Low Energy Lorentz Violation in Polymer Quantization
Revisited

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Abstract

In previous work, it had been shown that polymer quantized scalar field theory predicts that even an inertial observer can experience spontaneous excitations. This prediction was shown to hold at low energies. However, in these papers it was assumed that the polymer scale is constant. But it is possible to relax this condition and obtain a larger class of theories where the polymer scale is a function of momentum. Does the prediction of low energy Lorentz violation hold for all of these theories? In this paper we prove that it does. We also obtain the modified rates of radiation for some of these theories.

1. Introduction.

The problem of finding the correct quantum theory of gravity is one of the biggest challenges in physics today. While the solution to the puzzle remains out of reach, many promising approaches have been developed. On one hand there are ‘top down’ approaches like string theory and loop quantum gravity where one starts with a theory and tries to obtain experimental predictions. On the other there are ‘bottom up’ phenomenological approaches where one mainly tries to understand the consequence of Planck scale modifications of physics on the matter sector.

Polymer quantized scalar field theory\cite{1} is a phenomenological model inspired by loop quantum gravity\cite{2, 3}. Here one first decomposes a free scalar
field theory into uncoupled harmonic oscillators in momentum space and then quantizes each oscillator using polymer quantization [1] which introduces a polymer scale. This procedure yields a modified propagator which converges to the standard propagator in the limit of low energies.

Now to test any modified theory we try to find situations where its predictions conflict with the predictions of the standard theory, preferably at accessible energies. For polymer quantization, the prediction of Unruh Effect [5] (or lack thereof) has proven to be such a scenario where results obtained from polymer quantization differ significantly from standard results. Unruh Effect for polymer quantized fields in linearly accelerated frames have been studied in [6, 7]. The case of rotating frames have been studied in [8]. But perhaps the most striking results have been established for inertial frames.

To understand this result, we should first note that polymer quantization violates Lorentz symmetry and establishes a preferred frame. It was shown in [9] that a detector moving with constant velocity with respect to this frame can detect radiation, if it is coupled to a polymer quantized field. Furthermore it was found that such detection occurs at low energies. In [10] it was established that there is a critical velocity such that a detector moving above this velocity will detect radiation. The rates of radiation were calculated in this paper and it was found that they cannot be suppressed by increasing the polymer scale.

However, a restrictive assumption had been made while polymer quantizing the scalar field theory in [1]. It was assumed that the polymer scale is a constant. This need not be the case! Recall our description of polymer quantization of scalar field. First the field is decomposed into harmonic oscillators, one at each point in the space of spatial momenta. Then each harmonic oscillator is polymer quantized. As we will see in more detail later, this quantization requires the introduction of a scale, which we call the polymer scale. In [1], the polymer scale was assumed to be the same for all the oscillators. But clearly, this assumption can be relaxed. It is a natural extension of [1] to consider the polymer scale to be a function of $|\vec{k}|$. A running polymer scale is also natural from the perspective of renormalization group flow.

Making this extension, we arrive at a large class of polymeric theories, one for each possible $\lambda(|\vec{k}|)$. The only stipulation we must put on these theories is that they reproduce the standard field theory propagator at the low energy limit. We can now ask, do one or more of these theories not violate Lorentz symmetry at low energies. This is the question that we address in this paper.
Surprisingly, we find that *none* of these theories can evade the fate of the original, all of them predict that an inertial detector will click at a certain critical velocity. We obtain a proof of why this should be so. We perform numerical experiments to find out the critical velocities for different theories. Surprisingly, we find that critical velocities turn out to have the same value for very different polymeric theories. Further investigation is necessary to understand why this should be so. Our result strengthens the existing result of low energy violation for polymer quantized theories. We also obtain the rates of radiations for some of these theories.

The paper is organized as follows. In the following section we recall polymer quantization of scalar fields and then modify it by introducing a momentum dependent polymer scale. In section III we test some of these theories numerically to see if they predict the clicking of an inertial detector and find that they do. In section IV we give proof of why this must be so. Section V presents our numerical results of the rates of radiation in some of these theories. The final section summarizes our results.

