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Remarks on the Radiative Transfer Equations for Climatology

Claude Bardos¹, François Golse² and Olivier Pironneau³

Abstract
Using theoretical and numerical arguments we discuss some of the commonly accepted approximations for the radiative transfer equations in climatology.

Keywords: Radiative transfer, Electronic Temperature, Integral equation, Numerical analysis

AMS 3510, 35Q35, 35Q85, 80A21, 80M10

1. Introduction
Satellite, atmospheric and terrestrial measurements are numerous to support the global warming. Various hypothesis are made to explain the disastrous effect of the Green-House Gases (GHG). However a rigorous proof derived from the fundamental equations of physics is not available and one of the problems is the complexity of the physical system of planet Earth in its astronomical setting around the Sun.

In many references such as [8],[9],[2] and [4] it is explained that the Sun radiates light with a heat flux $Q = 1370 \text{Watt/m}^2$, in the frequency range $(0.5, 20) \times 10^{14} \text{Hz}$ corresponding approximately to a black body at temperature of 5800K; 70% of this light intensity reaches the ground because the atmosphere is almost transparent to this spectrum and about 30% is reflected back by the clouds or the ocean, snow, etc (albedo). The Earth behaves almost like a black body at temperature $T_e = 288 \text{K}$ and as such radiates rays of frequencies $\nu$ in the infrared spectrum $(0.03, 0.6) \times 10^{14} \text{Hz}$.

"The absorption coefficient variation with frequencies is such that the atmosphere is essentially transparent to solar radiation" ([4], p65).

"Solar radiation that is not absorbed or reflected by the atmosphere (for example by clouds) reaches the surface of the Earth. The Earth absorbs most of the energy reaching its surface, a small fraction is reflected". Carbone dioxide renders the atmosphere opaque to infrared radiations around 20THz . Hence increasing its proportion in air increases the absorption coefficient in that range. It is believed to be one of the causes of global warming. However it must be more complex because some experimental measurements show that increasing opacity decreases the temperature at high altitude, above the clouds [3].

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⁴https://public.wmo.int/en/sun%E2%80%99s-impact-earth

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In the framework of the radiative transfer equations, one can studies the conjectures cited above. To support theoretical claims, the numerical solver developed in [1], [10] and [7] is used. We will address 4 questions:

- What is the effect of increasing an altitude dependent absorption coefficient?
- Is it true that sunlight crosses the Earth atmosphere unaffected?
- Is 30% of albedo equivalent to 30% reduction of the Sun radiative energy?
- What is the effect on the temperature of increasing the absorption coefficient in a specific frequency range?

2. The Radiative Transfer Equations for a Stratified Atmosphere

Finding the temperature $T$ in a fluid heated by electromagnetic radiations is a complex problem because interactions of photons with atoms in the medium involve rather intricate quantum phenomena. Assuming local thermodynamic equilibrium leads to a well-defined electronic temperature. In that case, one can write a kinetic equation for the radiative intensity $I_\nu(x, \omega, t)$ at time $t$, at position $x$ and in the direction $\omega$ for photons of frequency $\nu$, in terms of the temperature field $T(x, t)$.

$$\frac{1}{c} \partial_t I_\nu + \omega \cdot \nabla I_\nu + \rho \bar{\kappa}_\nu a_\nu \left[ I_\nu - \frac{1}{4\pi} \int_{S^2} p(\omega, \omega') I_\nu(\omega') d\omega' \right] = \rho \bar{\kappa}_\nu (1 - a_\nu) [B_\nu(T) - I_\nu].$$  \hfill (1)

In this equation $c$ is the speed of light, $\rho$ is the density of the fluid, $\nabla$ designates the gradient with respect to the position $x$, while

$$B_\nu(T) = \frac{2\hbar \nu^3}{c^2 [e^{\frac{\hbar \nu}{kT}} - 1]}$$  \hfill (2)

is the Planck function at temperature $T$, with $\hbar$ the Planck constant and $k$ the Boltzmann constant. Recall the Stefan-Boltzmann identity,

$$\int_0^\infty B_\nu(T) d\nu = \bar{\sigma} T^4, \quad \bar{\sigma} = \frac{2\pi^4 k^4}{15 c^2 \hbar^3},$$  \hfill (3)

where $\pi \bar{\sigma}$ is the Stefan-Boltzmann constant.

The intricacy of the interaction of photons with the atoms of the medium is contained in the mass-absorption $\bar{\kappa}_\nu$, which is theoretically the result of vibration and rotation atomic resonance, but for practical purpose the fraction of radiative intensity at frequency $\nu$ that is absorbed by fluid per unit length. The coefficient $a_\nu \in (0, 1)$ is the scattering albedo, and $\frac{1}{4\pi} p(\omega, \omega') d\omega'$ is the probability that an incident ray of light with direction $\omega'$ scatters in the infinitesimal element of solid angle $d\omega$ centered at $\omega$.

The kinetic equation (1) is coupled to the temperature - or energy conservation - equations of the fluid. When thermal diffusion is small, the system decouples and energy balance becomes:

$$\int_0^\infty \rho \bar{\kappa}_\nu (1 - a_\nu) \left( \int_{S^2} I_\nu(\omega) d\omega - 4\pi B_\nu(T) \right) d\nu = 0.$$  \hfill (4)
When the wave source is far in the direction \( z \) the problem becomes one-dimensional in \( z \) and the radiative intensity scattered in the direction \( \omega \) depends only on \( \mu \), the cosine of the angle between \( \omega \) and \( Oz \). Consequently the system becomes,

\[
\mu \partial_z I_\nu + \rho \kappa_\nu I_\nu = \rho \kappa_\nu (1 - a_\nu) B_\nu(T) + \frac{\rho \kappa_\nu a_\nu}{2} \int_{-1}^{1} p(\mu, \mu') I_\nu(z, \mu') d\mu',
\]

\[
\int_{0}^{\infty} \rho \kappa_\nu (1 - a_\nu) \left( B_\nu(T) - \frac{1}{2} \int_{-1}^{1} I_\nu d\mu \right) d\nu = 0, \quad z \in (0, H), \quad |\mu| < 1, \quad \nu \in \mathbb{R}^+.
\]  

For this system to be mathematically well posed, the radiation intensity \( I_\nu \) must be given on the domain boundary where radiation enters. For example,

\[
I_\nu(H, -\mu) = Q^{-}(\mu), \quad I(0, \mu) = Q^{+}(\mu), \quad 0 < \mu < 1.
\]

2.1. Semi-analytical Solution when the scattering is isotropic

Isotropic scattering is modelled by taking \( p \equiv 1 \). By introducing an optical depth (in [4] and others, the signs are changed),

\[
\tau = \int_{0}^{z} \rho(\eta) d\eta, \quad \rho \text{ becomes 1 and } z \text{ becomes } \tau \in (0, Z), \quad Z = \int_{0}^{H} \rho(\eta) d\eta.
\]

Let us denote the exponential integral by

\[
E_{p}(X) := \int_{0}^{1} e^{-X/\mu} \mu^{p-2} d\mu.
\]

Define

\[
J_\nu(\tau) = \frac{1}{2} \int_{-1}^{1} I_\nu d\mu, \quad S_\nu(\tau) = \frac{1}{2} \int_{0}^{1} \left( e^{-\frac{\kappa_\nu \tau}{\mu}} Q_{\nu}^{+}(\mu) + e^{-\frac{\kappa_\nu (Z - \tau)}{\mu}} Q_{\nu}^{-}(\mu) \right) d\mu.
\]

