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Spectrally equivalent time-dependent double wells and unstable anharmonic oscillators

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ABSTRACT: We construct a time-dependent double well potential as an exact spectral equivalent to the explicitly time-dependent negative quartic oscillator with a time-dependent mass term. Defining the unstable anharmonic oscillator Hamiltonian on a contour in the lower-half complex plane, the resulting time-dependent non-Hermitian Hamiltonian is first mapped by an exact solution of the time-dependent Dyson equation to a time-dependent Hermitian Hamiltonian defined on the real axis. When unitary transformed, scaled and Fourier transformed we obtain a time-dependent double well potential bounded from below. All transformations are carried out npn-perturbatively so that all Hamiltonians in this process are spectrally exactly equivalent in the sense that they have identical instantaneous energy eigenvalue spectra.

1. Introduction

Anharmonic oscillators have a wide range of applications in quantum mechanics as they describe for instance delocalization and decoherence of quantum states, e.g. [1]. They also occur naturally in relativistic models, e.g. [2]. From a mathematical point of view their nonlinear nature make them ideal testing grounds for various approximation methods, such as perturbative approaches [3]. Based on a perturbative expansion of the energy eigenvalues it was shown in [4] that the quartic anharmonic oscillator with mass term is spectrally equivalent to a double well potential with linear symmetry breaking. The first hint about the fact that even the unstable quartic anharmonic oscillator posses a well defined bounded real spectrum, despite being unbounded from below on the real axis, was proved in [5, 6], where it was proven that its energy eigenvalues series is Borel summable. The spectral equivalence between an unstable anharmonic oscillator and a complex double well potential was then proven directly by Buslaev and Grecchi in [7].

Subsequently the unstable quartic anharmonic oscillator without mass term was treated in [8] as part of the general series of PT-symmetric potentials $V(x) = x^2(ix)^\varepsilon$, i.e. $\varepsilon = 2$, 


where it was shown numerically that the Hamiltonians in this series have real and positive spectra for $\varepsilon \geq 2$. Applying the techniques developed in this area of non-Hermitian $\mathcal{PT}$-symmetric quantum mechanics [9, 10] Jones and Mateo [11] showed that the two Hamiltonians
\begin{equation}
H = p^2 - gx^4, \quad \text{and} \quad h = \frac{p^4}{64g} - \frac{1}{2}p + 16gx^2,
\end{equation}
are spectrally equivalent. This was established by first defining $H$ on a suitable contour in the complex plane, $x \to -2i\sqrt{1 + ix}$, within the Stokes wedges where the corresponding wavefunctions decay asymptotically. Subsequently the resulting complex Hamiltonian was similarity transformed to a Hermitian Hamiltonian $h$ that is well defined on the real axis.

Here our central aim is to extend the analysis by making the Hamiltonian explicitly time-dependent $H \to H(t)$ through the inclusion of an explicit time-dependence into the coefficients. The similarity transformation acquires then the form
\begin{equation}
h(t) = \eta(t)H(t)\eta^{-1}(t) + i\partial_t \eta(t)\eta^{-1}(t),
\end{equation}
often referred to as the time-dependent Dyson equation [12, 13, 14, 15, 16, 17, 18, 19, 20], in which $H \neq H^\dagger$ is a non-Hermitian explicitly time-dependent Hamiltonian, $h = h^\dagger$ a Hermitian explicitly time-dependent Hamiltonian and $\eta(t)$ the time-dependent Dyson map. The latter can be used to define a time-dependent metric $\rho(t)$ via the relation $\rho(t) = \eta(t)\eta(t)$. Spectral equivalence is then understood on the level of the instantaneous energy eigenvalues for the operators $h(t)$ and the corresponding operator for the non-Hermitian system
\begin{equation}
\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\eta^{-1}(t)\partial_t \eta(t).
\end{equation}

Note while $\tilde{H}$ is observable it is not a Hamiltonian governing the time-evolution and satisfying the time-dependent Schrödinger equation. On the other hand the Hamiltonian $H(t)$ is not observable. Besides the aforementioned interest in the unstable anharmonic oscillator itself, there are not many known exact solutions [15, 17, 21, 18, 22, 19, 23, 24, 25, 26, 27, 28, 29, 30] to the time-dependent Dyson equation (1.2), so that any new exact solution provides valuable insights.

