Instability of 3-band Tomonaga-Luttinger liquids: renormalization group analysis and possible application to $K_2Cr_3As_3$

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Reference: arXiv:1603.03651

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Cr based superconductors under ambient pressure: \( \text{A}_2\text{Cr}_3\text{As}_3 \) (A=K,Rb,Cs)

Guang-Han Cao’s group in ZJU, Phys. Rev. X 5, 011013 (2015); Phys. Rev. B 91, 020506(R) (2015); Science China Materials, 58(1) 16-10 (2015).

\( \text{K}_2\text{Cr}_3\text{As}_3 \): \( T_c \sim 6.1 \text{K} \)

\( \text{Rb}_2\text{Cr}_3\text{As}_3 \): \( T_c \sim 4.8 \text{K} \)

\( \text{Cs}_2\text{Cr}_3\text{As}_3 \): \( T_c \sim 2.2 \text{K} \)
Quasi-1D crystal and electronic structure

Guang-Han Cao’s group, Phys. Rev. X 5, 011013 (2015). arXiv:1412.0067

Jiang, Cao, Cao, arXiv:1412.1309 (2014)
Specific heat: deviation from BCS

K$_2$Cr$_3$As$_3$

Large Sommerfeld coefficient

Guang-Han Cao's group, Phys. Rev. X 5, 011013 (2015); Phys. Rev. B 91, 020506(R) (2015); Science China Materials, 58(1) 16-10 (2015).

Rb$_2$Cr$_3$As$_3$

Cs$_2$Cr$_3$As$_3$
Magnetic field dependent $\gamma(H)$

$\gamma(H) \equiv C_\gamma / T \propto H^{1/2}$ indicates nodes in the gap function

Guang-Han Cao’s group, Phys. Rev. B 91, 020506(R) (2015).
Upper critical field $H_{c2}$

Guang-Han Cao’s group, Phys. Rev. X 5, 011013 (2015); Phys. Rev. B 91, 020506(R) (2015); Science China Materials, 58(1) 16-10 (2015).

Ames’ group, RRB 91, 020507(R) (2015).
Angle resolved upper critical field

ZengWei Zhu, Guang-Han Cao et. al., arXiv:1511.06169
Penetration depth: $K_2Cr_3As_3$

Linear $T$ dependent penetration depth is an evidence for line nodal superconducting gap.

H. Q. Yuan’s group in ZJU, PRB 91, 220502(R) (2015).
Unconventional superconducting states

- **Specific heat**
  - Large Sommerfeld coefficient: band renormalization
  - Large specific-heat jump: deviation from BCS scenario
    - G.H. Cao et al. (2014, 2015)

- **Upper critical field**
  - Exceeding Pauli limit: $2H_P$ for $H//c$; $3.4H_P$ for $H//ab$
    - G.H. Cao et al. (2014, 2015), Ames’ group (2015)
  - Three-fold or six-fold modulation of $H_{c2}$
    - Z.W. Zhu et al. (2015)

- **NMR/NQR**
  - Absence of Hebel-Slichter coherence peak
  - $1/T_1 T \propto T^4$? or $T^5$ (T$^5$ indicates point nodal gap)
    - T. Imai et al. (2015); G.q. Zheng et al. (2015)

- **Penetration depth**
  - $\Delta \lambda \propto T$ (line nodal gap)
    - H.Q. Yuan et al. (2015)

- **muSR**
  - Weak evidence of a spontaneous internal magnetic field above $T_c$
    - ISIS facility (2015)
Normal state: NMR / NQR

T. Imai, et al., PRL114, 147004 (2015)  
G.q. Zheng, et al., PRL115, 147002 (2015)

Polycrystalline samples

K$_2$Cr$_3$As$_3$

Tonomaga-Luttinger Liquid vs. Critical spin fluctuations

$1/T_1 T = a + b/(T + \theta_C)$, with $\theta_C \sim 0$ K
Fermi liquid: \( \rho_0 + AT^2 \)

\[
\begin{align*}
K_2Cr_3As_3 \\
\text{Polycrystalline samples}
\end{align*}
\]

Guang-Han Cao’s group, Phys. Rev. X 5, 011013 (2015); Phys. Rev. B 91, 020506(R) (2015); Science China Materials, 58(1) 16-10 (2015).
Normal state: smectic metal

\[ \rho_c = \rho_0 + AT^\alpha, \, \alpha \approx 3 \]

Guang-Han Cao’s group (unpublished)
Previous theoretical studies

Yi Zhou, Chao Cao, and Fu-Chun Zhang, arXiv:1502.03928

- Effective Hamiltonian in 3D
- Superconducting instability by RPA

Other groups

- Strong coupling approach
  Xianxin Wu, Fan Yang, Congcong Le, Heng Fan, Jiangping Hu, arXiv:1503.06707

- Effective Hamiltonian for a single-tube Cr2As3
  Hanting Zhong, Xiao-Yong Feng, Hua Chen, Jianhui Dai, arXiv:1503.08965
At least three orbitals per unit cell are required to catch the low energy electronic features.

A three-orbital model is sufficient to describe both 1D and 3D features for superconductivity.

Jiang, Cao, Cao, arXiv:1412.1309 (2014)
Three-orbital model

Selected by the principle of symmetry.
Too many atomic orbitals, more than 10 atomic orbitals per unit cell.
Three relevant molecular (per Cr$_6$As$_6$ cluster) orbitals: $A'_{1}$ and $E'$ states.
Neglect spin-orbit coupling at first for simplicity.

