Extracting analytic proofs from numerically solved
Shannon-type Inequalities

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I. INTRODUCTION

A class of information inequalities, called Shannon-type inequalities (STIs), can be proven via a computer software called ITIP \[1\]. In previous work \[2\], we have shown how this technique can be utilized to Fourier-Motzkin elimination algorithm for Information Theoretic Inequalities. Here, we provide an algorithm for extracting analytic proofs of information inequalities. Shannon-type inequalities are proven by solving an optimization problem. We will show how to extract a formal proof of numerically solved information inequality. Such proof may become useful when an inequality is implied by several constraints due to the PMF, and the proof is not apparent easily. More complicated are cases where an inequality holds due to both constraints from the PMF and due to other constraints that arise from the statistical model. Such cases include information theoretic capacity regions, rate-distortion functions and lossless compression rates. We begin with formal definition of Shannon-type information inequalities. We then review the optimal solution of the optimization problem and how to extract a proof that is readable to the user.

II. PRELIMINARIES AND NOTATIONS

We use the following notation. Calligraphic letters denote discrete sets, e.g., \( \mathcal{X} \). The empty set is denoted by \( \phi \), while \( \mathcal{N}_n \triangleq \{1, 2, \ldots, n\} \) is a set of indices. Lowercase letters, e.g. \( x \), represent variables. A vector of \( n \) variables \((x_1, \ldots, x_n)\) is denoted by \( x_{\mathcal{N}_n} \), and its substring as \( x_\alpha = (x_i \in \Omega \mid i \in \alpha, \phi \neq \alpha \subseteq \mathcal{N}_n) \), e.g., \( x_{\{1,2\}} = (x_1, x_2) \). Whenever the dimensions are clear from the context, the subscript is omitted. Vector inequalities, e.g., \( v \geq 0 \), are in element-wise sense. Random variables are denoted by uppercase letters, e.g., \( X \), with similar conventions for random vectors.

III. INFORMATION INEQUALITIES AND CONSTRAINTS

In \[3\], Yeung characterized a subset of information inequalities named Shannon-type inequalities (STIs), that are provable using a computer program called ITIP \[1\]. More work on the ITIP was done in \[4\]. This section consist of the mathematical review of this work. We establish a canonical form for linear combination of Shannon’s information measures, which uniquely represent the expression as a linear combination of joint entropies. By giving constraints of non-negativity on the information measures, we establish a region where linear information inequalities residue. A theorem provides a minimization problem, which can be solved by linear programming techniques, makes the identification of true information inequalities applicable.
A. Unconstrained inequalities

Given a random vector \( \mathbf{X} \) that takes values in \( \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \), define \( h_\ell \equiv \mathbf{H}(\mathbf{X}_\alpha | \phi \neq \alpha \subseteq \mathcal{N}_n) \). Let \( \mathcal{P} \) be the set of all probability mass functions (PMFs) over \( \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \). Moreover, for every \( p \in \mathcal{P} \), \( h_\ell(p) \in \mathbb{R}_n \) is a vector whose entries are the values of \( H(X_\alpha), \phi \neq \alpha \subseteq \mathcal{N}_n \), with respect to \( p \).

**Definition 1 (Basic information measure (BIM))** An information measure is called basic if it takes on one of the following forms:

\[
H(\mathbf{X}_\alpha | \mathbf{X}_\gamma)
\]

(1a)

\[
I(\mathbf{X}_\alpha; \mathbf{X}_\beta | \mathbf{X}_\gamma)
\]

(1b)

where \( \alpha, \beta, \gamma \subseteq \mathcal{N}_n \) and \( \alpha, \beta \neq \phi \).

