The Model of Multi Limits of the Quantity of Electrons in Central Coulomb force Field

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Abstract. Based on the theory of periodic bifurcation of iterative equation, a conjectural model of periodic bifurcation of number of electrons in a central Coulomb force field is proposed. After that with the help of the methods Zeng’s The Course of Quantum Mechanics and Wu’s Methods of Mathematical Physics, [1], [2] the wave function of the electrons under the approximate state is solved in the central Coulomb force field. By using the method of separating variables for solving partial differential equations and some transformation and construction techniques, the strict mathematical solution of the Schrödinger equation for the electron in the field of central Coulomb force is obtained, and the iterative formula of the level of electron number is given theoretically. And using MATLAB, the multi-limit model of electron number is simulated under different initial value problems, to explore the change of the limit with the initial value and the factors affecting the limit number to a certain extent. Some potential research value of this model is also proposed.

Keywords: Schrödinger equation, wave function, MATLAB, non-linear iteration, bifurcation, multiple limits of electron number, possible application.

1. Introduction of multi-limit model of electron number in Coulomb force field

It is supposed there is a central Coulomb force field in a space, where there are a bunch of atoms and electrons, and the central Coulomb force is created by a hole which is positively charged and in the middle region of this space. And two approximations are made in order to simplify the model, making its Schrödinger equation accessible to solved.

a. Since the number of electrons is enough to ignore the phase difference because of time, the static Schrödinger equation can be directly used to describe the distribution probability of electrons. The potential energy that this central Coulomb force field brings to each electron is shown as follows.

\[ V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r} \]  

(1).

b. The coulomb force of the hole is required to be large enough to ignore the force exerted on one electron by the other electrons in the nucleus. So that the Hamiltonian of the electron is shown as follows.
\[ H = -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \]  

(2).

At the same time using the concept of time quantum, with which the location of the particles in quantum mechanics in the next moment is defined as each particle's position after one quantum time, and apply the definition of the randomness of quantum mechanics and wave functions, namely every in a quantum time, electronic will appear at a random location, and its appearance in the probability of a certain spatial location can be determined by wave functions.

As assumed, there are many electrons revolving around this hole. If these electrons fall in the middle, they will be absorbed by the hole, at the same time, the energy will be released, and the electrons absorbed by the hole will be transferred. If an electron hits a nearby atom, it knocks out a new electron, and in this process, the total energy is conserved. There it is suppose that the energy of the hole which stay still and of neutral atoms add up to zero. Therefore, the whole energy of this system maintains a constant value. When the electron of an atom is expelled, it is no longer electrically neutral, but becomes a positive ion, which has a positive charge, and is repulsive by the hole, and escapes from the system. The positive ion escaping from the system can combine with the electron absorbed by the hole to form a new neutral atom, and the outside world will continuously replenish atoms to the system to keep the number of atoms in the system stable.

It can be known that the amount of newly appearing electron and that of swallowed electron is positively correlated with the current number of electrons. In this system, the number level of electrons can be defined as \( x \), which is strictly proportional to the number of electrons, and it is required that \( x \) is in the range as below.

\[ 0 \leq x \leq 1 \]  

(3)

The number level of electrons at the next moment should be proportional to the number of electrons at the present moment. But since the total energy remains constant, the more electrons there are, the less average energy of each electron there is, the probability that the electrons will be absorbed more easily by the black hole in the middle will be bigger. There we assume that the decrease is negatively correlated to the number of electrons, which can be inferred from the relationship between orbital radius and energy on the principal quantum number \( n \), which will be proved below. Therefore, while calculating the quantity level of electrons at the next moment through the number level of electrons at the present moment, the right side of the equation should be multiplied by factor \( \left[ 1 - \frac{1}{r_1^4} \left( 1 - \frac{u}{x} \right) \right] \) or \( \left[ 1 - \frac{1}{r_1^4} \left( 1 - \frac{u}{x_0} \right) \right] \), which will be discussed in the following chapters. The probability of each new electron emitted and those absorbed is only proportional to the amount level of electrons at that moment. As a result, after a Planck time, the number level of electrons after the process is proportional to \( x \left[ 1 - \frac{1}{r_1^4} \left( 1 - \frac{u}{x} \right) \right] \) or \( x \left[ 1 - v^2(1 - \frac{w}{x})^2 \right] \).

