\[ \eta \to \pi^0 \gamma \gamma \] to $O(p^6)$ in Chiral Perturbation Theory

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Abstract

The decay \( \eta \to \pi^0 \gamma \gamma \) is discussed in the framework of SU(3) chiral perturbation theory. The process is dominated by the $O(p^6)$ in the momentum expansion where tree-level amplitudes from the effective Lagrangian $\mathcal{L}_6$ enter together with one-loop contributions from $\mathcal{L}_4$ and two-loop contributions from $\mathcal{L}_2$. We estimate the 6 independent $\mathcal{L}_6$ coupling constants by resonance saturation consistent with the pion production process $\gamma \gamma \to \pi^0 \pi^0$ and calculate the pion-loop part of the one- and two-loop amplitude. Predictions for the total rate and spectrum of $\eta \to \pi^0 \gamma \gamma$ are given together with a discussion of the uncertainties involved.
I. INTRODUCTION

The reactions involving two neutral pseudoscalar mesons and two photons have received much attention during the past few years. In the framework of chiral perturbation theory, they are unique in that the tree–level amplitudes from the Lagrangians $L^2$ and $L^4$ both vanish. As a consequence, processes such as $\gamma\gamma \rightarrow \pi^0\pi^0$, $\eta \rightarrow \pi^0\gamma\gamma$ (or the reactions related to them by time reversal) test higher order and loop contributions in the chiral perturbation series. In fact it turns out that the $O(p^4)$ one-loop amplitude for $\gamma\gamma \rightarrow \pi^0\pi^0$ \[1,2\] underpredicts the Crystal Ball data \[3\] at low invariant mass of the two photons (where chiral perturbation theory is supposed to work best). The $O(p^4)$ result for the $\eta \rightarrow \pi^0\gamma\gamma$ decay–width is a tiny $3.89 \cdot 10^{-3}$ eV \[4\], to be compared with an experimental value of $0.84 \pm 0.2$ eV \[3\]. The $\eta$–decay amplitude is thus dominated by higher order contributions.

According to Weinberg’s power–counting formula \[6\], the $O(p^6)$–amplitude will contain tree-level contributions from the effective Lagrangian $L^6$ together with one–loop contributions from $L^4$ and two–loop contributions from $L^2$. The form of the complete $L^6$–Lagrangian has been given recently \[7\]; we will see below that a total of 6 linearly independent structures contribute to the processes considered here. In extending the chiral Lagrangian approach to $O(p^6)$, we therefore have to deal with two difficulties: A realistic estimate of the 6 low–energy constants involved and the solution of the 2–loop Feynman integrals, in particular for the case where the two loop momenta overlap.

There have been various attempts to determine the $L^6$ $\eta$–decay amplitude in the past either phenomenologically or in a bosonized Nambu-Jona-Lasinio model. In \[1\], only the leading terms in a $1/N_c$–expansion are considered, i. e. the Lagrangian is restricted to the single flavour trace terms

$$d_1 F_{\mu\nu} F^{\mu\nu} Tr(Q^2 \delta^\lambda U \delta_\lambda U^\dagger) + d_2 F_{\mu\alpha} F^{\mu\beta} Tr(Q^2 \delta^\alpha U \delta_\beta U^\dagger). \quad (1.1)$$

(For the notation, see Section II A). If the two coupling constants $d_1$ and $d_2$ are determined by comparison to a vector dominance (VMD) amplitude

$$-d_1 = \frac{1}{2} d_2 \sim 3.6 \cdot 10^{-3} \text{GeV}^{-2};$$

the tree level amplitude from the $O(p^6)$ Lagrangian (1.1) yields a total decay width of

$$\Gamma^6(\eta \rightarrow \pi^0\gamma\gamma) = 0.18 \text{eV}.\)

This is to be compared to the full VMD amplitude of \[3,10\]

$$\Gamma^{VMD}(\eta \rightarrow \pi^0\gamma\gamma) \sim 0.31 \text{eV}.$$

In the Nambu–Jona–Lasinio model, a further single trace term contributes to $L^6$, namely \[11,12\]

$$d_3 F_{\mu\nu} F^{\mu\nu} Tr(Q^2 (\chi^\dagger U + U^\dagger \chi)). \quad (1.2)$$

The determination of the constant $d_3$ is ambiguous. According to Ref. \[11\], the term (1.2) increases the $\eta$–decay width to

$$\Gamma^{ENJL}(\eta \rightarrow \pi^0\gamma\gamma) = 0.58 \pm 0.3 \text{eV}$$

2
whereas the authors of [12] advocate values of
\[ \Gamma^{\eta \rightarrow \pi^0 \gamma \gamma} = 0.11 \ldots 0.45 \text{ eV}. \]

If \( d_3 \) is calculated via resonance saturation in \( \gamma \gamma \rightarrow \pi^0 \pi^0 \) [13], the term (1.2) doesn’t affect the \( \eta \)-decay width very much. Conversely, an attempt to keep \( d_1 \) and \( d_2 \) as given by vector meson resonance saturation and then fit \( d_3 \) to the experimental value of the \( \eta \)-decay width [14] leads to a very large \( d_3 \) and implies that further terms in the Lagrangian are required for a realistic description of pion polarizabilities and the the process \( \gamma \gamma \rightarrow \pi^0 \pi^0 \). In summary, one is led to the conclusion [12] that a self-consistent, quantitative description of the decay \( \eta \rightarrow \pi^0 \gamma \gamma \) based on the Lagrangians (1.1) and (1.2) is problematic.

The numerical decay widths quoted so far are calculated from pure \( \mathcal{L}_6 \) tree-level amplitudes. A complete treatment of the \( \mathcal{L}_2 \) two-loop contributions has been achieved in an SU(2)-calculation of \( \gamma \gamma \rightarrow \pi^0 \pi^0 \) [13], where only pionic degrees of freedom are taken into account. Numerically, the two-loop amplitude is crucial for a fit of the experimental spectrum. For the \( \eta \)-decay, the contributions of the factorizable 2-loop diagrams have been given recently [12]. Although the effects on the \( \eta \)-decay are small, it is clearly desirable to carry out a complete calculation of the two neutral meson processes at two-loop level.

The purpose of the present paper is thus twofold: First, we explore the structure of the tree-level amplitudes if the most general Lagrangian \( \mathcal{L}_6 \) is used and compare the result to previous calculations. Second, we outline a method to derive the two-loop amplitude including overlapping graphs. We present the results for the case of pure pion loops and estimate the missing parts for \( \eta \rightarrow \pi^0 \gamma \gamma \). A method of solving the specific Feynman integrals occurring in the problem may be of general interest and is therefore discussed in the appendix.

II. THE \( \eta \rightarrow \pi^0 \gamma \gamma \) AMPLITUDE

A. General Formalism and Kinematics

Throughout the following, we will use the notation of [6] applied to our special case: The \( U(3) \)-matrix \( U = \exp(\frac{i}{F_\pi}(\Phi_8 + \Phi_1)) \) contains the nonet of pseudoscalar mesons

\[
\Phi_8(x) = \begin{pmatrix}
\pi^0 + \frac{\eta}{\sqrt{3}} & \frac{\sqrt{2}\pi^0}{\sqrt{3}} & \frac{\sqrt{2}K^+}{\sqrt{3}} \\
\frac{\sqrt{2}\pi^+}{\sqrt{3}} & -\pi^0 + \frac{\eta}{\sqrt{3}} & \frac{\sqrt{2}K^0}{\sqrt{3}} \\
\frac{\sqrt{2}K^-}{\sqrt{3}} & \frac{\sqrt{2}K^0}{\sqrt{3}} & -\frac{2\eta}{\sqrt{3}}
\end{pmatrix}, \quad \Phi_1(x) = \sqrt{\frac{2}{3}} \eta I
\]

\( F_\pi = 93.2 \text{ MeV} \) is the pion decay constant [8]. The covariant derivative for photons as external gauge fields reads

\[ D_\mu B = \delta_\mu B + ieA_\mu [Q, B] \]

where \( Q = \text{diag}(2/3, -1/3, -1/3) \) is the quark charge matrix and \( B \) stands for be any operator that transforms linearly under the chiral group \( U(3)_L \times U(3)_R: B \rightarrow B' = V_R B V_L^\dagger \)

We note that
\begin{equation}
[Q, \Phi] = \begin{pmatrix}
0 & \pi^+ & K^+ \\
\pi^- & 0 & 0 \\
K^- & 0 & 0
\end{pmatrix}
\end{equation}

so that the commutator \([Q, \Phi]\) couples the photon to charged mesons only. In Eq. \((1.1)\), coupling of the neutral mesons to photons is mediated by the field tensor \(F_{\mu\nu} = (\delta_\mu A_\nu - \delta_\nu A_\mu)\).

The physical \(\eta\) particle can be written as \([4]\)

\(\eta = \cos \theta_\eta \eta_8 - \sin \theta_\eta \eta_1 \sim \frac{2}{3} \sqrt{2} \eta_8 + \frac{1}{3} \eta_1.\)

In our normalization, the lowest order chiral Lagrangian reads

\begin{equation}
\mathcal{L}_2 = \frac{F^2}{4} Tr(D_\mu U D^\mu U^\dagger) + \frac{F^2}{4} Tr(\chi^\dagger U + U^\dagger \chi)
\end{equation}

where the mass terms contain the quark mass matrix \(\chi = 2B_0 \text{diag}(m_u, m_d, m_s)\) and the constant \(B_0\) relates the masses of quarks and pseudoscalar mesons via \([8,4]\)

\begin{equation}
B_0 = \frac{m_K^2}{m_u + m_s} = \frac{m_\pi^2}{m_u + m_d} = \frac{\Delta m_K^2}{m_d - m_u}.
\end{equation}

A numerical estimate for the electromagnetic mass split of the kaon yields the value of \(\Delta m_K \sim 6000\) MeV\(^2\) \([4]\).