Yi Zhou, Chao Cao, and Fu-Chun Zhang, arXiv:1502.03928
Superconducting pairing instability

Parameters:

\[ U_1 = U_1' = U, \quad U_2' = U_2 \quad \text{and} \quad J' = J \]

The results are similar when \( 0.5 < J'/J < 2 \).

- All the dominant states are spin-triplet states.
- For small \( U \), the pairing arises from 3D \( \gamma \)-band, and has spatial \( f \)-wave symmetry when \( J/U > 1/3 \).
  - Driving force: Hund’s coupling
  - Line nodes in the gap function
  - DOS at Fermi level at \( \gamma \)-band is the largest
- For large \( U \), a fully gapped \( p \)-wave state dominates at the quasi-1D \( \alpha \)-band.

Yi Zhou, Chao Cao, and Fu-Chun Zhang, arXiv:1502.03928
Issues

- How the superconducting state is related to the smectic metal normal state?

- Superconducting instability in a single-tube $\text{K}_2\text{Cr}_3\text{As}_3$?
  
  - Spin-orbit coupling will mix spin-singlet and spin-triplet SC pairing through broken inversion symmetry.
  
  - Which component (singlet or triplet) will be more important?

- Why Rb compound exhibit different temperature dependence in $1/T_1$ from K compound in the normal state?
1D three-band Hubbard model

Hamiltonian

\[ H^F = H_0^F + H_{\text{int}}^F \]

Non-interacting part

\[ H_0^F = \sum_{km\sigma} \xi_{km} c_{km\sigma}^\dagger c_{km\sigma} \]

\[ m = 0, \quad A_1' \text{ orbital} \]

\[ m = \pm 1, \quad \text{degenerate } E' \text{ orbitals} \]

On-site repulsion

\[ H_{\text{int}}^F = \frac{1}{2} \sum_{im} \sum_{\sigma\neq\sigma'} U n_{im\sigma} n_{im\sigma'} + \frac{1}{2} \sum_{i\sigma\sigma' m\neq m'} U' n_{im\sigma} n_{im'\sigma'} \\
- \sum_{im} \sum_{m\neq m'} J \left( \vec{S}_{im} \cdot \vec{S}_{im'} + \frac{1}{4} n_{im} n_{im'} \right) + \frac{1}{2} \sum_{i\sigma m\neq m'} J' c^\dagger_{im\sigma} c^\dagger_{im\bar{\sigma}} c_{im'\bar{\sigma}} c_{im'\sigma} \]

Irreducible representations for \( D_{3h} \) group

\[ \Rightarrow \quad U = U' + 2J \]

We also set

\[ J = J' \]
Scattering processes and “g-ology”

Single-band scattering processes

back scattering

forward scattering (intra-chirality)

Umklapp scattering

forward scattering (inter-chirality)
Three-band scattering processes

| chirality | band | spin |
|-----------|------|------|
| $g^{(1)}$ | $\psi_p^+\psi_p^+\psi_p^+\psi_p^+$ | $g_1$ | $\psi_m^+\psi_m^+\psi_m^+\psi_m^+$ | $f_1$ | $\psi_m^+\psi_m^+\psi_m^+\psi_m^+ + h.c.$ | $g_{||}$ |
| $g^{(2)}$ | $\psi_p^+\psi_p^+\psi_p^+\psi_p^+$ | $g_2$ | $\psi_m^+\psi_m^+\psi_m^+\psi_m^+$ | $f_2$ | $\psi_m^+\psi_m^+\psi_m^+\psi_m^+ + h.c.$ | $g_{\perp}$ |
| $g^{(3)}$ | $\psi_p^+\psi_p^+\psi_p^+\psi_p^+$ | $g_3$ | $\psi_m^+\psi_m^+\psi_m^+\psi_m^+$ | $f_3$ | $\psi_m^+\psi_m^+\psi_m^+\psi_m^+ + h.c.$ |
| $g^{(4)}$ | $\psi_p^+\psi_p^+\psi_p^+\psi_p^+$ | $g_4$ | $\psi_m^+\psi_m^+\psi_m^+\psi_m^+$ | $f_4$ | $\psi_m^+\psi_m^+\psi_m^+\psi_m^+$ |

Four dominant scattering processes at incommensurate filling

$g^{(2)}$ (SC) 

$g^{(1)}$ (SDW) 

$g^{(3)}$ (SDW) 

$f^{(1)}$ (SDW)
Bosonization for 1D systems

Abelian bosonization

\[ \psi_{pm\sigma} = \frac{\eta_{m\sigma}}{\sqrt{2\pi a}} e^{ipk_F m x} e^{-ip\varphi_{pm\sigma}} \]

\[ \{\eta_{m\sigma}, \eta_{m'\sigma'}\} = 2\delta_{mm'}\delta_{\sigma\sigma'} \]

Chiral and non-chiral fields

\[ \varphi_{pm\sigma} = \phi_{m\sigma} - p\theta_{m\sigma} \]

\[ \nabla \phi_{m\sigma} \propto n_{m\sigma} = \psi^\dagger_{Rm\sigma} \psi_{Rm\sigma} + \psi^\dagger_{Lm\sigma} \psi_{Lm\sigma} \]

\[ \nabla \theta_{m\sigma} \propto j_{m\sigma} = \psi^\dagger_{Rm\sigma} \psi_{Rm\sigma} - \psi^\dagger_{Lm\sigma} \psi_{Lm\sigma} \]

