Short proofs on $k$-extendible graphs

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Abstract

In this note, we give short inductive proofs of two known results on $k$-extendible graphs based on a property proved in [Qinglin Yu, A note on $n$-extendable graphs. Journal of Graph Theory, 16:349-353, 1992].

1 Introduction

A graph $G$ is $k$-extendible if it satisfies the following conditions:

- $|G| \geq 2k + 2$;
- $G$ is connected;
- $G$ has a perfect matching;
- for every matching $M_k$ of $G$ of size $k$, there is a perfect matching of $G$ containing $M_k$.

The notion of $k$-extendible graphs was first defined and studied by Plummer [5]. In particular, 2-extendible bipartite graphs play an important role in the study of Pólya’s permanent problem [6] whose solution was obtained by Robertsen, Seymour and Tomar [7] and independently by McCuaig [4]. We refer to the monograph of Lovász and Plummer [3] for a detailed account of 1-extendible graphs.

Our Contribution. Based on a property of $k$-extendible graphs proved by Yu [8], we give short inductive proofs of two known results on $k$-extendible graphs. Our proofs are much simpler than the existing proofs due to the fact that the property allows us to apply the inductive hypothesis on subgraphs of the given $k$-extendible graph rather than on the given graph itself.

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2 Preliminaries

Let $G = (V, E)$ be a graph. For $S \subseteq V$ and a subgraph $H$ of $G$, the neighborhood of $S$ in $H$, denoted by $N_H(S)$, is the set of vertices in $H$ that are adjacent to some vertex in $S$. The size of $N_H(S)$ is denoted by $d_H(S)$. If $S = \{v\}$, we simply write $N_H(v)$ and $d_H(v)$ instead of $N_H(\{v\})$ and $d_H(\{v\})$, respectively. For a subset $S \subseteq V$, we denote by $G[S]$ the subgraph induced by $S$. Given a matching $M$ of $G$, we denote by $V(M)$ the set of vertices that are endvertices of edges in $M$. The minimum degree and the matching number of $G$ are denoted by $\delta(G)$ and $\alpha'(G)$, respectively. For other standard terminology we refer to [1].

We start with two simple propositions of $k$-extendible graphs that were obtained by Plummer [5]. Since the proofs are short, we include them here for the sake of completeness.

**Proposition 1 ([5]).** Every $k$-extendible graph is $(k - 1)$-extendible.

**Proof.** Let $G$ be a $k$-extendible graph. By contradiction, let $M = \{a_1b_1, \ldots, a_{k-1}b_{k-1}\}$ be a matching of size $k - 1$ that is not contained in a perfect matching of $G$. Since $G$ is $k$-extendible, $M$ is a maximal matching of $G$. This implies that

$$S = V(G) \setminus \{a_1, b_1, \ldots, a_{k-1}, b_{k-1}\}$$

is independent. Since $|G| \geq 2k + 2$, it follows that $|S| \geq 4$. Since $M$ is not a maximum matching of $G$, it follows from Berge’s Theorem (see [1]) that there exists an $M$-augmenting path, that is, an $u$-$v$ path $P$ such that $u, v \notin V(M)$ and edges of $P$ are alternating between $E(G) \setminus M$ and $M$, starting with an edge not in $M$. Then $M' = (E(P) \setminus M) \cup (M \setminus E(P))$ is a matching of size $k$ with $V(M') = V(M) \cup \{u, v\}$. Since $G - V(M') = S \setminus \{u, v\}$ is an independent set of size at least 2, $M'$ cannot be extended to a perfect matching of $G$. This is a contradiction.

**Proposition 2 ([5]).** Every 1-extendible graph is 2-connected.

**Proof.** Suppose by contradiction that $G$ is a 1-extendible graph but not 2-connected. Then there exists a cut vertex $v$ such that $G - v$ has components $C_1, \ldots, C_t$ for some $t \geq 2$. Since $G$ is connected, $v$ has a neighbor $u_i \in C_i$ for each $1 \leq i \leq t$. Since $G$ is 1-extendible, there is a perfect matching containing $vu_1$ and this implies that $|C_1|$ is odd. On the other hand, there is a perfect matching containing $vu_2$ and this implies that $|C_1|$ is even. This is a contradiction.

The following property of $k$-extendible graphs was proved by Yu [8] whose proof used Theorem 1 below. Here we give a new proof that avoids the use of Theorem 1.

**Proposition 3 ([8]).** Let $G$ be a $k$-extendible graph with $k \geq 2$. Then for every edge $uv \in E(G)$, $G - \{u, v\}$ is $(k - 1)$-extendible.

