Lossless Convexification of Optimal Control Problems with Semi-continuous Inputs

Danylo Malyuta* Behçet Açıkmese*

* Dept. of Aeronautics & Astronautics, University of Washington, Seattle, WA 98195 USA (e-mail: {danylo, behcet}@uw.edu)

Abstract: This paper presents a novel convex optimization-based method for finding the globally optimal solutions of a class of mixed-integer non-convex optimal control problems. We consider problems with non-convex constraints that restrict the input norms to be either zero or lower- and upper-bounded. The non-convex problem is relaxed to a convex one whose optimal solution is proved to be optimal almost everywhere for the original problem, a procedure known as lossless convexification. This paper is the first to allow individual input sets to overlap and to have different norm bounds, integral input and state costs, and convex state constraints that can be activated at discrete time instances. The solution relies on second-order cone programming and demonstrates that a meaningful class of optimal control problems with binary variables can be solved reliably and in polynomial time. A rocket landing example with a coupled thrust-gimbal constraint corroborates the effectiveness of the approach.

Keywords: Optimal control, convex optimization, maximum principle, integer programming.

1 Introduction

We present a convex programming solution to a class of optimal control problems with semi-continuous control input norms. Semi-continuous variables are a particular type of binary non-convexity.

Definition 1. Variable $x \in \mathbb{R}$ is semi-continuous if $x \in \{0\} \cup [a, b]$ with $0 < a \leq b$ MOSEK ApS (2019).

The constraint $az \leq x \leq bz$ with $z \in \{0, 1\}$ models semi-continuity. Practical rocket landing and spacecraft rendezvous path planning problems include such constraints, and can take hours to solve using existing mixed-integer convex programming (MICP) methods. In this paper, we propose an algorithm based on lossless convexification that solves these problems to global optimality in seconds.

Lossless convexification is a method for finding the globally optimal solution of non-convex problems using convex optimization. The method relaxes the original problem to a convex one via a slack variable, enabling the use of second-order cone programming (SOCP). The maximum principle is used to prove that the solution of the relaxed problem is globally optimal for the original problem.

Classical lossless convexification deals with non-convexity in the form of an input norm lower-bound. The first result was introduced in Açıkmese and Ploen (2007) for minimum-fuel rocket landing and was later expanded to more general non-convex input sets Açıkmese and Blackmore (2011). Extensions of the method were introduced in Blackmore et al. (2010); Carson III et al. (2011); Açıkmese et al. (2013) to handle minimum-error rocket landing and non-convex pointing constraints. More recently, lossless convexification was shown to handle affine and quadratic state constraints Harris and Açıkmese (2013a,b), culminating in Harris and Açıkmese (2014).

A recurring assumption of classical lossless convexification is that there is a single input which cannot be turned off. Our interest is in problems that have multiple inputs which may be turned off. When active, the input norm is lower-bounded, making it semi-continuous in the sense of Definition 1. This is a richer binary non-convexity than what was handled by classical lossless convexification.

The concept of lossless convexification with binary variables implemented via MICP was explored in Blackmore et al. (2012); Zhang et al. (2017). However, the $\text{NP}$-hard nature of MICP generally makes the approach computationally expensive. Recently, a limited class of binary non-convexity was handled via lossless convexification in Malyuta et al. (2019), proving that a class of $\text{NP}$-hard problems is of $\mathcal{P}$ complexity. The approach is amenable to real-time onboard optimization for autonomous systems and for rapid design trade studies.

Our main contribution is to extend the lossless convexification result of Malyuta et al. (2019). The list of extensions that we introduce is as follows. We allow an input integral cost, a state integral cost, different norm lower- and upper-bounds for each input, overlapping pointing directions of the inputs, and state constraints.

The paper is organized as follows. Section 2 defines the class of optimal control problems that our method handles. Section 3 proposes our solution method based on lossless convexification. Section 5 proves that our method finds the globally optimal solution based on the necessary conditions of optimality presented in Section 4. Section 6 presents a rocket landing example which corroborates the method’s effectiveness for practical path planning applications. Section 7 outlines future work and Section 8 summarizes the result.