Charge and spin degrees of freedom

\[ \phi_{m\sigma} = \frac{1}{\sqrt{2}} (\phi_{cm} + \sigma\phi_{sm}) \]

\[ \theta_{m\sigma} = \frac{1}{\sqrt{2}} (\theta_{cm} + \sigma\theta_{sm}) \]

Gauge choice for Klein factors

\[ \eta_{m\sigma} \eta_{m\bar{\sigma}} = i m \sigma \]

\[ \eta_{m\sigma} \bar{\eta}_{m\bar{\sigma}} = i \sigma \]

\[ \eta_{m\sigma} \eta_{m\bar{\sigma}} = i m \]

\[ \eta_{0\sigma} \eta_{m\sigma} = i m \sigma \]

\[ \eta_{0\sigma} \eta_{0\bar{\sigma}} = i \sigma \]

\[ \eta_{0\sigma} \eta_{m\bar{\sigma}} = i m \]
Non-interacting bosonic Hamiltonian

\[ H_0^B = \frac{1}{2\pi} \int dx \sum_{\mu = c, s} v_{\mu \nu} \left[ K_{\mu \nu} \left( \nabla \theta_{\mu \nu} \right)^2 + \frac{1}{K_{\mu \nu}} \left( \nabla \phi_{\mu \nu} \right)^2 \right] \]

\[ \mu = c, s \]
\[ \nu = 0, \pm 1 \]

**Luttinger liquid fixed point:** gives rise to smectically metallic behaviors in normal state.

Renormalized Fermi velocities and Luttinger parameters

\[ v_{c(s)\pm 1} = v_F \left\{ 1 - \frac{\left( (+) g_{4\perp}^{(2)} - \left( g_{2\|}^{(2)} + (-) g_{2\perp}^{(2)} \right) \right)^2}{\left( 2\pi v_F \right)^2} \right\}^{1/2} \]

\[ v_{c(s)0} = v_F \left\{ 1 - \frac{\left( + (-) g_{4\perp}^{(2)} + 2 \left( g_{2\|}^{(2)} + (-) g_{2\perp}^{(2)} \right) \right)^2}{\left( 2\pi v_F \right)^2} \right\}^{1/2} \]

\[ K_{c(s)\pm 1} = \left\{ \frac{1 - \frac{1}{2\pi v_F} \left[ \left( + (-) g_{4\perp}^{(2)} - \left( g_{2\|}^{(2)} + (-) g_{2\perp}^{(2)} \right) \right) \right]}{\frac{1}{2\pi v_F} \left[ \left( + (-) g_{4\perp}^{(2)} - \left( g_{2\|}^{(2)} + (-) g_{2\perp}^{(2)} \right) \right) \right]} \right\}^{1/2} \]

\[ K_{c(s)0} = \left\{ \frac{1 - \frac{1}{2\pi v_F} \left[ \left( + (-) g_{4\perp}^{(2)} + 2 \left( g_{2\|}^{(2)} + (-) g_{2\perp}^{(2)} \right) \right) \right]}{1 + \frac{1}{2\pi v_F} \left[ \left( + (-) g_{4\perp}^{(2)} + 2 \left( g_{2\|}^{(2)} + (-) g_{2\perp}^{(2)} \right) \right) \right]} \right\}^{1/2} \]
Interacting bosonic Hamiltonian

\[ H_{int}^B = -g^{(1)}_{1\perp} \left( \frac{4}{(2\pi a)^2} \right) \int dx \cos \left( \frac{2}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_0 \right) \cos (2\tilde{\theta}_{s+1}) \]
\[ + g^{(1)}_{2\parallel} \left( \frac{4}{(2\pi a)^2} \right) \int dx \cos \left( \tilde{\phi}_{c+1} \right) \cos (2\tilde{\phi}_{s+1}) \]
\[ + g^{(1)}_{2\perp} \left( \frac{4}{(2\pi a)^2} \right) \int dx \cos \left( \tilde{\phi}_{c+1} \right) \cos \left( \frac{2}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_0 \right) \]
\[ + g^{(1)}_{3\parallel} \left( \frac{4}{(2\pi a)^2} \right) \int dx \cos \left( \tilde{\theta}_{c+1} \right) \cos (2\tilde{\theta}_{s+1}) \]
\[ + g^{(1)}_{3\perp} \left( \frac{4}{(2\pi a)^2} \right) \int dx \cos \left( \tilde{\theta}_{c+1} \right) \cos \left( \frac{2}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_0 \right) \]
\[ - g^{(2)}_{1\parallel} \left( \frac{4}{(2\pi a)^2} \right) \int dx \cos \left( \tilde{\phi}_{c+1} \right) \cos (2\tilde{\theta}_{s+1}) \]
\[ + g^{(2)}_{3\perp} \left( \frac{4}{(2\pi a)^2} \right) \int dx \cos \left( \tilde{\theta}_{c+1} \right) \cos (2\tilde{\phi}_{s+1}) \]
\[ - f^{(1)}_{1\parallel} \left( \frac{8}{(2\pi a)^2} \right) \int dx \left[ \cos \tilde{\phi}_{s+1} \cos \left( -\frac{1}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_0 \right) \cos \tilde{\theta}_{s+1} \cos \sqrt{3}\tilde{\phi}_{s-1} + (\cos \rightarrow \sin) \right] \]
\[ + f^{(1)}_{3\parallel} \left( \frac{8}{(2\pi a)^2} \right) \int dx \left[ \cos \tilde{\theta}_{c+1} \cos \sqrt{3}\tilde{\phi}_{c-1} \cos \tilde{\theta}_{s+1} \cos \sqrt{3}\tilde{\phi}_{s-1} + (\cos \rightarrow \sin) \right] \]
\[ + f^{(1)}_{3\perp} \left( \frac{8}{(2\pi a)^2} \right) \int dx \left[ \cos \tilde{\theta}_{c+1} \cos \sqrt{3}\tilde{\phi}_{c-1} \cos \tilde{\phi}_{s+1} \cos \left( -\frac{1}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_0 \right) \cos \right] \]
\[ + f^{(2)}_{3\perp} \left( \frac{8}{(2\pi a)^2} \right) \int dx \left[ \cos \tilde{\theta}_{c+1} \cos \sqrt{3}\tilde{\phi}_{c-1} \cos \tilde{\phi}_{s+1} \cos \sqrt{3}\tilde{\phi}_{s-1} + (\cos \rightarrow \sin) \right] \]
\[ + g^{(1)}_{1\perp} \left( \frac{2}{(2\pi a)^2} \right) \int dx \cos \left( -\frac{4}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_0 \right) \]
Order parameters

