Small ball probability and Dvoretzky Theorem

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Abstract

Large deviation estimates are by now a standard tool in the Asymptotic Convex Geometry, contrary to small deviation results. In this note we present a novel application of a small deviations inequality to a problem related to the diameters of random sections of high dimensional convex bodies. Our results imply an unexpected distinction between the lower and the upper inclusions in Dvoretzky Theorem.

1 Introduction

In probability theory, the large deviation theory (or the tail probabilities) and the small deviation theory (or the small ball probabilities) are in a sense two complementary directions. The large deviation theory, which is a more classical direction, seeks to control the probability of deviations of a random variable $X$ from its mean $M$, i.e. one looks for upper bounds on $\text{Prob}(|X - M| > t)$.

The small deviation theory seeks to control the probability of $X$ being very small, i.e. it looks for upper bounds on $\text{Prob}(|X| < t)$. There is a number of excellent texts on large deviations, see e.g. recent books [DZ] and [dH]. A recent exposition of the state of the art in the small deviation theory can be found in [LS].

A modern powerful approach to large deviations is via the celebrated concentration of measure phenomenon. One of the early manifestations of this idea was V. Milman’s proof of Dvoretzky Theorem in the 70s. Recall that Dvoretzky Theorem entails that any $n$-dimensional convex body has a section of dimension $c \log n$ which is approximately a Euclidean ball. Since Milman’s proof, the concentration of measure philosophy plays a major role in geometric functional analysis and in many other areas. A recent book of M. Ledoux [L] gives account on many ramifications of the method. A standard instance of the concentration of measure phenomenon is the case of a Lipschitz function on the unit Euclidean sphere $S^{n-1}$. In view of the geometric applications, we shall state it for a norm $\| \cdot \|$ on $\mathbb{R}^n$, or equivalently for its unit ball, which is a centrally symmetric convex body $K \subset \mathbb{R}^n$. Two parameters are responsible for many geometric properties of the convex body $K$ – the maximal and the average value of the norm on the sphere $S^{n-1}$, which is equipped with the probability
rotation invariant measure $\sigma$:

$$b = b(K) = \sup_{x \in S^{n-1}} \|x\|, \quad M = M(K) = \int_{S^{n-1}} \|x\| \, d\sigma(x).$$

(1)

The concentration of measure inequality, which appears e.g. in the first pages of [MS1] states that the norm is close to its mean $M$ on most of the sphere. For any $t > 1$,

$$\sigma \{ x \in S^{n-1} : \|x\| - M > tM \} < \exp \left( -ct^2k \right)$$

(2)

where

$$k = k(K) = n \left( \frac{M(K)}{b(K)} \right)^2.$$

Here and thereafter the letters $c, C, \tilde{c}, c_1, c_2$ etc. denote some positive universal constants, whose values may be different in various appearances. The symbol $\simeq$ denotes equivalence of two quantities up to an absolute constant factor, i.e. $a \simeq b$ if $ca \leq b \leq Ca$ for some absolute constants $c, C > 0$.

The concentration of measure inequality can of course be interpreted as a large deviation inequality for the random variable $\|x\|$, and the connection to probability theory becomes even more sound when one recalls an analogous inequality for Gaussian measures, see [L]. The quantity $k(K)$ plays a crucial role in high dimensional convex geometry, as it is the critical dimension in Dvoretzky Theorem. We will call this dimension $k(K)$ here the Dvoretzky dimension.

Milman’s proof of Dvoretzky Theorem [M1] (see also the book [MS1, 5.8]) provides accurate information regarding the dimension of the almost spherical sections of $K$. Milman’s argument shows that if $l < ck(K)$, then with probability larger than $1 - e^{-c l}$, a random $l$-dimensional subspace $E \in G_{n,l}$ satisfies

$$\frac{c}{M}(D^n \cap E) \subset K \cap E \subset \frac{c}{M}(D^n \cap E),$$

(3)

where $D^n$ denotes the unit Euclidean ball in $\mathbb{R}^n$ and the randomness is induced by the unique rotation invariant probability measure on the grassmannian $G_{n,l}$ of $l$-dimensional subspaces in $\mathbb{R}^n$.

