Double pendulum model for tennis stroke including a collision process.

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Abstract

By means of adding a collision process between the ball and racket in double pendulum model, we analyzed the tennis stroke. It is possible that the speed of the rebound ball does not simply depend on the angular velocity of the racket, and higher angular velocity sometimes gives lower ball speed. We numerically showed that the proper time lagged racket rotation increases the speed of the rebound ball by 20%. We also showed that the elbow should move in order to add the angular velocity of the racket.

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I. INTRODUCTION

The double pendulum has been studied as an example of the chaotic motion in physics[1-3]. If we consider the first cycle of the pendulum, the double pendulum model holds application in sports such as golf[4-7], baseball[8], and tennis[9]. Since the double pendulum is not a simple linear system, the motion of the pendulum can not be optimized as a simple analytic form. The swing pattern utilised to maximize the angular velocity of the hitting rod such as racket, bat, and club has been analyzed on the assumption that the angular velocity is the dominant factor for the speed of the rebound ball[9].

For a tennis stroke, the two rods for double pendulum are an arm and a forearm for first rod and a racket for the second rod. In our model, we added the collision process between the ball and the racket. Without the collision process[9], there is no criteria to attain high speed of the rebound ball except for the angular speed of the second rod, racket. If we set the impact angle of the first rod at which the ball hits the racket, the speed of the rebound ball is mainly dependent on the angular velocity of the second rod. On the other hand, if we release the impact angle of the first rod, the speed of the rebound ball does not remain as a simple function of the angular speed of the hitting rod. We showed that the speed of the rebound ball is different even though the angular velocities of the second rod at the contact time are same. Furthermore, considering the whole stroke, the maximum angular velocity for the lower speed of the rebound ball is greater than the maximum angular velocity for the higher speed of the rebound ball. Therefore, to get maximum speed of the rebound ball, it’s not sufficient to set the condition to generate high angular velocity of the hitting racket.

The collision between the racket and ball has been studied in various ways [11, 12]. In our simple collision model, we assumed that the racket is a simple one dimensional rod without any nodal motion and the collision occurs in one dimension. This assumption is valid if the racket and a ball moves in the same line for the short collision time. Although, our model does not give any detailed information on the collision about the effect of the tension of the string and the mass distribution of the racket, however provides some insights on the proper swinging pattern to get maximum speed of the rebound ball.

In this article, we also analyzed the tame lagged torque effect for the double pendulum system. By applying time independent constant torques on first rod and the second rod, the speed of the rebound ball can be calculated for certain initial conditions. In the same
condition, if we simply hold the racket for a short time without enforcing a torque and with subsequent application of torque, the naive intuition estimates decrease in the speed of the rebound ball. The reason behind this is application of less energy for the double pendulum system. However, the speed of the rebound ball increase by 20% by choosing the proper delay time. It’s mainly because the double pendulum system is not a simple linear system. Adding energy to the double pendulum system does not directly increase the speed of the rebound ball. We also analyzed movement of the elbow to which the first rod of the double pendulum is attached. At a first glance, if we add extra movement towards the rebound ball’s direction at contact time, the speed of the ball increases; however, with varying results. Apparently, it becomes clear that, the double pendulum system is really a nonlinear system.

The present paper is organized as follows: In Section II, we introduce a double pendulum system including the collision process. In Section III, the differential equations obtained in section II are solved, we numerically showed that the speed of the rebound ball does not simply depend on the angular speed of the racket. For some cases, the higher angular velocity gives lower speed of the rebound ball. In Section IV, we analyzed the dependence of the racket mass and length of the first rod. The general properties of swing system has been demonstrated. When we applied a time dependent torque on the racket, we could increase the speed of the rebound ball. The time lagged rotation of the racket was analyzed in section V. In section VI, the elbow movement is analyzed to add additional speed to the ball. In Section VII, we summarize the main results and discuss the application of our results.

