1 Introduction

Let $D$ be a reduced hypersurface in $\mathbb{C}^N$. In [18], K. Saito associated $D$ with the sheaf of logarithmic vector fields along $D$, denoted by $\text{Der}(-\log D)$, which is the subsheaf of the holomorphic vector fields on $(\mathbb{C}^N, 0)$. If $\text{Der}(-\log D)$ is a locally free $\mathcal{O}_{\mathbb{C}^N, 0}$-module then $D$ is said to be a free divisor (also known as Saito divisor in the literature). The basic examples of a free divisor are normal crossing divisors and reduced plane curve singularities ([18]). The example motivating K. Saito and attracting the attention of many other mathematicians on the subject is the freeness of discriminants in the base space of the versal deformation of isolated hypersurface singularities. In [4], Buchweitz and Mond introduced a new class of free divisors arising from the representation of quivers. The discriminant in the representation space of a quiver with given dimension vector is a linear free divisor, in the sense that, under some conditions on the quiver and the dimension vector, $\text{Der}(-\log D)$ is a free module; furthermore, generated by vector fields which only have linear function coefficients with respect to the standard basis. The first examples in this direction are Dynkin quivers; that is, a quiver whose underlying unoriented graph is a Dynkin diagram of type ADE. A Dynkin quiver with a real Schur root as a dimension vector determines a linear free divisor ([4]). It is well known that the Dynkin
diagrams appear as the dual graph of the minimal resolution of rational surface singularities of complex surfaces of multiplicity 2. Further examples of linear free divisors and a classification in dimension $N \leq 4$ was given in [9].

Recall that a singularity of a normal surface is rational if the geometric genus of the surface is unchanged by a resolution of the singularity. The rational singularities of surfaces are classified by their multiplicity $m$ which equals $-Z^2$ where $Z$ is the Artin cycle of the resolution. When $m = 2$, the dual graph of the minimal resolution of the rational singularity is one of the Dynkin diagrams, which are the underlying graphs of the Dynkin quivers. So we naturally ask “what are the other dual resolution graphs of singularities which may give linear free divisors?” Here we try to answer this question.

By [8, Theorem 3.10], linear free divisors can only be obtained from a quiver having the form of a tree. In this work, we are interested in rational singularities of surfaces as their dual resolution graphs are trees. We will refer to their resolution graphs as rational trees. In particular, we consider rational singularities for which, in the Artin cycle, the coefficients of the exceptional curves corresponding to the vertices with weight $\geq 3$ are all equal to one. The singularities having such resolution graphs are called rational quasi-determinantal singularities ([6]). The simplest quasi-determinantal singularities of surfaces, which can also be seen as a generalisation of rational surface singularities of multiplicity 2, are the rational singularities of multiplicity 3, which are called rational triple points (or RTP for short). In fact, they are determinantal ([23]). Moreover, their dual resolution graphs contain Dynkin diagrams as subtrees. These are the simplest rational trees (quivers) answering our question above.

We start by constructing a new root system for rational triple points using the results of [21] and determine the number of roots for each RTP-singularity (cf. [3]). In Section 5, we show that, for each root, we obtain a linear free divisor (Theorem 5.11). We also prove that this construction is independent of the orientation on the tree. Furthermore, using the results in [8] and [14], we deduce that linear free divisors defined by rational triple quivers satisfy the logarithmic comparison theorem. In Section 6, we generalize the results of Section 3 and 5 to rational quasi-determinantal singularities.

In the 1970’s Gabriel proved that a quiver is of finite representation type if and only if its underlying graph is a Dynkin diagram of type A, D or E ([7]). Then the classical McKay correspondence described the bijection between the set of isomorphism classes of nontrivial irreducible representations of finite subgroups of SL(2, $\mathbb{C}$) and the vertices of the corresponding Dynkin diagram (see [20]). In quasi-determinantal case, even in the special case of RTP’s, we do not have a Lie algebra corresponding to the singularity. In other words, the relation between the theory of representations of finite subgroups of our quivers and the minimal resolution of the singularity is yet to be discovered. Only some of the RTP’s appear as quotient singularities ([17]). After writing up this work, the authors realised that the geometric construction of positive roots were first discussed in Wahl’s article [22].

2 Rational Singularities of Complex Surfaces

Let $(X,0)$ be a germ of a normal surface singularity embedded in $\mathbb{C}^N$. Let $\pi: \tilde{X} \to (X,0)$ be the minimal resolution of $X$. The singularity of $X$ at 0 is
We will denote it by $Z$ supported on $E$. Coefficients are non-negative and at least one is positive; otherwise, it is called a negative divisor. A positive divisor $Z$ associated with $E$ a rational singularity, which can be read from $E$, (or equivalently on $Γ$), is called a positive divisor if all the coefficients are non-negative and at least one is positive; otherwise, it is called a negative divisor. A positive divisor $Z$ is called the Artin cycle of $π$ if it satisfies $(Z · E_i) ≤ 0$ for each $i$ and has arithmetic genus $p_a(Z)$ equal to zero (see [1]). We will denote it by $Z = \sum_{i=1}^{n} a_i E_i$. Additional numerical characterizations of a rational singularity, which can be read from $E$, or from $Γ$, are as follows:

1. $p_a(Y) := \frac{1}{2} \left[ Y \cdot Y + \sum_{i=1}^{n} m_i (w_i - 2) \right] + 1 \leq 0$ for all positive divisors $Y$ that are supported on $E$.

2. The Artin cycle $Z$ of $Γ$ satisfies $Z \cdot Z = -m$ where $m$ is the multiplicity of $X$ at the singularity $0$.

Conversely, for a given tree $Γ$ with vertices $E_1, \ldots, E_n$, let us assign a weight $w_i \in \mathbb{N}^*$ for each $E_i$. Consider the incidence matrix $M(Γ) = (c_{ij})_{(i,j)=1,\ldots,n}$, associated with $Γ$ such that $c_{ij}$ equals $-w_i$ and $e_{ij}$ is the number of edges between the vertices $E_i$ and $E_j$, which is 0 or 1 since $Γ$ is a tree. If $M(Γ)$ is negative definite then, by Grauert’s theorem in [10], $Γ$ is the dual graph of the exceptional fibre of a resolution of a germ of a normal surface singularity where $E_i^2 = -w_i$ for $i = 1, \ldots, n$. When $w_i \geq 2$ for all $i$, $Γ$ is the dual graph of the minimal resolution.

**Definition 2.1** ([1]). Following this construction, if, in addition, $Γ$ satisfies the two properties given above then the singularity is rational. In this case, $Γ$ is called a rational $m$-tuple tree.

For example, Dynkin diagrams are the rational $2$-tuple trees. These are the only rational trees having weight $2$ at each vertex.

**Proposition 2.2** ([1][13]). Any subtree of a rational $m$-tuple tree is a rational $m'$-tuple tree with $m' \leq m$.

Let $Γ$ be a rational $m$-tuple tree. Let us consider the vertices $E_i$ with weight $\geq 3$ and reindex them as $E_{i_0}, E_{i_1}, \ldots, E_{i_k}$. Then we have

$$Γ = \bigcap_j \Gamma_j$$

where $\Gamma_j$ is a Dynkin diagram for each $j$. This says that a subtree $Γ'$ of $Γ$ is of the following type if it contains a unique vertex with weight $w \geq 3$ (denoted by the symbol ■) where each $Γ_j$ is a Dynkin diagram. In addition, we have the following property of rational trees which bounds the number of the subtrees $Γ_j$.

**Proposition 2.3** ([19]). If $Γ$ is a rational tree, then each vertex $E_i$ satisfies

$$v_i \leq w_i + 1$$

where $v_i$ is the number of vertices adjacent to $E_i$ in $Γ$, called the valency of $E_i$ and, $w_i$ is the weight of $E_i$ given by $-E_i^2$.

Furthermore, any given tree can be weighted in such a way that it becomes rational. See [13] for the details and the gluing conditions of rational trees.
3 Triple Root System

The classification of rational 3-tuple trees is given by Artin in [1]. They are of the form given in Figure 1 such that \( w = 3 \) and \( \ell \leq 3 \) where \( \ell \) is the valency of the central vertex, i.e. the vertex whose weight is \( \geq 3 \) denoted by a ■. The corresponding rational singularities are singularities of surfaces embedded in \( \mathbb{C}^4 \) and defined by three equations as listed in [21].

In the sequel, a rational 3-tuple \( \Gamma \) will be called a rational triple tree. Since for a rational triple singularity, we have \( Z_2 = -3 \) and \( p_a(Z) = 0 \), a rational tree contains a unique vertex with weight 3 and attached to some Dynkin diagrams. Moreover, the condition \((Z \cdot E_i) \leq 0\) for each \( i \) implies that the coefficient of the vertex with weight 3 in the Artin cycle \( Z \) is equal to 1.

Definition 3.1. Let \( \Gamma \) be a rational triple tree and \( \mathcal{G} \) be the set of all positive and negative divisors supported on \( E \). Then

\[
R(\Gamma) := \{Y \in \mathcal{G} \mid (Y \cdot Y) = -2 \text{ or } (Y \cdot Y) = -3\}
\]

such that \( 3(Y \cdot Y')/(Y \cdot Y) \in \mathbb{Z} \) for all \( Y, Y' \in R(\Gamma) \) with \( (Y \cdot Y) = -3 \), is called the triple root system.

