The existence and blow-up criterion of liquid crystals system in critical Besov space

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Abstract

We consider the existence of strong solution to liquid crystals system in critical Besov space, and give a criterion which is similar to Serrin’s criterion on regularity of weak solution to Navier-Stokes equations.

Key words: Littlewood-Paley theory, liquid crystals, strong solution.

1 Introduction

In this paper, we study the following incompressible liquid crystals system:

\[
\begin{align*}
\text{div} u &= 0, \\
u_t + u \cdot \nabla u - \mu \Delta u + \nabla p &= -\nabla (\nabla d \odot \nabla d), \\
d_t + u \cdot \nabla d - \Delta d - |\nabla d|^2 d &= 0,
\end{align*}
\]

(1.1)

on \((0, T) \times \mathbb{R}^N\). Here \(u\) is the velocity, \(p\) is the pressure, \(d\) is the macroscopic average of molecular arrangement, \(\nabla d \odot \nabla d\) is a matrix, whose \((i, j)\) - th entry is \(d_{k,i}d_{k,j}\). Let’s consider system (1.1) with the following initial conditions

\[
\begin{align*}
u(0) &= u_0(x), \\
d(0) &= d_0(x), \\
|d_0(x)| &= 1,
\end{align*}
\]

(1.2)

and far field behaviors

\[
u \to 0, \quad d \to \bar{d}_0, \quad \text{as} \quad |x| \to \infty.
\]

(1.3)

Here \(\bar{d}_0\) is a constant vector with \(|\bar{d}_0| = 1\). As it is difficult to deal with the high order term \(|\nabla d|^2 d\), Lin-Liu [11, 12] proposed to investigate an approximate model of liquid crystals system by Ginzburg-Landau function. The approximate liquid crystals system reads as follows

\[
\begin{align*}
\text{div} u &= 0, \\
u_t + u \cdot \nabla u - \mu \Delta u + \nabla p &= -\nabla (\nabla d \odot \nabla d), \\
d_t + u \cdot \nabla d - \Delta d + \frac{1}{\epsilon}(1 - |d|^2) d &= 0.
\end{align*}
\]

(1.4)

The hydrodynamic theory of nematic liquid crystals was established by Ericksen [4] and Lieslie [8] in 1950s. Lin [10] introduced a simple model and studied it from mathematical

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viewpoint. Since then, many applauding results to system (1.4) with (1.2) and (1.3) have been established: 1. the global existence of weak solutions, see [12, 13]; 2. the global existence of classical solution for small initial data or large viscosity, see [13, 17]; the local existence of classical solution for general initial data, see [13]; 3. the partial regularity property of the weak solutions, see [11]; 4. the stability and long time behavior of strong solution, see [17].

From the viewpoint of partial differential equations, system (1.1) is a strongly coupled system between the incompressible Navier-Stokes equations and the transported heat flow of harmonic maps. It is difficult to get the global existence for general initial data, even the existence of weak solutions in three-dimensional space. However, Lin et al [14] and Hong [6] obtained the global weak solutions in two-dimensional space. In this paper, a solution is called to be a strong solution if the uniqueness holds. The well-posedness of system (1.1)–(1.3) has been studied by Li and Wang [9], Lin and Ding [15], Wang [16]. We are interested in the local and global existence of strong solution to system (1.1)–(1.3) in this paper. We will also consider a criterion for system (1.1), which is similar to serrin’s criterion on regularity of weak solution to Navier-Stokes equations. A natural way of dealing with uniqueness is to find a function space as large as possible, where the existence and uniqueness of solution hold. In a word, we need to find a “critical” space. This approach has been initiated by Fujita and Kato [5] on Navier-Stokes equations. We give the scaling of \((u, p, d)\) as follows

\[ u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x), \quad d_\lambda(t, x) = d(\lambda^2 t, \lambda x), \]

where \((u, p, d)\) is a solution of (1.1)–(1.3). We note that \((u_\lambda, p_\lambda, d_\lambda)\) still satisfies system (1.1) with initial data

\[ u_\lambda(0, \cdot) = \lambda u_0(\lambda x), \quad d_\lambda(0, \cdot) = d_0(\lambda x). \]

So it is natural to give the following definition: a space \(X\) is called critical space if

\[ \forall u \in X, \quad \|u_\lambda\|_X = \|u\|_X, \quad for \ all \ \lambda > 0. \]

In [5], Fujita and Kato used \(\dot{H}^{1/2}_{1/2}\) as the critical space on Navier-Stokes equations in three-dimensional space. In this paper, we consider the homogeneous Besov spaces \(\dot{B}^s_{p, r}\). Inspired by [2], we find

\[ \|u(\cdot, \cdot)\|_{\dot{B}^{s-1}_{2, 1}} = \|u_\lambda(\cdot, \cdot)\|_{\dot{B}^{s-1}_{2, 1}}, \quad \|d(\cdot, \cdot)\|_{\dot{B}^s_{2, 1}} = \|d_\lambda(\cdot, \cdot)\|_{\dot{B}^s_{2, 1}}, \]

where \(N\) denotes the spacial dimension. In addition, we suppose \(|d| = 1\) in physics and note that

\[ \dot{H}^{N-1}_{2, 1} \nsubseteq L^\infty, \quad \dot{B}^N_{2, 1} \hookrightarrow L^\infty. \]

So it is natural to choose \(\dot{B}^{N-1}_{2, 1}\) (for \(u\)) and \(\dot{B}^N_{2, 1}\) (for \(d\)) as the critical spaces in which we study liquid crystals system (1.1). We are able to prove the following theorems.