The Dvoretzky dimension $k(K)$ was proved in [MS2] to be the exact critical dimension for a random section to satisfy (3), in the following strong sense. If a random $l$-dimensional subspace $E \in G_{n,l}$ satisfies (3) with probability larger than, say, $1 - \frac{1}{n}$, then necessarily $l < Ck(K)$. Thus a random section of dimension $l < ck(K)$ is close to Euclidean with high probability, and a random section of dimension $l > Ck(K)$ is typically far from Euclidean. These arguments completely clarify the question of the dimensions in which random sections of a given convex body are close to Euclidean. Once $b(K)$ and $M(K)$ are calculated, the behavior of a random section is known. For instance, Dvoretzky dimension of the cube is $O(\log n)$, while the cross polytope $K = \{ x \in \mathbb{R}^n : \sum |x_i| \leq 1 \}$ has Dvoretzky dimension as large as $O(n)$.

In this note we investigate Dvoretzky Theorem from a different direction, which does not involve the standard large deviations inequality [4]. The second named author conjectured that a phenomenon similar to the concentration
of measure should also occur for the small ball probability, and he proved a weaker statement. The conjecture has been recently proved by R. Latala and K. Oleszkiewicz, using the solution to the B-conjecture by Cordero, Fradelizi and Maurey [CFM]:

**Theorem 1.1 (Small ball probability).** For every $0 < \varepsilon < \frac{1}{2}$,

$$\sigma \left\{ x \in S^{n-1} : \|x\| < \varepsilon M \right\} < \varepsilon^{ck(K)}$$

where $c > 0$ is a universal constant.

This theorem is related to the small ball probability (as a direction of the probability theory) in exactly the same way as the concentration of measure is related to large deviations. Here we apply Theorem 1.1 to study questions arising from Dvoretzky Theorem. We show that for some purposes, it is possible to relax the Dvoretzky dimension $k(K)$, replacing it by a quantity independent of the Lipschitzness of the norm (which is quantified by the Lipschitz constant $b(K)$). Precisely, we wish to replace $k(K)$ by

$$d(K) = \min \{ -\log \sigma \left\{ x \in S^{n-1} : \|x\| \leq \frac{1}{2} M \right\} , n \}.$$ 

Selecting $t = \frac{1}{2}$ in the concentration of measure inequality (2), we conclude that $d(K)$ must be at least of the same order as Dvoretzky dimension $k(K)$:

$$d(K) \geq C k(K).$$

The small ball Theorem 1.1 indeed holds with $d(K)$ (this is a part of the argument of Latala and Oleszkiewicz, reproduced below). The resulting inequality can be viewed as Kahane-Khinchine type inequality for negative exponents:

**Proposition 1.2 (Negative moments of a norm).** Assume that $0 < l < cd(K)$. Then

$$c M < \left( \int_{S^{n-1}} \|x\|^{-l} d\sigma(x) \right)^{-\frac{1}{l}} < CM$$

where $c, C > 0$ are universal constants.

For positive exponents, this inequality was proved in [LMS]: for $0 < k < ck(K)$,

$$c M < \left( \int_{S^{n-1}} \|x\|^k d\sigma(x) \right)^{\frac{1}{k}} < CM. \quad (4)$$

For negative exponents $-1 < k < 0$, inequality (4) follows from results of Guedon [G] that generalize Lovasz-Simonovits inequality [LoSi]. Proposition 1.2 extends (4) to the range $[-cd(K), ck(K)]$ (which of course includes the range $[-ck(K), ck(K)]$).

In Proposition 1.2, $\|x\|^{-1}$ can be regarded as the radius of the one-dimensional section of the body $K$. Combining this with the recent inequality for diameters
of sections due to the first named author $K$, we are able to lift the dimension of the section and thus compute the average diameter of $l$-dimensional sections of any symmetric convex body $K$. This formal “dimension lift” might be of an independent interest.