We call an element of \( R(\Gamma) \) a root. We also call each \( E_i \) a simple (positive) root of \( R(\Gamma) \). Note that for Dynkin diagrams, the set of divisors \( Y \) in \( \mathcal{G} \) with the property \((Y \cdot Y) = -2\) form the root system (16). Hence, all roots associated with Dynkin diagrams are obtained by \((Y \cdot Y) = -2\) (20), compare with [3].

From now on, we will denote the unique vertex in the triple tree with weight 3 by \( E_0 \) and the coefficient of \( E_0 \) in a root \( Y \) by \( a_0 \).

Lemma 3.2. If \( Y = \sum a_iE_i \) is a positive divisor on \( \Gamma \) then \((Y \cdot Y) \leq -2\).

Proof. It easily follows from the fact that \( p(Y) \leq 0 \) (see also [1]).

Lemma 3.3. If \( Y \in R(\Gamma) \), then \( Y \) is either a positive or negative divisor.

Proof. Suppose that \( Y = Y_1 - Y_2 \) such that \( Y_1 \) and \( Y_2 \) are positive and without common components. Then \((Y \cdot Y) = (Y_1 \cdot Y_1) - 2(Y_1 \cdot Y_2) + (Y_2 \cdot Y_2)\). Note that \((Y_1 \cdot Y_1) \leq 0\), \((Y_2 \cdot Y_2) \leq 0\), and \((Y_1 \cdot Y_2) \geq 0\). If \( Y_1 \neq 0 \) and \( Y_2 \neq 0 \), then \((Y_1 \cdot Y_1) \leq -2 \) and \((Y_2 \cdot Y_2) \leq -2 \) by Lemma 3.2. However this contradicts \((Y \cdot Y) = -2 \) or \(-3 \). Hence \( Y_1 = 0 \) or \( Y_2 = 0 \).

Lemma 3.4. If \( Y = \sum a_iE_i \in R(\Gamma) \), then \( \text{Supp}(Y) \) is connected where \( \text{Supp}(Y) \) is the support of \( Y \) which is the set of \( E_i \)'s for which \( a_i \neq 0 \).
Therefore we have
\[ (Y \cdot Y) = (Y_1 \cdot Y_1) + (Y_2 \cdot Y_2) \leq -4, \] which is a contradiction.

\[ \square \]

**Proposition 3.5.** Let \( Y \in R(\Gamma) \). We have \( (Y \cdot Y) = -2 \) if and only if \( a_0 = 0 \) and, \( (Y \cdot Y) = -3 \) if and only if \( a_0 = \pm 1 \).

**Proof.** It easily follows from the fact that \( p(Y) \leq 0 \) and \( w_i = 2 \) for a positive root \( Y \).

\[ \square \]

**Definition 3.6.** Let \( \Gamma \) be a rational triple tree. (From now on, we will also call it a triple Dynkin diagram.) Let \( \Gamma_0 \) denote the diagram obtained from \( \Gamma \) by changing a rational triple point of \( \Gamma \) into a rational double point. We say that \( \Gamma_0 \) is the underlying diagram of \( \Gamma \). The underlying diagram of \( \Gamma \) is classical if \( \Gamma_0 \) is of type \( A, D, E \).

Suppose that the underlying diagram of \( \Gamma \) is classical. We denote the simple roots of \( R(\Gamma) \) and \( R(\Gamma_0) \) by the same notation \( E_i \), while \( E_0 \) denotes the simple root corresponding to the rational triple point. We will write all roots by the simple root corresponding to the rational triple point. We will write all roots by the simple root corresponding to the rational triple point. We will write all roots by the simple root corresponding to the rational triple point.

**Proposition 3.7.** Let \( \Gamma \) be a triple Dynkin diagram. Suppose that the underlying diagram \( \Gamma_0 \) of \( \Gamma \) is classical. Then \( R(\Gamma) \subseteq R(\Gamma_0) \).

**Proof.** For each \( Y \in \sum a_i E_i \in \oplus \mathbb{Z} E_i \), we have \( (Y \cdot Y) = (Y \cdot Y)_{\Gamma_0} - a_0^2 \), where \( (Y \cdot Y)_{\Gamma_0} \) denotes the self-intersection of \( Y \) with respect to \( R(\Gamma_0) \). If \( (Y \cdot Y) = -2 \), then \( (Y \cdot Y)_{\Gamma_0} \geq -2 \), and hence \( (Y \cdot Y)_{\Gamma_0} = -2 \) and \( a_0 = 0 \). If \( (Y \cdot Y) = -3 \), then \( (Y \cdot Y)_{\Gamma_0} \geq -3 \). Because \( (Y \cdot Y)_{\Gamma_0} \) is even, \( (Y \cdot Y)_{\Gamma_0} = -2 \) and \( a_0 = \pm 1 \). Therefore \( R(\Gamma) \subseteq R(\Gamma_0) \).

\[ \square \]

**Remark 3.8.** The highest root in \( R(\Gamma) \) is exactly the Artin cycle of the given tree.

**Corollary 3.9.** Suppose that \( \Gamma_0 \) is classical. Let \( Z(\Gamma_0) = \sum a_i E_i \) be the highest root of \( R(\Gamma_0) \). If the coefficient \( a_0 = 1 \), then \( R(\Gamma) = R(\Gamma_0) \).

**Proof.** We only need to show that each positive root \( Y \) of \( R(\Gamma_0) \) is contained in \( R(\Gamma) \). Since \( Y \leq Z(\Gamma_0) \), the coefficient \( a_0 \) of \( Y \) is less than 1. Then \( (Y \cdot Y) = (Y \cdot Y)_{\Gamma_0} - a_0^2 \geq -2 - 1 = -3 \). On the other hand, \( (Y \cdot Y) \leq -2 \). Thus we have proved that \( Y \in R(\Gamma) \).

\[ \square \]

Let us introduce examples of triple Dynkin diagrams whose underlying diagrams are classical. For a diagram \( \Gamma \) of type \( A, D, E \) we denote by \( \Gamma_i \) the triple Dynkin diagram obtained by replacing \( E_i \) by a vertex of weight 3. For example, \( A_{n, 2} \) is the diagram obtained from \( A_n \) (see Figure 2) by replacing \( E_2 \) by a vertex of weight 3.

**Example 3.10.** Consider the Dynkin diagram of type \( A_n \) (Figure 2). It is easy to see that \( R(A_n) = R(A_{n, i}) \) for each \( i \).
Example 3.11. Consider the Dynkin diagram $D_n$ (see Figure 3). These triple Dynkin diagrams have the following relations.

$$R(D_{n,n-2}) \subseteq R(D_{n,n-3}) \subseteq \cdots \subseteq R(D_{n,2}) \subseteq R(D_{n,1}) = R(D_{n,n}) = R(D_n).$$

Example 3.12. Consider the Dynkin diagram $E_6$ pictured in Figure 4. Then we have

$$R(E_{6,3}) \subseteq R(E_{6,6}) \subseteq R(E_{6,2}) = R(E_{6,4}) \subseteq R(E_{6,1}) = R(E_6).$$

Example 3.13. Consider the Dynkin diagram $E_7$ in Figure 5.

We have the following inclusion of the root systems.

$$R(E_{7,3}) \hookrightarrow R(E_{7,2}) \hookrightarrow R(E_{7,7}) \hookrightarrow R(E_{7,1}) \hookrightarrow R(E_{7,0}) = R(E_7).$$
**Example 3.14.** Consider the Dynkin diagram $E_8$ shown in Figure 6. Then we have

$$
R(E_8,6) \to R(E_8,3) \to R(E_8,4) \to R(E_8,5) \to R(E_8,1) \to R(E_8,7) \to R(E_8,2) \to R(E_8,8)
$$

Now we will compute the number of roots in each triple root system corresponding to the rational triple trees classified by Artin ([1]) (see also Figures 7-15 where ■ is the vertex with weight 3 in each tree). Here we refer to them using the notation $A_{n,m,k}$, $B_{m,n}$, $C_{m,n}$, $D_{n,5}$, $F_n$, $H_n$ following Tjurina’s work ([21]) and $E_7,1$, $E_8,1$, $E_8,2$ for the rest which were unlabelled in either two articles. We will denote the number of elements in the set $R(\Gamma)$ by $|R(\Gamma)|$.

**Proposition 3.15.** The number of roots in $R(A_{n,m,k})$ is equal to

$$n^2 + m^2 + k^2 + n + m + k + 2(n + 1)(m + 1)(k + 1)$$

where the corresponding graph $A_{n,m,k}$ is a tree with $n + m + k + 1$ vertices as shown in Figure 7.

**Lemma 3.16.** Let us consider the Dynkin diagram $A_n$ (Figure 3). For any positive cycle $Y = \sum_{i=1}^{n} a_i E_i$, we have $(Y \cdot Y) \leq -a_n^2 - 1$. In particular, $(Y \cdot Y) \leq -2a_n$.

*Proof.* We prove the statement by induction on $n$. If $n = 1$, then $(Y \cdot Y) = -2a_1^2 \leq -a_1^2 - 1$. 

