Observables in quantum gravity

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October 22, 2018

Abstract

We study a family of physical observable quantities in quantum gravity. We denote them $W$ functions, or $n$-net functions. They represent transition amplitudes between quantum states of the geometry, are analogous to the $n$-point functions in quantum field theory, but depend on spin networks with $n$ connected components. In particular, they include the three-geometry to three-geometry transition amplitude. The $W$ functions are scalar under four-dimensional diffeomorphism, and fully gauge invariant. They capture the physical content of the quantum gravitational theory.

We show that $W$ functions are the natural $n$-point functions of the field theoretical formulation of the gravitational spin foam models. They can be computed from a perturbation expansion, which can be interpreted as a sum-over-four-geometries. Therefore the $W$ functions bridge between the canonical (loop) and the covariant (spinfoam) formulations of quantum gravity. Following Wightman, the physical Hilbert space of the theory can be reconstructed from the $W$ functions, if a suitable positivity condition is satisfied.

We compute explicitly the $W$ functions in a “free” model in which the interaction giving the gravitational vertex is shut off, and we show that, in this simple case, we have positivity, the physical Hilbert space of the theory can be constructed explicitly and the theory admits a well defined interpretation in terms of diffeomorphism invariant transition amplitudes between quantized geometries.

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1 Introduction

One of the hard problems in non-perturbative quantum gravity \[1\] is to construct a full set of physically meaningful observable quantities \[4\]. In this paper, we point out that there is a natural set of quantities that one can define in quantum general relativity, which are gauge invariant, have a natural physical interpretation, and could play the role played by the \(n\)-point functions in quantum field theory. We denote these quantity as \(W\) functions, or \(n\)-net functions. As the \(n\)-point functions in a quantum field theory, these quantities are not natural quantities of the corresponding classical field theory, namely in general relativity. Nevertheless, they capture the physical content of the quantum theory and are related to the classical theory.

The \(W\) functions are strictly related to the three-geometry to three-geometry transition amplitude studied by Hawking \[3\]. However, they are not transition amplitudes between states in which the classical three-geometry has an arbitrary sharp value, but rather transition amplitudes between eigenstates of the three-geometry. In loop quantum gravity \[4\], these eigenstates are characterized by discretized geometries and are labelled by abstract spin networks \[5, 6\], or s-knots. Thus the \(W\) functions are rather transition amplitudes between states with fixed amounts of “quanta of geometry”. This is analogous to the \(n\)-point functions in field theory, which are not transition amplitudes between field configurations, but rather transition amplitudes between states characterized by a fixed number of “quanta of field” – that is, particles. Furthermore, the \(W\) functions generalize the three-geometry to three-geometry amplitude (a 2-point function) to arbitrary \(n\)-point functions; more precisely, we define the \(W\) functions as a functional \(W(s)\) over an algebra \(\mathcal{A}\) of abstract (not necessarily connected) spin networks. In this respect, the \(W\) functions are analogous to the Wightman distributions \[7\] (hence the choice of the letter \(W\)).

We start from a general definition of the \(W\) functions, based on canonical quantum gravity. We show that the \(W\) functions are well defined diffeomorphism invariant observable quantities and we clarify their physical interpretation. In this paper we focus on the case in which the dynamics is “real”, in a sense defined below. The physical meaning of this reality and the extension of the formalism to the general case are discussed at the end of the paper.

A crucial property of the Wightman functions is the possibility of reconstructing the quantum field theory from them – a subtle application of the beautiful Gelfand-Naimark-Segel (GNS) representation theorem in the
theory of $C^*$ algebras. We show here that the $W$ functions have the same property: under appropriate conditions—in particular, a positivity condition—the physical Hilbert space of the theory and a suitable operator algebra can be reconstructed from the $W$ functions, using the GNS construction. In other words, we explore the extension to the generally covariant context of Wightman's remarkable intuition that the content of a quantum field theory is coded in its $n$-point functions. To this aim, we need to strip Wightman's theory from all the "details" that follow from the existence of a background Minkowski space (positivity of the energy, uniqueness of the vacuum, microlocality...) and show that the core idea remains valid even in the absence of a background spacetime. For a line of investigation similar in spirit, see [8].

Since the diffeomorphism invariant quantum field theory can be characterized by its $W$ functions, the way is open for defining a quantum theory of gravity by directly constructing its $W$ functions. Remarkably, spin foam models [9, 10] provide a natural perturbative definition of $W$ functions. In particular, it has been recently shown that general spin foam models can be obtained as the Feynman expansion of certain peculiar field theories over a group [11, 12, 13]. We show here that the gauge invariant $n$-point functions of these field theories are precisely $W$ functions. This construction provides a direct link between the field theoretical formulation of the spin foam models and canonical quantum gravity. The link is similar in spirit to the link between the operator definition quantum field theory and the construction of its $n$-point functions via a functional integral [7]. In particular, given the field theoretical formulation of a spin foam model, we can construct a quantum gravity physical Hilbert space of from its $W$ functions. On the one hand, loop quantum gravity provides the general framework and, in particular, the physical interpretation of the $W$ functions; on the other hand, the field theory over the group provides an indirect but complete definition of the dynamics. This is especially interesting on the light, in particular, of the recent construction of Lorentzian spin foam models [14, 15]. In turn, the perturbative expansion of the field theory defines a sum over spinfoams which can be directly interpreted as a sum over 4-geometries formulation of quantum gravity.

A very interesting paper by Alexandar Mikovic [16] has recently appeared, in which some of the idea presented here are independently derived. We present some comments on the relation between the two approaches in the conclusion section.