The problem is equivalent to a functional integral equation:

\[
\begin{aligned}
J_\nu(\tau) &= S_\nu(\tau) + \frac{\kappa_\nu}{2} \int_{0}^{Z} E_{1}(\kappa_\nu |\tau - t|) \left( (1 - a_\nu) B_\nu(T(t)) + a_\nu J_\nu(T(t)) \right) dt, \\
\int_{0}^{\infty} (1 - a_\nu) \kappa_\nu B_\nu(\tau(t)) dt &= \int_{0}^{\infty} (1 - a_\nu) \kappa_\nu J_\nu(\tau(t)) dt,
\end{aligned}
\]

Once \( J_\nu, T \) are known, \( I_\nu \) can be recovered by

\[
I_\nu(\tau, \mu) = e^{-\frac{\kappa_\nu \tau}{\mu}} I_\nu(0, \mu) 1_{\mu > 0} + e^{-\frac{\kappa_\nu (Z - \tau)}{\mu}} I_\nu(Z, \mu) 1_{\mu < 0}
\]

\[
+ 1_{\mu > 0} \int_{0}^{\tau} e^{-\frac{\kappa_\nu |\tau - t|}{|\mu|}} \left( (1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t) \right) dt
\]

\[
+ 1_{\mu < 0} \int_{\tau}^{Z} e^{-\frac{\kappa_\nu |\tau - t|}{|\mu|}} \left( (1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t) \right) dt.
\]

Remark 1. Notice that two solutions \((T, J_\nu), (T', J'_\nu)\) with \( J_\nu \equiv J'_\nu \) will have \( T \equiv T' \) but not necessarily \( I_\nu \equiv I'_\nu \).
3. Increasing an Altitude Dependent Absorption Coefficient Decreases the Temperature in the atmosphere

The following argument goes against a common belief which claims that increasing the absorption coefficient - which GHG do indeed - implies an increase of temperature in the atmosphere. Consider the case of an altitude dependent absorption coefficient which we write as $r(z)\kappa$ with $\kappa$ constant.

**Proposition 1.** If $\kappa$ is independent of $\nu$ but function of the altitude and $T$ is monotone decreasing with altitude, then, increasing $\kappa$ anywhere will decrease the temperature.

**Proof** The proof is a straightforward consequence of the concept of optical length (7). Here, let $\tau = \int_0^z r(\zeta)d\zeta$. Observe that $\partial_z = \frac{\partial r}{\partial \tau}\partial_\tau = r(z)\partial_z$; consequently $r$ disappears from (5), $H$ is replaced by $Z = \int_0^H r(\zeta)d\zeta$, and $z$ is replaced by $\tau$.

When $\kappa$ and $a$ are constant, $\bar{I}_r = \int_0^\infty I_r d\nu$ is solution of the ”grey model”:

$$\mu \partial_\tau \bar{I}_1 + \kappa \bar{I}_1 = \frac{1}{2} \int_{-1}^{1} \left( \kappa(1-a)\bar{I}_1 + \kappa a \int_{-1}^{1} p(\mu,\mu')\bar{I}_1(\mu,\mu')d\mu' \right) d\mu,$$

$$I_r(H,-\mu) = 0, \quad I_r(0,\mu) = Q^+(\mu), \quad 0 < \mu < 1.$$  \hspace{1cm} (12)

where the Stefan-Boltzmann identity has been used: $\sigma T^4 = \frac{1}{2} \int_{-1}^{1} \bar{I}_r d\mu$.

Let $\tilde{I}_1(\tau,\mu)$ be the solution of (12). We want to compare $I_{r+\delta r}$ with $I_r$ when $\delta r$ has a small support near altitude $y$. By (7),

$$I_{r+\delta r}(z) = \tilde{I}_1(\int_0^z (r(\zeta) + \delta r(\zeta))d\zeta) \approx \tilde{I}_1(\tau(z)) + \frac{d}{d\tau} \tilde{I}_1(\tau(y))\delta r(y)$$

$$\Rightarrow \quad \frac{1}{2} \int_{-1}^{1} \tilde{I}_{r+\delta r}(z)d\mu - \frac{1}{2} \int_{-1}^{1} \tilde{I}_r(z)d\mu \approx \frac{d}{d\tau} \left( \frac{1}{2} \int_{-1}^{1} \tilde{I}_1(\tau)d\mu \right) |_{\tau(y)}\delta r(y)$$

Finally from (3),

$$T_{r+\delta r}(z) - T_r(z) \approx 4\sigma T^3(y) \frac{\delta r(y)}{r(y)} \frac{dT_r}{dz}(y).$$  \hspace{1cm} (14)

\Box

4. Sunlight Crosses the Earth Atmosphere Unaffected?

It is true that the absorption coefficient $\kappa_\nu$ is small in the frequency range of visible light; but does it imply that the light source can be used as a boundary condition at altitude zero instead of altitude $H$, the top of the troposphere?

First eliminate the $z$ dependency of the coefficients by using the optical length $\tau$ defined in (7). The following result gives an interesting answer to this question.
Proposition 2. Assume $a_\nu = 0$ and assume that for some $\nu^*$,

$$
\epsilon := \max_{\nu > \nu^*} \kappa_\nu < 1 \quad \text{and} \quad \kappa_\nu Q^0_\nu|_{\nu > \nu^*} = O(1).
$$

Let $I_\nu, T$ and $I'_\nu, T'$ be solutions of (5),(6) respectively with

$$
Q^+_\nu(\mu) = 0, \quad Q^-_\nu(\mu) = \mu Q^0_\nu, \quad 0 < \mu < 1.
$$

Then $T'(\tau) = T(\tau) + O(\epsilon)$ and

$$
I'_\nu - I_\nu = \mu e^{-\frac{\eta_\nu Z}{\nu}} Q^0_\nu \left(1_{\mu > 0} e^{-\frac{\eta_\nu Z}{\nu}} + 1_{\mu < 0} e^{\frac{\eta_\nu Z}{\nu}}\right) + O(\epsilon).
$$

Proof. Note that

$$
\begin{align*}
\mu \partial_\tau I_\nu + \kappa_\nu (I_\nu - B_\nu(T)) & = 0, \quad \nu < \nu^*, \\
I_\nu(0)|_{\mu > 0} & = 0, \quad I_\nu(Z)|_{\mu < 0} = Q^-_\nu(-\mu), \quad \nu < \nu^*, \\
I_\nu & \approx I'_\nu(\mu) \text{ independent of } \tau, \quad \nu > \nu^*, \\
I'_\nu|_{\mu > 0} & = 0, \quad I'_\nu|_{\mu < 0} = Q^-_\nu(-\mu), \quad \nu > \nu^*.
\end{align*}
$$

(16)

Now $I'_\nu$ is defined by (5) but with

$$
I'_\nu(0)|_{\mu > 0} = Q^+_\nu(\mu), \quad I'_\nu(Z)|_{\mu < 0} = 0.
$$

By the same argument, when $\nu > \nu^*$, $I'_\nu \approx I'^*_{\nu}(\mu)$, independent of $\tau$, and

$$
\begin{align*}
\mu \partial_\tau I'_\nu + \kappa_\nu (I'_\nu - B_\nu(T')) & = 0, \quad \nu < \nu^*, \\
I'_\nu(0)|_{\mu > 0} = Q'^*_\nu(\mu), \quad I'_\nu(Z)|_{\mu < 0} = 0, \quad \nu < \nu^*, \\
\int_0^\infty \kappa_\nu B_\nu(T')d\nu & \approx \int_0^\nu \frac{\kappa_\nu}{2} \int_{-1}^1 I'_\nu d\mu d\nu + \int_{\nu^*}^\infty \frac{\kappa_\nu}{2} \int_{-1}^1 Q'^*_\nu(\mu) d\mu d\nu
\end{align*}
$$

(17)

