Nuclear matter as a liquid phase of spontaneously broken semi-classical \( SU(2)_L \times SU(2)_R \) chiral perturbation theory:

Static chiral nucleon liquids

Bryan W. Lynn\(^\dagger\) and Brian J. Coffey

Dept Physics/CERCA/ISO, CWRU, Cleveland, OH, 44106 USA

Kellen E. McGee

Dept Physics/Astronomy, Michigan State U., East Lansing MI, 48824

Glenn D. Starkman

Dept Physics/CERCA/ISO, CWRU, Cleveland, OH, 44106 USA and Dept Astronomy, CWRU, Cleveland, OH, 44106 USA

(Dated: December 17, 2021)

The Standard Model of particle physics (SM), augmented with neutrino mixing, is either the complete theory of interactions of known particles at energies accessible to Nature on Earth, or very nearly so. Starting with a Lagrangian symmetric under the global \( SU(2)_L \times SU(2)_R \) symmetry of two-massless-quark QCD, spontaneously broken to \( SU(2)_L + R \), using naive dimensional operator power counting that enables perturbation and truncation in inverse powers of \( \Lambda_{\chi SB} \sim 1\text{GeV} \), we show that, to \( O(\Lambda_{\chi SB}) \) and \( O(\Lambda_{\chi SB}^0) \), \( SU(2)_\chi PT \) of protons, neutrons and pions admits a liquid phase, with energy required to increase or decrease the nucleon density. We further show that in the semi-classical approximation – i.e., quantum nucleons and classical pions – "Pionless SU(2)_\chi PT" emerges in that chiral liquid: soft static infrared Nambu-Goldstone-Boson pions decouple from "Static Chiral Nucleon Liquids" (Static\( \chi NL \)). This vastly simplifies the derivation of saturated nuclear matter (the infinite liquid phase) and of finite microscopic liquid drops (ground-state heavy nuclides). Static\( \chi NL \) are made entirely of nucleons. They have even parity, total spin zero, even proton number \( Z \) and even neutron number \( N \). The nucleons are arranged so local expectation values for spin and momentum vanish. We derive the Static\( \chi NL \) effective Lagrangian from semi-classical \( SU(2)_\chi PT \) symmetries to order \( \Lambda_{\chi SB} \), \( \Lambda_{\chi SB}^0 \) including: all relativistic 4-nucleon operators that survive Fierz rearrangement in the non-relativistic limit; \( SU(2)_\chi PT \) fermion exchange operators and iso-vector exchange operators which are important when \( Z \neq N \).

Mean-field Static\( \chi NL \) non-topological solitons are true solutions of \( SU(2)_\chi PT \) semi-classical symmetries: e.g. they obey all CVC, PCAC conservation laws. They have zero internal and external pressure. The nuclear liquid-drop model and Bethe-von Weizsäcker semi-empirical mass formula emerge – with correct nuclear density and saturation and asymmetry energies – in an explicit Thomas-Fermi construction.

I. INTRODUCTION

In the Standard Model (SM) of particle physics, Quantum Chromodynamics (QCD) describes the strong interactions among quarks and gluons. At low energies, quarks and gluons are confined inside hadrons, concealing their degrees of freedom in such a way that we must employ an effective field theory (EFT) of hadrons. In doing so, we acknowledge as a starting point a still-mysterious experimental fact: Nature first makes hadrons and then assembles nuclei from them. \[1,4].\]

Since nuclei are made of hadrons, the fundamental challenge of nuclear physics is to identify the correct EFT of hadrons and use it to characterize all nuclear physics observations. (See the recent review by Hammer et al.\[5\].) Ultimately, the correct choice of EFT will both match the observations and be derivable from the SM, i.e., QCD.

Chiral perturbation theory (\( \chi PT \))\[6,11\] is a low-energy perturbative approach to identifying the operators in the EFT of hadrons that are allowed by the global symmetries of the SM. It builds on the observation that the up and down quarks \( (m_u \simeq 6 \text{MeV}, m_d \simeq 12 \text{MeV}) \), as well as the 3 pions \( (\pi^\pm, \pi^0, m_\pi \simeq 140 \text{MeV}) \)–which are pseudo-Nambu-Goldstone bosons (pNGBs) of the chiral symmetry–are all nearly massless compared to the cut-off energy scale in low-energy hadronic physics \( \Lambda_{\chi SB} \sim 1\text{GeV} \).

With naive power counting\[12\], the effective Lagrangian of \( SU(2)_L \times SU(2)_R \) \( \chi PT \) incorporates explicit breaking. The resultant perturbation expansion in the inverse of the chiral-symmetry-breaking scale \( \Lambda_{\chi SB}^{-1} \sim 1 \text{GeV}^{-1} \) renders \( SU(2)_\chi PT \)’s strong...
interaction predictions calculable in practice. Its low-energy dynamics of a proton-neutron nucleon doublet and three pions as a pNGB triplet are our best understanding, together with lattice QCD, of the experimentally observed low-energy dynamics of QCD strong interactions. The predictive power of $\chi$PT \cite{6,8,15} derives from its ability to maintain a well-ordered low-energy perturbation expansion that can be truncated.

B.W. Lynn \cite{16} first introduced the idea that SU(2)$\chi$PT could also admit a liquid phase and introduced the idea of an \textit{SU}(2)$_L \times \text{SU}(2)_R$ chiral liquid' as a statistically significant number of baryons interacting via chiral operators with an almost constant saturated density which can survive as liquid as a statistically significant number of baryons. The Lagrangian included all analytic SU(2)\textit{calized liquid drops at zero external pressure}. The fact that the angular momentum of each nucleon is a good quantum number. Ref. \cite{16} did not derive semi-classical pionless SU(2)$\chi$PT. Here, we focus our study of chiral liquids in \cite{16} focused on those explicit chiral symmetry breaking terms whose origin lies entirely in the non-zero light quark masses.

The result is a semi-classical nuclear picture, where Thomas-Fermi nucleons with contact interactions move in a mean spherical symmetric \textit{classical} pion field, which in turn generates a \textit{no-core} radial potential for nucleons. Finite saturating heavy nuclei, with well-defined surfaces, emerge as microscopic droplets of chiral liquid. Saturating infinite nuclear matter emerges as very large drops of chiral liquid, while neutron stars (Q-Stars) emerge as oceans of chiral liquid: These droplets emerge as non-topological-soliton semi-classical solutions of explicitly-broken SU(2)$\chi$PT. Lynn \cite{16} conjectured the possible emergence of shell structure in that no-core spherical potential based on the observation that the angular momentum of each nucleon is a good quantum number. Ref. \cite{16} did not derive semi-classical pionless SU(2)$\chi$PT. Here, we focus our study of chiral liquids in the chiral limit, and prove the emergence of semi-classical pionless SU(2)$\chi$PT solutions.

There is a long history of viewing nuclear matter as a non-topological soliton. In the mid 1970's T.D. Lee and co-workers \cite{17,19}, S.A. Chin & J.D. Walecka \cite{20}, and R. Serber \cite{21} first identified certain fermion non-topological solitons with the ground state of heavy nuclei (as well as possible super-heavy nuclei) in "normal" and "abnormal" phases, thus making a crucial connection to the ancient (but still persistently predictive) insight of nuclear liquids, such as G. Gamow’s nuclear liquid-drop model (NLDM) and H. Bethe & C.F. von Weizsäcker’s semi-empirical Mass-Formula (SEMF). Breaking all precedent, these workers proposed for the first time a theory of liquid nuclear structure composed entirely of nucleons and a static scalar field, with no pions!

Mathematically, such solutions emerge as a sub-species of non-topological solitons or Q-balls \cite{17,22–36}, a certain sub-set of which are composed of fermions along with the usual scalars. A practical goal was to identify mean-field nucleon non-topological solitons with the ground state of ordinary even-even spin-zero spherically symmetric heavy nuclei, such as $^{20}$Ca, $^{90}$Zr, and $^{208}$Pb.

Nuclear non-topological solitons identified as nuclear liquids became popular with the work of Chin & Walecka \cite{20}, carried forward by Serot \cite{36}. Walecka’s nuclear Quantum Hadrodynamics - 1 (QHD-1) models \cite{37,39} contain four dynamical particles: protons, neutrons, the Lorentz-scalar iso-scalar $\sigma$, and the Lorentz-vector iso-scalar $\omega$. Nucleons are treated as locally free-particles in Thomas-Fermi approximation. Finite-width nuclear surfaces are generated by dynamical attractive $\sigma$-particle exchange, allowing them to exist at zero external pressure. The empirical success of QHD-1 is based on balancing $\sigma$-boson-exchange attraction against $\omega$-boson-exchange repulsion. That that balance must be fine-tuned remains a famous mystery of the structure of the QHD-1 ground state. In the absence of long-ranged electromagnetic forces, infinite symmetric $Z = N$ nuclear matter, as well as finite microscopic ground state $Z = N$ nuclei, appear as symmetric nuclear liquid drops. These nuclear non-topological solitons of are to be classified as liquids because:

- they have no crystalline or other solid structure;
- it costs energy to either increase or decrease the density of the constituent nucleons compared to an optimum value;
- they survive at zero external pressure, e.g. in the absence of gravity, so they are not a “gas.”

Despite their successes, such topological soliton models suffer from the flaw that higher loop corrections do not necessarily decrease in size and importance which can significantly renormalize the parameters at each order. This was first demonstrated by Furnstahl et al. \cite{40} for the Walecka model in the two-loop case. (See also the discussion in \cite{41}.)

This paper cures those problems, and resurrects nuclear liquids as a good starting point toward understanding the properties of bound nuclear matter (with $Z$ and $N$ both even) by strict compliance with the requirements of SU(2)$\chi$PT effective field theory.
of protons, neutrons and pions. The static chiral nucleon liquids (Static $\chi_{NL}$) studied below are true solutions to semi-classical SU(2) $\chi$PT, and have all of the semi-classical symmetries of spontaneously broken SU(2) $\chi$PT found in Appendix A: they obey all CVC and PCAC Ward identities; they are dependent on just a few experimentally measurable chiral coefficients; and, by the symmetries of spontaneously broken SU(2) $\chi$PT, they restore (cf. Appendix A) theoretical predictive power over heavy nuclides.

II. THE EMERGENCE OF SEMI-CLASSICAL PIONLESS STATIC $\chi_{NL}$

In this paper we focus on the chiral limit and postpone treatment of departures from the chiral limit to future work. The SU(2) $\chi$PT Lagrangian with all terms of order $\Lambda_{\chi_{SB}}$ and $\Lambda^0_{\chi_{SB}}$ in the chiral limit is:

$$L_{\chi PT}^{N \gamma N} = L_{\chi PT}^{\pi \gamma N} + L_{\chi PT}^{N \pi N} + L_{\chi PT}^{\pi \pi N}$$

$$L_{\chi PT}^{\pi \gamma N} = \frac{f^2_\pi}{4} \text{Tr} \partial_\mu \Sigma \partial^\mu \Sigma^\dagger$$

$$L_{\chi PT}^{N \pi N} = \bar{N} (i \gamma_\mu (\partial_\mu + V_\mu) - m^N \mathbb{1}) N$$

$$- g_A \bar{N} \gamma_\mu \gamma_5 A_\mu N$$

$$= \bar{N} (i \gamma_\mu \partial_\mu - m^N \mathbb{1}) N + i \bar{\mu} \cdot \vec{V}_\mu$$

$$- g_A \bar{\mu} \cdot \vec{A}_\mu$$

$$L_{\chi PT}^{\pi \pi N} = C_{\omega} \frac{1}{2} \bar{\pi} \gamma_5 N \gamma_\mu N \gamma_\mu N + + +,$$

where the pion field is:

$$\Sigma = \exp(2i\pi \frac{t_o}{f_\pi}),$$

and we defined the fermion bi-linear and pionic currents:

$$\bar{\mu} = \bar{N} \gamma_\mu \vec{t} N, \quad \bar{\mu} = \bar{N} \gamma_\mu \vec{t} N,$$

$$\bar{\mu} = \bar{N} \gamma_\mu \vec{t} N, \quad \bar{\mu} = \bar{N} \gamma_\mu \vec{t} N,$$

$$V_\mu = \vec{t} \cdot \vec{V}_\mu, \quad \vec{V}_\mu = 2i \text{sinc}^2 \left( \frac{\pi}{f_\pi} \right) [\vec{\pi} \times (\partial_\mu \vec{\pi}^\dagger)]$$

$$A_\mu = \vec{t} \cdot \vec{A}_\mu,$$

$$\vec{A}_\mu = - \frac{2}{\pi} \left( \vec{\pi} \cdot (\partial_\mu \vec{\pi}) + \text{sinc} \left( \frac{\pi}{f_\pi} \right) \left( \vec{\pi} \times (\partial_\mu \vec{\pi} \times \vec{\pi}) \right) \right),$$

(3)

with $\vec{t} \equiv \frac{1}{2} \vec{\pi} \cdot \tau$ the Pauli iso-spin matrices, $\pi = |\vec{\pi}| = \sqrt{f_\pi^2}$, and $\text{sinc}(x) \equiv \sin(x)/x$. The pion→di-leptons decay constant is $F_\pi = 130.4 \pm 0.04 \pm 0.2$ MeV [42]. We use $f_\pi \equiv F_\pi / \sqrt{2} = 92.2$ MeV.

The parentheses in the four-nucleon Lagrangian indicate the order of SU(2) index contraction, while $+++$ indicates that one should include all possible combinations of such contractions. As usual, $\gamma_{\omega}^a \equiv (1, \gamma^\mu, i\sigma^{\mu\nu}, i\gamma^\mu \gamma^5, i\gamma^\nu \gamma^5)$, for $a = 1, \ldots, 16$ (with $\sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$). These are commonly referred to as scalar (S), vector (V), tensor (T), axial-vector (A), and pseudo-scalar (P) respectively. $C_{\omega}$ are a set of chiral constants.

