ON NUMBER AND EVENNESS OF SOLUTIONS OF THE SU(3) TODA SYSTEM ON FLAT TORI WITH NON-CRITICAL PARAMETERS

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ABSTRACT. We study the SU(3) Toda system with singular sources

\[ \begin{cases} \Delta u + 2e^u - e^v = 4\pi \sum_{k=0}^{m} n_{1,k} \delta_{p_k} & \text{on } E_\tau, \\ \Delta v + 2e^v - e^u = 4\pi \sum_{k=0}^{m} n_{2,k} \delta_{p_k} & \text{on } E_\tau, \end{cases} \]

where \( E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \) with \( \operatorname{Im} \tau > 0 \) is a flat torus, \( \delta_{p_k} \) is the Dirac measure at \( p_k \), and \( n_{1,k}, n_{2,k} \in \mathbb{Z}_{\geq 0} \) satisfy \( \sum_k n_{1,k} \neq \sum_k n_{2,k} \mod 3 \). This is known as the non-critical case and it follows from a general existence result of [3] that solutions always exist. In this paper we prove that

(i) The system has at most

\[ \frac{1}{3 \times 2^{m+1}} \prod_{k=0}^{m} (n_{1,k} + 1)(n_{2,k} + 1)(n_{1,k} + n_{2,k} + 2) \in \mathbb{N} \]

solutions. We have several examples to indicate that this upper bound should be sharp. Our proof presents a nice combination of the a priori estimates from analysis and the classical Bézout theorem from algebraic geometry.

(ii) For \( m = 0 \) and \( p_0 = 0 \), the system has even solutions if and only if at least one of \( \{ n_{1,0}, n_{2,0} \} \) is even. Furthermore, if \( n_{1,0} \) is odd, \( n_{2,0} \) is even and \( n_{1,0} < n_{2,0} \), then except for finitely many \( \tau \)'s modulo \( SL(2, \mathbb{Z}) \) action, the system has exactly \( \frac{n_{1,0} + 1}{2} \) even solutions.

Differently from [3], our proofs are based on the integrability of the Toda system, and also imply a general non-existence result for even solutions of the Toda system with four singular sources.

1. INTRODUCTION

Throughout the paper, we use the notations \( \omega_0 = 0, \omega_1 = 1, \omega_2 = \tau, \omega_3 = 1 + \tau \) and \( \Lambda_\tau = \mathbb{Z} + \mathbb{Z} \tau \), where \( \tau \in \mathbb{H} = \{ \tau \mid \operatorname{Im} \tau > 0 \} \). Define \( E_\tau := \mathbb{C}/\Lambda_\tau \) to be a flat torus and \( E_\tau[2] := \{ \frac{\omega_k}{2} \mid k = 0, 1, 2, 3 \} + \Lambda_\tau \) to be the set consisting of the lattice points and 2-torsion points of \( E_\tau \).

1.1. The SU(3) Toda system. In this paper, we study the following SU(3) Toda system with arbitrary singular sources

\[ \begin{cases} \Delta u + 2e^u - e^v = 4\pi \sum_{k=0}^{m} n_{1,k} \delta_{p_k} & \text{on } E_\tau, \\ \Delta v + 2e^v - e^u = 4\pi \sum_{k=0}^{m} n_{2,k} \delta_{p_k} & \text{on } E_\tau, \end{cases} \]

where \( m \geq 0, p_0, \ldots, p_m \) are \( m + 1 \) distinct points in \( E_\tau \), \( \delta_{p_k} \) is the Dirac measure at the point \( p_k \) and \( n_{j,k} \in \mathbb{Z}_{\geq 0} \) for all \( j, k \) with \( \sum_k n_{j,k} \geq 1 \). We
always use the complex variable \( z = x_1 + i x_2 \). Then the Laplace operator \( \Delta = 4 \partial_{zz} \).

The \( SU(3) \) Toda system (1.1) or its general \( SU(N + 1) \) version is closely related to the classical infinitesimal Plücker formula in algebraic geometry [16]; see e.g. [22, 23]. Besides, by denoting

\[
\mathcal{N}_j := \sum_{k=0}^{m} n_{j,k} \in \mathbb{Z}_{\geq 0}, \quad j = 1, 2,
\]

the \( SU(3) \) Toda system (1.1) can be also written as

\[
\begin{cases}
\Delta u + 2 \rho_1 \left( \frac{e^u}{\int e^u} - \frac{1}{|E_1|} \right) - \rho_2 \left( \frac{e^u}{\int e^u} - \frac{1}{|E_1|} \right) = 4 \pi \sum_{k=0}^{m} n_{1,1} (\delta_{p_k} - \frac{1}{|E_1|}) \text{ on } E_1, \\
\Delta v + 2 \rho_2 \left( \frac{e^v}{\int e^v} - \frac{1}{|E_1|} \right) - \rho_1 \left( \frac{e^v}{\int e^v} - \frac{1}{|E_1|} \right) = 4 \pi \sum_{k=0}^{m} n_{2,1} (\delta_{p_k} - \frac{1}{|E_1|}) \text{ on } E_1,
\end{cases}
\]

where \( \rho_1 = 4 \pi \sum_{j=0}^{\infty} n_{1,j} > 0, \quad \rho_2 = 4 \pi \sum_{j=0}^{\infty} n_{2,j} > 0, \)

which is a special case of the following general \( SU(3) \) Toda system of mean field type on a compact Riemann surface \( \Sigma \):

\[
\begin{cases}
\Delta u + 2 \rho_1 \left( \frac{h_1 e^u}{\int h_1 e^u} - \frac{1}{|E_1|} \right) - \rho_2 \left( \frac{h_2 e^u}{\int h_2 e^u} - \frac{1}{|E_1|} \right) = 4 \pi \sum_{k=0}^{m} a_{1,k} (\delta_{p_k} - \frac{1}{|E_1|}), \\
\Delta v + 2 \rho_2 \left( \frac{h_1 e^v}{\int h_1 e^v} - \frac{1}{|E_1|} \right) - \rho_1 \left( \frac{h_2 e^v}{\int h_2 e^v} - \frac{1}{|E_1|} \right) = 4 \pi \sum_{k=0}^{m} a_{2,k} (\delta_{p_k} - \frac{1}{|E_1|}).
\end{cases}
\]

Here \( |\Sigma| \) denotes the area of \( \Sigma \), \( h_1, h_2 \) are positive smooth functions on \( \Sigma \) and \( a_{i,j} > -1 \) for all \( i,j \). System (1.4) and its \( SU(N + 1) \) version arise in many geometric and physical problems. On the geometric side, the Toda system (1.4) has deep relations to holomorphic curves in \( \mathbb{C}P^2 \), flat \( SU(3) \) connection, complete integrability and harmonic sequences. See e.g. [5, 6, 11, 17, 21, 23] and references therein. While in mathematical physics it arises from the non-abelian Chern-Simons theory which describes the physics of high critical temperature superconductivity; see e.g. [9, 12, 13, 14, 25, 32, 33, 35] and references therein. The singularities represent the ramification points of the complex curves and the vortices of the wave functions respectively.

For the Toda system (1.4), the existence of solutions is a challenging problem and has been widely studied in the literature; see [2, 3, 4, 19, 20, 26, 27, 28] and references therein. Remark that in these works, the apriori estimates are needed to apply either the variational method or Leray-Schauder degree method. Due to this reason, for given singular parameters \( a_{i,j} \)'s, the parameters \( (\rho_1, \rho_2) \) are called non-critical (resp. critical) if the apriori estimates hold (resp. fail). For our purpose, let us consider the special case

\[
a_{i,j} \in \mathbb{Z}_{\geq 0} \quad \text{for all } i,j.
\]

Then by studying the bubbling phenomena of (1.4), it was proved in [22, 24] that \( (\rho_1, \rho_2) \) are non-critical as long as \( \rho_j \notin 4 \pi \mathbb{N} \) for \( j = 1, 2 \).

**Theorem A.** [22, 24] Let \( a_{i,j} \in \mathbb{Z}_{\geq 0} \) for all \( i,j \), and let \( K \subset \Sigma \setminus \{ p_k \}_{k=0}^{m} \) be any compact set. If \( \rho_j \notin 4 \pi \mathbb{N} \) for \( j = 1, 2 \), then there is \( C = C(K, \rho_1, \rho_2) \) such that
for any solution \((u, v)\) of (1.4),

\[(1.5) \quad |u(z)| + |v(z)| \leq C, \quad \forall z \in K.\]

By applying the a priori estimate established in Theorem A, it was proved via variational methods in [3] that

**Theorem B.** [3] Let \(a_{j,k} \in \mathbb{Z}_{>0}\) for all \(j, k\). If the genus of \(\Sigma\) is positive and \(\rho_j \notin 4\pi\mathbb{N}\) for \(j = 1, 2\), then (1.4) has solutions.

In view of Theorems A-B, the first basic question that interests us is:

**Question 1:** Let \(n_{j,k} \in \mathbb{R}_{>-1}\) for all \(j, k\) and \((\rho_1, \rho_2)\) be non-critical (i.e. the a priori estimates hold). Does the Toda system (1.1) or equivalently (1.3) have only finitely many solutions?

For general elliptic PDEs, it is known that the a priori estimates do not necessarily imply the finiteness of number of solutions (A well-known example is \(-\Delta u + u = u^p, u > 0\) in \(\mathbb{R}^N\) with \(1 < p < \frac{N+2}{(N-2)_+}\), for which the a priori estimate holds but the dimension of the solution space is \(N\) due to the invariance of translation). We strongly believe that the answer of Question 1 for the Toda system (1.1) is positive. However, even for such Toda system, the finiteness of the number of solutions is a highly non-trivial question from the analytic point of view. For a single equation with exponential non-linearity, there are only a few results to answer Question 1 (see e.g. [2]). In this paper, we initiate to study this question for the Toda system (1.1) with \(n_{j,k} \in \mathbb{Z}_{\geq 0}\).

Note for the Toda system (1.1) or equivalently (1.3), we have

\[\rho_1 = 4\pi \frac{2N_1 + N_2}{3} > 0, \quad \rho_2 = 4\pi \frac{N_1 + 2N_2}{3} > 0,\]

so

\[\rho_1, \rho_2 \notin 4\pi\mathbb{N} \quad \text{if and only if} \quad N_1 \not\equiv N_2 \pmod{3}.\]

Thus it follows from Theorem B that

**Theorem C.** Let \(n_{j,k} \in \mathbb{Z}_{\geq 0}\) for all \(j, k\) and \(N_1 \not\equiv N_2 \pmod{3}\), then the Toda system (1.1) always has solutions.

Our first result of this paper reads as follows.

**Theorem 1.1.** Let \(n_{j,k} \in \mathbb{Z}_{\geq 0}\) for all \(j, k\) and \(N_1 \not\equiv N_2 \pmod{3}\). Then the Toda system (1.1) has at most

\[N(\{n_{1,k}\}_k, \{n_{2,k}\}_k) := \frac{1}{3 \times 2^{m+1}} \prod_{k=0}^{m}(n_{1,k} + 1)(n_{2,k} + 1)(n_{1,k} + n_{2,k} + 2)\]

solutions.

It is elementary to see that \(N(\{n_{1,k}\}_k, \{n_{2,k}\}_k) \in \mathbb{N}\) because

\[2|(n_{1,k} + 1)(n_{2,k} + 1)(n_{1,k} + n_{2,k} + 2) \quad \forall k,\]
and $N_1 \not\equiv N_2 \mod 3$ implies $n_{1,k_0} \not\equiv n_{2,k_0} \mod 3$ for some $k_0$ and so exactly one of $\{n_{1,k_0} + 1, n_{2,k_0} + 1, n_{1,k_0} + n_{2,k_0} + 2\}$ is a multiple of 3, i.e.

$$6\left((n_{1,k_0} + 1)(n_{2,k_0} + 1)(n_{1,k_0} + n_{2,k_0} + 2)\right).$$

Theorem 1.1 not only answers Question 1 for the integer case $n_{i,k} \in \mathbb{Z}_{\geq 0}$, but also gives an explicit upper bound for the number of solutions. We will see from some explicit examples below (see Conjecture 1.7 and Theorem 1.9-1.10) that this upper bound should be optimal.

**Remark 1.2.** The topological Leray-Schauder degree is another notion in analysis to describe the “number” of solutions. Since the a priori estimates hold by Theorem A, the topological Leray-Schauder degree for the Toda system (1.1) under the assumption of Theorem 1.1 is well-defined (cf. [20]).

We propose

**Conjecture 1.3.** Under the same condition as Theorem 1.1, the topological Leray-Schauder degree for the Toda system (1.1) equals to $N\left(\{n_{1,k}\}, \{n_{2,k}\}\right)$.

Next we consider the Toda system (1.1) with singular sources at the lattice points and half periods (i.e. $m = 3$ and $p_k = \frac{\omega_k}{2}$):

\[
\begin{align*}
\Delta u + 2e^u - e^v &= 4\pi \sum_{k=0}^{3} n_{1,k} \delta_{\omega_k} \quad \text{on} \; E_{\tau}, \\
\Delta v + 2e^v - e^u &= 4\pi \sum_{k=0}^{3} n_{2,k} \delta_{\omega_k} \quad \text{on} \; E_{\tau}.
\end{align*}
\]

An interesting property of the Toda system (1.6) is that if $(u(z), v(z))$ is a solution, then so does $(u(-z), v(-z))$. Thus a natural question arises:

**Question 2:** Does the Toda system (1.6) have even solutions?

At the first sight Question 2 looks simple: One might think that the answer should be positive by restricting the associated functional on the even function subspace of the Sobolev space and applying the variational argument in [3]. However, it turns out that the method in [3] cannot work for even solutions in general.

Our second purpose in this paper is to study Question 2. We will see that Question 2 is not trivial either and its answer could be negative for some cases. Here is our second result.

**Theorem 1.4** (Nonexistence of even solutions). Let $N_1 \not\equiv N_2 \mod 3$. If $n_{1,k}, n_{2,k}$ are both odd for some $k \in \{0, 1, 2, 3\}$, then the Toda system (1.6) has no even solutions.

In other words, the solution obtained in Theorem C cannot be even and so the Toda system (1.6) has at least two solutions.

