Quantization and “theta functions”

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Abstract

Geometric Quantization links holomorphic geometry with real geometry, a relation that is a prototype for the modern development of mirror symmetry. We show how this treatment can be used to construct a special basis in every space of conformal blocks. This is a direct generalization of the basis of theta functions with characteristics in every complete linear system on an Abelian variety (see [Mum]). The same construction generalizes the classical theory of theta functions to vector bundles of higher rank on Abelian varieties and K3 surfaces. We also discuss the geometry behind these constructions.

1 Introduction

It is a fruitful question to ask for some special basis of the complete linear systems $\mathbb{P}H^0(X, L^k)$, where $X$ is a smooth complete algebraic variety and $L$ a polarization. After this, following Mumford, we can ask for special equations defining $X$ under its embedding in $\mathbb{P}H^0(X, L^k)^*$. Of course, this is a priori impossible (for example, for $\mathbb{P}H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(k))$), but it can be done after “rigidification” – that is, fixing some discrete structure on $X$. This is the subject of Invariant Theory in its pre-Hilbert form; however, any proposed “geometric” rigidification depends on the level $k$, and there is no universal way of doing it. The amazing fact is we can do it in many cases using the “classical” Geometric Quantization Procedure (GQP); but for this, we must leave algebraic geometry and go over to symplectic geometry instead. I would like to call this method the general theory of theta functions.

The starting point is that, together with a complex structure $I$ on $X$, a polarization $L$ gives us a quadruple $(X, \omega, L, a_L)$, where $\omega$ is the Kähler form
and a_L a Hermitian connection on L with curvature form F_a = 2\pi i \cdot \omega of Hodge type (1, 1), giving the holomorphic structure on L. The pair (X, \omega) is a symplectic manifold; we can thus view it as the phase space of some classical mechanical system, and the pair (L, a_L) as prequantization data of this system.

We should start by recalling the construction of spaces of wave functions for a pair (S, \omega), where S is a smooth symplectic manifold of dimension 2n with a given symplectic form \omega. To switch on any quantization procedure, we suppose that the cohomology class [\omega] of the symplectic form is integral, that is, there exists a complex line bundle L with c_1(L) = [\omega]. Moreover, suppose that L has a Hermitian connection a with curvature form F_a = 2\pi i \cdot \omega. Any quadruple of this type

\[(S, \omega, L, a)\] (1.1)

is called a prequantization of the classical mechanical system with phase space (S, \omega).

There are two approaches to the geometric quantization of (S, \omega, L, a) (1.1) (see [A], [S1] or [W]). We discuss here the simplest version of these constructions, avoiding questions such as the choice of metaplectic structures, densities and half densities specifying geometric conditions on the manifold S. (Roughly speaking, S should be a Calabi–Yau manifold). The usual slogan is that we have to choose “one half” of the set of all functions on S using some “polarization” conditions. The first approach is as follows:

**Complex polarizations**

To define a complex polarization, we give S a complex structure I such that S_I = X is a Kähler manifold with Kähler form \omega. Then the curvature form of the Hermitian connection a is of type (1, 1), hence for any level \(k \in \mathbb{Z}^+\), the line bundle L^k is a holomorphic line bundle on S_I. Complex quantization provides the space of wave functions of level \(k\):

\[\mathcal{H}_{L^k} = H^0(S_I, L^k),\] (1.2)

– that is, the space of holomorphic sections of L^k. Thus a complex polarization of (S, \omega, L, a) (1.1) returns to the algebraic geometry S = X we started from.

In particular, the spaces of wave functions (1.2) obtained in this way is the collection of complete linear systems in the usual sense. We will suppose
\( L \) to be an \textit{ample} holomorphic line bundle, and in particular,
\[
H^i(S, L) = 0 \quad \text{for all } i > 0.
\]

The second approach is the choice of a real polarization:

\textbf{Real polarizations}

A real polarization of \((S, \omega, L, a)\) is a Lagrangian fibration
\[
\pi : S \to B,
\]
(1.3)
such that
\[
\omega|_{\pi^{-1}(b)} = 0 \quad \text{for every point } b \in B,
\]
and the fibre \( \pi^{-1}(b) \) is a smooth Lagrangian submanifold for generic \( b \).

Thus a mechanical system admits a real polarization if and only if it is \textit{complete integrable}.

\textbf{Remark} \hspace{1em} Actually, for the ordinary technical tricks of the theory of geometric quantization to work, we should require that the fibration has regular geometric behavior (see, for example, [S2]). But beginning with Guillemin and Sternberg’s paper [GS2], it is reasonable to consider more general fibrations, namely, \textit{real polarizations with singularities}.

Then restricting \( L \) to a Lagrangian fibre gives a flat connection \( a|_{\text{fibre}} \) or equivalently, a character of the fundamental group
\[
\chi : \pi_1(\text{fibre}) \to \mathbb{U}(1).
\]

Let \( \mathcal{L}_\pi \) be the sheaf of sections of \( L \) that are covariant constant along fibres. Then we get the space
\[
\mathcal{H}_\pi = \bigoplus_i H^i(S, \mathcal{L}_\pi).
\]

In the regular case, Śniatycki proved that
\[
H^i(S, \mathcal{L}_\pi) = 0 \quad \text{for } i \neq n.
\]
Definition 1.1  

(1) A fibre of $\pi$ is a Bohr–Sommerfeld cycle of $(S, \omega, L, a)$ if $\chi = 1$.

(2) $BS \subset B$ is the subset of Bohr–Sommerfeld fibres.

(3) $k$-$BS \subset B$ is the subset of Bohr–Sommerfeld fibres for $(S, \omega, L^k, ka)$.

According to the general theory of real quantizations, we expect to get a finite number of Bohr–Sommerfeld fibres, and in the regular case,

$$H^n(S, L_\pi) = \bigoplus_{BS} \mathbb{C} \cdot s_i,$$

where $s_i$ is a nonzero covariant constant section of the restriction of $(L, a)$ to a Bohr–Sommerfeld fibre of the real polarization $\pi$.

In the general case, we can use this to define the new collection of spaces of wave functions (of level $k$):

$$H^k_\pi = \bigoplus_{k$-BS} \mathbb{C} \cdot s_i, \quad (1.4)$$

and use special tricks to compare (1.4) with (1.2).

There is a canonical way of describing the Bohr–Sommerfeld subset. For this, we must choose special coordinates on $B$, the so-called action coordinates, which are part of the action angle coordinates (see [A], [GS1], [GS2]).

Locally around a point $b \in B$, the action coordinates $c_i$ are given as periods along 1-cycles of the fibre $\pi^{-1}(b)$ of a 1-form $\alpha$ such that

$$d\alpha = \omega. \quad (1.5)$$

This system of coordinates $\{c_i\}$ is defined up to additive constants and an integral linear transformations. Thus, if $B$ is simply connected, the action coordinates map $B$ locally diffeomorphically to some open subset

$$B_c \subset \mathbb{R}^n_{(c_1, \ldots, c_n)} \quad (1.6)$$

with coordinates $\{c_i\}$. If $(0, \ldots, 0)$ is a Bohr–Sommerfeld point, then

$$BS = B_c \cap \mathbb{Z}^n \quad (1.7)$$

is the set of integral points in $B_c$.

Let us return to our collections of spaces of wave functions.
Remark An important observation, proved mathematically in a number of cases, is that the projectivization of the spaces (1.2) are given purely by the symplectic prequantization data and do not depend on the choice of complex structure on $S$. The same is true for the projectivization of the spaces (1.4). Moreover, these spaces do not depend on the real polarization $\pi$ (1.3), provided that we extend our prequantization data $(S, \omega, L, a, F)$ by adding some “half density” $F$ (see [GS1]).

Our main problem is to compare the spaces $\mathcal{H}_L^k$ and $\mathcal{H}_\pi^k$. If we are lucky enough to be able to construct a canonical isomorphism between these spaces, we get a special basis in the space of wave functions of a complex polarization, and in particular in any ample complete linear system. To distinguish this basis from others, we call it the system of theta functions of level $k$, with “characteristics” which are Bohr–Sommerfeld fibres.

