A Topological Field Theory with a Finite Number of Connected Feynman Diagrams

Franco Ferrari

Institute of Physics, University of Szczecin, ul. Wielkopolska 15, 70-451 Szczecin, Poland

November 1, 2018

Abstract

A new topological field theory is constructed, which is characterized by cubic interactions similar to those of non-abelian Chern-Simons field theories, but still retains the simplicity of the abelian case. The perturbative expansion of this theory contains in fact only two connected Feynman diagrams, the propagator and a three vertex. Apart from the Gauss linking number, the Wilson loop amplitudes generate a further topological invariant, whose physical and mathematical meaning is investigated.
1 Foreword

In several situations it has been experimentally observed that the topological properties of certain physical systems may influence their behavior to a relevant extent. This is for instance the case of vortex structures in nematic liquid crystals [1] and in $^3$He superfluids [2]. Other examples are provided by polymers [3] or by the lowest lying excitations of two-dimensional electron gases, which have topological non-trivial configurations at some filling fractions [4]. In the investigation of phenomena related to the presence of topological constraints in the system, the use of quantum or statistical mechanical models coupled to abelian Chern–Simons (C-S) field theories [5] has been particularly successful. One reason of this success is the fact that abelian models do not require a complex mathematical treatment as their non-abelian counterparts and thus their physical meaning is more transparent.

Motivated by possible applications in physics, the aim of this work is the construction of a topological field theory with non-trivial cubic interactions similar to those of non-abelian C-S field theories, but which still retains the simplicity of the abelian case. As a result of this effort an exactly solvable topological field theory is obtained, called hereafter truncated topological field theory or briefly TTFT, which contains only two connected Feynman diagrams in its perturbative expansion. The name of the theory is owing to the fact that any further expansion of the perturbative series, which could in principle generate new diagrams, has been truncated by the introduction of suitable constraints. From the computation of the Wilson loop amplitudes it turns out that, apart from the Gauss linking number that is already present in the abelian C-S field theory, the TTFT delivers a further topological invariant, which can be interpreted as an Hopf term.

The material presented in this paper is divided as follows. A naive topological field theory consisting of three BF–models [6] coupled together by cubic interaction terms is investigated in Section 2. Since this theory is topological, it is convenient to choose as its observables the so-called Wilson loops. Unfortunately, after the insertion of the Wilson loops in the partition function in order to compute their amplitudes, one observes that the constraints generated by the longitudinal components of the fields become inconsistent with the rest of the equations of motion. This problem is solved in Section 3 by enlarging the gauge group of the naive model via the addition of suitable topological terms to its action. In this way a well defined TTFT is obtained, in which the longitudinal components of the fields are harmless, because they correspond to pure gauge field configurations and are thus irrelevant. In Section 4 the Wilson loops amplitudes of the TTFT are computed in the Lorentz gauge, showing that they contain a single topological invariant apart from the Gauss linking number. The physical and mathematical meaning of this invariant is investigated. Finally, the Conclusions and a possible extension of the TTFT are presented in Section 5.
2 Problems with Cubic Interactions in Abelian BF–Models

In the quest for a topological field theory which generates only a finite number of topological invariants, it is natural to start from the naive action:

\[ S = \int d^3 x \left[ \frac{\kappa}{4\pi} \tilde{\Omega}^i \cdot (\nabla \times \Omega^j) + \Lambda \Omega^1 \cdot (\Omega^2 \times \Omega^3) + J_i \cdot \tilde{\Omega}^i \right] \quad (1) \]

For simplicity, \( S \) has been defined here on a three dimensional Euclidean space. In Eq. (1) \( \kappa \) and \( \Lambda \) denote real coupling constants, while \( \tilde{\Omega}^i \) and \( \Omega^i, i = 1, 2, 3 \), form a set of six abelian vector fields. The \( J_i \)'s are assumed to be conserved external currents, i.e. such that \( \nabla \cdot J_i = 0 \). Analogous sources coupled to the fields \( \Omega^i \) have been omitted for a reason which will be clear below. Summation over repeated indices is everywhere understood. The action \( S \) describes a BF–model with the addition of a cubic interaction term. In components, Eq. (1) becomes

\[ S = \int d^3 x \left[ \epsilon^{\mu\nu\rho} \left( \frac{\kappa}{4\pi} \tilde{\Omega}_\mu^i \partial_\nu \Omega_\rho^i + \Lambda \Omega_\mu^1 \Omega^2_\mu \Omega_\rho^3 \right) + J^\mu_i \tilde{\Omega}_\mu^i \right] \quad (2) \]

where, as a convention, greek letters label space indices, while roman letters distinguish different vector fields. Finally, \( \epsilon^{\mu\nu\rho} \) is the Levi-Civita tensor density defined so that \( \epsilon^{123} = 1 \).