Our next result shows that the assumption of Theorem 1.4 is sharp at least for the special case $n_{i,k} = 0$ for $k = 1, 2, 3$, i.e. the following system with one singular source

\[
\begin{align*}
\Delta u + 2e^u - e^v &= 4\pi n_1 \delta_0 \quad \text{on} \; E_{\tau}, \\
\Delta v + 2e^v - e^u &= 4\pi n_2 \delta_0 \quad \text{on} \; E_{\tau},
\end{align*}
\]

\[
\text{(1.7)}
\]
where \( n_1, n_2 \in \mathbb{Z}_{\ge 0} \) satisfy the non-critical condition \( n_1 \not\equiv n_2 \pmod{3} \). Furthermore, without loss of generality we always assume \( n_1 < n_2 \).

Theorem 1.4 asserts that the system (1.7) has no even solutions if \( n_1, n_2 \) are both odd. Our next result shows that (1.7) has even solutions if at least one of \( n_1, n_2 \) is even. Note that in this case, there is a unique even number among \( \{ n_1 + 1, n_2 + 1, n_1 + n_2 + 2 \} \); we denote it by

\[
2N_e := \text{the unique even number among } \{ n_1 + 1, n_2 + 1, n_1 + n_2 + 2 \}.
\]

Note that \( N_e = 1 \) if \( n_i = 1 \) for some \( i \).

**Theorem 1.5 (Existence of even solutions).** Let \( n_1 \not\equiv n_2 \pmod{3} \), \( n_1 < n_2 \) and at least one of \( n_1, n_2 \) be even, with \( N_e \) given by (1.8). Then the following hold.

(i) The Toda system (1.7) always has even solutions, and the number of even solutions is at most \( N_e \).

(ii) The even solution is unique if \( n_i = 1 \) for some \( i \).

(iii) If further one of the following holds:

1. \( n_1 \) is odd, i.e. \( N_e = \frac{n_1 + 1}{2} \),
2. \( n_2 \) is odd (i.e. \( N_e = \frac{n_2 + 1}{2} \)) and \( n_2 - n_1 \in \{1, 5\} \),
3. \( n_1 \) is even and \( n_2 = n_1 + 2 \), i.e. \( N_e = \frac{n_1 + n_2 + 2}{2} = n_1 + 2 \),

then except for finitely many \( \tau \)'s modulo \( SL(2, \mathbb{Z}) \) action, the number of even solutions for (1.7) is exactly \( N_e \).

Here a matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \) acting on \( \tau \) means the Mobius transformation \( A \cdot \tau := \frac{a \tau + b}{c \tau + d} \), and \( \tilde{\tau} \equiv \tau \) modulo \( SL(2, \mathbb{Z}) \) if \( \tilde{\tau} = A \cdot \tau \) for some \( A \in SL(2, \mathbb{Z}) \). It is known that such two tori \( E_\tau \) and \( E_{\tilde{\tau}} \) are conformally equivalent. We believe that the assertion of Theorem 1.5(iii) should holds without the additional conditions (1)-(3); see Remark 6.3 in Section 6.

**Remark 1.6.** It is interesting to compare Theorems 1.4, 1.5 with those results in our previous work [10], where we studied the Toda system (1.7) in the critical case \( n_2 = n_1 + 3l \) with \( n_1, l \in \mathbb{Z}_{\ge 0} \) and proved that

(i) When \( n_1 \) is odd and \( n_2 \) is even, the Toda system has no even solutions.

(ii) When \( n_1, n_2 \) are both odd, the Toda system has at least one family of 2-parametric even solutions \( (u_{\lambda, \mu}, v_{\lambda, \mu}) \), \( \lambda, \mu > 0 \).

(iii) When \( n_1 \) is even, the existence of even solutions depends on the choice of the period \( \tau \). Moreover, once the Toda system has an even solution, then it has a 1-parametric family of even solutions \( (u_{\lambda}, v_{\lambda}) \), \( \lambda > 0 \).

Therefore, the structures of even solutions are completely different between the critical case and the non-critical case.

For the Toda system (1.7), we propose the following conjecture.
Conjecture 1.7. Let \( n_1 \neq n_2 \) mod 3 and \( n_1 < n_2 \). Then the Toda system (1.7) has exactly
\[
N(n_1, n_2) = \frac{(n_1 + 1)(n_2 + 1)(n_1 + n_2 + 2)}{6}
\]
solutions except for finitely many \( \tau \)'s modulo \( \text{SL}(2, \mathbb{Z}) \).

We will give an explanation of Conjecture 1.7 in Remark 4.4. Motivated by Conjecture 1.7 and Theorem 1.9-1.10 below, we propose the following stronger conjecture.

Conjecture 1.8. Let \( n_1 \neq n_2 \) mod 3 and \( n_1 < n_2 \). Then except for finitely many \( \tau \)'s modulo \( \text{SL}(2, \mathbb{Z}) \), any solution of the Toda system (1.7) is non-degenerate.

It is easy to see that under the assumption of Theorem 1.5,
\[
N(n_1, n_2) > N_e \quad \text{unless} \quad (n_1, n_2) \in \{(0, 1), (0, 2)\}.
\]

This suggests that the Toda system (1.7) should have only even solutions for \( (n_1, n_2) \in \{(0, 1), (0, 2)\} \), while for other \( (n_1, n_2)'s \) even solutions and non-even solutions should exist simultaneously for generic \( \tau \)'s. Our next results confirm these assertions and Conjecture 1.7 for special \( (n_1, n_2)'s \).

Recall that \( \wp(z) = \wp(z; \tau) \) is the Weierstrass \( \wp \)-function with periods \( \omega_1 = 1 \) and \( \omega_2 = \tau \), and \( g_2 = g_2(\tau), g_3 = g_3(\tau) \) are well known invariants of the elliptic curve \( E_\tau \), given by
\[
\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.
\]

Theorem 1.9.
(1) If \( (n_1, n_2) = (0, 1) \), then the solution of the Toda system (1.7) is unique and so is even.
(2) If \( (n_1, n_2) = (0, 2) \), then for \( \tau \equiv e^{\pi i/3} \) modulo \( \text{SL}(2, \mathbb{Z}) \) (i.e. \( g_2 = 0 \)), the Toda system (1.7) has a unique solution which is even; for all other \( \tau \)'s (i.e. \( g_2 \neq 0 \)), the Toda system (1.7) has exactly 2 solutions which are both even.

Theorem 1.10. Let \( (n_1, n_2) = (0, 4) \) and note that the weight 12 modular form
\[
343g_3^2 - 6561g_2^3
\]
has a unique zero \( \tau_0 \) modulo \( \text{SL}(2, \mathbb{Z}) \). Then the following hold.
(1) For \( \tau \equiv i = \sqrt{-1} \) modulo \( \text{SL}(2, \mathbb{Z}) \) (i.e. \( g_3 = 0 \)), the Toda system (1.7) has exactly 3 solutions that are all even.
(2) For \( \tau \equiv \tau_0 \) modulo \( \text{SL}(2, \mathbb{Z}) \), the Toda system (1.7) has exactly 4 solutions, among which 2 of them are even solutions.
(3) For \( \tau \neq i, \tau_0 \) modulo \( \text{SL}(2, \mathbb{Z}) \), the Toda system (1.7) has exactly 5 solutions, among which 3 of them are even solutions.

Theorem 1.10 indicates that when \( \tau \to \tau_0 \), two distinct even solutions converge to the same degenerate even solution; while when \( \tau \to i \), three solutions (i.e. two non-even and one even) converge simultaneously to the same degenerate even solution. These statements can be easily seen from the proof of Theorem 1.10 in Section 4.
1.2. The approach: A third order linear ODE. Differently from [3], our approach of proving these results is based on the well-known integrability of the Toda system (1.1). Let

\[ U := \frac{2u + v}{3}, \quad V := \frac{u + 2v}{3}, \]
\[ \gamma_{1,k} := \frac{2n_{1,k} + n_{2,k}}{3}, \quad \gamma_{2,k} := \frac{n_{1,k} + 2n_{2,k}}{3}. \]

Then (1.1) is equivalent to

\[
\begin{cases}
\Delta U + e^{2U-V} = 4\pi \sum_{k=0}^{m} \gamma_{1,k} \delta_{p_k} & \text{on } E_\tau, \\
\Delta V + e^{2V-U} = 4\pi \sum_{k=0}^{m} \gamma_{2,k} \delta_{p_k} & \text{on } E_\tau.
\end{cases}
\]

Let \((U, V)\) be a solution of (1.9), then \(y(z) = e^{-U(z)}\) solves the following third order linear ODE

\[ (\partial_z - V_z)(\partial_z + V_z - U_z)(\partial_z + U_z)y(z) = 0. \]

Recall that

\[ \zeta(z) = \zeta(z; \tau) = \frac{1}{z} - \frac{\varphi}{8\pi} z^3 + \cdots \]

is the Weierstrass zeta function defined by \(\xi'(z) := -\varphi(z)\). Then we will see in Section 2 that ODE (1.10) has the following form

\[
y''' - \left( \sum_{k=0}^{m} a_k \varphi(z - p_k) + \sum_{k=0}^{m} B_k \zeta(z - p_k) + B \right) y' + \left( \sum_{k=0}^{m} \beta_k \varphi'(z - p_k) + \sum_{k=0}^{m} D_k \varphi(z - p_k) + \sum_{k=0}^{m} A_k \zeta(z - p_k) + D \right) y = 0,
\]

where \(a_k, \beta_k\) are explicitly determined by \(\gamma_{j,k}\)'s (see (2.3)) and \(A_k, B_k, D_k, B, D\) are some constants satisfying the residue condition

\[ \sum_{k=0}^{m} A_k = 0, \quad \sum_{k=0}^{m} B_k = 0. \]

This ODE (1.11) has regular singularities at \(p_k\) with local exponents

\[-\gamma_{1,k}, \quad -\gamma_{1,k} + (n_{1,k} + 1), \quad -\gamma_{1,k} + (n_{1,k} + n_{2,k} + 2),\]

so (1.11) might have solutions with logarithmic singularity at \(p_k\). Conventionally, the singularity \(p_k\) is called apparent if (1.11) has no solutions with logarithmic singularity at \(p_k\). It is a standard fact (cf. [23, 30]) that \(p_k\)'s are all apparent singularities of (1.11) if it comes from a solution \((U, V)\) of (1.9); see Section 2 for a brief explanation. Thanks to this fact, we will prove Theorem 1.4 in Section 5 by showing that at least one of \(\gamma_{j,k}\)'s can not be apparent of (1.11) under the assumption of Theorem 1.4.

Denote the set of \(3m + 5\) parameters of ODE (1.11) by

\[ \mathbf{B} := (A_0, \ldots, A_m, B_0, \ldots, B_m, B, D_0, \ldots, D_m, D) \]

for convenience. Our proof of Theorem 1.4 is based on the following result.
Theorem 1.11. Let $N_1 \not\equiv N_2 \mod 3$. Then there is a one-to-one correspondence between solutions of the Toda system (1.1) and those $\vec{B}$'s such that all $p_k$'s are apparent singularities of ODE (1.11)-(1.12).

By Theorem 1.11 to prove Theorem 1.1 we only need to count those $\vec{B}$'s such that all $p_k$'s are apparent singularities.

For the special case (1.7), ODE (1.11) has a simpler expression and reads as

\begin{equation}
y''' - (\alpha \zeta(z) + B)y' + (\beta \zeta'(z) + D_0 \zeta(z) + D)y = 0,
\end{equation}

where $B, D_0, D$ are complex constants and

\begin{align*}
\alpha &= \gamma_1 (\gamma_1 + 1) + \gamma_2 (\gamma_2 + 1) - \gamma_1 \gamma_2, \\
\beta &= \frac{-2\gamma_1 (\gamma_1 + 1) + \gamma_1 \gamma_2 (\gamma_1 - \gamma_2 - 1)}{2},
\end{align*}

with

\begin{align*}
\gamma_1 := \frac{2n_1 + n_2}{3}, \quad \gamma_2 := \frac{n_1 + 2n_2}{3}.
\end{align*}

See Section 2 for a proof. Then our proof of Theorems 1.5 and 1.9-1.10 is based on the following result.

Theorem 1.12. Let $n_1 \not\equiv n_2 \mod 3$ and $n_1 < n_2$. Then there is a one-to-one correspondence between even solutions of the Toda system (1.7) and those $(B, D_0, D)$'s with $D_0 = D = 0$ such that 0 is an apparent singularity of ODE (1.14).

Thanks to Theorems 1.11, 1.12 to prove Theorems 1.5 and 1.9-1.10 we only need to count those $(B, D_0, D)$'s (with $D_0 = D = 0$ for Theorem 1.5) such that 0 is an apparent singularity of (1.14).

The rest of this paper is organized as follows. In Section 2, we study the precise relation between the Toda system (1.9) and its associated third order linear ODE (1.11). In Section 3, we prove Theorems 1.11, 1.12. In Section 4, we prove Theorems 1.1 and 1.9-1.10. Theorems 1.4 and 1.5 will be proved in Sections 5 and 6 respectively. Parts of the above results can be generalized to the general $SU(N + 1)$ Toda system and will be studied in a forthcoming paper.

2. THE ASSOCIATED LINEAR ODE

In this section, we study the relation between solutions of the Toda system and the associated linear ODE.

