Abstract. Let $G$ be a connected, loopless multigraph. The sandpile group of $G$ is a finite abelian group associated to $G$ whose order is equal to the number of spanning trees in $G$. Holroyd et al. used a dynamical process on graphs called rotor-routing to define a simply transitive action of the sandpile group of $G$ on its set of spanning trees. Their definition depends on two pieces of auxiliary data: a choice of a ribbon graph structure on $G$, and a choice of a root vertex. Chan, Church, and Grochow showed that if $G$ is a planar ribbon graph, it has a canonical rotor-routing action associated to it, i.e., the rotor-routing action is actually independent of the choice of root vertex.

It is well-known that the spanning trees of a planar graph $G$ are in canonical bijection with those of its planar dual $G^*$, and furthermore that the sandpile groups of $G$ and $G^*$ are isomorphic. Thus, one can ask: are the two rotor-routing actions, of the sandpile group of $G$ on its spanning trees, and of the sandpile group of $G^*$ on its spanning trees, compatible under plane duality? In this paper, we give an affirmative answer to this question, which had been conjectured by Baker.

1. Introduction

Let $G$ be a connected multigraph with no loop edges. The sandpile group of $G$ is a finite abelian group whose order is equal to the number of spanning trees in $G$; it is the group of degree zero divisors of $G$ modulo the equivalence relation generated by lending moves. (We will recall all relevant definitions in Section 2.)

In [7], Holroyd, Levine, Meszaró, Peres, Propp, and Wilson use a dynamical process on graphs called rotor-routing to define a simply transitive action of the sandpile group of $G$ on its set of spanning trees. Rotor-routing itself was introduced in [9] under the name “Eulerian walkers” and has been rediscovered several times in different fields: see [7] for a concise history of the topic.

The definition of the rotor-routing action on $G$ given in [7] involves two pieces of auxiliary data. First, the action is defined with respect to a choice of a root vertex $v \in V(G)$, or basepoint. Second, it depends on a ribbon graph structure on $G$: a choice of a cyclic ordering of the set of edges incident to each vertex $v$. Note that such a choice of cyclic orders defines an embedding of $G$ on some closed, oriented surface $S$, in which all cyclic orders correspond to a positive orientation, say, with respect to $S$. We say that $G$ is a planar...
ribbon graph if \( S \) is just a sphere, i.e., if the chosen ribbon structure equips \( G \) with an embedding into the plane.

A recent paper of Chan, Church, and Grochow \([5]\) answers a question of J. Ellenberg \([8]\) by proving that the rotor-routing action does not depend on the choice of basepoint if and only if \( G \) is a planar ribbon graph. This result is somewhat surprising, and as a nice consequence of it, we may henceforth refer to \textit{the} rotor-routing action on a planar ribbon graph, without further reference to a choice of basepoint.

Any graph \( G \) embedded in the plane has a planar dual graph \( G^* \) whose spanning trees are in canonical bijection with those of \( G \). Moreover, the sandpile groups of \( G \) and \( G^* \) are, up to sign, canonically isomorphic \([1]\) (see also \([6]\)). Thus, one would hope that the two rotor-routing actions, of the sandpile group of \( G \) on the set \( T(G) \) of its spanning trees, and of the sandpile group of \( G^* \) on its spanning trees, are compatible.

This was, in fact, exactly the conjecture suggested to us by M. Baker. In this paper, we provide a proof of Baker’s conjecture on the compatibility of the rotor-routing action of the sandpile group with plane duality. See Theorem 3.1 for the precise statement, and see Figure 1 for an example illustrating the result.

We begin with preliminary definitions on the sandpile group and rotor-routing in Section 2. The proof of our main result occupies Section 3. The key idea of our proof is the \textit{angle} between two spanning trees \( T \) and \( T' \) of \( G \): see Definition 3.3. The angle from \( T \) to \( T' \) remembers the element of the sandpile group that takes \( T \) to \( T' \) under rotor-routing. On the other hand, we are able to show using a direct geometric argument that the angle is compatible with plane duality, so the main theorem follows.

