Microscopic and Macroscopic Behaviors of Palatini Modified Gravity Theories

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We show that, within modified gravity, the non-linear nature of the field equations implies that the usual naive averaging procedure (replacing the microscopic energy-momentum by its cosmological average) is invalid. We discuss then how the averaging should be performed correctly and show that, as a consequence, at classical level the physical masses and geodesics of particles, cosmology and astrophysics in Palatini modified gravity theories are all indistinguishable from the results of general relativity plus a cosmological constant. Palatini gravity is however a different theory from general relativity and predicts different internal structures of particles from the latter. On the other hand, and in contrast to classical particles, the electromagnetic field permeates in the space, hence a different averaging procedure should be applied here. We show that in general Palatini gravity theories would then affect the propagation of photons, thus changing the behaviour of a Universe dominated by radiation. Finally, Palatini theories also predict alterations to particle physics laws. For example, it can lead to sensitive corrections to the hydrogen energy levels, the measurements of which could be used to place very strong constraints on the properties of viable Palatini gravity theories.

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I. INTRODUCTION

Extensions of General Relativity (GR) have always received a great deal of attention. Such theories are motivated by quantum gravity models and by the wish to find phenomenological alternatives to the standard paradigm of dark matter and dark energy. Indeed, it’s possible for a theory that deviates significantly from GR at the level of the microscopic field equations to be indistinguishable from GR when correctly coarse-grained over macroscopic (e.g., astrophysical) scales. We illustrate this point for a class of modified gravity theories in which the Ricci scalar is a function of some independent of $\bar{\omega}$ and Palatini. In the alternative, Palatini approach, $R = R_{ab}g^{ab} \Gamma^c_{ab}$, where $R_{ab}$ is a function of some connection field $\Gamma^c_{ab}$, that is, a priori, treated as being independent of $\bar{g}_{ab}$. The field equations are then found by minimizing the action with respect to variations in $\bar{g}_{ab}$. In the alternative, Palatini approach, $R = R_{ab}g^{ab}$ where $R_{ab}$ is a function of some connection field $\Gamma^c_{ab}$, that is, a priori, treated as being independent of $\bar{g}_{ab}$. The field equations are then found by minimizing the action with respect to both $\Gamma^c_{ab}$ and $\bar{g}_{ab}$. If $f(R, R_{ab}R_{ab}) = R - 2\Lambda$ (i.e., GR with a cosmological constant) then the two approaches result in the same field equations. Otherwise they are generally different.

Before going into further details there is one point to be noticed. It has been argued that Palatini approach, as outlined above, swaps one theory for another when applied to $f(R)$ actions. Whether or not this is indeed the case, the Palatini $f(R)$ field equations are mathematically equivalent to a $\omega = -\frac{1}{2}$ Brans-Dicke theory with a

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In this article we show that such an approach cannot simply be applied to modified gravity theories without a detailed analysis of the energy-momentum microstructure. Indeed, the naively averaging over the microscopic structure will generally lead one to make incorrect predictions, and inaccurate conclusions as to the validity of the theory. Indeed, it’s possible for a theory that deviates significantly from GR at the level of the microscopic field equations to be indistinguishable from GR when correctly coarse-grained over macroscopic (e.g., astrophysical) scales. We illustrate this point for a class of modified gravity theories in which the Ricci scalar is a function of some independent of $\bar{\omega}$ and Palatini. In the alternative, Palatini approach, $R = R_{ab}g^{ab} \Gamma^c_{ab}$, where $R_{ab}$ is a function of some connection field $\Gamma^c_{ab}$, that is, a priori, treated as being independent of $\bar{g}_{ab}$. The field equations are then found by minimizing the action with respect to variations in $\bar{g}_{ab}$. In the alternative, Palatini approach, $R = R_{ab}g^{ab}$ where $R_{ab}$ is a function of some connection field $\Gamma^c_{ab}$, that is, a priori, treated as being independent of $\bar{g}_{ab}$. The field equations are then found by minimizing the action with respect to both $\Gamma^c_{ab}$ and $\bar{g}_{ab}$. If $f(R, R_{ab}R_{ab}) = R - 2\Lambda$ (i.e., GR with a cosmological constant) then the two approaches result in the same field equations. Otherwise they are generally different.
potential, and so certainly do correspond to a mathematically valid, and widely studied, modified gravity theory, even if it is not technically derivable from an $f(R)$ action.

This work is organized as follows. In § II we briefly introduce the main ingredients of Palatini modified gravity theories, derive the gravitational field equations for a general $f(R, R^{ab} R_{ab})$ Lagrangian and discuss their behaviour in vacuum. In § III we explain why the popular naive averaging procedure (in which one simply replaces the quantities in the field equations with some coarsely-grained averages) fails in some cases, and detail our new averaging method. We then reconsider the particle kinematics, cosmology, astrophysics and atomic physics thoroughly using the new approach and compare our results with old results in the literature in the subsequent sections. In § IV the motion of classical particles (clumps of energy density in tiny patches in between which there is vacuum) is considered, and we find that particles in Palatini theories move in exactly the same ways as they do in GR, their active gravitational, passive gravitational and inertial masses are all equal, and most importantly the predictions on cosmology and astrophysics are also the same as those of GR. § V is devoted to an analysis of the behaviour of electromagnetic field in Palatini theories: in contrast to classical particles the electromagnetic field permeates in the space and its averaging is a bit different. We find that in general Palatini theories (albeit not ones where $f(R, R^{ab} R_{ab}) = f(R)$) the propagation of photons is altered as compared with GR, and the universe dominated by radiation will also behave differently. § VI then considers the atomic physics. We argue that for atomic physics calculations it is more convenient to work in the Einstein frame metric and show that the matter Lagrangian is modified at the field theoretic level. In particular, the atomic energy levels now depend on the modification very sensitively and experimental data puts very strong constraints on any Palatini-type deviations from GR. Although the analysis is performed in the Einstein frame, we show that the resulting experimental constraints are independent of one’s frame choice. We finally summarize in § VII

II. PALATINI $f(R, R_{ab} R^{ab})$ THEORIES

In this section we briefly summarize the main ingredients of Palatini modified gravity theories.

In general, to modify gravity one could add functions of the curvature invariants $R, R^{ab} R_{ab}, R^{abcd} R_{abcd}$ to the standard Einstein-Hilbert action. In the Palatini variational approach, the case of $R^{abcd} R_{abcd}$ has not yet been explored up to date, so in this paper we shall focus on the special class of theories where $f = f(R, R^{ab} R_{ab})$, the cosmology of which has recently been the subject of much interest. We stress that, as mentioned in § II in these theories the Ricci tensor $R_{ab}$ is constructed from the connection $\Gamma^a_{bc}$ which is generally not the Levi-Civita connection of the matter metric $g_{ab}$, which is instead denoted by $\bar{\Gamma}^a_{bc}$. In what follows we shall use several different notations and for clarity we define them here. We use $g_{ab}$ to denote the metric whose Levi-Civita connection is $\Gamma^a_{bc}$ and as such we have $R_{ab} = R_{ab}(\Gamma) = R_{ab}(g)$, the Ricci scalar calculated from this metric is $R \equiv g^{ab} R_{ab}$; in a similar way for the matter metric $\bar{g}_{ab}$ we have $R_{ab} \equiv R_{ab}(\bar{\Gamma}) = R(\bar{g})$ and $R = \bar{g}^{ab} R_{ab}$. Besides this, we also need the mixed contractions $R = \bar{g}^{ab} R_{ab}, R^{ab} = g^{ab} \bar{g}^{cd} R_{cd}, R^a_b = \bar{g}^{ac} R_{cb}$. We further define the covariant derivatives $\nabla_c$ and $\bar{\nabla}_c$ to be compatible with the connections $\Gamma$ and $\bar{\Gamma}$ respectively, i.e., $\bar{\nabla}_c g_{ab} = \bar{\nabla}_c \bar{g}_{ab} = 0$. To be consistent with these conventions we shall rename the $f(R, R^{ab} R_{ab})$ theories as $f(\mathcal{R}, \bar{\mathcal{R}}^{ab} R_{ab})$ theories from now on, and these new notations clearly show that the theories at hand are neither metric ones nor pure affine ones. It will become clear below how these different quantities relate with each other. Note also that we assume $R_{ab}$ to be a symmetric tensor (if it contains antisymmetric parts then the field equation will be spoiled).

A. The Action

For Palatini $f(\mathcal{R}, \bar{\mathcal{R}}^{ab} R_{ab})$ gravity we start from the following action

$$S_{f(\mathcal{R}, R^{ab} R_{ab})} = \int d^4 x \sqrt{-g} \frac{1}{16\pi G} f(R, \bar{\mathcal{R}}^{ab} R_{ab}) + S_{\text{matter}}(g_{\mu\nu}, \psi_i),$$

(1)

where $\kappa = 8\pi G$ with $G$ being the gravitational constant; $c = \hbar = 1$. $S_{\text{matter}}$ is the matter action depending only on the matter metric $\bar{g}_{ab}$ and specific matter species $\psi_i$, and not on $\Gamma^a_{bc}$. This means that the energy momentum tensor, defined as:

$$T_{ab} = - \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{ab}},$$

(2)

is conserved with respect to $\bar{g}_{ab}$ and the particle geodesics are determined by the metric $\bar{g}_{ab}$. The conservation law is

$$\bar{\nabla}^a T_{ab} = 0,$$

where $\bar{\nabla}_a \bar{g}_{bc} = 0$ as defined above.

B. Field Equations

For the sake of convenience and clearness, we shall define $\Phi = \mathcal{R}, \chi = \bar{\mathcal{R}}^{ab} R_{ab}$ and $K^a_b = \bar{\mathcal{R}}^{ab}_{\ bc}$ for a general $f(\mathcal{R}, \bar{\mathcal{R}}^{ab} R_{ab})$ theory. Minimizing the action with respect to the variations in the connection, $\Gamma^a_{bc}$, gives

$$\sqrt{-g} g^{ab} = \sqrt{-\bar{g}} g^{ac} (f_{\Phi} \delta^b_c + 2 f_{\chi} K^b_c),$$

(3)

where $f_{\Phi} = \partial f(\Phi, \chi)/\partial \Phi, f_{\chi} = \partial f(\Phi, \chi)/\partial \chi, g_{ab}$ is the metric whose Levi-Civita connection is $\Gamma^a_{bc}$. In $f(\mathcal{R})$
theories where \( f_\chi = 0 \), or in the cases when \( K^b \propto \delta^b \), the two metrics are related conformally: \( g_{ab} = f_\chi \tilde{g}_{ab} \). More generally, the relationship between the two metrics is a disformal one.

We define the matrix \( \mathbf{K} \) by \( \mathbf{K} = K^a \) and minimize the action with respect to variations in \( \tilde{g}_{ab} \) to find

\[
f_\Phi \mathbf{K} + 2f_\chi \mathbf{K}^2 - \frac{1}{2} \mathbf{f} T = \kappa \mathbf{I},
\]

where \( T^a_{bc} = \tilde{g}^{ac}T_{bc} \) and \( \mathbf{I} \) is the \((4 \times 4)\) unit matrix. \( \Phi \) and \( \chi \) are then given by the following algebraic relations

\[
\Phi = \text{tr} \mathbf{K}, \quad \chi = \text{tr} \mathbf{K}^2
\]

The trace of Eq. (4) reads

\[
f_\Phi \Phi + 2f_\chi \chi - 2f = \kappa T,
\]

where \( T = T^a_{ab} \). Defining

\[
Q^2 = \sqrt{-\tilde{g}}/\sqrt{-\tilde{g}} = f_\Phi^2 \left[ \det \left( I + \frac{2f_\chi \mathbf{K}}{f_\Phi} \right) \right]^{1/2},
\]

and using Eq. (3) it is straightforward to check that

\[
Q^2R^a_{ab} = Q^2g^{ac}R_{cb} = \tilde{g}^{ac}f_\Phi R_{ab} + 2f_\chi \tilde{g}^{ac}g^{de}R_{cd} = f_\Phi K^a_{ab} + 2f_\chi K^a_cK^c_b,
\]

and so Eq. (4) is equivalent to

\[
G^a_{b}(g) = R^a_{b}(g) - \frac{1}{2} R(g)\delta^a_{b} = \kappa T^\mu_{\nu} = \frac{1}{Q^2} (\kappa g^{ac}T_{cb} - V(\Phi, \chi)\delta^a_{b})
\]

where the potential, \( V(\Phi, \chi) \), is given by

\[
V(\Phi, \chi) = \frac{f_\Phi \Phi + 2f_\chi \chi - f(\Phi, \chi)}{2\kappa}
\]

When written in terms of \( g_{ab} \), the Palatini field equations, Eq. (7), are essentially those of General Relativity but with a modified source term given in terms of the natural energy momentum tensor, \( T^a_{ab} \), together with a potential term Eq. (8). This correspondence between Palatini theories and modified source theories is well known.

