Improved Extension Protocols for Byzantine Broadcast and Agreement

Kartik Nayak\textsuperscript{1}, Ling Ren\textsuperscript{2}, Elaine Shi\textsuperscript{3}, Nitin H. Vaidya\textsuperscript{4}, and Zhuolun Xiang \textsuperscript{*5}

\textsuperscript{1}Duke University, kartik@cs.duke.edu
\textsuperscript{2, 5}University of Illinois at Urbana-Champaign, \{renling, xiangzl\}@illinois.edu
\textsuperscript{3}Cornell University, runting@gmail.com
\textsuperscript{4}Georgetown University, nitin.vaidya@georgetown.edu

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Abstract

Byzantine broadcast (BB) and Byzantine agreement (BA) are two most fundamental problems and essential building blocks in distributed computing, and improving their efficiency is of interest to both theoreticians and practitioners. In this paper, we study extension protocols of BB and BA, i.e., protocols that solve BB/BA with long inputs of \(l\) bits using lower costs than \(l\) single-bit instances. We present new protocols with improved communication complexity in almost all settings: unauthenticated BA/BB with \(t < n/3\), authenticated BA/BB with \(t < n/2\), authenticated BB with \(t < (1-\epsilon)n\), and asynchronous reliable broadcast and BA with \(t < n/3\). The new protocols are advantageous and significant in several aspects. First, they achieve the best-possible communication complexity of \(\Theta(nl)\) for wider ranges of input sizes compared to prior results. Second, the authenticated extension protocols achieve optimal communication complexity given the best available BB/BA protocols for short messages. Third, to the best of our knowledge, our asynchronous and authenticated protocols in the setting are the first extension protocols in that setting.

1 Introduction

This paper investigates extension protocols\textsuperscript{17} for Byzantine broadcast (BB) and Byzantine agreement (BA). The problem of Byzantine broadcast is for some designated party (sender) to send its message to all parties and let them output the same message, despite some malicious parties that may behave in a Byzantine fashion. Similarly, the goal of Byzantine agreement is to let all parties each with an input message output the same message. We are interested in the problem of efficient Byzantine broadcast and agreement with long input messages since such protocols are widely used as building blocks for other distributed computing problems such as multi-party computation\textsuperscript{30} and permissioned blockchain\textsuperscript{24}. For example, practical blockchain systems typically achieve agreement on large (e.g., 1MB) blocks.

\textsuperscript{*}lead author
A straightforward solution for BB/BA with \(l\)-bit long messages is to invoke the single-bit BB/BA oracle \(l\) times. This approach will incur at least \(\Omega(n^2l)\) communication complexity where \(n\) is the number of parties. This is because any deterministic single-bit BB/BA has cost \(\Omega(n^2)\) \[13\]. In fact, the \(\Omega(n^2)\) lower bound holds even for randomized protocols against a strongly adaptive adversary \[1\]. Another tempting solution is to run BB/BA on the hash digest and let parties disseminate the actual message to each other. However, if a linear fraction of parties can be Byzantine (which is the typical assumption), they can each ask all honest parties for the long message, again forcing the communication complexity to be \(\Omega(n^2l)\).

It turns out non-trivial techniques are needed to get better than \(\Omega(n^2l)\) or to achieve the optimal communication complexity of \(O(nl)\). These are known in the literature as extension protocols, which construct algorithms for BB/BA with long input messages using a small number of BB/BA primitives for short messages. Table 1 and 2 summarize the most related works and our new results in the synchronous and asynchronous setting, respectively. In the tables, \(n\) is the number of parties, \(t\) is the maximum number of Byzantine parties, \(l\) is the length of the input, \(\mathcal{A}(m)\) is the communication cost of \(m\)-bit BA oracle, and \(\mathcal{B}(m)\) is the communication cost of \(m\)-bit BB or reliable broadcast oracle. For cryptographically secure protocols, several cryptographic primitives have been employed in our work and prior works. To make the communication costs comparable, we assume that the output length of the involved cryptographic building blocks are on the same order. Specifically, let \(\lambda\) denote the security parameter, \(k_h = k_h(\lambda)\) denote the hash size, \(k_s = k_s(\lambda)\) denote the signature size, \(k_a = k_a(\lambda)\) denote the output size of the accumulator (see Section 4.1), and \(k = \max(k_h, k_s, k_a)\). We assume \(k = \Theta(k_h) = \Theta(k_s) = \Theta(k_a) = \Theta(\lambda)\) \[1\]. Therefore, in the tables, we use the same variable \(k\) to denote the hash size, signature size and accumulator size for ease of comparison. The expressions distinguishing the cryptographic primitives and the costs of state-of-the-art BB/BA oracle schemes can be found in Section 2.

Contributions. All our protocols achieve the optimal communication complexity \(\Theta(nl)\) for wider ranges of input sizes (see Table 1). In addition to that, our cryptographically secure extension protocols have the following advantages.

- They achieve best-possible communication complexity under the current best BB/BA protocols for short messages. The state-of-the-art BA/BB protocols \[14\] have communication cost \(\mathcal{A}(1) = \mathcal{B}(1) = O(kn^2 + n^3)\). Unless better primitives for short messages are invented, no extension protocol can have communication cost better than ours.
- They can be easily adapted to the asynchronous setting. To the best of our knowledge, these are the first asynchronous authenticated extension protocols and they significantly outperform the unauthenticated ones in the literature.
- Their simplicity makes them less error-prone and more appealing for practical adoption. On this note, in deriving our results, we discover a flaw in the prior best protocol \[17, 18\] and provide a simple fix (Appendix D).

\[1\] This assumption is reasonable since the signature scheme and accumulator scheme (see Appendix A) with the shortest output length are both based on pairing-friendly curves, which are believed to require \(\Theta(\lambda)\) bits for \(\lambda\)-bit security based on the state-of-the-art attack \[20\]. As for hash functions, it is common to model practical hash functions as random oracles, in which case \(\lambda\)-bit security requires \(\Theta(\lambda)\)-bit hash size.

\[2\] The complexity of our cryptographic BB extension protocol is in fact \(O(nl + \mathcal{B}(k) + \mathcal{A}(1) + kn^2)\). Due to the well-known transformation \[21\] from BA to BB, \(\mathcal{B}(k) \leq \mathcal{A}(k) + nl\). Hence, the result in the table also holds.
Table 1: Extension Protocols for Synchronous Byzantine Agreement and Broadcast

| Security  | Problem                  | Communication Complexity | Input range l to reach optimality | Reference |
|-----------|--------------------------|--------------------------|-----------------------------------|-----------|
| crypto.   | agreement/broadcast      | \(O(nl + B(k) + kn^2)\)  | \(\Omega(nl)\)                   | This paper |
|           |                          | \(O(nl + A(k) + kn^2)\)  | \(\Omega(kn)\)                   | This paper |

Table 2: Extension Protocols for Asynchronous Byzantine Agreement and Reliable Broadcast under \(t < n/3\)

| Security  | Problem                  | Communication Complexity | Input range l to reach optimality | Reference |
|-----------|--------------------------|--------------------------|-----------------------------------|-----------|
| error-free| reliable broadcast       | \(O(nl + B(nk) + n^2B(n \log n))\) | \(\Omega(n^2 \log n + kn^2 \log n)\) | This paper |
|           | agreement                | \(O(nl + B(1) + n^2B(1))\) | \(\Omega(n^3)\)                   | This paper |

2 Related Work

**Timing and setup assumptions.** With different security assumptions on the adversary and timing assumptions, Byzantine broadcast and agreement can be solved for different thresholds of the Byzantine parties. For the timing assumptions, protocols under both synchrony and asynchrony have been studied. If a trusted setup like public-key infrastructure (PKI) exists, it is called the authenticated setting; otherwise, it is the unauthenticated setting.

- In the synchronous setting, BB/BA can be solved under \(t < n/3\) without authentication [21]. With authentication, BA can be solved under \(t < n/2\) and BB can be solved under \(t < n\) [21, 14, 28].
- In the asynchronous setting, BB is impossible; BA (randomized) and reliable broadcast can be solved under \(t < n/3\) with or without authentication [8, 9]. The authenticated protocols in this paper are cryptographically secure, in which case the adversary is assumed to be computationally bounded and unable to comprise the cryptographic tools involved; the unauthenticated protocols in this paper are error-free.

**Previous extension protocols.** Table 1 summarizes the most related extension protocols in the synchronous setting. Here we describe the protocols in Table 1 and mention several other extension protocols. Let \(k_h\) denote output size of the collision-resistant hash function. For both Byzantine broadcast and agreement in the synchronous setting under \(t < n/2\), recent work proposes authenticated algorithms with communication cost \(O(nl + nB(k_h) + n^3k_h)\) [17, 18]. For the authenticated setting when \(t < n\), the state-of-the-art BB extension protocols have communication complexity \(O(nl + B(nk_h) + n^2B(n \log n))\) [17, 18]. Recent work also proposes authenticated extension protocols with information-theoretic security [16, 11]; these protocols have worse communication complexity than cryptographic ones. For the unauthenticated setting with \(t < n/3\), existing works mostly focus error-free protocols. Liang and Vaidya [22] propose the first optimal error-free Byzantine broadcast and agreement protocol with communication complexity \(O(nl + (n^2\sqrt{7} + n^4)B(1))\) for the synchronous case. Patra [27] improves the communication complexity to \(O(nl + n^2B(1))\).
Table 2 summarizes the most related extension protocols in the asynchronous setting. The error-free unauthenticated protocol of Patra [27] can be adapted to asynchrony to solve reliable broadcast and asynchronous BA; the communication complexity for reliable broadcast is $O(nl + n^2 \log nB(1))$, and $O(nl + n^3 \log nB(1) + nA(1))$ for BA. To the best of our knowledge, there exist no extension protocols in the authenticated and asynchronous setting. Of course, Patra’s unauthenticated protocol can be used in the authenticated setting, but the cost would be much higher than our new protocols. Cachin and Tessaro [10] adapt Bracha’s broadcast [8] to handle $l$-bit long messages with communication cost $O(nl + k_h n^2 \log n)$, but their method does not seem to generalize to other protocols and hence does not yield an extension protocol.

State-of-the-art oracle schemes. To better interpret the improvements we obtained for extension protocols, we provide a summary of the state-of-the-art broadcast and agreement protocols that can be used as the oracle in our extension protocol. For the synchronous setting, we focus on deterministic solutions in this paper. The best solution to authenticated BB for $t < n$ is the classic Dolev-Strong [14] protocol. After applying multi-signatures, the communication complexity to broadcast (up to) $k$ bits is $B(1) = B(k) = \Theta((k + k_s)n^2 + n^3)$ where $k_s$ is the signature size. The Dolev-Strong protocol can also be modified to solve authenticated BA for the $t < n/2$ case (BA is impossible if $t \geq n/2$). Using an initial all-to-all round with threshold signature to simulate the sender, the communication complexity remains as $A(1) = A(k) = \Theta((k + k_s)n^2 + n^3)$. In the unauthenticated setting, only $t < n/3$ Byzantine parties can be tolerated and Berman et al. [5] achieves $B(1) = A(1) = \Theta(n^2)$ (when $t = \Theta(n)$), matching the lower bound on communication complexity.

In the asynchronous setting, Bracha’s reliable broadcast [8] is deterministic and has communication complexity $B(1) = O(n^2)$. Randomization is necessary for asynchronous BA given the FLP impossibility [15]. State-of-art protocols rely on a “common coin” oracle to provide shared randomness but are deterministic otherwise. The most efficient unauthenticated asynchronous BA [25] achieves expected communication complexity $A(1) = O(n^2)$ assuming a magic common coin oracle. The most efficient authenticated asynchronous BA [2] achieves expected communication complexity $A(1) = A(k) = O(kn^2)$ and provides a construction for the common coin oracle.

