THIRD ORDER ODES AND FOUR-DIMENSIONAL SPLIT SIGNATURE EINSTEIN METRICS

MICHAL GODLIŃSKI AND PAWEŁ NUROWSKI

ABSTRACT. We construct a family of split signature Einstein metrics in four dimensions, corresponding to particular classes of third order ODEs considered modulo fiber preserving transformations of variables.

1. Introduction

Our starting point is a 3rd order ordinary differential equation (ODE)

\[ y''' = F(x, y, y', y''), \]

for a real function \( y = y(x) \). Here \( F = F(x, y, p, q) \) is a sufficiently smooth real function of four real variables \( (x, y, p = y', q = y'') \).

Given another 3rd order ODE

\[ \bar{y}''' = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'') \]

it is often convenient to know whether there exists a suitable transformation of variables \( (x, y, p, q) \to (\bar{x}, \bar{y}, \bar{p}, \bar{q}) \) which brings (2) to (1). Several types of such transformations are of particular importance. Here we consider fiber preserving (f.p.) transformations, which are of the form

\[ \bar{x} = \bar{x}(x), \quad \bar{y} = \bar{y}(x, y). \]

We say that two 3rd order ODEs, (1) and (2), are (locally) f.p. equivalent iff there exists a (local) f.p. transformation (3), which brings (2) to (1). The task of finding necessary and sufficient conditions for ODEs (1) and (2) to be (locally) f.p. equivalent, is called a f.p. equivalence problem for 3rd order ODEs. In the cases of (more general) point transformations and contact transformations, this problem was studied and solved by E. Cartan [1] and S-S. Chern [2] in the years 1939-1941. The interest in these studies has been recently revived due to the fact that important equivalence classes of 3rd order ODEs naturally define 3-dimensional conformal Lorentzian structures including Einstein-Weyl structures. This makes these equivalence problems applicable not only to differential geometry but also to the theory of integrable systems and General Relativity [3, 8, 11].

In this paper we show how to construct 4-dimensional split signature Einstein metrics, starting from particular ODEs of 3rd order. We formulate the problem of f.p. equivalence in terms of differential forms. Invoking Cartan’s equivalence method, we construct a 6-d manifold with a distinguished coframe on it, which encodes all information about original equivalence problem. For specific types of
the ODEs, the class of Einstein metrics can be explicitly constructed from this coframe. This result is a byproduct of the full solution of the f.p. equivalence problem, that will be described in [5].

We acknowledge that all our calculations were checked by the independent use of the two symbolic calculations programs: Maple and Mathematica.

2. Third order ODE and Cartan’s method

Following Cartan and Chern, we rewrite (1), using 1-forms
\[
\begin{align*}
\omega_1 &= dy - px, \\
\omega_2 &= dp - qx, \\
\omega_3 &= dq - F(x,y,p,q)dx, \\
\omega_4 &= dx.
\end{align*}
\]
These are defined on the second jet space \( J^2 \) locally parametrized by \((x, y, p, q)\).

Each solution \( y = f(x) \) of (1) is fully described by the two conditions: forms \( \omega_1, \omega_2, \omega_3 \) vanish on a curve \((t, f(t), f'(t), f''(t))\) and, as this defines a solution up to transformations of \( x \), \( \omega_4 = dt \) on this curve. Suppose now, that equation (1) undergoes fiber preserving transformations (3). Then the forms (4) transform by
\[
\begin{align*}
\bar{\omega}_1 &= \alpha \omega_1, \\
\bar{\omega}_2 &= \beta (\omega_2 + \gamma \omega_1), \\
\bar{\omega}_3 &= \epsilon (\omega_3 + \eta \omega_2 + \kappa \omega_1), \\
\bar{\omega}_4 &= \lambda \omega_4,
\end{align*}
\]
where functions \( \alpha, \beta, \gamma, \epsilon, \eta, \kappa, \lambda \) are defined on \( J^2 \), satisfy \( \alpha \beta \epsilon \lambda \neq 0 \) and are determined by a particular choice of transformation (3). A fiber preserving equivalence class of ODEs is described by forms (4) defined up to transformations (5).

Equations (1) and (2) are f.p. equivalent, iff their corresponding forms \( (\omega_i) \) and \( (\bar{\omega}_j) \) are related as above.