The general form of the amplitude for the decay \(\eta(P) \to \pi_0(p)\gamma(q_1, \epsilon_1)\gamma(q_2, \epsilon_2)\) is \([10,15]\)

\begin{equation}
M = \epsilon_{1\mu} \left\{ A(s, t) \cdot \left( \frac{g^{\mu \nu} s}{2} - q_2^{\mu} q_1^{\nu} \right) + B(s, t) \frac{2}{s} \cdot \left( g^{\mu \nu} P \cdot q_1 P \cdot q_2 + P^\mu P^\nu s^2 - q_2^{\mu} P^\nu P \cdot q_1 - P^\mu q_1^{\nu} P \cdot q_2 \right) \right\} \epsilon_{2\nu}
\end{equation}

where we have defined the kinematic invariants \(s = (q_1 + q_2)^2, t = (P - q_2)^2, u = (P - q_1)^2 = m_\eta^2 + m_\pi^2 - s - t\). The decay rate is calculated according to

\begin{equation}
\frac{d\Gamma}{ds} = \frac{1}{1024 m_\eta^3 m_\pi^3} \int_{t_1}^{t_2} dt \left( |A_s - m_\eta^2 B|^2 + \frac{|B|^2}{s^2} (m_\pi^2 m_\eta^2 - tu)^2 \right),
\end{equation}

where \(s\) is restricted to \(0 \leq s \leq (m_\eta^2 - m_\pi^2)^2\) and

\begin{equation}
t_{1,2} = \frac{1}{2} \left[ (m_\eta^2 + m_\pi^2 - s) \pm \sqrt{(m_\eta^2 + m_\pi^2 - s)^2 - 4 m_\eta^2 m_\pi^2} \right].
\end{equation}

**B. Complete \(\mathcal{L}^6\)-Amplitude**

The tree–level amplitude from the Lagrangian \(\mathcal{L}^6\) forms the dominant contribution of the \(\eta\)–decay width and will therefore be discussed separately from the one– and two–loop corrections.
The most general chiral Lagrangian at $O(p)^6$ has been derived for the $SU(3)$–case. $U = \exp(\frac{\mu}{F_H} \Phi_5)$. Introduction of $\eta - \eta'$ mixing according to Eq. (2.1) eliminates a symmetry constraint and potentially generates more terms that we have to neglect at this point (for an example see [16]). Keeping this simplification in mind, we can take the $L^6$–terms required from Table II, section no. 4 (terms involving $B_{30} - B_{30}$) of Ref. [7]. The number of independent structures contributing to the 2 photon/2 neutral meson processes can be determined in two steps: First make use of the simplifications explained in Sections IV B/C of Ref. [7] and note that, owing to Eq. (2.2), the commutator $[Q, \Phi]$ can be set to zero in the covariant derivative as well as under the trace of the $L^6$–expressions. As a result, for instance both the terms in $B_{35}$ and $B_{36}$ of Ref. [7], Table II reduce to the $d_2$–term in Eq. (1.1):

$$B_{35} Tr([D_\alpha U] - [D^\alpha U] - [G_\alpha^\beta] + [G_{\beta}^\alpha] +) + B_{36} Tr([D_\alpha U] - [D^\alpha U] - [G_\beta^\alpha] + [G^{\alpha\beta}] +) = d_2 F_{\mu\alpha} F^{\mu\beta} Tr(Q^2 \delta^\alpha U \delta_\beta U^\dagger), \quad d_2 = 4(B_{35} + B_{36}).$$  (2.9)

The terms resulting from this procedure are more symmetric than the original general $L^6$–terms. This implies that terms which are independent in the general Lagrangian $L^6$ can be redundant due to $[Q, \Phi] = 0$. Therefore, in a second step, we have to check the result of the simplified Lagrangian and reject terms that are linked by a trace relation of the type discussed in [7], App. A. For example, one of the terms

$$B_{34} Tr([D_\alpha U] - [D^\alpha U] - [G_\alpha^\beta] + [G_{\beta}^\alpha] +) = 4B_{34} F_{\alpha\beta} F^{\alpha\beta} Tr(Q^2 \delta^\alpha U \delta_\beta U^\dagger),$$
$$B_{37} Tr([D_\alpha U] - [D^\alpha U] - Tr([G^\alpha_\beta] + [G_{\beta}^\alpha] +) = 4B_{37} F_{\alpha\beta} F^{\alpha\beta} Tr(Q^2) Tr(\delta^\alpha U \delta_\beta U^\dagger),$$
$$B_{38} Tr([D_\alpha U] - [G_\alpha^\beta] +) Tr([D^\alpha U] - [G_{\beta}^\alpha] +) = 4B_{38} F_{\alpha\beta} F^{\alpha\beta} Tr(Q^2) Tr(Q\delta^\alpha U \delta_\beta U^\dagger)$$

is redundant after simplification due to the the trace relation

$$4 Tr(A^2 B^2) + 2 Tr(ABAB) - Tr(A^2) Tr(B^2) - 2(Tr(AB))^2 = 0$$

for any complex $3 \times 3$–matrices $A, B$ with $Tr(A) = Tr(B) = 0$. Alternatively, the $B_{34}$ and $B_{37}$ can be seen to be equivalent because of $Tr(Q^2) = 2/3$.

We are finally left with the 3 single trace terms of Eqs. (1.1) and (1.2) plus 3 double trace terms

$$d_4 F_{\mu\nu} F^{\mu\nu} Tr(Q^2 \delta^\alpha U \delta_\beta U^\dagger) + d_4 F_{\mu\nu} F^{\mu\nu} Tr(Q^2 \delta^\alpha U \delta_\beta U^\dagger) + d_4 F_{\mu\nu} F^{\mu\nu} Tr(Q^2 \delta^\alpha U \delta_\beta U^\dagger) + d_5 F_{\mu\nu} F^{\mu\nu} Tr(Q^2 \delta^\alpha U \delta_\beta U^\dagger) + d_5 F_{\mu\nu} F^{\mu\nu} Tr(Q^2 \delta^\alpha U \delta_\beta U^\dagger) + d_6 F_{\mu\nu} F^{\mu\nu} Tr(Q^2 \delta^\alpha U \delta_\beta U^\dagger).$$  (2.11)

and thus a total of 6 low–energy constants to be determined either from experiment or in some model[1]. The low–energy constants $d_1, \ldots, d_6$ are related to the $B$–constants of [7] as follows:

1Elimination of redundant terms in the Lagrangian is desirable for simplicity but not indispensable from a practical point of view. Keeping more terms would result in fixed linear combinations of coefficients appearing in the amplitudes; the fit procedure would ultimately not be affected. Moreover, we have some freedom as how to represent the Lagrangian and which terms to discard. A choice different from Eq. (2.11) might affect the expressions for the amplitudes and the values of the parameters quoted but of course must leave the physical content of the Lagrangian invariant.
Using Eq. (2.3), the $\eta$–decay amplitude from the full $\mathcal{L}^6$–Lagrangian can now be cast in the form (2.6) with

$$A_\eta(s, t) = \frac{2\sqrt{2}}{3\sqrt{3}F_\pi^2} \left\{ (4d_1 - 12d_4)m_\eta^2 - 2d_3m_\pi^2 - (4d_1 - 12d_4 + d_2 - 3d_5)P \cdot (q_1 + q_2) \right\}$$

$$B_\eta(s, t) = \frac{4\sqrt{2}}{3\sqrt{3}F_\pi^2} \frac{s}{2} \left\{ d_2 - 3d_5 \right\}$$ \hspace{1cm} (2.13)

Note that we used the isospin approximation $m_u = m_d$ neglecting a small contribution from the double trace mass term in Eq. (2.11) proportional to $d_6$. This does not change any of the conclusions of this chapter. The $d_6$–terms does affect, however, the $\gamma\gamma \to \pi^0\pi^0$ amplitude which can be written in analogy to Eq. (2.13)

$$A_\pi(s, t) = -4 \frac{F_\pi^2}{s} (s - 2m_\pi^2) \left( \frac{5}{9}d_1 - d_4 \right) - \frac{1}{F_\pi^2} s \left( \frac{5}{9}d_2 - d_5 \right) - \frac{4}{F_\pi^2} m_\pi^2 \left( \frac{5}{9}d_3 + \frac{12}{9}d_6 \right)$$

$$B_\pi(s, t) = 4 \frac{F_\pi^2}{2} \frac{5}{9} \left( d_2 - d_5 \right)$$ \hspace{1cm} (2.14)

Determination of the 6 constants $d_1 \ldots d_6$ from experimental data alone is now impeded by a combination of factors:

1) For the process $\eta \to \pi^0\gamma\gamma$ where the $\mathcal{L}^6$–amplitude (2.13) forms the leading contribution, only the total decay width has been measured [5].

2) The energy dependence of the cross section for neutral pion pair production is experimentally known. In this case, however, the amplitude (2.14) is small and interferes with the large, complex one–loop and two–loop contributions [13] so that the fit becomes ambiguous. Complementary data such as pion polarizabilities [13,17] don’t put sufficient constraints on the parameters.

3) There are further two neutral meson/two photon data, e. g. the production cross sections for $\gamma\gamma \to \eta\pi^0$ and $\gamma \to K^0\bar{K}^0$ [3], but the processes have too high energy thresholds.

On the other hand, the method of resonance saturation has been successfully applied in order to confirm the phenomenological constants of the Gasser–Leutwyler Lagrangian $\mathcal{L}^4$ [18]. Moreover, the pion production amplitude (2.14) with the constraint $d_4 = d_5 = d_6 = 0$ has been determined via resonance saturation [13] and compares well with the data. We proceed to discuss how the method can be applied to our general case.