**Definition 2 (Elemental information measure (EIM))** An information measure is called elemental if it takes on one of the following forms:

\[
H(\mathbf{X}_i | \mathcal{N}_n \setminus \{i\})
\]

(2a)

\[
I(\mathbf{X}_i; \mathbf{X}_j | \mathbf{X}_\mathcal{K})
\]

(2b)

where \( i, j \in \mathcal{N}_n, i \neq j, \mathcal{K} \subseteq \mathcal{N}_n \setminus \{i, j\} \)

**Lemma 1** Every BIM can be represented as a linear combination of EIMs with non-negative coefficients.

By the definition of mutual information and by the entropy chain rule, for every \( i, j \in \mathcal{N}_n, i \neq j \) and \( \mathcal{K} \subseteq \mathcal{N}_n \setminus \{i, j\} \), we have

\[
H(X_i | X_{\mathcal{N}_n \setminus \{i\}}) = H(X_{\mathcal{N}_n}) - H(X_{\mathcal{N}_n \setminus \{i\}}),
\]

(3a)

\[
I(X_i; X_j | X_{\mathcal{K}}) = H(X_i, X_{\mathcal{K}}) + H(X_j, X_{\mathcal{K}}) - H(X_{\mathcal{K}}) - H(X_i, X_j, X_{\mathcal{K}}).
\]

(3b)

Lemma 1 combined with (3) implies that every BIM is uniquely representable as a linear combination of unconditional joint entropies. This representation, which is called the canonical form, allows one to write every linear combination of BIMs as \( \mathbf{b}^\top \mathbf{h}_\ell \), where \( \mathbf{b} \) is a vector of coefficients.

1 We assume a lexicographical ordering of the elements of \( \mathbf{h}_\ell \).

2 For the proof of uniqueness see [3, Section 13.2].

3 Henceforth, an arbitrary linear combination of BIMs is denoted by \( \mathbf{b}^\top \mathbf{h}_\ell \).
Definition 3 An information inequality $\mathbf{b}^\top \mathbf{h}_\ell \geq 0$ always holds if $\mathbf{b}^\top \mathbf{h}_\ell(p) \geq 0$, for every $p \in \mathcal{P}$.

Proposition 1 An information inequality $\mathbf{b}^\top \mathbf{h}_\ell \geq 0$ always holds if and only if (iff)

$$\min_{p \in \mathcal{P}} \mathbf{b}^\top \mathbf{h}_\ell(p) = \min_{\mathbf{h}_\ell(p) \in \Gamma^*_n} \mathbf{b}^\top \mathbf{h}_\ell(p) = 0,$$

where

$$\Gamma^*_n = \bigcup_{p \in \mathcal{P}} \mathbf{h}_\ell(p).$$  \hspace{1cm} (4b)

Proposition 1 follows since there is always a $p \in \mathcal{P}$ for which $\mathbf{b}^\top \mathbf{h}_\ell(p) = 0$.

The optimization problem in (4) is infeasible as it involves optimizing over the set of all PMFs of $n$ discrete random variables. Therefore, an algorithm that numerically proves information inequalities requires an simpler alternative description of $\Gamma^*_n$. Such a description, being currently unknown, leads one to search for a different subspace of $\mathbb{R}^{2^n-1}$, that is, in a sense, similar to $\Gamma^*_n$, based on which numerical proofs can be implemented.

Definition 4 (Basic and elemental inequalities) Non-negativity inequalities on BIMs and EIMs are called basic inequalities (BIs) and elemental inequalities (EIs), respectively.

Every $\mathbf{h} \in \Gamma^*_n$ is a vector of entropies that is induced by some $p \in \mathcal{P}$, and, in particular, satisfies all BIs. Since BIs are linear constraints on $\mathbf{h}$, in [3],

$$\Gamma_n \triangleq \{ \mathbf{h} \in \mathbb{R}^{2^n-1} | \mathbf{h} \text{ satisfies all BIs} \}$$  \hspace{1cm} (5)

was proposed as an alternative for $\Gamma^*_n$.

Lemma 2 (Minimality of elemental inequalities) The set of EIs is minimal in sense that every BI is implied by a subset of EIs.

Remark 1 There are $n + \binom{n}{2} 2^{n-2}$ EIs while the amount of BIs is bounded by $\sum_{j=1}^{n} \binom{n}{j} \left(2^{n-j} + \sum_{i=1}^{n-j} \binom{n-j}{i} 2^{n-j-i} \right)$ from below.

Based on Remark 1 and Lemma 2 we write

$$\Gamma_n = \{ \mathbf{h} \in \mathbb{R}^{2^n-1} | \mathbf{Gh} \geq 0 \},$$  \hspace{1cm} (6)

where $\mathbf{G}$ is a matrix such that the elements of $\mathbf{Gh}_\ell$ are all EIMs.