Let $I''_\nu = I_\nu - I'_\nu$. It holds

$$
\begin{align*}
\mu \partial_\tau I''_\nu + \kappa_\nu (I''_\nu - (B_\nu(T) - B_\nu(T'))) & = 0, \quad \nu < \nu^*, \\
\int_0^\infty \kappa_\nu (B_\nu(T) - B_\nu(T'))d\nu & = \int_0^\nu \frac{\kappa_\nu}{2} \int_{-1}^1 I''_\nu d\mu d\nu \\
& \quad + \int_{\nu^*}^\infty \left(\frac{\kappa_\nu}{2} \int_0^1 (Q^-_\nu(\mu) - Q'^*_\nu(\mu)) d\mu\right) d\nu
\end{align*}
$$

(18)

We notice that $T = T'$ and $Q^-_\nu(\mu) = e^{-\frac{\eta_\nu Z}{\nu}} Q^+_\nu(\mu)$ implies

$$
\begin{align*}
\mu \partial_\tau I''_\nu + \kappa_\nu I''_\nu & = 0, \quad \nu < \nu^*, \\
\int_0^\nu \left(\frac{\kappa_\nu}{2} \int_{-1}^1 I''_\nu d\mu\right) d\nu & = \int_{\nu^*}^\infty \left(\frac{\kappa_\nu}{2} \int_0^1 (1 - e^{-\frac{\eta_\nu Z}{\nu}}) Q^+_\nu(\mu) d\mu\right) d\nu
\end{align*}
$$

(19)

$$
I''_\nu(0)|_{\mu > 0} = -Q^+_\nu(\mu), \quad I''_\nu(Z)|_{\mu < 0} = e^{\frac{\eta_\nu Z}{\nu}} Q'^*_\nu(-\mu)
$$
Let $Q'_\nu^+(\mu) = \mu Q^0_\nu$. Then

\[ I'_\nu(0)|_{\mu>0} = -\mu Q^0_\nu, \quad I'_\nu(Z)|_{\mu<0} = -\mu e^{\frac{a_\nu Z}{\mu}} Q^0_\nu. \]

Consequently, from (19) and (11), the solution is

\[ I''_\nu(\tau, \mu) = -\mu Q^0_\nu \left( \mu > 0 \right) e^{-\frac{a_\nu \mu \tau}{\mu}} + 1_{\mu < 0} e^{\frac{a_\nu \mu \tau}{\mu}}. \]

\[ \Rightarrow \int_{-1}^{1} I''_\nu(\tau) d\mu = -Q^0_\nu E_3(\kappa_\nu \tau) + Q^0_\nu E_3(\kappa_\nu \tau) = 0, \quad (20) \]

We have satisfied (19) with precision $O(\kappa_\nu^2)$ because

\[ \kappa_\nu (1 - e^{-\frac{a_\nu \mu Z}{\mu}}) \mu Q^0_\nu = \kappa_\nu Z Q^0_\nu + O(\kappa_\nu^3) \approx 0, \quad \nu > \nu^* \]

\[ \square \]

**Remark 2.** It is easy to see that applying $e^{-\frac{a_\nu \mu Z}{\mu}} \mu Q^0_\nu$ at $\tau = 0$, leads to change $E_3(\kappa_\nu \tau)$ in (11) into $E_3(\kappa_\nu (Z + \tau))$.

The change in temperature is shown on Figure 1. All variables are rescaled as in [6]§7.1. In particular temperatures are in Kelvin and divided by 4780. The air density is $1.22e^{-z}$. Here $Q^0 = 5.66 \cdot 10^{-5}$ and $T_{sun} = 1.2$.

**Conclusion:** For the specific case of sunlight on Earth It is a gross approximation to assume that sunlight crosses the atmosphere unaffected. However the theory works when $T_{sun} = 2.4$ and $\nu^* = 0.2$, probably because max $\nu B_\nu(T_{sun} \approx 18$ against 2 with $T_{sun} = 1.2$.

5. **Is the 30% Earth Albedo Equivalent to a 30% reduction of the Sun Radiative Energy?**

Note that (11) can be used to find the solution of (5)-a with generalized albedo (inspired by the model proposed in [3]):

\[ I_\nu(0, \mu) = \sum_k \alpha_k I_\nu(\tau_k, -\mu) + Q^0_\nu \mu, \quad I_\nu(Z, -\mu) = 0, \quad 0 < \mu < 1. \]

with $k = 1, 2, ..K$. Indeed by (11)

\[ I_\nu(\tau, \mu) = e^{-\frac{a_\nu \mu \tau}{\mu}} \left( \sum_k \alpha_k I_\nu(\tau_k, -\mu) + Q^0_\nu \mu \right) 1_{\mu>0} \]

\[ + 1_{\mu>0} \int_0^\tau e^{-\frac{a_\nu \mu (\tau-t)}{\mu}} \frac{a_\nu}{|\mu|} ((1-a_\nu)\kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)) \, dt \]

\[ + 1_{\mu<0} \int_\tau^Z e^{-\frac{a_\nu \mu (t-\tau)}{\mu}} \frac{a_\nu}{|\mu|} ((1-a_\nu)\kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)) \, dt \]

(21)
Figure 1: Scaled temperatures \( z \to T(z) \) versus altitude, computed with 0.5(0.1 + \( 1\nu<3 \)). The 3 curves \( T_1, T_2, T_3 \) corresponds to 1: \( Q^- = Q^0, Q^+ = 0, \) 2: \( Q^- = 0, I_\nu(0, \mu)|_{\mu>0} = \mu e^{-\frac{\mu Z}{\nu}} Q^0 \) and to 3: \( Q^- = 0, Q^+ = Q^0 \). These can be compared to the grey case \( T_0 \) with \( \kappa = 0.5 \). Note that \( T_1 \) is very different from \( T_2 \). However if the Sun temperature is multiplied by 2 and \( \kappa_\nu = 0.5(0.1 + \nu<0.2) \), then the conditions in Proposition 2 are met and confirmed by the numerical simulations: \( T_3 \) is computed like \( T_1 \) and \( T_3 \) is computed like \( T_2 \).

\[
\begin{align*}
&= 1_{\mu>0} e^{-\frac{\nu Z}{\mu}} \sum_k \alpha_k \int_{\tau_k}^{Z} e^{-\frac{\nu(t-\tau)}{\mu}} \left( (1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t) \right) dt \\
&+ 1_{\mu>0} \left( e^{-\frac{\nu Z}{\mu}} Q^0_\nu \mu \int_0^{\tau} e^{-\frac{\nu(t-\tau)}{\mu}} \left( (1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t) \right) dt \right) \\
&+ 1_{\mu<0} \int_0^{\tau} e^{-\frac{\nu(t-\tau)}{\mu}} \left( (1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t) \right) dt \\
\Rightarrow \\
J_\nu(\tau) &= \frac{1}{2} Q^0_\nu E_3(\kappa_\nu \tau) + \frac{1}{2} \int_0^{Z} E_1(\kappa_\nu |\tau - t|) \left( (1 - a_\nu) B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t) \right) dt \\
&+ \frac{1}{2} \sum_k \alpha_k \kappa_\nu \int_{\tau_k}^{Z} E_1(\kappa_\nu (t + \tau - \tau_k)) \left( (1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t) \right) dt.
\end{align*}
\]

5.1. Earth Albedo

Consider the case

\[
I_\nu(Z, \mu)|_{\mu<0} = 0, \quad I_\nu(0, \mu)|_{\mu>0} = Q^0_\nu + \alpha I_\nu(0, -\mu).
\]