We give an example of reconstruction in Section 5. We consider a sim-
ple “free” model, obtained by dropping the interaction term which gives the quantum gravity vertex. We prove positivity for this case, and thus the existence of a Hilbert space of spin networks for this quantum theory. Finally, in Section 6 we discuss the meaning of the reality assumption and the extension to the complex case.

All together, we obtain an attracting unified picture, in which canonical loop quantum gravity, covariant spin foam models, and a family of diffeomorphism invariant physical observables for quantum gravity fit into a unified scheme.

The ideas described in this paper have been first presented in the second conference on Quantum Gravity in Warsaw, in June 1999.

2 The 2-net function $W(s, s')$ in canonical quantum gravity

In loop quantum gravity [4], the Hilbert space $\mathcal{H}_{\text{diff}}$ of the states invariant under three-dimensional diffeomorphisms admits a discrete [20] basis of states $|s\rangle$, labelled by abstract spin networks (or $s$-knots) $s$. An abstract spin network $s$ is an abstract graph (not necessarily connected) $\Gamma_s$, with links labelled by $SU(2)$ representations and nodes labelled by $SU(2)$ intertwiners (and satisfying suitable conditions [19, 20]). There is a natural union operation $\cup$ defined on the abstract spin networks: $s \cup s'$ is the spin network defined by the the graph formed by the two disconnected components $\Gamma_s$ and $\Gamma_{s'}$.

Notice that the relation between $\mathcal{H}_{\text{diff}}$ and the extended Hilbert space $\mathcal{H}_{\text{ex}}$ formed by the unconstrained states [18, 19] can be expressed in terms of a “projection” operator $P_{\text{diff}} : \mathcal{H}_{\text{ex}} \mapsto \mathcal{H}_{\text{diff}}$ which sends an embedded spin network state of a suitable basis of $\mathcal{H}_{\text{ex}}$ into a an abstract spin network state, or $s$-knot, in $\mathcal{H}_{\text{diff}}$ [10]. Due to the infinite volume of the group of the diffeomorphisms (or to the fact that the zero eigenvalue of the diffeomorphism constraint is in the continuum spectrum) $\mathcal{H}_{\text{diff}}$ is not a proper subspace of $\mathcal{H}_{\text{ex}}$ and $P_{\text{diff}}$ is not a true projection operator (hence the quotation marks), but many alternative techniques for taking care of these technicalities are known, and the space $\mathcal{H}_{\text{diff}}$ and the operator $P_{\text{diff}}$ are nevertheless well defined.

The states $|s\rangle$ have a straightforward physical interpretation, which fol-
allows from the fact that they are projections on $\mathcal{H}_{\text{diff}}$ of eigenstates of the area and volume operators $[5, 6]$. The interpretation is the following. A state $|s\rangle$ represents a three-geometry. A three-geometry is an equivalence class of three-metrics under diffeomorphisms. The geometry represented by $|s\rangle$ is quantized, in the sense that it is formed by regions and surfaces having quantized values of volume and area. Intuitively, each node of $s$ represents a “chunk” of space, whose (quantized) volume is determined by the intertwiner associated to the node. Two of such chunks of space are adjacent if there is a link between the corresponding nodes. The (quantized) area of the surface that separates them is determined by the representation $j$ associated to this link, according to the now well known relation $[6]$

$$A = 8\pi\gamma\hbar G \sqrt{j(j + 1)} \quad (1)$$

where $\hbar$, $G$, $\gamma$ are the reduced Planck constant, the Newton constant and the Immirzi parameter (the dimensionless free parameter in the theory). This interpretation of the states $|s\rangle$ follows from the study of the area and volume operator on the Hilbert space of the non-diffeomorphism invariant states. Notice that the states $|s\rangle$ are not gauge invariant either, and do not represent physical gauge invariant notions. The same is true for the corresponding classical notion of three-geometry: a three-geometry is determined by an ADM surface, which is a non-gauge-invariant notion in general relativity.

The dynamics of the theory is given by the Hamiltonian constraint $H(x)$, which we assume here to be a symmetric operator. The space of the solutions of this constraint is the physical Hilbert space of the theory $H_{\text{ph}}$. Instead of using the Hamiltonian constraint, we can work with the linear operator $P : H_{\text{diff}} \rightarrow H_{\text{ph}}$ that projects on the Kernel of $H(x)$. (A suitable extension of $H_{\text{diff}}$ to its generalized states –or any other of the many techniques developed for this purpose– should be used in order to take care of the technical complications in defining the Hilbert eigenspace corresponding to an eigenvalue in the continuum spectrum.) For more details on this operator, and, in particular, a more precise definition as a three-diffeomorphism invariant object, see $[10]$. Instead of worrying about the explicit construction of $P$, we assume here that the operator $P : H_{\text{diff}} \rightarrow H_{\text{ph}}$ is given, and we consider the quantity

$$W(s, s') := \langle s | P | s' \rangle \quad (2)$$

We claim that this is a well-defined fully gauge invariant quantity, which represents a physical observable in quantum gravity and has a precise and
well-understood physical interpretation.

The gauge invariance of \( W(s, s') \) is immediate. All the objects on the r.h.s. of (2) are invariant under three-dimensional diffeomorphisms, therefore we need to check only invariance under time reparametrizations. An infinitesimal coordinate-time shift is generated by the Hamiltonian constraint. If we gauge transform (say) the bra state \( \langle s | \) we obtain

\[
\delta W(s, s') = \langle Hs | P | s' \rangle = \langle s | HP | s' \rangle = 0,
\]

because \( P \) is precisely the projector on the Kernel of \( H \). Therefore \( W(s, s') \) represents a gauge-invariant transition amplitude. In fact, this is precisely the physical three-geometry to three-geometry physical transition amplitude.

