Diffeomorphism-invariant Covariant Hamiltonians of a pseudo-Riemannian Metric and a Linear Connection

J. Muñoz Masqué†, M. Eugenia Rosado María‡

†Instituto de Física Aplicada, CSIC
C/ Serrano 144, 28006-Madrid, Spain
‡Departamento de Matemática Aplicada
Escuela Técnica Superior de Arquitectura, UPM
Avda. Juan de Herrera 4, 28040-Madrid, Spain

jaime@iec.csic.es, eugenia.rosado@upm.es

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Abstract

Let $M \to N$ (resp. $C \to N$) be the fibre bundle of pseudo-Riemannian metrics of a given signature (resp. the bundle of linear connections) on an orientable connected manifold $N$. A geometrically defined class of first-order Ehresmann connections on the product fibre bundle $M \times_N C$ is determined such that, for every connection $\gamma$ belonging to this class and every $\text{Diff}_N$-invariant Lagrangian density $\Lambda$ on $J^1(M \times_N C)$, the corresponding covariant Hamiltonian $\Lambda^\gamma$ is also $\text{Diff}_N$-invariant. The case of $\text{Diff}_N$-invariant second-order Lagrangian densities on $J^2M$ is also studied and the results obtained are then applied to Palatini and Einstein-Hilbert Lagrangians.
1 Introduction

In Mechanics, the Hamiltonian function attached to a Lagrangian density $\Lambda = L(t, q^i, \dot{q}^i) dt$ on $\mathbb{R} \times TQ$ is given by $H = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L$, but—as it was early observed in [16]—this is not an invariant definition if an arbitrary fibred manifold $t: E \to \mathbb{R}$ is considered (thus generalizing the notion of an absolute time) instead of the direct product bundle $\mathbb{R} \times Q \to \mathbb{R}$; e.g., see [7], [23], [25] for this point of view. In this case, an Ehresmann connection is needed in order to lift the vector field $\partial/\partial t$ from $\mathbb{R}$ to $E$, and the Hamiltonian is then defined by applying the Poincaré-Cartan form attached to $\Lambda$ to the horizontal lift of $\partial/\partial t$.

In the field theory—where no distinguished vector field exists on the base manifold—the need of an Ehresmann connection is even greater, in order to attach a covariant Hamiltonian to each Lagrangian density; e.g., see [24, 4.1], [23], and the definitions below.

Let $p: E \to N$ be an arbitrary fibred manifold over a connected manifold $N$, $n = \dim N$, $\dim E = m+n$, oriented by $v_n = dx^1 \wedge \cdots \wedge dx^n$. Throughout this paper, Latin (resp. Greek) indices run from 1 to $n$ (resp. $m$). An Ehresmann connection on a fibred manifold $p: E \to N$ is a differential 1-form $\gamma$ on $E$ taking values in the vertical sub-bundle $V(p)$ such that $\gamma(X) = X$ for every $X \in V(p)$ (e.g., see [23], [24], [32], [34]). Once an Ehresmann connection $\gamma$ is given, a decomposition of vector bundles holds $T(E) = V(p) \oplus \ker \gamma$, where $\ker \gamma$ is called the horizontal sub-bundle determined by $\gamma$. In a fibred coordinate system $(x^j, y^\alpha)$ for $p$, an Ehresmann connection can be written as

$$\gamma = (dy^\alpha + \gamma^\alpha_j dx^j) \otimes \frac{\partial}{\partial y^\alpha}, \quad \gamma^\alpha_j \in C^\infty(E).$$
According to [24], the covariant Hamiltonian $\Lambda^\gamma$ associated to a Lagrangian density on $J^1E$, $\Lambda = Lv_n, L \in C^\infty(J^1E)$, with respect to $\gamma$ is the Lagrangian density defined by,

\[(1) \quad \Lambda^\gamma = \left( (p_0^1)^* \gamma - \theta \right) \wedge \omega_\Lambda - \Lambda,
\]

where, $p_0^1: J^1E \to J^0E = E$ is the projection mapping, $\theta = \theta^\alpha \otimes \partial/\partial y^\alpha$, $\theta^\alpha = dy^\alpha - y^\alpha_i dx^i$ is the $V(p)$-valued 1-form on $J^1E$ associated with the contact structure, written on a fibred coordinate system $(x^i, y^\alpha)$, and $\omega_\Lambda$ is the Legendre form attached to $\Lambda$, i.e., the $V^*(p)$-valued $p^1$-horizontal $(n-1)$-form on $J^1E$ given by

\[
\omega_\Lambda = \left( -1 \right)^{i-1} \frac{\partial L}{\partial y_i^\alpha} i_{\partial/\partial x^i} v_n \otimes dy^\alpha,
\]

where $(x^i, y^\alpha; y^\alpha_i)$ is the coordinate system induced from $(x^i, y^\alpha)$ on the 1-jet bundle and $p_1: J^1E \to N$ is the projection on the base manifold. Locally,

\[(2) \quad \Lambda^\gamma = \left( \gamma_i^\alpha + y_i^\alpha \right) \frac{\partial L}{\partial y_i^\alpha} - L \right) dx^1 \wedge \cdots \wedge dx^n.
\]

From (1) we obtain the following decomposition of the Poincaré-Cartan form attached to $\Lambda$ (e.g., see [17], [23], [27]): $\Theta_\Lambda = \theta \wedge \omega_\Lambda + \Lambda = (p_0^1)^* \gamma \wedge \omega_\Lambda - \Lambda^\gamma$.

A diffeomorphism $\Phi: E \to E$ is said to be an automorphism of $p$ if there exists $\phi \in \text{Diff}\, N$ such that $p \circ \Phi = \phi \circ p$. The set of such automorphisms is denoted by Aut($p$) and its Lie algebra is identified to the space $\text{aut}(p) \subset \mathfrak{X}(E)$ of $p$-projectable vector fields on $E$. Given a subgroup $G \subseteq \text{Aut}(p)$, a Lagrangian density $\Lambda$ is said to be $G$-invariant if $(\Phi^1)^* \Lambda = \Lambda$ for every $\Phi \in G$, where $\Phi^1: J^1E \to J^1E$ denotes the 1-jet prolongation of $\Phi$. Infinitesimally, the $G$-invariance equation can be reformulated as $L_X \Lambda = 0$ for every $X \in \text{Lie}(G)$, $X^{(1)}$ denoting the 1-jet prolongation of the vector field $X$.

When a group $G$ of transformations of $E$ is given, a natural question arises:

- Determine a class—as small as possible—of Ehresmann connections $\gamma$ such that $\Lambda^\gamma$ is $G$-invariant for every $G$-invariant Lagrangian density $\Lambda$.

Below we tackle this question in the framework of General Relativity, i.e., the group $G$ is the group of all diffeomorphisms of the ground manifold $N$ acting in a natural way either on the bundle of pseudo-Riemannian metrics $p_M: M = M(N) \to N$ of a given signature $(n^+, n^-)$, $n^+ + n^- = n$, or on the product bundle $p: M \times_N C \to N$, where $p_C: C = C(N) \to N$ is the bundle of linear connections on $N$. Namely, we solve the following two problems:
(P): Determine a class—as small as possible—of Ehresmann connections $\gamma$ such that for every $\text{Diff}_N$-invariant first-order Lagrangian density $\Lambda$ on the bundle $J^1(M \times_N C)$, the corresponding covariant Hamiltonian $\Lambda^\gamma$ is also $\text{Diff}_N$-invariant.

Similarly to the problem (P), we formulate the corresponding problem on $J^2M$ as follows:

(P2): Determine a class of second-order Ehresmann connections $\gamma^2$ on $M$ such that for every $\text{Diff}_N$-invariant second-order Lagrangian density $\Lambda$ on the bundle $J^2M$, the corresponding covariant Hamiltonian $\Lambda^{\gamma^2}$—defined in (42)—is also $\text{Diff}_N$-invariant.

Essentially, a class of first-order Ehresmann connections on the bundle $M \times_N C$ is obtained, defined by the conditions $(C_M)$ and $(C_C)$ below (see Propositions 3.4 and 3.5), solving the problem (P). This class of connections also helps to solve (P2) by means of a natural isomorphism between $J^1M$ and $M \times_N C_{\text{sym}}$, where $C_{\text{sym}}$ denotes the sub-bundle of symmetric connections on $N$ (cf. Theorem 4.1). Finally, this approach is applied to Palatini and Einstein-Hilbert Lagrangians ([3], [4]), obtaining results compatible with their usual Hamiltonian formalisms.

2 Invariance under diffeomorphisms

2.1 Preliminaries

2.1.1 Jet-bundle notations

Let $p^k: J^kE \to N$ be the $k$-jet bundle of local sections of an arbitrary fibred manifold $p: E \to N$, with projections $p^k_l: J^kE \to J^lE$, $p^k_l(j^k_xs) = j^l_x s$, for $k \geq l$, $j^k_xs$ denoting the $k$-jet at $x$ of a section $s$ of $p$ defined on a neighbourhood of $x \in N$.

A fibred coordinate system $(x^i, y^\alpha)$ on $V$ induces a coordinate system $(x^i, y^\alpha_I)$, $I = (i_1, \ldots, i_N) \in \mathbb{N}^n$, $0 \leq |I| = i_1 + \cdots + i_N \leq r$, on $(p^k_0)^{-1}(V) = J^rV$ as follows: $y^\alpha_I(j^r_xs) = (\partial|I|(y^\alpha \circ s)/\partial x^I)(x)$, with $y^\alpha_0 = y^\alpha$.

Every morphism $\Phi: E \to E'$ whose associated map $\phi: N \to N'$ is a diffeomorphism, induces a map

$$
\Phi^{(r)}: J^rE \to J^rE',
\Phi^{(r)}(j^r_xs) = j^r_{\phi(s)}(\Phi \circ s \circ \phi^{-1}).
$$
If $\Phi_t$ is the flow of a vector field $X \in \text{aut}(p)$, then $\Phi_t^{(r)}$ is the flow of a vector field $X^{(r)} \in \mathfrak{X}(J^rE)$, called the infinitesimal contact transformation of order $r$ associated to the vector field $X$. The mapping

$$\text{aut}(p) \ni X \mapsto X^{(r)} \in \mathfrak{X}(J^rE)$$

is an injection of Lie algebras, namely, one has

$$(\lambda X + \mu Y)^{(r)} = \lambda X^{(r)} + \mu Y^{(r)},
\quad [X,Y]^{(r)} = [X^{(r)},Y^{(r)}],
\quad \forall \lambda, \mu \in \mathbb{R}, \forall X,Y \in \text{aut}(p).$$

In particular, for $r = 1$,

$$X = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha}, \quad u^i \in C^\infty(N), v^\alpha \in C^\infty(E),$$

$$X^{(1)} = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v^\alpha_i \frac{\partial}{\partial y^\alpha_i}, \quad v^\alpha_i = \frac{\partial v^\alpha}{\partial x^i} + y^\beta_i \frac{\partial v^\alpha}{\partial y^\beta} - y^k_i \frac{\partial u^k}{\partial x^i}.$$
2.2 Natural lifts

We think of $\text{gau} F(N)$ as the ‘Lie algebra’ of the gauge group $\text{Gau} F(N)$. Moreover, $p_C : C \to N$ is an affine bundle modelled over the vector bundle $\otimes^2 T^* N \otimes TN$. The section of $p_C$ induced tautologically by the linear connection $\Gamma$ is denoted by $s_\Gamma : N \to C$. Every $B \in \mathfrak{gl}(n, \mathbb{R})$ defines a one-parameter group $\varphi^B_t : U \times \text{Gl}(n, \mathbb{R}) \to U \times \text{Gl}(n, \mathbb{R})$ of gauge transformations by setting (cf. [5]), $\varphi^B_t (x, \Lambda) = (x, \exp(tB) \cdot \Lambda)$. Let us denote by $\tilde{B} \in \text{gau}(p_F)^{-1}(U)$ the corresponding infinitesimal generator. If $(E^i_j)$ is the standard basis of $\mathfrak{gl}(n, \mathbb{R})$, then $E^i_j = \sum_{h=1}^n x_h^i \partial / \partial x_h^j$, for $i, j = 1, \ldots, n$, is a basis of $\text{gau}(p_F)^{-1}(U)$. Let $E^i_j = E^i_j \mod G$ be the class of $E^i_j$ on $\text{ad} F(N)$. Unique smooth functions $A_{jk}^i$ on $(p_C)^{-1}(U)$ exist such that,

\begin{equation}
(6) \quad s_\Gamma \left( \frac{\partial}{\partial x^j} \right) = \frac{\partial}{\partial x^j} - (A^i_{jk} \circ \Gamma) E^i_k
\end{equation}

for every $s_\Gamma$ and $A^i_{jk}(\Gamma_x) = \Gamma^i_{jk}(x)$, where $\Gamma^i_{jk}$ are the Christoffel symbols of the linear connection $\Gamma$ in the coordinate system $(x^i)$, see [20, III, Proposition 7.4].

