NONARCHIMEDEAN SUPERRIGIDITY OF
SOLVABLE S-ARITHMETIC GROUPS

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Abstract. Let \( \Gamma \) be an \( S \)-arithmetic subgroup of a solvable algebraic \( \mathbb{Q} \)-group \( G \), where \( S \) is a nonempty, finite set of prime numbers. We construct a certain subgroup \( G_\Gamma^S \) of \( G_S \), such that if \( L \) is any local field and \( \alpha: \Gamma \to \text{GL}_n(L) \) is any homomorphism, then \( \alpha \) virtually extends (modulo a bounded error) to a continuous homomorphism defined on some finite-index subgroup of \( G_\Gamma^S \). If \( G \) has no nontrivial, \( \mathbb{R} \)-split tori, and \( \Gamma \) is Zariski-dense, then \( G_\Gamma^S \) can be taken to be all of \( G_S \).

We also prove generalizations that apply when \( G \) is an algebraic group over a finite extension of \( \mathbb{Q} \), or that do not require \( G \) to be solvable.

1. Introduction

Roughly speaking, a subgroup \( \Lambda \) of a topological group \( G \) is said to be “superrigid” if every finite-dimensional representation of \( \Lambda \) is the restriction of a continuous representation of \( G \). However, this need only be true up to finite-index subgroups and modulo a compact subgroup of the range:

Definition 1.1 (cf. [3, Thm. 2, p. 2]). Let \( \Gamma \) be a (countable) subgroup of a topological group \( G \), and let \( L \) be a local field. We say that \( \Gamma \) is \( L \)-superrigid in \( G \) if, for every homomorphism \( \alpha: \Gamma \to \text{GL}_n(L) \), there are

- a finite-index, open subgroup \( G_0 \) of \( G \),
- a finite-index subgroup \( \Gamma_0 \) of \( \Gamma \cap G_0 \),
- a finite, normal subgroup \( N \) of \( H \), where \( H \) is the group of \( L \)-points of the Zariski closure of \( \alpha(\Gamma_0) \),
- a continuous homomorphism \( \hat{\alpha}: G_0 \to H/N \), and
- a compact subgroup \( K \) of \( H/N \) that centralizes \( \hat{\alpha}(G_0) \),

such that \( \alpha(\gamma)N \in \hat{\alpha}(\gamma)K \) for all \( \gamma \in \Gamma_0 \).

It is known that \( S \)-arithmetic subgroups of solvable \( \mathbb{Q} \)-groups are \( \mathbb{R} \)-superrigid. (Hence, they are also \( \mathbb{C} \)-superrigid.) We now show that they are also superrigid over the other local fields. (That is, they are \( L \)-superrigid when \( L \) is a \( p \)-adic field \( \mathbb{Q}_p \) or a function field \( \mathbb{F}_q((T)) \).)

The precise statements require some terminology:

Definition 1.2. Suppose \( G \) is an algebraic group defined over \( \mathbb{Q} \), and \( S \) is a finite set of prime numbers. (All algebraic groups in this paper are assumed to be affine.)

- Subgroups \( \Gamma \) and \( \Lambda \) of \( G \) are commensurable if \( \Gamma \cap \Lambda \) has finite index in both \( \Gamma \) and \( \Lambda \).
- \( Z_S = \mathbb{Z}[1/p] \rvert p \in S \) = \{ \( a/b \in \mathbb{Q} \rvert \) every prime divisor of \( b \) is in \( S \) \}.
- A subgroup \( \Gamma \) of \( G \) is \( S \)-arithmetic if it is commensurable to \( G(Z_S) \).
• For any field \( F \) that contains \( \mathbb{Q} \), \( \text{rank}_F \mathbf{G} \) is the dimension of any maximal \( F \)-split torus in \( \mathbf{G} \).

• \( \mathbf{G}_S = \mathbf{G}(\mathbb{R}) \times \prod_{p \in S} \mathbf{G}(_p\mathbb{Q}) \), where \( _p\mathbb{Q} \) is the field of \( p \)-adic numbers.

**Remark 1.3.** In the definition of an \( S \)-arithmetic group, it is usually assumed that \( S \) contains all of the archimedean places, but, in our notation, \( S \) consists entirely of nonarchimedean places (and the archimedean factor is a separate term in the definition of \( \mathbf{G}_S \)).

Here is an archimedean superrigidity theorem that is easy to state:

**Theorem 1.4 (Witte [6 Thm. 1.6]).** Suppose \( \Gamma \) is a Zariski-dense, \( S \)-arithmetic subgroup of a solvable algebraic \( \mathbb{Q} \)-group \( \mathbf{G} \). If \( \text{rank}_\mathbb{Q} \mathbf{G} = 0 \), then \( \Gamma \) is \( \mathbb{R} \)-superrigid in \( \mathbf{G}_S \).

We prove that if the rank of \( \mathbf{G} \) is 0 over \( \mathbb{R} \), not just over \( \mathbb{Q} \), and \( S \) is not empty, then \( \Gamma \) is also superrigid over nonarchimedean local fields:

**Theorem 1.5.** Suppose \( \Gamma \) is a Zariski-dense, \( S \)-arithmetic subgroup of a solvable algebraic \( \mathbb{Q} \)-group \( \mathbf{G} \). If \( S \neq \emptyset \) and \( \text{rank}_\mathbb{R} \mathbf{G} = 0 \), then \( \Gamma \) is \( L \)-superrigid in \( \mathbf{G}_S \), for every local field \( L \).

The assumption that \( \text{rank}_\mathbb{R} \mathbf{G} = 0 \) can be removed if we replace \( \mathbf{G}_S \) with a certain subgroup, which we now define.

**Definition 1.6.** Suppose \( \mathbf{G} \) is a solvable algebraic group defined over \( \mathbb{Q} \), \( S \) is a finite set of prime numbers, and \( \Gamma \) is a subgroup of \( \mathbf{G}_S \). If \( \mathbf{U} \) is the unipotent radical of \( \mathbf{G} \), and \( \mathbf{C}/\mathbf{U}_S \) is the (unique) maximal compact subgroup of the abelian group \( (\mathbf{G}^0)_S/\mathbf{U}_S \), then \( \mathbf{G}^\Gamma_S = \Gamma \cdot \mathbf{C} \).

**Theorem 1.7.** Suppose \( \Gamma \) is a Zariski-dense, \( S \)-arithmetic subgroup of a solvable algebraic \( \mathbb{Q} \)-group \( \mathbf{G} \). If \( S \neq \emptyset \), then \( \Gamma \) is \( L \)-superrigid in \( \mathbf{G}^\Gamma_S \), for every local field \( L \).

**Remark 1.8.** Suppose \( \text{rank}_\mathbb{R} \mathbf{G} = 0 \). This implies that the subgroup \( \mathbf{C} \) in Definition 1.6 is open. It also implies that \( \mathbf{G}_S/\Gamma \) is compact [4 Thm. 5.5(1), p. 260]. Hence, \( \mathbf{G}^\Gamma_S = \Gamma \cdot \mathbf{C} \) is a finite-index subgroup of \( \mathbf{G}_S \). Therefore, Theorem 1.7 is a generalization of Theorem 1.5.

