On the density of the winding number of planar Brownian motion

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Abstract

We obtain a formula for the density \( f(\theta, t) \) of the winding number of a planar Brownian motion \( Z_t \) around the origin. From this formula we deduce an expansion for \( f(\log(\sqrt{t}) \theta, t) \) in inverse powers of \( \log \sqrt{t} \) and \( (1 + \theta^2)^{1/2} \) which in particular yields the corrections of any order to Spitzer’s asymptotic law (1.1). We also obtain an expansion for \( f(\theta, t) \) in inverse powers of \( \log \sqrt{t} \), which yields precise asymptotics as \( t \to \infty \) for a local limit theorem for the windings.

Keywords: planar Brownian motion, winding number, transition density, Spitzer’s law, local limit theorem, asymptotic expansions

1 Introduction

In his celebrated paper about 2–dimensional Brownian motion, F. Spitzer considered the transition probabilities of \( \Theta_t \), its continuous winding number around the origin, and showed in particular that the limiting distribution of \( \frac{\Theta_t}{\log \sqrt{t}} \) is the Cauchy distribution:

\[
\lim_{t \to \infty} \mathbb{P}\left( \frac{\Theta_t}{\log \sqrt{t}} \leq \alpha \right) = \frac{1}{\pi} \int_{-\infty}^{\alpha} \frac{d\theta}{1 + \theta^2}.
\]  

(1.1)

Several proofs of this result have since then been given, see for instance page 43 of the book [10] by L.G.C. Rogers and D. Williams and the discussion and references therein. Extensions to more general situations are known as well, for instance those considered by J.Pitman and M. Yor in [7], [8] and [9] in the frame of asymptotic laws for planar Brownian motion.

More recently, V. Bentkus, G.Pap and M. Yor in (11 and 6) obtained by Fourier methods an expansion of the distribution function of \( \Theta_t \), that yields detailed asymptotics, both in \( t \) and \( \alpha \), as they go to \( \infty \).
F. Delbaen, E. Kowalski and A. Nikeghbali proved in [3] (also using Fourier methods) the following local limit theorem for \( \Theta_t \): if \( \alpha < \beta \in \mathbb{R} \),

\[
\lim_{t \to \infty} \log \sqrt{t} \mathbb{P}(\alpha < \Theta_t < \beta) = \frac{\beta - \alpha}{\pi}.
\] (1.2)

In the present work, we obtain a formula for the density \( f(\theta, t; \rho) \) of \( \Theta_t \) when the initial condition \( Z_0 = \rho \neq 0, \rho \in \mathbb{R} \), that follows by integrating an expression for the joint density \( p(r, \theta, t; \rho) \) of \((|Z_t|, \Theta_t)\) that appears in [2] (and follows as well from results in [12], see also the comments there for older references). We compute \( f \) in the next section. Let us mention that in Chapter V of [5], R. Mansuy and M. Yor discuss some representations for the distribution of the winding number of the Brownian lace of length \( t \).

The expression for the density \( f \) is given in terms of an integral including a couple of fractions. In the last section, we expand these fractions in inverse powers of \( \log \sqrt{t} \) and obtain two asymptotic expansions, after term by term integration. The first one is an expansion for \( f(\log \sqrt{t} \theta, t; \rho) \) very close to that given in [6]. The difference is that the dependence of the coefficients on \( \theta \) is given more explicitly, and that we obtain also accurate estimates on the behaviour in \( t \) of those coefficients. A more precise comparison is discussed in Remark 4 after the statement of the corresponding result (Theorem 3.1). The second expansion is for \( f(\theta, t; 1) \) in inverse powers of \( \log \sqrt{t} \), that yields in particular the corrections of any order to (1.2). The result is stated as Theorem 3.2.

2 A formula for the density

We consider a planar Brownian motion \( Z_t \) starting at \( z_0 \neq 0, \Theta_t \) a continuous determination of its argument and \( R_t = |Z_t| \). Let us call \( p \) the joint density of \( \Theta_t \) and \( R_t \),

\[
p(r, \theta, t; \rho, \alpha) r dr d\theta = \mathbb{P}(R_t \in dr, \Theta_t \in d\theta | R_0 = \rho, \Theta_0 = \alpha)
\]

Since \( p \) is clearly a function of \( \theta - \alpha \), it suffices to consider \( \alpha = 0 \), what we do in the sequel. The following expression for \( p \) was deduced in [2] (see also [12]),

\[
p(r, \theta, t; \rho) = \frac{1}{\pi t} \exp \left( -\frac{r^2 + \rho^2}{2t} \right) \int_0^\infty \cos(\nu \theta) I_\nu \left( \frac{\rho r}{t} \right) d\nu
\]

(2.1)

Integration in \( r \) of the above formula yields an expression for the density of \( \Theta_t, f(\theta, t; \rho) d\theta = \mathbb{P}(\Theta_t \in d\theta | R_0 = \rho) \).
Denote by $1_I_A$ the indicator function of the set $A$. We have then:

**Proposition 2.1.** The density $f$ of $\Theta_t$ is given by

$$f(\theta, t; \rho) = \frac{\rho e^{-\frac{\rho^2}{4t}(1-\cos 2\theta)}}{\sqrt{2\pi t}} \cos \theta 1_I_{(-\pi/2,\pi/2)}(\theta)$$

$$+ \frac{\rho e^{-\frac{\rho^2}{4t} \cosh \omega}}{2\pi \sqrt{2\pi t}} \int_0^\infty e^{-\frac{\rho^2}{4t} \cosh \omega} \sinh(\omega/2)$$

$$\times \left( \frac{\omega/2}{(\omega/2)^2 + (\theta + \pi/2)^2} + \frac{\omega/2}{(\omega/2)^2 + (\theta - \pi/2)^2} \right) d\omega \quad (2.2)$$

Before proving the previous proposition, let us show another formula for the density $f$, that will prove more suitable to obtaining asymptotic expansions (as $t \to \infty$). It follows directly from the change of variables $\frac{\rho^2}{4t} \cosh \omega \to z$ in the integral above.

**Corollary 2.2.** The following formula for $f$ holds:

$$f(\theta, t; \rho) = \frac{\rho e^{-\frac{\rho^2}{4t}(1-\cos 2\theta)}}{\sqrt{2\pi t}} \cos \theta 1_I_{(-\pi/2,\pi/2)}(\theta)$$

$$+ \frac{1}{2\pi \sqrt{\pi}} \int_{\rho^2/4t}^\infty \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z + \rho^2/4t}} \times$$

$$\left( \frac{1}{2} \arccosh(\frac{4t z}{\rho^2}) \right)^2 + (\theta + \pi/2)^2 + \frac{1}{2} \arccosh(\frac{4t z}{\rho^2})^2 + (\theta - \pi/2)^2 \right) dz \quad (2.3)$$

**Proof of Proposition 2.1.** Integrate by parts once, recall that $2 I'_\nu(x) = I_{\nu-1}(x) + I_{\nu+1}(x)$ and use formula 6.118–4 of [4] to compute

$$\int_0^\infty e^{-\frac{\rho^2}{4t} I_\nu(\frac{\rho^2}{t})} r dr = \rho \int_0^\infty e^{-\frac{\rho^2}{2} I'_\nu(\frac{\rho^2}{t})} dr =$$

$$\frac{\rho}{2} \int_0^\infty e^{-\frac{\rho^2}{4t} \left[ I_{\nu-1}(\frac{\rho^2}{t}) + I_{\nu+1}(\frac{\rho^2}{t}) \right]} dr = \frac{\rho \sqrt{2\pi t}}{4} e^{\frac{\rho^2}{4t} I_{\nu-1}(\frac{\rho^2}{t}) + I_{\nu+1}(\frac{\rho^2}{4t})}$$

With the aid of this formula we can integrate (2.1), after interchanging the order of the integrals, obtaining

$$f(\theta, t; \rho) = \int_0^\infty p(r, \theta, t; \rho) r dr$$

$$= \frac{\rho e^{-\frac{\rho^2}{4t}}}{2\sqrt{2\pi t}} \int_0^\infty \cos(\nu \theta) \left[ I_{\nu-1}(\frac{\rho^2}{4t}) + I_{\nu+1}(\frac{\rho^2}{4t}) \right] d\nu \quad (2.4)$$
Recall next the integral representation for the Bessel function $I_\nu$ (formula 8.431–5 of \[4\]):

$$I_\nu(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos \omega} \cos(\nu \omega) \, d\omega - \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-x \cosh \omega} \, d\omega$$

Substitution of this last formula in the integral in (2.4) yields

$$\int_0^\infty \cos(\nu \theta) \left[ I_{\nu-\frac{1}{2}} \left( \frac{\rho^2}{4t} \right) + I_{\nu+\frac{1}{2}} \left( \frac{\rho^2}{4t} \right) \right] \, d\nu =$$

$$\frac{1}{\pi} \int_0^\infty \cos(\nu \theta) \int_0^\pi\int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} e^{\frac{\rho^2}{4t} \cos \omega} \cos((\nu - 1)\omega/2) + \cos((\nu + 1)\omega/2) \, d\omega \, d\nu \, d\nu$$

$$- \frac{1}{\pi} \int_0^\infty \cos(\nu \theta) \sin(\pi(\nu - 1)/2) \int_0^\infty e^{-\frac{\rho^2}{4t} \cosh(\omega) - (\nu - 1)\omega/2} \, d\omega \, d\nu$$

$$- \frac{1}{\pi} \int_0^\infty \cos(\nu \theta) \sin(\pi(\nu + 1)/2) \int_0^\infty e^{-\frac{\rho^2}{4t} \cosh(\omega) - (\nu + 1)\omega/2} \, d\omega \, d\nu$$

$$= S_1 + S_2 + S_3 \quad (2.5)$$

To compute the first integral $S_1$, use that $\cos((\nu - 1)\omega/2) + \cos((\nu + 1)\omega/2) = 2 \cos(\nu \omega/2) \cos(\omega/2)$, and recall that the integrand is an even function of $\omega$ and $\nu$ to write:

$$S_1 = \frac{1}{2\pi} \int_{-\infty}^\infty e^{\frac{\rho^2}{4t} \cos \omega} \cos(\nu \omega) \, d\omega \, d\nu$$

$$= 2 e^{\frac{\rho^2}{4t} \cos 2\theta} \cos \theta \mathbb{I}_{\nu \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)}(\theta). \quad (2.6)$$

The last identity follows from the Fourier inversion theorem, which applies since the inner integral $I(\nu)$ is an $L^1$ function. This in turn can be seen from the fact that both $|I(\nu)|$ and $|\nu^2 I(\nu)|$ are uniformly bounded. The last statement can be checked integrating by parts twice, using that $2 e^{\frac{\rho^2}{4t} \cos \omega}$ vanishes at $\pm \frac{\pi}{2}$.