Assume no scattering, \( \tau_1 = 0, K = 1 \). As in [7], the following iterative scheme is considered:

\[
\begin{align*}
J^{n+1}_\nu(\tau) &= \frac{1}{2} Q^0_\nu E_3(\kappa_\nu \tau) + \frac{\kappa_\nu}{2} \int_0^{Z} \left( E_1(\kappa_\nu |\tau - t|) + \alpha E_1(\kappa_\nu (t + \tau)) \right) B_\nu(T^n(t)) dt, \\
\int_0^\infty \kappa_\nu B_\nu(T^{n+1}_\nu) d\nu &= \int_0^\infty \kappa_\nu J^{n+1}_\nu(\tau) d\nu.
\end{align*}
\]
5.2. Numerical Example
Assume $a_\nu = 0$. Figure 2 compares the numerical solutions of (5) with $\kappa_\nu = 0.5[1_{\nu<6} + 0.1]$ and

$$Q^-_\nu = 0, \quad Q^+_\nu = \lambda Q^0_\nu, \quad \lambda = 1 \text{ or } 0.7.$$ 

with the solution of (5) with $\beta = 0$ or 1 in

$$Q^{-,\nu}_\nu = 0, \quad I_{\nu}'(0, \mu) = 0.3I_{\nu}'(0, -\mu) + \mu Q^0_\nu[1_{0.7} e^{-\frac{2Z}{\kappa}} \beta + 1 - \beta], \quad 0 < \mu < 1.$$ 

This numerical simulation is an attempt to replace the radiative source at $\tau = H$ by a radiative source at $\tau = 0$ which gives the same result. Notice alongside that adding 0.3 albedo induces a heating of the atmosphere (i.e. comparing both computations with $\lambda = 1, \beta = 1$ above).

6. Earth Albedo, General Statement, Accommodation Coefficient
As above one considers a stratified atmosphere (5) with $(z, \mu, \nu)$ in $(0, H) \times (-1, 1) \times \mathbb{R}^+$; the radiation intensity $I_{\nu}(z, \mu)$ satisfies on at $z = H$ an incoming condition

$$I_{\nu}(H, -\mu) = Q^-_\nu(\mu), \quad 0 < \mu < 1,$$

and on Earth, at $z = 0$, some “albedo” condition, denoted:

$$I_{\nu}(0, \mu) = \mathcal{A}_\nu(I_{\nu}(0, -\mu)), \quad 0 < \mu < 1,$$

with $\mathcal{A}$ representing the albedo effect of the earth, hence being an operator from the space of outgoing intensities into the space of incoming intensities.
Below will be given, based on energy estimates, some sufficient conditions on $A$ for existence, stability – hence uniqueness – of the solution, beginning with the grey model and then considering a natural operator which involves the effective temperature of Earth (see [4] page 66).

6.1. The energy type estimate for the grey problem

For clarity we present the case without scattering ($a_\nu = 0$) and with a constant $\kappa_\nu$ independent of the frequency $\nu$. By rescaling $z$ with $\kappa$, the problem can be formulated in term of $I(z, \mu) = \int_0^\infty I_\nu(z, \mu, \nu) d\nu$ as,

$$\mu \partial_z I(z, \mu) + I(z, \mu) - \frac{1}{2} \int_{-1}^{1} I(\mu', \mu) d\mu' = 0 , \quad (26a)$$

$$I(H, -\mu) = Q(\mu) := \overline{Q}_\nu(\mu), \quad I(0, \mu) = A(I(0, -\mu)), \quad 0 < \mu < 1 . \quad (26b)$$

Note that we have assumed that $A$ commutes with the integration in $\nu$. First observe that the operator $I \mapsto I - \frac{1}{2} \int_{-1}^{1} I(\mu') d\mu'$ is, in the space $L^2(-1, 1)$ the orthogonal projection on functions of mean value 0, because

$$\int_{-1}^{1} (I - \frac{1}{2} \int_{-1}^{1} I(\mu') d\mu') I(\mu) d\mu = \int_{-1}^{1} (I - \frac{1}{2} \int_{-1}^{1} I(\mu') d\mu')^2 d\mu . \quad (27)$$

This implies that the left hand side of (27) is nonnegative and equal to 0 iff $I = \frac{1}{2} \int_{-1}^{1} I(\mu') d\mu'$ is independent of $\mu$. Multiplying the equation (26a) by $I(z, \mu)$, integrating over $(0, H) \times (-1, 1)$, using Green’s formula and inserting in the computations the conditions (26b) one obtains:

$$\int_{-1}^{1} (\overline{T} - \frac{1}{2} \int_{-1}^{1} \overline{T}(\mu') d\mu')^2 d\mu + \frac{1}{2} \int_{-1}^{1} \mu(\overline{T}(H, -\mu))^2 d\mu + \frac{1}{2} \int_{-1}^{1} \mu(\overline{T}(0, \mu))^2 d\mu \quad (28)$$

This estimate indicates that any property which implies the positivity of

$$\int_{-1}^{1} \mu(\overline{T}(0, \mu))^2 - (A(\overline{T}(\mu, 0))^2) d\mu$$

will lead to a well posed problem with a unique nonnegative solution; for instance:

**Theorem 1.** Assume that the operator $\overline{T} \mapsto A(\overline{T})$ restricted to the positive cone $C^+(\overline{T}(\mu) \geq 0)$ on the space $(L^2(-1, 1), \mu, d\mu)$ then for any positive $\mu \mapsto Q(\mu)$ there is a unique nonnegative solution of the (26a),(26b).

**Remark 3.** Making use of the maximum principle one could prove similar results when the albedo operator is a contraction in the positive cone of $L^\infty(0, 1).$
Remark 4. The albedo condition which is a relation, for $0 < \mu < 1$, between $\mathcal{T}(0, \mu)|_{\mu>0}$ and $\mathcal{T}(0, \mu)|_{\mu<0}$ can also be expressed in term of $\mathcal{T}(0, \mu)|_{\mu>0}$ only.

Indeed, define the operator $\mathcal{T} : \mathcal{T}_+(\mu) \mapsto \mathcal{T}(\mathcal{T}_+(\mu)) = I(0, -\mu)|_{\mu>0}$ (defined for instance in $L^\infty(0, 1)$) by solving (26a) with

$$\mathcal{T}(H, -\mu) = Q(\mu), \quad \mathcal{T}(0, \mu) = \mathcal{T}_+(\mu), \quad 0 < \mu < 1. \quad (29)$$

Then one has:

$$I(0, \mu)|_{\mu>0} = A(I(0, -\mu)) \Leftrightarrow I(0, \mu)|_{\mu>0} = A(\mathcal{T}(I(0, \mu)|_{\mu>0})). \quad (30)$$

With $Q(\mu) > 0$ in $L^\infty(0, 1)$, $\mathcal{T}$ is an affine operator which preserves the positivity and the monotonicity, and this leads to the simple but useful proposition, which, in turn, is the key to the proof of Theorem 1.

Proposition 3. Consider two solutions $\{I_i\}_{i=1,2}$ of the (26a) with (26b) with the same $Q(\mu)$ but with two different albedo operators $\mathcal{A}_i$. Assume that both $\mathcal{A}_i$ are linear contractions and one of them is a strict linear contraction, which preserve positivity. If

$$\forall f \geq 0 \in L^\infty(0, 1) \quad \forall \mu \in (0, 1), \quad A_2(f)(\mu) \leq A_1(f)(\mu) \quad (31)$$

then one has the same ordering for the corresponding solutions :

$$\forall (z, \mu) \in (0, 1) \times (0, H) \quad I_1(z, \mu) \leq I_2(z, \mu). \quad (32)$$

Proof. Using the linearity of the operators $\mathcal{A}_i$ one observes that $R = I_1 - I_2$ solves (26a) with $R(H, -\mu) = 0$ and, for all $0 < \mu < 1$,

$$R(0, \mu) - \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)(\mathcal{T}(R(0, \mu))) = \frac{1}{2}(\mathcal{A}_1 - \mathcal{A}_2)(\mathcal{T}((I_1(0, \mu) + I_2(0, \mu)))). \quad (33)$$

Moreover, restricted to solutions which satisfy $R(H, -\mu) = 0, \mu > 0$ the operator $\mathcal{T}$ is a strict monotonicity preserving linear contraction; the same observation holds for $R \mapsto \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)(\mathcal{T}(R))$. As a consequence the operator $R \mapsto (I - \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)(\mathcal{T}))$ is invertible with inverse given by the Neumann series (also preserving the positivity):

$$R = \sum_{k \geq 0} \left(\frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)(\mathcal{T})\right)^k \frac{1}{2}(\mathcal{A}_1 - \mathcal{A}_2)(\mathcal{T}((I_1 + I_2)|_{z=0, \mu>0}). \quad (34)$$

$I_1$ and $I_2$ are positive intensities and for $\frac{1}{2}(\mathcal{A}_1 - \mathcal{A}_2)(\mathcal{T}((I_1 + I_2)|_{z=0, \mu>0}$ and the righthand-side of (34), (31) is also true. \hfill \Box

6.2. The frequency dependent case

Let us come back to (5) with (26b). Let us denote $f_+ = \max(f, 0)$. When not ambiguous, let us write $I_\nu(\mu)$ instead of $I_\nu(0, \mu)$. 

