Conformal Approach to Particle Phenomenology

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We propose the existence of a non-supersymmetric conformal field theory softly broken at the TeV scale as a new mechanism for solving the hierarchy problem. We find the imposition of conformal invariance to be very restrictive with many predictive consequences, including severe restrictions on the field content, the number of families as well as on the structure of inter-family Yukawa couplings. A large class of potentially conformal non-supersymmetric theories are considered and some general predictions are made about the existence of a rich spectrum of color and weak multiplets in the TeV range.

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1. Introduction

One of the basic facts about particle physics is the existence of (at least) two very different scales: The Planck scale $\sim 10^{19}\ GeV$ and the weak scale $\sim 10^2\ GeV$. The existence of these two very different scales and their stability under quantum correction is generally called the hierarchy problem. In fact there are two separate issues here: One is the existence of two very different scales, and the other one is the stability of such different scales under quantum corrections. As an example of the latter point generically the mass of the scalars should be artificially fine tuned to avoid getting the Planck scale masses by quantum corrections. One approach to solving both aspects of the hierarchy problem is to postulate the existence of supersymmetry which remains unbroken all the way down to near the weak scale. Supersymmetry is responsible for cancellation of many quantum corrections and this at least makes the hierarchy technically possible. The generation of widely different scales can also be explained if one assumes non-perturbative effects to be responsible. Indirect evidence for supersymmetry is mounting. These include the unification of the gauge coupling constants of electroweak and strong forces in the minimal supersymmetric extension of the standard model (but their lack of unification with the standard model particle content) and the successful prediction of the top quark mass. More theoretic arguments for supersymmetry have also been advanced, for example by appealing to its natural appearance in string theory.

Despite all this, there are some puzzles even in the supersymmetric context including the $\mu$-problem (i.e. why the Higgs mass is at the TeV scale as opposed to the Planck scale— in other words the first aspect of hierarchy problem) and the issue of proton stability. More fundamentally breaking supersymmetry would seem to lead to an unacceptably large value for the cosmological constant. Also the theoretic motivation for supersymmetry as being fundamental for string theory is questionable. In fact one does have solutions of string theories without any supersymmetry (see e.g. [1] for non-supersymmetric solutions with zero (or small) cosmological constant). Even in the supersymmetric compactifications one will have to deal with strings in non-supersymmetric contexts (for example when one considers thermal backgrounds). For these reasons, as well as because the issue is such a fundamental one for physics, one should attempt alternative routes for solving the hierarchy problem.

This paper is devoted to studying the possibility of replacing supersymmetry with conformal invariance near the TeV scale in an attempt to solve the hierarchy problem.
We will show that this idea is in principle possible and we present some toy models. We will mainly address the issue of the stability of the existence of two scales, and do not specifically discuss the mechanisms of breaking conformal invariance which should give rise to two very different scales.

We find that the principle of conformal invariance is more rigid than supersymmetry in that in many examples it predicts the number of generations as well as a rich structure for Yukawa couplings among the various families. This inter-family rigidity is a welcome feature of the conformal approach to particle phenomenology. In fact it is challenging to come up with a conformal model which satisfies all the known properties of the standard model.

The organization of this paper is as follows: In the next section we motivate the basic idea of considering conformal theories for particle phenomenology. In section 3 we review a large class of non-supersymmetric quantum field theories which are believed to lead to conformal theories. In fact we present further arguments, using string dualities, supporting the conjecture that the theories in question are conformal. In section 4 we consider some aspects of model building using the quantum field theories discussed in section 3. In section 5 we end with some concluding remarks.