Actually, this generalization of the theory of theta functions requires the final ingredient of the quantization procedure – the algebra of observables represented as an algebra of operators on spaces of wave functions (like the Heisenberg algebra on spaces of classical theta functions). We avoid using such algebras in this article, but they underlie our constructions, so it is reasonable to recall briefly the general shape of this ingredient.

Algebra of observables and its space of states

As a result of any quantization procedure, we get a $\mathbb{C}^*$-algebra of observables represented as some algebra $A$ of operators on the spaces of wave functions (1.2) or (1.4). As usual, this algebra is a noncommutative extension of some commutative $\mathbb{C}^*$-algebra $A_0 \subset A$. For example, if $S = T^*M$ for some manifold $M$ then $A_0$ is the algebra of continuous complex valued functions, so that $M$ is the space of maximal ideals of $A_0$.

A pair $A_0 \subset A$ gives us a space $\mathcal{H}$ of wave functions (1.2) or (1.4) as the subset of the space of states. Recall that a state is a map:

$$\psi: A \to \mathbb{C} \text{ such that } \psi(a^*a) \geq 0 \text{ and } \|\psi\| = 1.$$  \hspace{1cm} (1.8)

The set $\mathcal{S}(A)$ of all states of $A$ is a convex space and its boundary elements are called pure states (for example, in the previous example, delta functions...
of points are pure states). If our \( \mathbb{C}^* \)-algebra is represented on \( \mathcal{H} \) by bounded operators then every vector \( |\psi\rangle \) defines the state as the \textit{expectation value}.

The known strategy to identify spaces (1.2) and (1.4) is to represent both as irreducible representation spaces of some algebra admitting a \textit{unique irreducible representation}.

The constructions of Berezin, Toeplitz and Rawnsley (see for example [R]) are extremely useful for our geometric investigations, and we consider them in §6.

2 Model for our theory: the classical theory of theta functions

Let \( A \) be a principally polarized Abelian variety of complex dimension \( g \) with flat metric \( g \). Then the tangent bundle \( TA \) has the standard constant Hermitian structure (that is, the Euclidean metric, symplectic form and complex structure \( I \)). The Kähler form \( 2\pi i\omega \) gives a polarization of degree 1. We fix a \textit{smooth} Lagrangian decomposition of \( A \)

\[
A = T^g_+ \times T^g_-, \tag{2.1}
\]

such that both tori are Lagrangian with respect to \( \omega \). (In the smooth category, \( A \) is the standard torus \( \mathbb{R}^{2g} / \mathbb{Z}^{2g} \) with the standard constant integral form \( \omega \), and the decomposition (2.1) just consists of putting \( \omega \) in normal form.) Let \( L \) be a holomorphic line bundle with holomorphic structure given by a Hermitian connection \( a \) with curvature form \( F_a = 2\pi i \cdot \omega \), and \( L = \mathcal{O}_A(\Theta) \), where \( \Theta \) is the classical \textit{symmetric} theta divisor. The decomposition (2.1) induces a decomposition

\[
H^1(A, \mathbb{Z}) = \mathbb{Z}_+^g \times \mathbb{Z}_-^g, \tag{2.2}
\]

and a Lagrangian decomposition

\[
A_k = (T^g_+)_k \times (T^g_-)_k \tag{2.3}
\]

of the group of points of order \( k \). Any smooth “irreducible” \( g \)-cycle in \( A \) is the image \( \varphi(T^g) \) of a smooth linear embedding \( \varphi : T^g \to A \).
Complex quantization

This is nothing other than the classical theory of theta functions. Indeed, the decomposition (2.2) defines the collection of compatible theta structures of every level \( k \): the decomposition (2.3) defines a Lagrangian decomposition \( A_k = (\mathbb{Z}^g)_k^+ \times (\mathbb{Z}^g)_k^- \), and a decomposition of the spaces of wave functions

\[
\mathcal{H}_{L^k} = H^0(A, L^k) = \bigoplus_{w \in (\mathbb{Z}^g)_k^-} \mathbb{C} \cdot \theta_w, \quad \text{with rank } \mathcal{H}_{L^k} = k^g, \tag{2.4}
\]

where \( \theta_c \) is the theta function with characteristic \( c \) (see \([\text{Mum}]\)).

The decomposition (2.4) is given by the following recipe: we identify the torus \( T^g_k \) with the dual torus, and consider vectors \( w \in (T^g_k)_k \) as (periodic) linear differential forms on \( T^g_k \). Applying the symplectic form \( \omega \) gives a collections of linear vector fields \( \xi_w \) on \( A \) parallel to the fibration by the tori \( T^g_k \). Finally, the translations \( t_w \) on \( A \) obtained as the exponentials of these vector fields give a finite subgroup of the translations group of \( A \).

Now by choosing \( \theta_0 \in H^0(A, L^k) \) to be a very symmetric section (actually, the section with divisor the sum of all the translates of the theta divisor \( \Theta \) by points of \( (T^g_\pm)_k \)), we get a basis of \( H^0(A, L^k) \):

\[
\{\theta_w = t_w^*(\theta_0)\}. \tag{2.5}
\]

Real polarization

The projection of the direct product (2.1) gives us a real polarization

\[
\pi: A \to T^g_\pm = B. \tag{2.6}
\]

Remark that in this case the action coordinates (1.6) are just flat coordinates on \( T^g_\pm = B \), and under this identification

\[
k\text{-BS} = (T^g_\pm)_k \tag{2.7}
\]

is the subgroup of points of order \( k \).

Now we can consider the dual fibration

\[
\pi': A' = \text{Pic}(A/T^g_k) \to T^g_\pm = B, \tag{2.8}
\]

with fibres

\[
(\pi')^{-1}(p) = \text{Hom}(\pi_1(\pi^{-1}(p)), U(1)).
\]
This fibration admits the section

\[ s_0 \in A \quad \text{with} \quad s_0 \cap (\pi')^{-1}(p) = \text{id} \in \text{Hom}(\pi_1(\pi^{-1}(p), U(1)), \quad (2.9) \]

so that we have a decomposition

\[ A' = (T^g)' \times T^g = B. \quad (2.10) \]

**Remark** An amazing fact recently proved by Golyshev, Lunts and Orlov [GLO] is that the $2g$-torus $A'$ is canonically equipped with

1. a symplectic form $\omega'$;
2. a complex structure $I'$.

Now we can apply geometric quantization to the real polarization (2.6) of the phase space $(A, \omega, L^k, a_k)$, where $a_k$ is the Hermitian connection defining the holomorphic structure on $L^k$. Sending the line bundle $L^k$ to the character of the fundamental group of a fibre gives a section

\[ s_{L^k} \subset A' = \text{Pic}(A/T^g); \quad (2.11) \]

and the Bohr–Sommerfeld subset of $B = T^g$ is

\[ s_0 \cap s_{L^k} \subset s_0 = B = T^g. \]

Under the identification $s_0 = T^g_+ = U(1)^g$, the intersection points

\[ s_0 \cap s_{L^k} = (U(1)^g)_k \]

are elements of order $k$ in $T^g = U(1)^g$. We thus get a decomposition

\[ \mathcal{H}^k = \bigoplus_{\rho \in U(1)^g} \mathbb{C} \cdot s_{\rho}. \quad (2.12) \]

**Corollary 2.1** (1) rank $\mathcal{H}_{L^k} = \text{rank} \mathcal{H}^k_{\pi}$.

(2) Moreover, there exists a canonical isomorphism

\[ \mathcal{H}_{L^k} = \mathcal{H}^k_{\pi}, \]

up to a scaling factor.
We get already this isomorphism up to the action of the $k^g$-torus $(\mathbb{C}^*)^{k^g}$ (compare decompositions (2.3) and (2.12)). But the canonical isomorphism is defined by the action of the Heisenberg group $H_k$ on holomorphic sections of the line bundle $L^k$ (= the theory of theta functions, see [Mum]) and the natural extension of the action of $H_k$ on the collection of Bohr–Sommerfeld orbits. Each of these representations is irreducible; thus the uniqueness of the irreducible representation of $H_k$ gives a canonical identification of these spaces up to scaling.

The functions making up the special bases of these spaces are called \textit{classical theta functions with characteristics of level $k$}.