We now apply Cartan’s equivalence method [9, 10]. Its key idea is to enlarge the space \( J^2 \) to a new manifold \( \tilde{P} \), on which functions \( \alpha, \beta, \gamma, \epsilon, \eta, \kappa, \lambda \) are additional coordinates. The coframe \( (\omega_i) \) defined up to transformations (5), is now replaced by a set of four well defined 1-forms \( \theta_1, \theta_2, \theta_3, \theta_4 \)
on \( \tilde{P} \). If, in addition, the following f.p. invariant condition [3, 4]
\[
F_{qq} \neq 0
\]
is satisfied then, there is a geometrically distinguished way of choosing five parameters \( \beta, \epsilon, \eta, \kappa, \lambda \) to be functions of \((x,y,p,q,\alpha,\gamma)\). Then, on a 6-dimensional manifold \( P \) parametrized by \((x,y,p,q,\alpha,\gamma)\) Cartan’s method give a way of supplementing the well defined four 1-forms \( (\theta_i) \) with two other 1-forms \( \Omega_1, \Omega_2 \) so that the set \( (\theta_1, \theta_2, \theta_3, \theta_4, \Omega_1, \Omega_2) \) constitutes a rigid coframe on \( P \). According to
the theory of G-structures [7, 10], all information about a f.p. equivalence class of equation (1) satisfying $F_{qq} \neq 0$ is encoded in the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega^1, \Omega^2)$. Two equations (1) and (2) are f.p. equivalent, iff there exists a diffeomorphism $\psi : \mathcal{P} \to \mathcal{P}$, such that $\psi^* \theta^i = \theta^i$, $\psi^* \Omega^A = \Omega^A$, where $i = 1, 2, 3, 4$ and $A = 1, 2$. The procedure of constructing manifold $\mathcal{P}$ and the coframe $(\theta^i, \Omega^A)$ is explained in details in [7, 10] for a general case and in [4, 5] for this specific problem. Here we omit the details of this procedure, summarizing the results on f.p. equivalence problem in the following theorem.

**Theorem 2.1.** A 3rd order ODE $y''' = F(x, y, y', y'')$, satisfying $F_{qq} \neq 0$, considered modulo fiber preserving transformations of variables, uniquely defines a 6-dimensional manifold $\mathcal{P}$, and an invariant coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega^1, \Omega^2)$ on it. In local coordinates $(x, y, p = y', q = y'', \alpha, \gamma)$ this coframe is given by

\[
\begin{align*}
\theta^1 &= \alpha \omega^1, \\
\theta^2 &= \frac{1}{6} F_{qq} (\omega^2 + \gamma \omega^1), \\
\theta^3 &= \frac{1}{36a} F_{qq} (\omega^3 + (\gamma - \frac{1}{6}) F_{q qq} \omega^2 + (\frac{1}{2} \gamma^2 + K) \omega^1), \\
\theta^4 &= \frac{\partial}{\partial q} F_{qq} \omega^4, \\
\Omega^1 &= \frac{1}{F_{qq}} (-F_{qq} \gamma^2 + (\frac{2}{3} F_{qq} F_q + \frac{1}{3} F_{qq}^2 + 2 F_{qq} \gamma) \\
&+ F_{qq} K_q + 2 F_{qq} K - 2 F_{qq} \omega^1 - \frac{\partial}{\partial q} d \alpha, \\
\Omega^2 &= -\frac{\partial}{\partial q} F_{qq} (\frac{1}{2} \gamma^2 + \frac{1}{3} F_q \gamma + K) \omega^4, \\
&+ \frac{1}{6a} (-\frac{1}{2} F_{qq} \gamma^2 + (\frac{1}{3} F_{qq} F_q + F_{qq q}) \gamma + F_{qq q} - F_{qq} \omega^2), \\
&+ \frac{1}{6a} (-\frac{1}{2} F_{qq} \gamma^3 + (\frac{2}{3} F_{qq}^2 + \frac{1}{3} F_{qq} F_q + F_{qq q}) \gamma^2 + (F_{qq} K_q - F_{qq} + F_{qq} K) \gamma, \\
&- \frac{1}{2} F_{qq} F_q - F_{qq} K_p - \frac{1}{2} F_{qq} F_q K_q + \frac{1}{3} F_{qq}^2 K \omega^1 + \frac{\partial}{\partial q} F_{qq} d \gamma,
\end{align*}
\]

where $K$ denotes

$$K = \frac{1}{6} (F_{xx} + p F_{xy} + q F_{qp} + F F_{qq}) - \frac{1}{6} F_q^2 - \frac{1}{2} F_p$$

and $\omega^i, i = 1, 2, 3, 4$ are defined by the ODE via (4).

Exterior derivatives of the above invariant forms read

\[
\begin{align*}
d \theta^1 &= \Omega^1 \wedge \theta^1 + \theta^1 \wedge \theta^2, \\
d \theta^2 &= \Omega^2 \wedge \theta^2 - \Omega^1 \wedge \theta^3 + (2 - 2c) \theta^3 \wedge \theta^2 + e \theta^1 \wedge \theta^1 + 2 b \theta^2 \wedge \theta^2, \\
d \theta^3 &= \Omega^3 \wedge \theta^3 + f \theta^4 \wedge \theta^1 + (c - 2) \theta^4 \wedge \theta^2 + a \theta^2 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\
d \theta^4 &= \Omega^4 \wedge \theta^4 + g \theta^5 \wedge \theta^1 + h \theta^2 \wedge \theta^2 + k \theta^4 \wedge \theta^3, \\
d \Omega^1 &= (2c - 2) \Omega^2 \wedge \theta^1 - \Omega^2 \wedge \theta^1 + g \theta^1 \wedge \theta^1 + h \theta^2 \wedge \theta^2 + k \theta^1 \wedge \theta^4 - f \theta^2 \wedge \theta^4, \\
d \Omega^2 &= \Omega^2 \wedge \Omega^1 - a \Omega^2 \wedge \theta^3 - b \Omega^2 \wedge \theta^4 + l \theta^1 \wedge \theta^2 + m \theta^1 \wedge \theta^3 + n \theta^1 \wedge \theta^4 + f \theta^3 \wedge \theta^3, \\
d \Omega^3 &= \Omega^2 \wedge \theta^3 - s \theta^2 \wedge \theta^4 - f \theta^3 \wedge \theta^3, \\
d \Omega^4 &= \Omega^3 \wedge \theta^2 - s \theta^2 \wedge \theta^4 - f \theta^3 \wedge \theta^3,
\end{align*}
\]