2. Conformal Invariance as a Solution to the Hierarchy Puzzle

The weak scale and the QCD scale, as well as the masses of observed quarks and leptons are all so small compared to the Planck scale, that it is reasonable to believe that in some approximation they are exactly zero. If so, then the quantum field theory which would be describing the massless fields should be a conformal theory (as it has no mass scales). This simple observation suggests that the gauge particles and the quarks and leptons, together with some yet unseen degrees of freedom may combine to give a quantum field theory with non-trivial realization of conformal invariance. In such a scenario the fact that there are no large mass corrections follows by the condition of conformal invariance. In other words 't Hooft's naturalness condition is satisfied, namely in the absence of masses there is an enhanced symmetry which is conformal invariance. We thus imagine the actual theory to be given by an action

\[ S = S_0 + \int d^4x \, \alpha_i O_i \]
where $S_0$ is the lagrangian for the conformal field theory in question, and $O_i$ are certain operators of dimension less than 4, breaking conformal invariance softly. The $\alpha_i$ represent the “mass” parameters. Their mass dimension is $4 - \Delta_i$ where $\Delta_i$ is the dimension of the field $O_i$ at the conformal point. Note that the breaking should be soft, in order for the idea of conformal invariance to be relevant for solving the hierarchy problem. This requires that the operators $O_i$ have dimension less than 4.

Let $M$ denote the mass scale determined by the parameters $\alpha_i$. This is the scale at which the conformal invariance is broken. In other words, for energies $E \gg M$ the couplings will not run while they start running for $E < M$. We will assume that $M$ is sufficiently near the TeV scale in order to solve the hierarchy problem using conformal invariance.

This is our basic setup. The main ingredient to fill is to supply examples for conformal theories and see how close can we come to the standard model. In the next section we review a large class of non-supersymmetric theories in 4 dimensions which have been argued to be conformal at least to the leading order in the large $\mathcal{N}$ expansion. We extend these arguments (making some plausible assumptions) to finite order in $\mathcal{N}$. In section 4 we apply them to model building.

### 3. Examples of Conformal Theories in 4 Dimensions

In this section we review the construction of a large class of quantum field theories in 4 dimensions, one for each discrete subgroup of $SU(4)$ and each choice of integer $\mathcal{N}$, motivated from string theory considerations \cite{2} \cite{3}. It has been proven that these theories have vanishing beta function to leading order in $\mathcal{N}$ using ‘t Hooft’s planar diagrams embedded in string theory \cite{4} or its translation to Feynman diagrams \cite{5}. Below we argue for the existence of at least one fixed point even for finite $\mathcal{N}$ (under some technical assumptions). The vanishing of the beta function at large $\mathcal{N}$ was also argued in \cite{2} using AdS/CFT correspondence.

We will now describe the prescription for constructing the above mentioned gauge theories. We refer the reader to \cite{2} \cite{3} for the motivation for the prescription. Roughly speaking what the prescription does is to start with an $\mathcal{N} = 4$ gauge theory and get rid of some fields in the theory and identify some of the other ones together in such a way that the resulting theory is conformal. One ends up with theories with equal number of bosons and fermions (if one includes the $U(1)$ parts) but with bosons and fermions in
different representations of the gauge group. In particular one typically ends up with a non-supersymmetric theory. The prescription is as follows: We start with \( \Gamma \subset SU(4) \) which denotes a discrete subgroup of \( SU(4) \) (the global symmetry for \( N = 4 \) supersymmetric gauge theory). Consider irreducible representations of \( \Gamma \). Suppose there are \( k \) irreducible representations \( R_i \), with dimensions \( d_i \) with \( i = 1, \ldots, k \). The gauge theory we construct has gauge symmetry

\[
SU(Nd_1) \times SU(Nd_2) \times \ldots \times SU(Nd_k)
\]

for an arbitrary choice of integer \( N \). The fermions in the theory are given as follows. Consider the 4 dimensional representation of \( \Gamma \) induced from its embedding in \( SU(4) \). It may or may not be an irreducible representation of \( \Gamma \). We consider the tensor product of \( 4 \) with the representations \( R_i \):

\[
4 \otimes R_i = \bigoplus_j n_i^j R_j
\]  

The chiral fermions are in bifundamental representations

\[
(1, 1, \ldots, Nd_i, 1, \ldots, Nd_j, 1, \ldots)
\]

with multiplicity \( n_i^j \) defined above. For \( i = j \) the above is understood in the sense that we obtain \( n_i^i \) adjoint fields plus \( n_i^i \) neutral fields of \( SU(Nd_i) \). Note that we can equivalently view \( n_i^j \) as the number of trivial representations in the tensor product

\[
(4 \otimes R_i \otimes R_j^*)_{\text{trivial}} = n_i^j
\]

