THE NOWICKI CONJECTURE
FOR FREE METABELIAN LIE ALGEBRAS

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Abstract. Let $K[X_d] = K[x_1, \ldots, x_d]$ be the polynomial algebra in $d$ variables over a field $K$ of characteristic 0. The classical theorem of Weitzenböck from 1932 states that for linear locally nilpotent derivations $\delta$ (known as Weitzenböck derivations) the algebra of constants $K[X_d]^{\delta}$ is finitely generated. When the Weitzenböck derivation $\delta$ acts on the polynomial algebra $K[X_d, Y_d]$ in $2d$ variables by $\delta(y_i) = x_i$, $\delta(x_i) = 0$, $i = 1, \ldots, d$, Nowicki conjectured that $K[X_d, Y_d]^{\delta}$ is generated by $X_d$ and $x_iy_j - y_ix_j$ for all $1 \leq i < j \leq d$. There are several proofs based on different ideas confirming this conjecture. Considering arbitrary Weitzenböck derivations of the free $d$-generated metabelian Lie algebra $F_d$, with few trivial exceptions, the algebra $F_d^{\delta}$ is not finitely generated. However, the vector subspace $(F_d')^{\delta}$ of the commutator ideal $F_d'$ of $F_d$ is finitely generated as a $K[X_d]$-module. In this paper we study an analogue of the Nowicki conjecture in the Lie algebra setting and give an explicit set of generators of the $K[X_d, Y_d]^{\delta}$-module $(F_d')^{\delta}$.

1. Introduction

A linear operator $\delta$ of a (not necessarily commutative or associative) algebra $R$ over a field $K$ is called a derivation if

$$\delta(uv) = \delta(u)v + u\delta(v) \text{ for all } u, v \in R.$$ 

The kernel $R^{\delta}$ of $\delta$ is called the algebra of constants of $\delta$.

In the sequel $K$ will be a field of characteristic 0. Let $K[X_d] = K[x_1, \ldots, x_d]$, $d \geq 2$, be the polynomial algebra in $d$ variables over $K$. A derivation $\delta$ of $K[X_d]$ acting as a nonzero nilpotent linear operator of the vector space $KX_d$ with basis $X_d$ is called a Weitzenböck derivation. The Jordan normal form $J(\delta) = (J_1, \ldots, J_s)$ of the matrix of $\delta$ considered as a linear operator acting on $KX_d$ consists of Jordan cells $J_i$, $i = 1, \ldots, s$, with zero diagonals. In 1932 Weitzenböck [17] proved that the algebra of constants

$$K[X_d]^{\delta} = \ker \delta = \{u \in K[X_d] \mid \delta(u) = 0\}$$

is finitely generated. For more information on Weitzenböck derivations one can see the books by Nowicki [14], Derksen and Kemper [4], and Sturmfels [16]. The algebra of constants $K[X_d]^{\delta}$ can be considered also from the point of view of classical

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invariant theory. The linear operator $\alpha \delta$ of $KX_d$ is nilpotent for all $\alpha \in K$. The exponent
\[
\exp(\alpha \delta) = 1 + \frac{\alpha \delta}{1!} + \frac{\alpha^2 \delta^2}{2!} + \cdots
\]
is a well defined invertible linear operator of $KX_d$ and this defines a $d$-dimensional representation of the unitriangular group
\[
UT_2(K) = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in K \right\}.
\]
This action can be extended diagonally on the whole algebra $K[X_d]$ and $K[X_d]^{\delta}$ is equal to the algebra of invariants $K[X_d]^{UT_2(K)}$.

Let $K[X_d, Y_d] = K[x_1, \ldots, x_d, y_1, \ldots, y_d]$, be the polynomial algebra in $2d$ variables and let $\delta$ be the Weitzenböck derivation defined by $\delta(y_i) = x_i$, $\delta(x_i) = 0$, $i = 1, \ldots, d$. In the language of invariant theory $K[X_d, Y_d]^{\delta}$ is the algebra of invariants for the action of the additive group $(K, +)$ on $K[X_d, Y_d]$ by
\[
\alpha : x_i \to x_i, y_i \to y_i + \alpha x_i, \quad i = 1, \ldots, d, \quad \alpha \in K.
\]
In 1994 Nowicki conjectured [14, p. 76, Conjecture 6.9.10] that the algebra of constants $K[X_d, Y_d]^{\delta}$ is generated by $x_1, \ldots, x_d$ and the determinants
\[
(1) \quad u_{pq} = \begin{vmatrix} x_p & y_p \\ x_q & y_q \end{vmatrix}, \quad 1 \leq p < q \leq d.
\]

