Abstract. A subset $X$ of a C*-algebra $A$ is called irredundant if no $A \in X$ belongs to the C*-subalgebra of $A$ generated by $X \setminus \{A\}$. Separable C*-algebras cannot have uncountable irredundant sets and all members of many classes of nonseparable C*-algebras, e.g., infinite dimensional von Neumann algebras have irredundant sets of cardinality continuum.

There exists a considerable literature showing that the question whether every AF commutative nonseparable C*-algebra has an uncountable irredundant set is sensitive to additional set-theoretic axioms and we investigate here the noncommutative case.

Assuming ♦ (an additional axiom stronger than the continuum hypothesis) we prove that there is an AF C*-subalgebra of $B(\ell_2)$ of density $\omega = \omega_1$ with no nonseparable commutative C*-subalgebra and with no uncountable irredundant set. On the other hand we also prove that it is consistent that every discrete collection of operators in $B(\ell_2)$ of cardinality continuum contains an irredundant subcollection of cardinality continuum.

Other partial results and more open problems are presented.

1. Introduction

Definition 1.1. Let $A$ be a C*-algebra. A subset $X \subseteq A$ is called irredundant if and only if for every $A \in X$, the C*-subalgebra of $A$ generated by $X \setminus \{A\}$ does not contain $A$. We define

$$\text{irr}(A) := \sup\{|X| : X \text{ is an irredundant set in } A\}.$$ 

Recall that the density of a C*-algebra $A$, denoted $d(A)$ is the least cardinality of a norm dense subset of $A$, i.e., $A$ is separable if and only if $d(A)$ is countable. It is easy to see that $\text{irr}(A) \leq d(A)$ for every C*-algebra, as irredundant sets must be norm discrete. When $A$ is an infinite dimensional C*-algebra, then $\text{irr}(A)$ is infinite, because then $A$ contains an infinite dimensional abelian C*-subalgebra ([41]) and locally compact infinite Hausdorff spaces contain pairwise disjoint infinite collections of open sets which yield infinite irredundant sets (Proposition 3.12). In this article, we are interested in uncountable irredundant sets in (C*-subalgebras of) the algebra $B(\ell_2)$ of all linear bounded operators on a separable Hilbert space.

Irredundant sets have been considered in the context of other structures. For example, a subset of a Boolean algebra is called irredundant if none of its elements belongs to the Boolean subalgebra generated by the remaining elements. We call such sets Boolean irredundant (Definition 3.8). In Banach spaces irredundant sets, i.e., where no element belongs to the closed subspace spanned by the
remaining elements correspond exactly to biorthogonal systems ([20], see [30] for some comparisons between this type of notions). Examples of Boolean irredundant sets include independent families, ideal independent families or (almost) disjoint families, but there are Boolean algebras of uncountable irredundance with no uncountable families of the above-mentioned classes (see Remark 3.7). A collection \((x_\alpha, x^*_\alpha)_{\alpha<\kappa} \subseteq \mathcal{B} \times \mathcal{B}^*\) of a Banach space \(\mathcal{B}\) is biorthogonal if \(x^*_\alpha(x_\alpha) = 1\) and \(x^*_\alpha(x_\beta) = 0\) for all \(\alpha < \beta < \kappa\). As usually linear functionals on a C*-algebra \(\mathcal{A}\) are not multiplicative there are many more biorthogonal systems than irredundant sets in \(\mathcal{A}\), one can even consistently have a commutative C*-algebra \(C(K)\) with countable irredundance but with uncountable biorthogonal systems ([9]).

One of our main motivations are consistent constructions of uncountable Boolean algebras with no uncountable irredundant sets. They were first obtained by Rubin ([46]) under the assumption of \(\diamond\) and then by Kunen ([40]) under the continuum hypothesis CH (improved further by Todorcevic to a \(b = \omega_1\) construction from 2.4 of [51]). Also some versions of the classical Ostaszewski’s construction assuming \(\diamond\) from [43] have these properties as further constructions assuming \(\clubsuit\) from [20] as well as forcing constructions from [6], [9], [29].

Some of the above constructions are of Boolean algebras and other of (locally) compact Hausdorff totally disconnected spaces. Using the Stone duality one translates one language to the other easily. The fact that the Kunen or Ostaszewski types of constructions mentioned above correspond to superatomic Boolean algebras or equivalently their Stone spaces are scattered spaces (every subset has a relative isolated point) yields the equality between the Boolean irredundance of the Boolean algebra and the irredundance of the commutative C*-algebra of continuous functions (Corollary 3.10). In particular the corresponding \(C(K)\)s have no uncountable irredundant sets. In fact the scatteredness can be exploited further to prove that the Banach spaces \(C(K)\) have no uncountable biorthogonal systems ([40], [20]).

The first question we considered was whether such phenomena can take place if the C*-algebra is made considerably noncommutative. One of our main results is:

Theorem 1.2. Assume \(\diamond\). There is a fully noncommutative nonseparable scattered C*-algebra (of operators in \(\mathcal{B}(\ell^2)\)) with no nonseparable commutative subalgebra and with no uncountable irredundant set.

Proof. Apply Theorems 2.12 and 6.2. \(\square\)

Here scattered C*-algebras are the noncommutative analogues of the scattered locally compact spaces. The condition of being fully noncommutative means that these algebras are “maximally noncommutative” among scattered algebras. These notions are reviewed in Section 2.1.

\(\diamond\) is an additional axiom (introduced by R. Jensen) which is true in the universe of constructible sets. It says that there is a sequence \((S_\alpha)_{\alpha < \omega_1}\) which “predicts” all subsets of \(\omega_1\) in the sense that for any \(X \subseteq \omega_1\) the set \(\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}\) meets every closed an unbounded subset of \(\omega_1\), for details see [25] or [33]. \(\diamond\) has been recently successfully applied in the context of nonseparable C*-algebras by Akemann, Farah, Hirschberg and Weaver ([3], [4], [17]). We will not use the \(\diamond\) directly but will apply its consequence from Theorem 2.12 which was developed by S. Todorcevic in [54].
Another motivation for our project was the result of Todorcevic ([52], [53]) that assuming Martin’s axiom MA and the negation of the CH every uncountable Boolean algebra has an uncountable irredundant set. Here the main question remains open:

**Question 1.3.** Is it consistent that every nonseparable (AF, scattered) C*-algebra (of operators in \(B(\ell_2)\)) contains an uncountable irredundant set?

It should be added that even the commutative general case is open, since the result of Todorcevic provides uncountable irredundant sets in \(C(K)\) only for \(K\) totally disconnected and there can be nonmetrizable compact spaces with no totally disconnected nonmetrizable compact subspace and similar examples (see [31]). So it is natural to restrict initially the attention in the noncommutative problem to C*-algebras corresponding to totally disconnected spaces, namely to approximately finite dimensional C*-algebras (AF), i.e., where there is a dense subset which is the union of a directed family of finite dimensional C*-subalgebras (see [16] for diverse notions of approximate finite-dimensionality in the nonseparable context). Another natural narrowing of the question is to consider only the scattered C*-algebras since one of the conditions equivalent to being scattered for a C*-algebra of density \(\omega_1\) is that each of its C*-subalgebras is AF.

Attempting to answer Question 1.3 we obtained several results which shed some light on it. Let us discuss them below.

If \(A\) is AF C*-algebra of density equal to the first uncountable cardinal \(\omega_1\), then it can be written as \(A = \bigcup_{\xi < \omega_1} A_\xi\) where \(A_\xi \subseteq A_\eta\) for all \(\xi < \xi' < \omega_1\) and each \(A_\xi\) is separable and AF. It follows from the result of Thiel in [48] (cf. [42], [49]) that each \(A_\xi\) is singly generated by one element \(A_\xi \in A_\xi\). Hence in the set \(\{A_\xi : \xi < \omega_1\}\) irredundant subsets are at most singletons. So there is no chance to extract (possibly using some additional forcing axioms) an uncountable irredundant set from an arbitrary norm discrete set of cardinality \(\omega_1\) of operators in \(B(\ell_2)\).

The AF hypothesis allows nevertheless to avoid sets of operators as above. Namely, if \(A = \bigcup_{D \in \mathbb{D}} A_D\), where all \(A_D\)s are finite-dimensional and \(A_D \subseteq A_{D'}\) whenever \(D \leq D'\) for \(D \in \mathbb{D}\) and \((\mathbb{D}, \leq)\) is directed, then given any norm discrete \(\{A_\xi : \xi < \omega_1\} \subseteq \bigcup_{D \in \mathbb{D}} A_D\), which exists by the nonseparability of \(A\), for every finite \(F \subseteq \omega_1\) the set

\[
X_F = \{\xi < \omega_1 : A_\xi \in A_F\}
\]

is a finite superset of \(F\), where \(A_F\) is the C*-subalgebra generated by \(\{A_\eta : \eta \in F\}\). So, the search for an uncountable irredundant set among \(\{A_\xi : \xi < \omega_1\}\) is equivalent to the search for an uncountable \(X \subseteq \omega_1\) such that \(X_F \cap X = F\) for every \(F \subseteq X\).

However this combinatorial problem for a general function from finite subsets of \(\omega_1\) to themselves has the negative solution\(^2\). Nevertheless passing to the second uncountable cardinal \(\omega_2\) allows for a very general consistency result which follows from Theorem 3.20:

\(^2\)It is enough to take \(X_F\) to be of the form \(Y \cap [(\max F) + 1]\) where \(Y \in \mu\) is of minimal rank which contains \(F\) and where \(\mu\) is an \((\omega, \omega_1)\)-cardinal as in [32], \(\mu\) is originally due to Velleman ([55]). A positive result for general functions is that given \(n \in \mathbb{N}\) and a function \(\phi\) from finite subsets of the \(n\)-th uncountable cardinal \(\omega_n\) into countable subsets of \(\omega_n\) there is an \(n\)-element set \(X \subseteq \omega_n\) such that \(\xi \notin \phi(X \setminus \{\xi\})\) for any \(\xi \in X\). In particular, this gives that any norm discrete subset of cardinality \(\omega_n\) in any C*-algebra has an irredundant subset of cardinality \(n\).
This is not a mere consequence of $B(\ell_2)$ having density $\omega_2$ because by a result of Brech and Koszmider ([8]) it is consistent that there exists a commutative C*-subalgebra of $\ell_\infty$ of density $2^\omega = \omega_2$ with no uncountable irredundant set. The cardinal $\omega_2$ in Theorem 1.4 can be replaced by any regular cardinal bigger than $\omega_1$ but it is not known if the result of [8] can be generalized to bigger cardinals than $\omega_2$. Combining 1.4, 1.2 and knowing that $\diamondsuit$ implies CH we obtain:

**Corollary 1.5.** It is independent from ZFC whether there is a norm discrete collection of operators (projections) $(A_\xi : \xi < 2^\omega)$ in $B(\ell_2)$ with no uncountable (of cardinality $2^\omega$) irredundant subcollection of size $2^\omega$.

It is independent from ZFC whether there is C*-subalgebra of $B(\ell_2)$ of density $2^\omega$ with no uncountable (of size $2^\omega$) irredundant set.

The commutative results mentioned above, in fact, are most often of topological nature, where the compact Hausdorff space under the consideration is the Stone space $K_A$ of a Boolean algebra $A$. For example, the reason the above-mentioned Boolean algebras have countable irredundance is that the spread of $K_A \times K_A$ is countable as the finite powers of the mentioned $K_A$s are hereditarily separable. Namely, in general we have $\text{irr}(A) \leq s(K_A \times K_A)$ which was first noted in [23] and easily follows from the characterization of irredundant sets in the commutative case (Lemma 3.4). Also the Urysohn Lemma gives the inequality $s(K) \leq \text{irr}(C(K))$ for any locally compact Hausdorff $K$. This argument cannot be transferred to the noncommutative setting since we do not have so general noncommutative Urysohn Lemma (for noncommutative Uryshon Lemma see [2]). That is for constructing an irredundant set of cardinality $\kappa$ in a C*-algebra $A$ it is enough to construct a sequence of states $(\tau_\alpha : \alpha < \kappa)$ and a sequence of positive elements $(A_\alpha : \alpha < \kappa)$ of $A$ such that $\tau_\alpha(A_\alpha) > 0$ for all $\alpha < \kappa$ and $\tau_\alpha(A_\beta) = 0$ for all distinct $\alpha, \beta < \kappa$ (Lemma 3.14), but a weak* discrete set of pure states does not produce the elements $A_\alpha$ as above due to the lack of the Urysohn Lemma for nonorthogonal closed projections. In fact assuming the Proper Forcing Axiom, PFA every nonseparable scattered C*-algebra has an uncountable weak* discrete set of pure states (Corollary 3.17), but this does not help us in constructing an uncountable irredundant set and answering Question 1.3 in the positive in the scattered case.

A bolder approach to Question 1.3 would be to try to answer the following question in the positive:

**Problem-Abelian** Question 1.6. Is it consistent (with MA and the negation of CH) that every nonseparable scattered (or even AF) C*-algebra has a nonseparable commutative subalgebra in one of its quotients?

Note that the class of scattered C*-algebras is closed under quotients and subalgebras and every locally compact scattered Hausdorff space is totally disconnected, so the positive answer to the question above and the MA result of Todorcevic mentioned above would give the positive answer to Question 1.3 in the scattered case.

Known ZFC examples of nonseparable C*-algebras with no nonseparable commutative subalgebras are the reduced group C*-algebra of an uncountable free group as shown by Popa in [45] and the algebras of Akemann and Doner as shown.

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3The spread of a topological space $K$, denoted by $s(K)$, is the supremum over the cardinalities of discrete subspaces of $K$. 
in [7]. However the former is not AF (and has an uncountable irredundant set corresponding to the free generators of the group) and the latter has a nonseparable commutative quotient \( \omega(\omega_1) \) (which also has an obvious uncountable irredundant set). Perhaps the algebra of [19] could provide the negative answer to Question 1.6.

The reason our algebra from Theorem 1.2 does not contain a nonseparable commutative C*-subalgebra is that given any discrete sequence of projections in certain dense subalgebra there are two of them which have maximal commutator equal to \( 1/2 \) (the fact that \( 1/2 \) is the maximal value is proved in [47]). However in such an arbitrary sequence there are also two projections which almost commute (see Theorem 6.2), so in this sense our algebra is quite random, that is no pattern repeats on any uncountable norm discrete subset of elements. In fact such behaviour is already sensitive to infinitary combinatorics beyond ZFC determined by \( \Diamond \) and Open Coloring Axiom (OCA)\(^4\), namely we have:

**Theorem 1.7.** Assume OCA. For every \( 0 < \varepsilon < 1/2 \) among any sequence of operators \((A_\xi : \xi < \omega_1)\) in \( B(\ell_2) \) there is an uncountable \( X \subseteq \omega_1 \) such that

- for every distinct \( \xi_1, \xi_2 \in X \) we have \( [A_{\xi_1}, A_{\xi_2}] > 1/2 - \varepsilon \), or
- for every \( \xi_1, \xi_2 \in X \) we have \( [A_{\xi_1}, A_{\xi_2}] < \varepsilon \).

However, assuming \( \Diamond \) there is a scattered C*-algebra \( A \subseteq B(\ell_2) \) (it is in particular AF) such that for every \( 0 < \varepsilon < 1/2 \) among any discrete sequence of projections \((P_\xi : \xi < \omega_1)\) in \( A \)

- there are \( \xi_1 < \xi_2 < \omega_1 \) such that \( [P_{\xi_1}, P_{\xi_2}] > 1/2 - \varepsilon \), or
- there are \( \xi_1 < \xi_2 < \omega_1 \) such that \( [P_{\xi_1}, P_{\xi_2}] < \varepsilon \).

*Proof.* Apply Corollary 4.6 and Theorem 6.2 \( \Box \)

Another natural question related to uncountable irredundant sets in general C*-algebras is the following:

**Question 1.8.**

1. Is it true that \( d(A) \leq 2^{\text{irr}(A)} \) holds for every C*-algebra (every C*-algebra of type I)?
2. Can there be arbitrarily big C*-algebras with no uncountable irredundant sets?

This is motivated by a Boolean result of McKenzie (see 4.2.3 of [28]) which says that a Boolean algebra has a dense subalgebra not bigger than its irredundance. This result has been generalized by Hida in [24] to all commutative C*-algebras which implies that \( \text{irr}(A) \leq 2^{d(A)} \) holds for commutative \( A \). We prove this inequality answering Question 1.8 for scattered C*-algebras in our Theorem 3.18.

In Section 2 we review scattered C*-algebras and constructions schemes which is an elegant framework to deal with some constructions using \( \Diamond \) recently introduced by Todorcevic in [54]. It was already applied in several functional analytic, topological and combinatorial contexts in [54], [36], [37]. In Section 3 we prove basic facts concerning irredundant sets in commutative and noncommutative setting. In Section 4 we prove the OCA part of Theorem 1.7. Section 5 is devoted to defining and investigating the partial order of finite dimensional approximations to our algebra from Theorem 1.2. In the final Section 6 we use the appropriate construction schemes described in Section 2 to construct the algebra from Theorem 1.2.

\(^4\)For the statement of OCA see Definition 4.4.
Notation and the terminology of this paper should be standard, however, it
draws from diverse parts of mathematics like Boolean algebras, operator theory,
set-theory, logic and general topology. When in doubt one could refer to textbooks
like [28], [39], [25], [33], [14]. In particular by an embedding (isomorphism onto
its image) we mean ∗-monomorphism (∗-isomorphism) of C∗-algebras which is not
necessarily unital, ℓ2(X) denotes the Hilbert space of square summable complex
functions defined on a set X, B(ℓ2(X)) denotes the C∗-algebra of all bounded
operators on ℓ2(X), ⟨·, ·⟩ denotes the scalar product, A+ denotes the set of positive
elements of a C∗-algebra A, 1A denotes the unit of A and ˜A the
unitization of A, B∗(X) the dual ball of the algebra A, [A, B] = AB − BA
for A, B ∈ B(ℓ2), Mn denotes the C∗-algebra of n × n matrices for n ∈ N, C(K)
denotes the C∗-algebra of complex valued continuous functions on a compact K
and C0(X) the C∗-algebra of complex valued continuous functions vanishing at infinity
on a locally compact X, χU denotes the characteristic function of a set U, Clop(K)
denotes the family of clopen subsets of a space K, ωn denotes the n-th uncountable
cardinal for n ∈ N, [X]n denotes the family of all n-element subsets of a set X,
[X]<ω denotes the family of all finite subsets of a set X; X < Y means that x < y
for all x ∈ X and y ∈ Y where X, Y are sets of ordinals.

