Stochastic Inversion of Gaussian Random Media Using Transverse Coherence Functions for Reflected Waves: Theory and Method

Hao Hu\textsuperscript{1} and Yingcai Zheng\textsuperscript{1}

\textsuperscript{1}Department of Earth and Atmospheric Sciences, The University of Houston, Houston, TX, USA

Abstract The transverse coherence functions (TCFs) of phase and amplitude fluctuations of a seismic wave are powerful to estimate the spatial distribution, length scales, and strength of random heterogeneities. However, TCFs have been formulated for transmitted waves only, not for reflected waves. In this paper, we derive reflection TCFs for Gaussian random media using the mirror reflection principle. Furthermore, we propose to invert for Gaussian random media using the reflection TCFs based on a grid search. We validate the new reflection TCF formulas using 2-D elastic finite-difference numerically modeled seismic data. We also show the feasibility and efficiency of the inverse problem. The stochastic inversion using reflected waves can be used in both exploration and global seismology.

1. Introduction

Earth is heterogeneous across multiple scales and the heterogeneities may be due to variation in rock composition, porosity, fluid content, or thermal states. Information of random heterogeneities in the global Earth can be used to infer dynamics and mixing processes (e.g., Anderson, 2006; Li & Zheng, 2019; Xu et al., 2008). A full deterministic description of the heterogeneities is neither possible nor desirable and a statistical description can be much more practical and useful. In exploration geophysics, the knowledge of small-scale heterogeneities is also important to assess the oil/gas/geothermal volume in the reservoir evaluation and production (e.g., Huang et al., 2012; Meng et al., 2017).

Many seismic scattering methods have been developed to characterize random media in terms of their statistical parameters in different regions of the Earth. Examples include well-logging analysis (e.g., Fukushima et al., 2003; Sivaji et al., 2002; Wu et al., 1994), seismic tomography (e.g., Meschede & Romanowicz, 2015; Nakata & Beroza, 2015), seismic envelope analysis for the mantle (e.g., Aki & Chouet, 1975; Emoto et al., 2017; Hedlin & Shearer, 2002; Jannaud et al., 1991; Sato, 1984; Wu, 1982), and the core (e.g., Peng et al., 2008; Vidale & Earle, 2000, 2005). The transmission fluctuation analysis using seismic array data is another method and is the focus of our paper (e.g., Chen & Aki, 1991; Flatte & Wu, 1988; Sivaji et al., 2001; Wu & Flatte, 1990; Yoshimoto et al., 2015; Zheng, 2012; Zheng & Wu, 2005, 2008).

In geophysics, Aki (1973) pioneered the inference of the spatial spectrum of velocity heterogeneity from measuring seismic wave amplitude and phase fluctuations using the Chernov theory (Chernov, 1960). Flatte and Wu (1988) developed the angular coherence functions (ACFs), to estimate the depth-dependent heterogeneity spectra. Wu and Flatte (1990) further formulated the joint transverse-ACFs (JTACF) using the Rytov and parabolic approximation. Later Wu and Xie (1991) used JTACF to invert for the heterogeneity spectra with a better depth resolution. Chen and Aki (1991) independently derived the JTACF, but using the Born approximation. Line et al. (1998) tried to use reflected waves from active sources in the stochastic inversion of transverse coherence functions (TCFs) but still based on the transmission TCF formulas. Zheng and Wu (2005) discussed the measurements of phase and amplitude and pointed out that the unwrapped phase and spectral amplitude should be used in forming TCFs. Zheng et al. (2007) proposed to invert for the heterogeneity spectrum of a single layer of stationary random heterogeneities via the Fourier transform of the sum of the log-amplitude and phase TCFs. Up to this point, all previous TCF theories were formulated for heterogeneities with a constant background medium. Zheng and Wu (2008) further extended the TCF theory to arbitrary depth-dependent background media.

However, the commonly used TCF methods were formulated for transmitted waves. The reflection case has not been done analytically. Transmission TCFs are useful for teleseismic or cross-well observations. The
reflection TCFs will be useful not only in earthquake seismology but also for exploration seismology where surface seismic surveys are readily available due to active sources.

In this paper, we first review the TCF method and propose the TCFs of reflected waves for a heterogeneous layer of a finite thickness, containing Gaussian random heterogeneities. The theory is derived based on the scalar wave equation. However, due to similarity in the forward scattering between acoustic wave and the elastic $P$ wave, the theory can be used for elastic waves. We then validate the accuracy of reflection TCFs using numerical experiments and finite-difference modeling for both acoustic and elastic wave fields. Finally, we show how to invert for the Gaussian heterogeneity spectrum using reflection TCFs.

2. Reflection TCFs Theory in Random Media

Here, we first formulate the reflection TCFs in the context of acoustic waves. The applications to elastic waves will be validated using elastic full-wave modeling later. We consider the reflection geometry shown in Figure 1 and a plane wave is incident upon a random medium layer and then reflected up to the receivers, by a reflector at the bottom of the random layer (Figure 1).

We measure the logarithmic amplitude ($\logA(x)$), and phase, $\phi(x)$, of the reflected wave, recorded at the surface receiver located at $x$. Usually, the measurement is done around a particular frequency. The amplitude is the spectral amplitude and the phase is the unwrapped phase (Zheng & Wu, 2005), for the reflected arrival. The TCFs for the $\logA$ ($\langle uu \rangle$) and the phase ($\langle \phi \phi \rangle$) are defined as follows:

$$
\langle uu \rangle = \langle u(x_1)u(x_2) \rangle, \quad \langle \phi \phi \rangle = \langle \phi(x_1)\phi(x_2) \rangle.
$$

(1)

$\langle \rangle$ means the ensemble average; $x_1$ and $x_2$ are two receiver locations. For a stationary and statistically isotropic (i.e., no preferred directions in the statistical sense) random medium, both $\langle uu \rangle$ and $\langle \phi \phi \rangle$ depend only on the station transverse distance, $|x_2 - x_1|$.

