ADE SURFACES AND THEIR MODULI

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Abstract. We define a class of surfaces and surface pairs corresponding to
the ADE root lattices and construct compactifications of their moduli spaces,
generalizing Losev-Manin spaces of curves.

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1. INTRODUCTION

There are two sources of motivation for this work. One is the Losev-Manin
spaces [LM00]. Let $L_{n+3}$ be the moduli space parameterizing weighted stable curves
$(C, Q_0 + Q_\infty + \epsilon \sum_{i=1}^{n+1} P_i)$ of genus 0 with $n + 3$ points, $n + 1$ of which are allowed to
collapse and the other two are not allowed to collide with any. One has $\dim L_{n+3} = n$.

Quite remarkably, $L_{n+3}$ is a projective toric variety corresponding to the Weyl
chamber fan (also called Coxeter fan) for the root lattice $A_n$, formed by the mirrors
to the roots. Of course, it comes with an action of the Weyl group $W(A_n) = S_{n+1}$
permuting the points $P_i$. There are other ways in which $L_{n+3}$ corresponds to the
root lattice $A_n$. For example, its interior, over which the fibers are $C \cong \mathbb{P}^1$, is the
torus $\text{Hom}(A_n, \mathbb{G}_m)$, and the discriminant locus, where some of the points $P_i, P_j$
coincide, is a union of root subtori $\cup_{\alpha} \{ e^\alpha = 1 \}$ with $\alpha = e_i - e_j$ going over the
roots of $A_n$. Additionally, the worst singularity that the divisor $\sum P_i$ can have is
$(x - 1)^{n+1} = 0$, which is an $A_n$-singularity.

Losev and Manin asked in [LM00] if similar moduli spaces existed for other root
lattices. This was partially answered by Batyrev and Blume in [BB11] where they
constructed compact moduli spaces for the $B_n$ and $C_n$ lattices as moduli of certain

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pointed rational curves with an involution. Batyrev-Blume’s method works only for infinite series of root lattices, such as $ABCD$, and it breaks down for $D_n$ where it leads to non-flat families (most fibers have dimension 1 but some have 2).

One of the main results of this paper is the construction of compact moduli spaces of stable surfaces $(X, D + \epsilon R)$ for the root lattices $A_n$, $D_n$, and $E_6, E_7, E_8$ which extend all the ways in which Losev-Manin spaces correspond to $A_n$: moduli, discriminant loci, singularities, and compactifications.

The second, and original source of motivation is the study of geometric compactifications of moduli spaces of K3 surfaces. Recall that the KSBA method provides a compact moduli space for the stable pairs $(X, B = \sum b_i B_i)$, where $X$ is a projective variety and $B$ a $\mathbb{Q}$-divisor satisfying two conditions: (1) $(X, B)$ has semi log canonical singularities, and (2) the divisor $K_X + B$ is ample. In particular, if $M$ is a moduli space of mildly singular varieties $X$ with $K_X \equiv 0$, or more generally of log pairs $(X, D)$ with $K_X + D \equiv 0$ that come with an intrinsic choice of an effective ample divisor $R$ (not an equivalence class!) then the moduli $\mathcal{M}$ of stable pairs $(X, D + \epsilon R)$ provides a natural functorial compactification of $M$.

One notable application of this idea is principally polarized abelian varieties, considered in [Ale02]. Another example is the moduli space $F_{2d}$ of polarized K3 surfaces $(X, L)$ with $ADE$ singularities. There are many ways to choose an intrinsically defined divisor for such a pair. One popular way is to take the sum of all rational curves in $|L|$. For example, for K3 surfaces of degree $L^2 = 2$ one gets a divisor in $|32L|$ this way. Frequently, however, there is a much smaller choice.

K3 surfaces of degree $2$ come with a canonical involution $\iota$ and the quotient surface $Y = X/\iota$ is either $\mathbb{P}^2$ or the cone $\mathbb{P}(1, 1, 4)$, with branch curve $B \in |-2K_Y|$. In this case the ramification divisor $R$ of the involution is in $|3L|$. What is the compactification provided by the KSBA theory in this case? Is it toroidal? If so, which fan does it correspond to?

The pair $(X, \iota, \epsilon R)$ is equivalent to the pair $(Y, (\epsilon + \frac{1}{\epsilon})B)$, so the compactification we seek is the same as the compactification for the pairs $(\mathbb{P}^2, \epsilon B_i)$ of the moduli space of plane curves considered by Hacking in [Hac04a, Hac04b], in the special case $d = 6$. Unfortunately Hacking’s description of the compactification for $d = 6$ is incomplete. Still, [Hac04b] contains many examples of degenerations.

Recall that the moduli space $F_2$ of degree-2 K3 surfaces is an arithmetic quotient of a Hermitian symmetric domain of type IV whose toroidal compactifications correspond to fans in the hyperbolic lattice $N_2 = \mathbb{E}_8^{\oplus 2} \oplus A_1 \oplus H$ of signature $(18, 1)$, supported on the rational closure $\overline{C}_Q$ of the light cone $C = \{v \mid v^2 < 0\}$. By analogy with the Losev-Manin space, for a possible modular toroidal compactification $\overline{F}^{tor}$ we consider the Coxeter fan, formed by the mirrors to the roots (vectors with $\alpha^2 = 2$ in $N_2$). This fan is described by a Vinberg (also called Coxeter, or Dynkin) diagram shown in Fig. 1 on the left.

It is well-known that the Baily-Borel compactification of $F_2$ has four 1-cusps of types $\tilde{E}_8 A_1$, $D_{10} E_7$, $D_{16} A_1$, $A_{17}$, in bijection with the rays $\mathbb{R}_{\geq 0} v \in \overline{C}_Q$ with $v^2 = 0$, modulo the isometry group $O(N_2) = s_3$-extension of the Weyl group $W(N_2)$. They correspond to maximal parabolic subdiagrams of the Vinberg diagram modulo $S_4$. For the proof, see e.g. [Fri84, Sec.5] or [Sca87, p. 82].

We now make the following interesting observation: Superimposing the Vinberg diagram on the lattice of monomials in $H^0(\mathbb{P}^2, \mathcal{O}(6))$, resp. in $H^0(\mathbb{P}(1, 1, 4), \mathcal{O}(12))$ it is easy to find toroidal degenerations of type II of $\mathbb{P}^2$, resp. $\mathbb{P}(1, 1, 4)$ and of their
K3 surface double covers that match these diagrams. We picture a toric degeneration \((\mathbb{P}^2, \mathcal{O}(6)) \to (\mathbb{P}^2, \mathcal{O}(4)) \cup (\mathbb{P}^1, \mathcal{O}(2s_1 + 6f))\) corresponding to the subdiagram \(\tilde{D}_{10}\tilde{E}_7\) as an example. In [AT15] we also find many toric degenerations of type III corresponding to the unions of ADE diagrams in the Vinberg diagram. The latter classify all the other cones in the Coxeter fan minus the origin, mod \(W(N_2)\). Therefore, many degenerations are indeed related to cones in the Coxeter fan and thus to strata in the toroidal compactification of \(F_2\) for this fan.

We unify the above motivations in a single picture as follows. In Section 2, we define \((K + D)\)-trivial polarized involution pairs and prove that they come in three types, in parallel to the theory of K3 surfaces and their degenerations.

Pairs of type I are K3 surfaces \(X\) with a non-symplectic involution \(\iota\); the boundary divisor \(D\) is zero in this case. They are double covers of klt del Pezzo surfaces \(Y = X/\iota\) of index \(\leq 2\) branched in a divisor \(B \in |-2K_Y|\). The classification of such del Pezzo surfaces was obtained by Alexeev-Nikulin in [AN88, AN89, AN06]. There are many types but only 10 overall families in which a generic element is a smooth del Pezzo surface of degree \(1 \leq K_Y^2 \leq 9\); there are two types for \(K_X^2 = 8\).

Pairs \((X, D)\) of types II and III are double covers of log canonical non-klt del Pezzo surfaces \((Y, C)\) of index \(\leq 2\) with a reduced, possibly empty boundary \(C\). On the minimal resolution \(\tilde{X}\) of \(X\) the boundary \(\tilde{D}\) is nonempty and consists of one or two elliptic curves in type II, and a cycle of rational curves in type III. The pairs \((X, D)\) of type III are closely related to Looijenga pairs, except that they may be singular and there is an involution with an ample ramification divisor.

In Section 3 we introduce \(ADE\) and \(\tilde{A}\tilde{D}\tilde{E}\) surfaces \((Y, C)\) with boundary. We also define \(ADE\) and \(\tilde{A}\tilde{D}\tilde{E}\) pairs \((Y, C, \frac{1}{2}B)\) by adding a divisor \(B \in |-2(K_Y + C)|\) that intersects the boundary \(C\) transversally at smooth points of \(Y\) and \(C\). We call the double covers \((X, D)\) branched in \(B\) the \(ADE\) and \(\tilde{A}\tilde{D}\tilde{E}\) \(\tilde{A}\tilde{D}\tilde{E}\) covers. Many of the \(ADE\) and \(\tilde{A}\tilde{D}\tilde{E}\) surfaces are toric, and and their defining \(ADE\) and \(\tilde{A}\tilde{D}\tilde{E}\) polytopes correspond to the \(ADE\) and \(\tilde{A}\tilde{D}\tilde{E}\) Dynkin diagrams in a very obvious way. The boundary divisor \(C\) is part of the toric boundary. We divide all these surfaces into \textit{shapes}, some \textit{pure} and some \textit{primed}. “Priming” is a natural operation on surfaces \((Y, C)\) and on pairs \((Y, C, \frac{1}{2}B)\) which removes some of the intersection points \(B \cap C\) and contracts the components of \(C\) that are no longer connected to \(B\).
We prove that every ADE and \( \tilde{A} \tilde{D} \tilde{E} \) surface \((Y, C)\) is a log canonical non-klt del Pezzo surface with boundary of index \( \leq 2 \). In Section 4 we establish the opposite: that the log canonical non-klt del Pezzo surfaces \((Y, C)\) with a reduced (possibly empty) boundary are precisely the \( \tilde{A} \tilde{D} \tilde{E} \) surfaces with boundary. We deduce this from the classification of log del Pezzo surfaces with boundary of index \( \leq 2 \) given by Nakayama in [Nak07] where it was done in very different terms; additional arguments are required. In §4.5 we outline how this classification can also be deduced from the analysis of involutions in the Cremona group, which are classically of three types: De Jonquières, Geiser, and Bertini.

In Section 5 we lay a foundation for the rest of the paper by introducing two-dimensional models of the ADE lattices. Such a model is a projection from the weight lattice \( \Lambda^* \simeq \mathbb{Z}^n \) of the corresponding Lie algebra down to \( \mathbb{Z}^2 \), mimicking the definition of the \( \tilde{A} \tilde{D} \tilde{E} \) polytopes. One of its remarkable properties is that it faithfully represents the partial order between the dominant weights lying below the fundamental weights.

In Section 6 for every ADE shape we provide a normal form for the equation \( f \) of the divisor \( B \in | -2(K_Y + C)| \) in an ADE pair. The ADE Dynkin diagrams show up in this description, too: the nodes correspond to the monomials in the normalized equation of \( f \). As a consequence, we show that the moduli space of ADE pairs is a quotient of \( \mathbb{A}^n \) by a finite abelian diagonalizable group.

In Section 7 we switch to a toric point of view on the families, replacing \( \mathbb{A}^n \) by a torus \( T_{\Lambda^*} \) whose character lattice is the weight lattice for the corresponding Lie algebra. The quotient \( T_{\Lambda^*} / W \) of this torus by the Weyl group is \( \mathbb{A}^n \) by the standard representation theory. For each shape, we write down an explicit \( W \)-equivariant family of equations \( f \) and of ADE pairs \((Y, C, \frac{1+\epsilon}{2} B)\) over \( T_{\Lambda^*} \). By construction, the discriminant locus of \( B \) (and of the double cover \( X \)) in this family is a union of root subtori, and the singularity of the fiber over \( 1 \in T_{\Lambda^*} \) is of the same ADE type as the shape.

We note that for the \( E_6, E_7, E_8 \) cases such families were previously discovered by Etingof- Oblomkov-Rains in [EOR07] as families of centers of certain non-commutative algebras. (The former are the coordinate rings \( k[X \setminus D] \) on the double covers in our situation). Our derivation for the \( E_n \) families is independent, it comes from the study of pencils of cubic curves following Tjurina’s method [Tju70].

In §7.10 we descend the family over \( T_{\Lambda^*} \) to a smaller torus \( T_{\Lambda'} \) for a lattice \( \Lambda' \) which in most cases is the root lattice \( \Lambda \) but in several primed cases is its overlattice of index 2 or 4.

In Section 8 we relate the families of ADE covers to the work of Gross-Hacking-Keel [GHK15] where families of Looijenga pairs over certain tori \( T_{D_{2\cdot}} \) naturally appear. In the cases when \( \Lambda' \neq \Lambda \), the torus \( T_{\Lambda'} \) is a proper subtorus of \( T_{D_{2\cdot}} \), of lower dimension, since not all of the corresponding Looijenga pairs admit an involution.

In Section 9 we show that the curves \( B \in | -2(K_Y + C)| \) and the double covers \( X \) have Du Val singularities away from the boundary, and we determine these singularities explicitly. It follows that the ADE pairs are precisely the \( \mathbb{Z}_2 \)-quotients of the \((K + D)\)-trivial polarized involution pairs \((X, \iota, R)\) of type III that we introduced in Section 2.
Switching to a toric point of view has many advantages. The most important one for us is that families over tori are more amenable to compactifications. In Section 10 for each \( ADE \) shape, pure or primed, we construct a compact \( W \)-invariant family of stable \( ADE \) pairs over a projective toric variety whose fan is the Coxeter fan for the corresponding Weyl group. We spell out the consequences for the coarse moduli space of stable pairs in \( \S 10.7 \). For several toric shapes we descend the family to a projective toric variety corresponding to a coarser Wythoff fan that is refined by the Coxeter fan.

As a special case, in \( \S 10.9 \) we obtain a geometric compactification of a 12:1 cover of the moduli space of rational elliptic surfaces which is quite different from the ones appearing in the literature, e.g. Heckman-Looijenga’s compactification [HL02].

We leave generalizations to the non simply laced Dynkin diagrams and to lattices of type I (i.e. corresponding to K3 surfaces with involution, e.g. \( N_2 = E_8^2 \oplus A_1 \oplus H \)) to subsequent papers. In Section 11 we briefly touch on these topics.

Throughout the paper, we work over a field \( k = \bar{k} \) of characteristic 0 for simplicity. It is obvious however that most constructions and results hold over \( \mathbb{Z}[1/N] \), where \( N = 2|\Lambda^*/\Lambda| \), for an \( ADE \) lattice of type \( \Lambda \). Working over \( \mathbb{Z}[1/N] \) only complicates the language but not the substance.

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2. \((K + D)\)-trivial polarized involution pairs

**Definition 2.1.** A \((K + D)\)-trivial polarized involution pair \((X, D, \iota)\) consists of a normal surface \( X \) with an effective reduced divisor \( D \), and an involution \( \iota: X \to X \), \( \iota(D) = D \) such that

1. \( K_X + D \sim 0 \) is a Cartier divisor linearly equivalent to 0,
2. the ramification divisor \( R \) is Cartier and ample, and
3. the pair \((X, D + \epsilon R)\) has log canonical (lc) singularities for \( 0 < \epsilon \ll 1 \).

Let \( \omega \) be a global generator of the 1-dimensional space \( H^0(\mathcal{O}_X(K_X + D)) \). The ramification divisor \( R \) is nonempty by ampleness and has no components in common with \( D \) by the lc condition. For a generic point \( x \in R \) there are local parameters \((u, v)\) such that \( \iota(u, v) = (u, -v) \). Then \( \iota^*(du \wedge dv) = -du \wedge dv \). Thus, the involution \( \iota \) is non-symplectic, meaning \( \iota(\omega) = -\omega \).

Let \( \pi: X \to Y = X/\iota \) be the quotient map, \( C = \pi(D) \) the boundary and \( B = \pi(R) \) the branch divisors. By Hurwitz formula, \( K_X + D = \pi^*(K_Y + C + \frac{1}{2}B) \).

**Lemma 2.2.** There is a one-to-one correspondence between \((K + D)\)-trivial polarized involution pairs \((X, D, \iota)\) and pairs \((Y, C + \frac{1}{2}B)\) such that

1. \( Y \) is a normal surface and \( C, B \) are reduced effective Weil divisors on it.
2. \((Y, C)\) is a del Pezzo surface with boundary of index \( \leq 2 \), i.e. \(-2(K_Y + C)\)
   is an ample Cartier divisor.
3. \( B \in |-2(K_Y + C)|; \) in particular \( B \) is Cartier.
4. The pair \((Y, C + \frac{1}{2}B)\) has lc singularities for \( 0 < \epsilon \ll 1 \).

Moreover, if (1)–(4) hold then one also has
(5) For any singular point \( y \in Y \): if \( y \in B \) then \( y \) is Du Val and \( y \not\in C \).

Proof. Suppose (1)–(4) hold and \( y \in B \) is a non Du Val singularity of \( Y \) or a Du Val singularity with \( y \in C \). Then on a minimal resolution \( g: \tilde{Y} \to Y \) there exists an exceptional divisor \( E \) whose discrepancy with respect to \( K_Y + C \) is \( < 0 \). Since \( 2(K_Y + C) \) is Cartier, one has \( a_E(K_Y + C) \leq -\frac{1}{2} \). But \( B \) is Cartier, so

\[
a_E \left( K_Y + C + \frac{1+\epsilon}{2}B \right) \leq -\frac{1}{2} - \frac{1+\epsilon}{2} < -1,
\]

and the pair \((Y, C + \frac{1+\epsilon}{2}B)\) is not lc, a contradiction. This proves (5).

Now let \((X, D, \iota)\) be a \((K + D)\)-trivial polarized involution pair. Using \( \iota^*(\omega) = -\omega \), it follows by [Kol13, Prop.2.62(4)] that for any \( x \in X \) étale-locally \((X, x) \to (Y, \pi(x))\) is the index-1 cover for the pair \((Y, C + \frac{1}{2}B)\). Thus, \( \pi_*O_X = O_Y \oplus \omega_Y(C) \), the divisor \( 2(K_Y + C) \) is Cartier, and \( B = (s) \), \( s \in H^0(O_Y(-2(K_Y+C))) \). From the identity \( K_X + D + \epsilon R \equiv \pi^*(K_Y + C + \frac{1+\epsilon}{2}B) \) it follows that the divisor \( K_Y + C + \frac{1+\epsilon}{2}B \) is ample and the pair \((Y, C + \frac{1+\epsilon}{2}B)\) has lc singularities.

Vice versa, let \((Y, C + \frac{1+\epsilon}{2}B)\) be a pair as above, and let \( X := \text{Spec}_{Y} O_Y \oplus \omega_Y(C) \) be the double cover corresponding to a section \( s \in H^0(O_Y(-2(K_Y+C))) \), \( B = (s) \). Thus, étale-locally it is the index-1 cover for the pair \((Y, C + \frac{1}{2}B)\). Then \( K_X + D \sim 0 \), \( K_X + D + \epsilon R \) is ample and lc, and \( 2R = \pi^*(B) \) is an ample Cartier divisor.

We claim that \( R \) itself is Cartier. Pick a point \( x \in R \) and let \( y = \pi(x) \in B \). The cover \( \pi \) corresponds to the divisorial sheaf \( O_Y(K_Y + C) \), which is locally free at \( y \) by (5). Then the double cover is given by a local equation \( u^2 = s \), and \( R \) is given by one local equation \( u = 0 \), so it is Cartier.

Thus, the classification of \((K + D)\)-trivial polarized involution pairs is reduced to that of del Pezzo surfaces \((Y, C)\) with reduced boundary of index \( \leq 2 \) plus a divisor \( B \in |-2(K_Y + C)| \) satisfying the lc singularity condition. In the case when \( C = 0 \), del Pezzo surfaces of index \( \leq 2 \) with log terminal singularities were classified by Alexeev-Nikulin in [AN88, AN89, AN06]. There are 50 main cases which are further subdivided into 73 cases according to the singularities of \( Y \). However, all these surfaces are smoothable, which follows either by using the theory of K3 surfaces or by [HP10, Prop. 3.2]. Thus, there are only 10 overall families, with a generic element a smooth del Pezzo surface of degree \( 1 \leq K_Y^2 \leq 9 \) (for \( K_Y^2 = 8 \) there are two families, for \( F_0 \) and \( F_1 \)). The dimension of the family of pairs \((Y, B)\), equivalently of the double covers \((X, \iota)\), is \( 10 + K_Y^2 \).

Del Pezzo surfaces with a half-integral boundary \( C \) of index \( \leq 2 \) were classified by Nakayama in [Nak07]. An important result of Nakayama is the Smooth Divisor Theorem [Nak07, Cor.3.20] generalizing that of [AN06, Thm.1.4.1]. It says that for any del Pezzo surface \((Y, C)\) with boundary of index \( \leq 2 \) a general divisor \( B \in |-2(K_Y + C)| \) is smooth and in particular does not pass through the singularities of \( Y \). Thus, every such surface \((Y, C)\) produces a family of \((K + D)\)-trivial polarized involution pairs \((X, D, \iota)\).

**Remark 2.3.** The divisors \( C \) and \( B \) play a very different role: \( C \) is fixed, and \( B \) varies in a linear system. For this reason, we will refer to them differently. We will call \( C \) the boundary and say that \((Y, C)\) is a surface with boundary (and sometimes we will drop the words “with boundary”). We will call \((Y, C + \frac{1+\epsilon}{2}B)\) a pair, consisting of a surface with boundary \((Y, C)\) plus an additional choice of divisor \( B \) on it. In many cases, surfaces with boundary are rigid, but pairs have moduli.
Let \( f: \tilde{X} \to X \) be the minimal resolution of singularities, and let \( \tilde{D} \) be the effective \( \mathbb{Z} \)-divisor on \( \tilde{X} \) defined by the formula \( K_{\tilde{X}} + \tilde{D} = f^*(K_X + D) \sim 0 \). It follows from the lc condition that \( \tilde{D} \) is reduced.

**Lemma 2.4.** For the minimal resolution of a \((K+D)\)-trivial polarized involution pair, one of the following holds:

(I) \( D = 0 \), \( \tilde{D} = 0 \), and \( X \) is canonical. Then \( X \) is a K3 surface with ADE singularities and \( \iota \) is an non-symplectic involution.

(II) \( (X,D) \) is strictly log canonical and \( \tilde{D} \) is one or two isomorphic smooth elliptic curve(s).

(III) \( (X,D) \) is strictly log canonical and \( \tilde{D} \) is a cycle of \( \mathbb{P}^1 \)s.

Accordingly, we will say that the \((K+D)\)-trivial polarized involution pair \((X,D,\iota)\) and the corresponding del Pezzo surface \((Y,C)\) with boundary have type I, II, or III. In type I \((Y,C)\) is klt, and in types II, III it is not klt.

**Proof.** (I) (Compare [AN06, Sec. 2.1]) \( \tilde{X} \) is either a K3 surface or an Abelian surface. If \( \tilde{X} = X \) is an Abelian surface then the involution is different from \((-1)\) since \( R \neq 0 \). Thus, the induced involution \( \iota^* \) on \( H^0(\Omega_{X}^1) \) is different from \((-1)\) and there exists a nontrivial 1-differential on \( X \) which descends to a minimal resolution \( \tilde{Y} \) of \( Y \). But \( \tilde{Y} \) is a del Pezzo surface with log terminal singularities, so basic vanishing gives \( h^0(\Omega_{\tilde{Y}}^1) = h^1(\mathcal{O}_{\tilde{Y}}) = h^1(\mathcal{O}_{Y}) = 0 \). Thus, \( \tilde{X} \) is a K3 surface, and we already noted that the involution is non-symplectic.

(II, III) Since \( \omega_{\tilde{D}} \simeq \mathcal{O}_{\tilde{D}} \) by adjunction, every connected component of \( \tilde{D} \) is either a smooth elliptic curve or a cycle of \( \mathbb{P}^1 \)s. Since \( K_{\tilde{X}} = -\tilde{D} \) is not effective, \( \tilde{X} \) is birationally ruled over a curve \( E \) and \( \tilde{D} \) is a bisection. The curve \( E \) has genus 1 or 0 since it is dominated by \( \tilde{D} \). If one of the connected components of \( \tilde{D} \) is a cycle of \( \mathbb{P}^1 \)s then \( g(E) = 0 \) and \( X \) is rational. In that case from \( H^1(-\tilde{D}) = H^1(K_{\tilde{X}}) = 0 \) we get \( h^0(\mathcal{O}_D) = h^0(\mathcal{O}_{\tilde{X}}) = 1 \), so \( \tilde{D} \) is connected. If \( g(E) = 1 \) and \( \tilde{D} \) has more than one connected component then they all must be horizontal. Thus, there must be two of them, each a section of \( \tilde{X} \to E \), so they are both isomorphic to \( E \). \( \square \)

3. **ADE and \( \tilde{A}\tilde{D}\tilde{E} \) surfaces and pairs**

In this section, we introduce ADE and \( \tilde{A}\tilde{D}\tilde{E} \) surfaces \((Y,C)\) with boundary. In other words, \( C \) is an effective reduced divisor on \( Y \). We usually drop the words "with boundary". The ADE surfaces are of type III and they correspond to the irreducible Dynkin diagrams \( A_n, D_n, E_n \). The \( \tilde{A}\tilde{D}\tilde{E} \) surfaces are of type II and they correspond to the irreducible extended ("affine") Dynkin diagrams \( \tilde{A}_n, \tilde{D}_n, \tilde{E}_n \). We divide all these surfaces into shapes, and they come in two guises: pure and primed.

First, there are several pure shapes:

1. \( A_{2n-1}, A_{2n-2}, A_{2n-3}; \tilde{A}_{2n-1} \).
2. \( D_{2n}, D_{2n-1}; \tilde{D}_{2n} \).
3. \( -E_6, -E_7, -E_8; \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \).
4. two exotic type II shapes \( \tilde{A}_7^+ \) and \( \tilde{A}_8^- \).

The ADE surfaces have two "sides", the irreducible components of \( C \), which we call "left" and "right". The \( \tilde{A}\tilde{D}\tilde{E} \) surfaces have a single "side", corresponding to the fact that \( C \) is irreducible in these cases. The minuses in \( A^- \), \( \tilde{A} \), \( \tilde{A}^- \), etc.
signify the “short sides”, see Def. 3.3. We call the shapes \( A_{2n-1}, D_{2n}, \tilde{D}_{2n}, \tilde{E}_7 \) with only long sides ultrapure, these are especially easy.

In addition to these, there are shapes obtained from the pure shapes by a process which we call priming, see §3.10. The minuses in the superscripts above become convenient for denoting the primed shapes. But when speaking of pure shapes, frequently the minuses can be deduced and skipped, so sometimes we omit them and talk simply of \( E_0, E_7, E_8 \) for example.

We will see that all of the \( ADE \) and \( \tilde{ADE} \) surfaces \( (Y, C) \) are maximally log canonical del Pezzo surfaces with boundary of index \( \leq 2 \), i.e. the pair \( (Y, C) \) has log canonical non-klt singularities and the divisor \(-2(K_Y + C)\) is Cartier and ample. In the next Section 4 we will establish the opposite direction.

**Definition 3.1.** An \( ADE \) (resp. \( \tilde{ADE} \)) pair is a pair \((Y, C + \frac{r+2}{2}B)\), where \((Y, C)\) is an \( ADE \) (resp. \( \tilde{ADE} \)) surface and \( B \in |-2(K_Y + C)| \) is an effective divisor which intersects the boundary \( C \) transversally at points that are smooth for both \( Y \) and \( C \).

An \( ADE \) (resp. \( \tilde{ADE} \)) cover \((X, D + \epsilon R)\) is a double cover of an \( ADE \) (resp. \( \tilde{ADE} \)) pair branched along the divisor \( B \), where \( X \) is the double cover surface, \( D \) is the pull-back of the boundary \( C \), and \( R \) is the ramification divisor.

