On a general matrix–valued unbalanced optimal transport and its fully
discretization: dynamic formulation and convergence framework

Bowen Li∗ Jun Zou†

Abstract
In this work, we present a rather general class of transport distances over the space of positive–semidefinite
matrix–valued Radon measures, called the weighted Wasserstein–Bures distance, and consider the convergence
property of their fully discretized counterparts. These distances are defined via a generalization of Benamou–Brenier
formulation of the quadratic optimal transport, based on a new weighted action functional and an abstract matricial
continuity equation. It gives rise to a convex optimization problem. We shall give a complete characterization of
its minimizer (i.e., the geodesic) and discuss some topological and geometrical properties of these distances. Some
recently proposed models: the interpolation distance by Chen et al. [18] and the Kantorovich–Bures distance by
Brenier et al. [11], as well as the well studied Wasserstein-Fisher-Rao distance [43, 19, 40], fit in our model. The
second part of this work is devoted to the numerical analysis of the fully discretization of the new transport model.
We reinterpret the convergence framework proposed very recently by Lavenant [41] for the quadratic optimal
transport from the perspective of Lax equivalence theorem and extend it to our general problem. In view of this
abstract framework, we suggest a concrete fully discretized scheme inspired by the finite element theory, and show
the unconditional convergence under mild assumptions. In particular, these assumptions are removed in the case
of Wasserstein-Fisher-Rao distance due to the existence of a static formulation.

1 Introduction

Optimal transport (OT) [68, 69, 66], which defines a family of metrics and geometric structures between probability
distributions, is currently a very active research area with rich applications including functional inequalities [22, 51, 4],
umerical PDEs [36], and more recently, computer vision and image processing [55, 23, 53, 32]. The original formulation
was first proposed by Monge in 1781 [48], however, to which the minimizer (called the transport map) may not exist.
To remedy this, Kantorovich introduced the concept of the transport plan (a joint probability measure
$\gamma$ with given marginals) in 1942 [39] to relax the problem, which leads to a convex linear program. In particular, if we take the
ground cost to be the Euclidean distance, it gives the well known quadratic Wasserstein distance:

\[
W_2^2(\rho_0, \rho_1) = \min_{\gamma} \int |x - y|^2 \, d\gamma \quad \text{for probability measures } \rho_0, \rho_1 .
\]

which, although looks simple, enjoys beautiful mathematical structures. Brenier characterized in 1991 [9] that in
this case the optimal transport map $T$ exists and can be uniquely given by the gradient of a convex function $\varphi$
(i.e. the Brenier potential), which further gives the Monge–Ampère equation by the measure-preserving condition
of transport maps. Instead of dealing with such a fully nonlinear PDE, it is also possible to add dynamics to consider
the geodesic between the two probability measures, by first interpolating the identity and the transport map
$T = \nabla \varphi$ (i.e. McCann’s displacement interpolation [45]) and then evolving the density $\rho$ by $\rho_t := ((1-t)I + t\nabla \varphi) \# \rho_0$. This can be understood as a Lagrangian formulation of the geodesic line. In the Benamou and Brenier’s breakthrough paper [6],
they gave an equivalent Eulerian description of the transport path $\rho_t$ by showing that

\[
W_2^2(\rho_0, \rho_1) = \min_{\rho, \varphi} \left\{ \frac{1}{2} \int \rho |\varphi|^2 \, dt \, dx ; \, \partial_t \rho_t + \text{div} (\rho \varphi) = 0 \right\} ,
\]

and the $\rho_t$ induced by the McCann’s interpolation solves the above nonconvex problem. They also introduced a very
important change of variable via the momentum $m = \rho \varphi$ to relax (1.1) as a convex minimization problem subject to
linear constraints:

\[
\min_{\rho, m} \left\{ \frac{1}{2} \int \rho^{-1} |\varphi|^2 \, dt \, dx ; \, \partial_t \rho_t + \text{div} m = 0 \right\} , \quad (P_{W_2})
\]

∗Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. (bwli@math.cuhk.edu.hk).
†Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. The work of this author was substantially
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and propose an augmented Lagrangian method to numerically solve it. As we shall see below, such kind of formulation or point of view stimulates a large number of follow up research works in this area, including the current work.

From the application perspective, one of limitations of the Monge–Kantorovich transportation problem is that they are defined on the space of probability measures (the measures of equal total mass). However, in many interesting cases [11, 13, 29], we need to take into account the creation and destruction of mass, which leads to unbalanced densities under comparison. The early effort of extending the classical OT to this unbalanced case may date back to Kantorovich and Rubinshtein in 1950s [38, 37]. They considered a simple purely static formulation and introduced extended Kantorovich norms, which is further investigated and extended by [33, 39]. The underlying idea is also related to the optimal partial transport [27, 15], which admits a dynamic formulation that is closely linked to the generalized Wasserstein distance proposed in [57, 58]. Recently, a new transport distance on the space of positive Radon measures was introduce independently, almost simultaneously by three research groups from different perspectives and formulations [19, 21, 44, 43, 40]. In this work we adopt the name Wasserstein–Fisher–Rao used in [19, 21] to refer to this metric, since our new transport model is much inspired by theirs. In [19], a source term describing the mass variation was added in the continuity equation and also the action functional (cf. (P_{WFR}) and (P_{WFR})). The resulting metric can be regarded as an inf-convolution of the quadratic Wasserstein metric tensor and the Fisher–Rao metric tensor, as the name suggests. The subsequent work [21] considered a broader family of Lagrangians and presented a class of new models in a unified framework via both static and dynamic formulations, and further showed their equivalence. We refer the readers to [21, 14] and the reference therein for a detailed overview of recent developments on the unbalanced optimal transport.

Another limitation of OT is that it only applies to the scalar data. Recently there is an increasing need for comparing the distributions of manifold–valued data driven by the advances in imaging science. For instance in diffusion tensor (magnetic resonance) imaging [42, 70], at each spatial position the system generates a tensor field (typically a positive-semidefinite matrix) to encode the local diffusivity of water molecules in the brain which gives the distribution of the white matter. A natural question that arises is how to measure the differences between two brain tensor fields, or mathematically how to define a reasonable transport distance between matrix–valued measures. In the past few years there are many attempts in this direction. Chen et al. [17, 16] defined a matricial Wasserstein distance on the space of matrix–valued densities with unit mass in a dynamical way inspired by (P_{W}) and based on the Lindblad equation from the quantum mechanics. Chen et al. later extended it to the unbalanced case [18] in a manner similar to [19], by interpolating between the matricial Wasserstein distance [17] and the matricial Hellinger distance (cf. Remark [83]) or the Frobenius norm. In the deep works of Carlen and Maas [14, 15], a related approach, also originated from the quantum mechanics, to define the noncommutative quadratic Wasserstein distance on the state space of matrices (not spatially dependent) was suggested. Very recently, Brenier and Vorotnikov [11] proposed another different dynamic matricial transport distance called the Kantorovich-Bures metric (cf. (P_{KB})), which is motivated by the observation in [10] that the incompressible Euler equation allows a dual problem which is a concave maximization problem. We also want to point out that Peyré et al. [20] presented static distance between matrix–valued measures with an entropic regularization and a associated entropic scaling algorithm, which generalize the results in [20].

The initial motivation of this work is numerically studying the unbalanced matricial transport models in [18] and [11] in line with [6]. We will see that although their formulations look very different (cf. (P_{W}) and (P_{WFR})), these two models are actually the examples of a general matricial dynamic optimal transport problem that shall be introduced in Section 3 which reveals their similar mathematical structures. Our definition (cf. Definition [3.4]) is motivated by (P_{W2}) and based on the convex analysis. One of the starting points is the following abstract continuity equation (cf. Definition [3.3]):

$$\partial_t G_t + D q_t = R_t^{sym},$$

where $D$ is a first–order constant coefficient linear differential operator with $D^*(I) = 0$ (in fact, most of our analysis holds even if $D$ is an appropriately defined unbounded operator). Here $q_t$ is a matrix–valued measure and each column of it can be intuitively regarded as a momentum variable; $Dq$ is the generalized advection part (a matricial analogue of divm in (P_{W})) controlling the mass transportation in space and between components; $R_t^{sym}$ is the reaction part describing the generation and absorption of mass. The other point is a class of weighted Lagrangian (cf. [3.2]):

$$J_{\lambda}(G_t(x), q_t(x), R_t(x)) = \frac{1}{2}(q_t(x)\Lambda_1^\dagger \cdot G_t^\dagger(x)q_t(x)\Lambda_1^\dagger) + \frac{1}{2}(R_t(x)\Lambda_2^\dagger \cdot G_t^\dagger(x)R_t(x)\Lambda_2^\dagger),$$

which measures the infinitesimal cost for transferring $G_t$ via $q_t$ and $R_t$, where $\Lambda_1$ and $\Lambda_2$ are weighted matrices representing the contributions of each components of $q$ and $G$ in $J_{\lambda}$. These two objects readily lead to the central model (P2). Our first contribution is a self-contained and comprehensive presentation of the properties of such kind of a transport problem and the induced metric, which we call the weighted Wasserstein–Bures distance. In particular, we show that the space of positive–semidefinite matrix–valued Radon measures endowed with this metric is a complete geodesic metric space (cf. Proposition [3.23] and Corollary [3.26]) and we characterize the optimality conditions for the minimizing geodesics (cf. Theorems [3.16] and [3.20]). This allows us to view matrix–valued measure space as a (formal)
Riemannian manifold and develop the Otto calculus in the spirit of [50]. Section 3.1 presents how our model connects the goals in [18] and [11], as well as the WFR metric, which provides rigorous mathematical arguments for many results in [18] and suggest a framework to remove the dimensional restriction in [11].

To numerically solve this convex optimization problem [27], one may first discretize it in time and space which gives a finite dimensional convex optimization problem with linear constraint. Such kind of optimization problem can efficiently solved by a class of proximal splitting schemes [52], which includes the initial augmented Lagrangian method [6] as a specific instantiation. We refer the readers to [52] and the references therein for a very detailed review on this topic and to [31, 35] for the convergence analysis for these first order convex optimization algorithms.

Compared to the extensive literature focusing on solving this kind of large scale optimization problems, there few results on the convergence of the discrete transport problem to the continuous one. Section 4 contributes to the design of a fully discretized scheme for this rather general model [27] and the corresponding error analysis, which immediately provides convergent schemes for the matricial interpolation distance [18], the Kantorovich-Bures distance [19], as well as the WFR distance [19]. In [19] Section 5, a numerical scheme based on a staggered grid and the Douglas–Rachford splitting method has been implemented but without any convergence analysis. To achieve the goals, we shall first extend the framework proposed in [11] for the quadratic Wasserstein distance to our general transport problem (2) in Section 4.2 from the perspective of the Lax(-Richtmyer) equivalence theorem and in more abstract manner. The original Lax equivalence theorem, the so called fundamental theorem of numerical analysis, actually only applies to a small set of numerical schemes for linear differential equations. But it shades light on what kind of verifiable properties of a scheme we may expect in order to have convergence. In Section 4.2, we introduce the concept of the discrete approximations of a Radon measure space (cf. Definitions 4.11 and 4.22), which helps to define a discrete transport problem with families of reconstruction operators embedding the discrete minimizers to the continuous space. We then propose a set of consistency conditions inspired by the ones in [41], and it turns out that these conditions are sufficient to guarantee the existence of a subsequence of discrete minimizers that weak–star converging to the solution to (2) under some necessary assumptions for the model and that the given measures are absolutely continuous w.r.t. the Lebesgue measure (cf. Theorem 4.10). The necessity of the absolute continuity assumption is discussed in Remark 4.29. If we change our point of views and regard the reconstruction operators as auxiliary objects as in Section 4.1, we can see this abstract framework actually suggest a way to concretely construct a convergent numerical scheme. To be more precise, we need to design a scheme such that the resulting discrete problem can “support” families of consistent reconstruction operators. This is Section 4.3 mainly devoted to. In particular, in Section 4.4 we apply the scheme constructed in Section 4.3 to the WFR distance and show that in this case the absolute continuity assumption for the given measures actually can be removed, thanks to the existence of the equivalent static formulation.

The remainder of this work is organized as follows. Section 2 presents basic notations, conventions, and some preliminary results. In Section 3 we define a class of weighted Wasserstein–Bures distances via a dynamic formulation, and explore its topological and metric properties by leveraging tools from convex analysis. In Section 4 we construct a convergent fully discretized scheme for our transport problem by first proposing an abstract convergence framework. Some details and auxiliary proofs are given in Appendices A and B.

2 Notations and Preliminary results

In this section, we collect some definitions, notations and conventions, as well as related preliminary results from matrix theory, convex analysis and measure theory, which will be used for the subsequent exposition.

Matrix theory. We denote by $\mathbb{R}^{n \times m}$, $n, m \in \mathbb{N}$, the space of $n \times m$ real matrices, and particularly by $\mathbb{M}^n$ if $m = n$. $\mathcal{S}^n$, $\mathcal{S}^n_+$ and $\mathcal{S}^n_+$ are the subsets of $\mathbb{M}^n$ consisting of symmetric matrices, symmetric positive-semidefinite matrices and symmetric positive-definite matrices, respectively, while $\mathcal{A}^n$ denotes the space of $n \times n$ antisymmetric matrices. It is known that $\mathcal{S}^n_+$ is a closed pointed convex cone in $\mathcal{S}^n$ with the interior $\mathcal{S}^n_+$. We equip all the matrix spaces with the Frobenius inner product $A \cdot B = \text{tr}(A^T B)$ and denote the induced norm by $|A| = \sqrt{A \cdot A}$ (we will occasionally write $| \cdot |_F$ to emphasize it). We denote by $A^{\text{sym}} = \frac{1}{2}(A + A^T)$ and $A^{\text{ant}} = \frac{1}{2}(A - A^T)$ the symmetric and antisymmetric part of $A \in \mathbb{M}^n$, respectively. We shall also write $A \preceq B$ (resp. $A \prec B$) for $A, B \in \mathcal{S}^n$ if $B - A \in \mathcal{S}^n_+$ (resp. $B - A \in \mathcal{S}^n_+$).

We recall the definition of the pseudoinverse (Moore-Penrose inverse) of a matrix $A \in \mathbb{R}^{n \times m}$: the matrix $A^\dagger \in \mathbb{R}^{m \times n}$ is the pseudoinverse of $A$ if and only if the following four conditions hold:

$$A^\dagger A^T = A^\dagger A, \quad (AA^\dagger)^T = AA^\dagger, \quad A^\dagger AA^\dagger = A^\dagger, \quad \text{and} \quad AA^\dagger A = A.$$ 

One can easily check that for $A \in \mathcal{S}^n$ with the eigendecomposition $A = P \Sigma P^T$, its pseudoinverse is given by $A^\dagger = P \Sigma^\dagger P^T$ with $\Sigma^\dagger = \text{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_s^{-1}, \ldots, 0)$, where $P$ is an orthogonal matrix consisting of the eigenvectors of $A$ and $\Sigma = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_s, 0)$ is a diagonal matrix with $\{\lambda_i\}_{i=1}^s$ being the nonzero eigenvalues of $A$. 

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Lemma 2.1. The following properties hold:

1. If $A \succeq B \succeq 0$ and $\text{ran}(A) = \text{ran}(B)$, then $B^\dagger \succeq A^\dagger$.

2. The cone $S^n_+$ in $S^n$ is self-dual, that is, $(S^n_+)^* = \{ B \in S^n : \text{tr}(AB) \geq 0, \forall A \in S^n_+ \} = S^n_+$.

3. If $A, B \succeq 0$ and $A \cdot B = 0$, then $\text{ran}B \subset \text{ker} A$, equivalently, $\text{ran}A \subset \text{ker} B$.

4. For $A \in S^n_+, M \in \mathbb{R}^{n \times m}$, there holds

$$AM : M \leq \text{tr}(A)|M|^2.$$  \hfill (2.1)

The proof of the above lemma, except for the first statement, is direct. We will provide a short proof of the first statement in Appendix [3] and as we can see from the example: $A = \text{diag}(1, 1, 1, 0)$ and $B = \text{diag}(1, 1, 0, 0)$, the range condition in it is necessary.

Matrix–valued Radon measure. In this work, if not specified separately, $\mathcal{X}$ always denotes a compact (separable) metric space with a Borel $\sigma$–algebra $\mathscr{B}(\mathcal{X})$. Let $C(\mathcal{X}, \mathbb{R}^n)$ be the Banach space of $\mathbb{R}^n$–valued continuous functions on $\mathcal{X}$ with the supremum norm $\| \|_\infty$, which is separable. We denote by $\mathcal{M}(\mathcal{X}, \mathbb{R}^n)$ the Banach space of $\mathbb{R}^n$–valued Radon measures endowed with the total variation norm $\| \|$, which can also be identified with the dual space of $C(\mathcal{X}, \mathbb{R}^n)$. Another important topology on $\mathcal{M}(\mathcal{X}, \mathbb{R}^n)$ is the weak–star topology: a sequence of measures $\{ \mu_j \}$ weak–star converges to $\mu \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n)$ if and only if $(\mu_j, \phi)_{\mathcal{X}} \to (\mu, \phi)_{\mathcal{X}}$ as $j \to +\infty$, for all $\phi \in C(\mathcal{X}, \mathbb{R}^n)$. Here and in what follows, $(\cdot, \cdot)_{\mathcal{X}}$ shall always denote the duality pairing between $\mathcal{M}(\mathcal{X}, \mathbb{R}^n)$ and $C(\mathcal{X}, \mathbb{R}^n)$, if not specified. For $\mu \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n)$, there exists an associated variation measure $\| \mu \| \in \mathcal{M}(\mathcal{X}, \mathbb{R}^+)$ such that $d\mu = d\sigma|\mu|$ with $|\sigma(x)| = 1$ for $\mu$–a.e. $x \in \mathcal{X}$, where $\sigma : \mathcal{X} \to \mathbb{R}^n$ is the Radon–Nikodym derivative (or the density) of $\mu$ w.r.t. $|\mu|$ (cf. [24, 32]).

The matrix–valued measures may be introduced via the general operator–valued measures, which is typically seen in the quantum information theory [26, 46, 54]. However, in this work, we directly identify the space of matrix–valued Radon measures, denoted by $\mathcal{M}(\mathcal{X}, \mathbb{R}^{n \times m})$, with $\mathcal{M}(\mathcal{X}, \mathbb{R}^{nm})$ by vectorization. It can be shown that both the sets of $S^n$–valued Radon measures and $S^n_+$–valued Radon measures are closed in $\mathcal{M}(\mathcal{X}, \mathbb{M}^n)$ w.r.t. the weak–star topology (cf. [24, Theorem 3.5]). Moreover, we have the following characterizations:

$$(C(\mathcal{X}, S^n))^* \simeq (C(\mathcal{X}, \mathbb{M}^n)/(C(\mathcal{X}, \mathbb{A}^n))^* \simeq \mathcal{M}(\mathcal{X}, S^n),$$

where $\simeq$ means the isometric isomorphism. Indeed, it is easy to observe that $\mu \in \mathcal{M}(\mathcal{X}, S^n) \subset \mathcal{M}(\mathcal{X}, \mathbb{M}^n) \simeq C(\mathcal{X}, \mathbb{A}^n)^*$ if and only if the induced bounded linear functional on $C(\mathcal{X}, \mathbb{M}^n)$ has the kernel $C(\mathcal{X}, \mathbb{A}^n)$, that is, $\mathcal{M}(\mathcal{X}, S^n) = C(\mathcal{X}, \mathbb{A}^n)^\perp$, which further yields

$$(C(\mathcal{X}, \mathbb{M}^n)/(C(\mathcal{X}, \mathbb{A}^n))^* \simeq \mathcal{M}(\mathcal{X}, S^n),$$

by [12 Proposition 11.9]. Then, it suffices to check $C(\mathcal{X}, S^n) \simeq (C(\mathcal{X}, \mathbb{M}^n)/(C(\mathcal{X}, \mathbb{A}^n))$, which is a direct corollary of $S^n = (\mathbb{A}^n)^\perp$ and $S^n \simeq \mathbb{M}^n/\mathbb{A}^n$.

We now collect some technical preliminaries related to the measurability of matrix–valued functions for later use.

Proposition 2.2. Let $A(x)$ be a $S^n$–valued Borel measurable function on $\mathcal{X}$. Then it holds that

1. The pseudoinverse $A^1(x)$ of $A(x)$ is measurable.

2. The eigenvalues $\{ \lambda_{A,x}(x) \}_{n=1}^\infty$ of $A(x)$, arranged in nondecreasing order, are measurable on $\mathcal{X}$; and the corresponding eigenvectors $\{ u_{A,x}(x) \}_{n=1}^\infty$ can also be selected to be measurable and to form an orthonormal basis of $\mathbb{R}^n$ for every $x \in \mathcal{X}$.

3. If further $A(x) \in S^n_+$, then the unique square root function $A^{1/2}(x)$ is measurable.

The first and the second property are from [61] and [60], respectively. The last property follows from the fact that $A^{1/2}$ is a continuous function on $S^n_+$, which can be proved by functional calculus. In fact, the following estimate holds

$$\|A^{1/2} - B^{1/2}\|_2 \leq C\|A - B\|_2^{1/2}, \quad \forall A, B \in S^n_+,$$  \hfill (2.2)

where $\| \|_2$ is the induced 2–norm (cf. Powers–Størmer inequality [59]).

For $\mu \in \mathcal{M}(\mathcal{X}, S^n_+)$, we define the associated trace measure $\text{tr}\mu$ by the set function $E \to \text{tr}(\mu(E))$, $E \in \mathscr{B}(\mathcal{X})$. It is clear that $0 \leq \mu(E) \leq \text{tr}(\mu(E))I$ and $\text{tr}\mu$ is equivalent to $|\mu|$ in the sense that

$$|\mu| \ll \text{tr}\mu \quad \text{and} \quad \text{tr}\mu \ll |\mu|. \quad \hfill (2.3)$$

Moreover, it is also easy to check that for any $\lambda \in \mathcal{M}(\mathcal{X}, \mathbb{R}^+)$ such that $|\mu| \ll \lambda$, the density $\frac{d\mu}{d\lambda} \in S^n_+$ holds for $\lambda$–a.e. $x \in \mathcal{X}$. We remark that in what follows, we typically choose $\text{tr}\mu$ as the dominant measure for $\mu \in \mathcal{M}(\mathcal{X}, S^n_+)$ instead of the variation measure $|\mu|$ since it is linear w.r.t. $\mu$. 

We now consider the integrable functions w.r.t. a matrix–valued measure. For a positive measure \( \lambda \), we denote by \( L^1_\lambda(X, \mathbb{R}^n) \), \( p \geq 1 \), the Banach spaces of \( \mathbb{R}^n \)-valued functions which are \( p \)-integrable w.r.t. \( \lambda \). For \( G \in \mathcal{M}(X, \mathbb{S}_+^n) \), let \( L^2_\lambda(X, \mathbb{R}^n) \) be the space of \( \mathbb{R}^n \)-valued measurable functions endowed with the semi–inner product:

\[
\langle q, p \rangle_{L^2_\lambda(X)} := \langle G, q \otimes p \rangle_X^{1/2} = \left( \int_X p \cdot G \cdot q d\lambda \right)^{1/2},
\]

where \( \lambda \) is a reference measure such that \( |G| \ll \lambda \) and \( G_\lambda := \frac{dG}{d\lambda} \). Here and throughout this work,

- we use sans serif letterforms, e.g. \( G \), to denote vector–valued or matrix–valued measures,

- while letters with serifs are typically reserved for their densities w.r.t. some reference measures, e.g. \( G_\lambda \).

Note that \( \|q\|_{L^2_\lambda(X)} = 0 \) is equivalent to \( G_\lambda^2/q = 0 \) for \( \lambda \)-a.e. \( x \in X \). The kernel of the seminorm \( \| \cdot \|_{L^2_\lambda(X)} \) is given by

\[
F = \{ q \text{ is Borel measurable; } q \in \ker(G_\lambda), \ \lambda\text{-a.e.}\},
\]

which is independent of the reference measure. It allows us to define, by abuse of notation, the Hilbert space \( L^2_\lambda(X, \mathbb{R}^n) \) by the quotient space \( L^2_\lambda(X, \mathbb{R}^n)/F \). It is worth emphasizing that \( L^2_{\text{loc}}(X, \mathbb{R}^n) \subset L^2_\lambda(X, \mathbb{R}^n) \) holds by \( (2.1) \) while the converse is not true, see \( (2.4) \) for the counterexample and related discussions.

**Convex analysis.** Finally we recall some concepts and important results from convex analysis. Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) be an extended real–valued function on a Banach space \( X \). We say that \( f \) is proper if \( \text{dom}(f) = f^{-1}([0, \infty)) \neq \emptyset \); and that \( f \) is positively homogeneous of degree \( k \) if for all \( x \in X \) and \( \alpha > 0 \), \( f(\alpha x) = \alpha^k f(x) \). The conjugate function \( f^* \) of \( f \) is defined on the dual space \( X^* \) by

\[
f^*(x^*) = \sup_{x \in X} \langle x^*, x \rangle - f(x), \ \forall x^* \in X^*, \tag{2.4}
\]

which is convex and lower semicontinuous (l.s.c.) w.r.t. the weak–star topology of \( X^* \). The following two results \( (3.3) \) shall be useful in the following exposition.

**Lemma 2.3 (Subgradient).** Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) be a proper convex function on a Banach space \( X \). Then, the following three properties are equivalent: (i) \( x^* \in \partial f(x) \); (ii) \( f(x) + f^*(x^*) = \langle x^*, x \rangle \); (iii) \( f(x) + f^*(x^*) \leq \langle x^*, x \rangle \). In addition, if \( f \) is lower-semicontinuous, then all of these properties are equivalent to \( x \in \partial f^*(x^*) \).

**Proposition 2.4 (Fenchel–Rockafellar duality).** Let \( X \) and \( Y \) be two Banach spaces. Let \( L : X \to Y \) be a bounded linear operator and \( L^* : Y^* \to X^* \) be its adjoint operator. Let \( f \) and \( g \) be two proper l.s.c. convex functions defined on \( X \) and \( Y \), respectively, with values in \( \mathbb{R} \cup \{+\infty\} \). If there exists \( x \in \text{dom}(f) \) such that \( g \) is continuous at \( Lx \), then

\[
\sup_{x \in X} -f(-x) - g(Lx) = \inf_{y^* \in Y^*} f^*(L^*y^*) + g^*(y^*) \tag{2.5}
\]

and the inf in \( (2.5) \) can be attained. Moreover, the sup in \( (2.5) \) is attained at \( x \in X \) if and only if there exists a \( y^* \in Y^* \) such that \( Lx \in \partial g^*(y^*) \) and \( L^*y^* \in \partial f(-x) \), in which case \( y^* \) also achieves the infimum in \( (2.5) \).

### 3 Weighted distances on matrix–valued measures

In this section, we shall first introduce a general matrix–valued transport problem using the dynamic formulation, which is the central object in this work. It defines a new class of metrics on the measure space \( M(\Omega, \mathbb{S}^n) \). Sections \( 3.2 \) and \( 3.3 \) are mainly devoted to the investigation of the properties of the minimizer (i.e., the geodesic) and the topological, metric, and geometric properties of these new distances. Finally in Section \( 3.4 \) we connect our general model with several existing models in the literature. In what follows, \( \Omega \subset \mathbb{R}^d \), \( d \in \mathbb{N} \), denotes the closure of a bounded open convex set with the Borel \( \sigma \)-algebra \( \mathcal{B}(\Omega) \), and \( \mathbb{Q}^1_\alpha := [a, b] \times \Omega, b > a > 0 \), is the corresponding time-space domain in \( \mathbb{R}^{d+1} \). If \( a = 0 \) and \( b = 1 \), we shall simply write \( Q \).

#### 3.1 Basic definition and properties

**Action functional.** To define our dynamic transportation model over the space of \( \mathbb{S}^n \)-valued measures, the starting point is the definition of a weighted action functional. For this, let \( n, k, m \in \mathbb{N} \) be fixed positive integers and \( \Lambda \) be a couple of weighted matrices \( \Lambda := (\Lambda_1, \Lambda_2), \Lambda_1 \in \mathbb{S}^k_+, \Lambda_2 \in \mathbb{S}^m_+ \). We consider the following closed convex set

\[
\mathcal{O}_\Lambda = \{(A, B, C) \in \mathbb{S}^n \times \mathbb{R}^{n \times k} \times \mathbb{R}^{n \times m}; \ A + \frac{1}{2}B\Lambda_1 B^T + \frac{1}{2}C\Lambda_2 C^T \leq 0\}. \tag{3.1}
\]

It is clear that its indicator function \( \mathcal{I}_{\mathcal{O}_\Lambda} \) is proper l.s.c. and convex (cf. \( (5) \) Lemma 1.24)). Denote by \( J_\Lambda \) the conjugate function of \( \mathcal{I}_{\mathcal{O}_\Lambda} \) (a.k.a. the support functional of \( \mathcal{O}_\Lambda \)) define by \( (2.4) \). Our first task is to derive the explicit formulas for \( J_\Lambda \) and its subgradient \( \partial J_\Lambda \) for later use.
Proposition 3.1. $J_A$ is proper, positively homogeneous (of degree 1), l.s.c. and convex, and has the following representation formula:

$$J_A(X,Y,Z) = \frac{1}{2}(YA^T_1) \cdot X^T(YA_1^1) + \frac{1}{2}(ZA_2^1) \cdot X^T(ZA_2^1) \quad (3.2)$$

if $X \in \mathbb{S}_n^+$, $\text{ran}(Y^T) \subset \text{ran}(A_1)$, $\text{ran}(Z^T) \subset \text{ran}(A_2)$ and $\text{ran}([Y,Z]) \subset \text{ran}(X)$; otherwise $J_A(X,Y,Z) = +\infty$.

Moreover, the subgradient of $J_A$ at $(X,Y,Z) \in \text{dom}(J_A)$ is characterized as follows:

$$\partial J_A(X,Y,Z) = \{(A,B,C) \in \mathcal{O}_A; \ Y = XBA^2_1, \ Z = XCA^2_2, \ X \cdot (A + \frac{1}{2}BA^1_2B^T + \frac{1}{2}CA^1_2C^T) = 0\} \quad (3.3)$$

$\partial J_A(X,Y,Z)$ is a singleton if and only if $(X,Y,Z) \in \mathbb{S}_n^+ \times \mathbb{R}^{n \times k} \times \mathbb{R}^{n \times m}$ and $\Lambda_1 \in \mathbb{S}_+^{n+k}$, $\Lambda_2 \in \mathbb{S}_+^{m}$.

Proof. It is known that for any nonempty, closed and convex set $C$ in a Hilbert space, its support functional is proper l.s.c. convex and also positively homogeneous (cf. [3], Proposition 14.11). Then the first statement follows. To derive the representation formula (3.2) for $J_A$, by definition, we have, for $(X,Y,Z) \in \mathbb{S}_n^+ \times \mathbb{R}^{n \times k} \times \mathbb{R}^{n \times m}$,

$$J_A(X,Y,Z) = \sup_{(A,B,C) \in \mathcal{O}_A} X \cdot A + Y \cdot B + Z \cdot C. \quad (3.4)$$

We consider the following four cases.

Case I: $X \in \mathbb{S}_n^n \setminus \mathbb{S}_n^+$. We choose a vector $a \in \mathbb{R}^n$ such that $a \cdot Xa < 0$ and then take $A = -\lambda a \otimes a \leq 0$, $B = 0$ and $C = 0$ with $\lambda > 0$ in (3.3). Then it follows that

$$J_A(X,Y,Z) \geq \sup_{\lambda > 0} X \cdot (-\lambda a \otimes a) = +\infty.$$

Case II: $\text{ran}(Y^T) \not\subset \text{ran}(A_1)$ or $\text{ran}(Z^T) \not\subset \text{ran}(A_2)$. It suffices to consider the case $\text{ran}(Y^T) \not\subset \text{ran}(A_1)$, since the same argument applies to the other one: $\text{ran}(Z^T) \not\subset \text{ran}(A_2)$. W.l.o.g., we let $Y$ have the form: $Y = [y_1, \ldots, y_n]^T$, where $y_i \in \mathbb{R}^k$ for $i = 1, \ldots, n$ and $y_i \notin \text{ran}(A_1)$. Since $A_1 \in \mathbb{S}_+^n$, $y_i$ can be orthogonally decomposed as:

$$y_i = y_i^{(1)} + y_i^{(2)} \quad \text{with} \quad y_i^{(1)} \in \ker(A_1) \supseteq \text{ran}(A_1), \quad y_i^{(2)} \neq 0 \in \ker(A_1).$$

Considering $(A,B,C) \in \mathcal{O}_A$ of the form: $A = 0$, $B = \lambda y_i^{(2)} 0^T$, and $C = 0$, $\lambda \in \mathbb{R}$, in (3.3), we have

$$J_A(X,Y,Z) \geq \sup_{\lambda > 0} |\lambda y_i^{(2)}|^2 = +\infty.$$

Case III: $\text{ran}([Y,Z]) \not\subset \text{ran}(X)$. Similarly, it suffices to consider $\text{ran}(Y) \not\subset \text{ran}(X)$. We take $(A,B,C)$ of the form

$$A = -\frac{\lambda^2}{2}(P_{\ker(X)}Y A_1)(P_{\ker(X)}Y A_1)^T, \quad B = \lambda P_{\ker(X)}Y \quad \text{and} \quad C = 0$$

with $\lambda > 0$ in (3.3), where $P_{\ker(X)} := I - X^T X$ is the orthogonal projector onto $\ker(X)$. A direct computation leads to

$$J_A(X,Y,Z) \geq \sup_{(A,B,0) \in \mathcal{O}_A} X \cdot A + Y \cdot B \geq \sup_{\lambda > 0} -\frac{\lambda^2}{2}(P_{\ker(X)}Y A_1) \cdot (X P_{\ker(X)}Y A_1) + \lambda Y \cdot (P_{\ker(X)}Y) \geq \sup_{\lambda > 0} \lambda (P_{\ker(X)}Y) \cdot (P_{\ker(X)}Y) = +\infty,$$

since there holds $\text{ran}(Y) \not\subset \text{ran}(X)$, which gives $(P_{\ker(X)}Y) \cdot (P_{\ker(X)}Y) > 0$.

Case IV: $(X,Y,Z) \in \mathbb{S}_n^+ \times \mathbb{R}^{n \times k} \times \mathbb{R}^{n \times m}$ with $\text{ran}(Y^T) \subset \text{ran}(A_1)$, $\text{ran}(Z^T) \subset \text{ran}(A_2)$ and $\text{ran}([Y,Z]) \subset \text{ran}(X)$. For this case, we directly compute

$$X \cdot A + Y \cdot B + Z \cdot C = X \cdot (A + \frac{1}{2}BA_1^2B^T + \frac{1}{2}CA_2^1C^T) + Y \cdot B + Z \cdot C - X \cdot \frac{1}{2}BA_1^2B^T + \frac{1}{2}CA_2^1C^T, \quad (3.5)$$

and

$$Y \cdot B + Z \cdot C - X \cdot \left(\frac{1}{2}BA_1^2B^T + \frac{1}{2}CA_2^1C^T\right) = -\frac{1}{2}|\sqrt{X}BA_1 - \sqrt{X}^T Y A_1|^2 - \frac{1}{2}|\sqrt{X}CA_2 - \sqrt{X}^T ZA_2|^2 + \frac{1}{2}|\sqrt{X}^T Y A_1|^2 + \frac{1}{2}|\sqrt{X}^T ZA_2|^2, \quad (3.6)$$

(3.6)
where we have used

\[ Y \cdot B + Z \cdot C = (\sqrt{X} \sqrt{X}^\dagger Y A_1^\dagger A_1) \cdot B + (\sqrt{X} \sqrt{X}^\dagger Z A_2^\dagger A_2) \cdot C, \]

which is from the range relations: \( \text{ran}(Y^T) \subseteq \text{ran}(A_1) \), \( \text{ran}(Z^T) \subseteq \text{ran}(A_2) \) and \( \text{ran}([Y, Z]) \subseteq \text{ran}(X) \). By Lemma 2.3, \( X \cdot (A + \frac{1}{2} B A_1^\dagger B^T + \frac{1}{2} C A_2^\dagger C^T) \leq 0 \) always holds. Then we readily see from (3.5) and (3.6) that the maximizers to (3.4) is given by the set

\[
\{(A, B, C) \in \mathcal{O}_\lambda; \ Y = X BA_1^2, \ Z = X CA_2^2, \ X \cdot (A + \frac{1}{2} B A_1^\dagger B^T + \frac{1}{2} C A_2^\dagger C^T) = 0\},
\]

and the corresponding supremum is

\[
\lambda^*(X, Y, Z) = \frac{1}{2} (Y A_1^\dagger) \cdot X^\dagger (Y A_1^\dagger) + \frac{1}{2} (Z A_2^\dagger) \cdot X^\dagger (Z A_2^\dagger),
\]

Hence, the proof of (3.2) is complete.

Finally, to characterize the subgradient of \( \lambda^* \), by Lemma 2.3, we have that \( (A, B, C) \in \partial \lambda^*(X, Y, Z) \) if and only if \( (A, B, C) \in \mathcal{O}_\lambda \) and \( \lambda^*(X, Y, Z) = X \cdot A + Y \cdot B + Z \cdot C \) holds. Hence (3.2) readily follows from the above proof for (3.2). For the last statement, we note that \( \partial \lambda^*(X, Y, Z) \) is a singleton if and only if the equations in (3.8) for \( (A, B, C) \) are uniquely solvable if and only if \( \lambda_1 \in S^n_{++}, \lambda_2 \in S^n_+ \), and \( X \in S^n_{++} \).

