On Strong Superadditivity of the Entanglement of Formation

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Abstract: We employ a basic formalism from convex analysis to show a simple relation between the entanglement of formation $E_F$ and the conjugate function $E^*$ of the entanglement function $E(\rho) = S(\text{Tr}_A \rho)$. We then consider the conjectured strong superadditivity of the entanglement of formation $E_F(\rho) \geq E_F(\rho_I) + E_F(\rho_{II})$, where $\rho_I$ and $\rho_{II}$ are the reductions of $\rho$ to the different Hilbert space copies, and prove that it is equivalent with subadditivity of $E^*$. Furthermore, we show that strong superadditivity would follow from multiplicativity of the maximal channel output purity for quantum filtering operations, when purity is measured by Schatten $p$-norms for $p$ tending to 1.

1. Introduction

One of the central quantities in quantum information theory is the entanglement cost of a state, defined as the number of maximally entangled pairs (singlets) required to prepare this state in an asymptotic way. Calculating the entanglement cost of a general mixed state as such is, with the present state of knowledge, a formidable task because one has to consider an infinite supply of singlets and construct a protocol using local or classical (LOCC) operations only, such that the resulting (infinite-dimensional) state approximates an infinite supply of the required state to arbitrary precision. Furthermore, the protocol must have maximal yield, the number of states produced per singlet. The entanglement cost is the inverse of this yield.

An important theoretical breakthrough was achieved in [1], where the entanglement cost $E_C$ was shown to be equal to the regularised entanglement of formation: $E_C(\rho) = \lim_{n \to \infty} E_F(\rho^\otimes n)/n$. The entanglement of formation (EoF) (defined below in (4)) is defined in a mathematical and non-operational way and is therefore much more amenable to calculation. Moreover, for 2-qubit mixed states, a closed formula for the EoF exists [2]. Nevertheless, calculating the entanglement cost still requires calculations over

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infinite-dimensional states. For that reason one would hope for the additivity property to hold for the EoF: 
\[ E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2), \]
because then \( E_C = E_F \). Additivity of the EoF has been proven in specific instances \([3–8]\). Some of these additivity results are sufficiently powerful to allow calculating the entanglement cost for certain classes of mixed states \([5–7]\). The much sought-after general proof, however, remains elusive for the time being and, in fact, general additivity is still a conjecture.

It is very easy to show that the EoF is subadditive:
\[ E_F(\rho_1 \otimes \rho_2) \leq E_F(\rho_1) + E_F(\rho_2). \] (1)
Additivity would then follow from superadditivity:
\[ E_F(\rho_1 \otimes \rho_2) \geq E_F(\rho_1) + E_F(\rho_2). \] (2)

In \([4]\) a stronger property, which would imply (super)additivity, has been conjectured for the EoF, namely strong superadditivity:
\[ E_F(\rho) \geq E_F(\rho_I) + E_F(\rho_{II}). \] (3)
where \( \rho \) is a general state over a duplicated Hilbert space and \( \rho_I \) and \( \rho_{II} \) are its reductions to the different copies of that space.

In this paper we show that strong superadditivity of EoF is equivalent to subadditivity of a much simpler quantity, the so-called conjugate of the entanglement functional \( E(\rho) = S(\text{Tr}_A \rho) \). We then exploit this equivalence to show that strong superadditivity would follow as a consequence of multiplicativity of the maximal output purity, measured by a Schatten norm, for quantum filtering operations (this quantity will also be defined in due course).

The main results are stated in Theorems 1 and 2. To arrive at these results, we have made use of a basic formalism from convex analysis \([9, 10]\) and we hope that our results will stimulate usage of this elegant theory in other areas of quantum information.

2. Notations

Let us first introduce the basic notations. Let \( S(\rho) \) denote the von Neumann entropy \( S(\rho) = -\text{Tr} \rho \ln \rho \). For state vectors we will typically use lowercase Greek letters, \( \psi \), \( \phi \). For mixed states we will use lowercase Greek letters \( \rho \), \( \sigma \), \( \tau \). The identity matrix will be denoted by \( I \).

We shall denote the set of bounded Hermitian operators over the Hilbert space \( \mathcal{H} \) by \( \mathcal{B}'(\mathcal{H}) \), the set of non-negative elements in \( \mathcal{B}'(\mathcal{H}) \) by \( \mathcal{B}^+(\mathcal{H}) \), and the (convex) set of all states (trace 1 positive operators) over \( \mathcal{H} \) by \( \mathcal{S}(\mathcal{H}) \).

We will frequently slim down expressions like \( \max_{\rho \in \mathcal{S}} \{ \ldots \} \) to \( \max_{\rho} \{ \ldots \} \). When the domain of, say, a maximisation over states is missing it will be implicitly understood that the whole of state space \( \mathcal{S}(\mathcal{H}) \) is meant. The abovementioned naming convention for states and vectors will be adhered to exactly for that reason.

Any state \( \rho \) can be realised by an ensemble of pure states. An ensemble is specified by a set of pairs \( \{ (p_i, \psi_i) \}_{i=1}^N \), consisting of \( N \) state vectors \( \psi_i \) and associated statistical weights \( p_i \) (with \( p_i \geq 0 \) and \( \sum_i p_i = 1 \)). Here, \( N \) is called the cardinality of the ensemble. The entanglement of formation (EoF) of a bipartite state \( \rho \) (i.e., a state over the bi-partite Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \)), is defined by \([11]\)
\[ E_F(\rho) = \min_{\{ (p_i, \psi_i) \}} \left\{ \sum_i p_i S(\text{Tr}_A |\psi_i\rangle \langle \psi_i|) : \sum_i p_i |\psi_i\rangle \langle \psi_i| = \rho \right\}. \] (4)