Let \( P \) be a polytope with integral vertices, i.e. in the lattice \( M \cong \mathbb{Z}^r \). To this polytope in toric geometry one associates a projective toric variety \( Y_P \) with an ample line bundle \( L_P \). Many of our surfaces will be toric. For all of them the log canonical property will follow from the following:

**Lemma 3.2.** Let \( P \) be an integral polytope with a distinguished vertex \( p_* \), and \((Y, L)\) be the corresponding polarized projective toric variety. Let \( C \) be the torus-invariant divisor corresponding to the sides passing through \( p_* \). Suppose that all the other sides of \( P \) are at lattice distance 2 from \( p_* \). Then \(-2(K_Y + C) \sim L \) is ample, and the pair \((Y, C)\) has log canonical singularities.

**Proof.** Let \( C' = \sum C'_i \) be the divisor corresponding to the sides not passing through the vertex \( p_* \). The zero divisor of the section \( e^{p_*} \in H^0(Y, L) \) is \( \sum d_i C'_i \), where \( d_i \) are the lattice distances from \( p_* \) to the corresponding sides. This gives \( L \sim 2C' \).

Combining it with the identity \( K_Y + C + C' \sim 0 \) gives the first statement. It is well known that the pair \((Y, C + C')\) has log canonical singularities. Thus, the smaller pair \((Y, C)\) also has log canonical singularities. \( \square \)

**Definition 3.3.** As in the previous lemma, let \( L = -2(K_Y + C) \). For an \( ADE \) surface, we call a side \( C_k \), i.e. an irreducible component of the boundary, short if \( LC_k = 1 \) and long if \( LC_k = 2 \). For a \( \tilde{ADE} \) surface, we call a side \( C_k \) short if \( LC_k = 3 \) and long if \( LC_k = 4 \).

Thus, if \((Y, C)\) is a toric \( ADE \) (resp. \( \tilde{ADE} \)) surface as in Lemma 3.2 then a short side has lattice length 1 (resp. 3), and a long side has lattice length 2 (resp. 4).

**§3.1. Shapes \( A_{2n-1}, A_{2n-2}, \tilde{A}_{2n-3}, \tilde{A}_{2n-3}^\prime \).** The surfaces of these shapes are toric and correspond to the \( A \)-polytopes with the following vertices:

1. \( A_{2n-1} \): \((0, 2), (0, 0), (0, 2n)\), and \( Y = \mathbb{P}(1, 1, n) = \mathbb{P}^0_n \).
2. \( A_{2n-2} \): \((0, 2), (0, 0), (0, 2n - 1)\).
3. \( \tilde{A}_{2n-3} \): \((0, 2), (1, 0), (0, 2n - 1)\).
§3.2. Shapes $D_{2n}$ and $D_{2n-1}^-$. The surfaces $(Y,C)$ are toric and correspond to the $D$-polytopes with the following vertices.

1. $D_{2n}$: $(2,2), (0,2), (0,0), (2n-2,0)$, and $Y = \mathbb{F}_{n-2}$.
2. $D_{2n-1}^-$: $(2,2), (0,2), (0,0), (2n-3,0)$.

§3.3. Shapes $-E_6^-, -E_7^-, -E_8^-$. The surfaces $(Y,C)$ are toric and correspond to the $E$-polytopes with the following vertices. We picture the polytopes in Fig. 4.

1. $-E_6^-$: $(0,0), (0,3), (2,2), (3,0)$.
2. $-E_7$: $(0,0), (0,3), (2,2), (4,0)$.
3. $-E_8^-): (0,0), (0,3), (2,2), (5,0)$.

Next, we list the pure type II cases, corresponding to the extended Dynkin diagrams, and two exotic type II shapes.
§3.4. Shape $\tilde{A}_{2n-1}$. The surface $Y$ is a cone over an elliptic curve, and $C = 0$ so there is no boundary. More precisely, let $\mathcal{F}$ be a line bundle of degree $n > 0$ on an elliptic curve $E$, and let $\tilde{Y}$ be the surface $\text{Proj}_E(\mathcal{O} \oplus \mathcal{F})$. Let $s, s_\infty$ be the zero, resp. the infinity sections, and let $f: \tilde{Y} \to Y$ be the contraction of the zero section. Then $f^* K_Y = K_{\tilde{Y}} + s = -s_\infty$, so $-K_Y$ is ample with $K_Y^2 = n$. If $B \in |-2K_Y|$ is a generic section then $\text{p_a}(B) = n + 1$ and the map $B \to E$ has $2n$ points of ramification. Of course, the surface $Y$ is not toric. The double cover $X \to Y$ branched in $B$ is unramified at the singular point, and $X$ has two elliptic singularities. One has $R^2 = 2K_Y^2 = 2n$.

§3.5. Shape $\tilde{D}_{2n}$, $2n \geq 4$. For $2n \geq 6$, $Y$ is again a toric surface, corresponding to the polytope with the vertices $(0, 0), (0, 2), (4, 2), (2n - 4, 0)$, and the boundary $C$ corresponding to the side $(0, 2) - (4, 2)$, as shown in Fig. 5. Thus, $Y \simeq F_{n-4}$. For $2n \geq 8$, the curve $C$ is an exceptional section of this ruled surface, and for $2n = 6$ it is an infinite section. (We can formally set $Y \simeq F_{-1} := F_1$ in this case.)

![Figure 5. Shape $\tilde{D}_{10}$ and a degenerate subcase of $\tilde{D}_4$](image)

For $\tilde{D}_4$ the surface is $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and the boundary $C \sim s + f$ is a diagonal. It has a toric degeneration to the quadratic cone $\mathbb{F}_{1,2} := \mathbb{F}_2^0 = \mathbb{P}(1,1,2)$ which we also include in the $\tilde{D}_4$ shape. (The boundary $C$ did not change in this degeneration.)

§3.6. Shapes $\tilde{E}_7$ and $\tilde{E}_8^-$. The surfaces $Y$ are toric and have vertices

1. $\tilde{E}_7$: (0, 4), (0, 0), (4, 0), $C = (0, 4) - (4, 0)$, $Y = \mathbb{P}^2$.
2. $\tilde{E}_8^-$: (0, 3), (0, 0), (6, 0), $C = (0, 3) - (6, 0)$, $Y = \mathbb{P}(1,1,2) = \mathbb{F}_2^0$.

The polytopes are shown in Fig. 6. The ‘$-$’ in $\tilde{E}_8^-$ indicates that $LC = 3$ and not 4, so the boundary divisor $C$ is “short”. There is no $\tilde{E}_6$ shape!

![Figure 6. Shapes $\tilde{E}_7$ and $\tilde{E}_8^-$](image)
§3.7. **Exotic \( \tilde{A}_1 \) shape.** The surface is the projective plane \( Y = \mathbb{P}^2 \), the boundary \( C \) is a smooth conic, and the branch curve \( B \) is a possibly singular conic. If \( B \) is smooth then \( X = \mathbb{P}^1 \times \mathbb{P}^1 \); if \( B \) is two lines then \( X = \mathbb{P}_2^0 = \mathbb{P}(1,1,2) \) with \( R \) passing through the apex. We also include here as a degenerate subcase when \( \mathbb{P}^2 \) degenerates to \( Y = \mathbb{P}_1^0 \). Then \( X = \mathbb{P}_2^0 \) with \( R \) not passing through the apex.

§3.8. **Exotic \( \tilde{A}_0 \) shape.** The surface is the quadratic cone \( Y = \mathbb{P}(1,1,2) \) with the minimal resolution \( \tilde{Y} = \mathbb{P}_2 \). The strict preimage of \( C \) on \( \tilde{Y} \) is a divisor in the linear system \(|s + 3f|\). The curve \( C \) passes through the vertex of the cone and is smooth at that point. The branch curve \( B \) is a hyperplane section disjoint from the vertex. The double cover is \( X = \mathbb{P}^2 \) with an involution \((x,y,z) \mapsto (x,-y,z)\), and the boundary divisor is a smooth elliptic curve \( y^2z = f_3(x,z) \).

§3.9. **Basic invariants.**

**Definition 3.4.** The *volume* of a del Pezzo surface \((Y,C)\) with boundary of index \( \leq 2 \) is \( \text{vol}(Y,C) = L^2/2 \in \mathbb{N} \), where \( L = -2(K_Y + C) \). If \( B \in [L] \) is a branch divisor and \((X,D + eR) \to (Y,C + 1 + eB)\) is an ADE cover, then \( \text{vol}(Y,C) = B^2/2 = R^2 \). The *genus* is \( g(Y,C) = \frac{(K_Y + L)L}{2} + 1 \). This is the arithmetic genus of any curve \( B \in [L] \).

\[
\begin{array}{cccccccc}
& A_{2n-1} & A_{2n-2}^- & A_{2n-3}^- & D_{2n} & D_{2n-1}^- & E_6^- & E_7^- & E_8^- \\
\text{vol}(Y,C) & 2n & 2n-1 & 2n-2 & 2n & 2n-1 & 6 & 7 & 8 \\
\text{genus } g & n - 1 & n - 1 & n - 1 & n - 1 & n - 1 & 3 & 3 & 4 \\
\text{smallest} & A_1 & A_0^- & A_1^- & D_4 & D_5 \\
\end{array}
\]

\[
\begin{array}{cccccc}
& \tilde{A}_{2n-1} & \tilde{D}_{2n} & \tilde{E}_7^- & \tilde{E}_8^- & \tilde{A}_1 \tilde{A}_0 \\
\text{vol}(Y,C) & 2n & 2n & 8 & 9 & 2 & 1 \\
\text{genus } g & n + 1 & n - 1 & 3 & 4 & 0 & 0 \\
\text{smallest} & \tilde{A}_1 & \tilde{D}_4 \\
\end{array}
\]

Table 1. Invariants of pure ADE and \( \tilde{A}\tilde{D}\tilde{E} \) surfaces

We summarize the basic invariants for the pairs introduced so far in Table 1.

§3.10. **Priming.** Priming is a natural method for producing new del Pezzo surfaces from old ones.

**Definition 3.5.** Let \((Y,C)\) be an ADE or \( \tilde{A}\tilde{D}\tilde{E} \) surface and let \( P_1, \ldots, P_k \in C \) be distinct nonsingular points of \( Y \) and \( C \). Choose ideals \( I_i \simeq (y, x^2) \) with support at \( P_i \) whose directions are transversal to \( C \). A *priming* of \((Y,C)\) at \( P_1, \ldots, P_k \) is the weighted blowup \( f : Y' \to Y \) at \( I = \prod_{i=1}^k \) composed with a contraction \( g : Y' \to \overline{Y'} \) given by the line bundle \( L' = -2(K_{Y'} + C') \), where \( C' = f^*C - F \) is the strict preimage of \( C \) provided that \( L' \) is semiample.

Note that a priming depends not just on the points \( P_i \) but also on a choice of directions at each point, which must be distinct from the direction of \( C \). The proof of the following theorem mimics some of the arguments of [Nak07], which we consider in more detail in the next section.
Theorem 3.6 (Allowed primings). One has the following:

1. With each blowup the volume drops by 1: $(L')^2/2 = L^2/2 - k$.
2. A necessary condition for the existence of a priming is that the number of points $P_i$ lying on a component $C_j$ of $C$ is $\leq LC_j$, i.e. in the $ADE$ case it is at most 2 for a long side and 1 for a short side; and in the $\tilde{ADE}$ cases at most 4 for a long side and 3 for a short side.
3. If this necessary condition is satisfied and the new volume $L^2/2 - k$ is $> 0$ then $L'$ is nef, big, and semiample, and one has a priming.
4. If $(Y', C')$ is a log canonical non-klt del Pezzo surface with boundary of index $\leq 2$, then so is the primed surface with boundary $(\overline{Y'}, \overline{C'})$.
5. If $(Y', C')$ is an exceptional curve of a blowup $\overline{F}_i$ is a divisor with $F_i^2 = -\frac{1}{2}$, and that $L' = f^*L - 2F$. Thus, under each prime $LC_j$ drops by 1, and part (2) follows. We also easily check that $f^*(K_Y + L) = K_Y + L'$. Combined with $\frac{1}{2}L' \equiv (K_Y + L') + C$ this says that as long as $K_Y + L$ is nef and condition (2) is satisfied, the line bundle $L'$ is nef. If its volume is positive then it is also big. Since $\frac{1}{2}L'$ is of the form $-(K_Y + C')$, it is automatically semiample, see e.g. [Fuj12, Thm.6.1].

The only pure shapes for which $K_Y + L$ is not nef are $A_1, A_0$, and the exotic shapes $\tilde{A}_1$ and $\tilde{A}_0$. By the volume condition we can only prime $A_1$ and $\tilde{A}_1$, once, and the results are as in (4) by a simple computation. Part (5) is easy. □

Definition 3.7. Priming has a very simple geometric meaning for the pairs $(Y, C + \frac{1+\epsilon}{2}B)$ and the double covers $(X, D + \epsilon R)$. Let $B \in |L|$ be a curve vanishing on the ideal $I$. Then its strict preimage under the blowup is $B' \in L'$. In this case the directions of the ideals $I_i$ should be equal the directions of $B$ at $P_i$, so there are no choices. Recall that by our Definition 3.1 of $ADE$ and $\tilde{ADE}$ pairs $B$ must be transversal to $C$ at $P_i$.

But it is on the double cover where the priming operation becomes the most natural and easiest to understand. The double cover $X'$ of $Y'$ ramified in $B'$ is an ordinary smooth blowup of $X$ at the corresponding points $Q_1, \ldots, Q_k$. So upstairs we simply make $k$ ordinary blowups at points $Q_i \in D \cap R$ in the boundary which are fixed by the involution, and then apply the linear system $|NR'|, N \gg 0$, provided that $R'$ is big and nef, to obtain the primed pair $(X', D' + \epsilon R')$.

We will call the above priming of a pair $(Y, C + \frac{1+\epsilon}{2}B)$, resp. priming of a double cover $(X, D + \epsilon R)$.

Remark 3.8. An important observation is that priming does not change the number of moduli of pairs, aside from the redundant case (4) of Theorem 3.6 which we generally exclude. The only difference in the moduli for the primed pair versus the original pair is a finite choice for the points being blown up. For any family of unprimed pairs, on a $2^k : 1$ level cover there is a family of primed pairs.

It is insightful at the point to digress briefly to discuss the process by which the redundant cases $A_1' = A_0$ and $(\tilde{A}_1')' = \tilde{A}_0$ of Theorem 3.6(4) lose moduli under priming. In both cases, this process is easier to see on the double cover $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^3_2$. On $X$, priming corresponds to blowing-up one of the points $D \cap R$; call the exceptional curve $E$. The resulting surface contains a second $(-1)$-curve which is contracted by the priming operation. The resulting surface is $\overline{X'} = \mathbb{P}^2$, the
We denote simple nodes by the usual notation. In Tables 2, 3 we use the following notation for singularities.

Given in Theorem 4.12, where we find the non-generic ones, too.

S are Val singularities of type if is the self-intersection number of the curve in the chain; note that the resolution is not necessarily a chain of curves, we use the following notation: A corresponds to the Du Val singularity ∗ whose resolution is not necessarily a chain of curves, we use the following notation:

Each single prime partially disconnects the branch divisor $B$ (resp. the ramification divisor $R$) from the boundary $C$ (resp. $D$) by decreasing the number $BC = RD$ by 1. When an irreducible component $C_j$ of $C$ is maximally primed, i.e. $L'C_j = 0$, it is contracted on the primed surface $Y'$, which therefore has fewer boundary components.

Notation 3.9. All of our $ADE$ shapes have two sides: left and right, for the two irreducible components of the boundary $C$. For this reason, when priming a shape $S$ we put a mark on the left / right respectively. We denote priming on a long side by a single prime $S'$ or double prime $S''$ if one, resp. two points $P_i$ are blown up. The prime $'$ is a symbolic representation of the ideal $(x^2, y)$.

We denote priming a shape $S^−$ on a short side by $S^+$; visually, the vertical prime converts $−$ to $+$. In the $\tilde{A}DE$ case, there is only one side (or none for $\tilde{A}_{2n−1}$). If the side is long, the primed shapes are $S', S'', S'''$, $S''''$. If the side is short, they are $S^+, S'^+, S'''^+$.

In Tables 2 and 3 we list all the $ADE$ and $\tilde{A}DE$ surfaces, pure and primed, together with the singularities that a generic surface in this family has. Note that all of these singularities appear in the boundary $C$ or at the points where this boundary gets contracted after priming. The proof for the list of singularities is given in Theorem 4.12, where we find the non-generic ones, too.

Notation 3.10. In Tables 2, 3 we use the following notation for singularities. We denote simple nodes by the usual $A_1$. For cyclic quotient singularities, whose resolutions are a chain of curves, we use the notation $(n_1, n_2, \ldots, n_k)$, where $−n_i$ is the self-intersection number of the $i$th curve in the chain; note that $(2, 2, \ldots, 2)$ corresponds to the Du Val singularity $A_n$. For more complicated singularities, whose resolution is not necessarily a chain of curves, we use the following notation: $(n_1, n_2, \ldots, n_k; 2^2)$ denotes a singularity obtained by contracting a configuration of exceptional curves with the first dual graph in Fig. 7. Note that this includes Du Val singularities of type $D_n$, which are denoted by $(2, 2, \ldots, 2; 2^2)$.

Finally, we will use the expression $(2^2; n_1, n_2, \ldots, n_k; 2^2)$ to denote a singularity obtained by contracting a configuration of exceptional curves with the second dual graph in Fig. 7. Two apparently degenerate cases of this notation are $A_1 = (2)$ and $(n; 2^2) = (2, n, 2)$; we nonetheless use both notations, as it is useful to make a distinction when we discuss double covers. We will also often use $(n; 2^4)$ in place of $(n; 2^2; 2^2)$.
The double cover of a simple node in fact smooth points. Double cover of a cyclic quotient singularity $(n, 1)$ and $(n, 2)$ are in fact smooth points.

Separately note that for $n = 1$ the “singularities” $(n)$ and $(n, 2)$ are in fact smooth points.

For completeness, we also note the corresponding singularities on the double covers. The double cover of a simple node $A_1$ is always a smooth point, and the double cover of a cyclic quotient singularity $(n_1, n_2, \ldots, n_k)$ is always a pair of $(2^2; n; 2^2)$.
cyclic quotient singularities with the same resolution; this explains why we draw a distinction between $A_1$, which has smooth double cover, and (2), which has double cover a pair of (2) singularities.

The double cover of a $(n_1, n_2, \ldots, n_k; 2^2)$ singularity is a cyclic quotient singularity $(n_1, n_2, \ldots, n_k-1, 2n_k - 2, n_k-1, \ldots, n_1)$; this explains the second degenerate piece of notation, as $(2, n, 2)$ has double cover a pair of $(2, n, 2)$ singularities, and $(n; 2^2)$ has double cover a single $(2n - 2)$ singularity. Finally, the double cover of a $(2^2; n_1, n_2, \ldots, n_k; 2^2)$ singularity, for $k \geq 2$, is a cusp singularity whose resolution is a cycle of rational curves with self-intersections $(-2n_1 + 2), -n_2, \ldots, -n_k-1, (-2n_k + 2), -n_k-1, \ldots, -n_2$ ordered cyclically, and the double cover of an $(n; 2^4)$ singularity is a simple elliptic singularity whose resolution is a smooth elliptic curve with self-intersection $-2n + 4$.

§3.11. $ADE$ , $\tilde{ADE}$ surfaces with zero boundary. When priming a pure shape surface $(Y, C) \sim (\overline{Y}, \overline{C})$, the new boundary $\overline{C}$ is zero iff all sides of $Y$ are maximally primed, so that the entire boundary $C$ is contracted. This means that in type III for each long side the decoration is $n$ and for a short side $+$, and in type II for a long side $n_\pm$ and for a short side $+n_\mp$.

A complete list of $ADE$, $\tilde{ADE}$ surfaces without boundary is $\tilde{A}_2'$, $A_2''$, $A_2'''$, $A_2''', A_4'$, $A_4'', A_5'$, $A_5''$, $A_6'$, $A_6''$; $\tilde{D}_2'$, $\tilde{D}_2''$, $\tilde{D}_2'''$, $\tilde{D}_2''''$, $E_7'$, $E_7''$, $E_8'$, $E_8''$.

Example 3.11. In [Pro17, Thm.1.1] Prokhorov gave a list of $\mathbb{Q}$-Gorenstein smoothable strictly log canonical del Pezzo surfaces with Picard number 1. Those that have index $\leq 2$ are in our notation $A_{2n-1}$, $D_{2n}$ with extra $4A_1$, $D_{2n-1}$ with extra $2A$, $A_{2n-1}$, and among them there are only finitely many possibilities for $n$ for the smoothable ones. See Thm. 4.12 about non-generic singularities.

§3.12. Toric primed cases. Consider the primed $A_{2n-1}$ case. It is obtained by a weighted blowup at an ideal $I$ with support at $Q \in C_1$. If we choose an infinite section $s_\infty$ on the toric surface $Y = \mathbb{F}_n$ to contain the ideal $I$, so that the toric boundary is $C_1 \cup C_2 \cup s_\infty$, then $I$ will be torus-invariant, and the primed surface $Y'$ again toric. In this way, we see that the primed surfaces $A_{2n-1}$, $A_{2n-2}$, $A_{2n-1}$, $D_{2n}$ are all toric. Some of them are shown in Fig. 8.

![Figure 8. Toric primed shapes $A_{2-1}^+$, $A_{2-2}^+$, $D_{2n}$](image)

![Figure 9. Some special toric surfaces in shapes $D_{2-1}$, $D_{2n}$](image)
For other primed shapes, the surfaces are generally not toric but toric surfaces do appear for special directions of the ideals being blown up. Some of them are shown in Fig. 9.

4. LC non-klt del Pezzo surfaces with boundary of index $\leq 2$

The purpose of this Section is to prove the following:

**Theorem 4.1.** The log canonical non-klt del Pezzo surfaces $(Y, C)$ with reduced, possibly empty boundary are exactly the same as the $ADE$ and $\tilde{A}DE$ surfaces $(Y, C)$, pure and primed.

Log del Pezzo surfaces with boundary $(Y, C)$ such that $-2(K_Y + C)$ is ample and Cartier were classified by Nakayama in [Nak07], over fields of arbitrary characteristic. Some work is still required to extract Theorem 4.1 from his classification. First, in [Nak07] the divisor $C$ is half-integral, and in our case it should be integral. Secondly, the classification in [Nak07] for the case $g = 1$ is not easily applicable to our case. Using it directly, we would have to find all Gorenstein del Pezzo surfaces for which $K_Y$ is divisible by 2 as a Weil divisor, and there are many of them. Rather than doing that, we adapt the arguments from other parts of [Nak07] to this case.

For ease of the use of [Nak07], for this section only, we adopt the notation of the latter paper. The basic setup is as follows. The log Del Pezzo surface with boundary is denoted $(S, B)$, versus our $(Y, C)$. At the outset, let us mention an important general result [Nak07, Cor.3.20] generalizing that of [AN06, Thm.1.4.1]:

**Theorem 4.2 (Smooth Divisor Theorem).** Let $(S, B)$ be a log del Pezzo surface with boundary of index $\leq 2$. Then a general element of the linear system $|-2(K_S + B)|$ is smooth.

By [Nak07, 3.16, 3.10], the only pairs with irrational $S$ and integral $B$ are cones over elliptic curves which we call $\tilde{A}_{2n-1}$. So below we assume that $S$ is rational. The minimal resolution of singularities of $S$ is denoted by $\alpha : M \to S$. One defines:

1. An effective $\mathbb{Z}$-divisor $E_M$ on $M$ by the formula $K_M = \alpha^*(K_S + B) - \frac{1}{2}E_M$. Since we assume the pair $(S, B)$ to be lc, $E_M$ has multiplicities 1 and 2. If $B = 0$ and $S$ is log terminal then $E_M$ is reduced. Otherwise, there is at least one component of multiplicity 2.

2. A big and nef line bundle $L_M = \alpha^*(-2(K_S + B))$. Thus, one has $L_M = -2K_M - E_M$.

3. The genus $g(S, B) = \frac{1}{2}(K_M + L_M)L_M + 1$. This is the genus of a general element of $|-2(K_S + B)|$.

This is the standard notation used in [Nak07]:

- On $\mathbb{P}^2$, a line is denoted by $\ell$.
- On $\mathbb{F}_n$, a zero section is $\sigma$, an infinite section $\sigma_{\infty}$, and a fiber $\ell$.
- On $\mathbb{P}(1, 1, n)$, $\bar{\ell}$ is the image of a fiber from $\mathbb{F}_n$, i.e. a line through the apex.

The classification of log Del Pezzo surfaces with boundary is divided into three cases:

1. $K_M + L_M$ is not nef.
2. $K_M + L_M$ is nef and $g \geq 2$.
3. $K_M + L_M$ is nef and $g = 1$. 
§4.1. The case $K_M + L_M$ is not nef. By [Nak07, 3.11], the only cases for us are:

1. $S = \mathbb{P}^2$, deg $B = 2 \implies B$ is a smooth conic (our $\mathbb{A}_1^*$) or two lines ($A_1$).
2. $S = \mathbb{P}(1,1,n)$, $n \geq 2$ and $B \in |(\frac{n}{2} + 2)\ell|$, in particular $n$ is even.

Note that the smallest divisor not passing through the apex is $\sigma_\infty \sim n\ell$. We consider the subcases:

(a) $B \nparallel$ apex. We need $\frac{n}{2} + 2 \geq n \implies n = 2, 4$. If $n = 2$ then $B \in |3\ell|$ is not Cartier, a contradiction. If $n = 4$ then $B \in |4\ell| = |O(1)|$, $L_M = O(1)$.

This is a degenerate subcase of $\mathbb{A}_1^*$, when $\mathbb{P}^2$ degenerates to $\mathbb{P}_4^0 = \mathbb{P}(1,1,4)$.

(b) $B \ni$ apex and is smooth there. The strict preimage of $B$ is then $\tilde{B} \sim \ell + k\sigma_\infty$ for some $k \geq 0$. Then $B \sim (1 + kn)\ell \implies \frac{n}{2} + 2 = 1 + kn$. It follows that $n = 2$ and $k = 1$. If $B$ is irreducible then this is our $\mathbb{A}_0^*$ case; if $B = \ell + \sigma_\infty$ then this is $A_0^-$.

(c) $B \ni$ apex and has two branches there. Then $\tilde{B} \sim 2\ell + k\sigma_\infty$ and $B \sim (2 + kn)\ell \sim (\frac{n}{2} + 2)\ell$. This is impossible.

§4.2. $K_M + L_M$ is nef and $g \geq 2$. Nakayama defines a basic pair to be a projective surface $X$ and a nonzero effective $\mathbb{Z}$-divisor $E$ so that, for $L = -2K_X - E$ one has:

(C1) $K_X + L$ is nef,
(C2) $(K_X + L)L = 2g - 2 > 0$,
(C3) $LE_i \geq 0$ for any irreducible component $E_i$ of $E$.

So, the minimal resolution of a log del Pezzo surface with boundary of index $\leq 2$ is a basic pair, unless $B = 0$ and $S$ has Du Val singularities (because then $E = 0$). Vice versa, by [Nak07, 3.19], any basic pair is the minimal resolution of a log del Pezzo surface with boundary of index $\leq 2$, with the semiparallel line bundle $NL$, $N > 0$, providing the contraction.

The next step is to run MMP for the divisor $K_X + \frac{1}{2}L$. Namely, if for some $(-1)$-curve $\gamma$ one has $(2K_X + L)\gamma = -E\gamma < 0$ then $L\gamma = E\gamma = 1$, the curve $\gamma$ can be contracted $\tau$: $X \to Z$ to obtain a new basic pair $(Z, E_Z)$, and one has $K_X + L = \tau^*(K_Z + L_Z)$, $K_X + E = \tau^*(K_Z + E_Z)$. Here, $E_Z = \tau_*(E)$ and it is again nonzero.