**Theorem 1.1** Let \(N \geq 2, \bar{d}_0\) be a constant vector with \(|\bar{d}_0| = 1\), \(u_0 \in \dot{B}^{N-1}_{2, 1}, (d_0 - \bar{d}_0) \in \dot{B}^N_{2, 1}\) in (1.2), (1.3), and

\[ E_0 = \|u_0\|_{\dot{B}^{N-1}_{2, 1}} + \|(d_0 - \bar{d}_0)\|_{\dot{B}^N_{2, 1}}. \]
I. There exists a constant $T > 0$, such that system (1.1)-(1.3) has a unique strong solution $(u, d)$ on $[0, T] \times \mathbb{R}^N$,

$$(u, \ d - \tilde{d}_0) \in \tilde{L}_T^1(B_{2,1}^{\frac{N}{2}+1}) \cap C([0, T]; \tilde{B}_{2,1}^{\frac{N}{2}+1}) \times \tilde{L}_T^1(B_{2,1}^{\frac{N}{2}+2}) \cap C([0, T]; \tilde{B}_{2,1}^{\frac{N}{2}}).$$

In addition, for any $\rho_1, \rho_2 \in [1, \infty]$,

$$\|u\|_{\tilde{L}_T^{\rho_1}(B_{2,1}^{\frac{N}{2}+1})} + \|d - \tilde{d}_0\|_{\tilde{L}_T^{\rho_2}(B_{2,1}^{\frac{N}{2}+2})} \leq C(E_0)$$

holds.

II. There exists a constant $\delta_0 \geq 0$, such that if $E_0 \leq \delta_0$, system (1.1)-(1.3) has a unique global strong solution $(u, d)$ on $[0, \infty) \times \mathbb{R}^N$,

$$(u, \ d - \tilde{d}_0) \in \tilde{L}^1(B_{2,1}^{\frac{N}{2}+1}) \cap C([0, \infty); \tilde{B}_{2,1}^{\frac{N}{2}+1}) \times \tilde{L}^1(B_{2,1}^{\frac{N}{2}+2}) \cap C([0, \infty); \tilde{B}_{2,1}^{\frac{N}{2}}).$$

In addition, for any $\rho_1, \rho_2 \in [1, \infty]$,

$$\|u\|_{L^{\rho_1}(B_{2,1}^{\frac{N}{2}+1})} + \|d - \tilde{d}_0\|_{L^{\rho_2}(B_{2,1}^{\frac{N}{2}+2})} \leq C(\delta_0)$$

holds.

Space $\tilde{L}_T^{\rho}(B_{p,r})$ and $\tilde{L}^{\rho}(B_{p,r})$ will be introduced in section 2. We want to point out that if $(u, \ d)$ is smooth, then we can proof $|d| = |d_0| = 1$ by maximal principle. Wang[16], Lin and Ding[15] proved $|d| = |d_0| = 1$ by taking $|\nabla d|^2d$ as a second fundamental form. The following result is inspired by Kozono and Shimada[7].

**Theorem 1.2** Let $(u, \ d)$ be a strong solution of (1.1)-(1.3) on $[0, T] \times \mathbb{R}^N$ in Theorem 1.1. For any $T' > T$, such that if

$$u \in \tilde{L}_{T'}^{\rho_1}(B_{p_1,\infty}^{\frac{N}{2}+\frac{2}{p_1}}), \quad d - \tilde{d}_0 \in \tilde{L}_{T'}^{\rho_2}(B_{p_2,\infty}^{\frac{N}{2}}) \quad \text{and} \quad d - \tilde{d}_0 \in \tilde{L}_{T'}^{\rho_3}(B_{p_3,\infty}^{\frac{N}{2}+\frac{2}{p_3}}),$$

$$\frac{N}{2} + \frac{2}{p_1} + \frac{2}{p_2} + \frac{2}{p_3} - 2 > 0,$$

$$\|u\|_{\tilde{L}_{T'}^{\rho_1}(B_{p_1,\infty}^{\frac{N}{2}+\frac{2}{p_1}})} + \|d - \tilde{d}_0\|_{\tilde{L}_{T'}^{\rho_2}(B_{p_2,\infty}^{\frac{N}{2}})} + \|d - \tilde{d}_0\|_{\tilde{L}_{T'}^{\rho_3}(B_{p_3,\infty}^{\frac{N}{2}+\frac{2}{p_3}})} < \infty \quad (1.5)$$

hold, where $(\rho_1, \rho_2, \rho_3) \in (2, \infty)^3$, then $(u, \ d)$ is a strong solution on $[0, T'] \times \mathbb{R}^N$ in Theorem 1.1.

It is well-known that using Besov space, one can get the same results in a lower regularity. And the following Remark is very interested.