The random medium is characterized by a stationary random acoustic-wave velocity field, $v = v(x, z)$. We define the velocity perturbation as $\delta v(x, z) = \frac{v_0^2}{2} \left( \frac{v^2}{v_0^2} - 1 \right)$; $v_0$ is a constant background velocity of the medium. The random medium has a correlation function,

$$
W(|x'' - x'|, |z'' - z'|) = \langle \delta v(x'', z'') \delta v(x', z') \rangle.
$$

(2)

where $(x', z')$ and $(x'', z'')$ indicate two arbitrary perturbation positions in the random medium. Because the random medium is stationary, $W$ only depends on the relative position of these two locations. In the stochastic random medium inversion, we invert for $W$, not the velocity field $v(x, z)$.

In the following, we propose to use a “mirror reflection principle” to show how to derive the reflection TCF formula from the medium correlation function, $W$. We treat the reflector as a mirror, which creates a mirror image for both the random medium and the receivers (Figure 2). Now we have converted this problem into a transmission TCF problem.

Figure 1. A schematic geometry of receivers (blue triangles) and a heterogeneous layer with Gaussian random perturbations in the $P$ wave velocity. The density of the medium is constant. The thickness of the heterogeneous layer is $L$. With an incident plane wave (downward black dashed arrows), receivers will record waves (upward red dashed arrows) reflected from the lower-boundary of the heterogeneous layer.
2.1. Transmission TCFs

We first need to review the TCF theory for transmitted waves. Assume there is a Gaussian random heterogeneous layer with a total thickness of $H$. The incident wave is a downgoing plane wave. There are several assumptions in deriving the TCF formula: (1) The scattering is mainly forward (i.e., the parabolic approximation), (2) the medium velocity perturbation is weak and smooth, and (3) we ignore multiple scattering (Rytov approximation). With these assumptions, for a 2-D acoustic wave field, the amplitude and phase TCFs can be expressed as (Zheng et al., 2007):

$$
\langle uu \rangle = I_1(r_x) + I_2(r_x),
$$
$$
\langle \phi\phi \rangle = I_1(r_x) - I_2(r_x),
$$

where $\langle uu \rangle$ and $\langle \phi\phi \rangle$ are the logA and phase TCFs for a transverse lag between two receivers ($r_x = |x_2 - x_1|$), respectively; $I_1(r_x)$ and $I_2(r_x)$ are two auxiliary functions that can be obtained:

$$
I_1(r_x) = \frac{k^2}{4\pi^2} \int dk_x e^{ik_x r_x} \int_0^H W(\kappa_x, \eta) \cos\left(\frac{\eta k_x^2}{2k}\right) (H - \eta) d\eta
$$
$$
I_2(r_x) = \frac{k^3}{4\pi^2} \int dk_x e^{ik_x r_x} \int_0^H W(\kappa_x, \eta) \left[\sin\left(\frac{\eta k_x^2}{2k}\right) - \sin\left(\frac{2H - \eta}{2k}\right)\right] d\eta,
$$

where $k = \omega/\nu_0$ is the wavenumber of the measured frequency $\omega$ in the background velocity $\nu_0$;

$$
\bar{W}(\kappa_x, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik_x r_x} W(r_x, \eta) dr_x
$$

is the medium perturbation correlation function $W(r_x, \eta)$ with the correlation lags $|x'' - x'| = r_x$ and $|z'' - z'| = \eta$ at a lateral wavenumber of $\kappa_x$.

Additionally, the power spectrum density function (PSDF) $P(\kappa_x, \kappa_z)$ of the random medium could be estimated from the $W(r_x, \eta)$ by Fourier transform in an infinite medium (Tatarskii, 1971; Wu & Flatte, 1990):

$$
P(\kappa_x, \kappa_z) = \frac{1}{4\pi^2} \int dr_x d\eta e^{-ik_x r_x - i\kappa_z \eta} W(r_x, \eta).
$$

To evaluate integrals, $I_1$ and $I_2$, in TCFs (Equation 4), Zheng et al. (2007) assumed that $W(r_x, |z'' - z'|)$ is very small. In many cases, people just used the delta-correlation assumption, $W(r_x, |z'' - z'|) \approx 0$.

Figure 2. The geometry used in the derivation of reflection TCFs. (a) Reflected wave raypath in the random medium form the source to the receiver; (b) transmitted path in the mirror medium from the source to the mirror receiver. The thickness of the heterogeneous layer is $L$. In (b), the mirrored heterogeneous layer is symmetric about $z = 0$. 

10.1029/2020JB020385

Journal of Geophysical Research: Solid Earth

HU AND ZHENG 3 of 13
2.2. Reflection TCFs

However, in our reflection TCFs case, we consider the TCF for a mirror medium (Figure 2b). In this case, the TCF formulas are the same except we need to consider, $W(r_x, i(z'' - z'''))$ for $i(z'' - z''') > \varepsilon_z$ due to the mirror reflection of the random medium.