This conjecture attracted many mathematicians and was verified by several authors with proofs based on different ideas: In his Ph.D. thesis in 2004 Khoury [9, 10] gave a computational proof using Gröbner basis techniques. The unpublished proofs of Derksen and Panyushev applied ideas of classical invariant theory. Several proofs appeared in 2009. Drensky and Makar-Limanov [7] gave an elementary proof using easy arguments from undergraduate algebra and a simple induction only, without involving any invariant theory. In his proof Bedratyuk [2] reduced the Nowicki conjecture to a well known problem of classical invariant theory. Kuroda [12] gave a short proof based on the ideas of Kurano [11] in his study on the analogue in positive characteristic of the Roberts’ counterexample to the Hilbert fourteenth problem. As Kuroda mentioned Hashimoto informed him that Goto, Hayasaka, Kurano, and Nakamura [8] and Miyazaki [13] determined sets of generators for certain invariant rings where $K[X_d, Y_d]^{\delta}$ is included, and this gives one more proof of the Nowicki conjecture.

Let $K\langle X_d \rangle$, $d \geq 2$, be the free (unitary or nonunitary) associative algebra freely generated by $X_d$ and let, as above, $\delta$ be a nilpotent linear operator acting on the vector space $KX_d$. Then the action of $\delta$ on $KX_d$ can be extended to an action as a derivation on the whole algebra $K\langle X_d \rangle$. If $V$ is a $T$-ideal (or a verbal ideal) of $K\langle X_d \rangle$, i.e., an ideal which is invariant under all endomorphisms of $K\langle X_d \rangle$, then it is well known that $\delta(V) \subseteq V$ and $\delta$ induces a derivation on the factor algebra $K\langle X_d \rangle/V$. We shall use the same notation $\delta$ for this derivation of $K\langle X_d \rangle/V$ and again shall call it Weitzenböck. The factor algebra $F_d(\mathfrak{V}) = K\langle X_d \rangle/V$ is a relatively free algebra in the variety $\mathfrak{V}$ of associative algebras defined by the polynomial identities from $V$. As in the case of polynomial algebras, the kernel $F_d(\mathfrak{V})^{\delta}$ of $\delta$ is the algebra of constants of $\delta$. Similarly, if $L_d = L(X_d)$, $d \geq 2$, is the free Lie algebra freely generated by $X_d$ and $W$ is a $T$-ideal (or a verbal ideal) of $L_d$, then the action of $\delta$ on $KX_d$ defines a Weitzenböck derivation on the relatively free
algebra $F_d(\mathfrak{W}) = L_d/W$ in the variety of Lie algebras $\mathfrak{W}$ defined by the polynomial identities from $W$. See e.g., the book by Bahturin [1] for a background on varieties of Lie algebras, the book [5] by one of the authors for associative PI-algebras, and his paper [6] with Gupta for Weitzenb"ock derivations acting on free and relatively free algebras, and for the properties of their algebras of constants.

In the sequel, let

$$F_d = F_d(\mathfrak{A}^2) = L_d/L_d''$$

be the factor algebra of $L_d$ modulo the second term $L_d''$ of the derived series of $L_d$. This is the free metabelian Lie algebra generated by $X_d$. It is a relatively free algebra in the variety $\mathfrak{A}^2$ of the metabelian (solvable of class 2) Lie algebras defined by the identity $[[x_1, x_2], [x_3, x_4]] = 0$. The variety $\mathfrak{A}^2$ has a key position in the theory of varieties of Lie algebras. By the well-known dichotomy a variety $\mathfrak{W}$ of Lie algebras either satisfies the Engel condition and by the theorem of Zelmanov [18] is nilpotent or contains the metabelian variety $\mathfrak{A}^2$. Since finitely generated nilpotent algebras are finite dimensional, the algebra $F_d = F_d(\mathfrak{A}^2)$ is the minimal relatively free algebra which is not finite dimensional. If $\delta$ is a Weitzenb"ock derivation of $F_d$, then Drensky and Gupta [6] showed that $F_d^\delta$ is finitely generated only in the trivial case when the Jordan normal form of $\delta$ consists of one Jordan cell of size $2 \times 2$ and $d - 2$ Jordan cells of size $1 \times 1$, i.e., when the rank of the matrix of $\delta$ is equal to 1.