We could also use Eqs. (3) and (9) to rewrite the Palatini field equations entirely in terms of what we will refer to as the natural, or matter, metric \( \tilde{g}_{ab} \) (since it is the metric that naturally appears in the matter action). In this case the field equations for a general Palatini \( f(\mathcal{R}, R^a_{ab}) \) theory are fairly unwieldy and so we only present the form they take in the special case when \( f = f(\mathcal{R}) \). Defining \( F = f_\Phi \) we have:

\[
\tilde{G}_{ab} = \frac{1}{F} \kappa T_{ab} - \frac{1}{2} \tilde{g}_{ab} \left( \mathcal{R} - \frac{f}{F} \right) + \frac{1}{F} (\nabla_a \nabla_b - \tilde{g}_{ab} \Box) F - \frac{3}{2F^2} \left( \nabla_a F \nabla_b - \frac{1}{2} \tilde{g}_{ab} \nabla F \nabla F \right)
\]

where \( \tilde{G}_{ab} = \tilde{R}_{ab} - \frac{1}{2} \tilde{g}_{ab} \mathcal{R} \) is the Einstein tensor constructed from \( \tilde{g}_{ab} \) and \( \Box = \nabla^a \nabla_a \). Since this equation involves second order derivatives of \( F \) (or equivalently of \( \mathcal{R} \)), it is generally difficult to solve while it is often easier to solve the equations in terms of \( g_{ab} \). Additionally, the curvature of \( \tilde{g}_{ab} \) is often much larger, even over small scales, than one would have expected based on the behaviour of the metric in GR. Consider for instance the trace of Eq. (9):

\[
\tilde{R} = -\frac{\kappa T}{F} - 2 \left( \mathcal{R} - \frac{f}{F} \right) + \frac{3}{2F} F + \frac{3(\nabla F)^2}{2F^2}.
\]

Based on GR one might expect that if \( F = f_\Phi \approx 1 \), \( \tilde{R} \sim O(-\kappa T) \), however from Eq. (10) it is clear that unless \( F \), which depends algebraically on \( T \), is only varying very slowly, one may actually have \( \tilde{R} \gg -\kappa T \), and so that gravity, as described by curvature of \( \tilde{g}_{ab} \), is actually much stronger than one would naturally expect. The immediate upshot of this is that it may not be appropriate to take \( \tilde{g}_{ab} \approx \eta_{ab} \) over laboratory scales as one might normally expect to be possible. We discuss the important implications of this later in § [LVIII] and return to field equations in terms of the metric \( \bar{g}_{ab} \) in § [LVIII].

C. Behaviour in Vacuum

The behaviour of these theories in a vacuum is, as we shall see later, of great importance. In vacuum, \( g^{ac}T_{bc} = 0 \), or possibly \( g^{ac}T_{bc} = -\lambda_0 \delta^a_b \), \( \lambda_0 = \text{const} \). In either case Eq. (4) gives \( \mathbf{K} = \Phi_0 \mathbf{I}/4 \), and so \( \Phi = \Phi_0, \chi = \Phi_0^2/4 \). Where \( \Phi_0 \) is given by Eq. (6)

\[
f_\Phi \Phi_0 + \frac{1}{2} f_\chi \Phi_0^2 = -4\lambda_0 + 2f_0.
\]

where we defined \( f_0 = f(\Phi_0, \Phi_0^2/4) \), \( f_\Phi = f_\Phi(\Phi_0, \Phi_0^2/4) \) and \( f_\chi = f_\chi(\Phi_0, \Phi_0^2/4) \). Denoting \( Q_0 = Q(\Phi_0, \Phi_0^2/4) \) and \( V_0 = V(\Phi_0, \Phi_0^2/4) \), we have

\[
Q_0 = \frac{(2f_0 - 4\lambda_0)}{\Phi_0},
\]

\[
V_0 = \frac{f_0 - 4\lambda_0}{2},
\]

and so from Eq. (7)

\[
\kappa T^a_{b}(\tilde{g}^{ac}T_{bc} = -\lambda_0 \delta^a_{b}) = -\frac{\Phi_0^2}{8(2f_0 - 4\lambda_0)} \delta^a_{b} = -\Lambda_{\text{eff}}(\Phi_0) \delta^a_{b}
\]

It is very well known that the vacuum field equations of \( f(\mathcal{R}, R^a_{ab}) \) Palatini theories are equivalent to those of General Relativity with an effective cosmological constant \( \Lambda_{\text{eff}}(\Phi_0) \). One should appreciate however that, depending on the specific form of \( f, (1) \) there may be more than one value of \( \Phi_0 \) that satisfies Eq. (11) and so the vacuum may not be unique in these theories; (2) the value of \( \Lambda_{\text{eff}} \) could be either positive or negative.
III. WHY AVERAGING MATTERS?

It is important to stress that the equations we present in the above section are all *microscopic* field equations, which is to say that they are only certainly valid when one has taken into account all of the microscopic structures in the distribution of energy and momentum described by $T_{ab}$. *A priori* there is no reason to expect these equations to remain valid if $T_{ab}$ and $T$ are replaced by some coarse-grained average. In particular for non-relativistic baryonic matter, there is no reason to believe that we can coarsen grain over the peaks in density centered on each nuclei. In chameleon scalar field theories [34, 35] where there is a strong coupling between the scalar field and nuclei. In chameleon scalar field theories [34, 35] where there is a strong coupling between the scalar field and nuclei. In chameleon scalar field theories [34, 35] where there is a strong coupling between the scalar field and nuclei. In chameleon scalar field theories [34, 35] where there is a strong coupling between the scalar field and nuclei. In chameleon scalar field theories [34, 35] where there is a strong coupling between the scalar field and nuclei. In chameleon scalar field theories [34, 35] where there is a strong coupling between the scalar field and nuclei. In chameleon scalar field theories [34, 35] where there is a strong coupling between the scalar field and nuclei.

In standard General Relativity this problem does not emerge, because $R$ depends linearly on $\rho$, so the average value of $R$ could be calculated simply by replacing the microscopic value of $\rho$ with its average. This averaging only works, however, because of the linear dependence of $R$ on $\rho$. In the Palatini modified gravity theories however, as we have seen, $R_{ab}$ and $\mathcal{R}$ do not generally depend linearly on $T$ (and hence $\rho$). Furthermore because $f(\mathcal{R}, \mathcal{R}^{ab} R_{ab})$ is a nonlinear function of $\mathcal{R}$ and $\mathcal{R}^{ab} R_{ab}$, there is no reason to expect the averaged value of $f$ in a region of space to be equal to the $f$ of the averaged values of $\mathcal{R}$ and $\mathcal{R}^{ab} R_{ab}$; indeed this will generally *not* be the case. Even so, in almost all of the literature the microscopic field equations are simply assumed to apply on macroscopic scales. For instance, when the cosmology of these theories is discussed the microscopic field equations are solved with $T_{ab}$ replaced by its cosmological average. There is no reason, *a priori*, to expect an analysis conducted along these lines to be valid. In fact as we shall show below, when the averaging is performed properly, the actual coarse-grained behavior of the theory is very different from what one would find if the microscopic field equations were naïvely applied to macroscopic scales.

Now let us discuss in more detail how this naïve averaging could lead to incorrect results. Consider a body, or a region of space, that, microscopically, contains $N$ non-relativistic particles (e.g. nuclei) each with (microscopically) uniform density $\rho_c$ and each occupying a volume (or with an average volume) $V_p$. The body as a whole is taken to have volume $V_{tot}$. The space in between the particles is assumed to be filled with some diffuse (non-relativistic) substance with microscopic energy density $\rho_s$ (e.g. a diffuse electron cloud or, if $\rho_s = 0$, just empty space).

For clarity we take the $f(\mathcal{R})$ case as an example. By Eq. (11), inside the particle we have $\Phi = \Phi_c$ such that $$f_c(\Phi_c) \Phi_c - 2 f(\Phi_c) = \kappa T_c \approx -\kappa \rho_c$$ where we have assumed $\rho_c \approx \text{const.}$ and that the particles are non-relativistic. Similarly in the space between the particles we have instead $\Phi = \Phi_s$ where $$f_s(\Phi_s) \Phi_s - 2 f(\Phi_s) = \kappa T_s = -\kappa \rho_s.$$ It is now straightforward to work out the true volume averaged value of $\Phi, f(\Phi)$ and $f_s(\Phi)$. By volume averaging of a quantity $Q(x)$ we mean $$\langle Q \rangle = \frac{\int_{V_{tot}} d^3 x Q(x)}{V_{tot}}.$$ It is then clear that $$\langle \rho \rangle = \rho_s \left( 1 - \frac{NV_p}{V_{tot}} \right) + \rho_c \frac{NV_p}{V_{tot}} , \quad (14)$$ and similarly

$$\langle \Phi \rangle = \Phi_s + (\Phi_c - \Phi_s) \frac{NV_p}{V_{tot}} , \quad (15)$$

$$\langle f(\Phi) \rangle = f(\Phi_s) + [ f(\Phi_c) - f(\Phi_s) ] \frac{NV_p}{V_{tot}} , \quad (16)$$

$$\langle F(\Phi) \rangle = f_s(\Phi_s) + [ f_s(\Phi_c) - f_s(\Phi_s) ] \frac{NV_p}{V_{tot}} , \quad (17)$$

and so on. In general the averaged value of the quantity $Q(x)$ is

$$\langle Q \rangle = Q_s + (Q_c - Q_s) \frac{NV_p}{V_{tot}} \approx Q_s + (Q_c - Q_s) \frac{\langle \rho \rangle - \rho_s}{\rho_c - \rho_s} . \quad (18)$$

Generally the microscopic field equations give $Q$ as a function of $T \approx -\rho$, i.e., $Q(x) = Q(\rho(x)) = Q(\rho)$. If one simply replaced $\rho$ by $\langle \rho \rangle$ then one would think that $\langle Q \rangle = Q_s \equiv Q(\langle \rho \rangle)$. It is obvious however that unless $Q(\rho) = a_1 + a_2 \rho$ (where $a_{1,2}$ do not depend on the $T_{ab}$ components) $\langle Q \rangle \neq Q_s$.

Because in the Palatini $f(\mathcal{R})$ theories $f$ is usually a nonlinear function of $\mathcal{R}$, we conclude that, except for isolated values of $\mathcal{R}$ we have $\langle f(\mathcal{R}) \rangle \neq f(\langle \mathcal{R} \rangle)$. As such the effective macroscopic (i.e., coarse-grained) behavior of Palatini $f(\mathcal{R})$ theories will not, as is almost always assumed, be well described by the microscopic equations with $T_{ab}$ replaced by $\langle T_{ab} \rangle$. A generalization of the above argument to the Palatini $f(\mathcal{R}^{ab} R_{ab})$ case is straightforward if more complicated, due to the similar nonlinear and algebraic nature of the field equations. As a result the predictions made by applying the microscopic field equations in Palatini modified gravity theories to macroscopic (e.g., astrophysical and cosmological) settings cannot generally be trusted.

In the following sections we shall study the behaviors of classical particles and radiation fields in Palatini gravity theories when the averaging strategy Eq. (15) is used, and compare with the results obtained by naïve averaging.
IV. MOTION OF CLASSICAL PARTICLES

In this section we consider the motion of a number of (classical) microscopic particles in Palatini modified gravity theories. By a classical particle, we simply mean some localized distribution of energy and momentum, i.e., for particle \( I \), \( T_{ab} \neq 0 \) only inside some world tube \( W(I) \). We further require that outside that tube, gravity is weak, which means that gravity is, to a good approximation, Newtonian (or indeed post-Newtonian). For this to be the case, we must require that the typical separations between the particles are large compared to their sizes. We also assume that these separations are large enough so that the particles are effectively collisionless. In between the particles we have vacuum, i.e., \( T_{ab} = 0 \) and so particles only interact with each other gravitationally. In \( f(R) \) theories, we can relax this requirement to \( T = 0 \) and so allow for electromagnetic forces between the particles, however for the moment we shall not do this.

We begin by considering a very simple, and rather idealized version of this set-up, where all the particles are assumed to be spherically symmetric, and furthermore, gravity is assumed to be weak not only outside of the particle world tubes but also inside them. This analysis was first presented in [37] and is repeated here to serve as an illustration of the more general result we present later in this work.

A. Spherically symmetric particles

1. A single Particle

We begin by considering a single, isolated, static and spherically symmetric particle centered at \( r = 0 \) and has a radius \( r_p \). Outside the particle \( T^a_b = 0 \) and without loss of generality we work in coordinates where

\[
g_{ab} \frac{dr^a}{dx} \frac{dr^b}{dx} = ds^2 = -e^{A(r,t)} dt^2 + e^{B(r,t)} dr^2 + r^2 d\Omega^2.
\]

Because the particle is static, so \( T^0_0 = 0 \), and the spherical symmetry imposes the conditions \( T^\theta_\theta = T^\phi_\phi = p_t - p_A/2 \), say, and \( T^t_0 = T^\phi_0 = 0 \). We define \( T^r_r = p_A + p_t \), \( T^{00} = -\rho \). One could find the metric \( g_{ab} \) both inside and outside such a particle by directly solving Eq. (4), however, it is much simpler to find \( g_{ab} \) via Eq. (7), and then if necessary one can recover the results in terms of \( g_{ab} \) by some transformations, as we shall show in § IV E.