After the first quantum time, the change in quantity level of electrons can be displayed by the equation as follows. \( x_1 = ax_0 \left[ 1 - \frac{1}{r_1^4} \left( 1 - \frac{u}{x_0} \right) \right] \) or \( x_1 = ax_0 \left[ 1 - v^2(1 - \frac{w}{x_0})^2 \right] \) (\( a \) is a constant related to the size of the hole and the number density of atoms), and in the second process, there is \( x_2 = kx_1 \left[ 1 - \frac{1}{r_1^4} \left( 1 - \frac{u}{x_1} \right) \right] \) or \( x_2 = kx_1 \left[ 1 - v^2(1 - \frac{w}{x_1})^2 \right] \). by parity of reasoning, in the kth quantum time, the 2 equations as below is raised up.

\[ x_{k+1} = ax_k \left[ 1 - \frac{1}{r_1^4} \left( 1 - \frac{u}{x_k} \right) \right] \]  

(4)

\[ x_{k+1} = ax_k \left[ 1 - v^2(1 - \frac{w}{x_k})^2 \right] \]  

(5)
Define that.

\[ I = \lim_{k \to \infty} x_k \quad (6) \]

According to the limit splitting and chaos model of logistic iterative equation, the limit of the number level of electrons in this space will tend to one or \( 2n \) different limits or even chaos, and what the limit will be depends on the parameter \( a \). That is, after many quantum time, the number of electrons at the last time and the next time will suddenly change, which is periodic. About \( 2n \) different limits, for example, if \( n = 1 \), \( 2n = 2 \). When \( k \to \infty \), \( x_{2f+1} = x_{2f+3} = x_{2f+5} = \cdots = I_1 \), \( x_{2f+2} = x_{2f+4} = x_{2f+6} = \cdots = I_2 \), however, \( I_1 \neq I_2 \). In fact, when there are \( 2n \) multi-limits, the limit of the iterative will oscillate periodically among \( 2n \) various values.

2. Wave function of electrons in central Coulomb force field

2.1. Schrödinger equation of electrons in a centered Coulomb force field

Stationary Schrödinger equation:

\[ H\psi = E\psi \quad (7) \]

And the Hamiltonian of the electron is shown as formula (2). Within it, the potential energy of electrons is exhibited as \( V(r) \).

In the spherical coordinates the Laplace operator is as below.

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (8) \]

Insert (1), (2) and (8) into (7), the equation as below is shown.

\[ \left\{ -\frac{\hbar^2}{2\mu} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right\} \psi = E\psi \quad (9) \]

Transferred, the equation can be

\[ \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right\} + \frac{2\mu}{\hbar^2} (E + \frac{Ze^2}{4\pi\varepsilon_0 r}) \psi = 0 \quad (10) \]

2.2. The separation of Schrödinger equation by separating variables

Make a definition as follows.

\[ \psi = R(r)\Theta(\theta)\Phi(\phi) \quad (11) \]

Introduce it into, the equation can be gotten.

\[ \left[ \Theta(\theta)\Phi(\phi) \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + R(r)\Phi(\phi) \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + \right. \]

\[ \Theta(\theta)\Phi(\phi) \frac{1}{r^2 \sin^2 \theta} \frac{d^2 \Phi(\phi)}{d\phi^2} \left\{ \frac{2\mu}{\hbar^2} (E + \frac{Ze^2}{4\pi\varepsilon_0 r}) \right\} R(r)\Theta(\theta)\Phi(\phi) = 0 \quad (12) \]

After 2 sides divided by \( R(r)\Theta(\theta)\Phi(\phi) \), we get

\[ \frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{1}{\Theta(\theta)} \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + \frac{1}{\Phi(\phi)} \frac{1}{r^2 \sin^2 \theta} \frac{d^2 \Phi(\phi)}{d\phi^2} + \frac{2\mu}{\hbar^2} (E + \frac{Ze^2}{4\pi\varepsilon_0 r}) = 0 \quad (13) \]
Multiplying $r^2 \sin^2 \theta$ on both sides, we have

$$\frac{\sin^2 \theta}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{\sin \theta}{\theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\theta}{d\theta} \right) + \frac{1}{\Phi(\varphi)} \frac{d^2 \Phi(\varphi)}{d\varphi^2} + \frac{2\mu r^2 \sin^2 \theta}{h^2} (E + \frac{Z \epsilon^2}{4\pi \sigma r}) = 0$$

(14)

After a deformation, the equation as below can be attained.

$$\frac{\sin^2 \theta}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{\sin \theta}{\theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\theta}{d\theta} \right) + \frac{2\mu r^2 \sin^2 \theta}{h^2} (E + \frac{Z \epsilon^2}{4\pi \sigma r}) = -\frac{1}{\Phi(\varphi)} \frac{d^2 \Phi(\varphi)}{d\varphi^2} = m^2 = \text{const}$$

(15)

Divide it we can get 2 equations as below.

$$\frac{d^2 \Phi(\varphi)}{d\varphi^2} + m^2 \Phi(\varphi) = 0$$

(16)

$$\frac{\sin^2 \theta}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{\sin \theta}{\theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\theta}{d\theta} \right) + \frac{2\mu r^2 \sin^2 \theta}{h^2} (E + \frac{Z \epsilon^2}{4\pi \sigma r}) = m^2$$

(17)

After both sides of equation (17) are divided by $\sin^2 \theta$, we have

$$\frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{1}{\sin \theta \theta(\theta)} \frac{d}{d\theta} \left( \sin \theta \frac{d\theta}{d\theta} \right) + \frac{2\mu r^2}{h^2} (E + \frac{Z \epsilon^2}{4\pi \sigma r}) = \frac{m^2}{\sin^2 \theta}$$

