Convolution identities of poly-Cauchy numbers with level 2

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Abstract
Poly-Cauchy numbers with level 2 are defined by inverse sine hyperbolic functions with the inverse relation from sine hyperbolic functions. In this paper, we introduce the Stirling numbers of the first kind with level 2 in order to establish some relations with poly-Cauchy numbers with level 2. Then, we show several convolution identities of poly-Cauchy numbers with level 2. In particular, that of three poly-Cauchy numbers with level 2 can be expressed as a simple form.

Keywords: Poly-Cauchy numbers, hyperbolic functions, inverse hyperbolic functions, convolutions, Stirling numbers of the first kind

1 Introduction
Poly-Cauchy numbers (of the first kind) $c_n^{(k)}$ are defined as

$$\text{Lif}_k(\log(1 + t)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{t^n}{n!},$$

(1)

where $\text{Lif}_k(z)$ is the polylogarithm factorial or polyfactorial function, defined by

$$\text{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

([5, 6]). The concept of poly-Cauchy numbers with the polylogarithm factorial function is an analogue of that of poly-Bernoulli numbers with the polylogarithm function ([4]).
There are many papers on poly-xxx numbers and most of them are just generalizations for generalization’s sake, but this paper does not add another example. For, most generalizations or variations of so-called poly numbers or polynomials are just with level 1, but we consider poly numbers with level 2.

Poly-Cauchy numbers $C^{(k)}_n$ with level 2 [9] are defined by

$$L_{2,k}(\text{arcsinh} t) = \sum_{n=0}^{\infty} C^{(k)}_n \frac{t^n}{n!},$$

where $\text{arcsinh} t$ is the inverse hyperbolic sine function and

$$L_{2,k}(z) = \sum_{m=0}^{\infty} \frac{z^{2m}}{(2m)!((2m+1)k)},$$

which may be called the polylogarithm factorial function with level 2. Notice that poly-Cauchy numbers with level 2 are not simple generalizations of poly-Cauchy numbers or the original Cauchy numbers $c_n := c^{(1)}_n$ defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

because poly-Cauchy numbers with level 2 are based on hyperbolic functions but so-called poly-Cauchy (or the original Cauchy) numbers with level 1 are based upon logarithm functions. In this sense, most generalizations of poly-Cauchy numbers are still based upon the same logarithm functions, but our poly-Cauchy numbers are constructed by a different function.

The original Cauchy numbers, poly-Cauchy numbers and most of their generalizations are related with the Stirling numbers of the first kind. On the contrary, poly-Cauchy numbers with level 2 are related with Stirling numbers of the first kind with level 2, which are not simple generalizations but essentially different from the original Stirling numbers of the first kind.

In fact, $C^{(k)}_n$ has an expression in terms of $(2m+1)^k$ ($m = 1, 2, \ldots, n$) by using the Stirling numbers of the first kind with level 2. More properties of the Stirling numbers of the first kind with level 2 are discussed in [10] in an extensive way. See also [11] for the Stirling numbers of the second kind with level 2. Note that $C_n = 0$ for odd $n$.

**Theorem 1.** For integers $n$ and $k$ with $n \geq 1$,

$$C^{(k)}_{2n} = \sum_{m=1}^{n} \frac{(-4)^{n-m}}{(2m+1)^k} \left[ \begin{array}{c} n \\ m \end{array} \right],$$

where $\left[ \begin{array}{c} n \\ m \end{array} \right]$ is the Stirling number of the second kind.
where for \( m = 1, 2, \ldots, n \)

\[
\left\lbrack \frac{n}{m} \right\rbrack = \left\lbrack \frac{n}{m} \right\rbrack - 2 \left\lbrack \frac{n}{m-1} \right\rbrack + 2 \left\lbrack \frac{n}{m-2} \right\rbrack - \cdots + 2(-1)^{m-1} \left\lbrack \frac{n}{1} \right\rbrack + 2\left(\frac{n}{m-2} - 2 \left\lbrack \frac{n}{m-2} \right\rbrack + 1 \right) \] (3)

with \( \left\lbrack \frac{n}{0} \right\rbrack = 0 \).

Remark. Poly-Cauchy numbers can be expressed by using the Stirling numbers of the first kind:

\[
c_i^{(k)} = \sum_{m=0}^{n} \frac{(-1)^{n-m}}{(m+1)^k} \left\lbrack \frac{n}{m} \right\rbrack \] (5)

([5]), where \( \left\lbrack \frac{n}{m} \right\rbrack \) are the (unsigned) Stirling numbers of the first kind arising as coefficient of the rising factorial

\[
x(x+1)(x+2)\cdots(x+n-1) = \sum_{m=0}^{n} \left\lbrack \frac{n}{m} \right\rbrack x^m. \]

The Stirling numbers of the first kind \( \left\lbrack \frac{n}{m} \right\rbrack \) with level 2 arise as coefficient of the rising factorial

\[
x(x+1^2)(x+2^2)\cdots(x+(n-1)^2) = \sum_{m=0}^{n} \left\lbrack \frac{n}{m} \right\rbrack x^m \] (4)

(see, e.g., [12, p.213–217],[2]), So, they can also be written as

\[
\left\lbrack \frac{n}{m} \right\rbrack = \sum_{1 \leq i_1 < \cdots < i_{n-m} \leq n-1} (i_1 \cdots i_{n-m})^2. \]

