A Class of Generalized Hyperbolic Continuous Time Integrated Stochastic Volatility Likelihood Models

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This paper discusses and analyzes a class of likelihood models which are based on two distributional innovations in financial models for stock returns. That is, the notion that the marginal distribution of aggregate returns of log-stock prices are well approximated by generalized hyperbolic distributions, and that volatility clustering can be handled by specifying the integrated volatility as a random process such as that proposed in a recent series of papers by Barndorff-Nielsen and Shephard (BNS). Indeed, the use of just the integrated Ornstein-Uhlenbeck (INT-OU) models of BNS serves to handle both features mentioned above. The BNS models produce likelihoods for aggregate returns which can be viewed as a subclass of latent regression models where one has \( n \) conditionally independent Normal random variables whose mean and variance are representable as linear functionals of a common unobserved Poisson random measure. James (2005b) recently obtains an exact analysis for such models yielding expressions of the likelihood in terms of quite tractable Fourier-Cosine integrals. Here, our idea is to analyze a class of likelihoods, which can be used for similar purposes, but where the latent regression models are based on \( n \) conditionally independent models with distributions belonging to a subclass of the generalized hyperbolic distributions and whose corresponding parameters are representable as linear functionals of a common unobserved Poisson random measure. Our models are perhaps most closely related to the Normal inverse Gaussian/GARCH/A-PARCH models of Brandorff-Nielsen (1997) and Jensen and Lande (2001), where in our case the GARCH component is replaced by quantities such as INT-OU processes. It is seen that, importantly, such likelihood models exhibit quite different features structurally. Rather than Fourier-Cosine integrals, the exact analysis of these models yields characterizations in terms of random partitions of the integers which can be easily handled by Bayesian SIS/MCMC procedures similar to those which have been applied to Dirichlet/Gamma process mixture models. Importantly, these methods do not necessarily require the simulation of random measures. One nice feature of the model is that it allows for more flexibility in terms of modelling of external regression parameters. Our models may also be viewed as alternatives to closely related latent class GARCH models arising in financial economics.

1 Introduction

In financial economics, it is well known that Gaussian based models such as the Black-Scholes-Samuelson model for the log-stock prices of returns do not fit with empirical observations when returns are observed over moderately sized intervals. For example, on a
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daily basis. The Black-Scholes-Samuelson model may be described in terms of the following stochastic differential equation,

\[ dx^*(t) = (\mu + \beta \sigma^2)dt + \sigma dw(t) \]

where \( x^*(t) \) denotes the price level, \( \sigma^2 \) represents a constant volatility and \( w(t) \) is Brownian Motion. When observed over \( i = 1, \ldots, n \) equally spaced time intervals of length \( \Delta > 0 \), one has that the aggregate returns \( x^*(i\Delta) - x^*((i-1)\Delta) \) are iid Normal random variables with mean and variance \( (\mu \Delta + \sigma^2 \beta, \sigma^2) \). In terms of statistical inference, this produces a classical Normal likelihood where estimation of parameters \( (\beta, \mu) \) are straightforward. However, it is known that, while the model (1) is plausible for large \( \Delta \), when \( \Delta \) is of moderate size the aggregate returns exhibit behavior more like that of semi-heavy tailed distributions. Moreover, these models exhibit a feature known as volatility persistence or clustering. This suggests that \( \sigma^2 \) should be replaced by a dynamic random process which has correlated increments. See for instance Carr and Wu (2004), Carr, Geman, Madan and Yor (2003), Barndorff-Nielsen and Shephard (2001a,b), Duan (1995), and Engle (1982) for these points and various proposals to enhance (1). Here we shall focus on the model of Barndorff-Nielsen and Shephard (2001a, b) which we now describe.

1.1 BNS model and likelihood

A quite attractive model was introduced by Barndorff-Nielsen and Shephard (2001a, b). Their proposed continuous time stochastic volatility (SV) model is based on the following differential equation,

\[ dx^*(t) = (\mu + \beta v(t))dt + v^{1/2}(t)dw(t) \]

where \( x^*(t) \) denotes the price level, and \( v(t) \) is a stationary Ornstein-Uhlenbeck (OU) process which models the instantaneous volatility and is independent of \( w(t) \). The induced likelihood model, which is based on the integrated volatility \( \tau(t) = \int_0^t v(u)du \), can be described as follows. Let \( X_i := x^*(i\Delta) - x^*((i-1)\Delta) \), for \( i = 1, \ldots, n \) denote a sequence of the returns of the log price of a stock observed over intervals of length \( \Delta > 0 \). Additionally for each interval \([i\Delta, i\Delta]\), let \( \tau_i = \tau(i\Delta) - \tau((i-1)\Delta) \). Now the model in (2) implies that \( X_i | \tau_i, \beta, \mu \) are conditionally independent with

\[ X_i = \mu \Delta + \tau_i \beta + \tau_i^{1/2} \epsilon_i. \]

where \( \epsilon_i \) are independent standard Normal random variables. Hence if \( \tau \) depends on external parameters \( \theta \), one is interested in estimating \((\mu, \beta, \theta)\) based on the likelihood

\[ L_{BNS}(X | \mu, \beta, \theta) = \int_{\mathbb{R}_+^n} \prod_{i=1}^n \phi(X_i | \mu \Delta + \beta \tau_i, \tau_i) f(\tau_1, \ldots, \tau_n | \theta) d\tau_1, \ldots, d\tau_n \]

where, setting \( A_i = (X_i - \mu \Delta) \), and \( \bar{A} = n^{-1} \sum_{i=1}^n A_i \),

\[ \phi(X_i | \mu \Delta + \beta \tau_i, \tau_i) = e^{A_i \beta} \frac{1}{\sqrt{2\pi} \tau_i^{1/2}} e^{-A_i^2/(2\tau_i)} e^{-\tau_i \beta^2/2} \]
denotes a Normal density. We note that because of the complex dependence structure of the joint density \( f(\tau_1, \ldots, \tau_n|\theta) \), the likelihood was thought to be intractable. Thus inhibiting full likelihood based statistical inference, for models involving quite arbitrary \( \tau \). This is in contrast to the case of latent class GARCH models which are of considerable interest in financial economics[see for instance, Fiorentini, Sentana, and Shephard (2004)]. However, in a closely related recent paper James (2005b) shows that the likelihood (4), where the \( \tau_i \) are further generalized to be linear functionals of a Poisson random measure, is tractable and can be expressed exactly in terms of multi-dimensional Fourier-cosine transforms. The implication is that in general one can use classical numerical techniques to evaluate the likelihood. Moreover, these expressions are similar to quantities which regularly appear in the math finance literature on option pricing and related areas.

In this paper we offer another approach that still allows us to work with integrated OU processes and indeed more general objects. Our purpose is two fold. One to propose models which we believe are complementary to the above framework but exhibit quite different features, and in fact more flexibility in the sense of incorporating more general regression coefficients. Secondly, we believe that because these two models exhibit different features that this invites individuals of quite varying backgrounds to conduct research on similar topics. Of course, exact analysis of these two classes of models also allows one to more easily critique, compare and improve such models. One could also consider a third class of models based on a hybridization of the two models.

The difference is that our approach yields expressions of the likelihood in terms of random partitions of the integers which can considered relatives of the Blackwell and MacQueen (1973) Pólya urn distribution. Hence these models inherit many of the well-known features of Dirichlet/Gamma process mixture models and extensions addressed in James (2005a). Additionally, the posterior distribution of the random processes are also more in line with what happens for the case of Bayesian multiplicative intensity models, as it depends on the jumps of the underlying Poisson random measure. Moreover our models serve as an alternative to latent class GARCH models. In fact, one will see in the next section that our models are perhaps most closely related to the Normal inverse Gaussian/GARCH/A-PARCH models of Brandorff-Nielsen (1997) and Jensen and Lunde (2001), where we replace their GARCH/A-PARCH components by general \( \tau \).

