NOTE ON THE SUM OF THE SMALLEST AND LARGEST EIGENVALUES OF A TRIANGLE-FREE GRAPH

PÉTER CSIKVÁRI

Abstract. Let $G$ be a triangle-free graph on $n$ vertices with adjacency matrix eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$. In this paper we study the quantity $\mu_1(G) + \mu_n(G)$.

We prove that for any triangle-free graph $G$ we have

$$\mu_1(G) + \mu_n(G) \leq (3 - 2\sqrt{2})n.$$ 

This was proved for regular graphs by Brandt, we show that the condition on regularity is not necessary. We also prove that among triangle-free strongly regular graphs the Higman-Sims graph achieves the maximum of

$$\frac{\mu_1(G) + \mu_n(G)}{n}.$$ 

1. Introduction

In this paper every graph is simple. Motivated by the papers [2] and [4] we study the following problem. Let $\mathcal{G}_3$ be the family of triangle-free graphs, and for a graph $G$ on $v(G) = n$ vertices let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ be the eigenvalues of the adjacency matrix of $G$. The problem is to determine $c_3 = \sup_{G \in \mathcal{G}_3} \frac{\mu_1(G) + \mu_n(G)}{v(G)}$.

Brandt [2] proved that for regular triangle-free graphs we have

$$\mu_1(G) + \mu_n(G) \leq (3 - 2\sqrt{2})n.$$ 

Very recently Balogh, Clemen, Lidický, Norin and Volec proved that for regular triangle-free graphs we have $\mu_1(G) + \mu_n(G) \leq \frac{15}{94}n < 0.1596n$ and they mention in their paper that a similar but larger computation also gives the result $0.15467$ instead of $0.1596$. In fact, they study the smallest eigenvalue $q_n(G)$ of the so-called signless laplacian matrix $L = D + A$, where $D$ is the diagonal matrix consisting of the degrees of the vertices and $A$ is the adjacency matrix of the graph $G$. The quantity $q_n(G)$ coincides with $\mu_1(G) + \mu_n(G)$ if $G$ is regular. Balogh, Clemen, Lidický, Norin and Volec mentions that in case of regular graphs they can further improve their result to prove $q_n(G) \leq 0.15442n$. Our first result is to prove that in Brandt’s theorem one can drop the condition of regularity.

Theorem 1.1. Let $G$ be a triangle-free graph on $n$ vertices, and let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ be the eigenvalues of its adjacency matrix. Then

$$\mu_1(G) + \mu_n(G) \leq (3 - 2\sqrt{2})n.$$ 

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The proof of Theorem 1.1 heavily relies on the following lemma which might be of independent interest.

**Lemma 1.2.** Let $G$ be a triangle-free graph on $n$ vertices, and let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ be the eigenvalues of its adjacency matrix. Then

$$\mu_1(G) \leq \frac{-n\mu_n(G)}{\mu_1(G) - \mu_n(G)}.$$ 

Brandt [2] also realized that for the so-called Higman-Sims graph $H$ we have

$$\frac{\mu_1(H) + \mu_n(H)}{v(H)} = \frac{22 + (-8)}{100} = 0.14$$

which gives a rather good lower bound for $c_3$. Higman-Sims graph is the unique strongly regular graph with parameters $(100, 22, 0, 6)$. Recall that a graph $G$ is a strongly regular graph with parameters $(n, k, a, b)$ if it has $n$ vertices, $k$-regular, any two adjacent vertices have exactly $a$ common neighbors, and any two non-adjacent vertices have exactly $b$ common neighbors. In this paper we show that among the strongly regular graphs, it is indeed the Higman-Sims graph which maximizes the quantity

$$\frac{\mu_1(G) + \mu_n(G)}{v(G)}.$$ 

Note that only finitely many triangle-free strongly regular graphs are known currently, but we do not rely on this fact.

**Theorem 1.3.** Let $G$ be a triangle-free strongly regular graph on $n$ vertices. Then

$$\mu_1(G) + \mu_n(G) \leq 0.14n$$

with equality if and only if $G$ is the Higman-Sims graph.

### 2. Proof of Theorem 1.1

We begin with proving Lemma 1.2. Before we actually start it let us mention that for regular graphs this lemma is a simple consequence of the Hoffman-Delsarte bound for independent sets. Indeed, let $\alpha(G)$ denote the size of the largest independent set of a $d$-regular graph. Then by the Hoffman-Delsarte bound we have

$$\alpha(G) \leq \frac{-n\mu_n(G)}{d - \mu_n(G)}.$$ 

Since $G$ is triangle-free, the neighbors of a vertex determine an independent set, whence $d \leq \alpha(G)$. Since $d = \mu_1(G)$ we get that

$$\mu_1(G) = d \leq \alpha(G) \leq \frac{-n\mu_n(G)}{d - \mu_n(G)} = \frac{-n\mu_n(G)}{\mu_1(G) - \mu_n(G)}.$$ 

Based on this inequality Brandt proved that

$$\mu_1(G) + \mu_n(G) \leq (3 - 2\sqrt{2})n.$$ 

So after proving Lemma 1.2 we practically copy the proof of Brandt.

