Fluctuation Spectra of Tilted and Intersecting D-branes from the Born-Infeld Action

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Abstract

We consider the spectra of excitations around diagonal and intersecting D-brane configurations on tori. These configurations are described by constant curvature connections in a dual gauge theory description. The low-energy string fluctuation spectrum is reproduced exactly by the gauge theory in the case of vanishing field strength; however, this correspondence breaks down for fixed nonzero field strength. We show that in many cases the full Born-Infeld action correctly captures the low-energy spectrum in the case of non-vanishing field strength. This gives a field theory description of the low-energy physics of systems of diagonally wound branes and branes at angles as considered by Berkooz, Douglas and Leigh. This description extends naturally to non-supersymmetric configurations, where the tachyonic instability associated with brane-anti-brane systems appears as an instability around a saddle point solution of the corresponding Yang-Mills/Born-Infeld theory. In some cases, the field theory description requires a non-abelian generalization of the Born-Infeld action. We follow Tseytlin’s recent proposal for formulating such an action. In the case of intersecting branes, the non-abelian Born-Infeld theory produces a transcendental relation which comes tantalizingly close to reproducing the correct spectrum; however, a discrepancy remains which indicates that a further clarification of the non-abelian Born-Infeld action may be necessary.

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1 Introduction

D-branes [1] have recently emerged as a key ingredient driving the non-perturbative dynamics of string theory. These objects provide an exact description of string theory solitons, and have led to remarkable developments towards understanding the nature of stringy black holes [2, 3]. D-branes also provide a fascinating connection between string theory and gauge theories. The low-energy effective dynamics of $N$ parallel D-branes is precisely that of supersymmetric Yang-Mills theory with gauge group $U(N)$ [4]; such a low-energy effective theory has in turn been conjectured to provide a reformulation of string theory itself [5].

The goal of this paper is to extend further the range of phenomena in D-brane physics which can be described precisely in the language of Yang-Mills field theory or its extension to Born-Infeld theory. In particular, we consider the correspondence between the perturbative fluctuation spectra of D-brane configurations wrapped on a torus and gauge theories with a constant background field on the dual torus [1]. Constant background field configurations in Yang-Mills theory on the torus are T-dual to D-branes which are either diagonally wrapped (“tilted”) [8] or intersecting at an angle [9]. Related recent work on branes at an angle can be found in [10, 11, 12, 13, 14, 15, 16]. The fluctuation spectra of gauge theories in a constant background field have been investigated previously in [17]. We compare these results to what is expected from the D-brane side. The constant background field strength in the Yang-Mills theory gives rise to a new scale in the problem in addition to the string scale and the radius of the torus. In an appropriate scaling limit, we show that Yang-Mills theory is insufficient to reproduce the spectrum of open strings on the D-branes and must be replaced by the full Born-Infeld action. We find that in many cases the full Born-Infeld action reproduces in a field theory context the exact spectrum of low-energy string excitations around the D-brane background.

The story is slightly more subtle in the non-abelian case. The excitation spectra for intersecting D-branes and the corresponding dual gauge fields were discussed in [8] and [17] respectively. However, these spectra agree only to lowest order in the angle of intersection. We investigate the resolution of this discrepancy by including higher order terms in the non-abelian Born-Infeld action. Although the non-abelian generalization of the Born-Infeld action has yet to be understood in full, a formulation involving a symmetrized trace was proposed recently by Tseytlin [18]. We find that this prescription for non-abelian Born-Infeld fails to reproduce the exact fluctuation spectrum seen in string theory. Nonetheless, we encounter remarkable relations encoded in the general structure of Tseytlin’s non-abelian Born-Infeld action which provide a hint indicating how this discrepancy might ultimately be resolved. Since the relation which emerges from the analysis of the action is highly non-trivial, we believe that there is a strong element of truth in Tseytlin’s proposal. However, we do not completely resolve the discrepancy at finite angles, and we leave this as a problem for future investigations.

1Such a correspondence at the level of the vacuum state was considered previously using the boundary state formalism in [1, 2].
A particularly interesting class of systems of the type considered here are those corresponding to non-supersymmetric D-brane configurations. On $T^4$ constant gauge field backgrounds are stable if and only if the gauge field is (anti)-self-dual. In [9] it was argued that the (anti)-self-dual condition is equivalent to the condition that the dual D-brane configuration preserve enough supersymmetry to be a BPS state. Considering non-supersymmetric D-brane configurations, we find that the tachyonic instability of the string ground state appears naturally in the field theory language as an instability of a saddle point configuration in the Yang-Mills/Born-Infeld theory. This provides a natural framework for studying aspects of brane-anti-brane interactions in the context of field theory.

The organization of this paper is as follows: In section 2 we consider D-brane configurations which are tilted with respect to the torus; these configurations correspond to gauge theory backgrounds with constant field strengths in the central $U(1)$ part of $U(N)$. The necessity for replacing the Yang-Mills action with the Born-Infeld action in order to exactly reproduce the fluctuation spectrum emerges from this analysis. We also discuss the issue of fractional quantization from the gauge theory point of view in this section. In section 3 we consider systems of intersecting D-branes which correspond to reducible connections in the dual gauge theory. We discuss the discrepancy between the spectra and its possible resolution by the non-abelian Born-Infeld action. We also discuss here the D-brane/Born-Infeld correspondence for non-supersymmetric D-brane configurations and the associated tachyonic instability.

2 Tilted D-branes and the Born-Infeld Action

2.1 Review of D-branes and T-duality

We begin with a brief review of some salient features of D-branes and their behavior under T-duality; for more background see [8].

In this paper we are concerned with D-branes in type II string theory. In type IIA (IIB) string theory, D-branes with even (odd) dimension appear. When the theory is compactified on a torus $T^d$ there is a T-duality symmetry in each compactified direction which exchanges Neumann and Dirichlet string boundary conditions and which therefore changes the dimension of a D-brane by one. The dynamics of a single D-brane are controlled by the Born-Infeld action, which reduces in the case of small field strength and flat background metric to a supersymmetric $U(1)$ Yang-Mills action with adjoint scalars corresponding to transverse fluctuations [19]. When $N$ D-branes become coincident, the Yang-Mills action is extended to a supersymmetric non-abelian $U(N)$ Yang-Mills theory [1]. The generalization of the full Born-Infeld action to the non-abelian case has not yet been completely understood; however, recent progress in this direction was made by Tseytlin [18].

When D-branes are living in a space which has been compactified in a transverse direction, there are winding strings which wrap around the compactified direction an arbitrary number of times. Under T-duality, string winding and string momentum are exchanged
so that these winding strings naturally become momentum modes of the gauge field under duality in the compact direction. At the level of the Yang-Mills action, this duality can be made manifest. As discussed in [20, 21], the winding strings can be packaged in an infinite matrix which corresponds to the operator $2\pi\alpha'(i\partial + A)$ in the dual gauge theory. Under this correspondence, the fluctuation spectrum of strings on a D-brane which is unwrapped in some set of compact dimensions is precisely captured by the dual gauge theory.