Notation: sets are calligraphic, e.g. $\mathcal{S}$. Set $\mathbb{R}^n_+$ denotes the $n$-dimensional non-positive orthant. The operator $\circ$
denotes the element-wise product. Given a function \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p \), we use the shorthand \( f[t] \equiv f(x(t), y(t)) \). In text, functions are referred to by their letter (e.g. \( f \)) and conflicts with another variable are to be understood from context. The gradient of \( f \) with respect an argument \( x \) is denoted \( \nabla_x f \in \mathbb{R}^{p \times n} \). Similarly, if \( f \) is nonsmooth then its subdifferential with respect to \( x \) is \( \partial_x f \subseteq \mathbb{R}^{p \times n} \). The normal cone at \( x \) to \( S \subseteq \mathbb{R}^n \) is denoted \( \mathcal{N}_S(x) \subseteq \mathbb{R}^n \). When we refer to an interval, we mean some time interval \([t_1, t_2]\) of non-zero duration, i.e. \( t_1 < t_2 \). We call the Euclidean projection of \( y \in \mathbb{R}^n \) onto \( S \subseteq \mathbb{R}^n \) the magnitude of the 2-norm projection of \( y \):

\[
P_S(y) = \arg\min_{z \in S} \| y - z \|_2.
\]

2 Problem Definition

This section presents the class of optimal control problems that can be solved via convex optimization by our methods. We consider mixed-integer non-convex optimal control problems with linear time-invariant (LTI) dynamics and semi-continuous input norms:

**Problem \( O \).**

\[
\begin{align*}
\min_{u_i, \gamma_i, t_f} \quad & m(t_f, x(t_f)) + \int_0^{t_f} \ell(x(t)) \, dt \\
\text{s.t.} \quad & (O.a) \quad \dot{x}(t) = Ax(t) + Bu_i(t) + w, \quad x(0) = x_0, \\
& (O.b) \quad \gamma_i(t) \rho_1^i \leq |u_i(t)|_2 \leq \gamma_i(t) \rho_2^i \quad i = 1, \ldots, M, \\
& (O.c) \quad \gamma_i(t) \in (0, 1], \quad i = 1, \ldots, M, \\
& (O.d) \quad \sum_{i=1}^M \gamma_i(t) \leq K, \\
& (O.e) \quad C_i u_i(t) \leq 0 \quad i = 1, \ldots, M, \\
& (O.f) \quad x(t) \in \mathcal{X}, \\
& (O.g) \quad b(x(t_f)) = 0,
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the state, \( u_i(t) \in \mathbb{R}^m \) is the \( i \)-th input, and \( w \in \mathbb{R}^n \) is a known external input. Convex functions \( m : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}, \ell : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( b : \mathbb{R}^n \rightarrow \mathbb{R}^m \) define the terminal cost, the state running cost and the terminal manifold respectively. The binary coefficient \( \zeta \in \{0, 1\} \) toggles the input running cost. The state must lie in the convex set \( \mathcal{X} \subseteq \mathbb{R}^n \). The input directions are constrained to polytopic cones called input pointing sets:

\[
\mathcal{U}_i = \{ u \in \mathbb{R}^m : C_i u \leq 0 \},
\]

where \( C_i \in \mathbb{R}^{p \times m} \) is a matrix with \( C_{i,j} \) the \( j \)-th row.

**Assumption 1.** Matrices \( C_i \) in \((O.a)\) are full row rank.

**Assumption 2.** The control norm bounds in \((O.c)\) are distinct, i.e. \( \rho_1^i < \rho_2^i \).

Problem \( O \) extends the problem class in Malyuta et al. (2019) in several non-trivial ways. First, there are input and state integral costs in \((O.a)\). Second, the input norm bounds in \((O.c)\) can be different for each input. Third, the state can be constrained to a convex set in \((O.g)\). Last and most important, (Malyuta et al., 2019, Assumption 1) is removed, such that the input pointing sets can overlap arbitrarily. Figure 1 shows how this enables richer input set geometry than permitted in Malyuta et al. (2019).

3 Lossless Convexification

This section presents the two main results, Theorems 1a and 1b, which state that the convex Problem \( R \) finds the global optimum of Problem \( O \) under certain conditions.
Fig. 2. Problem \( R \) convexifies the input set of Problem \( O \), here shown for \( M = 2, K = 1 \) and \( m = 2 \). The relaxation consists of three steps: a) \((O,c)-(O,f)\) originally define a non-convex set of a binary nature; b) by relaxing \((O,c)\) to \((R,d)-(R,e)\), individual input sets are convexified; c) by relaxing \((O,d)\) to \((R,f)\), a convex hull is obtained.

(b) on any interval where \( \Gamma_i(t) = 0 \), there exist \( K \) inputs with \( \Gamma_i(t) > 0 \) or \( M - K \) inputs where \( \Gamma_i(t) < 0 \).