(18)

After the transposition, we get the following equation.

$$\frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{2\mu r^2}{h^2} (E + \frac{Z \epsilon^2}{4\pi \sigma r}) = \frac{m^2}{\sin^2 \theta} - \frac{1}{\sin \theta \theta(\theta)} \frac{d}{d\theta} \left( \sin \theta \frac{d\theta}{d\theta} \right) = \beta = \text{const}$$

(19)

Divide the equation (19)

$$\frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{2\mu r^2}{h^2} (E + \frac{Z \epsilon^2}{4\pi \sigma r}) = \beta$$

(20)

$$\frac{m^2}{\sin^2 \theta} - \frac{1}{\sin \theta \theta(\theta)} \frac{d}{d\theta} \left( \sin \theta \frac{d\theta}{d\theta} \right) = \beta$$

(21)

From equation (12), there are 3 independent equations we can list out.

Equation (16) is just an equation of variable $\varphi$, (20) is only an equation of variable $r$, (21) is merely an equation of variable $\theta$.

2.3. Solution of the equation about $\varphi$

Equation (16) is the one that is only related to variable $\varphi$, which is a kind of common second order linear homogeneous equation, and the solution is not complicated, as shown below.

$$\Phi_m(\varphi) = C_1 \exp(-im\varphi) + C_2 \exp(im\varphi) = -iC_1 \sin(m\varphi) + C_1 \cos(m\varphi) + iC_2 \sin(m\varphi) + C_2 \cos(m\varphi)$$

(22)

The selection of the initial value of $\varphi$ is arbitrary, so we could list out the equations as below.

$$\Phi_m(0) = 0$$

(23)

And via (23), we have following equation.

$$C_1 + C_2 = 0$$

(24)
So that

\[ C_1 = -C_2 \]  

(25)

Therefore, the following function can be obtained.

\[ \Phi_m(\varphi) = C_1[\exp(-im\varphi) - \exp(im\varphi)] = -2iC_1 \sin(m\varphi) = -iA \sin(m\varphi) \]  

(A = 2C_1)  

(26)

(27)

And there is the periodic condition of angle \( \varphi \), shown as follows.

\[ \Phi_m(\varphi) = \Phi_m(\varphi + 2\pi) \]  

(28)

That is exhibited as below.

\[ \sin(m\varphi) = \sin(m\varphi + 2m\pi) \]  

(29)

And it is for every value of \( \varphi \), so that the \( m \) could take different values.

\[ m = 0, \pm 1, \pm 2, \pm 3 \ldots \ldots \]  

(30)

Because of the condition of normalizing, the equation below is shown.

\[ \int_0^{2\pi} \Phi(\varphi)\Phi^*(\varphi)d\varphi = \int_0^{2\pi} A^2 \sin^2(m\varphi)d\varphi = 1 \]  

(31)

By (31), we can figure out this as follow. \[ A^2 \left( \pi - \frac{\sin(4mn)}{4m} \right) = 1, \]  

Because \( \sin(4m) = 0 \), \( m = 0, \pm 1, \pm 2, \pm 3 \ldots \ldots \), we have

\[ \frac{\sin(4m)}{4m} = 0 \]  

(32)

So, we have.

\[ \pi A^2 = 1 \]  

(33)

Then we get.

\[ A = \sqrt{\frac{1}{\pi}} \]  

(34)

Therefore, we derive that.

\[ \Phi_m(\varphi) = -i \sqrt{\frac{1}{\pi}} \sin(m\varphi) \]  

(35)

\[ m = 0, \pm 1, \pm 2, \pm 3 \ldots \ldots \]  

Because the square of the mode of the wave function has physical meaning, which is that the square of the mode of the wave function is equal to the probability density of the particle, so we can get the following function by removing the imaginary number factors \( i \) and \( -i \).
\[ \Phi_m(\varphi) = \frac{1}{\sqrt{\pi}} \sin(m\varphi) \]  

(36)

Similarly, for the different corresponding zero value point \( \varphi \), for example \( \Phi_m \left( \frac{\pi}{2} \right) = 0 \), we could substitute sine function with cosine function.

\[ \Phi_m(\varphi) = \frac{1}{\sqrt{\pi}} \cos(m\varphi) \]  

(37)

2.4. Solution of the equation about \( r \)

When it comes to figure out the radical wave function, we need to solve the equation about \( r \), which is exhibited as (20)

There we make a following definition.

\[ \beta = l(l + 1) \]  

(38)

After transposition and dividing both sides by \( r^2 \), the equation as below is obtained.

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d R(r)}{dr} \right) + \left[ \frac{2\mu}{\hbar^2} \left( E + \frac{Ze^2}{4\pi\varepsilon_0 r} \right) - \frac{l(l + 1)}{r^2} \right] R(r) = 0 \]  

(39)

The first term can be expanded as follows.