Definition

\[
O_{ph}^{ij} = \sum_{mm'} \lambda_i^m \sigma_j^{m'} \psi_R^{\sigma m} \psi_L^{m' \sigma'} \\
O_{pp}^{ij} = \sum_{mm'} \lambda_i^m \sigma_j^{m'} \psi_R^{\sigma m} \psi_L^{m' \sigma'} \\
\lambda^i : \text{ Gell-Mann matrices} \\
\sigma^j : \text{ Pauli matrices}
\]

Examples for bosonization

SDW

\[
O_{ph}^{13} \propto e^{-i2k_F x + \left( \frac{1}{\sqrt{3}} \tilde{\phi}_{c-1} + \frac{2}{\sqrt{6}} \tilde{\phi}_0 \right)} \left[ \cos \left( \frac{1}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{2}{\sqrt{6}} \tilde{\phi}_0 \right) \sin \tilde{\theta}_{c+1} \cos \tilde{\theta}_{s+1} + i (\cos \leftrightarrow \sin) \right]
\]

Spin-triplet superconducting state

\[
O_{pp}^{23} \propto e^{i \left( \frac{1}{\sqrt{3}} \tilde{\phi}_{c-1} + \frac{2}{\sqrt{6}} \tilde{\phi}_c \right)} \left[ \cos \left( \frac{1}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{2}{\sqrt{6}} \tilde{\phi}_0 \right) \cos \tilde{\phi}_{c+1} \cos \tilde{\phi}_{s+1} - i (\cos \leftrightarrow \sin) \right]
\]
Renormalization group

Operator product expansion (OPE)

\[ \frac{dg_k}{dl} = (d - \Delta_k)g_k - \sum_{ij} C_{ij}^k g_i g_j \]

tree level one-loop

Renormalized coupling constants

\[ y_i = \frac{g_i}{\pi v_F} \]
\[ x_i = \frac{f_i}{\pi v_F} \]
Tree-level RG equations

\[
\begin{align*}
\frac{d g^{(1)}_{1\perp}}{d l} &= \left[ 2 - \left( \frac{1}{3} K_{s-1} + \frac{2}{3} K_{s0} + K_{s+1}^{-1} \right) \right] g^{(1)}_{1\perp} \\
\frac{d g^{(1)}_{2\parallel}}{d l} &= \left[ 2 - (K_{c+1} + K_{s+1}) \right] g^{(1)}_{2\parallel} \\
\frac{d g^{(1)}_{2\perp}}{d l} &= \left[ 2 - \left( K_{c+1} + \frac{1}{3} K_{s-1} + \frac{2}{3} K_{s0} \right) \right] g^{(1)}_{2\perp} \\
\frac{d g^{(1)}_{3\parallel}}{d l} &= \left[ 2 - (K_{c+1}^{-1} + K_{s+1}^{-1}) \right] g^{(1)}_{3\parallel} \\
\frac{d g^{(1)}_{3\perp}}{d l} &= \left[ 2 - \left( K_{c+1}^{-1} + \frac{1}{3} K_{s-1} + \frac{2}{3} K_{s0} \right) \right] g^{(1)}_{3\perp} \\
\frac{d g^{(1)}_{4\perp}}{d l} &= \left[ 2 - \left( K_{c+1} + \frac{1}{3} K_{s-1} + \frac{2}{3} K_{s0} \right) \right] g^{(1)}_{4\perp} \\
\frac{d g^{(2)}_{1\perp}}{d l} &= \left[ 2 - (K_{c+1} + K_{s+1}^{-1}) \right] g^{(2)}_{1\perp} \\
\frac{d g^{(2)}_{3\perp}}{d l} &= \left[ 2 - (K_{c+1}^{-1} + K_{s+1}) \right] g^{(2)}_{3\perp} \\
\frac{d f^{(1)}_{1\perp}}{d l} &= \left[ 2 - \left( \frac{1}{4} K_{s+1} + \frac{1}{12} K_{s-1} + \frac{2}{3} K_{s0} + \frac{1}{4} K_{s+1}^{-1} + \frac{3}{4} K_{s-1}^{-1} \right) \right] f^{(1)}_{1\perp} \\
\frac{d f^{(1)}_{3\parallel}}{d l} &= \left[ 2 - \left( \frac{1}{4} K_{c+1}^{-1} + \frac{3}{4} K_{c-1}^{-1} + \frac{1}{4} K_{s+1}^{-1} + \frac{3}{4} K_{s-1}^{-1} \right) \right] f^{(1)}_{3\parallel} \\
\frac{d f^{(1)}_{3\perp}}{d l} &= \left[ 2 - \left( \frac{1}{4} K_{c+1}^{-1} + \frac{3}{4} K_{c-1}^{-1} + \frac{1}{4} K_{s+1}^{-1} + \frac{1}{12} K_{s-1} + \frac{2}{3} K_{s0} \right) \right] f^{(1)}_{3\perp} \\
\frac{d f^{(2)}_{3\perp}}{d l} &= \left[ 2 - \left( \frac{1}{4} K_{c+1}^{-1} + \frac{3}{4} K_{c-1}^{-1} + \frac{1}{4} K_{s+1}^{-1} + \frac{3}{4} K_{s-1}^{-1} \right) \right] f^{(2)}_{3\perp} \\
\frac{d g^{(1)}_{1\perp}}{d l} &= \left[ 2 - \left( \frac{4}{3} K_{s-1} + \frac{2}{3} K_{s0} \right) \right] g^{(1)}_{1\perp}
\end{align*}
\]