**Condition 4.** \( \ell[t] + \sum_{i=1}^{M} \sigma_i(t) + \nabla \alpha m[t] = 0 \forall t \in [0, t_f]. \)

We now state the two main results of this paper, which claim that Problem \( R \) solves Problem \( O \) under certain conditions. The theorems are proved in Section 5.

**Theorem 1a.** The solution of Problem \( R \) is globally optimal a.e. \([0, t_f]\) for Problem \( O \) if Conditions 1-4 hold and the state constraint \((O.g)\) is never activated.

**Theorem 1b.** The solution of Problem \( R \) is globally optimal a.e. \([0, t_f]\) for Problem \( O \) if Conditions 1-4 hold and the state constraint \((O.g)\) is activated at discrete times.

### 3.1 Discussion on Strong Observability

This section describes Condition 1 and its verification. Strong observability extends the concept of observability to the case of non-zero inputs. A strongly observable system does not have transmission zeroes. To be precise, let us state strong observability in the context of (3).

**Definition 2 (Trentelman et al., 2001, Definition 7.8).** A point \( \lambda_0 \in \mathbb{R}^n \) is weakly unobservable if there exists an interval \( \mathcal{T} = [\tau_1, \tau_2] \) and an input trajectory \( v(t) \in \partial \mathcal{E}[t] \) for \( t \in \mathcal{T} \) such that if \( \lambda(\tau_1) = \lambda_0 \) then the primal vector satisfies \( y(t) = 0 \forall t \in \mathcal{T} \). The set of all weakly unobservable points is denoted \( W \), which is called the weakly unobservable set.

**Theorem 2 (Trentelman et al., 2001, Theorem 7.16).** The adjoint system (3) is strongly observable if \( W = \{0\} \).

To verify Condition 1 via simple matrix algebra, it is sufficient to apply the algorithm for computing \( \mathcal{V} \) in (Trentelman et al., 2001, Section 7.3) using the following alternative to (3a):

\[
\dot{\lambda}(t) = -A^T \lambda(t) + Dv(t),
\]

where \( v(t) \in \mathbb{R}^n \) and range \( D = \text{span} \bigcup_{t \in [\tau_1, \tau_2]} \partial \mathcal{E}(x(t)) \). This conservative approximation assumes that the input can come from a subspace spanned by the subdifferentials. Section 6 uses this approximation to verify Conditions 1-3 for the rocket landing problem.

### 4 Nonsmooth Maximum Principle

This section states a nonsmooth version of the maximum principle that we shall use for proving Theorems 1a and 1b. Consider the following general optimal control problem:

**Problem \( G \).**

\[
\begin{align*}
\min_{u, t_f} & \quad m(t_f, x(t_f)) + \int_0^{t_f} \ell(t, u(t), x(t)) dt \\
\text{s.t.} & \quad x(t) = f(t, x(t), u(t)), \quad x(0) = x_0, \\
& \quad g(t, u(t)) \leq 0,
\end{align*}
\]

where the state trajectory \( x(\cdot) \) is absolutely continuous and the control trajectory \( u(\cdot) \) is measurable. The dynamics \( f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) are convex and continuously differentiable. The terminal cost \( m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \), the running cost \( \ell : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \), the input constraint \( g : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n \), and the terminal constraint \( b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m \) are convex. Define the terminal manifold as \( \mathcal{T} = \{ x \in \mathbb{R}^n : (G.d) \text{ holds} \} \) and the Hamiltonian function:

\[
H(t, x(t), u(t), \alpha, \psi(t)) = \alpha \ell(t) + \psi(t)^T f(t), \tag{6}
\]

where \( \alpha \leq 0 \) is the abnormal multiplier and \( \psi(\cdot) \) is the adjoint variable trajectory. We now state the nonsmooth maximum principle, due to (Vinter, 2000, Theorem 8.7.1) (see also Clarke (2010); Hartl et al. (1995)), which specifies the necessary conditions of optimality for Problem \( G \).

**Theorem 3 (Maximum Principle).** Let \( x(\cdot) \) and \( u(\cdot) \) be optimal on the interval \([0, t_f]\). There exist a constant \( \alpha \leq 0 \) and an absolutely continuous \( \psi(\cdot) \) such that the following conditions are satisfied:

1. **Non-triviality:**

\[
(\alpha, \psi(t)) \neq 0 \forall t \in [0, t_f]; \tag{7}
\]

2. **Pointwise maximum:**

\[
u(t) = \arg\max_{\psi \in (G.c)} H(t, x(t), v, \alpha, \psi(t)) \text{ a.e. } [0, t_f]; \tag{8}
\]

