Weierstrass Weight of Gorenstein Singularities with One or Two Branches

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Let \(X\) denote an integral, projective Gorenstein curve over an algebraically closed field \(k\). In the case when \(k\) is of characteristic zero, C. Widland and the second author (\([22], [21], [13]\)) have defined Weierstrass points of a line bundle on \(X\). In the first section, we extend this by defining Weierstrass points of linear systems in arbitrary characteristic. This definition may be viewed as a generalization of the definitions of Laksov [10] and Stöhr-Voloch [19] to the Gorenstein case. Recently Laksov and Thorup [11,12] have given a more general definition of Weierstrass points of “Wronski systems,” and our definition may be viewed as a concrete realization in our setting of their rather abstract definition.

In the second section, we give an example illustrating our definition. This example is a plane curve of arithmetic genus 3 in characteristic 2 such that the gap sequence at every smooth point (with respect to the dualizing bundle) is 1,2,5 and there are no smooth Weierstrass points. Since every smooth curve of genus 3 in characteristic 2 is classical, this gives us an example of a singular nonclassical curve that is the limit of nonsingular classical curves. We also compute the Weierstrass weights of points on a rational curve with a single unibranch singularity whose local ring has an especially simple form.

In the third section, we compute the Weierstrass weight of a unibranch singularity (on a not necessarily rational curve) in terms of its semigroup of values. In order to arrive at a nice formula, we make the assumption that the characteristic is zero. We also compute the number of smooth Weierstrass points on a general rational curve with only unibranch singularities.

In the final section, we compute the Weierstrass weight of a singularity with precisely two branches (again assuming that the characteristic is zero). This depends heavily on the structure of the semigroup of values of the singularity. These semigroups have been studied by the first author [4] and by F. Delgado [1, 2], among others. In the course of this argument, we construct a basis for the dualizing differentials that is analogous to a (Hermitian) basis of regular differentials adapted to a point in the smooth case.

1. Let \(k\) denote an algebraically closed field of arbitrary characteristic. Let \(X\) be an integral, projective Gorenstein curve over \(k\) of arithmetic genus \(g > 0\).

(Gorenstein curves include any curve that is locally a complete intersection; so any curve that lies on a smooth surface is Gorenstein.) Let \(K\) denote the field of rational functions on \(X\). Let \(\pi : Y \to X\) denote the normalization of \(X\). Let \(\omega = \omega_X\) denote the sheaf of dualizing differentials on \(X\) and let \(\mathcal{O}_{X,P}\) denote the local ring of the structure sheaf \(\mathcal{O}_X\) at the point \(P \in X\). We recall (cf. [18]) that if \(P \in X\), then \(\omega_P\) consists of all rational differentials \(\tau\) on \(X\) such that

\[
\sum_{Q \to P} \text{Res}_Q(f\tau) = 0 \quad \text{for all } f \in \mathcal{O}_{X,P},
\]

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where the sum is over all points on $Y$ lying over $P$. Since $X$ is Gorenstein, $\omega$ is an invertible sheaf.

Let $\mathcal{L}$ be an invertible sheaf on $X$. Assume that $\mathcal{L}$ has nontrivial global sections and let $V \subseteq H^0(X, \mathcal{L})$ be a subspace of dimension $s > 0$. Choose a basis $\phi_1, \phi_2, \ldots, \phi_s$ of $V$. Suppose $\{U_\alpha\}$ is an open covering of $X$ such that $\mathcal{L}(U_\alpha)$ and $\omega(U_\alpha)$ are free $O_X(U_\alpha)$-modules generated by $\psi_\alpha$ and $\tau_\alpha$, respectively. Write $\phi_j|_{U_\alpha} = f_j \psi_\alpha$ for some $f_j \in O_X(U_\alpha)$ and $j = 1, 2, \ldots, s$. Then $f_1, f_2, \ldots, f_s$ are linearly independent rational functions over $k$.

Suppose $t$ is a separating element of $K$ over $k$ and let $D$ denote the iterative (or Hasse-Schmidt) derivative with respect to $t$. We recall that the higher derivatives $D^{(i)}$ satisfy the property that

$$D^{(i)}(\sum_j c_j t^j) = \sum_j \binom{j}{i} c_j t^{j-i},$$

where $c_j \in k$ for all $j$. F.K. Schmidt [16] showed that there exist integers $0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_{s-1}$, minimal with respect to lexicographic ordering of $s$-tuples, such that

$$\det (D^{(\epsilon_i)} f_j) \neq 0,$$

where $i = 0, 1, \ldots, s-1$ and $j = 1, 2, \ldots, s$. These integers are independent of the choices of $\alpha$, the separating element $t$, and the basis of $V$. The sequence $\epsilon_0, \epsilon_1, \ldots, \epsilon_{s-1}$ is called the order sequence of the linear system $V$, and one refers to each $\epsilon_i$ as a $V$-order. If $V$ is a base-point-free linear system, then the order sequence of $V$ is also referred to as the order sequence of the morphism from $X$ to $\mathbf{P}^{s-1}$ associated to $V$ (cf. [19]), where $\mathbf{P}^{s-1} = \mathbf{P}^{s-1}_k$ denotes the projective space of dimension $s-1$ over $k$. If the characteristic of $k$ is 0, then the order sequence is $0, 1, \ldots, s-1$. Put $N = \sum_{i=0}^{s-1} \epsilon_i$.

When the characteristic of $k$ is $p > 0$, F. K. Schmidt [16] (also see [19, Cor. 1.9]) showed that the order sequence of a linear system satisfies the following property.

(1.1) Proposition. Suppose $\epsilon$ is a $V$-order. Let $\mu$ be an integer such that

$$\left( \frac{\epsilon}{\mu} \right) \not\equiv 0 \pmod{p}.$$

Then $\mu$ is also a $V$-order.

We note that $\left( \frac{\epsilon}{\mu} \right) \not\equiv 0 \pmod{p}$ if and only if $\mu \geq 0$ and $\mu$ is $p$-adically smaller than $\epsilon$, which means that each coefficient in the $p$-adic expansion of $\mu$ is less than or equal to the the corresponding coefficient in the $p$-adic expansion of $\epsilon$.

(1.2) Definition. Let $p$ be a prime. We say that a finite sequence $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ of nonnegative integers satisfies the $p$-adic criterion if whenever $\mu$ is $p$-adically smaller than $\epsilon_i$, for some $i = 0, 1, \ldots, n$, then $\mu$ is a term in the sequence (i.e., if the sequence has the property in Proposition (1.1)).
(1.3) Proposition. If \( a_0 < a_1 < \cdots < a_n \) is a sequence that satisfies the \( p \)-adic criterion, then these integers are the orders of the morphism from \( \mathbb{P}^1 \) to \( \mathbb{P}^n \) defined by
\[
t \mapsto (t^{a_0} : t^{a_1} : \cdots : t^{a_n}).
\]

Proof. [16, Satz 7] (also see M. Homma [8]).

Let \( \mathcal{C} = \text{Ann}(\pi_*\mathcal{O}_Y/\mathcal{O}_X) = \text{Ann}(\omega_X/\pi_*\omega_Y) \) denote the conductor sheaf. The dualizing differential \( \tau_\alpha \) is of the form
\[
\tau_\alpha = \frac{dt}{h_\alpha},
\]
where \( t \) is some separating element for \( K \) over \( k \) and where \( h_\alpha \in \mathcal{C}(U_\alpha) \). Put
\[
\rho_\alpha = \det (h_\alpha^i D(\epsilon_j) f_j) \psi^s \tau_\alpha^N,
\]
where \( i = 0, 1, \ldots, s - 1 \) and \( j = 1, 2, \ldots, s \). We note that although \( D(\epsilon_i) f_j \) may not be in \( \mathcal{O}_X(U_\alpha) \), it is in \( \pi_*\mathcal{O}_Y(U_\alpha) \), and the product \( h_\alpha^i D(\epsilon_j) f_j \) is thus in \( \mathcal{O}_X(U_\alpha) \). Proceeding in this way on each \( U_\alpha \), we obtain functions \( \rho_\alpha \) and it is not hard to show, using properties of determinants as in [19, Proposition 1.4], that the \( \rho_\alpha \) patch to define a section \( \rho \in H^0(X, \mathcal{L}^s \otimes \omega^{\otimes N}) \), which we refer to as a wronskian.

If \( P \in X \) and if \( \psi \) generates \( \mathcal{L}_P \) and \( \tau \) generates \( \omega_P \), then we may write \( \rho = f \psi^s \tau^N \) for some nonzero \( f \in \mathcal{O}_P = \mathcal{O}_{X,P} \). We define \( \text{ord}_P \rho \) to be \( \text{ord}_P f = \dim \mathcal{O}_P/(f) \). This order of vanishing is independent of the choices of the basis of \( V \) and the generators for \( \mathcal{L}_P \) and \( \omega_P \).

(1.4) Definitions. Put \( W_V(P) = \text{ord}_P \rho \) and call this number the \( V \)-Weierstrass weight of \( P \). The point \( P \) is called a \( V \)-Weierstrass point if \( W_V(P) > 0 \). If \( V = H^0(X, \mathcal{L}) \), then we write \( W_\mathcal{L}(P) \) for \( W_V(P) \) and a \( V \)-Weierstrass point is called an \( \mathcal{L} \)-Weierstrass point. The Weierstrass points of \( X \) are the \( \omega \)-Weierstrass points. We will write \( W_X(P) \), or simply \( W(P) \) if it is clear to what curve we are referring, instead of \( W_\omega(P) \).

