Dimension of Gibbs measures with infinite entropy

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Abstract
We study the Hausdorff dimension of Gibbs measures with infinite entropy with respect to maps of the interval with countably many branches. We show that under simple conditions, such measures are symbolic-exact dimensional, and provide an almost sure value for the symbolic dimension. We also show that the lower local dimension dimension is almost surely equal to zero, while the upper local dimension is almost surely equal to the symbolic dimension. In particular, we prove that a large class of Gibbs measures with infinite entropy for the Gauss map have Hausdorff dimension zero and packing dimension equal to 1/2, and so such measures are not exact dimensional.

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1. Introduction

In this paper we study the dimension of measures invariant under a certain class of maps of the unit interval $[0, 1]$: expanding Markov Renyi (EMR) maps. These maps $T : [0, 1] \to [0, 1]$ admit representations by means of symbolic dynamics, and satisfy smoothness properties that allow us to use ergodic theoretic methods to study their geometric properties. Given an ergodic $T$-invariant probability measure $\mu$, we are interested in the pointwise behaviour of the local dimension

$$d(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

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where $B(x, r)$ denotes the open ball of centre $x$ and radius $r$. This limit in general may not exist, in which case we study the corresponding limit superior and limit inferior. When the limit exists almost everywhere, we say that the measure is exact dimensional. If this is the case, by ergodicity of $\mu$ the value of the local dimension is constant almost everywhere. Knowledge of the almost sure behaviour of the local dimension yields information about the Hausdorff and the packing dimension of the measure.

There are two dynamical quantities which are particularly relevant when studying the local dimension of such measures: the metric entropy $h_\mu$ (or simply the entropy) and the Lyapunov exponent $\lambda_\mu$ of $(T, \mu)$ (see section 2 for the definitions of $h_\mu$, $\lambda_\mu$ and dimension). Formulae relating the dynamical invariants $h_\mu$, $\lambda_\mu$ and the local dimension have been extensively studied for the last few decades in the case $h_\mu < \infty$. For Bernoulli measures invariant under the Gauss map, Kinney and Pitcher proved in [KP66] that if the measure $\mu$ is defined by a probability vector $p = \{p_i\}$, the Hausdorff dimension of $\mu$ can be computed with the formula

$$\dim_H \mu = \frac{-\sum_{n=1}^{\infty} p_n \log p_n}{2\int_0^1 |\log x|d\mu(x)}$$

provided that $\sum_{n=1}^{\infty} p_n \log n < \infty$.

For more general maps of the interval, in [LM85] the authors proved that for a $C^1$ map $T : [0, 1] \to [0, 1]$ where $T$ and $T'$ are piecewise monotonic and the Lyapunov exponent $\lambda_\mu$ is positive, if $\mu$ is an invariant ergodic probability measure, then [LM85, corollary in the appendix] we have that the such measure is exact dimensional and

$$\lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \frac{h_\mu}{\lambda_\mu}$$

In particular, $\dim_H \mu = h_\mu/\lambda_\mu$. Other versions of the formula were proved by Young and Hofbauer, Raith in [You82] and [HR92], among others. In all of these examples, it is assumed $0 < \lambda_\mu < \infty$. In the context of countable Markov systems, Mauldin and Urbanski proved ([MU03, theorem 4.4.2]) the following theorem:

**Volume lemma.** Let $(X, T)$ be a countable Markov shift coded by the shift in countably many symbols $(\Sigma, \sigma)$. Suppose that $\mu$ is a Borel shift-invariant ergodic probability measure on $\Sigma$ such that at least one of the numbers $H(\mu, \alpha)$ or $\lambda_\mu$ is finite, where $H(\mu, \alpha)$ is the entropy of $\mu$ with respect to the natural partition $\alpha$ in cylinders of $\Sigma$. Then $\mu$ is exact dimensional and

$$\dim_H(\mu \circ \pi^{-1}) = \frac{h_\mu}{\lambda_\mu}$$

where $\pi : \Sigma \to X$ is the coding map.

The case when $\lambda_\mu = 0$ was studied by Ledrappier and Misiurewicz in [LM85], wherein they constructed a $C^r$ map of the interval and a non-atomic ergodic invariant measure which has zero Lyapunov exponent and is such that the local dimension does not exist almost everywhere. More precisely, they show that the lower local dimension and upper local dimension are not equal ([LM85, theorem 1]):

$$d_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} < \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \overline{d}_\mu(x)$$

almost everywhere. For this construction, the authors consider a class of unimodal maps (Feigenbaum’s maps).

The focus of the article is twofold: in the first place, we investigate the Hausdorff dimensions of invariant ergodic measures for piecewise expanding maps of the interval with countably
many branches. In particular, we focus on maps exhibiting similar properties to the Gauss map (EMR maps and Gauss-like maps, see definitions 2.1 and 2.2 respectively) and measures with infinite entropy and infinite Lyapunov exponent. In the second place, we show that the measures considered are not exact dimensional, by showing that the upper dimension is positive while the lower dimension is zero almost everywhere.

**Theorem 1.** Let \( T : [0, 1] \to [0, 1] \) be a Gauss-like map and \( \mu \) be an infinite entropy Gibbs measure of controlled decay, and such that the decay ratio \( s \) exists. Then \( d_{\mu}(x) = 0, d_{\mu}(x) = s \) \( \mu \)-almost everywhere.

This shows that there is a dimension gap for this class of maps and measures. For the Gauss map, \( s = 1/2 \). The Gibbs assumption on the measure implies that a certain sequence of observables can be seen as a non-integrable stationary ergodic process and allows us to use some tools of infinite ergodic theory developed by Aaronson and Nakada (see [Aar77, AN03]). In particular, the pointwise behaviour of the Birkhoff sums excluding the biggest term of such sums plays a fundamental role in our arguments. We remark that the methods used in the context of finite entropy fail, as they rely on the fact that the measure and diameter of the iterates of the natural Markov partition decrease at an exponential rate given by \( h_\mu \) and \( \lambda_\mu \) respectively, enabling the use of coverings by balls of different scales. To tackle this problem, we make use of more refined coverings of balls, which are capable of detecting the asymptotic interaction between the Gibbs measure and the Lebesgue measure.

The paper is structured as follows. In section 2 we introduce the notation used throughout the paper as well as the main objects of study. We also state the results of the paper. In section 3 we compute the symbolic dimension and characterize it in terms of the Markov partition. In section 4 we study the consequences of \( h_\mu = \lambda_\mu = \infty \) at the level of the asymptotic rate of contraction of the cylinders. In sections 5 and 6 we prove the results for the Hausdorff and the Packing dimension respectively. We finish the article stating some questions of interest that could not be answered with the methods used in this paper.

## 2. Notation and statement of main results

### 2.1. The class of maps

We start introducing the EMR (expanding-Markov–Renyi) maps of the interval.

**Definition 2.1.** We say that a map \( T : I \to I \) of the interval \( I = [0, 1] \) is an EMR map if there is a countable collection of closed intervals \( \{I(n)\} \), with disjoint interiors \( \text{int} I(n) \), such that:

(a) The map is \( C^2 \) on \( \bigcup_n \text{int} I(n) \),
(b) Some power of \( T \) is uniformly expanding, i.e., there is a positive integer \( r \) and a constant \( \alpha > 1 \) such that \(|T^r(x)| \geq \alpha \) for all \( x \in \bigcup_n \text{int} I(n) \),
(c) The map is Markov and can be coded by a full shift: \( \text{int} T(I(n)) = [0, 1] \) for all \( n \),
(d) The map satisfies Renyi’s condition: there is a constant \( E > 0 \) such that

\[
\sup_{n \in \mathbb{N}, y, z \in I(n)} \frac{|T''(x)|}{|T(y)||T'(z)|} \leq E.
\]

This class of maps was first introduced in [PW99] in the context the multifractal analysis of the Lyapunov exponent for the Gauss map. Renyi’s condition provides good estimates for the Lebesgue measure of the cylinders associated to the Markov structure of the map (see next
subsection). For simplicity, we will assume that the maps are orientation preserving (the orientation reversing case only differs from the orientation preserving case in the relative position of the cylinders). The set of branches must accumulate at least at one point, and we assume that it accumulates at exactly one point: we also assume that the branches accumulate on the left endpoint of $I$ (the case when the branches accumulate in the right endpoint of $I$ is analogous).

Re-indexing if necessary, we can assume that $I(n + 1) < I(n)$ for all $n$. Let $r_n = |I(n)|$.

**Definition 2.2.** We say that an EMR map $T$ is a Gauss-like map if it satisfies the following conditions:

(a) $r_n > 0$ for every $n \in \mathbb{N}$,

(b) $r_{n+1} \leq r_n$,

(c) $\sum n r_n = 1$,

(d) $0 < K \leq r_{n+1}/r_n < K' < \infty$ for some constants $K, K'$,

(e) $\{r_n\}$ decays polynomially as $n$ goes to infinity (see definition 3.7).

We want to keep in mind piecewise linear functions as the main example, as for this class of maps, calculations are simplified. We will also keep in mind the example of the Gauss map. In figure 1 we see an orientation preserving version of the Gauss map.