2 A class of generalised hyperbolic integrated stochastic volatility models

Before we present the model, we note that many authors have fitted semi-heavy tailed models in finance by specifying \( \sigma^2 \) in the Black-Scholes formula to be a generalized inverse Gaussian (GIG) distribution. Hence the aggregate returns are from a generalised Hyperbolic (GH) distribution. We pause to describe this density which we shall use later. Let \( \lambda, v \) and \( \delta \) be such that \( -\infty < \lambda < \infty \), while \( v \) and \( \delta \) are non-negative and not simultaneously 0. As in Barndorff-Nielsen and Shephard(2001a), \( T \) is GIG(\( \lambda, \delta, v \)) random variable if its density is of the form

\[
  f_{GIG}(t|\lambda, \delta, v) = \frac{(v/\delta)^\lambda}{2K_\lambda(\delta v)} t^{\lambda-1} \exp\left\{-\frac{1}{2}(\delta^2 t^{-1} + v^2 t)\right\}
\]
where $K_\lambda$ is a Bessel function. When $\delta = 0$ and $\lambda > 0, v > 0$, GIG($\lambda, 0, v$) equates with the Gamma distribution. When $\lambda < 0, \delta > 0$ and $v = 0$, then GIG($\lambda, \delta, 0$) is a reciprocal, or inverse Gamma distribution. Using the parametrization, $\lambda = -a$, for $a > 0$, and $b = \delta^2/2$, yields the density of an inverse Gamma distribution with parameters, $a, b$. A special case of this is when $\lambda = -1/2$ leading to a stable law of index $1/2$. The inverse Gaussian distribution defined by setting $\lambda = -1/2$, $\delta > 0$, and $v > 0$ that is a GIG($-1/2, \delta, v$). The Hyperbolic distribution coincides with the case of $\lambda = 1$. See Prause (1999) and Eberlein (2001) for some additional background and references. The additional innovation in, for instance, Barndorff-Nielsen and Shephard (2001a, b) is that modeling volatility as a random process, $\nu(t)$, rather than a random variable, not only allows for semi-heavy-tailed models, but additionally induces serial dependence.

2.1 The model and conditional likelihood

We now describe a model which is a direct variant of (3) but is otherwise a subclass of a considerably more flexible but still tractable proposal which is given in section 2.3. Let $X_{s,t} := x^*(t) - x^*(s)$ denote the aggregate return of the log stock price over some interval $[s, t]$ for $0 \leq s < t$. Furthermore, define $\tau_{s,t} = \tau(t) - \tau(s)$ and $A_{s,t} = (x - \mu(t - s))$. Then given some filtration $\mathcal{F}_{\tau(t)}$ determined by $\tau$, and further depending on $\beta$ and $\mu$, we assume that $X_{s,t}$ is conditionally independent of the past with density

$$f_{X_{s,t}}(x|\tau, \beta, \mu) = \left(\frac{2}{\pi}\right)^{1/2} \tau_{s,t}^{\lambda/2} e^{A_{s,t}\beta} \left(\frac{\beta^2}{2\tau_{s,t} + A_{s,t}^2}\right)^{\lambda+1/2} \frac{K_{\lambda+1/2}(\beta \sqrt{2\tau_{s,t} + A_{s,t}^2})}{\Gamma(\lambda)}$$

for $\lambda > 0$.

It follows that the density (5), for fixed $\tau$, represents a subclass of generalized Hyperbolic (GH) densities which reduces to the Student distribution when $\mu = 0$ and $\beta = 0$, but otherwise is one of the well-defined limiting cases of the (GH) model[see for instance Prause (1999)]. We point out further that although this density, for fixed $\tau$, does not contain the Normal inverse Gaussian or Hyperbolic distribution, Prause (1999, p. 7-11) gives examples where the densities in (5) provides a more plausible fit to the data than those models. However, due to the general distributional flexibility of $\tau$, these issues do not really concern us, and we shall further take $\lambda = 1$ for additional tractability. That is to say estimation and model fitting will depend on parameters such as $\theta, \beta$ and $\mu$ and the distributional features of $\tau$.

It would appear that the models in (5), and its corresponding likelihood model, are considerably more complex than (2), (3) and (4). However for each increment one may write,

$$X_{s,t} \overset{d}{=} \mu(t - s) + (\tau_{s,t}/Z)\beta + (\tau_{s,t}/Z)^{1/2}\epsilon_{s,t}$$

where $Z$ is a Gamma random variable with shape $\lambda$, which we set to 1, and scale 1, and $\epsilon_{s,t}$ is an independent standard Normal random variable. Moreover, by utilizing a change of variable $W = Z/\tau_{s,t}$, the density (5) can be written as

$$\int_0^\infty \phi(x|\mu(t - s) + \beta w^{-1}, w^{-1})e^{-w\tau_{s,t}}\tau_{s,t}dw$$
2.2 The likelihood

Let $Z_i$ be iid Gamma random variables with shape $\lambda = 1$ and scale 1. Then under the setting of (4) our model translates into the case where $X_i|\tau_i, \mu, \beta$ are conditionally independent and representable as

$$X_i \overset{d}{=} \mu \Delta + (\tau_i/Z_i)\beta + (\tau_i/Z_i)^{1/2} \epsilon_i$$

Now using a change of variable $w_i = z_i/\tau_i$ results in a likelihood of $X_i|\mu, \beta, \theta$ expressible as

$$L(X_i|\mu, \beta, \theta) = \int_{\mathbb{R}^n} \prod_{i=1}^n \phi(X_i|\mu \Delta + \beta w_i^{-1}, w_i^{-1}) \mathbb{E} \left[ \prod_{i=1}^n e^{-w_i \tau_i \tau_i} \right] \prod_{i=1}^n dw_i$$

where

$$\mathbb{E} \left[ \prod_{i=1}^n e^{-w_i \tau_i \tau_i} \right] = \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-w_i \tau_i \tau_i} f(\tau_1, \ldots, \tau_n|\theta) \prod_{i=1}^n d\tau_i$$

It is noteworthy that (9) also has the form,

$$L(X_i|\mu, \beta, \theta) = C(X_i|\mu, \beta) \int_{\mathbb{R}^n} \mathbb{E} \left[ \prod_{i=1}^n e^{-w_i \tau_i \tau_i} \right] \prod_{i=1}^n f_{GIG}(w_i|3/2, |\beta|, |A_i|) dw_i$$

where $C(X_i|\mu, \beta)$ is determined from the Normal density and the GIG density. As we shall show the expression in (10) is easily handled by applying the results James (2005a, 2002). In closing this section notice that once $\tau$ is integrated out in (9) that one has a model whereby $X_i|w_i, \mu, \beta$ are independent

$$X_i = \mu \Delta + w_i^{-1/2} \beta + w_i^{-1/2} \epsilon_i$$

Hence conditional on $W = (W_1, \ldots, W_n)$, the parameters $(\mu, \beta)$ are easily estimated by standard parametric methods.

Remark 1. Note that the likelihood models, (4), analyzed in James (2005b), depended on $\tau$ only through terms such as $E \left[ \prod_{i=1}^n e^{-w_i \tau_i} \right]$ rather than $\mathbb{E} \left[ \prod_{i=1}^n e^{-w_i \tau_i \tau_i} \right]$. In analogy to survival analysis, the first expression can be thought of as the likelihood of a model where one only observes right-censored observations, while the latter may represent the appearance of both complete and censored observations. This creates a fundamental difference in their respective marginal analysis and structure.