The asymmetry between \( i \) and \( j \) is manifest in the above formula. Thus in general we have

\[
n_i^j \neq n_j^i
\]

and so the theory in question is in general a chiral theory. However if \( \Gamma \) is a real subgroup of \( SU(4) \), i.e. if \( 4 = 4^* \) as far as \( \Gamma \) representations are concerned, then we have by taking the complex conjugate of (3.2):

\[
n_i^j = (4 \otimes R_i \otimes R_j^*)_{\text{trivial}} = (4 \otimes R_i \otimes R_j^*)^*_{\text{trivial}} =
\]

\[
(4^* \otimes R_i^* \otimes R_j)_{\text{trivial}} = (4 \otimes R_i^* \otimes R_j)_{\text{trivial}} = n_i^j.
\]

So the theory is chiral if and only if \( 4 \) is a complex representation of \( \Gamma \), i.e. if and only if \( 4 \neq 4^* \) as a representation of \( \Gamma \). If \( \Gamma \) were a real subgroup of \( SU(4) \) then \( n_i^i = n_j^j \).
If \( \Gamma \) is a complex subgroup, the theory is chiral, but it is free of gauge anomalies. To see this note that the number of chiral fermions in the fundamental representation of each group \( SU(Nd_i) \) plus \( Nd_i \) times the number of chiral fermions in the adjoint representation is given by

\[
\sum_j n_j^i N d_j = 4Nd_i
\]  

(3.4)

(where the number of adjoints is given by \( n_i^j \)). Similarly the number of anti-fundamentals plus \( Nd_i \) times the number of adjoints is given by

\[
\sum_j n_j^i N d_j = \sum_j Nd_j (4 \otimes R_j \otimes R_i^*)_{\text{trivial}} = \sum_j Nd_j (4^* \otimes R_j^* \otimes R_i)_{\text{trivial}} = 4Nd_i \tag{3.5}
\]

Thus, comparing with (3.4) we see that the difference of the number of chiral fermions in the fundamental and the anti-fundamental representation is zero (note that the adjoint representation is real and does not contribute to anomaly). Thus each gauge group is anomaly free.

In addition to fermions, we have bosons, also in the bi-fundamental representations. The number of bosons \( M_{ij} \) in the bi-fundamental representation of \( SU(Nd_i) \otimes SU(Nd_j) \) is given by the number of \( R_j \) representations in the tensor product of the representation \( 6 \) of \( SU(4) \) restricted to \( \Gamma \) with the \( R_i \) representation. Note that since \( 6 \) is a real representation we have

\[
M_{ij} = (6 \otimes R_i \otimes R_j^*)_{\text{trivial}} = (6 \otimes R_i^* \otimes R_j)_{\text{trivial}} = M_{ji}
\]

In other words for each \( M_{ij} \) we have a complex scalar field in the corresponding bi-fundamental representation, where complex conjugation will take us from the fields labeled by \( M^j_i \) to \( M_i^j \).

The fields in the theory are naturally summarized by a graph, called the quiver diagram, where for each gauge group \( SU(Nd_i) \) there corresponds a node in the graph, for each chiral fermion in the representation \((Nd_i, Nd_j)\), \( n_i^j \) in total, corresponds a directed arrow from the \( i \)-th node to the \( j \)-th node, and for each complex scalar in the bifundamental of \( SU(Nd_i) \times SU(Nd_j) \), \( M_{ij} \) in total, corresponds an undirected line between the \( i \)-th node and the \( j \)-th node (see Fig. 1).
3.1. Interactions

The interactions of the gauge fields with the matter is fixed by the gauge coupling constants for each gauge group. The inverse coupling constant squared for the \( i \)-th group combined with the theta angle for the \( i \)-th gauge group is

\[
\tau_i = \theta_i + \frac{i}{4\pi g_i^2} = \frac{d_i \tau}{|\Gamma|}
\]

where \( \tau = \theta + \frac{i}{4\pi g^2} \) is an arbitrary complex parameter independent of the gauge group and \( |\Gamma| \) denotes the number of elements in \( \Gamma \).

