A DETERMINISTIC ALGORITHM FOR INTEGER FACTORIZATION

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ABSTRACT. A deterministic algorithm for factoring $n$ using $n^{1/3+o(1)}$ bit operations is presented. The algorithm tests the divisibility of $n$ by all the integers in a short interval at once rather than integer by integer as in trial division. The algorithm is implemented.

1. INTRODUCTION

One can use trial division to factor an integer $n$ using $\leq n^{1/2}$ divisions on integers of size $\leq n$. There are several algorithms that improve the running time to $n^{1/3+o(1)}$ bit operations without using fast Fourier transform (FFT) techniques: Lehman’s method [6], which “uses a dissection of the continuum similar to the Farey dissection”; Lenstra’s algorithm [7], which looks for divisors of $n$ in residue classes; McKee’s algorithm [8], which is related to Euler’s factoring method; and an algorithm due to Rubinstein [11] that relies on estimates for Kloosterman sums. The Pollard-Strassen algorithm [9,12] uses an FFT precomputation to improve the time complexity to $n^{1/4+o(1)}$ bit operations and requiring $n^{1/4+o(1)}$ bits of storage (memory); see [1,3,4,13] for example. As far as we know, this is the fastest deterministic factoring algorithm with a fully proven complexity, though it has the practical disadvantage of requiring much memory space. The Coppersmith algorithm [2] for finding small roots of bivariate rational polynomials enables factoring $n$ using $n^{1/4+o(1)}$ operations. This algorithm uses lattice basis reduction techniques, and it has the advantage of requiring little memory space. (However, the $n^{o(1)}$ factor in the running time seems significant, involving a high power of $\log n$.) Shank’s class group method (see [4]) has a better complexity of $n^{1/5+o(1)}$ bit operations to factor $n$, but it assumes the generalized Riemann hypothesis, which is unproved so far.

In this paper, we present a new deterministic method for factoring $n$ in $n^{1/3+o(1)}$ time. Like the other exponential factoring methods mentioned before, this algorithm is mainly of theoretical interest. There are already probabilistic methods that far outperform it in practice and in heuristically subexponential time; see [4] for a survey of such methods. Our goal, rather, is to present a new deterministic approach for integer factorization that we hope can be improved in the future.
2. Main result

An integer \( n > 1 \) is composite if the equation \( n = xy \) has a non-trivial integer solution \((x, y)\). One can test whether \( n = xy \) holds by testing if \( n/x \equiv 0 \mod 1 \), which can be decided by dividing \( n \) by \( x \) directly, say. By looping through the integers \( 1 < x \leq \sqrt{n} \) this way, either one will find a non-trivial factor of \( n \) or, if no such factor is found, one will conclude that \( n \) is prime. This trial division procedure is guaranteed to terminate after \( \leq \sqrt{n} \) steps. The new algorithm that we present, Algorithm 1, enables a speed-up over trial division because it can test the equation \( n/(x + h) \equiv 0 \mod 1 \) for many integers \( h \in [-H, H] \) in basically a single step. The observation is that, locally (i.e. if \( H \) is small enough compared to \( x \)), one can approximate \( n/(x + h) \) by a linear polynomial in \( h \) with rational coefficients. The oscillations of this polynomial modulo 1 are easy to understand, due to linearity, which leads to the speed-up.

The main result is Theorem 2.1, which gives an upper bound on the complexity of Algorithm 1. The complexity is measured by the total number of the following operations consumed: +, −, ×, ÷, \( \exp \), \( \log \). This in turn can be routinely bounded in terms of bit operations since all the numbers that occur in Theorem 2.1 can be expressed using \( \ll \log n \) bits. We will make use of some basic algorithms such as how to generate the continued fraction (CF) convergents of a rational number and how to solve a quadratic equation. We will use the notation \([x] \) to denote the nearest integer to \( x \) (if \( x \) is half an integer, we take \([x] = \lfloor x \rfloor\)).

Algorithm 1 Given an integer \( n > 1 \), this algorithm finds a non-trivial factor of \( n \) or proves that \( n \) is prime.

