INFINITE DIMENSIONAL CHEVALLEY GROUPS AND KAC–MOODY GROUPS OVER \( \mathbb{Z} \)

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Abstract. Let \( A \) be a symmetrizable generalized Cartan matrix, which is not of finite or affine type. Let \( g \) be the corresponding Kac–Moody algebra over a commutative ring \( R \) with 1. We construct an infinite-dimensional group \( G_V(R) \) analogous to a finite–dimensional Chevalley group over \( R \). We use a \( \mathbb{Z} \)-form of the universal enveloping algebra of \( g \) and a \( \mathbb{Z} \)-form of an integrable highest-weight module \( V \). We construct groups \( G_V(\mathbb{Z}) \) analogous to arithmetic subgroups in the finite-dimensional case. We also consider a universal representation–theoretic Kac–Moody group \( \tilde{G} \) and its completion \( \tilde{G} \). For the completion we prove a Bruhat decomposition \( \tilde{G}(\mathbb{Q}) = \tilde{G}(\mathbb{Z})B(\mathbb{Q}) \) over \( \mathbb{Q} \), and that the arithmetic subgroup \( \Gamma(\mathbb{Z}) \) coincides with the subgroup of integral points \( \tilde{G}(\mathbb{Z}) \).

1. Introduction

A Kac–Moody group is an abstract group \( G = G(A) \) associated to an infinite–dimensional Kac–Moody algebra \( g = g(A) \) over \( \mathbb{C} \) with symmetrizable generalized Cartan matrix \( A \). We view the group \( G \) as an infinite–dimensional analog of a Lie group. Over a general commutative ring \( R \), there is still no widely–agreed–upon definition of a Kac–Moody group, except in the affine case ([Ga1]; see also [A]). There are numerous methods for constructing Kac–Moody groups over fields, however, using a variety of techniques and additional external data (see, for example, [CG, Ga1, GW, KP, Ku, Ma, Mar, MT, Sl, Ti1]).

Many constructions use some version of the Tits functor \( \check{G} = \check{G}(A) \) from commutative rings to groups ([Ti2]). Tits showed that if another group functor \( \mathfrak{G} \) satisfies certain axioms imitating the properties of Chevalley–Demazure group schemes, then there is a functorial homomorphism \( \mathfrak{G} \to \check{G} \), which is an isomorphism over every field.

Here we describe a construction of a representation–theoretic Kac–Moody group \( G_V(R) \) over any commutative ring \( R \) using a \( \mathbb{Z} \)-form \( U_\mathbb{Z} \) of the universal enveloping algebra of \( g \), an integrable highest–weight module \( V \) of \( g \), and a \( \mathbb{Z} \)-form \( V_\mathbb{Z} \) of \( V \). Tits refers to the possibility of making such a construction (item (b) on p. 554 of [Ti2]). Our construction is a natural generalization of the theory of elementary Chevalley groups over commutative rings ([Ch, St] or see, for example, [VP]) and builds on the seminal work of Garland ([Ga1]). This method was also used in [CG] to construct Kac–Moody groups over fields.

We give a detailed description of the integral form \( G_V(\mathbb{Z}) \) in Section 3. We prove a Bruhat decomposition \( G_V(\mathbb{Q}) = G_V(\mathbb{Z})B(\mathbb{Q}) \) over \( \mathbb{Q} \), where \( B \) is the analog of the Borel subgroup of \( G \).

We eventually consider a universal representation-theoretic Kac–Moody group \( G(\mathbb{Q}) = G_V(\mathbb{Q}) \), by taking the direct sum \( V \) of all the irreducible integrable highest weight modules \( V^\lambda \), as well as a natural completion \( \tilde{G}(\mathbb{Q}) \) of \( G(\mathbb{Q}) \).

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Let $\Gamma(\mathbb{Z}) = \{ g \in G(\mathbb{Q}) \mid g(\mathbb{Q}_\mathbb{Z}) = \mathbb{Q}_\mathbb{Z} \}$ and let $\Gamma(\mathbb{Z})$ be the completion of $\Gamma(\mathbb{Z})$ in $\widetilde{G}(\mathbb{Q})$. Let $G(\mathbb{Z})$ be the subgroup of integral points of $G(\mathbb{Q})$ and let $\widetilde{G}(\mathbb{Z})$ be the completion of $\widetilde{G}(\mathbb{Z})$ in $G(\mathbb{Q})$.

Applying our Bruhat decomposition extended to $\widetilde{G}(\mathbb{Q})$, we prove that $\Gamma(\mathbb{Z})$ and $\widetilde{G}(\mathbb{Z})$ coincide. As a corollary we obtain two finite generating sets of $A$.

Our group constructions depend on choice of lattice $\Lambda$ between the root lattice and weight lattice and an integrable highest weight module $V$. In [CW], the authors obtained some preliminary results about the dependence of $G_V(\mathbb{Z})$ on the irreducible highest weight module $V = V^\lambda$. The details are discussed in Subsection 3.5.

In the finite dimensional case, a choice of Chevalley basis for the underlying Lie algebra is a crucial tool for constructing Chevalley groups. This in turn allows the determination of structure constants for the Lie algebra. Construction of Chevalley bases and determining structure constants across a whole Kac–Moody algebra are challenging tasks which are being undertaken in several works in progress.

Our motivation comes in part from several conjectures regarding discrete duality groups in high energy theoretical physics, in particular in supergravity and M–theory, where integral forms of Kac–Moody symmetry groups are conjectured to arise as a result of quantum effects ([BC, DHN, Ju1, Ju2, We]).

2. Preliminaries

Let $I = \{1, 2, \ldots, \ell\}$ and let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix. That is,

$$a_{ij} \in \mathbb{Z};$$

$$a_{ij} \leq 0, \text{ for } i \neq j; \text{ and } a_{ii} = 2;$$

$$a_{ij} = 0 \iff a_{ji} = 0,$$

for all $i, j \in I$. We assume throughout that $A$ is symmetrizable. That is, there exist positive rational numbers $q_1, \ldots, q_\ell$ such that the matrix $\text{diag}(q_1, \ldots, q_\ell)A$ is symmetric. We say that $A$ is of finite type if $A$ is positive definite and that $A$ is of affine type if $A$ is positive-semidefinite, but not positive-definite. If $A$ is not of finite or affine type, we say that $A$ has indefinite type. In particular, $A$ is of hyperbolic type if $A$ is neither of finite nor affine type, but every proper, indecomposable submatrix is either of finite or of affine type.

For this paper, we assume that $A$ is a symmetrizable generalized Cartan matrix, which is not of finite or affine type. Let $\mathfrak{h}$ be a $\mathbb{C}$-vector space of dimension $2\ell - \text{rank}(A)$, and let $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ denote the natural nondegenerate bilinear pairing between $\mathfrak{h}$ and its dual. Fix a choice of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subseteq \mathfrak{h}^*$ and simple coroots $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subseteq \mathfrak{h}$ such that $\Pi$ and $\Pi^\vee$ are linearly independent and $\langle \alpha_j, \alpha_i^\vee \rangle = \delta_{ij}$. Then the Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{h}$ and the elements $(e_i)_{i \in I}$ and $(f_i)_{i \in I}$ subject to defining relations ([M1, M2] and [Ka, Theorem 9.11]):

$$[h, h'] = 0;$$

$$[h, e_i] = (\alpha_i, h)e_i;$$

$$[e_i, f_i] = \alpha_i^\vee;$$

$$(\text{ad } e_i)^{-a_{ij}+1}(e_j) = 0;$$

$$(\text{ad } f_i)^{-a_{ij}+1}(f_j) = 0;$$

for $h, h' \in \mathfrak{h}$, and $i, j \in I$ with $i \neq j$.

2.1. Roots and the Weyl group. Now $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. The roots of $\mathfrak{g}$ are the nonzero $\alpha \in \mathfrak{h}^*$ for which the corresponding root space

$$\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \}$$
is nontrivial. Every root $\alpha$ has an expression of the form $\alpha = \sum_{i=1}^\ell k_i \alpha_i$ where the $k_i$ are are integral; and either all $k_i \geq 0$, in which case $\alpha$ is called positive, or all $k_i \leq 0$, in which case $\alpha$ is called negative. Denote the set of roots by $\Delta$ and the set of positive (resp. negative) roots by $\Delta_+$ (resp. $\Delta_-$). The algebra $\mathfrak{g}$ has a triangular decomposition ([Ka, Theorem 1.2])

$$\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+$$

under the simultaneous adjoint action of $\mathfrak{h}$, where

$$n^+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \quad \text{and} \quad n^- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha.$$

Since $A$ is symmetrizable, the algebra $\mathfrak{g} = \mathfrak{g}(A)$ admits a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ preserved by the Lie product. This induces a bilinear form on $\mathfrak{h}^*$, which determines the squared length of a root ([Ka, Theorem 2.2]).

For each simple root $\alpha_i$ we define the simple reflection $w_i : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$w_i(v) = v - \langle v, \alpha_i^\vee \rangle \alpha_i.$$

It follows that $w_i(\alpha_i) = -\alpha_i$. The $w_i$ generate a subgroup $W = W(A) \subseteq \text{Aut}(\mathfrak{h}^*)$.

called the Weyl group of $A$ and the bilinear form on $\mathfrak{h}^*$ is $W$-invariant.