A real polarization without degenerate fibres such as $\pi$ in (2.6) is called \textit{regular}. Using more sophisticated techniques (as in [GS2]) we get a basis of the same type for real polarizations with degenerate fibres (see Remark after (1.3)). But if we start with \textit{any} polarized Kähler manifold $X$, the main question is the following:

\textit{how to find a real polarization like (1.3) on $X$ (possibly with degenerate fibres)?}

The amazing fact is that we can do it in many absolutely unpredictable cases. For example, we now show how to find a real polarization of complex projective space $\mathbb{P}^3$. \textbf{Warning:} We construct some real polarization of $\mathbb{P}^3$, but not a special theta basis in $\mathbb{P}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$!

For this, we consider a special presentation of the complex threefold $\mathbb{P}^3$ as a real 6-manifold: let $\Sigma_2$ be a Riemann surface of genus 2. Then as a 6-manifold,

$$\mathbb{P}^3 = \text{Hom}(\pi_1(\Sigma_2), \text{SU}(2))/ \text{PU}(2) = R_2$$

is the space of classes of SU(2)-representations of the fundamental group of a Riemann surface of genus 2.

Thus $\mathbb{P}^3$ is the first manifold of the collection of manifolds $R_g$. If we solve the problem of real polarizations of these, we get in particular a real polarization of $\mathbb{P}^3$. We do this in the following section, but we first extend the direct approach by giving a description in terms of general theories giving rise to these constructions.
3 Chern–Simons quantizations of $R_g$

According to the general procedure, we must present $R_g$ as the classical phase space of some mechanical system. We begin by recalling the full steps of this procedure.

A classical field theory on a manifold $M$ has three ingredients:

1. a collection $\mathcal{A}$ of fields on $M$, which are geometric objects such as sections of vector bundles, connections on vector bundles, maps from $M$ to some auxiliary manifold (the target space) and so on;

2. an action functional

$$S: \mathcal{A} \rightarrow \mathbb{C}$$

which is an integral of a function $L$ (the Lagrangian) of fields;

3. a collection of observable functionals on the space of fields,

$$\mathcal{W}: \mathcal{A} \rightarrow \mathbb{C}.$$

Our case is the following.

**Example: Chern–Simons functional**

Here $M$ is a 3-manifold,

$$\mathcal{A} = \Omega^1(M) \otimes \mathfrak{su}(2)$$

and

$$S(a) = \frac{1}{8} \pi^2 \int_M \text{tr}(ada + \frac{2}{3} a^3).$$

As observable, we can consider a Wilson loop, given by some knot $K \subset M$: 

$$\mathcal{W}_K(a) = \text{tr}(\text{Hol}_K(a))$$

the trace of the holonomy of a connection $a$ around the knot $K$.

Now let

$$R_g = \text{Hom}(\pi_1(\Sigma_g), \text{SU}(2))/\text{PU}(2)$$

(3.2)
be the space of classes of SU(2)-representations of the fundamental group of a Riemann surface of genus $g$. This space is stratified by the subspace of reducible representations
\[ R^\text{red}_g \subset R_g, \quad R^\text{irr}_g = R_g - R^\text{red}_g. \] (3.3)
To get this space as the phase space of some mechanical system, consider a compact smooth Riemann surface $\Sigma$ of genus $g > 1$ and the trivial Hermitian vector bundle $E_h$ of rank 2 on it. As usual, let $\mathcal{A}_h$ be the affine space (over the vector space $\Omega^1(\text{End} E_h)$) of Hermitian connections and $\mathcal{G}_h$ the Hermitian gauge group. This space admits a stratification:
\[ \mathcal{A}^\text{red}_h \subset \mathcal{A}_h \]
where the left-hand side is the subset of reducible connections. As usual, let
\[ \mathcal{A}^\text{irr}_h = \mathcal{A}_h - \mathcal{A}^\text{red}_h. \]
Sending a connection to its curvature tensor defines a $\mathcal{G}_h$-equivariant map
\[ F: \mathcal{A}(E_h) \to \Omega^2(\text{End} E_h) = \text{Lie}(\mathcal{G}_h)^* \] (3.4)
to the coalgebra $\text{Lie}$ of the gauge group.
We can consider this map as the moment map with respect to the action of $\mathcal{G}_h$. The subset
\[ F^{-1}(0) = \mathcal{A}_F \] (3.5)
is the subset of flat connections and
\[ \mathcal{A}^\text{irr}_F = \mathcal{A}_F \cap \mathcal{A}^\text{irr}_h \]
the subspace of irreducible flat connections.
For a connection $a \in \mathcal{A}_F$ and a tangent vector to $\mathcal{A}_h$ at $a$
\[ \omega \in \Omega^1(\text{End} E_h) = TA_h, \]
we have
\[ \omega \in TA_F \iff \nabla_a(\omega) = 0. \] (3.6)
The trivial vector bundle $E_h$ admits the trivial connection $\theta$, which is interesting and important from many points of view, and it provides in particular the possibility of identifying $A_h$ with $\Omega^1(\text{End } E_h)$ by sending a connection $a$ to the form $a - \theta$. We will identify forms and connections in this way.

The space $A_h = \Omega^1(\text{End } E_h)$ is the collection of fields of YM-QFT with the Yang–Mills functional

$$S(a) = \int_\Sigma |F_a|^2.$$  

Thus $A_F$ is a classical phase space, that is, the space of solutions of an Euler–Lagrange equation $\delta S(a) = 0$.

There exists a symplectic structure on the affine space $A_h$, induced by the canonical 2-form given on the tangent space $\Omega^1(\text{End } E_h)$ at a connection $a$ by the formula

$$\Omega_0(\omega_1, \omega_2) = \int_\Sigma \text{tr}(\omega_1 \wedge \omega_2). \quad (3.7)$$

This form is $G_h$-invariant, and its restriction to $A^\text{irr}_F$ is degenerate along $G_h$-orbits: at a connection $a$, for a tangent vector $\omega \in \Omega^1(\text{End } E_h)$, we have

$$\omega \in T\mathcal{G}_h \iff \omega = \nabla_a \varphi \quad \text{for } \varphi \in \Omega^0(\text{End } E_h) = \text{Lie}(G_h)^*,$$

and

$$\int_\Sigma \text{tr}(\nabla_a \varphi \wedge \omega) = \int_\Sigma \text{tr}(\varphi \wedge \nabla_a \omega) = 0.$$

Hence

$$\omega \in T A_F \iff \Omega_0(\nabla_a \varphi, \omega) = 0. \quad (3.8)$$

Interpreting (3.4) as a moment map and using symplectic reduction arguments, we get a nondegenerate closed symplectic form $\Omega$ on the space

$$A_F/G_h = R_g$$

of classes of SU(2)-representations of the fundamental group of the Riemann surface, and a stratification of this space. The form $\Omega$ defines a symplectic structure on $R^\text{irr}_g$ and a symplectic orbifold structure on $R_g$. 

On the other hand, the form $\Omega_0$ on $A_h$ is the differential of the 1-form $D$ given by the formula

$$D(\omega) = \int_\Sigma \text{tr}((a) \wedge \omega). \quad (3.9)$$

We consider this form as a unitary connection $A_0$ on the trivial principal $U(1)$-bundle $L_0$ on $A_h$.

To descend this Hermitian bundle and its connection to the orbit space, one defines the $\Theta$-cocycle (or $\Theta$-torsor) on the trivial line bundle (see [RSW]). This cocycle is the $U(1)$-valued function $\Theta$ on $A_h \times G_h$ defined as follows: for any triple $(\Sigma, a, g)$ where $(a, g) \in A_h \times G_h$, we can find a triple $(Y, A, G)$ where $Y$ is a smooth compact 3-manifold, $A$ a SU(2)-connection on the trivial vector bundle $E$ on $Y$ and $G$ a gauge transformation of it, such that

$$\partial Y = \Sigma, \quad a = A|_\Sigma \quad \text{and} \quad g = G|_\Sigma.$$  

Then

$$\Theta(a, g) = e^{i(CS(A^G) - CS(A))}.$$

Recall that the Chern–Simons functional on the space $A(E_h)$ of unitary connections on the trivial vector bundle is given by the formula

$$CS_Y(a_0 + \omega) = \int_Y \text{tr} \left( \omega \wedge F_{a_0} - \frac{2}{3} \omega \wedge \omega \wedge \omega \right). \quad (3.11)$$

It can be checked that the function (3.10) does not depend on the choice of the triple $(Y, A, G)$ (see [RSW], §2).

