Stability in the Marcinkiewicz theorem

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Dedicated to the memory of I. V. Ostrovskii

Abstract

Ostrovskii’s generalization of the Marcinkiewicz theorem implies that if an entire characteristic functions of a probability distribution satisfies \( \log \log M(r, f) = o(r) \) and is zero-free then the distribution is normal. We show that under the same growth condition, absence of zeros in a wide vertical strip implies that the distribution is close to a normal one. This generalizes and simplifies a recent result of Michelen and Sahasrabudhe.

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Following Linnik [3], an entire function \( f \) is called a ridge function if \( |f(z)| \leq |f(i \Im z)|, \ z \in \mathbb{C} \). This definition is justified by Probability theory: characteristic functions of random variables are ridge functions when they are entire. We will apply the same name to subharmonic functions \( u \) in \( \mathbb{C} \) satisfying

\[
  u(z) \leq u(i \Im z), \quad z \in \mathbb{C}. \tag{1}
\]

Classical theorem of Marcinkiewicz [5] says that all ridge entire functions of finite order without zeros are of the form \( \exp(-az^2 + biz + c) \), where \( a > 0, b \) is real, and \( c \) is complex. This was generalized by Ostrovskii [7] who proved a conjecture of Linnik that the condition of finite order can be relaxed to

\[
  \log^+ \log |f(z)| = o(|z|), \quad z \to \infty. \tag{2}
\]

This condition was further relaxed in [9] to

\[
  \lim \inf_{z \to \infty} \frac{\log^+ \log |f(z)|}{|z|} = 0. \tag{2}
\]
Paper [3] contains a survey of further generalizations of Ostrovskii’s result.

We prove a “stable version” of this theorem for entire functions which are free of zeros in vertical strips:

**Theorem 1.** If \( u \) is a ridge subharmonic function in \( \mathbb{C} \) satisfying

\[
\liminf_{r \to \infty} \frac{\log \max \{ u(ir), u(-ir) \}}{r} = 0, \tag{3}
\]

is harmonic in the strip

\[
S(\Delta) = \{ z : |\text{Re} z| < \Delta \} \tag{4}
\]

and normalized by \( u(0) = u_x(0) = u_y(0) = 0 \) and \( u_{yy}(0) = 1 \), then

\[
|u(z) + \text{Re} \left( \frac{z^2}{2} \right)| \leq c_0 |z|^3/\Delta, \quad |z| \leq \Delta/3, \tag{5}
\]

where \( c_0 \) is an absolute constant.

Example \( u(z) = \cosh y \cos y - 1 \) shows that the growth condition (3) is best possible. A new proof of Linnik’s conjecture is obtained by setting \( u = \log |f| \) and \( \Delta = \infty \).

As a corollary we obtain a generalization of the recent theorem by Michelen and Sahasrabudhe [6, Thm. 4.1]:

**Theorem 2.** Let \( X \) be a random variable with average \( \mu \) and standard deviation \( \sigma \). Suppose that the characteristic function \( f_X \) is entire, satisfies (2), and is free of zeros in the strip \( \{ z : |\text{Re} z| < \delta \} \). Then the distribution function \( F_{X^*} \) of the random variable \( X^* = (X - \mu)/\sigma \) satisfies

\[
|F_{X^*} - F_N|_{\infty} \leq \frac{c_1}{\sigma \delta},
\]

where \( c_1 \) is an absolute constant, and \( N \) is the standard normal distribution with characteristic function \( f_N(z) = \exp(-z^2/2) \).

This theorem was proved in [6] under the additional assumption that \( X \) takes values in the set \( \{ 0, 1, \ldots, n \} \). We generalize the result and propose a shorter proof. We will use the

**Phragmén–Lindelöf Theorem.** If a subharmonic function \( v \) in a strip \( S \) satisfies

\[
\liminf_{z \to \infty} \frac{\log^+ v(z)}{|z|} = 0, \tag{6}
\]
and $v(z) \leq 0$, $z \in \partial S$, then $v(z) \leq 0$ in $S$.

**Lemma 1.** If a harmonic function in a strip $S(\Delta)$ satisfies (3) and (1), then for all real $y$, the function $x \mapsto u(x + iy)$ is decreasing for $x \in [0, \Delta/2]$.

**Proof.** Let us fix $s \in (0, \Delta/2)$ and let $z \mapsto z^*$ be the reflection with respect to the line $\Re z = s$, that is $z^* = 2s - \overline{z}$. We define $u^*(z) = u(z^*)$, and

$$v(z) = \max\{u(z), u^*(z)\}, \quad 0 < \Re z < 2s.$$

On the lines $\Re z = 0$ and $\Re z = s$ we have $v(z) \leq u(z)$. For a ridge function $u$, condition (3) implies (6) so $u$ and $v$ satisfy (6), and by the Phragmén–Lindelöf theorem we conclude that $v(z) \leq u(z)$ in the strip $\{z : 0 < \Re z < s\}$. On the other hand $v(z) \geq u(z)$ by definition, so

$$v(z) = u(z), \quad 0 < \Re z < s. \quad (7)$$

On the lines $\Re z = s$ and $\Re z = 2s$ we have $v(z) \leq u^*(z)$, so by a similar application of the the Phragmén–Lindelöf theorem we conclude that $v(z) \leq u^*(z)$ in the strip $\{z : s < \Re z < 2s\}$. On the other hand, $v(z) \geq u^*(z)$ by definition, so

$$v(z) = u^*(z), \quad s < \Re z < 2s. \quad (8)$$

Since $v(z)$ is subharmonic, we have $v_x(s - 0) \leq v_x(s + 0)$, and in view of (7), (8) we have

$$v_x(s - 0) = u_x(s) \quad \text{and} \quad v_x(s + 0) = u^*_x(s) = -u_x(s),$$

and so we obtain that $u_x(s) \leq -u_x(s)$ that is $u_x(s) \leq 0$, which proves the Lemma.

