LATTICE DIAGRAMS OF KNOTS AND DIAGRAMS OF LATTICE STICK KNOTS

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ABSTRACT. We give a simple example showing that a knot or link diagram that lies in the $\mathbb{Z}^2$ lattice is not necessarily the projection of a lattice stick knot or link in the $\mathbb{Z}^3$ lattice, and we give a necessary and sufficient condition for when a knot or link diagram that lies in the $\mathbb{Z}^2$ lattice is in fact the projection of a lattice stick knot or link.

1. INTRODUCTION

Lattice stick knots and links, that is, knots and links that are in the $\mathbb{Z}^3$ lattice (which is the graph in $\mathbb{R}^3$ where the vertices are the points with integer coefficients, and the edges are unit length and parallel to the coordinate axes), have been studied by a number of authors, for example [Dia93], [EP02], [JvRP95], [UvRO+98], [SIA+09], [HHK+14], [DEPZ], [DEY04] and [HKON14]. There is some variation in terminology in these and other papers; for example, some authors use the term “cubic lattice” rather than $\mathbb{Z}^3$ lattice, and some use “step” to mean an edge in the $\mathbb{Z}^3$ lattice. The general goal is to find the minimum number of edges needed to represent a given knot or link as a lattice stick knot or link, and, when the minimum number of edges has not been found, to give an upper bound for it.

Here we address a more basic question, inspired by an analogous comment in the stick knot (but not lattice stick knot) case in [AS09], which defines the “projection stick index” of a knot $K$ as “the least number of sticks in any projection of a polygonal conformation of $K$,” and where they add “Note that there is no a priori reason that it is equal to the least number of sticks in polygonal projections of knot embeddings that are not themselves polygonal.”

An analogous question could be asked in the case of stick numbers and edge numbers of projections of lattice stick knots and links, but, more fundamentally, we raise the question of the relation between projections of lattice stick knots and links on the one hand, and, on the other hand, projections of knots and links in $\mathbb{R}^3$ that are not necessarily themselves lattice stick knots and links but where the projection is in the $\mathbb{Z}^2$ lattice.

For standard knots and links, there is no analogous question to be asked. That is, any knot or link diagram in $\mathbb{R}^2$ is the regular projection of a knot or link in $\mathbb{R}^3$. That implies, for example, that if a knot or link in $\mathbb{R}^3$ is projected onto $\mathbb{R}^2$, and if the resulting diagram is then manipulated (e.g. using the Reidemeister moves), then after the manipulation the diagram is still the projection of a knot or link in $\mathbb{R}^3$.

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The situation is not the same in the case of lattice stick knots and links. In particular, we give a simple example showing that of a knot or link diagram that lies in the $\mathbb{Z}^2$ lattice is not necessarily the projection of a lattice stick knot or link in the $\mathbb{Z}^3$ lattice. In Theorem 2.4, we give a necessary and sufficient condition for when a knot or link diagram that lies in the $\mathbb{Z}^2$ lattice is in fact the projection of a lattice stick knot or link.

2. PROJECTIONS OF LATTICE STICK KNOTS AND LINKS

We start with some terminology. An $x$-edge, $y$-edge and $z$-edge of the $\mathbb{Z}^2$ lattice or $\mathbb{Z}^3$ lattice is an edge that is parallel to the corresponding coordinate axes. A $z$-stick in the $\mathbb{Z}^3$ lattice is a line segment that is the union of finitely many $z$-edges; for convenience, we will sometimes consider vertices in the $\mathbb{Z}^2$ lattice to be trivial $z$-sticks.

Let $\pi_{xy}$ denote orthogonal projection from $\mathbb{R}^3$ onto the $xy$-plane.

The objects we wish to study are lattice stick knots and links, which are knots and links contained in the $\mathbb{Z}^3$ lattice. See Figure 1 for a lattice stick knot representing the trefoil knot.

![Figure 1](image)

Recall that when forming the diagram of a knot or link, it is required that the knot or link be positioned in $\mathbb{R}^3$ so that its projection onto the $xy$-plane is a regular projection, which means that the inverse image under projection onto the $xy$-plane of any point in the diagram that is not a crossing is a single point, and the inverse image at a crossing (of which there are only finitely many) is two points.

On the other hand, whereas the projection of a lattice stick knot or link onto the $xy$-plane is indeed a diagram of a knot of link (see, for example, the lattice stick knot in Figure 1 and its projection in Figure 2), such a projection is not a regular projection of a knot or link, because the inverse image of a vertex in the projection that is not crossing can be a non-trivial $z$-stick rather than a single point, and the the inverse image of a vertex in the projection that is a crossing can be one or two non-trivial $z$-sticks rather than two points.

