sections to the orbit (one at $x$, the other at $\varphi_t(x)$) and can be represented as a $(d - 1) \times (d - 1)$-dimensional matrix. By $\wedge^\ell A$ we mean the matrix associated to the standard $\ell$–th exterior product of $A$ – see more about dynamical determinants in [Ba16, BaTs08]. Given any $\phi$–invariant probability measure $\mu$ on $M$ with $h_\mu(\phi)$ being the measure theoretic entropy of $\phi_1$. The topological entropy $h_{\text{top}}(\phi_1)$ can be defined by

$$h_{\text{top}}(\phi_1) \equiv \sup \{ h_\mu(\phi) : \mu \text{ is a } \phi \text{–invariant probability measure} \}.  \hspace{1cm} (2.5)$$

Thus, given $0 \leq \ell \leq d - 1$, $\tau \in T$, we let

$$\chi_\ell(\tau) = \frac{\text{tr}(\wedge^\ell(D_{\text{hyp} \varphi - \lambda(\tau)}))}{\epsilon(\tau) |\det(1 - D_{\text{hyp} \varphi - \lambda(\tau)})|},$$

in order to write Equation (2.4) in a shorter way as

$$D_\ell(z) = \exp \left( -\sum_{\tau \in T} \frac{\chi_\ell(\tau)}{\mu(\tau)} e^{-z \lambda(\tau)} \right).  \hspace{1cm} (2.6)$$

Now let us define $C^r$ sections of $\wedge^\ell(T^*M)$ as the space $\Omega^\ell_v(M)$ of $\ell$–forms on $M$ for all $v, \ell \in \mathbb{N}$.

**Definition 2.2.** Let $\Omega^\ell_{0,v} \subset \Omega^\ell_v(M)$ be the subspace of forms null in the flow direction, such that

$$\Omega^\ell_{0,v}(M) = \{ h \in \Omega^\ell_v(M) : h(V, \ldots) = 0 \}.$$

For a detailed construction and proof of Definition 2.2 – see [GiLiPo, §3]. Gouëzel–Liverani [GoLi], defined Banach spaces adapted to Anosov systems and was adapted by [GiLiPo] in the following way.

**Definition 2.3.** For all $p \in \mathbb{N}, q \in \mathbb{R}_+, \ell \in \{0, \ldots, d - 1\}$ we define the spaces $B^{p,q,\ell}$ to be the closures of $\Omega^\ell_{0,v}(M)$ with respect to the norm $\| \cdot \|_{p,q,\ell}$ and the spaces $B^{p,q,\ell}_+$ to be the closures of $\Omega^\ell_{0,v}(M)$ with respect to the norm $\| \cdot \|_{p,q,\ell}^+$.  

The construction of the $B$ space in Definition 2.3 is detailed in [GiLiPo, §3.2]. The sum in Equation (2.4) is well–defined, provided Re($z$) is large enough and follows a product analogous [GiLiPo, Equation (2.5)] such that
\[
\prod_{\ell=0}^{d-1} \mathcal{D}_\ell(z) (-1)^{\ell+1} = \zeta_{\text{Ruelle}}(z).
\tag{2.7}
\]
Now, to take care of the $t \leq t_0$, we introduce the dynamical norm $\|\cdot\|_{p,q,\ell}$ – see [GiLiPo, §4]. For each $h \in \Omega^\ell_{r}(M)$, we set
\[
\|h\|_{p,q,\ell} = \sup_{s \leq t_0} \|\mathcal{L}_s^{(\ell)} h\|_{p,q,\ell},
\tag{2.8}
\]
where $\mathcal{L}_t^{(\ell)}$ is a linear operator such that $\mathcal{L}_t^{(\ell)} : \Omega^\ell_{0,r-1}(M) \rightarrow \Omega^\ell_{0,r-1}(M)$, for some $t \in \mathbb{R}_+$. Furthermore,
\[
\mathcal{L}_t^{(\ell)} h := \varphi_{-t}^* h,
\tag{2.9}
\]
for some $h \in \Omega^\ell_{0,r-1}(M)$ – see more [GiLiPo, §4.1]. Thus, we define
\[
\widetilde{B}^{p,q,\ell} = \overline{\Omega^\ell_{0,r}} \|\cdot\|_{p,q,\ell} \subset B^{p,q,\ell}.
\]
Let $\lambda_{i,\ell}$ be the eigenvalues of $X^{(\ell)}$. Then for each $z \in B(\xi,\rho_{p,q,\ell})$, we let
\[
\widetilde{B}(\xi - z, \xi) = \left( \prod_{\lambda_{i,\ell} \in B(\xi,\rho_{p,q,\ell})} \frac{z - \lambda_{i,\ell}}{\xi - \lambda_{i,\ell}} \right) \psi(\xi, z),
\tag{2.10}
\]
where $\psi(\xi, z)$ is analytic and nonzero for $z \in B(\xi,\rho_{p,q,\ell})$. Thus, the Equation (2.10) shows that the poles of $\zeta_{\text{Ruelle}}$ are a subset of the eigenvalues of the $X^{(\ell)}$.

**Definition 2.4.** Given an operator $A \in L(B^{p,q,\ell}, B^{p,q,\ell})$, we define the flat trace as
\[
tr^\flat(A) = \lim_{\epsilon \to 0} \int_M \sum_{\alpha,\beta} \langle \omega_{\alpha,\beta}, A(j_{\epsilon,\alpha,\beta, x}) \rangle x \omega_M(x),
\tag{2.11}
\]
where $\omega_{\alpha,\beta}$ is the dual of 1–forms such that $\omega_{\alpha,\beta}(\hat{e}_{\alpha,\beta}) = \delta_{i,j}$ and $j_{\epsilon,\alpha,\beta, x}(y)$ is defined in [GiLiPo, Equation (5.2)], then provided the limit exists.

For a detailed proof of Definition 2.4 – see [GiLiPo, §5].