2. The time-dependent unstable harmonic oscillator

The Hamiltonian we investigate here is similar to the one in equation (1.1), but with time-dependent coefficient functions and an additional mass term
\begin{equation}
H(z,t) = p^2 + \frac{m(t)}{4}z^2 - \frac{g(t)}{16}z^4, \quad m \in \mathbb{R}, g \in \mathbb{R}^+.
\end{equation}
Defining $H(z,t)$ now on the same contour in the lower-half complex plane $z = -2i\sqrt{1 + ix}$ as suggested by Jones and Mateo [11], it is mapped into the non-Hermitian Hamiltonian
\begin{equation}
H(x,t) = p^2 - \frac{1}{2}p + \frac{i}{2}\{x, p^2\} - m(t)(1 + ix) + g(t)(x - i)^2,
\end{equation}
with \(\{\cdot,\cdot\}\) denoting the anti-commutator. Next we attempt to solve the time-dependent Dyson equation (1.2) to find a Hermitian counterpart \(h\). Making the following general Ansatz for the Dyson map

\[
\eta(t) = e^{\alpha(t)x}e^{\beta(t)p^2+i\gamma(t)p^2+i\delta(t)p}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R},
\]

(2.3)

we use the Baker-Campbell-Hausdorff formula to compute the adjoint action of \(\eta(t)\) on all terms appearing in \(H(x, t)\)

\[
\eta xy^{-1} = x + \delta + 6\alpha\beta p + 2\gamma p + 3i\alpha^2\beta + 2i\alpha\gamma - 3i\beta p^2,
\]

(2.4)

\[
\eta y^{-1} = p + i\alpha,
\]

(2.5)

\[
\eta xx^{-1} = x^2 - 9\beta^2 p^4 - 12i\beta(3\alpha\beta + \gamma)p^3 + (54\alpha^2\beta^2 + 36\alpha\beta\gamma + 4\gamma^2 - 6i\beta\delta)p^2
\]

\[
+ 4(3\alpha\beta + \gamma)(\delta + ia(3\alpha\beta + 2\gamma))p + 2(\delta + 3i\alpha^2\beta + 2i\alpha\gamma)x
\]

\[
+ (6\alpha\beta + 2\gamma)\{x,p\} - 3i\beta\{x,p^2\} - (3\alpha^2\beta + 2\alpha\gamma - i\delta)^2,
\]

(2.6)

\[
\eta p^2 y^{-1} = p^2 - \alpha^2 + 2i\alpha p,
\]

(2.7)

\[
\eta \{x,p^2\} y^{-1} = \{x,p^2\} - 6i\beta p^4 + (24\alpha \beta + 4\gamma)p^3 + (36i\alpha^2\beta + 12i\alpha\gamma + 2\delta)p^2 - 2\alpha^2 x
\]

\[
+ 4(i\alpha\delta - 6\alpha^2\beta - 3\alpha^2\gamma)p - 2i\alpha^2(3\alpha^2\beta + 2\alpha\gamma - i\delta) + 4i\alpha\{x,p\},
\]

(2.8)

The gauge like terms in (1.2) and (1.3) are calculated to

\[
i\eta^{-1} = ix\dot{\alpha} + i\beta p^3 - \left(3\dot{\beta}\alpha + \gamma\right)p^2 - \left(3i\beta\alpha^2 + 2i\gamma\alpha + \delta\right)p + i\beta\alpha^3 + \gamma\alpha^2 - i\delta\alpha,
\]

(2.9)

\[
i\eta^{-1} = ix\dot{\alpha} + i\beta p^3 - \left(3\dot{\beta}\alpha + \gamma\right)p^2 - \left(2i\gamma\alpha + \delta\right)p - i\delta\alpha,
\]

(2.10)

where as commonly used we abbreviate partial derivatives with respect to \(t\) by an overdot.