In the chiral limit, where $\vec{\pi}$’s are massless, the presence of quantum nucleon sources could allow the massless NGB to build up, with tree-level interactions only, a non-linear quantum pion cloud. If we minimize the resultant action with respect to variations only, a non-linear quantum pion cloud. If we capture the part of the quantum cloud that is to be characterized as a classical soft-pion field, thus giving us the pion ground state in the presence of the ground state “Chiral Nucleon Liquid” (NL) with fixed baryon number $A = Z + N$:

$$0 = \left[ \partial_\mu \frac{\partial}{\partial (\partial_\mu \vec{\pi})} - \frac{\partial}{\partial \vec{\pi}} \right] L_{\chi PT}^{\pi \gamma N} + i \vec{J}_\mu \cdot \left[ \frac{\partial}{\partial (\partial_\mu \vec{\pi})} - \frac{\partial}{\partial \vec{\pi}} \right] \vec{V}_\mu$$

$$- g_A \vec{J}_\mu \cdot \left[ \frac{\partial}{\partial (\partial_\mu \vec{\pi})} - \frac{\partial}{\partial \vec{\pi}} \right] \vec{A}_\mu$$

$$- 2 \partial_\mu \bar{\mu} \cdot \vec{V}_\mu + \frac{2}{\pi} g_A \partial_\mu \bar{\mu} \cdot \vec{A}_\mu,$$

$$\times \left[ \vec{\pi} \cdot (\vec{m} \times \vec{m}) + \text{sinc} \left( \frac{\pi}{f_\pi} \right) \left( \vec{\pi} \times (\vec{m} \times \vec{m}) \right) \right].$$

We divide the classical pion field into “IR” and “non-IR” parts. By definition, only IR pions survive the internal projection operators associated with taking expectation values of the classical NGB $\vec{\pi}$'s in the $|\chi_{NL}\rangle$ quantum state:

$$\langle \chi_{NL} | F (\partial_\mu \vec{\pi}, \vec{\pi}) \rangle |\chi_{NL}\rangle = \langle \chi_{NL} | \text{IR part} [F (\partial_\mu \vec{\pi}, \vec{\pi})] |\chi_{NL}\rangle$$

$$= \{ F (\partial_\mu \vec{\pi}, \vec{\pi}) \}_\text{IR},$$

where $F$ is an unspecified function. The IR part does not change the NL. It could in principle play

1 This is a chiral-limit SU(2) $\chi$PT analogue of QED where, in the presence of quantum lepton sources, a specific superposition of massless infra-red photons builds up into a classical electromagnetic field. Important examples are the “exponentiation” of IR photons in $e^+ e^- \rightarrow \mu^+ \mu^-$ asymmetries, and $e^+ e^- \rightarrow e^+ e^-$ Bhabha scattering, at LEP1. Understanding the classical fields generated by initial-state and final-state soft-photon radiation [33] is crucial to disentangling high-precision electro-weak loop effects, such as the experimentally confirmed precise Standard Model predictions for the top-quark [34] and Higgs’ masses [35].
an important role in the excited states of the $\chi_{NL}$: a $\pi$ condensate, a giant resonance, a breathing mode, or a time-dependent flashing-pion mode. To ignore such classical IR $\pi$ would therefore be an incorrect definition of the excited states of $\chi_{NL}$.

We call these "IR pions" by keeping in mind a simple picture, where the $\pi$ wavelength is longer than the scale within the $\chi_{NL}$ over which the local mean values of nucleon spin and momentum vanish. Only such IR pions survive the internal projection operators associated with taking expectation values of the classical NGB $\pi$ in the $\chi_{NL}$ quantum state.

We now take expectation values of the $\pi$ equations of motion. In the presence of the quantum $\chi_{NL}$ source, the classical NGB $\pi$ cloud obeys

$$\langle \chi_{NL} | \partial_\mu J^\mu_{IR} | \chi_{NL} \rangle = 0 \quad \text{for all } \mu. \tag{10}$$

for all $\mu$. Note that (6)-(10) follow because the liquid ground state is assumed to have definite numbers of fully paired nucleons in a spherically symmetric, homogeneous, and isotropic arrangement. Current conservation enforces

$$\langle \chi_{NL} | \partial_\mu J^\mu_{IR} | \chi_{NL} \rangle = 0, \tag{11}$$

which leaves only a single non-vanishing current expectation value:

$$\langle \chi_{NL} | J^3_{IR} | \chi_{NL} \rangle \neq 0. \tag{12}$$

Equation (6), governing the classical pion cloud, is thus enormously simplified

$$0 \simeq \left\{ \left[ \partial_\nu \left( \frac{\partial}{\partial (\partial_\mu \pi^m)} - \frac{\partial}{\partial \pi^m} \right) L^\mu_{IR} \right] \langle \chi_{NL} | \chi_{NL} \rangle \right\}_{IR}$$

with

$$+ i \langle \chi_{NL} | J^{3,0}_{IR} | \chi_{NL} \rangle \left\{ \left[ \partial_\nu \frac{\partial}{\partial (\partial_\mu \pi^m)} - \frac{\partial}{\partial \pi^m} \right] V^3_0 \right\}_{IR} \tag{13}$$

We examine the following semi-classical nuclear current components:

$$J^\mu_\pm = J^\mu_{\pi} \pm i J^\mu_{\gamma} \equiv \left\{ \frac{\pi_{\gamma} \mu n}{\pi_{\gamma} \mu p}, \frac{\pi_{\gamma} \mu n}{\pi_{\gamma} \mu p} \right\};$$

$$J^3_\pm = \frac{1}{2} \left( \pi_{\gamma} \mu p - \pi_{\gamma} \mu n \right);$$

$$J^5_\pm = J^5_{\pi} + i J^5_{\gamma} \equiv \left\{ \frac{\pi_{\gamma} \mu 5n}{\pi_{\gamma} \mu 5p}, \frac{\pi_{\gamma} \mu 5n}{\pi_{\gamma} \mu 5p} \right\};$$

$$J^3_\pm = \frac{1}{2} \left( \pi_{\gamma} \mu 5p - \pi_{\gamma} \mu 5n \right);$$

and find that the ground-state expectation values of these currents and their divergences in (6) vanish:

$$\langle \chi_{NL} | J^\pm_{\mu} | \chi_{NL} \rangle = \langle \chi_{NL} | J^{5,5}_{\mu} | \chi_{NL} \rangle = 0,$$

$$\langle \chi_{NL} | \partial_\mu J^\mu_{\pi} | \chi_{NL} \rangle = \langle \chi_{NL} | \partial_\mu J^{5,5}_{\pi} | \chi_{NL} \rangle = 0, \tag{8}$$

because $J^\pm_{\pi}$ and $J^{5,5}_{\pi}$ change neutron and proton number. Since the liquid ground state is homogeneous, isotropic and spherically symmetric, spatial components of vector currents vanish, in particular

$$\langle \chi_{NL} | J^i_{\pi} | \chi_{NL} \rangle \simeq 0 \quad \text{for Lorentz index } i = 1, 2, 3. \tag{9}$$

Because there are separately equal numbers of left-handed and right-handed protons and neutrons in the nuclear ground state we have:

$$\langle \chi_{NL} | J^3_{\mu} | \chi_{NL} \rangle \simeq 0 \quad \text{for all } \mu. \tag{10}$$

for all $\mu$. Note that (6)-(10) follow because the liquid ground state is assumed to have definite numbers of fully paired nucleons in a spherically symmetric, homogeneous, and isotropic arrangement. Current conservation enforces

$$\langle \chi_{NL} | \partial_\mu J^\mu_{\pi} | \chi_{NL} \rangle = \langle \chi_{NL} | \partial_\mu J^{3,5}_{\mu} | \chi_{NL} \rangle = 0, \tag{11}$$

which leaves only a single non-vanishing current expectation value:

$$\langle \chi_{NL} | J^3_{IR} | \chi_{NL} \rangle \neq 0. \tag{12}$$

Equation (6), governing the classical pion cloud, is thus enormously simplified

$$0 \simeq \left\{ \left[ \partial_\nu \left( \frac{\partial}{\partial (\partial_\mu \pi^m)} - \frac{\partial}{\partial \pi^m} \right) L^\mu_{IR} \right] \langle \chi_{NL} | \chi_{NL} \rangle \right\}_{IR}$$

with

$$+ i \langle \chi_{NL} | J^{3,0}_{IR} | \chi_{NL} \rangle \left\{ \left[ \partial_\nu \frac{\partial}{\partial (\partial_\mu \pi^m)} - \frac{\partial}{\partial \pi^m} \right] V^3_0 \right\}_{IR} \tag{13}$$
\[
\left\{ \frac{\partial}{\partial (\partial_{\nu} \pi^m)} \right\}_{IR} \partial_{\nu} \frac{\partial}{\partial (\partial_{\mu} \pi^m)} V_0^3 = \\
2i \left[ (\partial_0 \vec{\pi}) \times \vec{m} + \vec{\pi} \times \vec{m} \partial_0 - \vec{m} \times (\partial_0 \vec{\pi}) - \vec{\pi} \times (\partial_0 \vec{\pi}) \frac{\partial}{\partial (\partial_{\mu} \pi^m)} \right]^3 \text{sinc}^2 \left( \frac{\pi}{2f_\pi} \right) \right\}_{IR}.
\]

Equations (17) and (19) show that, to order \(\Lambda_{\chi_{SB}}^0\) and \(\Lambda_{\chi_{SB}}^0\), Static\(\chi_{NL}\) are composed entirely of nucleons. That is also the basic premise of many empirical models and we have shown that that empirical nuclear premise can be (to good approximation) traced directly to the global SU(2)\(_L\)×SU(2)\(_R\) symmetries of 2-massless-quark QCD of the Standard Model.

The effective Lagrangian derived from SU(2)\(_L\)×SU(2)\(_R\) \(\chi\)PT governing Static\(\chi_{NL}\) can now be written:

\[
\langle L_{\chi_{PT}} \rangle_{\chi_{NL}} \equiv L_{\chi_{NL}}^L = L_{\chi_{NL}}^L + L_{\chi_{NL}}^N \\
L_{\chi_{NL}}^L = \langle \bar{N} i\gamma^\mu \partial_\mu - m^N \bar{\gamma}^0 N \rangle \\
L_{\chi_{NL}}^N = \left\{ \frac{1}{2f_\pi^2} C_{\gamma^0}^L (\bar{N} \gamma_{\gamma^0}^N)(\bar{N} \gamma_{\gamma^0}^N) + \text{++} \right\}.
\]

III. SEMI-CLASSICAL PIONLESS STATIC\(\chi_{NL}\) AS THE APPROXIMATE GROUND STATE OF CERTAIN NUCLEI

To further elucidate the properties of the static \(\chi_{NL}\), we must address the effects of the four-nucleon interactions. In this paper, we ignore fluctuations in all bi-linear nucleon operators. For our purposes this
is equivalent to ignoring any and all nuclear excited states.

*A priori* there are 10 possible contact interactions representing isosinglet and isotriplet channels for each of five spatial current types: scalar, vector, tensor, pseudo-scalar and axial-vector. There are therefore ten chiral coefficients parametrizing 4-nucleon contact terms: $C^K_T$ with $K \in \{S, V, T, A, P\}$ and $T \in \{0, 1\}$.

The inclusion of exchange interactions induces the isospin ($T = 1$) operators to appear [10], and potentially greatly complicates the effective chiral Lagrangian. Fortunately, we are interested here in the liquid limit of this Lagrangian. Spinor-interchange contributions are properly obtained by Fierz rearranging before imposing the properties of the semiclassical liquid (see Appendix B). The appropriate Static$\chi$NL Lagrangian, is given by

$$L_{S\chi NL} = \bar{N} \left( i \gamma^\mu \partial^\mu + \Theta \right) N + L^{4-N;BE}_{S\chi NL},$$  \hspace{1cm} (22)

where the contact interactions can be approximated by:

$$-L^{4-N;BE}_{S\chi NL} = \frac{C^S_{200}}{2f^2_\pi} \langle \bar{N}N \rangle \langle \bar{N}N \rangle - \frac{C^S_{200}}{4f^2_\pi} \left\{ \langle \bar{N}N \rangle \langle \bar{N}N \rangle + 4 \langle \bar{N}t_3N \rangle \langle \bar{N}t_3N \rangle \right\} + \frac{C^V_{200}}{2f^2_\pi} \left\{ \langle N^\dagger N \rangle \langle N^\dagger N \rangle \right\} - \frac{C^V_{200}}{4f^2_\pi} \left\{ \langle N^\dagger N \rangle \langle N^\dagger N \rangle + 4 \langle N^\dagger t_3N \rangle \langle N^\dagger t_3N \rangle \right\}$$  \hspace{1cm} (23)

with only four independent chiral coefficients:

$$C^S_{200} = C^{T=0}_{S}=0,$$

$$-C^S_{200} = \frac{1}{4} \left\{ C^{T=0} + 5C^{T=1} + 6 \left( C^{T=0}_T + C^{T=1}_T \right) + \left( C^{T=0}_P + C^{T=1}_P \right) \right\},$$

$$C^V_{200} = 2C^{T=0}_V,$$

$$-C^V_{200} = \frac{1}{2} \left[ -C^{T=0}_V + C^{T=0}_A + C^{T=1}_V + C^{T=1}_A \right].$$  \hspace{1cm} (24)

To simplify the notation and to retain the connection with previous work [20] we introduce:

$$C^2_V = \frac{1}{f^2_\pi} \left( C^{V}_{200} - \frac{1}{2}C^{V\dagger}_{200} \right),$$  \hspace{1cm} (25)

$$C^2_S = -\frac{1}{f^2_\pi} \left( C^{S}_{200} - \frac{1}{2}C^{S\dagger}_{200} \right).$$  \hspace{1cm} (26)

For brevity, we also define:

$$\overline{C}^2_V = \frac{1}{f^2_\pi} \overline{C}^{V}_{200},$$  \hspace{1cm} (27)

$$\overline{C}^2_S = \frac{1}{f^2_\pi} \overline{C}^{S}_{200}.$$  \hspace{1cm} (28)

In (22) the operator $\Theta$ is given by:

$$\Theta = -m^N - C^S_{200} - \overline{C}^V_{200}\gamma^0,$$  \hspace{1cm} (29)

with:

$$C^V_{200} \equiv C^2_V \left\{ \langle N^\dagger N \rangle - 2C^S_{200} \langle N^\dagger t_3N \rangle t_3 \right\},$$

$$C^S_{200} \equiv -C^2_S \left\{ \langle N\bar{N} \rangle - 2C^V_{200} \langle N^\dagger t_3N \rangle t_3 \right\},$$  \hspace{1cm} (30)

$$0 = \left[ t_3, C^S_{200} \right] = \left[ t_3, \overline{C}^V_{200}\gamma^0 \right] = [t_3, \Theta].$$

We have ignored possible excited states that contribute to fluctuations in the nuclear density and which are beyond the scope of this paper.