The commutator ideal $F_d'$ has a natural structure of a $K[X_d]$-module. Recently Dangovski and the authors [3] established that the vector space $(F_d')^\delta$ of the constants of $\delta$ in the commutator ideal $F_d'$ of $F_d$ is a finitely generated $K[X_d]^\delta$-module. Freely speaking, this means that the algebra of constants $F_d^\delta$ is very close to be finitely generated.

In the present paper we consider the free metabelian Lie algebra $F_{2d}$ of rank $2d$ generated by the set $X_d \cup Y_d$. We assume that $\delta$ is its Weitzenb"ock derivation acting similarly as in the Nowicki conjecture. We give a complete set of generators of the $K[X_d, Y_d]^\delta$-module $(F_{2d})^\delta$. This gives also an infinite set of generators of the Lie algebra $(F_{2d})^\delta$.

### 2. Preliminaries

Till the end of the paper we fix the notation $F_{2d} = L_{2d}/L_{2d}''$ for the free metabelian Lie algebra of rank $2d$ freely generated by $X_d \cup Y_d = \{x_1, \ldots, x_d, y_1, \ldots, y_d\}$. We assume that all Lie commutators are left normed, e.g.,

$$[z_1, z_2, z_3] = [[z_1, z_2], z_3] = [z_1, z_2]ad z_3$$

for all $z_1, z_2, z_3 \in F_{2d}$. The metabelian identity implies, see, e.g., [1], that

$$[z_{j_1}, z_{j_2}, z_{j_{\sigma(3)}}, \ldots, z_{j_{\sigma(k)}}] = [z_{j_1}, z_{j_2}, z_{j_3}, \ldots, z_{j_k}],$$

where $\sigma$ is an arbitrary permutation of $3, \ldots, k$. Thus the polynomial algebra $K[X_d, Y_d]$ acts on $F_{2d}'$ by the rule

$$uf(x_1, \ldots, x_d, y_1, \ldots, y_d) = uf(ad x_1, \ldots, ad x_d, ay_1, \ldots, ay_d),$$

where $u \in F_{2d}'$, $f(X_d, Y_d) = f(x_1, \ldots, x_d, y_1, \ldots, y_d) \in K[X_d, Y_d]$.

We construct the abelian wreath product due to Shmel’kin [15]. Let $A_{2d} = K(A_d \cup B_d)$ and $G_{2d} = K(P_d \cup Q_d)$ denote the abelian Lie algebras with linear bases

$$A_d \cup B_d = \{a_1, \ldots, a_d\} \cup \{b_1, \ldots, b_d\} \text{ and } P_d \cup Q_d = \{p_1, \ldots, p_d\} \cup \{q_1, \ldots, q_d\},$$
respectively, and let $C_{2d}$ be the free right $K[X_d,Y_d]$-module with free generators $A_d \cup B_d$. Equipping $C_{2d}$ with trivial multiplication we give it the structure of an abelian Lie algebra. The abelian wreath product $W_{2d} = \Lambda_{2d} \wr \Gamma_{2d}$ is equal to the semidirect sum $C_{2d} \times \Gamma_{2d}$. The elements of $W_{2d}$ are of the form

$$
\sum_{i=1}^{d} a_if_i(X_d,Y_d) + \sum_{i=1}^{d} b_ig_i(X_d,Y_d) + \sum_{i=1}^{d} \alpha_ip_i + \sum_{i=1}^{d} \beta_iq_i,
$$

where $\alpha_i, \beta_i \in K$. The multiplication in $W_{2d}$ is defined by

$$[C_{2d}, C_{2d}] = [\Gamma_{2d}, \Gamma_{2d}] = 0,$$

$$[a_if_i(X_d,Y_d), p_j] = a_if_i(X_d,Y_d)x_j, \quad [b_ig_i(X_d,Y_d), p_j] = b_ig_i(X_d,Y_d)x_j,$$

$$[a_if_i(X_d,Y_d), q_j] = a_if_i(X_d,Y_d)y_j, \quad [b_ig_i(X_d,Y_d), q_j] = b_ig_i(X_d,Y_d)y_j,$$