The minimal basic pairs, without the $(-1)$-curves as above are $\mathbb{P}^2$ and $\mathbb{P}_n$, and it is easy to list the possibilities for $E$ on them. Nakayama proves that the morphism $\phi: M \to X$ to a minimal basic pair is a sequence of blowups of the simplest type which can be conveniently locally encoded by a zero-dimensional subscheme $\Delta$ of a smooth curve, i.e. a subscheme given by an ideal $I = (y, x^k)$ for some local parameters $x, y$ and $k > 0$. If $\mu: Y \to X$ is a simple blowup then $I_Y = \mu^*I = (y, x^{k-1})\cap O_Y(-\Gamma)$, where $\Gamma$ is the exceptional $(-1)$-curve of $\mu$. Then one continues to eliminate $I_Y$ by induction, making $k$ blowups in total. Equivalently, one can blow up the ideal $I$ and then take the minimal resolution.

In this way, we obtain a triple $(X, E, \Delta)$ satisfying

(F1) $(X, E)$ is a minimal basic pair, $L = -2K_X - E$,
(F2) $\Delta$ is empty or a zero-dimensional subscheme of $X$ which is locally a sub-
scheme of a smooth curve,
(F3) $\Delta$ is a subscheme of $E$ considered as a subscheme of $X$ (recall that $E$ is an effective Cartier divisor with multiplicities 1 or 2) such that for every reduced irreducible component $E_i$ of $E$ one has $LE_i \geq \deg(\Delta \cap E_i)$.

Nakayama calls these quasi fundamental triplets. Vice versa, by [Nak07, 4.2] for any quasi fundamental triplet $(X, E, \Delta)$ the pair $(M, E_M)$ obtained by eliminating $\Delta$ is a basic pair, that is the minimal resolution of singularities of a log del Pezzo surface with boundary. Thus, one is reduced to enumerating quasi fundamental triplets.

For a given basic pair $(M, E_M)$, the sequence of blowdowns of $(-1)$-curves and thus the resulting quasi fundamental triplet $(X, E, \Delta)$ are not unique. To cure this, Nakayama defines a fundamental triplet that satisfies additional normalizing conditions [Nak07, Def. 4.3]. He then proves in [Nak07, 4.9] that the fundamental triplet exists and is unique in most cases, including all the cases when $(S, B)$ is strictly log canonical – the case that we operate in. For this case, the possible fundamental triplets are listed in [Nak07, 4.7(2)].

It remains to consider these fundamental triplets and the resulting minimal resolutions $M$. But first, we can narrow down the possibilities for $\Delta$ since our situation is restricted by the condition that $B$ is integral and not half-integral as in [Nak07].

**Definition 4.3.** We introduce the following simple subschemes $\Delta \subset E$.

| $E$          | $\Delta$ | $\deg(\Delta)$ | $\text{mult}_P(\Delta \cap E_i)$ |
|--------------|----------|-----------------|----------------------------------|
| $(\cdot)$    | $(y)$    | $(y, x)$        | 1                                |
| $(-)_{1}$    | $(y)$    | $(y, x^2)$      | 2                                |
| $(−)$        | $(y^2)$  | $(y, x^2)$      | 2                                |
| $(t)$        | $(y^2)$  | $(y^2, x)$      | 2                                |
| $(+)$        | $(y^2)$  | $(y^2, y + \epsilon x^2)$, $\epsilon \neq 0$ | 4                                |

An alternative description for the last subscheme is $(y + \epsilon x^2, x^4)$.

**Lemma 4.4.** The effect of eliminating the subschemes of (4.3) is as follows.

- $(\cdot)$ $E_M = 1E_1 + 0\Gamma_1$, $\Gamma_1 = -1$, $E_1 \Gamma_1 = 1$, $L_M \Gamma_1 = 1$.
- $(-)_{1}$ $E_M = 1E_1 + 0\Gamma_1 + 0\Gamma_2$, $\Gamma_1 = -1$, $\Gamma_2 = -2$, $E_1 \Gamma_1 = \Gamma_1 \Gamma_2 = 1$, $L_M \Gamma_1 = 1$.
- $(−)$ $E_M = 2E_1 + 2\Gamma_1 + 0\Gamma_2$, $\Gamma_1 = -1$, $\Gamma_2 = -2$, $E_1 \Gamma_1 = \Gamma_1 \Gamma_2 = 1$, $L_M \Gamma_1 = 1$.
- $(t)$ $E_M = 2E_1 + 2\Gamma_1 + 0\Gamma_2$, $\Gamma_1 = -2$, $\Gamma_2 = -1$, $E_1 \Gamma_1 = \Gamma_1 \Gamma_2 = 1$, $L_M \Gamma_2 = 1$.
- $(+)$ $E_M = 2E_1 + 2\Gamma_1 + 0\Gamma_2 + 0\Gamma_3$, $\Gamma_1 = -1$, $\Gamma_2 = 0$, $\Gamma_3 = 0$, $E_1 \Gamma_1 = \Gamma_1 \Gamma_2 = \Gamma_1 \Gamma_3 = \Gamma_3 \Gamma_4 = 1$, $L_M \Gamma_4 = 1$.

It is pictured in Fig. 10.

**Proof.** This is direct computation, following [Nak07, Sec.2].

**Notation 4.5.** In Fig. 10, the rectangle with label “d” denotes an irreducible component $E_i$ of $E$ with $E_i^2 = -d$. The empty vertices are $\mathbb{P}^1$’s of square $(-1)$, the filled ones of square $(-2)$. Rectangles and vertices are shown in blue, resp. red or black, if they appear in $E_M$ with multiplicity $2$, resp. $1$ or $0$. The half-edges denote $\text{mult}_P(\Delta \cap E_i)$, which are $2$ (double line) or $1$ (single line). When we are working with a geometric triple $(X, B + \frac{1}{2}K_S + D)$, where $D \in |-2K_S - E|$ is a section, these half edges are the local intersection numbers $DE_i$ at a point $P \in D \cap E_i$. The double edge means that $D$ is tangent to $E_i$ at $P$.

We note the following, see proof of [Nak07, 4.7]:
Lemma 4.6 (Nakayama). The pair $(S, B)$ is log canonical iff for every irreducible component $E_i$ of $E$ in the fundamental triplet $(X, E, \Delta)$, one has $\text{mult}_E(E_i) \leq 2$, $\Delta$ is disjoint from the nodes of the double part $\cup E_i$ of $E$, and $\text{mult}_P(\Delta \cap E_i) \leq 2$ for every irreducible component $E_i$ with $\text{mult}_E(E_i) = 2$ and all $P \in \Delta$.

Theorem 4.7. Let $(M, E_M)$ be a basic pair with $M$ the minimal resolution of singularities of a strictly log canonical log del Pezzo surface with boundary $(S, B)$ of index $\leq 2$ with integral $B$, and let $\phi: M \to X$ be a contraction to a minimal basic pair so that $(M, E_M)$ is obtained from a quasi fundamental triplet $(X, E, \Delta)$ by eliminating the 0-dimensional scheme $\Delta$. Then

1. If a component $E_i$ of $E$ has multiplicity 1 then its strict preimage on $M$ must be isomorphic to $\mathbb{P}^1$ and have $E^2_i \leq -2$.
2. Additionally, assume that $\Delta$ is disjoint from the singular part of $E_{\text{red}}$ and that for every irreducible component $E_i$ of $E$ with $\text{mult}_E(E_i) = 1$, one has $\text{mult}_P(\Delta \cap E_i) \leq 2$. Then the only connected components of $\Delta$ are the five subschemes of Def. 4.3.

Remark 4.8. Concerning the additional assumptions of (2), we note that they are satisfied for the strictly log canonical fundamental triplets by [Nak07, 4.6]. So we can ignore them in the case $g(S, B) \geq 2$.

Proof. (1) Our condition for the integrality of $B$ means that all components of $E_M$ of multiplicity 1 must be contracted by $\alpha: M \to S$. They are all $\mathbb{P}^1$'s with $E^2_i \leq -2$.

(2) We then go through the short list of subschemes with $\text{mult}_P(\Delta \cap E_i) \leq 2$, eliminating those that lead to $(-1)$-curves $\Gamma$ with $\text{mult}_E(\Gamma) = 1$. For example, the case $\Delta = (x, y) \subset E = (y^2)$ is eliminated.

Nakayama defined fundamental triplets $(X, E, \Delta)$ (without “quasi”) in order to obtain uniqueness for them, in most cases. We pick a different normalization: we pick $(X, E)$ to correspond to one of the pure shapes and all connected components of $\Delta$ to be of type $\ell$.

Theorem 4.9. Let $(S, B)$ be a log del Pezzo surface with boundary $(S, B)$ of index $\leq 2$ of genus $g(S, B) \geq 2$. Then it is one of the following shapes or is obtained from them by any allowable primings as in Theorem 3.6.

1. $\tilde{D}_{2n}$, $D_{2n}$, $D_{2n-1}$, $A_{2n-1}$, $A_{2n-2}$, $-A_{2n-3}$ for $2n \geq 6$. 

Figure 10. Effect of eliminating simple subschemes
Proof. We go through the complete list [Nak07, 4.7(2)] of fundamental triplets and see that they are as above.

Case $[n; 2, e]_2$ for $n \geq 0$, $e \leq \max(4, n+1)$ with $\text{mult}_i F \leq 2$ for any $\ell \leq F$. This means that $X = \mathbb{P}_n$ and $E = 2\sigma + F$, where $F \sim e\ell$ is a sum of several fibers, each with multiplicity $\leq 2$, and $\Delta \cap \sigma = \emptyset$. We have $L \sim 2\sigma_\infty + (4-e)\ell$ and $L\sigma = 4-e$.

If $e = 0$ then $\Delta = \emptyset$. This is $\tilde{D}_{2n+8}$, so we obtain $\tilde{D}_{2m}$ for $2m \geq 8$.

If $e = 1$ then we must have $\Delta = (\cdot)$, that is two disjoint copies of $(\cdot)$ contained in a fiber $F$, or $(\cdot)_1$ which is a degeneration of it. Let us use the extended notation $[n; 2, 0; \cdot \cdot \cdot ]$, resp. $[n; 2, 0; -1]$ by writing $\Delta$ at the end. Note that we must apply $(\cdot)$ twice, otherwise $F$ is a $(1)$-curve in $E_M$ with multiplicity 1, which is not allowed by Theorem 4.7.

Contracting one of the $(1)$-curves back and then $F_1$, we can view this as the quasi fundamental triplet $[n-1; 2, 0; \ell]$, which is $\tilde{D}_{2n+6}$. Thus, we get $\tilde{D}_{2m}$ for $2m \geq 6$.

In the degenerate case $\Delta = (\cdot)_1$ of $(\cdot)$, the direction of the “prime” coincides with the direction of the fiber $\ell$ on $\mathbb{P}_{n-1}$ for the triplet $[n-1; 2, 0; \ell]$. In that case the strict preimage of this fiber gives an extra $(1)$-curve, and the surface $Y$ acquires an extra $A_1$ singularity outside of $B$.

If $e = 2$ and $F = \ell_1 + \ell_2$ then we get $[n-2; 2, 0; \cdot \cdot \cdot ]$ this way, which is $\tilde{D}_{2n+4}''$. Since $n \geq 1$, we get $\tilde{D}_{2m}''$ for $2m \geq 6$. Similarly when $e = 3, 4$ and $F$ is the sum of $e$ distinct fibers, we get $\tilde{D}_{2m}'''$ and $\tilde{D}_{2m}''''$ for $2m \geq 6$. Similar to the above, for every priming the preimage of the corresponding fiber $\ell$ gives an $(1)$-curve which gives an additional singularity of $Y$.

Now consider the case when $e = 2$ and $F = 2\ell$ is a double fiber. If $\Delta = \emptyset$ then this is $\tilde{D}_{2n+4}$, i.e. $\tilde{D}_{2m}$ for $2m \geq 6$. For $\Delta = (-), (\cdot), +$ we get $\tilde{D}_{2n+1}$, $\tilde{D}_{2n+2}$, $\tilde{D}_{2n+2}$, $\tilde{D}_{2n+2}$, $\tilde{D}_{2n+2}$, $\tilde{D}_{2n+2}$ for $2m \geq 6$. Adding single fibers to $F$, i.e. $F = 2\ell + \ell_1$ or $2\ell + \ell_1 + \ell_2$, gives priming on the left side, all the cases $\tilde{D}$ and $\tilde{D}'$ for $2m \geq 6$.

Finally, $e = 4$, $F = 2\ell_1 + 2\ell_2$ and $\Delta = \emptyset$ gives $A_{2n+1}$. Adding $\Delta = --, +, -$ adds various decorations in the $A$ case, with each $-, +$ decreasing the index by 1.

Case $[1; 2, 2]_{2\infty}$: $Y = F_1$, $E = 2\sigma_\infty$, and $\Delta = \emptyset$. This is $\tilde{D}_6$.

Case $[2]_2$ with mult$_P(\Delta \cap \ell) \leq 2$ for any $P \in \ell$: $Y = \mathbb{P}^2$, $E = 2\ell$ and $L = \mathcal{O}(4)$. For $\Delta = \emptyset$, this is $\tilde{E}_7$. For $\Delta = (-)$, resp. $(\cdot)$, this is $-\tilde{E}_7$, $-\tilde{E}_6$. Considering various other possibilities for $\Delta$ leads to all the allowable primings of $-\tilde{E}_7$, $-\tilde{E}_7$, $-\tilde{E}_6$.

Case $[2; 1, 2]_2$ with mult$_P(\Delta \cap \ell) \leq 2$ for any $P \in \ell$: $Y = \mathbb{P}^2$, $E = \sigma + 2\ell$, deg$(\Delta \cap \ell) \leq 3$ and $\Delta \cap \sigma = \emptyset$. For $\Delta = \emptyset$ this is $\tilde{E}_8$, and for $\Delta = (\cdot)$ this is $-\tilde{E}_8$. Considering various other possibilities for $\Delta$ leads to all the allowable primings of $-\tilde{E}_8$ and $-\tilde{E}_8$.

Case $[0; 2, 1]_0$. This is a typo, this is a klt case so it does not appear.

§4.3. $K_M + L_M$ is nef and $g = 1$. In this case the main result of [Nak07] is (3.12) which says that $S$ must be a Gorenstein log del Pezzo surface and $2B \sim -K_S$. To apply it in our case, we would have to find all Gorenstein del Pezzo surfaces with Du Val singularities and $K_S$ divisible by 2 as a Weil divisor – of which there are many – and then consider all the possibilities for $B$. 

\[ \Box \]
Instead, we adopt a different strategy. Let us define a weak basic pair with the same definition as a basic pair but dropping the condition \((C2)\) that \(2g - 2 > 0\). Similarly, we define a weak quasi fundamental triplet \((X, E, \Delta)\) by asking that \(X\) in \((\mathcal{F}1)\) is merely a weak minimal basic pair. Then:

1. It is still true that \(K_M + L_M\) is nef for any weak basic pair obtained by eliminating a 0-dimensional scheme of a weak fundamental triple \((X, E, \Delta)\): the corresponding proofs in [Nak07, 4.2, 3.14 nefness] go through.
2. We have additional conditions \(K_M + L_M = K_M + E_M = 0\) by [Nak07, 3.12].
3. Our Theorem 4.7 still holds.
4. We have to check separately that \(L_M\) is big, this condition is no longer automatic. However, this is easy to do: \(L^2/2\) drops by \(\deg(\Delta)/2\), i.e. by 1 under the operations \((\eta), (\ldots), (\ldots), \) and by 2 under \((\eta)\).

**Lemma 4.10.** The weak fundamental triplets for strictly lc pairs \((S, B)\) are:

1. \(X = \mathbb{P}^2\), \(E = 2\ell_1 + \ell_2\).
2. \(X = \mathbb{P}_0\), and \((a)\) \(E = 2\sigma + 2\ell\), \((b)\) \(E = 2\sigma + \ell_1 + \ell_2\), \((c)\) \(E = 2D\), \(D \sim \sigma + \ell\).
3. \(X = \mathbb{P}_1\), and \((a)\) \(E = 2\sigma + 2\ell_1 + \ell_2\), \((b)\) \(E = 2\sigma + \ell_1 + \ell_2 + \ell_3\), \((c)\) \(E = \sigma + \sigma_\infty + 2\ell\), \((d)\) \(E = 2\sigma_\infty + \ell\).
4. \(X = \mathbb{P}_2\), and \((a)\) \(E = 2\sigma + 2\ell_1 + 2\ell_2\), \((b)\) \(E = 2\sigma + 2\ell_1 + \ell_2 + \ell_3\), \((c)\) \(E = 2\sigma + \ell_1 + \ell_2 + \ell_3 + \ell_4\), \((d)\) \(E = \sigma + 2\ell + \sigma_\infty\), \((e)\) \(E = 2\sigma_\infty\).

**Proof.** Immediate: \(X = \mathbb{P}^2\) or \(\mathbb{P}_n\), \(L = -K_X\) must be nef, and \(E = -2K_X - L\) must have at least one component of multiplicity 2. We simply list the possibilities. \(\square\)

**Theorem 4.11.** Let \((S, B)\) be a log del Pezzo surface with boundary \((S, B)\) of index \(\leq 2\) of genus \(g(S, B) = 1\). Then it is one of one of the shapes \(\overline{D}_4\), \(D_4\), \(A_3\), \(A_2^*\), \(\sim A_1^*\), or is obtained from one of them by any allowable primings as in Theorem 3.6.

**Proof.** The pairs of Lemma 4.10 in which all components of \(E\) have multiplicity 2 already appear in our classification: \((2a)\) \(D_4\), \((2c)\) \(\overline{D}_4\), \((4a)\) \(A_3\), \((4e)\) degenerate case of \(\overline{D}_4\). Our first step is to reduce all other cases to them.

Let us begin with case \((1)\). The line \(\ell_2\) must be blown up at least one by Theorem 4.7(1). Thus, we are reduced to case \((2)\).

Now consider for example case \((2b)\). The fiber \(\ell_1\) must be blown up at least once, again by \((4.7)(1)\). Let \(\tau: X' \to X\) be the first blowup at a point \(P \in E\) and let \(E_0\) be the exceptional \((\sim 1)\)-curve. We have \(K_{X'} + E' = \tau^*(K_X + E) = 0\). If \(P = \ell_1 \cap \sigma\) then \(E_0\) appears in \(E'\) with coefficient 2, otherwise it appears with coefficient 0; either way it is even. Let \(X' \to X''\) be the contraction of the strict preimage of \(\ell_1\), which is a \((\sim 1)\)-curve on \(X'\). We obtained another minimal model \(M \to X''\) for \(M\) which has fewer components of multiplicity 1 in \(E\).

This way, we reduce all cases to the purely even cases above except cases \((3c)\) and \((4d)\). Consider now \((3c)\). The curve \(\sigma_\infty\) has to be blown up at least once. Blowing up and contracting the strict preimage of a fiber reduces to the case \((3a)\) which was already considered. The case \((4d)\) reduces to \((3c)\) and then to \((3a)\).

So now we are reduced to the pairs of shapes \(D_4\), \(A_3\), \(\overline{D}_4\) and the pairs obtained from them by eliminating 0-dimensional subschemes \(\Delta\). The conditions of Theorem 4.7(2) hold, so the connected components of \(\Delta\) have types \((\eta), (\ldots), (\ldots)\). In the cases \(D_4\), \(A_3\) we also have \(\deg(\Delta \cap E_i) \leq 2\) for \(i = 1, 2\). In all three cases, \(\deg(\Delta) \leq 6\) by the condition \(L^2_M > 0\).
So let us now begin with $D_4$ and consider different possibilities for $\Delta$. If one or two components of $\Delta$ are $(-)$ then we get respectively $D_1^\prime = A_3'$ and $-D_2^\prime = -A_2'$. If the components are $(+)$ then we get respectively $D_2'' = A_3''$ and $-D_3'' = -A_3''$. When the components of $\Delta$ are $(t)$, we get the usual primings.

For $\tilde{D}_4$, $\Delta = (-)$ gives $D_1'$ and $\Delta = (--)$ gives $A_1'$, with other combinations of $(-), (+)$, $t$ giving primings of those. For $A_3$, it is easier: $\Delta = (-), (--), (+)$ etc. gives the usual $A_2'$, $A_1'$, $A_1''$ and adding $(t)$'s gives the usual primings. □

This completes the proof of Theorem 4.1. We now switch from the notation of [Nak07] to our notation $\pi: (X, D + \epsilon R) \rightarrow (Y, C + \frac{1+\epsilon}{2} B)$.

§4.4. Singularities of $ADE$ and $\tilde{ADE}$ surfaces.

**Theorem 4.12.** For a generic surface in each $ADE$ and $\tilde{ADE}$ family, the singularities are as in Tables 2, 3. Special fibers $Y$ in the families of type II and III shapes may also have the following additional singularities away from $C$:

1. $A_1'$: a singularity $(\frac{1+1}{4}) = (4)$ when $\mathbb{P}^2$ degenerates to $\mathbb{P}(1, 1, 4) = \mathbb{P}^1_A$, and $\tilde{D}_4$ and those cases obtained from it by priming: a singularity $A_1$ when $\mathbb{P}^1 \times \mathbb{P}^1$ degenerates to $\mathbb{P}(1, 1, 2)$.

2. For $2n \geq 6$, for each priming of $\tilde{D}_{2n}$ and for each priming on the left of $D_{2n}$, $D'_{2n}$, $D''_{2n}$, $D_{2n-1}'$, $D_{2n-1}''$, an extra $A_1$ may appear.

3. In the following shapes, one of these configurations of singularities is possible, which should be added to the generic ones for that shape:

| $\tilde{D}_4'$ | $D_2''$ | $D_4''$ | $\tilde{A}_3''$ |
| --- | --- | --- | --- |
| $A_2: 2A_1; A_1$ | $A_3: 2A_1; A_1; 3A_1$ | $A_3: 2A_1; A_1; 3A_1$ | $A_2; 2A_1; A_1$ |
| $D_4: A_1$ | $D_4': 2A_1; A_1$ | $D_4': A_2; A_1$ | $\tilde{A}_3' 2A_1; A_1$ |
| $D_4': A_2; A_1$ | $D_4': A_2; A_1$ | $D_4': 2A_1; A_1$ | $D_4': 3A_1; 2A_1; A_1$ |
| $\tilde{A}_3': A_1$ | $\tilde{A}_3': A_1$ | $\tilde{A}_3': A_1$ | $\tilde{A}_3': A_1$ |

**Proof.** In the situation of our §4.2 [Nak07, Lemma 4.3] describes all possibilities for the curves with $E^2 \leq -2$ on a minimal resolution of $Y$. With one exception, stated below, they are all obtained from the exceptional section and fibers of the minimal resolutions of the unprimed surfaces. We thus do a direct computation:

For $A_{2n-1}$ the surface is $Y = \mathbb{F}_n$ with a singularity $(n)$. Every $'-'$ makes this vertex into the chain $(n, 2, 1, 2)$, thus into singularities $(n, 2), A_1$. Similarly, every single prime produces $(n, 1, 2)$, double prime $(n, 2, 2)$, and $'+'$ gives $(n, 2, 2; 2^2)$. With these rules, the configurations in all the $A$-shapes follow. Similarly, for $D_{2n}$ the surface is $Y = \mathbb{F}_{n-2}$, $C_1$ is a $(2-n)$-section, and $C_2$ is a fiber. Priming on the left gives $'\mapsto (2, n-1)$, $''\mapsto (2^2, n)$. On the right we get $'\mapsto (n-2, 2), A_1$, $''\mapsto (n-2), A_1$, $''\mapsto (n-2), (2; 2^2), +\mapsto (n-2, 2, 2^2)$. This gives the list. The $E$-shapes are done similarly.

The one exception in our notation is as follows. When priming in the $\tilde{D}$ cases and priming on the left in the $D$ cases, we pick a direction for the weighted blowup at $(x^2, y)$. If this direction is the direction of a fiber $F$ then the strict preimage of $F$ is a $(-2)$-curve which gives an additional $A_1$-singularity.

We analyze the remaining cases, those corresponding to §4.1 and §4.3, case by case. We omit this tedious but elementary computation.
Finally, (1) is a consequence of our definition of a family. In the cases other than \( \tilde{A}_1 \) and \( \tilde{D}_4 \) by definition the surface \( Y \) is fixed and only the divisor \( B \) varies. These are exceptions. \( \Box \)

§4.5. Relation to involutions in the Cremona group. Classically, the involutions in the Cremona group \( \text{Cr}(\mathbb{P}^2) \), the group of birational automorphisms of \( \mathbb{P}^2 \), are of three types: De Jonquières, Geiser, and Bertini. For a nice modern treatment that uses equivariant MMP, see [BB00]. For a \((K+D)\)-trivial polarized involution pair \((X,D,\iota)\), if \( X \) is rational then \( \iota \) is an involution in \( \text{Cr}(\mathbb{P}^2) \).

**Theorem 4.13.** Let \((X,D,\iota)\) be a \((K+D)\)-trivial polarized involution pair with rational surface \( X \) and a smooth ramification curve \( R \). Then

1. If \((X,D,\iota)\) is of shape \( \tilde{D}, D, \text{ or } A \) (pure or primed) then \( \iota \) is De Jonquières.
2. If it is of shape \( \tilde{E}_7, E_7, \text{ or } E_6 \) (pure or primed) then \( \iota \) is Geiser.
3. If it is of shape \( \tilde{E}_8, E_8 \) (pure or primed) then \( \iota \) is Bertini.

**Proof.** By [BB00, Prop. 2.7], the type of the involution is uniquely determined by the normalization \( \tilde{R} \) of the ramification curve \( R \): for De Jonquières \( \tilde{R} \) is hyperelliptic, for Geiser it is non-hyperelliptic of genus 3, and for Bertini it is non-hyperelliptic of genus 4. In the \( \tilde{D}-D-A \) cases the branch curve \( B \cong R \) is a two-section of a ruling, so it is hyperelliptic. In the \( \tilde{E}_7-E_7-E_6 \) cases \( R \) is a quartic curve in \( \mathbb{P}^2 \), so a non-hyperelliptic curve of genus 3, and in the \( \tilde{E}_8-E_8 \) cases it is a section of \( \mathcal{O}(1) \) on the quadratic cone \( F_0^2 \), so a non-hyperelliptic curve of genus 4. \( \Box \)

**Remark 4.14.** When \( R \) has nodes, the involution may easily be of a different type. When it has \( \geq 2 \) nodes, the involution is always De Jonquières.

We can give an alternative proof for the classification of the double covers \((X,D) \to (Y,C)\) of log canonical non-klt surfaces using [BB00] in some cases:

**Theorem 4.15.** Let \((X,D,\iota)\) be a \((K+D)\)-trivial polarized involution pair with rational \( X \). Suppose that \( X \) is smooth outside of the boundary \( D \), and in particular that the ramification curve \( R \) is smooth. Then the quotient \((Y,C)\) of this pair is an ADE or \( \tilde{A}\tilde{D}E \) surface given in Section 3.

**Sketch of the proof.** Let \( \tilde{X} \) be the minimal resolution of \( X \), it comes with an induced involution \( \tilde{\iota} \). [BB00, Thm. 1.4] gives six possibilities for the pair \((\tilde{X},\tilde{\iota})\) when it is minimal, i.e. there does not exist one or two \((-1)\)-curves that can be equivariantly contracted to another smooth surface with an involution. In our case, \( \tilde{X} \) is obtained from such a minimal surface by a sequence of single or double blowups which satisfy two conditions:

1. They have to be involution-invariant.
2. There are no \((-2)\)-curves disjoint from \( B \).

It follows that \( \tilde{X} \) is obtained by blowups at the points \( B \cap R \), either one involution-invariant point or two points exchanged by the involution. We analyze them directly. The different cases of [BB00, Thm. 1.4] then lead to the following:

(i) impossible, i.e. does not lead to a \((K+D)\)-trivial polarized involution pair with an ample \( R \).

(ii) \((ii)_{\text{sm}}\) is impossible, and \((ii)_{g}\) gives the \( \tilde{D}-D-A \) shapes.

(iii) \( A_0^- \) and \( \tilde{A}_0^- \).
(iv) $\tilde{A}_1^*$, $A_1$.
(v) $E_7$, $-E_7$, $-E_6$ and the primed shapes.
(vi) $E_8$, $-E_8$ and the primed shapes.