1. [Initialize]
   set \( x_0 = \min\{[(17n)^{1/3}], [\sqrt{n}]\}, x = x_0 + 2, H = 1; \)
2. [Trial division]
   check if \( n \) has a divisor \( 1 < k \leq x_0 \), if so return \( k \);)
3. [Loop]
   while \((x - H \leq [\sqrt{n}] \) ) {
     generate the CF convergents of \( n/x^2 \), say \([b_0/q_0, \ldots, b_r/q_r]\),
     then find the convergent with the largest \( q_j \leq 4H \);
     set \( b = b_j, q = q_j, a = [qn/x]; \)
     solve \((qn-ax) + (bx-a)h + bh^2 = 0\); for each integer solution \( h \) test if \( x + h \) divides \( n \), if so return \( x + h \);
     increment \( x \leftarrow x + 2H + 1, \) set \( H = [(17n)^{-1/3}x]; \)
   }
   return \( n \) is prime;

Theorem 2.1. Algorithm 1 returns a non-trivial factor of \( n > 1 \) or proves that \( n \) is prime, using \( \ll n^{1/3} \log^2 n \) operations on numbers of \( \ll \log n \) bits.

The “while” loop in Algorithm 1 checks for divisors of \( n \) in successive blocks of the form \([x-H, x+H]\). The block size, \( 2H+1 \), increases as the loop progresses. Roughly speaking, as the algorithm searches through an interval like \([x, 2x]\), \( H \) doubles in size, increasing from \( H \approx x/(17n)^{1/3} \) at the beginning, to \( H \approx 2x/(17n)^{1/3} \) by the
end. This choice of \( H \) is not optimal, in that it can be chosen larger depending on \( a, b, \) and \( q \); see \(^1\) However, fixing the choice as we did simplifies the proof of Theorem 3.1 later.

One feature of Algorithm 1 is that, like the Pollard-Strassen method, it can be adapted to obtain partial information about the factorization of \( n \). For example, after small modifications, Algorithm 1 can rule out factors of \( n \) in a given interval \([z, z + w] \), \( z, w \in \mathbb{Z}^+ \), using \( \ll (w n^{1/3} / z + 1) \log(n + z + w) \) operations. To do so, one adjusts the trial division statement to cover the smaller range \( z \leq k \leq \min\{x_0, z + w\} \), then initializes \( x = x_0 + 2 \) or \( x = z + 1 \) depending on whether \( z < x_0 \) or not, and adjusts the loop statement to be \texttt{while}(\( x - H \leq z + w \)).

It is interesting to compare our method with the Coppersmith algorithm. Using the latter, one can factor \( n = pq \) in poly-log time in \( n \) if the high-order \( \frac{1}{3} \log_2 N \) bits of \( p \) are known.\(^2\) (Here, \( \log_2 \) is the logarithm to base 2.) By comparison, our method requires more, the high-order \( \frac{1}{2} \log_2 N \) bits of \( p \). Therefore, our method is of a comparable strength to the algorithm of Rivest and Shamir \(^{10}\), where this problem is set up in terms of integer programming in two dimensions.

3. Proof of Theorem 2.1

\textbf{Lemma 3.1.} Let \( n, x, \) and \( H \) be positive integers. Then there is a rational approximation of \( n/x^2 \) of the form

\[
\frac{n}{x^2} = \frac{b}{q} + \frac{e_2}{qq'}, \quad 0 < q \leq 4H \leq q', \quad |e_2| < 1.
\]

This approximation can be found using \( \ll \log(n + x) \) operations on integers of \( \ll \log(n + x) \) bits.