At any point \( P \in X \), one may consider the \( (V,P) \)-orders, where an integer \( \mu \) is a \( (V,P) \)-order if there exists \( f \in V \) such that \( \text{ord}_P f = \mu \). At a smooth point \( Q \) of \( X \), our definition of Weierstrass point restricts to the definitions in [17], [19], or [10], and one may consider the gaps at \( Q \), as usual. From [19, Theorem 1.5], we have that if the characteristic of \( k \) is \( p \) and if \( \epsilon_0(Q), \epsilon_1(Q), \ldots, \epsilon_{s-1}(Q) \) are the \( (V,Q) \)-orders, then
\[
W_V(Q) \geq \sum_{i=0}^{s-1} \epsilon_i(Q) - \epsilon_i,
\]
with equality holding if and if
\[
\det \left( \begin{array}{c} \epsilon_j(Q) \\ \epsilon_i \end{array} \right) \not\equiv 0 \pmod{p}.
\]
(Equality always holds in characteristic 0.)
(1.5) Proposition. The number of $V$-Weierstrass points, counting multiplicities, is $s \deg \mathcal{L} + (2g - 2)N$, where $N = \epsilon_0 + \epsilon_1 + \cdots + \epsilon_{s-1}$.

Proof. This number is the degree of $\mathcal{L}^\otimes s \otimes \omega^\otimes N$.

Let $\mathcal{O}$ denote the local ring at $P$ and let $\mathcal{O}$ denote the normalization of $\mathcal{O}$. Put $\delta = \delta_P = \dim \mathcal{O}/\mathcal{O}$. Suppose that $\tau \in H^0(X, \omega)$ generates $\omega_P$. Then, locally at $P$, we may write $\tau$ in the form $\tau = dt/h$, where $t$ is a rational function such that $\ord_{\mathcal{O}} t = 1$ for all $Q$ on $Y$ lying over $P$, and where $h$ is some generator (in $\mathcal{O}$) of the conductor of $\mathcal{O}$ in $\mathcal{O}$ (cf. [18]). Since $\mathcal{O}$ is a Gorenstein ring, we have $\ord_P h = 2\delta$.

(1.6) Proposition. $W_V(P) = 2\delta N + \ord_P \det (D^{(\epsilon_i)} f_j)$, where $i = 0, 1, \ldots, s-1$ and $j = 1, 2, \ldots, s$.

Proof. We have

$$W_V(P) = \ord_P \det (h^{\epsilon_i} D^{(\epsilon_i)} f_j)$$

$$= \ord_P h^{\delta N} + \ord_P \det (D^{(\epsilon_i)} f_j)$$

$$= 2\delta N + \ord_P \det (D^{(\epsilon_i)} f_j) \square$$

(1.7) Corollary. $W_V(P) \geq 2\delta N$. In particular, if $P$ is a singular point and if $s > 1$, then $P$ is a $V$-Weierstrass point.

Proof. If $s > 1$, then $N = \epsilon_0 + \epsilon_1 + \cdots + \epsilon_{s-1} \geq 0 + 1 = 1$. Hence $W_V(P) \geq 2\delta > 0$, since $P$ is singular. $\square$

2. As one might expect, phenomena can occur on Gorenstein curves in positive characteristic that do not occur on Gorenstein curves in characteristic zero. And there exist (singular) Gorenstein curves in positive characteristic that exhibit behavior that is not found on smooth curves. The following example serves to illustrate both of these points.

(2.1) Example. Suppose the characteristic of $k$ is 2. Let $X$ be the rational Gorenstein curve obtained from the projective line over $k$ by replacing the local ring $\mathcal{O}$ at 0 with the ring

$$\mathcal{O} = k + kt^3 + kt^4 + t^6 \mathcal{O},$$

where $t$ is a uniformizing parameter of $\mathcal{O}$ that generates the function field of $X$. Equivalently, $X$ is the plane quartic $y^3z = x^4$. Let $P$ denote the singular point of $X$.

We will find the Weierstrass points of $X$ (i.e., the $\omega$-Weierstrass points). It is easy to see that the rational differentials

$$\tau_1 = dt/t^6, \tau_2 = dt/t^3, \text{ and } \tau_3 = dt/t^2$$

form a basis of $H^0(X, \omega)$ and that $\tau_1$ generates $\omega_P$. Referring to the notation in §1, we have $f_1 = 1, f_2 = t^3, f_3 = t^4$, and $h = t^6$. Let $(D^{(i)} f_j)$ denote the triple whose components are the $i$-th iterative derivatives (with respect to $t$) of the functions $f_1, f_2$, and $f_3$. Then we have

$$(D^{(1)} f_j) = (0, t^2, 0)$$

$$(D^{(2)} f_j) = (0, t, 0)$$

$$(D^{(3)} f_j) = (0, 1, 0)$$

$$(D^{(4)} f_j) = (0, 0, 1)$$

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Therefore, the order sequence of $\omega$ is 0,1,4. The wronskian at $P$ is then

$$\det \begin{pmatrix} 1 & t^3 & t^4 \\ 0 & t^8 & 0 \\ 0 & 0 & t^{24} \end{pmatrix} = t^{32}.$$ 

It follows that $P$ has weight 32 and is the only Weierstrass point of $X$. All other points of $X$ have gap sequence 1,2,5. Thus $X$ is a nonclassical curve. (A curve of genus $g$ is called classical if the sequence of Weierstrass gaps at all but finitely many points of the curve is 1,2,\ldots,$g$.)

F.K. Schmidt [17] observed that there are no nonclassical smooth curves of genus 3 in characteristic 2, so this example shows that singular curves can exhibit behavior not found on smooth curves. Note that the one-parameter family of curves

$$\{X_u\} = y^3z + x^4 + uxz^3$$

over $k$ has the property that $X_u$ is nonsingular, and hence classical, if $u \neq 0$, but $X_0$ is nonclassical.

In characteristic zero, C. Widland [unpublished] has shown that every Gorenstein curve (of arithmetic genus at least two) must have at least two Weierstrass points. In particular, there cannot exist a single singular point that “uses up” all the Weierstrass weight as does the singularity in this example. This should not be too surprising though, since there also exist smooth curves in positive characteristic that have a single Weierstrass point (cf. [6]).

We can generalize the situation in the above example to obtain the following result, which describes the Weierstrass points on a rational curve with one unibranch singularity whose local ring is of a certain type. First, we recall some facts about numerical semigroups. A numerical semigroup is a subsemigroup of the nonnegative integers $\mathbb{N}$ that includes all but finitely many positive integers. The missing positive integers are called gaps. If $S$ is a numerical semigroup, then the conductor of $S$ is the least integer $c$ such that $c + \mathbb{N} \subseteq S$. A numerical semigroup $S$ with conductor $c$ is called symmetric if $m \in S$ if and only if $c - 1 - m \notin S$ for all integers $m$. If $S$ is symmetric and has $g$ gaps, then the conductor of $S$ is $2g$. E. Kunz [9] showed that a unibranch curve singularity is Gorenstein if and only if the semigroup of values associated to the singularity is a symmetric semigroup.

(2.2) Definition. Suppose $X$ is a rational curve and $P$ is a unibranch singularity on $X$ with semigroup of values $S$. Suppose the conductor of $S$ is $c$. Let $0,n_1,n_2,\ldots,n_r$ denote the nonnegative integers less than $c$ that are in $S$. We will call $P$ a monomial unibranch singularity if the local ring $\mathcal{O}$ at $P$ is of the form

$$\mathcal{O} = k + kt^{n_1} + kt^{n_2} + \cdots + kt^{n_r} + t^c\tilde{\mathcal{O}},$$

where $\tilde{\mathcal{O}}$ denotes the normalization of $\mathcal{O}$ and where $t$ is a uniformizing parameter of $\tilde{\mathcal{O}}$ that generates the function field of $X$. 

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(2.3) Theorem. Suppose \( S \) is a symmetric numerical semigroup with gaps \( l_1 = 1, l_2, \ldots, l_g \). Let \( n_0 = 0, n_1, \ldots, n_{g-1} \) be the nonnegative integers less than \( 2g \) that are in \( S \). Suppose \( X \) is a rational curve with a unique singular point \( P \) such that \( P \) is a unibranch singularity and the semigroup of values associated to \( P \) is \( S \). Denote by \( \epsilon_0 = 0, \epsilon_1 = 1, \epsilon_2, \ldots, \epsilon_{g-1} \) the orders of \( \omega \). The assertion about when the equality holds follows from [19, Theorem 1.5].

1) \( W(P) \geq \sum_{i=0}^{g-1} (n_i - \epsilon_i) + 2g \sum_{i=0}^{g-1} \epsilon_i \), and equality occurs if and only if \( \det \left( \binom{n_i}{\epsilon_i} \right) \neq 0 \) (mod \( p \)).

2) Suppose moreover that \( P \) is a monomial unibranch singularity with local ring

\[ \mathcal{O} = k + kt^{n_1} + \cdots + kt^{n_{g-1}} + t^{2g} \tilde{\mathcal{O}}. \]

Then

\[ W(P) = \sum_{i=0}^{g-1} (n_i - \epsilon_i) + 2g \sum_{i=0}^{g-1} \epsilon_i, \]

\[ W(P_{\infty}) = \sum_{i=0}^{g-1} (l_{i+1} - 1 - \epsilon_i), \]

where \( P_{\infty} \) represents the pole of the function \( t \), and there are no other Weierstrass points.