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Definition 1. An operator $\mathcal{A}$ defined in $L^1_\mu(0,1)$ is non-accretive if

$$\forall I^1, I^2 \in L^1_\mu(0,1), \int_0^1 \mu ((I^2 - I^1)_+) - (\mathcal{A}(I^2) - \mathcal{A}(I^1))_+ \, d\mu \geq 0. \quad (35)$$

Remark 5. Obviously (35) follows from the same pointwise property:

$$\forall (I^1, I^2) \quad ((\mathcal{A}(I^2) - \mathcal{A}(I^1))_+ \leq (I^2 - I^1)_+, \quad \mu > 0. \quad (36)$$

Let us uses the symbol $\langle f \rangle = \int_0^\infty \frac{1}{2} \int_{-1}^1 f_\nu(\mu) d\mu d\nu$.

Then for two solutions $(I^2_\nu, T^2)$, $(I^1_\nu, T^1)$, by (5), according to Theorem 4.1 in [7] (see also [5]), one finds by the same method that existence and uniqueness derives from

$$-\frac{d}{dz} \langle \mu (I^2_\nu - I^1_\nu)_+ \rangle = \langle \rho \kappa_\nu (1 - a_\nu)((I^2_\nu - I^1_\nu) - (B_\nu(T^2) - B_\nu(T^1)))1_{I^2_\nu > I^1_\nu} \rangle$$

$$+ \langle \rho \kappa_\nu a_\nu ((I^2_\nu - I^1_\nu) + \frac{1}{2} \int_{-1}^1 p(\mu, \mu')(I^2_\nu - I^1_\nu)(z, \mu') d\mu') 1_{I^2_\nu > I^1_\nu} \rangle \geq 0$$

and therefore $\frac{d}{dz} \langle \mu (I^2_\nu(z, \mu) - I^1_\nu(z, \mu))_+ \rangle \leq 0$. On the other hand

$$I^2_\nu(H, -\mu) \leq I^1_\nu(H, -\mu), \quad \mu > 0 \implies \langle \mu (I^2_\nu(H, \mu) - I^1_\nu(H, \mu))_+ \rangle = 0$$

while the non accretivity of $\mathcal{A}$ gives

$$\langle \mu (I^2_\nu(0, \mu) - I^1_\nu(0, \mu))_+ \rangle$$

$$= \frac{1}{2} \int_0^\infty d\nu \int_0^1 \mu (\mathcal{A}(I^2_\nu(0, -\mu)) - \mathcal{A}(I^1_\nu(0, -\mu)))_+ - (I^2_\nu - I^1_\nu)_+(0, -\mu) d\mu \leq 0 \quad (38)$$

Therefore, unless $\langle \mu (I^2_\nu(0, \mu) - I^1_\nu(0, \mu))_+ \rangle = 0$ there is a contradiction because it is a function which is decreasing, negative at $z = 0$ and zero at $z = H$. Then consider the difference of equations (5) with arguments $(I^1_\nu, T^1)$ $i = 1, 2$ multiplied by $\frac{\mu - 1}{\kappa_\nu}1_{I^2_\nu > I^1_\nu}$.

Following the end of the proof of Theorem 4.1 in [7] (see also [5]), one finds by the same method that existence and uniqueness derives from

$$\frac{d}{dz} \langle \frac{\mu^2}{\kappa_\nu} (I^2_\nu - I^1_\nu)_+ \rangle = 0, \quad 0 < z < H.$$

Eventually one has:

**Theorem 2.** Consider (5) with incoming data $0 \leq I_\nu(H, -\mu) \leq B_\nu(T_M)$, depending on a given (but possibly large) temperature $T_M$, and with outgoing data given by an accretive operator, of the form $I_\nu(0, \mu)|_{\mu > 0} = \mathcal{A}(I_\nu(0, -\mu))|_{\mu > 0}$. The problem has a unique well defined nonnegative solution.
6.3. Examples of Albedo operators

Below are given some examples of albedo operators which combine an accommodation parameter $\alpha$ and the reflection and the thermalisation effects of the Earth.

The simplest example would be given (cf. Remark 3) with $0 \leq \alpha \leq 1$ by the formula,

$$A(I(\theta, \mu)) = \alpha I(\theta, -\mu) + (1 - \alpha) \int_0^1 \mu^p I(\theta, -\mu')d\mu'.$$

(39)

For the more realistic frequency dependent cases, one should compare the reflection with the emission from the Earth as global black body under an effective temperature $T_e \simeq 288$ and this leads to try the albedo operator

$$A(I(\nu, \mu)) = \alpha I(\nu, -\mu) + (1 - \alpha) B(\nu(T_e)), \quad \mu > 0.$$ 

(40)

This obviously satisfies the hypothesis of non accretivity.

7. Calculus of Variations for the $\nu$-dependent Case

The following argument sheds some light on the conditions needed to obtain a cooling (resp. heating) from a local increase of the absorption coefficient.

**Conjecture 1.** Let $I^\epsilon(\tau, \mu), T^\epsilon(\tau)$ be the solution of (5) when

$$\kappa^\epsilon := \kappa + \epsilon \delta \kappa$$

(41)

Let $I^0(\nu, \mu), T^0(\tau)$ be the solution of (5) with $\kappa_{\nu} = \kappa$ constant. When $\delta \kappa \nu = \delta \nu^*$, the Dirac mass at $\nu^*$, and $\kappa$ is small, then the sign of $\frac{dT}{d\epsilon}(\tau)|_{\epsilon=0}$ is governed by the sign of

$$\frac{1}{2} \int_{-1}^1 I^0_{\nu^*}(\tau) d\mu - B(\nu^*(T^0(\tau))).$$

(42)

7.1. Calculus of Variations Support of the Conjecture

It will be convenient to define the Planck function in terms of the quantity given by the Stefan-Boltzmann law $\Phi := \sigma T^4$. Henceforth we denote

$$B(\nu(T)) = b(\Phi).$$

We seek to study how the average radiation intensity or the temperature is altered if the absorption is modified on various intervals in the frequency variable. We expect that the simplest situation corresponds to absorption of the form (41) with $\kappa > 0$ independent of $\nu$, while $0 < \epsilon \ll 1$. This is of course an extremely general formulation, but in practice one could think of

$$\delta \kappa_{\nu} := \kappa \chi_{\nu_1 < \nu < \nu_2}.$$ 

This corresponds to multiplying the absorption by $(1+\epsilon)$ in the frequency interval $(\nu_1, \nu_2)$ only, and leaving it invariant for all the other frequencies.

We assume that the scattering is isotropic with constant rate $\lambda := \kappa a \geq 0$. 