Let us note that the fields \( \tilde{\Omega}^i \) play in (1) the role of pure Lagrange multipliers, which constrain the fields \( \Omega^i \) and neutralize possible radiative corrections. Moreover, at the classical level there are only two connected Feynman diagrams, which are shown in Fig. 1. They correspond to the field propagators and to the three-vertex associated to the cubic interaction term present in Eq. (1). Higher order tree diagrams, which could in principle be generated by contracting together the legs of many three-vertices, are actually ruled out due to the off-diagonal structure of the propagators, which forbids any self-interaction among the fields \( \Omega^i \). For the same reason, the source term \( \int d^3 x J_i \cdot \Omega^i \) for the fields \( \tilde{\Omega}^i \), where the \( J_i \)'s, \( i = 1, 2, 3 \), are conserved external currents, has been omitted from Eq. (1). As a matter of fact, the addition of such term would not
change the dynamics of the fields $\Omega^i$ and, besides, it is easy to see that it can be eliminated by a shift of the Lagrange multipliers $\tilde{\Omega}^i$.

Clearly, the action $S$ is metric independent and its topological properties are not spoiled by quantum corrections, since the latter vanish identically as previously remarked. Thus, we are in presence of a topological field theory, which is also invariant under the following abelian gauge transformations

$$\tilde{\Omega}^i(x) \rightarrow \tilde{\Omega}^i(x) + \nabla x^i(x)$$  \hspace{1cm} (3)

As a consequence, one may choose as observables metric independent and gauge invariant operators like the Wilson loops:

$$W_i(C) = \exp \left[ i \oint_{\Gamma} dx \cdot \Omega^i \right]$$  \hspace{1cm} (4)

$\Gamma$ is defined here as a superposition of closed non-intersecting paths $\gamma, \gamma', \gamma''$, ..., i. e. $\Gamma = \gamma + \gamma' + \gamma'' + \ldots$, so that a generic correlation function of Wilson loops is given by

$$\langle W_1(\Gamma_1)W_2(\Gamma_2)W_3(\Gamma_3) \rangle = \int \prod_{i=1}^3 D\Omega^i D\tilde{\Omega}^i e^{-iS}$$  \hspace{1cm} (5)

In writing the above equation, the insertion of Wilson loops has been taken into account by making the special choice of external currents:

$$J_i^\mu = \oint_{\Gamma_i} dx_i^\mu \delta(x - x_i)$$  \hspace{1cm} (6)

$\delta(x)$ being the Dirac $\delta-$function.

Unfortunately, any attempt to compute the amplitude (5) runs into troubles due to the longitudinal components of the fields, whose role has not been discussed so far. In the case of the $\tilde{\Omega}^i$-fields, it is possible to get rid of them by fixing the gauge in such a way that:

$$\nabla \cdot \tilde{\Omega}^i = 0$$  \hspace{1cm} (7)

Yet, the undamped longitudinal components of the fields $\Omega^i$ remain in the cubic interaction term of Eq. (1) and introduce new constraints which, without any treatment, lead to inconsistencies in the theory. To see how the problem arises, we investigate the classical equations of motion associated to the action (1). A variation of $S$ with respect to the fields $\tilde{\Omega}_\mu^i$ produces the constraints:

$$\frac{\kappa}{4\pi} \nabla \times \Omega^i + J_i = 0$$  \hspace{1cm} (8)

\footnote{Actually, we will see later that, due to the simplicity of the theory under consideration, the only relevant correlation function of Wilson loops occurs when $\Gamma_i = \gamma_i$, $i = 1, 2, 3$.}
An analogous variation with respect to the fields $\Omega^i_{\mu}$ yields as a result the following relations:

$$\frac{\kappa}{4\pi} \nabla \times \tilde{\Omega}^1 + \Lambda \Omega^2 \times \Omega^3 = 0 \quad (9)$$

$$\frac{\kappa}{4\pi} \nabla \times \tilde{\Omega}^2 - \Lambda \Omega^1 \times \Omega^3 = 0 \quad (10)$$

$$\frac{\kappa}{4\pi} \nabla \times \tilde{\Omega}^3 + \Lambda \Omega^1 \times \Omega^2 = 0 \quad (11)$$