The Static$\chi$NL Lagrangian offers a significant improvement in the predictive power of the theory, while still providing sufficient free parameters to balance vector repulsive forces against scalar attractive forces when fitting (to order $A^0_{\beta\delta}$) non-topological-soliton and Skyrme nuclear models to the experimentally observed structure of ground state nuclei. Further simplification results for a sufficiently large number of nucleons: simple Hartree analysis of (23) is equivalent to more accurate Hartree-Fock analysis of the same Lagrangian without spinor-interchange terms.

We now see that, inside the Static$\chi$NL, a nucleon living in the self-consistent field of the other nucleons obeys the Dirac equation

$$0 = \left( i \gamma^\mu \partial^\mu + \Theta \right) N.$$  \hspace{1cm} (31)

Baryon-number and the third component of isospin are both conserved; i.e., the associated currents $J^\mu_{\text{Baryon}} \equiv \nabla \gamma^\mu N$ and $J^3_\gamma \equiv \nabla \gamma^3 N$ are both divergence-free. The neutral axial-vector current $J^5_\gamma = \frac{1}{\sqrt{3}} \nabla \gamma^5 N$, corresponding to the projection onto SU(2) of the NGB $\eta$ particle, part of the unbroken SU(3)$_L \times SU(3)_R$ meson octet, is also divergence free,

$$\frac{2}{\sqrt{3}} \left\langle i \partial^\mu J^5_\gamma \right\rangle = \left\langle \nabla \left\{ \Theta, \gamma^5 \right\} N \right\rangle = 2 \left\langle \nabla \left( -m^N - C^S_{200} \right) \gamma^5 N \right\rangle \approx 0.$$  \hspace{1cm} (32)

This result can be understood as a statement that the $\eta$ particle cannot survive in the parity-even interior of a Static$\chi$NL, since it is a NGB pseudo-scalar in the chiral limit. Similarly, the 3rd component of the axial vector current is divergence-free; i.e.,

$$\left\langle i \partial^\mu J^3_3 \right\rangle = \left\langle \nabla \left\{ \Theta, \gamma^3 \right\} t_3N \right\rangle = 2 \left\langle \nabla \left( -m^N - C^S_{200} \right) \gamma^3 t_3N \right\rangle \approx 0.$$  \hspace{1cm} (33)
because the SU(2)χPT π3 particle is also a NGB pseudo-scalar in the chiral limit, and cannot survive in the interior of a parity-even StaticχNL.

Even though explicit pion and eta fields vanish in StaticχNL, their quantum numbers reappear in its PCAC properties from nucleon bi-linears and four-nucleon terms in the divergences of axial vector currents. That these average to zero in StaticχNL plays a crucial role in the conservation of axial-vector currents within the liquid.

It is now straightforward to see that, in the liquid approximation, a homogeneous SU(2)χPT nucleon liquid drop with no meson condensate satisfies all relevant CVC and PCAC equations. As shown in Appendix C, most of the space-time components of the three SU(2)_{L+R} vector currents J_{\mu}^a and three axial vector currents J_{\mu}^{a\mu} vanish: only J_3^a is nonzero in StaticχNL.

The neutral SU(3)_L x SU(3)_R currents are conserved \( \langle \partial_\mu J_3^\mu \rangle = 0 \) and \( \langle \partial_\mu J_{\mu}^{a\mu} \rangle = 0 \) in the StaticχNL mean field. In addition, the neutral SU(3)_L x SU(3)_R vector current’s spatial components \( J_3^a \) and the axial-vector currents \( J_{\mu}^{a\mu} \) all vanish. Only \( J_3^a \), proportional to the baryon number density, survives in the StaticχNL mean field.

Since StaticχNL chiral nuclear liquids satisfy all relevant χPT CVC and PCAC equations in the liquid phase, they are true solutions of the all-orders-renormalized tree-level semi-classical liquid equations of motion truncated at \( \mathcal{O}(\Lambda_{\chi}^0) \).

IV. NUCLEI AND NEUTRON STARS AS MEAN-FIELD STATIC χNL

A. Thomas-Fermi non-topological solitons, liquid drops and the semi-empirical mass formula

Mean-field StaticχNL non-topological solitons are solutions of χPT semi-classical symmetries, obeying all CVC and PCAC conservation laws. They have zero internal and external pressure. The nuclear liquid-drop model and Bethe-von Weizsäcker SEMF emerge – with correct nuclear density, and saturation and asymmetry energies – in an explicit Thomas-Fermi construction.

In Appendix D, we construct explicit liquid mean field StaticχNL solutions based on \( \chi_{4}^{\epsilon} \), constrained to order \( 4\pi f_\pi \approx \Lambda_{\chi} \approx 1 \text{ GeV} \) and \( \Lambda_{\chi}^{0} \) naive power-counting, in an independent-nucleon model, using the Thomas-Fermi free-particle approximation.\(^3\)

\(^3\) An effective Lagrangian, built from \( \mathcal{O}(\Lambda_{\chi}^{0}) \) free nucleons and \( \mathcal{O}(\Lambda_{\chi}^{0}) \) point-coupling interaction-operators, was also identified by G. Gelmini and B. Ritzi [47]. However, it does not correspond to Chin-Walecka infinite symmetric \( Z = N \) nuclear matter, and the authors constructed no \( Z = N \) bound-state non-topological-solitons with zero internal and external pressure, which could therefore survive in an external vacuum.
In practice, there is very little sensitivity to our 4th independent chiral coefficient $C_{200}$: this in agreement with Niksic [50] et al., who argue that, although the total is-vector strength has a relatively well-defined value, the distribution between the isovector Lorentz-scalar $\vec{\delta}$ exchange channel, and the isovector Lorentz-vector $\vec{\rho}_\mu$ exchange channel, is not determined by ground state data. We have assumed $(\frac{Z}{N^2})^2 \ll 1$. In addition, we have

$$\langle N^1N \rangle - \langle \bar{N}N \rangle \simeq \frac{3}{10} \frac{k_F^2}{m_{ss}^2} \langle N^1N \rangle$$

$$= (0.0762) \langle N^1N \rangle \ll \langle N^1N \rangle$$

where $k_F = 279.7$ MeV and where $m_{ss} \equiv \frac{1}{2}(m_{sp} + m_{sn}) = 555$ MeV which follows from the Thomas Fermi solution in as found in Appendix D1. It follows that only the combination $(C_{200}^V + C_{200}^S)$ can strictly be fit to our $O(\Lambda_{\chi_B}^{-2})$ Static\$\chi_N$L accuracy. Therefore, for convenience and without loss of generality, we choose

$$C_{200}^S = 0.$$  

All coefficients in (37), (38), and (39) then obey naive $O(1)$ dimensional power counting, and so are legitimate natural chiral coefficients. Note the fine-tuning between $C_{200}^V = 2.198$ and $C_{200}^S = 2.580$ in (37) and (39) inherited from R. Serber’s and J.D. Walecka’s 1974 quadratic models [20], [51] and [21]. That fine-tuning is alleviated in (37) by the our inclusion of $\vec{\rho}_\mu$ exchange, necessary to Static\$\chi_N$Ls. Equations (37), (38), (39) and (41) all satisfy naive dimensional power-counting $O(1)$ naturalness, and so are legitimate chiral coefficients. The astute reader will notice that the difference (40) is of the same order as the next terms in the chiral expansion. Although we have calculated self-consistently in powers of $\Lambda_{\chi_{SB}}$ in chiral perturbation theory, terms of order $\Lambda_{\chi_{SB}}^{-1}$ must still play an important role in the nontopological soliton solutions. Indeed, it is inconsistent to neglect them. We hope to return to this question in future work.

The SEMF is closely associated with Gamow’s nuclear liquid-drop model (NLDM). Recall that, following Walecka’s infinite symmetric nuclear matter (and neutron matter), we have imposed on the Thomas-Fermi mean field the condition that the pressure vanish both internally and externally, not only at the surface of a finite “drop.” Our non-topological soliton nuclei therefore resemble ice cream balls scooped from an infinite vat [52], more than they do conventional liquid drops.

We clearly have no right to use the Thomas-Fermi approximation to calculate the surface and pairing energies, $E_{SB}^{Surf}$ and $E_{SB}^{Pair}$ of (44), at order $\Lambda_{\chi_{SB}}$ and $\Lambda_{\chi_{SB}}^{-1}$ in the spontaneously broken theory. Unsurprisingly, the surface energy calculated entirely as a change in density gives incorrect $\alpha_s$. However, there exist $O(\Lambda_{\chi_{SB}}^{-2})$ nuclear-surface SU(2)$\chi_{PT}$ terms that might replace the scalar $\sigma$ particle in the Chin-Walecka model in describing the nuclear surface [20] [53], namely

$$L_{\chi_{PT}}^{Surf} = -\frac{1}{2} \frac{C_{220}}{\Lambda_{\chi_{SB}}^2} \partial_\nu (N\bar{N}) \partial^\nu (N\bar{N}),$$

with an $O(1)$ constant $C_{220}$, obeying naturalness. $L_{\chi_{PT}}^{Surf}$ is invariant under non-linear SU(2)$_L$ x SU(2)$_R$ transformations including pions, but is automatically pionless, even without the liquid approximation. It contains no dangerous $\partial_\nu \sim m_N$ nucleon mass terms, so non-relativistic re-ordering is unnecessary. Nucleon-exchange and spinor-interchange interactions must also be included.

Meanwhile, calculation of $\alpha_{Pair}$ involves understanding low-level excited states, such as Z-odd N-odd states which we have ignored in our study of the Lagrangian (23), which are beyond the scope of this paper, and will likely require explicit pions lying outside semi-classical pionless SU(2)$\chi_{PT}$.
B. Neutron Stars

Putting aside exotica (i.e., quark condensates, pion condensates, strange-kaon condensates, etc.), we conjecture that much of the structure of neutron stars may be traced directly to 2-massless-quark QCD, and thus directly to the Standard Model. This will be explored further in a companion paper. Here we note only that the models of Harrison & Wheeler [53], Salpeter [54] and Baym, Pethic and Sutherland [55], are all based on the Bethe-von Weizsäcker semi-empirical mass formula [56]. They would therefore seem to follow from Static $\chi$NL; however, we do not yet know how well the observed chart of nuclides counts--classical SU(2) spherically-magic nuclides.

We conjecture here that non-topological Static $\chi$NL solitons could, with inclusion of explicit axial breaking terms, have naive operator power-counting Static breaking? and with experimental parameters:

\begin{align}
(a_1, a_2, a_3) &= (0.28, -0.56, 1.3 \pm 0.2) \\
(m_u, m_d, \sigma_{\pi N}) &= (6, 12, 60) \text{ MeV} \\
\beta &= 0.864 \pm 0.120
\end{align}

measured in SU(3)$_L \times$SU(3)$_R$ $\chi$PT processes [13] and [57].

Since $\langle L_{\chi PT}^{N\chi SB} \rangle > 0$, the explicit symmetry-breaking terms lower the effective nucleon mass inside a static $\pi = |\vec{\pi}|$ condensate.

We conjecture that semi-classical SU(2)$_L \times$SU(2)$_R$ $\chi$PT (i.e., including all $\mathcal{O}(A_{\chi SB})$ and $\mathcal{O}(A_{\chi SB}^0)$ non-strange analytic naive operator power-counting terms, both those from the chiral-limit and those from explicit $m_u, m_d \neq 0$ chiral symmetry breaking) applied to certain finite nuclei, nuclear and neutron matter and neutron stars will give a reasonable match to their structure.

C. Shell structure from chiral symmetry breaking?

We conjecture here that non-topological Static $\chi$NL solitons could, with inclusion of explicit axial symmetry breaking, be re-quantized to incorporate no-core nuclear shell structure and magic numbers, as imagined in [16]. Lynn first introduced the idea [16] that SU(2)$_L \times$SU(2)$_R$ $\chi$PT could admit a liquid phase. Like ours, his Lagrangian included only terms of $\mathcal{O}(A_{\chi SB})$ and $\mathcal{O}(A_{\chi SB}^0)$. Though he did not anticipate Static $\chi$NLs, he was careful to include only and all those terms that respect the SU(2)$_L \times$SU(2)$_R$ semi-classical symmetries - i.e., of quantum nucleons and classical pions - discussed in this paper. These included strong interaction terms that survive the chiral limit, as well as explicit axial breaking terms that do not.

The purpose of [16] was to generate a "no-core" classical static spherical central potential for $|\vec{\pi}|$, in which all of the quantum nucleons moved, and thus plausibly shell structure for certain heavy even-even ground state spin-zero spherical nuclei. It now seems advantageous to focus on doubly-magic or spherically-magic nuclides.

Such shell structure is plausibly in semi-classical SU(2)$_L \times$SU(2)$_R$ $\chi$PT because the explicit symmetry-breaking terms have naive operator power counting $m = 0$, $l = 1$, $n = 1$ in $(\Lambda, 14)$. Ignoring $\pi^\pm - \pi^0$ mass splitting, these are

$$
\langle L_{\chi PT}^{\chi NL} \rangle \simeq m(a_1 + 2a_3) \left[ 1 - \cos \frac{|\vec{\pi}|}{f_\pi} \right] \langle \tilde{N}N \rangle,
$$

\begin{equation}
\equiv \beta \sigma_{\pi N} \left[ 1 - \cos \frac{|\vec{\pi}|}{f_\pi} \right] \langle \tilde{N}N \rangle,
\end{equation}

with

$$
\overline{m} \equiv \frac{1}{2}(m_u + m_d),
$$

and with experimental parameters:

\begin{align}
(a_1, a_2, a_3) &= (0.28, -0.56, 1.3 \pm 0.2) \\
(m_u, m_d, \sigma_{\pi N}) &= (6, 12, 60) \text{ MeV} \\
\beta &= 0.864 \pm 0.120
\end{align}

measured in SU(3)$_L \times$SU(3)$_R$ $\chi$PT processes [13] and [57].