---

![Figure 6: Dynkin diagram of type $E_8$.](image)

![Figure 7: Triple diagram of type $A_{n,m,k}$.](image)
Assume that \( n \geq 2 \). Suppose that the statement is true in the case \( n - 1 \). Let \( Y' = \sum_{i=1}^{n-1} a_i E_i \) so that \( Y = Y' + a_n E_n \). By the hypothesis, \( (Y' \cdot Y') \leq -a_{n-1}^2 - 1 \). Hence

\[
(Y \cdot Y) = (Y' \cdot Y') + 2(Y' \cdot a_n E_n) - 2a_n^2 \\
\leq -a_{n-1}^2 - 1 + 2a_{n-1}a_n - 2a_n^2 \\
= -(a_{n-1} - a_n)^2 - a_n^2 - 1 \\
\leq -a_n^2 - 1.
\]

On the other hand, the second claim follows from the fact that \(-x^2 - 1 \leq -2x\) for any \( x \). This completes the proof.

**Proof of Proposition 3.15.** Let \( Y \) be a positive root in \( R(A_{n,m,k}) \). We can write \( Y = A + B + C + a_0 E_0 \), where \( A = \alpha_1 \alpha_2 + \cdots + \alpha_n A_n \), \( B = \beta_1 B_1 + \cdots + \beta_m B_m \), and \( C = \gamma_1 C_1 + \cdots + \gamma_k C_k \). Then \( a_0 = 0 \) or 1. If \( a_0 = 0 \), then \( Y = A, B, \) or \( C \) since \( \text{Supp}(Y) \) is connected. In this case the number of positive roots is equal to

\[
\frac{1}{2} \left( |R(A_n)| + |R(A_m)| + |R(A_k)| \right) = \frac{1}{2} (a^2 + n + m^2 + m + k^2 + k).
\]

(3.1)

Now let us assume that \( a_0 = 1 \). Note that

\[
(Y \cdot Y) = (A \cdot A) + 2\alpha_1 + (B \cdot B) + 2\beta_1 + (C \cdot C) + 2\gamma_1 - 3
\]

By Lemma 3.10, \((A \cdot A) + 2\alpha_1 \leq 0, (B \cdot B) + 2\beta_1 \leq 0, \) and \((C \cdot C) + 2\gamma_1 \leq 0 \). Since \( Y \) is a root of \( A_{n,m,k} \), we must have \((A \cdot A) + 2\alpha_1 = 0, (B \cdot B) + 2\beta_1 = 0 \) and \((C \cdot C) + 2\gamma_1 = 0 \). Furthermore, \(-2\alpha_1 = (A \cdot A) \leq -\alpha_1^2 - 1, \) and hence \( \alpha_1 = 1 \). Similarly, \( \beta_1 = \gamma_1 = 1 \). Therefore \((A \cdot A) = (B \cdot B) = (C \cdot C) = -2 \). Since \( A \in R(A_n), B \in R(A_m), \) and \( C \in R(A_k) \), we can easily verify that the number of positive roots with \( a_0 = 1 \) is equal to \((n + 1)(m + 1)(k + 1) \) since \( \text{Supp}(Y) \) is connected.

The sum in (3.1) and \((n + 1)(m + 1)(k + 1) \) add up to the number of positive roots. Clearly, we get the same number of negative roots. This concludes the proof.

**Proposition 3.17.** The number of roots in \( R(B_{m,n}) \) is

\[
n(n+1)(m+1) + m(m+1) + n(n+1)
\]

with \( m, n \geq 0 \).

Figure 8: Triple diagram of type \( B_{m,n} \).
Proof. We consider the diagram in Figure 8. Let $Y \in R(B_{m,n})$ be given by $Y = \sum_{i=0}^{m} a_i E_i + \sum_{j=1}^{n} b_j F_j$. First we study the case $a_0 = 0$. Since $\text{Supp}(Y)$ is connected, $Y \in R(A_m) := R(B_{m,n}) \cap \bigoplus_{i=1}^{m} \mathbb{Z} E_i$ or $Y \in R(A_n) := R(B_{m,n}) \cap \bigoplus_{j=1}^{n} \mathbb{Z} F_j$. The number of all roots with $a_0 = 0$ is then $|R(A_m)| + |R(A_n)| = m(m+1) + n(n+1)$. 

Next let us consider the case $a_0 = 1$. Put $D_1 = \sum_{i=1}^{m} a_i E_i$ and $D_2 = E_0 + \sum_{j=1}^{n} b_j F_j$. Note that 

$$-3 = (Y \cdot Y) = (D_1 + D_2)^2 = D_1^2 + 2a_1 + D_2^2$$

and that $D_1^2 + 2a_1 \leq 0$ by Lemma 3.10. Then $D_1^2 \geq -3$ and hence $D_2^2 = -3$.

Again by Lemma 3.10, we have $D_1^2 = -2$ or $D_1 = 0$ because $D_1^2 \leq -a_1^2 - 1 \leq -2a_1 = D_2^2$. Thus $D_1 \in R(A_m)$ or $D_1 = 0$, and $D_2 \in R(D_{n+1,1}) \cap \mathbb{Z} E_0 \oplus \bigoplus_{j=1}^{n} \mathbb{Z} F_j$. By the connectedness of $\text{Supp}(Y)$, we have 

$$\{ Y \in R(B_{m,n}) \mid a_0 = 1 \} = \left\{ D_1 + D_2 \mid D_1 = 0 \text{ or } D_1 \in R(A_m) \text{ with } a_1 > 0 \right\}.$$

Note that $|R(D_{n+1,1})| = |R(D_{n+1})|$ by Example 3.11. The number of elements in $\{ D_2 = E_0 + \sum_{j=1}^{n} b_j F_j \in R(D_{n+1,1}) \}$ is equal to the number of elements in $R(D_{n+1}) \cap \{ D \in R(D_{n+1}) \mid |\text{Supp}(D) \cap \{ F_j \mid j = 1, 2, \ldots, n \}| = 1 \}$. Here we denote by $R(\cdot)$ the positive roots in $R(\cdot)$. It is easy to check that the number of elements in $\{ D_1 \mid D_1 = 0 \text{ or } D_1 \in R(A_m) \text{ with } a_1 > 0 \}$ is $m + 1$. Therefore 

$$|\{ Y \in R(B_{m,n}) \mid a_0 = 1 \}| = (m+1)(|R(D_{n+1})| + |R(A_n)|) = (m+1)n(n+1)/2.$$

In the same way we see that the number of all roots with $a_0 = -1$ is equal to $n(n+1)(m+1)/2$. Summing up all of the numbers above, we obtain $|R(B_{m,n})|$. \hfill \Box

**Proposition 3.18.** The number of roots in $R(C_{m,n})$ is $2m^2 + 4mn + n^2 + 2m + n$.

![Figure 9: Triple diagram of type $C_{m,n}$.](image)

Proof. The tree $C_{m,n}$ (Figure 9) can be seen as the gluing of rational double trees of type $A_n$ and $D_m$ by the vertex $E_0$. Consider a root $Y = \sum_{i=0}^{m} a_i E_i + \sum_{j=1}^{n} b_j F_j \in R(C_{m,n})$. Then $a_0 = 0$ or $\pm 1$. If $a_0 = 0$ then $Y$ is supported on $A_n$ or $D_m$. If it is supported on $A_n$, we have $|R(A_n)| = n(n+1)$ roots and, if it is supported on $D_m$, we have $|R(D_m)| = 2m(m-1)$ roots.

Let us consider the case $a_0 = 1$. Note that the underlying diagram of $C_{m,n}$ is of type $D_{m+n+1}$. Comparing the bilinear forms of $C_{m,n}$ and $D_{m+n+1}$, we have $(Y \cdot Y) = (Y \cdot Y)_{D_{m+n+1}} - a_0^2$, where we denote by $(\cdot, \cdot)_{D_{m+n+1}}$ the bilinear form of $D_{m+n+1}$. Since $a_0 = 1$ and $(Y \cdot Y) = -3$, $(Y \cdot Y)_{D_{m+n+1}} = -2$. Hence $Y$ can be regarded as a positive root of $R(D_{m+n+1})$ with $a_0 = 1$. Now let us
calculate the number of such roots. Let \( \Delta = \{ \pm e_i \pm e_j \mid 1 \leq i \neq j \leq m + n + 1 \} \) be the set of roots in \( D_{m+n+1} \). The positive simple roots are \( \alpha_i = e_i - e_{i+1} \), \( 1 \leq i < m + n + 1 \), and \( \alpha_{m+n+1} = e_{n-1} + e_n \). The set of all positive roots satisfying the condition \( a_0 = 1 \) is \( \{ e_i \pm e_j \mid 1 \leq i \leq n + 1 < j \leq m + n + 1 \} \).

Hence the number of roots with \( a_0 = 1 \) is \( 2mn + 2m \). We can also obtain the same result for \( a_0 = -1 \). Then

\[
|R(C_{m,n})| = n(n+1) + 2m(m-1) + 4mn + 4m = 2m^2 + 4mn + n^2 + 2m + n.
\]

This concludes the proof. \( \square \)

**Proposition 3.19.** The number of roots in \( R(D_{n,5}) \) is \( n^2 + 33n + 72 \).

\[
\text{Figure 10: Triple diagram of type } D_{n,5}.
\]

**Proof.** The tree \( D_{n,5} \) (see Figure 10) can be seen as the glueing of the rational trees \( A_n \) and \( D_5 \) by the vertex \( E_0 \). If \( n = 0 \), by Example 3.12, we have \( R(D_{n,5}) = R(E_{0,1}) = R(E_0) \), so \( |R(D_{n,5})| = 72 \). Notice that the number of roots \( Y \) with \( (Y \cdot Y) = -2 \) is given by \( |R(D_3)| \) which is equal to 40.