Let $(U, V)$ be a solution of the Toda system (1.9), as in [23, 30] we consider the following linear differential operator

\begin{equation}
\mathcal{L} := (\partial_z - V_z)(\partial_z + V_z - U_z)(\partial_z + U_z) = \partial_z^3 + W_2 \partial_z + W_3,
\end{equation}

where

\begin{align*}
W_2 := U_{zz} - U_z^2 + V_{zz} - V_z^2 + U_z V_z, \\
W_3 := U_{zzz} - 2U_z U_{zz} + U_z V_{zz} + V_z U_z^2 - U_z V_z^2,
\end{align*}
are known as $W$-invariants or $W$-symmetries in the literature due to their relationship to the $W$-algebras; see e.g. \cite{23,29,30,31}. It is easy to see that they satisfy the following crucial property (cf. \cite{23,30})

$$W_{2,z} = W_{3,z} = 0, \quad \text{for } z \in E_{\tau} \setminus \{p_k\}_{k=0}^m,$$

so $W_2$ and $W_3$ are elliptic functions. Since (1.9) yields

$$U(z) = 2\gamma_{1,k} \ln |z - p_k| + O(1), \quad V(z) = 2\gamma_{2,k} \ln |z - p_k| + O(1), \quad \text{at } z = p_k,$$

we have

$$W_2(z) = \frac{-\alpha_k}{(z - p_k)^2} + O((z - p_k)^{-1}),$$

$$W_3(z) = \frac{-2\beta_k}{(z - p_k)^2} + O((z - p_k)^{-2}),$$

and $W_2, W_3$ have no other poles in $E_{\tau}$ except $\{p_k\}_{k=0}^m$, where

$$\alpha_k := \gamma_{1,k} (\gamma_{1,k} + 1) + \gamma_{2,k} (\gamma_{2,k} + 1) - \gamma_{1,k} \gamma_{2,k},$$

$$\beta_k := -\frac{2\gamma_{1,k} (\gamma_{1,k} + 1) + \gamma_{1,k} \gamma_{2,k} (\gamma_{1,k} - \gamma_{2,k} - 1)}{2}.$$ 

Then the Liouville theorem implies that

$$W_2(z) = -\left( \sum_{k=0}^m \alpha_k \wp(z - p_k) + \sum_{k=0}^m B_k \zeta(z - p_k) + B \right),$$

$$W_3(z) = \sum_{k=0}^m \beta_k \zeta'(z - p_k) + \sum_{k=0}^m D_k \wp(z - p_k) + \sum_{k=0}^m A_k \zeta(z - p_k) + D,$$

where $A_k, B_k, D_k, B, D$'s are some constants satisfying the residue condition (1.12). Indeed, there are $B_k$'s such that

$$W_2(z) + \left( \sum_{k=0}^m \alpha_k \wp(z - p_k) + \sum_{k=0}^m B_k \zeta(z - p_k) \right)$$

is holomorphic in $\mathbb{C}$. Since this function is doubly-periodic and so bounded in $\mathbb{C}$, it must be a constant by the Liouville theorem.

In conclusion, it follows from (2.1) that $e^{-U(z)}$ solves the following third order linear ODE

$$L y = y''' - \left( \sum_{k=0}^m \alpha_k \wp(z - p_k) + \sum_{k=0}^m B_k \zeta(z - p_k) + B \right) y'$$

$$+ \left( \sum_{k=0}^m \beta_k \zeta'(z - p_k) + \sum_{k=0}^m D_k \wp(z - p_k) + \sum_{k=0}^m A_k \zeta(z - p_k) + D \right) y = 0.$$ 

This ODE (2.6) is of Fuschian type and has regular singularities $p_k$'s. A direct computation shows that the local exponents $\rho_{k,j}, j = 1, 2, 3,$ of (2.6) at $z = p_k$ are

$$\rho_{k,1} = -\gamma_{1,k}, \quad \rho_{k,2} = -\gamma_{1,k} + (n_{1,k} + 1),$$

$$\rho_{k,3} = -\gamma_{1,k} + (n_{1,k} + n_{2,k} + 2).$$
Note that under our basic assumption $N_1 \not\equiv N_2 \mod 3$, we see that $\gamma_{1,k} \not\equiv 0 \mod Z$ for some $k$, so any solution of (2.6) has branch points at such $p_k$ and hence is multi-valued.

On the other hand, since the exponent differences are all integers, (2.6) might have solutions with logarithmic singularity at $p_k$'s. As recalled in Section 1, $p_k$ is called an apparent singularity of (2.6) if all solutions of (2.6) has no logarithmic singularity at $p_k$. In this case, the local monodromy matrix (denoted it by $M_k$) at the apparent singularity $p_k$ is given by

\begin{equation}
M_k = e^{-2\pi i \gamma_{1,k}} I_3,
\end{equation}

where $I_3 = \text{diag}(1,1,1)$ denotes the identity matrix.

**Remark 2.1.** Fix a base point $q_0 = -\epsilon_0(1 + \tau) = -\epsilon_0 \omega_3$ with $0 < \epsilon_0 \ll 1$ such that $q_0$ is close to 0 and its neighborhood $B(q_0, |q_0|) = \{z \in \mathbb{C}| |z - q_0| < |q_0|\}$ contains no singularities of (2.6). The monodromy representation of (2.6) is a group homomorphism $\rho : \pi_1(E, \{p_k\}_{k=0}^m, q_0) \to SL(3, \mathbb{C})$.

Let

\begin{equation}
\tilde{Y} := (y_1, y_2, y_3)^T = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}
\end{equation}

be a basis of local solutions in a neighborhood $B(q_0, |q_0|/2) = \{z \in \mathbb{C}| |z - q_0| < |q_0|/2\}$ of $q_0$. For any loop $\ell \in \pi_1(E, \{p_k\}_{k=0}^m, q_0)$, we denote by $\ell^* y_j(z)$ to be the analytic continuation of $y_j(z)$ along $\ell$, then there is a matrix $\rho(\ell) \in SL(3, \mathbb{C})$ such that

\begin{equation}
\ell^* \tilde{Y} = \rho(\ell) \tilde{Y}.
\end{equation}

We say that the monodromy is *unitary* if the monodromy group is conjugate to a subgroup of the unitary group $SU(3)$.

Recall $\omega_1 = 1$, $\omega_2 = \tau$. Let $\ell_j \in \pi_1(E, \{p_k\}_{k=0}^m, q_0)$, $j = 1, 2$, be two fundamental cycles of $E_r$ connecting $q_0$ with $q_0 + \omega_j$ and let $\ell_k \in \pi_1(E, \{p_k\}_{k=0}^m, q_0)$ be a simple loop encircling $p_k$ counterclockwise respectively such that

\begin{equation}
\ell_1 \ell_2 \ell_1^{-1} \ell_2^{-1} = \zeta_0 \zeta_1 \cdots \zeta_m \text{ in } \pi_1(E, \{p_k\}_{k=0}^m, q_0).
\end{equation}

Now suppose that $p_k$ are apparent singularities of (2.6) for all $k$. Then (2.8) implies $\rho(\ell_k) = e^{-2\pi i \gamma_{1,k}} I_3$ for all $k$, so we conclude from (2.10) that the monodromy matrices $N_j := \rho(\ell_j)$ satisfy

\begin{equation}
N_1 N_2 N_1^{-1} N_2^{-1} = \varepsilon I_3, \quad \text{where } \varepsilon := e^{-2\pi i \sum \gamma_{1,k}} = e^{-2\pi i \frac{N_1 + N_2}{3}}.
\end{equation}

Since $N_1 \not\equiv N_2 \mod 3$, we have $\varepsilon \neq \pm 1$ and $\varepsilon^3 = 1$.

**Lemma 2.2.** Suppose that $p_k$ are apparent singularities of (2.6) for all $k$. Then up to multiplying by a common constant $c \neq 0$, there is a unique basis of local
solutions $\vec{Y} = (y_1, y_2, y_3)^T$ in $B(q_0, |q_0|/2)$ such that the associated monodromy matrices $N_1, N_2$ are given by

$$
N_1 = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon^2 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
$$

where $\varepsilon = e^{-2\pi i \frac{2N_1+N_2}{3}}$. In particular, the monodromy is unitary.

Conversely, if the monodromy is unitary, then $p_k$ are apparent singularities of (2.6) for all $k$ and so (2.12) holds.

Proof. Suppose that $p_k$ are apparent singularities of (2.6) for all $k$. Then we have (2.11) holds, i.e. $N_1N_2 = \varepsilon N_2N_1$. Let $\lambda$ be an eigenvalue of $N_1$ with eigenvector $x \neq 0$, then $\lambda \neq 0$, $N_2 x \neq 0$ and

$$
N_1 N_2 x = \varepsilon N_2 N_1 x = \varepsilon \lambda N_2 x,
$$

so $\lambda, \varepsilon \lambda, \varepsilon^2 \lambda$ are all the eigenvalues of $N_1$. Consequently, $\lambda^3 = \lambda^3 \varepsilon^3 = \det N_1 = 1$, which implies $\lambda \in \{1, \varepsilon, \varepsilon^2\}$. Thus $1, \varepsilon, \varepsilon^2$ are all the eigenvalues of $N_1$, so up to common conjugation to $N_1, N_2$, we may assume

$$
N_1 = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon^2 & 1 \end{pmatrix}.
$$

Inserting this into $N_1 N_2 = \varepsilon N_2 N_1$, a direct computation leads to

$$
N_2 = \begin{pmatrix} b & a \\ c & \varepsilon \end{pmatrix}
$$

with $abc = 1$.

Letting $P = \text{diag}(1, b^{-1}, a)$, we obtain

$$
PN_1 P^{-1} = N_1 = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon^2 & 1 \end{pmatrix},
$$

$$
PN_2 P^{-1} = \begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix} \begin{pmatrix} b & a \\ c & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & \varepsilon^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
$$

This proves (2.12). In particular, $N_1, N_2 \in SU(3)$. Since the monodromy group is generated by $N_1, N_2$ and $\rho(\xi_k) = e^{-2\pi i \gamma_1 k} I_3 \in SU(3)$, $0 \leq k \leq m$, we conclude that the monodromy group is contained in $SU(3)$, namely the monodromy is unitary.

If there are two bases of local solutions $(y_1, y_2, y_3)^T$ and $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)^T$ such that the corresponding $N_j$’s are both given by (2.12), it follows from $\varepsilon \neq \pm 1$ and the expression of $N_1$ that $\tilde{y}_i = c_i y_i$ with $c_i \neq 0$ for all $i$, and then the expression of $N_2$ implies $c_1 = c_2 = c_3 =: c$, so $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)^T = c(y_1, y_2, y_3)^T$ for some constant $c \neq 0$.

Conversely, if the monodromy is unitary, then $p_k$ are apparent singularities of (2.6) for all $k$ because of the standard fact (see. e.g. [18, Section 1.3 in Chapter 1]): The local monodromy matrix at $p_k$ is of the form

$$
e^{-2\pi i \gamma_1 k \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}
$$

with $(a, b, c) \neq (0, 0, 0)$, and so can not be a unitary matrix if (2.6) has solutions with logarithmic singularity at $p_k$. □
Lemma 2.3. Suppose the Toda system (1.9) has a solution \((U, V)\). Then \(p_k\) are apparent singularities of the associated ODE (2.6) for all \(k\) and so Lemma 2.2 applies. In particular, there are a basis of local solutions \(\tilde{Y} = (y_1, y_2, y_3)^T\) in \(B(q_0, |q_0|/2)\) such that

\[
e^{-U(z)} = |y_1(z)|^2 + |y_2(z)|^2 + |y_3(z)|^2, \tag{2.13}
\]

and the monodromy matrices \(N_1, N_2\) with respect to \((y_1, y_2, y_3)^T\) are given by (2.12).

Proof. The proof can be adapted easily from [23] and we sketch it here for the reader’s convenience.

Let \(f_1, f_2, f_3\) be a basis of local solutions in \(B(q_0, |q_0|/2)\) for ODE (2.6) associated to \((U, V)\). Since \(e^{-U} > 0\) solves (2.6), we have

\[
e^{-U} = \sum_{i,j=1}^{3} m_{i,j} f_i f_j,
\]

where \((m_{i,j})\) is a Hermitian matrix. By similar arguments as in [23] Section 5.3, it follows that the Hermitian matrix \((m_{i,j})\) is positive, so we can write

\[
(m_{i,j}) = \overline{P^T} P
\]

for some invertible matrix \(P \in GL(3, \mathbb{C})\). Define a new basis \((y_1, y_2, y_3)^T := P(f_1, f_2, f_3)^T\), then (2.13) holds.

Let \(\ell \in \pi_1(E_{\tau}\setminus \{p_k\}_{k=1}^{m}, q_0)\) and \(\rho(\ell)\) be the corresponding monodromy matrix with respect to \(\tilde{Y} = (y_1, y_2, y_3)^T\), i.e. (2.9) holds. Since \(e^{-U(z)}\) is single-valued and doubly periodic, namely \(\ell^* e^{-U(z)} = e^{-U(z)}\), we see from (2.13) that

\[
|y_1(z)|^2 + |y_2(z)|^2 + |y_3(z)|^2 = (y_1(z), y_2(z), y_3(z)) \overline{\rho(\ell)} \rho(\ell)^T (y_1(z) \ y_2(z) \ y_3(z)).
\]

This implies \(\overline{\rho(\ell)^T} \rho(\ell) = I_3\), i.e. \(\rho(\ell) \in SU(3)\). So the monodromy group with respect to \((y_1, y_2, y_3)^T\) is contained in \(SU(3)\), namely the monodromy is unitary. Then Lemma 2.2 applies and so \(p_k\) are apparent singularities for all \(k\). By \(N_1, N_2 \in SU(3)\) and together with \(N_1 N_2 = \varepsilon N_2 N_1\) as done in Lemma 2.2 it is elementary to prove the existence of a unitary matrix \(P_1\) such that

\[
P_1 N_1 P_1^{-1} = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}, \quad P_1 N_2 P_1^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Then by replacing \((y_1, y_2, y_3)^T\) with \(P_1(y_1, y_2, y_3)^T\), we finally obtain (2.13) with the corresponding \(N_1\) given by (2.12). \(\Box\)

3. Proofs of Theorems 1.11 and 1.12

In this section, we prove Theorems 1.11 and 1.12.
3.1. **Proof of Theorem 1.11.** First we recall the notion of **dual equation** for later usage. Consider the following third order linear ODE

\[ y''' + p_1(z)y' + p_0(z)y = 0. \]  

For any two functions \( y_1, y_2 \), we denote by

\[ W(y_1, y_2) := y_1'y_2 - y_1y'_2 \]

to be the Wronskian. Then for any two linearly independent solutions \( y_1, y_2 \) of (3.1), a direct computation shows that \( W(y_1, y_2) \) solves the following equation

\[ h''' + p_1(z)h' + (p_1'(z) - p_0(z))h = 0. \]

Equation (3.3) is called the **dual equation** of (3.1) in the literature. By direct computations it is easy to prove that

\[ (\star) \text{ if } y_1, y_2, y_3 \text{ are linearly independent solutions of (3.1), then } W(y_1, y_2), W(y_2, y_3), W(y_3, y_1) \text{ are linearly independent solutions of the dual equation (3.3).} \]

The notion of dual equations appears naturally from the Toda system. As in (2.1), we consider the following linear differential operator

\[ (\partial_z - U_z)(\partial_z + U_z - V_z)(\partial_z + V_z) = \partial_z^3 + W_2\partial_z + \tilde{W}_3, \]

where

\[ \tilde{W}_3 := V_{zzz} - 2V_zV_{zz} + V_zU_{zz} + U_zV_z^2 - V_zU_z^2 = W_2' - W_3. \]

Therefore, \( e^{-V(z)} \) solves

\[ h''' - \left( \sum_{k=0}^{m} \alpha_k \psi(z - p_k) + \sum_{k=0}^{m} B_k \tilde{\zeta}(z - p_k) + B \right) h' \]
\[ - \left( \sum_{k=0}^{m} (\alpha_k + \beta_k) \psi'(z - p_k) \right. \]
\[ + \left. \sum_{k=0}^{m} (B_k + D_k) \psi(z - p_k) + \sum_{k=0}^{m} A_k \tilde{\zeta}(z - p_k) + D \right) h = 0, \]

which is exactly the dual equation of (2.6). Remark that the local exponents of the dual equation (3.4) at \( p_k \) are

\[ \gamma_{2,k}, \gamma_{2,k} + (n_{2,k} + 1), \gamma_{2,k} + (n_{1,k} + n_{2,k} + 2). \]

Recall Lemma 2.3 that if the Toda system (1.9) has a solution \((U, V)\), then there exist associated parameters \( \mathbf{B} \) such that \( p_k \) are apparent singularities of (2.6) with this \( \mathbf{B} \) for all \( k \). Therefore, to prove Theorem 1.11 we only need to prove the following result.