**Remark 1.1** By the embedding theory

$$\tilde{L}_T^{\rho_1}(B_{p_1,\infty}^{\frac{N}{2}+\frac{2}{p_1}}) \hookrightarrow \tilde{L}_T^{\rho_2}(B_{p_2,\infty}^{\frac{N}{2}}), \quad L^p \hookrightarrow B_{p,\infty}^0 \hookrightarrow B_{\infty,\infty}^{-\frac{N}{p}},$$

we have

$$L_T^{\rho_1}L^p \hookrightarrow \tilde{L}_T^{\rho_1}(B_{\infty,\infty}^{\frac{N}{2}}), \quad -1 + \frac{2}{\rho_1} = -\frac{N}{p},$$

which is the Serrin’s criterion on endpoint.
Remark 1.2 The proof of Theorem 1.2 implies that the life-span can be extended a little larger under the condition (1.5). Also we obtain that if \([0,T)\) is the life-span of strong solution to (1.1)-(1.3), then (1.5) fails, that is
\[
\lim_{a \to T} \left( \|u\|_{\dot{L}^{s_a,1}(B^{\rho}_{\infty,1})} + \|d - \tilde{d}_0\|_{\dot{L}^{s_a,2}(B^{\rho}_{\infty,2})} + \|d - \tilde{d}_0\|_{\dot{L}^{s_a,2}(B^{\rho}_{2,\infty})} \right) = \infty.
\]

The rest of our paper is organized as follows. In section 2, we give a short introduction on Besov space. In section 3, we prove Theorem 1.1 and Theorem 1.2.

2 Littlewood-Paley theory and Besov space

In this section, we present some well-known facts on Littlewood-paley theory, more details see [1, 3]. Let \(S(\mathbb{R}^N)\) be the Schwartz space and \(S'(\mathbb{R}^N)\) be its dual space. Let \((\chi, \varphi)\) be a couple of smooth functions such that
\[
\chi(x) = \begin{cases} 
1 & x \in B(0, \frac{4}{3}), \\
0 & x \in \mathbb{R}^N \setminus B(0, \frac{4}{3}),
\end{cases}
\]
\[
\varphi(x) = \chi \left( \frac{x}{2} \right) - \chi(x).
\]

Then we have \(\text{supp} \varphi \subset C(0, \frac{4}{3}, \frac{8}{3})\), \(\varphi \geq 0\), \(x \in \mathbb{R}^N\). Here we denote \(B(0, R)\) as an open ball with radius \(R\) centered at zero, and \(C(0, R_1, R_2)\) as an annulus \(\{x \in \mathbb{R}^N \mid R_1 \leq |x| \leq R_2\}\). Then
\[
\varphi_q(x) := \varphi \left( \frac{x}{2^q} \right), \quad \text{and} \quad \text{supp} \varphi_q \subset C \left( 0, \frac{3}{4}, \frac{8}{3} \right),
\]
For any \(u \in S(\mathbb{R}^N)\), the Fourier transform of \(u\) denotes by \(\hat{u}\) or \(\mathcal{F}u\). The inverse Fourier transform denotes by \(\mathcal{F}^{-1}\). Let \(h = \mathcal{F}^{-1} \varphi\). We define homogeneous dyadic blocks as follows
\[
\Delta_q u := \varphi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} h(2^qy)u(x - y)dy,
\]
\[
S_q u := \sum_{p \leq q - 1} \Delta_p u.
\]
Then for any \(u \in S'(\mathbb{R}^N)\), the following decomposition
\[
u = \sum_{q = -\infty}^{\infty} \Delta_q u
\]
is called homogeneous Littlewood-Paley decomposition.
Let \(u \in S'(\mathbb{R}^N)\), \(s\) be a real number, and \(1 \leq p, r \leq \infty\). We set
\[
\|u\|_{\dot{B}^s_{p,r}} := \left( \sum_{q = -\infty}^{\infty} (2^{qs}\| \Delta_q u \|_{L^p})^r \right)^{\frac{1}{r}}.
\]
The besov space \(\dot{B}^s_{p,r}\) is defined as follows:
\[
\dot{B}^s_{p,r} := \left\{ u \in S'(\mathbb{R}^N) \mid \|u\|_{\dot{B}^s_{p,r}} < \infty \right\}.
\]
Definition 2.1 Let \( u \in S'((0, T) \times \mathbb{R}^N) \), \( s \in \mathbb{R} \) and \( 1 \leq p, r, \rho \leq \infty \). We define

\[
\| u \|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} := \left( \sum_{q=-\infty}^{\infty} (2^{qs} \| \triangle_q u \|_{L^\rho_T})^r \right)^{\frac{1}{r}}
\]

and

\[
\tilde{L}^\rho_T(\dot{B}^s_{p,r}) := \left\{ u \in S'((0, T) \times \mathbb{R}^N) \mid \| u \|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} < \infty \right\}.
\]

Similarly, we set

\[
\| u \|_{\tilde{L}^\rho(\dot{B}^s_{p,r})} := \left( \sum_{q=-\infty}^{\infty} (2^{qs} \| \triangle_q u \|_{L^\rho(R^+; L^p)})^r \right)^{\frac{1}{r}},
\]

\[
\| u \|_{\tilde{L}^\rho(\dot{B}^s_{p,r}((T_1, T_2)))} := \left( \sum_{q=-\infty}^{\infty} (2^{qs} \| \triangle_q u \|_{L^\rho((T_1, T_2); L^p)})^r \right)^{\frac{1}{r}},
\]

and define the corresponding spaces \( \tilde{L}^\rho(\dot{B}^s_{p,r}) \), \( \tilde{L}^\rho(\dot{B}^s_{p,r}((T_1, T_2))) \) respectively. \( L^\rho_T(\dot{B}^s_{p,r}) \) is defined as usual.