**2.2. Markov structure and symbolic coding**

We describe now the Markov structure of the maps considered. Given a finite sequence of natural numbers $(a_1, \ldots, a_n) \in \mathbb{N}^n$, the $n$th level cylinder associated to $(a_1, \ldots, a_n)$ is the set $I(a_1, \ldots, a_n) = I_{a_1} \cap T^{-1}(I(a_2)) \cap \cdots \cap T^{-(n-1)}(I(a_n))$. Let $\mathcal{O} = \bigcup_k T^{-k}(\partial I(k))$, then given $x \in [0, 1] \setminus \mathcal{O}$ and $n \in \mathbb{N}$, there exists a unique sequence $(a_1(x), a_2(x), \ldots) \in \mathbb{N}^\mathbb{N}$ such that $x \in I(a_1(x), \ldots, a_n(x))$ for every $n$. We denote this sequence by $(a_1, a_2, \ldots)$ when $x$ is clear from the context. We also denote $I_n(x) = I(a_1, \ldots, a_n)$ and we say $x$ is coded by the sequence $(a_n)$. From now on, whenever we say $x \in I$, we mean $x \in I \setminus \mathcal{O}$.

Let $\Sigma = \mathbb{N}^\mathbb{N}$ and $\sigma : \Sigma \to \Sigma$ be the left shift over $\mathbb{N}$: $(\sigma(x))_n = x_{n+1}$. Then the cylinders in the symbolic space are defined by

$$C(a_1, a_2, \ldots, a_n) = \{(x_n) \in \Sigma | x_j = a_j \text{ for } j = 1, \ldots, n\}.$$
We endow the space $\Sigma$ with the topology generated by the cylinders defined above. Then the map $\pi : \Sigma \to I/\mathcal{O}$ given by $\pi((x_0)) = \bigcap_{n \geq 0} I(x_1, \ldots, x_n)$ is a continuous bijection.

Given $x \in \mathcal{P}(\mathcal{O})$ with coding sequence $(a_n)$ and $n \geq 1$, denote by $I'_n(x) = I(a_1, \ldots, a_{n-1}, a_n - 1)$ (resp $I'_n(x) = I(a_1, \ldots, a_{n-1}, a_n + 1)$ if $a_n \geq 2$) the level $n$ cylinder on the left (resp right) of $I_n(x)$. Also, denote by $I_n(x) = I_n(x) \cup I'_n(x) \cup I''_n(x)$. If there is no risk of confusion, we omit the dependence on $x$.

Renyi’s condition introduced in the previous subsection implies that the length of each cylinder is comparable to the derivative of the iterates of the map at any point of the cylinder. More precisely,

$$0 < D^{-1} \leq |(T^n)'(x)| \cdot |I(a_1, \ldots, a_n)| \leq D < \infty$$

for every finite sequence $(a_1, \ldots, a_n) \in \mathbb{N}^n$ and $x \in I(a_1, \ldots, a_n)$.

2.3. The class of measures

We start by defining Gibbs measures:

**Definition 2.3.** Let $\mu$ be an invariant probability measure with respect to $T$. Then we say that $\mu$ is a *Gibbs measure* associated to the potential $\varphi : \Sigma \to \mathbb{R}$, that is, there exists a constant $C > 0$ so that

$$C^{-1} \leq \frac{\mu(C(a_1, \ldots, a_n))}{\exp(-nP(\log \varphi) + S_n(\log \varphi)(x))} \leq C,$$

where $x$ is any point in $C(a_1, \ldots, a_n)$, $(a_1, \ldots, a_n, \ldots)$ is any sequence in $\Sigma$, $S_n(\varphi)$ is the Birkhoff sum of $\varphi$ at the point $x$, and $P(\log \varphi)$ is a constant (depending on the potential) called the **topological pressure** of $\log \varphi$.

Throughout this work we will assume that $P(\log \varphi) = 0$, otherwise we can take the zero pressure potential $\varphi = -P(\log \varphi)$. It is important to note that it is not trivial that this will not affect our computations, and we will show later how we can overcome that difficulty. The sequence $p_n = \mu(I(n))$ will be of particular relevance for our computations.

We can project this measure to $I$ by setting $\hat{\mu} = \pi^{-1} \circ \mu$. We assume these measures are invariant and ergodic with respect to $T$. We will denote by $\mu$ both the measure in the symbolic space and the projected measure.

We define the $n$th **variation** of the potential $\log \varphi$ by

$$\text{var}_n(\log \varphi) = \sup\{|\log \varphi(x) - \log \varphi(y)| : x, y \in I(a_1, \ldots, a_n), (a_1, \ldots, a_n) \in \mathbb{N}^n\}.$$

**Definition 2.4.** Let $x_n$ be the unique fixed point of $T$ in $I(n)$. We define then the **decay ratio** by

$$s = \lim_{n \to \infty} \frac{\log \varphi(x_n)}{\log r_n} = \lim_{n \to \infty} \frac{\log p_n}{\log r_n},$$

whenever any of these limits exists. Similarly, the **tail decay ratio** is defined by

$$\hat{s} = \lim_{n \to \infty} \frac{\log \sum_{m \geq n} \varphi(x_m)}{\log \sum_{m \geq n} r_m} = \lim_{n \to \infty} \frac{\log \sum p_m}{\log \sum r_m}.$$
Both definitions for $s$ and $\delta$ agree since $\mu$ is a Gibbs measure. Note also that the definitions above are independent of the choice of the point $x_n$ representing each cylinder if $\text{var}_1(\log \varphi) < \infty$. By the Cersaro–Stolz theorem (see [Fur13, appendix B]) we can write the decay ratio as

$$s = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} p_n \log p_n}{\sum_{k=1}^{n} r_n \log r_n}$$

**Definition 2.5.** Assume that $\text{var}_1(\log \varphi) < \infty$. Suppose that for the sequence sequence $q = \{q_n\}_{n \in \mathbb{N}} = \{\varphi(x_n)\}$ we have

$$0 < K \leq q_{n+1}/q_n \leq K' < \infty$$

for every $n \in \mathbb{N}$, for some constants $K, K'$. Then we say that $\mu$ has controlled decay.

This condition prevents the existence of large jumps for the potential along sufficiently sparse subsequences of $\{x_n\}$. By the Gibbs property, the properties hold if we replace $q_n$ by $p_n$.

**2.4. Entropy and Lyapunov exponent**

Our main assumption on the class of measures is that they have infinite entropy. This can be expressed by saying that the potential $-\log \varphi$ is not integrable with respect to $\mu$.

Our definition of entropy differs from the conventional (see [Wal82, chapter 4]), as we deal with partitions with infinite entropy. For this, recall the Shannon–McMillan–Breiman theorem adapted to our system, which in the case of Gibbs measures, is equivalent to the Ergodic theorem:

**Theorem 2.6 (Shannon–McMillan–Breiman).** For any Gibbs measure $\mu$ associated to a potential with finite first variation, the limit

$$\lim_{n \to \infty} -\frac{1}{n} \log(\mu(I_n(x))) \quad (2.1)$$

exists $\mu$-almost everywhere and is constant. If $\sum_n -p_n \log p_n < \infty$, then such constant is finite; otherwise, it is equal to infinity.

The proof for the case when the series is finite can be found in [VO16, section 9.3]. The proof for the infinite case follows from lemmas 2.7 and 3.2, using that the measures have the Gibbs property. We define then the entropy $h_\mu$ as the almost sure value of the limit (2.1).

**Lemma 2.7.** For a Gibbs measure with finite first variation, the entropy $h_\mu$ is finite if and only if any of the series

$$-\sum_{n=1}^{\infty} q_n \log q_n \quad , \quad -\sum_{n=1}^{\infty} p_n \log p_n$$

converges.

**Proof.** The partition of $[0, 1]$ by cylinders $\{I(n)\}$ is a generating partition, and hence Sinai’s generator theorem (see [VO16, corollary 9.2.5]) allows us to compute the entropy of $\mu$ using the entropy of this partition. The entropy of $\mu$ with respect to this partition is given by

$$H(\mu, \alpha) = -\sum_{n=1}^{\infty} p_n \log p_n.$$
The convergence of this series is equivalent to the convergence of \(-\sum_{n=1}^{\infty} q_n \log q_n\) since we have
\[
\exp(-\text{var}_1 \log \varphi) < \frac{q_n}{\varphi(x)} < \exp(\text{var}_1 \log \varphi),
\]
\[
C^{-1} \leq \frac{p_n}{\varphi(x)} \leq C
\]
for any \(x \in I(n)\).

We define the Lyapunov exponent as
\[
\lambda_\mu = \int_0^1 \log |T'(x)| \, d\mu(x).
\]
By the bounded distortion property, this integral converges if and only if the series \(-\sum_{n=1}^{\infty} q_n \log r_n\) converges.

2.5. Hausdorff and packing dimension

In this section we introduce the dimension theory elements we will study throughout this work. Recall the diameter of a set \(U \subset \mathbb{R}\) is given by
\[
|U| = \sup\{|x - y| | x, y \in U\}.
\]
For a cover \(\mathcal{U}\) of a set \(X \subset \mathbb{R}\), its diameter is given by
\[
\text{diam} \mathcal{U} = \sup\{|U| | U \in \mathcal{U}\}.
\]

**Definition 2.8.** Given \(X \subset \mathbb{R}\) and \(\alpha \in \mathbb{R}\), the \(\alpha\)-dimensional Hausdorff measure of \(X\) is given by
\[
m(X, \alpha) = \liminf_{\delta \to 0} \frac{1}{|U|} \sum_{U \in \mathcal{U}} |U|^\alpha,
\]
where the infimum is taken over finite or countable covers \(\mathcal{U}\) of \(X\) with \(\text{diam} \mathcal{U} \leq \delta\).

It is possible to prove that there exists a number \(s \in [0, \infty)\) such that \(m(X, \alpha) = \infty\) for \(t < s\) and \(m(X, \alpha) = 0\) for \(t > s\), since \(m(X, \alpha)\) is decreasing in \(\alpha\) for a fixed set \(X\).

**Definition 2.9.** The unique number
\[
\dim_H X = \inf \{\alpha \in [0, \infty] | m(X, \alpha) = 0\}
\]
is called the Hausdorff dimension of \(X\).

