Characterization theorem for best polynomial spline approximation with free knots

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In this paper, we derive a necessary condition for a best approximation by piecewise polynomial functions. We apply nonsmooth nonconvex analysis to obtain this result, which is also a necessary and sufficient condition for inf-stationarity in the sense of Demyanov-Rubinov. We start from identifying a special property of the knots. Then, using this property, we construct a characterization theorem for best free knots polynomial spline approximation, which is stronger than the existing characterisation results when only continuity is required.

1 Introduction

The problem of approximating a continuous function by a piecewise polynomial (polynomial spline) has been studied for over four decades [11]; yet, when the knots joining the polynomials are also variable, finding conditions for a best Chebyshev approximation remains an open problem [1] problem 1. We derive a necessary optimality condition for a best approximation which is stronger than the existing ones [7,9] when only continuity is required.

It is acknowledged in [1] that the existing optimisation tools are not adapted to this problem, due to its nonconvex and nonsmooth nature. Therefore, our motivation is to apply recently developed nonsmooth optimisation tools [2,3] not well-known outside of the optimisation community to improve the existing results.

In this paper we are concentrating on necessary optimality conditions. These conditions are important for the development of an algorithm for constructing a best polynomial spline approximation, since these conditions can be used as stopping criteria. Most local optimisation methods can improve a solution that is not locally optimal. Therefore, it is especially important to confirm that an iterate is at least locally optimal or use it as a starting point for local optimisation.

The paper is organised as follows. In section 2 we introduce necessary definitions and relevant results from the area of polynomials spline approximation. The last subsection of this section provides necessary results from the theory of quasidifferentials, developed by
Demyanov and Rubinov [2,3]. Some of these techniques allow us to overcome the difficulties highlighted in [1]. After all the necessary preliminaries, we proceed with obtaining the new results, formulated in section 3 (theorem 3.1). The proof of this result takes several steps. First, in section 4 we reformulate our approximation problem as an optimisation problem and show that the quasidifferential of its objective function is expressed in terms of extreme points. Second, in section 5 we introduce the notion of confined quasidifferential, a subset of the whole quasidifferential containing only the components based on the extreme points from a given subinterval. Third, in section 6 we define an invertible linear transformation which simplifies the vectors from the sub- and superdifferentials. Fourth, in section 7 we prove proposition 5.1. Our final step is to show that proposition 5.1 and theorem 3.1 are equivalent. Finally, in section 8 we conclude and highlight future research directions.

2 Preliminaries

2.1 Definitions and formulations

**Definition 2.1** (Polynomial Spline). A polynomial spline is a piecewise polynomial. Each polynomial piece lies on an interval $[\xi_i, \xi_{i+1}], i = 0, \ldots, N - 1$. The points $\xi_0$ and $\xi_N$ are the *external knots*, and the points $\xi_i, (i = 1, \ldots, N - 1)$ are the *internal knots* of the polynomial spline.

The spline is generally not infinitely differentiable at its knots. Denote the set of polynomials of degree $m$ by $\mathcal{P}_m$ and the set of piecewise polynomials of degree $m$ with $k$ knots by $\mathcal{P}_{m,k}$.

**Definition 2.2** (multiplicity). An integer $m_i \leq m + 1$ is called the *multiplicity* of the spline at the knot $\xi_i$ if the spline is $m - m_i$ times continuously differentiable at $\xi_i$.

In the case examined in this paper only continuity of the spline is required and hence $m_i = m, m = 1, \ldots, N - 1$. Therefore we consider the problem of finding a best approximation by a continuous piecewise polynomial as follows:

\[
\text{minimize } \Psi(s) \text{ subject to } s \in \mathcal{P}_{m,N} \cap C[a,b], \\
\Psi(s) = \sup_{t \in [\xi_0, \xi_N]} \max \{s(t) - f(t), -s(t) + f(t)\}. \tag{2.1a}
\]

**Definition 2.3.** The difference between the spline and the function to approximate is called the *deviation*.

We denote the deviation function at point $t$ by $\psi_t(s) = \psi_t(s, t) = s(t) - f(t)$.

Our aim is to minimize the maximal absolute deviation. This maximal deviation occurs at points in the interval $[\xi_0, \xi_N]$ which we call *extreme points*.

**Proposition 2.1.** If the function $f$ is continuous, problem (2.1a) admits a solution
Proof. Clearly, we can restrict our search to the set
\[ \bigcup_{(\xi_1, \ldots, \xi_N) \in [a, b]} \text{Argmin}\{\Psi(s) : s \in \mathcal{P}_m(\xi_1, \ldots, \xi_N)\} \]
where \( \mathcal{P}_m(\xi_1, \ldots, \xi_N) \) is the set of polynomial splines with knots at \( \xi_1, \ldots, \xi_N \). This set, as a union of sets of solutions to the problem of best Chebyshev approximation with fixed knots [9], is well defined. Its closure may contain discontinuous splines when two of the knots coincide. However, if any such discontinuous spline is optimal, then it is possible to construct a continuous spline at least as good. The proof is the same as the one of [11, theorem 3.3].

Polynomial splines can be constructed in different ways. In this paper we use the truncated power function [7, Appendix, p. 191]:
\[
(t - \tau)_+^j = \begin{cases} 
0, & \text{if } t < \tau \\
(t - \tau)^j, & \text{if } t \geq \tau 
\end{cases}
\]
Let \( X = (a_{00}, x_0, \xi_1, x_1, \ldots, \xi_{N-1}, x_{N-1}) \in \mathbb{R}^{(m+1)N} \), where \( x_i = (a_{i1}, \ldots, a_{im}) \in \mathbb{R}^m, i = 0, \ldots, N-1 \) and
\[
a = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_{N-1} \leq \xi_N = b,
\]
then
\[
s(t) = s[X](t) = a_{00} + \sum_{i=0}^{N-1} \sum_{j=1}^{m} a_{ij}(t - \xi_i)^{m+1-j}.
\]
On the \( l \)-th interval, between the knots \( \xi_{l-1} \) and \( \xi_l \), the spline \( s[X](t) \) coincides with a polynomial which we denote by \( P_l(X, t) \):
\[
P_l(X, t) = a_{00} + \sum_{i=0}^{l-1} \sum_{j=1}^{m} a_{ij}(t - \xi_i)^{m+1-j}.
\]
The formulation (2.3) allows for the straightforward handling of constraint (2.2): it suffices to re-order the knots of any given spline in increasing order to obtain a coinciding spline satisfying this constraint. If some knots lie outside of the interval \([a, b]\), they can simply be ignored (or replaced by knots of multiplicity 0) as they will not affect the values taken by the spline over this interval.