For our application, we restrict ourselves to the case \( m = n \) and \( \lambda_2 \in S^n_+ \) (cf. Remark 3.2 for the reasons). Since \( \lambda^* \) is positively homogeneous by Proposition 3.1 for a given triplet of measures \( \mu \in \mathcal{M}(\mathcal{X}, S^n \times \mathbb{R}^{n \times k} \times M^n) \), we can define another nonnegative measure \( \lambda^\ast \) on \( \mathcal{X} \) by

\[
\lambda^\ast(E) := \int_E \lambda \left( \frac{d\mu}{d\lambda} \right) d\lambda \text{ for } \forall E \in \mathcal{B}(\mathcal{X}),
\]

where \( \lambda \in \mathcal{M}(\mathcal{X}, \mathbb{R}) \) is a reference measure such that \( |\mu| \ll \lambda \). We remark that the definition of \( \lambda^\ast \) is independent of the choice of the reference measure by the positive homogeneity of \( \lambda^* \). Indeed, if \( \nu \) is another reference measure such that \( |\mu| \ll \nu \), then it holds that

\[
\lambda^\ast(E) = \int_E \lambda \left( \frac{d\mu}{d\nu} \right) d\lambda = \int_E \lambda \left( \frac{d\mu}{d\nu + \lambda} \right) d(\lambda + \mu) = \int_E \lambda \left( \frac{d\mu}{d\nu} \right) d\nu \text{ for } \forall E \in \mathcal{B}(\mathcal{X}),
\]

by the chain rule of Radon–Nikodym derivatives. To alleviate notations, we shall adopt the following conventions from now on.

- We define the space \( \mathcal{X} := S^n \times \mathbb{R}^{n \times k} \times M^n \) and hence write \( \mathcal{M}(\mathcal{X}, \mathcal{X}) = \mathcal{M}(\mathcal{X}, S^n \times \mathbb{R}^{n \times k} \times M^n) = C(\mathcal{X}, \mathcal{X})^* \) where \( C(\mathcal{X}, \mathcal{X}) = C(\mathcal{X}, S^n \times \mathbb{R}^{n \times k} \times M^n) \).
- We occasionally write \( \mu \) for \((G, q, R) \in \mathcal{M}(\mathcal{X}, \mathcal{X}) \) without definition, if it is clear from the context.
- We will use \( \lambda^\ast \) to denote its total variation \( \lambda^\ast(\mu)(\mathcal{X}) \) instead of the measure itself by abuse of notation, thanks to the observation \( \lambda^\ast(\mu)(E) = \lambda^\ast(E) \) for \( E \in \mathcal{B}(\mathcal{X}) \).
- In the sequel \( (G, q, R) \) denotes the density of \((G, q, R) \in \mathcal{M}(\mathcal{X}, \mathcal{X}) \) w.r.t. a reference measure \( \lambda \in \mathcal{M}(\mathcal{X}, \mathbb{R}) \) that satisfies \( |(G, q, R)| \ll \lambda \). We shall also often omit the subscript \( \lambda \) for simplicity, if the involved quantity is independent of the choice of \( \lambda \).

**Definition 3.2.** The \( \lambda \)-weighted action functional for a measure \( \mu \in \mathcal{M}(\mathcal{X}, \mathcal{X}) \) is defined as \( \lambda^\ast(\mu) \).

By Proposition 3.1 and definition of \( \lambda^\ast(\mu) \), we readily have the following useful lemma.

**Lemma 3.3.** For \( \mu \in \mathcal{M}(\mathcal{X}, \mathcal{X}) \) with \( \lambda^\ast(\mu) < +\infty \), we have \( G \in \mathcal{M}(\mathcal{X}, S^n_+) \) and \( |q, R| \ll \text{tr}G \) with

\[
\begin{align*}
G \in S^n_+ \quad &\text{and} \quad |q, R| \ll \text{tr}G \\
\text{ran}(q_1^\dagger) \subseteq \text{ran}(A_1), \quad &\text{ran}(R_2^\dagger) \subseteq \text{ran}(A_2), \quad \lambda \text{-a.e.}.
\end{align*}
\]

**Proof.** By \( \lambda^\ast(\mu) = \int_X \lambda(\mu) d\lambda \), we have \( \lambda^\ast(\mu)(x) \) is finite for \( \lambda \)-a.e. \( x \in \mathcal{X} \), where \( \lambda(\mu, x) = (G, q_1^\dagger(x), R_2^\dagger(x)) \). Equivalently, \( \mu(x) \in \text{dom}(\lambda) \) holds \( \lambda \)-a.e., which immediately yields the property (3.9). We next show the absolute continuity of \( |q| \) and \( |R| \) w.r.t. \text{tr}G, that is, for \( E \in \mathcal{B}(\mathcal{X}) \) with \( \text{tr}G(E) = 0 \), we show \( |q|(E) = |R|(E) = 0 \). For this, we consider two Borel subsets \( E_1 \) and \( E_2 \) of \( E \):

\[
E_1 = \{ x \in E; \ G(x) \in S^n_+ \{0\} \}, \quad E_2 = \{ x \in E; \ G(x) = 0 \},
\]

with \( E = E_1 \cup E_2 \). By \( \text{tr}G(E_1) = 0 \) and \( \text{tr}G \lambda > 0 \) on \( E_1 \) everywhere, we have \( \lambda(E_1) = 0 \). Then \( |q|(E_1) = 0 \) and \( |R|(E_1) = 0 \) follows from \( |q|, |R| \ll \lambda \). Moreover, since \( \lambda_G = 0 \) on \( E_2 \) and \( \text{tr}G \) holds, we readily have \( q_1(x) = 0 \) and \( R_2(x) = 0 \) for \( \lambda \)-a.e. \( x \in E_2 \). Then it follows that \( |q|(E_2) = 0 \) and \( |R|(E_2) = 0 \). The proof is complete.
Continuity equation. Another key ingredient for the dynamic formulation is the following general matricial continuity equation (cf. Definition 3.4). Let $D^* : C_c^\infty(\mathbb{R}^d, S^n) \rightarrow C_c^\infty(\mathbb{R}^d, \mathbb{R}^{n \times k})$ be a first–order constant coefficient linear differential operator with $D^*(I) = 0$. By Fourier transform, it can be alternatively characterized by

$$D^*(\Phi)(x) = \int_{\mathbb{R}^d} \hat{D}^*(\xi)\hat{\Phi}(\xi) e^{ix\cdot\xi} d\xi, \quad \Phi \in C_c^\infty(\mathbb{R}^d, S^n),$$  

(3.10)

where $\hat{\Phi}(\xi) = (2\pi)^d \int_{\mathbb{R}^d} \Phi(x) e^{-i\xi \cdot x} dx$ is the Fourier transform of $\Phi$ and $\hat{D}^*(\xi) : \mathbb{R}^d \rightarrow \mathcal{L}(S^n, \mathbb{R}^{n \times k})$ is the symbol of $D^*$ satisfying that for any $A \in S^n$ and $E \in \mathbb{R}^{n \times k}$, $E \cdot \hat{D}^*(\xi)(A)$ is a first–order polynomial in $\xi$. Moreover, if we write $\hat{D}^*(\xi)$ as the sum of its homogeneous components: $\hat{D}^*(\xi) = \hat{D}^*_0 + \hat{D}^*_1(\xi)$, where $\hat{D}^*_0 \in \mathcal{L}(S^n, \mathbb{R}^{n \times k})$, $\hat{D}^*_1(\xi)$ are homogeneous of degree zero and degree one, respectively, it is easy to see that the condition $D^*(I) = 0$ can be equivalently written as $D^*(0)(I) = D^*_0(I) = 0$ (recall that the Fourier transform of $I$ is $\delta_0 I$). In the sequel, for a function $\Phi$ of $(t, x) \in \mathbb{R}^{d+1}$, $D^*\Phi$ is defined by acting $\hat{D}^*$ on the spatial variable $x$.

**Definition 3.4.** A measure $(G, q, R) \in \mathcal{M}(Q^b_\alpha, \mathbb{X})$, $b > a > 0$, is said to be a curve connecting $G_a, G_b \in \mathcal{M}(\Omega, S^n)$ over the time interval $[a, b]$ if it satisfies the general matrix–valued continuity equation:

$$\partial_t G + Dq = R^{sym} + \delta_\alpha(t) \otimes G_a - \delta_b(t) \otimes G_b$$  

(3.11)

in the sense of distributions $\mathcal{D}'(\mathbb{R}^{d+1})$, where $D$ is the adjoint operator of $D^*$ defined by

$$\langle Dq, \Phi \rangle_{C^1(Q^b_\alpha, S^n)} = -\langle q, D^*\Phi \rangle_{Q^b_\alpha} \quad \forall \Phi \in C^1(Q^b_\alpha, S^n), \quad \forall q \in \mathcal{M}(Q^b_\alpha, \mathbb{R}^{n \times k}).$$  

(3.12)

We denote by $\mathcal{CE}([a, b]; G_a, G_b)$ the set of curves in $\mathcal{M}(Q^b_\alpha, \mathbb{X})$ connecting $G_a, G_b \in \mathcal{M}(\Omega, S^n)$ over $[a, b]$. The measures $G_a$ and $G_b$ are referred to as the initial and final distributions of $G$ respectively.

In the above definition of continuity equation, we have regarded Radon measures as distributions on $\mathbb{R}^{d+1}$. To be more precise, we have the following weak formulation of (3.11):

$$\int_{Q^b_\alpha} \partial_t \Phi \cdot dG + D^*\Phi \cdot dq + \Phi \cdot dR = \int_\Omega \Phi_b \cdot dG_b - \int_\Omega \Phi_a \cdot dG_a \quad \forall \Phi \in C^1(Q^b_\alpha, S^n).$$  

(3.13)

Before proceeding to investigate the properties of the continuity equation, let us first provide an intuitive interpretation of (3.11) here. We denote by $D^*_0$ and $D^*_1$ the homogeneous differential operators induced by the symbols $\hat{D}^*_0$ and $\hat{D}^*_1(\xi)$ respectively, and similarly define their adjoint operators $D_0$ and $D_1$ by (3.12). By definition, $D_0$ is a linear map from $\mathbb{R}^{n \times k}$ to $S^n$ while $D_1$ vanishes when acting on a constant function. It allows us to split $Dq$ in two parts: $D_0q$ and $D_1q$, where $D_0q$ and $D_1q$ describe the mass transportation between components of $G$ and the transportation in space respectively. Moreover, the condition $D^*(I) = 0$ can be thought as a conservativity condition in the (heuristic) sense: $\langle Dq, I \rangle = -\langle q, D^*(I) \rangle = 0$. In view of this observation, we may expect that the mass variation is purely controlled by $R$ (cf. Proposition 3.4).

It is worth pointing out that $q$ satisfies a homogeneous boundary condition on $\partial \Omega$ that is implicitly encoded into the dual relation (3.12). To have an explicit representation, we let $q$ admit a smooth density $\nu$ w.r.t. Lebesgue measure. The Stokes’ theorem gives

$$\int_\Omega Dq \cdot \Phi = \int_{\partial \Omega} q \cdot \hat{D}_1(\nu)\Phi = \int_{\partial \Omega} \hat{D}_1(\nu)(q) \cdot \Phi \quad \forall \Phi \in C^1(\Omega),$$

where $\hat{D}_1$ is the symbol of $D_1$ similarly defined by (3.10) and $\nu$ is the exterior unit normal vector of $\partial \Omega$. Then it readily follows that the spatial boundary condition is given by $\hat{D}_1(\nu)(q) = 0$ for a smooth density $\nu$. On the other hand, the temporal boundary condition for $G$ is enforced by the term $\delta_\alpha(t) \otimes G_a - \delta_b(t) \otimes G_b$. Recalling that the Dirac delta function is the distributional derivative of the Heaviside function, i.e., $\delta_a = \chi_{[a, +\infty)}$, the temporal boundary condition can be removed by considering the extended curve:

$$\tilde{\mu} = (G, \tilde{q}, \tilde{R}) := (G, q, R) + dt|_{(-\infty, a]} \otimes (G_a, 0, 0) + dt|_{[b, +\infty)} \otimes (G_b, 0, 0) \in \mathcal{M}(\mathbb{R}^{d+1}, \mathbb{X}),$$  

(3.14)

which satisfies $\partial_t \tilde{G} + D\tilde{q} = R^{sym}$ in $\mathcal{D}'(\mathbb{R}^{d+1})$. Moreover, we shall also see very soon that under very mild conditions, the temporal boundary condition is in fact redundant (see remarks after Proposition 3.5).

Let $\pi^1 : (t, x) \rightarrow t$ be the projection from the time–space domain $[a, b] \times \Omega$ to the time interval $[a, b]$. The following elementary lemma about the absolute continuity of the time marginal of $G$ is a direct corollary of (3.13).

**Lemma 3.5.** Suppose $\mu \in \mathcal{CE}([a, b]; G_a, G_b)$, then we have that $\pi^*_\# G \in \mathcal{M}([a, b], S^n)$ has the time distributional derivative $(\pi^*_\# R)^{sym} \in \mathcal{M}([a, b], S^n)$. If, further, $G \in \mathcal{M}(Q^b_\alpha, S^n)$, then $\pi^*_\# |G| \ll dt$. 

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Proof. It suffices to consider the time interval $[0, 1]$. It is clear that $\pi^t_#G$ and $\pi^t_#R$ are Radon measures since every finite Borel measure on $[0, 1]$ is regular. By the weak formulation of the continuity equation (3.13) with test functions $\Phi$ of the form $\Phi(t, x) = \phi(t) \in C_c^1((0, 1), \mathbb{R}^n)$, we have

$$
\int_0^1 \partial_t \phi \cdot \mathrm{d} \pi^t_# G + \phi \cdot \mathrm{d} \pi^t_# R = 0,
$$

which implies that $\pi^t_# R$ is the distributional derivative of $\pi^t_# G$. Suppose that $M(t)$ is the matrix-valued bounded variation function that generates the Radon measure $\pi^t_# R$ (cf. [28, Theorem 3.29]). Then, by (3.15) we have

$$
d\pi^t_# G = (M(t))^{\text{sym}} + C \mathrm{d} t
$$

holds for some constant symmetric matrix $C \in \mathbb{S}^n$ (cf. [28, Theorem 3.36]). If $G \in \mathcal{M}(Q, S^+_n)$, it readily follows from (3.10) and (2.3) that

$$
\text{tr} \pi^t_# G \sim |\pi^t_# G| \ll \mathrm{d} t,
$$

which further yields $\pi^t_# |G| \ll \mathrm{d} t$, since $\text{tr} \pi^t_# G = \pi^t_# \text{tr} G \sim |\pi^t_# |G||$ again by (2.3). This completes the proof. □

**Weighted Wasserstein–Bures distance.** We are now in a position to define a class of distances as functionals from $\mathcal{M}(\Omega, S^+_n) \times \mathcal{M}(\Omega, S^+_n)$ to $[0, +\infty)$, by minimizing the action functional $J_{\Lambda, Q}(\mu)$ over the solutions to the continuity equation (3.11).

**Definition 3.6.** The weighted Wasserstein–Bures distance between measures $G_0, G_1 \in \mathcal{M}(\Omega, S^+_n)$ is defined by

$$
\text{WB}^2_{\Lambda}(G_0, G_1) = \inf_{\mu \in \mathcal{CE}([0, 1]; G_0, G_1)} J_{\Lambda, Q}(\mu) \tag{P}
$$

We remark that the quantity $J_{\Lambda, Q}(\mu)$ may be understood as the energy of the curve $\mu \in \mathcal{CE}([0, 1]; G_0, G_1)$. The following priori estimate shows that $\mathcal{CE}([0, 1]; G_0, G_1)$ is nonempty and the distance $\text{KB} \Lambda(G_0, G_1)$ is always finite, and hence the problem (P) is well posed.

**Lemma 3.7.** For any $G_0, G_1 \in \mathcal{M}(\Omega, S^+_n)$, there exists a curve $\mu = (G, 0, R) \in \mathcal{CE}([0, 1]; G_0, G_1)$ with finite $J_{\Lambda, Q}(\mu)$. Moreover, it holds that

$$
\text{WB}^2_{\Lambda}(G_0, G_1) \leq \text{KB}^2_{\Lambda, \omega_2}(G_0, G_1) \leq 2|\Lambda^2_2|^{1/2} \int_{\Omega} \left| \sqrt{G_{1, \lambda}} - \sqrt{G_{0, \lambda}} \right| \mathrm{d} \lambda,
$$

where $G_{0, \lambda}$ and $G_{1, \lambda}$ are densities of $G_0$ and $G_1$ w.r.t. the reference measure $\lambda \in \mathcal{M}(\Omega)$ that satisfies $|G_0|, |G_1| \ll \lambda$.

**Proof.** We omit the subscript $\lambda$ of $G_{0, \lambda}$ and $G_{1, \lambda}$ for simplicity. We define measures

$$
G := (\sqrt{G_0} + t(\sqrt{G_1} - \sqrt{G_0}))^2 \mathrm{d} \lambda \in \mathcal{M}(Q, S^+_n),
$$

and

$$
R := 2(\sqrt{G_0} + t(\sqrt{G_1} - \sqrt{G_0}))(\sqrt{G_1} - \sqrt{G_0}) \mathrm{d} \lambda \in \mathcal{M}(Q, M^n).
$$

Then it is clear that $\text{ran}(\frac{\partial R}{\partial G_{0, \lambda}}) \subset \text{ran}(\frac{\partial G}{\partial G_{0, \lambda}})$, $\mathrm{d} \lambda \otimes \lambda$–a.e. We also note

$$
\text{ran}(\sqrt{G_1} - \sqrt{G_0}) \subset \text{ran}(\sqrt{G_0} + t(\sqrt{G_1} - \sqrt{G_0})),
$$

from the relation: $\ker(\sqrt{G_0} + t(\sqrt{G_1} - \sqrt{G_0})) = \ker(\sqrt{G_0}) \cap \ker(\sqrt{G_1}) \subset \ker(\sqrt{G_1} - \sqrt{G_0})$, for $t \in (0, 1)$. By definition and the above observations, we directly compute the energy $J_{\Lambda, Q}(\mu)$ of the curve $\mu$ as

$$
J_{\Lambda, Q}(\mu) = 2 \int_{\Omega} \left| (\sqrt{G_1} - \sqrt{G_0}) \Lambda^2_2 \right| \mathrm{d} \lambda.
$$

The proof is completed by Definition 3.6 and the submultiplicativity of the Frobenius norm. □

**Remark 3.8.** We see from the above proof of Lemma 3.7 that the assumption $\Lambda_2 \in S^m_{++}$ mainly imposes $\text{ran}(\Lambda_2) = \mathbb{R}^m$ so that there always a curve $\mu \in \mathcal{CE}([0, 1]; G_0, G_1)$ with finite $J_{\Lambda, Q}(\mu)$ (cf. Lemma 3.7), and $\text{KB} \Lambda$ can be defined on the whole space $\mathcal{M}(\Omega, S^+_n)$ (otherwise, it may only be able to be defined on a subset).
Remark 3.9. The bound provided in (3.17) for \( KB_{(0, A_2)} \), and for \( WB_\Lambda \) as well, may not be tight, although it is sufficient for our use. In fact, \( KB_{(0, A_2)} \) is essentially the matricial Hellinger distance \( d_H \) defined in [39, Definition 4.1], up to a transformation. To see this, recalling Lemma 3.3 we have that if \( \Lambda_1 = 0 \), then \( q \) must be zero (otherwise \( J_0^R Q(\mu) = +\infty \)). Then, (3.17) reduces to

\[
KB_{(0, A_2)}(G_0, G_1) = \inf \{ J_0(0, A_2), Q(\mu) : \mu = (G, 0, R) \in CE(\{0, 1\}; G_0, G_1) \}.
\]

(3.18)

Given \( S \in S_{+}^n \), let \( g_S \) be a linear map defined by \( g_S(A) := SAS : S_{+}^n \to S_{+}^n \) with the inverse map \( g_{S^{-1}} \). It is easy to see that \( (G, 0, R) \in CE(\{0, 1\}; G_0, G_1) \) if and only if \( (g_{A^{-1}}(G), 0, g_{A^{-1}}(R)) \in CE(\{0, 1\}; g_{A^{-1}}(G), g_{A^{-1}}(G_1)) \), and there holds \( J_0(0, A_2), Q((G, 0, R)) = J_0(0, A_2), Q(g_{A^{-1}}(G), 0, g_{A^{-1}}(R)) \). By consequence, we have

\[
KB_{(0, A_2)}(G_0, G_1) = KB_{(0, A_2)}(g_{A^{-1}}(G_0), g_{A^{-1}}(G_1)).
\]

It is shown in the proof of Proposition 3.20 that \( KB_{(0, A_2)} \) is nothing else than the convex formulation of the Hellinger distance \( d_H \) in [39, Definition 4.1], up to a constant. We refer the readers to [40, Lemma 4.3 and Theorem 2] for an explicit characterization of the Hellinger distance which is closely related to the Bures-Wasserstein distance on \( S_{+}^n \).

Thanks to the above lemma, we can change the admissible set \( CE(\{0, 1\}; G_0, G_1) \) in (2) to the set of curves with finite energy:

\[
CE(\{0, 1\}; G_0, G_1) := CE(\{0, 1\}; G_0, G_1) \cap \{ \mu \in M(Q) ; J_0, Q(\mu) < +\infty \},
\]

which has desired properties summarized in the following several results. Before stating them, let us prepare some tools. We introduce

\[
J_\Lambda^*(G, u, W) = \frac{1}{2} \| (uA_1, WA_2) \|_{L^2_\Lambda(X)}^2 \quad \text{on } M(\mathcal{X}, S_{+}^n) \times C(\mathcal{X}, \mathbb{R}^{n \times k} \times M^n).
\]

(3.19)

By an argument similar to the one for Lemma 3.15 we have that the conjugate of \( J_\Lambda^*(G, u, W) \) w.r.t. \( (u, W) \) is nothing but \( J_\Lambda^*(G, q, R) \). Moreover, it holds that

\[
J_\Lambda^*(G, q, R) = \sup_{(u, W) \in L^2_{(G, q, R)}} \langle (q, R), (u, W) \rangle_X - J_\Lambda^*(G, u, W).
\]

(3.20)

Since \( J_\Lambda^*(G, q, R) \) and \( J_\Lambda^*(G, u, W) \) are homogeneous of degree 2 w.r.t. \( (q, R) \) and \( (u, W) \) respectively, we have, by (3.20), that for \( (G, q, R) \in M(\mathcal{X}) \) and \( (u, W) \in L^2_{(G, q, R)}(\mathcal{X}, \mathbb{R}^{n \times k} \times M^n) \), there holds

\[
\langle (q, R), (u, W) \rangle_X \leq \gamma^{-2} J_\Lambda^*(G, q, R) + \gamma^2 J_\Lambda^*(G, u, W) \quad \forall \gamma > 0.
\]

(3.21)

We minimize the right-hand side of (3.21) w.r.t. \( \gamma \) and obtain the following analogue of Cauchy’s inequality:

\[
\langle (q, R), (u, W) \rangle_X \leq 2 \sqrt{J_\Lambda^*(G, q, R) J_\Lambda^*(G, u, W)},
\]

(3.22)

where we have also used non-negativity of \( J_\Lambda^*(G, q, R) \) and \( J_\Lambda^*(G, u, W) \).

We also observe from Lemma 3.3 and Proposition 2.2 that for \( \mu = (G, q, R) \in M(\mathcal{X}, X) \) with \( J_\Lambda, Q(\mu) < +\infty \), the functions \( G^* q \Lambda \) and \( G^* R \Lambda_{A_2}^{-1} \) are well defined, Borel measurable and independent of the reference measure \( \lambda \) (hence we omit the subscript \( \lambda \) in the sequel for simplicity), and there holds

\[
J_\Lambda, Q(\mu) = \frac{1}{2} \| G^* q \Lambda \|_{L^2_\Lambda(Q)}^2 + \frac{1}{2} \| G^* R \Lambda_{A_2}^{-1} \|_{L^2_\Lambda(\mathcal{E})}^2 < +\infty.
\]

(3.23)

We now give the a priori bounds for measures \( q \) and \( R \), which shall turn out very useful.

Lemma 3.10. For \( \mu = (G, q, R) \in M(\mathcal{X}, X) \) with \( J_\Lambda, Q(\mu) < +\infty \), it holds that

\[
|q|(E) \leq \sqrt{\text{tr} G(E)} |A_1|^2 \| G^* q \Lambda \|_{L^2_\Lambda(\mathcal{E})}^2, \quad |R|(E) \leq \sqrt{\text{tr} G(E)} |A_2|^2 \| G^* R \Lambda_{A_2}^{-1} \|_{L^2_\Lambda(\mathcal{E})}, \quad \forall E \in \mathcal{B}(Q).
\]

(3.24)

Proof. Recall that there exists bounded measurable functions \( \sigma_q \) and \( \sigma_R \) with \( |\sigma_q| = |\sigma_R| = 1 \) such that \( dq = \sigma_q d|q| \) and \( dR = \sigma_R d|R| \). Taking \( X = E, R = 0 \) and \( (u, W) = (\chi_E \sigma_q, 0) \) in (3.22), we obtain

\[
|q|(E) = \langle q, u \rangle_E \leq 2 \sqrt{J_\Lambda, E(G, q, 0) J_\Lambda, E(G, u, 0)} \leq \sqrt{\text{tr} G(E)} |A_1|^2 \| G^* q \Lambda \|_{L^2_\Lambda(\mathcal{E})},
\]

by (3.22) and the estimate

\[
J_\Lambda, E(G, u, W) \leq \frac{1}{2} \text{tr} G(E) |A_1|^2,
\]

which is derived from (3.19) and (2.1). Similarly, by taking \( q = 0 \) and \( (u, W) = (0, \chi_E \sigma_R) \) in (3.22), we obtain the estimate for \( R \) in (3.24).

\qed
With the help of the above lemma, the following proposition holds.

**Proposition 3.11.** For $\mu = (G, q, R) \in CE_{\infty}([0, 1]; G_0, G_1)$, the following properties hold:

(i) $G \in \mathcal{M}(Q, S^n_+)$ and the time marginal distribution of $|G|$ is absolutely continuous w.r.t. the Lebesgue measure $dt$. Moreover, $\mu$ can be disintegrated as:

$$
\mu = \int_0^1 \delta_t \otimes (G_t, q_t, R_t) dt.
$$

(ii) There exists a weak–star continuous curve $\{\bar{G}_t\}_{t \in [0, 1]}$ in $\mathcal{M}(\Omega, S^n)$ such that $G_t = \bar{G}_t$ for a.e. $t \in [0, 1]$ and, for any interval $[t_1, t_2] \subset [0, 1]$, it holds that

$$
\int_{t_1}^{t_2} \partial_t \Phi \cdot dG + D^* \Phi \cdot dq + \Phi \cdot dR = \int_{t_1}^{t_2} \Phi_{t_1} : d\bar{G}_{t_1} - \int_{t_1}^{t_2} \Phi_{t_0} : d\bar{G}_{t_0}, \quad \forall \Phi \in C^1(Q_{t_0}, S^n).
$$

Moreover, there holds

$$
\text{tr}G_t(\Omega) \leq C \left( \text{tr}G_0(\Omega) + \|G^1 R A_2^{-1} \|_{L^2(Q)}^2 |A_2|^2 \right) \forall t \in [0, 1].
$$

**Proof.** (i) By Proposition A.1, we can disintegrate $\mu$ w.r.t. $\nu = \pi^\dagger_x |\mu|$ as $\mu = \int_0^1 \delta_t \otimes \mu_t d\nu$ with $\mu_t \in \mathcal{M}(\Omega, X)$ for $\nu$-a.e. $t \in [0, 1]$. Then by Lemmas 3.3 and 3.5, we have $G \in \mathcal{M}(Q, S^n_+)$ and $\nu \ll \pi^\dagger_x |G| \ll dt$ on $[0, 1]$ with a Borel density $\frac{d\nu}{dt}$, which allows us to define another measurable family $\{\tilde{\mu}_t\}_{t \in [0, 1]} := \{\frac{m_t}{\mu_t} \}_{t \in [0, 1]}$ and disintegrate $\mu = \int_0^1 \delta_t \otimes \tilde{\mu}_t dt$.

(ii) We consider the test functions $\Phi$ in (3.13) of the form $a(t) \Psi(x)$ with $a(t) \in C^1([0, 1])$ and $\Psi(x) \in C^1(\Omega, S^n)$ and then obtain $\int_{t_1}^{t_2} \Psi \cdot dG_t \in W^{1, 1}([0, 1]) = AC([0, 1])$ (cf. 12) with the weak derivative

$$
\partial_t (G_t, \Psi)_{\Omega} = (q_t, D^* \Psi)_{\Omega} + (R_t, \Psi)_{\Omega}.
$$

For the estimate (3.27), we consider $\langle G_t, \Psi \rangle_{\Omega}$ with $\Psi = I$ and then obtain $\partial_t \text{tr}G_t(\Omega) = \text{tr}G_t^{\dagger}(\Omega)$ a.e. by $D^*(I) = 0$, which implies there exists a nonnegative function $m(t) \in C([0, 1])$ such that $\text{tr}G_t(\Omega) = m(t)$ a.e. on $[0, 1]$ and

$$
m(t) - m(s) = \int_s^t \text{tr}G_t^{\dagger}(\Omega) dt \quad \forall 0 \leq s \leq t \leq 1.
$$

By (3.24) in Lemma 3.10, it follows from (3.29) that

$$
|m(t) - m(s)| \leq C|R|(Q) \leq C \sqrt{\text{tr}G(Q) |A_2|}^2 \|G^1 R A_2^{-1} \|_{L^2(Q)}.
$$

We choose $t_0$ such that $m(t_0) = \max_{t \in [0, 1]} m(t)$ by continuity of $m(t)$. Then (3.30) immediately implies

$$
m(t_0) \leq m(0) + C \sqrt{m(t_0) |A_2|}^2 \|G^1 R A_2^{-1} \|_{L^2(Q)},
$$

which further yields, by an elementary calculation,

$$
(m(t_0))^{1/2} - \frac{C}{2} \|G^1 R A_2^{-1} \|_{L^2(Q)}^2 |A_2|^2 \leq m(0) + \frac{C^2}{4} \|G^1 R A_2^{-1} \|_{L^2(Q)}^2 |A_2|^2.
$$

Then we have

$$
m(t) \leq C(m(0) + \|G^1 R A_2^{-1} \|_{L^2(Q)}^2 |A_2|^2).
$$

The formula (3.28) can be proved by an argument similar to the one of [2, Lemma 8.1.2]. We sketch it here for the sake of completeness. By (3.24) and (3.27), as well as (3.29), we can conclude that there exists a subset $E \in [0, 1]$ of Lebesgue measure zero and depending on a countable dense subset of $C^1(\Omega, S^n)$ such that $\text{tr}G_t(\Omega) = m(t)$ on $[0, 1] \setminus E$, and there holds

$$
|\langle G_t, \Psi \rangle_{\Omega} - \langle G_s, \Psi \rangle_{\Omega}| \leq C \|\Psi\|_{1, \infty} \left( |\langle \{s, t \} \times \Omega \rangle + |R|([s, t] \times \Omega) \right)
$$

$$
\leq C|t - s|^{1/2}(m(0) + \|G^1 q A_1 |L^2(Q) |A_1|^2 + \|G^1 R A_2^{-1} \|_{L^2(Q)}^2 |A_2|^2)\|\Psi\|_{1, \infty},
$$

for any $t, s \in [0, 1] \setminus E$ with $s < t$ and $\Psi \in C^1(\Omega, S^n)$. The estimate (3.33) allows us to uniquely extend the restriction of $G_t$ on $[0, 1] \setminus E$ to a continuous curve $\{\tilde{G}_t\}_{t \in [0, 1]}$ in $C^1(\Omega, S^n)$. Then, by the density of $C^1(\Omega, S^n)$ in $C^1(\Omega, S^n)$ (cf. Stone–Weierstrass theorem and the boundedness of $\{\partial_\gamma G_t(\Omega)\}_{t \in [0, 1]}$ (cf. 3.3)), we have the curve $\{G_t\}_{t \in [0, 1]}$ is also continuous in $\mathcal{M}(\Omega, S^n)$ w.r.t. the weak-star topology. The formula (3.26) follows from taking test functions $\Phi(x, t) = \eta_t(t) \Phi(t, x)$ in (3.13) where $\Phi \in C^1(\Omega, S^n)$ and $\eta_t \in C_{\text{sym}}([0, 1] \times (t_0, t_1))$ with $0 \leq \eta_t \leq 1$, $\lim_{t \to 0} \eta_t(t) = \chi_{(t_0, t_1)}(t)$ pointwisely and $\lim_{t \to 0} \eta_t' = \delta_{t_0} - \delta_{t_1}$ in the distributional sense, and the fact that $D$ is a differential operator acting on the spatial variable $x$. By continuity, we have $\text{tr}G_t = m(t)$ and then (3.27) follows from (3.32). □
By the above proposition, we can always identify a curve \( \mu = (G, q, R) \in \mathcal{CE}_\infty([0,1]; G_0, G_1) \) with a measurable family \( \{\mu_t = (G_t, q_t, R_t)\}_{t \in [0,1]} \) in \( \mathcal{M}(\Omega, \mathbb{S}_+^d) \) via the disintegration (3.23), where \( G_t \) admits a weak–star continuous representative. In view of this fact, the initial and final distributions \( G_0 \) and \( G_1 \) in Definition 3.4 are actually superfluous since they can be recovered by the values of the curve \( G_t \) on the open interval \( (0,1) \) (see the estimate (3.33)). Moreover, we note that (3.24) can be rephrased as: for every \( \Psi \in C^1(\Omega, \mathbb{S}^d) \), \( \langle G_t, \Psi \rangle_{\Omega} \) is absolutely continuous on \([0,1]\) with the weak derivative given by

\[
\frac{d}{dt}(G_t, \Psi)_{\Omega} = \langle q_t, D^\ast \Psi \rangle_{\Omega} + \langle R_t, \Psi \rangle_{\Omega} \quad \text{for a.e. } t \in [0,1].
\] (3.34)

In fact, its converse is also true. That is, for a measurable family \( \mu_t = (G_t, q_t, R_t) \) such that (3.34) holds for any \( \Psi \in C^1(\Omega, \mathbb{S}^d) \), then \( \mu = \int_0^1 \delta_t \otimes \mu_t dt \in \mathcal{CE}([0,1]; G_0, G_1) \), which is a direct application of the fact that the tensor product \( C_c([0,1]) \otimes C_c(\mathbb{R}^d) \) is dense in \( \mathbb{D}'(\mathbb{R}^{d+1}) \), see [17] Proposition 2.73.

By writing \( J_{\lambda, Q}(\mu) = \int_0^1 J_{\lambda, Q}(\mu_t) dt \) for \( \mu \in \mathcal{CE}_\infty([0,1]; G_0, G_1) \), the following scaling formulas readily follow from the change of variables.

**Lemma 3.12.** Let \( s(t) : [0,1] \to [a,b] \) be a strictly increasing absolutely continuous map with an absolutely continuous inverse: \( t = s^{-1} \). Let \( T : \Omega \to T(\Omega) \) be a diffeomorphism, and suppose that there exists \( T_{0^*}(x) \) such that for each \( x \in \Omega, t \in \mathbb{D}^*_{T^*}(x) \in \mathbb{L}(\mathbb{R}^{n \times k}) \) and there holds

\[
T_{0^*}([D^* \Phi] \circ T) := D^* (\Phi \circ T).
\] (3.35)

Then we have

1. \( \mu \in \mathcal{CE}_\infty([0,1]; G_0, G_1) \) if and only if \( \bar{\mu} := \int_0^1 \delta_s \otimes (G_{s}, q_{s}, R_{s}) dt \in \mathcal{CE}_\infty([0,1]; G_0, G_1) \). Moreover, the following holds for the energy functional:

\[
\int_0^1 t'(s(t)) J_{\lambda, Q}(\mu_t) dt = \int_a^b J_{\lambda, Q}(\bar{\mu}_t) dt.
\] (3.36)

2. \( \mu \in \mathcal{CE}_\infty([0,1]; G_0, G_1) \) on \( \Omega \) if and only if \( \bar{\mu} := \int_0^1 \delta_s \otimes T_{\#}(G_s, q_s, R_s) dt \in \mathcal{CE}_\infty([0,1]; T_{\#}G_0, T_{\#}G_1) \) on \( T(\Omega) \), where \( T_{\#} \) is the transpose of \( T_{\#} \) defined via \( (T_{\#}q) \cdot p = q \cdot (T_{\#}p) \), \( \forall p, q \in \mathbb{R}^{n \times k} \).