2.3 Model flexibility

One important advantage of the present approach, over say the direct use of (4), is that we can more easily handle variations in the model. Briefly, we mention the following variation, which we believe helps address a question raised by M.C. Jones in the discussant section of Barndorff-Nielsen and Shephard (2001a, p. 225, 237),

$$X_i \overset{d}{=} \mu \Delta + (\tau_i/Z_i)^{1/2} \beta + (\tau_i/Z_i)^{1/2} \epsilon_i.$$
Owing to the same derivations above, this leads to a model whereby the $X_i|w_i, \mu, \beta$ are independent

$$X_i = \mu \Delta + w_i^{1/2}\beta + w_i^{-1/2}\epsilon_i.$$ 

More generally, for known real numbers $(a_0, a_1, \ldots, a_k)$ and possibly unknown $(\beta_1, \ldots, \beta_k)$ our approach extends to models of the type

$$X_i \overset{d}{=} \mu \Delta + \sum_{j=1}^{k} (\tau_i/Z_i)^{a_j}\beta_j + (\tau_i/Z_i)^{a_0}\epsilon_i$$

and beyond. This is seen by the fact the transformation $w_i = \tau_i/z_i$, yields the models $X_i|w_i, \mu, \beta$ are independent

$$X_i = \mu \Delta + \sum_{j=1}^{k} w_i^{-a_j}\beta_j + w_i^{-a_0}\epsilon_i$$

Note that the marginal distribution of $W|\mu, \beta_1, \ldots, \beta_k$ is the same as for the case of (11). To be quite clear, all our forthcoming results hold for this more general setting by replacing $\phi(X_i|\mu \Delta + \beta w_i^{-1}, w_i^{-1})$ with

$$\phi(X_i|\mu \Delta + \sum_{j=1}^{k} w_i^{-a_j}\beta_j, w_i^{-2a_0}).$$

Obviously in this case the density (5) has to be replaced by a more general Normal-Gamma mixture, but is otherwise just as easy to implement.

**Remark 2.** It is not difficult to deal with models where say $\tau_i e^{-\tau_i}$ is replaced by $\tau_i^\alpha e^{-\tau_i}$ for $0 < \alpha < 1$. Based on our approach one simply writes

$$\tau_i^\alpha = \tau_i^{\tau_i^{-(1-\alpha)}} = \frac{\tau_i}{\Gamma(1-\alpha)} \int_0^\infty e^{-yi/\tau_i}y_i^{-\alpha}dy_i$$

An augmentation reveals that the likelihood would involve an additional $n$ latent variables.

### 3 Evaluation of the likelihood for general $\tau$

Similar to James (2005b) we now evaluate the likelihood in the case where $\tau$ are more generally modeled as linear functionals of a Poisson random measure defined over Polish spaces. Let $N$ denote a Poisson random measure on some Polish space $\mathcal{V}$ with mean intensity, $E[N(dv)] = \nu(dv)$.

We denote the Poisson law of $N$ with intensity $\nu$ as $P(dN|\nu)$. The Laplace functional for $N$ is defined as

$$E[e^{-N(f)}] = \int_M e^{-N(f)}P(dN|\nu) = e^{-\Lambda(f)}$$

where $\Lambda(f)$ is the Laplace transform of $N$.

The Laplace transform of a random variable $X$ is defined as $\Lambda_X(f) = \mathbb{E}[e^{fX}]$. For a Poisson random measure $N$ with intensity $\nu$ on a Polish space $\mathcal{V}$, the Laplace functional $\Lambda(f)$ is given by

$$\Lambda(f) = \int_{\mathcal{V}} e^{-f(x)}\nu(dx).$$
where for any positive \( f \), \( N(f) = \int_{\mathcal{Y}} f(x)N(dx) \) and \( \Lambda(f) = \int_{\mathcal{Y}} (1 - e^{-f(x)})\nu(dx) \). \( \mathcal{M} \) denotes the space of boundedly finite measures on \( \mathcal{Y} \) [see Daley and Vere-Jones (1988)]. We suppose that \( \tau_i = N(f_i) \), for \( i = 1, \ldots, n \) where \( f_1, \ldots, f_n \) are positive measurable functions on \( \mathcal{Y} \). Notice now that the index \( i = 1, \ldots, n \) need not correspond to fixed intervals involving \( \Delta \). With this in mind, let \( (w_1, \ldots, w_n) \) denote arbitrary non-negative numbers. We shall assume throughout that \( f_1, \ldots, f_n \) are such that \( \Lambda(\sum_{i=1}^n w_i f_i) < \infty \). Notice first that one can write

\[
\prod_{i=1}^n \tau_i e^{-w_i \tau_i} = e^{-N(\sum_{i=1}^n w_i f_i)} \left[ \prod_{i=1}^n \int_{\mathcal{Y}} f(V_i)N(dx) \right].
\]

Removing the integrals, one can treat the \( \mathbf{V} = (V_1, \ldots, V_n) \) as missing values taking their values in \( \mathcal{Y} \). Similar to the case of the Blackwell-MacQueen distribution, which plays a fundamental role in Dirichlet and Gamma process mixture models [see Lo (1984) and Lo and Weng (1989), Ishwaran and James (2004) and James (2005a)], we can express the \( \mathbf{V} \) as follows. Let \( \mathbf{V}^* = (V_1^*, \ldots, V_n^*(\mathbf{p})) \) denote the \( n(\mathbf{p}) \leq n \), distinct values of \( \mathbf{V} \), where \( \mathbf{p} = \{C_1, \ldots, C_{n(\mathbf{p})}\} \) denotes a partition of the integers \( \{1, 2, \ldots, n\} \), with cells \( C_j = \{i : V_i = V_j^*\} \) for \( j = 1, \ldots, n(\mathbf{p}) \). Additionally, let \( e_j \), sometimes written as \( e_j \), denote the size, or cardinality, of the cell \( C_j \). Define \( \Omega_n(v) = \sum_{i=1}^{n(\mathbf{p})} w_i f_i(v) \). Let \( \mathbb{P}(dN|\nu_{\Omega_n}, \mathbf{V}) \) correspond to the law of the random measure \( N_{\Omega_n} + \sum_{j=1}^{n(\mathbf{p})} \delta_{V_j^*} \), where conditional on \( (\mathbf{V}, \mathbf{W}) \), \( N_{\Omega_n} \) is a Poisson random measure with mean intensity \( \nu_{\Omega_n}(dv) := e^{-\Omega_n(v)}\nu(v) \). Now an application of James (2005a, Proposition 2.3), augmenting (9), yields a joint distribution of \( (\mathbf{V}, \mathbf{W}, N, \mathbf{X}) \) which is expressible as,

\[
\mathbb{P}(dN|\nu_{\Omega_n}, \mathbf{V}) \left[ \prod_{i=1}^n f(V_i) \prod_{j=1}^{n(\mathbf{p})} e^{-\Omega_n(V_j^*)\nu}(dV_j^*) \right] e^{-\Lambda(\Omega_n)} \prod_{i=1}^n \phi(X_i|\mu\Delta + \beta W_i^{-1}, W_i^{-1})
\]

where

\[
e^{-\Lambda(\Omega_n)} = E \left[ e^{-N(\sum_{i=1}^n w_i f_i)} \right] = \int_{\mathcal{M}} e^{-N(\sum_{i=1}^n w_i f_i)} \mathbb{P}(dN|\nu).
\]

Note also that

\[
\mathcal{M}(d\mathbf{V}|\nu_{\Omega_n}) = \nu_{\Omega_n}(dV_1) \prod_{i=2}^n \left[ \nu_{\Omega_n}(dV_i) + \sum_{j=1}^{n(\mathbf{p})} \delta_{V_j^*}(dV_i) \right] = \prod_{j=1}^{n(\mathbf{p})} e^{-\Omega_n(V_j^*) \nu}(dV_j^*)
\]

corresponds to the \( n \)-th moment measure of a Poisson random measure with intensity \( \nu_{\Omega_n} \), and importantly has a structure similar to the Blackwell-MacQueen distribution. The expression \( \mathbf{p}_{i-1} \) corresponds to a partition of the integers \( \{1, \ldots, i-1\} \) for \( i \geq 2 \). Now integrating out \( (\mathbf{V}, \mathbf{W}, N) \) in (12) leads to an expression for the likelihood.