Summarising all the above, problem (2.1a) can be reformulated as follows:
\[
\text{minimize } \sup_{t \in [\xi_0, \xi_N]} \left| a_{00} + \sum_{i=0}^{N-1} \sum_{j=1}^{m} a_{ij}(t - \xi_i)^{m+1-j} - f(t) \right|,
\]
subject to \( X \in \mathbb{R}^{(m+1)N} \).

The problem is unconstrained and well defined, and the variables are the parameters of the polynomial pieces and the knots.
2.2 Existing work and motivation

The theory on polynomial and fixed-knots polynomial spline approximation is generally complete \[8, 10, 12, 13, 14\]. These results are thoroughly reviewed in \[7\]. In this subsection we review the main known results for free-knot polynomial spline approximation.

2.2.1 Characterization theorems and necessary optimality conditions

Most characterization theorems (optimality conditions for best Chebyshev approximation) are based on the notion of \textit{alternating extreme points} of the deviation function.

**Definition 2.4.** Points \(t_1, \ldots, t_p\) are called \textit{alternating extreme points} of a function \(g\) defined over an interval \([a, b]\) if there exists a sign \(\sigma \in \{-1, 1\}\) such that

\[
\sigma \cdot (-1)^i g(t_i) = \sup_{t \in [a, b]} |g(t)|.
\]

Traditionally, the set of polynomial splines with \(k\) free knots is defined as follows:

**Definition 2.5.** (\[7, Appendix, definition 1.1\]) Let integers \(N \geq 1, m \geq 1\) and \(k \geq 1\) be given. The set of polynomial splines of degree \(m\) with \(k\) free knots is

\[
S_{m,k} = \left\{ s : [\xi_0, \xi_N] \rightarrow \mathbb{R} : \text{there exist points } \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_N \text{ and integers } m_1 \ldots m_{N-1} \in \{1, \ldots, m+1\} \text{ with } \sum_{i=1}^{N-1} m_i \leq k \text{ such that} \\
 s|_{[\xi_i, \xi_{i+1}]} \in \mathcal{P}_m, \; i = 0, \ldots, N-2, \; s|_{[\xi_{N-1}, \xi_N]} \in \mathcal{P}_m \text{ and} \\
 s \text{ has at least } m - m_i \text{ continuous derivatives at } \xi_i, \; i = 1 \ldots N \right\},
\]

where \(m_i, i = 1, \ldots, N - 1\) are the multiplicities of the corresponding knots.

This definition is used when smoothness is desirable. Indeed, it was shown in \[11\] that it is not always possible to approximate a function by a spline with some degree of differentiability and definition 2.5 was introduced to address this problem. The number of knots is linked to the differentiability of the spline. A necessary condition for best approximation from this set is presented in \[7, theorem 1.6, Appendix 1\].

**Theorem 2.1.** Consider a continuous function \(f\) and a polynomial spline \(s_0\) of degree \(m\) with knots \(\xi_0, \ldots, \xi_N\) and the corresponding multiplicities \(m_1, \ldots, m_{N-1}\). The spline \(s_0\) is a best Chebyshev approximation in \(S_{m,k}\) to the function \(f\), then there exists an interval \([\xi_p, \xi_q]\) where the function \(s_0 - f\) admits at least \(q - p + m + 1 + \sum_{i=p+1}^{N-1} m_i\) alternating extreme points.

This result has been strengthened in some cases \[9\], but these improvements can only be applied to the case of smooth splines.

Definition 2.5 and theorem 2.1 allow for the number and multiplicities of knots to change outside of the interval \([\xi_p, \xi_q]\). As shown in example 2.1, this may lead to some suboptimal solutions to satisfy theorem 2.1.
Example 2.1. Consider the problem of approximating the piecewise linear function joining the points \([-2, 7], (-1, 2), (-\frac{1}{2}, -\frac{7}{8}), (0, 1), (\frac{1}{2}, -\frac{7}{8}), (1, 2), (2, 7)\]. Let \(m = 3\) and \(k = 1\) and consider the spline \(s(t) = |t^3|\). This spline is twice differentiable at its only knot \(t = 0\) and has 7 alternating extreme points. Therefore it satisfies theorem 2.1. Yet, it is not even optimal for a knot fixed at \(t = 0\).

A major obstacle to obtain a conclusive characterization theorem has been the problem’s nonconvexity [1, problem 1]. One tool from nonconvex analysis, the quasidifferential [2], was successfully applied to obtain several results on fixed-knots spline approximation [12, 13, 14]. We will extend these results to the case of polynomial splines with free knots.

2.2.2 Algorithms

Most existing algorithms for best Chebyshev approximation by free knot polynomial spline are heuristic. The following one works in two steps [5]: first it finds the knots by approximating the function with a discontinuous spline. Then, after fixing the knots, it applies a Remez-like algorithm [7] to find a spline with the required smoothness. Although the authors demonstrate that the algorithm works well in practice, it is not guaranteed to converge. Indeed, the following example shows that it may fail to reach even a locally optimal solution.

Example 2.2. Consider the problem of approximating the following function with a piecewise linear spline with only one internal knot:

\[
f(x) = \begin{cases} 
\sin(x) & \text{for } x \in [0, \pi] \\
-\sin(2x) & \text{for } x \in [\pi, 3\pi/2] 
\end{cases}
\]

As can be seen in figure [1], the algorithm from [5] does not converge towards a locally optimal solution. Indeed, the solution it reaches does not satisfy existing necessary conditions for a local best approximation [2] and [3, p.20].

If a heuristic algorithm terminates at a solution which is not locally optimal, the results can be improved by most local optimisation methods. Therefore, it is crucial to have strong necessary condition for (local) optimality validation. Our motivation is to develop necessary optimality conditions which are stronger than the existing ones [7].

2.3 Quasidifferentiable functions

Definition 2.6. A function \(f\) defined on an open set \(\Omega\) is quasidifferentiable [2, 3] at a point \(x \in \Omega\) if it is locally Lipschitz continuous, directionally differentiable at this point and there exists compact, convex sets \(\partial f(x)\) and \(\overline{\partial} f(x)\) such that the derivative of \(f\) at \(x\) in any direction \(g\) can be expressed as

\[
f'(x, g) = \max_{\mu \in \partial f(x)} \langle \mu, g \rangle + \min_{\nu \in \overline{\partial} f(x)} \langle \nu, g \rangle.
\]

The sets \(\partial f(x)\) and \(\overline{\partial} f(x)\) are called respectively the sub- and superdifferential of the function \(f\) at the point \(x\). The pair \([\partial f(x), \overline{\partial} f(x)]\) is called a quasidifferential of the function \(f\) at the point \(x\).
(a) Discontinuous spline obtained after the first step of the algorithm. Over each segment the polynomial satisfies Chebyshev’s best approximation condition.