When we consider D-branes which are not parallel to the generators of a perpendicular torus, however, or systems of intersecting D-branes, we must be more careful. It was pointed out in [22, 23] that nontrivial gauge configurations can carry D-brane charge. This follows from the Chern-Simons term in the D-brane action which couples higher powers of the gauge field strength $F$ to RR fields. When we have a system of tilted or intersecting D-branes on a torus, T-dualizing to a gauge theory on the torus will generally lead to a topologically nontrivial bundle whose Chern classes determine the D-brane charges on various subtori. Such bundles were discussed in the $SU(N)$ case in [24], and in the context of D-branes in [10]. The approach we will take in this paper is to consider fixed configurations of flat D-branes on the torus which may be tilted or intersecting. These D-brane configurations map under T-duality to constant field strength backgrounds in the dual gauge theory. We will compare the fluctuation spectra of the D-brane configurations as calculated in string theory to the fluctuation spectra of the gauge fields around the constant background. As we shall see, in many situations the action must be generalized to the full non-abelian Born-Infeld action to achieve an exact correspondence between these two spectra.

2.2 1-brane on $T^2$ with winding $(1, q)$

Let us begin with the simplest example of a gauge field configuration corresponding to the T-dual of a tilted D-brane. Consider a 2-brane wrapped on $T^2$ with periods $L_1$ and $L_2$. The fluctuation spectrum of this theory is precisely that of SUSY $U(1)$ Yang-Mills theory in 2+1 dimensions around a vanishing background field. There are gauge fields corresponding to the regular bosonic $U(1)$ theory, adjoint scalars corresponding to transverse fluctuations, and various fermionic fields. Throughout this paper we will restrict attention to the fluctuation spectra of the bosonic modes, although a similar analysis could easily be performed for the fermion modes. Consider turning on a background

$$\begin{align*}
A^0_1 &= 0 \\
A^0_2 &= F_0 x_1.
\end{align*}$$

The flux quantization condition requires that $F_0$ be an integer multiple of the flux quantum:

$$F_0 = F_0^{21} = \frac{2\pi}{L_1 L_2} q, \quad q \in \mathbb{Z}.$$  

Such a configuration corresponds to a $U(1)$ bundle with first Chern class $C_1 = q$. For concreteness, let us take $q = 1$. From the point of view of $U(1)$ gauge theory, it is straightforward
to compute the spectrum of fluctuations around this background. In terms of the fluctuation \( \delta A_{\mu} \) defined through

\[
A_{\mu} = A_{\mu}^0 + \delta A_{\mu},
\]

the Lagrangian is simply (up to an overall constant)

\[
\mathcal{L} = F_{\mu\nu} F^{\mu\nu} = F_{\mu\nu}^0 F^{\mu\nu\nu} + 2 F_{\mu\nu}^0 (\partial^{\mu} \delta A^{\nu} - \partial^{\nu} \delta A^{\mu}) + (\partial_{\mu} \delta A_{\nu} - \partial_{\nu} \delta A_{\mu}) (\partial^{\mu} \delta A^{\nu} - \partial^{\nu} \delta A^{\mu})
\]

We can ignore the additive constant and the total derivative terms, so one recovers the original Yang-Mills Lagrangian

\[
\mathcal{L} = (\partial_{\mu} \delta A_{\nu} - \partial_{\nu} \delta A_{\mu}) (\partial^{\mu} \delta A^{\nu} - \partial^{\nu} \delta A^{\mu})
\] (2)

whose physical spectrum is simply that of the quantized momenta

\[
E^2 = \left( \frac{2\pi}{L_1} n_1 \right)^2 + \left( \frac{2\pi}{L_2} n_2 \right)^2.
\] (3)

The spectrum for the transverse bosonic fields is clearly the same, as they obey the same periodic boundary conditions.

Let us now consider the D-brane configuration given after T-duality in the \( x_2 \) direction. In the gauge theory configuration described above, we have a single unit of 2-brane charge corresponding to the \( U(1) \) gauge group. The flux \( F_0 \) carries a single unit of 0-brane charge. Under T-duality, we get a D-brane configuration with one unit of 1-brane charge in each direction \( x_1, x_2 \). T-duality inverts the radius of the torus along the \( x_2 \) direction, so the dimensions of the dual torus are \( L_1 \times 4\pi^2 \alpha'/L_2 \). The background field given by (1) can be interpreted under T-duality as the transverse coordinate of a 1-brane on \( x_1 \),

\[
X_2 = 2\pi \alpha' A_0^2 = \frac{4\pi^2 \alpha'}{L_1 L_2} x_1.
\]

This shows that the 1-brane is wound diagonally on the dual torus, as expected of a \((1,1)\) bound state (see figure [II]). The field theory limit of this configuration corresponds to taking \( L_1 \) and \( L_2 \) to be of the same order of magnitude while sending \( \alpha'/L_1^2 \) to zero. In this limit, the dual torus becomes long and thin. The low-lying states in the T-dual description are
therefore given by momentum modes along the 1-brane and winding modes in the direction perpendicular to the 1-brane. The winding modes can also be thought of as strings stretching between adjacent 1-branes in the covering space of the dual torus. From the point of view of string theory in a background given by such a D-brane configuration, we expect the spectrum of low-lying excitations to be given by

$$E^2 = \left( \frac{2\pi}{L_1^\prime} n_1 \right)^2 + \left( \frac{L_2^\prime}{2\pi\alpha'} n_2 \right)^2,$$

where $L_1^\prime$ is the length of the 1-brane along the diagonal

$$L_1^\prime = L_1 \sqrt{1 + \left( \frac{4\pi^2\alpha'}{L_1 L_2} \right)^2} = L_1 \sqrt{1 + (2\pi\alpha' F_0)^2},$$

and $L_2^\prime$ is the distance between adjacent 1-branes

$$L_2^\prime = \frac{1}{\sqrt{1 + \left( \frac{4\pi^2\alpha'}{L_1 L_2} \right)^2}} \left( \frac{4\pi^2\alpha'}{L_2} \right) = \frac{1}{\sqrt{1 + (2\pi\alpha' F_0)^2}} \left( \frac{4\pi^2\alpha'}{L_2} \right).$$

In terms of the original variables, (4) becomes

$$E^2 = \frac{1}{1 + (2\pi\alpha' F_0)^2} \left( \left( \frac{2\pi}{L_1} \right)^2 + \left( \frac{2\pi}{L_2} \right)^2 \right).$$