Since $\langle L_{\chi PT}^{N\chi SB} \rangle > 0$, the explicit symmetry-breaking terms lower the effective nucleon mass inside a static $\pi = |\vec{\pi}|$ condensate.

We conjecture that semi-classical SU(2)$_L \times$SU(2)$_R$ $\chi$PT (i.e., including all $\mathcal{O}(A_{\chi SB})$ and $\mathcal{O}(A_{\chi SB}^0)$ non-strange analytic naive operator power-counting terms, both those from the chiral-limit and those from explicit $m_u, m_d \neq 0$ chiral symmetry breaking) applied to certain finite nuclei, nuclear and neutron matter and neutron stars will give a reasonable match to their structure.

V. RELATION OF STATIC $\chi$NL TO PIONLESS EFT

Our pionless static chiral nuclear liquid solution bears superficial resemblance to results from pionless EFT [5]: both are “pionless.” They are both pionless for different reasons, however. Pionless EFTs are pionless because the pions have been “integrated out” and so are valid for momenta less than the pion mass. Static $\chi$NL are pionless because the pionic source terms vanish in even-even, spin-zero spherical nuclei: here we work in the chiral limit of vanishing pion mass. The soliton solution has $k_F \geq 280$ MeV. In fitting the parameters $C_A$ in Eq. (2), we must fit to inferred infinite-nuclear-matter data. As pointed out by Hammer et al. [5] perturbation theory cannot be used to relate the coupling constants in the two theories. In future work one might hope to relate the coupling constants of Static $\chi$NL to those of pionless and of Halo/Cluster EFTs [5].

U. van Kolck and the Pionless EFT community like to reveal relationships among their results by plotting them on the complex $Re(k) - Im(k)$ momentum plane inside the circle $|k| \leq A^2 < m_\pi$. To that disc we add an orthogonal $A = Z + N$ axis – forming a 3-D cylindrical $Re(k) - Im(k) - A$ volume – and highlight some Pionless EFT results. In the $A = 2$ plane, $N - N$ elastic scattering is properly compared to Nijmegen data and lies along positive $Re(k)$. The $-2.2$ MeV bound deuteron is at $k^2_{Pole}$ on the positive $Im(k)$ axis, while the shallow resonance is at $k^2_{Pole}$ on the negative $Im(k)$ axis. The
$A = 4$ plane places the deeply bound ($-28.296$ Mev) \( \alpha \) particle (\( \sim 2H_e \)) at positive $Im(k)$. 

Halo/cluster EFT at $A \geq 5$ has no pions, and is mathematically similar to Pionless EFT, becoming Pionless EFT for light nuclei when the cores are nucleons. We plot only the classic example $\frac{3}{2}He_4$, where the energy required to remove the cluster (\( \alpha \) particle), or either of the two halo nucleons, is much less than to break up the cluster. It lies on the $A = 6$ plane at postive $Im(k)$.

In order to plot our Thomas-Fermi Static $\chi$NL results from Appendix D and show their position relative to Pionless EFT, we add an annulus to that Pionless EFT cylinder, extending the radius of its $Re(k) - Im(k)$ base to the region $\Lambda^A_{\chi_{\mathrm{NL}}} < |k| \leq \Lambda^A_{\chi_{\mathrm{SB}}} = \Lambda_{\chi_{\mathrm{SB}}} \sim 1$ GeV. Our bound-state Static $\chi$NL "ice-cream-scoop" nuclei are then horizontal lines along $Im(k)$, in the positive $Im(k) - A$ quarter-plane, with $0 \leq |k| \leq k_F \approx 280$ MeV: they intersect the $A$ axis at $A_{\mathrm{Even}Even} = Z_{\mathrm{Even}} + N_{\mathrm{Even}} \geq 4$. For visual simplicity, we plot symmetric $Z = N$ Static $\chi$NL nuclei only for $^{28}\text{Si}$ and $^{20}\text{Ca}$, at $A = 28, 40$. We show asymmetric Static $\chi$NL only for $^{20}\text{Ca}$, $^{40}\text{Ca}$, $^{90}\text{Zr}$ and $^{208}\text{Pb}$ with $X^2 \ll 1$. For further pedagogical simplicity, we have averaged $\frac{1}{2}(k_F^p + k_F^n) \approx k_F \approx 280$ MeV.

Going forward, an important challenge is to find an SU(2)$_L$ x SU(2)$_R$ $\chi$PT integration of the physics of Static $\chi$NL and that of Pionless EFT and halo/cluster EFT.

In the Summary of the 1985 Paris Conference on Nuclear Physics with Electromagnetic Probes, Torleif Ericson [58] showed just how many facets there are to the nuclear "truth" – different physical domains require different descriptions, each of which is the truth for that domain. If Static $\chi$NL as derived from the symmetries of QCD describes heavy (spin-zero even-even) spherical nuclei, its truth may be difficult to relate directly to accurate descriptions of other physical domains.

VI. CONCLUSIONS

In this paper, we have explored heavy symmetric nuclei in a semi-classical approach starting with Chiral EFT that respects the global symmetries of QCD. In this, we have been guided by two key observations: that nuclei are made of protons and neutrons, not quarks; and that the up and down quarks, which are the fermionic constituents of the protons and neutrons, are much lighter than the principal mass scales of QCD, such as the proton and neutron masses. Taken together, these strongly suggest that the full complexity of the Standard Model can largely be captured, for the purposes of nuclear physics, by an effective field theory (EFT) – SU(2)$_L$ x SU(2)$_R$ chiral perturbation theory (SU(2)$_L$\chi$PT)$ of protons and neutrons.

Building on this longstanding insight, we have studied the chiral limit of spontaneously broken SU(2)$_L$ x SU(2)$_R$ (i.e., SU(2)$_L$\chi$PT)$, including only operators of order $\Lambda_{\chi_{\mathrm{SB}}}$ and $\Lambda^2_{\chi_{\mathrm{SB}}}$. We find that SU(2)$_L$\chi$PT$ of protons, neutrons and three pseudo Nambu-Goldstone Bosons pions - admits a semi-classical liquid phase, a Static Chiral Nucleon Liquid (Static$\chi$NL). Static$\chi$NLs are made entirely of nucleons, with approximately zero anti-proton and antineutron content. They are parity even and time-independent. As we have studied them so far, not just the total nuclear spin $\vec{S} = 0$, but also the local expectation value for spin $< \vec{s} > \sim 0$. Similarly, the nucleon momenta vanish locally in the spherically symmetric Static$\chi$NL rest frame. For these reasons, our study of Static$\chi$NL is applicable to bulk ground state spin-zero nuclear matter, and to the ground state of appropriate spin-zero parity-even nuclei with an even number $Z$ of protons and an even number $N$ of neutrons.

We classify these solutions of SU(2)$_L$\chi$PT$ as "liquid" because energy is required both to pull the constituent nucleons further apart and to push them...
closer together. This is analogous with the balancing of the attractive Lorentz-scalar $\sigma$-exchange force and the repulsive Lorentz-vector $\omega_\mu$-exchange force in the Walecka model. The nucleon number density therefore takes a saturated value even in zero external pressure (e.g. in the absence of gravity), so the material is not a "gas." Meanwhile they are statistically homogeneous and isotropic, lacking the reduced symmetries of crystals or other solids.

We have shown that in this ground state liquid phase, the expectation values of many of the allowed operators of the most general SU(2)$\chi$PT Lagrangian vanish or are small. Going forward, it is imperative to understand the effects of of excited nucleon states to the spectra of heavy nuclei.

We have also shown that this spontaneously broken ground state liquid phase does not support a classical pion field – infrared pions decouple from this solution. We expect that this emergence of "semiclassical pionless SU(2)$\chi$PT" is at the heart of the apparent theoretical independence of much successful nuclear structure physics from pion properties such as the pion mass.

We have constructed explicit Static$\chi$NL’s in the Thomas-Fermi approximation, demonstrating the existence of zero-pressure non-topological soliton Static$\chi$NL solutions with macroscopic (infinite nuclear matter) and microscopic (heavy nuclear ground states).

ACKNOWLEDGMENTS

We thank David M. Jacobs for his early contributions to this work. BJC, BWL and GDS thank Bira van Kolck for crucial discussions. We would also like to thank the referee for suggestions that contributed significantly to the quality of the paper. BJC and BWL thank IPNO, Universite de Paris Sud, for their hospitality in summer 2019; while BJC thanks CWRU for its hospitality in fall 2019. GDS is partially supported by a grant from the US DOE to the particle-astrophysics theory group at CWRU.

[1] T. E. O. Ericson and W. Weise, *Pions and Nuclei* (Clarendon Press, Oxford, UK, 1988).
[2] S. Weinberg, Phys. Lett. B 251, 288 (1990).
[3] C. O. me and U. van Kolck, Phys. Lett. B 291, 459 (1992).  
[4] U. Van Kolck, L. J. Abu-Raddad, and D. M. Cardamone, AIP Conf. Proc. 631, 191 (2002), nucl-th/0205058.
[5] H.-W. Hammer, S. König, and U. van Kolck, Rev. Mod. Phys. 92, 025004 (2020).
[6] S. Weinberg, Phys. Rev. 166, 1568 (1968).
[7] S. Weinberg, Physica A96, 327 (1979).
[8] S. Coleman, J. Weiss, and B. Zumino, Phys. Rev. 177, 2239 (1969).
[9] C. G. Callan, S. Coleman, J. Weiss, and B. Zumino, Phys. Rev. 177, 2247 (1969).
[10] J. Gasser and H. Leutwyler, Annals of Physics 210, 142 (1984).
[11] J. Gasser, Nuclear Physics B 250, 465 (1985).
[12] A. Manohar and H. Georgi, Nuclear Physics B 234, 189 (1984).
[13] H. Georgi, *Weak Interactions and Modern Particle Theory* (Benjamin Cummings, 1984).
[14] B. Borasoy and B. R. Holstein, Eur. Phys. J. C6, 85 (1999), hep-ph/9805430.
[15] E. E. Jenkins and A. V. Manohar, *Baryon chiral perturbation theory*, in *Workshop on Effective Field Theories of the Standard Model* Dobogoko, Hungary, August 22-26, 1991, pp. 113–137, 1991.
[16] B. W. Lynn, Nuclear Physics B 402, 281 (1993).
[17] T. D. Lee and G. C. Wick, Phys. Rev. D9, 2291 (1974).
[18] T. D. Lee, Rev. Mod. Phys. 47, 267 (1975).
[19] T. D. Lee and M. Margulies, Phys. Rev. D11, 1591 (1975), [Erratum: Phys. Rev.D12, 4008(1975)].
[20] S. A. Chin and J. D. Walecka, Phys. Lett. 52B, 24 (1974).
[21] T. Lee, Phys. Rev. C14, 718 (1976).
[22] S. Bahcall, B. W. Lynn, and S. B. Selipsky, Nucl. Phys. B331, 67 (1990).
[23] S. Bahcall, B. W. Lynn, and S. B. Selipsky, Nucl. Phys. B325, 606 (1989).
[24] S. B. Selipsky, D. C. Kennedy, and B. W. Lynn, Nucl. Phys. B321, 430 (1989).
[25] B. W. Lynn, Nucl. Phys. B321, 465 (1989).
[26] S. R. Bahcall and B. W. Lynn, Nuovo Cim. B113, 959 (1998).
[27] G. Rosen, Journal of Mathematical Physics 9, 999 (1968).
[28] T. D. Lee, Phys. Rept. 23, 254 (1976).
[29] R. Friedberg, T. D. Lee, and A. Sirlin, Phys. Rev. D13, 2739 (1976).
[30] R. Friedberg, T. D. Lee, and A. Sirlin, Nucl. Phys. B115, 32 (1976).
[31] S. R. Coleman, Nucl. Phys. B262, 263 (1985), [Erratum: Nucl. Phys.B269, 744(1986)].
[32] R. Friedberg and T. D. Lee, Phys. Rev. D15, 1694 (1977).
[33] J. W. Lee, Phys. Lett. 71B, 367 (1977).
[34] T. F. Morris, Phys. Lett. 76B, 337 (1978).
[35] T. F. Morris, Phys. Lett. 78B, 87 (1978).
[36] B. D. Serot and J. D. Walecka, Phys. Lett. B87, 172 (1979).
[37] S. Chin, Ann. Phys. 108, 301 (1977).
[38] B. D. Serot and J. D. Walecka, Adv. Nucl. Phys. 16, 1 (1986).
Appendix A: SU(2)_L × SU(2)_R × PT of a nucleon doublet and a pion triplet in the spontaneously broken (i.e., chiral) limit

The chiral symmetry of two light quark flavors in QCD, together with the symmetry-breaking and Goldstone’s theorem, makes it possible to obtain an approximate solution to QCD at low energies using a SU(2)_L × SU(2)_R EFT, where the degrees of freedom are hadrons [6][13][39]. In particular, the non-linear SU(2)_χPT effective Lagrangian has been shown to successfully model the interactions of pions with nucleons, where a perturbation expansion (e.g., in soft momentum |k|/Λ_{χSB} ≪ 1, baryon number density ∂ N/N ∂ |k|/Λ_{χSB} ≪ 1, for chiral symmetry breaking scale Λ_{χSB} ≈ 1 GeV) has demonstrated predictive power. Such naïve power-counting in Λ_{χSB} includes all analytic quantum-loop effects into experimentally measurable coefficients of SU(2)_L × SU(2)_R current-algebraic operators obedient to the global symmetries of QCD, with light-quark masses generating additional explicit chiral-symmetry-breaking terms. Therefore, SU(2)_L × SU(2)_R χPT tree-level calculations with a naïve power-counting effective Lagrangian are to be regarded as true predictions of QCD and the Standard Model of elementary particles.