We extend the notion of Hausdorff dimension to finite Borel measures on \(\mathbb{R}\):
Definition 2.10. Let \( \mu \) be a finite Borel measure on \( \mathbb{R} \). The Hausdorff dimension of \( \mu \) is defined by
\[
\dim_H \mu = \inf \{ \dim_H(Z) | \mu(\mathbb{R} \setminus Z) = 0 \}.
\]
We define now the analogue notion of packing dimension

Definition 2.11. We say that a collection of balls \( \{ U_n \} \subset \mathbb{R} \) is a \( \delta \)–packing of the set \( E \subset \mathbb{R} \) if the diameter of the balls is less than or equal to \( \delta \), they are pairwise disjoint and their centres belong to \( E \). For \( \alpha \in \mathbb{R} \), the \( \alpha \)–dimensional pre-packing measure of \( E \) is given by
\[
P(E, \alpha) = \limsup_{\delta \to \alpha} \left\{ \sum_n \delta \log \mu(U_n) \right\}
\]
where the supremum is taken over all \( \delta \)–packings of \( E \). The \( \alpha \)–dimensional packing measure of \( E \) is defined by
\[
p(E, \alpha) = \inf \left\{ \sum_i p(E_i, \alpha) \right\}
\]
where the infimum is taken over all covers \( \{ E_i \} \) of \( E \). Finally, we define the packing dimension of \( E \) by
\[
\dim_p(E) = \sup \{ s | p(E, \alpha) = \infty \} = \inf \{ s | p(E, \alpha) = 0 \}.
\]
We extend the notion of packing dimension to finite Borel measures on \( \mathbb{R} \).

Definition 2.12. Let \( \mu \) be a finite Borel measure on \( \mathbb{R} \). The packing dimension of \( \mu \) is defined by
\[
\dim_p \mu = \inf \{ \dim_p(Z) | \mu(\mathbb{R} \setminus Z) = 0 \}.
\]

Bounding the Hausdorff dimension from above or the Packing dimension from below usually involves the use of a single suitable cover of the space, while for bounds from below and above respectively, we have to deal with every cover of the space. There are several tools to help with this problem, and we will make use of the so called (local) mass distribution principles. For this, we introduce the notion of local dimension.

Definition 2.13. The lower and upper pointwise dimensions of the measure \( \mu \) at a point \( x \in X \) is given by
\[
\underline{d}_p(x) = \liminf_{r \to 0} \frac{\mu(B(x, r))}{\log r}, \quad \overline{d}_p(x) = \limsup_{r \to 0} \frac{\mu(B(x, r))}{\log r}.
\]
When both limits coincide, we call the common value the pointwise dimension of \( \mu \) at \( x \) and denote it by \( d_p(x) \) and say that \( \mu \) is exact dimensional if \( \underline{d}_p(\cdot) = \overline{d}_p(\cdot) \) \( \mu \)-almost everywhere.

If \( d_p(x) = d \), then \( \mu(B(x, r)) \sim r^d \) for small values of \( r \). We state now the local version of the mass distribution principle.

Proposition 2.14. Let \( X \subset \mathbb{R} \) and \( \alpha \in (0, \infty] \), then
(a) If \( \underline{d}_p(x) \geq \alpha \) for \( \mu \)-almost every \( x \in X \), then \( \dim_H \mu \geq \alpha \);
(b) If \( \underline{d}_p(x) \leq \alpha \) for every \( x \in X \), then \( \dim_H X \leq \alpha \);
(c) If \( \overline{d}_p(x) \geq \alpha \) for \( \mu \)-almost every \( x \in X \), then \( \dim_p \mu \geq \alpha \);
(d) If \( \overline{d}_p(x) \leq \alpha \) for every \( x \in X \), then \( \dim_p X \leq \alpha \);
(e) We have
\[
\dim_H \mu = \text{ess sup } \{d_\mu(x) | x \in X\}, \\
\dim_P \mu = \text{ess sup } \{\mathcal{I}_\mu(x) | x \in X\},
\]

**Proof.** This follows from proposition 2.3 of [Fal97]. \(\square\)

In particular, if \(d_\mu(\cdot)\) is constant almost everywhere, then \(\dim_H \mu\) is equal to that constant value. Analogously, if \(\mathcal{I}_\mu(\cdot)\) is constant almost everywhere, then \(\dim_P \mu\) is equal to that constant value.

A notion of dimension which is more adapted to the underlying structure of our dynamical system is the symbolic dimension, which we proceed to define.

**Definition 2.15.** Given \(x \in I\), we define the **lower symbolic dimension** of \(\mu\) at \(x\) by

\[
\underline{\delta}(x) = \liminf_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|},
\]

and the **upper symbolic dimension** of \(\mu\) at \(x\) by

\[
\overline{\delta}(x) = \limsup_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|},
\]

If \(\overline{\delta}(x) = \underline{\delta}(x)\), then we define the **symbolic dimension** of \(\mu\) at \(x\) as the common value, denote it by \(\delta(x)\), and we say that \(\mu\) is **symbolic exact dimensional** if \(\underline{\delta}(x) = \overline{\delta}(x)\) almost everywhere.

This notion is fundamental as it provides a connection between symbolic dynamics and the geometric properties of the measures we are interested in.

### 3. Symbolic dimension

#### 3.1. Computation of the symbolic dimension

We prove now that under the above assumptions, the Gibbs measure \(\mu\) is symbolic exact dimensional, and this dimension coincides with the decay ratio. This result does not depend on the length decaying ratio of the partition of the interval.

In general the Lyapunov exponent majorizes the entropy. In a more general setting, this result is known as Ruelle’s inequality (see [Rue78]).

**Proposition 3.1.** If \(h_\mu = \infty\) then \(\lambda_\mu = \infty\).

**Proof.** This is an immediate consequence of the volume lemma (theorem 1): if \(\lambda_\mu \neq \infty\), then \(\dim_H \mu = \infty\) which is impossible. \(\square\)

We prove a well known fact about non-integrable observables.

**Lemma 3.2.** Let \(f : [0, 1] \to \mathbb{R}\) be a bounded below measurable function such that \(\int_0^1 f \ d\mu = \infty\). Then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \infty
\]

for \(\mu\) almost every point.
Theorem 3.4. Let $T$ be an EMR map and

$$\text{Proof.} \quad \text{The proof is an application of the monotone convergence theorem. Assume } f \text{ is positive (otherwise, decompose } f \text{ into its positive and negative part) and let } M > 0. \text{ Then}
$$

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) \geq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \min\{f \circ T^k, M\}(x)
$$

$$= \int_0^1 \min\{f, M\}(x) \, d\mu(x)
$$

by Birkhoff’s Ergodic theorem applied to $\min\{f, M\}$. By the monotone convergence theorem,

$$\lim_{n \to \infty} \int_0^1 \min\{f, M\}(x) \, d\mu(x) = \int_0^1 f \, d\mu(x) = \infty
$$

from where we conclude the result. \qed

This result implies in particular that we can assume that the pressure of our potential is zero, as $S_\phi(\log \varphi)$ dominates $-nP(\log \varphi)$ when $\log \varphi$ is not integrable.

We formulate a lemma regarding the metric and measure theoretic properties of the cylinders associated to the map. This will allow us to write geometric quantities in ergodic theoretic terms. Its proof is a standard applications of the bounded distortion and Gibbs properties.

**Lemma 3.3.** For every finite sequence $(a_1, \ldots, a_n) \in \mathbb{N}^n$ and $j \in \mathbb{N}$, we have that

(a) $| \log |I(a_1, \ldots, a_n)| - \sum_{k=1}^{n} \log r_k | \leq nD_1 + D_2,$

(b) $| \log \left| \bigcup_{m=0}^{j} I(a_1, \ldots, a_{n-1}, a_n + m) \right| - \sum_{k=1}^{n} \log r_k - \log \left( \sum_{m=j}^{\infty} r_{a_k + m} \right) | \leq nD_1 + D_2,$

(c) $| \log \left| \bigcup_{m=0}^{j} I(a_1, \ldots, a_{n-1}, a_n + m) \right| - \sum_{k=1}^{n-1} \log p_k - \log \left( \sum_{m=j}^{\infty} p_{a_k + m} \right) | \leq nG_1 + G_2.$

(d) $| \log \mu(I(a_1, \ldots, a_n)) - \sum_{k=1}^{n} \log r_k | \leq nD_1 + D_2,$

(e) $| \log \mu \left( \bigcup_{m=0}^{j} I(a_1, \ldots, a_{n-1}, a_n + m) \right) - \sum_{k=1}^{n-1} \log p_k - \log \left( \sum_{m=j}^{\infty} p_{a_k + m} \right) | \leq nG_1 + G_2.$

where $D_1, D_2$ are distortion constants and $G_1, G_2$ are constants arising from the Gibbs property.

We proceed to compute the symbolic dimension of our system. This result holds regardless of the decay rate of the sequence $(r_k)$.