**Theorem 3.1** Suppose that \( \tau_i = N(f_i) \) for \( i = 1, \ldots, n \) where \( N \) is a Poisson random measure on \( \mathcal{Y} \) with intensity \( \nu \). Then the likelihood (9) can be expressed as

\[
\mathcal{L}(\mathbf{X}|\mu, \beta, \theta) = \int_{\mathbb{R}_+^n} \mathcal{B}(\mathbf{w}) e^{-\Lambda(\Omega_n)} \prod_{i=1}^n \phi(X_i|\mu\Delta + \beta w_i^{-1}, w_i^{-1})\mathbb{P}(dN)\mathcal{M}(d\mathbf{V}|\nu_{\Omega_n})
\]

where \( \mathcal{B}(\mathbf{w}) = \int_{\mathcal{Y}} \prod_{i=1}^n f_i(V_i) \mathcal{M}(d\mathbf{V}|\nu_{\Omega_n}) = \sum_{\mathbf{p}} \prod_{j=1}^{n(\mathbf{p})} \int_{\mathcal{Y}} \left[ \prod_{i\in C_j} f_i(v) \right] \nu_{\Omega_n}(dv) \). Where \( \sum_{\mathbf{p}} \) denotes the sum over all partitions of the integers \( \{1, \ldots, n\} \). \( \square \)
Theorem 3.2 Suppose that $\tau_i = N(f_i)$ for $i = 1, \ldots, n$ where $N$ is a Poisson random measure on $\mathcal{Y}$ with intensity $\nu$. Then augmenting the likelihood in Theorem 3.1 yields the joint posterior distribution of $V, W|X$ given by,

$$\mathcal{P}(dv, dw) \propto \prod_{i=1}^{n} f_i(v_i) \mathcal{M}(dV|\nu_{\Omega_n}) e^{-\Lambda(\Omega_n)} \prod_{i=1}^{n} \phi(X_i|\mu \Delta + \beta w_i^{-1}, w_i^{-1})dw_i$$

In particular we have the following posterior distributions

(i) $\pi(dw|v, X) \propto e^{-\Lambda(\Omega_n)} \prod_{i=1}^{n} f_{GIG}(w_i|3/2, |\beta|, \sqrt{A_i^2 + 2 \sum_{j=1}^{n(p)} f_i(v_j^*)})$

(ii) $\pi(dw|w, X) \propto \left[ \prod_{i=1}^{n(p)} f_i(v_i^*) \right] \mathcal{M}(dw|\nu_{\Omega_n})$

(iii) $\mathcal{P}(dw) = \mathcal{B}(w)e^{-\Lambda(\Omega_n)} \prod_{i=1}^{n} \phi(X_i|\mu \Delta + \beta w_i^{-1}, w_i^{-1})dw_i/\mathcal{L}(X|\mu, \beta, \theta)$ is the posterior density of $W|X$. \qed

Proposition 3.1 The distribution of $V|W, X$ can be further described as follows. The posterior distribution of $V|p, W, X$, is such that the unique values $V^*$ are conditionally independent with distributions

$$P(V_j^* \in dv|p, w, X) \propto \prod_{i \in C_j} f_i(v) \nu_{\Omega_n}(dv) \text{ for } j = 1, \ldots, n(p).$$

The posterior distribution of $p|W, X$ is given by $\pi(p|w, X) \propto \prod_{j=1}^{n(p)} \int_{v \in \mathcal{Y}} \left[ \prod_{i \in C_j} f_i(v) \right] \nu_{\Omega_n}(dv) \Box$

3.1 Posterior distribution of parameters

Upon examining the likelihood, one sees that Bayesian inference for parameters ($\mu, \beta, \theta$) can be implemented in a straightforward manner, along the lines of methods outlined for the Dirichlet/Gamma process semi-parametric mixture models. See Ishwaran and James (2004) for these ideas and further pertinent references. Specifically, a straightforward application of Bayes rule yields the following results.

Proposition 3.2 Suppose that $\tau$ depends on a $d$-dimensional parameter $\theta$. Then if $q(d\theta)$, $q(d\beta)$, $q(\mu)$ denote independent priors for $(\beta, \mu, \theta)$, their posterior distributions can be written as follows

(i) $q(d\beta|\mu, w, X) \propto q(d\beta)e^{-|\beta|^2/2 \sum_{i=1}^{n} w_i - n A \beta}$

(ii) $q(d\mu|\beta, w, X) \propto q(d\mu)e^{-|\sum_{i=1}^{n} A_i^2 w_i^{-1} - n A \beta}$

(iii) $q(d\theta|w, V, X) \propto q(d\theta)e^{-\Lambda(\Omega_n)} \prod_{j=1}^{n(p)} \nu_q(dV_j^*)$ \qed
3.2 Posterior distribution of the process

The above results describe the behavior of the finite-dimensional likelihood and parameters. It is useful to also obtain a description of the underlying random process given the data. This allows one to see directly how the data affects the overall process. Moreover, combined with the results in James (2005a), it provides a calculus for more general functionals. For notational simplicity we suppose that \((\mu, \beta, \theta)\) are fixed. The next result also follows immediately from an application of Fubini’s theorem and (12).

**Theorem 3.3** Suppose that a likelihood of \(X\) and the specifications for \(\tau\) and \(N\) are defined by the specifications in Theorem 3.1. Let \(\Omega_n(x) = \sum_{i=1}^n W_i f_i(x)\). Let \(\mathcal{F} = \mathcal{V} \times (0, \infty)\). Then the posterior distribution of \(N | X\) is

\[
\int_{\mathcal{F}^n} \mathbb{P}(dN|\nu_{\Omega_n}, v) \mathcal{P}_X(dv, dw).
\]

In particular for any positive or integrable function \(g\) on \(\mathcal{M}\),

\[
\int_{\mathcal{F}^n} \left[ \int_{\mathcal{M}} g(N) \mathbb{P}(dN|\nu_{\Omega_n}, v) \right] \mathcal{P}_X(dv, dw) = \int_{\mathcal{F}^n} \left[ \int_{\mathcal{M}} g(N + \sum_{j=1}^{n(p)} \delta_{V_j^*}) \mathbb{P}(dN|\nu_{\Omega_n}) \right] \mathcal{P}_X(dv, dw)
\]

\[\square\]

3.3 A general posterior predictive density for the log price

We now define a random variable similar to (6) which can be thought of as representing the log-price and give an explicit expression for its posterior density given \(X\). The random variable is defined as,

\[\tilde{X} = \mu \tilde{\Delta} + (\tilde{\tau}/Z)\beta + (\tilde{\tau}/Z)^{1/2} \epsilon,\]

where \(\tilde{\Delta}\) is just some non-negative number, \(\tilde{\tau} = N(\hat{f})\) for some positive function \(\hat{f}\) such that Laplace transform of \(\tilde{\tau}\) exists. \(\epsilon\) is a standard Normal random variable. Now let \(\Omega_{n+1}(x) = \Omega_n(x) + w\tilde{f}(x)\).

