Superconformal Chern-Simons theories beyond leading order

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Abstract

We discuss higher-order corrections to superconformal invariance for a class of $\mathcal{N} = 2$ supersymmetric Chern-Simons theories including the ABJM model. We argue that corrections are inevitable for general theories in this class; but that it is probable that any corrections are of a particular “maximally transcendental” form.
1 Introduction

Chern-Simons gauge theories have attracted attention for a considerable time due to their topological nature [1–3] (in the pure gauge case) and their possible relation to the quantum Hall effect and high-$T_c$ superconductivity. More recently there has been substantial interest in $\mathcal{N} = 2$ supersymmetric Chern-Simons matter theories in the context of the AdS/CFT correspondence and in particular, a wide range of superconformal theories has been discovered [4]–[29], starting with the BLG [8, 9] and ABJ/ABJM [12, 24] models. Although a more familiar formulation is in terms of “quiver”-type gauge theories based on the gauge group $U(N) \times U(M)$, many of them may be understood in terms of an underlying “3-algebra” structure [8], [30]–[38]. It was shown in Ref. [39] that $\mathcal{N} = 3$ Chern-Simons theories (which include the ABJ, ABJM models as special cases) are exactly superconformal to all orders. Explicit perturbative computations to corroborate the superconformal property have been carried out in Refs. [32, 40, 41] at lowest order (two loops for a theory in three dimensions). Since the gauge coupling $\beta$-function is zero for any Chern-Simons theory [42] due to the topological nature of the theory (and indeed is quantised at certain values—the Chern-Simons “level”) it is only necessary to compute the anomalous dimensions of the chiral fields in order to check for superconformality (in view of the non-renormalisation theorem). Our purpose here is to attempt to extend the explicit check of superconformality beyond lowest order, and beyond the $\mathcal{N} = 3$ theories which are already known to be exactly superconformal. These $\mathcal{N} = 3$ superconformal theories involve a simple choice of the superpotential couplings in terms of the Chern-Simons level, and the fact that this choice renders the theory finite to all orders is analogous to the case of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetric theories in four dimensions, where the finiteness properties are manifest to all orders in the $\mathcal{N} = 1$ superfield description once the field content and superpotential have been specified (assuming a supersymmetric regulator such as DRED). However, an alternative possibility is that one might have to adjust the couplings order by order so as to achieve finiteness [43, 44]. This would be more analogous to the case of finite $\mathcal{N} = 1$ theories in four dimensions, where the finiteness is obtained through an order-by-order adjustment of the couplings. We might well expect this behaviour in theories where superconformality is achieved by solving a somewhat non-trivial condition at lowest order.

In odd spacetime dimensions, divergences only occur at even loop order, so to go beyond leading order we are driven to consider a four-loop calculation. The total number of diagrams is colossal; so here we report on what can be learned from the consideration of a subset of the full set of diagrams, namely those which have at least one (in fact at least two) Yukawa vertices. We were able to compute all the relevant diagrams with the exception of a single non-planar diagram. Our conclusions are as follows: firstly, we note that the contributions to the anomalous dimension at this order fall into two classes, proportional respectively to $F^4$ and $\pi^2 F^4$, where $F$ is the usual factor associated with loops in dimensional regularisation, in 3 dimensions $F = \frac{1}{8\pi}$. The latter class has been called “maximally transcendental” [45, 46], and we shall call the former “rational”. We then show that the maximally-transcendental contributions to the four-loop anomalous dimension in
general fail to vanish upon imposing lowest-order superconformality and hence require a coupling redefinition to restore superconformality. We shall consider in some detail the case of multi-trace deformations where the non-vanishing contributions where it is particularly clear that a redefinition will always be required. On the other hand, we shall show that (at least to leading order in \( N, M \), and probably to all orders) the “rational” contributions to the four-loop anomalous dimension do vanish, for a large class of theories, once the lowest-order superconformality conditions are imposed.

Finally, we discuss the possibility that any non-vanishing redefinition required might be expressible in a simple general form, analogous to experience in four dimensions; this turns out to require that the divergent contribution from a certain non-planar diagram, which we have been unable to compute explicitly, must take a certain value.

The paper is organised as follows: in Section 2 we describe the general \( \mathcal{N} = 2 \) supersymmetric \( \mathcal{U}(N) \times \mathcal{U}(M) \) Chern-Simons theory in three dimensions together with various choices of superpotential and couplings which render it superconformal; in Section 3 we describe our calculations and give our main results; and in Section 4 we discuss the issue of coupling redefinitions from the standpoint of a more general theory. Section 5 contains some brief conclusions, and we explain our conventions and list various useful basic results and identities in an Appendix.

## \( \mathcal{N} = 2 \) Chern-Simons theory in three dimensions

We consider an \( \mathcal{N} = 2 \) supersymmetric \( \mathcal{U}(N) \times \mathcal{U}(M) \) Chern-Simons theory with vector multiplets \( \mathcal{V}, \hat{\mathcal{V}} \) in the adjoint representations of \( \mathcal{U}(N) \) and \( \mathcal{U}(M) \) respectively, and we write

\[
\mathcal{V}^b_a = \mathcal{V}^A(R_A)^b_a, \quad \hat{\mathcal{V}}^\hat{b}_{\hat{a}} = \hat{\mathcal{V}}^\hat{A}({\hat{R}}_\hat{A})^\hat{b}_{\hat{a}},
\]

where \( R_A, A = 1, \ldots N^2 \) and \( \hat{R}_\hat{A}, \hat{A} = 1, \ldots M^2 \) are the generators for the fundamental representations of \( \mathcal{U}(N), \mathcal{U}(M) \) respectively.

The vector multiplets are coupled to chiral multiplets \((A^i)^a_{\dot{a}}\) and \((B_i)_{\dot{a}}^\dot{a}\), \( i = 1, 2 \) in the \((N, \overline{M})\) and \((\overline{N}, M)\) representations of the gauge group, respectively. The gauge matrices \( R_A \) satisfy

\[
[R_A, R_B] = i f_{ABC} R_C, \\
\text{Tr}(R_A R_B) = \delta_{AB},
\]

with similar expressions for \( \hat{R}_\hat{A} \) with structure constants \( \hat{f}_{ABC} \).