$j = 1, \ldots, d$. Hence $W_{2d}$ is a metabelian Lie algebra and every mapping

$\{x_1, \ldots, x_n, y_1, \ldots, y_n\} \to W_{2d}$

can be extended to a homomorphism $F_{2d} \to W_{2d}$. As a special case of the embedding theorem of Shmel’kin, the homomorphism $\varepsilon : F_{2d} \to W_{2d}$ defined by

$$\varepsilon(x_i) = a_i + p_i, \quad \varepsilon(y_i) = b_i + q_i, \quad i = 1, \ldots, d,$$

is a monomorphism. By this action of $\varepsilon$, the commutator ideal $F'_{2d}$ is embedded into the free right $K[X_d,Y_d]$-module

$$C_{2d} = a_1K[X_d,Y_d] \oplus \cdots \oplus a_dK[X_d,Y_d] \oplus b_1K[X_d,Y_d] \oplus \cdots \oplus b_dK[X_d,Y_d]$$

as follows:

$$\varepsilon([x_i, x_j]) = a_ix_j - a_jx_i, \quad \varepsilon([y_i, y_j]) = b_iy_j - b_jy_i, \quad \varepsilon([x_i, y_j]) = a_iy_j - b_jx_i,$$

and then, by induction, if $w \in F'_{2d}$, then

$$\varepsilon([w, x_j]) = \varepsilon(w)x_j, \quad \varepsilon([w, y_j]) = \varepsilon(w)y_j, \quad j = 1, \ldots, d.$$

As a consequence of this construction, we have the following result.

**Lemma 2.1.** [15] Theorem 2] An element

$$\sum_{i=1}^{d} a_if_i(X_d,Y_d) + \sum_{i=1}^{d} b_ig_i(X_d,Y_d)$$

from $C_{2d}$ is an image of an element from the commutator ideal $F'_{2d}$ if and only if

$$\sum_{i=1}^{d} x_if_i(X_d,Y_d) + \sum_{i=1}^{d} y_ig_i(X_d,Y_d) = 0.$$

It follows immediately from Lemma 2.1 that the image $\varepsilon(F'_{2d})$ of $F'_{2d}$ in $C_{2d}$ is a submodule of the $K[X_d,Y_d]$-module $C_{2d}$. In the sequel we shall identify the elements of $F'_{2d}$ with their images in $C_{2d}$.

If $\delta$ is a Weitzenböck derivation of $K[X_d,Y_d]$ we shall assume that it acts on $K[X_d,Y_d]$ by the rule

$$\delta(y_i) = x_i, \delta(x_i) = 0, \quad i = 1, \ldots, d,$$

and shall use without reference that the algebra of constants $K[X_d,Y_d]^\delta$ is generated by $x_1, \ldots, x_d$ and the determinants [1] as conjectured by Nowicki [14] and proved
in \([9, 10, 7, 2, 12]\). In this special case the Jordan normal form \(J(\delta)\) of \(\delta\) consist of 
\(2 \times 2\) Jordan cells only, i.e.,
\[
J(\delta) = \begin{pmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]
The action of \(\delta\) on \(\{a_i, b_i \mid i = 1, \ldots, d\}\) will be defined in the same way as on 
\(\{x_i, y_i \mid i = 1, \ldots, d\}\). Thus \(\delta\) is extended to a derivation of \(F_{2d}\) and \(W_{2d}\). The vector 
space \(C_{2d}^e\) of the constants of \(\delta\) in the free \(K[X_d, Y_d]\)-module \(C_{2d}\) is a \(K[X_d, Y_d]^\delta\)-module. The following theorem is a partial case of a result of \([3]\).