(b) Spline obtained after the knots were fixed. This spline satisfies the optimality condition for fixed-knot splines, but not the conditions of theorem 2.1, requiring at least 5 alternating points. Thus it is not a local minimizer.

Figure 1: An example where the algorithm fails to reach a locally optimal solution

At any local minimizer $x^* \in \Omega$ of a quasidifferentiable function $f$ we have

$$\partial f(x^*) \subset \partial^* f(x^*). \quad (2.8)$$

A point $x^*$ satisfying condition (2.8) is an inf-stationary point.

The only points where the spline function can be nonsmooth are its knots. The discontinuity of the derivative at the knot $\xi_l$ is determined by the value of $a_{lm}$: if $a_{lm} = 0$, then the spline is differentiable at the knot $\xi_l$. We say that the knot is a neutral knot. If $a_{lm} > 0$, then around the knot $\xi_l$ the spline behaves like the maximum of two linear functions. We call such a knot a max-knot. Finally, if $a_{lm} < 0$, then in the neighbourhood of knot $\xi_l$ the spline behaves like the minimum of two linear functions and the knot is called a min-knot.

We use the following notations.

• $E_{\text{smooth}}$ is the set of extreme points of the deviation function outside of internal knots of the spline;

• $E_{\text{neutral}}$ is the set of neutral knots;

• $E_{\text{max}}$ is the set of max-knots; and

• $E_{\text{min}}$ is the set of min-knots.

In the following section we will formulate the main result of this paper.

3 Characterization through quasidifferentiability

The necessary condition relies on the following definition:

**Definition 3.1.** Min-knots with positive maximal deviation and max-knots with negative maximal deviation are called unstable knots. Max-knots with positive maximal deviation and min-knots with negative maximal deviation are called stable knots.
The following theorem is our main result, which is an improvement to the existing ones in the case when only continuity is desired.

**Theorem 3.1.** A spline satisfies condition (2.8) over the interval $[\xi_0, \xi_N]$ if and only if there exists a subinterval $[\xi_p, \xi_q]$ containing a sequence of $(m(q-p)+2+l)$ alternating extreme points of the deviation function, where $l$ is the number of non-neutral internal knots inside $(\xi_p, \xi_q)$. The end-points $\xi_p$ and $\xi_q$ may be included in this sequence only if they are not unstable.

*Proof.* The proof of theorem 3.1 proceeds as follows:

- First, in section 4 we show that the quasidifferential of function (2.1b) is expressed in terms of extreme points.
- Second, in section 5 we introduce the notion of confined quasidifferential, a subset of the whole quasidifferential containing only the components based on the extreme points from a given subinterval.
- Third, in section 6 we define an invertible linear transformation which simplifies the vectors from the sub- and superdifferentials.
- Fourth, in section 7 we prove proposition 5.1 which is equivalent to (2.8) and therefore it gives another necessary and sufficient stationarity condition.
- Our final step is to show that proposition 5.1 and theorem 3.1 (our main result) are equivalent and therefore our main result is equivalent to the stationarity condition in the sense of Demyanov-Rubinov.

4 Quasidifferential of the objective function

4.1 Quasidifferential of the spline functions

The aim of this subsection is to analyse the function $\Psi$ as a function of $X$. As pointed above, when the point $t$ is a knot, the function $\psi_t(s[X])$ may not be differentiable.

4.1.1 Case when the point $t$ is not a knot

If there exists an index $l$ such that $\xi_l < t < \xi_{l+1}$, then the function $\psi_t$ is differentiable:

$$\nabla \psi_t(s[X]) = \nabla_X P_l(X,t),$$

(4.1)

where $\nabla_X P_l(X,t) = \partial P_l(X,t)/\partial X$.

4.1.2 Case when point $t$ is a knot

The function $\psi_t(s[X])$ can be nondifferentiable at $t = \xi_l$ for some $l \in \{1, \ldots, N-1\}$. In order to compute its quasidifferential we isolate the nonsmooth part by rewriting the function as
follows: \( \psi_t(s[X]) = \zeta_t(X) + \gamma_t(X) \), where

\[
\zeta_t(X) = a_{lm}(t - \xi_l)_+, \\
\gamma_t(X) = a_{00} + \sum_{i=0}^{l-1} \sum_{j=1}^{m} a_{ij}(t - \xi_i)_+^{m+1-j} + \sum_{j=1}^{m-1} a_{ij}(t - \xi_i)_+^{m+1-j} - f(t) \\
+ \sum_{i=l+1}^{N} \sum_{j=1}^{m} a_{ij}(t - \xi_i)_+^{m+1-j} \\
= P_l(X, t) + \sum_{j=1}^{m-1} a_{ij}(t - \xi_i)_+^{m+1-j} - f(t). \tag{4.2}
\]

**Quasidifferential of function** \( \gamma_t \): At \( t = \xi_l \), the function \( \gamma_t \) is differentiable with respect to \( X \), and its gradient coincides with the gradient of \( P_l(X, t) \):

\[
\nabla \gamma_t(X) = \nabla_X P_l(X, t). \tag{4.3}
\]

**Quasidifferential of function** \( \zeta_t \): The function \( \zeta_t \) may not be differentiable. As a product of a constant and a maximum of two linear functions its quasidifferential depends on the sign of \( a_{lm} \):

if \( a_{lm} < 0 \), then

\[
\tilde{\partial} \zeta_t(X) = \{0_{N(m+1)}\}, \\
\bar{\partial} \zeta_t(X) = \text{co}\{0_{N(m+1)}, (0_{l(m+1)-1}, -a_{lm}, 0_{N-l(m+1)})^T\}. \tag{4.4a}
\]

if \( a_{lm} = 0 \) then

\[
\tilde{\partial} \zeta_t(X) = \{0_{N(m+1)}\}, \\
\bar{\partial} \zeta_t(X) = \{0_{N(m+1)}\}. \tag{4.4b}
\]

Notice that in this case the function \( \psi_t \) is differentiable at \( t = \xi_l \).

if \( a_{lm} > 0 \), then

\[
\tilde{\partial} \zeta_t(X) = \text{co}\{0_{N(m+1)}, (0_{l(m+1)-1}, -a_{lm}, 0_{N-l(m+1)})^T\}, \\
\bar{\partial} \zeta_t(X) = \{0_{N(m+1)}\}. \tag{4.4c}
\]