II. RACKET SYSTEM INCLUDING THE COLLISION WITH A BALL

The geometry of the double pendulum model for the swing of a racket is shown in Fig 1(a). Though, this geometry is for the left-handed player if we see from the +z direction, it’s originally related to the real double pendulum problem in the gravitational field \[8\]. Our basic model and some notations are closely related with those in work reported by Rod Cross \[9\]. The elbow moves in the \(xy\) plane and the arm and the racket also moves in \(xy\) plane. We also modeled the arm and the forearm as a simple uniform rod with mass \(M_1\) and length \(L_1\). The racket including the hands is also treated as a uniform rod with mass \(M_2\) and length \(L_2\). The arm and the racket rotate in a clockwise direction in a plane at angular velocities \(\omega_1 = -d\theta/dt\) and \(\omega_2 = -d\phi/dt\), respectively. If we assume that the velocity of the
elbow is \((V_x^E, V_y^E)\) then the velocities of the center of the first rod (arm and forearm system) and the second rod (hand and racket system) becomes

\[
\begin{align*}
v_{x1} &= V_x^E - h_1 \omega_1 \cos \theta \\
v_{y1} &= V_y^E - h_1 \omega_1 \sin \theta \\
v_{x2} &= V_x^E - L_1 \omega_1 \cos \theta - h_2 \omega_2 \cos \phi \\
v_{y2} &= V_y^E - L_1 \omega_1 \sin \theta - h_2 \omega_2 \sin \phi
\end{align*}
\]

(1)

,where \((v_{x1}, v_{y1})\), \((v_{x2}, v_{y2})\) are the velocity of the center of the mass of the first rod and the second rod, respectively. The center of masses are located in the middle of the rod, \(h_1 = L_1/2\) and \(h_2 = L_2/2\), since we assumed uniform rods.

Let the force from the elbow to the first rod on joint point between the elbow and the first rod be \((F_{x1}, F_{y1})\), and the force from the first rod on joint point between two rods be \((F_{x2}, F_{y2})\), then the equations of the motion for two center of masses become

\[
\begin{align*}
M_1 \frac{dv_{x1}}{dt} &= F_{x1} - F_{x2} \\
M_1 \frac{dv_{y1}}{dt} &= F_{y1} - F_{y2}
\end{align*}
\]

(2)

FIG. 1: Schematic diagram of arm and racket system. (a) Length of the first rod (forearm and arm system) is \(L_1\), and the length of the racket is \(L_2\). The angles \(\theta\) and \(\phi\) are defined from the \(-y\) direction. (b) The Force on two rods. \((F_{x1}, F_{y1})\) is the force acted on the first rod at the joint between elbow and the first rod. \((F_{x2}, F_{y2})\) is the force acted on the second rod at the joint between the two rods. \(F_{col}\) is the force on the second rod by a ball.
\[
M_2 \frac{dv_{x2}}{dt} = F_{x2} + F_{col}
\]
\[
M_2 \frac{dv_{y2}}{dt} = F_{y2}
\]

The two forces \((-F_{x2}, -F_{y2})\) are reaction force of \((F_{x2}, F_{y2})\) from the second rod. We added the force \(F_H\) from the ball to the second rod at the point of \(h_H\) from the center of the mass. This force from the collision between the racket and the ball changes the torque equation in [9] and the two torque equations becomes

\[
I_{2,cm} \frac{d\omega_2}{dt} = C_2 + F_{x2}h_2 \cos \phi + F_{y2}h_2 \sin \phi - F_{col}h_H \cos \phi
\]
\[
I_{1,cm} \frac{d\omega_1}{dt} = C_1 - C_2 + F_{x1}h_1 \cos \theta + F_{y1}h_1 \sin \theta + F_{x2}(L_1 - h_1) \cos \theta + F_{y2}(L_1 - h_1) \sin \theta
\]

where \(C_1\) is the torque on joint point between the elbow and the first rod and \(C_2\) is the torque on the joint point between the two rods. From the Eqs. [2]-[5] we obtain two equations for the time derivative of two angular velocities as follows

\[
\frac{d\omega_1}{dt} = \frac{P(I_{2,cm} + M_2h_2^2) - QM_2h_2L_1 \cos(\phi - \theta) + S_1}{(I_{2,cm} + M_2h_2^2)(I_{1,cm} + M_1h_1^2 + M_2L_1^2) - M_2^2h_2^2L_1^2 \cos^2(\phi - \theta)}
\]
\[
\frac{d\omega_2}{dt} = \frac{Q(I_{1,cm} + M_1h_1^2 + M_2L_1^2) - PM_2h_2L_1 \cos(\phi - \theta) + S_2}{(I_{2,cm} + M_2h_2^2)(I_{1,cm} + M_1h_1^2 + M_2L_1^2) - M_2^2h_2^2L_1^2 \cos^2(\phi - \theta)}
\]

where \(a_x = \frac{dv_{x2}}{dt}, a_y = \frac{dv_{y2}}{dt}\), and

\[
P = C_1 - C_2 - M_2h_2L_1\omega_2^2 \sin(\phi - \theta) + (M_1h_1 + M_2L_1)[a_x \cos \theta + a_y \sin \theta]
\]
\[
Q = C_2 + M_2h_2L_1\omega_2^2 \sin(\phi - \theta) + M_2h_2[a_x \cos \phi + a_y \sin \phi]
\]