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2. Preliminaries

2.1. Scattered C∗-algebras. The reason why scattered C∗-algebras will play an
important role in our investigation of irredundant sets is that in such algebras irre-
dundant sets can easily be replaced by irredundant sets of projections (Proposition
3.3), in particular the Boolean results pass to the C∗-algebraic ones (Corollary
3.10). Moreover all commutative results culminate around the scattered case which
seems most basic.

Recall that a topological space is called scattered if it does not contain any perfect
subset or in other words if each (closed) nonempty subset has a relative isolated
point. The phenomena related to the scatteredness were already analysed by Cantor
which resulted in the notion of the Cantor-Bendixson derivative of a topological
space ([14]). The Boolean algebra manifestation of these phenomena was discovered
by Mostowski and Tarski in [38] as what is today known as superatomic Boolean
algebras. The importance of the class of Banach spaces of the form C(K), where
K is scattered, already implicitly known in the 30ties, was first systematically
revealed in [44]. Its generalization, Asplund Banach spaces, started to play an
important role in Banach space theory since the 60ties. It was Jensen in [26] who
first defined a scattered C∗-algebra but they were considered earlier by Tomiyama
[50] and Wojtaszczyk [56]. A recent survey [18] underlines the links of scattered
C∗-algebras with its Boolean algebraic and commutative predecessors. Recall that
a projection p in a C∗-algebra is called minimal if and only if pAp = Cp, i.e.,
minimal projections generalize isolated points. The ∗-subalgebra of A generated
by the minimal projections of A will be denoted IAt(A). We have the following
observation from [18]:

Proposition 2.1. Suppose that A is a C∗-algebra.

(1) IAt(A) is an ideal of A,
(2) $I^\omega(A)$ is isomorphic to a subalgebra of the algebra $K(H)$ of all compact operators on a Hilbert space $H$,
(3) $I^\omega(A)$ contains all ideals of $A$ which are isomorphic to a subalgebra of $K(H)$ for some Hilbert space $H$,
(4) if an ideal $I \subseteq A$ is essential and isomorphic to a subalgebra of $K(H)$ for some Hilbert space $H$, then $I = I^\omega(A)$.

A selected list of conditions equivalent to being scattered and which are relevant to our paper is given below. Any of these conditions can be taken as the definition of a scattered algebra.

**Theorem 2.2** ([26, 27, 56, 35, 34, 50, 18]). Suppose that $A$ is a $C^*$-algebra. The following conditions are equivalent:

1. Every non-zero $*$-homomorphic image of $A$ has a minimal projection.
2. There is an ordinal $ht(A)$ and a continuous increasing sequence of closed ideals $(I^\alpha(A))_{\alpha < ht(A)}$ called the Cantor-Bendixson composition series for $A$ such that $I_0 = \{0\}$, $I_{ht(A)} = A$, and
   \[ I^\alpha(A/I^\omega(A)) = \{ [a]I^\omega(A) : a \in I^\alpha_{\alpha+1}(A) \}, \]
   for every $\alpha < ht(A)$.
3. Every non-zero subalgebra of $A$ has a minimal projection.
4. Every non-zero subalgebra has a projection,
5. Every subalgebra of $A$ has real rank zero,
6. $A$ does not contain a copy of the $C^*$-algebra $C_0((0,1]] = \{ f \in C((0,1]) : \lim_{x \to 0} f(x) = 0 \}$.
7. The spectrum of every self-adjoint element is countable.

**Definition 2.3.** [18] A scattered $C^*$-algebra is called thin-tall if and only if $ht(A)$ from Theorem 2.2 (2) is equal $\omega_1$ and $I^\alpha_{\alpha+1}(A)/I^\omega(A)$ is separable for each $\alpha < \omega_1$.

In the nonseparable context we are especially interested in condition (2) which was introduced in [18] which gives an essential composition series corresponding to the Cantor-Bendixson derivative. A scattered $C^*$-algebra is called fully noncommutative if and only if for all $\alpha < ht(A)$ the algebra $I^\alpha(A/I^\omega(A))$ is $*$-isomorphic to the algebra of all compact operators on a Hilbert space. We have the following two observations from [18]:

**Proposition 2.4.** Suppose that $A$ is a scattered $C^*$-algebra. The following are equivalent:

1. $A$ is fully noncommutative,
2. the ideals of $A$ form a chain,
3. the centers of the multiplier algebras of any quotient of $A$ are all trivial.

**Proposition 2.5.** Every scattered $C^*$-algebra $A$ is atomic, i.e., the ideal $I^\omega(A)$ is essential.

Recall that in a topological space a sequence of points $\{ x_\xi : \xi < \kappa \}$ is called right-separated (left-separated) if and only if $x_\xi \not\in \{ x_\eta : \eta > \xi \}$ for all $\xi < \kappa$ ($x_\xi \not\in \{ x_\eta : \eta < \xi \}$ for all $\xi < \kappa$). Left and right separated sequences play an important role in commutative set-theoretic topology because a regular space is hereditarily Lindelöf (hereditarily separable) if it has no uncountable right-separated
(left-separated) sequences. Additional axioms like ♦, CH, MA, PFA have substantial impact on the existence of right or left separated sequences in regular topological spaces, for example PFA implies that there are no regular S-spaces, i.e., every regular topological space which has an uncountable right-separated sequence has an uncountable left-separated sequence as well (Theorem 8.9 of [51]).

**Proposition 2.6.** Suppose that $\mathcal{A}$ is a thin-tall $C^*$-algebra. Then the dual ball $B_{\mathcal{A}^*}$ of $\mathcal{A}^*$ contains an uncountable right-separated sequence of pure states in the weak$^*$ topology. In particular under the Proper Forcing Axiom (PFA) the dual ball $B_{\mathcal{A}^*}$ of $\mathcal{A}^*$ contains an uncountable discrete set consisting of pure states.

**Proof.** Let $(\mathcal{I}_\alpha)_{\alpha<\omega_1}$ be the Cantor-Bendixson composition series of 2.2 (3). As $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is an essential ideal of $\mathcal{A}/\mathcal{I}_\alpha$ which is *-isomorphic with the the algebra of all compact operators on $\ell_2$, we can embed $\mathcal{A}/\mathcal{I}_\alpha$ into $B(\ell_2)$ with the range containing all compact operators. Take $\tau_\alpha$ to be a vector pure state on $B(\ell_2)$ composed with the quotient map and the embedding. So $\tau_\alpha$ is a pure state on $\mathcal{A}$ which is zero on $\mathcal{I}_\alpha$ and there is $A_\alpha \in \mathcal{I}_{\alpha+1}$ such that $\tau_\alpha(A_\alpha) = 1$. Denote the set of all pure states on $\mathcal{A}$ by $P(\mathcal{A})$. Now consider

$$U_\alpha = \{ \tau \in P(\mathcal{A}) : \tau(A_\alpha) > 0 \}.$$  

Note that if $\tau_\beta \in U_\alpha$ then $\beta \leq \alpha$, so $\{ \tau_\alpha : \alpha < \omega_1 \}$ is right-separated in the weak$^*$ topology. So $\{ \tau_\alpha : \alpha < \omega_1 \}$ contains and uncountable left-separated sequence by PFA (Theorem 8.9 or [51]). It is clear that a sequence which is both left and right separated is discrete. \hfill $\Box$

2.2. **Construction schemes.** In this section we recall some definitions and results from [54].

**Definition 2.7.** Let $E, F \in [\omega_1]<\omega$.

1. $F < E$ whenever $\alpha < \beta$ for all $\alpha \in F$ and $\beta \in E$,
2. $F \sqsubseteq E$ whenever there is $\alpha \in \omega_1$ such that $E \cap \alpha = F$ (we say that $F$ is an initial fragment of $E$ or that $E$ end-extends $F$),
3. $F \sqsubseteq E$ whenever $F \subseteq E$ and $E \setminus F \neq \emptyset$.

**Definition 2.8.** Let $\eta$ be an ordinal and let $(F_\xi : \xi < \eta) = F \subseteq [\omega_1]<\omega$.

1. $F$ is cofinal if for all $E \in [\omega_1]<\omega$ there is $F \in F$ such that $E \subseteq F$,
2. $(F_\xi : \xi < \eta)$ is a $\Delta$-system of length $\eta$ with root $\Delta$ whenever $F_\xi \cap F_{\xi'} = \Delta$ for all $\xi < \xi' < \eta$,
3. A $\Delta$-system $(F_\xi : \xi < \eta)$ with root $\Delta$ is increasing whenever $F_\xi \setminus \Delta < F_{\xi'} \setminus \Delta$ for all $\xi < \xi' < \eta$,
4. A subset of a $\Delta$-system is called a subsystem,
5. $F|\mathcal{F} = \{ E \in \mathcal{F} : E \subseteq F \}$ for $F \subseteq \omega_1$.

**Definition 2.9.** A pair of sequences $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and $(r_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ is called allowed parameters if and only if

1. $n_0 = r_1 = n_0 = 0$
2. $n_k \geq 2$ for all $k \in \mathbb{N}$,
3. each natural value appears in the sequence $(r_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ infinitely many times

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$^5$For the statement of the Proper Forcing Axiom (PFA) or Martin’s Axiom (MA) we refer the reader, for example, to [55] or [51]. PFA implies among others MA, OCA and $2^\omega = \omega_2$.  

(4) \( r_{k+1} < m_k \) where \( m_0 = 1 \), \( m_{k+1} = r_{k+1} + n_{k+1}(m_k - r_{k+1}) \) for \( k > 0 \).

**Definition 2.10.** A construction scheme with a pair of allowed parameters \((n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N} \) and \((r_k)_{k \in \mathbb{N}} \subseteq \mathbb{N} \) is a cofinal family \( \mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \) satisfying

1. \( \mathcal{F}_0 = [\omega_1]^1 \),
2. If \( k > 0 \) and \( E, F \in \mathcal{F}_k \), then \( |E| = |F| \) and \( E \cap F \subseteq E, F \) and \( \{\phi_{F,E}[G] : G \in \mathcal{F}[E] \} = \mathcal{F}[F] \),

where \( \phi_{F,E} : E \to F \) is the order preserving bijection between \( E \) and \( F \).
3. If \( k \geq 0 \) and \( F \in \mathcal{F}_{k+1} \), then the maximal elements of \( \mathcal{F}[F] \) are in \( \mathcal{F}_k \) and they form an increasing \( \Delta \)-system of length \( n_{k+1} \) such that \( F \) is its union. The family of all these maximal elements is called the canonical decomposition of \( F \).

**Definition 2.11.** Given a construction scheme \( \mathcal{F} \), we say that an \( F \in \mathcal{F}_k \) for \( k > 0 \) captures a \( \Delta \)-system \( \{s_i : i < n\} \) of finite subsets of \( \omega_1 \) with root \( s \) if the canonical decomposition \( \mathcal{F}_i : i < n_k \) of \( F \) with root \( \Delta \) has the following properties:

1. \( n_k \geq n \), \( s \subseteq \Delta \), and \( s_i \setminus s \subseteq F_i \setminus \Delta \) for all \( i \leq n \).
2. \( \phi_{F_i,F_j}[s_i] = s_j \) for all \( i < j < n \).

When \( n = n_k \), we say that \( F \) fully captures the \( \Delta \)-system.

**Theorem 2.12** ([54]). Assume \( \diamond \). For any pair of allowed parameters \((n_k)_{k \in \mathbb{N}} \) and \((r_k)_{k \in \mathbb{N}} \) there is a construction scheme \( \mathcal{F} \) with these parameters and there is a partition \((P_n)_{n \in \mathbb{N}} \) of \( \mathbb{N} \) into infinitely many infinite sets such that for every \( n \in \mathbb{N} \) and every uncountable \( \Delta \)-system \( T \) of finite subsets of \( \omega_1 \) there exist arbitrarily large \( k \in P_n \) and \( F \in \mathcal{F}_{k+1} \) which fully captures a subsystem of \( T \).

3. Irredundant sets

3.1 Reducing irredundant sets to special ones. Because of Weierstrass-Stone theorem for unital commutative C*-algebras sometimes it is useful to consider a strengthening of being irredundant, where the subalgebras we generate are unital. However this does not affect the cardinalities of irredundant sets much:

**Lemma 3.1.** Suppose that \( \mathcal{A} \) is a unital C*-algebra and that \( \mathcal{X} \subseteq \mathcal{A} \) is its nonempty irredundant set. Then there is \( x_0 \in \mathcal{X} \) such that no element \( x \) of \( \mathcal{X} \setminus \{x_0\} \) belongs to the unital C*-subalgebra generated by \( \mathcal{X} \setminus \{x_0, x\} \).

**Proof.** If no element \( x \) of \( \mathcal{X} \) belongs to the unital C*-subalgebra generated by \( \mathcal{X} \setminus \{x_0\} \) we are done by taking any element of \( \mathcal{X} \) as \( x_0 \).

Otherwise let \( x_0 \in \mathcal{X} \) belong to the unital C*-subalgebra generated by \( \mathcal{X} \setminus \{x_0\} \) so \( x_0 = \lambda_1 + y \) where \( y \) is in the subalgebra generated by \( \mathcal{X} \setminus \{x_0\} \) and \( \lambda \in \mathbb{C} \setminus \{0\} \). Suppose that there is \( x \in \mathcal{X} \setminus \{x_0\} \) in the unital C*-subalgebra generated by \( \mathcal{X} \setminus \{x_0, x\} \), i.e., \( x = \lambda_1' + z \) where \( z \) is in the subalgebra generated by \( \mathcal{X} \setminus \{x_0, x\} \) and \( \lambda' \in \mathbb{C} \setminus \{0\} \). So 1 is in the algebra generated by \( \mathcal{X} \setminus \{x_0\} \), but this shows that \( x_0 = \lambda_1 + y \) is in the subalgebra generated by \( \mathcal{X} \setminus \{x_0\} \), a contradiction with the fact that \( \mathcal{X} \) is irredundant. \( \square \)

Clearly any two orthogonal one-dimensional projections in \( M_2 \) form an irredundant set, however each of them is in the unital C*-algebra generated by the other projection.
Proposition 3.2. Suppose that $\mathcal{A}$ is a C*-algebra, $\kappa$ is an infinite cardinal and $\{A_\xi : \xi < \kappa\}$ is an irredundant set in $\mathcal{A}$. Then there is an irredundant set $\{B_\xi : \xi < \kappa\}$ consisting of positive elements of $\mathcal{A}$.

Proof. Given $X \subseteq \kappa$ let $\mathcal{A}_X$ be the C*-subalgebra of $\mathcal{A}$ generated by $\{A_\xi : \xi \in X\}$. Clearly $(A_\eta + A_\eta^*)/2, (A_\eta - A_\eta^*)/2i \in \mathcal{A}_{\kappa \setminus \{\xi\}}$ for every $\eta \neq \xi$ and $(A_\xi + A_\xi^*)/2$ and $(A_\xi - A_\xi^*)/2i$ cannot both belong to $\mathcal{A}_{\kappa \setminus \{\xi\}}$. So $\{B_\xi : \xi < \kappa\}$ is irredundant set consisting of self-adjoint elements, where $B_\xi \in \{(A_\xi + A_\xi^*)/2, (A_\xi - A_\xi^*)/2i\}$ is such that $B_\xi \notin \mathcal{A}_{\kappa \setminus \{\xi\}}$.

To prove that we can obtain the same cardinality irredundant set consisting of all positive elements, by the above we may assume that the original $\mathcal{A}_{\xi}$s are self-adjoint. We have $A_\xi = A_{\xi +} - A_{\xi -}$. Note that there is $B_\xi \in \{A_{\xi +}, A_{\xi -}\}$ which does not belong to $\mathcal{A}_{\kappa \setminus \{\xi\}}$. But $A_{\eta +} = (|A_\eta| + A_\eta)/2$ and $A_{\eta -} = (|A_\eta| - A_\eta)/2$ for $|A_\xi| = \sqrt{A_\xi^* A_\xi}$ belong to $A_\xi$ for all $\eta \in \kappa \setminus \{\xi\}$. So $\{B_\xi : \xi < \kappa\}$ is irredundant, as required.

The following proposition shows the role of being scattered while extracting irredundant sets consisting of projections.

Proposition 3.3. Suppose that $\mathcal{A}$ is a scattered C*-algebra, $\kappa$ is an infinite cardinal and $\{A_\xi : \xi < \kappa\}$ is an irredundant set in $\mathcal{A}$. Then there is an irredundant set $\{P_\xi : \xi < \kappa\}$ consisting of projections.

Proof. By Lemma 3.2 we may assume that $\mathcal{A}_{\xi}$s are self-adjoint. Let us adopt the notation $\mathcal{A}_X$ for $X \subseteq \kappa$ from the proof of Lemma 3.2.

Since subalgebras of scattered algebras are scattered, $\mathcal{A}_{\xi}$s are scattered for each $\xi < \kappa$ and so of the form $C_0(K(\xi))$ for some locally compact scattered $K(\xi)$ which must be totally disconnected. It follows that linear combinations of projections of $\mathcal{A}_{\xi}$s are norm dense in $\mathcal{A}_{\xi}$s. Hence for each $\xi < \kappa$ there is a projection $P_\xi \in \mathcal{A}_\xi$ such that $P_\xi \notin \mathcal{A}_{\kappa \setminus \{\xi\}}$. It follows that $\{P_\xi : \xi < \kappa\}$ is irredundant.

3.2. Irredundant sets in commutative C*-algebras. The following two lemmas characterize irredundant sets in commutative C*-algebras.

Lemma 3.4. Suppose that $K$ is compact Hausdorff space and $X \subseteq C(K)$ is such that no $f \in X$ belongs to the unital C*-subalgebra of $C(K)$ generated by $X \setminus \{f\}$. Then for each $f \in X$ there are $x_f, y_f \in K$ such that $f(x_f) \neq f(y_f)$ but $g(x_f) = g(y_f)$ for all $g \in X \setminus \{f\}$.

Consequently if $X$ is a nonempty irredundant set in $C(K)$, then there is $h \in X$ such that $X \setminus \{h\}$ has the above property.