Remark 2.1 According to Minkowski inequality, we have

\[
\| u \|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} \leq \| u \|_{L^\rho_T(\dot{B}^s_{p,r})} \quad \text{if} \quad r \geq \rho,
\]

\[
\| u \|_{\tilde{L}^\rho(\dot{B}^s_{p,r})} \geq \| u \|_{L^\rho(\dot{B}^s_{p,r})} \quad \text{if} \quad r \leq \rho.
\]

Next let’s present some results of paradifferential calculus in homogeneous space, which is useful for dealing with nonlinear equations. For \( u, v \in S'((0, T) \times \mathbb{R}^N) \), we have the following formal decomposition:

\[
uv = \sum_{p,q} \triangle_p u \triangle_q v.
\]

Define the operators \( T, R \) as follows,

\[
T_v = \sum_{p \leq -2} \triangle_p u \triangle_q v = \sum_{q} S_{q-1} u \triangle_q v,
\]

\[
R(u, v) = \sum_{|p-q| \leq 1} \triangle_p u \triangle_q v.
\]

Then we have the following so-called homogeneous Bony decomposition:

\[
wv = Twv + Tv + R(u, v).
\]

In order to estimate the above terms, let’s recall some lemmas (see [1, 3]).

Lemma 2.1 There exists a constant \( C \), such that, for any couple of real numbers \((s, \sigma)\), with \( \sigma \) positive and any \((p, r, r_1, r_2)\) in \([1, \infty]^4\) with \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \), we have

\[
\| T \|_{L^\infty(\dot{B}^s_{p,r}; \dot{B}^s_{p,r})} \leq C \quad \text{if} \quad s < \frac{N}{p} \text{ or } s = \frac{N}{p}, \quad r = 1,
\]

\[
\| T \|_{L^\infty(\dot{B}^s_{p,r}; \dot{B}^s_{p,r})} \leq C \quad \text{if} \quad s - \sigma < \frac{N}{p} \text{ or } s - \sigma = \frac{N}{p}, \quad r = 1.
\]
Lemma 2.2. There exists a constant $C$, such that, for any real numbers $s_1, s_2, (p, p_1, p_2, r, r_1, r_2)$ in $[1, \infty]^6$ with

$$s_1 + s_2 > 0, \quad \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \quad \frac{1}{r} \leq \frac{1}{r_1} + \frac{1}{r_2} \leq 1,$$

we have

$$\|R\|_{L(\dot{B}^{s_1}_{p_1, r_1} \times \dot{B}^{s_2}_{p_2, r_2}; \dot{B}^s_{p, r})} \leq C, \quad \sigma = s_1 + s_2 - N\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right),$$

providing that $\sigma < \frac{N}{p}$, or $\sigma = \frac{N}{p}$ and $r = 1$.

The initial problem of heat equation reads

$$\begin{aligned}
    \begin{cases}
        v_t - a\Delta v = G, \\
        v|_{t=0} = v_0.
    \end{cases}
\end{aligned} \quad (2.7)$$

The following lemma can be found in [1, 3].

Lemma 2.3. Let $T > 0$, $s \in \mathbb{R}$, and $[\rho, p, r] \in [1, \infty]^3$. Assume that $v_0 \in \dot{B}^s_{p, r}$ and $G \in \dot{L}^{\rho}(\dot{B}^{s-2+\frac{6}{\rho}}_{p, r})$, Then (2.7) has a unique solution $v \in \dot{L}^1(\dot{B}^s_{p, r}) \cap \dot{L}^\infty(\dot{B}^{s-2}_{p, r})$, satisfying

$$a^{\frac{1}{\sqrt{2}}} \|v\|_{L(\dot{B}^{s+\frac{6}{\rho}}_{p, r})} \leq C \left(\|v_0\|_{\dot{B}^s_{p, r}} + a^{\frac{1}{\sqrt{2}}-1} \|G\|_{L(\dot{B}^{s-2+\frac{6}{\rho}}_{p, r})}\right), \quad \rho_1 \in [\rho, \infty],$$

in addition if $r$ is finite, then $v \in C([0, T]; \dot{B}^s_{p, r})$.