Renormalize coupling constants

\[
\begin{align*}
y_i &= \frac{g_i}{\pi v_F} \\
x_i &= \frac{f_i}{\pi v_F}
\end{align*}
\]

Expanding Luttinger parameters

\[
K_{\mu\nu} = 1 - y_{\mu\nu}
\]

with

\[
\begin{align*}
y_{c\pm 1} &= \frac{1}{2} \left[ \left( y_{4\perp}^{(2)} \right) - \left( y_{2\parallel}^{(2)} + y_{2\perp}^{(2)} \right) \right] \\
y_{c0} &= \frac{1}{2} \left[ \left( y_{4\perp}^{(2)} \right) + 2 \left( y_{2\parallel}^{(2)} + y_{2\perp}^{(2)} \right) \right] \\
y_{s\pm 1} &= \frac{1}{2} \left[ \left( - y_{4\perp}^{(2)} \right) - \left( y_{2\parallel}^{(2)} - y_{2\perp}^{(2)} \right) \right] \\
y_{s0} &= \frac{1}{2} \left[ \left( - y_{4\perp}^{(2)} \right) + 2 \left( y_{2\parallel}^{(2)} - y_{2\perp}^{(2)} \right) \right]
\end{align*}
\]
Tree-level RG equations

\[ \frac{dy^{(1)}_{1 \perp}}{dl} = \left( y^{(2)}_{2 \parallel} - y^{(2)}_{2 \perp} \right) y^{(1)}_{1 \perp}, \]
\[ \frac{dy_{2 \parallel}}{dl} = -y^{(2)}_{2 \parallel} y^{(1)}_{2 \parallel}, \]
\[ \frac{dy_{2 \perp}}{dl} = -y^{(2)}_{2 \perp} y^{(1)}_{2 \perp}, \]
\[ \frac{dy^{(1)}_{3 \parallel}}{dl} = y^{(2)}_{3 \parallel} y^{(1)}_{3 \parallel}, \]
\[ \frac{dy^{(1)}_{3 \perp}}{dl} = \left( -y^{(2)}_{4 \perp} + y^{(2)}_{2 \perp} \right) y^{(1)}_{3 \perp}, \]
\[ \frac{dy^{(2)}_{4 \perp}}{dl} = \left( y^{(2)}_{4 \perp} - y^{(2)}_{2 \perp} \right) y^{(2)}_{4 \perp}, \]
\[ \frac{dy^{(2)}_{1 \perp}}{dl} = \left( -y^{(2)}_{4 \perp} + y^{(2)}_{2 \perp} \right) y^{(2)}_{1 \perp}, \]
\[ \frac{dy^{(1)}_{3 \parallel}}{dl} = \left( y^{(2)}_{2 \parallel} - y^{(2)}_{2 \perp} \right) x^{(1)}_{3 \parallel}, \]
\[ \frac{dy^{(1)}_{3 \perp}}{dl} = \left( -y^{(2)}_{4 \perp} + y^{(2)}_{2 \perp} \right) x^{(1)}_{3 \perp}, \]
\[ \frac{dy^{(2)}_{3 \parallel}}{dl} = \left( -y^{(2)}_{4 \perp} + y^{(2)}_{2 \perp} \right) x^{(2)}_{3 \parallel}, \]
\[ \frac{dy^{(2)}_{3 \perp}}{dl} = \left( -y^{(2)}_{4 \perp} + y^{(2)}_{2 \perp} \right) x^{(2)}_{3 \perp}, \]
\[ \frac{dy^{(1)}_{1 \perp}}{dl} = -y^{(2)}_{4 \perp} y^{(1)}_{1 \perp}. \]

Relevant coupling constants

- \( J < U/3 \) \( x^{(1)}_{3 \parallel}, \ y^{(1)}_{3 \parallel} \) and \( y^{(2)}_{1 \perp} \)