C. Meson Resonance Amplitude

In the VMD model, the leading contribution to $\eta \to \pi^0\gamma\gamma$ is generated by exchange of internal vector mesons $\omega, \rho$. This generates the amplitudes [10,4,13]:

$$d_1 = 4B_{34} + \frac{8}{3}B_{37}, \quad d_4 = 4B_{38},$$

$$d_2 = 4(B_{35} + B_{36}) + \frac{8}{3}B_{39}, \quad d_5 = 4(B_{40} + B_{41}),$$

$$d_3 = 4B_{47}, \quad d_6 = 4B_{50}.$$ \hspace{1cm} (2.12)
\[ A^{VMD}_\eta(s, t) = - \sum_{V=\omega, \rho, \Phi} \frac{G_{\eta V}}{2} \left[ \frac{t + m^2_\eta}{t - m^2_V} + \frac{u + m^2_\eta}{u - m^2_V} \right] \rightarrow \sum_{V=\omega, \rho, \Phi} \frac{G_{\eta V}}{m^2_V} \left( 3m^2_\eta + m^2_\pi - s \right), \]

\[ B^{VMD}_\eta(s, t) = - \sum_{V=\omega, \rho} G_{\eta V}\frac{s}{2} \left[ \frac{t - m^2_\eta}{t - m^2_V} + \frac{u - m^2_\eta}{u - m^2_V} \right] \rightarrow \sum_{V=\omega, \rho} \frac{G_{\eta V} \cdot s}{m^2_V} \] (2.15)

The right hand side indicates the \( O(p^6) \) low energy limits of the VMD amplitudes. The VMD amplitude for neutral pion pair production \( \gamma \gamma \rightarrow \pi^0 \pi^0 \) is (2.13) with the substitutions \( G_{\eta V} \rightarrow G_{\pi V}, m_\eta \rightarrow m_\pi \). The coupling constants \( G_{\eta V}, G_{\pi V} \) can be extracted from the decay widths of the vector mesons. We list the experimental values of all the coupling constants used in the normalization of Eqs. (2.13)–(2.18) together with the meson masses in Table I.

The contributions of the C-odd axial-vector resonances are (13)

\[ A^B_\eta(s, t) = - \sum_{B=b_1, b_2} \frac{G_{\eta B}}{2} \left[ \frac{-t + m^2_\eta}{t - m^2_V} + \frac{-u + m^2_\eta}{u - m^2_V} \right] \rightarrow \sum_{B=b_1, b_2} \frac{G_{\eta B}}{m^2_V} \left( 3m^2_\eta + m^2_\pi - s \right), \]

\[ B^B_\eta(s, t) = - \sum_{B=b_1, b_2} G_{\eta V}\frac{s}{2} \left[ \frac{t - m^2_\eta}{t - m^2_V} + \frac{u - m^2_\eta}{u - m^2_V} \right] \rightarrow \sum_{B=b_1, b_2} \frac{G_{\eta V} \cdot s}{m^2_V} \] (2.16)

Again, there is an equivalent amplitude for pion pair production. The axial vector mesons interfere constructively with the vector meson amplitudes (2.15) and thus enhance the \( \eta \)-decay width while deteriorating the reproduction of the pion pair production data (13). We include only the measured \( b_1 \) resonance in our model.

The production cross section \( \gamma \gamma \rightarrow \eta \pi^0 \) (3) is dominated by the (scalar) resonance \( a_0(983) \) and the (tensor) resonance \( a_2(1318) \). Their contributions can be written as

\[ A^T_\eta(s, t) = - \frac{G^T_\eta m^2_\eta}{4 \left( m^2_T - s \right)}; \quad B^T_\eta(s, t) = \frac{G^T_\eta m^2_\eta}{4 \left( 2 m^2_T - s \right)} \] (2.17)

for the \( a_2 \) and

\[ A^S_\eta(s, t) = \frac{(G^{Sd}_\eta(s + m^2_\eta - 3m^2_\pi) + G^{Sm}_\pi 2m^2_\pi)}{m^2_S - s}; \quad B^S_\eta(s, t) = 0; \quad G^{Sd}_\pi, G^{Sm}_\pi \geq 0 \] (2.18)

for the \( a_0 \) respectively (18, 3). The \( \gamma \gamma \rightarrow \pi^0 \pi^0 \) spectrum shows the scalar resonance \( f_0(983) \) and the tensor resonance \( f_2(1275) \) with contributions analogous to Eq. (2.17), (2.18) (set \( m_\eta \rightarrow m_\pi \) and choose the appropriate couplings, see (13)).

The coupling constants for the vector and axial–vector mesons are positive, but the signs of the (s-channel and therefore not quadratic) coupling constants \( G^S \) and \( G^T \) relative to the VMD amplitude are ambiguous. Our choice is motivated by the following observation: Consider the one-loop and (approximate) analytic two–loop amplitude for \( \gamma \gamma \rightarrow \pi^0 \pi^0 \) from Ref. (13) as the QCD–background for the process and complement this amplitude by the full resonance amplitude of Eqs. (2.15), (2.16), (2.17) and (2.18). At high energy, parametrize the resonance widths \( m_T \) and \( m_S \) by a relativistic Breit–Wigner form given by Eqs. (5) and (6) of the original data analysis of the Crystal Ball experiment (20). The result is of course not accurate in the region of the \( f_0 \) and \( f_2 \) resonances but fits the shape of the data fairly well.
provided the signs of the couplings are chosen as in Table I (see Fig. 4). With the opposite signs, the measured cross section can not be reproduced. Our sign convention implies that the \( \eta \)–decay width is increased by the contribution of the scalar and tensor mesons. By the way, this result is consistent with the interference pattern found for the scalar resonances in the extended Nambu–Jona-Lasinio model of [11].

Having constructed a realistic meson exchange model for the two neutral meson/two photon processes (see Fig. 5), we can now proceed to test the prediction of the chiral Lagrangian \( \mathcal{L}_6 \) to both pion pair production and the \( \eta \)–decay width. To this end, the propagators of Eqs. (2.13) – (2.18) are taken to first order in the kinematic invariants \( s, u \) and \( t \) as indicated in Eqs. (2.15) and (2.16); the resulting \( \mathcal{O}(p^6) \) low–energy expressions are used to fit simultaneously the parameters of the chiral amplitudes Eqs. (2.13) and (2.14). The procedure yields a total \( \eta \)–decay width of

\[
\Gamma_6^{\mathcal{E}}(\eta \to \pi^0\gamma\gamma) = 0.652 \text{ eV},
\]

(2.19)

(The last digit has been quoted for the discussion of the small loop contributions, see below). As shown in Fig. 2, the spectrum obtained with the \( \mathcal{O}(p^6) \)–fit differs considerably from the original meson exchange spectrum, being smaller as \( s \to 0 \) and showing a broad maximum at \( \sqrt{s} = E_{\gamma\gamma} \sim \frac{3}{2}\sqrt{s_{\text{max}}} \). The difference at low invariant mass \( \sqrt{s} \) is induced by the higher order terms in the t–channel amplitudes Eqs. (2.13) and (2.16). As \( s \) grows, we increasingly neglect contributions from the s–channel amplitudes (2.17) and (2.18) as well. It is in fact easy to perform an "all order" fit of the meson–exchange amplitude, see Fig. 2. We discard this option, however, as not consistent with the spirit of the momentum expansion.

The \( \mathcal{O}(p^6) \) tree–level contributions are small for low–energy pion pair production so that details of the fit have only a minor effect on the cross section (see Fig. 3). \( \gamma\gamma \to \pi^0\pi^0 \) therefore doesn’t put further restrictions on the parameters of the model. Note, however, that it is in general not possible to obtain an accurate reproduction of the (all order) meson exchange amplitude by \( \mathcal{L}_6 \)–terms in the energy domain \( E_{\gamma\gamma} > 500 \text{ MeV} \).

The coupling constants \( d_1 \ldots d_6 \) derived from the fit and collected in Table II deserve some comments. At first glance, a comparison of the \( \mathcal{O}(p^6) \) and all order results for the kinetic constants \( d_1, d_2, d_4 \) and \( d_5 \) suggests that the fit procedure is highly unstable, changing the order of magnitude and even the signs of the constants as one passes from one scheme to another. This behavior is indicative for the fact that the ratio between the empirical meson coupling constants \( G_\pi \) and \( G_\pi \) used in the fit (Table II) is close to the ratio between the \( SU(3) \)–factors contained in Eqs. (2.13) and (2.14) – quark symmetry and therefore the physics of the process itself seems to exclude a stable inversion procedure. The results appear more trustworthy if one keeps in mind that in both amplitudes, only linear combinations of the coefficients \( d_1 \ldots d_6 \) enter. For the \( \eta \)–decay (Eq. (2.13)), these are

a) \( d_1 - 3d_4 = 4B_{34} + \frac{8}{3}B_{37} - 12B_{38} = -3.56 \cdot 10^{-3} \text{ GeV}^{-2} \) \(( -4.81 \cdot 10^{-3} \text{ GeV}^{-2} \) for the \( \mathcal{O}(p^6) \) (all order) fit, to be compared with the value of \( -3.6 \cdot 10^{-3} \text{ GeV}^{-2} \) derived in the simplest fit scheme (Eq. (II)).

b) \( d_2 - 3d_5 = 4(B_{35} + B_{36}) + \frac{8}{3}B_{39} - 12(B_{40} + B_{41}) = 9.59 \cdot 10^{-3} \text{ GeV}^{-2} \) \(( 12.4 \cdot 10^{-3} \text{ GeV}^{-2} \) for the \( \mathcal{O}(p^6) \) (all order) fit, to be compared with the value of \( 7.2 \cdot 10^{-3} \text{ GeV}^{-2} \) from Eq. (II).

The linear combinations \( \frac{5}{3}d_1 - d_4 \) and \( \frac{5}{3}d_2 - d_5 \) that appear the pion production amplitude are similarly stable with respect to the fit method chosen. Due to the similarity of our
meson exchange model with the one used in [13], they are doomed to be consistent with the constants found there.

The mass term constants $d_3$ and $d_6$ show a slightly different behavior. As $d_6$ doesn’t influence the $\eta$–decay amplitude, they have to be stable separately, which is indeed what we observe. The values found for $d_3$ are consistent with the value fitted to the experimental total decay width [14] ($d_3 = 45 \cdot 10^{-3}$ GeV$^{-2}$) within $\sim 10\%$, furthermore, [14] obtains a spectrum similar to the one shown in Fig. 2. Note, however, that we have derived our coupling constants from the general form of the Lagrangian $L^6$ and the meson exchange model alone: no assumption concerning the $\eta$–decay width or pion production cross section was made. The agreement between our model and the one discussed in [14] might indicate that the relatively high $\eta$–decay width measured is indeed correct.