Note that Eq. (7) is nothing but the standard Einstein equation with a modified energy momentum tensor:

\[
\hat{T}^a_b = \frac{1}{Q^2} [T^a_b - V(\Phi, \chi) \delta^a_b],
\]

We can see from this expression that \( T^0_0 = 0 \) if and only if \( T^r_r = 0 \), and that all the symmetries of \( \hat{T}^a_b \) are preserved by \( \hat{T}^a_b \). The solution for the metric functions in this case is well known. Indeed if one works in terms of \( \hat{T}^a_b \) rather than \( T^a_b \), the equations that must be solved are just those of static, spherically General Relativity where outside the particle,

\[
\hat{T}^a_b = -\Lambda_{\text{eff}}(\Phi_0) \delta^a_b.
\]

Taking the following ansatz for the metric functions

\[
e^A = W(r)e^{2C(r)}, \quad e^B = 1/W(r),
\]

then the \( tt \) and \( rr \) components of the Einstein equations are

\[
\frac{1}{r^2} \frac{d}{dr} [r(1 - W)] = \frac{\kappa}{Q^2} [\rho + V(\Phi, \chi)],
\]

\[
\frac{2}{r} \frac{dC}{dr} = \frac{1}{Q^2 W^2} \kappa (\rho + p_t + p_A).
\]

The solutions are

\[
W(r) = 1 - \frac{2GM(r)}{r} - \frac{\Lambda_{\text{eff}}(\Phi_0)}{3},
\]

\[
C(r) = \frac{\kappa}{2} \int_0^r dr' r'^2 \frac{p_t + p_A}{Q^2(\Phi, \chi)} W(r') + C',
\]

\[
M(r) = 4\pi \int_0^r dr' r'^2 \frac{\rho + \Delta V(\Phi, \chi)}{Q^2(\Phi, \chi)},
\]

where \( C' \) is a constant of integration and

\[
\Delta V = V(\Phi, \chi) - Q_0^2 V_0/Q^2(\Phi, \chi).
\]

Outside the particle \( C \) is a constant given by

\[
C = \frac{\kappa}{2} \int_0^{r_p} dr' r'^2 \frac{\rho + p_t + p_A}{Q^2(\Phi, \chi)} W(r') + C',
\]

and for convenience we can set this constant to be zero by redefining the time coordinate.

\[
C' = -\frac{\kappa}{2} \int_0^{r_p} dr' r'^2 \frac{\rho + p_t + p_A}{Q^2(\Phi, \chi)} W(r')
\]

and so finally \( C \) can be written as

\[
C = -\frac{\kappa}{2} \int_r^{r_p} dr' r'^2 \frac{\rho + p_t + p_A}{Q^2(\Phi, \chi)} W(r')
\]

Additionally, outside the particle \( \rho = \Delta V = 0 \) and so \( M(r) = M_p = M(r_p) = \text{const.} \) The angular components of Eq. (7) provide us the analogue of the Tolman-Oppenheimer-Volkoff (TOV) equation:

\[
\frac{dP_{\text{eff}}}{dr} + \frac{p_t + p_A}{Q^2 r W(r)} Y(r) = -\frac{3p_A}{Q^2},
\]

where

\[
P_{\text{eff}} = (p_t - p_A - \Delta V)/Q^2
\]

and

\[
Y(r) = (4\pi G P_{\text{eff}} r^3 + GM - \Lambda_{\text{eff}} r^3)/3.
\]
This last equation implies that \( P_{\text{eff}} \) is \( C_0 \) continuous at \( r = r_p \), and so \( P_{\text{eff}}(r_p) = 0 \). The key thing to note is that, no matter what the internal structure of the particle is, and no matter what the details of the Palatini theory are, outside such a particle the metric is simply Schwarzschild-de-Sitter, just as it is in GR with cosmological constant \( \Lambda \). This is of course precisely what one should expect given that we have assumed spherical symmetry and the vacuum field equations are the same as GR with \( \Lambda = \Lambda_{\text{eff}} \). As expected as it might be, this simple observations carries with it an important corollary which must be measured gravitationally by an external observer, and only \( \Lambda_{\text{eff}} \), since on scales that are very much smaller than the cosmological horizon, it is known to have negligible effect on particle motions.

To \( \mathcal{O}(\epsilon) \) the \( 0_0 \) component of Eq. (7) becomes:

\[
\frac{1}{2} h_{00,ii} = \kappa \left( T^0_0 - T^i_i + 2\Delta V \right) / 2Q^2,
\]

where, inside the \( K^{th} \) particle we define \( T^0_0 = -\rho^{(K)} \) and to leading order

\[
T^i_j = \left( p^{(K)}_l - \frac{1}{2} p^{(K)}_{ii} \right) \delta^i_j + \frac{3}{2} \kappa p^{(K)}_{ij} y^{(K)}_j y^{(K)}_i / r^2_{(K)}.
\]

Here \( y^{(K)}_j = x^i - x^{(K)}_j(t) \) and \( r^{(K)} = \sqrt{y^{(K)}_j y^{(K)}_i} \) is the radial coordinate centered at \( x^{(K)}_j(t) \). Defining \( 2S = h_{00} \) we then have:

\[
S_{ii} = -\frac{4\pi G}{Q^2} [\rho + 3p_l - 2\Delta V(\Phi, \chi)].
\]

Both inside the \( K^{th} \) and in the vacuum between the particles, Eq. (25) has the solution:

\[
S(x^i, t) = \frac{G M^{(K)}(x - x^{(K)}(t))}{|x - x^{(K)}|} - C^{(K)}(x - x^{(K)}(t)) + U^{(K)}(x^i, t),
\]

where

\[
U^{(K)}(x^i, t) = \sum_{j \neq K} \frac{G M^{(j)}}{|x - x^{(j)}(t)|},
\]

with \( C^{(K)}(r^{(K)}) \) and \( M^{(K)}(r^{(K)}) \) are given by Eqs. (28) and (22) with \( r \to r^{(K)} \), \( \rho \to \rho^{(K)} \), \( p_l \to p^{(K)}_l \) and \( p_A \to p^{(K)}_A \). \( M^{(j)}(t) \) is the total mass of the \( j^{th} \) particle and is also given by Eq. (22) with the appropriate substitutions in the limit \( r \to \infty \).

In a similar manner, from \( i' \) component of Eq. (7) we have, \( h_{ij} = 2\Psi \delta_{ij} \) where to \( \mathcal{O}(\epsilon) \) \( \Psi_{kk} = \kappa T^0_0/2 \):

\[
\Psi = U + \frac{G M^{(K)}(r^{(K)})}{r^{(K)}} - D(r^{(K)}),
\]

where for a particle with radius \( r_p \)

\[
D(r) = \frac{\kappa}{2} \int_r^{r_p} dr' r' \left( \frac{\rho + \Delta V}{Q^2} \right),
\]

so outside the particles \( D = 0 \). Finally for \( j_0 \) we fix the gauge so that:

\[
h_{0j,j} = 4\Psi_{0}.
\]
and then the Einstein equations give
\[ \rho_{0i, kk} = -2\kappa T_{0i} \]
We can now write down the metric outside all but the \( K^{th} \) particle:
\[
g_{ab} dx^a dx^b = -\left[ 1 - \frac{2GM(r(K))}{r(K)} + 2C(r(K)) - 2U(x^i, t) \right] dt^2 + \sum \frac{GM(J)/r(J)}{r(\mathcal{K})} J_{ab} \]
Because outside the particles we have \( C = D = 0 \), so the metric is given simply as
\[
g_{ab} dx^a dx^b = - (1 - 2\Psi_N) dt^2 + (1 + 2\Psi_N) dx^2, \quad (29)\]
where
\[
\Psi_N = U(K) + GM(K)/r(K) = \sum J_{ab} \]
3. Motion of Particles

It is clear from Eq. (7) that if we define
\[
\kappa T^a_b = \kappa \tilde{T}^a_b + \lambda_{\text{eff}} \delta^a_b
\]
we have
\[
\nabla_a \tilde{T}^{ab} = 0, \quad (30)
\]
in which \( \Gamma^{ab} = \sqrt{-g} \nabla_a \tilde{T}^{ab} \),
\[
t^a = \frac{1}{2\kappa} \left[ \mathbf{L} \delta^a_b + 2\sqrt{-g} \delta^a_b \lambda_{\text{eff}} - g^c_{ab} \frac{\partial \mathbf{L}}{\partial g^{cd}} \right]
\]
and
\[
\mathbf{g}^{ab} = -1 + 4\Psi, \quad \mathbf{g}^{ij} = \delta^{ij},
\]
and to leading order \( \mathbf{g}^{0i} = -\rho_{0i} \), so
\[
J^0_{(K)} = \frac{2}{\kappa} \int_{\Sigma_K} d^2 \Sigma_i \Psi_i.
\]
Similarly we find after some manipulation
\[
J^i_{(K)} = -\int_{\mathcal{V}_{(K)}} d^3 x \partial_i \tilde{T}^0_{0i} x^i
\]
and since \( \tilde{T}^0_{0i} \) depends on \( x^i \) and \( t \) only in the combination \( r = |x^i - \tilde{x}_i(K)| \) at leading order we have
\[
J^0_{(K)} = \int_{\mathcal{V}_{(K)}} d^3 x \tilde{T}^0_{0i} \tilde{x}^i_{(K)} x^i
\]
and
\[
\tilde{J}^0_{(K)} = \tilde{x}^i_{(K)} M_{(K)}
\]
by a constant conformal particles, the geodesics of the two metrics are the same metric. Thus outside the body, in general, \( \bar{\gamma} \frac{\partial}{\partial x^i} \), where \( \bar{\gamma} \) is the metric that appears in the matter equation of motion. Since \( \bar{\gamma}_{ab} \) is the metric that appears in the matter equation of motion, it is clear that small particles will move along geodesics in \( \bar{\gamma}_{ab} \). Now Eq. (37) tells us that small particles will also move along geodesics in \( \bar{\gamma}_{ab} \). Inside the body, the gravitational mass, \( \bar{\gamma}_{ab} \) and \( \bar{\gamma}_{ab} \) are only related disformally, and hence will have different geodesics, this result may seem rather counter-intuitive. It can, however, be understood in a fairly simple fashion. Although the two metrics are in general related disformally, we noted in Section II C, that in vacuum regions

\[
\mathbf{t}^{ij} = \frac{2}{\kappa} \left( \Psi_{;i} \Psi_{;j} - \frac{1}{2} \delta_{ij} \Psi_{;k} \Psi_{;k} \right),
\]

the \( i \) components gives

\[
\frac{d}{dt} J^i_{(K)} = M_{(K)} \frac{d^2 x^i_{(K)}(t)}{dt^2} = -\frac{2}{\kappa} \int_{V_{(K)}} d^3x \Psi_{;kk} \Psi_{;i},
\]

where the last line follows from expanding \( U_{,i} \) about \( x^i = x^i_{(K)} \), \( U_{,kk} = 0 \) inside the body, and \( \bar{T}^{0}_{0} \) being a function of \( r_{(K)} \) only at leading order. Thus to leading order in \( \epsilon \) we have

\[
\frac{d^2 x^i_{(K)}(t)}{dt^2} = U_{,i}(x^i_{(K)}),
\]

which is precisely the Newtonian equation of motion, and precisely the same result that we find in General Relativity. Since \( \bar{\gamma}_{ab} \) is the metric that appears in the matter action, it is clear that small particles will move along geodesics in \( \bar{\gamma}_{ab} \). Now Eq. (37) tells us that small particles will also move along geodesics in \( \bar{\gamma}_{ab} \). Since inside the body, the gravitational mass, \( \bar{\gamma}_{ab} \) and \( \bar{\gamma}_{ab} \) are only related disformally, and hence will have different geodesics, this result may seem rather counter-intuitive. It can, however, be understood in a fairly simple fashion. Although the two metrics are in general related disformally, we noted in Section II C that in vacuum the two metrics are related by a constant conformal factor, \( i.e. \), up to a rescaling of coordinates they are the same metric. Thus outside the particles, the geodesics of the two metrics are the same. Inside a particle then the only differences between the geodesics of the two metrics are due to essentially local effects, \( i.e. \), they are due only to the local matter content. Such local differences cannot lead to the particle developing an overall acceleration, as this would constitute a self-acceleration and hence a violation of energy and momentum conservation. Both the natural, \( \bar{T}^{0}_{0} \), and the effective, \( \bar{T}^{a}_{b} \), are conserved (with respect to \( \bar{\gamma}_{ab} \) and \( \bar{\gamma}_{ab} \) respectively). They also both vanish outside of the particle implying that there is no flux of energy or momentum in or out of the particle. Thus the total energy and momentum inside the particle must be conserved, and there can be no self-accelerations. It should therefore come as no surprise that particles move along geodesics in \( \bar{\gamma}_{ab} \).

B. Generalization

We have seen, as we first reported in [37], that the motion of spherically symmetric, classical particles in a general Palatini \( f(R, R^{ab} R_{ab}) \) theory is observationally indistinguishable from the motion of the same set of particles in pure General Relativity with a cosmological constant, \( \Lambda_{eff} \). The reason for this is remarkably simple, albeit unappreciated up to this point, and follows from a couple of very well known phenomenon, both illustrated above. The first is that the field equations of all Palatini theories are equivalent with those of GR with some modified source \( T^{a}_{b} \), when this is just an effective cosmological constant. The second, is that, quite remarkably, GR is in many ways holographic and specifically the equations of motions for a localised distribution of energy and momentum surrounded by vacuum can be derived by considering surface, rather than volume, integrals over curvature components [38, 39]. Indeed in General Relativity, even when a modified source is present, the gravitational mass of an isolated particle, as well as higher mass moments, can all be defined in terms of surface integrals outside the body and hence identified with parameters in the general vacuum solution [39].