2.1.2. On the proof. We start stablishing in what region $\zeta_{\text{Ruelle}}(z)$ is meromorphic.

**Lemma 2.5.** For any $C^r$ Anosov flow $\varphi_t$ with $r > 2$, then $\zeta_{\text{Ruelle}}(z)$ is meromorphic in the region
\[
\text{Re}(z) > h_{\text{top}}(\varphi_1) - \frac{\lambda}{2} \left| r - \frac{1}{2} \right|
\tag{2.12}
\]
where $\lambda$ is determined by the Anosov splitting.
Lemma 2.5 follows by the study of dynamical determinants (Definition 2.1). In fact, to study in what region \( \zeta_{\text{Ruelle}} \) is meromorphic, we must study in what region the dynamical determinants are so. Moreover, for \( \xi, z \in \mathbb{C} \) we let

\[
\tilde{D}(\xi, z) = \exp \left( - \sum_{n=1}^{\infty} \frac{\xi^n}{n!} \sum_{\tau \in T} \frac{\chi_\ell(\tau)}{\mu(\tau)} \lambda(\tau)^n e^{-z\lambda(\tau)} \right), \tag{2.13}
\]

Using (2.13), for \( \text{Re}(z) \) sufficiently large and \( |\xi - z| \) sufficiently small, we can write

\[
\tilde{D}(\xi - z, \xi) = \exp \left( - \sum_{n=1}^{\infty} \frac{(\xi - z)^n}{n!} \sum_{\tau \in T} \frac{\chi_\ell(\tau)}{\mu(\tau)} \lambda(\tau)^n e^{-z\lambda(\tau)} \right)
= \exp \left( - \sum_{\tau \in T} \frac{\chi_\ell(\tau)}{\mu(\tau)} \left( e^{-z\lambda(\tau)} - e^{-\xi\lambda(\tau)} \right) \right)
= \frac{D_\ell(z)}{D_\ell(\xi)}. \tag{2.14}
\]

**Theorem 1.** \( \zeta_{\text{Ruelle}}(z) \) is analytic for \( \text{Re}(z) > h_{\text{top}}(\varphi_1) \) and nonzero for \( \text{Re}(z) > \max\{h_{\text{top}}(\varphi_1) - \frac{\lambda}{2}, h_{\text{top}}(\varphi_1) - \lambda\} \). Furthermore, if the flow is topologically mixing then \( \zeta_{\text{Ruelle}}(z) \) has no poles on the line \( \{h_{\text{top}}(\varphi_1) + ib\}_{b \in \mathbb{R}} \) apart from the single simple pole at \( z = h_{\text{top}}(\varphi_1) \).

By Theorem 1, \( \zeta_{\text{Ruelle}}(z) \) is meromorphic in the entire complex plane for smooth geodesic flows on any manifold that asserts the assumptions (B1)–(B2). Moreover, \( \zeta_{\text{Ruelle}}(z) \) has no zeroes or poles on the line \( \{h_{\text{top}}(\varphi_1) + ib\}_{b \in \mathbb{R}} \), except at \( z = h_{\text{top}}(\varphi_1) \) where \( \zeta_{\text{Ruelle}}(z)^{-1} \) has a simple zero. From here, [GiLiPo] specializes to contact Anosov flows. Let \( \lambda_+ \geq 0 \) such that \( \|D\varphi_{-t}\|_{\infty} \leq C_0 e^{\lambda_+ t} \) for all \( t \geq 0 \).

**Theorem 2.** For a contact Anosov flow \( \varphi_1 \in C^r \) where \( r > 2 \), with \( \frac{\lambda}{\lambda_+} > \frac{1}{3} \) there exists \( \tau_* > 0 \) such that the Ruelle zeta function is analytic in \( \{ z \in \mathbb{C} : \text{Re}(z) \geq h_{\text{top}}(\varphi_1) - \tau_* \} \) apart from a simple pole at \( z = h_{\text{top}}(\varphi_1) \).

**Proof.** Equation (2.23) and Equation (2.14) show that the poles of \( \zeta_{\text{Ruelle}}(z) \) are a subset of the eigenvalues of \( X^{(\ell)} \).

**Lemma 2.6.** For any \( C^r \) Anosov flow \( \varphi_t \) with \( r > 2, \xi \in \mathbb{C} \) and \( z \in D_{\ell}(\xi) \), then it is analytic and nonzero in the region \( \text{Re}(\xi) > h_{\text{top}}(\varphi_1) - \lambda |d_s - \ell| \).

**Proof.** Let \( \Omega_{0,v}(M) \subset \Omega_{\ell}(M) \) as Definition 2.2, let \( \mathcal{B}^{p,q,\ell} \) such that \( p \in \mathbb{N} \) and \( q \in \mathbb{R}_+ \) as Definition 2.3 and let \( L_{\ell}(h) \) as Equation (2.8) for some \( h \in \Omega_{0,v}(M) \). By restricting the transfer operator \( L_{\ell}(h) \) to the space \( \Omega_{0,r}(M) \) we mimic the action of the standard
transfer operators on sections transverse to the flow. The operators \((2.9)\) generalize
the action of the transfer operator \(L_t\) on the spaces \(B_{p,q}\). Thus,
\[
\|L^t_\ell h\|_{p,q,\ell} \leq C_{p,q} e^{\sigma_t} \|h\|_{p,q,\ell}, \tag{2.15}
\]
\[
\|h\|_{p,q,\ell} = \sup_{s \leq t_0} \|L^s_\ell h\|_{p,q,\ell}. \tag{2.16}
\]
By Equations \((2.15)\) and \((2.16)\) imply that for some \(t < t_0\), then
\[
\|L^t_\ell h\|_{p,q,\ell} \leq \max \{\|h\|_{p,q,\ell}, C_{p,q} e^{\sigma t}\|h\|_{p,q,\ell}\} \leq C_{p,q} \|h\|_{p,q,\ell},
\]
while for \(t \geq t_0\) the required inequality holds trivially. The boundedness of \(L^t_\ell\) follows.
The second inequality follows directly from the above, for small times and Equation \((2.15)\) for larger times. On \(\tilde{B}_{p,q,\ell}\) the operators \(L^t_\ell\) form a strongly continuous semigroup
with generators \(X(\ell)\) by the above. We consider the resolvent \(R(\ell)(z) = (z1 - X(\ell))^{-1}\), then we have the following Lemma.

