Exponentially localized solutions of the Klein-Gordon equation

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Abstract. Exponentially localized solutions of the Klein-Gordon equation for two and three space variables are presented. The solutions depend on four free parameters. For some relations between the parameters, the solutions describe wave packets filled with oscillations whose amplitudes decrease in the Gaussian way with distance from a point running with group velocity along a straight line. The solutions are constructed using exact complex solutions of the eikonal equation and may be regarded as ray solutions with amplitudes involving one term. It is also shown that the multidimensional nonlinear Klein-Gordon equation can be reduced to an ordinary differential equation with respect to the complex eikonal.

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Introduction. The construction of various highly localized solutions of the wave equation

$$\phi_{tt} - \phi_{xx} - \phi_{yy} - \phi_{zz} = 0$$

is a subject of several publications (see [1-8]). In particular, in [1] a solution that is exponentially localized in the vicinity of a point running with velocity of light is given. In this paper, we generalize the result of [6] by presenting a family of localized solutions that includes the one given in [6] as a special case.

Based on the results obtained for the wave equation, we construct a family of particle-like solutions of the Klein-Gordon equation

$$h^2(u_{tt} - u_{xx} - u_{yy}) + u = 0, \quad h = \text{const.}$$

These solutions have finite energy and describe wave packets with central frequency $\omega$ and wave number $k$, where $\omega^2 = k^2 + 1/h^2$. Their amplitudes decrease exponentially with distance from a point running along a straight line with group velocity $v = d\omega/dk$. By analogy with the solutions of the wave equation, we call them Gaussian wave packets.

All the solutions of the Klein-Gordon equation from this class can be represented in the form

$$u = Af(iS/h),$$

where the function $S$ satisfies the eikonal (or Hamilton-Jacobi) equation

$$S_{t}^2 - S_{x}^2 - S_{y}^2 = 1,$$
the amplitude factor $A$ does not depend on the coordinates, $S$ and $A$ are independent on $h$, and $f$ is expressed in terms of the Hankel function.

The comparison of our results with those available in the literature shows that one of the solutions that we found for the three-dimensional Klein-Gordon equation coincides with the solution given in [7]. We also consider in detail another solution from the constructed family, which depends only on one variable $S$ that is one of the exact complex solutions of the eikonal equation. It turned out that the search for a solution dependent only on $S$ of the nonlinear Klein-Gordon equation is reduced to the solution of an ordinary differential equation. This is also true for nonlinear Klein-Gordon equations with arbitrary number of space variables.

In constructing particle-like solutions of the wave equation, we followed the idea by Ziolkowski [3] and sought such solutions in the form of a superposition of Gaussian beams, which are solutions localized near rays. The latter solutions were found for the first time in papers of Brittingham [1] and Kiselev [2] and belong to the class of relatively undistorted waves, in the terminology of Courant and Hilbert [10]. The construction of the solution of the Klein-Gordon equation employs the simple observation that, taking the Fourier transform of a solution of wave equation (1), say with respect to $z$,

$$u(x, y, t) = \int_{-\infty}^{\infty} dz \phi(x, y, z, t) e^{iz/h},$$

we obtain a solution of the Klein-Gordon equation (2).

**Generalized Gaussian packets for the wave equation.** We start from the solution of wave equation (1), described by the formula

$$\phi_b(x, y, z, t, q) = \exp \{iq\Theta_1\} / ((\beta - i\varepsilon_1)^{1/2}(\beta - i\varepsilon_2)^{1/2})$$

(see [1, 2, 6]), where the notation

$$\Theta_1 = x - t + \frac{y^2}{\beta - i\varepsilon_1} + \frac{z^2}{\beta - i\varepsilon_2}, \quad \beta = x + t,$$

is introduced. The function $\phi_b$ satisfies (1) for any $q$, $\varepsilon_1$, and $\varepsilon_2$, and in the case of $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and $q > 0$ it is a Gaussian beam, which means that it is localized in the Gaussian way in the vicinity of the $x$ axis. We seek particle-like solutions of Eq. (1) in the form of a superposition of Gaussian beams:

$$\phi^{(\nu)}_p(x, y, z, t) = \int_0^\infty dq F^{(\nu)}(q) \phi_b(x, y, z, t, q),$$