The differential of $\Theta$ at $(a, g)$ is given by the formula

$$d\Theta(\omega, \varphi) = \frac{\pi i}{4} \Theta \int_\Sigma \left( \text{tr}(g^{-1} dg \wedge g^{-1} \omega g) - \text{tr}(a \wedge \nabla_{a^\varphi} \varphi) + 2 \text{tr}(F_{a^\varphi} \wedge \varphi) \right), \quad (3.12)$$

where $\omega \in \Omega^1(\text{End} E_h)$ and $\varphi \in \Omega^0(\text{End} E_h) = \text{Lie}(G_h)$.

But the restriction of this differential to the subspace of flat connections is much simpler:

$$d\Theta(\omega, \varphi) = \frac{\pi i}{4} \Theta \int_\Sigma \text{tr}(g^{-1} dg \wedge g^{-1} \omega g), \quad (3.13)$$
and is independent of the second coordinate.

That this function is in fact a cocycle results from the functional equation

$$\Theta(a, g_1 g_2) = \Theta(a, g_1) \Theta(a^{g_1}, g_2).$$

(3.14)

Using this function as a torsor $\mathcal{A}_h \times \mathcal{G} \ U(1)$ we get a principal $U(1)$-bundle $S^1(L)$ on the orbit space $\mathcal{A}_h / \mathcal{G}_h$:

$$S^1(L) = (\mathcal{A}_h \times S^1) / \mathcal{G}_h,$$

(3.15)

where the gauge group $\mathcal{G}_h$ acts by

$$g(a, z) = (a^g, \Theta(a, g) z),$$

or the line bundle $L$ with a Hermitian structure.

Following [RSW], let us restrict this bundle to the subspace of flat connections $\mathcal{A}_F$. Then one can check that the restriction of the form $D$ (3.9) to $\mathcal{A}_F$ defines a $U(1)$-connection $A_{CS}$ on the line bundle $L$.

By definition, the curvature form of this connection is

$$F_{A_{CS}} = i \cdot \Omega.$$ 

(3.16)

Thus the quadruple

$$(R_g, \Omega, L, A_{CS})$$

(3.17)

is a prequantum system and we are ready to switch on the Geometric Quantization Procedure.

### 4 Complex polarization of $R_g$

The standard way of getting a complex polarization is to give a Riemann surface $\Sigma$ of genus $g$ a conformal structure $I$. We get a complex structure on the space of classes of representations $R_g$ as follows: let $E$ be our complex vector bundle and $\mathcal{A}$ the space of all connections on it. Every connection $a \in \mathcal{A}$ is given by a covariant derivative $\nabla_a : \Gamma(E) \to \Gamma(E \otimes T^*X)$, a first order differential operator with the ordinary derivative $d$ as the principal symbol and a complex structure gives the decomposition $d = \partial + \overline{\partial}$, so any covariant derivative can be decomposed as $\nabla_a = \partial_a + \overline{\partial} a$, where $\partial_a : \Gamma(E) \to$
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\Gamma(E \otimes \Omega^{1,0}) \text{ and } \bar{\partial}_a: \Gamma(E) \rightarrow \Gamma(E \otimes \Omega^{0,1}). \text{ Thus the space of connections admits a decomposition}

\[ \mathcal{A} = \mathcal{A}' \times \mathcal{A}'', \]

(4.1)

where \( \mathcal{A}' \) is an affine space over \( \Omega^{1,0}(\text{End } E) \) and \( \mathcal{A}'' \) an affine space over \( \Omega^{0,1}(\text{End } E) \).

The group \( \mathcal{G} \) of all automorphisms of \( E \) acts as the group of gauge transformations, and the projection \( \text{pr}: \mathcal{A} \rightarrow \mathcal{A}'' \) to the space \( \mathcal{A}'' \) of \( \bar{\partial} \)-operators on \( E \) is equivariant with respect to the \( \mathcal{G} \)-action.

Giving \( E \) a Hermitian structure \( h \), we get the subspace \( \mathcal{A}_h \subset \mathcal{A} \) of Hermitian connections, and the restriction of the projection \( \text{pr} \) to \( \mathcal{A}_h \) is one-to-one. Under this Hermitian metric \( h \), every element \( g \in \mathcal{G} \) gives an element \( \overline{g} = (g^*)^{-1} \) such that

\[ \overline{g} = g \iff g \in \mathcal{G}_h. \]

Now for \( g \in \mathcal{G} \), the action of \( \mathcal{G} \) on the component \( \mathcal{A}'' \) is standard:

\[ \bar{\partial}_{g(a)} = g \cdot \bar{\partial}_a \cdot g^{-1} = \bar{\partial}_a - (\bar{\partial}_a g) \cdot g^{-1}; \]

and the action on the first component \( \mathcal{A}' \) of \( \partial \)-operators is

\[ \partial_{g(a)} = \overline{g} \cdot \partial_a \cdot \overline{g}^{-1} = \partial_a - (\bar{\partial}_a g) \cdot g^{-1} \ast. \]

It is easy to see directly that the action just described preserves unitary connections:

\[ \mathcal{G}(\mathcal{A}_h) = \mathcal{A}_h, \]

(4.2)

and that the identification \( \mathcal{A}_h = \mathcal{A} \) is equivariant with respect to this action.

It is easy to see that \( \bar{\partial}_a^2 \in \Omega^{0,2}(\text{End } E) = 0. \) Thus the orbit space

\[ \mathcal{A}''/\mathcal{G} = \bigcup \mathcal{M}_i \]

(4.3)

is the union of all components of the moduli space of topologically trivial \( I \)-holomorphic bundles on \( \Sigma_I \). (This union doesn’t admit any good structure, as it contains all unstable vector bundles). Finally, the image of \( \mathcal{A}_F \in \mathcal{A}_h \) is the component \( \mathcal{M}^{ss} \) of maximal dimension \( (3g - 3) \) of s-classes of semistable vector bundles. Thus by classical technique of GIT of Kempf–Ness type we get:
Proposition 4.1 (Narasimhan–Seshadri)

\[ R_{\Sigma} = R_g = \mathcal{M}^{ss}. \]

Proposition 4.2 The form \( F_{A_{\text{CS}}} \) is a \((1,1)\)-form and the line bundle \( L \) admits a unique holomorphic structure compatible with the Hermitian connection \( A_{\text{CS}} \).

On the other hand, a complex structure \( I \) on \( \Sigma \) defines a Kähler metric on \( \mathcal{M}^{ss} \) (the so-called Weyl–Petersson metric) with Kähler form

\[ \omega_{\text{WP}} = i F_{A_{\text{CS}}} = i \cdot \Omega. \] (4.4)

This metric defines the Levi-Civita connection on the complex tangent bundle \( T\mathcal{M}^{ss} \), and hence a Hermitian connection \( A_{\text{LC}} \) on the line bundle

\[ \det T\mathcal{M}^{ss} = L^{\otimes 4}, \] (4.5)

and a Hermitian connection \( \frac{1}{4} A_{\text{LC}} \) on \( L \) compatible with the holomorphic structure on \( L \). Thus we have

Proposition 4.3

\[ \frac{1}{4} A_{\text{LC}} = A_{\text{CS}}. \]

Finally, considering \( \mathcal{M}^{ss} \) as a family of \( \overline{\partial} \)-operators, we get the Quillen determinant line bundle \( L \) having a Hermitian connection \( A_Q \) with curvature form

\[ F_{A_Q} = i \cdot \Omega. \] (4.6)

Hence we can extend the equality of Proposition 4.3:

Proposition 4.4

\[ \frac{1}{4} A_{\text{LC}} = A_{\text{CS}} = A_Q. \]
Summarizing, the result of the complex quantization procedure of the pre-quantum system (3.17) can be considered to be the spaces of wave functions of level \( k \), that is, the spaces of \( I \)-holomorphic sections
\[
\mathcal{H}_{L^k} = H^0(L^k)
\] (4.7)

One knows that this system of spaces and monomorphisms is related to the system of representations of \( \mathfrak{su}(2, \mathbb{C}) \) in the Weiss–Zumino–Novikov–Witten model of CQFT. Namely, for a half integer \( i \), consider the irreducible representation \( V_i \) of dimension \( 2i + 1 \) of \( \mathfrak{su}(2, \mathbb{C}) \). The tensor product of two such representations is given by the Clebsch–Gordan rule
\[
V_i \otimes V_j = V_{i+j} \oplus V_{i+j-1} \oplus \cdots \oplus V_{i-j} \quad \text{for } i \geq j,
\] (4.8)
and the level of \( V_i \) is \( 2i \).