To compute $S_2$, change the order of the integrals, recall that $\sin(\pi(\nu - 1)/2) = - \cos(\nu \pi/2)$ and integrate

$$\int_0^\infty \cos(\nu \theta) \cos(\nu \pi/2) e^{-\frac{\rho^2}{4t}} \, d\nu =$$

$$\frac{1}{2} \left( \frac{y/2}{(y/2)^2 + (\theta + \pi/2)^2} + \frac{y/2}{(y/2)^2 + (\theta - \pi/2)^2} \right).$$
The computation of $S_3$ is similar, and we obtain finally

$$S_2 + S_3 = \frac{1}{\pi} \int_0^\infty e^{-\frac{r^2}{4}} \cosh \omega \sinh(\omega/2) \times \left( \frac{\omega/2}{(\omega/2)^2 + (\theta + \pi/2)^2} + \frac{\omega/2}{(\omega/2)^2 + (\theta - \pi/2)^2} \right) d\omega$$

(2.7)

From (2.5), (2.6) and this last equation yield (2.2).

Despite formulae (2.2) and (2.3) may appear complicated, we can see that they give information on the behaviour of $\Theta_t$. In particular, it is not hard to deduce for instance detailed asymptotics as $t \to \infty$. We state and prove two precise results in the following section.

3 Two asymptotic results

**Theorem 3.1.** The density $f(\theta, t; \rho)$ admits the following expansion: for any natural number $N \geq 0$, $t > 2$ and $\delta \in (0, \frac{1}{2})$,

$$\log(\sqrt{t}) f(\theta \log(\sqrt{t}), t; \rho) = \frac{1}{2\pi \sqrt{\pi}} \sum_{n=0}^{N} \frac{(-1)^n A_n(\theta) C_n(t; \rho)}{2^{n-1}(1 + \theta^2)^{\frac{n+1}{2}} (\log(\sqrt{t}))^n} + R_N$$

(3.1)

The coefficients satisfy

$$A_n(\theta) = \sum_{k \text{ even}} \binom{n+1}{k} (-1)^\frac{k}{2} \theta^k \quad \text{and} \quad C_n(t; \rho) = c_n + o_n(t^{-\delta})$$

(3.2)

where the $c_n$ are constants given by

$$c_n = \sum_{k \text{ even}} \binom{n}{k} (-1)^\frac{k}{2} \pi^k \int_0^\infty \frac{e^{-z}}{\sqrt{z}} (\log(8z/\rho^2))^{n-k} dz.$$  

(3.3)

The remainder terms $R_N$ satisfy that for some positive constants $k_N$,

$$|R_N| \leq \frac{k_N}{(\log(\sqrt{t}))^{N+1}(1 + \theta^2)^{\frac{N}{2}+1}}$$

(3.4)

Let us make some observations on the previous result.
Remarks

(1) The dependence on the initial condition $\rho$ is made explicit in the coefficients $c_n$, but the constants $k_N$, and $o_n(t^{-\delta})$ are not uniform in $\rho$. We could have taken $\rho = 1$ as usual, but preferred to keep it explicit, as the dependence may be traced back in order to investigate the behaviour in $\rho$ also.

(2) For $N$ even, $A_N \sim \frac{1}{(1+\theta^2)^{N/2}}$ as $\theta \to \infty$, while for $N$ odd, $A_N \sim 1$ as $\theta \to \infty$. Then the order in $\theta$ of the coefficient $\frac{A_N(\theta)}{(1+\theta^2)^{N/2}}$ in the expansion is the same as that of $R_N$ for $N$ even.

(3) As $c_0 = \sqrt{\pi}$ and $A_0(\theta) = \frac{1}{(1+\theta^2)^{1/2}}$, the first term yields precisely Spitzer’s result (1.1). The $c_n$ can be expressed in terms of derivatives of the Gamma function $\Gamma$, using the formula

$$\int_0^\infty \frac{e^{-z}}{\sqrt{z}} \left( \log(z) \right)^n dz = \frac{d^n}{dl^n} \Gamma(l) \bigg|_{l=\frac{1}{2}}$$

(3.5)

(see for instance formula 4.358.5 on page 578 of [4]).

(4) As mentioned in the introduction, V. Bentkus, G.Pap and M. Yor in ([6] and [1]) obtained an expansion for $f$ (in the case $\rho = 1$), by different methods. The expression for the coefficients in $\theta$ in (3.1) is slightly more explicit here than the one they obtain. It is not difficult to see that the first three terms coincide with those they compute. We also provide accurate estimates on the rate of convergence to 0 of $C_n - c_n$ as $t \to \infty$.

As the density $f$ of $\Theta_t$ is a function of $\frac{\rho}{\sqrt{t}}$, we consider $\rho = 1$ in the next result, and omit the reference to the initial condition, keeping the notation $f(\theta, t)$.

