The running mass $m_s$ at low scale from the heavy-light meson decay constants

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It is shown that a 25(20)% difference between the decay constants $f_{D_s}(f_{B_s})$ and $f_D(f_B)$ occurs due to large differences in the pole masses of the $s$ and $d(u)$ quarks. The values $\eta_D = f_{D_s}/f_D \approx 1.23(15)$, recently observed in the CLEO experiment, and $\eta_B = f_{B_s}/f_B \approx 1.20$, obtained in unquenched lattice QCD, can be reached only if in the relativistic Hamiltonian the running mass $m_s$ at low scale is $m_s(\sim 0.5$ GeV$) = 170 - 200$ MeV. Our results follow from the analytical expression for the pseudoscalar decay constant $f_P$ based on the path-integral representation of the meson Green's function.

Relativistic potential models (RPM) have been successful in their description of light-light and heavy-light (HL) meson spectra [1, 2]. Still there exists a fundamental problem, which remains partly unsolved up to now. It concerns the choice of the quark masses in the kinetic term of the relativistic Hamiltonian, which in different RPMs vary in a wide range. For example, HL mesons were studied with the use of the Dirac equation, taking for the light quark mass $m_n(n = u, d) = 7$ MeV in [3] and 72 MeV in [4], and in the Salpeter equation for the strange quark mass the values $m_s = 419$ MeV in [5] and $m_s = 180$ MeV in [6] have been used. However, in contrast to constituent quark models, where the constituent mass can be considered as a fitting parameter, a fundamental relativistic Hamiltonian has to contain only conventional quark masses—the pole masses. These masses are now well established for heavy and light quarks [7]. They have been used in the QCD string approach giving a

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good description of meson spectra [6, 8, 9]. However, the strange quark mass $m_s$ is still not determined at low scale. At present, owing to the QCD sum rules calculations [10] and lattice QCD [11], $m_s$ is well established at a rather large scale: $m_s(\mu = 2\text{ GeV}) = 90 \pm 10\text{ MeV}$, while in the Hamiltonian approach the mass $m_s$ enters at a lower scale, which is evidently smaller that the scale $\mu_c \approx 1.2\text{ GeV}$ for the $c$ quark. Therefore it is very important to find physical quantities which are very sensitive to $m_s$ at low scale: $\mu_s \leq 1\text{ GeV}$. In this letter we show that such information can be extracted from the analysis of the decay constants of HL mesons, namely, from the ratios $f_{D_s}/f_D$ and $f_{B_s}/f_B$.

Recently, direct measurements of the leptonic decay constants in the processes $D(D_s) \rightarrow \mu \nu$ [12, 13], and $B \rightarrow \tau \mu$ [14, 15] have been reported. In Refs. [12, 13] the CLEO collaboration gives $f_{D_s} = 274(20)\text{ MeV}$ and $f_D = 222.6(20)\text{ MeV}$ with $\eta_D = f_{D_s}/f_D = 1.23(15)$, having reached an accuracy much better than in previous experiments [16]. This central value for $\eta_D$ appears to be larger than in many theoretical predictions which typically lie in the range $1.0 - 1.15$ [17, 18, 19, 20]. Therefore, one can expect that precise measurements of $\eta_D$ and $\eta_B$ in the future can become a very important criterium to distinguish different theoretical models and check their accuracy. In particular, relatively large values

$$\eta_D = 1.25(3), \quad \eta_B = 1.20(3),$$

(1)

have been obtained recently in lattice (unquenched) calculations [21, 22] and also in our paper [6]. In this letter we show that

1. The running mass $m_s(\mu_1)$ at a low scale, $\mu_1 \approx 0.5\text{ GeV}$, can be extracted from the values $\eta_D$ and $\eta_B$, if they are known with high accuracy, $\lesssim 5\%$.

2. The values $\eta_D$ and $\eta_B$, as given in Eq. (1), can be obtained only if the running mass $m_s(\mu_1)$ lies inside the range 170 – 200 MeV. In particular, in the chiral limit, $m_d = m_u = 0$, as well as for $m_d = 8\text{ MeV}$, and for $m_s(\mu_1) = 180\text{ MeV}$, the ratios $\eta_D$ and $\eta_B$ calculated here are

$$\eta_D = 1.25, \quad \eta_B = 1.19.$$

(2)

3. The value $m_s(0.5\text{ GeV})$ satisfies the relation

$$\frac{m_s(0.5\text{ GeV})}{m_s(2\text{ GeV})} \approx 1.97.$$  

(3)