\textbf{Proof.} This follows routinely from the classical theory of continued fractions; see \(^5\) for example. \( \square \)

\textbf{Lemma 3.2.} Let \( n \geq 400 \), \( x \) and \( H \) be positive integers with \( H/x \leq (17n)^{-1/3} \). For each integer \( |h| \leq H \), if \( n/(x + h) \equiv 0 \mod 1 \), then \( h \) must be a solution of the equation \( g_{n,x}(y) := c_0 + c_1 y + c_2 y^2 = 0 \) where, letting

\[
\frac{n}{x} = \frac{a}{q} + \frac{e_1}{q}, \quad a = \lfloor qn/x \rfloor,
\]

we have \( c_0 := xe_1 = qn - ax, c_1 := e_1 - xe_2/q' = bx - a, \) and \( c_2 := nq/x^2 - e_2/q' = b \).

(\text{Here, } q, q', \text{ and } e_2 \text{ are as in Lemma 3.1.) Moreover, } g_{n,x}(y) \text{ does not vanish identically, so there are at most two solutions of the equation } g_{n,x}(y) = 0. \)

\textbf{Proof.} Since \( x + h > 0 \) for \( |h| \leq H \), we have the identity \( n/(x + h) = n/x - nh/x^2 + nh^2/((x + h)x^2) \). Let us define \( \epsilon(h) := e_1 - he_2/q' + qnh^2/((x + h)x^2) \). Then

\[
\frac{n}{x + h} = \frac{a - bh}{q} + \frac{\epsilon(h)}{q}.
\]

Multiplying both sides by \( q \), we see that if \( n/(x + h) \equiv 0 \mod 1 \), then necessarily \( a - bh + \epsilon(h) \equiv 0 \mod q \). In particular, since \( a - bh \) is an integer, so must \( \epsilon(h); \)

\(^1\) It is puzzling that the Coppersmith method does not seem to be more well-known in the relevant number theory literature.

\(^2\) In fact, the Coppersmith algorithm leads to the same result if the low-order bits of \( p \) are known instead of the high-order bits. More generally, the algorithm can decide, in poly-log time in \( n \), whether \( p \) lies in a given residue class modulo an integer of size about \( n^{1/4} \).
i.e. $\epsilon(h) \equiv 0 \mod 1$. By the triangle inequality, the bound $|h| \leq H$, and the bound $H \leq x/2$, we have

\begin{equation}
|\epsilon(h)| \leq |\epsilon_1| + \frac{\epsilon_2 H^2}{q^2} + \frac{q n H^2}{x^2 H - H}.
\end{equation}

By construction, $|\epsilon_1| \leq 1/2$, $|\epsilon_2 H/q^2| < 1/4$, and $q \leq 4H$. Since also $H \leq (17n)^{-1/3} x$ by hypothesis, we obtain that $q n H^2/(x^2 (x - H)) \leq 4n (H/x)^3/(1- H/x) \leq (4/17)/(1- 6800^{-1/3}) < 1/4$, where we used the assumption $n \geq 400$. So we deduce that $|\epsilon(h)| < 1/2 + 1/4 + 1/4 = 1$. Therefore, in our situation, the congruence $\epsilon(h) \equiv 0 \mod 1$ is equivalent to the equation $\epsilon(h) = 0$. Last, since $(x + h)\epsilon(h) = g_{n,x}(h)$ and $x + h \neq 0$, we deduce that the equations $\epsilon(h) = 0$ and $g_{n,x}(h) = 0$ are equivalent.

For the second part of the lemma, note that if $g_{n,x}(y) \equiv 0$, then $c_0 = c_1 = c_2 = 0$. Since $c_0 = 0$ then $\epsilon_1 = 0$. And since $c_1 = 0$ also, we deduce that $\epsilon_2 = 0$. But then $c_2 = nq/x^2 - \epsilon_2/q^2 = nq/x^2 \neq 0$.