In Ref. [39], via a computational tour-de-force, the Post-2-Newtonian equations of motion for a set of \( N \) classical particles (with arbitrary internal structure and no assumed symmetry) where calculated using surface integrals. The motion of the particles was found to depend entirely on quantities defined outside the particles which are naturally interpreted as the different mass moments of the particles. Indeed the spherically symmetric particle analysis given above follows from a spherical case of the calculation done in Ref. [39], and so the results of Ref. [39] imply that all our conclusions apply equally well to particles with no symmetry.

On the largest scales, the higher mass moments play a relativity insignificant rôle and the motion of a set of non-relativistic classical particles is determined, to a good approximation, entirely by their initial positions, velocities and their gravitational masses. As should come of no surprise then, one does not need to know the internal structure of a particle to know how it moves, one does not, for instance, need to know the precise molecular or atomic structure of a clump of particle to predict its motion under gravitational and other external forces. This is true for any theory that is equivalent with General Relativity with a modified source provided that a vacuum form \( i.e. \), \( \propto \delta^{a}_{b} \), in the natural energy momentum tensor, \( T^{a}_{b} \), corresponds to a vacuum form in the effective one, \( \bar{T}^{a}_{b} \). In all these theories then the modification of the source term does not ultimately matter as classical particles \( i.e. \), clumps of matter surrounded by vacuum, still move as particles do in General Relativity. Particle motions are, in so far as particle motions under gravity and other external forces go, observationally indistinguishable.
C. The Physical Mass of a Classical Particle

In this section we consider the meaning of the physical mass of a classical particle. In principle, there can be a number of different quantities associated with a particle that will, in different situations, play the role of its mass. Firstly, we have the active gravitational mass which is a measure of the strength of the gravitational field induced by a body. Secondly, there is the passive gravitational mass, which determines the force that a body feels within a given gravitational field. Lastly, there is the inertial mass of a body, which determines how quickly a particle’s momentum changes when a force is applied. In General Relativity the latter two are manifestly equal. Furthermore, any theory in which the passive gravitational and inertial masses are equal is said to satisfy the weak equivalence principle, and in the absence of any non-gravitational external forces the trajectory of particles will depend only on their position and velocity and not on their composition. Additionally in the Newtonian limit of General Relativity, the two types of gravitational mass are also equal \([1]\), and any violations of this equality are tightly constrained by experiments \([1]\).

We saw above the Newtonian limit of the equation of motion for a classical, non-relativistic, particle in both General Relativity and Palatini theories is the same. Our analysis was for spherically symmetric particles although the results of Ref. \([30]\) readily extend this to all classical particles.

In general if one defines the center of mass, \(x_{\text{cm}}^i\), of the classical particle, inside which gravity is weak, with effective energy momentum tensor \(T^a_b\) in the standard way, we have:

\[
x_{\text{cm}}^i = \frac{M_p^i}{M_p},
\]

(38)

where

\[
M_p^i = - \int_{V_p} d^3x' \sqrt{-g} \tilde{T}_{00} x'^i,
\]

(39)

\[
M_p = - \int_{V_p} d^3x' \sqrt{-g} \tilde{T}_{00}.
\]

(40)

where \(V_p\) is the volume occupied by the particle. Then

\[
J^a = M_p \frac{dx_{\text{cm}}^a}{dt},
\]

where \(x_{\text{cm}}^a = (t, x_{\text{cm}}^i)\). In the absence of non-gravitational external forces, i.e., \(\nabla_a T^{ab}_0 = 0\), we have \(M_p = 0\) and:

\[
\frac{dJ^i}{dt} = M_p \frac{d^2x_{\text{cm}}^i}{dt^2} = M_p U_{,i}
\]

(43)

where \(U_{,i}\) is the Newtonian potential due to other particles:

\[
U = \sum_I \frac{GM_I(r(t))}{r(t)},
\]

(41)

where \(r(t)\) is the distance from \(x_{\text{cm}}^i\) to the center of mass of the \(I\)th, and \(M_{(I)}\) is that particle’s mass. It is clear from this relation that \(M_p\) and the \(M_{(I)}\) are the active gravitational mass of their respective particles. It is also clear that

\[
\frac{d^2x_{\text{cm}}^i}{dt^2} = U_{,i}.
\]

(42)

Thus the trajectory of a classical particle is independent of its mass and hence its composition, and hence two particles at the same point would feel the same acceleration. If the difference in the acceleration of two bodies at the same point vanishes in one frame with one choice of coordinates then it obviously must vanish in all frames. Furthermore it is independent of which of the metrics, \(g_{ab}\) or \(\tilde{g}_{ab}\), one is working with when performing the calculation \([1]\). The absence of any differential acceleration is precisely the statement of the weak equivalence principle which in turn is equivalent to saying that the passive gravitational mass and inertial mass of any body are equal.

We should not be surprised by this. The weak equivalence principle certainly holds in GR, irrespective of the composition or internal structure of the particles one considers, and so it must also hold in any theory which is equivalent to GR up to a modified source and in which a vacuum form for the natural energy momentum tensor, \(\tilde{T}^a_b\), corresponds to a vacuum form for the modified source term, \(\tilde{T}^a_b\). Since, in so far as particle motions go, the only difference between such a theory and GR are in the internal composition of the particles.

If non-gravitational forces are present and associated with an effective energy momentum tensor \(T_j^{ab}\), then generally \(\nabla_a T^{ab} = -\nabla_a T_{a}^{bb} = f^b\). Assuming that the gravitational field is still dominated by the matter in the particles, we find by repeating the analysis of Section \([1]\) that

\[
\frac{dJ^i}{dt} = M_p \frac{d^2x^i}{dt^2} = M_p U_{,i} + \int_{V_p} \sqrt{-g} f^i
\]

(43)

Thus \(f^i\) is then identified as the total, non-gravitational, external force on the system. It is clear from Eq. \([43]\) that \(M_p\) is, in addition to being the active gravitational

\[\text{[1] In fact one could apply } \nabla^a T_{ab} = 0 \text{ to a classical particle which is in static configuration, and show that the TOV equation is equivalent to the statement that the particle has no self-acceleration. If one forgets the internal structure (pressure gradients etc.) of the particle, then the gradient of } \Phi \text{ inside the particle would not be balanced and would appear in the geodesic equation of the particle. As } \Phi \text{ depends on the local energy density of the particle, this would lead to the conclusion that the particle feels a self-force that depends on its energy density or materials and so WEP is violated. However, with TOV equation taken into account, one can find that the pressure gradients and } \Phi \text{ gradients cancel, leaving no self-force on the particle.} \]
mass, the inertial mass of the particle. Thus we have that inertial and both types of gravitational mass are equal to $M_p$ in Palatini theories. In all the usual senses, i.e., gravitational and inertial, then $M_p$ is the physical mass of the particle. However $M_p$ is not, in general, equal to the particle mass in the matter action, $S_m(\Psi^i, \bar{g}_{ab})$. If it where then $M_p$ would depend only on $T^a_b$, but $M_p$ is given by Eq. (22), and this manifestly depends not only on $T^a_b$ but also on $V(\Phi, \chi)$ and $Q(\Phi, \chi)$.

D. The Physical Energy Momentum Tensor and Metric

In the study of Palatini theories, it is common practice to refer to $T^a_b$ as the ‘physical’ energy momentum tensor, and $\bar{g}_{ab}$ as the ‘physical’ metric. This is purely a convention inherited from GR: because if one selects a local inertial frame defined at some point with respect to $g_{ab}$, then in a region around this point physics is well-described by the special relativistic limit of the matter action, i.e., $S_m(\Psi^i, \eta_{ab})$. The length of this region is essentially equivalent to the length scale over which the spacetime appears to be flat.

In a general scalar tensor theory, it is usually the case that although gravity is modified, its strength is still roughly the same and hence the length scale of the curvature of spacetime is no smaller than it usually is. Assuming that $\bar{g}_{ab} \approx \eta_{ab}$ in a laboratory is therefore no more or less valid that it is in General Relativity. Under these circumstances attaching the label ‘physical’ to $T^a_b$ and $\bar{g}_{ab}$ seems reasonable. In Palatini theories (and as $\omega \to -3/2$ in Brans-Dicke theories), however, things are different. Here, if we treat $\bar{g}_{ab}$ as the physical metric, then gravity is much stronger over very small scales. Indeed one could view these theories as containing an additional component to the gravitational force which is infinitely strong but has zero range. The upshot of this is that the curvature of $\bar{g}_{ab}$ is much larger than one might naively expect, and so the length scale over which one can treat $\bar{g}_{ab} \approx \eta_{ab}$ is much smaller. Indeed, depending on the composition of the particles considered, it may even be smaller than the spatial extent of the particles themselves.

The presence of a new strong component to the gravitational force means that even over laboratory scales we can not be sure that physics will be well described by $S_m(\Psi^i, \eta_{ab})$. Attaching the label ‘physical’ to $T^a_b$ and $\bar{g}_{ab}$ is therefore misleading. Indeed we saw above that in the Newtonian limit, the physical mass of a particle was not given by a volume integral over $\rho = -T^{00}_{\Sigma}$ but by a volume integral over $-\hat{T}^{a}_b$.

It is, arguably, more straightforward to treat $g_{ab}$ and $T^a_b$ as being ‘physical’. The gravitational side of the theory is then simple General Relativity as evidenced by Eq. (7).

In this case we could make the definition $\hat{S}_m(\bar{\Psi}^i, \bar{g}_{ab}) = S_m(\Psi^i, \bar{g}_{ab})$, where $\bar{\Psi}^i$ are some redefinitions of the original matter fields. Since now the (new) matter Lagrangian depends on the metric $\bar{g}_{ab}$ whose Einstein tensor behaves exactly like in GR [cf. Eq. (4)], and on laboratory scales we are justified in taking $g_{ab} \approx \eta_{ab}$, so on these scales special relativity applies as in GR and gravity plays negligible role in microscopic physics. However, due to the redefinition of matter fields, the microscopic physics (e.g., field theoretic) itself is now described by some modified action $\hat{S}_m$, which will generally include new interactions between fundamental particles.

Throughout we have endeavored not to ascribe the label ‘physical’ to either metric or energy-momentum tensor, although in Palatini theories one should be aware that in the frame of a laboratory here on Earth it is $g_{ab}$ and $T^a_b$, rather than $\bar{g}_{ab}$ and $\hat{T}^a_b$ which behave as we would expect given our initiation for how things work in General Relativity. Ultimately, all truly measurable, and hence physical, quantities should be independent of which names one gives to which metrics or which frame one works in, and so the names which one gives to the metrics and energy momentum tensors should only be seen as a guide to intuition rather than having any deeper meaning.

E. Coarse-graining the Energy Momentum tensor of Particles

We now consider the coarse-grained form of the effective energy momentum tensor, $\hat{T}^{a}_b$ when microscopically matter is clumped into classical particles surrounded by a vacuum. Given that we have seen that particle motions in GR and Palatini theories are the same, we should expect the coarse-grained effective energy momentum to have the form of the energy momentum tensor for collisionless dust, as it would in General Relativity. We show this explicitly below. It should be stressed that the simple calculation present below is not new in the context of general relativity [50], however its consequences for modified source theories have not been appreciated so far.

Defining $\hat{T}^{ab} = \sqrt{-\hat{g}} \hat{T}^{ab} + \Lambda_{cd}^{ab}$ we have

$$\nabla_a \hat{T}^{ab} = 0 \Leftrightarrow \partial_i \hat{T}^{0b} = -\partial_0 \hat{T}^{0b} - \Gamma^{b}_{cd} \hat{T}^{cd}. \quad (44)$$

We now average $\hat{T}^{ab}$ over a region with fixed volume $V$ and surface $\Sigma$. $\Sigma$ is chosen to lie in the vacuum region between the particles such that on $\Sigma$ we have $T^{a}_b = 0$. After some algebra we find

$$\int d^3 x \hat{T}^{ij} = \frac{1}{2} \frac{d^2}{d\ell^2} \int d^3 x x^i x^j \hat{T}^{00}$$

$$+ \frac{1}{2} \int d^3 x \left[ x^i x^j (\Gamma^{0}_{cd} \hat{T}^{cd})_0 + 2 x^i (\Gamma^{0}_{ij} \hat{T}^{cd}) \right]. \quad (45)$$

For non-relativistic particles, in a weak gravitational field we have $\Gamma^{a}_{bc} \sim O(\epsilon)$ and each time derivative introduces a factor of $\epsilon^{1/2}$ so

$$\int d^3 x \sqrt{-g} \hat{F}^{ij} = -\Lambda_{cd} V \delta^{ij} + O(\epsilon) \int d^3 x \hat{T}^{00}. \quad (46)$$
Indeed, using the metric found in Section IV and assuming that the internal structure of the particles is in equilibrium, Eq. 15 gives to \( \mathcal{O}(\epsilon) \) for particles with masses \( M_i \) and centers of mass \( x_i(t) \):

\[
\int d^3x T^{ij} = \sum_{(K)} \bar{x}_i^{(K)} \bar{x}_j^{(K)} M_i(K). \tag{47}
\]

Where the sum is over all of the particles inside the volume \( V \). Similarly, one can show that

\[
\int d^3x T^{0i} = \sum_{(K)} \bar{x}_i^{(K)} M_i(K). \tag{48}
\]

This is precisely the same as what one finds in general relativity, which is unsurprising given that, up to a modified source, the two theories are equivalent, and we have shown that the modified source does not affect the motion of classical particles. It is straightforward to show that the same is true in a cosmological background provided the peculiar velocities of the particles are also small. Thus it is clear that the coarse-grained energy momentum tensor, \( T^{ab} \), is that of collisionless dust with a cosmological constant \( \Lambda \).