**Theorem 3.1 (\( = \)Theorem 1.11).** Suppose \( p_k \) are apparent singularities of (2.6) for some \( \mathbf{B} \). Then the Toda system (1.9) has a unique solution \((U, V)\) corresponding to this \( \mathbf{B} \).
Proof. Under our assumption, it follows from Lemma 2.2 that (2.6) has a basis of local solutions \( \tilde{Y} = (y_1, y_2, y_3)^T \) in \( B(q_0, |q_0|/2) \) such that the associated monodromy matrices \( N_1, N_2 \) are given by (2.12).

**Step 1.** Define the Wronskian matrix of \((y_1, y_2, y_3)\):

\[
W := \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix}.
\]

Then \(|W| := |\det W|\) is a positive constant. Define a positive definite matrix

\[
R := |W|^{-\frac{1}{2}}WW^T.
\]

Clearly \(\det R = 1\). For \(1 \leq i \leq 3\), we let \(R_i\) denote the leading principal minor of \(R\) of dimension \(i\), and define

\[
e^{-U(z)} := \frac{1}{4}R_1 = \frac{1}{4}|W|^{-\frac{1}{2}} \sum_{j=1}^{3} |y_j(z)|^2, \ z \in B(q_0, |q_0|/2),
\]

\[
e^{-V(z)} := \frac{1}{4}R_2, \ z \in B(q_0, |q_0|/2).
\]

Since \(R\) is positive definite, we have \(R_1 > 0\) and so \(e^{-U}, e^{-V}\) are well-defined in \(B(q_0, |q_0|/2)\). Since \(y_j's\) are all holomorphic in \(B(q_0, |q_0|/2)\), it is easy to prove (cf. [23, 29]) that

\[
R_i(\partial_{zz} R_i) - (\partial_z R_i)(\partial_z R_i) = R_{i-1}R_{i+1}, \ \text{for} \ i = 1, 2,
\]

where \(R_0 := 1\). Note that \(R_3 = \det R = 1\). Letting \(i = 1\) in (3.7) leads to

\[
4e^{-V} = R_2 = 16[e^{-U}(\partial_{zz} e^{-U}) - (\partial_z e^{-U})(\partial_z e^{-U})]
\]

\[
= -16e^{-2U}\partial_{zz} U = -4e^{-2U}\Delta U,
\]

and letting \(i = 2\) in (3.7) leads to

\[
4e^{-U} = R_1 = 16[e^{-V}(\partial_{zz} e^{-V}) - (\partial_z e^{-V})(\partial_z e^{-V})]
\]

\[
= -16e^{-2V}\partial_{zz} V = -4e^{-2V}\Delta V.
\]

This proves that \((U, V)\) is a solution of the Toda system (1.9) in \(B(q_0, |q_0|/2)\).

Now since the monodromy group with respect to \(\tilde{Y} = (y_1, y_2, y_3)^T\) is contained in \(SU(3)\), it follows from (3.6) that for any loop \(\ell \in \pi_1(E_T \setminus \{0\}, q_0)\), the analytic continuation of \(e^{-U(z)}\) along \(\ell\) is invariant: \(\ell^* e^{-U(z)} = e^{-U(z)}\), so after analytic continuation \(e^{-U(z)}\) is well-defined on \(E_T\) and then so does \(e^{-V(z)}\). Therefore, \((U, V)\) is a solution of the Toda system (1.9) in \(E_T \setminus \{p_k\} \cup \{q_0\}\) (because \(e^{-U(z)}, e^{-V(z)} < +\infty\) in \(E_T \setminus \{p_k\} \cup \{q_0\}\)).

**Step 2.** We prove that \((U, V)\) is a solution of (1.9) in \(E_T\). Clearly this assertion is equivalent to prove that for each \(k\),

\[
U(z) = 2\gamma_{1,k} \ln |z - p_k| + O(1) \quad \text{as} \ z \to p_k,
\]

\[
V(z) = 2\gamma_{2,k} \ln |z - p_k| + O(1) \quad \text{as} \ z \to p_k.
\]
Thus (3.10) holds. This proves that solution of the dual equation (3.4). Since follows from (3.5) that for some constant \(a\)

\[
(3.13)
\]

Graph of Step 1, there is a small neighborhood (2.7) that as pointed out in (3.2), (3.13)

\[
\text{Step 3.}
\]

\[
(3.11)
\]

\[
\frac{1}{4} e^{-U} = \frac{1}{4} |W|^{-\frac{2}{3}}(|\tilde{y}_1|^2 + |\tilde{y}_2|^2 + |\tilde{y}_3|^2) \quad \text{in } B(p_k, \varepsilon_k).
\]

Since \((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)^T\) is a basis of local solutions in \(B(p_k, \varepsilon_k)\), it follows from (2.7) that

\[
(3.12) \quad \tilde{y}_j(z) \sim (z - p_k)^{-\gamma_{jk}} \quad \text{as } z \to p_k, \quad \text{for some } j \in \{1, 2, 3\},
\]

\[
(3.13) \quad (z - p_k)^{\gamma_{ik}} \tilde{y}_i(z) = O(1) \quad \text{as } z \to p_k, \quad \text{for all } j.
\]

From here and (3.11) we obtain (3.9).

To study the asymptotic behavior of \(V\) near 0, we insert (3.11) into (3.8), which leads to

\[
\frac{1}{4} e^{-V} = e^{-U}(\partial_{zz} e^{-U}) - (\partial_z e^{-U})(\partial_{\bar{z}} e^{-U})
\]

\[
= \frac{1}{16} |W|^{-\frac{2}{3}}[(\sum |y_j|^2)(\sum |y_j|^2) - (\sum y_j \bar{y}_j)(\sum y_j \bar{y}_j)]
\]

\[
= \frac{1}{16} |W|^{-\frac{2}{3}}[|W(\tilde{y}_1, \tilde{y}_2)|^2 + |W(\tilde{y}_2, \tilde{y}_3)|^2 + |W(\tilde{y}_3, \tilde{y}_1)|^2],
\]

where as pointed out in (3.2), \(W(\tilde{y}_i, \tilde{y}_j) := \tilde{y}_i \tilde{y}_j - \tilde{y}_j \tilde{y}_i, i \neq j,\) is a nontrivial solution of the dual equation (3.4). Since \(W(\tilde{y}_i, \tilde{y}_j)\)'s form a basis of (3.4), it follows from (3.5) that

\[
W(\tilde{y}_i, \tilde{y}_j)(z) \sim (z - p_k)^{-\gamma_{ij}} \quad \text{as } z \to p_k, \quad \text{for some } (i, j),
\]

\[
(z - p_k)^{\gamma_{ij}} W(\tilde{y}_i, \tilde{y}_j)(z) = O(1) \quad \text{as } z \to p_k, \quad \text{for all } (i, j).
\]

Thus (3.10) holds. This proves that \((U, V)\) is a solution of (1.9).

**Step 3.** We prove that this \((U, V)\) is the unique solution of (1.9) corresponding to this \(B\).

Let \((\hat{U}, \hat{V})\) be a solution of (1.9) corresponding to this \(B\). We need to show \((\hat{U}, \hat{V}) = (U, V)\). By Lemmas 2.2 and 2.3, we have

\[
e^{-\hat{U}(z)} = a(|y_1|^2 + |y_2|^2 + |y_3|^2) \quad \text{in } B(q_0, |q_0|/2)
\]

for some constant \(a > 0\), so we need to prove \(a = \frac{1}{4} |W|^{-\frac{2}{3}}\).

Indeed, we see from (3.6) that

\[
R_1 = |W|^{-\frac{2}{3}} a^{-1} e^{-\hat{U}}.
\]

Inserting this into (3.7) with \(i = 1\), we obtain

\[
R_2 = |W|^{-\frac{2}{3}} a^{-2} [e^{-\hat{U}}(\partial_{zz} e^{-\hat{U}}) - (\partial_z e^{-\hat{U}})(\partial_{\bar{z}} e^{-\hat{U}})]
\]

\[
= -|W|^{-\frac{2}{3}} a^{-2} e^{-2\hat{U}}(\partial_{zz} e^{-\hat{U}}) = \frac{1}{4} |W|^{-\frac{2}{3}} a^{-2} e^{-\hat{V}}.
\]
Inserting these into (3.7) with \( i = 2 \), we obtain
\[
|W|^{-\frac{2}{3}}a^{-1}e^{-\bar{U}} = R_1 = \frac{|W|^{-\frac{2}{3}}a^{-4}}{16} e^{-\bar{V}} (\partial_{zz} e^{-\bar{V}}) = -\frac{|W|^{-\frac{2}{3}}a^{-4}}{16} e^{-\bar{U}},
\]
so \( a = \frac{1}{4} |W|^{-\frac{2}{3}} \), i.e. \( e^{-\bar{U}} = e^{-\bar{V}} \) and so \( (\bar{U}, \bar{V}) = (U, V) \). \( \square \)

### 3.2. Proof of Theorem 1.12

In this section, we consider the Toda system (1.7), which is equivalent to
\[
\begin{align*}
\Delta U + e^{2U-V} &= 4\pi \gamma_1 \delta_0 \quad \text{on} \quad E_\tau, \\
\Delta V + e^{2V-U} &= 4\pi \gamma_2 \delta_0 \quad \text{on} \quad E_\tau,
\end{align*}
\]
(3.14)

Note that for this system, we have \( m = 0 \) and \( p_0 = 0 \), so \( \mathcal{N}_j = n_j \) and \( n_1 \not\equiv n_2 \mod 3 \), so \( \epsilon = e^{-2\pi i \gamma_1} \). Recall that we always assume \( n_1 < n_2 \).

Clearly the associated ODE (2.6) becomes (note (1.12))
\[
y''' - (\alpha \varphi(z) + B)y' + (\beta \varphi'(z) + D_0 \varphi(z) + D)y = 0,
\]
(3.15)

where
\[
\begin{align*}
\alpha &= \gamma_1 (\gamma_1 + 1) + \gamma_2 (\gamma_2 + 1) - \gamma_1 \gamma_2, \\
\beta &= -\frac{2\gamma_1 (\gamma_1 + 1) + \gamma_1 \gamma_2 (\gamma_1 - \gamma_2 - 1)}{2}.
\end{align*}
\]

Furthermore, if \( (U, V) \) is an even solution of (3.14), then \( W_3 = \beta \varphi'(z) + D_0 \varphi(z) + D \) is odd, so \( D_0 = D = 0 \), i.e. the associated ODE (3.15) of an even solution becomes
\[
y''' - (\alpha \varphi(z) + B)y' + \beta \varphi'(z) y = 0.
\]
(3.16)

The (3.15) has a regular singularity 0 with local exponents \( \rho_j \not\in \mathbb{Z} \):
\[
\rho_1 = -\gamma_1, \quad \rho_2 = -\gamma_1 + (n_1 + 1), \quad \rho_3 = -\gamma_1 + (n_1 + n_2 + 2),
\]
(3.17)

so any solution of (3.15) has a branch point at 0. Now the monodromy representation of (3.15) is a group homomorphism \( \rho : \pi_1(E_\tau \setminus \{0\}, q_0) \rightarrow SL(3, \mathbb{C}) \) and
\[
\ell_1 \ell_2 \ell_1^{-1} \ell_2^{-1} = \zeta_0 \quad \text{in} \quad \pi_1(E_\tau \setminus \{0\}, q_0).
\]

Again if 0 is an apparent singularity, then the local monodromy matrix \( M_0 \) is given by
\[
M_0 = e^{-2\pi i \gamma_1} I_3 = \epsilon I_3,
\]
(3.18)

which implies \( \rho(\zeta_0) = \epsilon I_3 \) and again
\[
N_1 N_2 N_1^{-1} N_2^{-1} = \epsilon I_3, \quad \text{i.e.} \quad N_1 N_2 = \epsilon N_2 N_1.
\]
(3.19)
Proof of Theorem 3.12. By Theorem 3.11 and the above argument, we only need to prove that the solution \((U, V)\) constructed in Theorem 3.1 is even if \(D_0 = D = 0\).