**Proof.** Let $e = uv \in E(G)$ and $G' = G - \{u, v\}$. By Proposition 1, $G$ is 1-extendible and so 2-connected by Proposition 2. Since $G$ is $k$-extendible, every matching of size $k - 1$ of $G'$ can be extended to a perfect matching of $G'$. So it remains to show that $G'$ is connected. Suppose by contradiction that $G'$ has components $C_1, C_2, \ldots, C_t$ for
Proposition 1. Now suppose that $t \geq 2$. Since $G$ is 2-connected, each of $u$ and $v$ has a neighbor in each component $C_i$. Let $s \in C_1$ be a neighbor of $u$ and $t \in C_2$ be a neighbor of $v$. Since $G$ is $k$-extendible with $k \geq 2$, there is a perfect matching of $G$ containing $\{us, vt\}$ by Proposition 1. This implies that $|C_1|$ is odd. Then there is no perfect matching of $G$ containing $uw$. This contradicts that $G$ is 1-extendible. Therefore, $G - \{u, v\}$ is $(k-1)$-extendible. □

3 New Proofs

In this section, we present our new proofs of two known results on $k$-extendible graphs. The first result was proved by Plummer [5] on the connectivity of $k$-extendible graphs. The overall strategy of Plummer [5] was to apply the inductive hypothesis on the input graph (due to Proposition 1) and then use a variation of Menger’s Theorem. Our proof below, on the other hand, is simpler due to the fact that we were able to apply the inductive hypothesis on subgraphs of the input graph due to Proposition 3.

Theorem 1 ([5]). Every $k$-extendible graph is $(k + 1)$-connected.

Our Proof. Let $G$ be a $k$-extendible graph. We prove by induction on $k$. The base case is Proposition 2. Now suppose that $k \geq 2$ and the statement is true for $(k-1)$-extendible graphs. By Proposition 3, $G - \{u, v\}$ is $(k-1)$-extendible for every edge $uv \in E(G)$ and so is $k$-connected by the inductive hypothesis. By Proposition 1 and Proposition 2, it follows that $\delta(G) \geq 2$. For any vertex $v \in V(G)$, let $u$ be a neighbor of $v$. Since $d(u) \geq 2$, $u$ has a neighbor other than $v$. Since $H = G - \{u, w\}$ is $k$-connected, $d_H(v) \geq k$ and thus $d_G(v) \geq k + 1$. This shows that $\delta(G) \geq k + 1$.

Now let $S \subseteq V(G)$ be an arbitrary set with $|S| = k$. Let $s \in S$ and $t$ be a neighbor of $s$. We show that $G - S$ is connected.

Case 1. $t \in S$. Then $G - S = (G - \{s, t\}) - (S \setminus \{s, t\})$ is connected, since $G - \{s, t\}$ is $k$-connected.

Case 2. $t \notin S$. Let $G' = G - (S \cup \{t\})$. Note that $G' = (G - \{s, t\}) - (S \setminus \{s\})$.

Since $G - \{s, t\}$ is $k$-connected, $G'$ is connected. Since $\delta(G) \geq k + 1$, $t$ has a neighbor in $G'$. Therefore, $G - S = G'[V(G') \cup \{t\}]$ is connected. □

The second result is on $k$-extendible bipartite graphs. The celebrated Hall’s Theorem gives a necessary and sufficient condition for a balanced bipartite graph to have a perfect matching. It turns out that $k$-extendible bipartite graphs have a similar characterization.

Theorem 2 ([2]). Let $G = (X, Y)$ be a connected bipartite graph with a perfect matching and $|G| \geq 2k + 2$. Then $G$ is $k$-extendible if and only if $|N(A)| \geq |A| + k$ for every subset $A \subseteq X$ with $1 \leq |A| \leq |X| - k$.  

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Theorem 2 was first stated and proved by Brualdi and Perfect [2] in the language of matrices (Theorem 2.1 in [2]). Here we give two graph-theoretical proofs. The first one relies on Proposition 3 while the second one is based on the König-Ore Formula.