It is well known that if \( \text{rank}_\mathbb{Q} \mathbf{G} \neq 0 \) (and \( \mathbf{G} \) is solvable), then \( S \)-arithmetic subgroups of \( \mathbf{G} \) are not lattices in \( \mathbf{G}_S \). Rather, they are lattices in a certain subgroup that is usually denoted \( \mathbf{G}^{(1)}_S \) [4 pp. 263–264]. In general, a finite-index subgroup of \( \mathbf{G}^\Gamma_S \) is contained in \( \mathbf{G}^{(1)}_S \), and may be much smaller, but the groups are commensurable if \( \text{rank}_\mathbb{R} \mathbf{G} = \text{rank}_\mathbb{Q} \mathbf{G} \). Therefore:

**Corollary 1.9.** Suppose \( \Gamma \) is a Zariski-dense, \( S \)-arithmetic subgroup of a solvable algebraic \( \mathbb{Q} \)-group \( \mathbf{G} \), and \( S \) is not empty. Assume, by passing to a finite-index subgroup if necessary, that \( \Gamma \subseteq \mathbf{G}^{(1)}_S \). If \( \text{rank}_\mathbb{R} \mathbf{G} = \text{rank}_\mathbb{Q} \mathbf{G} \), then \( \Gamma \) is \( L \)-superrigid in \( \mathbf{G}^{(1)}_S \), for every local field \( L \).

We state and prove our main theorem in Section 2. By specializing this general result to solvable groups, Section 3 obtains Theorem 1.7, and also some generalizations that apply to solvable algebraic groups that are defined over finite extensions of \( \mathbb{Q} \). A consequence for non-solvable groups is stated in Section 4.

**Remark 1.10.** Our results assume that \( \mathbf{G} \) is defined over a field of characteristic zero, so we have nothing to say about \( S \)-arithmetic subgroups of solvable groups that are defined over a global field of positive characteristic. That seems to be a much more difficult problem, and we merely point out that the paper [2] proves a rigidity result (but not superrigidity) in a very special case.
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2. A NONARCHIMEDEAN SUPERRIGIDITY THEOREM

We now state and prove our main result. Later sections of the paper explain that super-rigidity results for various $S$-arithmetic groups are special cases of this theorem.

Notation 2.1. We use $\overline{X}$ to denote the Zariski closure of a matrix group $X$. We emphasize that this is the Zariski closure, not the closure in the ordinary topology.

Definition 2.2. Let $L$ be a local field. A subgroup $\Gamma$ of a topological group $G$ is semisimply $L$-superrigid in $G$ if $\Gamma$ is $L$-superrigid in $G$, and, for every homomorphism $\alpha : \Gamma \to \text{GL}_n(L)$:

1. if $\text{char} L = 0$, then $\overline{\alpha(\Gamma)}$ is semisimple, and
2. if $\text{char} L \neq 0$, then $\overline{\alpha(\Gamma)}$ is contained in a compact subgroup of $\text{GL}_n(L)$.

Theorem 2.3. Let

- $G$ be a connected algebraic group over $\mathbb{Q}$,
- $S$ be a finite set of prime numbers,
- $\Gamma$ be a Zariski-dense subgroup of $G$, and
- $L$ be a nonarchimedean local field.

Write

- $G = M\Gamma T U_S$, where
  - $M$ is a semisimple $\mathbb{Q}$-group,
  - $T$ is a $\mathbb{Q}$-torus that centralizes $M$, and
  - $U = \text{unip}_G$,
- $M_\Gamma = \Gamma \cap M$, $T_\Gamma = \Gamma \cap T$, and $U_\Gamma = \Gamma \cap U$, and
- $G_S^\Gamma = T_\Gamma M_S K_T U_S$, where $K_T$ is the (unique) maximal compact subgroup of $T_S$.

Assume:

1. $G(\mathbb{Z}) \subseteq \Gamma \subseteq G(\mathbb{Z}_S)$, where $\subseteq$ means “has a finite-index subgroup that is contained in,”
2. $M_\Gamma T_\Gamma U_\Gamma$ has finite index in $\Gamma$,
3. $M_\Gamma$ is semisimply $L$-superrigid in $M_S$, and
4. for every finite-index subgroup $\Gamma'$ of $\Gamma$, the group $(\Gamma' \cap U)/(\Gamma' \cap U) \cap U$ has no infinite, cyclic quotient.

Then $\Gamma$ is $L$-superrigid in $G_S^\Gamma$.

Before proving Theorem 2.3, we record a few observations, mostly about unipotent groups. First of all, note that every subgroup of a unipotent group is nilpotent, and therefore has a well-defined (Hirsch) rank, which is the supremum of the ranks of its finitely generated subgroups [5, Defn. 2.9, p. 32].

Lemma 2.4 (cf. [5] Thm. 2.10, p. 32]). If $\Gamma$ is any subgroup of a unipotent algebraic group $U$ (over a field of characteristic zero), then $\text{rank} \Gamma \geq \text{dim} \Gamma$.

Proof. Let $\pi : U \to U/[U, U]$ be the natural homomorphism. There is no harm in assuming $U = \overline{\Gamma}$. Then $\Gamma \cap [U, U]$ is Zariski-dense in $[U, U]$, so, by induction on $\text{dim} U$, we know that
rank(\Gamma \cap [U, U]) \geq \dim[U, U]. Also, \pi(\Gamma) is Zariski-dense in the abelian unipotent group \pi(U), so it is obvious that rank \pi(\Gamma) \geq \dim \pi(U). Therefore
\begin{align*}
rank \Gamma &= \text{rank}(\Gamma \cap \ker \pi) + \text{rank} \pi(\Gamma) = \text{rank}(\Gamma \cap [U, U]) + \text{rank} \pi(\Gamma) \\
&\geq \dim[U, U] + \dim(U/[U, U]) = \dim U = \dim \Gamma.
\end{align*}

Corollary 2.5 (cf. [5, Thm. 2.11, p. 33]). If
\begin{itemize}
\item U is a unipotent algebraic group over \mathbb{Q}_p, for some prime p,
\item \Gamma is a subgroup of U(\mathbb{Q}_p),
\item rank \Gamma = \dim \Gamma,
\item \alpha: \Gamma \to \text{GL}_n(\mathbb{Q}_p) is a finite-dimensional representation of \Gamma, such that \alpha(\Gamma) is unipotent,
\end{itemize}
then \alpha extends uniquely to a rational representation \hat{\alpha}: \Gamma \to \text{GL}_n. (Furthermore, \hat{\alpha} is defined over \mathbb{Q}_p.)

Proof. The uniqueness of the extension \hat{\alpha} is immediate from the Zariski density of \Gamma in \Gamma.
Also, \hat{\alpha} must be defined over \mathbb{Q}_p, since it maps the Zariski-dense set \Gamma of \mathbb{Q}_p-points of \Gamma into the \mathbb{Q}_p-points of \text{GL}_n. Therefore, we need only prove the existence of \hat{\alpha}.