Because of the mirror, we need to modify $W$ in Equation 5. A detailed derivation can be found in Appendix A1 (Equations A11 and A12). For a mirrored heterogeneous layer with isotropic Gaussian random perturbations, we can obtain the following correlation function and its PSDF in 2-D:

$$W_{\text{mirror}}(r_x, r_z) = \varepsilon_z^2 \exp\left(-\frac{r_x^2}{a^2}\right) \left\{ \exp\left(-\frac{r_z^2}{a^2}\right) + \frac{1}{2\sqrt{\pi}a} \text{erf}\left(\frac{r_z}{a}\right) - r_z \exp\left(-\frac{r_z^2}{a^2}\right) \right\}, \tag{7}$$

$$P_{\text{mirror}}(\kappa_x, \kappa_z) = \varepsilon_z^2 a^2 \exp\left(-\frac{1}{4} \kappa_x^2 a^2\right) \left\{ \exp\left(-\frac{1}{4} \kappa_z^2 a^2\right) + i \frac{1}{4L} \left(a^2 \kappa_z - \frac{2}{\kappa_z}\right) \right\}, \tag{8}$$

where $r_x$ and $r_z$ are the horizontal and vertical correlation lags; $\kappa_x$ and $\kappa_z$ are their corresponding wave-numbers; $\varepsilon$ and $a$ are the Gaussian perturbation strength and scale, respectively; $2L$ is the total thickness of the mirrored random medium (Figure 2b).

To compute the reflection TCFs, we just need to plug Expression 7 into Equations 3–5 to replace $W$ and its spectrum $\tilde{W}$.

3. Modeling TCFs for Gaussian Random Media

To validate the new formulas of reflection TCFs, we build a numerical model containing a Gaussian random layer (Figure 3) to model the reflected waves. The perturbation correlation functions follow the Gaussian function with a correlation length $a = 100$ m and a perturbation strength $\varepsilon = 0.01$. The thickness of the heterogeneous layer is 500 m (Figure 3a). We use a staggered-grid finite-difference (FD) numerical solution of the two-way acoustic wave equations to model the wave propagation in the numerical model (e.g., Graves, 1996; Virieux, 1986). The wavelet of the incident vertical plane wave is a Ricker wavelet with a dominant frequency at 20 Hz. The wavelength is 200 m. One example showing the recorded waveforms is in Figure 3b. The measured phase and logA at 20 Hz are in Figures 3c and 3d. After 100 random realizations, we can obtain the ensemble averages for phase and logA TCFs (in Figure 4). Meanwhile, in Figure 4 we can compare the measured TCFs from reflected waves and the theoretical reflection TCFs (Equations 3–5 and 7). Overall, the measured TCFs fit well with the theoretically predicted TCFs. It also validates our new formulas for reflected waves.

There are some small discrepancies between the measured TCFs and theoretical TCFs due to the limited number of random realizations and model discretization in the numerical modeling of wave propagations. It can also be due to the use of the mirror assumption that neglects the interference of the backscattering effect because it uses transmission geometry for the reflected wave (Line et al., 1998).

In our mirror approach, it is implicitly assumed that the waves are totally reflected from the interface (Figure 2). In reality, part of the energy is transmitted and cannot be recorded. The mirror principle gives higher recorded amplitudes. However, this is not an issue for our TCF so long as the background model is layered. TCF uses the relative phase and logA (not the absolute amplitudes) fluctuations of the reflected $P$ waves. The phase fluctuation is mainly accumulated along the total travel path. Likewise, the reflection coefficient due to the interface is a constant shift in logA and this constant is removed in logA fluctuation, which is mainly due to focusing and defocusing along the path. Since we are using plane waves, we do not need to consider the angle (offset)-dependent reflection coefficient.

4. Inversion of the Heterogeneous Medium Using Reflected Waves

Once we have obtained the phase and logA TCF from data, we can do a grid search for the Gaussian random medium parameters (the perturbation strength $\varepsilon$ and the scale $a$) that can yield theoretical TCFs to best fit the measured TCFs. We assume we know the random layer thickness, $L$. We can formulate this inverse problem by minimizing the following misfit function:
\[ F(r_0|a, \varepsilon) = |\langle uu \rangle(r_0|a, \varepsilon) - \langle uu \rangle_{data}(r_0)| \\
+ |\langle \phi \phi \rangle(r_0|a, \varepsilon) - \langle \phi \phi \rangle_{data}(r_0)|, \]

where \( \langle uu \rangle(r_0|a, \varepsilon) \) and \( \langle \phi \phi \rangle(r_0|a, \varepsilon) \) are theoretical TCFs at a correlation lag, \( r_0 \), for a trial proposal for \( a \) and \( \varepsilon \), using Equations 3–5 and 7. \( \langle uu \rangle_{data} \) and \( \langle \phi \phi \rangle_{data} \) are the TCFs measured from the recorded reflected waves.

Figure 3. Numerical modeling of the reflected wavefield fluctuation. (a) The model and geometry for modeling the wave. The incident wave is a vertical plane wave initiated at the depth of 500 m (red dots). The depth of receivers is 500 m (represented by black triangles) and they are horizontally deployed from 250 to 7,750 m at an interval of 5 m. In the heterogeneous layer, the background \( P \) velocity is 4,000 m/s and the density is \( \rho = 2,000 \text{ kg/m}^3 \). The layer beneath the heterogeneous layer has \( V_p = 2,000 \text{ m/s} \) and \( \rho = 1,000 \text{ kg/m}^3 \) to generate strong reflected waves. The recorded reflected waves are in (b)-(d) are the measured phase and logA, respectively. The time window length in the TCF measurements is 0.08 s centered at the predicted traveltime.
waves. We will validate the inversion of Gaussian random medium parameters ($a$ and $\varepsilon$) using the numerical data from the previous section.