Comparison of eq. (3) and (5) indicates that they disagree by a factor of $(1 + (2\pi\alpha' F_0)^2)$. We will now show that the spectrum of excited states arising from the Born-Infeld action in 2+1 dimensions gives rise to the correct factor of $(1 + (2\pi\alpha' F_0)^2)$, correcting (3) to precisely match the string theory result found in equation (5). The Born-Infeld action in 2+1 dimensions is given by (up to an overall constant)

$$S = \int d^3x \sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}$$

where $\eta_{\mu\nu}$ is the Minkowski metric $\text{Diag}\{-1, 1, 1\}$, which will henceforth be used to raise and lower the Lorentz indices. Just as in the case of the original Yang-Mills theory, the dynamics of small fluctuations is described by expanding the action around a fixed background. Substituting

$$A_\mu = A_\mu^0 + \delta A_\mu$$

into (3) and expanding to quadratic order in $\delta A_\mu$, one finds that up to an overall scaling factor the quadratic term in the action is given by

$$\mathcal{L}_2 = \sqrt{1 + (2\pi\alpha' F_0)^2} \left( (1 + 2\pi\alpha' F_0)^{-1} \tilde{F}^{\mu\lambda} (1 + 2\pi\alpha' F_0)^{-1} \tilde{F}^\mu_{\nu}\lambda - \frac{1}{2} (1 + 2\pi\alpha' F_0)^{-1} \tilde{F}^\mu_{\nu}\right)^2.$$
Figure 2: A 2-brane with $q$ flux quanta and its T-dual

where

$$\tilde{F}_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu$$

is the fluctuation. It will be convenient to separate $(1 + 2\pi\alpha'F_0)^{-1}$ into its symmetric and antisymmetric components:

$$(1 + 2\pi\alpha'F_0)^{-1}_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}.$$  

Using this decomposition and ignoring the overall factor of $\sqrt{1 + (2\pi\alpha'F_0)^2}$, the action takes the form

$$\mathcal{L}_2 = g^{\mu\nu} \tilde{F}_{\nu\lambda} g^{\lambda\sigma} \tilde{F}_{\sigma\mu} + \left( B^{\mu\nu} \tilde{F}_{\nu\lambda} B^{\lambda\sigma} \tilde{F}_{\sigma\mu} - \frac{1}{2} (B^{\mu\nu} \tilde{F}_{\mu\nu})^2 \right).$$

Further computation shows that the terms containing $B_{\mu\nu}$ vanish, and $g^{\mu\nu}$ is a diagonal matrix $g^{\mu\nu} = \text{Diag}\{-1, g_{(12)}, g_{(12)}\}$ with

$$g_{(12)} = \frac{1}{1 + (2\pi\alpha'F_0)^2}.$$  

Compared to the dynamics of small fluctuations in Yang-Mills theory given by equation (4), the Born-Infeld action has effectively rescaled the spatial coordinates by a factor of $\sqrt{1 + (2\pi\alpha'F_0)^2}$. This properly rescales the spectrum of the momentum modes so that there is a precise agreement with the spectrum of low-lying open string states (5) seen from the D-brane point of view.

This exercise of establishing the necessity for considering the Born-Infeld action over the Yang-Mills action in order to exactly match the string theory spectrum might seem purely academic in light of the fact that in the field theory limit $2\pi\alpha'F_0$ goes to zero and the factor in question, $1 + (2\pi\alpha'F_0)^2$, is identically 1 to leading order in $\alpha'/L_2^2$. However, there is a way to take a field theory limit keeping $2\pi\alpha'F_0$ constant. Consider the same background as before

$$F_0 = \frac{2\pi}{L_1 L_2} q, \quad q \in \mathbb{Z}.$$  

with $q$ not equal to 1. From the D-brane point of view, this corresponds to a 1-brane with wrapping number $(1, q)$. (See figure 3) Again, we expect the spectrum to be given by (4),
and just as before, we obtain

\[ E^2 = \frac{1}{1 + (2\pi\alpha' F_0)^2} \left( \left( \frac{2\pi}{L_1} n_1 \right)^2 + \left( \frac{2\pi}{L_2} n_2 \right)^2 \right) \]  

(8)

Now, we can take the field theory limit by keeping \( L_1, L_2 \) and \( qa'/L_1L_2 \) constant and sending \( q \) to infinity. This time, the difference between Born-Infeld and Yang-Mills theory is non-trivial even at leading order in \( \alpha'/L_1^2 \).

The fact that the Born-Infeld action captures the dual string spectrum more correctly than Yang-Mills theory is very natural in light of the fact that the dynamics on the world-volume of a D-brane is indeed described by the Born-Infeld action and Yang-Mills theory is only its leading approximation in the field theory limit for a fixed background. In fact, the same type of system as the one we have discussed in this section was analyzed in [25, 26, 27] from the point of view of string theory in a background gauge field, giving essentially the same results. The interesting point about this type of configuration is that by scaling the background as above it is possible to retain the structure of Born-Infeld dynamics even in the field theory limit. The extra terms arising from the Born-Infeld action are usually subleading in the \( \alpha' \) expansion and are associated with the non-locality of string theory. We can in fact see that the winding strings of minimum length attached to the tilted branes are not local objects from the world-volume observer’s point of view, as is illustrated in figure 3. The results from this section indicate that it is possible to enhance the non-locality of string theory even in the field theory limit by scaling up the background field strength.

In the presence of such a background, the interaction terms in the Born-Infeld action will induce a mass shift in the spectrum of excited states to precisely reproduce the string theory spectrum.

### 2.3 1-brane on \( T^2 \) with winding \((p, q)\)

In the previous subsection we considered only configurations with a single 2-brane in the gauge theory picture, and all our analysis was done in the context of the abelian Born-Infeld action. In this subsection we discuss a more general configuration which is T-dual to a D1-brane on \( T^2 \) with wrapping numbers \((p, q)\) with \( p, q \) relatively prime. To describe such a configuration we need to define a \( U(p) \) bundle over \( T^2 \) whose connections have total flux \( 2\pi q/p \). The flux is quantized in units of \( 2\pi/p \) since the gauge group is \( U(p) = (U(1) \times \ldots \right. \)
SU(p)/\mathbb{Z}_p$. We can describe a $U(p)$ bundle in the fundamental representation by giving the boundary overlap functions for sections

\[ \phi(x_1 + L_1, x_2) = \Omega_1(x_2)\phi(x_1, x_2) \]
\[ \phi(x_1, x_2 + L_2) = \Omega_2(x_1)\phi(x_1, x_2). \]

The condition that these boundary conditions give a well-defined $U(p)$ bundle is

\[ \Omega_2^{-1}(L_1)\Omega_1^{-1}(0)\Omega_2(0)\Omega_1(L_2) = 1. \] (9)