- \( J > U/3 \) \( y^{(1)}_{2 \parallel} \) and \( y^{(2)}_{1 \perp} \)

Fixed points (hypersurface)

- \( J < U/3 \)
  \[ y^{(1)}_{3 \parallel} = y^{(1)}_{3 \parallel} \]
  \[ y^{(2)}_{1 \perp} = y^{(2)}_{1 \perp} \]
  \[ x^{(1)}_{3 \parallel} = x^{(1)}_{3 \parallel} \]

- \( J > U/3 \)
  \[ y^{(1)}_{2 \parallel} = y^{(1)}_{2 \parallel} \]
  \[ y^{(2)}_{1 \perp} = y^{(2)}_{1 \perp} \]

- Insufficient to determine the ground state.
- One-loop correction will change these results.
One-loop RG equations (spin rotational symmetry has been applied for simplicity)

\[
\begin{align*}
\frac{dy_{1\perp}}{dl} &= -\left(y_{1\perp}^{(1)}\right)^2 - y_{2\perp}^{(1)} y_{1\perp}^{(2)} + y_{3\parallel}^{(1)} y_{3\perp}^{(1)}, \\
\frac{dy_{2\parallel}}{dl} &= \frac{1}{2} y_{1\perp}^{(1)} y_{2\perp}^{(1)} - y_{2\perp}^{(1)} y_{4\perp}^{(1)}, \\
\frac{dy_{2\perp}}{dl} &= -\frac{1}{2} y_{1\perp}^{(1)} y_{2\perp}^{(1)} - y_{1\perp}^{(2)} y_{2\parallel}^{(1)} - y_{2\parallel}^{(1)} y_{4\perp}^{(1)}, \\
\frac{dy_{3\parallel}}{dl} &= -\frac{1}{2} y_{1\perp}^{(1)} y_{3\parallel}^{(1)} + y_{1\perp}^{(1)} y_{3\perp}^{(1)}, \\
\frac{dy_{3\perp}}{dl} &= -\left(y_{4\perp}^{(1)} + \frac{1}{2} y_{1\perp}^{(1)}\right) y_{3\perp}^{(1)} + y_{4\perp}^{(1)} y_{3\perp}^{(2)} - y_{4\perp}^{(1)} y_{3\perp}^{(1)}, \\
\frac{dy_{4\perp}}{dl} &= \frac{1}{2} y_{1\perp}^{(1)} y_{4\perp}^{(1)} - y_{2\parallel}^{(1)} y_{2\perp}^{(1)} - y_{3\parallel}^{(1)} y_{3\perp}^{(2)}, \\
\frac{dy_{1\perp}^{(2)}}{dl} &= \left(y_{4\perp}^{(1)} - \frac{1}{2} y_{1\perp}^{(1)}\right) y_{1\perp}^{(2)} - y_{1\perp}^{(1)} y_{2\perp}^{(1)}, \\
\frac{dy_{3\perp}^{(2)}}{dl} &= \left(-y_{4\perp}^{(1)} + \frac{1}{2} y_{1\perp}^{(1)}\right) y_{3\perp}^{(2)} - y_{3\perp}^{(1)} y_{4\perp}^{(1)}, \\
\frac{dx_{1\perp}^{(1)}}{dl} &= -\left(x_{1\perp}^{(1)}\right)^2 + x_{3\parallel}^{(1)} x_{3\perp}^{(1)}, \\
\frac{dx_{1\perp}^{(3)}}{dl} &= -\frac{1}{2} x_{1\perp}^{(1)} x_{3\parallel}^{(1)} + x_{1\perp}^{(1)} x_{3\perp}^{(1)}, \\
\frac{dx_{3\perp}^{(1)}}{dl} &= -\left(y_{1\perp}^{(1)} + \frac{1}{2} x_{1\perp}^{(1)}\right) x_{3\perp}^{(1)} + x_{1\perp}^{(1)} x_{3\parallel}^{(1)}, \\
\frac{dx_{3\perp}^{(2)}}{dl} &= \left(-y_{1\perp}^{(1)} + \frac{1}{2} x_{1\perp}^{(1)}\right) x_{3\perp}^{(2)}, \\
\frac{dy_{1\perp}^{(1)}}{dl} &= -\left(y_{1\perp}^{(1)}\right)^2.
\end{align*}
\]
## Relevant scaling field and corresponding ground states

