Stochastic treatment of finite-$N$ effects in mean-field systems and its application to the lifetimes of coherent structures

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A stochastic treatment yielding to the derivation of a general Fokker-Planck equation is presented to model the slow convergence towards equilibrium of mean-field systems due to finite-$N$ effects. The thermalization process involves notably the disintegration of coherent structures that may sustain out-of-equilibrium quasistationary states. The time evolution of the fraction of particles remaining close to a mean-field potential trouch is analytically computed. This indicator enables to estimate the lifetime of coherent structures and thermalization timescale in mean-field systems.

Many physical systems may be considered as isolated assemblies of $N$ bodies interacting via long-range pair interactions. This is the case for systems ranging from charged particles interacting via Coulomb interaction to self-gravitating massive objects like globular clusters or stars in galaxies and this may even include suitably prepared Bose-Einstein condensates in a close future. The physically relevant issue of the dynamics of those systems in the large-$N$ limit forms the subject of kinetic theory. Long-range systems are prone to collective behavior that may be largely dominating before binary collisional effects set on. This hierarchy between collective and collisional behavior is responsible for the unusual properties of the relaxation process towards equilibrium as well as for the richness and complexity of the physics of long-range systems. These are motivations for the present consideration raised by long-range systems in various fields such as plasma physics, astrophysics, statistical physics or applied mathematics.

Collective behavior of long-range systems as well as the intricacies of the relationships between their dynamics, kinetic theory and equilibrium statistical properties may be more conveniently unveiled through models that are already of mean-field type for finite $N$. These are Hamiltonian models describing e.g. wave-particle interaction, which is an ubiquitous phenomenon in hot and dilute plasmas, or the all-to-all coupling of $N$ bodies evolving in the phase space $S^N$ under the dynamics deriving from the Hamiltonian

$$
\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{n=1}^{s} \sum_{i,j=1}^{N} V_n \cos[k_n(q_j - q_i)],
$$

where $q_i \in S_L$ is the position of particle $i$ on the circle $S_L = \mathbb{R}/L$, $p_i$ its conjugate momentum, and where only the first $s$ long-range components with wave numbers $k_n = 2\pi n/L$, for $1 \leq n \leq s$, are retained in the potential term. When $V_n \propto n^{-2}$, model amounts to the long-range truncation of the one-dimensional Newtonian potential with space-periodic boundary conditions, that describes Coulomb or gravitational interaction depending on the potential sign. Various systems covered by have been discussed in Ref. 8. An extension to a spatial dimension $d > 1$ should not be a conceptual problem. Introducing the set of collective observables $\{M_n\}$ through

$$
M_n = \frac{1}{N} \sum_{j=1}^{N} (\cos(k_n q_j), \sin(k_n q_j)) = M_n(\cos \phi_n, \sin \phi_n)
$$

yields the equation of motion of any particle $i$ as

$$
\dot{q}_i + \sum_{n=1}^{s} k_n V_n M_n \sin(k_n q_i - \phi_n) = 0.
$$

The Letter is organized as follows: first, a stochastic treatment of finite $N$-effects in mean field systems will be proposed leading to the establishment of a Fokker-Planck equation. In order to test this model, we then shall consider an Hamiltonian model of $N$ particles in self-consistent interaction via a cosine potential. Starting from configurations where $O(N)$ particles are trapped into their self-potential well, an analytic expression giving the fraction of the particles that remain trapped as a function of time will be successfully tested against numerical results. The relevance of this indicator to the thermalization issue will be shortly discussed.

Consider $N$ particles evolving in the phase space $S^N \times \mathbb{R}^{N}$ under the dynamics deriving from the Hamiltonian

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Therefore, the collective variables $M_n$ behave as mean fields that, as well as the phases $\phi_n$, depend on time through the self-consistency relations [2].

