THE LONGEST EXCURSION  
OF A RANDOM INTERACTING POLYMER  

JANINE KÖCHER¹ and WOLFGANG KÖNIG¹²  

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Abstract: We consider a random $N$-step polymer under the influence of an attractive interaction with the origin and derive a limit law – after suitable shifting and norming – for the length of the longest excursion towards the Gumbel distribution. The embodied law of large numbers in particular implies that the longest excursion is of order $\log N$ long. The main tools are taken from extreme value theory and renewal theory.

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1. Introduction and main results

Let $(S_n)_{n \in \mathbb{N}_0}$ be a random walk on the lattice $\mathbb{Z}^d$ starting at the origin and having steps of mean zero. By $\mathbb{P}$ and $\mathbb{E}$ we denote the corresponding probability and expectation, respectively. We conceive the walk $(n, S_n)_{n=0,\ldots,N}$ as an $N$-step polymer in the $(d + 1)$-dimensional space. We introduce an attractive interaction with the origin by introducing the Gibbs measure $\mathbb{P}_{\beta,N}$ via the density

$$
\frac{d\mathbb{P}_{\beta,N}}{d\mathbb{P}} = \frac{e^{\beta L_N}}{Z_{\beta,N}}
$$

with $Z_{\beta,N} = \mathbb{E}[e^{\beta L_N}]$, (1.1)

where $\beta \in (0, \infty)$ is a parameter and

$$
L_N = |\{k \in \{1,\ldots,N\} : S_k = 0\}|
$$

(1.2)

denotes the walker’s local time at the origin, i.e., the number of returns to the origin. The properties of the polymer under $\mathbb{P}_{\beta,N}$ have been studied a lot [dH09, G07]. In particular, the free energy

$$
F(\beta) = \lim_{N \to \infty} \frac{1}{N} \log Z_{\beta,N} \in (0, \beta)
$$

(1.3)

has been shown to exist and to be positive and strictly increasing in $\beta$. Furthermore, it has been shown that the polymer is localised in the sense that $L_N$ is of order $N$ under $\mathbb{P}_{\beta,N}$, and

¹Institute for Mathematics, TU Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany, JanineKoecher@aol.com

²Weierstrass Institute Berlin, Mohrenstr. 39, 10117 Berlin, koenig@wias-berlin.de
the density of the set of hits of the origin has been characterised. In particular, the constrained
version, i.e., the polymer under
\[
P^{(c)}_{\beta,N}(\cdot) = \frac{1}{Z^{(c)}_{\beta,N}} \mathbb{E}[e^{\beta L_N} \mathbb{1}_{\{S_N = 0\}}],
\]
where
\[
Z^{(c)}_{\beta,N} = \mathbb{E}[e^{\beta L_N} \mathbb{1}_{\{S_N = 0\}}],
\]
(1.4)
has been studied.

In this paper, we consider the length of the longest excursion of the polymer under \(P^{(c)}_{\beta,N}\). To introduce this object, we denote by \(\tau = \{\tau_i : i \in \mathbb{N}_0\}\) the set of return times to the origin, where \(\tau_0 = 0\) and, inductively, \(\tau_{i+1} = \inf\{n > \tau_i : S_n = 0\}\), \(i \in \mathbb{N}_0\).

Then \(P^{(c)}_{\beta,N}\) is the conditional distribution of the polymer given \(\{N \in \tau\}\). The length of the longest excursion is now given as
\[
\text{maxexc}_N = \max\{\tau_i - \tau_{i-1} : i \in \mathbb{N}, \tau_i \leq N\}. 
\]
(1.6)
According to [dH09, Theorem 7.3], \(\text{maxexc}_N\) is of order \(\log N\) under \(P^{(c)}_{\beta,N}\), in the sense that the distribution of \(\text{maxexc}_N / \log N\) under \(P^{(c)}_{\beta,N}\) is tight in \(N\). The proof gives the upper bound \(2/F(\beta)\), which is not sharp, as we will see below. It is the main goal of this note to derive not only the law of large numbers for \(\text{maxexc}_N\), but also a non-trivial limit law for \(\text{maxexc}_N\) after suitable shifting, in the spirit of extreme value theory.

To formulate our main result, we need to fix our assumptions first.

**Assumption (\(\tau\)).** There are \(D \in (0, \infty)\) and \(\alpha \in (1, \infty)\) such that
\[
K(n) := \mathbb{P}(\tau_1 = n) \sim Dn^{-\alpha}, \quad n \to \infty.
\]
This assumption is fulfilled for most of the aperiodic random walks \((S_n)_{n \in \mathbb{N}_0}\) under consideration in the literature. For random walks with period \(p \in \mathbb{N}\), one has to work with \(K(pn)\) instead of \(K(n)\) and with \(pN\)-step polymers and obtains analogous results. Assumption (\(\tau\)) can be relaxed with the help of slowly varying functions, on cost of a more cumbersome formulation and proof of the main result.