2.2 Natural lifts

Let $f_M : M \to M$, cf. [30] (resp. $\tilde{f} : F(N) \to F(N)$, cf. [20, p. 226]) be the natural lift of $f \in \text{Diff} N$ to the bundle of metrics (resp. linear frame bundle); namely $f_M(g_x) = (f^{-1})^* g_x$ (resp. $\tilde{f}(X_1, \ldots, X_N) = (f_* X_1, \ldots, f_* X_N)$, where $(X_1, \ldots, X_N) \in F_x(N)$); hence $p_M \circ f_M = f \circ p_M$ (resp. $p_F \circ \tilde{f} = f \circ p_F$), and $f_M : M \to M$ (resp. $\tilde{f} : F(N) \to F(N)$) have a natural extension to jet bundles $f^{(r)}_M : J^r(M) \to J^r(M)$ (resp. $\tilde{f}^{(r)} : J^r(FN) \to J^r(FN)$) as defined in the formula (3), i.e.,

\begin{equation}
\tilde{f}^{(r)}_M (j^r_x g) = j^r_x(f_M \circ g \circ f^{-1}) \quad \text{(resp. } \tilde{f}^{(r)}_M (j^r_x s) = j^r_x(f_M (\tilde{f} \circ s \circ f^{-1})))
\end{equation}

As $\tilde{f}$ is an automorphism of the principal $\text{Gl}(n, \mathbb{R})$-bundle $F(N)$, it acts on linear connections by pulling back connection forms, i.e., $\gamma' = \tilde{f} (\gamma)$ where $\omega_{\gamma'} = (\tilde{f}^{-1})^* \omega_\gamma$ (see [20, II, Proposition 6.2-(b)], [5, 3.3]). Hence, there exists a unique diffeomorphism $f_C : C \to C$ such that,

1) $p_C \circ f_C = f \circ p_C$, and
2) \( f_C \circ s_\Gamma = s_{f(\Gamma)} \) for every linear connection \( \Gamma \).

If \( f_t \) is the flow of a vector field \( X \in \mathfrak{X}(N) \), then the infinitesimal generator of \((f_t)_M \) (resp. \( f_t \), resp. \((f_t)_C)) \) in \( \text{Diff} \) \( M \) (resp. \( \text{Diff} \) \( F(N) \), resp. \( \text{Diff} \) \( C \)) is denoted by \( X_M \) (resp. \( \tilde{X} \), resp. \( \tilde{X}_C \)) and the following Lie-algebra homomorphisms are obtained:

\[
\begin{align*}
\mathfrak{X}(N) &\rightarrow \mathfrak{X}(M), \quad X \mapsto X_M \\
\mathfrak{X}(N) &\rightarrow \mathfrak{X}(F(N)), \quad X \mapsto \tilde{X} \\
\mathfrak{X}(N) &\rightarrow \mathfrak{X}(C), \quad X \mapsto \tilde{X}_C
\end{align*}
\]

If \( X = u^i \partial/\partial x^i \in \mathfrak{X}(N) \) is the local expression for \( X \), then

1) From \([30, \text{eqs. (2)}-\text{(4)}]\) we know that the natural lift of \( X \) to \( M \) is given by,

\[
X_M = u^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left( \frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} \in \mathfrak{X}(M).
\]

and its 1-jet prolongation,

\[
X_M^{(1)} = u^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left( \frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} - \sum_{i \leq j} \left( \frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} + \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} + \frac{\partial u^h}{\partial x^j} y_{hij} + \frac{\partial u^h}{\partial x^k} y_{hij} \right) \frac{\partial}{\partial y_{ij,k}}.
\]

2) From \([10, \text{Proposition 3}]\) (also see \([20, \text{VI, Proposition 21.1}]\)) we know that the natural lift of \( X \) to \( F(N) \) is given by,

\[
\tilde{X} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^j} \frac{x_j}{x^i} \frac{\partial}{\partial x^i},
\]

and its 1-jet prolongation,

\[
\tilde{X}^{(1)} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^j} x_j \frac{\partial}{\partial x^i} + v^j_{ik} \frac{\partial}{\partial x^j},
\]

\[
v^j_{ik} = \frac{\partial u^i}{\partial x^k} x_{j,k} - \frac{\partial u^i}{\partial x^j} x_{i,k} + \frac{\partial^2 u^i}{\partial x^k \partial x^l} x^j_l.
\]

3) Finally,

\[
\tilde{X}_C = u^i \frac{\partial}{\partial x^i} - \left( \frac{\partial^2 u^i}{\partial x^j \partial x^k} - \frac{\partial u^i}{\partial x^l} A^i_{jk} + \frac{\partial u^l}{\partial x^j} A^i_{lk} + \frac{\partial u^l}{\partial x^k} A^i_{lj} \right) \frac{\partial}{\partial A^i_{jk}}.
\]
\[
\tilde{X}_C^{(1)} = u^i \frac{\partial}{\partial x^i} + w^j_{hk} \frac{\partial}{\partial A^i_{jk}} + w^j_{jkh} \frac{\partial}{\partial A^i_{jkh}},
\]
(7) \[ w^j_{jk} = -\frac{\partial^2 u^i}{\partial x^j \partial x^k} + \frac{\partial u^i}{\partial x^j} A^i_{jk} - \frac{\partial u^i}{\partial x^k} A^i_{jk} - \frac{\partial u^i}{\partial x^j} A^i_{jk}, \]
(8) \[ w^j_{jkh} = -\frac{\partial^2 u^i}{\partial x^j \partial x^k \partial x^l} + \frac{\partial^2 u^i}{\partial x^j \partial x^k} A^i_{jk} - \frac{\partial^2 u^i}{\partial x^j \partial x^l} A^i_{jk} - \frac{\partial^2 u^i}{\partial x^k \partial x^l} A^i_{jk} + \frac{\partial u^i}{\partial x^j} A^i_{jk,h} - \frac{\partial u^i}{\partial x^k} A^i_{jk,h} - \frac{\partial u^i}{\partial x^l} A^i_{jk,h} - \frac{\partial u^i}{\partial x^k} A^i_{jkh}. \]

Let \( p: M \times_N C \rightarrow N \) be the natural projection. We denote by \( \tilde{f} = (f_M, \tilde{f}_C) \) (resp. \( \tilde{X} = (X_M, \tilde{X}_C) \) in \( \mathfrak{X}(M \times N C) \)) the natural lift of \( f \) (resp. \( X \)) to \( M \times N C \). The prolongation to the bundle \( J^1(M \times N C) \) of \( \tilde{X} \) is as follows:

\[
\tilde{X}^{(1)}(1) = \left( X_M^{(1)}, \tilde{X}_C^{(1)} \right)
= u^i \frac{\partial}{\partial x^i} + \sum_{i \leq j} v_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{i \leq j} v_{ijk} \frac{\partial}{\partial y_{ijk}} + w^j_{jk} \frac{\partial}{\partial A^i_{jk}} + w^j_{jkh} \frac{\partial}{\partial A^i_{jkh}},
\]
where

\[
v_{ij} = \frac{\partial u^h}{\partial x^i} y_{hj} - \frac{\partial u^h}{\partial x^j} y_{hi}, \quad \text{and} \quad v_{ijk} = \frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} - \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} - \frac{\partial u^h}{\partial x^j} y_{hj,k} - \frac{\partial u^h}{\partial x^i} y_{hi,k} - \frac{\partial u^h}{\partial x^k} y_{ij,h},
\]
(10) and \( w^j_{jk}, w^j_{jkh} \) are given in the formulas (7), (8), respectively.

### 2.3 Diff\(N\)- and \(\mathfrak{X}(N)\)-invariance

A differential form \( \omega_r \in \Omega^r(J^1(M \times N C)), r \in \mathbb{N} \), is said to be Diff\(N\)-invariant— or invariant under diffeomorphisms— (resp. \(\mathfrak{X}(N)\)-invariant) if the following equation holds: \((f^{(1)})^* \omega_r = \omega_r, \forall f \in \text{Diff } N \) (resp. \( L_X \omega_r = 0, \forall X \in \mathfrak{X}(N) \)). Obviously, “Diff\(N\)-invariance” implies “\(\mathfrak{X}(N)\)-invariance” and the converse is almost true (see [14], [28]). Because of this, below we consider \(\mathfrak{X}(N)\)-invariance only.

A linear frame \((X_1, \ldots, X_N)\) at \(x\) is said to be orthonormal with respect to \(g_x \in M_x(N)\) (or simply \(g_x\)-orthonormal) if \(g_x(X_i, X_j) = 0\) for \(1 \leq i < j \leq n\), \(g(X_i, X_i) = 1\) for \(1 \leq i \leq n^+\), \(g(X_i, X_i) = -1\) for \(n^++1 \leq i \leq n\).
As \( N \) is an oriented manifold, there exists a unique \( p \)-horizontal \( n \)-form \( \mathbf{v} \) on \( M \times N \) such that, \( \mathbf{v}_{(g_{x}, \Gamma_{x})} (X_{1}, \ldots, X_{N}) = 1 \), for every \( g_{x} \)-orthonormal basis \( (X_{1}, \ldots, X_{N}) \) belonging to the orientation of \( N \). Locally \( \mathbf{v} = \rho \mathbf{v}_{n} \), where \( \rho = \sqrt{(-1)^{n-\text{det}(y_{ij})}} \) and \( \mathbf{v}_{n} = dx^{1} \wedge \cdots \wedge dx^{n} \). As proved in [30, Proposition 7], the form \( \mathbf{v} \) is \( \text{Diff}N \)-invariant and hence \( \mathcal{X}(N) \)-invariant. A Lagrangian density \( \Lambda \) on \( J^{1}(M \times N, C) \) can be globally written as \( \Lambda = \mathcal{L} \mathbf{v} \) for a unique function \( \mathcal{L} \in C^{\infty}(J^{1}(M \times N, C)) \) and \( \mathcal{L} \) is \( \mathcal{X}(N) \)-invariant if and only if the function \( \mathcal{L} \) is. Therefore, the invariance of Lagrangian densities is reduced to that of scalar functions.

**Proposition 2.1.** A function \( \mathcal{L} \in C^{\infty}(J^{1}(M \times N, C)) \) is \( \mathcal{X}(N) \)-invariant if and only if the following system of partial differential equations hold:

\[
\begin{align*}
0 &= X^{i}(\mathcal{L}), \quad \forall i, \\
0 &= X^{j}_{h}(\mathcal{L}), \quad \forall h, i, \\
0 &= X^{ik}_{h}(\mathcal{L}), \quad \forall h, i \leq k, \\
0 &= X^{jkh}_{i}(\mathcal{L}), \quad \forall i, j \leq k, h,
\end{align*}
\]

where

\[
X^{i} = \frac{\partial}{\partial x^{i}}, \quad \forall i,
\]

\[
X^{j}_{h} = -y_{h,j} \frac{\partial}{\partial y_{j}} - y_{h,i} \frac{\partial}{\partial y_{i,j}} - y_{h,k} \frac{\partial}{\partial y_{i,j,k}} - \sum_{s \leq j} y_{s,j,h} \frac{\partial}{\partial y_{s,j,i}} + A^{i}_{jk} \frac{\partial}{\partial A^{h}_{jk}} - A^{r}_{hj} \frac{\partial}{\partial A^{r}_{jk}} - A^{i}_{h,k} \frac{\partial}{\partial A^{i}_{jk}} + \frac{\partial}{\partial A^{i}_{jk}}, \quad \forall h, i,
\]

\[
X^{ik}_{h} = -y_{h,k} \frac{\partial}{\partial y_{i,k}} - y_{h,j} \frac{\partial}{\partial y_{j,k}} - y_{h,j} \frac{\partial}{\partial y_{j,k}} - \sum_{s \leq j} y_{s,j,h} \frac{\partial}{\partial y_{s,j,i}} + A^{i}_{jk} \frac{\partial}{\partial A^{h}_{jk}} - A^{r}_{hj} \frac{\partial}{\partial A^{r}_{jk}} - A^{i}_{h,k} \frac{\partial}{\partial A^{i}_{jk}} + \frac{\partial}{\partial A^{i}_{jk}}, \quad \forall h, i \leq k,
\]

\[
X^{jkh}_{i} = -y_{h,k} \frac{\partial}{\partial y_{i,k}} - y_{h,j} \frac{\partial}{\partial y_{j,k}} - y_{h,j} \frac{\partial}{\partial y_{j,k}} - \sum_{s \leq j} y_{s,j,h} \frac{\partial}{\partial y_{s,j,i}} + A^{i}_{jk} \frac{\partial}{\partial A^{h}_{jk}} - A^{r}_{hj} \frac{\partial}{\partial A^{r}_{jk}} - A^{i}_{h,k} \frac{\partial}{\partial A^{i}_{jk}} + \frac{\partial}{\partial A^{i}_{jk}} + \frac{\partial}{\partial A^{i}_{jk}}, \quad \forall i, h \leq j \leq k.
\]