### 3 Preliminaries

We consider $n$ parties $P_1, ..., P_n$ connected by a reliable, authenticated all-to-all network, where up to $t$ parties may be corrupted by an adversary $A$ and behave in a Byzantine fashion. All the results in the main paper consider the synchronous model, where there exists a known upper bound on the message delay and computation delay, and some results can be extended to asynchrony (in Appendix C.1 and C.2 where such upper bounds do not exist. We consider a static adversary which decides the set of corrupted parties at the beginning of the execution. We denote parties that are not corrupted by the adversary as honest parties. Two types of the adversary are considered. The computationally bounded adversary, considered in cryptographically secure protocols, is restricted on some computational assumptions such as cryptographic accumulators and public key infrastructure (introduced in Section 4.1 later). On the other hand, the computationally unbounded adversary is considered in the error-free protocols. The communication complexity $M$ of the protocol is measured by the worst-case or expected number of bits transmitted by the honest parties according to the protocol specification over all possible executions under any adversary strategy. Here, we provide the formal definition of Byzantine broadcast (BB) and Byzantine agreement (BA).
Definition 1 (Byzantine Broadcast). A protocol for a set of parties $\mathcal{P} = \{P_1, ..., P_n\}$, where a distinguished party called the sender $P_s \in \mathcal{P}$ holds an initial $l$-bit input $m$, is a Byzantine broadcast protocol tolerating an adversary $A$, if the following properties hold:

- Termination. Every honest party eventually outputs a message.
- Agreement. All the honest parties output the same message.
- Validity. If the sender is honest, all honest parties output the message $m$.

Definition 2 (Byzantine Agreement). A protocol for a set of parties $\mathcal{P} = \{P_1, ..., P_n\}$, where each party $P_i \in \mathcal{P}$ holds an initial $l$-bit input $m_i$, is a Byzantine agreement protocol tolerating an adversary $A$, if the following properties hold:

- Termination. Every honest party eventually outputs a message.
- Agreement. All the honest parties output the same message.
- Validity. If every honest party $P_i$ holds the same input message $m$, then all honest parties output the message $m$.

We introduce two basic tools we will use.

Linear Error Correcting Code [29]. We will use standard Reed-Solomon (RS) codes [29] in our protocols, which is a $(n, b)$ RS code in Galois Field $\mathbb{F} = GF(2^a)$ with $n \leq 2^a - 1$. This code encodes $b$ data symbols from $GF(2^a)$ into codewords of $n$ symbols from $GF(2^a)$, and can decode the codewords to recover the original data.

- ENC. Given inputs $m_1, ..., m_b$, an encoding function $\text{ENC}$ computes $(s_1, ..., s_n) = \text{ENC}(m_1, ..., m_b)$, where $(s_1, ..., s_n)$ are codewords of length $n$. By the property of the RS code, knowledge of any $b$ elements of the codeword uniquely determines the input message and the remaining of the codeword.

- DEC. The function $\text{DEC}$ computes $(m_1, ..., m_b) = \text{DEC}(s_1, ..., s_n)$, and is capable of tolerating up to $c$ errors and $d$ erasures in codewords $(s_1, ..., s_n)$, if and only if $n - b \geq 2c + d$. In our protocol, We will invoke $\text{DEC}$ with specific values of $c, d$ satisfying the above relation, and $\text{DEC}$ will return correct output.

Multi-signature Scheme [7]. Multi-signature scheme can aggregate $n$ signatures into one signature, therefore reduce the size of signatures. Given $n$ signatures $\sigma_i = \text{Sign}(sk_i, m)$ on the same message $m$ with corresponding public keys $pk_i$ for $1 \leq i \leq n$, a multi-signature scheme can combine the $n$ signatures above into one signature $\Sigma$ where $|\Sigma| = |\sigma_i|$. The combined signature can be verified by anyone using a verification function $\text{Ver}(PK, \Sigma, m, \mathcal{L})$, where $\mathcal{L}$ is the list of signers and $PK$ is the union of $n$ public keys $pk_i$.

4 Cryptographically Secure Extension Protocols under $t < n/2$ Faults

In this section, we consider the computationally bounded adversary, which is restricted on computational assumptions such as the cryptographic accumulator and public-key infrastructure. Under such assumption, we design cryptographically secure extension protocols with improved communication complexity. Recall that we use $k$ to denote the size of all cryptographic primitives used in the extension protocol (see Section 1).

For $t < n/2$, we propose an extension protocol for synchronous Byzantine agreement with communication complexity $O(nl + A(k) + kn^2)$. A similar extension protocol for synchronous Byzantine broadcast has communication complexity $O(nl + B(k) + A(1) + kn^2)$.
The above protocols are adapted to asynchronous Byzantine agreement and reliable broadcast under $t < n/3$ in Appendix C.

### 4.1 Building Blocks: Encode, Distribute and Reconstruct

In this section, we define several subprotocols that will be used in our extension protocols in Section 4.2 and 5. In addition to the linear error correcting code introduced in Section 3, the subprotocols also use a cryptographic tool called accumulator defined as follows.

**Cryptographic Accumulators** [3, 12]. We present the definition of cryptographic accumulators proposed by Barić and Pfitzmann [3]. Intuitively, the cryptographic accumulator constructs an accumulation value for a set of values and can produce a witness for each value in the set. Given the accumulation value and a witness, any party can verify if a value is indeed in the set.

Given a parameter $k$, and a set $\mathcal{D}$ of $n$ values $d_1, ..., d_n$, an accumulator has the following components:

- **Gen**$(1^k, n)$: This algorithm takes a parameter $k$ represented in unary form $1^k$ and an accumulation threshold $n$ (an upper bound on the number of values that can be accumulated securely), returns an accumulator key $ak$. The accumulator key $ak$ is part of the trusted setup and therefore is public to all parties.
- **Eval**(ak, $\mathcal{D}$): This algorithm takes an accumulator key ak and a set $\mathcal{D}$ of values to be accumulated, returns an accumulation value $z$ for the value set $\mathcal{D}$.
- **CreateWit**$(ak, z, d_i)$: This algorithm takes an accumulator key $ak$, an accumulation value $z$ for $\mathcal{D}$ and a value $d_i$, returns $\bot$ if $d_i/\not \in \mathcal{D}$, and a witness $w_i$ if $d_i \in \mathcal{D}$.
- **Verify**$(ak, z, w_i, d_i)$: This algorithm takes an accumulator key $ak$, an accumulation value $z$ for $\mathcal{D}$, a witness $w_i$ and a value $d_i$, returns **true** if $w_i$ is the witness for $d_i \in \mathcal{D}$, and **false** otherwise.

In our extension protocols, the input message is encoded as a set of values, and the cryptographic accumulator is used for verifying the correctness of the values received from other parties. Note that for simplicity, our definition of the cryptographic accumulator above omits the auxiliary information aux that appears in the standard definition [3] because the the bilinear accumulator we will use (in Appendix A) does not use aux. We also assume that the function Eval is deterministic, since the bilinear accumulator implementation has a deterministic construction. The bilinear accumulator satisfies the following properties.

**Lemma 1** (Collision-free accumulator [26]). The bilinear accumulator is collision-free. That is, for any set size $n$, there is only a probability negligible in $k$. for a probabilistic polynomial-time adversary to find an accumulator key $ak$, a set $\mathcal{D} = \{d_1, ..., d_n\}$, an accumulation value $z$ for $\mathcal{D}$, a value $d' \not \in \mathcal{D}$, and a witness $w'$ such that $\text{Verify}(ak, z, w', d') = \text{true}$.

**Corollary 1.** The bilinear accumulator satisfies that, for any set size $n$, there is only a probability negligible in $k$ for a probabilistic polynomial-time adversary to find two sets $\mathcal{D} \neq \mathcal{D}'$ that have the same accumulation value.

**Explanation of the building blocks.** Now we define three subprotocols Encode, Distribute and Reconstruct that will be used as building blocks for our extension protocols, listed in Figure 1.

- **Encode** first divides a message $m$ into $b$ blocks, then compute $n$ coded values $(s_1, ..., s_n)$ using RS codes (defined in Section 3), and attaches an index $j$ for each value $s_j$. The purpose of Encode is to introduce resilience by encoding the message into fault-tolerant coded values –
Our extension protocols in Section 4 and 5 will use Encode, Distribute the input message to coded values, use Lemma 2. After receiving the coded values, Reconstruct m such that m', b = Eval(ak, Encode(m, b)) computes a witness one can recover the message from the remaining coded values. First removes any invalid value together with the value. is to recover the message, despite the presence of at most d accumulation value in a robust yet efficient manner – if at least one honest party that has the correct message j any honest party from the remaining values with at most d accumulation value z will be detected by the accumulator scheme and thus erased.

Figure 1: Building Blocks

after applying Encode to a message m, even if n – b coded values in (s1, ..., sn) are erased, one can recover the message from the remaining coded values.

- **Encode**
  
  Input: a message m, a number b
  Output: n coded values s1, ..., sn
  Divide m into b blocks evenly, m1, ..., mb, each has l/b bits where l is the length of m. Compute (s1, ..., sn) = ENC(m1, ..., mb) using RS codes, where ENC is defined in Section 4.1. Add an index to every value in (s1, ..., sn), i.e., D = ((1, s1), ..., (n, sn)), and return D.

- **Distribute**
  
  Input: a set of indexed values D = ((1, s1), ..., (n, sn)), an accumulator key ak, an accumulation value z
  Compute wj = CreateWit(ak, z, ⟨j, sj⟩) for every ⟨j, sj⟩ ∈ D. Send (sj, wj) to party Pj for every j ∈ [n].

- **Reconstruct**
  
  Input: S = (((1, s1), w1), ..., ((n, sn), wn)) where each ((i, si), wi) is a pair of indexed value and witness, an accumulator key ak, an accumulation value z, a number d0
  Output: a message m
  For every j ∈ [n], if Verify(ak, z', wj, ⟨j, sj⟩) = false, let sj = ⊥. Apply DEC on the codewords (s1, ..., sn) with c = 0 and d = d0, where DEC is defined in Section 4.1. Return m = m1|...|mb where m1, ..., mb are the data returned by DEC.

Our extension protocols in Section 4 and 5 will use Encode at the beginning of the protocol to encode the input message to coded values, use Distribute in the middle to let every party distribute their coded values with the witnesses, and use Reconstruct to reconstruct the original input message after receiving the coded values.

**Lemma 2.** For any message m, let z = Eval(ak, Encode(m, b)). The adversary cannot generate m' such that m' ≠ m and z = Eval(ak, Encode(m', b)) with non-negligible probability in k.
Input of every party $P_i$: An $l$-bit message $m_i$

Primitive: Byzantine agreement oracle, cryptographic accumulator with $\text{Eval}$, $\text{CreateWit}$, $\text{Verify}$

Protocol for party $P_i$:

1. Compute $D_i = (\langle 1, s_1 \rangle, ..., \langle n, s_n \rangle) = \text{Encode}(m_i, b)$. Compute the accumulation value $z_i = \text{Eval}(ak, D_i)$. Input $z_i$ to an instance of $k$-bit Byzantine agreement oracle.

2. When the above BA outputs $z$, if $z = z_i$, set $\text{happy}_i = 1$, otherwise set $\text{happy}_i = 0$. Input $\text{happy}_i$ to an instance of 1-bit Byzantine agreement oracle.

3. • If the above BA outputs 0, output $o_i = \bot$ and abort.
   • If the above BA outputs 1 and $\text{happy}_i = 1$, invoke $\text{Distribute}(D_i, ak, z_i)$.

4. For the set of pairs $\{(s_i, w_i)\}$ received from the previous step, if there exists a $(s_i, w_i)$ pair such that $\text{Verify}(ak, z, w_i, \langle i, s_i \rangle) = \text{true}$, then send $(s_i, w_i)$ to all other parties.

5. If $\text{happy}_i = 1$, set $o_i = m_i$. Otherwise, let $(s_j, w_j)$ be the message received from party $P_j$ from the previous step and $S_i = (((1, s_1), w_1), ..., ((n, s_n), w_n))$, and set $o_i = \text{Reconstruct}(S_i, ak, z, t)$.