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d R(r)}{dr} \right) = \frac{1}{r^2} \left[ r^2 R''(r) + 2r R'(r) \right] = R''(r) + \frac{2}{r} R'(r) \]  

(40)

Inserting (40) into (39), the function below is obtained.

\[ R''(r) + \frac{2}{r} R'(r) + \left[ \frac{2\mu}{\hbar^2} \left( E + \frac{Ze^2}{4\pi\varepsilon_0 r} \right) - \frac{l(l + 1)}{r^2} \right] R(r) = 0 \]  

(41)

Within equation (41), the potential energy term is displayed as formula \( V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r} \), and the product of \( V(r) \) and \( r^2 \) is given as follows.

\[ r^2 V(r) = -\frac{Ze^2 r}{4\pi\varepsilon_0} \]  

(42)

When \( r \to 0 \). \( r^2 V(r) \to 0 \).

So that when \( r \to 0 \), the equation (41) can be displayed asymptotically as follows. [1]

\[ R''(r) + \frac{2}{r} R'(r) - \frac{l(l + 1)}{r^2} R(r) = 0 \]  

(43)

Make an assumption as below.

When \( r \to 0 \), \( R_I \propto r^s \)

Define \( O(n) \) as follows, when.

\[ r \to 0, O(n) \propto r^n \]  

(44)

Which means when \( r \) is tending to zero, \( O(n) \) has same order with \( r^n \). Therefore, when \( r \) is very close to zero, \( R_I \) can be substituted by \( O(s) \). [1].
Introduce (44) into (43), the following equation can be obtained.

\[ s(s - 1)O(s - 2) + \frac{2}{r}sO(s - 1) - \frac{l(l+1)}{r^2}O(s) = 0 \] (45)

According to the definition of \( O(s) \), we have the equation below.

\[ s(s - 1)O(s - 2) + 2sO(s - 2) - l(l + 1)O(s - 2) = 0 \] (46)

Then it can be transferred as follows.

\[ s(s - 1) + 2s - l(l + 1) = 0 \] (47)

Then it can be simplified as

\[ s(s + 1) - l(l + 1) = 0 \] (48)

The two different solutions of equation (48) are shown as below.

\[ s = l \quad \text{or} \quad s = -l - 1 \]

Which mean that When \( r \to 0 \), \( R_l \propto r^l \) or \( R_l \propto r^{-l-1} \)

When \( l \geq 1, -l - 1 \leq 0 \), as a result when \( r \to 0 \), \( R_l \) does not tend to 0. So that, the solution \( s = -l - 1 \) is rejected. While the solution \( s = l \) is able to satisfy the condition above, so the solution \( s = l \) is reserved.

\[ s = l, \text{ when } r \to 0 \quad R_l \propto r^l. \]

Making the definition as follows, we have equation (50). [1].

\[ P(r) = rR(r) \] (49)

\[ \frac{d^2P(r)}{dr^2} + \left[ \frac{2\mu}{\hbar^2} \left( \frac{Ze^2}{4\pi\epsilon_0r} + E \right) - \frac{l(l+1)}{r^2} \right] P(r) = 0 \] (50)

When \( r \to 0 \). \( P_l \propto r^{l+1} \), so we have the inference as follows.

\[ P(0) = 0 \] (51)

When \( r \to \infty \), equation (50) is similar to the following differential equation.

\[ \frac{d^2P(r)}{dr^2} + \frac{2\mu}{\hbar^2} EP(r) = 0 \] (52)

When \( r \to \infty \), according to the solution of differential equation (48), we have the following result.

\[ P_l(r) \to C_1 \exp \left( -\sqrt{\frac{2\mu E}{\hbar^2}} r \right) + C_2 \exp \left( \sqrt{\frac{2\mu E}{\hbar^2}} r \right) \] (53)

And it is required that \( r \to \infty \), \( P_l \to 0 \), thus we have \( C_2 = 0 \).

\[ P_l(r) \to C_1 \exp \left( -\sqrt{\frac{2\mu E}{\hbar^2}} r \right) \] (54)

In order to satisfy the two boundary conditions when \( r \) tends to 0 and \( \infty \), and introduce the definition (55), the solution of equation (50) can be exhibited as (56).
\[
\frac{\sqrt{2\mu E}}{\hbar} = \alpha \tag{55}
\]

\[P_l(r) = r^{l+1}e^{-\alpha r}u(r) \tag{56}\]

Then we define as below.

\[
\frac{Ze^2}{4\pi\varepsilon_0} = \gamma \tag{57}
\]

With (57), the equation (50) can be displayed as follows.

\[
\frac{d^2 P_l(r)}{dr^2} + \left[\alpha^2 \left(\frac{Y}{r} + E\right) - \frac{l(l+1)}{r^2}\right] P_l(r) = 0 \tag{58}\]