**Theorem 2.2.** Let \(\delta\) be a Weitzenböck derivation of the free metabelian Lie algebra \(F_{2d}\). Then the vector space \((F_{2d})^\delta\) of the constants of \(\delta\) in the commutator ideal \(F_{2d}\) of \(F_{2d}\) is a finitely generated \(K[X_d, Y_d]^\delta\)-module.

Since it will not cause misunderstanding with the notation for the polynomial algebra we shall use the notation
\[
K[X_d, Y_d]^\delta = K[X_d, U] = K[X_d, u_{pq} \mid 1 \leq p < q \leq d]
\]
for the algebra generated by \(X_d\) and the elements \(U = \{u_{pq} \mid 1 \leq i < j < k \leq d\}\) defined in \([\Pi]\). Drensky and Makar-Limanov \([7]\) showed that the algebra \(K[X_d, Y_d]^\delta\) has the following defining relations
\[
(2) \quad x_i u_{jk} - x_j u_{ik} + x_k u_{ij} = 0, \quad 1 \leq i < j < k \leq d.
\]
\[
(3) \quad u_{ij} u_{kl} - u_{ik} u_{jl} + u_{il} u_{jk} = 0, \quad 1 \leq i < j < k < l \leq d.
\]
and gave a canonical linear basis consisting of the elements of the form
\[
(4) \quad x_{i_1} \cdots x_{i_m} u_{k_1 l_1} \cdots u_{k_s l_s}
\]
such that the generators \(u_{k_1 l_1}\) and \(u_{k_3 l_3}\) do not intersect each other and \(u_{k_s l_s}\),
does not cover \(x_{i}\), for any \(\alpha, \beta, \gamma\). Here each \(u_{k_1 l_1}\) is identified with the open
interval \((k_1, l_1)\) on the real line. The generators \(u_{k_1 l_1}\) and \(u_{k_3 l_3}\) intersect each other if the intervals \((k_1, l_1)\) and \((k_3, l_3)\) have a nonempty intersection and are not contained in each other. We say also that \(u_{k_1 l_1}\) covers \(x_{i}\) if \(i\) belongs to the open
interval \((k_1, l_1)\). The order of generators in this basis is assumed to be as follows:
\(p_1 \leq \cdots \leq p_s\) and if \(p_n = p_{n+1}\), then \(q_n \leq q_{n+1}\); and the order among \(x_{i}\) is such
that \(i_1 \leq \cdots \leq i_m\).

As a direct consequence of the affirmative answer to the Nowicki Conjecture the algebra of constants \(K[A_d, B_d, X_d, Y_d]^\delta\) of the derivation \(\delta\) acting on the polynomial
algebra \(K[A_d, B_d, X_d, Y_d]\) is generated by \(A_d, X_d\) and the determinants
\[
(5) \quad w_{pq} = a_p y_q - b_p x_q = \begin{vmatrix}
a_p & b_p \\
a_q & b_q
\end{vmatrix}, \quad p, q = 1, \ldots, d.
\]
Hence the \(K[X_d, U]\)-module \(C_{2d}^\delta\) is generated by the elements \(a_1, \ldots, a_d\) and the
determinants \([5]\), and as a vector space \(C_{2d}^\delta\) is spanned by the elements of the form
\[
(6) \quad a_{i_1} x_{i_1} \cdots x_{i_m} u_{k_1 l_1} \cdots u_{k_s l_s}
\]
for $i_0, p_0, q_0 = 1, \ldots, d$. Ordering the elements $A_d \cup B_d \cup X_d \cup Y_d$ and assuming that the elements from $A_d$ and $B_d$ precede, respectively, the elements from $X_d$ and $Y_d$, we obtain as an application of (2) and (3) that the $K[X_d, U]$-module $C^d_{2d}$ has the following defining relations
\begin{align*}
    a_i u_{jk} - w_{ik} x_j + w_{ij} x_k &= 0, \quad 1 \leq i \leq d, 1 \leq j < k \leq d, \\
    w_{ij} u_{kl} - w_{ik} u_{jl} + w_{il} u_{jk} &= 0, \quad 1 \leq i \leq d, 1 \leq j < k < l \leq d.
\end{align*}

In order to fix a basis of $C^d_{2d}$ as a vector space, the factors $x_i \cdot \ldots \cdot x_{i_m}, u_{k_1 l_1} \cdot \ldots \cdot u_{k_s l_s}$ and $x_j \cdot \ldots \cdot x_{j_r}, u_{p_1 q_1} \cdot \ldots \cdot u_{p_r q_r}$ of the elements (6) and (7) have to satisfy the restrictions in (4). Additionally, for the elements in (7) we require $q_0 \leq p_1$.