**Theorem 3.4.** Let $T$ be an EMR map and $\mu$ a Gibbs measure with controlled decay and infinite entropy. Then if the decay ratio exists, we have that $\mu$ is symbolic-exact dimensional and for $\mu$-almost every $x \in I$,

$$\delta(x) = s.$$  

**Proof.** By lemma 3.2 applied to the observables $\log \varphi$ and $\log r_{a_i}$ and lemma 3.3, we have

$$\hat{\delta}(x) \leq \liminf_{n \to \infty} \frac{S_n(\log \varphi)(x)}{-nD_1 - D_2 + S_n(\log r_{a_i})(x)} = \liminf_{n \to \infty} \frac{\log(q_{a_1} \cdots q_{a_n})}{\log(r_{a_1} \cdots r_{a_n})},$$

$$\hat{\delta}(x) \geq \limsup_{n \to \infty} \frac{S_n(\log \varphi)(x)}{-nD_1 - D_2 + S_n(\log r_{a_i})(x)} = \limsup_{n \to \infty} \frac{\log(q_{a_1} \cdots q_{a_n})}{\log(r_{a_1} \cdots r_{a_n})}.$$
for almost every $x \in I$, and analogously for the upper symbolic dimension
\[
\overline{\delta}(x) = \limsup_{n \to \infty} \frac{\log(q_{a_1} \cdots q_{a_n})}{\log(r_{a_1} \cdots r_{a_n})}
\]
where $(a_1, a_2, \ldots)$ is the sequence coding $x$. With a similar argument, we can also show that the same holds true if we switch $q_k$ for $p_k$:
\[
\underline{\delta}(x) = \liminf_{n \to \infty} \frac{\log(p_{a_1} \cdots p_{a_n})}{\log(r_{a_1} \cdots r_{a_n})},
\]
and analogously for the upper symbolic dimension.
For $x \in I$ and $n, k \geq 1$, define
\[
f_{n,k}(x) = \#\{i \in \{1, \ldots, n\} | a_i(x) = k\},
\]
that is, the number of times the orbit of $x$ visits the interval $I_k$ in the first $n$ steps. Recall that from the Birkhoff theorem, we have that for every $k$,
\[
\lim_{n \to \infty} \frac{f_{n,k}}{n} = p_k
\]
for $\mu$—almost every $x \in I$. In particular, the orbit of almost every $x \in I$ visits every cylinder $I(n)$ infinitely many times. Fix $x$ in the set where the convergence holds, and then define $m : \mathbb{N} \to \mathbb{N}$ by $m(n) = \max\{k(x) | i \in \{1, \ldots, n\}\}$. The previous remark shows that $m$ is unbounded, and it is clearly non-decreasing. Thus, we can write
\[
-\log(r_{a_1} \cdots r_{a_n}) = -\sum_{j=1}^{n} \log r_{a_j} = -\sum_{j=1}^{m(n)} f_{n,j} \log r_j.
\]
Given $\epsilon > 0$, there exists $n_1$ such that
\[
\left| \frac{\log p_k}{\log r_k} - s \right| < \epsilon
\]
for every $k \geq n_1$, that is, $(-\log p_k) < (\epsilon + s)(-\log r_k)$ for $k \geq n_1$. For $n$ large enough so that $m(n) > n_1$, we write
\[
\frac{\log(p_{k_1} \cdots p_{k_n})}{\log(r_{k_1} \cdots r_{k_n})} = \frac{\sum_{k=1}^{n_1} f_{n,k}(-\log p_k) + \sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log p_k)}{\sum_{k=1}^{n_1} f_{n,k}(-\log r_k) + \sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log r_k)}.
\]
We split the sum in two different parts:
\[
A(n) = \frac{\sum_{k=1}^{n_1} f_{n,k}(-\log p_k)}{\sum_{k=1}^{n_1} f_{n,k}(-\log r_k) + \sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log r_k)},
\]
\[
B(n) = \frac{\sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log p_k)}{\sum_{k=1}^{n_1} f_{n,k}(-\log r_k) + \sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log r_k)}.
\]
For $k = 1, \ldots, n_1$ taking $c_k = p_k/2$ there exists $n_2 \geq n_1$ such that
\[
\frac{np_k}{2} \leq f_{n,k} \leq \frac{3np_k}{2}
\]
for every $n \geq n_3$. Thus, the terms $\sum_{k=1}^{n_1} f_{n,k}(-\log p_k)$ and $\sum_{k=1}^{n_1} f_{n,k}(-\log r_k)$ grow linearly in $n$ for $n$ large enough. We will show that $\sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log r_k)$ grows faster than linear as a function of $n$.

Given $M > 0$, since the Lyapunov exponent is infinite, there exists $n_3$ such that

$$\sum_{k=n_1+1}^{m(n)} p_k(-\log r_k) > 2M$$

for every $n \geq n_3$. Now, for $k = n_1 + 1, \ldots, m(n_3)$, take $\epsilon_k = p_k/2$ and so there exists $n_4 \geq n_3$ such that

$$f_{n,k} \geq \frac{n p_k}{2}$$

for every $n \geq n_4$ and $k = n_1 + 1, \ldots, m(n_3)$. Thus

$$\frac{1}{n} \sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log r_k) = \frac{1}{n} \sum_{k=n_1+1}^{m(n_3)} f_{n,k}(-\log r_k) + \frac{1}{n} \sum_{k=m(n_3)+1}^{m(n)} f_{n,k}(-\log r_k) \geq \frac{1}{n} \sum_{k=n_1+1}^{m(n_3)} \frac{n p_k}{2}(-\log r_k)$$

$$= \frac{1}{2} \sum_{k=n_1+1}^{m(n_3)} p_k(-\log r_k) > M$$

for every $n \geq n_4$. This shows that $A(n) \to 0$ as $n \to \infty$. To estimate $B(n)$, we note that

$$B(n) \leq (s + \epsilon) \cdot \frac{\sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log r_k)}{\sum_{k=1}^{n_1} f_{n,k}(-\log r_k) + \sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log r_k)}$$

Using the same argument as above, we can show that $\sum_{k=n_1+1}^{m(n)} f_{n,k}(-\log r_k)$ grows faster than linear, so $\lim B(n) \leq s + \epsilon$. This shows that

$$\delta(x) \leq s.$$

The proof of the opposite inequality is analogous. \hfill \Box

### 3.2. The decay ratio

Now we proceed to study the properties of the decay ratio. In fact, we show that for infinite entropy measures, it is completely determined by the properties of the partition $\{I(n)|n \in \mathbb{N}\}$:

**Definition 3.5.** The convergence exponent of the partition $\{r_n\}$ of $I$ is defined by

$$s_{\infty} = \inf \left\{ s \geq 0 | \sum_{n=1}^{\infty} r_n^s < \infty \right\}.$$

**Proposition 3.6.** In general, we have that $s_{\infty} \leq s$. Under the assumption that $h_{\mu} = \infty$, we also have $s \leq s_{\infty}$.
Proof. Given $\epsilon > 0$, there exists $n_1$ such that
$$(s + \epsilon) \log r_n < \log p_n < (s - \epsilon) \log r_n$$
for every $n \geq n_1$, and thus $r_n^{s+\epsilon} < p_n$ for every $n \geq n_1$. Summing over $n$ we get
$$\sum_{n=1}^{\infty} r_n^{s+\epsilon} = \sum_{n=1}^{n_1-1} r_n^{s+\epsilon} + \sum_{n=n_1}^{\infty} r_n^{s+\epsilon} < \sum_{n=n_1}^{\infty} p_n < \infty.$$ Hence, $s_\infty \leq s + \epsilon$ for every $\epsilon > 0$ and so $s_\infty \leq s$.

Now, suppose that $s_\infty < s$, and hence, there is $\alpha > 0$ such that $s_\infty \leq s_\infty + \alpha < s$ and
$$\sum_{n=1}^{\infty} r_n^{s_\infty + \alpha} < \infty.$$ Let $\epsilon = (s - s_\infty - \alpha)/2 > 0$, then there is an integer $n_0$ such that
$$r_n^{s+\epsilon} \leq p_n \leq r_n^{s-\epsilon}$$
for all $n \geq n_0$. This implies that
$$\sum_{n=n_0}^{\infty} p_n(- \log p_n) \leq (s + \epsilon) \sum_{n=n_0}^{\infty} r_n^{s-\epsilon}(- \log r_n).$$
Recall the one sided limit criterion for convergence of series: let $a_n, b_n > 0$ sequences such that
$$\limsup_{n \to \infty} \frac{a_n}{b_n} = c \in [0, \infty)$$
and $\sum b_n < \infty$. Then $\sum a_n < \infty$.

Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by
$$f(x) = \begin{cases} 0, & \text{for } x = 0, \\ x(- \log x), & \text{for } x > 0. \end{cases}$$
It is easy to see that $f$ is continuous. Taking $a_n = r_n^{s-\epsilon}(- \log r_n)$ and $b_n = r_n^{s_\infty + \alpha}$ and using the continuity of $f$, we get that
$$\limsup_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} f_n(- \log r_n) = 0.$$ We conclude that
$$\sum_{n=n_0}^{\infty} p_n(- \log p_n) \leq (s + \epsilon) \sum_{n=n_0}^{\infty} r_n^{s-\epsilon}(- \log r_n) < \infty,$$
contradicting the fact that the entropy is infinite. \hfill \square

We give now a definition for the asymptotic decay of the sequence $\{r_n\}$.

Definition 3.7. The asymptotic rate of the sequence $\{r_n\}$ is defined as
$$\alpha = \sup \{t \geq 0 \mid \lim_{n \to \infty} n^t r_n < \infty\}.$$
We say that \( \{ r_n \} \) decays polynomially if \( \alpha > 1 \), and we say that \( \{ r_n \} \) decays superpolynomially if \( \alpha = \infty \).

Note that if \( r_n \) has polynomial decay with asymptotic \( \alpha \), then \( s_\infty = 1/\alpha \). If we know the asymptotic of \( \{ r_n \} \), we can compute the asymptotic of the tail of the series of \( \{ r_n \} \):

**Lemma 3.8.** If the asymptotic of \( \{ r_n \} \) is \( \alpha > 1 \), then the asymptotic of \( \{ R_n = \sum_{m>n} r_n \} \) is \( \alpha - 1 \).

