Petrov-Galerkin Method on Modified Symm’s Integral Equation from Interior and Exterior Dirichlet Problem

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Abstract. In two dimension, for bounded simply connected domain $\Omega$ of infinitely smooth class, the boundary value problem of Laplace equation can be transformed into Symm’s integral equation, the convergence and divergence analysis are done respectively. In this paper, we follow the procedure to handle the modified Symm’s integral equation which solves the interior and exterior Dirichlet problem of Laplace equation, give the convergence and divergence result, including optimal convergence rate and divergence rate.

1. Introduction

Integral equation method plays an important role in solving the (BVP) of Laplace equations. Let $\Omega \subseteq \mathbb{R}^2$ be bounded and simply connected with boundary $\partial \Omega$ of class $C^2$ and $f \in C(\partial \Omega)$. To solve Dirichlet problem of Laplace equation

$$\Delta u = 0 \text{ in } \Omega, \quad u = f \text{ on } \partial \Omega,$$

When $f \in C^{1,\alpha}(\partial \Omega)$, the solution $u$ can be represented as single-layer potential

$$u(x) := -\frac{1}{2\pi} \int_{\partial \Omega} \psi(y) \ln |x - y| ds(y), \quad x \in \mathbb{R}^2,$$

provided that the density $\psi \in C^{0,\alpha}(\partial \Omega)$ solves

$$S\psi := -\frac{1}{2\pi} \int_{\partial \Omega} \psi(y) \ln |x - y| ds(y) = f(x), \quad x \in \partial \Omega,$$  \hspace{1cm} (1.1)

(1.1) is known as Symm’s integral equation of the first kind. There exists numerous work on numerical solution of (SIE). Frequently used method is Galerkin method (mainly on finite element basis, also on Fourier or wavelet basis), see [2-5,11,12]. Notice that, when numerically handling (SIE), it will generally require some geometrical assumption on the boundary for the uniqueness of (SIE), for example,

- There exists an interior point $z_0$ of $\Omega$ such that $|x - z_0| \neq 1$ for all $x \in \partial \Omega$
We mainly focus on the Petrov-Galerkin method with Fourier basis to handle (SIE). In past, assuming \( \partial \Omega \) to be analytic with nonzero pointwise tangent, that is, \( \partial \Omega \) possesses the regular parameterizations
\[
\partial \Omega := \{ \gamma(t) : t \in [0, 2\pi] \}
\]
and \( |\gamma'(t)| > 0, \ t \in [0, 2\pi) \). Inserting (1.2) into (1.1), (SIE) is transformed into integral equation of 1 D:
\[
-\frac{1}{2\pi} \int_0^{2\pi} \Psi(s) \ln |\gamma(t) - \gamma(s)| ds = g(t), \ \ x \in [0, 2\pi],
\]
for the transformed density \( \Psi(s) := \psi(\gamma(s)) |\dot{\gamma}(s)| \) and \( g(t) := f(\gamma(t)), \ s \in [0, 2\pi] \). As a classical result in this topic, the convergence and error are analyzed in [1, Chapter 3] for different Petro-Galerkin methods to \( f \in H^r(\partial \Omega) \), \( r \geq 1 \) under \( L^2 \) setting where optimal convergence rate are obtained for Least squares and Bubnov-Galerkin methods. Further, weakening \( \partial \Omega \) to be \( C^3 \) with nonzero pointwise tangent, divergence and rates are analyzed for the same Petro-Galerkin methods to \( f \in H^r(\partial \Omega), 0 \leq r < 1 \) under the same setting.

Similar to interior Dirichlet problem, to solve combined interior and exterior problem
\[
\Delta u = 0 \text{ in } \Omega, \ u = f \text{ on } \partial \Omega,
\]
\[
\Delta u = 0 \text{ in } \mathbb{R}^2 \setminus \Omega, \ u = f \text{ on } \partial \Omega, \ \ u(x) = O(1), \text{ for } |x| \to \infty.
\]
When \( f \in C^{1,\alpha}(\partial \Omega) \), with introduction of mean value operator \( M \) defined by
\[
M : \varphi \mapsto \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi ds,
\]
the solution \( u \) can be represented as the modified single-layer potential
\[
u(x) := -\frac{1}{2\pi} \int_{\partial \Omega} (\varphi(y) - M \varphi) \ln |x - y| ds(y) + M \varphi, \ \ x \in \mathbb{R}^2,
\]
provided that the density \( \varphi \in C^{\alpha,\alpha}(\partial \Omega) \) solves the integral equation
\[
S_0 \varphi := -\frac{1}{2\pi} \int_{\partial \Omega} (\varphi(y) - M \varphi) \ln |x - y| ds(y) + M \varphi = f(x), \ \ x \in \partial \Omega. \ (1.4)
\]
Notice that the modified single-layer approach solves the Dirichlet problem in \( \mathbb{R}^2 \) with no necessary geometric condition. Rewrite (1.4) as
\[
S_0 \varphi = \int_{\partial \Omega} G(x, y) \varphi(y) ds(y)
\]
where
\[
G(x, y) := -\frac{1}{2\pi} \ln |x - y| + \frac{1}{2\pi |\partial \Omega|} \int_{\partial \Omega} \ln |x - z| ds(z) + \frac{1}{|\partial \Omega|}. \ (1.6)
\]
In this paper, we apply Petrov-Galerkin method with trigonometric basis to the modified Symm’s integral equation with \( \Omega \subseteq \mathbb{R}^2 \) being bounded and simply connected, and boundary \( \partial \Omega \) of class \( C^3 \) with nonzero pointwise tangent. Following the method in [], we give the convergence and error analysis under \( L^2 \) setting to \( f \in H^r(\partial \Omega), r \geq 1 \), and divergence analysis to \( f \in H^r(\partial \Omega), 0 \leq r < 1 \).
Now the regular parameterization
\[ \partial \Omega := \{ \gamma(t) : t \in [0, 2\pi) \} \]
is three times continuously differentiable. Inserting it into (1.3), the modified (SIE) takes the form
\[ S_0 \varphi := \int_0^{2\pi} G(t, s) \Psi(s) ds = g(t), \quad t \in [0, 2\pi) \]  \hspace{1cm} (1.7)
with the transformed kernel
\[ G(t, s) := -\frac{1}{2\pi} \ln |\gamma(t) - \gamma(s)| + \frac{1}{2\pi} \frac{1}{|\partial \Omega|} \int_0^{2\pi} \ln |\gamma(t) - \gamma(\sigma)| |\gamma'\sigma| d\sigma + \frac{1}{|\partial \Omega|}. \]
the transformed density \( \Psi(s) := \varphi(\gamma(s)) |\gamma'(s)| \) and \( g(t) := f(\gamma(t)), \quad s \in [0, 2\pi] \).