Based on the measured TCFs of reflected waves (in Figure 4), we perform a grid search to calculate the misfit function between the measured reflection TCFs and the theoretical reflection TCFs using Equation 9. The trial $a$ varies from 50 to 200 m at an interval of 5 m while $\varepsilon$ from 0.05 to 0.015 at an interval of 0.005. The receiver correlation lag $r_0$ varies from 10 to 250 m at an interval of 10 m. The misfit map is shown in Figure 5 as a function of $\varepsilon$ and $a$. The global minimum occurs at $a = 110$ m and $\varepsilon = 0.095$, which are very close to the true values of the Gaussian random model. This numerical experiment illustrates that the inversion using reflection TCFs base on grid search is feasible and efficient.

5. Discussion and Conclusions

5.1. Reflection TCFs and Transmission TCFs

Line et al. (1998) tried to characterize the random heterogeneities in the crust using phase and logA measured from reflection data. However, they used the transmission TCF formula on reflection data to invert for the random medium parameters. Because the reflected path is two way, in their TCF inversion the random layer thickness is doubled, but they did not use the mirror technique we proposed here. Therefore, it would be worthwhile to investigate the error in using their proposed approach.

We consider two cases:

1. Case 1 (Reflection). Similar to the case in Figure 3a, we use FD to model the reflected wave and measure the phase and logA fluctuations and we then compute the TCFs (i.e., measured reflection TCFs) using Equation 9. The trial $a$ varies from 50 to 200 m at an interval of 5 m while $\varepsilon$ from 0.05 to 0.015 at an interval of 0.005. The receiver correlation lag $r_0$ varies from 10 to 250 m at an interval of 10 m. The misfit map is shown in Figure 5 as a function of $\varepsilon$ and $a$. The global minimum occurs at $a = 110$ m and $\varepsilon = 0.095$, which are very close to the true values of the Gaussian random model. This numerical experiment illustrates that the inversion using reflection TCFs base on grid search is feasible and efficient.
We use FD to model the transmitted wave and measure its phase and logA fluctuations to form the TCFs (i.e., measured transmission TCFs). We also use the transmission TCF formula to compute the theoretical transmitted TCFs. This case corresponds to Line et al. (1998).

We can see in both Case 1 and Case 2, the measured and the theoretically predicted TCFs agree well with each other (Figures 6b and 6c). We can also see the phase TCF for the reflection case is overall larger than the transmission phase TCF, up to ~30%. Our proposed mirror approach did a much better job in predicting the measured reflection TCFs than the transmission theory (i.e., thickness doubled). The strength of the logA TCF for the reflection wave in Case 1 is similar to the transmission wave in Case 2. These differences demonstrate clearly that in order to invert for the random medium parameters using the measured reflection TCFs, our newly derived formulas using a mirror approach could achieve more accurate inversion results. To be clear, in our numerical examples, we use Gaussian random medium that has one dominant scale. The other types of random medium with more complex spectrum are not tested in this paper.

5.2. Validity of Applying the Acoustic Theory to Elastic Waves

We derived the reflection TCF formulas in the context of acoustic fluid media. However, the Earth is elastic. The reason why the acoustic theory may be valid for elastic data is the following. For most part of the reflected elastic P wave path (except around the reflection boundary), it is forward scattering which only depends on the P wave velocity perturbation strength and scale (e.g., Wu & Aki, 1985). Because of this scattering property, the acoustic theory works well in the regime of weak scattering and forward scattering. To further investigate whether the acoustic formulas can be indeed used to describe the stochastic properties of an elastic wave in a random medium, we did numerical elastic modeling. In the elastic model, we adopt the same model configuration, including the source, receivers, P wave velocity (Vp) model and density model...
The S wave velocity ($V_s$) model is converted from the background $P$ wave velocity modeling using a constant $V_p/V_s$ ratio of 1.7. We also use a staggered-grid FD numerical solution of the full-wave elastic wave equations to model the elastic wave propagation in the elastic model. The recorded waveforms are vertical-component particle velocity. The measured reflected $P$ wave phase and logA TCFs of are shown in Figure 7. The time window length used in the 20 Hz TCF measurements is 0.08 s. The measurement time window should contain the reflected wavelet but exclude other noise (e.g., mode conversions). We can see the measured TCFs are similar to the acoustic cases and our acoustic theory predicts the measured elastic wave TCFs. The elastic phenomena, such as mode conversions and $S$ wave scattering, have slight influence on the measured $P$ wave reflection TCFs. This numerical test demonstrates that our reflection TCF formulas based on the acoustic assumption are appropriate to be used for the elastic wave.

5.3. Other Types of Random Media

The newly proposed reflection TCF formula is derived for Gaussian random media. For other non-Gaussian media, such as the multiscale von Karman-type (e.g., Klimeš, 2002; Sato, 2019; Sato et al., 2012; Wu, 1982), we can also modify the random medium correlation function using the mirror reflection. Moreover, our formulas can also be extended to 3-D by changing the form of the perturbation correlation function.

In our numerical validation, we assume the random medium is Gaussian with one dominant scale that could benefit the accuracy of acoustic TCFs using Rytov approximation within a certain scale range. The other types of random medium may have multiple-scale perturbations and could weaken the accuracy of acoustic assumption. The receiver geometry may also affect the measurement accuracy of TCFs if they are not regularly and densely distributed. The effects of other types of random medium and receiver geometry need further investigations.