The main result of this paper is the following.

**Theorem 1.1.** Suppose that Assumption (\(\tau\)) is satisfied, and fix \(\beta \in (0, \infty)\). Then, as \(N \to \infty\), the distribution of
\[
F(\beta)\text{maxexc}_N - \log \frac{N}{\mu_{\beta}} + \alpha \log \log \frac{N}{\mu_{\beta}} - C
\]
under \(P^{(c)}_{\beta,N}\) weakly converges towards the standard Gumbel distribution, where
\[
\mu_{\beta} = e^\beta \sum_{n \in \mathbb{N}} nK(n)e^{-nF(\beta)} \quad \text{and} \quad C = \log \left(F(\beta)\alpha D \frac{e^\beta - F(\beta)}{1 - e^{-F(\beta)}}\right).
\]
(1.7)
(1.8)
Explicitly, it is stated that, for any \(x \in \mathbb{R}\),
\[
\lim_{N \to \infty} P^{(c)}_{\beta,N}(\text{maxexc}_N \leq \gamma_x(N/\mu_{\beta})) = e^{-e^{-x}},
\]
where \(\gamma_x(N) = \frac{x + C + \log N - \alpha \log \log N}{F(\beta)}\).
(1.9)
In particular, we have the law of large numbers: \(\text{maxexc}_N / \log N \to 1/F(\beta)\) in \(P^{(c)}_{\beta,N}\)-probability as \(N \to \infty\).
2. The proof

It is well-known that the free energy $F(\beta)$ is characterized by the equation

$$e^\beta = \sum_{n \in \mathbb{N}} K(n)e^{-nF(\beta)}, \quad (2.1)$$

and that it actually holds that $Z_{\beta,N}^{(c)} \sim e^{NF(\beta)}\frac{1}{\mu_\beta}$ as $N \to \infty$. In particular, $F(\beta)$ is also the exponential rate of $Z_{\beta,N}^{(c)}$. The first step, which is basic to all investigations of the polymer, is a change of measure to the measure $Q_\beta$, under which the excursion lengths $T_k = \tau_{k+1} - \tau_k$, are i.i.d. in $k \in \mathbb{N}_0$ with distribution

$$Q_\beta(T_1 = n) = e^{-\beta}K(n)e^{-nF(\beta)}, \quad n \in \mathbb{N}.$$

Since maxexc$_N$ is measurable with respect to the family of the $T_k$'s, it is easy to see from the technique explained in [G07, p. 9] that

$$P_\beta N(\text{maxexc}_N \leq \gamma_N, N \in \tau), \quad N \to \infty, \quad (2.2)$$

for any choice of the sequence $(\gamma_N)_{N \in \mathbb{N}}$, where $\mu_\beta = \sum_{n \in \mathbb{N}} nQ_\beta(T_1 = n) \in [1, \infty)$ is the expectation of the length of the first excursion under $Q_\beta$. Introducing

$$M_n = \max_{k=1}^n T_k \quad \text{and} \quad \sigma_N = \inf\{k \in \mathbb{N}: \tau_k \geq N\}, \quad (2.3)$$

we see that maxexc$_N = M_{\sigma_N}$ on $\{N \in \tau\}$ for any $N \in \mathbb{N}$. (Note that $\sigma_N = L_N$ on the event $\{N \in \tau\}$.) Hence, Theorem 1.1 is equivalent to

$$\lim_{N \to \infty} Q_\beta(M_{\sigma_N} \leq \gamma_x(N/\mu_\beta), N \in \tau) = \frac{1}{\mu_\beta}e^{-e^{-x}}, \quad x \in \mathbb{R}. \quad (2.4)$$

The proof of this consist of a combination of three fundamental ingredients:

1. an extreme value theorem for $M_n$ under $Q_\beta$,
2. a law of large numbers for $\sigma_N$ under $Q_\beta$,
3. a renewal theorem for $\tau$ under $Q_\beta$.