3. **The differential equations and inclusions:**

\[
\dot{x}(t) = \nabla \psi H[t]^T \text{ a.e. } [0, t_f], \tag{9a}
\]

\[
\dot{\psi}(t) \in -\partial \alpha H[t]^T \text{ a.e. } [0, t_f], \tag{9b}
\]

\[
H[t] \in \partial \alpha H[t] \text{ a.e. } [0, t_f]; \tag{9c}
\]
5 Lossless Convexification Proof

This section proves Theorems 1a and 1b. The general outline is as follows. We first prove Theorem 1a by showing that (step 1) the solution of Problem $\mathcal{R}$ is feasible for Problem $\mathcal{O}$, and (step 2) the solution is globally optimal. We then show Theorem 1b via a proof by contradiction in which Theorem 1a is applied on each interval where the state constraint is inactive.

**Lemma 1.** The solution of Problem $\mathcal{R}$ is feasible a.e. $[0, t_f]$ for Problem $\mathcal{O}$ if $x(t) \in \text{int}(X)$ and Conditions 1-4 hold.

**Proof.** The proof uses the maximum principle from Theorem 3. Since there are two states, partition the adjoint variable as $\psi(t) = (\lambda(t) \in \mathbb{R}^3, \eta(t) \in \mathbb{R})$. For Problem $\mathcal{R}$ and $x(t) \in \text{int}(X)$, the adjoint and Hamiltonian dynamics follow from (9b) and (9c):

\[
\dot{\lambda}(t) = -A^\lambda \lambda(t) - \alpha(t), \quad \dot{\eta}(t) = 0 \quad \text{a.e.} \ [0, t_f],
\]

\[
\dot{H}[t] = 0 \quad \text{a.e.} \ [0, t_f],
\]

for some $\beta \in \mathbb{R}^{n_u}$. Due to (11b)-(11c), (12b)-(12c) and absolute continuity, we can set $c = 0$ and $d = -\alpha$. For this reason, $\lambda_1 = 0$ and $\lambda_2 \geq 0$ are necessary for optimality. Thus, (19) simplifies to:

\[
\|y - C_1^T \lambda_3^f\|_2^2 = \zeta + \lambda_3.
\]
is sufficient to ensure that it is optimal to set
\[ \gamma_i = \begin{cases} 1 & \text{if } i \leq K''', \\ 0 & \text{otherwise}. \end{cases} \] (25)

The lemma holds if (24) holds a.e. on \([0, t_f]\). This is assured by Conditions 2 and 3. Condition 2 case (a) assures \(\Gamma_{r''} > 0\) a.e. on \([0, t_f]\). If on some interval \(\Gamma = 0\), Condition 2 case (b) assures that \(k > K''\). If \(K'' < K\) then due to \(\Gamma_{r''} > 0\) and the definition of \(K''\), it must be that \(\Gamma_{r''+1} = 0 \Rightarrow \Gamma_{r''} > \Gamma_{r''+1}\). Else if \(K'' = K\), Condition 3 case (a) assures \(\Gamma_K > \Gamma_{K+1}\) a.e. on \([0, t_f]\). If on some interval \(\Gamma_k = \Gamma_{k+1}\), Condition 3 case (b) assures that \(k \neq K\).

Thus, (24) holds a.e. on \([0, t_f]\) and the lemma is proved. From (25), the structure of the optimal solution is bang-bang with at most \(K\) inputs active a.e. on \([0, t_f]\).

Lemma 1 guarantees that Problem \(\mathcal{R}\) produces a feasible solution of Problem \(\mathcal{O}\). We will now show that this solution is globally optimal, thus proving Theorem 1a.

**Proof of Theorem 1a.** The solution of Problem \(\mathcal{R}\) is feasible a.e. on \([0, t_f]\) for Problem \(\mathcal{O}\) due to Lemma 1. Furthermore, if \(c = 0\) then the costs functions of Problems \(\mathcal{O}\) and \(\mathcal{R}\) are the same. This is also true when \(\zeta = 1\) because Lemma 1 guarantees that \(\|u_i(t)\| = \sigma_i(t)\). The optimal costs thus satisfy \(J_{\mathcal{O}}^* \leq J_{\mathcal{R}}^*\). However, any solution of Problem \(\mathcal{O}\) is feasible for Problem \(\mathcal{R}\) by setting \(\sigma_i(t) = \|u_i(t)\|_2\), thus \(J_{\mathcal{R}} \leq J_{\mathcal{O}}^*\). Therefore \(J_{\mathcal{R}} = J_{\mathcal{O}}^*\), so the Problem \(\mathcal{R}\) solution is globally optimal for Problem \(\mathcal{O}\) a.e. on \([0, t_f]\).