There are two other kinds of interactions: Yukawa interactions and quartic scalar field interactions. The Yukawa interactions are in 1-1 correspondence with triangles in the quiver diagram with two directed fermionic edges and one undirected scalar edge, with compatible directions of the fermionic edges (see Fig. 1):

\[
S_{Yukawa} = \frac{1}{4\pi g^2} \sum_{\text{directed triangles}} d^{abc} \text{Tr} \psi^a_{ij} \cdot \phi^b_{jk} \cdot \psi^c_{ki}^* \]

where \( a, b, c \) denote a degeneracy label of the corresponding fields. \( d^{abc} \) are flavor dependent numbers determined by Clebsch-Gordan coefficients as follows: \( a, b, c \) determine elements \( u, v, w \) (the corresponding trivial representation) in \( 4 \otimes R_i \otimes R_j^* \), \( 6 \otimes R_j \otimes R_k^* \) and \( 4 \otimes R_k \otimes R_i^* \). Then

\[
d^{abc} = u \cdot v \cdot w
\]

where the product on the right-hand side corresponds to contracting the corresponding representation indices for \( R_m \)’s with \( R_m^* \)’s as well as contracting the \( (4 \otimes 6 \otimes 4) \) according to the unique \( SU(4) \) trivial representation in this tensor product.

Similarly the quartic scalar interactions are in 1-1 correspondence with the 4-sided polygons in the quiver diagram, with each edge corresponding to an undirected line (see Fig. 1). We have

\[
S_{Quartic} = \frac{1}{4\pi g^2} \sum_{4\text{-gons}} f^{abcd} \psi_{ij}^a \cdot \phi_{jk}^b \cdot \psi_{kl}^c \cdot \phi_{li}^d
\]

where again the fields correspond to lines \( a, b, c, d \) which in turn determine an element in the tensor products of the form \( 6 \otimes R_m \otimes R_n^* \). \( f^{abcd} \) is obtained by contraction of the corresponding element as in the case for Yukawa coupling and also using a \( [\mu, \nu][\mu, \nu] \) contraction in the \( 6 \otimes 6 \otimes 6 \otimes 6 \) part of the product.
Fig. 1: A part of the Quiver Diagram. The edges with arrow represent fermionic matter and the edges without arrows represent scalar matter. The Yukawa coupling in this diagram comes from the ACB triangle and the quartic coupling comes from the ABCDA square. The edges whose beginning and end are the same point correspond to adjoint matter.

