PECULIARITIES OF INCLINED FORCE AFFECTION ON HOMOGENEOUS ONE-DIMENSIONAL ELASTIC LUMPED-PARAMETERS LINE

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Abstract

In this paper are analysed exact analytic solutions for one-dimensional elastic lumped-parameters line under affection of external force inclined to the axe of line. It is shown that in this case an inclined wave being described by implicit function propagates along the line. The conclusions are extended to unforced vibration in the line and to distributed parameters line vibration. Solution in the form of implicit function is proved as a generalizing one for the wave equation.

1 Introduction

In papers [1,2] have been considered longitudinal vibrations in one-dimensional elastic lumped-parameters line. However in such line vibrations of more general form are possible, if we attribute the idea of one-dimensionality only to general shape of line, not to degree of freedom of vibrating elements of line.

This paper investigates the peculiarities of exact analytic solutions for this type of problems.
2 Vibrations in semi-infinite elastic line under inclined external force affecting on start of line

Regarding stated above, supposing two degrees of vibration freedom for elastic line elements, the model can be presented in form shown in fig.1. In case when amplitude of vibration is small (linear vibration), this model can be described by two sets of equations - with respect to x- and y-projections of external force affection correspondingly:

\[
\begin{cases}
    m\frac{d^2\Delta_1}{dt^2} = F(t) \cos \alpha + s (\Delta_2 - \Delta_1) \\
    m\frac{d^2\Delta_2}{dt^2} = s (\Delta_3 + \Delta_1 - 2\Delta_2) \\
    m\frac{d^2\Delta_3}{dt^2} = s (\Delta_{n+1} + \Delta_{n-1} - 2\Delta_n) \\
\end{cases}
\]

\[
\begin{cases}
    m\frac{d^2y_1}{dt^2} = F(t) \sin \alpha + s (y_2 - y_1) \\
    m\frac{d^2y_2}{dt^2} = s (y_3 + y_1 - 2y_2) \\
    m\frac{d^2y_3}{dt^2} = s (y_{n+1} + y_{n-1} - 2y_n) \\
\end{cases}
\]

where $\alpha$ is angle of external force inclination to the axe of line.

Each of these sets of equations is similar to ones having been investigated in [1]. Consequently, we can write at once exact analytic solutions for every of them.

For x-component of vibration: periodic regime ($\beta < 1$)

\[
\Delta_n = -j \frac{F_0 \cos \alpha}{\omega \sqrt{sm}} e^{j[\omega t - (2n-1)\tau]} 
\]

aperiodic one ($\beta > 1$)

\[
\Delta_n = (-1)^n \frac{F_0 \cos \alpha}{\omega \sqrt{sm}} \gamma^{2n-1} e^{j\omega t} 
\]

critical one ($\beta = 1$)

\[
\Delta_n = (-1)^n \frac{F_0 \cos \alpha}{2s} e^{j\omega t} 
\]
For $y$-component of vibration we obtain correspondingly: periodic regime ($\beta < 1$)

$$y_n = -\frac{jF_0 \sin \alpha}{\omega \sqrt{s \rho m}} e^{j(\omega t - (2n - 1)\tau)}$$  \hspace{1cm} (6)

aperiodic one ($\beta > 1$)

$$y_n = (-1)^n \frac{F_0 \sin \alpha}{\omega \sqrt{s \rho m}} \gamma^{2n-1} e^{j\omega t}$$  \hspace{1cm} (7)

and critical one ($\beta = 1$)

$$y_n = (-1)^n \frac{F_0 \sin \alpha}{2s} e^{j\omega t}$$  \hspace{1cm} (8)

As a result of superposition, there forms an inclined wave propagating in positive direction of axe $x$; this is confirmed by diagram of vibration shown in fig.2.

It is characteristic that inclined pattern of vibration remains as under unforced vibrations in lumped-parameters line as under limiting process to distributed-parameters line.

Really, basing on results presented in [1], solution, e.g. for unforced vibration, has the following form: for $x$-component of vibration

$$\Delta_n = \frac{X_k \cos (2i - 1) \tau}{\cos (2k - 1) \tau} e^{j\omega t}$$  \hspace{1cm} (9)

for $y$-component

$$y_n = \frac{Y_k \cos (2i - 1) \tau}{\cos (2k - 1) \tau} e^{j\omega t}$$  \hspace{1cm} (10)

where $X_k$ and $Y_k$ are $x$- and $y$-components of vibration amplitude of $k$th element which parameters are specified; $k$ is number of element which vibration is specified.

