An SIR epidemic model with free boundary

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Abstract. An SIR epidemic model with free boundary is investigated. This model describes the transmission of diseases. The behavior of positive solutions to a reaction-diffusion system in a radially symmetric domain is investigated. The existence and uniqueness of the global solution are given by the contraction mapping theorem. Sufficient conditions for the disease vanishing or spreading are given. Our result shows that the disease will not spread to the whole area if the basic reproduction number $R_0 < 1$ or the initial infected radius $h_0$ is sufficiently small even that $R_0 > 1$. Moreover, we prove that the disease will spread to the whole area if $R_0 > 1$ and the initial infected radius $h_0$ is suitably large.

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1 Introduction

Recently epidemic model has been received a great attention in mathematical ecology. To describe the development of an infectious disease, compartmental models have been given to separate a population into various classes based on the stages of infection \[2\]. The classical SIR model is described by partitioning the population into susceptible, infectious and recovered individuals, denoted by $S, I$ and $R$, respectively. Assume that the disease incubation period is negligible so that each susceptible individual becomes infectious and later recovers with a permanently or temporarily acquired immunity, then the SIR model is governed by the following system of differential equations:

\[
\begin{align*}
\dot{S}(t) & = -\beta S(t)I(t) - \mu_1 S(t) + b, \\
\dot{I}(t) & = \beta S(t)I(t) - \mu_2 I(t) - \alpha I(t), \\
\dot{R}(t) & = \alpha I(t) - \mu_3 R(t),
\end{align*}
\]

(1.1)

where the total population size has been normalized to one and the influx of the susceptible comes from a constant recruitment rate $b$. The death rate for the $S, I$ and $R$ class is, respectively, given by $\mu_1, \mu_2$ and $\mu_3$. Biologically, it is natural to assume that $\mu_1 < \min\{\mu_2, \mu_3\}$. The standard incidence of disease is denoted by $\beta SI$, where $\beta$ is the constant effective contact rate, which is the average number of contacts of the infectious per unit time. The recovery rate of the infectious is
denoted by $\alpha$ such that $1/\alpha$ is the mean time of infection.

In [13], the threshold behavior was given. The authors showed that the basic reproduction number $R_0 = \frac{b\beta}{\mu_1(\mu_2 + \alpha)}$ determines whether the disease dies out ($R_0 < 1$) or remains endemic ($R_0 > 1$). In [12], a complete analysis of the global dynamics of an ordinary differential equation model with multiple infectious stages was presented, showing the same threshold behavior. For other works on various types of SIR epidemic model, interested readers may refer to [1, 3, 11, 17, 20, 22, 26] and the references therein.

There are other compartmental combinations for modelling some other diseases. For example, the SI model describes a disease, such as herpes or HIV, with two stages, where individuals are infectious for life and never removed. The SIS model describes the case when individuals recover from the disease but there is no immunity, and they return to the susceptible class. Examples for this SIS model include sexually transmitted diseases, plague and meningitis. Unlike SIR models, SEI models [9, 19] assume that a susceptible individual first goes through a latent (exposed) period before becoming infectious. An example of this model is the transmission of SARS [25], which is one of the serious diseases that human beings face at present.

When the distribution of the distinct classes is in different spatial locations, the diffusion terms should be taken into consideration and thus an extended version
of the above SIR system (1.1) can be described as the following:

\[
\begin{align*}
S_t - d_1 \Delta S &= -\beta S(t)I(t) - \mu_1 S(t) + b, & x \in \Omega, & t > 0, \\
I_t - d_2 \Delta I &= \beta S(t)I(t) - \mu_2 I(t) - \alpha I(t), & x \in \Omega, & t > 0, \\
R_t - d_3 \Delta R &= \alpha I(t) - \mu_3 R(t), & x \in \Omega, & t > 0, \\
\partial_\eta S = \partial_\eta I = \partial_\eta R &= 0, & x \in \partial \Omega, & t > 0, \\
S(x, 0) = S_0(x), & I(x, 0) = I_0(x), & R(x, 0) = R_0(x), & x \in \Omega,
\end{align*}
\]  

(1.2)

where \( \Omega \) is a fixed and bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), and \( \eta \) is the outward unit normal vector on the boundary. Here the homogeneous Neumann boundary condition implies that the above system is self-contained and there is no emigration across the boundary. The positive constants \( d_i \) \((i = 1, 2, 3)\) are the diffusion coefficients.

It must be pointed out that the solution of system (1.2) is always positive for any time \( t > 0 \) no matter what the nonnegative nontrivial initial date is. It means that the disease spreads to the whole area immediately even when the infectious is confined to a small part of the area in the beginning. It doesn’t match the observed fact that disease always spreads gradually. Recently the free boundary has been introduced in many areas, especially the well-known Stefan condition has been used to describe the spreading process. For example, it was used in describing the melting of ice in contact with water \([24]\), in the modeling of oxygen in the muscle \([6]\), and in the dynamics of population \([14, 18, 21, 23]\). There is a vast literature on the Stefan problem, and some recent and theoretically advanced results can be found in \([4]\).

Motivated by the statements mentioned above, we are attempting to consider a
SIR epidemic model with a free boundary, which describes the spreading frontier of the disease. For simplicity, we assume the environment is radially symmetric. We will investigate the behavior of the positive solution \((S(r, t), I(r, t), R(r, t); h(t))\) with \(r = |x|\) and \(x \in \mathbb{R}^n\) in the following problem:

\[
\begin{align*}
S_t - d_1 \Delta S &= b - \beta S(r, t)I(r, t) - \mu_1 S(r, t), & r > 0, t > 0, \\
I_t - d_2 \Delta I &= \beta S(r, t)I(r, t) - \mu_2 I(r, t) - \alpha I(r, t), & 0 < r < h(t), t > 0, \\
R_t - d_3 \Delta R &= \alpha I(r, t) - \mu_3 R(r, t), & 0 < r < h(t), t > 0, \\
S_r(0, t) = I_r(0, t) = R_r(0, t) &= 0, & t > 0, \\
I(r, t) = R(r, t) &= 0, & r \geq h(t), t > 0, \\
h'(t) &= -\mu I_r(h(t), t), & h(0) = h_0 > 0, t > 0, \\
S(r, 0) = S_0(r), I(r, 0) = I_0(r), R(r, 0) = R_0(r) &= r \geq 0,
\end{align*}
\]

where \(\Delta w = w_{rr} + \frac{n-1}{r} w_r\), \(r = h(t)\) is the moving boundary to be determined, \(h_0, d_i\) and \(\mu\) are positive constants. The initial functions \(S_0, I_0\) and \(R_0\) are nonnegative and satisfy

\[
\begin{align*}
S_0 &\in C^2([0, +\infty)), I_0, R_0 \in C^2([0, h_0]), \\
I_0(r) = R_0(r) &= 0, r \in [h_0, +\infty) \text{ and } I_0(r) > 0, r \in [0, h_0).
\end{align*}
\]

Ecologically, this model means that beyond the free boundary \(r = h(t)\), there is only susceptible, no infectious or recovered individuals. The equation governing the free boundary, \(h'(t) = -\mu I_r(h(t), t)\), is a special case of the well-known Stefan condition, which has been established in [21] for the diffusive populations.