There is no harm in assuming U = \Gamma. Let
\begin{align*}
\text{graph}(\alpha) = \{ (\gamma, \alpha(\gamma)) \mid \gamma \in \Gamma \} \subseteq U \times \alpha(\Gamma),
\end{align*}
and
\begin{align*}
\pi: \text{graph}(\alpha) \to U \text{ be the composition } \text{graph}(\alpha) \hookrightarrow U \times \alpha(\Gamma) \to U.
\end{align*}
Then \pi(\text{graph}(\alpha)) = \Gamma is Zariski-dense in U, so \pi(\text{graph}(\alpha)) = U. On the other hand, we have graph(\alpha) \cong \Gamma, so, from Lemma 2.4, we know
\begin{align*}
\dim \text{graph}(\alpha) \leq \text{rank graph}(\alpha) = \text{rank} \Gamma = \dim \Gamma = \dim U.
\end{align*}
Therefore \dim \ker \pi = 0. Since the unipotent group \text{graph}(\alpha) has no nontrivial subgroups that are 0-dimensional (in other words, finite), this implies that \pi is an isomorphism of algebraic groups. Therefore, \text{graph}(\alpha) is the graph of a rational homomorphism \hat{\alpha}: U \to \alpha(\Gamma). Namely, \hat{\alpha} is the composition
\begin{align*}
U \xrightarrow{\pi^{-1}} \text{graph}(\alpha) \hookrightarrow U \times \alpha(\Gamma) \twoheadrightarrow \alpha(\Gamma).
\end{align*}

Lemma 2.6. Let
\begin{itemize}
\item p be a prime number,
\item G be a connected algebraic group defined over \mathbb{Q}_p,
\item G = M \ltimes R, where M is a semisimple group defined over \mathbb{Q}_p, and R is the solvable radical,
\item K be a compact subgroup of M(\mathbb{Q}_p), and
\item K_1, \ldots, K_n be compact subgroups of R(\mathbb{Q}_p).
\end{itemize}
Then the p-adic closure of \langle K, K_1, \ldots, K_n \rangle is compact.

Proof. Write R = T \ltimes U, where T is a torus that centralizes M, and U is unipotent, and let C be the unique maximal compact subgroup of the abelian group T(\mathbb{Q}_p). For each i, let
\begin{align*}
U_i = (KC_i K_i CK) \cap U(\mathbb{Q}_p),
\end{align*}
so $U_i$ is a compact subset of $U(Q_p)$ that is normalized by both $K$ and $C$. The maximality of $C$ implies that $K_i \subseteq C \cdot U(Q_p)$, so $K_i \subseteq CU_i$. Then, since every compactly generated subgroup of $U(Q_p)$ is bounded [Prop. 2.6.3, p. 46], we conclude that the closure of $KC \cdot \{U_1, \ldots, U_n\}$ is a compact subgroup of $G(Q_p)$ that contains $\langle K, K_1, \ldots, K_n \rangle$. □

We now prove the main theorem:

**Proof of Theorem 2.3.** By passing to a (torsion-free) subgroup of finite index in $\Gamma$, we may assume $T_{\Gamma} \cap M_S K_T U_S = \{e\}$ (and also that $\Gamma = M_T T_{\Gamma}$). Let us assume $\text{char } L = 0$. (See Remark 2.8 for the case where $\text{char } L \neq 0$.) Then, since $L$ is nonarchimedean, we may assume $L = Q_p$, for some prime number $p$, so we are given a homomorphism $\alpha: \Gamma \to \text{GL}_n(Q_p)$. We consider two cases.

**Case 1. Assume** $\alpha \in S$. Write $\overline{\alpha(\Gamma)} = N \times C \times H$ (after modding out a finite subgroup), where $N$ is semisimple, $C$ is a torus, and every reductive subgroup of $H$ acts faithfully by conjugation on $\text{unip} H$. Then $\alpha$ can be decomposed into a homomorphism $\alpha_N$ into $N$, a homomorphism $\alpha_C$ into $C$, and a homomorphism $\alpha_H$ into $H$. We consider these three components of $\alpha$ separately.

Since $\alpha_N(T_{\Gamma} U_{\Gamma})$ is a solvable, normal subgroup of the semisimple group $N$, we know that it is finite. By modding it out, we may assume $\alpha_N$ is trivial on $T_{\Gamma} U_{\Gamma}$. Assumption 2.3(3) provides a continuous homomorphism $\hat{\alpha}_N: M_S \to N(Q_p)$ whose restriction to $M_T$ agrees with $\alpha_N|_{M_T}$ up to a bounded error. By modding out a finite subgroup of $\overline{\alpha(\Gamma)}$, we may assume $\hat{\alpha}_N$ is trivial on $M_S \cap T_S$. Then $\hat{\alpha}_N$ can be extended to a continuous homomorphism $\hat{\alpha}_N: G_S \to N(Q_p)$, by specifying that the extension is trivial on $T_S U_S$. Then $\hat{\alpha}_N$ agrees with $\alpha_N$ up to a bounded error, so this deals with the part of $\alpha$ that maps into $N$. We may therefore assume, henceforth, that $N$ is trivial.

We can extend $\alpha_C|_{T_{\Gamma}}$ to a continuous homomorphism $\hat{\alpha}_C: T_{\Gamma} M_S K_T U_S \to C(Q_p)$, by specifying that the extension is trivial on the open, normal subgroup $M_S K_T U_S$. Now, let $K_C$ be the maximal compact subgroup of $C(Q_p)$, so $C(Q_p)/K_C$ is a finitely generated, torsion-free, abelian group. Then Assumption 2.3(1) implies that the image of $\alpha_C(U_{\Gamma})$ in $C(Q_p)/K_C$ must be trivial, which means $\alpha_C(U_{\Gamma}) \subseteq K_C$. Also, since $C$ is abelian, Assumption 2.3(3) implies that $\alpha_C(M_{\Gamma})$ is trivial (after passing to a finite-index subgroup). Then, for all $m \in M_T$, $t \in T_{\Gamma}$, and $u \in U_{\Gamma}$, we have

$$\alpha_C(mt) = \alpha_C(m) \cdot \alpha_C(t) \cdot \alpha_C(u) \in e \cdot \hat{\alpha}_C(mt) \cdot K_C = \hat{\alpha}_C(mt) K_C,$$

so this deals with the part of $\alpha$ that maps into $C$. We may therefore assume, henceforth, that $C$ is trivial.