A root $\alpha \in \Delta$ is called a real root if there exists $w \in W$ such that $w\alpha$ is a simple root. A root $\alpha$ which is not real is called imaginary. We denote by $\Delta^{\text{re}}$ the real roots, $\Delta^{\text{im}}$ the imaginary roots. It follows that $\Delta^{\text{re}} = W\Pi$.

2.2. Root lattice and weight lattice. Define the root lattice $Q := \mathbb{Z} \Pi \subseteq \mathfrak{h}^*$ and coroot lattice $Q^\vee := \mathbb{Z} \Pi^\vee \subseteq \mathfrak{h}$. The roots lie on the root lattice, i.e. $\Delta \subseteq Q$. For $\alpha = \sum_{i=1}^\ell k_i \alpha_i \in Q$ the height of $\alpha$ is given by $ht(\alpha) := \sum_{i=1}^\ell k_i$. The weight lattice $P \subseteq \mathfrak{h}^*$ is the dual lattice

$$P = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \ i \in I \}.$$

The dominant weights are

$$P_+ = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}, \ i \in I \}.$$

The weight lattice has a basis of fundamental weights $\varpi_1, \ldots, \varpi_\ell$ such that

$$\langle \varpi_i, \alpha_j^\vee \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

So $P = \mathbb{Z} \varpi_1 \oplus \cdots \oplus \mathbb{Z} \varpi_\ell$. Since $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij} \in \mathbb{Z}$, $i, j \in I$, we get $\langle \alpha, \alpha_i^\vee \rangle \in \mathbb{Z}$ for $\alpha \in \Delta$, and so all roots are weights, i.e. $Q \subseteq P$. As in the finite dimensional case, the index of the root lattice $Q$ in the weight lattice $P$ is finite and is given by $|\det A|$, since the generalized Cartan matrix $A$ is just the matrix for the change of basis from the fundamental weights to the simple roots [CS].

2.3. Integrable representations. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac–Moody algebra of $A$. Let $\mathcal{U}_C = \mathcal{U}_C(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.

Fix a $\mathfrak{g}$-module $V$ with defining homomorphism $\rho$. Throughout the paper, we write $\exp(t x_\alpha)$ for $\exp(\rho(t x_\alpha))$.

The weight space of $V$ with weight $\mu \in P$ is

$$V_\mu = \{ v \in V \mid x \cdot v = \mu(x)v \text{ for all } x \in \mathfrak{h} \}.$$
The set of weights of the representation $V$ is
\[
\text{wts}(V) = \{ \mu \in \mathfrak{h}^* \mid V_\mu \neq 0 \}.
\]

The $\mathfrak{g}$-module $V$ is diagonalizable if it is a direct sum of its weight spaces, that is,
\[
V = \bigoplus_{\mu \in \text{wts}(V)} V_\mu.
\]

We say $x \in \mathfrak{g}$ acts locally nilpotently on $V$ if, for each $v \in V$, there is a natural number $n$ such that $x^n \cdot v = 0$. A $\mathfrak{g}$-module $V$ is called integrable if it is diagonalizable and each generator $e_i$ and $f_i$ acts locally nilpotently on $V$.

### 2.4. Highest weight modules.

A $\mathfrak{g}$-module $V$ is called a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there exists $0 \neq v_\lambda \in V$ such that
\[
\begin{align*}
    h \cdot v_\lambda &= \lambda(h) v_\lambda & \text{if } h \in \mathfrak{h}, \\
    \mathfrak{n}^+ \cdot v_\lambda &= 0, \quad \text{and} \\
    \mathcal{U}(\mathfrak{g}) \cdot v_\lambda &= V.
\end{align*}
\]

Since $\mathfrak{n}^+$ annihilates $v_\lambda$ and $\mathfrak{h}$ acts as scalar multiplication on $v_\lambda$, we have $V = \mathcal{U}(\mathfrak{n}^-) \cdot v_\lambda$.

If $V$ is a highest weight module with highest weight $\lambda$, then all weights of $V^\lambda$ have the form
\[
\lambda - \sum_{i=1}^n k_i \alpha_i,
\]
for integers $k_i \geq 0$.

For all $\lambda \in \mathfrak{h}^*$, $\mathfrak{g}$ has a highest weight module $V$ with highest weight $\lambda$ ([MP, Prop. 2.3.1]). The highest weight vector $0 \neq v_\lambda \in V$ is unique up to nonzero scalar multiple. We have
\[
V = \bigoplus_{\mu \in \text{wts}(V)} V_\mu, \quad \dim(V_\mu) < \infty, \quad V_\lambda = \mathbb{C}v_\lambda.
\]

Let $L_V$ be the $\mathbb{Z}$-lattice generated by $\text{wts}(V)$.

Among all modules with highest weight $\lambda \in \mathfrak{h}^*$, there is a unique one that is irreducible as a $\mathfrak{g}$-module ([Ka, Prop. 9.3]), which we denote by $V^\lambda$. The module $V^\lambda$ is integrable if and only if $\lambda \in P_+$, that is, $\lambda$ is a dominant integral weight ([Ka, Lemma 10.1]). In this case we have $L^\lambda := L_{V^\lambda} \subseteq P$.

Note that, for simple Kac–Moody algebras $\mathfrak{g}$, all nontrivial $\mathfrak{g}$-modules are faithful.

**Lemma 2.1.** If $V^\lambda$ is faithful then the lattice $L^\lambda$ generated by $\text{wts}(V^\lambda)$ contains the root lattice $Q$.

**Proof.** Since $V^\lambda$ is faithful, no $e_i$ or $f_i$ acts trivially on $V^\lambda$. So there exists $\mu \in \text{wts}(V^\lambda)$ such that $e_i$ does not act trivially on $V^\lambda_\mu$. We have $e_i : V^\lambda_\mu \to V^\lambda_{\mu + \alpha_i}$, so $\mu + \alpha_i \in \text{wts}(V^\lambda)$, and thus $\alpha_i \in L^\lambda$. It follows that the lattice $Q = \mathbb{Z}\Pi \subseteq L^\lambda$.

Hence, for a faithful $V^\lambda$, we have $Q \subseteq L^\lambda \subseteq P$. 

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[4]
Let $\mathfrak{g}(A)$ be the Kac–Moody algebra over $\mathbb{C}$ of a symmetrizable generalized Cartan matrix $A$ that is neither finite nor affine. Recall that the Chevalley involution $\omega$ is an automorphism of $\mathfrak{g}$ with $\omega^2 = 1$, $\omega(h) = -h$ for all $h \in \mathfrak{h}$, and $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for all $\alpha \in \Delta$.

Let $V^\lambda$ be the unique irreducible highest weight module for $\mathfrak{g}$ corresponding to dominant integral weight $\lambda$. Recall that if $\alpha \in \Delta^\text{re}$, then the root space $\mathfrak{g}_\alpha$ is one dimensional. For each $\alpha \in \Delta^\text{re}$ we choose a root vector $x_\alpha \in \mathfrak{g}_\alpha$ and $x_{-\alpha} := -\omega(x_\alpha)$, normalized so that $[x_\alpha, x_{-\alpha}] = \alpha^\vee$. In particular, take $x_{e_i} = e_i$ and $x_{-e_i} = f_i$.

3.1. The $\mathbb{Z}$-form of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$. Let $\mathcal{U}_\mathbb{C} = \mathcal{U}_\mathbb{C}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let

- $\mathcal{U}^+_{\mathbb{Z}}$ be the $\mathbb{Z}$-subalgebra generated by $x_\alpha^m$ for $\alpha \in \Delta^\text{re}$ and $m \geq 0$,
- $\mathcal{U}^-_{\mathbb{Z}}$ be the $\mathbb{Z}$-subalgebra generated by $x_{-\alpha}^m$ for $\alpha \in \Delta^\text{re}$ and $m \geq 0$,
- $\mathcal{U}^0_{\mathbb{Z}} \subseteq \mathcal{U}_\mathbb{C}(\mathfrak{h})$ be the $\mathbb{Z}$-subalgebra generated by $h^m := h(h-1)\ldots(h-m+1)/m!$ for $h \in \mathfrak{h}$ and $m \geq 0$.
- $\mathcal{U}^\text{Z}_{\mathbb{C}} \subseteq \mathcal{U}_\mathbb{C}$ be the $\mathbb{Z}$-subalgebra generated by $\mathcal{U}^+_{\mathbb{Z}}$, $\mathcal{U}^-_{\mathbb{Z}}$, and $\mathcal{U}^0_{\mathbb{Z}}$.

We set

$$g = g_{\mathbb{Z}} \cap \mathcal{U}^\text{Z}_{\mathbb{C}}, \quad \mathfrak{h} = \mathfrak{h}_{\mathbb{Z}} \cap \mathcal{U}^\text{Z}_{\mathbb{C}}.$$ 

Then $g = g_{\mathbb{Z}} \otimes \mathbb{Z} \subset \mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h} = \mathfrak{h}_{\mathbb{Z}} \otimes \mathbb{C}$.