**Proposition 3.3** The posterior density of the log stock price given \(X\) is, \(f_{\tilde{X}}(x|\beta, \mu, X)\) equal to,

\[
\int_{\mathbb{R}_+} \left[ \int_0^\infty q(w|v, w)e^{-[\Lambda(\Omega_{n+1})-\Lambda(\Omega_n)]} \phi(x|\mu \tilde{\Delta} + \beta w^{-1}, w^{-1}) dw \right] \mathcal{P}_X(dv, dw),
\]

where \(q(w|V, w) = \left[ f_{\tilde{\mathcal{F}}}(v) e^{-w\hat{f}(v)} \nu_{\Omega_n}(dv) + \sum_{j=1}^{n(p)} \hat{f}(V_j^*) \right] e^{-\sum_{j=1}^{n(p)} w\hat{f}(V_j^*)}\).

**Proof.** The result follows from Theorem 3.3, and (7), using the following fact,

\[
\int_{\mathbb{M}} N(\hat{f}) e^{-N(w\hat{f})} \mathbb{P}(dN|\nu_{\Omega_n}, V) = e^{-\sum_{j=1}^{n(p)} w\hat{f}(V_j^*)} \int_{\mathbb{M}} [N(\hat{f}) + \sum_{j=1}^{n(p)} \hat{f}(V_j^*)] e^{-N(w\hat{f})} \mathbb{P}(dN|\nu_{\Omega_n}),
\]

where \(wN(\hat{f}) = w\tilde{\tau}\), and \(\int_{\mathbb{M}} e^{-N(w\hat{f})} \mathbb{P}(dN|\nu_{\Omega_n}) = e^{-[\Lambda(\Omega_{n+1})-\Lambda(\Omega_n)]}\). \(\square\)
Hyperbolic Integrated Volatility

3.4 Generalized Chinese Restaurant and Pólya Urn procedures

The results in the previous sections show that, viewing $W$ as a parameter, these models are structurally similar to Bayesian semi-parametric mixture models based on multiplicative intensity likelihoods. As such, one can import computational sampling schemes described in Ishwaran and James (2004) and James (2005a) and references therein. One can deduce the necessary modifications for the general nonparametric setting in James (2005a) from the methods for the semi-parametric Gamma process setting described in Ishwaran and James (2004). In particular, this includes general semi-parametric analogues of Pólya Urn Gibbs samplers and SIS procedures given by Escobar (1994), Liu (1996), and West, Müller and Escobar (1994), and the Gibbs sampling/SIS procedures based on a generalized weighted Chinese restaurant process [see Lo, Brunner and Chan (1996) and Ishwaran and James (2003)]. However, since these schemes are phrased primarily for completely random measures, which are a subclass of the models we look at here, we mention a few details for clarification in the general Poisson case. First note that sampling from the distribution of $W, \theta, \beta, \mu | V, X$, described by Theorem 3.2 and Proposition 3.2, proceeds along the lines of well-known parametric procedures such as random walk Metropolis-Hastings. The task then remains to approximately sample $V | W, \theta, \beta, \mu$. The key fact, is that structurally these models are not markedly different than the Dirichlet/Gamma process mixture models based on the Blackwell-MacQueen Pólya Urn distribution. In fact, a weighted Chinese restaurant SIS algorithm to sample Urn distributions derived from general Poisson random measures, such as that of $V$, has already been given in James (2002, section 2.3). This of course translates into dual Gibbs sampling procedures. Here we sketch out the relevant probabilities to implement these type of schemes for the simulation from the distribution,

$$
\pi(dV|W, \beta, \theta, \mu, X) \propto \prod_{i=1}^{n} f_i(V_i) \cdot \mathcal{M}(dV|\nu_{\Omega_n}) = \prod_{j=1}^{n(p)} \prod_{i \in C_j} f_i(V_i^*) \cdot \mathcal{M}(dV|\nu_{\Omega_n})
$$

Note that this distribution will typically not depend on $(\beta, \mu)$ except through $W$. Similar to James (2005a, equation 40), define for $r = 0, \ldots, n-1$ conditional probabilities,

$$
\mathbb{P}(V_{r+1} \in dx|V_r) = \frac{l_{0,r} \lambda_r(dx)}{c_r} + \sum_{j=1}^{n(p_r)} \frac{l_{j,r}(V_j^*)}{c_r} \delta_{V_j^*}(dx)
$$

where $V_r = \{V_1, \ldots, V_r\}$, $\lambda_r(dx) \propto f_{r+1}(x)\nu_{\Omega_n}(dx)$ and $l_{0,r} = \int f_r f_{r+1}(x)\nu_{\Omega_n}(dx)$ and $l_{j,r}(x) = f_{r+1}(x)$ Additionally $c_r = l_{0,r} + \sum_{j=1}^{n(p_r)} l_{j,r}(V_j^*)$. Examining James (2005a, section 4.4.) we see these are the ingredients to implement general analogues of the Pólya Urn type methods described in James (2005a, section 4.4.) which involve integrating the jump components in the $V$ vector should be employed if possible. To get the Chinese restaurant type procedures
one samples partitions \( p \) based on probabilities derived from \( l_{0,r} \) and

\[
l_{j,r} = \int_{\mathcal{Y}} l_{j,r}(x) \left[ \prod_{i \in C_{j,r}} f_i(x) \right] \nu_{\Omega_n}(dx) \text{ for } j = 1, \ldots, n(p_r)
\]

where \( p_r \) denotes a partition of the integers \( \{1, \ldots, r\} \) and each \( C_{j,r} = \{ i \leq r : V_i = V_j^* \} \) denotes the corresponding cells. Additionally, \( l(r) = l_{0,r} + \sum_{j=1}^{n(p_r)} l_{j,r} \). See James (2002, section 2.3) for justification of these procedures.

Remark 3. The main distinction, structurally, between the nonparametric multiplicative intensity models described in James (2005a) and the present semi-parametric setting, is the non-cancelation of the Laplace transform \( e^{-\Lambda(\Omega_n)} \) in the likelihood as it depends on external parameters \( (W, \theta) \). This point is addressed in Ishwaran and James (2004) for Gamma processes.

4 Results for completely random measures

Many processes \( \tau \) will be directly expressible as functionals of completely random measures, say \( \mu \). This is the case for models based on the integrated OU processes of Barndorff-Nielsen and Shephard (2001a,b), where \( \mu \) is the Background Driving Lévy Process (BDLP). As such, refinements of the above results in that case can be deduced from James (2005a, section 4). Note that a homogeneous completely random measure, say \( \mu \), with no drift, has the representation \( \mu(dy) = \int_0^\infty uN(du,dy) \). Where, \( \mathcal{Y} = (0, \infty) \times \mathcal{Y} \) for some Polish space \( \mathcal{Y} \), \( \nu(du,dy) := \rho(du)\psi(dy) \). The measure \( \rho \) is the Lévy density of a non-negative infinitely divisible random variable, \( T \), with Laplace transform

\[
E[e^{-\omega T}] = e^{-\psi(\omega)}
\]

where \( \psi(\omega) = \int_0^\infty (1 - e^{-\omega u})\rho(du) \). One may then write \( V_i = (J_i, Y_i) \) and \( V_j^* = (J_{j,n}, Y_j^*) \), where \( (J_{j,n}) \) denotes the unique jumps of the process \( \mu \), picked by a type of biased sampling. Moreover it follows that for measureable functions \( f_i(u,y) = u g_i(y) \), \( \tau_i := N(f_i) = \mu(g_i) \). Additionally for any \( f \) and \( g \) such that \( f(u,y) = u g(y) \), one has \( \Lambda(f) = \int_{\mathcal{Y}} \psi(g(y))\eta(dy) < \infty \). Define, \( \rho_{\Omega_n}(du|y) = e^{-u \sum_{i=1}^n u g_i(y)\rho(du)} \), and for \( l = 1, \ldots, n \) conditional cumulants

\[
\kappa_l(\rho_{\Omega_n}|y) = \int_0^\infty u^l \rho_{\Omega_n}(du|y).
\]

Theorem 4.1 Suppose that \( N \) is a Poisson random measure with intensity \( \nu(du,dy) = \rho(du)\eta(dy) \) on \( \mathcal{Y} = (0, \infty) \times \mathcal{Y} \). Suppose that \( \tau_i := N(f_i) = \mu(g_i) \) for \( i = 1, \ldots, n \). Then according to the model (9), one has the following results.