The action for the theory can be written

\[
S = S_{\text{SUSY}} + S_{\text{GF}}
\]
where $S_{\text{SUSY}}$ is the usual supersymmetric action [47]

$$
S_{\text{SUSY}} = \int d^3x \int d^4\theta \int_0^1 dt \left\{ K_1 \text{Tr}[D^\alpha (e^{-tV} D_\alpha e^{tV})] + K_2 \text{Tr}[\overline{D}^\alpha (e^{-t\overline{V}} D_\alpha e^{t\overline{V}})] \right\}
+ \int d^3x \int d^4\theta \text{Tr} \left( \overline{A}_i e^V A_i e^{-\overline{V}} + \overline{B}_i e^\overline{V} B_i e^{-V} \right)
+ \left( \int d^3x \int d^2\theta W(A_i, B_i) + \text{h.c.} \right). \tag{4}
$$

Here the superpotential (quartic for renormalisability in three dimensions) $W(A_i, B_i)$ is given by

$$
W(A_i, B_i) = \text{Tr}[h_1(A_i^1 B_i^1)^2 + h_2(A_i^2 B_i^2)^2 + h_3 A_i^1 B_i^1 A_i^2 B_i^2 + h_4 A_i^2 B_i^1 A_i^1 B_i^2]
+ \frac{1}{2} H_1[\text{Tr}(A_i^1 B_i^1)]^2 + H_12 \text{Tr}(A_i^1 B_i^1) \text{Tr}(A_i^2 B_i^2) + \frac{1}{2} H_2[\text{Tr}(A_i^2 B_i^2)]^2 \tag{5}
$$

Gauge invariance requires $2\pi K_1$ and $2\pi K_2$ to be integers.

A variety of interesting theories may be obtained by specialising the superpotential in Eq. (5) and the gauge group and associated Chern-Simons levels in various ways. For $H_1, H_2 = H_{12} = 0$, $h_1 = h_2 = \frac{1}{2} \left( \frac{1}{K_1} + \frac{1}{K_2} \right)$, $h_3 = \frac{1}{K_1}$, $h_4 = \frac{1}{K_2}$, we obtain the $\mathcal{N} = 3$ superconformal theory described in Ref. [20]. Specialising to $K_1 = -K_2 = K$, so that $h_1 = h_2 = 0$, $h_3 = -h_4 = h$, we obtain the $\mathcal{N} = 2$ ABJM/ABJ-like theories studied in Ref. [40]. In particular, for $h = \frac{1}{K}$ one obtains the $\mathcal{N} = 6$ superconformal ABJ theory and for $N = M$ the ABJM theory. Additional more general superconformal theories may be found by solving the lowest order finiteness conditions (see later). Further superconformal theories may also be obtained by adding flavour matter [41].

We now consider the details of gauge fixing and quantisation for our Chern-Simons theory. In each gauge sector we choose a gauge-fixing term $S_{\text{GF}}$ in Eq. (3) given by [41]

$$
S_{\text{GF}} = \frac{K}{2\alpha} \int d^3x d^2\theta \text{tr}[ff] - \frac{K}{2\alpha} \int d^3x d^2\theta \text{tr}[\overline{ff}] \tag{7}
$$

and we introduce into the functional integral a corresponding ghost term

$$
\int Df D\overline{f} \Delta(V) \Delta^{-1}V \tag{8}
$$

with

$$
\Delta(V) = \int d\Lambda d\overline{\Lambda} \delta(F(V, \Lambda, \overline{\Lambda}) - f) \delta(F(V, \Lambda, \overline{\Lambda}) - \overline{f}), \tag{9}
$$

with $\overline{F} = D^2V, F = \overline{D}^2V$. With $\alpha = 0$ this results in a gauge propagator for $V$ of the form

$$
\langle V^A(1)V^B(2) \rangle = -\frac{1}{K_1} \frac{1}{\partial^2} \frac{1}{\partial^2} D_\alpha \delta^A \delta^B \delta^{AB}, \tag{10}
$$

\[ \text{3} \]
with a similar propagator for $\hat{V}$. The gauge vertices are obtained by expanding $S_{SUSY} + S_{GF}$ as given by Eqs. (4), (7):

$$S_{SUSY} + S_{GF} \rightarrow -\frac{i}{6} K_1 f^{ABC} \int d^3 x d^4 \theta D^A V^A D_\alpha V^B V^C$$

$$-\frac{i}{6} K_2 \hat{f}^{ABC} \int d^3 x d^4 \theta D^A \hat{V}^A D_\alpha \hat{V}^B \hat{V}^C + \ldots$$

(11)

The ghost action resulting from Eq. (9) has the same form as in the four-dimensional $\mathcal{N} = 1$ case [48, 49]

$$S_{gh} = \int d^3 x d^4 \theta \text{tr}\{\bar{c}' c - c' \bar{c} + \frac{1}{2} (c + \bar{c})[V, c + \bar{c}] + \frac{1}{12} (c + \bar{c})[V, [V, c - \bar{c}]]\} + \ldots$$

(12)

leading to ghost propagators

$$\langle \bar{c}'(1) c(2) \rangle = -\langle c'(1) \bar{c}(2) \rangle = -\frac{1}{\partial^2} \delta^4(\theta_1 - \theta_2),$$

(13)

(together with similar expressions involving $\hat{V}$ and its own ghosts), and cubic and higher-order vertices which may easily be read off from Eq. (12). Finally the chiral propagator and chiral-gauge vertices are readily obtained by expanding Eq. (4); the chiral propagators are given by:

$$\langle \tilde{A}_i^a A^{jb}_b \rangle = -\frac{1}{\partial^2} \delta^4(\theta_1 - \theta_2) \delta_a^b \delta^i_j,$$

(14)

with a similar expression for the $B$-propagator.

The regularisation of the theory is effected by replacing $V, \hat{V}, A, B, h_i, H_i$ (and the various ghost fields) by corresponding bare quantities $V_B, \hat{V}_B, A_B, B_B, h_{Bi}, H_{Bi}$ (and similarly for the ghost fields) with the bare and renormalised fields related by

$$V_B = Z^\frac{1}{2}_V V, \quad Z_A = Z^\frac{1}{2}_A A,$$

(15)