3.2. Conformality

As mentioned before it has been shown that the above theories are conformal to leading order in a $1/N$ expansion for arbitrary values of $\tau$. The question is whether this can be pushed to go beyond the leading order in the $1/N$ expansion. If $\Gamma$ resides in an $SU(2)$ subgroup of $SU(4)$ then one obtains an $N = 2$ superconformal theory and the conformality is automatic for finite $N$ as well, by non-renormalization theorems of $N = 2$. In fact it has been shown that all the finite $N = 2$ theories with matter representations in the bi-fundamental representation of the groups are captured by the above construction \cite{7}, and are thus in 1-1 correspondence with discrete subgroups of $SU(2)$. In particular the corresponding reduced quiver diagram corresponds, in this case, to the affine $A - D - E$ Dynkin diagrams (for cyclic, dihedral and exceptional subgroups of $SU(2)$ respectively) with each link of the Dynkin diagram corresponding to hypermultiplet matter, and the rank of each gauge group being proportional to the corresponding Dynkin numbers. In this case one can actually change the coupling of each gauge group independently and still obtain a conformal theory. If $\Gamma$ resides in an $SU(3)$ subgroup, one obtains an $N = 1$ supersymmetric theory. In this case one can use $N = 1$ non-renormalization theorems, along the lines of \cite{8} to prove for the existence of conformal theories to all orders. In these cases there would be at least a one dimensional line of fixed points.
If $\Gamma$ resides in the full $SU(4)$ group, then we have a non-supersymmetric theory, and it is natural to ask to what extent one can expect conformal theories for finite values of $N$. It has been conjectured in [3] that this continues to be the case, at least for some choice of the coupling constants. Here we will give a plausibility argument for this conjecture. At the leading order in $1/N$, as we have indicated above, all the coupling constants in the Lagrangian are determined by a single coupling constant $\tau$, and we thus effectively have only one coupling constant. Under the Renormalization group flow it is not necessarily true that all the couplings in the theory will still be determined by a single coupling constant; however let us assume this continues to be correct. These theories have an additional property that has not been mentioned so far, and that is the $\tau \to -1/\tau$ strong-weak duality. This duality exchanges $4\pi g^2 \leftrightarrow 1/4\pi g^2$ (at $\theta = 0$). This follows from their embedding in the type IIB string theory which enjoys the same symmetry. In fact this gauge theory defines a particular type IIB string theory background [2] and so this symmetry must be true for the gauge theory as well. In the leading order in $N$ the beta function vanishes. Let us assume at the next order there is a negative beta function, i.e., that we have an asymptotically free theory. Then the flow towards infrared increases the value of the coupling constant. Similarly, by the strong-weak duality, the flow towards the infrared at large values of the coupling constant must decrease the value of the coupling constant. Therefore we conclude that the beta function must have at least one zero for a finite value of $g$.

This argument is not rigorous for three reasons: One is that we ignored the flow for the $\theta$ angle. This can be remedied by using the fact that the moduli space is the upper half-plane modulo $SL(2, \mathbb{Z})$ which gives rise to a sphere topology and using the fact that any vector field has a zero on the sphere (“you cannot comb your hair on a sphere”). The second reason is that we assumed asymptotic freedom at the first non-vanishing order in the large $N$ expansion. This can in principle be checked by perturbative techniques and at least it is not a far-fetched assumption. More serious, however, is the assumption that there is effectively one coupling constant. It would be interesting to see if one can relax or verify this assumption, which is valid at large $N$. 
4. Applications to Particle Phenomenology

In section 2 we have outlined the basic strategy of connecting conformal theory to particle phenomenology and in section 3 we have given a large list of quantum field theories which are potentially conformal. In this section we would like to indicate by general arguments and also through examples, how particle phenomenology may arise from such a picture. Even though there presumably are many more conformal theories without supersymmetry, we will mainly concentrate on the ones that we discussed in section 3. For an extension of these models, see [9]. Hopefully more examples will be known in the near future, which one may apply to particle phenomenology.

As discussed in section 2 we assume that the Lagrangian describing the particles and their interaction is nearly conformal, i.e. it is a soft-breaking of a conformal theory. In the examples provided in section 3 the soft breaking terms would involve arbitrary additions to the Lagrangain involving quadratic scalar mass terms, fermionic mass terms and cubic scalar mass terms consistent with gauge interactions. From the perspective of quiver diagrams this means considering 2-gons with two compatibly directed edges (corresponding to fermion mass term) or 2-gons with two undirected edges, corresponding to scalar mass terms, or triangles with three undirected edges, corresponding to cubic scalar couplings:

\[ S = S_0 + \int \alpha_{ab} \text{Tr} \Psi^a_{ij} \Psi^b_{ji} + \alpha^2_{cd} \text{Tr} \Phi^c_{ij} \Phi^d_{ji} + \alpha_{efg} \text{Tr} \Phi^e_{ij} \Phi^f_{jk} \Phi^g_{ki} + c.c. \quad (4.1) \]