Let us turn back to the proof of Theorem 3.1. Since \(Z + Z\tau\) is the set of all branch points of ODE (3.15), we consider

\[
L := [-1, 1] \cup \bigcup_{i=0}^{+\infty} ([2 + 3i, 4 + 3i] \cup [-4 - 3i, -2 - 3i]) \subset \mathbb{R},
\]

and

\[
\Gamma := \mathbb{C}\setminus (L + Z\tau).
\]

Then \(\Gamma\) has no intersection with \(Z + Z\tau\), and is path-connected, symmetric with respect to \(z \leftrightarrow -z\) and \(z \leftrightarrow z + \tau\). Note that each line in \(\Gamma\) contains exactly 3 points of \(Z + Z\tau\). By (3.18) (i.e. the local monodromy matrix at each point of \(Z + Z\tau\)) is \(e L_3\) and \(e^3 = 1\), it follows that the local solutions \(y_j(z)\) in \(B(q_0, ||q_0||/2\) can be extended to be single-valued holomorphic functions (still denoted by \(y_j(z)\)) in \(\Gamma\). Then

\[
e^{-U} = \frac{1}{4} |W|^{-\frac{3}{2}} (|y_1|^2 + |y_2|^2 + |y_3|^2) \quad \text{in} \quad \Gamma.
\]

Since \(D_0 = D = 0\), ODE (3.15) becomes (3.16), i.e. is invariant with respect to \(z \leftrightarrow z - \tau\). Since \(\Gamma = \mathbb{C}\setminus (L + Z\tau)\) is symmetric with respect to \(z \leftrightarrow -z\), \(y_j(-z)\) is also a well-defined solution of (3.15) in \(\Gamma\). So there is a invertible matrix \(M\) such that

\[
\tilde{Y}(-z) = M\tilde{Y}(z), \quad \forall z \in \Gamma.
\]

It is easy to see \(\det M = -1\). It suffices to prove \(M \in U(3)\), i.e. \(M^T M = I_3\). Once this holds, it follows immediately from (3.20) that \(U(z) = U(-z)\) and so \(V(z) = V(-z)\).

Since \(\Gamma\) is symmetric with respect to \(z \leftrightarrow z + \tau\), \(y_j(z + \tau)\) is also a well-defined solution of (3.15) in \(\Gamma\). So there is a invertible matrix \(\tilde{N}_2\) such that

\[
\tilde{Y}(z + \tau) = \tilde{N}_2\tilde{Y}(z), \quad \forall z \in \Gamma.
\]

By the definition of \(\Gamma\), the choice of \(q_0\) and the fundamental circle \(\ell_2\) in Remark 2.1, we have \(\tilde{N}_2 = \varepsilon^2 N_2\), i.e.

\[
\tilde{Y}(z + \tau) = \varepsilon^2 N_2\tilde{Y}(z), \quad \forall z \in \Gamma.
\]

Define

\[
\Omega_+ := \{a + b\tau \mid b \in (0, 1), a \in \mathbb{R}\} \subset \Gamma,
\]

\[
\Omega_- := \{a + b\tau \mid b \in (-1, 0), a \in \mathbb{R}\} \subset \Gamma,
\]

Since \(\Omega_\pm = \Omega_\pm + 1, y_j(z + 1)\) is also a well-defined solution for \(z \in \Omega_\pm\), so there are invertible matrices \(N_\pm\) such that

\[
\tilde{Y}(z + 1) = N_\pm\tilde{Y}(z), \quad \forall z \in \Omega_\pm.
\]
By \( q_0 \in \Omega_- \) and the definition of the fundamental circle \( \ell_1 \), we have \( N_- = N_1 \), i.e.
\[
(3.24) \quad \tilde{Y}(z + 1) = N_1 \tilde{Y}(z), \quad \forall z \in \Omega_-.
\]
Recalling the definition of \( \Omega \) and \( (3.18) \), it is easy to see that \( N_+^{-1}N_1 = M_0 = \varepsilon I_3 \), so \( N_+ = \varepsilon^{-1}N_1 \), i.e.
\[
(3.25) \quad \tilde{Y}(z + 1) = \varepsilon^{-1}N_1 \tilde{Y}(z), \quad \forall z \in \Omega_+.
\]
Remark that \( (3.25) \) can be also proved in a different way: Since \( z - \tau \in \Omega_- \) for \( z \in \Omega_+ \), we have for \( z \in \Omega_+ \) that
\[
\begin{align*}
\tilde{Y}(z + 1) &= \tilde{Y}(z - \tau + 1 + \tau) = \varepsilon^2 N_2 \tilde{Y}(z - \tau + 1) \quad \text{by (3.23)} \\
&= \varepsilon^2 N_2 N_1 \tilde{Y}(z - \tau) \quad \text{by (3.24)} \\
&= N_2 N_1 N_2^{-1} \tilde{Y}(z) \quad \text{by (3.23)} \\
&= \varepsilon^{-1} N_1 \tilde{Y}(z) \quad \text{by (3.19)}.
\end{align*}
\]
Therefore, for \( z \in \Omega_- \) we have
\[
\tilde{Y}(-(z + 1)) = MN_1 \tilde{Y}(z) \quad \text{by (3.24)},
\]
and (note \( -z \in \Omega_+ \))
\[
\begin{align*}
\tilde{Y}(-(z + 1)) &= \tilde{Y}(-z - 1) = \varepsilon N_1^{-1} \varepsilon N_1 \tilde{Y}(-z) \quad \text{by (3.25)} \\
&= \varepsilon N_1^{-1} \tilde{Y}(z) \quad \text{by (3.21)},
\end{align*}
\]
so \( MN_1 = \varepsilon N_1^{-1}M \). From here and \( (2.12) \), a direct computation gives
\[
M = \begin{pmatrix} b & a \\ c & \end{pmatrix} \quad \text{with } abc = -\det M = 1.
\]
On the other hand, by applying \( (3.21) \) and \( (3.23) \) to \( \tilde{Y}(-(z + \tau)) \), we obtain
\[
M \varepsilon^2 N_2 = \varepsilon^{-2} N_2^{-1} M,
\]
from which and \( (2.12) \) we easily obtain \( c = eb = \varepsilon^2 a \), so \( 1 = abc = \varepsilon^3 a^3 = a^3 \), i.e. \( a \in \{1, \varepsilon, \varepsilon^2\} \) and
\[
M = a \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}.
\]
This proves \( M \in U(3) \) and so \((U(z), V(z))\) is an even solution.

The proof is complete. \( \square \)

4. Proofs of Theorems 1.1 and 1.9-1.10

In this section, we prove Theorems 1.1, 1.9-1.10 and also give an explanation of Conjecture 1.7.
4.1. **Proof of Theorem 1.1.** By Theorem 1.1, we need to study the condition on

\[ \mathcal{B} = (A_0, \cdots, A_m, B_0, \cdots, B_m, B, D_0, \cdots, D_m, D) \]

such that \( p_k \) are apparent singularities of (1.11) for all \( k \). Here we apply the standard Frobenius’ method (see e.g. [18, Section 1.3 in Chapter 1]) to study the apparent condition. In the sequel we use the notations

\[ \zeta_{kl} = \zeta(p_k - p_l), \quad \varphi_{kl}^{(n)} = \varphi^{(n)}(p_k - p_l), \quad \forall n \geq 0, \]

whenever \( k \neq l \). We also use the notation \( Q[g_2, g_3, \zeta_{kl}, \varphi_{kl}^{(n)}][\mathcal{B}] \) to denote

\[ Q[g_2, g_3, \{\zeta_{kl} : l \neq k\}, \{\varphi_{kl}^{(n)} : l \neq k, n \geq 0\}] [\mathcal{B}] \]

just for convenience.

**Lemma 4.1.** Fix any \( k \). Then there are three polynomials

\[ P_{k,1}(\mathcal{B}) = r_1 D_k^{n_1,k+1} + \cdots \in Q[g_2, g_3, \zeta_{kl}, \varphi_{kl}^{(n)}][\mathcal{B}], \quad r_1 \in \mathbb{Q} \setminus \{0\}, \]

\[ P_{k,2}(\mathcal{B}) = r_2 D_k^{n_2,k+1} + \cdots \in Q[g_2, g_3, \zeta_{kl}, \varphi_{kl}^{(n)}][\mathcal{B}], \quad r_2 \in \mathbb{Q} \setminus \{0\}, \]

\[ P_{k,3}(\mathcal{B}) = r_3 A_k D_k^{n_1,k+n_2,k} + \cdots \in Q[g_2, g_3, \zeta_{kl}, \varphi_{kl}^{(n)}][\mathcal{B}], \quad r_3 \in \mathbb{Q} \setminus \{0\}, \]

with homogeneous weights

\[ (4.1) \text{ Weig } P_{k,1} = n_{1,k} + 1, \text{ Weig } P_{k,2} = n_{2,k} + 1, \text{ Weig } P_{k,3} = n_{1,k} + n_{2,k} + 2, \]

such that \( p_k \) is an apparent singularity of ODE (1.11) if and only if

\[ P_{k,1}(\mathcal{B}) = P_{k,2}(\mathcal{B}) = P_{k,3}(\mathcal{B}) = 0. \]

Here the weights of \( B_i, D_i, \zeta_{kl}, A_j, B, D, g_2, g_3, \varphi_{kl}^{(n)} \) are \( 1, 1, 2, 2, 3, 4, 6, n + 2 \) respectively.

**Proof.** Recalling the local exponents \( \rho_{k,i} \) in (2.7), it is standard by Frobenius theory that \( p_k \) is an apparent singularity if and only if ODE (1.11) has local solutions of the form

\[ y(z) = u^{\rho_{k,1}} \sum_{j=0}^{+\infty} c_j u^j, \quad u := z - p_k \]

with \( c_0, c_{n_{1,k}+1}, c_{n_{1,k}+n_{2,k}+2} \) being arbitrary (this corresponds to the dimension of solutions being 3).

Recall the well-known Laurent expansions of \( \varphi(z), \zeta(z) \):

\[ \varphi(z - p_k) = \sum_{j=0}^{+\infty} b_j u^{j-2}, \quad \zeta(z - p_k) = -\sum_{j=0}^{+\infty} b_j u^{j-1}, \]

where \( b_0 = 1, b_2 = 0, b_j = 0 \) for all odd \( j \) (here \( \frac{b_j}{j-1} := 0 \) for \( j = 1 \)) and

\( b_j \in \mathbb{Q}[g_2, g_3] \) is of homogeneous weight \( j \) for all even \( j \geq 4 \). For example,
where under our setting of the weights for
neous weight $i$
4.4
0
inserting (4.2) and these Laurent or Taylor expansions into ODE (1.11), a
slightly complicate but direct computation leads to
\[
\phi_j c_j + \sum_{j=0}^{\infty} \left( \frac{\phi_j}{2^j} \right) c_{j-1}
\]
Inserting (4.2) and these Laurent or Taylor expansions into ODE (1.11), a
slightly complicate but direct computation leads to
\[
\sum_{j=0}^{\infty} \left( \phi_j c_j + \sum_{l=0}^{\infty} \frac{\phi_j}{2^j} \right) c_{j-1}
\]
Remark that in (4.3), the knowledge of the explicit complicate-looking ex-
pressions of the coefficient of $c_0$ is not necessary in our following
argument; the only important thing is that the coefficient of $c_{j-i}$ is of homoge-
neous weight $i$ under our setting of the weights for $B_t, D_t$ etc.
Therefore, $y(z)$ is a solution of (1.11) if and only if
\[
\phi_j c_j = - \left( D_k - (j + \rho_{k,1} - 1)B_k \right) c_{j-1} + \left[ \left( j + \rho_{k,1} - 2 \right) B - A_k \right] c_{j-2} + \sum_{l=0}^{\infty} R_l c_{j-l}, \forall j \geq 0,
\]
where \( R_i \in \mathbb{Q}[g_2, g_3, \zeta_{kl}, \varphi_{kl}^{(n)}][\mathbf{B}] \) is of homogeneous weight \( i \) and its expression can be determined by (4.3).

Note that \( \varphi_i = 0 \) if and only if \( j \in \{0, n_{1,k} + 1, n_{1,k} + n_{2,k} + 2\} \), and (4.4) with \( j = 0 \) holds automatically. By (4.4) and the induction argument, for \( 1 \leq j \leq n_{1,k} \), \( c_j \) can be uniquely solved as

\[
\frac{c_j}{c_0} = c_j(\mathbf{B}) = \bar{r}_j D_k^j + \cdots \in \mathbb{Q}[g_2, g_3, \zeta_{kl}, \varphi_{kl}^{(n)}][\mathbf{B}]
\]

with degree \( j \) in \( \mathbf{B} \) and homogeneous weight \( j \), where \( \bar{r}_j \in \mathbb{Q} \setminus \{0\} \).

Consequently, there is a polynomial \( P_{k,1}(\mathbf{B}) \) as stated in this lemma such that the RHS of (4.4) with \( j = n_{1,k} + 1 \) equals \( c_0 P_{k,1}(\mathbf{B}) \). Since \( c_0 \) can be arbitrary, we obtain \( P_{k,1}(\mathbf{B}) = 0 \).

Again by (4.4) and the induction argument, for \( n_{1,k} + 2 \leq j \leq n_{1,k} + n_{2,k} + 1 \), \( c_j \) can be uniquely solved as

\[
c_j = c_{n_{1,k} + 1} c_{j,1}(\mathbf{B}) + c_0 c_{j,2}(\mathbf{B}),
\]

where

\[
c_{j,1}(\mathbf{B}) = r_{j,1} D_k^{j-n_{1,k}-1} + \cdots \in \mathbb{Q}[g_2, g_3, \zeta_{kl}, \varphi_{kl}^{(n)}][\mathbf{B}]
\]

is of homogeneous weight \( j - n_{1,k} - 1 \), and

\[
c_{j,2}(\mathbf{B}) = r_{j,2} A_k D_k^{j-2} + \cdots \in \mathbb{Q}[g_2, g_3, \zeta_{kl}, \varphi_{kl}^{(n)}][\mathbf{B}]
\]

is of homogeneous weight \( j \), and \( r_{j,1}, r_{j,2} \in \mathbb{Q} \setminus \{0\} \).

Inserting these into (4.4) with \( j = n_{1,k} + n_{2,k} + 2 \), there are two polynomials \( P_{k,s}(\mathbf{B}), s = 2, 3 \) as stated in this lemma such that the RHS of (4.4) with \( j = n_{1,k} + n_{2,k} + 2 \) equals

\[
c_{n_{1,k} + 1} P_{k,2}(\mathbf{B}) + c_0 P_{k,3}(\mathbf{B}).
\]

Since \( c_0 \) and \( c_{n_{1,k} + 1} \) can be arbitrary, we obtain \( P_{k,s}(\mathbf{B}) = 0 \) for \( s = 2, 3 \).