For smooth potentials like in [1], the convergence of the finite-$N$ dynamics to Vlasov equation is rigorously proved on arbitrary finite-time intervals [3]. Vlasov equation being time reversible, its solution $f(p,q,t)$ cannot approach an equilibrium [10], yet macroscopic quantities, involving phase space integrals of $f$, such as the mean-fields, can converge to stationary values associated to QSSs. In the realistic finite-$N$ Hamiltonian framework, finite-$N$ effects will in the long term induce the thermalization process and disintegration of coherent structures possibly sustaining those QSSs.

Modelling this process, we assume that the system [1] is trapped in a QSS, such that one can write the mean fields as $M_n(t) = M_n^0(t) + \delta M_n(t)$ with $\delta M_n \ll M_n^0$, where $\delta M_n$ varies on a time scale which is very much smaller than the characteristic time scale of $M_n^0$, the latter being comparable to a local average. We now make the hypothesis that during the QSS regime, the fluctuations around the local mean value have a variance decreasing as $N^{-1}$. The central idea behind this is to replace the deterministic but yet very chaotic fluctuations of $\delta M_n$ by stochastic processes, whose variances are suitably chosen. Numerical observations support this modelling. For instance, as shown in Fig.1 for some special case, the mean fields clearly exhibit two different timescales, in agreement with the decomposition suggested earlier. Moreover, both histograms and the autocorrelation function shown in Fig.1 suggest that one can model the fluctuations $\delta M_n$ by a Gaussian (white) noise.

\begin{equation}
\frac{\partial f}{\partial t} + \frac{\partial}{\partial q}(p f) - \sum_{n=1}^{s} k_n V_n M_n^0 \sin(k_n q - \phi_n) \frac{\partial f}{\partial p} = \sum_{n=1}^{s} \frac{k_n^2 V_n^2}{2} - \frac{1}{2} (\xi_n^2)^2 \sin^2(k_n q - \phi_n) \frac{\partial^2 f}{\partial p^2}
\end{equation}

This equation may be interpreted as a Vlasov equation supplemented with a r.h.s. of order $N^{-1}$, consistently with the argument presented in Ref. [3], coming here not from binary collisions but from the fluctuations of the mean fields. This differs from other FPEs derived in mean-field systems in other places [11, 12]. Moreover, this equation was derived without any need to invoke dissipation (see e.g. [12]).

In what follows, we shall consider the case of a single resonance ($s = 1$) in which coherent structures may survive for long times close to the potential trough (see e.g. Fig. 2). The FPE [4] may be further simplified by looking for solutions in separate variables $q$ and $p$ writing $f(p,q,t) = g(q,t) f(p,t)$. Assuming that $g$ is even in the wave frame and that $\int_0^L g(q,t) dq$ is constant, one obtains a simple diffusion equation

\begin{equation}
\frac{\partial \tilde{f}}{\partial t} = D(t) \frac{\partial^2 \tilde{f}}{\partial p^2},
\end{equation}

with diffusion coefficient

\begin{equation}
D(t) = \frac{k^2 V^2}{2} \frac{\langle \xi^2 \rangle}{\sin^2(kq - \phi)},
\end{equation}

and $\sin^2(kq - \phi) \equiv \frac{L^2}{2} \int_0^L \sin^2(kq - \phi) g(q,t) dq / \int_0^L g(q,t) dq$. Numerical evidence [3] supports the fact that $p$ and $q$ may be treated as separate variables and that $q$ may be considered as a fast variable compared to $p$, meaning that the distribution function in $q$ approaches much more quickly its Boltzmann-Gibbs shape than the $p$ one consistently with basic dimensional arguments [13]. The forthcoming numerical tests will show that the average of $\sin^2(kq - \phi)$ may be effectively replaced by its ensemble average.