We are now assuming that $N$ and $C$ are trivial, so $\overline{\alpha(\Gamma)} = H$, which means that

\begin{equation}
\text{(2.7) every reductive subgroup of } \overline{\alpha(\Gamma)} \text{ acts faithfully on } \text{unip} \overline{\alpha(\Gamma)}.
\end{equation}

We know that $\overline{\alpha(U_{\Gamma})}$ is nilpotent (since $U_{\Gamma}$ is nilpotent), so it has a unique maximal torus $R$. We also know that $\overline{\alpha(U_{\Gamma})}$ is a normal subgroup of $\overline{\alpha(\Gamma)}$ (since $U_{\Gamma}$ is a normal subgroup of $\Gamma$). Therefore $R$ is a normal subgroup of $\overline{\alpha(\Gamma)}$. Since any normal torus in a connected algebraic group is central, we conclude that $R$ is contained in the center of $\overline{\alpha(\Gamma)}$. Then (2.7) implies that $R$ is trivial. This means that $\alpha(U_{\Gamma})$ is unipotent. Also, since $U$ is a unipotent $Q$-group, and Assumption 2.3(1) tells us that $U(Z) \subseteq U_{\Gamma} \subseteq U(Z_S)$, we know that $U_{\Gamma}$ is Zariski-dense in $U$ and rank $U_{\Gamma} = \dim U$. (To establish the equality, note that if $\Gamma^+$ is any finitely generated subgroup of $U(Q)$ that contains $U(Z)$, then the proof of [5]. Thm. 2.10,
p. 32] shows that $\Gamma^+$ is a lattice in $U(\mathbb{R})$, so [19, Thm. 2.10, p. 32] implies $\text{rank}(\Gamma^+) = \text{dim } U$.) Therefore, Corollary 2.5 tells us that $\alpha|_{U_\Gamma}$ extends to a (unique) rational homomorphism $\alpha_U : U \to \alpha(U_\Gamma)$. (Furthermore, $\alpha_U$ is defined over $\mathbb{Q}_p$.)

**Claim.** $\{e\} \times V$ is the unipotent radical of $\text{graph}(\alpha|_{M_\Gamma_{T_\Gamma}})$, and $V$ is in the center of $\alpha(\Gamma)$, so

$$\text{graph}(\alpha|_{M_\Gamma_{T_\Gamma}}) = B \cdot (\{e\} \times V) \cong B \times V.$$ 

To verify this, first note that, since the reductive group $MT$ has no nontrivial normal unipotent subgroups, the unipotent radical of $\text{graph}(\alpha|_{M_\Gamma_{T_\Gamma}})$ must be contained in the kernel of the natural projection from $\text{graph}(\alpha|_{M_\Gamma_{T_\Gamma}})$ to $MT$. This kernel is $\{e\} \times V$. We now prove the reverse inclusion. Since $M_\Gamma T_\Gamma$ normalizes $U_\Gamma$, we know that $\text{graph}(\alpha|_{M_\Gamma_{T_\Gamma}})$ normalizes $\text{graph}(\alpha|_{U_\Gamma})$. Then the uniqueness of $\alpha_U$ implies that $\text{graph}(\alpha|_{M_\Gamma_{T_\Gamma}})$ also normalizes $\text{graph}(\alpha_U)$. Since $\text{graph}(\alpha_U)$ is Zariski closed (because the homomorphism $\alpha_U$ is rational), we conclude that $\text{graph}(\alpha|_{M_\Gamma_{T_\Gamma}})$ normalizes $\text{graph}(\alpha_U)$. So $\{e\} \times V$ normalizes $\text{graph}(\alpha_U)$. Hence, for any $v \in V$ and $u \in U_\Gamma$, we have

$$(u, v^{-1} \alpha_U(u) v) = (e, v)^{-1} (u, \alpha_U(v))(e, v) \in \text{graph}(\alpha_U),$$

so $v^{-1} \alpha_U(u) v = \alpha_U(u)$. This implies that $V$ centralizes $\alpha(U_\Gamma)$.

Also, since $T_\Gamma$ is central in $M_\Gamma T_\Gamma$, and Assumption 2.3 tells us that $\alpha(M_\Gamma)$ has no unipotent radical, we know that $\text{unip } \alpha(M_\Gamma T_\Gamma)$ is central in $\alpha(M_\Gamma T_\Gamma)$. Therefore, $V$ centralizes $\alpha(U_\Gamma) \cdot \text{unip } \alpha(M_\Gamma T_\Gamma) = \text{unip } \alpha(\Gamma)$, so 2.3 tells us that $V$ is unipotent. Since $V$ is normal, this completes the proof that $\{e\} \times V$ is the unipotent radical of $\text{graph}(\alpha|_{M_\Gamma_{T_\Gamma}})$. In addition, $V$ is central in $\alpha(\Gamma)$, because it centralizes both $\alpha(U_\Gamma)$ and $\alpha(M_\Gamma T_\Gamma)$. (The latter is because $V \subseteq \text{unip } \alpha(M_\Gamma T_\Gamma)$.) This completes the proof of the claim.

Since $B \cap (\{e\} \times V)$ is trivial, the projection from $B$ onto $MT$ has trivial kernel, so it is an isomorphism of algebraic groups. Therefore, $B$ is the graph of a rational homomorphism $\tilde{\alpha}_{MT} : MT \to \alpha(M_\Gamma T_\Gamma)$ that is defined over $\mathbb{Q}_p$.

Define $\alpha_V : M_\Gamma T_\Gamma \to V(\mathbb{Q}_p)$ by $\tilde{\alpha}_{MT}(g) = \alpha(g)\alpha_V(g)$. (This is a homomorphism, since $V$ is central.) Since $V$ is abelian, Assumption 2.3 tells us that $\alpha_V(T_\Gamma)$ is trivial (after passing to a finite-index subgroup). Also, since $T_\Gamma$ is finitely generated [4, Thm. 5.12, p. 176], and $V$ is unipotent, we know that $\alpha_V(T_\Gamma)$ is contained in a compact subgroup $K_V$ of $V(\mathbb{Q}_p)$ [4, Prop. 2.6.3, p. 46]. Now, since $B \subseteq \text{graph}(\alpha|_{M_\Gamma_{T_\Gamma}})$ normalizes $\text{graph}(\alpha_U)$, the product
\( B \cdot \text{graph}(\alpha_U) \) is a subgroup of \( G \times \alpha(\Gamma) \). It is the graph of a rational homomorphism \( \tilde{\alpha}: G \to \alpha(\Gamma) \) that is defined over \( \mathbb{Q}_p \), and satisfies
\[
\tilde{\alpha}(gu) = \tilde{\alpha}_{MT}(g) \alpha_U(u) \quad \text{for} \quad g \in MT \text{ and } u \in U.
\]
Then, for \( g \in M_T T_T \) and \( u \in U \), we have
\[
\tilde{\alpha}(gu) = \tilde{\alpha}_{MT}(g) \cdot \alpha_U(u) = \alpha(g) \alpha_V(g) \cdot \alpha(u) \in \alpha(g) K_V \cdot \alpha(u) = \alpha(gu) \cdot K_V.
\]
Since \( K_V \subset V \) is central in \( \alpha(\Gamma) \), this completes the proof of this case.