To clarify why the three-geometry to three-geometry transition amplitude is a physical gauge-invariant quantity, consider a simple analogy with a well known system. Consider a free relativistic particle in three spatial dimensions. Its physical description is given by its position \( \vec{x}(t) \) at each time \( t \). To have explicit Lorentz invariance in the formalism, the dynamics can be represented as a constrained reparametrization invariant dynamical system, by promoting the time variable \( t \) to the role of dynamical variable \( x_0 = t \), and introducing an unphysical parameter “time” \( \tau \). The dynamics is then entirely determined by the constraints \( p^2 - m^2 = 0 \) and \( p^0 > 0 \). The corresponding constraints in the quantum theory are the Klein Gordon equation and the restriction to its positive frequency solutions. The Hilbert space \( \mathcal{H}_{{\text{K}}\!\!\!\!\!\!\text{G}} \) of the unconstrained states is formed by the square integrable functions on Minkowski space. The physical Hilbert space \( \mathcal{H}_{{\text{P}}\!\!\!\!\!\!\text{h}} \) of the physical states is formed by the positive frequency solutions of the Klein Gordon equation. There is a well defined projection operator \( P \), which restricts any state in \( \mathcal{H}_{{\text{P}}\!\!\!\!\!\!\text{h}} \) (more precisely, in the extension of \( \mathcal{H}_{{\text{P}}\!\!\!\!\!\!\text{h}} \) which includes its generalized states) to its mass shell, positive frequency, component. Now, consider the (generalized) state \( |\vec{x}, x^0\rangle \) in \( \mathcal{H} \). This is the eigenstate of both the position \( \vec{x} \) and the time \( x^0 \) operators, which are well defined self-adjoint operators on \( \mathcal{H} \). The interpretation of \( |\vec{x}, x^0\rangle \) is clear: it is a particle at the Minkowski spacetime point \( (\vec{x}, x^0) \). On the other hand, this is clearly not a physical state: there is no physical particle that can “stay” in a single point of spacetime (where is it after a second?). It is a state that does not satisfies the dynamics. Notice also that in \( \mathcal{H} \) two such states at two different points of Minkowski space are orthogonal. However, given the state \( |\vec{x}, x^0\rangle \) in \( \mathcal{H} \), we
can project it down to \( H_{Ph} \) and define the **physical** state

\[
|\vec{x}, x^0\rangle_{Ph} = P|\vec{x}, x^0\rangle.
\]

In momentum space, this amounts to restrict it to its mass shell positive frequency components. In coordinate space, this amount to spread out the delta function to a full solution of the Klein Gordon equation, which –as its happens– at time \( x^0 \) is concentrated around \( \vec{x} \), but at other times is spread around the future and past light cones of \( (\vec{x}, x^0) \). The state \( |\vec{x}, x^0\rangle_{Ph} \) is a physical state, and has a physical interpretation consistent with the dynamics: it is a (Heisenberg) state in which the particle is in \( \vec{x} \) at time \( x^0 \), and has appropriately moved around in space at other times. The transition amplitude between two such states is a physically meaningful quantity. Indeed, it is nothing else that the familiar propagator in Minkowski space. But notice that

\[
W(\vec{x}, x^0; \vec{x}', x'^0) = p_{Ph} \langle \vec{x}, x^0 | \vec{x}', x'^0 \rangle_{Ph} = \langle \vec{x}, x^0 | P | \vec{x}', x'^0 \rangle,
\]

Namely the propagator is nothing but the matrix element of the projector operator \( P \) between the unphysical states \( |\vec{x}, x^0\rangle \).

It is clear that the structure illustrated is the precisely the same as in quantum gravity. A classical three-geometry is determined by three degrees of freedom per space point. Two of these correspond to physical degrees of freedom of the gravitational field, in analogy with the dependent variable \( \vec{x} \) above. The third is the independent temporal variable, in analogy with the \( x^0 \) variable in the example above.\(^1\) Therefore \( s \), precisely as \( (\vec{x}, x^0) \) includes the dependent as well as the independent (time) variables. The states \( |s\rangle \) are quantum states concentrated at a single three-geometry. Precisely as the states \( |\vec{x}, x^0\rangle \), these are unphysical, because spacetime cannot be concentrated on a unique three-geometry, in the very same sense in which a particle cannot be at a unique point of Minkowski space. The projector \( P \) project a state \( |s\rangle \) into a physical state which spreads across three-geometries, and the transition amplitude (3) gives the amplitude of measuring the three-geometry corresponding to \( s \) after we have measured the three geometry corresponding to \( s' \). This amplitude is well defined and diffeomorphism invariant.

\(^1\)Of course, there is no a priori physical distinctions between the two sets. This is because the dynamics of general relativity is relational: it provides relations between equal-footing quantities, not a preferred temporal variable. The advantage of the formalism we are considering here is that it does not require such a distinction to be made. It does not require to single out a preferred time variable.
3 Reality of $P$ and $W$ functions

Let us now return to the gravitational theory. We assume in this section that $P$ has the following property, which we call (for reason that will become clear later on) “reality”

$$\langle s_1 \cup s_3 | P | s_2 \rangle = \langle s_1 | P | s_2 \cup s_3 \rangle.$$  \hspace{1cm} (6)

The physical meaning of this property, as well as the extension of the formalism to the case in which this property does not hold are discussed in Section 6.