Here \( \text{co} \) represents the convex hull of the corresponding set, \( 0_n \) is an \( n \)-dimensional column vector.
Combine $\gamma'_l$ and $\zeta'_l$: Finally, notice that at $t = \xi_l$

$$\nabla_X P_{l+1}(X, t) - \nabla_X P_l(X, t) = \nabla_X (P_{l+1} - P_l)(X, t)$$

$$= (0_{(m+1)-1}, -a_{lm}, 0_{(N-l)(m+1)})^T.$$ 

Therefore, combining equation (4.3) with (4.4) using quasidifferential calculus [3] Chapter 1, p.11, we obtain

if $a_{lm} < 0$, then

$$\partial \psi_t(s[X]) = \{0_{N(m+1)}\};$$

$$\partial^q \psi_t(s[X]) = \{0_{N(m+1)}\}.$$ (4.5a)

if $a_{lm} = 0$ then

$$\partial \psi_t(s[X]) = \{\nabla_X P_l(X, t), \nabla_X P_{l+1}(X, t)\};$$

$$\partial^q \psi_t(s[X]) = \{0_{N(m+1)}\}.$$ (4.5b)

if $a_{lm} > 0$, then

$$\partial \psi_t(s[X]) = \{\nabla_X P_l(X, t), \nabla_X P_{l+1}(X, t)\};$$

$$\partial^q \psi_t(s[X]) = \{0_{N(m+1)}\}.$$ (4.5c)

4.2 Continual maximum

The function $\Psi(s)$ defined in (2.1b) is a continual maximum of functions. The quasidifferential properties and calculus of such functions have been studied in [4]. According to [4], theorem 1, the function $\Psi(s)$ is quasidifferentiable at a point $s_0$ if:

1. the function under the supremum, $|\psi(s, t)|$, is continuous in $t$ for any $s$ from the neighbourhood of $s_0$;

2. the function $|\psi(s, t)|$ is uniformly directionally differentiable at the point $s_0$ for any $t \in [\xi_0, \xi_N]$;

3. the function $|\psi(s, t)|$ is quasidifferentiable with respect to $s$ at the point $s_0$ and for any extreme point $t$ there exists a pair of convex compact sets $B(s_0)$ and $A_t(s_0)$ such that $B(s_0) = \partial^q |\psi_t(s_0)| + A_t(s_0)$.

The first condition is verified whenever the function $f(t)$ is continuous. The second condition is also verified, because the function $\psi(s, t)$ is locally Lipschitz continuous. To verify the third condition, we need to calculate the quasidifferential of the function $\psi_t(s[X])$, and therefore of the spline $s[X]$. We will verify this condition in subsection 4.3.
4.3 Quasidifferential of the objective function

4.3.1 Explicit formulation of the quasidifferential of the objective function

We denote the index of the interval containing the extreme point \( t \) by \( j_t \). If \( t \) is a knot, then it joins the \( j_t \)-th and \((j_t + 1)\)-st intervals.

We apply quasidifferential calculus [3, Chapter 1, formula (5.3)] to obtain the quasidifferential of function \( |\psi_t(s)| \) from that of function \( \psi_t(s) \). Since there are a finite number of points (all of them are knots) where the superdifferential is not zero, it is easy to construct the sets \( B(s) \) and \( A_t(s) \) required to fulfil the third requirement of [4, theorem 1]. One way to construct these sets is as follows:

- \( t \) is not an internal knot, then
  \[
  A_t(s_0) = \sum_{i=1}^{N} \partial |\psi_{\xi_i}(s_0)|; \\
  B(s_0) = A_t(s_0) = \sum_{i=1}^{N} \partial |\psi_{\xi_i}(s_0)|; \\
  \]

- \( t = \xi_j, 1 < j < N \) is an internal knot, then
  \[
  B(s_0) = \sum_{i=1}^{N} \partial |\psi_{\xi_i}(s_0)|, \\
  A_t(s_0) = \sum_{i=1}^{j-1} \partial |\psi_{\xi_i}(s_0)| + \sum_{i=j+1}^{N} \partial |\psi_{\xi_i}(s_0)|. \\
  \]

Let us denote the sets of indices of the extreme points of the deviation function respectively with positive and negative deviation by \( K^+ = \{ t : \psi(s, t) = \Psi(s) \} \) and \( K^- = \{ t : \psi(s, t) = -\Psi(s) \} \).

Function \( \Psi \) admits the following superdifferential:

\[
\partial \Psi(s[X]) = \{0_{2N} \} + \text{co}(\Sigma^+ - \Sigma^-), \tag{4.6a}
\]

where

\[
\Sigma^+ = \sum_{t \in K^+ \cap E_{\min}} \text{co} \{\nabla_X P_{j_t}(X, t), \nabla_X P_{j_t+1}(X, t)\}, \tag{4.6b}
\]

\[
\Sigma^- = \sum_{t \in K^- \cap E_{\max}} \text{co} \{\nabla_X P_{j_t}(X, t), \nabla_X P_{j_t+1}(X, t)\}. \tag{4.6c}
\]
and the corresponding subdifferential:

\[
\partial \Psi(s[X]) = \text{co}\left\{ \bigcup_{t \in K^+ \cap (E_{\text{smooth}} \cup E_{\text{neutral}})} (\nabla_X P_{j_t}(X, t) - \overline{\partial} \Psi(s[X])), \right. \\
\bigcup_{t \in K^- \cap (E_{\text{neutral}} \cup E_{\text{neutral}})} (-\nabla_X P_{j_t}(X, t) - \overline{\partial} \Psi(s[X])), \\
\left. \bigcup_{t \in K^+ \cap E_{\text{max}}} (\text{co}\{\nabla_X P_{j_t}(X, t), \nabla_X P_{j_t+1}(X, t)\} - \overline{\partial} \Psi(s[X])), \right. \\
\left. \bigcup_{t \in K^- \cap E_{\text{min}}} (-\text{co}\{\nabla_X P_{j_t}(X, t), \nabla_X P_{j_t+1}(X, t)\} - \overline{\partial} \Psi(s[X])), \right. \\
\left. \bigcup_{t \in K^+ \cap E_{\text{max}}} (0 \sum_{\tau \neq t \in K^+ \cap E_{\text{min}}} \text{co}\{\nabla_X P_{j_t}(X, \tau), \nabla_X P_{j_t+1}(X, \tau)\} + \Sigma^-), \right. \\
\left. \bigcup_{t \in K^- \cap E_{\text{max}}} (0 \sum_{\tau \neq t \in K^+ \cap E_{\text{max}}} \text{co}\{\nabla_X P_{j_t}(X, \tau), \nabla_X P_{j_t+1}(X, \tau)\} - \Sigma^+) \right\}. 
\] (4.7f)

Remark 4.1. Only unstable extreme points contribute to the superdifferential.