Applying the following approach, the equation (62) can be obtained, finally we can insert the relevant units. [1]

\[Z = \hbar = e = \mu = 4\pi\varepsilon_0 = 1 \tag{59}\]

As a result, we have following two formulas.

\[\alpha^2 = 2 \tag{60}\]

\[\gamma = 1 \tag{61}\]

\[
\frac{d^2 P_l(r)}{dr^2} + \left[2\left(\frac{l+1}{r} + E\right) - \frac{l(l+1)}{r^2}\right] P_l(r) = 0 \tag{62}\]

The double differential expression is expanded as (63) shows.

\[
\frac{d^2 P_l(r)}{dr^2} = e^{-\alpha} \{r^{l+1}u'' + 2[(l+1)r^l - \alpha r^{l+1}]u' + [(l+1)l r^{l-1} - 2\alpha (l+1)r^l + \alpha^2 r^{l+1}]u\} \tag{63}\]

Introduce (63) to (62), the following equation is obtained.

\[
e^{-\alpha} \{r^{l+1}u'' + 2[(l+1)r^l - \alpha r^{l+1}]u' + [(l+1)l r^{l-1} - 2\alpha (l+1)r^l + \alpha^2 r^{l+1}]u\} + \left[2\left(\frac{l+1}{r} + E\right) - \frac{l(l+1)}{r^2}\right] r^{l+1}e^{-\alpha r}u(r) = 0 \tag{64}\]

After simplified, the equation (60) can be transferred as below.

\[ru'' + [2(l+1) - 2\alpha r]u' + [-2\alpha (l+1) + 2]u = 0 \tag{65}\]

Make it definition as below.

\[\rho = 2\alpha r \tag{66}\]

\[u' = \frac{du}{dr} = \frac{du}{d\rho} \times \frac{d\rho}{dr} = 2\alpha \frac{du}{d\rho} \tag{67}\]

\[u'' = \frac{d}{dr} \left(2\alpha \frac{du}{d\rho}\right) = 2\alpha \frac{d}{dr} \left(\frac{du}{d\rho}\right) = 2\alpha \frac{d}{d\rho} \left(\frac{du}{d\rho}\right) \times \frac{d\rho}{dr} = 4\alpha \frac{d^2 u}{d\rho^2} \tag{68}\]
Introduce (67) and (68), and the equation is displayed as follows.

\[ \rho \frac{d^2u}{d\rho^2} + \left[ 2(l + 1) - \rho \right] \frac{du}{d\rho} - \left[ (l + 1) - \frac{1}{a} \right] u = 0 \]  

Equation (69) is the confluent hyper geometric equation, and the general form of confluent hyper geometric equation is shown as below. [2]

\[ \rho \frac{d^2u}{d\rho^2} + (M - \rho) \frac{du}{d\rho} - Nu = 0 \]  

Comparing (69) and (70) it can be found as below.

\[ M = 2(l + 1) \geq 2 \text{ (Positive integer)} \]  

\[ N = l + 1 - \frac{1}{a} \]  

Mathematically, there are two solutions of the equation (70). [2]

\[ u_1 = F(N, M, \rho) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(N+k)}{\Gamma(N)} \frac{\Gamma(M)}{\Gamma(M+k)} \rho^k \]  

\[ u_2 = \rho^{1-M} F(N-M+1, 2-M, \rho) \]  

About (72), it could be seen as follows.

\[ \rho^{1-M} \propto r^{-2l-1} \]  

As what has been proved above, when \( r \rightarrow 0, R_1 \propto r^l, P_l \propto r^{l+1} \). For \( u_2 \), we have that.

\[ P_l(r) = r^{l+1} e^{-ar} u_2(r) \]  

When \( r \rightarrow 0, P_l(r) \propto r^{-l} \), it is inconsistent with the above conclusion \( P_l \propto r^{l+1} \), hence \( u_2 \) is supposed to be given up.

To (71), when \( \rho \rightarrow \infty \), we gain what is shown as below.

\[ \frac{1}{n!} \frac{\Gamma(N+n)}{\Gamma(N)} \frac{\Gamma(M)}{\Gamma(M+n)} \rho^n \rightarrow \rho^n \]  

According to the expansion of McLaughlin series of \( e^x \), the following conclusion is gained.

\[ F(N, M, \rho) \sim e^{\rho} = e^{2\alpha} \]  

Introduce it to (56), we have the following result.

\[ P_l(r) \sim r^{l+1} e^{ar} \]  

About (78), when \( r \rightarrow \infty \), it does not tend to zero. Thus, it is demanded that \( F(N, M, \rho) \) interrupts as a polynomial. More than that, it is demanded that \( N \) is a nonnegative integer. [1], [2]

\[ N = l + 1 - \frac{1}{\alpha} = -n_r, n_r = 0,1,2 \ldots \]
Define that.

\[ n = n_r + l + 1, \quad n = 0, 1, 2 \cdots \quad (81) \]

So that, we see that.

\[ \alpha = \frac{1}{n} \quad (82) \]

And we have defined that \( \frac{\sqrt{2\mu E}}{h} = \alpha \).