The remainder of this paper is organized as follows. In the next section, we first apply a contraction mapping theorem to prove the global existence and uniqueness of the solution to the problem (1.3). Then we make use of the Hopf Lemma to give the monotonicity of the free boundary. Section 3 is devoted to prove that
the disease will vanish if the basic reproduction number $R_0 < 1$. In Section 4, we discuss the case $R_0 > 1$. Our results show that for the case $R_0 > 1$, the disease will spread to the whole area if $h_0$ is suitably large; while the disease will vanish if $h_0$ is sufficiently small. Our arguments are based on the comparison principle and the construction of appropriate supper solution of (1.3). Finally, we give a brief discussion in Section 5.

2 Existence and uniqueness

In this section, we first prove the following local existence and uniqueness result by the contraction mapping theorem. We then use suitable estimates to show that the solution is defined for all $t > 0$.

**Theorem 2.1** For any given $(S_0, I_0, R_0)$ satisfying (1.4) and any $\gamma \in (0, 1)$, there is a $T > 0$ such that problem (1.3) admits a unique bounded solution

$$(S, I, R; h) \in C^{1+\gamma,(1+\gamma)/2}(D^\infty_T) \times [C^{1+\gamma,(1+\gamma)/2}(D_T)]^2 \times C^{1+\gamma/2}([0, T]);$$

moreover,

$$\|S\|_{C^{1+\gamma,(1+\gamma)/2}(D^\infty_T)} + \|I\|_{C^{1+\gamma,(1+\gamma)/2}(D_T)} + \|R\|_{C^{1+\gamma,(1+\gamma)/2}(D_T)} + \|h\|_{C^{1+\gamma/2}([0,T])} \leq C(2.1)$$

where $D^\infty_T = \{(r,t) \in \mathbb{R}^2 : r \in [0, +\infty), t \in [0, T]\}$ and $D_T = \{(r,t) \in \mathbb{R}^2 : r \in [0, h(t)], t \in [0, T]\}$. Here $C$ and $T$ only depend on $h_0, \gamma, \|S_0\|_{C^2([0,\infty))}, \|I_0\|_{C^2([0,h_0])}$ and $\|R_0\|_{C^2([0,h_0])}$. 

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Proof: We first straighten the free boundary as in [5]. Let $\xi(s)$ be a function in $C^3[0, \infty)$ satisfying

$$
\xi(s) = 1 \text{ if } |s - h_0| < \frac{h_0}{8}, \quad \xi(s) = 0 \text{ if } |s - h_0| > \frac{h_0}{2}, \quad |\xi'(s)| < \frac{5}{h_0} \text{ for all } s.
$$

Consider the transformation

$$(y, t) \rightarrow (x, t), \text{ where } x = y + \xi(|y|)(h(t) - h_0y/|y|), \quad y \in \mathbb{R}^n,$$

which leads to the transformation

$$(s, t) \rightarrow (r, t) \text{ with } r = s + \xi(s)(h(t) - h_0), \quad 0 \leq s < \infty.$$ 

As long as

$$|h(t) - h_0| \leq \frac{h_0}{8},$$

the above transformation $x \rightarrow y$ is a diffeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$ and the transformation $s \rightarrow r$ is also a diffeomorphism from $[0, +\infty)$ onto $[0, +\infty)$. Moreover, it changes the free boundary $r = h(t)$ to the line $s = h_0$. Now, direct calculations show that

$$
\frac{\partial s}{\partial r} = \frac{1}{1 + \xi'(s)(h(t) - h_0)} := \sqrt{A(h(t), s)},
$$

$$
\frac{\partial^2 s}{\partial r^2} = -\frac{\xi''(s)(h(t) - h_0)}{[1 + \xi'(s)(h(t) - h_0)]^3} := B(h(t), s),
$$

$$
-\frac{1}{h'(t)} \frac{\partial s}{\partial t} = \frac{\xi(s)}{1 + \xi'(s)(h(t) - h_0)} := C(h(t), s),
$$

$$
\frac{(n - 1)\sqrt{A}}{s + \xi(s)(h(t) - h_0)} := D(h(t), s).
$$
Now, if we set

\[ S(r, t) = S(s + \xi(s)(h(t) - h_0), t) := u(s, t), \]

\[ I(r, t) = I(s + \xi(s)(h(t) - h_0), t) := v(s, t), \]

\[ R(r, t) = R(s + \xi(s)(h(t) - h_0), t) := w(s, t), \]

then the free boundary problem (1.3) becomes

\[
\begin{aligned}
&u_t - Ad_1 \Delta_s u - (Bd_1 + h'C + Dd_1)u_s = b - \beta uv - \mu_1 u, \quad s > 0, \ t > 0, \\
v_t - Ad_2 \Delta_s v - (Bd_2 + h'C + Dd_2)v_s = \beta uv - \mu_2 v - \alpha v, \quad 0 < s < h_0, \ t > 0, \\
w_t - Ad_3 \Delta_s w - (Bd_3 + h'C + Dd_3)w_s = \alpha v - \mu_3 w, \quad 0 < s < h_0, \ t > 0, \\
u_s(0, t) = v_s(0, t) = w_s(0, t) = 0, \quad t > 0, \\
v(s, t) = w(s, t) = 0, \quad s \geq h_0, \ t > 0, \\
h'(t) = -\mu v_s(h_0, t), \quad h(0) = h_0, \\
u(s, 0) = u_0(s), \quad v(s, 0) = v_0(s), \quad w(s, 0) = w_0(s), \\
\end{aligned}
\]