**Remark 3.13.** The condition (3.33) is nontrivial and necessary for the second statement, which depends on the properties of both \( D^* \) and \( T \). To be precise, let us use Fourier transform and obtain

\[
D^* (\Phi \circ T) = \int_{\mathbb{R}^d} \hat{\Phi}(\xi) e^{i\xi \cdot T(x)} d\xi = \int_{\mathbb{R}^d} \hat{D}^* (\xi \cdot \nabla T(x))|\hat{\Phi}(\xi)| e^{i\xi \cdot T(x)} d\xi = (D^* \Phi) \circ T,
\]

where \( (\xi \cdot \nabla T(x))_j = \xi_j \cdot \partial_j T(x), D^* (\xi) \) is the symbol of \( D^* \) and \( D^*_T \) is the differential operator with variable coefficients induced by the symbol \( D^* (\xi \cdot \nabla T(x)) \). Then we readily see that the condition (3.33) is, in fact, equivalent to the property that \( D^* (\xi \cdot \nabla T(x)) \) admits a separation of variables: \( D^* (\xi \cdot \nabla T(x))(A) = T_{\#}D^* (\xi)(A) \), \( A \in \mathbb{S}^d \). A sufficient condition to ensure that (3.33) holds is that \( D^* \) is homogeneous of degree zero , or \( D^*_T \) is homogeneous of degree one and \( T \) is of the form \( T(x) = ax + b \) with \( a \in \mathbb{R}, b \in \mathbb{R}^d \). A detailed discussion on the necessary condition is beyond the scope of this work.

Using Lemma 3.12 with \( s(t) = (b-a)t + a : [0,1] \to [a,b], b > a > 0 \), we see that for every \( \mu \in \mathcal{CE}_\infty([0,1]; G_0, G_1) \), there exists \( \bar{\mu} \in \mathcal{CE}_\infty([a,b]; G_0, G_1) \) such that

\[
\int_0^1 J_{\lambda, Q}(\mu_t) dt = (b-a) \int_a^b J_{\lambda, Q}(\bar{\mu}_t) dt,
\]

and vice versa, which immediately leads to the following equivalent characterization of \( WB \):

\[
KB_{A}^2(G_0, G_1) = \inf_{\mathcal{CE}_\infty([a,b]; G_0, G_1)} (b-a) \int_a^b J_{\lambda, Q}(\mu_t) dt, \quad G_0, G_1 \in \mathcal{M}(\Omega, \mathbb{S}_+^d). \quad (P')
\]

We end the discussion of basic properties of the set \( \mathcal{CE}_\infty([0,1]; G_0, G_1) \) with a compactness result.

**Proposition 3.14.** Suppose that \( \mu^n \in \mathcal{CE}_\infty([0,1]; G_0^n, G_1^n) \) is a sequence of curves with

\[
m := \sup_{n \in \mathbb{N}} \text{tr}(G_0^n) < +\infty, \quad M := \sup_{n \in \mathbb{N}} J_{\lambda, Q}^k(\mu^n) < +\infty \quad (3.37)
\]

Then there exists a subsequence, still denoted by \( \mu^n \), and a curve \( \mu \in \mathcal{CE}_\infty([0,1]; G_0, G_1) \) such that for every \( t \in [0,1], G_t^n \) weak–star converges to \( G_t \) in \( \mathcal{M}(\Omega, \mathbb{S}^d) \), and \( (q^n, R^n) \) weak–star converges to \( (q, R) \) in \( \mathcal{M}(Q, \mathbb{R}^{n \times k} \times \mathbb{M}^n) \). Moreover, it holds that, for \( 0 \leq a < b \leq 1 \),

\[
J_{\lambda, Q}^k(\mu) \leq \liminf_{n \to \infty} J_{\lambda, Q}^k(\mu^n).
\] (3.38)
Proof. By \( (3.37) \), up to a subsequence, we can let \( G^n_0 \) weak–star converge to some \( G_0 \in M(\Omega, S^n_+) \). It is also clear from the a priori estimates \( (3.24) \) and \( (3.27) \), as well as the assumption \( (3.37) \), that \( \{\mu^n\}_{n \in \mathbb{N}} \) is bounded in \( M(Q, \mathbb{X}) \). Hence, there exists a subsequence of \( \{\mu^n\}_{n \in \mathbb{N}} \), still indexed by \( n \), weak–star converging to some \( \mu \in M(\Omega, \mathbb{X}) \). We next prove that the restriction of \( \mu_n \) on \( Q^b_\varepsilon \), i.e., \( \mu_n|_{Q^b_\varepsilon} \), weak–star converges to \( \mu|_{Q^b_\varepsilon} \) in \( M(Q^b_\varepsilon, \mathbb{X}) \) for any \( 0 \leq a \leq b \leq 1 \). For this, again by estimates \( (3.24) \) and \( (3.27) \), we have that there exists a constant \( C \) depending on \( m \) and \( M \) such that
\[
|\mu^n|(\{t_0, t_1\} \times \Omega) \leq C|t_1 - t_0|^{1/2} \quad \forall 0 \leq t_0 \leq t_1 \leq 1,
\]
which also holds for \( \mu \). Let \( \eta(t) \) be a smooth function, compactly supported in \([a, b]\), with \( |\eta(t)| \leq 1 \) on \( \mathbb{R} \) and \( \eta = 1 \) on \([a + \varepsilon, b - \varepsilon]\) for some small \( \varepsilon \). Then for any \( \Xi \in C(Q^b_\varepsilon, \mathbb{X}) \), we can define \( \Xi(t, x) = \eta(t)\Xi(t, x) \) which belongs to \( C(Q, \mathbb{X}) \). The following estimate readily follows from the properties of \( \eta \) and the estimate \( (3.39) \):
\[
|\langle \Xi, \mu^n\rangle_{Q^b_\varepsilon} - \langle \Xi, \mu\rangle_{Q^b_\varepsilon}| \leq |\langle \Xi, \mu^n\rangle_{\mathbb{Q}} - \langle \Xi, \mu\rangle_{\mathbb{Q}}| + C\varepsilon^{1/2},
\]
where the constant \( C \) depends on \( m, M \) and \( \Xi \). Since \( \mu^n \) weak–star converges to \( \mu \) in \( M(Q, \mathbb{X}) \) and \( \varepsilon \) is arbitrary, we have
\[
\lim_{n \to 0} |\langle \Xi, \mu^n\rangle_{Q^b_\varepsilon} - \langle \Xi, \mu\rangle_{Q^b_\varepsilon}| = 0 \quad \forall \Xi \in C(Q^b_\varepsilon, \mathbb{X}),
\]
as desired. Hence \( (3.33) \) follows by the lower–semi-continuity of \( J_{\Lambda, Q^b_\varepsilon} \). We now show the weak–star convergence of \( G^n_t \) for every \( t \in [0, 1] \). We note
\[
\int_0^t \left( \int_\Omega D^*\Psi \cdot dq^n_s + \int_\Omega \Psi \cdot dR^n_s \right) ds = \int_\Omega \Psi \cdot dG^n_t - \int_\Omega \Psi \cdot dG^n_0 \quad \forall \Psi \in C^1(\Omega, \mathbb{S}^n),
\]
by taking \( \Phi(s, x) = \chi_{[0, t]}(x)\Psi(x) \) in \( (3.20) \) with \( \Psi(x) \in C^1(\Omega, \mathbb{S}^n) \). Then, using the weak–star convergences of \( G^n_0 \) in \( M(\Omega, \mathbb{S}^n) \) and \( (q^n_s, R^n_s)|_{Q^b_\varepsilon} \) in \( M(Q^b_0, \mathbb{R}^{n \times k} \times \mathbb{M}^n) \), we get the convergence of \( (G^n_t, \Psi) \) as \( n \to \infty \). The proof is completed by the boundedness of \( C^1(\Omega, \mathbb{S}^n) \) in \( C(\Omega, \mathbb{S}^n) \) and the uniform boundedness of \( \text{tr} G^n_t(\emptyset) \) w.r.t. \( n \) (cf. (3.24)). \( \square \)

### 3.2 Existence and optimality

In this section, we investigate the convex optimization problem with a linear constraint \( (2) \). We shall first use the Fenchel-Rockafellar theorem to show the existence of the minimizer and derive the corresponding optimality condition. We then explore the conditions of changing the order of inf and sup in more detail, which essentially leads to the Riemannian interpretation in Section 3.3. Finally, we consider the effect of varying weighted matrices and show the convergence of the minimizers to \( (2) \) to the one to \( (3.18) \).

For our purpose, let us first introduce the Lagrange multiplier \( \Phi \in C^1(Q, \mathbb{S}^n) \) and define the Lagrangian of \( (2) \):
\[
\mathcal{L}(\mu, \Phi) := J_{\Lambda, Q}(\mu) - \langle \mu, (\partial_t \Phi, D^*\Phi, \Phi) \rangle_Q + \langle G_1, \Phi \rangle_{\Omega} - \langle G_0, \Phi \rangle_{\Omega},
\]
which allows us to write
\[
KB^2_\Lambda(G_0, G_1) = \inf_{\mu \in M(\Omega, \mathbb{X})} \sup_{\Phi \in C^1(Q, \mathbb{S}^n)} \mathcal{L}(\mu, \Phi).
\]
If we change the order of sup and inf, then a formal calculation via integration by parts leads to the dual problem:
\[
\text{WB}^2_\Lambda(G_0, G_1) \geq \sup_{\mu \in \mathcal{M}(\Omega, \mathbb{X})} \inf_{\Phi \in C^1(Q, \mathbb{S}^n)} \mathcal{L}(\mu, \Phi)
\]
\[
= \sup_{\Phi} \left\{ \langle G_1, \Phi \rangle_{\Omega} - \langle G_0, \Phi \rangle_{\Omega} : \partial_t \Phi + \frac{1}{2} D^*\Phi \Lambda_1^2(D^*\Phi)^T + \frac{1}{2} \Phi \Lambda_2^2 \Phi \leq 0 \right\}.
\]
We next use the Fenchel–Rockafellar theorem to show that the duality gap is zero (i.e., the strong duality), which shall also give us the existence of the minimizer to \( (2) \) and the optimality conditions. For this, let \( C(\Omega, \mathcal{O}_\varepsilon) \) be the closed convex subset \{\( \varphi \in C(\Omega, X) : \varphi(x) \in \mathcal{O}_\varepsilon, \forall x \in Q \} \) of \( C(\Omega, \mathbb{X}) \). We then define proper l.s.c. convex functions: \( f(\Phi) = \langle G_1, \Phi \rangle_{\Omega} - \langle G_0, \Phi \rangle_{\Omega}, \Phi \in C^1(Q, \mathbb{S}^n) \), and \( g(\Xi) = \iota_{C(\Omega, \mathcal{O}_\varepsilon)}(\Xi), \Xi \in C(\Omega, \mathbb{X}) \). We also introduce the bounded linear operator: \( L : \Phi \in C^1(Q, \mathbb{S}^n) \to (\partial_t \Phi, D^*\Phi, \Phi) \in C(\Omega, \mathbb{X}) \) with the adjoint operator \( L^* : (G, q, R) \to -\partial_t G - Dq + \text{Re} \varepsilon \text{in} C^1(Q, \mathbb{S}^n)^* \), which is defined by the duality: \( \langle L^* \mu, \Phi \rangle_{C^1(Q, \mathbb{S}^n)} := \langle \mu, L \Phi \rangle_{\Omega} \). These notions help us to write \( (3.30) \) as sup\{\( f(\Phi) - g(L\Phi) : \Phi \in C^1(Q, \mathbb{S}^n) \}\). We now verify the condition in Proposition \( 2.3 \). We consider \( \Phi = -\varepsilon I + \tilde{\varepsilon} I \in C^1(Q, \mathbb{S}^n) \). It is clear that \( f(\Phi) \) is finite and \( L\Phi = (-\varepsilon I, 0, -\varepsilon I + \tilde{\varepsilon} I) \) by \( D^*(I) = 0 \). By a direct calculation, we have
\[
\partial_t \Phi + \frac{1}{2} D^*\Phi \Lambda_1^2(D^*\Phi)^T + \frac{1}{2} \Phi \Lambda_2^2 \Phi = -\varepsilon I + \frac{1}{2} \varepsilon^2 (-t + \frac{1}{2})^2 \Lambda_2^2 \leq -\varepsilon I + \frac{1}{8} \varepsilon^2 \Lambda_2^2,
\]
which implies that for small enough \( \varepsilon \), \( (L\Phi)(t,x) \) is in the interior of \( \mathcal{O}_\Lambda \): \( \{ (A,B,C) \in \mathcal{O}_\Lambda : -A \in \mathbb{S}_+^n \} \) for every \( (t,x) \in Q \), and hence \( g \) is continuous at \( L\Phi \). Then Proposition 2.3 readily gives

\[
\min_{\mu \in \mathcal{M}(Q,S)} f^*(L\mu) + g^*(\mu) = \sup_{\Phi \in C^1(Q,S^n)} f(\Phi) - g(L\Phi),
\]

where \( f^*(L\mu) = \sup\{ \langle \mu, L\Phi \rangle_Q - f(\Phi) : \Phi \in C^1(Q,S^n) \} \) can be easily computed to be \( \iota_{\mathcal{E}}([0,1];G_0,G_1) \) by linearity of \( f \) while \( g^*(\mu) \) is nothing but \( J_{\Lambda,Q}(\mu) \) by the following lemma, which is a direct corollary of general results in [8] 02.

For the sake of completeness and the reader’s convenience, we state it here and provide a simple proof in Appendix B.

**Lemma 3.15.** Let \( \mathcal{X} \) be the compact separable metric space and \( C(\mathcal{X},\mathcal{O}_\Lambda) \) be defined as above. Then we have

\[
\iota_{C(\mathcal{X},\mathcal{O}_\Lambda)}^* = \sup_{\Xi \in L^\infty_{\text{loc}}(\mathcal{X},\mathcal{O}_\Lambda)} \langle \mu, \Xi \rangle_{\mathcal{X}} = J_{\Lambda,\mathcal{X}}(\mu),
\]

which is proper, convex and l.s.c. with respect to the weak–star topology of \( \mathcal{M}(\mathcal{X},\mathcal{X}) \). Moreover, we can characterize the subgradient \( \partial J_{\Lambda,\mathcal{X}}(\mu)_{|C(\mathcal{X},\mathcal{X})} \) as follows:

\[
\partial J_{\Lambda,\mathcal{X}}(\mu)_{|C(\mathcal{X},\mathcal{X})} = \{ \Xi \in C(\mathcal{X},\mathcal{O}_\Lambda) : \Xi(x) \in \partial J_{\Lambda}(\mu)(x) \, \lambda–a.e. \},
\]

which is independent of the choice of the reference measure \( \lambda \) which satisfies \( |\mu| \ll \lambda \).

By the above arguments, we have shown the following result.

**Theorem 3.16.** The optimization problem (7) always admits a minimizer \( \mu \in \mathcal{C}\mathcal{E}([0,1];G_0,G_1) \) and a dual formulation with zero duality gap:

\[
WB_\Lambda^2(G_0,G_1) = \sup_{\Phi \in C^1(Q,S^n)} \{ \langle G_1, \Phi \rangle_{\Omega} - \langle G_0, \Phi \rangle_{\Omega} - \iota_{C(\mathcal{X},\mathcal{O}_\Lambda)}(\partial_\Phi, D^*\Phi, \Phi) \},
\]

where the supremum is attained at a field \( \Phi \in C^1(Q,S^n) \) if and only if there exists \( \mu \in \mathcal{M}(Q,\mathcal{X}) \) such that \( L\Phi \in \partial J_{\Lambda,Q}(\mu) \) and \( L^*\mu = \delta_1(t) \otimes G_1 - \delta_0(t) \otimes G_0 \), equivalently, there exists \( \mu \in \mathcal{C}\mathcal{E}([0,1];G_0,G_1) \) such that

\[
q_\lambda = G_\lambda(D^*\Phi)\Lambda_1^2, \quad R_\lambda = G_\lambda\Phi\Lambda_2^2,
\]

and

\[
G_\lambda \cdot (\partial_\Phi + \frac{1}{2}D^*\Phi)\Lambda_1^2 + \frac{1}{2}D^*\Phi\Lambda_2^2 = 0,
\]

for \( \lambda–a.e. (t,x) \in Q \). In this case, \( \mu \) is also the minimizer to the problem (7).

**Remark 3.17.** The optimality conditions in Theorem 3.16 is not necessary since the optimal \( \Phi \) to the dual problem may not have \( C^1 \)–regularity as required. It is an interesting and nontrivial problem to find the suitable space (larger than \( C^1(Q,S^n) \)) so that the relaxed dual problem can always admit a maximizer.

A simple application of Lemma 3.15 and the dual formulation (3.44) gives the sublinearity and the weak–star lower semicontinuity of \( WB_\Lambda^2(\cdot,\cdot) \).

**Corollary 3.18.** The function \( WB_\Lambda^2(\cdot,\cdot) \) on \( \mathcal{M}(\Omega,S_+^n) \times \mathcal{M}(\Omega,S_+^n) \) is sublinear: for \( \alpha > 0 \), \( G_0, G_1, \tilde{G}_0, \tilde{G}_1 \in \mathcal{M}(\Omega,S_+^n) \), there holds

\[
WB_\Lambda^2(\alpha G_0, \alpha G_1) = \alpha WB_\Lambda^2(G_0, G_1), \quad WB_\Lambda^2(G_0 + \tilde{G}_0, G_1 + \tilde{G}_1) \leq WB_\Lambda^2(G_0, G_1) + WB_\Lambda^2(\tilde{G}_0, \tilde{G}_1).
\]

Moreover, \( WB_\Lambda \) is lower semicontinuous w.r.t. the weak–star topology, i.e., for any sequences \( \{G_0^n\}_{n \in \mathbb{N}} \) and \( \{G_1^n\}_{n \in \mathbb{N}} \) in \( \mathcal{M}(\Omega,S_+^n) \) that weak–star convergence to measures \( G_0, G_1 \in \mathcal{M}(\Omega,S_+^n) \), respectively, there holds

\[
WB_\Lambda(G_0, G_1) \leq \liminf_{n \to \infty} WB_\Lambda(G_0^n, G_1^n).
\]

**Proof.** The sublinearity follows from the stability of the continuity equation under the linear combination and the sublinearity of \( J_{\Lambda,Q}(\mu) \) guaranteed by Lemma 3.15. For the weak–star lower semicontinuity, by (3.44), for any \( \Phi \in C^1(Q,S^n) \) with \( \iota_{C(\mathcal{X},\mathcal{O}_\Lambda)}(\partial_\Phi, D^*\Phi, \Phi) = 0 \), it holds that

\[
\liminf_{n \to \infty} WB_\Lambda^2(G_0^n, G_1^n) \geq \liminf_{n \to \infty} \langle G_1^n, \Phi \rangle_{\Omega} - \langle G_0^n, \Phi \rangle_{\Omega} = \langle G_1, \Phi \rangle_{\Omega} - \langle G_0, \Phi \rangle_{\Omega},
\]

where \( G_0^n \) and \( G_1^n \) weak–star converge to \( G_0 \) and \( G_1 \) respectively. Then (3.47) follows by taking the supremum of (3.48) over the admissible \( \Phi \).
On the other hand, a simple consequence of optimality conditions (3.45) and (3.46) is the following explicit characterization of the minimizer (i.e., the geodesic) between inflating measures.

**Proposition 3.19.** Let $G^* \in \mathcal{M}(\Omega, S^+_n)$ and $g$ be a linear map from $S^+_n$ to $\mathcal{M}(\Omega, S^+_n)$ defined by $g(A) = AG^*A$ for $A \in S^+_n$. Then for commuting matrices $A_0, A_1 \in S^+_n$ that also commute with $\Lambda$, we have

$$KB^2_\Lambda((g(A_0), g(A_1)) = 2[(A_1 - A_0)\Lambda^{-1}_2] \cdot g_*(\Omega) [(A_1 - A_0)\Lambda^{-1}_2],$$

(3.49)

and the geodesic between $g(A_0)$ and $g(A_1)$ is given by $G_t = g(tA_1 + (1-t)A_0)$, $0 \leq t \leq 1$.

**Proof.** Let us first assume that $A_0$ and $A_1$ are invertible. By a direct calculation, we have

$$\partial_t G_t = (A_1 - A_0)G^*(tA_1 + (1-t)A_0) + (tA_1 + (1-t)A_0)G^*(A_1 - A_0).$$

We define

$$\Phi = 2(tA_1 + (1-t)A_0)^{-1}(A_1 - A_0)\Lambda^{-2}_2$$

and $R_t = G_t \Phi \Lambda^2_t$, $0 \leq t \leq 1$.

Then it is clear that $\Phi \in C^3(Q, \mathbb{R})$ and $\mu = (G, 0, R) \in CE([0, 1]; g(A_0), g(A_1))$, since $A_0$, $A_1$ and $\Lambda$ commute pairwise. Moreover, by using the formula: $(A^{-1})'(H) = -A^{-1}HA^{-1}$ for an invertible matrix $A$ and any matrices $H$, we have

$$\partial_t \Phi = -2(tA_1 + (1-t)A_0)^{-1}(A_1 - A_0)(tA_1 + (1-t)A_0)^{-1}(A_1 - A_0)\Lambda^{-2}_2 = -\frac{1}{2} \Phi \Lambda^2 \Phi.$$ 

We have now verified the optimality conditions (3.45) and (3.46), which implies that the curve $\mu$ defined as above is a minimizer to (32) with $G_i = g(A_i)$, $i = 0, 1$. Then by definition it readily follows that

$$KB^2_\Lambda((g(A_0), g(A_1)) = \frac{1}{2} \int_0^1 \int_\Omega \Phi \cdot G_t \Phi \Lambda^2_t dt = 2(A_1 - A_0) \cdot g_*(\Omega)(A_1 - A_0)\Lambda^{-2}_2.$$ 

For the general case $A_0, A_1 \in S^+_n$, it is easy to see that $\mu = (G, 0, R)$ with $R = 2(tA_1 + (1-t)A_0)G^*(A_1 - A_0)$ as above still satisfies the continuity equation and its action $J_{A,Q}(\mu)$ is given by the right-hand side of (3.49). To finish the proof, it suffices to show that the equality in (3.49) holds. For this, we consider $A^\varepsilon_i = A_i + \varepsilon I \in S^+_n$ for $i = 0, 1$. Then by triangle inequality of $KB_\Lambda$ (see Proposition 3.28 below) and Lemma 3.7 we have $KB_\Lambda((g(A^\varepsilon_0), g(A^\varepsilon_1)) \to KB_\Lambda((g(A_0), g(A_1))$ as $\varepsilon \to 0$. It is also clear that

$$KB^2_\Lambda((g(A^\varepsilon_0), g(A^\varepsilon_1)) = 2[(A^\varepsilon_1 - A^\varepsilon_0)\Lambda^{-1}_2] \cdot g_*(\Omega) [(A^\varepsilon_1 - A^\varepsilon_0)\Lambda^{-1}_2]$$

$$\to 2[(A_1 - A_0)\Lambda^{-1}_2] \cdot g_*(\Omega) [(A_1 - A_0)\Lambda^{-1}_2], \quad \varepsilon \to 0.$$ 

Hence, the proof is complete.

We proceed to study in more depth the optimality conditions by viewing $G$ as the main variable and $(q, R)$ as the control variable. We first observe

$$WB^2_\Lambda(G_0, G_1) = \inf_{G} \inf_{q,R} \{J_{A,Q}(\mu); \mu = (G, q, R) \in CE([0, 1]; G_0, G_1)\},$$

(3.50)

by taking the infimum in (32) over $G$ and $(q, R)$ separately. Recall the representation of $J_{A,Q}(\mu)$ in (3.23), which motivates us to introduce a weighted semi-inner product:

$$\langle (u, W), (u', W') \rangle_{L^2_\Lambda(Q)} = \langle uA_1^\dagger, u'A_1^\dagger \rangle_{L^2_\Lambda(Q)} + \langle W A_2^{-1}, W' A_2^{-1} \rangle_{L^2_\Lambda(Q)},$$

and the associated weighted seminorm $\| \cdot \|_{L^2_\Lambda(Q)}$ on the space of measurable vector fields valued in $\mathbb{R}^{n\times k} \times \mathbb{R}^n$. The corresponding Hilbert space, denoted by $L^2_\Lambda(Q, \mathbb{R}^{n\times k} \times \mathbb{M}^n)$, is defined as the quotient space by the subspace of vector fields of seminorm $\| \cdot \|_{L^2_\Lambda(Q)}$ zero, and by consequence we can write $2J_{A,Q}(\mu) = \| (G^t q, G^t R) \|^2_{L^2_\Lambda(Q)}$ if $J_{A,Q}(\mu)$ is finite. Moreover, we define the set

$$\mathcal{AC}([0, 1]) := \{ G \in \mathcal{M}(Q, S^+_n); \exists (q, R) \in \mathcal{M}(Q, \mathbb{R}^{n\times k} \times \mathbb{M}^n) \text{ s.t. } (G, q, R) \in CE([0, 1]; G_0, G_1)\},$$

(3.51)

and the energy functional associated with $\mathcal{AC}([0, 1])$: for $G \in \mathcal{AC}([0, 1]),$

$$J^A_{G_0, G_1}(G) := \inf_{(q, R)} \left\{ \frac{1}{2} \| (G^t q, G^t R) \|^2_{L^2_\Lambda(Q)}; (G, q, R) \in CE([0, 1]; G_0, G_1) \right\}.$$ 

(3.52)
Here we choose the notation $\mathcal{AC}([0,1])$ deliberately to suggest the close relation between the set $\mathcal{AC}([0,1])$ and the set of absolutely continuous curves in the metric space $(\mathcal{M}(\Omega, S_n^2), WB_\Lambda)$, which will be more clear in the next section. With the help of these notions, (3.30) can be reformulated in the compact form:

$$WB_\Lambda^2(G_0, G_1) = \inf_{\mu \in \mathcal{AC}([0,1])} J^\Lambda_{G_0, G_1} (\mu).$$

(3.53)

To further understand the structure of the functional $J^\Lambda_{G_0, G_1} (\mu)$, by Lemma 3.3, we note that for $\mu = (G, q, R) \in C^\infty([0,1]; G_0, G_1)$, the weak formulation (3.13) can be equivalently represented as

$$\langle D^* \Phi \Lambda_1, G^1 q_1 \Lambda_1 \rangle_{L^2_\Lambda (Q)} + \langle \Phi \Lambda_2, G^2 R \Lambda_2^{-1} \rangle_{L^2_\Lambda (Q)} = l_G (\Phi) \quad \forall \Phi \in C^1(Q, S^n),$$

(3.54)

where $l_G, G \in \mathcal{AC}([0,1])$, is a linear functional defined on $C^1(Q, S^n)$ by

$$l_G (\Phi) = \langle G_1, \Phi \rangle_{\Omega} - \langle G_0, \Phi \rangle_{\Omega} - \langle G, \partial_t \Phi \rangle_{Q}.$$  

(3.55)

In view of (3.54), $l_G$ is uniquely extended to a bounded linear functional on the space

$$H_{G, \Lambda}(D^*) := \{ \langle D^* \Phi \Lambda_1^2, \Phi \Lambda_2^2 \rangle; \Phi \in C^1(Q, S^n) \} \| \|_{L^2_\Lambda (Q)},$$

(3.56)

with the norm estimate

$$\| l_G \|_{H^*_{G, \Lambda} (D^*)} \leq \| (G^1 q, G^1 R) \|_{L^2_\Lambda (Q)}.$$  

(3.57)

It is worth emphasizing that such kind of extension is independent of the choice of $(q, R)$.

Next, we show that (3.52) admits a unique minimizer $(q, R)$ which also satisfies the equality in (3.57). For this, we note that every Borel vector fields $(u, W) \in L^2_{G, \Lambda}(Q; \mathbb{R}^{n \times k} \times M^n)$ such that $\langle (D^* \Phi \Lambda_1^2, \Phi \Lambda_2^2), (u, W) \rangle_{L^2_\Lambda (Q)} = l_G (\Phi)$ holds for all $\Phi \in C^1(Q, S^n)$ induces a measure $(q, R) := (G u, \partial_t W, GW)$, which is well defined for the equivalent classes in $L^2_{G, \Lambda}(Q; \mathbb{R}^{n \times k} \times M^n)$, such that $(G, q, R) \in C^\infty([0,1]; G_0, G_1)$, where $\mathcal{P}_{\Lambda_1} = \Lambda_1^* \Lambda_1$ is the orthogonal projection matrix on $\text{ran}(\Lambda_1)$. This observation leads us to the fact that $J^\Lambda_{G_0, G_1} (\mu)$ is actually a minimum norm problem in a Hilbert space with the constraint set being an affine space:

$$J^\Lambda_{G_0, G_1} (\mu) = \inf \left\{ \frac{1}{2} \| (u, W) \|_{L^2_\Lambda (Q)}^2; \langle (D^* \Phi \Lambda_1^2, \Phi \Lambda_2^2), (u, W) \rangle_{L^2_\Lambda (Q)} = l_G (\Phi) \quad \forall \Phi \in C^1(Q, S^n) \right\}$$

$$(u, W) \in L^2_{G, \Lambda}(Q; \mathbb{R}^{n \times k} \times M^n).$$

(3.58)

Such kind of problem is known to be uniquely solvable. The unique minimizer $(u^*_\Lambda, W^*_\Lambda)$ is given by the orthogonal projection of 0 on the constraint set, which is also the Riesz representation of the functional $l_G$ on the space $H^*_{G, \Lambda}(D^*)$. It then follows that $(q^*_\Lambda, R^*_\Lambda) = (G u^*_\Lambda, \partial_t W^*_\Lambda, GW^*_\Lambda)$ is the desired minimizer to (3.52) and there holds

$$\| l_G \|_{H^*_{G, \Lambda} (D^*)} = \| (u^*_\Lambda, W^*_\Lambda) \|_{L^2_\Lambda (Q)} = \| (G^1 q^*_\Lambda, G^1 R^*_\Lambda) \|_{L^2_\Lambda (Q)}.$$  

(3.59)

Based on the above facts, we can easily show the following useful theorem.

**Theorem 3.20.** $WB_\Lambda^2(G_0, G_1)$ has the following representation:

$$WB_\Lambda^2(G_0, G_1) = \inf_{\mu \in \mathcal{AC}([0,1])} J^\Lambda_{G_0, G_1} (\mu),$$

(3.60)

where $J^\Lambda_{G_0, G_1} (\mu)$ is given by

$$J^\Lambda_{G_0, G_1} (\mu) = \frac{1}{2} \| l_G \|_{H^*_{G, \Lambda} (D^*)}^2 = \frac{1}{2} \| (u^*_\Lambda, W^*_\Lambda) \|_{L^2_\Lambda (Q)}^2.$$  

(3.61)

Here $\mathcal{AC}([0,1])$ and $l_G$ with $G \in \mathcal{AC}([0,1])$ are the set and the bounded linear functional defined in (3.30) and (3.34) respectively; and $(u^*_\Lambda, W^*_\Lambda)$ is the Riesz representation of $l_G$ in $H^*_{G, \Lambda}(D^*)$ which uniquely solves the minimum norm problem (3.58). Moreover, $J^\Lambda_{G_0, G_1} (\mu)$ admits the following dual formulation:

$$J^\Lambda_{G_0, G_1} (\mu) = \sup \left\{ l_G (\Phi) - \frac{1}{2} \| (D^* \Phi \Lambda_1^2, \Phi \Lambda_2^2) \|_{L^2_\Lambda (Q)}^2; \Phi \in C^1(Q, S^n) \right\}.$$  

(3.62)

**Proof.** It suffices to derive the dual formulation of $J^\Lambda_{G_0, G_1} (\mu)$ (3.62). For this, we first note

$$\frac{1}{2} \| (u, W) \|_{L^2_\Lambda (Q)}^2 = \sup_{(u', W') \in L^2_\Lambda (Q; \mathbb{R}^{n \times k} \times M^n)} \langle (u, W), (u', W') \rangle_{L^2_\Lambda (Q)} - \frac{1}{2} \| (u', W') \|_{L^2_\Lambda (Q)}^2,$$
which further implies, by \((u^*_λ, W^*_λ) ∈ H_{G,λ}(D^*)\),

\[
J^λ_{G_0,G_1}(G) = \frac{1}{2} \|(u^*_λ, W^*_λ)\|^2_{L^2_λ(Q)} \geq (D^*Φ_λ^2, u^*_λ)_{L^2_λ(Q)} + (Φ_λ^2, W^*_λ)_{L^2_λ(Q)} - \frac{1}{2} \|(D^*Φ_λ^2, Φ_λ^2)\|^2_{L^2_λ(Q)}
\]

\[
= l_G(Φ) - \frac{1}{2} \|(D^*Φ_λ, Φ_λ)\|^2_{L^2_λ(Q)} \quad ∀Φ ∈ C^1(Q, S^n) .
\]

Then, by \(\text{[3.59]}\), choosing a sequence of fields \((D^*Φ_n, Φ_n)\) with \(Φ_n ∈ C^1(Q, S^n)\) to approximate \((u^*_λ, W^*_λ)\) gives \(\text{[3.62]}\) as desired.

Note that the definition of \(l_G\) (cf. \(\text{[3.11]}\)) is independent of the weighted matrices \(Λ_1, Λ_2\) while its Riesz representation depends on the space \(H_{G,λ}(D^*)\), and hence also \(λ = (Λ_1, Λ_2)\). It is natural to regard \(WB_λ\) as a family of distances indexed by \(λ\) and investigate the behaviors of \(WB_λ\) and the minimizer \(μ_λ\) to \([P]\) when \(λ\) varies, in particular, when \(|Λ_1|\) or \(|Λ_2|\) tends to zero or infinity. We give a partial answer to this question as follows. For ease of representation, we introduce

\[
J^λ_1(μ) = J_{λ,Q}((G, q, 0)), \quad J^λ_2(μ) = J_{λ,Q}((G, 0, R)) \quad \text{for } μ ∈ M(Q, X) .
\]

**Proposition 3.21.** Given \(G_0, G_1 ∈ M(Ω, S^n)\), let \(μ_λ\) be the family of minimizers to \(KB^2_{G_0, G_1}(G_0, G_1)\). Then there exists a sequence \(Λ_{1,j} ∈ S^n, j ∈ \mathbb{N}\), such that when \(j → ∞, |Λ_{1,j}|\) tends to zero, and \(μ_{Λ_j} = (Λ_{1,j}, Λ_2)\) weak-star converges to a minimizer \(μ^*\) to \(KB^2_{G_0, G_1}(G_0, G_1)\), and \(KB^2_{G_{1,j}}(G_0, G_1) → KB^2_{G_{0,λ}}(G_0, G_1)\).

**Proof.** We first show that \(|Λ_1|^2 J^λ_1(μ_λ)\) and \(|Λ_2|^2 J^λ_2(μ_λ)\) are bounded when \(|Λ_1| → 0\), which, by the a priori estimates \(\text{[3.24]}\) and \(\text{[3.27]}\), shall imply that \(μ_λ\) is bounded in \(M(Q, X)\). For this, we consider the set

\[
CE_{Λ_1, q} := \arg\min\{J^λ_1(μ) : μ ∈ CE([0, 1]; G_0, G_1)\} .
\]

By the proof of Lemma \(\text{[3.7]}\), it is easy to see that \(CE_{Λ_1, q}\) is nonempty and contains at least one element with \(q = 0\); and min\{\(J^λ_1(μ) : μ ∈ CE([0, 1]; G_0, G_1)\}\} = 0. This observation, along with the fact that \(μ_λ\) is the minimizer to \([P]\), gives

\[
J_{λ,Q}(μ_λ) = J^λ_1(μ_λ) + J^λ_2(μ_λ) ≤ J_{λ,Q}(μ) = J^λ_1(μ) \quad ∀μ = (G, 0, R) ∈ CE_{Λ_1, q} ,
\]

which readily yields that \(J^λ_2(μ_λ)\) is bounded by a constant independent of \(Λ_1\). Moreover, multiplying \(|Λ_1|^2\) on both sides of \(\text{[3.65]}\) and then letting \(|Λ_1|\) tend to zero, we obtain

\[
\lim_{|Λ_1| → 0} |Λ_1|^2 J^λ_1(μ_λ) = 0 .
\]

Then the boundedness of \(|Λ_1|^2 J^λ_1(μ_λ)\) for small enough \(|Λ_1|\) follows. We complete the proof of the claim.

By the boundedness of \(|μ_λ|\|\) as \(|Λ_1| → 0\), we are allowed to take a sequence \(Λ_{1,j}\) such that \(|Λ_{1,j}|\) tends to zero and \(μ_{Λ_j} = (Λ_{1,j}, Λ_2)\) weak-star converges to a measure \(μ^* ∈ M(Q, X)\) when \(n → ∞\). It is clear that \(μ^* ∈ CE([0, 1]; G_0, G_1)\) by the weak formulation \(\text{[3.13]}\). The weak-star lower semicontinuity of \(J^λ_2\) and \(\text{[3.65]}\) gives us

\[
J^λ_2(μ^*) ≤ \liminf_{j \to ∞} J^λ_2(μ_{Λ_j}) ≤ \limsup_{j \to ∞} KB^2_{Λ_j}(G_0, G_1) ≤ \inf\{J^λ_2(μ) : μ = (G, 0, R) ∈ CE_{Λ_1, q}\} .
\]

The right-hand side of \(\text{[3.67]}\) can be recognized as the weighted Bures metric \(KB_{G_0, G_1}(G_0, G_1)\) (cf. Remark \(\text{[3.9]}\), and by consequence the infimum is attained, by noting \(\{(G, 0, R) ∈ CE_{Λ_1, q}\} = \{(G, 0, R) ∈ CE([0, 1]; G_0, G_1)\}\). Moreover, \(\text{[3.66]}\) and \(\text{[3.24]}\) guarantee that the limit measure \(μ^* ∈ CE([0, 1]; G_0, G_1)\) is of the form \((G^*, 0, R^*)\). The proof is completed by \(\text{[3.67]}\).