For $R$ a commutative ring, set $g_{R} = g_{\mathbb{Z}} \otimes \mathbb{Z} R$. Now $g_{R}$ is the infinite-dimensional analog of the Chevalley algebra over $R$. Note that $g_{\mathbb{C}}$ can be identified with $g$. We have triangular decomposition

$$g_{R} = n_{R}^{-} \oplus h_{R} \oplus n_{R}^{+},$$

where

$$n_{R}^{\pm} = \bigoplus_{\alpha \in \Delta_{\pm}} g_{\alpha, R} \quad \text{and} \quad n_{R}^{0} = \bigoplus_{\alpha \in \Delta_{0}} g_{\alpha, R}$$

and $g_{\alpha, R} = (g_{R} \cap \mathcal{U}_{\mathbb{Z}}) \otimes \mathbb{Z} R \subseteq g_{R}$. In particular, for $\alpha \in \Delta_{\text{im}}$, $g_{\alpha, \mathbb{Z}} = g_{\alpha} \cap \mathcal{U}_{\mathbb{Z}}$ is a free abelian group of rank $n_\alpha := \dim g_{\alpha}$. Choose a basis $x_{\alpha, 1}, \ldots, x_{\alpha, n_\alpha}$ of $g_{\alpha, \mathbb{Z}}$ for $\alpha \in \Delta_{\text{im}}$ and put $x_{-\alpha, i} = -\omega(x_{\alpha, i}) \in g_{-\alpha, R}$ such that

$$[x_{\alpha, i}, x_{-\alpha, j}] = \delta_{ij} \alpha^\vee, \quad i, j = 1, \ldots, n_\alpha.$$ 

To unify notations, for $\alpha \in \Delta^\text{re}$ in which case $n_\alpha = 1$, we put $x_{\alpha, 1} = x_\alpha$ as above.

Let $V^\lambda$ be the irreducible integrable highest weight module corresponding to a dominant integral weight $\lambda$. Fix a highest weight vector $v_\lambda$ of $V^\lambda$ and consider the orbit of $v_\lambda$ under $\mathcal{U}_{\mathbb{Z}}$. We have

$$\mathcal{U}^{+}_{\mathbb{Z}} \cdot v_\lambda = \mathbb{Z} v_\lambda$$

since every $x_\alpha$ with $\alpha \in \Delta^\text{re}$ annihilates $v_\lambda$. Also

$$\mathcal{U}^{0}_{\mathbb{Z}} \cdot v_\lambda = \mathbb{Z} v_\lambda$$

since $\mathcal{U}^{0}_{\mathbb{Z}}$ acts on $v_\lambda$ as scalar multiplication by integers. Therefore

$$\mathcal{U}^{\text{Z}}_{\mathbb{Z}} \cdot v_\lambda = \mathcal{U}^{-}_{\mathbb{Z}} \cdot v_\lambda.$$ 

This is the infinite dimensional analog of a hyperalgebra as in [C] (see also [Hu]). We set

$$V^\lambda_{\mathbb{Z}} = \mathcal{U}^{\text{Z}}_{\mathbb{Z}} \cdot v_\lambda = \mathcal{U}^{-}_{\mathbb{Z}} \cdot v_\lambda,$$
and $V_R^\lambda := V^\lambda_Z \otimes Z R$. The module $V_R^\lambda$ is a free $R$-module and a module over $g_R$, called the Weyl module. Recall that $V_\mu^\lambda$ is the weight space corresponding to a weight $\mu$ of $V^\lambda$. Set

$$V_{\mu,Z}^\lambda = V_\mu^\lambda \cap V^\lambda_Z \quad \text{and} \quad V_{\mu,R}^\lambda = V_\mu^\lambda \otimes Z R,$$

so that

$$V_Z^\lambda = \bigoplus_{\mu} V_{\mu,Z}^\lambda \quad \text{and} \quad V_R^\lambda = \bigoplus_{\mu} V_{\mu,R}^\lambda.$$

For $t \in R$ and $\alpha \in \Delta^{re}$, set

$$\chi_\alpha(t) = \exp(t x_\alpha) = \sum_{m=0}^{\infty} \frac{t^m x_\alpha^m}{m!},$$

which is a well-defined operator acting on $V_R^\lambda$ since the action of $x_\alpha$ is locally nilpotent. Thus these are elements of $\text{Aut}_R(V_R^\lambda)$.

We let $G^\lambda(R) \subseteq \text{Aut}_R(V_R^\lambda)$ be the group generated by $\chi_\alpha(t)$ for $\alpha \in \Delta^{re}$ and $t \in R$. We refer to $G^\lambda(R)$ as a representation-theoretic Kac–Moody group, or a Kac–Moody Chevalley group. We summarize the construction in the following.

**Definition 3.1.** Let $g = g(A)$ be a symmetrizable Kac–Moody algebra with $A$ neither finite nor affine. Let $R$ be a commutative ring $R$. For each $\alpha \in \Delta^{re}$ choose a root vector $x_\alpha \in g_\alpha$ and $x_{-\alpha} := -\omega(x_\alpha)$, normalized so that $[x_\alpha, x_{-\alpha}] = \alpha^\vee$. Let $V_R^\lambda$ be an $R$-form of an integrable highest weight module $V^\lambda$ for $g$, corresponding to dominant integral weight $\lambda$. Then

$$G^\lambda(R) = \langle \chi_\alpha(t) \mid \alpha \in \Delta^{re}, t \in R \rangle \subseteq \text{Aut}_R(V_R^\lambda)$$

is a Kac–Moody Chevalley group associated to $g$ and $V^\lambda$.

For $V$ a direct sum of integrable highest weight modules, we similarly define the Kac–Moody group $G_V(R) \subseteq \text{Aut}_R(V_R)$. In particular, $G^\lambda(R) = G_{V^\lambda}(R)$. We also mention that the same construction for $G^\lambda$ was used in [CG] to construct Kac–Moody groups over arbitrary fields.

For the reason which will be clear later in the construction of complete Kac–Moody groups, we also extend the definition of the operator $\chi_\alpha(t)$ for $\alpha \in \Delta^{im}$ as follows. Recall that we have chosen a basis $x_{\alpha,1}, \ldots, x_{\alpha,n_\alpha}$ of $g_\alpha, Z$. For $t = (t_1, \ldots, t_{n_\alpha}) \in R^{n_\alpha}$ and $m = (m_1, \ldots, m_{n_\alpha}) \in \mathbb{N}^{n_\alpha}$, put

$$m! = m_1! \cdots m_{n_\alpha}!, \quad |m| = m_1 + \cdots + m_{n_\alpha},$$

$$t^m = t_1^{m_1} \cdots t_{n_\alpha}^{m_{n_\alpha}}, \quad x_\alpha^m = x_{\alpha,1}^{m_1} \cdots x_{\alpha,n_\alpha}^{m_{n_\alpha}},$$

and define

$$\chi_\alpha(t) := \exp(t_1 x_{\alpha,1}) \cdots \exp(t_{n_\alpha} x_{\alpha,n_\alpha}) = \sum_{|m| \geq 0} \frac{t^m x_\alpha^m}{m!},$$

which has a well-defined action on $V_R^\lambda$.

**3.2. The torus.** Let $V$ be a direct sum of integrable highest weight modules of $g = g(A)$, and let $G_V(R)$ be the corresponding Kac–Moody group. Let $\tilde{W}$ be the subgroup of $G^\lambda(R)$ generated by the elements

$$\tilde{w}_i(t) = \chi_{\alpha_i}(t) \chi_{-\alpha_i}(-t^{-1}) \chi_{\alpha_i}(t),$$
for \( i \in I, \ t \in \mathbb{R}^\times \). We call \( \tilde{W} \) the extended Weyl group. There is a surjective map to the Weyl group \( \tilde{W} \rightarrow W, \tilde{w}_i(t) \mapsto w_i \). Set \( \tilde{w}_\alpha = \tilde{w}_\alpha (1) \) for each \( i \in I \). For \( t \in \mathbb{R}^\times, \ i \in I \), define
\[
h_\alpha (t) = \tilde{w}_\alpha (t) \tilde{w}_\alpha (1)^{-1},
\]
and let \( H_V(R) \subseteq G_V(R) \) be the subgroup generated by these elements.

**Theorem 3.2.** Assume that the set of weights of \( V \) contains all the fundamental weights. Then the map \( t \mapsto h_\alpha (t) \) is an injective homomorphism from \( \mathbb{R}^\times \) into \( H_V(R) \), for each \( i \in I \). The map
\[
(R^\times)^I \rightarrow H^\lambda (R), \quad (t_1, t_2, \ldots, t_\ell) \mapsto \prod_{i \in I} h_\alpha (t_i)
\]
is also an injective homomorphism.

**Proof.** Since each fundamental weight \( \varpi_i, \ i \in I \), is a weight of \( V \), \( h_\alpha (t) \) acts as scalar multiplication on the weight space \( V_{\varpi_i} \), by
\[
t^{\langle \varpi_i, \alpha \rangle} = t.
\]
Hence for each \( i \in I \), if \( h_\alpha (t) = 1 \) then \( t = 1 \). The last assertion is then obvious. \( \square \)

As Theorem 3.2 shows, the Kac–Moody Chevalley group has desirable properties when we choose a direct sum of integrable highest weight modules whose set of weights contains all the fundamental weights. For example, \( L_V = P \) if \( V = V^{\varpi_1} \oplus \cdots \oplus V^{\varpi_1} \), since \( \varpi_i \) is a heighest weight of \( V \) for all \( i \in I \).