It is not difficult to find out the relations between these two sets of coordinates, the main observation being that Eqs. 51, 52 should be equivalent to each other. Because there is no dependence on the angular coordinates, we could set, from the comparison between Eqs. 51, 52, \( d\Omega^2 = d\bar{\Omega}^2 \) and thus

\[
\bar{r}^2 = \frac{r^2}{F}. \tag{53}
\]

Similarly we set \( dt = \bar{dt} \). Then we just need to use

\[
\frac{1}{F} e^{A(r)} = e^{\bar{A}(r)}, \quad \frac{1}{F} e^{B(r)} dr^2 = e^{B(r)} d\bar{r}^2 \tag{54}
\]

and Eq. 53 to find out the relations between \( A, \bar{A} \) and \( B, \bar{B} \) in this way calculate explicitly the matter metric in Eq. 52.

To do this, note that \( F \) is a function of \( T \) and thus of \( r \). Equivalently it could also be expressed as a function of \( \bar{r} \). This is because there is a relationship between \( r \) and \( \bar{r} \) by Eq. 53, i.e., we could write \( \bar{r}(r) \) or \( r(\bar{r}) \). So once the form of \( f(R) \) is known and \( F(r) \) solved as in Sec. IV A 1, Eq. 53 can be used to find out \( r(\bar{r}) \) and \( \bar{r}(r) \), then \( F(\bar{r}) = F[r(\bar{r})] \) could be calculated, at least numerically.

In what follows we shall use a prime (star) to denote the derivative with respect to \( r(\bar{r}) \), i.e., \( F' = \frac{dF(r)}{dr} \) and \( F^* = \frac{dF(\bar{r})}{d\bar{r}} \). Then Eq. 53 can be written as

\[ r = \sqrt{F(\bar{r})} \Rightarrow dr = \sqrt{F(\bar{r})} (1 + \gamma) d\bar{r} \]

where \( \gamma = \bar{r} F^*/2F \), and from Eq. 54 we finally obtain

\[
\text{exp} \left[ \frac{A(\bar{r})}{F(\bar{r})} \right] = \frac{1}{F(\bar{r})} \exp \left[ A \left( \sqrt{F(\bar{r})} \right) \right] \tag{55}
\]

\[
\text{exp} \left[ \frac{B(\bar{r})}{F(\bar{r})} \right] = \exp \left[ B \left( \sqrt{F(\bar{r})} \right) \right] (1 + \gamma)^2. \tag{56}
\]

There are several points to be noted about these results:

1. We manage to calculate the metric Eq. 52 without explicitly solving the complicated modified Einstein equation Eq. 19. This method, when generalized appropriately, should be very useful when one deals with the \( f(R^{ab} R_{ab}) \) gravity theories, in which case, as we discussed in Sec. III the modified Einstein equation calculated with the physical metric \( \bar{g}_{ab} \) should be very complicated.
2. Because \( F^* \) is involved in Eq. (55), \( \bar{B} \) and thus the metric \( \bar{g}_{\mu\nu} \) could be discontinuous even though \( g_{\mu\nu} \) is continuous. This discontinuity happens when there is a sudden change of energy density distribution, for example from \( \rho = \rho_0 = \text{const.} \) inside the particle to \( \rho = 0 \) outside it. Other cases when there will be singularity can be found in [31]. Furthermore, if the size of a particle with some fixed mass is tiny, then \( \gamma \) could be very large, making the metric \( \bar{g}_{ab} \) deviate from \( g_{ab} \) significantly. 

3. Outside the particle where \( F = \text{const.} \), for some choices of \( f(R) \), e.g. \( f(R) = R^\alpha \) with \( \alpha \geq 1, R = 0 \) is a vacuum solution to Eq. (55) and so outside the particle \( F = 1, \gamma = 0 \). In this case obviously \( \bar{r} = r \) and \( \bar{A} = A, \bar{B} = B \) outside the particle so that the spacetime there is exactly Schwarzschild. In other cases \( F \neq 1, \gamma \neq 0 \), the outside spacetime is Schwarzschild-de-Sitter. Consequently the claim that the spacetime outside a particle is Schwarzschild-de-Sitter also holds after changing to the matter metric \( \bar{g}_{ab} \). Indeed the claim in Sec. [IV.C] that for \( f(R, R^a R^b) \) gravity theories the active gravitational mass of a particle is equal to its inertial mass is correct no matter we use \( \bar{g}_{ab} \) or \( g_{ab} \), which is as expected because these two are ultimately equivalent.

As a further example, we could use the above method to obtain explicitly the equations that \( \bar{A}(\bar{r}) \), \( \bar{B}(\bar{r}) \) must satisfy. From the second equation in Eq. (55) we have

\[
\bar{B}(\bar{r}) = B(r) + 2 \log(1 + \gamma)
\]

the derivative of which with respect to \( \bar{r} \) gives

\[
\bar{B}^* = B' \frac{dr}{d\bar{r}} + \frac{2}{1 + \gamma} \gamma^* = -e^B \frac{d}{dr} e^{-B} \sqrt{F(\bar{r})} (1 + \gamma) + \frac{F^*}{F} \frac{\bar{F}^*}{\bar{F}} - \frac{\bar{F}^*}{\bar{F}} \frac{d}{d\bar{r}} \left( \frac{F^*}{\bar{F}} \right)^2 + \frac{1}{1 + \gamma} \left[ \frac{\bar{B}}{\bar{F}} (\kappa \rho + V) \bar{r} + 1 - e^{\bar{B}} \right] + \frac{1}{1 + \gamma} \left[ \bar{r} F^* + 2 F^* \frac{d}{d\bar{r}} \left( \frac{F^*}{\bar{F}} \right)^2 \right]
\]  

where in the second step we have used the relations

\[
\gamma = \bar{r} F^*/2F, \quad dr = \sqrt{F(\bar{r})} (1 + \gamma) d\bar{r}
\]

and

\[
B' = -e^B \frac{d}{dr} e^{-B},
\]

and in the third step we have used \( e^B = \frac{1}{W} \) and Eq. (20). Similarly from the first of Eq. (55) we have

\[
\bar{A}(\bar{r}) = A(r) - \log F
\]

G. Discussion and Summary

Having directly considered both the motion of classical particles and the coarse-graining of the energy momentum tensor for matter consisting of classical particles, we are now ready to discuss how the coarse-grained averaging in the Palatini modified gravity theories leads to effects which are distinct from what have been claimed using the naive averaging.

On the microscopic level, the matter in our Universe, whether it is dark or baryonic (radiation will be discussed in the next section), is made up of small particles. Consequently our previous analysis of the motion of particles in Palatini modified gravity theories is directly applicable to this setting. As we have seen, the effect of the Palatini modification to General Relativity is a change of the internal configuration and structure of the particle, while outside the particle the spacetime is Schwarzschild-de-Sitter with a cosmological constant \( \Lambda_{\text{eff}} \) determined by the model itself. The motion of these particles in Palatini theories follows the geodesics which is precisely the same as in general relativity with \( \Lambda = \Lambda_{\text{eff}} \). There are no new extra dynamical degrees of freedom in the Palatini theories, which in turn means that there is no new long-range forces. Any new effective force must be non-dynamical and act only at points; so it is entirely local and cannot be felt inter-particles. The closest analogue to this in particle physics would be Fermi’s original proposal for a theory of the weak force.

We could also see this from the viewpoint of averaging. We showed above in Section [IV.E] that if \( T^a_b \) describes matter clumped into classical particles in some
vacuum then, when properly coarse-grained over scales much larger than the inter-particle separation, the modified source term, $\bar{T}^{ab}$, has the form of the energy momentum tensor for collisionless dust plus some effective cosmological constant $\Lambda_{\text{eff}}$. This is entirely as one would expect given our analysis of particle motions in Palatini theories. For $N$ particles each with physical mass $m_p$ in a volume $V_{\text{tot}}$, the effective energy density of the dust is $\rho_{\text{eff}}^{(ab)} = N m_p / V_{\text{tot}}$.

Now we compare our above conclusion with that one might make when using the naïve averaging by considering some specific situations (for simplicity we take $f(\mathcal{R})$ as an example, the case of $f(\mathcal{R}^{ab} R_{ab})$ is similar).

1. Consider the cosmological setting. In the literature it has been extensively claimed that if $f(\mathcal{R})$ is chosen so that the deviation from $f(\mathcal{R}) = \mathcal{R}$ grows at small values of $\mathcal{R}$, e.g., $f(\mathcal{R}) = \mathcal{R} - \frac{\alpha}{\mathcal{R}}$, then the model could lead to a phase of accelerating expansion of the Universe at late times which is different from that predicted by general relativity plus a cosmological constant. There is nothing wrong with this if we can model the matter distribution as smooth even at the smallest scales. Since in this case both $F(\Phi)$ and $V(\Phi)$ vary as $\Phi$ varies with time so that from Eq. (7) we could expect the evolution to be different from $\Lambda$CDM. However, as we have emphasized several times, it is more realistic to model the matter as made up of small particles on microscopic scales and only becoming a fluid on large scales after coarse-grained averaging. What happens after this averaging? Take $F$ for example, according to Eq. (18) we have

$$\langle F \rangle = F_0 + [F(\rho_0) - F_0] \frac{NV_p}{V_{\text{tot}}}$$

where $F_0 \equiv F(\rho_0) = F(0)$. As $V_p \ll V_{\text{tot}}$ it can be shown that $[F(\rho_0) - F_0] \frac{NV_p}{V_{\text{tot}}} \ll F_0$ because $F$ does not depend on $\rho$ linearly $[F(\rho_0) \sim F_0 \sim O(1)]$, so essentially we have $\langle F \rangle = F_0$. Similarly $\langle V \rangle = V_0$. Thus according to Eq. (7) the model is indistinguishable from $\Lambda$CDM.

2. In astrophysical environments such as the Solar System, the matter density is so low that $NV_p \ll V_{\text{tot}}$. Again our analysis for the cosmological setting applies here, namely the Palatini theories behave indistinguishably from General Relativity plus a cosmological constant after coarse-grained averaging. This in particular means that the Parametrized Post-Newtonian (PPN) parameters we measure should be the same as those in General Relativity. Note however that in some exceptionally high-density astrophysical systems, such as neutron stars, we have $NV_p \sim V_{\text{tot}}$ and so the naïve averaging may give a reasonably good description; in this case one could expect the model predictions of Palatini theories and general relativity to be different.

3. As a last example, consider the matter metric $\bar{g}_{ab}$ in Eq. (52), Eq. (55) tells us that the metric function $A$ depends on the local value of $F$. Now suppose we use the naïve averaging. Since the function $A$ determines the results of the Rebka-Pound experiment, we can do two such experiments, one in the normal atmosphere and the other in a vacuum chamber. Obviously in these two experiments the values of $F$ are very different, i.e., $F(\rho_{\text{atm}}) \neq F_0$ where $\rho_{\text{atm}}$ is the energy density of the earth atmosphere and so one may expect that such an experiment could distinguish between Palatini theory and general relativity. However, according to our above analysis, after coarse-grained averaging the relevant value of $F$ in the atmosphere is nothing but $F_0$ because $NV_p \ll V_{\text{tot}}$. This means that the above thinking experiments will not work.

So, in conclusion, at classical level the motion of particles, cosmology and astrophysics in Palatini modified gravity theories are indistinguishable from the results of general relativity plus a cosmological constant. However, it must be emphasized that these are not equivalent theories. As we mentioned above, the internal structure of a particle in Palatini theories is generally different from that in general relativity (though we cannot measure the differences in masses classically). This point will become relevant when we consider the structures of atoms and neutron stars in these two types of gravitational theories (see § VI). Furthermore, in Palatini theories the evolution of a region of space where the natural microscopic energy momentum tensor, $T^{ab}$, truly was that of a continuously-distributed pressureless dust would be different from that of a region with the same coarse-grained density but where the matter was microscopically contained in small particles.

V. THE ELECTROMAGNETIC FIELD

In the previous section we have considered the dynamics of classical particles in Palatini theories. In this section we investigate how radiation, especially that due to the electromagnetic (EM) field (i.e., photons), behaves in these theories. This is particularly important since it is necessary to understand the propagation of light in order to correctly interpret cosmological observations.

We start again from the microscopic theory and will be careful about the averaging procedure. While in the previous sections we have modelled the classical particles as occupying tiny portions of the space and in between of them there is vacuum, the electromagnetic field permeates in the space diffusively and can be treated as a continuum. Consequently its averaging procedure is somehow like the naïve one and slightly different from what we have met for classical particles.

