A new proof of subcritical Trudinger-Moser inequalities on the whole Euclidean space

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Abstract
In this note, we give a new proof of subcritical Trudinger-Moser inequality on $\mathbb{R}^n$. All the existing proofs on this inequality are based on the rearrangement argument with respect to functions in the Sobolev space $W^{1,n}$. Our method avoids this technique and thus can be used in the Riemannian manifold case and in the entire Heisenberg group.

Key words: Trudinger-Moser inequality, Adams inequality
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1. Introduction
It was proved by Cao [4], Panda [9] and do´O [5] that

**Theorem A** Let $\alpha_n = n\omega_{n-1}^{1/n}$, where $\omega_{n-1}$ is the measure of the unit sphere in $\mathbb{R}^n$. Then for any $\alpha < \alpha_n$ there holds

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{W^{1,n}} \leq 1} \int_{\mathbb{R}^n} \left( e^{\alpha |u|^n} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^k}{k!} \right) dx < \infty. \quad (1.1)$$

This result has various extensions, among which we mention Adachi and Tanaka [1], Ruf [11], Li-Ruf [7], Adimurthi-Yang [3]. To the authors’ knowledge, all the existing proofs of such an inequality are based on rearrangement argument with respect to functions in the Sobolev space $W^{1,n}(\mathbb{R}^n)$. The purpose of this short note is to provide a new method to reprove Theorem A. Namely, we use a technique of the analogy of unity decomposition. More precisely, for any $u \in W^{1,n}(\mathbb{R}^n)$, we first take a cut-off function $\phi_i \in C^\infty_0(B_{R}(x_i))$ such that $0 \leq \phi_i \leq 1$ on $B_{R}(x_i)$, $\phi_i \equiv 1$ on $B_{R/2}(x_i)$. Then, using the usual Trudinger-Moser inequality [8, 10, 13] for bounded domain, we prove a key estimate

$$\int_{\mathbb{R}^n} \left( e^{\alpha |\phi_i u|^n} - \sum_{k=0}^{n-2} \frac{\alpha^k |\phi_i u|^k}{k!} \right) dx \leq \int_{\mathbb{R}^n} |\nabla (\phi_i u)|^n dx \quad (1.2)$$

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under the condition that

$$\int_{\mathbb{R}^n} |\nabla (\phi_i u)|^p dx \leq 1.$$  

The power of (1.2) is evident. It permits us to approximate $u$ by $\sum \phi_i u$, where every $\phi_i$ is supported in $B_R(x_i)$, $\mathbb{R}^n = \cup_{i=1}^\infty B_{R/2}(x_i)$, and any fixed $x \in \mathbb{R}^n$ belongs to at most $c(n)$ balls $B_R(x_i)$ for some universal constant $c(n)$. If we further take $\phi_i$ such that $|\nabla \phi_i| \leq 4/R$. Note that for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$\int_{\mathbb{R}^n} |\nabla (\phi_i u)|^p dx \leq (1 + \epsilon) \int_{\mathbb{R}^n} |\nabla u|^p dx + \frac{C(\epsilon)}{R^n} \int_{\mathbb{R}^n} |u|^n dx.$$  

Selecting $\epsilon > 0$ sufficiently small and $R > 0$ sufficiently large, we get the desired result.

Similar idea was used by the first named author to deal with similar problems on complete Riemannian manifolds [14] or the entire Heisenberg group [16]. Note that due to the complicated geometric structure, we have not obtained Theorem A on manifolds, but a weaker result. Namely

**Theorem B** Let $(M, g)$ be a complete noncompact Riemannian $n$-manifold. Suppose that its Ricci curvature has lower bound, namely $\text{Rc}_{(M, g)} \geq Kg$ for some constant $K \in \mathbb{R}$, and its injectivity radius is strictly positive, namely $\text{inj}(M, g) \geq i_0$ for some constant $i_0 > 0$. Then we have

(i) for any $0 \leq \alpha < \alpha_0$ there exists positive constants $\tau$ and $\beta$ depending only on $n$, $\alpha$, $K$ and $i_0$ such that

$$\sup_{u \in W^{1, n}(M), ||u||_1 \leq 1} \int_M \left( e^{\|u\|^\alpha} - \sum_{k=0}^{n-2} \frac{\alpha^k ||u||^\alpha}{k!} \right) dv \leq \beta,$$  

(1.3)

where

$$||u||_{1, \tau} = \left( \int_M |\nabla u|^p dv \right)^{1/p} + \tau \left( \int_M |u|^n dv \right)^{1/n}.$$  

(1.4)

As a consequence, $W^{1, n}(M)$ is embedded in $L^q(M)$ continuously for all $q \geq n$;

(ii) for any $\alpha > \alpha_0$ and any $\tau > 0$, the supremum in (1.3) is infinite;

(iii) for any $u \in W^{1, n}(M)$ and any $\alpha > 0$, the integrals in (1.3) are still finite.