**Proof of Lemma 1.2.** Let $\mu_s, \ldots, \mu_n$ be the set of non-positive eigenvalues. Then

$$0 = 6 \cdot \text{number of triangles} = \sum_{i=1}^{n} \mu_i^3 \geq \mu_1^3 + \sum_{i=s}^{n} \mu_i^3.$$
Hence
\[ \sum_{i=s}^{n} (-\mu_i)^3 \geq \mu_1^3. \]

On the other hand, we have
\[ \sum_{i=s}^{n} (-\mu_i)^3 \leq (-\mu_n) \sum_{i=s}^{n} (-\mu_i)^2 \leq (-\mu_n)(2e(G) - \mu_1^2) \leq (-\mu_n)(n\mu_1 - \mu_1^2). \]

Hence \( \mu_1^3 \leq (-\mu_n)(n\mu_1 - \mu_1^2) \), thus \( \mu_1^2 \leq (-\mu_n)(n - \mu_1) \), or in other words,
\[ \mu_1 \leq \frac{-n\mu_n}{\mu_1 - \mu_n}. \]

\( \square \)

**Proof of Theorem 1.1** As we mentioned earlier this proof practically follows the argument of [2].

We only need to solve the constrained maximization problem:
\[ \max \left\{ \frac{\mu_1 + \mu_n}{n} \mid \mu_1 \leq \frac{-n\mu_n}{\mu_1 - \mu_n} \right\}. \]

Let \( a = \mu_1 \), \( b = -\mu_n \) then we have \( a \leq \frac{nb}{a-b} \) which is equivalent to \( \frac{a^2}{n-a} \leq b \). Hence
\[ \frac{a-b}{n} \leq \frac{1}{n} \left( a - \frac{a^2}{n-a} \right) = \frac{an - 2a^2}{n(n-a)}. \]

So with the notation \( \alpha = a/n \) we need to maximize \( f(\alpha) := \frac{a^2}{n-a} \). Its derivative is
\( \frac{2a - 2a^2}{(n-a)^2} \) which is 0 at \( \alpha = 1 \pm 1/\sqrt{2} \). Note that \( \mu_1 \leq \Delta \leq n-1 \), where \( \Delta \) is the largest degree, so \( 0 \leq \alpha \leq 1 \). So we only need to consider \( \alpha = 1 - 1/\sqrt{2} \) and the extreme points of the interval, \( \alpha = 0 \) and \( 1 \), to see that \( f(\alpha) \) is indeed maximal at \( 1 - 1/\sqrt{2} \) and \( f(\alpha) = 3 - 2\sqrt{2} \).

Hence \( \mu_1 + \mu_n \leq (3 - 2\sqrt{2})n \).

\( \square \)

3. **Proof of Theorem 1.3**

In this section we prove Theorem 1.3. **Proof of Theorem 1.3** Suppose for contradiction that \( G \) is a strongly regular graph with eigenvalues \( (k, \mu_2^{(m_2)}, \mu_n^{(m_n)}) \) such that \( \frac{k+\mu_n}{n} > 0.14 \). Let \( -\mu_n = r \) and \( r/k = x \). Again we use that \( k \leq \alpha(G) \leq \frac{\sqrt{\mu_2}}{k-\mu_n} \). Hence \( x/k \geq \frac{k+r}{k} \). Then
\[ 0.14 < \frac{k-r}{n} = \frac{k-r}{k+r} \cdot \frac{k+r}{n} \leq \frac{k-r}{k+r} \cdot \frac{r}{k} = \frac{x(1-x)}{1+x}. \]