Now, assume that \( n > 0 \). Let \( Y = \sum_{i=0}^{n} a_i E_i + \sum_{j=1}^{5} b_j F_j \) be a root in \( R(D_{n,5}) \). By Proposition 3.5, we have \( a_0 = 0 \) or \( a_0 = \pm 1 \) in \( Y \). If \( a_0 = 0 \), \( Y \) is supported either on \( A_n \) or on \( D_5 \). So we have \( |R(A_n)| + |R(D_5)| = n(n+1) + 40 \) roots, in total, for \( a = 0 \).

Assume that \( a_0 = 1 \). Put \( D = \sum_{i=1}^{n} a_i E_i \) and \( F = E_0 + \sum_{j=1}^{5} b_j F_j \). Then we have \( -3 = (Y \cdot Y) = (D + F)^2 = (D \cdot D) + 2a_1 + (F \cdot F) \). Since \( (D \cdot D) + 2a_1 \leq (D \cdot D) + a_1^2 + 1 \leq 0 \) and \( (F \cdot F) \leq -3 \), we see that \( (D \cdot D) = -2 \) and \( (F \cdot F) = -3 \). By the connectedness of \( \text{Supp}(Y) \), \( D = 0 \) or \( D = \sum_{j=1}^{k} E_j \) for some \( 1 \leq k \leq n \), and \( F = E_0 \) or \( b_1 > 0 \). The number of elements in \( \{ F \mid F = E_0 \text{ or } b_1 > 0 \} \) is equal to \( (72 - 40)/2 = 16 \) since \( |R(D_{n,5})| = 72 \) and \( |R(D_5)| = 40 \). Hence there exist 16\((n+1) \) roots with \( a_0 = 1 \). Similarly, there exist 16\((n+1) \) roots with \( a_0 = -1 \). Therefore we have \( |R(D_{n,5})| = n(n+1) + 40 + 2 \cdot 16(n+1) = n^2 + 33n + 72 \). \( \square \)

**Proposition 3.20.** The number of roots in \( R(F_n) \) is \( n^2 + 55n + 126 \).

\[
\text{Figure 11: Triple diagram of type } F_n.
\]

**Proof.** The idea is the same as in the preceding proof. The tree \( F_n \) (Figure 11) is the glueing of the rational trees \( A_n \) and \( E_6 \). In case \( n = 0 \), we obtain \( |R(F_0)| = |R(E_{7,1})| = |R(E_7)| \) which is equal to 126.
Lemma 3.22. Proposition [3.21] is a consequence of the following two lemmas.

Proof. If $n = 5$, then $H_5 = D_{5,5}$ (see Figure 12). The fact that $R(D_{5,5}) = R(D_5)$ implies that $|R(H_5)| = 40$. This completes the proof. 

Lemma 3.23. For each $n \geq 6$, we have $|R(H_n)| - |R(H_{n-1})| = n^2 - n$. In particular, $|R(H_n)| = (n^3 - n)/3$ for $n \geq 5$.

Proof. We have $R(H_n) \supset R(H_{n-1}) := R(H_n) \cap \oplus_{i=1}^{n-1} \mathbb{Z}E_i$. We only need to show that the number of the elements in $R(H_n) \setminus R(H_{n-1})$ is $n^2 - n$. If $Y = \sum_{i=0}^{n-1} a_i E_i \in R(H_n) \setminus R(H_{n-1})$, then $Y$ is one of the following three types:

1. $Y \in R(A_{n-1}) = R(H_n) \cap \oplus_{i=1}^{n-1} \mathbb{Z}E_i$ and $a_{n-1} \neq 0$,
2. $Y \in R(D_{n-1,1}) = R(H_n) \cap \mathbb{Z}E_0 \oplus (\oplus_{i=2}^{n-1} \mathbb{Z}E_i)$, $a_0 \neq 0$, and $a_{n-1} \neq 0$,
3. the case that $a_0 \neq 0$, $a_1 \neq 0$, and $a_{n-1} \neq 0$.

It is easy to check that the number of all roots of type (1) is $2(n - 1)$ and that the number of all roots of type (2) is $2(n - 2)$. We claim that the number of all roots of type (3) is $(n - 2)(n - 3)$. If the claim is true, then $|R(H_n) \setminus R(H_{n-1})| = 2(n - 1) + 2(n - 2) + (n - 2)(n - 3) = n^2 - n$, and hence the proof is completed. For proving this claim, we have two lemmas.

Lemma 3.24. If $Y = \sum_{i=0}^{n-1} a_i E_i$ is a positive root of type (3), then $Y$ satisfies the following conditions:

(a) $a_0 = a_1 = a_{n-1} = 1$,
(b) \(a_1 \leq a_2 \leq a_3 \geq a_4 \geq a_5 \geq \cdots \geq a_{n-1} \) and \(a_3 \leq 3\),

(c) \(|a_i - a_{i+1}| \leq 1\) for each \(1 \leq i \leq n - 2\).

**Proof of Lemma 3.24** We prove the lemma by induction. If \(n = 5\), then the statement is true. Suppose that it is also true for \(n - 1\). Now let us consider the \(n\) case. Put \(D = E_1 + E_2 + \cdots + E_{n-1}\). Since \(\text{Supp}(Y)\) is connected, \(Y - D > 0\). Now we show that \(Y - D \in R(H_n)\). Indeed, by easy calculation we have \((Y - D)^2 = Y^2 - 2YD + D^2 = -3 + 2(a_1 + a_{n-1} - 1) - 2 = -7 + 2(a_1 + a_{n-1})\).

Here remark that \(Y^2 = -3\) and \(a_0 = 1\) by Proposition 5.3. Lemma 3.2 says that \((Y - D)^2 \leq -2\). On the other hands, the assumption \(a_1 > 0\) and \(a_{n-1} > 0\) implies that \((Y - D)^2 \geq -3\). Since \((Y - D)^2\) is odd, \((Y - D)^2 = -3\) and \(a_1 = a_{n-1} = 1\). Hence we have shown \(Y - D \in R(H_n)\) and the condition (a).

Since \(a_{n-1} = 1\), we see that \(Y - D \in R(H_{n-1}) = R(H_n) \cap \bigoplus_{i=0}^{n-2} E_i\). By induction, \(Y - D\) is a root of type (2) or \(Y - D\) satisfies the conditions (a), (b), and (c). It is easy to check that \(Y\) satisfies the conditions (a), (b), and (c) except \(a_3 \leq 3\). Finally we see that \(a_3 \leq a_2 + 1 \leq a_1 + 2 = 3\). This completes the proof of the lemma.

**Lemma 3.25.** The number of all roots of type (3) in Lemma 3.24 is equal to \((n - 2)(n - 3)\).

**Proof.** It is easy to see that any cycle satisfying the conditions (a), (b), and (c) is a root. Therefore we only need to show that the number of all positive cycles satisfying (a)-(c) is equal to \((n - 2)(n - 3)/2\). For listing up all roots we denote a root by \((a_2, a_3, a_4, \ldots, a_{n-2})\) since \(a_0 = a_1 = a_{n-1} = 1\). All roots can be grouped into the following forms

1. \((1, 1, 1, \ldots, 1)\),
2. \((1, 2, 2, \ldots, 1, \ldots, 1)\),
3. \((2, 2, 2, \ldots, 1, \ldots, 1)\),
4. \((2, 3, 3, \ldots, 3, 2, \ldots, 2)\), where \(i \geq 1, j \geq 0\) with \(i + j = n - 4\), and
5. \((2, 3, 3, \ldots, 3, 2, \ldots, 2, 1, \ldots, 1)\) where \(i, j \geq 1, k \geq 0\) with \(i + j + k = n - 4\).

We can easily check that the number of roots in the 5 cases above are \(1, (n - 4), (n - 4), (n - 4),\) and \((n - 5)(n - 6)/2\), respectively. Summing up these, we obtain the number of all roots of type (3) which is \((n - 2)(n - 3)\).

Let us return the proof of Proposition 3.23. Now we have \(|R(H_n) \setminus R(H_{n-1})| = 2(n - 1) + 2(n - 2) + (n - 2)(n - 3) = n^2 - n\) By Lemma 3.22 we also have \(|R(H_n)| = (n^3 - n)/3\) for \(n \geq 5\). Therefore we have finished the proof of Proposition 3.23.

**Proposition 3.26.** The number of roots in \(R(E_7,1)\) is 124.
Proof. Note that $R(E_7) \supset R(E_{7,1})$. Let $Z(E_7)$ be the Artin cycle of $R(E_7)$. The coefficient of $E_1$ in $Z(E_7)$ is 2. For each positive root in $R(E_7)$ except $Z(E_7)$ the coefficient of $E_1$ is less than 2. Hence $R(E_7) \setminus \{ \pm Z(E_7) \} = R(E_{7,1})$. Therefore $\#R(E_{7,1}) = \#R(E_7) - 2 = 124$.