Moreover, the vector fields \( X^{i}, X^{j}_{h}, X^{ik}_{h}, X^{jkh}_{i} \) are linearly independent and they span an involutive distribution on \( J^{1}(M \times N, C) \) of rank \( n \left( \binom{n+3}{3} \right) \). Hence, the number of functionally invariant Lagrangians on \( J^{1}(M \times N, C) \) is

\[
\frac{1}{6} \left( 5n^{4} + 3n^{3} - 5n^{2} + 3n \right).
\]
Proof. According to the formula (9), $\mathcal{L}$ is invariant if and only if,

$$u^i \frac{\partial \mathcal{L}}{\partial x^i} + \sum_{i \leq j} v_{ij} \frac{\partial \mathcal{L}}{\partial y_{ij}} + \sum_{i \leq j} v_{ijk} \frac{\partial \mathcal{L}}{\partial y_{ijk}} + w^j_k \frac{\partial \mathcal{L}}{\partial A_{jk}} + w^j_{kh} \frac{\partial \mathcal{L}}{\partial A_{jk,h}} = 0,$$

$\forall u^i \in C^\infty(N)$,

and expanding on this equation by using the formulas (10), (11), (7), and (8) we obtain

$$0 = u^i \frac{\partial \mathcal{L}}{\partial x^i} + \frac{\partial u^b}{\partial x^j} \left( -y_{ih} \frac{\partial \mathcal{L}}{\partial y_{ij}} - y_{ij,k} \frac{\partial \mathcal{L}}{\partial y_{ijk}} - y_{hj,k} \frac{\partial \mathcal{L}}{\partial y_{ijk}} \right) - \sum_{s \leq j} y_{sj,h} \frac{\partial \mathcal{L}}{\partial y_{sj,i}} + A^i_{jk,s} \frac{\partial \mathcal{L}}{\partial A_{jk,s}^i} - A^a_{i,jh,k} \frac{\partial \mathcal{L}}{\partial y_{ijk}} + \sum_{s \leq j} y_{sj,h} \frac{\partial \mathcal{L}}{\partial y_{sj,i}} + A^i_{jk,s} \frac{\partial \mathcal{L}}{\partial A_{jk,s}^i}

+ A^i_{jk,s} \frac{\partial \mathcal{L}}{\partial A_{jk,s}^i} - A^a_{i,jh,k} \frac{\partial \mathcal{L}}{\partial y_{ijk}} - A^r_{h,k} \frac{\partial \mathcal{L}}{\partial A_{jk,i}^r} - A^r_{h,k} \frac{\partial \mathcal{L}}{\partial A_{jk,k}^r}

+ \frac{\partial^2 u^h}{\partial x^i \partial x^k} \left( -y_{ih} \frac{\partial \mathcal{L}}{\partial y_{ij}} - y_{ij,k} \frac{\partial \mathcal{L}}{\partial y_{ijk}} - y_{hj,k} \frac{\partial \mathcal{L}}{\partial y_{ijk}} \right)

+ A^i_{jk,s} \frac{\partial \mathcal{L}}{\partial A_{jk,s}^i} - A^a_{i,jh,k} \frac{\partial \mathcal{L}}{\partial y_{ijk}} + A^a_{i,jh,k} \frac{\partial \mathcal{L}}{\partial A_{jk,i}^a}

- \frac{\partial^3 u^j}{\partial x^i \partial x^j \partial x^k} \frac{\partial \mathcal{L}}{\partial A_{jk,h}^i}.$$

This equation is equivalent to the system of the statement as the values for $u^b$, $\partial u^h / \partial x^i$, $\partial^2 u^h / \partial x^i \partial x^j$ ($i \leq j$), and $\partial^3 u^h / \partial x^i \partial x^j \partial x^k$ ($i \leq j \leq k$) at a point $x \in N$ can be taken arbitrarily. Moreover, assume a linear combination holds

$$\lambda^a X^a + \lambda^b X^b + \sum_{b \leq c} \lambda^a_{bc} X^a + \sum_{b \leq c \leq d} \lambda^a_{bcd} X^a = 0,$$

$\lambda^a, \lambda^b, \lambda^c, \lambda^d \in C^\infty(J^1(M \times N C)).$

By applying (15) to $x^a$ (resp. $y_{ab}$) we obtain $\lambda_a = 0$ (resp. $\lambda^a_b = 0$); again by applying (15) to $A^a_{bc}$, $b \leq c$ (resp. $A^a_{bc}$, $c \leq b$) and taking the expressions of the vector fields (13) and (14) into account, we obtain $\lambda^a_{bc} = 0$, $b \leq c$ (resp. $\lambda^a_{bc} = 0$, $c \leq b$). Hence, (15) reads $\sum_{b \leq c \leq d} \lambda^a_{bcd} X^a = 0$, and by applying it to $A^a_{bc,d}$ and taking the expressions of the vector fields (14) into account, we finally obtain $\lambda^a_{bcd} = 0$. The distribution

$$\mathcal{D}_{M \times N C} = \left\{ X^{(1)}_{(j^1_g, j^1_s^r)} : X \in \mathcal{X}(N), (j^1_g, j^1_s^r) \in J^1(M \times N C) \right\}.$$
in \( T(J^1(M \times_N C)) \), where \( \bar{X}^{(1)} \) is defined in (9), is involutive as
\[
\left[ \bar{X}^{(1)}, \bar{Y}^{(1)} \right] = [X, Y]^{(1)}, \quad \forall X, Y \in \mathfrak{X}(N),
\]
and it is spanned by \( X^i, X_h^i, X_h^{ik}, X^{jkh} \), as proved by the formulas above. The rest of the statement follows from the following identities:
\[
\# \left\{ X^i; X^i_h; X^{ik}_i, i \leq k; X^{jkh}_i, h \leq j \leq k : h, i, j, k = 1, \ldots, n \right\} = n + n^2 + n \binom{n+1}{2} + n \binom{n+2}{3} = n \binom{n+3}{3},
\]
\[
\dim J^1(M \times_N C) - n \binom{n+3}{3} = \frac{1}{6} (5n^4 + 3n^3 - 5n^2 + 3n).
\]
\[ \square \]

3 Invariance of covariant Hamiltonians

3.1 Position of the problem

On the bundle \( E = M \times_N C \), an Ehresmann connection can locally be written as follows:
\[
\gamma = \sum_{i \leq j} (dy_{ij} + \gamma_{ijk} dx^k) \otimes \frac{\partial}{\partial y_{ij,k}} + \left( dA^i_j + \gamma^i_{jkl} dx^l \right) \otimes \frac{\partial}{\partial A^i_j},
\]
\( \gamma_{ijk}, \gamma^i_{jkl} \in C^\infty(M \times_N C). \)

In particular, for a Lagrangian density \( \Lambda \) on \( J^1(M \times_N C) \) we obtain
\[
\Lambda^\gamma = \left( \sum_{i \leq j} (\gamma_{ijk} + y_{ij,k}) \frac{\partial L}{\partial y_{ij,k}} + \left( \gamma^i_{jkl} + A^i_{jkl} \right) \frac{\partial L}{\partial A^i_{jkl}} - L \right) dx^1 \wedge \cdots \wedge dx^n,
\]
or equivalently,
\[
L^\gamma = D^\gamma (L) - L,
\]
where
\[
D^\gamma = \sum_{i \leq j} \left( \gamma_{ijk} + y_{ij,k} \right) \frac{\partial}{\partial y_{ij,k}} + \left( \gamma^i_{jkl} + A^i_{jkl} \right) \frac{\partial}{\partial A^i_{jkl}}.
\]

Remark 3.1. The horizontal form \((p^1_0)^* \gamma - \theta = (\gamma_i^\alpha + y_i^\alpha) dx^i \otimes \partial/\partial y^\alpha\) can also be viewed as the \( p^1_0 \)-vertical vector field
\[
D^\gamma = (\gamma_i^\alpha + y_i^\alpha) \frac{\partial}{\partial y^\alpha},
\]
taking the natural isomorphism \( V(p^1_0) \cong (p^1_0)^*(p^*T^*N \otimes V(p)) \) into account (cf. \cite{23}, \cite{24}, \cite{32}, \cite{34}).
According to the previous formulas, this means: If the system (12) holds for a Lagrangian function $L$, then it also holds for the covariant Hamiltonian $L\gamma$.

If $X \in \{X^i, X^i_h, X^i_h, X^j_{kh}\}$, then $X (L\gamma) = X (D\gamma (L))$, as $L$ is assumed to be invariant and hence $X (L) = 0$. Therefore

$$X (L\gamma) = X (D\gamma (L)) = [X, D\gamma] (L),$$

and we conclude the following:

**Proposition 3.2.** The property (P) holds for an Ehresmann connection $\gamma$ on $M \times N C$ if and only if the vector field $D\gamma$ transforms the sections of the distribution $D_{M \times N C}$ into themselves, namely, $[D\gamma, \Gamma(D_{M \times N C})] \subseteq \Gamma(D_{M \times N C})$.

The problem thus reduces to compute the brackets $[X^i, D\gamma]$, $[X^i_h, D\gamma]$, $[X^i_h, D\gamma]$, and $[X^j_{ikh}, D\gamma]$. We have

\begin{equation}
[X^h, D\gamma] = \sum_{i \leq j} \frac{\partial \gamma_{ijk}}{\partial x^h} \frac{\partial}{\partial y_{ij,k}} + \frac{\partial \gamma^i_{jkl}}{\partial x^h} \frac{\partial}{\partial A^i_{jkl}},
\end{equation}

\begin{equation}
[X^i_{hda}, D\gamma] = X^i_{hda}, \quad \forall b, c \leq d \leq a,
\end{equation}

\begin{equation}
[X^i_h, D\gamma] = \sum_{a \leq b} Y^i_h (\gamma_{abc}) \frac{\partial}{\partial y_{ab,c}} + Y^i_h (\gamma^d_{abc}) \frac{\partial}{\partial A^d_{ab,c}} + X^i_h - Y^i_h,
\end{equation}

where

$$Y^i_h = -y_h \frac{\partial}{\partial y_{ii}} - y_n \frac{\partial}{\partial y_{ij}} + A^i_{jk} \frac{\partial}{\partial A^h_{jk}} - A^i_{jk} \frac{\partial}{\partial A^r_{ji}} - A^r_{hk} \frac{\partial}{\partial A^i_{rk}},$$

$$Y^i_{jk} = -\frac{\partial}{\partial A^h_{ik}} - \frac{\partial}{\partial A^h_{ki}}.$$
and the following formula has been used:

\[
\frac{\partial y_{rs,k}}{\partial y_{ij,h}} = \delta^k_h \left( \delta^r_i \delta^s_j + \delta^r_j \delta^s_i - \delta^i_j \delta^r_s \delta^j_s \right).
\]

### 3.2 The class of the Ehresmann connections defined

Let \( p: M \times_N C \rightarrow N \), \( \text{pr}_1: M \times_N C \rightarrow M \), \( \text{pr}_2: M \times_N C \rightarrow C \) be the natural projections. By taking the differential of \( \text{pr}_1 \) and \( \text{pr}_2 \), a natural identification is obtained \( T(M \times_N C) = TM \times T_N TC \). Hence

\[
V(p) = V(p_M) \times_N V(p_C)
\]

and two unique vector-bundle homomorphisms exist

\[
\gamma_M: \text{pr}_1^*TM \rightarrow \text{pr}_1^*V(p_M), \quad \gamma_C: \text{pr}_2^*TC \rightarrow \text{pr}_2^*V(p_C),
\]

such that,

\[
\gamma(X) = (\gamma_M(\text{pr}_1^*X), \gamma_C(\text{pr}_2^*X)), \quad \forall X \in T(M \times_N C),
\]

\[
\gamma_M(Y) = Y, \quad \forall Y \in \text{pr}_1^*V(p_M),
\]

\[
\gamma_C(Z) = Z, \quad \forall Z \in \text{pr}_2^*V(p_C).
\]

If \( \gamma \) is given by the local expression of the formula (16), then

\[
\gamma_M = \sum_{i,j} (dy_{ij} + \gamma_{ijk} dx^k) \otimes \frac{\partial}{\partial y_{ij}}, \quad \gamma_C = \left( dA_{jk}^i + \gamma_{jkl}^i dx^l \right) \otimes \frac{\partial}{\partial A_{jk}^i},
\]

\( \gamma_{ijk}, \gamma_{jkl}^i \in C^\infty(M \times_N C) \).