Consider the vector in $H_{ph}$

$$|0\rangle_{ph} \equiv P|0\rangle.$$ \hspace{1cm} (7)

and, in general,

$$|s\rangle_{ph} \equiv P|s\rangle.$$ \hspace{1cm} (8)

(See the particle analogy discussed at the end of last section.) The 2-net function $W(s, s')$, defined in (2), can then be written also as

$$W(s, s') = \langle s | s' \rangle_{ph}.$$ \hspace{1cm} (9)

Clearly the states $|s\rangle_{ph}$ form an overcomplete basis of $H_{ph}$. In particular, there will be relations between them, of the form

$$\sum_s c_s |s\rangle_{ph} = 0$$ \hspace{1cm} (10)

for appropriate complex numbers $c_s$. Notice that (10) is equivalent to $P \sum_s c_s |s\rangle = 0$, or $\langle s'P \sum_s c_s |s\rangle = 0, \forall s'$. This can also be rewritten as $\langle s' \cup s'' P \sum_s c_s |s\rangle = 0, \forall s'$ and, because of the reality (3) of $P$, as $\langle s'P \sum_s c_s |s \cup s''\rangle = 0, \forall s'$. Therefore

$$\sum_s c_s |s \cup s''\rangle_{ph} = 0$$ \hspace{1cm} (11)

for all $s''$, whenever (10) holds. Using this fact, we define on $H_{ph}$ the operator

$$\hat{\phi}_s |s'\rangle_{ph} = |s' \cup s\rangle_{ph}.$$ \hspace{1cm} (12)

This definition is well posed, in spite of the overcompleteness of the vectors $|s\rangle_{ph}$, because of (11); that is, $\hat{\phi}_s$ sends the vanishing linear combinations (10).
of states into the linear combinations (11) which are still vanishing. Also, notice that $\hat{\phi}_s$ is self-adjoint, again because of the reality of $P$

$$\langle s_1|\phi_s\rangle ph = \langle \phi_s|s_2\rangle ph = \langle s_1 \cup s|P|s_2\rangle ph = \langle s_1|P|s_2 \cup s\rangle ph = \langle \phi_s|s_2\rangle ph,$$

(13)

and it commutes with itself

$$[\hat{\phi}_s, \hat{\phi}_{s'}] = 0,$$

(14)

since

$$\hat{\phi}_s \hat{\phi}_{s'} = \hat{\phi}_{s'} \hat{\phi}_s = \hat{\phi}_{s \cup s'}.$$  

(15)

The 2-net function $W(s, s')$, defined in (12), can be written now as

$$W(s, s') = \langle \phi_s|\phi_{s'}\rangle 0 ph.$$

(16)

More in general, we can define

$$W(s) = \langle \phi_s|\phi_s\rangle 0 ph,$$

(17)

so that

$$W(s, s') = W(s \cup s').$$

(18)

Now, consider the free linear space $A$ formed by the (formal) linear combinations of spin networks, with complex coefficients

$$A = \sum_s c_s s.$$  

(19)

There is a natural product defined on $A$ by $s \cdot s' = s \cup s'$, and a natural star operation defined by $s^* = s$ (Here we refer to spin networks labeled by $SU(2)$ representations and each representation of $SU(2)$ is conjugate to itself. When spin networks are labeled by representations of groups which are not self-conjugate the star operation should replaces representations with dual representations.) We define the norm $||A|| = sup_s |c_s|$. We obtain in this way a $C^*$ algebra structure on $A$. The quantity $W(s)$, defined in (17), defines a linear functional on $A$. A straightforward calculation shows that the functional is positive

$$W(A^*A) \geq 0.$$  

(20)
We can thus apply the Gelfand-Naimark-Segal construction to the $C^*$ algebra $\mathcal{A}$ and the positive linear functional $W$, obtaining a Hilbert space $\mathcal{H}$, a “vacuum” state $|0\rangle$ and a representation $\hat{\phi}$ of $\mathcal{A}$ in the Hilbert space, such that

$$W(s) = \langle 0| \hat{\phi}(s)|0\rangle.$$  \hspace{1cm} (21)

But it is clear that in doing so we have simply reconstructed the Hilbert space $\mathcal{H}_{ph}$, the “vacuum” state $|0\rangle_{ph}$ and the algebra of the operators $\hat{\phi}_s$. In other words, the content of the canonical theory of quantum gravity can be coded, in the spirit of Wightman, in the positive linear functional $W(s)$ over the algebra $\mathcal{A}$ of the spin networks.

We can thus determine the dynamics of the theory by giving $W(s)$, instead of explicitly giving the projector $P$, or the Hamiltonian constraint, and reconstruct the physical Hilbert space from $W(s)$. In particular, the main physical gauge-invariant observable, namely the three-geometry to three-geometry transition amplitude is simply the value of $W(s)$ on the spin networks $s$ formed by two disjoint components.

We close this section with a comment about locality. The sense in which general relativity is a local theory is far more subtle that in ordinary field theory. For a detailed discussion of this issue see for instance \[2\]. In particular, physical gauge invariant observables are independent from the spacetime coordinates $\vec{x}, t$, and therefore they are not localized on the spacetime manifold, which is coordinatized by $\vec{x}, t$. Nevertheless, the dynamics of general relativity is still local in an appropriate sense. This locality should be reflected in a general property of the $W$ functions. Roughly, we expect that if a spin network $s$ can be cut in two parts (connected to each other) $s_{ext}$ and $s_{in}$, and a second spin network $s'$ can be cut in two parts (connected to each other) $s'_{ext}$ and $s'_{in}$, and if $s_{ext} = s'_{ext}$, then $W(s, s')$ should be independent from $s_{ext}$. In other words, the local evolution in apart of the spin network should be independent from what happens elsewhere on the spin network. A precisely formulation of this property and its consequences deserve to be studied.