(2.2)

where \( A = A(h(t), s), \ B = B(h(t), s), \ C = C(h(t), s), \ D = D(h(t), s) \) and \( u_0 = S_0, v_0 = I_0, w_0 = R_0. \)

We denote \( h^* = -\mu v_0'(h_0), \) and for \( 0 < T \leq \frac{h_0}{s(1 + h^*)}, \) set

\[ H_T = \left\{ h \in C^1([0, T]) : h(0) = h_0, \ h'(0) = h^*, \ ||h' - h^*||_{C([0, T])} \leq 1 \right\}, \]

\[ U_T = \left\{ u \in C([0, +\infty) \times [0, T]) : u(s, 0) = u_0(s), \ ||u - u_0||_{L^\infty([0, +\infty) \times [0, T])} \leq 1 \right\}, \]

\[ V_T = \left\{ v \in C([0, \infty) \times [0, T]) : v(s, t) \equiv 0 \ for \ s \geq h_0, \ 0 \leq t \leq T, \quad v(s, 0) = v_0(s) \ for \ 0 \leq s \leq h_0, \ ||v - v_0||_{L^\infty([0, \infty) \times [0, T])} \leq 1 \right\}, \]

\[ W_T = \left\{ w \in C([0, \infty) \times [0, T]) : w(s, t) \equiv 0 \ for \ s \geq h_0, \ 0 \leq t \leq T, \quad w(s, 0) = w_0(s) \ for \ 0 \leq s \leq h_0, \ ||w - w_0||_{L^\infty([0, \infty) \times [0, T])} \leq 1 \right\}. \]
Noticing the fact that for \( h_1, h_2 \in H_T \), due to \( h_1(0) = h_2(0) = h_0 \), we have

\[
\|h_1 - h_2\|_{C([0, T])} \leq T\|h'_1 - h'_2\|_{C([0, T])},
\]

(2.3)

it is not difficult to see that \( \Gamma_T := U_T \times V_T \times W_T \times H_T \) is a complete metric space with the metric

\[
D((u_1, v_1, w_1; h_1), (u_2, v_2, w_2; h_2)) = \|u_1 - u_2\|_{L^\infty([0, +\infty) \times [0, T])} + \|v_1 - v_2\|_{L^\infty([0, +\infty) \times [0, T])} + \|w_1 - w_2\|_{L^\infty([0, +\infty) \times [0, T])} + \|h'_1 - h'_2\|_{C([0, T])}.
\]

Next, we shall prove the existence and uniqueness result by using the contraction mapping theorem. Applying standard \( L^p \) theory and the Sobolev imbedding theorem \[15\], we can find that for any \((u, v, w; h) \in \Gamma_T \), the following initial boundary value problem

\[
\begin{cases}
\tilde{u}_t - Ad_1 \Delta_s \tilde{u} - (Bd_1 + h'C + Dd_1)\tilde{u}_s = b - \beta vw - \mu_1 u, & s > 0, \ t > 0, \\
\tilde{v}_t - Ad_2 \Delta_s \tilde{v} - (Bd_2 + h'C + Dd_2)\tilde{v}_s = \beta uv - \mu_2 v - \alpha w, & 0 < s < h_0, \ t > 0, \\
\tilde{w}_t - Ad_3 \Delta_s \tilde{w} - (Bd_3 + h'C + Dd_3)\tilde{w}_s = \alpha v - \mu_3 w, & 0 < s < h_0, \ t > 0, \\
\tilde{u}_s(0, t) = \tilde{v}_s(0, t) = \tilde{w}_s(0, t) = 0, & t > 0, \\
\tilde{v}(s, t) = \tilde{w}(s, t) = 0, & s \geq h_0, \ t > 0, \\
\tilde{u}(s, 0) = u_0(s), \tilde{v}(s, 0) = v_0(s), \tilde{w}(s, 0) = w_0(s), & s \leq 0
\end{cases}
\]

(2.4)

admits a unique solution

\[
(\tilde{u}, \tilde{v}, \tilde{w}) \in \left[C^{1+\gamma, (1+\gamma)/2}([0, +\infty) \times [0, T])\right]^3
\]

and

\[
\|\tilde{u}\|_{C^{1+\gamma, (1+\gamma)/2}([0, +\infty) \times [0, T])} \leq K_1,
\]

(2.5)

\[
\|\tilde{v}\|_{C^{1+\gamma, (1+\gamma)/2}([0, h_0] \times [0, T])} \leq K_1,
\]

(2.6)

\[
\|\tilde{w}\|_{C^{1+\gamma, (1+\gamma)/2}([0, h_0] \times [0, T])} \leq K_1,
\]

(2.7)
where $K_1$ is a constant depending on $\gamma, h_0, \|S_0\|_{C^2[0, +\infty)}, \|I_0\|_{C^2[0, h_0]}$ and $\|R_0\|_{C^2[0, h_0]}$.

Now, we define $\tilde{h}(t)$ by the sixth equation in (2.2) as the following:

$$\tilde{h}(t) = h_0 - \mu \int_0^t \tilde{v}_s(h_0, \tau)d\tau,$$

then we have $\tilde{h}'(t) = -\mu \tilde{v}_s(h_0, t)$, $\tilde{h}(0) = h_0$ and $\tilde{h}'(0) = -\mu \tilde{v}_0(h_0) = h^*$. Hence $\tilde{h}'(t) \in C^{\gamma/2}([0, T])$ with

$$\|\tilde{h}'(t)\|_{C^{\gamma/2}([0, T])} \leq K_2 := \mu K_1.$$

In what follows, we define a map

$$\mathcal{F} : \Gamma_T \longrightarrow [C([0, +\infty) \times [0, T])]^3 \times C^1([0, T])$$

by $\mathcal{F}(u(s, t), v(s, t), w(s, t); h(t)) = (\tilde{u}(s, t), \tilde{v}(s, t), \tilde{w}(s, t); \tilde{h}(t))$. It is obvious that $(u(s, t), v(s, t), w(s, t); h(t)) \in \Gamma_T$ is a fixed point of $\mathcal{F}$ if and only if it solves (2.2).