### $J < U/3$

| Scaling field | $x^{(1)}_{1\perp} + x^{(1)}_{3\perp}$ | $y^{(1)}_{1\perp} + y^{(1)}_{3\perp}$ | $y^{(1)}_{1\perp} - y^{(1)}_{2\perp}$ |
|---------------|--------------------------------------|--------------------------------------|--------------------------------------|
| Instability   | SDW                                  | SDW                                  | SSC                                  |
| Order parameter | $O_{ph}^{43}, O_{ph}^{63}$           | $O_{ph}^{13}$                        | $O_{pp}^{20}$                        |
| Saddle point  | $\frac{1}{2}\tilde{\phi}_{s+1} - \frac{1}{\sqrt{12}}\tilde{\phi}_{s-1} + \frac{2}{\sqrt{6}}\tilde{\phi}_{s0} = 0 \left(\frac{\pi}{2}\right)$ | $\frac{1}{\sqrt{3}}\tilde{\phi}_{s-1} + \frac{2}{\sqrt{6}}\tilde{\phi}_{s0} = 0 \left(\frac{\pi}{2}\right)$ | $\frac{1}{\sqrt{3}}\tilde{\phi}_{s-1} + \frac{2}{\sqrt{6}}\tilde{\phi}_{s0} = 0 \left(\frac{\pi}{2}\right)$ |
|               | $\frac{1}{2}\tilde{\theta}_{c+1} + \frac{3}{\sqrt{12}}\tilde{\theta}_{c-1} = \frac{\pi}{2} \left(0\right)$ | $\tilde{\theta}_{c+1} = \frac{\pi}{2} \left(0\right)$ | $\tilde{\phi}_{c+1} = \frac{\pi}{2} \left(0\right)$ |
|               | $\frac{1}{2}\tilde{\theta}_{s+1} + \frac{3}{\sqrt{12}}\tilde{\theta}_{s-1} = 0 \left(\frac{\pi}{2}\right)$ | $\tilde{\theta}_{s+1} = 0 \left(\frac{\pi}{2}\right)$ | $\tilde{\theta}_{s+1} = 0 \left(\frac{\pi}{2}\right)$ |

### $J > U/3$

| Scaling field | $y^{(1)}_{2\perp} + y^{(1)}_{4\perp}$ | $y^{(1)}_{1\perp} - y^{(1)}_{2\perp}$ |
|---------------|--------------------------------------|--------------------------------------|
| Instability   | SDW                                  | TSC                                  |
| Order parameter | $O_{ph}^{03} + \sqrt{3}O_{ph}^{83}$ | $O_{pp}^{23}$                        |
| Saddle point  | $\frac{1}{\sqrt{3}}\tilde{\phi}_{s-1} + \frac{2}{\sqrt{6}}\tilde{\phi}_{s0} = 0 \left(\frac{\pi}{2}\right)$ | $\frac{1}{\sqrt{3}}\tilde{\phi}_{s-1} + \frac{2}{\sqrt{6}}\tilde{\phi}_{s0} = 0 \left(\frac{\pi}{2}\right)$ |
|               | $\tilde{\phi}_{c+1} = \frac{\pi}{2} \left(0\right)$ | $\tilde{\phi}_{c+1} = 0 \left(\frac{\pi}{2}\right)$ |
|               | $\tilde{\phi}_{s+1} = \frac{\pi}{2} \left(0\right)$ | $\tilde{\theta}_{s+1} = 0 \left(\frac{\pi}{2}\right)$ |
Determined by initial coupling constants, say, the microscopic model.
Normal state: TLL fixed point

NMR: Spin-lattice relaxation rate

\[ \frac{1}{T_1} = A^2 T \sum_q \frac{\text{Im} \chi(q, \omega)}{\omega} \]

Three-band model

\[ \frac{1}{T_1} \propto A_1 T \]
\[ + A_2 T^{1/2} \left[ (K_{c+1} + \frac{1}{3} K_{c-1} + \frac{2}{3} K_{c0}) + (K_{s+1} + \frac{1}{3} K_{s-1} + \frac{2}{3} K_{s0}) \right] - 1 \]
\[ + A_3 T^{1/2} \left[ \left( \frac{4}{3} K_{c-1} + \frac{2}{3} K_{c0} \right) + \left( \frac{4}{3} K_{s-1} + \frac{2}{3} K_{s0} \right) \right] - 1. \]  \quad (C3)

Spin-rotational symmetric system

\[ \frac{1}{T_1} \propto A T + B T^{1- \frac{U}{2\pi \nu_F}} \]

U>0 and U<0 will result in different low temperature behaviors!
### Possible superconducting ground states

#### $0 < J < U/3$

|                   | $y_{1\perp}^{(1)} - y_{2\perp}^{(1)}$ |
|-------------------|----------------------------------------|
| **SSC**           |                                        |
| $O_{pp}^{20}$     |                                        |

\[
\frac{1}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{2}{\sqrt{6}} \tilde{\phi}_{s0} = 0 \left( \frac{\pi}{2} \right) \\
\tilde{\phi}_{c+1} = \frac{\pi}{2} (0) \\
\tilde{\theta}_{s+1} = \frac{\pi}{2} (0)
\]

**spin-singlet, odd parity, orbital antisymmetric**

#### $U/3 < J < U/2$

|                   | $y_{1\perp}^{(1)} - y_{2\perp}^{(1)}$ |
|-------------------|----------------------------------------|
| **TSC**           |                                        |
| $O_{pp}^{23}$     |                                        |

\[
\frac{1}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{2}{\sqrt{6}} \tilde{\phi}_{s0} = 0 \left( \frac{\pi}{2} \right) \\
\tilde{\phi}_{c+1} = 0 \left( \frac{\pi}{2} \right) \\
\tilde{\theta}_{s+1} = 0 \left( \frac{\pi}{2} \right)
\]