In our analysis we use the analytical expression for the leptonic decay constant in the pseudoscalar (P) channel, derived in Ref. [6] with the use of the path-integral representation
for the correlator $G_P$ of the currents $j_P(x)$: $G_P(x) = \langle j_P(x)j_P(0) \rangle_{\text{vac}}$:

$$J_P = \int G_P(x)d^3x = 2N_c \sum_n \frac{\langle Y_P \rangle_n |\varphi_n(0)|^2}{\omega_{qn}\omega_{Qn}} e^{-M_n T},$$

(4)

where $M_n$ and $\varphi_n(r)$ are the eigenvalues and eigenfunctions of the relativistic string Hamiltonian \cite{23, 24, 25}, while $\omega_{qn}(\omega_{Qn})$ is the average kinetic energy of a quark $q(Q)$ for a given $nS$ state:

$$\omega_{qn} = \langle \sqrt{m_q^2 + p^2} \rangle_{nS}, \quad \omega_{Qn} = \langle \sqrt{m_Q^2 + p^2} \rangle_{nS}.$$

(5)

In Eq. (5), $m_q(m_Q)$ is the pole mass of the lighter (heavier) quark in a heavy-light meson. The matrix element $\langle Y_P \rangle_n$ refers to the P channel (with exception of the $\pi$ and $K$ mesons where additional chiral terms occur) and was calculated in Ref. \cite{6}:

$$\langle Y_P \rangle_n = m_q m_Q + \omega_{qn}\omega_{Qn} - \langle p^2 \rangle_{nS}.$$

(6)

On the other hand, for the integral $J_P$ (4) one can use the conventional spectral decomposition:

$$J_P = \int G_P(x)d^3x = \sum_n \frac{1}{2M_n} (f^n_P)^2 e^{-M_n T}.$$

(7)

Then from Eqs. (4) and (7) one obtains that

$$(f^n_P)^2 = \frac{2N_c \langle Y_P \rangle_n |\varphi_n(0)|^2}{\omega_{qn}\omega_{Qn}M_n}.$$

(8)

All necessary factors in Eq. (8) for the ground state ($n = 1$) and the first radial excitation ($n = 2$) are calculated in Ref. \cite{6} but here we consider only ground states and omit the index $n$ everywhere. Our calculations are performed with the static potential $V_0(r) = \sigma r - \frac{4}{3} \frac{\alpha_B(r)}{r}$ \cite{9}, and the hyperfine and self-energy contributions are considered as a perturbation. It is important that our input parameters contain only fundamental values: the string tension $\sigma$, the QCD constant $\Lambda(n_f = 3)$ in $\alpha_B(r)$, and the conventional pole quark masses. For the $s$-quark mass $m_s(\mu_1)$ one can expect that the scale $\mu_1$ is close to the characteristic momentum $\mu_1 \approx \sqrt{(p^2)} \sim 0.5 - 0.6$ GeV. This scale also corresponds to the r.m.s. radii $R_M(1S)$ of the meson we consider. For HL mesons

$$R_D \approx R_{D_s} = 0.55(1) \text{ fm}, \quad R_B \approx R_{B_s} = 0.50(1) \text{ fm},$$

(9)

so that $\mu_1 \sim R_M^{-1} \approx 0.4 - 0.5$ GeV. We show here that this mass $m_s(\mu_1)$ is strongly correlated with the values of $\eta_D$ and $\eta_B$. For other quarks we take $m_c = 1.40$ GeV and $m_b = 4.78$ GeV \cite{8}.
It is of interest to notice that for HL mesons the ratios

$$\xi_D = \xi_{D_s} = \frac{|R_{1D}(0)|^2}{\omega_q\omega_c} = 0.347(3)$$

are equal for the $D$ and $D_s$ mesons with an accuracy better than 1%, and also that these fractions for $B$ and $B_s$ mesons coincide with $\xi_B = \xi_{B_s}$ with an accuracy better than 2% accuracy ($\varphi_1^2(0) = R_1^2(0)/4\pi$):

$$\xi_B = \xi_{B_s} = \frac{|R_{1B}(0)|^2}{\omega_q\omega_b} = 0.146(2).$$

(11)

It is important that the equalities $\xi_D = \xi_{D_s}$ and $\xi_B = \xi_{B_s}$ practically do not depend on the details of the interaction in HL mesons. Therefore, in the ratios $\eta_D(\eta_B)$ the factors given in Eq. (10), $\xi_D(\xi_B)$ cancel and one obtains

$$\eta_{D(B)}^2 = \left(\frac{m_sm_c(b)}{\langle Y_P \rangle_{D(B)}} + \frac{\omega_s\omega_c(b) - \langle p^2 \rangle_{D_s(B_s)}}{\langle Y_P \rangle_{D(B)}}\right) \frac{M_{D(B)}}{M_{D_s(B_s)}}.$$  (12)