Using the expressions in (2.4)-(2.9) for the evaluation of (1.2) and demanding the right hand side to be Hermitian yields the following constraints for the coefficient functions in the Dyson map

\[
\alpha = \frac{\dot{g}}{6g}, \quad \beta = \frac{1}{6g}, \quad \gamma = \frac{12g^2 + 6mg^2 + \ddot{g}^2 - g\ddot{g}}{4\dot{g}^2}, \quad \delta = c_1 \frac{g}{\dot{g}} - \frac{g \ln g}{2\dot{g}},
\]

(2.11)

with \(c_1 \in \mathbb{R}\) being an integration constant. Moreover, the time-dependent coefficient functions in the Hamiltonian (2.1) must be related by the third order differential equation

\[
9g^2(\ddot{g} - 6g\dot{m}) + 36g\dot{g}(gm - \dot{g}) + 28\dot{g}^3 = 0.
\]

(2.12)

Integrating once and introducing a new parameterization function \(\sigma(t)\), we solve this equation by

\[
g = \frac{1}{4\sigma^3}, \quad \text{and} \quad m = \frac{4c_2 + \dot{\sigma}^2 - 2\sigma\ddot{\sigma}}{4\sigma^2},
\]

(2.13)

with \(c_2 \in \mathbb{R}\) denoting the integration constant corresponding to the only integration we have carried out. The time-dependent Hermitian Hamiltonian in equation (1.2) then results to

\[
h(x, t) = \sigma^3 p^4 + f_{pp}(t)p^2 + f_x(t)x + f_p(t)p + f_{xp}(t)\{x, p\} + f_{xx}(t)x^2 + C(t).
\]

(2.14)
with
\[ f_{pp} = \sigma \left\{ \sigma \left[ 2 (\sigma (\dot{\sigma}^2 - 4c_2^2) - 2) \dot{\sigma} + 16c_2^2 + \dot{\sigma}^4 \right] + 16c_2 \right\} + 4, \quad f_{xp} = \frac{\left( \sigma (\dot{\sigma}^2 - 4c_2^2) - 2 \right)}{4\sigma^4 \dot{\sigma}}, \]
\[ f_p = \frac{2c_1 \left[ \sigma (4c_2 + \dot{\sigma}^2 - 2\sigma \dot{\sigma}) + 2 \right]}{12\sigma \dot{\sigma}^2} + \ln \left( 4\sigma^3 \right), \quad f_x = \frac{-2c_1 + \ln \left( 4\sigma^3 \right)}{12\sigma \dot{\sigma}^2}, \quad f_{xx} = \frac{1}{4\sigma^3}, \]
\[ C = \frac{(2c_1 + \ln \left( 4\sigma^3 \right))^2}{144\sigma \dot{\sigma}^2} + \frac{36\dot{\sigma}^2 \left( 4c_2^2 + \dot{\sigma} \right)}{144\sigma \dot{\sigma}^2} + \frac{1}{8} \left( \dot{\sigma}^2 - 4c_2 \right) \frac{1}{\dot{\sigma} - \frac{\dot{\sigma}^2}{4\sigma^2}}. \]

We may choose to set \( c_1 = c_2 = 0 \) and reintroduce the original time-dependent coefficient functions \( g(t), m(t) \) so that the Hamiltonian simplifies to

\[ h(x, t) = \frac{p^4}{4g} + \left( \frac{18g^2(2g + m)}{\dot{g}^2} - \frac{\dot{g}^2}{72g^3} - \frac{2g + m}{4g} \right) p^2 - \frac{3 \left( \dot{g}^2 \sigma^2 + g^3 \right) \ln g}{\dot{g}} p + \frac{\dot{g}^2 \ln(g)}{\dot{g}} x^2 + \frac{1296g^6 \ln^2 g + \dot{g}^6 - 36\dot{g}^4 g^2 (2g + m)}{1296g^6 \dot{g}^2} - \frac{m}{2}. \quad (2.15) \]

Notice that \( \sigma(t) \) can be any function, but the coefficient functions \( g(t) \) and \( m(t) \) must be related by (2.12) that is (2.13).

The massless case for \( m(t) = 0 \) is more restrictive and leads to \( \sigma(t) \) being a second order polynomial \( \sigma(t) = \kappa_0 + \kappa_1 t + \kappa_2 t^2 \) with real constants \( \kappa_i \). This case is consistently recovered from (2.13) with the choice \( c_2 = \kappa_1 \kappa_3 - \kappa_2^2 / 4 \). The solution found for the time-independent case in [11], would be obtained from (2.3) in the limits \( \alpha \to 0, \beta \to 1 / 6g, \gamma \to 0, \delta \to i \) and \( m \to 0 \). While this limit obviously exists for \( \alpha \) and \( \beta \), the constraints for \( \gamma \) and \( \delta \) are different from those reported in (2.11). In fact, setting \( \delta(t) \to i \delta(t) \) enforces \( g \) to be time-independent and there is no time-dependent solution corresponding to that choice. The energy operator \( \hat{H} \) defined in (1.3) is obtained directly by adding \( H(x, t) \) in (2.2) and the gauge-like term in (2.10).