**Lemma 2.7.** \(R(\ell)(z)\) is a quasi–compact operator on \(\tilde{B}_{p,q,\ell}\).

**Proof.** This follows by [GiLiPo, Lemma 3.8].

Although the operator \(X(\ell)\) is an unbounded closed operator on \(\tilde{B}_{p,q,\ell}\), we can access to its spectrum thanks to Lemma 2.7. Now, let us make \(\ell = d_s\) and let \(\tilde{\omega}_s\) be a
volume. A form on \(E_s\) normalized so that \(\|\tilde{\omega}_s\| = 1\) and it is globally continuous. Let \(\pi_s(x) = T_x M \rightarrow E_s(x)\) be the projections on \(E_s(x)\) along \(E_u(x) \oplus E_0(x)\) such that
\[
\omega_s(v_1, \ldots, v_{d_s}) = \tilde{\omega}_s(\pi_s v_1, \ldots, \pi_s v_{d_s})
\]
by construction \(\omega_s \in \Omega_{d_s}^{d_s}\). Note that \(\varphi_{-t}\omega_s = J_s \varphi_{-t}\) where \(J_s \varphi_{-t}\) is the Jacobian
restricted to the stable manifold. Note that, \(\omega_{s,\varepsilon} = M_{\varepsilon}\omega_s\), for \(\varepsilon\) small enough, we have
\[
\langle \omega_{s,\varepsilon}, \omega_s \rangle \geq \frac{1}{2}. \text{ Hence,}
\]
\[
\int_{W_{\alpha,G}} \langle \omega_{s,\varepsilon}, L_t \omega_s \rangle \geq \int_{W_{\alpha,G}} \frac{J_s \varphi_{-t}}{2} \geq C_\# \int_{W_{\alpha,G}} J_W \varphi_{-t} \geq C_\# \text{vol}(\varphi_{-t}W_{\alpha,G}). \tag{2.17}
\]
Then the Equation \((2.17)\) implies that the spectral radius of \(R(d_s)(a)\) on \(\tilde{B}_{q,\tilde{\omega},d_s}\) is exactly \((a - \sigma_{d_s})^{-1}\). Thus
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (a - \sigma_{d_s})^k (R(d_s)(a))^k = \left\{ \begin{array}{ll}
\prod \text{if}(a - \sigma_{d_s})^{-1} \in \sigma_{\tilde{B}_{p,q,\ell}}(R(d_s)(a)), & 0 \text{ otherwise,}
\end{array} \right. \tag{2.18}
\]

\(\Box\)
where $\prod$ is the eigenprojector on the associated eigenspace and the convergence takes places in the strong operator topology of $L(\widetilde{B}^{p,q,d_s}, \widetilde{B}^{p,q,d_s})$. Thus, by Equation (2.17),

$$\int_{W_{a,G}} \langle W_{s,\varepsilon}, \prod W_s \rangle > 0,$$

we have that $\prod \neq 0$ and $(a - \sigma_{d_s})^{-1}$ belongs to the spectrum. This implies, that if $\ell = d_s$, then $h_{\text{top}}(\varphi_1)$ is an eigenvalue of $X$ and if the flow is topologically transitive $h_{\text{top}}(\varphi_1)$ is a simple eigenvalue. Moreover, if the flow is topologically mixing, then $h_{\text{top}}(\varphi_1)$ is the only eigenvalue on the line $\{h_{\text{top}}(\varphi_1) + ib\}_{b \in \mathbb{R}}$. Thus, this proves Lemma 2.6. \hfill \Box

In the same time, Theorem 2 follows by Lemma 2.6. \hfill \Box

**Theorem 3** ([Po85, Theorem 2]). Let $\varphi : \Lambda \to \Lambda$ be a weak–mixing Axiom A flow, then the Fourier transform $\hat{\rho}_{f,g}(z)$ has a meromorphic extension to a strip $|\mathcal{F}(z)| \leq \varepsilon$, which is analytic on the real line. Furthermore, $\hat{\rho}_{f,g}(t)$ tends to zero exponentially fast (for all Hölder continuous functions $f, g : \Lambda \to \mathbb{R}$) only if $\zeta(s, F)$ has an analytic extension to some strip $R(s) > P(F) - \varepsilon$, except for the simple pole at $s = P(F)$.

**Theorem 4** ([ReSi, Paley-Wiener Theorem]). Let $\rho$ be in $\mathcal{S}'(\mathbb{R})$. Suppose that $\hat{\rho}$ is a function with an analytic continuation to the set $\{\zeta \mid \text{Im} \zeta < a\}$ for some $a > 0$. Suppose further that for each $\eta \in \mathbb{R}^n$ with $|\eta| < a$, $\hat{\rho}(\cdot + i\eta) \in L^1(\mathbb{R}^n)$ and for any $b < a$, $\sup_{|\eta| < b} \|\hat{\rho}(\cdot + i\eta)\|_1 < \infty$. Then $\rho$ is a bounded continuous function and for any $b < a$, there is a constant $C_b$ such that

$$|\rho(x)| \leq C_b e^{-b|x|}. \quad (2.19)$$

**Theorem 5.** Let $\xi$ as in Lemma 2.6, then the function $\tilde{\mathcal{D}}_{\ell}(\xi - z, \xi)$ is analytic and nonzero for $z$ in the region

$$|\xi - z| < \text{Re}(\xi) - h_{\text{top}}(\varphi_1) + |d_s - \ell| |\lambda| \quad (2.20)$$

and analytic in $z$, in the region

$$|\xi - z| < \text{Re}(\xi) - h_{\text{top}}(\varphi_1) + |d_s - \ell| |\lambda| + \frac{1}{2} \left| \frac{r - 1}{2} \right| \quad (2.21)$$

**Proof.** We can write the spectral decomposition $R^{(\ell)}(z) = P^{(\ell)}(z) + U^{(\ell)}(z)$ where $P^{(\ell)}(z)$ is a finite rank operator and $U^{(\ell)}(z)$ has spectral radius arbitrarily close to $\rho_{\text{ess}}(R^{(\ell)}(z))$. Let $\text{tr}^{\mathcal{H}}(R^{(\ell)}(z)^n) < \infty$ as in Definition 2.4, then it can be written as

$$\text{tr}^{\mathcal{H}}(R^{(\ell)}(z)^n) = \frac{1}{(n - 1)!} \sum_{\tau \in \mathcal{T}} \frac{\chi_{\ell}(\tau)}{\mu(\tau)} \lambda(\tau)^n e^{-z\lambda(\tau)}. \quad (2.22)$$
Then, we substitute Equation (2.22) in Equation (2.4) and we get that \( \tilde{D}_\ell(\xi, z) \) can be interpreted as the “determinant” of \((1 - \xi R(\ell)(z))^{-1}\), while \( D_\ell(z) \) can be interpreted as the “determinant” of \( zI - X(\ell) \). Thus,

\[
\tilde{D}(\xi - z, z) = \exp \left( - \sum_{i=0}^{\infty} \frac{(\xi - z)^n}{n} \text{tr}^\ell(R(\ell)(\xi)^n) \right) = \left( \prod_{\lambda_i, \ell \in B(\xi, \rho_{p,q,\ell})} \frac{z - \lambda_i, \ell}{\xi - \lambda_i, \ell} \right) \psi(\xi, z)
\]

(2.23)

where \( \psi(\xi, z) \) is analytic and nonzero for \( z \in B(\xi, \rho_{p,q,\ell}) \).