**Proposition 3.27.** The number of roots in $R(E_{8,1})$ is 238.

Proof. Note that $R(E_8) \supset R(E_{8,1})$. Let $Z(E_8)$ be the Artin cycle of $R(E_8)$. The coefficient of $E_1$ in $Z(E_8)$ is 2. For each positive root in $R(E_8)$ except $Z(E_8)$, the coefficient of $E_1$ is less than 2. Hence $R(E_8) \setminus \{ \pm Z(E_8) \} = R(E_{8,1})$. Therefore $\#R(E_{8,1}) = \#R(E_8) - 2 = 238$.

**Proposition 3.28.** The number of roots in $R(E_{8,2})$ is 212.
We have 14 positive roots, so 28 roots, which are not contained in $R(E_{8,2})$.
Then $\#R(E_{8,2}) = R(E_8) - 28 = 212$.

For each triple root case we can easily verify the following statements.

**Theorem 3.29.** Let $\Gamma$ be a triple Dynkin diagram. Let $Z(\Gamma)$ be the Artin cycle of $\Gamma$. Then any root $Y$ in $R(\Gamma)$ satisfies $Y \leq Z(\Gamma)$.

**Proof.** For each case, we can verify the statement from the discussions above.

**Remark 3.30.** Let $\Gamma_0$ be a Dynkin diagram of type $A, D$ or $E$. Let $\Gamma$ be a triple Dynkin diagram obtained by replacing a vertex $E_i$ of $\Gamma_0$ by a vertex with weight 3. Then there exists the highest root $Z(\Gamma) = \sum a_i E_i$ among the roots in $R(\Gamma_0)$ with $a_i = 1$. Furthermore $R(\Gamma) = \{Y \in R(\Gamma_0) \mid -Z(\Gamma) \leq Y \leq Z(\Gamma)\}$ and $Z(\Gamma)$ is the highest root of $R(\Gamma)$.

4 Construction of the triple roots

Let $\Gamma$ be a rational triple tree with vertices $E_0, \ldots, E_n$ where $E_0$ denotes the vertex with weight 3. First we use Laufer’s algorithm to construct the highest root as follows.

Let $Z(\Gamma)$ denote the Artin cycle of $\Gamma$. Consider $Z_1 = \sum_{i=1}^n E_i$. If $(Z_1 \cdot E_i) \leq 0$ for all $i$, then $Z_1 = Z$. If else, there exists an $E_i$ such that $(Z_1 \cdot E_i) > 0$; in this case, we put $Z_2 = Z_1 + E_i$, and we see whether $(Z_2 \cdot E_i) \leq 0$ for all $i$. The term $Z_j$, ($j \geq 1$), of the sequence satisfies, either $(Z_1 \cdot E_i) \leq 0$ for all $i$, then we put $Z = Z_j$, or there is an irreducible component $E_i$ such that $(Z_1 \cdot E_i) > 0$, then we put $Z_{j+1} = Z_j + E_i$. Thus the Artin cycle (equivalently, the highest root) of $\Gamma$ is the first cycle $Z_k$ of this sequence such that $(Z_k \cdot E_i) \leq 0$ for all $i$.

Now, we can construct the elements of the set of positive roots $R^+(\Gamma)$.

Denote by $E_i^{(0)}$ the vertices of $\Gamma$ adjacent to $E_0$ and $a_i^{(0)}$ their coefficients in a given root. Let $v_\Gamma$ denotes the valency of $E_0$ in $\Gamma$. After computing the highest root by using Laufer’s algorithm, we continue as follows:

1. If $\sum_{i=1}^n E_i^{(0)} = 1$ where $a_i^{(0)}$ represents the coefficient of $E_i^{(0)}$ in $Z(\Gamma)$, take all the vertices $E_i$ of $\Gamma$ such that $(Z(\Gamma) \cdot E_i) < 0$; if not, take only the vertices with weight 2 of $\Gamma$ such that $(Z(\Gamma) \cdot E_i) < 0$. By reindexation, let us denote these vertices by $F_1, \ldots, F_k$, ($k \leq n$).

2. Let $Y^{(0)} = Z(\Gamma)$. Put, for each $j$, $(j = 1, \ldots, k)$, $Y_j^{(1)} = Y^{(0)} - F_j$. We have $Y_j^{(1)} \in R^+(\Gamma)$ since $(Y_j^{(1)} \cdot Y_j^{(1)}) = -3$ or $(Y_j^{(1)} \cdot Y_j^{(1)}) = -2$ depending on the case (1) above.

3. Now, for each $j$, replace $Z(\Gamma)$ by $Y_j^{(1)}$ in the case (1) and repeat the same process for $Y_j^{(1)}$, $(j = 1, \ldots, k)$.

This process stops when $Y_j^{(l)}$ for some $l$ equals one of the simple roots $E_0, E_1, \ldots, E_n$ in $R^+(\Gamma)$.

**Lemma 4.1.** With preceding notation, let $Y$ be a root in $R^+(\Gamma)$. If $(Y \cdot E_i) = 0$, then $Y - E_i$ is not a root.
Proof. It is obvious since \((Y-E_1) \cdot (Y-E_i) = 0\).

Lemma 4.2. With preceding notation, if \(\sum_{i=0}^{\nu}(E_0) \ a_i^{(0)} = 1\) in a given root \(Y\) in \(R^+(\Gamma)\), then \(Y-E_0\) is a root in \(R^+(\Gamma)\).

Proof. We know that, in a rational triple tree \(\Gamma\), \(v(E_0)\) can be at most 3, so \(\sum_{i=0}^{\nu} a_i^{(0)} \leq 3\). Hence, by lemma 22 and 23, it is sufficient to see only the cases when \(a_1^{(0)} = 1\) or 2 and \(a_2^{(0)} = a_3^{(0)} = 0\). If \(a_1^{(0)} = 2\) in \(Y \in R^+(\Gamma)\), we have \((Y \cdot E_0) = -1\); this implies \((Y-E_0) \cdot (Y-E_0) = -4\). If \(a_1^{(0)} = 1\) in \(Y \in R^+(\Gamma)\), we have \((Y \cdot E_0) = -2\); this implies \((Y-E_0) \cdot (Y-E_0) = -2\), so \(Y-E_0 \in R^+(\Gamma)\).

Finally, we have the following:

Theorem 4.3 (see [20]). Let \(R^+(E) = \{Y_0, \ldots, Y_k\}\) with \(Y_k = Z(\Gamma)\). Then, for each \(j = 0, \ldots, k-1\), there exists an element \(Y_j\) in \(R^+(E)\) such that \((Y_j \cdot E_i) < 0\) and \(Y_j = Y_i - E_i\) for some \(i\). Inversely, for each simple positive root \(E_i\) in \(R^+(\Gamma)\), \(Y_i + E_i\) with \((Y_i \cdot E_i) = 1\), \((i = 1, \ldots, n)\), is a root in \(R^+(E)\).

Proof. The existence of at least one vertex \(E_i\) in each \(Y_j\) such that \((Y_j \cdot E_i) < 0\) is due to negative definiteness of the intersection matrix of \(\Gamma\). Then, theorem follows from the construction above.

5 Linear free divisors

A reduced hypersurface \(D \subset (\mathbb{C}^n, 0)\) is called a free divisor if the module \(\text{Der}(-\log D)\) of logarithmic vector fields along \(D\) is a locally free module of rank \(n\) over \(O_{\mathbb{C}^n, 0}\). By Saito’s criterion ([13]), \(D\) is a free divisor if and only if there exists a basis \(\chi_1, \ldots, \chi_n\) of \(\text{Der}(-\log D)\) such that the determinant of the matrix formed by the coefficients of \(\chi_i\) is a reduced equation defining \(D\). If, in particular, each \(\chi_i\) is a weight zero vector field, i.e. of the form \(\sum_{\ell = 0}^{m} \xi_{ij} x_j \frac{\partial}{\partial x_i}\), for some \(\xi_{ij} \in \mathbb{C}\), then \(D\) is a linear free divisor. In this section, we will recall the examples of linear free divisors arising in representation theory.

Let \(\alpha : GL_n(\mathbb{C}) \times \mathbb{C}^n \to \mathbb{C}^n\) be a group action given by the right multiplication. Consider the restriction

\[
\alpha_x : GL_n(\mathbb{C}) \times \{x\} \to \mathbb{C}^n
\]

for some \(x \in \mathbb{C}^n\). Then each element \(v \in \mathfrak{gl}_n\) gives rise to a vector field \(\chi_n\) on \(\mathbb{C}^n\) defined by

\[
\chi_n(x) = d_x \alpha_x(v).
\]

In particular, the elementary matrix \(\epsilon_{ij}\), which has 1 in the \((i,j)\)-th entry, 0 everywhere else, corresponds to the vector field \(x_j \frac{\partial}{\partial x_i}\).

More generally, let \(G \subset GL_n(\mathbb{C})\) be a connected algebraic subgroup of dimension \(n\). Let \(\chi_1, \ldots, \chi_n\) be vector fields generating the infinitesimal action of \(G\) induced by \(\gamma\). Then, each \(\chi_i\) is of the form \(\sum_{\ell = 0}^{m} \xi_{ij} x_j \frac{\partial}{\partial x_i}\), for some \(\xi_{ij} \in \mathbb{C}\), and corresponds to the matrix \(\sum_{\ell = 0}^{m} \xi_{ij} \epsilon_{ij}\). Let \(\Delta\) be the matrix of the coefficients \(\{\chi_1, \ldots, \chi_n\}\) with respect to the standard basis \(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\) of \(\text{Der}_\mathbb{C} \mathbb{C}^n\). Then, the determinant of \(\Delta\) defines the discriminant \(D\) which consists of points \(x\) where \(\chi_1, \ldots, \chi_n\) fail to span the tangent space \(T_x \mathbb{C}^n\). Furthermore, \(D\) is a
homogeneous divisor in $\mathbb{C}^n$ of degree $n$. By Saito’s criterion, if the determinant of $\Delta$ is reduced then $D$ is a free divisor (necessarily linear) and $\chi_1, \ldots, \chi_n$ form a basis of $\text{Der}(\log D)$ ([9, Lemma 2.4]).