Proof. By the Gelfand representation we may assume that $C(K)$ is the unital C*-algebra generated by $X$. By the complex Stone-Weierstrass theorem the proper C*-subalgebra generated by $X \setminus \{f\}$ does not separate a pair of points of $K$, say $x_f, y_f$. But they must be separated by $f$ by the fact that $X$ generated $C(K)$.

The last part of the lemma follows from Lemma 3.1.

Lemma 3.5. Suppose that $X$ is locally compact noncompact Hausdorff space and $X \subseteq C_0(X)$ is irredundant then for every $f \in X$ there are $x_f, y_f \in X$ such that either

- $f(x_f) \neq 0$ and $g(x_f) = 0$ for all $g \in X \setminus \{f\}$, or
• \( f(xf) \neq f(yf) \) but \( g(xf) = g(yf) \) for all \( g \in X \setminus \{ f \} \).

Points \( xf \) satisfying the first case form a discrete subspace of \( X \).

**Proof.** Let \( K = X \cup \{ \infty \} \) be the one-point compactification of \( X \). We will identify \( C_0(K) \) with a \( C^* \)-subalgebra of \( C(K) \). Note that \( X \) satisfies the hypothesis of Lemma 3.4, because if \( f = \lambda 1 + g \) for \( f, g \in C_0(X) \), the unit would be in \( C_0(X) \) which contradicts the hypothesis that \( X \) is noncompact. So we obtain the pairs of points \( xf, yf \) \( x, y \in K \) as in Lemma 3.4. The first case of the lemma corresponds to the situation when one of the points \( xf, yf \) is \( \infty \), say \( yf \). But then \( h(yf) = 0 \) for all \( h \in C_0(X) \) which implies \( f(xf) \neq 0 \) and \( g(xf) = 0 \) for all \( g \in X \setminus \{ f \} \).

Considering open sets \( U_f = \{ x \in X_0 : f(x) \neq 0 \} \) we obtain neighbourhoods witnessing the discretness of the set of \( xf \)s satisfying the first case. \( \square \)

In fact discrete subsets of \( K \) provide strong irredundant subsets in \( C(K) \):

**Remark 3.6.** Suppose that \( K \) is compact Hausdorff space and \( D \subseteq K \) is discrete. For each \( d \in D \) consider \( f_d \in C(K) \) such that \( f_d(d) \neq 0 \) and \( f_d(d') = 0 \) for all \( d' \in D \setminus \{ d \} \). Then \( f_d \) does not belong to the ideal generated by \( \{ f_d' : d' \in D \setminus \{ d \} \} \). In particular \( \{ f_d : d \in D \} \) is irredundant.

One should, however, note that there could be dramatic gap between the sizes of discrete subsets and the sizes of irredundant sets:

**Remark 3.7.** Let \( K \) be the split interval, i.e., \( \{ 0, 1 \}^{\mathbb{N}} \times \{ 0, 1 \} \) with the order topology induced by the lexicographical order. Then \( K \) has no uncountable discrete subset (in fact, \( K \) is hereditarily separable and hereditarily Lindelöf) but \( C(K) \) has an irredundant set \( \{ \chi_{0^\infty, 0} : x \in \{ 0, 1 \}^\infty \} \) of cardinality continuum.

Most of the literature concerning implicitly or explicitly irredundant sets are related to Boolean algebras. As shown in the following lemma the relationship between Boolean irredundance and irredundance for \( C^* \)-algebras is very close in the light of lemma 3.1.

**Def-boolean-irr**

**Definition 3.8.** A subset \( X \) of a Boolean algebra \( A \) is called Boolean irredundant if for every \( x \in X \) the element \( x \) does not belong to the Boolean subalgebra generated by \( X \setminus \{ x \} \).

**Lifting-boolean**

**Lemma 3.9.** Suppose that \( A \) is a unital \( C^* \)-algebra and \( B \subseteq A \) is a Boolean algebra of projections in \( A \) and \( X \subseteq B \) is Boolean irredundant. Then \( X \) is irredundant in \( A \).

Suppose that \( K \) is a totally disconnected space and \( X \subseteq C(K) \) consists of projections and no element of \( x \in X \) belongs to the unital \( C^* \)-algebra generated by \( X \setminus \{ x \} \). Then \( X \) is Boolean irredundant in the Boolean algebra \( \{ \chi_U : U \in \text{Clop}(K) \} \).

**Proof.** Let \( C \) be the \( C^* \)-subalgebra of \( A \) generated by \( B \). It is abelian, so it is of the form \( C(K) \) where \( K \) is the Stone space of \( B \). It is enough to prove that \( X \) is irredundant in \( C \). But given a proper Boolean subalgebra, there are distinct ultrafilters on the superalgebra which coincide on the subalgebra. These ultrafilters are the points of \( K \) witnessing the irredundance of \( X \) as in Lemma 3.15. \( \square \)

For commutative scattered \( C^* \)-algebras the relationship between Boolean and \( C^* \)-algebraic irredundance is even closer:
Corollary 3.10. Suppose that $\mathcal{A}$ is an infinite superatomic Boolean algebra. Then the Boolean irredundance of $\mathcal{A}$ is the same as $\text{irr}(C(K_\mathcal{A}))$, where $K_\mathcal{A}$ is the Stone space of $\mathcal{A}$.

Proof. As $\mathcal{A}$ is infinite, its Boolean irredundance is infinite (just take an infinite pairwise disjoint collection). By the first part of Lemma 3.9 the Boolean irredundance of $\mathcal{A}$ is not bigger than $\text{irr}(C(K_\mathcal{A}))$. On the other hand consider any infinite irredundant subset $X$ of $C(K_\mathcal{A})$. $K_\mathcal{A}$ is scattered as $\mathcal{A}$ is superatomic and so $C(K_\mathcal{A})$ is a scattered C*-algebra. By Proposition 3.3 there is an irredundant subset $Y$ of $C(K_\mathcal{A})$ of the same cardinality as $X$ and consisting of projections in $C(K_\mathcal{A})$. By removing at most one element of $Y$, by the second part of Lemma 3.9 and Lemma 3.1 it is a Boolean irredundant set in the Boolean algebra $\{\chi_U : U \in \text{Clop}(K_\mathcal{A})\}$ and this yields a Boolean irredundant set in $\mathcal{A}$. □

The above positively answers Question 3.10 (3) of [12] in the case of a scattered space.

Corollary 3.11. Suppose that $K$ is an infinite Hausdorff compact space. Then $\text{irr}^*(C_0(K)) = \text{irr}(C(K))$ where $\text{irr}^*(C_0(K))$ is the supremum over the cardinalities of sets $X$ of real-valued continuous functions on $K$ such that no $f \in X$ belongs to the real unital Banach algebra generated by $X \setminus \{f\}$. In particular the $\pi$-weight of $K$ is bounded by $\text{irr}(C(K))$ and the density of $C(K)$ is bounded by $2^{2^{\text{irr}(C(K))}}$.

Proof. Let $X \subseteq C(K)$ be an infinite irredundant set. By Lemma 3.2 we may assume that it consists of real-valued (non-negative) functions. As in the proof of Lemma 3.1, by removing at most one element we may assume that no $f \in X$ belongs to the real unital C*-algebra generated by $X \setminus \{f\}$. So $\text{irr}^*(C_0(K)) \geq \text{irr}(C(K))$.

Now given a set $X$ as in the lemma, by the real unital Weierstrass-Stone theorem there are pairs of points $x_f, y_f \in K$ such that $f(x_f) \neq f(y_f)$ but $g(x_f) = g(y_f)$ for any $g \in X \setminus \{f\}$ (cf. [30]). hence $X$ is an irredundant set in the C*-algebra $C(K)$ by Lemma 3.4. The last part of the corollary follows from Theorem 10 of [24] where $\pi(K) \leq \text{irr}^*(C_0(K))$ is proved and from the fact that the weight of a regular space is bounded by the exponent of its $\pi$-weight (Theorem 3.3. of [21]). □

3.3. Irredundant sets in general C*-algebras. Having developed the motivations in the previous section now we move to the irredundant sets in general, possibly noncommutative C*-algebras.

Proposition 3.12. Every infinite pairwise orthogonal collection of self-adjoint elements in a C*-algebra is irredundant. In particular, every infinite dimensional C*-algebra contains an infinite irredundant set.

Proof. This follows from the fact that given a self-adjoint element $A$ of a C*-algebra $\mathcal{A}$ the set $\{B \in \mathcal{A} : AB = BA = 0\}$ is a C*-subalgebra of $\mathcal{A}$. □

Proposition 3.13. Suppose that an infinite dimensional C*-algebra $\mathcal{A}$ is a von Neumann algebra. Then $\mathcal{A}$ has an irredundant set of cardinality continuum.

Proof. An infinite dimensional von Neumann algebra has an infinite pairwise orthogonal collection of projections and so it contains the commutative C*-algebra $\ell_\infty$ which is $\pi$-isomorphic to $C(\mathbb{B})$. The Boolean algebra $\phi(\mathbb{N})$ is isomorphic to $\{\chi_U : U \in \text{Clop}(\mathbb{B})\}$ and so the Boolean irredundance of $\phi(\mathbb{N})$ is equal to the
irredundance of $\ell_\infty$ by Lemma 3.10. By considering an almost disjoint family (or an independent family) of cardinality continuum of subsets of $\mathbb{N}$ we obtain an irredundant set of cardinality continuum.

The following Proposition corresponds to Remark 3.6.

**Lemma 3.14.** Suppose that $\mathcal{A}$ is a C*-algebra, $\kappa$ a cardinal, $\{A_\xi : \xi < \kappa\} \subseteq \mathcal{A}$, and $\{\tau_\alpha : \alpha < \kappa\}$ a family of states such that

- $\tau_\alpha(A_\alpha) > 0$
- $\tau_\alpha(A_\xi) = 0$ for $\xi \neq \alpha$.

Then $\{A_\xi : \xi < \kappa\}$ is irredundant.

**Proof.** As in the GNS construction one proves that $L_\alpha = \{A \in \mathcal{A} : \tau_\alpha(A^*A) = 0\}$ is a left-ideal in $\mathcal{A}$, and in particular a C*-subalgebra. So $X_\xi \in L_\alpha$, where $A_\xi = X_\xi^*X_\xi$ and so $A_\xi \in L_\alpha$ for all $\xi \neq \alpha$. However by Theorem 3.3.2. of [39] we have

$$0 < \tau_\alpha(A_\alpha) \leq \|\tau_\alpha\|\tau_\alpha(A_\alpha^*A_\alpha),$$

so $A_\alpha \notin L_\alpha$. □

In the noncommutative case, for pure states $\{\tau_\alpha : \alpha < \kappa\}$ being discrete in the weak* topology does not yield in general the existence of positive elements $A_\alpha$ like in the lemma above, as the noncommutative Urysohn lemmas require extra hypotheses ([2]).

The following proposition is a version of the commutative characterizations in Lemmas 3.4 and 3.5. It could be interesting to remark that a version of the following proposition where "representations" are replaced by "irreducible representations" implies the noncommutative Stone-Weierstrass theorem, which remains a well-known open problem. One should note that below one of the possibilities of the representation is to be constantly zero.

**Proposition 3.15** ([22]). Suppose that $\mathcal{A}$ is a C*-algebra and $X \subseteq \mathcal{A}$ is an irredundant set. Then for all $a \in X$ there are Hilbert spaces $H_a$ and representations $\pi_1^a, \pi_2^a : \mathcal{A} \to \mathcal{B}(H_a)$ such that $\pi_1^a(a) \neq \pi_2^a(a)$ but

$$X \setminus \{a\} \subseteq \{b \in \mathcal{A} : \pi_1^a(b) = \pi_2^a(b)\}.$$

### 3.4. Irredundance in scattered C*-algebras

The following proposition shows that thin-tall algebras play a special role in the context of uncountable irredundant sets.

**Proposition 3.16.** If there is a nonseparable scattered C*-algebra with no uncountable irredundant set, then it contains a thin-tall scattered C*-algebra.

**Proof.** First note that by the characterization of subalgebras of the algebra of compact operators ([5]) a C*-algebra which is isomorphic to a subalgebra of the algebra of all compact operators on a Hilbert space $\mathcal{H}$ but not isomorphic to a subalgebra of the algebra of all compact operators on the separable Hilbert space $\mathcal{H}$ must contain an uncountable pairwise orthogonal set which is irredundant by 3.12. So if a scattered $\mathcal{A}$ has no uncountable irredundant set, then all the algebras $\mathcal{L}^M(\mathcal{A}/\mathcal{I}_\alpha)$ are *-isomorphic to a subalgebra of the algebra of all compact operators on the separable or finite dimensional Hilbert space, but as $\mathcal{A}$ is nonseparable $ht(\mathcal{A}) \geq \omega_1$ and so $\mathcal{I}_{\omega_1}$ is the required thin-tall subalgebra of $\mathcal{A}$. □
**Corollary 3.17.** Assume PFA. Suppose that $A$ is a nonseparable scattered C*-algebra. Then there is an uncountable weak* discrete set of pure states of $A$.

**Proof.** First suppose that $A$ has a quotient that contains an uncountable orthogonal set of projections. Then it is clear that we can find pure states which form a weak* discrete set. Otherwise using the argument as in the proof of Proposition 3.16 we may assume that $A$ is thin-tall. By Proposition 2.6 $A$ has an uncountable weak* discrete set of pure states. \qed

Below we prove a simple noncommutative version of a Theorem of McKenzie (see 4.2.3 of [28]):

**Theorem 3.18.** If $A$ is a scattered C*-algebra, then

$$d(A) \leq 2^{irr(A)}.$$  

**Proof.** Let $\kappa$ be the minimal cardinal such that $I_{At}(A)$ is a subalgebra of the algebra of all compact operators on $\ell_2(\kappa)$. By the characterization of subalgebras of the algebra of compact operators ([5]) $A$ must contain pairwise orthogonal set of cardinality $\kappa$ which is irredundant by 3.12. So $\kappa \leq irr(A)$. By the essentiality of $I_{At}(A)$ which follows from Proposition 2.5 we can embed $A$ into $B(\ell_2(\kappa))$, so $d(A) \leq 2^{\kappa} \leq 2^{irr(A)}$ as required. \qed

### 3.5. Extracting irredundant sets from a given collection of operators.

**Proposition 3.19.** There is a collection of operators $(A_\xi : \xi < \omega_1)$ in $B(\ell_2)$ which generates a nonseparable C*-subalgebra of $B(\ell_2)$ with no two-element irredundant subset. Any fully noncommutative thin-tall C*-algebra is generated by such a sequence.

**Proof.** Construct a fully noncommutative thin-tall C*-algebra $A$ as in Theorem 7.6. of [18], in particular with Cantor-Bendixson decomposition $(T_{At}^A(\mathcal{A}))_{\alpha<\omega_1}$ (see 2.2 (2)), where $T_{At}^A(\mathcal{A})$ is *-isomorphic to $T_{At}^A(\mathcal{A}) \otimes K(\ell_2)$.

By Theorem 8 of [42] any C*-algebra of the form $B \otimes K(\ell_2)$ is singly generated if $B$ is separable and unital. So for each $\alpha < \omega_1$ pick $A_\alpha$ to be a single generator of $T_{At}^A(\mathcal{A})$.

An alternative approach which gives the final statement of the Proposition is to use the fact that scattered C*-algebras are locally finite dimensional (see [16] for more on these notions in the nonseparable context) in the sense that each of its finite subsets can be approximated from a finite dimensional C*-subalgebra ([34], [35]). So $T_{At}^A(\mathcal{A})$ is locally finite dimensional and separable for each $\alpha < \omega_1$ and so AF. Thus the result of [48] implies that $T_{At}^A(\mathcal{A})$ is singly generated for each $\alpha < \omega_1$. So pick $A_{\alpha+1}$ as before. This completes the proof of the theorem. \qed

Using the free set lemmas like in [13] one can prove that given a discrete set of operators $(A_n)_{n<\omega_n}$ for $n \in \mathbb{N}$ there is an $n$-element irredundant set. However there is much stronger consistent extraction principle:

**Theorem 3.20.** It is relatively consistent that whenever $(A_\xi : \xi < 2^\omega)$ is a collection of operators in $B(\ell_2)$ which generates a C*-algebra of density continuum, then there is a set $I \subseteq 2^\omega$ of cardinality continuum such that $(A_\xi : \xi \in I)$ is irredundant.
Proof. To obtain the relative consistency we will use the method of forcing (see [33]). We start with the ground model \( V \) satisfying the generalized continuum hypothesis (GCH) and we will consider the generic extension \( V[G] \) where \( G \) is a generic set in the forcing \( \mathbb{P} = Fn(\omega_2, 2) \) for adding \( \omega_2 \) Cohen reals (see Chapter VIII §2 of [33]).

Fix a ground model orthonormal basis \( (e_n : n \in \mathbb{N}) \) for \( \ell_2 \) in \( V \). In \( V[G] \) let \( (A_\xi : \xi < 2^\omega) \) be as in the theorem. By passing to a subset of cardinality \( 2^\omega = \omega_2 \) and using the hypothesis that \( (A_\xi : \xi < 2^\omega) \) is a collection of operators in \( B(\ell_2) \) which generates a C*-algebra of density continuum, we may assume that \( A_\xi \) does not belong to the C*-algebra generated by the operators \( (A_\eta : \eta < \xi) \) for each \( \xi < \omega_1 \). Moreover by passing to a subsequence we may assume that there is a rational \( \varepsilon > 0 \) such that \( \|A - A_\xi\| > \varepsilon \) for every \( A \) in the C*-algebra generated by the operators \( (A_\eta : \eta < \xi) \) for each \( \xi < \omega_1 \).