That said, just as a regular projection of a knot or link has restrictions on the possible inverse images of points, so too when we we project a lattice stick knot or link onto the $xy$-plane.
Definition 2.1. Let $K$ be a lattice stick knot or link.

1. The lattice stick knot or link $K$ is proper if the inverse image under projection onto the $xy$-plane of any point in the diagram that is not a vertex of the $\mathbb{Z}^2$ lattice is a single point; the inverse image of a vertex in the diagram that is not a crossing is a single $z$-stick (possibly trivial); and the inverse image of a vertex in the diagram that is at a crossing is two $z$-sticks (again, possibly trivial).

2. Suppose that $K$ is proper. Let $R$ be the projection of $K$, and let $a$ be a vertex of $R$. Then $\pi_{xy}^{-1}(a)$ is the union of two $z$-sticks, one of which has larger $z$-values for all its points than the $z$-values of the other $z$-stick; the former of the two $z$-sticks is called the upper $z$-stick of $K$ over $a$, and the other $z$-stick is called the lower $z$-stick. △

We note that whereas the knot diagram seen in Figure 2 is naturally thought of as created by a (non-regular) projection of a proper lattice stick knot or link, this knot diagram can also be obtained by a regular projection of a non-lattice knot, as seen in Figure 3.

Figure 2.

The question we address is the reverse of the above observation, which we state using the following terminology.

Definition 2.2. A lattice diagram of a knot or link is a diagram of a knot or link, such that the diagram is contained the $\mathbb{Z}^2$ lattice, and that all crossings in the knot or link diagram are at vertices of the $\mathbb{Z}^2$ lattice. △

For example, the knot diagram in Figure 2 is a lattice diagram of a knot.
We then ask, is every lattice diagram of a knot or link the projection of a proper lattice stick knot or link? It might be thought that the answer is trivially yes, because we could simply do a lattice analog of what we did in the non-lattice context in Figure 3. However, the following example shows that that is not always possible.

Consider the lattice diagram \( R \) in Figure 4, which is the knot \( 5_2 \). Suppose that \( R \) is the projection of a lattice stick knot \( K \). The edge of \( K \) that projects onto \( \langle a, b \rangle \) is connected to the lower \( z \)-stick of \( K \) over \( b \), and the edge of \( K \) that projects onto \( \langle b, c \rangle \) is connected to the upper \( z \)-stick of \( K \) over \( b \). Hence the edge of \( K \) that projects onto \( \langle a, b \rangle \) has smaller \( z \)-value than the edge of \( K \) that projects onto \( \langle b, c \rangle \). The same argument shows that the edge of \( K \) that projects onto \( \langle b, c \rangle \) has smaller \( z \)-value than the edge of \( K \) that projects onto \( \langle c, d \rangle \), that the edge of \( K \) that projects onto \( \langle c, d \rangle \) has smaller \( z \)-value than the edge of \( K \) that projects onto \( \langle d, a \rangle \), and that the edge of \( K \) that projects onto \( \langle d, a \rangle \) has smaller \( z \)-value than the edge of \( K \) that projects onto \( \langle a, b \rangle \), which leads to the obvious contradiction. Hence \( R \) is not the projection of a proper lattice stick knot \( K \).

![Figure 4](image)

Clearly, any lattice diagram of a knot or link with the same type of configuration as the four vertices \( a, b, c \) and \( d \) in Figure 4 is not be a projection of a proper lattice stick knot or link. As we will see in Theorem 2.4 below, such a configuration is the only obstacle to a lattice diagram of a knot or link being a projection of a proper lattice stick knot or link.

**Definition 2.3.** A Celtic configuration is a subset of a lattice diagram of a knot or link that is equivalent to either of the configurations in Figure 5, where the crossings in the figure are at adjacent vertices in the \( \mathbb{Z}^2 \) lattice.

The name “Celtic configuration” is due to the fact that this configuration occurs regularly (albeit often on the diagonal) in Celtic interlace patterns, as in [Fis], for example; it also occurs in interlace patterns in other cultures, for example China.

Our theorem is the following.

**Theorem 2.4.** Let \( R \) be a lattice diagram of a knot or link. Then \( R \) is the projection of a proper lattice stick knot or link if and only if \( R \) does not have a Celtic configuration.

The proof of Theorem 2.4 will be given in Section 4 after some preliminaries.
3. CROSSING GRAPH AND PROBLEM CROSSING GRAPH

We now define two graphs that arise from lattice diagrams of knots and links, using the following terminology. A lattice graph is a subgraph the $\mathbb{Z}^2$ lattice; a lattice arc, respectively lattice simple closed curve, is a lattice graph that is a path graph, respectively cycle graph.