Furthermore, Theorem 5 implies Theorem 1. \( \square \)

**Theorem 6 ([GiLiPo, Corollary 2.7]).** The geodesic flow \( \varphi_t : M \to M \) for a compact manifold \( M \) with better than \( \frac{1}{9} \)-pinched negative section curvatures is exponentially mixing with respect to the Bowen–Margulis measure \( \mu \); that is; there exists \( \alpha \) such that for \( f, g \in C^\infty(T^1M) \) there exists a \( C > 0 \) for which the correlation function

\[
\rho(t) = \int f \circ \varphi_t g d\mu - \int f d\mu \int g d\mu,
\]

satisfies \( |\rho(t)| \leq C\# e^{-\alpha|t|} \), for all \( t \in \mathbb{R} \).

**Proof.** Consider the Fourier transform \( \hat{\rho}(s) = \int_{-\infty}^{\infty} e^{ist} \rho(t) dt \) of the correlation function \( \rho(t) \). By Theorem 3 and [Ru87, Theorem 4.1], the analytic extension of \( \zeta_{Ruelle}(z) \) in Theorem 2 implies that there exists \( 0 < \eta \leq \tau_* \) such that \( \hat{\rho}(s) \) has an analytic extension to a strip \( |\text{Im}(s)| < \eta \). Now, without the loss the generality, we fixed each value \( -\eta < t < \eta \), such we have that the function \( \sigma \mapsto \hat{\rho}(\sigma + it) \) is in \( L^1(\mathbb{R}) \). Finally we apply the Paley–Wiener Theorem 4 and result follows. \( \square \)

As related to the Equation (1.2), the poles in \( \hat{\rho}(s) \) of Theorem 6 are the Pollicott–Ruelle resonances on a compact manifold which asserts the assumptions \((B1)–(B2)\).

**2.2. A short microlocal proof.** Unlike [GiLiPo]; whose proofs only work for contact flows; Dyatlov-Zworski [DyZw13] proved in 2013, the meromorphic continuation of the Ruelle zeta function for \( C^\infty \) Anosov flows under the perspective of microlocal analysis, using semiclassical and scattering tools, and based on the study of the generator of the flow as a semiclassical differential operator. The proofs applies to any Anosov flow for which linearized Poincaré maps \( P_\gamma \), where \( \gamma \) is a closed orbit such that

\[
|\det(I - P_\gamma)| = (-1)^q \det(I - P_\gamma), \quad \text{with } q \text{ independent of } \gamma.
\]

Furthermore, the assumptions \((B1)–(B2)\) still hold in this subsection. Let us first list some important definitions, for a major literature – see more on microlocal analysis [HöI-II, HöIII-IV, Ve, Mea, Iv], semiclassical analysis [Zwa, GuSt, EvZw, Be] and scattering theory [Meb].
2.2.1. Definitions. Victor Guillermin [Gu], defined a Trace formula using distributional operations of pullback by some \( \iota(t, x) = (t, x, x) \) and some pushforward \( \pi : (t, x) \to t \) such that

\[
\text{tr}^\flat e^{-itP} := \pi_* \iota^* K e^{-itP},
\]

where \( K \) denotes the distributional kernel of operator [Gu, Theorem 6]. As Lars Hörmander claimed in [HöI-II, Theorem 8.2.4], the pullback is well-defined in the sense of distributions since

\[
\text{WF}(K e^{-itP}) \cap N^*(\mathbb{R}_t \times \Delta(X)) = \emptyset, \quad t > 0,
\]

where \( \Delta(X) \subset X \times X \) is the diagonal and \( N^*(\mathbb{R}_t \times \Delta(X)) \subset T^*(\mathbb{R}_t \times X \times X) \) is the conormal bundle. Thus, we define

**Definition 2.8** (Guillemin’s Trace Formula).

\[
\text{tr}^\flat e^{-itP} = \sum_{\gamma} T^\sharp_\gamma \delta(t - T_\gamma) \left| \det(I - P_\gamma) \right|,
\]

for some \( t > 0 \), (2.25)

where \( T_\gamma \) is the period of the orbit \( \gamma \), \( T^\sharp_\gamma \) is the primitive period, \( P_\gamma \) is the linearized Poincaré map and \( \delta(\bullet) \) is the Dirac delta function.

For a detailed proof of Equation (2.25) – see [DyZw13, §Appendix B] and [Gu, §II].

Let \( \text{WF}(u) \) be the wavefront set for some \( u \in D'(M) \) distribution. Since we do need a more robust measure of semiclassical regularity of functions, we define the semiclassical wavefront set \( \text{WF}_h \) in the sense of [Zwa, §8.4.2], for a parameter \( h \) such \( h \)-tempered families of distributions \( \{u(h)\}_{0 < h < 1} \).

**Definition 2.9.** The semiclassical wavefront set \( \text{WF}_h \subset \overline{T^*M} \) is a subset from the fiber-radially compactified cotangent–bundle (i.e., a manifold with interior \( T^*M \) and boundary \( \partial T^*M = S^*M = (T^*M\backslash 0)/\mathbb{R}^+ \), the cosphere bundle – see [DyZw, §E.1]). Furthermore, \( \text{WF}_h \) measures oscillations on the \( h \)–scale and if \( u \) is an \( h \)–independent distribution, then

\[
\text{WF}(u) = \text{WF}_h(u) \cap (T^*M\backslash 0).
\]

(2.26)

Now, we consider the semiclassical operator \( P \in \Psi^k_h(M, \text{Hom}(\mathcal{E})) \) where \( \mathcal{E} \) is a vector bundle over \( M \) such that it is acting on \( h \)–tempered families of distributions \( u(h) \in D'(M, \mathcal{E}) \). From Definition 2.9, we denote the natural projection

\[
\kappa : T^*M \to S^*M = \partial T^*M.
\]

(2.27)

Let \( L \subset T^*M \) be a closed conic invariant set under the flow \( e^{iH_p} \) such there is an open neighbourhood \( U \) of \( L \) – see more [HöI-II, §18.3]. Then, the Equation (2.27) asserts
that
\[ d(\kappa(e^{-tH_p}(U)), \kappa(L)) \to 0 \quad \text{as} \quad t \to +\infty; \]
\[(x, \xi) \in U \implies |e^{-tH_p}(x, \xi)| \geq C^{-1}e^{\theta t}|\xi|, \text{for any norm on the fibers and some } \theta > 0. \]

(2.28)

**Definition 2.10.** Let \( \kappa \) as in Equation (2.27) and \( L \) be a closed conic invariant that asserts Equation (2.28). Then, we say that \( L \) is called a radial source and if we reverse the direction of the flow, then \( L \) is called a radial sink.