**spin-triplet, even parity, orbital antisymmetric**
Discussion: Lifted degeneracy

- The two-fold degenerate of $E'$ bands will be lifted by *inter-chain coupling*.
Interacting bosonic Hamiltonian when $k_{F-1} \neq k_{F1}$

$$H_{int}^B = -g_{1\perp}^{(1)} \frac{4}{(2\pi)^2} \int dx \cos \left( \frac{2}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_s \right) \cos \left( 2\tilde{\theta}_{s+1} \right)$$

$$+ g_{2\parallel}^{(1)} \frac{4}{(2\pi)^2} \int dx \cos \left( 2\Delta k_F x + 2\tilde{\phi}_{c+1} \right) \cos \left( 2\tilde{\phi}_{s+1} \right)$$

$$+ g_{2\perp}^{(1)} \frac{4}{(2\pi)^2} \int dx \cos \left( 2\Delta k_F x + 2\tilde{\phi}_{c+1} \right) \cos \left( \frac{2}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_s \right)$$

$$+ g_{3\parallel}^{(1)} \frac{4}{(2\pi)^2} \int dx \cos \left( 2\tilde{\theta}_{c+1} \right) \cos \left( 2\tilde{\theta}_{s+1} \right)$$

$$+ g_{3\perp}^{(1)} \frac{4}{(2\pi)^2} \int dx \cos \left( 2\tilde{\theta}_{c+1} \right) \cos \left( \frac{2}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_s \right)$$

$$+ g_{4\perp}^{(1)} \frac{4}{(2\pi)^2} \int dx \cos \left( 2\tilde{\phi}_{s+1} \right) \cos \left( \frac{2}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_s \right)$$

$$- g_{1\perp}^{(2)} \frac{4}{(2\pi)^2} \int dx \cos \left( 2\Delta k_F x + 2\tilde{\phi}_{c+1} \right) \cos \left( 2\tilde{\phi}_{s+1} \right)$$

$$+ g_{3\perp}^{(2)} \frac{4}{(2\pi)^2} \int dx \cos \left( 2\tilde{\phi}_{c+1} \right) \cos \left( 2\tilde{\phi}_{s+1} \right)$$

$$+ \tilde{f}_{1\perp}^{(1)} \frac{8}{(2\pi)^2} \int dx \left[ \cos \tilde{\phi}_{s+1} \cos \left( \frac{1}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_s \right) \cos \tilde{\theta}_{s+1} \cos \sqrt{3}\tilde{\theta}_{s-1} + (\cos \rightarrow \sin) \right]$$

$$+ \tilde{f}_{3\parallel}^{(1)} \frac{8}{(2\pi)^2} \int dx \left[ \cos \tilde{\theta}_{c+1} \cos \sqrt{3}\tilde{\theta}_{c-1} \cos \tilde{\phi}_{s+1} \cos \sqrt{3}\tilde{\phi}_{s-1} + (\cos \rightarrow \sin) \right]$$

$$+ \tilde{f}_{3\perp}^{(1)} \frac{8}{(2\pi)^2} \int dx \left[ \cos \tilde{\theta}_{c+1} \cos \sqrt{3}\tilde{\theta}_{c-1} \cos \tilde{\phi}_{s+1} \cos \left( \frac{1}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_s \right) + (\cos \rightarrow \sin) \right]$$

$$+ \tilde{f}_{3\perp}^{(2)} \frac{8}{(2\pi)^2} \int dx \left[ \cos \tilde{\theta}_{c+1} \cos \sqrt{3}\tilde{\theta}_{c-1} \cos \tilde{\phi}_{s+1} \cos \sqrt{3}\tilde{\phi}_{s-1} + (\cos \rightarrow \sin) \right]$$

$$+ \tilde{g}_{1\perp}^{(1)} \frac{2}{(2\pi)^2} \int dx \cos \left( \frac{4}{\sqrt{3}} \tilde{\phi}_{s-1} + \frac{4}{\sqrt{6}} \tilde{\phi}_s \right),$$

Fixed points (hypersurface)

- $J < U / 3$
  - SDW
  - SSC
  - SDW

- $J > U / 3$
  - SDW
  - TSC

$$y_{3\parallel}^{(1)} = y_{3\parallel}^{(1)*}$$
$$y_{1\perp}^{(2)} = y_{1\perp}^{(2)*}$$
$$x_{3\parallel}^{(1)} = x_{3\parallel}^{(1)*}$$
$$y_{2\parallel}^{(1)} = y_{2\parallel}^{(1)*}$$
$$y_{1\perp}^{(2)} = y_{1\perp}^{(2)*}$$
Consequences of lifted degeneracy

- SSC be suppressed, SDW have chance to dominate for $J < U/3$
- Interband pairing will be modulated by a phase factor
  - Possible FFLO state?
  - Unstable when the degeneracy lift becomes significant.
Phase diagrams

Degenerate bands $k_{F-1} = k_{F1}$

Nearly-degenerate bands $k_{F-1} \neq k_{F1}$
Take home message

- One-loop RG analysis to 1D three-band Hubbard model
  - Assumption 1: Two of the three bands are (nearly) degenerate
  - Assumption 2: Incommensurate electron filling

- Results:
  - \(0 < J < U/3\), spin-singlet SC (degenerate bands) or spin density wave (non-degenerate bands)
  - \(U/3 < J < U/2\), spin-triplet SC (nearly degenerate bands)
  - \(J > U/2\) (unphysical region), SDW

- Possible application to superconductor \(K_2Cr_3As_3\)
  - Take Luttinger liquid normal state as the starting point
  - Inter-chain coupling will lift band degeneracy and SDW will dominate over SSC when \(J/U < 1/3\); while TSC will always dominate over SDW when \(J/U > 1/3\).
  - Inter-chain coupling will determine the spatially pairing symmetry
Thank you!