In case of limiting process to distributed-parameters line, we can present

$$\rho = \frac{m}{a} ; \quad s = \frac{T}{a} ; \quad n = \frac{x_0}{a}$$

where $\rho$ is density; $T$ is tension in line; $x_0$ is distance from start of line to the point of rest of investigated element of line; $m$ is mass of element of line.
With it solutions (3)÷(8) transform to the set of equations

$$\begin{cases}
x = -j \frac{F_0 \cos \alpha}{\omega \sqrt{T \rho}} e^{j \omega (t-x_0 \sqrt{\rho/T})} + x_0 \\
y = -j \frac{F_0 \cos \alpha}{\omega \sqrt{T \rho}} e^{j \omega (t-x_0 \sqrt{\rho/T})}
\end{cases} \quad (11)$$

Obtained set of equations describes parametrically the inclined wave propagating along the axe $x$, as shown in fig.3.

We can see of carried out investigation that inclined vibrations arise far from always as a consequence of nonlinear processes in elastic system, as it was supposed before. Inclined waves can arise quite naturally under affecion of force inclined to the direction of wave process propagation. And this conclusion can be quite simply extended to a most wide spectrum of vibration process.

### 3 Elements-of-line motion trajectory

Paying attention to a separate element motion trajectory, we can see easy, this trajectory has form of ellipse circumscribed around the point of rest of element. And inclination of wave forms, at the cost of shift phase of motion along element-to-element elliptic trajectories. Presented structure of vibration is well-known in physics, particularly in wave processes in unbounded volumes of liquid. "In a wave, motion of liquid is non-stationary. So trajectories of separate particles are far from coinciding with lines of current in time. They have absolutely other form. Under small amplitudes they are circumferences in a great approximation. We find these circular trajectories as on surface as in depth of liquid. Only in the most upper layers the diameters of circular ways are the most large" [3, pp.300-301].

Indeed, vibration processes in space have their peculiarities. None the less, it is characteristic that basic regularities are run down already in one-dimensional model. It also follows of obtained solutions that ellipsoidal pattern of vibrations of elements remains as in critical as in aperiodic regimes. Consequently, in last case in the line forms compound wave fast-decaying along the line, and this is one more peculiarity that exact analytic solutions demonstrate.
4 On new class of functions being the solution of wave equation

Foregoing generalizations can be extended also to solution of wave equation in the whole.

It is known that differential equation of hyperbolic type

\[ \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{k^2}{\omega^2} \frac{\partial^2 x(x,t)}{\partial t^2} = 0 \] (12)

has general solution [5, p.300]

\[ \Phi(x,t) = \Phi_1(kx - \omega t) + \Phi_2(kx + \omega t) \] (13)

where \( c = \omega/k \) is velocity of wave propagation, i.e. in the form of two explicit functions with respect to \((x - ct)\) and \((x + ct)\) correspondingly. Till now this solution was considered the only and complete, due to theorem of uniqueness of solution of differential equation. None the less, there exists one more class of functions being the solution of differential equation (12) but not taken into account by solution (13). We can present general form of this class of functions in the form

\[ y(x,t) = \Phi_1(kx - \omega t + \psi_1(y)) + \Phi_2(kx + \omega t + \psi_2(y)) \] (14)

where \( \psi_1(y) \) and \( \psi_2(y) \) are some twice-differentiable functions. In other words, given solution (14) belongs to the class of implicit functions whose regularities of behavior and technique of differentiation and integration essentially differ from such for explicit functions. Important that, while for explicit functions definite systematization of differential equations has been created and for definite class of these equations the regularities and schemes to obtain solutions have been defined, for implicit functions all these developments are absent. Naturally, for today correspondence of expression (14) to differential equation (12) can be checked only by the most simple way - by straight substitution (14) into (12).