Henceforth, we consider the radiative transfer equation on \((0, Z) \times (-1, 1) \times \mathbb{R}\),

\[
\mu \partial_{\tau} I^0_{\nu} + \kappa^e_{\nu}(I^e_{\nu} - b_\nu(\Phi^e)) + \lambda(I^e_{\nu} - \bar{I}^e_{\nu}) = 0, \quad \langle \kappa^e_{\nu}(\bar{I}^e_{\nu} - b_\nu(\Phi^e)) \rangle = 0,
\]

with the boundary conditions

\[
I^e_{\nu}(0, \mu) = \mu Q^0_{\nu}, \quad I^e_{\nu}(Z, -\mu) = 0, \quad 0 < \mu < 1,
\]

and the notations

\[
\bar{\Phi} := \frac{1}{2} \int_{-1}^{1} \phi(\mu) d\mu, \quad \langle \psi \rangle := \int_{0}^{\infty} \psi(\nu) d\nu.
\]

First we study the case \(\epsilon = 0\): since \(\kappa\) is constant, one can average in frequency, and set

\[
\bar{I}^0(\tau, \mu) := \langle I^0(\tau, \mu) \rangle, \quad \bar{I}^0(\tau) = \langle b_\nu(\Phi^0) \rangle = \Phi^0(\tau).
\]

One finds

\[
\mu \partial_{\tau} \bar{I}^0(\tau, \mu) + \kappa(\bar{I}^0(\tau, \mu) - \Phi^0) + \lambda(\bar{I}^0(\tau, \mu) - \bar{I}^0(\tau)) = 0,
\]

or, equivalently

\[
\mu \partial_{\tau} \bar{I}^0(\tau, \mu) + (\kappa + \lambda)(\bar{I}^0(\tau, \mu) - \bar{I}^0(\tau)) = 0, \quad \tau > 0,
\]

\[
\bar{I}^0(0, \mu) = \mu \langle Q^0_{\nu} \rangle, \quad 0 < \mu < 1.
\]

Once \(\bar{I}^0\) is known, one recovers \(I^0_{\nu}\) by using the semi-analytical formula (11), which in this case is

\[
I^0_{\nu}(\tau, \mu) = 1_{0 < \mu < 1} e^{-\frac{(\kappa + \lambda)\tau}{\mu}} Q^0_{\nu} + 1_{0 < \mu < 1} \int_{0}^{\tau} e^{-\frac{(\kappa + \lambda)(\tau - \tau')}{\mu}} \frac{\kappa b_\nu(\bar{I}^0(y)) + \lambda \bar{I}^0_{\nu}(y)}{\mu} dy
\]

\[
+ 1_{-1 < \mu < 0} \int_{\tau}^{\infty} e^{-\frac{(\kappa + \lambda)(\tau - \tau')}{|\mu|}} \frac{\kappa b_\nu(\bar{I}^0(y)) + \lambda \bar{I}^0_{\nu}(y)}{|\mu|} dy.
\]

This is precisely the solution of the Milne problem in the case of radiative transfer in a grey atmosphere.

In order to study the effect of the perturbation of the absorption defined above, we seek to compute

\[
I^e_{\nu}(\tau, \mu) = \frac{\partial I^e_{\nu}(\tau, \mu)}{\partial \epsilon} \bigg|_{\epsilon = 0}, \quad \Phi'(\tau) = \frac{\partial \Phi(\tau)}{\partial \epsilon} \bigg|_{\epsilon = 0}, \quad u'(\tau, \mu) = \frac{\partial u(\tau, \mu)}{\partial \epsilon} \bigg|_{\epsilon = 0}.
\]

Let \(\dot{b}_\nu(\Phi) = \partial_\Phi b_\nu(\Phi)\). Let us differentiate in \(\epsilon\) both sides of the equations (43) at \(\epsilon = 0\): one easily finds that

\[
\mu \partial_{\tau} I^e_{\nu} + \kappa(I^e_{\nu} - \dot{b}_\nu(\Phi^0)\Phi'_{\nu}) + \lambda(I^e_{\nu} - \bar{I}^e_{\nu}) + \delta \kappa b_\nu(I^0_{\nu} - b_\nu(\Phi^0)) = 0,
\]

\[
\kappa \langle \bar{I}^e_{\nu} \rangle - \kappa \langle \dot{b}_\nu(\Phi^0) \rangle \Phi' + \langle \delta \kappa b_\nu(I^0_{\nu} - b_\nu(\Phi^0)) \rangle = 0,
\]

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and we can recast the second equality as

\[ \Phi'(\tau) = \tilde{I}_0' (\tau) + \frac{1}{\bar{\kappa}} \left( \delta \kappa_\nu (\tilde{I}_0(\tau) - b_\nu (\Phi^0(\tau))) \right) = \tilde{I}_0' (\tau) + \frac{1}{\bar{\kappa}} \left( \delta \kappa_\nu (\tilde{I}_0(\tau) - b_\nu (\tilde{I}_0(\tau))) \right). \]  

(46)

Averaging in frequency the RT equation, we find that

\[ \mu \partial_\tau \tilde{I}_0'(\tau, \mu) + \kappa (\tilde{I}_0'(\tau, \mu) - \tilde{I}_0'(\tilde{I}_0(\tau) - b_\nu (\Phi^0(\tau)))) = 0, \]

and we can further eliminate \( \Phi'(\tau) \) between the last two equations, to find,

\[ \mu \partial_\tau \tilde{I}_0'(\tau, \mu) + (\kappa + \lambda) (\tilde{I}_0'(\tau, \mu) - \tilde{I}_0'(\tilde{I}_0(\tau))) = -\left( \delta \kappa_\nu (\tilde{I}_0'(\tau, \mu) - \tilde{I}_0'(\tilde{I}_0(\tau))) \right). \]  

(47)

If \( \delta \kappa \) is small everywhere but large near \( \nu^* \) and if \( \kappa + \lambda \ll 1 \) so that the righthandside dominates the second term on the left, then (47) becomes

\[ \mu \partial_\tau \tilde{I}_0'(\tau, \mu) = -\delta \kappa (\tilde{I}_0'(\tau, \mu) - \tilde{I}_0'(\tilde{I}_0(\tau))). \]

Together with (46) it gives the sign of \( T'(z) \) by,

\[ \Phi'(\tau) = -\delta \kappa \left| \int_\tau \frac{\tilde{I}_0'(t, \mu)}{\mu} dt \right| + \frac{\delta \kappa}{\kappa} (\tilde{I}_0'(\tau) - b_\nu (\tilde{I}_0(\tau))). \]  

(48)

when \( \kappa \ll 1 \) the second terms dominate the first one and it is possible to obtain a positive sign when \( \tilde{I}_0'(\tau) > b_\nu (\tilde{I}_0(\tau)) \).

7.2. Numerical Simulations

We study the sign of (42). As it is a complex surface of \( \nu \) and \( \tau \), we look at the frequencies and altitudes for which (42) changes sign when one of the two variable varies. Figure 3 shows the frequencies and altitudes where there is a change of sign, for the grey case \( \kappa = 0.5 \).

![Figure 3: Switch sign points: left when \( \nu \) is increased at fixed \( z \). Right: when altitude is increase at fixed \( \nu \).](image-url)
Consider now $\kappa_\nu = 0.5 + \delta \kappa 1_{(\nu^1,\nu^2)}$. We use an AD (automatic differentiation) extension of the computer program for (45) and display the sensitivity of $T$ with respect to $\delta \kappa$ for 3 ranges of support $(\nu^1,\nu^2)$.

On Figure 4 it is seen that when $\nu^1 = 0.2$, $\nu^2 = 0.3$ the sensitivity of $T$ is negative, meaning that increasing $\kappa_\nu$ in the range $(\nu^1,\nu^2)$ triggers a decrease of temperature at all altitudes. Conversely with $\nu^1 = 0.6$, $\nu^2 = 0.8$ the sensitivity is positive above 800m meaning that an increase of $\kappa_\nu$ in that range triggers a temperature decrease below 800m and an increase above 800m. The same is true for $T'_1$ but at 1500m instead of 800m.