Note that if $G$ is a lattice graph, and if $a$ and $b$ are vertices of $G$, we say that $a$ and $b$ are “adjacent” if they are joined by an edge of $G$, not if they are joined only by an edge in the $\mathbb{Z}^2$ lattice.

**Definition 3.1.** Let $G$ be a lattice graph, and let $v$ be a vertex of $G$.

1. The lattice graph $G$ is **deleted-square free** if it does not have three edges that are part of a unit square in the $\mathbb{Z}^2$ lattice.
2. Let $a$ and $b$ be distinct vertices of $G$. Suppose that $a$ and $b$ are both adjacent to $v$. The vertices $a$ and $b$ are **opposing neighbors**, respectively **nearby neighbors**, of $v$ if the edges $\langle v, a \rangle$ and $\langle v, b \rangle$ are parallel, respectively perpendicular.
3. The vertex $v$ is a **corner** of $G$ if $v$ is adjacent to precisely two other vertices of $G$ and the two vertices adjacent to $v$ are nearby neighbors.
4. The lattice graph $G$ is **2-near regular** if the degree of every vertex is at most 2. △

Clearly a graph being 2-near regular is equivalent to the graph having components that are isolated vertices, path graphs and cycle graphs.

**Definition 3.2.** Let $R$ be a lattice diagram of a knot or link, and let $c$ be a crossing of $R$.

1. The x-strand, respectively y-strand, of $R$ at $c$ is the union of two x-edges, respectively two y-edges, of $R$ that have $c$ as an endpoint.
2. The crossing $c$ is an x-crossing, respectively y-crossing, if the upper strand of the crossing is the x-strand, respectively y-strand, as seen in Figure 6. △

**Definition 3.3.** Let $R$ be a lattice diagram of a knot or link. The **crossing graph** of $R$, denoted $CG_R$, is the lattice graph with a vertex at every crossing of $R$ and an edge between any two vertices that are adjacent in the $\mathbb{Z}^2$ lattice. △

For example, let $R$ be the lattice diagram of a link seen in Figure 7. The crossing graph of $R$ is shown in Figure 8.
Definition 3.4. Let $R$ be a lattice diagram of a knot or link, and let $c$ be a crossing of $R$. The crossing $c$ is a **problem crossing** if, thought of as a vertex in $CG_R$, it has a pair of nearby neighbors, called **bad neighbors** of $c$, which both have the opposite crossing type as $c$.

For example, each of the crossings labeled $a$, $b$, $c$, and $d$ in Figure 4 are problem crossing, where the two labeled crossings next to any of these four vertices are its bad neighbors.

Observe that a problem crossing can have more than two bad neighbors.

The following remark is straightforward, and we omit the details.

Remark 3.5. Let $R$ be a lattice diagram of a knot or link.

1. Let $d$ be a crossing of $R$. Then $d$ is a not a problem crossing if and only if at least one pair of opposing neighbors of $d$ has the property that each of the two opposing neighbors either is not a crossing or is a crossing having the same crossing type as $d$.

2. Suppose that four crossings of $R$ are the vertices of a unit square in the $\mathbb{Z}^2$ lattice. These crossings form a Celtic configuration if and only if each crossing of the four is a problem crossing and its two adjacent crossings among the four are among its bad neighbors.

Lemma 3.6. Let $R$ be a lattice diagram of a knot or link.

1. Suppose that two problem crossings in $R$ are adjacent. Then either the two problem crossings have the same crossing type and neither is a bad neighbor of the other, or they have the opposite crossing type and each is a bad neighbor of the other.
(2) Suppose that four crossings in $\mathcal{R}$ are the vertices of a unit square. If at least two of these four crossings is a problem crossing such that its two adjacent crossings among the four are among its bad neighbors, then all four of the crossings are problem crossings, and the four crossings form a Celtic configuration.

(3) If $\mathcal{R}$ has a problem crossing with more than two bad neighbors that are problem crossings, then $\mathcal{R}$ has a Celtic configuration.

Proof. For Part (1), let $c$ and $d$ be adjacent problem crossings. If $c$ and $d$ have the same crossing type, then clearly neither is a bad neighbor of the other, so suppose that $c$ and $d$ have the opposite crossing type. Because $d$ is a problem crossing, then it must have at least two bad neighbors, all of which have the opposite crossing type as $d$. If $c$ is not a bad neighbor of $d$, then one of the bad neighbors of $d$, say $e$, is a nearby neighbor of $d$ together with $c$, but that is a contradiction, because $e$ and $c$ both have the opposite crossing type as $d$, making $c$ a bad neighbor of $d$. Hence $c$ is a bad neighbor of $d$. A similar argument shows that $d$ is a bad neighbor of $c$.