1. Non-linear transformation properties

We present the Lagrangian of SU(2)_L × SU(2)_R χPT of a nucleon doublet and a pNGB triplet. We employ the defining SU(2) strong isospin representation of unitary 2 × 2 Pauli matrices τ_a, with asymmetric structure constants f_{abc} = ε_{abc}

\[ t_a = \frac{\tau_a}{2}, \quad a = 1, 3 \]

\[ \text{Tr}(t_at_b) = \frac{\delta_{ab}}{2} \]

\[ [t_a, t_b] = i f_{abc} t_c \]

\[ \{t_a, t_b\} = \frac{\delta_{ab}}{2} \]

The vector and axial-vector charges obey the algebra:

\[ [Q_a^{L+R}, Q_b^{L+R}] = i f_{abc} Q_c^{L+R} \]

\[ [Q_a^{L-R}, Q_b^{L-R}] = i f_{abc} Q_c^{L-R} \]

\[ [Q_a^{L+R}, Q_b^{L-R}] = i f_{abc} Q_c^{L-R} \]  

We consider a triplet representation of NGBs,

\[ \pi_a t_a = \frac{1}{\sqrt{2}} \begin{bmatrix} a^0 & \pi^+ \\ \pi^- & -a^0 \end{bmatrix} \]
and a doublet of nucleons,
\[ N = \begin{bmatrix} p \\ n \end{bmatrix}. \] (A.4)

For pedagogical simplicity, representations of higher mass are neglected, even though the SU(2)_L \times SU(2)_R baryon decuplet (especially \Delta_{1232}) is known to have important nuclear structure and scattering effects.

Since SU(2)\chi PT matrix elements are independent of representation [8, 9], we choose a representation [12, 13, 59] where the NGB triplet has only derivative couplings,
\[ \Sigma \equiv \exp(2i\pi_a t_a f_\pi). \] (A.5)

Under a unitary global SU(2)_L \times SU(2)_R transformation, given by \( L \equiv \exp(it_a t_a) \) and \( R \equiv \exp(i\pi_a t_a) \),
\[ \Sigma \to \Sigma' = L \Sigma R^\dagger. \] (A.6)

It also proves useful to introduce the “square root” of \( \Sigma \)
\[ \xi \equiv \exp(i\pi_a t_a f_\pi), \] (A.7)
which transforms as
\[ \xi \to \xi' = \exp(i\pi_a t_a f_\pi). \] (A.8)

We observe that
\[ \xi' = L \xi U^\dagger = U \xi R^\dagger, \] (A.9)
for a certain unitary local transformation matrix \( U(L, R, \pi_a(t, x)) \).

The vector and axial-vector NGB currents
\begin{align*}
V_\mu &\equiv \frac{1}{2}(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger), \\
A_\mu &\equiv \frac{i}{2}(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger),
\end{align*}
(A.10)
transform straightforwardly as:
\begin{align*}
V_\mu \to V' &= U V_\mu U^\dagger + U \partial_\mu U^\dagger, \\
A_\mu \to A' &= U A_\mu U^\dagger.
\end{align*}
(A.11)

Meanwhile the nucleons transform as
\[ N \to N' = U N, \] (A.12)
and
\[ D_\mu N \equiv \partial_\mu N + \pi_\mu N \to U(D_\mu N). \] (A.13)

2. Naive \( \Lambda_{\chi SB} \) operator power counting

The SU(2)\chi PT Lagrangian, including all analytic quantum-loop effects for soft momenta (\( \ll 1 \) GeV), can be written [12, 59]:
\begin{align*}
L_{\chi PT} &= - \sum_{l,m,n} C_{lmn} f_\pi^2 \Lambda_{\chi SB}^2 \left( \frac{\partial_\mu}{\Lambda_{\chi SB}} \right)^m \left( \frac{\sqrt{N}}{f_\pi} \right)^l \left( \frac{m_{\text{quark}}}{\Lambda_{\chi SB}} \right)^n f_{lmn} \left( \frac{\pi_a}{f_\pi} \right), \quad (A.14)
\end{align*}
where \( f_{lmn} \) is an analytic function, and the dimensionless constants \( C_{lmn} \) are \( O(\Lambda_{\chi SB}^0) \) and, presumably, \( \sim 1 \). As a power series in \( \Lambda_{\chi SB} \) we take, self-consistently, \( \Lambda_{\chi SB} \approx 1 \) GeV and, in higher orders, reorder the non-relativistic perturbation expansion in \( \partial_\mu \) to converge with large nucleon mass \( m^N \approx \Lambda_{\chi SB} \) [2, 01, 62].

3. The Chiral Symmetric Limit

For the purposes of this paper, we retain from [A.14] only terms of order \( \Lambda_{\chi SB} \) and \( \Lambda_{\chi SB}^0 \), i.e., \( 1 \leq m + l + n \leq 2 \). We can further divide \( L_{\chi PT} \) into a symmetric piece (i.e., with spontaneous breaking and massless Goldstones) and a symmetry-breaking piece (i.e., explicit breaking, arising from non-zero quark masses) generating three massive pNGB:
\[ L_{\chi PT} = L_{\chi PT}^{\text{Sym}} + L_{\chi PT}^{\text{Sym-Breaking}}. \] (A.15)

In this paper, we are interested only in unbroken SU(2)\chi PT and so take \( n = 0 \) in (A.14)
\[ L_{\chi PT}^{\text{Sym-Breaking}} = 0. \] (A.16)
We separate \( L_{\chi PT}^{\text{Sym}} \) into pure meson terms, terms quadratic in baryons (i.e., nucleons), and four-baryon terms:
\begin{align*}
L_{\chi PT}^{\text{Sym}} &= L_{\chi PT}^{\text{Sym}:2} + L_{\chi PT}^{N:\text{Sym}} + L_{\chi PT}^{4-N:\text{Sym}}. \quad (A.17)
\end{align*}
with (as in (1)):

\[
L^{\pi;}_{\chi PT}^{\text{Sym}} = \frac{f_\pi^2}{4} \text{Tr} \partial_\mu \Sigma \partial^\mu \Sigma^\dagger,
\]

\[
L^{N;}_{\chi PT}^{\text{Sym}} = \mathcal{N} (i\gamma^\mu D_\mu - mN^\dagger) N - g_A N\gamma^\mu A_\mu N,
\]

\[
L^{4-N;}_{\chi PT}^{\text{Sym}} \sim \frac{1}{f_\pi^2} \left( \mathcal{N} \gamma^a N \right) \left( \mathcal{N} \gamma^a N \right) + \ldots,
\]

(A.18)

As described below (1), the parentheses in the four-nucleon Lagrangian indicate the order of SU(2) index contraction, and +++ indicates that one should include all possible combinations of such contractions. As usual, \( \gamma^a \equiv \{1, \epsilon, \sigma^{\mu\nu}, i\sigma^5, \gamma^5, \gamma^5\} \), for \( a = 1, \ldots, 16 \) (with \( \sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu] \)). These are commonly referred to as scalar (S), vector (V), tensor (T), axial-vector (A), and pseudo-scalar (P) respectively.

In this paper, we will focus on the Semi-Classical Symmetries of chiral (i.e., spontaneously broken) SU(2) \( \chi PT \). Nucleons are treated as quantized fermions. Pions are classical fields: i.e., \( \xi, V_\mu, A_\mu, U, \Sigma, \pi_a, R, L \) defined in Subsection (A.1) are not quantized: their non-trivial commutation properties are entirely due to strong isospin.

4. SU(2)\(_L\)xSU(2)\(_R\) invariant 4-nucleon contact interactions

We focus on the 4-fermion terms in (A.18). We use the completeness relation for \( 2 \times 2 \) matrices:

\[
\delta_{cf} \delta_{cd} = 2 \sum_{B=0}^{3} t^B_{cd} t^B_{cf};
\]

(A.19)

\[
\left[ \tilde{t}, U \left( \tilde{\pi}(x), r, \ell \right) \right] \neq 0;
\]

with \( t^B = (\frac{1}{2} I, \tilde{t}) \). (We use Greek letters for relativistic spinor indices, and Roman letters for isospin indices.) Both iso-scalar and iso-vector 4-nucleon contact interactions appear in the SU(2)\(_L\)xSU(2)\(_R\) invariant Lagrangian:

\[
L^{4-N;}_{\chi PT}^{\text{Sym}} = \frac{C^{T=0}}{f_\pi^2} \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \left( \mathcal{N} \gamma^a \mathcal{N} N \right) + \frac{C^{T=1}}{f_\pi^2} \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \left( \mathcal{N} \gamma^a \mathcal{N} N \right) + \ldots
\]

\[
+ \frac{C^{T=0}}{f_\pi^2} \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \delta_{cf} \delta_{cd}
\]

\[
+ \frac{C^{T=1}}{f_\pi^2} \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \delta_{cf} \delta_{cd}
\]

\[
+ 2 \sum_{B=0}^{3} \frac{C^{T=1}}{f_\pi^2} \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \delta_{cf} \delta_{cd}
\]

(B.20)

Appendix B: 4-nucleon contact interactions in Static\( \chi \)NL's

1. Boson-exchange-inspired 4-nucleon contact interactions

We wish to study the expectation value of \( L^{4-N;}_{\chi PT}^{\text{Sym}} \) in the ground state of the chiral nuclear liquid (which we continue to represent with \( \langle \rangle \)). Using (A.20) we find:

\[
-L^{BE}_{\chi \chi \chi \chi NL} \equiv \langle - L^{4-N;}_{\chi PT}^{\text{Sym}} \rangle = \sum_{L=0}^{3} \frac{C^{T=0}}{f_\pi^2} \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \left( \mathcal{N} \gamma^a \mathcal{N} N \right) + \sum_{L=0}^{3} \frac{C^{T=1}}{f_\pi^2} \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \delta_{cf} \delta_{cd}
\]

(B.1)

In what follows we ignore any and all excited states and consider the effective Lagrangian:

\[
-L^{BE}_{\chi \chi \chi \chi NL} = \frac{1}{2f_\pi^2} \sum_{L=0}^{3} \left[ \mathcal{N} \gamma^a \mathcal{N} N \right] \left( \mathcal{N} \gamma^a \mathcal{N} N \right) + 2 \sum_{B=0}^{3} \frac{C^{T=1}}{f_\pi^2} \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \left( \mathcal{N} \gamma^a \mathcal{N} N \right) \delta_{cf} \delta_{cd}
\]

(B.2)
A useful identity is:

\[
\frac{1}{2} \langle \bar{N}_e^\alpha \gamma^{\rho \sigma \alpha} N^\beta \rangle \langle \bar{N}_e^\lambda \gamma^{\rho \sigma \lambda} N^\sigma_e \rangle \\
+ \langle \bar{N}_e^\alpha t_{3;cd} \gamma^{\rho \sigma \alpha} N^\beta_d \rangle \langle \bar{N}_e^\lambda t_{3;ef} \gamma^{\rho \sigma \lambda} N^\sigma_f \rangle \\
= \frac{1}{2} \langle \bar{T}^e_\rho \gamma^{\rho \sigma \alpha} \bar{T}^e_\rho \rangle \langle \bar{T}^e_\rho \gamma^{\rho \sigma \lambda} \bar{T}^e_\rho \rangle \\
+ \frac{1}{2} \langle \bar{N}_e^\rho \gamma^{\rho \sigma \alpha} \bar{N}_e^\rho \rangle \langle \bar{N}_e^\rho \gamma^{\rho \sigma \lambda} \bar{N}_e^\rho \rangle.
\]

(3)

2. Contact interactions that mimic hadronic boson-exchange

Taking expectation values inside the Static NL as in \( B.3 \), we obtain:

\[- L_{SxNL}^{BE} \simeq \frac{1}{2 f_p^2} (L_{Sx0}^{T=0} + L_{Vx0}^{T=0} + L_{Sx1}^{T=1} + L_{Vx1}^{T=1})\]

where

\[L_{Sx0}^{T=0} = C_{Sx0}^{T=0} \langle \bar{N}_e^\alpha N^\beta_e \rangle \langle \bar{N}_e^\lambda N^\sigma_e \rangle,\]

\[L_{Vx0}^{T=0} = C_{Vx0}^{T=0} \langle \bar{N}_e^\alpha \gamma^{0 \beta \alpha} N^\beta_e \rangle \langle \bar{N}_e^\lambda \gamma^{0 \lambda \sigma} N^\sigma_e \rangle,\]

\[L_{Sx1}^{T=1} = 2 C_{Sx1}^{T=1} \left\{ \frac{1}{4} \langle \bar{N}_e^\alpha N^\alpha_e \rangle \langle \bar{N}_e^\lambda N^\lambda_e \rangle \right. \]

\[+ \left. \langle \bar{N}_e^\alpha t_{3;cd} N^\beta_d \rangle \langle \bar{N}_e^\lambda t_{3;ef} N^\sigma_f \rangle \right\},\]

\[L_{Vx1}^{T=1} = 2 C_{Vx1}^{T=1} \left\{ \frac{1}{4} \langle \bar{N}_e^\alpha \gamma^{0 \beta \alpha} N^\beta_e \rangle \langle \bar{N}_e^\lambda \gamma^{0 \lambda \sigma} N^\sigma_e \rangle \right. \]

\[+ \left. \langle \bar{N}_e^\alpha t_{3;cd} \gamma^{0 \alpha \beta} N^\beta_d \rangle \langle \bar{N}_e^\lambda t_{3;ef} \gamma^{0 \lambda \sigma} N^\sigma_f \rangle \right\}.\]

(4)

The factorization in \( L_{SxNL}^{BE} \), and its name, are inspired by a simple picture of forces carried by heavy hadronic-boson exchange, which is commonly envisioned in Walecka-like, nuclear-Skyrme and density-functional models; i.e., we have integrated out the auxiliary fields:

- Lorentz-scalar isoscalar \( \sigma \), with chiral coefficient \( C_S^{T=0} \);
- Lorentz-vector isoscalar \( \omega_\mu \), with chiral coefficient \( C_V^{T=0} \);
- Lorentz-scalar isovector \( \delta \), with chiral coefficient \( C_S^{T=1} \);
- Lorentz-vector isovector \( \rho_\mu \), with chiral coefficient \( C_V^{T=1} \).