It is easy to check that the general solutions of the constraints (8) are:

$$\Omega^i_{\mu} = b^i_{\mu} + \partial_{\mu} \omega^i \quad (12)$$

In the above equation we have put

$$b^i_{\mu}(x) = \frac{1}{\kappa} \epsilon_{\mu\alpha\beta} \int d^3y \frac{(x-y)^\alpha}{|x-y|^3} J^\beta_1(y) = \frac{1}{\kappa} \epsilon_{\mu\alpha\beta} \oint_{\Gamma_i} dx^\beta \frac{(x-x_i)^\alpha}{|x-x_i|^3} \quad (13)$$

while the $\omega^i(x)$ represent differentiable functions, which take into account the longitudinal components of the vectors $\Omega^i_{\mu}(x)$. The form of the $\omega^i(x)$'s cannot be determined from Eqs. (8). At this point, it is possible to solve also Eqs. (9–11) exactly with respect to the $\tilde{\Omega}^i$. However, this is not the end of the story, because there are further constraints which can be obtained by applying the differential operator $\partial_{\mu}$ to Eqs. (9–11). Exploiting Eqs. (8) in order to evaluate the curls of the $\Omega^i$–fields, one finds the following relations:

$$\Omega^1 \cdot J_2 - \Omega^2 \cdot J_1 = 0 \quad (14)$$

$$\Omega^2 \cdot J_3 - \Omega^3 \cdot J_2 = 0 \quad (15)$$

$$\Omega^3 \cdot J_1 - \Omega^1 \cdot J_3 = 0 \quad (16)$$

The solutions of Eqs. (14–16):

$$\Omega^i = J_i \quad (17)$$

are inconsistent with Eqs. (8) if $J_i \neq 0$. In fact, since the currents $J_i$ are conserved by assumption, Eqs. (13) and (17) give two different and clearly incompatible expressions for the transverse components of the $\Omega^i$ fields.

In the next Section it will be shown how to solve this problem.

### 3 Solving The Problems: The Truncated Topological Field Theory

We have seen in the previous Section that the abelian BF–model with a cubic interaction term defined in Eq. (1) is inconsistent if the external currents $J_i$ are different from zero. The difficulties come from the undamped longitudinal components of the $\Omega^i_{\mu}$-fields. In fact, these components are responsible for the constraints (14–16), which are incompatible with the other classical equations of motion of the theory. A possible strategy to overcome this problem is to add
suitable terms to the action $S$, so that the new action is invariant with respect to gauge transformations of the kind:

$$\Omega^i(x) \rightarrow \Omega^i(x) + \nabla \lambda^i(x)$$  \hspace{1cm} (18)

The idea behind this strategy is that, once gauge invariance is established, the longitudinal components of the $\Omega^i$'s become irrelevant. For instance, they may be easily eliminated choosing a gauge condition in which the fields are purely transverse.

To start with, we compute first of all the variations $\delta \lambda_i S$ of the action $S$ under the gauge transformations (18). After a few calculations one finds:

$$\delta \lambda_1 S = \Lambda \int d^3x \nabla \lambda^1 \cdot (\Omega^2 \times \Omega^3)$$ \hspace{1cm} (19)

$$\delta \lambda_2 S = -\Lambda \int d^3x \nabla \lambda^2 \cdot (\Omega^1 \times \Omega^3)$$ \hspace{1cm} (20)

$$\delta \lambda_3 S = \Lambda \int d^3x \nabla \lambda^3 \cdot (\Omega^1 \times \Omega^2)$$ \hspace{1cm} (21)

In deriving the above equations it has not been taken into account the fact that there is no real dynamics in our theory and, for this reason, the expression of the variations $\delta \lambda_i S$ is unnecessarily complicated. Indeed, in the right hand sides of Eqs. (19–21) there is a linear dependence on the fields $\Omega^i$ which is hidden. To show that, we remember that the transverse components of the $\Omega^i$'s are bounded to live in the subspace of all classical field configurations determined by the constraints (8) and coincide with the vectors $b^i$ given in Eq. (13). Exploiting these constraints, it is possible to rewrite the variations $\delta \lambda_i S$ in the form:

$$\delta \lambda_1 S = \frac{4\pi \Lambda}{\kappa} \int d^3x \lambda^1 \left[ J_2 \cdot \Omega^3_c - J_3 \cdot \Omega^2_c \right]$$ \hspace{1cm} (22)

$$\delta \lambda_2 S = \frac{4\pi \Lambda}{\kappa} \int d^3x \lambda^2 \left[ J_3 \cdot \Omega^1_c - J_1 \cdot \Omega^3_c \right]$$ \hspace{1cm} (23)

$$\delta \lambda_3 S = \frac{4\pi \Lambda}{\kappa} \int d^3x \lambda^3 \left[ J_1 \cdot \Omega^2_c - J_2 \cdot \Omega^1_c \right]$$ \hspace{1cm} (24)

Here the symbols $\Omega^i_c$ have been introduced to remember that in Eqs. (22–24) the transverse degrees of freedom of the fields $\Omega^i$ have been fixed by means of Eq. (8).