We also have the following

**Proposition 3.3.** [?, Proposition 6.3.12] Let \( \lambda \) be a regular dominant integral weight, and \( g \in G^\lambda (R) \). If \( g \) acts by scalar multiplication on \( V^\lambda (R) \), then \( g \in H^\lambda (R) := H_{V^\lambda} (R) \).

### 3.3. Arithmetic subgroup of a Kac–Moody group.

Suppose that \( R \) is a subring of a field \( K \). Then we define an analog of the arithmetic subgroup \( \Gamma_V(R) \) to be the group:
\[
\Gamma_V(R) = \{ g \in G_V(K) | g(V_R) = V_R \} \subseteq \text{Aut}_R(V_R).
\]
For example, if \( R = \mathbb{Z} \) we have
\[
\Gamma_V(\mathbb{Z}) = \{ g \in G_V(K) | g(V_{\mathbb{Z}}) = V_{\mathbb{Z}} \} \subseteq \text{Aut}_{\mathbb{Z}}(V_{\mathbb{Z}}).
\]

### 3.4. Adjoint Kac–Moody group.

Our construction of the representation–theoretic Kac–Moody group in the previous subsections will give the following analog of the adjoint group. Choose \( V \) to be the adjoint module of \( g \). That is \( V = g \) as a vector space, with the representation
\[
ad_x : g \mapsto \text{End}(g)
\]
\[
y \mapsto [x, y].
\]
This in turn gives a representation
\[
\text{Ad}_{\exp(x)} : G_{ad}(R) \rightarrow GL(g)
\]
where \( G_{ad}(R) := G_V(R) \) is our Kac–Moody Chevalley group for \( V = g \) with the adjoint action. Note that \( V \) is integrable but not a highest weight module. This representation has the property that
\[
\exp(ad_x(y)) = \text{Ad}_{\exp(x)}(\exp(y))
\]
for all \( x \in n_R \) and moreover
\[
\frac{d}{dt} \bigg|_{t=0} \text{Ad}_{\exp(tx)}(y) = [x, y].
\]
3.5. Dependence on choices. In the case where $V^\lambda$ is faithful, our group construction depends on

1. a choice of lattice $Q \subseteq \Lambda \subseteq P$ between the root lattice $Q$ and weight lattice $P$, and
2. a Z-form $V^\lambda_{\mathbb{Z}}$ of the integrable highest weight module $V^\lambda$.

The lattice $\Lambda$ can be realized as $L_{V^\lambda}$, the lattice of weights of the representation $V^\lambda$, which is between $Q$ and $P$ ([St, Lemma 27]).

In what follows however, we will primarily consider only the simply connected Kac–Moody group $G = G_{V}(R)$, which assumes a choice of integrable highest weight module $V^\lambda$ satisfying $L_{V^\lambda} = P$. This depends on the parameter $\lambda$, a dominant integral weight.

When $R$ is a field, Garland constructed affine Kac–Moody groups as central extensions of loop groups and characterized the dependence of a completion of $G_{V}(R)$ on $\lambda$ in terms of the Steinberg cocycle [Ga1].

Recent work of Rousseau [Ro16] shows that over fields of characteristic zero, completions of Kac–Moody groups (as in [RR, CG], see also [Ma88, Ma89]) are isomorphic as topological groups. The complete Kac–Moody groups of [CG] are constructed with respect to a choice of dominant integral weight $\lambda$. Thus over fields of characteristic zero, the complete Kac–Moody groups of [CG] are independent of $\lambda$ up to isomorphism, as topological groups.

For Kac–Moody groups over rings, in [CW] the authors gave some preliminary results about the dependence of $G_{V}(\mathbb{Z})$ on $\lambda$ when $\mathfrak{g}$ is simply laced and hyperbolic, by comparison with a finite presentation for the Tits functor $\tilde{\mathfrak{G}}$ obtained by [AC]. In [CW] there is a homomorphism $\rho_{V,\mathbb{Z}} : \tilde{\mathfrak{G}}(\mathbb{Z}) \to G_{V}(\mathbb{Z})$ defined on generators and the authors prove that the kernel $K_{V}(\mathbb{Z})$ of the map $\rho_{V,\mathbb{Z}}$ lies in the complex torus $H(\mathbb{C})$ and if the natural group homomorphism $\phi : \tilde{\mathfrak{G}}(\mathbb{Z}) \to \tilde{\mathfrak{G}}(\mathbb{C})$ is injective, then $K_{V}(\mathbb{Z}) \subseteq H(\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\text{rank}(A)}$. Injectivity of the natural map $\phi : \tilde{\mathfrak{G}}(\mathbb{Z}) \to \tilde{\mathfrak{G}}(\mathbb{C})$ is not currently known and depends on functorial properties of Tits’ group [Ti12].

3.6. Integrality. Some of the difficulties of constructing integral forms of Kac–Moody groups can already be seen in finite dimensional Chevalley groups, including $\text{SL}_2$.

For example, if we have an element \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) written in terms of the generators \( \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \chi_{\pm}(s) \chi_{\pm}(t) \ldots \), then it is not necessarily the case that the scalars $t_i$ are all integers. For example,

\[
\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \chi_{\pm}(s) \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \chi_{-}(s) \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}.
\]

However, since $\text{SL}_2(\mathbb{Z})$ is generated by $\chi_{\pm}(s)$ and $\chi_{-}(t)$ for $s, t \in \mathbb{Z}$, we may look for another decomposition of \( \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \) in terms of generators whose scalars are integer valued, and we find

\[
\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \chi_{-}(1) \chi_{\alpha}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Thus given \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \), there exist $t_1, \ldots, t_k \in \mathbb{Z}$ such that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_{\pm}(t_1) \chi_{\pm}(t_2) \ldots \chi_{\pm}(t_k).
\]

Since these arguments will reappear in later sections, we recall the following important but standard fact.
Proposition 3.4. The subgroup $\text{SL}_2(\mathbb{Z})$ of $\text{SL}_2(\mathbb{Q})$ is the stabilizer of $V_\mathbb{Z} \subset V_\mathbb{Q}$ of the standard representation $V_\mathbb{Q}$.

Proof. Take $V_\mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$ and $V_\mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}$. Then $\text{SL}_2(\mathbb{Q})$ acts on $V_\mathbb{Q}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Suppose now that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ stabilizes $V_\mathbb{Z}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot V_\mathbb{Z} \subseteq V_\mathbb{Z},$$

that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

where $ad - bc = 1$, $x, y \in \mathbb{Z}$ and $u = ax + by \in \mathbb{Z}$, $v = cx + dy \in \mathbb{Z}$. Take $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $u = ax + by$ implies $b \in \mathbb{Z}$ and $v = cx + dy$ implies $d \in \mathbb{Z}$. Take $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $u = ax + by$ implies $a \in \mathbb{Z}$ and $v = cx + dy$ implies $c \in \mathbb{Z}$. Thus if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot V_\mathbb{Z} \subseteq V_\mathbb{Z}$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. □

4. Bruhat decomposition of $G(\mathbb{Q})$

Let $\mathfrak{g} = \mathfrak{g}(A)$ be symmetrizable Kac–Moody algebra over $\mathbb{Q}$, with $A$ neither finite not affine. Take simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ indexed over $I = \{i, \ldots, \ell\}$ and Cartan subalgebra $\mathfrak{h}$. Let $V^\lambda = V^\lambda_\mathbb{Q}$ denote the irreducible integrable highest weight module corresponding to a dominant integral weight $\lambda$. We assume that $L_{V^\lambda} = P$, the weight lattice. To simplify notation, in this section we shall omit the subscript $\lambda$ and denote $G = G^\lambda$. Thus $G(\mathbb{Q}) = \langle \chi_\alpha(t) \mid t \in \mathbb{Q}, \alpha \in \Delta^\text{re} \rangle$. Define subgroups

$$H(\mathbb{Q}) := \langle h_\alpha(t) \mid t \in \mathbb{Q}^x, i \in I \rangle,$$

$$U_\alpha := \langle \chi_\alpha(t) \mid t \in \mathbb{Q} \rangle \text{ if } \alpha \in \Delta^\text{re},$$

$$U(\mathbb{Q}) := \langle U_\alpha \mid \alpha \in \Delta^\text{re} \rangle = \langle \chi_\alpha(t) \mid t \in \mathbb{Q}, \alpha \in \Delta^\text{re} \rangle,$$

$$B(\mathbb{Q}) := \langle H(\mathbb{Q}), U(\mathbb{Q}) \rangle.$$

We note that our generating set for $G(\mathbb{Q})$ is highly redundant and can be reduced to $\{\chi_\alpha(t), \chi_{-\alpha}(s)\}$.