(7)

where $F^{(\nu)}(q)$ is a particular function depending on the parameter $\nu$. We put

$$F^{(\nu)}(q) \equiv a q^{-\nu-1} e^{-q^2/(\beta - i\varepsilon_1)^{1/2}}$$

(8)

where $\nu$, $\sigma$, and $\varepsilon$ are arbitrary constants, $\sigma > 0$, $\varepsilon > 0$, and $a = (4\varepsilon\sigma^2)^{\nu}/(2\sqrt{\pi})$. It can easily be shown that (7) is reduced to an integral representation of the Hankel function $H^{(1)}_\nu$ of the first kind [12] and

$$\phi^{(\nu)}_p(x, y, z, t) = C \frac{s^\nu H^{(1)}_\nu(s)}{\sqrt{(\beta - i\varepsilon_1)(\beta - i\varepsilon_2)}}, s = 2i\sigma \varepsilon \left(1 - i\Theta_1 / \varepsilon\right)^{1/2},$$

(9)
where \( C = i2^{\nu-1}\sqrt{\pi} \). It is worth noting that \( s \) satisfies the Hamilton-Jacobi equation 
\[ s_t^2 = s_x^2 + s_y^2 + s_z^2 \] 
for wave equation (1). We note that for \( \nu = 1/2 \), formula (9) yields a solution of the wave equation presented earlier in [6]: 
\[ \phi(1/2)_p(x, y, z, t) = \exp \left\{ -2\sigma \varepsilon \sqrt{1 - i\Theta_1/\varepsilon} \right\} \]
\[ \sqrt{(\beta - i\varepsilon_1)(\beta - i\varepsilon_2)} \]
This solution depends on four free parameters \( \varepsilon, \varepsilon_1, \varepsilon_2 \), and \( \sigma \). It is established in [6] that if all these parameters are positive, it is localized in the Gaussian way near the point \( x = y = 0 \) and \( z = ct \) that runs with velocity of light \( c = 1 \) along the \( x \) axis. The asymptotics of the solutions of (1) of the form (9) with respect to the large argument have the same exponential factor as (10). Arguments similar to those adduced in [6] prove their localization. Therefore, (9) is a localized solution generalizing (10).

**Gaussian beams for the Klein-Gordon equation.** Here we give a solution \( u_b \) of the Klein-Gordon equation, which has a Gaussian localization near a ray, in order to use it in construction of particle-like solutions of the Klein-Gordon equation. To find such a \( u_b \), we calculate the Fourier transform (4) with respect to \( z \) of expression (5):
\[ u_b(x, y, t; q) = \sqrt{\pi e^{i\pi/4 - \varepsilon_2/(4qh^2)}} \exp \left\{ i\Theta q - i\beta/(4qh^2) \right\} \sqrt{\beta - i\varepsilon_1}, \]
where
\[ \Theta = x - t + \frac{y^2}{\beta - i\varepsilon_1}. \]
This solution was found first in [8]. We call such solutions of the Klein-Gordon equation *Gaussian beams*, by analogy with solutions of wave equations.