eq.$
where \( \mu \) is the usual dimensional regularisation mass scale (introduced to preserve dimensions of couplings away from \( d = 3 \) dimensions). Using the fact that \( Z_{A^1} \) on the left-hand side of Eq. (16) is \( \mu \)-independent, while the renormalised couplings in \( Z_{A^1}^{(L,m)} \) are \( \mu \)-dependent, implies that \( \gamma_{A^1} \) is determined by the simple poles in \( Z_{A^1} \) according to
\[
\gamma_{A^1}^{(L)} = L Z_{A^1}^{(L,1)};
\]
and the higher order poles in \( Z_{A^1} \) are determined by consistency conditions, the one relevant for our purposes being
\[
16Z_{A^1}^{(4,2)} = 2 \left( \gamma_{A^1}^{(2)} \right)^2 - \sum_r \beta_{\lambda^r}^{(2)} \frac{\partial}{\partial \lambda^r} \gamma_{A^1}^{(2)}
\]
where \( \{ \lambda^r, r = 1 \ldots 14 \} = \{ h_i, \overline{h_i}, h_i, \overline{H_i}, H_{12}, \overline{H_{12}} \} \). The \( \beta \)-functions \( \beta_{\lambda^r} \) in Eq. (19) are defined as usual by (picking \( h_3 \) for instance)
\[
\beta_{h_3} = \mu \frac{d}{d\mu} h_3
\]
and measure the scale dependence of the renormalised couplings. For a superconformal theory all the \( \beta \)-functions must therefore vanish. Since the \( \beta \)-functions for the Chern-Simons levels \( K_{1,2} \) are expected to vanish for a generic Chern-Simons theory due to the topological nature of the theory (so that \( K_{B1,2} = K_{1,2} \)), superconformality will be determined purely by the vanishing of the \( \beta \)-functions for the superpotential couplings. For a general theory the \( \beta \)-functions are given in terms of the simple poles in the corresponding bare coupling, analogously to Eq. (18). However for \( \mathcal{N} = 2 \) supersymmetric theories in three dimensions (as for \( \mathcal{N} = 1 \) supersymmetric theories in four dimensions), the \( \beta \)-functions can be expressed according to the non-renormalisation theorem in terms of the anomalous dimensions of the fields associated with each coupling; for instance
\[
\beta_{h_3} = (\gamma_{A^1} + \gamma_{B_1} + \gamma_{A^2} + \gamma_{B_2}) h_3,
\]
with similar expressions for the other superpotential couplings (the \( \beta \)-function for any coupling is the same as that for its conjugate). At lowest order (two loops) it was found that superconformality (i.e. the vanishing of \( \beta_{\lambda^r} \)) was equivalent to the vanishing of all the corresponding anomalous dimensions (for the fields involved in the \( \lambda^r \) coupling) in all the cases considered.

### 3 Perturbative Calculations

In this section we review the two-loop calculation and describe in detail our four-loop results.

The renormalisation constants of the chiral superfields \( A^{1,2}, B_{1,2} \) are given at two loops by
\[
F^{-2} \gamma_{A^1}^{(2)} = 2(\rho_{A^1} - \rho_k)
\]

(22)
Figure 1: Two-loop diagrams

(with similar expressions for \( A^2, B_{1,2} \)) where \( F = \frac{1}{8\pi} \) as defined before and

\[
\begin{align*}
\rho_{A^1} &= \rho_{B_1} = 4|h_1|^2(MN + 1) + (|h_3|^2 + |h_4|^2)MN + (h_3h_4 + h_4h_3) \\
&\quad + MN(|H_1|^2 + |H_{12}|^2) + |H_1|^2, \\
\rho_{A^2} &= \rho_{B_2} = 4|h_2|^2(MN + 1) + (|h_3|^2 + |h_4|^2)MN + (h_3h_4 + h_4h_3) \\
&\quad + MN(|H_2|^2 + |H_{12}|^2) + |H_2|^2, \\
\rho_k &= (k_1^2 + k_2^2)(2MN + 1) + 2(MN + 2)k_1k_2,
\end{align*}
\]

(23)

with

\[
k_1 = \frac{1}{K_1}, \quad k_2 = \frac{1}{K_2}.
\]

(24)

This result may readily be obtained by \( \mathcal{N} = 2 \) superfield methods \[32,41,50,51\] from

the two-loop two-point diagrams depicted in Fig. 1; see the Appendix for our \( \mathcal{N} = 2 \)

superfield conventions. Here and later we do not distinguish in the diagrams between the

different chiral or gauge fields, so that each diagram in Fig. 1 is a schematic representation

of several distinct Feynman diagrams. \( \rho_{A^1} \) etc correspond to Fig. 1(a) while it may easily

be checked that

\[
\rho_k = \rho_b + \rho_c
\]

(25)

where the contributions \( \rho_{b,c} \) corresponding to Fig. 1(b,c) are given by

\[
\begin{align*}
\rho_b &= \frac{1}{2}(C_1 + C_2) = \frac{1}{2}\{(N^2 + 1)k_1^2 + (M^2 + 1)k_2^2 + 4MNk_1k_2\}, \\
\rho_c &= \frac{1}{2}[X_1k_1^2 + X_2Mk_2^2 + X_{12}k_1k_2],
\end{align*}
\]

(26)

with

\[
\begin{align*}
C_1 &= N^2k_1^2 + M^2k_2^2 + 2MNk_1k_2, \\
C_2 &= k_1^2 + k_2^2 + 2MNk_1k_2, \\
X_1 &= 4M - \frac{N^2 - 1}{N}, \\
X_2 &= 4N - \frac{M^2 - 1}{M}, \\
X_{12} &= 8.
\end{align*}
\]

(27)
$C_{1,2}$ correspond to the two different symmetrisations of the gauge lines in Fig. 1(b), while the $X_{1,2}, \chi_{12}$ correspond to the contributions from the “blob” in Fig. 1(c) which represents the three one-loop diagrams depicted in Fig. 2 (the dashed line representing a ghost propagator).

(We note here that the two-loop results for general Chern-Simons theories obtained in Ref. [52] are not directly comparable since they were computed in the $N = 1$ framework.)

As mentioned in the Introduction, we shall consider two classes of model in some detail; the first without, and the second with, multitrace deformations. We shall call these Class I, Class II theories respectively. Class I corresponds to taking $H_{1,2} = H_{12} = 0$ in Eq. (5); and in fact we start with the even simpler example of

$$H_{1,2} = H_{12} = 0, \quad h_1 = h_2 = 0, \quad h_3 + h_4 = 0,$$

(28)

with $h_3 = -h_4 = h$ real; we shall call this Class Ia. This is a class of theories considered in Ref. [20], which reduces to the ABJ model on setting $K_1 = -K_2$ (or $k_1 = -k_2$) and to the ABJM model on further setting $M = N$.