For Part (2), let $a$, $b$, $c$ and $b$ be four crossings in $\mathcal{R}$ that are the vertices of a unit square. Suppose that at least two of these four vertices is a problem crossing such that its two adjacent crossings among the four are among its bad neighbors. Without out loss of generality, suppose that $a$ is a problem crossing, and that $a$ is an $x$-crossing such that its two adjacent crossings among the four are among its bad neighbors. See Figure 9. We know that $b$ and $d$ must be $y$-crossings; the crossing shown by a dot is unspecified as of yet. By hypothesis at least one of $b$, $c$ or $d$ is also a problem crossing such that its two adjacent crossings among the four are among its bad neighbors, and in any of these cases it is clear that $c$ must be an $x$-crossing, which makes all four of the vertices problem crossings, and make the four crossings into a Celtic configuration, by Remark 3.5 (2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{Figure 9.}
\end{figure}

For Part (3), suppose that $PG_\mathcal{R}$ has a problem crossing $a$ with more than two bad neighbors that are problem crossings. We consider the case where $a$ has exactly three bad neighbors that are problem crossings; the case where $a$ has four bad neighbors that are problem crossings is similar. Without loss of generality, suppose that $a$ and its three bad neighbors that are problem crossings, denoted $b$, $c$ and $d$, are as seen in Figure 10 and that $a$ is a $y$-crossing. Hence $b$, $c$ and $d$ are $x$-crossings; the crossings shown by dots are unspecified as of yet. Clearly $z$ and $w$, as shown in the figure, are also crossings.

By Part (1) of this lemma we know that $a$ is a bad neighbor of $d$, and hence at least one of $z$ or $w$ is also a bad neighbor of $d$. Hence either the four vertices $a$, $d$,
z and b, or the four vertices a, d, w and c, would have at least two vertices being a problem crossing such that its two adjacent crossings among the four are among its bad neighbors, and it follows from Part (2) of this lemma that R has a Celtic configuration.

The following definition makes sense by Lemma 3.6 (1).

**Definition 3.7.** Let R be a lattice diagram of a knot or link. The **problem crossing graph** of R, denoted PG_R, is the subgraph of CG_R with a vertex at every problem crossing of R, and an edge between any two vertices that are bad neighbors of each other.

If R is the lattice diagram of a link seen in Figure 7, the problem crossing graph of R is shown in Figure 11, where the problem crossing graph itself consists of the edges and vertices in black, and where the gray arrows point from each problem crossing to its bad neighbors (some of which are in the crossing graph but not the problem crossing graph).

The following lemma is derived straightforwardly from Lemma 3.6, combined with the fact that a deleted-square free lattice simple closed curve must have a vertex that is not a corner; the proof is omitted.

**Lemma 3.8.** Let R be a lattice diagram of a knot or link.

1. If two problem crossings of R are adjacent in CG_R but are not adjacent in PG_R, then neither is a bad neighbor of the other, and they have the same crossing type.
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(2) If $R$ does not have a Celtic configuration, then $PG_R$ is 2-near regular and deleted-square free, and every component of $PG_R$ that is a lattice simple closed curve has a vertex that is not a corner of $PG_R$.

4. PROOF OF THE THEOREM

To prove the non-trivial part of Theorem 2.4, the idea is that we start with a lattice diagram of a knot or link, and we then modify it one crossing at a time, so that after each modification, the strands at that crossing do not intersect, and such that the crossings that were previously modified remain with their strands not intersecting; it is the latter that necessitates some care, which we accomplish by doing the modification at the crossings in a specific order. After we complete all the modifications, we will end up with a proper lattice stick knot or link that projects onto the lattice diagram of a knot or link.

Whereas a knot or link is an embedding of a simple closed curve or simple closed curves in $\mathbb{R}^3$, it turns out that in the process of modification mentioned above we will have self-intersections of the knot or link, and hence we need a version of proper lattice stick knots and links that is not necessarily embedded. We start with the following preliminary.

Definition 4.1. Let $C$ be a 2-regular graph.

(1) A function $f : C \to \mathbb{R}^3$ is a lattice map if it is continuous, it maps every vertex of $C$ to a vertex of the $\mathbb{Z}^3$ lattice, and it maps every edge of $C$ onto a single edge of the $\mathbb{Z}^3$ lattice.