The mass term difference $\Delta m = \frac{5}{9}d_3 + \frac{12}{9}d_6 = \frac{4}{9}(5B_{47} + 12B_{50}) = 1.44 \cdot 10^{-3}$ GeV$^{-2}$ entering the pion production amplitude (2.14) is more than an order of magnitude below the value of $d_3$ (and thus of the order of magnitude of the kinetic constants). Again, our values of $\Delta m$ have to be consistent with the fit of [13] by construction. However, identifying $\Delta m$ with the mass term $d_3$ entering the $\eta$–decay amplitude would inevitably lead to an underprediction of the $\eta$–decay width.

It seems therefore that any attempt to describe $\eta \to \pi^0\gamma\gamma$ and $\gamma\gamma \to \pi^0\pi^0$ simultaneously in a model containing only 3 $L^6$ constants as in Eqs. (1.1) and (1.2) is bound to be troubled by the incompatibility of the mass terms. This explains some of the inconsistencies about the constants $d_1$, $d_2$ and $d_3$ found in the literature.

Because of the relative insensitivity of the $\gamma\gamma \to \pi^0\pi^0$ cross section, a measurement of the $\eta$–decay spectrum would discriminate between different sets of $L^6$ coupling constants. However, in order to pin things further down, we need an accurate estimate of the loop contributions generated in chiral perturbation theory.

D. Effect of Chiral Loops to $O(p^6)$

The following section is devoted to a discussion of the size of the various loop contributions occurring up to $O(p^6)$ and the basic technicalities of their evaluation. For the formal aspects of an two–loop calculation and the implications for the amplitude, we refer to earlier publications [13,20].

The leading loop contributions appearing in chiral perturbation theory are the $O(p^4)$ one–loop diagrams with vertices generated by the Lagrangian $L^2$ of Eq. (2.4). A detailed calculation of this contribution for the $\gamma\gamma \to \pi^0\pi^0$ amplitude has been given in [1]; the $\eta$–decay can be treated analogously, employing expressions similar to Eq. (4.8) without the tadpole term. For the result see Ref. [4]. The $O(p^4)$–amplitude involves charged pion and kaon loops and strongly interferes with the tree–level amplitude from $L^6$ so that its effect

\footnote{From the $\eta\pi^0\pi^+\pi^-$–vertex displayed below (Eq. (4.1)), we obtain only the first (leading) term of the one pion loop amplitude Eq. (14) of Ref. [4]. We agree, however, with the numerical result Eq. (16) of this reference.}
increases with the latter amplitude. Adding it to the $L^6$ result Eq. (2.19), the decay width increases by about 10% to

$$\Gamma^{L^6+O(p^4)}(\eta \to \pi^0\gamma\gamma) = 0.733 \text{ eV}. \quad (2.20)$$

We note that about $3/4$ of the one–loop effect stems from charged kaon loops. Inclusion of pion loops alone would increase the decay width to 0.668 eV.

At order $O(p^6)$, there are further one–loop diagrams generated by the Gasser–Leutwyler Lagrangian $L^4$ [8]. Besides the recalculation of vertices for all the $L^4$–terms, these contributions introduce no new technical difficulties. For a pion loop only, the task is simplified because, like in the case of the Lagrangian $L^2$, $\eta\pi^0\pi^+\pi^−$–vertices can only be generated by the mass terms. This leaves the terms in $L_4$ and $L_5$ as possible candidates where the renormalized coupling constant $L^r_4$ is $N_c$–suppressed and empirically consistent with 0. The $L_5$–amplitude invokes the loop function $F$ defined in the Appendix and is compact enough to be displayed

$$A_{L_5}^{\pi^+\pi^-}(s) = -L^r_5 \frac{8\sqrt{2}}{3\sqrt{3}} \frac{e^2(\Delta m_K)^2}{(4\pi)^2 F_\pi^4} \left(24s - 36m^2_\pi - 4m^2_\eta\right) \frac{1}{s} \left(\frac{1}{2} + \frac{m^2}{s} F(s)\right)$$

$$B_{L_5}^{\pi^+\pi^-}(s) = 0; \quad L^r_5 = 2.2 \times 10^{-3}. \quad (2.21)$$

The decay width from the amplitude (2.21) turns out to be very small ($\Gamma(L^4, \pi^+\pi^-) = 0.45 \times 10^{-4} \text{ eV}$). In superposition to the amplitude corresponding to the decay width (2.20), its effect is totally negligible.

The calculation of charged kaon loops from $L^4$ is analogous to the pion loop calculation but invokes the terms in the Gasser–Leutwyler Lagrangian proportional to $L_4$, $L_5$, $L_8$ and $L_{10}$. By interference of all terms, we gain a factor of 25 ($\Gamma(L^4, \pi^+\pi^- + k^+K^-) = 1.9 \times 10^{-3} \text{ eV}$), the complete $L^4$–contribution reduces the decay width (2.20) to

$$\Gamma^{L^6+O(p^4)+L^4}(\eta \to \pi^0\gamma\gamma) = 0.673 \text{ eV}. \quad (2.22)$$

The values quoted for the $L^4$–loops seem to match the $Z^3_2 = 0.62$ results of Ref. [12]. We come to the conclusion that the one–loop contributions from $L^2$ and $L^4$ mostly cancel each other.

The next group of diagrams are the factorizable 2–loop diagrams derived from Fig. 4 (a) and (b). Again, we first consider the case where only pions are propagating in the loops. If a loop has no photons attached to it, there are also neutral pions allowed. Thus, the 6–meson vertex diagram of Fig. 4 (a) contains two charged pion loops plus the combination of one charged and one neutral loop.

It is a straightforward procedure to determine the Feynman diagrams and derive the corresponding integral expressions. Solving these integrals is largely simplified by the fact that they effectively reduce to a product of two one–loop integrals. On the other hand, as the masses of external and loop particles are comparable, we need to solve the integrals exactly. This can be done by putting propagators with different loop momenta together by means of Feynman parameters and performing the four dimensional integration in the scheme of dimensional regularization [21]. One is left with an (at most) two–dimensional integration in the Feynman parameters space that is analytic due to the fact that the external photons
are on their mass shell. It is further simplified because the masses of the loop particles are all equal. The task is nevertheless a bit tedious because the one–loop integrals generate divergent terms proportional to \( \lim_{n \to 4} \frac{2n^4}{2^n} \) if the number \( n \) of space–time dimensions is taken to 4 so that one is forced to expand all the expressions to linear order in \( \epsilon = \frac{4-n}{2} \).

Details of the calculation are discussed in the Appendix, subsection B.

It is convenient to express everything as sums of products of the elementary integrals \( I_{\text{tad}}, I_{\text{bub}}, \) etc. (Eqs. (4.2), (4.3), (4.6)) and then do the multiplication of one–loop integrals and subtraction of singularities numerically. In order to be consistent with the scheme of chiral perturbation theory and the renormalization of the \( \mathcal{L}^4 \) coupling constants, we have to subtract terms proportional to powers of \( R \equiv \frac{1}{\epsilon} - 1 - \ln(4\pi) + \gamma \), where \( \gamma \) is the Euler constant. Also, the renormalization constant is to be chosen as \( \mu = m_\eta \) in order to be consistent with the numerical values quoted for the \( \mathcal{L}^4 \) low–energy constants. The finite end result has to be gauge invariant, i. e. proportional to \( g^{\mu\nu}s^2 - q^2 q^\nu \). (As in the case of the one–loop amplitudes, there is no contribution \( B(s,t) \).) Gauge invariance can be shown analytically and, on the other hand, serves as a check to the calculation.

It turns out that the various terms in the factorizable two pion–loop amplitude largely cancel each other. Our result for the decay width is \( \Gamma = 2.7 \times 10^{-4} \) eV, about a factor of 3 smaller than the one pion–loop amplitude from \( \mathcal{L}^2 \) and well in agreement with the result of [12]. The effect is also very small in superposition

\[ \Gamma^{\mathcal{L}^0 + O(p^4) + \mathcal{L}^4 + \text{fact.}} (\eta \to \pi^0\gamma\gamma) = 0.682 \text{ eV}. \] (2.23)

Due to the crucial role of cancellations between various terms in the amplitude, it is problematic to predict contributions without actually performing the calculation. We may, however, observe the following: Replacement of a pion loop by a kaon loop lifts the G–parity suppression expressed by the factor \((\Delta m_K)^2/m_K^2\) but introduces a mass factor from the propagators which, in the low–energy limit, amounts to \( m_\pi^2/m_K^2 \). A very rough estimate would therefore be that the amplitude from a kaon loop is larger than the corresponding pion loop amplitude by a factor of \( m_\pi^2/(\Delta m_K)^2 \sim 3 \), a factorizable two–loop diagram containing two kaon loops should be suppressed by \( m_\pi^4/(\Delta m_K)^2 m_K^2 \sim 1/3 \). The values 3 and 1/3 quoted are the relevant ratios because only the interference terms between loop and tree–level contributions affect the result for the decay width noticeably. Our estimate about correct for the \( \mathcal{L}^2 \) one–loop contributions and seems to be confirmed by the two–loop results quoted in [12]. (Estimating the kaon loops from \( \mathcal{L}^4 \) is more problematic because more structures in the Lagrangian can generate kaon loops.) Applying it to the factorizable two–loop amplitude, we would conclude that the two–kaon loops should be totally negligible whereas the pion–kaon loop amplitude might change the result (2.23) by \( \pm 0.02 \) eV, a value that is markedly lower than the uncertainties introduced by the meson exchange model.

The technically challenging part of the amplitude consists in evaluating the overlapping two–loop diagrams Fig. 7. Again, we need the exact result and can not resort, e. g. to developments in the external momenta that have been successfully applied for QCD calculations [25]. Feynman diagrams of the type required have been evaluated for the \( \gamma\gamma \to \pi^0\pi^0 \) amplitude. As outlined in Ref. [13], the expressions corresponding to Fig. 7 (a) and (b) can indeed be transformed into two–dimensional integrals by first integrating over one of the loop momenta and then transforming the result into a dispersion integral over a parameter–dependent box diagram that in turn can be represented as a dispersion integral. We note
that the graph (a) leads to direct \(s-t\) channel box diagrams suggesting a fixed-\(t\) dispersion relation whereas the "master"–diagram (b) yields crossed \(u-t\) channel box diagrams. In our case, the analyticity properties of the graphs are unfortunately more complex because the external \(\eta\) is unstable against decay and thus introduces anomalous thresholds \[22\]. Correspondingly, we will find that the overlapping amplitude has a complex part for arbitrary provided that \(m_\eta \geq 3m_\pi\).