Each \( A_\xi \) can be identified with an \( \mathbb{N} \times \mathbb{N} \) complex valued matrix \( ((A_\xi(e_n), e_m))_{m,n \in \mathbb{N}} \). Let \( A_\xi \) be \( \mathbb{P} \)-names in \( V \) for these matrices. Using the standard argument of nice names, the countable chain condition for \( \mathbb{P} \) and passing to a subsequence using the \( \Delta \)-system lemma for countable sets which follows from the GCH, we may assume that there are permutations \( \sigma_{\xi,\eta} : \omega_2 \to \omega_2 \) which lift to the automorphisms of \( \mathbb{P} \) and the permutations \( \sigma'_{\xi,\eta} \) of \( \mathbb{P} \)-names such that
\[
\sigma_{\xi,\eta}(A_\eta) = A_\xi,
\]
and for every \( \xi, \eta \in \omega_2 \) we have that
\[
\mathbb{P} \models \phi(\dot{x}_1, \ldots, \dot{x}_k) \text{ if and only if } \mathbb{P} \models \phi(\sigma'_{\xi,\eta}(\dot{x}_1), \ldots, \sigma'_{\xi,\eta}(\dot{x}_k))
\]
for any formula \( \phi \) in \( k \in \mathbb{N} \) free variables and any sequence \( \dot{x}_1, \ldots, \dot{x}_k \) of \( \mathbb{P} \)-names for \( k \in \mathbb{N} \) (7.13 [33]). Using this for the formulas which say that the distance of \( A_\xi \) from any element of the C*-algebra generated by the operators \( (A_\eta : \eta \in F) \) for any finite \( F \subseteq \xi \) is bigger than \( \varepsilon \), we conclude that \( \mathbb{P} \) forces that no \( A_\xi \) belongs the C*-algebra generated by any countable collection from \( \{A_\eta : \eta \neq \xi\} \) (by considering a permutation of \( \omega_2 \) which moves \( \xi \) above the countable set). This means that \( \mathbb{P} \) forces that no \( A_\xi \) belongs the C*-algebra generated by the remaining operators \( \{A_\eta : \eta \neq \xi\} \), i.e., that the collection is irredundant as required.

The above is a version of applying a standard argument as in [32] in the context of Boolean irredundance.

4. Commutators under OCA

The main consistent construction of this paper presented in the following sections has a strong randomness properties. In this section we show that this randomness does not take place for any uncountable collection of operators in \( B(\ell_2) \) under the assumption of Open Coloring Axiom, OCA. We will follow the approach to the strong operator topology from the book [11] of Davidson. Thus we have:

**Definition 4.1.** Let \( H \) be a Hilbert space. The strong operator topology (SOT) on \( B(H) \) is defined as the weakest topology such that the sets
\[
S(a, x) := \{b \in B(H) : \|(b - a)(x)\| < 1\}
\]
are open for each \( a \in \mathcal{B}(H) \) and \( x \in H \). We denote by \((\mathcal{B}(H), \tau_{sot})\) and \((\mathcal{B}(H)_1, \tau_{sot})\) respectively the space \( \mathcal{B}(H) \) and the unit ball of \( \mathcal{B}(H) \) with the strong operator topology.

**Proposition 4.2.** If \( H \) is a separable Hilbert space, then \((\mathcal{B}(H)_1, \tau_{sot})\) is metrizable and separable in the strong operator topology.

**Proof.** For metrizability see [11], Proposition I.6.3. For the separability fix some orthonormal basis \((e_n)_{n \in \mathbb{N}}\) and consider finite rank operators in the linear span of one dimensional operators of the form \( v \otimes w \) where \( v, w \) have finitely many nonzero rational coordinates with respect to \((e_n)_{n \in \mathbb{N}}\). It is clear that such operators are SOT dense in \( \mathcal{B}(l_2) \), as required. \( \square \)

By the remarks on page 16 and 17 of [11] we have the following:

**Lemma 4.3.** The multiplication on \( (\mathcal{B}(H)_1)_1 \) is jointly continuous in the SOT topology and so every polynomial\(^{6}\) is SOT continuous on \((\mathcal{B}(H)_1)_1 \).

We will follow the approach to the Open Coloring Axiom (OCA) from [15], page 55. Its weaker version was discovered by Abraham, Rubin and Shelah ([1]) and the final form was introduced by Todorcevic in [51]. It is consistent with ZFC. In fact, it is a consequence of the Proper Forcing Axiom (PFA). See Theorem 8 of [51]. Recall that

\[
[X]^2 = \{ \{x, y\} : x \neq y \}.
\]

It is well known that the original form of OCA from [51] for subsets of the reals is equivalent to the version for separable metric spaces as in [15]:

**Definition 4.4 (Todorcevic [51]).** OCA denotes de following statement: If \( X \) is a separable metric Hausdorff space and \([X]^2 = K_0 \cup K_1 \) is a partition with \( K_0 \) open\(^\prime\), then either there is an uncountable \( Y \subseteq X \) such that \([Y]^2 \subseteq K_0 \), or else \( X = \bigcup_{n \in \mathbb{N}} X_n \), where \([X_n]^2 \subseteq K_1 \) for each \( n \in \mathbb{N} \).

**Theorem 4.5 (OCA).** Let \((A_n)_{\alpha < \omega_1}\) be an uncountable family in \( \mathcal{B}(l_2) \) and \( P(x, y) \) be a polynomial satisfying \( \|P(A, B)\| = \|P(B, A)\| \) for all \( A, B \in \mathcal{B}(l_2) \). Then given \( \varepsilon > 0 \), either there is an uncountable \( \Gamma_0 \subseteq \omega_1 \) such that \( \|P(A_\alpha, A_\beta)\| \leq \varepsilon \) for every distinct \( \alpha, \beta \in \Gamma_0 \) or else there is an uncountable \( \Gamma_1 \subseteq \omega_1 \) such that \( \|P(A_\alpha, A_\beta)\| > \varepsilon \) for every distinct \( \alpha, \beta \in \Gamma_1 \).

**Proof.** By passing to an uncountable subset, we may assume that there is \( M > 0 \) such that \( \|A_\alpha\| \leq M \) for all \( \alpha < \omega_1 \). Let \( X = \{ A_\alpha : \alpha < \omega_1 \} \subseteq MB(l_2)_1 \) and note that \( MB(l_2)_1 \) is metric and separable by Proposition 4.2. Define

\[ K_0 = \{ \{A, B\} \in [X]^2 : \|P(A, B)\| > \varepsilon \} \]

and \( K_1 = [X]^2 \setminus K_0 \).

First note that the separability is hereditary for metric spaces, so \( X \) is metric separable as a subspace of \((MB(l_2)_1, \tau_{sot})\).

Now note that \( K_0 \) is open. Indeed if \( \|P(A, B)\| > \varepsilon \), then there is \( x \in l_2 \) of norm one and \( \delta > 0 \) such that \( \|P(A, B)(x)\| > \varepsilon + \delta \). Now if \( P(A', B') \in S(P(A, B), x/\delta) \), we have \( \|P(A', B')(x) - P(A, B)(x)\| < \delta \) and so \( \|P(A', B')\| > \varepsilon \).

\(^6\)By a polynomial \( P(x, y) \) we mean an expression in the form \( P(x, y) = \sum a_i x^i + \sum b_i y^i + \sum c_{i,j} x^i y^j + \sum d_{i,j} x^i y^j + e_0 \).

\(^7\)We call \( K_0 \subseteq [X]^2 \) open if the symmetric set \( \{ (x, y) \in X \times X : (x, y) \in K_0 \} \) is open in \( K \times K \setminus \Delta \) in the product topology, where \( \Delta \) denotes the diagonal of \( X \times X \).
Hence \( \{ A', B' \} \in [X]^2 : P(A', B') \in P(A(B), x/\delta) \} \subseteq K_0 \). But \((A, B) \in P^{-1}[S(P(A, B), x/\delta)]\) is open in \(X \times X\) with the product SOT topology by the continuity of \(P\) (Lemma 4.3).

So we are in the position of applying the OCA. From 4.4 we obtain the required uncountable set \( \Gamma_0 \) or \( \Gamma_1 \).

\[\square\]

**Corollary 4.6 (OCA).** Let \((A_{\alpha})_{\alpha<\omega_1}\) be an uncountable family in \(B(l_2)\). Then given \(\varepsilon > 0\), either there is an uncountable \(\Gamma_0 \subseteq \omega_1\) such that \(||A_{\alpha}, A_{\beta}|| \leq \varepsilon\) for every \(\alpha, \beta \in \Gamma_0\) or else there is an uncountable \(\Gamma_1 \subseteq \omega_1\) such that \(||A_{\alpha}, A_{\beta}|| > \varepsilon\) for every \(\alpha, \beta \in \Gamma_1\).

**Proof.** Consider \(P(x, y) = xy - yx\) and apply Theorem 4.5. \(\square\)

**Remark 4.7.** Let us remark on two trivial versions of the above results. First let \((A_{\alpha})_{\alpha<\omega_1}\) be an infinite family in a separable 

\[\text{C*-algebra of } B(l_2)\]. Then given \(\varepsilon > 0\), either there is an infinite \(\Gamma_0 \subseteq \mathbb{N}\) such that \(||A_{\alpha}, A_{m}|| \leq \varepsilon\) for every \(n, m \in \Gamma_0\) or else there is an infinite \(\Gamma_1 \subseteq \mathbb{N}\) such that \(||A_{\alpha}, A_{m}|| > \varepsilon\) for every \(n, m \in \Gamma_1\). This follows from the Ramsey theorem whose consistent generalization is the OCA.

Secondly note that if \((A_{\alpha})_{\alpha<\omega_1}\) is an uncountable family in a separable 

\[\text{C*-subalgebra of } B(l_2)\] then by its second countability in the norm topology it follows that for every \(\delta > 0\) there is an uncountable \(\Gamma_0 \subseteq \omega_1\) such that \(||A_{\alpha} - A_{\beta}|| < \delta\) for every \(\alpha, \beta \in \Gamma_0\) and so given any polynomial \(P\) satisfying \(P(x, x) = 0\) and \(\varepsilon > 0\), by the norm continuity of \(P\) there is an uncountable \(\Gamma_0 \subseteq \omega_1\) such that \(||P(A_{\alpha}, P_{\beta})|| < \varepsilon\) for every \(\alpha, \beta \in \Gamma_0\).

In fact, in the nontrivial cases of Theorem 4.5 and Corollary 4.6 when \((A_{\alpha})_{\alpha<\omega_1}\) generates a nonseparable 

\[\text{C*-subalgebra of } B(l_2)\] we may assume that \((A_{\alpha})_{\alpha<\omega_1}\) forms a norm discrete set.

### 5. The partial order of finite dimensional approximations

#### 5.1. Notation

The C*-algebras that we consider in the rest of this paper are subalgebras of \(B(\ell_2(\omega_1 \times \mathbb{N}))\). In fact, the subspaces \(\ell_2(\{\xi\} \times \mathbb{N})\) of \(\ell_2(\omega_1 \times \mathbb{N})\), which we call columns will be invariant for all our algebras, so our algebras could be identified with subalgebras of \(\Pi_{\xi<\omega_1} B(\ell_2(\{\xi\} \times \mathbb{N}))\). Also the map \(\pi_\alpha : \Pi_{\xi<\omega_1} \rightarrow \Pi_{\alpha<\omega_1} \rightarrow \ell_2(\{\alpha\} \times \mathbb{N})\) applied to the appropriate quotients, will be faithful (see Lemma 5.24 (3)). Thus the purpose of this presentation of the algebras is related to the transparent structure of the Cantor-Bendixson composition series (see Proposition 5.25 (3)).

For \(X \subseteq \omega_1 \times \mathbb{N}\), we introduce the following notation:

- \(\{e_{\xi,n} : \xi < \omega_1, n \in \mathbb{N}\}\) is the canonical orthonormal basis of \(\ell_2(\omega_1 \times \mathbb{N})\),
- the family of all operators \(A\) in \(B(\ell_2(\omega_1 \times \mathbb{N}))\) such that \(- \ell_2(X \cap (\{\xi\} \times \mathbb{N}))\) is \(A\)-invariant for all \(\xi < \omega_1\), \(- A(e_{\xi,n}) = 0\) whenever \((\xi, n) \not\in X\),
- the unit of the C*-algebra \(B_X\) will be denoted by \(P_X\),
- \(1_{\xi,m,n}\) is the operator in \(B_{\omega_1 \times \mathbb{N}}\) satisfying

\[1_{\xi,m,n}(e_{\eta,k}) = \begin{cases} e_{\xi,m} & \text{if } k = n, \xi = \eta \\ 0 & \text{otherwise} \end{cases} \]
• if $A \in \mathcal{B}_{\omega_1 \times \mathbb{N}}$ we define $A|X = AP_X$,
• if $A \in \mathcal{B}_{\omega_1 \times \mathbb{N}}$ and $a \subseteq \omega_1$ we define $A|a$ as $A(a \times \mathbb{N})$,
• $A|X = \{A|X : A \in A\}$ for $A \subseteq \mathcal{B}_{\omega_1 \times \mathbb{N}}$ and $X \subseteq \omega_1 \times \mathbb{N}$.

5.2. The definition of the partial order of finite-dimensional approximations.

**Definition 5.1.** We define a partial order $\mathbb{P}$ consisting of elements $p = (a_p, \{n^p_\xi : \xi \in a_p\}, \{A^p_{\xi,m,n} : \xi \in a_p, n,m \in [0,n^p_\xi]\})$, where
\begin{enumerate}
  \item $a_p$ is a finite subset of $\omega_1$,
  \item $n^p_\xi \in \mathbb{N}$ for each $\xi \in a_p$,
  \item $A^p_{\xi,m,n} \in \mathcal{B}_{X_p}$ for each $\xi \in a_p$ and $n,m \in [0,n^p_\xi]$, where $X_p = \{(\xi,n) : \xi \in a_p; n \in [0,n^p_\xi]\}$, the order $\leq$ on $\mathbb{P}$ is defined by declaring $p \leq q$ if and only if:
    \begin{enumerate}
      \item $a_q \supseteq a_p$,
      \item $n^q_\xi \geq n^p_\xi$ for $\xi \in a_q$,
      \item there is a (nonunital) $^*$-embedding $i_{pq} : \mathcal{B}_{X_q} \rightarrow \mathcal{B}_{X_p}$ such that $i_{pq}(A^p_{\xi,m,n}) = A^q_{\xi,m,n}$ for all $\xi \in a_p$ and $m,n \in [0,n^p_\xi]$,
      \item $i_{pq}(A)|X_q = A$ for all $A \in \mathcal{B}_{X_q}$.
    \end{enumerate}
\end{enumerate}

**Lemma 5.3.** For every $\alpha \in \omega_1$ and every $p \in \mathbb{P}$ we have $A^p_{X_p \cap (\alpha \times \mathbb{N})} = \mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})}$. In particular $A^p_{X_p} = \mathcal{B}_{X_p}$.

Proof. We will prove it by induction on $|a_p \cap \alpha|$. If $a_p \cap \alpha = \emptyset$, then both of the algebras are $\{0\}$. Suppose $|a_p \cap \alpha| = n + 1$ and we have proved the Lemma for every $q \in \mathbb{P}$ and $\alpha < \omega_1$ such that $|a_q \cap \alpha| = n$. Let $\xi = \max (a_p \cap \alpha)$. By the definition of $X_p$, we have that $\mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})}$ is $^*$-isomorphic to $\mathcal{B}_{X_p \cap (\xi \times \mathbb{N})} \otimes \mathcal{B}_{\xi \times [0,n^p_\xi]}$. By the inductive hypothesis, $\mathcal{B}_{X_p \cap (\xi \times \mathbb{N})}$ is generated by $\{A^p_{\eta,m,n} : \eta \in a_p \cap \xi; m,n \in [0,n^p_\xi]\}$. But by (4) in Definition 5.1, we have that $1_{\xi,m,n} = A^p_{\xi,m,n} - A$ for some $A \in \mathcal{B}_{X_p \cap (\xi \times \mathbb{N})}$ and all $m,n \in [0,n^p_\xi]$. In particular, $\mathcal{B}_{\xi \times [0,n^p_\xi]}$ is included in the algebra generated by $\{A^p_{\eta,m,n} : \eta \in a_p \cap \xi; m,n \in [0,n^p_\xi]\}$. This together with the inductive hypothesis completes the proof. \hfill $\square$

**Lemma 5.4.** Suppose that $\alpha < \omega_1$ and $p,q \in \mathbb{P}$ satisfy $p \leq q$ and $A = i_{p,q}(B)$, where $B \in A^p_{X_q}$. Then
\[ \|A|_{[\alpha,\omega_1]}\| = \|B|_{[\alpha,\omega_1]}\|.
\]

Proof. Since $B|\alpha$ and $B|_{[\alpha,\omega_1]}$ are in $A^p_{X_q}$ by Lemma 5.3, we have
\[ \|A|_{[\alpha,\omega_1]}\| = \|i_{p,q}(B)|_{[\alpha,\omega_1]}\| = \|i_{p,q}(B)|_{[\alpha,\omega_1]} + i_{p,q}(B|[\alpha,\omega_1])|_{[\alpha,\omega_1]}\|.\]
But $B|\alpha \in A^q_{\alpha,(\alpha \times N)}$ by Lemma 5.3, and this generation must be preserved by the isomorphism $i_{p,q}$, i.e., $i_{p,q}(B(\alpha))|_{\alpha,\omega_1} = 0$ and so
\[ \|A|_{\alpha,\omega_1}\| = \|i_{p,q}(B|_{\alpha,\omega_1})|_{\alpha,\omega_1}\| \leq \|i_{p,q}(B)|_{\alpha,\omega_1}\|. \]
Since $i_{p,q}$ is an embedding (in particular an isometry), we conclude that
\[ \|A|_{\alpha,\omega_1}\| \leq \|B|_{\alpha,\omega_1}\|. \]
The other inequality follows from Definition 5.1 (c-d).

5.3. Density Lemmas. In the terminology related to partial orders occurring in the theory of forcing a subset $D$ of a partial order $Q$ is said to be dense if for every $p \in Q$ there is $d \in D$ satisfying $d \leq p$. In what follows we usually need stronger information for $Q = P$ namely that $a_d = a_p$.

**Lemma 5.5.** Suppose that $\xi < \omega_1$. Then
\[ D_\xi = \{ p \in P : \xi \in a_p \} \]
is a dense subset of $P$.

**Proof.** Let $q \in P$ be such that $\xi \notin a_q$. Define $p$ as follows:
- $a_p = a_q \cup \{ \xi \}$,
- $n^p_\eta = n^q_\eta$ for $\eta \in a_q$ and $n^p_\xi = 1$,
- $A^p_{\eta,m,n} = A^q_{\eta,m,n}$ for $\eta \in a_q$ and $A^p_{\xi,0,0} = 1_{\xi,0,0}$.

It is clear that $p \in P$. Also $p \leq q$ as $Id_{B_{X_q}} : B_{X_q} \to B_{X_p}$ is a *-embedding good for $i_{p,q}$ in Definition 5.1 (c).