**Case 2.** Assume \( p \notin S \). Write \( \alpha(T_T) = R \times V \), where \( R \) is a torus and \( V \) is unipotent. Since \( \mathbb{Q}_p^\times = \langle p \rangle \times \text{compact} \), we may write \( R(\mathbb{Q}_p) = Z \times E \), where \( Z \) is free abelian, \( E \) is compact, and every eigenvalue of every element of \( Z \) is a power of \( p \). Let \( \alpha_Z \) be the projection of \( \alpha|_{T_T} \) to \( Z \), and extend it to a continuous homomorphism \( \tilde{\alpha}: T_T M_S K_T U_S \to Z \), by defining \( \tilde{\alpha} \) to be trivial on the open, normal subgroup \( M_S K_T U_S \).

Since \( M_T \subseteq M(\mathbb{Z}_S) \) (up to finite index) and \( p \notin S \), we know that \( M_T \) is contained in the compact subgroup \( M(\mathbb{Z}_p) \) of \( M(\mathbb{Q}_p) \). Hence, Assumption \([2.3][1]\) implies that \( \alpha(M_T) \) is contained in a compact subgroup \( K_M \) of \( \text{GL}_n(\mathbb{Q}_p) \). Of course, we may assume \( K_M \subseteq \alpha(M_T) \). Also, since \( T_T \) is finitely generated \([1]\) Thm. 5.12, p. 176], the \( p \)-adic closure of the projection of \( \alpha(T_T) \) to \( V(\mathbb{Q}_p) \) is a compact subgroup \( K_V \) \([1]\) Prop. 2.6.3, p. 46].

Furthermore, we now show that the \( p \)-adic closure of \( \alpha(U_T) \) is a compact subgroup \( K_U \). Write \( \alpha(U_T) = C \times W \), where \( C \) is a torus and \( W \) is unipotent, and let \( \alpha_C \) and \( \alpha_W \) be the projections of \( \alpha|_{U_T} \) to the two direct factors. Just as in Case \([1]\) we see that \( \alpha_C(U_T) \) is contained in a compact subgroup \( K_C \) of \( C(\mathbb{Q}_p) \). Now, let \( K \) be the \( p \)-adic closure of \( \alpha_W(U(\mathbb{Z}) \cap \Gamma) \) in \( W(\mathbb{Q}_p) \). Since \( U(\mathbb{Z}) \) is finitely generated, we know that \( K \) is compact \([1]\) Prop. 2.6.3, p. 46]. Let
\[
K_W \text{ be the } p \text{-adic closure of } \{ \sqrt[m]{w} \mid w \in K, \ m \in \mathbb{Z}^+, \ p \nmid m \} \subseteq W(\mathbb{Q}_p).
\]
Then \( K_W \) is also a compact subgroup of \( W(\mathbb{Q}_p) \). (It is a subgroup by \([1]\) Prop. 2.4.3(a), p. 34]. It is compact because dividing an element of the Lie algebra of \( W(\mathbb{Q}_p) \) by a scalar that is coprime to \( p \) does not change the \( p \)-adic norm of the vector.) After passing to a finite-index subgroup, Assumption \([2.3]\) tells us that \( U_T \subseteq W(\mathbb{Z}_S) \). Then, since \( p \notin S \), we have \( \alpha_W(U_T) \subseteq K_W \), so \( \alpha(U_T) \subseteq K_C K_W \) is contained in a compact subgroup, as desired.

For \( m \in M_T \), \( t \in T_T \) and \( u \in U_T \), we have
\[
\tilde{\alpha}(mtu) = \alpha_Z(t) \in \alpha(t) \cdot EK_V = \alpha(m)^{-1} \cdot \alpha(m) \alpha(t) \alpha(u) \cdot \alpha(u^{-1}) EK_V \\
\subseteq K_M \cdot \alpha(mt) \cdot K_U EK_V = \alpha(mt) \cdot K_M K_V EK_V.
\]
Lemma \([2.6]\) tells us that the closure of \( \langle K_M K_U EK_V \rangle \) is compact.

To complete the proof, all that remains is to show that \( \langle K_M K_U EK_V \rangle \) is centralized by \( \tilde{\alpha}(G_S^\times) \). Since \( T \) normalizes \( U \), we know that \( \alpha(T_T) \) normalizes \( K_U \). It also normalizes (in fact, centralizes) \( K_M, E, \) and \( K_V \), since all three groups are contained in \( \alpha(M_T T_T) \), whose center contains \( \alpha(T_T) \). Therefore, \( \alpha(T_T) \) normalizes \( \langle K_M K_U EK_V \rangle \), which is also normalized by \( EK_V \). Since \( \alpha_Z(T_T) \subseteq \alpha(T_T) EK_V \), we conclude that \( \alpha_Z(T_T) \) normalizes the closure of \( \langle K_M K_U EK_V \rangle \), which is compact (as was already noted at the end of the preceding paragraph). However, any element \( z \) of \( Z \) is diagonalizable (since it is in a torus) and all of its eigenvalues are in \( \mathbb{Q}_p \) (indeed, they are powers of \( p \)), so \( z \) is diagonalizable over \( \mathbb{Q}_p \). Since all of its eigenvalues are are powers of \( p \), this implies that \( z \) must centralize any compact
subgroup of $\text{GL}_n(\mathbb{Q}_p)$ that it normalizes. We conclude that $\widehat{\alpha}(G^t_S) = \alpha_Z(T)$ centralizes $\langle K_M K_U E K_V \rangle$, as desired.

\[ \square \]

Remark 2.8. To complete the proof of Theorem 2.3, we treat the case where the characteristic of $L$ is nonzero, by adapting Case 2 of the above proof. Since $T_\Gamma$ and $U_\Gamma$ have finite Hirsch rank, and unipotent $L$-groups are torsion, we may assume, after passing to a subgroup of finite index, that $\alpha(T_\Gamma U_\Gamma)$ is a torus $\mathbb{R}$. Letting $p$ be a uniformizer of $L$, we have $L^\times = \langle p \rangle \times \text{compact}$, so we may write $\mathbb{R}(L) = Z \times E$, where $Z$ is free abelian, $E$ is compact, and every eigenvalue of every element of $Z$ is a power of $p$. Let $\alpha_Z$ be the projection of $\alpha|_{T_\Gamma}$ to $Z$, and extend it to a continuous homomorphism $\alpha: T_\Gamma \cdot M_S K_U U_S \to Z$, by defining $\alpha$ to be trivial on the open, normal subgroup $M_S K_U U_S$. From Assumption 2.3(4), we see that $\alpha(U_\Gamma) \subseteq E$. Also, since $\text{char } L \neq 0$, Assumption 2.3(4) tells us that $\alpha(M_\Gamma)$ is contained in a compact subgroup $K_M$ of $\text{GL}_n(L)$. Then, for $m \in M_\Gamma$, $t \in T_\Gamma$, and $u \in U_\Gamma$, we have

\[ \widehat{\alpha}(mtu) = \alpha_Z(t) \in \alpha(t) \cdot E = \alpha(m)^{-1} \cdot \alpha(m) \alpha(t) \alpha(u) \cdot \alpha(u^{-1}) E \subseteq K_M \cdot \alpha(mtu) \cdot E = \alpha(mtu) \cdot K_M E. \]

Furthermore, we may assume $K_M$ is contained in $\overline{\alpha(M_\Gamma)}$, which centralizes $\mathbb{R}(L) = Z \times E$ (assuming, as we may, that $\overline{\alpha(M_\Gamma)}$ is connected), so $K_M E$ is a compact subgroup that centralizes $\alpha(\Gamma)$ (since $\alpha(\Gamma) \subseteq Z$).