Similarly as in [7], there is a $T > 0$ such that $\mathcal{F}$ is a contraction mapping in $\Gamma_T$. It follows from the contraction mapping theorem that there is a $(u(s, t), v(s, t), w(s, t); h(t))$ in $\Gamma_T$ such that

$$\mathcal{F}(u(s, t), v(s, t), w(s, t); h(t)) = (u(s, t), v(s, t), w(s, t); h(t)).$$

In other words, $(u(s, t), v(s, t), w(s, t); h(t))$ is the solution of the problem (2.2) and thereby $(S(r, t), I(r, t), R(r, t); h(t))$ is the solution of the problem (1.3). Moreover, by using the Schauder estimates, we have additional regularity of the solution, $h(t) \in C^{1+\gamma/2}([0, T]), S \in C^{2+\gamma, 1+\gamma/2}((0, +\infty) \times (0, T))$ and $I, R \in C^{2+\gamma, 1+\gamma/2}((0, h(t)) \times (0, T))$.
(0, T]). Thus \((S(r, t), I(r, t), R(r, t); h(t))\) is the classical solution of the problem \((1.3)\).

To show that the local solution obtained in Theorem 2.1 can be extended to all \(t > 0\), we need the following estimate.

**Lemma 2.2** Let \((S, I, R; h)\) be a bounded solution to problem \((1.3)\) defined for \(t \in (0, T_0)\) for some \(T_0 \in (0, +\infty]\). Then there exist constants \(C_1\) and \(C_2\) independent of \(T_0\) such that

\[0 < S(r, t) \leq C_1 \text{ for } 0 \leq r < +\infty, \ t \in (0, T_0).\]

\[0 < I(r, t), R(r, t) \leq C_2 \text{ for } 0 \leq r < h(t), \ t \in (0, T_0).\]

**Proof:** It is easy to see that \(S \geq 0, I \geq 0\) and \(R \geq 0\) in \([0, +\infty) \times [0, T_0]\) as long as the solution exists.

Using the strong maximum principle to the equations in \([0, h(t)] \times [0, T_0]\), we immediately obtain

\[S(r, t), I(r, t), R(r, t) > 0 \text{ for } 0 \leq r < h(t), \ 0 < t < T_0.\]

The upper bounds of the solution are followed from the maximum principle, we omit the proof here. \(\square\)

The next lemma shows that the free boundary for problem \((1.3)\) is strictly monotone increasing.

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Lemma 2.3  Let \((S, I, R; h)\) be a solution to problem (1.3) defined for \(t \in (0, T_0)\) for some \(T_0 \in (0, +\infty]\). Then there exists a constant \(C_3\) independent of \(T_0\) such that
\[
0 < h'(t) \leq C_3 \quad \text{for } t \in (0, T_0).
\]

Proof: Using the Hopf Lemma to the equation of \(I\) yields that
\[
I_r(h(t), t) < 0 \quad \text{for } 0 < t < T_0.
\]
Hence \(h'(t) > 0\) for \(t \in (0, T_0)\) from the Stefan condition.

Next we show that \(h'(t) \leq C_3\) for all \(t \in (0, T_0)\) and some \(C_3\) independent of \(T_0\). As in [21], we define
\[
\Omega = \Omega_M := \{(r, t) : h(t) - M^{-1} < r < h(t), \ 0 < t < T_0\}
\]
and construct an auxiliary function
\[
w(r, t) := C_2[2M(h(t) - r) - M^2(h(t) - r)^2].
\]
We will choose \(M\) so that \(w(r, t) \geq I(r, t)\) holds over \(\Omega\).

Direct calculations show that, for \((r, t) \in \Omega\),
\[
w_t = 2C_2Mh'(t)(1 - M(h(t) - r)) \geq 0,
\]
\[-\Delta w = 2C_2M^2, \quad \beta SI - (\mu_2 + \alpha)I \leq \beta C_1C_2,
\]
and then
\[
w_t - d_2\Delta w \geq 2d_2C_2M^2 \geq \beta C_1C_2 \text{ in } \Omega.
\]
if $M^2 \geq \frac{\beta C_1}{2d_2}$. On the other hand, we have

$$w(h(t) - M^{-1}, t) = C_2 \geq I(h(t) - M^{-1}, t), \quad w(h(t), t) = 0 = I(h(t), t).$$

Hence, if we can choose $M$ such that $I_0(r) \leq w(r, 0)$ for $r \in [h_0 - M^{-1}, h_0]$, then we can apply the maximum principle to $w - I$ over $\Omega$ to deduce that $I(r, t) \leq w(r, t)$ for $(r, t) \in \Omega$. It would then follow that

$$I_r(h(t), t) \geq w_r(h(t), t) = -2MC_2, \quad h'(t) = -\mu I_r(h(t), t) \leq C_3 := 2MC_2\mu.$$

To complete the proof, we only have to find some $M$ independent of $T_0$ such that $I_0(r) \leq w(r, 0)$ for $r \in [h_0 - M^{-1}, h_0]$. We calculate

$$w_r(r, 0) = -2C_2M[1 - M(h_0 - r)] \leq -C_2M \text{ for } r \in [h_0 - (2M)^{-1}, h_0].$$

Then upon choosing

$$M := \max \left\{ \sqrt{\frac{\beta C_1}{2d_2}}, \frac{4\|I_0\|_{C^1([0, h_0])}}{3C_2} \right\},$$

we have

$$w_r(r, 0) \leq -MC_2 \leq -\frac{4}{3}\|I_0\|_{C^1} \leq I'_0(r) \text{ for } r \in [h_0 - (2M)^{-1}, h_0].$$

Since $w(h_0, 0) = I_0(h_0) = 0$, the above inequality implies that

$$w(r, 0) \geq I_0(r) \text{ for } r \in [h_0 - (2M)^{-1}, h_0].$$

Moreover, for $r \in [h_0 - M^{-1}, h_0 - (2M)^{-1}]$, we have

$$w(r, 0) \geq \frac{3}{4}C_2, \quad I_0(r) \leq \|I_0\|_{C^1([0, h_0])}M^{-1} \leq \frac{3}{4}C_2.$$

Therefore $I_0(r) \leq w(r, 0)$ for $r \in [h_0 - M^{-1}, h_0]$. This completes the proof. \qed
Theorem 2.4 The solution of the problem (1.3) exists and is unique for all \( t \in (0, \infty) \).