where $a, b, c, e, f, g, h, k, l, m, n, r, s$ are functions on $\mathcal{P}$, which can be simply calculated due to formulae (4). The simplest and the most symmetric case, when all the functions $a, b, c, e, f, g, h, k, l, m, n, r, s$ vanish, corresponds to the f.p. equivalence class of equation

$$y''' = \frac{3}{2} y''^2.$$
the \( \text{so}(2,2) \)-valued flat Cartan connection on \( \mathcal{P} = \text{SO}(2,2) \). Since the Levi-Civita connection for the split signature metrics in four dimensions also takes value in \( \text{so}(2,2) \), we ask under which conditions on f.p. equivalence classes of ODEs (1), the equations (7) may be interpreted as the structure equations for the Levi-Civita connection of a certain 4-dimensional split signature metric \( G \).

3. The construction of the metrics

It is convenient to change the basis of 1-forms \( \theta^1, \theta^2, \theta^3, \theta^4, \Omega^1, \Omega^2 \) on \( \mathcal{P} \) to

\[
\tau^1 = 2\theta^1 + \theta^4, \quad \tau^2 = \Omega^2, \quad \tau^3 = \Omega^2 + 2\theta^1, \quad \tau^4 = \theta^4, \\
\Gamma_1 = \Omega^1, \quad \Gamma_2 = \Omega^1 + 2\theta^2.
\]

After this change, equations (7) yield the formulae for the exterior differentials of \( \tau^1, \tau^2, \tau^3, \tau^4, \Gamma_1, \Gamma_2 \). These are the formulae (23) of the appendix. They can be used to analyze the properties of the following bilinear tensor field

\[
\tilde{G} = \tilde{G}_{ij} \tau^i \tau^j = 2\tau^1 \tau^2 + 2\tau^3 \tau^4
\]
on \( \mathcal{P} \). The first question we ask here is the following: under which conditions on \( a, b, c, e, f, g, k, l, m, n, r, s \) the first four of equations (23) may be identified with

\[
d\tau^i + \Gamma^i_j \wedge \tau^j = 0,
\]
where the 1-forms \( \Gamma^i_j, i, j = 1, 2, 3, 4 \) satisfy

\[
\Gamma_{(ij)} = 0, \quad \text{and} \quad \Gamma_{ij} = \tilde{G}_{ik} \Gamma_{kj}.
\]
This happens if and only if

\[
c = 0, \quad l = 0, \quad r = 0, \quad s = 0.
\]
Now, we call 1-forms \( \Gamma_1, \Gamma_2 \) as \emph{vertical} and 1-forms \( \tau^1, \tau^2, \tau^3, \tau^4 \) as \emph{horizontal}. To be able to interpret

\[
R^i_j = d\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j
\]
as a curvature, we have to require that it is horizontal, i.e contains no \( \Gamma_1, \Gamma_2 \) terms. This is equivalent to

\[
m = 0, \quad a = 0, \quad g = 0, \quad f = -b.
\]
If these conditions are satisfied then the exterior derivatives of (23) give also

\[
b = 0, \quad h = 0.
\]
Concluding, having conditions (10), (11), and (12) satisfied, we have the following differentials of the coframe \( (\theta^1, \theta^2, \theta^3, \theta^4, \Gamma_1, \Gamma_2) \):

\[
d\tau^1 = \Gamma_1 \wedge \tau^1, \\
d\tau^2 = -\Gamma_1 \wedge \tau^2 + \frac{1}{2} n \tau^1 \wedge \tau^4, \\
d\tau^3 = -\Gamma_2 \wedge \tau^3 + \left( \frac{1}{2} n - e \right) \tau^1 \wedge \tau^4, \\
d\tau^4 = \Gamma_2 \wedge \tau^4, \\
d\Gamma_1 = \tau^1 \wedge \tau^2 + \frac{1}{2} k \tau^1 \wedge \tau^4, \\
d\Gamma_2 = \frac{1}{2} k \tau^1 \wedge \tau^4 - \tau^3 \wedge \tau^4,
\]
and the following formulae for the matrix of 1-forms

\[
\Gamma^i_j = \begin{pmatrix}
-\Gamma_1 & 0 & 0 & 0 \\
0 & \Gamma_1 & 0 & -\frac{1}{2}n\tau^1 + (e - \frac{1}{2}n)\tau^4 \\
0 & 0 & 0 & -\Gamma_2 \\
\end{pmatrix}.
\]