**Proof.** It suffices to show that the sets

\[
A = \{ t \geq 1 | \lim_{n \to \infty} n^t r_n < \infty \}
\]

and

\[
A' = \{ t \geq 0 | \lim_{n \to \infty} n^t R_n < \infty \}
\]

are the same. Let \( t \in A \), then \( \lim_{n \to \infty} n^t r_n = d \), and so given \( \epsilon \), there is \( n_0 \in \mathbb{N} \) such that

\[
(d - \epsilon) n^t < r_n < (d + \epsilon) n^t.
\]

for \( n \geq n_0 \). Hence, for \( n \geq n_0 \),

\[
\frac{(d - \epsilon)}{n^t} \leq \sum_{m=n}^{\infty} n^{-1}(d - \epsilon) \leq n^{-1}R_n \leq \sum_{m=n}^{\infty} \frac{n^{-1}(d + \epsilon)}{m^t} \leq \frac{n^{-1}(d + \epsilon)}{(n + 1)^{-1}(t - 1)}
\]

from which follows that \( t - 1 \in A' \). Now, if \( t \in A' \), we have that \( \lim_{n \to \infty} n^{t-1}R_n = d' < \infty \), and thus, given \( \epsilon > 0 \), there is \( n_1 \in \mathbb{N} \) such that

\[
-\epsilon + d' \leq \sum_{m < n} r_n \leq \epsilon + d'.
\]

This implies that

\[
\frac{(-\epsilon + d')}{n^t} \leq \frac{\epsilon + d'}{(n + 1)^t} \leq \frac{r_n}{n^t} \leq \frac{\epsilon + d'}{(n + 1)^t}.
\]

from which follows that \( t + 1 \in A \), proving the assertion. \( \square \)

### 4. Infinite ergodic theory

In this section we explore the consequences of the non-integrability of the functions \( -\log r_{a_1} \) and \( -\log p_{a_1} \) (or equivalently, \( h_\mu = \lambda_\mu = \infty \)). Using tools of infinite ergodic theory we can prove that the diameter of the cylinders decreases faster than exponentially from a given level to the next.

#### 4.1. Finite Lyapunov exponent argument

We proceed to show now one of the usual arguments used to compute Hausdorff dimensions and remark how it fails in our case.

**Lemma 4.1.** Let \( T \) be an EMR map and \( \mu \) a Gibbs measure. Then for almost every \( x \in I \) and every \( r > 0 \) there exists \( n \) such that

\[
\frac{\log \mu(B(x, r))}{\log r} \leq \frac{\log \mu(I_n(x))}{\log |I_{n-1}(x)|}.
\]  (4.1)
Proof. This is a well known argument and can be found for instance in [Pes08]. Given \( r > 0 \), there exists a unique integer \( n = n(r) \) such that

\[
|I_n(x)| < r \leqslant |I_{n-1}(x)|
\]

so then

\[
I_n(x) \subseteq B(x, |I_n(x)|) \subseteq B(x, r) \subseteq B(x, |I_{n-1}(x)|).
\]

Then

\[
\log \mu(I_n(x)) \leqslant \log \mu(B(x, r)),
\]

and since \( \log r \leqslant \log |I_{n-1}(x)| \), we obtain

\[
\frac{\log \mu(B(x, r))}{\log r} \leqslant \frac{\log \mu(I_n(x))}{\log |I_{n-1}(x)|}
\]

as we wanted. \( \square \)

In a similar way, it is possible to show that

\[
\log \frac{C_1 \mu(I_{n-1}(x))}{C_2 |I_n(x)|} \leqslant \log \frac{\mu(B(x, r))}{\log |I_{n-1}(x)|},
\]

where \( C_1, C_2 \) are constants arising from the controlled decay property and Renyi’s property respectively. Note that if \( \lambda_\mu < \infty \), then inequalities (4.1) and (4.2), and the Ergodic theorem would immediately imply that \( s = \dim_H \mu = \dim_P \mu \). However, since in our case \( \lambda_\mu = \infty \), the previous argument does not work. In fact, here lies the main difficulty of the infinite entropy and Lyapunov exponent case. The following theorem shows that the situation is as bad as it can get: for almost every point, the diameter of the cylinders decreases arbitrarily from one level to the next.

**Theorem 4.2.** Let \( T \) be a Gauss-like map and \( \mu \) an infinite entropy Gibbs with controlled decay. Then for almost every \( x \in I \), we have that

\[
\lim \inf_{n \to \infty} \frac{\log |I_n(x)|}{\log |I_{n-1}(x)|} = 1,
\]

and

\[
\lim \sup_{n \to \infty} \frac{\log |I_n(x)|}{\log |I_{n-1}(x)|} = \infty.
\]

The proof of the first equality is an immediate consequence of recurrence. We postpone the proof of the second equality. We will return to this issue once we set up the appropriate tools to prove this result.

**Corollary 4.3.** For almost every \( x \in I \), we have that \( d(x) \leqslant s \) and hence \( \dim_H \mu \leqslant s \).

The main tool that we will use to prove theorem 4.2 are results about the pointwise behaviour of trimmed sums.
4.2. Trimmed convergence

In this section we introduce some infinite ergodic theory notions and results. Define \( \{g_n = -\log r_1 \circ T^{n-1}\} \). The cumulative distribution function of \( g_1 \) is \( \mathcal{F}(t) = \mu(g_1 \geq t) \), and it can be seen that \( \mu(g_1) = \int_0^1 g_1 d\mu = \lambda_\mu \). By invariance of the measure, the cumulative distribution of \( g_n \) is the same as \( \mathcal{F} \). As we saw in lemma 3.2, the Ergodic theorem fails to provide non-trivial information. This result was vastly generalized by Robbins and Chow for i.i.d. random variables in [CR61] and in the ergodic stationary case by Aaronson in [Aar77] who proved the following theorem:

**Theorem 4.4.** [Aar77, theorem 1]. Let \( f : [0, 1] \to \mathbb{R} \) be a non-negative measurable function. If \( \mu(f) = \infty \) then for any sequence \( \{b_n\} \) of positive numbers, either

\[
\limsup_{n \to \infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ T^k = \infty \quad \text{a.e.}
\]

or

\[
\liminf_{n \to \infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ T^k = 0 \quad \text{a.e.}
\]

It is possible to prove that the lack of convergence in the previous theorem is due to a finite number of terms which are not comparable in size to the rest of the terms of the sum. This was proved in the i.i.d. case by Mori in [Mor76, Mor77] and in the stationary ergodic case by Aaronson and Nakada in [AN03]. We formulate the result by Aaronson and Nakada in a setting appropriate for our purposes.

We denote the ergodic sum of a function \( f \) by \( S_n(f)(x) \) and define \( S'_n(f)(x) = S_n(f)(x) - \max\{f, \ldots, f \circ T^{n-1}\}(x) \). When the dependence of \( S_n(f)/S'_n(f) \) on \( f \) is clear, we drop it from the notation and write \( S_n \). We refer to \( S'_n \) as the trimmed ergodic sum of \( f \).

**Definition 4.5.** We say that the sequence \( \{f \circ T^k\} \) has trimmed convergence if there exists a sequence \( \{b_n\} \) such that

\[
\lim_{n \to \infty} \frac{S'_n(x)}{b_n} = 1
\]

almost surely.

**Theorem 4.6.** [AN03, theorem 1.1]. Let \( (X_1, X_2, \ldots) \) be a non-negative, ergodic stationary process with \( L(t) = \mu(\min\{X, t\}) \), and set \( \varepsilon(t) := t(\log L)'(t) \). Suppose that the process is continued fraction mixing with exponential rate (see [AN03]), and that

\[
\sum_{n=1}^{\infty} \frac{\varepsilon^2(n)}{n} < \infty.
\]

Then \( \{X_n\} \) has trimmed convergence.
As remarked in [AN03], any Gibbs–Markov map is CF-mixing with exponential rate. For our particular sequence, the series in the previous theorem can be explicitly expressed in terms of the sequences \( \{ p_n \} \) and \( \{ r_n \} \):

**Lemma 4.7.** Suppose that

\[
\sum_{n=1}^{\infty} (\log r_n)^2 (p_n^2 + 2 p_n p_{n+1}) < \infty.
\]

Then the sequence \( \{ g_n = -\log r_1 \circ T^{n-1} \} \) has trimmed convergence.

**Proof.** We show that if

\[
\sum_{n=1}^{\infty} (\log r_n)^2 (p_n^2 + 2 p_n p_{n+1}) < \infty,
\]

then

\[
\sum_{n=1}^{\infty} \frac{e^2(n)}{n} < \infty.
\]

Let \( F(t) = \mu(X \geq t) \) and note that

\[
(\log L)'(t) = \frac{F(t)}{L^2(t)},
\]

and hence

\[
\sum_{n=1}^{\infty} \frac{e^2(n)}{n} = \sum_{n=1}^{\infty} \frac{nF^2(n)}{L^2(n)} \leq \sum_{n=1}^{\infty} nF^2(n).
\]

We compare the above sum to the corresponding integral. We can then see that if \( x \in [0, -\log r_1) \) then \( F(x) = 1 \), while if \( x \in [-\log r_n, -\log r_{n+1}) \) for \( n \geq 1 \) then

\[
F(x) = \sum_{k=n+1}^{\infty} p_k,
\]

so then the integral is

\[
\int_{0}^{\infty} x (F(x))^2 \, dx = \int_{0}^{-\log r_1} x \left( \sum_{k=1}^{\infty} p_k \right)^2 \, dx + \sum_{n=1}^{\infty} \left( \int_{-\log r_n}^{-\log r_{n+1}} x \left( \sum_{k=n}^{\infty} p_k \right)^2 \, dx \right)
\]

\[
= \frac{(\log r_1)^2}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left( \int_{-\log r_n}^{-\log r_{n+1}} x \left( \sum_{i,j=n}^{\infty} p_i p_j \right) \, dx \right)
\]

\[
= \frac{(\log r_1)^2}{2} + \frac{1}{2} \sum_{n=1}^{\infty} ( (\log r_{n-1})^2 - (\log r_n)^2 ) \left( \sum_{i,j=n}^{\infty} p_i p_j \right).
\]

Call now

\[
a_n = (\log r_n)^2, \quad b_n = \sum_{i,j=n}^{\infty} p_i p_j.
\]
Then, the above expression has the form
\[ \sum_{n=1}^{\infty} (a_{n+1} - a_n)b_n \]
which can be written as
\[ -a_1b_1 + \sum_{n=1}^{\infty} a_{n+1}(b_n - b_{n+1}). \]
Note that
\[ b_{n+1} - b_n = 2p_n p_{n+1} + p_n^2 \]
\[ b_1 = 1. \]
With this, the integral becomes
\[ \int_{0}^{\infty} x (F(x))^2 \, dx = \frac{(\log r_1)^2}{2} + \frac{1}{2} \left( - (\log r_1)^2 + \sum_{n=1}^{\infty} (\log r_n)^2 (p_n^2 + 2p_n p_{n+1}) \right) \]
\[ = \sum_{n=1}^{\infty} (\log r_n)^2 (p_n^2 + 2p_n p_{n+1}) \]
as we wanted to prove.