As to the arrangement of the rest contents. In section 2, we introduce necessary preliminaries, such as periodic Sobolev space, basic properties of modified Symm’s integral operator. In section 3, we introduce unified Petrov-Galerkin setting and three special cases: least squares, dual least squares, Bubnov-Galerkin methods. In section 4,5,6, we analyze the divergence and prove the first order rate for three specific projectional settings respectively. In section 7, we give an example to confirm the first order to be uniformly optimal.

2. Preliminaries

2.1. Periodic Sobolev space \( H^r(0, 2\pi) \), trace space \( H^k(\Gamma) \) and estimates

Throughout this paper, we denote the \( 2\pi \) periodic Sobolev space of order \( r \in \mathbb{R} \) by \( H^r(0, 2\pi) \) (refer to [1,4]). Notice that, for \( r > s \), the Sobolev space \( H^r(0, 2\pi) \) is a dense subspace of \( H^s(0, 2\pi) \). The inclusion operator from \( H^r(0, 2\pi) \) into \( H^s(0, 2\pi) \) is compact.

Let \( \Gamma \) be the boundary of a simply connected bounded domain \( D \subset \mathbb{R}^2 \) of class \( C^k, k \in \mathbb{N} \). With the aid of a regular and \( k \) times continuously differentiable \( 2\pi \) periodic parameter representation
\[ \Gamma = \{ z(t) : t \in [0, 2\pi) \} \]
for \( 0 \leq p \leq k \) we can define the trace space \( H^p(\Gamma) \) as the space of all functions \( \varphi \in L^2(\Gamma) \) with the property that \( \varphi \circ z \in H^p(0, 2\pi) \). By \( \varphi \circ z \), we denote the \( 2\pi \) periodic function given by \( (\varphi \circ z)(t) := \varphi(z(t)), t \in \mathbb{R} \). The scalar product and norm on \( H^p(\Gamma) \) are defined through the scalar product on \( H^p(0, 2\pi) \) by
\[ (\varphi, \psi)_{H^p(\Gamma)} := (\varphi \circ z, \psi \circ z)_{H^p(0, 2\pi)}. \]

**Lemma 2.1** Let \( P_n : L^2(0, 2\pi) \longrightarrow X_n \subset L^2(0, 2\pi) \) be an orthogonal projection operator, where \( X_n = \text{span}\{e^{ikt}\}_{k=-n}^n \). Then \( P_n \) is given as follows
\[ (P_n x)(t) = \sum_{k=-n}^n a_k e^{ikt}, \quad x \in L^2(0, 2\pi), \]
where

\[ a_k = \frac{1}{2\pi} \int_0^{2\pi} x(s) \exp(-iks) ds, \quad k \in \mathbb{N}, \]

are the Fourier coefficients of \( x \). Furthermore, the following estimate holds:

\[ \|x - P_n x\|_{H^r} \leq \frac{1}{n^{r-s}} \|x\|_{H^r}, \quad x \in H^r(0, 2\pi), \]

where \( r \geq s \).

**Proof 1** See [1, Theorem A.43].

**Lemma 2.2** (Inverse inequality): Let \( r \geq s \). Then there exists a \( c > 0 \) such that

\[ \|\psi_n\|_{H^r} \leq cn^{r-s}\|\psi_n\|_{H^s}, \quad \forall \psi_n \in X_n \]

for all \( n \in \mathbb{N} \).

**Proof 2** See [1, Theorem 3.19].

2.2. Integral operator and regularity

**Lemma 2.3** Let \( r \in \mathbb{N} \) and \( k \in C^r([0, 2\pi] \times [0, 2\pi]) \) be \( 2\pi \)-periodic with respect to both variables. Then the integral operator \( K \), defined by

\[ (Kx)(t) := \int_0^{2\pi} k(t,s)x(s)ds, \quad t \in (0, 2\pi), \]

can be extended to a bounded operator from \( H^p(0, 2\pi) \) into \( H^r(0, 2\pi) \) for every \( -r \leq p \leq r \).

**Proof 3** See [1, Theorem A.45].

2.3. Modified Symm’s integral equation of the first kind

Throughout this paper, we denote the modified Symm’s integral operator in (1.5) by \( S_0 \).

\[ (S_0\Psi)(t) := \int_0^{2\pi} G(t,s)\Psi(s)ds = g(t), t \in [0, 2\pi) \quad (2.1) \]

with the transformed kernel

\[ G(t,s) := -\frac{1}{2\pi} \ln |\gamma(t) - \gamma(s)| + \frac{1}{2\pi} \frac{1}{|\partial\Omega|} \int_0^{2\pi} \ln |\gamma(t) - \gamma(\sigma)||\gamma'(\sigma)|d\sigma + \frac{1}{|\partial\Omega|}. \]

Utilizing the common decomposition technique on kernel (see [1, Chapter 3.3]) in Symm’s integral equation of the first kind, we split kernel \( G(t,s) \) into three parts:

\[ G(t,s) = G_1(t,s) + G_2(t,s) + G_3(t), \quad (2.2) \]

where

\[ G_1(t,s) := -\frac{1}{4\pi}(\ln(4\sin^2\frac{t-s}{2}) - 1) \quad (t \neq s) \quad (2.3) \]
\[ G_2(t, s) := -\frac{1}{2\pi} \ln |\gamma(t) - \gamma(s)| + \frac{1}{4\pi} (\ln(4 \sin^2 \frac{t-s}{2}) - 1) \quad (t \neq s) \quad (2.4) \]

\[ G_3(t) := \frac{1}{2\pi} \frac{1}{|\partial \Omega|} \int_0^{2\pi} \ln |\gamma(t) - \gamma(\sigma)||\gamma'(\sigma)||d\sigma + \frac{1}{|\partial \Omega|}. \quad (2.5) \]