Finally we introduce the following notation. Let

\[
\beta_{j_t}(X, t) = \text{sign}(\Psi(s[X], t)) \cdot \nabla_X P_{j_t}(X, t). 
\] (4.8)

To the smooth extreme points (non internal knots), neutral knots and stable knots we associate the set

\[
\mathcal{S} = \{ \beta_{j_t}(X, t) : t \in E_{\text{smooth}} \cup E_{\text{neutral}} \cup (K^+ \cap E_{\text{max}}) \cup (K^- \cap E_{\text{min}}) \}.
\]

To the extreme points coinciding with unstable knots we associate the following sets:

\[
\Delta_{j_t} = \text{co}\{\beta_{j_t}(X, t), \beta_{j_t+1}(X, t)\};
\]

\[
C_\Delta = - \sum_{\Delta_{j_t} \neq \Delta} \Delta_{j_t}.
\]

Define \( \mathcal{U} = \{ \Delta_{j_t} : t \in (K^+ \cap E_{\text{min}}) \cup (K^- \cap E_{\text{max}}) \} \). The quasidifferential of the function \( \Psi \) is

\[
\overline{\partial} \Psi = \sum_{\Delta_{j_t}} \Delta_{j_t} \text{ and } \partial \Psi = \text{co}\{\mathcal{S} - \overline{\partial} \Psi, \mathcal{U} = \mathcal{U} \cup \mathcal{U} \cup (0 + C_\Delta)\}.
\]

5 Confined quasidifferential

In this section we show that the inclusion (2.8) is verified when it is verified on a subinterval. We start by introducing the confined quasidifferential. Let \( p \) and \( q \) be given indices. Consider the following notation. We define \( \mathcal{S}_p = \{ \beta_{j_t}(X, t) \in \mathcal{S} : t \in [\xi_{p}, \xi_{q}] \} \) and \( \mathcal{U}_p = \{ \Delta_{v} \in \mathcal{U} : \xi_{v} \in (\xi_{p}, \xi_{q}) \} \) the respective subsets of elements of \( \mathcal{S} \) and \( \mathcal{U} \) corresponding to the extreme points lying in the interval \([\xi_{p}, \xi_{q}]\). Similarly, \( C_{p, \Delta} = - \sum_{\Delta_{j_t} \in \mathcal{U}_p, \Delta_{j_t} \neq \Delta} \Delta_{j_t} \).
Definition 5.1. The quasidifferential \( \overline{\partial}_p^q \psi \) confined to the interval \([\xi_p, \xi_q]\) is defined by
\[
\overline{\partial}_p^q \psi = \sum_{\Delta \in \mathcal{U}_p^q} \Delta, \quad (5.1a)
\]
\[
\partial_p^q \psi = \text{co}\left\{ \mathcal{Q}_p^q - \overline{\partial}_p^q \psi, \bigcup_{\Delta \in \mathcal{U}_p^q} C_p^q \Delta \right\}. \quad (5.1b)
\]

5.1 Stationary subintervals

Definition 5.2. An interval \([\xi_p, \xi_q]\) is stationary if
\[
-\overline{\partial}_p^q \psi(s) \subset \partial_p^q \psi(s). \quad (5.2)
\]

Denote by \( \prod_{i=1}^q A_i \) the collection of sets composed of one element per set \( A_i \):
\[
\{ \delta_1 \ldots \delta_q \} \in \prod_{i=1}^q A_i \iff (\delta_1, \ldots, \delta_q) \in \prod_{i=1}^q A_i
\]
and \( \prod \) represents the Cartesian product.

Proposition 5.1. The interval \([\xi_p, \xi_q]\) is stationary if and only if
\[
0 \in \text{co}\{ \mathcal{Q}_p^q, \mathcal{C} \}, \forall \mathcal{C} \in \prod_{\Delta \in \mathcal{U}_p^q} \Delta, \quad (5.3)
\]

Before proceeding to prove this proposition we recall the following result:

Lemma 5.1. Let \( A \) and \( B \) be convex sets and \( C \) a compact set. If \( B + C \subset A + C \), then \( B \subset A \).

Proof. It suffices to show that no hyperplane can separate the set \( A \) from any subset of \( B \). Consider a nonzero vector \( u \in \mathbb{R}^n \), \( u \neq 0 \) and a scalar \( \alpha \in \mathbb{R} \) such that \( \langle u, a \rangle > \alpha \), for all \( a \in A \) and take any point \( b \in B \). Define \( c_0 \in C \) such that \( \langle u, c_0 \rangle = \min \{ \langle u, c \rangle : c \in C \} \) and let \( d = b + c_0 \). Since \( d \in B + C \subset A + C \), we can find \((a, c) \in A \times C \) such that \( d = a + c \), and so
\[
\langle u, b \rangle = \langle u, d - c_0 \rangle + (\langle u, a \rangle + (\langle u, c - c_0 \rangle > \alpha). \quad \square
\]

Proof of Proposition 5.1. First assume that the interval \([\xi_p, \xi_q]\) is stationary and take
\[
\mathcal{C} = \bigcup_{\Delta \in \mathcal{U}_p^q} \{ \delta_{\Delta} \} \in \prod_{\Delta \in \mathcal{U}_p^q} \Delta, \text{ where } \delta_{\Delta} \text{ is a convex combination of } \beta_{j_1} \text{ and } \beta_{j_1+1}.
\]
Let \( b = -\overline{\partial}_p^q \psi \in \partial_p^q \Psi \). There exist \( d_\Delta \in C_p^q \Delta \) for each \( \Delta \in \mathcal{U}_p^q \), \( b_d \in \overline{\partial}_p^q \psi \) for each \( d \in \mathcal{Q}_p^q \), and associated \( a_d \geq 0, \alpha_\Delta \geq 0, \sum_{d \in \mathcal{Q}_p^q} a_d + \sum_{\Delta \in \mathcal{U}_p^q} \alpha_\Delta = 1 \) such that
\[
b = \sum_{d \in \mathcal{Q}_p^q} a_d (d - b_d) + \sum_{\Delta \in \mathcal{U}_p^q} \alpha_\Delta d_\Delta.
\]
\[
= \sum_{d \in \mathcal{Q}_p^q} a_d d + \sum_{\Delta \in \mathcal{U}_p^q} \alpha_\Delta \delta_{\Delta} - \sum_{\Delta \in \mathcal{U}_p^q} \alpha_\Delta (\delta_{\Delta} - d_\Delta) - \sum_{d \in \mathcal{Q}_p^q} a_d b_d.
\]
Since $\overline{\partial_p^q \Psi} = \Delta - C_p^q \Delta$ for any $\Delta \in \mathcal{U}_p^q$, and by the convexity of the confined subdifferential,

$$
\sum_{\Delta \in \mathcal{U}_p^q} \alpha_{\Delta} (\delta_{\Delta} - d_{\Delta}) + \sum_{d \in \mathcal{S}^q_p} \alpha_d b_d \in \overline{\partial_p^q \Psi}.
$$

Hence, since $b$ was arbitrary, $-\overline{\partial_p^q \Psi} \subset \text{co}\{\mathcal{S}^q_p, \mathcal{C} \} - \overline{\partial_p^q \Psi}$. Applying lemma 5.1 we conclude that $0 \in \text{co}\{\mathcal{S}^q_p, \mathcal{C} \}$.