We show now that the trimmed convergence condition is satisfied by systems for which \( \{r_n\} \) decays polynomially or slower.

**Lemma 4.8.** Suppose that
\[ \lim_{n \to \infty} \frac{1}{n} (\log r_n)^2 = c \in [0, \infty). \]
Then the sequence \( \{g_n = -\log r_1 \circ T^{\circ -1}\} \) has trimmed convergence.

**Proof.** Since \( p_n \) and \( p_{n+1} \) are comparable, it suffices to prove that
\[ \sum_{n=1}^{\infty} (\log r_n)^2 p_n^2 < \infty. \]
Note that \( \{p_n\} \subset \ell^2 \) and we have that
\[ 1 = \left( \sum_{n=1}^{\infty} p_n \right)^2 = \sum_{i,j=1}^{\infty} p_i p_j. \]
Since the sequence \( \{p_n\} \) is decreasing, we have that
\[ \sum_{j=2}^{\infty} p_j^2 (j-1) = \sum_{j=2}^{\infty} p_j \sum_{i=1}^{j-1} p_j \leq \sum_{j=2}^{\infty} p_j \sum_{i=1}^{j-1} p_i \leq \sum_{i=1}^{\infty} p_i \sum_{i=1}^{\infty} p_i < \infty. \]
Comparing in the limit the series of the left hand side to the series \( \sum_n p_n (\log r_n)^2 \), we get that this series converge. \qed
Corollary 4.9. If $T$ is a Gauss-like map, then it has trimmed convergence.

Now we are in position to prove theorem 4.2: proof of theorem 4.2. By theorem 4.6 and, there exists a sequence \( \{b_n\} \) such that

\[
\lim_{n \to \infty} \frac{S_n'(x)}{b_n} = 1 \text{ a.e.}
\]

Now, by theorem 4.4 we also have that

\[
\limsup_{n \to \infty} \frac{S_n(x)}{b_n} = \infty \text{ a.e.}
\]

or

\[
\liminf_{n \to \infty} \frac{S_n(x)}{b_n} = 0 \text{ a.e.}
\]

Since the trimmed sum is $o(b_n)$, the first condition must hold in a set of full measure. Let \( (a_n) \) be the coding sequence of \( x \). With an argument analogue to the one used in the proof of theorem 3.4, the limit in question is equivalent to

\[
\limsup_{n \to \infty} \frac{1}{b_n} \sum_{k=1}^{n} \log r_{a_k} = 1 + \limsup_{n \to \infty} \frac{\log r_{a_n}}{\log(r_{a_1} \cdots r_{a_{n-1}})} = 1 + \limsup_{n \to \infty} \frac{g_n(x)}{S_{n-1}(x)}
\]

Given $1 > \varepsilon > 0$, there exists $n_0$ such that

\[
\frac{|S_n'(x)|}{b_n} - 1 < \varepsilon
\]

for every $n \geq n_0$ at $x$. Since $\limsup \frac{S_n'}{b_n} = \infty$, given an integer $M > 0$ there exists $n_1 \geq n_0$ such that

\[
\frac{S_{n1}(x)}{b_{n1}} > 2M + 1
\]

at $x$. Combining these two inequalities, we obtain

\[
\left| \frac{\max\{g_1, \ldots, g_{n1}\}(x)}{b_{n1}} \right| = \left| \frac{S_{n1}(x)}{b_{n1}} - \frac{S_n'(x)}{b_n} \right| > 2M.
\]

Now, there exists an index $j \in \{1, \ldots, n1\}$ such that $g_j = \max\{g_1, \ldots, g_{n1}\}$ at $x$, and so $S_j(x) = S_{j-1}(x)$. Since the $g_i$ are positive, we have that

\[
S_{j-1}(x) = S_j'(x) \leq S_{n1}'(x) < b_{n1}(1 + \varepsilon) < 2b_{n1} < \frac{\max\{g_1, \ldots, g_{n1}\}(x)}{M} = \frac{g_j(x)}{M}.
\]

and hence

\[
M < \frac{g_j(x)}{S_{j-1}(x)}.
\]

This implies that

\[
\limsup_{n \to \infty} \frac{g_n(x)}{S_{n-1}(x)} = \infty
\]
and so
\[
\limsup_{n \to \infty} \frac{\log |I_n|}{\log |I_{n-1}|} = \infty
\]
as we wanted to prove. □

5. Computing the Hausdorff dimension

With the tools developed in the previous sections, we proceed with the dimension computations.

Now we prove an upper bound for \( \dim_H \mu \). This bound is related to the tail decay ratio \( \hat{s} \).
We prove two necessary lemmas to give the desired bound. The first lemma shows that \( \{ p_n \} \) decays slower than any polynomial, while the second lemma, shows the existence of \( \hat{s} \) and that \( \hat{s} = 0 \) for Gauss-like maps.

Lemma 5.1. Suppose that the decay ratio exists and it is equal to \( s \), the sequence \( \{ r_n \} \) decays polynomially and the measure \( \mu \) has infinite entropy. Then for all \( \delta > 0 \), there exist constants \( C, n_0 \) such that
\[
p_n \geq \frac{C}{n^{1+\delta}}
\]
for all \( n \geq n_0 \).

Proof. Let \( \alpha > 0 \) be the polynomial decay of \( r_n \). Then by proposition 3.6, \( s = s_{\infty} = 1/\alpha \), we can take \( \epsilon > 0 \) small enough so that \( \epsilon \alpha + \epsilon s + \epsilon^2 < \delta \). Then there exists \( C > 0 \) and \( n_0 \in \mathbb{N} \) such that
\[
\frac{C}{n^{\alpha + \epsilon}} \leq r_n
\]
\[
\log r_n^{\epsilon + \epsilon} \leq \log p_n
\]
for all \( n \geq n_0 \). This implies that
\[
\frac{C^{\epsilon + \epsilon}}{n^{1+\delta}} \leq \frac{C^{\epsilon + \epsilon}}{n^{\alpha + \epsilon + \epsilon + \epsilon}} \leq p_n
\]
for all \( n \geq n_0 \) as we wanted. □

Lemma 5.2. Under the same assumptions of the previous lemma, the tails decay ratio \( \hat{s} \) (definition 2.4) exists and is equal to zero.

Proof. By the lemma above, for \( \delta > 0 \), there are constants \( C, n_0 \) such that
\[
p_n \geq \frac{C}{n^{1+\delta}}
\]
for all \( n \geq n_0 \). This implies that
\[
\sum_{n=n_0}^{\infty} p_n \geq \frac{C}{\delta n^\delta}
\]
for $n \geq n_0$. On the other hand, if we take $\epsilon < \alpha - 1$, there exists $n_1$ such that

$$r_n \leq \frac{C}{n^{\alpha - \epsilon}}$$

for $n \geq n_1$ and consequently,

$$\sum_{m=n}^{\infty} r_m \leq \frac{C}{(\alpha - \epsilon - 1)n^{\alpha - \epsilon - 1}}$$

for $n \geq n_1$. Hence

$$\log \sum_{m=n}^{\infty} p_m \leq \frac{\log C - \log \delta - \delta \log n}{\log C - \log(\alpha - \epsilon - 1) - (\alpha - \epsilon - 1) \log n}$$

for $n \geq \max\{n_0, n_1\}$. This implies that

$$\limsup_{n \to \infty} \frac{\log \sum_{m=n}^{\infty} p_m}{\log \sum_{m=n}^{\infty} f_m} \leq \frac{\delta}{(\alpha - \epsilon - 1)}.$$

Letting $\delta \to 0$ we conclude the result.

Now we can compute the lower local dimension, and consequently, obtain the Hausdorff dimension of the measure.

**Proposition 5.3.** Suppose $T$ is a Gauss-like map and $\mu$ is an infinite entropy Gibbs measure with controlled decay. Then

$$\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = 0$$

for $\mu$ almost every $x \in I$.