At this point we denote with the symbol $S$ the gauge invariant extension of the action $S$ and we try for it the ansatz:

$$S = S + S_1^b + S_2^b$$ \hspace{1cm} (25)

where $S_1^b$ and $S_2^b$ contain respectively terms which are linear and quadratic in the fields $\Omega^i$:

$$S_1^b = \frac{\Lambda}{\kappa} \int d^3x \Omega^1_{\mu}(x) \left[ \int_{\Gamma_3} dx_3 (x - x_3)^\mu b^3_{\mu}(x_3) - \int_{\Gamma_2} dx_2 (x - x_2)^\mu b^2_{\mu}(x_2) \right]$$
To verify the validity of Eq. (28), it is sufficient to compute the variations \( \delta S \) of Eqs. (25) if the external \( \mu \) have been written directly for the special case in which the currents \( J_i \) are given by Eq. (18). The generalization to currents of general form is straightforward. The action \( S \) of Eq. (25) defines what we call here truncated topological field theory or TTFT.

It is now possible to check that \( S_b^1 \) and \( S_b^2 \) satisfy the conditions listed below:

i) The variations of \( S_b^1 \) and \( S_b^2 \) under the gauge transformations (18) satisfy the relations:

\[
\delta_{\lambda i} (S_b^1 + S_b^2) = 0
\]

for \( i = 1, 2, 3 \). This condition guarantees the gauge invariance of the action \( S \).

ii) The addition of the counterterms \( S_b^1 \) and \( S_b^2 \) to the action \( S \) does not affect the equations of motion (26) and (27) for what is concerning the transverse components of the fields \( \Omega^i \) and \( \hat{\Omega}^i \).

iii) \( S_b^1 \) and \( S_b^2 \) consists of topological terms, so that the topological properties of the action \( S \) are not spoiled.

To verify the validity of Eq. (28), it is sufficient to compute the variations \( \delta_{\lambda i} (S_b^1 + S_b^2) \). A straightforward calculation yields the following result:

\[
\delta_{\lambda i} (S_b^1 + S_b^2) = \frac{4\pi}{\kappa} \left[ \oint_{\Gamma_2} dx_2 \cdot \lambda_1^i (x_2) \Omega^2_{\mu} (x_2) - \oint_{\Gamma_3} dx_3 \cdot \lambda_1^i (x_3) \Omega^2_{\mu} (x_3) \right] \\
\delta_{\lambda 2} (S_b^1 + S_b^2) = \frac{4\pi}{\kappa} \left[ \oint_{\Gamma_3} dx_3 \cdot \lambda_2^i (x_3) \Omega^1_{\mu} (x_3) - \oint_{\Gamma_1} dx_1 \cdot \lambda_2^i (x_1) \Omega^1_{\mu} (x_1) \right] \\
\delta_{\lambda 3} (S_b^1 + S_b^2) = \frac{4\pi}{\kappa} \left[ \oint_{\Gamma_1} dx_1 \cdot \lambda_3^i (x_1) \Omega^0_{\mu} (x_1) - \oint_{\Gamma_2} dx_2 \cdot \lambda_3^i (x_2) \Omega^0_{\mu} (x_2) \right]
\]

(29)

It is easy to realize that the right hand sides of Eqs. (29) coincide exactly, apart from a sign, with the gauge variations \( \delta_{\lambda i} S \) of Eqs. (22, 24) if the external
currents \( J_i \) are given by Eq. (3). This proves condition i) and thus the gauge invariance of the action \( S \).

At this point we note that \( S^1_b \) and \( S^2_b \) contain only the longitudinal components \( \Omega^i_L \) of the fields \( \Omega^i \). Due to this fact, condition ii) is automatically satisfied and, moreover, it is possible to perform the following substitutions in Eqs. (26–27):

\[
\Omega^i(x) \equiv \Omega^i_L(x) = \nabla(\Omega^i(x) - \Omega^i(x_0))
\]  
(30)

where the \( \Omega^i(x) \) are singlevalued scalar fields defined by the relations:

\[
\Omega^i(x) - \Omega^i(x_0) = \int_{x_0}^x dx'^i \cdot \Omega^i_L(x')
\]  
(31)