Thus, the following relation holds:

\[
\left\lbrack \frac{n}{m} \right\rbrack = \left\lbrack \frac{n-1}{m-1} \right\rbrack + (n-1)^2 \left\lbrack \frac{n-1}{m} \right\rbrack \] (5)

\[1\text{There is a relation } \left\lbrack \frac{n}{m} \right\rbrack = (-1)^{n-m}t(2n,2m), \text{ where } t(n,m) \text{ are the central factorial numbers of the first kind, defined by } x(x + \frac{m}{2} - 1)(x + \frac{m}{2} - 2)\cdots(x - \frac{m}{2} + 1) = \sum_{m=0}^{n} t(n,m)x^m. \]
(Cf. \([3]^2\)). In this sense, the numbers \(^{m}n\) are suitable to be called the \textit{Stirling numbers of the first kind with level 2}, because the (unsigned) Stirling numbers of the first kind satisfy the recurrence relation

\[
^{m}n = \binom{n - 1}{m - 1} + (n - 1) \binom{n - 1}{m}.
\]

Notice that concerning the Stirling numbers of the first kind we see

\[
^{0}n = 0 \quad (n \geq 1), \quad ^{1}n = (n - 1)!, \quad ^{2}n = (n - 1)!H_{n-1},
\]

\[
^{n}n = 1, \quad ^{n}n - 1 = \binom{n}{2}, \quad ^{n}n - 2 = \frac{3n - 1}{4} \binom{n}{3}, \quad ^{n}n - 3 = \binom{n}{2} \binom{n}{4},
\]

\[
^{n}n - 4 = \frac{15n^3 - 30n^2 + 5n - 2}{48} \binom{n}{5}, \quad ^{n}n - 5 = \frac{3n^2 - 7n - 2}{8} \binom{n}{2} \binom{n}{6},
\]

\[
^{n}n - 6 = \frac{63n^5 - 315n^4 + 315n^3 + 91n^2 - 42n - 16}{576} \binom{n}{7}.
\]

In particular, for \(m = 1, 2, 3\)

\[
^{n}1 = ((n - 1)!)^2,
\]

\[
^{n}2 = ((n - 1)!)^2 H^{(2)}_{n - 1}[13, A001819],
\]

\[
^{n}3 = ((n - 1)!)^2 \frac{(H^{(2)}_{n - 1})^2 - H^{(4)}_{n - 1}}{2}[13, A001820],
\]

where

\[
H^{(k)}_n = \sum_{j=1}^{n} \frac{1}{j^k}
\]

is the generalized harmonic number of order \(k\). The numbers \(^{n}n\) can be found in [13, A001821]. Since \(^{n}k = 0\) for \(k > n > 0\),

\[
^{n}n = \binom{n}{n}^2 = 1,
\]

\[
^{n}n - 1 = \binom{n}{n - 1}^2 - 2 \binom{n}{n - 2} \binom{n}{n} = \frac{1}{2^n} \binom{2n}{3},
\]

\[
^{n}n - 2 = \binom{n}{n - 2}^2 - 2 \binom{n}{n - 3} \binom{n}{n - 1} + 2 \binom{n}{n - 4} \binom{n}{n}.
\]

\(^2\)In [3] \(u(n, m)\) are used as \(u(n, m) = (-1)^{n-m} \binom{n}{m}\).
\[
\begin{align*}
\left[ n - 3 \right] &= \left[ \frac{n}{n - 3} \right]^2 - 2 \left[ \frac{n}{n - 4} \right] \left[ \frac{n}{n - 2} \right] + 2 \left[ \frac{n}{n - 5} \right] \left[ \frac{n}{n - 1} \right] - 2 \left[ \frac{n}{n - 6} \right] \left[ \frac{n}{n} \right] \\
&= \frac{35n^2 + 21n + 4}{9 \cdot 2^4} \left( \frac{2n}{7} \right), \\
\left[ n - 4 \right] &= \frac{(5n + 2)(35n^2 + 28n + 9)}{15 \cdot 2^5} \left( \frac{2n}{9} \right), \\
\left[ n - 5 \right] &= \frac{385n^4 + 770n^3 + 671n^2 + 286n + 48}{9 \cdot 2^6} \left( \frac{2n}{11} \right).
\end{align*}
\]

Proof of Theorem 1. First, notice that the expression of \( \left[ \frac{n}{m} \right] \) in terms of \( \left[ \frac{n}{k} \right] \) can satisfy the same recurrence relation in (5). From (3), since

\[
\begin{align*}
\left[ \frac{n - 1}{m} \right] \left[ \frac{n - 1}{m - 1} \right] &- \left[ \frac{n - 1}{m - 1} \right] \left[ \frac{n - 1}{m} \right] + \left[ \frac{n - 1}{m - 2} \right] \left[ \frac{n - 1}{m + 1} \right] - \cdots \\
&- \left[ \frac{n - 1}{m + 1} \right] \left[ \frac{n - 1}{m - 2} \right] + \left[ \frac{n - 1}{m + 2} \right] \left[ \frac{n - 1}{m - 3} \right] - \cdots = 0,
\end{align*}
\]