Conversely, assume that $D$ is a linear free divisor and consider the group

$$G_D := \{ A \in \text{GL}_n(\mathbb{C}) \mid A(D) = D \}.$$ 

Let $G^0_D$ be the connected component of $G_D$ containing the identity element. Then $\mathbb{C}^n \setminus D$ is a single orbit of $G^0_D$ ([9, Lemma 2.3]).

In the following section, we study examples of free divisors arising in quiver representations.

5.1 Rational triple quivers and representations

A quiver $Q$ is an oriented graph together with sets $Q_0$ and $Q_1$ consisting of the vertices and arrows, respectively. A representation $V$ of a quiver over a field $k$ consists of a vector space $V_v$ for each vertex $v \in Q_0$ and a $k$-linear map $V(\alpha) \colon V_{t\alpha} \to V_h\alpha$ for each arrow $\alpha \in Q_1$, where $t\alpha$ is the start and $h\alpha$ is the end of the arrow $\alpha$.

Let $a := (a_v)_{v \in Q_0} \in \mathbb{N}^{|Q_0|}$ be the dimension vector assigned to $Q$. Then, the $k$-vector space of representations of $Q$ is defined by

$$\text{Rep}(Q, a) := \prod_{\alpha \in Q_1} \text{Hom}_k(V_{t\alpha}, V_{h\alpha}) \cong \prod_{\alpha \in Q_1} \text{Hom}_k(k^{a_\alpha}, k^{a_{h\alpha}}).$$

The group $\text{GL}(Q, a) := \prod_{v \in Q} \text{GL}_{a_v}(k)$ acts on $\text{Rep}(Q, a)$ by

$$((g_v)_{v \in Q_0}, V)_{\alpha \in Q_1} \mapsto (g_{h\alpha} \cdot A_\alpha \cdot g_{t\alpha}^{-1})_{\alpha \in Q_1},$$

where $(g_v)_{v \in Q_0} \in \text{GL}(Q, a)$.

A morphism $\phi \colon V \to W$ of representations is well-defined if there exists a commutative diagram

$$
\begin{array}{ccc}
V_{t\alpha} & \xrightarrow{V(\alpha)} & V_{h\alpha} \\
\phi_{t\alpha} \downarrow & & \downarrow \phi_{h\alpha} \\
W_{t\alpha} & \xrightarrow{W(\alpha)} & W_{h\alpha}
\end{array}
$$

where $k$-linear maps $\phi_{t\alpha}$ and $\phi_{h\alpha}$, for each $\alpha \in Q_1$. Moreover, $\phi$ is an isomorphism if $\phi_v$ is an isomorphism for all $v \in Q_0$. On the other hand, the direct sum of two representations $V \in \text{Rep}(Q, a)$ and $W \in \text{Rep}(Q, b)$ is $V \oplus W \in \text{Rep}(Q, a + b)$ with $(V \oplus W)_v := V_v \oplus W_v$ and

$$(V \oplus W)(\alpha) := \begin{bmatrix} V_\alpha & 0 \\ 0 & W_\alpha \end{bmatrix}$$

for all $v \in Q_0$ and $\alpha \in Q_1$.

**Definition 5.1** ([9]). A representation $V' \in \text{Rep}(Q, a')$ is decomposable if it is isomorphic to the direct sum of two nontrivial representations, that is, $V' = V \oplus W$ for $V \in \text{Rep}(Q, a)$ and $W \in \text{Rep}(Q, b)$ and $a' = a + b$. Otherwise, $V'$ is called indecomposable. A quiver is of finite representation type if it has only finitely many indecomposable representations, up to isomorphism.
The Tits form of a dimension vector $a$ is given by

$$\langle a, a \rangle := \sum_{v \in Q_0} a_v^2 - \sum_{\alpha \in Q_1} a_{t \alpha} a_{h \alpha} = \dim_k \prod_{v \in Q_0} \text{Hom}_k(V_v, V_v) - \dim_k \prod_{\alpha \in Q_1} \text{Hom}_k(V_{t \alpha}, V_{h \alpha}).$$

**Proposition 5.2.** Let $\Gamma$ be a rational triple quiver. Then $\langle a, a \rangle = 1$ if and only if $Y \in R(\Gamma)$ where $Y = \sum a_i E_i$.

**Proof.** It follows from definitions of the Tits form and the intersection matrix associated to the rational triple quiver. \hfill \Box

Therefore, from now on, we will use the term root for a dimension vector assigned to any rational triple quiver.

**Corollary 5.3.** For each root $a$, $\text{Rep}(Q, a)$ contains an indecomposable representation.

**Definition 5.4.** A root is called sincere if its support is $\bigcup_{i=1}^n E_i$.

**Theorem 5.5** (Corollary 5.5, [4]). Let $Q$ be a Dynkin quiver and $d$ be a real root. Then the discriminant $D$, of the action of $\text{GL}(Q, a)$ on $\text{Rep}(Q, a)$ is a linear free divisor.

Let $\Gamma$ denotes one of the trees studied in Proposition 3.15-3.21. We will denote a rational triple tree $\Gamma$ by $Q$ after assigning an orientation and call it a rational triple quiver.

**Lemma 5.6.** Any subquiver of a rational triple quiver is a rational triple or a Dynkin quiver.

This comes from Proposition 2.2.

### 5.2 Rational triple quivers and linear free divisors

A vertex $v \in Q_0$ is called a source (resp. sink) if there is no arrow ending (resp. starting) at $v$. Assume that $a = (a, a, \ldots, a) \in R(Q)$ (means in fact $a \in R(\Gamma)$).

**Definition 5.7** ([2]). Let $V \in \text{Rep}(Q, a)$ be a quiver representation. Let $v \in Q_0$ be a source. The reflection functor with respect to $v$ is a transformation which produces a representation $V^* \in \text{Rep}(Q, a^*)$ where $a^*_v = \sum_{v \rightarrow v_i \in Q_1} a_{v_i} - a_v$ and $a^*_v = a_v$ for $v_i \neq v$, all arrows involving $v$ are reversed, and $V^*_v$ is the cokernel of the map

$$(f_{v \rightarrow v_i})_{v \rightarrow v_i \in Q_1} : V_v \rightarrow \bigoplus_{v \rightarrow v_i \in Q_1} V_{v_i}$$

and $V_{v_i} = V^*_v$ for $v_i \neq v$. A similar definition exists in the case of a sink. If $v \in Q_0$ is a sink, then $V^* \in \text{Rep}(Q, a^*)$ is obtained by reversing all arrows involving $v$, setting $a^*$ as above and $V^*_v$ as the kernel of the map

$$(f_{v \leftarrow v_i})_{v \leftarrow v_i \in Q_1} : \bigoplus_{v \leftarrow v_i \in Q_1} V_{v_i} \rightarrow V_v$$

and $V_{v_i} = V^*_v$ for $v_i \neq v$. \hfill 17
It was shown by Sato and Kimura that reflection functors (or castling transformations as they call them) give a one-to-one correspondence between relative invariants of representations. Moreover, if $D$ is a linear free divisor coming from a quiver representation $V$ then the discriminant $D^*$ of $V^*$ is also a linear free divisor ([8, Proposition 2.10]). Using this result and Theorem 5.5 we will prove that if $Q$ is a rational triple quiver then its representation also yields a linear free divisor. First we prove the following lemma to simplify our claims.

**Lemma 5.8.** Let $Q$ be a quiver of type $A_n$ (see Figure 2) with any chosen orientation. Then any representation $V \in \text{Rep}(Q, a)$ can be transformed into the trivial representation of the subquiver by successive reflection functors; in other words, into $\{0\} = \text{Rep}(Q^*, a^*)$ where $Q^*$ is the quiver consisting of just one vertex $v_n$ with $a^* = a$.

**Proof.** For any given orientation, the vertex $v_1$ at the left end is either a sink or a source. In either case, by applying the appropriate reflection with respect to $v_1$, we get a subrepresentation $V^*$ of $V$ with $V^*_v = 0$, $V^*_v = V_v$, and $a_{v_1} = 0$, $a_{v_i} = a$ for all $i = 2, \ldots, n$. Therefore, the claim easily follows by applying reflection functors at the vertices $v_1$, $v_2$ and $v_{n-1}$ in the given order.

**Example 5.9.** Let us consider the rational triple quiver of type $B_{m,n}$ with any given orientation and the root $a = (a_1, \ldots, a_n, 1, \ldots, 1)$. Then the discriminant in $\text{Rep}(Q, a)$ is a linear free divisor. See Figure 16 for the quiver with an arbitrary orientation.

![Figure 16: A representation of a rational triple quiver of type $B_{m,n}$.](image)

By Lemma 5.9, $Q$ can transformed into a Dynkin quiver $Q^*$ of type $D_{n+1}$ with the root $a^* = (a_1, \ldots, a_n, 0, \ldots, 0)$. For example, the quiver in Figure 16 is transformed into the quiver in Figure 17.