6. Output $o_i$.

Figure 2: Protocol Synchronous Crypto. $\frac{n}{2}$-BA

Proof. For any $m'$ such that $z = \text{Eval}(ak, \text{Encode}(m', b))$, let $D = \text{Encode}(m, b)$ and $D' = \text{Encode}(m', b)$. According to Corollary 1, the adversary cannot find $D' \neq D$ that have the same accumulation value with non-negligible probability in $k$. Therefore the same accumulation value implies that the two accumulated sets are identical, that is, $D = D'$. By the RS code, the same codewords correspond to the same message, and thus $m = m'$.

4.2 Byzantine Agreement under $< \frac{n}{2}$ faults

Without loss of generality, we assume the number of Byzantine parties is $t = \lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 1$ in this section. Let $b = n - t = \lfloor \frac{n}{2} \rfloor + 1$.

The protocol is presented in Figure 2. We briefly describe each step of Protocol Synchronous Crypto. $\frac{n}{2}$-BA. First each party encodes its message using RS codes and computes the accumulation value for the set of coded values. With a deterministic $\text{Eval}$, any honest party with the same accumulator key and set will produce the same accumulation value. The RS codes can recover the message with up to $t$ coded values being erased, and the accumulation value uniquely corresponds to the set of coded values (by Corollary 1) and equivalently the original message. Then every party runs an instance of $k$-bit Byzantine agreement with the accumulation value as the input. After the above agreement terminates, each party checks whether the agreement output matches its accumulation value, and inputs the result to an 1-bit Byzantine agreement instance. If the above agreement outputs 0, then no honest party has a message corresponding to an accumulation value matching the agreement output $z$, therefore all parties output $\bot$ and abort. If the above agreement outputs 1, then at least one honest party has the accumulation value $z_i$ matching with
the agreement output \( z \), and every honest party will agree on the message corresponding to \( z \). Then in Distribute, all parties send the \( j \)-th coded value to party \( P_j \). After that, each honest party \( P_j \) will send a valid \( j \)-th coded value to all other parties, from which the correct message can be obtained in Reconstruct. One nice property of our protocol is that, if at least one honest party with message \( m \) invokes Distribute, then all honest parties can obtain \( m \) from Reconstruct (see the proof of Lemma 4). We prove the validity and agreement and properties and analyze the communication complexity below.

Lemma 3. If every honest party has the same input message \( m_i = m \), all honest parties output the same message \( m \).

Proof. If all honest parties have the same input message \( m_i = m \), they compute and input the same accumulation value \( z \) to the instance of Byzantine agreement in step 1. Then in step 2, the BA outputs \( z \) by the validity condition, and any honest party sets \( \text{happy}_i = 1 \). Therefore, every honest party \( P_i \) inputs 1 to the 1-bit Byzantine agreement oracle in step 2. By the validity of the Byzantine agreement oracle, the agreement will output 1. Then any honest party \( P_i \) sets \( o_i = m \) in step 5 since \( \text{happy}_i = 1 \). Hence, all honest parties output \( m \) when the protocol terminates.

Lemma 4. All honest parties output the same message.

Proof. If the Byzantine agreement in step 3 outputs 0, then all honest parties output the same message \( \bot \). If the agreement agreement in step 3 outputs 1, then by the validity of the Byzantine agreement, some honest party \( P_i \) must input 1 and thus has \( z_i = z \). By Lemma 2 any honest party \( P_i \) with \( \text{happy}_i = 1 \) has the identical message \( m \) corresponding to \( z \), and sets the output to be \( m \) at step 5. In step 3, any honest party \( P_i \) with \( \text{happy}_i = 1 \) invokes Distribute to compute witness \( w_j \) for each index value \( \langle j, s_j \rangle \), and sends the valid \( (s_j, w_j) \) pair computed from message \( m \) to party \( P_j \) for every \( P_j \). By Lemma 1 the Byzantine parties cannot generate a different pair \( (s'_j, w'_j) \) that can be verified. Therefore, in step 4, every honest party \( P_j \) receives at least one valid \( (s_j, w_j) \) pair, and forwards it to all other parties. Since there are at least \( n - t \) honest parties, in step 5, each honest party will receive at least \( n - t \) valid coded values. In Reconstruct, using the accumulation value associated with the coded value, any party \( P_i \) can detect the corrupted values and remove them. By the property of RS codes, since \( n - b = t \geq 2c + d = t \), any honest party \( P_i \) with \( \text{happy}_i = 0 \) is able to recover the message \( m \), and any honest party \( P_i \) with \( \text{happy}_i = 1 \) already has the message \( m \). Therefore all honest parties outputs \( m \).

Theorem 1. Protocol Synchronous Crypto. \( \frac{n}{2} \)-BA satisfies Termination, Agreement and Validity, and has communication complexity \( O(nl + A(k) + kn^2) \).

Proof. Termination is clearly satisfied. By Lemma 3 agreement is satisfied. By Lemma 4 validity is satisfied.

Step 1 has cost \( O(nl + A(k)) \), where \( k \) is the size of the cryptographic accumulator. Step 2 has cost \( A(1) \leq A(k) \). Step 3 has cost \( O(nl + kn^2) \), since each honest party invokes an instance of Distribute, which leads to an all-to-all communication with each message of size \( O(l/b + k) = O(l/n + k) \). For step 4, it also has cost \( O(nl + kn^2) \) similarly as step 3. Hence the total cost is \( O(nl + A(k) + kn^2) \).
4.3 Lower Bound on BA for Long Messages

We start by showing a straightforward lower bound of $\Omega(nl + A(k) + n^2)$ for $l$-bit BA as follows. First of all, $\Omega(nl)$ is a lower bound for Byzantine agreement protocol with $l$-bit inputs according to [16]. We briefly mention the proof idea from [16] for completeness. Let a set $A$ of $n - t$ parties have input $m$ and a set $B$ of the rest $t$ parties have input $m' \neq m$. In scenario 1, let parties in $B$ be Byzantine but behave as if they are honest. Then by the validity condition, all parties in $A$ will output $m$. In scenario 2, let parties in $B$ be honest. To parties in $A$, the scenario 2 is indistinguishable from scenario 1, and thus they will output $m$. By the agreement condition, parties in $B$ also need to output $m$, which leads to a lower bound on the communication cost of $\Omega(tl) = \Omega(nl)$. Secondly, since $A(m)$ denotes the communication complexity of a BA oracle with $m$-bit inputs, it is clear that $A(k)$ is a lower bound for $l \geq k$. Finally, according to [13], $\Omega(n^2)$ is a lower bound on the communication complexity for any Byzantine agreement protocol tolerating $t = \Theta(n)$ faults. The above lower bounds together imply a lower bound of $\Omega(nl + A(k) + n^2)$ for $l$-bit BA.

By Theorem 1, our Protocol Synchronous Crypto. $\frac{n}{2}$-BA has cost $O(nl + A(k) + kn^2)$, which is very close to the lower bound. Although it does not meet the lower bound, we remark that further improvements seem challenging. Notice that if $A(k) = \Omega(kn^2)$, then a lower bound of $\Omega(nl + A(k) + kn^2)$ follows, matching our upper bound. Thus, improving upon our upper bound requires a $k$-bit BA oracle whose communication complexity is $o(kn^2)$. In fact, if we were to design such a more efficient BA protocol, we have to follow a very particular paradigm. The $\Omega(n^2)$ lower bound by Dolev and Reischuk is a lower bound on the number of messages [13]. If every message is signed, then $\Omega(kn^2)$ communication must be incurred. Yet, we know authentication is necessary for tolerating minority faults. Thus, such a protocol must use $\Omega(n^2)$ messages in total but only authenticate a small subset of them. We are not aware of any work exploring this direction, and closing this gap is an interesting open problem. For reference, as mentioned in Section 2, the state-of-the-art BA with minority faults has a communication complexity of $A(k) = \Theta(kn^2 + n^3)$.

4.4 Byzantine Broadcast under $t < n/2$

The extension protocol for synchronous Byzantine broadcast under $< \frac{n}{2}$ is very similar to that of synchronous Byzantine agreement, as presented in Figure 3. The main difference is that the sender uses a $k$-bit Byzantine broadcast oracle to broadcast its accumulation value, instead of every party inputting its accumulation value to a $k$-bit Byzantine agreement oracle.

Lemma 5. If the sender is honest and has input $m_s$, all honest parties output the same message $m_s$.

Proof. If the sender is honest, every honest party will receive $m_s$ and the Byzantine broadcast outputs the corresponding accumulation value $z_s$. Then in step 2, any honest party $P_i$ computes a matching accumulation value $z_i = z_s$ with $m_s$, sets $happy_i = 1$ and inputs 1 to an 1-bit Byzantine agreement oracle. By the validity of the Byzantine agreement, the agreement will output 1. Then any honest party $P_i$ sets $o_i = m$ in step 5 since $happy_i = 1$. Hence, all honest parties output $m$ when the protocol terminates.

Lemma 6. All honest parties output the same message.

Proof. The proof is identical to that of Lemma 4.
Input of the sender $P$: An $l$-bit message $m_s$

Primitive: Byzantine broadcast oracle, Byzantine agreement oracle, cryptographic accumulator with $\text{Eval}$, $\text{CreateWit}$, $\text{Verify}$

Protocol for party $P_i$:

1. If $i = s$, perform the following. Compute $D_s = ((1, s_1), ..., (n, s_n)) = \text{Encode}(m_s, e)$. Compute the accumulation value $z_s = \text{Eval}(ak, D_s)$. Send $m_s$ to every party, and broadcast $z_s$ by invoking a $k$-bit Byzantine broadcast oracle.

2. When receiving the message $m$ from the sender, and the Byzantine broadcast above output $z$, perform the following. Compute $D_i = ((1, s_1), ..., (n, s_n)) = \text{Encode}(m, e)$. Compute the accumulation value $z_i = \text{Eval}(ak, D_i)$. If $z_i = z$, set $\text{happy}_i = 1$, otherwise set $\text{happy}_i = 0$. Input $\text{happy}_i$ to an instance of single-bit Byzantine agreement oracle.

3. Steps 3 to 6 are identical to that of Protocol Synchronous Crypto. $\frac{n}{2}$-BA.

Theorem 2. Protocol Synchronous Crypto. $\frac{n}{2}$-BB satisfies Termination, Agreement and Validity. The protocol has communication complexity $O(nl + B(k) + A(1) + kn^2)$.

Proof. Termination is clearly satisfied. By Lemma 6, agreement is satisfied. By Lemma 5, validity is satisfied.

Step 1 has cost $O(nl + B(k))$, where $k$ is the size of the cryptographic accumulator. Step 2 has cost $O(A(1))$. Step 3 has cost $O(nl + kn^2)$, since each honest party invokes an instance of $\text{Distribute}$, which leads to an all-to-all communication with each message of size $O(l/b + k) = O(l/n + k)$. For step 4, it also has cost $O(nl + kn^2)$. Hence the total cost is $O(nl + B(k) + A(1) + kn^2)$. \hfill \Box

5 Cryptographically Secure Extension Protocol under $t < (1 - \varepsilon)n$ Faults

In this section, we propose an extension protocol for synchronous BB with communication complexity $O(nl + B(k) + kn^2 + n^3)$ under $t < (1 - \varepsilon)n$ where $\varepsilon > 0$ is some constant. Our protocol does not achieve the best fault tolerance of $t < n$ as state-of-art solutions [17, 18] did, but it is much more efficient than state-of-art solutions. Also note that BA is impossible under $t \geq n/2$ faults even with authentication [21]. Without loss of generality, we assume the number of Byzantine parties is $t = \lfloor (1 - \varepsilon)(n - 1) \rfloor$ in this section. Let $b = n - t$.