We would also like to refer the reader to the recent preprint \([3]\), which arrives at another proof of Theorem 3.1 via a completely different route. In that paper, Baker and Wang prove that the bijections obtained by Bernardi in \([4]\) Theorem 45] give rise to another simply transitive action of the sandpile group on the spanning trees of a ribbon graph \( G \) with a fixed root vertex. They show that this action is compatible with plane duality and that it coincides with the rotor-routing action when \( G \) is planar. It would be interesting to study the relationship between these two approaches further.

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2. Preliminaries

2.1. The sandpile group. Let $G = (V, E)$ be a finite connected loopless multigraph with vertex set $V$ and edge multiset $E$. The set of divisors on $G$ is the free abelian group on the vertices: $\text{Div}(G) = \mathbb{Z}^V$. We imagine a divisor $D = \sum_{v \in V} a_v v$ to be an assignment of $D(v) := a_v$ chips to each vertex $v$, keeping in mind that this number may be negative. We write $\text{Div}_0(G)$ for the subgroup of divisors whose net number of chips $\sum D(v)$ is zero.

A lending move by a vertex $v$ consists of removing $\deg(v)$ chips from $v$ and distributing them along incident edges to the vertices neighboring $v$. In other words, letting $n(v, w)$ denote the number of edges between $v$ and $w$, a lending move by $v$ performed on a divisor $D$ produces a divisor $D'$ given by

$$D'(w) = \begin{cases} 
D(w) + n(v, w) & \text{if } w \neq v \\
D(v) - \deg(v) & \text{if } w = v.
\end{cases}$$

Notice that lending moves do not change the total number of chips in a divisor. Divisors $D$ and $D'$ are linearly equivalent, denoted $D \sim D'$, if one can be obtained from the other by a sequence of lending moves at various
vertices. The sandpile group of $G$ is
\[ S(G) = \text{Div}^0(G)/\sim. \]
The sandpile group of a graph is also variously known as the Jacobian of $G$, the Picard group $\text{Pic}^0(G)$, or the critical group of $G$.

2.2. Integral cuts and cycles. Fix an arbitrary orientation on the edges $E$, and let $\mathbb{Z}E$ be the free abelian group on these oriented edges. If $e = \{u, v\} \in E$ is given the orientation $(u, v)$, we write $e^+ = \text{head}(e) = v$ and $e^- = \text{tail}(e) = u$. We identify $-e$ with the oppositely oriented edge $(v, u)$.

Each directed cycle on the underlying undirected graph $G$ may be thought of as an element of $\mathbb{Z}E$, and the $\mathbb{Z}$-linear span of these cycles in $\mathbb{Z}E$ is the integral cycle space for $G$, which we denote by $\mathcal{C}$.

Next, for any subset $U \subset V$, the collection of all edges joining a vertex of $U$ to a vertex of $V \setminus U$ is called a cut. By directing all of these edges from vertices in $U$ to vertices in $V \setminus U$, we can identify this cut with an element of $\mathbb{Z}E$. If $U$ consists of single vertex $v$, this cut is called a vertex cut at $v$. The integer span of all cuts is the integral cut space for $G$ and is denoted by $\mathcal{C}^*$. Note that the vertex cuts generate the cut space.

Define
\[ \mathcal{E}(G) = \mathbb{Z}E/(\mathcal{C} + \mathcal{C}^*). \]
We now identify $\mathcal{E}(G)$ with the sandpile group $S(G)$, as follows. Define the boundary map $\mathbb{Z}E \to \text{Div}^0(G)$ by sending each edge $e$ to $e^+ - e^-$. The boundary map is surjective since $G$ is connected, and its kernel is exactly the cycle space of $G$, so it identifies $\text{Div}^0(G)$ with $\mathbb{Z}E/\mathcal{C}$. Now given $D \in \text{Div}^0(G)$, let $D_v$ be the boundary of a vertex cut at the vertex $v$. Then $D + D_v$ is the divisor obtained from $D$ by performing a lending move at $v$. Therefore the boundary map induces an isomorphism
\[ \partial_G: \mathcal{E}(G) \xrightarrow{\cong} S(G) \]
\[ e \mapsto [e^+ - e^-], \]
as was proved in [1] Proposition 8. We will sometimes write $\partial$ instead of $\partial_G$ for short.