To have a total flux of $2\pi q/p$ we can decompose these boundary conditions into abelian and non-abelian components. The abelian component should have flux $2\pi q/p$, while the non-abelian component should have an 't Hooft [24] twist giving a flux $2\pi \tilde{q}/p$ where $0 \leq \tilde{q} < p$ and $\tilde{q} \equiv q(\text{mod } p)$. A standard choice of gauge for defining a bundle of this type is to pick

\[ \Omega_1(x_2) = e^{2\pi i x_2/L_2}U^q \]
\[ \Omega_2(x_1) = V \]

where $U$ and $V$ are the diagonal and shift matrices

\[ U = \begin{pmatrix} 1 & e^{2\pi i/p} & \cdots & e^{2\pi i(p-1)/p} \\ & & & \\ & & & \\ & & & \end{pmatrix}, \quad V = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \] (10)

Choosing this set of boundary conditions, however, complicates the discussion of T-duality in the $x_2$ direction. We will therefore instead choose the boundary conditions

\[ \Omega_1(x_2) = e^{2\pi i x_2/L_2}TV^q \]
\[ \Omega_2(x_1) = 1 \]

with $T$ being the diagonal matrix

\[ T = \frac{q}{p}I + \text{Diag}(-\tilde{q}/p, \ldots, -\tilde{q}/p, 1 - \tilde{q}/p, \ldots, 1 - \tilde{q}/p) \]
\[ = \text{Diag}(n, \ldots, n, n + 1, \ldots, n + 1) \]

where $n$ is the integral part of $q/p$, and where the multiplicities of the diagonal elements of $T$ are $p - q$ and $q$ respectively. Since $T$ has integral diagonal elements, we have

\[ \Omega_1(L_2) = V^q = \Omega_1(0) \]

so (9) is clearly satisfied. The abelian part of $T$ is just $(q/p)I$ so this bundle has the correct flux. Since the boundary conditions are trivial in the $x_2$ direction, we can discuss T-duality in this direction without encountering undue complications.
Figure 4: $p$ 2-branes with $q$ flux quanta and the T-dual $(p, q)$ 1-brane

The gauge fields $A_\mu$ on this bundle will satisfy the boundary conditions

\begin{align}
A_1(x_1 + L_1, x_2) &= e^{2\pi i (x_2 / L_2)T} V^q A_1(x_1, x_2) V^{-q} e^{-2\pi i (x_2 / L_2)T} \\
A_1(x_1, x_2 + L_2) &= A_1(x_1, x_2) \\
A_2(x_1 + L_1, x_2) &= e^{2\pi i (x_2 / L_2)T} V^q A_2(x_1, x_2) V^{-q} e^{-2\pi i (x_2 / L_2)T} + \left(\frac{2\pi}{L_2}\right) T \\
A_2(x_1, x_2 + L_2) &= A_2(x_1, x_2)
\end{align}

(11)

The adjoint scalar fields $X^i$ will satisfy the same boundary conditions as $A_1$. The constant curvature background corresponding to these boundary conditions is

\begin{align}
A_1^0 &= 0 \\
A_2^0 &= F_0 x_1 I + \frac{2\pi}{L_2} \text{Diag}(0, 1/p, \ldots, (p-1)/p)
\end{align}

where

$$F_0 = \frac{2\pi q}{L_1 L_2 p}.$$ 

The T-dual along $x_2$ gives rise to the type of D1-brane configuration illustrated in figure 4, with

$$X_2 = 2\pi \alpha' F_0 x_1 + \frac{4\pi^2 \alpha'}{L_2} \text{Diag}(0, 1/p, \ldots, (p-1)/p)$$

It is straightforward to derive the spectrum of open strings around this D-brane background:

$$E^2 = \frac{1}{p^2 + \left(\frac{4\pi^2 \alpha'}{L_1 L_2}\right)^2 q^2 \left(\frac{2\pi}{L_1 n_1}\right)^2 + \left(\frac{2\pi}{L_2 n_2}\right)^2}.$$  

(12)

In order to reproduce this spectrum in field theory, we need the non-abelian generalization of the Born-Infeld action. Although there are some ambiguities in defining the non-abelian Born-Infeld action, a procedure for defining this action was recently proposed by Tseytlin in [18]. In the following, we will simply follow this proposal, which gives an action of the form

$$\mathcal{L} = \text{STr} \sqrt{-\det(\eta_{\mu\nu} + 2\pi \alpha' F_{\mu\nu})}$$
where “STr” indicates that the trace is to be taken after we symmetrize all the non-commuting products. Our analysis is simplified drastically by the fact that the background gauge field is proportional to the identity and commutes with everything. Expanding the non-abelian Born-Infeld action to second order will therefore lead to

\[
\mathcal{L}_2 = \sqrt{1 + (2\pi \alpha' F_0)^2} \text{Tr}[g^{\mu\nu} \tilde{F}_{\nu\lambda} g^{\lambda\sigma} \tilde{F}_{\sigma\mu}]
\]

(13)

with the same \(g^{\mu\nu}\) we encountered previously. Just as in the previous case, we see that the effect of including the correction arising from the full Born-Infeld type interaction is the rescaling of the spectrum by a factor of \(1 + (2\pi \alpha' F_0)^2\). Thus, we have a spectrum given by

\[
E^2 = \frac{1}{1 + (2\pi \alpha' F_0)^2} \left( k_1^2 + k_2^2 \right).
\]

(14)

where \(k_i\) is the eigenvalue of the covariant derivative operator \(D_i\) acting on the modes satisfying the boundary conditions. Due to the twisting in the boundary conditions, the quantization conditions of these modes are complicated slightly, and the \(p^2\)-fold degeneracy corresponding to the number of independent components of matrices in the adjoint of \(U(N)\) is lifted. Let us now consider the effects of the boundary conditions on the periodicity of the various modes. The inhomogeneous term in the boundary condition for \(A_2\) does not affect the periodicity structure of the fluctuation modes since it is taken care of by the background field. Thus, the fluctuation spectrum will be the same for each of the gauge fields and the transverse scalars. Each of these fields is described by a matrix containing \(p^2\) elements.