By Definition 2.10 and letting \( E_s^*, E_0^*, E_u^* \) be the duals of \( E_s, E_0, E_u \), respectively, then by Equation (2.28), we say that \( E_s^* \) and \( E_u^* \) are a radial source and a radial sink, respectively. Let \( P \) as before such that \( P : C^\infty(M; \mathcal{E}) \to C^\infty(M; \mathcal{E}) \), besides

\[ P(u) = \frac{1}{i}L_vu, \quad \mathcal{E} = \bigoplus_{j=0}^n \Lambda^j(T^*M), \]

where \( V \) is the generator of the flow \( \varphi_t \), \( L \) denotes the Lie derivate and \( u \) is a differential form on \( M \).

**Definition 2.11 (Anisotropic Sobolev Spaces).** The Anisotropic Sobolev spaces are defined using the exponential weight – see [Zwa, Lemma 7.6] and [Zwa, Theorem 7.7]

\[ H_{sG} := \exp(-sG)(L^2(M)), \quad \|u\|_{H_{sG}} := \|\exp(sG)u\|_{L^2}, \]

(2.29)

where \( G \in \Psi^{0+}(M) \) satisfying

\[ \sigma(G)(x, \xi) = m_G \log|\xi|, \]

where \( m_G = 1 \) near \( E_s^* \) and \( m_G = -1 \) near \( E_u^* \).

For more about Anisotropic Sobolev spaces – see Duistermaat [Du], Unterberger [Un], Zworski [Zwa, §8.3] and Baladi–Tsujii [BaTs07].

2.2.2. *On the proof.* Now, let us use some essentials theorems from [DyZw13] in order to prove the existence of Pollicott–Ruelle resonances on the compact case via microlocal analysis.

**Theorem 7 ([DyZw13]).** Suppose \( M \) is a compact manifold and \( \varphi_t : M \to M \) is a \( C^\infty \) Anosov flow with orientable and unstable bundles. Let \( \{\gamma^\#\} \) denote the set of primitive orbits of \( \varphi_t \), with \( T^\#_\gamma \) their periodics. Then the Ruelle zeta function,

\[ \zeta_{Ruelle}(\lambda) = \prod_{\gamma^\#}(1 - e^{i\lambda T^\#_\gamma}), \]

(2.30)

which converges for \( \Im \lambda \gg 1 \), has a meromorphic continuation to \( \mathbb{C} \).

One of the main Propositions in [DyZw13], is:
Theorem 8 ([DyZw13, Proposition 3.4]). Fix a constant $C_0 > 0$ and $\varepsilon > 0$. Then for $s > 0$ large enough depending on $C_0$ and $h$ small enough, the operator

$$P_\delta(z) : D_{sG(h)} \to H_{sG(h)},$$

$$-C_0h \leq \text{Im} z \leq 1, \quad |\text{Re} z| \leq h^{\varepsilon},$$

is invertible, and the inverse, $R_\delta(z)$, satisfies

$$\|R_\delta(z)\|_{H_{sG(h)}\to H_{sG(h)}} \leq C h^{-1},$$

$$\text{WF}_h'(R_\delta(z)) \cap T^*(M \times M) \subset \Delta(T^*M) \cup \Omega_+,$$

with $\Delta(T^*M), \Omega_+$ defined in [DyZw13, Propostion 3.3], and $\text{WF}_h'(\bullet) \subset T^*(M \times M)$ is defined for an $h$–tempered family of operators $\bullet(h) : C^\infty(M) \to \mathcal{D}'(M)$.

Proof. The proof of Theorem 8 is assumed by $\|u\|_{H_{sG(h)}} \leq 1$ such that

$$\|u\|_{H_{sG(h)}} \leq C h^{-1} \|f\|_{H_{sG(h)}}, \quad u \in D_{sG(h)}, \quad f = P_\delta(z)u. \quad (2.31)$$

Then for some $A \in \Psi_0^0(M)$, we can get bounds on $Au$ as are detailed in [DyZw13, Proposition 3.4] which arrive to the Equation (2.31). \qed

From Theorem 8, we can deduce that:

1. $H_{sG}$ and $D_{sG}$ are topologically isomorphic to $H_{sG(h)}$ and $D_{sG(h)}$, respectively. And $Q_\delta : D_{sG} \to H_{sG}$ is smoothing and thus compact – see [DyZw13, Proposition 3.1]).

2. If $\text{Im} \lambda > C_1, u \in H_{sG} \subset H^{-s}$ and $(P - \lambda)u = f \in H_{sG}$, then

$$u = -\int_0^\infty \partial_t(e^{i\lambda t}\varphi_{-t}^* u)dt = i \int_0^\infty e^{i\lambda t} \varphi_{-t}^* f dt,$$

where the integrals converge in $H^{-s}$. This also implies that $(P - \lambda)$ is injective and invertible $D_{sG} \to H_{sG}$. Then

$$(P - \lambda)^{-1} = i \int_0^\infty e^{i\lambda t} \varphi_{-t}^* dt, \quad (2.32)$$

where $\varphi_{-t}^* : C^\infty(X; \mathcal{E}) \to C^\infty(X; \mathcal{E})$ is the pullback operator by $\varphi_{-t}$ on differential forms and the integral on the right–hand side converges in operator norm $H^s \to H^s$ and $H^{-s} \to H^{-s}$ – see [DyZw13, Proposition 3.2].