**Theorem 1.3 (Diameters of random sections).** Assume that $0 < l < cd(K)$. Select a random $l$-dimensional subspace $E \in G_{n,l}$. Then with probability larger than $1 - e^{-c' l}$,

$$K \cap E \subset \frac{c}{M} (D^n \cap E).$$

Furthermore,

$$\frac{\bar{c}}{M} < \left( \int_{G_{n,l}} \text{diam}(K \cap E)^l \text{d}\mu(E) \right)^{\frac{1}{l}} < \frac{\bar{C}}{M}$$

where $c, c', \bar{c}, C, \bar{C} > 0$ are universal constants.

The relation between Theorem 1.3 and Dvoretzky Theorem is clear. We show that for dimensions which may be much larger than $k(K)$, the upper inclusion in Dvoretzky Theorem holds with high probability. This reveals an intriguing point in Dvoretzky Theorem. Milman’s proof of Dvoretzky Theorem focuses on the left-most inclusion in (3). Once it is proved that the left-most inclusion in (3) holds with high probability, the right-most inclusion follows almost automatically in his proof.

Furthermore, Milman-Schechtman’s argument [MS2] implies in fact that the left-most inclusion does not hold (with large probability) for dimensions larger than the Dvoretzky dimension. The reason that a random $l$-dimensional section is far from Euclidean when $l > ck(K)$ is that a typical section does not contain a sufficiently large Euclidean ball. In comparison to that, we observe that the upper inclusion in (3) is true in a much wider range of the dimensions.

There are cases, such as the case of the cube, where the Dvoretzky dimension $k(K)$ is $O(\log n)$, while $d(K)$ is a polynomial in $n$. Hence, while sections of the cube of dimension $n^c$ are already contained in the appropriate Euclidean ball (for any fixed $c < 1$, independent of $n$), only when the dimension is $O(\log n)$ the sections start to “fill from inside”, and an isomorphic Euclidean ball is observed. The fact that $d(K)$ is typically larger than $k(K)$ is a bit unexpected. It implies that the correct upper bound for random sections of a convex body appears sometimes in much larger dimensions than those in which we have the lower bound.

In the last decade, diameters of random lower-dimensional sections of convex bodies attracted a considerable amount of attention, see in particular [GM1, GM2, GMT]. Theorem 1.3 is a significant addition to this line of results. It implies that diameters of random sections are equivalent for a wide range of dimensions – starting from dimension one, when the random diameter simply equals $\frac{1}{M(k)}$, and up to the critical dimension $d(K)$. 
Remark. The proof shows that \( d(K) \) can be further relaxed in all our results. For any fixed \( u > 1 \) it can be replaced by

\[
d_u(K) = \min \{- \log \sigma \{ x \in S^{n-1} : \|x\| \leq \frac{1}{u} M \}, n \}.
\]

The rest of the paper is organized as follows. In Section 2 we discuss the negative moments of the norm, proving Proposition 1.2 and Theorem 1.1 by the Latala-Oleszkiewicz’s argument. In Section 3 we do the lift of dimension and compute the average diameters of random sections, proving Theorem 1.3.

2 Concentration of measure and the small ball probability

We start by proving Proposition 1.2. It is a reformulation of the “small ball probability conjecture” due to the second named author. It was recently deduced by R. Latala and K. Oleszkiewicz from the B-conjecture proved by Cordero, Fradelizi and Maurey [CPM]. We will reproduce Latala-Oleszkiewicz argument here. We start with a standard and well-known lemma, on the close relation between the uniform measure \( \sigma \) on the sphere \( S^{n-1} \) and the standard gaussian measure \( \gamma \) on \( \mathbb{R}^n \). For the convenience of the reader, we include its proof.

Lemma 2.1. For every centrally-symmetric convex body \( K \),

\[
\frac{1}{2} \sigma(S^{n-1} \cap \frac{1}{2} K) \leq \gamma(\sqrt{n} K) \leq \sigma(S^{n-1} \cap 2K) + e^{-cn}
\]

where \( c > 0 \) is a universal constant.