Then the fusion ring \( R_k(\mathfrak{su}(2, \mathbb{C})) \) of level \( k \) is the quotient
\[
R_k(\mathfrak{su}(2, \mathbb{C})) = R(\mathfrak{su}(2, \mathbb{C}))/\langle V_{(k+1)/2} \rangle
\] (4.9)
of the representation ring \( R(\mathfrak{su}(2, \mathbb{C})) \) by the ideal generated by \( V_{(k+1)/2} \).

Moreover, every character of the ring \( R(\mathfrak{su}(2, \mathbb{C})) \) is given by a complex number \( z \in \mathbb{C} \) which we can consider as a diagonal \( 2 \times 2 \) matrix \( \text{diag}(iz, -iz) \). This matrix acts on \( \mathfrak{su}(2, \mathbb{C}) \) and \( V_i \) and
\[
\chi_z(V_i) = \text{tr}(\exp(\text{diag}(iz, -iz))) = \frac{\sin((2i+1)z)}{\sin z}.
\] (4.10)

Thus
\[
\chi_z(V_i) = 0 \iff z = \frac{n\pi}{k+2} \text{ for } 1 \leq n \leq 2i+1.
\] (4.11)

In these terms we get:
\[
\mathcal{H}_{L^k} = \frac{(k+2)^{g-1}}{2^{g-1}} \sum_{n=1}^{k+1} \frac{1}{(\sin(n\pi/(k+2)))^{2g-2}}.
\] (4.12)

See [3] for a mathematical derivation of this formula.
5 Real polarization of $R_g$

The collection of real polarizations of the prequantum system

$$(R_g, \Omega, L, A_{CS})$$

is given in a very geometric way in the set-up of perturbation theory of 3-dimensional Chern–Simons theory. The crucial point is a trinion decomposition of a Riemann surfaces, given by a choice of a maximal collection of disjoint, noncontractible, pairwise nonisotopic smooth circles on $\Sigma$. An isotopy class of such a collection of circles is called a marking of the Riemann surface. It is easy to see ([HT]) that any such system contains $3g - 3$ simple closed circles

$$C_1, \ldots, C_{3g-3} \subset \Sigma_g,$$

and the complement is the union

$$\Sigma_g - \{C_1, \ldots, C_{3g-3}\} = \bigcup_{i=1}^{2g-2} P_i$$

of $2g - 2$ trinions $P_i$, where every trinion is a 2-sphere with 3 disjoint discs deleted:

$$P_i = S^2 \setminus (D_1 \cup D_2 \cup D_3) \quad \text{with} \quad \overline{D_i} \cap \overline{D_j} = \emptyset \quad \text{for} \quad i \neq j.$$

A collection $\{C_i\}$ with these conditions is called a trinion decomposition of $\Sigma$. The invariant of such a decomposition by marking class is given by its 3-valent dual graph $\Gamma(\{C_i\})$, associating a vertex to each trinion $P_i$, and an edge linking $P_i$ and $P_j$ to a circle $C_l$ ([5.1]) such that

$$C_l \subset \partial P_i \cap \partial P_j.$$

If we fix the isotopy class of a trinion decomposition $\{C_i\}$, we get a map

$$\pi_{\{C_i\}}: R_g \to \mathbb{R}^{3g-3}$$

with fixed coordinates $(c_1, \ldots, c_{3g-3})$ such that

$$c_i(\pi_{\{C_i\}}(\rho)) = \frac{1}{\pi} \cos^{-1}\left(\frac{1}{2} \text{tr} \rho([C_i])\right) \in [0, 1].$$

We see that
**Proposition 5.1**  
(1) The map $\pi_{\{C_i\}}$ is a real polarization of the system $(R_g, k \cdot \omega, L^k, k \cdot A_{CS})$.

(2) The coordinates $c_i$ are action coordinates for this Hamiltonian system (see (1.6) and [D]).

These functions $c_i$ are continuous on all $R_g$ and smooth over $(0, 1)$. Moreover, Goldman [G] constructed $U(1)$-actions on the open dense set

$$U_i = c_i^{-1}(0, 1)$$

for which the function $c_i$ is the moment map, and all these $U(1)$-actions commute with each other. Hence we get:

**Proposition 5.2**  
(1) The restriction $\pi_{\{C_i\}}|_{\bigcap_i U_i} : \bigcap_i U_i \to (0, 1)^{3g-3}$ is the moment map for the $U(1)^{3g-3}$-action on $\bigcap_i U_i$.

(2) The image of $R_g$ under $\pi_{\{C_i\}}$ is a convex polyhedron

$$I_{\{C_i\}} \subset [0, 1]^{3g-3}.$$  

(5.4)

(3) The symplectic volume of $R_g$ equals the Euclidean volume of $I_{\{C_i\}}$:

$$\int_{R_g} \omega^{3g-3} = \text{Vol} I_{\{C_i\}}.$$  

(5.5)

(4) The expected number of Bohr–Sommerfeld orbits of the real polarization $\{C_i\}$

$$N_{BS}(\pi_{\{C_i\}}, R_g, \omega, L, A_{CS})$$  

equals the number of half integer points in the polyhedron $I_{\{C_i\}}$, and

$$\lim_{k \to \infty} k^{3-3g} \cdot N_{k-BS} = \int_{R_g} \omega^{3g-3} = \text{Vol} I_{\{C_i\}}.$$  

(5.7)
From the combinatorial point of view, the number \( N_{k,BS} \), or more generally the numbers \( N_{k,BS} \) of \( k \)-BS fibres, is determined as follows: consider functions

\[
w: \{C_i\} \to \frac{1}{2k}\{0, 1, 2, \ldots, k\}
\]

on the collection of edges of the 3-valent graph \( \Gamma(\{C_i\}) \) to the collection of \( \frac{1}{2k} \) integers, such that, for any three edges \( C_l, C_m, C_n \) meeting at a vertex \( P_i \), the following 3 conditions hold:

1. \( w(C_l) + w(C_m) + w(C_n) \in \frac{1}{k} \cdot \mathbb{Z} \);
2. \( w(C_l) + w(C_m) + w(C_n) \leq 1 \);
3. for any ordering of the triple \( C_l, C_m, C_n \),
   \[
   |w(C_l) - w(C_m)| \leq w(C_n) \leq w(C_l) + w(C_m).
   \]

Such a function \( w \) is called an admissible integer weight of level \( k \) on the graph \( \Gamma(\{C_i\}) \).

**Proposition 5.3**

1. The number \( |W^k_g| \) of admissible weights of level \( k \) is independent of the graph \( \Gamma(\{C_i\}) \);
2. \( |W^k_g| = N_{k,BS} \).

The conditions (5.9) are called Clebsh–Gordan conditions for \( \text{su}(2, \mathbb{C}) \), for obvious reasons. We can view the space of all real functions with values in \( [0, 1] \) subject to these conditions to get a complex \( I_{\{C_i\}} \).