5.4. Source Time Function and Wavelet Variations in Practical Applications of Reflection TCFs

Our reflection TCFs could be applied in exploration geophysics and global seismology to characterize the random heterogeneities. TCF is a weak-scattering theory and is applicable when the recorded wavelets are similar. The variations in source time functions could impact the measurement of reflection TCFs in phase and logA. Such variabilities can directly come from the source itself due to azimuthal dependent radiation or due to long-distance travel paths in strongly scattering media (Fehler et al., 2000). In exploration seismology, we may use the locally stacked data after move-out correction or migration to mitigate the effect of source time function variations. In global earthquake seismology, we may use array seismograms from teleseismic whose wave front could be approximated as plane wave and radiation pattern may have negligible
effects on the recorded wavelets. In practice, the choice of frequency also matters. We may choose ~1.0 Hz or lower in frequency to get similar wavelets in global seismology. To address the amplitude variation, Cormier et al. (2020) used the inverted moment tensor source for each earthquake to compute the background wavefield and then obtain the fluctuations.

5.5. Conclusions

We derived new formula for reflection TCFs for the phase and logA fluctuations using a mirror reflection principle using acoustic wave theory and parabolic forward scattering as well as single-scattering Rytov approximation. We validate the new formula using 2-D numerical experiments via full-wave finite-difference modeling of wave propagation in random media. The measured phase and logA TCFs of reflected elastic waves match well with those theoretical TCFs. Our TCF theory is applicable for weak-scattering and similar recorded wavelets. We also showed how to use the measured reflection TCFs from data to invert for Gaussian medium parameters, including the heterogeneity scale and the random perturbation strength using a grid-search scheme. Our newly derived formula and our stochastic inversion of TCFs using reflected waves have many potential applications in exploration geophysics for resource characterization and in global seismology.

Appendix A: Correlation Functions of Mirrored Random Media

A1. Correlation Functions of Mirrored Random Media: Theoretical Derivation

For a 2-D isotropic Gaussian random medium, the perturbation correlation function and its PSDF can be expressed as (e.g., Zheng & Wu, 2008):

\[
W(\tau_x, \tau_z) = \varepsilon^2 \exp \left( -\frac{\tau_x^2}{a^2} \right) \exp \left( -\frac{\tau_z^2}{a^2} \right),
\]

\[
P(\kappa_x, \kappa_z) = \varepsilon^2 \pi a^2 \exp \left( -\frac{1}{4} \kappa_x^2 a^2 \right) \exp \left( -\frac{1}{4} \kappa_z^2 a^2 \right),
\]

where \(\tau_x\) and \(\tau_z\) are the horizontal and vertical correlation lags between two arbitrary perturbations; \(\kappa_x\) and \(\kappa_z\) are their corresponding wavenumbers; \(\varepsilon\) is the perturbation strength; \(a\) is the Gaussian correlation length that controls the perturbation scale. The 2-D Gaussian correlation function can be decoupled by multiplication of two 1-D Gaussian functions, \(\exp \left( -\frac{\tau_x^2}{a^2} \right)\) in the \(x\) direction and \(\exp \left( -\frac{\tau_z^2}{a^2} \right)\) in the \(z\) direction. For a mirrored Gaussian random medium with a symmetry axis at \(z = 0\) (see Figure 2b), the correlation function along the \(x\) axis at a constant depth is still Gaussian. However, the correlation function along the \(z\) axis needs to be reevaluated.

To make the derivation easier to follow, we first consider the correlation function for a 1-D mirror medium. Assume we have a 1-D random medium (no mirror yet) with a limited thickness \(L\). The velocity perturbation correlation function is

\[
\langle \delta v(z) \delta v(z + r) \rangle = \frac{1}{L} \int_0^L \delta v(z) \delta v(z + r) dz = W(r),
\]

\[
W(r) = \varepsilon^2 \exp \left( -\frac{r^2}{a^2} \right),
\]

where \(\langle \rangle\) implies the statistical ensemble averaging and an ensemble is defined as a set of random media with the same correlation function and each copy of the medium is called a random realization; \(\delta v = \frac{1}{2} \left( \frac{v^2}{v_0^2} - 1 \right)\) is the velocity perturbation; \(v_0\) and \(v\) are the background (i.e., no heterogeneities) and true velocity fields (i.e., with heterogeneities), respectively; \(r\) is the vertical distance between two perturbations.

We can extend the random layer \(\delta v(z)\) from \(z \in [0, L]\) to \(z \in [-L, 0]\) by a mirror reflection (see Figure 2b). The mirrored random layer \(\delta v_{\text{mirror}}(z)\) is now a medium with a thickness of \(2L\) and can be expressed as follows:
Consequently, the 1-D correlation functions \( W_{\text{mirror}}(r) \) could be written as follows:

\[
W_{\text{mirror}}(r) = \langle \delta v_{\text{mirror}}(z) \delta v_{\text{mirror}}(z + r) \rangle = \frac{1}{2L} \int_{-L}^{L} \langle \delta v_{\text{mirror}}(z) \delta v_{\text{mirror}}(z + r) \rangle dz \\
= \frac{1}{2L} \left[ \int_{-L}^{0} \langle \delta v(z) \delta v(z + r) \rangle dz + \int_{0}^{L} \langle \delta v(z) \delta v(z + r) \rangle dz \right] \\
= \frac{1}{2L} \left[ \int_{-L}^{0} \langle \delta v(z) \delta v(z + r) \rangle dz + \int_{0}^{L} \langle \delta v(z) \delta v(z + r) \rangle dz \right] + \frac{1}{2} W(r) \\
= \frac{1}{2L} \int_{0}^{L} \langle \delta v(z) \delta v(z - r) \rangle dz + \frac{1}{2L} \int_{r}^{L} \langle \delta v(z) \delta v(z - r) \rangle dz + \frac{1}{2} W(r),
\]