We will also use the following refinement of Theorem 2.3:

Corollary 2.9. Assume the notation and hypotheses of Theorem 2.3. For each $p \in S$, choose $\mathbb{Q}_p$-subgroups $A_p$ and $B_p$ of $G$, such that

(i) $G = A_p \times B_p$ (up to finite index), and

(ii) the projection of $\Gamma$ to $B_p$ is contained in a compact subgroup $K_p$ of $B_p(\mathbb{Q}_p)$.

Let

- $A_S = G(\mathbb{R}) \times \times_{p \in S} A_p(\mathbb{Q}_p)$, and
- $A^t_S$ be the image of $G^t_S$ under the natural projection $\pi_A: G_S \to A_S$ with kernel $B^t_S = \times_{p \in S} B_p(\mathbb{Q}_p)$.

Then $\Gamma$ (or, more precisely, $\pi_A(\Gamma)$) is $L$-superrigid in $A^t_S$.

Proof. Suppose $\alpha: \pi_A(\Gamma) \to \text{GL}_n(L)$. Composing with $\pi_A$ yields a homomorphism $\alpha': \Gamma \to \text{GL}_n(L)$. The proof of Theorem 2.3 constructs only two kinds of extensions of $\alpha'$. Namely, if we ignore a bounded error (and ignore passing to finite-index subgroups), and if, in Case 1, we consider only a single component $\alpha_N^*, \alpha_C^*$, or $\alpha_H^*$, then either:

(a) $\alpha'$ extends to a continuous homomorphism $\widehat{\alpha}: G^t_S \to \text{GL}_n(L)$, or

(b) $\alpha'$ factors through the projection $\pi_T: G^t_S \to T_\Gamma$.

In situation (a), let

- $\widehat{\alpha}$ be the restriction of $\alpha$ to $A_S$, and
- $K$ be the closure of the projection of $\Gamma$ to $B^t_S$.

For $\gamma \in \Gamma$ we have (up to bounded error):

\[ \alpha(\pi_A(\gamma)) = \alpha'(\gamma) = \widehat{\alpha}(\gamma) \in \widehat{\alpha}(\pi_A(\gamma)) \cdot \widehat{\alpha}(K) = \widehat{\alpha}(\pi_A(\gamma)) \cdot \widehat{\alpha}(K). \]

From Assumption 2.9(3), we know that $K$ is a compact subgroup of $B^t_S$. Since $\widehat{\alpha}$ is continuous, this implies that $\widehat{\alpha}(K)$ is compact. So $\alpha$ agrees with $\widehat{\alpha}$ up to a bounded error on $\pi_A(\Gamma)$. 
We now consider situation (b). Write $A_S = M_A T_A U_A$, where $M_A$ is semisimple, $T_A$ is a torus, and $U_A$ is the unipotent radical. Let $C_A/(M_A U_A)$ be the (unique) maximal compact subgroup of the abelian group $A_S/(M_A U_A)$. Then we have

\[ A^\Gamma_S = \pi_A(G^\Gamma_S) = \pi_A(T^\Gamma) \cdot C_A. \]

Assume $T^\Gamma$ is torsion free (by passing to a subgroup of finite index). Then, since Assumption 2.9(ii) implies that $\pi_A(T^\Gamma)$ is discrete in $T^\Gamma$, we see that $\pi_A(T^\Gamma) \cap C_A = \{e\}$. Also note that the restriction of $\pi_A$ to $\Gamma$ is bijective (since $\Gamma$ embeds in $G(\mathbb{R})$, which is one of the factors in the definition of $A_S$). Since we are in situation (b), this implies that $\alpha$ must factor through the projection $A^\Gamma_S \to \pi_A(T^\Gamma)$. Therefore, $\alpha$ can be extended to a continuous homomorphism defined on $A^\Gamma_S$, by specifying that the extension is trivial on the open, normal subgroup $C_A$.

Finally, we remark that $\pi_A(\Gamma)$ can be identified with $\Gamma$, since the restriction of $\pi_A$ to $\Gamma$ is bijective (as was noted above). □

3. Solvable groups

In this section, we use Theorem 2.3 to establish several results on the superrigidity of $S$-arithmetic subgroups of solvable groups. We begin with the following slight generalization of Theorem 1.7.

**Theorem 3.1.** Suppose $\Gamma$ is a Zariski-dense, $S$-arithmetic subgroup of a connected, solvable algebraic $\mathbb{Q}$-group $G$. If either $S \neq \emptyset$ or $[G, G] = \text{unip} G$, then $\Gamma$ is $L$-superrigid in $G^\Gamma_S$, for every local field $L$.

**Proof.** Since [6, Thm. 1.10] treats the case where $L$ is archimedean, we may assume $L$ is nonarchimedean. Therefore, it suffices to verify the hypotheses of Theorem 2.3. Since $\Gamma$ is $S$-arithmetic, Assumptions 2.3(1) and 2.3(2) are immediate. The semisimple group $M$ is trivial, since $G$ is solvable, so Assumption 2.3(3) is trivially true. See Lemma 3.2 below for Assumption 2.3(4) if $S \neq \emptyset$. On the other hand, if $S = \emptyset$, then we must have $[G, G] = \text{unip} G$, so $[\Gamma, \Gamma]$ is Zariski-dense in unip $G$, and must therefore have finite index in $U^\Gamma$, so Assumption 2.3(4) is immediate. □

**Lemma 3.2 (¶ Lem. 7.5.4, p. 164).** Let

- $F$ be an algebraic number field,
- $U$ be a unipotent algebraic group defined over $F$,
- $S$ be a finite set of nonarchimedean places of $F$, and
- $\Gamma$ be an $S$-arithmetic subgroup of $U$.

If $S \neq \emptyset$, then $\Gamma$ has no infinite, cyclic quotients.

**Remark 3.3.** Note that if $\Gamma$ is superrigid in $G$, then $\Gamma$ is also superrigid in $G \times H$, for any group $H$. Therefore, in the statement of Theorem 3.1 (and in many of our other results), the assumption that $\Gamma$ is superrigid in $G$ can be replaced with the weaker assumption that $\Gamma$ is Zariski-dense in the quotient $G/\mathbb{Z}$, where $\mathbb{Z}$ is the center of $G$. In other words, $\Gamma$ is Zariski-dense in $\text{Ad} G$.

If $U$ is any unipotent $\mathbb{Q}$-group, then $\text{rank}_\mathbb{R} U = 0$ (since $U$ has no nontrivial tori), so Theorem 1.5 applies. However, we can say a bit more in this unipotent case.