we can also see that

\[
\begin{align*}
\left[ \frac{n}{m} \right] &= \left( \left[ \frac{n - 1}{m} \right] + \left[ \frac{n - 1}{m - 1} \right] \right)^2 \\
&\quad - 2 \left( \left[ \frac{n - 1}{m - 1} \right] + \left[ \frac{n - 1}{m - 2} \right] \right) \left( \left[ \frac{n - 1}{m - 1} \right] + \left[ \frac{n - 1}{m} \right] \right) \\
&\quad + 2 \left( \left[ \frac{n - 1}{m - 1} \right] + \left[ \frac{n - 1}{m - 2} \right] \right) \left( \left[ \frac{n - 1}{m - 2} \right] + \left[ \frac{n - 1}{m + 1} \right] \right) - \cdots \\
&= \left[ \frac{n - 1}{m - 1} \right] + \left[ \frac{n - 1}{m} \right]^2 \\
&\quad + 2(n - 1) \left( \left[ \frac{n - 1}{m} \right] \left[ \frac{n - 1}{m - 1} \right] - \left[ \frac{n - 1}{m - 1} \right] \left[ \frac{n - 1}{m} \right] \right) + \left[ \frac{n - 1}{m} \right] \left[ \frac{n - 1}{m - 1} \right] - \cdots \\
&\quad - \left[ \frac{n - 1}{m + 1} \right] \left[ \frac{n - 1}{m - 2} \right] + \left[ \frac{n - 1}{m + 2} \right] \left[ \frac{n - 1}{m - 3} \right] - \cdots \\
&= \left[ \frac{n - 1}{m - 1} \right] + (n - 1)^2 \left[ \frac{n - 1}{m} \right].
\end{align*}
\]

Now, because

\[
\frac{(\text{arcsinh}t)^{2m}}{(2m)!} = \sum_{n=m}^{\infty} (-1)^n \left[ \frac{n}{m} \right] \frac{t^{2n}}{(2n)!}
\]
(see, e.g., [2, (4.1.4)]), we have
\[
\sum_{n=0}^{\infty} \mathcal{C}_{2n} \frac{t^{2n}}{(2n)!} = \sum_{m=0}^{\infty} \frac{(\text{arcsinh})^{2m}}{(2m)!(2m+1)^k}
\]
\[
= \sum_{m=0}^{\infty} \frac{1}{(2m+1)^k} \sum_{n=m}^{\infty} (-4)^{n-m} \binom{n}{m} \frac{t^{2n}}{(2n)!}
\]
\[
= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{(-4)^{n-m}}{(2m+1)^k} \binom{n}{m} \frac{t^{2n}}{(2n)!}.
\]

Comparing the coefficients on both sides, we get the result. \qed

Poly-Cauchy numbers have an expression of integrals
\[
c_{n}^{(k)} = n! \int_{0}^{1} \cdots \int_{0}^{1} \left( \frac{x_1 x_2 \cdots x_k}{n} \right)^{2m} dx_1 dx_2 \cdots dx_k
\]
([5]). Poly-Cauchy numbers with level 2 also have a similar expression (or a kind of definition).

**Corollary 1.** For \( n \geq 0 \) and \( k \geq 1 \), we have
\[
\mathcal{C}_{2n}^{(k)} = (-4)^n (n!)^2 \int_{0}^{1} \cdots \int_{0}^{1} \left( \frac{x_1 x_2 \cdots x_k}{n} \right) \left( -\frac{x_1 x_2 \cdots x_k}{n} \right) dx_1 dx_2 \cdots dx_k.
\]

**Proof.** By Theorem 1 and the expression in (4),
\[
(-4)^n (n!)^2 \int_{0}^{1} \cdots \int_{0}^{1} \left( \frac{x_1 x_2 \cdots x_k}{n} \right) \left( -\frac{x_1 x_2 \cdots x_k}{n} \right) dx_1 dx_2 \cdots dx_k
\]
\[
= \int_{0}^{1} \cdots \int_{0}^{1} \sum_{m=0}^{n} (-4)^{n-m} \binom{n}{m} \frac{n}{(2m+1)^k} dx_1 dx_2 \cdots dx_k
\]
\[
= \sum_{m=1}^{n} \frac{(-4)^{n-m}}{(2m+1)^k} \binom{n}{m} = \mathcal{C}_{2n}^{(k)}.
\]

\( ^3 \)This proof is based upon the inverse relation between sin and arcsin with the orthogonal property of the central factorial numbers of both kinds.

6
2 Convolution

When \( k = 1 \), several initial values of \( C_n = C_n^{(1)} \) are as follows.

\[
\{ C_{2n} \}_{n \geq 0} = 1, \left\{ \begin{array}{c} \frac{1}{3} \\ -\frac{17}{15} \\ \frac{27859}{21} \\ \frac{1295803}{45} \\ \frac{5329242827}{33} \end{array} \right\} , \ldots .
\]

In [14], a convolution identity for Cauchy numbers is given as

\[
\sum_{k=0}^{n} \binom{n}{k} c_k c_{n-k} = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \geq 0).
\]

A more general case \( \sum_{k=0}^{n} \binom{n}{k} c_{k+l} c_{n-k+m} \) for some fixed nonnegative integers \( l \) and \( m \) is treated in [7]. In [8], the convolution identities for Cauchy numbers of the second kind \( \hat{c}_n \), defined by

\[
t(1 + t) \log(1 + t) = \sum_{n=0}^{\infty} \hat{c}_n t^n n!
\]

have been studied. In this section, we give the convolution identities for Cauchy numbers with level 2. For simplicity, we use the conventional convolution notation

\[
(\mathcal{C}_{2i_1} + \cdots + \mathcal{C}_{2i_k})^\circ := \sum_{i_1 + \cdots + i_k = n \atop i_1, \ldots, i_k \geq 0} \binom{2n}{2i_1, \ldots, 2i_k} \mathcal{C}_{2i_1 + 2i_2} \cdots \mathcal{C}_{2i_k + 2i_k},
\]

where

\[
\binom{2n}{2i_1, \ldots, 2i_k} = \frac{(2n)!}{(2i_1)! \cdots (2i_k)!}
\]

is the multinomial coefficient.