Proof. (1) Put \( c = 2g \). Suppose \( \sigma \in H^0(X, \omega) \). Locally at \( P \), write \( \sigma = dt/h \), where \( t \) is a local coordinate on the normalization of \( X \) centered at the point lying over \( P \). Suppose \( \text{ord}_P h = m + 1 \). If \( m \in S \), then there would be a function \( f \in \mathcal{O}_P \) such that \( \text{ord}_P f = m \); but then the residue of \( f \sigma \) would not be zero at 0. Therefore, \( m \) must be a gap of \( S \). It follows that there exist linearly independent dualizing differentials \( \tau_1, \tau_2, \ldots, \tau_g \) such that if we write \( \tau_j = dt/h_j \), then \( \text{ord}_P h_j = l_j + 1 \). Note that \( l_g + 1 = c \). For a generator of \( \omega_p \), we may take \( \tau = \tau_g = dt/h_g \). With notation as in \( \S 1 \), the functions \( f_1, f_2, \ldots, f_g \) used to construct the wronskian are, after renumbering,

\[ f_1 = 1, f_2 = h_g/h_{g-1}, \ldots, f_g = h_g/h_1. \]

Note that we have

\[ \text{ord}_P f_j = c - (l_{g-j+1} + 1) = n_{j-1} \quad \text{for} \quad j = 1, 2, \ldots, g, \]

since \( S \) is symmetric. At \( P \), the wronskian vanishes to order

\[ c(\epsilon_0 + \epsilon_1 + \cdots + \epsilon_{g-1}) + \text{ord}_P \det(D^{(\epsilon_i)}f_j) \geq \]

\[ 2g \sum_{i=0}^{g-1} \epsilon_i + \sum_{i=0}^{g-1} (n_i - \epsilon_i). \]

The assertion about when the equality holds follows from [19, Theorem 1.5].

(2) From the assumption that \( P \) is a monomial unibranch singularity, it follows that \( dt/t^{l+1} \) is in fact a dualizing differential, so we may take \( \tau_j = dt/t^{l+1} \) and \( \tau = dt/t^c \) generates \( \omega_P \). At \( P_{\infty} \), a local coordinate is \( u = 1/t \) and \( \tau_1 = -du \) generates \( \omega_{P_{\infty}} \). Therefore,
at $P_\infty$, our basis of dualizing differentials can be written $\tau_j = u^{l_j - 1}\tau_1$ for $j = 1, 2, \ldots, g$. Hence the Weierstrass gap sequence at $P_\infty$ is $l_1, l_2, \ldots, l_g$. Thus, we have

$$\textstyle W(P_\infty) \geq \sum_{i=0}^{g-1} (l_{i+1} - 1 - \epsilon_i).$$  (**)

Now, if we add the right sides of equations (*) and (**) and use the fact that $n_i + l_{g-i} = 2g - 1$ for each $i = 0, 1, \ldots, g - 1$, then we obtain

$$\textstyle 2g \sum_{i=0}^{g-1} \epsilon_i + \sum_{i=0}^{g-1} (n_i - \epsilon_i) + \sum_{i=0}^{g-1} (l_{i+1} - 1 - \epsilon_i)$$

$$\textstyle = \sum_{i=0}^{g-1} (n_i + l_{g-i}) - g + (2g - 2) \sum_{i=0}^{g-1} \epsilon_i$$

$$\textstyle = g(2g - 1) - g + (2g - 2) \sum_{i=0}^{g-1} \epsilon_i$$

$$\textstyle = (\sum_{i=1}^{g-1} \epsilon_i + g)(2g - 2).$$

Since this is the total Weierstrass weight of all Weierstrass points of $\omega$, we must have equality in (*) and in (**) and the Theorem follows.

Notice that, for $X$ as in part (2) of Theorem (2.3), the symmetric semigroup $S$ is the Weierstrass semigroup of nongaps at the point $P_\infty$. Thus, as was previously noted (in characteristic zero) by K.-O. Stöhr [20], it is easy to see that every numerical symmetric semigroup occurs as the Weierstrass semigroup at a point on some Gorenstein curve. It is still unknown if every such semigroup occurs as the Weierstrass semigroup at a point on some nonsingular curve.

(2.4) Corollary. Suppose that $X$ is as in part (2) of Theorem (2.3). If the characteristic of $k$ does not divide the integer

$$\textstyle \prod_{i>j}(l_i - l_j)/(i - j),$$

then $X$ is classical.

Proof. The $\omega$-orders at $P_\infty$ are $l_1 - 1, l_2 - 1, \ldots, l_g - 1$. The Corollary then follows from the proof of Corollary 1.7 in [19].

In order to completely characterize the rational curves with one monomial unibranch singularity that uses up all the Weierstrass weight, we need the following result. For the remainder of this section we assume that $k$ has characteristic $p > 0$. 7
(2.5) Proposition. Suppose $V$ is a linear system on $X$ of (affine) dimension $s$. Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_{s-1}$ be the order sequence of $V$. Suppose $Q \in X$ is a smooth point and let

$$0 = \epsilon_0(Q) < \epsilon_1(Q) < \cdots < \epsilon_{s-1}(Q)$$

be the $(V, Q)$-orders. Then the following are equivalent.

(i) The $V$-Weierstrass weight of $Q$ is $\sum_{i=0}^{s-1} (\epsilon_i(Q) - \epsilon_i)$.

(ii) The sequence $\epsilon_0, \epsilon_1, \ldots, \epsilon_{s-1}$ is the minimal sequence, in the lexicographic order, such that

$$\det \left( \begin{pmatrix} \epsilon_j(Q) \\ \epsilon_i \end{pmatrix} \right) \not\equiv 0 \pmod{p}.$$ 

(iii) The orders of the morphism from $\mathbb{P}^1$ to $\mathbb{P}^{s-1}$ defined by

$$t \mapsto (1 : t^{\epsilon_1(Q)} : t^{\epsilon_2(Q)} : \cdots : t^{\epsilon_{s-1}(Q)})$$

are $\epsilon_0, \epsilon_1, \ldots, \epsilon_{s-1}$.

(iv) The sequence $\epsilon_0, \epsilon_1, \ldots, \epsilon_{s-1}$ is the minimal sequence, in the lexicographic order, such that

$$\det \left( \begin{pmatrix} \epsilon_{s-1}(Q) - \epsilon_{s-1-j}(Q) \\ \epsilon_i \end{pmatrix} \right) \not\equiv 0 \pmod{p}.$$

Proof. The equivalence of (i) and (ii) follows from [19, Theorem 1.5 and Proposition 1.6]. The equivalence of (ii) and (iii) follows from [19, Proposition 1.6 and subsequent Remark]. To see that (iv) is equivalent to the other statements, first note that if $t \neq 0$, then

$$(1 : t^{\epsilon_1(Q)} : t^{\epsilon_2(Q)} : \cdots : t^{\epsilon_{s-1}(Q)}) = (1 : (1/t)^{-\epsilon_1(Q)} : (1/t)^{-\epsilon_2(Q)} : \cdots : (1/t)^{-\epsilon_{s-1}(Q)})$$

$$=((1/t)_{s-1}(Q) : (1/t)_{s-1}(Q)-\epsilon_1(Q) : \cdots : 1).$$

Thus, the orders of the morphism in (iii) are the same as the orders of the morphism from $\mathbb{P}^1$ to $\mathbb{P}^{s-1}$ defined by

$$1/t \mapsto (1 : (1/t)_{s-1}(Q)-\epsilon_{s-2}(Q) : \cdots : (1/t)_{s-1}(Q)-\epsilon_1(Q) : (1/t)_{s-1}(Q)).$$

The equivalence of (iv) with the other statements now follows from the equivalence of (ii) and (iii). \]

As one consequence of Proposition (2.5), we obtain the following general corollary, which is a new result in the theory of Weierstrass points in positive characteristic.
(2.6) Corollary. Suppose $Q$ is a smooth $V$-Weierstrass point of $X$ with the $(V,Q)$-orders being $\epsilon_i(Q), i = 0, 1, \ldots, s-1$. If the sequence $\epsilon_0(Q), \epsilon_1(Q), \ldots, \epsilon_{s-1}(Q)$ satisfies the $p$-adic criterion, then

$$W_V(Q) > \sum_{i=0}^{s-1} (\epsilon_i(Q) - \epsilon_i),$$

where $\epsilon_i, i = 0, 1, \ldots, s-1$ are the orders of $V$.

Proof. It follows from Propositions (1.3) and (2.5) that if

$$W_V(Q) = \sum_{i=0}^{s-1} (\epsilon_i(Q) - \epsilon_i),$$

then the $\epsilon_i(Q)$ would be the orders of $V$. But then $Q$ would not be a $V$-Weierstrass point.

(2.7) Corollary. Let $X$ be a curve as in part (2) of Theorem (2.3). Then the $\omega$-orders $\epsilon_0, \epsilon_1, \ldots, \epsilon_{g-1}$ are minimal, in the lexicographic order, such that

$$\det \left( \begin{pmatrix} l_{j+1} - 1 \\ \epsilon_i \end{pmatrix} \right) \not\equiv 0 \pmod{p}.$$

Proof. In the proof of Theorem (2.3), it was shown that $\epsilon_i(P_\infty) = l_i + 1 - 1$ for $i = 0, 1, \ldots, g-1$. The Corollary then follows from Theorem (2.3) and Proposition (2.5), taking $V = H^0(X, \omega)$ and $Q = P_\infty$.

Finally, we obtain a characterization of rational curves with a single monomial unibranch singularity that uses up all the Weierstrass weight.

(2.8) Corollary. Let $X$ be a curve as in part (2) of Theorem (2.3). Then the singularity $P$ uses up all the Weierstrass weight if and only if the sequence

$$l_1 - 1, l_2 - 1, \ldots, l_g - 1$$

satisfies the $p$-adic criterion.

Proof. By Theorem (2.3), $P$ uses up all the Weierstrass weight if and only if $W(P_\infty) = 0$; i.e., if and only if $l_{i+1} - 1$ are the orders of $\omega$. If these are the orders of $\omega$, then, by Proposition (1.1), they satisfy the $p$-adic criterion. Conversely, if the sequence $l_1 - 1, l_2 - 1, \ldots, l_g - 1$ satisfies the $p$-adic criterion, then by Proposition (1.3) these integers are the orders of the morphism from $\mathbb{P}^1$ to $\mathbb{P}^{g-1}$ defined by

$$t \mapsto (1 : t^{l_2 - 1} : \ldots : t^{l_g - 1}).$$

Therefore, from Proposition (2.5), these integers must be the orders of $\omega$ and the weight of $P_\infty$ is 0.