Theorem 4.1 ([T12, CG]). The data $(G(\mathbb{Q}), B(\mathbb{Q}), \widetilde{W}, \{w_i \mid i \in I\})$ defines a BN-pair for $G(\mathbb{Q})$ with $B = B(\mathbb{Q})$ and $N = \widetilde{W}$. We have Bruhat decomposition

$$G(\mathbb{Q}) = \bigsqcup_{w \in \widetilde{W}} B(\mathbb{Q})wB(\mathbb{Q}).$$

Recall here that $B(\mathbb{Q})wB(\mathbb{Q})$ actually denotes the double coset $B(\mathbb{Q})wB(\mathbb{Q})$, where $\tilde{w} \in \widetilde{W}$ is an element of the extended Weyl group that maps to $w \in W$. This double coset is independent of the choice of $\tilde{w}$.

Lemma 4.2. We have $\text{SL}_2(\mathbb{Q}) = \text{SL}_2(\mathbb{Z})B(\text{SL}_2(\mathbb{Q}))$, where $B(\text{SL}_2(\mathbb{Q}))$ denotes the standard Borel subgroup of $\text{SL}_2(\mathbb{Q})$. 

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The following theorem gives an analog of Lemma 4.5.

We prove the result using the Bruhat decomposition from Theorem 4.7. We have

\[ (p \ q) = (a \ b)^{-1} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \in SL_2(Z)B(SL_2(Q)). \]

\[ B \]

For each \( i \in I \), there is a homomorphism ([Ga1, Lemma 16.3])

\[ \varphi_i : SL_2(Z) \to G \]

satisfying

\[ \varphi_i \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \chi_{\alpha_i}(s), \quad \varphi_i \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \chi_{-\alpha_i}(s), \]

\[ \varphi_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = w_{\alpha_i}(1), \quad \varphi_i \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = h_{\alpha_i}(a). \]

**Lemma 4.3.** For fixed \( i \in I \), \( \varphi_i(SL_2(Z)) = \langle \chi_{\alpha_i}(t), \chi_{-\alpha_i}(t) \mid t \in Z \rangle \).

**Lemma 4.4.** We can choose a set coset representatives \( Y_i \) for \( Bw_iB/B \) such that \( Y_i \subseteq \varphi_i(SL_2(Z)) \).

Proof. By Lemma 4.2, we can write an element of \( SL_2(Q) \) in \( SL_2(Z)B(SL_2(Q)) \). Modulo \( B \), the \( Y_i \) can thus be chosen such that \( Y_i \subseteq \varphi_i(SL_2(Z)) \).

It follows that for each \( i \), \( Y_i \subseteq \varphi_i(SL_2(Z)) = \langle \chi_{\alpha_i}(t), \chi_{-\alpha_i}(t) \mid t \in Z \rangle \subseteq G(Z) \).

**Lemma 4.5.** For each \( i \in I \), \( Bw_iB = Y_iB \).

**Corollary 4.6.** For each \( i \in I \), we have \( Bw_iB = Y_iB \subseteq G(Z)B(Q) \).

The following theorem gives an analog of Lemma 4.2 for \( G(Q) \).

**Theorem 4.7.** We have \( G(Q) = G(Z)B(Q) \).

Proof. We prove the result using the Bruhat decomposition from Theorem 4.1. That is, we prove by induction on the length of \( w \in W \) that each Bruhat cell \( B(Q)wB(Q) \) is contained in \( G(Z)B(Q) \). The case \( \ell(w) = 0 \) gives \( B(Q) \subseteq G(Z)B(Q) \). The case \( \ell(w) = 1 \) follows from Corollary 4.6, which gives a base for induction.

Assume inductively that

\[ B(Q)w_{i_1}w_{i_2} \ldots w_{i_k}B(Q) \subseteq G(Z)B(Q), \]

for all \( k \geq 1 \), where \( w_{i_1}w_{i_2} \ldots w_{i_k} \) has length \( k \), and each \( w_{i_j} \) is a simple reflection.

Let \( w_{i_{k+1}} \) be a simple reflection and assume that \( w_{i_1}w_{i_2} \ldots w_{i_k}w_{i_{k+1}} \) has length \( k + 1 \). Then the theory of BN-pair gives

\[ B(Q)w_{i_1}w_{i_2} \ldots w_{i_k}w_{i_{k+1}}B(Q) = B(Q)w_{i_1}w_{i_2} \ldots w_{i_k}B(Q)w_{i_{k+1}}B(Q). \]

By the inductive hypothesis, \( B(Q)w_{i_1}w_{i_2} \ldots w_{i_k}B(Q) \subseteq G(Z)B(Q) \). Since \( w_{i_{k+1}} \) is a simple root reflection, the inductive hypothesis also implies that

\[ B(Q)w_{i_{k+1}}B(Q) \subseteq G(Z)B(Q). \]
Lemma 5.2. If \( h \in H \), then the conclusion follows directly from the definitions of \( V \).

Proof. The conclusion follows directly from the definitions of \( V \).

5. Integality and the torus

To simplify notation we omit the subscript \( V \) and write \( G = G_V \). Recall that we have defined the arithmetic subgroup of \( G \) to be \( \Gamma(\mathbb{Z}) = \{ g \in G(\mathbb{Q}) \mid g(V_\mathbb{Z}) = V_\mathbb{Z} \} \).

Lemma 5.1. Suppose that the set wts(\( V \)) of weights of \( V \) contains all the fundamental weights. In \( G(\mathbb{Q}) \) we have

\[
G(\mathbb{Z}) \cap H(\mathbb{Q}) = \Gamma(\mathbb{Z}) \cap H(\mathbb{Q}) = H(\mathbb{Z}) = \langle h_{\alpha_i}(-1) \mid i \in I \rangle.
\]

Proof. It is immediate that \( G(\mathbb{Z}) \cap H(\mathbb{Q}) = \langle h_{\alpha_i}(-1) \mid i \in I \rangle \). The element \( h_{\alpha_i}(t) \) acts on the weight space \( V_{\omega_i} \) as scalar multiplication by \( t_i \). Hence \( h_{\alpha_i}(t_i)(V_\mathbb{Z}) = V_\mathbb{Z} \) only if \( t_i \in \mathbb{Z}^\times = \{ \pm 1 \} \). Thus \( \Gamma(\mathbb{Z}) \cap H(\mathbb{Q}) = \langle h_{\alpha_i}(-1) \mid i \in I \rangle \).

Recall that \( h_\mathbb{Z} = h \cap g_\mathbb{Z} \). We now define \( g_\mathbb{Z} \) to be the stabilizer of \( V_\mathbb{Z} \) in \( g \), and

\[
h_\mathbb{Z} = h \cap g_\mathbb{Z}.
\]

If \( V \) is finite dimensional, the discrepancy between \( g_\mathbb{Z} \) and \( g_\mathbb{Z} \) is given by the difference between \( h_\mathbb{Z} \) and \( h_\mathbb{Z} \). In general, \( g_\mathbb{Z} \subseteq g_\mathbb{Z} \). For \( h \in h \) we have \( h \cdot V_\mathbb{Z} \subseteq V_\mathbb{Z} \) if and only if \( \mu(h) \in \mathbb{Z} \) for all \( \mu \in L_V \).

Thus

\[
h_\mathbb{Z} = \{ h \in h \mid \mu(h) \in \mathbb{Z} \text{ for all } \mu \in L_V \}.
\]

If \( V \) is the adjoint representation, then \( L_V = Q \) and we have

\[
h_{ad} = \{ h \in h \mid \mu(h) \in \mathbb{Z} \text{ for all } \mu \in Q \},
\]

which is the coweight lattice. On the other hand \( h_\mathbb{Z} \) is the coroot lattice, a free abelian group with basis \( \alpha_i^\vee \), \( i \in I \).

Lemma 5.2. If \( Q = L_V = P \) then \( h_\mathbb{Z} = h_\mathbb{Z} = h_{ad} \). If \( [P : Q] = 2 \) then \( h_\mathbb{Z} = h_\mathbb{Z} \) or \( h_\mathbb{Z} = h_{ad} \).

Proof. The conclusion follows directly from the definitions of \( h_{ad} \) and \( h_\mathbb{Z} \).

6. Actions on \( V^\lambda \) and \( V_\mathbb{Z}^\lambda \)

Let \( V = V^\lambda \) be an irreducible integral highest weight \( g \)-module and fix a highest weight vector \( v_\lambda \). Recall that \( V := U_\mathbb{Z} : v_\lambda \) is a lattice in \( V \) and a \( U_\mathbb{Z} \)-module. For \( \alpha \in \Delta \), recall that

\[
g_{\alpha, Z} := g_{\alpha} \cap U_\mathbb{Z} = \mathbb{Z}\alpha_{\lambda,1} + \cdots + \mathbb{Z}\alpha_{\lambda,n}.
\]

So for \( x \in g_{\alpha, Z} \) we have

\[
\frac{x^n}{n!} (g_{\alpha, Z}) \subseteq g_{\alpha, Z}.
\]

Let \( \langle \rangle \) be the order on \( Q_+ := \{ \sum_{i \in I} k_i \alpha_i \in Q \mid k_i \geq 0 \text{ for all } i \in I \} \) defined by

1. if \( ht(\alpha) < ht(\beta) \), then \( \alpha \prec \beta \);
2. if \( ht(\alpha) = ht(\beta) \), then use lexicographic order on the coordinates of \( \alpha \) and \( \beta \) with respect to the simple roots.