Protocol Synchronous Crypto. $(1 - \varepsilon)n$-BB. The protocol is presented in Figure 4 and we briefly explain each step of the protocol. First the sender encodes its message and computes the accumulation value using the coded values. Then the sender broadcasts the accumulation value via an instance of $k$-bit Byzantine broadcast oracle. By the agreement condition, all honest replicas output the same value for BB. The remaining of the protocol runs in iterations $r = 1, 2, ..., t + 1$. Each iteration consists 3 steps. The Distribution step, Sharing step and Reconstruction step are analogous to steps 3 – 5 in Protocol Synchronous Crypto. $\frac{n}{2}$-BA in Figure 2 but here each step is examined in every iteration for execution, and is executed only once. The Distribution step aims to
Input of the sender $P_s$: An $l$-bit message $m_s$

Primitive: Byzantine broadcast oracle, cryptographic accumulator with $\text{Eval}$, $\text{CreateWit}$, $\text{Verify}$

Protocol for party $P_i$:

1. The sender $P_s$ initializes $o_s = m_s, \text{happy}_s = 1$, and other party $P_i$ initializes $o_i = \perp, \text{happy}_i = 0$. The sender computes $D_s = \text{Encode}(m_s, b)$, the accumulator value $z_s = \text{Eval}(ak, D_s)$, and broadcasts $z_s$ by invoking an instance of $k$-bit Byzantine broadcast oracle. Let $z_i$ denote the output of the Byzantine broadcast at party $P_i$.

2. For iterations $r = 1, \ldots, t + 1$:

   **Distribution step:**
   
   If $\text{happy}_i = 1$, then sign the HAPPY message using the multi-signature scheme, send the multi-signature signed by $r$ distinct parties to all other parties, invoke $\text{Distribute}(D_i, ak, z_i)$, and skip the Distribution step in all future iterations.

   **Sharing step:**
   
   If a valid $(s_i, w_i)$ pair is received from the Distribution step such that $\text{Verify}(ak, z_i, w_i, \langle i, s_i \rangle) = \text{true}$, then send $(s_i, w_i)$ to all other parties and skip the Sharing step in all future iterations.

   **Reconstruction step:** (not a communication step)
   
   Let $(s_j, w_j)$ be the first message received from party $P_j$ from the Sharing step (possibly from previous iterations). Let $S_i = (((1, s_1), w_1), \ldots, ((n, s_n), w_n))$. Compute $M_i = \text{Reconstruct}(S_i, ak, z_i, t)$ and $D_i = \text{Encode}(M_i, b)$. If $\text{Eval}(ak, D_i) = z_i$ and a HAPPY message signed by $r$ distinct parties excluding $P_i$ was received in the Distribution step of this iteration, then set $\text{happy}_i = 1$, set $o_i = M_i$, and skip the Reconstruction step in all future iterations.

3. Output $o_i$.

Figure 4: Protocol Synchronous Crypto. $(1 - \varepsilon)$-BB

distribute the indexed coded values to other parties. The Sharing step forwards the correct coded value to other parties. The Reconstruction step aims to reconstruct the original message from the coded values received from other parties and set the output. Similar to Protocol Scynchronous Crypto. $\frac{k}{2}$-BA, the above steps provide a nice guarantee that if at least one honest party with message $m$ invokes $\text{Distribute}$ in the Distribution step, then all honest parties can obtain $m$ in the Reconstruction step (see the proof of Lemma 7).

Now we give a more detailed description. A party becomes happy (i.e., sets $\text{happy}_i = 1$) if it is ready to output a message that is not $\perp$. In the first iteration, only the sender is happy; it invokes $\text{Distribute}$ and also signs and sends a message HAPPY of a constant size. The role of the message HAPPY is to be signed by the rest of the parties using multi-signatures to form a signature chain, similar to the Dolve-Strong Byzantine broadcast algorithm [14]. An honest party becomes happy at the end of iteration $r$, if it reconstructs the correct message (matching the agreed upon accumulation value) in the Reconstruction step of iteration $r$ and has received a HAPPY message.
signed by \( r \) parties in the Distribution step of iteration \( r \). When an honest party becomes happy, it will set its output to be the reconstructed message \( M_i \); then, in the Distribution step of the next iteration (if there is one), it will also send its own signature of \textsc{HAPPY} to all other parties, and invoke \textsc{Distribute}. This way, if an honest party becomes happy in the last iteration \( r = t + 1 \), it can be assured that some honest party has invoked \textsc{Distribute}, so that all honest parties will be ready to output the correct message. We reiterate that each step is executed at most once in the entire protocol. Finally, after \( t + 1 \) iterations, every party outputs the message.

**Lemma 7.** If any honest party \( P_i \) invokes \textsc{Distribute} with message \( m \), then every honest party \( P_j \) outputs \( o_j = m \).

**Proof.** By the agreement condition of the Byzantine broadcast, the output \( z_i \) of the BB at every honest party \( P_i \) is identical. If an honest party \( P_i \) invokes \textsc{Distribute} with message \( m \), \( m \) satisfies \( z_i = \text{Eval}(ak, \text{Encode}(m, b)) \). If any other honest party \( P_j \) sets \( o_j = m' \) after initialization, it must satisfy \( \text{Eval}(ak, \text{Encode}(m', b)) = z_j = z_i \). By Lemma 2, \( m = m' \). Thus, we only need to show that every other honest party \( P_j \) sets \( o_j \).

Suppose that \( P_i \) invokes \textsc{Distribute} in some iteration \( r \). According to the subprotocol \textsc{Distribute}, \( P_i \) computes a witness \( w_j \) for each indexed value \( \langle j, s_j \rangle \) and sends the pair \( (s_j, w_j) \) to each party \( P_j \). According to Lemma 1, the adversary cannot generate \( d' \notin D_i \) and a witness \( w' \) such that \( \text{Verify}(ak, z_i, w', d') = \text{true} \). Then, in Sharing step of iteration \( r \), every honest party \( P_j \) can identify and forward the valid pair \( (s_j, w_j) \) to all other parties, unless it has already done that in previous iterations. Since there are at least \( n - t = b \) honest parties, in the Reconstruction step of iteration \( r \), every honest party \( P_j \) receives at least \( n - t = b \) correct coded values. In \textsc{Reconstruct}, using the witness associated with the indexed coded value, every party \( P_j \) can identify the corrupted values and remove them. The number of erased values is at most \( t \). By the property of RS codes, since \( n - b \geq 2c + d = t \), \( P_j \) with \text{happy}_j = 0 is able to recover the message \( m \).

Furthermore, we will show that each party will receive a \text{HAPPY} message signed by \( r \) distinct parties in the Reconstruction step of iteration \( r \). If \( r = 1 \), then \( P_i = P_s \) and every \( P_j \) will receive a signature for \text{HAPPY}. If \( r > 1 \), then \( P_i \) has received a multi-signature of \text{HAPPY} signed by \( r - 1 \) distinct parties excluding \( P_i \) in the Reconstruction step of iteration \( r - 1 \); \( P_i \) adds its own signature of \text{HAPPY} in iteration \( r \), so each honest \( P_j \) will receive a multi-signature of \text{HAPPY} signed by \( r \) distinct parties in the Reconstruction step of iteration \( r \).

Therefore, if \text{happy}_j = 0 up till now, then an honest \( P_j \) will set \text{happy}_j = 1 and \( o_j = m \) in the Reconstruction step of iteration \( r \). If \text{happy}_j = 1, then \( P_j \) has already set \( o_j = m \). Note that an honest sender does not set its output again in the Reconstruction step, since the \text{HAPPY} message always contains its signature. Once \( P_j \) sets \( o_j \), it will skip the Reconstruction step in all future iterations, and \( o_j \) will not be changed. Therefore, all honest parties output \( m \) when they terminate.

**Lemma 8.** If the sender is honest and has input \( m_s \), every honest party outputs \( m_s \).

**Proof.** If the sender is honest and has input \( m_s \), in step 1 it computes and broadcasts the accumulation value \( z_s \). In iteration \( r = 1 \), the sender sends a signed \text{HAPPY} to all other parties and invokes \textsc{Distribute}. By Lemma 7, every honest parties output \( m_s \).

**Lemma 9.** Every honest party outputs the same message.
Proof. If all honest parties output ⊥, then the lemma is true. Otherwise, suppose some honest party \( P_i \) outputs \( o_i = m \) where \( m \neq ⊥ \). If \( P_i \) is the sender, then by Lemma 8 all honest parties output \( m \). Now consider the case where \( P_i \) is not the sender. According to the protocol, if \( P_i \neq P_s \) sets \( o_i = m \neq ⊥ \) in the Reconstruction step of iteration \( 1 \leq r \leq t \), \( P_i \) will invoke \texttt{Distribute} with \( m \) in iteration \( r + 1 \). By Lemma 7 all honest parties output \( m \). If the honest party \( P_i \) sets \( o_i = m \) in iteration \( r = t + 1 \), according to the protocol, \( P_i \) receives a \texttt{HAPPY} signed by \( t + 1 \) distinct parties. Since there are at most \( t \) Byzantine parties, there exists at least one honest party \( P_j \neq P_i \) that has signed \texttt{HAPPY} and invoked \texttt{Distribute} with \( o_j = m' \) in a previous iteration \( 1 \leq r' \leq t \). Then, by Lemma 7 all honest parties including \( P_i \) output \( m' \). Therefore, \( m' = m \), and all honest parties output \( m \). \qed

**Theorem 3.** Protocol Synchronous Crypto. \((1 - \varepsilon)n\text{-BB}\) satisfies Termination, Agreement and Validity. The protocol has communication complexity \( O(nl + B(k) + kn^2 + n^3) \).

Proof. Termination is clearly satisfied. By Lemma 9 agreement is satisfied. By Lemma 8 validity is satisfied.

Step 1 has cost \( B(k) \) for the \( k \)-bit BB oracle. The Distribution step has total communication cost \( O(nl + kn^2 + n^3) \), since each honest party executes the Distribution step at most once, where invoking \texttt{Distribute} has cost \( O(lkn) \), and sending the signed \texttt{HAPPY} message has cost \( O((k+n)n) \) where the \((k+n)\) term is due to the signature size and the list of signers in the multi-signature scheme. The Sharing step is also performed at most once for every honest party, and has total cost \( O(nl + kn^2) \) since each honest party in the Sharing step sends a message of size \( O(l/n + k) \) to all other parties. The Reconstruction step has no communication cost. Hence, the total communication complexity is \( O(nl + B(k) + kn^2 + n^3) \).

Optimality with the current best BB oracle. By Theorem 3 our Protocol Synchronous Crypto. \((1 - \varepsilon)n\text{-BB}\) has communication complexity \( O(nl + B(k) + kn^2 + n^3) \). From Section 2, the classic Dolev-Strong [14] protocol is the best solution for BB, with cost \( B(k) = \Theta((k+k_s)n^2 + n^3) \) for \( k \)-bit inputs where \( k_s \) is the signature size. Assuming \( \Theta(k_s) = \Theta(k) \), then our protocol is optimal with the current best BB oracle, since any extension protocol cannot have communication cost better than \( O(nl + kn^2 + n^3) \) unless BB protocol better than the Dolev-Strong protocol is proposed.

6 Error-free Extension Protocols under \( t < \frac{n}{3} \) Faults

In this section, we present improved error-free extension protocols for synchronous Byzantine agreement/broadcast. Error-free extension protocols for the asynchronous case are presented in Appendix C.2. We consider computationally unbounded adversary in this section. Without loss of generality, we assume the number of Byzantine parties is \( t = \lfloor \frac{n-1}{3} \rfloor \) in this section.

For error-free multi-valued synchronous Byzantine agreement, the state-of-art extension protocol has communication complexity \( O(nl + n^2B(1)) = O(nl + n^4) \) [27][18]. Here, we propose a protocol under the same setting with improved communication complexity \( O(nl + n^3 + nB(1)) = O(nl + n^3) \). For \( l \geq O(n^2) \), our protocol is optimal in communication complexity, and for \( l \leq O(n^3) \), our protocol has communication cost \( O(n^3) \), which is better than the previous work [27][18] by a factor of \( n \). For the synchronous case, Byzantine broadcast can be constructed from Byzantine agreement with the same asymptotic communication complexity, by first letting the sender send the message
1. Let $G = (P, E)$ be the input graph. Let $H = \overline{G} = (P, \overline{E})$ be the complementary graph of $G$. Find a maximum matching in $H$ using any deterministic algorithm such as [6]. Let $M$ be the matching and $N$ be the set of matched nodes. Let $\overline{N} = P \setminus N$.