Three other examples of curves as in part (2) of Theorem (2.3) such that the singularity uses up all the Weierstrass weight are:
(1) \( p = 2, g = 7, S = \langle 4, 6, 11 \rangle = 4N + 6N + 11N. \)
(2) \( p = 3, g = 4, S = \langle 3, 5 \rangle = 3N + 5N. \)
(3) \( p = 5, g = 6, S = \langle 4, 5 \rangle = 4N + 5N. \)

We conclude this section with an example of a rational curve with a single (nonmonomial) unibranch singularity and two smooth Weierstrass points.

**Example.** Let \( X \) be the rational Gorenstein curve obtained from the projective line over \( k \) by replacing the local ring \( \hat{O} \) at 0 with the ring
\[
\mathcal{O} = k + k(t^3 + t^5) + kt^4 + t^6\hat{O},
\]
where \( t \) is a uniformizing parameter of \( \hat{O} \) that generates the function field of \( X \). Let \( P \) denote the singular point of \( X \). The semigroup of values at \( P \) is \( 3N + 5N \), the same as the semigroup of values in Example (2.1). But notice here that \( dt/t^6 \) is not a dualizing differential since
\[
\text{Res}_0(t^3 + t^5) \frac{dt}{t^6} = 1.
\]
It is not hard to see that a basis for \( H^0(X, \omega) \) is
\[
\tau_1 = (1 - t^2)dt/t^6, \tau_2 = dt/t^2, \tau_3 = dt/t^3
\]
and \( \tau_1 \) generates \( \omega_P \). One may check easily that the order sequence at infinity is 0, 1, 2, so \( X \) is classical (in all characteristics).

If the characteristic is not 2, then the wronskian, on the open subset of \( X \) obtained by excluding the point at infinity and the zeros of \( 1 - t^2 \), is
\[
\frac{t^{22}(6 - t^2)}{(1 - t^2)^6},
\]
up to a nonzero constant. Thus, if the characteristic is not 2 or 3, \( P \) has weight 22 and there are two weight one Weierstrass points at the two square roots of 6. If the characteristic is 3, then \( P \) has weight 24 and is the unique Weierstrass point of \( X \). A computation of the wronskian in characteristic 2 shows that \( P \) has weight 24 in this case as well.

3. In this section, we will consider rational curves with several unibranch singularities. We will assume for the remainder of this article that \( k \) has characteristic 0. We do this so that our formulas are not overly complicated. One can modify these formulas, taking into account the order sequences of the canonical bundles involved, and obtain analogous results in positive characteristic.

**Definition.** If \( S \) is a numerical semigroup with gaps \( l_1, l_2, \ldots, l_\delta \), then we define the **weight of \( S \)**, denoted \( wt(S) \), by
\[
wt(S) = \sum_{j=1}^{\delta} (l_j - j).
\]
We define the weight of \( N \) to be 0.

**Remark.** We always have \( l_\delta \leq 2\delta - 1 \) (and the equality occurs if and only if \( S \) is symmetric). To see this, note that if \( c \) is the conductor of \( S \) and if \( x + y = c - 1 = l_\delta \), then either \( x \) or \( y \) must be a gap. Hence among the nonnegative integers less than \( c \), there are at least as many gaps as there are nongaps.
Lemma. If $S$ is a numerical semigroup with $\delta$ gaps and if $0 = n_0, n_1, \ldots, n_{\delta} - 1$ are the elements of $S$ less than $2\delta$, then

$$\sum_{i=0}^{\delta-1} (n_i - i) = (\delta - 1)\delta - \text{wt}(S).$$

Proof. Let $l_1, l_2, \ldots, l_\delta$ denote the gaps of $S$. We have

$$\sum_{j=1}^{\delta} l_j + \sum_{i=0}^{\delta-1} n_i = \sum_{k=0}^{2\delta-1} k = \delta(2\delta - 1).$$

Hence

$$\text{wt}(S) = \sum_{j=1}^{\delta} l_j - \delta(\delta + 1)/2$$

$$= \delta(2\delta - 1) - \sum_{i=0}^{\delta-1} n_i - \delta(\delta + 1)/2$$

$$= \delta(2\delta - 1) - \sum_{i=0}^{\delta-1} n_i - (\delta^2 - (\delta - 1)\delta)/2$$

$$= (\delta - 1)\delta - \sum_{i=0}^{\delta-1} (n_i - i).$$

The next theorem treats the weight of a unibranch singularity on an arbitrary (in particular, not necessarily rational) Gorenstein curve. This theorem generalizes a result of C. Widland [21] in the case of a simple cusp.

We recall that if $P$ is a singular point of $X$, then by the partial normalization of $X$ at $P$ one means the curve obtained from $X$ by desingularizing only the singularity $P$.

Theorem. Suppose $X$ is a (not necessarily rational) Gorenstein curve of arithmetic genus $g$. Suppose $P \in X$ is a unibranch singularity. Put $\delta = \delta_P$. Let $S$ denote the semigroup of values at $P$. Let $Y$ denote the partial normalization of $X$ at $P$ and let $Q$ denote the point of $Y$ that lies over $P$. Then

$$W_X(P) = \delta(g - 1)(g + 1) - \text{wt}(S) + W_Y(Q).$$

Proof. Let $\sigma_1, \sigma_2, \ldots, \sigma_{g-\delta}$ be a basis of $H^0(Y, \omega_Y)$. Locally at $Q$, write $\sigma_i = f_i \, dt$, for $i = 1, \ldots, g - \delta$, where $t$ is a local coordinate on $Y$ centered at $Q$. Put $r_i = \text{ord}_Q f_i$. We will assume that the basis of differentials has been chosen so that $r_1 < r_2 < \cdots < r_{g-\delta}$.

Dualizing differentials on $Y$ are also dualizing differentials on $X$, so we may extend the above differentials to a basis

$$\tau_1, \tau_2, \ldots, \tau_\delta, \sigma_1, \sigma_2, \ldots, \sigma_{g-\delta}.$$
of $H^0(X,\omega_X)$. Let $l_1, l_2, \ldots, l_\delta$ denote the gaps of the semigroup of values at $P$. As in Theorem (2.3), we may assume that, locally at $P$, we have $\tau_i = dt/h_i$, where $\text{ord}_P h_i = l_i + 1$ for $i = 1, 2, \ldots, \delta$. Since the semigroup of values at $P$ is symmetric, we have $l_\delta + 1 = 2\delta = c$, the conductor of the semigroup of values. The differential $\tau_\delta$ generates $\omega_X, P$. Put $h = h_\delta$.

Since we are assuming the characteristic is 0, the orders of $\omega_X$ are $0, 1, \ldots, g - 1$ and the sum of the orders is $N = (g - 1)g/2$. As in the proof of Proposition (1.6), we have

$$W_X(P) = \delta(g - 1)g + \text{ord}_Q W_t(1, h/h_\delta - 1, h/h_\delta - 2, \ldots, h/h_1, h f_1, h f_2, \ldots, h f_{g-\delta}),$$

where $W_t$ denotes the ordinary wronskian (obtained by differentiating with respect to $t$) of the given functions and where we have used the fact that the order of a function in $\mathcal{O}_P$ is the same at $P$ as it is at $Q$. Notice that each of the functions $1, h/h_\delta - 1, \ldots, h/h_1, h f_1, \ldots, h f_{g-\delta}$ has a different order at $Q$. Indeed, we have $\text{ord}_Q h/h_\delta - i = c - (l_\delta - i + 1) = n_i$, where $n_0, n_1, \ldots, n_{\delta - 1}$ are the elements in $S$ that are less than $2\delta$, and $\text{ord}_Q h f_j = c + r_j = 2\delta + r_j$. Hence the order of the wronskian of these functions at $Q$ may be easily computed as in [3, p.82]. We have

$$\text{ord}_Q W_t(1, h/h_\delta - 1, \ldots, h f_{g-\delta}) = \sum_{i=0}^{\delta - 1} (n_i - i) + \sum_{j=1}^{g-\delta} (2\delta + r_j - (\delta + j - 1))$$

$$= \sum_{i=0}^{\delta - 1} (n_i - i) + \delta (g - \delta) + \sum_{j=1}^{g-\delta} r_j - j + 1$$

$$= \sum_{i=0}^{\delta - 1} (n_i - i) + \delta (g - \delta) + W_Y(Q).$$

The Theorem now follows from Lemma (3.3).}

Using Theorems (2.3) and (3.4), one may compute the weight of each singularity on a rational curve with precisely two monomial unibranch singularities.

(3.5) Theorem. Suppose $X$ is a rational Gorenstein curve of arithmetic genus $g$ with two monomial unibranch singularities $P_1$ and $P_2$. For $i = 1, 2$ put $\delta_i = \delta_{P_i}$, let $S_i$ denote the semigroup of values at $P_i$, and let $t_i$ denote the uniformizing parameter at $P_i$ that generates the function field of $X$ used to define the local ring at $P_i$ as in Definition (2.2). Let $Q_1$ and $Q_2$ be the points on $\mathbb{P}^1$ lying over $P_1$ and $P_2$, respectively. Then

1) If $Q_1$ and $Q_2$ are the poles of the functions $t_2$ and $t_1$, respectively (i.e., if $t_1 t_2$ is a nonzero constant), then

$$W_X(P_1) = \delta_1 (g - 1)(g + 1) - \text{wt}(S_1) + \text{wt}(S_2)$$

$$W_X(P_2) = \delta_2 (g - 1)(g + 1) - \text{wt}(S_2) + \text{wt}(S_1),$$

and there are no smooth Weierstrass points on $X$.

2) If $Q_1$ is the pole of the function $t_2$, but $Q_2$ is not the pole of the function $t_1$, then

$$W_X(P_1) = \delta_1 (g - 1)(g + 1) - \text{wt}(S_1) + \text{wt}(S_2)$$

$$W_X(P_2) = \delta_2 (g - 1)(g + 1) - \text{wt}(S_2),$$
and there are $wt(S_1)$ smooth Weierstrass points on $X$, counting multiplicities.  