**Remark** Following this combinatorial approach, we can construct a two dimensional complex \( Y_g \): the set of vertices is the set of all dual graphs associated with all types of markings of \( \Sigma \). Two vertices are joined by an edge if and only if the two graphs are related by an elementary fusion operation. The 2-cells correspond to pentagons, and so on (see [MS]). The topology of this complex reflects the combinatorial properties of real polarizations of this type.
The geometric meaning of this combinatorial description is as follows: consider the space $\mathbb{R}^{3g-3}$ with action coordinates $c_i$ (5.3). This space contains the integer sublattice $\mathbb{Z}^{3g-3} \subset \mathbb{R}^{3g-3}$, and we can consider the action torus:

$$T^A = \mathbb{R}^{3g-3} / \mathbb{Z}^{3g-3}.$$  \hfill (5.11)

In particular, we get a map

$$\pi_A : R_g \to T^A$$  \hfill (5.12)

which glues at most points of the boundary of $I_{\{C_i\}}$.

Now every integer weight $w$ (5.8) satisfying (1) and (2), but a priori without the Clebsch–Gordan conditions (5.9), defines a point of order $2k$ on the action torus

$$w \in T^A_{2k}.$$  

In particular, the collection $W_g^k$ of admissible integer weights (subject to (5.9)) can be considered as a subset of points of order $2k$ on the action torus:

$$W_g^k \subset T^A_{2k}.$$  \hfill (5.13)

On the other hand, every vector $w \in \mathbb{R}^{3g-3}$ can be interpreted as a differential 1-form on $\mathbb{R}^{3g-3}$, and by the usual construction using the symplectic form $\Omega$, this defines a vector field $\xi_w$ tangent to the fibres of $\pi$. Integrating such vector fields defines the collection of transformations

$$\{t_w\} = e^{\xi_w} \subset \text{Diff}^+(R_g).$$  \hfill (5.14)

These transformations preserve the curvature form $A_{CS}$ of the connection. Thus (because $R_g$ is simple connected), there exists a collection of gauge transformations $\alpha_w \in \mathcal{G}_L$ of $L$ such that

$$(t_w)^*(A_{CS}) = A_{CS}^{\alpha_w}.$$  \hfill (5.15)

We can view such gauge transformations as $U(1)$-torsors, just as in describing the formulas for classical theta functions for Abelian varieties in §2.

Moreover, if $R_\Sigma$ is given the Kähler structure induced from $\Sigma$ and $s \in H^0(R_\Sigma, L^k)$ is a holomorphic section, then we have the following.
Proposition 5.4

\[(t_w)^*(s) \in H^0(R_\Sigma, L^k)\]  \hspace{1cm} (5.16)

is also a holomorphic section.

Corollary 5.1 If \(s_0\) is a sufficiently symmetric holomorphic section of \(L^k\), then the system

\[\{s_w = t_w^*(s_0)\} \subset H^0(R_\Sigma, L^k)\]  \hspace{1cm} (5.17)

is a special theta basis of some subspace of \(H^0(L^k)\).

Comparing (5.17) and (2.6), we see that the recipe to construct the theta basis is the same as for Abelian varieties with the action space \(T^A\) (see §2) but instead of the full collection \(T^A_{2k}\) of points of order \(2k\), we only use the subset \(W_g^k \subset T^A_{2k}\).

In our realistic situation, the prequantum system \((R_g, \Omega, L, A_{CS})\) is far from the regular “theoretical” case. But in the fundamental papers [JW1] and [JW2] there is a well-defined correction to the “theoretical” situation. Here we only explain what we must do at a maximally degenerate Bohr–Sommerfeld fibre. We get proofs of the central statements of Proposition 5.4 and Corollary 5.1 by a quite fruitful method: we give new definitions making the statements almost obvious. We do this in the following special section.

Unitary Schottky representations

Every oriented trinion \(P_i\) defines a handle \(HP_i\), and all these handles glue together to give a handlebody \(H_{\{C_i\}}\), a compact 3-manifold such that

\[\partial H_{\{C_i\}} = \Sigma.\]  \hspace{1cm} (5.18)

We get a surjection

\[\varphi: \pi_1(\Sigma) \to \pi_1(H_{\{C_i\}}),\]  \hspace{1cm} (5.19)

which defines the subspace

\[B_{\{C_i\}} = \{\rho \in R_g \mid \rho|_{\ker \varphi} = 1\}.\]  \hspace{1cm} (5.20)
Proposition 5.5  (1) $B_{\{C_i\}}$ is a Lagrangian subspace of $R_g$.

(2) More precisely, it is a fibre of the real polarization $\pi_{\{C_i\}}$ (5.3):
$$B_{\{C_i\}} = \pi^{-1}_{\{C_i\}}(1, \ldots, 1).$$

(3) Moreover, $B_{\{C_i\}}$ is a Bohr–Sommerfeld orbit of $\pi_{\{C_i\}}$.

This Lagrangian subspace $B_{\{C_i\}}$ is singular:
$$B^\text{red}_{\{C_i\}} = B_{\{C_i\}} \cap R^\text{red}_{g} = \text{Sing } B_{\{C_i\}};$$

and $B^\text{irr}_{\{C_i\}} = B_{\{C_i\}} \cap R^\text{irr}_{g}$ is a nonsingular Lagrangian subvariety.

Under the identification of $R_g$ with the moduli space of $s$-classes of semi-stable vector bundles on the algebraic curve $\Sigma$, the subspace $B_{\{C_i\}}$ is called the subset of \textit{unitary Schottky subbundles}.

Obviousely for this Bohr–Sommerfeld fibre $w_{US} \in T^4_k$, we must use a special description of the symplectomorphism $t_{w_{US}}$. This was done in the papers $[JW1]$ and $[JW2]$.

Returning to the general geometric quantization procedure and summarizing these results, we get two spaces of wave functions: complex quantization gives the spaces
$$\mathcal{H}^k_\Sigma = \mathcal{H}^0(L^k)$$
of $I$-holomorphic sections of $L^k$, and real quantization gives the direct sum
$$\mathcal{H}^k_\pi = \sum_{k\text{-BS fibres}} \mathbb{C} \cdot s_{k\cdot BS}$$
of lines generated by covariant constant sections of restrictions of our pre-quantum line bundle $(L^k, k \cdot A_{CS})$ to the Bohr–Sommerfeld fibres of $\pi$ of level $k$.

The amazing fact is the following:

Proposition 5.6  For any level $k$, any complex Riemann surface $\Sigma$, and any trinion decomposition $\{C_i\}$ with the real polarization $\pi$ of $R_g$ we have
$$\text{rank } \mathcal{H}_\Sigma = \mathcal{H}^0(L^k) = \text{rank } \mathcal{H}_\pi,$$
and these ranks can be computed by the Verlinde calculus.
Corollary 5.2 Our construction gives a distinguished theta basis of the first space $H^0(L^k)$.

This isomorphism between spaces of wave functions underlies all the “modular” behavior of gauge theory invariants in dimensions 2, 3 and 4.

The final “classical” question concerns the Fourier decomposition of our non-Abelian theta functions $s_w$ (5.17). It can be done using the Fourier decomposition along coordinates $\{c_i\}$ of the action torus $T^A$ (5.11) twisting by the system of torsors $\{\alpha_w\}$ (5.15). Roughly speaking, the theta functions $s_w$ (5.17) are truncated theta functions on the $(6g-6)$-dimensional “Fourier torus”

$$T_F = U(1)^{3g-3} \times T^A.$$ 

Namely all coefficients of Fourier decompositions not satisfying the Clebsch–Gordan conditions (5.9) must go to zero. Can this condition be interpreted in terms of the heat equation?

6 Other definition of a theta basis

We must first recall the main constructions of GQP. Let $h$ be the Hermitian form on $L$, and

$$\mu = \frac{1}{(3g-3)!} \omega^{3g-3} \quad (6.1)$$

the volume form on $R_g$. Then we have a scalar product and norm on the space $\Gamma(L^k)$ of global differentiable sections of $L^k$:

$$\langle s_1, s_2 \rangle = \int_{R_g} h(s_1, s_2) \cdot \mu \quad \text{and} \quad \|s\| = \sqrt{\langle s, s \rangle}. \quad (6.2)$$

Let $L^2(L^k)$ be the $L^2$-completion of $\Gamma(L^k)$ and

$$P_k: L^2(L^k) \to H^0(L^k) \quad (6.3)$$

the orthogonal projection to the finite dimensional subspace of holomorphic sections $H^0(L^k) \subset L^2(L^k)$. 
The ring $C^\infty(R_g)$ of smooth functions on $R_g$ acts on $L^2(L^k)$ by multiplication $s \to f \cdot s$, and acts on the space $H^0(L)$ as a Toeplitz operator:

$$T_f = P \circ f \in \text{End}(H^0(L^k));$$

the map $C^\infty(R_g) \to \text{End}(H^0(L^k))$ is called the Berezin–Toeplitz map.