Proof: It follows from the uniqueness of the solution that there is a number \( T_{\text{max}} \) such that \([0, T_{\text{max}})\) is the maximal time interval in which the solution exists. Now we prove that \( T_{\text{max}} = \infty \) by the contradiction argument. Assume that \( T_{\text{max}} < \infty \). Then it follows from Lemma 2.2 that there exist \( C_1, C_2 \) and \( C_3 \) independent of \( T_{\text{max}} \) such that for \( t \in [0, T_{\text{max}}) \) and \( r \in [0, h(t)] \),

\[
0 \leq S(r, t) \leq C_1, \quad (r, t) \in [0, +\infty) \times [0, T_{\text{max}}),
\]

\[
0 \leq I(r, t), R(r, t) \leq C_2, \quad (r, t) \in [0, h(t)] \times [0, T_{\text{max}}),
\]

\[
h_0 \leq h(t) \leq h_0 + C_3t, \quad 0 \leq h'(t) \leq C_3, \quad t \in [0, T_{\text{max}}).
\]

We now fix \( \delta_0 \in (0, T_{\text{max}}) \) and \( M > T_{\text{max}} \). Then by the standard parabolic regularity, we can find \( C_4 > 0 \) depending only on \( \delta_0, M, C_1, C_2 \) and \( C_3 \) such that

\[
||S(\cdot, t)||_{C^{1+\gamma}[0, +\infty)}, \quad ||I(\cdot, t)||_{C^{1+\gamma}[0, h(t)]}, \quad ||R(\cdot, t)||_{C^{1+\gamma}[0, h(t)]} \leq C_4
\]

for \( t \in [\delta_0, T_{\text{max}}) \). It then follows from the proof of Theorem 2.1 that there exists a \( \tau > 0 \) depending only on \( C_i (i = 1, 2, 3, 4) \) such that the solution of problem (1.3) with initial time \( T_{\text{max}} - \tau/2 \) can be extended uniquely to the time \( T_{\text{max}} - \tau/2 + \tau \).

But this contradicts the assumption and thereby the proof is complete. \( \square \)

Remark 2.1 It follows from the uniqueness of the solution to (1.3) and some standard compactness arguments that the unique solution \((S, I, R, h)\) depends continu-
ously on the parameters appearing in \( L^3 \). This fact will be used in the following sections hereafter.

We next decide when the transmission of diseases is spreading or vanishing. We need to divide our discussion into two cases: \( R_0 < 1 \) and \( R_0 > 1 \).

### 3 The case \( R_0 < 1 \)

It follows from Lemma 2.3 that \( r = h(t) \) is monotonic increasing and therefore there exists \( h_\infty \in (0, +\infty] \) such that \( \lim_{t \to +\infty} h(t) = h_\infty \). The following theorem shows that the transmission of diseases is vanishing in the case that \( R_0 < 1 \).

**Theorem 3.1** If \( R_0 =: \frac{b\beta}{\mu_1(\mu_2 + \alpha)} < 1 \), then \( \lim_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} = 0 \) and \( h_\infty < \infty \). Moreover, \( \lim_{t \to +\infty} ||R(\cdot, t)||_{C([0, h(t)])} = 0 \) and \( \lim_{t \to +\infty} S(r, t) = \frac{b}{\mu_1} \) uniformly in any bounded subset of \([0, \infty)\).

**Proof:** It follows from the comparison principle that \( S(r, t) \leq \overline{S}(t) \) for \( r \geq 0 \) and \( t \in (0, +\infty) \), where

\[
\overline{S}(t) := \frac{b}{\mu_1} + (||S_0||_{\infty} - \frac{b}{\mu_1})e^{-\mu_1 t},
\]

which is the solution of the problem

\[
\frac{d\overline{S}}{dt} = b - \mu_1 \overline{S}, \quad t > 0; \quad \overline{S}(0) = ||S_0||_{\infty}. \tag{3.1}
\]

Since \( \lim_{t \to \infty} \overline{S}(t) = \frac{b}{\mu_1} \), we deduce that

\[
\limsup_{t \to +\infty} S(r, t) \leq \frac{b}{\mu_1} \quad \text{uniformly for } r \in [0, \infty).
\]
Recalling the condition $R_0 < 1$, there exists $T_0$ such $S(r, t) \leq \frac{b}{\mu_1} \frac{1+R_0}{2R_0}$ in $[0, \infty) \times [T_0, +\infty)$. Now $I(r, t)$ satisfies

$$
\begin{align*}
  & I_t - d_2 \Delta I \leq \left[ \frac{b}{\mu_1} \frac{1+R_0}{2R_0} - \mu_2 - \alpha \right] I(r, t), \quad 0 < r < h(t), \ t > T_0, \\
  & I(r, t) = 0, \quad r = h(t), \ t > 0, \\
  & I(r, T_0) > 0, \quad 0 \leq r \leq h(T_0).
\end{align*}
$$

Therefore $\|I(\cdot, t)\|_{C([0, h(t)])} \to 0$ as $t \to \infty$, since that $\beta \frac{b}{\mu_1} \frac{1+R_0}{2R_0} - \mu_2 - \alpha < 0$. We then have $\|R(\cdot, t)\|_{C([0, h(t)])} \to 0$ as $t \to \infty$ from the third equation of (1.3).

Next we show that $h_{\infty} < +\infty$. In fact, direct calculation yields

$$
\frac{d}{dt} \int_0^{h(t)} r^{n-1} I(r, t)dr = \int_0^{h(t)} r^{n-1} I_t(r, t)dr + h'(t)h^{n-1}(t)I(h(t), t)
$$

$$
= \int_0^{h(t)} d_2 r^{n-1} \Delta I dr + \int_0^{h(t)} I(r, t)(\beta S(r, t) - \mu_2 - \alpha) r^{n-1} dr
$$

$$
= \int_0^{h(t)} d_2 (r^{n-1} I_r(r, t)) dr + \int_0^{h(t)} I(r, t)(\beta S(r, t) - \mu_2 - \alpha) r^{n-1} dr
$$

$$
= - \frac{d_2}{\mu} h^{n-1} h'(t) + \int_0^{h(t)} I(r, t)(\beta S(r, t) - \mu_2 - \alpha) r^{n-1} dr.
$$

Integrating from $T_0$ to $t (> T_0)$ yields

$$
\int_0^{h(t)} r^{n-1} I(r, t)dr = \int_0^{h(T_0)} r^{n-1} I(r, T_0)dr + \frac{d_2}{n\mu} h^n(T_0) - \frac{d_2}{n\mu} h^n(t)
$$

$$
+ \int_{T_0}^{t} \int_0^{h(s)} I(r, s)(\beta S(r, s) - \mu_2 - \alpha) r^{n-1} dr ds, \quad t \geq T_0. \tag{3.3}
$$

Since $0 < S(r, t) \leq \frac{b}{\mu_1} \frac{1+R_0}{2R_0}$ for $r \in [0, h(t))$ and $t \geq T_0$, we have $\beta S(r, t) - \mu_2 - \alpha \leq 0$ for $t \geq T_0$.

$$
\int_0^{h(t)} r^{n-1} I(r, t)dr \leq \int_0^{h(T_0)} r^{n-1} I(r, T_0)dr + \frac{d_2}{n\mu} h^n(T_0) - \frac{d_2}{n\mu} h^n(t) \text{ for } t \geq T_0,
$$
which in turn gives that \( h_\infty < \infty \).

