Hidden Topological Order in $^{23}$Na ($F=1$) Bose-Einstein Condensates

Fei Zhou

ITP, Utrecht University, Minnaert Building, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands

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We show the existence of a new hidden topological order in $^{23}$Na ($F=1$) Bose-Einstein condensates (BEC) with antiferromagnetic interactions. Occurrence of this order is due to the confinement of hedgehogs in spin ordered BEC where a spin Josephson effect takes place. However, a topological long range order is also argued to coexist with a short range spin correlation, as a result of topological order from disorder.

$^{23}$Na ($F=1$) BEC with antiferromagnetic interactions turn out to have many extremely fascinating properties in the presence of antiferromagnetic interactions. One of them is the topology of the order parameter space. It was pointed out in previous works that $^{23}$Na BEC in optical traps are in Quantum Spin Nematic States, with the order parameter space being $[S^1 \times S^2]/Z_2$ [1,2]. In this article we will report a new hidden long range order of pure topological nature and explore the possibility of having spin correlated states based on a topological consideration. To study spin correlated BEC of spin-1 interacting atoms, it is important to properly choose a set of collective coordinates. These variables are to completely characterize the nonlinear spin dynamics in the whole parameter space and to yield a simplest representation for two-body spin dependent scatterings. For this reason we employ a vector $\mathbf{n}$ living on a unit sphere for the description of the nonlinear spin dynamics in the presence of antiferromagnetic interactions. The low energy spin dynamics in the system can be mapped into an $O(3)$ nonlinear sigma model(NLSM) [3,4],

$$\mathcal{L} = \frac{1}{2f}(\partial_\mu \mathbf{n})^2, \mathbf{n}^2 = 1,$$

$$f = (16\pi)^{1/2}(\rho \Delta a^3)^{1/6}, \Delta a = \frac{a_2 - a_0}{3} \quad (1)$$

$a_{0,2}$ are the scattering lengths in $F = 0, 2$ channels and $\rho$ is the density. We also have introduced dimensionless length and time: $r \rightarrow r/\rho^{1/3}, \tau \rightarrow \tau/\nu_s a_0^{1/3}$, and $\nu_s = \sqrt{16\pi \hbar^2 \Delta \rho/\rho M^2}$. Derivatives $\partial_\mu$ are defined as $r, \partial_x, \partial_y, \partial_z$. A reduction from $[S^1 \times S^2]/Z_2$ to $S^2$ is possible when $Z_2$ fields discussed in [3] are effectively frozen, or $Z_2$ strings are gapped. Moreover, the local coupling between the $U(1)$ phase and spin order involves higher derivative terms and becomes important only at a rather high frequency $\hbar \nu_s^{2/3}/2M$.

Following Eq.1, the energy of the system consists of two parts: a) the potential energy $\hbar^2 \rho \langle \nabla \mathbf{n} \rangle^2/2M$, which is the energy cost in the presence of a slow variation of $\mathbf{n}$, and the zero point kinetic(rotation) energy $I_0 = \hbar^2 \Delta a^2/2$, where $I_0 = (\hbar^2 \Delta \rho/\rho M)^{-1}$ can be considered as the effective inertial of an individual atom. This inertial originates from two-body scatterings and is inversely proportional to scattering lengths. Zero point rotations tend to disrupt the order of $\mathbf{n}$ between different atoms. $f$ is a measure of the amplitude of quantum fluctuations in $^{23}$Na BEC.

In weakly interacting limit $f$ is much less than unity, the symmetry is broken in the ground state and the BEC have a spin order, with $\mathbf{n}(r,t) = \mathbf{n}_0$. Just as the Josephson effect occurs in a superfluid where a phase long range order is established, a spin Josephson effect occurs in spin ordered BEC. A spinor condensate in a weak external magnetic field $\mathbf{B} \mathbf{e}_z$ exhibits the following time dependence:

$$\Psi(t) = \frac{1}{\sqrt{2}}[\exp(i\mu_+ t)[1,1] + \exp(i\mu_- t)[1,-1]], \quad (2)$$