3. By [Zwa, §D.3], $R(\lambda) = R_H(\lambda) + \sum_{j=1}^{J(\lambda_0)} A_j/(\lambda - \lambda_0)^j$ where $\lambda_0$ is a near pole and $A_j$ are operators of finite rank such that

$$\Pi := -A_1 = \frac{1}{2\pi i} \oint_{\lambda_0} (\lambda - P)^{-1}d\lambda,$$

where $[\Pi, P] = 0$. Thus $A_j = -(P - \lambda_0)^j \Pi$ and $(P - \lambda_0)^{J(\lambda_0)} \Pi = 0$ – see [DyZw13, Proposition 3.3].
(4) Since $Q_\delta$ is pseudodifferential and supposing the fact that

$$R(\lambda) = h(R_\delta(z) - iR_\delta(z)Q_\delta R_\delta(z)) - R_\delta(z)Q_\delta R(\lambda)Q_\delta R_\delta(z),$$

we get that

$$WF'_h(R_\delta(z) - iR_\delta(z)Q_\delta R_\delta(z)) \cap T^*(M \times M) \subset \Delta(T^*M) \cup \Omega_+$$

– see [DyZw13, Proposition 3.3].

The Pollicott–Ruelle resonances are the poles of Re($\lambda$) in the region Im $\lambda > -C_0$ of the meromorphic continuation of the Schwartz Kernel of the operator given by the right-hand side of (2.32), and thus are independent of the choice of $s$ and the weight $G$. For the microlocal proof of the meromorphic continuation of Theorem 7 – see [DyZw13, §4].

2.2.3. Further developments. Many applications had been development since the proof of those methods:

- Dyatlov–Zworski [DyZw15], showed that Pollicott–Ruelle resonances are the limits of eigenvalues of $V/i + i\varepsilon\delta_g$, as $\varepsilon \to 0^+$, where $-\delta_g$ is any Laplace–Beltrami operator on $X$.

- Jin–Zworski [JiZw], proved that for any Anosov flows there exists a strip with infinitely many resources and a counting function which cannot be sublinear.

- Colin Guillarmou [G1], studied regularity properties of cohomological equations and provides applications. Guillarmou [G2] also established a deformation lens rigidity for a class of manifolds including manifolds with negative curvature and strictly convex boundary.

- Dyatlov–Guillarmou [DyGu14], proved meromorphic continuation for $(P - \lambda)^{-1}$ and zeta functions for non–compact manifolds with compact hyperbolic trapped sets.

- Dyatlov–Faure–Guillarmou [DyFaGu], described the complex poles of the power spectrum of correlations for the geodesic flow on compact hyperbolic manifolds in terms of eigenvalues of the Laplacian on certain natural tensor bundles.

- Dyatlov used [DyZw13, Proposition 2.4] and [DyZw13, Proposition 2.5] in [Dya] as part to establish a resonance free strip for condimension 2 symplectic normally hyperbolic trapped sets. To see a major literature about resonances for infinite–area hyperbolic surfaces – see [Bo16].

- Dyatlov–Zworski [DyZw15], used microlocal methods similar to [DyZw13] in order to show stochastic stability of Pollicott–Ruelle resonances, more precisely, let $P_\mathcal{E} = \frac{1}{\tau}V + i\mathcal{E}\Delta_\mathcal{E}$ and let $\{\lambda_j(\mathcal{E})\}_0^\infty$ be the set of its $L^2$–eigenvalues. Furthermore, let $\{\lambda_j\}_{j=0}^\infty$ be the set of the Pollicott–Ruelle resonances of the flow $\varphi_t$, then $\lambda_j(\mathcal{E}) \to \lambda_j$ as $\mathcal{E} \to 0^+$ with convergence uniform for $\lambda_j$ in a compact set – see the proof in [DyZw15, §5].
• Similar to [DyZw15], Zworski [Zwc] showed scattering resonances of $-\Delta + V$ where $V \in L^\infty_c(\mathbb{R}^n)$, are the limits eigenvalues of $-\Delta + V - i\varepsilon x^2$ as $\varepsilon \to 0^+$ via complex scaling method [Zwc, §2] – to see more about scattering resonances [DyZw, Zwb].

• Alexis Drouot [Dr], showed that for a compact manifold and negatively curved $M$, the $L^2$–spectrum of the infinitesimal generator of the Kinetic Brownian motion on the cosphere bundle as a stochastic process modeled by the geodesic equation perturbed with a random force of size $\varepsilon$, converges to the Pollicott–Ruelle resonances as $\varepsilon$ goes to 0.

3. ON THE OPEN SYSTEMS CASE

In 2014, Dyatlov–Guillemard [DyGu14] defined Pollicott–Ruelle resonances for open systems, more precisely, geodesic flows on noncompact asymptotically hyperbolic negatively curved manifolds, as well as for more general open hyperbolic systems related to Axiom A flows. They used many generalized microlocal tools from [DyZw13, FaSj] and functional analysis tools from [GiLiPo], and used anisotropic Sobolev spaces to control the singularities at fiber infinity, and using complex absorbing potentials on the boundary and complex absorbing pseudodifferential operators beyond the boundary to obtain a global Fredholm problem for the extension of $X$ to a compact manifold without boundary $\mathcal{M}$ – see more [DyGu14, §4].

3.1. Definitions. We use the same notation as in [DyGu14], given a $n$–dimensional compact manifold $\overline{U}$ with interior $U$ and boundary $\partial U$, then $X$ is a smooth $C^\infty$–nonvanishing vector field on $\overline{U}$ such that for some $t$, the corresponding flow is defined as $\varphi^t = e^{tX}$. Furthermore, $\partial U$ is strictly convex (i.e., for some $x \in U$ then $X \rho(x) = 0 \implies X^2 \rho(x) < 0$ where $\rho \in C^\infty(\overline{U})$).

**Definition 3.1.** The incoming ($\Gamma_+$) and outgoing ($\Gamma_-$) tails are subsets from $\overline{U}$ such that

$$\Gamma_\pm = \bigcap_{t \geq 0} \varphi^t(\overline{U}).$$  \hspace{1cm} (3.1)

From Definition 3.1, let $K = \Gamma_+ \cap \Gamma_-$ be the trapped set such that for some $x \in K$ there is a splitting in $T_x\mathcal{M}$ in the sense of Equation (2.1) and Equation (2.2), where $\mathcal{M}$ is a compact manifold without boundary such that $\overline{U}$ is embedded in $\mathcal{M}$ – see more about dynamical assumptions of the trapped set in [Dyb, §3.5.1]. Now, let $\mathcal{E}$ be the smooth complex vector bundle over $\overline{U}$ and the first order differential operator $X : C^\infty(\overline{U}; \mathcal{E}) \longrightarrow C^\infty(\overline{U}; \mathcal{E})$ such that

$$X(fu) = (Xf)u + f(Xu), \quad f \in C^\infty(\overline{U}), \, u \in C^\infty(\overline{U}; \mathcal{E}).$$  \hspace{1cm} (3.2)
Fixing a smooth measure $\mu$ on $\mathcal{M}$ and the norm $L^2(\mathcal{M}; \mathcal{E})$, we define the transfer operator $e^{-t\mathbf{X}} : L^2(\mathcal{M}; \mathcal{E}) \to L^2(\mathcal{M}; \mathcal{E})$ and by Equation (3.2) we have that the support of $e^{-t\mathbf{X}}$ is:

$$e^{-t\mathbf{X}}(f\mathbf{u}) = (f \circ \varphi^{-t})e^{-t\mathbf{X}}\mathbf{u}, \quad f \in C^\infty(\mathcal{M}), \mathbf{u} \in C^\infty(\mathcal{M}; \mathcal{E}).$$  \hspace{1cm} (3.3)