\(P\) and \(Q\) are similar to the results in [9] with setting \(g = 0\), and the collision force from the ball add following two terms

\[
S_1 = F_{col}L_1[(2I_{2,cm} + h_2 - h_H)M_2] \cos \theta - h_2(h_2 + h_H)M_2 \cos(\theta - 2\phi)]
\]
\[
S_2 = F_{col}[(2h_H(I_{1,cm} + M_1h_1^2 + M_2L_1^2)
\]
\[
+ h_2(2I_{1,cm} + 2M_1h_1^2 + M_2L_1^2)) \cos \phi - h_2M_2L_1^2 \cos(2\theta - \phi)]
\]

The collision force defines the motion of the ball as follow,

\[
m_b\frac{d^2x_b}{dt^2} = F_{col}
\]
\[
= -f_k(x_H - x_b)
\]
\[
x_H = x_E + L_1 \sin \theta + (h_2 + h_H) \sin \phi
\]
where $x_H$ is the $x$ component of the hitting part of the racket which locates $h_2 + h_H$ from the bottom of the racket, and $x_E$ is the $x$ component of the joint between the elbow and arm.

The force between the racket and the ball should be repulsive, so that the force form becomes

$$f_k(x_H - x_b) = \begin{cases} k(x_H - x_b) & \text{if } x_H > x_b \\ 0 & \text{if } x_b \geq x_H. \end{cases}$$

(11)

If we consider this harmonic force between the ball and the second rod, the period of the oscillation is

$$T = \frac{2\pi}{\sqrt{\frac{k(m_b + M_2)}{m_b M_2}}}.$$  

(12)

Since the half of this period is the collision duration between the ball and the racket, we controlled the $k$ value from 188 $N/m$ to 19000 $N/m$. Subsequently, the collision time varies from 50 $ms$ to 5 $ms$. However, if the contraction length is large, the Hook’s model is not valid for the ball and the racket system. Then the force can be rewritten as $kx_m \sin(\frac{x x_m}{x_m})$ [10], or $\frac{2k x_m}{\pi} \tan(\frac{x x_m}{2x_m})$ in order to limit the maximum contraction length. However, in our numerical calculation we restricted our system in order to follow the Hook’s rule.

In this article, we assumed a simple model for the ball and racket collision. In our model, the ball is assumed to hit the racket when the racket is parallel to y axis ($\phi = 0$). In addition to this, the ball is assumed to be moving in x axis. In this case, we numerically calculated $\theta(t)$ and $\phi(t)$ till the time $t_0$, when the racket is parallel to the y axis ($\phi(t_0) = 0$). With these numerical results, we set new initial conditions just before the collision. We assumed that the collision occurs at $t = t_0 - T/2$, where $T$ is the period of the harmonic oscillator system between the racket and the ball. At this time, we numerically solved the ball and two rod system attached to the moving elbow numerically till the ball is rebound from the racket. We determined the speed of the ball at that moment. Based on our simple model, the coupling constant $k$ determines the collision duration, but the speed of the rebound ball is not altered very much. The reason for such a behavior is consideration of one dimensional collision. However, the purpose of this article was to find the optimum path for the swing of the racket, we did not extend our model to two dimension. We only restricted our numerical conditions to render presence of our setups in the valid region.
III. DIFFERENT BALL SPEED WITH THE SAME ANGULAR VELOCITY OF THE RACKET

We assumed that an arm and a forearm forms a simple rod and the moment of inertia about the center of mass is \( I_{1,cn} = M_1 L_1^2 / 2 \). The moment of inertia about the center of mass of the hand-racket system is also assumed as \( I_{2,cn} = M_2 L_2^2 / 2 \). Although these assumptions are not enough to study the tennis stroke in detail, we focused our attention on the double pendulum model, which has its application in the tennis stroke. In this section, the torques applied to the first forearm system \( C_1 \) and to the racket system \( C_2 \) were set by 25\( N \) and 2.5\( N \) as described in Rod Cross work [9]. The velocity (\( m/s \)) of the elbow is assumed as follows [9],

\[
\begin{align*}
v_x^E(t) &= -33t + 69t^2 \\
v_y^E(t) &= -42t + 174t^2.
\end{align*}
\]