To order \( \Lambda_{SB}^0 \), the only 4-nucleon contact terms allowed by local SU(2) \( \chi PT \) symmetry are exhibited in \( B.3 \) and \( B.4 \). Note that isospin operators \( \tilde{t} = \frac{1}{2} \tau \) have appeared. However, quantum-loop naive power counting requires inclusion of nucleon Lorentz-spinor-interchange interactions, in order to enforce anti-symmetrization of fermion wavefunctions; these are the same order as direct interactions; i.e., \( O(\Lambda_{SB}^0) \). The empirical nuclear models of Manakos and Mannel \([63, 64]\) were specifically built to include such spinor-interchange terms.

Explicit inclusion of spinor-interchange terms yields a great technical advantage for the liquid approximation: it allows us to treat Static NLs in Hartree-Fock approximation, i.e., including fermion wave function anti-symmetrization, rather than in less-accurate Hartree approximation.

3. Contact-interactions, including spinor-interchange terms enforcing effective anti-symmetrization of fermion wavefunctions in the Hartree-Fock approximation

In this section, we write an effective Static NL Lagrangian for the four-nucleon contact interactions in terms of the ten independent chiral coefficients: \( C_T^K \) with \( K \in \{S, V, T, A, P\} \) and \( T \in \{0, 1\} \).

For pedagogical simplicity, we first focus on the “boson-exchange-inspired” terms, with power-counting contact-interactions of order \( (\Lambda_{SB}^0) \). “Direct” terms depend only on \( C_T^{I=0}, C_T^{I=0}, C_T^{I=1}, \) and \( C_T^{I=1} \), because isoscalar \( (C_T^{I=0}, C_T^{I=0}, C_T^{I=0}) \), and isovector \( (C_T^{I=1}, C_T^{I=1}, C_T^{I=1}) \) vanish when evaluated in the liquid. “Spinor-interchange” terms depend all 10 coefficients after Fierz rearrangement. (Such terms do not appear in the SU(2) \( \chi PT \) analysis of the deuterium ground state, because it only has 1 proton and 1 neutron.) The combination of direct and spinor-interchange terms (which we refer to below as “Total”) depends on all 10 coefficients.

Because of the inclusion of spinor exchange terms, Hartree treatment of the Static NL Lagrangian is equivalent to Hartree-Fock treatment of the liquid. When building the semi-classical liquid quantum state, this enforces the anti-symmetrization of the fermion wavefunctions. A crucial observation is that the resultant liquid depends on only four independent chiral coefficients: \( C_S^2, C_V^2, C_S^2, \) and \( C_T^2 \). These provide sufficient free parameters to balance the scalar attractive force carried by \( C_S^2 \) and \( C_T^2 \) against the vector repulsive force carried by \( C_V^2 \) when fitting to the experimentally observed structure of ground state nuclei (as reflected, e.g., in the different signs in definition of \( C_S^2 \) and \( C_T^2 \) in \([25]\) and \([26]\)).

Motivated by the empirical success of Non-topological Soliton models we conjecture that excited-nucleon-inspired contact-interaction terms are small, and that the simple picture of scalar attraction balanced against vector repulsion persists when including them. Such analysis is beyond the
scope of this paper.

a. **Lorentz Vector (V) and Axial-vector (A) forces**

Proceeding in a similar manner for the vector and axial vector terms we find:

\[
-L_{S\chi NL}^{V,A} = - \langle L^4 - N^2 \rangle V,A \equiv \frac{1}{2f_\pi^2} \sum_{\sigma d = V,A} \left\{ 2 C^{T=0}_d \left\langle N^\alpha \gamma^{\sigma\alpha\beta} N^\beta \right\rangle \left\langle \bar{N}^\lambda \gamma^{\lambda\sigma} N^\sigma \right\rangle + C^{T=1}_d \left\langle N^\alpha t^B \gamma^{\sigma\alpha\beta} N^\beta \right\rangle \left\langle \bar{N}^\lambda t^B \gamma^{\lambda\sigma} N^\sigma \right\rangle \right\} + 2 \sum_B C^{T=1}_B \left\langle N^\alpha \gamma^{\sigma\alpha\beta} N^\beta \right\rangle \left\langle \bar{N}^\lambda \gamma^{\lambda\sigma} N^\sigma \right\rangle \right\} \]

which is

\[
-L_{S\chi NL}^{V,A} = \frac{1}{2f_\pi^2} \sum_{\sigma d = V,A} \left\{ 2 C^{T=0}_d \left\langle \bar{p}_e^\gamma \gamma^{\sigma\alpha\beta} p^\beta \right\rangle \left\langle \bar{p}_e^\lambda \gamma^{\lambda\sigma} n^\sigma \right\rangle \right\} + \left[ C^{T=0}_d + C^{T=1}_d \right] \left\{ \left\langle \bar{p}_e^\gamma \gamma^{\sigma\alpha\beta} p^\beta \right\rangle \left\langle \bar{p}_e^\lambda \gamma^{\lambda\sigma} n^\sigma \right\rangle + \left\langle \bar{p}_e^\gamma \gamma^{\sigma\alpha\beta} n^\beta \right\rangle \left\langle \bar{p}_e^\lambda \gamma^{\lambda\sigma} n^\sigma \right\rangle \right\} \]

\[
(B.6)
\]

\[
(B.7)
\]

a. **Direct terms:** The properties of Static\(\chi\)NLs enable this expression to be written as:

\[
-L_{S\chi NL,D}^{V,A} = \frac{1}{2f_\pi^2} \left\{ 2 \left\langle p^\dagger p \right\rangle \left\langle n^\dagger n \right\rangle \right\} + \frac{1}{2f_\pi^2} \left[ C^{T=0}_V + C^{T=1}_V \right] \left\{ \left\langle p^\dagger p \right\rangle^2 + \left\langle n^\dagger n \right\rangle^2 \right\} \]

(B.8)

where \(\left\langle p^\dagger p \right\rangle\) and \(\left\langle n^\dagger n \right\rangle\) represent \(\left\langle p^\dagger_c p^\alpha_c \right\rangle\) and \(\left\langle n^\dagger_c n^\lambda_c \right\rangle\), respectively.

b. **Spinor-interchange terms:** After interchanging the appropriate spinors, normal ordering creation and annihilation operators, and Fierz re-arrangement, spinor-interchange contributions depend on \(C^{T=0}_V, C^{T=0}_A, C^{T=1}_V, \text{ and } C^{T=1}_A\).

\[
-L_{S\chi NL;Ex}^{V,A} = \frac{1}{2f_\pi^2} \left[ \left( C^{T=0}_V + C^{T=1}_V \right) + \left( C^{T=0}_A + C^{T=1}_A \right) \right] \times \left\{ \left\langle p^\dagger L_p L \right\rangle^2 + \left\langle p^\dagger R p R \right\rangle^2 + \left\langle n^\dagger L n L \right\rangle^2 + \left\langle n^\dagger R n R \right\rangle^2 \right\} \]

(B.9)

where we have expanded the spinors \(p\) and \(n\) into left-handed and right-handed components via \(p = p_L + p_R\) and \(n = n_L + n_R\).

c. **Total direct and spinor-interchange terms:** Combining the direct and exchange terms yields:

\[
-L_{S\chi NL;Total}^{V,A} = \frac{1}{2f_\pi^2} \left\{ 2 \left\langle p^\dagger p \right\rangle \left\langle n^\dagger n \right\rangle \right\} + \frac{C^0_V + C^1_V}{2f_\pi^2} \left\{ \left\langle p^\dagger L p L \right\rangle \left\langle p^\dagger R p R \right\rangle + \left\langle n^\dagger L n L \right\rangle \left\langle n^\dagger R n R \right\rangle \right\}

+ \frac{C^0_A + C^1_A}{2f_\pi^2} \sum_{h = L,R} \left\{ \left\langle p^\dagger_h p_h \right\rangle^2 + \left\langle n^\dagger_h n_h \right\rangle^2 \right\} \]

(B.10)

The reader should note the cancellation of the term

\[
(C^{T=0}_V + C^{T=1}_V) \sum_{h = L,R} \left\{ \left\langle p^\dagger_h p_h \right\rangle^2 + \left\langle n^\dagger_h n_h \right\rangle^2 \right\} \]

(B.11)

showing that vector-boson exchange cannot carry forces between same-handed fermion protons, or between same-handed fermion neutrons.

Significant simplification follows because Static\(\chi\)NLs are defined to have equal left-handed and right-handed densities; i.e.,

\[
\left\langle p^\dagger L p L \right\rangle = \left\langle p^\dagger R p R \right\rangle = \frac{1}{2} \left\langle p^\dagger p \right\rangle

\left\langle n^\dagger L n L \right\rangle = \left\langle n^\dagger R n R \right\rangle = \frac{1}{2} \left\langle n^\dagger n \right\rangle \]

(B.12)
Using (25) the contribution of (B.10) to the Lorentz-spinor-interchange Lagrangian can be written:

\[-L^{V,A}_{S\chi NL:total} = \frac{1}{2} C^V \langle N^1 N \rangle^2 - C^V \langle N^1 t_3 N \rangle^2\]  
(B.13)

with

\[C^V_{200} = C^{T=0}_V\]
\[-C^V_{200} = \frac{1}{2} \left( -C^{T=0}_V + C^{A}_V + C^{T=1}_V + C^{A}_V \right).\]

(B.14)

The crucial observation is that (B.13) and (B.14) depend on just two independent chiral coefficients, \(C^S_0\) and \(\overline{C}^V_0\), (or equivalently \(C^S_{200}\) and \(\overline{C}^V_{200}\)), instead of four, while still providing sufficient free parameters to fit the vector repulsive force (i.e., within Non-topological Soliton, Density Functional and Skyrme nuclear models) up to naive power-counting order \((A^0_{\chi SB})\), to the experimentally observed structure of ground state nuclei.

b. Lorentz Scalar (S), Tensor (T) and Pseudo-scalar (P) forces

Proceeding in a similar manner we define:

\[L^{STP}_{S\chi NL} \equiv \left\{ L^{A-N;STP}_\chi \right\}\]  
(B.15)

with

\[-L^{STP}_{S\chi NL} = \frac{1}{2 f^2_{\pi}} \sum_{s=T,P} \left\{ C^{T=0}_{st} \langle N^c_\gamma \gamma^{d\alpha\beta} N^d_\sigma \rangle \langle \hat{\gamma}_{\alpha\beta} \hat{\gamma}_{\sigma} \rangle \right\} \]
\[+ 2 \sum_B C^{T=1}_{st} \langle N^c_\gamma \gamma^{d\alpha\beta} N^d_\sigma \rangle \langle \hat{\gamma}_{\alpha\beta} \hat{\gamma}_{\sigma} \rangle \]
\[+ \left( C^{T=0}_{st} + C^{T=1}_{st} \right) \left\{ \langle \hat{p}_e \gamma^{d\alpha\beta} p_c \rangle \langle \hat{\gamma}_{\alpha\beta} \hat{\gamma}_{\sigma} \rangle \right\} \]
\[+ \left( C^{T=0}_{st} + C^{T=1}_{st} \right) \left\{ \langle \hat{p}_e \gamma^{d\alpha\beta} n_c \rangle \langle \hat{\gamma}_{\alpha\beta} \hat{\gamma}_{\sigma} \rangle \right\} \]
(B.16)

which is:

\[-L^{STP}_{S\chi NL} = \frac{1}{2 f^2_{\pi}} \sum_{s=T,P} \left\{ C^{T=0}_{st} \langle N^c_\gamma \gamma^{d\alpha\beta} N^d_\sigma \rangle \langle \hat{\gamma}_{\alpha\beta} \hat{\gamma}_{\sigma} \rangle \right\} \]
\[+ \left( C^{T=0}_{st} + C^{T=1}_{st} \right) \left\{ \langle \hat{p}_e \gamma^{d\alpha\beta} p_c \rangle \langle \hat{\gamma}_{\alpha\beta} \hat{\gamma}_{\sigma} \rangle \right\} \]
\[+ \left( C^{T=0}_{st} + C^{T=1}_{st} \right) \left\{ \langle \hat{p}_e \gamma^{d\alpha\beta} n_c \rangle \langle \hat{\gamma}_{\alpha\beta} \hat{\gamma}_{\sigma} \rangle \right\} \]
(B.17)

a. Direct terms: The properties of Static\(\chi NLs\)

\[-L^{STP}_{S\chi NL:D} = \frac{1}{2 f^2_{\pi}} C^{T=0}_{st} \langle \hat{N} N \rangle \langle \hat{N} N \rangle \]
\[+ \frac{1}{2 f^2_{\pi}} C^{T=1}_{st} \left( \langle \hat{p} p \rangle \langle \hat{p} p \rangle + \langle \hat{\pi} n \rangle \langle \hat{\pi} n \rangle \right).\]
(B.18)

b. Spinor-interchange terms: Spinor-interchange contributions depend on six chiral coefficients: isoscalars \(C^S_T=0\), \(C^S_T=0\), \(C^P_T=0\) and isovectors \(C^S_T=1\), \(C^S_T=1\), \(C^P_T=1\).

\[-L^{STP}_{\chi NL:Ex} = \frac{1}{4 f^2_{\pi}} \left[ \left( C^{T=0}_T + C^{T=1}_T \right) \right. \]
\[+ 6 \left( C^{T=0}_T + C^{T=1}_T \right) + \left( C^{T=0}_P + C^{T=1}_P \right) \]
\[\times \left\{ \langle \hat{\gamma}_{LPR} \rangle^2 + \langle \hat{\gamma}_{RPL} \rangle^2 + \langle \hat{\pi}_{LNL} \rangle^2 + \langle \hat{\pi}_{RNL} \rangle^2 \right\} \]
(B.19)

c. Total direct and spinor-interchange terms: As above, since Static\(\chi NLs\) have equal left-handed and right-handed scalar densities by definition, the total direct and spinor-interchange contribution is considerably simplified:

\[-L^{STP}_{S\chi NL:total} = -\frac{1}{2} C^2_\chi \langle \hat{N} N \rangle^2 - C^2_\chi \langle \hat{N} t_3 N \rangle^2\]  
(B.20)

where in (20) and (28) we have:

\[C^S_{200} = C^{T=0}_T,\]
\[-C^S_{200} = \frac{1}{2} \left( C^{T=0}_T + C^{T=1}_T \right) + \left( C^{P=0}_P + C^{P=1}_P \right) \]
\[+ 3 \left( C^{T=0}_T + C^{T=1}_T \right) + \left( C^{P=0}_P + C^{P=1}_P \right) \]
(B.21)

Once again we find that (B.20) and (B.21) depend on just two independent chiral coefficients, \(C^S_{200}\) and \(\overline{C}^V_{200}\), instead of six, while still providing sufficient free parameters to fit the scalar attractive force (i.e., within Non-topological Soliton, Density Functional and Skyrme nuclear models) up to naive power-counting order \((A^0_{\chi SB})\), to the experimentally observed structure of ground state nuclei.