We note that the logarithmic singularities at \( t = s \) in \( G(t, s) \) is separated to \( G_1 \), and \( G_1 \) corresponds to the regular representation of disc with center 0 and radius \( a = e^{-\frac{1}{2}} \), that is,

\[ \gamma_a(s) = a(\cos s, \sin s), \quad s \in [0, 2\pi). \]

The second part \( G_2 \) has a \( C^2 \) continuation onto \([0, 2\pi] \times [0, 2\pi]\) (See Lemma 8.2) since \( \gamma \) is three times continuously differentiable. The third part

\[ G_3(t) = -\frac{1}{|\partial \Omega|} h(t) + \frac{1}{|\partial \Omega|}, \quad t \in [0, 2\pi], \]

where

\[ h(t) = -\frac{1}{2\pi} \int_0^{2\pi} \ln |\gamma(t) - \gamma(\sigma)||\gamma'(\sigma)||d\sigma \]

is the single layer potential of constant function 1 on \( \partial \Omega \) which is three times continuously differentiable. By Lemma 2.4, \( G_3(t) \in C^2[0, 2\pi] \)

Now we define integral operators respectively as

\[ (S_1 \Psi)(t) := \int_0^{2\pi} G_1(t, s) \Psi(s) ds \quad (2.6) \]

\[ (S_2 \Psi)(t) := \int_0^{2\pi} (G_2(t, s) + G_3(t)) \Psi(s) ds. \quad (2.7) \]

\[ (K_2 \Psi)(t) := \int_0^{2\pi} (G_2(t, s) + G_3(s)) \Psi(s) ds. \quad (2.8) \]

\[ (K \Psi)(t) := \int_0^{2\pi} (G_1(t, s) + G_2(t, s) + G_3(s)) \Psi(s) ds. \quad (2.9) \]

\[ S_0 = S_1 + S_2, \quad K = S_1 + K_2. \quad (2.10) \]

**Lemma 2.4** It holds that

\[ \frac{1}{2\pi} \int_0^{2\pi} e^{int} \ln(4 \sin^2 \frac{s}{2}) ds = \begin{cases} 
\frac{1}{|n|}, & n \in \mathbb{Z}, n \neq 0, \\
0, & n = 0.
\end{cases} \]

This gives that the functions

\[ \hat{\psi}_n(t) := e^{int}, \quad t \in [0, 2\pi], \quad n \in \mathbb{Z}, \]

are eigenfunctions of \( S_1 \):

\[ S_1 \hat{\psi}_n = \frac{1}{2|n|} \hat{\psi}_n \quad \text{for } n \neq 0 \text{ and} \]

\[ S_1 \hat{\psi}_0 = \frac{1}{2} \hat{\psi}_0. \]

**Proof** See [1, Theorem 3.17]
Lemma 2.5 Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected bounded domain with $\partial \Omega$ be its boundary belongs to class of $C^3$. Then

(a) $S_0$ is compact in $L^2(0,2\pi)$ and $K = S_0^*$ when we see $K, S_0$ both as operator on $L^2(0,2\pi)$.

(b) The operator $S_1$ is bounded injective from $H^{s-1}(0,2\pi)$ onto $H^s(0,2\pi)$ with bounded inverses for every $s \in \mathbb{R}$, the same assertion also holds for $S_0, K$ when $-1 \leq s < 2$.

(c) The operator $S_1$ is coercive from $H^{-\frac{1}{2}}(0,2\pi)$ into $H^{\frac{1}{2}}(0,2\pi)$.

(d) The operator $S_2, K_2$ is compact from $H^{s-1}(0,2\pi)$ into $H^s(0,2\pi)$ for every $-1 \leq s < 2$.

Proof 5 See [1, Theorem A.33 and Theorem 3.18] for (a), the former part of (b), (c).

Following the main idea in [1, theorem 3.18], we prove the latter part of (b) and (d).

Since the $G_2(t,s) + G_3(t,s)$ has a $C^2$ continuation on $[0,2\pi] \times [0,2\pi]$, by Lemma 2.3, $S_2$ defines a bounded operator from $H^p(0,2\pi)$ to $H^2(0,2\pi)$ with $-2 \leq p \leq 2$. Composing with compact embedding $H^2(0,2\pi) \subseteq H^s(0,2\pi), (s < 2),$ (d) follows.

For the latter part of (b) it is sufficient to prove the injectivity of $S_0,K$ from $H^{s-1}(0,2\pi)$ to $H^s(0,2\pi)$ with $-1 \leq s < 2$. Let $\Psi \in H^{s-1}(0,2\pi)$ with $S_0\Psi = 0$. From $S_1\Psi = -S_2\Psi$ and the mapping properties (Lemma 2.3) of $S_2$, we know $S_1\Psi \in H^2(0,2\pi)$ and thus, $\Psi \in H^1(0,2\pi)$. This implies that $\Psi$ is continuous and the transformed function $\varphi(\gamma(t)) = \frac{\psi(t)}{|\gamma(t)|}$ satisfies (1.2) for $g = 0$. Lemma 2.4 gives $\varphi = 0$.

Notice that when $K, S_0$ is defined on $L^2(0,2\pi)$, $\mathcal{N}(K) = \mathcal{N}(S_0) = \mathcal{R}(S_0)^\perp = 0$. Let $\Psi \in H^{s-1}(0,2\pi)$ with $K\Psi = 0$. From $S_1\Psi = -K_2\Psi$ and the mapping properties (Lemma 2.3) of $K_2$, we know $S_1\Psi \in H^2(0,2\pi)$ and thus, $\Psi \in H^1(0,2\pi) \subseteq L^2(0,2\pi)$. Thus, $\Psi = 0$.

2.4. Gelfand triple, coercivity and G"arding’s inequality

Let $V$ be reflexive Banach space with dual space $V^*$. We denote the norms in $V$ and $V^*$ by $\| \cdot \|_V$ and $\| \cdot \|_{V^*}$, respectively. A linear bounded operator $A : V^* \to V$ is called coercive if there exists a $\gamma > 0$ such that

$$\Re \langle x, Ax \rangle \geq \gamma \| x \|_{V^*}^2$$

for all $x \in V^*$, with dual pairing $\langle \cdot, \cdot \rangle$ in $(V^*,V)$. The operator $A$ satisfies G"arding’s inequality if there exists a linear compact operator $C : V^* \to V$ such that $K + C$ is coercive, that is,

$$\Re \langle x, Ax \rangle \geq \gamma \| x \|_{V^*}^2 - \Re \langle x, Cx \rangle$$

for all $x \in V^*$.