In Eq. (12) the second term is close to 1.05, while the first term, proportional to $m_s$, is not small, giving about 30-60% for different $m_s$ (below we take $m_s$ from the range 140 ± 60 MeV/$c^2$). With an accuracy of $\approx 2%$:

$$\eta_D^2 = 2.708 \times m_s(\text{GeV}) + 1.07(1), \text{ if } m_d = m_u = 0,$$

$$\eta_{D^+}^2 = 2.648 \times m_s(\text{GeV}) + 1.05(1), \text{ if } m_d = 8 \text{ MeV},$$

(13)

i.e., in the chiral limit

$$\eta_D = 1.14 \text{ (} m_s = 85 \text{ MeV)} , \quad 1.25 \text{ (} m_s = 180 \text{ MeV)} , \quad 1.27 \text{ (} m_s = 200 \text{ MeV)},$$

(14)

and for $m_d = 8 \text{ MeV}, \eta_D = 1.13, 1.24,$ and 1.26, respectively, for the same values of $m_s$, so decreasing only by $\sim 1\%$.

For the $B$ and $B_s$ mesons

$$\eta_{B}^2 = 1.90 \times m_s + 1.07(1) \quad (m_d = m = 0);$$

$$\eta_{B_s}^2 = 1.871 \times m_s + 1.07(1) \quad (m_d = 8 \text{ MeV}),$$

(15)

which practically coincide, and in the chiral limit $(m_d = m_u = 0)$

$$\eta_B = 1.11 \text{ (} m_s = 85 \text{ MeV)}, \quad 1.19 \text{ (} m_s = 180 \text{ MeV)} \quad 1.21 \text{ (} m_s = 200 \text{ MeV)},$$

(16)

These values of $\eta_B$ appear to be only $3-5\%$ smaller than $\eta_D$.  


Thus for \( m_s = 180 \) MeV and \( m_d = 8 \) MeV we have obtained

\[
\eta_{D^+} = 1.25(2), \quad \eta_B = 1.19(1),
\]

(17)
in good agreement with the CLEO data: \( \eta_D(\text{exp}) = 1.23(15) \) [13].

To check our choice of \( m_s = 180 \) MeV, we estimate the ratio \( m_s(0.5 \text{ GeV})/m_s(2 \text{ GeV}) \) using the conventional perturbative (one-loop) formula for the running mass [26]

\[
m(\mu^2) = m_0 \left( \frac{1}{2} \ln \frac{\mu^2}{\Lambda^2} \right)^{-d_m} \left[ 1 - d_1 \frac{\ln \ln \frac{\mu^2}{\Lambda^2}}{\ln \frac{\mu^2}{\Lambda^2}} \right].
\]

(18)

Here \( m_0 \) is an integration constant and the other constants are

\[
d_1 = \frac{8}{\beta_0^3} \left( \frac{51}{3} - \frac{19}{3} n_f \right), \quad \beta_0 = 11 - \frac{2}{3} n_f, \quad d_m = \frac{4}{\beta_0}.
\]

(19)

To calculate \( m_s(2 \text{ GeV}) \) we take \( n_f = 4, \Lambda(n_f = 4) = 250 \) MeV (\( \beta_0 = \frac{25}{3}, \quad d_m = \frac{12}{25}, \quad d_1 = 0.3548 \)). We take \( n_f = 3, \Lambda(n_f = 3) = 280 \) MeV to determine \( m_s(1 \text{ GeV}) \) and \( m_s(0.5 \text{ GeV}) \) (for \( n_f = 3, \quad \beta_0 = 9, \quad d_m = \frac{4}{3}, \quad d_1 = 0.3512 \)). Then, from Eqs. (18,19) \( m_s(2 \text{ GeV}) = 0.618 \) \( m_0, \quad m_s(1 \text{ GeV}) = 0.7825 \) \( m_0, \) and \( m_s(0.5 \text{ GeV}) = 1.217 \) \( m_0 \) and therefore

\[
\frac{m_s(1 \text{ GeV})}{m_s(2 \text{ GeV})} = 1.27, \quad \frac{m_s(0.5 \text{ GeV})}{m_s(2 \text{ GeV})} = 1.97.
\]

(20)

It means that \( m_s(0.5 \text{ GeV}) = 180 \) MeV, which we used in our calculations, corresponds to \( m_s(2 \text{ GeV}) = 91 \) MeV which coincides with the conventional value of \( m_s(2 \text{ GeV}) = 90 \pm 15 \) MeV [7]. Thus our estimate of \( m_s(0.5 \text{ GeV}) = 180 \) MeV supports our choice of this value in the relativistic string Hamiltonian, which provides a good description of the HL meson spectra and decay constants, and gives rise to the relatively large values of \( \eta_D \) and \( \eta_B \) in Eq. (1).