#### 3.2.1 The first geometric condition on \( \gamma \)

Let \( q: F(N) \rightarrow M \) be the projection given by

\[
q(X_1, \ldots, X_N) = g_x = \varepsilon_h w^h \otimes w^h,
\]

where \((w^1, \ldots, w^n)\) is the dual coframe of \((X_1, \ldots, X_N) \in F_x(N)\), i.e., \( g_x \) is the metric for which \((X_1, \ldots, X_N)\) is a \( g_x \)-orthonormal basis and \( \varepsilon_h = 1 \) for \( 1 \leq h \leq n^+ \), \( \varepsilon_h = -1 \) for \( n^+ + 1 \leq h \leq n \). As readily seen, \( q \) is a principal \( G \)-bundle with \( G = O(n^+, n^-) \).

Given a linear connection \( \Gamma \) and a tangent vector \( X \in T_x N \), for every \( u \) in \( p^{-1}(x) \) there exists a unique \( \Gamma \)-horizontal tangent vector \( X_u^{hr} \in T_u(FN) \)
such that, \((p_F)_*X^h_{u_f} = X\). The local expression for the horizontal lift is known to be ([20, Chapter III, Proposition 7.4]),

\[
\left(\frac{\partial}{\partial x^j}\right)^{h_{u_f}} = \frac{\partial}{\partial x^j} - \Gamma^i_{jk} \frac{\partial}{\partial x^i}. 
\]

**Lemma 3.3.** Given a metric \(g_x \in p^{-1}_M(x)\), let \(u \in F^{-1}(x)\) be a linear frame such that \(q(u) = g_x\). The projection \(q_*(X^h_{u_f})\) does not depend on the linear frame \(u\) chosen over \(g_x\).

**Proof.** In fact, any other linear frame projecting onto \(g_x\) can be written as \(u \cdot A\), \(A \in G\). As the horizontal distribution is invariant under right translations (see [20, II, Proposition 1.2]), the following equation holds:

\[
(R_A)_* (X^h_{u_f}) = X^h_{u \cdot A}. 
\]

\(\square\)

**Proposition 3.4.** An Ehresmann connection \(\gamma\) on \(M \times_N C\) satisfies the following condition:

\[(C_M): \gamma_M ((g_x, \Gamma_x), X) = X - q_* \left( ((p_M)_*(X))^{h_{u_f}} \right), \]

\(\forall X \in T_{g_x}M, \ u \in q^{-1}(g_x), \) (which does not depend on the linear frame \(u \in q^{-1}(g_x)\) chosen, according to Lemma 3.3) if and only if the following equations hold:

\[
\gamma_{klj} = - \left( y_{al} A^a_{jk} + y_{ak} A^a_{jl} \right), 
\]

where the functions \(\gamma_{klj}\) (resp. \(y_{ij}\), resp. \(A^a_{ij}\)) are defined in the formula (16) (resp. (4), resp. (6)).

**Proof.** Letting \((\chi^h_j)^{n}_{i,j=1} = \left((x^h_i)^{n}_{i,j=1}\right)^{-1}\), the dual coframe of the linear frame \(u = (X_1, \ldots, X_N) \in F_x(N)\) given in (5) is \((w^1, \ldots, w^n)\), \(w^h = \chi^h_k(u) (dx^k)_x\), \(1 \leq h \leq n\), and the projection \(q\) is given by

\[
q(u) = g_x = \sum_{h=1}^n \varepsilon_h \chi^h_k(u) \chi^h_l(u) (dx^k)_x \otimes (dx^l)_x. 
\]
Therefore the equations of the projection (21) are as follows:

\[ x^i \circ q = x^i, \]
\[ y_{kl} \circ q = \sum_{h=1}^{n} \varepsilon_h \chi_i^h \chi_i^h. \]

Hence

\[ q^* \left( \frac{\partial}{\partial x_b^a} \right)_u = \sum_{k \leq l} \varepsilon_h \left\{ \frac{\partial \chi_i^h}{\partial x_b^a} \chi_i^h + \chi_i^h \frac{\partial \chi_i^h}{\partial x_b^a} \right\} (u) \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x}. \]

Taking derivatives with respect to \( x_a^b \) on the identity \( \chi_i^h x_i^r = \delta_i^h \), multiplying the outcome by \( \chi_i^k \), and summing up over the index \( i \), the following formula is obtained:

\[ \frac{\partial \chi_i^h}{\partial x_a^b} = -\chi_i^a \chi_i^b. \]

Replacing this equation into the expression for \( q^* \left( \frac{\partial}{\partial x_b^a} \right)_u \) above, we have

\[ q^* \left( \frac{\partial}{\partial x_b^a} \right)_u = -\sum_{k \leq l} \{ \chi_i^k (u) y_{al} (g_x) + \chi_i^l (u) y_{ak} (g_x) \} \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x}. \]

From (22), evaluated at \( u \in q^{-1}(g_x) \), we deduce

\[ q^* \left( \frac{\partial}{\partial x^j} \right)_u = \left( \frac{\partial}{\partial x^j} \right)_{g_x} - \Gamma^a_{jc}(x) x_b^c (u) q^* \left( \frac{\partial}{\partial x_b^a} \right)_{g_x} \]

\[ = \left( \frac{\partial}{\partial x^j} \right)_{g_x} + \sum_{k \leq l} \Gamma^a_{jc}(x) x_b^c (u) \{ \chi_i^k (u) y_{al} (g_x) + \chi_i^l (u) y_{ak} (g_x) \} \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x}. \]

The condition \((C_M)\) holds automatically whenever \( X \in V(p_M) \). Hence, \((C_M)\) holds if and only if it holds for \( X = (\partial/\partial x^j)_{g_x} \), namely,

\[ \sum_{k \leq l} \gamma_{kj}(g_x, \Gamma_x) \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x} = \gamma_M \left( g_x, \Gamma_x, \left( \frac{\partial}{\partial x^j} \right)_{g_x} \right) \]

\[ = \left( \frac{\partial}{\partial x^j} \right)_{g_x} - q^* \left( \frac{\partial}{\partial x^j} \right)_u \]

\[ = -\sum_{k \leq l} \{ \Gamma^a_{jk}(x) y_{al} (g_x) + \Gamma^a_{jl}(x) y_{ak} (g_x) \} \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x}, \]

thus proving the formula (23) in the statement. \( \square \)
3.2.2 The canonical covariant derivative

As is known (e.g., see [20, III, section 1], [23, pp. 157–158]) every connection $\Gamma$ on a principal $G$-bundle $P \to N$ induces a covariant derivative $\nabla^\Gamma$ on the vector bundle associated to $P$ under a linear representation $\rho: G \to \text{Gl}(m, \mathbb{R})$ with standard fibre $\mathbb{R}^m$. In particular, this applies to the principal bundle of linear frames, thus proving that every linear connection $\Gamma$ on $N$ induces a covariant derivative $\nabla^\Gamma$ on every tensorial vector bundle $E \to N$.

The bundles $(p_C)^*E$, where $E$ is a tensorial vector bundle, are endowed with a canonical covariant derivative $\nabla^E$ completely determined by the formula:

\[(\nabla^E)_X (f\xi) \quad (\Gamma_x) = \left(\left(\frac{\partial}{\partial x^i}\right) (p_C)^* (f\xi) \quad (\Gamma_x) + f \left(\nabla^\Gamma_{(p_C)_x} X \xi\right)\right)(x),\]

for all $X \in T_{\Gamma C}$, $f \in C^\infty(C)$, and every local section $\xi$ of $E$ defined on a neighbourhood of $x$. The uniqueness of $\nabla^E$ follows from (24) as the sections of $E$ span the sections of $(p_C)^*E$ over $C^\infty(C)$, see [8, 0.3.6]. Below, we are specially concerned with the cases $E = TN$ and $E = \wedge^2 T^*N \otimes TN$.

3.2.3 The 2-form associated with $\gamma_C$

As $p_C: C \to N$ is an affine bundle modelled over $\otimes^2 T^*N \otimes TN$, there is a natural identification

\[V(p_C) \cong (p_C)^* (\otimes^2 T^*N \otimes TN)\]

and consequently, an Ehresmann connection $\gamma_C$ on $C$ can also be viewed as a homomorphism $\gamma_C: TC \to \otimes^2 T^*N \otimes TN$. If $\gamma_C$ is locally given by

\[\gamma_C = \left(\left(dA^i_{jk} + \gamma^i_{jkl} dx^l\right) \otimes \frac{\partial}{\partial A^i_{jk}}\right), \quad \gamma^i_{jkl} \in C^\infty(C),\]

then

\[\gamma_C = \left(\left(dA^i_{jk} + \gamma^i_{jkl} dx^l\right) \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}\right),\]

and $\gamma_C$ induces a 2-form $\tilde{\gamma}_C$ taking values in $(p_C)^* (T^*N \otimes TN)$ as follows:

\[\tilde{\gamma}_C(X, Y) = c_1^i ((p_C)_x (Y) \otimes \gamma_C (X)) - c_1^i ((p_C)_x (X) \otimes \gamma_C (Y)),\]

\[\forall X, Y \in T_x C,\]

where

\[c_1^i: TN \otimes T^*N \otimes T^*N \otimes TN \to T^*N \otimes TN,\]

\[c_1^i (X_1 \otimes w_1 \otimes w_2 \otimes X_2) = w_1(X_1)w_2 \otimes X_2,\]

$X_1, X_2 \in T_x N$, $w_1, w_2 \in T^*_x N$. 
If $\gamma_C$ is given by (25), then from the very definition of $\tilde{\gamma}_C$ the following local expression is obtained:

$$\tilde{\gamma}_C = (dA^c_{ih} + (\gamma^c_{ha} - \gamma^c_{ahl}) dx^a) \wedge dx^j \otimes dx^h \otimes \frac{\partial}{\partial x^c}.$$  

### 3.2.4 The second geometric condition on $\gamma$

Let $\text{alt}_{12} : \otimes^2 T^*N \otimes TN \to \wedge^2 T^*N \otimes TN$ be the operator alternating the two covariant arguments.

The vector bundle $(p_C)^\ast (\otimes^2 T^*N \otimes TN)$ admits a canonical section

$$\tau_N : C \to \wedge^2 T^*N \otimes TN,$$

$$\tau_N (\Gamma_x) = T^\Gamma_x, \forall \Gamma_x \in C,$$

where $T^\Gamma_x$ is the torsion of $\Gamma_x$. Locally,

$$\tau_N = \sum_{j<k} (A^i_{jk} - A^i_{kj}) dx^j \wedge dx^k \otimes \frac{\partial}{\partial x^i}.$$  

From the previous formulas the next result follows:

**Proposition 3.5.** Let $\gamma$ be an Ehresmann connection on $M \times_N C$, let $\nabla^{(1)} = \nabla^{E_1}$ with $E_1 = TN$, let $R^{\gamma^{(1)}}$ be its curvature form, and finally, let $\nabla^{(2)} = \nabla^{E_2}$ with $E_2 = \wedge^2 T^*N \otimes TN$.