Conversely, if \( P_{k,s}(\mathbf{B}) = 0 \) for \( s = 1, 2, 3 \), then the standard Frobenius theory shows that (1.11) has local solutions of the form (4.2) with \( c_0, c_{n_{1,k} + 1}, c_{n_{1,k} + n_{2,k} + 2} \) being arbitrary, so \( p_k \) is apparent. This completes the proof. \( \square \)

As a consequence of Theorem 1.11 and Lemma 4.1, we obtain

**Corollary 4.2.** \( p_k \) are apparent singularities of ODE (1.11) with (1.12) for all \( 0 \leq k \leq m \) if and only if \( \mathbf{B} \in \mathbb{C}^{3m+5} \) is a solution of the following \( 3m + 5 \) polynomials system

\[
\begin{align*}
P_1(\mathbf{B}) & := \sum_{k=0}^m B_k = 0, \\
P_2(\mathbf{B}) & := \sum_{k=0}^m A_k = 0, \\
P_{k,1}(\mathbf{B}) & = P_{k,2}(\mathbf{B}) = P_{k,3}(\mathbf{B}) = 0, \quad 0 \leq k \leq m.
\end{align*}
\]

Consequently, the number of solutions of the Toda system (1.11) equals to the number of solutions \( \mathbf{B} \)'s of the polynomial system (4.5).

Now we are in the position to prove Theorem 1.1.
Proof of Theorem 1.1. The key point is to show the existence of a constant $C > 1$ such that for any solution $\vec{B}$ of (4.5), there holds

\[(4.6) \quad \sum_{k=0}^{m} |B_k| + \sum_{k=0}^{m} |A_k| + \sum_{k=0}^{m} |D_k| + |B| + |D| \leq C.\]

This assertion seems difficult to prove from the viewpoint of the polynomials. Here we use its connection with the Toda system. By Theorem 3.1, we know that for any solution $\vec{B}$ of (4.5), there is a solution $(U, V)$ of the Toda system (1.9) corresponding to this $\vec{B}$ via (2.2) and (2.5).

Let $z_0 \in E_T \setminus \{p_k\}_{k=0}^{m}$ and fix small $\epsilon > 0$ such that the neighborhood $B(z_0, \epsilon) \subset E_T \setminus \{p_k\}_{k=0}^{m}$. Then by Theorem A, there is a constant $C_1$ such that for any solution $(U, V)$ of the Toda system (1.9), we have

$$|U(z)| + |V(z)| \leq C_1, \quad \forall z \in B(z_0, \epsilon).$$

Then by applying standard gradient estimates to the system (1.9), there is a constant $C_2$ such that for any solution $(U, V)$ of (1.9), we have

$$|U_z| + |V_z| + |U_{zz}| + |V_{zz}| + |U_{zzz}| \leq C_2, \quad \forall z \in B(z_0, \epsilon/2).$$

From here, (2.2) and (2.5), we see that there is a constant $C_3$ such that for any solution $\vec{B}$ of (4.5),

\[(4.7) \quad \left| \sum_{k=0}^{m} B_k \zeta(z - p_k) + B \right| \leq C_3, \quad \forall z \in B(z_0, \epsilon/2),
\]

\[\left| \sum_{k=0}^{m} D_k \psi(z - p_k) + \sum_{k=0}^{m} A_k \zeta(z - p_k) + D \right| \leq C_3, \quad \forall z \in B(z_0, \epsilon/2).\]

Here we used the fact that both $\sum_{k=0}^{m} a_k \zeta(z - p_k)$ and $\sum_{k=0}^{m} \beta_k \psi'(z - p_k)$ are uniformly bounded for $z \in B(z_0, \epsilon/2)$ because $B(z_0, \epsilon) \subset E_T \setminus \{p_k\}_{k=0}^{m}$.

By taking $m + 2$ distinct points $z_j \in B(z_0, \epsilon/2), 1 \leq j \leq m + 2$, such that the $(m + 2) \times (m + 2)$ matrix

\[
\begin{pmatrix}
\zeta(z_1 - p_0) & \cdots & \zeta(z_1 - p_m) & 1 \\
\vdots & \vdots & \vdots & \vdots \\
\zeta(z_{m+2} - p_0) & \cdots & \zeta(z_{m+2} - p_m) & 1
\end{pmatrix}
\]

is invertible, we conclude from (4.7) that $\sum_{k=0}^{m} |B_k| + |B|$ is uniformly bounded. The similar argument also yields the uniform estimates for $\sum_{k=0}^{m} |A_k| + \sum_{k=0}^{m} |D_k| + |D|$. This proves (4.6).

Now we let

\[(4.8) \quad B = \vec{B}^2, \quad D = \vec{D}^2, \quad A_k = \vec{A}_k^2, \quad \forall 0 \leq k \leq m,
\]

and define

$$\vec{B} := (\vec{A}_0, \cdots, \vec{A}_m, \vec{B}, \vec{D}), \quad \vec{p_j}(\vec{B}) := p_j(\vec{B}), \quad j = 1, 2,$$
Then it follows from Lemma 4.1 and Corollary 4.2 that as polynomials of $\vec{B}$, there hold
\[
\deg \vec{p}_1 = 1, \quad \deg \vec{p}_2 = 2,
\]
\[
\deg \vec{p}_{k,1} = n_{1,k} + 1, \quad \deg \vec{p}_{k,2} = n_{2,k} + 1, \quad \deg \vec{p}_{k,3} = n_{1,k} + n_{2,k} + 2.
\]
Furthermore, (4.6) implies that
\[
\sum_{k=0}^{m} |B_k| + \sum_{k=0}^{m} |\vec{A}_k| + \sum_{k=0}^{m} |D_k| + |\vec{B}| + |\vec{D}| \leq C
\]
holds for any solution $\vec{B}$ of the new polynomial system
\[
\begin{cases}
\vec{p}_1(\vec{B}) = \sum_{k=0}^{m} B_k = 0, \\
\vec{p}_2(\vec{B}) := \sum_{k=0}^{m} \vec{A}_k = 0, \\
\vec{p}_{k,1}(\vec{B}) = \vec{p}_{k,2}(\vec{B}) = \vec{p}_{k,3}(\vec{B}) = 0, \quad 0 \leq k \leq m.
\end{cases}
\]
Denote $X := \{ \vec{B} \in C^{3m+5} | \vec{B} \text{ is a solution of (4.10)} \}$, then $X$ is an affine variety. Since an affine variety over $C$ can not be bounded in the standard topology except it is a finite set (see e.g. [13, p.35]), we conclude from (4.9) that $X$ consists of finite points, namely the polynomial system (4.10) has only finitely many solutions. Then it follows from the Bézout theorem in algebraic geometry (see e.g. [34, p.246]) that the polynomial system (4.10) has at most
\[
\text{deg} \vec{p}_1 \times \text{deg} \vec{p}_2 \times \prod_{k=0}^{m} \prod_{j=1}^{3} \text{deg} \vec{p}_{k,j}
\]
\[
= 2 \prod_{k=0}^{m} (n_{1,k} + 1)(n_{2,k} + 1)(n_{1,k} + n_{2,k} + 2)
\]
\[
= 3 \times 2^{m+2} N(\{n_{1,k}\}_k, \{n_{2,k}\}_k)
\]
solutions by counting multiplicities. Thus we conclude from (4.8) that the polynomial system (4.5) has at most $N(\{n_{1,k}\}_k, \{n_{2,k}\}_k)$ solutions and so does the Toda system (1.1).

4.2. Proofs of Theorem 1.9, 1.10

In this section, we study the special Toda system (1.7) and the associated ODE (1.14) via Theorem 1.12 and prove Theorem 1.9, 1.10. In this case, since $m = 0$, the polynomial system (4.5) with 5 polynomials reduces to a polynomial system with 3 polynomials of $(B, D_0, D)$ because $B_0 = A_0 = 0$. We summarize this fact as follows.

Lemma 4.3. There are three polynomials
\[
P_{0,1}(B, D_0, D) = r_1 D_0^{n_1+1} + \cdots \in Q[g_2, g_3][B, D_0, D], \quad r_1 \in Q \setminus \{0\},
\]
\[
P_{0,2}(B, D_0, D) = r_2 D_0^{n_2+1} + \cdots \in Q[g_2, g_3][B, D_0, D], \quad r_2 \in Q \setminus \{0\},
\]
\[
P_{0,3}(B, D_0, D) = r_3 B D_0^{n_1+n_2} + \cdots \in Q[g_2, g_3][B, D_0, D], \quad r_3 \in Q \setminus \{0\},
\]
with homogeneous weights

\[ \text{Weig } P_{0,1} = n_1 + 1, \quad \text{Weig } P_{0,2} = n_2 + 1, \quad \text{Weig } P_{0,3} = n_1 + n_2 + 2, \]

such that 0 is an apparent singularity of ODE (1.14) if and only if

\[ P_{0,1}(B, D_0, D) = P_{0,2}(B, D_0, D) = P_{0,3}(B, D_0, D) = 0. \]

Consequently, the number of solutions of the Toda system (1.7) equals to the number of solutions \((B, D_0, D)\)’s of the polynomial system (4.11), and the number of even solutions equals to the number of solutions of the form \((B, 0, 0)\).

**Proof.** Comparing to Lemma 4.1, we only need to explain the term \(BD_0^{n_1+n_2}\) in the expression of \(P_{0,3}\) (because \(A_0 = 0\) makes the term \(A_0D_0^{n_1+n_2} = 0\) in the original expression of \(P_{0,3}\) stated in Lemma 4.1).

Recalling the local exponents \(\rho_j\) in (3.17), for ODE (1.14) the formula (4.3) becomes simpler and reads as

\[ \sum_{j=0}^{\infty} \left( \phi_j c_j + D_0 c_{j-1} - (j + \rho_1 - 2) B c_{j-2} + D c_{j-3} - \sum_{i=4}^{j} [(j + \rho_1 - i) \alpha - (i - 2) \beta] b_i c_{j-i} + D_0 \sum_{i=4}^{j-1} b_i c_{j-1-i} \right) z^{j + \rho_1 - 3} = 0, \]

with

\[ \phi_j = j(j - n_1 - 1)(j - n_1 - n_2 - 2). \]

so the recursive formula (4.4) becomes

\[ \phi_j c_j = -D_0 c_{j-1} + (j + \rho_1 - 2) B c_{j-2} - D c_{j-3} + \sum_{i=4}^{j} [(j + \rho_1 - i) \alpha - (i - 2) \beta] b_i c_{j-i} - D_0 \sum_{i=4}^{j-1} b_i c_{j-1-i} \quad \forall j \geq 0. \]

Now we have \(\rho_1 \notin \mathbb{Z}\), so \(j + \rho_1 - 2 \neq 0\) for all \(j \geq 0\). Then the induction argument as in Lemma 4.1 implies \(P_{0,3}(B, D_0, D) = r_3 BD_0^{n_1+n_2} + \cdots \) with \(r_3 \neq 0\).

**Remark 4.4.** Conjecture 1.7 is equivalent to assert that the number of solutions of the polynomial system (4.11) is exactly

\[ N(n_1, n_2) = \frac{(n_1 + 1)(n_2 + 1)(n_1 + n_2 + 2)}{6} \]

except for finitely many \(\tau\)’s modulo \(SL(2, \mathbb{Z})\). Counting the number of solutions of polynomial equations is not easy in general. Here we have some further discussions about Conjecture 1.7.

Recalling Lemma 4.3 that \(P_{0,j}(B, D_0, D) \in \mathbb{Q}[g_2, g_3][B, D_0, D]\), we define

\[ \hat{P}_{0,j}(B, D_0, D) := P_{0,j}(B, D_0, D) \bigg|_{g_2 = g_3 = 0} \quad j = 1, 2, 3, \]
and consider the corresponding system
\[(4.14) \quad \hat{P}_{0,1}(B, D_0, D) = \hat{P}_{0,2}(B, D_0, D) = \hat{P}_{0,3}(B, D_0, D) = 0.\]

Since letting \(g_2 = g_3 = 0\) is equivalent to replacing \(\varphi(z)\) with \(\frac{1}{z^2}\) in ODE \((1.14)\), we consider the corresponding ODE
\[(4.15) \quad y'''' - (\frac{a}{z^2} + B)y''' + (\frac{-2B}{z^2} + \frac{D_0}{z} + D)y = 0.\]

Then 0 is an apparent singularity of \((4.15)\) if and only if \((4.14)\) holds.

Conjecture 4.5. 0 is an apparent singularity of \((4.15)\) if and only if \(B = D_0 = D = 0\), or equivalently the polynomial system \((4.14)\) has only trivial solution \(B = D_0 = D = 0\).

Conjecture 4.5 can be proved for small values of \((n_1, n_2)\) via direct computations. It remains open for general \((n_1, n_2)\) satisfying \(n_1 \neq n_2 \mod 3\).

Now we turn back to the proof of Theorem 1.9 in Section 4.1: Let \(B = \tilde{B}^2\), \(D = \tilde{D}^3\) and define
\[\hat{P}_{0,j}(\tilde{B}, D_0, \tilde{D}) := P_{0,j}(B, D_0, D), \quad j = 1, 2, 3.\]

Suppose Conjecture 4.5 holds. Then the associated homogenized system of \((4.16)\)
\[\hat{P}_{0,1}(\tilde{B}, D_0, \tilde{D}) = \hat{P}_{0,2}(\tilde{B}, D_0, \tilde{D}) = \hat{P}_{0,3}(\tilde{B}, D_0, \tilde{D}) = 0\]
has no solutions at infinity in \(\mathbb{C}P^3\), so it follows from the Bézout theorem (see e.g. [34, p.246]) that the polynomial system \((4.16)\) has exactly \(\prod_{j=1}^{3} \deg \hat{P}_{0,j} = 6N(n_1, n_2)\) solutions by counting multiplicities.

Therefore, Conjecture 1.7 is equivalent to assert that Conjecture 4.5 holds and the multiplicity of any solution of \((4.16)\) is 1 except for finitely many \(\tau's\) modulo \(SL(2, \mathbb{Z})\), which we strongly believe to be true.