**Lemma 5.6.** Suppose that $\xi < \omega_1$; $k \in N$ and $q \in P$ is such that $\xi \in a_q$. Then there is
\[ p \in E_{\xi,k} = \{ p \in P : \xi \in a_p, n^p_\xi \geq k \} \]
such that $p \leq q$ and $a_p = a_q$.

**Proof.** Consider $q \in P$ such that $\xi \in a_q$ but $n^q_\xi < k$. Define $p$ as follows:
- $a_p = a_q$,
- $n^p_\eta = n^q_\eta$ for $\eta \in a_q \setminus \{ \xi \}$ and $n^p_\xi = k$,
- $A^p_{\eta,m,n} = A^q_{\eta,m,n}$ for $\eta \in a_q \setminus \{ \xi \}$,
- $A^p_{\xi,m,n} = A^q_{\xi,m,n}$ for $n, m \in [0, n^q_m)$,
- $A^p_{\xi,m,n} = 1_{\xi,m,n}$ if $n, m \in [0, k)$ and $\{ n, m \} \cap \{ n^q_m, k \} \neq \emptyset$.

It is clear that $p \in P \cap E_{\xi,k}$. Also $p \leq q$ as $Id_{B_{X_q}} : B_{X_q} \to B_{X_p}$ is a *-embedding good for $i_{p,q}$ in Definition 5.1 (c).

**Lemma 5.7.** Suppose that $q \in P$ and $X \subseteq X_q$ and that $\alpha \in a_q$. Then there is $p \leq q$ such that $p \in P_{X,\alpha}$, where
\[ F_{X,\alpha} = \{ p \in P : \alpha \in a_p, X \subseteq X_p, \forall A \in A^p_X \ | A|\{\alpha}\| \geq |A|_{\alpha,\omega_1} \}. \]
Moreover, $a_p = a_q$ and $n^p_\xi = n^q_\xi$ whenever $\xi \in a_p \setminus \{ \alpha \}$.
Proof. Let \( q \in \mathcal{P} \). We may assume that \( X = X_q \). If \( \alpha = \max(a_q) \), then there is nothing to prove. So let \( a_q \setminus (\alpha + 1) = \{\xi_1, \ldots, \xi_k\} \) for some \( k \in \mathbb{N} \) and put
\[
l = \sum \{n^q_i : 1 \leq i \leq k\}.
\]
Consider \( Y = X_q \cap ((\alpha, \omega_1) \times \mathbb{N}) \). Let \( \phi : Y \to [n^q_0, n^q_0 + l) \) be any bijection. We obtain a \(*\)-homomorphism \( i : B_{X_q} \to B_{X_q, i}(\alpha) \times [n^q_0, n^q_0 + l) \) given by \( i(A) = A + i(A) \) where \( i : B_{X_q} \to B_{(\alpha) \times [n^q_0, n^q_0 + l)} \) satisfies
\[
(i_{i}(A)(e_{\alpha, n^q_0 + \phi(\xi, k)}), e_{\alpha, n^q_0 + \phi(\xi, k) + 1}) = (A(e_{\xi, k}), e_{\xi, k + 1})
\]
for all \( (\xi, k) \in Y \) and every \( A \in B_{X_q} \). Define \( p \) in the following way
\[
\begin{align*}
& a_p = a_q, \\
& n^p_\xi = n^q_\xi \text{ if } \xi \in a_p \setminus \{\alpha\} \text{ and } n^p_\alpha = n^q_\alpha + l, \\
& A^p_\xi, m, n \equiv i(A^q_\xi, m, n) \text{ for } (\xi, m, (\xi, n)) \in X_q. \\
& A^p_{\alpha, m, n} = 1_{a, m, n} \text{ if } \{m, n\} \cap [n^p_\alpha, [n^p_\alpha + n)) \neq \emptyset.
\end{align*}
\]
It is clear from the construction that \( p \in \mathcal{P} \) as condition (4) of Definition 5.1 is satisfied due to the fact that we change only \( A^q_{\xi, m, n} \) for \( \xi > \alpha \) on \( \{\alpha\} \times \mathbb{N} \), and that (a), (b) of Definition 5.1 are satisfied.

If we put \( i_{p, q} = i \), condition (c) follows from the fact that \( i \) is a \(*\)-embedding since \( \{\alpha\} \times [n^q_0, n^q_0 + l) \cap X_q = \emptyset \). We also have \( i_{p, q}(A^q_{\xi, m, n}) = A^p_{\xi, m, n} \) for \( (\xi, m, (\xi, n)) \in X_q \). The construction yields (d) of Definition 5.1.

Finally to check the main assertion of the lemma note that by Lemma 5.4 for any \( A \in B_{X_q} \) we have
\[
|\{i_{p, q}(A)\}| = \max(|\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|) = \max(|\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|) = \max(|\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|) = |\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|, |\{i_{p, q}(A)\}|)
\]
for any \( A \in B_{X_q} \) as required since \( X \subseteq X_q \).

\[\Box\]

Lemma 5.8. Let \( X \subseteq \omega_1 \times \mathbb{N} \) be finite and \( \alpha \in X \). If \( q \in F_{X, \alpha} \) and \( p \leq q \), then \( p \in F_{X, \alpha} \).

Proof. Let \( A \in A^q_X \). As \( X \subseteq X_q \) we have that \( A = i_{p, q}(B) \) for some \( B \in A^q_X \subseteq A^q_{X_q} \). First note that by Lemma 5.4
\[
|A|, |\alpha, \omega_1| = |B|, |\alpha, \omega_1|.
\]
Now \(|B|, |\alpha, \omega_1| \leq |B|, |\alpha, \omega_1| \) by the hypothesis that \( q \in F_{X, \alpha} \). But \( |B|, |\alpha, \omega_1| \) \leq |\alpha, \omega_1| by the fact that \( A[X^\omega] = B \) by Definition 5.1 (d). So \(|A|, |\alpha, \omega_1| \leq |A|, |\alpha, \omega_1| \) as required.

\[\Box\]

5.4. Basic amalgamations.

Definition 5.9. We say that two elements \( p, q \in \mathcal{P} \) are in the convenient position (as witnessed by \( \sigma : a_p \to a_q \)) if and only if
\[
\Delta := a_p \setminus a_q < a_p \setminus \Delta < a_q \setminus \Delta
\]
and there is an order preserving bijection \( \sigma : a_p \to a_q \) such that
\[
\begin{align*}
& n^p_\xi = n^q_{\sigma(\xi)} \text{ for } \xi \in a_p,
\end{align*}
\]
and the $*$-isomorphism of $B_{X_\xi}$ onto $B_{X_\eta}$ induced by $\sigma$, denoted by $j_\sigma$, which is given by
\[
(j_\sigma(A)(e_{\xi,k}), e_{\xi,l}) = \langle A(e_{\sigma(\xi),k}), e_{\sigma(\xi),l}\rangle
\]
for every $(\xi, k), (\xi, l) \in X_p$ and $A \in B_{X_\xi}$ satisfies
\begin{itemize}
  \item $j_\sigma(A^\eta_{\xi,n,m}) = A^\eta_{\xi,n,m}$ for every $\xi \in a_p$, $n, m \in [0, n^\eta_\xi)$.
\end{itemize}

Lemma 5.10. Suppose that two elements $p, q \in \mathbb{P}$ are in the convenient position as witnessed by $\sigma : a_p \to a_q$ and that $\xi \in \Delta = a_p \cap a_q$. Then $A^q_{\xi,n,m} = A^p_{\xi,n,m}$ for every $n, m \in [0, n^p_\xi)$. Proof. Note that in Definition 5.9 the bijection $\sigma$ must be the identity on $\Delta$ because it is order-preserving and $\Delta$ is the initial fragment of both $a_p$ and $a_q$ and so any $\xi \in \Delta$ must have the same position in both $a_p$ and $a_q$. So $j_\sigma(A^q_{\xi,n,m}) = A^p_{\xi,n,m}$ and it is enough to prove that $j_\sigma(A^q_{\xi,n,m}) = A^q_{\xi,n,m}$.

For $\eta \in a_p \setminus \Delta$ we have
\[
(j_\sigma(A^q_{\xi,n,m})(e_{\eta,k}), e_{\eta,l}) = \langle A^q_{\xi,n,m}(e_{\sigma(\eta),k}), e_{\sigma(\eta),l}\rangle = 0
\]
for every $k, l \in \mathbb{N}$ such that $(\eta, k), (\eta, l) \in X_p$ as $\sigma(\eta) \in a_q \setminus \Delta$.

On the other hand for $\eta \in \Delta$ we have $\sigma(\eta) = \eta$ and so
\[
(j_\sigma(A^q_{\xi,n,m})(e_{\eta,k}), e_{\eta,l}) = \langle A^q_{\xi,n,m}(e_{\eta,k}), e_{\eta,l}\rangle
\]
for every $k, l \in \mathbb{N}$ such that $(\eta, k), (\eta, l) \in X_p$ as $\sigma(\eta) = \eta$ by Definition 5.9. Using Definition 5.1 (4) this proves the required $A^q_{\xi,n,m} = j_\sigma(A^q_{\xi,n,m}) = A^p_{\xi,n,m}$.

Lemma 5.11. Suppose that $p, q \in \mathbb{P}$ are in the convenient position as witnessed by $\sigma : a_p \to a_q$. Then there is $r \leq p, q$ such that
\begin{itemize}
  \item $a_r = a_q \cup a_q$,
  \item $n^r_\xi = n^q_\eta$ if $\xi \in a_p$ and $n^r_\xi = n^q_\eta$ if $\xi \in a_q$,
  \item $i_{r,p} = Id_{B_{X_p}}, i_{r,q} = Id_{B_{X_q}}$.
\end{itemize}

In particular,
\begin{itemize}
  \item $A^r_{\xi,n,m} = A^p_{\xi,n,m}$ for each $\xi \in a_p$ and $n, m \in [0, n^p_\xi)$,
  \item $A^r_{\xi,n,m} = A^q_{\xi,n,m}$ for each $\xi \in a_q$ and $n, m \in [0, n^q_\xi)$.
\end{itemize}
The element $r$ will be called the disjoint amalgamation of $p$ and $q$.

Proof. Define $r$ as in the lemma. As $p, q \in \mathbb{P}$, it is easy to see that $r \in \mathbb{P}$. To see that $r \leq p, q$ note that $Id_{B_{X_p}}$ and $Id_{B_{X_q}}$ are $*$-embeddings into $B_{X_r}$.

Lemma 5.12. Suppose that $p, q$ are two elements of $\mathbb{P}$ in the convenient position as witnessed by $\sigma_{q,p} : a_p \to a_q$. Let $U \in B_{X_\xi}, X_\eta$ be a partial isometry satisfying $UU^* = U^*U = P_{X_\xi \setminus X_\eta}$, where $P_{X_\xi \setminus X_\eta}$ is the projection on the space spanned by $\{e_{\xi,k} : (\xi, k) \in X_p \setminus X_q\}$. Then there is $r_U = r \leq p, q$ such that
\begin{itemize}
  \item $a_r = a_q \cup a_q$,
  \item $n^r_\xi = n^q_\xi$ if $\xi \in a_p$, $n^r_\xi = n^q_\xi$ if $\xi \in a_q$,
  \item $i_{r,p} = Id_{B_{X_p}}$,
  \item $i_{r,q}(A) = A + U j_{\sigma_{q,p}}(A)U^*$ for all $A \in B_{X_q}$,
\end{itemize}
in particular,
\begin{itemize}
  \item $A^r_{\xi,n,m} = A^p_{\xi,n,m}$ for $\xi \in a_p$ and $n, m \in [0, n^p_\xi)$,
  \item $A^r_{\xi,n,m} = U A^p_{\sigma_{q,p}(\xi),n,m} U^* + A^q_{\xi,n,m}$ for $\xi \in a_q \setminus a_p$ and $n, m \in [0, n^q_\xi)$.
\end{itemize}
The element \( r_U \) will be called the \( U \)-including amalgamation of \( p \) and \( q \); if \( U = P_{X_q \setminus X_p} \), then \( r_U \) is called the including amalgamation.

**Proof.** Define \( r_U \) as in the lemma. It is clear by Definition 5.1 applied to \( p \) and \( q \) that \( r_U \in \mathbb{P} \). \( r_U \leq p \) because \( 1d_{X_q} : B_{X_q} \to B_{X_p} \) is a \(*\)-embedding. For \( r_U \leq q \) we note that \( A_{p,m,n}^q \mid X_q = A_{p,m,n}^q \) as \((UAP_{m,n}^q|_{\mathcal{L}})^*\mid X_q = 0 \) since \( UU^* = U^*U = P_{X_q} \setminus X_q \) and that the formula \( \tau_{r,q}(A) = A + Uj_{r,q}(A)U^* \) for all \( A \in B_{X_q} \) defines a \(*\)-embedding from \( B_{X_q} \) to \( B_{X_p} \). This is follows from the fact that sending \( A \) to \( Uj_{r,q}(A)U^* \) is a \(*\)-homomorphism since \( B_{X_q \setminus X_p} \) is \( A_{p}^q \)-invariant, so \( \tau_{r,q} \) is a \(*\)-homomorphism. But its kernel is null since \( Uj_{r,q}(A)U^* = (Uj_{r,q}(A)U^*)(X_p \setminus X_q) \) for all \( A \in B_{X_q} \).

**Lemma 5.13.** Suppose that \( v_1, v_2 \) are two orthogonal unit vectors of \( \mathbb{C}^n \) for \( n > 1 \). Then there is a unitary \( U \in M_n \) such that

\[
\| [UAU^*, A] \| = 1/2
\]

for every nonexpanding linear \( A \in M_n \) satisfying \( A(v_1) = v_1 \) and \( A(v_2) = 0 \).

**Proof.** Choose an orthonormal basis \( v_1, \ldots, v_n \) of \( \mathbb{C}^n \) starting with \( v_1, v_2 \) and consider the orthogonal projection \( P \in M_n \) onto the line containing \( v_1 \), so in particular we have \( P(v_1) = v_1 \) and \( P(v_2) = 0 \). Let \( U = V \oplus I_{n-2} \), \( U^* = V^* \oplus I_{n-2} \), where

\[
V = V^* = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

So we obtain that

\[
UPU^* = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix} \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix} \oplus 0_{n-2} = \begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{pmatrix} \oplus 0_{n-2}.
\]

Hence

\[
[UPU^*, P] = UPU^*P - PUU^* = \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix} \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix} \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{pmatrix} \oplus 0_{n-2} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

And so \( \| [UPU^*, P] \| = 1/2 \) and in particular

\[
[UPU^*, P](v_1) = (-1/2)v_2 \quad \text{and} \quad [UPU^*, P](v_2) = (1/2)v_1.
\]

Since \( P \) equals \( A \) on the space spanned by \( v_1 \) and \( v_2 \), and \( U, U^* \) leave this space invariant, we have the same equalities for \( A \) instead of \( P \), hence \( \| [UAU^*, A] \| \geq 1/2 \). The other inequality follows from the fact that \( \| [B,C] \| \leq 1/2 \) for any two \( B, C \) satisfying \( 0 \leq B, C \leq 1 \) by a result of Stampfli (Corollary 2 of [47]).

**Lemma 5.14.** Suppose that \( p, q \) are two elements of \( \mathbb{P} \) in the convenient position as witnessed by \( \sigma : a_p \to a_q \) such that \( \Delta < a_p \setminus \Delta < a_q \setminus \Delta \). Suppose that \( n^1 \) is \( n \geq 1 \) for every \( \xi \in a_q \setminus a_p \) and that \( v_1 = (v_1^1, \ldots, v_1^{n-1}) \), \( v_2 = (v_2^0, \ldots, v_2^n) \) are two orthogonal unit vectors of \( \mathbb{C}^n \). Then there is \( r \leq p, q \) such that

\[
\begin{align*}
ar_r &= a_q \cup a_q, \\
n^1_r &= n^1 \text{ if } \xi \in a_p, \quad n^2_r &= n^2 \text{ if } \xi \in a_q,
\end{align*}
\]

and
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• \[ \| [r_{q}(A), i_{rp}(j_{\sigma}(A))] \| = 1/2 \]
  for every nonexpanding \( A \in \mathcal{B}_{X_{q}} \) such that there is \( \xi \in a_{q} \setminus a_{p} \), with \( A(\sum_{k<n} v_{1}^{k} \epsilon_{\xi,k}) = \sum_{k<n} v_{1}^{k} \epsilon_{\xi,k} \) and \( A(\sum_{k<n} v_{2}^{k} \epsilon_{\xi,k}) = 0 \). We call \( r \) the \( (v_{1}, v_{2}) \)-anticommuting amalgamation of \( p \) and \( q \).

Proof. By Lemma 5.13 for each \( \xi \in a_{q} \setminus a_{p} \) for \( \eta_{k} = \sigma^{-1}(\xi) \) there is a unitary \( U_{k} \in \mathcal{B}_{(\eta_{k}) \times \{0, n\}} \) such that

\[ \| [U_{k}(j_{\sigma}(A))(\eta_{k})]U_{k}^{*}, j_{\sigma}(A)(\eta_{k}) \| = 1/2 \]

whenever \( A \in \mathcal{B}_{X_{q}} \) is nonexpanding such that

\[ A(\sum_{k<n} v_{1}^{k} \epsilon_{\xi,k}) = \sum_{k<n} v_{1}^{k} \epsilon_{\xi,k} \] and \( A(\sum_{k<n} v_{2}^{k} \epsilon_{\xi,k}) = 0 \).

Let \( U \in \mathcal{B}_{X_{q} \setminus X_{v}} \) be a partial isometry such that \( U([\{\eta_{k} \times \{0, n\}]) = U_{\xi} \) and \( U \) is zero on the columns not in in \( X_{p} \setminus X_{v} \), and \( U^* = P_{X_{q} \setminus X_{v}} \). Consider the \( U \)-including amalgamation \( r_{U} \leq p, q \) as in Lemma 5.12.