3. MAIN RESULTS

In this section we give the generators of the $K[X_d, U]^d$-module of constants $(F^d_{2d})^d$ in the commutator ideal $F^d_{2d}$ of the free metabelian Lie algebra $F^d_{2d}$. Since $(F^d_{2d})^d$ is canonically embedded in $C^d_{2d}$ we shall work in $C^d_{2d}$ instead of directly in $(F^d_{2d})^d$.

**Definition 3.1.** We define the $K[X_d, U]$-submodule $L$ of $C^d_{2d}$ generated by the elements
\begin{align*}
    w_{ii}, & \quad 1 \leq i \leq d, \\
    w_{ij} + w_{ji}, & \quad 1 \leq i < j \leq d, \\
    a_i x_j - a_j x_i, & \quad 1 \leq i < j \leq d, \\
    a_i u_{pq} - w_{pq} x_i, & \quad 1 \leq i \leq d, 1 \leq p < q \leq d, \\
    a_j u_{ik} - a_j u_{ik} + a_k u_{ij}, & \quad 1 \leq i < j < k \leq d, \\
    w_{ij} u_{pq} - w_{pq} u_{ij}, & \quad 1 \leq i < j \leq d, 1 \leq p < q \leq d.
\end{align*}

By Lemma 2.1 one can easily observe that the generating elements (10)–(15) of $L$ are Lie elements, i.e., images of elements in the commutator ideal $F^d_{2d}$ of $F^d_{2d}$.

**Lemma 3.2.** The following elements span the quotient space $C^d_{2d}/L$.
\begin{align*}
    w_{p_0 q_0} u_{p_1 q_1} \cdot \ldots \cdot u_{p_r q_r}, \\
    a_{i_0} x_{i_1} \cdot \ldots \cdot x_{i_m} u_{k_1 l_1} \cdot \ldots \cdot u_{k_s l_s}
\end{align*}

where $u_{p_0 q_0} u_{p_1 q_1} \cdot \ldots \cdot u_{p_r q_r}$ and $x_{i_0} x_{i_1} \cdot \ldots \cdot x_{i_m} u_{k_1 l_1} \cdot \ldots \cdot u_{k_s l_s}$ are elements of the form (4); i.e., they are canonical basis elements of the algebra $K[X_d, U]$.

**Proof.** We shall work in the vector space $C^d_{2d}$ modulo the subspace $L$. It is sufficient to handle the basis elements of $C^d_{2d}$ of the form (16) and (17). Starting with the element $w_{p_0 q_0} x_{j_1} \cdot \ldots \cdot x_{j_r} u_{p_1 q_1} \cdot \ldots \cdot u_{p_r q_r}$ in (7) we apply the relation $w_{pq} x_i \equiv a_i u_{pq}$ (mod $L$) from (13) and bring the element from (7) to an element from (10). If the element from (7) is of the form $w_{p_0 q_0} u_{p_1 q_1} \cdot \ldots \cdot u_{p_r q_r}$, then (10) and (11) imply that we may assume that $p_0 < q_0$. Then using the relation (9) we obtain that the interval $(p_0, q_0)$ does not intersect with $(p_1, q_1), \ldots, (p_r, q_r)$, and the generator (15) fixes the order among $u_{p_0 q_0}, u_{p_1 q_1}, \ldots, u_{p_r q_r}$. This closes the case (7). Now we consider the
element \(a_0, x_1, \ldots, x_m, u_{k_1, l_1}, \ldots, u_{k_s, l_s}\) in\(^6\). By assumption, the integers \(i_1, \ldots, i_m\) do not belong to the open intervals \((p_l, q_l)\). If \(i_0 \in (p_l, q_l)\) for some \(l = 1, \ldots, s\), then the relation \(a_j u_{k_l} \equiv a_l u_{k_j} + a_k u_{i_j} \mod L\) from \(^{14}\) replaces \(a_0 u_{p_0 q_0}\) with \(a_0 u_{p_0 q_0}\) and \(a_0 u_{p_0 q_0}\). Since the intervals \((i_0, q_0)\) and \((p_l, i_0, p_l)\) are shorter than the interval \((p_l, q_l)\), the integers \(i_1, \ldots, i_m\) are not covered by the intervals \((p_1, q_1), \ldots, (p_s, q_s)\). In finite number of steps the same holds for the integer \(i_0\). Finally the generator \(^{12}\) fixes the order among \(x_i, t = 0, 1, \ldots, m\). \(\square\)