\section*{4 \ W(s) in field theories over a group}

In the last few years, intriguing developments in quantum gravity have been obtained using the spin foam \[9\] formalism. Recently, it has been shown that any spin foam model can be derived from an auxiliary field theory over
a group manifold \([11, 12]\). Several spin foam models defined from auxiliary theories defined over a group have been developed. They are covariant, have remarkable finiteness properties \([13]\), exist in Lorentzian form \([15]\) and represent intriguing covariant models for a quantum theory of the gravitational field. In this section, we illustrate the emergence of a \(W(s)\) functional over \(A\) in the context of these field theories over a group manifolds. For a similar derivation see \([16]\).

For concreteness, and simplicity of the presentation, let us consider a specific model. Consider a real field theory for a scalar field defined over a group manifold \(\phi(g_1, g_2, g_3, g_4)\), where \(g_i \in G\), which we chose for the moment to be a compact Lie group \([11, 13]\). The field \(\phi(g_1, g_2, g_3, g_4)\) is defined to be symmetric under permutation of its four arguments and \(G\) invariant in the sense that it satisfies \(P_g \phi = \phi\), where the operator \(P_g\) is defined by

\[
P_g \phi(g_1, g_2, g_3, g_4) \equiv \int_G dg \, \phi(g_1 g, g_2 g, g_3 g, g_4 g),
\]

The dynamics is given by an action \(S[\phi]\), which we do not specify for the moment. The \(n\)-point functions of the theory have the form

\[
W(g_1^{i_1}, \ldots, g_4^{i_4}) = \int [D\phi] \phi(g_1^{i_1}), \ldots, \phi(g_4^{i_4}) e^{iS[\phi]}.
\]

Let us work in momentum space. Using Peter-Weyl theorem, we expand the field in terms of the matrix elements of the irreducible representations \(D^{(N)}_{\alpha\beta}\) of \(G\).

\[
\phi(g_1, \ldots, g_4) = \sum_{N_1 \ldots N_4} \Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4) \beta_1 \ldots \beta_4} D^{(N_1)\beta_1}_{\alpha_1}(g_1) \ldots D^{(N_4)\beta_4}_{\alpha_4}(g_4).
\]

We denote as \(C^{N_1 \ldots N_4 \Lambda}_{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4}\) a normalized basis in the space of the intertwiners between the representations \(N_1 \ldots N_4\). Imposing the \(G\) invariance of the field on the momentum space components, and using the relation

\[
\int_G dg \, D^{(N_1)}_{\alpha_1 \beta_1}(g) \ldots D^{(N_4)}_{\alpha_4 \beta_4}(g) = \sum_\Lambda C^{N_1 \ldots N_4 \Lambda}_{\alpha_1 \ldots \alpha_4} C^{N_1 \ldots N_4 \Lambda}_{\beta_1 \ldots \beta_4},
\]

we can write the field as

\[
\phi(g_1, \ldots, g_4) = \sum_{N_1 \ldots N_4} \Phi_{(N_1 \ldots N_4) \beta_1 \ldots \beta_4} D^{(N_1)\gamma_1}_{\alpha_1}(g_1) \ldots D^{(N_4)\gamma_4}_{\alpha_4}(g_4) \sum_\Lambda C_{\gamma_1 \ldots \gamma_4 \Lambda} C^{\beta_1 \ldots \beta_4}_{\Lambda}.
\]
or, defining (for later convenience)

\[ \phi^{\alpha_1 \ldots \alpha_4}_{\Lambda} := \frac{\Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4)\beta_1 \ldots \beta_4} C^{N_1 \ldots N_4 \Lambda}}{\Delta_{N_1} \Delta_{N_2} \Delta_{N_3} \Delta_{N_4}}, \]  

(27)
as

\[ \phi(g_1, \ldots, g_4) = \sum_{N_1 \ldots N_4, \Lambda} \phi^{\alpha_1 \ldots \alpha_4}_{N_1 \ldots N_4, \Lambda} \left( \Delta_{N_1} \ldots \Delta_{N_4} D_{\alpha_1}^{(N_1)\gamma_1}(g_1) \cdots D_{\alpha_4}^{(N_4)\gamma_4}(g_4) C^{\gamma_1 \ldots \gamma_4}_{\Lambda} \right). \]  

(28)

We can take the quantities \( \phi^{\alpha_1 \ldots \alpha_4}_{N_1 \ldots N_4, \Lambda} \) as the independent “Fourier components” of the field, and therefore write the \( W \) functions, in momentum space as

\[ W^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}_{N_1 N_2 N_3 N_4, \Lambda} \cdots \alpha_1^0 \alpha_2^0 \alpha_3^0 \alpha_4^0 N_1 N_2 N_3 N_4, \Lambda^n = \int [D\phi] \phi^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}_{N_1 N_2 N_3 N_4, \Lambda^n} \cdots \phi^{\alpha_1^0 \alpha_2^0 \alpha_3^0 \alpha_4^0}_{N_1 N_2 N_3 N_4, \Lambda^n} e^{iS[\phi]}. \]  

(29)

However, the measure and the action are \( G \) invariant. Therefore the only nontrivial independent \( W \) functions are given by \( G \) invariant combinations of fields, where \( G \) acts on each index \( \alpha_i^n \) by the representation \( N_i^n \). There is only one way of obtaining \( G \) singlets: to have the indices \( \alpha_i^n \) all paired –with the two indices of the pair sitting in the same representation– and to sum over the paired indices. Each independent \( W \) function is determined by a choice of indices and their pairing.