Theorem 1a implies that Problem \(\mathcal{O}\) is solved in polynomial time by an SOCP solver applied to Problem \(\mathcal{R}\). This can be done efficiently with several numerically reliable SOCP solvers Dueri et al. (2014). Therefore the class of \(NP\)-hard problems defined by Problem \(\mathcal{O}\) is in fact of \(P\) complexity if \(x(t) \in \text{int}(\mathcal{X})\) and Conditions 1-4 hold.

### 5.1 The Case of Active State Constraints

So far it has been assumed that the state constraint \((\mathcal{O}_g)\) is inactive. This section guarantees lossless convexification in a limited setting when \((\mathcal{O}_g)\) is activated at a discrete set of times. To begin, define the *interior time* and *contact time* sets as follows:

\[
\mathcal{T}_c \triangleq \{ t \in (0, t_f) : x(t) \in \text{int}(\mathcal{X}) \}, \quad \mathcal{T}_e \triangleq [0, t_f] \setminus \mathcal{T}_c. \quad (26a)
\]

A point \(t \in \mathcal{T}_e\) is called an isolated point if there exists a neighborhood of \(t\) not containing other points of \(\mathcal{T}_c\). Stein and Shakarchi (2005). A set of isolated points is called a discrete set and any discrete subset of an Euclidian space has measure zero Aćuzkmeşe and Blackmore (2011). We can now prove Theorem 1b.

**Proof of Theorem 1b.** The proof is similar to (Aćuzkmeşe and Blackmore, 2011, Corollary 3). To begin, let \(\Sigma_O = \{t_f, x_f, \bar{x}_f, u_{i_f}^*, \bar{y}_f, \bar{\sigma}_f^*\}\) be the original solution returned by Problem \(\mathcal{R}\), which achieves the optimal cost value \(J_{\mathcal{R}}^*\). Since \(\mathcal{T}_c\) is a discrete set, for any consecutive contact times \(\tau_i \leq \bar{\tau}_2\) there exists a large enough real \(a > 0\) such that \(\tau_1 + 1/a < \bar{\tau}_2 - 1/a\) and \(\bar{\tau}_f = \tau_1 + 1/a\) and \(\bar{\tau}_f = \bar{\tau}_2 - 1/a\). Now consider solving Problem \(\mathcal{R}\) over \([\tau_c, \tau_c + \Delta \tau]\) with \(t_f = \Delta \tau, \bar{x}_0 = x(\tau_c), b[t_f] = x(\Delta \tau) - x(\tau_f)\). Call the solution to this problem the *subproblem solution* \(\Sigma_S = \{\Delta \tau, \bar{x}_f, \bar{y}_f, \bar{\sigma}_f^*\}\), and let \(J_S^*\) be the achieved optimal cost. We claim that the corresponding portion of \(\Sigma_O\) must also achieve \(J_S^*\). If it does not, the modified solution \(\Sigma_M = \{t_f, \bar{x}_f, \bar{y}_f, \bar{\sigma}_f^*\}\) such that \(\bar{t}_f = t_f^* + \Delta \tau - (\tau_f - \tau_c)\) and \(\bar{\eta}_f, \bar{\bar{y}}_f, \bar{\sigma}_f^*\) is also feasible for Problem \(\mathcal{R}\) and achieves a lower cost than \(J_{\mathcal{R}}^*\), which contradicts that the \([\tau_c^*, \tau_f^*]\) segment of \(\Sigma_O\) is optimal. Thus, \(\Sigma_S^*\) must be optimal for the original problem. By Theorem 1a, \(\Sigma_S^*\) must be globally optimal for Problem \(\mathcal{O}\). Since \(a\) is arbitrarily large, \(\Sigma_S^*\) must be optimal for Problem \(\mathcal{O}\) over \(t \in (t_1, t_2)\). Let \(\mathcal{T}_c^* = \{\tau_i, i = 1, 2, \ldots\}\), \(\tau_i < \tau_{i+1}\) and let \(\Sigma_O^* \triangleq \text{int}(\mathcal{T}_c^*)\) and \(\Sigma_O^*\) is globally optimal for Problem \(\mathcal{O}\) a.e. on \(\mathcal{T}_c^*\). Since \(\mathcal{T}_c^*\) is a discrete set, \(c(\mathcal{T}_c^*) \in [0, t_f]\) and so the Problem \(\mathcal{R}\) solution is globally optimal for Problem \(\mathcal{O}\) a.e. on \([0, t_f]\).