**Theorem 2.** For \( n \geq 0 \)

\[
(\mathcal{C}_0 + \mathcal{C}_0)^\circ = (2n)! \sum_{l=0}^{n} \frac{(-1)^{n-l}(2n-2l-3)!!(2l-1)}{2^{n-l}(n-l)!(2l)!} \mathcal{C}_{2l}.
\]

Here \((2i-1)!! = (2i-1)(2i-3)\cdots1 \quad (i \geq 1) \) with \((- (2i+1) )!! = \frac{(-1)^{i}}{(2i-1)!!} \quad (i \geq 1) \) and \((-1)!! = 1\).

**Proof.** For simplicity, put

\[
L(t) := \frac{t}{\text{arcsinh} t} = \sum_{n=0}^{\infty} \mathcal{C}_{2n} t^{2n} (2n)!
\]

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Since
\[ L'(t) = \frac{1}{\text{arcsinh} \ t} - \frac{t}{\sqrt{1 + t^2}} \]
\[ = \frac{1}{t} L(t) - \frac{1}{t\sqrt{1 + t^2}} L(t)^2, \]
we have
\[ L(t)^2 = -t\sqrt{1 + t^2} L'(t) + \sqrt{1 + t^2} L(t). \]  \hfill (6)

Because
\[ \sqrt{1 + t^2} = \sum_{j=0}^{\infty} \left( \frac{1}{2} \right)^j (t^2)^j \]
\[ = \sum_{j=0}^{\infty} \frac{(-1)^{j-1}(2j - 3)!!}{2^j \cdot j!} t^{2j} \]
and
\[ tL(t)' = \sum_{l=0}^{\infty} (2n)C_{2n} \frac{t^{2n}}{(2n)!}, \]
we have
\[ t\sqrt{1 + t^2} L(t)' = \left( \sum_{j=0}^{\infty} \frac{(-1)^{j-1}(2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=0}^{\infty} (2n)C_{2n} \frac{t^{2n}}{(2n)!} \right) \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(-1)^{n-l-1}(2n - 2l - 3)!!(2l)C_{2l}}{2^{n-l}(n-l)!} \right) t^{2n} \]
and
\[ t\sqrt{1 + t^2} L(t) = \left( \sum_{j=0}^{\infty} \frac{(-1)^{j-1}(2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=0}^{\infty} C_{2n} \frac{t^{2n}}{(2n)!} \right) \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(-1)^{n-l-1}(2n - 2l - 3)!! C_{2l}}{2^{n-l}(n-l)!} \right) t^{2n} \]
Comparing the coefficients with
\[ L(t)^2 = \sum_{n=0}^{\infty} (C_0 + C_0)^n \frac{t^{2n}}{(2n)!}, \]
we get the result. \qed
Since
\[ L(t) L''(t) = \left( \frac{1}{2(1+t^2)^{3/2}} - \frac{1}{6\sqrt{1+t^2}} \right) L(t) \]
\[ + \left( \frac{\sqrt{1+t^2}}{6t} + \frac{1}{2t(1+t^2)^{3/2}} - \frac{2}{3t\sqrt{1+t^2}} \right) L'(t) \]
\[ + \frac{1}{2} \left( \frac{1}{\sqrt{1+t^2}} - \sqrt{1+t^2} \right) L''(t) - \frac{t\sqrt{1+t^2}}{3} L^{(3)}(t), \quad (7) \]
together with
\[ \frac{1}{\sqrt{1+t^2}} = \sum_{j=0}^{\infty} (-1)^j (2j - 1)!! \frac{t^{2j}}{2^j \cdot j!} \]
and
\[ \frac{1}{(1+t^2)^{3/2}} = \sum_{j=0}^{\infty} (-1)^j (2j + 1)!! \frac{t^{2j}}{2^j \cdot j!} \]
we have
\[ \left( \frac{1}{2(1+t^2)^{3/2}} - \frac{1}{6\sqrt{1+t^2}} \right) L(t) \]
\[ = \left( \sum_{j=0}^{\infty} (-1)^j (3j + 1)(2j - 1)!! \frac{t^{2j}}{3 \cdot 2^j \cdot j!} \right) \]
\[ \times \left( \sum_{l=0}^{\infty} \mathcal{C}_{2l} \frac{t^{2l}}{(2l)!} \right) \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \left( -1 \right)^{n-l} (2n - 2l - 3)!! \frac{(n - l)(3n - 3l - 2)}{3 \cdot 2^{n-l} (n - l)! (2l)!} \mathcal{C}_{2l} \right) t^{2n}, \]
\[ \left( \frac{\sqrt{1+t^2}}{6t} + \frac{1}{2t(1+t^2)^{3/2}} - \frac{2}{3t\sqrt{1+t^2}} \right) L'(t) \]
\[ = \left( \sum_{j=0}^{\infty} (-1)^j (-1 + 3(2j + 1)(2j - 1) - 4(2j - 1)) (2j - 3)!! \frac{t^{2j}}{6 \cdot 2^j \cdot j!} \right) \]
\[ \times \left( \sum_{l=0}^{\infty} \mathcal{C}_{2l+2} \frac{t^{2l}}{(2l + 1)!} \right) \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \left( -1 \right)^{n-l} (2n - 2l - 3)!! \frac{(n - l)(3n - 3l - 2)}{3 \cdot 2^{n-l} (n - l)! (2l + 1)!} \mathcal{C}_{2l+2} \right) t^{2n}, \]
\[
\frac{1}{2} \left( \frac{1}{\sqrt{1+t^2}} - \sqrt{1+t^2} \right) L''(t) \\
= \left( \sum_{j=0}^{\infty} \frac{(-1)^j j(2j-3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=0}^{\infty} \mathcal{C}_{2l+2} \frac{t^{2l}}{(2l)!} \right) \\
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(-1)^{n-l}(2n-2l-3)!!(n-l)}{2^{n-l}(n-l)!(2l)!} \mathcal{C}_{2l+2} \right) t^{2n}
\]