A Gelfand triple $(V, X, V^*)$ consists of a reflexive Banach space $V$, a Hilbert space $X$, and the dual space $V^*$ of $V$ such that

(a) $V$ is dense subspace of $X$, and

(b) the embedding $J : V \to X$ is bounded.

We write $V \subseteq X \subseteq V^*$ because we can identify $X$ with a dense subspace of $V^*$. This identification is given by the dual operator $J^* : X \to V^*$ of $J$, where we identify the dual of the Hilbert space $X$ by itself and $(x, y) = \langle J^* x, y \rangle$ for all $x \in X$ and $y \in V$. 
3. Unified projection setting and its divergence result

Let \( X, Y \) be Hilbert spaces over the complex scalar field, \( \{X_n\} \) and \( \{Y_n\} \) be sequences of closed subspaces of \( X \) and \( Y \) respectively, \( P_n := P_{X_n} \) and \( Q_n := Q_{Y_n} \) be orthogonal projection operators which project \( X \) and \( Y \) onto \( X_n \) and \( Y_n \) respectively. Let the original operator equation of the first kind be
\[
Ax = b, \quad A \in B(X, Y), \quad x \in X, \quad b \in Y \tag{3.1}
\]
Its unified projection approximation setting is
\[
A_n x_n = b_n, \quad A_n \in B(X_n, Y_n), \quad x_n \in X_n, \quad b_n \in Y_n, \tag{3.2}
\]
where
\[
A_n := Q_n A P_n : X_n \to Y_n, \quad \mathcal{R}(A_n) \text{ closed}.
\]
Specifically, three different projectional setting is arranged as
1. Least squares method: Finite-dimensional \( X_{LS}^n \subseteq X \) such that \( \bigcup_{n \in \mathbb{N}} X_{LS}^n \) is dense in \( X \) with \( Y_{LS}^n = A(X_{LS}^n) \) and \( b_{LS}^n := Q_{LS}^n b \), where \( Q_{LS}^n := Q_{Y_{LS}^n} \);
2. Dual least squares method: Finite-dimensional \( Y_{DLS}^n \subseteq Y \) such that \( \bigcup_{n \in \mathbb{N}} Y_{DLS}^n \) is dense in \( Y \) with \( X_{DLS}^n = A^*(Y_{DLS}^n) \) and \( b_{DLS}^n := Q_{DLS}^n b \), where \( Q_{DLS}^n := Q_{Y_{DLS}^n} \);
3. Bubnov-Galerkin method: Background Hilbert spaces \( X = Y \) with finite-dimensional \( Y_{BG}^n \subseteq X \) such that \( \bigcup_{n \in \mathbb{N}} Y_{BG}^n \) is dense in \( X \) and \( b_{BG}^n := Q_{BG}^n b \), where \( Q_{BG}^n := Q_{Y_{BG}^n} \).

The Unified divergence result for general projection setting is illustrated as follows.

**Lemma 3.1** For projection setting (3.1), (3.2), if \( \{\{X_n\}_{n \in \mathbb{N}}, \{Y_n\}_{n \in \mathbb{N}}\} \) satisfies the completeness condition, that is,
\[
P_n \overset{s}{\to} I_X, \quad Q_n \overset{s}{\to} I_Y,
\]
and
\[
\sup_n \|A_n^+ Q_n A\| < \infty \tag{3.3}
\]
where \( A^+ \) denotes the Moore-Penrose inverse of linear operator (See [2, Definition 2.2]), then, for \( b \notin \mathcal{D}(A^+) = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \),
\[
\lim_{n \to \infty} \|A_n^+ Q_n Q_{\mathcal{R}(A)^\perp} b\| = \infty
\]

**Proof 6** See [3, Theorem 2.2 (c)]

4. Analysis for Least square method

4.1. Convergence and error analysis

Set \( X = Y = L^2(0, 2\pi) \) and
\[
X_{LS}^n = \text{span}\{e^{ikt}\}_{k = -n}^n, \quad Y_{LS}^n = S_0(X_{LS}^n) \tag{4.1}
\]
To prepare the convergence and error analysis of least squares method for (1.5), we list some basic results as follows:
Lemma 4.1 Let $A : X \to Y$ be a linear, bounded, and injective operator between Hilbert spaces and $X_n^{LS} \subseteq X$ be finite-dimensional subspaces such that $\bigcup_{n \in \mathbb{N}} X_n^{LS}$ is dense in $X$. Let $x \in X$ be the solution of $Ax = y$ and $x_n^\delta$ be the least square solution from (3.2) with $b$ being replaced by $b^\delta$ and $\|b^\delta - b\| \leq \delta$. Define
\[
\sigma_n^{LS} = \sigma_n^{LS}(A) := \max \{ \| z_n \| : z_n \in X_n^{LS}, \| Az_n \| = 1 \},
\]
let there exists a constant $\tau_n^{LS} > 0$, independent of $n$, such that
\[
\min_{z_n \in X_n^{LS}} \{ \| x - z_n \| + \sigma_n \| A(x - z_n) \| \} \leq \tau_n^{LS} \| x \| \text{ for all } x \in X. \tag{4.2}
\]
Then the least square method is uniquely solvable, that is, $A_n^{LS} := Q_{Y_n^{LS}} A P_{X_n^{LS}} : X_n^{LS} \to Y_n^{LS}$ is invertible, where $Y_n^{LS} = A(X_n^{LS})$, and convergent, that is,
\[
A_n^{LS}^{-1} Q_n^{LS} b \to A^{-1} b, \quad (b \in \mathcal{R}(A)) \tag{4.3}
\]
with $\| R_n^{LS} \| \leq \sigma_n^{LS}$, where $R_n^{LS} := A_n^{LS}^{-1} Q_n^{LS} : Y \to X_n^{LS} \subseteq X$. In this case, we have the error estimate
\[
\| A^{-1} b - A_n^{LS}^{-1} Q_n^{LS} b^\delta \| \leq \sigma_n^{LS} \delta + c^{LS} \min \{ \| x - z_n \| : z_n \in X_n^{LS} \}
\]
where $c^{LS} := \tau_n^{LS} + 1$, Notice that $(\{X_n^{LS}\}, \{Y_n^{LS}\})_{n \in \mathbb{N}}$ are all not specifically chosen.