Our solution avoids dispersion relations and is inspired by treatments such as \[23\] where the graphs are transformed into expressions of the type \(I(m, n, Z)\) (see Eq. (4.12)). The hierarchy of \(I(m, n, Z)\) is interrelated by partial fractions, differentiations and partial integrations so that, in the case \(k = 0\) considered in Ref. \[23\], all integrals can be derived from the basic formula for \(I(2, 1, 1)\) which itself can be written as an analytic expression.

For the problem considered here, we have to extend the formalism by introducing a four momentum vector \(k\) in the expressions (4.12) which depends on the external momenta plus up to 2 additional Feynman parameters. This implies no major changes in the Feynman parametrization and evaluation of the four momentum integrals but markedly complicates the subsequent Feynman parameter integrations: in general, we obtain a four dimensional integral where, due to the \(\eta\) instability mentioned above, the integrand contains logarithmic singularities within the parameter domain and is complex for arbitrary values of the kinematic variables. It is essential to work out the analytic structure of the integrand by performing at least two integrations analytically; the remaining two–dimensional integral can then be estimated numerically with reasonable accuracy. We will point out the nontrivial steps of the procedure in the appendix, sect. C.

Because of the amount of algebra involved, it is essential to do the whole calculation as systematically as possible. As far as the derivation of integral expressions and the reduction and simplification of the numerators \(Z(l_1, l_2, q_1, q_2, P, x_1, \ldots, x_m)\) is concerned, we are helped by the fact that the \(\eta 3\pi\)–vertices always produce a constant that factors out. The basic integrals \(I(m, n, Z)\) can be checked independently and various relations between them help to exclude errors in the derivation. Finally, at least in the subthreshold region, all analytic one– or two–dimensional integral formulas can be easily checked on the computer.

Unless the loop corrections discussed so far, the overlapping diagrams contribute to the amplitude part \(B(s, t)\) as well as to \(A(s)\). This implies that there is a contribution from these graphs even at \(s = 0\) where \(A(s)\) has no influence (see Eq. (2.7) and Fig. 8). The analogy with the pion production process where nonanalytical contributions were found to be small in the energy range \(2m_\pi \leq \sqrt{s} \leq 400\ MeV\ \[13\] can therefore not be used to argue that the effect of the overlapping diagrams on the \(\eta\)–decay rate is negligible \[12\]. Indeed, we find a decay width from the overlapping pion diagrams alone that is comparable to the one found for the factorizable diagrams \(\Gamma = 4.1 \times 10^{-4}\ eV\), this changes the decay width (2.23) to

\[
\Gamma^{L_6+O(p^4)+L_4+\text{fact.}+\text{overl.}}(\eta \to \pi^0 \gamma \gamma) = 0.696\ eV. \tag{2.24}
\]

Again, we might estimate an uncertainty of \(\pm 0.04\ eV\) due to the kaon loops. If \(\gamma\) as in the case of the \(O(p)^4\) one–loops – the interference of kaon loops and pion loops is equal, the prediction of chiral perturbation theory to \(O(p)^6\) should be am \(\eta\)–decay width of about \(0.76\ eV\).
As pointed out in Ref. [4], the anomalous part of the $L^4$ Lagrangian generates one–loop terms that are $O(p)^8$ in the momentum expansion but of a size comparable to the $O(p)^4$ loops because there is no G–parity or kaon mass suppression. Adding these contributions (Eq. (27)/(28) of Ref. [4]) to the amplitude of Eq. (2.24), we get the final result

$$\Gamma^{L^6+O(p^4)+L^4+\text{fact.}} (\eta \to \pi^0 \gamma \gamma) = 0.765 \text{ eV.}$$  \hspace{1cm} (2.25)

The coupling constants of the vector mesons used to determine the $L^6$–constants are subject to errors. Based on the uncertainties quoted in [4] for the $\omega$ and $\rho$ couplings, we calculate a relative error of the tree level amplitude (2.19) of about 20 %. Together with the missing kaon loops, this implies an uncertainty to the $O(p)^6$ amplitude of

$$\Gamma^{L^6+O(p^4)+L^4+\text{fact.}} (\eta \to \pi^0 \gamma \gamma) = 0.77 \pm 16 \text{ eV.}$$  \hspace{1cm} (2.26)

This is in close agreement with the experimental value and the result of the quark box model calculation of [27] but is markedly higher than the prediction of the meson exchange model used to determine the Lagrangian.

Addition of the loop contributions to the ”all order fit” amplitude of Table II does not alter the interference scheme described so far but would result in a total decay width of

$$\Gamma^{\text{VMD+loops}} (\eta \to \pi^0 \gamma \gamma) = 0.439 \pm 9 \text{ eV.}$$  \hspace{1cm} (2.27)

The difference between Eqs. (2.25) and (2.27) gives an estimate of the uncertainty of the model induced by higher order terms in the momentum expansion. The various contributions to the decay width in both the $O(p)^6$ and the all order fit scheme are collected in Table 3; the influence of the loops to the decay spectrum is shown in Fig. 8. The plot shows again that contributions to $B(s, t)$ affect the spectrum especially at low energy whereas the $A(s)$–contributions are cut off at $s = 0$.

### III. CONCLUSIONS

We have considered the decay $\eta \to \pi^0 \gamma \gamma$ as an example for the application of the general chiral Lagrangian $L^6$. A total of 6 structures contribute to the process; the corresponding low–energy constants can be determined in a meson exchange model if the analogous process $\gamma \gamma \to \pi^0 \pi^0$ is considered simultaneously. To be consistent at $O(p)^6$, the $L^6$ tree–level amplitude must be complemented by one– and two–loop amplitudes. The result is in agreement with the experimental data for both the neutral pion production and the $\eta$–decay process.

We have given arguments for how to choose the relative signs of the coupling constants in the meson exchange amplitude. As this amplitude contains terms of $O(p)^8$ and higher, there remains a sizeable difference between the predictions of the full meson exchange model and $O(p)^6$ chiral perturbation theory. The total decay widths calculated in both models differ by a factor of 1.7 ($\Gamma(\text{VMD}) = 0.45 \text{ eV}$, $\Gamma(\chi PT) = 0.77 \text{ eV}$. Furthermore, unlike the meson exchange spectrum, the differential width in chiral perturbation theory is small at low invariant energy $s$ and reaches a maximum at about 80 % of the maximum $s$ kinematically allowed. An experiment measuring the decay spectrum would discriminate between both predictions and thus shed light on the details of the mechanism.
The loop contributions for the \(\eta\)-decay are all individually small. However, as we know from the contribution of the \(\mathcal{O}(p)^4\)-loops [1], there can be sizeable interference effects with the leading tree–level amplitude. After a detailed calculation we could show that such interference effects occur (at the 5-10% level) in parts of the amplitude but that the complete two–loop result is suppressed by destructive interference, in particular between the \(\mathcal{L}^4\) kaon–loops and the rest. The two kaon–loop amplitudes have not been calculated but rather estimated to be sufficiently small. Our results confirm the convergence of the chiral loop expansion.

The relative size of the two–loop contributions, in particular the overlapping graphs displayed in Fig. 4 are not quite large enough to justify the amount of work required for their calculation. Nevertheless, we are confident that the strategy for solving the graphs outlined in the appendix may be useful for related calculations. In particular, it would be worthwhile to treat the production amplitude \(\gamma\gamma \rightarrow \pi^0\pi^0\) in \(SU(3)\)-chiral perturbation theory and compare our formalism with the alternative dispersion–theoretical approach.

IV. APPENDIX: A RECIPE FOR SOLVING TWO–LOOP INTEGRALS IN \(\chi PT\)

A. Restrictions, Remaining Vertices and Diagrams

In this section, we develop a method for solving the two–loop contributions for a two neutral meson/two photon process applicable for equal masses of the loop–mesons and real photons but arbitrary external mesons. (The general mass case would only introduce some modifications in the analytic integrations of the overlapping diagrams treated in subsection C.) The general diagrams contributing at \(\mathcal{O}(p^6)\) in the chiral expansion are shown in Fig. 2 of Ref. [13]. For the \(\eta\)-decay, the diagrams involving \(\mathcal{L}^4\) tree–level interactions are cancelled; the only diagrams involving \(\mathcal{L}^4\)-terms are thus one–loop diagrams with a single 4–meson vertex.

As \(\mathcal{L}^2\) can only generate vertices with an even number of mesons, the two–loop configurations we are left with are those displayed in Fig. 4 where the squares denote \(\mathcal{L}^2\)-interactions and photons are to be attached in all possible ways. If we restrict ourselves to internal pions, the diagrams contain each a vertex function \(\Gamma_{\pi\pi\pi\pi} = \text{const.}\) (see Eq. (4.1) below). By a symmetric loop integration argument, one can show that this excludes the diagrams of the type (c) of Fig. 4. We are thus left with the factorizable diagrams (a) and (b) plus the overlapping diagrams of type (d).