and
\[
\frac{t \sqrt{1+t^2}}{3} L^{(3)}(t) \\
= \left( \sum_{j=0}^{\infty} \frac{(-1)^j j^{-1}(2j-3)!!}{3 \cdot 2^j \cdot j!} t^{2j} \right) \left( \sum_{l=0}^{\infty} (2l)! \mathcal{C}_{2l+2} \frac{t^{2l}}{(2l)!} \right) \\
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(-1)^{n-l-1}(2n-2l-3)!!(n-l)(2n) \mathcal{C}_{2l+2}}{3 \cdot 2^{n-l}(n-l)!(2l)!} \right) t^{2n}.
\]

Thus, the right-hand side of (7) is equal to
\[
\sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(-1)^{n-l}(2n-2l-1)!!(3n-3l+1)}{3 \cdot 2^{n-l}(n-l)!(2l)!} \mathcal{C}_{2l} \right) t^{2n} \\
+ \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{(-1)^{n-l}(6n^2 - 6nl + 4l^2 - n + 3l)(2n-2l-3)!!}{3 \cdot 2^{n-l}(n-l)!(2l+1)!} \mathcal{C}_{2l+2} t^{2n} \\
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(-1)^{n-l}(2n-2l-1)!!(3n-3l+1) \mathcal{C}_{2l}}{3 \cdot 2^{n-l}(n-l)!(2l)!} \right) t^{2n} \\
+ \sum_{n=0}^{\infty} \sum_{l=1}^{n+1} \frac{(-1)^{n-l-1}(6n^2 - 6nl + 4l^2 + 5n - 5l + 1)(2n-2l-1)!!}{3 \cdot 2^{n-l+1}(n-l+1)!(2l-1)!} \mathcal{C}_{2l} t^{2n} \\
= \sum_{n=0}^{\infty} \sum_{l=0}^{n+1} \frac{(-1)^{n-l-1}(2l-1)(3n^2 - 3nl + 2l^2 + 4n - 3l + 1)(2n-2l-1)!!}{3 \cdot 2^{n-l}(n-l+1)!(2l)!} \mathcal{C}_{2l} t^{2n}.
\]

Since the left-hand side of (7) is equal to
\[
\sum_{n=0}^{\infty} (\mathcal{C}_0 + \mathcal{C}_2)^n \frac{t^{2n}}{(2n)!},
\]
comparing the coefficients on both sides, we get the result of \((\mathcal{C}_0 + \mathcal{C}_2)^n\).
**Theorem 3.** For $n \geq 0$,

$$
(C_0 + C_2)^n = (2n)! \sum_{l=0}^{n+1} \frac{(-1)^{n-l}}{30} \frac{(2l - 1)(3n^2 - 3nl + 2l^2 + 4n - 3l + 1)(2n - 2l - 1)!!}{3 \cdot 2^{n-l}(n-l)!(2l)!} C_{2l}.
$$

Similarly, the convolution of $(C_2 + C_2)^n$ can be given as follows.

**Theorem 4.** For $n \geq 0$,

$$
(C_2 + C_2)^n = \frac{(2n)!}{30} \sum_{l=0}^{n} \frac{(-1)^{n-l}(10n - 8l + 5)(2n - 2l - 3)!!}{2^{n-l}(n-l)!(2l)!} C_{2l+4}
$$

$$
- \frac{(2n)!}{3} \sum_{l=0}^{n} \frac{(-1)^{n-l}(6l + 1)(2n - 2l + 1)!!}{2^{n-l}(n-l)!(2l)!} C_{2l+2}
$$

$$
- \frac{(2n)!}{30} \sum_{l=0}^{n} \frac{(-1)^{n-l}(160l^3 - 220l^2 + 72l - 1)(2n - 2l + 1)!!}{2^{n-l}(n-l)!(2l)!} C_{2l}.
$$

**Proof.** We know that

$$
(L''(t))^2 = -\frac{t\sqrt{1 + t^2}}{30} L^{(5)}(t) + \frac{1}{6} \left( \frac{1}{\sqrt{1 + t^2}} - 2\sqrt{1 + t^2} \right) L^{(4)}(t)
$$

$$
- \frac{3t + 2t^2}{3(1 + t^2)^{3/2}} L^{(3)}(t) - \frac{2 + t^2}{6(1 + t^2)^{3/2}} L''(t) - \frac{t}{30(1 + t^2)^{3/2}} L'(t)
$$

$$
+ \frac{1}{30(1 + t^2)^{3/2}} L(t).
$$

Since

$$
- \frac{t\sqrt{1 + t^2}}{30} L^{(5)}(t)
$$

$$
= -\frac{1}{30} \sum_{j=0}^{\infty} \frac{(-1)^{j-1}(2j - 3)!!}{2^j \cdot j!} \sum_{l=0}^{\infty} C_{2l} \frac{t^{2l-4}}{(2l - 5)!}
$$