**Gaussian packets for the Klein-Gordon equation.** Taking the Fourier transform (4) with respect to \( z \) of both sides of Eq. (7), we obtain solutions of the Klein-Gordon equation in the form of an expansion in the beam solutions \( u_b \) from (11):
\[ u^{(\nu)}_p(x, y, t) = \int_0^\infty dq F^{(\nu)}(q) u_b(x, y, t; q). \]
Substituting \( u_b \) of the form (11) and \( F^{(\nu)}(q) \) of the form (8) into (13), we arrive at
\[ u^{(\nu)}_p(x, y, t) = C_1 \frac{S_p^{(\nu+1/2)} H^{(1)}_{\nu+1/2}(S_p/h)}{(\beta - i\varepsilon_1)^{1/2}(\beta - 4i\varepsilon\sigma^2h^2 - i\varepsilon_2)^{\nu+1/2}} \]
with
\[ S_p = i \left[ (\Theta + i\varepsilon)(\beta - 4i\varepsilon\sigma^2h^2 - i\varepsilon_2) \right]^{1/2}. \]
The function \( S_p \) does not depend on \( \nu \). Formula (14) yields a family of exact solutions of the Klein-Gordon equation, which are all particle-like as will be seen in what follows. This family depends on four free parameters \( \nu, \varepsilon, \varepsilon_1, \) and \( \sigma \). In what follows we use a different and simpler parametrization of (14). The constant factor in (14) is equal to \( C_1 = \pi (8\varepsilon\sigma^2h^2 + 2\varepsilon_2)^{\nu-1/2}h^{\nu+3/2}e^{i\pi(\nu+1)/\sqrt{2}}. \)
The ray interpretation of Gaussian beams and Gaussian packets. The introduction of a new free parameter $\kappa$ by the relation $\kappa = qh$ enables us to rewrite (11) in the form of a ray series, which is reduced to only one term

$$u_b = c_b \exp \left\{ iS_b/h \right\}, \quad c_b = \text{const},$$

with

$$S_b = \kappa \Theta - \frac{\beta}{4\kappa}. \quad (17)$$

Here, the complex phase function $S_b$ is independent of $h$ and satisfies eikonal equation (3). We also note that $u_b$ may be regarded as a reference solution for the asymptotic construction presented by Maslov in [11].

The solutions $u_p^{(\nu)}$ can conveniently be written in terms of a new free parameter $\gamma = 4\varepsilon_2^2 h^2 + \varepsilon_2$ instead of $\sigma$ and $\varepsilon_2$ and $\mu = \nu + 1/2$ instead of $\nu$. Now,

$$u_p^{(\mu - 1/2)} = C_1 \frac{S_p^\mu H_\mu(1)(S_p/h)}{(\beta - i\varepsilon_1)^{1/2}(\beta - i\gamma)^\mu}.$$  

(18)

The complex phase function

$$S_p = i \left[ (\Theta + i\varepsilon)(\beta - i\gamma) \right]^{1/2} \quad (19)$$

is also independent of $h$ and satisfies eikonal equation (3). In the case of a half-integer $\mu$, the Hankel functions are reduced to elementary functions. For the values $\mu = 1/2$ and $\mu = -1/2$, we have

$$u_p^{(0)} = C_2 \frac{\exp \left\{ iS_p/h \right\}}{(\beta - i\varepsilon_1)^{1/2}(\beta - i\gamma)^{1/2}}$$

and

$$u_p^{(-1)} = C_3 \frac{\exp \left\{ iS_p/h \right\}}{S_p} \sqrt{\frac{\beta - i\gamma}{\beta - i\varepsilon_1}},$$

(20)

respectively, where $C_2$ and $C_3$ are constants. These solutions, as well as (16), are ray solutions reduced to their zero-order term.

**Properties of the solutions.** First we show that the solution $u_b$ in (16) is localized near the $x$ axis. Separating the real and imaginary parts in the complex phase function $S_b$ from (17) and introducing the notation

$$\tilde{\kappa} = \left( \kappa - \frac{1}{4\kappa} \right), \quad \tilde{\omega} = \sqrt{\tilde{\kappa}^2 + 1}, \quad \Delta_y = \sqrt{\frac{2\varepsilon_1^2 + \beta^2}{\varepsilon_1(\tilde{\kappa} + \tilde{\omega})}}, \quad (21)$$

we obtain

$$iS_b = i(\tilde{\kappa} x - \tilde{\omega} t) - \frac{y^2}{\Delta_y^2} + i \frac{y^2}{\Delta_y^2} \cdot \frac{\beta}{\varepsilon_1}.$$  

(22)

For $|\beta| \ll \varepsilon_1$, the solution $u_b$ describes a wave with frequency $\omega = \tilde{\omega}/h$ and wave number $k = \tilde{\kappa}/h$, which propagates along the $x$ axis and decays in the Gaussian way with distance from the axis. The degree of localization near the $x$ axis is determined by
the parameter $h \Delta_y$. The solution shows the Gaussian localization near the $x$ axis for any finite values of $\beta$, and the degree of localization decreases as $\beta$ grows.