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For each weight $\mu$ of $V$, $\mu = \lambda - \alpha$, where $\alpha \in Q_+$. Define the depth of $\mu$ to be
$$\text{depth}(\mu) = \text{ht}(\alpha) = \text{ht}(\lambda) - \text{ht}(\mu).$$

A $\mathbb{Z}$-basis $\Xi = \{v_1, v_2, \ldots \}$ of $V_{\mathbb{Z}}$ is called coherently ordered if
\begin{enumerate}
  \item $\Xi$ consists of weight vectors; and
  \item if $v_i \in V_{\mu, \mathbb{Z}}$, $v_j \in V_{\nu, \mathbb{Z}}$ and $\mu \prec \nu$, then $i < j$.
\end{enumerate}

If $\text{depth}(\mu) = 0$, then $\mu = \lambda$ is a highest weight vector, so we can take $v_1 = v_{\lambda}$. Since $\prec$ is a linear ordering, for $\mu$ a weight, $\Xi \cap V_{\mu, \mathbb{Z}}$ consists of a consecutive subsequence $v_k, v_{k+1}, \ldots, v_{k+n}$. If $v \in V_{\mu, \mathbb{Z}}^\lambda$ with $\text{depth}(\mu) = m$, then $v$ is a linear combination of vectors of the form $f_i f_{i_1} \cdots f_{i_m} \cdot v_{\lambda}$, for some $i_1, \ldots, i_m \in I$.

**Theorem 6.1** ([Ga1] Theorem 11.3; [CG]). There is a coherently ordered $\mathbb{Z}$-basis $\Xi = \{v_1, v_2, \ldots \}$ for $V_{\mathbb{Z}}$, where each $v_i = \xi_i v_1$ for some $\xi_i \in U_{\mathbb{Z}}$.

We define $V_{\mu, \mathbb{Z}}$ to be the lattice with $\mathbb{Z}$-basis $\Xi \cap V_{\mu} = \{v_k, v_{k+1}, \ldots, v_{k+n}\}$.

**Lemma 6.2.** Let $v \in V_{\mu, \mathbb{Z}}^\lambda$ and $\alpha \in \Delta$. Then, for all $t \in \mathbb{Q}^{n_\alpha}$,
$$\chi_\alpha(t) \cdot v = v + \sum_{m \in \mathbb{N}^{n_\alpha}, |m| > 0} t^m u_m,$$
for some $u_m \in V_{\mu + |m| \alpha, \mathbb{Z}}^\lambda$, only finitely many of which are nonzero. If moreover $v \in V_{\mu, \mathbb{Z}}^\lambda$, then $u_m \in V_{\mu + |m| \alpha, \mathbb{Z}}^\lambda$.

**Proof.** The first assertion is given by [St, Lemma 7.2]. If $v \in V_{\mu, \mathbb{Z}}^\lambda$, taking $t = (1, \ldots, 1)$ shows that $u_m \in V_{\mu + |m| \alpha, \mathbb{Z}}$ for each $m \in \mathbb{N}^{n_\alpha}$.

**Lemma 6.3.** Let $\alpha \in \Delta_+$ and $t \in \mathbb{Q}^{n_\alpha}$. Then $t \in \mathbb{Z}^{n_\alpha}$ if and only if $\chi_\alpha(t)$ preserves $V_{\mathbb{Z}}^\lambda$ for every dominant integral weight $\lambda$.

**Proof.** If $t \in \mathbb{Z}^{n_\alpha}$, then $\chi_\alpha(t)$ clearly preserves $V_{\mathbb{Z}}^\lambda$. Conversely, assume that $\chi_\alpha(t)$ preserves $V_{\mathbb{Z}}^\lambda$ for each $\lambda$. We need to show that this implies that $t \in \mathbb{Z}^{n_\alpha}$. Recall that $v_{\lambda}$ is the highest weight vector of $V_{\lambda}^\nu$ we have chosen to define the $\mathbb{Z}$-form $V_{\mathbb{Z}}^\lambda = U_{\mathbb{Z}} \cdot v_{\lambda}$.

Recall that $x_{\alpha, i}, i = 1, \ldots, n_\alpha$ are chosen such that $[x_{\alpha, i}, x_{-\alpha, j}] = \delta_{ij} \alpha^\vee$, where $x_{-\alpha, i} = -\omega(x_{\alpha, i})$. Write $t = (t_1, \ldots, t_{n_\alpha}) \in \mathbb{Q}^{n_\alpha}$. Since $\chi_\alpha(t)$ preserves $V_{\lambda}^\nu$ and $x_{-\alpha, i} \cdot v_{\lambda} \in V_{\lambda - \alpha, \mathbb{Z}}^\lambda$, from the expansion
$$\chi_\alpha(t) = \sum_{m \in \mathbb{N}^{n_\alpha}, |m| \geq 0} \frac{t^m}{m!} x_{\alpha, i}^m (x_{-\alpha, i} \cdot v_{\lambda}),$$
and $x_{\alpha, i}^m (x_{-\alpha, i} \cdot v_{\lambda}) = 0$ for $|m| > 1$, we deduce that
$$\chi_\alpha(t) \cdot (x_{-\alpha, i} \cdot v_{\lambda}) = x_{-\alpha, i} \cdot v_{\lambda} + \sum_{j=1}^{n_\alpha} t_j x_{\alpha, j} \cdot (x_{-\alpha, i} \cdot v_{\lambda})$$
$$= x_{-\alpha} \cdot v_{\lambda} + \sum_{j=1}^{n_\alpha} t_j \delta_{ij} \alpha^\vee v_{\lambda}$$
$$= x_{-\alpha} \cdot v_{\lambda} + t_i \alpha^\vee v_{\lambda}$$
$$= x_{-\alpha} \cdot v_{\lambda} + t_i (\lambda, \alpha^\vee) v_{\lambda}$$
$$\in V_{\mathbb{Z}}^\lambda.$$
Hence \( t_i \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \) for each \( i = 1, \ldots, n_\alpha \). We claim that we may choose different dominant integral weights \( \lambda \) and \( \lambda' \) such that \( \langle \lambda, \alpha^\vee \rangle \) and \( \langle \lambda', \alpha^\vee \rangle \) are relatively prime. This in turn implies that \( t_i \in \mathbb{Z} \). To prove the claim, we note that there exists \( w \in W \) such that \( \alpha = w\alpha_i \) for some simple root \( \alpha_i \). Recall that \( \varpi_i \) is the \( i \)-th fundamental weight so that \( \langle \varpi_i, \alpha_\vee \rangle = 1 \). We may choose a dominant integral weight \( \lambda \) sufficiently large such that \( \lambda' := \lambda + w\varpi_i \) is dominant as well. Applying the above argument for the two highest weight modules \( V^\lambda \) and \( V^{\lambda'} \), we obtain that \( t \langle \lambda, \alpha^\vee \rangle \) and \( t \langle \lambda', \alpha^\vee \rangle \) are both integers. But

\[
\langle w\varpi_i, \alpha^\vee \rangle = \langle w\varpi_i, w\alpha_i^\vee \rangle = \langle \varpi_i, \alpha_i^\vee \rangle = 1,
\]

from which it follows that

\[
\langle \lambda', \alpha^\vee \rangle = \langle \lambda, \alpha^\vee \rangle + 1.
\]

In particular \( \langle \lambda, \alpha^\vee \rangle \) and \( \langle \lambda', \alpha^\vee \rangle \) are relatively prime, which implies that \( t \in \mathbb{Z} \). \( \square \)

**Proposition 6.4.** Let \( u = \chi_{\beta_1}(t_1) \cdots \chi_{\beta_k}(t_k) \in U(\mathbb{Q}) \), where \( \beta_1 > \cdots > \beta_k \) are roots in \( \Delta_+ \), and \( t_j \in \mathbb{Q}^{n_{\beta_j}} \), \( j = 1, \ldots, k \). Then \( t_j \in \mathbb{Z}^{n_{\beta_j}} \), \( j = 1, \ldots, k \) if and only if \( u \) preserves \( V^\lambda \) for every dominant integral weight \( \lambda \).

**Proof.** Assume that \( u \) preserves \( V^\lambda \) for each \( \lambda \) and we need to show that each \( t_j \in \mathbb{Z}^{n_{\beta_j}} \). We use induction on \( k \). The case \( k = 1 \) is Lemma 6.3. Assume that the assertion is true for \( k - 1 \). The order we have chosen on \( \Delta_+ \subset Q_+ \) implies that \( V^\lambda_{\lambda - \alpha + \beta} = 0 \) whenever \( \alpha \prec \beta \).