Therefore, we figure out the relation between the energy of every electron and the number \( n \).

\[ E_n = \frac{\hbar^2}{2\mu n^2}, \quad n = 0, 1, 2 \cdots \quad (83) \]

Introduce the term of the number of charges, the energy can be displayed as follows.

\[ E_n = \frac{z^2e^4\hbar^2}{2\mu n^2} \quad (84) \]

Because the initial value selection of scalar is not fixed, it is reasonable to assume that the energy of electrons at infinity is zero, consequently, the value of the energy of electrons in finite distance is negative.

\[ E_n = -\frac{z^2e^4\hbar^2}{2\mu n^2}, \quad n = 0, 1, 2 \cdots \quad (85) \]

Multiplying the constant term, we could make an assumption about the value of \( P_{nl}(r) \) as below.

\[ P_{nl}(r) = A_{nl} r^{l+1} e^{-\alpha r} u_1(r) = A_{nl} r^{l+1} e^{-\alpha r} F(N, M, \rho) = A_{nl} r^{l+1} e^{-\alpha r} F(-n + l + 1, 2l + 2, \rho) \quad (86) \]

\( A_{nl} \) is a constant related to \( n \) and \( l \) while independent of \( r \).

And based on formulas above in this chapter, the following formula as below can be obtained.

\[ R_{nl}(r) = r^{l+1} e^{-\alpha r} u_1(r) = A_{nl} r^{l} e^{-\alpha} F(-n + l + 1, 2l + 2, \rho) \quad (87) \]

According to normalization, the following integral is shown.

\[ \int_0^\infty [R_{nl}(r)]^2 r^2dr = 1 \quad (88) \]

We can figure out that.

\[ A_{nl} = \frac{2\mu^{3/2}e^3}{\hbar^3n^2(2l+1)!} \sqrt{\frac{(n+l)!}{(n-l-1)!}} \quad (89) \]

2.5. Solution of the equation about \( \theta \)

To figure out the function about \( \theta \), the equation (19) is needed to solve.

And base of the definition above \( l(l + 1) \), we have following equation.

\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + [l(l + 1) - \frac{m^2}{\sin^2 \theta}] \Theta(\theta) = 0 \quad (90) \]
Equation (82) is an associated Legendre equation, the solution of which is associated Legendre polynomial as follows.

\[ \Theta_{lm}(\theta) = B_{lm}P_{lm} \cos(\theta) \quad (91) \]

And \( B_{lm} \) is a constant that is only related to \( l \) and \( |m| \) with nothing to do with \( \theta \).

There is a normalized integral equation for it.

\[ \int_0^\pi \Theta_{lm}^2 \sin(\theta) \, d\theta = 1 \quad (92) \]

We get that.

\[ B_{lm} = \left( \frac{(2l+1)(l-|m|)!}{2(l+|m|)!} \right)^{1/2} \quad (93) \]

2.6. Solution of Schrödinger equation of electrons in central Coulomb force field

According to the solution of the 3 independent equations about the three components of spherical coordinate’s \( r, \theta \) and \( \phi \) respectively in the charters 2.3, 2.4 and 2.5, the solution of equation (10) is obtained as below.

\[ \psi_{nlm}(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi) = A_{nl}r^l e^{-\alpha r}F(-n+l+1,2l+2,\alpha r)B_{lm}P_{lm} \cos(\theta) \sqrt{\frac{1}{\alpha}} \sin(m\phi) \quad (94) \]

\( A_{nl} \) and \( B_{lm} \) Which are irrespective of \( r, \theta \) and \( \phi \), have been figured out above.

3. The Deduction of Multi Limits of the Quantity of Electrons

From \( E_n = \frac{Z^2 e^4 h^2}{2\mu n^2} \), \( n = 0,1,2 \ldots \) in charter 2.4, we obtain the following relation between the energy of every electron and principal quantum number \( n \).

\[ E_n \sim -\frac{1}{n^2} \quad (95) \]

\[ E_n = \frac{1}{n^2} E_1 \quad (96) \]

\[ E_n - E_1 = -E_1 \left( 1 - \frac{1}{n^2} \right) = \frac{Z^2 e^4 h^2}{2\mu} \left( 1 - \frac{1}{n^2} \right) = b \left( 1 - \frac{1}{n^2} \right) \quad (97) \]

\[ b = \frac{Z^2 e^4 h^2}{2\mu} \quad (98) \]

If there we no longer hypothesize infinity as zero energy, instead we define that \( E_1 \) is zero of energy. Thus, \( E_n \) could be defined as follows.

\[ E_n = b \left( 1 - \frac{1}{n^2} \right) \quad (99) \]

And because the whole energy of this system remains a constant. It is hypothesized that \( \overline{E_n} \) is the average energy of electrons and \( \overline{n} \) is the average of all electrons’ principal quantum number. The product of number level of electrons \( x_k \) and \( \overline{E_n} \) should be a constant.
\[ x_kE_n^* = \text{const} \] (100)

\[ x_k \left( 1 - \frac{1}{n^2} \right) = \text{const} \] (101)

In addition, because the value of \( x_k \) can be various, the value of \( E_n^* \) is better to be continuous approximately. And because when the \( n \to \infty \), the value of \( \frac{(n+1)^2}{n^2} \to 1 \), and \( \frac{E_{n+1}}{E_n} \to 1 \), consequently it demands that the principal number should be large.