From which we get that \( x > 1/5 \). Secondly, \( m_n \geq \alpha(G) \geq k \) since we can assume that \( \mu_2 > 0 \). (Note that \( \mu_2 > 0 \) if \( G \) is not a blow-up of a complete graph.) Hence \( kn = 2e(G) \geq rn\mu_n^2 \geq kr^2 \). So we have \( n > r^2 \). So we have two inequalities: \( 0.14 < \frac{k-r}{n} = \frac{k-r}{k+r} \) and \( n > r^2 \). Then \( n > r^2 = (kx)^2 > x^2 \frac{0.14^2}{(1-x)^2} n^2 \). Thus \( \frac{1}{0.14^2} \frac{(1-x)^2}{x^2} > n \). Since \( x > 1/5 \) we have \( \frac{(1-x)^2}{x^2} < 16 \). Hence \( n < \frac{16}{0.14^2} \approx 816.33 \). Now we can finish the proof since we know all possible strongly regular graph parameters up to 816, see Brouwer’s website [3] and for the triangle-free strongly regular graph parameters the table on the next page. One can check that indeed the Higman-Sims graph achieves the maximum of \( (\mu_1(G) + \mu_n(G))/n \).
Remark 3.1. An interesting thing arises from the table on Andries Brouwer’s website. If there were a strongly regular graph $G$ with parameters $(28, 9, 0, 4)$ then for this graph $G$ we would have

$$\frac{\mu_1(G) + \mu_n(G)}{v(G)} = \frac{9 + (-5)}{28} = \frac{1}{7} > 0.14.$$  

It is known that there is no such strongly regular graph just as there is no strongly regular graph with parameters $(64, 21, 0, 10)$. For this graph we would have

$$\frac{\mu_1(G) + \mu_n(G)}{v(G)} = \frac{21 + (-11)}{64} = \frac{10}{64} > \frac{1}{7} > 0.14.$$  

| $n$ | $k$ | $a$ | $b$ | $\vartheta_1$ | $\vartheta_2$ | $m_1$ | $m_2$ | $\frac{k + \vartheta_1}{n}$ | Appr. | Existence |
|-----|-----|-----|-----|---------------|---------------|-------|-------|-----------------|-------|-----------|
| 5   | 2   | 0   | 1   | $\sqrt{\frac{\vartheta_1 - 1}{2}}$ | 2             | 2     |       | 0.076           | Yes   |           |
| 10  | 3   | 0   | 1   | $-2$ | 4             | 5     |       | 0.1             | Yes   |           |
| 16  | 5   | 0   | 2   | $-3$ | 10            | 5     |       | 0.125           | Yes   |           |
| 28  | 9   | 0   | 4   | $-5$ | 21            | 6     |       | 0.142           | No    |           |
| 50  | 7   | 0   | 1   | $-3$ | 28            | 21    |       | 0.08            | Yes   |           |
| 56  | 10  | 0   | 2   | $-4$ | 35            | 20    |       | 0.106           | Yes   |           |
| 64  | 21  | 0   | 10  | $-11$ | 56          | 7    |       | 0.156           | No    |           |
| 77  | 16  | 0   | 4   | $-6$ | 55            | 21    |       | 0.129           | Yes   |           |
| 100 | 22  | 0   | 6   | $-8$ | 77            | 22    |       | 0.14            | Yes   |           |
| 162 | 21  | 0   | 3   | $-6$ | 105           | 56    |       | 0.092           | ?     |           |
| 176 | 25  | 0   | 4   | $-7$ | 120           | 55    |       | 0.102           | ?     |           |
| 210 | 33  | 0   | 6   | $-9$ | 154           | 55    |       | 0.114           | ?     |           |
| 266 | 45  | 0   | 9   | $-12$ | 209          | 56    |       | 0.124           | ?     |           |
| 324 | 57  | 0   | 12  | $-15$ | 266        | 57    |       | 0.129           | No    |           |
| 352 | 26  | 0   | 2   | $-6$ | 208           | 143   |       | 0.056           | ?     |           |
| 352 | 36  | 0   | 4   | $-8$ | 231           | 120   |       | 0.079           | ?     |           |
| 392 | 46  | 0   | 6   | $-10$ | 276         | 115   |       | 0.091           | ?     |           |
| 552 | 76  | 0   | 12  | $-16$ | 437        | 114   |       | 0.108           | ?     |           |
| 638 | 49  | 0   | 4   | $-9$ | 406           | 231   |       | 0.062           | ?     |           |
| 650 | 55  | 0   | 5   | $-10$ | 429        | 220   |       | 0.076           | ?     |           |
| 667 | 96  | 0   | 16  | $-20$ | 551        | 115   |       | 0.113           | ?     |           |
| 704 | 37  | 0   | 2   | $-7$ | 407           | 296   |       | 0.042           | ?     |           |
| 784 | 116 | 0  | 20  | $-24$ | 667         | 116   |       | 0.117           | ?     |           |
| 800 | 85  | 0  | 10  | $-15$ | 595        | 204   |       | 0.087           | ?     |           |
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Alfréd Rényi Institute of Mathematics & Eötvös Loránd University, Department of Computer Science

Email address: peter.csikvari@gmail.com