**Theorem 3.3.** The \(K[X_d, U]\)-module \(L\) consists of all Lie elements in \(C_{2d}^\delta\).

**Proof.** Let

\[
\sum \xi_{ikl} a_{i_0} x_{i_1} \cdots x_{i_m} u_{k_1, l_1} \cdots u_{k_s, l_s} + \sum \psi_{pq} w_{p_0 q_0} u_{p_1, q_1} \cdots u_{p_r, q_r}
\]

be a Lie element in the vector space \(C_{2d}^\delta / L\). Then by Lemma 2.1 and Lemma 3.2 we have that

\[
\sum \xi_{ikl} x_{i_0} x_{i_1} \cdots x_{i_m} u_{k_1, l_1} \cdots u_{k_s, l_s} + \sum \psi_{pq} w_{p_0 q_0} u_{p_1, q_1} \cdots u_{p_r, q_r} = 0,
\]

where \(x_{i_0} x_{i_1} \cdots x_{i_m} u_{k_1, l_1} \cdots u_{k_s, l_s}\) and \(w_{p_0 q_0} u_{p_1, q_1} \cdots u_{p_r, q_r}\) are basis elements of the algebra \(K[X_d, U]\). Clearly each element from the first sum is linearly independent from the elements of the second sum, since there is at least one multiplier of the form \(x_{i_0}\) in each summand of the first sum which does not appear in the second sum. This implies that

\[
\sum \xi_{ikl} x_{i_0} x_{i_1} \cdots x_{i_m} u_{k_1, l_1} \cdots u_{k_s, l_s} = \sum \psi_{pq} w_{p_0 q_0} u_{p_1, q_1} \cdots u_{p_r, q_r} = 0.
\]

Thus \(\xi_{ikl} = 0 = \psi_{pq}\) for all \(i, k, l, p, q\), because \(x_{i_0} x_{i_1} \cdots x_{i_m} u_{k_1, l_1} \cdots u_{k_s, l_s}\) and \(w_{p_0 q_0} u_{p_1, q_1} \cdots u_{p_r, q_r}\) uniquely determine the monomials \(a_0 x_{i_0} x_{i_1} \cdots x_{i_m} u_{k_1, l_1} \cdots u_{k_s, l_s}\) and \(w_{p_0 q_0} u_{p_1, q_1} \cdots u_{p_r, q_r}\), respectively. \(\square\)

Finally the generators of \((F_{2d}^\delta)^5\) are obtained by computing the inverse images of generators of \(L\).

**Corollary 3.4.** The \(K[X_d, U]\)-module \((F_{2d}^\delta)^5\) is generated by the following elements

\[
[x_i, y_i], \quad 1 \leq i \leq d,
\]

\[
[x_i, x_j], \quad 1 \leq i < j \leq d,
\]

\[
[x_i, y_j] + [x_j, y_i], \quad 1 \leq i < j \leq d,
\]

\[
[x_i, x_p, y_q] - [x_i, y_p, x_q], \quad 1 \leq i \leq d, 1 \leq p < q \leq d,
\]

\[
[x_i, x_j, y_k] - [x_i, x_k, y_j] + [x_j, x_k, y_i], \quad 1 \leq i < j < k \leq d,
\]

and

\[
[x_i, x_p, y_j, y_q] + [y_i, y_p, x_j, x_q] - [x_i, y_p, y_j, x_q] - [y_i, x_p, x_j, y_q],
\]

where \(1 \leq i < j \leq d, 1 \leq p < q \leq d\).

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