2. Compute output as follows:
   2.1. Find the set $T = \{P_i \in \overline{N} \mid \exists P_j, P_k \text{ s.t. } (P_j, P_k) \in M \text{ and } (P_i, P_j), (P_i, P_k) \in \overline{E}\}$. Let $C = \overline{N} \setminus T$.
   2.2. Find the set $B \subseteq N$ of matched nodes that have neighbors in $C$ in $H$. That is, $B = \{P_j \in N \mid \exists P_k \in C \text{ s.t. } (P_j, P_k) \in \overline{E}\}$. Let $D = P \setminus B$.
   2.3. If $|C| \geq 2n - t$ and $|D| \geq n - t$, output $(C, D)$. Otherwise, output noSTAR.

Figure 5: Protocol STAR

to all parties and then perform a Byzantine agreement to reach agreement [23]. Therefore, we only present the Byzantine agreement protocol.

**Building Block: STAR protocol [4].** The following building block is adopted from [18]. A sub-protocol (Figure 5) called STAR [4] is used to find an $(n, t)$-star in a given undirected graph.

**Definition 3 ((n, t)-star [4]).** For a given undirected graph $G = (P, E)$, an $(n, t)$-star is a pair $(C, D)$ of sets with $C \subseteq D \subseteq P$ that satisfies

1. $|C| \geq n - 2t$, $|D| \geq n - t$

2. There exists an edge $(P_i, P_j) \in E$ for $\forall P_i \in C, P_j \in D$

Another tool that we will use is the Linear Error Correcting Code introduced in Section 4.1.

**Protocol Synchronous Error-free $n/3$-BA.** Now we present an improved extension protocol for error-free multi-valued synchronous Byzantine agreement when $t < n/3$, as presented in Figure 6.

The algorithm is inspired by the error-free protocol from [27, 18], and has a similar structure. Here we briefly describe each step of the protocol. Initially, each party divides and encodes its message into $n$ blocks via RS code, and sends blocks to the corresponding party. Then each party compares its block and the block received from others, and constructs a vector to record whether the corresponding blocks are identical. After all the parties exchange their vectors, each party $P_i$ constructs a graph $G_i$ from which a set $E_i$ of parties is derived. Then each party $P_i$ broadcasts whether it has successfully obtained $E_i$, and sends $E_i$ to all other parties. When there are enough parties successfully obtaining the set, each honest party can extract a correct piece of codewords $maj$ and send to others. From all the codewords received, each honest party is able to reconstruct the message and thus reach an agreement.

Comparing to the protocol in [27, 18], the novelty of our protocol is that, instead of every party $P_i$ broadcasting a length $n$ vector $v_i$ via Byzantine broadcast in step 2 which will result in communication complexity $O(n^2 B(1)) = O(n^4)$, we only let each party to send the vector to all other parties and thus reduce the cost to $O(n^3)$. As a result, different parties may receive different vectors from the Byzantine parties, which leads to different constructions of set $C_i, D_i, F_i, E_i$ at different party $P_i$ instead of the identical sets $C, D, F, E$ at all parties as in Protocol $n/3$-BA from [27, 18]. How to resolve such conflicting information and still obtain useful information to reconstruct an identical message at all honest parties is the main contribution of our protocol. The observation
Input of every party $p_i$: An $l$-bit message $m_i$

Primitive: Broadcast oracle for a single bit, $\text{STAR}$

Protocol for party $P_i$:

1. Divide the $l$-bit message $m_i$ into $t+1$ blocks, $m_{i0}, \ldots, m_{it}$, each has $l/(t+1)$ bits. Compute $(s_{i1}, \ldots, s_{in}) = \text{ENC}(m_{i0}, \ldots, m_{it})$. Send $s_{ii}$ to every party. Send $s_{ij}$ to $P_j$ for $j = 1, \ldots, n$.

2. Construct a binary vector $v_i$ of length $n$. Assign $v_i[j] = 1$, if $s_{ij} = s_{jj}$ and $s_{ii} = s_{ji}$ where $s_{jj}$ and $s_{ji}$ are received from $P_j$. Otherwise assign $v_i[j] = 0$. Send $v_i$ to every party.

3. Construct an undirected graph $G_i$ with parties in $P$ as vertices, and add an edge $(P_x, P_y)$ if $v_x[y] = v_y[x] = 1$. Invoke $\text{STAR}(G_i)$.

4. If $(C_i, D_i)$ is returned by $\text{STAR}$, find $F_i$ of size at least $2t + 1$ as the set of parties who have at least $t+1$ neighbours in $C_i$ in graph $G_i$, and find $E_i$ of size at least $2t + 1$ as the set of parties who have at least $2t + 1$ neighbours in $F_i$ in graph $G_i$. Any party $P_j$ is viewed as its neighbor for the purpose of finding $E_i$. Obtain the set $E_i$ as above if possible, otherwise let $E_i = \emptyset$.

5. Broadcast a single bit of 1 using the single-bit Byzantine broadcast primitive if $E_i \neq \emptyset$. Send $E_i$ as an $n$-bit vector to every party. Otherwise, broadcast a single bit of 0 using the single-bit Byzantine broadcast primitive.

6. After the above Byzantine broadcasts finish, perform the following.

   - If the above Byzantine broadcast delivers $\geq 2t + 1$ 1’s: let $E$ contain the corresponding set of $E$’s that are received by $P_i$. For each $E_x \in E$, let $\text{maj}_x$ be the value $s_{jj}$ received from the majority of the parties in $E_x$. That is, $\text{maj}_x$ satisfies that $|\{j \in E_x \mid s_{jj} = \text{maj}_x\}| \geq [(E_x + 1)/2]$. If such a majority does not exist, let $\text{maj}_x = \perp$. Find a subset $E' \subseteq E$, such that $|E'| \geq t + 1$ and for any $E_x, E_y \in E'$, $\text{maj}_x = \text{maj}_y \neq \perp$. Denote the above value as $\text{maj}$. Send the value $\text{maj}$ to every party.

   - If the above Byzantine broadcast delivers $< 2t + 1$ 1’s: agree on some predefined message $m'$ of length $l$ and abort.

7. Let $(\text{maj}_1, \ldots, \text{maj}_n)$ be the vector where $\text{maj}_j$ is received from $P_j$ in the above step. Apply $\text{DEC}$ on $(\text{maj}_1, \ldots, \text{maj}_n)$ with $c = t$ and $d = 0$. Let $m_0, m_1, \ldots, m_t$ be the data returned by $\text{DEC}$. Output $m = m_0 \cdots m_t$.

Figure 6: Protocol Synchronous Error-free $\frac{t}{3}$-BA

is that, as will be shown later in the proofs, honest parties are the majority in any set $E_i$ and all have the same message, which can be used to extract enough pieces of identical codewords for reconstructing the message. The above procedure is done in step 6 and 7.

**Lemma 10.** For any honest party $P_i$, if it obtains nonempty set $E_i$ in step 4 of the protocol, then the honest parties in $E_i$ hold the same message of length $l$.
Lemma 11. If all honest parties start with the same input \( m \), then every honest party \( P_i \) will obtain a set \( E_i \neq \emptyset \).

The proofs for the above lemmas are similar to those of Lemma 3.1 and 3.2 in [18], and are deferred to Appendix [2].

Lemma 12. All honest parties output the same message.

Proof. After the reliable broadcast of a single bit, all honest parties will deliver an identical set of \{0, 1\}. Therefore if one honest party delivers \(< 2t + 1 \) 1’s and outputs the predefined message \( m' \), all honest parties will output \( m' \).

Now consider the case where all honest parties deliver \( \geq 2t + 1 \) 1’s for the reliable broadcast of a single bit. This implies that at least \( t + 1 \) honest parties \( P_i \) send the set \( E_i \) to all parties. Denote the above set of honest parties as \( H \). By Lemma 10 we have all honest parties in \( E_i \) have the same message \( m \) of length \( l \). Also, for any two honest parties \( P_i, P_j \) above and their set \( E_i, E_j \), we know that \( E_i \cap E_j \) contains at least one honest party, since \(|E_i| \geq 2t + 1 \) and \(|E_j| \geq 2t + 1 \). Both facts above imply that all honest parties in \( E_i \cup E_j \) have the same message \( m \).

Since at least \( t + 1 \) honest parties send their \( E \) set to all parties, the algorithm can find the feasible set \( E' \) that contains all honest parties that are in \( H \). The conditions "\(|E'| \geq t + 1 \), for any \( E_x, E_y \in E' \), maj_x = maj_y \neq \bot" in step 6a can be satisfied: Honest parties are the majority in any \( E_x \in E' \) since \(|E_x| \geq 2t + 1 \), and they have the same message \( m \). Thus for any \( E_x \in E' \), maj_x is the same. Let \((s_1, ..., s_n) = ENC(m_0, m_1, ..., m_t) \) where \( m = m_0|m_1|...|m_t \). We know that \( maj_x = s_x \) for all \( E_x \in E' \). This implies that \( maj_j = s_j \) for all honest party \( P_j \) in step 7. Hence at least \( 2t + 1 \) values received by any honest party in step 7 are identical to the corresponding elements in \((s_1, ..., s_n)\). Since the codewords used in the protocol are \((n, t + 1)\) RS code which corrects at most \( t \) failures, after step 7, all honest parties will recover and output the same message \( m \).

Lemma 13. If all honest parties start with the same input \( m \), then all honest parties output \( m \).

Proof. When all honest parties start with the same input \( m \), by Lemma 11 every honest party \( P_i \) will obtain its set \( E_i \neq \emptyset \). Then, all honest parties will broadcast a single bit of 1, and deliver at least \( 2t + 1 \) 1’s for the broadcast. By the same proof of Lemma 12 all honest parties will output the same message \( m \).

Theorem 4. Protocol Synchronous Error-free \( \frac{2n}{3} \)-BA satisfies Termination, Agreement and Validity. The protocol has communication complexity \( O(nl + n^3 + nB(1)) \).

Proof. Termination is clearly satisfied. Agreement is proved by Lemma 12, and Validity is proved by Lemma 13.

Step 1 has communication cost \( O(n^2l/(t + 1)) = O(nl) \). Step 2 has communication cost \( O(n^3) \). Step 5 has communication cost \( O(n^3 + nB(1)) \). Step 6 has communication cost \( O(nl) \).

7 Extension Protocols Under Asynchrony

As mentioned, our results on synchronous extension protocols can be extended to the asynchronous case. The cryptographically secure extension protocols in Section 4 for BA and BB under \(< n/2 \) faults can be adapted to solve BA and reliable broadcast (RB) under \(< n/3 \) faults in the asynchronous setting. Note that under asynchronous, BA or RB is possible under \(< n/3 \) faults. No
extension protocol has been proposed for this case to the best of our knowledge. Our error-free extension protocols in Section 6 can also be adapted to solve BA and RB under asynchrony. Our results under asynchrony have been summarized in Table 2 and details are presented in Appendix C.

8 Conclusion

We investigate and propose several extension protocols with improved communication complexity for solving Byzantine broadcast and agreement under various settings. We propose simple yet efficient cryptographically secure extension protocols with improved communication complexity, for both synchronous Byzantine broadcast and agreement under $t < n/2$, and for synchronous Byzantine broadcast under $t < (1 - \varepsilon)n$ where $\varepsilon > 0$ is a constant. For $t < n/3$, we improve the communication complexity of existing error-free extension protocols for both Byzantine/reliable broadcast and Byzantine agreement under synchronous or asynchronous settings.