Proof. We will use the two estimates on the Gaussian measure of the Euclidean ball,

\[
\gamma(2\sqrt{n}D^n) > \frac{1}{2}, \quad \gamma(\frac{1}{2}\sqrt{n}D^n) < e^{-cn}.
\]

The first estimate is just Chebychev’s inequality, and the second follows from standard large deviation inequalities, e.g. Cramer’s Theorem [V]. Since \( K \) is star-shaped,

\[
\gamma(\sqrt{n} K) \geq \gamma(2\sqrt{n}D^n \cap \sqrt{n} K) \geq \gamma(2\sqrt{n}D^n) \sigma_1(2\sqrt{n}S^{n-1} \cap \sqrt{n} K)
\]

where \( \sigma_1 \) denotes the probability rotation invariant measure on the sphere \( 2\sqrt{n}S^{n-1} \)

\[
\geq \frac{1}{2} \sigma(S^{n-1} \cap \frac{1}{2} K).
\]

This proves the lower estimate in the Lemma.
For the upper estimate, note that no points of \( \sqrt{n}K \) can lie outside both the ball \( \frac{1}{2}\sqrt{n}D^n \) and the positive cone generated by \( \frac{1}{2}\sqrt{n}S^{n-1} \cap \sqrt{n}K \). Adding the two measures together, we obtain
\[
\gamma(\sqrt{n}K) \leq \gamma(\frac{1}{2}\sqrt{n}D^n) + \sigma_2(\frac{1}{2}\sqrt{n}S^{n-1} \cap \sqrt{n}K)
\]
where \( \sigma_2 \) denotes the probability rotation invariant measure on the sphere \( \frac{1}{2}\sqrt{n}S^{n-1} \)
\[
\leq e^{-cn} + \sigma(S^{n-1} \cap 2K).
\]
This completes the proof.

**Proof of Proposition 1.2.** As usual, \( K \) will denote the unit ball of the norm \( \| \cdot \| \). The B-conjecture, proved in [CFM], asserts that the function \( t \mapsto \gamma(e^tK) \) is log-concave. This means that for any \( a, b > 0 \) and \( 0 < \lambda < 1 \),
\[
\gamma(a^\lambda b^{1-\lambda}K) \geq \gamma(aK)^\lambda \gamma(bK)^{1-\lambda}.
\]
(6)

Let \( \text{Med} = \text{Med}(K) \) be the median of the norm \( \| \cdot \| \) on the unit sphere \( S^{n-1} \).

By Chebychev’s inequality, \( \text{Med} \leq 2\text{M}(K) \). Set \( L = \text{Med} \cdot \sqrt{n}K \). According to Lemma 2.1,
\[
\gamma(2L) \geq \frac{1}{2} \sigma(S^{n-1} \cap \text{Med} \cdot K) \geq \frac{1}{4}
\]
by the definition of the median. On the other hand, again by Lemma 2.1
\[
\gamma(\frac{1}{8}L) \leq \sigma(S^{n-1} \cap \frac{1}{8}\text{Med} \cdot K) + e^{-cn}
\]
\[
= \sigma(x \in S^{n-1} : \|x\| \leq \frac{1}{8}\text{Med}) + e^{-cn}
\]
\[
\leq \sigma(x \in S^{n-1} : \|x\| \leq \frac{1}{2}\text{M}(K)) + e^{-cn} \leq e^{-d(K)} + e^{-c'n} < 2e^{-Cd(K)}
\]
(8)
because \( d(K) \leq n \). We may assume that \( \varepsilon < e^{-3} \), and apply (6) for \( a = \varepsilon, b = 2, \lambda = \frac{3}{\log \frac{1}{\varepsilon}} \). This yields
\[
\gamma(\varepsilon L)^{\frac{3}{\log(1/\varepsilon)}} \gamma(2L)^{1-\frac{3}{\log(1/\varepsilon)}} \leq \gamma\left(\varepsilon^{\frac{3}{\log(1/\varepsilon)}} 2^{1-\frac{3}{\log(1/\varepsilon)}} L\right) \leq \gamma(\frac{1}{8}L).
\]
Combining this with (7) and (8), we obtain that
\[
\gamma(\varepsilon L) \leq 8e^{Cd(K) \log \varepsilon} \leq 8e^{cd(K)} < (c'\varepsilon)^{cd(K)}
\]
and according to Lemma 2.1 we can transfer this to the spherical measure, obtaining
\[
\sigma(x \in S^{n-1} : \|x\| < \varepsilon M) < (C\varepsilon)^{cd(K)}.
\]
By integration by parts, this yields
\[
\int_{S^{n-1}} \|x/M\|^{-cd(K)/10} d\sigma(x) \leq C,
\]
which implies the left hand side of the inequality in Proposition 1.2. The right hand side follows easily by Hölder’s inequality.
By Chebychev’s inequality, Proposition 1.2 yields the desired tail inequality for the small ball probability:

**Corollary 2.2 (The small ball probability).** For every $0 < \varepsilon < \frac{1}{2}$,

$$\sigma \left\{ x \in S^{n-1} : \|x\| < \varepsilon M \right\} < \varepsilon^{c\ell(K)} < \varepsilon^{c'\ell(K)}.$$

where $c, c' > 0$ are universal constants.

Theorem 1.1 is contained in Corollary 2.2. Let us give some interpretation of the expression in Proposition 1.2. For a subspace $E \subset \mathbb{R}^n$, let $S(E) = S^{n-1} \cap E$ and $\sigma_E$ be the unique rotation invariant probability measure on the sphere $S(E)$. We will use the fact that $\text{Vol}(K) = \text{Vol}(D^n) \int_{S^{n-1}} \|x\|^{-n} d\sigma(x)$. The *volume radius* of a $k$-dimensional set $T$ is defined as

$$v.\text{rad.}(T) = \left( \frac{\text{Vol}(T)}{\text{Vol}(D^k)} \right)^{\frac{1}{k}}.$$

Thus

$$v.\text{rad.}(K) = \left( \int_{S^{n-1}} \|x\|^{-n} d\sigma(x) \right)^{1/n}.$$

By the rotation invariance of all the measures (as in [K]), we conclude that

$$\int_{S^{n-1}} \|x\|^{-k} d\sigma(x) = \int_{G_{n,k}} \int_{S(E)} \|x\|^{-k} d\sigma_E(x) d\mu(E)$$

$$= \int_{G_{n,k}} v.\text{rad.}(K \cap E)^k d\mu(E). \quad (9)$$

Thus Proposition 1.2 asymptotically computes the average volume radius of random sections. This fits perfectly to the estimates for diameters of sections in [K], to be applied next.

### 3 Diameters of random sections

In this section we prove the main result of the paper, Theorem 1.3. We regard $\|x\|^{-1}$ as the radius of the one-dimensional section spanned by $x$; thus Proposition 3.1 is an asymptotically sharp bound on the diameters of random one-dimensional sections. Theorem 1.3 extends this bound to $k$-dimensional sections, for all $k$ up to the critical dimension $d(K)$. We start with a “lift of dimension”, which is a variation of the “low $M$ estimate”, Proposition 3.9 of [K] (the case $\lambda = \frac{1}{2}$ there). The difference to that Proposition, is that here we estimate the $L_k$ norm, rather than just the tail probability as in [K].

**Proposition 3.1 (Dimension lift for diameters of sections).** Let $1 \leq k < k_0 < n$. Then for any integer $k < k_0/4$,

$$\left( \int_{G_{n,k}} \text{diam}(K \cap E)^k d\mu(E) \right)^{\frac{1}{k}} \leq CM(K) \left( \int_{S^{n-1}} \|x\|^{-k_0} d\sigma(x) \right)^{\frac{2}{k_0}}.$$
Remarks.