We say more words about this method. For Sobolev inequalities on complete noncompact Riemannian manifolds, unity decomposition was employed by Hebey et al. [6]. In the case of Trudinger-Moser inequality, it is not evidently applicable. We are lucky to find its analogy ([14], Lemma 4.1).

2. Preliminary lemmas

We first give a local estimate concerning the Trudinger-Moser functional. Precisely we have

**Lemma 1** For any $x_0 \in \mathbb{R}^n$ and any $u \in W^{1, n}_0(B_R(x_0))$, $\int_{B_R(x_0)} |\nabla u|^p dx \leq 1$, we have

$$\int_{B_R(x_0)} \left( e^{\|u\|^\alpha} - \sum_{k=0}^{n-2} \frac{\alpha^k ||u||^\alpha}{k!} \right) dx \leq C(n) R^n \int_{B_R(x_0)} |\nabla u|^p dx,$$  

(2.1)
where \( C(n) \) is a constant depending only on \( n \).

**Proof.** Essentially this is the same as ([14], Lemma 4.1). For reader’s convenience we give the details here. It is well known \([8,10,13]\) that

\[
\sup_{u \in W^{1,m}_0(B(x_0)), \int_{B(x_0)} |\nabla u|^m dx \leq 1} \int_{B(x_0)} e^{a_0|u|^{m-1}} dx \leq C(n)R^n. \tag{2.2}
\]

Letting \( \bar{u} = \frac{u}{\|u\|_{L^m(B(x_0))}} \) for any \( u \in W^{1,m}_0(B(x_0)) \setminus \{0\} \), we have

\[
\int_{B(x_0)} \left( e^{a_0|\bar{u}|^{m-1}} - \sum_{k=0}^{n-2} \frac{a_k^k|\bar{u}|^{m-1}}{k!} \right) dx \geq \frac{1}{\|\nabla \bar{u}\|_{L^m(B(x_0))}} \int_{B(x_0)} \sum_{k=n-1}^{\infty} \frac{a_k^k|\bar{u}|^{m-1}}{k!} dx
\]

\[
= \frac{1}{\|\nabla \bar{u}\|_{L^m(B(x_0))}} \int_{B(x_0)} \left( e^{a_0|\bar{u}|^{m-1}} - \sum_{k=0}^{n-2} \frac{a_k^k|\bar{u}|^{m-1}}{k!} \right) dx. \tag{2.3}
\]

Combining (2.2) and (2.3), we get the desired result. \( \square \)

Also we need a covering lemma of \( \mathbb{R}^n \), see for example ([6], Lemma 1.6).

**Lemma 2** For any \( R > 0 \), there exists a sequence \( \{x_i\}_{i=1}^{\infty} \subset \mathbb{R}^n \) such that

(i) \( \bigcup_{i=1}^{\infty} B_{R/2}(x_i) = \mathbb{R}^n \);

(ii) \( \forall i \neq j, B_{R/4}(x_i) \cap B_{R/4}(x_j) = \emptyset \);

(iii) \( \forall x \in \mathbb{R}^n, x \text{ belongs to at most } N \text{ balls } B_{R}(x_i) \text{ for some integer } N. \)

### 3. Proof of Theorem A

We shall obtain a global inequality (3.1) by gluing local estimates (2.1).

**Proof of Theorem A.** Let \( R > 0 \) to be determined later. Let \( \phi_i \) be the cut-off function satisfies the following conditions: (i) \( \phi_i \in C_0^\infty(B_R(x_i)) \); (ii) \( 0 \leq \phi_i \leq 1 \) on \( B_R(x_i) \) and \( \phi_i \equiv 1 \) on \( B_{R/2}(x_i) \); (iii) \( |\nabla \phi_i(x)| \leq 4/R \). For \( u \in W^{1,m}(\mathbb{R}^n) \) satisfying

\[
\int_{\mathbb{R}^n} (|\nabla u|^m + |u|^m) dx \leq 1, \tag{3.1}
\]

we have \( \phi_i u \in W^{1,m}_0(B_R(x_i)) \), using Cauchy inequality with \( \epsilon \) term we obtain