References

[1] Ittai Abraham, TH Hubert Chan, Danny Dolev, Kartik Nayak, Rafael Pass, Ling Ren, and Elaine Shi. Communication complexity of byzantine agreement, revisited. In 38th ACM Symposium on Principles of Distributed Computing, PODC 2019, pages 317–326. Association for Computing Machinery, 2019.

[2] Ittai Abraham, Dahlia Malkhi, and Alexander Spiegelman. Validated asynchronous byzantine agreement with optimal resilience and asymptotically optimal time and word communication. arXiv preprint arXiv:1811.01332, 2018.

[3] Niko Barić and Birgit Pfitzmann. Collision-free accumulators and fail-stop signature schemes without trees. In International Conference on the Theory and Applications of Cryptographic Techniques, pages 480–494. Springer, 1997.

[4] Michael Ben-Or, Ran Canetti, and Oded Goldreich. Asynchronous secure computation. In Proceedings of the twenty-fifth annual ACM symposium on Theory of computing, pages 52–61. ACM, 1993.

[5] Piotr Berman, Juan A Garay, and Kenneth J Perry. Bit optimal distributed consensus. In Computer science, pages 313–321. Springer, 1992.

[6] Norbert Blum. A new approach to maximum matching in general graphs. In International Colloquium on Automata, Languages, and Programming, pages 586–597. Springer, 1990.

[7] Dan Boneh, Craig Gentry, Ben Lynn, and Hovav Shacham. Aggregate and verifiably encrypted signatures from bilinear maps. In International Conference on the Theory and Applications of Cryptographic Techniques, pages 416–432. Springer, 2003.

[8] Gabriel Bracha. Asynchronous byzantine agreement protocols. Information and Computation, 75(2):130–143, 1987.
[9] Christian Cachin, Klaus Kursawe, Frank Petzold, and Victor Shoup. Secure and efficient asynchronous broadcast protocols. In Annual International Cryptology Conference, pages 524–541. Springer, 2001.

[10] Christian Cachin and Stefano Tessaro. Asynchronous verifiable information dispersal. In 24th IEEE Symposium on Reliable Distributed Systems (SRDS’05), pages 191–201. IEEE, 2005.

[11] Wutichai Chongchitmate and Rafail Ostrovsky. Information-theoretic broadcast with dishonest majority for long messages. In Theory of Cryptography Conference, pages 370–388. Springer, 2018.

[12] David Derler, Christian Hanser, and Daniel Slamanig. Revisiting cryptographic accumulators, additional properties and relations to other primitives. In Cryptographers’ Track at the RSA Conference, pages 127–144. Springer, 2015.

[13] Danny Dolev and Ruediger Reischuk. Bounds on information exchange for byzantine agreement. In Proceedings of the first ACM SIGACT-SIGOPS symposium on Principles of distributed computing, pages 132–140. ACM, 1982.

[14] Danny Dolev and H. Raymond Strong. Authenticated algorithms for byzantine agreement. SIAM Journal on Computing, 12(4):656–666, 1983.

[15] Michael J Fischer, Nancy A Lynch, and Michael S Paterson. Impossibility of distributed consensus with one faulty process. Journal of the ACM (JACM), 32(2):374–382, 1985.

[16] Matthias Fitzi and Martin Hirt. Optimally efficient multi-valued byzantine agreement. In Proceedings of the twenty-fifth annual ACM symposium on Principles of distributed computing, pages 163–168. ACM, 2006.

[17] Chaya Ganesh and Arpita Patra. Broadcast extensions with optimal communication and round complexity. In Proceedings of the 2016 ACM Symposium on Principles of Distributed Computing, pages 371–380. ACM, 2016.

[18] Chaya Ganesh and Arpita Patra. Optimal extension protocols for byzantine broadcast and agreement. 2017.

[19] Aniket Kate, Gregory M Zaverucha, and Ian Goldberg. Constant-size commitments to polynomials and their applications. In International Conference on the Theory and Application of Cryptology and Information Security, pages 177–194. Springer, 2010.

[20] Taechan Kim and Razvan Barbulescu. Extended tower number field sieve: A new complexity for the medium prime case. In Annual International Cryptology Conference, pages 543–571. Springer, 2016.

[21] Leslie Lamport, Robert Shostak, and Marshall Pease. The byzantine generals problem. ACM Transactions on Programming Languages and Systems, 4(3):382–401, 1982.

[22] Guanfeng Liang and Nitin Vaidya. Error-free multi-valued consensus with byzantine failures. In Proceedings of the 30th annual ACM SIGACT-SIGOPS symposium on Principles of distributed computing, pages 11–20. ACM, 2011.
[23] Nancy A Lynch. *Distributed algorithms*. Elsevier, 1996.

[24] Andrew Miller, Yu Xia, Kyle Croman, Elaine Shi, and Dawn Song. The honey badger of bft protocols. In *Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security*, pages 31–42. ACM, 2016.

[25] Achour Mostéfaoui, Hamouma Moumen, and Michel Raynal. Signature-free asynchronous binary byzantine consensus with $t < n/3$, $o(n^2)$ messages, and $o(1)$ expected time. *Journal of the ACM (JACM)*, 62(4):1–21, 2015.

[26] Lan Nguyen. Accumulators from bilinear pairings and applications. In *Cryptographers’ Track at the RSA Conference*, pages 275–292. Springer, 2005.

[27] Arpita Patra. Error-free multi-valued broadcast and byzantine agreement with optimal communication complexity. In *International Conference On Principles Of Distributed Systems*, pages 34–49. Springer, 2011.

[28] Birgit Pfitzmann and Michael Waidner. *Information-theoretic pseudosignatures and byzantine agreement for $t \geq n/3$.*

[29] Irving S Reed and Gustave Solomon. Polynomial codes over certain finite fields. *Journal of the society for industrial and applied mathematics*, 8(2):300–304, 1960.

[30] Andrew C Yao. Protocols for secure computations. In *Proceedings of the 23rd Annual Symposium on Foundations of Computer Science*, pages 160–164. IEEE Computer Society, 1982.

[31] Andrew Chi-Chih Yao. Some complexity questions related to distributive computing (preliminary report). In *Proceedings of the eleventh annual ACM symposium on Theory of computing*, pages 209–213. ACM, 1979.
A An Implementation of Accumulator: the Bilinear Accumulator

To satisfy our assumption in Section 1 on the security parameters being the same order, we can choose an accumulator implementation called the bilinear accumulator [26]. The bilinear accumulator is collision-free under the Strong Diffie-Hellman assumption (q-SDH) [25, 19].

**Bilinear Pairing.** Let $G_1, G_2$ be two cyclic multiplicative groups of prime order $p$. Let $g_1, g_2$ be the corresponding generator, and there exists an isomorphism $\phi : G_2 \rightarrow G_1$ such that $\phi(g_2) = g_1$. Let $G_M$ also be a cyclic multiplicative group of prime order $p$, and $e : G_1 \times G_2 \rightarrow G_M$ is a bilinear pairing if satisfies the following properties:

1. Bilinearity: $e(P^a, Q^b) = e(P, Q)^{ab}$ for all $P \in G_1$, $Q \in G_2$ and $a, b \in \mathbb{Z}_p$;
2. Non-degeneracy: $e(g_1, g_2) \neq 1$;
3. Computability: There is an efficient algorithm to compute $e(P, Q)$ for all $P \in G_1$ and $Q \in G_2$.

**Accumulator Construction.** For accumulator construction, we can have $G_1 = G_2 = G$ and $g_1 = g_2 = g$. The bilinear accumulator works for elements in $\mathbb{Z}_p^*$ and the accumulation value is an element in $G$. Therefore, we assume a function $f : U \rightarrow \mathbb{Z}_p^*$ that maps any value in the input domain $U$ to an value in $\mathbb{Z}_p^*$. Let $\mathcal{D} = \{d_1, ..., d_n\}$ be a set of $n$ values in $\mathbb{Z}_p^*$ after applying the function $f$. Let $s$ denote a trapdoor that is hidden from all parties participating in the extension protocol. Before the extension protocol starts, all parties obtain a set of q-SDH public parameters $(g^a, g^{a^2}, ..., g^{a^n})$, either via the trusted setup or MPC protocols [26]. Let $C_{\mathcal{D}}(x) = (x + d_1)(x + d_2)\cdot \cdot \cdot (x + d_n)$ denote the characteristic polynomial of $\mathcal{D}$ with coefficients $c_0, c_1, ..., c_n$, so that $C_{\mathcal{D}}(x) = (x + d_1)(x + d_2)\cdot \cdot \cdot (x + d_n) = c_0 + c_1x + ... + c_nx^n$. Let $q_i(x) = \frac{C_{\mathcal{D}}(x)}{x + d_i} = \prod_{j \neq i}(x + d_j) = c_{0}^{(i)} + c_{1}^{(i)}x + ... + c_{n}^{(i)}x^{n-1}$ denote the quotient polynomial.

- **Gen**$(1^k, n)$ returns a uniformly random tuple $ak = (p, G, G_M, e, g)$ of bilinear pairings parameters, where $p$ is of size $k$.
- **Eval**$(ak, \mathcal{D})$ computes and returns the accumulation value as $z = g^{C_{\mathcal{D}}(s)} = g^{c_0 + c_1s + ... + c_ns^n} = g^{c_0(g^a)^{c_1}\cdot \cdot \cdot (g^{a^n})^{c_n}}$.
- **CreateWit**$(ak, z, d_i)$ computes and returns the witness $w_i$ as $w_i = \frac{C_{\mathcal{D}}(x)}{x + d_i} = g^{c_0^{(i)} + c_1^{(i)}x + ... + c_{n-1}^{(i)}x^{n-1}}$ if $d_i \in \mathcal{D}$, and $w_i = \bot$ if $d_i \notin \mathcal{D}$.
- **Verify**$(ak, z, w_i, d_i)$ tests whether $e(g^{d_i} \cdot g^a, w_i) = e(z, g)$ where $e$ is the bilinear pairing, and return the result.

Since $p$ is of size $k$, and the accumulation value $z$ or any witness $w_i$ is an element in the group $\mathbb{Z}_p$, they all have size $k$ bits. It is believed that this gives $\Theta(k)$ bits of security.

B Omitted Proofs

**Proof of Lemma 10.** We show the following properties are satisfied if $P_i$ is able to obtain its $E_i$ in step 3.
• The honest parties in \( C_i \) hold the same message of length \( l \).

By the definition of \( (n,t) \)-star, \( |D_i| \geq n-t \) and any two parties \( P_j \in C_i, P_k \in D_i \) are connected by an edge. This implies that \( D_i \) contains at least \( n-2t \geq t+1 \) honest parties \( \{P_{i_1}, P_{i_2}, ..., P_{i_q}\} \), and every honest party \( P_j \in C_i \) connects to all those honest parties. Then for any honest party \( P_j \in C_i \), by definition we have \( s_{jix} = s_{iix} \) for all \( x = 1, 2, ..., q \) where \( q \geq t+1 \). This means that the codewords of each honest party in \( C \) have at least \( t+1 \) elements in common. Since the codewords used in the protocol are \( (n,t+1) \) RS code, all honest parties in \( C_i \) hold the same message of length \( l \). Let the common message be \( m \), and let \( (s_1, ..., s_n) = \text{ENC}(m_0, m_1, ..., m_t) \) where \( m = m_0|m_1|...|m_t \).

• Every honest party \( P_j \in F_i \) holds \( s_j \).

Recall every honest party \( P_j \in F_i \) has at least \( t+1 \) neighbors in \( C_i \), and therefore at least \( 1 \) honest neighbor \( P_k \) in \( C_i \). Since \( P_j, P_k \) are neighbors, \( s_{jj} \) of \( P_j \) equals \( s_{kj} \) of \( P_k \). Since \( P_k \) holds message \( m \), \( s_{kj} = s_j \), which implies that \( P_j \) holds \( s_j \).

• The honest parties in \( E_i \) hold the same message of length \( l \).