(C$C$) Assume the component $\gamma_C$ of $\gamma$ is defined on $C$. Then, the equations

(26) $$\tilde{\gamma}_C = R^{\gamma^{(1)}},$$  

(27) $$\text{alt}_{12} \circ \gamma_C = \nabla^{(2)} \tau_N,$$

are locally equivalent to the following ones:

(28) $$\gamma^h_{str} - \gamma^h_{rts} = A^m_{vm} A^m_{st} - A^m_{sm} A^m_{rt},$$

(29) $$\gamma^h_{rst} - \gamma^h_{srt} = A^h_{tm} (A^m_{rs} - A^m_{sr}) + A^m_{ls} \left( A^h_{mr} - A^h_{rm} \right) + A^m_{tr} \left( A^h_{sm} - A^h_{ms} \right).$$

### 3.3 Solution to the problem (P)

**Theorem 3.6.** If the connection $\gamma$ on $M \times N C$ satisfies the conditions $(C_M)$ and $(C_C)$ introduced above, then the vector field $D^\gamma$ satisfies the property stated in Proposition 3.2 and, accordingly the covariant Hamiltonian with respect to $\gamma$ of every $\mathcal{X}(N)$-invariant Lagrangian is also $\mathcal{X}(N)$-invariant.
Proof. When $\gamma_M$ satisfies the condition $(C_M)$ the brackets (18), (19), and (20) are respectively given by

\begin{equation}
\left[ X^h, D^\gamma \right] = \frac{\partial \gamma^i_{jkl}}{\partial A^i_{jk,l}},
\end{equation}

\begin{equation}
\left[ X^h_i, D^\gamma \right] = \left( Y^i_h (\gamma^a_{bcr}) - \delta^i_a \gamma^a_{bcr} + \delta^c_i \gamma^a_{bhr} + \delta^b_i \gamma^a_{hcr} + \delta^r_i \gamma^a_{bch} \right) \frac{\partial}{\partial A^a_{bc,r}},
\end{equation}

\begin{equation}
\left[ X^h_k, D^\gamma \right] = \left( -\frac{\partial \gamma_{abc}}{\partial A^i_{hk}} + \delta^i_k \left( \delta^b_i A^a_{ab} - \delta^a_i A^b_{ah} - \delta^a_i A^d_{hb} \right) - \frac{\partial \gamma_{abc}}{\partial A^i_{ki}} + \delta^i_j \left( \delta^b_i A^a_{ai} - \delta^a_i A^b_{ia} - \delta^a_i A^d_{ih} \right) \right) \frac{\partial}{\partial A^i_{ab,c}}.
\end{equation}

In addition, if $\gamma_C$ satisfies the condition $(C_C)$, then taking derivatives with respect to $x^h$ in (28) and (29) we obtain

\[ \frac{\partial \gamma^i_{klj}}{\partial x^h} = \frac{\partial \gamma^i_{jkl}}{\partial x^h}, \quad \frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{klj}}{\partial x^h}. \]

and renaming indices we deduce

\[ \frac{\partial \gamma^i_{jjk}}{\partial x^h} = \frac{\partial \gamma^i_{kkj}}{\partial x^h} = \frac{\partial \gamma^i_{kjj}}{\partial x^h} = \frac{\partial \gamma^i_{jkk}}{\partial x^h} = \frac{\partial \gamma^i_{jij}}{\partial x^h} = \frac{\partial \gamma^i_{ijj}}{\partial x^h} = \frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{klj}}{\partial x^h} = \frac{\partial \gamma^i_{lkj}}{\partial x^h} = \frac{\partial \gamma^i_{jlk}}{\partial x^h} = \frac{\partial \gamma^i_{lkj}}{\partial x^h} = \frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{klj}}{\partial x^h}. \]

From (30) we obtain

\[ \left[ X^h, D^\gamma \right] = \sum_{j<k<l} \frac{\partial \gamma^i_{jkl}}{\partial x^h} X^i_{jkl} + \frac{1}{2} \sum_{j<k} \frac{\partial \gamma^i_{jkk}}{\partial x^h} X^i_{jjk} + \frac{1}{6} \sum_{j<k} \frac{\partial \gamma^i_{jij}}{\partial x^h} X^i_{jjj}, \]

and consequently the values of $[X^h, D^\gamma]$ belong to the distribution $D_{M \times N C}$.

Moreover, as $\gamma_C$ is assumed to be defined on $C$, we have

\[ Y^i_h (\gamma^a_{bcr}) = \left( \delta^a_h A^i_{jk} - \delta^i_k A^a_{jh} - \delta^a_j A^i_{hk} \right) \frac{\partial \gamma^a_{bcr}}{\partial A^i_{jk}}. \]

For the sake of simplicity, below we set

\[ (T^i_h)^a_{bc} = A^i_{jk} \frac{\partial \gamma^a_{bc}}{\partial A^i_{jk}} - A^i_{ji} \frac{\partial \gamma^a_{bc}}{\partial A^i_{ji}} - A^i_{kh} \frac{\partial \gamma^a_{bc}}{\partial A^i_{ik}} - A^i_{hi} \frac{\partial \gamma^a_{bc}}{\partial A^i_{ih}} + \delta^i_h \gamma^a_{bc} + \delta^a_i \gamma^a_{bc} + \delta^a_i \gamma^a_{bc}. \]
Hence and the proof is complete.

Taking derivatives with respect to \( A^i_{jk} \), the equations (28) and (29) yield

\[
\frac{\partial \gamma^a_{bcr}}{\partial A^i_{jk}} - \frac{\partial \gamma^a_{rcb}}{\partial A^i_{jk}} = \delta^a_i \delta^a_k A^i_{cr} - \delta^a_i \delta^a_s A^i_{rc} + \delta^a_i \delta^a_k A^a_{rc} - \delta^a_i \delta^a_s A^a_{cr} - \delta^a_i \delta^a_k A^a_{bs} - \delta^a_i \delta^a_s A^a_{bs},
\]

\[
\frac{\partial \gamma^a_{rbc}}{\partial A^i_{jk}} - \frac{\partial \gamma^a_{rbc}}{\partial A^i_{jk}} = \delta^a_i \delta^a_s A^i_{rb} - \delta^a_i \delta^a_k A^i_{rb} - \delta^a_i \delta^a_k A^i_{cs} + \delta^a_i \delta^a_k A^i_{cs} + \delta^a_i \delta^a_k A^a_{cs} - \delta^a_i \delta^a_k A^a_{cs} - \delta^a_i \delta^a_k A^a_{bs} - \delta^a_i \delta^a_k A^a_{bs}.
\]

From these expressions, the following symmetries of indices are obtained:

\[
(T^i_h)^a_{bcb} = (T^i_h)^a_{cb} = (T^i_h)^a_{cb} (b < c),
\]

\[
(T^i_h)^a_{bec} = (T^i_h)^a_{cbe} = (T^i_h)^a_{ccb} (b < c),
\]

\[
(T^i_h)^a_{bcd} = (T^i_h)^a_{dcb} = (T^i_h)^a_{dcb} = (T^i_h)^a_{db} (b < c < d),
\]

and from (31) we obtain

\[
[X^i_h, D^a] = \sum_{b < c < d} (T^i_h)^a_{bcd} X^b_{a} + \frac{1}{2} \sum_{b < c} (T^i_h)^a_{bcb} X^b_{a} + \frac{1}{2} \sum_{b < c} (T^i_h)^a_{ccb} X^b_{a} + \frac{1}{6} (T^i_h)^a_{bbb} X^b_{a}.
\]

Hence \([X^i_h, D^a]\) also takes values into the distribution \( D_M \times N_C \).

The proof for the third bracket is similar to the previous two cases but longer. Letting

\[
(T^{ik}_h)^a_{rbc} = -\frac{\partial \gamma^a_{rbc}}{\partial A^i_{ik}} - \frac{\partial \gamma^a_{rbc}}{\partial A^i_{ki}} + \delta^a_i \left( \delta^a_h A^i_{rb} - \delta^a_b A^i_{rh} - \delta^a_r A^a_{hb} \right) + \delta^a_i \left( \delta^a_h A^i_{rb} - \delta^a_b A^i_{rh} - \delta^a_r A^a_{hb} \right),
\]

the following symmetries are obtained:

\[
(T^{ik}_h)^a_{bcb} = (T^{ik}_h)^a_{cb} = (T^{ik}_h)^a_{cb} (b < c),
\]

\[
(T^{ik}_h)^a_{bec} = (T^{ik}_h)^a_{cbe} = (T^{ik}_h)^a_{ccb} (b < c),
\]

\[
(T^{ik}_h)^a_{bcd} = (T^{ik}_h)^a_{dcb} = (T^{ik}_h)^a_{dcb} = (T^{ik}_h)^a_{db} (b < c < d).
\]

Hence

\[
[X^{ik}_h, D^a] = \sum_{b < c < d} (T^{ik}_h)^a_{bcd} X^b_{a} + \frac{1}{2} \sum_{b < c} (T^{ik}_h)^a_{bcb} X^b_{a} + \frac{1}{2} \sum_{b < c} (T^{ik}_h)^a_{ccb} X^b_{a} + \frac{1}{6} (T^{ik}_h)^a_{bbb} X^b_{a},
\]

and the proof is complete. \( \square \)
Theorem 3.7. The Ehresmann connections on \( C \) satisfying the equations (26) and (27) are the sections of an affine bundle over \( C \) modelled over the vector bundle \((p_C)^* (S^3T^*N \otimes TN)\). Consequently, there always exist Ehresmann connections on \( M \times_N C \) fulfilling the conditions (\( C_M \)) and (\( C_C \)) introduced above.

Proof. If two Ehresmann connections \( \gamma_C, \gamma_C' \) satisfy the equations (26) and (27), then the difference tensor field \( t = \gamma_C' - \gamma_C \), which is a section of the bundle \((p_C)^* (\otimes^3 T^*N \otimes TN)\), satisfies the following symmetries:

\[
\begin{align*}
(32) & \quad t(X_1, X_2, X_3) = t(X_3, X_2, X_1), \\
(33) & \quad t(X_1, X_2, X_3) = t(X_2, X_1, X_3),
\end{align*}
\]

according to (28), (29), respectively, for all \( X_1, X_2, X_3 \in T_x N, \Gamma_x \in C_x(N) \).

Hence

\[
\begin{align*}
t(X_1, X_3, X_2) & \stackrel{(32)}{=} t(X_2, X_3, X_1) \stackrel{(33)}{=} t(X_3, X_2, X_1) \stackrel{(32)}{=} t(X_1, X_2, X_3),
\end{align*}
\]

thus proving that \( t \) is totally symmetric. The second part of the statement thus follows from the fact that an affine bundle always admits global sections, e.g., see [20, I, Theorem 5.7]. \( \square \)

Remark 3.8. The results obtained above also hold if the bundle of linear connections is replaced by the subbundle \( C_{sym} = C_{sym}(N) \subset C \) of symmetric linear connections; the only difference to be observed between both bundles is that in the symmetric cases the equation (27), or equivalently (29), holds automatically.

4 The second-order formalism

In this section we consider the problem of invariance of covariant Hamiltonians for second-order Lagrangians defined on the bundle of metrics, i.e., for functions \( \mathcal{L} \in C^\infty(J^2M) \), where \( M \) denotes, as throughout this paper, the bundle of pseudo-Riemannian metrics of a given signature \((n^+, n^-)\) on \( N \).

4.1 Second-order Ehresmann connections

A second-order Ehresmann connection on \( p: E \rightarrow N \) is a differential 1-form \( \gamma^2 \) on \( J^1E \) taking values in the vertical sub-bundle \( V(p^1) \) such that \( \gamma^2(X) = X \) for every \( X \in V(p^1) \). (We refer the reader to [29] for the basics on Ehresmann connections of arbitrary order.) Once a connection \( \gamma^2 \) is given, we have a decomposition of vector bundles \( T(J^1E) = V(p^1) \oplus \ker \gamma^2 \), where \( \ker \gamma^2 \) is called the horizontal sub-bundle determined by \( \gamma^2 \). In the
coordinate system on $J^1E$ induced from a fibred coordinate system $(x^j, y^\alpha)$ for $p$, a connection form can be written as

$$\gamma^2 = (dy^\alpha + \gamma^i_\alpha dx^j) \otimes \frac{\partial}{\partial y^\alpha} + (dy^\alpha + \gamma^i_\alpha dx^j) \otimes \frac{\partial}{\partial y^i}.$$  

As in the first-order case, the action of the group $\text{Aut}(p)$ on the space of second-order connections is defined by the formula

$$\Phi \cdot \gamma^2 = \left(\Phi^{(1)}\right)_* \circ \gamma^2 \circ \left(\Phi^{(1)}\right)_*^{-1}, \quad \forall \Phi \in \text{Aut}(p).$$

As $\Phi^{(1)}: J^1M \rightarrow J^1M$ is a morphism of fibred manifolds over $N$, $(\Phi^{(1)})_*$ transforms the vertical subbundle $V(p^1)$ into itself; hence the previous definition makes sense.

### 4.2 A remarkable isomorphism

**Theorem 4.1.** Let $\Gamma^g$ be the Levi-Civita connection of a pseudo-Riemannian metric $g$ on $N$. The mapping $\zeta_N: J^1M \rightarrow M \times_N C^\text{sym}$, $\zeta_N(j^1_g) = (g_x, \Gamma^g_x)$, is a diffeomorphism. There is a natural one-to-one correspondence between first-order Ehresmann connections on the bundle $p: M \times_N C^\text{sym} \rightarrow N$ and second-order Ehresmann connections on the bundle $p_M: M \rightarrow N$, which is explicitly given by,

$$\gamma = ((\zeta_N^v)_*)^{-1} \circ \gamma \circ (\zeta_N)_*,$$

where $\gamma: T(M \times_N C^\text{sym}) \rightarrow V(p)$ is a first-order Ehresmann connection,

$$(\zeta_N)_*: T(J^1M) \rightarrow T(M \times_N C^\text{sym})$$

is the Jacobian mapping induced by $\zeta_N$, and $(\zeta_N^v)_*: V(p^1_M) \rightarrow V(p)$ is its restriction to the vertical bundles.