Proof 7 This is an operator equation version of [1, Theorem 3.10].

Lemma 4.2 (Stability estimate): There exists a $c > 0$, independent of $n$, such that
\[
\| S_n \|_{L^2} \leq C^{LS} n \| S_0 \Psi_n \|_{L^2} \text{ for all } \Psi_n \in X_n^{LS}. \tag{4.4}
\]
This yields that $\sigma_n^{LS}(S_0) \leq C^{LS} n$. The assertion also holds for the adjoint operator $S_0^*$, that is,
\[
\| S_n \|_{L^2} \leq C^{QLS} n \| S_0^* \Psi_n \|_{L^2} \text{ for all } \Psi_n \in X_n^{LS}. \tag{4.5}
\]
Notice that the constants $C^{LS}, C^{QLS}$ depend on $\| S_0^{-1} \|_{L^2 \to H^{-1}}$ and $\| K^{-1} \|_{L^2 \to H^{-1}}$ respectively which are unknown in whole computation process.

Proof 8 Similar to [1, Lemma 3.19], for $\Psi_n = \sum_{k=-n}^n a_k e^{ikt} \in X_n^{LS},$
\[
\| S_1 \Psi_n \|_{L^2}^2 = \frac{\pi}{2} \| a_0 \|^2 + \sum_{|j| \leq n, j \neq 0} \frac{1}{j^2} |a_j|^2 \geq \frac{1}{n^2} \| \Psi_n \|^2.
\]
which proves estimate (4.4) for $S_1$. The estimate for $S_0$ follows from the observation that $S_0 = (S_0 S_1^{-1}) S_1$ and that $(S_0 S_1^{-1})$ is bounded with bounded inverse in $L^2(0, 2\pi)$ by Lemma 2.6 (b). As to the adjoint case, $S_0^* \Psi_n = K \Psi_n, \forall \Psi_n \in X_n$, again using above observation, (4.5) follows.

Proof 9 Choosing $z_n = P_n^{LS} x$, we have
\[
\min_{z_n \in X_n^{LS}} \{ \| x - z_n \| + \sigma_n \| S_0(x - z_n) \| \}
\leq \| x - P_n^{LS} x \| + \sigma_n(S_0) \| S_0(x - P_n^{LS} x) \|
\leq 2 \| x \| + C^{LS} n \| S_0(x - P_n^{LS} x) \| \quad \text{by Lemma 4.2,} \tag{4.6}
\]
where \( c > 0 \) is a constant independent of \( n \). Applying Lemma 2.6 (b) with \( s = 0 \), we know that \( S_0 \) is bounded from \( H^{-1}(0, 2\pi) \) onto \( L^2(0, 2\pi) \), thus,
\[
\|S_0(x - P_{n,0}^{LS}x)\|_{L^2} \leq \|S_0\|_{H^{-1} \to L^2}\|x - P_{n,0}^{LS}x\|_{H^{-1}} (L^2(0, 2\pi) \subseteq H^{-1}(0, 2\pi))
\]
\[
\leq \|S_0\|_{H^{-1} \to L^2} \frac{1}{n} \|x\|_{L^2} \quad \text{for all } x \in L^2(0, 2\pi).
\]

with Lemma 2.1 of \( r = 0 \) and \( s = -1 \). Together with (4.6), it yields that
\[
\min_{z_n \in X_{n,0}^{LS}} \{ \|x - z_n\| + \sigma_n \|S_0(x - z_n)\| \} \leq (2 + C^{LS} \|S_0\|_{H^{-1} \to L^2}) \|x\|_{L^2}.
\]
This complete the proof.

Thus we have error estimate for least squares method
\[
\|S_0^{-1}b - S_{n,0}^{LS-1}Q_n^{LS}b\|_{L^2} \leq C^{LS} n\delta + C_1^{LS} \min\{\|S_0^{-1}b - z_n\|_{L^2} : z_n \in X_{n,0}^{LS}\}
\]
where \( C_1^{LS} := 3 + C^{LS} \|S_0\|_{H^{-1} \to L^2} \). With further regularity assumption on exact solution \( S_0^{-1}b \in H^r(0, 2\pi) \), \( (r \leq 2) \), that is, \( b \in H^{r+1}(0, 2\pi) \), by Lemma 2.1,
\[
\|S_0^{-1}b - S_{n,0}^{LS-1}Q_n^{LS}b\|_{L^2} \leq C^{LS} n\delta + \frac{C_1^{LS}}{n^{r}} \|x\|_{H^r}
\]
Choosing \( n = \delta^{-\frac{2}{r+1}} \), we have
\[
\|S_0^{-1}b - S_{n,0}^{LS-1}Q_n^{LS}b\|_{L^2} = O(\delta^{\frac{2}{r+1}}).
\]
This is optimal since we can examine that the rate \( O(\delta^{\frac{2}{r+1}}) \) is obtained for \( S_0^{-1}b \in \mathcal{R}(S_0(\mu)) \subseteq H^{2\mu}(0, 2\pi) \), \( \mu = \frac{1}{2} \) or 1.

### 4.2 Divergence analysis

In the following, we utilize Lemma 3.1 to analyze the divergence of least squares method of equation (1.5). The completeness condition for \( \{X_{n,0}^{LS}\}, \{Y_{n}^{LS}\} \) is verified in Appendix A. The (3.3) are transformed into
\[
\sup_n \|S_{n,0}^{LS-1}Q_n^{LS}S_0\| < \infty \tag{4.7}
\]
where \( S_{0,n}^{LS} := Q_n^{LS}S_0P_{n,0}^{LS} : X_n^{LS} \to Y_n^{LS} \). Notice that (4.3) holds for \( S_0 \), inserting \( b = S_0x \), \( x \in X \) into (4.3), we have
\[
S_{0,n}^{LS-1}Q_n^{LS}S_0x \xrightarrow{s} x = S_0^{-1}S_0x, \quad x \in L^2(0, 2\pi)
\]
The Banach-Steinhaus theorem gives (4.5). Thus, by Lemma 3.1, we have

**Theorem 4.1** For \( b \in L^2(0, 2\pi) \setminus H^1(0, 2\pi) \), the least squares method with Fourier basis for (1.5) diverges.