2.3. Rotor-routing action on spanning trees. Fix a ribbon graph structure on $G$, i.e., for each vertex $v$, fix a cyclic ordering of the edges incident to $v$. Fix a vertex $q$. A rotor configuration with basepoint $q$ is the choice for each vertex $v \neq q$ of an edge, $\rho(v)$, incident to $v$. We orient each edge $\rho(v)$ so that its tail is $v$.

Let $D$ be a divisor on $G$, thought of as a chip configuration on $G$, and let $\rho$ be a rotor configuration with basepoint $q$. We now recall the rotor-routing process, by which a divisor $D$ transforms $\rho$ into a new rotor configuration $\rho'$. Firing a vertex $v$ consists of updating $\rho$ by replacing $\rho(v)$ with the next edge in the cyclic ordering of edges at $v$, then removing a chip from $v$ and placing it at the other end of the new edge $\rho(v)$. Note that firing $v$ a total of $\deg(v)$
times does not change the original rotor configuration, but transforms $D$ by a lending move at $v$. Now, every divisor $D$ on $G$ is linearly equivalent to a divisor $D'$ with $D'(v) \geq 0$ for all $v \neq q$, see e.g. [2, Proposition 3.1]. From that point, [7] shows that solely through vertex firings, all chips may be routed into $q$, and the rotor configuration at the end of this process depends solely on the divisor class of $D$.

Let $\mathcal{T}(G)$ denote the set of spanning trees of $G$. Rooting $T \in \mathcal{T}(G)$ at $q$ uniquely determines a rotor configuration $\rho_T$: for each vertex $v \neq q$, set $\rho_T(v)$ to be the edge incident to $v$ on the path in $T$ from $v$ to $q$. Given a divisor class $[D] \in \mathcal{S}(G)$, use the rotor routing process to route all chips into $q$ (at which point, all chips will be gone since $\deg(D) = 0$). It is shown in [7] that the resulting rotor configuration is a spanning tree, directed into $q$. Call the underlying undirected spanning tree $[D] \cdot T$. Then according to [7], the resulting map

$$\mu_G: \mathcal{S}(G) \times \mathcal{T}(G) \to \mathcal{T}(G),$$

$$([D], T) \mapsto [D] \cdot T$$

is a simply transitive action of $\mathcal{S}(G)$ on $\mathcal{T}(G)$.

2.4. Planar duality. Now suppose that $G = (V, E)$ is a planar ribbon graph, and let $G^* = (V^*, E^*)$ be its planar dual graph, whose vertices are the faces of $G$ and whose edges cross the edges of $G$. We shall assume throughout that both $G$ and $G^*$ are loopless, i.e., $G$ has neither bridges nor loops. We write $e^*$ for the edge of $G^*$ crossing the edge $e$ of $G$. Each spanning tree of $G$ determines a spanning tree of $G^*$: namely, there is a natural bijection

$$\delta: \mathcal{T}(G) \xrightarrow{\cong} \mathcal{T}(G^*),$$

sending $T$ to the tree $T^* = \{e^* \in E^*: e \in E \setminus T\}$.

Let us call the orientation of the plane that agrees with the cyclic orderings of $G$ clockwise. Then we fix once and for all the following planar dual ribbon graph structure on $G^*$: take the cyclic orderings of the edges at the vertices of $G^*$ to be counter-clockwise with respect to the plane.