Theorem 3.2. For any natural number $N \geq 0$ and $t > 2$, the following expansion holds:

$$\log(\sqrt{t}) f(\theta, t) = \frac{1}{2\pi \sqrt{\pi}} \sum_{n=0}^N \frac{g_n(\theta)}{(\log(\sqrt{t}))^n} + E_N.$$  (3.6)
The \( g_n(\theta) \) are polynomials in \( \theta \) given by

\[
g_n(\theta) = (-1)^n \sum_{k \text{ even}, k \leq n} \binom{n}{k} (-1)^k \left( \left( \frac{\pi}{2} + \theta \right)^k + \left( \frac{\pi}{2} - \theta \right)^k \right) \int_0^\infty \frac{e^{-z}}{\sqrt{z}} \left( \log(8z) \right)^{n-k} dz
\]

and the remainder \( E_N \) satisfies

\[
|E_N| \leq \frac{S_{N+2}(|\theta|)}{\left( \log(\sqrt{t}) \right)^{N+1}}.
\]

\( S_{N+2} \) above is a polynomial of degree \( N + 2 \), whose coefficients may depend on \( N \).

**Remark** It is easy to compute \( g_0 = 2\sqrt{\pi} \), and then the local limit theorem \((1.2)\) follows at once from Theorem \((3.2)\). Moreover, corrections of any order to \((1.2)\) in inverse powers of \( \log\sqrt{t} \) follow from \((3.6)\). Simple integration of the polynomials \( g_n \) provide the coefficients, let us say \( G_n \), for the expansion of \( \log\sqrt{t} \mathbb{P}(\alpha < \Theta_t \leq \beta) = \sum_{n \geq 0} \frac{G_n}{(\log\sqrt{t})^n} \). For instance, computing with the aid of \((3.5)\) and \((3.7)\),

\[
g_1 = -2 \int_0^\infty \frac{e^{-z}}{\sqrt{z}} (\log 8 + \log z) \, dz = 2\sqrt{\pi} (\gamma + \log \frac{1}{2}),
\]

where \( \gamma \) is the Euler constant, we have

\[
\log\sqrt{t} \mathbb{P}(\alpha < \Theta_t \leq \beta) = \frac{\beta - \alpha}{\pi} + \frac{1}{\log\sqrt{t}} \frac{(\beta - \alpha)(\gamma + \log \frac{1}{2})}{\pi} + O\left(1/\left(\log\sqrt{t}\right)^2\right)
\]

The proofs of both theorems are given after stating and proving the next three lemmas.

Let us shorthand

\[
x = \frac{1}{\log\sqrt{t}} \quad \text{and} \quad b = \frac{1}{2} \log \left( \frac{4z}{\rho^2} + \sqrt{\left( \frac{4z}{\rho^2} \right)^2 - \frac{1}{t^2}} \right)
\]

(3.9)
Lemma 3.3. The density $f$ satisfies:

$$\log(\sqrt{t}) f(\theta \log(\sqrt{t}), t; \rho) = N(\theta, t; \rho) + \frac{1}{2\pi \sqrt{\pi}} \int_{\rho^2/4t}^{\infty} e^{-\left(z + \rho^2/4t\right)} \sqrt{z + \rho^2/4t} dz$$

$$\times \left(\frac{1 + bx}{(1 + bx)^2 + (\theta + \frac{\pi}{2} x)^2} + \frac{1 + bx}{(1 + bx)^2 + (\theta - \frac{\pi}{2} x)^2}\right) dz,$$  \hspace{1cm} (3.10)

The term $N$ is negligible as $t \to \infty$, uniformly in $\theta$ in the following sense: for any given $K \in \mathbb{N}$ and $\delta < 1/2$, there exists a positive constant $C$, independent of $\rho$ and $t$ such that

$$\sup_{\theta \in \mathbb{R}} (1 + \theta^2)^K |N(\theta, t; \rho)| \leq C \rho t^{-\delta} \text{ for any } t > 2$$  \hspace{1cm} (3.11)

Proof. Consider the expression for $\log(\sqrt{t}) f(\theta \log(\sqrt{t}), t; \rho)$ obtained from (2.3), and denote by $N$ the term coming from the first term on the r.h.s.. It is clear that it satisfies (3.11).

The other term comes simply from organizing the fractions in the integral, (3.9) and the formula $\text{arcosh} y = \log(y + \sqrt{y^2 - 1})$.

In the next lemmas, we show how to obtain an explicit expansion in powers of $x$ of the fractions in the last line in (3.10). Let us denote by $\Re(z)$ the real part of $z \in \mathbb{Z}$.