Now assume formula (5.3) and let $b \in \overline{\partial_p^q \Psi}$. There exists $\delta_{\Delta} \in \Delta, \Delta \in \mathcal{U}_p^q$ such that $b = \sum_{\Delta \in \mathcal{U}_p^q} \delta_{\Delta}$. Then, by assumption there exist $d \in \text{co} \mathcal{S}^q_p$ and associated $\alpha_d, \alpha_{\Delta} (\Delta \in \mathcal{U}_p^q)$, such that $\alpha_d + \sum_{\Delta \in \mathcal{U}_p^q} \alpha_{\Delta} = 1$,

$$
-b = \alpha_d d + \sum_{\Delta \in \mathcal{U}_p^q} \alpha_{\Delta} \delta_{\Delta} - b = \alpha_d (d - b) + \sum_{\Delta \in \mathcal{U}_p^q} \alpha_{\Delta} (\delta_{\Delta} - b) \in \overline{\partial_p^q \Psi}.
$$

Therefore the interval $[\xi_p, \xi_q]$ is stationary.

**Corollary 5.1.** A spline $s[X]$ is an inf-stationary solution to the problem (2.1a) if and only if there exists a stationary subinterval.

**Proof.** By definition, a spline $s[X]$ is an inf-stationary solution if and only if the interval $[\xi_0, \xi_m]$ is stationary. Since for any $\mathcal{C} = \bigcup_{\Delta \in \mathcal{U}} \{\delta_{\Delta}\} \in \bigcup_{\Delta \in \mathcal{U}} \Delta$ and any $0 \leq p \leq q \leq m$

$$
\text{co}\{\mathcal{S}^q_p, \bigcup_{\Delta \in \mathcal{U}^q_p} \delta_{\Delta}\} \subset \text{co}\{\mathcal{S}, \mathcal{C}\},
$$

the proof immediately follows proposition 5.1.

### 6 Auxiliary linear transformation and necessary inclusion

The quasidifferential of the function (2.1b) is based on the gradients of the polynomials $P_i(X, t)$:

$$
\nabla_X P_i(X, t) =
\begin{pmatrix}
1 \\
\eta_1(t) \\
\mu_2(t) \\
\eta_2(t) \\
\vdots \\
\mu_i(t) \\
\eta_i(t) \\
0
\end{pmatrix},
$$

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where
\[
\eta_l(t) = (t - \xi_{l-1}, \ldots, t - \xi_{l-1}^m)^T, \ l = 1, \ldots, N, \quad (6.1)
\]
\[
\mu_1(t) = 1
\]
\[
\mu_l(t) = -\sum_{j=1}^{m} (m - j + 1)a_{l-1} j (t - \xi_{l-1})^{m-j}
\]
\[
= -a_{l-1} m - \sum_{j=1}^{m-1} (m - j + 1)a_{l-1} j (t - \xi_{l-1})^{m-j}, \ l = 2, \ldots, N
\]
\[
= -a_{l-1} m - \langle v_l^T, \eta_l(t) \rangle, \ l = 2, \ldots, N, \quad (6.3)
\]

where \( v_l = (2a_{l-1} m-1, 3a_{l-1} m-2, \ldots, m a_{l-1}, 0) \).

Since the coefficients of \( \eta_l(t) \) are powers of \( (t - \xi_i) \) and \( \mu_l(t) \) is a linear combination of these powers, it is possible to use binomial expansion to define a linear transform which sets most coefficients to 0 and only leaves nonzero the elements corresponding to the block interval to which \( t \) belongs. We provide details below.

Define the matrices:
\[
V_1 = I_{m+1}, \quad V_l = \begin{pmatrix} -1 & -v_l \\ 0 & I_m \end{pmatrix} \in \mathbb{R}^{(m+1)\times(m+1)}, \ l = 2, \ldots, N,
\]
\[
V = \begin{pmatrix} V_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & V_n \end{pmatrix} \in \mathbb{R}^{n(m+1)\times n(m+1)}.
\]

Then we find that
\[
V_l \begin{pmatrix} \mu_1(t) \\ \eta_1(t) \end{pmatrix} = \begin{pmatrix} a_{l-1} m \\ \eta_l(t) \end{pmatrix}
\]
and
\[
V \nabla_X P_l(X, t) = \begin{pmatrix} 1 \\ \eta_1(t) \\ a_{2m} \\ \eta_2(t) \\ \vdots \\ a_{l m} \\ \eta_l(t) \\ 0 \end{pmatrix}.
\]

Consider an arbitrary point \( t \). Let us first notice the following equality
\[
(t - \xi_p)^j = (t - \xi_q + \xi_q - \xi_p)^j = \sum_{k=0}^{j} \binom{j}{k}(\xi_q - \xi_p)^j-k(t - \xi_q)^k, \ j = 1, \ldots, m.
\]
Consider the following vectors:

\[
\begin{align*}
\mathbf{w}_p^0 &= \left(\frac{a_{pm}}{\alpha_{qm}}, 0, \cdots, 0\right), \\
\mathbf{w}_p^j &= \left(\frac{1}{\alpha_{qm}}(\xi_q - \xi_p)^j, \left(\frac{1}{j}\right)(\xi_q - \xi_p)^{j-1}, \cdots, \left(\frac{1}{j}\right)(\xi_q - \xi_p), 1, 0, \cdots, 0\right).
\end{align*}
\]

For each index \( p \) define the corresponding index \( q \) in the following way:

\[ q = k(p) = \min\{l > p : \alpha_{lm} \neq 0\}. \tag{6.4} \]

That is, for each \( \xi_p \) we find the index \( q \) of the next nonsmooth (non-neutral) knot.

Define the matrices \( W_{pr} \in \mathbb{R}^{(m+1) \times (m+1)} \) \( p = 1 \ldots N, \ r = p + 1 \ldots N \) as follows:

- the rows of \( W_{pr} \) are the row vectors \( w_{pr}^j \) \( (j = 0 \ldots m) \) if \( r = q = k(p) \)
- \( W_{pr} = 0 \) otherwise.