The four-loop diagrams with Yukawa couplings contributing to the anomalous dimensions are depicted in Figs. 3, 4. As we explained before, we have not considered the much larger set of diagrams with no Yukawa couplings, but we shall still be able to draw some
Figure 3: Four-loop diagrams
Figure 4: Four-loop diagrams (continued)
conclusions. The contributions to $F^{-4}Z_A^{(4)}$ from these diagrams are given by

$$G_a = 3\rho_h^2 I_4,$$
$$G_b = 2(MN^3 + NM^3 - 4M^2 - 4N^2 + 10MN - 4)h^4 I_{4bb},$$
$$G_c = -3\rho_h\rho_b I_4,$$
$$G_d = 3\rho_c C_2 I_{4bb},$$
$$G_e = -\rho_c h I_4,$$
$$G_f = 2\rho_c C_2 I_5,$$
$$G_g = -T_2 I_5,$$
$$G_h = -T_2 I_{4bb},$$
$$G_i = -T_1 I_5,$$
$$G_j = 4T_2 \left(I_4 - \frac{1}{2} I_{4bb}\right),$$
$$G_k = -2T_2 I_4,$$
$$G_l = -2T_1(-2I_4 + 2I_{4bb} + I_5),$$
$$G_m = -2\rho_c \left(I_4 - \frac{1}{2} I_{4bb}\right),$$
$$G_n = 3\rho_c I_{22},$$
$$G_o = -3\rho_c I_{22},$$
$$G_p = 2\rho_c T_3 (I_4 - J_4 - I_{42bc}),$$
$$G_q = -2\rho_c T_3 (I_4 - J_4),$$
$$G_r = \frac{1}{\varepsilon} T_1 (a + b\pi^2)$$

(29)

where

$$\rho_h = 2(MN - 1)h^2$$

(30)

is the common value of $\rho_{A^{1,2}}$, $\rho_{B^{1,2}}$ upon imposing Eq. [28] and

$$T_1 = h^2[(N^2 - 2MN + 1)k_1^2 + (M^2 - 2MN + 1)k_2^2$$
$$+ 2(M^2N^2 + 2MN - M^2 - N^2 - 1)k_1k_2],$$
$$T_2 = h^2[(N^3M + 5MN - 3N^2 - 3)k_1^2 + (M^3N + 5MN - 3M^2 - 3)k_2^2$$
$$+ 4(M^2 + N^2 - 3MN + 1)k_1k_2],$$
$$T_3 = 4[(k_1^2 + k_2^2)MN + 2k_1k_2],$$
$$T_4 = h^2[(3MN - N^2 - 2)k_1^2 + (3MN - M^2 - 2)k_2^2$$
$$+ 2(N^2 + M^2 - 3MN + 1)k_1k_2].$$

(31)

These quantities are not all independent and in fact (as a consequence of gauge invariance) satisfy the identities

$$2T_1 + T_2 = \rho_h \rho_b,$$
$$T_1 + T_4 = \frac{1}{2} \rho_h C_2,$$

(32)
where $\rho_b$, $C_2$ and $\rho_h$ are defined in Eqs. (26), (27) and (30). The results are expressed in terms of a basis of momentum integrals defined and computed in Ref. [45]. The divergent contributions from these momentum integrals are listed in the Appendix. We have not been able to compute the momentum integral corresponding to Fig. 4(r), and therefore $a$, $b$ in $G_r$ in Eq. (29) are unknown. This momentum integral is depicted in Fig. 5(g) in the Appendix. The contributions from Fig. 4(s)-(v) are all finite or zero and therefore not listed explicitly.

The full result obtained by summing the individual contributions in Eq. (29):

$$F^{-4}Z_{A_1}^{(4)} = G^{(4)},$$
$$G^{(4)} = G_a + \ldots + G_r + R,$$

where $R$ represents the (currently unknown) contribution from graphs with no Yukawa couplings, may be divided into transcendental and rational contributions (according to whether the contribution contains a factor of $\pi^2$ or not, coming from the right-hand sides of Eqs. (92) ) as

$$G^{(4)} = G_{\text{rat}} + G_{\text{trans}}\pi^2$$

The transcendental contribution is given (using Eq. (32)) by

$$G_{\text{trans}} = \frac{1}{\epsilon}\left\{h^4(MN^3 + NM^3 - 4M^2 - 4N^2 + 10MN - 4) + \frac{1}{6}[(5 + 3b)\rho_hC_2 + 3\rho_h\rho_k - 10\rho_h\rho_b + 2(4 - 3b)T_1]\right\} + R_{\text{trans}},$$

with obvious definitions for $R_{\text{trans}}$, $R_{\text{rat}}$. We shall postpone comment on this until later, and focus on the rational contribution, which is given (again using Eq. (32)) by

$$G_{\text{rat}} = 3\rho_h^2I_4 - 2\rho_h\rho_kI_4 - 2\rho_hT_3I_{42b} + \frac{a}{\epsilon}T_4 + R_{\text{rat}}$$

where $\rho_k$ is given by Eq. (23). We have used here the fact that $I_5$ as defined in Eq. (92) gives only a transcendental simple pole. Since $T_4$ is $O(N^2)$, the $a$ term from the non-planar graph $G_r$ certainly gives no contribution at leading order $O(N^4)$; and based on experience with non-planar graphs, we believe it is likely that $G_r$ gives a purely transcendental divergent contribution and hence $a = 0$. Upon imposing the two-loop superconformality condition $\gamma^{(2)} = 0$ in the form

$$\rho_h = \rho_k,$$

(37)

(using Eqs. (22), (23), (30)), we find

$$G_{c,\text{rat}} = \rho_k^2I_4 - 2\rho_kT_3I_{42b} + \frac{a}{\epsilon}T_4 + R_{\text{rat}},$$

where $G_c$ denotes the value of $G$ upon imposing leading-order superconformality. The value of $h = h_c(k_1, k_2)$ implicit in the 3rd term in Eq. (38) (according to Eq. (31)) will be determined by solving Eq. (37) and clearly depends on the particular form of the
superpotential. However the remaining terms in Eq. (38) are independent of $h$ and thus (since we see from Eq. (31) that the 3rd term is subleading in $N, M$) the form of $G_{\text{rat}}$ is independent of the form of the superpotential to leading order. In fact, it is straightforward to see that this result is more general and applies to any Class I theory, as we shall proceed to show. Firstly, the $I_4$ terms in Eq. (29) supply the double pole contributions of the form $h^4$ and $h^2k^2$; and this will remain the case for any Class I theory. The $h^4$ terms are given according to Eqs. (19), (23) and (22) by

$$\frac{1}{2} \rho_{A_1}^2 - 4(\rho_{A_1} + \rho_{B_1})h^2(MN + 1) - \frac{1}{2}(\rho_{A_1} + \rho_{B_1} + \rho_{A_2} + \rho_{B_2})[(|h_3|^2 + |h_4|^2)MN + h_3h_4 + h_4h_3]$$

(39)

which reduces to $-\frac{3}{2} \rho_k^2$ upon imposing the two-loop superconformal invariance condition, now from Eqs. (22)

$$\rho_{A_1,2} = \rho_{B_1,2} = \rho_k.$$  

(40)

This reproduces exactly the contribution of the first term in Eq. (36) to Eq. (38). The $h^2k^2$ terms are given according to Eq. (19) by $\rho_k \rho_{A_1}$ which of course reduces to $\rho_k^2$ upon imposing $\rho_{A_1} = \rho_k$. This reproduces exactly the contribution of the second term in Eq. (36) to Eq. (38). Furthermore, in the general case, the coefficient in $G_p$ in Eq. (29) becomes

$$- 2(\rho_{A_1} + \rho_{A_2} + \rho_{B_1} + \rho_{B_2})[(k_1^2 + k_2^2)MN + 2k_1k_2]$$