Lemma 3.4. Consider, for $b, c, d \in \mathbb{R}$ such that $c^2 - 4d < 0$, the rational function

$$g(x) = \frac{1 + bx}{1 + cx + dx^2}$$

For any given natural $N \geq 0$, the following finite $N$-expansion with remainder holds:

$$g(x) = \sum_{n=0}^{N} a_n x^n + q_N(x),$$

where

$$a_n = \frac{(-1)^n}{2^n} \Re\left((1 + i \frac{2b - c}{\sqrt{4d - c^2}})(c + i \sqrt{4d - c^2})^n\right) \quad n \geq -1$$  \hspace{1cm} (3.12)

$$q_N(x) = -x^{N+1} \left(\frac{d a_{N-1} + c a_N + x d a_N}{1 + cx + dx^2}\right)$$  \hspace{1cm} (3.13)
Proof. The sequence \( a_n \) has to satisfy the recurrence \( a_{n+2} + c a_{n+1} + d a_n = 0 \) for \( n \geq 0 \), with \( a_0 = 1 \) and \( a_1 = b - c \). To solve it, let us introduce the function

\[
F(x) = \sum_{n \geq 0} \frac{a_n x^n}{n!}
\]

(3.14)

From the recurrence for the \( a_n \)'s, it follows that \( F \) has to satisfy the differential equation

\[
F''(x) + c F'(x) + d F(x) = 0,
\]

with boundary conditions \( F(0) = 1 \), \( F'(0) = b - c \), whose solution is

\[
F(x) = A e^{rx} + \bar{A} e^{\bar{r}x}
\]

(3.15)

with \( A = \frac{1}{2} + i \frac{2b-c}{2\sqrt{4d-c^2}} \) and \( r = -\frac{c+i\sqrt{4d-c^2}}{2} \). Expanding the right hand side of (3.15) in its Taylor series, and equating coefficients with the expression (3.14), we get \( a_n = 2 \Re(e^{r \theta}) \), which is exactly (3.12).

The formula (3.13) for the remainder is proven easily by induction for \( N \geq 1 \). Indeed, compute

\[
g(x) - a_0 - (b - c)x = -x^2 \left( \frac{d + c(b - c) + xd(b - c)}{1 + cx + dx^2} \right)
\]

to prove it for \( N = 1 \). Suppose it true for \( N \), and write next

\[
g(x) - \sum_{n=0}^{N} a_n x^n - a_{N+1} x^{N+1} = q_N(x) - a_{N+1} x^{N+1}.
\]

Use the recurrence relation for the \( a_n \) to conclude that the r.h.s is \( q_{N+1} \).

A direct computation shows that the formula is also valid for \( N = 0 \), with \( a_{-1} \) given by taking \( n = -1 \) in (3.12).

Lemma 3.5. The fractions on the second line of (5.10) can be expanded in powers of \( x \) as follows: for any given integer \( N \geq 0 \) and \( x \leq 4 \),

\[
\frac{1 + bx}{(1 + bx)^2 + (\theta + \frac{\pi}{2} x)^2} + \frac{1 + bx}{(1 + bx)^2 + (\theta - \frac{\pi}{2} x)^2} = \sum_{n=0}^{N} \frac{(-1)^n P_n(b) \cos ((n + 1)\beta)}{2^{n-1}(1 + \theta^2)^{\frac{n+1}{2}}} x^n + Q_N,
\]

(3.16)

where

\[
e^{i\beta} = \frac{1 + i\theta}{(1 + \theta^2)^{\frac{1}{2}}} \quad P_n(b) = \sum_{k \leq n, k \text{ even}} \binom{n}{k} (2b)^{n-k} (-1)^{\frac{k}{2}} \pi^k,
\]

(3.17)
and \[ Q_N = \frac{x^{N+1}}{(1 + \theta^2)^{\frac{N}{2} + 1}} E_N(b, \theta) \]
with
\[ |E_N(b, \theta)| \leq K \frac{(4b^2 + \pi^2)^{\frac{N}{2} + 1}}{2^{N-1}} \left( \begin{array}{c} \mathbb{I}_{|\theta| \leq 20} \left( 1 + bx \right)^2 + \mathbb{I}_{|\theta| > 20} \end{array} \right) \]
for some constant \( K > 0 \) independent of \( x, b, N \) and \( \theta \).

Proof. Let \( N \geq 0 \) given, and set
\[ \frac{1 + bx}{(1 + bx)^2 + (\theta + \frac{\pi}{2})^2} = \frac{1 + bx}{1 + \theta^2} \left( \frac{1 + bx}{1 + c x + d x^2} \right), \]
for \( c = \frac{2b + \pi \theta}{1 + \theta^2} \) and \( d = \frac{b^2 + \pi^2 / 4}{1 + \theta^2} \). Observe that \( c^2 - 4d = \frac{(2\theta - \pi)^2}{1 + \theta^2} \), so the conditions of Lemma 3.4 are satisfied and we may apply formulae (3.12) and (3.13) to obtain, after some algebraic manipulations,
\[ \frac{1 + bx}{1 + c x + d x^2} = \sum_{n=0}^{N} a^{(1)}_n x^n + q^{(1)}_N, \]
with \( a^{(1)}_n = \frac{(-1)^n}{2^n(1 + \theta^2)^n} \Re \left( (1 + i\theta)^{n+1} (2b - i\pi)^n \right) \) and
\[ q^{(1)}_N = -\frac{1}{1 + \theta^2} \left( (b^2 + \pi^2 / 4) a^{(1)}_{N-1} + x a^{(1)}_N \right) + (2b + \pi \theta) a^{(1)}_N \]

Since the second fraction in (3.16) is just the first one evaluated at \( -\theta \), the expansion for the former is:
\[ \frac{1 + bx}{(1 + bx)^2 + (\theta - \frac{\pi}{2})^2} = \frac{1}{1 + \theta^2} \left( \sum_{n=0}^{N} a^{(2)}_n x^n + q^{(2)}_N \right), \]
where the following expression for \( a^{(2)}_n \) results from evaluating (3.21) at \( -\theta \) and conjugating the complex inside \( \Re \),
\[ a^{(2)}_n = \frac{(-1)^n}{2^n(1 + \theta^2)^n} \Re \left( (1 + i\theta)^{n+1} (2b + i\pi)^n \right) \]
and \( q^{(2)}_N(\theta) = q^{(1)}_N(-\theta) \).