Remark 4.16. If we are to classify families of log del Pezzo pairs, in which the surface $Y$ may acquire singularities away from the boundary, then there are two ingredients to make this result for generic pairs into an alternative proof of the classification. We need to know that:

1. The branch divisor $B$ can be smoothed. This is the Smooth Divisor Theorem [Nak07, Cor. 3.20].
2. The singular points of the surface $Y$ away from the boundary can be smoothed. For surfaces without the boundary, this is [HP10, Prop. 3.2]. For the pairs $(Y, C)$ with the boundary this does not seem to be easy to prove directly. This follows a posteriori from the classification of all log del Pezzo surfaces with boundary given in the previous two sections.

5. Two-dimensional models of $ADE$ lattices

Notation 5.1. Let $\Lambda$ be an irreducible $A_n$, $D_n$, or $E_n$ root lattice with a set of positive roots $\Phi^+$, and let $\Lambda^* \supset \Lambda$ be its dual, the weight lattice. Fix simple roots $\alpha_1, \ldots, \alpha_n$ and fundamental weights $\varpi_1, \ldots, \varpi_n$, so that $\langle \alpha_i, \varpi_j \rangle = \delta_{ij}$. (Note that the roots and coroots are the same in the $ADE$ cases.)

Let $W = W(\Lambda)$ be the Weyl group. It is well known that the ring of invariants $k[\Lambda^*]^W$ is the polynomial ring $k[\chi_1, \ldots, \chi_n]$, where $\chi_i$ are the characters of the fundamental weights $\varpi_i$. This ring also has an additive basis consisting of the characters $\chi(\lambda)$ of dominant weights $\lambda \in \Lambda^+_+ = \bigoplus_{i=1}^n N\varpi_i$.

On the set of dominant weights there is a natural partial order: $\lambda \geq \lambda'$ if the difference $\lambda - \lambda' = \sum n_i \alpha_i$ is a nonnegative linear combination of the simple roots. The character $\chi(\lambda)$ is the sum of monomials $e^\mu$, appearing with positive multiplicities, of all dominant weights $\mu \leq \lambda$ plus their $W$-translations $e^{w.\mu}$, so that the entire sum is $W$-invariant. The multiplicities can be computed e.g. from the Weyl character formula or Kostant multiplicity formula.

Notation 5.2. The $ADE$ Dynkin diagrams that appeared in Figs. 2, 3, 4 come with natural coordinates for the nodes. Each integral point $(i, j)$ in the polytope corresponds to a monomial $x^iy^j$. It will be convenient to denote the nodes of the diagram in a similar way to the monomial: $p_i$ for the monomial $x^i$ with $i \geq 0$, $p'_j$ for $y^j$ with $j > 0$. To these, we add the points $p''_j$ for the monomial $xy$ (appearing only in the $D_n$ and $E_n$ cases) and $p_*$ for the special central vertex. We also include the two endpoints for the edges out of $p_*$. Fig. 11 illustrates this notation in the $A_4$, $D_4$, and $E_8$ cases.

To keep track of various families and degenerations it will be very convenient to denote the fundamental weights $\varpi$, their characters $\chi$, simple roots $\alpha$, and certain coefficients $c$ appearing below using the same sub- and superscripts as the points in the polytope: $\varpi_i$, $\varpi'_j$, $\varpi''$ etc. From now on, we adopt this suggestive notation.

When we need to refer to all the fundamental weights, we write it as $\{\varpi \in \Pi\}$.

Definition 5.3. We define the following vectors in $\mathbb{Z}^3$, pictured in Fig. 11.
Forgetting the end points, i.e. under the projection to $\Lambda$, define a linear map $\phi: \Lambda^* \oplus \mathbb{Z} \to \mathbb{Z}^3$ by sending $\varpi_i \mapsto u_i$, $\varpi_j \mapsto u_j$, $\varpi'' \mapsto v''$, $\varpi_\ast \mapsto v_\ast$. This linear map defines the following commutative diagram, in which $j$ is the natural embedding of the root lattice into the weight lattice:

$$
\begin{array}{ccc}
0 & \longrightarrow & \Lambda \\
& \searrow & \downarrow \phi \\
& & \Lambda^* \oplus \mathbb{Z}^3 \\
& \nearrow & \downarrow (\text{id}, 0) \\
& & \mathbb{Z}^3 \\
& \swarrow & \downarrow 0 \\
& & 0
\end{array}
$$

Moreover, let $\lambda$ be one of the fundamental weights (i.e. $\lambda = \varpi_i$, $\varpi_j$, or $\varpi''$) and let $\mu = \sum_{\varpi \in \Pi} m(\varpi) \varpi$ be a dominant weight. Then one has $\lambda \geq \mu \iff \psi(\lambda - \mu)$ is a nonnegative combination of the vectors $u_i$, $u_j$ (including the endpoints), $v''$, and $-v_\ast$, with one exception: $u_0 = 2v''$ but $\varpi_0 \not\geq 2\varpi''$. There are no exceptions if we define $\overline{v}'' = (1 - \epsilon, \rho'' - p_\ast)$ for $0 < \epsilon \ll 1$ (but then $\psi \not\in \mathbb{Z}^3$).

**Proof.** Case $A_n$. The kernel of $\phi$ has a basis of vectors $-\varpi_{i-1} + 2\varpi_i - \varpi_{i+1}$ for $1 \leq i \leq n$. Forgetting the end points, i.e. under the projection to $\Lambda^*$, they go to $2\varpi_0 - \varpi_1$, $-\varpi_{i-1} + 2\varpi_i - \varpi_{i+1}$ for $2 \leq i \leq n - 1$, and $-\varpi_{n-1} + 2\varpi_n$. These are precisely the simple roots $\alpha_n$ of $A_n$ when expressed in the basis of fundamental weights. The fundamental characters of $A_n$ are miniscule, i.e. there are no dominant weights below any fundamental weight. And indeed, no $\overline{u}_i$ is a nonnegative linear combination of other vectors $\overline{u}_k$, pictured in Fig. 11.

Case $D_n$. A basis of $\ker \phi$ is given by

$-\varpi_{i-1} + 2\varpi_i + \varpi_{i+1}$, (1 $\leq i \leq n - 3$), $-\varpi_2' + 2\varpi_1' - \varpi_0$, $2\varpi'' - \varpi_0$, $2\varpi_0 - \varpi_1 - \varpi_1'' + \varpi_\ast$.

Forgetting $\varpi_\ast$ and the endpoints $\varpi_{n-2}$, $\varpi_{n-1}'$, these are exactly the simple roots of $D_n$. The partial order on the fundamental weights is $\varpi_i \geq \varpi_{i+2k}$ for $i \geq 0, i + 2k \leq n - 3$.

**Figure 11.** The $A_1^*$ and $-E_6$ shapes with coordinates

(1) $\overline{u}_i = (2, p_i - p_\ast)$, $\overline{u}_j' = (2, p_j' - p_\ast)$.

(2) $v'' = (1, p'' - p_\ast)$, $v_\ast = (1, 0, 0)$.

The first coordinate is the degree (equal to 2 or 1), and the second and third coordinates give a vector in $\mathbb{Z}^2$, with the special point $p_\ast$ chosen as the origin.

**Definition 5.4.** We define an extended weight lattice to be $\Lambda^* \oplus \mathbb{Z}^3$. Here, $\Lambda^* = \oplus_{\varpi \in \Pi} \mathbb{Z} \varpi$ is an ADE weight lattice with a basis of fundamental weights, to which we add three additional basis vectors of $\mathbb{Z}^3$: $\varpi_\ast$ for the special point $p_\ast$ and two vectors for the endpoints. In the $A_n$ case the two basis vectors for the endpoints are $\varpi_0, \varpi_{n+1}$, for $D_n$ they are $\varpi_2', \varpi_{n-2}$, and for $E_n$ they are $\varpi_3', \varpi_{n-3}$.

**Theorem 5.5.** For each of the $A_n$, $D_n$, $E_n$ root lattices, define a linear map $\phi: \Lambda^* \oplus \mathbb{Z} \to \mathbb{Z}^3$ by sending $\varpi_i \mapsto \overline{u}_i$, $\varpi_j \mapsto \overline{u}_j$, $\varpi'' \mapsto v''$, $\varpi_\ast \mapsto v_\ast$. This linear map defines the following commutative diagram, in which $j$ is the natural embedding of the root lattice into the weight lattice:

$$
\begin{array}{ccc}
0 & \longrightarrow & \Lambda \\
& \searrow & \downarrow \phi \\
& & \Lambda^* \oplus \mathbb{Z}^3 \\
& \nearrow & \downarrow (\text{id}, 0) \\
& & \mathbb{Z}^3 \\
& \swarrow & \downarrow 0 \\
& & 0
\end{array}
$$

Moreover, let $\lambda$ be one of the fundamental weights (i.e. $\lambda = \varpi_i$, $\varpi_j$, or $\varpi''$) and let $\mu = \sum_{\varpi \in \Pi} m(\varpi) \varpi$ be a dominant weight. Then one has $\lambda \geq \mu \iff \psi(\lambda - \mu)$ is a nonnegative combination of the vectors $\overline{u}_i$, $\overline{u}_j$ (including the endpoints), $v''$, and $-v_\ast$, with one exception: $\overline{u}_0 = 2v''$ but $\varpi_0 \not\geq 2\varpi''$. There are no exceptions if we define $\overline{v}'' = (1 - \epsilon, \rho'' - p_\ast)$ for $0 < \epsilon \ll 1$ (but then $\psi \not\in \mathbb{Z}^3$).
For a later use, we spell it out:

\[ \varpi_i + 2 = \varpi_i - \alpha'' - \alpha'_1 - 2 \sum_{s=0}^{i} \alpha_s - \alpha_{i+1}. \]

This corresponds to the only convex relations for the vectors: \( \tilde{u}_i = \eta_{i+2} + \bar{u}_2 - 2\bar{\nu}_i. \)

| higher coords | lower coords | \( \alpha'' \) | \( \alpha' \) | \( \alpha'' \) | \( \alpha' \) | \( \alpha'' \) | \( \alpha' \) | \( \alpha'' \) | \( \alpha' \) | \( \alpha'' \) | \( \alpha' \) |
|---------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( \varpi_0 \) | 6 10 14     | \( \varpi'_0 + \varpi_2 \) | 6 10 12     | 1 1 2 2 1 1 | D_4         |
| \( \varpi'_2 + \varpi_2 \) | 6 10 12 | \( \varpi''_2 + \varpi_3 \) | 6 9 12      | 1 1 1 1 1 1 | A_5         |
| 2\( \varpi'_2 + \varpi_4 \) | 6 10 10 | \( \varpi'_1 + \varpi_4 \) | 6 9 11 1     | 1 1 1 1 1 1 | A_3         |
| \( \varpi'' + \varpi_3 \) | 6 9 12. | \( \alpha'' + \varpi_3 \) | 6 9 11 1     | 1 1 1 1 1 1 | A_3         |
| \( \varpi'' + \varpi_3 \) | 6 9 12. | \( \varpi'' + \varpi_4 \) | 6 9 11 1     | 1 1 1 1 1 1 | A_3         |
| \( \varpi'_1 + \varpi_4 \) | 6 9 11 | \( \varpi'_1 + \varpi_4 \) | 6 9 11 1     | 1 1 1 1 1 1 | A_4         |
| \( \varpi'_1 + \varpi_4 \) | 6 9 11 | \( \varpi'_1 + \varpi_4 \) | 6 9 11 1     | 1 1 1 1 1 1 | A_4         |
| \( \varpi'' + \varpi'_4 \) | 5 9 11. | \( \varpi'' + \varpi'_4 \) | 5 8 11 1     | 1 1 1 1 1 1 | A_4         |
| \( \varpi_1 \) | 5 8 11 | \( \varpi'_1 + \varpi_3 \) | 5 8 9 1     | 1 1 2 2 1 1 | D_5         |
| \( \varpi'_2 + \varpi_3 \) | 5 8 9 | \( \varpi'' + \varpi_4 \) | 5 7 9 1     | 1 1 1 1 1 1 | A_6         |
| \( \varpi'_2 + \varpi_3 \) | 5 8 9 | \( \varpi'' + \varpi_4 \) | 5 7 9 1     | 1 1 1 1 1 1 | A_6         |
| \( \varpi'' + \varpi_3 \) | 5 7 9 | \( \varpi'' + \varpi_4 \) | 5 6 7 2 1 2 3 2 1 | E_6         |
| \( \varpi'' + \varpi_4 \) | 5 7 9 | \( \varpi'' + \varpi_4 \) | 5 6 7 2 1 2 3 2 1 | E_6         |
| \( \varpi'_3 + \varpi_4 \) | 5 6 7 | \( \varpi'_2 + \varpi_4 \) | 4 6 8 1 1 1 1 1 1 | A_2         |
| \( 2\varpi'_2 \) | 1 2 3 | \( \varpi'_2 + \varpi_4 \) | 4 6 8 1 1 1 1 1 1 | A_2         |
| \( \varpi'_1 \) | 4 7 9 | \( \varpi'_2 + \varpi_4 \) | 4 6 8 1 1 1 1 1 1 | A_2         |
| \( \varpi'_2 \) | 4 6 8 | \( \varpi'_2 + \varpi_4 \) | 4 6 8 1 1 1 1 1 1 | A_2         |
| \( \varpi'_2 + \varpi_4 \) | 4 6 6 | \( 2\varpi_4 \) | 4 4 4 2 2 2 2 2 2 1 | E_7         |
| \( \varpi_2 + \varpi_4 \) | 4 6 6 | \( \varpi'' + \varpi_4 \) | 3 5 7. 1 1 1 1 1 1 1 1 1 | A_7         |
| \( 2\varpi_4 \) | 4 4 4 | \( \varpi'' + \varpi_4 \) | 3 4 5 1 1 1 1 1 1 1 1 1 | A_7         |
| \( \varpi'' \) | 3 5 7 | \( \varpi'_3 \) | 3 4 5 2 1 2 3 2 2 1 | E_6         |
| \( \varpi'_3 \) | 3 4 5 | \( \varpi'_2 + \varpi_4 \) | 2 4 4 1 1 2 2 2 2 2 2 1 | E_7         |
| \( \varpi'_2 \) | 2 4 4 | \( \varpi'_2 + \varpi_4 \) | 2 2 2 2 2 3 4 3 2 2 1 | E_7         |
| \( \varpi_4 \) | 2 2 2 | \( \varpi'_2 + \varpi_4 \) | 0 0 0 0 3 2 4 6 5 4 3 2 | E_6         |

Table 4. Partial order on dominant weights of \( E_8 \) below \( \varpi_0 \)

Case \( E_n \). In the same way as for \( D_n \), \( \ker \phi \) is given by the vectors that project to the simple roots in \( \Lambda^\ast \). This proves the first part. For the “moreover” part, we need to know the partial order on the dominant weights below the fundamental weights. This time it is quite nontrivial.

For the hardest case \( E_8 \) we computed it in Table 4. Each highlighted line is for a fundamental weight. The lines below it correspond exactly to the dominant weights lying below it; in addition, 0 is also always a dominant weight below it. In particular,
\[\varpi_0 \text{ dominates all the other fundamental weights. For each dominant weight } \lambda \text{ we give the coordinates of the vector } \psi(\lambda) \in \mathbb{Z}^3 \text{ using the basis } \{\vec{u}_5, \vec{u}_3, -\vec{u}_4\}. \text{ All other } \vec{u}_i, \vec{u}'_i, \text{ are positive linear combinations of these three.} \]

Every line is a “cover”, a minimal step in the partial order, and we write the difference as a positive combination of simple roots. The difference in a cover is known to be equal to the highest root of some connected Dynkin subdiagram, see e.g. [Ste98, Thm.2.6]. We give this diagram in the last column.

The posets of dominant weights for \(E_7\) and \(E_6\) are easily deduced from the \(E_8\) poset. For the step \(E_8 \to E_7\): in the weights below a given weight \(\varpi \neq \varpi_4\), the symbol \(\varpi_4\) is formally set to zero, and all the weights \(\lambda\) in the \(E_8\) table for which \(\varpi - \lambda\) contains \(\alpha_4\) disappear. The same rule works for the \(E_7 \to E_6\) step with \(\varpi_3, \alpha_3\) instead of \(\varpi_4, \alpha_4\). The proof of the “moreover” part of the theorem in the \(E_n\) cases is now done by a direct observation.

\[\text{Remark 5.6. Perhaps even better is a 3-dimensional model of the } D_n \text{ and } E_n \text{ lattices, using an extended weight lattice } \Lambda^* \oplus \mathbb{Z}^4 \text{ and the coordinates } \varpi \mapsto (1, q(\varpi) - q_*) \text{, where } q_* = (1, 1, 1) \text{ and } q_i = (i, 0, 0) \text{ for } 0 \leq i \leq 5, q'_i = (0, i, 0) \text{ for } i = 1, 2, 3, \text{ and } q''_i = (0, 0, i) \text{ for } i = 1, 2. \]

In this model the exception for \(\varpi_0 \not\geq 2\varpi^n\) disappears, and the order between all the dominant weights \(\lambda \leq \varpi_0\) is preserved, with only one tie-break: \(2\varpi_3 > \varpi_2 + \varpi_4\) but their coordinates are equal. Our 2-dimensional model is obtained from this 3-dimensional model by a projection corresponding to the double cover.

6. Normal forms and moduli of \(ADE\) pairs

Below, we will consider the \(ADE\) pairs \((Y, C_1 + C_2 + \frac{1+\epsilon}{2}B)\), obtained by fixing the left and the right sides of \(C = C_1 + C_2\). In most cases these sides are uniquely determined, cf. Figs. 2, 3, 4. The exceptions are \(A_{2n-1}, -A_{2n-1}, D_4\) and \(-E_6\), where forgetting the order divides the moduli space by an involution.

\[\text{Theorem 6.1. Let } (Y, C_1 + C_2 + \frac{1+\epsilon}{2}B) \text{ be an ADE pair of pure shape. Then there exists a change of coordinates such that the equation } f \text{ has zero coefficients for all monomials except at those corresponding to the nodes of the Dynkin diagram, the endpoints, and the special vertex } p_+. \text{ Such a change of coordinates is unique except for rescaling } x \mapsto ax, y \mapsto by. \]

\[\text{Proof. Write the equation as } f = \sum c_{i,j} x^i y^j, \text{ with the monomials } x^i y^j \text{ going over the integral points } (i, j) \text{ in the polytope of Figs. 2, 3, 4.} \]

Recall from §3.10 that \(B\) intersects each long side \(C_k\) of \(Y\) at two points, and a short side at one smooth point. Thus, without loss of generality we can assume that the monomials for the endpoints have coefficient 1, and the one for the special vertex \(p_+\) coefficient \(-\frac{1}{4}\). (We could have chosen it to be 1 as well; \(-\frac{1}{4}\) will come in handy in the next Section.) The complement \(S = Y \setminus C\) is an affine surface, and automorphisms of \((Y, C_1, C_2)\) are obtained by changes of affine coordinates of \(S\).

\[\text{Case } A_n. \text{ One has } f = -\frac{1}{4} y^2 + y \sum c_{i,1} x^i + \sum c_{i,0} x^i. \text{ The substitution } x \mapsto x, y \mapsto y + 2 \sum c_{i,1} x^i \text{ completing the square makes the coefficients of } y x^i \text{ vanish.} \]

\[\text{Case } D_n. \text{ The substitution is } x \mapsto x, y \mapsto y + 2 \sum c_{i,1} x^i\text{.} \]

\[\text{Case } E_n. \text{ For } E_6 \text{ and } E_7 \text{ the substitution is } x \mapsto x + 2c_{2,1}, y \mapsto y + 2c_{1,2}. \text{ In the } E_8 \text{ case it is } x \mapsto x + 2c_{2,1}, y \mapsto y + 2c_{1,2} + 2c_{2,2} x. \]
It is easy to see that essentially these are the only allowable changes of variables. Any other substitutions introduce monomials outside of the polytope, once plugged into the leading monomial \( y^2 \), resp. \( x^2 y^2 \) respectively. The only freedom is rescaling \( x, y \) keeping the ratios of the fixed corner monomials.  

We now make precise the allowed rescalings \( x \mapsto ax, \; y \mapsto by \). Recall that the Cartier dual of a finite abelian group \( H \) is defined as \( \hat{H} = \text{Hom}(H, \mathbb{G}_m) \). This is a finite abelian group of multiplicative type. For example, the dual of \( \mathbb{Z}_n \) is \( \mu_n \), the group of \( n \)-th roots of unity. The difference between \( H \) and \( \hat{H} \) is only really felt over a non closed field, or in positive characteristic, or over a more general base. We mostly work over a field \( k = \mathbb{C} \) of characteristic 0. Still, it is good to make a distinction, notationally and ideologically.

**Theorem 6.2.** The moduli stack of ADE pairs \((Y, C_1 + C_2 + [\frac{1+\epsilon}{2}] B)\) of a pure \( A_n \), \( D_n \), \( E_n \) shape is the quotient stack \([h^n/\hat{H} \times \hat{A}]\), where \( \hat{A} \) is \( \mathbb{Z}_2 = \mu_2 \) acting trivially for the \( A_n \) shape and is trivial for \( D_n, E_n \), and \( H = \Lambda^*/\Lambda \), i.e.:

1. for \( A_n \): \( H = \mathbb{Z}_{n+1} \),
2. for \( D_n \): \( H = \mathbb{Z}_2^2 \) for even \( n \) and \( \mathbb{Z}_4 \) for odd \( n \),
3. for \( E_n \): \( H = \mathbb{Z}_3 \) for \( E_6 \), \( H = \mathbb{Z}_2 \) for \( E_7 \), and \( H \) is trivial for \( E_8 \).

The moduli stack stack of ADE pairs \((Y, C + [\frac{1+\epsilon}{2}] B)\), without marking the sides as “left” and “right” is the same except in the shapes \( A_{2n-1}, A_{2n-3}, D_4, E_6 \), where it is an additional quotient by \( \mathbb{Z}_2 \).

**Proof.** The allowable substitutions are a subgroup of the torus \( \mathbb{G}_m^2 \) that multiply the leading monomial for the point \( p \), and the two endpoints by a common multiple. This group is \( \hat{H} \), where \( H \) is the quotient of \( \mathbb{Z}^2 \) by the sublattice generated by the above three points. We now observe that this group is as stated for the ADE polytopes. This is also essentially part of the statement of Theorem 5.5.

In the \( A_n \) cases the \( \mu_2 \) subgroup \( y \mapsto \pm y \) leaves the equation fixed. The consequence is that every element in this family has a \( \mu_2 \)-automorphism. In other cases, a generic element in the family has a trivial automorphism group.  

| Shape | \( \Lambda'/\Lambda \) | Generators of \( \Lambda'/\Lambda \) |
|-------|----------------|-----------------
| \( A'_{2n-1} \), \( A''_{2n-1} \), \( A''_{2n} \) | \( \mathbb{Z}_2 \) | \( \varpi_n \) |
| \( D_{2n} \), \( D''_{2n} \), \( D_{2n} \) | \( \mathbb{Z}_2 \) | \( \varpi_{2n-3} \) |
| \( D_{2n} \), \( D''_{2n} \), \( D_{2n} \) | \( \mathbb{Z}_2 \) | \( \varpi' \) |
| \( D'_{2n} \), \( D''_{2n} \), \( D'_{2n} \), \( D''_{2n} \) | \( \mathbb{Z}_2^2 \) | \( \varpi', \varpi_{2n-3} \) |
| \( D_{2n-1} \), \( D'_{2n-1} \), \( D''_{2n-1} \), \( D''_{2n-1} \) | \( \mathbb{Z}_2 \) | \( \varpi_{2n-4} \) |
| \( -E_7^1, -E_7^1, +E_7^1, +E_7^1 \) | \( \mathbb{Z}_2 \) | \( \varpi_3 \) |

Table 5. All cases where \( \Lambda' \neq \Lambda \)

**Theorem 6.3.** The moduli stack of ADE pairs \((Y, C_1 + C_2 + [\frac{1+\epsilon}{2}] B)\) of a primed \( A_n \), \( D_n \), \( E_n \) shape is the quotient stack \([h^n/\hat{H} \times \hat{A}]\), where \( \hat{A} \) is \( \mu_2 \) acting trivially for the pure \( A \)-shapes and is trivial otherwise, and \( H = \Lambda^*/\Lambda' \) for a certain intermediate lattice \( \Lambda \subset \Lambda' \subset \Lambda^* \). In all the cases except those listed in Table 5, one has \( \Lambda' = \Lambda \).
Proof. The subgroup of $G_m^2$ of allowable rescalings $x \mapsto ax$, $y \mapsto by$ for a primed shape is smaller: it has to preserve points in $B \cap C_k$ being blown up, at which we “prime”. For a short side $C_k$ there is a unique point, so it is not a condition. For a long side, we do indeed a get a proper subgroup.

In the $A_n$ case, the first priming removes the extra $\mu_2$ summand, and leaves $H = \Lambda^*/\Lambda$. The second priming on the same side does not add a condition. If there is a second long side then we are in the $A_{2n-1}$ case. A simple computation gives $\Lambda' = \Lambda + \varpi_n$. Here, $\varpi_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n + (n-1)\alpha_{n+1} + \cdots + \alpha_{2n-1})$ is the fundamental weight for the middle node. The other cases are done entirely similarly. The only sides to check are the long sides, and they appear in $D_{2n}$ (two of them), $D_{2n-1}^-$, and $^-E_7$. Priming on a short side is “free”. □

7. Weyl group equivariant families of $ADE$ pairs over tori

As we already mentioned in Section 5, a basic result of representation theory is that the ring $k[\Lambda^*]^W$ of invariants of a group algebra of the weight lattice for the action of the Weyl group $W = W(\Lambda)$ is the polynomial ring $k[\chi(\varpi), \varpi \in \Pi]$ generated by the characters of the fundamental weights. So, the quotient of the torus $T_\Lambda = \text{Hom}(\Lambda, G_m)$ by the Weyl group action is $T_\Lambda^*/W = \mathbb{A}^n$, over which by Theorem 6.2 there is a family of $ADE$ surfaces.

It is an old idea that a Weyl cover cures many ills and simplifies the situation dramatically. Applications of this idea include Brieskorn-Tjurina-Arnold’s versal deformations of $ADE$ singularities, a Brieskorn-Grothendieck resolution, and a Springer resolution, to name a few.

In our case, the $\hat{H}$- and $W$-actions on $T_\Lambda^*$ commute, and we have the following commutative diagram:

$$
\begin{array}{ccc}
T_\Lambda^* & \xrightarrow{\hat{H}} & T_\Lambda \\
\downarrow /W & & \downarrow /W \\
\mathbb{A}^n & \xrightarrow{\hat{H}} & \mathbb{A}^n/\hat{H} = T_\Lambda/W
\end{array}
$$

Lifting the family from $\mathbb{A}^n$ to $T_\Lambda^*$ has many advantages. The discriminant locus becomes very easy: it is a union of root subtori $e^\alpha = 1$ for the roots $\alpha \in \Phi^+$ of $\Lambda$. But most importantly for us, a family over a torus can be naturally extended to a family over a compactification, a projective toric variety, which we do in Section 10.

Construction 7.1. For each $ADE$ lattice, we will construct an explicit family of Laurent polynomials $f(x,y)$ over the torus $T_\Lambda^* = \text{Hom}(\Lambda^*, G_m)$, i.e. with coefficients in the ring $k[\Lambda^*]$, which has the following properties:

1. The Newton polytope of $f(x,y)$ is an $ADE$ polytope as in Figs. 2, 3, 4.
2. The nonzero coefficients of $f(x,y)$ are in $k[\Lambda^*]^W$. There are three constant terms. The others are labeled by the fundamental weights $\varpi \in \Pi$ of $\Lambda$, and are of the form $c(\varpi) = \chi(\varpi) + \text{(lower terms)}$,

where “lower terms” are the characters of smaller dominant weights $\mu \leq \varpi$, i.e. $\mu = \varpi - \sum n(\alpha)\alpha$ for simple roots $\alpha$ and integers $n(\alpha) \geq 0$. (Cf. Theorem 5.5.)
(3) for \( t \in T_\Lambda \), the zero set \( B_t \) of \( f_i \) is singular \( \iff \) \( t \) lies in the union of root loci \( \cup_{\alpha \in \Phi^+} \{ e^\alpha = 1 \} \). For each \( t \) the singularities of \( B_t \) are Du Val, and for \( t = 1 \) the curve \( B_1 \) has a unique singularity of type exactly \( \Lambda \).

