Competing orders in high-$T_c$ superconductors

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Abstract

Within a (2+1)-dimensional U(1) gauge field theory, after calculating the Dyson-Schwinger equation for fermion self-energy we find that chiral symmetry breaking (CSB) occurs if the gauge boson has a very small mass but is suppressed when the mass is larger than a critical value. In the CSB phase, the fermion acquires a dynamically generated mass, which leads to antiferromagnetic (AF) long-range order. Since in the superconducting (SC) state the gauge boson acquires a finite mass via Anderson-Higgs mechanism, we obtain a field theoretical description of the competition between the AF order and the SC order. As a compromise of this competition, there is a coexistence of these two orders in the bulk material of cuprate superconductors.
Understanding the competition of various ground states of cuprate superconductors is one of the central problems in modern condensed matter physics. At half-filling, the cuprate superconductor is a Mott insulator with antiferromagnetic (AF) long-range order. When the doping concentration increases, AF order rapidly disappears and superconducting (SC) order emerges as the ground state. How to describe this evolution from AF order to SC order upon doping in a simple way is a very interesting problem.

Since the discovery of high temperature superconductors, extensive theoretical and experimental work appeared to investigate the possibility of spin-charge separation [1,2], which states that the spin and charge degrees of freedom might be carried by different quasiparticles, i.e. charge carrying holons and spin carrying spinons. The recently observed breakdown of the Wiedemann-Franz (WF) law in underdoped cuprates [3] confirms its existence [4]. In spin-charge separated theories, the pseudogap in the normal state is the spin gap arising from pairing of spinons. Superconductivity is achieved when the holons Bose condense into a macroscopic quantum state, which indicates that the vortex of high temperature superconductors carries a double flux quantum $hc/e$. However, so far experiments found only $hc/2e$ vortices [5]. Recently Lee and Wen [6] showed that SU(2) slave-boson theory [7] can naturally lead to a stable $hc/2e$ vortex inside which a finite pseudogap exists [8].

In this Letter, we propose that the competition between the AF order and the SC order can be modeled by a competition between chiral symmetry breaking (CSB) and the mass of a gauge boson within an effective theory of the SU(2) formulation. After calculating the Dyson-Schwinger (DS) equation for fermion self-energy, we find that CSB happens if the gauge boson mass is zero or very small but is destroyed when the gauge boson mass is larger than
a finite critical value. We then show that CSB corresponds to the formation of long-range AF order by calculating AF correlation function. Because the gauge boson mass is generated via the Anderson-Higgs mechanism in the superconducting state, there is a competition between AF order and SC order.

We start our discussion from the staggered flux state in the SU(2) slave-boson treatment of the $t$-$J$ model [7]. In this paper we adopt the following effective model of the underdoped cuprates that consists of massless fermions, bosons and a U(1) gauge field [4]

$$\mathcal{L}_F = \sum_{\sigma=1}^{N} \bar{\psi}_\sigma v_{\sigma,\mu} (\partial_\mu - ia_\mu) \gamma_\mu \psi_\sigma + ||(\partial_\mu - ia_\mu) b||^2 + V(|b|^2). \quad (1)$$

The Fermi field $\psi_\sigma$ is a $4 \times 1$ spinor. The $4 \times 4$ $\gamma_\mu$ matrices obey the algebra, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and for simplicity, we let $v_{\sigma,\mu} = 1 \ (\mu, \nu = 0, 1, 2)$. $b = (b_1, b_2)$ is a doublet of scalar fields representing the holons.

In Lagrangian (1) the fermions are massless, so it respects the chiral symmetries $\psi \rightarrow \exp(i\theta \gamma_3,5)\psi$, with $\gamma_3$ and $\gamma_5$ two $4 \times 4$ matrices that anticommute with $\gamma_\mu \ (\mu = 0, 1, 2)$. If the fermion flavor is below a critical number $N_c$ the strong gauge field can generate a finite mass for the fermions [9-11], which breaks the chiral symmetries. This phenomenon is called chiral symmetry breaking (CSB), which has been studied for many years in particle physics as a possible mechanism to generate fermion mass without introducing annoying Higgs particles. Previous study found that CSB happens when the holons are absent [9-11] and when the holons are not Bose condensed [4].