Moreover, introducing the frame of the vector fields \((X_1, X_2, X_3, X_4, Y_1, Y_2)\) dual to the coframe \(\tau^1, \ldots, \tau^4, \Gamma_1, \Gamma_2\) we get the following non-vanishing 2-forms \(R^i_j\):

\[
R_1^1 = -\tau^1 \wedge \tau^2 - \frac{1}{2}k\tau^1 \wedge \tau^4,
R_2^2 = \tau^1 \wedge \tau^2 + \frac{1}{2}k\tau^1 \wedge \tau^4,
R_4^4 = \frac{1}{2}k\tau^1 \wedge \tau^4 - \tau^2 \wedge \tau^4,
R_3^3 = \frac{1}{2}k\tau^1 \wedge \tau^4 - \tau^3 \wedge \tau^4.
\]

Here \(f_i\) denotes \(X_i(f)\). It further follows that \(Ric_{ij} = R^k_{ikj}\) satisfies

\[
Ric_{ij} = -\tilde{G}_{ij}.
\]

These preparatory steps enable us to associate with each f.p. equivalence class of ODEs \((1)\) satisfying conditions \((10)-(12)\) a 4-manifold \(\mathcal{M}\) equipped with a split signature Einstein metric \(G\). This is done as follows.

- The system \((13)\) guarantees that the distribution \(\mathcal{V}\) spanned by the vector fields \(Y_1, Y_2\) is integrable. The leaf space of this foliation is 4-dimensional and may be identified with \(\mathcal{M}\). We also have the projection \(\pi : \mathcal{P} \to \mathcal{M}\).
- The tensor field \(\tilde{G}\) is degenerate, \(\tilde{G}(Y_1, \cdot) = 0, \tilde{G}(Y_2, \cdot) = 0\), along the leaves of \(\mathcal{V}\). Moreover, equations \((13)\) imply that

\[
L_{\tau^i} \tilde{G} = 0, \quad L_{\tau^i} \tilde{G} = 0.
\]

Thus, \(\tilde{G}\) projects to a well defined split signature metric \(G\) on \(\mathcal{M}\).

- The Levi-Civita connection 1-form for \(G\) and the curvature 2-form, pulled-back via \(\tau^i\) to \(\mathcal{P}\), identify with \(\Gamma^i_j\) and \(R^i_j\) respectively.
- Thus, due to equations \((13)\), the metric \(G\) satisfies the Einstein field equations with cosmological constant \(\Lambda = -1\).

Below we find all functions \(F = F(x, y, p, q)\) which solve conditions \((10)-(12)\). This will enable us to write down the explicit formulae for the Einstein metrics \(G\) associated with the corresponding equations \(y''' = F(x, y, y', y'')\).

The conditions \(b = 0, c = 0\) in coordinates \(x, y, p, q, \alpha, \gamma\) read

\[
F_{qp} + \frac{1}{3}F_{qq} + 3K_q = 0, \quad F_{qqq} - F_{qqp} - \frac{1}{3}F_{qqq}F_q + \frac{1}{9}F^2_{qq} = 0.
\]

The most general function \(F(x, y, p, q)\) defining 3rd order ODEs satisfying these constraints is

\[
F = \frac{q^2}{p + \sigma(x, y)} + 3\frac{\sigma_x(x, y) + p\sigma_y(x, y)}{p + \sigma(x, y)}q + \xi(x, y, p),
\]

where \(\sigma, \xi\) are arbitrary functions of two and three variables, respectively. Since the equations are considered modulo fiber preserving transformations, we can put
\( \sigma = 0 \) by transformation \( \bar{x} = x \) and \( \bar{y} = \bar{y}(x, y) \) such that \( \bar{y}_x = -\sigma(x, \bar{y}(x, y)) \). Condition \( l = 0 \) now becomes
\[
p^3 \xi_{ppp} - 3p^2 \xi_{pp} + 6p \xi_p - 6 \xi = 0,
\]
with the following general solution
\[
\xi = A(x, y)p^3 + C(x, y)p^2 + B(x, y)p.
\]
Hence \( F \) is given by
\[
F = \frac{3}{2} \frac{q^2}{p} + A(x, y)p^3 + C(x, y)p^2 + B(x, y)p.
\]
It further follows that it fulfills the remaining conditions \( a = f = g = h = m = r = s = 0 \) and that
\[
k = -\frac{C}{4\alpha^2 p}, \quad n = \frac{C_y - zC - 2Ax}{8\alpha^3 p}, \quad e = \frac{1}{2} n + \frac{tC + 2By - C_y}{16\alpha^3 p^2}.
\]
A straightforward application of Theorem 2.1 leads to the following expressions for the ‘null coframe’ \( (\tau^1, \tau^2, \tau^3, \tau^4) \):
\[
\begin{align*}
\tau^1 &= 2\alpha \ dy \\
\tau^2 &= (4\alpha)^{-1} [ C \ dx + (2A - z^2) \ dy + 2dz ] \\
\tau^3 &= (4\alpha p)^{-1} [ -(t + 2B) \ dx - C \ dy + 2dt ] \\
\tau^4 &= 2\alpha p \ dx,
\end{align*}
\]
where the new coordinates \( z \) and \( t \) are
\[
z = \frac{\gamma}{p}, \quad t = \frac{q}{p} + \gamma.
\]
This brings
\[
\tilde{G} = 2(\tau^1 \tau^2 + \tau^3 \tau^4)
\]
on \( P \) to the form that depends only on coordinates \( (x, y, z, t) \). Thus, \( \tilde{G} \) projects to a well defined split signature metric
\[
G = -[t^2 + 2B(x, y)]dx^2 + 2dt dx + [2A(x, y) - z^2]dy^2 + 2dz dy
\]
on a 4-manifold \( M \) parametrized by \( (x, y, z, t) \).