(i) Setting \( V_j^* = (J_{j,n}, Y_j^*) \), for \( j = 1, \ldots, n(p) \), conditional on \( p, W, X \), the pairs of random variables on \( \mathcal{Y} \) are independent with distributions

\[
P(J_{j,n} \in du, Y_j^* \in dy|W, X) \propto \frac{\nu^{\bar{e}_j}\rho_{\Omega_n}(du|y)}{\kappa_{\bar{e}_j}(\rho_{\Omega_n}|y)} \frac{\kappa_{e_j}(\rho_{\Omega_n}|y)}{\kappa_{e_j}(\rho_{\Omega_n}|y)} \left[ \prod_{i \in C_j} g_i(y) \right] \eta(dy),
\]
The following remarks address specifically the models in this section, but clearly have extensions to the general setting. Suppose the infinitely divisible random variable \( f_\kappa \) is important to note that it has density,

Now defining the first \( l \) cumulants may be calculated using the result of Theile. That is,

The posterior distribution of \( \mu | p, W, X \), is such that \( \mu(dx) \overset{d}{=} \mu_{\Omega_n}(dx) + \sum_{j=1}^{n(p)} J_{j,n} \delta_{Y_j^*}(dx) \)

where given \( p, W, X \), \( \mu_{\Omega_n} \) is a completely random measure determined by the law \( P(dN|\nu_{\Omega_n}) \), and the pairs \( (J_{j,n}, Y_j^*) \) are conditionally independent of \( \mu_{\Omega_n} \) with distribution described in [(i)]

(iii) The distribution of \( p|W, X \) is proportional to \( \prod_{j=1}^{n(p)} \vartheta(C_j|w) \).

(iv) The density of \( W|p, X \) is

\[
f(w|p, X) \propto e^{-\int \omega \psi(\sum_{i=1}^{n} w_i g_i(y)) \eta(dy)} \prod_{j=1}^{n(p)} \vartheta(C_j|w) \prod_{i=1}^{n} \phi(X_i|\mu \Delta + \beta w_i^{-1}, w_i^{-1})
\]

All the above results hold, in an obvious way, for the inhomogeneous case of \( \nu(du,dy) = \rho(du|y)\eta(dy) \). □

4.1 Remarks on implementation

The following remarks address specifically the models in this section, but clearly have extensions to the general setting. Suppose the infinitely divisible random variable \( T \) has density \( f_T \), and hence for a unique \( \rho \), it has Laplace transform \( \int_0^\infty e^{-\omega t} f_T(t)dt = e^{-\psi(\omega)} \). It is then important to note that \( \kappa_l(\rho_{\Omega_n}|y) \) are for fixed \( y \), the first \( l = 1, \ldots, n \) cumulants of an infinitely divisible random variable with density,

\[
f_{\Omega_n}(t|y) := e^{-\sum_{i=1}^{n} w_i g_i(y)} f_T(t) e^{\psi(\sum_{i=1}^{n} w_i g_i(y))}.
\]

Now defining the first \( l = 1, \ldots, n \) moments as \( m_l(y) = \int_0^\infty t^l f_{\Omega_n}(t|y)dt \), it follows that the cumulants may be calculated using the result of Theile. That is,

\[
k_l(\rho_{\Omega_n}|y) = m_l(y) - \sum_{k=1}^{l-1} \binom{l-1}{k-1} k_{k}(\rho_{\Omega_n}|y)m_{l-k}(y).
\]

This indicates quite clearly, the important fact, that one need not have the specific form of the Lévy density \( \rho \) to implement estimation procedures for our models. An interesting case would be where \( T \) has a log Normal distribution. Of course for models such as stable laws where \( \rho \) has a simple form and the probability density is generally complex, the converse is also true.

4.2 Example: Generalized Gamma processes

We now provide some details for one of the most tractable classes of models. An interesting class of measures are the family of generalized Gamma random measures discussed in Brix (1999). Using the description of Brix (1999), these are \( \mu \) processes with Lévy measure

\[
\rho(du) = \frac{1}{\Gamma(1-\alpha)} u^{-\alpha-1} e^{-bu} du.
\]
The values for $\alpha$ and $b$ are restricted to satisfy $0 < \alpha < 1$ and $0 \leq b < \infty$ or $-\infty < \alpha \leq 0$ and $0 < b < \infty$. Different choices for $\alpha$ and $b$ yield various subordinators. These include the stable subordinator when $b = 0$, the Gamma process subordinator when $\alpha = 0$ and the inverse-Gaussian subordinator when $\alpha = 1/2$ and $b > 0$. When $\alpha < 0$ this results in a class of Gamma compound Poisson processes. It follows that, for $\alpha \neq 0$, and $b \geq 0$,
\[
\psi(\sum_{i=1}^{n} w_i g_i(y)) = \frac{1}{\alpha}(b + \sum_{i=1}^{n} w_i g_i(y))^\alpha - b^\alpha
\]
The case of the Gamma process, $\alpha = 0$, $b > 0$ is a limiting case and results in the well-known expression
\[
\psi(\sum_{i=1}^{n} w_i g_i(y)) = \ln(1 + \sum_{i=1}^{n} w_i g_i(y)/b)
\]
Now, for all $\alpha$ and $b$, conditional on $Y_j^*, \mathbf{p}, \mathbf{W}, \mathbf{X}$, each $J_{j,n}$ is Gamma distributed with shape and scale parameters $(e_j - \alpha, b + \sum_{i=1}^{n} w_i g_i(Y_j^*))$. It follows that the joint moment measure of $Y|\mathbf{W}, \mathbf{X}$ is,
\[
\left[\prod_{j=1}^{n(p)} \frac{\Gamma(e_{j,n} - \alpha)}{\Gamma(1 - \alpha)}\right]^{n(p)} \prod_{j=1}^{n(p)} \left(b + \sum_{i=1}^{n} w_i g_i(Y_j^*)\right)^{-(e_{j,n} - \alpha)} \eta(dY_j^*)
\]
which is the key component in the sampling algorithms described in James (2005a). See Ishwaran and James (2004) for many details, including the usage of Blocked Gibbs Samplers, in the Gamma process semi-parametric setting which easily translates to the generalized Gamma class.

5 Example: BNS-OU model

For proper comparison with the models (4) as derived in James (2005b), it is interesting to look at how the Integrated OU model of Barndorff-Nielsen and Shephard (2001a,b, 2003) behave in this scenario. We shall refer to this model as the BN S-OU model. Throughout this section we shall take $\mathcal{Y} = (-\infty, \infty)$. One may express the Barndorff-Nielsen and Shephard (2001 a, b) integrated OU process $\tau$ as

\[
\tau(t) = \lambda^{-1}[(1 - e^{-\lambda t}) \int_{-\infty}^{0} e^y \mu(dy) + \int_{0}^{t} (1 - e^{-\lambda(t-y)}) \mu(dy)]
\]

where $v(0) := v_0 = \int_{-\infty}^{0} e^y \mu(dy)$, denotes the instantaneous volatility at time 0. The form in (14) is taken from Carr, Geman, Madan and Yor (2003, p. 365). It follows that for any $s < t$, $[\tau(t) - \tau(s)] = \mu(g_{s,t}) = N(f_{s,t})$ where $f_{s,t}(u,y) = u g_{s,t}(y)$ and $\lambda g_{s,t}(y)$ equals,

\[
e^{-\lambda s}(1 - e^{-\lambda(t-s)})e^{y}I_{\{y\leq 0\}} + (1 - e^{-\lambda(t-y)})I_{\{s < y \leq t\}} + e^{-\lambda s}(1 - e^{-\lambda(t-s)})e^{\lambda y}I_{\{0 < y \leq s\}}.
\]