Appendix C: Nucleon bi-linears and semi-classical nuclear currents in Static\(\chi NLs\)

The structure of Static\(\chi NLs\) suppresses various nucleon bi-linears:

- Vectors’ space-components: because it is a 3-vector, parity odd and stationary

\[\langle \hat{N}^c_\gamma \gamma^{d\alpha\beta} N^d_\sigma \rangle \sim \langle \hat{k} \rangle \approx 0\]  
(C.1)

- Tensors: because the local expectation value of nuclear spin \(<\hat{s}> = \frac{1}{3} <\hat{d}> \approx 0\)
1. $\sigma^{ij}$:
\[
\langle N_{c}^{\alpha} \sigma^{ij,\alpha \beta} N_{c}^{\beta} \rangle \\
= \langle N_{L}^{\alpha} \sigma^{ij} N_{R}^{\alpha} \rangle + \langle N_{R}^{\alpha} \sigma^{ij} N_{L}^{\alpha} \rangle \\
= 2\langle N_{L}^{\alpha} \begin{bmatrix} 0 & s_{j} \\ s_{j} & 0 \end{bmatrix} N_{R}^{\alpha} \rangle \\
+ 2\langle N_{R}^{\alpha} \begin{bmatrix} 0 & s_{j} \\ s_{j} & 0 \end{bmatrix} N_{L}^{\alpha} \rangle \\
\simeq 0 
\] (C.2)

2. $\sigma^{ij}$:
\[
\langle N_{c}^{\alpha} \sigma^{ij,\alpha \beta} N_{c}^{\beta} \rangle \\
= \langle N_{L}^{\alpha} \sigma^{ij} N_{R}^{\alpha} \rangle + \langle N_{R}^{\alpha} \sigma^{ij} N_{L}^{\alpha} \rangle \\
= -2i\epsilon_{ijk} \langle N_{L}^{\alpha} s_{k} N_{R}^{\beta} \rangle \\
- 2i\epsilon_{ijk} \langle N_{R}^{\alpha} s_{k} N_{L}^{\beta} \rangle \\
\simeq 0
\] (C.3)

- Axial-vectors: because $p_{L}, p_{R}$ are equally represented in Static $\chi_{NL}$, as are $n_{L}, n_{R}$:
\[
\langle \bar{N}_{c}^{\alpha} \gamma^{A,\alpha \beta} N_{c}^{\beta} \rangle \\
= \langle N_{L}^{\alpha} \gamma^{A} N_{R}^{\alpha} \rangle + \langle N_{R}^{\alpha} \gamma^{A} N_{L}^{\alpha} \rangle \\
= -\langle N_{L}^{\alpha} \gamma^{A} N_{L}^{\alpha} \rangle + \langle N_{R}^{\alpha} \gamma^{A} N_{R}^{\alpha} \rangle \\
\simeq 0
\] (C.4)

- Pseudo-scalars: because Static $\chi_{NL}$ are of even parity
\[
\langle N_{c}^{\alpha} \gamma^{P,\alpha \beta} N_{c}^{\beta} \rangle \\
= \langle N_{R}^{\alpha} \gamma^{P} N_{L}^{\alpha} \rangle + \langle N_{L}^{\alpha} \gamma^{P} N_{R}^{\alpha} \rangle \\
\simeq 0
\] (C.5)

Therefore, various Lorentz and isospin representations are suppressed in Static $\chi_{NL}$. In summary: Isoscalars
\[
\langle N_{c}^{\alpha} N_{c}^{\alpha} \rangle \neq 0, \\
\langle N_{c}^{\alpha} \gamma^{0,\alpha \beta} N_{c}^{\beta} \rangle \neq 0, \\
\langle N_{c}^{\alpha} \gamma^{\alpha \beta} N_{c}^{\beta} \rangle \simeq 0, \\
\langle N_{c}^{\alpha} \gamma^{T,\alpha \beta} N_{c}^{\beta} \rangle \simeq 0, \\
\langle N_{c}^{\alpha} \gamma^{A,\alpha \beta} N_{c}^{\beta} \rangle \simeq 0, \\
\langle N_{c}^{\alpha} \gamma^{P,\alpha \beta} N_{c}^{\beta} \rangle \simeq 0.
\] (C.6)

Now form the semi-classical nuclear currents
\[
J_{k}^{\mu} = \bar{N} \gamma^{\mu} t_{k} N, \quad k = 1, 2, 3
\]
\[
J_{3}^{\mu} = \frac{1}{2} (\bar{p} \gamma^{\mu} p - \bar{\gamma} \gamma^{\mu} p),
\]
\[
J_{5}^{\mu} = \frac{\sqrt{3}}{2} (\bar{p} \gamma^{\mu} p + \bar{\gamma} \gamma^{\mu} p),
\]
\[
J_{QED}^{\mu} = \frac{1}{\sqrt{3}} J_{5}^{\mu} + J_{3}^{\mu} = \bar{p} \gamma^{\mu} p
\] (C.8)

SU(2)$_{L}$×SU(2)$_{R}$ nuclear currents within Static $\chi_{NL}$ are obedient to its semi-classical symmetries. Thus we have:
\[
\langle J_{\pm}^{\mu,5} \rangle = \langle \partial_{\mu} J_{\pm}^{\mu,5} \rangle = \langle \partial_{\mu} J_{\pm}^{\mu,5} \rangle = 0
\] (C.9)

and
\[
\langle \partial_{\mu} J_{3}^{\mu} \rangle, \langle J_{3}^{\mu} \rangle, \langle J_{5}^{\mu} \rangle \simeq 0,
\]
\[
\langle J_{3}^{\mu=1,2,3} \rangle, \langle J_{5}^{\mu=1,2,3} \rangle \simeq 0,
\]
\[
\langle \partial_{\mu} J_{5}^{\mu} \rangle, \langle \partial_{\mu} J_{Baryon}^{\mu} \rangle, \langle \partial_{\mu} J_{QED}^{\mu} \rangle \simeq 0,
\]
\[
\langle J_{5}^{\mu=1,2,3} \rangle, \langle J_{QED}^{\mu=1,2,3} \rangle \simeq 0,
\] (C.10)

\[
\frac{1}{\sqrt{3}} \langle \partial_{\mu} J_{5}^{\mu,5} \rangle \propto \langle (m^{N} + C_{200}^{\tilde{S}})^{5} \rangle \sim \eta \simeq 0,
\]
\[
\frac{1}{2} \langle \partial_{\mu} J_{3}^{\mu,5} \rangle \propto \langle (m^{N} + C_{200}^{\tilde{S}})^{5} \rangle \sim \pi \simeq 0.
\] (C.11)
The remaining non-zero contributions to the currents are:

\[ \langle J^0_{\text{baryon}} \rangle \neq 0; \]
\[ \langle J^0_n \rangle \neq 0; \]
\[ \langle J^0_p \rangle \neq 0; \]
\[ \langle J^0_{\text{QED}} \rangle \neq 0; \]  \hspace{1cm} (C.12)

Appendix D: Thomas-Fermi non-topological solitons and the semi-empirical mass formula

We are interested here in semi-classical solutions to (31), identifiable as quantum chiral nucleon liquids, that are, for reasons laid out in the main body of the paper: in the ground state, spin zero, spherically symmetric, and even-even (i.e., have an even number of protons and of neutrons). We employ relativistic mean-field point-coupling Hartree-Fock and Thomas-Fermi approximations, ignoring the anti-nucleon sea.

We seek solutions that are static, homogeneous and isotropic. Given the absence of any surface terms at the order \( A^0_{SB} \) in chiral symmetry breaking to which we are working, we avoid the ad hoc imposition of such terms. We therefore impose the condition that the pressure vanishes everywhere, rather than just at the surface of a finite “liquid drop.” Our finite Static NL nuclei therefore resemble “ice cream balls scooped from an infinite vat” [65], more than they do conventional liquid drops (which have surface tension).

The Thomas-Fermi approximation replaces the neutrons and protons with homogeneous and isotropic expectation values over free neutron and proton spinors, with (for \( j = n \) and \( p \)) effective reduced mass \( m^j \), 3-momentum \( \vec{k}^j \), energy \( E^j = \sqrt{(\vec{k}^j)^2 + (m^j)^2} \), and zero spin. Most of these vanish because of the absence of any preferred direction for spin or momenta in Static NL:

\[ \vec{p} \rightarrow \langle \vec{p} \rangle = \frac{m^*}{E_n^*}; \]
\[ \pi (\gamma^0, \vec{\gamma}) n \rightarrow \langle \pi (\gamma^0, \vec{\gamma}) n \rangle = (1, 0), \]
\[ \pi (\sigma^{0j}, \sigma^{ij}) n \rightarrow \langle \pi (\sigma^{0j}, \sigma^{ij}) n \rangle = 0, \]
\[ \pi (\gamma^0, \vec{\gamma}) \gamma^5 n \rightarrow \langle \pi (\gamma^0, \vec{\gamma}) \gamma^5 n \rangle = 0, \]
\[ \pi \gamma^5 n \rightarrow \langle \pi \gamma^5 n \rangle = 0, \]

and similarly for the proton. To simplify our notation, we drop the \( \langle \cdots \rangle \) in the remainder of this appendix.

Within the liquid drop, the baryon number density

\[ N^1 = p^+ p + n^+ n, \]  \hspace{1cm} (D.2)

and scalar density

\[ \overline{N} N = \overline{p} p + \overline{n} n. \]  \hspace{1cm} (D.3)

The neutron contributions to these densities are:

\[ n^1 n = 2 \int_0^{k^p_n} \frac{d^3 k}{(2\pi)^3} \frac{(k^p_n)^3}{3\pi^2}, \]
\[ \overline{n} n = 2 \int_0^{k^p_n} \frac{d^3 k}{(2\pi)^3} \frac{m^*}{\sqrt{k^2 + (m^*^2)}}, \]
\[ = \frac{m^*}{2\pi^2} \left( k^p_n \mu^* - \frac{1}{2} (m^*^2) \right. \ln \left( \frac{\mu^* + k^p_n}{\mu^* - k^p_n} \right), \]

with

\[ m^* \equiv m_n + C_n^2 N - \frac{1}{2} \frac{c_{SB}^2}{200} (\overline{n} n - \overline{p} p), \]
\[ \mu^* \equiv \sqrt{(k^p_n)^2 + (m^*^2)}. \]  \hspace{1cm} (D.4)

The equivalent proton contributions are obtained by straightforward substitution of \( n \leftrightarrow p \).

It is convenient to define:

\[ \epsilon^f n = 2 \int_0^{k^p_n} \frac{d^3 k}{(2\pi)^3} (k^2 + (m^*^2)), \]
\[ \epsilon^f n = 3 \mu^* n^+ n + \frac{1}{4} m^* \overline{n} n, \]  \hspace{1cm} (D.6)
\[ P^f n = 2 \int_0^{k^p_n} \frac{d^3 k}{(2\pi)^3} \frac{k^2}{3\sqrt{k^2 + (m^*^2)}}, \]
\[ = \frac{1}{4} \mu^* n^+ n + \frac{1}{4} m^* \overline{n} n, \]  \hspace{1cm} (D.7)

and equivalently for protons. These look conveniently like the neutron and proton energy density and pressure, and indeed:

\[ \epsilon^f n - 3 P^f n = m^* \overline{n} n, \]
\[ \epsilon^f n + P^f n = \mu^* n^+ n. \]  \hspace{1cm} (D.8)

The actual nucleon energy density and pressure are properly constructed from the stress-energy tensor:

\[ (T^{\mu \nu}_{\chi PT})^n = \frac{\partial L^n_{\chi PT}}{\partial (\partial^\mu N)} \partial^\nu N - g^{\mu \nu} L^n_{\chi PT}, \]  \hspace{1cm} (D.9)

with

\[ c^N \equiv (T^{\mu \nu}_{\chi PT})^{00}, \]
\[ P^N \equiv \frac{1}{3} (T^{\mu \nu}_{\chi PT})^{ij}. \]  \hspace{1cm} (D.10)

The total nucleon energy and pressure are thus:

\[ (T^{\mu \nu}_{\chi PT})^N = \frac{\partial L^N_{\chi PT}}{\partial (\partial^\mu N)} \partial^\nu N - g^{\mu \nu} L^N_{\chi PT}. \]
Using
\[ \mu_B^p = \mu_s^p + C_V^2 N^\dagger N - C_V^2 (n^\dagger n - p^\dagger p), \]
\[ \mu_B^n = \mu_s^p + C_V^2 N^\dagger N + C_V^2 (n^\dagger n - p^\dagger p), \]

it follows that \( \epsilon^N \) and \( P^N \) are related by the baryon number densities:
\[ \epsilon^N + P^N = \mu_B^p p^\dagger p + \mu_B^p n^\dagger n. \]  

The objects of our calculations are therefore the six quantities: \( \mu_B^p, \mu_B^p, m_s^p, m_s^p, \) and \( k_F^p, k_F^n \). These are, respectively, the chemical potential, reduced mass, and Fermi-momentum for neutrons and protons.