From (3.17) and (3.21) we have
\[ a^{(1)}_n + a^{(2)}_n = \frac{(-1)^n}{2^n(1 + \theta^2)^n} \Re \left( (1 + i\theta)^{n+1} \left( (2b - i\pi)^n + (2b + i\pi)^n \right) \right) \]
\[ = \frac{(-1)^n P_n(b)}{2^{n-1}(1 + \theta^2)^n} \Re \left( (1 + i\theta)^{n+1} \right) = \frac{(-1)^n P_n(b)}{2^{n-1}(1 + \theta^2)^n} \cos ((n + 1)\beta). \]
Then, (3.16) follows from (3.19), (3.20) and (3.23), with \( Q_N = \frac{1}{1 + \theta^2} (q_N^{(1)} + q_N^{(2)}) \). To complete the proof, we only need to show (3.18). From (3.21),

\[
|a_n^{(1)}| \leq \left| \frac{(1 + i\theta)^n(2b - i\pi)^n}{2^n(1 + \theta^2)^n} \right| = \frac{(4b^2 + \pi^2)^{\frac{n}{2}}}{2^n(1 + \theta^2)^{\frac{n}{2} + 1}},
\]

and then from (3.22), there is a constant \( K_1 > 0 \) such that

\[
|q_N^{(1)}| \leq x^{N+1} K_1 \frac{(4b^2 + \pi^2)^{\frac{N}{2} + 1}}{D^{2N-1}(1 + \theta^2)^{\frac{N}{2} - 1}}, \tag{3.24}
\]

where \( D \) is the denominator in (3.22). Recall that \( x \leq 4 \) to estimate

\[
D = 1 + \theta^2 + (2b + \pi\theta)x + (2b^2 + \pi^2/4)x^2 = (1 + \theta^2)\left( \frac{(1 + b^2 x)^2}{1 + \theta^2} + \frac{(\theta + \pi x)^2}{1 + \theta^2} \right) \leq (1 + \theta^2)\left( \frac{(1 + b^2 x)^2}{1 + \theta^2} + \frac{(\theta + \pi x)^2}{1 + \theta^2} \right) \leq (1 + \theta^2)\left( \frac{(1 + b x)^2}{401} \mathbb{I}_{|\theta|\leq20} + \frac{1}{3} \mathbb{I}_{|\theta|>20} \right), \tag{3.25}
\]

so, from (3.24),

\[
|q_N^{(1)}| \leq K_2 x^{N+1} \frac{(4b^2 + \pi^2)^{\frac{N}{2} + 1}}{D^{2N-1}(1 + \theta^2)^{\frac{N}{2} - 1}} \left( \mathbb{I}_{|\theta|\leq20} + \mathbb{I}_{|\theta|>20} \right)
\]

The same estimate holds for \( |q_N^{(2)}| \), and (3.18) follows.

Proof of Theorem 3.4

The proof consists in substituting the fractions in (3.10) by the \( N \)-expansion (3.16) and then integrate term by term, recalling that only \( b \) depends on the integration variable \( z \). Let \( N \geq 0 \) be given, recall that \( x = \frac{1}{\log \sqrt{t}} \), and write then

\[
\log(\sqrt{t}) f(\theta \log(\sqrt{t}), t; \rho) = N(\theta, t; \rho) + \frac{1}{2\pi \sqrt{\pi}} \sum_{n=0}^{N} (-1)^n x^n \cos \left( \frac{(n + 1)\beta}{2} \right) \int_{\rho^2/4t}^{\infty} \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z + \rho^2/4t}} P_n(b) dz + \frac{1}{2\pi \sqrt{\pi}} \int_{\rho^2/4t}^{\infty} \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z + \rho^2/4t}} Q_N dz. \tag{3.26}
\]
Call

\[ C_n = \int_{\rho^2/4t}^{\infty} \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z + \rho^2/4t}} P_n(b) \, dz \]  

(3.27)

\[ R_N = N(\theta, t; \rho) + \frac{1}{2\pi} \int_{\rho^2/4t}^{\infty} \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z + \rho^2/4t}} Q_N \, dz. \]  

(3.28)

From (3.17), \(\cos((n + 1)\beta) = A_n(\theta)\), so to prove Theorem 3.1 it is enough to show that \(C_n\) and \(R_N\) satisfy (3.2) and (3.4) respectively.