For small values of \((n_1, n_2)\), Conjecture 1.7 can be proved by direct computations as done below for \(n_1 = 0\) and \(n_2 = 1, 2, 4\).

Proof of Theorem 1.9
1. Let \((n_1, n_2) = (0, 1)\). Then \(\rho_1 = -\frac{1}{3}\) and a direct computation via the recursive formula \((4.13)\) gives
\[P_{0,1} = -D_0, \quad P_{0,2} = -\frac{D_0^2}{2} + \frac{2B}{3}, \quad P_{0,3} = -D - \frac{BD_0}{6}.\]

So \((4.11)\) has a unique solution \((0, 0, 0)\), namely the Toda system \((1.7)\) has a unique solution which is even.

2. Let \((n_1, n_2) = (0, 2)\). Then \(\rho_1 = -\frac{2}{3}\) and so
\[P_{0,1} = -D_0, \quad P_{0,2} = -\frac{D_0^3}{24} + \frac{7BD_0}{18} - D, \quad P_{0,3} = -\frac{BD_0^2}{36} - \frac{D_0D}{6} + \frac{2B^2}{9} - \frac{2g_2}{27}.\]

So \((4.11)\) has solutions \(D_0 = D = 0\) and \(B = \pm \sqrt{g_2/3}\).

Consequently, when \(g_2 = 0\) (i.e. \(\tau \equiv e^{\pi i/3} \mod SL(2, \mathbb{Z})\)), the Toda system \((1.7)\) has a unique solution which is even; when \(g_2 \neq 0\) (i.e. \(\tau \neq e^{\pi i/3}\))
\[ e^{\pi i/3} \text{ mod } SL(2,\mathbb{Z}), \] the Toda system (1.7) has exactly 2 solutions that are both even. \[ \square \]

**Proof of Theorem 1.10** Let \((n_1, n_2) = (0, 4)\). Then \(\rho_1 = -\frac{4}{3}\). Again \(P_{0,1} = -D_0\) while the expressions of \(P_{0,2}, P_{0,3}\) are slightly complicated. Instead we may insert \(D_0 = 0\) in the recursive formula (4.13) and obtain

\[
P_{0,2}(B, 0, D) = \frac{5}{54} BD, \quad P_{0,3}(B, 0, D) = -\frac{1}{486}(6B^3 + 27D^2 - 56g_2B + 288g_3).
\]

It follows from \(P_{0,2}(B, 0, D) = 0\) that \(B = 0\) or \(D = 0\).

If \(D = 0\), we have \(6B^3 - 56g_2B + 288g_3 = 0\), the discriminant of which is \(\Delta(\tau) := 343g_2^3 - 6561g_2^2\) up to a nonzero constant. Since this \(\Delta(\tau)\) is a modular form of weight 12 with respect to \(SL(2,\mathbb{Z})\), it is standard by the theory of modular forms that it has a unique zero \(\tau_0\) modulo \(SL(2,\mathbb{Z})\) action. Therefore, \(6B^3 - 56g_2B + 288g_3 = 0\) has 3 (resp. 2) distinct roots for \(\tau \not\equiv \tau_0\) modulo \(SL(2,\mathbb{Z})\) (resp. \(\tau \equiv \tau_0\) modulo \(SL(2,\mathbb{Z})\)), namely the Toda system has exactly 3 (resp. 2) even solutions for \(\tau \not\equiv \tau_0\) modulo \(SL(2,\mathbb{Z})\) (resp. \(\tau \equiv \tau_0\) modulo \(SL(2,\mathbb{Z})\)).

If \(D \neq 0\) then \(B = 0\) and \(27D^2 + 288g_3 = 0\), so the Toda system has exactly 2 solutions which are not even as long as \(g_3 \neq 0\). Noting that \(g_3(\tau_0) \neq 0\), we obtain the desired statements of Theorem 1.10. \[ \square \]

### 5. Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. Consider the special case \(m = 3\) and \(p_k = \frac{\omega_k}{2}\), i.e. the Toda system (1.6) or equivalently

\[
\begin{aligned}
\Delta U + e^{2u} - V &= 4\pi \sum_{k=0}^{3} \gamma_{1,k} \delta \frac{\omega_k}{2} \quad \text{on } E_\tau, \\
\Delta V + e^{2v} - U &= 4\pi \sum_{k=0}^{3} \gamma_{2,k} \delta \frac{\omega_k}{2} \quad \text{on } E_\tau.
\end{aligned}
\]

If the solution \((U, V)\) of the Toda system (5.1) is even, then \(W_5\) is even and \(W_3\) is odd, so the associated ODE (2.6) takes a simpler form (Note \(A_0 = -A_1 - A_2 - A_3\))

\[
Ly = y''' - \left( \sum_{k=0}^{3} \alpha_k \varphi(z - \frac{\omega_k}{2}) + B \right) y' \\
+ \left( \sum_{k=0}^{3} \beta_k \varphi'(z - \frac{\omega_k}{2}) + \sum_{k=1}^{3} A_k \frac{\varphi'(z)}{2(\varphi(z) - e_k)} \right) y = 0,
\]

where \(e_k = \varphi(\frac{\omega_k}{2})\) and we used the formula

\[
\zeta(z - \frac{\omega_k}{2}) - \zeta(z) + \zeta(\frac{\omega_k}{2}) = \frac{\varphi'(z)}{2(\varphi(z) - e_k)}.
\]

**Proof of Theorem 1.4** By changing variable \(z \to z + \frac{\omega_k}{2}\) if necessary, we may assume that \(n_{1,0}\) and \(n_{2,0}\) are both odd.

Assume by contradiction that the Toda system (5.1) has an even solution \((U, V)\). Then it follows from Lemma 2.3 that \(\frac{\omega_k}{2}\)'s are apparent singularities...
of the associated ODE (5.2) for all \( k \), and there is a basis of local solutions \( \vec{Y} = (y_1, y_2, y_3)^T \) in \( B(q_0, |q_0|/2) \) such that the monodromy matrices \( N_1, N_2 \) with respect to \( \vec{Y} \) are given by (2.12).

**Step 1.** Recalling the local exponents \( \rho_{0,j} \) in (2.7), since 0 is an apparent singularity and ODE (5.2) is invariant under \( z \leftrightarrow -z \), the standard Frobenius’ method implies that there is a basis of local solutions \( (\eta_1, \eta_2, \eta_3)^T \) of the form

\[
\eta_j(z) = z^{\rho_{0,j} + \infty} \sum_{l=0}^{\infty} c_{jl} z^{2l}, \quad c_{jl} \neq 0, \quad j = 1, 2, 3
\]

in a small neighborhood \( B(0, \delta_0) = \{z||z| < \delta_0 \} \) of 0, where we can take \( \delta_0 \in (0, \frac{1}{2(1 + |\im\eta|)}) \). Since \( n_{1,0}, n_{2,0} \) are both odd, we see from (2.7) that \( \rho_{0,j} - \rho_{0,1} \) are positive even integers for \( j = 2, 3 \), so \( \eta_j(z) \) are all of the form

\[
\eta_j(z) = z^{\rho_{0,j} + \infty} \sum_{l=0}^{\infty} \tilde{c}_{jl} z^{2l}, \quad z \in B(0, \delta_0), \quad j = 1, 2, 3.
\]

On the other hand, we may take \( \varepsilon_0 < \frac{\delta_0}{4|1+|\im\eta|} \) such that \( B(q_0, |q_0|/2) \subset B(0, \delta_0) \). Then there is a matrix \( P \in GL(3, \mathbb{C}) \) such that

\[
\vec{Y} = (y_1, y_2, y_3)^T = P(\eta_1, \eta_2, \eta_3)^T \quad \text{in} \quad B(q_0, |q_0|/2).
\]

**Step 2.** We consider simply-connected domains that have no intersections with the set \( E, \{2\} \) of singularities of ODE (5.2):

\[
\Omega := \{a + b\tau|a \in \mathbb{R}, b \in (-\frac{1}{2}, \frac{1}{2})\} \setminus ((-\infty, -\frac{1}{2}] \cup [0, +\infty]),
\]

\[
\Omega_\pm := \{a + b\tau|a \in \mathbb{R}, b \in (0, \frac{1}{2})\},
\]

\[
\Omega := \{a + b\tau|a \in \mathbb{R}, b \in (-\frac{1}{2}, 0)\} = -\Omega_\pm.
\]

Since \( B(q_0, |q_0|/2) \subset \Omega_\pm \subset \Omega \), by analytic continuation the local solutions \( \tilde{Y} = (y_1, y_2, y_3)^T \) in \( B(q_0, |q_0|/2) \) can be extended to be single-valued holomorphic functions (still denoted by \( \tilde{Y} = (y_1, y_2, y_3)^T \)) in \( \Omega \). Then by (5.3)-(5.4) we have

\[
\tilde{Y} = P(\eta_1, \eta_2, \eta_3)^T \quad \text{in} \quad B(0, \delta_0) \cap \Omega = B(0, \delta_0) \setminus [0, +\infty).
\]

**Step 3.** Since \( \Omega_\pm + 1 = \Omega_\pm \), \( y_j(z + 1) \) is well-defined and also a solution of ODE (5.2) for \( z \in \Omega_\pm \). So there are invertible matrices \( N_\pm \) such that \( \tilde{Y}(z + 1) = N_\pm \tilde{Y}(z) \), \( \forall z \in \Omega_\pm \).

Recalling the definition of \( \Omega \) and (2.8), it is easy to see

\[
N_+^{-1} N_- = M_0 M_1 = e^{-2\pi i (\gamma_{1,0} + \gamma_{1,1})} I_3.
\]

On the other hand, we see from \( B(q_0, |q_0|/2) \subset \Omega_- \), the definition of the fundamental cycle \( \ell_1 \) and \( N_1 = \rho(\ell_1) \) that \( N_- = N_1 \), so we conclude

\[
\tilde{Y}(z + 1) = N_1 \tilde{Y}(z), \quad \forall z \in \Omega_-,
\]

\[
\tilde{Y}(z + 1) = e^{2\pi i (\gamma_{1,0} + \gamma_{1,1})} N_1 \tilde{Y}(z), \quad \forall z \in \Omega_+.
\]
On the other hand, since (5.3) and (5.5) together imply
\[ B \]
we study directly the condition on
\[ \text{counting the solutions of the form} \]
(5.9)
\[ \tilde{\vec{Y}}(z) = \tilde{N}\tilde{\vec{Y}}(z) \quad \forall z \in \Omega_-. \]
On the other hand, since (5.3) and (5.5) together imply
\[ \tilde{\vec{Y}}(z) = \tilde{Y}(e^{-\pi i z}) = e^{-\pi ip_{0,1}}\tilde{\vec{Y}}(z) \quad \text{for} \ z \in B(q_0, |q_0|/2), \]
we have \[ \tilde{N} = e^{-\pi ip_{0,1}}I_3, \]
namely
(5.9)
\[ \tilde{\vec{Y}}(z) = e^{-\pi ip_{0,1}}\tilde{\vec{Y}}(z) \quad \forall z \in \Omega_. \]
Step 5. By (5.7)-(5.8) and (5.9), we have for \( z \in \Omega_- \)
\[ \tilde{\vec{Y}}(-(z + 1)) = e^{-\pi ip_{0,1}}\tilde{\vec{Y}}(z + 1) = e^{-\pi ip_{0,1}}N_1\tilde{\vec{Y}}(z), \]
and
\[ \tilde{\vec{Y}}(-(z + 1)) = \tilde{\vec{Y}}(-z - 1) \]
\[ = e^{-2\pi i(\gamma_{1,0} + \gamma_{1,1})}N_1^{-1}\tilde{\vec{Y}}(-z) \]
\[ = e^{-2\pi i(\gamma_{1,0} + \gamma_{1,1})}e^{-\pi ip_{0,1}}N_1^{-1}\tilde{\vec{Y}}(z), \]
so we obtain \[ N_1 = e^{-2\pi i(\gamma_{1,0} + \gamma_{1,1})}N_1^{-1}, \]
i.e.
\[ e^{-2\pi i(\gamma_{1,0} + \gamma_{1,1})}I_3 = N_1^2 = \left( \begin{array}{cc} 1 & \varepsilon^2 \\ \varepsilon & 1 \end{array} \right)^2 = \left( \begin{array}{cc} 1 & \varepsilon^4 \\ \varepsilon^2 & 1 \end{array} \right), \]
clearly a contradiction because \( \varepsilon = e^{-2\pi i 2N_1/3} \neq \pm 1. \)
Therefore, (5.1) has no even solutions.

6. PROOF OF THEOREM 1.5

This section is devoted to the proof of Theorem 1.5. Here instead of
counting the solutions of the form \((B, 0, 0)\) for the polynomial system (4.11),
we study directly the condition on \( B \) such that 0 is an apparent singularity
of (3.16). Again we apply the standard Frobenius’ method.
Recall that since at least one of \( n_1, n_2 \) is even, the integer \( N_\varepsilon \) is well
defined by (1.8).

Lemma 6.1. Suppose at least one of \( n_1, n_2 \) is even. Then there is a monic poly-
momial \( P_{N_\varepsilon}(B) \in \mathbb{Q}[g_2, g_3][B] \) with degree \( N_\varepsilon \) in \( B \) and also homogeneous weight \( N_\varepsilon \)
such that 0 is an apparent singularity of (3.16) if and only if \( P_{N_\varepsilon}(B) = 0 \). Here
the weights of \( B, g_2, g_3 \) are 1, 2, 3 respectively.

Proof. Since ODE (3.16) is invariant with respect to \( z \leftrightarrow -z \), it is more con-
venient for us to descend (3.16) to \( \mathbb{P}^1 \) under the double cover \( \varphi : E_\varepsilon \to \mathbb{P}^1 \).
Let \( x = \varphi(z), \tilde{y}(x) = y(z) \) and denote
\[ p(x) := 4x^3 - g_2x - g_3 = \varphi'(z)^2. \]
Then
\[ \frac{dy}{dz} = \varphi'(z)\frac{d\tilde{y}}{dx}, \quad \frac{d^2y}{dz^2} = p(x)\frac{d^2\tilde{y}}{dx^2} + \frac{1}{2}p'(x)\frac{d\tilde{y}}{dx}, \]
\[
\frac{d^3 y}{dz^3} = \varphi'(z) \left[ p(x) \frac{d^3 \tilde{y}}{dx^3} + \frac{3}{2} p'(x) \frac{d^2 \tilde{y}}{dx^2} + 12x \frac{d\tilde{y}}{dx} \right],
\]
so
\begin{equation}
(6.1) \quad y''' - (\alpha \varphi(z) + B)y' + \beta \varphi'(z)y = \varphi'(z)\tilde{L}y,
\end{equation}
where
\[
\tilde{L} := p(x)D^3 + 3(6x^2 - \frac{\Phi}{2})D^2 + [(12 - \alpha)x - B]D + \beta, \quad D := \frac{d}{dx}.
\]
Namely \( y(z) \) solves (3.16) if and only if \( \tilde{L} \tilde{y}(x) = 0 \).