In terms of the \( \Omega^i(x) \)'s, \( S^1_b \) and \( S^2_b \) can be written in a form which is explicitly metric independent in agreement with condition iii):

\[
S^1_b = \frac{4\pi\Lambda}{\kappa} \left[ \int_{\Gamma_1} dx_3^\mu \Omega^1(x_3) b_\rho^2(x_3) - \int_{\Gamma_2} dx_2^\mu \Omega^1(x_2) b_\rho^3(x_2) \right]
+ \frac{4\pi\Lambda}{\kappa} \left[ \int_{\Gamma_1} dx_3^\mu \Omega^2(x_3) b_\rho^3(x_3) - \int_{\Gamma_2} dx_2^\mu \Omega^2(x_2) b_\rho^1(x_2) \right]
+ \frac{4\pi\Lambda}{\kappa} \left[ \int_{\Gamma_3} dx_3^\mu \Omega^2(x_3) b_\rho^3(x_3) - \int_{\Gamma_2} dx_2^\mu \Omega^1(x_2) b_\rho^3(x_2) \right]
\]  
(32)

\[
S^2_b = \frac{4\pi\Lambda}{\kappa} \left[ \int_{\Gamma_1} dx_3^\mu \Omega^3(x_3) \partial_\rho \Omega^1(x_3) - \int_{\Gamma_2} dx_2^\mu \Omega^3(x_2) \partial_\rho \Omega^1(x_2) \right]
+ \int_{\Gamma_1} dx_3^\mu \Omega^3(x_3) \partial_\rho \Omega^2(x_3)
\]  
(33)

We stress the fact that in the above formulas the undesired presence of the variable \( x_0 \) introduced in Eqs. (30) and (31) has disappeared.

Before concluding this Section, let us show that the TTTF of Eq. (25) is free from the inconsistences which affected the naive BF–model with cubic interactions of Eq. (1). To this purpose, we need to study the classical equations of motion of the fields. First of all, the variation of \( S \) with respect to \( \Omega^i \) produces again the constraints (8). The solution of these equations has been already given in Eq. (13). Varying instead \( S \) with respect to the \( \Omega^i_{\nu}(x) \)'s one obtains the following relations:

\[
\frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \partial_\nu \tilde{\Omega}^1_{\rho}(x) + \Lambda \epsilon^{\mu\nu\rho} \Omega^2_\nu(x) \Omega^3_\rho(x)
+ \frac{\Lambda}{\kappa} \left[ \int_{\Gamma_1} dx_3^\mu b_\rho^3(x_3) \partial_\mu \frac{1}{|x - x_3|} - \int_{\Gamma_2} dx_2^\mu b_\rho^3(x_2) \partial_\mu \frac{1}{|x - x_2|} \right]
+ \frac{\Lambda}{4\pi\kappa} \int d^3y \partial^\nu \Omega^3_\nu(x) \int_{\Gamma_1} dx_3^\mu \partial^\mu \frac{1}{|x - x_3|} \partial_\nu \frac{1}{|y - x_2|}
- \frac{\Lambda}{4\pi\kappa} \int d^3y \partial^\nu \Omega^3_\nu(x) \int_{\Gamma_1} dx_3^\mu \partial^\mu \frac{1}{|x - x_3|} \partial_\nu \frac{1}{|y - x_3|} = 0
\]  
(34)

\footnote{Let us notice however that, similarly to what happens in the usual Chern-Simons field theories, the gauge invariance of \( S \) is realized only up to terms of the kind \( \int d^3 x \nabla \cdot [\chi \cdot (\nabla \times \tilde{\Omega})] \), which can be discarded only if the theory is defined on manifolds without boundary.}

8
Here the average with respect to the fields
amplitudes of the TTFT, which are given by:
\[
S
\]
To realize that the contributions coming from the cubic interactions present in
the action \( S \) cancel exactly against the new contributions coming from
\( S \) and \( \tilde{\Omega} \).

We note that the differences between Eqs. (34–36) and Eqs. (29) are limited to
purely longitudinal terms, so that there is no effect on the transverse components of the fields, in agreement with condition ii). Moreover, the longitudinal components of the fields \( \Omega^i \) do not generate further constraints, contrarily to what happens in the naive BF–model of the previous Section. As a matter of fact, if one applies the operator \( \partial_\mu \) to both sides of Eqs. (34–36), it is easy to realize that the contributions coming from the cubic interactions present in
the action \( S \) cancel exactly against the new contributions coming from \( S \) and \( S \).

The reason is that now only the transverse components of the fields are
physical, while the longitudinal components are associated to gauge degrees of
freedom and remain thus undetermined by the equations of motion. In this
way, the extension of the gauge symmetry to include the transformations (18)
has solved the consistency problems discussed in Section 2.