We claim that \( r = r_{U} \) satisfies the lemma we are proving. Let \( A \in \mathcal{B}_{X_{q}} \) be nonexpanding and \( \xi \in a_{q} \setminus a_{p} \) be such that \( A(\sum_{k<n} v_{1}^{k} \epsilon_{\xi,k}) = \sum_{k<n} v_{1}^{k} \epsilon_{\xi,k} \) and \( A(\sum_{k<n} v_{2}^{k} \epsilon_{\xi,k}) = 0 \). Since \( \ell_{2}([\xi] \times \{0, n\}) \) for \( \xi \in a_{q} \setminus a_{p} \) are invariant for \( \mathcal{B}_{X_{q}} \), the operator \( A(\xi) \) is nonexpanding as well and so is \( j_{\sigma}(A)(\eta_{k}) \). By Lemma 5.12 we have \( i_{r_{q}}(A) = A + U j_{\sigma}(A)U^{*} \) and \( i_{r_{p}}(j_{\sigma}(A)) = j_{\sigma}(A) \), so for \( \eta_{k} = \sigma^{-1}(\xi) \) we have

\[ [i_{r_{q}}(A), i_{r_{p}}(j_{\sigma}(A))](\{\eta_{k} \times \{0, n\})) = [U_{k}(j_{\sigma}(A))(\eta_{k})]U_{k}^{*}, j_{\sigma}(A)(\eta_{k}) \].

So by (*) we have \( \| [i_{r_{q}} (A), i_{r_{p}} (j_{\sigma}(A))] \| \geq 1/2 \). The other inequality follows from the maximality of \( 1/2 \) (Corollary 2 of [47]).

\[ \square \]

5.5. Types of 3-amalgamations.

Lemma 5.15. Suppose that \( p_{1}, p_{2}, p_{3} \) are distinct elements in \( \mathcal{P} \) which are pairwise in the convenient position as witnessed by \( \sigma_{j,i} : a_{p_{i}} \rightarrow a_{p_{j}} \) for \( 1 \leq i < j \leq 3 \) such that \( \Delta < a_{p_{i}} \setminus a_{p_{j}} \Delta < a_{p_{j}} \setminus a_{p_{i}} \Delta \). Then there is \( r \leq p_{1}, p_{2}, p_{3} \) satisfying

- \( a_{r} = a_{p_{1}} \cup a_{p_{2}} \cup a_{p_{3}} \);
- there is \( n \in \mathbb{N} \) such that for each \( \xi \in a_{r} \) we have \( n' > n \), \( n' = \max\{n_{k}^{\xi} : \xi \in a_{p_{i}}, 1 \leq i \leq 3\} \),

- \( r \in \bigcap\{F_{X,\alpha} : X \in \{X_{p_{1}}, X_{p_{2}}, X_{p_{3}}\}, \alpha \in X\} \).

The element \( r \) is called the amalgamation of \( p_{1}, p_{2}, p_{3} \) of type 1.

Proof. Let \( a_{p_{1}} = \{a_{1}, \ldots, a_{k}\} \) in the increasing order. Using Lemma 5.7 find \( p_{1} \geq p_{1} \geq \ldots \geq p_{k}^{\alpha} \) such that \( a_{p_{1}} = a_{p_{i}} \) and \( a_{p_{1}} \in F_{X_{p_{1}} \setminus a_{i}} \) for \( 1 \leq i \leq 1 \) \( a_{p_{i}} \). Now using Lemma 5.6 several times find \( q_{1} \leq p_{i}^{\alpha} \) such that \( a_{q_{1}} = a_{p_{1}} \) and \( n_{1}^{\alpha} = n > n' \) for every \( \xi \in a_{q_{1}} \).

Now find \( q_{2}, q_{3} \in \mathcal{P} \) such that \( q_{2} \leq p_{2} \) and \( q_{3} \leq p_{3} \) and "isomorphic" with \( q_{1} \) i.e., with \( a_{q_{2}} = a_{p_{2}}, a_{q_{3}} = a_{p_{3}} \) and where \( q_{1}, q_{2}, q_{3} \) are pairwise in the convenient position as witnessed by \( \sigma_{j,i} : a_{q_{i}} \rightarrow a_{q_{j}} \) for \( 1 \leq i < j \leq 3 \). Note that by Lemma 5.8 we have

\[ q_{2} \in \bigcap\{F_{X_{p_{1}} \setminus a_{i}} : \alpha \in X_{p_{1}}\} \]
Now let $s_1 \leq q_1, q_2$ and $s_2 \leq q_1, q_3$ be the disjoint amalgamations as in Lemma 5.11. Note that $s_1$ and $s_2$ are in the convenient position as witnessed by $I_{ap_1} \cup \sigma_{q_1,2} : a_{p_1} \cup a_{p_2} \to a_{p_1} \cup a_{p_3}$ where $a_{s_1} \cap a_{s_2} = a_{p_1}$. So now let $r \leq s_1, s_2$ be the disjoint amalgamation of $s_1$ and $s_2$ as in Lemma 5.11. Note that we have the final statement of the lemma by Lemma 5.8.

**Lemma 5.16.** Suppose that $p_1, p_2, p_3$ are distinct elements in $P$ which are pairwise in the convenient position as witnessed by $\sigma_{j,i} : a_{p_i} \to a_{p_j}$ for $1 \leq i < j \leq 3$ such that $\Delta < a_{p_i} \setminus \Delta < a_{p_3} \setminus \Delta$. Then there is $r \leq p_1, p_2, p_3$ satisfying

- $a_r = a_{p_1} \cup a_{p_2} \cup a_{p_3}$,
- $n^p_\xi = n^p_\xi$ if $\xi \in a_{p_i}$ for $1 \leq i \leq 3$,
- $i_{r,p_1} = Id_{a_{p_1}}$,
- $i_{r,p_2}(A) = A + j_{s_2,1}(A)(X_{p_1} \setminus X_{p_2})$ for all $A \in \mathcal{B}_{X_{p_2}}$,
- $i_{r,p_3}(A) = A + j_{s_3,1}(A)(X_{p_1} \setminus X_{p_3})$ for all $A \in \mathcal{B}_{X_{p_3}}$.

In particular

$$i_{r,p_2}(A)i_{r,p_3}(j_{s_2,3}(A)) = i_{r,p_1}(j_{s_3,1}(A))^2$$

for every $A \in \mathcal{B}_{X_{p_1}}$. The element $r$ is called the amalgamation of $p_1, p_2, p_3$ of type 2.

**Proof.** First consider $s_2 \leq p_1, p_2$ and $s_3 \leq p_1, p_3$ which are the including amalgamations of $p_1, p_2$ and $p_1, p_3$ as in Lemma 5.12. It is clear that $s_1$ and $s_2$ are in the convenient position as witnessed by $I_{ap_1} \cup \sigma_{q_1,2} : a_{p_1} \cup a_{p_2} \to a_{p_1} \cup a_{p_3}$. Now let $r$ be the disjoint amalgamation of $s_1$ and $s_2$ as in Lemma 5.11. The properties of $r$ follow from Lemma 5.12 and Definition 5.1.

To prove the last statement of the lemma note that $i_{r,p_2}(A)i_{r,p_3}(j_{s_2,3}(A)) = (A + j_{s_2,1}(A)(X_{p_1} \setminus X_{p_2}))(j_{s_3,1}(A)(X_{p_1} \setminus X_{p_3})) = (j_{s_3,1}(A))^2 = i_{r,p_1}(j_{s_3,1}(A))^2$.

**Lemma 5.17.** Suppose that $p_1, p_2, p_3$ are distinct elements in $P$ which are pairwise in the convenient position as witnessed by $\sigma_{j,i} : a_{p_i} \to a_{p_j}$ for $1 \leq i < j \leq 3$ such that $\Delta < a_{p_i} \setminus \Delta < a_{p_3} \setminus \Delta$ and $n^p_\xi = n$ for some $n > 1$ and each $i \in \{1, 2, 3\}$ and that $v_1 = (v_{1,0}^1, \ldots, v_{1,n-1}^1), v_2 = (v_{1,0}^2, \ldots, v_{2,n-1}^2)$ are two orthogonal unit vectors of $C^n$. Then there is $r \leq p_1, p_2, p_3$ satisfying

- $a_r = a_{p_1} \cup a_{p_2} \cup a_{p_3}$,
- $n^p_\xi = n^p_\xi$ if $\xi \in a_{p_i}$ for $1 \leq i \leq 3$,
- $||[i_{r,p_1}(A), i_{r,p_1}(j_{s_m,1}(A))]] = 1/2$,
- for $m = 2, 3$ and for every nonexpanding $A \in \mathcal{B}_{X_{p_m}}$ such that there is $\xi \in a_n \setminus a_{p_1}$ with $A(\sum_{k<n} v_k^i e_{\xi,k}) = \sum_{k<n} v_k^i e_{\xi,k}$ and $A(\sum_{k<n} v_k^i e_{\xi,k}) = 0$.

The element $r$ is called the amalgamation of $p_1, p_2, p_3$ of type 3 for vectors $v_1$ and $v_2$.

**Proof.** First consider $s_2 \leq p_1, p_2$ and $s_3 \leq p_1, p_3$ which are the $(v_1, v_2)$-anti-commuting amalgamations of $p_1, p_2$ and $p_1, p_3$ as in Lemma 5.14. It is clear that $s_1$ and $s_3$ are in the convenient position as witnessed by $I_{ap_1} \cup \sigma_{q_1,2} : a_{p_1} \cup a_{p_2} \to a_{p_1} \cup a_{p_3}$. Now let $r$ be the disjoint amalgamation of $s_1$ and $s_2$ as in Lemma 5.11. The properties of $s_1$ and $s_3$ from the $(v_1, v_2)$-anti-commuting amalgamations $s_1$ and $s_2$ pass to $r$ by Definition 5.1 (d).
5.6. Inductive limits of directed families in \( \mathbb{P} \). In this section we adopt the terminology where a directed set is a partial order \( (X, \leq) \) where for any two \( x, y \in X \) there is \( z \in X \) such that \( z \leq x, y \). In this section we will consider inductive limits \( A^G \) of systems \( (A^p_X : p \in G) \) where \( G \subseteq \mathbb{P} \) is a directed subset of \( \mathbb{P} \) with the order \( \leq \). Here for \( p \leq q \) the embeddings \( i_{pq} : A^p_X \to A^q_X \) are given by Definition 5.1 (c), i.e., they satisfy \( i_{pq}(A^p_{\xi,n,m}) = A^p_{\xi,n,m} \) for \( \xi \in a_q \) and \( n, m \in [0, n]^1 \). Formally we define \( A^G \) differently in order to work with its convenient representation in \( \mathcal{B}_{\omega_1 \times \mathbb{N}} \) but then, in Lemma 5.22 we prove that the constructed algebra is the corresponding inductive limit.

**Definition 5.18.** We say that \( G \subseteq \mathbb{P} \) is covering if and only if \( \omega_1 \times \mathbb{N} \subseteq \bigcup \{ X_p : p \in G \} \).

**Definition 5.19.** Suppose that \( G \subseteq \mathbb{P} \) is directed and covering. Then \( A^G_{\xi,n,m} \in \mathcal{B}_{\omega_1 \times \mathbb{N}} \) is given by \( (A^G_{\xi,n,m}(e_{\eta,k}), e_{\eta,l}) = (A^p_{\xi,n,m}(e_{\eta,k}), e_{\eta,l}) \) for any (all) \( p \in G \) such \( (\eta, k), (\eta, l), (\xi, n), (\xi, m) \in X_p \).

Note that \( A^G_{\xi,n,m} \) are well-defined if \( G \) is directed and covering. This is because given two \( p, p' \in G \) such \( (\eta, k), (\eta, l), (\xi, n), (\xi, m) \in X_p, X_{p'} \) there is \( q \leq p, p' \) which implies that \( X_p, X_{p'} \subseteq X_q \) and so \( (A^G_{\xi,n,m}(e_{\eta,k}), e_{\eta,l}) = (A^q_{\xi,n,m}(e_{\eta,k}), e_{\eta,l}) = (A^q_{\xi,n,m}(e_{\eta,k}), e_{\eta,l}) \) by Definition 5.1 (c-d). The following definition is parallel to Definition 5.2.

**Definition 5.20.** Suppose that \( G \subseteq \mathbb{P} \) is directed and covering. \( A^G \) is the subalgebra of \( \mathcal{B}_{\omega_1 \times \mathbb{N}} \) generated by the operators \( A^G_{\xi,n,m} \) for all \( \xi \in \omega_1 \) and \( n, m \in \mathbb{N} \).

Let \( X \) be a subset of \( \omega_1 \times \mathbb{N} \). We define \( A^G_X \) to be the \( \mathcal{C}^* \)-subalgebra of \( A^G \) generated by \( (A^G_{\xi,n,m} : (\xi, n), (\xi, m) \in X) \). In particular, for every \( \alpha < \omega_1 \), by \( A^G_{\alpha} \) we mean the \( \mathcal{C}^* \)-subalgebra of \( A^G \) generated by \( \{ A^G_{\xi,n,m} : \xi < \alpha, n, m \in \mathbb{N} \} \).

**Lemma 5.21.** Suppose that \( G \subseteq \mathbb{P} \) is directed and covering and \( p \in G \). There is a \( * \)-embedding \( i_{G, p} : A^G_X \to A^{G}_{X_p} \) such that

1. \( i_{G, p}(A^G_{\xi,n,m}) = A^G_{\xi,n,m} \) and
2. \( i_{G, p}(A^G_{\xi,n,m})|X_p = A^p_{\xi,n,m} \)

for every \( \xi, n, m \) such that \( (\xi, n), (\xi, m) \in X \).

**Proof.** By Definitions 5.1 and 5.19, a map sending \( A^G_{\xi,n,m} \) to \( A^G_{\xi,n,m} \) extends to a \( * \)-homomorphism of \( A^G_X \) into \( A^{G}_{X_p} \). Its kernel must be null as the kernels of \( i_{G, p} \) for \( q \leq p \) are null.

To prove the second part of the lemma, use the first part and Definition 5.19.

**Lemma 5.22.** Suppose that \( G \subseteq \mathbb{P} \) is directed and covering. There is a \( * \)-isomorphism \( j \) of \( A^G \) and the inductive limit \( \lim_{p \in G} A^p_{\xi,n,m} \) of the system \( (A^p_{\xi,n,m} : p \in G) \) with maps \( (i_{G, p} : p \leq q) \) such that

\[
j(A^G_{\xi,n,m}) = \lim_{p \in G} A^p_{\xi,n,m}
\]

for each \( \xi \in \omega_1 \) and \( m, n \in \mathbb{N} \).
Proof. As in Ex. 1. Chapter 6 of [39] it is enough to prove that for every \( p, q \in \mathbb{G} \) satisfying \( p \leq q \) the diagram

\[
\begin{array}{ccc}
A^q_{X_q} & \xrightarrow{i_{q,q}} & A^q_{X_q} \\
\downarrow{i_{p,q}} & & \downarrow{i_{p,q}} \\
A^p_{X_p} & \xrightarrow{i_{p,q}} & A^p_{X_p}
\end{array}
\]

commutes. This follows from the fact that by Definition 5.19 we have \( i_{q,p}(A^p_{X_{q,n,m}}) = A^q_{X_{q,n,m}} \) for \( \xi, m, n \) such that \( (\xi, m), (\xi, n) \in X_q \). But these elements generate \( A^q_{X_q} \).

\[\square\]

**Definition 5.23.** A family \( G \subseteq \mathbb{F} \) is called \( \mathbb{F} \)-rich if and only if \( G \) is directed, covering and \( \mathbb{F}_{X,\alpha} \cap G \neq \emptyset \) for every finite \( X \subseteq \omega_1 \times \mathbb{N} \) and \( \alpha \in X \), where \( \mathbb{F}_{X,\alpha} \)'s are defined in Lemma 5.7.

**Lemma 5.24.** Let \( G \subseteq \mathbb{P} \) be an \( \mathbb{F} \)-rich family. Then for every \( \alpha < \omega_1 \) the following hold:

1. \( A^G_\alpha \) is an ideal of \( A^G_\alpha \) equal to \( \{ A \subseteq A^G_\alpha : A[\alpha, \omega_1) = 0 \} \).
2. there is a *-isomorphism \( j_{\alpha} : A^G_\alpha / A^G_\alpha \to A^G[\alpha, \omega_1) \).
3. the representation \( \pi_{\alpha} : A^G[\alpha, \omega_1) \to A^G[\alpha] \) given by \( \pi_{\alpha}(A) = A[\alpha] \) is faithful.

**Proof.** As \( \ell_2(\{\xi\} \times \mathbb{N}) \) are \( A^G \)-invariant, it is clear that sending \( A \in A^G_\alpha \) to \( A[\alpha, \omega_1) \) is a *-homomorphism. So for (1) and (2) we are left with proving that its kernel is equal to \( A^G_\alpha \).

First note that the kernel contains every generator \( A^G_{\xi, n, m} \) for \( \xi < \alpha \) and \( m, n \in \mathbb{N} \) of \( A^G_\alpha \) and so includes \( A^G_\alpha \). This is true by Definition 5.1 (4).

For the other inclusion let \( A \in A^G_\alpha \) satisfy \( A[\alpha, \omega_1) = 0 \). Since \( A^G_\alpha \) is the inductive limit of \( A^G_{\xi, n, m} \) for \( \alpha \in G \) by Lemma 5.22, for every \( \varepsilon > 0 \) there is \( p \in G \) and \( B \in A^G_{\xi, n, m} \) such that \( \|i_{p,q}(B) - A\| < \varepsilon \) and so \( \|i_{p,q}(B)[\alpha, \omega_1)\| < \varepsilon \). By Lemma 5.3, \( B[\alpha, \omega_1) \subseteq A^G_{\xi, n, m} \), and so we can apply \( i_{\xi, p} \) to them. By Lemma 5.4 and Definition 5.19 we have that

\[
\|i_{\xi, p}(B)[\alpha, \omega_1)\| = \|i_{\xi, p}(B)[\alpha, \omega_1)\|
\]

So we have

\[
\|A - i_{\xi, p}(B[\alpha])\| = \|A - i_{\xi, p}(B) + i_{\xi, p}(B[\alpha, \omega_1))\| \leq \|A - i_{\xi, p}(B[\alpha, \omega_1)\| \leq 2\varepsilon.
\]

But \( i_{\xi, p}(B[\alpha]) \in A^G_{\alpha} \) since \( B[\alpha] \in A^G_{\alpha} \). As \( \varepsilon > 0 \) was arbitrary and \( i_{\xi, p}(B[\alpha]) \in A^G_{\alpha} \), we conclude that \( A \in A^G_{\alpha} \), completing the proof of (1) and (2).