3) If $Q_1$ and $Q_2$ are not the poles of the functions $t_2$ and $t_1$, respectively, then

$$W_X(P_1) = \delta_1(g - 1)(g + 1) - wt(S_1)$$
$$W_X(P_2) = \delta_2(g - 1)(g + 1) - wt(S_2),$$

and there are $wt(S_1) + wt(S_2)$ smooth Weierstrass points on $X$, counting multiplicities.

**Proof.** By Theorem (3.4) we have

$$W_X(P_1) = \delta_1(g - 1)(g + 1) - wt(S_1) + W_{Y_1}(Q_1),$$

where $Y_1$ is the partial normalization of $X$ at $P_1$ and $Q_1$ is the point on $Y_1$ that lies over $P_1$. Now, $Y_1$ is a rational curve with the unique monomial unibranch singularity $P_2$. Hence we see from Theorem (2.3) that $W_{Y_1}(Q_1) = wt(S_2)$ if $Q_1$ is the pole of the function $t_2$ and is 0 otherwise. A similar argument holds with regard to $W_X(P_2)$.

The assertions about the number of smooth Weierstrass points on $X$ follow by adding the weights of $P_1$ and $P_2$ and subtracting from $g^3 - g$, which is the total of all the weights. Note that $g = \delta_1 + \delta_2$.

**(3.6) Example.** Suppose that $X$ is the rational curve obtained from $\mathbb{P}^1$ by creating two monomial unibranch singularities $P_0$ and $P_1$, each with semigroup of values generated by 3 and 4, at 0 and 1. Then $X$ has arithmetic genus 6 and the Weierstrass weight of each singularity is 103. The total Weierstrass weight is 210, and it may be seen, by computing the wronskian on the smooth locus of $X$, that there are four distinct smooth Weierstrass points (each of weight one). We note that the point at infinity is not a Weierstrass point on $X$, but it is a Weierstrass point on $Y_1$, the partial normalization of $X$ at $P_1$. The existence of a function with a zero of order 3 at $P_0$ shows that 3 is a nongap at infinity on the curve $Y_1$, but this function is not regular at $P_1$ on $X$ and 3 is not a nongap at infinity on $X$.

The situation in part (3) of Theorem (3.5) may be generalized as follows.

**(3.7) Theorem.** Suppose that $X$ is a rational Gorenstein curve of arithmetic genus $g$ with unibranch singularities $P_1, P_2, \ldots, P_n$ as its only singularities. For $i = 1, 2, \ldots, n$, let $S_i$ denote the semigroup of values at $P_i$, and assume that the point $Q_i$ that lies over $P_i$ is not a Weierstrass point of the partial normalization $Y_i$ of $X$ at $P_i$. Then the number of smooth Weierstrass points on $X$, counting multiplicities, is $\sum_{i=1}^n wt(S_i)$.

**Proof.** Since by hypothesis $W_{Y_i}(Q_i) = 0$, it follows from Theorem (3.4) that the number of smooth Weierstrass points on $X$, counting multiplicities, is

$$g^3 - g - \sum_{i=1}^n W_X(P_i) = g^3 - g - \sum_{i=1}^n (\delta_i(g - 1)(g + 1) - wt(S_i))$$
$$= \sum_{i=0}^n wt(S_i),$$

since $g = \sum_{i=0}^n \delta_i$. 

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4. We now consider singularities with two branches. Suppose $X$ is an integral, projective Gorenstein curve over an algebraically closed field of characteristic zero. Suppose $P \in X$ is a singularity with two branches. Let $\pi : Y \to X$ denote the partial normalization of $X$ at $P$ and let $Q_1, Q_2 \in Y$ denote the points lying over $P$. Let $\nu_1$ and $\nu_2$ denote the discrete valuations associated to $Q_1$ and $Q_2$, respectively. The value semigroup $S$ at $P$ is given by

$$S = \{(\nu_1(f), \nu_2(f)) \in \mathbb{N} \times \mathbb{N} : f \in \mathcal{O}_P, f \neq 0\}.$$  

Such semigroups have been studied by, among others, the first author [4] in the case of plane curves and by F. Delgado in the cases of plane curves [1] and Gorenstein curves [2] with an arbitrary number of branches. Let $\xi = (\xi_1, \xi_2)$ denote the conductor of $S$; i.e., $\xi$ is the minimum element in $S$ with respect to the product order on $\mathbb{N} \times \mathbb{N}$, such that $\xi + \mathbb{N} \times \mathbb{N} \subseteq S$.

We think of the semigroup $S$ as being a set of points in the plane. Let $S_i = \pi_i(S)$, for $i = 1, 2$, denote the projections of $S$ onto the coordinate axes. For $i = 1, 2$, let $\delta_i$ denote the number of gaps of $S_i$ and let $c_i$ denote the conductor of $S_i$. We note that $S_1$ and $S_2$ need not be symmetric semigroups (see Example (4.10) below).

Since $\mathcal{O} = \mathcal{O}_P$ is a Gorenstein ring, the semigroup $S$ also has certain symmetry properties, which we now recall.

(4.1) Definitions. For $x \in \mathbb{N}$, the vertical fiber at $x$, denoted $\text{VF}(x)$, is defined by

$$\text{VF}(x) = \{(x, y') \in \mathbb{N} \times \mathbb{N} : (x, y') \in S\}.$$  

For $y \in \mathbb{N}$, the horizontal fiber at $y$, denoted $\text{HF}(y)$, is defined similarly. A point $(x_1, y_1)$ is said to be above (resp. to the right of) another point $(x_2, y_2)$ if $x_1 = x_2$ and $y_1 > y_2$ (resp. if $y_1 = y_2$ and $x_1 > x_2$). Put

$$\Delta((x, y)) = \{(x', y') \in S : (x', y') \text{ is either above or to the right of } (x, y)\}.$$  

A point $(x, y) \in S$ is called a maximal point (or simply a maximal) if $\Delta((x, y)) = \emptyset$.

(4.2) Lemma.

1) Suppose $n \in S_1$ and $n < \xi_1$. Then

$$(n, \xi_2) \in S \Leftrightarrow \xi_1 - 1 - n \notin S_1 \Leftrightarrow \text{VF}(n) \text{ is infinite}.$$  

2) Suppose $n \in S_2$ and $n < \xi_2$. Then

$$(\xi_1, n) \in S \Leftrightarrow \xi_2 - 1 - n \notin S_2 \Leftrightarrow \text{HF}(n) \text{ is infinite}.$$  

Proof. [2, Lemma (1.8) and Theorem (2.3)]

Put $\mu = (\xi_1 - 1, \xi_2 - 1)$. From [2, Corollary (2.7)], we have that $\mu$ is a maximal point in $S$. This point plays a role in $S$ analogous to the number $c - 1$ in a symmetric numerical semigroup. More precisely, one has the following result.
(4.3) Proposition. (Symmetry properties of $S$). The semigroup $S$ has the following symmetry properties.

1) For any $(x, y) \in \mathbb{Z} \times \mathbb{Z}$,

$$(x, y) \in S \leftrightarrow \Delta(\mu - (x, y)) = \emptyset.$$ 

2) For any $(x, y) \in \mathbb{N} \times \mathbb{N}$,

$(x, y)$ is a maximal of $S \Leftrightarrow \mu - (x, y)$ is a maximal of $S$.

Proof. Delgado [2, Theorem (2.8)] establishes the first property above and notes ([2, Remark (2.9)]) that the second property also holds.

(4.4) Lemma.

1) Suppose $n < \xi_1$. Then $VF(n)$ is infinite if and only if $\xi_1 - 1 - n$ is a gap of $S_1$.
2) Suppose $n < \xi_2$. Then $HF(n)$ is infinite if and only if $\xi_2 - 1 - n$ is a gap of $S_2$.

Proof. Suppose $VF(n)$ is infinite, with $n < \xi_1$. Then there exists a point $(n, y) \in S$ with $y > \xi_2$. By adding the function corresponding to this point with the function corresponding to $(\xi_1, \xi_2)$, we see that $(n, \xi_2) \in S$. Therefore, by Lemma (4.2), we have that $\xi_1 - 1 - n \notin S_1$. Conversely, if $n < \xi_1$ and $\xi_1 - 1 - n \notin S_1$, then we claim that $n \in S_1$. For consider the point $\alpha = (n, \xi_2 - c_2)$. Then $\mu - \alpha = (\xi_1 - 1 - n, c_2 - 1)$. But $\xi_1 - 1 - n \notin S_1$ and $c_2 - 1 \notin S_2$, so $\Delta(\mu - \alpha) = \emptyset$. It follows from Proposition (4.3) that $\alpha \in S$ and so $n \in S_1$. Hence, if $n < \xi_1$ and $\xi_1 - 1 - n \notin S_1$, we can conclude from Lemma (4.2) that $VF(n)$ is infinite.

The proof of (2) is similar.

(4.5) Proposition. The symmetry properties in Proposition (4.3) are equivalent.

Proof. (1) $\Rightarrow$ (2): By the symmetrical form of statement (2) in Proposition (4.3), it suffices to show that if $(x, y)$ is a maximal of $S$, then $\mu - (x, y)$ is also a maximal of $S$. Now, if $(x, y)$ is a maximal of $S$, then $(x, y) \in S$ and $\Delta((x, y)) = \emptyset$. But then, by applying (1) of Proposition (4.3) in both directions, we see that $\Delta(\mu - (x, y)) = \emptyset$ and $\mu - (x, y) \in S$. Therefore, $\mu - (x, y)$ is a maximal of $S$.

(2) $\Rightarrow$ (1): Assume $(x, y) \in S$. Suppose there exists a point in $S$ above $\mu - (x, y)$. (A similar argument applies if there exists a point in $S$ to the right of $\mu - (x, y)$.) But then $(x, y) + (\mu_1 - x, \mu_2 - y + z)$, for some $z > 0$, is in $S$, contradicting the fact that $\mu$ is a maximal point.