It is straightforward to show that the homogeneous part of the boundary condition is satisfied by modes given by

\[
\delta A(x_1, x_2) = \Lambda e^{2\pi i (m_1 x_1 / L_1 + m_2 x_2 / L_2)}
\]

where \(\Lambda\) is a \(p \times p\) matrix

\[
\Lambda = \text{Diag}\{1, \omega, \omega^2, \ldots, \omega^{p-1}\} = \begin{pmatrix}
1 & e^{-2\pi i x_2 / L_2} & \cdots & e^{-2\pi i p x_2 / L_2} \\
\vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & e^{-2\pi i (p-1) x_2 / L_2}
\end{pmatrix}
\]

where \(\omega = e^{2\pi i m_1 s}\) for \(qs \equiv 1\) (mod \(p\)). The boundary condition further implies that

\[
A(x_1 + pL_1, x_2) = A(x_1, x_2)
\]

indicating that \(m_1\) takes on values quantized in units of \(1/p\). Since \(A_1^0\) vanishes, the covariant derivative in the \(x_1\) direction is simply the partial derivative, and it follows that

\[
k_1 = \frac{2\pi m_1}{L_1} = \frac{2\pi n_1}{p}\frac{1}{L_1}; \quad n_1 \in \mathbb{Z}.
\]

(15)
The boundary condition in the $x_2$ direction is trivial, indicating that $m_2 \in \mathbb{Z}$. Due to the presence of the background, however, the covariant derivative is modified by the commutator term. One finds that

$$D_2 \delta A(x_1, x_2) = \frac{1}{i} \partial_2 \delta A(x_1, x_2) - \left[ A^0, \delta A(x_1, x_2) \right] = \frac{2\pi}{L_2} \left( m_2 - \frac{r}{p} \right) \delta A(x_1, x_2).$$

Since $r$ could be any integer between 0 and $p - 1$, we find that $k_2$ is also quantized in units of $1/p$:

$$k_2 = \frac{2\pi n_2}{p L_2}, \quad n_2 \in \mathbb{Z} \quad (16)$$

Thus, in general we find that the $p^2$ degrees of freedom in the matrix appear as modes with momentum quantized in units of $1/p$ in both the $x_1$ and $x_2$ directions. Substituting the expressions for $k_1$ and $k_2$ in equations (15) and (16) back into the spectrum (14), we recover

$$E^2 = \frac{1}{p^2 + \left( \frac{4\pi^2 \alpha'}{L_1 L_2} \right)^2 q^2 \left( \left( \frac{2\pi}{L_1} n_1 \right)^2 + \left( \frac{2\pi}{L_2} n_2 \right)^2 \right)^2} \quad (17)$$

in agreement with the string theory answer. It is interesting to note that the fractional quantization of momentum in the $x_1$ direction appears naturally as a periodicity on a $p$-fold covering, whereas the fractional quantization in the $x_2$ direction appears because there are $p$ different “types” of excitations, labeled by which diagonal of the matrix they appear in. These two distinct mechanisms for fractional quantization correspond naturally to the string geometry in the D-brane picture, where the $x_1$ excitations correspond to momentum on a $p$-fold wrapped brane while the $x_2$ excitations correspond to strings connecting branes separated by a distance proportional to $i - j$.

This explicit construction in the context of gauge theory gives an interesting point of view on fractional momentum states, which have been considered in a number of contexts [28, 29, 30]. There is a close correspondence between this discussion and the description of fractional momentum states as arising from the twisted sector in an orbifold by the center of the gauge group [31, 32].

3 Intersecting Branes and Non-Abelian Born-Infeld

In the previous section, we examined the D-brane/gauge theory correspondence for tilted D-branes, corresponding to a gauge theory background in the central $U(1)$ of $U(N)$. In this section, we examine systems of intersecting D-branes which correspond to constant curvature backgrounds with reducible connections. When the constant background field can be expressed in terms of a background connection $A^0$ with components which all commute, the T-dual system corresponds to a system of intersecting branes with well-defined positions. For concreteness, we will concentrate on systems corresponding to gauge theories with $N = 2$. The constant curvature $U(2)$ backgrounds which describe dual configurations of intersecting
D-branes are precisely those background fields which can be described in terms of reducible connections. Thus, in the gauge theory language, we will be considering connections which decompose as a direct sum of $U(1)$ connections. This restriction simplifies the analysis somewhat. A complete classification of constant curvature connections on $SU(2)$ bundles over $T^4$ was given by van Baal in [17]; in this work he also calculated the Yang-Mills fluctuation spectra around these backgrounds. In the D-brane language, the corresponding backgrounds describe a pair of branes intersecting at angles; the spectrum of low-lying open strings in such backgrounds have been studied in [9]. Although the spectra computed in [17] and [9] have striking qualitative similarities, they are distinct in certain details. We will investigate the possibility of resolving this discrepancy by considering the full non-abelian Born-Infeld action. We also discuss the dynamics of non-supersymmetric backgrounds.

3.1 Branes at angles

We begin our discussion by reviewing the D-brane construction of [9]. One of the main points made in [9] is the fact that there exist supersymmetric configurations of branes intersecting at angles. Here, we consider a general class of configurations of intersecting branes which includes non-supersymmetric brane configurations.

The configurations we are interested in correspond to intersecting D-branes living on tori. As discussed above, these configurations are dual to gauge theory backgrounds with reducible connections. We will here restrict attention to pairs of intersecting branes, concentrating in particular on the case of two 2-branes on $T^4$. There are several kinds of string excitations around such a background. There are strings which stretch from each of the two 2-branes back to itself. These strings carry momentum and winding numbers. These excitations correspond to modes living in one of the reducible components of the dual bundle and are essentially equivalent to those discussed in the previous section. Thus, we will not consider these modes further here but will concentrate on the strings connecting the two different branes. Unlike the strings connecting a brane to itself, the strings connecting the different branes are unaffected by the compactification of the space in which the branes are living. This can be seen by going to the covering space, where all strings connecting the two branes are homotopically connected (as long as they are associated with the same intersection point). Of course, at higher order there are diagrams connecting these strings with winding strings connecting a brane to itself. However, these diagrams will not affect the fluctuation spectrum. Thus, to understand the fluctuation spectrum of the strings connecting the distinct 2-branes, it is sufficient to understand the spectrum of strings connecting two 2-branes in noncompact space. This is precisely the situation considered in [9], and we recapitulate their discussion here with the additional feature that we do not insist on the supersymmetry of the D-brane configuration.

A sufficiently general configuration of two 2-branes in $\mathbb{R}^4$ can be defined by beginning with two 2-branes oriented along the 14 axis, and rotating one of them by an angle $\theta_1$ in the 12 plane and by an angle $\theta_2$ in the 34 plane. For $\theta_1 = \theta_2$, this configuration was shown to
be supersymmetric in \cite{9}. Following \cite{9}, we use complex coordinates $Z_1$ and $Z_2$ instead of $X^1 + iX^2$ and $X^4 + iX^3$, respectively.