Then it follows from the first equation of (1.3) that \( \lim_{t \to +\infty} S(r, t) = \frac{b}{\mu_1} \)
uniformly in any bounded subset of \([0, \infty)\).

4 The case \( R_0 > 1 \)

In order to study the case that the reproduction number \( R_0 > 1 \), and for later applications, we need a comparison principle, which can be used to estimate \( S(r, t) \), \( I(r, t) \), \( R(r, t) \) and the free boundary \( r = h(t) \). As in [7], the following comparison lemma can be obtained analogously.

**Lemma 4.1** Suppose that \( T \in (0, \infty) \), \( \bar{h} \in C^1([0, T]) \), \( \bar{S} \in C([0, \infty) \times [0, T]) \) \( \cap \)
\( C^{2, 1}((0, \infty) \times (0, T]) \), \( \bar{I}, \bar{R} \in C(\bar{D}_T^+) \cap C^{2, 1}(D_T^*) \) with \( D_T^* = \{(r, t) \in \mathbb{R}^2 : 0 < r < \bar{h}(t), 0 < t \leq T \} \), and

\[
\begin{align*}
\bar{S}_t - d_1 \Delta \bar{S} & \geq b - \mu_1 \bar{S}, \\
\bar{I}_t - d_2 \Delta \bar{I} & \geq (\beta \bar{S} - \mu_2 - \alpha) \bar{I}, \\
\bar{R}_t - d_3 \Delta \bar{R} & \geq \alpha \bar{I} - \mu_3 \bar{R}, \\
\bar{S}_r(0, t) & \geq 0, \quad \bar{I}_r(0, t) \geq 0, \quad \bar{R}_r(0, t) \geq 0, \\
\bar{I}(r, t) = \bar{R}(r, t) & = 0, \quad 0 < r < \bar{h}(t), 0 < t \leq T, \\
\bar{I}(r, 0) & = \bar{R}(r, 0) = 0, \quad 0 \leq r \leq \bar{h}(0). \\
\end{align*}
\]

Then the solution \((S, I, R; h)\) of free boundary problem (1.3) satisfies

\[
S(r, t) \leq \bar{S}(r, t), \quad h(t) \leq \bar{h}(t) \quad \text{for } r \in (0, \infty) \text{ and } t \in (0, T],
\]

\[
I(r, t) \leq \bar{I}(r, t), \quad R(r, t) \leq \bar{R}(r, t) \quad \text{for } r \in (0, h(t)) \text{ and } t \in (0, T].
\]
Next we show that if $h_0$ and $\mu$ are sufficiently small, the disease is vanishing for the case $R_0 > 1$.

**Theorem 4.2** If $R_0(\equiv \frac{b\beta}{\mu_1(\mu_2+\alpha)}) > 1$, $h_0 \leq \min\left\{\sqrt{\frac{d_2}{16k_0}}, \sqrt{\frac{d_4}{16\alpha}}\right\}$ and $\mu \leq \frac{d}{8M}$, then $h_\infty < \infty$. Where $k_0 = \beta C_1 - \mu_2 - \alpha > 0$, $C_1 = \max\left\{||S_0||_\infty, \frac{b}{\mu_1}\right\}$, and $M = \frac{4}{3} \max\left\{||I_0||_\infty, ||R_0||_\infty\right\}$.

**Proof:** We are going to construct a suitable upper solution to (1.3) and then apply Lemma 4.1. As in [7], we define

\[
\bar{S}(r, t) = C_1,
\]

\[
\bar{I} = \begin{cases} 
Me^{-\gamma t}V(r/h(t)), & 0 \leq r \leq h(t), \\
0, & r > h(t),
\end{cases}
\]

\[
\bar{R} = \begin{cases} 
Me^{-\gamma t}V(r/h(t)), & 0 \leq r \leq h(t), \\
0, & r > h(t),
\end{cases}
\]

and

\[
\bar{h}(t) = 2h_0(2 - e^{-\gamma t}), \quad t \geq 0; \quad V(y) = 1 - y^2, \quad 0 \leq y \leq 1,
\]

where $C_1 = \max\left\{||S_0||_\infty, \frac{b}{\mu_1}\right\}$, $\gamma$ and $M$ are positive constants to be chosen later.
Denoting $k_0 = \beta C_1 - \mu_2 - \alpha$, we have $k_0 > 0$ since $R_0 > 1$. Direct computations yield

\[ \bar{S}_t - d_1 \Delta \bar{S} = 0 \geq b - \mu_1 \bar{S}, \]
\[ \bar{I}_t - d_2 \Delta \bar{I} - (\beta \bar{S} - \mu_2 - \alpha) \bar{I} \]
\[ = \bar{I}_t - d_2 \Delta \bar{I} - k_0 \bar{I} \]
\[ = Me^{-\gamma t} [-\gamma V - \gamma V' - \frac{d_2}{h} \bar{h}^{-2} V' - \frac{d_2}{r} \bar{h}^{-1} V' - k_0 V] \]
\[ \geq Me^{-\gamma t} \left[ \frac{d_2}{8h_0^2} - \gamma - k_0 \right], \]
\[ \bar{R}_t - d_3 \Delta \bar{R} - (\alpha \bar{I} - \mu_3 \bar{R}) \geq Me^{-\gamma t} \left[ \frac{d_3}{8h_0^2} - \gamma - \alpha \right] \]

for all $0 < r < \bar{h}(t)$ and $t > 0$. On the other hand, we have $\bar{h}(t) = 2h_0 \gamma e^{-\gamma t}$ and $-\mu \bar{I}_r(\bar{h}(t), t) = 2M \mu \bar{h}^{-1} (t) e^{-\gamma t}$. Moreover, it follows that $\bar{S}(r, 0) \geq S_0(r)$, $\bar{I}(r, 0) = M(1 - \frac{r^2}{4h_0^2}) \geq \frac{3}{4} M$, $\bar{R}(r, 0) = M(1 - \frac{r^2}{4h_0^2}) \geq \frac{3}{4} M$ for $r \in [0, h_0]$. Noting that $\bar{h}(t) \leq 4h_0$, we now choose