If \( p \) and \( q \) satisfy equation (6.4), then

\[
\begin{pmatrix}
\alpha_{pm} \\
\eta_p(t)
\end{pmatrix} = W_{pq} \begin{pmatrix}
\alpha_{qm} \\
\eta_q(t)
\end{pmatrix}.
\]

Define also

\[
W = \begin{pmatrix}
I_{m+1} & -W_{i2} & \cdots & -W_{im} \\
0 & \ddots & \ddots & \vdots \\
0 & \ddots & -W_{m-1,m} & I_{m+1} \\
0 & \cdots & 0 & I_{m+1}
\end{pmatrix} \in \mathbb{R}^{nm \times nm}.
\]

Note that for each index \( p \) there exists only one index \( q = k(p) \), such that the corresponding \( W_{pq} \) is nonzero.

Both \( V \) and \( W \) are triangular matrices with nonzero elements on the diagonal, and so they are both full rank. Furthermore, for any \( t \) we have:

\[
WV\nabla_X P_l(X, t) = \begin{pmatrix}
0 \\
\alpha_{pm} \\
\eta_p(t) \\
\vdots \\
\alpha_{qm} \\
\eta_q(t) \end{pmatrix},
\]

where \( q \) is defined as in (6.4) and therefore all the knots between \( \xi_p \) and \( \xi_q \) are neutral.

Let us introduce the following definitions.

**Definition 6.1.** Subintervals \([\xi_{i-1}, \xi_i], i = 1, \ldots, n\) are called **unit subintervals**. Intervals delimited by non-neutral external knots and whose internal knots are neutral are called **block subintervals**.
Consider now the linear transformation $M = W V$ and define the following sets for any block subinterval $[\xi_p, \xi_q]$

\[
\mathcal{A}_p^q \text{ is the block of nonzero coordinates of } \bigcup_{d \in \mathcal{P}_p^q} (Md) \quad (6.5)
\]

\[
\mathcal{B}_p^q \text{ is the block of nonzero coordinates of } \bigcup_{d \in \Delta} (Md) \quad (6.6)
\]

The following lemma holds.

Lemma 6.1. Formula (5.3) is equivalent to

\[
0 \in \text{co}\{\mathcal{A}_p^q, \mathcal{C}\}, \forall \mathcal{C} \in \bigsqcup_{\Delta \in \mathcal{R}_p^q} \mathcal{B}_p^q. \quad (6.7)
\]

Proof. The proof is straightforward: it suffices to apply the linear transformation on both sides of the equation. □

7 Identifying stationary subintervals

In this section we obtain the main results of this paper. By proposition 5.1, we only need to find a stationary interval $[\xi_p, \xi_q]$.

7.1 Optimality conditions through block subintervals

We start with the following lemma.

Lemma 7.1. Given two intervals $[\xi_l, \xi_p]$ and $[\xi_q, \xi_r]$, such that $p \leq q$ and $\xi_l, \xi_p, \xi_q, \xi_r$ are not neutral, at least one of these intervals is stationary if and only if

\[
0 \in \text{co}\{\mathcal{A}_l^p, \mathcal{C}_l^p \cup \mathcal{C}_r^q\}, \forall \mathcal{C}_l^p \in \bigsqcup_{\Delta \in \mathcal{R}_l^p} \mathcal{B}_p^q, \forall \mathcal{C}_q^r \in \bigsqcup_{\Delta \in \mathcal{R}_q^r} \mathcal{B}_p^q. \quad (7.1)
\]

Proof. First, since both $\text{co}\{\mathcal{A}_l^p, \mathcal{C}_l^p \}$ and $\text{co}\{\mathcal{A}_q^r, \mathcal{C}_q^r \}$ are subsets of $\text{co}\{\mathcal{A}_l^p, \mathcal{A}_q^r, \mathcal{C}_l^p \cup \mathcal{C}_q^r\}$, the stationarity of either $[\xi_l, \xi_p]$ or $[\xi_q, \xi_r]$ implies that inclusion.

Now, assume that equation (7.1) is true and that the interval $[\xi_q, \xi_r]$ is not stationary. Then, we can find a set $\mathcal{C}_q^r \in \bigsqcup_{\Delta \in \mathcal{R}_q^r} \mathcal{B}_q^r$ such that

\[
0 \notin \text{co}\{\mathcal{A}_l^p, \mathcal{C}_q^r\}. \quad (7.2)
\]

Applying the linear transformation to the above formula, we obtain that for any $\mathcal{C}_l^p \in \bigsqcup_{\Delta \in \mathcal{R}_l^p} \mathcal{B}_l^p$, there exists $\lambda \in [0, 1]$ such that

\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \lambda \text{co}\left\{\begin{pmatrix} \mathcal{A}_l^p \\ \mathcal{C}_l^p \end{pmatrix}, \bigcup_{\delta \in \mathcal{C}_l^p} \begin{pmatrix} 0 \\ \delta \end{pmatrix}\right\} + (1 - \lambda) \text{co}\left\{\begin{pmatrix} 0 \\ \mathcal{A}_l^p \end{pmatrix}, \bigcup_{\delta \in \mathcal{C}_l^p} \begin{pmatrix} 0 \\ \delta \end{pmatrix}\right\}.
\]

Formula (7.2) ensures that $\lambda \neq 0$, which implies that $0 \in \text{co}\{\mathcal{A}_l^p, \mathcal{C}_l^p\}$ and so the interval $[\xi_l, \xi_p]$ is stationary. □
We can now proceed with the proof of theorem 3.1. The application of lemma 7.1 is straightforward.

**Proof.** If the knot $\xi_r$ is stable or not an extreme point, then $\mathcal{V}_p^q = \mathcal{V}_p^q \cup \mathcal{V}_r^q$. The system (7.4) is equivalent to solving a linear system involving a block triangular matrix where each block is a Vandermonde matrix. It was studied in [14] (see also [12] where a similar system was considered). There it is proved that the interval $[\xi_{i_j}, \xi_{i_{j+1}}]$ contains a sequence of

**Corollary 7.1.** If a stationary interval $[\xi_p, \xi_q]$ contains a stable internal knot or an internal knot which is not an extreme point $\xi_r \in (\xi_p, \xi_q)$, then either $[\xi_p, \xi_r]$ or $[\xi_r, \xi_q]$ is stationary.