$$
= \frac{1}{30} \sum_{n=0}^{\infty} \left( \frac{(-1)^{n-l}(2n - 2l - 3)!!(2l)}{2^{n-l}(n-l)!(2l)!} C_{2l+4} \right) t^{2l},
$$
\[
\frac{1}{6} \left( \frac{1}{\sqrt{1 + t^2}} - 2 \sqrt{1 + t^2} \right) L^{(4)}(t) = -\frac{1}{6} \sum_{j=0}^{\infty} \frac{(-1)^j (2j - 1)!!}{2^j \cdot j!} t^{2j} - \sum_{j=0}^{\infty} \frac{(-1)^j (2j - 3)!!}{2^j \cdot j!} \frac{t^{2j}}{\mathcal{C}_{2l}(2l - 1)!} \sum_{l=2}^{\infty} \mathcal{C}_{2l} t^{2l-4} \]

\[
= \frac{1}{6} \sum_{n=0}^{\infty} \left( \frac{(-1)^{n-l} (2n - 2l - 3)!! (2n - 2l + 1)}{2^{n-l} (n-l)! (2l)!} \mathcal{C}_{2l} \right) t^{2l},
\]

\[
- \frac{3t + 2t^2}{3(1 + t^2)^{3/2}} L^{(3)}(t) = -\sum_{j=0}^{\infty} \frac{(-1)^j (2j + 1)!!}{2^j \cdot j!} t^{2j} \sum_{l=2}^{\infty} \mathcal{C}_{2l} \frac{t^{2l-2}}{(2l - 3)!} - \frac{2}{3} \sum_{j=0}^{\infty} \frac{(-1)^j (2j + 1)!!}{2^j \cdot j!} t^{2j} \sum_{l=2}^{\infty} \mathcal{C}_{2l} \frac{t^{2l}}{(2l - 3)!}
\]

\[
= -\sum_{n=0}^{\infty} \left( \frac{(-1)^{n-l} (2n - 2l + 1)!!}{2^{n-l} (n-l)! (2l)!} \mathcal{C}_{2l+2} \right) t^{2l} - \frac{2}{3} \sum_{n=0}^{\infty} \left( \frac{(-1)^{n-l} (2n - 2l + 1)!! (2l) (2l - 1) (2l - 2)}{2^{n-l} (n-l)! (2l)!} \mathcal{C}_{2l} \right) t^{2l},
\]

\[
- \frac{2 + t^2}{6(1 + t^2)^{3/2}} L''(t)
\]

\[
= -\frac{2}{6} \sum_{j=0}^{\infty} \frac{(-1)^j (2j + 1)!!}{2^j \cdot j!} t^{2j} \sum_{l=1}^{\infty} \mathcal{C}_{2l} \frac{t^{2l-2}}{(2l - 2)!} - \frac{1}{6} \sum_{j=0}^{\infty} \frac{(-1)^j (2j + 1)!!}{2^j \cdot j!} t^{2j} \sum_{l=2}^{\infty} \mathcal{C}_{2l} \frac{t^{2l}}{(2l - 2)!}
\]

\[
= -\frac{1}{3} \sum_{n=0}^{\infty} \left( \frac{(-1)^{n-l} (2n - 2l + 1)!!}{2^{n-l} (n-l)! (2l)!} \mathcal{C}_{2l+2} \right) t^{2l} - \frac{1}{6} \sum_{n=0}^{\infty} \left( \frac{(-1)^{n-l} (2n - 2l + 1)!! (2l) (2l - 1)}{2^{n-l} (n-l)! (2l)!} \mathcal{C}_{2l} \right) t^{2l},
\]
and
\[
- \frac{t}{30(1 + t^2)^{3/2}} L'(t) + \frac{1}{30(1 + t^2)^{3/2}} L(t) = \frac{1}{30} \sum_{j=0}^{\infty} \frac{(-1)^j (2j + 1)!!}{2^j \cdot j!} \sum_{l=1}^{\infty} \frac{\mathcal{C}_{2l} t^{2l}}{(2l - 1)!}
\]
\[+ \frac{1}{30} \sum_{j=0}^{\infty} \frac{(-1)^j (2j + 1)!!}{2^j \cdot j!} \sum_{l=1}^{\infty} \frac{\mathcal{C}_{2l} t^{2l}}{(2l)!}
\]
\[= - \frac{1}{30} \sum_{n=0}^{\infty} \left( \frac{(-1)^{n-l} (2n - 2l + 1)! (2l - 1)! \mathcal{C}_{2l}}{2^{n-l} (n-l)! (2l)!} \right) t^{2l},
\]
we have
\[
(L''(t))^2 = \frac{1}{30} \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(-1)^{n-l} (10n - 8l + 5) (2n - 2l - 3)!! \mathcal{C}_{2l+4}}{2^{n-l} (n-l)! (2l)!} \right) t^{2l}
\]
\[- \frac{1}{3} \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(-1)^{n-l} (6l + 1) (2n - 2l + 1)!! \mathcal{C}_{2l+2}}{2^{n-l} (n-l)! (2l)!} \right) t^{2l}
\]
\[- \frac{1}{30} \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(-1)^{n-l} (160l^3 - 220l^2 + 72l - 1) (2n - 2l + 1)!! \mathcal{C}_{2l}}{2^{n-l} (n-l)! (2l)!} \right) t^{2l}.
\]

Since
\[
(L''(t))^2 = \sum_{n=0}^{\infty} (\mathcal{C}_2 + \mathcal{C}_2)^n \frac{t^{2n}}{(2n)!},
\]
comparing the coefficients on both sides, we get the desired result. \(\square\)

### 2.1 Higher-order convolutions

The convolution identity for three Cauchy numbers with level 2 can be given as follows.