It follows from the construction that metric \( G \) is f.p. invariant. However, it does not yield all the f.p. information about the corresponding ODE. It is clear, since the function \( C \) which is proportional to the f. p. Cartan’s invariant \( k \) of (13), is not appearing in the metric \( G \). From the point of view of the metric, function \( C \) represents a ‘null rotation’ of coframe \( (\tau^i) \). Thus it is not a geometric quantity. Therefore \( G \), although f.p. invariant, can not distinguish between various f.p. nonequivalent classes of equations such as, for example, those with \( C \equiv 0 \) and \( C \neq 0 \). To fully distinguish all non-equivalent ODEs with (15) one needs additional structure than the metric \( G \). This structure is only fully described by the bundle \( \pi : P \to M \) together with the coframe \( (\tau^1, \tau^2, \tau^3, \Gamma_1, \Gamma_2) \) of (13) on \( P \). An alternative description, more in the spirit of the split signature metric \( G \), is presented in section 5.

Now, equations (14) imply that the metric \( G \) is Einstein with cosmological constant \( \Lambda = -1 \). The anti-selfdual part of its Weyl tensor is always of Petrov-Penrose type D. The selfdual Weyl tensor is of type II, if the functions \( A \) and \( B \) are generic.
If $A = A(y)$ and $B = B(x)$ the self-dual Weyl tensor degenerates to a tensor of type D. Summing up we have following theorem.

**Theorem 3.1.** Third order ODE

\[ y''' = \frac{3}{2} \frac{yy''}{y'} + A(x, y)y'^3 + C(x, y)y^2 + B(x, y)y' \]

defines, by virtue of Cartan’s equivalence method, a 4-dimensional split signature metric

\[ G = -[t^2 + 2B(x, y)]dx^2 + 2dtdx + [2A(x, y) - z^2]dy^2 + 2dzdy \]

which is Einstein

\[ \text{Ric}(G) = -G \]

and has Weyl tensor $W = W^{ASD} + W^{SD}$ of Petrov type $D+II$, with the exception of the case $A = A(y)$, $B = B(x)$, when it is of type $D+D$. The metric $G$ is invariant with respect to f.p. transformations of the variables of the ODE.

4. **Uniqueness of the Metrics**

In this section we prove the following theorem.

**Theorem 4.1.** The metrics of theorem 3.1 are the unique family of metrics $G$, which are defined by f.p. equivalence classes of 3rd order ODEs and satisfy the following three conditions.

- The metrics are split signature, Einstein: $\text{Ric}(G) = -G$, and each of them is defined on 4-d manifold $M$, which is the base of the fibration $\pi: P \to M$.
- The family contains a metric corresponding to equation $y''' = \frac{3}{2} \frac{yy''}{y'}$.
- The tensor $\tilde{G} = \pi^*G = \mu_{ij} \theta^i \theta^j + \nu_{iA} \theta^i \Omega^A + \rho_{AB} \Omega^A \Omega^B$, on $P$, when expressed by the invariant coframe $(\theta^i, \Omega^A)$ associated with the respective f.p. equivalence class, has the coefficients $\mu_{ij}, \nu_{iA}, \rho_{AB}; i,j = 1,\ldots, 4; A,B = 1,2$ constant and the same for all classes of the ODEs for which $G$ is defined.

To prove the theorem, it is enough to show the uniqueness of $G$ in the simplest case of equation $y''' = \frac{3}{2} \frac{yy''}{y'}$, and to repeat the calculations from pp. 5 – 6 of Section 3 for a generic equation. The following trivial proposition holds.

**Proposition 4.2.** Let $\tilde{G}$ be a bilinear symmetric form of signature $(+ + - - 00)$ on $P$, such that for a vector field $N$

\[ \tilde{G}(N, \cdot) = 0 \quad \text{then} \quad L_N \tilde{G} = 0. \]

A distribution spanned by such vector fields $N$ is integrable and defines a 4-d manifold $M$ as a space of its integral leaves. There exists exactly one bilinear form $G$ on $M$ with the property $\pi^*G = \tilde{G}$, where $\pi: P \to M$ is the canonical projection assigning a point of $M$ to an integral leaf of the distribution.