\begin{proofof}
We choose integers $1 < x_0 < x_1 \ldots$ and $1 \leq H_1 \leq H_2 \leq \ldots$ and define the following sequence of intervals: $B_0 := [2, x_0]$, $B_1 := [x_1 - H_1, x_1 + H_1]$, $B_2 := [x_2 - H_2, x_2 + H_2],\ldots$. Specifically, we choose $x_0 := \lceil (17n)^{1/3} \rceil$, $x_1 := x_0 + 2$, $H_1 = 1$, and, for $j \geq 2$, we let $x_j := x_{j-1} + 2H_{j-1} + 1$ where $H_j := \lceil (17n)^{-1/3} x_j \rceil$. So $1 \leq H_1 \leq H_2 \leq \ldots$, and therefore $B_0 \cup \ldots \cup B_j$ covers the integers in the interval $[2, x_j]$ completely. We use trial division to search for a factor of $n$ in $B_0$ using $\ll n^{1/3}$ operations. If a factor is found, then it is returned and the algorithm reaches an end point. Otherwise, we successively search for a factor in the intervals $B_j = [x_j + H_j, x_j - H_j]$. We note at this point that if $n < 400$, then the algorithm reaches an end after searching $B_0$. This is because $x_0 = \lceil (17n)^{1/3} \rceil \geq \lceil \sqrt{n} \rceil$ for $n < 400$, as can be checked by direct computation, and this implies that the algorithm will not enter the “while” loop. So, in analyzing the Loop phase of the algorithm, we may assume that $n \geq 400$. Furthermore, we observe that $H_j/x_j \leq (17n)^{-1/3}$ by construction, and so $n$, $x_j$, and $H_j$ satisfy the hypothesis of Lemma 3.2. Thus, applying the lemma to $B_j$, one can quickly locate all the divisors of $n$ in that block (if any) using $\ll \log(n + x_j)$ operations on numbers of $\ll \log(n + x_j)$ bits. This is mainly the cost of finding the rational approximation in Lemma 3.1 via the continued fraction representation of $n/x^2$, then solving the resulting quadratic equation. Last, we only need to search for $B_j$ that satisfy $B_j \cap [2, \lceil \sqrt{n} \rceil] \neq \emptyset$; i.e. $x_j - H_j \leq \sqrt{n}$. This is because if no factor is found in these $B_j$, then one will have proved $n$ prime. Given this, it is easy to show that the total number of blocks that need to be searched is $\ll n^{1/3} \log(2 + n/x_0)$. Since $n \geq 2$ by hypothesis, this is $\ll n^{1/3} \log n$, which yields the result.
\end{proofof}

4. Implementation

We implemented Algorithm 1 in Mathematica. The implementation is available at https://people.math.osu.edu/hiary.1/factorTest.nb. We were able to reduce the running time by about 20% by choosing the block size asymmetrically about $x$. From the left we set $H_L = \lceil (17n)^{-1/3} x \rceil$, which is the same as in Algorithm 1 and from the right we set $H_R = \min\{H_{R,1}, H_{R,2}\}$ where $H_{R,1} = \lfloor 0.4(1 - |\epsilon_1|)q^2/|\epsilon_2| \rfloor$ and $H_{R,2} = \lfloor \sqrt{0.6(1 - |\epsilon_1|)x^3/(qn)} \rfloor$. Also, we required that $q \leq 4H_L$. Together, this ensured that $|\epsilon(h)| < 1$ for $-H_L \leq h \leq H_R$, as needed,
and it allowed a larger block size. This is because $H_R$ will be at least the size of $H_L$, but it can get much larger if $|e_2/q'|$ and $q$ happen to be small, e.g. if $n/x^2$ can be approximated well by a rational with a small denominator. To take advantage of the larger block size in the implementation, we incremented $x \leftarrow x + H_L + H_R + 1$ instead of $x \leftarrow x + 2H_L + 1$.

Our implementation of Lemma 3.1 became faster, on average, than trial division when $H_L \gtrsim 50$. So we used trial division in the interval $[2, \lceil 50(17n)^{1/3} \rceil]$. The running time of the full algorithm started to beat trial division when $n \gtrsim 10^{14}$, with $n$ a product of two primes of roughly equal size. The algorithm is about two times faster than trial division when $n \approx 10^{18}$. This running time can be expected to improve using a more careful implementation; e.g. one need not generate all the continued fraction convergents of $n/x^2$, as done now, but only the convergents with denominator $\leq 4H_L$.

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