4 The Wilson Loop Amplitudes of the TTFT

Summarizing the results of the previous Section, the action \( S \) describes a well
defined topological field theory coupled to a set of Wilson loops. Since the
inconsistencies of the original action \( S \) have been eliminated by the introdution of the terms \( S \) and \( S \), we are now ready to compute the Wilson loop amplitudes of the TTFT, which are given by:

\[
\langle W_1(\Gamma_1)W_2(\Gamma_2)W_3(\Gamma_3)\rangle_b = \int \left[ \prod_{i=1}^{3} D\Omega^i D\tilde{\Omega}^i \right] e^{-iS} \tag{37}
\]

Here the average with respect to the fields \( \Omega^i \) and \( \tilde{\Omega}^i \) has been written with the symbol \( \langle \ldots \rangle_b \) to distinguish it from the analogous average of Eq. (37), in which
the fields’ behavior is governed by the action $S$. Eq. (37) represents the most

general correlator of Wilson loops. In the following, we suppose that none of

the Wilson loop operators is trivial, i.e.

$$W_i(\Gamma_i) \neq 1 \quad i = 1, 2, 3$$

(38)

This condition is useful to rule out simpler subcases which are not of particular

interest in the present context. As a matter of fact, it is easy to see that, in

the computation of amplitudes of the kind $\langle W_i(\Gamma_i)W_j(\Gamma_j) \rangle_b, \ 1 \leq i \neq j \leq 3,$

the contributions coming from the cubic interaction terms present in $S$ are

irrelevant. Thus, the TTFT behaves as a standard abelian BF–model and
delivers as topological invariants only the Gauss linking numbers of the set of

trajectories $\Gamma_i$ and $\Gamma_j$.

Let us now come back to the evaluation of Eq. (37) under the assumption (38). To eliminate the gauge freedom with respect to the transformations (3) and (18), we choose the Lorentz gauge fixing, in which the fields are purely

transverse:

$$\nabla \cdot \Omega^i = \nabla \cdot \tilde{\Omega}^i = 0$$

(39)

An immediate consequence of the Lorentz gauge is that $S^1_b$ and $S^2_b$ vanish

identically because they contain only the longitudinal components of the fields.

Thus, the amplitude (37) can be written as follows:

$$\langle W_1(\Gamma_1)W_2(\Gamma_2)W_3(\Gamma_3) \rangle_b = \int \prod_{i=1}^{3}\mathcal{D}\Omega^i \mathcal{D}\tilde{\Omega}^i e^{-iS_q}$$

(40)

where the “quantum” action $S_q$ is given by:

$$S_q = S - iS_{gf}$$

(41)

$S_{gf}$ being the gauge fixing term:

$$S_{gf} = \int d^3x \left( \nabla \varphi^i \cdot \Omega^i + \nabla \tilde{\varphi}^i \cdot \tilde{\Omega}^i \right)$$

(42)

In the above equation we have introduced the scalar fields $\varphi^i, \tilde{\varphi}^i$, which are

Lagrange multipliers imposing the gauge constraint (39).

We stress the fact that the disastrous effects caused by the longitudinal

components of the fields if one uses the naive action $S$ are brilliantly removed

by the presence of the gauge fixing term $S_{gf}$. Indeed, one may easily check

\footnote{Since the gauge group is abelian, the contribution of the ghost fields to the Wilson loop amplitude \footnote{Let us note that, as it happens in C-S field theories, the topological actions $S$ and $S$ keep the complex factor $i$ in the Feynman path integral even in spaces equipped with Euclidean metrics. For this reason, in agreement with our conventions, the gauge fixing term appears in Eq. (41) with a $-i$ factor in front.}}
that the classical equations of motion coming from the action \((\Pi)\) are free of inconsistencies and admit the non-trivial solutions given below:

\[
\begin{align*}
\Omega^i(x) &= b^i(x) \\
\tilde{\Omega}^i(x) &= \frac{\Lambda}{2\kappa} \epsilon^{ijk} \int d^3y \left( \nabla \frac{1}{|x-y|} \right) \times \left[ b^j(y) \times b^k(y) \right] \\
\varphi^i(x) &= 0 \\
\varphi^1(x) &= \frac{i\Lambda}{\kappa} \int d^3y \frac{1}{|x-y|} \left[ b^2(y) \cdot J^3(y) - b^3(y) \cdot J^2(y) \right] \\
\varphi^2(x) &= \frac{i\Lambda}{\kappa} \int d^3y \frac{1}{|x-y|} \left[ b^3(y) \cdot J^1(y) - b^1(y) \cdot J^3(y) \right] \\
\varphi^3(x) &= \frac{i\Lambda}{\kappa} \int d^3y \frac{1}{|x-y|} \left[ b^1(y) \cdot J^2(y) - b^2(y) \cdot J^1(y) \right]
\end{align*}
\]