Depending on the sign of quadratic terms for the scalars in the above action, the conformal breaking terms in (4.1) could induce gauge symmetry breaking using the Higgs mechanism. Consider for example two gauge groups \( SU(Nd_i) \times SU(Nd_j) \) and let us suppose that the above terms are such that \( \langle \phi_{ij} \rangle \neq 0 \). For the sake of example let us assume \( d_i = d_j = d \). Then we can represent the expectation value of \( \phi_{ij} \) as a square matrix with diagonal entries. Depending on the eigenvalues of the matrix we get various patterns of gauge symmetry breaking. For example if we have 2 equal non-vanishing eigenvalues and the rest zero we get the breaking pattern

\[ SU(Nd) \times SU(Nd) \rightarrow SU(2)_{\text{diagonal}} \times U(1) \times SU(Nd - 2) \times SU(Nd - 2). \]

If we have more scalars in the bifundamentals of the \( SU(Nd) \times SU(Nd) \) we can break the groups even further, by various alignments of the expectation values. Thus in general we can have a rich pattern of gauge symmetry breakings.
4.1. Some General Predictions

We would like to discuss how $SU(3) \times SU(2) \times U(1)$ standard model can be imbedded in the conformal theories under discussion. In other words we consider some embedding

$$SU(3) \times SU(2) \times U(1) \subset \otimes_i SU(N_{d_i})$$

in the set of conformal theories discussed in section 3. Each gauge group of the standard model may lie in a single $SU(N_{d_i})$ group or in some diagonal subgroup of a number of $SU(N_{d_i})$ gauge groups in the conformal theory. The first fact to note, independently of the embeddings, is that the matter representations we will get in this way are severely restricted. This is because in the conformal theories we only have bi-fundamental fields (including adjoint fields), and thus any embedding of the standard model in the conformal theories under discussion will result in matter in bi-fundamentals (including adjoints), and no other representation. For example we cannot have a matter field transforming according to representation of the form $(8, 2)$ of $SU(3) \times SU(2)$. That we can have only fundamental fields or bi-fundamental fields is a strong restriction on the matter content of the standard model which in fact is satisfied and we take it as a check (or evidence!) for the conformal approach to phenomenology. The rigidity of conformal invariance in this regard can be compared to other approaches, where typically we can have various kinds of representations.

Another fact to note is that there are no $U(1)$ factors in the conformal theories (having charged $U(1)$ fields is in conflict with conformality) and in particular the existence of quantization of hypercharge is automatic in our setup, as the $U(1)$ has to be embedded in some product of $SU$ groups. This is the conformal version of the analogous statement in the standard scenarios to unification, such as $SU(5)$ GUT.

The rigidity of conformal invariance goes much further. Embedding the standard model in the conformal theories predicts a rich spectrum of additional unobserved particles charged under $SU(3)$ and $SU(2)$. These extra particles should have mass in the TeV range (the conformal breaking scale) in order for conformal invariance to be relevant for the resolution of the hierarchy puzzle. The lower bound on the number of extra particles arises if we assume each group
of the standard model is embedded in a single $SU(Nd_i)$ group of the same size. This is because the number of extra fermions transforming under fundamental representation (plus $Nd_i$ times the number of adjoint fields) is given by $4Nd_i$ for each $SU(Nd_i)$ and taking a diagonal subgroup of them will only give rise to more matter fields. Moreover the minimal extra matter will come by assuming $Nd_i$ is equal to 3 and 2 for the $SU(3)$ and $SU(2)$ gauge groups respectively. To obtain the lower bound on the number of extra color charged matter fields we thus assume that one of the gauge groups in $S_0$ is the $SU(3)$ color. Then from the constructions discussed in the previous section we see that the number of the triplets $N_3$ plus 3 times the number of adjoints $N_8$ is bounded by

$$N_3 + 3N_8 \geq 4 \cdot 3 = 12$$

Subtracting the observed number of quarks $N_3 = 6 + \Delta N_3$ we see that

$$\Delta N_3 + 3N_8 \geq 6$$

In other words, if all the extra color charged fermions were in the fundamental representations, we would be predicting at least 6 more quarks in the TeV range. Of course the actual number of families will depend on the rest of the gauge groups. For example if the breaking goes through an $SU(3) \times SU(3) \times SU(3)$ gauge group, as in the trinification scenario [10] each family would contain 3 quarks (one of which must be much heavier than the other two to be phenomenologically realistic) and in the case of the lower bound we would end up with 4 families. Similarly we predict the existence of extra colored scalars. If we denote the number of complex triplet scalars by $M_3$ and the number of adjoint scalars by $M_8$ we would predict