\[\text{[1]} \quad \text{E.S. Swanson, Phys. Rep. 429, 243 (2006) and references therein.}\]
\[\text{[2]} \quad \text{T. Barnes, S. Godfrey, and E.S. Swanson, Phys. Rev. D 72, 054026 (2005); P. Colangelo, E. De Fazio, R. Ferrandes, Mod. Phys. Lett. A 19, 2083 (2004).}\]
\[\text{[3]} \quad \text{Yu.A. Simonov, J.A. Tjon, Phys. Rev. D 70, 114013 (2004).}\]
\[\text{[4]} \quad \text{M. DiPierro, E. Eichten, Phys. Rev. D 64, 114004 (2001).}\]
\[\text{[5]} \quad \text{S. Godfrey, N. Isgur, Phys. Rev. D 32, 189 (1985).}\]

[6] A.M. Badalian, B.L.G. Bakker, Yu.A. Simonov, Phys. Rev. D 75, 116001 (2007).
[7] Particle Data Group, S. Eidelman et al., Phys. Lett. B 592, 1 (2004).
[8] A.M. Badalian, A.I. Veselov, B.L.G. Bakker, Phys. Atom. Nucl. 67, 1367 (2004).
[9] A.M. Badalian and B.L.G. Bakker, Phys. Rev. D 66, 034025 (2002); A.M. Badalian, B.L.G. Bakker, and Yu.A. Simonov, Phys. Rev. D 66, 034026 (2002);
[10] K.G. Chetyrkin, A. Khodjamiran, Eur. Phys. J. C 46, 721 (2006); S. Narison, Phys. Rev. D 74, 034013 (2006); Phys. Lett. B 605, 319 (2005). E. Gamiz, M. Jamin, A. Pich, J. Prades, F. Schwab, Phys. Rev. Lett. 94, 011803 (2005).
[11] M. Gockler et al., Phys. Rev. D 73, 054508 (2006) and references therein.
[12] (CLEO Collaboration) M. Artuso et al., Phys. Rev. Lett. 95, 251801 (2005); hep-ex/0508057; G. Bonvicini et al., Phys. Rev. D 70, 112004 (2004).
[13] (CLEO Collaboration) M. Artuso et al., arXiv:0704.0629; (CLEO Collaboration) T.K. Pedlar et al., arXiv: 0704.0437;
[14] (BELLE Collaboration) K. Ikado et al., Phys. Rev. Lett. 97, 251802 (2006); hep-ex/0604018.
[15] (BaBar Collaboration) B. Aubert et al., hep-ex/0607094, hep-ex/0608019; (BaBar Collaboration) L.A. Corwin, hep-ex/0611019.
[16] (OPAL Collaboration) G. Albiendi et al., Phys. Lett. B 516, 236 (2001); (ALEPH Collaboration) A. Heister et al., Phys. Lett. B 528, 1 (2002); hep-ex/0201024; (BES Collaboration) M. Ablikim et al., Phys. Lett. B 610, 183 (2005).
[17] Z.G. Wang et al., Nucl. Phys. A 744, 156 (2004).
[18] C. Cvetič, C.S. Kim, G.-L. Wang, W. Namgung, Phys. Lett. B 596, 84 (2004).
[19] D. Ebert, R.N. Faustov, and V.O. Galkin, Phys. Lett. B 635, 93 (2006); Mod. Phys. Lett. A 17, 803 (2002).
[20] S. Narison, hep-ph/0202200; Phys. Lett. B 520, 115 (2001).
[21] C. Aubin et al., Phys. Rev. Lett. 95, 122002 (2005); hep-lat/0506030; C. Aubin et al., Phys. Rev. D 70, 114501 (2004).
[22] A. Gray et al., Phys. Rev. Lett. 95, 212001 (2005); hep-lat/0507015; M. Wingate, C.T.H. Davies, A. Gray, G.P. Lepage, and J. Shigemitsu, Phys. Rev. Lett. 92, 162001 (2004).
[23] A.Yu. Dubin, A.B. Kaidalov, Yu.A. Simonov, Phys. Atom. Nucl. 56, 1745 (1993) [Yad. Fiz. 56, 213 (1993)]; hep-ph/9311344; Phys. Lett. B 323, 41 (1994); E.L. Gubankova, A.Yu. Dubin, Phys. Lett. B 334, 180 (1994); Yu.A. Simonov, hep-ph/9911237.
[24] Yu.S. Kalashnikova, A.V. Nefediev, Yu.A. Simonov, Phys. Rev. D 64, 014037 (2001).

[25] Yu.A. Simonov, J.A. Tjon, Annals Phys. 300, 54 (2002) and references therein.

[26] F.Y. Yndurain, “The Theory of Quark and Gluon Interactions”, fourth edition, Springer-Verlag Berlin Heidelberg, 2006, p. 81.