The vertex functions required can be calculated in a standard way by inserting the expansion \(U = 1 + \frac{i}{F_\pi^2}\Phi - \ldots\) in the Lagrangian \(\mathcal{L}^2\) and collecting all terms with a given number of meson and photon fields. The procedure implies the trace of a product of up to six \(3 \times 3\)-matrices and is therefore most conveniently done on the computer. Neglecting the kaon sector, one obtains the following vertex functions:

\[
\Gamma_{\pi^+\pi^-} = -ie(p_+ - p_-); \quad \Gamma_{\pi^+\pi^-}^{\mu\nu} = -2ieg^{\mu\nu};
\]

\[
\Gamma_{\pi^+\pi^-\eta^0} = -\frac{i\sqrt{2}}{3\sqrt{3}F_\pi^2}(\Delta m_K)^2 \quad \Gamma_{\pi^\alpha\pi^\alpha\eta^0} = -\frac{i\sqrt{2}}{\sqrt{3}F_\pi^2}(\Delta m_K)^2
\]

\[
\Gamma_{\pi^+\pi^-\eta^0\eta^0} = \frac{i}{3F_\pi^2}\left(2(p_1p_2 + p_+p_-) + (p_1 + p_2)^2 + m_\pi^2\right)
\]
\[
\Gamma_{\pi^+ \pi^- \pi^+ \pi^-} = \frac{i}{3F_\pi^2} \left( p_1^2 + p_2^2 + p_3^2 + p_4^2 - 3(p_+ - p_-)^2 + 2m_\pi^2 \right)
\]
\[
\Gamma^{\mu \pi^+ \pi^- \pi^+ \pi^-} = \frac{-2ie}{3F_\pi^2} (p_+ - p_-)^\mu; \quad \Gamma^{\mu \pi^+ \pi^- \pi^+ \pi^-} = \frac{-8i}{3F_\pi^2} (p_+ - p_-)^\mu; \quad \Gamma^{\mu \pi^+ \pi^- \pi^+ \pi^-} = \frac{-4ie^2}{3F_\pi^2} g^{\mu\nu}; \quad \Gamma^{\mu \pi^+ \pi^- \pi^+ \pi^-} = \frac{-16ie^2}{3F_\pi^2} g^{\mu\nu};
\]
\[
\Gamma_{\pi^+ \pi^- \pi^+ \pi^- \pi^+ \pi^-} = \frac{i\sqrt{2}}{15\sqrt{3}F_\pi^4}(\Delta m_K)^2; \quad \Gamma_{\pi^+ \pi^- \pi^+ \pi^- \eta \pi^0} = \frac{i\sqrt{2}}{10\sqrt{3}F_\pi^4}(\Delta m_K)^2; \quad (4.1)
\]

All other vertex functions vanish. The conventions used for these expressions, in particular the orientation of the four momenta, are displayed in Fig. 4. Statistical factors for the case that there are two or more identical particles have been included in Eqs. (4.1). For vertices involving two \(\pi^+ \pi^-\) pairs, the momenta \(p_+, p_-\) have to be assigned symmetrically. Note that there are no photon vertices with neutral mesons only and that all the vertices involving a single \(\eta\)-meson are generated by the mass term of the Lagrangian \(\mathcal{L}^2\) and are constant. In analogy to the one–loop amplitude \[4\], the total internal pion two–loop amplitude is G-parity suppressed.

**B. Factorizable Diagrams**

The Diagrams (a) and (b) of Fig. 4 contain no propagators containing both loop momenta and therefore effectively factorize into a product of two one–loop expressions. According to the scheme of dimensional regularization \[21\], one–loop integrals contain singularities linear in \(\frac{1}{\epsilon} = \frac{1}{2-n}\) where \(n\) is the continuous dimension parameter. In order to retrieve all nonvanishing terms as \(n \to 4\), we are forced to develop the one–loop expressions to \(\mathcal{O}(\epsilon)\). The divergence of the resulting two–loop amplitude is at most quadratic in \(\frac{1}{\epsilon}\).

For clarity, we display the explicit expansions of the simplest elementary one–loop diagrams. A tadpole loop (see Fig. 4 (a)) generates \[23\]:

\[
I_{tad} \equiv \int \frac{d^4l}{(2\pi)^4} \frac{1}{|l|} = -im^2 \Gamma(2 - \frac{n}{2}) \left(\frac{4\pi\mu^2}{m^2}\right)^{2-n} \frac{1}{\epsilon} \equiv \frac{im^2}{(4\pi)^2} \left(\frac{1}{\epsilon} + \left[1 - \gamma - \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right]\right) + \epsilon \left[1 + \delta - \gamma(1 - \ln\left(\frac{m^2}{4\pi\mu^2}\right)) - \ln\left(\frac{m^2}{4\pi\mu^2}\right) + \frac{1}{2} \ln^2\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^2)\right], \quad (4.2)
\]

where we have denoted the propagator as \(|l| = l^2 - m^2 + i\epsilon\), expanded the \(\Gamma\)–function

\[
\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \epsilon \delta + \mathcal{O}(\epsilon^2); \quad \delta = \frac{\pi^2}{12} + \frac{\gamma^2}{2},
\]

with the Euler constant \(\gamma = 0.577215665\) and introduced the renormalization constant \(\mu\). The small imaginary factor in the propagator will not be written explicitly in the following but should be kept in mind in order to retrieve the imaginary parts of the loop integrals. Also, in Eq. (4.2) as well as in the following, the limits \(n \to 4\) and equivalently \(\epsilon \to 0\) are always understood.

A symmetric bubble–diagram (Figure 4 (b)) with photon momenta \(q_1 + q_2\) where \(s = (q_1 + q_2)^2\) yields after Feynman parametrization
where the loop integrals

\[ G(s) = \int_0^1 dx \ln(1 - \frac{s}{m^2}x(1 - x)); \quad G(0) = 0; \]

\[ G^2(s) = \int_0^1 dx \ln^2(1 - \frac{s}{m^2}x(1 - x)); \quad G^2(0) = 0 \]  

are analytic functions cut along the real axis as \( s > 4m^2 \). The integrals \( G(s) \) and

\[ F(s) = \int_0^1 dx \frac{dx}{x} \ln(1 - \frac{s}{m^2}x(1 - x)); \quad F(0) = 0 \]  

can be easily reduced to analytic expressions \([4][5]\); moreover, loop integrals with a single logarithm but additional factors \( x, x^2 \) etc. can be expressed in terms of \( G(s) \) and \( F(s) \). The loop functions of the form \( G^2(s), F^2(s) \) etc. can be expressed in terms of polylogarithms \([24]\) or be solved numerically.

The scalar vertex diagram (scalar meaning numerator equal to 1, see Fig. 6 (c)) is finite and written as

\[ I_\Delta = \int \frac{d^4l}{(2\pi)^4 l[l-q_1][l+q_2]} = \frac{i}{(4\pi)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{dx_1 dx_2 \Gamma(3 - \frac{n}{2})}{s x_1 x_2 - m^2} \]

\[ = \frac{i}{(4\pi)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[ 1 + \epsilon \left( -\gamma - \ln\left(\frac{m^2}{4\pi\mu^2}\right) - \ln\left(1 - \frac{s}{m^2}x_1 x_2\right) \right) \right] \quad (4.6) \]

The scalar integrals considered so far can be easily generalized to the case where loop momenta appear in the numerator of the integrand. For the case of a linear momentum in the numerator, the formula for dimensional regularization are applicable. Integrals with quadratic and higher order loop momenta in the numerator are reduced to simpler integrals with formula such as

\[ \int \frac{d^4l}{(2\pi)^4 (l^2 + 2kl - m^2)^\alpha} = \frac{i}{16\pi^{\frac{\alpha}{2}}} \frac{4k^\mu k^\nu - \frac{4}{n} g^{\mu\nu} k^2}{(-k^2 - m^2)^{\alpha - \frac{\alpha}{2}}} \]  

(4.7)

The vertex and box diagrams appearing in the amplitude generate further loop functions of the types

\[ \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{s x_1 x_2 - m^2}; \quad \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{s x_1 x_2 - m^2}; \quad \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{s x_1 x_2 - m^2}; \quad \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{s x_1 x_2 - m^2}; \quad \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{s x_1 x_2 - m^2}; \]
with powers $0 \leq m, n \leq 3$. The loop functions are symmetric in m and n. In all cases, the $x_2$–integration is easy to perform analytically; partial integration and simple algebraic manipulations allow to express everything in terms of $F(s)$, $G(s)$, $F^2(s)$ (defined as $F(s)$ but with a quadratic logarithm in the integrand) and $G^2(s)$. Due to the similarity of the loop functions, we find that the factorizable amplitude has a unique s–channel threshold $s = 4m^2$.

Let us illustrate the method in a simple example, the 6-meson vertex diagram of Fig. 4 (a). In order to get a nonvanishing amplitude, both photon lines have to be attached to one charged meson loop; the second loop can be charged or neutral. From the vertex expressions in Eq. (4.1), one finds that the amplitude can be written as

$$M^{\mu \nu} = -\frac{3i\sqrt{2}e^2(\Delta m_K)^2}{2 \cdot 9\sqrt{3}F^4_\pi} \int \frac{d^4l_1}{(2\pi)^4} \frac{d^4l_2}{(2\pi)^4} \frac{(2l_1 - q_1)^{\mu}(2l_1 + q_2)^{\nu} - g^{\mu \nu}(l_1^2 - m^2)}{[l_1][l_1 - q_1][l_1 + q_2][l_2]} + (q_1, \mu \leftrightarrow q_2, \nu)$$

(4.8)

This is gauge invariant because of

$$q_{1 \mu}M^{\mu \nu} \sim \int \frac{d^4l_1}{(2\pi)^4} q_{1 \mu} \frac{[(2l_1 - q_1)^{\mu}(2l_1 + q_2)^{\nu} - g^{\mu \nu}(l_1^2 - m^2)]}{[l_1][l_1 - q_1][l_1 + q_2]} \begin{array}{c} \\ \end{array} = \int \frac{d^4l_1}{(2\pi)^4} \frac{2l_1^{\mu}}{[l_1 - \frac{q_1 + q_2}{2}][l_1 + \frac{q_1 + q_2}{2}]} - \int \frac{d^4l_1}{(2\pi)^4} \frac{2l_1^{\nu}}{[l_1 - \frac{q_1 + q_2}{2}][l_1 + \frac{q_1 + q_2}{2}]} = 0$$

(4.9)

With the expressions (4.2), (4.3) and (4.6) for the elementary diagrams, the amplitude (4.8) reads

$$M^{\mu \nu} = -\frac{i\sqrt{2}e^2(\Delta m_K)^2}{3\sqrt{3}F^4_\pi} \left(g^{\mu \nu} \frac{s}{2} - q_2^{\mu} q_1^{\nu}\right) I_{tad} \times \frac{i}{(4\pi)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1x_2}{s x_1 x_2 - m^2} \left[1 + \epsilon \left(\frac{\gamma - \ln(\frac{m^2}{2s}) - \ln(1 - \frac{s}{m^2}x_1 x_2)}{s x_1 x_2 - m^2}\right)\right].$$

(4.10)

The loop integrals are readily solved and yield

$$\begin{align*}
\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{s x_1 x_2 - m^2} &= \frac{1}{s} \left(\frac{1}{2} + \frac{m^2}{s} F(s)\right) \\
\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2 \ln(1 - \frac{m^2}{2s})}{s x_1 x_2 - m^2} &= \frac{1}{s} \left(-\frac{1}{2} - \frac{m^2}{s} F(s) + \frac{1}{2} G(s) + \frac{m^2}{2s} F^2(s)\right)
\end{align*}$$

(4.11)

The 6-meson diagrams contribute to the form factor $A(s,t)$ of Eq. (2.4) only. The same is true for all factorizable diagrams.