To prove (3) first note that since \( \ell_2(\{\alpha\} \times \omega_1) \) is \( A^G \)-invariant, it is clear that \( \pi_{\alpha} \) is a representation of \( A^G[\alpha, \omega_1) \). Now suppose that \( A \in A^G_{\alpha} \). By Lemma 5.21 there is \( B \in A^G_{\alpha} \) such that \( i_{\xi, p}(B) = A \). Since \( G \) is assumed to be \( \mathbb{F} \)-rich, by Lemmas 5.7 and 5.8 there is \( p \in \mathbb{F}_{X,\alpha} \) such that \( p \leq q \). By Lemma 5.4 and Definition 5.19 we have \( \|A[\alpha, \omega_1)\| = \|i_{p,q}(B)[\alpha, \omega_1)\| \). By the fact that \( p \in \mathbb{F}_{X,\alpha} \) we have that

\[
\|A[\alpha]\| \geq \|i_{p,q}(B[\alpha, \omega_1))\| \geq \|i_{p,q}(B)[\alpha, \omega_1)\| = \|A[\alpha, \omega_1)\|.
\]
This shows that \( \pi_\alpha \) is an isometry when restricted to \( \bigcup_{q \in G} A^G_{q, \alpha} \) which is dense in \( A^G|\alpha, \omega_1 \) by Lemma 5.22, and so the representation is faithful. □

**Proposition 5.25.** Suppose that \( G \subseteq \mathcal{P} \) is an \( \mathcal{F} \)-rich family. Then \( A^G \) is a scattered thin-tall fully noncommutative \( C^* \)-algebra such that

1. \( I_0(\mathcal{A}^G) = A^G_\alpha \),
2. there is a \( * \)-isomorphism \( j_\alpha : \mathcal{A}^G/I_0(\mathcal{A}^G) \rightarrow A^G|\alpha, \omega_1 \) satisfying

\[
j_\alpha([A]_{I_0(\mathcal{A})}) = A|\alpha, \omega_1, \]

3. the collection \( \{[A_{\alpha, m, n}]_{I_0(\mathcal{A})} : n, m \in \mathbb{N} \} \) satisfies the matrix units relations and generates the essential ideal \( \mathcal{H}(\mathcal{A}^G/I_0(\mathcal{A}^G)) \).

**Proof.** By Theorem 1.4 of [18] it is enough to prove (1) - (3) to conclude that \( A \) is a scattered thin-tall fully noncommutative \( C^* \)-algebra.

The proof of (1) - (3) is by induction on \( \alpha < \omega_1 \). For \( \alpha = 0 \) we have that \( I_\alpha = \{0\} \) and so (1) and (2) are trivial. Also \( A_{0, n, m} = 0_{0, n, m} \) by Definition 5.1, so these elements satisfy the matrix unit relations. Moreover they generate the algebra of all compact operators on \( l^2(\{0\} \times \mathbb{N}) \) which is an essential ideal in \( \mathcal{B}(l^2(\{0\} \times \mathbb{N})) \). Since \( \pi_0 \) from Lemma 5.24 is faithfull, the collection \( \{A_{0, n, m} : n, m \in \mathbb{N} \} \) generates an essential ideal isomorphic to an algebra of all compact operators on a Hilbert space, so by Theorem 1.2 (4) of [18] this ideal is \( I_0(\mathcal{A}^G) \) as required.

Now suppose we are done for \( \beta < \alpha < \omega_1 \).

(1) If \( \alpha \) is a limit ordinal, then by 1.A of [18] and the inductive hypothesis we have

\[
I_\alpha = \bigcup_{\beta < \alpha} I_\beta = \bigcup_{\beta < \alpha} A_\beta = A_\alpha.
\]

If \( \alpha = \beta + 1 \), then (3) of the inductive hypothesis implies (1).

(2) follows from Lemma 5.24.

(3) Is proved like in the case \( \alpha = 0 \). □

6. AN OPERATOR ALGEBRA ALONG A CONSTRUCTION SCHEME

In this section we adopt the terminology and the notation of Section 5. We will use the constructions scheme of [54] described in Section 2.2 to build appropriate \( \mathcal{F} \)-rich families \( G \) in the partial order \( \mathcal{P} \) of approximations whose inductive limit \( A^G \) will have interesting properties described in the introduction. To prove the main theorem of this section we need one more general lemma:

**Lemma 6.1.** Suppose that \( A \) is an AF \( C^* \)-algebra where \( \{A_D : D \in \mathcal{D} \} \) is a directed family of finite-dimensional subalgebras with dense union. Let \( P \in A \) be a projection. Then for every \( 0 < \varepsilon < 1 \) there is \( D \in \mathcal{D} \) and a projection \( Q \in A_D \) such that \( \|Q - P\| < \varepsilon \).

**Proof.** Let \( D \in \mathcal{D} \) be such that there is \( A \in A_D \) satisfying \( \|A - P\| < \varepsilon/6 \). By considering \( (A + A^*)/2 \) instead of \( A \) we may assume that \( A \) is self-adjoint and \( \|A - P\| < \varepsilon/6 \). As \( A_D \) is finite dimensional, it is *-isomorphic to the direct sum of full matrix algebras. Let \( \pi \) be the isomorphism. The matrix \( \pi(A) \) is self-adjoint, so it can be diagonalized. As \( \|A - P\| < \varepsilon/6 \) we have that \( \|\pi(A) - \pi(P)\| < \varepsilon/2 \) and so the distance of each entry on the diagonal of the diagonalized \( \pi(A) \) from 0 or 1 cannot
be bigger than $\varepsilon/2$, so there is a projection $Q \in \mathcal{A}_p$ such that $\|\pi(Q) - \pi(A)\| < \varepsilon/2$ and hence $\|Q - A\| < \varepsilon/2$ and $\|Q - F\| < \varepsilon$ as required.

**Theorem 6.2.** Suppose that there exists a construction scheme $\mathcal{F}$ with allowed parameters $(r_k)_{k \in \mathbb{N}}$ and $(n_k)_{k \in \mathbb{N}}$, where $n_k = 3$ for each $k \in \mathbb{N} \setminus \{0\}$ and a partition $(P_m)_{m \in \mathbb{N}}$ of $\mathbb{N}$ into infinite sets such that for every $m \in \mathbb{N}$ and every uncountable $\Delta$-system $T$ of finite subsets of $\omega_1$ there exist $F \in \mathcal{F}$ of arbitrarily large rank in $P_m$ which fully captures a subsystem of $T$.

Then there is an $\mathcal{F}$-rich family $\mathcal{G}$ of elements of $\mathcal{P}$ such that the scattered thin-tall fully noncommutative $C^*$-algebra $\mathcal{A}_G$ has the following properties:

1. There is a nondecreasing unbounded sequence $(l_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and a directed family of finite dimensional algebras $\{\mathcal{A}^*_G : X = F \times [0,l_k) : F \in \mathcal{F}_k, k \in \mathbb{N}\}$ whose union $\mathcal{B}$ is dense in $\mathcal{A}$ such that whenever $(P_\xi : \xi < \omega_1) \subseteq \mathcal{B}$ is a family of projections which generate a nonseparable subalgebra of $\mathcal{A}^*_G$, then

   (a) there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $\|P_\xi_1 - P_\xi_2 P_\xi_3\| < \varepsilon$,
   
   (b) there are $\xi_1 < \xi_2 < \omega_1$ such that $\|P_{\xi_1}, P_{\xi_2}\| < \varepsilon$,

   (c) there are $\xi_1 < \xi_2 < \omega_1$ such that $\|P_{\xi_1}, P_{\xi_2}\| > 1/2 - \varepsilon$.

2. $\mathcal{A}^*_G$ has no uncountable irredundant subset.

3. $\mathcal{A}^*_G$ has no nonseparable abelian subalgebra.

**Proof.** Fix an enumeration $((v^m, w^m)) : m \geq 3$, with possible repetitions, of all pairs of orthonormal complex vectors with finitely many coordinates, all of them rational, such that $v^m, w^m \in \mathbb{C}^m$ for each $m \geq 3$ (we abuse notation and identify $\mathbb{C}^m$ with a subset of $\mathbb{C}^{m'}$ for $m' \leq m$).

We construct the sequence $(l_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and $G = \{p_F : F \in \mathcal{F}\} \subseteq \mathcal{P}$ by induction with respect to $k \in \mathbb{N}$ such that $F \in \mathcal{F}_k$. Moreover, for each $k \in \mathbb{N}$ we require that whenever $F, F' \in \mathcal{F}_k$ are such that $F \setminus F' < F' \setminus F$ (cf. Definition 2.10 (2)), then

(1) $p_F$ and $p_{F'}$ are in the convenient position as witnessed by $\phi_{F',F}$.

(2) $a_{FF'} = F'$.

(3) $n_{F'}^F = l_k$ for all $\xi \in F$ and $F \in \mathcal{F}_k$ and $k \in \mathbb{N}$.

For $k = 0$ we have that $\mathcal{F}_1 = [\omega_1]^1$ by Definition 2.10 (1), so we define $p_F$ for $F = \{\xi\}$ to be the element of $\mathcal{P}$ such that

- $a_{FF} = \{\xi\}$,
- $n_{FF}^F = l_0 = 1$,
- $A_{\xi,0,0}^F = 1_{\xi,0,0}$.

Suppose that we have constructed $p_F$s for all $F \in \mathcal{F}_k$ for $k' \leq k$ satisfying (*) - (***)+. Now we need to define the $p_F$s for $F \in \mathcal{F}_{k+1}$. Since $n_{k+1} = 3$, each $F \in \mathcal{F}_{k+1}$ is the union of the maximal elements $G_1, G_2, G_3$ of $\mathcal{F}|F$ which form an increasing $\Delta$-system by Definition 2.10 (3). If $k \in P_1$, then we define $p_F$ as the amalgamation of $p_{G_1}, p_{G_2}, p_{G_3}$ of type 1 from Lemma 5.15. If $k \in P_2$, then we define $p_F$ as the amalgamation of $p_{G_1}, p_{G_2}, p_{G_3}$ of type 2 from Lemma 5.16. If $k \in P_3$ for $m \geq 3$ and $l_k < m$, then we define $p_F$ as the amalgamation of $p_{G_1}, p_{G_2}, p_{G_3}$ of type 1 from Lemma 5.15. If $k \in P_m$ for $m \geq 3$, and $l_k \geq m$, then we define $p_F$ as the amalgamation of $p_{G_1}, p_{G_2}, p_{G_3}$ of type 3 for vectors $(v^m, w^m)$ from Lemma 5.17.

Observe that amalgamation of type 1 increases $l_k$, so $l_k \to \infty$ when $k \to \infty$.
First let us note that our inductive hypothesis (*) - (****) is preserved when we pass from \( k \in \mathbb{N} \) to \( k + 1 \). Let \( F, F' \in \mathcal{F}_{k+1} \) be such that \( F \setminus F' < F' \setminus F \). By Definition 2.10 (2) there is an order preserving bijection \( \phi_{F', F} : F \to F' \) and \( F \cap F' < F \setminus F' \setminus F \). In particular Definition 2.10 (2) implies that the maximal elements of \( \mathcal{F}|F \) are sent by \( \phi_{F', F} \) onto the maximal elements of \( \mathcal{F}|F' \), on the other hand (3) of 2.10 implies that these maximal elements form the canonical decomposition consisting of elements in \( \mathcal{F}_{k} \), which is in fact used in the construction of \( p_F \) or \( p_F' \). Now to verify Definition 5.9 in order to check (*) we note that the amalgamations described in Lemmas 5.15, 5.16, 5.17 consist of the constructions of operators which depend only on the place of the involved objects in \( F \), so Definition 5.9 and (*) are satisfied for \( p_F \) and \( p_F' \). (***) follow from the descriptions of the amalgamations from Lemmas 5.15, 5.16, 5.17. We have \( l_{k+1} = l_k \) if \( k \in \mathbb{N} \setminus P_1 \) and \( l_{k+1} > l_k \) if \( k \in P_1 \) and Definition 2.10 guarantees that the amalgamations which follow Lemma 5.15 can be done in “the same way” up to the bijection \( \phi_{F', F} \) and so obtaining \( n_{\beta_F'} = l_{k+1} = n_{\beta_F} \) for any \( F, F' \in \mathcal{F}_{k+1} \) and \( \xi \in F \) and \( \xi' \in F' \). This completes the construction of \( G = \{p_F : F \in \mathcal{F}\} \) and determines completely the C*-algebra \( A^G \) as in Definition 5.20.

Now note that \( G \) is \( \mathcal{F} \)-rich as in Definition 5.23. First note that \( p_F \leq p_F' \) whenever \( F' \subseteq F \) and \( F, F' \in \mathcal{F} \). This can be proved by induction on \( k \in \mathbb{N} \) such that \( F \in \mathcal{F}_k \). Note that it is true if \( F' \) is a maximal element of \( \mathcal{F}|F \), because then \( F' \) is in the canonical decomposition of \( F \) by Definition 2.10 (3) and we use \( p_F \) in the construction of \( p_F \) obtaining \( p_F \leq p_F' \) by the Lemmas 5.15, 5.16, 5.17. Now we proceed with the inductive argument, given \( F' \subseteq F \) either \( F' \) is below a maximal element \( G \) of \( \mathcal{F}|F \) or it is one of the maximal elements. The latter case is proved above and the former follows from the inductive assumption for the pair \( F', G \) and from the transitivity of the order in \( \mathcal{F} \).

To prove the directedness of \( G \) take \( F, F' \in \mathcal{F} \) and use the cofinality of \( \mathcal{F} \) in \( [\omega_1]^{<\omega} \) (Definition 2.10) to find \( F'' \in \mathcal{F} \) such that \( F \cup F' \subseteq F'' \). By the above arguments we have \( p_F, p_{F'} \leq p_{F''} \).

Now let \( X = a \times [0, 1] \in [\omega_1 \times \mathbb{N}]^{<\omega} \) and \( \alpha \in \omega_1 \) and aim at proving further parts of the \( \mathcal{F} \)-richness. Consider the \( \Delta \)-system \( T = \{a \cup \{\alpha, \xi \} : \max(a \cup \{\alpha\}) < \xi < \omega_1\} \) of finite subsets of \( \omega_1 \). By the hypothesis there is \( k \in P_1 \) with \( l_k \geq l \) and \( F \in \mathcal{F} \) such that \( F \) fully captures a subsystem of \( T \). In particular \( F = G_1 \cup G_2 \cup G_3 \) for some \( G_1, G_2, G_3 \in \mathcal{F}_k \) and \( X \subseteq X_{\alpha_{G_1}} \) and \( \alpha \in a_{\alpha_{G_1}} \). By the construction, we do the amalgamation of type 1 like in Lemma 5.15 while constructing \( p_F \) and so \( p_F \) is in \( \mathcal{F}_{X_{\alpha_{G_1}}} \) but this implies that it is in \( \mathcal{F}_{X, \alpha} \) as required for \( \mathcal{F} \)-richness in Definition 5.23.

Proposition 5.25 implies that \( A^G \) as in Definition 5.20 is a thin-tall fully non-commutative scattered C*-algebra.

To prove (1) the directed family of finite dimensional subalgebras of \( A^G \) is \( \{A^G_{X} : p \in G\} \) as in Definition 5.20. By Lemma 5.21 the algebras \( A^G_{X} \) are *-isomorphic to the algebras \( A^G_{X_a} \) and they are finite dimensional since they are equal to \( B_{X_a} \) by Lemma 5.3. Let \( B = \bigcup\{A^G_{X} : p \in G\} \).

Suppose that \( \{P_\xi : \xi < \omega_1\} \subseteq B \) is a collection of projections which generate a nonseparable subalgebra of \( A^G \). So, there must be distinct \( \alpha_\xi \in \omega_1 \) such that \( P_\xi|((\alpha_\xi \times N) \setminus \mathbb{N}) \neq 0 \). Since \( B_{((\alpha_\xi \times N) \setminus \mathbb{N})} \) is invariant for \( A^G \) it follows that \( P_\xi|((\alpha_\xi \times N) \setminus \mathbb{N}) \) is a non-zero projection. Moreover it is not the unit of \( B_{((\alpha_\xi \times N) \setminus \mathbb{N})} \) because such a unit
would produce a unit of $A^G/F\alpha_{\xi}(A^G)$ by Lemma 5.24 and Theorem 5.25, which is impossible because $A^G$ is the union of proper ideals $I_{\alpha_{\xi}}(A^G)$ for $\alpha < \omega_1$.

Let $F_{\xi} \in \mathcal{F}$ be such that $\alpha_{\xi} \in F_{\xi}$, $P_{\xi} \in A^G_{\alpha_{\xi}} = A^G_{\alpha_{\xi} \times [0,l_\xi)}$ for each $\xi \in \omega_1$, where $l_{\xi} = l_{\xi_k}$ for $F_{\xi} \in \mathcal{F}_k$ and $P_{\xi_1}((\alpha_{\xi_1} \times [0,l_{\xi_1}))$ is a nonzero projection which is not the unit of $B_{(\alpha_{\xi_1}) \times [0,l_{\xi_1})}$. This can be obtained from the cofinality of $\mathcal{F}$ and the fact that $l_{\xi} \to \infty$ when $k \to \infty$.

Let $Q_{\xi} \in A^F_{\alpha_{\xi} \times [0,l_{\xi}]}$ be such that $i_{G, p_{\xi}}(Q_{\xi}) = P_{\xi}$. Note that by Lemma 5.21 $Q_{\xi}G$ are projections and $Q_{\xi}((\alpha_{\xi} \times [0,l_{\xi})))$ is a nonzero projection which is not the unit of $B_{(\alpha_{\xi}) \times [0,l_{\xi}]}$ for each $\xi < \omega_1$.