Let us now eliminate the terms in \( h(x, t) \) proportionate to \( x \) and \( \{x, p\} \) by means of a unitary transformation

\[ U = e^{-i \int f_{pp}x^2 - i \int f_{xp}x}, \quad (2.16) \]

which leads to the unitary transformed Hamiltonian

\[ \hat{h}(x, t) = \sigma^3 p^4 + \left( f_{pp} - \frac{f_{xp}^2}{f_{xx}} \right) p^2 - f_p + f_{xx} x^2 + C - \frac{f_{xx}^2}{4f_{xx}}. \quad (2.17) \]

Similarly as in the time-independent case [11], we may scale this Hamiltonian, albeit now with a time-dependent function, \( x \to (f_{xx})^{-1/2} x \). Subsequently we Fourier transform \( \hat{h}(x, t) \) so that it is viewed in momentum space. In this way we obtain a spectrally equivalent Hamiltonian with a time-dependent potential

\[ \hat{h}(y, t) = \frac{g^2}{4} y^2 + \sigma^3 f_{xx}^2 y^4 + \left( f_{xx} f_{pp} - f_{xx}^2 \right) y^2 + \left( \sqrt{f_{xx}} f_p - \frac{f_{xx} f_{xp}}{\sqrt{f_{xx}}} \right) y + C - \frac{f_{xx}^2}{4f_{xx}}, \quad (2.18) \]

\[ = \frac{g^2}{4} y^2 \left( y^2 + \frac{\dot{g}^2}{36g^3} + \frac{72g^2 m}{g^2} - \frac{m}{g} + 2 \right) + \frac{\left( 36g^2 m + \dot{g}^2 \right)}{12g^2} \sqrt{g} \ln g y \quad (2.19) \]

\[ + \frac{\dot{g}^4}{5184g^9} - \frac{\dot{g}^2 m}{144g^3} - \frac{\dot{g}^2}{72g^2} - \frac{m}{2}, \]

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where for simplicity we have set $c_1 = c_2 = 0$ in (2.19). The potential in $\tilde{h}(y, t)$ is a double well that is bounded from below. We illustrate this for a specific choice of $\sigma(t)$, that is $g(t)$ and $m(t)$, in figure 1.

![Figure 1: Spectrally equivalent time-dependent anharmonic oscillator potential $V(z, t)$ in (2.1) and time-dependent double well potential $\tilde{V}(y, t)$ in (2.19) for $\sigma(t) = \cosh t$, $g(t) = 1/4 \cosh^3 t$, $m(t) = (\tanh^2 t - 2)/4$ at different values of time.](image)

3. Conclusions

We have proven the remarkable fact that the time-dependent unstable anharmonic oscillator is spectrally equivalent to a time-dependent double well potential that is bounded from below. The transformations we carried out are summarized as follows:

$$H(z, t) \xrightarrow{z \to x} H(x, t) \xrightarrow{\text{Dyson}} h(x, t) \xrightarrow{\text{unitary transform}} \hat{h}(x, t) \xrightarrow{\text{Fourier}} \tilde{h}(y, t).$$

We have first transformed the time-dependent anharmonic oscillator $H(z, t)$ from a complex contour in a Stokes wedge to the real axis $H(x, t)$. The resulting non-Hermitian Hamiltonian $H(x, t)$ was then mapped by mean of a time-dependent Dyson map $\eta(t)$ to a time-dependent Hermitian Hamiltonian $h(x, t)$. It turned out that the Dyson map cannot be obtained by simply introducing time-dependence into the known solution for the time-independent case [11], but it required to complexify one of the constants and the inclusion of two additional factors. In order to obtain a potential Hamiltonian we have unitary transformed $h(x, t)$ into a spectrally equivalent Hamiltonian $\hat{h}(x, t)$, which when Fourier transformed leads to $\tilde{h}(y, t)$ that involved a time-dependent double well potential.

A detailed analysis of the spectra and eigenfunctions using approximation methods for time-dependent potential [31] is left for future investigations. Moreover, it is well known that Dyson maps are not unique, in the time-dependent as well as time-independent case, and it might therefore be interesting to explore whether additional spectrally equivalent Hamiltonians to $H(z, t)$ can be found in the same fashion for new type of maps.

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