In the $\bar{g}_{ab}$ frame the energy momentum tensor of the
EM field is:

\[
T^a_{\text{EM}} = - \bar{g}^{ac} \bar{g}^{de} F_{cd} F_{eb} + \frac{1}{4} \delta^a_{\ b} \bar{g}^{cf} \bar{g}^{de} F_{cd} F_{ef},
\]

where \( F_{ab} = 2 \partial_t A_b \). We begin by considering the simple case of \( f(R) \) theories. In these theories, with \( \Phi = R \), Eq. (6) gives:

\[
f_r(\Phi)\Phi - 2f(\Phi) = 0
\]

so \( \Phi \) takes its constant vacuum value and does not depend on \( F_{cd} \), and Eq. (2) gives:

\[
\sqrt{-\bar{g}} \bar{g}^{ab} f_r(\Phi) = \sqrt{-g} g^{ab} F_{cd} F_{cd},
\]

and so \( \bar{g}^{ab} = f_r g^{ab} \). We also have that \( Q^2 = f_r^2 \). Thus the effective energy momentum tensor in the \( g_{ab} \)-frame microscopic Einstein equation, Eq. (7), is:

\[
\tilde{T}^a_{\text{EM}} = - \bar{g}^{ac} \bar{g}^{de} F_{cd} F_{eb} + \frac{1}{4} \delta^a_{\ b} \bar{g}^{cf} \bar{g}^{de} F_{cd} F_{ef} - \frac{V(\Phi)}{f_r^2(\Phi)} \delta^a_{\ b},
\]

and \( V(\Phi)/f_r^2 = \text{const} \). Apart from the addition of a vacuum energy term \( -V(\Phi)/f_r^2(\Phi) \delta^a_{\ b} \), the effective energy momentum tensor in the \( g_{ab} \) frame has precisely the same form as it does in the \( \bar{g}_{ab} \) frame. It follows that in vacuum:

\[
\nabla_c T^{ab} = 0
\]

where \( \nabla a g_{bc} = 0 \) and we have raised the indices of \( F^{ab} \) using \( g_{ab} \). The vacuum energy term generates an effective cosmological constant, but other than that, the combined system the Palatini microscopic vacuum Einstein Maxwell equations in both metric frames is precisely the same as it is in GR. There is therefore no need to consider further the effect of averaging in these theories, or the way light propagates, in \( f(R) \) theories since the they will be precisely as they are in unmodified GR. This comes about because in these theories the two metrics are conformally related, and the electromagnetic action is conformally invariant. As such the energy-momentum of the EM field does not source the extra Palatini degree of freedom encoded by \( \Phi \), and the evolution of the EM field is not affected by the modification of gravity.

More generally, however, in \( f(R, R^{ab} R_{ab}) \) the two metrics \( g_{ab} \) and \( \bar{g}_{ab} \) would instead be disformally related. In these theories then the microscopic behaviour of the electromagnetic field will be altered, and hence its large scale behaviour may also change. It is most straightforward to see how the EM field behaves on macroscopic scales if we would in a frame where gravity is certainly no stronger than it is in GR over microscopic scales i.e. the \( g_{ab} \) frame. We therefore consider the form of the energy momentum tensor, \( \tilde{T}^a_{\text{b}} \), in the \( g_{ab} \) frame. Because of the complicated relationship between \( g^{ab} \) and \( \bar{g}^{ab} \), as given by Eqs. (3) (4), it is not possible, in general, to write down a simple expression, in terms of \( F_{ab} \), for \( \tilde{T}^a_{\text{b}} \). We therefore consider the relatively simple, but cosmologically interesting case, where the EM field is microscopically disordered and hence describes a bath of radiation.

Choosing an appropriate coordinate system \( (t, x, y, z) \) so that at some point \( (t_0, x_0) \), \( g_{ab} = \eta_{ab} \), we may write the components of this \( T^{ab} \) explicitly, for example,

\[
T^{00} = \frac{1}{2} \sum_{i=x, y, z} (E_i^2 + H_i^2)
\]

\[
T^{01} = E_y H_z - H_y E_z
\]

\[
T^{11} = \frac{1}{2} \sum_{i=x, y, z} (E_i^2 + H_i^2) - \frac{1}{2} (E_x^2 + H_x^2)
\]

\[
T^{12} = -(E_x E_y + H_x H_y)
\]

These are respectively the energy density, heat flux, pressure and anisotropic stress terms, in which \( E, H \) are the strengths of the electric and magnetic fields.

Since a totally disordered electromagnetic field necessarily, on average, has no preferred direction, when microscopic fluctuations in the field are averaged over we must have:

\[
\langle E_x^2 \rangle = \langle E_y^2 \rangle = \langle E_z^2 \rangle, \quad \langle H_x^2 \rangle = \langle H_y^2 \rangle = \langle H_z^2 \rangle
\]

Also

\[
\langle E_y H_z - H_y E_z \rangle = (y \to x) = (z \to x) = 0
\]

and

\[
\langle E_y E_z \rangle = \langle E_x E_z \rangle = \langle E_x E_z \rangle = 0,
\]

\[
\langle H_y H_z \rangle = \langle H_x H_z \rangle = \langle H_x H_z \rangle = 0
\]

because of the lack of phase relations amongst the different components of the field strengths in the disordered EM field. The above results imply that on macroscopic scales \( T^{ab} \) for a disordered EM field behaves like a perfect fluid with \( \rho = 3p \) and vanishing heat flux \& anisotropic stress. This is precisely how EM field is treated cosmologically in standard GR. However, in Palatini theories, \( T^{ab} \) only represents the energy momentum tensor of the \( g_{ab} \) frame. There gravity can be very strong over small scales, and hence behaves in a significantly different fashion to how it behaves in GR. We prefer to consider the \( g_{ab} \) frame energy momentum tensor \( \tilde{T}_{ab} \), since in this frame gravity behaves no differently to how it behaves in GR.

How the averaging actually works in the case of Palatini \( f(R, R^{ab} R_{ab}) \) gravity depends on the microscopic field equations Eq. (7), and here comes the difference between the \( f(R) \) and \( f(R^{ab} R_{ab}) \) cases. For convenience we decompose the symmetric tensors \( R_{ab} \) and \( \tilde{T}_{ab} \) respectively as

\[
R_{ab} = K_{ab} = \Delta \tilde{u}_a \tilde{u}_b + \Xi \tilde{e}_{ab} + 2 \tilde{u}_a \Upsilon_b + \Sigma_{ab},
\]

\[
\tilde{T}_{ab} = \rho \tilde{u}_a \tilde{u}_b + p \tilde{e}_{ab} + 2 \tilde{u}_a \tilde{y}_b + \pi_{ab}.
\]
where \( u_a \) is the four velocity of the observer \((\hat{g}^{ab}u_au_b = -1)\), \( \xi^a = g_{ab}u_a\xi^b \) is the projection tensor to the hypersurface perpendicular to \( u \), and \( \rho, \pi, \rho_a, \pi_{ab} \) are the energy density, isotropic stress, heat flux and anisotropic stress. Note that \( \hat{u}_a\xi^a = u^a\pi_a = u^a\pi_{ab} = 0 \) and \( R = -\Delta + 3\Xi \). Throughout we use the convention that \( R^{ab} = g^{ac}R_{cb} \) but \( K^{ab} = \hat{g}^{ac}K_{cb} = \hat{g}^{ac}R_{cb} \).

We note that Eq. (5) gives:

\[
R = R_{ab}g^{ab} = \frac{K_{ab}}{Q^2} (f,\Phi g^{ab} + 2f,\chi K^{ab}) ,
\]

\[
= \frac{f,\Phi + 2f,\chi}{Q^2} = 3\kappa\rho - \kappa\rho + 2f,
\]

where the last two equalities follow from Eq. (4) and the definitions: \( \Phi = K_{ab}\hat{g}^{ab} \), \( \chi = K_{ab}K^{ab} \). Eq. (14) then gives (where all indices are raised w.r.t. \( g_{ab} \)):

\[
\kappa\rho = \Delta (f,\Phi - 2f,\chi) + 2f,\chi T_c^c + \frac{1}{2} f, (64)
\]

\[
\kappa\rho_a = (f,\Phi + 2f,\chi (\Xi - \Delta)) T_a^c + 2f,\chi \Sigma_{ca}, (65)
\]

\[
\kappa\pi_{ab} = (f,\Phi + 4f,\chi \Xi) \Sigma_{ab} + 2f,\chi (\Sigma_a \Sigma_{cb} - \Sigma_c \Sigma_{ab} - T_a T_b),
\]

\[
- \frac{2f,\chi}{3} \xi_{ab} (\Sigma_{ca} \Sigma_{cd} - \Sigma_c \Sigma_{ab}), (67)
\]

We also have:

\[
\Phi = 3\Xi - \Delta, (68)
\]

\[
\chi = \Delta^2 + 3\Xi^2 - 2\chi T_c^c + \Sigma^{ab} \Sigma_{ab}, (69)
\]

and from Eq. (15): \( f,\Phi + 2f,\chi \chi - 2f = \kappa(3\rho - \rho). (70) \)

We are concerned primarily with the form of the effective energy momentum tensor \( \hat{T}_{ab} \) which is defined by Eq. (15). Defining \( U_a = \lambda\hat{u}_a \) and \( \xi_{ab} = g_{ab} + U_a U_b \) so that \( U_a U_b g^{ab} = -1 \) i.e.,

\[
\frac{Q^2}{\lambda^2} = \frac{f,\Phi - 2f,\chi}{\lambda^2} = \frac{1}{\lambda^2},
\]

we decompose \( \hat{T}_{ab} \) thus:

\[
\hat{T}_{ab} = \hat{\rho} U_a U_b + \hat{\pi}_{ab} + 2U_a \xi_b + \hat{\pi}_{ab}, (71)
\]

where (raising indices with \( g^{ab} \)) we have \( U^a \xi_a = 0 \), \( U^a \hat{\pi}_{ab} = 0 \) and \( g^{ab} \hat{\pi}_{ab} = 0 \).

We note that \( \kappa \hat{T}_{ab} = \frac{1}{Q^2} (T_{ab} - V(\Phi,\chi)\delta_{ab}) \) and so:

\[
\hat{\rho} = \frac{\rho}{\rho} - \frac{2a^2 f,\chi T_a^a + f,\Phi + 2f,\chi}{2\kappa Q^2} ,
\]

\[
\hat{\pi} = \frac{\pi}{\rho} - \frac{2a^2 f,\chi T_a^a + f,\Phi + 2f,\chi}{2\kappa Q^2} ,
\]

\[
\hat{\pi} = \frac{\lambda}{Q^2} [f,\Phi q_a + 2f,\chi \Xi q_a + 2f,\chi \Sigma_{ab} q^b]
\]

\[
- \frac{4f^2 a^2}{Q^2} \xi_T T_c q^c , (72)
\]

A similar expression to the above can be found for \( \hat{\pi}_{ab} \) although for our purposes it shall not be needed. We note that if the average values of \( q_a \) and \( \Sigma_{ab} \) vanish, then so do the average values of \( \hat{\pi}_{ab} \) and \( \hat{\pi}_{ab} \).

The symmetries of \( T^a_{cb} \), i.e., \( \langle q_a \rangle = \langle \pi_{ab} \rangle = 0 \) imply that \( \langle q_a \rangle = \langle \hat{\pi}_{ab} \rangle = 0 \), and that:

\[
\langle \kappa\hat{\rho} - 3\kappa\hat{\pi} \rangle = \left\langle \frac{2(f,\Phi + 2f,\chi \chi - f)}{Q^2} \right\rangle ,
\]

and by Eq. (6) we have \( f,\Phi + 2f,\chi \chi = 2f \) so:

\[
\langle \kappa\hat{\rho} - 3\kappa\hat{\pi} \rangle = \left\langle \frac{2(f,\Phi,\chi)}{Q^2} \right\rangle , (73)
\]

The average form of \( \hat{T}^a_{ab} \) would only be equivalent to radiation with some cosmological constant if its trace (as given by the above equation) was constant. It is evident from the above equations, that even if we (incorrectly) replace any quantity \( Q \) with \( (\Xi) \) in the above equations, e.g., \( \pi_{ab} \rightarrow \langle \pi_{ab} \rangle \), \( \hat{Y}_a \hat{Y}_b \rightarrow \langle \hat{Y}_a \hat{Y}_b \rangle \) and use the averaged values \( \langle \rho \rangle = 3\rho \) and \( \langle q_a \rangle = \langle \pi_{ab} \rangle = 0 \), then we would find \( \langle Y_a \rangle = \langle \Sigma_{ab} \rangle = 0 \) and conclude that all averaged quantities involving \( Y_a \) or \( \Sigma_{ab} \) also vanish. If we accept this as true, we still have by Eqs. (61) and (66):

\[
\langle \Delta \rangle = H_{\perp} \langle \kappa\rho - \frac{1}{2} f \rangle, (74)
\]

\[
\langle \Xi \rangle = H_{\perp} \langle \kappa\rho + \frac{1}{2} f \rangle, (75)
\]

where

\[
H_{\perp}(x) = \pm \left[ \frac{f,\Phi - f,\Phi}{4f,\chi} + \frac{\sqrt{f,\Phi} \pm \frac{1}{2f,\chi}}{16f,\chi} \right] = \frac{x}{2f,\chi}.
\]

It is then clear that generally when \( f,\chi \neq 0 \), \( \langle \Phi \rangle = \langle 3\Xi - \Delta \rangle \neq \text{const} \). Similarly \( \langle \chi \rangle = \langle \Delta \rangle^2 + 3\langle \Xi \rangle^2 \neq \text{const} \), for \( \rho \neq 0 \). Hence unless \( f,\chi = 0 \), we have that \( \langle f,\Phi,\chi \rangle \) and then also \( (\Xi) \) depend on \( \rho \) and hence on \( \hat{\rho} \). This means that in general \( f(R, R^{ab} R_{ab}) \) theories (in contrast to the \( f(R) \) models) a radiation dominated Universe will not obey the expansion law \( a \propto t^2 \), and moreover it will not even behave as a Universe with radiation and some cosmological constant. These deviations will be particularly significant for early time cosmology if \( f(R, R^{ab} R_{ab}) \) is chosen to make the deviation from GR more significant at high densities. This represents the first complication associated with the behaviour of the EM field in Palatini theories.