Conversely, suppose that $\Delta(\mu - (x, y)) = \emptyset$. Since $VF(\mu_1 - x)$ and $HF(\mu_2 - y)$ are then finite, it follows, from Lemma (4.4), that $x \in S_1$ and $y \in S_2$. We have then two possibilities: either $VF(\mu_1 - x)$ is empty or nonempty. If $VF(\mu_1 - x)$ is empty, it follows from Lemma (4.4) that $VF(x)$ is infinite. If $V(x)$ is nonempty, then the maximal point of this fiber, call it $(\mu_1 - x, z)$, satisfies $z \leq \mu_2 - y$. Hence, applying (2) of Proposition (4.3), we have a maximal point of the form $(x, \mu_2 - z)$ with $\mu_2 - z \geq y$. So, in any case, one has a point in the semigroup $S$ of the form $(x, y')$ with $y' \geq y$. Similarly, one has a
point in $S$ of the form $(x', y)$ with $x' \geq x$. If $y' = y$ or $x' = x$, we are finished. We can then assume that $y' > y$ and $x' > x$. Then the sum of the functions in the local ring $\mathcal{O}_P$ corresponding to $(x', y)$ and $(x, y')$ is a function $f$ satisfying $\nu_1(f) = x$ and $\nu_2(f) = y$, showing that $(x, y) \in S$.

Put $I$ equal to the number of maximal points in $S$. From [4], we have that if $X$ is a plane curve, then $I$ is also equal to the intersection number of the two branches and the conductor of $S$ is $(I + 2\delta_1, I + 2\delta_2)$. We now show that these results also hold for any Gorenstein curve.

**4.6 Proposition.** The coordinates of the conductor of $S$ are $\xi_1 = I + 2\delta_1, \xi_2 = I + 2\delta_2$.

**Proof.** Consider the vertical fibers $\text{VF}(x)$ for $0 \leq x < \xi_1$. We will count how many of these fibers are infinite, empty, or finite and nonempty. From Lemma (4.4), we see that the number of these vertical fibers that are infinite is $\delta_1$. The number of empty vertical fibers is also equal to $\delta_1$. The number of nonempty finite fibers is equal to $I$, the number of maximal points. Therefore, $\xi_1 = I + 2\delta_1$. A similar argument using horizontal fibers shows that $\xi_2 = I + 2\delta_2$.

**4.7 Corollary.** $\delta = \delta_P = I + \delta_1 + \delta_2$ and $I$ is the intersection number of the two branches at $P$.

**Proof.** From Proposition (4.6) and the fact that $\mathcal{O}_P$ is Gorenstein, we have $2\delta_P = 2I + 2\delta_1 + 2\delta_2$. Therefore, $\delta_P = I + \delta_1 + \delta_2$. It then follows from [7, Proposition 4] that $I$ is the intersection number of the two branches at $P$.

**4.8 Corollary.** Suppose the maximal points of $S$ are $(a_0, b_0), (a_1, b_1), \ldots, (a_{I-1}, b_{I-1})$.

Then we have

$$\sum_{i=0}^{I-1} a_i = I(I - 1)/2 + \delta_1 I$$

$$\sum_{i=0}^{I-1} b_i = I(I - 1)/2 + \delta_2 I.$$ 

**Proof.** By Propositions (4.3) and (4.6), $a_i$ is the first coordinate of a maximal point if and only if $I + 2\delta_1 - 1 - a_i$ is also the first coordinate of a maximal. Hence we have

$$\sum_{i=0}^{I-1} a_i = I(I + 2\delta_1 - 1) - \sum_{i=0}^{I-1} a_i,$$

and the first equality in the statement of the Corollary follows. A similar argument applies to the second coordinates of the maximal points.

Consider the rectangle (with one vertex deleted)

$$R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x \leq \xi_1 \text{ and } y < \xi_2 \text{ or } x < \xi_1 \text{ and } y \leq \xi_2\}.$$
It follows from Lemma (4.4) that the points in $S$ that are on the top edge of $R$ are of the form $(\xi_1 - l, \xi_2)$, where $l$ is a gap of $S_1$, and the points in $S$ that are on the right edge of $R$ are of the form $(\xi_1, \xi_2 - l')$, where $l'$ is a gap of $S_2$. If $S_1$ and $S_2$ are symmetric, then one can write these points in a nicer form.

(4.9) Corollary.

1) If $S_1$ is symmetric and if $m_0, m_1, \ldots, m_{\delta_1 - 1}$ are the elements in $S_1$ that are less than $c_1$, then the points that are in $S$ and on the upper edge of the rectangle $R$ are the points $(I + m_j, \xi_2)$, $j = 0, 1, \ldots, \delta_1 - 1$.

2) If $S_2$ is symmetric and if $n_0, n_1, \ldots, n_{\delta_2 - 1}$ are the elements in $S_2$ that are less than $c_2$, then the points that are in $S$ and on the right edge of $R$ are the points $(\xi_1, I + n_k)$, $k = 0, 1, \ldots, \delta_2 - 1$.

Proof. Suppose $S_1$ is symmetric. Then $c_1 = 2\delta_1$ and $c_1 - 1 - n \in S_1$ if and only if $n \notin S_1$. Therefore, from Lemma (4.4) and Proposition (4.6),

$$n \notin S_1 \iff \text{VF}(I + c_1 - 1 - n) \text{ is infinite.}$$

Thus, the infinite vertical fibers are of the form $\text{VF}(I + m)$, where $m \in S_1$. The analogous statement for $S_2$ is proved similarly.

We thank Professor K.-O. Stöhr for the following example of a two-branch Gorenstein singularity having one branch that is not Gorenstein.

(4.10) Example. Let $H$ be a numerical semigroup with $g$ gaps $l_1, l_2, \ldots, l_g$ such that $l_g = 2g - 2$. Clearly, $(g - 1) \notin H$. Take $S_1 = (g - 1)\mathbb{N} + H$. We claim that

$$S_1 = H \cup \{g - 1, 2g - 2\};$$

i.e., that $S_1$ has $g - 2$ gaps. In fact, if $\alpha$ belongs to $S_1$ but not to $H \cup \{g - 1, 2g - 2\}$, then we can write $\alpha = (g - 1) + h$, for some $h \in H$. Since $\alpha \notin g - 1$ and $\alpha$ is a gap of $H$ we have, from [15, Prop. 1.2], that

$$2g - 2 - \alpha = g - 1 - h = h_1 \in H.$$

Hence $g - 1 = h + h_1 \in H$, a contradiction.

Take $\mathcal{O} \subseteq \tilde{\mathcal{O}} = k[[t]] \times k[[u]]$ to be the local ring given by:

$$\mathcal{O} = \left\{ \sum_{i=0}^{\infty} a_i t^i, \sum_{i=0}^{\infty} b_i u^i : a_0 = b_0, a_{g-1} = b_1, a_{2g-2} = b_2, \text{ and } a_i = 0 \text{ for all } i \notin S_1 \right\}.$$

Since $S_1$ has $(g - 2)$ gaps, we have

$$\delta = \dim_k \tilde{\mathcal{O}}/\mathcal{O} = (g - 2) + 3 = g + 1.$$ 

Clearly, the conductor ideal $\mathcal{C}$ of $\mathcal{O}$ in $\tilde{\mathcal{O}}$ is

$$\mathcal{C} = t^{2g-1} k[[t]] \times u^3 k[[u]].$$

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\[ \dim \mathcal{O}/\mathcal{C} = 2g - 1 + 3 = 2g + 2. \] This shows that \( \mathcal{O} \) is Gorenstein. However, the semigroup of the first branch, namely \( S_1 \), is not symmetric if \( 3 \notin H \). In fact, if \( S_1 \) is symmetric, then its conductor \( c_1 \) satisfies \( c_1 = 2(g - 2) = 2g - 4 \). Hence \( 2g-5 \) is a gap of \( S_1 \) and of \( H \). Again by [15, Prop. 1.2], we have \( 2g - 2 - (2g - 5) = 3 \in H \).

We now want to describe how to find a basis of dualizing differentials on \( X \) that have certain orders at \( P \). This process is analogous to finding a “Hermitian” basis (or basis “adapted to a point”) of regular differentials at a point on a smooth curve. We want to show that we can choose linearly independent dualizing differentials on \( X \) whose orders at \( P \) are related to the maximal points of \( S \) and the points of \( S \) that lie on the upper edge and right edge of the rectangle \( R \). These differentials will be those dualizing differentials in a “Hermitian” basis at \( P \) that are not regular on \( Y \) either at \( Q_1 \) or at \( Q_2 \) (or at both points if the differential corresponds to a maximal point of \( S \)).

We will use the following Riemann-Roch Theorem for zero-dimensional subschemes on a Gorenstein curve, which was proved in [22] (also cf. [5]). If \( J \) is a proper ideal of \( \mathcal{O}_P \), we let \( \mathcal{I}(J) \) denote the sheaf of \( \mathcal{O}_X \)-ideals defined by \( \mathcal{I}(J)_P = J \) and \( \mathcal{I}(J)_Q = \mathcal{O}_Q \) for all \( Q \neq P \). Put
\[
\begin{align*}
  h(J) &= \dim_k \text{Hom}_{\mathcal{O}_X}(\mathcal{I}(J), \mathcal{O}_X) \\
  \iota(J) &= \dim_k H^0(X, \mathcal{I}(J) \otimes \omega) \\
  d(J) &= \dim_k \mathcal{O}_P/J.
\end{align*}
\]

The elements of \( \text{Hom}_{\mathcal{O}_X}(\mathcal{I}(J), \mathcal{O}_X) \) may be identified with rational functions \( f \) on \( X \) such that \( fJ \subseteq \mathcal{O}_P \) and \( f \in \mathcal{O}_Q \) for all \( Q \neq P \).