2. Polymer quantization with variable polymer scale

In this section we briefly review polymer quantization of scalar field and then extend it by introducing momentum dependence in the polymer scale. First, let’s recall polymer quantization of a harmonic oscillator. In polymer Hilbert space, the position operator $\hat{x}$ and translation operator $\hat{U}(\lambda)$ are considered to be basic operators. Since the translation operator is not weakly continuous in the parameter $\lambda$, the momentum operator does not exist in polymer Hilbert space. However, one can define the momentum operator as $\hat{p}_\lambda = 1/(2i\lambda)(\hat{U}(\lambda) - \hat{U}(-\lambda))$ and one can recover usual momentum operator by taking the limit $\lambda \to 0$. In polymer Hilbert space the limit $\lambda \to 0$ does not exist and $\lambda$ is considered as a fundamental scale. By choosing $\lambda$ to be $\lambda_*$, the Hamiltonian of simple harmonic oscillator can be expressed as:

$$\hat{H} = \frac{1}{8m\lambda_*^2}(2 - \hat{U}(2\lambda_*) - \hat{U}(-2\lambda_*)) + \frac{m\omega^2\hat{x}^2}{2}.$$  

(1)

Note that it is at this step that the polymer scale enters the theory. This modifies the Schrodinger equation to:

$$\frac{1}{8m\lambda_*^2}(2 - 2\cos(2\lambda_*p))\psi - \frac{m\omega^2}{2} \frac{\partial^2 \psi}{\partial p^2} = E\psi.$$  

(2)
This can be mapped to a Mathieu equation through the following redefinitions:

\[ u = \lambda \ast p + \pi/2, \quad \alpha = 2E/g\omega - 1/2g^2, \quad g = m\omega \lambda^2. \quad (3) \]

With these redefinitions the above equation takes the standard form of the Mathieu equation:

\[ \psi''(u) + (\alpha - \frac{1}{2}g^{-2})\cos(2u)\psi(u) = 0. \quad (4) \]

This equation admits periodic solutions for certain values of \( \alpha \):

\[ \psi_{2n}(u) = \pi^{-1/2}cc_n(1/4g^2, u), \quad \alpha = A_n(1/4g^2), \quad (5) \]

\[ \psi_{2n+1}(u) = \pi^{-1/2}se_{n+1}(1/4g^2, u), \quad \alpha = B_n(1/4g^2), \quad (6) \]

where \( cc_n, se_n (n = 0, 1 \ldots) \) are respectively the elliptic cosine and sine functions and \( A_n, B_n \) are the Mathieu characteristic value functions. The energy eigenvalues of the polymer harmonic oscillator are given by:

\[ \frac{E_{2n}}{\omega} = \frac{2g^2A_n(1/4g^2) + 1}{4g}, \quad (7) \]

\[ \frac{E_{2n+1}}{\omega} = \frac{2g^2B_{n+1}(1/4g^2) + 1}{4g}. \quad (8) \]

Now let us recall polymer quantization of scalar fields. Here the starting point is the free Klein Gordon field. First one takes decomposes this field into uncoupled harmonic oscillators with Hamiltonians:

\[ H_{|\vec{k}|} = \frac{\pi^2}{2|\vec{k}|} + \frac{|\vec{k}|^2}{2}. \quad (9) \]

Now each of these harmonic oscillators can be polymer quantized by introducing some polymer scale \( \lambda \). In \[ 1 \] each of these oscillators were quantized using the same polymer scale. This gives the polymer Wightman function:

\[ \langle 0|\hat{\phi}(t, \vec{x})\hat{\phi}(t', \vec{x}')|0 \rangle = \sum_{n=0}^{\infty} \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}|c_{4n+3}|^2 e^{-i\Delta E_{4n+3}(t-t')}, \quad (10) \]
where
\[ \Delta E_n \equiv E_n(g) - E_0(g) , \] (11)

and \( c_n(g) = \langle n | \hat{\phi}_n | 0, \vec{k} \rangle \) and \( g = \lambda^2 | \vec{k} | \).

Using the asymptotic expansions for Mathieu value functions, one can obtain the propagator for low momenta \((g \ll 1)\):

\[ D_p = \frac{i(1 - 2\lambda^2 | \vec{k} |)}{p^2 - \lambda^2 | \vec{k} |^3 - i\epsilon} . \] (12)

This can be seen to go to the usual limit as \( g \to 0 \). This completes the review of standard polymer quantization. Now we note that all the oscillators need not be polymer quantized using the same polymer scale \( \lambda_* \). Oscillators corresponding to different momenta can have different polymer scales\(^1\). In other words, the polymer scale can be a function of momenta. In particular, since only \( | \vec{k} | \) enters the oscillator Hamiltonian, the polymer scale should be taken to be a function \( \mu(| \vec{k} |) \). With this modification we now have a large class of polymeric theories, one for each possible \( \mu(| \vec{k} |) \). The new formula for the Wightman function and propagator will have the same form as above, with the only modification that constant \( \lambda_* \) will be replaced by \( \mu(| \vec{k} |) \) wherever it appears. So far we have not imposed any restrictions on \( \mu(| \vec{k} |) \). We will now demand that it reduces to the standard field theory propagator at the limit of low momenta. The modified polymer propagator at low energy is given by:

\[ D^\mu_p = \frac{i(1 - 2| \vec{k} | \mu(| \vec{k} |)^2)}{p^2 - | \vec{k} |^3 \mu(| \vec{k} |)^2 - i\epsilon} . \] (13)