(4) Let \( Y \) is a toric surface corresponding to the Newton polytope of \( f(x, y) \), \( C = C_1 + C_2 \) the curve corresponding to the part of the boundary as in Figs. 2, 3, 4, and let \( B = \{ f = 0 \} \). Then \( (Y, C_1 + C_2 + \frac{1}{2}B) \) is an \( ADE \) pair.

**Corollary 7.2.** A family given by Construction 7.1 is a lift of the family in Thms. 6.2, 6.3.

**Proof.** Indeed, it differs from a family over \( \mathbb{A}^n = \text{Spec } k[\varpi] \) by an automorphism of \( \mathbb{A}^n \), choosing a different set of coordinates \( c(\varpi) \) instead of \( \chi(\varpi) \). \( \square \)

§7.1. Two notions of discriminant. At the heart of this construction are two notions of the discriminant:

1. Discriminant \( \text{Discr}(f) \) of a polynomial \( f(x, y) \). This is a polynomial in the coefficients \( c \) of \( f \) for which the zero set of \( f \) is singular.
2. Discriminant of the lattice \( \text{Discr}(\Lambda) \), the square of the expression

\[
\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w \in W} \epsilon(w)e^{w \cdot \rho}, \quad \text{where } \rho = \sum_{\varpi \in \Pi} \varpi = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.
\]

appearing in the Weyl character formula. The zero set of \( \text{Discr}(\Lambda) \) is obviously the union of the root subtori \( e^\alpha = 1 \).

**Theorem 7.3.** For each of the ADE cases, there is a choice \( c(\varpi) = \chi(\varpi) + (\text{lower terms}) \) of the coefficients \( c(\varpi) \) of the polynomials \( f(x, y) \) as in Construction 7.1 for which \( \text{Discr}(f) = \text{Discr}(\Lambda) \).

This theorem will be proved case-by-case in the rest of this section. We will use Notation 5.2 for the characters \( \chi \) and the coefficients \( c \).

§7.2. \( A_n \) families of curves. The simplest case of this phenomenon is for the \( A_n \) lattice. The \( A_n \) root lattice is \( \langle e_i - e_j \rangle \subset \mathbb{Z}^{n+1} \) and the dual weight lattice is \( A_n^* = \langle f_i = e_i - p \rangle \), where \( p = \frac{1}{n+1} \sum e_i \), so that \( \sum_{i=1}^{n+1} f_i = 0 \). In these coordinates, \( k[\Lambda^*] = k[t_1^\pm, \ldots, t_{n+1}^\pm]/(\prod t_k - 1) \) and \( k[\Lambda] = k[t_i/t_j] \). The characters of the fundamental weights are the symmetric polynomials \( \sigma_i(t_k) \).

Consider a polynomial in one variable with the roots \( t_k \) such that \( \prod_{k=1}^{n+1} t_k = 1 \):

\[
f(x) = \prod_{i=1}^{n+1} (x + t_i) = 1 + t_1 x + \ldots + t_n x^n + x^{n+1}.
\]

The zero set of \( f \) is singular \( \iff \) \( f(x) \) has a double root \( \iff \prod_{i>j} (\frac{t_i}{t_j} - 1) = 0 \). This expression is exactly \( \text{Discr}(A_n) \) since the positive roots are \( \alpha = e_i - e_j \) for \( i > j \), and \( e^\alpha = t_i/t_j \). So in this case the two discriminants coincide by setting simply \( c_i = \chi_i \). And indeed the fundamental characters of \( A_n \) are miniscule, so there is no room for "lower terms".

The point of Theorem 7.3 is that there is a similar interpretation for \( \text{Discr}(D_n) \) and \( \text{Discr}(E_n) \) if one uses a polynomial \( f(x, y) \) in two variables.
§7.3. $A_n$ families of surfaces. In the two-dimensional case, we take $f(x, y)$ to be
\[ -\left(\frac{y}{2}\right)^2 + c(x) \quad \text{and} \quad -\left(\frac{y}{2}\right)^2 + xc(x), \]
where $c(x) = 1 + \sum_{i=1}^{n} \chi_i x^i + x^{n+1},$
where the first (resp. second) equation is for the case when the left side of the polytope is “long” (resp. “short”). This is essentially the one-dimensional case considered above, with an extra variable added so that the Newton polytope becomes two-dimensional and as in Fig. 2. One can replace $y$ by $xy$ for uniformity with the $D, E$ cases.

§7.4. $D_n$ families. We take the affine equation of the surface $X$ upstairs to be
\[ xyz = z^2 + c''z + y^2 + c'y + c(x), \quad c(x) = c_0 + c_1 x + \ldots + c_{n-3} x^{n-3} + x^{n-2}. \]
Here, we abbreviated $c'_i$ to $c'$. Completing the square and introducing the variable $w = z - \frac{xy - c''}{2}$, the equation of $X$ becomes $w^2 + f(x, y) = 0$ with the involution acting as $i: (x, y, w) \mapsto (x, y, -w)$. The equation of the branch curve $B$ on the quotient $Y = X/\iota$ is then
\[
(7.1) \quad f(x, y) = -\left(\frac{xy - c''}{2}\right)^2 + y^2 + c'y + c(x)
\]
We have the following root lattice, weight lattice, Weyl group, fundamental roots $\alpha_i$, and fundamental weights $\varpi_i$:
\[
\Lambda = \{(a_i) \in \mathbb{Z}^n \oplus \mathbb{Z}e_i \mid \sum a_i \text{ is even}\}, \quad \Lambda^* = \mathbb{Z}^n + \frac{1}{2} \sum e_i.
\]
\[
W = \mathbb{Z}^{n-1}_2 \rtimes S_n, \quad \alpha_{n-2-i} = e_i - e_{i+1} \text{ for } 1 \leq i \leq n-1, \quad \alpha'' = e_{n-1} + e_n.
\]
\[
\varpi_{n-2-i} = \sum_{k=1}^{i} e_k \text{ for } i \leq n-2, \quad \varpi' = \frac{1}{2} \left(-e_n + \sum_{i=1}^{n-1} e_i\right), \quad \varpi'' = \frac{1}{2} \left(\sum_{i=1}^{n} e_i\right).
\]
Denoting by $\sigma_k$ the $k$-th symmetric polynomial, one has
\[
\chi_i = \sigma_{n-2-i}(t_k^x) \text{ for } i \leq n-2, \quad \chi' = \frac{\sum_{s \geq 0} \sigma_{2s+1}(t_k)}{\sqrt{\prod t_k}}, \quad \chi'' = \frac{\sum_{s \geq 0} \sigma_{2s}(t_k)}{\sqrt{\prod t_k}}.
\]
Let $f_k(x)$ be the polynomials defined recursively by $f_0 = 1$, $f_1 = x$, and $f_{k+2} = x f_{k+1} - f_k$. These are the Fibonacci polynomials, except for the signs and a shift in degrees by 1. One has $f_2 = x^2 - 1$, $f_3 = x^3 - 2x$, etc.

**Theorem 7.4.** For the $D_n$ lattice, one has $c' = \chi'$, $c'' = \chi''$, and the expression for $c(x)$ can be obtained from the generating function
\[
c(x, \chi) = \sum_{i, j \geq 0} c_{ij} x^i \chi^j = \sum_{i \geq 0} c_i(\chi) x^i = \sum_{j \geq 0} p_j(x) \chi^j
\]
by substituting $\chi_j$ for $\chi^j$ and setting $\chi_{n-2} = 1$ and $\chi_j = 0$ for $j > n-2$. One has

1. $c(x, \chi) = \frac{1}{(1 - \chi^2)(1 - x \chi + \chi^2)}$ and $c_i(\chi) = \frac{\chi^i}{(1 - \chi^4)(1 + \chi^2)^i}$.
2. $p_{2k}(x) = f_k^2$ and $p_{2k+1} = f_k f_{k+1}$.

The central fiber has a $D_n$ singularity at the point $(x, y, z) = (-2, -2^{n-3}, -2^{n-3})$.

**Example 7.5.** For $D_7$ we obtain for the following expressions for $c(x)$:

$$
egin{align*}
\chi_0 + \chi_1 x + \chi_2 x^2 + \chi_3 x(x^2 - 1) + \chi_4 (x^2 - 1)^2 + (x^2 - 1)(x^3 - 2x) = \\
(\chi_0 + \chi_4) + (\chi_1 - \chi_3 + 2)x + (\chi_2 - 2\chi_4)x^2 + (\chi_3 - 3\chi_5)x^3 + \chi_4 x^4 + x^5
\end{align*}
$$

and for any lower $D_n$ the formulas can be obtained from these by truncation.

**Proof of Thm. 7.4.** We start with the polynomial $f(x, y)$ in equation (7.1). As a quadratic polynomial in $y$, it represents a curve which is a double cover of $\mathbb{A}^1$. This curve is singular when the following polynomial in $x$

$$
\text{Discr}_y(f) = (x^2 - 4)c(x) + c''x + c'^2 + c''^2
$$

has a double root. Equate Discr$_y(f)$ with the polynomial $p(x) = \prod_{i=1}^{n}(x + t_i t_i^{-1})$. It has a double root iff some $t_i + t_i^{-1} = t_j + t_j^{-1}$, i.e., $t_i t_j^{-1} = 1$ or $t_i t_j = 1$. These are exactly the root subtori for the root lattice $D_n$. The coefficients of $p(x)$ are $\sigma_i(t_k t_k^{-1})$, and they are invariant under the $W(D_n)$-action, so they are polynomials in the fundamental characters $\chi_i$ listed above. The rest of the proof is a combinatorial manipulation to get the exact formula, left to the reader as an exercise. \(\square\)

§7.5. $E_n$ families. Similarly to the $D_n$ case, we take the following affine equations for the surface $X$ upstairs and the branch curve $B \subset Y$ downstairs:

$$
xyz = z^2 + c''z + y^3 + c_2 y^2 + c_1 y + c(x), \quad c(x) = c_0 + \ldots + c_{n-4} x^{n-4} + x^{n-3}
$$

(7.2) $f(x, y) = -\left(\frac{xy - c''}{2}\right)^2 + y^3 + c_2 y^2 + c_1 y + c(x)$

**Theorem 7.6.** The following are the solutions for Theorem 7.3 in the $E_n$ cases:

- **$E_6$:**

  $$
  \begin{align*}
  c'' &= \chi'' - 6 \\
  c_2 &= \chi_2' \\
  c_1 &= \chi_1 - \chi_2 \\
  c_0 &= \chi_0 - 3\chi'' + 9 \\
  c_1 &= \chi_1 - \chi_2' \\
  c_2 &= \chi_2
  \end{align*}
  $$

  The central fiber with an $E_6$ singularity at $(x, y, z) = (-6, -6, -18)$ is

  $$
  xyz = z^2 + 72z + y^3 + 27y^2 + 324y + 2700 + 324x + 27x^2 + x^3.
  $$

- **$E_7$:**

  $$
  \begin{align*}
  c'' &= \chi'' - 6\chi_3 \\
  c_2 &= \chi_2' - 25 \\
  c_1 &= \chi_1 - \chi_2 - 16\chi_2' + 206 \\
  c_0 &= \chi_0 - 3\chi(\chi'' + \chi_3) + \chi(2\chi_3) - 12\chi_1 + 9\chi(2\chi_3) + 16\chi_2 + 69\chi_2' - 548 \\
  c_1 &= \chi_1 - \chi(\chi_2' + \chi_3) - 6\chi'' + 28\chi_3 \\
  c_2 &= \chi_2 - 2\chi_2' + 23 \\
  c_3 &= \chi_3
  \end{align*}
  $$

  The central fiber with an $E_7$ singularity at $(x, y, z) = (-12, -24, -144)$ is

  $$
  xyz = z^2 + 576z + y^3 + 108y^2 + 5184y + 193536 + 17280x + 1296x^2 + 56x^3 + x^4.
  $$
Lemma 7.8. Let $\Delta$ over $G$

The discriminant locus of this family is a union of affine hyperplanes $c$ cubic the roots $\alpha$ Ccuspidal cubic

This basis is also $\tilde{\text{Dynkin}}$ diagrams, which are the centers of certain non-commutative algebras associated to affine star-shaped
dynkin diagrams.$^{|\Lambda|}$

Remark 7.7. There are two natural choices for a basis in the ring of invariants $k[\Lambda^\vee]^W$: (1) an additive basis of characters of dominant weights, and (2) a polynomial basis of the fundamental characters $\chi_1, \ldots, \chi_n$. Using the method described below and [Sage], we found the answer in the additive basis. This basis is also more convenient for Section 10. Once we recomputed the answer in the polynomial basis, we did a web search on the largest coefficient 1484285983. There was one mathematical match, to the paper [EOR07]. That paper gives the formulas for the $E_6$, $E_7$, $E_8$ cases in the polynomial basis. The equations appear as relations for the centers of certain non-commutative algebras associated to affine star-shaped Dynkin diagrams, which are $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.

Our proof is different. In [Tju70] Tjurina constructed a versal deformation of an $E_8$ singularity as a family over $\mathbb{A}^8$ the parameter space for 8 smooth points on a cuspidal cubic $C$ (note that one has $C \setminus \text{cusp} \simeq \mathbb{A}^1$). See also [DPT80, p.190]. The discriminant locus of this family is a union of affine hyperplanes $e^\alpha = 0$ for the roots $\alpha \in E_8$. Our observation is that replacing the cuspidal cubic by a nodal cubic $C$ (so that $C \setminus \text{node} \simeq \mathbb{G}_m$) gives a multiplicative version of Tjurina’s family over $\mathbb{G}_m^8$ that we are after.

The lattice $E_8$ can be realized as an intermediate sublattice of index 3 in $A_8 \subset E_8 \subset A_8^*$. The lattice $A_8^*$ is generated by $e_i - p$, where $p = \frac{1}{5} \sum_{i=1}^9 e_i$. The lattice $A_8$ is generated by $e_i - e_j$, and the intermediate lattice $E_8$ is obtained by adding $\ell = e_i - e_j$, where $\ell = 3p$.

Now let $C$ be an irreducible curve of genus 1, so $C$ is either smooth, or has a node, or a cusp. Let $G = \text{Pic}^0 C$, so either an elliptic curve (with a choice of 0), or $\mathbb{G}_m \cong G$, or $\mathbb{G}_a \cong G$. The nonsingular locus $C^0$ is a $G$-torsor.

Lemma 7.8. Let $A_n$, $E^\vee_8$ be the standard root lattices, $A_8^\vee$ be the dual lattice. Then:

(1) $\text{Hom}(A_n, G) = A_n^\vee \otimes G = G^{n+1}/\text{diag} G = (C^0)^{n+1}/G$ parameterizes $(n+1)$ nonsingular points $P_i$ on $C$ modulo translations by $G$. 

$E_8$:  
\[ c'' = \chi'' - 6\chi_3 - 35\chi_2 + 920\chi_4 - 57505 \quad c_2' = \chi_2' - 25\chi_4 + 2325 \]
\[ c_1' = \chi_1 - \chi_2 - 16\chi(\varphi_2 + \varphi_4) - 44\chi'' + 206\chi(2\varphi_4) + \]
\[ + 360\chi_3 + 2196\chi_2' - 51246\chi_4 + 2401900 \]
\[ c_0 = \chi_0 - 3\chi(\varphi'' + \varphi_3) + \chi(2\varphi_2 + \varphi_4) - 12\chi(\varphi_2 + \varphi_4) - 28\chi(\varphi_2 + \varphi'') + \]
\[ + 9\chi(2\varphi_3) + 16\chi(\varphi_2 + \varphi_4) - 68\chi_1 + 69\chi(\varphi_2 + 2\varphi_4) + 212\chi(\varphi_2 + \varphi_3) + \]
\[ + 1024\chi(\varphi'' + \varphi_4) + 236\chi(2\varphi_2) + 2453\chi_1 - 548\chi(3\varphi_4) - \]
\[ - 5228\chi(\varphi_3 + \varphi_4) - 1507\chi_2 - 42656\chi(\varphi_2 + \varphi_4) - 107636\chi'' + \]
\[ + 488553\chi(2\varphi_4) + 640064\chi_3 + 2988404\chi_2' - 52027360\chi_4 + 1484779780 \]
\[ c_1 = \chi_1 - \chi(\varphi_2 + \varphi_3) - 6\chi(\varphi'' + \varphi_4) + 2\chi(2\varphi_2) - 17\chi_1 + \]
\[ + 28\chi(\varphi_3 + \varphi_4) - 79\chi_2 + 383\chi(\varphi_2 + \varphi_4) + 1429\chi'' - \]
\[ - 4414\chi(2\varphi_4) + 84\chi_3 - 49768\chi_2' + 271931\chi_4 + 4528192 \]
\[ c_2 = \chi_2 - 2\chi(\varphi_2 + \varphi_4) - 9\chi'' + 23\chi(2\varphi_4) - 114\chi_3 + 601\chi_2' + 7673\chi_4 - 955955 \]
\[ c_3 = 3\chi_3 - 3\chi_2' - 170\chi_4 + 23405 \quad c_4 = \chi_4 - 248 \]

The central fiber with an $E_8$ singularity at $(0,0,0)$ is $xyz = z^2 + y^3 + x^5$. 


(2) $\text{Hom}(A^*_n, G) = A_n \otimes G = \{(g_1, \ldots, g_{n+1}) \mid \sum g_i = 0\}$ parameterizes the choice of an origin $P_0 \in C^0$ plus $(n + 1)$ nonsingular points $P_i \in C^0$ such that $(n + 1)P_0 \sim \sum P_i$.

(3) $\text{Hom}(E_8, G) = E_8 \otimes G$ parameterizes embeddings $C \subset \mathbb{P}^2$ as a cubic curve plus 8 points $P_i \in C^0$, or equivalently embeddings $C \subset \mathbb{P}^2$ plus 9 points $P_i \in C^0$ such that $\prod_{i=1}^9 P_i = 1$ in the group law of $C^0$. Thus, $P_0$ is the 9th base point of the pencil $|C|$ of cubic curves on $\mathbb{P}^2$ through $P_1, \ldots, P_8$.

Proof. We have $A^*_n = \mathbb{Z}^{n+1}/\text{diag} \mathbb{Z}$ and $A_n = \{(a_1, \ldots, a_{n+1}) \in \mathbb{Z}^{n+1} \mid \sum a_i = 0\}$, so (1) and (2) follow. Hence, $\text{Hom}(A^*_8, G)$ parameterizes embeddings $C \subset \mathbb{P}^2$ with 8 points and a choice of a flex, and $E_8 \otimes G = \text{Hom}(A^*_8, G)/G[3]$ forgets the flex. □

Thus, the torus $T_{A^*_8}$ parameterizes 8 smooth points $P_1, \ldots, P_8$ on a nodal cubic curve $C \subset \mathbb{P}^2$ with a chosen flex, and the torus $T_{E_8}$ – the same, but forgetting the flex. We now take a concrete rational nodal cubic $C \subset \mathbb{P}^2$ given by the equation $g_9 = -uvw + v^3 + w^3$ with a rational parametrization $(u : v : w) = (t^3 - 1 : t : -t^2)$, so that the singular point of $C$ is $(1 : 0 : 0)$ corresponding to $t = 0$ or $\infty$. Now consider a family over $A^8$ of cubics $g_1 = \sum_{i,j \geq 0, i+j \leq 3} a_{ij} u^{3-i-j} v^i w^j$, $a_{00} = 1$, $a_{11} = 0$.

Then any pencil of cubic curves, parameterized by $x$, with a smooth generic fiber which has $C$ at $x = \infty$ has a unique representation by a polynomial $g(x; u, v, w) = xg_0 + g_1$. It is a simple exercise to put this pencil into the Weierstrass form $\varphi^2 = y^3 + A(x)y + B(x)$ using Nagell’s algorithm or simply by using the [Sage] function \texttt{WeierstrassForm}. The polynomials $A(x)$, $B(x)$ have degrees 4 and 5 (not 6 since $C$ is singular). The following is an easy explicit computation:

**Lemma 7.9.** There is a unique change of coordinates of the form $x \mapsto x + d$, $y \mapsto y + ax^2 + bx + c$ which leaves the fiber $C$ at $x = \infty$ in the pencil intact and takes the polynomial $f(x, y)$ into the form of the equation (7.2) for $E_8$.

We now build a family over $\mathbb{G}^8_m$ with the required properties. We pick $t_1, \ldots, t_8$ in $\mathbb{G}_m$ arbitrarily and then also $t_9$ so that $\prod_{i=1}^9 t_i = 1$. Using the rational parametrization of the nodal cubic $C$, this gives 9 smooth points $P_1, \ldots, P_9 \in C$.

**Lemma 7.10.** The pencil $g(x; u, v, w)$ passes through the points $P_1, \ldots, P_9$ iff $a_{10} = \sigma_8$, $a_{01} = \sigma_1$, $a_{21} = -\sigma_2 + \sigma_5 - \sigma_8$, $a_{12} = -\sigma_1 + \sigma_4 - \sigma_7$, $a_{30} = -3 + \sigma_6$, $a_{03} = -3 + \sigma_3$, $a_{20} = -\sigma_1 + \sigma_7$, $a_{02} = \sigma_2 - \sigma_8$,

where $\sigma_i$ are the elementary symmetric polynomials in $t_1, \ldots, t_9$.

Proof. We plug the rational parametrization $(u : v : w) = (t^3 - 1 : t : -t^2)$ into $g_1(u, v, w)$ to obtain a monic polynomial of degree 9 with constant coefficient $-1$ which we set equal to $\prod_{k=1}^9 (x - t_k) = \sum_{n=0}^9 (-1)^{n+1} \sigma_n x^n$. Then we solve the resulting linear equations for $a_{ij}$. □

The rest of the proof of Thm. 7.6. By the above procedure, we get a family of polynomials $f(x, y)$ as in the equation (7.2) parameterized by the torus $T(A^*_8) = \text{Hom}(A^*_8, \mathbb{G}_m)$, i.e. with coefficients $c_i \in k[A^*_8]^{S_9}$. The explicit expression, obtained by applying Lemmas 7.9 and 7.10, is easy and quick to compute but comes out to be quite enormous. It is in terms of $\sigma_i(t_k)$, the fundamental characters
of \( A_8 \). The final and very computationally intensive step is to write it in terms of the characters of \( E_8 \). For this, using [Sage] we computed the restrictions to \( A_8 \) of the fundamental characters \( \chi_i \) of \( E_8 \) and of their “lower terms”, the dominant weights lying below \( \varpi_1 \), listed in Table 4. We then computed \( c_i \) recursively as their linear combinations, and wrote the answer in the statement of Thm. 7.6.

We now prove that the discriminant \( \text{Discr}(f) \) of this family of polynomials coincides with the discriminant \( \text{Discr}(E_8) \). We have a trivial family \( \tilde{X}^0 = \mathbb{P}^2 \times T_{A_8^*} \rightarrow T_{A_8^*} \) with 9 sections, call them \( s_1, \ldots, s_9 \), corresponding to the points \( P_i \in C^0 \). Let \( \tilde{X}^n, 1 \leq n \leq 9 \), be the family obtained by performing a smooth blowup of \( \tilde{X}^{n-1} \) along the strict preimage of \( s_n \).

On each fiber the points \( P_1, \ldots, P_9 \in \mathbb{P}^2 \) are in an almost general position because they lie on an irreducible cubic (see [DPT80, p.39]). This means that \( -K_{X^9} \) is relatively nef and semiample, and defines a contraction to a family \( X^8 \rightarrow T(A_8^*) \) of del Pezzo surfaces with relatively ample \( -K_{X^8} \) and with Du Val singularities.

On the other hand, \( X^9 \) is a family of Jacobian elliptic surfaces with a section \( s_9 \) corresponding to the last point \( P_9 \). The linear system \( |N s_9|, N > 0 \) gives a contraction \( \tilde{X}^9 \rightarrow X^9 \) to a family of surfaces with ADE singularities. Let \( \tilde{I}_9 \) be an elliptic involution \( w \mapsto -w \) for this choice of a zero section. It descends to an involution \( \tilde{I}_8 \) of \( \tilde{X}^9 \) which in turn descends to an involution \( I_8 \) of \( X^8 \). It is easy to see that the quotients are families of the surfaces \( X^9/\tilde{I}_9 = \mathbb{P}_2 \) and \( Y^8 = X^8/\tilde{I}_8 = \mathbb{P}_2 = \mathbb{P}(1,1,2) \). The families of the polynomials \( f(x,y) \) written above are just the equations of the branch curves. On each fiber, the ramification curve passes through the singular point of the nodal cubic \( C \). Blowing up the image of this point on \( Y^8 \) finally gives the toric \( E_8 \)-surface \( Y \) as in Fig. 4 corresponding to the Newton polytope of \( f(x,y) \).

The branch curve \( f = 0 \) is singular iff the double cover \( X^8 \) is singular. This happens precisely when the points \( P_1, \ldots, P_8 \) are not in general position:

1. some 3 out of 9 points \( P_i, P_j, P_k \) lie on a line \iff the complementary 6 points lie on a conic \iff \( t_i t_j t_k = 1 \).
2. some 2 out of 9 points \( P_i = P_j \) \( (i > j) \) coincide \iff the complementary 7 points lie on a cubic which also has a node at \( P_j \) \iff \( t_i = t_j \).

These are precisely the root loci for the roots of \( E_8 \) in terms of the lattice \( A_8^* \). For our explicit parametrization of the nodal cubic \( C \) this can be seen from

\[
\det \begin{vmatrix} t_i^3 - 1 & t_i & -t_i^2 \\ t_j^3 - 1 & t_j & -t_j^2 \\ t_k^3 - 1 & t_k & -t_k^2 \end{vmatrix} = (t_i t_j t_k - 1)(t_i - t_j)(t_i - t_k)(t_j - t_k).
\]

This shows that \( \text{Discr}(f) \) is a product of the equations \( (e^\alpha - 1) \) of the root loci, and it is easy to see that they appear with multiplicity 1. Thus, \( \text{Discr}(f) = \pm \text{Discr}(E_8) \).

This completes the proof in the \( E_8 \) case. The \( E_7 \) and \( E_6 \) cases are obtained as degenerations of this construction. In the \( E_7 \) we blow up 7 smooth points of the cubic \( C \) and the node \( P_9 \). Then there exists a unique point \( P_9 \) which is infinitely near to \( P_8 \) such that all the cubics in the pencil \( |C - P_1 - \cdots - P_8| \) pass through \( P_9 \). In other words, \( P_9 \) is a point on the exceptional divisor \( E_9 \) of the blowup at \( P_9 \) corresponding to a direction \( t_9 \neq 0, \infty \) at \( P_8 \) for which we can write an explicit equation. Blowing up at \( P_9 \) gives an elliptic surface \( \tilde{X}^9 \rightarrow \mathbb{P}^1 \) with a zero section and an elliptic involution. The preimage of \( C \) on \( \tilde{X}^9 \) is an \( I_2 \) Kodaira fiber, instead
of an $I_1$ fiber in the $E_8$ case. In the same way as above, the discriminant locus is a union of root loci for the roots of $E_7$.

The $E_6$ case is a further degeneration. We pick 6 smooth points on $C$ plus the node $P_7$ plus an infinitely near point $P_8 \to P_7$ corresponding to one of the directions at the node. Then there exists a unique infinitely near point $P_9 \to P_8$ such that all the cubics in the pencil $|C - P_1 - \cdots - P_8|$ pass through $P_9$. Blowing up at $P_9$ gives an elliptic surface $\tilde{X}^9 \to \mathbb{P}^1$ with a zero section and an elliptic involution. The preimage of $C$ on $\tilde{X}^9$ is an $I_3$ Kodaira fiber. In the same way as above, the discriminant locus is a union of root loci for the roots of $E_6$.

As we noted in §3.12, the primed shapes $A'_n$, $A'_{2n-1}$, $D'_{2n}$ can be represented torically. For these cases we write down families of Laurent polynomials whose Newton polytopes are as in §3.12.