(A6)

where there are three terms. The \( z \) and \( r - z \) in the first term in Equation A6 are ≥0. Equation A6 can be calculated as follows:

\[
\frac{1}{2L} \int_{0}^{r} \langle \delta v(z) \delta v(r - z) \rangle dz = \frac{1}{2L} \int_{0}^{r} e^z \exp \left( -\frac{(z - r + z)^2}{a^2} \right) dz = \frac{1}{2L} e^z \cdot \frac{1}{2} \sqrt{\pi a} \text{erf} \left( \frac{r}{a} \right), 
\]

(A7)

where \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt \) is the Gauss error function. This term is the correction of the correlation near the boundary (\( z \in [0, r] \)) caused by the mirrored perturbations.

In the second term of Equation A6, the \( z \geq 0 \) while \( r - z \leq 0 \). The second term could be rewritten as follows:

\[
\frac{1}{2L} \int_{r}^{L} \langle \delta v(z) \delta v(z - r) \rangle dz = \frac{1}{2L} \int_{0}^{L} \langle \delta v(z) \delta v(z - r) \rangle dz - \frac{1}{2L} \int_{0}^{r} \langle \delta v(z) \delta v(z - r) \rangle dz \\
= \frac{1}{2} W(r) - \frac{r}{2L} e^z \exp \left( -\frac{r^2}{a^2} \right).
\]

(A8)

By inserting Equations A7 and A8, Equation A6 can be rewritten as follows:

\[
W_{\text{mirror}}(r) = W(r) + \frac{1}{2L} e^z \left[ \frac{1}{2} \sqrt{\pi a} \text{erf} \left( \frac{r}{a} \right) - \exp \left( -\frac{r^2}{a^2} \right) \right].
\]

(A9)

Here we can see the mirrored Gaussian random has a different correlation function compared to the one-layer Gaussian random. There are two terms as a result of mirroring. In addition, these two correction terms are related to the random medium thickness \( L \) as well as the correlation lag \( r \). If the layer thickness \( L \) is infinite, \( W_{\text{mirror}} \) will be identical to \( W \). If the correlation lag \( r \) is comparable with \( L \), the correction terms could not be ignored.

We can also get the PSDF for the mirrored medium using the Fourier transform:

\[
P_{\text{mirror}}(k_r) = \int W_{\text{mirror}}(r) e^{-i2\pi k_r r} dr = P(k_r) + \frac{1}{2L} e^z \left\{ \frac{\sqrt{\pi a}}{2} \text{FT} \left[ \text{erf} \left( \frac{r}{a} \right) \right] + \frac{1}{2} \sqrt{\pi a} k_r \exp \left( -\frac{1}{4} a^2 k_r^2 \right) \right\} \\
= P(k_r) + \frac{1}{2L} e^z \left\{ \frac{2 \exp \left( -\frac{1}{4} a^2 k_r^2 \right)}{\sqrt{\pi a} k_r} \right\} + \frac{1}{2} \sqrt{\pi a} k_r \exp \left( -\frac{1}{4} a^2 k_r^2 \right) \\
= P(k_r) + \frac{i}{4L} \left( a^2 k_r - \frac{2}{k_r} \right) e^z \sqrt{\pi a} \exp \left( -\frac{1}{4} a^2 k_r^2 \right) \\
= e^z \sqrt{\pi a} \exp \left( -\frac{1}{4} a^2 k_r^2 \right) \left[ 1 + \frac{i}{4L} \left( a^2 k_r - \frac{2}{k_r} \right) \right],
\]

(A10)

where \( k_r \) is the wavenumber corresponding to the correlation lag, \( r \).
For the 2-D case, the mirrored medium with a horizontal symmetry axis (such as Figure 2b), we can obtain the correlation function and the associated spectrum in 2-D:

$$W_{\text{mirror}}(r_x, r_z) = \varepsilon^2 \exp\left(-\frac{r_x^2}{a^2}\right) \left\{ \exp\left(-\frac{r_z^2}{a^2}\right) + \frac{1}{2L} \left[ \frac{1}{\sqrt{\pi}} \text{erf}\left(\frac{r_z}{a}\right) - r_z \exp\left(-\frac{r_z^2}{a^2}\right) \right] \right\}, \quad (A11)$$

$$P_{\text{mirror}}(\kappa_x, \kappa_z) = \varepsilon^2 \pi a^2 \exp\left(-\frac{1}{4} \kappa_x^2 a^2\right) \left[ \exp\left(-\frac{1}{4} \kappa_z^2 a^2\right) + i \frac{4L}{a} a^2 \kappa_z - \frac{2}{\kappa_z^2} \right]. \quad (A12)$$

A2. Numerical Validation of the Correlation Function of Mirrored Random Media for 1-D Case