Items (2) and (3) are immediate: We have from renewal theory that $\sigma_N/N \to 1/\mu_\beta$ in $Q_\beta$-probability and $\lim_{N \to \infty} Q_\beta(N \in \tau) = 1/\mu_\beta$. The first item needs a bit more care:

**Lemma 2.1.**

$$\lim_{N \to \infty} Q_\beta(M_N \leq \gamma_x(N)) = e^{-e^{-x}}, \quad x \in \mathbb{R}.$$

**Proof.** Note that $M_N$ is the maximum of $N$ independent random variables with the same distribution as $T_1 = \tau_1$ under $Q_\beta$. Observe that the tails of this distribution are given by

$$Q_\beta(\tau_1 > k) = e^\beta \sum_{n > k} K(n)e^{-nF(\beta)} \sim e^\beta D \sum_{n > k} n^{-\alpha}e^{-nF(\beta)}$$

$$= e^\beta De^{-kF(\beta)}k^{-\alpha} \sum_{n \in \mathbb{N}} (1 + \frac{n}{k})^{-\alpha}e^{-nF(\beta)}$$

$$\sim e^{-kF(\beta)}k^{-\alpha} D \frac{e^{\beta-F(\beta)}}{1 - e^{-F(\beta)}}, \quad k \to \infty,$$
where in the last step we used the monotonous convergence theorem and the geometric series. Hence, replacing $k$ by $\gamma_x(N)$, we see that, as $N \to \infty$,

$$Q_{\beta}(\tau_1 > \gamma_x(N)) \sim e^{-\gamma_x(N)F(\beta)}\gamma_x(N)^{-\alpha}D\frac{e^{\beta-F(\beta)}}{1-e^{-F(\beta)}}$$

$$= \frac{1}{N}e^{C-x}(\log N)^\alpha\left(\frac{x+C+\log N-\alpha \log \log N}{F(\beta)}\right)^{-\alpha}e^{C}F(\beta)^{-\alpha}$$

$$\sim \frac{e^{-x}}{N}.$$

From this the assertion easily follows. \hfill \Box

Hence, Theorem 1.1 is easily seen to follow from the above three ingredients, as soon as one shows that $\sigma_N$ may asymptotically be replaced by $N/\mu_\beta$ and that the two events in (2.4) are asymptotically independent. This is what we show now. First we show that $M_{\sigma_N}$ and $M_{N/\mu_\beta}$ have the same limiting distribution.

**Lemma 2.2.**

$$\lim_{N \to \infty} Q_{\beta}(M_{\sigma_N} \leq \gamma_x(N/\mu_\beta)) = e^{-e^{-x}}, \quad x \in \mathbb{R}.$$

**Proof.** The upper bound is proved as follows. Fix a small $\varepsilon > 0$, then we have, as $N \to \infty$,

$$Q_{\beta}(M_{\sigma_N} \leq \gamma_x(N/\mu_\beta)) \leq Q_{\beta}\left(M_{\sigma_N} \leq \gamma_x(N/\mu_\beta), \sigma_N \geq \frac{N}{\mu_\beta + \varepsilon}\right) + Q_{\beta}\left(\sigma_N < \frac{N}{\mu_\beta + \varepsilon}\right)$$

$$\leq Q_{\beta}\left(M_{N/(\mu_\beta + \varepsilon)} \leq \gamma_x(N/\mu_\beta)\right) + o(1).$$

Observe that, as $N \to \infty$,

$$\gamma_x(N/\mu_\beta) - \gamma_x(N/\mu_\beta + \varepsilon) = \frac{1}{F(\beta)}\log(1 + \frac{\varepsilon}{\mu_\beta}) + \frac{\alpha}{F(\beta)}\log \frac{\log N - \log (\mu_\beta + \varepsilon)}{\log N - \log \mu_\beta}$$

$$= \frac{1}{F(\beta)}\log(1 + \frac{\varepsilon}{\mu_\beta}) + o(1).$$

Hence, we may replace, as an upper bound, $\gamma_x(N/\mu_\beta)$ on the right of (2.5) by $\gamma_x(N_B(N/\mu_\beta + \varepsilon))$ for some suitable $B \in \mathbb{R}$, use Lemma 2.1 for $N$ replaced by $N/(\mu_\beta + \varepsilon)$ and $x$ replaced by $x + B\varepsilon$ and make $\varepsilon \downarrow 0$ in the end. This shows that the upper bound of the assertion holds. The lower bound is proved in the same way. \hfill \Box