The above proposition tells us that if we give too large weight on the transportation part, the measure \(q\) is forced to be nearly zero. We may have similar effect if we consider the problem “in large scale”. To make this statement more precise, let us consider \(μ ∈ CE_{∞}(0, 1]; G_0, G_1)\) with a special \(D^*\) being homogeneous of degree one for instance. By Lemma \(\text{[3.12]}\) with \(T(x) = ax\) that maps \(Ω\) to \(aΩ\), we have \(T_D = aI\) and \(μ̃ := \int_0^1 δ_t ⊗ T_#(G_t, q, R_t)dt ∈ CE_{∞}(0, 1]; T_#G_0, T_#G_1)\). The a direct computation leads to

\[
J_{Λ,[0,1]} × aΩ(μ̃) = \int_0^1 J_{(a^{-1}Λ_1, Λ_2), aΩ}(T_#(G_t, q, R_t))dt = \int_0^1 J_{(a^{-1}Λ_1, Λ_2), Ω}(μ)dt = J_{(a^{-1}Λ_1, Λ_2), Ω}(μ) ,
\]

which connects the change of the weighted matrix \(Λ_1\) and the space scaling. It may be possible and interesting to consider other limiting models, e.g., \(|Λ_1| → ∞, |Λ_2| → 0\), or even only let part of eigenvalues of \(Λ_1\) vanish. However, in view of our general framework, some necessary assumptions may be needed to guarantee the well posedness of these limiting models, and we choose not to further pursue this direction here.
3.3 Properties of the weighted Wasserstein-Bures metric

In this section we shall study the space $\mathcal{M}(\Omega, S^+_1)$ equipped with the distance function $\text{WB}_\Lambda(\cdot, \cdot)$ from the metric point of view. For this, let us first show that $\text{WB}_\Lambda(\cdot, \cdot)$ is indeed a metric on $\mathcal{M}(\Omega, S^+_1)$, which is a simple corollary of the following alternative characterization of $\text{WB}_\Lambda(\cdot, \cdot)$.

**Proposition 3.22.** Let $G_0, G_1 \in \mathcal{M}(\Omega, S^+_1)$ be two $S^+_1$-valued Radon measures. For any $b > a > 0$, we have the following equivalent definition of $\text{WB}_\Lambda$:

$$\text{WB}_\Lambda(G_0, G_1) = \inf_{\mu \in \mathcal{E}([a,b]; G_0, G_1)} \int_a^b J_{\Lambda, \Omega}(\mu)^{1/2} dt. \quad (3.68)$$

Moreover, every minimizer $\mu$ to the problem (P) has “constant speed”, namely it satisfies

$$(b - a) J_{\Lambda, \Omega}(\mu)^{1/2} = \text{WB}_\Lambda(G_0, G_1) \quad \text{for a.e. } t \in [a, b]. \quad (3.69)$$

The above proposition is an analogue to the fundamental geometric fact that minimizing the energy of a parametric curve is the same as minimizing its length with an additional constraint that the speed is constant in time. The proof of Proposition 3.22 is established via the standard reparameterization technique [2]. We provide it in Appendix [B] for the sake of completeness. Based on Proposition 3.22 and the results established in the previous sections, we have the following fundamental result that summarizes some basic properties of the distance function $\text{WB}_\Lambda(\cdot, \cdot)$.

**Proposition 3.23.** $(\mathcal{M}(\Omega, S^+_1), \text{WB}_\Lambda)$ is a complete metric space. Moreover, the topology induced by the metric $\text{WB}_\Lambda$ is stronger than the weak–star topology, i.e., $\lim_{n \to \infty} \text{WB}_\Lambda(G^n, G) = 0$ implies the weak–star convergence of $G^n$ to $G$.

For this, we need the following lemma which is a direct consequence of Lemma 3.7 and the a priori estimates (3.24) and (3.25).

**Lemma 3.24.** A subset of $\mathcal{M}(\Omega, S^+_1)$ is bounded w.r.t. the metric $\text{KB}_\Lambda$ if and only if it is bounded w.r.t. the total variation norm. Hence a bounded set in $(\mathcal{M}(\Omega, S^+_1), \text{KB}_\Lambda)$ is weak–star relatively compact.

**Proof of Proposition 3.23.** It is clear that $\text{WB}_\Lambda$ is a function from $\mathcal{M}(\Omega, S^+_1) \times \mathcal{M}(\Omega, S^+_1)$ to $[0, +\infty)$ by definition and Lemma 3.7. It is also easy to check the symmetry $\text{WB}_\Lambda(G_0, G_1) = \text{WB}_\Lambda(G_1, G_0)$ by Lemma 3.7 and the triangle inequality by (3.68). To show $\text{WB}_\Lambda$ is a metric, it suffices to prove that $\text{WB}_\Lambda(G_0, G_1) = 0$ implies $G_0 = G_1$. For this, suppose that $\mu = (G, q, R)$ is the associated minimizer to $\text{WB}_\Lambda(G_0, G_1)$ with $J_{\Lambda, \Omega}(\mu) = 0$. By definition of $J_{\Lambda, \Omega}(\mu)$, we have $(q, R) = 0$. Then, taking test functions $\Phi(t, x) = \Psi(x)$ with $\Psi(x) \in C^1(\Omega, S^+_n)$ in (3.13), we obtain $(G_1 - G_0, \Psi) = 0$ for any $\Psi \in C^1(\Omega, S^+_n)$, which further implies $G_0 = G_1$.

Next, we show that the metric space $(\mathcal{M}(\Omega, S^+_1), \text{WB}_\Lambda)$ is complete. Let $\{G^n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $(\mathcal{M}(\Omega, S^+_1), \text{WB}_\Lambda)$, and hence also bounded. By Lemma 3.24, we have $G^n$, up to a subsequence, weak–star converges to a measure $G \in \mathcal{M}(\Omega, S^+_1)$. Then, by Corollary 3.18 and the fact that $G^n$ is a Cauchy sequence, for $\varepsilon > 0$, there holds

$$\liminf_{n \to \infty} \text{WB}_\Lambda(G^n, G^m) \geq \text{WB}_\Lambda(G, G^m),$$

for large enough $m$, which immediately gives us $\text{WB}_\Lambda(G, G^m) \to 0$ as $m \to \infty$. To finish the proof, we proceed to show that $G^n$ weak–star converges to $G$ if $G^n$ converges to $G$ in $(\mathcal{M}(\Omega, S^+_1), \text{KB}_\Lambda)$. By a similar argument as above, we can have that every subsequence of $G^n$ has a weak–star convergent (sub-)subsequence to $G$, which readily guarantees the weak–star convergence of the original sequence $G^n$ to $G$.

The main aim of this section is to show that $(\mathcal{M}(\Omega, S^+_1), \text{WB}_\Lambda)$ is a geodesic space and then equip it with a “differential structure”, which is consistent with the metric structure. For the reader’s convenience, we recall some related important concepts here, which only depend on the metric structure of the space. Let $(X, d)$ be a metric space and $\{\omega_t\}_{t \in [a, b]}$, $b > a > 0$, is a curve in $(X, d)$ (i.e., a continuous map from $[0, 1]$ to $X$). We say that a curve $\{\omega_t\}_{t \in [a, b]}$ is absolutely continuous if there exists a function $g \in L^1([a, b])$ such that $d(\omega_s, \omega_t) \leq \int_s^t g(r) dr$ for any $a \leq s \leq t \leq b$. Moreover, the curve is said to have finite $p$-energy if $g \in L^p([a, b])$. The metric derivative $|\omega_t'|$ of $\{\omega_t\}_{t \in [a, b]}$ at the time point $t$ is defined by $|\omega_t'| := \lim_{\delta \to 0} |\delta|^{-1} d(\omega_{t+\delta}, \omega_t)$, if the limit exists. It can be shown [2, Theorem 1.1.2] that for an absolutely continuous curve $\omega_t$, the metric derivative $|\omega_t'|$ is well defined and satisfies $|\omega_t'| \leq g(t)$ for a.e. $t \in [a, b]$.

The length $L(\omega_t)$ of an absolutely continuous curve $\{\omega_t\}_{t \in [a, b]}$ can be defined as $L(\omega_t) = \int_a^b |\omega_t'| dt$, which is clearly invariant w.r.t. the reparameterization. We recall that $(X, d)$ is a geodesic space if for any $x, y \in X$, there holds

$$d(x, y) = \min \{L(\omega_t) : \{\omega_t\}_{t \in [0, 1]} \text{ is absolutely continuous with } \omega(0) = x, \omega(1) = y\}.$$
By taking a by using Lebesgue differentiation theorem in (3.74) The uniqueness of (equation w.r.t. the variables (q,Gk)
Proposition 3.14. On the other hand, since
\[ \lVert \omega \rVert = (G, q^* R^*) \] a.e.. Hence, we can always assume that the geodesic is constant-speed (i.e. \( \omega' \) is constant a.e.). Then by definition it is clear that a curve \( \{ \omega_t \}_{t \in [0,1]} \) is a constant-speed geodesic if and only if it satisfies \( d(\omega_s, \omega_t) = |t-s| d(\omega_0, \omega_1) \) for any \( 0 < s < t < 1 \).

With the help of these notions, we see that to achieve our main aims, the key step is to characterize the absolutely continuous curves \( \{ G_t \}_{t \in [a, b]} \) by Proposition 3.14. On the other hand, since \( G_t \) has finite 2-energy, then the unique field \( (q^*, R^*) \) given by \( (G^*, W^*_\Lambda) \), where \( (u^*_\Lambda, W^*_\Lambda) \) is the same as in Theorem 3.25.

**Proof.** For notational simplicity, we limit ourselves to the case \([a, b] = [0,1]\). We first consider if part, which is quite direct. For \( \mu \in CE([0,1]; G_0, G_1) \) with the property (3.70), it follows from (3.65) that

\[ WB(G_{t^*}, G_t) \leq \int_a^t \int_a^b |G'_t|^2 dt \quad a.e. t \in [0,1]. \]

which, by definition, immediately implies that \( \{ G_t \}_{t \in [0,1]} \) is absolutely continuous and (3.71) holds. We now consider the only if part. Let \( \{ G_t \}_{t \in [0,1]} \) be an absolutely continuous curve, which we further assume to be Lipschitz with the Lipschitz constant \( \text{Lip}(G_t) \) by reparameterization. We shall approximate it by piecewise constant speed curves. More precisely, we fix an integer \( N \in \mathbb{N} \) with the step size \( \tau = 2^{-N} \). Let \( \{ \mu_{t,k}^{N} \}_{t \in [k-1, k]} \) be a minimizer to (3.77) with \( a = (k-1) \tau \) and \( b = k \tau \), which satisfies

\[ \tau^{1/2} \int_a^b |G'_{t,k}^{N}|^2 dt \leq WB(G_{(k-1)\tau}, G_{k\tau}) \leq \left( \int_a^b |G'_t|^2 dt \right)^{1/2} \quad a.e. t \in [(k-1)\tau, k\tau]. \]

where we have used Proposition 3.22 and the absolutely continuity of \( G_t \). We glue the curves \( \{ \mu_{t,k}^{N} \}_{t \in [(k-1)\tau, k\tau]}, k = 1, \ldots, 2^N \), together and obtain a new curve \( \{ \mu_{t}^{N}, G_i^{N} \}_{t \in [0,1]} \in CE([0,1], G_0, G_1) \).

To proceed, we note that for any \((a, b) \subset [0, T] \), there exists \( k_1, k_2 \in \mathbb{N} \) with \( N \) large enough such that \( (k_1 \tau, k_2 \tau - \tau) \subset (a, b) \subset [k_1 \tau, k_2 \tau] \), and it holds that, by squaring (3.72) and summing it from \( k = k_1 \) to \( k = k_2 \),

\[ \int_a^b \int_a^b |J_{\Lambda, \Omega}(\mu_t)|^2 dt \leq \sum_{k=k_1+1}^{k_2} \int_a^b |G'_t|^2 dt + 2\tau \text{Lip}(G_t)^2. \]

By taking \( a = 0, b = 1 \) in (3.73), we observe that the curves \( \{ \mu_{t}^{N} \}_{t \in [0,1]}, N \in \mathbb{N} \), have actions \( \int_a^b |J_{\Lambda, \Omega}(\mu_t)|^2 dt \) that are bounded by \( \int_a^b \sum_{k=k_1+1}^{k_2} |G'_t|^2 dt \). By Proposition 3.14 up to a subsequence, \( \{ \mu_{t}^{N} \}_{t \in [0,1]} \) weak-star converges to a \( \tilde{\mu} = (\tilde{G}, \tilde{q}, \tilde{R}) \in CE([0,1], G_0, G_1) \), and it follows from (3.68) and (3.72) that

\[ \int_a^b |J_{\Lambda, \Omega}(\tilde{\mu}_t)|^2 dt \leq \liminf_{N \to +\infty} \int_a^b |J_{\Lambda, \Omega}(\mu_t)|^2 dt \leq \int_a^b |G'_t|^2 dt. \]

We now show that \( \tilde{G}_t \) is nothing else than \( G_t \). For this, note that for \( t \in [0,1] \), there exists a sequence of integers \( \{ n \}_{N \in \mathbb{N}} \) such that \( s_N = k_N 2^{-N} \) tends to \( t \) as \( N \to \infty \), which implies \( G_{s_N} \) weak-star converges to \( G_t \) by Proposition 3.14. On the other hand, since \( \{ G_t \}_{t \in [0,1]} \) is absolutely continuous w.r.t. the metric \( WB_\Lambda \), we also have \( G_{s_N} \) weak-star converges to \( G_t \) by Proposition 3.22. Hence, we have \( \tilde{G}_t = G_t, 0 \leq t \leq 1. \) Then it follows that \( (\tilde{G}, \tilde{q}, \tilde{R}) \in CE([0,1], G_0, G_1) \) and

\[ J_{\Lambda, \Omega}(\tilde{\mu}_t) = J_{\Lambda, \Omega}((G_t, \tilde{q}_t, \tilde{R}_t)) = |G'_t|^2, \]

by using Lebesgue differentiation theorem in (3.74) The uniqueness of \( (\tilde{q}, \tilde{R}) \) follows from the linearity of the continuity equation w.r.t. the variables \( (q, R) \) and the strict convexity of the \( L^2_\Omega \) norm.
To complete the proof, it suffices to show that \( \mu = (G, Gu^\ast, GW^\ast) \in CE_\infty([0,1]; G_0, G_1) \) satisfies \( J_{\Lambda, \Omega}(\mu_t)^{1/2} \leq |\mathcal{G}_t| \) for a.e. \( t \in [0,1] \), where \( G_t \) is absolutely continuous with finite 2-energy and \( (u^\ast, W^\ast) \) is the Riesz representation of \( \mathcal{I}_G \) in \( H_{G, \Lambda}(D^*) \) (cf. Theorem 3.22). Let \( (a,b) \) be an open interval in \([0,1]\), \( \eta \in C_c^\infty((a,b)) \) with \( 0 \leq \eta \leq 1 \), and \((D^*\Phi_n, \Phi_n)\) with \( \Phi_n \in C^1(Q, S^n) \) be a sequence approximating \((u^\ast, W^\ast)\) in \( H_{G, \Lambda}(D^*) \). Then by using (3.74), we have

\[
\|(\eta u^\ast, \eta W^\ast)\|^2_{L^2_{\mathcal{I}, \Lambda}(Q)} = \lim_{n \to +\infty} \|(\eta u^\ast, \eta W^\ast, (D^*\Phi_n, \Phi_n))\|_{L^2_{\mathcal{I}, \Lambda}(Q)} = \lim_{n \to +\infty} \mathcal{I}_G(\eta \Phi_n). \tag{3.75}
\]

Again by (3.74) with \((G, q, \tilde{R})\) constructed as above, we have

\[
|\mathcal{I}_G(\eta \Phi_n)| \leq \|(G^I\tilde{q}, G^I\tilde{R})\|^2_{L^2_{\mathcal{I}, \Lambda}((a,b) \times \Omega)} \|(D^*\eta \Phi_n, \eta \Phi_n)\|^2_{L^2_{\mathcal{I}, \Lambda}((a,b) \times \Omega)} \leq \left( \int_a^b |G_t|^2 \mathrm{d}s \right)^{1/2} \|(D^*\Phi_n, \Phi_n)\|^2_{L^2_{\mathcal{I}, \Lambda}((a,b) \times \Omega)}. \tag{3.76}
\]

Combining (3.75) with (3.76) and letting \( \eta \) approximate \( \chi_{[a,b]} \), we obtain

\[
\|(u^\ast, W^\ast)\|_{G, \Lambda,(a,b) \times \Omega} \leq \left( \int_a^b |G_t|^2 \mathrm{d}s \right)^{1/2}.
\]

Then since \((a,b)\) is arbitrary, Lebesgue differentiation theorem gives that \( \mu = (G, Gu^\ast, GW^\ast) \) satisfies \( J_{\Lambda, \Omega}(\mu_t)^{1/2} \leq |\mathcal{G}_t| \) as desired. The proof is complete.

As a consequence of Proposition 3.22 and Theorem 3.26, we have

\[
K_{\Lambda}(G_0, G_1) = \inf_{G} \inf_{(q, R)} \left\{ \int_0^1 J_{\Lambda, \Omega}(\mu_t)^{1/2} \mathrm{d}t \mid \mu = (G, q, R) \in CE([0,1]; G_0, G_1) \right\} = \inf \left\{ \int_0^1 |\mathcal{G}_t| \mathrm{d}t \mid \{G_t\}_{t \in [0,1]} \text{ is absolutely continuous with } |\mathcal{G}_t|_{t=0} = G_0, |\mathcal{G}_t|_{t=1} = G_1 \right\}. \tag{3.77}
\]

Moreover, note that the property of being a minimizer to (3.2) is stable w.r.t. the restriction of the time interval: if \( \{\mu_t\}_{t \in [0,1]} \in CE_\infty([0,1]; G_0, G_1) \) minimizes the energy cost \( \int_0^1 J_{\Lambda, \Omega}(\mu_t)^{1/2} \mathrm{d}t \), then for any \( 0 \leq a < b \leq 1 \), \( \{\mu_t\}_{t \in [a,b]} \) is a minimizer to (3.2) with \( G_{\mathcal{G}} = G_{\mathcal{G}}|_{t=a} \) and \( G_{\mathcal{G}} = G_{\mathcal{G}}|_{t=b} \). Then by the constant speed property (3.69) in Proposition 3.22 we see that the curve \( \{G_t\}_{t \in [0,1]} \) associated with the minimizer \( \{\mu_t\}_{t \in [0,1]} \) is the desired constant speed geodesic:

\[
W_{\lambda}(G_0, G_1) = |t-s|W_{\lambda}(G_0, G_1) \quad \forall 0 \leq s \leq t \leq 1.
\]

It helps us to conclude that the infimum in (3.77) is attained, and the following important result follows.

**Corollary 3.26.** \((M(\Omega, S^n)), W_{\lambda})\) is a geodesic space. The constant speed geodesic between \( G_0, G_1 \in M(\Omega, S^n) \) is given by the minimizer to (3.2).

Another important application of Theorem 3.25 is that we can view the set of \( S^n \)-valued measures as a pseudo-Riemannian manifold in the spirit of (3.2). We have seen in Theorem 3.25 that among all the vector measures \((q, R)\) that produce the same absolutely continuous flow \( G_t \), there is an unique optimal one \((q^o, R^o)\) with \( \mu^o = (G, q^o, R^o) \) such that \( |\mathcal{G}_t| = J_{\Lambda, \Omega}(\mu^o) \) a.e. \( t \in [0,1] \). It is natural to regard this optimal vector field \((q^o, R^o)\) as the tangent vector to the curve \( G_t \). In particular, if \( G_t \) is absolutely continuous with finite 2-energy (i.e., an element in \( AC([0,1]) \); see (3.61)), recalling the definition of \( H_{G, \Lambda}(D^*) \) in (3.70), we have

\[
(q^o, R^o) = (G_t u^\ast_{\Lambda, t}, G_t W^\ast_{\Lambda, t}) \in \text{Tang}_{G_0}(K_{\Lambda}) := \text{Tang}_{G_0}(H_{G, \Lambda}(D^*), G_0, G_1, G_1 \Phi_n \in C^1(\Omega, S^n))^{L^2_{\mathcal{I}, \Lambda}(\Omega, R^{n \times k} \times M^n)}. \tag{3.79}
\]

Moreover, as we have recalled above, by reparametrizing in time, we can assume that an absolutely continuous curve is Lipschitz and hence also has finite 2-energy. These observations motivates us to define the tangent plane of an absolutely continuous curve by the space \( \text{Tang}_{G_0}(K_{\Lambda}) \), introduced in (3.79), and further to rephrase as the following corollary, which provides Riemannian interpretation of the total distance function on \( M_\lambda \).

**Corollary 3.27.** Let \( \{G_t\}_{t \in [0,1]} \) be an absolutely continuous curve in \((M(\Omega, S^n)), K_{\Lambda}) \) and \((q_t, R_t)\) be the Borel family of measures in \( M(\Omega, R^{n \times k} \times M^n) \) such that \( (G, q, R) \in CE([0,1]; G_0, G_1) \) and \( J_{\Lambda, \Omega}(G_t, q_t, R_t) \) is finite. Then \( |\mathcal{G}_t| = (1/\sqrt{2})\|(G_t q_t, G_t R_t)\|^2_{L^2_{\mathcal{I}, \Lambda}(\Omega)} \) holds for a.e. \( t \in [0,1] \) if and only if \((q_t, R_t)\) is in \( \text{Tang}_{G_0}(K_{\Lambda}) \) a.e. Moreover, for the absolutely continuous curve \( \{G_t\}_{t \in [0,1]} \), there always exists a unique family of measures \( \{(q_t, R_t)\}_{t \in [0,1]} \) such that \( (G, q, R) \in CE([0,1]; G_0, G_1) \) and \((q_t, R_t)\) are in \( \text{Tang}_{G_0}(K_{\Lambda}) \) a.e.
We end this section with a regularization result which allows to approximate a curve $μ ∈ C\mathcal{E}_∞([0, 1]; G_0, G_1)$ by smooth ones with controlled actions and end points. It will be useful in Section 4. For this, we need the following set of assumptions, which are in some sense necessary, see Remark 3.29 below.

**A.1** $Ω$ is a compact domain which is star shaped w.r.t. a set of points with nonempty interior.

**A.2** $D^*$ admit a decomposition: $D^*(Φ) = (D_0^*(Φ), D_1^*(Φ))$ for $Φ ∈ C^∞(\mathbb{R}^d, S^n)$, where $D_0^*$ and $D_1^*$ are the homogeneous components of degree zero and degree one of $D^*$, which satisfy $D_0^* : L(S^n, \mathbb{R}^{n×k_1})$ and $D_1^* : C^∞(\mathbb{R}^d, S^n) → C^∞(\mathbb{R}^d, \mathbb{R}^{n×k_2})$ with $k = k_1 + k_2$.

**A.3** $G_0, G_1 ∈ \mathcal{M}(Ω, S^n)$ are absolutely continuous w.r.t. the Lebesgue measure.

**Proposition 3.28.** Let $μ = (G, q, R) ∈ C\mathcal{E}_∞([0, 1]; G_0, G_1)$ be a curve connecting given measures $G_0, G_1 ∈ \mathcal{M}(Ω, S^n)$. Suppose that the assumptions A.1 and A.2 hold. Then there exist sequences of measures: $\{μ^n = (G^n, q^n, R^n)\} n ∈ \mathbb{N}$ in $\mathcal{M}(Q, X)$, and $\{G^n_0\} n ∈ \mathbb{N}, \{G^n_1\} n ∈ \mathbb{N}$ in $\mathcal{M}(Ω, S^n)$, such that $μ^n ∈ C\mathcal{E}_∞([0, 1]; G^n_0, G^n_1)$ with the following properties:

1. $G^n_0, G^n_1$ and $μ^n = (G^n, q^n, R^n)$ admit $C^∞$-smooth densities $G^n_0, G^n_1$ and $(G^n, q^n, R^n)$, respectively, w.r.t. the Lebesgue measure. Moreover, $G^n$ is uniformly elliptic on $Q$.

2. $μ^n$ weak–star converges to $μ$, and there holds

$$\lim_{n → ∞} J_{λ, Q}(μ^n) = J_{λ, Q}(μ).$$

3. $G^n_0, G^n_1$ weak–star converge to $G_0$ and $G_1$ respectively. If the assumption A.3 holds, $G^n_0, G^n_1$ can further satisfy

$$\lim_{n → ∞} \text{WB}_λ(G_0, G^n_0) = 0, \lim_{n → ∞} \text{WB}_λ(G_1, G^n_1) = 0.$$

**Proof.** We extend the curve $\{μ_t\}_{t ∈ [0, 1]} ∈ C\mathcal{E}_∞([0, 1], G_0, G_1)$ as in (3.14):

$$μ_t := μ t(x(0, 1), (0, 0), e(0, 1, +∞), t) ∈ \mathcal{M}(Ω, X).$$

We define the $\varepsilon$–neighborhood of $Ω$ by $Ω_ε := \{x ∈ \mathbb{R}^d : \text{dist}(x, Ω) ≤ ε\}$. The proof proceeds in four steps.

**Step 1 (lift).** Let $G ∈ C^∞(Ω, X)$ be a smooth density supported in $Ω_1$ and uniformly elliptic on $Ω_{1/2}$ with constant 1, namely $G(x) ≥ 1$ for $x ∈ Ω_{1/2}$. For small $ε > 0$, we define

$$μ^δ := (G^δ, q^δ, R^δ) := ((1 − δ)G + δG^*, (1 − δ)q, (1 − δ)R) ∈ \mathcal{M}(Ω, X).$$

It is clear that $μ^δ$ satisfies the continuity equation: $∂_t G^δ + Dq^δ = (R^δ)^{sym}$ in the distributional sense and $G^δ(E) ≥ δ I|E|$ for any $E ∈ \mathcal{B}(Ω × \mathbb{R})$. Moreover, by the convexity of $J_{λ, X}$, we have

$$J_{λ, R}^d(μ^δ) ≤ (1 − δ)J_{λ, R}^d(μ) + δ J_{λ, R}^d((G^*, 0, 0)) = (1 − δ)J_{λ, Q}(μ) < +∞.$$

**Step 2 (time-space regularization).** Let $θ : \mathbb{R} → \mathbb{R}_+$ be a function which is infinitely differentiable, radial and supported in $B_1(0, 1)$ with $∫_{B_1(0, 1)} θ(t, x)dx = 1$, and $θ(t, x) := \varepsilon − θ(t, x/ε)$ be the associated family of convolution kernels. We then introduce the convolution kernels on $\mathbb{R}^{d+1}$ by $ρ_ε(t, x) := θ(t, x)$. Alleviating notations, in the sequel we denote by $μ$ the measure $μ^δ$ obtained in Step 1. We define $μ^ε := (ρ_ε * μ)|_{[−ε, ε]} × Ω$. Hence, $μ^ε$ admits a $C^∞$–smooth density $(G^ε, q^ε, R^ε)$ on $[−ε, 1] × Ω$, w.r.t. $dt ⊗ dx$ and $G^ε$ is uniformly elliptic with constant $δ$. Moreover, by linearity of the continuity equation and basic properties of convolution, it is clear that $μ^ε ∈ \mathcal{C}(\mathcal{M}(Ω × \mathbb{R}))$ satisfies $μ^ε ∈ C\mathcal{E}_∞([0, b]; G^ε|_{[a, b]}, G^ε|_{[a, b]}$) on $Ω_ε$ for any $−ε ≤ a < b ≤ 1 + ε$.

**Step 3 (compression).** By assumption A.1, for $Ω$, w.l.o.g., we let 0 be the interior of the set of points w.r.t. which $Ω$ is star shaped. By [63, Theorem 5.3], we have that the Minkowski functional $p_λ(x) := \inf\{λ > 0 : x ∈ λΩ\}$ is Lipschitz with the Lipschitz constant denoted by $L$. By definition, it follows that $p_λ(x) ≤ 1 + Lε$ for any $x ∈ Ω_ε$. Hence, we are allowed to define the time-space scaling: $T^ε(t, x) = (s^ε(t), T^ε(x))$ with

$$s^ε(t) = (1 + 2ε)^{−1}(t + ε) : [−ε, 1 + ε] → [0, 1], \quad T^ε(x) = (1 + Lε)^{−1}x : Ω_ε → Ω.$$

By assumption A.2 for $D^*$, the linear map $T^ε_0$, associated with $T^ε$ is well defined and given by $T^ε_0(p) = (p_1, (1 + Lε)^{−1}p_2)$ for $p = (p_1, p_2)$ with $p ∈ \mathbb{R}^{n×k_1}$, $p ∈ \mathbb{R}^{n×k_2}$ and $k = k_1 + k_2$. Then by Lemma 3.12 we can define

$$μ^ε := T^ε_0(G^ε(t, x), (1 + 2ε)T^ε_0(q^ε(t, x)), (1 + 2ε)T^ε_0(R^ε(t, x))), \quad s ∈ [0, 1],$$

which satisfies the continuity equation, where $T^ε$ is the inverse of $s^ε$ and $T^ε_0$ is the transpose of $T^ε_0$.  

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We now consider To do so, it suffices to consider the weak–star convergence of $\epsilon$. A similar estimate as above shows the weak–star convergence of $\alpha$. For this, by estimates (2.2) and (3.17), we have

$$
(3.85)
$$
where the constant $C$ depends on $L$ and $d$; (1) is from $\epsilon$; (2) is from the change of variables: $x = T^\epsilon(y)$ and the fact that $(\bar{G}^\epsilon, T^\epsilon_D(\bar{q}^\epsilon_D), \bar{R}^\epsilon_D)$ admits a smooth density with $\bar{G}^\epsilon$ being uniformly elliptic; (3) is from the explicit formulas of $T^\epsilon_D$ and $J^\epsilon_{\alpha};$ (4) is from the fact that $\tilde{\mu}$ is the restriction of $\mu_\phi * \mu$ on $[-\epsilon, 1 + \epsilon] \times \Omega$. It is easy to show that the action functional $J^\epsilon_{\alpha,2B^\epsilon+1}$ is monotone w.r.t. the smoothing (See [2], Lemma 8.1.10]). Then recalling $\epsilon$, by $\epsilon$, we arrive at

$$
J^\epsilon_{\alpha,2B^\epsilon+1} \leq (1 + C\epsilon)(1 - \epsilon)J^\epsilon_{\alpha,2B^\epsilon+1}.
$$

We now consider $\delta = \epsilon$ and show that $\mu^\epsilon$, constructed as above, weak–star converges to the measure $\mu$ (given in the proposition), which, by $\epsilon$ and the lower semicontinuity of $J^\epsilon_{\alpha,2B^\epsilon+1}$, readily implies

$$
\lim_{\epsilon \to 0} J^\epsilon_{\alpha,2B^\epsilon+1} = J^\epsilon_{\alpha,2B^\epsilon+1}.
$$

To do so, it suffices to consider the weak–star convergence of $G^\epsilon$, since the same analysis applies to $q^\epsilon$ and $R^\epsilon$ as well. By the above construction, a direct computation leads to

$$
(3.87)
$$
for $\Phi \in C(\Omega, \mathbb{S}^n)$, where the constant $C$ depends on $\Omega$, $\|\Phi\|_\infty$ and $\|\tilde{G}\|_\infty$. Note that for any $(t, x) \in Q$ and $(s, y) \in B_1(t, \epsilon) \times B_d(x, \epsilon)$, there hold $T^\epsilon(s, y) \in Q$ and

$$
|T^\epsilon(s, y) - (t, x)| \leq \frac{s + \epsilon - t - 2\epsilon t}{1 + 2\epsilon} + \frac{|y - x - L\epsilon x|}{1 + \epsilon} \leq C|\epsilon|.
$$

Then by uniform continuity of $\Phi$ on $Q$, we have the pointwise convergence:

$$
\lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon} \sup_{(t, x) \in (\Omega, \mathbb{S}^n)} |\Phi(T^\epsilon(s, y)) - \Phi(t, x)| = \lim_{\epsilon \to 0} \int_{(\Omega, \mathbb{S}^n)} \rho_\phi^\epsilon(t, x) |\Phi(T^\epsilon(s, y)) - \Phi(t, x)| d\mu = 0,
$$

which, along with (3.87), implies $\lim_{\epsilon \to 0} |(G^\epsilon, \Phi) - (G, \Phi)| = 0$.

For the third statement in the proposition, we only consider the initial distribution since the argument for the final distribution is the same. We first calculate $G^\epsilon|_{t=0}$ as

$$
(3.88)
$$
A similar estimate as above shows the weak–star convergence of $G^\epsilon|_{t=0}$. We next estimate $K_B^\epsilon(G^\epsilon|_{t=0}, 0)$ under the assumption $\alpha$. For this, by estimates (2.2) and (3.17), we have

$$
\begin{align*}
W_B^\epsilon(G^\epsilon|_{t=0}, 0) & \lesssim \int_{(\Omega, \mathbb{S}^n)} |(1 + L\epsilon)^d(\theta^\epsilon * G_0) \circ (T^\epsilon)^{-1} - G_0| dx + \epsilon \\
& \lesssim \int_{(\Omega, \mathbb{S}^n)} |(\theta^\epsilon * G_0)(x) - (1 + L\epsilon)^dG_0(T^\epsilon(x))| dx + \epsilon \\
& \lesssim \|\theta^\epsilon * G_0 - G_0\|_{L^1(B^\epsilon)} + \|G_0 - G_0\|_{L^1([-\epsilon, 1+\epsilon] \times \Omega)} + \epsilon. \quad (3.89)
\end{align*}
$$

Then the approximation property of conventional kernels gives $\|\theta^\epsilon * G_0 - G_0\|_{L^1(B^\epsilon)} \to 0$ as $\epsilon \to 0$, while by the density of $C_c^\infty(\mathbb{R}^d, \mathbb{S}^n)$ in $L^1(\mathbb{R}^d, \mathbb{S}^n)$, it is easy to show that $\|G_0 - (1 + L\epsilon)^dG_0 \circ T^\epsilon\|_{L^1([-\epsilon, 1+\epsilon] \times \Omega)}$ also tends to zero when $\epsilon \to 0$. Hence, (3.89) gives (3.81) as desired and completes the proof. \qed
For our later use, we will refer to the measures $G_0 = G^t|_{t=0}$ and $G_1 = G^t|_{t=1}$ constructed as in (3.88) as the $\varepsilon$-regularizations of the end points $G_0$ and $G_1$.

**Remark 3.29.** In this remark we illustrate that these assumptions are, in some sense, necessary. \[A.3\] mainly allows us to define a scaling operator to compress the supports of measures that are smoothed by the standard mollifier, so that the whole regularization procedure can preserve the domain $\Omega$, which is where our transport problem is defined. Then, in view of Lemma 3.72 as a consequence, \[A.2\] is needed to control the effect of space scaling (cf. Remark 5.13). It is clear that if we consider the whole space $\mathbb{R}^d$ or the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, these two assumptions can be removed since a single mollification step is enough to generate the desired regularizing families of measures. We next consider the assumption \[A.3\] and show that (3.81) may fail in the case $\Lambda = (0, \Lambda_2)$ if \[A.3\] does not hold. For this, let us consider $\Omega = \mathbb{R}^d$ and estimate $\text{KB}_{0,\Lambda_2}(0, \Lambda_2, G_0 \ast \Lambda_2, \Omega, G_0)$ with $G_0 = \delta_{0} I$ for simplicity. We have connected $\text{KB}_{0,\Lambda_2}$ with the matrical Hellinger distance in Remark 3.9. Moreover, it was shown in [19 Theorem 3] that the Hellinger distance, and hence $\text{KB}_{0,\Lambda_2}$ as well, is topologically equivalent to the total variation distance. These observations reduce the estimate of $\text{KB}_{0,\Lambda_2}(0, \Lambda_2, G_0 \ast \Lambda_2, \Omega, G_0)$ to the estimate of $\|\theta^*_{\varepsilon} \|_{\Omega, G_0}$, which is quite direct:

\[
\|\theta^*_{\varepsilon} \|_{\Omega, G_0} = \|\theta^*_{\varepsilon}(x) dx - \delta_0 I\| = \sup\{\|\theta^*_{\varepsilon}(x) dx - \delta_0 I\|_{\mathcal{R}^d}; \Phi \in C_c(\mathbb{R}^d, \mathbb{R}^n) \text{ with } \|\Phi\|_{\infty} \leq 1\} \\
\leq \sup\{\|\theta^*_{\varepsilon}(x) dx - \delta_0 I\|_{\mathcal{R}^d}; \Phi \in C_c(\mathbb{R}^d, \mathbb{R}) \text{ with } \|\Phi\|_{\infty} \leq 1\} = 1,
\]

where in the last step we consider a sequence of smooth functions $\phi_k$ defined by $\phi_k := \exp((|kx|^2 - 1)^{-1} + 1)$ for $|x| < k^{-1}$ and $\phi_k := 0$ for $|x| \geq |k|^{-1}. This confirms the necessity of \[A.3.\] However, considering Lemma 3.77 it is possible to avoid this assumption in some concrete models, for instance, the Wasserstein–Fisher–Rao distance.