![Figure 17: A representation of a rational triple quiver of type $B_{m,n}$.](image)

An easy calculation shows that $(a, a) = 1$ if and only if $(a^*, a^*) = 1$; in other words, $a$ is a root if and only if $a^*$ is. Therefore, the discriminant $D$ in
$\text{Rep}(Q, a)$ is a linear free divisor since $D^*$ in $\text{Rep}(Q^*, a^*)$ is a linear free divisor by Theorem 5.5 and [8, Proposition 2.10]). Moreover, we have

$$\text{Rep}(Q, a) \cong \text{Rep}(Q^*, a^*) \times \mathbb{C}^m,$$

and if $D^* = V(f)$ then $D = V(A_1 A_2 \cdots A_m f)$ where $A_i$ are the maps $\mathbb{C} \to \mathbb{C}$ in the representation. So, the divisors $D$ and $D^*$ are related as follows

$$D = (D^* \times \mathbb{C}^m) \bigcup \left( \bigcup_{i=1}^m \mathbb{C}^{\dim D^* + 1} \times V(A_i) \right) \quad (5.2)$$

(cf. [8, Proposition 3.32]).

**Theorem 5.10** (Theorem 3.10, [8]). Let $Q$ be any quiver and $a$ a root associated with $Q$. If the discriminant in the representation space of $Q$ is a linear free divisor then $Q$ is a tree.

In the following theorem, we present a new case where the converse of Theorem 5.10 is true.

**Theorem 5.11.** Let $Q$ be a rational triple quiver and $a$ be a sincere triple root in $R(Q)$. Then, the discriminant in the representation space $\text{Rep}(Q, a)$ is a linear free divisor independent of the orientation.

**Proof.** We notice that the underlying graphs $E_{7,1}$, $E_{8,1}$ and $E_{8,2}$ are of type $E_7$, $E_8$ and $E_8$, respectively. Since $a$ is a root, the claim follows from Theorem 5.5.

As for the series $A_{n,m,k}, B_{m,n}, C_{m,n}, D_{n,5}$ and $F_n$ (see Figures 7-11), the general forms of the roots are indicated in Table 1. Each of them necessarily consists of a subquiver of type $A_{n-1}$ with 1 at its vertices for some $n$ (cf. the results in Section 3). In other words, $a = a_1 \oplus a_2$ for $a_1 = (1, \ldots, 1) \in \mathbb{N}^n$ and $a_2 \in \mathbb{N}^\ell$ for some $\ell \geq 3$. We have already observed that a rational triple quiver of type $B_{m,n}$ with such dimension vector yields a linear free divisor in Example 5.9. Similarly, by Lemma 5.8, the representations of $A_{m,n,k}, C_{m,n}, D_{n,5}$ and $F_n$ can be transformed into the representations of the Dynkin quivers of types $A_{m+n}, D_m, E_6$ and $E_7$, in the given order. See Table 1 for a summary in which $\downarrow$ represents the vertex of weight $-3$. Moreover, $a$ is a root if and only if $a_2$ is a root. Therefore, in each case the discriminant is linear free divisor by Theorem 5.5 and [8, Proposition 2.10]).
Table 1: Rational triple quivers

| Label       | \( \mathbf{a} \) | \( \mathbf{a}^* \) | Dynkin type |
|-------------|------------------|-------------------|------------|
| \( A_{m,n,k} \) | 1 ... 1 \( \begin{array}{l}1 \\ \vdots \\ 1 \end{array} \) | 0 ... 0 \( \begin{array}{l}1 \\ \vdots \\ 1 \end{array} \) | \( A_{n+k} \) |
| \( B_{m,n} \) | 1 \( \begin{array}{l}a_1 \\ a_2 \\ \vdots \\ a_n \end{array} \) | 0 ... 0 \( \begin{array}{l}a_1 \\ a_2 \\ \vdots \\ a_n \end{array} \) | \( D_{n+1} \) |
| \( C_{m,n} \) | 1 \( \begin{array}{l}a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_m \end{array} \) | 0 ... 0 \( \begin{array}{l}a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_m \end{array} \) | \( D_{m+1} \) |
| \( D_{n,5} \) | 1 \( \begin{array}{l}a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \) | 0 ... 0 \( \begin{array}{l}a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \) | \( E_6 \) |
| \( F_n \) | 1 \( \begin{array}{l}a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array} \) | 0 ... 0 \( \begin{array}{l}a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array} \) | \( E_7 \) |

Now, we prove the claim for \( H_n \) explicitly by considering the sincere roots which are listed in the proof of Lemma 3.25. Note that we do not indicate the directions of the arrows in the figures as our claim is independent of the given orientation.

1. \( \mathbf{a} = (1, 1, \ldots, 1) \in \mathbb{N}^n. \) A quiver representation \( \text{Rep}(Q, \mathbf{a}) \) is given in Figure 18 where \( Q_1 = \{A_0, A_1, \ldots, A_{n-1}\}. \) It is easy to see that the discriminant is the normal crossing divisor

\[
D = \{A_0 A_1 \cdots A_{n-1} = 0\} \subset \text{Rep}(Q, \mathbf{a}) \cong \mathbb{C}^n.
\]

Hence, it is a linear free divisor.

![Figure 18: A quiver representation of type \( H_n \) – orientation not shown.](image)

2. \( \mathbf{a} = (1, 1, 2, 2, \ldots, 2, 1, \ldots, 1, 1) \in \mathbb{N}^n \) where \( i \geq 1, j \geq 0 \) with \( i + j = n - 4. \)

The corresponding quiver representation \( \text{Rep}(Q, \mathbf{a}) \) is given in Figure 19.
Notice that a reflection with respect to \( v_1 \) transforms the representation into a representation as shown in Figure 20 whose underlying graph is of the Dynkin type \( D_{n-1} \), with the root \( \alpha' = (1,0,1,2,\ldots,2,1,\ldots,1,1) \). Hence the claim follows.

Let us consider the subquiver \( Q' \) consisting of the vertices \( v_1, v_2, v_3 \). There are four possible choices of orientation for \( Q' \).

(a)
Notice that if the orientation between $v_1, v_2$ and $v_3$ is as in (c) or (d), a reflection with respect to $v_1$, reverses the arrow between $v_1$ and $v_2$ but leaves the dimensions unchanged. Therefore, it transforms the representation into one of the cases (a) or (b). If the orientation between $v_1, v_2$ and $v_3$ is as in (a) or (b), a reflection with respect to $v_2$ transforms the representation into a representation given in Figure 19. Therefore the claim follows.

4. $a = (1, 1, 2, 3, \ldots, 3, 2, \ldots, 2, 1)$, where $i \geq 1, j \geq 0$ with $i + j = n - 4$.

Let us consider the representation depicted in Figure 22. We will show that it can be transformed into a quiver representation studied in Step 3. Let us first study the subquiver consisting of the vertices $v_{n-1}, v_{n-2}, v_{n-3}$. Notice that there are four possible choices of orientation between $v_{n-1}, v_{n-2}, v_{n-3}$ similar to the cases (a)-(d) above.

![Figure 22: A quiver representation of type $H_n$ with $i \geq 1, j \geq 0$ and $i + j = n - 4$ – orientation not shown.](image)

Therefore, by a certain number of reflection functors, we can deduce that a representation of $H_n$ with a root as in Figure 22 can be transformed into a representation shown in Figure 23.

![Figure 23: A quiver representation of type $H_n$ with $1 \leq i \leq n - 4$ – orientation not shown.](image)

Moreover, another set of reflections with respect to the vertices $v_{i+2}, \ldots, v_4$ in the given order, possibly combined with another set of reflections with respect
to the vertices coming before them to reverse arrows if needed, yields a representation of the quiver given in Figure 24.

Figure 24: A quiver representation of type $H_n$ with $1 \leq i \leq n-4$ – orientation not shown.

Finally, if $v_3$ is not a sink (or a source), by a series of reflections with respect to other vertices (except $v_0$ so that the dimension at $v_0$ stays 1), we can transform it into a sink (or a source). Then a reflection with respect to $v_3$ yields a quiver representation shown in Figure 21. Therefore, the claim follows by the discussion in Step 3.

5. $(2, 3, 3, \ldots, 2, 2, 1, \ldots, 1)$ where $i, j \geq 1, k \geq 0$ with $i + j + k = n - 4$.

By Lemma 5.3 and the discussion for step 4, such quiver also defines a linear free divisor.

This concludes the proof of the proposition for $H_n$ whence for all of the series.

A linear free divisor in the representation space of a Dynkin quiver is locally quasihomogeneous, that is, it can locally be defined by a quasihomogeneous polynomial with respect to positive weights. Moreover, if $D^*$ is a linear free divisor in the representation space of a Dynkin quiver, then $D^*$ is locally quasihomogeneous ([8, Theorem 3.22]). Therefore, $D^*$ also satisfies the logarithmic comparison theorem by [14, Remark 1.7.4].

Corollary 5.12. All linear free divisors arising from rational triple quivers of type $A_{n,m,k}, B_{m,n}, C_{m,n}, D_{n,5}$ and $F_n$, together with $H_n$ with dimension vectors of type 1 and 2 in Lemma 3.25, are locally weakly quasihomogeneous and satisfy the logarithmic comparison theorem.