(41)

which reduces to

$$8\rho_k[(k_1^2 + k_2^2)MN + 2k_1k_2] = 2\rho_k T_3$$

(42)

upon imposing Eq. (40); now reproducing the contribution of the third term in Eq. (36) to Eq. (38). Finally, the contribution from the non-planar graph $G_r$ is subleading in $N, M$ for any theory with superpotential of the form Eq. (5) with $H_{1,2} = H_{12} = 0$; in fact the only reason we have had to exclude multi-trace deformations from the definition of Class I is that otherwise this is no longer true. Therefore the form of $G_{\text{rat}}$ in Eq. (38) is in general independent of the form of the potential at leading order in $M, N$ upon imposing the conformal invariance condition, as long as multi-trace deformations are excluded. Since we believe it likely that $a = 0$, this result may well also hold at lower orders and in the presence of multi-trace deformations.

The results from the remaining diagrams with no Yukawa couplings are of course also independent of the form of the potential, and the rational contribution from these graphs must take the form

$$R_{\text{rat}} = \rho_k^2 I_4 + \frac{1}{\epsilon} \delta(k_1, k_2)$$

(43)

in order that the total double pole contribution to $G_{c,\text{rat}}$ cancels, as it must due to the lower order superconformal invariance. We therefore obtain from Eqs. (32), (38), (43)

$$F^{-1} Z^{(4)}_{\text{rat}, A^4} = -2\rho_k T_3 I_{42bc} + \frac{a}{\epsilon} T_4 + \frac{1}{\epsilon} \delta(k_1, k_2).$$

(44)
We are left (using Eqs. (18), (92)) with an expression for the residual four-loop rational contribution to the anomalous dimension after imposing two-loop superconformal invariance, valid for any Class I theory (and any field, so we therefore suppress the field label):

\[ F^{-4} \gamma^{(4)}_{c,rat} = -16\rho_k T_3 + 4a\tilde{T}_4 + 4\delta(k_1, k_2). \]  

(45)

Here \( \tilde{T}_4 \) represents the generalisation to a general Class I theory of the expression in Eq. (31), which will now depend on \( h_{1-4} \). Now we know that \( \gamma^{(4)}_{rat} \) must vanish when \( h_{1-4} \) take the values given in Eq. (6) corresponding to \( \mathcal{N} = 3 \) supersymmetry, since this theory is superconformal without any renormalisation [39]. However in Eq. (45), \( h_{1-4} \) only appear in the \( a \) term, and in particular in the subleading part of \( \tilde{T}_4 \). If \( a = 0 \), we can immediately deduce that \( \delta(k_1, k_2) = 4\rho_k T_3 \) and hence (since this conclusion is independent of the values of \( h_{1-4} \)) that \( \gamma^{(4)}_{c,rat} = 0 \) for any Class I theory. However, if \( a \neq 0 \), the most we can say is that the leading term in \( \gamma^{(4)}_{c,rat} \) vanishes for any Class I theory; but a “rational” coupling redefinition may in general be required to restore superconformal invariance for the subleading terms–see later for the general form of this redefinition.

We believe that our result will also extend to the superconformal theories with flavour matter discussed in Ref. [41]; and, if \( a = 0 \) in \( G_r \) in Eq. (29), to theories with multi-trace deformations as well. We would be able to apply the same arguments in the case with flavour as in the situation just discussed, since the wide class of theories with flavour discussed in Ref. [41] contains an \( \mathcal{N} = 3 \) theory with flavour as a special case for particular choices of coupling; and once again this \( \mathcal{N} = 3 \) theory is exactly superconformal [22].

We shall not consider further here the transcendental contribution for the Class I models, since we can draw a more striking conclusion in the case of the Class II models; suffice it to say that the expression given in Eq. (35) for the Class 1a models clearly gives a model-dependent result upon imposing two-loop superconformality, Eq. (37) (in that the \( h^4 \) terms, and the \( h^2 \) terms in \( T_1 \), are a consequence of the choice of superpotential). Therefore although \( G_{trans} \) must vanish for \( h = k = k_1 = -k_2 \) (corresponding to \( \mathcal{N} = 3 \) supersymmetry) it cannot vanish for general \( h \) satisfying Eq. (37) and hence a “transcendental” coupling redefinition will be required to restore superconformality.

Before leaving the Class I models, we point out that by using \( \mathcal{N} = 3 \) superconformality we have determined the “rational” remainder term \( R_{rat} \) in Eq. (36), up to the unknown value of \( a \); and we could also obtain in a similar way the “transcendental” remainder term \( R_{trans} \) in Eq. (35), up to the unknown value of \( b \), and for \( k_2 = -k_1 \). We have tried to derive our conclusions with the minimum effort; but one could straightforwardly extend our calculation to the case of general \( h_{1-4} \), whereupon imposing \( \mathcal{N} = 3 \) supersymmetry would enable one to derive \( R_{trans} \) for general \( k_{1,2} \) (\( b \) of course remaining as an unknown).

We now turn to the Class II models, containing multi-trace deformations. We consider the simplest example of such a model, taking in Eq. (5)

\[ M = N, \quad k_1 = -k_2 = k, \quad h_3 = -h_4 = h, \quad H_{12} = H_1 = H_2 = H. \]  

(46)

In this case the two-loop result in Eq. (22) reduces to

\[ F^{-2} \gamma^{(2)}_{A} = 2(\rho_H - \rho_k). \]  

(47)