**Proof of Theorem 1.1**

It is convenient to introduce a Markov chain $(Y_n)_{n \in \mathbb{N}_0}$ with

$$Y_n = (Y_n^{(1)}, Y_n^{(2)}) = (T_{\sigma_n}, \tau_{\sigma_n} - n)$$

on the state space $I = \{(i, j) \in \mathbb{N} \times \mathbb{N}_0; j \leq i\}$, where we recall (2.3). In words, the first component is the size of the step over $n$, and the last is the size of the overshoot. This Markov chain is ergodic and positiv recurrent with invariant distribution $\pi(i, j) = Q_{\beta}(\tau_1 = i)/\mu_\beta$ for $(i, j) \in I$. We denote by $\tilde{Q}_{i,j}$ the distribution of this chain given that it starts in $Y_0 = (i, j)$; note that $Q_{\beta} = \tilde{Q}_{i,0}$ with an unspecified value of $i$, which we put equal to 1 by default. The event $\{N \in \tau\}$ is identical to $\{Y_N^{(2)} = 0\} = \{Y_N \in \mathbb{N} \times \{0\}\}$; by ergodicity, its probability under $\tilde{Q}_{i,j}$ converges, as $N \to \infty$, to $\pi(\mathbb{N} \times \{0\}) = \frac{1}{\mu_\beta}$, for any $(i, j) \in I$, which is one way to prove the renewal theorem.

Now let $\varepsilon > 0$ be given. Pick $K_\varepsilon \in \mathbb{N}$ so large that $\pi(I_{K_\varepsilon}^\varepsilon) < \varepsilon/2$, where $I_k = \{(i, j) \in I; i \leq k\}$ for any $k \in \mathbb{N}$. Furthermore, pick $R_\varepsilon \in \mathbb{N}$ with $R_\varepsilon > K_\varepsilon$ so large that $\tilde{Q}_{i,j}(R_\varepsilon \in \tau) \leq \frac{1}{\mu_\beta} + \varepsilon$ for
any \((i, j) \in I_{K\varepsilon}\). Now pick \(N_{\varepsilon} \in \mathbb{N}\) so large that \(N_{\varepsilon} > R_{\varepsilon}\) and \(\tilde{Q}_{1,0}(Y_{N-R_{\varepsilon}} \in I_{K\varepsilon}^c) < \pi(I_{K\varepsilon}^c) + \varepsilon/2\) for any \(N \geq N_{\varepsilon}\). The latter is possible, since \(\tilde{Q}_{1,0}(Y_{N-R_{\varepsilon}} \in I_{K\varepsilon}^c) = 1 - Q_{1,0}(Y_{N-R_{\varepsilon}} \in I_{K\varepsilon})\) converges towards \(1 - \pi(I_{K\varepsilon}) = \pi(I_{K\varepsilon}^c)\) as \(N \to \infty\) by ergodicity.

Recall that we only have to prove (2.4). We calculate, with the help of the Markov property at time \(N - R_{\varepsilon}\), for \(N > N_{\varepsilon}\),

\[
Q_{\beta}(\sigma_N \leq \gamma_x(N/\mu_\beta), N \in \tau) = \tilde{Q}_{1,0}\left(\max_{k=1}^{N-R_{\varepsilon}} Y_k(1) \leq \gamma_x(N/\mu_\beta), Y_N(2) = 0\right)
\]

\[
\leq \tilde{Q}_{1,0}\left(\max_{k=1}^{N-R_{\varepsilon}} Y_k(1) \leq \gamma_x(N/\mu_\beta), Y_N(2) = 0\right)
+ \tilde{Q}_{1,0}(Y_{N-R_{\varepsilon}} \in I_{K\varepsilon})
\]

\[
\leq \sum_{(i,j) \in I_{K\varepsilon}} \tilde{Q}_{1,0}\left(\max_{k=1}^{N-R_{\varepsilon}} Y_k(1) \leq \gamma_x(N/\mu_\beta), Y_N(2) = 0\right)
+ \tilde{Q}_{i,j}(Y_{N-R_{\varepsilon}} = 0) + \pi(I_{K\varepsilon}) + \varepsilon/2
\]

\[
\leq \tilde{Q}_{1,0}\left(\max_{k=1}^{N-R_{\varepsilon}} Y_k(1) \leq \gamma_x(N/\mu_\beta)\right)\left(\frac{1}{\mu_\beta} + \varepsilon\right) + \varepsilon
\]

Now apply Lemma 2.2 for \(N\) replaced by \(N - R_{\varepsilon}\) and observe that \(\lim_{N \to \infty}(\gamma_x(N/\mu_\beta) - \gamma_x((N - R_{\varepsilon})/\mu_\beta)) = 0\). Afterwards letting \(\varepsilon \downarrow 0\) shows that the upper bound in (2.4) holds. The proof of the corresponding lower bound is similar, and we omit it. \(\square\)

**References**

[G07] G. Giacomin, *Random Polymer Models*, Imperial College Press (2007).

[dH09] F. den Hollander, *Random Polymers*, Springer (2009).