**Definition 3.2.** Let $\mathbf{R}$ be the restricted resolvent defined as

$$\mathbf{R}(\lambda) = 1_{\mathcal{U}}(\mathbf{X} + \lambda)^{-1}1_{\mathcal{U}} : C^\infty_0(\mathcal{U}; \mathcal{E}) \to \mathcal{D}'(\mathcal{U}; \mathcal{E}), \quad \text{Re} \lambda > C_0,$$  \hspace{1cm} (3.4)

where $\lambda \in \mathbb{C}$. Then, for each $j \geq 1$ the space of generalized resonant states

$$\text{Res}_\mathbf{X}(\lambda) = \{ \mathbf{u} \in \mathcal{D}'(\mathcal{U}; \mathcal{E}) \mid \text{supp} \mathbf{u} \subset \Gamma_+, \text{WF}(\mathbf{u}) \subset E^*_+, (\mathbf{X} + \lambda)^j(\mathbf{u}) = 0 \},$$  \hspace{1cm} (3.5)

where $E^*_+ \supset E^*_u$ is the extended unstable bundle over $\Gamma_+$. For a detailed construction of $E^*_+$ in Definition 3.2, see [DyGu14, Lemma 2.10]. Furthermore, the subbundle $E^*_+$ is a generalized radial sink and $E^*_+ \supset E^*_u$ is a generalized radial source; which is a modification from Equation (2.28). As related to Definition 2.11, [DyGu14] defined the anisotropic Sobolev space $\mathcal{H}^r_h$; in order to control the singularities at fiber infinity; as

$$\mathcal{H}^r_h = \exp(-rG(h))(L^2(\mathcal{M}; \mathcal{E})), \quad \Vert \mathbf{u} \Vert_{\mathcal{H}^r_h} = \Vert \exp(rG(h))\mathbf{u} \Vert_{L^2(\mathcal{M}; \mathcal{E})},$$  \hspace{1cm} (3.6)

where $G$ is the operator defined as $G(h) \in \bigcap_{\lambda > 0} \Psi^h_k(\mathcal{M})$ — see more [DyGu14, §4.1] and for the propagation of singularities [DyZw13, Proposition 2.5]. Now, let $V \in C^\infty(\mathcal{U}; \mathbb{C})$, then $\gamma^\varphi : [0, T_{\gamma^\varphi}] \to K$ of $\varphi_t$ of period $T_{\gamma^\varphi}$, thus

$$V_{\gamma^\varphi} = \frac{1}{T_{\gamma^\varphi}} \int_0^{T_{\gamma^\varphi}} V(\gamma^\varphi(t))dt$$  \hspace{1cm} (3.7)

be the average of $V$ over $\gamma^\varphi$. Thus, we define the Ruelle zeta function as the product over all primitive closed trajectories of $\varphi^t$ on $K$:

$$\zeta_{\text{Ruelle}}(\lambda) = \prod_{\gamma^\varphi}(1 - \exp(-T_{\gamma^\varphi}^\varphi(\lambda + V_{\gamma^\varphi}))), \quad \text{Re} \lambda \gg 1.$$  \hspace{1cm} (3.8)

For express Pollicott–Ruelle resonances of $\mathbf{X}$ as poles, let $\mathcal{E}_0$ be the vector bundle over $\mathcal{U}$ by

$$\mathcal{E}_0(x) = \{ \eta \in T^*_x\mathcal{M} \mid \langle X(x), \eta \rangle = 0 \}, \quad x \in \mathcal{U},$$

and let $\mathcal{P}_{x,t} : \mathcal{E}_0(x) \to \mathcal{E}_0(\varphi^t(x))$ be the Poincaré map such that $\mathcal{P}_{x,t} = (d\varphi^t(x))^{-1}\mathcal{P}_{x,0}(x)$. Now, for each $\mathbf{u} \in C^\infty(\mathcal{M}; \mathcal{E})$, we put $\alpha_{x,t}(\mathbf{u}(x)) = e^{-t\mathbf{X}}\mathbf{u}(\varphi^t(x))$ where $\alpha_{x,t}$ is the parallel transport defined as $\alpha_{x,t} : \mathcal{E}(x) \to \mathcal{E}(\varphi^t(x))$, then if $\mathbf{u}(x) = 0$ implies that $e^{-t\mathbf{X}}\mathbf{u}(\varphi^t(x)) = 0$ by Equation (3.3). Thus, for the operator $\alpha_{\varphi^t(x_0), T} : \mathcal{E}(\varphi^t(x_0)) \to \mathcal{E}(\varphi^t(x_0))$ where $T > 0$ we have that:

$$\text{tr} \alpha_{\varphi^t(x_0)} = \text{tr} \alpha_{\varphi^t(x_0), T}, \quad \det(I - \mathcal{P}_{\gamma}) = \det(I - \mathcal{P}_{\varphi^t(x_0), T}) \neq 0.$$  \hspace{1cm} (3.9)
Definition 3.3. Using the wavefront set $WF$ in the sense of Definition 2.9, of any $u \in \mathcal{D}'(\mathcal{M})$ and considering wavefront sets $WF^i(B) \subset T^*(\mathcal{M} \times \mathcal{M}) \setminus 0$, where $B : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ are operators, we define

$$WF^i(B) = \{(x, \xi, y, -\eta) \mid (x, \xi, y, \eta) \in WF(K_B)\},$$

(3.10)

where the Schwartz Kernel $K_B \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ is given by

$$Bf(x) = \int_{\mathcal{M}} K_B(x, y)f(y)dy, \quad f \in C^\infty(\mathcal{M}).$$

(3.11)

Let $V, W \in \overline{T}^*\mathcal{M}$ be open sets, such that $e^{-THp}(x, \xi) \in V$ and $e^{-tHp}(x, \xi) \in W$ for $t \in [0, T]$. We denote the open subset

$$Con_p(V; W) \subset \overline{T}^*\mathcal{M},$$

the set of such points – see [DyGu14, Proposition 2.5]. Let $A, B, B_1 \in \Psi_0^0(\mathcal{M})$ be operators such that $q \geq 0$ near $WF_h(B_1)$, where $WF_h \subset Con_p(\ell_h(B); \ell_h(B_1))$, thus the trajectories of $e^{-THp}$ starting on $WF_h(A)$ either pass through $\ell_h(B)$ or converge to some closed set $L$, while staying on $\ell_h(B_1)$ – see [DyGu14, Definition 3.3].