3.1. Derivation under Bohr’s theory about radius orbit of electron in hydrogen atoms (Model 1)

Although Bohr’s theory about the track of electrons is to describe the relation between the radius of orbital electrons and the principal quantum number \( n \), but it works to electrons in hydrogen like atoms. When it comes to the model raised in this paper, the charged hole in the center can be regarded a nucleus charged positively as much as enough to ignore the interaction among electrons, consequently to describe the probability of the distribution of every single electron, we only consider the interaction between the positively charged hole and one single electron. As a result, there Bohr’s theory about the radius of orbits is accessible.

Firstly, with classical quantum theory, we quote Bohr’s radius theory, which tells the radius of orbit of electrons is related with principal quantum number \( n \), [3] shown as follows.

\[ r_n' = n^2 r_1' \] (102)

In addition, also when the \( n \to \infty \), the value of \( \frac{(n+1)^2}{n^2} \to 1 \), \( \frac{r_{n+1}'}{r_n'} \to 1 \), which results in that the value of \( r_n' \) tends to be successive approximately.

The so-called radius of electrons actually refers to the position with the maximum distribution probability of electron cloud in \( r \) direction. [3] Define that \( \bar{r}_n' \) is the average radius of orbit of electrons. The probability of distribution of electrons is closer to the hole, so the possibility of its being swallowed by the hole (there it defined as \( P \) ) is greater. Approximately we have the following formula.

\[ P = \frac{1}{r_n} \] (103)

And simply

\[ \bar{r}_n' = \bar{r}^2 r_1' \] (104)

Then the formula can be gained, with formula \( x_k \left( 1 - \frac{1}{n^2} \right) = \text{const} \)

\[ x_k \left( 1 - \frac{r_1'}{r_n'} \right) = \text{const} \] (105)

After that, (106) can be written down.

\[ x_k (1 - r_1' P) = \text{const} = u \] (106)

\[ P = \frac{1}{r_1'} \left( 1 - \frac{u}{x_k} \right) \] (107)
So after a quantum time, to calculate the number level of electrons, because $P$ of the total electrons would be absorbed, we need to multiply the term $(1 - P)$. Thus after a quantum time, the iteration of the electron number level can be shown as integration (4).

3.2. Derivation by analysis of radial wave function (Model 2)
From the wave function of electrons, it is known that the distribution of electrons in $r$ direction only depends on the radial wave function which has been figured out in the charter 2.4, displayed as (87) and from formula (89) with $\rho = 2\alpha r$ and $\alpha = \frac{1}{n}$.

$$R_{nl}(r) = \frac{2\mu^{3/2}\rho^3}{\hbar^3n^2(2l+1)!}\sqrt{\frac{(n+l)!}{(n-l-1)!}}r^le^{-\frac{r}{\pi}}\left(-n + l + 1,2l + 2\frac{2r}{n}\right)$$ (108)

When $n \to \infty$, it can be seen that.

$$\sqrt{\frac{(n+l)!}{(n-l-1)!}} \to 1$$ (109)

$$F(N,M,\rho) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(N+k)}{\Gamma(N)} \frac{\Gamma(M)}{\Gamma(M+k)} \rho^k$$ (110)

Because according to the features of Gamma Function, when $x$ is a positive integer, we have the formula as below. [2]

$$\Gamma(x) = (x - 1)!$$ (111)

Then insert (111) into (110), formula (112) can be obtained.

$$F(N,M,\rho) = \frac{1}{N} + \frac{N(N+1)}{M(M+1)}\rho^2 + \frac{2N(N+2)}{3M(M+1)(M+2)}\rho^3 + \cdots$$ (112)

When $n \to \infty$, $\rho = 2\alpha r = \frac{2r}{n} \to 0$, and as to satisfy the rules of wave function that cannot be divergent, the value of $\sum_{k=1}^{\infty} \frac{1}{k!} \frac{\Gamma(N+k)}{\Gamma(N)} \frac{\Gamma(M)}{\Gamma(M+k)} \rho^k$ is finite, therefore $\rho \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\Gamma(N+k)}{\Gamma(N)} \frac{\Gamma(M)}{\Gamma(M+k)} \to 0$, and $\sum_{k=1}^{\infty} \frac{1}{k!} \frac{\Gamma(N+k)}{\Gamma(N)} \frac{\Gamma(M)}{\Gamma(M+k)} \rho^k \to 0$, and finally $F(N,M,\rho) \to 1$, which refers to $F \left(-n + l + 1,2l + 2\frac{2r}{n}\right) \to 1$.