Theorem 1 Let $\mathbf{b}^\top \mathbf{h}_\ell \geq 0$ be an information inequality, and let

$$\rho^* = \min_{\mathbf{h}_\ell} \mathbf{b}^\top \mathbf{h}.$$  \hspace{1cm} (7)
If $\rho^* = 0$ then $b^\top h_\ell \geq 0$ always holds.

The proof of Theorem \[\text{Proposition}\] follows from Proposition \[\text{Proposition}\] since every $h \in \Gamma^*_n$ satisfies all ELs, which implies that $\Gamma^*_n \subseteq \Gamma^*_3$. The optimization problem \[\text{Proposition}\] is solvable using linear programming (LP) optimization methods \[\text{Proposition}\]. Information inequalities that are provable by Theorem \[\text{Proposition}\] form a subset of inequalities called *unconstrained Shannon-type inequalities (STIs)*.

B. Constrained STIs

Some information inequalities (respectively, identities) hold only when a certain structure is imposed on the PMFs. Such a structure may account for independencies between random variables, Markov chains and functional dependencies. We formulate these constraints on the PMF domain as linear constraints on entropy and mutual information terms.

**Lemma 3** Let \{X\}_i be a collection of random variables.

1) \{X\}_i are mutually independent iff $H(X_{N_n}) = \sum_{i=1}^n H(X_i)$.

2) \{X\}_i are pairwise independent iff $I(X_i; X_j) = 0$, for every $1 \leq i \neq j \leq n$.

3) Let $\alpha, \beta \subseteq N_n$ and $\alpha, \beta \neq \phi$. $X_\alpha$ is a function of $X_\beta$ iff $H(X_\alpha|X_\beta) = 0$.

4) Let $\alpha, \beta, \gamma, \delta \subseteq N_n$ and $\alpha, \beta, \gamma, \delta \neq \phi$. $X_\alpha - X_\beta - X_\gamma - X_\delta$ forms a Markov chain iff $I(X_\alpha; X_\gamma, X_\delta|X_\beta) = 0$ and $I(X_\alpha, X_\beta; X_\delta|X_\gamma) = 0$.

Every set of constraints as in Lemma \[\text{Proposition}\] is representable by

$$Qh_\ell = 0$$

where $Q$ is a matrix whose rows are the coefficients that correspond to each constraint.

**Proposition 2** Given a PMF defined on discrete random variables, the constraints which induced by reading the joint PMF defines all probabilistic relations between the random variables.

For instance, given a PMF $P(X, Y, Z)$ where $X - Y - Z$ forms a Markov chain, we can write

$$P(X, Y, Z) = P(X)P(Y|X)P(Z|Y)$$

where $H(Z|Y) = H(Z|Y, X)$. The induced constraint is equivalent to $I(X; Z|Y) = 0$, which implies that $X - Y - Z$ forms a Markov chain. By Lemma \[\text{Proposition}\] all such constraints imply the joint PMF.

$^4$ If $\Gamma^*_2 = \Gamma^*_2$ but $\Gamma^*_3 \neq \Gamma^*_3$ [Section 15.1].

$^5$ Property 4 of Lemma \[\text{Proposition}\] is an extension of [Section 13.3.2] and can be generalized to account for longer Markov chains.
Theorem 2 Let $b^\top h_\ell \geq 0$ be an information inequality and

$$
\rho^* = \min_{\substack{h \geq 0 \\
Q h = 0}} b^\top h.
$$

(10)

If $\rho^* = 0$ then $b^\top h_\ell \geq 0$ always holds under the constraints $Qh_\ell = 0$.

Constrained information inequalities that are captured by Theorem 2 are called constrained STIs.

IV. Optimization problems and optimal solution

A. The Lagrangian dual function

An LP problem is a private case of convex optimization problems. The reader may refer to [6] for further study about convex optimization (here we use lemmas from Chapter 5). Consider an optimization problem in the standard form

$$
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
$$

(11)

with $x \in \mathbb{R}^n$. The function $f_0(x)$ is called the objective function.