In order to define $Z\mathcal{E}$, we fixed an arbitrary orientation of the edges of $G$. To define $Z\mathcal{E}^*$, we will now choose a compatible orientation on the edges of $G^*$. For an oriented edge $e$ of $G$, let $e'$ (respectively $e''$) denote the edge at $v = e^-$ before (respectively after) $e$ in the cyclic order at $v$. Now, call the face between $e'$ and $e$ at $v$ the face before $e$, and call the face between $e$ and $e''$ at $v$ the face after $e$. Then we orient $e^*$ so that its head is the face of $G$ before $e$, and its tail is the face of $G$ after $e$. For example, in Figure 4 with the rotors of $G$ oriented clockwise relative to the page, suppose $e$ is the directed edge from $x$ to $y$. Then $e^*$ is the directed edge in $G^*$ from $b$ to $a$.

Since directed cycles of $G$ are directed cuts of $G^*$ and vice versa, mapping each edge to its dual produces an isomorphism $\mathcal{E}(G) \cong \mathcal{E}(G^*)$, and hence we get an isomorphism $\phi$ of sandpile groups labeled as in the following
3. Compatibility of rotor-routing with duality

Let $G$ be any planar ribbon graph such that both $G$ and its dual $G^*$ are loopless. In the previous section, we established an isomorphism $\phi: S(G) \to S(G^*)$ that depended on a single global choice of orientation of the $E^*$ derived from the orientation $E$. With respect to this choice, we may now state the main theorem of the paper:

**Theorem 3.1.** The diagram

\[
\begin{array}{ccc}
S(G) \times T(G) & \xrightarrow{\phi \times \delta} & T(G) \\
\phi \times \delta & \downarrow & \delta \\
S(G^*) \times T(G^*) & \xrightarrow{\mu_{G^*}} & T(G^*)
\end{array}
\]

commutes. In other words, the rotor-routing action is compatible with plane duality.

In the rest of this section, we prove Theorem 3.1. We begin with a topological definition of the angle between two spanning trees; this definition applies to all ribbon graphs, not just planar ones, and is the key idea in our proof of Theorem 3.1.

Suppose $G$ is any ribbon graph, and let $e$ and $e'$ be directed edges emanating from a vertex $u$. Suppose that in the cyclic order starting from $e = e_0$, the edges between $e$ and $e'$ are $e_0, e_1, \ldots, e_k$ where $e_k = e'$, all directed outward from $u$. Define the angle between $e$ and $e'$ at $u$ by

\[
\angle_u(e, e') = \sum_{i=1}^{k} \partial e_i \in S(G).
\]

Recall that $\partial$ denotes the boundary map sending a directed edge $e$ to the element $[e^+ - e^-] \in S(G)$. Note that the sum includes $e'$ but not $e$.

**Lemma 3.2.** Suppose $G$ is a planar ribbon graph, and let $e_0, \ldots, e_k$ be consecutive outgoing edges from some vertex $u$ in the cyclic order at $u$. For $i = 0, \ldots, k$, let $r_i$ be the face of $G$, equivalently the vertex of $G^*$, lying to the right of $e_i$ (with respect to the cyclic order at $u$). Then

\[
\phi(\angle_u(e_0, e_k)) = [r_0 - r_k] \in S(G^*).
\]

**Proof.** By definition, $\phi(\partial e_i) = [r_{i-1} - r_i]$. By linearity, it follows that $\phi(\angle_u(e_0, e_k))$ is the telescoping sum $\left[ (r_0 - r_1) + (r_1 - r_2) + \cdots + (r_{k-1} - r_k) \right]$, proving the claim. \qed
Definition 3.3. Let $G$ be an arbitrary ribbon graph, and let $T$ and $T'$ be two spanning trees of $G$. Let $v \in V$ be any vertex. As in §2.3 let $\rho_T$ and $\rho_{T'}$ be the rotor configurations based at $v$ arising from orienting $T$ and $T'$ towards $v$.