For this to reduce to the standard propagator we must have \( \mu(| \vec{k} |)^2 | \vec{k} | \to 0 \) in the low momentum limit. Thus we have our only condition on \( \mu(| \vec{k} |) \):

\[ \mu(| \vec{k} |)^2 | \vec{k} | \to 0 \text{ when } | \vec{k} | \to 0 . \] (14)

\(^1\text{We note that another possible extension of polymer quantization could come from making the energy spacings field dependent. In this case the oscillators won’t be governed by Mathieu equations. This would be an interesting avenue to pursue in future.}\)
3. Low energy Lorentz violation in extended polymeric theories

In this section we investigate whether one or more of these theories can avoid violating Lorentz symmetry at low energies. First let us recall the criterion for an inertial detector to click given in [10]. In [10] it was shown that the rate of radiation for an inertial detector with energy gap $\Omega$ coupled to a polymer scalar field is given by:

$$F(\Omega) = \frac{1}{2\pi} \sinh \beta \sum_{n=0}^{\infty} \int d|\vec{k}| \ |\vec{k}| \ |c_{4n+3}|^2$$

$$\theta \left( \lambda^2 |\vec{k}| \sinh \beta - |\lambda^2 \Omega + \lambda^2 |\vec{k}| \frac{\Delta E_{4n+3} \lambda^2 |\vec{k}|}{|\vec{k}|} \cosh \beta \right) .$$ (15)

where $\beta$ is the rapidity of the detector with respect to the preferred frame. From the above expression one can see that the rate will vanish if

$$\frac{\Delta E_{4n+3}}{|\vec{k}|} |\vec{k}| \geq 1$$

for all $|\vec{k}|$ and all $n$. But if $\frac{\Delta E_{4n+3}}{|\vec{k}|}$ dips below 1 for any range of $|\vec{k}|$ and for any $n$, we will have an inertial detector registering radiation. In [10] it was shown that for the standard polymer theory $\frac{\Delta E_{4n+3}}{|\vec{k}|}$ never dips below 1 for $n > 0$. But $\frac{\Delta E_{3}}{|\vec{k}|}$ does dip below 1. Now for the extended class of polymeric theories that we have introduced, the above expression for rate holds with the replacement of constant $\lambda$ by $\mu(|\vec{k}|)$ wherever the former occurs. Thus the criterion for an inertial detector not clicking for these theories is

$$\frac{\Delta E_{4n+3}}{|\vec{k}|} |\vec{k}| \geq 1$$

for all $|\vec{k}|$. Here $\Delta E_{4n+3}^{\mu}$ is obtained by replacing $\lambda$ by $\mu(|\vec{k}|)$ in the equations (7), (8) and (11). We will prove that for any $\mu(|\vec{k}|)$ that satisfies (14) $\frac{\Delta E_{3}^{\mu}}{|\vec{k}|}$ must dip below unity. We will also show that the dip below unity for all these theories occurs at low momenta. Thus all these theories exhibit low energy Lorentz violation. Before going into the proof let us pause to look at some numerical evidence for this claim. In FIG.[11] we have plotted $\frac{\Delta E_{3}^{\mu}}{|\vec{k}|}$ for different functions $\mu(|\vec{k}|)$. We have considered the following functions $\mu(|\vec{k}|)$:

Case:(i) $\mu^2(|\vec{k}|) = \lambda^2$ (the standard polymer scale case)

(ii) $\mu^2(|\vec{k}|) = \lambda^4 |\vec{k}|$  (iii) $\mu^2(|\vec{k}|) = \lambda^2 \left( 1 - e^{-\lambda^2 |\vec{k}|} \right)$

(iv) $\mu^2(|\vec{k}|) = \lambda^2 e^{\lambda^2 |\vec{k}|}$ and (v) $\mu^2(|\vec{k}|) = \frac{\lambda^2}{\sqrt{|\vec{k}|}}$. We see that for all of them, there is a dip below unity at low momentum. Now we proceed to the proof.
The first step is to write $\frac{\Delta E_3^\mu}{|\vec{k}|}$ in terms of Mathieu characteristic value functions:

$$\frac{\Delta E_3^\mu}{|\vec{k}|} = \frac{\mu^2(|\vec{k}|)|\vec{k}|}{2} \left( B_2\left(\frac{1}{4\mu^4(|\vec{k}|)|\vec{k}|^2}\right) - A_0\left(\frac{1}{4\mu^4(|\vec{k}|)|\vec{k}|^2}\right) \right) . \quad (16)$$