From now on, in order to simplify expressions, we shall put $V = 1$ and $L = 2\pi$, which gives $k = 1$. This amounts to the well-known Hamiltonian Mean Field model [3]. Figure 2 shows numerical results obtained starting from a monokinetic beam [6]. The upper panel shows the one-particle phase space plots at three different stages of the evolution. Deep inside the mean-field potential trough, particles move almost regularly forming a clear coherent structure pattern that progressively dissipates. It is interesting to note the similarity of these figures with the phase space plots for the 1-D finite-$N$ cold dark matter simulations of Ref. [14]. The lower panel shows the evolution of the mean-field for four different numbers of particles, ranging from $N = 10^3$ to $N = 10^4$. The dashed line marks its equilibrium ensemble average. It is clear from
induced by the mean fields fluctuations, these particles will eventually escape, and we shall now estimate the characteristic time needed by the system to evacuate a fraction \( 1 - \delta \) of the \( N_0 \) particles initially contained in the band of momenta \([-\lambda; \lambda]\). Using the linearity of the diffusion equation \( \alpha \), one can focus on the contribution of particles whose momenta remain up to time \( t \) within the band \([-\lambda; \lambda]\) and solve \( \alpha \) by imposing the cancelation of \( f \) at \( p = \pm \lambda \). Finally, integrating over \( p \) yields the solution

\[
\frac{n(t)}{N_0} = \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}c_n}{(2n+1)\pi} \exp \left( -\frac{(2n+1)^2\pi^2}{4\lambda^2}Dt \right),
\]

with \( D \) given by Eq. \( \beta \) and

\[
c_n = \frac{\int_{-\lambda}^{\lambda} \tilde{f}(p, t = 0) \cos \left( \frac{(2n+1)\pi p}{2\lambda} \right) dp}{\int_{-\lambda}^{\lambda} \tilde{f}(p, t = 0) dp}.
\]

As confirmed by the numerical simulations shown in Fig. 3 the dynamics arising from Eq. \( \alpha \) proves to correctly depict the escape process through the validation of Eqs. \( \gamma \) - \( \delta \). In these numerical simulations, particles were initially distributed according to a so-called waterbag distribution

\[
f_0(p, q) = \Theta(\Delta p - |p|)\Theta(\Delta q - |q|)/(4\Delta p\Delta q)
\]

where \( \Theta \) stands for the Heaviside step function. It is interesting to note that when starting from these initial waterbag conditions, the phase-space distribution eventually exhibits a core-halo structure which has been recently investigated in [18]. Once again, it is visible on Fig. 3 that, at the time when all the particles that where initially in the momentum band \([-\lambda; \lambda]\) have at least once escaped this domain, the system has seemingly reached its thermal equilibrium. In order to test the escape model given by Eqs. \( \theta \) - \( \phi \), one needs to know the diffusion coefficient \( D \) given by Eq. \( \iota \). As already

\[\text{FIG. 2: (Upper panel) Phase-space snapshots at increasing times (from left to right, } t = 6, t = 100 \text{ and } t = 2000 \text{) for an initial cold-beam condition of energy } U = 0.5 \text{. The simulation was performed using } N = 10^4 \text{ particles. The instantaneous separatrix is plotted in bold red. (Lower panel) Time evolution of the magnetization } M \text{ and fraction } n/N \text{ for } N = 10^3 \text{ (green), } 2.10^3 \text{ (red), } 5.10^3 \text{ black and } 10^4 \text{ (blue) particles.}\]

\[\text{FIG. 3: (Left) Snapshot of the one-particle phase space at } t = 12 \text{ for } N = 5000 \text{ particles initially distributed in a waterbag configuration } \| \text{ with } \Delta p = 0.848 \text{ and } \Delta q = 2.16 \text{. (Right) Time evolution of the mean-field } M \text{ (red curve) and of } n/N \text{ (black curve). The blue dashed curve is the analytic expression for } n/N \text{ as deduced from Eqs. \( \gamma \) - \( \phi \). The initial time has been chosen at time } t = 10 \text{ and } N_0 = 0.95N.}\]
discussed, \( \sin^2(q) \) may be estimated from \( \langle \sin^2 q \rangle_c = I_1(\beta \langle M \rangle_c)/[\beta \langle M \rangle_c I_0(\beta \langle M \rangle_c)] \). A priori \( \langle \xi^2 \rangle \) has to be determined from numerical simulations since the system is not at equilibrium. However, in the cases that were considered, the numerically computed variance \( \langle \xi^2 \rangle \) was almost indistinguishable from its canonical value given by