In world-sheet language, strings stretching between the two branes are described by open strings with boundary conditions

\[
\begin{align*}
\text{Re } & \frac{\partial}{\partial \sigma} Z^i|_{\sigma=0} = 0, \\
\text{Im } & Z^i|_{\sigma=0} = 0, \\
\text{Re } & e^{i\theta_1} \frac{\partial}{\partial \sigma} Z^i|_{\sigma=\pi} = 0, \\
\text{Im } & e^{i\theta_1} Z^i|_{\sigma=\pi} = 0.
\end{align*}
\]

The mode expansion for the complex bosonic field is given by

\[
Z^i(w, \bar{w}) = \sum_{m \in \mathbb{Z}} \left\{ x^i_{\alpha+i+m} e^{i(m-\alpha_i)w} + \bar{x}^i_{\alpha+i+m} e^{-i(m+\alpha_i)\bar{w}} \right\}
\]

where $w = \sigma + \tau$, $\bar{w} = \sigma - \bar{\tau}$ and $\alpha_i = \theta_i/\pi$. The vacuum energy

\[
\Delta E = -\frac{1}{2} + \frac{\alpha_1}{2} + \frac{\alpha_2}{2}
\]

can be computed either from the $\zeta$-function regularization of the sum

\[
\begin{align*}
\alpha + (\alpha + 1) + (\alpha + 2) + & \ldots = -\frac{1}{12} + \frac{1}{8}\alpha, \\
\left(\frac{1}{2} + \alpha\right) + \left(\frac{3}{2} + \alpha\right) + & \ldots = \frac{1}{24} - \frac{1}{8}\alpha
\end{align*}
\]

or from the dimensions of the world sheet operators needed to twist the boundary condition \cite{9, 33, 34}. The low-lying open string excitations in the NS sector described in equation (3.5) of \cite{1} generalize to

\[
\begin{align*}
\bar{\psi}^i_{\alpha_1 - 1/2}(x^1_{-\alpha_1})^{n_1}(x^2_{-\alpha_1})^{n_2}|0\rangle; & \quad \bar{\psi}^i_{\alpha_2 - 1/2}(x^1_{-\alpha_1})^{n_1}(x^2_{-\alpha_2})^{n_2}|0\rangle \\
\psi^i_{\alpha_1 - 1/2}(x^1_{-\alpha_1})^{n_1}(x^2_{-\alpha_1})^{n_2}|0\rangle; & \quad \psi^i_{\alpha_2 - 1/2}(x^1_{-\alpha_1})^{n_1}(x^2_{-\alpha_2})^{n_2}|0\rangle \\
\psi^\mu_{1/2}(x^1_{-\alpha_1})^{n_1}(x^2_{-\alpha_2})^{n_2}|0\rangle
\end{align*}
\]

and their contribution to the masses are, respectively,

\[
\begin{align*}
(-\alpha_1 + 1/2) + n_1 \alpha_1 + n_2 \alpha_2 + \Delta E & = (n_1 - 1/2)\alpha_1 + (n_2 + 1/2)\alpha_2 \\
(-\alpha_2 + 1/2) + n_1 \alpha_1 + n_2 \alpha_2 + \Delta E & = (n_1 + 1/2)\alpha_1 + (n_2 - 1/2)\alpha_2 \\
(\alpha_1 + 1/2) + n_1 \alpha_1 + n_2 \alpha_2 + \Delta E & = (n_1 + 3/2)\alpha_1 + (n_2 + 1/2)\alpha_2 \\
(\alpha_2 + 1/2) + n_1 \alpha_1 + n_2 \alpha_2 + \Delta E & = (n_1 + 1/2)\alpha_1 + (n_2 + 3/2)\alpha_2 \\
1/2 + n_1 \alpha_1 + n_2 \alpha_2 + \Delta E & = (n_1 + 1/2)\alpha_1 + (n_2 + 1/2)\alpha_2
\end{align*}
\]

This gives a complete characterization of the fluctuation spectra of strings stretching between two 2-branes at angles $\theta_1, \theta_2$, including cases where the brane configuration is not BPS. Note that when $\alpha_1 \neq \alpha_2$ the spectrum contains a tachyon.
3.2 Yang-Mills fluctuation spectra

We will now discuss a gauge theory configuration whose dual corresponds to two 2-branes on $T^4$. As stated, we will restrict attention to $U(2)$ bundles with connections which are reducible to a direct sum of two $U(1)$ connections. The discussion could be generalized in a straightforward way to arbitrary reducible $U(N)$ bundles. It was pointed out by 't Hooft [24] that for $N > 2$ there are constant curvature connections which do not satisfy abelian boundary conditions; these correspond to D-brane configurations with fewer than $N$ D-branes, with some of the D-branes wrapped multiple times around the base space as in Section 2.3.

We will consider here a constant background field given by a $U(2)$ connection $A^0_\mu$ on $T^4$ with the following components

\[
A^0_1 = 0 \\
A^0_2 = \frac{\pi}{L_2 L_1} n_{21} x_1 \tau_3 \\
A^0_3 = \frac{\pi}{L_3 L_4} n_{34} x_4 \tau_3 \\
A^0_4 = 0.
\]

where $L_\mu$ are the dimensions of the 4-torus and $n_{\mu\nu}$ is an anti-selfdual matrix with with $n_{21} = -n_{12} = n_{34} = -n_{43} = 2$ and zero otherwise. $\tau_3$ is the SU(2) generator $\text{Diag}\{1, -1\}$. This configuration has an instanton number $\frac{C_2 = n_{\mu\nu}\tilde{n}_{\mu\nu}}{8} = 2$.

Because the field strength is proportional to $\tau_3$ the first Chern class vanishes. Thus, this configuration corresponds to a (singular) point in the moduli space of $U(2)$ instantons on $T^4$ with instanton number $k = 2$. (Note that while a single instanton on $T^4$ shrinks to a point, there is a regular moduli space of instantons with $k = 2$.) Performing a T-duality transformation along directions 2 and 3 will invert the periods along these directions, and will give rise to a pair of 2-branes embedded according to the equations (see figure 5)

\[
X_2 = 2\pi\alpha' A^0_2 = \pm \frac{2\pi^2 \alpha'}{L_1 L_2} n_{21} x_1 \\
X_3 = 2\pi\alpha' A^0_3 = \pm \frac{2\pi^2 \alpha'}{L_3 L_4} n_{34} x_4
\]

The field strength $F_{\mu\nu}$ can immediately be related to the angles $\theta_1, \theta_2$ between the branes:

\[
\tan(\theta_1/2) = 2\pi\alpha' F_{21} \\
\tan(\theta_2/2) = 2\pi\alpha' F_{34}
\]