and $\mu_\pm = \mu_1 \pm \omega_B, \omega_B = g \mu_B B$ ($g$ is the Lande factor of atoms). Notice that in the presence of a spin stiffness, a well defined relative phase between two components of the condensates is developed. To observe the spin coherence, one introduces a reference condensate which is at a chemical potential $\mu_2$ but experiences no magnetic fields. The temporal dependence of the interference between these two condensates would be

$$\delta \rho(t) \propto \cos ((\mu_1 - \mu_2)t) \cos (\omega_B t). \quad (3)$$

At $\omega_B = 0$, Eq. 3 reflects to a usual AC Josephson effect and when $\mu_1 = \mu_2$, one observes an AC spin Josephson effect due to a spin order in BEC, a sinusoidal-time dependence with frequency determined by $\omega_B$ instead of difference in chemical potentials. The alternative way to observe the spin coherence without introducing a reference condensate is to transform $[1,1] >$ component into $[1,-1] >$ after the magnetic field is switched on for time $t$. This results in an interference between the induced $[1,-1] >$ component and the original $[1,1] >$ component. The condensate density in the overlapped region exhibits a sinusoidal dependence on the waiting time $t$ with a frequency $2\omega_B$.

Quantum fluctuations of the nematic order can be estimated in a lowest order approximation. Introducing $\delta \mathbf{n} = \mathbf{n} - \mathbf{n}_0$ and assuming the fluctuations are weak, we obtain, $<\delta \mathbf{n}^2> \propto f$. As $f$ increases, spin wave excitations start to interact strongly and Eq.2 becomes invalid. The renormalization group (RG) equation of $f$ is determined by the interactions between collective modes. In $3+1$ dimension, the RG equation...
takes a form \( df/dl = \beta(f) \), with the \( \beta \)-function given as 
\[ \beta(f) = -2 f(f_c - f) \]. Within the frame work of NLqM, 
\( f_c = 8 \pi^2 \) in \( d = 3 \). \( f \) characterizes spin correlations in 
the ground state of \( ^{23}\text{Na} \) BEC.

At a high density limit \( f > f_c \), zero point kinetic 
ergy dominates and the spin stiffness is renormalized 
to zero at a long wave length limit. Especially, in an 
extremely quantum disordered phase, the potential 
ergy at interatomic scale \( E_T = \hbar^2 \rho^2/2M \) is negligible 
compared with zero point rotation energy \( E_o = \hbar^2/2I_0 \) 
of an individual atom. \( n \) of each atom fluctuates indepen-
dently and spins of atoms only correlate at an inter-
atomic distance. The excitation spectrum has a gap of 
\( I \) \( n \) \( t \) \( \tau \) \( f \) \( \xi \) \( \beta \) \( F \) \( \tau \) \( t \) \( V \) \( \hbar \) \( \pi \) \( \mu \) \( \nu \) \( m \) \( \tau \) \( \eta \) \( C \) \( \xi \) \( \nu \) \( I \) \( H \) \( \partial \) \( Q \) \( \Omega \) \( \sigma \) \( T \) \( \partial \) \( \psi \) \( \xi \) \( \gamma \) \( \delta \) \( \eta \) \( \zeta \) \( \chi \) \( \psi \) \( \theta \) \( \vartheta \) \( \varphi \) \( \kappa \) \( \lambda \) \( \mu \) \( \nu \) \( \xi \) \( \omicron \) \( \pi \) \( \rho \) \( \varsigma \) \( \tau \) \( \upsilon \) \( \phi \) \( \chi \) \( \psi \) \( \omega \) \( \alpha \) \( \beta \) \( \theta \) \( \vartheta \) \( \varphi \) \( \kappa \) \( \lambda \) \( \mu \) \( \nu \) \( \xi \) \( \omicron \) \( \pi \) \( \rho \) \( \varsigma \) \( \tau \) \( \upsilon \) \( \phi \) \( \chi \) \( \psi \) \( \omega \) \( \alpha \) \( \beta \) \( \theta \) \( \vartheta \) \( \varphi \) \( \kappa \) \( \lambda \) \( \mu \) \( \nu \) \( \xi \) \( \omicron \) \( \pi \) \( \rho \) \( \varsigma \) \( \tau \) \( \upsilon \) \( \phi \) \( \chi \) \( \psi \) \( \omega \)
The probability of finding deconfined space-time monopoles depends on the energy of Skyrmions with $C_m = 1$ with respect to $C_m = 0$ configuration. In polar coordinates $(r, \phi)$, a static Skyrmion is a configuration with $\mathbf{n}(\rho, \phi) = (\sin \theta(r) \cos \phi, \sin \theta(r) \sin \phi, \cos(r))$; $\theta(r)$ varies from 0 at $r = 0$ to $\pi$ at $r \gg \xi$, with $\xi$ an arbitrary parameter. For spin ordered BEC where spin Josephson effects should be observed, the energy of the Skyrmion is proportional to $8\pi\hbar^2/2M$ and is scale invariant. For a Skyrmion of given $\xi$ \cite{12}, the connection field is concentrated in a region of the size $\xi$, $\mathbf{H}_c = r^{-1} \sin \theta(r) \partial \theta(r)/\partial r$, and gets screened at a length scale larger than $\xi$. In spin ordered BEC, a Skyrmion is nondegenerate with respect to a trivial vacuum. As we will see, this leads to the confinement of space-time monopoles.