3 Proof of main results

Let $P$ denote the Leray projector on solenoidal vector fields which is defined by

$$P = I - \nabla \Delta^{-1} \text{div}.$$

By operator $P$, we can project the second equation of (1.1) onto the divergence free vector field. Then the pressure $p$ can be eliminated. It is easy to notice that $P$ is a homogeneous multiplier of degree zero. Denote $\tau = d - \bar{d}_0$. We only need to consider the following equations

$$\begin{aligned}
    \begin{cases}
        u_t - \mu \Delta u + P [u \cdot \nabla u + \nabla (\nabla \tau \circ \nabla \tau)] = 0, \\
        \tau_t - \Delta \tau + u \cdot \nabla \tau - |\nabla \tau|^2 \tau - |\nabla \tau|^2 \bar{d}_0 = 0,
    \end{cases}
\end{aligned} \quad (3.8)$$

with initial conditions

$$\begin{aligned}
    u|_{t=0} = u_0(x), \quad \tau|_{t=0} = \tau_0 = d_0(x) - \bar{d}_0, \quad (3.9)
\end{aligned}$$

and far field behaviors

$$u \to 0, \quad \tau \to 0, \quad \text{as} \quad |x| \to \infty. \quad (3.10)$$

Instead of proving Theorem 1.1 and Theorem 1.2, we should only consider the following theorems.

Theorem 3.1. Let $N \geq 2$, $u_0 \in \dot{B}^{\frac{N}{2} - 1}_{2, 1}$, $\tau_0 \in \dot{B}^{\frac{N}{2}}_{2, 1}$ in (3.9), and

$$E_0 = \|u_0\|_{\dot{B}^{\frac{N}{2} - 1}_{2, 1}} + \|\tau_0\|_{\dot{B}^{\frac{N}{2}}_{2, 1}}.$$
I. There exists a constant $T > 0$, such that system (3.8)-(3.10) has a unique strong solution $(u, \tau)$ on $[0, T] \times \mathbb{R}^N$, 

$$(u, \tau) \in \tilde{L}^1_T(B_2^{\frac{n+1}{2}}; B_2^{\frac{n}{2}-1}) \cap C([0, T]; \tilde{L}^1_T(B_2^{\frac{n+2}{2}})) \cap C([0, T]; \tilde{B}_2^{\frac{n}{2}}).$$

In addition, for any $\rho_1, \rho_2 \in [1, \infty]$, 

$$\|u\|_{\tilde{L}^p_\rho(B_2^{\frac{n-1+p_1}{p_1}})} + \|\tau\|_{\tilde{L}^p_\rho(B_2^{\frac{n-1+p_2}{p_2}})} \leq C(E_0) \quad (3.11)$$

holds.

II. There exists a constant $\delta_0 \geq 0$, such that if $E_0 \leq \delta_0$, system (3.8)-(3.10) has a unique global strong solution $(u, \tau)$ on $[0, \infty) \times \mathbb{R}^N$, 

$$(u, \tau) \in \tilde{L}^1_T(B_2^{\frac{n+1}{2}}) \cap C([0, \infty); \tilde{L}^1_T(B_2^{\frac{n+2}{2}})) \cap C([0, \infty); \tilde{B}_2^{\frac{n}{2}}).$$

In addition, for any $\rho_1, \rho_2 \in [1, \infty]$, 

$$\|u\|_{\tilde{L}^p_\rho(B_2^{\frac{n-1+p_1}{p_1}})} + \|\tau\|_{\tilde{L}^p_\rho(B_2^{\frac{n-1+p_2}{p_2}})} \leq C(\delta_0) \quad (3.12)$$

holds.

**Theorem 3.2** Let $(u, \tau)$ be a strong solution of (3.8)-(3.10) on $[0, T] \times \mathbb{R}^N$ in Theorem 3.1. For any $T' > T$, such that if 

$u \in \tilde{L}^p_\rho(B_2^{\frac{n-1+p_1}{p_1}}(\rho_1), \tau \in \tilde{L}^p_\rho(B_2^{\frac{n-1+p_2}{p_2}}(\rho_2))$ and $\tau \in \tilde{L}^p_\rho(B_2^{\frac{n-1+p_3}{p_3}}(\rho_3))$, 

$$\|u\|_{\tilde{L}^p_\rho(B_2^{\frac{n-1+p_1}{p_1}})} + \|\tau\|_{\tilde{L}^p_\rho(B_2^{\frac{n-1+p_2}{p_2}})} + \|\tau\|_{\tilde{L}^p_\rho(B_2^{\frac{n-1+p_3}{p_3}})} < \infty \quad (3.13)$$

hold, where $(\rho_1, \rho_2, \rho_3) \in (2, \infty)^3$, then $(u, \tau)$ is a strong solution on $[0, T'] \times \mathbb{R}^N$ in Theorem 3.1.

Next let us introduce the heat semi-group operator $e^{at\Delta}$. Let 

$$v = e^{at\Delta}v_0,$$

then $v$ solves problem (2.7) with $G$ replaced by zero.