The connection $A^0_\mu$ is gauge equivalent to one considered in [17],

\[
A_\mu(x) = -\frac{1}{2} \frac{\pi}{L_\mu L_\nu} n_{\mu\nu} x_\nu \tau_3.
\]
Van Baal investigated the spectrum of $SU(2)$ Yang-Mills fluctuations around backgrounds of this type. He explicitly expanded the Yang-Mills action around $A^0$ using

$$A_\mu = A^0_\mu + \left( \frac{b^1_\mu}{\sqrt{2}c^*_\mu} \right)$$

to second order in $b_\mu$ and $c_\mu$. Here, $b_\mu$ and $c_\mu$ are real and complex fields respectively; in $SU(2)$, we would have $b^1_\mu = -b^2_\mu$. There are also ghost fields $\psi$ living in the adjoint representation. The action, expressed in these variables, takes the form

$$S = S_0 + \int d^4x \left[ \frac{1}{2} b^a_\mu (M_0 \delta^{\mu\nu} \delta^{ab}) b^b_\nu + c^*_\mu (M_n \delta^{\mu\nu} - 4\pi i F^{\mu\nu}) c_\nu + \text{Tr} (\psi^\dagger M_{gh} \psi) + \mathcal{O}(\delta A^3_\mu) \right]$$

with

$$M_0 = \left( \frac{\partial}{\partial x^\mu} \right)^2$$

$$M_n = \left( \frac{\partial}{\partial x^\mu} - \pi F_{\mu\nu} x^\nu \right)^2$$

$$M_{gh} = -D^2$$

As discussed in Section 3.1, the fields $b^i$ correspond to strings in the dual picture connecting a brane with itself, and have spectra corresponding to the discussion in Section 2. Thus, we concentrate now on the spectrum for the field $c$ corresponding to strings stretching between the two branes. The computation of this spectrum is described in detail in [17], so we simply summarize the results of that investigation here\footnote{Although van Baal was studying the purely bosonic four-dimensional theory, and thus did not include the adjoint scalars associated with transverse fluctuations of the D-brane world-volume in his analysis, the quadratic forms in the ghost kinetic terms are identical in structure to the quadratic forms of these adjoint scalars and can be used to compare their spectra.}. The boundary conditions on $c$ indicate that it is a theta function on $T^4$. The operator $M_n$ acts on the appropriate space of theta functions as a Laplace-Beltrami operator up to a constant. The eigenfunctions of this operator were classified in [36], and are given by a sum over the lattice of harmonic oscillator eigenfunctions rotated by linear phases. The low-lying eigenfunctions are localized in the vicinity of the
Figure 6: (12)-plane cross section of two 4-branes with a constant field strength many-instanton background and the T-dual intersecting 2-branes with multiple winding.

string intersection points. We will discuss a specific example of such eigenfunctions more explicitly in Section 3.4. The spectrum of eigenvalues for the four-dimensional system of interest here is given in equation (3.25) of [17]:

\[
M_n - 4\pi iF : \begin{align*}
2\pi (2m_1 - 1) f_1 + 2\pi (2m_2 + 1) f_2 \\
2\pi (2m_1 + 1) f_1 + 2\pi (2m_2 - 1) f_2 \\
2\pi (2m_1 + 3) f_1 + 2\pi (2m_2 + 1) f_2 \\
2\pi (2m_1 + 1) f_1 + 2\pi (2m_2 + 3) f_2
\end{align*}
\]

\[
M_{gh} : 2\pi (2m_1 + 1) f_1 + 2\pi (2m_2 + 1) f_2
\]

where

\[
f_1 = \frac{1}{\pi} F_{21} = \frac{\tan(\theta_1/2)}{2\pi^2 \alpha'}
\]

\[
f_2 = \frac{1}{\pi} F_{34} = \frac{\tan(\theta_2/2)}{2\pi^2 \alpha'}
\]

We therefore see that the spectrum of fluctuations around this constant field strength background has precisely the structure of the oscillator excitations found from the string theory point of view in [18] up to the identification

\[
\frac{2\tan(\theta_1/2)}{\pi} \leftrightarrow \alpha_1 = \frac{\theta_1}{\pi}
\]

\[
\frac{2\tan(\theta_2/2)}{\pi} \leftrightarrow \alpha_2 = \frac{\theta_2}{\pi}
\]

Thus, the fluctuation spectra computed from these two points of view agree up to a discrepancy between \(\theta/2\) and \(\tan(\theta/2)\). For small \(\theta\) these spectra agree exactly.

### 3.3 Resolving the discrepancy using non-abelian Born-Infeld

The difference in the spectra resulting from the discrepancy between \(\theta/2\) and \(\tan(\theta/2)\) might seem minor in light of the fact that they are in perfect agreement at small angles, and that in the field theory limit keeping the background constant, \(\theta\) goes to zero. However, just as
in Section 2, one can take a field theory limit while scaling the background to keep $\theta$ finite but small compared to 1. To be more specific, we can consider the background

$$A_\mu(x) = -\frac{1}{2} \frac{\pi}{L_\mu L_\nu} q n_{\mu \nu} x_\nu \tau_3, \quad q \in \mathbb{Z}$$

and take the limit $\alpha'/L_1^2 \to 0$ keeping the ratios of $L_\mu$'s and $\sqrt{q\alpha'}$ constant. After a gauge transformation to set $A_1 = A_4 = 0$, this will lead in the T-dual picture to 2-branes wrapping multiply around the 23 cycles as illustrated in figure 6.

We saw in Section 2 that scaling the field strength enhances the terms ordinarily subleading in $\alpha'$ and that the full Born-Infeld action was needed to recover the exact correspondence in the case of tilted branes. It is therefore natural to suspect that the same mechanism underlies the resolution of the $\theta/2$ vs. $\tan(\theta/2)$ discrepancy. We investigate this possibility in this subsection. The analysis is complicated by the fact that the background field does not commute with all fluctuations. The field theory analysis therefore depends on the full structure of the non-abelian Born-Infeld action. Recently, a concrete formulation of the non-abelian Born-Infeld action was proposed by Tseytlin [18]. Using Tseytlin’s Born-Infeld action, we compute the spectrum of fluctuations around the same background as was considered in the previous subsection. A remarkable relation emerges in the analysis, bringing the field theory spectrum tantalizingly close to the string theory spectrum of section 3.1. However, we fall slightly short of reproducing the string theory spectrum in its entirety. We interpret this situation in the following manner: Due to the highly non-trivial nature of the partial success in resolving the $\theta/2$ vs. $\tan(\theta/2)$ discrepancy, there must be a strong element of truth in Tseytlin’s formulation of the non-abelian Born-Infeld action, but in as much as it fails to reproduce the string theory spectrum precisely, it is still in some sense incomplete. Reformulating the Born-Infeld action to resolve this discrepancy is likely to be a rather tricky enterprise. Nonetheless, we feel that reproducing the spectrum of a system of D-branes intersecting at an angle will serve as a useful test and could possibly be used as a guiding principle in formulating the elusive non-abelian Born-Infeld action.