Recalling (3.17), the local exponents of \( \tilde{L} \tilde{y}(x) = 0 \) at \( x = \infty \) are \( \frac{\rho_i}{2} \)'s:
\begin{equation}
(6.2) \quad \frac{\rho_1}{2} = -\frac{\gamma_1}{2} < \frac{\rho_2}{2} = -\frac{\gamma_1}{2} + \frac{n_1 + 1}{2} < \frac{\rho_3}{2} = -\frac{\gamma_1}{2} + \frac{n_1 + n_2 + 2}{2}.
\end{equation}
Clearly 0 is an apparent singularity of (3.16) if and only if \( \infty \) is an apparent singularity of \( \tilde{L} \tilde{y}(x) = 0 \). Moreover, the standard Frobenius’ method says that \( \infty \) is an apparent singularity if and only if \( \tilde{L} \tilde{y}(x) = 0 \) has solutions of the form
\begin{equation}
(6.3) \quad \tilde{y}_k(x) = x^{-\frac{\rho_i}{2}} \sum_{j=0}^{+\infty} C_j x^{-j}, \quad C_0 = 1
\end{equation}
for those \( k \in \{1, 2\} \) satisfying that \( \frac{\rho_i}{2} - \frac{\rho_k}{2} \in \mathbb{N} \) for some \( i > k \).

Inserting (6.3) into \( \tilde{L} \tilde{y}_k(x) = 0 \) we easily obtain
\[
0 = \tilde{\tilde{L}} \tilde{y}_k(x) = \sum_{j=0}^{+\infty} \Phi_j x^{-j-\frac{\rho_j}{2}},
\]
where
\[
\Phi_j := \phi_j C_j - B(-j - \frac{\rho_k}{2} + 1)C_{j-1}
- g_2(-j - \frac{\rho_k}{2} + 2) (-j - \frac{\rho_k}{2} + \frac{3}{2}) (-j - \frac{\rho_k}{2} + 1)C_{j-2}
- g_3(-j - \frac{\rho_k}{2} + 3) (-j - \frac{\rho_k}{2} + 2) (-j - \frac{\rho_k}{2} + 1)C_{j-3},
\]
\( C_{-3} = C_{-2} = C_{-1} := 0 \) and
\begin{equation}
(6.4) \quad \phi_j := 4(-j - \frac{\rho_k}{2})(-j - \frac{\rho_k}{2} - 1)(-j - \frac{\rho_k}{2} - 2)
+ 18(-j - \frac{\rho_k}{2})(-j - \frac{\rho_k}{2} - 1) + (12 - \alpha)(-j - \frac{\rho_k}{2}) + \beta
= -4 \prod_{i=1}^{3} (j + \frac{\rho_k}{2} - \frac{\rho_i}{2}).
\end{equation}
Therefore, \( \tilde{\tilde{L}} \tilde{y}_k(x) = 0 \) if and only if
\begin{equation}
(6.5) \quad \phi_j C_j = B(-j - \frac{\rho_k}{2} + 1)C_{j-1}
+ g_2(-j - \frac{\rho_k}{2} + 2) (-j - \frac{\rho_k}{2} + \frac{3}{2}) (-j - \frac{\rho_k}{2} + 1)C_{j-2}
+ g_3(-j - \frac{\rho_k}{2} + 3) (-j - \frac{\rho_k}{2} + 2) (-j - \frac{\rho_k}{2} + 1)C_{j-3}, \forall j \geq 0.
\end{equation}
Note that \( -j - \frac{\rho_k}{2} + 1 \neq 0 \) for any \( j \geq 0 \) because \( \frac{\rho_k}{2} \notin \mathbb{Z} \).

**Case 1.** \( n_1 \) is odd and \( n_2 \) is even.
Then $N_e = \frac{n_1+1}{2}$ and $\frac{\phi_j}{2} - \frac{\phi_k}{2} \in \mathbb{N}$ if and only if $(k,i) = (1,2)$. Let $k = 1$ in (6.4)-(6.5), we have

\begin{equation}
(6.6) \quad \phi_j = -4j(j - \frac{n_1+1}{2})(j - \frac{n_1+n_2+2}{2}) = -4j(j - N_e)(j - \frac{n_1+n_2+2}{2}),
\end{equation}

so $\phi_j = 0$ with $j \geq 0$ if and only if $j \in \{0,N_e\}$.

Clearly (6.5) with $j = \phi$ holds automatically. For $1 \leq j \leq N_e - 1$, by (6.5) and the induction argument, $C_j$ can be uniquely solved as $C_j = C_j(B) \in \mathbb{Q}[g_2,g_3][B]$ with degree $j$ in $B$ and homogeneous weight $j$. Consequently, the RHS of (6.5) with $j = N_e$ is a polynomial in $\mathbb{Q}[g_2,g_3][B]$ with degree $N_e$ and homogenous weight $N_e$. Define $P_{N_e}(B)$ to be the corresponding monic polynomial. Then the standard Frobenius theory shows that $\tilde{L}y_1 = 0$ has a local solution $\tilde{y}_1(z)$ of the form (6.4) if and only if $P_{N_e}(B) = 0$. This proves that $\tilde{y}_1$ is an apparent singularity if and only if $P_{N_e}(B) = 0$.

**Case 2.** $n_1$ is even and $n_2$ is odd.

Then $N_e = \frac{n_1+1}{2}$ and $\frac{\phi_j}{2} - \frac{\phi_k}{2} \in \mathbb{N}$ if and only if $(k,i) = (2,3)$. Let $k = 2$ in (6.4)-(6.5), we have

\begin{equation}
(6.7) \quad \phi_j = -4j(j + \frac{n_1+1}{2})(j + \frac{n_1+n_2+2}{2}) = -4j(j + \frac{n_1+1}{2})(j - N_e),
\end{equation}

so $\phi_j = 0$ with $j \geq 0$ if and only if $j \in \{0,N_e\}$. The rest proof is the same as Case 1.

**Case 3.** $n_1$ and $n_2$ are both even.

Then $N_e = \frac{n_1+n_2+2}{2}$ and $\frac{\phi_j}{2} - \frac{\phi_k}{2} \in \mathbb{N}$ if and only if $(k,i) = (1,3)$. Let $k = 1$ in (6.4)-(6.5), we have

\begin{equation}
(6.8) \quad \phi_j = -4j(j - \frac{n_1+1}{2})(j + \frac{n_1+n_2+2}{2}) = -4j(j - \frac{n_1+1}{2})(j - N_e),
\end{equation}

so $\phi_j = 0$ with $j \geq 0$ if and only if $j \in \{0,N_e\}$. Again the rest proof is the same as Case 1.

\[\square\]

**Lemma 6.2.** If one of the following holds:

1. $n_1$ is odd, i.e. $N_e = \frac{n_1+1}{2}$,
2. $n_2$ is odd (i.e. $N_e = \frac{n_2+1}{2}$) and $n_2 - n_1 \in \{1,5\}$,
3. $n_1$ is even and $n_2 = n_1 + 2$, i.e. $N_e = \frac{n_1+n_2+2}{2} = n_1 + 2$,

then the polynomial $P_{N_e}(B)$ in Lemma 6.1 has $N_e$ distinct roots expect for finitely many $\tau$'s modulo $SL(2,\mathbb{Z})$ action.

**Proof.** **Step 1.** We consider the special case $\tau = i = \sqrt{-1}$ and prove that $P_{N_e}(B)$ has $N_e$ real distinct roots. We only prove this assertion for the condition (1). The cases for conditions (2)-(3) can be proved similarly and we leave the details to the interested reader.

For $\tau = i$, it is well known that $g_3 = 0$ and $g_2 > 0$. Then under condition (1), we see from (6.5)-(6.6) and $\rho_1 = -\frac{2n_1+n_2}{3}$ that

\begin{equation}
(6.7) \quad \phi_jC_j = B(-j + \frac{2n_1+n_2}{6} + 1)C_{j-1} + g_2(-j + \frac{2n_1+n_2}{6} + 2)
\cdot (-j + \frac{2n_1+n_2}{6} + \frac{1}{2})(-j + \frac{2n_1+n_2}{6} + 1)C_{j-2}, \forall j \geq 0,
\end{equation}

as expected.
Since \( N_e = \frac{n_1 + 1}{2} \) and \( n_1 < n_2 \), we have
\[
\phi_j = -4j(n - N_e)j - \frac{n_1 + n_2 + 2}{2}.
\]

Recall that \( C_0 = 1 \), \( C_{-3} = C_{-2} = C_{-1} = 0 \), and \( C_j = C_j(B) \) are polynomials in \( Q[g]_2[B] \subset \mathbb{R}[B] \) with degree \( j \) for \( 1 \leq j \leq N_e - 1 \). For convenience we also denote by \( -C_{N_e}(B) \) to be the RHS of (6.7) with \( j = N_e \), i.e.
\[
-C_{N_e}(B) := B\left(-N_e + \frac{2n_1 + n_2}{6} + 1\right)C_{N_e - 1} + g_2\left(-N_e + \frac{2n_1 + n_2}{6} + 2\right)\left(-N_e + \frac{2n_1 + n_2}{6} + 1\right)C_{N_e - 2}.
\]

Then \( C_{N_e}(B) = cP_{N_e}(B) \) for some \( c \neq 0 \).

In view of the above argument, it is easy to see that the following properties hold for \( 1 \leq j \leq N_e \):

**P1** Up to a positive constant, the leading term of \( C_j(B) \) is \((-1)^j B^j\).

**P2** If \( C_{j-1}(B) = 0 \) and \( C_{j-2}(B) \neq 0 \) for \( B \in \mathbb{R} \), then \( C_j(B)C_{j-2}(B) < 0 \).

Then by an induction argument, it is easy to prove that for \( 1 \leq j \leq N_e \),
\( C_j(B) \) has \( j \) real distinct roots, denoted by \( r_1^j < \cdots < r_j^j \), such that
\[
r_1^j < r_1^{j-1} < r_2^j < \cdots < r_{j-1}^j < r_j^{j-1} < r_j^j.
\]

See e.g. [8]. We give the details here for completeness.

The case \( j = 1 \) is trivial because \( \deg C_1(B) = 1 \). Let \( 2 \leq m \leq N_e - 1 \) and assume that the statement is true for any \( j \leq m - 1 \). We prove it for \( j = m \). From the assumption of the induction, we have
\[
r_1^{m-1} < r_1^{m-2} < r_2^{m-1} < \cdots < r_{m-2}^{m-1} < r_{m-1}^{m-2} < r_{m-1}^{m-1}.
\]

Recall (P1) that
\[
\lim_{B \to -\infty} C_{m-2}(B) = +\infty, \quad \lim_{B \to +\infty} C_{m-2}(B) = (-1)^{m-2}\infty.
\]

Since \( r_j^{m-2}, 1 \leq j \leq m - 2, \) are all the roots of \( C_{m-2}(B) \), it follows from (6.10) and (6.11) that
\[
C_{m-2}(r_j^{m-1}) \sim (-1)^{j-1}, \quad \forall j \in [1, m - 1].
\]

Here \( c \sim (-1)^j \) means \( c = (-1)^j\tilde{c} \) for some \( \tilde{c} > 0 \). Then we see from (P2) that
\[
C_m(r_j^{m-1}) \sim (-1)^j, \quad \forall j \in [1, m - 1].
\]

On the other hand, (P1) implies
\[
\lim_{B \to -\infty} C_m(B) = +\infty, \quad \lim_{B \to +\infty} C_m(B) = (-1)^m\infty.
\]

From here, it follows from the intermediate value theorem that the polynomial \( C_m(B) \) has \( m \) real distinct roots \( r_j^m (1 \leq j \leq m) \) such that
\[
r_1^m < r_1^{m-1} < r_2^m < \cdots < r_{m-1}^m < r_{m-1}^{m-1} < r_m^m.
\]
This proves (6.9) for all 1 \leq j \leq N_e. In particular, P_{N_e}(B) has N_e real distinct roots because C_{N_e}(B) = cP_{N_e}(B) for some c \neq 0.

**Step 2.** We complete the proof.

Recall that g_2 = g_2(\tau), g_3 = g_3(\tau) are modular forms of weights 4, 6 respectively, with respect to SL(2, \mathbb{Z}). By Lemma 6.1 and Step 1, the discriminant of P_{N_e}(B) is a nonzero modular form with respect to SL(2, \mathbb{Z}) and so has only finitely many zeros \tau's modulo SL(2, \mathbb{Z}) action. This implies that P_{N_e}(B) has N_e distinct roots expect for finitely many \tau's modulo SL(2, \mathbb{Z}) action. □

**Remark 6.3.** We believe that the assertion of Lemma 6.2 should hold without the conditions (1)-(3). For other cases than (1)-(3), the above induction argument via the recursive formula does not apply. For example, let us consider the case that n_2 is odd, n_1 is even and n_2 - n_1 \geq 7. Then for \tau = i, we will obtain from the recursive relation (6.5) that C_2(B) = d_1B^2 + d_2g_2 with some d_1, d_2 \in \mathbb{Q} \setminus \{0\} satisfying d_1d_2 > 0, namely C_2(B) has no real roots and so the induction argument fails. Thus different ideas are needed to settle this problem, which remains open.

**Proof of Theorem 1.5** Clearly by Theorem 1.12 and Lemma 6.1, we see that the number of even solutions of the Toda system (1.9) equals to the number of distinct roots of the polynomial P_{N_e}(B). Therefore, Theorem 1.5 follows directly from Lemma 6.2. □

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