(4.11) Theorem. \( h(J) - \iota(J) = d(J) + 1 - g \).

Let \( C \) denote the conductor of \( \mathcal{O}_P \) in its normalization \( \tilde{\mathcal{O}}_P \).

(4.12) Lemma. \( h(C) = 1 \).

Proof. This follows from the fact that \( \text{Hom}_{\mathcal{O}_P}(C, \mathcal{O}_P) = \tilde{\mathcal{O}}_P \) (cf. the proof of Proposition (2.2) of [14]).

(4.13) Proposition. Suppose \( \tau \in H^0(X, \omega) \) generates \( \omega_P \). Suppose that
\[ \mathcal{O}_P = J_0 \supset J_1 \supset \cdots \supset J_{\delta-1} \supset J_\delta = C \]
is a strictly decreasing chain of \( \mathcal{O}_P \)-ideals. Then there exist \( \delta \) linearly independent dualizing differentials \( \tau_1, \tau_2, \ldots, \tau_\delta \in H^0(X, \omega) \) such that, locally at \( P \), we have \( \tau_i = f_i \tau \) with \( f_i \in J_{i-1} \setminus J_i \) for \( i = 1, 2, \ldots, \delta \).

Proof. Note that \( d(J_i) = i \) since \( \dim_k \mathcal{O}_P/C = \delta \). Since \( J_i \supseteq C \) and \( h(C) = 1 \), it follows that \( h(J_i) = 1 \) for all \( i \). Therefore, from Theorem (4.11), we have \( \iota(J_i) = \iota(J_{i-1}) - 1 \) for \( i = 1, 2, \ldots, \delta \). Thus, there exists \( \tau_i \in H^0(X, \mathcal{I}(J_{i-1}) \otimes \omega) \setminus H^0(X, \mathcal{I}(J_i) \otimes \omega) \) for \( i = 1, 2, \ldots, \delta \). Then, locally at \( P \), we have \( \tau_i = f_i \tau \) for some \( f_i \in J_{i-1} \setminus J_i \).

(4.14) Definition. For \( (x, y) \in \mathbb{N} \times \mathbb{N} \), put
\[ J(x, y) = \{ f \in \mathcal{O}_P : \nu_1(f) \geq x \text{ and } \nu_2(f) \geq y \}. \]

(4.15) Definition. Suppose \( \sigma \in H^0(X, \omega) \) and \( \tau \) generates \( \omega_P \). Locally at \( P \), write \( \sigma = f \tau \), with \( f \in \mathcal{O}_P \). Then put \( \nu_1(\sigma) = \nu_1(f) \) and \( \nu_2(\sigma) = \nu_2(f) \).
(4.16) Theorem. There exist $\delta$ linearly independent dualizing differentials

$$\tau_0, \tau_1, \ldots, \tau_{\delta-1} \in H^0(X, \omega)$$

such that

1) For each maximal point $(a, b) \in S$, there is a $\tau_i$, $0 \leq i \leq I - 1$, such that $\nu_1(\tau_i) = a$, and $\nu_2(\tau_i) = b$.

2) For each point in $S$ of the form $(r, \xi_2)$, with $r < \xi_1$, there is a $\tau_j$, $I \leq j \leq I + \delta_1 - 1$, such that $\nu_1(\tau_j) = r$, and $\nu_2(\tau_j) \geq \xi_2$.

3) For each point in $S$ of the form $(\xi_1, s)$, with $s < \xi_2$, there is a $\tau_k$, $I + \delta_1 \leq k \leq \delta - 1$, such that $\nu_1(\tau_k) \geq \xi_1$ and $\nu_2(\tau_k) = s$.

Proof. Let $0 = x_1, x_2, \ldots, x_{I+\delta_1}$ be the nonnegative integers such that $x_k < \xi_1$ and $VF(x_k) \neq \emptyset$. Let $(\xi_1, s_0), (\xi_1, s_1), \ldots, (\xi_1, s_{\delta_2-1})$ denote the points in $S$ on the right edge of the rectangle $R$. (These points correspond to infinite horizontal fibers that lie below the line $Y = \xi_2$.) Let $C$ denote the conductor of $\mathcal{O}_P$ in its normalization. Consider the following chain of ideals in $\mathcal{O}_P$:

$$J(0, 0) \supset J(x_2, 0) \supset \cdots \supset J(x_{I+\delta_1}, 0) \supset J(\xi_1, s_0) \supset J(\xi_1, s_1) \supset \cdots \supset J(\xi_1, s_{\delta_2-1}) \supset C.$$  \hspace{1cm} (*)

This is a proper chain of ideals as in Proposition (4.13). Hence there exist $\delta$ linearly independent dualizing differentials $\sigma_0, \sigma_1, \ldots, \sigma_{\delta-1}$ as in Proposition (4.13). The last $\delta_2$ of these differentials, call them $\tau_{I+\delta_1}, \ldots, \tau_{\delta-1}$, satisfy condition (3) in the statement of the Theorem.

In a similar manner, we may find $\delta_1$ differentials, call them $\tau_I, \ldots, \tau_{I+\delta_1-1}$, satisfying condition (2) in the statement of the Theorem.

Suppose the maximal points of $S$ are

$$(a_0, b_0) < (a_1, b_1) < \cdots < (a_{I-1}, b_{I-1}),$$

ordered lexicographically. One of the differentials, call it $\sigma$, that we found using the chain (*) above satisfies

$$\nu_1(\sigma) = a_{I-1}, \nu_2(\sigma) \leq b_{I-1}.$$ 

If $\nu_2(\sigma) \neq b_{I-1}$, then $\nu_2(\sigma) = s_k$ for some $k$, $0 \leq k \leq \delta_2 - 1$. In that case, a suitable linear combination of $\sigma$ and the differential $\tau_{I+\delta_1+k}$ will yield a differential $\bar{\sigma}$ such that

$$\nu_1(\bar{\sigma}) = a_{I-1} \text{ and } \nu_2(\bar{\sigma}) > \nu_2(\sigma).$$

If $\nu_2(\sigma) = b_{I-1}$, then $\bar{\sigma}$ is one of the differentials we need to satisfy condition (1) in the statement of the Theorem and we will put $\tau_{I-1} = \bar{\sigma}$. If not, then $\nu_2(\bar{\sigma}) = s_{k'}$ for some $k'$ with $k < k' \leq \delta_2 - 1$. Then, by adding a suitable multiple of $\tau_{I+\delta_1+k'}$, we obtain a differential with a greater order on the second branch (while leaving the order on the
first branch unchanged). In this way, we obtain a differential, call it \( \tau_{I-1} \), such that
\[ \nu_1(\tau_{I-1}) = a_I - 1 \] and \( \nu_2(\tau_{I-1}) = b_I - 1 \).

We continue by induction, assuming that we have found the differentials \( \tau_{I-t+1}, \ldots, \tau_{I-1} \) corresponding to the maximal points \((a_{I-t+1}, b_{I-t+1}), \ldots, (a_{I-1}, b_{I-1})\). Consider the maximal point \((a_{I-t}, b_{I-t})\). One of the differentials we found above using chain (\( \ast \)), call it \( \rho \), satisfies \( \nu_1(\rho) = a_{I-t} \). If \( \nu_2(\rho) \neq b_{I-t} \), then we add to \( \rho \) a suitable multiple of either \( \tau_{I+t-k} \) if \( \nu_2(\rho) = s_k \) for some \( k \), or \( \tau_{I-u} \) if \( \nu_2(\rho) = b_{I-u} \) for some \( u, 1 \leq u \leq t - 1 \).

Continuing in this way, we can increase the order of the differential on the second branch, without changing the order on the first branch, until we obtain a differential \( \nu_1(\tau_{I-t}) = a_{I-t} \) and \( \nu_2(\tau_{I-t}) = b_{I-t} \). By this inductive process, we obtain differentials \( \tau_0, \ldots, \tau_{I-1} \) satisfying condition (1) in the statement of the Theorem.

The differentials \( \tau_0, \tau_1, \ldots, \tau_{\delta-1} \) are easily seen to be linearly independent by considering their orders on the two branches at \( P \).

A basis of \( g \) linearly independent dualizing differentials on \( X \) may be obtained by taking the union of a basis of \( g - \delta \) dualizing differentials on \( Y \) and the \( \delta \) differentials in Theorem (4.16). We will divide such a basis into four subsets and will use the following notation. Let

\[ \tau_0, \tau_1, \ldots, \tau_{I-1} \]

denote the differentials corresponding, as in (1) of Theorem (4.16), to the maximal points of \( S \). Let

\[ \zeta_0, \zeta_1, \ldots, \zeta_{\delta_1-1} \]

denote the differentials corresponding, as in (2) of Theorem (4.16), to certain points in \( S \) with first coordinate \( \xi_1 - 1 - l \), where \( l \) is a gap of \( S_1 \). Note that on \( Y \) each of the \( \zeta_j \)'s is regular at \( Q_2 \) and has a pole at \( Q_1 \). Let

\[ \eta_0, \eta_1, \ldots, \eta_{\delta_2-1} \]

denote the differentials corresponding, as in (3) of Theorem (4.16), to certain points in \( S \) with second coordinate \( \xi_2 - 1 - l' \), where \( l' \) is a gap of \( S_2 \). On \( Y \), each of the \( \eta_k \)'s is regular at \( Q_1 \) and has a pole at \( Q_2 \). Finally, let

\[ \sigma_0, \sigma_1, \ldots, \sigma_{g-\delta-1} \]

be a basis of the dualizing differentials on \( Y \).

To state the main result of this section, we must also introduce two linear systems on \( Y \). Let

\[ V_1 \subseteq H^0(Y, \omega_Y(-c_2 Q_2)) \]

be the linear system generated by

\[ \eta_0, \ldots, \eta_{\delta_2-1}, \sigma_0, \ldots, \sigma_{g-\delta-1} \].