The angle between $T$ and $T'$ based at $v$, denoted $\angle_v(T, T')$, is the sum of the angles between their edges at each non-root vertex. That is,

$$\angle_v(T, T') := \sum_{u \in V \setminus \{v\}} \angle^u(\rho_T(u), \rho_{T'}(u)) \in S(G).$$

Lemma 3.4. Let $G$ be any ribbon graph, and let $T$ be a spanning tree of $G$. For any vertex $v$ and any $[D] \in S(G)$, we have

$$\angle_v(T, [D] \cdot T)) = [\pm D].$$

Here, the rotor-routing action of $[D]$ on $T$ is computed with respect to the basepoint $v$.

Proof. Without loss of generality, we may choose $D$ to be a chip configuration that is nonnegative at vertices other than $v$. Consider the rotor-routing process that calculates $[D] \cdot T$. We will say that the directed edge $(x, y)$ is activated if a chip is sent from vertex $x$ to vertex $y$ during this process. Note that when the chip is fired, the chip configuration on the graph changes by $\partial(x, y) = y - x$. Since at the end of the rotor-routing process there are no chips left on the graph, it follows that

$$[D] + \sum_e \partial e = 0,$$

where the sum is over the multiset of edges that have been activated during the process.

Next, we claim that the angle between $T$ and $[D] \cdot T$ is in fact equal to $\sum_e \partial e$, where the sum is again over the multiset of activated edges. This is because at each vertex $u \neq v$, the sum of the boundaries of all outgoing edges $e$ at $u$ is $0 \in S(G)$; after all, this sum corresponds to a lending move at $u$. So the sum over all activated edges leaving $u$ is exactly the angle at $u$ between the edge of $T$ leaving $u$ and that of $T'$, and the claim follows.
Summarizing, we have

\[ \angle_v(T, [D] \cdot T) = \sum_e \partial e = [-D]. \]

\[ \square \]

**Corollary 3.5.** Let \( G \) be any planar ribbon graph, and let \( T \) and \( T' \) be spanning trees of \( G \) rooted at the same vertex \( v \). Then \( \angle_v(T, T') = 0 \) if and only if \( T = T' \).

**Proof.** Assume that \( \angle_v(T, T') = 0 \), and let \( [D] \in \mathcal{S}(G) \) take \( T \) to \( T' \) under the rotor-routing action with basepoint \( v \). It follows from \([7, \text{Lemma 3.17}]\) that the element \([D]\) exists and is unique. Then by Lemma 3.4 \([D] = 0\), so \( T = T' \). The converse is clear. \[ \square \]

**Remark 3.6.** It follows from Lemma 3.4 and from \([5, \text{Theorem 2}]\) that the notion of angle between trees for \( G \) is independent of the choice of root vertex for the trees if and only if \( G \) is a planar ribbon graph. Indeed, Lemma 3.4 shows that \( \angle_v(T, T') \) is exactly the element of \( \mathcal{S}(G) \) sending \( T' \) to \( T \) in the rotor-routing action based at \( v \), and the rotor-routing action is basepoint-independent if and only if \( G \) is a planar ribbon graph by \([5]\). Thus, if \( G \) is planar, we will henceforth write \( \angle(T, T') \) for the angle between \( T \) and \( T' \), computed with respect to any vertex.

We can now prove our main lemma.

**Lemma 3.7.** Let \( G \) be a planar ribbon graph, and let \( T \) and \( T' \) be spanning trees of \( G \). Then

\[ \phi(\angle(T, T')) = \angle(T^*, T'^*). \]

**Proof.** Given a spanning tree \( T \) and an edge \( e \) not in \( T \), we call the unique cycle \( C(e) \) in \( T \cup \{e\} \) the fundamental cycle of \( e \) with respect to \( T \). We first note that there is a sequence of trees \( T = T_0, T_1, \ldots, T_r = T' \) such that for each \( j \), the trees \( T_{j+1} \) and \( T_j \) have exactly \( n - 1 \) edges in common. If \( T = T' \) this statement is vacuously true. Otherwise, pick \( e' \in T' \setminus T \); then the fundamental cycle of \( e' \) with respect to \( T \) must contain some edge \( e \in T \setminus T' \). Set \( T_1 = T \cup \{e'\} \setminus \{e\} \). Then \( T_1 \) and \( T' \) have smaller symmetric difference, so repeating, we produce a sequence of spanning trees as desired. It follows by induction that we may assume \( T' = T \cup \{e'\} \setminus \{e\} \).