**Lemma 3.1** Let $v = e^{at\Delta}v_0$, $v_0 \in \dot{B}_2^s$, $s \in R$.

I. Assume $\|v_0\|_{\dot{B}_2^s} \leq C_0$, $\rho \in [1, \infty)$. For any small $\varepsilon_0 > 0$, there exists $T_0$, such that the following estimate holds 

$$\|v\|_{\tilde{L}^p_\rho(\dot{B}_2^s)} \leq \varepsilon_0, \quad \text{for any } T \leq T_0.$$ 

II. Assume $\rho \in [1, \infty]$. For any small $\varepsilon_0 > 0$, there exists $\delta_0$, such that if $\|v_0\|_{\dot{B}_2^s} \leq \delta_0$, the following estimate holds 

$$\|v\|_{\tilde{L}^p_\rho(\dot{B}_2^s)} \leq \varepsilon_0.$$
Proof. Firstly, we know that
\[ \hat{v}(t, \xi) = e^{-at|\xi|^2} \hat{v}_0(\xi). \]
Then we have
\[ \Delta_q v(t) = \mathcal{F}^{-1} (e^{-at|\xi|^2} \varphi_q(\xi) \hat{v}_0(\xi)) = e^{at} \Delta_q v_0, \]
and
\[ \| \Delta_q v(t) \|_{L^2} \leq e^{-kaT|\xi|^2} \| \Delta_q v_0 \|_{L^2}. \]
(3.16)
where \( k \) is a constant.

Case I. By using (3.16), we obtain
\[ \| \Delta_q v(t) \|_{L^p L^2} \leq \left( \frac{1 - e^{-kaT|\xi|^2}}{kappa} \right)^{\frac{1}{\rho}} \| \Delta_q v_0 \|_{L^2}. \]
(3.17)
It is easy to find \( N > 0 \), such that
\[ \sum_{|q| > N} \left( \frac{1 - e^{-kaT|\xi|^2}}{kappa} \right)^{\frac{1}{\rho}} 2^{qs} \| \Delta_q v_0 \|_{L^2} \leq \frac{\varepsilon_0}{2}. \]
(3.18)
Combining (3.17) and (3.18), we obtain (3.14).

Case II. By using (3.16), we have
\[ \| \Delta_q v(t) \|_{L^p L^2} \leq \left( \frac{1}{kappa} \right)^{\frac{1}{\rho}} \| \Delta_q v_0 \|_{L^2}. \]
Then
\[ \| v \|_{L^p(\hat{B}_{2,1}^{s+\frac{2}{p}})} \leq C v_0 \|_{\hat{B}_{2,1}^s}. \]
By choosing \( \delta_0 \leq \frac{\varepsilon_0}{C} \), we finish the proof. \( \square \)

Proof of Theorem 3.1.

In this part, we will use the iterative method to prove Theorem 3.1 which is separated into three steps. Firstly we point out that the case I in the following proof corresponds to I in Theorem 3.1 and case II corresponds to II in Theorem 3.1. Also we should keep in mind that \( \hat{B}_{2,1}^s, s > \frac{1}{2} \) is not a Banach space. Then let us consider the following linear equations,
\[ \begin{cases} 
\partial_t u_n - \mu \Delta u_n = -\mathcal{P} [u_{n-1} \cdot \nabla u_{n-1} + \nabla (\nabla \tau_{n-1} \circ \nabla \tau_{n-1})], \\
\partial_t \tau_n - \Delta \tau_n = -u_{n-1} \cdot \nabla \tau_{n-1} + |\nabla \tau_{n-1}|^2 \tau_{n-1} + |\nabla \tau_{n-1}|^2 \delta_0, 
\end{cases} \]
(3.19)
with initial conditions

\[ u_n|_{t=0} = u_0 , \quad \tau_n|_{t=0} = \tau_0 . \]

Let us set \( u_1 = e^{ot\Delta} u_0 \), \( \tau_1 = e^{ot\Delta} \tau_0 \) and begin our proof.

**First step: Uniform boundedness**

**Case I.** We claim that the following estimates hold for some \( T > 0 \),

\[
\|u_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} + \|\tau_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} \leq (C + 1)\epsilon_0, \quad n = 1, 2, 3, \ldots , \tag{3.20}
\]

\[
\|\tau_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} \leq (C + 1)E_0, \quad n = 1, 2, 3, \ldots . \tag{3.21}
\]

Here \( C \) is an absolute constant. Indeed by Lemma 2.1 and Lemma 2.3 we can choose \( \epsilon_0 \leq \frac{1}{(C+1)\sqrt{1+(\frac{1}{\epsilon_0})}} \) to obtain

\[
\|\tau_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} \leq CE_0 + (C + 1)^2 \epsilon_0^2(C + 1)E_0 + 2(C + 1)^2 \epsilon_0^2
\]

\[
\leq E_0\left(C + (C + 1)^3 \epsilon_0^2 + \frac{2(C + 1)^2 \epsilon_0^2}{E_0}\right)
\]

\[
\leq (C + 1)E_0,
\]

where \( C \) is the constant appeared in Lemma 2.1, Lemma 2.3. Then by Lemma 2.1, Lemma 2.3, we have

\[
\|u_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} + \|\tau_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} \leq C\epsilon_0 + 4(C + 1)^2 \epsilon_0^2 + (C + 1)^3 \epsilon_0^2 E_0
\]

\[
\leq (C + 1)\epsilon_0,
\]

by choosing \( \epsilon_0 \leq \min\left\{\frac{1}{(C+1)\sqrt{1+(\frac{1}{\epsilon_0})}}, \frac{1}{4(C+1)[1+(C+1)E_0]}\right\} \). So (3.20) and (3.21) are proved.