\[ M = \frac{4}{3} \max \{ ||I_0||_\infty, ||R_0||_\infty \} \]

and take

\[ \gamma = \frac{d}{16h_0^2}, \quad \mu \leq \frac{d}{8M} \]

and take
where \( d := \min \{d_1, d_2 \} \) and \( h_0 \leq \min \left\{ \sqrt{\frac{d}{16h_0}}, \sqrt{\frac{d}{16\mu_1}} \right\} \). Then we have

\[
\begin{cases}
\bar{S}_t - d_1 \Delta \bar{S} \geq b - \beta \bar{S}I - \mu_1 \bar{S}, & 0 < r, \ t > 0, \\
\bar{I}_t - d_2 \Delta \bar{I} \geq (\beta \bar{S} - \mu_2 - \alpha) \bar{I}, & 0 < r < \bar{h}(t), \ t > 0, \\
\bar{R}_t - d_3 \Delta \bar{R} \geq \alpha \bar{I} - \mu_3 \bar{R}, & 0 < r < \bar{h}(t), \ t > 0, \\
\bar{S}_r(0, t) = 0, \ \bar{I}_r(0, t) \geq 0, \ \bar{R}_r(0, t) \geq 0, & t > 0, \\
\bar{I}(r, t) = \bar{R}(r, t) = 0, & r \geq \bar{h}(t), \ t > 0, \\
\bar{\alpha}(t) \geq -\mu \bar{I}_r(\bar{\alpha}(t), t), \ \bar{\alpha}(0) = 2h_0 > h_0, & t > 0, \\
\bar{S}(r, 0) \geq S_0(r), \ \bar{I}(r, 0) \geq I_0(r), \ \bar{R}(r, 0) \geq R_0(r), & r \geq 0.
\end{cases}
\]

Hence we can apply Lemma 4.1 to conclude that \( h(t) \leq \bar{h}(t) \) for \( t > 0 \). Therefore, we have \( h_\infty \leq \lim_{t \to \infty} \bar{h}(t) = 4h_0 < \infty \). \( \Box \)

For the case that \( R_0 > 1 \), we next prove that if \( h_0 \) is suitably large, the disease is spreading.

**Lemma 4.3** If \( h_\infty < \infty \), then \( \lim_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} = 0 \). Moreover, we have \( \lim_{t \to +\infty} ||R(\cdot, t)||_{C([0, h(t)])} = 0 \) and \( \lim_{t \to +\infty} S(r, t) = \frac{h}{\mu_1} \) uniformly in any bounded subset of \([0, \infty)\).

**Proof**: Assume \( \limsup_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} = \delta > 0 \) by contradiction. Then there exists a sequence \((r_k, t_k)\) in \([0, h(t))] \times (0, \infty)\) such that \( I(r_k, t_k) \geq \delta/2 \) for all \( k \in \mathbb{N} \), and \( t_k \to \infty \) as \( k \to \infty \). Since that \( 0 \leq r_k < h(t) < h_\infty < \infty \), we then have that a subsequence of \( \{r_n\} \) converges to \( r_0 \in [0, h_\infty) \). Without loss of generality, we assume \( r_k \to r_0 \) as \( k \to \infty \).

Define \( S_k(r, t) = S(r, t_k + t) \), \( I_k(r, t) = I(r, t_k + t) \) and \( R_k(r, t) = R(r, t_k + t) \) for \( r \in (0, h(t_k + t)), t \in (-t_k, \infty) \). It follows from the parabolic regularity
that \(\{(S_k, I_k, R_k)\}\) has a subsequence \(\{(S_{k_i}, I_{k_i}, R_{k_i})\}\) such that \((S_{k_i}, I_{k_i}, R_{k_i}) \to (\tilde{S}, \tilde{I}, \tilde{R})\) as \(i \to \infty\) and \((\tilde{S}, \tilde{I}, \tilde{R})\) satisfies
\[
\begin{aligned}
\tilde{S}_t - d_1 \Delta \tilde{S} &= b - \beta \tilde{S} \tilde{I} - \mu_1 \tilde{S}, & 0 < r < h_\infty, & t \in (-\infty, \infty), \\
\tilde{I}_t - d_2 \Delta \tilde{I} &= \beta \tilde{S} \tilde{I} - \mu_2 \tilde{I} - \alpha \tilde{I}, & 0 < r < h_\infty, & t \in (-\infty, \infty), \\
\tilde{R}_t - d_3 \Delta \tilde{R} &= \alpha \tilde{I} - \mu_3 \tilde{R}, & 0 < r < h_\infty, & t \in (-\infty, \infty).
\end{aligned}
\]
Since \(\tilde{I}(r_0, 0) \geq \delta/2\), we have \(\tilde{I} > 0\) in \([0, h_\infty)\times (-\infty, \infty)\). Recalling that \((\beta \tilde{S} - \mu_2 - \alpha)\) is bounded by \(M := \beta \max\{\frac{b}{\mu_1}, ||S_0||_{L^\infty}\} + \mu_2 + \alpha\). Applying Hopf lemma to the equation \(\tilde{I}_t - d_2 \Delta \tilde{I} \geq -M \tilde{I}\) at the point \((0, h_\infty)\) yields that \(\tilde{I}_{r}(h_\infty, 0) \leq -\sigma_0\) for some \(\sigma_0 > 0\).

On the other hand, \(h(t)\) is increasing and bounded. Moreover, for any \(0 < \alpha < 1\), there exists a constant \(\tilde{C}\), which depends on \(\alpha, h_0, ||I_0||_{C^{1+\alpha}[0,h_0]}\) and \(h_\infty\), such that
\[
||I||_{C^{1+\alpha}(0, h(t))} + ||h||_{C^{1+\alpha/2}(0, h(t))} \leq \tilde{C}. \tag{4.1}
\]
In fact, let us straighten the free boundary in a way different from that in Theorem 2.1.