In fact, we may further assume, again by induction, that \( e \) and \( e' \) are edges incident to a common face of \( G \). Indeed, since \( T^* \cup \{e^*\} \setminus \{e'^*\} = T'^* \) is acyclic, the fundamental cycle \( C(e^*) \) of \( e^* \) with respect to \( T^* \) contains \( e'^* \). Now starting at \( e^* \) and proceeding along the cycle \( C(e^*) \) in either direction, let \( e^* = e_0^*, e_1^*, \ldots, e_s^* = e'^* \) be the sequence of edges traversed. Then

\[ T^*, (T^* \cup \{e^*_1\}) \setminus \{e_1^*\}, (T^* \cup \{e^*_2\}) \setminus \{e_2^*\}, \ldots, (T^* \cup \{e^*_s\}) \setminus \{e_s^*\} \]

is a sequence of trees in \( G^* \) such that the symmetric difference of any consecutive pair of trees consists of two edges of \( G^* \) adjacent to the same vertex.
Now passing to $G$, we conclude that
\[
T, \ (T \cup \{e_1\}) \setminus \{e\}, \ (T \cup \{e_2\}) \setminus \{e\}, \ldots, \ (T \cup \{e'\}) \setminus \{e\}
\]
is a sequence of trees in $G$ such that the symmetric difference of any consecutive pair of trees consists of two edges of $G$ incident to the same face.

Thus from here on, we assume that $T' = (T \cup \{e'\}) \setminus \{e\}$, where $e, e' \in E(G)$ are incident to a common face, which we call $f$. Write $e = xy$ and $e' = x'y'$ for vertices $x, y, x', y'$ of $V(G)$, such that $f$ is to the left of the edge $e$ when it is traversed in the direction $x \to y$, and $f$ is to the right of the edge $e'$ when it is traversed in the direction $x' \to y'$. Write $C$ for the fundamental cycle in $T \cup \{e'\}$; it is illustrated in Figure 3. (Here and throughout the rest of the proof, we assume a clockwise orientation on the rotors of $G$ simply in order to talk about the left and right sides of an edge freely. For example, the face to the right of an oriented edge $e = (x, y)$ should be interpreted as the face coming in between $e$ and the edge after $e$ in the cyclic order at $x$.)

By Remark 3.6, the calculation of the angle $\angle(T, T') \in S(G)$ is independent of the choice of root vertex. Choose $x'$ as the root and orient $T$ and $T'$ towards $x'$. We wish to study the sum of the angles at each vertex $v \neq x'$ of $G$ between the edges of $T$ and $T'$ that are outgoing from $v$.

Having rooted the trees at $x'$, we start by observing that the path between $y$ and $y'$ in $T$ is directed from $y'$ to $y$, whereas in $T'$ it has the opposite orientation. This is illustrated in Figure 4. Furthermore, all other edges shared by $T$ and $T'$ have the same orientation. Indeed, consider a vertex $v$ not on $C$ and say its unique path in $T$ to $x'$ first meets $C$ at $v'$; then the same path $v-v'$ in $T'$ must be an initial subpath of the unique path in $T'$ from $v$ to $x'$, so in particular the edge leaving $v$ is unchanged.

Let us fix some notation before going further. Write
\[
y' = y_0, e_1, y_1, e_2, \ldots, y_{m-1} = y
\]
Figure 4. Parts of the trees $T$ and $T'$, rooted at the vertex $x'$.

for the sequence of vertices and directed edges in the $y'-y$ path in $T$. For each directed edge $e_i$, we write $f_i$ (respectively $h_i$) for the face of $G$ to the right (respectively left) of $e_i$.