We now would like to consider the superconducting phase where boson $b$ acquires a nonzero vacuum expectation value, i.e., $\langle b \rangle \neq 0$. The nonzero $\langle b \rangle$ spontaneously breaks gauge symmetry of the theory and the gauge boson acquires a finite mass $\xi$ via Anderson-Higgs mechanism. CSB is a low-energy
phenomenon because (2+1)-dimensional U(1) gauge field theory is asymptotically free and only in the infrared region the gauge interaction is strong enough to cause fermion condensation. This requires the fermions be apart from each other. In the superconducting state, the gauge boson becomes massive and can not mediate a long-range interaction. Intuitively, a finite gauge boson mass is repulsive to CSB which is achieved by the formation of fermion-anti-fermion pairs. To determine whether CSB still occurs in the SC state, quantitative calculations should be carried out.

CSB is a nonperturbative phenomenon and can not be obtained within any finite order of the perturbation expansion. The standard approach to this problem is to solve the self-consistent DS equation for the fermion self-energy.

The inverse fermion propagator is written as $S^{-1}(p) = i \gamma \cdot p A(p^2) + \Sigma(p^2)$, $A(p^2)$ is the wave-function renormalization and $\Sigma(p^2)$ the fermion self-energy. If the DS equation has only trivial solutions, the fermions remain massless and the chiral symmetries are not broken. However, not all nontrivial solutions correspond to a dynamically generated fermion mass [12]. If a nontrivial solution of the DS equation satisfy an additional squarely integral condition [12,13], it then signals the appearance of CSB.

The DS equation is

$$\Sigma(p^2) = \int \frac{d^3k}{(2\pi)^3} \frac{\gamma^\mu D_{\mu\nu}(p-k)\Sigma(k^2)\gamma^\nu}{k^2 + \Sigma^2(k^2)}. \tag{2}$$

In this paper, we use the following propagator of the massive gauge boson

$$D_{\mu\nu}(p-k) = \frac{8}{(N+1)(|p-k|+\eta)} \left( \delta_{\mu\nu} - \frac{(p-k)_{\mu}(p-k)_{\nu}}{(p-k)^2} \right), \tag{3}$$

where $\eta$ reflects the effect of the gauge boson mass $\xi$ ($\eta = 8\xi^2/(N+1)$). After performing angular integration and introducing an ultraviolet cutoff $\Lambda$, we have

$$\Sigma(p^2) = \lambda \int_0^\Lambda dk \frac{k\Sigma(k^2)}{k^2 + \Sigma^2(k^2)}.$$
where \( \lambda = 4/(N + 1)\pi^2 \) serves as an effective coupling constant. Here, for simplicity, we assumed that \( A(p^2) = 1 \). For \( \eta = 0 \), this assumption leads to \( N_c = 32/\pi^2 \) [9]. More careful treatment [10] calculated the DS equation for \( \Sigma(p^2)/A(p^2) \) considering higher-order corrections and found that the critical behaviour is qualitatively unchanged. Since assuming that \( A(p^2) = 1 \) can significantly simplify the calculations and the higher-order corrections are small we expect the result derived from DS equation (4) is reliable.

If we do not introduce an ultraviolet cutoff (\( \Lambda \to \infty \)), the critical behavior of Eq.(4) is completely independent of \( \eta \), which can be easily seen by making the scale transformation, \( p \to p/\eta \), \( k \to k/\eta \) and \( \Sigma \to \Sigma/\eta \). This scale invariance is destroyed by an ultraviolet cutoff \( \Lambda \) which is natural because the theory (1) was originally defined on lattices. Once an ultraviolet cutoff is introduced, the solution \( \Sigma(p^2) \) then automatically satisfies the squarely integrable condition [13] and hence is a symmetry breaking solution. Theoretical analysis implies that the critical fermion number \( N_c \) of Eq.(4) should depend on \( \Lambda/\eta \). Actually, we have showed that when the gauge boson has a very large mass, say \( \eta \gg \Lambda \), the DS equation has no physical solutions. If the gauge boson is massless, the last term in the kernel of Eq.(4) can be dropped off, leaving an equation that has a critical number \( 32/\pi^2 \). However, at present we do not have a detailed dependence of the critical number \( N_c \) on \( \Lambda/\eta \). In particular, it is not clear whether \( N_c \) is a monotonous function of \( \Lambda/\eta \) or not. To settle this problem, we should solve the DS equation quantitatively.

The most intriguing property of the DS equation is that it is a (or a set of) nonlinear integral equation which exhibits many interesting phenomena and at the same time is very hard to be studied analytically and numerically.
In this paper, based on bifurcation theory we are able to obtain the phase transition point of the nonlinear DS equation by studying the eigenvalue problem for its associated Frèchet derivative. This scheme [14,15] simplifies the numerical computation and also can lead to a reliable bifurcation point. We will calculate the eigenvalues of the linearized integral equation using parameter imbedding method, which can eliminate the uncertainty that is unavoidable when carrying out numerical computation in the vicinity of a singularity. The details of the computation will be given elsewhere.