(13)

In Fig. 2, we plotted the speed of the rebound ball from the racket as a function of the initial angles \( (\theta_0, \delta_0) \). The angle \( \delta \equiv \phi - \theta \) is defined as the angle between the first rod and the racket. \( \delta_0 \) is the initial angle at \( t = 0 \). We assumed that the length of the first rod (an arm and a forearm system) as \( L_1 = 0.3m \) and the mass of the first rod as \( M_1 = 2.0kg \). And the length of the racket system as \( L_2 = 0.7 \) and the mass of the racket system as \( M_2 = 0.3kg \). We assumed that the initial velocity of the ball just before the contact is \( 5m/s \) to the positive \( x \) direction. The \( x_H \) , a hitting position from the center of mass assumed to be as \( 0.15m \). The length of the first rod (an arm and forearm system) assumed to be as \( 0.3m \). This length is projection length in \( x - y \) planes. We plotted initial conditions which gives maximum speed of the rebound ball. As \( \theta_0 \) changes from \( 60^\circ \) to \( 120^\circ \), the \( \delta_0 \) becomes smaller and reaches \( 20^\circ \). In Fig. 3, we also plotted the speed of the rebound ball from the racket as a function of the initial angles \( (\theta_0, \delta_0) \). We only changed the length of the first rod, in other words we extended the distance between the elbow and the hand. At this time the \( \delta_0 \) is smaller as compared to the case when \( L_1 = 0.3m \). When the initial angle \( \theta_0 \) is \( 120^\circ \), the racket and the first rod should be in the same line so as to get the maximum speed of the ball. In our model, the speed of the rebound ball is not a simple function of the angular velocity \( \omega_2 \). In Fig. 4 we plotted the angular velocities \( \omega_1 \) and \( \omega_2 \) for two cases. For the case a) the initial angle \( (\theta_0, \delta_0) \) is \( (120.0^\circ, 81.5^\circ) \) and the contact time \( t_a = 0.198s \) and the angular
FIG. 2: The speed of the ball in $km/h$ unit as a function of the initial angle of the racket and arm with respect to the $-y$ axis. $\theta_0$: initial angle of the arm. $\delta_0$: initial angle between the racket and the first rod. The length of the first rod is $0.3m$. Red line indicates two initial angles which gives the maximum speed.

FIG. 3: The speed of the ball in $km/h$ unit as a function of the initial angle of the racket and arm with respect to the $-y$ axis. $\theta_0$: initial angle of the arm. $\delta_0$: initial angle between the racket and the first rod. The length of the first rod is $0.4m$. Red line indicates two initial angles which gives the maximum speed.
velocity is \( \omega_1 = 11.34 \text{rad/s} \). For the case b), the initial angle \((\theta_0, \delta_0)\) is \((75.0^\circ, 39.0^\circ)\) and the contact time \(t_b = 0.220s\) and the angular velocity \(\omega_1 = 17.9 \text{rad/s} \). However, the angular velocities \(\omega_2\) at the contact times have the similar value as \(37.98 \text{rad/s}\). Furthermore, the angular velocity \(\omega_{2b}\) increases even after the collision time. The interesting thing is that the speed of the rebound ball for the case b) is smaller than the speed of the ball for the case a). The speeds are \(156 \text{km/h}\) and \(141 \text{km/h}\), respectively.

If we only check the angular velocity \(\omega_2\), we may conclude that the case b) gives higher speed of the rebound ball. But the angular velocity is not the entire factor to determine the speed of the ball. This can be explained if we examine the angle \(\theta_c\) of the first rod when the racket contacts the ball. \(\theta_c\)'s are \(11.5^\circ\) and \(87.1^\circ\) for the cases a) and b), respectively. The speed of the ball \(v_{out}\) is also a functions of \(\theta_c\) as well as \(\omega_1\) and \(\omega_2\). In order to get the maximum speed of the rebound ball, the double pendulum system should be examined as a whole system.

![Image](image_url)

**FIG. 4:** The angular velocities \(\omega_1\) and \(\omega_2\) for two cases. The initial angle \((\theta_0, \delta_0)\) is \((120.0^\circ, 81.5^\circ)\) for the case a) and \((75.0^\circ, 39.0^\circ)\) for the case b). \(t_a\) and \(t_b\) are the contact time.