3.2. On the proof. Dyatlov–Guillarmour used sharp Gårding inequality – see [Zwa, §4.7], in [DyGu14, Lemma 3.4] to show that the definition of real part Re $P$ is not trivial, that is, assume that $P \in \Psi_h^{2m+1}(\mathcal{M}; \mathcal{E})$ is principally scalar, $A \in \Psi_h^0(\mathcal{M})$, and Re $\sigma_h(P) \leq 0$ in a neighborhood $U \subset \overline{T}^*\mathcal{M}$ of $WF_h(A)$. Then, there exist a constant $C$ such that for each $N$ and $u \in H_{h, n}^{m+1/2}(\mathcal{M}, \mathcal{E})$,

$$\text{Re}(PAu, Au)_{L^2} \leq Ch \|Au\|_{H_{h, n}^m}^2 + O(h^\infty) \|u\|_{H_{h, n}^{-N}}^2.$$  

(3.12)

Furthermore, if $L$ and $P$ satisfies that $\text{Im}(P - iQ) \lesssim -h$ on $H_{h, n}^m$ near $L$, for all $s$, where $Q \in \Phi_h^m(\mathcal{M})$, $\text{Im} \sigma_h(P) \leq 0$ near $L$ and $\text{Re} \sigma_h(Q) > 0$ on $L$. Then, for some aditional $p := \text{Re} \sigma_h(P) \in \text{Hom}^1(T^*\mathcal{M}; \mathbb{R})$ and assuming that $L \subset \partial \mathcal{T}^*\mathcal{M}$, where $L$ is invariant under $e^{THp}$. Fix a metric $|\cdot|$ on the fibers of $T^*\mathcal{M}$. Then,

1. Assume that there exist $c, \gamma > 0$ such that

$$\frac{|e^{\xi T Hp(x, \xi)}|}{|\xi|} \geq ce^{\gamma |t|} \quad \text{for } (x, \xi) \in L, t \leq 0.$$  

(3.13)

Then there exists $s_0$ such that for all $s > s_0$, $\text{Im} P \lesssim -h$ near $L$ on $H_{h, n}^s$.

2. Assume that there exist $c, \gamma > 0$ such that

$$\frac{|e^{\xi T Hp(x, \xi)}|}{|\xi|} \geq ce^{\gamma |t|} \quad \text{for } (x, \xi) \in L, t \geq 0.$$  

(3.14)

Then there exists $s_0$ such that for all $s < s_0$, $\text{Im} P \lesssim -h$ near $L$ on $H_{h, n}^s$.

For the proofs of Equations (3.12), (3.13) and (3.14) – see [DyGu14, §3].
Theorem 9. The family of \( \{ \mathbf{R}(\lambda) \} \), defined in the sense of Equation (3.4), continues meromorphically to \( \lambda \in \mathbb{C} \), with poles of finite rank.

Theorem 10 ([DyGu14, Theorem 4]). Define for \( \Re \lambda \gg 1 \)

\[
F_X(\lambda) = \sum_{\gamma} e^{-\lambda T_\gamma T_\gamma^*} \frac{\text{tr} \alpha_\gamma}{|\det(I - P_\gamma)|},
\]

(3.15)

where the sum is over all closed trajectories \( \gamma \) inside \( K, T_\gamma > 0 \) is the period of \( \gamma \), and \( T_\gamma^* \) is the primitive period. Then \( F(\lambda) \) extends meromorphically to \( \lambda \in \mathbb{C} \). The poles of \( F(\lambda) \) are the Pollicott–Ruelle resonances of \( X \) and the residue at a pole \( \lambda_0 \) is equal to the rank of \( \Pi_{\lambda_0} \).

Proof. We define the flat trace in the sense of the operator \( A : C^\infty(M; U) \to \mathcal{D}'(M; U) \) such that \( \text{WF}'(A) \cap \Delta(T^*M\backslash 0) = \emptyset \), then

\[
\text{tr}^b A = \int_M \text{tr}_{\text{End}(E)} K_A(x,x) dx.
\]

(3.16)

Making

\[
F_X(\lambda) = \text{tr}^b(\chi e^{-t_0(X+\lambda)}\mathbf{R}(\lambda)\chi)
\]

(3.17)

for \( \lambda > C_1, C_1 > 0, \chi \in C^\infty_0(U) \) and \( t_0 > 0 \) is small enough so that \( t_0 < T_\gamma \) for all \( \gamma \). Then, by [DyGu14, Theorem 2] and Equation (3.17) we have that

\[
\text{tr}^b \sum_{j=1}^{J(\lambda_0)} (-1)^{j-1} \frac{\chi e^{-t_0(X+\lambda)}(X + \lambda_0)^j \Pi_{\lambda_0} \chi}{(\lambda - \lambda_0)^j} = \frac{\text{rank} \Pi_{\lambda_0}}{\lambda - \lambda_0} + \text{Hol}(\lambda),
\]

(3.18)

where \( \text{Hol}(\lambda) \) is holomorphic near \( \lambda_0 \).

The proof (and in fact the work of Dyatlov–Guillarmou) is really complex, for a full detailed and several particular cases of Theorem 10 – see [DyGu14, §4]. By [DyGu14, Lemma 4.3] and [DyGu14, Lemma 3.3], the operator \( -ih \Pi_{\lambda_0} \mathbf{R}_q(ih\lambda) \Pi_{\lambda_0} \) gives the meromorphic continuation of \( \mathbf{R}(\lambda) \) in the region \([-C_1, h^{-1}] + i[-C_2, C_2]\) for \( h \) small enough. Since \( C_1 \) and \( C_2 \) can be chosen arbitrary and \( h \) can be arbitrarily small, we obtain the continuation to the entire complex plane and Theorem 9 follows.

\[
\square
\]

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