1. Theorem 1.3 follows immediately from Proposition 1.2 and Proposition 3.1. Indeed, the right hand side in Proposition 3.1 is bounded by

$$CM(K) \left( \frac{C}{M(K)} \right)^2 \leq \frac{C}{M(K)}.$$ 

2. Note the special normalization in Proposition 3.1 (compare with Proposition 3.9 in [K]). Actually, as follows from the proof, for any $\lambda > 0$, the right hand side in Proposition 3.1 may be replaced with

$$C(\lambda)M(K)^\lambda \left( \int_{S^{n-1}} \|x\|^{-k_0} d\sigma(x) \right)^{\frac{1+\lambda}{k_0}}.$$ 

in the price of increasing $k_0$ (just replace in the proof Cauchy-Schwartz with the appropriate Hölder inequality). However, we cannot have $\lambda = 0$, as is demonstrated by the example of $K = \mathbb{R}^{n-1}$. The average diameter of one-dimensional sections of $K$ is zero, while the diameter of any section of dimension larger than one is infinity. Thus there can not be any formal dimension lift unless one has an extra factor which must be infinity for “flat” bodies. Such a factor is $M(K) = \int_{S^{n-1}} \|x\| d\sigma(x)$.

In order to prove Theorem 1.2 we need a standard lemma that claims that the average norm $M$ is stable under the operation of passing to a random subspace. We are unaware of a reference for the exact statement we need (a similar result appears e.g. in Lemma 6.6 of [M2]), so a proof is provided. The average norm on a subspace $E \in G_{n,k}$ is denoted by $M_E = \int_{S(E)} \|x\| d\sigma_E(x)$.

**Lemma 3.2.** For every norm $\| \cdot \|$ on $\mathbb{R}^n$ and every integer $0 < k < n$,

$$cM < \left( \int_{G_{n,k}} (M_E)^{2k} d\mu(E) \right)^{\frac{1}{k}} < CM$$

where $c, C > 0$ are universal constants.

**Proof.** The first inequality in (10) follows easily from Hölder’s inequality. In the proof of the second inequality, we will use a variant of Raz’s argument (see [R, MW]). We normalize so that $M = 1$. Let $X_1, ..., X_k$ be $k$ independent random vectors, distributed uniformly on $S^{n-1}$. It is well known that a norm of a random vector on the sphere has a subgaussian tail (e.g. [LMS 3.1]):

$$\mathbb{E} \exp (s \|X_i\|) < \exp (cs^2)$$

for all $i$ and all $s > 1$

which by independence implies

$$\mathbb{E} \exp \left( s \cdot \frac{1}{k} \sum_{i=1}^k \|X_i\| \right) < \exp \left( \frac{Cs^2}{k} \right)$$

for $s > 1$. 

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Using Chebychev’s inequality and optimizing over $s$ (e.g. [MS1, 7.4]), we get

$$
\Pr\left\{ \frac{1}{k} \sum_{i=1}^{k} \|X_i\| > Ct \right\} < \exp\left(-t^2k\right) \quad \text{for } t > 1.
$$

(11)

Let $E$ be the linear span of $X_1, \ldots, X_k$. Then $E$ is distributed uniformly in $G_{n,k}$ (up to an event of measure zero). Since for any two events one has $\Pr(A) \leq \frac{\Pr(B)}{\Pr(B|A)}$, we conclude that

$$
\Pr\{M_E > 2ct\} \leq \frac{\Pr\left\{ \frac{1}{k} \sum_{i=1}^{k} \|X_i\| > ct \right\}}{\Pr\left\{ \frac{1}{k} \sum_{i=1}^{k} \|X_i\| > ct \mid M_E > 2ct \right\}}.
$$

(12)

The enumerator in (12) is bounded by (11). To bound below the denominator note that $\|X_i\| < C\sqrt{k}M_E$ pointwise for all $i$. This is a consequence of a simple comparison inequality for the Gaussian analogs of $M$, $M_E$ (see e.g. [MS1, 5.9]). Note that each $X_i$ is distributed uniformly in $S(E)$. For a fixed $E \in G_{n,k}$, we can estimate $P_E = \left\{ \frac{1}{k} \sum_{i=1}^{k} \|X_i\| > \frac{M_E}{2} \mid \text{span}\{X_1, \ldots, X_k\} = E \right\}$ via Chebychev’s inequality as

$$
M_E = E\left( \frac{1}{k} \sum_{i=1}^{k} \|X_i\| \mid \text{span}\{X_1, \ldots, X_k\} = E \right) \leq C\sqrt{k}M_EP_E + \frac{M_E}{2}(1 - P_E).
$$