§7.6. ‘$A'_n$’ families. For the cases ‘$A_n$’ (for $n$ odd) and ‘$A'_n$’ (for $n$ even), we have either of the following equivalent formulas for $f(x, y)$:

$$f = -\left(\frac{y - \chi_1 x}{2}\right)^2 + y + \sum_{i=2}^{n} \chi_i x_i^2 + x_i^{n+1}, -\left(\frac{xy - \chi_1}{2}\right)^2 + y + \sum_{i=0}^{n-2} \chi_{i+2} x_i^2 + x_i^{n-1}$$

To derive them, start with the first $A_n$ equation in §7.3. The boundary curve $C_1$ corresponding to the left side of the polytope is $x = 0$ and $B \cap C_1 = (0, \pm 2)$. The substitution $y \mapsto y - 2 - \chi_1 x$ moves the point $(0, 2)$ to $(0, 0)$ so that $x, y$ are the tangent directions of $C_1, B$. After this change, we get the first equation above. We now perform a weighted blowup at the ideal $(x^2, y)$ corresponding to the Newton polytope of this equation. The result is an ‘$A_n$’ family. The second equation above is obtained by making a further change of variables $y \mapsto x^2 y$ and dividing by $x^2$.

§7.7. ‘$A'_{2n-1}$’ families. In a similar way, we can prime the $A_{2n-1}$ family on both sides, by changing the coordinates so that the infinite section passes through both points blown up. This amounts to completing the square by making a substitution $y \mapsto y - 2 - \chi_1 x - 2x^n - \chi_{2n-1} x^{n-1}$, and the resulting equation is

$$f = -\left(\frac{y - \chi_1 x - \chi_{2n-1} x^{n-1}}{2}\right)^2 + y + y x^n + \sum_{i=2}^{2n-2} \chi_i x_i^2 - \chi_{2n-1} x^{n-1} - 2x^n - \chi_1 x^{n+1}.$$ 

The two weighted blowups of the toric $A_{2n-1}$ surface correspond to taking the Newton polytope of this equation, obtained from the Newton polytope of the $A_{2n-1}$ equation by a slant cut of the corners $(0, 0)$ and $(2n, 0)$.

§7.8. ‘$D'_{2n}$’ families. Entirely similarly, the equation is obtained from the $D_{2n}$ equation by making a substitution $y \mapsto y - 2x^{n-1} - \chi_{2n-2} x^{n-2}$ and completing the square. The Newton polytope of this equation is obtained from that of the $D_{2n}$ equation by a slant cut of the $(2n - 2, 0)$ corner.

§7.9. All the primed cases. By this point, for each pure $ADE$ shape we have constructed a family of toric pairs $(Y, C + \frac{1}{m} B)$ over the torus $\mathbb{T}_x$. We note that the set $C \cap B$ is fixed in this family because the monomials on the sides $C_1, C_2$ have constant coefficients. We have two, resp. one point for each long, resp. short curve $C_i$ in $C = C_1 + C_2$. By blowing up the sections corresponding to some of these points we thus obtain families for all the primed type III cases.
§7.10. Descending families to a smaller torus.

**Theorem 7.11.** For each pure or primed shape, the family over the torus $T_\Lambda$ descends to a family over the torus $T_{\Lambda'}$, where $\Lambda \subset \Lambda' \subset \Lambda^*$ is an intermediate lattice as in Theorem 6.3.

**Proof.** The proof is the same as for Theorems 6.2, 6.3, except that the action of $\hat{H}$ on $T_\Lambda$ is free (unlike the action on $\mathbb{A}^n$!), so the family descends. $\square$

**Corollary 7.12.** For any fixed ADE shape, one has the following commutative diagram of stacks, in which $H = \Lambda^*/\Lambda'$.

$$
\begin{array}{ccc}
T_\Lambda & \longrightarrow & T_{\Lambda'} \\
\downarrow /\hat{H} & & \downarrow /H \\
\mathbb{A}^n & \longrightarrow & [\mathbb{A}^n/\hat{H}] = [T_{\Lambda'}/W]
\end{array}
$$

The map $T_{\Lambda'} \to [T_{\Lambda'}/W]$ is ramified along the union of root subtori $T_\alpha = \{e^\alpha = 1\}$. In particular, for a point $t \in T_\Lambda$ the automorphism group of the corresponding ADE surface $(Y, C_1 + C_2 + \frac{1}{2\pi B})$ is $G_t/W_t$, where $G_t \subset W$ is the stabilizer of $t$, and $W_t$ is the subgroup generated by reflections in the roots $\alpha$ such that $t \in T_\alpha$.

Note that $\Lambda'$ a $W$-invariant sublattice of $\Lambda^*$. Indeed, for any weight $\lambda$ and a root $\alpha$, one has $w_\alpha(\lambda) = \lambda - (\alpha, \varpi)\alpha \in \lambda + \Lambda$.

**Remark 7.13.** The coarse moduli space, the variety $\mathbb{A}^n/\hat{H}$ is generally singular, see [Pop11, Thm.3.2].

8. **Connection with Looijenga pairs**

A Looijenga pair is a smooth rational surface $(\tilde{X}, \tilde{D})$ such that $K_{\tilde{X}} + \tilde{D} \sim 0$ and $\tilde{D}$ is a cycle of rational curves. If $(X, D, \iota)$ is an ADE cover, i.e. a $(K + D)$-trivial polarized involution pair then its minimal resolution is a Looijenga pair. In [GHK15], Gross-Hacking-Keel construct moduli of Looijenga pairs of a fixed type (the configuration of the rational curves $\tilde{D}$). The result is as follows. First, one defines the lattice $\tilde{D}^\perp \subset \text{Pic} \tilde{X}$ as the orthogonal to the irreducible components of $\tilde{D}$, and the torus $T = T_{\tilde{D}^\perp}$. It is proved that the surface $\tilde{D}$ is obtained from another Looijenga pair called a “toric model” $(X', D')$ by blowing up several smooth points of $D'$, which may include infinitely near points. (The surface $X'$ may not be actually toric but it becomes toric after several blowups at the nodes of $D'$ and then some blowdowns.) One also defines a “small” Weyl group $W$ generated by the $(-2)$-classes in $\tilde{D}^\perp$.

Next, one constructs a non-separated scheme $U$ that is a union of several copies of $T$ glued along open subsets. The moduli space is the quotient $U/W$. The non-separatedness is expected since $\tilde{X}$ are smooth surfaces without a polarization. It arises from the order of the points on $D'$ to be blown up when they collide. The separated quotient of $U/W$ is $T/W$, a quotient of a torus by a certain “small” Weyl group whose rank may be smaller than $\dim T$. So, this is an object which is quite similar to the quotient $T_\Lambda/W$, except that $W_\Lambda$ is a “large” Weyl group.

**Theorem 8.1.** For a pure ADE shape the lattice $\tilde{D}^\perp$ is isomorphic to the corresponding root lattice $\Lambda$. The same is true for all the primed cases except when those
listed in Table 5 when $\mathcal{N} \neq \Lambda$. In the exceptional cases of Table 5, the lattice $\tilde{D}^\perp$ has higher rank $n + 1 \leq r \leq n + 4$. The ADE covers with an involution correspond to a subtorus $T_{\tilde{D}^\perp/L}$ for a certain sublattice $L \subset \text{Pic} \tilde{X}$, and one has $\tilde{D}^\perp/L \simeq \mathcal{N}$, where $\mathcal{N}$ is as given in Theorem 6.3.

Proof. This is a direct computation. We sketch details in a representative selection of cases, and leave the remainder to the reader.

$A$-shapes. We begin with the pure shape $A_{2n-1}$ and those cases derived from it by priming. In this case $\tilde{X}$ is isomorphic to a copy of $\mathbb{P}^n$, that has been blown-up in $2n$ points, all of which lie in a section. Let $s$ and $f$ in Pic($\tilde{X}$) denote the classes of the pull-backs of the $(-n)$-section and a general fiber in $\mathbb{P}^n$, and let $e_1, \ldots, e_{2n}$ denote the classes of the exceptional curves; these classes generate Pic($\tilde{X}$). The divisor $\tilde{D}$ is the union of four curves in the classes $s$, $f$, $(s + nf - \sum_{i=1}^{2n} e_i)$, and $(f-e_{2n+1})$ (ordered cyclically), so the lattice $\tilde{D}^\perp$ is generated by the classes $(e_i - e_{i+1})$ for $1 \leq i \leq 2n - 1$. We thus see that $\tilde{D}^\perp$ is a root lattice of type $A_{n-1}$. Priming once to obtain $'A_{2n-1}$ does not alter $\tilde{D}^\perp$.

The primed shape $'A_{2n-1}^2$ is obtained from the surface described above by blowing up two of the points where $\tilde{D}$ meets the ramification curve, with these points lying on different components of $\tilde{D}$. Let the classes of the resulting exceptional curves be $e_{2n+1}$ and $e_{2n+2}$ and label the other classes as above. The divisor $\tilde{D}$ is now the union of four curves in the classes $s$, $(f-e_{2n+1})$, $(s + nf - \sum_{i=1}^{2n} e_i)$, and $(f-2e_{2n+2})$ (ordered cyclically), so $\tilde{D}^\perp$ is generated by the classes $(e_i - e_{i+1})$ for $1 \leq i \leq 2n - 1$ and $(s + nf - \sum_{i=1}^{2n} e_i - e_{2n+1} - e_{2n+2})$. This is a lattice of rank $2n$, but in this case not all deformations of the pair $(\tilde{X}, \tilde{D})$ also admit involutions.

To see this, consider the quotient $\tilde{Y}$ of $\tilde{X}$ by the involution, and let $\overline{C} \subset \tilde{Y}$ denote the image of $\tilde{D}$. Then $\tilde{Y}$ is a singular ruled surface with two nodes. The fiber through each node consists of a pair of $(-\frac{1}{2})$-curves, one of which is a component of $\overline{C}$. Contract the $(-\frac{1}{2})$-curves that are not components of $\overline{C}$ to two points $P_1$ and $P_2$. The result is the ruled surface $\mathbb{F}_n$. Let $S$ denote a section of the ruling on $\mathbb{F}_n$ that passes through both $P_1$ and $P_2$ and does not intersect the $(-n)$-section. Then the strict transform of $S$ on $\tilde{Y}$ does not intersect $\overline{C}$, so the double cover $\overline{S} \subset \tilde{X}$ is an effective curve orthogonal to $\tilde{D}$. The curve $\overline{S}$ lies in the class $\gamma = 2s + 2nf - \sum_{i=1}^{2n} e_i - 2e_{2n+1} - 2e_{2n+2} \in \tilde{D}^\perp$, so this class is effective.

As $\gamma \in \tilde{D}^\perp$ is effective, its restriction to $\tilde{D}$ is trivial. So by the definition of the period map [GHK15, Def. 1.5] the period point $\phi \in T_{\tilde{D}^\perp}$ of $(\tilde{X}, \tilde{D})$ must satisfy $\phi(\gamma) = 1$. So the existence of an involution forces the period point of our surface to lie in the codimension $1$ sublocus of the period domain defined by the equation $\phi(\gamma) = 1$. Writing $\gamma$ in terms of the generators of $\tilde{D}^\perp$, we see that $\tilde{D}^\perp/\gamma$ is the overlattice $\mathcal{N}'$ given in Theorem 6.3.

Next we turn our attention to the primed case $''A_{2n-1}$. In this case the surface $\tilde{X}$ is obtained from the surface described at the start of this section by blowing up two of the points where $\tilde{D}$ meets the ramification curve, with these points lying on the same component of $\tilde{D}$. Let the classes of the resulting exceptional curves be $e_{2n+1}$ and $e_{2n+2}$ and label the other classes in Pic($\tilde{X}$) as before. The divisor $\tilde{D}$ is now the union of four curves in the classes $s$, $f$, $(s + nf - \sum_{i=1}^{2n} e_i)$, and $(f-2e_{2n+1} - e_{2n+2})$ (ordered cyclically), so $\tilde{D}^\perp$ is the lattice $A_{2n-1} \oplus A_1$ generated.
by the classes \((e_i - e_{i+1})\) for \(1 \leq i \leq 2n - 1\) and \((e_{2n+1} - e_{2n+2})\). However, we note that the existence of the involution implies that the four points on the component of \(\tilde{D}\) with class \((f - e_{2n-1} - e_{2n+2})\) gives its intersections with the remaining components of \(\tilde{D}\) and \(e_{2n+1}\) and \(e_{2n+2}\) are in harmonic correspondence; this implies that the period point \(\phi \in T(\tilde{D}^\perp)\) of \((\tilde{X}, \tilde{D})\) must satisfy \(\phi(e_{2n+1} - e_{2n+2}) = -1\). Thus we see that the period domain for stable pairs with involution in this case is \(T_{A_{2n-1}}\), as predicted in Theorem 6.3. The remaining cases of \(A\) shape are computed analogously.

**D-shapes.** Next we consider the pure case \(D_{2n}\) and those cases obtained from it by priming. In this case \(\tilde{X}\) may be constructed from a copy of \(\mathbb{F}_1\) by blowing up \(2n\) points on a chosen bisection in the class \(2s + 2f\), where \(s\) denotes the class of the \((-1)\)-section and \(f\) the class of a fiber. Label the resulting exceptional curves \(e_1, \ldots, e_{2n}\). Then the classes \(s, f, e_1, \ldots, e_{2n}\) generate \(\text{Pic}(\tilde{X})\). The divisor \(\tilde{D}\) is the union of two curves in the classes \(f\) and \((2s + 2f - \sum_{i=1}^{2n} e_i)\) (ordered cyclically), so the lattice \(\tilde{D}^\perp\) is now generated by the classes \((e_i - e_{i+1})\) for \(1 \leq i \leq 2n - 1\) and \(f - e_1 - e_2\). We thus see that \(\tilde{D}^\perp\) is a root lattice of type \(D_{2n}\).

The primed case \(D'_{2n}\) is obtained by blowing up one of the intersections between the component of \(\tilde{D}\) in the class \(f\) and the ramification curve, to obtain a new exceptional curve \(e_{2n+1}\). The lattice \(\tilde{D}'^\perp\) is now generated by the classes \((e_i - e_{i+1})\) for \(1 \leq i \leq 2n - 1\), \((f - e_1 - e_2)\), and \((s - e_{2n+1})\). However, in case \(A'_{2n-1}\), the existence of the involution forces the class \(\gamma := 2s + nf - \sum_{i=1}^{2n} e_i - 2e_{2n+1} \in \tilde{D}^\perp\) to be effective, so if \(\phi \in T_{\tilde{D}^\perp}\) is the period point of \((\tilde{X}, \tilde{D})\), then \(\phi(\gamma) = 1\). As before, we find that \(\tilde{D}^\perp/\gamma\) is the overlattice \(\Lambda'\) given in Theorem 6.3.

The case \(D_{2n}\) is analogous to \(D'_{2n}\), and \(D'_{2n}\) may be obtained by combining these two cases. The remaining computations of \(D\) shape are similar to ones already completed above.

**E-shapes.** The only difficult case of \(E\) shape is \(-E_7\). In this case, \(\tilde{X}\) is constructed by blowing up seven general points and the node on a nodal cubic curve in \(\mathbb{P}^2\). Call the resulting exceptional divisors \(e_1, \ldots, e_7\) (from the general points) and \(e_8\) (from the node). \(\text{Pic}(\tilde{X})\) is generated by the classes \(l\) (the class of a line in \(\mathbb{P}^2\)) and \(e_1, \ldots, e_8\). The divisor \(\tilde{D}\) is the union of the strict transform of the nodal cubic, which lies in the class \((3l - \sum_{i=1}^{7} e_i - 2e_8)\), and the exceptional curve \(e_8\), so the lattice \(\tilde{D}^\perp\) is generated by the classes \((e_i - e_{i+1})\) for \(1 \leq i \leq 6\) and \((l - e_1 - e_2 - e_3)\). We thus see that \(\tilde{D}^\perp\) is a root lattice of type \(E_7\).

Now consider the case \(-E_7'\), obtained from the surface above by blowing up one of the intersections between the strict transform of the nodal cubic and the ramification curve. Call the resulting exceptional curve \(e_9\). The lattice \(\tilde{D}^\perp\) is now generated by the classes \((e_i - e_{i+1})\) for \(1 \leq i \leq 6\), \((l - e_1 - e_2 - e_3)\), and \((e_7 - e_9)\). However, if \(\tilde{X}\) admits an involution, then the class \(\gamma = (3l - \sum_{i=1}^{7} e_i - 2e_9) \in \tilde{D}^\perp\) is the pull-back under the double cover of the class of a line in the \(-E_7\) surface which lies tangent to the branch divisor at the point where the priming operation occurs; \(\gamma\) is thus an effective class. Thus if \(\phi \in T(\tilde{D}^\perp)\) is the period point of \((\tilde{X}, \tilde{D})\), then \(\phi(\gamma) = 1\). As before, we find that \(\tilde{D}^\perp/\gamma\) is the overlattice \(\Lambda'\) given in Theorem 6.3. The remaining cases of \(E\) shape follow by similar computations to the ones above.

□
9. Singularities of \textit{ADE} pairs

In all the \textit{ADE} and \textit{\bar{AD}E} cases, the singularities of the surface \(Y\) lying in the boundary \(C\) and the singularities of \(Y \setminus (C \cup B)\) (resp. singularities of the double cover \(X\) lying in the boundary \(D\) and on \(X \setminus (D \cup R)\)) are provided by Theorem 4.12.

In this Section we describe explicitly the singularities of the curves \(B \cap (Y \setminus C)\) away from the boundary. Equivalently, these are the singularities of the surfaces \(X \setminus D\) near \(R\). The families of the curves \(B_t\) were constructed in \$7.3, \$7.4, \$7.5.

Since the primed cases are obtained by blowing up points in the boundary, we can assume that we work with the pure shapes, and by \$7.10 the family is parameterized by the torus \(T_\Lambda = \text{Hom}(\Lambda, \mathbb{G}_m)\).

\textbf{Definition 9.1.} Let \(\Lambda\) be an \textit{ADE} lattice with a root system \(\Phi\) and Dynkin diagram \(\Delta\), and let \(G\) be some abelian group which we will write multiplicatively.

Let \(t \in \text{Hom}(\Lambda, G)\) be a homomorphism. Define the sublattice

\[ A_t = \{ \alpha \mid t(\alpha) = 1 \} \subset \Lambda \text{ generated by the roots } \alpha \in \Phi \cap \ker(t). \]

It is well known that a sublattice of an \textit{ADE} lattice generated by some of the roots is a direct sum of root lattices corresponding to smaller \textit{ADE} Dynkin diagrams. All such root sublattices can be obtained by the Dynkin-Borel-de Siebenthal (DBS) algorithm, see [Dyn52, Thms. 5.2, 5.3], as follows. Make several of the steps (DBS1): replace a connected component of the Dynkin diagram by an extended Dynkin diagram and then remove a node; and then several of the steps (DBS2): remove a node. Below, we determine which of these lattices are realizable as \(A_t\).

All root sublattices are listed in [Dyn52, Tables 9–11]. The answer is as follows. Recall that the lattice \(A_n \subset \mathbb{Z}^{n+1}\) is generated by the roots \(e_i - e_j\). All root sublattices of \(A_n\) are of the form \(A_{|I_1|-1} \oplus \cdots \oplus A_{|I_s|-1}\), where \(I_1 \sqcup \cdots \sqcup I_s = \{1, \ldots, n+1\}\) is a partition, \(|I_i| \geq 1\). Here, \(A_{|I_i|-1} = 0\) if \(|I_i| = 1\).

The lattice \(D_n \subset \mathbb{Z}^n\) is generated by the roots \(e_i \pm e_j\). All root sublattices of \(D_n\) are of the form \(A_{|I_1|-1} \oplus \cdots \oplus A_{|I_s|-1} \oplus D_{|J_1|} \oplus \cdots \oplus D_{|J_r|}\), where \(I_1 \sqcup \cdots \sqcup I_s \sqcup J_1 \sqcup \cdots \sqcup J_r = \{1, \ldots, n\}\) is a partition, \(|I_i| \geq 1\) and \(|J_j| \geq 2\). \(D_2\) and \(D_3\) are a special case. They are isomorphic to \(2A_1\) and \(A_3\) respectively as abstract lattices, but they are different as sublattices of \(D_n\).

The sublattices of \(E_6, E_7, E_8\) are listed in [Dyn52, Table 11] but note the typos: in the \(E_6\) table one of the two \(A_7 + A_1\) is \(E_7 + A_1\), and \(A_6 + A_2\) should be \(E_6 + A_2\).

\textbf{Definition 9.2.} Let \(M \subset \Lambda\) be two \textit{ADE} lattices. Let \(\text{Tors}(\Lambda/M)\) be the torsion subgroup of \(\Lambda/M\) and \(\text{im}(\Phi \cap M_R) \subset \text{Tors}(\Lambda/M)\) be the image of the set of roots \(\alpha \in \Phi \cap M_R\). We define the closure \(\overline{\text{im}(\Phi \cap M_R)}\) to be the subset of \(\text{Tors}(\Lambda/M)\) consisting of the elements \(x \neq 0\) such that \(0 \neq nx \in \text{im}(\Phi \cap M_R)\) for some \(n \in \mathbb{N}\); plus \(x = 0\). Both \(\text{im}(\Phi \cap M_R)\) and \(\overline{\text{im}(\Phi \cap M_R)}\) are finite sets, and a priori neither of them has to be a group.

\textbf{Lemma 9.3.} Let \(M \subset \Lambda\) be two \textit{ADE} lattices. Let \(G\) be an abelian group containing \(\mathbb{Z}^r\), where \(r = \text{rk} \Lambda - \text{rk} M\). Then \(M = \Lambda_t\) for some \(t \in \text{Hom}(\Lambda, G)\) iff there exists a homomorphism \(\phi:\ \text{Tors}(\Lambda/M) \to G\) such that for any \(0 \neq x \in \overline{\text{im}(\Phi \cap M_R)}\) one has \(\phi(x) \neq 0\).

\textbf{Proof.} Of course one must have \(M \subset \ker(t)\), so the question is whether there exists a homomorphism \(\Lambda/M \to G\) which does not map any roots not lying in \(M\) to zero. We have \(\Lambda/M = \mathbb{Z}^r \oplus \text{Tors}(\Lambda/M)\). An embedding \(\mathbb{Z}^r \to G\) can always be adjusted
by an element of GL(r, \mathbb{Z}) so that the images of roots not in Tors(\Lambda/M) do not map to zero. So the only condition is on im(\Phi \cap M_k) in Tors(\Lambda/M) or, equivalently, on its closure.

\[\square\]

**Corollary 9.4.** Let \( M \subset \Lambda \) be two ADE lattices and let \( k \) be an algebraically closed field of characteristic zero. If the group Tors(\Lambda/M) is cyclic then \( M = \Lambda_t \) for some \( t \in \text{Hom}(\Lambda, \mathbb{G}_m(k)) \). In the opposite direction, if \( \text{im}(\Phi \cap M_k) \) contains a non-cyclic subgroup then \( M \neq \Lambda_t \) for any \( t \in \text{Hom}(\Lambda, \mathbb{G}_m(k)) \).

\[\text{Proof.} \] This follows from the fact that any finite cyclic group can be embedded into \( \mathbb{G}_m(k) \), and there are no non-cyclic finite subgroups in \( \mathbb{G}_m(k) \).

\[\square\]

**Theorem 9.5.** Let \( \Lambda \) be an irreducible ADE lattice and \( M \) be an ADE root sublattice. Assume that the field \( k \) is algebraically closed of characteristic zero. Then \( M = \Lambda_t \) for some \( t \in \text{Hom}(\Lambda, \mathbb{G}_m(k)) \) iff any of the following equivalent conditions holds:

1. Tors(\Lambda/M) is cyclic.
2. \( M \) is obtained from \( \Lambda \) by a single DBS1 step and then some DBS2 steps.
3. \( M \) corresponds to a proper subdiagram of the extended Dynkin diagram \( \tilde{\Delta} \).
4. \( M \) corresponds to a subdiagram \( \Delta \) of the following Dynkin diagrams:
   - \( A_n: A_n \); \( D_n: D_n \) or \( D_n D_6 \subset D_n \) with \( a + b = n, a, b \geq 2 \).
   - \( E_6: E_6, A_5 A_1, 3A_2 \); \( E_7: E_7, D_6 A_1, A_7, A_5 A_2, 2A_3 A_1 \);
   - \( E_8: E_8, E_7 A_1, E_6 A_2, D_8, D_5 A_3, A_8, A_7 A_1, 2A_4, A_5 A_2 A_1 \).
5. \( M \) is not one of the following forbidden sublattices:
   - \( D_n: a \) sublattice with \( \geq 3 \) D-blocks;
   - \( E_7: D_4, 3A_1, 7A_1, 6A_1 \);
   - \( E_8: 4A_2, 2D_4, D_6 A_2, 2D_4, D_4 A_1, 2A_3 A_1, 8A_1, D_4 A_1, 7A_1, 2A_3 A_1, 6A_1 \).

\[\text{Proof.} \] We first prove the equivalence of the conditions (1-5). For one direction, the identity \( \sum_{\alpha \in \Delta} m_{\alpha} \alpha = 0 \) implies that if the Dynkin diagram \( \Delta(M) \) is obtained from \( \tilde{\Delta} \) by removing one node (i.e., by a single DBS1 step) then the cotorsion group is cyclic of the order equal to the multiplicity \( m_{\alpha} \) of the removed node in the highest root of \( \Delta \). Any sublattice of these lattices obtained by DBS2 steps also has cyclic cotorsion. The lists in (4) are simply the lattices obtained by one DBS1 step. To complete the equivalence of (1-5) for \( E_n \) we use Dynkin’s lists of sublattices together with [Per90, Table 1] which gives the torsion groups, and check the finitely many cases. The \( D_n \) case is easy.

Now let \( M \) be a sublattice as in (1). Then \( M = \Lambda_t \) for some \( t \in \text{Hom}(\Lambda, \mathbb{G}_m(k)) \) by Cor. 9.4. Vice versa, let \( M \) be one of the sublattices with a non-cyclic Tors(\Lambda/M), which are listed in (5). If \( \Lambda = D_n \) and \( M \) has \( r \geq 3 \) D-blocks then Tors(\Lambda/M) = \( \mathbb{Z}_2^{r-1} \) and we easily calculate \( \text{im}(\Phi \cap M_k) \) to be \( \{0, e_i, e_i + e_j \mid 1 \leq i, j \leq r-1\} \). This set contains a non-cyclic subgroup \( \mathbb{Z}_2^2 = \{0, e_1, e_2, e_1 + e_2\} \), so \( M \neq \Lambda_t \) by Cor. 9.4.

For each sublattice of \( E_7 \) and \( E_8 \) listed in (5) we explicitly compute \( \text{im}(\Phi \cap M_k) \). We have \( (\Lambda \cap M_k)/M \subset M^*/M \), so we find the images of the roots \( \alpha \in \Phi \cap M_k \) in \( M^*/M \). The result is as follows. For \( 8A_1 \) the set \( \text{im}(\Phi \cap M_k) \) has 15 elements and contains \( \mathbb{Z}_4^2 \); for \( 2A_3 \) it has 7 elements and its closure is \( \mathbb{Z}_4 \oplus \mathbb{Z}_2 \); for \( 2A_2 \) it has 8 elements and its closure is \( \mathbb{Z}_8^2 \). In all the other cases, one has \( \text{im}(\Phi \cap M_k) = \text{Tors}(\Lambda/M) \). We conclude that \( M \neq \Lambda_t \) by Cor. 9.4.

\[\square\]

**Theorem 9.6.** Consider a family of pure shape ADE surfaces, as in \$7.3, 7.4, 7.5, 7.10$. Then for a point \( t \in T \), the singularities of the curve \( B_t \cap (Y_t \setminus C_t) \)
and of the double cover $X_t \setminus D_t$ near $R_t$ are Du Val of the type corresponding to the lattice $A_t$. In particular, a curve is singular iff it lies in a union of root subtori $\{e^\alpha = 1\}$, and for $t = 1$ there is a unique singularity of the same Du Val type as the root lattice.

**Proof.** The $A_n$ case is obvious: the curve curve $-y^2/4 + c(x), c(x) = \prod(x + t_i)$ has singularities $A_{m_1-1}, \ldots, A_{m_n-1}$, each occurring when some $m_k$ of the $t_i$’s coincide, i.e. when several of the monomials $e^{t_i - t_j}$ vanish at the same time.