**Theorem 5.** For \(n \geq 1\),
\[
(\mathcal{C}_0 + \mathcal{C}_0 + \mathcal{C}_0)^n = (2n - 1)(n-1)\mathcal{C}_{2n} + n(2n - 1)(2n - 3)\mathcal{C}_{2n-2}.
\]
Proof. From (6),

\[ L(t)L(t)' = \frac{t}{2\sqrt{1+t^2}}L(t) - \frac{t^2}{2\sqrt{1+t^2}}L(t)' - \frac{t\sqrt{1+t^2}}{2}L(t)''. \]

So,

\[ L(t)^3 = -\sqrt{1+t^2}L(t)L(t)' + \sqrt{1+t^2}L(t)^2 \]

\[ = -\sqrt{1+t^2}\left(\frac{t}{2\sqrt{1+t^2}}L(t) - \frac{t^2}{2\sqrt{1+t^2}}L(t)' - \frac{t\sqrt{1+t^2}}{2}L(t)''\right) \]

\[ + \sqrt{1+t^2}(-t\sqrt{1+t^2}L(t)' + \sqrt{1+t^2}L(t)) \]

\[ = \left(1 + \frac{t^2}{2}\right)L(t) - \left(t + \frac{t^3}{2}\right)L(t)' + \frac{t^2(1+t^2)}{2}L(t)'' \]

\[ = \sum_{n=0}^\infty \mathcal{C}_{2n} \frac{t^{2n}}{(2n)!} + \frac{1}{2} \sum_{n=0}^\infty (2n+2)(2n+1)\mathcal{C}_{2n} \frac{t^{2n+2}}{(2n+2)!} \]

\[ - \frac{1}{2} \sum_{n=0}^\infty (2n+2)(2n+1)(2n-1)\mathcal{C}_{2n+2} \frac{t^{2n+2}}{(2n+2)!} \]

\[ + \frac{1}{2} \sum_{n=0}^\infty (2n+2)(2n+1)(2n)(2n-1)\mathcal{C}_{2n} \frac{t^{2n}}{(2n-2)!} \]

\[ + \frac{1}{2} \sum_{n=0}^\infty (2n+2)(2n+1)(2n)(2n-1)\mathcal{C}_{2n} \frac{t^{2n+2}}{(2n+2)!} \]

\[ = \sum_{n=0}^\infty \left(1 - 2n + \frac{(2n)(2n-1)}{2}\right)\mathcal{C}_{2n} \frac{t^{2n}}{(2n)!} \]

\[ + \frac{1}{2} \sum_{n=1}^\infty (2n)(2n-1)(1 - (2n-2) + (2n-2)(2n-3))\mathcal{C}_{2n-2} \frac{t^{2n}}{(2n)!} \]

\[ = \sum_{n=0}^\infty (2n-1)(n-1)\mathcal{C}_{2n} \frac{t^{2n}}{(2n)!} + \sum_{n=1}^\infty n(2n-1)(2n-3)^2\mathcal{C}_{2n-2} \frac{t^{2n}}{(2n)!}. \]

Comparing the coefficients with

\[ L(t)^3 = \sum_{n=0}^\infty (\mathcal{C}_0 + \mathcal{C}_0 + \mathcal{C}_0)^n \frac{t^{2n}}{(2n)!}, \]

we get the result. \( \square \)
The convolution identity for four Cauchy numbers with level 2 can be given as follows.

**Theorem 6.** For \( n \geq 1 \),

\[
(C_0 + C_0 + C_0 + C_0)^n = \frac{(2n)!}{6} \sum_{l=0}^{n} \frac{(-1)^{n-l}(2n-2l-3)!!(2l)!}{2^{n-l}(n-l)!(2l)!} C_{2l}
\]

\[
+ \frac{(2n)!}{6} \sum_{l=1}^{n} \frac{(-1)^{n-l}(2n-2l-3)!!(2l)(2l-1)(2l-3)^3}{2^{n-l}(n-l)!(2l)!} C_{2l-2}.
\]

**Proof.** From the proof of Theorem 5 and Theorem 3, we get

\[
L(t)^4 = \left(1 + \frac{t^2}{2}\right) L(t) - \left(t + \frac{t^3}{2}\right) L(t)' + \frac{t^2(1 + t^2)}{2} L(t)''
\]

\[
= \frac{(6 + t^2)\sqrt{1+t^2}}{6} L(t) - \frac{t(6 + t^2)\sqrt{1+t^2}}{6} L'(t)
\]

\[
+ \frac{t^2\sqrt{1+t^2}}{6} L''(t) - \frac{t^3 + t^5}{6} \sqrt{1+t^2} L^{(3)}.
\]

Since

\[
\frac{(6 + t^2)\sqrt{1+t^2}}{6} L(t)
\]

\[
= \left( \sum_{j=0}^{\infty} \frac{(-1)^{j} (2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=0}^{\infty} \frac{C_{2l} t^{2l}}{(2l)!} \right)
\]

\[
+ \frac{1}{6} \left( \sum_{j=0}^{\infty} \frac{(-1)^{j} (2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=1}^{\infty} (2l)(2l-1) C_{2l-2} \frac{t^{2l}}{(2l)!} \right),
\]

\[
- \frac{t(6 + t^2)\sqrt{1+t^2}}{6} L'(t)
\]

\[
= - \left( \sum_{j=0}^{\infty} \frac{(-1)^{j} (2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=0}^{\infty} \frac{C_{2l} t^{2l}}{(2l)!} \right)
\]