13
where
\[ \rho_H = 2h^2(N^2 - 1) + H^2(2N^2 + 1) \] (48)
with \( \rho_k \) given according to Eq. (26) but with now in Eq. (27)
\[ C_1 = 0, \quad C_2 = -2(N^2 - 1)k^2, \] (49)
so that
\[ \rho_b = -(N^2 - 1)k^2, \quad \rho_c = 3(N^2 - 1)k^2, \]
\[ \rho_k = 2(N^2 - 1)k^2. \] (50)
The results for the diagrams in Figs 3, 4 are now given by
\[
G'_a = 3\rho_H^2 I_4, \\
G'_b = [4(N^2 - 1)(N^2 + 2)h^4 + 36(N^2 - 1)h^2H^2 \]
\[ + (2N^2 + 1)(4N^4 + 6N^2 + 5)H^4]I_{4bbb}, \]
\[ G'_c = -3\rho_b\rho_H I_4, \]
\[ G'_d = 3C_2\rho_H I_{4bbb}, \]
\[ G'_e = -\rho_b\rho_H I_4, \]
\[ G'_f = 2C_2\rho_H I_5, \]
\[ G'_g = -T'_2 I_5, \]
\[ G'_h = -T'_2 I_{4bbb}, \]
\[ G'_i = -T'_1 I_5, \]
\[ G'_j = 4T'_2 \left( I_4 - \frac{1}{2} I_{4bbb} \right), \]
\[ G'_k = -2T'_2 I_4, \]
\[ G'_l = -2T'_1(-2I_4 + 2I_{4bbb} + I_5), \]
\[ G'_m = -2\rho_H \rho_c \left( I_4 - \frac{1}{2} I_{4bbb} \right), \]
\[ G'_n = 3\rho_c\rho_H I_{22}, \]
\[ G'_o = -3\rho_c\rho_H I_{22}, \]
\[ G'_p = 2\rho_H T'_3(I_4 - I_{4206c}), \]
\[ G'_q = -2\rho_H T'_3(I_4 - J_4), \]
\[ G'_r = \frac{1}{\epsilon} T'_4(a + b\pi^2) \] (51)
where
\[
T'_1 = -(N^2 - 1)k^2[2(N^2 + 2)h^2 + 3H^2], \\
T'_2 = (N^2 - 1)k^2[2(N^2 + 5)h^2 - (2N^2 - 5)H^2], \\
T'_3 = 8(N^2 - 1)k^2, \\
T'_4 = 2(N^2 - 1)[3h^2 - (N^2 - 1)H^2]k^2. \] (52)
The quantities $T_{1,2,4}'$ satisfy identities similar to Eq. (32), namely

\begin{align}
2T_1' + T_2' &= \rho_H \rho_b,
T_1' + T_4' &= \frac{1}{2} \rho_H C_2,
\end{align}

but where $\rho_H$, $C_2$ and $\rho_b$ are now as defined in Eqs. (48), (49) and (50). The case $M = N$ and $k_1 = -k_2$ can be expressed in terms of the 3-algebra formalism [32]; this lends itself to automation and the results in Eq. (51) were obtained using FORM [53].

For this class of models we shall start by discussing the transcendental contributions to the anomalous dimension, since the results are more striking than for the rational case. The transcendental contribution is given by summing the contributions involving $I_{4bbb}$ and $I_5$ in Eq. (51) together with $G_{\epsilon}^r$ and using Eqs. (53), (92) (and including the contribution $R_{\epsilon}^{\text{trans}}$ from graphs with no Yukawa couplings):

\begin{align}
G_{\epsilon}^{\text{trans}} &= \frac{1}{2\epsilon} \left\{ 4(N^2 - 1)(N^2 + 2)h^4 + 36(N^2 - 1)h^2 H^2 \\
&\quad+ (2N^2 + 1)(4N^4 + 6N^2 + 5)H^4 \\
&\quad+ \frac{1}{3}[(5 + 3b)\rho_H C_2 + 3\rho_H \rho_k - 10\rho_H \rho_b + 2(4 - 3b)T_1'] \right\} + R_{\epsilon}^{\text{trans}}.
\end{align}

To lowest order the vanishing of the anomalous dimensions now requires (using Eqs. (47), (48), (50)) that the couplings $h$ and $H$ must be chosen to satisfy

\begin{align}
F^{-2}\gamma^{(2)} &= 2[2(N^2 - 1)h^2 + (2N^2 + 1)H^2 - 2(N^2 - 1)k^2] = 0.
\end{align}

(We suppress the field label $A^{1,2}B_{1,2}$ on $\gamma$ here and later, since for this class of models every field has the same anomalous dimension.) In order for $G_{\epsilon}^{\text{trans}}$ to adopt a universal form upon imposing two-loop superconformal invariance as in Eq. (55), we would require Eq. (54) to adopt the form

\begin{align}
G_{\epsilon}^{\text{trans}} &= f(2(N^2 - 1)h^2 + (2N^2 + 1)H^2).
\end{align}

This is clearly not the case. We shall therefore consider the two cases $H = 0$ and $H \neq 0$ separately, and find that they are very different. If $H = 0$ (so that we are considering the ABJM model) then Eq. (51) reduces to

\begin{align}
G_{\epsilon}^{\text{trans}} &= \frac{1}{\epsilon}(N^2 - 1) \left[ 2(N^2 + 2)h^4 - \frac{2}{3}[N^2 + 11 - 9b]h^2 k^2 \right] + R_{\epsilon}^{\text{trans}}.
\end{align}

For $H = 0$ the lowest-order superconformal invariance condition $\gamma^{(2)} = 0$ simply implies (see Eq. (55)) that

\begin{align}
h = h_c = k.
\end{align}

If we assume

\begin{align}
R_{\epsilon}^{\text{trans}} &= \frac{1}{\epsilon}(N^2 - 1)R k^4
\end{align}
then using Eq. (18) the total transcendental contribution to the anomalous dimension at this order is

\[ F^{-4}\gamma_{\text{trans}}^{(4)} = 4(N^2 - 1)\sigma \]  \tag{60}

where

\[ \sigma = 2(N^2 + 2)h_c^4 - \frac{2}{3}(N^2 + 11 - 9b)h_c^2k^2 + R_kk^4. \]  \tag{61}

This is easily derived by using Eqs. (48), (49), (50) in Eq. (54), in conjunction with Eq. (47). (The factor \(N^2 - 1\) in Eq. (59) may be inferred from the fact that for \(K_1 = -K_2\), all the contributions vanish identically in the abelian case due to the “quiver” structure.) We have refrained from making the replacement \(h_c = k\) in Eq. (61) for later reference. Now since we know that the ABJM model is exactly superconformal when \(h_c = k\), we can deduce that \(\sigma\) must vanish for this value of \(h_c\) and so

\[ R_k = -\frac{2}{3}(2N^2 - 5 + 9b). \]  \tag{62}

Returning to the case of \(H \neq 0\), it is clear that the exact superconformal invariance will no longer persist; if we substitute the superconformal invariance condition in the form \(\rho_H = \rho_k\) into Eq. (54) and use Eq. (62), we obtain a non-vanishing result for \(G_{c,\text{trans}}\) and hence for the transcendental contribution to \(\gamma_c^{(4)}\). We write this as

\[ \gamma_{c,\text{trans}}^{(4)} = \alpha_1h_c^4 + \alpha_2h_c^2H_c^2 + \alpha_3H_c^4 + (\alpha_4h_c^2 + \alpha_5H_c^2)k^2 + \alpha_6(N^2 - 1)k^4, \]

\[ \alpha_1 = 8(N^2 - 1)(N^2 + 2), \]

\[ \alpha_2 = 72(N^2 - 1), \]

\[ \alpha_3 = 2(2N^2 + 1)(4N^4 + 6N^2 + 5), \]