**Proof** 10 By Lemma 3.1, we have, for every \( b \notin \mathcal{D}(S_0^1) = \mathcal{R}(S_0) \oplus \mathcal{R}(S_0)^\perp \),
\[
\lim_{n \to \infty} \|S_{0,n}^{LS}Q_n^{LS}Q_{\mathcal{R}(S_0)^\perp}b\|_{L^2} = \infty.
\]
Since application of Lemma 2.6 (b) with \( s = 1 \) gives \( \mathcal{R}(S_0) = H^1(0, 2\pi) \), with the fact that \( H^1(0, 2\pi) \) is dense in \( L^2(0, 2\pi) \), we have \( \mathcal{R}(S_0)^\perp = \mathcal{R}(S_0)^\perp = 0 \) and \( Q_{\mathcal{R}(S_0)^\perp} = I_{L^2} \). This yields that, for \( b \in L^2(0, 2\pi) \setminus H^1(0, 2\pi) \),
\[
\lim_{n \to \infty} \|S_{0,n}^{LS}Q_n^{LS}b\|_{L^2} = \infty.
\]
Using the third item in Lemma 4.1 with Lemma 4.2 gives that $\|S_{0,n}^{-1}Q_n^{LS}\|_{L^2 \rightarrow L^2} \leq cn$. Together with Theorem 4.1, it leads to the divergence rate result.

**Theorem 4.2** For $b \in L^2(0,2\pi) \backslash H^1(0,2\pi)$, the Least squares method for (1.5) diverges with $\|S_{0,n}^{-1}Q_n^{LS}b\|_{L^2} = O(n)$.

5. Divergence analysis for Dual least square method

5.1. Convergence and error analysis

For dual least square method with $X = Y = L^2(0,2\pi)$, set

$$Y_n^{DLS} = \text{span}\{e^{ikt}\}_{k=-n}^{n}, \quad X_n^{DLS} = S_0^*(Y_n^{DLS}),$$

(5.1)

To prepare convergence and error analysis, we introduce a basic result:

**Lemma 5.1** Let $X$ and $Y$ be Hilbert spaces and $A : X \rightarrow Y$ be a linear, bounded, and injective such that the range $\mathcal{R}(A)$ is dense in $Y$. Let $Y_n^{DLS} \subseteq Y$ be finite-dimensional subspaces such that $\bigcup_{n\in\mathbb{N}} Y_n^{DLS}$ is dense in $Y$. Then the dual least square method is uniquely solvable, that is, $A_n^{DLS} := Q_n^{DLS}AP_{X_n^{DLS}} : X_n^{QLS} \rightarrow Y_n^{QLS}$ is invertible, where $X_n^{DLS} = A^*(Y_n^{DLS})$, and convergent, that is,

$$A_n^{QLS}^{-1}Q_n^{QLS}b \rightarrow A^{-1}b, \quad (b \in \mathcal{R}(A))$$

(5.2)

with $\|P_n^{QLS}\| \leq \sigma_n^{QLS}$, where

$$\sigma_n^{QLS} := \max\{\|z_n\| : z_n \in Y_n^{DLS}, \|A^*(Y_n^{DLS})\| = 1\}$$

and $P_n^{QLS} := A_n^{QLS}^{-1}Q_n^{QLS} : Y \rightarrow X_n^{QLS} \subseteq X$. Furthermore, we have error estimate

$$\|A^{-1}b - A_n^{DLS}^{-1}Q_n^{DLS}b\| \leq \sigma_n^{DLS}\delta + c\min\{\|A^{-1}b - z_n\| : z_n \in A^*(Y_n)\}$$

where $c = 2$. Notice that $(\{X_n^{QLS}\}, \{Y_n^{QLS}\})_{n\in\mathbb{N}}$ are all not specifically chosen.

**Proof 11** This is an operator equation version of [1, Theorem 3.11] with [1, Theorem 3.7].

The (4.5) with $Y_n^{QLS} = X_n^{LS}$ yields that $\sigma_n^{QLS}(S_0) \leq c^{QLS}n$. Thus, we have error estimate for dual least squares method

$$\|S_0^{-1}b - S_{0,n}^{QLS}^{-1}b\|_{L^2} \leq c^{QLS}n\delta + 2\min\{\|S_0^{-1}b - z_n\|_{L^2} : z_n \in S_0^*(Y_n^{DLS})\}$$

5.2. Divergence analysis

Now, by the same sake in least squares method, there holds that

$$\sup_n \|S_{0,n}^{DLS}Q_n^{DLS}S_0\| < \infty, \quad S_n^{DLS} := Q_n^{DLS}S_0P_n^{DLS} : X_n^{DLS} \rightarrow Y_n^{DLS}$$

where $Q_n^{DLS} := Q_n^{Y,DLS}$ and $P_n^{DLS} := P_{X_n^{DLS}}$. By Lemma 8.2, one can verify that $(\{X_n^{DLS}\}, \{Y_n^{DLS}\})$ satisfies the completeness condition. Thus, similar to least squares method, we obtain divergence result for dual least squares method as
Theorem 5.1 For $b \in L^2(0, 2\pi) \setminus H^1(0, 2\pi)$, the dual least square method with Fourier basis diverges for (1.5), that is,
\[
\lim_{n \to \infty} \| S_n^{DLS} Q_n^{DLS} b \|_{L^2} = \infty.
\]

Remark 5.1 Furthermore, the assertion holds for arbitrary $L^2(0, 2\pi)$ basis $\{\xi_k\}_{k=1}^\infty$, for instance, wavelet, piecewise constant, Legendre polynomials and so on. For $b \in H^1(0, 2\pi)$, the dual least square method with arbitrary $L^2(0, 2\pi)$ basis converges with the same proof. Thus, we give complete division to all $b \in L^2(0, 2\pi)$ for convergence or divergence in dual least square method with arbitrary $L^2(0, 2\pi)$ basis.