At this point it is possible to compute the generic Wilson loop amplitude \((\Pi)\). As in the case of the naive BF-model discussed in Section 2, there are only two connected Feynman diagrams, which are represented in Fig. [1]. The path integrals in Eq. \((\Pi)\) may be easily evaluated integrating first over the \(\tilde{\Omega}^i\) fields and then exploiting the constraints \((\Pi)\) obtained in this way to perform the integration over the fields \(\Omega^i\). Alternatively, one can derive the analytic expression of \(\langle W_1(\Gamma_1)W_2(\Gamma_2)W_3(\Gamma_3) \rangle_b \) by means of successive Gaussian integrations. In both cases the result is:

\[
\langle W_1(\Gamma_1)W_2(\Gamma_2)W_3(\Gamma_3) \rangle_b = N \exp \left[ -i\Lambda \int d^3x b^1(x) \cdot (b^2(x) \times b^3(x)) \right]
\]

where the \(b^i\)'s have been defined in \((\Pi)\) and \(N\) is a normalization constant given by:

\[
N = \int \left[ \prod_{i=1}^{3} D\Omega^i \right] e^{-i \int d^3x \epsilon^{\mu\nu\rho} \partial_\mu \Omega_i^\nu \partial_\rho \Omega_i^\rho}
\]

We recall that the symbols \(\Gamma_i\) in Eq. \((\Pi)\) denote an ensemble of closed, non-intersecting paths \(\gamma_i, \gamma_i', \ldots\): \(\Gamma_i = \gamma_i + \gamma_i' + \ldots\). Due to the linearity properties of the exponent in the right hand side of Eq. \((\Pi)\), however, it is clear that the amplitude \(\langle W_1(\Gamma_1)W_2(\Gamma_2)W_3(\Gamma_3) \rangle_b\) can be decomposed into a product of correlation functions of three Wilson loops. For this reason, it will be sufficient to consider from now on only the fundamental three loop correlation function \(\langle W_1(\gamma_1)W_2(\gamma_2)W_3(\gamma_3) \rangle_b\), putting \(\Gamma_i = \gamma_i, i = 1,2,3\) in Eq. \((\Pi)\). As an upshot, the TTFT \((\Pi)\) contains in practice a single topological invariant, which appears in the exponent of the right hand side of Eq. \((\Pi)\) and it is given by:

\[
\mathcal{H} = \frac{1}{3} \int d^3x \epsilon^{\mu\nu\rho} \epsilon^{ijk} b^j_\mu(x) b^k_\nu(x) b^i_\rho(x)
\]

To conclude this Section, we study the topological term \(\mathcal{H}\). In the following, it will be convenient to interpret the vector fields \(b^i_\mu(x)\) of Eq. \((\Pi)\) as magnetic fields \(\Pi\)

\[
b^i(x) = -\frac{1}{\kappa} \int_{\Gamma_i} dx_i \times \frac{(x - x_i)}{|x - x_i|^3}
\]
generated by the currents $j_i = -\frac{1}{\kappa}J_i$. Indeed, it is possible to see that the $b^i_\mu(x)$'s satisfy the relations:

$$
\nabla \times b^i = j_i, \quad \nabla \cdot b_i = 0 \\
\nabla \times a^i = b_i, \quad \nabla \cdot a^i = 0
$$

(53)

where the $a^i$'s are their associated electromagnetic potentials:

$$
a^i = -\frac{1}{\kappa} \oint_{\Gamma^i} \frac{1}{|x-x_i|} dx_i
$$

(54)

Moreover, one can introduce the multivalued magnetic potentials $v^i(x)$:

$$
v^i(x) - v^i(x_0) = -\int_{x_0}^x d\mathbf{x}' \cdot \mathbf{b}^i(x')
$$

(55)

defined in such a way that $b^i(x) = -\nabla v^i(x)$.

The most straightforward interpretation of $\mathcal{H}$ is that of an Hopf invariant of the underlying gauge group $U(1) \otimes U(1) \otimes U(1) \equiv [U(1)]^3$. To show that, we build the $[U(1)]^3$ group element

$$
g(x) = e^{-i \sum_{i=1}^3 v^i(x)}
$$

(56)

It is now easy to check that, apart from a proportionality factor, $\mathcal{H}$ has exactly the form of the desired Hopf term:

$$
\mathcal{H} \propto \epsilon^{\mu\nu\rho} \int d^3x \frac{\partial v^1(x)}{\partial x^\mu} \frac{\partial v^2(x)}{\partial x^\nu} \frac{\partial v^3(x)}{\partial x^\rho} g^{-1} \frac{\partial g}{\partial v^1} g^{-1} \frac{\partial g}{\partial v^2} g^{-1} \frac{\partial g}{\partial v^3}
$$

(57)

Another form of $\mathcal{H}$ may be derived introducing the Pauli matrices $\sigma^i$ and the vector fields

$$
b_\mu(x) = \sigma^i b^i_\mu(x)
$$

(58)

The expression of $\mathcal{H}$ as a function of $b_\mu(x)$ becomes:

$$
\mathcal{H} = \frac{1}{6} \int d^3x \text{Tr} [\mathbf{b}(x) \cdot (\mathbf{b}(x) \times \mathbf{b}(x))]
$$

(59)

Here the symbol Tr denotes trace over the Pauli matrices. To go back to the original formulation of $\mathcal{H}$ given in Eq. (5) it is sufficient to use the relation $\text{Tr}[\sigma^i \sigma^j \sigma^k] = 2\epsilon^{ijk}$. Apparently, from the above equation $\mathcal{H}$ coincides with an Hopf term for the group $SU(2)$, which is not a symmetry group of our theory, but of course one should remember that $\mathbf{b}(x)$ is not a pure $SU(2)$ gauge field configuration.