In addition, we can get from (3.20), (3.21) and Lemma 2.3 that

\[
\|u_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} + \|\tau_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} \leq CE_0,
\]

and more precisely,

\[
\|u_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} + \|\tau_n\|_{L^\infty_T(B^\frac{\alpha}{2} + \frac{\beta}{2})} \leq (C(\rho + 1)\epsilon_0), \quad \rho \in [1, \infty). \tag{3.22}
\]

**Case II.** For the small initial data, it is simple to prove

\[
\|u_n\|_{L^\rho(B^\frac{\alpha}{2} + \frac{\beta}{2})} + \|\tau_n\|_{L^\rho(B^\frac{\alpha}{2} + \frac{\beta}{2})} \leq (C + 1)\epsilon_0, \quad \rho \in [1, \infty],
\]

providing \( \delta_0 \) is small enough. Here we omit the details.

**Second step: Convergence**

**Case I.** We will prove \( \{(u_n, \tau_n)\}_{n=1, 2, \ldots} \) is a Cauchy sequence. Firstly, let’s consider

\[
\|u_{m+n+1} - u_{n+1}\|_{L^\infty_T(B^{\frac{\alpha}{2}})} \quad \text{and} \quad \|\tau_{m+n+1} - \tau_{n+1}\|_{L^\infty_T(B^{\frac{\alpha}{2}})}.
\]
According to the proof of Lemma 2.3, we need to estimate the following terms:

\[ I_1(t,x) = \int_0^t e^{-\mu(t-s)} \Delta \mathcal{P} \left[ (u_{m+n} \cdot \nabla u_{m+n}) - (u_n \cdot \nabla u_n) \right], \]

\[ I_2(t,x) = \int_0^t e^{-\mu(t-s)} \Delta \mathcal{P} \left[ (\nabla (\nabla \tau_{m+n} \circ \nabla \tau_{m+n}) - (\nabla (\nabla \tau_n \circ \nabla \tau_n)) \right], \]

\[ I_3(t,x) = \int_0^t e^{-(t-s)} \Delta \left[ (u_{m+n} \cdot \nabla \tau_{m+n}) - (u_n \cdot \nabla \tau_n) \right], \]

\[ I_4(t,x) = \int_0^t e^{-(t-s)} \Delta \left[ (|\nabla \tau_{m+n}|^2 \tau_{m+n}) - (|\nabla \tau_n|^2 \tau_n) \right], \]

\[ I_5(t,x) = \int_0^t e^{-(t-s)} \Delta \left[ (|\nabla \tau_{m+n}|^2 \bar{d}_0) - (|\nabla \tau_n|^2 \bar{d}_0) \right]. \]

By using (3.22), Lemma 2.1 and Lemma 2.2, we can obtain

\[ \|I_1(t,x)\|_{L^2_T(\mathbb{B}_{2,1}^{\frac{n}{2}})} \leq C \|u_{m+n} - u_n\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}-1})} \left( \|u_{m+n}\|_{L^2_T(\mathbb{B}_{2,1}^\frac{n}{2})} + \|u_n\|_{L^2_T(\mathbb{B}_{2,1}^\frac{n}{2})} \right), \]

\[ \|I_2(t,x)\|_{L^2_T(\mathbb{B}_{2,1}^{\frac{n}{2}})} \leq C \|\tau_{m+n} - \tau_n\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}+1})} \left( \|\tau_{m+n}\|_{L^2_T(\mathbb{B}_{2,1}^\frac{n}{2})} + \|\tau_n\|_{L^2_T(\mathbb{B}_{2,1}^\frac{n}{2})} \right), \]

\[ \|I_3(t,x)\|_{L^2_T(\mathbb{B}_{2,1}^{\frac{n}{2}})} \leq C \|u_{m+n} - u_n\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}+1})} \|\tau_{m+n}\|_{L^2_T(\mathbb{B}_{2,1}^\frac{n}{2})} + C \|u_n\|_{L^0_T(\mathbb{B}_{2,1}^{\frac{n}{2}})} \|\tau_{m+n} - \tau_n\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}+1})}, \]

\[ \|I_4(t,x)\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}+1})} \leq C \|\tau_{m+n} - \tau_n\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}+1})} \|\tau_{m+n} + \tau_n\|_{L^2_T(\mathbb{B}_{2,1}^\frac{n}{2})} \|\tau_{m+n}\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}+1})} + C \|\tau_n\|_{L^2_T(\mathbb{B}_{2,1}^{\frac{n}{2}})} \|\tau_{m+n} - \tau_n\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}+1})}, \]

\[ \|I_5(t,x)\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}+1})} \leq C \|\tau_{m+n} - \tau_n\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}+1})} \|\tau_{m+n} + \tau_n\|_{L^2_T(\mathbb{B}_{2,1}^\frac{n}{2})}. \]

By Lemma 2.3, we obtain

\[ \|u_{m+n+1} - u_{n+1}\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}-1})} + \|\tau_{m+n+1} - \tau_{n+1}\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}})} \leq \frac{1}{2} \left( \|u_{m+n} - u_n\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}-1})} + \|\tau_{m+n} - \tau_n\|_{L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}})} \right), \]

by choosing \( \varepsilon_0 \) small enough. Then \( \{(u_m, \tau_m) | m = 1, 2 \cdots \} \) is a Cauchy sequence in Banach space \( L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}-1}) \times L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}}) \). Let

\[ u_m \rightarrow u \quad \text{in} \quad L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}-1}), \]

\[ \tau_m \rightarrow \tau \quad \text{in} \quad L^\infty_T(\mathbb{B}_{2,1}^{\frac{n}{4}}). \]