When $n \to \infty$, $e^{-\frac{r}{\pi}} \to 1$.

In conclusion, when $n \to \infty$, we get the approximation of $R_{nl}(r)$.

$$R_{nl}(r) = \frac{2\mu^{3/2}\rho^3}{\hbar^3n^2(2l+1)!}\sqrt{\frac{(n+l)!}{(n-l-1)!}}r^l e^{-\frac{r}{\pi}} = \frac{C}{n^2} r^l$$ (113)

$$C = \frac{2\mu^{3/2}\rho^3}{\hbar^3(2l+1)!}$$ (114)

The passivity of distribution between 0 and $r_0$ in radial direction of one single electron could be calculated by integral as follows.

$$P = \int_0^{r_0} [R_{nl}(r)]^2 r^2 dr = \int_0^{r_0} C^2 \frac{1}{n^2} r^{2l+2} dr = \frac{1}{n^2} \int_0^{r_0} C^2 r^{2l+2} dr$$ (115)
So that

\[ P = \frac{1}{n^4} \]  

(116)

Closely, it is written.

\[ P = \frac{v^2}{n^4} \]  

(117)

(v is a constant)

Inserting, \( x_k \left( 1 - \frac{1}{n^2} \right) = \text{const} \) (92) we have following equation.

\[ x_k \left( 1 - \frac{\sqrt{P}}{v} \right) = \text{const} = w \]  

(118)

\[ P = v^2 \left( 1 - \frac{w}{x_k} \right)^2 \]  

(119)

So that we need to multiply the term \((1 - P)\), which equals to \( \left( 1 - v^2 \left( 1 - \frac{w}{x_k} \right)^2 \right) \). Thus, after a quantum time, a non-linear iteration of the electron number level can be shown as (5).

4. Simulation through MATLAB

Mathematically, the MATLAB is applied to see if the limit of \( x_k \) would be multiple, when the parameter \( a \) Meet certain conditions. Obviously, for both formula (4) and (5), the limits of the iterative is zero when \( a < 1 \), hence there we only discuss the situation when \( a > 1 \). The limits of the iterative depends on the parameter, instead of the initial value of the number level of electrons \( x_0 \). [4] Thus, it is supposed that the initial number level \( x_0 = 0.5 \).

4.1. Simulation of the first model

To the first situation.

We hypothesize that \( \frac{1}{r_1} = 0.25 \) and \( u = 1 \), then correspondence between the limits of the iterative (4) and parameter \( a \) could be seen by MATLAB. The figure about that is displayed as fellow.

![Figure 1. Diagram of I—a (Model 1).](image-url)
Zooming the figure 1, we could see one value of $a$ could correspond to various limits $I$ shown as figure 2.

![Figure 2](image)

**Figure 2.** Partial enlarged view of diagram of $I-a$ (Model 1).

### 4.2. Simulation of the second model

When it comes to the second model acquired from reduction by analysis of radial wave function, integration (5) is raised up. Hypothesizing that $v = 0.5$ and $w = 1$, via MATLAB, we have the 2 diagrams as below.

![Figure 3](image) ![Figure 4](image)

**Figure 3.** Diagram of $I-a$ (Model 2).  **Figure 4.** Partial enlarged view of diagram of $I-a$ (Model 2).

### 4.3. Conclusion on the Simulation of the two Models

Successfully, under the 2 situation, we manage to see the multi-limits of the quantity of electrons in the system raised in this paper.

### 5. Discussion of the Non-Approximate Situation

After researches, scientists have found the phenomena of bifurcation and chaos is universal for nonlinear mapping. [4] It is by now known that many non-linear iterations can lead to bifurcation and chaos. When we strictly raise up the non-approximate model, the formula that describe the correspondence between the number level of electrons at present an that after a quantum time, the multi-limits could be seen as well.
6. **The possible value of this model**

Because the conjectural model uses the concept of quantum as a precondition to apply iterative function to the variance of the quantity of electrons. According to the research of logistics iteration, when the parameters are controlled within a certain range, the period of the limit of the iterative function \((2n)\) will be very large, therefore the cycle of time of the quantity change of electrons would be very long, which could be enough to be observed. And based on the theories, with various parameters that lead to period-doubling bifurcations, the value of every limit of the iteration are accessible to be calculated and the length of the periods are determined by the parameters, so that via mathematical methods, we are able to predict the number of the electrons, as well as the time of changes. If the number of electrons can be detected to vary in periods and the changing figures follow this model, the concept of quantum is going to be proved.

A latent application of this model could be timer with higher accuracy. When there are \(2n\) bifurcation of limits of the quantity of electrons, the level of the number of the electrons oscillates at the \(2n\) different values, and the length of time it takes to jump from one value to another one is a quantum time. Using this, by recording the level of the number of electrons, time could be recorded.

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