We define $\mathcal{D}$ to be the domain of this problem,

$$
\mathcal{D} \triangleq \left\{ \bigcap_{i=0}^m \text{dom}(f_i) \right\} \cap \left\{ \bigcap_{i=1}^p \text{dom}(h_i) \right\}
$$

where $\text{dom}(\cdot)$ is the domain of its arguments. We assume nonempty domain, i.e., $\mathcal{D} \neq \emptyset$ and that there is an optimal solution for this problem.

Definition 5 (Tha Lagrangian) The Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ associated with the problem in (11) is

$$
L(x, \lambda, \nu) \triangleq f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)
$$

(12)

where $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^p$. We refer $\lambda_i$ as the Lagrange multiplier of the $i$-th inequality constraint and $\nu_i$ as the Lagrange multiplier of the $i$-th equality constraint.

Definition 6 (Lagrangian dual function) The Lagrangian dual function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is the infimum of the Lagrangian over $x$:

$$
g(\lambda, \nu) \triangleq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)
$$
Lemma 4  For any \( \lambda \geq 0 \) and any \( \nu \),

\[
g(\lambda, \nu) \leq p^*
\]

(13)

where \( p^* \) is the optimal solution of the problem in (11).

Let \( d^* \) be optimal solution of the following optimization problem

\[
\begin{align*}
\text{maximize:} & \quad g(\lambda, \nu) \\
\text{subject to:} & \quad \lambda \geq 0.
\end{align*}
\]

Note that since the problem in (11) is convex, (14) is also convex. We refer \( \lambda^*, \nu^* \) as the optimal Lagrange multipliers and \( b^* \) as the dual optimal solution of the problem in (14). In general, by Lemma 4 we know that \( d^* \leq p^* \). Furthermore, under specific conditions we achieve equality.

B. Strong duality

When \( d^* = p^* \), the solutions of both the dual and original optimization problems coincide. If that is the case, we say we have strong duality. We here assume all equality constraints are affine. Thus, the equality constraints in (11) can be replaced with \( Ax = 0 \).

Definition 7 (Slater’s condition (for affine constraints)) Assume that \( f_i(x) \) are affine functions of \( x \) for \( i = 1, \ldots, k \).

If there exists \( x \in \text{relint}(D) \), where \( \text{relint}(D) \) is the relative interior of \( D \), such that

\[
\begin{align*}
&f_i(x) \leq 0, \quad i = 1, \ldots, k \\
&f_i(x) < 0, \quad i = k + 1, \ldots, m \\
&Ax = 0
\end{align*}
\]

(15a)  \hspace{1cm} (15b) \hspace{1cm} (15c)

then we say that Slater’s condition holds.

Lemma 5  If Slater’s condition holds, then strong duality exists.

Remark 2  In LP problems, all constraints are affine, and therefore Slater’s condition reduce to weak inequality in all constraints. Moreover, if the solution of the problem is feasible, then Slater’s condition holds and we have strong duality.

\(^6\)In many information theoretic problems, the inequalities are affine functions of the entropies.
Consider an LP problem of the form

\[
\begin{align*}
\text{minimize:} & \quad c^\top x \\
\text{subject to:} & \quad Ax = b \\
& \quad B \leq d
\end{align*}
\]

The Lagrangian of this problem is

\[
L(x, \lambda, \nu) = c^\top x + \lambda^\top (Bx - d) + \nu^\top (Ax - b)
\]

and the dual function is

\[
g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)
\]

\[
= (-b^\top \nu - d^\top \lambda) + \inf_x \{(c + B\lambda + A^\top \nu)^\top x\}
\]

subject to \(\lambda \geq 0\).

**Lemma 6 (Optimal Lagrange multipliers of an LP problem)** If a solution to an LP problem with linear constraints exists, then

\[
A^\top \nu^* + B^\top \lambda^* + c = 0
\]

The proof of Lemma follows directly from the definition of the dual function, since

\[
g(\lambda, \nu) = \begin{cases} 
-b^\top \nu - d^\top \lambda & A^\top \nu + B^\top \lambda + c = 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

Recall that since Slater’s condition holds, \(g(\lambda^*, \nu^*) = p^*\). Consequently, using the optimal Lagrange multipliers, we can represent the linear objective function by means of linear combination of the constraints.