**Definition 3.4.** A (countable) subgroup $\Gamma$ of a topological group $G$ is strictly $L$-superrigid in $G$ if $\Gamma$ is $L$-superrigid in $G$ and the subgroup $K$ in Definition 1.1 can always be taken to be trivial.
Proposition 3.5. Let

- $U$ be a unipotent algebraic group defined over $\mathbb{Q}$,
- $S$ be a finite set of prime numbers,
- $\Gamma$ be an $S$-arithmetic subgroup of $U$, and
- $L$ be a local field.

If $S \neq \emptyset$, then $\Gamma$ is $L$-superrigid in $U_S$.

More precisely, suppose $\alpha: \Gamma \to \text{GL}_n(L)$ is any homomorphism. Then:

1. If either $\text{char} \ L \neq 0$, or $L$ is a finite extension of $\mathbb{Q}_p$, with $p \notin S$, then $\alpha(\Gamma)$ is contained in a compact subgroup of $\text{GL}_n(L)$.
2. If $L = \mathbb{Q}_p$, for some $p \in S$, then there exists a unique rational homomorphism $\hat{\alpha}: U \to \text{GL}_n$, defined over $\mathbb{Q}_p$, such that, for some finite-index subgroup $\Gamma_0$ of $\Gamma$, we have $\alpha(\gamma) = \hat{\alpha}(\gamma)$ for all $\gamma \in \Gamma_0$. Therefore, $\Gamma$ is strictly $\mathbb{Q}_p$-superrigid.

Proof. (1) This is implicit in Remark 2.8 and Case 2 of the proof of Theorem 2.3.

(2) Recall that a group $G$ is said to be “$p$-radicable” if every element of $G$ has a $p$th root in $G$. Since $p \in S$, it is obvious that the additive abelian group $\mathbb{Z}_S$ is $p$-radicable. Since $U/[U, U]$ is a direct sum of 1-dimensional unipotent groups, we conclude that $\Gamma$ has a finite-index subgroup $\Gamma'$ whose abelianization is $p$-radicable. By a straightforward induction on the nilpotence class of $\Gamma'$, this implies that $\Gamma'$ is $p$-radicable [1 Prop. 2.4.2, p. 33].

By passing to a finite-index subgroup of $\Gamma$, we may assume $\alpha(\Gamma)$ is connected. Then, since $\alpha(\Gamma)$, like $\Gamma$, is nilpotent, we may write $\alpha(\Gamma) = C \times V$, where $C$ is a torus and $V$ is unipotent. Let $\alpha_C: \Gamma \to C$ be the projection of $\alpha$ to $C$. As in Case 1 of the proof of Theorem 2.3, we see that $\alpha_C(\Gamma)$ is contained in the maximal compact subgroup $E$ of $C(\mathbb{Q}_p)$.

However:

- $E$ has a finite-index subgroup that is pro-$p$ (cf. [4 Lem. 3.8, p. 138]), and
- no $p$-radicable group has a nontrivial homomorphism to any pro-$p$ group.

Therefore, $\alpha_C(\Gamma)$ must be finite. Hence, by passing to a finite-index subgroup, we may assume that $\alpha_C(\Gamma)$ is trivial. Then $C$ is trivial, so $\Gamma = V$ is unipotent. Hence, Corollary 2.3 provides an extension of $\alpha$ to a rational homomorphism $\hat{\alpha}: U \to \text{GL}_n$. □

Remark 3.6. The assumption that $S \neq \emptyset$ is necessary in Proposition 3.5 (unless $U$ is trivial or $L$ is archimedean). To see this, suppose $S = \emptyset$ and $U$ is nontrivial. Then $U_S = U_\emptyset = U(\mathbb{R})$ is connected, so there is no nontrivial, continuous homomorphism from $U_S$ to the totally disconnected group $\mathbb{Q}_p^\times = \text{GL}_1(\mathbb{Q}_p)$. Therefore, if $\Gamma$ were $\mathbb{Q}_p$-superrigid in $U_S$, then every homomorphism from $\Gamma$ to $\mathbb{Q}_p^\times$ would have bounded image. However, to the contrary, $\Gamma = U(\mathbb{Z})$ is a nontrivial, finitely generated, torsion-free, nilpotent group, so there exists a nontrivial homomorphism from $\Gamma$ to $\mathbb{Z}$. (That is, Assumption 2.3(4) fails.) Hence, there is a homomorphism $\alpha: \Gamma \to \mathbb{Q}_p^\times$ with unbounded image, for any prime $p$.

We now consider groups that are defined over extensions of $\mathbb{Q}$.

Notation 3.7. Suppose $F$ is an algebraic number field, $S$ is a set of nonarchimedean places of $F$, and $G$ is an algebraic group over $F$.

- char $v$ denotes the residue characteristic of a nonarchimedean place $v$ of $F$ (so $F_v$ is a finite extension of $\mathbb{Q}_{\text{char } v}$).
- char $S = \{ \text{char } v \mid v \in S \}$.
- For any prime number $p$, we let $S_p = \{ v \in S \mid \text{char } v = p \}$. 

\[ \begin{align*}
\bullet \ G_p &= \times_{v \in S_p} G(F_v). \\
\bullet \ S_\infty &\text{ denotes the set of archimedean places of } F. \\
\bullet \ G_S &= \left( \times_{v \in S_\infty} G(F_v) \right) \times \left( \times_{p \in \text{char } S} G_p \right). 
\end{align*} \]

Note that \( G_p \) can be thought of as (the \( \mathbb{Q}_p \)-points of) an algebraic group over \( \mathbb{Q}_p \).

We have the following generalization of Theorem 1.7.

**Theorem 3.8.** Suppose

- \( F \) is a finite extension of \( \mathbb{Q} \),
- \( G \) is a connected, solvable algebraic over \( F \),
- \( S \) is a finite set of nonarchimedean places of \( F \), and
- \( \Gamma \) is a Zariski-dense, \( S \)-arithmetic subgroup of \( G \).

If either \( S \neq \emptyset \) or \([G, G] = \text{unip } G\), then \( \Gamma \) is \( L \)-superrigid in \((\text{Res}_{F/\mathbb{Q}} G)^{\Gamma}_{\text{char } S}\), for every local field \( L \).

**Proof.** This is essentially the same as the proof of Theorem 3.1, but with \( \text{Res}_{F/\mathbb{Q}} G \) in the role of \( G \), and \( \text{char } S \) in the role of \( S \). \( \square \)

The above theorem is somewhat unsatisfactory, because the hypotheses deal with objects \( G \) and \( S \) that are defined over the number field \( F \), but the conclusion replaces them with corresponding objects over \( \mathbb{Q} \). The following result eliminates this shortcoming, at the expense of a Zariski-density assumption.

**Corollary 3.9.** Assume \( F, G, S, \) and \( \Gamma \) are as in Theorem 3.8. Let

- \( C/US \) be the (unique) maximal compact subgroup of the abelian group \( G_S/US \), where \( U \) is the unipotent radical of \( G \), and
- \( \overline{G}^\Gamma_S = \Gamma \cdot C \).