Then \((u, \tau)\) is a solution of (3.8)-(3.10) on \((0, T) \times \mathbb{R}^3\) satisfying (3.11). By Lemma 2.3, we can get \((u, \tau) \in C([0, T]; \mathbb{B}_{2,1}^{\frac{n}{4}-1}) \times C([0, T]; \mathbb{B}_{2,1}^{\frac{n}{4}}). \]

**Case II.** For the small initial data, we can obtain that

\[ \|u_{m+n+1} - u_{n+1}\|_{L^\infty(\mathbb{B}_{2,1}^{\frac{n}{4}-1})} + \|\tau_{m+n+1} - \tau_{n+1}\|_{L^\infty(\mathbb{B}_{2,1}^{\frac{n}{4}})} \]
Lemma 2.3 and the estimates in the second step of the proof of Theorem 3.1, we can get
\[ \delta > 0. \]
Here we should guarantee 
\[ T \]
where 0 < \( \delta \) (some small constant \( \delta > 0 \)), we can prove \( w = 0, \Gamma = 0 \) on \( [0, \delta] \). By repeating the procedure, we can obtain \( w = 0, \Gamma = 0 \) on \( [0, T] \) (or \( [0, \infty) \)). We finish the proof of Theorem 3.1.

**Proof of Theorem 3.2.**

We note that \( (u, \tau) \in C([0, \infty); \mathbb{B}^{rac{3}{2} - 1}_{2,1}) \times C([0, \infty); \mathbb{B}^{rac{3}{2}}_{2,1}) \). By Lemma 2.3, we can find \( T_1 \) such that \( T_1 > T \) and \( (u, \tau) \) is a solution on \( [0, T_1] \times \mathbb{R}^3 \). Since (3.13) holds, for any small \( \varepsilon_0 > 0 \), we can find \( \delta > 0 \) which depends only on \( T' \), such that
\[
\|u\|_{L^p_{[T_2, T_2 + \delta]}(\mathbb{B}^{rac{3}{2} - 1}_{2,1})} + \|\tau\|_{L^p_{[T_2, T_2 + \delta]}(\mathbb{B}^{rac{3}{2}}_{2,1})} \leq \varepsilon_0. \tag{3.23}
\]
Here we should guarantee \( [T_2, T_2 + \delta] \in [T, T'] \). Let \( T_2 \) be the initial time. Then by using Lemma 2.3 and the estimates in the second step of the proof of Theorem 3.1, we can get
\[
\|u\|_{L^p_{[T_2, T_2 + \delta]}(\mathbb{B}^{rac{3}{2} - 1}_{2,1})} + \|\tau\|_{L^p_{[T_2, T_2 + \delta]}(\mathbb{B}^{rac{3}{2}}_{2,1})} \leq \theta \left( \|u(T_2, \cdot)\|_{\mathbb{B}^{rac{3}{2} - 1}_{2,1}} + \|\tau(T_2, \cdot)\|_{\mathbb{B}^{rac{3}{2}}_{2,1}} \right),
\]
where \( 0 < \theta < 1 \), and providing \( \varepsilon_0 \) is small enough. Then we get
\[
\|u\|_{L^p_{[T_2, T_2 + \delta]}(\mathbb{B}^{rac{3}{2} - 1}_{2,1})} + \|\tau\|_{L^p_{[T_2, T_2 + \delta]}(\mathbb{B}^{rac{3}{2}}_{2,1})} \leq C \left( \|u(T_2, \cdot)\|_{\mathbb{B}^{rac{3}{2} - 1}_{2,1}} + \|\tau(T_2, \cdot)\|_{\mathbb{B}^{rac{3}{2}}_{2,1}} \right), \tag{3.24}
\]
where the constant \( C \) is independent of \( T_2 \). Let's repeat above procedure, we claim that \( T_1 \geq T' \). Indeed, if it is not true, we can find \( \bar{T} \in [T_1, T'] \), such that \( (\bar{T} \) is the first blow-up time)
\[
\lim_{t \to \bar{T}} \left( \|u(t, \cdot)\|_{\mathbb{B}^{rac{3}{2} - 1}_{2,1}} + \|\tau(t, \cdot)\|_{\mathbb{B}^{rac{3}{2}}_{2,1}} \right) = \infty, \tag{3.25}
\]
and \( \left( \|u(\bar{T} - \frac{\delta}{2}, \cdot)\|_{\mathbb{B}^{rac{3}{2} - 1}_{2,1}} + \|\tau(\bar{T} - \frac{\delta}{2}, \cdot)\|_{\mathbb{B}^{rac{3}{2}}_{2,1}} \right) \) is bounded. But we find that (3.24) with \( T_2 \) replaced by \( \bar{T} - \frac{\delta}{2} \) is contradicting with (3.25). So we get \( T_1 \geq T' \), and finish the proof of Theorem 3.2.
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