**Figure 4:** Scaled temperatures sensitivities $z \to T'(z)$ versus altitude, computed with $\kappa_\nu = 0.5 + \delta \kappa 1_{(\nu^1,\nu^2)}$ and $\delta \kappa$, at $\delta \kappa = 0$ when case 0: $\nu^1 = 0.2$, $\nu^2 = 0.3$. Case 1: $\nu^1 = 0.3$, $\nu^2 = 0.4$. Case 2 is with $\nu^1 = 0.6$, $\nu^2 = 0.8$. Notice the change of signs in the sensitivity $T'_2$.

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#include <iostream>
#include <fstream>
#include <cmath>
#include <string>
#include <time.h>
using namespace std;

#define sqr(x) (x*x)

#define NOAUTODIF

#if defined(NOAUTODIF)
#define ddouble double
#define ffabs(x) (fabs(x))
const ddouble dknu=0;
#else
#define ”ddouble.h"
#define ffabs(x) (fabs(x.val[0]))
const ddouble dknu(0.,1.); // means dknu=0.1 and AD computes d_T/d_dknu
#endif
const bool greycase=true; // see beginning of main() for parameters. If false use below
const string kappa012="kappa0f"; // defines 3 runs with kappa vs nu in columns 2,3,4
const bool verbose=false;
const int Ntau=60; // nb points in tau
const int kmax=16; // nb fixed point iterations
const double Z=1-exp(-12.); // max tau after change of var
const double SBsun =2.03e-5*(2*sqr(2.))/0.7; // scaled sunlight power
const double Tsun = 1.209; // scaled sun temperature
const int jmaxmax=600; // max of max nb of points for integration in nu range
const int newton = 50; // to compute the temperature from int k*Botlzmann= int k*mean
const double epsdycho=0.01, epsnewton=1.e-12; // precision for dychotomy before Newton
const double tmin=1.e-10; // min t in ExpInt(t)
const double dtt = 0.005; //min integration step size in ExpInt integrals
const double kappamin = 0.001; // if kappa read is too small max it with kappamin
const int nt = 5; // min nb of integration step in anal formula
const double ealb = 0.3; // and Earth albedo
const double alpha = 1; // proportion of light affected at altitude 0 vs Z
const double ais = 0., ars = 0.; // max isotropic and Rayleigh scattering
const double tm1 = Z * 0.6, tm2 = Z * 0.9; // altitudes for scattering
const double tm1 = Z * 0.6, tm2 = Z * 0.9; // altitudes for scattering

double nu[jmaxmax], // uneven discretization of [numin, numax]
a[i][jmaxmax], ar[jmaxmax], aux[jmaxmax]; // isotropic and Rayleigh scattering

ddouble Inut[Ntau], // mu integral of I
Snut[Ntau], // mu integral of nu^2 I
G[jmaxmax][Ntau], // J(nu, tau)
S[jmaxmax][Ntau], // K(nu, tau)
T[Ntau], kappanu[jmaxmax]; // T0 with kappanu0[]

string basedir("/Users/pironneau/Dropbox/afaire/Golse–Bardos/greenhouse4/");
string myresulttemperature("temperature");
string myresultmeanintensity("imean");
string mykappafile(basedir+kappa012+".txt"); // has nu, kappa0, kappa1, kappa2
int jmax;

struct ddouble2 { ddouble d1, d2; };

// don't use if kappa >18/Z

double expint_E1(const ddouble t = 1) {
    // if your compiler has it or if you can link to gsl you may adapt this function
    // it computes E1(t)*B when t<18
    double abst = fabs(t);
    const int K = 9 + (abst - 1) * 4; // precision in the exponential integral function E1
    const double gaNtau = 0.577215664901533; // special integration for log(t)
    if (abst < tmin) return 0.; // because integral_0^t (logx)dx ˜ 0
    if (abst > 18) { cout << "value of E_1 is incorrect with " << t << "\n" << endl;
        return 0.;
    }
    double ak = (t < 0) ? -t : t, soNtau = gaNtau - log(abst) + ak;
    for (int k = 2; k < K; k++) {
        ak += -(abst * (k - 1) / sqrt(k));
        soNtau += ak;
    }
    return soNtau;
}

double expint_E2(const ddouble t = 1) {
    ddouble aux = exp(-t) - t * expint_E1(t);
    return aux;
}

double expint_E3(const ddouble t = 1) {
    return (exp(-t) - t * expint_E2(t)) / 2;
}

double expint_E4(const ddouble t = 1) {
    return (exp(-t) - t * expint_E3(t)) / 3;
}
ddouble expint_E5(const ddouble t=1) {
    return (exp(-t) - t*expint_E4(t))/4;
}

int readkappa(string mykappafile, int which=0) {
    // on each line of kappafile: nu kappa0 & optional kappa1 kappa2
    ifstream kappafile(mykappafile);
    int j = -1;
    double kappaux=0, dummy, nuj =0.01;
    while((j++<jmaxmax) && (!kappafile.eof())) {
      if (which==0) kappafile >> nuj >> kappaux >> dummy >> dummy;
      else if (which==1) kappafile >> nuj >> dummy >> kappaux >> dummy;
      else if (which==2) kappafile >> nuj >> dummy >> dummy >> kappaux >>
      dummy;
      kappaux = fmax(kappaux, kappamin);
      nu[j] = nuj;
    }
    kappafile.close();
    jmax = j;
    return 0;
}

double Bsun(const double nu) {
    return SBsun *sqr(nu)*nu/(exp(nu/Tsun) -1);
    // Boltzmann
}

double BB(const double nu, const ddouble T) {
    if(T<1.e-7) return 0;
    if(nu<1.e-10) return T*sqr(nu);
    return sqr(nu)*nu/(exp(nu/T) -1);  // Boltzmann
}

double dBB(const double nu, const ddouble T) {
    if(T<1.e-7) return 0;
    if(nu<1.e-10) return sqr(nu);
    double a = exp(nu/T); return a*sqr(nu*nu/(a -1)/T);
}

double2 intBS(const int jnu, const double tau, const double tmin, const
double2 Tmax) {
    // returns the convolution t-integral for the mean in nu of kappa*I_nu
    // and nu^2 kappa*I_nu
    ddouble Imean =0, Imu2mean =0;
    ddouble kappa=kappanu[jnu];
    const double dt=fmin(dtt, nt/(Tmax-tmin));
    for (double t=tmin; t<Tmax; t+=dt) {
      int it = int((Ntau-1)*t/Z);  // parabolic length for frequency step
      double annu4=ar[it]*sqr(nu[jnu]-0.8)*sqr(nu[jnu]-1.2)*(nu[jnu]>0.8)
      *(nu[jnu]<1.2)*40;
      ddouble H0 = kappa*(BB(nu[jnu],T[it]))*(1-annu4) +
          (ai[it]+1.125*annu4)*G[jnu][it] -1.125*annu4*S[jnu][it]);
      ddouble H2 = -0.375*annu4*kappa*(G[jnu][it] -3*S[jnu][it]);
      if (kappa*(t-tau)!=0) {
        Imean += dt*H0*(expint_E1(kappa*fabs(tau-t))+ealb*expint_E1( kappa*(tau+t)))/2;
        Imu2mean += dt*H0*(expint_E3(kappa*fabs(tau-t))+ealb*expint_E3(}
kappa*tau+t)/2;
if (H2 !=0) {
  Imean += dt*H2*(expint_E3(kappa*fabs(tau-t))+ealb*expint_E3
(kappa*(tau+t)))/2;
  Imu2mean += dt*H2*(expint_E5(kappa*fabs(tau-t))+ealb*
expint_E5(kappa*(tau+t)))/2;
}
}
double2 mean; mean.d1=Imean; mean.d2 = Imu2mean;
return mean;
}