Recall the definition of \(b\) in (3.9), and set

\[ b_0 = \lim_{t \to \infty} b = \frac{1}{2} \log \left( \frac{8z}{\rho^2} \right) \Rightarrow c_n = \lim_{t \to \infty} C_n = \int_{0}^{\infty} \frac{e^{-z}}{\sqrt{z}} P_n(b_0) \, dz \]  

(3.29)

as follows from (3.17) and (3.3). The \(c_n\)'s are clearly finite, and we have

\[ C_n = c_n + \int_{\rho^2/4t}^{\infty} \frac{P_n(b) e^{-(z+\rho^2/4t)}}{\sqrt{z + \rho^2/4t}} \, dz - \frac{P_n(b_0) e^{-z}}{\sqrt{z}} \int_{0}^{\rho^2/4t} \frac{P_n(b_0) e^{-z}}{\sqrt{z}} \, dz \]  

(3.30)

To estimate the integrals above, let us shorthand \(w = \frac{1}{2} \left( 1 - \sqrt{1 - \left( \frac{\rho^2}{4zt} \right)^2} \right)\) and note that \(0 \leq w \leq \rho^2/8tz \leq 1/2\) for \(z \geq \rho^2/4t\) to obtain

\[ |b - b_0| = -\frac{1}{2} \log (1 - w) = \frac{1}{2} \log \left( 1 + \frac{w}{1-w} \right) \leq w \leq \frac{\rho^2}{8zt} \]  

(3.31)

Denote by \(k_n\) positive constants depending only on \(n\), that may change from line to line. From (3.17) and the previous inequality, it follows that

\[ |P_n(b) - P_n(b_0)| \leq k_n |b - b_0| (|b_0|^{n-1} + 1), \]  

(3.32)
Then, let $0 < \delta_0 < 1/2$, and estimate with the aid of (3.31) and (3.32),

$$\left| \int_0^\infty \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z+\rho^2/4t}} P_n(b) - \frac{e^{-z}}{\sqrt{z}} P_n(b_0) \, dz \right| \leq$$

$$\int_0^\infty \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z+\rho^2/4t}} |P_n(b) - P_n(b_0)| \, dz +$$

$$\int_0^\infty \frac{e^{-z}}{\sqrt{z}} |P_n(b_0)| \left(1 - \frac{e^{-\rho^2/4t}\sqrt{z}}{\sqrt{z+\rho^2/4t}}\right) \, dz$$

$$\leq \frac{k_n}{t} \int_0^\infty \frac{e^{-z}z^{-(1/2)-\delta_0} \, dz + 1}{\sqrt{z}} + \frac{k_n}{t} \left( \int_0^1 z^{-(1/2)-\delta_0} \, dz + 1 \right)$$

$$+ \frac{k_n}{\sqrt{t}} \left( \int_0^1 z^{1-\delta_0} \, dz + 1 \right) \leq k_n t^{-(1/2)+\delta_0} \quad (3.33)$$

To obtain the last inequality, we have split the integrals according to $z < 1$ or $z \geq 1$, after observing that, for $S_n$ a polynomial of degree $n$, $|S_n(b_0)|z^{\delta_0} \leq k_n$ for $0 < z < 1$. For $z \geq 1$, the integrals are clearly uniformly bounded. The estimation of the last term in (3.30) is similar:

$$\left| \int_0^{\rho^2/4t} \frac{e^{-z}}{\sqrt{z}} P_n(b_0) \, dz \right| \leq k_n t^{-1/2+\delta_0}, \quad (3.34)$$

which together with (3.33) and (3.30) concludes the proof of (3.2).

It remains to show that $R_N$ satisfies (3.4). Consider first the case $|\theta| > 20$. From (3.11) and (3.18), it is enough in this case to show that

$$\int_0^\infty \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z+\rho^2/4t}} \left(4b^2 + \pi^2\right)^{N+1} 2^{N-1} \, dz \leq k_N, \quad (3.35)$$

which can be seen by the same procedure as that to estimate (3.21) (just considering $(4b^2 + \pi^2)^{N+1}$ instead of $P_n(b)$).

In the case $|\theta| \leq 20$, let us split the integral in (3.28) according to $z \in (\rho^2/4t, \rho^2/4\sqrt{t})$, or $z > \rho^2/4\sqrt{t}$. For the latter, observe that

$$1 + bx = \frac{\log \left(4zt/\rho^2 + \sqrt{(4zt/\rho^2)^2 - 1}\right)}{\log t} \geq \frac{\log \sqrt{t}}{\log t} = \frac{1}{2} \quad (3.36)$$
so from (3.18), again a procedure as that to estimate the integral in (3.27) yields that this integral is bounded. For the case \( z \in (\rho^2/4t, \rho^2/4\sqrt{t}) \), use the expression for \( Q_N \) coming from (3.16) and take \( 0 < \delta_1 < 1/4 \)

\[
\left| \int_{\rho^2/4t}^{\rho^2/4\sqrt{t}} \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z+\rho^2/4t}} \left( \frac{1 + bx}{(1 + bx)^2 + (\theta + \frac{\pi}{2}x)^2} + \frac{1 + bx}{(1 + bx)^2 + (\theta - \frac{\pi}{2}x)^2} \right) \, dz \right| \leq \int_{\rho^2/4t}^{\rho^2/4\sqrt{t}} \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z+\rho^2/4t}} \sum_{n=0}^N (-1)^n P_n(b) \cos \left( (n + 1)\frac{\pi}{2} \right) x^{n} \, dz \leq \frac{1}{\sqrt{t}} \log t \leq 1
\]