In the $D_n$ case, we use the notations of §7.4. Let $\text{Discr}_y(f) = \prod_{i=1}^n(x + t_i + t_i^{-1})$ as in the proof of Thm. 7.4. It is easy to see that for every root $x \neq \pm 2$ of $\text{Discr}_y$ of multiplicity $m$, the curve $f = 0$ has an $A_{m-1}$-singularity, and if $x = \pm 2$ is a root of $\text{Discr}_y$ of multiplicity $m$ then $f$ has a $D_m$-singularity. This includes $D_3 = A_3$, $D_2 = 2A_1$, and $D_1 = \text{smooth}$. On the other hand, the root tori are of the form $\{t_i = 1\}$. The irreducible components of $A_t$ correspond to the disjoint subsets $I \subset \{1, \ldots, n\}$ of indices for which $t_i = t_j^{-1}$ for $i, j \in I$. If $t_i \neq \pm 1$, i.e. $t_i + t_i^{-1} \neq \pm 2$, then the component is of the $A_i|I|^{-1}$-type; otherwise it is of the $D_i|I|$-type.

In the $E_n$ cases the singularities are Du Val by construction in §7.5. Using notation as in the proof of Theorem 7.6, let us fix a linear function $\varphi$ on $E_8 \subset A_8^*$ such that $\varphi(p) > \varphi(e_1) > \cdots > \varphi(e_8)$, and let the positive roots $\alpha$ be those with $\varphi(\alpha) > 0$. Then for any subroot system of $E_8$ the simple roots are exactly the roots that are realizable by irreducible $(-2)$-curves on $\overline{X}_8$: $e_i - e_j$ for $i > j$ (preimages of the exceptional divisors $E_i$ of blowups at $P_1$), $\ell - e_i - e_j - e_k$ (preimages of lines passing through 3 points $P_i, P_j, P_k$, $2\ell - \sum_{k=1}^6 e_{ik}$ (preimages of conics through 6 points), and $3\ell - 2e_j - \sum_{k=1}^7 e_{ik}$ (preimages of nodal cubics through 8 points). So for every $t \in \text{Hom}(E_8, \mathbb{G}_m)$, the simple roots in the lattice $A_t$ are realized by $(-2)$-curves on $\overline{X}_t$ which contract to a configuration of singularities on $X_8$ with the same Dynkin diagram as $A_t$. The $E_7$ and $E_6$ cases are done similarly. 

**Remark 9.7.** By the proof of Theorem 7.6, the surfaces in the $E_6, E_7, E_8$ families correspond to rational elliptic fibrations with an $I_3, I_2, I_1$ fiber respectively. The singularity type of the double cover $X_t \setminus D_t$ is obtained from the Kodaira type of the elliptic fibration by dropping one $I_3, I_2, I_1$ fiber respectively (it gives a singularity of $X_t$ lying in the boundary $D_t$; of type $A_2, A_1$, or none resp.) and converting the other Kodaira fibers into the $ADE$ singularities.

As a self-check, we note that the list of maximal sublattices in Theorem 9.5(4) is equivalent to the list of the rational extremal non-isotrivial elliptic fibrations in [MP86, Thm. 4.1], and that the full list of sublattices in Theorem 9.5 is consistent with the full list of Kodaira fibers of rational elliptic fibrations in [Per90]. Persson’s list contains 6 surfaces with an $I_m$ fiber for which the corresponding sublattice of $E_8$ has non-cyclic cotorsion: $I_3^2 2I_2 (D_6, 2A_1), I_0^3 3I_2 (D_4, 3A_1), 2I_4 2I_2 (2A_3, 2A_1), I_1 4I_2 (A_3, 4A_1), 4I_3 (4A_2), 6I_2 (6A_1)$. But $D_6A_1, D_4 2A_1, 2A_3 A_1, A_3 3A_1$ and $5A_1$ are sublattices of $E_7$ and $3A_2$ is a sublattice of $E_6$, all with cyclic cotorsion.

Next, recall that part of the definition of an $ADE$ pair is that the divisor $B$ intersects the boundary transversally at smooth points of $Y$ and $C$. This is equivalent to saying that the pair $(Y, C + \frac{1+r}{2}B)$ is log canonical in a neighborhood of $C$. We now prove that this holds everywhere:
Theorem 9.8. Any ADE (i.e. of type III) pair \((Y, C + \frac{1+\epsilon}{2}B)\) is log canonical and any ADE cover \((X, D + \epsilon R)\) is log canonical for some \(0 < \epsilon \ll 1\) depending on the volume \(\text{vol}(Y, C) = \frac{B^2}{2} = R^2\).

First proof. \((Y, C + \frac{1+\epsilon}{2}B)\) is lc iff \((X, D + \epsilon R)\) is lc, cf. Lemma 2.2.

For a pure shape family, the pair is lc near the boundary by definition and \(Y \setminus C\) is nonsingular. By Thm. 9.6, \(B \cap (Y \setminus C)\) is Du Val, so \(X \setminus D\) is Du Val. Thus, \((X \setminus D, \epsilon R)\) is lc for \(0 < \epsilon \ll 1\). The log canonicity of the pair is preserved by the priming, so the statement for all the primed pairs follows. \(\square\)

Second proof. (Proposed to us by V.V. Shokurov.) The non-klt locus of the pair \((Y, C + \frac{1+\epsilon}{2}B)\) for \(0 < \epsilon \ll 1\) is the same as for \(\epsilon = 0\). For the latter, we have \(K_Y + C + \frac{1}{2}B \equiv 0\). In this case, by [Sho92, 6.9] the non-klt locus must be connected, with a single exception when it may have two components, both of them simple, i.e. on a resolution there must be a unique curve with discrepancy \(-1\). For an ADE surface, i.e. in type III, the curve \(C\) is connected with two irreducible components, so it is not simple. Thus, this does not appear as an exception. Consequently, there are no non-lc points in \(Y \setminus C\). \(\square\)

Remark 9.9. Note also that the fact that in type III the curve \(C\) on a resolution is connected with more than one component can be easily proved directly, without going through a classification.

The above proofs of Theorem 9.8 do not work in types I (klt) and II (\(\tilde{A} \tilde{D} \tilde{E}\)). And indeed the statement is false in both of these cases. Moreover, the cases of failure in type II are explained by the above exceptional case in Shokurov’s connectedness.

10. Compact moduli of stable ADE pairs

Recall the following basic definitions for the KSBA stable pairs.

**Definition 10.1.** A pair \((X, B = \sum b_iB_i)\) is said to have semi log canonical (slc) singularities if

1. \(X\) is a seminormal variety satisfying Serre’s \(S_2\) condition, with at worst double crossings in codimension 1.
2. \(B = \sum b_iB_i, 0 < b_i \leq 1\) is a \(\mathbb{Q}\)-Weil divisor not containing any irreducible components of the double locus.
3. The \(\mathbb{Q}\)-Weil divisor \(K_X + B\), well defined due to (1) and (2), is \(\mathbb{Q}\)-Cartier.
4. Denoting \(\nu: X \to X\) the normalization, the pair \((\tilde{X}, \nu^{-1}B+(\text{double locus}))\) has log canonical singularities.

**Definition 10.2.** A pair \((X, B = \sum b_iB_i)\) is said to be stable if

1. \((X, B)\) is slc.
2. \(K_X + B\) is ample.

We now define:

**Definition 10.3.** A stable ADE pair is a pair \((Y, C + \frac{1+\epsilon}{2}B)\) satisfying the conditions (1-3) of Definition 10.1 and such that the normalization \((\tilde{Y}, \tilde{C}+(\text{double locus})+\frac{1+\epsilon}{2}\tilde{B})\) is a disjoint union of ADE pairs.

A stable ADE cover is a pair \((X, D + \epsilon R, \iota)\) with an involution \(\iota\) satisfying the conditions (1-3) of Definition 10.1 and such that the normalization \((\tilde{X}, \tilde{D}+(\text{double locus}), \tilde{\iota})\) is a disjoint union of ADE covers.
Thus, stable \( ADE \) pairs (resp. covers) are essentially just surfaces glued from several ordinary \( ADE \) pairs (resp. covers) along some of their boundary curves \( C_k \) (resp. \( D_k \)).

**Lemma 10.4.** Stable \( ADE \) pairs \((Y, C + t \Delta B)\) and stable \( ADE \) covers \((X, D + t R)\) are stable pairs in the sense of Definition 10.2 for \( 0 < \epsilon < 1 \).

**Proof.** Immediate from the above Definitions 10.1, 10.2 and Theorem 9.8. \( \Box \)

§10.1. **Compact families.** We fix a root lattice \( \Lambda = A_n, D_n \) or \( E_n \) and a pure shape for this lattice. As usual, \( \Lambda^* \) denotes the dual lattice of weights.

**Definition 10.5.** In toric geometry there are two standard lattices: lattice \( M \simeq \mathbb{Z}^n \) of monomials, characters of the torus \( T \), and \( N = M^* \) of 1-parameter subgroups of \( T \). We take \( M = \Lambda \) and \( N = \Lambda^* \). The Coxeter fan on \( N_{\mathbb{R}} = N \otimes \mathbb{R} \) is formed by the system of hyperplanes \( \alpha^\perp \) for all the (positive) roots \( \alpha \in \Phi^+ \) of \( \Lambda \). Thus, a typical maximal cone of \( \Sigma \) is the positive Weyl chamber

\[
C^+ = \{ n \in N_{\mathbb{R}} \mid (n, \alpha) \geq 0 \text{ for } \alpha \in \Delta \},
\]

where \( \Delta \) is the set of simple roots. All other maximal cones are obtained from \( C^+ \) by reflections in the Weyl group \( W = W(\Lambda) \). We will denote by \( V \) (resp. \( V_2 \)) the projective variety for \( M = \Lambda \) (resp. \( M = \frac{1}{2}\Lambda \)) and the Coxeter fan \( \Sigma \).

The main result of this section is the following:

**Theorem 10.6.** Over the variety \( V_2 \) there exists a \((W \times \mu_n^0)\)-invariant family of stable \( ADE \) surfaces extending the pullback under \( T_{\frac{1}{2}\Lambda} \to T_\Lambda \) of the family of \( ADE \) surfaces of the chosen pure shape over the torus \( T_\Lambda \) from Section 7.

We will then descend this result to a moduli stack whose coarse moduli space is the quotient \( V/W \). We will also construct similar families in the primed cases, with the same Coxeter fan but for the torus \( T_{\Lambda'} \), with \( \Lambda' \) as in Theorem 6.3.

To begin with, we note that \( V_2 \) is covered by copies of the affine space \( \mathbb{A}^n \). Indeed, if we define

\[
\Lambda_- = \oplus_{\alpha \in \Delta} \mathbb{N}(-\alpha) \simeq \mathbb{N}^n \subset \Lambda,
\]

then the affine chart corresponding to the positive chamber \( C^+ \) is \( \text{Spec} \mathbb{C}[\frac{1}{2}\Lambda_-] \simeq \mathbb{A}^n \), and the other charts are isomorphic to it. We construct a family over this chart directly, in coordinates. The definition is strongly motivated by the two-dimensional model of \( ADE \) lattices given in Section 5 and the equations of families in Section 7. We then glue these families over different charts together.

The following statement will be convenient to prove the flatness of our families.

**Lemma 10.7.** Let \( Y \) be a connected reduced scheme, \( \mathbb{P} \) a weighted projective space, and let \( Y \subset \overline{Y} \subset \mathbb{P} \times V \) be two closed subschemes. Let \( p(d) \in \mathbb{Q}[d] \) be a polynomial. Suppose that the Hilbert polynomials of all fibers of \( Y \) over \( V \) are equal to \( p(d) \) and a generic fiber of \( Y \) over \( V \) has the Hilbert polynomial \( p(d) \). Then \( Y = \overline{Y} \) and \( Y \to V \) is flat.

**Proof.** Let \( L \) be a fixed ample invertible sheaf on \( \mathbb{P} \). A projective morphism \( Y \to V \) over a reduced connected base is flat iff its Hilbert polynomial function \( h_Y : V \to \mathbb{Q}[d], t \to \chi(Y_t, L^d) \) is constant. Therefore, \( Y \to V \) is flat. The Hilbert function is upper semi continuous in families. For any fiber we have \( h_Y(t) \leq h_{\overline{Y}}(t) \) for a generic fiber we have the equality. This implies that \( h_Y(t) = h_{\overline{Y}}(t) = p(d) \) everywhere, so \( Y = \overline{Y} \) and \( Y \to V \) is flat. \( \Box \)
§10.2. $A_n$ families of curves. We begin with the curve case (cf. §7.2), i.e. the Losev-Manin space $L_{n,3}$. We will call a pair $(\mathbb{P}^1, Q_0 + Q_\infty + \epsilon D)$, deg $D = n + 1$ a curve of shape $A_n$. In this case we construct the family directly over $V$, without an additional $\mu_p^2$-cover to $V_2$. The equations below can be well understood using the two-dimensional model for the $A_n$ lattice in Section 5. In the curve case, however, one can define the vectors there to be $\bar{u}_i = (1, p_i - p_s)$ instead of $\bar{u}_i = (2, p_i - p_s)$. The extra factor of 2 appears only for surfaces.

Consider a $k[\Lambda_-]$-subalgebra $S \subset k[\Lambda^*][x, \xi]$ generated by the $(n + 2)$ variables $u_i = e^{\varpi_i}x^i\xi$ for $0 \leq i \leq n + 1$, where we set $e^{\varpi_0} = e^{\varpi_{n+1}} = 1$ for convenience. The algebra is graded by the powers of $\xi$. We define the family $F$: $(\mathcal{Y}, C + \epsilon B) \to \mathbb{A}^n$ as follows:

1. $\mathcal{Y} = \text{Proj } S$ with a natural morphism to $\text{Spec } k[\Lambda_-] = \mathbb{A}^n$.
2. the boundary $C = C_1 + C_2$, $C_1 = Z(u_i, i > 0)$, $C_2 = Z(u_i, i < n + 1)$.
3. the divisor $B = Z(f)$, where $f = \xi \sum c_i x^i$ and $c_i = \chi_i$.

Note that $f \in S^{(1)}$. Indeed, the character $\chi_i$ is a sum of monomials in $k[\Lambda^*]$, the largest of which is $e^{\varpi_i}$, and all others are of the form $e^{\lambda}$, where $\lambda = \varpi_i - \sum n_k \alpha_k$ for some $n_k \geq 0$, so $\xi e^{\lambda}$ are multiples of $u_i$ with coefficients in $k[\Lambda_-]$. Moreover, the leading coefficient is 1 and all other coefficients are divisible by $e^{-\alpha_i}$. Explicitly:

$$\chi_{n+1-i} = \sigma_i(t_k) = t_1 t_2 \cdots t_i (1 + (t_{i+1}/t_i) \cdot \text{polynomial}(t_2/t_1, \ldots, t_n/t_{n+1})).$$

Thus, $f \in H^0(\text{Proj } S, \mathcal{O}(1))$ and $B$ is a Cartier divisor. Denote $a_i = e^{-\alpha_i}$. These are the parameters in our family. We have the following primary and secondary relations between the variables:

$$u_{i-1}u_{i+1} = a_i u_i^2, \ (1 \leq i \leq n), \quad \text{for } k \leq k' + 2: \ u_k u_{k'} = u_{k+1} u_{k'-1} \prod_{k < i < k'} a_i.$$

The primary relations correspond to the simple roots, cf. the proof of Thm. 5.5.

Lemma 10.8. One has the following:

1. For a set of indices $0 < i_1 < i_2 < \cdots < i_N < n + 1$, the fiber $Y$ over a point with $a_{i_1} = \cdots = a_{i_N} = 0$ and other $a_i \neq 0$ corresponds to the subdivision of $[0, n + 1]$ into intervals $[i_s, i_{s+1}]$ of lengths $d_s = i_{s+1} - i_s$. It is is a chain of $N + 1$ curves $Y_s \cong \mathbb{P}^1$. One has $Q_0 \in Y_0$ and $Q_\infty \in Y_N$.

2. The curve $Y_s$ together with the endpoints $(Q_0$ or the double locus of $Y$) is a curve of the curve shape $A_{d_s-1}$.

3. The family $F$: $\mathcal{Y} \to \mathbb{A}^n$ is flat.

Proof. For each coordinate with $a_i = 0$ the primary relation becomes $u_{i-1}u_{i+1} = 0$. We will call this the breaking relation. Visually, it breaks the interval $[0, n+1]$ into two intervals $[0, i]$ and $[i, n+1]$. The secondary relations, for any $k < i < k'$, become $u_k u_{k'} = 0$. These are the zero relations.

For each of the intervals $[i_s, i_{s+1}]$, add to the zero relations the toric relations, homogeneous relations between the variables $u_i$ with $i_s \leq i \leq i_{s+1}$, in the algebra $k[\Lambda^*][a_{i_s+1}^{-1}, \ldots, a_{i_{s+1}-1}^{-1}][x, \xi]$.

Together, they define a reduced subscheme $\overline{Y}$ of $\bigcup_{s=1}^{N+1} \mathbb{P}^{d_s} \subset \mathbb{P}^n$ that is isomorphic to a chain of $\mathbb{P}^1$s. The Hilbert polynomial of this variety w.r.t. $\mathcal{O}(1)$ on $\text{Proj } S$ is the same as the Hilbert polynomial of the generic fiber. The flatness of the family now follows by Lemma 10.7.
The restriction of the equation \( f \) to an irreducible component \( Y_s \) is obtained by setting all \( u_i \) with \( i \not\in [i_s, i_{s+1}] \) to zero. We get the definition of a curve of curve shape \( A_{d-1} \).

**Example 10.9.** The central fiber is a chain of \((n + 1) \mathbb{P}^1\)'s with \( B \cap Y_s \) a single point. These are the curves of curve shape \( A_0 \). Note that the number \( n \) in \( A_n \) is the number of moduli. So a curve of shape \( A_0 \) is rigid.

The local charts glue into a global family over \( V \). To see this, it is sufficient to look at two neighboring chambers, say \( C^+ \) and \( w_p.C^+ \), where \( w_p \) is the reflection in the root \( \alpha_p \). The chart for \( w_p.C^+ \) is \( \text{Spec} k[\alpha^{-\alpha_p}; \alpha \in \Delta^+] \). Here, \( w_p(\alpha) = \alpha - (\alpha, \alpha_p)\alpha_p \), in particular \( w_p(\alpha_p) = -\alpha_p \). So the intersection of the two charts is \( \text{Spec} k[\Lambda^-[1/a_p]] \).

We now observe that the variables \( u_i \) for the old chart and the new chart differ by a power of \( a_p \), which is invertible. Hence, they generate the same \( k[\Lambda^-][1/a_p]\)-subalgebra of \( k[\Lambda^+][1/a_p][x, \xi] \), and the families over the two charts glue. Of course, the divisor \( \mathcal{C} \) is the same, and the divisor \( \mathcal{B} \) is also the same since it was defined globally, independent of the charts; the equation \( f \) was \( W \)-invariant.

What we just described is the Losev-Manin moduli space \( V = L_{n+3} \) and the family of stable curves \((Y, Q_0 + Q_\infty + \epsilon \sum_{i=1}^{n+1} P_i)\) over it.

**Remark 10.10.** We can divide this family by the action of Weyl group \( W(A_n) = S_{n+1} \) forgetting the order of the points, to obtain the moduli stack \([V/W]\) of pairs \((Y, C + \frac{1+x}{2} B)\) where \( B \) is a divisor of degree \( n + 1 \) on the curve \( Y \). The map \( V \to [V/W] \) is ramified along the union of the reflection hyperplanes \( t_i = t_j \), the closures of the root tori. The ramifications do not contribute to the automorphism groups. So for a point \( t \in L_{n+3} \), for the corresponding unordered pair one has

\[
\text{Aut}(Y, Q_0 + Q_\infty + \epsilon B) = G_t/W_t,
\]

where \( G_t \subset W \) is the stabilizer group, and \( W_t \subset G_t \) is the subgroup generated by reflections.

For example, for \( t = 1 \) one has \( f = (x + 1)^{n+1} \), \( B \) is one point with multiplicity \( n + 1 \), \( G_t = W_t = S_{n+1} \), and the automorphism group is trivial. On the other hand for \( f = x^{n+1} + 1 \) the divisor \( B \) is \( n + 1 \) distinct point permuted by \( W \) cyclically. We get \( G_t = Z_{n+1} \) and \( W_t = 1 \), and the automorphism group is \( \mu_{n+1} \).

§10.3. **\( A_n \) families of surfaces.** The computation below is for the cases \( A_n^n \) with a long left side. The right side is arbitrary: \( ? = 0 \) or \( '−1' \). The cases \( −A_n^n \) with a short left side are obtained immediately by shifting the indices by 1.

Following the two-dimensional model of \( A_n \) from Section 5, we define the following \( n + 3 \) elements of the ring \( k[\frac{1}{2}\Lambda^\pm][x, y, \xi] \); deg \( u_i \) = deg \( u_s = 2 \).

1. \( u_i = e^{\pm_i x^i \xi^2} \) for \( 0 \leq i \leq n + 1 \), where we set formally \( e^{\pm_0} = e^{\pm_{n+1}} = 1 \).
2. \( u_s = y^2 \xi^2 \) and \( v_s = \sqrt{v_s} = y \xi \), deg \( v_s = 1 \).

Further, for even \( i \) we define the degree-1 variables \( v_i = \sqrt{u_i} = e^{\pm_i^/2 x^i/2} \xi \).

Now we define a \( k[\frac{1}{2}\Lambda^-] \)-algebra \( S \subset k[\frac{1}{2}\Lambda^+] \) as the one generated by the variables \( v_i \) for even \( i \), \( u_i \) for odd \( i \), and \( v_s \). We take our family over \( A^n \) to be \( \text{Proj} S \to \text{Spec} k[\frac{1}{2}\Lambda^-] \). The divisor \( C_1 \) (resp. \( C_2 \)) is the zero set of \( u_i, v_i \) with \( i > 0 \) (resp. with \( i < n + 1 \)). The relative Cartier divisor \( B \) is defined by the formula
from §7.3:

\[ f = - \left( \frac{y}{2} \right)^2 \xi^2 + \sum_{i=0}^{n+1} c_i x^i \xi^2 \in S^{(2)}, \quad \text{where } c_i = \chi_i. \]

Setting \( a_i = e^{-\alpha_i/2} \), the primary relations between the variables are

\[ u_{i-1} u_{i+1} = a_i^2 v_i^2 \quad \text{for even } i, \quad v_{i-1} v_{i+1} = a_i u_i \quad \text{for odd } i. \]

The secondary relations include the following three types depending on the parity:

\[ v_k v_{k'} = v_{k+1} v_{k'-1} \cdot a_{k+1} \prod_{i=k+2}^{k'-1} a_i^2, \quad u_k u_{k'} = v_k^2 v_{k'-1} \prod_{i=k+1}^{k'-1} a_i^2, \]

\[ v_k v_{k+2n} = u_{k+n} \cdot a_{k+1} a_2 \cdots a_k a_{k+n} a_{k+n+1} \cdots a_{k+2n}. \]

**Lemma 10.11.** One has the following:

1. For a set of indices \( 0 < i_1 < i_2 < \cdots < i_N < n + 1 \), the fiber \( Y \) over a point with \( u_{i_1} = \cdots a_{i_N} = 0 \) and other \( a_i \neq 0 \) corresponds to the subdivision of the \( A_n \) triangle into \( N + 1 \) triangles = cones with the vertex \((0,2)\) over the intervals \([s, s+1]\) of lengths \(d_s = s_{s+1} - i_s\). It is is a chain of \( N + 1 \) surfaces \( Y_s \).

2. Taking the boundaries to be the boundaries of \( Y \) and the double loci, the irreducible components \( Y_s \) are surfaces of shape \( \Lambda_1^{\dagger} \), where \( \dagger \) (resp. \( \ddagger \)) is 0 if \( i_s \) (resp. \( i_{s+1} \)) is even, so the side is long, and - if \( i_s \) (resp. \( i_{s+1} \)) is odd, so the side is short.

3. The family \( F: Y \to \mathbb{A}^n \) is flat.

4. There are natural compatible \( \mu_2^n \)-actions on the family and on the base.

**Example 10.12.** Over the origin, i.e. when all \( a_i = 0 \), the fiber \( Y \) is a union of \( n + 1 \) surfaces \( \mathbb{P}(v_s, u_i, v_{i\pm 1}) = \mathbb{P}(1,1,2) \) and the equation \( f \) restricts on each component to \( f = -\left( \frac{y}{2} \right)^2 + v_i^2 + u_{i\pm 1} \). This is a chain of \( n + 1 \) copies of the surface of shape \( A_n \). It is a kind of “weighted cone” over the chain of \( n + 1 \) copies of \( \mathbb{P}^1 \).

**Proof.** The proof of this Lemma is exactly the same as that of (10.8), using the new secondary relations, and adding a new independent variable \( v_s \) which does not appear in the relations. The \( \mu_2^n \)-action in part (4) is \( a_i \mapsto -a_i, u_i \mapsto u_i, v_i \mapsto -v_i \).

\[ \square \]

**§10.4. \( D_n \) families.** Following the two-dimensional model of \( D_n \) of Section 5, we define the following \( n + 3 \) elements of the ring \( k[1/2\Lambda_-]|x, y, \xi| \):

1. \( u_i = e^{\pi/2} x^i \xi^2 \) for \( 0 \leq i \leq n - 2 \), \( u_j' = e^{\pi/2} y^j \xi^2 \) for \( j = 1, 2 \), where we set formally \( e^{\pi/2} = e^{\pi/2} \).

2. Further, for even \( i \) we define the degree-1 variables \( v_i = \sqrt{u_i} = e^{\pi/2} x^i \xi^2 \) and \( v_j' = \sqrt{u_j'} = e^{\pi/2} y^j \xi \).

3. We also define a special non-monomial variable \( v_s = (xy - c^n) \xi \).

As before, \( u, v \)'s are of degree 2 and \( v \)'s are of degree 1. We define a \( k[1/2\Lambda_-]\)-subalgebra \( S \subset k[1/2\Lambda_-]|x, y, \xi| \) as the one generated by the variables \( v_i, v_j' \) for even \( i, u_i, u_j' \) for odd \( i \), and \( v_s \). We take our family over \( \mathbb{A}^n \) to be \( \text{Proj} S \to \text{Spec} k[1/2\Lambda_-] \).

**Definition 10.13.** We define normalized characters and coefficients by

\[ \hat{\chi}_i = e^{-\pi/2} \chi_i, \quad \hat{\chi}_j' = e^{-\pi/2} \chi_j', \quad \hat{c}_i = e^{-\pi/2} c_i, \quad \hat{c}_j' = e^{-\pi/2} c_j', \]

where we recall that \( c_i = \chi_i + \text{(lower terms)} \), and similarly for \( c_j', c^n \). We also define the variables \( a_i = e^{-\alpha_i/2} \). Thus, each \( \hat{\chi} = 1 + \ldots \) and each \( \hat{c} = 1 + \ldots \).
We define the relative Cartier divisor $B$ by the equation from Section §7.4:
\[
f = -\left(\frac{v_x}{2}\right)^2 + \sum_{j=1}^{2} c_j y_j^2 + \sum_{i=0}^{n-2} c_i x_i^2 = -\left(\frac{v_x}{2}\right)^2 + \sum_{j=1}^{2} \tilde{c}_j u_j' + \sum_{i=0}^{n-2} \tilde{c}_i u_i
\]
Since the monomials appearing in $c_j, c_i$ are of the form $\lambda - \sum_{\varpi \in \Pi} \lambda_{\varpi} \varpi$ with $\lambda_{\varpi} \geq 0$, each of the expressions $\tilde{c}$ is a polynomial in $k[\Lambda_-]$. Thus, $f \in S(2)$, it defines a section of $O_{\text{Proj} S}(2)$, and its zero scheme is a Cartier divisor.

The primary relations between the variables again correspond to the simple roots:
\[
u_i - v_{i+1} = a_i^2 v_i^2 \quad \text{for even } i, \quad v_i - v_{i+1} = a_i v_i \quad \text{for odd } i, \quad v_0 v_2 = a_1' u_1',
\]
and the corner relation $u_1 u_1' = a_0 v_0^3 (\xi'' v_0 + a'' v_s)$.