\[
- \frac{1}{6} \left( \sum_{j=0}^{\infty} \frac{(-1)^{j} (2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=1}^{\infty} (2l)(2l-1) C_{2l-2} \frac{t^{2l}}{(2l)!} \right),
\]
\[
\frac{t^2 \sqrt{1 + t^2}}{2} L''(t)
\]
\[
= \frac{1}{2} \left( \sum_{j=0}^{\infty} \frac{(-1)^j (2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=0}^{\infty} \frac{(2l)!}{(2l - 1)!} C_{2l} \frac{t^{2l}}{(2l)!} \right)
\]
and
\[
- \frac{(t^3 + t^5) \sqrt{1 + t^2}}{6} L^{(3)}
\]
\[
= -\frac{1}{6} \left( \sum_{j=0}^{\infty} \frac{(-1)^j (2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=1}^{\infty} \frac{(2l)!}{(2l - 5)!} C_{2l-2} \frac{t^{2l}}{(2l)!} \right)
\]
\[
- \frac{1}{6} \left( \sum_{j=0}^{\infty} \frac{(-1)^j (2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=1}^{\infty} \frac{(2l)!}{(2l - 3)!} C_{2l-3} \frac{t^{2l}}{(2l)!} \right)
\]
we have
\[
L(t)^4 = \frac{1}{6} \left( \sum_{j=0}^{\infty} \frac{(-1)^j (2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=0}^{\infty} \frac{(2l - 1)(2l - 2)(2l - 3)C_{2l} t^{2l}}{(2l)!} \right)
\]
\[
+ \frac{1}{6} \left( \sum_{j=0}^{\infty} \frac{(-1)^j (2j - 3)!!}{2^j \cdot j!} t^{2j} \right) \left( \sum_{l=1}^{\infty} \frac{(2l)(2l - 1)(2l - 3)C_{2l-2} t^{2l}}{(2l)!} \right)
\]
\[
= \frac{1}{6} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{(-1)^n (-2n - 2l - 3)!!(2l - 1)(2l - 2)(2l - 3)C_{2l} t^{2n}}{2^{n-l}(n-l)!}(2l)!
\]
\[
+ \frac{1}{6} \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{(-1)^n (-2n - 2l - 3)!!(2l)(2l - 1)(2l - 3)^3C_{2l-3} t^{2n - 2l}}{2^{n-l}(n-l)!}(2l)!
\]
Comparing the coefficients with
\[
L(t)^4 = \sum_{n=0}^{\infty} (C_0 + C_0 + C_0 + C_0)^n \frac{t^{2n}}{(2n)!}
\]
we get the result. \(\square\)

As seen, convolution identities of the odd number of Cauchy numbers are simpler than those of the even number. Similarly, from
\[
L(x)^5 = \frac{x^4(1 + x^2)^2}{4!} L^{(4)}(x) + \frac{x^3(x^2 - 2)(1 + x^2)}{12} L^{(3)}(x)
\]
\[ + \frac{x^2(x^4 + 10x^2 + 12)}{4!} L''(x) - \frac{x(x^4 + 20x^2 + 24)}{4!} L'(x) + \frac{x^4 + 20x^2 + 24}{4!} L(x) , \]

we have for \( n \geq 2 \)

\[
(\mathcal{C}_0 + \mathcal{C}_0 + \mathcal{C}_0 + \mathcal{C}_0)^n = \left(\frac{2n-1}{4}\right) \mathcal{C}_{2n} + \frac{4n^2 - 16n + 17}{3} \left(\frac{2n}{2}\right) \left(\frac{2n-3}{2}\right) \mathcal{C}_{2n-2} + \left(\frac{2n}{4}\right) (2n - 5)^4 \mathcal{C}_{2n-4} .
\]

From

\[
L(x)^7 = \frac{x^6(1 + x^2)^3}{6!} L^{(6)}(x) + \frac{x^5(3x^2 - 2)(1 + x^2)^2}{2 \cdot 5!} L^{(5)}(x) + \frac{x^4(4x^4 + x^2 + 6)(1 + x^2)}{3! 4!} L^{(4)}(x) + \frac{x^3(2x^2 + 3)(x^4 - 4x^2 - 8)}{4! 3!} L^{(3)}(x) + \frac{x^2(x^6 + 91x^4 + 420x^2 + 360)}{6!} L''(x) + \frac{x(x^6 + 182x^4 + 840x^2 + 720)}{6!} L'(x) + \frac{x^6 + 182x^4 + 840x^2 + 720}{6!} L(x),
\]

we have for \( n \geq 3 \)

\[
(\mathcal{C}_0 + \cdots + \mathcal{C}_0)^n = \left(\frac{2n-1}{6}\right) \mathcal{C}_{2n} + \frac{12n^2 - 60n + 83}{15} \left(\frac{2n}{2}\right) \left(\frac{2n-3}{4}\right) \mathcal{C}_{2n-2} + \frac{(4n^2 - 24n + 39)(12n^2 - 72n + 109)}{15} \left(\frac{2n}{4}\right) \left(\frac{2n-5}{2}\right) \mathcal{C}_{2n-4} + \left(\frac{2n}{6}\right) (2n - 7)^6 \mathcal{C}_{2n-6} .
\]

Nevertheless, the higher-order cases seem to be more complicated when the number of Cauchy numbers increases. It is expected that for any integer
\[ r \geq 1 \]

\[
\left( c_0 + \cdots + c_0 \right)^n = \sum_{k=0}^{r} P_{r,2k}(n) \binom{2n}{2k} \binom{2n - 2k - 1}{2r - 2k} c_{2n-2k},
\]

where \( P_{r,2k}(n) \) are the polynomials of \( n \) with degree \( 2k \) (\( 0 \leq k \leq r \)). In particular, \( P_{r,0}(n) = 1 \) and \( P_{r,2r} = (2n - 2r - 1)^{2r} \).

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