To obtain the last inequality above, start by estimating the first integral in the previous line. Observe that \( 1 + bx = \frac{\arccosh(\sqrt{t})}{\log t} \), and change variables \( y = \arccosh(\sqrt{t}) \) to obtain:

\[
\int_{\rho^2/4t}^{\rho^2/4\sqrt{t}} \frac{e^{-(z+\rho^2/4t)}}{\sqrt{z+\rho^2/4t}} \left( \frac{1}{1 + bx} \right) \, dz \leq \log t \int_{\rho^2/4t}^{\rho^2/4\sqrt{t}} \frac{1}{\sqrt{z} \arccosh(\sqrt{t})} \, dz = \frac{\rho}{\sqrt{t}} \left( \int_{0}^{1} \sinh y \, dy + \int_{1}^{\sqrt{t}} \frac{\sinh y}{y \cosh y} \, dy \right) \leq \frac{\rho}{\sqrt{t}} (k_1 + k_2 t^{1/2})
\]

The last integral in (3.37) is bounded by \( t^{-\frac{1}{4} + \delta_1} \), as can be seen from estimates similar to those used to estimate (3.27), which finishes the proof of Theorem 3.1.

Proof of Theorem 3.2 Recall that \( \rho = 1 \) and \( x = 1/\log t \). From (2.3) and (3.9), we have

\[
\log(\sqrt{t}) f(\theta, t) = \frac{\log(\sqrt{t})}{\sqrt{2\pi t}} \int_{-\pi/2}^{\pi/2} e^{-\frac{1}{4}(1-\cos 2\theta)} \cos \theta \, \mathbf{1}_{(-\pi/2, \pi/2)}(\theta) + \frac{1}{2\pi \sqrt{t}} \int_{1/4t}^{\infty} e^{-(z+1/4t)/\sqrt{z+1/4t}} \, dz \times \left( \frac{1 + bx}{(1 + bx)^2 + (\theta + \frac{\pi}{2}x)^2} + \frac{1 + bx}{(1 + bx)^2 + (\theta - \frac{\pi}{2}x)^2} \right) \, dz.
\]

Let us denote by \( k \) positive constants that may change from line to line, and split the integral above according to \( z \in (1/4t, 1/4\sqrt{t}) \) or \( z > 1/4\sqrt{t} \).
Proceeding as in the estimation of (3.38), it follows that

\[
\frac{\log(\sqrt{t})}{\sqrt{2\pi t}} e^{-\frac{1}{4}(1-\cos 2\theta)} \cos \theta \int_{(-\pi/2,\pi/2)} d\theta + \frac{1}{2\pi \sqrt{t}} \int_{1/4t}^{1/4\sqrt{t}} \frac{e^{-(z+1/4t)}}{\sqrt{z+1/4t}} \times \nabla \quad (3.38)
\]

Expanding the fractions in (3.39) with the aid of Lemma 3.4, we have

\[
\frac{1 + bx}{(1 + bx)^2 + (\theta + \frac{\pi}{2})^2x^2} + \frac{1 + bx}{(1 + bx)^2 + (\theta - \frac{\pi}{2})^2x^2} dz \leq k t^{-1/4}(3.40)
\]

From (3.42), it is easy to see that

\[
|A_n(b, \pm \theta)| \leq (b^2 + (\frac{\pi}{2} \pm \theta)^2)^{n/2}. (3.44)
\]

For \( z > \frac{1}{1/4\sqrt{t}} \) the denominators in \( T_n^\pm \) are \( > (\frac{1}{2})^2 \), as follows from (3.36). Therefore

\[
|T_n^\pm| \leq x^{N+1} k (b^2 + (\frac{\pi}{2} \pm \theta)^2)^{\frac{N+1}{2}},
\]

and proceeding as in the estimation of (3.35), we obtain

\[
\frac{1}{2\pi \sqrt{t}} \int_{1/4\sqrt{t}}^{\infty} \frac{e^{-(z+1/4t)}}{\sqrt{z+1/4t}} (T_n^\pm + T_n^\mp) dz \leq k_n \frac{1 + |\theta|^{N+2}}{(\log \sqrt{t})^{N+1}} (3.45)
\]

On the other hand, from (3.42), it is easy to see that

\[
|A_n(b, \pm \theta) - A_n(b_0, \pm \theta)| \leq |b - b_0| k_n (|b_0|^n + |\theta|^n + 1) (3.46)
\]

Next, set

\[
g_n = \int_0^\infty \frac{e^{-z}}{\sqrt{z}} ((A_n(b_0, \theta) + A_n(b_0, -\theta)) dz,
\]
which from (3.42) is precisely (3.7). Note that the \( g_n' s \) are polynomials in \( \theta \) of degree \( n \). To finish the proof, from (3.39), (3.40), (3.41) and (3.45) it is enough to show that for \( \delta_1 < \frac{1}{4} \),

\[
\int_{1/4\sqrt{t}}^{\infty} \frac{(A_n(b, \theta) + A_n(b, -\theta)) e^{-(z+1/4t)}}{\sqrt{z+1/4t}} \, dz - \int_0^{\rho^2/4t} \frac{(A_n(b_0, \theta) + A_n(b_0, -\theta)) e^{-z}}{\sqrt{z}} \, dz \leq k_n (1 + |\theta|^n) t^{-\frac{1}{4} + \delta_1},
\]

(3.47)

what follows from estimates similar to those in (3.33) and (3.34), with the aid of (3.44) and (3.46). 

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