\[
\int_{B_R(x_i)} |\nabla (\phi_i u)|^m dx \leq (1+\epsilon) \int_{B_R(x_i)} \phi_i^m |\nabla u|^m dx + C(\epsilon) \int_{B_R(x_i)} |\nabla \phi_i|^m |u|^m dx
\]

\[
\leq (1+\epsilon) \int_{B_R(x_i)} |\nabla u|^m dx + \frac{C(\epsilon)}{R^n} \int_{B_R(x_i)} |u|^m dx
\]

\[
\leq (1+\epsilon) \int_{B_R(x_i)} (|\nabla u|^m + |u|^m) dx, \tag{3.2}
\]
where in the last inequality we choose a sufficiently large \( R \) to make sure \( \frac{C(n)}{R^{n}} \leq (1 + \epsilon) \). Let \( \alpha_{\epsilon} = \frac{\alpha}{(1 + \epsilon)^{n-1}} \) and \( \phi_{\epsilon}u = \phi_{\epsilon} \). Noting that \( \phi_{\epsilon}u \in W^{1,1}_{0}(B_{R}(x_{i})) \), we have by (3.2) and Lemma 1

\[
\int_{B_{R}(x_{i})} \left( e^{\alpha_{\epsilon}|u|_{R}^{-n}} - \sum_{k=0}^{n-2} \frac{\alpha_{\epsilon}^{k}|u|_{R}^{-k}}{k!} \right) \, dx \leq \int_{B_{R}(x_{i})} \left( e^{\alpha_{\epsilon}|u|_{R}^{-n}} - \sum_{k=0}^{n-2} \frac{\alpha_{\epsilon}^{k}|\phi_{\epsilon}u|_{R}^{-k}}{k!} \right) \, dx
\]

\[
= \int_{B_{R}(x_{i})} \left( e^{\alpha_{\epsilon}|u|_{R}^{-n}} - \sum_{k=0}^{n-2} \frac{\alpha_{\epsilon}^{k}|\phi_{\epsilon}u|_{R}^{-k}}{k!} \right) \, dx
\]

\[
\leq C(n)R^{n} \int_{B_{R}(x_{i})} |\nabla (\phi_{\epsilon}u)|^{n} \, dx
\]

\[
\leq C(n)R^{n} \int_{B_{R}(x_{i})} (|\nabla u|^{n} + |u|^{n}) \, dx. \tag{3.3}
\]

By Lemma 2 and (3.3), we have

\[
\int_{\mathbb{R}^{n}} \left( e^{\alpha_{\epsilon}|u|_{\mathbb{R}^{n}}^{-n}} - \sum_{k=0}^{n-2} \frac{\alpha_{\epsilon}^{k}|u|_{\mathbb{R}^{n}}^{-k}}{k!} \right) \, dx \leq \int_{\bigcup_{i=1}^{N} B_{R}(x_{i})} \left( e^{\alpha_{\epsilon}|u|_{R}^{-n}} - \sum_{k=0}^{n-2} \frac{\alpha_{\epsilon}^{k}|\phi_{\epsilon}u|_{R}^{-k}}{k!} \right) \, dx
\]

\[
\leq \sum_{i=1}^{N} \int_{B_{R}(x_{i})} \left( e^{\alpha_{\epsilon}|u|_{R}^{-n}} - \sum_{k=0}^{n-2} \frac{\alpha_{\epsilon}^{k}|\phi_{\epsilon}u|_{R}^{-k}}{k!} \right) \, dx
\]

\[
\leq \sum_{i=1}^{N} C(n)R^{n} \int_{B_{R}(x_{i})} (|\nabla u|^{n} + |u|^{n}) \, dx
\]

\[
\leq C(n)R^{n}N \int_{\mathbb{R}^{n}} (|\nabla u|^{n} + |u|^{n}) \, dx
\]

\[
\leq C(n)R^{n}N. \tag{3.4}
\]

For any \( \alpha < \alpha_{\epsilon} \), we can choose \( \epsilon > 0 \) sufficiently small such that \( \alpha < \alpha_{\epsilon} \). This ends the proof of Theorem A.

\[\square\]

4. Concluding remarks

Using the same idea to prove Theorem A, we can also prove the subcritical Adams inequality in \( \mathbb{R}^{n} \) \([2,12,15]\), which strengthen ([14], Theorem 2.6). Since the proof is completely analogous to our proof of Theorem A, we leave it to the reader.

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