The first component in (15) represents the contribution from $v_0$. Specializing this to $s = (i - 1)\Delta$ and $t = i\Delta$ one has $\tau_i = \mu(g_i) = N(f_i)$ where $f_i(u,y) = u g_i(y)$ and further $g_i(y) = g_{i,1}(y) + g_{i,2}(y)$ with,

\[
g_{i,1}(y) = \lambda^{-1}[(1 - e^{-\lambda(i\Delta-y)}) I_{\{(i-1)\Delta < y \leq i\Delta\}} + e^{-\lambda(i-1)\Delta} (1 - e^{-\lambda\Delta}) e^{\lambda y}I_{\{y\leq 0\}}]
\]

and

\[
g_{i,2}(y) = \lambda^{-1} e^{-\lambda(i-1)\Delta} (1 - e^{-\lambda\Delta}) e^{\lambda y}I_{\{0 < y \leq (i-1)\Delta\}}.
\]
Now for $i = 1, \ldots, n$, set $r_i = \lambda^{-1}[\sum_{k=i}^{n} w_k e^{-\lambda(k-1)\Delta}](1 - e^{-\lambda\Delta})$, and define $r_{n+1} = 0$.

Now notice that for any sequence of numbers, the simplest expression will be obtained by utilizing the following facts.

\begin{equation}
\sum_{j=1}^{n} w_j[g_{j,1}(y) + g_{j,2}(y)] = r_1 e^{y} \text{ for } y \leq 0
\end{equation}

and for $i = 1, \ldots, n$

\begin{equation}
\sum_{j=1}^{n} w_j[g_{j,1}(y) + g_{j,2}(y)] = \zeta(y|w_i, r_{i+1}) \text{ for } (i - 1)\Delta < y \leq i\Delta.
\end{equation}

Where for each $i$, $\zeta(y|w_i, r_{i+1}) = [\lambda^{-1}w_i(1 - e^{-\lambda(i\Delta - y)}) + r_{i+1}e^{y}]$. Then one has the following result which is a generalized version of James (2005b, Proposition 3.1).

**Proposition 5.1** For $0 \leq s < t$, let $\tau(t) - \tau(s)$ be defined by (14) and (15) Then the results of Theorem 4.1 hold with $f_i(u, y) = u[g_{i,1}(y) + g_{i,2}(y)]$, as described in (16) and (17). Suppose that $\eta(dy) := \eta(y)dy$, and define $\eta_i(i\Delta, u) = \eta(i\Delta + \lambda^{-1}\ln(1-u))$. Now, in particular, using a change of variable,

(i) $e^{-\Lambda(\sum_{i=1}^{n} w_i f_i)} = e^{-\Phi_0(r_1)} e^{-\Phi_n(w_n)} \prod_{i=1}^{n-1} e^{-\Phi_i(w_i|r_{i+1})}$

(ii) $\Phi(w_i|r_{i+1}) = \int_{1-e^{-\lambda\Delta}}^{1} \lambda^{-1}\psi(r_{i+1})e^{\lambda(i\Delta - (1 - u)) + \lambda^{-1}w_i u}\frac{\eta_i(i\Delta, u)du}{1-u}$, for $i = 1, \ldots, n - 1$

(iii) $\Phi(w_n) = \int_{1-e^{-\lambda\Delta}}^{1} \lambda^{-1}\psi(\lambda^{-1}w_n u)\frac{\eta_n(i\Delta, u)du}{1-u}$

(iv) $\Phi_0(r_1) = \int_{0}^{1} \psi(r_1 u)\eta(\ln u)\frac{du}{u}$, where $e^{-\Phi_0(r_1)} = \mathbb{E}[e^{-r_1\psi_0}]$.

\[\Box\]

Additionally one has the following features which do not play a role in the analysis of James (2005b). First statements (18) and (19) imply that

\[\rho_{\Omega_n}(du|y) = \begin{bmatrix} e^{-ur_1 e^{y}} I_{\{y \leq 0\}} + \sum_{i=1}^{n} e^{-u\zeta(y|w_i, r_{i+1})} I_{\{(i-1)\Delta < y \leq i\Delta\}} \end{bmatrix} \rho(du).\]

Now with some abuse of notation write

\[\rho_{r_1}(du|y) = e^{-ur_1 e^{y}} \rho(du) \text{ and } \rho_{w_i, r_{i+1}}(du|y) = e^{-u\zeta(y|w_i, r_{i+1})} \rho(du).\]

Hence for $l = 1, \ldots, n$, one can write in an obvious way,

\[\kappa_l(\rho_{\Omega_n}|y) := \kappa_l(\rho_{r_1}|y) I_{\{y \leq 0\}} + \sum_{i=1}^{n} \kappa_l(\rho_{w_i, r_{i+1}}|y) I_{\{(i-1)\Delta < y \leq i\Delta\}}.\]

Let $i^*_j = \min_{i \in C_j}$, that is the minimal index in a cell $C_j$, then,

\[\prod_{i \in C_j} \left(g_{i,1}(y) + g_{i,2}(y)\right) = a(j, \Delta) \left(q(y|i^*_j, e_j) I_{\{(i^*_j-1)\Delta < y \leq i^*_j\Delta\}} + e^{\epsilon_j y} I_{\{y \leq 0\}} + e^{|\lambda| y} I_{\{0 < y \leq (i^*_j-1)\Delta\}}\right)\]
where
\[
q(y|i_j^*, e_j) = \frac{(1 - e^{\lambda(i_j^* - y)}) e^{\lambda e_j y}}{e^{-\lambda i_j^*} (1 - e^{-\lambda})} \quad \text{and} \quad a(j, \Delta) = \prod_{i \in C_j} \frac{1}{\lambda} e^{-\lambda(i-1)\Delta} (1 - e^{-\lambda\Delta}).
\]

Now one has \( \kappa_{e_j} (\rho_{\Delta n} | y) \prod_{i \in C_j} (g_{i,1} (y) + g_{i,2} (y)) / a(j, \Delta) \) is equal to
\[
r(y|i_j^*, e_j, w) = e^{\rho_{\Delta n} y} \kappa_{e_j} (\rho_{\Delta n} | y) I_{\{y \leq 0\}} + q(y|i_j^*, e_j) \kappa_{e_j} (\rho_{\Delta n, r_{i_j^*+1}} | y) I_{\{(i_j^* - 1) \Delta < y \leq i_j^* \Delta\}} + e^{\rho_{\Delta n} y} \sum_{k=1}^{i_j^*-1} \kappa_{e_j} (\rho_{w_k, r_{k+1}} | y) I_{\{(k-1) \Delta < y \leq k \Delta\}}.
\]

One can check these points easily by looking at \([g_{i,1}(y) + g_{i,2}(y)] / [g_{i,1}(y) + g_{i,2}(y)]\) for any pair \(i < l\). Additionally,
\[
an := \prod_{j=1}^{n(p)} a(j, \Delta) = \lambda^{-n} (1 - e^{\lambda \Delta})^n e^{-\lambda n(n-1)\Delta/2}.
\]

The above derivations yield the necessary specifications for the characterization of the pertinent features of this model via Theorem 4.1. However, our derivations above importantly reveal a much more refined structure which has been associated with Bayesian mixture models for monotone hazards and densities. That is the works of Dykstra and Laud (1981), Lo and Weng (1989), Brunner and Lo (1989), Ho (2002) and Ho (2005).