In deriving the above, we assumed incorrectly that because \( \langle Y_a \rangle = \langle \Sigma_{ab} \rangle = 0 \), we could take the average values of quantities such as \( Y_a Y^a \) to vanish as well. Even though \( \langle q_a \rangle = 0 \), in general for a disorder electromagnetic field:

\[
\langle \kappa^2 q^a q_a \rangle \sim \mathcal{O}(E^2 H^2) \kappa^2
\]

and

\[
\langle \kappa^2 \pi_{ab} \pi_{ab} \rangle \sim \mathcal{O}(E^4 + H^4) \kappa^2
\]
are obviously nonzero. It then follows from Eqs. 61 - 67 that $\kappa \rho \kappa, \kappa^2 q^2 q_{\alpha}, \kappa^2 \pi_{ab} \pi_{ab}$ and $\kappa^2 a^2 a^2 a_{ab}$ are related to the five (scalar) unknowns $\Delta, \Xi, \Gamma^a \Gamma_a, \Sigma_{ab} \Sigma_{ab}$ and $\Upsilon^a \Upsilon^b \Upsilon_{ab}$. Also unless $q_{\alpha} = \pi_{ab} = 0$, we do not generally have $\Upsilon_a \Gamma^a = 0$, $\Sigma_{ab} \Sigma_{ab}$ or $\Upsilon_a \Upsilon^a \Upsilon_{ab}$. Indeed since $\Upsilon_a \Gamma^a, \Sigma_{ab} \Sigma_{ab} \geq 0$, the average values of these quantities could be non-vanishing even though the average values of $q_{\alpha}, \pi_{ab}, \Upsilon_a$ and $\Sigma_{ab}$ do vanish.

Now in application to cosmology suppose we have $f \sim R \sim \Delta \sim \Xi$ (merely of order, $f \sim R$ simply means that the correction term is significant), $F \sim \Delta, \Xi \sim 1$. Then it could be seen that $\langle \Delta \rangle \sim \langle \Xi \rangle \sim O(\kappa \rho)$ and $\langle \Upsilon_a \Gamma^a \rangle \sim \langle \Sigma_{ab} \Sigma_{ab} \rangle \sim O(\kappa^2 \rho^2)$, so that

$$\langle \Delta^2 \rangle \sim \langle \Xi^2 \rangle \sim \langle \Upsilon_a \Gamma^a \rangle \sim \langle \Sigma_{ab} \Sigma_{ab} \rangle,$$

and the terms involving $\Upsilon_a \Gamma^a, \Sigma_{ab} \Sigma_{ab}$ in Eqs. 64 - 67 are equally important as other modification terms. So when calculating the cosmology in a radiation dominated Universe we must take them into account. This is the second complication of the Palatini $f(R) R_{ab}$ model. We would still, however, find that generally $\langle \Delta \rangle$ depends on $\rho$, signalling a deviation from the GR behaviour, when $f(\chi) \neq 0$.

These two complications make the Palatini $f(R) R_{ab}$ model less trivial than its $f(R)$ counterpart: for the latter the cosmologies for both radiation and matter dominated Universes are the same as in ΛCDM, while for the former, as we referred to in above, the radiation dominated Universe could behave rather differently from ΛCDM, even violating the $a \propto t^\frac{2}{3}$ law of expansion. It will be interesting to consider such models in more details, but due to its complexity this is beyond the scope of this paper and will be further investigated elsewhere.

But how does this difference from Palatini $f(R)$ gravity and from the behaviour of classical particles arise? In the case of classical particles, there are no interactions between separated particles other than gravity and this is why the cosmology is like that of GR with a cosmological constant; the modification to GR simply alters the internal structure of the particles. The radiation field, as we said above, could be treated as a continuum and the new ‘interaction’ due to the modification to GR (which acts at a point!) exists everywhere; so photons feel the modification everywhere and the corresponding cosmology is changed: this is essentially the argument of a naive averaging. The Palatini $f(R)$ is immune to this effect because in this model the interaction depends on $\rho - 3 \rho$ which is zero identically. In general $f(R, R_{ab} R_{ab})$ models the interaction does affect the propagation of photons.

VI. CONSTRAINTS FROM ATOMIC PHYSICS

In this §we provide a particular example of how microscopic physics tightly constrains the properties of Palatini theories (see also 40). We consider how the Palatini modification alters energies of the photons that are emitted when an electron transitions from one atomic energy level to a lower one. This analysis is particular to $f(R)$ theories. A similar calculation could be performed for a general $f(R, R_{ab} R_{ab})$ theory, but this would be significantly more complicated due to the fact that the light propagation is also altered in these theories.

Consider the total action for a charged fermion field $\Psi_F$:

$$S_{tot} = S_{grav} + S_m = \bar{S}_{grav} + S_m$$

where

$$\bar{S}_{grav} = \int -\sqrt{-\bar{g}} d^4 x \frac{1}{2 \kappa} f(\bar{R}),$$

$$S_m = \int -\sqrt{-\bar{g}} d^4 x \left\{ \bar{\Psi}_F \left( \bar{g}^{\alpha \beta} \nabla_{\alpha} m_0 - iq q^a A_a \right) \bar{\Psi}_F - \frac{1}{4} \bar{g}^{ab} \bar{g}^{cd} F_{ac} F_{bd} \right\},$$

where $A_a$ is the electromagnetic field, $q$ is the charge of the particle, $\bar{g}^{ab}$ are the curved space-time analogue of the Dirac $\gamma$ matrices and $q^a g^{ab} + q^b g^{ba} = 2 g^{ab}$. We will perform our calculations in the Einstein frame, i.e., where the metric is $g_{ab}$. We do this because, as we noted above, in this frame the local curvature of space-time is certainly of a similar magnitude to that which one would expect from general relativity. As such the approximation $g_{ab} \approx \eta_{ab}$ in the frame of a laboratory experiment will be equally as valid in Palatini theories as it is in standard general relativity. The same is not necessarily true for the Jordan frame where $\bar{g}_{ab}$ is the metric. Working in the Einstein frame should be viewed merely as a computational convenience and should not be viewed as attaching any special physical meaning to this frame. The observable quantities we will extract from our calculation will be independent of the choice of frame.

We now convert the action, as given by Eq. 76, to the Palatini frame. In this frame we have $g_{ab} = f(\Phi) g_{ab}, f, \Phi = R, \sqrt{-g} = \sqrt{-\bar{g}},$ and so

$$\sqrt{-\bar{g}} f(\Phi) = \sqrt{-\bar{g}} f(\Phi) + \sqrt{-\bar{g}} (f(\Phi) - f, \Phi)$$

$$= \frac{1}{f^{(2)}(\Phi)} \sqrt{-\bar{g}} \left[ R - 2 \kappa V(\Phi) \right],$$

where again $\Phi = R$ and $V(\Phi) = (f, \Phi - f)/2 \kappa$. Thus Eq. 76 becomes:

$$S_{tot} = S_{grav} + S_{m}^{(eff)},$$

where

$$S_{grav} = \int -\sqrt{-\bar{g}} d^4 x \frac{R}{2 \kappa f, \Phi},$$

$$S_{m}^{(eff)} = S_m - \int \sqrt{-\bar{g}} d^4 x \frac{V(\Phi)}{f^{(2)}(\Phi)}.$$
We rewrite $S_m$ in the Einstein frame as follows: we define new fields
\[ \varphi = f_\Phi^{-3/8} \Psi_F, \quad \tilde{\varphi} = f_\Phi^{-9/8} \Psi_F, \]
and
\[ \gamma^a = f_\Phi^{-1/2} \bar{e}^a \]
so that
\[ \gamma^a \gamma^b + \gamma^b \gamma^a = 2g^{ab}. \]
We then have:
\[
S_m = \int \sqrt{-g} \, d^4x \left\{ \varphi \left[ \gamma^a \nabla_a - m(\Phi) - i q \gamma^a A_a \right] \varphi - \frac{1}{4} g^{ab} g^{cd} F_{ac} F_{bd} \right\},
\tag{83}
\]
in which $\nabla_a$ satisfies $\nabla_a g_{bc} = 0$ and $m(\Phi) = m_0 f_\Phi^{-1/2}$. Varying this action with respect to $\varphi$, we arrive at the modified Dirac equation obeyed by electrons in these theories:
\[
\gamma^a \partial_a \varphi = m(\Phi) \varphi + i q \gamma^a A_a \varphi,
\tag{84}
\]
Varying the action with respect to $\Phi$ gives us:
\[
\frac{2 f(\Phi) - f_\Phi f_\Phi}{f_\Phi^2} = \kappa m(\Phi) \tilde{\varphi} \varphi.
\tag{85}
\]
With the equations written in this form, the effect of the Palatini modifications is manifest: it is to make the mass of the electron, $m(\Phi)$, a $\Phi$-dependant quantity (see e.g. [47, 48, 49] for considerations of other local density dependent quantities). Since the mass of the electron depends on $\Phi$, it also depends on the local density of matter, and hence also on the local electron density. In an atom the peak local electron density generally decreases as the energy of the orbit decreases, and so the effective mass of the electron will be different for different energy orbits. This will lead to potentially detectable deviations from the standard model of particle physics. We now quantify these deviations.

We wish to consider the energy eigenstates of an electron orbiting an a hydrogen nucleus. We therefore take $\varphi \propto \exp(-i E_c t)$ where $E_c$ is the electron energy, and for the hydrogen atom $q A_0 = \alpha_{EM}/r = -V(r)$ where $\alpha_{EM}$ is the fine structure constant. From Eq. (84), the electron obeys:
\[
E_c \varphi = -i \tilde{\alpha}_k \partial_k \varphi - \frac{\alpha_{EM}}{r} \varphi + m(\Phi) \beta \varphi,
\]
where $\tilde{\alpha}_i \beta = -\beta \tilde{\alpha}_i$, $\beta^2 = 1$ and $\tilde{\alpha}_i (\tilde{\alpha}_j) = \delta_{ij}$. We define:
\[
\varphi = \left( \begin{array}{c} F(x) \\ i G(x) \end{array} \right),
\tag{86}
\]
and then:
\[
\left[ E_c - m(\Phi) + \frac{\alpha_{EM}}{r} \right] F = \sigma \cdot \nabla G,
\tag{87}
\]
\[
\left[ E_c + m(\Phi) + \frac{\alpha_{EM}}{r} \right] G = -\sigma \cdot \nabla F,
\tag{88}
\]
where $\sigma_i$ are the Pauli matrices. $E$ is the total energy of the electron and includes the contribution from the electron rest mass, we therefore write $E_c = m_0 + E'_c$.

We calculate the modifications to the energies of the photon that is emitted when an electron changes from one energy level to another by assuming that the Palatini modification is small enough so that we may write $f(\Phi) = \Phi(1 + \varepsilon(\Phi))$ where $\varepsilon(\Phi) \ll 1$. Not all Palatini theories can be written in this may, but considering we will find that any changes in $\varepsilon(\Phi)$ are constrained to be very small, we believe that is it highly unlikely that any theory, which cannot be written thus, could be experimentally viable.

Assuming this form for $f(\Phi)$ we have:
\[
m(\Phi) = m_0 / \sqrt{f_\Phi} \approx m_0 \left[ 1 - \frac{1}{2} (\varepsilon(\Phi)) \right] = m_0 + \delta m(\Phi),
\]
where $\delta m(\Phi)/m_0 \ll 1$. We solve Eqs. (87, 88) perturbatively in the small parameter $\delta m(\Phi)/m_0$.