Secondary relations include
\[
u_i' v_2 = a_0'^2 a_1 (\xi'' v_0 + a'' v_s) v_0^2, \quad v_2 v_2' = a_1'' a_0'^2 a_1 (\xi'' v_0 + a'' v_s) v_0.
\]
We observe that these relations can be formally obtained from the $A_{n-1}$-relations by setting $u_{-i} = u_i'$ and replacing one of the $v_0$’s in the right-hand-side by $\xi'' v_0 + a'' v_s$.

To understand the degenerations, we will need the following fact. We recall that for a fundamental weight $\varpi$ we defined $\hat{\chi} (\varpi) \in k[\Lambda_-]$ via $\chi(\varpi) = e^{\varpi} \hat{\chi} (\varpi)$.

**Lemma 10.14.** Let $\Lambda$ be a an irreducible ADE root lattice with Dynkin diagram $\Delta$ and Weyl group $W = \langle w_\alpha \mid \alpha \in \Delta \rangle$. Let $\beta \in \Delta$ be a simple root, and $\Lambda'$ be the lattice (not necessarily irreducible) corresponding to $\Delta \setminus \beta$, with the Weyl group $W' = \langle w_\alpha \mid \alpha \neq \beta \rangle$. Let $r$ be the natural restriction homomorphism
\[
r: k[\Lambda_-] \to k[\Lambda'_-], \quad e^{-\beta} \mapsto 0, \quad e^{-\alpha} \mapsto e^{-\alpha} \text{ for } \alpha \neq \beta.
\]
Then for a fundamental character $\varpi_\alpha$ corresponding to a simple root $\alpha$ one has
\[
r(\hat{\chi}(\varpi_\alpha)) = \begin{cases} 1 & \text{for } \alpha = \beta \\ \hat{\chi}(\varpi_\alpha) & \text{for } \alpha \neq \beta \end{cases}
\]
**Proof.** Consider a dominant weight $\mu \in \Lambda^*$. We first make an elementary observation about the weight diagram of the highest weight representation $V(\mu)$. The weight diagram is obtained by starting with the highest weight $\mu = \sum m_\nu \varpi_\nu$ and subtracting simple roots $\alpha_s$ if the corresponding coordinate $m_s$ of $\mu$ is positive.

Thus, for $\mu = \varpi_\beta$, the first and only move down is to the weight $\mu - \beta$. This says that $\hat{\chi}(\varpi_\beta) = 1 + e^{-\beta} (\ldots)$. Therefore, $r(\hat{\chi}(\varpi_\beta)) = 1$.

For $\alpha \neq \beta$, the moves down in the weight diagram of $V(\varpi_\alpha)$ not involving $\beta$ are the same as the moves in the Dynkin diagram $\Delta \setminus \beta$. So the monomials appearing in $r(\hat{\chi}(\varpi_\alpha))$ for the Dynkin diagram $\Delta$ and the monomials appearing in $\hat{\chi}(\varpi_\alpha)$ for the Dynkin diagram $\Delta \setminus \beta$ are the same.

We have to show that the coefficients of these monomials are also the same. This follows from the Weyl character formula
\[
\chi(\lambda) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) e^{w(\rho)}}, \quad \text{where } \rho = \sum_{\varpi \in \Pi} \varpi.
\]
Isolating the terms $e^\mu$ on the top and the bottom where the linear function $(\beta, \mu)$ takes the maximum, and setting other terms to zero gives the same Weyl Character formula expression for the Weyl group $W'$. This concludes the proof.  \[\square\]
We now analyze the codimension-1 degenerations.

**Lemma 10.15.** The fibers of the $D_n^2$ family over a point with a single $a = 0$ are all reduced, have one or two irreducible components, and are of the shapes:

\[ a'' = 0: A_{n-1}^2, \quad a'_1 = 0: A_{n-1}^3, \quad a_0 = 0: A_1 A_{n-3}^2, \quad a_1 = 0: A_3^2 A_{n-4}, \quad a_i = 0 \text{ for } i \geq 2: D_{i+2}^1 A_{n-i-3}^2, \]

where $\hat{1} = \pm$ for $i$ even or odd.

Proof. Case $a'' = 0$. By Lemma 10.14 the expression $c'' v_0 + a'' v_*$ restricts to $v_0$. Thus, the $D_n$ relations between our variables restrict to the $A_{n-1}$ relations since they were obtained from the $A_{n-1}$ relations by substituting a single $v_0$ by $c'' v_0 + a'' v_*$. The equation of $f$ for $D_n$ also restricts to the $A_{n-1}$ equation. Indeed, by Eq. 5.1, all the lower terms are divisible by $a''$, so they vanish.

In all the other cases we have some breaking relations which split the $D_n^2$ polytope into two polytopes of shapes $A_{n-1}^2 A_{n-1}^1$, $A_1 A_{n-3}^2$, $A_3^2 A_{n-4}$, or $D_{i+2}^1 A_{n-i-3}^2$ respectively. The secondary relations imply that all products between the variables not lying in a single polytope are zero.

Further, if $a'' \neq 0$ then the generator $v_*$ can be replaced by a monomial generator $xy = v_0 + \frac{1}{a''} v_0$. Then the algebra $S$ becomes the algebra of a stable toric variety, a seminormal union of two irreducible toric varieties for the above polytopes glued along a common part of the toric boundary, isomorphic to $\mathbb{P}^1$.

The fact that the equation $f$ of $B$ on each component restricts to the equation of $A_0^2$, $A_{n-1}^2$, $A_1 A_{n-3}^2$, $A_3 A_{n-4}^2$, $D_{i+2}^1$, or $A_{n-i-3}^2$ again is an immediate consequence of Eq. 5.1 which tells us which “lower terms” vanish when we set some of the $a$-variables to zero. \(\square\)

**Remark 10.16.** The degeneration $a'' = 0$ is the most interesting one here. This corresponds to “smoothing of the corner”, a truly non-toric behavior. The quintessential example, corresponding to $D_4 \to A_3$ in our notation, is the well-known degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ to the quadric cone $\mathbb{F}_2^2$.

On the other hand, the case $a_0 = 0$ is special in that it loses moduli: the family has codimension 2, whereas the other families have codimension 1.

**Lemma 10.17.** The other fibers of the family $F$ are unions of further degenerations of the shapes listed in Lemma 10.15, with the parts degenerating independently. The family $F$ is flat and it comes with a natural $\mu_2^n$-action.

Proof. Setting other $a$-variables to zero just adds more zero relations. Together with the toric relations for the variables in the same polytope, they define a reduced union of toric varieties which has the same Hilbert polynomial as the generic fiber. The flatness now follows by Lemma 10.7.

The $\mu_2^n$-action is given by $a_i \mapsto -a_i$, $u_i \mapsto u_i'$, $v_i \mapsto v_i'$, $v_* \mapsto -v_*$. \(\square\)

§10.5. $E_n$ families. We explain the $n = 8$ case. The $E_7$ and $E_6$ shapes are done entirely similarly but are easier. Following the two-dimensional model of $E_8$ of Section 5, we define the following 8 + 3 elements of the ring $k[x, y, \xi]$: \begin{enumerate}
  \item $u_i = e^{\omega_i} x^{i+1} \xi^2$ for $0 \leq i \leq n - 2$, $u'_j = e^{\omega_j^2} y^j \xi^2$ for $j = 1, 2, 3$, where we set formally $e^{\omega_0} = e^{\omega_3} = 1$. Further, for even $i$ we define the degree-1 variables $v_i = \sqrt{w_i} = e^{\omega_i/2} x^{i/2} \xi$ and $v'_j = \sqrt{w'} = e^{\omega_j/2} y^j \xi$.
  \item We also define a special non-monomial variable $v_* = (xy - c') \xi$.
\end{enumerate}
As before, $u$'s are of degree 2 and $v$'s are of degree 1. We define a $k[\frac{1}{2}\Lambda_-]$-subalgebra $S \subset k[\frac{1}{2}\Lambda_-][x, y, \zeta]$ as the one generated by the variables $v_i, v_i'$ for even $i$, $u_i, u_i'$ for odd $i$, and $v_r$. We take our family over $\mathbb{A}^n$ to be $\text{Proj} \ S \to \text{Spec} \ k[\frac{1}{2}\Lambda_-]$.

The divisor $B$ is given by the formula from section §7.5:

$$f = -\left(\frac{v_r}{2}\right)^2 + \sum_{j=1}^3 c_j y^j \zeta^2 + \sum_{i=0}^{n-2} c_i x^i \zeta^2 = -\left(\frac{v_r}{2}\right)^2 + \sum_{j=1}^3 c_j y^j + \sum_{i=0}^{n-2} c_i u_i \in S^{(2)}$$

**Lemma 10.18.** The fibers of the $E_8$ family over a point with a single $a = 0$ are all reduced, have one or two irreducible components, and are of the shapes:

- $a'' = 0$: $-A_7^-$, $a'_2 = 0$: $-A_9 D_7^-$, $a'_1 = 0$: $-A_7^-$ $\Gamma_6^-$, $a_0 = 0$: $-A_2 A_4^-$,
- $a_1 = 0$: $-A_4^- A_3^-$, $a_2 = 0$: $D_5^+ A_2^-$, $a_2 = 0$: $E_6^- A_1^-$, $a_3 = 0$: $E_7 A_0^-$.

All fibers are unions of further degenerations of the shapes listed in Lemma 10.15, with the parts degenerating independently. The family $F$ is flat and it comes with a natural $\mu_2^\infty$-action.

**Proof.** The proof is the same as in the $D_n$ case, so we do not repeat it. The only new ingredient is to check that the equation of $f$ in the $E_8$ case restricts on every irreducible component to the corresponding equation for the $A, A', D,$ or $E$ case. This follows right away from Table 4. For each $a$-variable set to zero, if the corresponding root $\alpha$ appears below a certain fundamental weight $\varpi$, then all the terms from that weight down vanish. Checking the 8 cases is a little tedious but elementary, so we do not write it down here in detail. \(\square\)

### §10.6. Priming Calculus and families in the primed cases.

Let $(Y, C + \frac{1 + e}{2}B)$ be an $ADE$ pair of a pure shape $S$. Recall from Theorem 3.6 that all primings of it are allowed for as long as the volume of the resulting pair is positive and we prime no more than two times on a long side and once on a short side; and that there are two redundant cases $'A_1 = -A_0, ^+A_1 = A_0^-$. Let us now consider some extreme cases when we prime at $k = \text{vol}(Y, C)$ points so that $\text{vol}(Y', C') = 0$.

**Lemma 10.19.** Let $(Y, C + \frac{1 + e}{2}B)$ be an $ADE$ pair of shape $A_0^-$, resp. $A_1$. When priming it once, resp. twice to obtain the cases $A_0^+, A_0^-, A_1^+$, the line bundle $L' = -2(K_{Y'} + C')$ on $Y'$ is nef and semiample and gives a contraction to a $\mathbb{P}^1$.

**Proof.** An elementary direct computation. \(\square\)

We now state the rules which we call the Priming Calculus:

$$'A_1 = -A_0, ^+A_1 = A_0^-, ^+A_0 = 'A_0 = A_0^- = A_1 = 0 \ "(A_0^- \ "S) \to ^+S$$

where $-S$ is any shape with a short side on the left.

**Remark 10.20.** We will only need the cases when priming happens on one side. There are other extreme priming cases of volume 0 when priming happens on both sides: $'A_1', ^+A_1', "A_2', "A_3', "D_4'$. Let $S$ be a pure $ADE$ shape, and $S'$ be a shape obtained from it by 1 $\leq k \leq 4$ primings. In the sections above, we constructed a family of stable $ADE$ pairs $f: \mathcal{V} \to V_2$ of shape $S$. We are now going to modify it to obtain a family of pairs of shape $S'$. Let $V_2'$ be a projective toric variety for the Coxeter fan $\Sigma$ and the character group $\Lambda'$ from Theorem 6.3.
**Theorem 10.21.** Over $V'_2$ there exists a family of stable ADE pairs whose restriction to the torus $T_{V'}^*$ is the family of ADE pairs of shape $S'$. The fibers of this family are obtained from those of the shape $S$ family according to Priming Calculus Rules (10.2).

**Proof.** Let $f': Y' = Y \times V_2' \to V_2'$ be the pullback of the $S$ shape family to $V_2'$. It has disjoint sections $s_1, \ldots, s_k$ ($k \leq 4$), one for each point to be primed. We blow up these sections to obtain a family $\tilde{f'}: \tilde{Y}' = \text{Bl}_{s_1, \ldots, s_k} Y' \to V_2'$. A fiber $Y'$ of this family is computed by Lemma 10.19 with one exception: priming $A_0$ on the left twice. In all the other cases by Lemma 10.19 the divisor $K_Y + C + \frac{1}{\ell} B'$ on $Y$ is nef (we need to look at primings on one side only.) Then the divisor $K_{Y'} + C' + \frac{1}{\ell} B'$ is $\tilde{f}'$-nef and semiample, and its relative canonical model $\tilde{f}': \tilde{Y} \to V_2'$ is the required family of stable ADE surfaces of the primed shape $S'$ with the fibers as stated.

We now treat the exceptional case $''A_0$. Working upstairs, $X = \mathbb{P}^2$, the long side $D_1$ is a conic, the short side $D_2$ is a line, the ramification divisor $R$ is a line and there is another fixed point of involution on $D_2$. Blowing up at the two points $D_1 \cap R$ makes $R'$ into a $(-1)$-curve, and so it is no longer nef.

From the description of the family $f$ in the previous section we see that this case occurs over a smooth divisor in the base, in the cases when there is a long side and the end $a$-variable is set to zero. On the double cover $\tilde{X}'$ we have a family of $(-1)$ curves, call it $\mathfrak{K}$, which is smooth and of codimension 2 in $\tilde{X}'$. We now make a flip in $\mathfrak{K}$ over $V_2'$. On the resulting family, on the fibers that were previously of shape $''A_0$ the divisor $K + D' + \epsilon R' = \epsilon R'$ is nef with $(R')^2 = 0$, so they get contracted. The effect on the neighboring irreducible component of shape $''S$ (which necessarily has a short left side) is priming, to make it $''S$. In this new family the log canonical divisor is nef and semiample, and its relative canonical model is the required family. Note that the shape $''S$ may get contracted as well, e.g. $''(A_0^- A_0^-) \to + A_0^- A_0^- = 0$. $\square$

§10.7. Consequences for the moduli spaces. For stable surface pairs $(Y, B = \sum b_i B_i)$ of a fixed numerical type over $\mathbb{C}$ one has a compact and projective moduli space, see [Kol15] for more details. There are technical difficulties with this construction if one works with arbitrary coefficients $b_i$; they disappear if the coefficients are $> \frac{1}{2}$.

Let us now fix an ADE shape, pure or primed, and a concrete $0 < \epsilon < 1$ for which Theorem 9.8 holds. For this type we then have a moduli space of stable pairs $(Y, C + \frac{1}{\ell} \epsilon B)$, call it $\mathcal{M}$. The classifying morphism for our $\mu_2^n$-equivariant family of stable ADE pairs over $V_2'$ to $\mathcal{M}$ obviously factors through $V_2'/\mu_2^n = V'$. Theorem 10.21 then immediately implies the following:

**Theorem 10.22.** There exists a proper morphism $\psi: V'/W \to \mathcal{M}$ such that

1. $\psi$ is a proper birational morphism to the main irreducible component of $\mathcal{M}$, and isomorphism on the open subset $T_{V'/W} = k^n/H$ of Theorem 6.3.

2. Toric strata of $V'$ have the same image in $\mathcal{M}$ iff they are equivalent under the Priming Calculus Rules (10.2).

§10.8. Wythoff fan and contracting the base. Consider a $D_n$- or $E_n$-family over the toric variety $V_2$ for the Coxeter fan. The maximal Weyl chamber corresponds to the maximal degeneration to a union of $\mathbb{P}(1,1,2)$’s, when all the $a$-variables are set to 0. Its codimension-1 walls correspond to “almost maximal degenerations”, when all but one of the $a$-variables are zero. However, when we set
all the $a$-variables except for $a''$ to zero, instead of the expected 1-dimensional family we get the maximal degeneration again, since the corner relation gives $u_1u_1' = a_0v_0^2(C''v_0 + a''v_0)$ and the right hand side is zero already, no matter what $a''$ is.

This gives a strong hint that our family is in fact a pullback from a smaller toric variety obtained by a contraction $V_2 \rightarrow V_2'$, corresponding to a coarser fan obtained by gluing some Weyl chambers together. This is exactly what we show in this section.

The following is Wythoff’s construction of uniform polytopes, so named by Coxeter in the 1930s, see [Cox35], [Cox73, 5-7]. In the more recent literature these polytopes are frequently called permutahedra. Also, in [BGW03, 6.8] they are called Coxeter matroid polytopes.

**Definition 10.23.** Consider a point $p \neq 0$ in the positive chamber $C^+ \subset \Lambda_Q^* = \Lambda_Q$, see Eq. 10.1. A permutahedron is the convex hull of the vertices $Wp$, where $W = W(\Lambda)$ is the Weyl group. We will consider this to be a polytope in the space $M_R = \Lambda \otimes \mathbb{R}$ of characters and denote by $\Sigma_p$ its normal fan in the space $N = \Lambda^*$. We call $\Sigma_p$ the Wythoff fan.

**Example 10.24.** If $p$ lies in the interior of $C^+$ then $\Sigma_p$ is the Coxeter fan of $\Lambda$.

In general, the set $\Delta$ of simple roots $\{ \alpha \}$ of $\Lambda$ is divided into the set of active mirrors such that $p \notin \alpha^\perp$ and inactive mirrors such that $p \in \alpha^\perp$. For the Coxeter fan, all mirrors are active. The reflections $v_\alpha \in W$ in the inactive mirrors $\alpha$ leave the point $p$ fixed, so the Wythoff fan is coarser than $\Sigma$. Thus, for the corresponding toric varieties we get a proper birational contraction $V \rightarrow V_p$.

**Definition 10.25.** We will define a decorated Dynkin diagram $(\Delta, \Delta^0)$ to be a Dynkin diagram $\Delta$ with a proper subset $\Delta^0 \neq \Delta$ of inactive nodes. We will draw the inactive nodes as unfilled, and active nodes as filled. The fan $\Sigma_p$ depends only on the pair $(\Delta, \Delta^0)$, so we will denote it by $\Sigma(\Delta, \Delta^0)$.

**Example 10.26.** From the start, our Dynkin diagrams in Figs. 3, 4 for the pure $D_n$ and $E_n$ shapes were semi-secretly decorated. The node for the interior point $p''$ is inactive. In the toric primed cases $'A_n$, $'A'_n$, $D'_n$ of Fig. 8 the Dynkin diagrams were also decorated.

We now define the decorated Dynkin diagrams for all the primed cases.

**Definition 10.27.** For a single priming, we make the node of the Dynkin diagram next to that side inactive (cf. the $'A_n$, $'A'_n$, $D'_n$ shapes). For a double priming, on a long side, we make the two end nodes next to it inactive.

**Definition 10.28.** We call the following shapes toric shapes: $A_n$, $D_{2n}$ $(2n \geq 4)$, $D_{2n-1}$ $(2n - 1 \geq 5)$, $E_n$ $(n = 6, 7, 8)$, $'A_{2n-1}$ $(2n - 1 \geq 3)$, $'A'_{2n-2}$ $(2n - 2 \geq 2)$, $'A''_{2n-1}$ $(2n - 1 \geq 3)$, $D''_{2n}$ $(2n \geq 4)$. Recall that for each of these shapes the $ADE$ surface $Y$ is toric and that for all of them we gave an explicit equation of the family in toric coordinates.

**Theorem 10.29.** For the toric shapes, over the toric variety $V^*_2(\Delta, \Delta^0)$ for the Wythoff fan $\Sigma(\Delta, \Delta^0)$ and the lattice $\frac{1}{2}\Lambda'$ (as in Table 5) there exists a projective family of stable $ADE$ varieties extending the family of $ADE$ surfaces over the interior torus. This family is $W \times \mu^2_{2}$-invariant. The family over the Coxeter fan toric variety $V^*_2(\Delta)$ is a pullback of this family.
Proof. We give the proof in the representative $D_n$ case; the same argument applies to all of them. The Wythoff fan in the $D_n$ case is obtained by gluing the positive chamber $C^+$ with a neighboring chamber $\alpha''.C^+$ through the wall $(\alpha''')^-$ and then taking all the $W$-translates of this union. The union $\hat{C} = C^+ \cup \alpha''.C^+$ is again simplicial, and the normals to its sides are the roots $\alpha'_1$, $\alpha_0$, $\alpha'_0 := \alpha_0 + \alpha''$, $\alpha_i$ ($i > 0$) with the Dynkin diagram

where the dashed line means that $(\alpha_0, \alpha'_0) = 1$ unlike $-1$ for the solid edges.

Accordingly, the corresponding affine chart is isomorphic to $\mathbb{A}^n$ with the coordinates $a_0$, $a'_0 := a_0 a''$, $a'_1$, and $a_i$ for $i > 0$. The original $\Lambda_n$ with the coordinates $a''$, $a_0$, $a'_1$, and $a_i$ for $i > 0$ is an affine blowup of this $\mathbb{A}^n$, corresponding to the ring inclusion

$$R' = k[a_0, a_0 a'', a'_1, a_i] \subset k[a'', a_0, a'_1, a_i] = R.$$  

We need to show that the equation $f$ of the divisor $B$ is in fact already in the subring $R'[u, v]$, where $u, v$ go over all our generators. This is equivalent to saying that for each of the fundamental characters $\lambda \neq \lambda''$, the coefficients of $\tilde{\lambda}$ are in $R'$ and not just in $R$. This follows by Lemma 10.30(1).

The relations between the $u, v$ variables are also defined over the subring $R'$. Indeed, the only relations involving $a_0$ are the corner relation and those that are derived from it. The corner relation becomes

$$u_1 u'_1 = a_0 v_0^3 (\tilde{c}'' v_0 + a'' v_*) = a_0 \tilde{c}'' v_0^4 + a'_1 v_0^3 v_*$$

We are thus reduced to showing that $a_0 \tilde{c}'' \in R'$. This follows from Lemma 10.30(2).

**Lemma 10.30.** Let $\Delta$ be a $D_n$ or $E_n$ Dynkin diagram with root lattice $\Lambda$ and weight lattice $\Lambda^\ast$. In the $A_n$ case, in order to use the same notation, we rename $\alpha_0 \mapsto \alpha''$, and $\alpha_i \mapsto \alpha_{i-1}$ for $i \geq 1$. Let $\Lambda_{++} \subset \Lambda^\ast$ be the following semigroup:

$$\Lambda_{++} = \mathbb{N} \alpha' + \mathbb{N} \alpha_0 + \mathbb{N} (\alpha_0 + \alpha'') + \mathbb{N} \alpha_i$$

Let $\omega$ be a fundamental weight and $\lambda = \omega - \sum_{\alpha \in \Delta} n_{\alpha} \alpha$ be a weight appearing in the fundamental character $\chi(\omega)$. Then:

1. if $\omega \neq \omega''$ then $\omega - \lambda \in \Lambda_{++}$;
2. if $\omega = \omega''$ then $\omega - \lambda + a_0 \in \Lambda_{++}$.

**Proof.** This is an elementary property of weight diagrams of highest weight representations. Let $\mu = \sum m_i \omega_i + m' \omega' + m'' \omega'' \in \Lambda^\ast_{++}$ be a dominant weight. The weight diagram of the representation $V(\mu)$ is computed by starting from $\mu$ and subtracting simple roots $\alpha$, written in the weight basis, when the corresponding coordinate is positive. Now suppose that the coordinate $m'' = 0$. Since $\alpha'' = (0, \ldots, 0, -1, 2)$ and $\alpha_0 = (0, \ldots, -1, 2, -1)$, we see that in order to get to a weight $\nu$ with a positive $\omega''$-coordinate from which we can subtract $\alpha''$, one has to
subtract \( \alpha_0 \) first. That is true because \( \alpha'' \) is an end vertex in the graph, and so a path from any other vertex to it goes through \( \alpha_0 \). So we can combine the two moves into \( \alpha_0 + \alpha'' \), and then apply induction. This implies the statement for all the fundamental weights \( \varpi \neq \varpi'' \). For \( \varpi'' \), the first and only move in the weight diagram of \( \mathcal{V}(\varpi'') \) is \( \varpi'' - \alpha'' = -\varpi'' + \varpi_0 \). From there, the same argument as above implies that for every \( \alpha'' \)-move there is a preceding \( \alpha_0 \)-move.

\[\square\]

§10.9. Compact moduli of rational elliptic surfaces. Let \( M_{\text{ell}} \) be the moduli space of smooth rational elliptic relatively minimal surfaces \( S \to \mathbb{P}^1 \) with a section \( E \). Let \( M_{\text{ell}}(I_1) \) be the moduli space of such surfaces \( (S, E, F) \) together with a fixed \( J_1 \) Kodaira fiber (i.e. a rational nodal curve). This is a \( 12:1 \) cover of a dense open subset of \( M_{\text{ell}} \) since a generic rational elliptic surface has 12 \( I_1 \) fibers.

Let \( j: S \to S \) be the elliptic involution w.r.t. the section \( E \) and \( E \sqcup R_8 \) be the fixed locus of \( j \). Contracting the \((-2)\)-curves in the fibers which are disjoint from the section \( E \) and then \( E \) itself gives a pair \( (X, D + \epsilon R) \) which is an \( ADE \) pair of shape \( E_8 \). Vice versa, any pair \( (X, D + \epsilon R) \) of \( E_8 \) shape is a del Pezzo surface of degree 1 with Du Val singularities. Blowing up the unique base point of \( |−K_X| \) and resolving the \( ADE \) singularities gives a rational elliptic fibration \( S \to \mathbb{P}^1 \) and the strict preimage of \( D \) is an \( I_1 \) fiber of this fibration.

In §7.5 we constructed the explicit family of pairs \( (X, D + \epsilon R) \) of \( E_8 \) shape over the torus \( T_{E_8} = \mathbb{G}_m^8 \). This gives \( M_{\text{ell}}(I_1) = T_{E_8}/W_{E_8} \). Recall that in Section 9 we described the singularities of the surfaces \( X \) appearing in this family. Thus, a particular special case of the results of the present section is a geometrically meaningful compactification \( \overline{M}_{\text{ell}}(I_1) \) of the moduli space \( M_{\text{ell}}(I_1) \).

Similarly, the \( E_7 \) compactified family gives a moduli compactification of the moduli space \( M_{\text{ell}}(I_2) \) of rational elliptic surfaces with an \( I_2 \) Kodaira fiber; the \( E_6 \) family gives \( \overline{M}_{\text{ell}}(I_3) \); the \( D_5 \) family gives \( \overline{M}_{\text{ell}}(I_4) \); and the \( A_4 \) family gives \( \overline{M}_{\text{ell}}(I_5) \).

11. Extensions and generalizations

§11.1. Non simply laced root lattices \( B_n, C_n, F_4, G_2 \). In the Slodowy correspondence [Slo80, 6.2] the non simply laced root lattices correspond to pairs of root lattices together with a finite group of symmetries: \( B_n = (A_{2n-1}, \mathbb{Z}_2) \), \( C_n = (D_{n+1}, \mathbb{Z}_2) \), \( F_4 = (E_6, \mathbb{Z}_2) \). Using it, one can define moduli spaces for these cases as moduli spaces of \( ADE \) pairs with an involution. Batyrev-Blume’s generalization [BB11] of Losev-Manin spaces in the case of curves to \( B_n \) and \( C_n \) lattices can also be understood in these terms. There is also the case \( G_2 = (D_4, S_3) \) but \( S_3 \) does not naturally act on our \( D_4 \) surfaces.

§11.2. Compact moduli for the affine \( \tilde{ADE} \) root lattices and K3 surfaces. For each connected subdiagram \( R' \subset R \), surfaces of type \( R' \) appear as irreducible components of degenerations of surfaces of type \( R \). By analogy, many of our \( ADE \) surfaces should appear as irreducible components of degenerations of K3 surfaces with involution, as predicted by the Vinberg diagram in Fig. 1. And indeed, many of them do, cf. [AT15]. We will consider the moduli and compactifications for the surfaces of types II and I in a separate paper. Those cases are very similar in many respects but require rather more technology, such as Kac-Moody algebras.
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