Our aim now is to find all the metrics $\tilde{G}$ of proposition 4.2 which, when expressed by the coframe $\theta^i, \Omega^A$ (or, equivalently, by $\tau^i, \Gamma_A$), have constant coefficients. Let us consider the simplest case, corresponding to equation $y''' = \frac{3}{2} \frac{yy''}{y'}$, for which all the invariant functions appearing in (16) and (23) vanish. $P$ is now the Lie
group $SO(2, 2)$, $\tilde{G}$ is a form on Lie algebra $so(2, 2)$, the distribution spanned by
the degenerate fields $N$ is a 2-d subalgebra $\mathfrak{h} \subset so(2, 2)$. Finding $\tilde{G}$ is now a purely
algebraic problem. In our case the basis $(\tau^i, \Gamma_A)$ satisfies
\begin{align}
\mathrm{d}\tau^1 &= \Gamma_1 \wedge \tau^1, & \mathrm{d}\tau^3 &= -\Gamma_2 \wedge \tau^3, \\
\mathrm{d}\tau^2 &= -\Gamma_1 \wedge \tau^2, & \mathrm{d}\tau^4 &= \Gamma_2 \wedge \tau^4,
\end{align}
(18)
which agrees with a decomposition $so(2, 2) = so(1, 2) \oplus so(1, 2)$. A group of trans-
f ormations preserving equations (18) is $O(1, 2) \times O(1, 2)$, that is the intersection
of the orthogonal group $O(2, 4)$ preserving the Killing form $\kappa$ of $so(2, 2)$ and the
group $GL(3) \times GL(3)$ preserving the decomposition $so(2, 2) = so(1, 2) \oplus so(1, 2)$.
Each coframe $(\tilde{\tau}^i, \tilde{\Gamma}_A)$, satisfying (18) is obtained by a linear transformation:
\begin{align}
\begin{pmatrix}
\tilde{\tau}^1 \\
\tilde{\tau}^2 \\
\tilde{\Gamma}_1
\end{pmatrix} &= A
\begin{pmatrix}
\tau^1 \\
\tau^2 \\
\Gamma_1
\end{pmatrix}, & \begin{pmatrix}
\tilde{\tau}^3 \\
\tilde{\tau}^4 \\
\tilde{\Gamma}_2
\end{pmatrix} &= B
\begin{pmatrix}
\tau^3 \\
\tau^4 \\
\Gamma_2
\end{pmatrix}, & A, B \in O(1, 2).
\end{align}
(19)
We use transformations (19) to obtain the most convenient form of the basis
$(\eta_1, \eta_2)$ of the subalgebra $\mathfrak{h} \subset so(2, 2)$. We write down the metric $G$ in the
corresponding coframe $(\tilde{\tau}^1, \tilde{\tau}^2, \tilde{\tau}^3, \tilde{\tau}^4, \tilde{\Gamma}_1, \tilde{\Gamma}_2)$ and impose conditions (17). This con-
ditions imply that the most general form of the metric is $G = 2u\tilde{\tau}^1 \tilde{\tau}^2 + 2v\tilde{\tau}^3 \tilde{\tau}^4$,
where $u, v$ are two real parameters. In such case, $[\eta_1, \eta_2] = 0$ and $\kappa(\eta_1, \eta_1) < 0$, \quad \kappa(\eta_2, \eta_2) < 0. When written in terms of the coframe $(\tau^i, \Gamma_A)$, $G$ involves six
real parameters $u, v, \mu, \phi, \nu, \psi$, however it appears, that only parameters $u$ and $v$ are essential; different choices of $\mu, \phi, \nu, \psi$ define different degenerate distributions spanned by $\eta_1, \eta_2$ and hence spaces $\mathcal{M}$ are different, but metrics $G$ on them are isometric. Thus we can choose $G = 2u\tau^1 \tau^2 + 2v\tau^3 \tau^4$. Computing $\tilde{G}$ for $F = \frac{1}{2}z^2$
we have, in a suitable coordinate system $(x, y, z, t)$,
\begin{align*}
G &= -v[t^2 + 2B(x, y)]d\tau^1 \wedge d\tau^2 + 2vdt \wedge dx + u[2A(x, y) - z^2]d\tau^1 \wedge d\tau^2 + 2ud\tau^1 \wedge dy.
\end{align*}
Parameters $u, v$ can be also fixed, if we demand $G$ to be Einstein with cosmological constant $\Lambda = -1$. This is only possible if $u = 1$, $v = 1$. The tensor field $\tilde{G}$ defined in this way is unique and has the form
\begin{align*}
\tilde{G} = 2\tau^1 \tau^2 + 2\tau^3 \tau^4 = 2\Omega^2(2\theta^1 + \theta^4) + 2\theta^4(2\theta^3 + \Omega^2).
\end{align*}
This formula is used in the generic case explaining our choice of the coframe (8)
and the metric (9). This finishes the proof of theorem (14).

5. The Cartan connection and the distinguished class of ODEs

Here we provide an alternative description of the f.p. equivalence class of third
order ODEs corresponding to $F = F(x, y, p, q)$ of (15). We consider a 4-dimensional
manifold $\mathcal{M}$ parametrized by $(x, y, z, t)$. Then the geometry of a f.p. equivalence
class of ODEs (15) is in one to one correspondence with the geometry of a class of
coframes
\begin{align*}
\tau^1 &= dy, \\
\tau^2 &= \frac{1}{2} \left[ C \, dx + (2A - z^2) \, dy + 2dz \right]
\end{align*}
\( \tau_0^3 = \frac{1}{2} \left[ -(t + 2B) \, dx - C \, dy + 2dt \right] \)

\( \tau_0^4 = dx \),
on \( M \) given modulo a special \( SO(2, 2) \) transformation

\( \tau_i^0 \mapsto h^i_j \tau_j^0 \), where \( (h^i_j) = \begin{pmatrix} 2^\alpha & 0 & 0 & 0 \\ 0 & (2^\alpha)^{-1} & 0 & 0 \\ 0 & 0 & (2\alpha_p)^{-1} & 0 \\ 0 & 0 & 0 & 2\alpha_p \end{pmatrix} \).