To $O((\delta m/m_0)^0)$ we have
\[
F = \tilde{F}, \quad G = \tilde{G} \quad \text{and} \quad E'_c = \tilde{E}_c = \alpha_{EM}^2 m_0 \varepsilon(\alpha_{EM})
\]
where having defined $y = x/a_0$, $a_0 = 1/\alpha_{EM} m_0$:
\[
\alpha_{EM} \left[ \varepsilon + \frac{1}{|y|} \right] \tilde{F} = \sigma \cdot \nabla (\frac{1}{|y|} \tilde{G}),
\tag{89}
\]
\[
\left[ 2 + \alpha_{EM}^2 \varepsilon + \frac{\alpha_{EM}^2}{|y|} \right] \tilde{G} = -\alpha_{EM} \sigma \cdot \nabla (\frac{1}{|y|} \tilde{F}),
\tag{90}
\]
thus $\tilde{G} \sim O(\alpha_{EM} \tilde{F})$. The unmodified energy levels, $\tilde{E}_c$ are just what they would be in the standard model with an electron mass $m_0$. We do not repeat a full calculation of those energy levels here. For the purposes of calculating the large contribution perturbation to the energy levels we need only work to leading order in $\alpha_{EM}$. To the leading order in $\alpha_{EM}$, $\tilde{F}$ satisfies the Schrödinger equation:
\[
-\frac{1}{2} \nabla^2 \tilde{F} = \varepsilon + \frac{1}{|y|} \tilde{F} (1 + O(\alpha_{EM}^2)).
\tag{91}
\]
To this order, for a state with energy level $n$ and angular momentum $(lm)$ we have to leading order in $\alpha_{EM}$:
\[
\varepsilon = \varepsilon_n = -1/2n^2 (1 + O(\alpha_{EM}^2))
\]
and we write
\[
\tilde{F}_{nlm} = \tilde{R}_{nl}(r) Y_{lm}(\theta, \phi)
\]
where $Y_{lm}(\theta, \phi)$ are spherical harmonics, $r$ is the distance from the nucleus and $\theta$ and $\phi$ are angular coordinates. We normalize $\tilde{F}_{nlm}$ so that:
\[
\int d^3 x \tilde{F}_{nlm}^2 = 1.
\]
To leading order in both $\alpha_{\text{EM}}$ and $\delta m/m_0$ the electron number density, $n_e$, is given by $n_e = \bar{F}^2$ and, so to this order by Eq. (52), where for a state with energy level $n$ and angular momentum $(lm)$, we have $\Phi = \bar{\Phi}_{nlm}$ and

$$\bar{\Phi} = \kappa m_0 \bar{F}^2_{nlm}.$$  

(92)

To $\mathcal{O}(\delta m/m_0)$ we write

$$F = \bar{F} + \delta F, \quad G = \bar{G} + \delta G$$

and $E'_e = \bar{E}_e + \Delta m$.

We therefore have:

$$-\nabla_y(\delta F) = \sigma \cdot \nabla_y \frac{\delta m}{m_0} \bar{G} + \frac{\Delta m + \delta m}{m_0} \left( \mathcal{E} + \frac{1}{|y|} \right) \overline{F}$$

$$+ 2 \alpha_{\text{EM}} \left( \frac{\Delta m - \delta m}{m_0} \right) F + \left( \mathcal{E} + \frac{1}{|y|} \right) \delta F,$$  

(93)

and so $\delta G \sim \mathcal{O}(\alpha_{\text{EM}} \delta F)$. To leading order in $\alpha_{\text{EM}}$ then we have $\delta F = \lambda \bar{F}/\alpha_{\text{EM}}^2$ where:

$$-\nabla_y(\bar{F}^2 \nabla_y \lambda) = 2 \left( \frac{\Delta m - \delta m}{m_0} \right) |\bar{F}|^2,$$  

(94)

integrating this expression gives, for a state with an energy level $n$, and angular momentum $(lm)$:

$$\Delta m_{nlm} = \int d^3 x \delta m_{nlm} |\bar{F}_{nlm}|^2 = - \frac{m_0}{2} \langle \epsilon + \epsilon, \Phi \rangle_{nlm},$$  

(95)

where $\delta m_{nlm} = \delta m(\Phi_{nlm})$ and we have defined:

$$\langle Q \rangle_{nlm} = \int d^3 x Q(x)n_e(nlm)(x).$$

In a local inertial frame we ignore the energy stored in the gravitational field, however we can see from Eq. (52) that even when $g_{ab} = \eta_{ab}$, the effective matter action includes a contribution from the effective potential $V(\Phi)$. Only when the contribution from $V(\Phi)$ is included is the energy and momentum conserved with respect to $g_{ab}$. Thus (excluding the energy of the nucleus which we assume to be independent of the electron energy level), the total conserved energy in this set-up is given by:

$$E_{\text{tot}} = E_e + E_{V(\Phi)} + E_\gamma,$$

where $E_\gamma$ is the energy stored in the photon field and when $g_{ab} = \eta_{ab}$:

$$E_{V(\Phi)} = \int d^3 x \frac{V(\Phi)}{f_{\Phi}^2}.$$  

(96)

Assuming as we have that $f(\Phi) = \Phi(1 + \epsilon(\Phi))$, we have:

$$\frac{V(\Phi)}{f_{\Phi}^2} \approx \frac{\bar{\Phi}^2 \epsilon_\Phi(\Phi)}{2\kappa}.$$  

Thus taking $\Phi = \bar{\Phi}_{nlm}$ we have:

$$E_{V(\Phi)} = \frac{m_0}{2} \langle \epsilon, \bar{\Phi}^2 \rangle_{nlm}. $$  

(97)

where we have used Eq. (62). Thus:

$$E_e + E_{V(\Phi)} = m_0 + \bar{E}_e(nlm) - \frac{m_0}{2} \langle \epsilon \rangle_{nlm} (1 + \mathcal{O}(\alpha_{\text{EM}}^2)).$$

Since $E_{\text{tot}}$ is a conserved quantity in this frame, when an electron changes from a level with quantum numbers $(nl)$ to a lower energy level with quantum numbers $(n'l')$, a photon with energy, $E_\gamma$ must be emitted where:

$$E_\gamma = \bar{E}_e(nl) - \bar{E}_e(n'l')$$

$$- m_0 [\langle \epsilon \rangle_{nl} - \langle \epsilon \rangle_{n'l'\gamma}] (1 + \mathcal{O}(\alpha_{\text{EM}}^2)).$$

(98)

We define $\bar{E}_\gamma(nl,n'l')$ be the energy of the photon to zero order in $\epsilon$. We then have:

$$E_\gamma = \left[1 + \Delta_{nl}^{n'l'}\right] \bar{E}_\gamma(nl,n'l'),$$

(99)

where, using $\bar{E}_\gamma(nl) = -\alpha_{\text{EM}}^2 m_0/2n^2$ to leading order in $\alpha_{\text{EM}}$, we have to leading order in $\alpha_{\text{EM}}$:

$$\Delta^{n'l'}_{nl} = \frac{\langle \epsilon \rangle_{nl} - \langle \epsilon \rangle_{n'l'}}{\alpha_{\text{EM}}^2 (n-2-n'-2)}.$$  

(100)

If $\Delta$ were the same for all transitions then this could be accounted for simply by a slight alternation of the electron mass. To constrain $\Delta$ we consider therefore how the energy of a photon released due to one transition changes relative to that emitted in another transition; this is independent of the electron mass i.e. we consider the ratio of $E_{\gamma}$ for one transition with that for another. It is also independent of one’s choice of frame. If $t$ is the time in a local inertial frame in the Einstein frame, and $\tilde{t}$ the time in a local inertial frame (LIF) defined with respect to the Jordan metric, then, since $g_{ab} = f_\Phi g_{ab}$, we have $dt = f_\Phi d\tilde{t}$. Thus if a photon has energy $E_{\gamma}$ in LIF of the Einstein frame metric, $g_{ab}$, it has energy $E_{\gamma} f_\Phi^{1/2}$ in a LIF of the Jordan frame metric, $\bar{g}_{ab}$. If the energies of two photons are measured at the same place, the ratio of those two energies is a frame independent quantity, as the frame dependant scaling factors $(f_\Phi^{1/2})$ cancel. Thus, even though we have performed our calculation in the Einstein frame, the quantity which we will constrain is independent of this frame choice.

We now consider two transitions for which measurements have been shown to agree to the standard theoretical prediction to a high accuracy [42]. Firstly we have the $1S_{1/2} - 2S_{1/2}$ transition. For this transition $(nl) = (20)$ and $(n'l') = (10)$. Secondly, we consider the $2S_{1/2} - 8D_{5/2}$ transition for this transition $(nl) = (83)$.
and \((n' \ell') = (20)\). Thus:

\[
\frac{E_\gamma(83; 20)E_\gamma(20; 10)}{E_\gamma(20; 10)E_\gamma(83; 20)} - 1 = \frac{64}{15\alpha EM} \left[ (\langle \epsilon \rangle_{83} - \langle \epsilon \rangle_{20}) + \frac{4}{3\alpha EM} (\langle \epsilon \rangle_{20} - \langle \epsilon \rangle_{10}) \right] = \frac{28}{5\alpha EM} \left[ \langle \epsilon \rangle_{20} - \frac{16}{21} \langle \epsilon \rangle_{83} - \frac{5}{21} \langle \epsilon \rangle_{10} \right].
\]

The measurements of Ref. [42] provide the following constraint: the magnitude of the left hand side of the above equation to be smaller than \(8 \times 10^{-10}\) and so:

\[
\left| \langle \epsilon \rangle_{20} - \frac{16}{21} \langle \epsilon \rangle_{83} - \frac{5}{21} \langle \epsilon \rangle_{10} \right| < 8 \times 10^{-16}. \tag{102}
\]

This represents a very strong constraint on the properties of Palatini \(f(R)\) theories. In particular we can see that, writing \(f(\Phi) \approx \Phi(1 + \epsilon(\Phi))\) when \(\Phi/k\) is of the order of the electron cloud density in hydrogen, then changes in \(\epsilon(\Phi)\) are constrained to be very small. If we expand \(\epsilon\) about some appropriate value of \(\Phi\) and find that to linear order we have

\[
\epsilon(\Phi) \approx \text{const} + \epsilon_0 \Phi/bH_0^2,
\]

where \(H_0^2\) is the value of the cosmological constant today, Eq. (102) gives the very strong constraint:

\[
|\epsilon_0| \approx |f''(\Phi)H_0^2/f'(\Phi)| \lesssim 4 \times 10^{-40}.
\]

**VII. DISCUSSIONS AND CONCLUSIONS**

Much of our intuition about how the microscopic behaviour of gravity affects physics on large scales is based upon Einstein’s general relativity. In this article we show that such intuition cannot simply be applied to modified gravity theories without a detailed analysis of the energy-momentum microstructure. Indeed, na"ively averaging over the microscopic structure will generally lead one to make incorrect predictions, and inaccurate conclusions as to the validity of the theory. In particular, the naive averaging procedure is invalid in Palatini theories.

For classical particles, we show that the relative motion of particles in Palatini theories is indistinguishable from that predicted by GR plus a cosmological constant. This means that the cosmology and astrophysics (except in some extreme environments such as neutron stars) of Palatini \(f(R, \mathcal{R}^{ab}R_{ab})\) models are identical to that of GR plus a cosmological constant. The above result can also been shown using our correct averaging procedure, as given in [IV]. This particularly means that the fine tuning problems associated with the cosmological constant are not alleviated in Palatini theories. It should be stressed that although the Palatini theories predict the same cosmology and astrophysics as GR, they are completely different theories: not only because they predict different internal structures of particles, but also because they behave differently from GR in the presence of electromagnetic field and in the atomic physics.

When coming to electromagnetic fields in Palatini theories, things becomes a bit different. In contrast to classical particles, which are tiny clumps of energy density in between of which there is vacuum, the EM field permeates in the space and the naive averaging actually works. However, when performing the averaging one should also take into account the fact that the field equations are microscopic and that at microscopic level the EM field is random and disordered. This make the Palatini \(f(R, \mathcal{R}^{ab}R_{ab})\) model less trivial than its \(f(R)\) counterpart: for the latter the cosmologies for both radiation and matter dominated Universes are the same as in ΛCDM, while for the former the radiation dominated Universe could behave rather differently from ΛCDM, even violating the \(a \propto t^2\) law of expansion. This difference from Palatini \(f(R)\) gravity and from the behaviour of classical particles arises because, in the case of classical particles, there are no interactions between separated particles other than gravity. This is why the cosmology is like that of GR with a cosmological constant. The modification to GR simply alters the internal structure of the particles. The radiation field can be treated as a continuum and the new ‘interaction’ due to the modification to GR (which act at point!) exists everywhere. So photons feel the modification everywhere and the corresponding cosmology is changed. The Palatini \(f(R)\) is immune to this effect because in this model the interaction depends on \(\rho - 3p\) which is zero identically. In general \(f(R, \mathcal{R}^{ab}R_{ab})\) models the interaction does affect the propagation of photons. In summary, the Palatini modifications can affect the propagation of photons (EM field) and even change the cosmic expansion during radiation domination. Observational data on Big Bang Nucleosynthesis could then place some constraints on these models.

Interestingly, although Palatini \(f(R)\) theories were designed to modify gravity on large scales, they actually modify physics on the smallest scales (e.g., the energy levels of electrons) while leaving the larger scales practically unaltered. We show that the observables in atomic physics, such as the energy levels, can be very sensitive to the Palatini modification to GR, and indeed experimental data places extremely stringent constraints on any deviation from GR. In general, before considering any astrophysical consequences of a modified gravity theory, it is important then to check that it does not make unrealistic predictions for atomic physics.

One may wonder why the same averaging problem does not appear in the metric \(f(R)\) gravity models. The answer is that, within the metric approach, averaging over microscopic scales is generally no less straightforward than it is in GR; this is because in both cases all degrees of freedom are dynamical. These dynamics generally ensure that the field equations, for all degrees of freedom, are approximately linear for small-scale structures. In contrast, averaging in Palatini models is not so trivial.
since the new degree of freedom is non-dynamical, and so its field equation remains non-linear even on the smallest scales. This non-linearity introduces an averaging problem that is specific to Palatini theories. And this explains why the cosmological behaviour of the theory from that which might have otherwise been expected from the microscopic equations. As much as the importance of any back-reaction from averaging remains an open problem in General Relativity, as we have illustrated in this work, it is likely to be an even more significant problem in modified theories of gravity.

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