**IV. DEPENDENCE OF THE RACKET MASS AND THE DISTANCE BETWEEN THE ELBOW AND THE HAND**

The mass of the racket-hand system is assumed to be \(0.3kg\), but the mass of the racket may be changed. All the movements are assumed to be in the \(xy\) plane in our model, the
projected length between the elbow and the hand in $xy$ plane can be changed by controlling the angle between the forearm and arm in actual tennis stroke. In Fig. 5, we plotted the speed of the rebound ball from the racket as a function of the racket mass ($M_2$) and the projected length in $xy$ plane between the elbow and the hand $L_1$. If the mass of the racket is about 0.2$kg$, the speed of the rebound ball is decreased as the projected length between the elbow and the hand is increased. However, if the racket mass is getting heavier, the speed of the rebound ball increases. Since an actual mass of the tennis racket is around 0.3$kg$, and $L_1$ is limited, the speed of the ball is restricted.

In Fig. 6, we plotted the angle ($\theta_c$) of the forearm system at the impact time as a function of the racket mass ($M_2$) and the projected length in $xy$ plane between the elbow and the hand $L_1$. Regardless of the racket mass ($M_2$), the forearm angle $\theta_c$ decrease to 20° as the length $L_1$ increase. This results shows that for the player with folded arm whose effective projected length of the forearm system is small, the contact angle ($\theta_0$) should be about 90°. If the length $L_1$ and the contact angle $\theta_0$ are reduced the ball should be hit relatively close to the body. This tendency may explain the impact point of the ball in tennis stroke and golf.

FIG. 5: The speed of the rebound ball from the racket as a function of the racket mass ($M_2$) and the projected length in $xy$ plane between the elbow and the hand $L_1$. The unit of $v_{out}$ is $km/h$. 

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V. TIME LAGGED TORQUE VS. CONSTANT TORQUE.

In actual tennis stroke, most of the players use time lagged racket movement. They intentionally keep the racket back as forearm rotates then they start to move the racket to get a high angular velocity $\omega_2$. The main difference of double pendulum when compared to the single pendulum, is the separate movement of the first rod (arm and forearm system) and the second rod (the racket system). In Fig. 7, we plotted the speed of the rebound ball as a function of increase in the torque $C_2$ and the delay time $\tau$. We set the projected length in $xy$ plane between the elbow and the hand $L_1$ as 0.4m. The torque applied to the racket at the joint of two rods, varies from 0 to 10N. The time delay $\tau$ is the starting time at which we applied the torque to the racket. After waiting for $\tau$ second, the torque suddenly changes from zero to a certain value $C_2$ till 0.5ms in our numerical calculation. This time is far behind the contact time (around 0.2ms). At first glance, if we start to apply the torque early, the angular velocity of the racket may accelerate little bit more. It’s simply because the acceleration of the angular velocity depends on the torque. If the torque $C_2$ is less than 1.56N, the maximum speed of the rebound ball is obtained when the delay time $\tau$ is zero as we expected. However, for the higher torque $C_2 > 1.56N$, the numerical results in Fig. 7 are different from our simple intuition. For a given torque value $C_2 > 1.56N$, there always exist a certain time delay $\tau$ at which the speed of the rebound ball is maximum and the $\tau$
is not zero. In other words, the important thing to get high speed of the rebound is not the total amount of impulse (torque × (applied time)), but the timing when the torque starts.

If the torque $C_2$ is lagged, the initial angle $\delta = 26.1^\circ$ to get a maximum speed of the rebound ball for the fixed initial angle $\theta_0 = 90^\circ$ is no more the optimum condition. In order to see the time lag effect clearly, we set the torque $C_2 = 5N$. In Fig. 8 we plotted the speed of the rebound ball as a function of the time lag $\tau$ and initial angle $\delta_0$ with an initial angle $\theta_0 = 90^\circ$. The new optimum condition to get maximum speed of the rebound ball is the time lag $\tau = 0$. In Fig. 9, we plotted the speed of the rebound ball as a simple function of time delay $\tau$. If the time lag $\tau$ of the torque $C_2$ is $\tau_L = 0$, the rebound ball speed is $v_L = 136.6 km/h$. And $v_{out}$ has its maximum ($v_H = 164.4 km/h$) at $\tau_H = 0.185 s$. The speed increases by about 20% by adjusting the time delay with the same magnitude of the torque $C_2$. The angular velocity of the forearm system and the racket system is shown in Fig. 10. For two time delay $\tau_L$ and $\tau_H$, the difference in the contact time is less then 10 ms. The dashed line in Fig. 10 indicates the time at which the ball collides with the racket. We note that the racket hit the ball before the racket has its maximum angular velocity. In earlier work [9], the authors have analyzed the double pendulum system in order to get the maximum angular velocity $\omega_2$. However, this may mislead the double
pendulum system. Considering Fig. 4 it is clear that higher angular velocity gives lower speed of the rebound ball.