In order to describe these index choices and pairings, let us associate to each field \( \phi^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}_{N_1 N_2 N_3 N_4, \Lambda} \) in the integrand a four valent node; we associate to this node the intertwiner \( \Lambda^n \), and to each of its four links a representation \( N_i^n \). We then connect the links between two fields with paired indices. We obtain a graph, with nodes labelled by intertwiners and links labelled by representations (satisfying Clebsch-Gordan like relations), namely a spin network \( s \) (in the group \( G \)). Thus, independent \( W \) functions are labelled by spin networks!

In other words, to each spin network \( s \), with nodes \( n \) labelled by intertwiners \( \Lambda_n \) and links \( l \) labelled by representations \( N_l \), we can associate a gauge invariant product of field operators \( \phi_s \)

\[ \phi_s = \sum_{\alpha_1} \prod_{n} \phi^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}_{N_1 N_2 N_3 N_4, \Lambda^n} \]  

(30)

where \( \alpha_1^n \) is the index associated to the link \( l \) which is the \( i \)-th link of the node \( n \). And we define

\[ W(s) = \int [D\phi] \phi_s e^{iS[\phi]}. \]  

(31)
Therefore the field theory over the group defines a $W$ functional over the spin networks algebra $\mathcal{A}$. If $W(s)$ is positive, we have then immediately, thanks to the GNS theorem, a Hilbert space and an algebra of field operators whose vacuum expectation value is $W(s)$. Under suitable conditions, this could be identified with the physical Hilbert space of quantum gravity. For this, the group $G$ has to be $SU(2)$, or, alternatively, the representations and the intertwiners should be in correspondence with the ones of $SU(2)$. This is the case in particular for the gravitational $SO(4)$ and $SO(3,1)$ models \[14, 15\] in which the dynamics restricts the representations to the simple, or balanced, representations, which can be identified with the irreducible $SU(2)$ representations.

The relation between the field theory and quantum gravity becomes much more transparent by expressing $W(s)$ explicitly as a perturbation expansion. Indeed, as shown in Refs. \[11, 12, 13\], the standard field theoretical perturbation expansion of $W(s)$ in Feynman graphs turns out to be a sum over spin foams. In particular $W(s, s') = W(s \cup s')$ is given by a sum over all spin foams $\sigma$ bounded by the spin networks $s$ and $s'$

$$W(s, s') = \sum_{\sigma, \partial \sigma = s \cup s'} A(\sigma). \tag{32}$$

A spin foam $\sigma$ admits an interpretation as a (discretized) 4-geometry. In particular, $\sigma$ can be the complex dual to a four dimensional cellular complex, and the representations and intertwiners are naturally related to areas and volumes of the elementary 2 and 3 cells. Therefore \[32\] is a (precise) implementation of the representation of quantum gravity as a sum over geometries, introduced by Wheeler and Misner \[17\], and developed by Hawking and collaborators \[3\]. The generation of the spacetimes summed over as Feynman diagrams is a four-dimensional analog of the two-dimensional quantum gravity models developed sometime ago in the context of the string theory in zero dimensions \[22\].

5 A “free” theory

As a simple example, we sketch here the structure of a very simple model in which the action $S[\phi]$ contains only a kinetic part and no interaction part.

$$S[\phi] = i/2 \int dg^1 \ldots dg^4 \phi^2(dg^1 \ldots dg^4). \tag{33}$$
A straightforward calculation yields the action in momentum space.

\[
S[\phi] = \frac{i}{2} \delta_{N_1 \ldots N_4, A} \phi^{\alpha_1 \ldots \alpha_4} \phi^{\beta_1 \ldots \beta_4} \left( \Delta_{N_1} \cdots \Delta_{N_4} \right) \delta_{\alpha_1 \beta_1} \cdots \delta_{\alpha_4 \beta_4} \delta_{N_1 \alpha_1} \cdots \delta_{N_4 \alpha_4} \delta_{M_1 \beta_1} \cdots \delta_{M_4 \beta_4} \Lambda \tilde{\Lambda},
\]

where sum over repeated indices is understood.

Every \( n \)-point function of the field theory can be calculated as functional derivatives of the generating function \( W(J) \) defined as:

\[
W(J) = \int D\phi \exp \left( iS[\phi] + J^{\alpha_1 \ldots \alpha_4} N_{1 \ldots N_4, A} \phi^{\alpha_1 \ldots \alpha_4} \right),
\]

which can be easily computed using standard Gaussian integration:

\[
W(J) = C \exp \left( \frac{1}{2} J^{\alpha_1 \ldots \alpha_4} N_{1 \ldots N_4, A} J_{\alpha_1 \ldots \alpha_4} \right),
\]