$$M_3 + 3M_8 \geq 6 \cdot 3 = 18$$

Going over the same exercise for the $SU(2)$ group we would predict the following bound on the extra weak charged fermions and bosons in the TeV range:

$$\Delta N_2 + 4N_3 \geq 4$$

$$\Delta M_2 + 2M_3 \geq 11$$
(where we subtracted 12 from \(N_2\) and 1 from \(M_2\) corresponding to the matter content in the standard model). These bounds clearly demonstrate the rigidity of the conformal approach to phenomenology.

As far as other rigidities that the conformal assumption introduces, we should note that the structure of the Yukawa and quartic couplings is untouched by the soft-breaking terms and is determined completely in terms of the fixed point values at the conformal point given in \(S_0\). As we saw in the last section, these Yukawa terms have rich flavor dependent structure which is dictated by conformal invariance. This is in sharp contrast to standard phenomenology or the MSSM where the Yukawa couplings are put in by hand. We thus see that conformal symmetry is far more rigid than supersymmetry.

4.2. Coupling Unification

One of the successful ideas about GUTs is the prediction of the meeting of the three gauge coupling constants of the standard model at a high energy scale (\(\sim 10^{16}\text{GeV}\)) which is a beautiful feature of the minimal supersymmetric extension of the standard model. In the case at hand, however we are assuming the existence of a conformal theory in the TeV scale. Beyond the conformal symmetry breaking scale the gauge group \(\otimes SU(N_{d_i})\) is restored and their couplings will not run. Even though the gauge groups do not unify into a single gauge group, recall from section 3 that the coupling constants of the gauge groups are related to each other, which is also a feature of GUT theories. Matching the coupling constant of the \(SU(3) \times SU(2) \times U(1)\) gauge theories at the conformal breaking scale with the fixed point values for the coupling constants at the conformal point, we see that the coupling constants of the \(SU(3)\) and \(SU(2)\) and \(U(1)\) will be strongly correlated, though not necessarily equal. Even if the gauge coupling constants of the \(SU(N_{d_i})\) groups were all equal (which is in general not true) we cannot deduce that the gauge couplings of the \(SU(3) \times SU(2) \times U(1)\) gauge theories are equal at the conformal breaking scale. In particular if \(SU(3)\) embeds into a single \(SU(N_{d_i})\) gauge group, and \(SU(2)\) in two such groups and \(U(1)\) in 6 such groups, we would have gotten the ratios of the coupling constants \(\tau_3/\tau_2/\tau_1 \sim 1/3/6\) which would be close to the observed ratios in the weak scale. We thus see that in the present scenario the coupling constants are strongly correlated, yet they do not have to be equal unlike the case in the standard GUTs.
4.3. Some Examples

In this section we will consider some illustrative examples. In order to specify an $S_0$ in the class of Lagrangian we considered in the previous section, we need to start with a discrete subgroup $\Gamma \subset SU(4)$. Moreover if we wish to have a chiral theory we should consider complex subgroups of $SU(4)$. The simplest (though not necessarily the most interesting) subgroups are the cyclic ones. Let us look for a non-supersymmetric chiral model. The $Z_2$ subgroup, consisting of $4(-1)$ eigenvalues, is real and so is not a chiral theory. For the $Z_3$ subgroup we have two choices: one is three equal eigenvalues and one eigenvalue $1$, which corresponds to an $N = 1$ supersymmetric chiral theory (for the rank 3 case we would get 3 families of the supersymmetric version of trinification without any Higgs multiplets). Another choice of $Z_3$ consists of two pairs of conjugate eigenvalues. This is a real representation and is not chiral. The next case is $Z_4$ and the only complex representation without supersymmetry is the embedding with four eigenvalues of $i$. This gives a gauge group $SU(N)^4$ for some choice of $N$ with fermion and bosons given by the quiver diagram below:

![Fig. 2](image.png)

**Fig. 2**: The quiver diagram for the $Z_4$ theory. Fermions correspond to the edges and the diagonals correspond to the scalars. The labels next to the lines denote the degeneracy of each line.