Finally, one can give a physical meaning to $\mathcal{H}$ exploiting the electromagnetic analogy established by Eqs. (52–54) and the fact that the $b^i_\mu(x)$'s satisfy

---

5Mathematically, each $v^i(x)$ is the solid angle under which the trajectory $\Gamma_i$ appears as seen from a point $x$ (see e.g., [8], Ch. 16).
the classical equations of motion (34–36) in the Lorentz gauge. After some calculations one finds:

\[ H = -\frac{4\pi}{3\kappa} \epsilon_{ijk} \iint_{\Sigma_i} dS_i \cdot (b^j \times b^k) \]  

(60)

Here \( \Sigma_i \) denotes an arbitrary surface whose boundary is given by the contour \( \Gamma_i \), while \( dS_i \) is the projection of the infinitesimal area element of \( \Sigma_i \) along the normal direction with respect to the surface. From Eq. (60) it turns out that \( H \) measures the sum for \( i = 1, 2, 3 \) of the fluxes of the vector fields \( \epsilon_{ijk} b^j \times b^k \) through the surfaces \( \Sigma_i \).

5 Conclusions

In this work a new topological field theory has been constructed, the TTFT of Eq. (25), with the property that its perturbative series contains only the finite set of Feynman diagrams given in Fig. 1. The TTFT is exactly solvable and, besides the Gauss link invariant which already appears in abelian C-S field theories, it produces the further topological invariant \( H \) of Eq. (51). The latter has been interpreted as an Hopf term in Eq. (57). Another form of \( H \) has been given in (59). This equation suggests also an interesting generalization of the TTFT, consisting in the replacement of the cubic interaction present in the naive action \( S \) with a new interaction of the kind \( \int d^3x f^{ijk} \Omega^i \cdot (\Omega^j \times \Omega^k) \), where \( f^{ijk} \) denotes the structure constants of a compact Lie group.

As a final remark, let us note that the derivation of the action of the TTFT starting from the naive BF–model of Eq. (1) has some analogies with the way in which gauge invariance is implemented in C-S based models of the quantum Hall effect. In our case a fictitious one-dimensional “boundary”, which lies on the trajectories \( \Gamma_i \), appears due to the introduction of the Wilson loops. The inconsistent constraints (14–16) arising in the naive BF–model are all concentrated along these trajectories because of the particular form of the currents \( J_i \) defined in Eq. (6). The analogue of the edge state action of the quantum Hall effect is given here by the boundary terms \( S_1^a \) and \( S_2^a \), which restore gauge invariance in the fields \( \Omega^i \) and eliminate in this way the inconsistences of the action (1).

References

[1] M. J. Bowick, L. Chander, E. A. Schiff and A. M. Srivastava, Science 263 (1994), 943.
[2] C. Bäuerle et al., Nature 382 (1996), 332; V. M. H. Ruutu et al., Nature 382 (1996), 334.
[3] E. Wasserman, Jour. Am. Chem. Soc. 82 (1960), 4433.
[4] D. R. Leadly et al., Phys. Rev. Lett. 79 (1997), 4246.
[5] A. S. Schwarz, *Lett. Math. Phys.* 2 (1978), 247; S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* 48 (1982), 975; S. Deser, R. Jackiw and S. Templeton, *Annals Phys.* 140 (1982), 372; C. R. Hagen, *Annals Phys.* 157 (1984), 342.

[6] G. Horowitz, *Comm. Math. Phys.* 125 (1989), 417; A. S. Schwartz, *Comm. Math. Phys.* 67 (1979), 1; M. Blau and G. Thompson, *Ann. Phys.* (NY) 205 (1991), 130.

[7] S. F. Edwards, *Proc. Phys. Soc.* 91 (1967), 513; J. Phys. A1 (1968), 15.

[8] H. Kleinert, *Path Integrals*, (World Scientific Publishing, 2nd Ed., Singapore, 1995).

[9] F. Wilczek, *Fractional Statistics and Anyon Superconductivity*, (World Scientific, New Jersey 1990), Ch. 9; X. G. Wen, *Phys. Rev.* B43 (1991), 11025.