We validate the newly derived correlation functions for mirrored random perturbations by numerical experiments for the 1-D case. We first generate a 1-D Gaussian random heterogeneous model in the spectral domain (e.g., Shapiro & Kneib, 1993), called one-layer Gaussian medium. In the spectral domain, for each wavenumber, we set its spectral amplitude according to a Gaussian function and assign it with a random phase within $[-\pi, \pi]$. Then we transform it to the space domain to form one random medium realization. When assigning the random phase, we need to honor the complex conjugate properties such that the space domain function is real valued. We then generate the mirrored Gaussian medium by mirror extending the one-layer Gaussian medium. To show the relationship between the correlation function of the mirrored Gaussian medium and the perturbation scale, we chose four cases with Gaussian correlation length $a = 25, 50, 75, \text{ and } 100 \text{ m}$. The perturbation strength $\varepsilon$ in four case is the same 0.01. The thickness of one-layer Gaussian medium is 500 m hence the mirrored Gaussian random medium is 1,000 m. After 100 random realizations, we can calculate the statistical ensemble averaged correlation functions and compare them with the theoretical formula we just derived in Equations A4 and A9 (Figure A1). From Figure A1, we can see...
the our new formula for the mirrored Gaussian medium fit the measurements very well, from zero lag distance to a few times of a. For the one-layer Gaussian medium, the correlation functions converge to zero for lag beyond 2a while the mirrored Gaussian medium does not. The difference of correlation function between the one-layer Gaussian medium and the mirrored Gaussian medium becomes more obvious when the perturbation scale is large. There are some small negligible discrepancies between the theoretical correlation functions and measured ones due to limited number of realizations. These numerical experiments validate our new formula of the perturbation correlation function for the mirrored Gaussian medium is accurate. We can use the new perturbation correlation function to calculate the TCFs of reflected waves.

Data Availability Statement

The synthetic data were generated using a standard staggered-grid finite-difference forward modeling scheme (e.g., Graves, 1996; Virieux, 1986). We provide a Matlab package (called TCF_Ref2D) that can generate random medium models and calculate theoretical TCFs. We have deposited the package at the Texas Data Repository (https://doi.org/10.18738/T8/5WJQIU) for public access and long term preservation.

References

Aki, K. (1973). Scattering of P waves under the Montana Lasa. Journal of Geophysical Research, 78(8), 1334–1346.

Aki, K., & Chouet, B. (1975). Origin of coda waves—Source, attenuation, and scattering effects. Journal of Geophysical Research, 80(23), 3322–3342. https://doi.org/10.1029/JB080i023p03322

Anderson, D. L. (2006). Speculations on the nature and cause of mantle heterogeneity. Tectonophysics, 416(1–4), 7–22. https://doi.org/10.1016/j.tecto.2005.07.011

Chen, X., & Aki, K. (1991). General coherence functions for amplitude and phase fluctuations in a randomly heterogeneous medium. Geophysical Journal International, 105(1), 155–162. https://doi.org/10.1111/j.1365-246X.1991.tb03451.x

Cernov, L. A. (1960). Wave propagation in a random medium. New York: McGraw-Hill.

Cormier, V. F., Tian, Y., & Zheng, Y. (2020). Heterogeneity spectrum of Earth’s upper mantle obtained from the coherence of teleseismic P waves. Communications in Computational Physics, 28, 74–97. https://doi.org/10.4208/cicp.oa-2018-0079

Emoto, K., Saito, T., & Shiomi, K. (2017). Statistical parameters of random heterogeneity estimated by analysing coda waves based on finite difference method. Geophysical Journal International, 211, 1575–1584. https://doi.org/10.1093/gji/ggx387

Fehler, M., Sato, H., & Huang, L. J. (2000). Envelope broadening of outgoing waves in 2D random media: A comparison between the Markov approximation and numerical simulations. Bulletin of the Seismological Society of America, 90(4), 914–928. https://doi.org/10.1785/0119990143

Flatte, S. M., & Wu, R. S. (1988). Small-scale structure in the lithosphere and asthenosphere deduced from arrival time and amplitude fluctuations at norsar. Journal of Geophysical Research, 93(B6), 6601–6614. https://doi.org/10.1029/JB093iB06p06601

Fukushima, Y., Nishizawa, O., Sato, H., & Ohtake, M. (2003). Laboratory study on scattering characteristics of shear waves in rock samples. Bulletin of the Seismological Society of America, 93(1), 253–263. https://doi.org/10.1785/0120020074

Graves, R. W. (1996). Simulating seismic wave propagation in 3D elastic media using staggered-grid finite differences. Bulletin of the Seismological Society of America, 86(4), 1091–1106.

Hedlin, M. A. H., & Shearer, P. M. (2002). Probing mid-mantle heterogeneity using pkp coda waves. Physics of the Earth and Planetary Interiors, 130(3–4), 195–208. https://doi.org/10.1016/S0031-9201(02)00007-9

Huang, J.-W., Belleur, G., & Mikkereit, B. (2012). Application of conditional simulation of heterogeneous rock properties to seismic scattering and attenuation analysis in gas hydrate reservoirs. Journal of Applied Geophysics, 77, 83–96. https://doi.org/10.1016/j.jappgeo.2011.12.002

Jannaud, L. R., Adler, P. M., & Jacquin, C. G. (1991). Spectral-analysis and inversion of coda. Journal of Geophysical Research, 96(B11), 18,215–18,231. https://doi.org/10.1029/91JB01427

Klimel, L. (2002). Correlation functions of random media. Pure and Applied Geophysics, 159(7–8), 1811–1831. https://doi.org/10.1007/s00024-002-8710-2

Li, J., & Zheng, Y. (2019). Generation of a stochastic binary field that fits a given heterogeneity power spectrum. Geophysical Journal International, 217, 294–300. https://doi.org/10.1093/gji/ggz024

Line, C. E. R., Hobbs, R. W., Hudson, J. A., & Snyder, D. B. (1998). Statistical inversion of controlled-source seismic data using parabolic wave scattering theory. Geophysical Journal International, 132(1), 61–78.