Now we describe the asymptotic behavior of $u_p^{(\mu-1/2)}$. As has already been mentioned, all the functions $S_p$ are independent of $\mu$. The asymptotic expressions of the Hankel function for large values of $|S_p|/h$ differ only in the phase factor, and all solutions of our family behave as

$$u_p^{(\mu-1/2)}(x, y, t) \sim C_3 \frac{S_p^{\mu-1/2} \exp \left\{ iS_p/h \right\}}{(\beta - i\epsilon_1)^{1/2}(\beta - i\gamma)^\mu} \quad (23)$$

with $C_3 = C_1 \sqrt{2/\pi} e^{-\pi \mu/2 - \pi i/4}$.

Now we investigate the dependence of $S_p$ on the space variables and time. Separating the real and imaginary parts in the subradical expression from (19), we obtain

$$iS_p = - \left[ \gamma \epsilon + x^2 - t^2 + y^2 \frac{\beta^2 + \gamma \epsilon_1}{\beta^2 + \epsilon_1^2} + 2i \left( \sqrt{\gamma \epsilon}(\tilde{\omega} t - \tilde{\kappa} x) + \frac{y^2 \beta (\epsilon_1 - \gamma)}{\beta^2 + \epsilon_1^2} \right) \right]^{1/2}, \quad (24)$$

where $\tilde{\kappa}$ and $\tilde{\omega}$ are defined by (21) with $\kappa = \sqrt{\gamma}/4 \epsilon$. We now prove a rough estimate valid for any value of $t$, which implies that any solution from the family described by (14) has finite energy. First we note the simple fact that for real $a_1$ and $a_2$, $\text{Re} \sqrt{a_1 + ia_2} = \left( (a_1 + \sqrt{a_1^2 + a_2^2})/2 \right)^{1/2}$, whence

$$\text{Re} \sqrt{a_1 + ia_2} \geq a_1 \text{ for } a_1 \geq 0$$

and thus

$$|\text{Re}(iS_p)| \geq \left[ \gamma \epsilon + x^2 - t^2 + y^2 \frac{\beta^2 + \gamma \epsilon_1}{\beta^2 + \epsilon_1^2} \right]^{1/2}. \quad (25)$$

Estimate (25) shows that for a fixed time and sufficiently large values of the space coordinates, the asymptotic expression for the Hankel functions is applicable and the behavior of solution (18) is described by formula (23). Therefore, the absolute value of any solution $|u_p^{(\mu-1/2)}|$ decreases exponentially with a rise in the coordinates, and therefore all these solutions have finite energy. This estimate is valid for all values of the space coordinates in case of $|t| \leq \gamma \epsilon$.

Now we prove that, under certain restrictions on $x$, $y$, and $t$ and some conditions on their parameters, the solutions $u_p^{(\mu-1/2)}$ describe wave packets localized in the Gaussian way. We expand $iS_p$ in a series, assuming that all subradical terms dependent on $x$, $y$, and $t$ are small in comparison with $\epsilon \gamma$. Keeping only the terms linear and quadratic in $x$, $y$, and $t$, we find

$$iS_p \sim i(\tilde{\kappa} x - \tilde{\omega} t) - \sqrt{\gamma \epsilon} - \frac{(x - vt)^2}{\Delta_x^2} - \frac{y^2}{\Delta_y^2}, \quad (26)$$

where $v$ stands for the group velocity $v = d\tilde{\omega}/d\tilde{\kappa} = \tilde{\kappa}/\tilde{\omega}$ and

$$\Delta_x = \frac{\sqrt{2\kappa \epsilon}}{\tilde{\omega}}, \quad \Delta_y = \sqrt{\frac{\epsilon_1}{\kappa}}, \quad (27)$$
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Figure 1. Real part of $u_p(0)$ in conventional units: $t = 0$ (left), $t = 10$ (right).