Notice that $Y_n^{DLS} = X_n^{LS}$, by Lemma 4.2, we have $\sigma_n^{DLS}(S_0) \leq cn$. This yields that:

Theorem 5.2 For $b \in L^2(0, 2\pi) \setminus H^1(0, 2\pi)$, the dual least square method for (1.5) with Fourier basis diverges with rate $O(n)$, that is, $\| S_n^{DLS} Q_n^{DLS} b \|_{L^2} = O(n)$

Although the dual least squares method with Fourier basis diverges for $b \in H^r$, $r < 1$ in $L^2$ norm, we can further consider the convergence in norm of $H^{-1}$. Now we consider (1.4) in setting $S_0 : H^{-1}(0, 2\pi) \to L^2(0, 2\pi)$ (Lemma 2.6 (b) $s = 0$)

6. Analysis for Bubnov-Galerkin method

6.1. Convergence and error analysis

Set $X = Y = L^2(0, 2\pi)$ and $X_n^{BG} = Y_n^{BG} = \text{span}\{e^{ikt}\}_{k=-n}^n$. To prepare this, we first introduce some technical lemmas:

Lemma 6.1 Let $(V, X, V^*)$ be a Gelfand triple, and $X_n^{BG} \subseteq V$ be finite-dimensional subspaces such that $\bigcup_{n \in \mathbb{N}} X_n^{BG}$ is dense in $X$. Let $A : V^* \to V$ be one-to-one and satisfies Gårding’s inequality with some compact operator $C : V^* \to V$, that is, there exists $\gamma > 0$ such that

\[
\Re\langle x, Ax \rangle \geq \gamma \|x\|_{V^*}^2 - \Re\langle x, Cx \rangle, \quad \text{(for all } x \in V^*).\]

Then the Bubnov-Galerkin system is uniquely solvable, that is, $A_n^{BG} := P_n^{BG} A P_n^{BG} : X_n^{BG} \to X_n^{BG}$ is invertible, where $X = Y$ and $X_n^{BG} = Y_n^{BG}$, and convergent in $X$, that is,

\[
A_n^{BG} A_n^{-1} P_n^{BG} b \to A^{-1} b, \quad (b \in \mathcal{R}(A)) \tag{6.1}
\]

if there exists $c > 0$ with

\[
\|u - P_n^{BG} u\|_{V^*} \leq \frac{c}{\rho_n} \|u\| \quad \text{for all } u \in X \tag{6.2}
\]

Furthermore, in this case, we have $\|P_n^{BG}\| \leq \frac{1}{\gamma} \rho_n^2$, where

\[
\rho_n := \max\{\|z_n\| : z_n \in X_n^{BG}, \|z_n\|_{V^*} = 1\},
\]

$P_n^{BG} := A_n^{BG}^{-1} P_n^{BG} : X \to X_n^{BG} \subseteq X$ and error estimate

\[
\|A^{-1} b - A_n^{BG} A_n^{-1} P_n^{BG} b\| \leq \frac{1}{\gamma} \rho_n^2 \delta + c^{BG} \min\{\|x - z_n\| : z_n \in X_n^{BG}\},
\]
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where

\[ c := \tau + 1, \tau := \sup_n \| A_n^{BG}P_n^{BG}A \| \]

Notice that \( \rho_n \) can be seen as a local inverse embedding constant and \((X, \{X_n^{QLS}\}_{n \in \mathbb{N}})\) are all not specifically chosen.

**Proof 12** This is the operator equation version of [1, Theorem 3.15] of no noise case \( \delta = 0 \).

Following [1, Theorem 3.20], set \( V = H^{\frac{1}{2}}(0, 2\pi) \) and \( V^* = H^{-\frac{1}{2}}(0, 2\pi) \), with Lemma 2.6 (c) and (d) of \( s = \frac{1}{2} \), we know \( S_0 : H^{-\frac{1}{2}}(0, 2\pi) \rightarrow H^{\frac{1}{2}}(0, 2\pi) \) satisfies Gårding inequality with \(-S_2\) defined in (2.7). Again following [1, theorem 3.20], with application of Lemma 2.2 of \( r = 0, s = -\frac{1}{2} \), we have

\[ \rho_n = \max\{\|\psi_n\|_{L^2} : \psi_n \in X_n, \|\psi_n\|_{H^{-\frac{1}{2}}} = 1\} \leq c\sqrt{n}. \]

By Lemma 2.1, we have

\[ \|u - P_n^{BG}u\|_{H^{-\frac{1}{2}}} \leq c\sqrt{n}\|u\|_{L^2} \quad \text{for all } u \in L^2(0, 2\pi) \]

that is, (6.2) holds for Bubnov-Galerkin method. Now, by Lemma 6.1, we have

\[ \|S_0^{-1}b - S_0^{BG}P_n^{BG}b\|_{L^2} \leq cn\delta + c\|(I - P_n^{BG})S_0^{-1}b\|_{L^2}. \]  \hfill (6.3)

If we further assume \( S_0^{-1}b \in H^r(0, 2\pi), r \leq 2 \), then, by Lemma 2.1, we have

\[ \|S_0^{-1}b - S_0^{BG}P_n^{BG}b\|_{L^2} \leq cn\delta + c\frac{1}{n^r}\|S_0^{-1}b\|_{H^r}. \]

As pointed in Least squares case, \( n = \delta^{-\frac{1}{1+r}} \), we have optimal convergence rate for Bubnov-Galerkin method

\[ \|S_0^{-1}b - S_0^{BG}Q_n^{BG}b\|_{L^2} = O(\delta^{-\frac{1}{1+r}}). \]

**6.2. Divergence analysis**

\((\{X_n^{BG}\}, \{Y_n^{BG}\})\) satisfies the completeness condition, the uniform boundedness holds for Bubnov-Galerkin method, that is,

\[ \sup_n \|S_0^{BG}P_n^{BG}S_0\| < \infty \]

where \( S_0^{BG} := P_n^{BG}S_0P_n^{BG} : X_n^{BG} \rightarrow X_n^{BG} \) and \( P_n^{BG} := P_n^{BG} \). By Lemma 3.1, we have

**Theorem 6.1** For \( b \in L^2(0, 2\pi) \setminus H^1(0, 2\pi) \), the Bubnov-Galerkin method with Fourier basis diverges for (1.5), that is,

\[ \lim_{n \rightarrow \infty} \|S_0^{BG}P_n^{BG}b\|_{L^2} = \infty. \]

Since \( \|S_0^{BG}P_n^{BG}\|_{L^2 \rightarrow L^2} \leq c\rho_n^2 \leq cn \), we have

**Theorem 6.2** For \( b \in L^2(0, 2\pi) \setminus H^1(0, 2\pi) \), the Bubnov-Galerkin method with trigonometric basis diverges with rate \( O(n) \), that is, \( \|S_0^{BG}P_n^{BG}b\|_{L^2} = O(n) \).
7. An example