Since \( v_\lambda \) is the highest weight vector, we have

\[
\chi_{\beta}(t) \cdot v_\lambda = v_\lambda
\]

for any \( \beta \in \Delta_+ \), and

\[
\chi_{-\beta}(t) \cdot (x_{-\alpha,i} \cdot v_\lambda) = x_{-\alpha,i} \cdot v_\lambda
\]

for any \( \alpha, \beta \in \Delta_+ \) with \( \alpha \prec \beta \), and \( i = 1, \ldots, n_\alpha \). Write \( t_k = (t_{k,1}, \ldots, t_{k,n_{\beta_k}}) \in \mathbb{Q}^{n_{\beta_k}} \). Then we have

\[
u \cdot (x_{-\beta_k,i} \cdot v_\lambda) = \chi_{\beta_k}(t_1) \cdots \chi_{\beta_{k-1}}(t_{k-1}) (x_{-\beta_k,i} \cdot v_\lambda + t_{k,i} \langle \lambda, \beta_k^\vee \rangle v_\lambda)
\]

\[
= x_{-\beta_k,i} \cdot v_\lambda + t_{k,i} \langle \lambda, \beta_k^\vee \rangle v_\lambda \in V^\lambda_Z.
\]

Hence \( t_{k,i} \langle \lambda, \beta_k^\vee \rangle \in \mathbb{Z} \), \( i = 1, \ldots, n_{\beta_k} \). Similar to the proof of Lemma 6.3, by choosing different weights \( \lambda \) we may deduce that \( t_{k,i} \in \mathbb{Z} \). It follows that \( \chi_{\beta_k}(t_k)^{-1} = \exp(-t_{k,n_{\beta_k}} x_{\beta_k,i}) \cdots \exp(-t_{k,1} x_{\beta_k,1}) \)

preserves each \( V^\lambda_Z \), and therefore

\[
\chi_{\beta_k}(t_1) \cdots \chi_{\beta_{k-1}}(t_{k-1}) = u \cdot \chi_{\beta_k}(t_k)^{-1} \in U(\mathbb{Q})
\]

preserves each \( V^\lambda_Z \) as well. We may now apply the inductive hypothesis to complete the proof. \( \square \)

7. **Completed Kac–Moody groups**

Let \( g \) be our Kac–Moody algebra. In this section, we first give a representation theoretic description of the complete pro–nilpotent subalgebra \( \widehat{n}^+ \) of \( g \) generated by the positive root spaces and its complete pro–unipotent pro–algebraic group \( \widehat{U}^+ \). Then we define a (universal) completed Kac-Moody group \( \widehat{G}(\mathbb{Q}) \) acting on the direct sum \( V \) of all integrable highest weight modules. These results are used in the next section to give a finite topological generating set for \( \widehat{G}(\mathbb{Z}) \).

The definitions given below are a minor extension of those of Kumar [Ku] and coincide with Rousseau [Ro16] in characteristic 0.
7.1. **The pro-nilpotent pro-subalgebra** $\hat{n}^+$. Recall that $n^+ = \bigoplus_{\alpha \in \Delta^+} g_\alpha$. Let

$$\hat{n}^+ = \prod_{\alpha \in \Delta^+} g_\alpha$$

be the formal completion of $n^+$. For $k \geq 1$, let

$$\hat{n}^+_k = \prod_{\alpha \in \Delta^+, \, ht(\alpha) \geq k} g_\alpha,$$

where for $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i \in Q^+$ one has the height

$$ht(\alpha) = \sum_{i=1}^{\ell} n_i.$$

We also have

$$\hat{n}^+ = \prod_{k \in \mathbb{N}} g_k$$

for a suitable $\mathbb{N}$-grading of $\hat{n}^+$, that is,

$$g_k := \bigoplus_{\alpha \in \Delta^+, \, ht(\alpha) = k} g_\alpha.$$

Then $\hat{n}^+$ is a pro-nilpotent pro-Lie algebra with respect to the family of ideals of $\hat{n}^+$ containing $\hat{n}^+_k$ for some $k \geq 1$ [Ku].

7.2. **Pro-representation** $\mathcal{V}$. Let

$$\hat{g} = n^- \oplus h \oplus \hat{n}.$$

Let $V^\lambda$ be an integrable highest weight $g$–module with highest weight $\lambda$. Then $V^\lambda$ is also a $\hat{g}$–module. Recall that $V^\lambda_{\lambda - \alpha}$ denotes the weight space of $V^\lambda$ of weight $\lambda - \alpha$, $\alpha \in Q^+$.

**Lemma 7.1.** For each $k > 1$, the ideal

$$\hat{n}^+_k = \prod_{\alpha \in \Delta^+, \, ht(\alpha) \geq k} g_\alpha$$

acts trivially on the subspace

$$V^\lambda_k = \bigoplus_{\alpha \in Q^+, \, ht(\alpha) < k} V^\lambda_{\lambda - \alpha}.$$

Let $\mathcal{V}$ denote the direct sum of all the integrable highest weight modules for $g$. That is,

$$\mathcal{V} = \bigoplus_{\lambda} V^\lambda,$$

where $\lambda$ runs over all dominant integral weights. Then $\mathcal{V}$ is a $\hat{g}$–module. Let

$$\rho : \hat{n}^+ \to \text{End}(\mathcal{V})$$

be the diagonal action of $\hat{n}^+$ on $\mathcal{V}$. Then for all $y \in \hat{n}^+$ and $v \in \mathcal{V}$, $\exp(\rho(y))(v)$ reduces to a finite sum.
7.3. **Pro–unipotent pro–algebraic subgroup $\hat{U}^+$.** For each $k \geq 1$, we set
\[
\hat{U}_k^+ = \exp \rho(g_k),
\]
\[
\hat{U}^+ = \exp \rho(\hat{n}^+).
\]

Then $\hat{U}_k^+$ is a finite dimensional unipotent group and
\[
\hat{U}^+ = \prod_{i=1}^{\infty} \hat{U}_k^+ = \left\{ \prod_{i=1}^{\infty} \exp(\rho(x_i)) \mid x_i \in g_k \right\}.
\]

Any element $x \in \hat{n}^+$ is an infinite (formal) sum, but for all $g \in \hat{U}^+$ and $v \in V$, $g \cdot v$ is a finite sum. We also have
\[
\hat{U}^+ = \left\{ \prod_{x \in g} \exp(\rho(x_{\alpha})) \mid x_{\alpha} \in g_{\alpha} \right\}.
\]

The following lemma is clear (see also [Ro16, Proposition 3.2] and [Ro17, Section 1.9]).

**Lemma 7.2.** With respect to a fixed order on the positive roots and a fixed choice of basis for the root spaces $g_{\alpha}$ of $\hat{g}$, every element $g \in \hat{U}^+$ has a unique expression of the form
\[
g = \prod_{\alpha \in \Delta^+} \exp(\rho(x_{\alpha}))
\]
for $x_{\alpha} \in g_{\alpha}$, where the product is taken in the fixed order on the positive roots.

With respect to a chosen basis $B_{\alpha} = \{x_1, \ldots, x_{n_{\alpha}}\}$ of $g_{\alpha}$ and an ordering $x_1 < \cdots < x_{n_{\alpha}}$ of the basis elements where $n_{\alpha} = \dim g_{\alpha}$, we have $x_{\alpha} = t_1 x_1 + \cdots + t_d x_d$, for $t_i \in \mathbb{C}$. Thus when restricted to $g_{\alpha}$, we have
\[
\exp(\rho(x_{\alpha})) = \exp(t_1 \rho(x_1)) \cdots \exp(t_{n_{\alpha}} \rho(x_{n_{\alpha}})) = \prod_{x \in B_{\alpha}} \exp(t_i \rho(x_i)).
\]

Expanding $\exp(\rho(x_{\alpha}))$ for all positive roots, we have
\[
g = \prod_{\alpha \in \Delta^+} \exp(\rho(x_{\alpha})) = \prod_{\alpha \in \Delta^+} \prod_{x \in B_{\alpha}} \exp(t_i \rho(x_i))
\]
and this decomposition is unique relative to the above choices.

From now on, we fix the order on $\Delta^+$ given by the restriction of the order $\prec$ on $Q^+$ defined in Section 6.

7.4. **Completed Kac-Moody group $\tilde{G}(\mathbb{Q})$.** The operator $\chi_{\alpha}(t)$, $t \in \mathbb{Q}$, $\alpha \in \Delta^{re}$ acts on $V$ by its diagonal action on the summands $V^\lambda$, and we have the universal representation-theoretic Kac–Moody group
\[
G(\mathbb{Q}) := G_{\mathbb{Q}}(\mathbb{Q}) = \langle \chi_{\alpha}(t) \mid t \in \mathbb{Q}, \alpha \in \Delta^{re} \rangle \subset \text{GL}(V).
\]

By restriction to $V^\lambda$ we have the natural homomorphism $G(\mathbb{Q}) \to G^\lambda(\mathbb{Q})$ for each $\lambda$. Recall that we have fixed a highest weight vector $v_{\lambda}$ and define the $\mathbb{Z}$-form $V^\lambda_{\mathbb{Z}} = U_{\mathbb{Z}} \cdot v_{\lambda}$ of $V^\lambda$ for each $\lambda$. Then we have the following $\mathbb{Z}$-form of $V$,
\[
V^\lambda_{\mathbb{Z}} := \bigoplus_{\lambda} V^\lambda_{\mathbb{Z}}.
\]

Let $\tilde{G}(\mathbb{Q})$ be the subgroup of $\text{GL}(V)$ generated by $G(\mathbb{Q})$ and $\hat{U}^+(\mathbb{Q})$. By Lemma 7.2, every $u \in \hat{U}^+(\mathbb{Q})$ can be uniquely written as
\[
u = \prod_{\alpha \in \Delta^+} \chi_{\alpha}(t_{\alpha}), \quad t_{\alpha} \in \mathbb{Q}^{\mathbb{N}_{\mathbb{Q}}}.
\]
where the product is taken in the order \(<\) on \(\Delta^+\), and \(\chi_\alpha(t_\alpha)\) is defined in Section 3.