We recall that $\kappa = \sqrt{\gamma}/4\varepsilon$.

Under the condition $\sqrt{\gamma\varepsilon} = 2\kappa\varepsilon \gg h$, the asymptotic form of the Hankel function can be used in (18) for all values of $x$, $y$, and $t$. Formula (26) can be applied for not-too-large $x$, $y$, and $t$. However, for some values of the parameters $\kappa$, $\varepsilon$, and $\varepsilon_1$, the domain of its applicability can be larger than the widths $\sqrt{h}\Delta_x$ and $\sqrt{h}\Delta_y$ of the packet. In this domain, all the solutions from the family $u_p^{(\mu-1/2)}$ demonstrate Gaussian localization near a point running along the $x$ axis with group velocity $v$. Such a behavior may occur for a certain relation between all the parameters, which we write below in an interesting special case.

On the Fig. 1. the real part of $u_p(0)$ given by (20) is presented in conventional units: $t = 0$ (left), $t = 10$ (right); in both cases, $h = 0.3$, $\varepsilon = 10$, and $\kappa = 1$.

A simple example of a Gaussian packet. An interesting solution of the Klein-Gordon equation (2) is obtained from (1) by setting $\varepsilon_1 = \gamma \equiv 4\kappa^2\varepsilon$. The solution $u_p^{(-1)}$ depends only on one variable $S_p$, and, up to a constant factor, we have

$$u_p^{(-1)} = C_p \frac{e^{iS_p/h}}{S_p}. \quad (28)$$

In this case, the phase $S_p$ is given by

$$S_p = i\sqrt{b^2 + x^2 + y^2 - t^2 + 2ib(\tilde{\omega}t - \tilde{\kappa}x)}$$

$$= i\sqrt{(x - ib\tilde{\kappa})^2 + y^2 - (t - ib\tilde{\omega})^2} \quad (29)$$

with $b = \sqrt{\gamma\varepsilon} \equiv 2\kappa\varepsilon$. Here, $\tilde{\kappa}$ can be expressed in terms of $\kappa$, see (21). However, it is more convenient to regard $\tilde{\kappa}$ and $b$ as independent parameters. Then, $\tilde{\omega} = \sqrt{\tilde{\kappa}^2 + 1}$.

The expansion of $S_p$ takes the form (26) with $\Delta_x = \sqrt{b}/\tilde{\omega}$ and $\Delta_y = \sqrt{2b}$. We describe the conditions for its validity. The expansion of (29) is justified for $x \ll b$, $y \ll b$, $t \ll b/\tilde{\omega}$, and $x \ll b/\tilde{\kappa}$. Since $\tilde{\omega} \geq 1$, the conditions $t \ll b/\tilde{\omega}$ and $x \ll \min(b, b/\tilde{\kappa})$ are more restrictive. It can be seen from (29) that the solution is localized in the Gaussian way near a point running with group velocity if the transverse and longitudinal widths $\sqrt{h}\Delta_x$ and $\sqrt{h}\Delta_y$, as well as the time width $\sqrt{h}\Delta_x/v$, are all far less that the domain where the root in (29) can be expanded, namely, for
\[ \sqrt{\hbar \bar{\omega}} \ll \min(b, b/\bar{\kappa}), \sqrt{2\hbar b} \ll b, \text{ and } \sqrt{\hbar b/\bar{\kappa}} \ll b/\bar{\omega}. \] Here, the most restrictive condition is
\[ h \ll b \frac{\bar{\kappa}^2}{\bar{\omega}^2}. \] (30)