Here we give an example to verify the divergence result for the three projection methods and further confirm the first order rate to be optimal. Let us consider the modified Symm’s integral equation with $\Omega$ is the disc with center at origin and radius $r > 0$ such that

$$A(r) := -\frac{1}{2\pi} \ln r + \frac{1}{|\partial \Omega|} r \ln r + \frac{1}{|\partial \Omega|} \neq 0,$$

then

$$G_1(t, s) + G_2(t, s) = -\frac{1}{2\pi} \ln |\gamma(t) - \gamma(s)|$$

$$= -\frac{1}{2\pi} \left\{ \frac{1}{2} \ln(\frac{4 \sin^2 \frac{t-s}{2}}{2}) + \ln r \right\},$$

$$G_3(t) = \frac{1}{2\pi |\partial \Omega|} \int_0^{2\pi} \ln |\gamma(t) - \gamma(\sigma)||\gamma'(\sigma)| d\sigma + \frac{1}{|\partial \Omega|} = \frac{1}{|\partial \Omega|}(1 + r \ln r),$$

$$G(t, s) = -\frac{1}{4\pi} (\ln(\frac{4 \sin^2 \frac{t-s}{2}}{2}) - 4\pi A(r))$$

Now $S_0 = S_0^*$, using Lemma 2.5, we have

$$Y_{LS}^n = S_0(X_{LS}^n) = X_{LS}^n = Y_{DLS}^n$$

$$= S_0^n(Y_{DLS}^n) = S_0^n(Y_{DLS}^n) = X_{DLS}^n = X_{BG}^n = Y_{BG}^n$$

This implies that the three Petrov-Galerkin setting coincides. Thus, we only need to test Bubnov-Galerkin method.

Set

$$b(t) = 1 + \sum_{0 \neq k \in \mathbb{Z}} \frac{1}{|k|^\frac{1}{2} + \alpha} e^{ikt} \in L^2(0, 2\pi) \setminus H^1(0, 2\pi). \quad (\alpha \in (0, \frac{1}{2}))$$

we can deduce that

$$\Psi_n^\dagger = S_{BG}^{n-1} P_{BG} b = \frac{1}{A(r)} + \sum_{k=1}^{n} 2|k|^\frac{1}{2} - \alpha e^{ikt} + \sum_{k=-n}^{-1} 2|k|^\frac{1}{2} - \alpha e^{ikt}$$

and thus,

$$c_1 n^{2-2\alpha} \leq ||\Psi_n^\dagger||_L^2 \leq c_2 n^2 \quad (\alpha \in (0, \frac{1}{2})).$$

This result verifies the divergence result and further confirm the first order divergence rate to be optimal by letting $\alpha \to 0^+$. 
8. Conclusion

Appendix A

**Lemma 8.1** The \((\{X_n^{LS}\}_{n \in \mathbb{N}}, \{Y_n^{LS}\}_{n \in \mathbb{N}})\) defined in (4.1) satisfies the completeness condition, that is,
\[
P_n^{LS} \xrightarrow{s} I_{L^2}, \quad Q_n^{LS} \xrightarrow{s} I_{L^2},
\]

**Proof 13** See [6, Lemma 8.1].

**Lemma 8.2** The \((\{X_n^{QLS}\}_{n \in \mathbb{N}}, \{Y_n^{QLS}\}_{n \in \mathbb{N}})\) defined in (5.1) satisfies the completeness condition, that is,
\[
P_n^{QLS} \xrightarrow{s} I_{L^2}, \quad Q_n^{QLS} \xrightarrow{s} I_{L^2},
\]

**Proof 14** It is sufficient to prove that
\[
\bigcup_{n \in \mathbb{N}} X_n^{QLS} = \bigcup_{n \in \mathbb{N}} S_0^*(Y_n^{QLS}) = L^2(0,2\pi)
\]
With closed range theorem, \(\mathcal{R}(S_0^*) = \mathcal{N}(S_0) = L^2(0,2\pi)\) (Lemma 2.6 (b) case \(s = 1\)).

Since
\[
\mathcal{R}(S_0^*) = S_0^* \left( \bigcup_{n \in \mathbb{N}} Y_n^{QLS} \right) \subseteq \bigcup_{n \in \mathbb{N}} S_0^*(Y_n^{QLS}) \quad (S_0^* \in B(L^2(0,2\pi)))
\]
we have
\[
L^2(0,2\pi) = \mathcal{R}(S_0^*) \subseteq \bigcup_{n \in \mathbb{N}} S_0^*(Y_n^{QLS}) \subseteq L^2(0,2\pi)
\]

Appendix B

**Lemma 8.3** Let \(\gamma = \gamma(s) = (a(s), b(s))\) be three times continuously differentiable, then \(k = k(t,s)\) defined in (2.1) can be extended to \(C^2([0,2\pi] \times [0,2\pi])\), that is, \(2\pi\)-periodic, two times continuously differentiable with respect to both variables. In particular,
\[
\lim_{s \to t} k(t,s) = -\frac{1}{\pi} \left( \ln |\gamma'(t)| + \frac{1}{2} \right), \quad \lim_{s \to t} \frac{\partial}{\partial t} k(t,s) = -\frac{1}{2\pi} \frac{\gamma'(t) \cdot \gamma''(t)}{|\gamma'(t)|^2},
\]
\[
\lim_{s \to t} \frac{\partial^2}{\partial t^2} k(t,s) = -\frac{1}{\pi} \frac{1}{|\gamma'(t)|^4} \times \left[ \frac{1}{12} |\gamma'(t)|^4 + |\gamma'(t)|^2 \left( \frac{1}{3} \gamma'''(t) \cdot \gamma'(t) + \frac{1}{4} |\gamma''(t)|^2 \right) + \frac{1}{2} (\gamma'(t) \cdot \gamma''(t))^2 \right].
\]

**Proof 15** See [6, Lemma 8.2]

**Lemma 8.4** Let \(\partial \Omega\) be the boundary of bounded simply connected domain \(\Omega \subseteq \mathbb{R}^2\). If \(\partial \Omega\) is of class \(C^{m+1,\alpha}\) and \(\phi\) of \(C^{m,\alpha}\) with \(m \in \mathbb{N}\) and \(0 < \alpha < 1\), then the interior single layer potential defined by \(\phi\) is of class \(C^{m+1,\alpha}\) on \(\partial \Omega\).

**Proof 16** See [5, Page 303]
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