### 3.4 Examples

In this section, we discuss some examples that fit into the framework developed in the previous sections. We will try to follow their notations for reader's convenience.

**Example 1 (Kantorovich-Bures metric).** We set the dimension parameter: $n = m = d$, $k = 1$ and the weighted matrices $\Lambda_i = I$ for $i = 1, 2$, and consider the differential operator: $D = \nabla_q$ where $\nabla_q$ is the symmetric gradient defined by $\nabla_q(q) = \frac{1}{\mu} \nabla (q + q^T)$ for a smooth vector field $q \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Then (3.22) readily gives us the convex formulation of the Kantorovich-Bures metric $d_{KB}$ on $\mathcal{M}(\Omega, \mathbb{S}^d_+)$ proposed in [11 Definition 2.1], up to a constant (here we change the definition domain): for given distributions $G_0, G_1 \in \mathcal{M}(\Omega, \mathbb{S}^d_+),$

\[
W^2_{KB}(G_0, G_1) = \frac{1}{2} d^2_{KB}(G_0, G_1) := \inf \{ \mathcal{J}_{A, Q}(\mu); \mu \in (G, q, R) \in \mathcal{M}(Q, X) \text{ satisfies } \}
\]

\[
\partial \Phi G = \{- \nabla q(t) + R(t)\}^{\text{sym}} \text{ with } G_t|_{t=0} = G_0, G_t|_{t=1} = G_1, \quad (* \mathcal{P}_{KB})
\]

where $\mathcal{J}_{A, Q}(\mu)$ with $\Lambda = (I, I)$ is given by (3.22):

\[
\mathcal{J}_{A, Q}(\mu) = \frac{1}{2} \|G^t q\|^2_{L^q_2(Q)} + \frac{1}{2} \|G^t R\|^2_{L^q_2(Q)}.
\]

**Example 2 (Wasserstein-Fisher-Rao metric).** If we set $n = m = 1$, $k = d$ and $\Lambda_1 = \sqrt{\Sigma}^{-1} I$, $\Lambda_2 = \sqrt{\Sigma}$, and consider the differential operator $D = \text{div}$, then (3.22) gives us the Wasserstein-Fisher-Rao metric: for given distributions $\rho_0, \rho_1 \in \mathcal{M}(\Omega, \mathbb{R}_+),$

\[
W^{2}_{FR}(\rho_0, \rho_1) = \inf \{ \int Q^t (\omega)^2 + \frac{1}{4} \chi^2; (\rho(t), \omega(t)) \in \mathcal{M}(Q, R \times \mathbb{R} \times \mathbb{R}) \text{ satisfies } \}
\]

\[
\partial \rho(t) + \text{div}(\omega) = \chi \text{ with } \rho(t)|_{t=0} = \rho_0, \rho(t)|_{t=1} = \rho_1. \quad (* \mathcal{P}_{WFR})
\]

**Example 3 (Interpolation distance in [18]).** Let $N$ be a positive integer and $(\mathbb{M}^n)^N$ denote the space of block-row vectors consisting of $N$ elements in $\mathbb{M}^n$. The spaces $(\mathbb{S}^n)^N$ and $(\mathbb{A}^n)^N$ are defined similarly. For $M \in (\mathbb{M}^n)^N$, we define its component transpose $M^t_k$ by $(M^t_k)_{k} = M^T_k$, $k = 1, \ldots, N$. We consider a sequence of symmetric matrices $L_k$, $k = 1, \ldots, N$, and define the linear operator $\nabla L: \mathbb{S}^n \to (\mathbb{A}^n)^N$ by $\nabla L X_k = L_k X - X L_k$. We denote by $\nabla L^t$ its dual operator w.r.t. the Frobenius inner product. We now set the dimension parameter $k = n(d + N)$ and by vectorization write $q \in \mathcal{M}(Q, \mathbb{R}^{n \times (n \times (n \times n \times (n \times N)))})$ for $\text{vec}([Q_0, Q_1])$ with $Q_0 \in \mathcal{M}(Q, \mathbb{R}^{n \times (n \times (n \times n \times (n \times N)))})$ and $Q_1 \in \mathcal{M}(Q, \mathbb{R}^{n \times (n \times (n \times n \times (n \times N)))})$. With the help of these preparations, we define

\[
Dq := \frac{1}{2} \text{div}(Q_0 + Q_1) - \frac{1}{2} \nabla^T_L (Q_1 - Q_0) .
\]

Then it is clear that (3.22) with weighted matrices $\Lambda_i = I$ for $i = 1, 2$ gives the problem in [18 (5.7a)–(5.7c)]:

\[
W_{2,FR}(G_0, G_1)^2 = \min \frac{1}{2} \|G^T Q_0\|^2_{L^q_2(Q)} + \frac{1}{2} \|G^T Q_1\|^2_{L^q_2(Q)} + \frac{1}{2} \|G^R R\|^2_{L^q_2(Q)} \quad (* \mathcal{P}_{2,FR})
\]
over $\mathcal{M}(Q, X)$, subject to
\[ \partial_t G = -\frac{1}{2} \text{div}(Q_0 + Q'_0) + \frac{1}{2} \nabla_L (Q_1 - Q'_1) + R^\text{sym} \]
with $G_t|_{t=0} = G_0$ and $G_t|_{t=1} = G_1$.

4 A convergent fully discretized scheme

This section mainly devotes to the numerical analysis of the transport problem [2]. For ease of exposition, we let the weighted matrices $(A_1, A_2)$ be $(I, I)$ and omit the corresponding subscripts. We first propose a convergence framework in the abstract level extended from the one in [11] via the perspective of the Lax equivalence theorem, which later helps us to construct a convergent scheme for [2]. We will also pay a special attention to the Wasserstein–Fisher–Rao metric, where a sharper result can be obtained.

4.1 Brief review of Lax equivalence theorem

Before introducing our convergence framework, let us first briefly review, following closely [67], how the Lax equivalence theorem works for a general linear operator equation. We consider Banach spaces $X$ and $Y$, and let $A \in \mathcal{L}(X, Y)$ be a bounded linear operator with a bounded linear inverse $A^{-1}$ so that the equation $Ax = y$ is well defined, where $y \in Y$ is given. We also assume that this linear problem admits a family of discrete approximations as follows. For each discretization parameter $h$, suppose that $X_h$ and $Y_h$ are discrete spaces (finite-dimensional vector spaces) used to approximate $X$ and $Y$, respectively; and $A_h$ is an invertible linear operator from $X_h$ to $Y_h$. To formulate the discrete problem and further consider the error analysis, we define interpolation operators $I_h^x \in \mathcal{L}(X, X_h)$, $I_h^y \in \mathcal{L}(Y, Y_h)$ and an embedding operator $P_h \in \mathcal{L}(X_h, X)$. The finite-dimensional linear problem reads: given $y \in Y$, solve $A_h x_h = I_h^y y$.

We are interested in the approximation error of $\|x - P_h x_h\|_X$ and the convergence of $P_h x_h$ to $x$. The Lax equivalence theorem tells us that to have a convergent scheme, it suffices to require it to be stable and consistent. To be precise, we need to the following concepts: consistency: $\|I_h^y A - A_h I_h^y\|_{Y_h} \to 0$ as $h \to 0$, which measures the extent to which $I_h^y A$ commutes with $A_h I_h^y$; stability: $\|A_h^{-1}\| \leq C$ for a constant $C$ independent of $h$; approximability: $\|x - P_h x_h\|_X \to 0$ as $h \to 0$, which measures the extent to which the discrete space $X_h$ approximates $X$. By definition, an one-line proof gives that for a scheme (i.e., the mathematical object $(A_h, I_h^x, I_h^y, P_h)$), if the consistency, stability and approximability hold, then the convergence: $\|x - P_h x_h\|_X \to 0$ as $h \to 0$ follows. If we are satisfactory with the convergence w.r.t. the discrete norm: $\|I_h^x x - x_h\|_{X_h} \to 0$ as $h \to 0$, then there is no need to require the approximability for the scheme, which is the common form of the Lax equivalence theorem.

It is worth pointing out that to define the discrete problem, we only need $(A_h, I_h^x)$ while $(I_h^y, P_h)$ are mainly introduced to measure the approximation error. For a given discrete problem, the design of suitable $I_h^x$ and $P_h$ to connect the discrete model and the continuous model is not unique and actually the key of error analysis. For instance, one may consider the Laplace equation and its discretization on a uniform grid, it is known that the finite element method, the finite difference method and the finite volume method may lead to the same coefficient matrix, up to a scaling factor. However, the numerical analysis of these methods are very different. By this observation, an insightful interpretation of the Lax equivalence theorem may be as follows: given the discrete system (equivalently $A_h$ and $I_h^x$), if there exists $I_h^y$ and $P_h$ satisfying the consistency and the approximability conditions, then we can have the convergence of $x_h$ to $x$. We shall see in the next section how this point of view guide us to design a convergent fully discretized scheme for [2].

4.2 Abstract framework

Approximation of Radon measures. To fully discretize the problem [2] and further consider its convergence, let us first prepare some concepts related to the approximation of a Radon measure. Let $\mathcal{X}$ be the closure of a bounded open convex set in an Euclidean space and $h$ denote a discretization parameter that belongs to a subset $\mathcal{H}$ of $\mathbb{R}_+^k$, $k \in \mathbb{N}$, with 0 being its accumulation point.

Definition 4.1. We say that $\{ (M_h, R_{M_h}) \}_{h \in \mathcal{H}}$ is a family of discrete approximations of the Radon measure space $\mathcal{M}(\mathcal{X}, \mathbb{R}^n)$ if, for any $h \in \mathcal{H},$

1. $M_h$ is a finite-dimensional vector space equipped with a norm $\|\cdot\|_{1, M_h}$ and an inner product $(\cdot, \cdot)_{M_h}$.
2. $R_{M_h} : (M_h, \|\cdot\|_{1, M_h}) \to \mathcal{M}(\mathcal{X}, \mathbb{R}^n)$, called the reconstruction operator, is a bounded linear operator with the operator norm uniformly bounded w.r.t. $h$, namely, there exists a constant $C$ independent of $h$ such that

\[ \| R_{M_h}(f^h) \| \leq C \| f^h \|_{1, M_h} \quad \forall f^h \in M_h. \] (4.1)
Here $∥f∥_{1,M_h}$, as the notation suggests, should be thought as the discrete analogue of the $L^1$-norm of integrable functions (which coincides with the total variation norm if we regard integrable functions as measures). Then the associated dual norm on $M_h$ defined by

$$
∥f^h∥_{*,M_h} := \max \left\{ (f^h, g^h)_{M_h} : ∥g^h∥_{1,M_h} \leq 1 \right\}, \quad f^h \in M_h,
$$

(4.2)
can be interpreted as the discrete analogue of the $L^\infty$-norm. These observations will be more apparent after the next definition. It is clear that given a family of discrete spaces $M_h$, there are many families of reconstructions mapping it to the measure space $\mathcal{M}(\mathcal{X}, \mathbb{R}^n)$ (since the only requirement is the uniform boundedness (1.1)). Meanwhile, note that our main aim is to show the weak–star convergence of (the subsequence of) the numerical solutions to a minimizer to (2), and these reconstruction operators are only used to help us to perform the error analysis. It is natural to ask when the two families of reconstructions $\mathcal{R}_{M_h}(f^h)$ and $\mathcal{R}_{\tilde{M}_h}(f^h)$ of the same discrete sequence $f^h$ tend to the same limit in the weak–star topology. For this, we introduce more properties on the families of reconstruction operators, which helps to give a satisfying answer to this question (cf. Lemma 4.3) and will be very useful in the subsequent exposition. Let $\mathcal{R}_{M_h} : C(\mathcal{X}, \mathbb{R}^n) \to (M_h, ∥∥_{*,M_h})$ be the adjoint operator of $\mathcal{R}_{M_h}$ defined by

$$
(f^h, \mathcal{R}_{M_h}(\varphi))_{M_h} := (\mathcal{R}_{M_h}(f^h), \varphi)_X \quad \forall f^h \in M_h, \forall \varphi \in C(\mathcal{X}, \mathbb{R}^n).
$$

(4.3)

**Definition 4.2.** We say that two families of reconstruction operators $\{\mathcal{R}_{M_h}\}_{h \in \mathcal{H}}$ and $\{\mathcal{R}_{\tilde{M}_h}\}_{h \in \mathcal{H}}$ are equivalent if there exists $\{\epsilon_h\}_{h \in \mathcal{H}}$, with $\epsilon_h \in \mathbb{R}_+$ and $\epsilon_h$ tending to 0 as $|h| \to 0$, such that there holds

$$
∥(\mathcal{R}_{M_h} - \mathcal{R}_{\tilde{M}_h})(\varphi)∥_{*,M_h} \leq \epsilon_h ∥\varphi∥_{1,\infty} \quad \forall \varphi \in C^1(\mathcal{X}, \mathbb{R}^n).
$$

(4.4)

Moreover, a family of reconstruction operators $\{\mathcal{R}_{M_h}\}_{h \in \mathcal{H}}$ is said to be quasi–isometric if there exists a constant $C$ independent of $h$ such that there holds

$$
C^{-1}∥f^h∥_{1,M_h} \leq ∥\mathcal{R}_{M_h}(f^h)∥ \leq C ∥f^h∥_{1,M_h} \quad \forall f^h \in M_h.
$$

(4.5)

It is easy to check that the condition (4.4) indeed defines an equivalence relation (i.e., a binary relation with reflexivity, symmetry and transitivity) on the set of all the families of reconstruction operators. In what follows, we will write $\mathcal{R}_{M_h} \sim \mathcal{R}_{\tilde{M}_h}$ if $\mathcal{R}_{M_h}$ and $\mathcal{R}_{\tilde{M}_h}$ are equivalent in the sense of (4.4), and denote by $\{[(\mathcal{R}_{M_h})]\}_{h \in \mathcal{H}}$ (sometimes, the subset of) the equivalence class of $\{\mathcal{R}_{M_h}\}_{h \in \mathcal{H}}$, for ease of representation. We also define the set $\{[(\mathcal{R}_{M_h})(f^h)]\}_{h \in \mathcal{H}} := \{\{\mathcal{R}_{M_h}(f^h)\}_{h \in \mathcal{H}} : \{\mathcal{R}_{M_h}\}_{h \in \mathcal{H}} \in \{[(\mathcal{R}_{M_h})]\}_{h \in \mathcal{H}}\}$ and adopt the following convention:

- We say that a statement holds for $\{[(\mathcal{R}_{M_h})]\}_{h \in \mathcal{H}}$ (resp. $\{[(\mathcal{R}_{M_h})(f^h)]\}_{h \in \mathcal{H}}$), if the statement holds for any $\{\mathcal{R}_{M_h}\}_{h \in \mathcal{H}}$ (resp. $\{\mathcal{R}_{M_h}(f^h)\}_{h \in \mathcal{H}}$) in the set $\{[(\mathcal{R}_{M_h})]\}_{h \in \mathcal{H}}$ (resp. $\{[(\mathcal{R}_{M_h})(f^h)]\}_{h \in \mathcal{H}}$). For instance, “$\{M_h, \mathcal{R}_{M_h}\}$ is a family of discrete approximations of $\mathcal{M}(\mathcal{X}, \mathbb{R}^n)$” means that for any $\{\mathcal{R}_{M_h}\}_{h \in \mathcal{H}}$ from the set $\{[(\mathcal{R}_{M_h})]\}_{h \in \mathcal{H}}$, $(M_h, \mathcal{R}_{M_h})$ is a family of discrete approximations.

It is worth pointing out that the quasi–isometric property is not preserved by this equivalence relation, namely an element $\{\mathcal{R}_{M_h}\}_{h \in \mathcal{H}}$ in the equivalence class of the family of quasi–isometric reconstruction operators $\{[(\mathcal{R}_{M_h})]\}_{h \in \mathcal{H}}$, $(M_h, \mathcal{R}_{M_h})$ may not be quasi–isometric. By these notions and conventions, we have the following useful lemma.

**Lemma 4.3.** Let $\{(M_h, \mathcal{R}_{M_h})\}_{h \in \mathcal{H}}$ be a family of discrete approximations of $\mathcal{M}(\mathcal{X}, \mathbb{R}^n)$ and $\{\mathcal{R}_{M_h}(f^h)\}_{h \in \mathcal{H}}$ be a sequence of reconstructions of $f^h \in M_h$ where $∥f^h∥_{1,M_h}$ is uniformly bounded. Then it holds that

1. $\{[(\mathcal{R}_{M_h})(f^h)]\}_{h \in \mathcal{H}}$ is bounded in $\mathcal{M}(\mathcal{X}, \mathbb{R}^n)$.
2. There exists a sequence $\{h_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ such that $|h_n|$ tends to zero and $[\mathcal{R}_{M_{h_n}}](f^{h_n})$ weak–star converges to a measure $\mu \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n)$, when $n$ tends to infinity.

**Proof.** By (1.1), it readily follows that every sequence $\{\mathcal{R}_{M_h}(f^h)\}_{h \in \mathcal{H}}$ with $\mathcal{R}_{M_h} \sim \mathcal{R}_{M_h}$ is bounded, i.e., the first statement holds. Then the boundedness of the sequence $\{\mathcal{R}_{M_h}(f^h)\}_{h \in \mathcal{H}}$ allows us to extract a subsequence $\mathcal{R}_{M_{h_n}}(f^{h_n})$ (still indexed by $h$) weak–star converging to a measure $\mu \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n)$. We proceed to show that every sequence $\{\mathcal{R}_{M_h}(f^h)\}_{h \in \mathcal{H}}$ with $\mathcal{R}_{M_h} \sim \mathcal{R}_{M_h}$ weak–star converges to the same limit. For this, by (4.4), we have

$$
∥((\mathcal{R}_{M_h} - \mathcal{R}_{M_h})(f^h), \varphi)_X∥ \leq \epsilon_h ∥\varphi∥_{1,\infty} ∥f^h∥_{1,M_h} \quad \forall \varphi \in C^1(\mathcal{X}, \mathbb{R}^n),
$$

Then taking the limit of both sides of the above formula gives us

$$
\lim_{n \to \infty} (\mathcal{R}_{M_{h_n}}(f^{h_n}) - \mu, \varphi)_X = 0 \quad \forall \varphi \in C^1(\mathcal{X}, \mathbb{R}^n),
$$

where we have used the boundedness of $∥f^h∥_{1,M_h}$ and the weak–star convergence of $\{\mathcal{R}_{M_h}(f^h)\}_{h \in \mathcal{H}}$. The proof is completed by the density of $C^1(\mathcal{X}, \mathbb{R}^n)$ in $C(\mathcal{X}, \mathbb{R}^n)$ and the boundedness of $\{\mathcal{R}_{M_h}(f^h)\}_{h \in \mathcal{H}}$. □
Corollary 4.4. Let \( \{(M_h, \hat{R}_{M_h})\}_{h \in \mathcal{H}} \) be a family of discrete approximations of \( \mathcal{M}(\mathcal{X}, \mathbb{R}^n) \). If the quasi-isometric reconstructions \( \hat{R}_{M_h}(f^h) \) is bounded, then so is \( R_{M_h}(f^h) \) for any equivalent reconstruction operators \( R_{M_h} \sim \hat{R}_{M_h} \); if \( \hat{R}_{M_h}(f^h) \) weak-star converges, then for \( R_{M_h}(f^h) \) weak-star converges to the same limit for any \( R_{M_h} \sim \hat{R}_{M_h} \).

Fully discretized transport problem. We are now in position to consider the discretization of the matrix-valued dynamical optimal transport problem \([7]\). We mainly focus on the spatial discretization, since, as we shall see very soon, the discretization of time variable is explicit and quite direct. For this, we introduce the following basic modules adapted from \([11]\) Definition 2.5 but with new features based on Definitions \([4,11]\) and \([12]\). Let \( \sigma \) be a parameter, thought as the spatial mesh size, from the parameter set \( \Sigma \subset (0, +\infty) \) (i.e., \( \mathcal{H} = \Sigma \) with \( k = 1 \)).

Definition 4.5 (Modules). We define the following modules for the discrete matrix-valued optimal transport:

- \( \{ \hat{X}_\sigma, \hat{R}_{\hat{X}_\sigma} \}_{\sigma \in \Sigma} \) and \( \{ \hat{Y}_\sigma, \hat{R}_{\hat{Y}_\sigma} \}_{\sigma \in \Sigma} \) are the families of discrete approximations of \( \mathcal{M}(\Omega, \mathbb{M}^n) \) and \( \mathcal{M}(\Omega, \mathbb{R}^{n \times k}) \), respectively, in the sense of Definition 4.1 where \( \hat{R}_{\hat{X}_\sigma} \) and \( \hat{R}_{\hat{Y}_\sigma} \) are quasi-isometric reconstruction operators. Moreover, \( X_\sigma \) is a subspace of \( \hat{X}_\sigma \) and contains a cone \( X_{\sigma, +} \) and \( X_{\sigma}^\perp \) is the orthogonal complement of \( X_\sigma \) in \( (\hat{X}_\sigma, \langle \cdot, \cdot \rangle_{\hat{X}_\sigma}) \).

The sets of all the admissible families of spatial reconstructions on \( \hat{X}_\sigma \) and \( Y_\sigma \), denoted by \( \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} \} \) and \( \{ [\hat{R}_{\hat{Y}_\sigma}]_{\sigma \in \Sigma} \} \) respectively by abuse of notation, are defined by \( \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} := \{ \hat{R}_{\hat{X}_\sigma} \}_{\sigma \in \Sigma} \sim \hat{R}_{\hat{X}_\sigma}, \hat{R}_{\hat{X}_\sigma} \) maps \( X_\sigma \) to \( M(\Omega, \mathbb{S}^n) \), \( X_{\sigma, +} \) to \( M(\Omega, \mathbb{S}^n_+) \) and \( X_{\sigma, +} \) to \( M(\Omega, \mathbb{S}^n_{++}) \), and \{ \hat{R}_{\hat{Y}_\sigma} \}_{\sigma \in \Sigma} \sim \hat{R}_{\hat{Y}_\sigma}, \hat{R}_{\hat{Y}_\sigma} \). The set of admissible family of spatial reconstructions \( \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} \} \) on \( X_\sigma \) is defined as the restriction of \( \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} \} \) on \( X_\sigma \): \( \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} := \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} \cap \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} \} \cap \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} \} \}_{\sigma \in \Sigma} \}_{\sigma \in \Sigma} \). For clarity, we denote the norm and the inner product on \( X_\sigma \) induced from those of \( X_\sigma \) by \( \| \cdot \|_{X_\sigma} \) and \( \langle \cdot, \cdot \rangle_{X_\sigma} \), respectively.

- \( \mathcal{J}_\sigma : (X_\sigma, Y_\sigma, \hat{X}_\sigma) \to [0, +\infty] \) is a proper, l.s.c., convex function with positive homogeneity of degree \(-1\) in its first variable and homogeneity of degree \(2\) in its second and third variables.

The discrete derivation \( \mathcal{D}_\sigma : Y_\sigma \to X_\sigma \) is a linear operator with its adjoint operator \( \mathcal{D}_\sigma^* : X_\sigma \to Y_\sigma \) defined by
\[
\langle y_\sigma, \mathcal{D}_\sigma x_\sigma \rangle_{Y_\sigma} := -\langle \mathcal{D}_\sigma y_\sigma, x_\sigma \rangle_{X_\sigma} \quad \forall x_\sigma \in X_\sigma, \forall y_\sigma \in Y_\sigma.
\]

Remark 4.6. The operator \( \mathcal{D}_\sigma \), viewed as the discrete version of the derivation \( D \), is automatically bounded since it is a linear map from a finite-dimensional vector space. However, the family of operators \( \{ \mathcal{D}_\sigma \}_{\sigma} \) is typically not uniformly bounded w.r.t. \( \sigma \), which is analogous to the fact that the derivative operator is an unbounded operator on the space of continuous functions.

Remark 4.7. By definition, it is clear that \( \hat{R}_{X_\sigma} = \hat{R}_{\hat{X}_\sigma} \cap X_\sigma \) is quasi-isometric. It is also easy to check that the elements in \( \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} \} \) are indeed equivalent to \( \{ [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} \} \) (hence the notation is reasonable) and that \( \{ X_\sigma, [\hat{R}_{\hat{X}_\sigma}]_{\sigma \in \Sigma} \} \) are families of discrete approximations of \( \mathcal{M}(\Omega, \mathbb{S}^n) \). One may interpret these admissible families of reconstructions as all the possible and acceptable spatial discretization schemes of the variables, in which the quasi-isometric reconstructions play the special roles in view of Corollary 4.4.

We also introduce the conjugate of \( \mathcal{J}_\sigma(G^\sigma, q^\sigma, R^\sigma) \) w.r.t. \( (q^\sigma, R^\sigma) \) by
\[
\mathcal{J}_\sigma^*(G^\sigma, u^\sigma, W^\sigma) = \sup_{(u^\sigma, W^\sigma) \in Y_\sigma \times X_\sigma} \langle q^\sigma, u^\sigma \rangle_{Y_\sigma} + \langle R^\sigma, W^\sigma \rangle_{X_\sigma} - J_\sigma(G^\sigma, q^\sigma, R^\sigma),
\]
for our later use. Since \( J_\sigma \) is homogeneous of degree two in \( (q^\sigma, R^\sigma) \), it is easy to check that \( J_\sigma^* \) is also homogeneous of degree two in \( (u^\sigma, W^\sigma) \). Then a same scaling argument as the one to \([39, 21]\) and \([39, 22]\) leads to
\[
\langle q^\sigma, b^\sigma \rangle_{Y_\sigma} + \langle R^\sigma, C^\sigma \rangle_{X_\sigma} \leq 2 \sqrt{J_\sigma(G^\sigma, q^\sigma, R^\sigma) J_\sigma^*(G^\sigma, u^\sigma, W^\sigma)}.
\]

Let \( \mathbb{F}_{X_\sigma} \) be the orthogonal projection from the Hilbert space \( (\hat{X}_\sigma, \langle \cdot, \cdot \rangle_{\hat{X}_\sigma}) \) to its subspace \( X_\sigma \). By the self-adjointness of \( \mathbb{F}_{X_\sigma} \) and the property that \( \hat{R}_{\hat{X}_\sigma} \) maps \( C(\Omega, \mathbb{S}^n) \) to \( X_\sigma \) (recall \( \hat{R}_{\hat{X}_\sigma} : X_{\sigma, +} \to M(\Omega, \mathbb{S}^n_{++}) \)), we have the following easily observed but very useful relation:
\[
\langle \hat{R}_{\hat{X}_\sigma}(\mathbb{F}_{X_\sigma} R^\sigma), \Phi \rangle_\Omega = \langle R^\sigma, \mathbb{F}_{X_\sigma} \hat{R}_{\hat{X}_\sigma}^*(\Phi) \rangle_\Omega = \langle R^\sigma, \hat{R}_{\hat{X}_\sigma}^*(\Phi) \rangle_\Omega.
\]

Moreover, the modules defined in Definition 4.3 readily lead to the following semi-discrete transport problem:

minimizing \( \int_0^1 J_\sigma(\mu^\sigma) dt \) over \( \mu^\sigma \in (G^\sigma_t, q^\sigma_t, R^\sigma_t) \in X_\sigma \times Y_\sigma \times \hat{X}_\sigma, t \in [0, 1] \)
subject to \( \partial_t G^\sigma_t + D_\sigma q^\sigma_t = \mathbb{F}_{X_\sigma} R^\sigma \) with \( G^\sigma_t |_{t=0} = G^\sigma_0, G^\sigma_t |_{t=1} = G^\sigma_1 \),

26
where $G_i^\tau \in X_{\tau,+,i} = 1, 2$, are a given approximation of $G_i \in \mathcal{M}(\Omega, S^n_+)$, $i = 1, 2$. We proceed to consider the time discretization. In this work, we adopt the staggered grid on the time interval $[0, 1]$. More precisely, let $N \geq 1$ be the number of time steps and $\tau := 1/N$ be temporal step size. We will approximate the values of $G_i^\tau$ at the time levels $t = i\tau, i = 0, \ldots, N$, while approximating the values of $q_i^\tau$ and $R_i^\tau$ at the time levels $t = i\tau - \sigma/2, i = 1, \ldots, N$.

A family of fully discretized matrix-valued transport problem is formulated as follows. Let $(\tau, \sigma)$ denote the time–space discretization parameter from the subset $\Sigma_{N} := \Sigma \times \{1/2, \ldots, 1/N, \ldots\}$ of $\mathbb{R}^2_+$, and define the product space $M_{\tau,\sigma} = X_{\tau,\sigma} \times Y_{\tau,\sigma} \times \tilde{X}_{\tau,\sigma} := (X_{\sigma})^{N+1} \times (Y_{\sigma})^N \times (\tilde{X}_{\sigma})^N$ for each $(\tau, \sigma) = (1/N, \sigma)$ in $\Sigma_{N}$.

**Definition 4.8.** Let $\{(X_{\tau}, Y_{\tau}, \tilde{X}_{\tau})\}_{\sigma \in \Sigma}, \{(J_{\tau})_{\sigma \in \Sigma}\}$ and $\{(\sigma_{\tau})_{\sigma \in \Sigma}\}$ be as defined in Definition 4.5, and $G_i^0, G_i^\tau \in X_{\tau,\sigma}$ be the given discrete initial and final distributions that satisfy

$$
\lim_{\sigma \to 0} \langle R_{X_{\sigma}}(G_i^\tau), \Phi \rangle_{\Omega} = \langle G_i, \Phi \rangle_{\Omega}, \quad i = 1, 2.
$$

We write $\mu^{\tau,\sigma}$ for $(G_{\sigma}^{\tau,\sigma}, q^{\tau,\sigma}, R^{\tau,\sigma}) \in M_{\tau,\sigma}$ and define the discrete action functional $J_{\tau,\sigma}$ by

$$
J_{\tau,\sigma}(\mu^{\tau,\sigma}) = \frac{\tau}{2} \sum_{k=1}^{N} J_{\sigma}(G_{k-1}^{\tau,\sigma} + \frac{1}{2} G_k^{\tau,\sigma}, q_k^{\tau,\sigma}, R_k^{\tau,\sigma}).
$$

We denote by $CE^{\tau,\sigma}([0, 1]; G_0, G_1)$ the subset of $M_{\tau,\sigma}$ consisting of the discrete curves $\mu^{\tau,\sigma}$ that satisfies the discrete continuity equation:

$$
\tau^{-1}(G_{k-1}^{\tau,\sigma} - G_k^{\tau,\sigma}) = -D_{\sigma} q_k^{\tau,\sigma} + P_{X_{\sigma}} R_k^{\tau,\sigma}, \quad k = 1, 2, \ldots, N,
$$

with $G_0^{\tau,\sigma} = G_0^\tau \in X_{\tau,\sigma}$ and $G_N^{\tau,\sigma} = G_1^\tau \in X_{\tau,\sigma}$. Then we introduce the discrete matrix–valued optimal transport problem as follows:

$$
\inf \left\{ J_{\tau,\sigma}(\mu^{\tau,\sigma}); \mu^{\tau,\sigma} \in CE^{\tau,\sigma}([0, 1]; G_0^\sigma, G_1^\sigma), G_i^{\tau,\sigma} \in X_{\tau,\sigma}, i = 1, \ldots, N \right\}.
$$

**Remark 4.9.** By the direct method in the calculus of variations, we can show the existence of the minimizer to $P_{\tau,\sigma}$.

If there exists a discrete curve $\mu^{\tau,\sigma} \in CE^{\tau,\sigma}$ with $J_{\tau,\sigma}(\mu^{\tau,\sigma}) < +\infty$ and $G_i^{\tau,\sigma} \in X_{\tau,\sigma}$, then the discrete transport problem $P_{\tau,\sigma}$ always admits a minimizer. Indeed, it is clear from Definition 4.11 that $J_{\tau,\sigma}$ is a proper l.s.c. convex functional. Moreover, by estimates similar to the ones in Theorem 4.4 for showing the boundedness of $\|\mu^{\tau,\sigma}\|_{1, M_{\tau,\sigma}}$, we can prove that a minimizing sequence for $P_{\tau,\sigma}$ must be bounded. Then a standard argument applies to finish the proof, thanks to the finite dimensionality of $P_{\tau,\sigma}$.

Before proceeding to consider the convergence of $\mu^{\tau,\sigma}$, we introduce the time-space reconstruction operators based on admissible families of spatial reconstructions $\{R_{X_{\tau}}\}_{\sigma \in \Sigma}, \{R_{Y_{\tau}}\}_{\sigma \in \Sigma}$ and $\{R_{\tilde{X}_{\tau}}\}_{\sigma \in \Sigma}$ as follows.

- We define the piecewise linear (in time) reconstruction $R_{X_{\tau}}^{\tau}(G^{\tau,\sigma})$ and the piecewise constant reconstruction $R_{\tilde{X}_{\tau}}^{\tau}(G^{\tau,\sigma})$ for $G^{\tau,\sigma} \in X_{\tau,\sigma}$, respectively, by: $\forall \Phi \in C(Q, S^n)$,

$$
\langle R_{X_{\tau}}^{\tau}(G^{\tau,\sigma}), \Phi \rangle_Q := \sum_{k=1}^{N} \int_{(k-1)\tau}^{k\tau} \left( \frac{t - \tau}{\tau} \langle R_{X_{\sigma}}(G_{k-1}^{\tau,\sigma}), \Phi \rangle_{\Omega} + \frac{t}{\tau} \langle R_{X_{\sigma}}(G_k^{\tau,\sigma}), \Phi \rangle_{\Omega} \right) dt,
$$

and

$$
\langle R_{\tilde{X}_{\tau}}^{\tau}(G^{\tau,\sigma}), \Phi \rangle_Q := \sum_{k=1}^{N} \int_{(k-1)\tau}^{k\tau} \frac{1}{2} \langle R_{X_{\sigma}}(G_{k-1}^{\tau,\sigma}), \Phi \rangle_{\Omega} + \frac{1}{2} \langle R_{X_{\sigma}}(G_k^{\tau,\sigma}), \Phi \rangle_{\Omega} dt.
$$

- We define the piecewise constant reconstructions $R_{Y_{\tau}}^{\tau}(q^{\tau,\sigma})$ and $R_{\tilde{X}_{\tau}}^{\tau}(R^{\tau,\sigma})$ for $q^{\tau,\sigma} \in Y_{\tau,\sigma}$ and $R^{\tau,\sigma} \in \tilde{X}_{\tau,\sigma}$, respectively, by

$$
\langle R_{Y_{\tau}}^{\tau}(q^{\tau,\sigma}), \Phi \rangle_Q := \sum_{k=1}^{N} \int_{(k-1)\tau}^{k\tau} \langle R_{Y_{\sigma}}(q_k^{\tau,\sigma}), \Phi \rangle_{\Omega} dt, \quad \forall \Phi \in C(Q, \mathbb{R}^{n \times k}),
$$

and

$$
\langle R_{\tilde{X}_{\tau}}^{\tau}(R^{\tau,\sigma}), \Phi \rangle_Q := \sum_{k=1}^{N} \int_{(k-1)\tau}^{k\tau} \langle R_{\tilde{X}_{\sigma}}(R_k^{\tau,\sigma}), \Phi \rangle_{\Omega} dt, \quad \forall \Phi \in C(Q, M^n).
$$

We next equip the finite-dimensional spaces $X_{\tau,\sigma}, Y_{\tau,\sigma}$ and $\tilde{X}_{\tau,\sigma}$ with norms and inner products so that they, together with the reconstruction operators defined above, form the families of discrete approximations of $\mathcal{M}(Q, S^n)$, $\mathcal{M}(Q, \mathbb{R}^{n \times k})$ and $\mathcal{M}(Q, M^n)$, respectively, in the sense of Definition 4.1.
We define the norm on $X_{\tau,\sigma}$ by $\|G^{\tau,\sigma}\|_{1,X_{\tau,\sigma}} = \sum_{k=0}^{N} \tau \|G_{k}^{\tau,\sigma}\|_{1,X_{\tau,\sigma}}$ for $G^{\tau,\sigma} \in X_{\tau,\sigma}$ and the inner product by $(G^{\tau,\sigma}, Q^{\tau,\sigma})_{X_{\tau,\sigma}} = \sum_{k=0}^{N} \tau (G_{k}^{\tau,\sigma}, Q_{k}^{\tau,\sigma})_{X_{\tau,\sigma}}$ for $G^{\tau,\sigma}, Q^{\tau,\sigma} \in X_{\tau,\sigma}$. By definition, the following estimate holds

$$\|\mathcal{R}_{X_{\tau,\sigma}}^{l/a}(G^{\tau,\sigma})\| \leq \tau \sum_{i=0}^{N} \|\mathcal{R}_{X_{\tau,\sigma}}(G_{i}^{\tau,\sigma})\| \quad \text{for } G^{\tau,\sigma} \in X_{\tau,\sigma}. \quad (4.17)$$

Then we can readily check that $\{(X_{\tau,\sigma}, \mathcal{R}_{X_{\tau,\sigma}}^{l/a})\}_{(\tau,\sigma) \in \Sigma_{h}}$ is a family of discrete approximations of $\mathcal{M}(\Omega, \mathbb{S}^{n})$. Here and in what follows, $\mathcal{R}_{X_{\tau,\sigma}}^{l/a}$ means two cases: $\mathcal{R}_{X_{\tau,\sigma}}^{l}$ and $\mathcal{R}_{X_{\tau,\sigma}}^{k}$.