**Proof.** As a computation shows, the equations of $\zeta_N$ in the coordinate systems introduced in the section 2.1.2, are as follows:

$$x^i \circ \zeta_N = x^i, \quad y^i_j \circ \zeta_N = y^i_j, \quad A^h_{ij} \circ \zeta_N = 1 \frac{\partial y^h}{\partial y^i_j}(y_{k,j} + y_{j,k} - y_{i,j}), \quad i \leq j,$$

where $(y^{ij})_{i,j=1}^n$ is the inverse mapping of the matrix $(y_{ij})_{i,j=1}^n$ and the functions $y_{ij}$ are defined in (4). Hence

$$x^i \circ \zeta_N^{-1} = x^i, \quad y^i_j \circ \zeta_N^{-1} = y^i_j, \quad y_{i,j,k} \circ \zeta_N^{-1} = y_{i,k} A^h_{j,k} + y_{k,j} A^h_{i,k}, \quad i \leq j.$$
As the diffeomorphism $\zeta_N$ induces the identity on the ground manifold $N$, it follows that the definition of $\gamma^2$ in (35) makes sense and the following formulas are obtained:

$$\gamma^2 \left( \frac{\partial}{\partial x'} \right) = \sum_{a \leq b} (\gamma_{abr} \circ \zeta_N) \frac{\partial}{\partial y_ab} + \sum_{i \leq j} \gamma_{ijkr} \frac{\partial}{\partial y_{ij,k}},$$

$$\gamma_{ijkr} = \frac{1}{2} \sum_{a \leq b} \delta_{ah} \delta_{bh} \frac{\partial}{\partial y_{ij,k}} \left( \gamma_{abr} \circ \zeta_N \right) y^{hl} (y_{jl,k} + y_{kl,j} - y_{jk,l}) + \frac{1}{2} \sum_{a \leq b} \delta_{ah} \delta_{bh} \frac{\partial}{\partial y_{ij,k}} \left( \gamma_{abr} \circ \zeta_N \right) y^{hl} (y_{il,k} + y_{kl,i} - y_{ik,l}) + \sum_{j \leq a} \delta_{ah} \delta_{bjh} \left( \gamma_{jkr} \circ \zeta_N \right) y_{hi} + \sum_{a \leq j} \delta_{ah} \delta_{ajh} \left( \gamma_{a}^{h} \circ \zeta_N \right) y_{hi} + \sum_{a \leq i} \delta_{ah} \delta_{ah} \left( \gamma_{a}^{h} \circ \zeta_N \right) y_{hj},$$

where

$$\gamma = \sum_{i \leq j} (dy_{ij} + \gamma_{ij, k} dx^k) \otimes \frac{\partial}{\partial y_{ij}} + \sum_{j \leq k} (dA_{jk}^i + \gamma_{jkl} dx^l) \otimes \frac{\partial}{\partial A_{jk}^i},$$

or equivalently,

$$\gamma = \frac{1}{2 - \delta_{ij}} \left( dy_{ij} + \gamma_{ij, k} dx^k \right) \otimes \frac{\partial}{\partial y_{ij}} + \frac{1}{2 - \delta_{jk}} \left( dA_{jk}^i + \gamma_{jkl} dx^l \right) \otimes \frac{\partial}{\partial A_{jk}^i},$$

assuming $\gamma_{hir} = \gamma_{ihr}$ for $h > i$, and $\gamma_{hkr} = \gamma_{hkr}$ for $j > k$. Taking the symmetry $A_{jk}^i = A_{kj}^i$ into account, we obtain

$$\gamma_{ijkr} = \frac{1}{2} \left( \gamma_{hkr} \circ \zeta_N \right) y^{hl} (y_{jl,k} + y_{kl,j} - y_{jk,l}) + \frac{1}{2} \left( \gamma_{hjr} \circ \zeta_N \right) y^{hl} (y_{il,k} + y_{kl,i} - y_{ik,l}) + \left( \gamma_{jkr} \circ \zeta_N \right) y_{hi} + \left( \gamma_{ikr} \circ \zeta_N \right) y_{hj}.$$

Hence

$$\gamma_{ijkr} \circ \zeta_N^{-1} = \gamma_{hkr} A_{jk}^h + \gamma_{hjr} A_{ik}^h + \gamma_{jkr} y_{hi} + \gamma_{ikr} y_{hj}, \quad i \leq j.$$

Permuting the indices $i, j, k$ cyclically on the previous equation, we have

$$\gamma_{ijr} = -\gamma_{hkr} A_{ij}^h y^{ks} - \frac{1}{2} \left( \gamma_{ijkr} \circ \zeta_N^{-1} - \gamma_{jkr} \circ \zeta_N^{-1} - \gamma_{ikr} \circ \zeta_N^{-1} \right) y^{ks},$$

thus proving that the mapping $\gamma \mapsto \gamma^2$ defined in the statement, is bijective.
4.3 Covariant Hamiltonians for second-order Lagrangians

The Legendre form of a second-order Lagrangian density \( \Lambda = Lv_n \) on the bundle \( p: E \to N \) is the \( V^*(p^1) \)-valued \( p^3 \)-horizontal \( (n-1) \)-form \( \omega_\Lambda \) on \( J^3E \) locally given by (e.g., see [17], [26], [35]),

\[
\omega_\Lambda = i_{\partial/\partial x^i} v_n \otimes (L^i_\alpha dy^\alpha + L^{ij}_\alpha dy^\alpha_j),
\]

where

\[
L^{ij}_\alpha = \frac{1}{2} \delta^{ij} \frac{\partial L}{\partial y^\alpha_{ij}}, \quad (40)
\]

\[
L^i_\alpha = \frac{\partial L}{\partial y^\alpha_i} - \sum_j \frac{1}{2} \delta^{ij} D_j \left( \frac{\partial L}{\partial y^\alpha_{ij}} \right), \quad (41)
\]

\[
D_j = \frac{\partial}{\partial x^j} + \sum_{I \in \mathbb{N}^n, |I|=0} y^\alpha_I (j) \frac{\partial}{\partial y^\alpha_I}
\]

denotes the total derivative with respect to the variable \( x^j \).

The Poincaré-Cartan form attached to \( \Lambda \) is then defined to be the ordinary \( n \)-form on \( J^3E \) given by, \( \Theta_\Lambda = (p^3_1)^* \theta^2 \wedge \omega_\Lambda + \Lambda \), where \( \theta^2 \) is the second-order structure form (cf. [33, (0.36)]) and the exterior product of \( (p^3_2)^* \theta^2 \) and the Legendre form, is taken with respect to the pairing induced by duality, \( V(p^1) \times_{F^1} V^*(p^1) \to \mathbb{R} \). The most outstanding difference with the first-order case is that the Legendre and Poincaré-Cartan forms associated with a second-order Lagrangian density are generally defined on \( J^3E \), thus increasing by one the order of the density.

Similarly to the first-order case (see [11], [24]), given a second-order Lagrangian density \( \Lambda \) on \( p: E \to N \) and a second-order connection \( \gamma^2 \) on \( p: E \to N \), by subtracting \( (p^3_2)^* \theta^2 \) from \( (p^3_1)^* \gamma^2 \) we obtain a \( p^3 \)-horizontal form, and we can define the corresponding covariant Hamiltonian to be the Lagrangian density \( \Lambda^{\gamma^2} \) of third order,

\[
\Lambda^{\gamma^2} = ((p^3_1)^* \gamma^2 - (p^3_2)^* \theta^2) \wedge \omega_\Lambda - \Lambda. \quad (42)
\]

Expanding on the right-hand side of the previous equation, we obtain a decomposition of \( \Theta_\Lambda \) that generalizes the classical formula for the Hamiltonian in Mechanics; namely, \( \Theta_\Lambda = (p^3_1)^* \gamma^2 \wedge \omega_\Lambda - \Lambda^{\gamma^2} \). With the same notations as in the formulas (34), (40), (41) the following formula is deduced:

\[
L^{\gamma^2} = (\gamma^\alpha_i + y^\alpha_i) L^i_\alpha + (\gamma^{hi}_h + y^{hi}_h) L^{ih}_\alpha - L. \quad (43)
\]

Because of the equation (41), \( \Theta_\Lambda \) and \( L^{\gamma^2} \) are generally defined on \( J^3E \).
4.4 Invariant covariant Hamiltonians on $J^2 \mathcal{M}$

**Lemma 4.2.** If $\gamma$ is a first-order Ehresmann connection on $M \times_N C^\text{sym}$ satisfying the conditions $(C_M)$, then the following equation holds for the second-order Ehresmann connection $\gamma^2$ on $M$ given in the formula (35):

$$\gamma_{abr} \circ \zeta_N = -y_{ab,r}.$$  

**Proof.** Actually, from the formulas (23) and (36) we obtain

$$
\gamma_{abr} \circ \zeta_N = - \left( y_{mb} (A_{ra}^m \circ \zeta_N) + y_{ma} (A_{rb}^m \circ \zeta_N) \right)
= - \frac{1}{2} \left\{ y_{mb} y^{mk} (y_{rk,a} + y_{ak,r} - y_{ra,k}) + y_{ma} y^{mk} (y_{rk,b} + y_{bk,r} - y_{rb,k}) \right\}
= -y_{ab,r}.
$$

□

**Lemma 4.3.** If a first-order connection $\gamma$ on $M \times_N C^\text{sym}$ satisfies the condition $(C_C)$ introduced above, then the following formulas for its components hold:

$$(44) \quad \gamma_{rts}^h - \gamma_{rst}^h = A_{sm}^h A_{rt}^m - A_{tm}^h A_{rs}^m.$$  

**Proof.** As the bundle under consideration is that of symmetric connections, the following symmetry holds: $\gamma_{abc}^h = \gamma_{bac}^h$, and we have

$$
\gamma_{rts}^h = \gamma_{str}^h - (A_{rm}^h A_{st}^m - A_{sm}^h A_{rt}^m) \quad [\text{by virtue of (28)}]
= \gamma_{tsr}^h - (A_{rm}^h A_{st}^m - A_{sm}^h A_{rt}^m)
= \left( \gamma_{rst}^h + A_{rm}^h A_{st}^m - A_{sm}^h A_{rt}^m \right) - (A_{rm}^h A_{st}^m - A_{sm}^h A_{rt}^m)
= \gamma_{rst}^h + (A_{sm}^h A_{rt}^m - A_{tm}^h A_{rs}^m).
$$

□

**Proposition 4.4.** Let

$$\zeta_N^2 = \zeta_N^{(1)} \big|_{J^2 \mathcal{M}} : J^2 \mathcal{M} \to J^1 (M \times_N C^\text{sym})$$

be the restriction to the closed submanifold $J^2 \mathcal{M} \subset J^1 (J^1 \mathcal{M})$ of the prolongation $\zeta_N^{(1)} : J^1 (J^1 \mathcal{M}) \to J^1 (M \times_N C^\text{sym})$ of the mapping $\zeta_N$ defined in Theorem 4.1. For every $(j_{x}^1 g, j_{x}^1 \Gamma) \in J^1 (M \times_N C^\text{sym})$ there exists a unique $j_{x}^2 g' \in J^2 \mathcal{M}$ such that, $j_{x}^2 g' = j_{x}^1 g$ and $j_{x}^2 \Gamma g' = j_{x}^1 \Gamma$ and the mapping $\kappa : J^1 (M \times_N C^\text{sym}) \to J^2 \mathcal{M}$ defined by $\kappa (j_{x}^1 g, j_{x}^1 \Gamma) = j_{x}^2 g'$ is a Diff$N$-equivariant retract of $\zeta_N^2$. 