(2) Let $f : C \to \mathbb{R}^3$ be a lattice map. An x-edge, y-edge or z-edge, respectively, of $C$ with respect to $f$ is an edge of $C$ the image of which under $f$ is an x-edge, y-edge or z-edge, respectively, of the $\mathbb{Z}^3$ lattice. An x-arc, y-arc or z-arc of $C$ with respect to $f$ is a maximal arc of $C$ that is the union of x-edges, y-edges or z-edges, respectively. △

Observe that if $C$ is a 2-regular graph and $f : C \to \mathbb{R}^3$ is a lattice map, then $C$ is the union of x-arcs, y-arcs, z-arcs, where such arcs intersect only in their endpoints. To avoid special cases, a vertex in $C$ that is the endpoint of x-edges and/or y-edges, but not z-edges, can be considered to be a degenerate z-arc.

We now define our immersed version of proper lattice stick knots and links, in relation to a given lattice diagram of a knot or link.

Definition 4.2. Let $R$ be a lattice diagram of a knot or link, let $C$ be a 2-regular graph and let $f : C \to \mathbb{R}^3$ be a lattice map. The lattice map $f$ is a lattice pre-image for $R$ if the following four conditions hold.

(a) $\pi_{xy}(f(C)) = R$.

(b) The inverse image under $\pi_{xy} \circ f$ of any point in $R$ that is not a vertex in the $\mathbb{Z}^2$ lattice is a single point in $C$.

(c) The inverse image under $\pi_{xy} \circ f$ of any vertex in $R$ that is not a crossing is a single z-arc in $C$ (possibly degenerate).
The inverse image under $\pi_{xy} \circ f$ of any vertex in $R$ that is a crossing is two $z$-arcs in $C$ (each possibly degenerate), where one $z$-arc intersects only $x$-edges in $C$ and the other $z$-arc intersects only $y$-edges in $C$. \[ \triangle \]

We note that a lattice diagram of a knot or link can be thought of as a lattice pre-image for itself.

**Definition 4.3.** Let $R$ be a lattice diagram of a knot or link. Let $f : C \to \mathbb{R}^3$ be a lattice pre-image for $R$. Let $v$ be a crossing of $R$.

1. The **$x$-facing $z$-arc**, respectively **$y$-facing $z$-arc**, of $f$ at $v$, is the $z$-arc of $C$ in $(\pi_{xy} \circ f)^{-1}(v)$ that intersects only $x$-edges, respectively $y$-edges, in $C$.
2. The **$x$-facing $z$-stick**, respectively **$y$-facing $z$-stick**, of $f$ at $v$, is the image under $f$ of the $x$-facing $z$-arc, respectively $y$-facing $z$-arc, of $f$ at $v$. A $z$-**stick** of $f$ at $v$ is either an $x$-facing $z$-stick or a $y$-facing $z$-stick.
3. The **$x$-pre-strand**, respectively **y-pre-strand**, of $f$ at $v$ is the image under $f$ of the $x$-facing $z$-arc and the two $x$-edges of $C$ that the $x$-facing $z$-arc intersects, respectively the image under $f$ of the $y$-facing $z$-arc and the two $y$-edges of $C$ that the $y$-facing $z$-arc intersects.
4. The crossing $v$ is **resolved with respect to** $f$ if the following two conditions hold: (a) the $x$-facing $z$-stick and $y$-facing $z$-stick at $v$ are disjoint, and (b) if $v$ is an $x$-crossing, respectively $y$-crossing, then every point in the $x$-facing $z$-stick at $v$ has larger, respectively smaller, $z$-value than every point in the $y$-facing $z$-stick at $v$. \[ \triangle \]

We note from Definition 4.3 that $x$-facing $z$-sticks and $y$-facing $z$-sticks are literal $z$-sticks in the $\mathbb{Z}^3$ lattice. Additionally, we observe that the projection by $\pi_{xy}$ of the $x$-pre-strand, respectively $y$-pre-strand, of $f$ at $v$ is the $x$-strand, respectively $y$-strand, of $R$ at $v$. Note also that the $x$-pre-strand and $y$-pre-strand of $R$ at $v$ each contains one of the two $z$-sticks over $v$.

**Remark 4.4.** Let $R$ be a lattice diagram of a knot or link. Let $f : C \to \mathbb{R}^3$ be a lattice pre-image for $R$. Then the image of $f$ is a proper lattice stick knot or link that projects onto $R$ if and only if each crossing of $R$ is resolved with respect to $f$. \[ \diamond \]

We now turn to the type of modification we use at the crossing of a lattice diagram of a knot or link.

We consider a lattice diagram of a knot or link to be in the $\mathbb{Z}^2$ lattice, and so prior to modification every crossing in a lattice diagram of a knot or link has $z$-value 0.