Similarly, we endow the space $Y_{\tau,\sigma}$ with the norm defined by $\|q^{\tau,\sigma}\|_{1,Y_{\tau,\sigma}} = \tau \sum_{i=1}^{N} \tau q_{i}^{\tau,\sigma}\|_{Y_{\tau,\sigma}}$ for $q^{\tau,\sigma} \in Y_{\tau,\sigma}$ and the inner product defined by $(p^{\tau,\sigma}, q^{\tau,\sigma})_{Y_{\tau,\sigma}} = \tau \sum_{i=1}^{N} (p_{i}^{\tau,\sigma}, q_{i}^{\tau,\sigma})_{Y_{\tau,\sigma}}$ for $p^{\tau,\sigma}, q^{\tau,\sigma} \in Y_{\tau,\sigma}$. The norm $\|\cdot\|_{1,Y_{\tau,\sigma}}$ and the inner product $(\cdot, \cdot)_{Y_{\tau,\sigma}}$ on $Y_{\tau,\sigma}$ are defined in the same way as the ones on $Y_{\tau,\sigma}$ by replacing $Y_{\tau,\sigma}$ by $\tilde{X}_{\tau,\sigma}$ and $Y_{\tau,\sigma}$ by $\tilde{Y}_{\tau,\sigma}$. Then we can see that $\{(Y_{\tau,\sigma}, \mathcal{R}_{Y_{\tau,\sigma}})\}_{(\tau,\sigma) \in \Sigma_{h}}$ and $\{(\tilde{X}_{\tau,\sigma}, \mathcal{R}_{\tilde{X}_{\tau,\sigma}})\}_{(\tau,\sigma) \in \Sigma_{h}}$ are the families of discrete approximations of $\mathcal{M}(Q, \mathbb{R}^{n \times k})$ and $\mathcal{M}(Q, \mathbb{M}^{n})$, respectively, with the following estimates

$$\|\mathcal{R}_{Y_{\tau,\sigma}}(q^{\tau,\sigma})\| = \tau \sum_{i=1}^{N} \|\mathcal{R}_{Y_{\tau,\sigma}}(q_{i}^{\tau,\sigma})\| \lesssim \|q^{\tau,\sigma}\|_{1,Y_{\tau,\sigma}}; \quad \|\mathcal{R}_{\tilde{X}_{\tau,\sigma}}(R^{\tau,\sigma})\| = \tau \sum_{i=1}^{N} \|\mathcal{R}_{\tilde{X}_{\tau,\sigma}}(R_{i}^{\tau,\sigma})\| \lesssim \|R^{\tau,\sigma}\|_{1,\tilde{X}_{\tau,\sigma}}. \quad (4.18)$$

For ease of exposition, we further endow the product space $\mathcal{M}_{\tau,\sigma}$ with the norm: $\|\mu^{\tau,\sigma}\|_{1,\mathcal{M}_{\tau,\sigma}} = \|G^{\tau,\sigma}\|_{1,X_{\tau,\sigma}} + \|q^{\tau,\sigma}\|_{1,Y_{\tau,\sigma}} + \|R^{\tau,\sigma}\|_{1,\tilde{X}_{\tau,\sigma}}$ for $\mu_{\tau,\sigma} = (G^{\tau,\sigma}, q^{\tau,\sigma}, R^{\tau,\sigma}) \in \mathcal{M}_{\tau,\sigma}$ (the inner product can be defined in the same way). By the time–space reconstructions defined above, we consider

$$\mathcal{R}_{\tau,\sigma}^{l/a}(G^{\tau,\sigma}, q^{\tau,\sigma}, R^{\tau,\sigma}) = (\mathcal{R}_{X_{\tau,\sigma}}^{l/a}(G^{\tau,\sigma}), \mathcal{R}_{Y_{\tau,\sigma}}(q^{\tau,\sigma}), \mathcal{R}_{\tilde{X}_{\tau,\sigma}}(R^{\tau,\sigma})) : \mathcal{M}_{\tau,\sigma} \rightarrow \mathcal{M}(Q, \mathbb{X}), \quad (4.19)$$

and introduce the set

$$\{[\mathcal{R}_{\tau,\sigma}^{l/a}]_{(\tau,\sigma) \in \Sigma_{h}} = \{[\mathcal{R}_{\tau,\sigma}^{l/a}]_{(\tau,\sigma) \in \Sigma_{h}} : \mathcal{R}_{X_{\tau,\sigma}}^{l/a}, \mathcal{R}_{Y_{\tau,\sigma}}^{l/a}, \text{ and } \mathcal{R}_{\tilde{X}_{\tau,\sigma}}^{l/a} \text{ are defined in (4.13)–(4.16) based on admissible families of spatial reconstructions } \mathcal{R}_{X_{\tau,\sigma}}, \mathcal{R}_{Y_{\tau,\sigma}}, \text{ and } \mathcal{R}_{\tilde{X}_{\tau,\sigma}}\}. \quad (4.20)$$

Moreover, by definition, a direct computation leads to

$$\|\mu^{\tau,\sigma}\|_{*,\mathcal{M}_{\tau,\sigma}} = \max \{\|G^{\tau,\sigma}\|_{*,X_{\tau,\sigma}}, \|q^{\tau,\sigma}\|_{*,Y_{\tau,\sigma}}, \|R^{\tau,\sigma}\|_{*,\tilde{X}_{\tau,\sigma}}\},$$

which helps us to see that $\{[\mathcal{R}_{\tau,\sigma}]_{(\tau,\sigma) \in \Sigma_{h}}$ is the subset of the equivalence class of $\mathcal{R}_{\tau,\sigma}^{l/a} = (\mathcal{R}_{X_{\tau,\sigma}}^{l/a}, \mathcal{R}_{Y_{\tau,\sigma}}^{l/a}, \mathcal{R}_{\tilde{X}_{\tau,\sigma}}^{l/a})$. The Lax equivalence principle reviewed in Section 3.1 suggests that if there exists some reconstructions that are “stable” and “consistent” in some suitable sense, then the convergence may be expected. It turns out that it is indeed the case, and in fact, a set of consistency conditions are sufficient to imply the convergence with the stability being automatically guaranteed.

The remaining is devoted to proving the main expected result in this section: Theorem 4.10 which gives the convergence of $\mu^{\tau,\sigma}$ in a reasonable sense. The proof shall be divided into two parts: the asymptotic lower bound in Theorem 4.11 and the asymptotic upper bound in Theorem 4.14 which are also interesting in themselves. We start with the consideration of the asymptotic lower bound. For this, we need to introduce the consistency between $D^{\sigma}$ and $D^{\sigma}_{+}$, and the one-side consistency between $J^{\sigma}$ and $J^{\sigma}_{+}$.

**Definition 4.10 (Consistent reconstructions).** Let $\{[\tilde{X}_{\tau,\sigma}]_{\sigma \in \Sigma}\}$ and $\{[\tilde{Y}_{\tau,\sigma}]_{\sigma \in \Sigma}\}$ be the admissible families of reconstruction operators. We say that they are consistent if the following conditions hold

- $D^{\sigma}$ and $D_{+}^{\sigma}$ are consistent via $\{[\tilde{X}_{\tau,\sigma}]_{\sigma \in \Sigma}\}$ and $\{[\tilde{Y}_{\tau,\sigma}]_{\sigma \in \Sigma}\}$ there exists $\{\varepsilon_{\sigma}\}_{\sigma \in \Sigma}$, with $\varepsilon_{\sigma} \in \mathbb{R}_{+}$ and $\varepsilon_{\sigma}$ tending to 0 as $\sigma \rightarrow 0$, and $\{\tilde{X}_{\tau,\sigma}\}_{\sigma \in \Sigma} \in \{[\tilde{X}_{\tau,\sigma}]_{\sigma \in \Sigma}\}$ such that there holds

$$\|\mathcal{R}_{\tau,\sigma}(q^{\sigma})|D^{\sigma}\|_{\Omega} + \|\mathcal{R}_{\tau,\sigma}(D_{+}^{\sigma}q^{\sigma})|\phi\|_{\Omega} \leq \varepsilon_{\sigma} \|\phi\|_{2,\infty} \|q^{\sigma}\|_{1,Y_{\tau,\sigma}} \forall q^{\sigma} \in Y_{\tau,\sigma}, \forall \phi \in C^{2}(\Omega, \mathbb{S}^{n}). \quad (4.21)$$

- $J^{\sigma}$ and $J_{+}^{\sigma}$ are one-sided consistent via $\{[\tilde{X}_{\tau,\sigma}]_{\sigma \in \Sigma}\}$ and $\{[\tilde{Y}_{\tau,\sigma}]_{\sigma \in \Sigma}\}$ in the following sense. There exists $\{\varepsilon_{\sigma}\}_{\sigma \in \Sigma}$, with $\varepsilon_{\sigma} \in \mathbb{R}_{+}$ and $\varepsilon_{\sigma}$ tending to 0 as $\sigma \rightarrow 0$, such that there holds

$$J_{+}^{\sigma}(G^{\sigma}, [\tilde{X}_{\tau,\sigma}](u), [\tilde{Y}_{\tau,\sigma}](W)) \leq J_{+}^{\Omega}(\mathcal{R}_{\tau,\sigma}(G^{\sigma}), u, W) + C\|u, W\|_{1,\infty} \|G^{\sigma}\|_{1,\varepsilon_{\sigma} \sigma}, \quad (4.22)$$
for some \( \{ \widehat{R}_{X_\tau} \}_{\tau \in \Sigma} \subseteq \{ \widehat{R}_{Y_\tau} \}_{\tau \in \Sigma} \), \( \{ \widehat{R}_{Y_\tau} \}_{\tau \in \Sigma} \subseteq \{ \widehat{R}_{X_\tau} \}_{\tau \in \Sigma} \), and for any 
\( G^\sigma \in X_\tau \), \((u, W) \in C^1(\Omega, \mathbb{R}^{n \times k} \times S^n) \). Moreover, there holds
\[
J^*_Q(G^\sigma, \widehat{R}_{Y_\tau}^\sigma(u), \widehat{R}_{X_\tau}^\sigma(W)) \lesssim (\|u\|^2_\infty + \|W\|^2_\infty) \|G^\sigma\|_{1,X_\tau}, \tag{4.23}
\]
for any \( G^\sigma \in X_\tau \) and \((u, W) \in C(\Omega, \mathbb{R}^{n \times k} \times S^n) \).

**Theorem 4.11** (Asymptotic lower bound). Let \( \mu^{\tau, \sigma} = (G^{\tau, \sigma}, q^{\tau, \sigma}, R^{\tau, \sigma}) \in \mathcal{CE}^{\tau, \sigma}([0, 1]; \hat{G}_0, \hat{G}_1), (\tau, \sigma) \in \Sigma_\Omega \), be a sequence of discrete curves such that
\[
E := \sup_{(\tau, \sigma) \in \Sigma_\Omega} J_{\tau, \sigma}(\mu^{\tau, \sigma}) < +\infty. \tag{4.24}
\]
Suppose that there exists the admissible family of spatial reconstructions \( \{ \widehat{R}_{X_\tau} \}_{\tau \in \Sigma} \) and \( \{ \widehat{R}_{Y_\tau} \}_{\tau \in \Sigma} \) which is consistent in the sense of Definition 4.10. Then \( \mu^{\tau, \sigma} \|1, M_{\tau, \sigma} \) is uniformly bounded w.r.t. \((\tau, \sigma) \). Up to a subsequence, \( \widehat{R}_{\tau, \sigma}(\mu^{\tau, \sigma}) \) weak–star converges to a curve \( \mu \in \mathcal{CE}_\infty([0, 1]; \hat{G}_0, \hat{G}_1) \) with \( J(\mu) \) satisfying
\[
J_Q(\mu) \leq \liminf_{(\tau, \sigma) \to 0} J_{\tau, \sigma}(\mu^{\tau, \sigma}), \tag{4.25}
\]
where \( \{ \widehat{R}_{\tau, \sigma} \}_{(\tau, \sigma) \in \Sigma_\Omega} \) is defined by \( \mathbf{1.20} \).

**Proof.** First, we show the boundedness of \( \|\mu^{\tau, \sigma}\|_{1,M_{\tau,\sigma}} \) in a manner similar to the estimates \( \mathbf{3.24} \) and \( \mathbf{3.27} \) in the continuous case. By definition and the estimate \( \mathbf{1.18} \), it is clear that \( \{ \widehat{R}_{Y_{\tau,\sigma}} \}_{(\tau, \sigma) \in \Sigma_\Omega} \) and \( \{ \widehat{R}_{X_{\tau,\sigma}} \}_{(\tau, \sigma) \in \Sigma_\Omega} \) are the families of quasi–isometric reconstruction operators on the spaces \( Y_{\tau,\sigma} \) and \( X_{\tau,\sigma} \), respectively. Then, for any 
\( u \in C(Q, \mathbb{R}^{n \times k}) \) and \( W \in C(Q, M^n) \) with \( \|u\|_\infty \leq 1 \) and \( \|W\|_\infty \leq 1 \), we have
\[
\langle \widehat{R}_{Y_{\tau,\sigma}}^\sigma(q^{\tau,\sigma}), u \rangle_Q + \langle \widehat{R}_{X_{\tau,\sigma}}^\sigma(R^{\tau,\sigma}), W \rangle_Q = \sum_{i=1}^N \tau \langle q_{i}^{\tau,\sigma}, \widehat{R}_{Y_{\tau,\sigma}}^\sigma(u_i) \rangle_{Y_{\sigma}} + \tau \langle R_{i}^{\tau,\sigma}, \widehat{R}_{X_{\tau,\sigma}}^\sigma(W_i) \rangle_{X_{\sigma}},
\]
where \( u_i(\cdot) \) and \( W_i(\cdot) \) are defined by \( \tau^{-1} \int_{i-1}^{i} u(s, \cdot) ds \) and \( \tau^{-1} \int_{i-1}^{i} W(s, \cdot) ds \) for \( i = 1, \ldots, N \), and satisfy \( \|u_i\|_\infty \leq 1 \) and \( \|W_i\|_\infty \leq 1 \) and we have used \( \mathbf{4.28} \) and Cauchy’s inequality in \( (1) \), and \( \mathbf{4.23} \) and the assumption \( \mathbf{1.21} \) in \( (2) \). It then follows from the definitions of the total variation norm and \( \|\cdot\|_{1,X_{\tau,\sigma}} \) that
\[
\|\widehat{R}_{Y_{\tau,\sigma}}^\sigma(q^{\tau,\sigma})\| + \|\widehat{R}_{X_{\tau,\sigma}}^\sigma(R^{\tau,\sigma})\| \lesssim \sqrt{E\|G^{\tau,\sigma}\|_{1,X_{\tau,\sigma}}}. \tag{4.26}
\]
We proceed to act the quasi–isometric reconstruction operator \( \widehat{R}_{X_{\sigma}} \) on both sides of \( \mathbf{4.12} \) and take the test function \( I \), then obtain
\[
(\tau^{-1}\widehat{R}_{X_{\tau}}(G_{i}^{\tau,\sigma} - G_{k}^{\tau,\sigma}), I)_\Omega = (-\widehat{R}_{X_{\tau}}(D_{\tau} q_{k}^{\tau,\sigma}) + \widehat{R}_{X_{\tau}}(\mathcal{F}_{X_{\tau}} R_{k}^{\tau,\sigma}), I)_\Omega, \quad k = 1, \ldots, N, \tag{4.27}
\]
which implies that for \( i, j \in \{1, \ldots, N\} \) with \( i < j \), there holds
\[
|\text{tr} \widehat{R}_{X_{\tau}}(G_{j}^{\tau,\sigma})(\Omega) - \text{tr} \widehat{R}_{X_{\tau}}(G_{i}^{\tau,\sigma})(\Omega)| \leq C \sum_{k=i+1}^{j} \tau \langle \varepsilon_{\sigma} ||q_{k}^{\tau,\sigma}||_{1,Y_{\sigma}} + ||\widehat{R}_{X_{\tau}}(R_{k}^{\tau,\sigma})|| \rangle, \tag{4.27}
\]
by the relation \( \mathbf{1.9} \) and the estimate \( \mathbf{1.21} \) with \( \phi = I \) (recall \( D^*(I) = 0 \) in Definition 3.4):
\[
|\langle \widehat{R}_{X_{\tau}}(D_{\tau} q_{k}^{\tau,\sigma}), I \rangle_\Omega| \lesssim \|q_{k}^{\tau,\sigma}\|_{1,Y_{\sigma}} \varepsilon_{\sigma}. \tag{4.27}
\]
Then, by \( \mathbf{1.27} \), it follows from \( G_{i}^{\tau,\sigma} \in X_{\sigma,+} \) and \( \widehat{R}_{X_{\tau}} : X_{\sigma,+} \to \mathcal{M}(\Omega, S^n) \), as well as the quasi–isometric property of \( \widehat{R}_{X_{\tau}} \), that
\[
\|G_{j}^{\tau,\sigma}\|_{1,X_{\tau}} \lesssim \|G_{0}^\sigma\|_{1,X_{\tau}} + \|\widehat{R}_{Y_{\tau,\sigma}}(q^{\tau,\sigma})\| + \|\widehat{R}_{X_{\tau,\sigma}}(R^{\tau,\sigma})\| \quad \forall j = 1, \ldots, N, \tag{4.27}
\]
for any \( G^\sigma \in X_\tau \) and \((u, W) \in C(\Omega, \mathbb{R}^{n \times k} \times S^n) \).
which, along with (4.20), yields
\[ \|G^{\tau,\sigma}\|_{1, X_{\tau,\sigma}} = \sum_{i=0}^{N} \tau \|G_{i}^{\tau,\sigma}\|_{1, X_{\tau,\sigma}} \lesssim \|G_{0}^{\tau,\sigma}\|_{1, X_{\tau,\sigma}} + \sqrt{E} \|G^{\tau,\sigma}\|_{1, X_{\tau,\sigma}}. \]

Note that \(\|G_{0}^{\tau,\sigma}\|_{1, X_{\tau,\sigma}}\) is bounded by the weak–star convergence in (4.10). An elementary estimate as the one for (3.31) gives that \(\|G^{\tau,\sigma}\|_{1, X_{\tau,\sigma}}\) is bounded by a constant independent of \((\tau, \sigma)\), and so is \(\|\hat{R}_{X_{\tau,\sigma}}(q^{\tau,\sigma})\| + \|\hat{R}_{X_{\tau,\sigma}}(R^{\tau,\sigma})\|\) by (4.26). Therefore, we have \(\|\mu^{\tau,\sigma}\|_{1, M_{\tau,\sigma}}\) is bounded. Then, by Lemma 4.3 up to a subsequence, \([\hat{R}_{X_{\tau,\sigma}}(\mu^{\tau,\sigma})]\) weak–star converges to a measure \(\mu \in \mathcal{M}(Q, X)\),

Next, we show that the limit measure \(\mu = (G, q, R) \in \mathcal{C}(Q, X)\). For this, we consider a special family of reconstructions \(\mathcal{R}_{\tau,\sigma} = (\hat{R}_{X_{\tau,\sigma}}, \hat{R}_{Y_{\tau,\sigma}}, \hat{R}_{\hat{X}_{\tau,\sigma}})\) in \([\mathcal{R}_{\tau,\sigma}]_{(\tau,\sigma) \in \Sigma_{0}}\) where \(\mathcal{R}_{\tau,\sigma}\) is defined by (4.15) based on the reconstructions \(\mathcal{R}_{\tau,\sigma}\) such that (4.21) holds, and show
\[
\lim_{(\tau,\sigma) \to 0} \langle \hat{R}_{X_{\tau,\sigma}}(G^{\tau,\sigma}), \partial_{t} \Phi \rangle_{Q} + \langle \hat{R}_{Y_{\tau,\sigma}}(q^{\tau,\sigma}), D^{*} \Phi \rangle_{Q} + \langle \hat{R}_{\hat{X}_{\tau,\sigma}}(R^{\tau,\sigma}), \Phi \rangle_{Q} - \langle G_{1}, \Phi_{1} \rangle_{\Omega} - \langle G_{0}, \Phi_{0} \rangle_{\Omega} = 0 \quad \forall \Phi \in C^{2}(Q, S^{1}),
\]

For this, we start with the following calculation:
\[
\langle \hat{R}_{X_{\tau,\sigma}}(G^{\tau,\sigma}), \partial_{t} \Phi \rangle_{Q} + \langle \hat{R}_{Y_{\tau,\sigma}}(q^{\tau,\sigma}), D^{*} \Phi \rangle_{Q} + \langle \hat{R}_{\hat{X}_{\tau,\sigma}}(R^{\tau,\sigma}), \Phi \rangle_{Q} = \langle \hat{R}_{X_{\tau,\sigma}}(G^{\tau,\sigma}) - \hat{R}_{X_{\tau,\sigma}}(G_{k}^{\tau,\sigma}), \partial_{t} \Phi \rangle_{Q} + \langle \hat{R}_{Y_{\tau,\sigma}}(q^{\tau,\sigma}) - \hat{R}_{Y_{\tau,\sigma}}(q_{k}^{\tau,\sigma}), D^{*} \Phi \rangle_{Q} + \langle \hat{R}_{\hat{X}_{\tau,\sigma}}(R^{\tau,\sigma}) - \hat{R}_{\hat{X}_{\tau,\sigma}}(R_{k}^{\tau,\sigma}), \Phi \rangle_{Q} - \langle \hat{R}_{X_{\tau,\sigma}}(G_{k}^{\tau,\sigma}), \partial_{t} \Phi \rangle_{Q} - \langle \hat{R}_{Y_{\tau,\sigma}}(q_{k}^{\tau,\sigma}), D^{*} \Phi \rangle_{Q} - \langle \hat{R}_{\hat{X}_{\tau,\sigma}}(R_{k}^{\tau,\sigma}), \Phi \rangle_{Q} + \langle G_{1}, \Phi_{1} \rangle_{\Omega} + \langle G_{0}, \Phi_{0} \rangle_{\Omega} + O(\varepsilon_{\tau} \|q^{\tau,\sigma}\|_{1, Y_{\tau,\sigma}} \|\Phi\|_{2, \infty}),
\]

where in (1) we have used the definitions of reconstructions \(\hat{R}_{X_{\tau,\sigma}}, \hat{R}_{Y_{\tau,\sigma}}\) and \(\hat{R}_{\hat{X}_{\tau,\sigma}}\) and the integration by parts; and in (2) we have used the following estimate: for \(k = 1, \ldots, N\),
\[
0 = \langle \hat{R}_{X_{\tau}}(\tau^{-1}(G_{k}^{\tau,\sigma} - G_{k-1}^{\tau,\sigma})), \Phi_{t} \rangle_{\Omega} + \langle \hat{R}_{X_{\tau}}(D_{\tau} q_{k}^{\tau,\sigma}), \Phi_{t} \rangle_{\Omega} - \langle \hat{R}_{X_{\tau}}(R_{k}^{\tau,\sigma}), \Phi_{t} \rangle_{\Omega} = \langle \hat{R}_{X_{\tau}}(\tau^{-1}(G_{k}^{\tau,\sigma} - G_{k-1}^{\tau,\sigma})), \Phi_{t} \rangle_{\Omega} - \langle \hat{R}_{X_{\tau}}(q_{k}^{\tau,\sigma}), D^{*} \Phi_{t} \rangle_{\Omega} - \langle \hat{R}_{X_{\tau}}(R_{k}^{\tau,\sigma}), \Phi_{t} \rangle_{\Omega} + O(\varepsilon_{\tau} \|q_{k}^{\tau,\sigma}\|_{1, Y_{\tau,\sigma}} \|\Phi\|_{2, \infty}),
\]

which is obtained by acting \(\hat{R}_{X_{\tau}}\) on (4.12) with test function \(\Phi \in C^{2}(\Omega, S^{1})\) and using (4.21). Then, by (4.10), the boundedness of \(\|q^{\tau,\sigma}\|_{1, Y_{\tau,\sigma}}\) and the weak–star convergence of \(\mathcal{R}_{\tau,\sigma}(\mu^{\tau,\sigma})\), taking the limit on both sides of (4.29) gives (4.28) as desired.

Finally, we derive the estimate (4.25). By definition of \(\mathcal{J}_{Q}\), for any \(\varepsilon > 0\), we can choose \(u \in C^{1}(Q, \mathbb{R}^{n \times k})\) and \(W \in C^{1}(Q, S^{n})\) such that
\[
\mathcal{J}_{Q}(\mu) \leq \langle (q, R), (u, W) \rangle_{Q} - \mathcal{J}_{Q}(G, u, W) + \varepsilon,
\]

where \(\mu = (G, q, R)\) is the limit measure. Then, let \(\hat{R}_{X_{\tau,\sigma}}, \hat{R}_{Y_{\tau,\sigma}}\) and \(\hat{R}_{\hat{X}_{\tau,\sigma}}\) be the reconstruction operators defined in (4.14)–(4.16) based on admissible families such that (4.22) holds. By (4.31) and the weak–star convergence of the reconstructed measures (up to a subsequence), we have
\[
\mathcal{J}_{Q}(\mu) \leq \lim_{(\tau,\sigma) \to 0} \left\{ \langle (\hat{R}_{X_{\tau,\sigma}}(q^{\tau,\sigma}), \hat{R}_{Y_{\tau,\sigma}}(R^{\tau,\sigma})), (u, W) \rangle_{Q} + \varepsilon - \sum_{i=1}^{N} \int_{(i-1)\tau}^{i\tau} \mathcal{J}_{\Omega} \left( \hat{R}_{X_{\tau}} \left( \frac{G_{i-1}^{\tau,\sigma} + G_{i}^{\tau,\sigma}}{2} \right), u_{i}, W_{i} \right) \right\}.
\]

Since \(\mathcal{J}_{\Omega}\) is convex w.r.t. \((u, W)\), by Jensen’s inequality and (4.22) we have
\[
\sum_{i=1}^{N} \int_{(i-1)\tau}^{i\tau} \mathcal{J}_{\Omega} \left( \hat{R}_{X_{\tau}} \left( \frac{G_{i-1}^{\tau,\sigma} + G_{i}^{\tau,\sigma}}{2} \right), u_{i}, W_{i} \right) \geq \sum_{i=1}^{N} \tau \mathcal{J}_{\Omega} \left( \hat{R}_{X_{\tau}} \left( \frac{G_{i-1}^{\tau,\sigma} + G_{i}^{\tau,\sigma}}{2} \right), u_{i}, W_{i} \right) \geq \sum_{i=1}^{N} \tau \mathcal{J}_{\Omega} \left( \frac{G_{i}^{\tau,\sigma}}{2}, \hat{R}_{Y_{\tau}}(u_{i}), \hat{R}_{\hat{X}_{\tau}}(W_{i}) \right) - C \|u, W\|_{1, \infty} \|G^{\tau,\sigma}\|_{1, X_{\tau,\sigma}} \varepsilon \sigma.
\]
Combining the above inequality with (4.31), we arrive at

\[ J_q(\mu) \leq \liminf_{\sigma,\epsilon \to 0} \left\{ \sum_{i=1}^{N} T((R_{Y_\epsilon}(q_i, R_{X_\epsilon}(R)), (u_i, W_i))\Omega + \epsilon + C\|u,W\|_{1,\infty}\|G^{r,\sigma}\|_{1,X_{r,\sigma}} \epsilon \sigma \right\} \\
- \liminf_{\tau,\sigma \to 0} \left\{ J_{\tau,\sigma}(\mu^{r,\sigma}) + \epsilon + C\|u,W\|_{1,\infty}\|G^{r,\sigma}\|_{1,X_{r,\sigma}} \epsilon \sigma \right\} . \]

Since \( \epsilon \) is arbitrary and \( \|G^{r,\sigma}\|_{1,X_{r,\sigma}} \) is bounded, the proof is complete. \( \square \)

We next consider the asymptotic upper bound. For this, we introduce the consistent sampling operators as follows.

**Definition 4.12.** Let \( S_{X_\epsilon} \) and \( S_{Y_\epsilon} \) be the bounded linear operators from \( C(\Omega, \mathbb{M}^n) \) to \( (\tilde{X}_\sigma, \|\cdot, X_\epsilon\|) \) and \( (Y_\epsilon, \|\cdot, Y_\epsilon\|) \), respectively, such that \( S_{X_\epsilon} \) maps \( C(\Omega, \mathbb{S}^n) \) to \( X_\sigma \) and \( (\Omega, \mathbb{K}^n) \) to \( Y_\epsilon \). We say that they are families of consistent sampling operators if the following conditions hold:

- \( D \) and \( D_\sigma \) are consistent via \( \{S_{X_\epsilon}\}_{\epsilon \in \Sigma} \) and \( \{S_{Y_\epsilon}\}_{\epsilon \in \Sigma} \) in the sense that there exists \( \{\epsilon_\sigma\}_{\epsilon \in \Sigma} \) with \( \epsilon_\sigma \in \mathbb{R}_+ \) and \( \epsilon_\sigma \) tending to zero as \( \sigma \to 0 \) such that
  \[ \|S_{X_\epsilon}D(\phi) - D_\sigma S_{Y_\epsilon}(\phi), X_\epsilon\| \leq \|\phi\|_{2,\infty}\epsilon_\sigma \quad \forall \phi \in C^2(\Omega, \mathbb{R}^n) ; \]

- \( J_\Omega \) and \( J_\sigma \) are consistent via \( \{S_{X_\epsilon}\}_{\epsilon \in \Sigma} \) and \( \{S_{Y_\epsilon}\}_{\epsilon \in \Sigma} \) in the sense that there exists \( \{\epsilon_\sigma\}_{\epsilon \in \Sigma} \) with \( \epsilon_\sigma \in \mathbb{R}_+ \) and \( \epsilon_\sigma \) tending to zero as \( \sigma \to 0 \) such that
  \[ J_\sigma(S_{X_\epsilon}(G), S_{Y_\epsilon}(q), S_{X_\epsilon}(R)) + \epsilon_\sigma \mu \leq J_\Omega(\mu) + C\||\mu\|_{1,\infty}(\epsilon_\sigma + \|\epsilon_\sigma \|_{\sigma,\epsilon_\sigma}) ; \]

for \( \mu = (G, q, R) \) in a bounded set of \( C^1(\Omega, \mathbb{S}^n \times \mathbb{R}^{n \times k} \times \mathbb{M}^n) \) with \( G \) being uniformly elliptic with constant \( c \), where \( \epsilon_\sigma \mu \in \tilde{X}_\sigma \) is an error term, which depends on \( \mu \) and satisfies \( \|\epsilon_\sigma \mu\|_{\epsilon, \tilde{X}_\sigma} \to 0 \) as \( \sigma \to 0 \).

**Remark 4.13.** In practice, the sampling operators are typically chosen as the dual of reconstructions. Here we choose to introduce them independently mainly because Theorem 4.14 is mathematically independent of Theorem 4.11 and there is no need for these sampling operators satisfying the properties of reconstructions. It is also worth noting that compared to the consistency conditions (4.21) - (4.23) in Definition 4.10 added on the dual objects, here the consistency conditions are added on the primal objects.

**Theorem 4.14 (Asymptotic upper bound).** Let \( \mu \in CE_{\infty}([0,1]; G_0, G_1) \) be a curve connecting measures \( G_0, G_1 \) in \( \mathcal{M}(\Omega, \mathbb{S}_+^n) \). Suppose that the assumptions \( A.1 \) and \( A.2 \) hold, and that there exist families of consistent sampling operators \( \{S_{X_\epsilon}\}_{\epsilon \in \Sigma} \) and \( \{S_{Y_\epsilon}\}_{\epsilon \in \Sigma} \) in the sense of Definition 4.12 and that there exist discrete curves \( \mu^{r,\sigma}_i \in CE^{r,\sigma}([0,1]; G^{r,\sigma}_i, S_{X_\epsilon}(G^{r,\sigma}_i) \}, \) for \( i = 0, 1 \), such that

\[ \limsup_{(\tau,\sigma) \to 0} J_{\tau,\sigma}(\mu^{r,\sigma}_i) = \omega_{\epsilon,i}, \quad (*) \]

with \( \omega_{\epsilon,i} \) tending to zero as \( \epsilon \to 0 \), where \( G^{r,\sigma}_i \) is the \( \epsilon \)-regularization of \( G_i \in \mathcal{M}(\Omega, \mathbb{S}_+^n) \) in Proposition 4.20. Then for any \( \eta > 0 \), we can construct a discrete curve \( \mu^{r,\sigma} \in CE^{r,\sigma}([0,1]; G_0, G_1) \) such that

\[ \limsup_{(\tau,\sigma) \to 0} J_{\tau,\sigma}(\mu^{r,\sigma}) \leq J_Q(\mu) + \eta . \]

For the proof, we need the following lemma.