double thefunc(const double rhs, const double T0){
double myeq=-rhs;
  for(int j=1; j<jmax; j++){
    myeq += kappanu[j]*BB(nu[j],T0)*(nu[j]-nu[j-1]);
  }
  return myeq;
}

double getTbydycho(const int i, const double Tstart){
double T0=Tstart;
if (Tstart <0.1) T0=0.1;
double Taux, myeq0=1,myeq1=-1, T1=T0, rhs =0;
  for(int j=1; j<jmax; j++){
    rhs+=kappanu[j]*G[j][i]*(nu[j]-nu[j-1]);
  }
myeq0 =-rhs; myeq1=-rhs;
  while (myeq0>0){
    T0=T0/2;
    myeq0=thefunc(rhs,T0);
    if (verbose) cout<<T0<<" down "<<myeq0<<endl;
  }
  while (myeq1<0){
    T1=2*T1; myeq1=thefunc(rhs,T1);
    if (verbose) cout<<T1<<" up "<<myeq1<<endl;
  }
  while (T1-T0 > epsdycho){
    Taux=(T1+T0)/2;
    myeq0=thefunc(rhs,Taux);
    if (myeq0>0) T1=Taux; else T0=Taux;
    if (verbose) cout<<T0<<" middle "<<T1<<" "<<myeq0 << endl;
  }
  return (T1+T0)/2;
}

int genT(){
  for(int i=0;i<Ntau;i++){
    T[i] = getTbydycho(i,T[i]);
    double presfunc = 1;
    int inewton =0;
    while(inewton+1<newton & & ffabs(presfunc) > epsnewton){
      double T0 = T[i];
      double left=0, rhs=0, deriv=0;
      double nul1=0;
```cpp
for (int j=1; j<max; j++){
    double dnu=nu[j]-nu[j-1]; // variable integral increment
    nu1=(nu[j]+nu[j-1])/2;
    rhs+=kappanu[j]*G[j][i]*dnu;
    left += kappanu[j]*BB(nu1,T0)*dnu;
    deriv += kappanu[j]*dBB(nu1,T0)*dnu;
}
presfunc = rhs - left;
if (fabs(deriv) > 1e-10) T[i] = T0 + presfunc / deriv;
if (verbose) cout << T[i] << newton << presfunc << endl;
}
if (inewton >= newton) cout << "Newton precision doubtful " << endl;
return 0;
}

int getT() { // return temperature by Kirchhoff’s law when kappa is constant
    const double pi = 4*atan(1.);
    for (int i=0; i<max; i++){
        ddouble rhs=0;
        for (int j=1; j<max; j++)
            rhs+=G[j][i]*(nu[j]-nu[j-1]);
        T[i] = sqrt(sqrt(15*rhs))/pi;
    }
    return 0;
}

int getISnu(const int j) { // returns mean I_nu & mean mu^2 I_nu
    double nu1 = nu[j];
    ddouble kappa=kappanu[j];
    if (kappa < 0.01) kappa = 0.01;
    for (int i=0; i<Ntau; i++)
        double x=i*Z/(Ntau-1);
        ddouble2 aux = intBS(j,x,0,Z);
        Inut[i] = aux.d1 + Bsun(nu1)*(alpha*expint_E3(kappa*(x+Z)) + (1-alpha)*expint_E3(kappa*(Z-x)))/2; // +ealb*alpha*(expint_E3(kappa*(Z+x)))/2;
        Snut[i] = aux.d2 + Bsun(nu1)*(alpha*expint_E5(kappa*x) + (1-alpha)*expint_E5(kappa*(Z-x)))/2; // +ealb*alpha*(expint_E5(kappa*(Z+x)))/2;
    }
    return 0;
}

int multiBlock(double initT)
{
    for (int i=0; i<Ntau; i++) T[i] = initT; // initialize
    for (int j=0; j<max; j++) G[j][i] = 0; S[j][i] = 0; }
    for (int k=0; k<kmax; k++) { // fixed point loop: first update F
        for (int i=0; i<Ntau; i++)
            Inut[i] = 0; Snut[i] = 0;
    }
    return 0;
}
```

for (int j=0; j<jmax; j++)
    getISnu(j);
    for (int i=0; i<Ntau; i++)
    { G[j][i]=Inut[i];
      S[j][i]=Snut[i];
    }
if (greycase)
#elif defined(NOAUTODIF)
genT(); // replace by getT() to speed-up but know what you do!
#else
/genT(); // Then update T. AD need genT() and false with getT() because kappa varies
#endif
else
    genT();
double normG=0, normS=0;
    for (int j=0; j<jmax; j++)
        for (int i=0; i<Ntau; i++)
        { normG+=fabs(G[j][i]); normS+=fabs(S[j][i]);
        }
cout<<"k= "<<k<<" ||T[2]|| ||G|| and ||S||\n";
    // ofstream tempfile (basedir+"tempHistScatLow"+std::to_string(k)+".txt") ; // study convergence
    // for (int i=1; i<Ntau; i++) tempfile<<-log(1-i*Z/(Ntau-1))<<"\t"<<T[i]<<endl;
    return 0;
}

int main(int argc , const char * argv[])
{
    for(int it=0; it<Ntau; it++)
    { // defines scattering values vs tau
        double tm = it*Z/Ntau;
        a[i] = ais*fmax(tm-tm1,0.)*fmax(tm2-tm,0.)*4/sqr(tm2-tm1);
        ar[i] = ars*fmax(tm-tm2,0.)/(Z-tm2);
    }
    for(int which=0; which<1; which++)
    { // change to which<1 for a single computation
        if(greycase){
            const double kappa0=0.5, numin = 0.05 , numax = 15;
            jmax = 400;
            for (int j=0; j<jmax; j++)
            { nu[j]= numax/(1+(jmax-j)*(numax-numin)/numin/jmax); // uniform in wavelength
                kappanu[j]=kappa0;+dknu*(nu[j]>0.2)*(nu[j]<0.3);
                kappanu[j]=kappa0*(nu[j]<6) + 0.1;
            }
        }
        else{
            readkappa(mykappafile , which);
            for(int j=0; j<jmax; j++)
                if((nu[j]<3./4)*(nu[j]>3./7)) kappanu[j]=fmax(0.5,1.5*kappanu[j]);
        }
    }
    ofstream resultfile =
        ofstream ( basedir+myresulttemperature+to_string ( which+10*(ais>0))+".txt" ) ;
    cout<<"\n iterations \t T near earth \t T far \t||G|| and \t||S||\n";
double t0 = clock();
multiBlock(0.); // "tempmin");
printf(" Time CPU = %10.6f\n", (clock() - t0)/CLOCKS_PER_SEC);
cout<<"\n tau\t[T]:"<<endl;
for(int i=1;i<Ntau;i++){
    cout << -log(1-i*Z/(Ntau-1))<<"\t"<<T[i] <<endl;
    resultfile << -log(1-i*Z/(Ntau-1))<<"\t" // altitude
    <<T[i]<<endl; // T(kappa) Milne by multigroup
}
ofstream imeanz0 =
    ofstream( basedir+myresultmeanintensity+to_string(10*(ais >0)+
    which)+".0.txt");
    ofstream imeanzZ =
    ofstream( basedir+myresultmeanintensity+to_string(10*(ais >0)
    +which)+"Z.txt");
    for(int j=10; j<jmax;j++){
        imeanz0 << 3/nu[j] << " " <<1e5*G[j][1] <<endl; // mean intensity
    imeanzZ<< 3/nu[j] << " " <<1e5*G[j][Ntau-1]<<endl; // mean intensity at max altitude
}    resultfile.close();
    imeanz0.close();
    return 0;