To illustrate the point of confinement, we consider $C_m$ as a function of time at some discrete unit and find the following binary representation for monopoles.

a) $\ldots0000000000000000\ldots$

b) $\ldots0000000111111111\ldots$

c) $\ldots0000011111100000\ldots$

Fig.2. In a binary representation, monopole-like instantons are represented by kinks living on a string, with $C_m$, the topological charge as the order parameter.

Each bit shown in Fig.2 labeled as 0, or 1 represents the topological charge read out at certain moment; a 1-bit corresponds to a Skyrmion while a 0-bit is for a $C_m = 0$ configuration. The a) string represents a trivial vacuum where $C_m$ remains zero at all time. In the second binary string b), a 1-string stands for any scale invariant Skyrmion created at the time $t_0$ defined by the domain wall separating 0-string and 1-string. So a domain wall in our binary representation represents a monopole $Q_m = 1$, terminating a Skyrmion configuration ($C_m = 1$) at the interface and changing $C_m$ by one unit. A domain wall-anti domain wall pair in the c) line corresponds to a monopole-antimonopole pair with $Q_m = \pm 1$; a 1-string here, or a Skyrmion, is terminated at a monopole at one end $t_1$ and at an antimonopole at the other end $t_2$.

Since the energy of a Skyrmion in spin ordered BEC is higher than that of a trivial configuration, the energy of a 1-bit is positive with respect to a 0-bit. The energy of a domain wall in Eq.1 which is the number of 1-bits in the structure, defines the action of the monopole and is proportional to the length of the 1-string. The action of an isolated space-time monopole, or creation of a Skyrmion therefore is proportional to $L_t\hbar^2/2M$, being infinity. ($L_t$ is the perimeter along a temporal direction.)

The action of having monopole-anti-monopole pairs is proportional to $(t_1 - t_2)$, the time interval between creation and annihilation of Skyrmions, thus monopoles are confined. For this reason, Eq.8 vanishes and a topological long range order prevails.

The conclusion arrived so far can be extended to 3D straightforwardly. The change of the topological density is due to the quantum nucleation of monopoles instead of Skyrmions. In the binary representation in Fig.2, a 1-bit stands for a static monopole and a 0-bit for a trivial configuration. A domain-wall represents a termination of a static monopole at certain time. Since the energy of a monopole is proportional to the system size $L$ in spin ordered BEC, the monopole is nondegenerate with respect to the trivial vacuum. Following the same argument carried out in 2D, the action to have a monopole nucleated is proportional to $LL_t\hbar^2/2M$, and is infinity. This result has a profound impact on the monopole confinement. The interaction between a monopole and an antimonopole has to be linear in terms of distance between them. The action of a monopole pair nucleated at $t_1$ and annihilated at $t_2$ with largest spatial separation $|r_1 - r_2|$ is proportional to $|r_1 - r_2|/\xi_1$ and monopoles are confined because of spin stiffness. Therefore the right hand side of Eq.8 has to be zero because of the confinement, and topological charge $C_m$ is a conserved quantity. We once again arrive at Eq.5 in a spin ordered BEC and topological charge is conserved.