\[ \alpha_4 = \frac{8}{3}(3b - 4)(N^2 - 1)(N^2 + 2), \]

\[ \alpha_5 = 4(3b - 4)(N^2 - 1), \]

\[ \alpha_6 = \frac{8}{3}(1 - 3b)(N^2 + 2), \]  \tag{63}

where \(h_c\) and \(H_c\) together are solutions of Eq. (55). Since \(\alpha_3\) is \(O(N^6)\) while all other terms at this loop order are at most \(O(N^4)\), it is particularly clear that we need to make a redefinition in Eq. (47) to restore superconformal invariance at this order for this theory. A change \(\delta h_c, \delta H_c\) produces a change in \(\gamma_c^{(2)}\) given according to Eq. (47) by

\[ F^{-2}\delta\gamma_c^{(2)} = 4[2(N^2 - 1)h_c\delta h_c + (2N^2 + 1)H_c\delta H_c] \]  \tag{64}

and we can therefore cancel the non-zero terms in \(\gamma_{c,\text{trans}}^{(4)}\) by taking

\[ F^{-2}\delta_{\text{trans}}h_c = -\frac{1}{8(N^2 - 1)}(\alpha_1h_c^2 + \alpha_4k^2)h_c - \kappa\frac{1}{8(N^2 - 1)}\alpha_2h_cH_c^2 - \frac{1}{8}Ck^2h_c, \]

\[ F^{-2}\delta_{\text{trans}}H_c = -\frac{1}{4(2N^2 + 1)}(\alpha_3H_c^2 + \alpha_5k^2)H_c - (1 - \kappa)\frac{1}{4(2N^2 + 1)}\alpha_2h_c^2H_c \]

\[ -\frac{1}{8}Ck^2H_c, \]  \tag{65}
where \( h_c \) and \( H_c \) together solve Eq. (65), and \( \kappa \) is arbitrary. For the \( \alpha_6 \) terms we have again applied Eq. (65). Note that in Eq. (65) we are still suppressing the "transcendental" factor of \( \pi^2 \). We therefore conclude that a "transcendental" redefinition is inevitably required as soon the ABJM model is supplemented by multitrace deformations. We note that Eqs. (63), (65) (with \( H_c = 0 \)) could easily be adapted to construct the redefinitions required in the Class I case. Finally turning to the rational contribution to the anomalous dimension for these models, the discussion would largely follow that for the previous class of models. However as mentioned there, for \( H_{1,2}, H_{12} \neq 0 \) (and unless \( a = 0 \)) we would find a model-dependent contribution from \( G'_r \) upon imposing two-loop superconformal invariance, and this would require a model-dependent redefinition of \( H_{1,2}, H_{12} \neq 0 \) akin to Eq. (65).

4 General coupling redefinitions

In this section we set the discussion of coupling redefinitions in a more general context. We have postponed this discussion until now, since it was easier for the purposes of exposition to define our models of interest explicitly \textit{ab initio} than to start with a general theory and then specialise. In this section we shall maintain the discussion at a general level throughout without including too much detail.

In a general supersymmetric theory in three dimensions with chiral fields \( \Phi_i \), a general renormalizable superpotential would take the form

\[
W(\Phi) = Y^{ijkl} \Phi_i \Phi_j \Phi_k \Phi_l. \tag{66}
\]

To lowest order the change in the \( \beta \)-function

\[
\beta_Y^{ijkl} = \mu \frac{d}{d\mu} Y^{ijkl} \tag{67}
\]

resulting from a change \( \delta Y^{ijkl} \) is given by

\[
\delta \beta_Y^{ijkl} = \left( \beta_Y J \frac{\partial}{\partial Y} \right) \delta Y^{ijkl} - \left( \delta Y J \frac{\partial}{\partial Y} \right) \beta_Y^{ijkl}, \tag{68}
\]

where

\[
\delta Y J \frac{\partial}{\partial Y} \equiv \delta Y^{mnpq} \frac{\partial}{\partial Y^{mnpq}}. \tag{69}
\]

In the case of superconformal invariance at lowest order (we shall assume as before that this is equivalent to \( \gamma = 0 \)), this reduces to

\[
\delta \beta_Y^{ijkl} = \delta \gamma^{(i} m \gamma^{jkl)m} \tag{70}
\]

where

\[
\delta \gamma^i_j = \delta Y \frac{\partial}{\partial Y} \gamma^i_j. \tag{71}
\]
The two-loop anomalous dimension is of the form
\[ \gamma^{(2)}_{ij} = \frac{1}{3} Y^{iklm} Y_{jklm} - C^i_j \tag{72} \]
with the convention that \( Y_{ijkl} = (Y^{ijkl})^* \) and where \( C^i_j \) is a function only of the Chern-Simon level(s). After imposing lowest-order superconformal invariance, \( O(k^4) \) terms in the four-loop anomalous dimension may be removed by a redefinition
\[ \delta Y^{ijkl} = \frac{1}{2} \lambda Y^{ijkl} c \tag{73} \]
where \( Y^{ijkl} c \) is a solution of the lowest order superconformal invariance condition \( \gamma^{(2)}_{ij} = 0 \) so that from Eq. (72)
\[ \frac{1}{3} Y^{iklm} Y_{cijklm} = C^i_j \tag{74} \]
and therefore using Eqs. (71), (73)
\[ \delta \gamma^{(2)}_{ij} = \lambda C^i_j. \tag{75} \]

The redefinition in Eq. (65) which removes the \( \alpha_6 \) term in Eq. (63) is clearly of this type.

In terms of the general superpotential, the contribution to the anomalous dimension from Fig. 3(b) (corresponding to \( G_b, G'_b \) in Eqs. (29), (51) respectively) is
\[ \frac{1}{4} \pi^2 Y^{ikl} Y_{klmn} Y^{mpq} Y_{pqj} \tag{76} \]
and this may be removed completely by a redefinition
\[ \delta Y^{ijkl} = -\frac{1}{8} \pi^2 Y^{mn(ij} Y_{mpq} Y^{kl)pq}. \tag{77} \]

This reproduces (for a particular value of \( \kappa \)) the effect of the terms cubic in \( h_c, H_c \) in Eq. (65) in removing the quartic \( (\alpha_1, \alpha_2) \) terms in Eq. (63). We also note that if \( b = \frac{4}{3} \) (thus removing the \( T'_1 \) term in Eq. (54) and setting \( \alpha_4 = \alpha_5 = 0 \)) then no further redefinition is required; whereas if \( b \neq \frac{4}{3} \) there is no such simple general form for the \( \alpha_4, \alpha_5 \) terms in Eq. (65).