In Fig. 10, when the angular velocity $\omega_{2H}$ has its maximum, the angular velocity $\omega_1$ is almost zero. This is explained for the double pendulum model of tennis stroke. The angular momentum and energy of the first rod (arm and forearm system) were totally transferred to the racket system in order to get maximum angular velocity of the racket.

We plotted the first rod (forearm and arm system) and the racket system for the time delays represented as $\tau_H$ and $\tau_L$ in Fig. 11 and 12 respectively. When the first rod starts to rotate, the racket stayed back in Fig. 11 till time $\tau_H$. After applying the torque $C_2$, the racket suddenly starts to rotate, and hits the ball with the angle $\theta_c = 53.3^\circ$. The racket and the first rod are shown in red at the contact time. Comparing the stroke with a constant torque (Fig. 12), the racket rotates more rapidly at the contact time. In double pendulum model for the stroke, this phenomena was expected qualitatively. In our model we quantitatively demonstrated that why time lagged stroke is needed and the extent to which the velocity can be increased.
FIG. 9: The speed of the rebound ball as a simple function of time delay $\tau$. At $\tau = 0.185$ s, the $v_{out}$ has its maximum.

FIG. 10: The angular velocity of the forearm system $\omega_1$ and the racket system ($\omega_2$). $(\omega_{1L}, \omega_{2L})$ indicates the angular velocities when the time delay is $\tau_L$, and $(\omega_{1H}, \omega_{2H})$ indicates the angular velocities when the time delay is $\tau_H$.

VI. NEED MORE IN ORDER TO INCREASE THE SPEED OF THE REBOUND BALL.

In the previous section, we noted that the important thing is not the total impulse to accelerate the racket, but the timing in order to increase the speed of the rebound ball.
FIG. 11: Time dependent trace of the tennis racket system (black bar) and the arm and forearm system (blue bar), when the time delay is $\tau_H$. The motions are captured from $t = 0$ to $t = 0.32s$ evenly. The red one is the racket position at the contact time.

FIG. 12: Time dependent trace of the tennis racket system (black bar) and the arm and forearm system (blue bar), when the time delay is $\tau_L = 0$. The motions are captured from $t = 0$ to $t = 0.32s$ evenly. The red one is the racket position at the contact time.

Now, we analyze the influence of the elbow movement on the speed of the rebound ball. We made a simple model to add an additional movement of the elbow. The added force around
the time $\tau_H$ as follows,

$$F_x(t) = F_0 \cos \Psi_F \exp\left(-\frac{(t - (\tau_H + \tau_F))^2}{t_d^2}\right)$$

$$F_y(t) = F_0 \sin \Psi_F \exp\left(-\frac{(t - (\tau_H + \tau_F))^2}{t_d^2}\right)$$

(14)

where $\Psi_F$ is the angle measured from the $+x$ axis counterclockwise, we also set $F_0 = 20N$, and the width $t_d = 0.05s$. Since the ball is rebound to $-x$ axis, the proper direction of the force seems to be in the $-x$ direction. We plotted the speed of the rebound ball when the extra force is applied in the direction of $\Psi_F$ and with extra time delay $\tau_F$ in Fig. 13. Numerical results show that we can obtain the maximum speed of the rebound ball when the angle is $76^\circ$. When the direction of the force $\Psi_F = 180^\circ$, the speed of the rebound ball is almost same to the speed of the ball without the additional force. In other words, the external movement towards $-x$ direction does not give any additional speed to the rebound ball. In Fig. 14, we plotted the trajectory of the elbow from the time $t = 0$ till the contact time. $P_{NF}$ is the path when no additional force is added. $P_{max}, P_{min}$ are elbow’s trajectory when the direction of the force $\Psi_F$ are $76^\circ$ and, $275^\circ$, respectively. Table I demonstrates the $v_{out}$ and the angular velocity $\omega_2$ for 4-cases. The movement with $\Psi_F = 76^\circ$ is slightly backward and mainly perpendicular to the direction of the rebound ball. From this result, we conclude that the main factor for the increased speed of the rebound ball is not the linear momentum added to the racket but the angular momentum added to the racket system.