**Theorem 3** (*The König-Ore Formula*). Let $G = (X, Y)$ be a bipartite graph. Then

$$\alpha'(G) = |X| - \max_{S \subseteq X}(|S| - |N(S)|).$$

**Our First Proof of Theorem 2.** We first prove the sufficiency. Take a matching

$$M = \{x_1y_1, \ldots, x_ky_k\}$$

of size $k$. Let $X' = X \setminus \{x_1, \ldots, x_k\}$ and $Y' = Y \setminus \{y_1, \ldots, y_k\}$. Denote by $H$ the subgraph of $G$ induced by $X' \cup Y'$. Note that every nonempty subset $A$ of $X'$ has $1 \leq |A| \leq |X| - k$. It follows from the assumption that $|N_G(A)| \geq |A| + k$. Thus implies that $|N_H(A)| \geq |A|$. By the König-Ore Formula, $H$ has a perfect matching $M_H$. It follows that $M_H \cup M$ is a perfect matching of $G$ containing $M$. This shows that $G$ is $k$-extendible.

We now prove the necessity by induction on $k$.

**Base Case:** $k = 1$. By contradiction, let $A$ be a subset of $X$ with $1 \leq |A| \leq |X| - 1$ such that $|N(A)| < |A| + 1$. Since $G$ has a perfect matching, $|N(A)| \geq |A|$. It then follows that $|N(A)| = |A|$. Since $G$ is connected, there is an edge $e = xy$ between $N(A)$ and $X \setminus A$. So there is no perfect matching of $G$ containing $e$, simply because there are not enough vertices in $G - \{x, y\}$ to match vertices in $A$.

**Inductive Step:** We assume that $k \geq 2$ and the statement is true for $k - 1$. Let $A$ be an arbitrary subset of $X$ with $1 \leq |A| \leq |X| - k$. If $N(A) = Y$, then

$$|N(A)| = |Y| = |X| \geq |A| + k.$$

So we may assume that $N(A) \neq Y$. Since $G$ is connected, there is an edge $xy \in E(G)$ such that $y \in N(A)$ and $x \in X \setminus A$. Let $G' = G - \{x, y\} = (X \setminus \{x\}, Y \setminus \{y\})$. By Proposition 3, $G'$ is $(k - 1)$-extendible. On the other hand, $A \subseteq X \setminus \{x\}$ has

$$1 \leq |A| \leq |X| - k = (|X| - 1) - (k - 1).$$

By the inductive hypothesis, $|N_{G'}(A)| \geq |A| + (k - 1)$. Since $N_G(A) = N_{G'}(A) \cup \{y\}$, it follows that $|N_G(A)| \geq |A| + k$.

**Our Second Proof of Theorem 2.** The difference lies in the inductive step of the necessity. We assume that $k \geq 2$ and the statement is true for $k - 1$. By contradiction, let $A$ be a subset of $X$ with $1 \leq |A| \leq |X| - k$ such that $|N(A)| < |A| + k$. By Proposition 1, $G$ is $(k - 1)$-extendible. By the inductive hypothesis, $|N(A)| \geq |A| + (k - 1)$. It follows that

$$|N(A)| = |A| + (k - 1).$$

(1)

Let $X' = X \setminus A$ and write $B = N(A)$. Note that $|B| \geq k$ for otherwise Equation 1 would be contradicted. Denote by $H$ the subgraph induced by $B \cup X'$. If $H$ has a matching of size $k$, then it cannot be extended to a perfect matching of $G$ (because
there are not enough vertices to match vertices in $A$). So the matching number of $H$ is at most $k - 1$. By the König-Ore Formula,

$$\alpha'(H) = |B| - \max_{S \subseteq B}(|S| - |N_H(S)|) \leq k - 1.$$  

So there exists a subset $S \subseteq B$ such that $|S| - |N_H(S)| \geq |B| - (k - 1)$. Since $|B| \geq k$, $|S| \geq 1$. Moreover, $|N_H(S)| \leq |S| + (k - 1) - |B|$. Therefore,

$$|N_G(S)| \leq |A| + |N_H(S)| \leq |A| + |S| + (k - 1) - |B| = |S|,$$

where the last equality follows from Equation 1. Since $|A| \leq |X| - k$, it follows that $|B| = |N(A)| = |A| + (k - 1) \leq |X| - 1$. Hence, $S \subseteq B$ violates the condition for $G$ to be 1-extendible.

\[ \square \]

4 Concluding Remarks

The fact that our proof of Proposition 3 does not use Theorem 1 makes our new proof of Theorem 1 self-contained. To the best of our knowledge, our first proof of Theorem 2 is new and self-contained. The second proof, in essence, is the graph counterpart of the proof given in [2] stated in matrix language. That proof used the Frobenius-König Theorem which is the matrix counterpart of the the König-Ore Formula. However, we feel that it may be convenient for graph theorists to have a graph-theoretical proof. So we include our second proof here as well.

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