**Definition 4.5.** Let $R$ be a lattice diagram of a knot or link. Let $f : C \to \mathbb{R}^3$ be a lattice pre-image for $R$. Let $v$ be a crossing of $R$.

1. Let $p \in \mathbb{Z}$. An **$(x, p)$-lift** at $v$ is a modification of $f$ defined as follows. Let $s$ and $t$ be the two endpoints of the $x$-strand of $R$ at $v$. First, remove the $x$-facing $z$-stick at $v$, and remove the $z$-stick at each of $s$ and $t$ that is connected to the $x$-strand at $v$. Second, move the images under $f$ of the two $x$-edges of $C$ in the $x$-pre-strand at $v$ to the $z$-value $p$. Finally, add in the necessary $z$-sticks to replace the three that were removed, in order to make
the modified map be a lattice map (doing so may entail adding or removing edges from \( C \)).

(2) An \( x \)-lift at \( v \) is an \((x, p)\)-lift for some \( p \in \mathbb{Z} \).

(3) A \((y, p)\)-lift at \( v \) for some \( p \in \mathbb{Z} \), and a \( y \)-lift at \( v \), are defined similarly.

(4) A lift at \( v \) is either an \( x \)-lift or a \( y \)-lift.

(5) A lift at \( v \) is \textit{proper} if after the lift, the crossing \( v \) is resolved with respect to \( f \).

An example of a proper \((y, 1)\)-lift is seen in Figures 12 and 13, where the former shows the crossing prior to the lift, and the latter shows the crossing after the lift.

![Figure 12](image1.png)  
![Figure 13](image2.png)

Clearly, if \( v \) is an \( x \)-crossing of \( R \), then doing an \((x, p)\)-lift or a \((y, -p)\)-lift for any sufficiently large \( p \in \mathbb{N} \) will be a proper lift, and similarly for a \( y \)-crossing.

We note that a proper lift at \( v \), while making \( v \) be resolved or maintaining \( v \) being resolved, might cause a neighboring crossing to go from being resolved to being not resolved. For example, we see in Figure 9 the crossings \( a \) and \( b \), which are an \( x \)-crossing and a \( y \)-crossing, respectively. Suppose we did a \((y, -1)\)-lift at \( a \), which is a proper lift. If we then did an \((x, -1)\)-lift at \( b \), then \( b \) would now be resolved, but \( a \) would no longer be resolved. Of course, if we did a \((y, 1)\)-lift at \( b \), that would make \( b \) be resolved and would leave \( a \) resolved; the problem would also be avoided if we had planned ahead and started with a \((y, -2)\)-lift at \( a \), followed by an \((x, -1)\)-lift at \( b \). To avoid such problems, we use the following terminology.

**Definition 4.6.** Let \( R \) be a lattice diagram of a knot or link. Let \( f : C \to \mathbb{R}^3 \) be a lattice pre-image for \( R \). Suppose that a proper lift is done at a crossing \( v \) of \( R \).

1. Let \( w \) be another crossing of \( R \). The lift at \( v \) is \textit{compatible} with \( w \) if the lift at \( v \) did not change \( w \) from resolved to not resolved.

2. The lift at \( v \) is \textit{backwards compatible} if it is compatible with all other crossings.

Note that in Definition 4.6, it does not matter what happens at a non-resolved crossing other than \( v \). Note also that if a lift at \( v \) is an \( x \)-lift, respectively \( y \)-lift, then the only two crossings that might change from resolved to not resolved would be at the two endpoints of the \( x \)-strand, respectively \( y \)-strand, at \( v \) (and only if those endpoints are crossings); in particular, if \( w \) is a crossing of \( R \) that is adjacent to \( v \), and if the lift at \( v \) is perpendicular to \( \langle v, w \rangle \), then \( w \) would not change from resolved to not resolved.
We note that it is not always possible to do a lift at a crossing that is backwards compatible. The issue occurs at problem crossings. For example, the crossing $a$ in Figure 9 is a problem crossing, and suppose that a $(y, +1)$-lift is done at each of $b$ and $d$, which makes these two crossings resolved, as seen in Figure 14. We then observe that a proper lift at $a$ could be either an $(x, p)$-lift for some $p \geq 2$, would change $b$ from resolved to not resolved, or a $(y, -q)$-lift for some $q \geq 1$, which would change $d$ from resolved to not resolved. Hence, no proper lift is possible at $a$ that is compatible with both $b$ and $d$.

The above example is why the Proof of Theorem 2.4, to which we now turn, is structured as it is.