**V. Extracting formal proof from the optimal solution**

Recall from Section III-B that non-negativity of linear combination of information measures can be proven by solving an LP problem. Assume we want to prove the following inequality in the canonical form

\[
f_L^\top h_\ell(p) \leq f_R^\top h_\ell(p), \quad p \in Q
\]

where \(Q\) is a subspace of \(P\) where the constraints due to the PMF hold. Define

\[
f_D \triangleq f_R - f_L
\]
The corresponding LP problem we solve to check in it is an unconstrained STI is

\[
\begin{align*}
\text{minimize:} & \quad f_D^T h \\
\text{subject to:} & \quad -Gh \leq 0 \\
& \quad Qh = 0
\end{align*}
\]  

(23a)

(23b)

(23c)

where \(Gh_\ell \geq 0\) represent the elemental inequality and \(Qh_\ell = 0\) the constraints due to the PMF. Note that both inequality and equality constraints in (23) are affine. If a solution to that problem exists, by Lemma 6, we have

\[
f_D = G^T \lambda^* - Q^T \nu^*
\]  

(24)

Thus, we can represent the coefficients of the objective by rows of \(G\) and \(Q\). Define \(G_\ell(p)\) and \(Q_\ell(p)\) to be vectors with labels correspond to \(G\) and \(Q\), respectively. The labels in the components are information measures which are represented by rows of the corresponding matrix. For instance, assume a case where there are only two variables, \((X_1, X_2)\). By our definitions,

\[
h_\ell(p)^T = [H(X_1), \ H(X_2), \ H(X_1, X_2)]
\]  

(25a)

\[
G_\ell^T = [H(X_1|X_2), \ H(X_2|X_1), \ I(X_1; X_2)]
\]  

(25b)

\[
G = \begin{bmatrix}
0 & -1 & 1 \\
-1 & 0 & 1 \\
1 & 1 & -1
\end{bmatrix}
\]  

(25c)

Note that \(h_\ell^T(p)f_D\) is the canonical form the RHS minus the LHS of the inequality we aim to prove. Similarly, the components of \(h_\ell^T(p)G\) and \(h_\ell^T(p)Q\) are the canonical forms of \(G_\ell^T(p)\) and \(Q_\ell^T(p)\), respectively. From (24) we have

\[
h_\ell^T(p)(f_R - f_L) = h_\ell^T(p)G\lambda^* - h_\ell^T(p)Q\nu^*
\]  

(26a)

\[
= G_\ell^T(p)\lambda^* - Q_\ell^T(p)\nu^*
\]  

(26b)

We then obtain a representation of the difference between R.H.S and L.H.S by elemental inequality and PMF constraints. From this representation, proving the inequality is more apparent. Since \(\lambda \geq 0\) and all elemental information measures are nonnegative, it is easy to see that \(G_\ell^T(p)\lambda^*\) is nonnegative. As for \(Q_\ell^T(p)\nu^*\), it is a sum of information measures that are each equal to zero due Markov chains induced by the PMF. We refer this representation as the elemental form of the expression. Equality between the original representation and the elemental form can be shows either using information theoretic identities or by showing that the canonical forms of both are the same.
We provide an illustration of how the algorithm works and what is the extracted proof. Let \((A, B, C, D)\) be 4 random variables such that \(A \leftrightarrow B \leftrightarrow C \leftrightarrow D\) is a Markov chain. The PMF of those variables factorizes as follows

\[
P(a, b, c, d) = P(a, b)P(c|b)P(d|c)
\]

(27)

Following Proposition 2, we obtain the following constraints

\[
I(C; A|B) = 0
\]

(28)

\[
I(D; A, B|C) = 0.
\]

(29)

Since \(A \leftrightarrow B \leftrightarrow C \leftrightarrow D\) is a Markov chain,

\[
I(A; D) \leq I(B; C)
\]

(30)

Fig. 1: Demonstration of the algorithm
always holds due to the PMF factorization. Using the canonical form of the R.H.S minus the L.H.S, it can be verified that

\[ I(B; C) - I(A; D) = I(A; C|D) + I(B; C|A) + I(B; D|A, C)I(C; A|B) - I(D; A, B|C). \]  

(31a)

This representation clarify that \( I(B; C) - I(A; D) \geq 0 \) because of the Markov chains. An implementation\(^7\) of this algorithm is demonstrated in Fig. 1.

REFERENCES

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\(^7\)The algorithm was implemented in MATLAB.