We saw earlier that redefinitions such as Eq. (77) are not universal in the sense that their form depends on the nature of the superpotential. Nevertheless it would be satisfying if all the necessary redefinitions could be expressed in a general form such as Eqs. (73), (77). This would point to the existence of a “superconformal renormalisation scheme” in which superconformality properties were manifest for the whole class of superconformal theories; akin to the “NSVZ” scheme [54] in which the gauge \( \beta \) function for an \( \mathcal{N} = 1 \) supersymmetric theory in four dimensions adopts the simple NSVZ form [55]. Unfortunately it is not clear without further calculation whether the present discussion can be extended to include the presence of flavour fields, in particular whether a redefinition such as Eq. (73) could simultaneously remove residual \( O(k^4) \) terms from the four-loop anomalous dimensions for both bifundamental and flavour matter; and while it is tempting to speculate that indeed \( b = \frac{4}{3} \) (and \( a = 0 \), which simplifies Eq. (38) in a similar way), this must remain a hypothesis for the moment.
5 Conclusions

We have shown that on the one hand, superconformal invariance of Chern-Simons theories in general requires transcendental corrections beyond leading order, except for special cases such as $\mathcal{N} = 3$ supersymmetry; and on the other hand, that at leading order (and likely beyond) in $N, M$, no rational corrections are required for a wide class of theories (Class I in our terminology). Our conclusions could be extended beyond leading order in $N$ by the computation of the non-planar diagram in Fig. 4(r); which would confirm or disprove our speculation that $a = 0$ and $b = \frac{4}{3}$ and hence $G_r = \frac{1}{2} \pi^2 T_4$ (with $T'_4$ replacing $T_4$ in the Class II case with multitrace deformations). If this speculation is correct, then at least for the theories considered, and possibly more generally, we can restore superconformal invariance at four-loop order by a combination of transformations of the form Eq. (77) (in which we simply have to substitute the particular form of the superpotential) and Eq. (73). In the latter equation, $\lambda$ will be determined purely by the field content and could be specified by extending our calculation to the case of non-zero $h_{1-4}$ so that we could use the exact superconformality of the $\mathcal{N} = 3$ theories to determine $R$ in Eq. (33). It would be interesting to attempt to extend all our calculations to the case with flavour matter, especially to see if general expressions can still be given for the coupling redefinitions required.

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Appendix

In this appendix we list our superspace and supersymmetry conventions, which follow those of Ref. [41]. We use a metric signature $(+ - -)$ so that a possible choice of $\gamma$ matrices is $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_3$, $\gamma^2 = i\sigma_1$ with

$$ (\gamma^\mu)_{\alpha}^\beta = (\sigma_2)_\alpha^\beta, \quad (78) $$

etc. We then have

$$ \gamma^\mu \gamma^\nu = \eta_{\mu\nu} - i\epsilon_{\mu\nu\rho}\gamma^\rho. \quad (79) $$

We have [41] two complex two-spinors $\theta^\alpha$ and $\bar{\theta}^\alpha$ with indices raised and lowered according to

$$ \theta^\alpha = C^{\alpha\beta} \bar{\theta}_\beta, \quad \theta_\alpha = \bar{\theta}^\beta C_{\beta\alpha}, \quad (80) $$

with $C^{12} = -C_{12} = i$. We then have

$$ \theta_\alpha \theta_\beta = C_{\beta\alpha} \theta^2, \quad \theta^\alpha \bar{\theta}^\beta = C^{\beta\alpha} \theta^2, \quad (81) $$

where

$$ \theta^2 = \frac{1}{2} \theta^\alpha \theta_\alpha. \quad (82) $$

The supercovariant derivatives are defined by

\[
D_\alpha = \partial_\alpha + i\frac{\gamma^\beta}{2} \partial_{\alpha\beta},
\]

(83)

\[
\overline{D}_\alpha = \overline{\partial}_\alpha + i\frac{\gamma^\beta}{2} \partial_{\alpha\beta},
\]

(84)

where

\[
\partial_{\alpha\beta} = \partial_{\mu} (\gamma^\mu)_{\alpha\beta},
\]

(85)

satisfying

\[
\{ D_\alpha, \overline{D}_\beta \} = i\partial_{\alpha\beta}.
\]

(86)

We also define

\[
d^2\theta = \frac{1}{2} d\theta^\alpha d\theta_\alpha, \quad d^2\overline{\theta} = \frac{1}{2} d\overline{\theta}^\alpha d\overline{\theta}_\alpha, d^4\theta = d^2\theta d^2\overline{\theta},
\]

(87)

so that

\[
\int d^2\theta d^2\overline{\theta} = -1.
\]

(88)

The vector superfield \( V(x, \theta, \overline{\theta}) \) is expanded in Wess-Zumino gauge as

\[
V = i\theta^\alpha \overline{\theta}_\alpha \sigma + \theta^\alpha \overline{\theta}_\beta A_{\alpha\beta} - \theta^2 \overline{\theta}^\alpha \lambda_\alpha - \overline{\theta}^2 \theta^\alpha \lambda_\alpha + \theta^2 \overline{\theta}^2 D,
\]

(89)

and the chiral field is expanded as

\[
\Phi = \phi(y) + \theta^\alpha \psi_\alpha(y) - \theta^2 F(y),
\]

(90)

where

\[
y^\mu = x^\mu + i\theta^\gamma \gamma^\mu \overline{\theta}.
\]

(91)

In the main text, our results were given in terms of a basis of momentum integrals. The results for their divergences were computed in Ref. [45, 46] and are listed below

\[
I_4 = -\frac{1}{2\epsilon^2} + \frac{2}{\epsilon}
\]

\[
I_{22} = -\frac{1}{\epsilon^2}
\]

\[
I_{4\beta\beta} = \frac{\pi^2}{2\epsilon}
\]

\[
I_{4\beta\beta\gamma} = \frac{2}{\epsilon}
\]

\[
I_{422q\gamma\gamma\gamma} = \frac{1}{4\epsilon^2} + \frac{1}{\epsilon} \left( \frac{5}{4} - \frac{\pi^2}{12} \right)
\]

\[
I_5 = \frac{1}{4}I_4 - \frac{5}{8}I_{22} - I_{4\beta\beta} + I_{4\beta\beta\gamma} - 2I_{422q\gamma\gamma\gamma} = -\frac{\pi^2}{3\epsilon}
\]

(92)

Note that our definitions of \( I_4 \) etc differ by a factor of \( F^{-4} \) from those of Ref. [45].

Figs. 5(a)-(f) depict \( I_4, I_{4\beta\beta}, I_{22}, I_{4\beta\beta\gamma}, I_{422q\gamma\gamma\gamma} \) and \( J_4 \) respectively. In the momentum integral for the so far uncomputed Fig. 5(g), there is a trace over a series of \( \rho^\mu \gamma_\mu \) in order around the perimeter.
Figure 5: Momentum integrals
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