5.1 A combinatorial reduction in terms of s-paths

Similar to Brunner and Lo (1989, p.1553), let \( m = (m_1, \ldots, m_n) \) denote a vector of non-negative integers taking values in \( \Xi := \{m \mid \sum_{i=1}^{j} m_i \geq j, 1 \leq j \leq n - 1, \sum_{i=1}^{n} m_i = n\} \). Then each \( m_i \) denotes the size of the cell whose minimal index is \( i \). We also define binary random variables \( \{\xi_1, \ldots, \xi_n\} \) where \( \xi_1 := 1 \) and in general \( \xi_i = 1 \) if \( i \) is the minimal index of a cell, otherwise it is 0. Viewed in terms of sequentially sampling the random variables \( Y \), the \( \xi_i = 1 \) if \( Y_i \) is distinct from the previous \( Y_1, \ldots, Y_{i-1} \) random variables. This idea is a generalization of the Bernoulli random variables associated with the Blackwell-MacQueen distribution as discussed in Korwar and Hollander (1973). Hence, from the derivations in the previous section, one has
\[
\prod_{j=1}^{n(p)} \varphi(C_j | w) = a_n \prod_{j=1}^{n(p)} \int_{\mathcal{Y}} r(y|i_j^*, e_j, w) \eta(dy) = a_n \prod_{i=1}^{n} \varphi(i, m_i | w)
\]
where \( \varphi(i, m_i) := \int_{\mathcal{Y}} r(y|i, m_i, w) \eta(dy) \) if \( m_i > 0 \) and otherwise, \( \varphi(i, 0) := 1 \). Note importantly that the size of the space \( \Xi \) is considerably smaller than the space of all partitions \( p \) of the integers \( \{1, \ldots, n\} \). Specifically a vector \( m \) contains information about the number of unique values or non-empty cells \( n(p) := \sum_{i=1}^{n} \xi_i \), the size of each cell \( m_i \), and the minimal index of each cell. However one does not know precisely the indices \( l \in C_j \) for each \( j = 1, \ldots, n(p) \). This leads to a more simplified version of the results in Theorem 4.1.
Proposition 5.2 Suppose that $N$ is a Poisson random measure with intensity $\nu(du,dy) = \rho(du)\eta(dy)$ on $\mathcal{Y} = (0,\infty) \times \mathcal{Y}$. Suppose that $\tau$ is defined by (14). Then the likelihood (9) can be expressed as

$$
\mathcal{L}(X|\mu,\beta,\theta) = a_n \int_{\mathbb{R}_+^n} \left[ \sum_{m \in \Xi} \prod_{i=1}^n \varphi(i,m_i|\mathbf{w}) \right] e^{-\Phi_0(r_1)} \prod_{i=1}^n e^{-\Phi_i(w_{i+1})} \phi(X_i|\mu \Delta + \beta w_i^{-1}, w_i^{-1}) dw_i
$$

where $\Phi_j$ for $j = 0, \ldots, n$ are given in Proposition 5.1. \(\square\)

Proposition 5.3 Suppose that $N$ is a Poisson random measure with intensity $\nu(du,dy) = \rho(du)\eta(dy)$ on $\mathcal{Y} = (0,\infty) \times \mathcal{Y}$. Suppose that $\tau$ is defined by (14). Let $\mathbf{M} = (M_1, \ldots, M_n)$ denote the random vector corresponding to the observations $\mathbf{m}$. Then, one has the following results.

(i) The posterior distribution of $(\mathbf{J}, \mathbf{Y})|\mathbf{M}, \mathbf{W}, \mathbf{X}$ consists of $n(p) = \sum_{i=1}^n \xi_i$ unique values $\{(\tilde{J}_i, \tilde{Y}_i) : \xi_i = 1, 1 \leq i \leq n\}$ which are conditionally independent with respective distributions,

$$
\mathbb{P}(\tilde{J}_i \in du, \tilde{Y}_i \in dy|\mathbf{m}, \mathbf{w}, \mathbf{X}) = \frac{n^{m_i} \rho_{\mathbf{m}}(du|y) \varphi(y|\mu \Delta + \beta w_i^{-1}, w_i^{-1})}{\kappa_{m_i}(\rho_{\mathbf{m}}|y) \varphi(i,m_i|\mathbf{w})}.
$$

(ii) The distribution of $\mathbf{M}|\mathbf{W}, \mathbf{X}$ is given by

$$
\mathbb{P}(M_1 = m_1, \ldots, M_n = m_n|\mathbf{w}, \mathbf{X}) \propto \prod_{i=1}^n \varphi(i,m_i|\mathbf{w}) \text{ for } \mathbf{m} \in \Xi.
$$

(iii) The density of $\mathbf{W}|\mathbf{M}, \mathbf{X}$ is

$$
f(\mathbf{w}|\mathbf{m}, \mathbf{X}) \propto e^{-\Phi_0(r_1)} \prod_{i=1}^n e^{-\Phi_i(w_{i+1})} \varphi(i,m_i|\mathbf{w}) \phi(X_i|\mu \Delta + \beta w_i^{-1}, w_i^{-1}).
$$

\(\square\)

Proof. The proof is straightforward. It essentially follows from a relabeling of the components in Theorem 4.1, combined with the form of the likelihood in Proposition 5.2. \(\square\)

We next describe the posterior distribution of the integrated OU process.

Proposition 5.4 Suppose that $\tau$ is defined by (14), then the posterior distribution of $\tau|\mathbf{M}, \mathbf{W}, \mathbf{X}$ is equivalent to the conditional distribution of the random measure,

$$
\tau_n(t) = \lambda^{-1}[(1 - e^{-\lambda t})v_{0,n} + \int_0^t (1 - e^{-\lambda(t-y)}) \mu_{\mathbf{m}}(dy) + \sum_{i=1}^n \xi_i \tilde{J}_i (1 - e^{-\lambda(t-\tilde{Y}_i)}) I_{[\tilde{Y}_i \leq t]}]
$$

where $v_{0,n} := \int_{-\infty}^0 e^y \mu_{\mathbf{m}}(dy) + \sum_{i=1}^n \xi_i \tilde{J}_i e^{\tilde{Y}_i} I_{[\tilde{Y}_i \leq 0]}$ has the posterior distribution of $v_0$. \(\square\)
Proof. This result follows from Theorem 4.1 using the fact that \( \sum_{j=1}^{n} J_{j,n} \delta_{Y_j}^{*} \overset{d}{=} \sum_{i=1}^{n} \xi_i \tilde{J}_i \delta_{\tilde{Y}_i} \). See Ho (2005) for a similar argument. \( \square \)

The structure \( m \in \Xi \) is in one to one correspondence to what are called s-paths by Brunner and Lo (1989). See that work, in particular Brunner and Lo (1989, Lemma 2.1, Theorem 2.1) for slightly different representations. We close by noting that we are rather surprised that the BNS-OU model used in our likelihood structure generates s-paths. As mentioned earlier, s-paths are known to be generated by representing monotone hazard rates as \( \int_{-\infty}^{\infty} I_{\{t<u\}} \mu(du) \) where \( \mu \) is a general completely random measure. This is the formulation recently investigated by Ho (2005) where the Gamma process results of Dykstra and Laud (1981), and Lo and Weng (1989) are special cases. Naturally, these are closely connected to the symmetric unimodal density Dirichlet mixture models considered by Brunner and Lo (1989). The BNS-OU models represent the first non-trivial mixture models outside the above mentioned class where inference can be based on sampling solely s-paths or equivalently \( m \), rather than \( p \). This is an important fact, since while indeed sampling partitions \( p \) is not difficult, these models are significantly less complex than models which can be minimally expressed in terms of partitions. That is, the space of partitions of the integers \( \{1, \ldots, n\} \) is known to contain Bell’s number of terms which is approximately \( n! \) and is considerably larger than \( \Xi \). Ho (2005), has devised efficient computational procedures for sampling s-path models which can be easily imported to the present setting. See also Ho (2002).

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