**Lemma 2.** Let $Q$ be the square,

$$Q = \{x + iy : 0 < x < 2, \ |y| < 1\}, \quad (9)$$

and let $P(z, \zeta)$ be the Poisson kernel of $Q$, where $z = x + iy \in Q$, and $\zeta \in \partial Q$. Then for $\zeta \in \partial Q \setminus (-i, i)$ we have

$$P_x(0, \zeta) \geq c_2,$$

where $c_2$ is an absolute constant.
Lemma 3. The family of harmonic functions in a vertical strip \( S(\Delta) \) as in (4) satisfying (3), (1) and normalized both conditions

\[
    u(0) = u_y(0) = 0, \quad u_{yy}(0) = 1,
\]

is uniformly bounded from above on every compact set \( K \subset S(\Delta/2) \) by a constant depending only on \( K \) and \( \Delta \).

Proof. By Lemma 1, harmonic functions \(-u_x\) are positive in the right half of the strip, and \( u_x(0, y) = 0 \) in view of (1). Applying to them the Poisson representation in rectangles \( cQ \) where \( Q \) is defined in (9) and using Lemma 2, we obtain that the total measure in this representation is bounded. So \( u_x \) are uniformly bounded on compacts. We conclude that the analytic functions \( u_x - iu_y \), are uniformly bounded on compacts. Since \( u_x(0) = 0 \) by the ridge property and \( u_y(0) = 0 \) by assumption, we conclude that functions \( u \) are uniformly bounded on compacts in \( S(\Delta/2) \). This proves Lemma 3.

Proof of Theorem 1. We may assume without loss of generality that \( \Delta \geq 1 \). Consider the expansion at 0:

\[
    u(z) = \text{Re} \left( -\frac{z^2}{2} + \sum_{n=3}^{\infty} a_n z^n \right).
\]

Let

\[
    u_\Delta = \Delta^{-2} u(z\Delta) = \text{Re} \left( -\frac{z^2}{2} + \sum_{n=3}^{\infty} a_n \Delta^{n-2} z^n \right), \quad z \in S(1).
\]

By Lemma 3, its coefficients are uniformly bounded, therefore \(|a_n| \leq c_3 \Delta^{2-n}\), and

\[
    \sum_{n=3}^{\infty} |a_n||z^n| \leq c_3 \Delta^{-1} \frac{|z|^3}{1 - |z|/\Delta} \leq c_0 |z|^3/\Delta, \quad \text{when } |z| \leq \Delta/3.
\]

This proves Theorem 1.

Derivation of Theorem 2 from Theorem 1. Following [6] and [2], we use the Berry–Esseen inequality

\[
    \sup_{t \in \mathbb{R}} |F_{X^*}(t) - F_Z(t)| \leq \frac{1}{\pi} \int_{-T}^{T} \left| \frac{f_{X^*}(x) - e^{-x^2/2}}{x} \right| dx + \frac{c}{T}, \quad (10)
\]

where \( c \) is an absolute constant.
This estimate can be found in [1, Ch. XVI, 3, Lemma 2] and in [4, Lemma 8.2.2].

We set $\Delta = \delta \sigma$. The statement of Theorem 2 is meaningful only when $\Delta$ is large, so we assume that $\Delta > c_0$, where $c_0$ is the constant in Theorem 1.

We are going to apply Theorem 1 to $u = \log |f_{X^*}|$, where $f_{X^*}$ is the characteristic function of $X^*$. Since $X^*$ is normalized, $u$ is normalized as required in Theorem 1. Since by assumption the characteristic function $f_X$ has no zeros in the strip $S(\delta)$, the function $f_{X^*}$ has no zeros in the strip $S(\Delta)$. Then Theorem 1 implies that

$$f_{X^*}(x) = \exp(-x^2/2 + R(x)), \quad \text{where } |R(x)| \leq c_0|x|^3/\Delta, \quad |x| < \Delta/2.$$

Set $T = \Delta/(4c_0)$ in (10). To estimate the integral in (10) we break it into two parts:

Let $a := (\Delta/c_0)^{1/3} \geq 1$.

When $|x| < a$, we have $|R(x)| \leq 1$, so $|e^{R(x)} - 1| \leq 2|R(x)| \leq 2c_0|x|^3/\Delta$, so

$$\int_{-a}^{a} \left| \frac{f_{X^*}(x) - e^{-x^2/2}}{x} \right| dx = \frac{2c_0}{\Delta} \int_{-\infty}^{\infty} e^{-x^2/2} x^2 dx \leq c_5/\Delta.$$

When $|x| \in [a, T]$ we use

$$f_{X^*}(x) = \exp(-x^2/2 + R(x))$$

and

$$x^2(-1/2 + |x|c_0/\Delta) \leq x^2(-1/2 + 1/4) = -x^2/4.$$ 

So

$$\int_{|x| \in [a, T]} \left| \frac{f_{X^*}(x) - e^{-x^2/2}}{x} \right| dx \leq 4 \int_{a}^{\infty} e^{-x^2/4} dx \leq c_6/\Delta.$$

This completes the proof.

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