**Proof.** Let $x$ be a point where theorems 3.4, 4.2 and 4.4 hold (such set is of full measure). Given such $x$ and $n \in \mathbb{N}$, take

$$r_n = \left| \bigcup_{m=0}^{\infty} f_n^m(x) \right|,$$

where $f_n^m(x) = I(a_1(x), \ldots, a_{n-1}(x), a_n(x) + m)$. Then

$$\bigcup_{m=0}^{\infty} f_n^m(x) \subseteq B(x, r_n)$$

and so

$$\frac{\log \mu(B(x, r_n))}{\log r_n} \leq \frac{\log \mu \left( \bigcup_{m=0}^{\infty} f_n^m(x) \right)}{\log \left( \bigcup_{m=0}^{\infty} f_n^m(x) \right)}.$$

Note now that the above inequality can be expressed in terms of the sequences $\{p_n\}, \{r_n\}$ using lemma 3.3.
We get that the tail decay
\[ \log \mu \left( \bigcup_{m=0}^{\infty} I_n^m(x) \right) \geq \sum_{k=1}^{n-1} \log p_{a_k} + \log \left( \sum_{m=0}^{\infty} p_{a_n+m} \right) - nG_1 - G_2 \]
and
\[ \log \left| \bigcup_{m=0}^{\infty} I_n^m(x) \right| \leq \sum_{k=1}^{n-1} \log r_{a_k} + \log \left( \sum_{m=0}^{\infty} r_{a_n+m} \right) + nD_1 + D_2 \]
where \( G_1, G_2 \) are constants arising from the Gibbs property and the finite first variation of the potential, and \( D_1, D_2 \) are constants arising from the bounded distortion property. Thus, we have
\[ \log \frac{\mu(B(x, r_n))}{\log r_n} \leq \sum_{k=1}^{n-1} \log p_{a_k} + \log \left( \sum_{m=0}^{\infty} p_{a_n+m} \right) - nG_1 - G_2. \]

For \( n \) large enough, we have that
\[ -\log \sum_{m=0}^{\infty} p_{a_n+m} < \epsilon \]
and
\[ -\epsilon + s < -\sum_{k=1}^{n-1} \log r_{a_k} < s + \epsilon. \]

Thus, if \( a_n \) is large enough, we have
\[ \log \frac{\mu(B(x, r_n))}{\log r_n} \leq (s + \epsilon) \left( \sum_{k=1}^{n-1} \log r_{a_k} \right) + \epsilon \log \left( \sum_{m=0}^{\infty} r_{a_n+m} \right) + nD_1 + D_2. \]

If \( \alpha > 1 \) is the polynomial decaying ratio of \( \{r_n\} \), then by lemma 3.8 we get that the tail decay asymptotic of \( \sum_{m=0}^{\infty} r_{a+n} \) is \( \alpha - 1 \). We can then rewrite the above inequality as
\[ \log \frac{\mu(B(x, r_n))}{\log r_n} \leq (s + \epsilon) \left( \sum_{k=1}^{n-1} \log r_{a_k} + \log K(\alpha - 1) \log (r_{a_k}) + nD_1 + D_2. \]
where \( K \) is the constant implied in the tail asymptotic for \( \{r_n\} \). By theorems 4.4 and 4.2, we can take an increasing subsequence \( a_{n_k} \) so that
\[ \lim_{k \to \infty} -\log r_{a_{n_k}} = \infty, \]
\[ \lim_{k \to \infty} -\frac{1}{n_k} \log r_{a_{n_k}} = \infty. \]

We get then
\[ \lim_{k \to \infty} \log \frac{\mu(B(x, r_{n_k}))}{\log r_{n_k}} \leq \epsilon \]
Letting \( \epsilon \to 0 \) we conclude that \( d(x) = 0 \) as we wanted. \( \square \)

From the above result, we can conclude that for such measures, \( \dim_H \mu = 0. \)
6. Packing dimension

In the previous section we completely determined the Hausdorff dimension of the measures of our interest. Now we proceed to compute the packing dimension. First we give a lower bound for the upper local dimension. The proof uses similar ideas to the proof of proposition 5.3: we choose a particular cover of the ball and use that the Birkhoff sums for the potentials \(- \log \mu_{\alpha 1}, - \log r_{\alpha 1}\) grow faster than linear.

**Proposition 6.1.** Suppose \(T\) is a Gauss-like map and \(\mu\) is an infinite entropy Gibbs measure with controlled decay. Then

\[
\limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq s
\]

for \(\mu\) almost every \(x \in I\).

**Proof.** By Birkhoff’s Ergodic theorem, we have that

\[
\lim_{n \to \infty} \frac{f_{n,k}}{n} = p_1
\]

almost everywhere, where \(f_{n,k}\) is as defined in the proof of theorem 3.4. Lemma 3.2 and theorem 3.4 hold in a set of full measure as well. We pick a point \(x\) where the three results hold. Since \(p_1 < 1\), we can pick a subsequence \(k_n \to \infty\) such that \(\alpha_{k_n} \neq 1\) for every \(n\). Then, for all \(n\), take \(r_n = \min\{|I_{k_n}|, |I'_{k_n}|, |I''_{k_n}|\} = |I''_{k_n}|\). Here we denote \(I'_n = I(a_n, a_{n-1}, a_n + 1)\) and \(I''_n = I(a_n, a_{n-1}, a_n - 1)\) whenever \(a_n > 1\). This choice of \(r_n\) implies that \(B(x, r_n) \subseteq I'_{k_n} \cup I_{k_n} \cup I''_{k_n}\). From the Gibbs property and the fact that \(\varphi(x_n), \varphi(x_{n+1})\), and \(r_n, r_{n+1}\) are comparable, it follows that there are constants \(C_1, C_2 > 0\) such that \(\mu(I'_n \cup I_{k_n} \cup I''_n) \leq C_1 \mu(I_n)\) and \(|I''_n| \geq C_2 |I_n|\) for every \(n\). Using this and lemma 3.3 we have that

\[
\frac{\log \mu(B(x, r_n))}{\log r_n} \geq \frac{\log(C_1 \mu(I_{k_n}))}{\log(C_2 |I_{k_n}|)}
\]

\[
= \frac{\log C_1 + k_n G_1 + G_2 \sum_{i=1}^{k_n} \log \mu_{\alpha_1}}{\log C_2 - D_2 - k_n \log D_1 + \sum_{i=1}^{k_n} \log r_{\alpha_i}}
\]

By lemma 3.2 and theorem 3.4, the last expression converges to \(s\) as desired. \(\square\)

Giving an upper bound for the upper local dimension requires a more involved analysis of the geometric structure of the partition and its relation to the geometry of the balls. We will need the following lemma:

**Lemma 6.2.** Suppose that \(\{r_n\}\) decays polynomially with degree \(\alpha > 1\). Then, for every \(0 < \delta < \min\{1/3, (\alpha - 1)/(\alpha + 1)\}, 0 < \eta < 1/2\) there exists \(k_0 \in \mathbb{N}\) such that

\[
\frac{\log \sum_{m=k}^{n+k} p_m}{\log \sum_{m=k+1}^{n+k+1} r_m} \leq \frac{1 + \delta}{\alpha - \delta} + \eta
\]

for all \(k \geq k_0\) and \(n \in \mathbb{N}\).

**Proof.** Recall that for such sequence \(\{r_n\}\), we have that \(s = 1/\alpha\). Fix \(0 < \delta < \min\{1/3, s(\alpha - 1)/(\alpha + 1)\}, 0 < \eta < 1/2\). Note that this implies that

\[
\frac{\delta}{\alpha - 1 - \delta} < s = \frac{1}{\alpha} < \frac{1 + \delta}{\alpha - \delta}.
\]
Now, since
\[
\lim_{k \to \infty} \frac{(1 + \delta) \log 2 + \delta \log k}{\log(\alpha - 1 - \delta) + (\alpha - 1 - \delta) \log(k - 2)} = \frac{\delta}{\alpha - 1 - \delta} < \frac{1 + \delta}{\alpha - \delta}
\]
and
\[
\lim_{k \to \infty} \frac{(1 + \delta) \log(2k)}{(\alpha - \delta) \log(k - 1) - \log 3} = \frac{1 + \delta}{\alpha - \delta},
\]
we can find \(k_0 \in \mathbb{N}\) such that
\[
\frac{(1 + \delta) \log 2 + \delta \log k}{\log(\alpha - 1 - \delta) + (\alpha - 1 - \delta) \log(k - 2)} < \frac{1 + \delta}{\alpha - \delta} + \eta
\]
and
\[
\frac{(1 + \delta) \log(2k)}{(\alpha - \delta) \log(k - 1) - \log 3} < \frac{1 + \delta}{\alpha - \delta} + \eta
\]
for all \(k \geq k_0\). It can be proved using calculus that for \(\delta < (\alpha - 1)/2\), the inequality
\[
(1 + \delta) \log(2k) \leq (\alpha - \delta) \log(k - 1) - \log 3
\]
holds for sufficiently large \(k\), so we can take \(k_0\) large enough so that this holds. Finally, we can take \(k_0\) large enough so that we also have
\[
r_k \leq \frac{1}{k^{1+\delta}}
\]
\[
\frac{1}{k^{1+\delta}} \leq p_k
\]
for all \(k \geq k_0\). Let \(n \in \mathbb{N}\). We divide in two cases:

**Case 1**: \(n \geq k\). Then
\[
\sum_{m=k}^{n+k} p_m \geq \frac{n}{(2k)^{1+\delta}} \geq \frac{1}{2^{1+\delta}k^\delta}
\]
and
\[
\sum_{k-1}^{n+k+1} r_m \leq \sum_{m=k-1}^{n+k+1} \frac{1}{m^{\alpha-\delta}} \leq \sum_{m=k-1}^{n+k+1} \frac{1}{m^{\alpha-\delta}} \leq \frac{1}{\alpha - 1 - \delta} \left( \frac{1}{(k - 2)^{\alpha - 1 - \delta}} \right)
\]
for all \(k \geq k_0\). Then
\[
\frac{\log \sum_{m=k}^{n+k} p_m}{\log \sum_{m=k-1}^{n+k+1} r_m} \leq \frac{(1 + \delta) \log 2 + \delta \log k}{\log(\alpha - 1 - \delta) + (\alpha - 1 - \delta) \log(k - 2)} \leq \frac{1 + \delta}{\alpha - \delta} + \eta
\]
for all \(k \geq k_0\).