Recall every honest party \( P_j \in E \) has at least \( 2t+1 \) neighbors in \( F \), and therefore at least \( t+1 \) honest neighbors \( \{P_{i_1}, P_{i_2}, ..., P_{i_q}\} \) in \( F \) where \( q \geq t+1 \). Since \( P_i \) and \( P_x \) are connected, \( s_{ix} = s_{ix} \) for \( 1 \leq x \leq q \). Recall that \( s_{ix} = s_{ix} \). Therefore the codewords of \( P_i \) has at least \( t+1 \) elements identical to the elements of \( (s_1, ..., s_n) \), where \( (s_1, ..., s_n) = \text{ENC}(m_0, m_1, ..., m_t) \) where \( m = m_0|m_1|...|m_t \). Since the codewords used in the protocol are \( (n,t+1) \) RS code, all honest parties in \( E_i \) hold the same message of length \( l \).

Proof of Lemma 6.2 When all honest parties have the same input \( m \), they will generate the same codewords. This implies all honest parties will connect to each other in \( G_i \), which forms a clique of size \( 2t+1 \).

First we show that \( C_i \) contains at least \( t+1 \) honest parties. Recall that in the protocol \( \text{STAR} \), \( C_i = (\mathcal{P} \setminus N) \setminus T \), where \( N \) is the set of matched parties in the complementary graph of \( G \), and \( T \) is the set of parties that connects to both endpoints of a matching. For any honest party \( P_j \in N \) such that \( (P_j, P_k) \in M \), it is ensured that \( P_k \) is Byzantine since \( P_j, P_k \) have conflicting messages. Similarly, for any honest party \( P_j \in T \) such that \( (P_x, P_y) \in M, (P_j, P_x) \in \overline{F} \) and \( (P_j, P_y) \in \overline{E} \), both \( P_x \) and \( P_y \) are ensured to be Byzantine since they have conflicting messages with \( P_j \). Therefore, for any honest party excluded from \( C_i = (\mathcal{P} \setminus N) \setminus T \), there exists at least one corresponding Byzantine party excluded from \( C_i \) as well. Since there are at most \( t \) Byzantine parties, at most \( t \) honest parties are excluded from \( C_i \). Hence \( C_i \) contains at least \( t+1 \) honest parties. Since \( C_i \) contains at least \( t+1 \) honest parties, \( F_i, E_i \) will subsequently contain all honest parties and have size \( \geq 2t+1 \).

C Asynchronous Extension Protocols

In this section, we present extended results for the asynchronous settings, which are summarized earlier in Table 2 in Section 7. We present asynchronous extension protocols for reliable broadcast and Byzantine agreement, for both cryptographically secure case (Appendix C.1) and error-free case (Appendix C.2). The definition of asynchronous reliable broadcast is the following.

Definition 4 (Asynchronous Reliable Broadcast). A protocol for a set of parties \( \mathcal{P} = \{P_1, ..., P_n\} \), where a distinguished party called the sender \( P_s \in \mathcal{P} \) holds an initial \( l \)-bit input \( m \), is an asynchronous broadcast protocol tolerating an adversary \( A \), if the following properties hold
• **Termination.** If the sender is honest, then every honest party eventually outputs a message. Otherwise, if some honest party outputs a message, then every honest party eventually outputs a message.

• **Agreement.** If some honest party outputs a message \( m' \), then every honest party eventually outputs \( m' \).

• **Validity.** If the sender is honest, all honest parties eventually output the message \( m \).

### C.1 Asynchronous Cryptographically Secure Extension Protocols

Without loss of generality, we assume the number of Byzantine parties is \( t = \lfloor \frac{n-1}{3} \rfloor \) in this section. Let \( b = n - t \).

**Input of the sender \( P_s \):** An \( l \)-bit message \( m_s \)

**Primitive:** asynchronous Byzantine agreement oracle, asynchronous reliable broadcast oracle, cryptographic accumulator with \( \text{Eval}, \text{CreateWit}, \text{Verify} \)

**Protocol for party \( P_i \):**

1. If \( i = s \), perform the following. Compute \( D_s = (\langle 1, s_1 \rangle, ..., \langle n, s_n \rangle) = \text{Encode}(m_s, b) \). Compute the accumulation value \( z_s = \text{Eval}(ak, D_s) \). Send \( m_s \) to every party, and broadcast \( z_s \) by invoking a \( k \)-bit asynchronous reliable broadcast oracle.

2. When receiving the message \( m \) from the sender, and the reliable broadcast above outputs \( z \), perform the following. Compute \( D_i = (\langle 1, s_1 \rangle, ..., \langle n, s_n \rangle) = \text{Encode}(m, b) \). Compute the accumulation value and auxiliary information \( z_i = \text{Eval}(ak, D_i) \). If \( z_i = z \), set \( \text{happy}_i = 1 \), otherwise set \( \text{happy}_i = 0 \).

3. If \( \text{happy}_i = 1 \), invoke \( \text{Distribute}(D_i, ak, z) \).

4. Wait for the first valid \( (s_i, w_i) \) pair such that \( \text{Verify}(ak, z, w_i, \langle i, s_i \rangle) = \text{true} \), then send \( (s_i, w_i) \) to all other parties.

5. If \( \text{happy}_i = 1 \), set \( o_i = m_i \). Otherwise, perform the following. Wait for at least \( n - t \) valid pairs \( \{(s_j, w_j)\} \) such that \( \text{Verify}(ak, z, w_j, \langle j, s_j \rangle) = \text{true} \) for each \( (s_j, w_j) \) from \( P_j \). Let \( S_i = \{(1, s_1), w_1), ..., (n, s_n), w_n)\} \), where \( (s_j, w_j) \) is the pair received from party \( P_j \). Compute \( o_i = \text{Reconstruct}(S_i, ak, z, t) \). Compute \( D'_i = \text{Encode}(o_i, b) \), and invoke \( \text{Distribute}(D'_i, ak, z) \).

6. Output \( o_i \).

**Figure 7:** Protocol Asynchronous Crypto. \( \frac{2}{3} \)-RB

**Lemma 14.** If an honest party \( P_i \) invokes \( \text{Distribute} \) with \( D_i = \text{Encode}(m, b) \), then any honest party \( P_j \) eventually output \( o_j = m \).

**Proof.** By the agreement condition of asynchronous reliable broadcast used in step 1, if any honest party obtains \( z \), then any honest party also eventually obtains \( z \). Then at step 2, by Lemma 2 any
Input of every party $P_i$: An $l$-bit message $m_i$

Primitives: asynchronous Byzantine agreement oracle, cryptographic accumulator with $\text{Eval}$, $\text{CreateWit}$, $\text{Verify}$

Protocol for party $P_i$:

1. Compute $D_i = (\langle 1, s_1 \rangle, ..., \langle n, s_n \rangle) = \text{Encode}(m_i, b)$. Compute the accumulation value $z_i = \text{Eval}(ak, D_i)$. Input $z_i$ to an instance of $k$-bit asynchronous Byzantine agreement oracle.

2. When the above ABA outputs $z$, if $z = z_i$, set $\text{happy}_i = 1$, otherwise set $\text{happy}_i = 0$. Input $\text{happy}_i$ to an instance of 1-bit asynchronous Byzantine agreement oracle.

3. • If the above ABA outputs 0, output $o_i = \perp$ and abort.
   • If the above ABA outputs 1 and $\text{happy}_i = 1$, invoke $\text{Distribute}(D_i, ak, z)$.

4. Wait for a valid $(s_i, w_i)$ pair such that $\text{Verify}(ak, z, w_i, \langle i, s_i \rangle) = \text{true}$, then send $(s_i, w_i)$ to all other parties.

5. If $\text{happy}_i = 1$, set $o_i = m_i$. Otherwise, perform the following. Wait for at least $n - t$ valid pairs $\{(s_j, w_j)\}$ from the previous step that satisfies $\text{Verify}(ak, z, w_j, \langle j, s_j \rangle) = \text{true}$. Let $S_i = (\{(1, s_1), w_1\}, ..., (\langle n, s_n \rangle, w_n))$, where $(s_j, w_j)$ is the pair received from party $P_j$. Compute $o_i = \text{Reconstruct}(S_i, ak, z, t)$.

6. Output $o_i$.

Figure 8: Protocol Asynchronous Crypto. $\frac{n}{3}$-BA

honest party $P_j$ with $\text{happy}_j = 1$ has the identical message $m$ corresponding to $z$, and sets $o_j = m$ at step 5. For other honest parties, the honest party $P_i$ with $\text{happy}_i = 1$ invokes $\text{Distribute}$ to compute witness $w_j$ for each index value $\langle j, s_j \rangle$, and sends the valid $(s_j, w_j)$ pair computed from message $m$ to party $P_j$ for every $P_j$. By Lemma 1, the Byzantine parties cannot generate a different pair $(s'_j, w'_j)$ that can be verified. Therefore, in step 4, every honest party $P_j$ eventually receives at least one valid $(s_j, w_j)$ pair, and forwards it to all other parties. Since there are at least $n - t$ honest parties, in step 5, each honest party will eventually receive at least $n - t$ valid coded values. In $\text{Reconstruct}$, using the accumulation value associated with the coded value, any party $P_j$ can detect the corrupted values and remove them. By the property of RS codes, since $n - b = t \geq 2c + d = t$, any honest party $P_j$ with $\text{happy}_j = 0$ is able to recover the message $m$, and any honest party $P_j$ with $\text{happy}_j = 1$ already has the message $m$. Therefore all honest parties output $m$.

Lemma 15. If the sender is honest and has input $m_s$, all honest parties eventually output the same message $m_s$.

Proof. If the sender is honest, every honest party eventually receive $m_s$, and the asynchronous reliable broadcast eventually outputs the corresponding accumulation value $z_s$ according to the termination condition of asynchronous reliable broadcast. Then in step 2, any honest party $P_i$ computes a matching accumulation value $z_i = z_s$ with $m_s$, and sets $\text{happy}_i = 1$. Then any honest
party $P_i$ sets $o_i = m$ in step 5 since $\text{happy}_i = 1$. Hence, all honest parties output $m$ when the protocol terminates.

**Lemma 16.** If some honest party outputs a message $m$, then every honest party eventually outputs $m$.

**Proof.** Suppose any honest party $P_i$ outputs $o_i = m$.

If $P_i$ has $\text{happy}_i = 1$ at step 5, it invokes Distribute at step 3. Then by Lemma 14, all honest parties eventually output $m$. If $P_i$ has $\text{happy}_i = 0$ at step 5, then it reconstructs the message $m$ from Reconstruct, and invokes Distribute. Then by Lemma 14, all honest parties eventually output $m$.

**Theorem 5.** Protocol Asynchronous Crypto. $\frac{n}{3}$-RB satisfies Termination, Agreement and Validity. The protocol has communication complexity $O(nl + \mathcal{B}(k) + kn^2)$.

**Proof.** Termination is proved by Lemma 15 and 16. Agreement is proved by Lemma 16. Validity is proved by Lemma 15.

Step 1 has cost $O(nl + \mathcal{B}(k))$, where $k$ is the size of the cryptographic accumulator. Step 3 and 5 in total have cost $O(nl + kn^2)$, since each honest party invokes at most one instance of Distribute, which leads to an all-to-all communication with each message of size $O(l/b + k) = O(l/n + k)$. For step 4, it also has cost $O(nl + kn^2)$. Hence the total cost is $O(nl + \mathcal{B}(k) + kn^2)$.

**Theorem 6.** Protocol Asynchronous Crypto. $\frac{n}{3}$-BA satisfies Termination, Agreement and Validity, and has communication complexity $O(nl + \mathcal{A}(k) + kn^2)$.