The \( n \)-point function depends on the boundary 4-valent spin network defined by the action of the \( J^{\alpha_1 \ldots \alpha_4} \)'s which can be represented by a 4-valent node carrying the representations \( N_1, N_2, N_3, N_4 \) respectively and an intertwiner colored by \( \Lambda \). Contraction of their indices \( \alpha_i \) is represented by the connection of the corresponding links. Let's illustrate with an example the computation of the Wightman functions for this free theory. The two point function \( W(s_1, s_2) \) is given by

\[
W(s_1, s_2) = \left\{ \frac{\delta}{\delta J^{\alpha_1 \ldots \alpha_4} N_{1 \ldots N_4, A}} \frac{\delta}{\delta J^{\alpha_1 \ldots \alpha_4} N_{1 \ldots N_4, A}} \frac{\delta}{\delta J^{\beta_1 \ldots \beta_4} M_{1 \ldots M_4, \Gamma}} \frac{\delta}{\delta J^{\beta_1 \ldots \beta_4} M_{1 \ldots M_4, \Gamma}} W(J) \right\}_{J=0},
\]

where the boundary spin networks \( s_1 \) and \( s_2 \) are given by the corresponding contraction of the \( J \)'s functional derivatives, namely:

\[
\frac{\delta}{\delta J^{\alpha_1 \ldots \alpha_4} N_{1 \ldots N_4, A}} \frac{\delta}{\delta J^{\alpha_1 \ldots \alpha_4} N_{1 \ldots N_4, A}} \rightarrow \Lambda \tilde{\Lambda} \quad (38)
\]

and

\[
\frac{\delta}{\delta J^{\beta_1 \ldots \beta_4} M_{1 \ldots M_4, \Gamma}} \frac{\delta}{\delta J^{\beta_1 \ldots \beta_4} M_{1 \ldots M_4, \Gamma}} \rightarrow \Gamma \tilde{\Gamma} \quad (39)
\]
A straightforward calculation gives

\[ W(S_1, S_2) = 1 + \delta_{N_1 M_1} \ldots \delta_{N_4 M_4} \delta_{\Lambda \Lambda}. \]  

(40)

The \( C^* \) algebra \( \mathcal{A} \) is defined as the algebra as the free sum of (non necessarily connected) 4-valent spin networks over \( SO(4) \) as in (19). The \( * \) operation is simply defined by complex conjugation of the components of \( \mathcal{A} \). We define the functional \( W \) over the algebra by means of the corresponding Wightman functions of our spin foam model. The fact that the positivity condition holds for \( W \) (namely, \( W(A^* A) \geq 0 \)) can be easily seen from the form of the functional measure. We can explicitly construct an orthonormal basis in \( \mathcal{H}_{ph} \) as follows. There are two kind of situations: spin network which do not interfere with the vacuum \( |0\rangle \) (the empty spin network), and those which do. In the first case the projection is trivial and the elements of the physical Hilbert space \( \mathcal{H}_{ph} \) are the simply the original spin network states. Some examples are the following

\[ |0\rangle = 1, \]

\[ |ij, \lambda\rangle = \begin{array}{c}
\includegraphics{example1.png}
\end{array}, \]

\[ |ijkl, \lambda\gamma\rangle = \begin{array}{c}
\includegraphics{example2.png}
\end{array} \quad \text{for} \quad \lambda \neq \gamma, \]

\[ |ijkl, \lambda\gamma\rangle = \begin{array}{c}
\includegraphics{example3.png}
\end{array} \quad \text{for} \quad \lambda \neq \gamma \text{ or } i \neq l, \]

\[ |ijklmn, \lambda\gamma\delta\rangle = \begin{array}{c}
\includegraphics{example4.png}
\end{array}. \]
The states which interfere with the vacuum are those for which there are closed bubble diagrams from the given spin network to ‘nothing’. In those orthonormal states in the physical state can constructed by simply subtracting the vacuum part using the standard Gram-Schmidt procedure. For example

\[ |ijkl, \lambda \rangle = -1 + \lambda \]

\[ |ijk, \lambda \rangle = -1 + i \]

\[ |ij, \lambda \rangle = -1 + \]

and so on. Other states can be turn out to be just the tensor product of the previous ones, namely

\[ |ijklmn, \lambda \delta \rangle = - \delta \]

\[ |ijkmn, \lambda \delta \rangle = - \delta \]

\[ |ijmn, \lambda \delta \rangle = - \delta \]

and so on. The procedure can be clearly continued to construct an orthonormal basis of \( \mathcal{H}_{ph} \).
6 Complex $P$

An important ingredient in the construction above is the assumption that $P$ is real, equation (6). This assumption greatly simplifies the construction, allowing a simple definition of the $\phi_s$ operators. Here we discuss the meaning of this assumption and the extension of the formalism to the case in which $P$ is non-real, or “complex”.

Figure 1: The three-geometry to three-geometry transition amplitude $W(s, s')$ between two connected three-geometries.

To clarify the meaning of the reality condition, let us represent graphically the 2-net function $W(s, s')$ as in Figure 1, when $s$ and $s'$ are connected. If $s'$ is formed by two connected components $s_1$ and $s_2$, we represent it as in Figure 2. Then the reality of $P$ is expressed by the equation in Figure 3. That is, it represents an a priori indistinguishability between past and future boundaries of spacetime. This property is strictly connected with crossing symmetry [10], which is essentially the analogous property at the level of the Hamiltonian constraint. The property is natural from the perspective of Atiyah’s topological quantum field theory axiomatic framework [21]. It is perhaps natural to expect this property for Euclidean quantum gravity. Whether we should expect Lorentzian quantum gravity to have the same property, on the other hand, is not clear to us. On the one hand, the causal structure of the Lorentzian four-geometries seems to suggest that one should distinguish past and future boundaries. Notice also that (6) implies that the transition...
amplitudes $W(s, s')$ are real, because
\begin{equation}
W(s, s') = \langle s Ps' \rangle = \langle 0 Ps' \cup s \rangle = \langle s' Ps \rangle = \overline{\langle s Ps' \rangle} = \overline{W(s, s')}, \tag{41}
\end{equation}
which would prevent quantum mechanical interference between the $|s\rangle$ basis states (but not between generic states). On the other hand, however, temporal relations between boundaries may be induced a posteriori by the dynamics, instead of being a priori given in the structure of the formalism itself.