Proof of Theorem 2.4 If $R$ is has a Celtic configuration, then the same argument used in regard to Figure 4 shows that $R$ is not the projection of a proper lattice stick knot or link.

Now suppose that $R$ does not have a Celtic configuration. We can view $R$ as a lattice pre-image for itself. We will do a proper lift at one crossing of $R$ at a time, where each lift is backwards compatible. After doing all the lifts, the resulting lattice pre-image will have all crossings resolved, and so it will be a embedding, and its image will be a proper lattice stick knot or link that projects onto $R$.

We start with two preliminary observations about lifts. Let $v$ be a crossing of $R$. Observation (1): Let $w$ be a crossing of $R$ that is adjacent to $v$. If $v$ and $w$ have the same crossing type, then any proper $x$-lift or proper $y$-lift at $v$ with sufficiently large $z$-value in absolute value is compatible with $w$.

To see why this observation is true, if $w$ is not resolved, there is nothing to prove, so suppose that $w$ is resolved. Suppose further, without loss of generality, that $\langle v, w \rangle$ is an $x$-edge. If any $y$-lift is performed at $v$, then $w$ would not change from resolved to not resolved. Hence, we need to consider only $x$-lifts at $v$. There are two cases.

First, suppose that $v$ and $w$ are both $x$-crossings. Because $w$ is assumed to be resolved, then we note that the $x$-facing $z$-stick at $w$ is higher than the $y$-facing $z$-stick at $w$. If an $(x, n)$-lift is performed at $v$ where $n$ is larger than the highest point in the $x$-facing $z$-stick at $w$, then such an $x$-lift at $v$ would not change $w$ from resolved to not resolved. The case where $v$ and $w$ are $y$-crossings is similar, except that the $x$-lift at $v$ has negative height.
Observation (2): Suppose \( v \) is not a problem crossing. Then there is a lift at \( v \) that is backwards compatible.

To see why this observation is true, we first note that by Remark 3.5 (1), at least one pair of opposing neighbors of \( v \) has the property that each of these two opposing neighbors either is not a crossing or is a crossing having the same crossing type as \( v \); let \( a \) and \( c \) be such opposing neighbors of \( v \). If \( a \) and/or \( c \) is not a crossing, or is a crossing that is not resolved, then there is nothing to be considered regarding that vertex, so assume that \( a \) and \( c \) are crossings that are resolved. Observing that \( a \), \( v \) and \( c \) all have the same crossing type. Without loss of generality, we assume that the strand at the crossing \( v \) that is in the direction of \( a \) and \( c \) is the upper strand at all three of \( a \), \( v \) and \( c \). A sufficiently high upward lift of the upper strand at \( v \) will cause \( v \) to be resolved, and will not cause \( a \) or \( c \) to change from resolved to not resolved.

We now return to doing proper lifts at one crossing of \( R \) at a time, starting with the problem crossings; if \( R \) has no problem crossings, then skip this step.

By Lemma 3.8 (2) we know that \( PG_R \) is 2-near regular. Hence, the components of \( PG_R \) are isolated vertices, lattice arcs and lattice simple closed curves.

We note that if two problem crossing of \( R \) are adjacent in \( CG_R \) but are in different components of \( PG_R \), or are in the same component of \( PG_R \) but are not adjacent in \( PG_R \), then by Lemma 3.8 (1) we know that the two problem crossing have the same crossing type, and hence we can apply Observation (1), which tells us that any proper \( x \)-lift or proper \( y \)-lift with sufficiently large \( z \)-value in absolute value at each of these crossings is compatible with the other. That tells us that we can do proper lifts for each component of \( PG_R \) separately without worrying about the impact on the other components of \( PG_R \), and also that within a single component of \( PG_R \), when we do a proper lift at a vertex, we need only be concerned about being compatible with the adjacent vertices in \( PG_R \), if there are any.

We proceed one component of \( PG_R \) at a time, in any order. Let \( M \) be a component of \( PG_R \).

First, suppose \( M \) is an isolated vertex (in \( PG_R \), not in \( CG_R \)). Let \( v \) be the single vertex in \( M \). Then by a previous observation, there are proper \( x \)-lifts and proper \( y \)-lifts at \( v \) that are compatible with all other problem crossings; we do any such lift at \( v \).