### 6 Numerical Example

This section shows how rocket landing trajectories can be generated much faster via Problem \(\mathcal{R}\) than MIPC. Python source code for this example is available online. Consider the in-plane rocket dynamics:

\[
x(t) = A(x) + B \sum_{i=1}^{M} u_i(t) + w, \quad (27)
\]

where the vehicle is treated as a point mass with \(x(t) = (r(t), \dot{v}_c(t)) \in \mathbb{R}^4\) the position and velocity state and \(\omega \in \mathbb{R}\) the planet rotation rate, which is assumed to be constant and perpendicular to the trajectory plane. The input \(u_i(t) \in \mathbb{R}^2\) represents an acceleration imparted on the rocket by a gimbaled thruster. The LTI matrices are:

\[
A(x) = \begin{bmatrix} 0 & I \\ \omega^2 I & 2 \omega S \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 \\ \omega^2 I + g \end{bmatrix}, \quad (28)
\]

where \(S = [0 \ldots 1] \in \mathbb{R}^{2 \times 2}, I \in \mathbb{R}^{2 \times 2}\) is identity, \(l \in \mathbb{R}^2\) is the landing pad position with respect to the planet’s center of rotation, and \(g \in \mathbb{R}^2\) is the gravity vector. Note that the dynamics assumes constant mass and gravity for concision, but both can be made variable within the lossless convexification framework Aćuzkmeşe and Ploen (2007); Blackmore et al. (2012).

The rocket is equipped with a single gimbaled thruster which operates in two modes: 1) low-thrust high-gimbal, and 2) high-thrust low-gimbal. A maximum gimbal angle range of \(\theta_i \in (0, \pi)\) is enforced via \((\mathcal{O}_I)\) by setting:

\[
C_i = \begin{bmatrix} -\cos(\theta_i/2) - \sin(\theta_i/2) \\ \cos(\theta_i/2) - \sin(\theta_i/2) \end{bmatrix}. \quad (29)
\]

We also impose a glide slope constraint as in Blackmore et al. (2010) which prevents the rocket from approaching the ground too closely prior to touchdown:

\[
\mathcal{X} = \{ x = (r, \dot{v}_c, \eta) \in \mathbb{R}^4 : \epsilon_{gr} \geq 0 \} \sin(\gamma_{gr}), \quad (30)
\]

where \(\epsilon_{gr} = (0, 1) \in \mathbb{R}^2\) is the unit vector along the altitude axis. We choose the following parameters, corresponding

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1. https://github.com/danlyuta/1cvx

2. This is done for simplicity in order to keep the motion planar. A general 3-dimensional angular velocity vector can also be considered.
to a Martian divert maneuver similar to A¸cıkme¸se and Ploen (2007):

\[ M = 2, \ K = 1, \ \omega = 2\pi/88775 \text{ rad s}^{-1}, \ \rho_1 = 4 \text{ m s}^{-2}, \]
\[ \rho_1^2 = 8 \text{ m s}^{-2}, \ \rho_2 = 8 \text{ m s}^{-2}, \ \rho_2^2 = 12 \text{ m s}^{-2}, \ \theta_1 = 120 ^\circ, \]
\[ \gamma_{gs} = 10 ^\circ, \ \ell = (0.33962.6) \text{ km}, \ \zeta \in \{0, 1\}, \]
\[ g = (0, -3.71) \text{ m s}^{-2}, \ m(t_f) = (1 - \zeta)t_f \xi_{max}/t_f \max, \]
\[ \ell(x(t)) = 10^{-3} \xi_{max} |r_1(t)| \tan(\gamma_{gs}) + |r_2(t)|/h_0, \]
\[ r(0) = (1500, h_0) \text{ m}, \ v(0) = (50, -70) \text{ m s}^{-1}, \]
\[ r(t_f) = (0, 0) \text{ m}, \ v(t_f) = (0, 0) \text{ m s}^{-1}, \]

where \( t_f \max = 100 \text{ s} \) is the time of flight upper-bound and \( \xi_{max} = t_f \max \rho_2^2 \) is the maximum input integral cost. The optimal cost is verified to be unimodal in \( t \) almost everywhere, i.e. \( r \in \mathbb{R}^2 \) is too conservative.

Assumption 3. The downrange and altitude are non-zero almost everywhere, i.e. \( r_1(t) \neq 0 \) and \( r_2(t) \neq 0 \) a.e. \([0, t_f] \). 

Leveraging Assumption 3 yields a piecewise constant input with \( \xi(t) \). To check Condition 1, we need to make the following assumption because replacing \( \partial_\xi |t|^T \) with \( \mathbb{R}^2 \) is too conservative.