Making Frèchet derivative of the nonlinear equation (4), we have the following linearized equation

\[
\Sigma(p^2) = \lambda \int_0^{\Lambda/\eta} dk \Sigma(k^2) \frac{1}{pk} \left( p + k - |p - k| - \ln \left( \frac{p + k + 1}{|p - k| + 1} \right) \right)
\]  

(5)

where for calculational convenience we made the transformation \( p \rightarrow \frac{p}{\eta}, k \rightarrow \frac{k}{\eta} \) and \( \Sigma \rightarrow \frac{\Sigma}{\eta} \). The smallest eigenvalue \( \lambda_c \) of this equation is just the bifurcation point from which a nontrivial solution of the DS equation (4) branches off. For \( \lambda > \lambda_c \), the DS equation has nontrivial solutions and CSB happens. The ultraviolet cutoff \( \Lambda \) is provided by the lattice constant and hence is fixed. We can obtain the relation of \( N_c \) and \( \eta \) by calculating the critical coupling \( \lambda_c \) for different values of \( \Lambda/\eta \).

In order to get the smallest eigenvalue \( \lambda_c \), we first use parameter imbedding method [14,15] to convert Eq.(5) into two differential-integral equations with \( \lambda \) their variable. After choosing an appropriate contour in the complex \( \lambda \)-plane and integrating with respect to the parameter \( \lambda \), we finally obtain the exact eigenvalue \( \lambda_c \).

Our numerical result is presented in Fig.(1). The most important result is that the critical fermion number \( N_c \) is a monotonously increasing function of \( \Lambda/\eta \). It conforms our expectation that a finite mass of the gauge boson is repulsive to CSB. For small \( \Lambda/\eta \) the critical number \( N_c \) is smaller than
physical fermion number 2, so CSB does not happen. When \( \Lambda/\eta \) increases, the critical number \( N_c \) increases accordingly and finally becomes larger than 2 at about \( \Lambda/\eta_c = 10^8 \). When \( \Lambda/\eta \) continues to increase, \( N_c \) increases more and more slowly and finally approaches a constant value 2.15. Thus we can conclude that CSB takes place when the gauge boson mass is zero and very small but is destroyed when the gauge boson mass is larger than a critical value.

We next show that CSB corresponds to the formation of AF long-range order [16] by calculating the AF spin correlation function. At the mean field level, the AF correlation function is defined as [17]

\[
\langle S^+ S^- \rangle_0 = -\frac{1}{4} \int \frac{d^3k}{(2\pi)^3} Tr \left[ G_0(k)G_0(k + p) \right],
\]

where \( G_0(k) \) is the fermion propagator. For massless fermions, \( G_0(k) = -\frac{i}{\gamma \mu k^\mu} \), we have

\[
\langle S^+ S^- \rangle_0 = -\frac{|p|}{16},
\]

and the AF correlation is largely lost. This result is natural if we look back to our starting point, i.e., the staggered flux mean field phase which is based on the resonating valence bond (RVB) picture proposed by Anderson [1]. The RVB state is actually a liquid of spin singlets, so it has only short range AF correlation. Since a Néel order was observed in experiments, we should find a way to get back the long-range spin correlation. One possible way is to go beyond the mean field treatment and include the gauge fluctuations which is necessary to impose the no double occupation constraint. As discussed above, when strong gauge interaction is taken into account, the system undergoes a chiral instability and the massless fermions acquire a finite mass. Although the dynamically generated fermion mass depends on the 3-momentum, here, for simplicity, we assume a constant mass \( \Sigma_0 \) for the fermions. This approx-
imation is valid because we only care about the low-energy property and \( \Sigma(0) \) is actually a constant. Then the propagator for the massive fermions is written as

\[
G_0(k) = \frac{-i\gamma_\mu k^\mu}{k^2 + \Sigma_0^2},
\]

which leads to

\[
\langle S^+ S^- \rangle_0 = -\frac{1}{4\pi} \left( \Sigma_0 + \frac{p^2 + 4\Sigma_0^2}{2|p|} \arcsin \left( \frac{p^2 + 4\Sigma_0^2}{p^2 + 4\Sigma_0^2} \right)^{1/2} \right).
\]

This spin correlation behaves like \(-\Sigma_0^2/2\pi \) as \( p \to 0 \) and we have long-range AF correlation when CSB happens.