Now, let

$$p: L^\star \to R_g$$

be the principal $C^\ast$-bundle of $L$. Every point $x \in L^k$ defines a linear form

$$l_x: H^0(L^k) \to \mathbb{C}, \quad \text{given by} \quad s(p(x)) = l_x(s) \cdot x,$$

and the coherent state vector $s_x \in H^0(L^k)$ associated to $x$, which is uniquely determined by the equation

$$\langle s_x, s \rangle = l_x(s).$$

Thus we get a map

$$\varphi_k: R_g \to \mathbb{P}H^0(L^k),$$

which is nothing other than the Hermitian conjugate of the standard algebraic geometric map by a complete linear system to the dual space, because of the equality

$$s_{\alpha \cdot x} = \alpha^{-1} \cdot s_x \quad \text{for} \ \alpha \in C^\ast.$$  

Now, following John Rawnsley, we can define coherent projectors $P_{p(x)}$ and the Rawnsley epsilon function $\varepsilon: R_g \to \mathbb{R}^+$ in such a way that:

$$\varepsilon(p(x)) = |x|^2 \cdot \langle s_x, s_x \rangle \quad \text{and} \quad h(s_1, s_2)_{p(x)} = \varepsilon(p(x)) \cdot \langle s_1, P_{p(x)}s_2 \rangle.$$  

Since $\varepsilon > 0$ we can modify the old measure on $R_g$:

$$\mu_\varepsilon = \varepsilon \cdot \mu,$$

where $\mu$ is (6.1). This measure gives an integral representation of Toeplitz operators:

$$T_f(s) = \int_{R_g} f(p(x)) \cdot P_{p(x)}(s) \cdot \mu_\varepsilon.$$  

Up to now, we have been working with a complex polarization. Let us return to the real polarization $\pi (5.3)$. We can identify the target real space $\mathbb{R}^{3g-3}$ of $\pi$ with the dual space $\mathbb{R}^{3g-3} = (\mathbb{R}^{3g-3})^*$ and we can consider our vectors $w \in W^k_s \subset T^A_{2k}$ as linear functions on the target space $\mathbb{R}^{3g-3}$. Thus we get a collection of functions

$$\pi^* w: R_g \to \mathbb{R} \quad (6.12)$$

and a collection of Toeplitz operators

$$T_{\pi^* w} \in \text{End}(H^0(L^k)). \quad (6.13)$$

Let us choose one (very symmetric) section $s_0$ in the following way: for $k = 1$, the space $H^0(L)$ is the space of ordinary theta functions (see, for example, [BL]) and every semistable bundle $E$ defines a theta divisor

$$\Theta_E = \{ L \in \text{Pic}_{g-1}(\Sigma) \mid h^0(E \otimes L) > 0 \}.$$  

Let $s_E$ be the section with this divisor as its zero set. Then one has the section

$$s_0 = s^k_{\mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma} \in H^0(R_\Sigma, L^k), \quad (6.14)$$

and the collection of sections

$$\{ T_{\pi^* w}(s_0) = s_w \} \subset H^0(R_\Sigma, L^k). \quad (6.15)$$

Using the integral representation (6.11), Rawnsley’s localization, and Proposition 5.6, we get immediately

**Theorem 6.1** The sections $\{ T_{\pi^* w}(s_0) = s_w \}$ form a basis of $H^0(R_\Sigma, L^k)$.

The reader not wishing to check the following statement may take (6.15) as the definition of the theta basis:

**Proposition 6.1** The basis $(5.17)$ coincides with the basis $(6.15)$.

Thus, we do indeed get a generalization of theta functions.
7 What next?

The theory of theta functions outlined above is just a small sample of the applications of the Geometric Quantization Procedure to algebraic geometry. Here we extend the list, mentioning applications which are natural generalizations of the above constructions.

Generalization to vector bundles of higher rank

This construction is new even in the classical set-up. Let us return to a real polarization of an Abelian variety \( \pi: A \to T^g_B \), and its dual fibration

\[
\pi': A' = \text{Pic}(A/T^g_B) \to T^g_B = B,
\]

with fibres

\[
(\pi')^{-1}(p) = \text{Hom}(\pi_1(\pi^{-1}(p)), U(1))
\]

and section

\[
s_0 \in A' \quad \text{with} \quad s_0 \cap (\pi')^{-1}(p) = \text{id} \in \text{Hom}(\pi_1(\pi^{-1}(p)), U(1)).
\]

Every stable holomorphic vector bundle \( E \) on a generic principally polarized Abelian variety \( A \) carries a Hermitian–Einstein connection \( a_E \) that defines a holomorphic structure on \( E \) with curvature

\[
F_{a_E} = \Lambda i \cdot \omega,
\]

where \( \Lambda \) is any constant element of \( U(\text{rank}(E)) \), for example, id. Hence the restriction of \( a_E \) to every fibre of \( \pi \) is a flat Hermitian connection on a \( g \)-torus, and thus

\[
(a_E|_{\pi^{-1}(b)}) = \chi_1 \oplus \cdots \oplus \chi_{\text{rank}E}, \quad \text{where} \quad \chi_i \in \text{Hom}(\pi_1(\pi^{-1}(b)), U(1))) = \text{Pic}(\pi^{-1}(b)) = (\pi')^{-1}(b).
\]

Definition 7.1 For vector bundles of higher rank, an \( E \)-BS fibre is a fibre \( \pi^{-1}(b) \) such that the restriction \( (E, a_E)|_{\pi^{-1}(b)} \) admits a covariant constant section.
Suppose that $E$ is ample, and in particular that
\[ H^i(A, E) = 0 \quad \text{for} \quad i > 0. \]
Then, alongside the complex “space of wave functions” $H^0(A, E)$, we get a new space of wave functions
\[ \mathcal{H}_E = \bigoplus_{E=BS} \mathbb{C} \cdot s_i, \quad (7.3) \]
where $s_i$ is a covariant constant section of the restriction of $E$ to a $E$-BS fibre.
We again have the problem of comparing the spaces
\[ \mathcal{H}_E = H^0(A, E) \quad \text{and} \quad \mathcal{H}_E^\pi. \quad (7.4) \]
This problem can be solved by analogous (but more sophisticated) methods from GQP. In particular

**Proposition 7.1** The space $H^0(A, E)$ admits a canonical theta basis.

Of course, if $X$ is any Kähler manifold with some real polarization (1.3) and stable holomorphic vector bundle $E$ admitting an Hermitian connection with curvature of the form (7.1), we get two spaces
\[ \mathcal{H}_E = H^0(X, E) \quad \text{and} \quad \mathcal{H}_E^\pi \]
to compare.
In particular if $X = R_{\Sigma_g}$ one has

**Proposition 7.2** For a stable vector bundle of higher rank $E$ on $R_{\Sigma}$,
\[ H^0(R_{\Sigma_g}, E) = \bigoplus_{E=BS} \mathbb{C} \cdot s_i. \]
This holds in particular for all the symmetric powers of the Poincaré bundles.

**K3 surfaces**
If the transcendental lattice $T_S$ of a K3 surface $S$ contains an even unimodular sublattice $H$ of rank 2, then $S$ admits a real polarization (see for example [G1], [G2] or [T]). Then every ample complete linear system on $S$ admits a special theta basis.
Geometry behind these constructions: mirror reflection of holomorphic geometry

If our real polarization (1.3) is regular, that is, the differential of the map \( \pi: X \to B \) is surjective then all fibres are \( n \)-tori, and the second fibration \( \pi': X' \to B \) can be defined fibrewise in the usual way:

\[
(\pi'^{-1})^{-1}(b) = \text{Hom}(\pi_1(\pi^{-1}(b)), U(1)) \quad \text{for any point } b \in B;
\]

that is, the fibre of \( \pi' \) is the space of classes of flat connections on the trivial line bundle on \( \pi^{-1}(b) \). This is a fibration in groups, and we want to consider its zero section \( s: B \to X' \) as a submanifold \( s_0 \subset X' \).