**Lemma 4.15.** Let \( \mu \in CE_{\infty}([0,1]; G_0, G_1) \) be a curve with a \( C^\infty \)-smooth density \( (G, q, R) \) w.r.t. \( dt \otimes dx \) with \( G \) being uniformly elliptic with constant \( c \). Then we can construct a discrete curve

\[ \mu^{r,\sigma} \in CE^{r,\sigma}([0,1]; S_{X_\epsilon}(G_0), S_{X_\epsilon}(G_1)) ; \]

with the following estimate

\[ J_{\tau,\sigma}(\mu^{r,\sigma}) \leq J_Q(\mu) + C(\sigma + \tau) . \]
Proof. By the smoothness of the density \((G, q, R)\), we define \(\mu^{\tau, \sigma} = (G^{\tau, \sigma}, q^{\tau, \sigma}, R^{\tau, \sigma})\) by \(G_0^{\tau, \sigma} := \mathcal{S}_{X_s}(G_t|_{\tau = 0})\) and 
\(G_N^{\tau, \sigma} = \mathcal{S}_{X_s}(G_t|_{\tau = 1})\) and
\[
\left( G^{\tau, \sigma}, q^{\tau, \sigma}, R^{\tau, \sigma} \right)_i = \left( \mathcal{S}_{X_s}(G_t), \mathcal{S}_{Y_s}(\int_{(i-1)\tau}^{i\tau} q_t), \mathcal{S}_{X_s}(\int_{(i-1)\tau}^{i\tau} R_t) + e_\sigma(D, \int_{(i-1)\tau}^{i\tau} q_t) \right), \quad i = 1, \ldots, N, \tag{4.36}
\]
where \(e_\sigma(D, \int_{(i-1)\tau}^{i\tau} q_t) := D_\sigma \mathcal{S}_{Y_s}(\int_{(i-1)\tau}^{i\tau} q_t) - \mathcal{S}_{X_s}(D \int_{(i-1)\tau}^{i\tau} q_t) \in \mathcal{X}_s\) is the consistency error satisfying the estimate
\[
\left\| e_\sigma(D, \int_{(i-1)\tau}^{i\tau} q_t) \right\|_{\mathcal{X}_s, \mathcal{H}} \leq \| q \|_{L^2} \varepsilon_s, \tag{4.37}
\]
by \((1.32)\). We further observe
\[
\left\langle \mathcal{P}_{X_s} \mathcal{S}_{X_s}(\int_{(i-1)\tau}^{i\tau} R_t), M^\sigma \right\rangle_{\mathcal{X}_s} = \left\langle \int_{(i-1)\tau}^{i\tau} R_t, \mathcal{S}_{X_s}(M^\sigma) \right\rangle_{\Omega} = \left\langle \mathcal{S}_{X_s}(\int_{(i-1)\tau}^{i\tau} R_{sym}^\sigma), M^\sigma \right\rangle_{\mathcal{X}_s} \quad \forall M^\sigma \in \mathcal{X}_s, \tag{4.38}
\]
since \(\mathcal{S}_{X_s}^*\) maps from \(\mathcal{X}_s\) to \(\mathcal{M}(\Omega, \mathcal{S}_{\mathcal{H}})\) by Definition \((1.12)\). Then we have \(\mu^{\tau, \sigma} \in \mathcal{CE}^{\tau, \sigma}([0, 1]; \mathcal{S}_{X_s}(G_0), \mathcal{S}_{X_s}(G_1))\) by the definition \((4.36)\) and the relation \((4.38)\). We proceed to estimate the energy functional. By \((1.33)\), we have that for \(k = 1, \ldots, N\), there holds
\[
\mathcal{J}_\sigma \left( \frac{G_{k-1}^{\tau, \sigma} + G_k^{\tau, \sigma}}{2}, q_k^{\tau, \sigma}, R_k^{\tau, \sigma} \right) \leq \mathcal{J}_\Omega \left( \frac{G_{k-1}^{\tau} + G_k^{\tau}}{2}, \int_{(k-1)\tau}^{k\tau} q_t, \int_{(k-1)\tau}^{k\tau} R_t \right) + C_{\|\mu\|_{L^2}_\infty} \left( \varepsilon_s + e_\sigma(D, \int_{(k-1)\tau}^{k\tau} q_t) \right).
\]
Then by \((4.37)\), definition of \(\mathcal{J}_{\tau, \sigma}(\mu^{\tau, \sigma}) \tag{4.11}\) and a direct estimation, we have
\[
\mathcal{J}_{\tau, \sigma}(\mu^{\tau, \sigma}) \leq \mathcal{J}_{\Omega}(G_{i-1}^{\tau} + G_i^{\tau}) \leq \sum_{i=1}^{N} \mathcal{J}_{\Omega} \left( G_{i-1}^{\tau} + G_{i}^{\tau} \right) + C_{\|\mu\|_{L^2}_\infty} \left( \varepsilon_s + \varepsilon_\tau \right),
\]
since \(\mathcal{J}_{\Omega}(\mu_t)\) is a smooth function of variable \(t\) and \(G\) is uniformly elliptic. The proof is complete. \(\square\)

**Proof of Theorem \((4.14)\)** We fix a small \(t_0^* \in (0, 1)\) and denote \(t_0 = n\tau\) for small enough \(\tau\), where \(n\) is uniquely determined by \(n\tau \leq t_0^* < (n+1)\tau\). For \(i = 0, 1\), let \(G_i^{\tau, \sigma}\) be the \(\varepsilon\)-regularization of the measure \(G_i\) and \(\mu^{\tau, \sigma, (i)}\) be the discrete curve given by Assumption \(\mathbb{A}\) which satisfies
\[
\mathcal{J}_{\tau, \sigma}(\mu^{\tau, \sigma, (i)}) \leq \omega_\varepsilon + \varepsilon_s + \frac{\tau}{t_0^*}, \tag{4.39}
\]
for \((\tilde{\omega}, \tilde{\sigma}) \in \Sigma_N\) with \(|(\tilde{\omega}, \tilde{\sigma})|\) small enough, by \((4.3)\). By Lemma \((4.13)\) we can construct the discrete curve \(\mu^{\tau, \sigma, (2)} \in \mathcal{CE}^{\tau, \sigma}_{\infty}([0, 1]; \mathcal{S}_{X_s}(G_0^\tau), \mathcal{S}_{X_s}(G_1^\tau))\) with the following estimate
\[
\mathcal{J}_{\tau, \sigma}(\mu^{\tau, \sigma, (2)}) \leq \mathcal{J}_{Q}(\mu^\tau) + C \left( \varepsilon_s + \frac{\tau}{1 - 2t_0^*} \right). \tag{4.40}
\]
Here the constant \(C\) is independent of \(\sigma, \tau\) and \(t_0\). We now glue the above discrete curves and define \(\mu^{\tau, \sigma}\) as follows:
\[
\mu^{\tau, \sigma} = \begin{cases}
\left( G_i^{\tau, \sigma, (0)} \frac{1}{t_0^*} q_i^{\tau, \sigma, (0)}, \frac{1}{t_0^*} R_i^{\tau, \sigma, (0)} \right), & i = 1, \ldots, t_0^*, \\
\left( G_i^{\tau, \sigma, (1)} \frac{1}{t_0^*} q_i^{\tau, \sigma, (1)}, \frac{1}{t_0^*} R_i^{\tau, \sigma, (1)} \right), & i = \frac{t_0^*}{\tau} + 1, \ldots, 1, \\
\left( G_i^{\tau, \sigma, (2)} - 2G_{i-1}^{\tau, \sigma, (2)} \frac{1}{1 - 2t_0^*} q_i^{\tau, \sigma, (2)} - 2G_{i-1}^{\tau, \sigma, (2)} \frac{1}{1 - 2t_0^*} R_i^{\tau, \sigma, (2)} \right), & i = \frac{t_0^*}{\tau} + 1, \ldots, 1.
\end{cases}
\]
It is clear that \(\mu^{\tau, \sigma} \in \mathcal{CE}^{\tau, \sigma}([0, 1]; G_0^\tau, G_1^\tau)\). By definition of \(\mathcal{J}_{\tau, \sigma}\) in \((4.11)\) and estimates \((4.39)\) and \((4.40)\), we have
\[
\mathcal{J}_{\tau, \sigma}(\mu^{\tau, \sigma}) \leq \frac{1}{t_0^*} \mathcal{J}_{\tau, \sigma}(\mu^{\tau, \sigma, (0)}) + \frac{1}{1 - 2t_0^*} \mathcal{J}_{\tau, \sigma}(\mu^{\tau, \sigma, (2)}) + \frac{1}{t_0^*} \mathcal{J}_{\tau, \sigma}(\mu^{\tau, \sigma, (1)}) \leq \frac{2}{t_0^*} \left( \omega_\varepsilon + \varepsilon_s + \frac{\tau}{t_0^*} + \frac{1}{1 - 2t_0^*} \left( \mathcal{J}(\mu^\tau) + C \left( \varepsilon_s + \frac{\tau}{1 - 2t_0^*} \right) \right) \right).
\]
It immediately follows that
\[
\limsup_{(\tau, \sigma) \to 0} J_{\tau, \sigma}(\mu^{\tau, \sigma}) \leq \frac{2}{t_0^2} \omega_\varepsilon + \frac{J(\mu^*) - J(\mu) + 2t_0^* J(\mu)}{1 - 2t_0^*} + J(\mu),
\]
(4.41)
which holds for the arbitrary (small) \(\varepsilon\) and \(t_0^*\). For any given \(0 < \eta < 1\), we can first choose \(t_0^*\) small enough such that
\[
\frac{2t_0^*}{1 - 2t_0^*} J(\mu) \leq \frac{\eta}{3},
\]
and then, by Proposition 3.28 choose \(\varepsilon\) small enough such that
\[
\frac{2}{t_0^2} \omega_\varepsilon \leq \frac{\eta}{3}, \quad \frac{J(\mu^*) - J(\mu)}{1 - 2t_0^*} \leq \frac{\eta}{3}.
\]
Hence, (4.41) readily gives (4.34) as desired. 

We are now well prepared to the following abstract convergence result which guarantees the existence of a subsequence of discrete minimizers \(\mu^{\tau, \sigma}\) to \(\{P_{\tau, \sigma}\}\) to a minimizer \(\mu\) to (2).

**Theorem 4.16.** Let \(\hat{\mu}_{*}^{\tau, \sigma}\) be a minimizer to \(\{\hat{P}_{\tau, \sigma}\}\) for \((\tau, \sigma) \in \Sigma_n\). Suppose that the assumptions 4.14, 4.13 and 4.1 hold; and that there exists families of consistent spatial reconstructions \(\{[\hat{R}_{X_\sigma}]_{\sigma \in \Sigma}\} \in CE\) and families of consistent sampling operators \(\{S_{X_\sigma}\}_{\sigma \in \Sigma}\) in the sense of Definition 4.16 and families of consistent sampling operators \(\{S_{Y_\sigma}\}_{\sigma \in \Sigma}\) in the sense of Definition 4.12. Then, up to the extraction of a subsequence, \([\hat{R}_{\tau, \sigma}]((\mu_{*}^{\tau, \sigma})^*)\) weak-star converges to a minimizer \(\mu_*\) to (2), where \([\{[\hat{R}_{\tau, \sigma}]\}_{(\tau, \sigma) \in \Sigma_n}\) is defined as in 4.20. Moreover, there holds
\[
\lim_{(\tau, \sigma) \to 0} J_{\tau, \sigma}(\mu_{*}^{\tau, \sigma}) = J_{Q}(\mu_*).
\]

**Proof.** For any given \(\mu \in CE_{\infty}([0, 1]; G_0, G_1)\) and \(\varepsilon > 0\), by Theorem 4.14 we can construct \(\mu^{\tau, \sigma}\) satisfying
\[
\limsup_{(\tau, \sigma) \to 0} J_{\tau, \sigma}(\mu^{\tau, \sigma}) \leq \limsup_{(\tau, \sigma) \to 0} J_{\tau, \sigma}(\mu^{\tau, \sigma}) \leq J_{Q}(\mu) + \varepsilon.
\]
It follows that \(\sup_{(\tau, \sigma) \in \Sigma_n} J_{\tau, \sigma}(\mu_{*}^{\tau, \sigma}) < +\infty\). Then by Theorem 4.11 there exists a subsequence of \([\hat{R}_{\tau, \sigma}]((\mu_{*}^{\tau, \sigma})^*)\) (still indexed by \(\tau, \sigma\)) weak-star converging to a \(\mu_* \in CE([0, 1]; G_0, G_1)\). Moreover, it holds that
\[
J_{Q}(\mu^*) \leq \liminf_{(\tau, \sigma) \to 0} J_{\tau, \sigma}(\mu_{*}^{\tau, \sigma}) \leq \limsup_{(\tau, \sigma) \to 0} J_{\tau, \sigma}(\mu_{*}^{\tau, \sigma}) \leq J_{Q}(\mu) + \varepsilon.
\]
Since \(\varepsilon\) is arbitrary, we readily have from the above formula that
\[
J_{Q}(\mu^*) = \lim_{(\tau, \sigma) \to 0} J_{\tau, \sigma}(\mu_{*}^{\tau, \sigma}) \leq J_{Q}(\mu).
\]
for any \(\mu \in CE_{\infty}([0, 1]; G_0, G_1)\). Hence the proof is complete. 

An importance of the above abstract framework is that it actually suggests us the following procedures for designing a concrete convergent scheme, which is the main task of the next section.

1. Given a family of meshes for the computation domain with the mesh size indexed by \(\sigma\) and converging to zero. Construct discrete approximations \((\hat{X}_\sigma, \hat{R}_{X_\sigma})\) and \((Y_\sigma, \hat{R}_{Y_\sigma})\) of \(M(\Omega, \mathbb{M}^n)\) and \(M(\Omega, \mathbb{R}^{n \times k})\), respectively, in the sense of Definition 4.11 where \(\hat{R}_{X_\sigma}\) and \(\hat{R}_{Y_\sigma}\) are the quasi–isometric reconstructions satisfying mapping properties in Definition 4.5.

2. Construct the discrete derivations \(D_\sigma\) and (if necessary) another family of reconstructions \(\hat{R}_{Y_\sigma}\) on \(Y_\sigma\) equivalent to \(\hat{R}_{Y_\sigma}\), and two families of sampling operators \(S_{X_\sigma}\) and \(S_{Y_\sigma}\) (typically, chosen as the adjoint of reconstructions) on \(\hat{X}_\sigma\) and \(Y_\sigma\) respectively with mapping properties in Definition 4.12 such that (4.21) and (4.32) hold.

3. Construct the discrete functional \(J_{\sigma}\) with properties in Definition 4.3 such that (4.22), (4.23) and (4.33) hold (if necessary, we can construct more families of reconstructions to meet the requirements while the sampling operators must be chosen as the second step).
4.3 A concrete scheme

In this section, we apply the abstract framework proposed in Section 4.2 to design a convergent scheme for the problem \((P)\). We assume that Assumptions [A.1] and [A.3] hold, while Assumptions [A.1] and [A.2] have been required in Theorem 4.14 while [A.3] is needed in the verification of \((P)\).

We will limit our discussion to a family of conforming and shape regular triangulations \((T^\sigma, V^\sigma)\) of \(\Omega \subset \mathbb{R}^d\) with \(\sigma\) being the mesh size of a triangulation: \(\sigma = \max_{K \in T^\sigma} \text{diam}(K)\), where \(T^\sigma\) denotes the set of simplices and \(V^\sigma\) is the set of vertices. We further introduce the following notations: \(|K|\) denotes the volume of a simplex \(K\) in \(T^\sigma\); \(T^\sigma_v\) is the set of simplices that contains \(v\) as a vertex, while \(V^\sigma_K\) is the set of vertices of a simplex \(K\). Moreover, we denote by \(\delta_v\) the nodal basis function associated with the vertex \(v\), that is, \(\delta_v\) is piecewise linear (w.r.t. the triangulation) such that \(\delta_v(v) = 1\) and \(\delta_v(v') = 0\) for any \(v' \in V^\sigma\) with \(v' \neq v\). By definition, we readily have the following useful relations:

\[
|T_v| := \int_\Omega \delta_v = \frac{1}{d+1} \sum_{K \in T^\sigma_v} |K| \quad \text{and} \quad \sum_{v \in V^\sigma_K} |T_v| = |\Omega|.
\]

(4.42)

For our purpose, let us follow the procedures proposed at the end of Section 4.2. We first introduce some modules for the approximation of \((P)\) with basic properties required by Definition 4.5. We start with the construction of the discrete approximations of \(M(\Omega, M^n)\) and \(M(\Omega, \mathbb{R}^{n \times k})\) with the quasi–isometric reconstructions. Let the discrete space \(\bar{X}_\sigma\) be defined by \(\bar{X}_\sigma = \{(R_v)_{v \in V^\sigma}; \ R_v \in M^n\}\) equipped with the inner product \(\langle \cdot, \cdot \rangle_{V^\sigma}\) and the “discrete \(L^1\)” norm \(\|\cdot\|_{1,V^\sigma}\): for \(A, B \in \bar{X}_\sigma\),

\[
(A, B)_{V^\sigma} = \sum_{v \in V^\sigma} |T_v| A_v \cdot B_v, \quad \|A\|_{1,V^\sigma} = \sum_{v \in V^\sigma} |A_v||T_v|.
\]

while the space \(Y_\sigma\) be given by \(Y_\sigma := \{(q(K))_{K \in T^\sigma}; \ q_K \in \mathbb{R}^{n \times k}\}\) equipped with the inner product \(\langle \cdot, \cdot \rangle_{T^\sigma}\) and the “discrete \(L^\infty\)” norm \(\|\cdot\|_{1,T^\sigma}\): for \(p, q \in Y_\sigma\),

\[
(p, q)_{T^\sigma} := \sum_{K \in T^\sigma} |K| p_K \cdot q_K, \quad \|p\|_{1,T^\sigma} := \sum_{K \in T^\sigma} |K| |p_K|,
\]

(4.43)

The dual norm on \(\bar{X}_\sigma\) associated with \(\|\cdot\|_{1,V^\sigma}\) w.r.t. the inner product \(\langle \cdot, \cdot \rangle_{V^\sigma}\), denoted by \(\|\cdot\|_{*,V^\sigma}\), can be computed as follows: for \(B \in \bar{X}_\sigma\),

\[
\|B\|_{*,V^\sigma} := \sup_{\|A\|_{1,V^\sigma} \leq 1} |(A, B)_{V^\sigma}| = \max_{v \in V^\sigma} |B_v|.
\]

Similarly, we can introduce the dual norm on \(Y_\sigma\). The subspace \(X_\sigma\) and the cone \(X_\sigma^+\) of \(\bar{X}_\sigma\) are defined by

\[X_\sigma = \{(G_v)_{v \in V^\sigma}; \ G_v \in \mathbb{S}_n\} \subset \bar{X}_\sigma := \{(S_v)_{v \in V^\sigma}; S_v \in \mathbb{S}_n\}.\]

We proceed to introduce the quasi–isometric reconstruction operators \(\hat{R}_{X_\sigma}\) on \(\bar{X}_\sigma\), and \(\hat{R}_{Y_\sigma}\) on \(Y_\sigma\) by

\[
\hat{R}_{X_\sigma}(R) = \sum_{v \in V^\sigma} R_v|T_v| \delta_v \quad \text{for} \ R \in \bar{X}_\sigma, \quad \hat{R}_{Y_\sigma}(q) = \sum_{K \in T^\sigma} q_K \chi_K \quad \text{for} \ q \in Y_\sigma
\]

(4.44)

where \(\chi_K\) is the characteristic function of a set \(K \in T^\sigma\), which is naturally identified with the measure \(dx|_K; \ |T^\sigma_v|\) is defined by (4.42) and \(\delta_v\) is the Dirac measure of mass 1 supported at the vertex \(v\). By definition, it follows that

\[
\|\hat{R}_{X_\sigma}(R)\| = \|R\|_{1,V^\sigma} \quad \text{for} \ R \in \bar{X}_\sigma, \quad \text{and} \quad \|\hat{R}_{Y_\sigma}(q)\| = \|q\|_{1,T^\sigma} \quad \text{for} \ q \in Y_\sigma.
\]

Moreover, it is clear that the objects defined above satisfy the conditions in Definition 4.5.

We next define the discrete differential operator \(D_\sigma\) and consider the consistency conditions (4.21) and (4.32). For this, we need to first introduce another family of reconstructions \(R_{X_\sigma}^l\) on \(X_\sigma\) equivalent to \(\hat{R}_{X_\sigma}\). Motivated by the finite element theory, we define the piecewise linear reconstructions (w.r.t. the triangulation) for \(R \in \bar{X}_\sigma\) by

\[
R_{X_\sigma}^l(R) = \sum_{v \in V^\sigma} R_v \hat{\delta}_v,
\]

with the norm estimate \(\|R_{X_\sigma}^l(R)\| \leq \sum_{v \in V^\sigma} |R_v||T_v| = \|R\|_{1,V^\sigma}\), where \(R_v \hat{\delta}_v\) is defined component-wisely. Here again we identify an integrable field with a vector–valued Radon measure. The equivalence between \(R_{X_\sigma}^l\) and \(\hat{R}_{X_\sigma}\) follows from the observation:

\[
\left((R_{X_\sigma}^l)\circ (\Phi)\right)_v = \frac{1}{|T_v|} \int_\Omega \Phi \hat{\delta}_v = (\hat{R}_{X_\sigma}\circ (\Phi))_v + O(\|\nabla \Phi\|_\infty \sigma) \quad \forall \Phi \in C^1(\Omega, \mathbb{S}_n),
\]

(4.45)
which is easily verified by definition (4.3) and Taylor expansion. Then we can introduce $D^*_\sigma$ by
\[ D^*_\sigma G = \tilde{R}_{X_\sigma}^* D^* R_{X_\sigma}^l(G) \quad \text{for} \ G \in X_\sigma, \]
noting that $D^*$ is a first order linear differential operator and $R_{X_\sigma}^l(G)$ has the $H^1$–regularity. Its adjoint operator is the desired discrete derivation defined via (4.16) for $G \in X_\sigma$, $q \in Y_\sigma$,
\[ (G, D_{\sigma} q)_{Y_\sigma} = -(D^*_\sigma G, q)_{T^\sigma} = -(\tilde{R}_{X_\sigma}^* D^* R_{X_\sigma}^l(G), q)_{T^\sigma}. \tag{4.46} \]
We shall simply take the sampling operators as the adjoint of the reconstructions: $S_{X_\sigma} := R_{X_\sigma}^\star$ and $S_{Y_\sigma} := \tilde{R}_{Y_\sigma}^\star$.

Instead of verifying these conditions (4.21) and (4.22) here, we proceed to introduce the discrete functional $J_\sigma$ as:
for $\mu^\sigma := (G^\sigma, q^\sigma, R^\sigma) \in X_\sigma \times Y_\sigma \times \bar{X}_\sigma$,
\[ J_\sigma(\mu^\sigma) := \frac{1}{2} \sum_{K \in T_\sigma} |K| q^\sigma_K \cdot \left( \sum_{v \in V^\sigma_K} G^\sigma_v / (d + 1) \right)^{1/2} q^\sigma_K + \frac{1}{2} \sum_{v \in V^\sigma} |T_v| R^\sigma_v \cdot (G^\sigma_v)^{1/2} R^\sigma_v \tag{4.47} \]
which completes the construction of the basic modules. Then the fully discretized transport problem $P_{\sigma,\gamma}$ readily follows from Definition 4.8. Moreover, by Proposition 4.3 we have that $J_\sigma$, as the composition of a proper, l.s.c., convex function and a linear function, is still proper, l.s.c., convex.

We have now constructed the modules satisfying the conditions in Definition 4.5. The following theorem verifies the set of consistency conditions in Definitions 4.10 and 4.12.

**Lemma 4.17.** \{\{\tilde{R}_{X_\sigma}^\star \}_{\sigma \in \Sigma} and \{\tilde{R}_{Y_\sigma}^\star \}_{\sigma \in \Sigma} are the families of consistent reconstruction operators in the sense of Definition 4.10, while \{S_{X_\sigma} \}_{\sigma \in \Sigma} and \{S_{Y_\sigma} \}_{\sigma \in \Sigma} are the families of consistent sampling operators in the sense of Definition 4.12.**

**Proof.** We first consider the consistency condition (4.21) with quasi–isometric reconstruction operators $\tilde{R}_{X_\sigma}$ and $\tilde{R}_{Y_\sigma}$. By definitions (4.14) and (4.16), we have, for $q^\sigma \in Y_\sigma$ and $\Phi \in C^2(\Omega, \mathbb{R}^n)$,
\[ \langle \tilde{R}_{Y_\sigma}(q^\sigma), D^* \Phi \rangle_{\Omega} + \langle \tilde{R}_{X_\sigma}(D_{\sigma} q^\sigma), \Phi \rangle_{\Omega} = \sum_{K \in T_\sigma} |K| q^\sigma_K \cdot \int_K D^* \Phi - \sum_{K \in T_\sigma} |K| q^\sigma_K \cdot \sum_{v \in V^\sigma_K} D^*(\hat{\phi}_v | K \Phi(v)). \tag{4.48} \]
We recall that $D^*$ can be decomposed as $D^* = D_0^* + D_1^*$, where $D_0^* \in \mathcal{L}(\mathbb{S}^n, \mathbb{R}^{n \times k})$ and $D^*$ is a homogeneous first order differential operator. Since $\Phi \in C^2$–smooth and $\hat{\phi}_v | K = 1$, a simple calculation by Taylor expansion gives
\[ \left| \int_K D_0^* \Phi - \sum_{v \in V^\sigma_K} D_0^*(\hat{\phi}_v | K \Phi(v)) \right| = \left| \sum_{v \in V^\sigma_K} \hat{\phi}_v | K D_0^*(\Phi(v) - \int_K \Phi) \right| \lesssim |D_0^*| ||\nabla \Phi||_{\infty^\sigma}. \]
On the other hand, we estimate
\[ \left| \int_K D_1^* \Phi - \sum_{v \in V^\sigma_K} D_1^*(\hat{\phi}_v | K \Phi(v)) \right| \lesssim ||\Phi||_{2, \infty^\sigma}, \]
by linearity the interpolation results from finite element theory.

Next we check (4.32). It suffices to estimate
\[ \langle \tilde{R}_{X_\sigma}(A^\sigma), D \varphi \rangle_{\Omega} - \langle A^\sigma, D_{\sigma} \tilde{R}_{Y_\sigma}(\varphi) \rangle_{Y_\sigma} \tag{4.49} \]
with test vectors $A^\sigma \in X_\sigma$. Since $R_{X_\sigma}^l$ is equivalent to $\tilde{R}_{X_\sigma}$ (cf. (4.43)), we can replace $\tilde{R}_{X_\sigma}$ in (4.49) by $R_{X_\sigma}^l$ with an error term bounded by $\sigma ||A_{\sigma}||_{1, X_\sigma} ||\varphi||_{2, \infty^\sigma}$. Noting that $\varphi$ satisfies the boundary condition, we can proceed to calculate
\[ \langle R_{X_\sigma}^l(A^\sigma), D \varphi \rangle_{\Omega} - \langle A^\sigma, D_{\sigma} \tilde{R}_{Y_\sigma}(\varphi) \rangle_{Y_\sigma} = -(D^* R_{X_\sigma}^l(A^\sigma), \varphi)_{\Omega} + \langle \tilde{R}_{Y_\sigma}^* D^* R_{X_\sigma}^l(A^\sigma), \tilde{R}_{Y_\sigma}^*(\varphi) \rangle_{Y_\sigma} \tag{4.50} \]
by definition of $D_{\sigma}$ (4.16). By decomposing $D^* = D_0^* + D_1^*$ as above, we have
\[ -(D_0^* R_{X_\sigma}^l(A^\sigma), \varphi)_{\Omega} + \langle \tilde{R}_{Y_\sigma}^* D_1^* R_{X_\sigma}^l(A^\sigma), \tilde{R}_{Y_\sigma}^*(\varphi) \rangle_{Y_\sigma} = 0, \tag{4.51} \]
since $D^*_1 R^1_{X,s}(A^\sigma)$ is a piecewise constant function, while a direct estimate by definition yields
\[
- \left( D^*_0 R^1_{X,s}(A^\sigma), \varphi \right)_\Omega + \left( \tilde{R}^*_V D^*_0 R^1_{X,s}(A^\sigma), \tilde{R}^*_V(\varphi) \right)_{V_s}
\leq \sum_{K \in T^*_\sigma} \int_K |D^*_0 R^1_{X,s}(A^\sigma) \cdot (\varphi - \bar{\varphi})|
\leq |D^*_0|_F \| \nabla \varphi \|_{\infty, \sigma} \sum_{K \in T^*_\sigma} \int_K |R^1_{X,s}(A^\sigma)|
\leq |D^*_0|_F \| \nabla \varphi \|_{\infty, \sigma} \| A^\sigma \|_{1, X_s}.
\]
(4.52)

Combining these two facts with (4.51) and (4.50), we have (4.52) as desired.

We now consider the consistency conditions (4.24) and (4.23) for the conjugate functional $J^*_\sigma$ of $J_\sigma$. For this, we first compute $J^*_\sigma$ by definitions (2.24) and (4.17): for $(G^\sigma, u^\sigma, W^\sigma) \in X_\sigma \times Y_\sigma \times \hat{X}_\sigma$,
\[
J^*_\sigma(G^\sigma, u^\sigma, W^\sigma) = \frac{1}{2} \sum_{K \in T^*} |K| |u^\sigma_K| \left( \sum_{v \in V^*_K} G^\sigma_v/(d+1) \right) u^\sigma_K + \frac{1}{2} \sum_{v \in V^*} |T_v| W^\sigma_v \cdot G^\sigma_v W^\sigma_v.
\]
To estimate
\[
J^*_\sigma(G^\sigma, \hat{R}^*_V, \tilde{R}^*_V(W)) = \frac{1}{2} \sum_{K \in T^*} |K| \int_K u \left( \sum_{v \in V^*_K} G^\sigma_v/(d+1) \right) \int_K u + \frac{1}{2} \sum_{v \in V^*} |T_v| W^\sigma v \cdot G^\sigma_v W^\sigma v,
\]
we have
\[
\frac{1}{2} \sum_{K \in T^*} |K| \left( \sum_{v \in V^*_K} G^\sigma_v/(d+1) \right) \left( \int_K u \int_K u^T \right)
= \frac{1}{2} \sum_{v \in V^*} G^\sigma_v \cdot \sum_{K \in T^*} |K|/(d+1) \left( \int_K u \int_K u^T \right)
\leq \frac{1}{2} \sum_{v \in V^*} G^\sigma_v \cdot |T_v| |u(v) u(v)^T| + C_{\| u \|_{1, \infty}} G^\sigma_v \| G^\sigma_v \|_{1, V^*_\sigma},
\]
by (4.42) and the $C^1$–smoothness of $u$. Then (4.53) readily gives
\[
J^*_\sigma(G^\sigma, \hat{R}^*_V, \tilde{R}^*_V(W)) \leq \frac{1}{2} \sum_{v \in V^*} |T_v| G^\sigma_v \cdot (u(v) u(v)^T + W(v) W(v)^T) + C_{\| u \|_{1, \infty}} \| G^\sigma_v \|_{1, V^*_\sigma},
\]
where the first term in the right–hand side is nothing else than $J^*_\Omega(\hat{R}_{X,s}, u, W)$ (cf. (3.19)). The estimate (4.28) is a direct consequence of the expression (4.53) by definition.

Finally, we consider $L^*_\Omega$. Suppose that $\mu = (G, q, R)$ is in a bounded set of $C^1(\Omega, X)$ with $G$ being uniformly elliptic with constant $\eta$. We first compute
\[
J_\sigma(S_{X,s}(G), S_{Y,s}(q), S_{X,s}(R) + \varepsilon_{\sigma, \mu})
= \frac{1}{2} \sum_{K \in T^*} |K| \int_K q \cdot \left( \sum_{v \in V^*_K} G(v)/(d+1) \right) \int_K q + \frac{1}{2} \sum_{v \in V^*} |T_v| (R(v) + \varepsilon_{\sigma, \mu}) \cdot G(v)^\dagger (R(v) + \varepsilon_{\sigma, \mu})
= \frac{1}{2} \sum_{K \in T^*} |K| (q(v) + O(\| q \|_{1, \infty}) \cdot \left( \sum_{v \in V^*_K} G(v)/(d+1) \right) (q(v) + O(\| q \|_{1, \infty}))
+ \frac{1}{2} \sum_{v \in V^*} |T_v| (R(v) + \varepsilon_{\sigma, \mu}) \cdot G(v)^\dagger (R(v) + \varepsilon_{\sigma, \mu})
= \frac{1}{2} \sum_{K \in T^*} |K| \left( \sum_{v \in V^*_K} G(v)/(d+1) \right) (q(v) q(v)^T + O(\| q \|_{1, \infty}))
+ \frac{1}{2} \sum_{v \in V^*} |T_v| (R(v) + \varepsilon_{\sigma, \mu}) \cdot G(v)^\dagger (R(v) + \varepsilon_{\sigma, \mu}),
\]
(4.54)
by using $\int_K q = q(v) + O(\| q \|_{1, \infty})$ due to $q \in C^1(\Omega, \mathbb{R}^{n \times k})$. Since $G$ is uniformly elliptic with consistent $\eta$, we have that the pseudoinverse is in fact the inverse and there holds
\[
\left( \sum_{v \in V^*_K} G(v)/(d+1) \right)^{-1} = \sum_{v \in V^*_K} G(v)^{-1} / (d+1) + O(\eta^{-1}) \| q \|_{1, \infty}.
\]
when $\sigma$ is small enough. Then it follows that
\[
\frac{1}{2} \sum_{v \in V^\sigma} |K| \left( \sum_{v \in V^\sigma} G(v)/(d+1)^{1/2} \cdot (q(v)q(v)^T + O(\|q\|^{2,\infty}_\sigma)) \right)
\leq \frac{1}{2} \sum_{v \in V^\sigma} |T_v|q(v) \cdot G(v)^{-1}q(v) + C\eta^{-1}\|G\|_{1,\infty}\|q\|^{2,\infty}_\sigma.
\] (4.55)

Similarly, by the assumption for $e_{\sigma,\mu}$, we have
\[
\frac{1}{2} \sum_{v \in V^\sigma} |T_v|((R(v) + e_{\sigma,\mu}) \cdot G(v)^{1/2}R(v) + e_{\sigma,\mu}) \leq \frac{1}{2} \sum_{v \in V^\sigma} |T_v|R(v) \cdot G(v)^{1/2}R(v) + C\eta^{-1}\|R\|_{\infty}\|e_{\sigma,\mu}\|_{*,V^\sigma},
\] (4.56)
where $\|\cdot\|_{*,V^\sigma}$ is the dual norm on $\tilde{X}_\sigma$ defined in (4.43). With the help of (4.56) and (4.54) implies
\[
\mathcal{J}_\mathcal{R}(S_{X_\sigma}(G), S_{V^\sigma}(q), S_{X_\sigma}(R) + e_{\sigma,\mu}) \leq \frac{1}{2} \sum_{v \in V^\sigma} |T_v|G(v)^{-1} \cdot (q(v)q(v)^T + R(v)R(v)^T) + C\mu(\sigma + \|e_{\sigma,\mu}\|_{*,V^\sigma}).
\] (4.57)

Since $G(v)^{-1} \cdot (q(v)q(v)^T + R(v)R(v)^T)$ is $C^1$–smooth and its gradient is bounded by a constant depending on $\|\mu\|_{1,\infty}$ and $\eta$, we can conclude $\mathcal{J}_{\mathcal{R}}$ by (4.57).

\[\square\]

To complete, we need to show that the condition \(\square\) in Theorem 4.14 holds. For this, we have the following lemma, which is a discrete analogue of Lemma 4.17.

**Lemma 4.18.** For any $G^\sigma_0, G^\sigma_1 \in X_\sigma$, there exists a discrete curve $\mu^{\tau,\sigma} \in \mathcal{C} \mathcal{E}_{\tau,\sigma}([0, 1]; G^\sigma_0, G^\sigma_1)$ satisfying
\[
\mathcal{J}_{\tau,\sigma}(\mu^{\tau,\sigma}) \leq \frac{1}{2} \sum_{v \in V^\sigma} |T_v||\sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}}|^2.
\]

**Proof.** We mimic the construction of the curve in Lemma 4.17. We define $G^{\tau,\sigma}_{i,v} \in X_{\tau,\sigma}$ by
\[
G^{\tau,\sigma}_{i,v} := \left( \frac{1}{2} \frac{G^\sigma_0 + \sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}}}{2} \right)^2, \quad i = 0, 1, \ldots, N,
\]
and
\[
R^{\tau,\sigma}_{i,v} = \{(2i-1)\tau(\sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}}) + 2\sqrt{G^\sigma_{1,v}} \left( \sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}} \right), \quad i = 1, \ldots, N,
\]
where $G^\sigma_0 = (G^\sigma_{0,v})_{v \in V^\sigma} \in X_\sigma$ and $G^\sigma_1 = (G^\sigma_{1,v})_{v \in V^\sigma} \in X_\sigma$ are the given discrete distributions. It is easy to check that $\mu^{\tau,\sigma} := (G^{\tau,\sigma}, 0, R^{\tau,\sigma})$ satisfies the discrete continuity equation, by noting
\[
(A + i\tau(B - A))^2 - (A + (i - 1)\tau(B - A))^2 = (2i - 1)\tau^2(B - A)(B - A) + \tau A(B - A) + \tau(B - A)A \quad \text{for } A, B \in \mathbb{M}^n.
\]
We next estimate the discrete action functional defined by (4.11) and (4.47):
\[
\mathcal{J}_{\tau,\sigma}(\mu^{\tau,\sigma}) = \frac{\tau}{2} \sum_{i=1}^N \sum_{v \in V^\sigma} |T_v|R^{\tau,\sigma}_{i,v} \cdot \left( \frac{G^{\tau,\sigma}_{i,v} + G^{\tau,\sigma}_{i-1,v}}{2} \right)^4 R^{\tau,\sigma}_{i,v}.
\] (4.58)

For this, we first compute by definition
\[
\frac{1}{2}(G^{\tau,\sigma}_{i,v} + G^{\tau,\sigma}_{i-1,v}) = \left( \frac{1}{2} \frac{1}{2} \tau(\sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}}) + \sqrt{\sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}}} \right)^2 + \frac{1}{4} \tau^2 \left( \sqrt{\sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}}} \right)^2.
\] (4.59)

Note that for $i = 1, \ldots, N$, there holds
\[
\text{ran} \left( \left( \sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}} \right)^2 \right) \subset \text{ran} \left( \left( \frac{1}{2} \frac{1}{2} \tau(\sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}}) + \sqrt{\sqrt{G^\sigma_{1,v}} - \sqrt{G^\sigma_{0,v}}} \right)^2 \right) = \text{ran}(G^{\tau,\sigma}_{i,v} + G^{\tau,\sigma}_{i-1,v}),
\]
and
\[
\text{ran}(G^{\tau,\sigma}_{i,v} + G^{\tau,\sigma}_{i-1,v}) = \left( \ker \left( \sqrt{G^\sigma_{0,v}} \right) \cap \ker \left( \sqrt{G^\sigma_{1,v}} \right) \right) \perp
\]
by observations similar to the ones in the proof of Lemma 3.7. Then by (4.58) and (4.59), as well as Lemma 2.1, we have the following estimates

\[ \mathcal{J}_{\tau, \sigma}(\hat{\mu}^{\tau, \sigma}) \leq \frac{T}{2} \sum_{i=1}^{N} \sum_{v \in \mathcal{V}^\sigma} |T_v| R_{i,v}^{\tau, \sigma} \cdot \left( \left( (i - \frac{1}{2}) \tau (\sqrt{G_{1,v} - \sqrt{G_{0,v}}} + \sqrt{G_{0,v}}) \right)^2 \right) ^{\frac{1}{2}} R_{i,v}^{\tau, \sigma} \]

\[ \leq 2 \tau \sum_{i=1}^{N} \sum_{v \in \mathcal{V}^\sigma} |T_v| \left( \sqrt{G_{1,v}^\sigma - \sqrt{G_{0,v}^\sigma}} \right) \cdot \left( \sqrt{G_{1,v} - \sqrt{G_{0,v}}} \right) \]

\[ \leq 2 \sum_{v \in \mathcal{V}^\sigma} |T_v| \left| \sqrt{G_{0,v}^\sigma - \sqrt{G_{0,v}}^2} \right| , \]

and hence finish the proof. \[ \square \]

We next check the condition \( \mathcal{A} \). It suffices to consider the case \( i = 0 \). By above Lemma with \( \mathcal{G}^\varepsilon \) being the given discrete initial distribution \( S_{X,\varepsilon}^\varepsilon(G_0) \) and \( G_1^\varepsilon = S_{X,\varepsilon}(G_0^\varepsilon) \), where \( G_0^\varepsilon = G_0^\varepsilon \) is the \( \varepsilon \)-regularization of \( G_0 \), there exists \( \hat{\mu}^{\tau, \sigma} \in C\mathcal{E}^{\tau, \sigma}([0, 1]; G_0^\varepsilon, (G_0^\varepsilon(v))_{v \in \mathcal{V}^\sigma} \) such that

\[ \mathcal{J}_{\tau, \sigma}(\hat{\mu}^{\tau, \sigma}) \leq 2 \sum_{v \in \mathcal{V}^\sigma} |T_v| \left| \sqrt{G_{0,v}^\sigma - \sqrt{G_{0,v}}^2} \right| . \]

Then by estimate (4.22) and definitions of \( G_0^\varepsilon \) and \( G_0^\varepsilon \), it follows that

\[ \mathcal{J}_{\tau, \sigma}(\hat{\mu}^{\tau, \sigma}) \lesssim \sum_{v \in \mathcal{V}^\sigma} |T_v| \left| G_{0,v}^\sigma - G_{0,v}^\varepsilon(v) \right| \]

\[ \lesssim \sum_{v \in \mathcal{V}^\sigma} \int \hat{\phi}_v |G_0 - G_0^\varepsilon| + \sum_{v \in \mathcal{V}^\sigma} | \int \hat{\phi}_v(x)(G_0^\varepsilon(x) - G_0^\varepsilon(0)) dx | \]

\[ \lesssim \| G_0 - G_0^\varepsilon \| + \sum_{v \in \mathcal{V}^\sigma} \int \hat{\phi}_v(x) dx \| \nabla G_0^\varepsilon \|_{\infty, \sigma}, \]

which further yields

\[ \limsup_{(\tau, \sigma) \to 0} \mathcal{J}_{\tau, \sigma}(\hat{\mu}^{\tau, \sigma}) \lesssim \| G_0 - G_0^\varepsilon \|. \]

(4.60)

Proposition 3.28 helps to conclude that the right-hand side of (4.60) tends to zero as \( \varepsilon \to 0 \), by which we have verified the condition \( \mathcal{A} \) and hence completed the construction of the convergent scheme.