The four gauge groups are identified with the vertices of the square. The fermions are four copies of the edges of the square and the scalars are 6 complex fields identified with the diagonals. The scalar content of this theory is too rigid to give a breaking pattern which could lead to the standard model spectrum. The simplest non-supersymmetric chiral model which contains the spectrum of the standard model and can be broken to it is a $Z_5$ subgroup of $SU(4)$ where
we choose the four eigenvalues to be \((\alpha, \alpha, \alpha, \alpha^2)\) where \(\alpha = \exp(2\pi i/5)\). For the gauge group we take \(SU(3)^5\) and it is convenient to identify them with 5 points on the circle, labeled in order by 0 through 4 with the integers defined mod 5. The theory at the conformal point has a \(Z_5\) cyclic permutation of the gauge groups (along with other symmetries which we will not consider here).

There are three chiral fermions in the bifundamental of \((i, i + 1)\) gauge groups and one bi-fundamental in the \((i, i + 2)\) gauge groups where \(i\) runs over all the gauge groups. There are in addition 3 complex scalars in the bifundamental of the \((i, i + 2)\) and 3 complex scalars in the bifundamental of \((i, i + 3)\). See the quiver diagram below.

**Fig. 3**: The quiver diagram for the \(Z_5\) conformal theory. In order to simplify the diagram we have drawn only the edges emanating from the 0 node. The rest are found by the cyclic symmetry of the theory under the cyclic permutation (01234). The \(SU(3)\) color is embedded in the gauge group at node 0, and the \(SU(2)\) weak is embedded in the diagonal subgroup of nodes 1 and 3, and the third gauge group in the trinification (which we have rather loosely called hyper) comes from the diagonal embedding in the gauge groups 2 and 4.
We identify the $SU(3)$ color group with the gauge group at node 0. We assume that the conformal breaking deformation breaks the groups $(1,3)$ to a diagonal $SU(3)$ (which contains the $SU(2)$ weak) and the groups $(2,4)$ to a diagonal $SU(3)$. We end up with 3 families of fermions in the standard trinification representation

$$(3,\overline{3},1) + (1,3,\overline{3}) + (\overline{3},1,3)$$

and one family in the conjugate representation. In addition we have one fermionic fields in the adjoint of the second and third $SU(3)$’s as well as 3 copies of $[(1,3,\overline{3}) + c.c.]$. In addition there are a number of scalars in various bi-fundamental representations. There are enough scalars to induce the breaking of the trinification group to the standard $SU(3) \times SU(2) \times U(1)$, which require only two bi-fundamental field in the last two $SU(3)$’s which is available. The point of this exercise was to show how the standard model may arise, and not to overemphasize this particular model.

5. Concluding Remarks

It should be clear from the various examples presented how restrictive the assumption of conformal invariance is for particle phenomenology. It is precisely for this reason that it is potentially useful as an organizing principle for particle physics.

Here we have concentrated on a large class of conformal field theories (which have been conjectured to be the complete list if one restricts attention to theories with bi-fundamental matter). However there may well be other interesting conformal theories for particle phenomenology and this issue should be further investigated. Even within the class of models considered, we would need a complete classification of subgroups of $SU(4)$ and their representation ring—this is mathematically within reach, but has not been done yet (for the $SU(3)$ case this is already known—see \cite{11,12}). Some progress in this direction has been made in \cite{13}.

A key issue to understand in connection with applying the ideas of the present paper to particle physics is to identify natural mechanisms to softly break conformal invariance. In particular one would like to get an explanation
of the generation of a small scale compared to the Planck scale. It is tempting to speculate that quantum gravity effects can play an interesting role here.

Within the class of models introduced, we have found a lower bound on the number of new charged fields in the TeV scale. Thus if conformal symmetry is realized near the TeV scale, we will be witnessing discovery of many new particles in the accelerators in the near future.

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