Define
\[
s = \frac{h_0 r}{h(t)}, \quad u(s, t) = S(r, t), \quad v(s, t) = I(r, t), \quad w(s, t) = R(r, t),
\]
then direct calculations yield that
\[
I_t = v_t - \frac{h'(t)}{h(t)} s v_s, \quad I_r = \frac{h_0}{h(t)} v_s, \quad \Delta_s I = \frac{h_0^2}{h^2(t)} \Delta_s v,
\]
therefore, \(v(s, t)\) satisfies
\[
\begin{aligned}
&v_t - \frac{h_0^2}{h^4(t)} \Delta_s v - \frac{h'(t)}{h(t)} s v_s = v(\beta u - \mu_2 - \alpha), & 0 < s < h_0, & t > 0, \\
v_s(0, t) = v(h_0, t) = 0, & t > 0, \\
v(s, 0) = I_0(s) \geq 0, & 0 \leq s \leq h_0.
\end{aligned} \tag{4.2}
\]
This transformation changes the free boundary $r = h(t)$ to the fixed line $s = h_0$, at the expense of making the equation more complicated. It follows from Lemmas 2.2 and 2.3 that

$$||v(βu - µ_2 - α)||_{L^∞} \leq M_1, \quad ||\frac{h'(t)}{h(t)}s||_{L^∞} \leq M_3.$$  

Applying standard $L^p$ theory and then the Sobolev imbedding theorem ([15]), we obtain that

$$\|v\|_{C^{1+α, \frac{1+α}{2}([0,h_0] \times [0,∞))}} \leq C_4,$$

where $C_4$ is a constant depending on $α, h_0, M_1, M_2, M_3$ and $\|I_0\|_{C^2([0,h_0])}$. This immediately leads to (4.1).

Since $\|h\|_{C^{1+α/2}([0,∞))} \leq \tilde{C}$ and $h(t)$ is bounded, we then have $h'(t) \to 0$ as $t \to ∞$, that is, $I_r(h(t_k), t_k) \to 0$ as $t_k \to ∞$ by the free boundary condition. Moreover, it follows from the inequality $\|I\|_{C^{1+α, \frac{1+α}{2}([0,h(t)] \times [0,∞))}} \leq \tilde{C}$ that $I_r(h(t_k), t_k + 0) = (I_k)_r(h(t_k), 0) \to \tilde{I}_r(h_∞, 0)$ as $k \to ∞$, which leads to a contradiction to the fact that $\tilde{I}_r(h_∞, 0) \leq -σ_0 < 0$. Thus $\lim_{t \to +∞} \|I(·, t)\|_{C([0,h(t)])} = 0$, and thereby $\lim_{t \to +∞} \|R(·, t)\|_{C([0,h(t)])} = 0$ and $\lim_{t \to +∞} S(r, t) = \frac{h}{µ_1}$ uniformly in any bounded subset of $[0, ∞)$.

Let $λ_1(R)$ be the principal eigenvalue of the operator $−Δ$ in $B_R$ (open ball with radius $R$) subject to homogeneous Dirichlet boundary condition. It is well-known that $λ_1(R)$ is a strictly decreasing continuous function and

$$\lim_{R \to 0^+} λ_1(R) = +∞ \quad \text{and} \quad \lim_{R \to +∞} λ_1(R) = 0.$$
Theorem 4.4 If $R_0 := \frac{b\beta}{\mu_1(\mu_2 + \alpha)} > 1$, then $h_\infty = \infty$ provided that $h_0 > h_0^*$, where $
abla_1(h_0^*) = \frac{(\mu_2 + \alpha)}{d_2}(R_0 - 1)$.

Proof: Assume that $h_\infty < +\infty$ by contradiction. It follows from Lemma 4.3 that $\lim_{t \to +\infty} ||I(r, t)||_{C([0, h(t)])} = 0$. Moreover, $\lim_{t \to +\infty} S(r, t) = \frac{b}{\mu_1}$ uniformly in the bounded subset $B_{h_0}$. Therefore, for $\varepsilon > 0$, there exists $T^* > 0$ such that

$$S(r, t) \geq \frac{b}{\mu_1} - \varepsilon$$

for $t \geq T^*$, $r \in [0, h(t))$. We then have that $I(r, t)$ satisfies

$$\begin{cases}
I_t - d_2 \Delta I \geq (\beta(\frac{b}{\mu_1} - \varepsilon) - \mu_2 - \alpha)I, & 0 < r < h_0, \ t > T^*, \\
I_r(0, t) = 0, \ I(h_0, t) \geq 0, & t > T^*, \\
I(r, T^*) > 0, & 0 \leq r < h_0.
\end{cases}$$

(4.3)

It is easy to see that $I(r, t)$ has a lower solution $\underline{I}(r, t)$ satisfying

$$\begin{cases}
\underline{I}_t - d_2 \Delta \underline{I} = (\beta(\frac{b}{\mu_1} - \varepsilon) - \mu_2 - \alpha)\underline{I}, & 0 < r < h_0, \ t > T^*, \\
\underline{I}_r(0, t) = 0, \ \underline{I}(h_0, t) = 0, & t > T^*, \\
\underline{I}(r, T^*) = \underline{I}(r, T^*), & 0 \leq r < h_0.
\end{cases}$$

(4.4)

Since $h_0 > h_0^*$, we can choose $\varepsilon$ sufficiently small such that $(\beta(\frac{b}{\mu_1} - \varepsilon) - \mu_2 - \alpha) > d_2 \nabla_1(h_0)$, it follows from well-known result that $\underline{I}$ is unbounded in $(0, h_0) \times (T^*, \infty)$, which leads to a contradiction that $\lim_{t \to +\infty} ||I(\cdot, t)||_{C([0, h(t)])} = 0$. 

□

5 Discussion

In this paper, we have considered the SIR epidemic model describing the transmission of diseases and examined the dynamical behavior of the population $(S, I, R)$ with spreading front $r = h(t)$ determined by (1.3). We have obtained the asymptotic behavior results.
The basic reproduction number $R_0 = \frac{b\beta}{\mu_1(\mu_2 + \alpha)}$ is important but not unique factor to determine whether the disease dies out or remains endemic. It is shown that if $R_0 < 1$, vanishing always happens or the disease dies out (Theorem 3.1). If $R_0 > 1$, spreading happens provided that $h_0$ is sufficiently large (Theorem 4.4) and vanishing is possible provided that $h_0$ is small (Theorem 4.2).

We feel it is reasonable to conclude that (1.3) is promising alternatives to (1.1) and (1.2) for the modeling of disease spreading, and there is still some works to do for the model (1.3). The first one is that, what is the asymptotic spreading speed when spreading happens? Since there is no other choice except spreading and vanishing, the second one is that we want to know the necessary condition for the disease to spread or to vanish.

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