If we drop the reality condition on $P$, the main difficulty is that the definition (12) of the field operator becomes inconsistent. However, can still retain a (partial) characterization of the field operator $\hat{\phi}_s$ by requiring only
\begin{equation}
\hat{\phi}_s |0\rangle_{ph} = |s\rangle_{ph}. \tag{42}
\end{equation}
This is certainly consistent. Notice that in general we have then
\begin{equation}
\hat{\phi}_s \neq \phi_s. \tag{43}
\end{equation}
And the 2-net function is now given by
\begin{equation}
W(s, s') = p_h \langle 0 | \hat{\phi}_s^\dagger \hat{\phi}_{s'} |0\rangle_{ph}. \tag{44}
\end{equation}
That is, we have to add the adjoint operation to equation (16).
The relevant abstract $C^*$-algebra has now a non trivial star operation, different from $s^* = s$. If $\hat{\phi}_s$ and $\hat{\phi}_{s'}$ are independent, the $C^*$-algebra is generated by products of $s$’s and $s^*$’s, and

$$W(s, s') = W(s^* \cup s').$$  \hspace{1cm} (45)

Starting from the field theory, we may generate a $W$ functional on the complex algebra $\mathcal{A}$ by using a complex field, instead than a real one. The resulting structure will be explored in detail elsewhere.

A strictly related problem is whether the $n$-net $W$ functions should be thought as analogous to the Wightman distributions (the vacuum expectation values of products of field operators), to the Feynman distributions (the vacuum expectation values of time ordered products of field operators) or rather to the Schwinger functions (the appropriate analytic continuations of the Wightman distributions to imaginary time). We recall that on Minkowski space one can directly apply the GNS reconstruction theorem to the Wightman distributions. On the other hand, one obtains directly the Feynman distributions as functional integrals of products of fields (with suitable “prescriptions” at the poles), while one can obtain the Schwinger functions as momenta of a well defined stochastic process \cite{7}. The Osterwalder-Schrader reconstruction theorem \cite{23} that allows the reconstruction of the Wightman distributions from the Schwinger functions requires a duality “star” operation to be defined, corresponding to the inversion of the time variable. Presumably, the distinction between these different families of $n$-point functions...
makes no sense in the generally covariant context. The peculiar analytic structure of the $n$-point functions of field theory on Minkowski space is a consequence of the positivity of the energy (the Fourier transform of a function with support on positive numbers is analytic in the upper complex plane.), while there is no notion of positivity of the energy in quantum gravity – indeed, there is no notion of energy at all–, and one should be careful in trying to generalize standard quantum field theoretical prejudices to the generally covariant context.

7 Conclusion

We have studied the family of quantities $W(s)$, which we propose as main physical observables of a quantum theory of gravity. We have proposed a general framework, based on these quantities, that ties the canonical (loop) and the covariant (spin foam) approaches to quantum gravity. The connection between the two formalisms is provided by the GNS reconstruction theorem, and parallels the connection between the Hilbert space and the functional formulations of conventional quantum field theory, which one obtains from the properties of the $n$-point functions.

Many issues deserve to be clarified. Among these are the reality of $P$ and the complex $\mathcal{A}$ algebra; the locality property of $W(s)$ mentioned at the end of Section 3; and the connection between $W(s)$ and the $S$ matrix when spacetime admits asymptotic regions. An explicit construction of the $C^*$ algebra and the its GNS construction in the case of the 2 dimensional theory would also be of great interest and presumably not too hard to do.

As mentioned in the introduction, an independent but related approach has recently appeared in Ref. [16] by Alexandar Mikovic. Mikovic studies the field theories of a group, and constructs a the Hilbert space of the theory –in fact, a Fock space– directly from the theory’s field operators, instead of using the intermediary step of the transition functions as we do here. He observes, as we do here, that this Hilbert space has a natural basis of spin network states, where the spin network correspond to a triangulation of the spacetime boundary, and that in perturbation theory the transition amplitudes between spin network amplitudes are given by the state-sum amplitudes for triangulated manifolds with boundaries. This structure parallels what is done here. It may be useful on the other hand, to emphasize a difference in attitude: Mikovic sees the discrete aspects of the geometry as a starting postulate;
here, on the contrary, we view them as a result of the canonical quantization of the continuum theory.

In addition, Mikovic considers evolution in a fixed number of simplices, and proposes to interpret the number of simplices between two spin networks as a time parameter. Transition amplitudes in this time are then finite. The same idea of fixing the number of Planck steps to get a time variable and finite transition amplitudes has been explored by Markopoulou and Smolin [25], and can be traced to Sorkin’s suggestion of using the 4d volume as a natural time variable [27]. See also Teitelboim’s [26]. The resulting transition amplitudes are analogous to Feynman’s proper time propagator $P(x, y; \tau)$ obtained integrating over all paths in Minkowski space that go from $x$ to $y$ in the proper time $\tau$. The physical propagator $P(x, y)$ is obtained from $P(x, y; \tau)$ by integrating over $\tau$. Indeed, it is this integration that amounts to impose the Hamiltonian constraint. As far as we understand, the unintegrated proper time propagator does not have a simple physical interpretation in the context of the dynamics of the particle. The analogy suggests that in the gravitational context we cannot confine ourselves to the transition amplitudes in a fixed number of Planck steps, but the issue deserves to better understood.

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