The Cartan equivalence method applied to the question if two coframes are transformable to each other via gives the full system of invariants of this geometry. These invariants consist of (i) a fibration \( \pi: P \to M \) of Section 3, which now becomes a Cartan bundle \( H \to P \to M \) with the 2-dimensional structure group \( H \) generated by \( h^i_j \), and (ii) of an \( so(2, 2) \)-valued Cartan connection \( \omega \) described by the coframe \( (\tau^1, \tau^2, \tau^3, \tau^4, \Gamma_1, \Gamma_2) \) of \( \mathcal{P} \). Explicitly, the connection \( \omega \) is given by

\[
\omega^i_j = \begin{pmatrix}
-\frac{1}{2}(\Gamma_1 + \Gamma_2 + \tau^4) & 0 & 0 & -\frac{1}{2} \tau^4 \\
0 & \frac{1}{2}(\tau^2 + \Gamma_2) & \frac{1}{2}(\Gamma_1 - \Gamma_2 - \tau^4) & 0 \\
\Gamma_2 - \tau^3 - \frac{1}{2} \tau^4 & \frac{1}{2}(\Gamma_1 + \Gamma_2 + \tau^4) & -\Gamma_2 + \tau^3 - \frac{1}{2} \tau^4 & -\frac{1}{2} \tau^2 \\
-\tau^1 & \frac{1}{2}(\Gamma_1 - \Gamma_2 - \tau^4) & 0 & \frac{1}{2}(-\Gamma_1 + \Gamma_2 + \tau^4)
\end{pmatrix}.
\]

To see that this is an \( so(2, 2) \) connection it is enough to note that \( g_{ij} \omega^j_k + g_{kj} \omega^i_k = 0 \) with the matrix \( g_{ij} \) given by

\[
g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

Now, equations are interpreted as the requirement that the curvature \( \Omega = d\omega + \omega \wedge \omega \) of this connection \( \omega \) has a very simple form

\[
\Omega = \begin{pmatrix} -\frac{1}{2}k & 0 & 0 & 0 \\ 0 & \frac{1}{2}k & \frac{1}{2}(-k + n - 2e) & -\frac{1}{2}n \\ \frac{1}{2}(k - n + 2e) & 0 & 0 & 0 \\ \frac{1}{2}(k - n + 2e) & 0 & 0 & 0 \end{pmatrix} \tau^1 \wedge \tau^4,
\]

where \( n, e \) and \( k \) are given by . The connection \( \omega \) and its curvature \( \Omega \) yields all the f.p. information of the equation corresponding to . In particular, all the equations with \( k = n = e = 0 \) are f.p. equivalent, all having the vanishing curvature of their Cartan connection \( \omega \).

It is interesting to search for a split signature 4-metric \( H \) for which the connection \( \omega \) is the Levi-Civita connection. The general form of such metric is

\[
H = g_{ij} T^i T^j,
\]

where \( (T^1, T^2, T^3, T^4) \) are four linearly independent 1-forms on \( \mathcal{P} \) which satisfy

\[
dT^i + \omega^i_j \wedge T^j = 0.
\]
Thus, for such $H$ to exist, the 1-forms $(T^1, T^2, T^3, T^4)$ must also satisfy the integrability conditions of (22),

$$\Omega^i_j \wedge T^j = 0,$$

which are just the Bianchi identities for $\omega$ to be the Levi-Civita connection of metric $H$. These identities provide severe algebraic constraints on the possible solutions ($T^i$). Using them, under the assumption that $C(x, y) \neq 0$ in the considered region of $\mathcal{P}$, we found all ($T^i$)s satisfying (22). Thus, with every triple $C \neq 0, A, B$ corresponding to an ODE given by $F$ of (13), we were able to find a split signature metric $H$ for which connection $\omega$ is the Levi-Civita connection. Surprisingly, given $A, B$ and $C \neq 0$ the general solution for ($T^i$) involves four free real functions. Two of these functions depend on 6 variables and the other two depend on 2 variables. Thus, each f.p. equivalence class of ODEs representd by $F$ of (13) defines a large family of split signature metrics $H$ for which $\omega$ is the Levi-Civita connection.

Writing down the explicit formulæ for these metrics is easy, but we do not present them here, due to their ugliness and due to the fact that, regardless of the choice of the four free functions, they never satisfy the Einstein equations. The proof of this last fact is based on lengthy calculations using the explicit forms of the general solutions for ($T^i$).