Hence $P_E \geq \frac{\bar{c}}{\sqrt{k}}$ for every $E \in G_{n,k}$. Thus

$$
\text{denominator in (12)} \geq \Pr\left\{ \frac{1}{k} \sum_{i=1}^{k} \|X_i\| > \frac{M_E}{2} \mid M_E > 2ct \right\}
$$

$$
\geq \min_{E \in G_{n,k}} P_E \geq \frac{\bar{c}}{\sqrt{k}}.
$$

Combining with (11) and (12) we get

$$
\Pr\{M_E > 2ct\} < \frac{c}{\sqrt{k}}e^{-t^2k} < e^{-Ct^2k} \quad \text{for } t > 1.
$$

Integrating by parts we get the desired estimate. ■

Proof of Proposition 3.1. By Hölder inequality, the right hand side increases with $k_0$, hence we may assume that $k_0 = 4k$. We shall rely on the main result of [K], which claims that for any centrally-symmetric convex body $T \subset \mathbb{R}^n$, and every integers $0 < k \leq l < n,$

$$
v.\text{rad.}(T) > C \left( \int_{G_{n,k}} v.\text{rad.}(T \cap E)^l \text{diam}(T \cap E)^{n-l} \text{d}\mu(E) \right)^{1/n}.
$$

(13)
We are going to apply (13) to $T = K \cap E$, for subspaces $E \in G_{n,k_0}$. Denote by $G_{E,k}$ the Grassmanian of all $k$-dimensional subspaces of $E$, equipped with the unique probability rotational invariant measure. Then by (13), (9) and the rotational invariance of all measures,

$$
\int_{E \in G_{n,k}} \text{v.rad.}(K \cap E)^{2k} \text{diam}(K \cap E)^{2k} \text{d}\mu(E)
$$

$$
= \int_{E \in G_{n,k}} \int_{F \in G_{E,k}} \text{v.rad.}(K \cap F)^{2k} \text{diam}(K \cap F)^{2k} \text{d}\mu_{E}(F) \text{d}\mu(E)
$$

$$
\leq C^{k_0} \int_{E \in G_{n,k_0}} \text{v.rad.}(K \cap E)^{k_0} \text{d}\mu(E) = C^{k_0} \int_{S^{n-1}} \|x\|^{-k_0} \text{d}\sigma(x).
$$

Also, by Cauchy-Schwartz inequality,

$$
\left( \int_{E \in G_{n,k}} \text{diam}(K \cap E)^{k} \text{d}\mu(E) \right)^{\frac{1}{k}}
$$

$$
\leq \left( \int_{E \in G_{n,k}} \text{v.rad.}(K \cap E)^{2k} \text{diam}(K \cap E)^{2k} \text{d}\mu(E) \right)^{\frac{1}{2k}} \left( \int_{E \in G_{n,k}} \frac{1}{\text{v.rad.}(K \cap E)^{2k}} \text{d}\mu(E) \right)^{\frac{1}{2k}}.
$$

We will use the standard inequality $\frac{1}{\text{v.rad.}(K \cap E)^{2k}} \leq M_{E}$, which follows directly from Hölder inequality (e.g. [MS1]). Then,

$$
\left( \int_{E \in G_{n,k}} \text{diam}(K \cap E)^{k} \text{d}\mu(E) \right)^{\frac{1}{k}}
$$

$$
\leq C \left( \int_{S^{n-1}} \|x\|^{-k_0} \text{d}\sigma(x) \right)^{\frac{1}{k_0}} \left( \int_{E \in G_{n,k}} (M_{E})^{2k} \text{d}\mu(E) \right)^{\frac{1}{2k}}
$$

and the proposition follows by Lemma [19].

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