**Proof.** If the knot $\xi_r$ is stable or not an extreme point, then $\mathcal{V}_p^q = \mathcal{V}_p^q \cup \mathcal{V}_r^q$ and $\mathcal{V}_r^q = \mathcal{V}_p^q \cup \mathcal{V}_r^q$. Noticing that the coefficients of all the vectors in the above set not corresponding to interval $[\xi_p, \xi_q]$ are 0, one can see that

- 0 from (7.3) can be formed without $\xi_p$ and $\xi_q$ if they are unstable;

- for every block subinterval $[\xi_{i_j}, \xi_{i_{j+1}}]$, containing $k_i$ unit subintervals (ignoring rows with all zeros):

\[
\begin{align*}
0_{m_{k_i+1}} & \in \alpha_{i_j} \cap \sigma(t) \left\{ \begin{array}{c}
\sigma(\xi_{i_j})(1 - \lambda_{i_j}) \\
\eta_{i_j+1}(\xi_{i_j}) \\
\vdots \\
0 \\
0 \\
n_j(t) \\
\vdots \\
\eta_{i_j+k_i}(t) \\
\end{array} \right\}, \\
0_{m_{k_i+1}} & \in \alpha(\xi_{i_j+k_i}) \left\{ \begin{array}{c}
1 \\
\eta_{i_j+1}(\xi_{i_{j+1}}) \\
\vdots \\
0 \\
0 \\
n_{i_{j+1}}(\xi_{i_{j+1}}) \\
\vdots \\
\eta_{i_{j+k_i}}(\xi_{i_{j+k_i}}) \\
\end{array} \right\},
\end{align*}
\]

where $\sigma(t) = \text{sign}(s(t) - f(t))$, that is the sign of the deviation function of $f(t)$ from spline $s(t)$, $\lambda_{i_j} = 1$, $\lambda_{i_{j+1}} = 0$.

The system (7.4) is equivalent to solving a linear system involving a block triangular matrix where each block is a Vandermonde matrix. It was studied in [14] (see also [12] where a similar system was considered). There it is proved that the interval $[\xi_{i_j}, \xi_{i_{j+1}}]$ contains a sequence of

7.2 Characterization using alternating extreme points

We can now proceed with the proof of theorem 3.1.

**Proof of theorem 3.1.** We know from proposition 5.1 that a spline is inf-stationary if and only if there exists a stationary subinterval. Let $[\xi_p, \xi_q]$ be such a subinterval not containing any strict stationary subintervals. Equivalently, by proposition 5.1 and applying the linear transformation described in section 6, for any $\Delta \in \mathcal{C}_{\lambda}$, for any $\Delta \in \mathcal{V}_p^q \cup \Delta$

\[
0_{N(m+1)} \in \text{co} \{ M_d : d \in \mathcal{V}_p^q \cup \Delta \}. 
\]

Let $\xi_{i_1}, \ldots, \xi_{i_{k-1}}$ be the nonsmooth knots located in $(\xi_p, \xi_q)$, that is $\xi_{i_1}, \ldots, \xi_{i_{k-1}}$ separate the corresponding block subintervals. Noticing that the coefficients of all the vectors in the above set not corresponding to interval $[\xi_p, \xi_q]$ are 0, one can see that

- 0 from (7.3) can be formed without $\xi_p$ and $\xi_q$ if they are unstable;

- for every block subinterval $[\xi_{i_j}, \xi_{i_{j+1}}]$, containing $k_i$ unit subintervals (ignoring rows with all zeros):

\[
\begin{align*}
0_{m_{k_i+1}} & \in \alpha_{i_j} \cap \sigma(t) \left\{ \begin{array}{c}
\sigma(\xi_{i_j})(1 - \lambda_{i_j}) \\
\eta_{i_j+1}(\xi_{i_j}) \\
\vdots \\
0 \\
0 \\
n_j(t) \\
\vdots \\
\eta_{i_j+k_i}(t) \\
\end{array} \right\}, \\
0_{m_{k_i+1}} & \in \alpha(\xi_{i_j+k_i}) \left\{ \begin{array}{c}
1 \\
\eta_{i_j+1}(\xi_{i_{j+1}}) \\
\vdots \\
0 \\
0 \\
n_{i_{j+1}}(\xi_{i_{j+1}}) \\
\vdots \\
\eta_{i_{j+k_i}}(\xi_{i_{j+k_i}}) \\
\end{array} \right\},
\end{align*}
\]

where $\sigma(t) = \text{sign}(s(t) - f(t))$, that is the sign of the deviation function of $f(t)$ from spline $s(t)$, $\lambda_{i_j} = 1$, $\lambda_{i_{j+1}} = 0$.
\( \tau_i \) unit subintervals \((\tau_i \leq k_i)\) with at least \( m \tau_i + 2 \) alternating extreme points and none of these points coincides with any of the internal knots of the block of \( \tau_i \) unit subintervals.

Consider the system of all the alternating extreme points, taking the same point twice if this point appears in both adjacent block subintervals. The total number of such points is

\[
2j_k + m \sum_{i=1}^{j_k} \tau_i.
\]

If each unstable internal knot of \([\xi_p, \xi_q]\) is represented twice in this sequence, then (since \( j_k = l + 1 \)) the total number of distinct alternating extreme points is exactly \( m(q - p) + 2 + l \), where \( l \) is the number of non-neutral (by corollary 7.1 these knots have to be unstable) internal knots inside of \([\xi_p, \xi_q]\). In this case all the conditions of our main theorem are satisfied.

If there exists an unstable internal knot \( \xi_j \) of \([\xi_p, \xi_q]\), which is not included twice, that is, not included to the subsequence of alternating extreme points for one or both adjacent block subintervals, then \( 0 \) from (7.3) can be formed on a shorter interval. Namely, if the block subinterval \([\xi_j - 1, \xi_j]\) does not include \( \xi_j \), then \( 0 \) from (7.3) can be formed by \([\xi_p, \xi_j]\). This proves the theorem.

8 Discussion and conclusive remarks

In this paper we obtained a characterization theorem for the approximation of continuous functions by continuous polynomial splines with free knots. This result is equivalent to the Demyanov-Rubinov stationarity.

Although our result does not address exactly the same problem as existing characterisation theorem for polynomial splines with free knots, several observations can be made:

1. In [9], the number of alternating extreme points depends on the multiplicity of the knots. In particular, knots of multiplicity higher than 2 decrease the number of extreme points. This may result in suboptimal solutions, as illustrated in example 2.1. In contrast, the results provided in this paper demonstrate that the characterization of continuous splines only depends on whether the spline is differentiable at the knot or not (that is its multiplicity is 0 or positive). This eliminates many splines, such as the one shown in that example.

2. The present characterization theorem also requires the first and last extreme points of the sequence to not be unstable. This improvement can have an interesting use in an algorithm for constructing a best polynomial spline approximation. Indeed, as there are a variety of efficient algorithms for best approximation by polynomial splines with fixed knots, it is natural to develop algorithms alternating between finding the polynomial coefficients and the knots (such as [5]). Our results offer an interesting strategy to improve on local minimisers by moving knots to coincide with the first and last extreme points of the stationary interval. This approach will be studied in details in our future work.
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