By passing to an uncountable subset, we may assume that $T = \{F_{\xi} : \xi < \omega_1\}$ forms an increasing $\Delta$-system of elements of $\mathcal{F}$, for a fixed $k' \in \mathbb{N}$ and that
\[
\left|\langle Q_{\xi}(e_{\eta,l}), e_{\eta,l'}\rangle - \langle Q_{\xi}(e_{\phi_{\epsilon_{\xi}}, r_{\phi_{\epsilon_{\xi}}}((\eta,l))), e_{\phi_{\epsilon_{\xi}}, r_{\phi_{\epsilon_{\xi}}}((\eta,l))}\rangle \right| < \varepsilon/2l_{\xi}.
\]
for every $(\eta,l), (\eta,l') \in F_{\xi} \times [0,l_{\xi})$ and every $\xi < \xi' < \omega_1$. This guarantees that
\[
(+) \quad \|j_{\phi_{\epsilon_{\xi}}, r_{\phi_{\epsilon_{\xi}}}((\eta,l))}(Q_{\xi}) - Q_{\xi}\| < \varepsilon/2
\]
for every $\xi < \xi' < \omega_1$. Now let us prove item (a) of (1). By the hypothesis on $\mathcal{F}$ there is $k \in P_2$ bigger than $k'$ and $F \in \mathcal{F}_{k+1}$ which fully captures $T$, i.e. the canonical decomposition of $F$ is $[G_1, G_2, G_3]$ and there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $F_{\xi_j} \leq G_i$ and $\phi_{\epsilon_{\xi_j}, G_i}(F_{\xi_j}) = F_{\xi_j}$ for all $1 \leq i,j \leq 3$. As $\phi_{\epsilon_{\xi_j}, G_i}$ are order preserving, they must agree with $\phi_{F_{\xi_j}, F_{\xi_j}}$ on $F_{\xi_j}$, so $(+)$ implies that
\[
\|j_{\phi_{\epsilon_{\xi_j}, G_i}}(Q_{\xi_j}) - Q_{\xi_j}\| < \varepsilon/2
\]
holds for $i = 1, 2$. Since we use amalgamation of type 2 at the construction of $p_F$ for $k \in P_2$ by Lemma 5.16 we have
\[
i_{p_F, p_{\xi_2}}(Q_{\xi_2})i_{p_F, p_{\xi_1}}(j_{\phi_{\epsilon_{\xi_2}, G_i}}(Q_{\xi_2})) = i_{p_F, p_{\xi_1}}(j_{\phi_{\epsilon_{\xi_2}, G_i}}(Q_{\xi_2}))^2,
\]
and so
\[
\|i_{p_F, p_{\xi_2}}(Q_{\xi_2})i_{p_F, p_{\xi_1}}(j_{\phi_{\epsilon_{\xi_2}, G_i}}(Q_{\xi_2})) - i_{p_F, p_{\xi_1}}(Q_{\xi_1})^2\| < \varepsilon
\]
and hence $\|P_{\xi_1} - P_{\xi_2}\| < \varepsilon/2$ since
\[
i_{G, p_{\xi_2}} \circ i_{p_F, p_{\xi_1}}(Q_{\xi_2}) = i_{G, p_{\xi_2}}(Q_{\xi_2}) = i_{G, p_{\xi_1}}(i_{p_F, p_{\xi_1}}(Q_{\xi_2})) = i_{G, p_{\xi_1}}(Q_{\xi_2}) = P_{\xi_1},
\]
by Definition 5.19 and Lemma 5.21. This completes the proof of (a) of (1). Item (b) follows from (a) for $\varepsilon/2$ and by taking the adjoints.

Now let us prove item (c) of (1). For $\xi < \omega_1$ let $Q_{\xi} \in B_{p_F \times [0,l_{\xi})}$ be such projections that $\|Q_{\xi} - Q_{\xi}^G\| < \varepsilon/8$ and there is an orthonormal basis in $B_{(\alpha_{\xi}) \times [0,l_{\xi})}$ of eigenvectors for $Q_{\xi}$ consisting only of vectors with all rational coordinates with respect to our canonical basis $(e_{\alpha_{\xi}, l} : 0 \leq l < l_{\xi'})$. Note that by Lemma 5.3 we have that $Q_{\xi} \in A^F_{\alpha_{\xi} \times [0,l_{\xi})}$. Since $Q_{\xi}((\alpha_{\xi} \times [0,l_{\xi})))$ is a nonzero projection which is not the unit of $B_{(\alpha_{\xi}) \times [0,l_{\xi}]}$ for each $\xi < \omega_1$, $Q_{\xi}$ may be assumed to have the same rank as $Q_{\xi}$ and so there are orthogonal unit vectors $v^\xi, w^\xi \in \ell^\omega$ with all rational coordinates such that
\[
Q_{\xi}(\sum_{l<l_{\xi'}} v_l^\xi e_{\alpha_{\xi}, l}) = \sum_{l<l_{\xi'}} v_l^\xi e_{\alpha_{\xi}, l}, \quad Q_{\xi}(\sum_{l<l_{\xi'}} w_l^\xi e_{\alpha_{\xi}, l}) = 0.
\]
As there are only countably many such vectors we may assume that all of them are equal to a pair \((v, w)\) and moreover that
\[
\|P_{\phi_{\xi_1, \omega}}(Q_{\xi_1}) - Q_{\xi_1}\| < \varepsilon/4
\]
for every \(\xi < \xi' < \omega_1\).

By the hypothesis on \(F\) there is \(k \in P_0\), bigger than \(k'\) such that \(v_m = v = (v_1, \ldots, v_\omega)\) and \(w_m = w = (w_1, \ldots, w_\omega)\) and there is \(F \in F_{k+1}\) which fully captures \(T\), i.e., the canonical decomposition of \(F\) is \(\{G_1, G_2, G_3\}\) and there are \(\xi_1 < \xi_2 < \xi_3 < \omega_1\) such that \(F_{\xi_i} \subseteq G_i\) and \(\phi_{\xi_i, \xi_{i+1}}(F_{\xi_i}) = F_{\xi_i}\) for all \(1 \leq i, j \leq 3\). Note that \(\alpha_{\xi_i}\)s are not in the root of \(\{G_1, G_2, G_3\}\) as they are not in the root of \(F_{\xi_i}\). As \(\phi_{\xi_i, \xi_j}\)s are order preserving, they must agree with \(\phi_{F_{\xi_i}, F_{\xi_j}}\) on \(F_{\xi_i}\), so \((++)\) implies that
\[
\|P_{\phi_{\xi_1, \omega}}(Q_{\xi_1}) - Q_{\xi_1}\| < \varepsilon/4
\]
holds for \(i = 2, 3\). Since we use amalgamation of type 3 at the construction of \(p_F\) for \(k \in P_0\) by Lemma 5.17 we have
\[
\|i_{p_F, p_{\omega_1}}(Q_{\xi_1}); i_{p_F, p_G}(\phi_{\xi_1, \omega}(Q_{\xi_1}))\| = 1/2
\]
and so
\[
\|i_{p_F, p_{\omega_1}}(Q_{\xi_1}); i_{p_F, p_G}(Q_{\xi_1})\| \geq 1/2 - \varepsilon/2
\]
and hence
\[
\|i_{p_F, p_{\omega_1}}(Q_{\xi_1}); i_{p_F, p_{\omega_2}}(Q_{\xi_1})\| \geq 1/2 - \varepsilon
\]
as \(\|Q_{\xi} - Q_{\xi'}\| < \varepsilon/8\) for each \(\xi < \xi' < \omega_1\), and finally
\[
\|[P_{\xi_1}, P_{\xi_2}]\| \geq 1/2 - \varepsilon
\]
since
\[
i_{G_{\xi_1}, p_{\omega_1}}(Q_{\xi_1}) = i_{G_{\xi_1}, p_{\omega_1}}(\phi_{\xi_1, \omega}(Q_{\xi_1})) = i_{G_{\xi_1}, p_{\omega_1}}(Q_{\xi_1}) = P_{\xi_1}
\]
by Definition 5.19 and Lemma 5.21. This completes the proof of \((c)\) of \((1)\).

The proof of \((2)\) will be based on \((1)\) \((a)\) and Lemma 6.1. Suppose that \(A^G\) contains an uncountable irrepresentable set \(\{Q_{\xi} : \xi < \omega_1\}\). By Lemma 3.3 we may assume that all \(Q_{\xi}\)s are projections. For each \(\xi\) let \(A_{\omega_1 \setminus \{\xi\}}\) be the \(C^*\)-subalgebra of \(A^G\) generated by the set \(\{Q_{\eta} : \eta \in \omega_1 \setminus \{\xi\}\}\). By passing to an uncountable subset we may assume that there is \(\varepsilon > 0\) such that for each \(\xi < \omega_1\) we have \(\|A - Q_{\xi}\| \geq \varepsilon\) for each \(A \in A_{\omega_1 \setminus \{\xi\}}\). Let \(P_{\xi} \in B\) be a projection satisfying \(\|P_{\xi} - Q_{\xi}\| < \varepsilon/4\) which is obtained using Lemma 6.1. By \((1)\) \((a)\) there are \(\xi_1 < \xi_2 < \xi_3 < \omega_1\) such that \(\|P_{\xi_1} - P_{\xi_2} P_{\xi_3}\| < \varepsilon/4\). This implies that \(\|Q_{\xi_1} - Q_{\xi_2} Q_{\xi_3}\| < \varepsilon\) which contradicts the defining property of \(\varepsilon\) and completes the proof of \((2)\).

The proof of \((3)\) will be based on \((1)\) \((c)\) and Lemma 6.1. Suppose that \(A^G\) contains a nonseparable abelian subalgebra. As subalgebras of scattered algebras are scattered, and scattered locally compact spaces are totally disconnected, it follows that \(A^G\) contains an uncountable Boolean algebra of \((\text{commuting})\) projections \(\{Q_{\xi} : \xi < \omega_1\}\). In particular \(\|Q_{\xi} - Q_{\xi'}\| = 1\) for all \(\xi < \xi' < \omega_1\).

Let \(P_{\xi} \in B\) for \(\xi < \omega_1\) be projections satisfying \(\|P_{\xi} - Q_{\xi}\| < 1/10\) for each \(\xi < \omega_1\) which is obtained using Lemma 6.1. In particular \(\|P_{\xi} - P_{\xi'}\| \geq 8/10\) for all \(\xi < \xi' < \omega_1\) and so they generate a nonseparable \(C^*\)-algebra.

We have \(\|P_{\xi_1} P_{\xi_2} - Q_{\xi_1} Q_{\xi_2}\| < 1/5\) and \(\|P_{\xi_2} P_{\xi_3} - Q_{\xi_2} Q_{\xi_3}\| < 1/5\) for each \(\xi_1 < \xi_2 < \omega_1\), so \(\|P_{\xi_2} - 2/5\) for each \(\xi_1 < \xi_2 < \omega_1\). But by \((1)\) \((c)\) there are \(\xi_1 < \xi_2 < \omega_1\) such that \(\|P_{\xi_2} - P_{\xi_2}\| \geq 2/5\), a contradiction.
References

1. U. Abraham, M. Rubin, S. Shelah, On the consistency of some partition theorems for continuous colorings, and the structure of $\mathfrak{N}_1$-dense real order types. Ann. Pure Appl. Logic 29 (1985), no. 2, 123–206.

2. C. Akemann, Left ideal structure of C*-algebras. J. Functional Analysis 6 1970, 305–317.

3. C. Akemann, N. Weaver, Consistency of a counterexample to Naimark’s problem. Proc. Natl. Acad. Sci. USA 105 (2008), no. 14, 5313–5314.

4. C. Akemann, N. Weaver, $B(H)$ has a pure state that is not multiplicative on any masa. Proc. Natl. Acad. Sci. USA 105 (2008), no. 14, 5313–5314.

5. W. Arveson, An invitation to C*-algebras. Graduate Texts in Mathematics, No. 39. Springer-Verlag, New York-Heidelberg, 1976.

6. M. Bell, J. Ginsburg, S. Todorčević, Countable spread of $\exp Y$ and $\lambda Y$. Topology Appl. 14 (1982), no. 1, 1–12.

7. T. Bice, P. Koszmider, A note on the Akemann-Doner and Farah-Wofsey constructions. Proc. Amer. Math. Soc. 145 (2017), no. 2, 681–687.

8. C. Brech, P. Koszmider, Thin-very tall compact scattered spaces which are hereditarily separable. Trans. Amer. Math. Soc. 363 (2011), no. 1, 501–519.

9. C. Brech, P. Koszmider, On biorthogonal systems whose functionals are finitely supported. Fund. Math. 213 (2011), no. 1, 43–66.

10. C. Brech, N. Weaver, $B(H)$ has a pure state that is not multiplicative on any masa. Proc. Natl. Acad. Sci. USA 105 (2008), no. 14, 5313–5314.

11. W. Arveson, An invitation to C*-algebras, Graduate Texts in Mathematics, No. 39. Springer-Verlag, New York-Heidelberg, 1976.

12. M. Dzamonja, I. Juhasz, CH, a problem of Rolewicz and bidiscrete systems. Topol. Appl. 158 (18) (2011) 2458–2494.

13. P. Enflo, H. Rosenthal, Some results concerning $L_p(\mu)$-spaces. J. Functional Analysis 14 (1973), 325–348.

14. R. Engelking, General topology. Translated from the Polish by the author. Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989.

15. I. Farah, Analytic quotients: theory of liftings for quotients over analytic ideals on the integers. Mem. Amer. Math. Soc. 148 (2000), no. 702.

16. I. Farah, T. Katsura, Nonseparable UHF algebras I: Dixmier’s problem. Adv. Math. 225 (2010), no. 3, 1399–1430.

17. I. Farah, I. Hirshberg, Simple nuclear C*-algebras not isomorphic to their opposites. Proc. Natl. Acad. Sci. USA 114 (2017), no. 24, 6244–6249.

18. S. Ghaisem, P. Koszmider, Noncommutative Cantor-Bendixson derivatives and scattered C*-algebras. Topology Appl. 240 (2018), 181–209.

19. S. Ghaisem, P. Koszmider, A non-stable C*-algebra with an elementary essential composition series. arXiv:1712.02090

20. P. Hajek, V. Montesinos Santalucia, J. Vanderwerff, V. Zizler, Biorthogonal systems in Banach spaces. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 26. Springer, New York, 2008.

21. R. Hodel, Cardinal functions. I. Handbook of set-theoretic topology, 1–61, North-Holland, Amsterdam, 1984.

22. K. Hofmann, K. Neeb, Epimorphisms of C*-algebras are surjective. Arch. Math. (Basel) 65 (1995), no. 2, 134–137.

23. L. Heindorf, A note on irredundant sets. Algebra Universalis 26 (1989), no. 2, 216–221.

24. C. Hida, Two cardinal inequalities about bidigenerate systems. Topology Appl. 212 (2016), 71–80.

25. T. Jech, Set theory. The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

26. H. Jensen, Scattered C*-algebras. Math. Scand. 41 (1977), no. 2, 308–314.

27. H. Jensen, Scattered C*-algebras. II. Math. Scand. 43 (1978), no. 2, 308–310 (1979).

28. S. Koppelberg, Handbook of Boolean algebras. Vol. 1. Edited by J. Donald Monk and Robert Bonnet. North-Holland Publishing Co., Amsterdam, 1989.
29. P. Koszmider, On a problem of Rolewicz about Banach spaces that admit support sets. J. Funct. Anal. 257 (2009), no. 9, 2723–2741.

30. P. Koszmider, Some topological invariants and biorthogonal systems in Banach spaces. Extr. Math. 26 (2) (2011) 271–294.

31. P. Koszmider, On the problem of compact totally disconnected reflection of nonmetrizability. Topology Appl. 213 (2016), 154–166.

32. P. Koszmider, On constructions with 2-cardinals. Arch. Math. Logic 56 (2017), no. 7-8, 849–870.

33. K. Kunen, An introduction to independence proofs. Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam-New York, 1980.

34. M. Kusuda, C*-algebras in which every C*-subalgebra is AF. Q. J. Math. 63 (2012), no. 3, 675–680.

35. H. X. Lin, The structure of quasi-multipliers of C*-algebras, Trans. Amer. Math. Soc. 315 (1989), no. 1, 147–172.

36. F. Lopez, S. Todorcevic, Trees and gaps from a construction scheme. Proc. Amer. Math. Soc. 145 (2017), no. 2, 871–879.

37. F. Lopez, Banach spaces from a construction scheme. J. Math. Anal. Appl. 446 (2017), no. 1, 426–435.

38. A. Mostowski, A. Tarski, Boolean Rings mit ordner basis. Fund. Math. 32, 69–86.

39. G. Murphy, C*-algebras and operator theory. Academic Press, Inc., Boston, MA, 1990.

40. S. Negrepontis, Banach spaces and topology. Handbook of set-theoretic topology, 1045–1142, North-Holland, Amsterdam, 1984.

41. T. Ogasawara, Finite-dimensionality of certain Banach algebras. J. Sci. Hiroshima Univ. Ser. A. 17, (1954). 359–364.

42. C. Olsen, W. Zame, Some C*-algebras with a single generator. Trans. Amer. Math. Soc. 215 (1976), 205–217.

43. A. Ostaszewski, On countably compact, perfectly normal spaces. J. London Math. Soc. (2) 14 (1976), no. 3, 505–516.

44. A. Pelczyński, Z. Semadeni, Spaces of continuous functions. III. Spaces C(Ω) for Ω without perfect subsets. Studia Math. 18 1959 211–222.

45. S. Popa. Orthogonal pairs of ∗-subalgebras in finite von Neumann algebras. J. Operator Theory 9 (1983), no. 2, 253–268.

46. M. Rubin, A Boolean algebra with few subalgebras, interval Boolean algebras and retraction-ness. Trans. Amer. Math. Soc. 278 (1983), no. 1, 65–89.

47. J.C. Stampfli, The norm of a derivation. Pacific J. Math. 33, 1970, 737–747.

48. H. Thié, The generator rank for C*-algebras. arXiv:1210.6608

49. H. Thié, W. Winter. The generator problem for Z-stable C*-algebras. Trans. Amer. Math. Soc. 366 (2014), no. 5, 2327-2343.

50. J. Tomiyama, A characterization of C*-algebras whose conjugate spaces are separable. Tohoku Mathematical Journal, Second Series 15.1 (1963): 96-102.

51. S. Todorcevic, Partition problems in topology. Contemporary Mathematics, 84. American Mathematical Society, Providence, RI, 1989.

52. S. Todorcevic, Irredundant sets in Boolean algebras, Trans. Am. Math. Soc. 339 (1) (1993) 35–44.

53. S. Todorcevic, Biorthogonal systems and quotient spaces via Baire category methods. Math. Ann. 335 (3) (2006), 687–715.

54. S. Todorcevic, A construction scheme for non-separable structures. Adv. Math. 313 (2017), 564–589.

55. D. Velleman, ω-morasses, and a weak form of Martin’s axiom provable in ZFC. Trans. Am. Math. Soc. 265 (2) 617–627 (1981)

56. P. Wojtaszczyk, On linear properties of separable conjugate spaces of C*-algebras. Studia Math. 52 (1974), 143–147.
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