**Appendix**

In this appendix we give the formulæ for the differentials of the transformed Cartan invariant coframe ($\tau^1, \tau^2, \tau^3, \tau^4, \Gamma_1, \Gamma_2$) on $\mathcal{P}$. These are:

(23a) $d\tau^1 = \Gamma_1 \wedge \tau^1 + \frac{1}{2} \theta \Gamma_1 \wedge \tau^2 - \frac{1}{2} \theta \Gamma_2 \wedge \tau^2 + \frac{1}{4} f \tau^4 \wedge \tau^1 - \frac{1}{2} g \tau^4 \wedge \tau^2$

+ $\frac{1}{2} a \tau^4 \wedge \tau^2$,  

(23b) $d\tau^2 = \frac{1}{4} \Gamma_1 \wedge \tau^1 + (\frac{1}{4} - 1) \Gamma_1 \wedge \tau^2 - \frac{1}{4} \theta \Gamma_1 \wedge \tau^3 - \frac{1}{4} \theta \Gamma_2 \wedge \tau^3 + (\frac{1}{4} + \frac{1}{2} s) \Gamma_1 \wedge \tau^4$

+ $\frac{1}{2} m \tau^2 \wedge \tau^2 - \frac{1}{2} m \tau^3 \wedge \tau^3 - \frac{1}{2} m \tau^4 \wedge \tau^3 + \frac{1}{2} a \tau^3 \wedge \tau^2$

+ $\left(\frac{1}{2} m - \frac{1}{2} f + b\right) \tau^4 \wedge \tau^2 + \left(\frac{1}{2} f - \frac{1}{2} m\right) \tau^4 \wedge \tau^3$,  

(23c) $d\tau^3 = \frac{1}{4} \Gamma_1 \wedge \tau^1 + \left( c + \frac{1}{4} \theta \right) \Gamma_1 \wedge \tau^2 - \left( c + \frac{1}{4} \theta \right) \Gamma_1 \wedge \tau^3 - \left( \frac{1}{4} l + \frac{1}{2} s \right) \Gamma_1 \wedge \tau^4$

+ $\frac{1}{2} \Gamma_2 \wedge \tau^1 - \left( c + \frac{1}{4} \theta \right) \Gamma_2 \wedge \tau^2 + \left( c + \frac{1}{4} \theta - 1 \right) \Gamma_2 \wedge \tau^3$

+ $\left( \frac{1}{4} l + \frac{1}{2} s \right) \Gamma_2 \wedge \tau^4 + \frac{1}{2} m \tau^2 \wedge \tau^2 - \frac{1}{2} m \tau^3 \wedge \tau^3 + \left( e - \frac{1}{2} n \right) \tau^4 \wedge \tau^1$

+ $\frac{1}{2} a \tau^3 \wedge \tau^2 + \left( \frac{1}{2} m - b - \frac{1}{2} f \right) \tau^4 \wedge \tau^2 + \left( \frac{1}{2} l + \frac{1}{2} f - \frac{1}{2} m \right) \tau^4 \wedge \tau^3$,  

(23d) $d\tau^4 = \frac{1}{2} c \Gamma_1 \wedge \tau^4 + (1 - \frac{1}{4} c) \Gamma_2 \wedge \tau^4 + \frac{1}{2} f \tau^4 \wedge \tau^1 - \frac{1}{2} a \tau^4 \wedge \tau^2$

+ $\frac{1}{2} a \tau^4 \wedge \tau^2$,  

(23e) $d\Gamma_1 = \frac{1}{2} g \Gamma_1 \wedge \tau^1 + \left( \frac{1}{2} f - \frac{1}{2} g \right) \Gamma_1 \wedge \tau^4 - \frac{1}{2} g \Gamma_2 \wedge \tau^1 + \left( \frac{1}{2} g - \frac{1}{2} f \right) \Gamma_2 \wedge \tau^4$

+ $\left( \frac{1}{2} h + c - 1 \right) \tau^2 \wedge \tau^1 - \frac{1}{2} h \tau^3 \wedge \tau^1 + \frac{1}{2} k \tau^4 \wedge \tau^1$

+ $\left( \frac{1}{2} h + c \right) \tau^4 \wedge \tau^2 - \frac{1}{2} h \tau^4 \wedge \tau^3$,  

(23f) $d\Gamma_2 = \frac{1}{2} g \Gamma_1 \wedge \tau^1 - \frac{1}{2} g \Gamma_2 \wedge \tau^2 + \frac{1}{2} g \Gamma_1 \wedge \tau^3 + \left( b + \frac{1}{2} f - \frac{1}{2} g \right) \Gamma_1 \wedge \tau^4$

+ $\frac{1}{2} g \Gamma_2 \wedge \tau^1 + \frac{1}{2} g \Gamma_2 \wedge \tau^2 - \frac{1}{2} g \Gamma_2 \wedge \tau^3 + \left( \frac{1}{2} g - b - \frac{1}{2} f \right) \Gamma_2 \wedge \tau^4$

\footnote{The 4-manifold on which each of these metrics resides is the leaf space of the 2-dimensional integrable distribution on $\mathcal{P}$ which annihilates forms ($T^1, T^2, T^3, T^4$).}
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$$+ \left( \frac{1}{4} h + c \right) \tau^2 \wedge \tau^1 - \frac{1}{4} h \tau^3 \wedge \tau^1 - \frac{1}{2} k \tau^4 \wedge \tau^1 + \left( \frac{1}{4} h + c \right) \tau^4 \wedge \tau^2$$

$$+ \left( 1 - \frac{1}{4} h \right) \tau^4 \wedge \tau^3.$$

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**Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoża 69, Warszawa, Poland**

E-mail address: godlinsk@fuw.edu.pl

**Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoża 69, Warszawa, Poland**

E-mail address: nurowski@fuw.edu.pl