Meng, X. C., Wang, S. X., Tang, G. Y., Li, J. N., & Sun, C. (2017). Stochastic parameter estimation of heterogeneity from crosswell seismic data based on the Monte Carlo radiative transfer theory. Journal of Geophysics and Engineering, 14, 621–633. https://doi.org/10.1088/1742-2140/aab130

Meschede, M., & Romanowicz, B. (2015). Lateral heterogeneity scales in regional and global upper mantle shear velocity models. Geophysical Journal International, 200, 1076–1093. https://doi.org/10.1093/gji/ggv424

Nakata, N., & Beroza, G. C. (2015). Stochastic characterization of mesoscale seismic velocity heterogeneity in Long Beach, California. Geophysical Journal International, 203, 2049–2054. https://doi.org/10.1093/gji/ggv421

Peng, Z. G., Koper, K. D., Vidale, J. E., Leyton, F., & Shearer, P. (2008). Inner-core fine-scale structure from scattered waves recorded by LASA. Journal of Geophysical Research, 113, B09312. https://doi.org/10.1029/2007JB005412

Sato, H. (1984). Attenuation and envelope formation of 3-component seismograms of small local earthquakes in randomly inhomogeneous lithosphere. Journal of Geophysical Research, 89(B2), 1221–1241. https://doi.org/10.1029/JB089iB02p1221

Sato, H. (2019). Power spectra of random heterogeneities in the solid earth. Solid Earth, 10, 275–292. https://doi.org/10.5194/se-10-275-2019

Sato, H., Fehler, M. C., & Maeda, T. (2012). Seismic wave propagation and scattering in the heterogeneous earth. Berlin Heidelberg: Springer Science & Business Media.
Shapiro, S. A., & Kneib, G. (1993). Seismic attenuation by scattering—Theory and numerical results. Geophysical Journal International, 114(2), 373–391. https://doi.org/10.1111/j.1365-246X.1993.tb03925.x

Sivaji, C., Nishizawa, O., & Fukushima, Y. (2001). Relationship between fluctuations of arrival time and energy of seismic waves and scale length of heterogeneity: An inference from experimental study. Bulletin of the Seismological Society of America, 91(2), 292–303. https://doi.org/10.1785/0120000046

Sivaji, C., Nishizawa, O., Kitagawa, G., & Fukushima, Y. (2002). A physical-model study of the statistics of seismic waveform fluctuations in random heterogeneous media. Geophysical Journal International, 148(3), 575–595. https://doi.org/10.1046/j.1365-246X.2002.01606.x

Tatarskii, V. I. (1971). The effects of the turbulent atmosphere on wave propagation. Retrieved from http://articles.adsabs.harvard.edu/full/1971etaw.book.....T/0000003.000.html

Vidale, J. E., & Earle, P. S. (2000). Fine-scale heterogeneity in the Earth’s inner core. Nature, 404(6775), 273–275. https://doi.org/10.1038/35005059

Vidale, J. E., & Earle, P. S. (2005). Evidence for inner-core rotation from possible changes with time in PKP coda. Geophysical Research Letters, 32, L01309. https://doi.org/10.1029/2004GL021240

Virieux, J. (1986). P-SV-wave propagation in heterogeneous media—Velocity-stress finite-difference method. Geophysics, 51(4), 889–901. https://doi.org/10.1190/1.1442147

Wu, R., & Aki, K. (1985). Scattering characteristics of elastic-waves by an elastic heterogeneity. Geophysics, 50(4), 582–595. https://doi.org/10.1190/1.1441934

Wu, R. S. (1982). Attenuation of short period seismic waves due to scattering. Geophysical Research Letters, 9(1), 9–12. https://doi.org/10.1029/gl009i001p00009

Wu, R. S., & Flatte, S. M. (1990). Transmission fluctuations across an array and heterogeneities in the crust and upper mantle. Pure and Applied Geophysics, 132(1–2), 175–196. https://doi.org/10.1007/BF00674362

Xu, W. B., Lithgow-Bertelloni, C., Stixrude, L., & Ritsema, J. (2008). The effect of bulk composition and temperature on mantle seismic structure. Earth and Planetary Science Letters, 275(1–2), 70–79. https://doi.org/10.1016/j.epsl.2008.08.012

Zheng, Y. (2012). Scale lengths of heterogeneities under Tibet. Earthquake Science, 25(5–6), 409–414. https://doi.org/10.1007/s11589-012-0866-y

Zheng, Y. C., & Wu, R. S. (2005). Measurement of phase fluctuations for transmitted waves in random media. Geophysical Research Letters, 32, L14314. https://doi.org/10.1029/2005GL023179

Zheng, Y. C., & Wu, R. S. (2008). Theory of transmission fluctuations in random media with a depth-dependent background velocity structure. Advances in Geophysics, 50(50), 21–41. https://doi.org/10.1016/S0065-2687(08)00002-2

Zheng, Y. C., Wu, R. S., & Lay, T. (2007). Inverting the power spectrum for a heterogeneous medium. Geophysical Journal International, 168(3), 1005–1010. https://doi.org/10.1111/j.1365-246X.2006.03241.x