For convenience, we extend the notation above as follows. We denote by $h_0$ the face of $G$ to the left of $e'$ when oriented from $x'$ to $y'$, and we denote by $h_m$ the face of $G$ to the left of $e$ when oriented from $y$ to $x$. Next, consider the path from $x$ to $x'$ that bounds $f$ and such that $f$ lies on its right. Call the faces on the left side of this $x-x'$ path $h_{m+1}, \ldots, h_N$. See Figure 3.

Letting $e_0 = e'$ and $e_m = e$, the angle between $T$ and $T'$ then is given by

$$\angle(T, T') = \sum_{i=0}^{m-1} \angle_y(e_{i+1}, e_i) \in S(G),$$

where in each expression in the sum, we regard each edge as being oriented away from $y_i$ in turn. Then by Lemma 3.2, we have

$$\phi(\angle(T, T')) = (f_1-h_0)+(f_2-h_1)+\cdots+(f_{m-1}-h_{m-2})+(f-h_{m-1}) \in S(G^*).$$

The angle between $T$ and $T'$ is shown in Figure 5. The signs indicate $\phi(\angle(T, T')) \in S(G^*)$.

Next, consider the oriented cycle $C$ running from $x'$ to $y'$, then along edges of $T$ from $y'$ to $x$, then along edges of $f$ back to $x'$, as shown in Figure 6. The dual $C^*$ of $C$ is a cut of $G^*$, so $\partial_{G^*}(C^*) = 0 \in S(G^*)$. On the other
Figure 5. $\angle(T, T') \in S(G)$ and $\phi(\angle(T, T')) \in S(G^*)$, the former drawn with arrows and the latter drawn with plus and minus signs.

hand,

$$\partial_{G^*}(C^*) = (h_0 - f) + (h_1 - f_1) + \cdots + (h_{m-1} - f_{m-1}) + \sum_{i=m}^{N} (h_i - f).$$

The signs in Figure 6 indicate $\partial_{G^*}(C^*) \in S(G^*)$.

Figure 6. The cycle $C$ in black and $\partial_{G^*}(C^*)$.

Summing, we have

$$\phi(\angle(T, T')) + \partial_{G^*}(C^*) = \sum_{i=m}^{N} (h_i - f).$$

This sum is shown in Figure 7.

But this sum is exactly $\angle(T^*, T'^*)$. To see this, root the trees $T^*$ and $T'^*$ at a vertex $u$ of $G^*$ on the cycle in $T^*\cup\{e^*\}$ but different from $f$, as illustrated
Figure 7. $\phi(\angle(T, T')) + \partial_{G^*}(C^*) \in \mathcal{S}(G^*)$.

in Figure 8. Then the only nonzero vertex angle contributing to $\angle(T^*, T'^*)$ is the angle at the vertex $f$, and by definition, this angle is $\sum_{i=m}^{N} (h_i - f)$, as shown in Figure 9. So we are done.

□

Figure 8. Parts of the trees $T^*$ and $T'^*$, rooted at $u$. 
We now prove our main result.

Proof of Theorem 3.1. Given \([D] \in S(G)\) and \(T \in T(G)\), let \(T' = [D] \cdot T\), and let \(T'' = \phi([D]) \cdot T^*\). We would like to show that \(T'' = T'^*\). By Lemma 3.4,

\[
\phi(\angle(T, T')) = \phi(\angle(D)) = \angle(T^*, T'').
\]

By Lemma 3.7,

\[
\phi(\angle(T, T')) = \angle(T^*, T'^*).
\]

Hence, \(\angle(T^*, T'') = \angle(T^*, T'^*)\). Therefore, \(\angle(T'', T'^*) = 0\), and the result then follows from Corollary 3.5. 

\[\square\]

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