**The case of three space variables.** All of the above considerations can be generalized to three or more space variables. Here we present the results for the three-dimensional case and compare them with the results found in [7]. We turn to the equation
\[ h^2(u_{tt} - u_{xx} - u_{yy} - u_{zz}) + u = 0, \quad h = \text{const}. \] (31)

The construction of its localized solutions is based on the expressions for localized solutions of the four dimensional wave equation
\[ \phi_{tt} - \phi_{xx} - \phi_{yy} - \phi_{zz} - \phi_{z1z1} = 0. \] (32)

Axially symmetric Gaussian beams for this equation are described by the expression
\[ \phi_b(x, y, z, z_1, t, q) = \exp\left\{iq\Theta_1\right\} \left(\beta - i\varepsilon_1\right) \left(\beta - i\varepsilon_2\right)^{1/2}. \] (33)

In contrast to (6), here
\[ \Theta_1 = x - t + \frac{y^2}{\beta - i\varepsilon_1} + \frac{z^2}{\beta - i\varepsilon_1} + \frac{z_1^2}{\beta - i\varepsilon_2}, \quad \beta = x + t. \] (34)

We construct generalized Gaussian packets for Eq. (32) by a formula similar to (7), starting from the Gaussian beams and using weight function (8) in the same way as we did this previously. Then we take the Fourier transform
\[ u(x, y, z, t) = \int_{-\infty}^{\infty} dz_1 \phi(x, y, z, z_1, t)e^{iz_1/h} \] (35)

of the both sides of a formula analogous to (7). In doing so, we find solutions of the Klein-Gordon equation (31) describing the Gaussian particles
\[ u_{p}^{(\mu-1/2)} = C_4 \frac{S_p^\mu H^{(1)}_\mu(S_p/h)}{\left(\beta - i\varepsilon_1\right)\left(\beta - i\gamma\right)\mu}, \] (36)

where \( C_4 = \text{const} \) and \( S_p \) is defined by (19) in which
\[ \Theta = x - t + \frac{y^2}{\beta - i\varepsilon_1} + \frac{z^2}{\beta - i\varepsilon_1}. \] (37)

The Gaussian particles \( u_{p}^{(\mu-1/2)} \) in the 3D case differ from the corresponding solutions in the 2D case (18) by a power of \( \beta - i\varepsilon_1 \) in the denominator and by the additional term involving the variable \( z \) in phase (37). The solution
\[ u_{p}^{(-1)} = C_5 \frac{(\beta - i\gamma)^{1/2}}{(\beta - i\varepsilon_1)S_p} \exp\left\{iS_p/h\right\} = C_5 \frac{\exp\left\{iS_p/h\right\}}{(\beta - i\varepsilon_1)(\Theta + i\varepsilon)^{1/2}} \] (38)

with \( C_5 = \text{const} \) was found earlier in [7]. This is a Gaussian particle from (36) for \( \mu = -1/2 \). In the three-dimensional case, as well as in the two-dimensional case, a
solution dependent only on the variables $S_p$ exists. This is $u_p^{-3/2}$, where we must set $\varepsilon_1 = \gamma$:

$$u_p^{-3/2} = C_6 \frac{H_1^{(1)}(iS_p/h)}{S_p}, \quad (39)$$

where $C_6 = \text{const}$.

**On nonlinear Klein-Gordon equations.** The above construction can be used in solving nonlinear Klein-Gordon equations

$$h^2(u_{tt} - u_{xx} - u_{yy}) + \phi(u) = 0 \quad \text{(40)}$$

in the two-dimensional case and

$$h^2(u_{tt} - u_{xx} - u_{yy} - u_{zz}) + \phi(u) = 0 \quad \text{(41)}$$

in the three-dimensional case. Seeking solutions in the form $u = F(s)$, where $s = S_p/h$, we arrive at nonlinear differential equations

$$F_{ss} + \frac{2}{s}F_s + \phi(F) = 0 \quad \text{(42)}$$

and

$$F_{ss} + \frac{3}{s}F_s + \phi(F) = 0 \quad \text{(43)}$$

in the case of (40) and (41), respectively.

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