FIG. 13: The speed of the rebound ball as a functions of the extra force with a direction of $\Psi_F$ and with a extra time delay $\tau_F$
FIG. 14: The trajectory of the elbow for four cases from the time $t = 0$ till the contact time. $P_{NF}$ is the path when no additional force is added. $P_{\text{forward}}, P_{\text{max}}, P_{\text{min}}$ are elbow’s trajectory when the direction of the force $\Psi_F$’s are $180^\circ, 76^\circ, 275^\circ$, respectively.

| $F_0$ (N) | $\Psi_F (^\circ)$ | $v_{out}$ (km/h) | $\omega_2$ rad/s |
|-----------|-----------------|-----------------|-----------------|
| 20        | 180             | 163.3           | 39.1            |
| 20        | 275             | 158.8           | 39.2            |
| 20        | 76              | 170.6           | 52.2            |
| 0         | 0               | 164.4           | 44.7            |

VII. CONCLUSION AND DISCUSSION.

The double pendulum model is applied to the baseball, tennis, and golf. It analyzes the swing pattern to maximize the angular velocity of the hitting rod such as racket, bat, and club on the assumption that the angular velocity is the dominant factor to attain speed of the rebound ball. If we set, $\theta_c$, the angle of the first rod at the impact time, it seems to be obviously reasonable. On the other hands, if we release the angle $\theta_c$, the speed of the rebound ball is not a simple function of the angular speed of the hitting rod. The speed of the rebound ball is different even though the angular velocities at the contact time are same in Fig. 4. Furthermore, considering the whole stroke, the maximum angular velocity $\omega_2$ for
the lower speed of the rebound ball is greater than the maximum angular velocity $\omega_2$ for the higher speed of the rebound ball. Therefore, to attain maximum speed of the rebound ball, it’s not sufficient to set the condition to generate high angular velocity $\omega_2$ of the hitting rod.

In the double pendulum, the efficient way to generate high angular velocity of the hitting rod is the energy transfer from the first rod to the second hitting rod. In other words, when the angular velocity $\omega_1$ of the forearm system is zero, the angular velocity of the hitting racket has maximum value as shown in Fig. 10. We analyzed the time lagged torque effect for the double pendulum system. With applying constant torques $C_1, C_2$ on the forearm system and on the racket, respectively, the speed of the rebound ball can be calculated. For the same condition, if we simply hold the racket for a short time without enforcing a torque $C_2$, then applying the torque $C_2$ at the proper time ($t = \tau_H$), the speed of the rebound ball increases by 20% as can be seen in Fig. 9. The reason is mainly because the double pendulum system is not a simple linear system. Adding the energy to the double pendulum system does not directly increase the speed of the rebound ball.

We also analyzed the elbow movement effect. In addition to the velocity of the elbow for the medium pace forehand, we added extra movement of the elbow. At a first glance, if we add extra movement towards the rebound ball’s direction, the speed of the ball is increased. But the double pendulum system is not a simple linear system. When the direction of the elbow movement is perpendicular to the ball’s direction, the speed of the rebound increases. Actually the direction is towards the center of the elbow’s circular movement. In other words, the added centripetal force does not add not the linear momentum of the racket, but the angular velocity of the racket.

Although our collision model is applied in one dimension, this collision process allows us to analyze the double pendulum system in a more realistic manner. We showed that the speed of the rebound ball does not simply depend on the angular velocity of the racket. The increase in the ball speed by the proper time lagged racket rotation was numerically studied. The elbow movement for adding the ball’s speed was counter intuitive. The addition of simple linear momentum to the elbow is not important; however, the elbow should move in order to add angular velocity to the racket.

In actual tennis stroke, the motion occurs in three dimensions and the magnitudes of the forces and torques are dependent on the muscle shape and movement. We did not include any bio-mechanical information such as pronation and we did not include the spin of the
ball in any way. The numerical data may also be not suitable for some players. However, our study on the double pendulum system for tennis stroke provides some insights to attain an efficient way to stroke a tennis ball.

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