\[
\langle \delta^2 M \rangle_c = \frac{2}{N} \frac{\partial}{\partial \beta} \log \left[ \frac{v^*}{\beta} \sqrt{\frac{2\pi N}{\beta}} e^{-N\psi(v^*)} \right] - \frac{v^*}{\beta^2},
\]

where \( \psi(v) = v^2/(2\beta) - \log I_0(v) \) and \( v^* = \beta \langle M \rangle_c \) satisfies the self-consistency equation \( \partial_r \psi|_{v^*} = 0 \). Numerically, this gives \( \langle \xi^2 \rangle \approx 0.43/N \), which is consistent with the fit obtained from numerical simulations. As shown in Fig. 3, the agreement between the numerically computed time evolution of \( n_\ell/N \) and its analytic modeling \( 7 \sim 8 \) is quite satisfactory.

Let us finally estimate the time needed to destroy the inner coherent structure by the means of Eq. \( 7 \). Considering that \( N_0 = O(N) \) and \( D = O(N^{-1}) \), a rough estimate of the time \( \tau_\ell(N) \) needed for \( n_\ell/N \) to reach down a sufficiently small fraction \( \delta \) gives

\[
\tau_\ell(N) \propto -N \log \delta. \tag{11}
\]

When \( D \) depends on time, Eq. \( 11 \) follows from the mean value theorem. This first result recovers the linear \( N \) scaling found in the abundant literature of long-range interacting systems \( 3 \sim 13, 16 \), where the numerical evidence is extracted from thresholds equivalent to the \( \delta \) criterion imposed here. One does not expect the continuum approach behind Eq. \( 7 \) to remain valid for vanishingly small values of \( \delta \). However, going up to the limit of validity of this model, one may infer that the sweeping of phase space is sufficient to reach a complete thermalization at a time \( \tau_{QSS} \) when \( n_\ell = O(1) \). Then Eq. \( 11 \) would give the maximal scaling \( \tau_{QSS} \propto N \log N \). This corresponds to the scaling recently suggested in Ref. \( 17 \) for the \( s = 1 \) case. Eq. \( 11 \) predicts a linear behavior with respect to \( N \), which we found to be correct over the whole range of values of \( N \) studied here, independently from the chosen threshold \( \delta \). We also checked the scaling with the latter parameter. Figure 4 shows a very good agreement between Eq. \( 11 \) and the numerical simulations.

These scalings contrast with the numerically obtained \( N^{1.7} \) scaling for the QSS lifetime starting from two special initial conditions \( 14 \sim 20 \). These cases are however not in contradiction with the results presented here since, even if this is less obvious for \( 19 \), they both correspond to QSS about a vanishing mean-field, a case that is excluded from the present framework since the phase would be no longer defined. In the intermediate cases, where the QSS magnetization is clearly above zero, but yet far from the equilibrium expectation, this method provides a good estimation of the time needed to destroy the coherent structures, but fails to predict the QSS lifetime, since the effective model does not capture the average growth of the separatrix with time.

The present framework and results are expected to be easily transposable to wave-particle models in which finite-\( N \) effects eventually drive the system towards equilibrium in contradiction with the Vlasov approach \( 21 \). This discreteness effect may be more than a numerical concern for simulations since some physical effects \( 22 \) cannot be explained in the Vlasov limit.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4}
\caption{(Left) Plot of \( \tau_\ell(N) \) in log-log scale with respect to \( N \). As predicted by Eq. 11, the behavior is linear with \( N \). (Right) Plot of the numerically measured \( \tau_\ell \) with respect to log \( \delta \). The behavior is linear over a wide range of \( \delta \). As expected, the measured times for very low threshold values are lower than the logarithmic prediction of Eq. 11, since the latter is obtained in the continuous limit.}
\end{figure}

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