We should emphasize that CSB is necessary for producing AF long-range order. If we only include gauge fluctuations into the staggered spin correlation (6) while keeping the fermions massless, then [17]

\[
\langle S^+ S^- \rangle_{GF} = -\frac{8}{12\pi^2(N+1)} |p| \ln \left( \frac{\Lambda^2}{p^2} \right)
\]

which approaches zero at the limit \( p \to 0 \). Wen and coworkers [17] used to claim that long-range AF correlation can be obtained by reexponentiating the spin correlation function based on the conclusion that the gauge field cannot generate a finite mass for fermions and hence is a marginal perturbation. This result is derived by means of perturbation expansion. However, CSB is a nonperturbative phenomenon and whether the gauge field generates a finite mass for the massless fermions can only be settled by investigating the self-consistent DS equation for the fermion self-energy. Studies of the DS equation show that gauge field is strong enough to generate a mass for fermions, so it is not a marginal perturbation. Moreover, the AF long-range order breaks the rotational symmetry accompanying a massless Goldstone boson (spin wave). This is difficult to understand if the AF order is induced by a marginally perturbative gauge field, rather than by a spontaneous symmetry breaking.
In the superconducting state, the gauge boson mass $\xi$ is proportional to the superfluid density $\rho_s$, i.e., $\xi \sim \rho_s$. Since $\eta \sim \xi^2$ and the superfluid density in high-$T_c$ superconductors is proportional to doping concentration $\delta$, we have $\eta \sim \delta^2$. Then whether CSB exists at $T \to 0$ depends on the doping level and the knob that tunes the different orders is the holon degree of freedom. At half-filling and very low $\delta$ CSB occurs because the holons are not Bose condensed and the gauge boson is massless. As $\delta$ increases the cuprate becomes a superconductor which gives the gauge boson a finite mass. When $\delta$ is larger than a critical value $\delta_c$, the mass of the gauge boson ($\xi_c \sim \delta_c$) is large enough to suppress CSB. Thus we obtain a competition between the AF order, which dominates at half-filling and low $\delta$, and the SC order, which dominates at high $\delta$. As a compromise of this competition, there is a coexistence of these two orders in the bulk material of cuprate superconductors for doping concentration between $\delta_c$ and the critical point $\delta_{sc}$ at which superconductivity begins to emerge.

It was generally claimed that the AF order is destroyed by the moving holes because no long-range AF correlation has been observed in cuprates at high doping $\delta$. However, our result [4] indicated that CSB and hence the AF order can coexist with free holons. It seems to us that the AF order is actually destroyed by Bose condensation of holons (or spontaneous gauge symmetry breaking) at low temperature and thermal fluctuations at high temperature (above $T_c$). To find out the true mechanism that destroys the AF order, experiments, preferably scanning tunnelling microscopy (STM) or neutron scattering, should be performed at the $T \to 0$ limit when the superconductivity is suppressed, for example, by strong magnetic fields.

Based on spin-charge separation and CSB, we now have a clear picture of the evolution of zero temperature ground states upon increasing the doping
concentration. For doping concentration less than $\delta_c$, CSB occurs and leads to long-range AF correlation, which is then destroyed by superconducting order for doping concentration higher than $\delta_c$. On the other hand, when superconductivity is completely suppressed by strong magnetic fields, CSB occurs in underdoped cuprates and gives the nodal spinons (originally gapless due to the $d$-wave symmetry of the spin-gap) a finite mass which provides a gap for free fermions to be excited at low temperatures, causing the breakdown of the WF law [4]. Thus the combination of spin-charge separation and CSB gives a unified field theoretical description of both the breakdown of WF law and the competition between long-range AF order and long-range SC order. This is the most notable advantage of our scenario.

Recently, much experimental [18,19] and theoretical [20] effort has been made to the magnetic field induced AF order in the vortex state of cuprate superconductors. In particular, neutron scattering and STM experiments found that AF order appears around the vortex cores where the superfluid is suppressed by magnetic field locally. Our result is consistent with these experiments. Qualitatively, the AF correlation is enhanced in regions where the superfluid density and hence the gauge boson mass becomes smaller than their critical value. Within our scenario, since CSB can coexist with a small gauge boson mass, the length scale for AF order to appear should be larger than the vortex scale, which is consistent with STM experiments [19]. However, to quantitatively explain the experimental data, several subtle problems should be made clear, especially the ratio of gauge boson mass to superfluid density and the detailed distribution of the superfluid density in the vortex state, which are beyond the scope of this paper.

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Figure 1: The dependence of the critical number \( N_c \) on \( \log_{10}(\Lambda/\eta) \).