Then \( V_1 \) has dimension \( g - \delta + \delta_2 \) and \( \dim_k H^0(Y, \omega_Y(-c_2 Q_2)) = g - \delta + c_2 - 1 \), assuming \( c_2 > 0 \). If \( c_2 = 0 \), then \( V_1 = H^0(Y, \omega_Y) \), while if \( c_2 > 0 \), then the codimension of \( V_1 \)
in $H^0(Y, \omega_Y(-c_2Q_2))$ is $c_2 - 1 - \delta_2$. Hence $V_1 = H^0(Y, \omega_Y(-c_2Q_2))$ if and only if the semigroup $S_2 = \{n \in \mathbb{N} : n = 0 \text{ or } n \geq c_2\}$ (e.g., if $P$ is a simple cusp on the second branch).

Let

$$V_2 \subseteq H^0(Y, \omega_Y(-c_1Q_1))$$

be the linear system generated by

$$\zeta_0, \ldots, \zeta_{\delta_1-1}, \sigma_0, \ldots, \sigma_{g-\delta-1}.$$  

Similar remarks to those made just above also hold concerning $V_2$ and $H^0(Y, \omega_Y(-c_1Q_1))$.

**Theorem.** Suppose $X$ is a Gorenstein curve of arithmetic genus $g$. Suppose $P$ is a singularity with precisely two branches. Let $Q_1$ and $Q_2$ be the two points on the partial normalization $Y$ of $X$ at $P$ that correspond to the branches at $P$. Let $V_1$ and $V_2$ denote the linear systems on $Y$ defined above. Then we have

$$W_X(P) = \delta(g - 1)(g + 1) - I(g - 1) - wt(S_1) - wt(S_2) + W_{V_1}(Q_1) + W_{V_2}(Q_2).$$

**Proof.** Locally at $P$, write

$$\tau_i = F_i \tau, \quad i = 0, 1, \ldots, I - 1$$

$$\zeta_j = G_j \tau, \quad j = 0, 1, \ldots, \delta_1 - 1$$

$$\eta_k = H_k \tau, \quad k = 0, 1, \ldots, \delta_2 - 1$$

$$\sigma_l = M_l \tau, \quad l = 0, 1, \ldots, g - \delta - 1.$$

Put

$$(\hat{F}, \hat{G}, \hat{H}, \hat{M}) = (F_0, \ldots, F_{I-1}, G_0, \ldots, G_{\delta_1-1}, H_0, \ldots, H_{\delta_2-1}, M_0, \ldots, M_{g-\delta-1}).$$

Then, as follows from Proposition (1.6), we have

$$W_X(P) = \delta(g - 1)g + \text{ord}_{Q_1} W_t(\hat{F}, \hat{G}, \hat{H}, \hat{M}) + \text{ord}_{Q_2} W_t(\hat{F}, \hat{G}, \hat{H}, \hat{M}),$$

where $t$ is a local coordinate at $Q_1$ and $Q_2$ and $W_t$ denotes the ordinary Wronskian (obtained by differentiating with respect to $t$). Notice that each of the functions $F_0, \ldots, F_{I-1}, G_0, \ldots, G_{\delta_1-1}$ has a different order at $Q_1$. By forming linear combinations of the $H_k$’s and $M_l$’s, if necessary, we may assume that each of the functions $H_0, \ldots, H_{\delta_2-1}, M_0, \ldots, M_{g-\delta-1}$ also has a different order at $Q_1$ and that, of these functions, $H_0$ has the lowest order at $Q_1$, with that order being $I + 2\delta_1$. Then we have

$$\text{ord}_{Q_1} W_t(\hat{F}, \hat{G}, \hat{H}, \hat{M}) = \sum_{i=0}^{I-1} \text{ord}_{Q_1} F_i + \sum_{j=0}^{\delta_1-1} \text{ord}_{Q_1} G_j$$

$$+ \sum_{k=0}^{\delta_2-1} \text{ord}_{Q_1} H_k + \sum_{l=0}^{g-\delta-1} \text{ord}_{Q_1} M_l - \sum_{n=0}^{g-1} n$$

$$= \sum_{i=0}^{I-1} (\text{ord}_{Q_1} F_i - i) + \sum_{j=0}^{\delta_1-1} (\text{ord}_{Q_1} G_j - (I + j))$$

$$+ \sum_{k=0}^{\delta_2-1} (\text{ord}_{Q_1} H_k - (I + \delta_1 + k)) + \sum_{l=0}^{g-\delta-1} (\text{ord}_{Q_1} M_l - (\delta + l))$$

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We then have

\[ (4.8) \]

\[ \delta_2 - 1 \]

\[ \sum_{k=0}^{\delta_2-1} (\text{ord}_{Q_1} H_k - (I + \delta_1 + k)) + \sum_{l=0}^{g-\delta-1} (\text{ord}_{Q_1} M_l - (\delta + l)) \]

\[ = \delta_1 I + (\delta_1 - 1)\delta_1 - wt(S_1) \]

\[ + \sum_{k=0}^{\delta_2-1} (\text{ord}_{Q_1} H_k - (I + \delta_1 + k)) + \sum_{l=0}^{g-\delta-1} (\text{ord}_{Q_1} M_l - (\delta + l)), \]

where \( l_1, l_2, \ldots, l_{\delta_1} \) are the gaps of \( S_1 \) and where, in the last equality, we have used Corollary (4.8) and the fact that

\[ \sum_{j=0}^{\delta_1-1} (I + 2\delta_1 - 1 - l_{j+1} - (I + j)) = (\delta_1 - 1)\delta_1 - \sum_{j=1}^{\delta_1}(l_j - j). \]

Now, to compute \( W_{V_1}(Q_1) \), we must express \( \eta_0, \ldots, \eta_{\delta_2-1}, \sigma_0, \ldots, \sigma_{g-\delta-1} \) in terms of a generator of \( \omega_Y(-c_2 Q_2) \) at \( Q_1 \). Let \( \eta \) be a generator of \( \omega_Y(-c_2 Q_2) \) at \( Q_1 \). Then \( \eta \) has order 0 at \( Q_1 \). Note that the rational function \( H = \eta/\tau \) has a zero of order \( I + 2\delta_1 \) at \( Q_1 \). We then have

\[ W_{V_1}(Q_1) = \text{ord}_{Q_1} W_t(H_0/H, \ldots, H_{\delta_2-1}/H, M_0/H, \ldots, M_{g-\delta-1}/H) \]

\[ = \sum_{k=0}^{\delta_2-1} ((\text{ord}_{Q_1} H_k - (I + 2\delta_1)) - k) + \sum_{l=0}^{g-\delta-1} ((\text{ord}_{Q_1} M_l - (I + 2\delta_1)) - (\delta_2 + l)) \]

\[ = \sum_{k=0}^{\delta_2-1} (\text{ord}_{Q_1} H_k - (I + \delta_1 + k)) - \delta_2 \delta_1 \]

\[ + \sum_{l=0}^{g-\delta-1} (\text{ord}_{Q_1} M_l - (\delta + l)) - (g - \delta)\delta_1. \]

Thus,

\[ \text{ord}_{Q_1} W_t(\mathcal{F}, \mathcal{G}, \mathcal{H}, M) \]

\[ = \delta_1 I + (\delta_1 - 1)\delta_1 - wt(S_1) + \delta_2 \delta_1 + (g - \delta)\delta_1 + W_{V_1}(Q_1) \]

\[ = \delta_1(I + \delta_1 - 1 + \delta_2 + g - \delta) - wt(S_1) + W_{V_1}(Q_1) \]

\[ = \delta_1(g - 1) - wt(S_1) + W_{V_1}(Q_1), \]

since \( \delta = I + \delta_1 + \delta_2 \).

Similarly, we have

\[ \text{ord}_{Q_2} W_t(\mathcal{F}, \mathcal{G}, \mathcal{H}, M) = \delta_2(g - 1) - wt(S_2) + W_{V_2}(Q_2). \]
The Theorem now follows by adding these two orders and using the fact that $\delta_1 + \delta_2 = \delta - I$.

In the case of an ordinary node, we have $I = 1, \delta_1 = \delta_2 = 0$ and Theorem (4.17) reduces to the following result of Widland [21].

(4.18) Corollary. If $P$ is an ordinary node, then

$$W_X(P) = (g - 1)g + W_Y(Q_1) + W_Y(Q_2).$$

We will call a singularity $P$ overweight if its Weierstrass weight is greater than the “expected” number. A unibranch singularity $P$ is not overweight if the point lying over $P$ on the partial normalization at $P$ is not a Weierstrass point (see Theorem (3.4)). A singularity $P$ with two branches is not overweight if, with the notation of Theorem (4.17), $W_{V_1}(Q_1) = W_{V_2}(Q_2) = 0$. We can now state a result in the case of a two-branch singularity that is analogous to (3) of Theorem (3.5).

(4.19) Proposition. Suppose $X$ is a rational Gorenstein curve of arithmetic genus $g$ with a single singularity $P$. Suppose that $P$ has precisely two branches and let $I, S_1$, and $S_2$ be as defined above. If $P$ is not overweight, then the number of smooth Weierstrass points on $X$, counting multiplicities, is

$$I(g - 1) + wt(S_1) + wt(S_2).$$

Proof. Since $X$ is rational, we have $g = \delta$. The weight of $P$ is then $g^2 - g - I(g - 1) - wt(S_1) - wt(S_2)$, as follows from Theorem (4.17) since $P$ is not overweight.

One can also prove a result similar to Theorem (3.7) in the case of a rational curve with unibranch and two-branch singularities that are not overweight. The result is that the number of smooth Weierstrass points, counting multiplicities, is given as the sum of local contributions from the singularities, with each unibranch singularity contributing the weight of its semigroup and each two-branch singularity contributing $I(g - 1) + wt(S_1) + wt(S_2)$.

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