### C. Overlapping Diagrams

The overlapping diagrams derived from Fig. 4 (d) translate into sums of integrals of the type
\[ I(m, n, Z) = \int_0^1 dx_1 \cdots \int_0^{1-x_1-\cdots-x_{n-1}} dx_n \int_0^1 dx_{n+1} \cdots \int_0^{1-x_{n+1}-\cdots-x_{m-1}} dx_m \]
\[ \times \frac{1}{(2\pi)^4 (2\pi)^4} \frac{d^4 l_1 d^4 l_2}{[l_1][l_2][l_2 - l_1 - k(x_1, \ldots, x_m)]} \]
\[ \int \frac{d^4 l_1 d^4 l_2}{(2\pi)^4} Z(l_1, l_2, q_1, q_2, P, x_1, \ldots, x_m) \]
where \( 1 \leq n, m - n \leq 3; \ m \leq 4 \). Here, we have used Feynman parameters in order to put together propagators with equal loop momenta and obtain more symmetric expressions. The numerators \( Z \) can be largely simplified by factoring out \( l \)-dependent terms; the remainder is at most an expression cubic in the loop momenta \( l^4 l^4 l^4 \). With further Feynman parametrization, all the integrals \( I(m, n, Z) \) can be cast in a form that makes the analytic structure of the loop integrals explicit and leaves a two-dimensional numerical integration over a well-behaved, complex function. We demonstrate the method for the scalar integrals \( I(2, 1, 1), I(2, 2, 1) \) and \( I(3, 1, 1) \).

The integral \( I(2, 1, 1) \) corresponds to the scalar part of the diagram Fig. 7 (c). The analytic case where \( k = 0 \) has been considered in [23]. In extension to the method outlined there, we apply subsequent Feynman parametrization and integration over the loop momenta and write

\[
I(2, 1, 1) \equiv \int_0^1 dx \int \frac{d^4 l_1 \ d^4 l_2}{(2\pi)^4 (2\pi)^4} \frac{1}{[l_1][l_2][l_2 - l_1 - k(x)]} \]
\[
= \int_0^1 dx \int \frac{d^4 l_1 \ d^4 l_2}{(2\pi)^4 (4\pi)^2} \int_0^1 dz \frac{\Gamma(2 - \frac{n}{2})(4\pi\mu^2)^{2-\frac{n}{2}}}{(z(1-z))^{2-\frac{n}{2}}} \frac{1}{[l_1][l_1 + k(x)]} \frac{1}{((l_1 + k(x))^2 - \frac{m^2}{z(1-z)})^{2-\frac{n}{2}}} \]
\[
= \int_0^1 dx \frac{1}{(4\pi)^4} \int_0^1 dz \frac{z(1-z)}{\epsilon} \left( \frac{(4\pi\mu^2)^2}{z(1-z)} \right)^{2-\frac{n}{2}} \int_0^1 dy (1-y)^{2-\frac{n}{2}} \]
\[
times \left\{ \frac{m^2}{z(1-z)} \left( k^2 y(1-y) - 2 k^2 (1-y) + k^2(1-y)^2 \right) \frac{\Gamma(5-n)}{(k^2 y(1-y) - m^2 y - \frac{m^2}{z(1-z)})^{5-n}} \right\} \]
\[
+ \frac{n}{2} \left( k^2 y(1-y) - m^2 y - \frac{m^2}{z(1-z)} \right)^{4-n} \]
\[ \Gamma(4-n) \]

After expanding \((1-y)^{2-\frac{n}{2}}\) in powers of \( \epsilon = 2 - \frac{n}{2} \), the second term in this expression can be partially integrated over \( y \). One obtains a sum of terms where the \( z \)-integration plus some trivial \( x \) - and \( y \)-integrations can be done analytically. The final result can be cast in the form

\[
I(2, 1, 1) = \frac{1}{(4\pi)^4} \left\{ \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} \left( 1 - 2\gamma - 2 \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right) \right. \\
\left. + \frac{\pi^2}{12} + \frac{5}{2} - \gamma + \gamma^2 - (1 - 2\gamma) \ln \left( \frac{m^2}{4\pi\mu^2} \right) + 2 \ln^2 \left( \frac{m^2}{4\pi\mu^2} \right) \right. \\
\left. + \int_0^1 dy G(a) - \int_0^1 \frac{dy}{y} \left( G(a) + \ln(y) + 2 \right) \right\} ; \\
a \equiv \frac{k^2(x)(1-y) - m^2(1-y)}{m^2 y} \]

(4.13)

with the loop function \( G \) from Eq. (1.4). We note that for the \( \eta \)-decay diagram Fig. 7 (c), the external momentum variable can be written as \( k = P - q_1 x \). Owing to the properties
of $G$ and the ratio of the $\eta$– and $\pi$–masses, the integrand in Eq. (4.1) is a complex function for arbitrary $s \geq 0$. The integral is finite with a logarithmic end point singularity in the real part of the integrand as $y \to 0$.

Feynman integrals with a numerator depending on the loop momenta can be solved analogously. One obtains

$$I(2, 1, l_1^\mu) \equiv \int_0^1 dx \int \frac{d^4l_1}{(2\pi)^4} \frac{d^4l_2}{(2\pi)^4} \frac{l_1^\mu}{[l_1^2][l_2^2]l_2 - l_1 - k(x)} = \frac{-1}{(4\pi)^4} \int_0^1 dx (k(x))^\mu \left\{ \frac{1}{4\epsilon} - \frac{1}{4} \ln \left( \frac{m^2}{4\pi \mu^2} \right) + \int_0^1 dy (1 - y)G(a) \right\};$$

$$I(2, 1, \Delta l_\mu^\nu) \equiv \int_0^1 dx \int \frac{d^4l_1}{(2\pi)^4} \frac{d^4l_2}{(2\pi)^4} \frac{4l_1^\mu l_2^\nu - \frac{4}{n}l_1^2 g^\mu_2}{[l_1^2][l_2^2]l_2 - l_1 - k(x)} = \frac{-1}{(4\pi)^4} \int_0^1 dx (4k(x)^\mu k(x)^\nu - \frac{4}{n}k(x)^2 g^\mu_2) \times \left\{ \frac{1}{12\epsilon} - \frac{1}{24} + \frac{1}{18} - \frac{\gamma}{6} - \frac{1}{6} \ln \left( \frac{m^2}{4\pi \mu^2} \right) + \int_0^1 dy y (1 - y)G(a) \right\};$$

(4.15)

The last equation shows how to reduce integrals with higher powers of loop momenta in the numerator to the integral class $I(1, 1, Z)$. The latter integral can in turn be related to $I(2, 1, Z)$ by means of the ‘partial’ operation [22, 23].

The ‘master’ diagram Fig. 7 (b) leads to a scalar integral $I(2, 2, 1)$ with $k(x_1, x_2) = P - q_1 x_1 - q_2 x_2$. The integral is finite so that we don’t have to worry about the regularization procedure:

$$I(2, 2, 1) \equiv \int_0^1 dx_1 \int_0^1 dx_2 \int \frac{d^4l_1}{(2\pi)^4} \frac{d^4l_2}{(2\pi)^4} \frac{1}{[l_1^2][l_2^2]l_2 - l_1 - k(x_1, x_2)}$$

$$= \frac{-1}{(4\pi)^4} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx \int_0^1 dy \frac{(k^2 y (1 - y) - m^2 y) z (1 - z) - m^2 (1 - y)}{z y}$$

$$= \frac{-1}{(4\pi)^4} \int_0^1 dy_1 \int_0^{1 - y_1} dy_2 \int_0^1 dz_1 \int_0^{1 - z_1} dz_2 \frac{1}{\tilde{r}^2 z_1 z_2 + \tilde{s}^2 z_1 (1 - z_1) - \tilde{m}^2}$$

$$= \frac{-1}{(4\pi)^4} \int_0^1 dy_1 \int_0^{1 - y_1} dy_2 \frac{1}{\tilde{r}^2} \left( F \left( \frac{\tilde{r}^2 + \tilde{s}^2}{\tilde{m}^2} \right) - F \left( \frac{\tilde{s}^2}{\tilde{m}^2} \right) \right),$$

(4.16)

where $\tilde{r}^2$, $\tilde{s}^2$ and $\tilde{m}^2$ depend on the Feynman parameters $y_1$, $y_2$ and the invariant kinematic variables

$$\tilde{r}^2 \equiv s y_1 y_2 - (m^2 - u) y_1 (1 - y_1),$$
$$\tilde{s}^2 \equiv - (m^2 - t) y_1 y_2 - m^2 (1 - y_1)^2,$$
$$\tilde{m}^2 \equiv m^2 y_1.$$

The analytic integration is largely simplified by the fact that both photons are real ($q_1^2 = q_2^2 = 0$) and the masses in the propagators are identical. The general mass case can still be treated in the same way, but with polylogarithms occurring from the analytical $z$–integration.

The integral $I(2, 2, 1)$ is linked to $I(2, 1, 1)$ by
\[ I(2, 2, 1) = \lim_{M^2 \to m^2} \frac{\partial}{\partial M^2} \int_0^1 dx_1 \int_0^1 dx_2 \int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4} \frac{1}{[l_1^2](l_2^2 - M^2)[l_2 - l_1 - k(x_1, x_2)]}. \] 

(4.17)

Relations such as Eq. (4.17) can be used to check the calculation.