Proof. From the formulas (36) and (37) we obtain
\[ \frac{\partial g'_{ij}}{\partial x^k} = g'_{hi} \left( \Gamma^g_{jik} \right)_h + g'_{hj} \left( \Gamma^g_{ikh} \right)_h, \]
\[ \left( \Gamma^g' \right)_{ij} = \frac{1}{2} g_{hk} \left( \frac{\partial g'_{ik}}{\partial x^j} + \frac{\partial g'_{jk}}{\partial x^i} - \frac{\partial g'_{ij}}{\partial x^k} \right). \]

for every non-singular metric \( g' \) on \( N \). Hence the second partial derivatives of \( g'_{ij} \) are completely determined, namely
\[ \frac{\partial^2 g'_{ij}}{\partial x^k \partial x_l} = \frac{\partial g_{hi}}{\partial x^l} \Gamma^h_{jik} + g_{hi} \frac{\partial \Gamma^h_{jk}}{\partial x^l} + \frac{\partial g_{hj}}{\partial x^l} \Gamma^h_{ik} + g_{hj} \frac{\partial \Gamma^h_{ik}}{\partial x^l}. \]

Moreover, the Levi-Civita connection of a metric depends functorially on the metric, i.e., \( \phi \cdot \Gamma^g = \Gamma^\phi g \) for every \( \phi \in \text{Diff} N \). Hence, by transforming the equations \( j_x^1 g' = j_x^1 g \) and \( j_x^1 \Gamma^g = j_x^1 \Gamma^g \) by \( \phi \) we can conclude.

\[ \square \]

**Theorem 4.5.** If a first-order Ehresmann connection \( \gamma \) on \( M \times_N C^\text{sym} \) satisfies the conditions \((C_M)\) and \((C_C)\) introduced above, then the covariant Hamiltonian \( \Lambda^\gamma \) attached to every \( \text{Diff} N \)-invariant second-order Lagrangian density \( \Lambda \) on \( M \) with respect to the second-order Ehresmann connection \( \gamma^2 \) on \( M \) defined in the formula (35), is defined on \( J^2 M \) and it is also \( \text{Diff} N \)-invariant.

Proof. Given a \( \text{Diff} N \)-invariant second-order Lagrangian density \( \Lambda = \mathcal{L}v \) on \( M \), let \( \Lambda' = \mathcal{L}'v \) be the first-order Lagrangian density on \( M \times_N C^\text{sym} \) given by \( \Lambda' = \varpi^* \Lambda \), which is also \( \text{Diff} N \)-invariant as \( \varpi \) is a \( \text{Diff} N \)-equivariant mapping according to Proposition 4.4. Moreover, as \( \varpi \) is a retract of \( \zeta_N^2 \), we have \( (\zeta_N^2)^* \Lambda' = (\zeta_N^2)^* \varpi^* \Lambda = \Lambda \), i.e., \( \Lambda = (\zeta_N^2)^* \Lambda' \). This formula is equivalent to saying \( \mathcal{L} = \mathcal{L}' \circ \zeta_N^2 \), as the \( n \)-form \( v \) is \( \text{Diff} N \)-invariant, and it is even equivalent to \( L = L' \circ \zeta_N^2 \) because \( \zeta_N^2 \) induces the identity on \( N \).

We claim \( \mathcal{L}^{\gamma^2} = (\mathcal{L}')^\gamma \circ \zeta_N^2 \). This formula will end the proof as the mapping \( \zeta_N^2 \) is \( \text{Diff} N \)-equivariant and \( (\mathcal{L}')^\gamma \) is \( \text{Diff} N \)-invariant by virtue of Theorem 3.6.

To start with, we observe that the formula (40) for \( \Lambda \) can be written, in the present case, as follows:
\[ L^{abij} = \frac{1}{2 - \delta_{ij}} \frac{\partial L}{\partial y_{ab,ij}}, \]
or equivalently, letting $\mathcal{L}^{abij} = \rho^{-1} L^{abij}$,

$$\mathcal{L}^{abij} = \frac{1}{2-\delta_{ij}} \frac{\partial \mathcal{L}}{\partial y_{ab,ij}}.$$  

(45)

Taking the formula in Lemma 4.2 into account, the formula (43) for $\Lambda$ reads as

$$L^\gamma = \sum_{a \leq b} \left( \gamma_{abij} + y_{ab,ij} \right) L^{abij} - L,$$

or even

$$L^\gamma = \sum_{a \leq b} \left( \gamma_{abij} + 2 \gamma_{ab,ij} \right) L^{abij} - L,$$

where $L^\gamma = \rho^{-1} L^\gamma$. Hence $\mathcal{L}^\gamma$ is defined over $J^2 M$. As $y_{ab,ij} = y_{ab,ji}$, we obtain

$$L^\gamma = \sum_{a \leq b} \sum_{i \leq j} \left( \frac{1}{2} \left( \gamma_{abij} + \gamma_{abji} \right) + y_{ab,ij} \right) L^{abij} - L,$$

Moreover, we have

$$(L')^\gamma = \sum_{a \leq b} \left( \gamma_{abc} + y_{ab,c} \right) \frac{\partial (L' \circ \zeta_N^2)}{\partial y_{ab,c}} + \sum_{a \leq b} \left( \gamma_{abl} + A^i_{ab,l} \right) \frac{\partial (L' \circ A^h_{kl,q})}{\partial A^i_{ab,l}} - L'.$$

Hence

$$(L')^\gamma \circ \zeta_N^2 = \sum_{k \leq l} \left( \gamma^h_{klq} \circ \zeta_N + A^h_{kl,q} \circ \zeta_N \right) \left( \frac{\partial (L' \circ A^h_{kl,q})}{\partial A^h_{kl,q}} \circ \zeta_N^2 \right) - L' \circ \zeta_N^2,$$

$$= \sum_{k \leq l} \left\{ -\frac{1}{2} \left( \gamma_{klr} - \gamma_{lrk} + \gamma_{rkl} \right) y_{rhl}^h 
+ \frac{1}{2} \left( y_{klr} - y_{lkr} + y_{rkl} - y_{rkl} \right) \right\} \left( \frac{\partial (L' \circ A^h_{kl,q})}{\partial A^h_{kl,q}} \circ \zeta_N^2 \right) - L' \circ \zeta_N^2.$$

Consequently, the proof reduces to state that the following equation

$$\frac{1}{2} \left( \gamma_{krql} + \gamma_{krlq} + \gamma_{rlkq} + \gamma_{lkrq} - \gamma_{klqr} - \gamma_{rklq} \right) = -\frac{1}{2} \left( \gamma_{krql} - \gamma_{lkrq} + \gamma_{rklq} \right)$$

holds true, or equivalently,

$$0 = \left( \gamma_{ijkr} - \gamma_{ijrk} + \gamma_{irjk} - \gamma_{rjki} \right) + \left( \gamma_{rjki} - \gamma_{rjik} \right).$$

(46)
According to the formulas (38) and (23) we obtain
\[ \gamma_{ijkr} \odot \zeta_{N}^{-1} = \left( \gamma_{jkr}^h - A_{ra}^h A_{jk}^r \right) y_{hi} + \left( \gamma_{ikr}^h - A_{ra}^h A_{ik}^r \right) y_{hj} \]
\[ - \left( A_{ij}^a A_{ik}^a + A_{ri}^a A_{jk}^r \right) y_{ah}. \]

The third term on the right-hand side of this equation is symmetric in the indices \( k \) and \( r \), as \( A_{bc}^a = A_{cb}^a \). Hence
\[ (\gamma_{ijkr} - \gamma_{ijrk}) \odot \zeta_{N}^{-1} = \left( \gamma_{jkr}^h - \gamma_{jrk}^h - A_{ra}^h A_{jk}^r + A_{ra}^h A_{jr}^a \right) y_{hi} \]
\[ + \left( \gamma_{ikr}^h - \gamma_{irk}^h - A_{ra}^h A_{ik}^r + A_{ra}^h A_{ir}^a \right) y_{hj}. \]

By composing the right-hand side of the equation (46) and \( \zeta_{N}^{-1} \), and taking the previous formula and the formulas (28) and (44) into account, we conclude that this expression vanishes indeed.

\[ \square \]

5 Palatini and Einstein-Hilbert Lagrangians

Let us compute the covariant Hamiltonian density attached to the Palatini Lagrangian. Following the notations in [20], the Ricci tensor field attached to the symmetric connection \( \Gamma \) is given by \( S^\Gamma(X,Y) = \text{tr}(Z \mapsto R^\Gamma(Z,X)Y) \), where \( R^\Gamma \) denotes the curvature tensor field of the covariant derivative \( \nabla^\Gamma \) associated to \( \Gamma \) on the tangent bundle; hence \( S^\Gamma = (R^\Gamma)_{jldx^l \otimes dx^d} \), where
\[ (R^\Gamma)_{jldx^l \otimes dx^d}, \]
\[ (R^\Gamma)_{jldx^l \otimes dx^d} = \partial \Gamma^i_{jl}/\partial x^k - \partial \Gamma^i_{jk}/\partial x^l + \Gamma^m_{jl} \Gamma^i_{km} - \Gamma^m_{jk} \Gamma^i_{lm}. \]

The Lagrangian is the function on \( J^1(M \times_N C^{\text{sym}}) \) thus given by,
\[ \mathcal{L}_P(j^1_x g, j^1_x \Gamma) = g^{ij}(x)(R^\Gamma)_{ij}(x) \]
and local expression
\[ \mathcal{L}_P = g^{ij}(A_{ij,k}^k - A_{i,k,j}^k + A_{ij}^m A_{km}^k - A_{ik}^m A_{jm}^k). \]

As a computation shows, for every first-order connection \( \gamma \) on \( M \times_N C^{\text{sym}} \) satisfying (44) and taking the formula (2) into account, we obtain \( \mathcal{L}_P = 0 \). This result is essentially due to the fact that the P-C form of the P density \( \Lambda_P = \mathcal{L}_P \nu = L_P v_n \) projects onto \( M \times_N C^{\text{sym}} \). In fact, the following general characterization holds:

**Proposition 5.1.** Let \( p : E \to N \) be an arbitrary fibred manifold and let \( \gamma \) be a first-order Ehresmann connection on \( E \). The equation \( L^\gamma = 0 \) holds true for a Lagrangian \( L \in C^\infty(J^1E) \) if and only if, i) the Poincaré-Cartan form of the density \( \Lambda = L v_n \) projects onto \( J^0E \) and, ii) \( L = \left( (p_0^1)^* \gamma - \theta, dL|_X(p_0^1) \right) \).
Proof. The equation $L^\gamma = 0$ is equivalent to the equation $D^\gamma L = L$, where $D^\gamma$ is the $p_0^1$-vertical vector field defined in the formula (17), and the general solution to the latter is $L = f(x^i, y^\alpha, \gamma_i^\alpha + y_i^\alpha)$, $f(x^i, y^\alpha, y_i^\alpha)$ being a homogeneous smooth function of degree one in the variables $(y_i^\alpha)$, $1 \leq \alpha \leq m$, $1 \leq i \leq n$, according to Euler’s homogeneous function theorem. As $f$ is defined for all values of the variables $(y_i^\alpha)$, $1 \leq \alpha \leq m$, $1 \leq i \leq n$, we conclude that the functions $L_i^\alpha = \partial L/\partial y_i^\alpha$ must be defined on $E$. Hence $L$ is written as $L = L_i^\alpha(x^j, y_i^\beta) y_i^\alpha + L_0(x^j, y_i^\beta)$, but this is exactly the condition for the P-C form of $\Lambda$ to be projectable onto $J^0E = E$, as follows from the local expression of this form, namely,

$$\Theta_\Lambda = \frac{\partial L}{\partial y_i^\alpha} y_i^\alpha \wedge i_{\partial/\partial x^i} v_n + L v_n$$

$$= \frac{\partial L}{\partial y_i^\alpha} dy_i^\alpha \wedge i_{\partial/\partial x^i} v_n + \left( L - y_i^\alpha \frac{\partial L}{\partial y_i^\alpha} \right) v_n.$$

Moreover, by imposing the condition $D^\gamma L = L$ we obtain $L_0 = L_i^\alpha \gamma_i^\alpha$, or in other words $L = (\gamma_i^\alpha + y_i^\alpha) \partial L/\partial y_i^\alpha$, which is equivalent to the equation ii) in the statement. □

The corresponding result for the second-order formalism is similar but the computations are more cumbersome. Let us compute the covariant Hamiltonian density attached to the Einstein-Hilbert Lagrangian. As a matter of notation, we set $S^9(X,Y) = S^{T^9}(X,Y)$ for the metric $g$, $\Gamma^9$ being its Levi-Civita connection, and similarly, $(R^9)^i_{jkl} = (R^{T^9})^i_{jkl}$.