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Leveraging Assumption 3 yields a piecewise constant input to the adjoint system:

\[ \partial \xi(t) = \partial_\xi |t|^T, \quad \xi \triangleq \begin{bmatrix} 0 \\ r \end{bmatrix}. \] (31)

Following the discussion in Section 3.1, we confirm that the LTI system \( \{-A', D, B'\} \) is strongly observable, hence Condition 1 holds. To check Conditions 2 and 3, we need to make the following assumption because replacing \( \partial_\xi |t|^T \) with \( \mathbb{R}^2 \) is too conservative.

Condition 2. \( \partial \xi(t) \) is strongly observable, hence Condition 1 holds. 

Condition 3. Conditions 2 and 3 can thus be violated only if \( \xi_{max} = \max \{\xi(t)\} \).

The downrange acceleration \( \sum_{i=1}^M u_{i,1}(t) \) changes sign at least once over \([0, t_f] \). 

The problem satisfies Conditions 1-4 under a few light assumptions. Because the glide slope (30) maintains the rocket above zero altitude, \( \ell(t) > 0 \) \( \forall t \in [0, t_f] \) such that Condition 4 holds irrespective of \( m \). To check Condition 1, recognize that for our choice of \( t \):

\[ \partial_\xi |t|^T = D \partial_\xi |t|^T, \quad \xi \triangleq \begin{bmatrix} 0 \\ r \end{bmatrix}. \] (32)

Leveraging (32), consider the following LTI system where a constant input is modelled as a static state, yielding an augmented state \( \lambda(t) \in \mathbb{R}^6 \):

\[ \lambda'(t) = \begin{bmatrix} -A' & D \\ 0 & 0 \end{bmatrix} \lambda(t) = A' \lambda(t), \] (33a)
\[ y(t) = [B' \ 0] \lambda(t) = C' \lambda(t). \] (33b)

When \( \zeta = 0 \), checking Conditions 2 and 3 restricts to ensuring that \( \dot{y}(t) = C' A' \lambda(t) \) cannot evolve perpendicular to certain constant vectors \( \hat{n} \in \mathbb{R}^2 \). The values of \( \hat{n} \) that need to be checked are illustrated in Figure 3a. To verify Conditions 2 and 3, we check the observability properties of the pair \( \{A', \hat{n}', C' \hat{A}'\} \), as in Malyuta et al. (2019). Let \( V_{n} \) be a matrix whose columns span the unobservable subspace. It turns out for the rocket landing problem that \( A V_{n} = 0 \forall \hat{n} \). Conditions 2 and 3 can thus be violated only by a constant primer vector. If this occurs, the input is constrained to point in the directions shown in Figure 3b. Notice that this constrains the downrange acceleration to always have the same sign. The following assumption requires the rocket to experience both acceleration and deceleration. The assumption is satisfied if, for example, the rocket is initially travelling away from the landing site and has to reverse its velocity.

Assumption 4. The downrange acceleration \( \sum_{i=1}^M u_{i,1}(t) \) changes sign at least once over \([0, t_f] \).

The assumption is sufficient for Theorem 1a but not Theorem 1b, because a discontinuity in \( y(t) \) may occur at \( t \in T_c \) (26b) Hartl et al. (1995). If state constraints are activated, a “sufficiently rich” gimbal history may be assumed or Conditions 2 and 3 may be verified \( a \ posteriori \), i.e. the solution is lossless if they hold.

When \( \zeta = 1 \), Condition 2 requires \( \|y(t)\|_2 \neq 1 \) a.e. \([0, t_f] \). 

Modal shape analysis for the pair \( \{A', C'\} \) reveals that, given a constant input in (32), \( \|y(t)\|_2 \) is 1 for an interval that is only possible if \( y(t) \) is constant. This is eliminated by Assumption 4 with the same caveat about state constraint activation. Checking Condition 3 is not possible \( a \ priori \) when \( \zeta = 1 \). The condition is verified \( a \ posteriori \).

The dynamics (27) are discretized via zeroth-order hold on a uniform temporal grid of 150 nodes. Python 2.7.15 and ECOS 2.0.7.post1 Domahidi et al. (2013) are used on a Ubuntu 18.04.1 64-bit platform with a 2.5 GHz Intel Core i5-7200U CPU and 8 GB of RAM. The solution and runtime are compared to a MICP formulation where \( (\mathcal{O}, \mathcal{D}) \) is implemented directly as a binary constraint using Gurobi 8.1 Gurobi Optimization (2018).