**Proof.** The proof is analogous to that of Theorem 1.

**C.2 Asynchronous Error-free Extension Protocols**

We can extend Protocol Synchronous Error-free $\frac{n}{3}$-BA from Section 6 to an asynchronous reliable broadcast, as presented in Figure 9. For brevity, we only present the difference. Since the system is asynchronous and the channel is reliable, each party can only expect to receive the messages from honest parties eventually. Thus, instead of receiving all vectors and then constructing the sets as in the step 3, 4 of the synchronous protocol, each party can only try to construct the sets every time a new message is received as in step 3, 4. Similar to the synchronous protocol, the parties can obtain enough pieces of correct codewords to reconstruct the message as in step 6 and 7.

**Lemma 17.** For any honest party $P_i$, if it obtains nonempty set $E_i$ in step 4 of the protocol, then the honest parties in $E_i$ hold the same message of length $l$.

**Proof.** Same proof of Lemma 10 applies.

**Lemma 18.** If the sender is honest, then every honest party $P_i$ will eventually obtain a set $E_i \neq \emptyset$.

**Proof.** The proof of this Lemma is similar to that of Lemma 11. If the sender is honest, all honest parties eventually receive the same message $m$ in step 1, and they will generate the same codewords. This implies all honest parties eventually will be connected to each other in $G_i$, which forms a clique of size $\geq 2t + 1$. By the same argument from the proof of Lemma 11, $C_i$ contains at least $t + 1$ honest parties. Since $C_i$ contains at least $t + 1$ honest parties and honest parties will eventually form a clique of size $\geq 2t + 1$ in $G_i$, $F_i, E_i$ will subsequently contain all honest parties and have size $\geq 2t + 1$.
Input of the sender $P_i$: An $l$-bit message $m_s$

Primitive: Asynchronous reliable broadcast oracle for a single bit, STAR

Protocol for party $P_i$:

1. If $i = s$, send $m_s$ to every party. Wait until receiving the message $m_i$ from the sender.

2. Same as step 1 of Protocol Synchronous Error-free $\frac{n}{2}$-BA.

3. When receiving $s_{jj}$ and $s_{ji}$ from $P_j$, send $0K(P_i, P_j)$ to every party if $s_{ij} = s_{jj}$. Construct an undirected graph $G_i$ with parties in $P$ as vertices. Add an edge $(P_x, P_y)$ every time when $0K(P_x, P_y)$ is received from $P_x$ and $0K(P_y, P_x)$ is received from $P_y$. If the edge $(P_x, P_y)$ is new, invoke STAR($G_i$).

4. Same as step 4 of Protocol Synchronous Error-free $\frac{n}{2}$-BA.

5. If the set $E_i$ is obtained for the first time, broadcast a single bit of 1 using the single-bit Byzantine broadcast primitive and send $E_i$ to every party. Stop updating $G_i$.

6. When the above reliable broadcast delivers $\geq 2t + 1$ 1’s, perform the step 6(a) of Protocol Synchronous Error-free $\frac{n}{2}$-BA.

7. On receiving $2t + 1 + r$ values $m_{ajj}$’s where $m_{ajj}$ is sent by $P_j$, apply $\text{DEC}$ with $c = r$ and $d = t - r$. If $\text{DEC}$ returns ‘failure’, wait for more values. If $\text{DEC}$ returns the data $m_0, m_1, \cdots, m_t$, output $m = m_0 | \cdots | m_t$.

Figure 9: Protocol Asynchronous Error-free $\frac{n}{2}$-RB

Lemma 19. If some honest party outputs a message $m'$, then every honest party eventually outputs $m'$.

Proof. If some honest party outputs a message $m'$, then the reliable broadcast in step 6 delivers $\geq 2t + 1$ 1’s. By the definition of reliable broadcast, eventually all honest parties will deliver an identical set of $\{0, 1\}$. Therefore eventually, every honest party will deliver $\geq 2t + 1$ 1’s.

By the same argument from the proof of Lemma 12, at least $t + 1$ honest parties $P_j$ send the set $E_i$ to all parties, and for any honest parties $P_i, P_j$ above, all honest parties in $E_i \cup E_j$ have the same message $m$. Then the conditions for the set $E'$ in step 6 is satisfied, which leads to identical $maj_x$. Let $(s_1, \ldots, s_n) = \text{ENC}(m_0, m_1, \ldots, m_t)$ where $m = m_0 | m_1 | \cdots | m_t$. We know that $maj_x = s_x$ for all $E_x \in E'$. This implies that $maj_j = s_j$ for all honest party $P_j$ in step 7. On receiving $2t + 1 + r$ values where $0 \leq r \leq t$, party $P_j$ try to decode the message with $c = r$ and $d = t - r \geq 0$ which satisfies that $2t + 1 + r - (t + 1) \geq 2c + d$. If there are more than $r$ corrupted values, $\text{DEC}$ will return ‘failure’, and wait for more values. Eventually every honest party $P_i$ will receive enough values to successfully decode $m$. Since one honest party outputs the message $m'$, we have $m = m'$.

Lemma 20. If the sender is honest, all honest parties eventually output the message $m$.

Proof. If the sender is honest, by Lemma 18, every honest party $P_i$ will eventually obtain its set $E_i \neq \emptyset$. Then, all honest parties will broadcast a single bit of 1, and eventually deliver at least $2t + 1$ 1’s for the broadcast. Then by the same proof of Lemma 19, all honest parties will output the same message $m$.
Theorem 7. Protocol Asynchronous Error-free $\frac{n}{3}$-RB satisfies Termination, Agreement and Validity. The protocol has communication complexity $O(nl + n^3 \log n + nB(1))$.

Proof. Termination is proven by Lemma 19 and 20. Agreement is proved by Lemma 19, and Validity is proved by Lemma 20.

Step 1 and 2 has communication cost $O(nl + n^2l / (t + 1)) = O(nl)$. Step 3 has communication cost $O(n^3 \log n)$. Step 5 has communication cost $O(n^3 + nB(1))$. Step 6 has communication cost $O(nl)$.

Error-free extension protocol for asynchronous Byzantine agreements under $t < n/3$. From the improved asynchronous error-free reliable broadcast protocol, we can obtain a better asynchronous error-free Byzantine agreement (ABA) protocol, directly via the same construction from [27]. The new protocol has communication complexity $O(nl + n^4 \log n + n^2B(1) + nA(1))$.

D A Note on Prior Results

The $(\frac{n}{2})$-BA protocol in [18] has a small flaw that the adversary can exploit to increase the communication complexity to $\Omega(n^2l)$. In this section, we will describe the issue and provide a simple fix.

For completeness, we provide the original protocol $(\frac{n}{2})$-BA in Figure 3 of [18]. From the protocol, the step 6 asks any party to send a $O(l + 1 + nk)$-bit message to some party $P_j \in P \setminus P_{sm}$, if $d_j$ is received from $P_j$. Since $d_j = \lceil(|P_{hmsm}| + 1)/2 \rceil$, $|P_{hmsm}| \geq n - 2t$ and $n \geq 2t + 1$, it is possible that $d_j = 1$. Therefore $t$ Byzantine parties can send to all other parties the message $d_j = 1$ in step 5, which will trigger all other parties to send back messages each of length $O(l + 1 + nk) = O(l + nk)$ bits. Therefore, step 6 will have communication complexity $O(nt(l + nk)) = O(n^2l + n^3k)$ instead of $O(nl + n^3k)$ as claimed in [18].

Here we provide a simple fix to resolve the issue above. Basically, we cannot allow Byzantine parties to deceive all other parties by requiring a large block of $O(l)$-bits. Then the key step of our fix is to change the “Send happy to all parties in $P$” in step 4 to “Broadcast happy using the single-bit broadcast oracle”. After the broadcast, in step 5, any honest party $P_i$ can construct identical sets $P_{conflict}^i = P_{conflict}$ and $P_{hmsm}^i = P_{hmsm}$, and compute an identical value $d_i = \lceil(|P_{hmsm}| + 1)/2 \rceil$. Then in step 6, any honest party $P_i$ in $P_{hmsm}$ will perform the same encoding, and send $(y_i, H_i)$ to all parties in $P \setminus P_{sm}$. Rest of the steps remain the same.

To see the communication complexity is correct after the fix, notice that only $|P_{hmsm}|$ honest parties will send a $O(l + 1 + nk)$-bit message to each party in $P \setminus P_{sm}$. Since $d = \lceil(|P_{hmsm}| + 1)/2 \rceil$, we have the communication complexity of step 6 equals $O(|P \setminus P_{sm}| \cdot |P_{hmsm}|(l + 1 + nk)) = O(nl + n^3k)$. Also, the broadcast in step 4 incurs $O(nB(1))$ cost, and rest of the protocol has the same cost, thus in total $O(nl + n^3k + nkB(1))$. 

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Input of every party $P_i$: An $l$-bit message $m_i$

Primitive: Byzantine broadcast oracle for a single bit, cryptographic collision-resistant hash function $\text{Hash}$

Checking Phase. Every party $P_i$ does the following:

1. Compute a hash of the message $m_i$ as $h_i = \text{Hash}(m_i)$ and broadcast $h_i$.
2. Check if at least $n - t$ broadcasted hashes are equal. If no $n - t$ broadcasted hashes are equal, output $o_i = \bot$ and terminate. Otherwise, let $h$ denote the common hash value broadcasted by at least $n - t$ parties. Then form $P_{sm}$ as teh set of parties broadcasting $h$.

Agreement Phase. Every party $P_i$ does the following:

1. If $P_i \in P_{sm}$, set output message $o_i = m_i$.
2. Form an injective function from $P \setminus P_{sm}$ to $P_{sm}$, by say, mapping the party with the smallest index in $P \setminus P_{sm}$ to the party with the smallest index in $P_{sm}$, i.e., $\phi : P \setminus P_{sm} \to P_{sm}$.
3. If $P_i \in P_{sm}$ and $P_i = \phi(P_j)$, then send $o_j$ to $P_j$.
4. If $P_i \in P \setminus P_{sm}$ and received a value say, $o'_j$ from $P_j \in P_{sm}$ in the previous round such that $P_j = \phi(P_i)$, then check if $\text{Hash}(o'_j) = h$. If the test passes, set $\text{happy}_i = 1$ and assign output message $o_i = o'_j$, else set $\text{happy}_j = 0$. Send $\text{happy}_j$ to all parties in $P$.
5. If $P_i \in P \setminus P_{sm}$ and $\text{happy}_i = 0$, then construct a set $P_{\text{conflict}}^i$ consisting of the parties $P_j, \phi(P_j)$ such that $\text{happy}_j$ received from $P_j$ in the previous step is 0 and $P_j$ belongs to $P \setminus P_{sm}$. Set $P_{\text{hmsi}}^i = P \setminus P_{\text{conflict}}^i$, $d_i = \lceil(|P_{\text{hmsi}}^i| + 1)/2 \rceil$ and send $d_i$ to all the parties belonging to $P_{\text{hmsi}}^i$ and nothing to all the others.
6. If $d_j$ is received from $P_j \in P \setminus P_{sm}$,
   
   6.1. Transform the message $o_i$ into a polynomial over $GF(2^c)$, for $c = \lceil l + 1/d_j \rceil$ denoted by $f_i$ with degree $d_j - 1$.
   
   6.2. Compute the $c$-bit piece $y_i = f_i(i)$, $H_i = (\text{Hash}(f_i(1)), \cdots, \text{Hash}(f_i(n)))$ and sends $(y_i, H_i)$ to $P_j$.

7. If $P_i \in P \setminus P_{sm}$ and $\text{happy}_i = 0$, check each piece $y_i$ received from each $P_j \in P_{\text{hmsi}}^i$ against the $j$th entry of every hash value vector $H_k$ received from $P_k \in P_{\text{hmsi}}^i$. If at least $d_i$ of the hash values match a piece $y_i$, then accept $y_i$, otherwise reject it. Interpolate the polynomial $f$ from the $d_i$ accepted pieces $y_i$, and compute the message $m$ corresponding to the polynomial $f$. Set $o_i = m$.

8. Output $o_i$ and terminate.

Figure 10: Protocol Synchronous Crypto. $\frac{n}{2}$-BA from [18]