For it, on the grounds of known laws of implicit functions differentiation, find first and second particular derivatives of expression (14) with respect to \( x \) and \( t \). To simplify calculation, consider a half of right part of expression (14)

\[ y(x,t) = \Phi_1(kx - \omega t + \psi_1(y)) = \Phi_1(A) \] (15)

\[ \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{k^2}{\omega^2} \frac{\partial^2 x(x,t)}{\partial t^2} = 0 \]
where $A = kx - \omega t + \psi_1(y)$. First derivatives have the form

$$
\frac{\partial y}{\partial x} = \frac{k \frac{d\Phi_1}{dA}}{1 - \frac{d\psi_1}{dy} \frac{d\Phi_1}{dA}} \quad ; \quad \frac{\partial y}{\partial t} = \frac{\omega \frac{d\Phi_1}{dA}}{1 - \frac{d\psi_1}{dy} \frac{d\Phi_1}{dA}}
$$

(16)

Second derivatives after transformation and substitution of expressions (16) take form

$$
\frac{\partial^2 y}{\partial x^2} = k^2 \left[ \frac{d^2\Phi_1}{dA^2} + \frac{d^2\psi_1}{dy^2} \left( \frac{d\Phi_1}{dA} \right)^3 \right]
$$

$$
\left( 1 - \frac{d^2\psi_1}{dy^2} \frac{d\Phi_1}{dA} \right)^3
$$

(17)

$$
\frac{\partial^2 y}{\partial t^2} = \omega^2 \left[ \frac{d^2\Phi_1}{dA^2} + \frac{d^2\psi_1}{dy^2} \left( \frac{d\Phi_1}{dA} \right)^3 \right]
$$

$$
\left( 1 - \frac{d^2\psi_1}{dy^2} \frac{d\Phi_1}{dA} \right)^3
$$

Substituting (17) into (12), we obtain required. Similarly we can prove the correspondence of second part of expression (14) to equation (12).

Thus, solution (17) defines a whole class of implicit functions satisfying the linear wave equation. And presence of new class of functions being the solution of equation (12) does not violate a least the theorem of uniqueness of solution of differential equation, because under definite conditions

$$
\psi_1(y) \equiv 0 \quad ; \quad \psi_2(y) \equiv 0
$$

expression (14) degenerates into (13). hereby it is proved that solution known before is a particular case of more general class of functions.

The found class of implicit functions defines nonlinear wave; its degree of deformation depends on form of functions $\psi_1(y)$ and $\psi_2(y)$. For example, in particular case of expression (14) (see fig.4)

$$
y = c \sin (kx - \omega t + y \cot \alpha)
$$

(18)

solution of wave equation (12) describes progressive wave propagating along axe $x$ and inclined by angle $\alpha$, what completely corresponds with inclined vibration in one-dimensional line investigated above.

5 Conclusions

1. Analyzing solutions for semi-infinite model on free end of which affects a force inclined to the axe, we have found that as a result of this affection, in the line propagate inclined waves described by implicit function.
2. Solutions of wave equation in the form of implicit functions are generalizing for known solution being superposition of running waves.

3. We have ascertained that under affection of inclined force the elements of line follow elliptic trajectories.

6 Symbols

$F(t)$ is external force affecting on the line; $F_0$ is amplitude of external force; $T$ is tension in the line; $X_i$, $Y_i$ are $x$- and $y$-components of vibration amplitude of $k$th element whose parameters of vibration are given; $a$ is distance between the elements of line; $f$ is frequency of vibration in the line; $i$, $k$, $n$ are indexes; $m$ is mass of element of line; $k$ is wave number; $s$ is stiffness coefficient of line; $t$ is time parameter; $x_0$ is distance from start of line to the point of rest of last element of line; $y_i$ is displacement of $i$th element of elastic line in vertical plane; $\Delta_i$ is instantaneous longitudinal displacement of $k$th element of line; $\alpha$ is angle of affecting force inclination to the axe of elastic line; $\beta$, $\gamma_+$, $\gamma_-$, $\tau$ are parameters of line; $\omega$ is circular frequency of affecting force.

References

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Fig. 1. Model of inclined vibration in one-dimensional elastic lumped-parameters line

\[ F(t) \]

\[ \alpha \]

\[ y_1, y_2, y_3, y_4, y_n \]

\[ \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_n \]
Fig. 2. Diagram of vibration in semi-infinite line under force inclined to the axe of line $m = 0,01 \text{ kg}; \ s = 100 \text{ N/m}; \ a = 0,01 \text{ m}; \ \alpha = 60^\circ; \ F_o = 2 \text{ N}; \ f = 15 \text{ Hz}$
Fig. 3. Vibration in distributed-parameters line under external inclined force affecting on start of line. The angle of inclination $\alpha=60^\circ$
Fig. 4. Plot of inclined vibration described by implicit function being the solution of wave equation, with $\alpha_1=45^\circ; \alpha_2=60^\circ$. 