8. Integrality in \(\tilde{G}(\mathbb{Q})\)

Recall that

\[ G(\mathbb{Z}) = \langle \chi_\alpha(t) \mid t \in \mathbb{Z}, \alpha \in \Delta^+ \rangle, \]

and

\[ \Gamma(\mathbb{Z}) = \{ g \in G(\mathbb{Q}) \mid g(\mathbb{V}_\mathbb{Z}) = \mathbb{V}_\mathbb{Z} \} \]

\[ = \{ g \in G(\mathbb{Q}) \mid g(\mathbb{V}_\mathbb{Z}^\lambda) = \mathbb{V}_\mathbb{Z}^\lambda \text{ for each dominant integral weight } \lambda \}. \]

Define \(\tilde{G}(\mathbb{Z})\) to be the subgroup of \(\tilde{G}(\mathbb{Q})\) generated by \(G(\mathbb{Z})\) and \(\tilde{U}^+(\mathbb{Z})\), and

\[ \tilde{\Gamma}(\mathbb{Z}) = \{ g \in \tilde{G}(\mathbb{Q}) \mid g(\mathbb{V}_\mathbb{Z}) = \mathbb{V}_\mathbb{Z} \}. \]

It is clear that \(\tilde{G}(\mathbb{Z}) \subseteq \tilde{\Gamma}(\mathbb{Z})\) and in general \(\tilde{G}(R) \subseteq \tilde{\Gamma}(R)\) for any commutative ring \(R\) (see Subsection 3.3). Our objective is to prove that \(\tilde{G}(\mathbb{Z}) = \tilde{\Gamma}(\mathbb{Z})\) (Theorem 8.3).

Since the completion \(\tilde{G}(\mathbb{Q})\) has a BN-pair structure ([CG]), Theorem 4.7 holds for the completion \(\tilde{G}(\mathbb{Q})\).

**Lemma 8.1.** We have \(\tilde{G}(\mathbb{Q}) = \tilde{G}(\mathbb{Z})B(\mathbb{Q})\).

Applying the results in the previous sections, we obtain the following.

**Theorem 8.2.** Let \(u^{-1} = \prod_{\alpha \in \Delta^+} \chi_\alpha(t_\alpha) \in \tilde{U}^+(\mathbb{Q})\), with the product taken in the order \(<\) on \(\Delta^+\) and \(t_\alpha \in \mathbb{Q}^{n_\alpha}\). Then \(t_\alpha \in \mathbb{Z}^{n_\alpha}\) for all \(\alpha \in \Delta^+\) if and only if \(u\) preserves \(\mathbb{V}_\mathbb{Z}\), i.e. preserves \(\mathbb{V}_\mathbb{Z}^\lambda\) for every dominant integral weight \(\lambda\).

**Proof.** The proof is a simple modification of that of Proposition 6.4. For convenience, let us label \(\Delta^+\) as \(\{\alpha_k, k = 1, 2, \ldots\}\) such that \(\alpha_j \prec \alpha_k\) if \(j < k\). Then we may rewrite

\[ u^{-1} = \chi_{\alpha_1}(t_1)\chi_{\alpha_2}(t_2) \cdots, \]

where \(t_k \in \mathbb{Q}^{n_{\alpha_k}}\).

For \(k \geq 1\), define \(u_k \in \tilde{U}^+(\mathbb{Q})\) by

\[ u_k^{-1} = \chi_{\alpha_k}(t_k)\chi_{\alpha_{k+1}}(t_{k+1}) \cdots. \]

Assume by induction that \(t_j \in \mathbb{Z}^{n_{\alpha_j}}, j = 1, \ldots, k - 1\). Then

\[ u_k = u \cdot \chi_{\alpha_1}(t_1) \cdots \chi_{\alpha_{k-1}}(t_{k-1}) \]

preserves \(\mathbb{V}_\mathbb{Z}^\lambda\) for each \(\lambda\). Write \(t_k = (t_{k,1}, \ldots, t_{k,n_{\alpha_k}}) \in \mathbb{Q}^{n_{\alpha_k}}\). Since \(u_k = u_{k+1} \cdot \chi_{\alpha_k}(t_k)^{-1}\), we have

\[ u_k \cdot (x_{-\alpha_k,i} \cdot v_\lambda) = u_{k+1} \cdot (x_{-\alpha_k,i} \cdot v_\lambda - t_{k,i}(\lambda, \alpha_k^\vee))v_\lambda \]

\[ = x_{-\alpha_k,i} \cdot v_\lambda - t_{k,i}(\lambda, \alpha_k^\vee)v_\lambda \in \mathbb{V}_\mathbb{Z}^\lambda, \]

which implies that \(t_{k,i}(\lambda, \alpha_k^\vee) \in \mathbb{Z}, i = 1, \ldots, n_{\alpha_k}\). By varying \(\lambda\) as before we conclude that \(t_{k,i} \in \mathbb{Z}\). This finishes the induction.

We now gather the above results and prove our main theorem.

**Theorem 8.3.** We have \(\tilde{G}(\mathbb{Z}) = \tilde{\Gamma}(\mathbb{Z})\). That is, for any element \(g \in \tilde{G}(\mathbb{Q})\) satisfying \(g(\mathbb{V}_\mathbb{Z}) = \mathbb{V}_\mathbb{Z}\), we have \(g \in \tilde{G}(\mathbb{Z})\).
Proof. Let $\tilde{B}(Q)$ be the subgroup of $\tilde{G}(Q)$ generated by $H(Q)$ and $\tilde{U}^+(Q)$. From Lemma 5.1 and Theorem 8.2, one can deduce that $\Gamma(Z) \cap B(Q) = \tilde{G}(Z) \cap B(Q)$. Also $\Gamma(Z) \subseteq G(Q) = \tilde{G}(Z)\tilde{B}(Q)$ by Lemma 8.1.

Let $\gamma \in \Gamma(Z)$. Then
$$\gamma = \gamma_0b \in \tilde{G}(Q)$$
with $\gamma_0 \in \tilde{G}(Z)$ and $b \in \tilde{B}(Q)$. Then $\gamma_0^{-1}\gamma = b \in \tilde{B}(Q)$. Since $\tilde{G}(Z) \subseteq \tilde{G}(Z)$, we have
$$\gamma_0^{-1}\gamma \in \Gamma(Z) \cap \tilde{B}(Q) = \tilde{G}(Z) \cap \tilde{B}(Q).$$

Thus $\gamma_0^{-1}\gamma \in \tilde{G}(Z)$, so $\gamma \in \tilde{G}(Z)$. □

We conjecture that Theorem 8.3 holds for any integrable highest weight module $V^\lambda$ such that $L_{V^\lambda} = P$. That is, $G_{V^\lambda}(Z) = \tilde{G}_{V^\lambda}(Z)$ for each fixed such $\lambda$. Our methods only yield a proof for the universal Kac–Moody group and its arithmetic subgroup. However, some refinement of our arguments should give the result for fixed $\lambda$.

**Corollary 8.4.** We have $\tilde{G}(Q) = \tilde{\Gamma}(Z)\tilde{B}(Q)$.

We are grateful to Ralf Koehl who informed us of the following direct proof of Corollary 8.4.

**Proposition 8.5.** The group $\tilde{\Gamma}(Z)$ acts transitively on each half of the twin building of $\tilde{G}(Q)$.

**Proof.** The group $SL_2(Z)$ acts transitively on $\mathbb{P}^1(Q)$ by the Euclidean algorithm. The claim follows by applying the local to global argument Proposition 3.10 and Corollary 3.11 of [DGH], as the panel stabilizers in $\tilde{\Gamma}(Z)$ contain $SL_2(Z)$.

□

**Corollary 8.6.** The group $\tilde{\Gamma}(Z)$ has the following topological generating sets:

1. $\chi_{\alpha_i}(1)$ and $\chi_{-\alpha_i}(1)$, $i = 1, \ldots, \ell$, or
2. $\chi_{\alpha_i}(1)$ and $w_{\alpha_i} = \chi_{\alpha_i}(1)\chi_{-\alpha_i}(-1)\chi_{\alpha_i}(1)$, $i = 1, \ldots, \ell$.

**Proof.** (1) is an immediate consequence of the exponential rule $\chi_{\alpha_i}(t) = \chi_{\alpha_i}(1)^t$ for $t \in \mathbb{Z}$. (2) follows immediately from (1). □

**Corollary 8.7.** Let $g \in \Gamma(Z)$. Then there are simple roots $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}$ such that
$$g = \chi_{\pm\alpha_{i_1}}(1)\chi_{\pm\alpha_{i_2}}(1)\cdots\chi_{\pm\alpha_{i_k}}(1).$$

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