Next we note from condition (14) that as $k \to 0$, $\frac{1}{4\mu^4(|\vec{k}|)|\vec{k}|^2} \to \infty$. This allows us to use the asymptotic expansion of Mathieu characteristic value functions at low momenta. Using the asymptotic expansion we get that:

$$B_2\left(\frac{1}{4\mu^4(|\vec{k}|)|\vec{k}|^2}\right) - A_0\left(\frac{1}{4\mu^4(|\vec{k}|)|\vec{k}|^2}\right) = 2\frac{1}{|\vec{k}|\mu^2(|\vec{k}|)} - |\vec{k}|\mu^2(|\vec{k}|) . \quad (17)$$

Combining this with (16) we obtain:

$$\frac{\Delta E_3^\mu}{|\vec{k}|} = \frac{\mu^2(|\vec{k}|)|\vec{k}|}{2} \left( 2\frac{1}{|\vec{k}|\mu^2(|\vec{k}|)} - |\vec{k}|\mu^2(|\vec{k}|) \right)$$

$$= 1 - \frac{|\vec{k}|\mu^2(|\vec{k}|)}{2}$$

$$= < 1 . \quad (18)$$

This proves that all theories with variable polymer scales that satisfy the consistency condition (14) will exhibit low energy Lorentz violation. But can we make a statement about the magnitude of critical velocity from this? In (Husain and Louko) the critical velocity had been found to be well within experimental reach. However for general polymeric theories it is possible that the critical speed is high enough to avoid detection by existing experiments.

Unfortunately, the asymptotic analysis we employed cannot determine the critical velocity and we must resort to numerical experiments. As for a given polymeric theory, the critical velocity needs to be determined numerically. In the following section we consider several examples of polymeric theories and test them numerically to obtain the critical velocity.

4. Rates of radiation for different polymeric theories

In this section we present our numerical estimation of critical velocities and radiation rates for different polymeric theories with different $\mu(|\vec{k}|)$ s.
Firstly, surprisingly numerical estimation shows that the minimum value of $\Delta E_3/|\vec{k}| \approx 0.8781$ is same for all polymeric theories which are considered here. From the Eq. (15) one can see that the critical rapidity (for $\Omega > 0$, above which rapidity the detector’s excitation rate is non-zero) depends only on the $\Delta E_3/|\vec{k}|$. Thus the critical rapidity $\beta_c = \text{arctanh}((\Delta E_3/|\vec{k}|)_{\text{min}}) \approx 1.3675$ remains same for all these polymeric theories. We have plotted the rate of radiation $F(\Omega)$ with the rapidity $\beta$. We find that, for a given value of $h = \lambda^2 \Omega$, the detector starts clicking at different rapidity in different polymeric theories. This is due to the fact that the occurrence of minima of $\Delta E_3/|\vec{k}|$ at different value of $g$ in different polymeric theories though it has same value for all polymeric theories. Secondly, the rates of radiations vary somewhat, but are not suppressed compared to the standard case with constant polymer scale.

Our numerical experiments suggest that violation at low velocity is a robust feature of polymeric theories. Further investigation is required to provide an explanation for the critical velocity being same in all cases.

In all the figures the three lines denote different values of the parameter $h$. The red (the highest pick), blue and green (the lowest pick) lines denote $h = 0.01, 0.05$ and 0.1 respectively.

Figure 1: Plot of $\Delta E_3^h/|\vec{k}|$ with $g = \lambda^2 |\vec{k}|$. The solid blue line represents the standard polymer case and other dashed lines represent for different functions $\mu(|\vec{k}|)$. 

\[ \Delta E_3^h/|\vec{k}| \]
Figure 2: Radiation rate for the standard case with constant polymer scale.

Figure 3: The radiation rate for different functions $\mu(|\vec{k}|)$. The sub-figures (a), (b), (c) and (d) represent the case (ii), (iii), (iv) and (v) respectively.

5. Summary

In this paper we first extended polymer scalar field theory into a larger class of theories by allowing the polymer scale to vary with momentum. Then we investigated whether all of these theories also exhibit low energy Lorentz violation. We gave a proof that they all do indeed violate Lorentz symmetry
at low energies (as demonstrated by the clicking of an inertial detector travelling at low velocities). We also verified this with numerical evidence for several possible cases. We also found through numerical experiments that the critical rapidity \( \beta_c = 1.3675 \) remains same for several possible polymeric theories. This is an intriguing feature which requires further investigation. Finally, we obtained the rates of radiation for different polymeric theories.

It should be possible to test our predictions at the Relativistic Heavy Ion Collider (RHIC). The dipole moment interaction between atoms and electromagnetic fields closely resembles the Unruh-DeWitt detector. Since the critical rapidity is much below than \( \beta \approx 3 \) attained by ions at RHIC [11], polymeric theories should be verifiable at RHIC.

Our result thus strengthens previous work on low energy Lorentz violation in polymer quantized theories, closing a possible loophole.

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