For an accurate numerical estimate of two–loop Feynman integrals, it is crucial to evaluate the first complex integration analytically. Due to the \( x \)–parametrization of loop momenta and the presence of \( l \)–dependent terms in the numerator \( Z \), the master diagram contains integrals of the form

\[ I^M(l, m, n) = \int_0^1 dz_1 \int_0^1 dz_2 \frac{z_1 z_2^m}{(1 - z_1)^n a z_1 z_2 + b z_1 (1 - z_1) - c}; \quad 0 \leq l, m, n \leq 3. \] 

(4.18)

The case \( m > 0 \) leads to cancelling singular terms after the first integration and requires formula such as

\[
\int_0^1 \frac{dz}{az^2 - az + 1} = \frac{2}{4 - a} (G(a) + 2).
\]

It turns out that all integrals \( I^M(l, m, n) \) can be expressed in terms of the loop functions \( F \) and \( G \). As in Eq. (4.14), we are left with a two–dimensional integral over a complex function that can be easily done numerically.

The diagram Fig. [a] yields \( k = P - q_1(1 - x_1) - q_2 x_2 \) and the modified integral

\[
\bar{I}(3, 1, 1) \equiv \int_0^1 dx_1 \int_0^1 dx_2 \int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4} \frac{1}{(l_1^2 + s x_1 x_2 - m^2)[l_2^2 - l_1 - k(x_1, x_2)]} \]

\[
= \left( \frac{1}{(4\pi)^4} \right) \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dz \frac{1}{\epsilon} \left( \frac{(4\pi \mu^2)^2}{z(1 - z)} \right)^{2 - \frac{n}{2}} \int_0^1 dy \frac{y^2(1 - y)^{2 - \frac{n}{2}}}{z(1 - z)}
\]

\[
\times \left\{ \frac{(k^2 y - \frac{m^2}{z(1 - z)})}{\Gamma(6 - n)} \frac{\Gamma(5 - n)}{\Gamma(6 - n)} \left[ \frac{k^2 y(1 - y) + (s x_1 x_2 - m^2) y - \frac{m^2(1 - y)}{z(1 - z)}}{6 - n} \right] \right\}
\]

(4.19)

At this point, it is convenient to write the denominators as

\[
\left[ k^2 y(1 - y) + (s x_1 x_2 - m^2) y - \frac{m^2(1 - y)}{z(1 - z)} \right]^{-1}
\]

\[
= \frac{1}{y(t - 1)} \lim_{M^2 - m^2} \frac{\partial}{\partial M^2} \left[ k^2 y(1 - y) + (s x_1 x_2 - M^2) y - \frac{m^2(1 - y)}{z(1 - z)} \right]^{-1},
\]

20
Note that finally reperform the partial derivation. The result can be represented as partially integrate the second term as in Eq. (4.14), expand and rearrange all terms and finally reperform the partial derivation. The result can be represented as

\[
\tilde{I}(3, 1, 1) = \frac{-1}{(4\pi)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left\{ \int_0^1 dz \int_0^1 dy \frac{y^2}{k^2 y(1-y) + (sx_1 x_2 - m^2)y - \frac{m^2(1-y)}{z(1-z)}} - \frac{1}{sx_1 x_2 - m^2} \right\}
\]

Note that \(\tilde{I}(3, 1, 1)\) contains simple s-channel threshold terms in \(sx_1 x_2 - m^2\) as well as an anomalous threshold term. The cancellation of the parameter singularities in the first two contributions to Eq. (4.20) can be used as a test for the analytic integration. The x-integrations in the s-threshold terms can be performed according to subsection B. In the first term of Eq. (4.20), the \(x_2\)- and \(z\)-integrations can be done analytically yielding a form analogous to Eq. (4.16). As in the case of the master diagram (see Eq. (4.18)), the diagram (a) involves a whole set of analytic parameter integrals.

In analogy to the integrals of the class \(I(2, 1, Z)\), singularities may cancel in the presence of \(l\)-dependent numerators. For instance, we obtain

\[
\tilde{I}(3, 1, l_1^\mu) \equiv \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{dl_1}{(2\pi)^4} \frac{dl_2}{(2\pi)^4} \frac{l_1^\mu}{(l^2 + sx_1 x_2 - m^2)^3 l_2 [l_2 - l_1 - k(x)]} \frac{y^2}{k^2 y(1-y) + (sx_1 x_2 - m^2)y - \frac{m^2(1-y)}{z(1-z)}}
\]

\[
\tilde{I}(3, 1, \Delta l_1^{\mu\nu}) \equiv \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{dl_1}{(2\pi)^4} \frac{dl_2}{(2\pi)^4} \frac{4\epsilon_1^{\mu\nu} - \epsilon_1 l_{1\mu} l_{1\nu}}{4 l_1^2 g^{\mu\nu}} \frac{y^2(1-y)(k^\mu k^\nu - g^{\mu\nu} k^2)}{k^2 y(1-y) + (sx_1 x_2 - m^2)y - \frac{m^2(1-y)}{z(1-z)}}
\]

This closes the list of basic integrals required for the type of processes we are considering.

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TABLE I. Meson parameters used for the meson exchange amplitudes of Eqs. (2.15) – (2.18). For the scalar mesons, both coupling constants $G^\text{Sd}$ and $G^\text{Sm}$ are given. The last column contains the (cumulative) total $\eta$–decay width.

| Meson | $m$ [MeV] | $G_\pi$ [GeV$^{-2}$] | $G_\eta$ [GeV$^{-2}$] | $\Gamma_\eta$ [eV] |
|-------|-----------|---------------------|---------------------|-----------------|
| $\rho$ | 768       | 0.084               | 0.17                | 0.11            |
| $\omega$ | 782     | 0.49                | 0.098               | 0.28            |
| $\Phi$  | 1019      | 0.018               | 0.0089              | 0.29            |
| $b_1$   | 1232      | 0.39                | 0.18                | 0.32            |
| $a_0$   | 983       | 0                   | -0.018/-0.022       | 0.35            |
| $a_2$   | 1318      | 0                   | 0.216               | 0.37            |
| $f_0$   | 974       | -0.007/-0.009       | 0                   | -               |
| $f_2$   | 1275      | 1.28                | 0                   | -               |

TABLE II. $\mathcal{L}^6$ low–energy constants determined in two different fit schemes (see text). All constants are in units of $10^{-3}$ GeV$^{-2}$. The relation between $d_1, \ldots, d_6$ and the low energy constants of Ref. [7] are given in Eq. (2.12).

| Model | $d_1$ | $d_4$ | $d_2$ | $d_5$ | $d_3$ | $d_6$ |
|-------|-------|-------|-------|-------|-------|-------|
| $\mathcal{O}(p^6)$ | -1.60 | 0.654 | 3.80  | -1.93 | 38.9  | -16.1 |
| All Order | 0.108 | 1.64  | 0.0486| -4.13 | 40.3  | -16.7 |

TABLE III. Cumulative contributions of the loop contributions to the total $\eta$–decay width. The $\mathcal{L}^6$–fit parameters are those given in Table II, the loop corrections are denoted as in the text. The last column contains the estimate of the missing $\mathcal{O}(p)^6$ kaon loops.

| Model | $\mathcal{L}^6$ | $\mathcal{O}(p)^4$ | $\mathcal{L}^4$ | fact. | overl. | anom. | (K–loops) |
|-------|-----------------|---------------------|-----------------|-------|--------|--------|-----------|
| $\mathcal{O}(p^6)$ | 0.652 | 0.733 | 0.673 | 0.682 | 0.696 | 0.765 | (0.83)   |
| All Order | 0.371 | 0.411 | 0.381 | 0.383 | 0.386 | 0.439 | (0.46)   |
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FIGURES

FIG. 1. The $\gamma \gamma \rightarrow \pi^0 \pi^0$ spectrum as measured by the Crystal Ball Collaboration [20]. The curves are results of chiral perturbation theory complemented by a $O(p^6)$ meson exchange amplitude: Pure chiral one- and two-loop contributions in the SU(2)-model (Eqs. (7.2) and (7.7) of Ref. [13]) (dashed), loop amplitude plus full meson exchange model Eqs. (2.15) - (2.18) (dotted) and loop amplitude with meson exchange model but without axial mesons Eq. (2.16).

FIG. 2. Predictions for the $\eta \rightarrow \pi^0 \gamma \gamma$ spectrum. The prediction of the full meson exchange model (solid) and its best fit according to the amplitude Eq. (2.13) (dotted) yields a total decay width of $\Gamma_{\text{tot}} = 0.37$ eV. For comparison, the predictions of the $O(p^6)$ fit (dashed, $\Gamma_{\text{tot}} = 0.65$ eV) and the 2-parameter fit according to [4], Eq. (1.1) (short dashed, $\Gamma_{\text{tot}} = 0.18$ eV) are also shown. Loop corrections are not taken into account.

FIG. 3. $\gamma \gamma \rightarrow \pi^0 \pi^0$ cross section from [20] and predictions from chiral perturbation theory. The results have been obtained with the chiral one- and two-loop contributions (Eqs. (7.2) and (7.7) of Ref. [13]) complemented with the $O(p^6)$-contributions as in Fig. 2. Axial vector mesons are not taken into account.

FIG. 4. The two types of factorizable two-loop diagrams contributing to the $\eta \rightarrow \pi^0 \gamma \gamma$ amplitude and considered in this paper.

FIG. 5. Vertices and vertex functions $\Gamma_{\pi^+ \pi^-}^{\mu}$, $\Gamma_{\pi^0 \pi^+ \pi^-}^{\mu \nu}$ and $\Gamma_{\pi^+ \pi^- \pi^0 \pi^0 \eta}$, as used in Eq. (4.1).

FIG. 6. Basic one-loop diagrams calculated in Appendix, subsection B: Tadpole (a), bubble (b) and vertex (c) diagram.

FIG. 7. The four types of overlapping diagrams occurring in the process $\eta \rightarrow \pi^0 \gamma \gamma$.

FIG. 8. Effect of the loop contributions on the $\eta$-decay spectrum. The $L^6$ tree-level results according to Table III, first column are solid (compare Fig. 2), the dashed lines include the $O(p)^4$ one-loop result (Table III, second column), the dotted lines correspond to the full result (Table III, second last column).