Proof. By the assumption, any representation is given by a root which involves a certain number of 1s, i.e. given by $(1, \ldots, 1, a_1, \ldots, a_r)$ for some $r \geq 0$. So, if $D$ is a linear free divisor defined by a rational triple quiver of any of those types, it is related to a linear free divisor $D^*$ coming from a Dynkin quiver by an equation of the form (5.2). Therefore the claim follows by the remarks above.

Remark 5.13. By [8, Theorem 3.19], all linear free divisors arising from rational triple quiver representations are Euler homogeneous, i.e. $\text{Der}(-\log D)$ contains an Euler vector field.
6 Generalisation to quasi-determinantal quivers

In preceding sections, our reference point for constructing root systems and linear free divisors was the vertex with weight 3 and of coefficient 1 in the Artin cycle $Z$ of the rational triple tree. Hence a generalisation of these trees will naturally be the rational $m$-tuple trees having multiple vertices with weight $\geq 3$ and their coefficients 1 in the Artin cycle of the tree. Such a tree is of the form given in Figure 25 below

where each $\Gamma^k_i$ and $R_i$ is one of the Dynkin diagrams and $w_1, \ldots, w_{\ell+1}, \ldots \geq 3$.

Clearly, $m = -Z^2$ depends on the combination of the subtrees $\Gamma_i$ and $R_j$. The subtrees $R_i$ connecting the vertices of weight $\geq 3$ can be only of type $A_n$ or $D_n$ as stated in [5, Proposition 1.3]. But note that [5] misses one case, which is shown here in Figure 26.

\[ \begin{array}{cccccccccccccccc}
\ldots & \Gamma^1_1 & \ldots & \Gamma^1_2 & \ldots & \Gamma^1_3 & \ldots & \Gamma^1_{\ell-1} & \ldots & \Gamma^1_{\ell} & \ldots & \Gamma^{\ell+1}_1 & \ldots & \Gamma^{\ell+1}_2 & \ldots & \ldots \\
& w_1 & \ldots & w_2 & \ldots & w_{\ell} & \ldots & w_{\ell+1} & \ldots & w_{\ell+2} & \ldots & w_{\ell+3} & \ldots & \ldots & \ldots & \ldots \\
\end{array} \]

Figure 25: Generalization of a rational triple tree.

\[ \begin{array}{cccccccccccccccc}
1 & 2 & \ldots & k & k + 1 & k + 1 & k + 1 & k & k - 1 & \ldots & 3 & 2 & 1 \\
w_i & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
t - 2 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
w_s & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
k - 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
w_j \\
\end{array} \]

Figure 26: $A_{k,t}$ – The missing case for $R_i$’s in [5] with its highest root.

**Definition 6.1.** The rational trees having the form given in Figure 25 are called rational $m$-tuple quasi-determinantal trees. The rational singularities having such a dual resolution graph are called rational quasi-determinantal singularities ([5]).

**Definition 6.2.** Let $\Gamma$ be a rational $m$-tuple quasi-determinantal tree. Consider the set $R(\Gamma)$ of divisors $Y = \sum m_i E_i$ satisfying the following conditions.

(i) $\text{Supp}(Y)$ is a connected tree.

(ii) The coefficients in $Y$ are all either positive or negative.

(iii) $-2 \geq (Y \cdot Y) \geq -Z^2$. 

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(iv) $3(Y \cdot Y')/ (Y \cdot Y) \in \mathbb{Z}, \ldots, (-Z^2)(Y \cdot Y')/ (Y \cdot Y) \in \mathbb{Z}$ for all $Y, Y' \in R(\Gamma)$.

The set $R(\Gamma)$ is called the quasi-determinantal root system.

We call an element of $R(\Gamma)$ a root. We also call each $E_i$ a simple (positive) root of $R(\Gamma)$. Note that all the roots associated with Dynkin diagrams are obtained by $(Y \cdot Y) = -2$ (20, cf. 3).

It is cumbersome to show all computations above for this general case; however, the methods we used in the preceding sections to calculate number of roots and to discuss linear free divisors can be applied to rational quasi-determinantal trees and quivers. It is crucial to note that Lemma 3.3 and 3.4 which are proved for rational triple root systems, may not be true for every quasi-determinantal root system. Hence we have two additional conditions (i) and (ii) in Definition 6.2.

For example, let us consider the case given in Figure 1 which is the most basic form of a rational quasi-determinantal tree. Note that $\ell$, the total number of subtrees, is the valency of the central vertex and $\ell \leq w$. The highest root is given by the sum of the highest roots of each Dynkin subtree and the coefficient 1 on the central vertex. The contribution of each $\Gamma_i$ of type $D_n$, $E_6$ and $E_7$ to $m$ is $-2$ and, type $A_n$, $A_{k,t}$ and $D^s_q$ is $-1$, $-k$ and $r-1$, respectively, when $q = 2r$ or $q = 2r + 1$ (see 15). Here, $D^s_q$ is a tree of type $D_q$ which can be glued to a vertex of weight $\geq 3$ along one of its short tail. Then we obtain

$$Z^2 = -w^2 + \sum_i \left( (-1)\#A_n_i + (-2)(\#D_n_i' + \#E_6 + \#E_7) \right)$$

$$+ \sum_i \left( (-k_i)\#A_{k_i,t_i} + (r_i - 1)\#D^s_{q_i} \right) \quad (6.1)$$

where $\#$ denotes the number of trees of the indicated type appearing in the configuration.

A quiver whose underlying tree is of type given in Figure 1 which is the most basic form of a rational quasi-determinantal tree. Note that $\ell$, the total number of subtrees, is the valency of the central vertex and $\ell \leq w$. The highest root is given by the sum of the highest roots of each Dynkin subtree and the coefficient 1 on the central vertex. The contribution of each $\Gamma_i$ of type $D_n$, $E_6$ and $E_7$ to $m$ is $-2$ and, type $A_n$, $A_{k,t}$ and $D^s_q$ is $-1$, $-k$ and $r-1$, respectively, when $q = 2r$ or $q = 2r + 1$ (see 15). Here, $D^s_q$ is a tree of type $D_q$ which can be glued to a vertex of weight $\geq 3$ along one of its short tail. Then we obtain

$$Z^2 = -w^2 + \sum_i \left( (-1)\#A_n_i + (-2)(\#D_n_i' + \#E_6 + \#E_7) \right)$$

$$+ \sum_i \left( (-k_i)\#A_{k_i,t_i} + (r_i - 1)\#D^s_{q_i} \right) \quad (6.1)$$

where $\#$ denotes the number of trees of the indicated type appearing in the configuration.

A quiver whose underlying tree is of type given in Figure 24 is called a rational quasi-determinantal quiver. The complexity of the study of linear free divisors arising from a ($-Z^2$)-tuple rational quasi-determinantal quivers depends on the number of the Dynkin subquivers. Here we will not attempt to study all cases. However, by the methods we used in Section 5 it is possible to show that any rational quasi-determinantal quiver gives rise to a linear free divisor.

Example 6.3. Let $Q$ be the rational quasi-determinantal quiver in Figure 27 which is formed by the quivers of type $A_{4,1}$, $D^s_5$ and $E_6$. We note that $Z^2 = -u_1 - w_2 - w_3 + 4$ with $w_1 \geq 3$, $w_2 \geq 4$ and $w_3 \geq 5$ for the underlying rational tree. Then $Q$ with the assigned root determines a linear free divisor. This is shown by using the reflection functors.
First, we apply reflection functors with respect to the vertices
\[ v_2, v_3, v_1, v_2, v_4, w_1, v_1, v_9, v_6, v_5, v_7, v_9, v_6, v_10, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{10}, v_{11}, v_{12}, v_{10}, v_{11}, v_{15}, v_{13}, v_{12}, \]
in the given order, to transform \( Q \) into the quiver \( Q' \) shown in Figure 28.

We notice that the subquiver of \( Q' \) formed by the vertices \( v_2, v_3, v_4 \) and \( w_3 \) is of Dynkin quiver type \( D_4 \) with the root \((1, 2, 1, 1)\). Let \( D \) be the linear free divisor which is defined by the subquiver. Then, \( Q' \) defines a linear free divisor given by \( D \cup N \subset \text{Rep}(Q', a') \cong \mathbb{C}^{18} \) where \( N \) is the normal crossing divisor in \( \mathbb{C}^{12} \) corresponding to the twelve arrows \( \mathbb{C} \to \mathbb{C} \) appearing in \( Q' \). Therefore, by the results of [8], the quiver \( Q \) also defines a linear free divisor.

Therefore, we are motivated to state the following theorem.

**Theorem 6.4.** If \( Q \) is a rational quasi-determinantal quiver and \( a \) is a root in \( R(Q) \) then \((Q, a)\) defines a linear free divisor.

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Kazunori Nakamoto, Department of Mathematics, Yamanashi University, Yamanashi, Japan
E-mail address: nakamoto@yamanashi.ac.jp

Ayşe Sharland, Institute for Mathematical Sciences, Stony Brook University, Stony Brook NY 11794-3660, US
E-mail address: aysealtintas@gmail.com

Meral Tosun, (Corresponding author) Department of Mathematics, Galatasaray University, Ortaköy 34357, Istanbul, Turkey
E-mail address: mritosun@gmail.com