The situation in disordered limit is more delicate and depends on dimensionalities. In spin disordered BEC, the spin stiffness is renormalized to zero and spin fluctuations are gapped. In the absence of spin stiffness, the energy of a Skyrmion is determined by its interaction with spin fluctuations. Upon integration of spin wave excitations, based on a general consideration of the gauge invariance and the parity invariance, we conclude Eq.1 should be reduced to $\mathcal{L}_s(\mathbf{F}_{\mu\nu}) = \mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}/2g + \ldots$; $g^{-1}(\Delta_s)$ is a function of the spin gap $\Delta_s$ measured in units of $(E_T E_0)^{1/2} = \hbar^2(\Delta a_r^{5/3})^{1/2}/2M$. In a leading approximation, $g^{-1}(x)$ is a logarithmic function of $x$, i.e. $\ln^{-1} x$ in 3D and is a linear function of $x$ in 2D \cite{13}. We will analysis the topological order based on $\mathcal{L}_s$. The energy of $C_m$-configuration is $\alpha(L)C_m^2$ if quantum tunneling is neglected, with $\alpha(L)$ a function of the system size.

In 2D, with induced interactions, Skyrmion energy is no longer scale invariant and is minimized at $\xi = \infty$. The connection field of a Skyrmion spreads over the whole 2D sheet. The energy of a Skyrmion scales as $L^{-2}$ and vanishes as the system size $L$ goes to infinity. This implies that skyrmion configurations become degenerate with a trivial vacuum and monopoles are deconfined. This mechanism of deconfinement resembles the liberation of fractionalized quasiparticles in one dimension polymers \cite{11}. In one dimensional polyacetylene, the ground state
has twofold degeneracy because of the Peierls instability and domain wall solitons become free excitations.

When instantons are liberated, the energy of a configuration $C_m$ is ill-defined. A more serious consideration beyond the mean field approach outlined above involves the evaluation of monopole-like instanton action. In $(2 + 1)d$ the partition function of a monopole configuration $\{x^n\}$, is $\sum_n P^n/n! \exp(-S_{in})$. And $P = L^2 L_s \exp(-\pi/g)$ is the quantum tunneling amplitude; $S_{in} = g^{-1} \sum_{m} \pi Q_m Q_m |x^n-x^m|$ represents the Coulomb interaction between space-time monopoles. The above result suggests that Skyrmions always condense in the spin disordered BEC and topological charge $C_m$ is not a conserved quantity. It is important to realize that in the absence of $Z_2$ strings, $\pm 1$ Skyrmions can be physically and homotopically distinguished because of the coherence of quantum spin nematic BEC, unlike the situation in a classical nematic liquid crystal. By moving a Skyrmion around a $\pi$-disclination, the condensate acquires a $\pi$-phase with respect to the original one, which can manifest itself in a Josephson type of effect. One can also distinguish $\pm 1$ Skyrmions by looking at the local connection field. In a positive Skyrmion configuration, a spin-$\frac{1}{2}$ collective excitation, which carries half charge with respect of connection fields, experiences a connection field of an opposite sign compared to that of a negative Skyrmion [4].

So we are able to show that the change of $C_m$ has a short range temporal correlation because of instanton effects,

$$< \frac{\partial C_m(\tau)}{\partial \tau} \frac{\partial C_m(0)}{\partial \tau} > \exp(-\frac{tS}{\tau_m}).$$

(9)

And $\tau_m$ is proportional to $\zeta^2 \tau_s \exp(\pi/g)$. Immediately, one recognizes that Eq.9 simply indicates a random walk of $C_m$ as a function of time in spin disordered 2D BEC. Therefore, we conclude Eq.6 holds.