If \( \Gamma \) is Zariski-dense in \( \text{Res}_{F/\mathbb{Q}} G \) and either \( S \neq \emptyset \) or \([G, G] = \text{unip } G\), then \( \Gamma \) is \( L \)-superrigid in \( \overline{G}^\Gamma_S \), for every local field \( L \).

**Proof.** For each \( p \in \text{char } S \), let

\[ S_p = \{ \text{nonarchimedean valuations } v \text{ of } F \mid \text{char } v = p \}, \]

and \( \hat{G}_p = \times_{v \in S_p \setminus \overline{S}_p} G(F_v) \). Since [3, Thm. 1.10] treats the case where \( L \) is archimedean, we may assume \( L \) is nonarchimedean. Then the hypotheses of Theorem 2.3 can be verified as in the proof of Theorem 3.1, so Corollary 2.9 applies, because

\[ (\text{Res}_{F/\mathbb{Q}} G)(\mathbb{Q}_p) \cong G(F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cong G \left( \bigoplus_{v \in S_p} F_v \right) \cong \times_{v \in S_p} G(F_v) \cong G_p \times \hat{G}_p, \]

and the image of \( \Gamma \) in \( \hat{G}_p \) is compact by definition. \( \square \)

**Corollary 3.10.** If \( U \) is a unipotent algebraic group over an algebraic number field \( F \), and \( S \) is any nonempty, finite set of nonarchimedean places of \( F \), then every \( S \)-arithmetic subgroup of \( U \) is \( L \)-superrigid in \( US \), for every local field \( L \).
Proof. Every \((S\text{-})\text{arithmetic subgroup of } U\) is Zariski-dense in \(\text{Res}_{F/Q} U\). (Indeed, if \(O\) is the ring of integers of \(F\), then it is well known that \(U(O) \cong (\text{Res}_{F/Q} U)(\mathbb{Z})\) is a lattice in \((\text{Res}_{F/Q} U)(\mathbb{R})\), and is therefore Zariski-dense.) Hence, the desired conclusion is immediate from Corollary 3.9.

\[\square\]

Remark 3.11.

(A) In the special case of Corollary 3.10 in which \(S\) contains every valuation \(v\) of \(F\) with \(\text{char } v = p\), the group \(\Gamma\) is strictly \(\mathbb{Q}_p\)-superrigid in \(U_S\). (In fact, \(\Gamma\) is strictly \(\mathbb{Q}_p\)-superrigid in \(U_p\).) To prove this, note that \(1/p\) is an \(S\)-integer (in other words, \(1/p \in O_S\)), so the conclusion follows from the argument in the first and third paragraphs of the proof of Proposition 3.5(2) (with \(O_S\) in the place of \(Z_S\)).

(B) On the other hand, there do exist cases of Corollary 3.10 in which the superrigidity of \(\Gamma\) is not strict. For example, let \(F = \mathbb{Q}[i]\) and \(p = 5\). We have \(p = ab\) with \(a = 2 + i\) and \(b = 2 - i\). Let \(v\) be the valuation corresponding to the prime ideal \((a)\), so \(\Gamma = \mathbb{Z}[i, 1/a]\) is a \(\{v\}\)-arithmetic subgroup of the one-dimensional unipotent group \(F\). Since \(p\Gamma = (1/a)p\Gamma = b\Gamma\), and the norm of \(b\) is \(p\), we see that \(\Gamma/p\Gamma\) is cyclic of order \(p\), so \(\mathbb{Z} + p\Gamma = \Gamma\). By induction on \(k\), this implies \(\mathbb{Z} + p^k\Gamma = \Gamma\) for all \(k \in \mathbb{Z}^+\), so \(\Gamma/p^k\Gamma \cong \mathbb{Z}/p^k\mathbb{Z}\). Passing to the projective limit yields a surjective homomorphism \(\alpha\) from \(\Gamma\) onto \(\mathbb{Z}_p\). The resulting composite homomorphism

\[\Gamma \xrightarrow{\alpha} \mathbb{Z}_p \hookrightarrow \mathbb{Z} \times \mathbb{Z}_p \times \frac{\mathbb{Z}}{(p-1)\mathbb{Z}} \cong \mathbb{Q}_p^\times\]

does not extend to a continuous homomorphism defined on all of \(\mathbb{Q}_p\) (since \(\mathbb{Q}_p\) is \(p\)-radicable).

We remark that the Zariski-density assumption in Corollary 3.9 can be replaced with a restriction on \(S\):

Corollary 3.12. Assume \(F\), \(G\), \(S\), and \(\Gamma\) are as in Corollary 3.9, and either \(S = \emptyset\) or \([G, G] = \text{unip } G\). If

\[S = \{ \text{valuations } v \text{ of } F \mid \text{char } v \in \text{char } S \},\]

then \(\Gamma\) is \(L\)-superrigid in \(G^\Gamma_S\), for every local field \(L\).

Proof. The assumption on \(S\) implies that \(G^\Gamma_S \cong (\text{Res}_{F/Q} G)^\Gamma_{\text{char } S}\), so this is a special case of Theorem 3.8. \(\square\)

4. Groups that are not solvable

Let us recall the famous Margulis Superrigidity Theorem, which tells us that Assumption 2.3(3) is often true.

Definition 4.1. Suppose \(G\) is an algebraic group over an algebraic number field \(F\), and \(S\) is a finite set of nonarchimedean places of \(F\). The \(S\)-rank of \(G\) is \(\sum_{v \in S \cup S_\infty} \text{rank}_{F_v} G\), where \(S_\infty\) is the set of archimedean places of \(F\).

Theorem 4.2 (Margulis [3, 8.B, pp. 258–259]). Suppose

- \(M\) is a connected, semisimple algebraic group over a number field \(F\),
- \(S\) is a finite set of nonarchimedean places of \(F\), and
- the \(S\)-rank of every \(F\)-simple factor of \(M\) is at least two.

Then every \(S\)-arithmetic subgroup of \(M\) is semisimply superrigid in \(M_S\).
Combining this with Theorem 2.3 and Corollary 2.9 (and the arguments of Section 3) yields:

**Corollary 4.3.** Let

- $G$ be a connected algebraic group over a number field $F$,
- $S$ be a finite set of nonarchimedean places of $F$,
- $\Gamma$ be an $S$-arithmetic subgroup of $G$ that is Zariski-dense, and
- $L$ be a local field.

Assume:

- the $S$-rank of every $F$-simple factor of $G/\text{Rad}G$ is at least two, and
- either $S \neq \emptyset$ or $\text{unip}G \subseteq [G,G]$.

Then:

1. $\Gamma$ is $L$-superrigid in $(\text{Res}_{F/Q} G)^\Gamma_{\text{char } S}$.
2. If $\Gamma$ is Zariski-dense in $\text{Res}_{F/Q} G$, then $\Gamma$ is $L$-superrigid in $\mathfrak{g}^\Gamma_S$.

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