The restriction of a pair \((E, a_E)\) to any fibre \( \pi^{-1}(b) \) defines a finite set of points \( (\pi')^{-1}(b) \), and hence a multisection

\[
\text{GFT}(E) \subset X',
\]

which we again consider as a middle dimensional submanifold of \( X' \). This cycle is called the Geometric Fourier Transformation of \( E \).

Under the identification \( s_0 = B \), the set of \( E \)-BS fibres is defined now as the set of intersection points

\[
E-\text{BS} = s_0 \cap \text{GFT}(E),
\]

and under some geometric conditions, we expect that the number

\[
\#E-\text{BS} = [s_0] \cap [\text{GFT}(E)]
\]

where \([\ ]\) is the cohomology class of a submanifold.

In the general case of a polarization with degenerate fibres, this construction can be performed over the open subset \( B_0 \subset B \) of smooth tori and a number of questions arise:

1. to construct a smooth compactification \( S' \);
2. to construct a symplectic form \( \omega' \) and extend it to \( S' \), in such a way that \( \pi' \) is a new real polarization;
3. to construct a complex polarization of \( S' \) such that the fibration \( \pi \) is given by construction we have described, starting from \((S', \omega', L', a')\).
In full generality these problems are very hard (see for example [G1], [G2]). The ideal picture is described by the mirror diagram

\[
\begin{array}{c}
S \\
\downarrow \pi \\
B \\
\uparrow \pi' \\
\end{array} \quad E
\]

(7.9)

with holomorphic objects (vector bundles) corresponding to the top of (7.9) and special Lagrangian cycles to the bottom.

**The inverse problem**

Every stable holomorphic vector bundle \( E \) on an \( S_I \) (top of (7.9)) is a point in the moduli space of stable holomorphic vector bundles

\[
E \in \mathcal{M}^{s}_{[E]}
\]

(7.10)
of topological type \([E]\).

But the cycle \( \text{GFT}(E) \) (bottom of (7.9)) is a point in the moduli space of special Lagrangian cycles (see [HI])

\[
\text{GFT}(E) \in \mathcal{M}^{[\text{GFT}(E)]}
\]

(7.11)
of topological type \([\text{GFT}(E)]\). Thus we get a map

\[
\text{GFT}: \mathcal{M}^{s}_{[E]} \rightarrow \mathcal{M}^{[\text{GFT}(E)]}
\]

(7.12)
sending \( E \) to \( \text{GFT}(E) \).

However, we have not used all the information contained in \( E \). Namely, \( \text{GFT}(E) \) can be defined as a *supercycle* (or *brane*). It’s easy to see that any cycle \( \text{GFT}(E) \) admits a tautological topologically trivial line subbundle \( L \) with Hermitian connection \( s_\tau \). A pair

\[
(\text{GFT}(E), a_\tau) = s\text{GFT}(E)
\]

(7.13)
of this type is called a *supercycle* (or brane).
The attempt to reconstruct the vector bundle \( E \) (top of (7.9)) from the supercycle \( s\text{GFT}(E) \) (bottom of (7.9)) is called the inverse problem. More formally, let \( S\mathcal{M}^{\text{GFT}(E)} \) be the moduli space of supercycles of topological type \([\text{GFT}(E)]\). Then in many special cases, one can prove that the map

\[
s\text{GFT}: \mathcal{M}^g \rightarrow S\mathcal{M}^{([\text{GFT}(E)])}
\]

is an embedding at the general point. That is, a general stable vector bundle \( E \) (top of (7.9)) is uniquely determined by the supercycle \( s\text{GFT}(E) \) on \( S' \) (bottom of (7.9)).

For example, if the fibration \( \pi: X \rightarrow B \) is the family of all deformations (with degenerations) of the general fibre \( \pi^{-1}(b) = T^n \) as a torus with special structure inside \( S \), then

\[
X' = S\mathcal{M}^{[T^n]}
\]

is the family of all deformations (with degenerations) of the pair \((T^n, \tau_0)\), where \( \tau_0 \) is the trivial connection.

At present this program is only realized in part (see, for example, [T]). We must first use the experience of the geometric quantization procedure, and apply it in the Calabi–Yau realm of simply connected Kähler manifolds with canonical class zero. But in this paper, we want to emphasize that there exists the collection of singular Fano varieties \( R_g \) for which these constructions are very important, although this is an extremely irregular case.

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References

[A] V. I. Arnol’d, Mathematical methods of classical mechanics, 2nd edition, Springer-Verlag 1989
[B] A. Beauville, Vector bundles on Riemann surfaces and conformal field theory, in “Algebraic and geometric methods in math. physics” (Kacively, 1993), Kluwer Acad. Publ., Dordrecht, 1996, pp. 145–166

[BL] A. Beauville and Y. Laszlo, Conformal blocks and generalized theta functions, Comm. Math. Phys. 164 (1994), 385–419

[D] J. J. Duistermaat, On global action-angle coordinates, Comm. Pure Appl. Math. 33 (1980), 687–706

[DH] J. J. Duistermaat and G. J. Heckman, On the variation in the cohomology of the symplectic form of the reduced phase space, Invent. Math., 69 (1982), 259–268; Addendum same J., 72 (1983), 153–158

[G] W. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, Invent. Math. 85 (1986), 263–302.

[G1] M. Gross, Special Lagrangian fibrations I: Topology, in “Integrable systems and algebraic geometry”, eds. Saito, Shimizu and Ueno, World Scientific, 1998, pp. 156–193 (preprint alg-geom 9710006, 27 pp.)

[G2] M. Gross, Special Lagrangian fibrations II: to appear in J. Diff. Geom., preprint alg-geom 9809073, 71 pp.

[GS1] V. Guillemin and S. Sternberg, Symplectic techniques in physics, CUP (1983)

[GS2] V. Guillemin and S. Sternberg, The Gel’fand–Cetlin system and quantization of the complex flag manifolds, J. Func. Analysis, 52 (1983), 106–128

[GLO] V. Golyshev, V. Lunts and D. Orlov, Mirror symmetry for Abelian varieties., preprint, alg-geom 9812003, 39 pp.

[HT] A. Hatcher and W. Thurston, A presentation for the mapping class group of a closed oriented surface, Topology 19 (1980), 221–237

[HL] R. Harvey and H. B. Lawson, Calibrated geometries, Acta Math. 148 (1982), 47–157
Quantization and “theta functions” 33

[JW1] L. C. Jeffrey and J. Weitsman, Half density quantization of the moduli space of flat connections and Witten’s semiclassical invariants, Topology 32 (1993), 509–529

[JW2] L. C. Jeffrey and J. Weitsman, Bohr–Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula, Commun. Math. Phys. 150 (1992), 593–630.

[MS] G. Moore and N. Seiberg, Classical and quantum conformal field theory, Comm. Math. Phys. 123 (1989), 177–254

[Mum] D. Mumford, Tata lectures on theta. I: Progr. Math 28, Birkhäuser (1983). II. Jacobian theta functions and differential equations: Progr. Math 43, Birkhäuser (1984). III. Progr. Math 97, Birkhäuser (1991)

[R] J. H. Rawnsley, Coherent states and Kähler manifolds, Quart. J. Math. 28 (1977), 403-415

[RSW] T. R. Ramadas, L. M. Singer and J. Weitsman, Some comments on Chern–Simons gauge theory, Commun. Math. Phys. 126 (1989), 409–420

[S1] J. Śniatycki, Geometric quantization and quantum mechanics, Applied Math Sciences 30, Springer (1980)

[S2] J. Śniatycki, Bohr–Sommerfeld conditions in Geometric quantization, Reports in Math. Phys. 7, (1974), 127–135

[T] Andrei Tyurin, Geometric quantization and mirror symmetry, Warwick preprint 22/1999, \texttt{alg-geom 9902027}, 53 pp.

[W] N. Woodhouse, Geometric Quantization, Oxford Math Monographs, OUP (1980)

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