1. **Z = N heavy nuclei in the chiral symmetric limit**

To calculate binding energies, we work in the chiral symmetric limit, \( m_p = m_n \): e.g. zero electromagnetic breaking, and \( m_s = \frac{1}{2} (m_p + m_n) \). We first study the case \( Z = N \), so \( m_s^p = m_s^p \equiv m_s \) for equal numbers of protons and neutrons. We search for a solution of the chiral-symmetric liquid equations that has \( P^N = 0 \). In this simple case, \( \mu_B^p = \mu_B^p \equiv \mu_B, \mu_s^p = \mu_s^p \equiv \mu_s, m_p = m_n \equiv m_N, \) and \( k_F^n = k_F^p = k_F \). Thus
\[ k_F = \sqrt{\mu_s^2 - m_s^2}. \]  

We also have \( n^\dagger n = p^\dagger p = \frac{1}{2} N^\dagger N, \) and \( \pi n = \pi p = \frac{2}{\sqrt{2}} N N, \) as well as equal fermion densities as:
\[ N^\dagger N = \frac{\mu_B - \mu_s}{C_V^2}, \]
\[ \tilde{N} N = \frac{m_N - m_s}{C_S^2}, \]

where, to make connection to Walecka’s model of nuclear matter, we use \( C_V^2 \) and \( C_S^2 \) defined in [26] and [20], respectively. The baryon number and scalar densities are simply twice the values in (D.4); i.e.:
\[ N^\dagger N = \frac{2k_F^2}{3\pi^2}, \]
\[ \tilde{N} N = \frac{m_s}{\pi^2} \left( \mu_s k_F - m_s^2 \ln \frac{\mu_s + k_F}{m_s} \right). \] 

The fermion pressure is now:
\[ P^N = \frac{1}{4} \left[ \mu_B N^\dagger N + C_V^2 \right. \]
\[ \left. + \frac{C_S^2}{2} (\pi p - \pi n)^2 \right]. \] 

To these six equations [D.14]-[D.18] in the seven variables \( k_F, \mu_s, \delta_\mu \equiv \mu - \mu_s, m_N, N^\dagger N \) and \( P^N \), we add the physical condition that the Static\( CHL \) non-topological soliton pressure vanish internally, in order that it remain stable when immersed in the physical vacuum:
\[ P^N = 0, \]

eliminating \( P^N \) as a free variable. Equations [D.14]-

![Graph showing \( \Delta_{SN} \) as a function of baryon chemical potential \( \mu_B \)]

**FIG. 2.** \( \Delta_{SN} \) (cf. [D.22]) as a function of baryon chemical potential \( \mu_B \) for \( C_V^2 = 222.65 \text{ GeV}^{-2} \) and \( C_S^2 = 303.45 \text{ GeV}^{-2} \), the Chin and Walecka values [20] equivalent to ours. A solution of the complete set of \( Z = N \) chiral-symmetric pressure-less liquid equations must have \( \Delta_{SN} = 0 \), and thus is found at \( \mu_B \simeq 923.17 \text{ MeV} \), where the curve intersects the \( \mu_B \) axis. This value equals the Chin-Walecka value shown as a black dot.

(D.17) can be solved analytically to give \( k_F, \mu_s, N^\dagger N \) and \( \tilde{N} N \) as functions of \( m_s \) and \( \delta_\mu \):
\[ k_F = \left( \frac{3\pi^2 \delta_\mu}{2 C_V^2} \right)^{1/3}, \]
\[ \mu_s = \sqrt{k_F^2 + m_s^2}. \]
Equation (D.18), with $P^N = 0$, then becomes a quartic equation for $m_*$ in terms of $\delta \mu$:

$$0 = m_*^2 + \left( \frac{3\pi^2 \delta \mu}{2C_V^2} \right)^{2/3} - \left[ \frac{C_V^2}{C_S^2} (m_N - m_*) (2m_N - m_*) - 2\delta \mu \right]^2,$$

which has up to four roots $m'_*(\delta \mu; C_S^2, C_V^2)$, for every value of $\delta \mu$, $C_S^2$, and $C_V^2$. To be an actual solution of the complete set of $Z = N$ chiral-symmetric pressure-less liquid equations, the root must also satisfy (D.16) and the second of (D.17); i.e.,

$$\Delta_{NN} = 1 - \frac{C_S^2}{m_N - m_*} NN = 0 \quad (D.22)$$

where we use (D.20) for $k_F(\delta \mu; C_V^2)$ and $\mu_*(\delta \mu; C_S^2, C_V^2)$.

4. The vanishing of pressure to second order provides an additional equation which allows all variations to be expressed in terms of the first order change in density only;

5. Since there appears no way to infer separately the value of $C_S^2$ we follow Niskic and co-workers [50] and set this constant to zero. This leads to significant simplification. In particular, changes in the proton and neutron reduced masses are equal in first order.

6. We then solve for $C_V^2$ by setting the asymmetry energy of the liquid model to the second order variation in the Thomas-Fermi energy.

In this section we use the following notation for the number and scalar densities:

$$\rho_p \equiv p^+_p; \quad \rho_n \equiv n^+_n; \quad \rho_\pm = \rho_p \pm \rho_n; \quad \rho_{sp} \equiv \bar{p}^+_p; \quad \rho_{sn} \equiv \bar{n}^+_n; \quad \rho_{sp} = \rho_{sp} \pm \rho_{sn} \quad (D.23)$$

We define the changes in densities as follows:

$$d \rho_p - d \rho_n = e^2 d \rho_+$$

$$d \rho_p + d \rho_n = e^2 d \rho_-$$

FIG. 3. Plot of $f_s^2 C_s^2$ and $f_v^2 C_v^2$ against the Fermi level in inverse $F$. The calibration used a bulk binding energy $E_{Vol} = 15.75$ MeV.

ably, we can now understand Chin and Walecka’s nuclear matter to be a pressure-less chiral-symmetric nuclear liquid. We also perhaps thereby gain some insight into the relative insensitivity of nuclear properties to pion properties.

2. $Z \neq N$ heavy nuclei in the chiral-symmetric limit

Here we outline the analytic and numerical treatment of the case where $Z \neq N$ in the chiral limit. The approach may be summarized as follows:

1. The starting point is the zeroth order solution for the case $Z = N$ which determines the coupling constants $C_S^2$ and $C_V^2$ for a given Fermi level and binding energy as in the previous section.

2. All proton and neutron specific quantities are expanded in a Taylor series;

3. The general rule is: quantities vanishing in zeroth order have a first order variation while those not vanishing in zeroth order have only a second order variation; thus all terms up to second order must be retained;

4. The vanishing of pressure to second order provides an additional equation which allows all variations to be expressed in terms of the first order change in density only;

5. Since there appears no way to infer separately the value of $C_S^2$ we follow Niskic and co-workers [50] and set this constant to zero. This leads to significant simplification. In particular, changes in the proton and neutron reduced masses are equal in first order.

6. We then solve for $C_V^2$ by setting the asymmetry energy of the liquid model to the second order variation in the Thomas-Fermi energy.
where \( \epsilon \) is merely a placeholder for the order of the variation. It then follows that:

\[
\rho_p = \frac{1}{2} \rho_+ + \frac{\epsilon}{2} d\rho_+ + \frac{\epsilon^2}{2} d\rho_+ , \quad \rho_n = \frac{1}{2} \rho_+ - \frac{\epsilon}{2} d\rho_+ + \frac{\epsilon^2}{2} d\rho_+ . \tag{D.25}
\]

Since the number density for each species is given by the first of (D.4), we get the following expansions for the Fermi levels:

\[
\delta k_{Fp} - \delta k_{Fn} = \frac{2k_F}{3\rho_+} \delta \rho_-, \quad \delta k_{Fp} + \delta k_{Fn} = \epsilon^2 \left( \frac{2k_F}{3\rho_+} \delta \rho_+ - \frac{2k_F}{9\rho_+^2} \delta \rho_-^2 \right) . \tag{D.26}
\]

It follows that:

\[
m_{s8} = \frac{1}{2} (m_{s8} + m_{s8}) = m_N - C_S^2 \rho_{s8}, \tag{D.27}
\]

\[
m_{s3} = \frac{1}{2} (m_{s8} - m_{s8}) = \frac{C_S^2}{2} \rho_{s8} ,
\]

where we used the second of (26). We also define:

\[
\mu_{s8} = \frac{1}{2} (\mu_{s8} + \mu_{s8}), \quad \mu_{s3} = \frac{1}{2} (\mu_{s8} - \mu_{s8}), \tag{D.28}
\]

with \( \mu_{s8} \) as in (D.5). We now enforce \( C_S^2 = 0 \): it follows immediately from the second of (D.27) that \( m_{s3} = \delta m_{s3} = 0 \) with considerable simplification. First, \( \delta \mu_{s3} \) is a linear function of \( \delta \rho_- \) only; i.e.,

\[
\delta \mu_{s3} = -\frac{\pi^2}{2k_F \mu_{s8}} \delta \rho_- . \tag{D.29}
\]

Second, \( \delta \mu_{s8} \) is also simplified:

\[
\delta \mu_{s8} = \frac{m_{s8}}{\mu_{s8}} \delta m_{s8} + \frac{\pi^2}{2k_F \mu_{s8}} \delta \rho_+ - \frac{\pi^4}{8k_F^4 \mu_{s8}^2} (m_{s8}^2 + 2k_0^2) \delta \rho_-^2 . \tag{D.30}
\]

(As noted, \( \delta \mu_{s3} \) is first order, while \( \delta \mu_{s8} \) is second order.) The variation in the first of (D.27) gives:

\[
\delta \rho_+ = -\frac{\delta m_{s8}}{C_S^2}, \tag{D.31}
\]

where the variation in \( \rho_{s8} \) is obtained using:

\[
\delta \rho_{s8,p,n} = 3 \left( \frac{\rho_{s8,p,n}}{\mu_{s8,p,n}} - \rho_{p,n} \right) \delta m_{s8,p,n} + \frac{m_{s8,p,n} k_0^2}{\mu_{s8,p,n} \pi^2} \delta k_{Fp,n} . \tag{D.32}
\]

After some algebra and substituting the variations in \( \mu_{s3} \) and \( \mu_{s8} \) from (D.29) and (D.30), we find:

\[
3 \left( \frac{\rho_{s8} - \rho_+}{\mu_{s8} + \rho_{s8}} + \frac{1}{C_S^2} \right) \delta m_{s8} + \frac{m_{s8} \delta \rho_+ - m_{s8} \pi^2 \delta \rho_-^2}{\mu_{s8} k_F} = 0 . \tag{D.33}
\]

We must also enforce the vanishing of the Fermi pressure. The first order variation of the Fermi pressure vanishes identically. The second order term is:

\[
\delta P_{S8}^2 = \frac{1}{4} \left( \rho_+ \delta m_{s8} + \mu_{s8} \delta \rho_+ + \delta \mu_{s3} \delta \rho_- \right) + C_S^2 \rho_+ \delta \rho_+ - \frac{C_S^4}{4 \pi^2} \delta \rho_-^2 + \frac{1}{4C_S^2} (3m_N - 2m_{s8}) \delta m_{s8} . \tag{D.34}
\]

After using (D.29) and (D.30), the zero pressure equation becomes:

\[
\left( \frac{3m_N - 2m_{s8}}{4C_S^2} + \frac{\rho_+ m_{s8}}{2 \mu_{s8}} \right) \delta m_{s8} + \left( \frac{\mu_{s8}}{4} + C_S^2 \rho_+ + \frac{k_F}{12 \mu_{s8}} \right) \delta \rho_+ \tag{D.35}
\]

\[
+ \frac{\pi^2 (5m_{s8}^2 - 4k_0^2)}{48 k_F \mu_{s8}^2} - \frac{C_S^4}{4} \delta \rho_-^2 = 0 .
\]

Equations (D.33) and (D.35) are solved to express \( \delta m_{s8} \) and \( \delta \rho_+ \) in terms of \( \delta \rho_-^2 \). To determine \( C_S^2 \) we need the second variation in the energy density \( \delta \epsilon \). This quantity is discussed below.

### 3. Calibration of \( C_S^2 \)

We start with the vanishing of the pressure and the relationship:

\[
\epsilon^N + P^N = \mu_p \rho_p + \mu_n \rho_n = \mu_{s8} \rho_+ + \mu_3 \rho_-, \tag{D.36}
\]

where

\[
\mu_{s8} = \mu_{s8} + C_S^2 \rho_+, \quad \mu_3 = \mu_{s3} - \frac{1}{2} C_S^2 \rho_- . \tag{D.37}
\]

The zeroth order energy density when \( Z = N \) follows at once:

\[
\epsilon_0^N = \mu_{s8} \rho_+ + C_S^2 \rho_+^2 . \tag{D.38}
\]

The first order energy term vanishes. The second order term is:

\[
\delta \epsilon_2^N = \rho_+ \delta \mu_{s8} + \mu_{s8} \delta \rho_+ + \delta \mu_{s3} \delta \rho_- + 2C_S^2 \rho_+ \delta \rho_+ - \frac{1}{2} C_S^2 \delta \rho_-^2 . \tag{D.39}
\]
Finally, we can express $\delta \rho_-$ in terms of the relative neutron excess as:

$$\delta \rho_- = \frac{Z - N}{Z + N} \rho_+. \tag{D.40}$$

The parameter $\overline{C_V^2}$ can be calibrated in two ways. In the first, we merely ascribe all of the second order energy to the asymmetry term in the liquid drop formula (34) for $\overline{C_V^2}$:

$$\delta \epsilon_2^N = a_{Asym} \left( \frac{Z - N}{Z + N} \right)^2 \rho_+ = a_{Asym} \frac{\delta \rho_+^2}{\rho_+}. \tag{D.41}$$

where $a_{Asym}$ is fit to SEMF observation. In the second approach, we calibrate directly to the binding energies of isotopes, possibly using the liquid drop formula to correct for effects that we have ignored in this paper such as the Coulomb and surface terms. Both approaches give comparable results. Figure 4 shows the behaviour of $\overline{C_V^2}$ for different values of $k_F$.

---

**FIG. 4.** Plot of $\overline{C_V^2} \equiv f^2 \overline{C_V^2}$ against the Fermi level in fm$^{-1}$. The behavior is roughly linear in the range considered and corresponds to a one-third power of the number density.