Figure 4 shows the resulting state, input and input gain trajectories. Let us first discuss Figures 4a and 4b. The top row shows the overall trajectory, from which we note that Assumptions 3 and 4 are satisfied. The second and third rows show that the input norm is feasible almost everywhere for Problem \( \mathcal{O} \). In particular, the thrust magnitude is bang-bang as predicted in Lemma 1. The intermediate thrusts occurring at the rising and falling edges in the third row are discretization artifacts. Recall that the lossless convexification guarantee is only “almost
Tables 1. Optimal cost and solver runtime when solving Problem $\mathcal{R}$ versus MICP. Dashes show when MICP took too long to converge (> 10 min per iteration).

| $h_0$ [m] | $\zeta$ | $J_R$ | $J_{\text{MICP}}$ | $t_R$ [s] | $t_{\text{MICP}}$ [s] |
|------------|--------|-------|-----------------|----------|----------------|---|
| 650        | 0      | 636.2 | -               | 2.9      | -              |   |
| 650        | 1      | 374.5 | -               | 2.4      | -              |   |
| 800        | 0      | 577.7 | 577.8           | 2.4      | 232.3          |   |
| 800        | 1      | 350.8 | 350.9           | 2.3      | 269.9          |   |
| 1000       | 0      | 548.9 | -               | 3.9      | -              |   |
| 1000       | 1      | 335.7 | 333.7           | 2.3      | 566.8          |   |
| 1500       | 0      | 493.4 | -               | 2.5      | -              |   |
| 1500       | 1      | 316.1 | 316.1           | 2.2      | 177.3          |   |
| 3000       | 0      | 558.0 | 558.0           | 2.5      | 73.1           |   |
| 3000       | 1      | 323.0 | 323.1           | 1.8      | 505.9          |   |

everywhere” in nature. These artifacts have been observed since the early days of lossless convexification theory by Açıkmeşe and Ploen (2007). Note the kink that occurs in the $g(t)$ trajectory in the second row, which coincides with the glide slope state constraint activation as highlighted by the red dot in the first row. Looking at the third row, $\sigma_i(t) \neq \|u_i(t)\|_2$ as expected when $\zeta = 0$ and both inputs are off, since there is no cost incentive to minimize $\sigma_i(t)$. Note that optimality nevertheless requires $u_i(t) = 0$, as predicted by Lemma 1. Finally, the fourth row shows the $\Gamma_j(t)$ trajectories. As predicted by (25), when $\Gamma_j(t) > \Gamma_i(t)$, optimality forces input $\gamma_i(t) = 1$ and $\gamma_j(t) = 0$.

Table 1 compares the achieved optimal cost and solver runtimes of lossless convexification versus a direct MICP implementation of (O,d). One can see that the optimal cost values are quasi-identical, with some slightly lower values for lossless convexification due to the “intermediate thrusts” discussed above. More importantly, solving Problem $\mathcal{R}$ is up to two orders of magnitude faster than using MICP. This is expected, since SOCP has polynomial time complexity in the problem size while MICP has exponential time complexity. Furthermore, MICP was not
able to find a trajectory in several cases (the computation was aborted when runtime exceeded 10 min for a single golden search iteration). The third column of Figure 4 shows a sequence of 50 landing trajectories for a sweep over $h_0 \in [650, 6000]$ m AGL. Computing this sequence of 50 trajectories with $N = 150$ takes 130 s, which is less than the average MICP solution time for a single trajectory.

7 Future Work

Future work consists of expanding the class of problems that can be handled. This includes considering different input norm types in ($O.a$) and ($O.c$), time-varying dynamics in ($O.b$), a lower-bound $L \leq \sum_{i=1}^{M} \gamma_i(t)$ in ($O.e$), a constraint on the input rate of change $\dot{u}_i(t)$, persistently active state constraints in ($O.g$), and removing the discretization artifacts observed in Section 6. A minor caveat of the Lemma 1 proof is that conditions which are proven to hold “almost everywhere” are assumed not to fail on nowhere dense sets of positive measure (e.g. the fat Cantor set) Morgan II (1990). We do not expect this pathology to occur for any practical problem, and in the future we seek to rigorously eliminate this pathology.

8 Conclusion

This paper presented a lossless convexification solution for a more general class of optimal control problems with semi-continuous input norms than the one handled in Malyuta et al. (2019). By relaxing the problem to a convex one and proving that the relaxed solution is globally optimal for the original problem, solutions can be found via convex optimization in polynomial time. The resulting algorithm is amenable to real-time onboard implementation and can also be used to accelerate design trade studies.

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