4.4 Special case: Wasserstein–Fisher–Rao metric

In this section, we are going to deal with the Wasserstein–Fisher–Rao metric and try to obtain a sharper result for the scheme constructed in Section 4.3. To be precise, we shall establish the following proposition which show that for WFR metric the condition \( \mathcal{A} \) actually holds without the assumption \( \mathcal{A} \).

Lemma 4.19. Let \( \rho^\varepsilon \) be the \( \varepsilon \)-regularization of \( \rho \in \mathcal{M}(\Omega, \mathbb{R}^+) \) as in Proposition 4.28. Then it holds that

\[ \lim_{\varepsilon \to 0} \text{WFR}^2(\rho, \rho^\varepsilon) = 0. \]

(4.61)

Moreover, there exists a sequence of curves \( \hat{\mu}^{\tau, \sigma} \in C\mathcal{E}^{\tau, \sigma}([0, 1]; S_{X,\varepsilon}^\varepsilon(\rho), S_{X,\varepsilon}^\varepsilon(\rho^\varepsilon)) \) such that

\[ \limsup_{(\tau, \sigma) \to 0} \mathcal{J}_{\tau, \sigma}(\hat{\mu}^{\tau, \sigma}) \leq \hat{\omega}_{\text{WFR}(\rho, \rho^\varepsilon)}, \]

(4.62)

where \( \hat{\omega}_{\text{WFR}(\rho, \rho^\varepsilon)} \) is a real number depending on \( \text{WFR}(\rho, \rho^\varepsilon) \) and tending to zero as \( \varepsilon \to 0 \).

Recall the \( \varepsilon \)-regularization defined in Proposition 3.28:

\[ \rho^\varepsilon = (1 - \varepsilon)\hat{\rho}^\varepsilon + \varepsilon \bar{\rho}^\varepsilon, \]

where \( \hat{\rho}^\varepsilon = T^\varepsilon_\#(\theta_\varepsilon^\ast \rho) \) and \( \bar{\rho}^\varepsilon = T^\varepsilon_\#((\theta_\varepsilon^\ast \bar{\rho})|_{\Omega_1}) \) with \( \bar{\rho} \) being a smooth density bounded below by constant one on \( \Omega_1 \).

The proof relies on the the following static formulation of \( \text{WFR}^2(\cdot, \cdot) \) [1] Theorem 5.6]: for \( \rho_0, \rho_1 \in \mathcal{M}(\Omega, \mathbb{R}^+) \),

\[ \text{WFR}^2(\rho_0, \rho_1) = \inf \left\{ \int_{\Omega^2} c(x, \frac{d\gamma_0}{d\gamma}, \frac{d\gamma_1}{d\gamma}) d\gamma; \ (\gamma_0, \gamma_1) \in \Gamma(\rho_0, \rho_1) \right\}, \]

(4.63)
where $\gamma$ is a reference measure such that $\gamma_0, \gamma_1 \ll \gamma$, $\Gamma(\rho_0, \rho_1) := \{ (\gamma_0, \gamma_1) \in M(\Omega^2, \mathbb{R}_+) \colon (\pi_\# \gamma_0, \pi_\# \gamma_1) = (\rho_0, \rho_1) \}$ is the set of semi-couplings, and the cost $c$ is given by

\[
c(x, m_0, y, m_1) = WFR^2(m_0 \delta_x, m_1 \delta_y) = m_0 + m_1 - 2\sqrt{m_0 m_1} \cos(\min\{\text{dist}(x, y), \pi/2\})
\]

Here $\pi^1$ and $\pi^2$ are the projections from $\Omega^2$ to its first factor and second factor respectively. Moreover, by the above explicit expression of the cost $c$, (4.63) can be reformulated as

\[
WFR^2(\rho_0, \rho_1) = \rho_0(\Omega) + \rho_1(\Omega) - 2\sup\{ \int_{|y-x|<\pi/2} \cos(|y-x|) \, d(\sqrt{\gamma_0^1}) (x, y) ; (\gamma_0, \gamma_1) \in \Gamma(\rho_0, \rho_1) \}, \tag{4.64}
\]

where $d(\sqrt{\gamma_0^1})$ is defined by $\frac{d\gamma_0^1}{\gamma_0^1} \frac{d\gamma_1}{\gamma_1}$ which is independent of the reference measure $\gamma$. Let us recall more facts related to the geodesic line between two Dirac measures $m_0 \delta_x$ and $m_1 \delta_y$ with $m_0, m_1 > 0$. It has been shown in [13] that if $|x - y| < \pi/2$, the geodesic is unique and of the form $m(s) \delta_{\gamma_2(s)}$, $0 \leq s \leq 1$, where $m$ is a quadratic function:

\[
m(s) = (1 - s)^2 m_0 + s^2 m_1 + 2 s (1 - s) \sqrt{m_0 m_1} \cos(|x - y|), \tag{4.65}
\]

and $\gamma(s)$ is a straight line connecting $x$ and $y$ with

\[
|\dot{\gamma}(s)| = |x - y| H(m)/m(s). \tag{4.66}
\]

Here $H(m)$ is a constant given by $(\int_1^1 1/m(s) \, ds)^{-1}$. If $|y - x| > \pi/2$, the unique geodesic is the same as the one to the Hellinger distance: $\rho(s) = (1 - s)^2 m_0 \delta_x + s^2 m_1 \delta_y$. If $|y - x| = \pi/2$, we have an uncountable number of geodesics.

**Proof of (4.01).** To estimate $WFR^2(\rho, \tilde{\rho}^\varepsilon)$, we first note

\[
WFR^2(\rho, \tilde{\rho}^\varepsilon) \leq (1 - \varepsilon) WFR^2(\rho, \rho^\varepsilon) + \varepsilon WFR^2(0, \rho^\varepsilon) \tag{4.67}
\]

by sublinearity of $WFR^2(\cdot, \cdot)$ (cf. Corollary 3.13). By Lemma 4.64, we can bound the second term in (4.67) by $C\varepsilon$ where $C$ is independent of $\varepsilon$. We next estimate $WFR^2(\rho, \tilde{\rho}^\varepsilon)$ by using the static formulation 4.64. For this, by definition we have

\[
\tilde{\rho}^\varepsilon(\Omega) = ((\theta^\varepsilon \ast \rho)((T^\varepsilon)^{-1} \Omega) = ((\theta^\varepsilon \ast \rho)(\mathbb{R}^d)) = \rho(\Omega). \tag{4.68}
\]

Then we consider the semi-coupling measures

\[
d\gamma_0(x,y) = d\gamma_1(x,y) = \theta^\varepsilon_y(x - (T^\varepsilon)^{-1} y) \, d\rho(x) \, d((T^\varepsilon)^{-1} y)
\]

and the reference measure $d\gamma_1(x,y) = d\rho(x) \, dy$ such that $\pi^1 \gamma_0 = \rho$ and $\pi^2 \gamma_1 = \tilde{\rho}^\varepsilon$, and $\gamma_0, \gamma_1 \ll \gamma$ with $d\gamma_0 = d\gamma_1 = (1 + L\varepsilon)^4 \theta^\varepsilon_y(x - (1 + L\varepsilon) y) \, d\gamma$. It immediately follows from (4.65) that

\[
WFR^2(\rho, \tilde{\rho}^\varepsilon) \leq 2 \rho(\Omega) - 2 \int_{|y-x|<\pi/2} \cos(|y-x|) (1 + L\varepsilon)^4 \theta^\varepsilon_x(x - (1 + L\varepsilon) y) \, d\gamma
\]

\[
\leq 2 \rho(\Omega) - 2 \rho(\Omega) - 2 (1 - C\varepsilon) \int_{\mathbb{R}^d} (1 + L\varepsilon)^4 \theta^\varepsilon_y(x - (1 + L\varepsilon) y) \, d\rho(\Omega),
\]

by noting that the choice of $L$ (the Lipschitz constant of the Minkowski functional) has guaranteed $\theta^\varepsilon_y(x - (1 + L\varepsilon) y) = 0$ if $x \in \Omega$ and $y \notin \Omega$, or $|y-x| \geq C\varepsilon$. Here $C$ and $C$ are some constants depending on $L$. Combining (4.67) and (4.69), we achieve the desired (4.01).

**Proof of (4.02).** By linearity of the discrete continuity equation, we can construct the discrete curves connecting $S^1_{X_\tau}(\rho)$ and $S^1_{X_\tau}(\tilde{\rho}^\varepsilon)$, and $0$ and $S^1_{X_\tau}(\tilde{\rho}^\varepsilon)$, and estimate their (discrete) actions respectively. Note that $\tilde{\rho}^\varepsilon$ is absolutely continuous w.r.t. the Lebesgue measure. Then by Lemma 4.13, there exists $\tilde{\mu}^{\tau, \sigma} \in CE^{\tau, \sigma}([0, 1])$, $0$, $S^1_{X_\sigma}(\tilde{\rho}^\varepsilon)$ with

\[
\mathcal{J}_{\tau, \sigma}(\tilde{\mu}^{\tau, \sigma}) \leq \sum_{v \in \mathcal{V}^\sigma} |T_v| |S^1_{X_\sigma}(\tilde{\rho}^\varepsilon)|_v \lesssim \|\tilde{\rho}\|_{L^1}. \tag{4.70}
\]

Hence it suffices to consider (4.02) with $\rho$ replaced by $\tilde{\rho}^\varepsilon$. Indeed, if it holds with discrete curve $\mu^{\tau, \sigma}$, then by (4.67) and (4.70), $(1 - \varepsilon) \mu^{\tau, \sigma} + \varepsilon \tilde{\mu}^{\tau, \sigma}$ gives the curves connection $S^1_{X_\sigma}(\rho)$ and $S^1_{X_\sigma}(\tilde{\rho}^\varepsilon)$ with the desired properties.
Moreover, we claim that the proof of \((4.62)\) can further reduce to the case: there exists discrete curves \(\mu_{(x,y)}^{\tau,\sigma} \in C_{\mathcal{E}^{\tau,\sigma}}([0,1]; S_{X_x}^l(m_0(x,y)\delta_x), S_{X_x}^l(m_1(x,y)\delta_y))\) with

\[
|x - (1 + L\varepsilon)y| \leq \varepsilon \quad \text{and} \quad m_0(x,y) = m_1(x,y) := \frac{d\gamma_0}{d\gamma} = \frac{d\gamma_1}{d\gamma} = (1 + L\varepsilon)^{\partial_{(x,y)}}(x - (1 + L\varepsilon)y),
\]

such that

\[
\mathcal{J}_{\tau,\sigma}(\mu_{(x,y)}^{\tau,\sigma}) \lesssim c(x, m_0, y, m_1) + \varepsilon_{\tau,\sigma},
\]

where \(\gamma_0, \gamma_1\) and \(\gamma\) are the semi-couplings and the reference measure defined as in the proof of \((4.61)\); \(\varepsilon_{\tau,\sigma}\) is independent of \((x, y)\) and tends to zero as \((\tau, \sigma) \to 0\).

To see this, we define (the measurability of \(\mu_{(x,y)}^{\tau,\sigma}\)) is guaranteed by its concrete construction as we shall see below

\[
\mu_{(x,y)}^{\tau,\sigma} := \int_{\Omega^2} \mu_{(x,y)}^{\tau,\sigma} d\gamma(x, y)
\]

which, by linearity, still satisfies the discrete continuity equation with the end points given by

\[
\int_{\Omega^2} \mathcal{S}_{X_x}^l \frac{d\gamma_0}{d\gamma}(x, y) d\gamma(x, y) = \frac{1}{|T|} \int_{\Omega^2} \frac{d\gamma_0}{d\gamma}(x, y) \hat{\phi}_v(x) d\gamma(x, y) = \frac{1}{|T|} \int_{\Omega} \hat{\phi}_v(x) d\rho_0(x) = \mathcal{S}_{X_x}^l(\rho_0)_v \quad \text{for} \; v \in \mathcal{V}^\sigma,
\]

and

\[
\int_{\Omega^2} \mathcal{S}_{X_x}^l \frac{d\gamma_1}{d\gamma}(x, y) d\gamma(x, y) = \mathcal{S}_{X_x}^l(\rho_1)_v \quad \text{for} \; v \in \mathcal{V}^\sigma.
\]

Then by sublinearity of \(\mathcal{J}_{\tau,\sigma}\) and Jensen’s inequality, we have

\[
\mathcal{J}_{\tau,\sigma}(\mu_{(x,y)}^{\tau,\sigma}) \leq \int_{\Omega^2} \mathcal{J}_{\tau,\sigma}(\mu_{(x,y)}^{\tau,\sigma}) d\gamma(x, y) \lesssim \int_{\Omega^2} c(x, m_0, y, m_1) d\gamma(x, y) + \gamma(\Omega^2)\varepsilon_{\tau,\sigma},
\]

which completes the proof of the claim, thanks to the fact that \(\int_{\Omega^2} c(x, m_0, y, m_1) d\gamma(x, y) \to 0\) as \(\varepsilon \to 0\), which was shown in the proof of \((4.61)\).

We next devoted ourselves to the proof of \((4.72)\). By the same argument as the one to \((4.75)\), it suffices to construct the “mid–states” \(\mu_i^\sigma = (\tilde{\rho}_i^\sigma, q_i^\sigma, r_i^\sigma) \in X_\sigma \times Y_\sigma \times X_\sigma, \; i = 1, 2\), such that

\[
-\text{div}_\sigma q_i^\sigma + r_i^\sigma = \tilde{\rho}^\sigma - \mathcal{S}_{X_x}^l(m_0\delta_x).
\]

and

\[
\mathcal{J}_\sigma(\mu_i^\sigma) \lesssim c(x, m_0, y, m_1) + \varepsilon_{\sigma},
\]

where \(\varepsilon_{\sigma}\) is independent of \((x, y)\) and tends to zero as \(\sigma \to 0\). W.o.l.g., we only prove the case \(i = 0\). For this, we start with the construction at the continuous level. Suppose that \(\rho = \{m(t)\delta_{\gamma(t)}\}_{t \in [0,1]}\) is the unique geodesic between \(m_0\delta_x\) and \(m_1\delta_y\) defined by \((4.65)-(4.66)\) (recall \((4.71)\)). We define the measure \(\tilde{\rho} \in \mathcal{M}(\Omega)\) by \(\langle \tilde{\rho}, \gamma \rangle_\Omega = \int_0^1 m(t)\phi(\gamma(t)) dt\) for \(\phi \in C(\Omega)\), which is a \(\mathcal{H}^1\) Hausdorff measure supported on \(\gamma([0,1])\). By absolute continuity of \(m(t)\) and \(\gamma(t)\), a direct calculation leads to

\[
\langle \tilde{\rho} - m_0\delta_x, \phi \rangle_\Omega = \int_0^1 m(t)\phi(\gamma(t)) - m(0)\phi(0) dt = \int_0^1 \int_0^t \frac{d}{ds}(m(s)\phi(\gamma(s)))ds dt = \int_0^1 (1 - s) \frac{d}{ds}(m(s)\phi(\gamma(s))) ds = \int_0^1 (1 - s)(\dot{m}(s)\phi(\gamma(s)) + m(s)\nabla \phi(\gamma(s)) \cdot \dot{\gamma}(s)) ds,
\]

By further introducing \(q_0\) and \(r_0\) by

\[
\langle r_0, \phi \rangle_\Omega := \int_0^1 (1 - s)(\dot{m}(s)\phi(\gamma(s)) \cdot \dot{\gamma}(s)) ds \quad \text{and} \quad \langle q_0, \psi \rangle_\Omega := \int_0^1 (1 - s)m(s)\psi(\gamma(s)) \cdot \dot{\gamma}(s) ds
\]
for $\phi \in C(\Omega)$ and $\psi \in C(\Omega, \mathbb{R}^d)$, respectively, it follows from (4.75) that

$$-\text{div} q_0 + r_0 = \bar{\rho} - m_0 \delta_x.$$ 

Noting that the densities of $q_0$ and $r_0$ w.r.t. $\bar{\rho}$ are given by $\frac{dq_0}{d\rho} = (1 - s)\hat{\gamma}(s)$ and $\frac{dr_0}{d\rho} = (1 - s)\frac{\hat{m}(s)}{m(s)}$, we can compute the associated action $J_\Omega(\bar{\rho}, q_0, r_0)$ as

$$J_\Omega(\bar{\rho}, q_0, r_0) = \int_0^1 (1 - s)\left(\frac{\hat{\gamma}(s)}{m(s)}\right)^2 + \frac{1}{4} \frac{\hat{m}(s)}{m(s)} \right)^2 \right) m(s) ds = \frac{1}{2} c(x, m_0, y, m_1).$$

We shall see that $\mu^\sigma = (S^\sigma_{\mathcal{X}_\sigma}(\bar{\rho}), S^\sigma_{\mathcal{Y}_\sigma}(q_0), S^\sigma_{\mathcal{X}_\sigma}(r_0))$ is the desired discrete "mid-state". In fact, by $\text{div}_\sigma S^\sigma_{\mathcal{Y}_\sigma}(q_0) = S^\sigma_{\mathcal{X}_\sigma} \left(\text{div} q_0\right)$, it is clear that $\mu^\sigma$ satisfies the discrete equation (4.75). We next estimate the discrete action:

$$J_\sigma(\mu^\sigma) := \frac{1}{2} \sum_{K \in T^\sigma} |K| \left( \sum_{v \in V^\sigma_K} \rho^v_K / (d + 1) \right)^2 |q^v_K|^2 + \frac{1}{2} \sum_{v \in V^\sigma} |T_v| \left( \rho^v_K \right)^2 |r^v_K|^2$$

(4.76)

Before doing so, let us write down some formulas by definition:

$$\rho^\sigma_K = \frac{1}{|K|} \int_{\{s; \: \gamma(s) \in K\}} m(s) ds,$$

(4.77a)

$$\rho^v_K = \frac{1}{|T_v|} \int m(s) \hat{\phi}_v(\gamma(s)) ds,$$ 

(4.77b)

$$q^v_K = \frac{1}{|K|} \int_{\{s; \: \gamma(s) \in K\}} (1 - s) m(s) \hat{\gamma}(s) ds,$$ 

(4.77c)

$$r^v_K = \frac{1}{|T_v|} \int_{\{s; \: \gamma(s) \in K\}} - (1 - s) \left( \frac{\hat{m}(s)}{m(s)} \right)^2 m(s) \hat{\phi}_v(\gamma(s)) ds,$$ 

(4.77d)

and introduce some auxiliary quantities or notations:

$$T^\sigma = \{ K \in T^\sigma; \: \gamma \text{ interacts with } K \},$$

(4.78a)

$$s_K := \text{arg max}\{ |\hat{\gamma}(s)|, \: \gamma(s) \in K \} \text{ for } K \in T^\sigma,$$ 

(4.78b)

$$s_v := \text{arg max}\{ |\hat{m}(s)|, \: \gamma(s) \in \bigcup_{K \in T^\sigma} K \} \text{ if } \{ s; \: \gamma(s) \in \bigcup_{K \in T^\sigma} K \} \neq \emptyset,$$ 

(4.78c)

$$c_{\text{min}} = \min_{s \in [0, 1]} |\hat{\gamma}(s)|, \: c_{\text{max}} = \max_{s \in [0, 1]} |\hat{\gamma}(s)|, \: C_{\text{Max}} = \max_{s \in [0, 1]} |\hat{m}(s)|$$ 

(4.78d)

for ease of exposition. We also note that $\{ s; \: \gamma(s) \in K \}$ is a closed interval with the length

$$l_K := |\{ s; \: \gamma(s) \in K \}| \leq c^{-1}_{\text{min}} \text{diam}(K) \leq c^{-1}_{\text{min}} \sigma.$$ 

(4.79)

We start with the estimation of the first term in (4.76). By the mesh regularity and (4.32), we see that there exists a constant $C$ such that $C^{-1} |K| \leq |T_v| \leq C |K|$ for any $v \in |K|$, which gives

$$\frac{1}{d + 1} \sum_{v \in V^\sigma_K} \rho^v_K \geq \frac{1}{|K|} \sum_{v \in V^\sigma_K} \langle \bar{\rho}, \hat{\phi}_v \rangle_{\Omega} \geq C^{-1} \bar{\rho}_K.$$ 

Then it follows that

$$\frac{1}{2} \sum_{K \in T^\sigma} |K| \left( \sum_{v \in V^\sigma_K} \rho^v_K / (d + 1) \right)^2 |q^v_K|^2 \leq \sum_{K \in T^\sigma} |K| \rho^\sigma_K |q^\sigma_K|^2.$$ 

(4.80)

By (4.77a) and (4.78a), we have

$$|q^\sigma_K|^2 \leq \frac{1}{|K|} |\hat{\gamma}(s_K)| \int_{\{ s; \: \gamma(s) \in K\}} m(s) ds = |\hat{\gamma}(s_K)| |\rho^\sigma_K|, \: K \in T^\sigma,$$ 

(4.81)

while by (4.77a) and (4.79), we have

$$|K| \rho^\sigma_K \leq \int_{\{ s; \: \gamma(s) \in K\}} m(s_K) + \text{Lip}(m)|s - s_K| ds \leq m(s_K) l_K + \text{Lip}(m) l_K c^{-1}_{\text{min}} \sigma,$$ 

(4.82)
where $\text{Lip}(m)$ is the Lipschitz constant of $m(s)$ on $[0, 1]$. Substituting (4.81) into (4.80) and then using (4.82) and (4.78d) gives

$$
\frac{1}{2} \sum_{K \in T^e} |K| \left( \sum_{v \in V_K^m} \rho_v^2/(d+1) \right) |q_K^e|^2 \lesssim \sum_{K \in T^e} |K| |\gamma(s_K)|^2 \rho_v^2
\lesssim \sum_{K \in T^e} |\gamma(s_K)|^2 m(s_K) l_K + \sum_{K \in T^e} c_{\text{max}} \text{Lip}(m) |l_K| c_{\text{min}}^{-1} \sigma.
$$

We next consider the second term in (4.76): $1$. By above estimates, substituting (4.83) and (4.84) into (4.76) gives

$$
|r_v^\sigma| \leq \frac{1}{|T_v|} \int m(s) \phi_v(\gamma(s)) ds = \frac{|m(s_v)|}{m(s_v)} |\rho_v|.
$$

Then similarly to the estimates above, it follows that

$$
\frac{1}{2} \sum_{v \in V^e} |T_v| |\rho_v^*|^2 |r_v^\sigma|^2 \lesssim \sum_{v \in V^e} |T_v| \left( \frac{|m(s_v)|}{m(s_v)} \right)^2 |\rho_v|^2
\lesssim \sum_{v \in V^e} \sum_{K \in T^e} \langle \rho_v, \hat{\phi}_v \rangle_K \left( \frac{|m(s_K)|}{m(s_K)} + \text{Lip}(\frac{\hat{m}}{m}) |s_v - s_K| \right)^2
\lesssim \sum_{K \in T^e} \sum_{v \in V_K^e} \langle \rho_v, \hat{\phi}_v \rangle_K \left( \frac{|m(s_K)|}{m(s_K)} \right)^2 + \|\rho\| \left( \text{Lip} \left( \frac{\hat{m}}{m} \right) c_{\text{min}}^{-1} \sigma \right)^2,
$$

where $\text{Lip}(\frac{\hat{m}}{m})$ is the Lipschitz constant of $\frac{\hat{m}}{m}$ on $[0, 1]$ and $\|\rho\|$ is the total variation of $\rho$. Noting $\sum_{v \in V_K^e} \langle \rho_v, \hat{\phi}_v \rangle_K = |K| \rho_K^2$, by (4.82) and (4.78c), we arrive at

$$
\frac{1}{2} \sum_{v \in V^e} |T_v| |\rho_v^*|^2 |r_v^\sigma|^2 \lesssim \sum_{K \in T^e} \left( m(s_K) l_K + \text{Lip}(m)|l_K| c_{\text{min}}^{-1} \sigma \left( \frac{|m(s_K)|}{m(s_K)} \right)^2 + \|\rho\| \left( \text{Lip} \left( \frac{\hat{m}}{m} \right) c_{\text{min}}^{-1} \sigma \right)^2 \right)
\leq \sum_{K \in T^e} \left( \frac{|m(s_K)|}{m(s_K)} \right)^2 l_K + \sum_{K \in T^e} C_{\text{Max}} \text{Lip}(m) l_K c_{\text{min}}^{-1} \sigma + \|\rho\| \left( \text{Lip} \left( \frac{\hat{m}}{m} \right) c_{\text{min}}^{-1} \sigma \right)^2.
$$

To proceed, we need to estimate the constants involved in (4.83) and (4.84).

1. Since $m_0 = m_1$, by definition (4.35), $m(s)$ can be written as $m(s) = m_0 f(s)$ with $f(s)$ being a quadratic function:

$$
(f(s) = 1 + (2s^2 - 2s)(1 - \cos(\|x - y\|)).
$$

Moreover, by (4.71), there holds $|x - y| \leq C\varepsilon$ with $C$ being a constant depending only on $\Omega$. Hence we are allowed to write the following estimates

$$
\begin{cases}
\text{Lip}(m) = m_0 \|f\|_\infty \lesssim m_0 |x - y|,
\text{Lip} \left( \frac{\hat{m}}{m} \right) = \text{Lip} \left( \frac{f'' f - (f')^2}{f^2} \right) \lesssim |x - y|^2,
C_{\text{Max}} \lesssim |x - y|.
\end{cases}
$$

2. By (4.86) and above item, we have

$$
c_{\text{min}}^{-1} \lesssim |x - y|^{-1} \quad \text{and} \quad c_{\text{max}} \lesssim |x - y|.
$$

3. By definition $\|\rho\| = m_0 \int_0^1 f(s) ds$. Since the mesh is regular, the number of the simplices that a line segment \( \{\gamma(s); \gamma(s) \in K\}, K \in T^e_g \), can belongs to is bounded by a constant $C$ independent of $\sigma$. Then it follows that

$$
\sum_{K \in T^e_g} l_K \leq C,
$$

for a $C$ independent of $\sigma$.

By above estimates, substituting (4.83) and (4.84) into (4.76) gives

$$
J_{\sigma}(\mu^\sigma) \lesssim c(x, m_0, y, m_1) + m_0 \sigma.
$$

as desired.
5 Concluding remarks

In this work, we have considered an abstract generalization [12] of the existing unbalanced matrix-valued optimal transport problems based on a dynamic formulation and convex analysis. It allows to define a class of transport problems and we have shown that the measure space $(\mathcal{M}(\Omega, \mathbb{S}^d_+), \mathbb{P}_B)$, which is called the weighted Wasserstein-Bures metric. We have shown that the measure space $(\mathcal{M}(\Omega, \mathbb{S}^d_+))$ equipped with this distance function $WB_\Lambda(\cdot, \cdot)$ is a complete geodesic measure space. On the other hand, we have considered the fully discretization of [12], which leads to a finite dimensional convex optimization problem. We have proposed an abstract convergence framework and suggested a concrete discretization scheme. Our results provide a unified framework for designing new matrix-valued optimal transport models.

Appendix A. Disintegration

Suppose that $(E, \mathcal{E}, \mu)$ is a finite measure space and $X$ is a Hausdorff topological space with the Borel $\sigma$-algebra $\mathcal{B}(X)$. A set of finite measures $\{\nu_\omega\}_{\omega \in E}$ on $(X, \mathcal{B}(X))$ is called a $(\mathcal{E}, \mathcal{B}(X))$-measurable family if $\mu_\omega(B)$ is a $\mathcal{E}$-measurable function for each $B \in \mathcal{B}(X)$. Moreover, we can define an associated measure $\gamma$ on the product measurable space $(E \times X, \mathcal{E} \otimes \mathcal{B}(X))$ by

$$\gamma(A) = \int_E \int_X \chi_A(\omega, x) d\nu_\omega d\mu, \quad \text{for } A \in \mathcal{E} \otimes \mathcal{B}(X),$$

which is simply written as $\gamma = \nu_\omega \otimes \mu$, by abuse of notation. The measurable family $\{\nu_\omega\}_{\omega \in E}$ is also referred to as the disintegration of $\gamma$ w.r.t. $\mu$. The converse result is the so-called disintegration theorem [11, 12].

Proposition A.1 (Disintegration). Let $X, Y$ be locally compact and separable metric spaces and let $\pi : X \to Y$ be a Borel map. Let $\lambda \in \mathcal{M}(X, \mathbb{R}^d)$ and define $\eta = \pi_*|\lambda|$, which is a non-negative Radon measure on $Y$. Then there exists measures $\lambda_y \in \mathcal{M}(X, \mathbb{R}^d)$ such that

1. $\{\lambda_y\}_{y \in Y}$ is a Borel measurable family of measures, and $|\lambda_y|$ is a probability measure in $X$ for $\eta$-a.e. $y \in Y$;
2. $\lambda = \lambda_y \otimes \eta$ in the sense that $\lambda(A) = \int_A \lambda_y(A) d\eta(y)$ for $A \in \mathcal{B}(X)$;
3. $|\lambda_y|(X \setminus \pi^{-1}(y)) = 0$ for $\eta$-a.e. $y \in Y$.

For any Borel map $f : X \to Z$, where $Z$ is any other compact metric space

$$f_\#(\lambda_y \otimes \eta)(A) = \int_Y \lambda_y(f^{-1}(A)) d\eta(y).$$

Appendix B. Auxiliary proofs

Proof of the first statement in Lemma 3.15. By $A - B \succeq 0$, we have $\sqrt{B}^\dagger A \sqrt{B}^\dagger - \mathbb{P}_B \succeq 0$, where $\mathbb{P}_B = \sqrt{B}^\dagger B^\sqrt{B}^\dagger$ is the orthogonal projector onto $\text{ran}(B)$. It follows that all the eigenvalues of the positive-semidefinite matrix $\sqrt{B}^\dagger B^\sqrt{B}^\dagger$ restricted on its invariant subspace $\text{ran}(B)$ is greater or equal to one. Recall that commuting the product of two matrices will not change the set of eigenvalues. We have $\sigma(\sqrt{B}^\dagger A \sqrt{B}^\dagger) = \sigma(\sqrt{AB}^\dagger \sqrt{A})$. Moreover, by $\text{ran}(A) = \text{ran}(B)$, it follows that $\sqrt{AB}^\dagger \sqrt{A} - \mathbb{P}_A \succeq 0$, which further gives $B^\dagger \succeq A^\dagger$ by conjugating with $\sqrt{A}^\dagger$. \qed

Proof of Lemma 3.19. For $\mu \in \mathcal{M}(\mathcal{X}, \mathcal{O}_\Lambda)$, by definition, we have $t_{\mu}(\mathcal{X}, \mathcal{O}_\Lambda) = \sup\{\langle \mu, \Xi \rangle_{\mathcal{X}} : \Xi \in C(\mathcal{X}, \mathcal{O}_\Lambda)\}$. To show that we can relax the admissible set $C(\mathcal{X}, \mathcal{O}_\Lambda)$ to $L^\infty_{\mu \mid |}(\mathcal{X}, \mathcal{O}_\Lambda)$, it suffices to prove

$$\sup_{\Xi \in L^\infty_{\mu \mid |}(\mathcal{X}, \mathcal{O}_\Lambda)} \langle \mu, \Xi \rangle_{\mathcal{X}} \leq \sup_{\Xi \in C(\mathcal{X}, \mathcal{O}_\Lambda)} \langle \mu, \Xi \rangle_{\mathcal{X}},$$

since the other direction is trivial. For this, we consider an essentially bounded measurable triplet $\Xi \in L^\infty_{\mu \mid |}(\mathcal{X}, \mathcal{O}_\Lambda)$. W.l.o.g., we assume that it is bounded by $\|\Xi\|_{\infty}$ everywhere. By Lusin’s theorem, for any $\varepsilon > 0$, there exists a triplet of continuous functions with compact support $\Xi \in C_{c}(\mathcal{X}, \mathcal{X})$ such that

$$|\mu|\{x \in \mathcal{X} : \Xi(x) \neq \Xi(x)\} \leq \varepsilon.$$
Define $\mathbb{P}_{O,\lambda}$ as the $L^2$-projection from $X$ to the closed convex set $O,\lambda$. By abuse of notation, we still denote by $\tilde{\Xi}$ the composite function $\mathbb{P}_{O,\lambda} \circ \Xi$ which belongs to the set $C(X, O,\lambda)$ and satisfies $\|\tilde{\Xi}\|_\infty \leq \|\Xi\|_\infty$. Moreover, (3.42) still holds. Then it follows that $|\langle \mu, \Xi \rangle_X - \langle \mu, \tilde{\Xi} \rangle_X | \leq 2\varepsilon \|\Xi\|_\infty$, which further implies
\[
\langle \mu, \Xi \rangle_X \leq \langle \mu, \tilde{\Xi} \rangle_X + 2\varepsilon \|\Xi\|_\infty \leq \sup_{\Xi \in C(X, O,\lambda)} \langle \mu, \Xi \rangle_X + 2\varepsilon \|\Xi\|_\infty.
\]
Since $\varepsilon$ is arbitrary, we complete the proof of the bound (B.1). By approximation, we can take the pointwise supremum in (3.42) and obtain the desired formula $\iota_{C(X, O,\lambda)}^*(\mu) = J_{\lambda, X}(\mu)$ by Proposition 6.4. Next, we characterize the subgradient $\partial J_{\lambda, X}(\mu)$. By Lemma 3.2 we have $\Xi \in \partial J_{\lambda, X}(\mu)_{C(X, X)}$ if and only if $\langle \mu, \Xi \rangle_X = g(\Xi) + g^*(\mu) = \iota_{C(X, O,\lambda)}(\Xi) + J_{\lambda, X}(\mu)$, which yields $\Xi \in C(X, O,\lambda)$ and
\[
\int_X \langle \mu, \Xi \rangle_X - J_{\lambda}(\mu) \rangle d\lambda = 0,
\]
where $\lambda$ is a reference measure such that $|\mu| \ll \lambda$ and $\mu,\lambda$ denotes the density of $\mu$ w.r.t. $\lambda$. We also note from $J_{\lambda} = \iota_{O,\lambda}$, and $\Xi(x) \in O,\lambda$ that $\langle \mu, \Xi \rangle_X - J_{\lambda}(\mu) \leq 0 \lambda$–a.e.. Then the equality in (B.3) must hold $\lambda$–a.e., which gives us (3.43) as desired.

**Proof of Proposition 3.22.** We denote by $D$ and $\tilde{D}$ the values of the right–hand side and the left-hand side of (3.68), respectively. By Hölder’s inequality and the characterization (P), we have $\tilde{D} \leq D$. For the other direction, we consider $\{\mu_t\}_{t \in [0,T]} \subset \mathcal{CE}_\infty([0,T])$ and reparameterize it by the $\varepsilon$-arc length function $s = s_\varepsilon(t)$ defined by
\[
s = s_\varepsilon(t) = \int_0^t \left\{ \mathcal{J}_\alpha(\mu_r) \frac{1}{2} + \varepsilon \right\} dr : [0,T] \to [0, L(\mu_t) + \varepsilon T],
\]
where $L(\mu_t) := \int_0^T \mathcal{J}_\alpha(\mu_t) \frac{1}{2} ds$. It is clear that $s_\varepsilon(t)$ is strictly increasing and absolutely continuous and has an absolutely continuous inverse. Then, by Lemma 3.12 and writing $\tilde{\mu} = \mu_{s^{-1}}(s)$ for short, we have
\[
D^2 \leq (L(\mu_t) + \varepsilon T) \int_0^{L(\mu_t) + \varepsilon T} \mathcal{J}_\alpha(\tilde{\mu}) ds \leq \left( \int_0^T \mathcal{J}_\alpha(\mu_t) \frac{1}{2} ds \right)^2 + \varepsilon TL(\mu_t),
\]
where we have also used (P) with $T$ being $L(\mu_t) + \varepsilon T$. If we assume that the curve $\mu$ is a minimizer to the problem (P), then by letting $\varepsilon$ tend to zero in (3.43) we have
\[
D = T \frac{1}{2} \left( \int_0^T \mathcal{J}_\alpha(\mu_t) dt \right)^{\frac{1}{2}} = \int_0^T \mathcal{J}_\alpha(\mu_t) \frac{1}{2} dt,
\]
which implies $\mathcal{J}_\alpha(\mu_t)$ is a constant function a.e.. Then (3.69) immediately follows.

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