The E-H Lagrangian is thus given by $\mathcal{L}_{EH} \circ j^2 g = (y^{ij} \circ g)(R^9)^{h}_{ij}$. As the Levi-Civita connection $\Gamma^9$ depends functorially on $g$, $\mathcal{L}_{EH}$ is readily seen to be Diff$N$-invariant; it is in addition linear in the second-order variables $y_{ij,kl}$. By using the third formula in (36) the following local expression for $\mathcal{L}_{EH}$ is obtained:

$$\mathcal{L}_{EH} = \frac{1}{2} y^{ij} y^{hd} (y_{dj,hi} - y_{dij,hi} + y_{dji,h} + y_{hij,d}) + \mathcal{L}'_{EH},$$

$$\mathcal{L}'_{EH} = \frac{1}{2} y^{ij} \left\{ y^{hm} y_{mn} y^{rd} (y_{id,h} + y_{hd,i} - y_{ih,d}) 
- y^{hm} y_{mr,h} y^{rd} (y_{id,j} + y_{jd,i} - y_{ij,d}) 
+ \frac{1}{2} y^{hr} y^{md} (y_{id,j} + y_{jd,i} - y_{ij,d}) (y_{hr,m} + y_{mr,h} - y_{hm,r}) 
- \frac{1}{2} y^{hr} y^{md} (y_{id,h} + y_{hd,i} - y_{ih,d}) (y_{jr,m} + y_{mr,j} - y_{jm,r}) \right\}.$$
According to (45), for every first-order connection form \( \gamma \) on \( M \times N \) satisfying the conditions \((C_M)\) and \((C_C)\) above, we have

\[
L_{EH}^2 = \sum_{a \leq b} \frac{1}{2 - \delta_{ij}} (\gamma_{abij} + y_{abij}) \frac{\partial L_{EH}}{\partial y_{abij}} - L_{EH},
\]

and as a computation shows,

\[
L_{EH}^2 = \frac{1}{2} y^{ij} \left( \gamma_{idjh} + \gamma_{jdh} - \gamma_{jdih} - \gamma_{hdi} - \gamma_{hijd} \right) y^{jd} + \frac{1}{2} y^{ij} \left( y^{hm} y_{mr} y^{rd} (y_{id,j} + y_{jd,i} - y_{ij,d}) - y^{hm} y_{mr} y^{rd} (y_{id,h} + y_{hd,i} - y_{ih,d}) - y^{hr} y^{md} (y_{id,h} + y_{hd,i} - y_{ih,d}) (y_{jr,m} + y_{mr,j} - y_{jm,r}) + \frac{1}{2} y^{hr} y^{md} (y_{id,h} + y_{hd,i} - y_{ih,d}) (y_{jr,m} + y_{mr,j} - y_{jm,r}) \right)
\]

\[= 0,
\]

where the formulas (39), (44), (36), and Lemma 4.3 have been used. In this case, the P-C form of the E-H density \( \Lambda_{EH} = L_{EH} v = L_{EH} v_n \),

\[
(47) \quad \Theta_{\Lambda_{EH}} = \sum_{k \leq l} \left( L_{EH}^{i,k} dy_{kl} + L_{EH}^{ij,kl} dy_{kl,i,j} \right) \wedge i_{\partial / \partial x^i} v_n + H v_n,
\]

\[
H = L_{EH}' - \sum_{k \leq l} L_{EH}^{i,k} y_{kl,i},
\]

\[
L_{EH}^{i,k} = \frac{\partial L_{EH}'}{\partial y_{kl,i}} - \frac{1}{2 - \delta_{ij}} y_{ab,j} \frac{\partial^2 L_{EH}}{\partial y_{ab} \partial y_{kl,ij}},
\]

\[
L_{EH}^{ij,kl} = \frac{1}{2 - \delta_{ij}} \frac{\partial L_{EH}'}{\partial y_{kl,ij}},
\]

(cf. (40), (41)) is not only projectable onto \( J^2 M \) but also on \( J^1 M \) (e.g., see [13]), although there is no first-order Lagrangian on \( J^1 M \) admitting (47) as its P-C form. This fact is strongly related to a classical result by Hermann Weyl ([39], Appendix II], also see [22], [18]) according to which the only Diff\(N\)-invariant Lagrangians on \( J^2 M \) depending linearly on the second-order coordinates \( y_{ab,ij} \) are of the form \( \lambda L_{EH} + \mu \), for scalars \( \lambda, \mu \). This also explains why a true first-order Hamiltonian formalism exists in the Einstein-Cartan gravitation theory, e.g., see [37], [38]. In fact, if

\[
L_{EH}' = \frac{1}{2 - \delta_{ij}} \frac{\partial L_{EH}}{\partial y_{kl,ij}} y_{kl,j} \quad \text{(hence } I_{EH}^{ij,kl} = \frac{\partial L_{EH}'}{\partial y_{kl,ij}} \text{)}
\]

and the momentum functions are defined as follows:

\[
p_{kl,i} = L_{EH}^{i,k} y_{kl,i} - \frac{\partial L_{EH}'}{\partial y_{kl}},
\]

then

\[
d \Theta_{\Lambda_{EH}} = dp_{kl,i} \wedge dy_{kl} \wedge i_{\partial / \partial x^i} v_n + dH \wedge v_n,
\]
and from the Hamilton-Cartan equation (e.g., see [13, (1)]) we conclude that a metric \( g \) is an extremal for \( \Lambda_{EH} \) if and only if,

\[
0 = \frac{\partial (p_{ab,i} \circ j^1 g)}{\partial x^i} - \frac{\partial H}{\partial y_{ab}} \circ j^1 g,
\]

\[
0 = \frac{\partial (y_{ab} \circ g)}{\partial x^i} + \frac{\partial H}{\partial y_{ab,i}} \circ j^1 g.
\]

On the other hand, it is no longer true that the covariant Hamiltonians of the non-linear Lagrangians of the form \( f(\mathcal{L}_{EH}), f'' \neq 0 \), considered in some cosmological models (e.g., see [1], [6], [9], [12], [19], [21], [31]) and those in higher dimensions (e.g., see [15], [36]) vanish. In fact, as a computation shows, one has \( f(\mathcal{L}_{EH})^2 = f'(\mathcal{L}_{EH})\mathcal{L}_{EH} - f(\mathcal{L}_{EH}), \forall f \in C^\infty(\mathbb{R}). \)

References

[1] A. Borowiec, M. Ferraris, Marco, M. Francaviglia, I. Volovich, Almost-complex and almost-product Einstein manifolds from a variational principle, J. Math. Phys. 40 (1999), no. 7, 3446–3464.
[2] U. Bruzzo, The global Utiyama theorem in Einstein-Cartan theory, J. Math. Phys. 28 (1987), no. 9, 2074–2077.
[3] H. Burton, R. B. Mann, Palatini variational principle for an extended Einstein-Hilbert action, Phys. Rev. D (3) 57 (1998), no. 8, 4754–4759.
[4] —, Palatini variational principle for \( N \)-dimensional dilaton gravity, Classical Quantum Gravity 15 (1998), no. 5, 1375–1385.
[5] M. Castrillón López, J. Muñoz Masqué, The geometry of the bundle of connections, Math. Z. 236 (2001), 797–811.
[6] S. Cotsakis, J. Miritzis, L. Querella, Variational and conformal structure of nonlinear metric-connection gravitational Lagrangians, J. Math. Phys. 40 (1999), no. 6, 3063–3071.
[7] M. Crampin, E. Martínez, W. Sarlet, Linear connections for systems of second-order ordinary differential equations, Ann. Inst. H. Poincaré, section A, 65 (1996), no. 2, 223–249.
[8] A. De Paris, A. Vinogradov, Fat manifolds and linear connections, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009.
[9] J. P. Durrusseau, R. Kerner, The effective gravitational Lagrangian and the energy-momentum tensor in the inflationary universe, Classical Quantum Gravity 3 (1986), no. 5, 817–824.
[10] F. Etayo Gordejuela, J. Muñoz Masqué, Gauge group and G-structures, J. Phys. A 28 (1995), no. 2, 497–510.
[11] Antonio Fernández, Pedro L. García, J. Muñoz Masqué, Gauge-invariant covariant Hamiltonians, J. Math. Phys. 41 (2000), 5292–5303.

[12] É. É. Flanagan, Palatini form of $1/R$ gravity, Phys. Rev. Lett. 92 (2004), no. 7, 071101, 4 pp.

[13] Pedro L. García, J. Muñoz Masqué, Le problème de la régularité dans le calcul des variations du second ordre, C. R. Acad. Sci. Paris 301 Série I (1985), 639–642.

[14] —, Differential invariants on the bundles of linear frames, J. Geom. Phys. 7, no. 3 (1990), 395–418.

[15] B. Giorgini, R. Kerner, Cosmology in ten dimensions with the generalised gravitational Lagrangian, Classical Quantum Gravity 5 (1988), no. 2, 339–351.

[16] H. Goldschmidt, S. Sternberg, The Hamilton-Cartan formalism in the Calculus of Variations, Ann. Inst. Fourier, Grenoble 23 (1973), 203–267.

[17] M.J. Gotay, An Exterior Differential Systems Approach to the Cartan form, Symplectic Geometry and Mathematical Physics (Eds.: P. Donato, C. Duval, J. Elhadad, G.M. Tuynman) Boston: Birkhäuser 1991, pp. 160–188.

[18] P. Von der Heyde, A generalized Lovelock theorem for the gravitational field with torsion, Phys. Lett. A (3) 51 (1975), 381–382.

[19] R. Kerner, Cosmology without singularity and nonlinear gravitational Lagrangians, Gen. Relativity Gravitation 14 (1982), no. 5, 453–469.

[20] S. Kobayashi, K. Nomizu, Foundations of differential Geometry, Volume I, John Wiley & Sons, Inc. N.Y., 1963.

[21] T. Koivisto, H. Kurki-Suonio, Cosmological perturbations in the Palatini formulation of modified gravity, Classical Quantum Gravity 23 (2006), no. 7, 2355–2369.

[22] D. Lovelock, The Einstein Tensor and Its Generalizations, J. Mathematical Phys. 12 (1971), 498–501.

[23] L. Mangiarotti, G. Sardanashvily, Connections in Classical and Quantum Field Theory, World Scientific Publishing Co. Inc. River Edge, NJ, 2000.

[24] J. Marsden, S. Shkoller, Multisymplectic geometry, covariant Hamiltonians, and water waves, Math. Proc. Cambridge Phil. Soc. 125 (1999), 553–575.

[25] E. Massa, E. Pagani, Jet bundle geometry, dynamical connections, and the inverse problem of Lagrangian mechanics, Ann. Inst. H. Poincaré Phys. Théor. 61 (1994), no. 1, 17–62.
[26] J. Muñoz Masqué, An axiomatic characterization of the Poincaré-Cartan form for second-order variational problems, Lecture Notes in Math. 1139, Springer-Verlag 1985, pp. 74–84.

[27] J. Muñoz Masqué, L. M. Pozo Coronado, Parameter Invariance in Field Theory and the Hamiltonian Formalism, Fortschr. Phys. 48 (2000), no. 4, 361–405.

[28] J. Muñoz Masqué, M. Eugenia Rosado, Invariant variational problems on linear frame bundles, J. Phys. A: Math. Gen. 35 (2002), 2013–2036.

[29] —, The Problem of Invariance for Covariant Hamiltonians, Rend. Sem. Mat. Univ. Padova 120 (2008), 1–28.

[30] J. Muñoz Masqué, A. Valdés Morales, The number of functionally independent invariants of a pseudo-Riemannian metric, J. Phys. A: Math. Gen. 27 (1994) 7843–7855.

[31] N. Poplawski, The cosmic snap parameter in $f(R)$ gravity, Classical Quantum Gravity 24 (2007), no. 11, 3013–3020.

[32] G. A. Sardanashvily, Gauge Theory in Jet Manifolds, Hadronic Press Monographs in Applied Mathematics, Hadronic Press, Inc., Palm Harbor, FL, U.S.A., 1993.

[33] G. Sardanashvily, O. Zakharov, Gauge Gravitation Theory, World Scientific Publishing Co. Inc. River Edge, NJ, 1992.

[34] D. J. Saunders, The Geometry of Jet Bundles, Cambridge University Press, Cambridge, UK, 1989.

[35] D. J. Saunders, M. Crampin, On the Legendre map in higher-order field theories, J. Phys. A: Math. Gen. 23 (1990), 3169–3182.

[36] B. Shahid-Saless, Palatini variation of curvature-squared action and gravitational collapse, J. Math. Phys. 32 (1991), no. 3, 694–697.

[37] W. Szczyrba, The canonical variables, the symplectic structure and the initial value formulation of the generalized Einstein-Cartan theory of gravity, Comm. Math. Phys. 60 (1978), no. 3, 215–232.

[38] —, Field equations and contracted Bianchi identities in the generalized Einstein-Cartan theory, Lett. Math. Phys. 2 (1977/78), no. 4, 265–274.

[39] H. Weyl, Space-Time-Matter, translated by H. L. Brose, Dover Publications, Inc., 1952.