**Case 2**: \(n < k\). Then
\[
\sum_{m=k}^{n+k} p_m \geq \frac{n + 1}{(2k)^{1+\delta}}
\]
\[ \sum_{k=1}^{n+k+1} r_m \leq \sum_{m=k-1}^{n+k+1} \frac{1}{m^{\alpha - \delta}} \leq \frac{n+3}{(k-1)^{\alpha - \delta}} \leq 3 \frac{(n+1)}{(k-1)^{\alpha - \delta}}. \]

Hence

\[ \frac{\log \sum_{m=k}^{n+k+1} \rho_m}{\log \sum_{m=k-1}^{n+k+1} r_m} \leq \frac{(1+\delta) \log (2k) - \log (n+1)}{(\alpha - \delta) \log (k-1) - \log 3 - \log (n+1)}. \]

We use the following lemma:

**Lemma 6.3.** For \( a, b, c > 0 \) such that \( a - c, b - c > 0 \), we have that

\[ \frac{a-c}{b-c} \leq \frac{a}{b} \]

if and only if \( b \geq a \).

We can use this with \( a = (1+\delta) \log (2k), b = (\alpha - \delta) \log (k-1) - \log 3 \) and \( c = \log (n+1) \).

This implies that

\[ \frac{\log \sum_{m=k}^{n+k+1} \rho_m}{\log \sum_{m=k-1}^{n+k+1} r_m} \leq \frac{(1+\delta) \log (2k)}{(\alpha - \delta) \log (k-1) - \log 3} \leq \frac{1 + \delta}{\alpha - \delta} + \eta. \]

for all \( k \geq k_0 \), as we wanted to prove. \( \square \)

With the previous lemma, we can now prove the upper bound for the upper local dimension. The proof is based on carefully choosing the covers of the balls; such covers must be fine enough so they are not affected by theorem 4.2. This means that we want to cover the ball with cylinders of the same scale, otherwise, the cover would yield trivial bounds.

**Proposition 6.4.** Suppose \( T \) is a Gauss-like map and \( \mu \) is an infinite entropy Gibbs measure with controlled decay. Then

\[ \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \leq s \]

for \( \mu \) almost every \( x \in I \).

**Proof.** Let \( x \) be a point where theorem 3.4 and lemma 3.2 applied to \( f = -\log r_{a_1} \) hold. Given \( r > 0 \), there exists a unique natural number \( n = n(r) \) such that

\[ |I_n(x)| < r \leq |I_{n-1}(x)|. \]

Note that \( n \to \infty \) as \( r \to 0 \). Let \( \delta > 0 \) and \( \eta \) as in lemma 6.2. Then there exists \( k_0 \in \mathbb{N} \) such that

\[ \frac{\log \sum_{m=k}^{n+k+1} \rho_m}{\log \sum_{m=k-1}^{n+k+1} r_m} \leq \frac{1 + \delta}{\alpha - \delta} + \eta. \]

for all \( k \geq k_0 \). Recall that by \( I_{n,m}^{\infty}(x) \) we denote the cylinder \( I(a_1, \ldots, a_{n-1}, a_n - m) \), where \( (a_n) \) is the sequence coding \( x \) and \( m < a_n \). We separate the proof in two cases:

Case 1:

\[ I(a_1, \ldots, a_{n-1}, k_0) \subset B(x,r). \]

5379
In this case, using lemma 3.3 we have that

\[
\log(\mu(B(x, r))) \geq \log(\mu(I(a_1, \ldots, a_{n-1}, k_0))) \geq \sum_{k=1}^{n-1} \log p_{a_k} + \log p_{k_0} - nG_1 - G_2.
\]

We get then

\[
\frac{\log \mu(B(x, r))}{\log r} \leq \frac{\sum_{k=1}^{n-1} \log p_{a_k} + \log p_{k_0} - nG_1 - G_2}{\sum_{k=1}^{n-1} \log r_{a_k} + nD_1 + D_2} \leq \frac{(s + \delta) \sum_{k=1}^{n-1} \log r_{a_k} + \log p_{k_0} - nG_1 - G_2}{\sum_{k=1}^{n-1} \log r_{a_k} + nD_1 + D_2}.
\]

(6.1)

Case 2:

\[
I(a_1, \ldots, a_{n-1}, k_0) \not\subset B(x, r).
\]

This implies that there exists \(k_1 \in \mathbb{N}\) such that

\[
\bigcup_{m=1}^{k_1-1} f_n^m(x) \subset B(x, r),
\]

\[
\bigcup_{m=0}^{k_1} I_n^m(x) > r
\]

and consequently

\[
\log(\mu(B(x, r))) \geq \sum_{k=1}^{n-1} \log p_{a_k} + \log \left(\sum_{k=1}^{k_1-1} p_{a_{n-k}}\right) - nG_1 - G_2
\]

\[
\log r \leq \sum_{k=1}^{n-1} \log r_{a_k} + \log \left(\sum_{k=1}^{k_1} r_{a_{n-k}}\right) + nD_1 + D_2.
\]

We obtain then

\[
\frac{\log(\mu(B(x, r)))}{\log r} \leq \frac{\sum_{k=1}^{n-1} \log p_{a_k} + \log \left(\sum_{k=1}^{k_1-1} p_{a_{n-k}}\right) - nG_1 - G_2}{\sum_{k=1}^{n-1} \log r_{a_k} + \log \left(\sum_{k=1}^{k_1} r_{a_{n-k}}\right) + nD_1 + D_2}
\]

Using inequality (6.2)

\[
\frac{\log(\mu(B(x, r)))}{\log r} \leq \frac{\sum_{k=1}^{n-1} \log p_{a_k} + \left(\frac{1+\delta}{\alpha - \delta} + \eta\right) \log \left(\sum_{k=0}^{k_1} r_{a_{n-k}}\right) - nG_1 - G_2}{\sum_{k=1}^{n-1} \log r_{a_k} + \log \left(\sum_{k=1}^{k_1} r_{a_{n-k}}\right) + nD_1 + D_2}.
\]

For \(\delta > 0\), there exist \(n_0 \in \mathbb{N}\) such that

\[
\frac{-\sum_{k=1}^{n-1} \log q_{a_k}}{\sum_{k=1}^{n-1} \log r_{a_k}} < s + \delta
\]
for all $n \geq n_0$. We obtain

$$\frac{\log(\mu(B(x,r)))}{\log r} \leq \frac{(s+\delta)\sum_{k=1}^{n-1} \log r_{a_k} + \left(\frac{1+\delta}{\alpha - \delta} + \eta\right) \log \left(\sum_{k=0}^{2^k} r_{a_{n-k}}\right) - n G_1 - G_2}{\sum_{k=1}^{n-1} \log r_{a_k} + \log \left(\sum_{k=0}^{2^k} r_{a_{n-k}}\right) + n D_1 + D_2}$$

$$\leq \max \left\{(s+\delta), \left(\frac{1+\delta}{\alpha - \delta} + \eta\right)\right\} \times \frac{\sum_{k=1}^{n-1} \log r_{a_k} + \log \left(\sum_{k=0}^{2^k} r_{a_{n-k}}\right) - n G_1 - G_2}{\sum_{k=1}^{n-1} \log r_{a_k} + \log \left(\sum_{k=0}^{2^k} r_{a_{n-k}}\right) + n D_1 + D_2}.

(6.2)$$

By lemma 3.2 we have that the right hand side of (6.1) and (6.2) converge to

$$(s+\delta), \max \left\{(s+\delta), \left(\frac{1+\delta}{\alpha - \delta} + \eta\right)\right\}$$

respectively. We conclude that

$$\limsup_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r} \leq \max \left\{(s+\delta), \left(\frac{1+\delta}{\alpha - \delta} + \eta\right)\right\}.$$

Letting $\delta \to 0$ and $\eta \to 0$, we obtain the desired result. □

Corollary 6.5. For an infinite entropy Gibbs measure $\mu$ with infinite entropy and controlled decay, associated to a Gauss-like map, we have that $0 = \underline{d}(x) < s = \overline{d}(x)$ for almost every point, and hence $\mu$ is not exact dimensional.

With this we have found the almost sure behaviour of the local dimensions, and hence, we have obtained values for both the packing and the Hausdorff dimension.

7. Final remarks

Theorem 1 implies that for maps such that $\{r_n\}$ decays polynomially, the Hausdorff dimension of ergodic invariant measures with infinite entropy is equal to zero under mild independence and regularity assumptions on the measure.

**Question 1.** Is there an ergodic invariant measure $\mu$ for a Gauss-like map with $h_\mu = \lambda_\mu = \infty$, and $\dim_H \mu > 0$?

We believe that the infinite entropy condition and the polynomial decay of the size of the partition forces the Hausdorff dimension to drop to zero. We also formulate two questions for a more general case:

**Question 2.** What can be said about the almost sure value of the symbolic dimension when $\mu$ is only assumed to be ergodic?

**Question 3.** What can be said about $\dim_H \mu$ when $\mu$ is only assumed to be ergodic?

The main difficulty with questions 2 and 3 is that our methods rely on the asymptotic independence of the digits in the symbolic space. This implies that we can write the measure and diameter of cylinders in the form of Birkhoff sums, allowing us to use ergodic theoretic methods to study the almost sure behaviour of such sums.

For measures which do not satisfy any kind of independence assumption, we are not able to use such techniques.
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