Second, suppose \( M \) is a lattice arc. Let \( v_1, \ldots, v_n \) be the vertices of \( M \) in order from one end of the arc to the other. First, do a proper lift at \( v_1 \) that is perpendicular to \( \langle v_1, v_2 \rangle \). Next, do a proper lift at each of \( v_2, \ldots, v_n \), in that order, such that the lift at \( v_i \) is perpendicular to \( \langle v_{i-1}, v_i \rangle \) for each \( i \in \{2, \ldots, n\} \). For each \( i \in \{2, \ldots, n\} \), we note that the perpendicularity implies that it is always possible to do such a lift at \( v_i \) that is compatible with \( v_{i-1} \); because \( v_i \) is not adjacent in \( PG_R \) to any of \( v_1, \ldots, v_{i-2} \), then the lift at \( v_i \) is compatible with \( v_1, \ldots, v_{i-1} \). Hence, we can do proper lifts at all vertices of \( M \) that are compatible with all other problem crossings.

Third, suppose \( M \) is a lattice simple closed curve. By Lemma 3.8 (2) we know that \( M \) has a vertex that is not a corner. Let \( w_1, \ldots, w_m \) be the vertices of \( M \) in order around the cycle, where \( w_m \) is not a corner of \( PG_R \). We then proceed exactly as in the case where \( M \) was a lattice arc. The only difference between the present
case and the lattice arc case is that in the present case, we need to ask whether the lift at $w_m$ is compatible with $w_1$. By construction we know that the lift at $w_m$ is perpendicular to $\langle w_{m-1}, v_m \rangle$. However, because $w_m$ is not a corner of $PG_R$, then we also know that the lift at $w_m$ is perpendicular to $\langle w_m, v_1 \rangle$, and that means that the lift at $w_m$ is compatible with $w_1$. Again, we see that we can do proper lifts at all vertices of $M$ that are compatible with all other problem crossings.

By doing the above to each of the components of $PG_R$, we have done lifts so that all the problem crossings are resolved. Finally, we do a lift at one non-problem crossing at a time, which by Observation (2) can always be done in a way that is backwards compatible.

As stated above, we have now found a proper lattice stick knot or link that projects onto $R$.

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\section*{References}

[AS09] Colin Adams and Todd Shayler, \textit{The projection stick index of knots}, J. Knot Theory Ramifications \textbf{18} (2009), no. 7, 889–899.

[Dia93] Yuanan Diao, \textit{Minimal knotted polygons on the cubic lattice}, J. Knot Theory Ramifications \textbf{2} (1993), no. 4, 413–425.

[DEPZ] Yuanan Diao, Claus Ernst, Attila Por, and Uta Ziegler, \textit{The ropelengths of knots are almost linear in terms of their crossing numbers}. arXiv:0912.3282v1.

[DEY04] Yuanan Diao, Claus Ernst, and Xingxing Yu, \textit{Hamiltonian knot projections and lengths of thick knots}, Topology Appl. \textbf{136} (2004), no. 1-3, 7–36.

[EP02] C. Ernst and M. Phipps, \textit{A minimal link on the cubic lattice}, J. Knot Theory Ramifications \textbf{11} (2002), no. 2, 165–172.

[Fis] Gwen Fisher, \textit{On the Topology of Celtic Knot Designs}, http://www.mi.sanu.ac.rs/vismath/fisher/index.html

[HKON14] Kyungpyo Hong, Hyoungjun Kim, Seungsang Oh, and Sungjong No, \textit{Minimum lattice length and ropelength of knots}, J. Knot Theory Ramifications \textbf{23} (2014), no. 7, 1460009, 10.

[HNO13] Kyungpyo Hong, Sungjong No, and Seungsang Oh, \textit{Upper bounds on the minimum length of cubic lattice knots}, J. Phys. A \textbf{46} (2013), no. 12, 125001, 7.

[HHK+14] Youngsik Huh, Kyungpyo Hong, Hyoungjun Kim, Sungjong No, and Seungsang Oh, \textit{Minimum lattice length and ropelength of 2-bridge knots and links}, J. Math. Phys. \textbf{55} (2014), no. 11, 113503, 11.

[JvRP95] E. J. Janse van Rensburg and S. D. Promislow, \textit{Minimal knots in the cubic lattice}, J. Knot Theory Ramifications \textbf{4} (1995), no. 1, 115–130.

[SIA+09] R. Scharein, K. Ishihara, J. Arsuaga, Y. Diao, K. Shimokawa, and M. Vazquez, \textit{Bounds for the minimum step number of knots in the simple cubic lattice}, J. Phys. A \textbf{42} (2009), no. 47, 475006, 24.

[UJvRO98] R. Uberti, E. J. Janse van Rensburg, E. Orlandini, M. C. Tesi, and S. G. Whittington, \textit{Minimal links in the cubic lattice}, Topology and
geometry in polymer science (Minneapolis, MN, 1996), IMA Vol. Math. Appl., vol. 103, Springer, New York, 1998, pp. 89–100.

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