But in 3D weakly disordered BEC where stiffness has vanished, a static monopole still carries a finite energy because of spin fluctuations induced interactions. In 3D, as illustrated in $\mathcal{L}_s(F_{\mu\nu})$, spin fluctuations discriminate topologically different configurations. Particularly, the fluctuations are strongest in $C_m = 0$ configuration so that the energy is lowest in the topological trivial configuration. The situation differs from 2D spin disordered limit where the energy of topological nontrivial configuration with $C_m = 1$ is the same as that of $C_m = 0$ one. For a monopole, each spherical shell of radius $R$ can be viewed to be a Skyrmion squeezed into a 2D sheet of size $R$. The smaller shells, or textures of a finite size, dominate the energy of a monopole. Spin fluctuations interacting with the singularity lift a degeneracy between the monopole configuration and a trivial vacuum in the absence of spin stiffness. It also appears to us that this is independent of the form of $\mathcal{L}_s(F_{\mu\nu})$ we introduced. The energy of a monopole here is inversely proportional to $g$, i.e., $(E_0 E_T)^{1/2} \ln \Delta^{-1}$. The action of having a monopole, which leads to a change in $C_m$ by one unit, is proportional to $L_t$ and is infinite for this reason.

So the monopoles remain confined even after spin correlation becomes short ranged, with the action to have a pair of monopoles being proportional to $(t_1 - t_2)$. Only pair production which conserves $C_m$ is allowed. The topological long range order thus coexists with a short range spin correlation in 3D. This can be considered as a case of order from disorder phenomena. And, the confinement of monopoles in this case is driven purely by the spin fluctuations, instead of spin stiffness in spin ordered BEC. However, by increasing the spin disorder further, the degeneracy between $C_m = 1$ and $C_m = 0$ could be established, which signifies deconfinement of monopoles and breakdown of the long range topological order.

It is also possible to demonstrate the topological order/disorder in terms of space-time correlations of connection fields or Wilson loop integrals. The topological order is vital for the quantum number fractionalization of excitations. In 3D spin disordered BEC, where a topological long range order is established, spin-$\frac{1}{2}$ excitations are elementary ones. On the other hand, when hedgehogs are liberated or condense, only spin-$1$ excitations exist in the spectrum. Finally, quantum hidden orders in spin liquids of strongly correlated electrons are recently reviewed by Wen [4]. Topological orders in spin triplet superconducting liquids are also investigated by Demler et al. [4]. I would like to thank ASPEN center of Physics and Amsterdam summer workshop on "Flux, charge, topology and statistics" for their hospitalities. Discussions with A. Abanov, Duncan Haldane, N. Read, T. Senthil, O. Starykh and Paul Wiegmann are greatly acknowledged.

[1] C. J. Myatt, et.al., Phys. Rev. Lett. 78, 586(1997).
[2] D. M. Stamper-Kurn, et.al., Phys. Rev. Lett. 80, 2027(1998).
[3] J. Stenger, et.al., Phys. Rev. Lett. 81, 742(1998).
[4] T. Ohmi and K. Machinda, J. Phys. Soc. Jpn. 67, 1822(1998).
[5] C. K. Law, et.al., Phys. Rev. Lett. 81, 5257(1998).
[6] Y. Castin and C. Herzog, cond-mat/0012040.
[7] F. Zhou and F. D. Haldane, ITP-UU-00/51(2000); F. Zhou, cond-mat/0104233.
[8] E. Demler, F. Zhou and F. D. Haldane, ITP-UU-01/09(2001); E. Demler and F. Zhou, cond-mat/0104409.
[9] This limit is difficult to approach in single traps because of fast recombination processes between alkali atoms. However, the difficulty can be avoided in an optical lat-
tice where $f = \sqrt{E_0/E_T}$; $E_0$ is the zero point rotation energy and $E_T$ is the exchange energy. See discussions in Ref.9.

[11] For an excellent review, see A. J. Heeger, S. Kivelson, W. P. Su and J. R. Schrieffer, Rev. Mod. Phys. 60, 781 (1988).

[12] Additional high derivative terms usually stabilize Skyrmions at a finite $\xi$.

[13] This ansatz can be justified in the context of $CP^N$ fields, in a large-$N$ expansion. See A. M. Polyakov, *Gauge Fields and Strings*, Hardwood academic publishers (1987).

[14] Fei Zhou, cond-mat/0106133, ITP-UU-01/23(2001).

[15] X. G. Wen, cond-mat/0107071. Examples of topological spin fluids in strongly correlated electron systems can also be found in P. Wiegmann, Phys. Rev. Lett. B 60, 821(1988); X. G. Wen, F. Wilczek and A. Zee, Phys. Rev. B 39, 11413(1990).

[16] E. Demler et al., cond-mat/0105446.