The key feature of Tseytlin’s non-abelian Born-Infeld action is its resolution of product ordering ambiguities through symmetrization

$$\mathcal{L} = \text{STr} \sqrt{-\det(\eta_{\mu \nu} + 2\pi \alpha' F_{\mu \nu})}.$$ 

Our strategy here is to perform formal manipulations of quantities inside the symmetric trace as if they were abelian. At the last stage of the analysis, we will re-insert the non-commuting factors and perform the symmetrized trace, thereby promoting the abelianized action back up to its original non-abelian counterpart. In our analysis we consider the symmetrized trace term by term in an expansion with respect to the background as well as the fluctuation. For example, a term in the expansion corresponding to an abelianized expression,

$$(F_0)^n X^2,$$
where \( F_0 = F_0^3 \tau_3 \) and \( X = \sum X^a \tau_a \) are in the adjoint of \( SU(2) \), is replaced by

\[
\text{STr}(F_0 \tau_3)^m(\tau_a X^a)(\tau_b X^b) = F_0^m X^a X^b \left[ \frac{1}{m+1} \sum_{i=0}^m \text{Tr}(\tau_3^i \tau_a \tau_3^{m-i} \tau_b) \right]
\]

upon reinsertion of non-commuting factors. Let us consider in particular the case where \( a \) and \( b \) are restricted to the range of values \( \{1, 2\} \). Simple algebraic manipulations of the Pauli matrices show that

\[
\text{Tr}(\tau_3^i \tau_a \tau_3^{m-i} \tau_b) = \text{Tr}((-1)^i \tau_3^m \tau_b \tau_a)
\]

At this point, it is convenient to consider the cases of \( m \) even and \( m \) odd separately. For \( m \) odd,

\[
\sum_{i=0}^m \text{Tr}((-1)^i \tau_3^m \tau_b \tau_a) = \sum_{i=0}^m \text{Tr}((-1)^i \tau_3 \tau_b \tau_a) = 0
\]

whereas for \( m \) even,

\[
\sum_{i=0}^m \text{Tr}((-1)^i \tau_3^m \tau_b \tau_a) = \sum_{i=0}^m \text{Tr}((-1)^i \tau_b \tau_a) = \delta_{ab}
\]

Therefore, if in the abelian Born-Infeld action the terms quadratic in fluctuations had an expansion consisting solely of terms with even powers in the background:

\[
S_{BI}(F_0, \tilde{F}) = (\tilde{F}^2) \left( a_0 + a_1 F_0^2 + a_2 F_0^4 + \ldots a_m F_0^{2m} + \ldots \right)
\]

then its non-abelian generalization for \( \tau_1 \) and \( \tau_2 \) fields will give rise to an expansion of the form

\[
S_{NBI}(F_0, \tilde{F}) = \delta_{ab} (\tilde{F}^a \tilde{F}^b) \left( a_0 + \frac{a_1}{3} F_0^2 + \frac{a_2}{5} F_0^4 + \ldots + \frac{a_m}{2m+1} F_0^{2m} + \ldots \right).
\]

These two expansions are related by

\[
S_{NBI}(F_0, \tilde{F}) = \frac{1}{F_0} \int_0^{F_0} dF_0 S_{BI}(F_0, \tilde{F})
\]

The goal at this point is to examine the fluctuations of

\[
\mathcal{L} = \text{STr} \sqrt{- \det(\eta_{\mu\nu} + 2\pi \alpha' F_{\mu\nu} + 2\pi \alpha' \tilde{F}_{\mu\nu})}
\]

in a background defined by

\[
A_\mu = A_\mu^0 + \delta A_\mu
\]

and

\[
\tilde{F}_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu - i[\delta A_\mu, A_\nu^0] - i[A_\mu^0, \delta A_\nu] - i[\delta A_\mu, \delta A_\nu]
\]

We expand the action to quadratic order using the standard identity

\[
\sqrt{\det(M_0 + \delta M)} = \sqrt{\det(M_0)} \left( 1 + \frac{1}{2} \text{Tr}[M_0^{-1} \delta M] + \frac{1}{8} (\text{Tr}[M_0^{-1} \delta M])^2 - \frac{1}{4} \text{Tr}[M_0^{-1} \delta M M_0^{-1} \delta M] \right)
\]

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In terms of our variables, the quadratic action reads

$$L_2 = \text{Str} \left[ \sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}^0)} \left\{ \frac{1}{2} B^{\mu\nu} \tilde{F}_{\nu\mu} - \frac{1}{4} g^{\mu\nu} \tilde{F}_{\nu\lambda} g^{\lambda\sigma} \tilde{F}_{\sigma\mu} \right. ight. $$

$$- \frac{1}{4} \left. \left( B^{\mu\nu} \tilde{F}_{\nu\lambda} B^{\lambda\sigma} \tilde{F}_{\sigma\mu} - \frac{1}{2} \left( B^{\mu\nu} \tilde{F}_{\nu\mu} \right)^2 \right) \right\} \right]$$

(21)

where now,

$$g^{\mu\nu} = (\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}^0)^{-1} = \text{Diag}\{-1, g_{(12)}, g_{(12)}, g_{(34)}, g_{(34)}\}$$

with

$$g_{(12)} = \frac{1}{1 + (2\pi\alpha' F_{21})^2}$$

$$g_{(34)} = \frac{1}{1 + (2\pi\alpha' F_{34})^2}$$

and

$$B^{\mu\nu} = -g^{\mu\lambda} F_{\lambda\rho} \eta_{\rho\nu}.$$ 

The term linear in $\tilde{F}$ contributes at quadratic order because the commutator term in $\tilde{F}$ is quadratic in the fluctuations. This time, the terms quadratic in $B^{\mu\nu}$ turn out not to vanish but give a topological contribution

$$B^{\mu\nu} \tilde{F}_{\nu\lambda} B^{\lambda\sigma} \tilde{F}_{\sigma\mu} - \frac{1}{2} \left( B^{\mu\nu} \tilde{F}_{\nu\mu} \right)^2 = \frac{2F_{21} F_{34}}{(1 + (2\pi\alpha' F_{21})^2)(1 + (2\pi\alpha' F_{34})^2)} \epsilon^{\mu\nu\lambda\sigma} \tilde{F}_{\mu\nu} \tilde{F}_{\lambda\sigma}$$

which can be ignored in the discussion of the fluctuation spectrum.

In order to compute the fluctuation spectrum we now must apply the integral prescription (20) to the action in order to deal with the lack of commutativity between the fluctuations and the background. Tseytlin’s symmetrized trace prescription indicates that we should associate a single $U(2)$ generator with each factor of the field strength $F$. This gives a well-defined formulation of a non-abelian action. With this prescription, the action as a function of the components of the field strength and its off-diagonal fluctuations should be computed by applying (20) to the terms quadratic in $\tilde{F}$ in (21). The term linear in $\tilde{F}$ is not modified, since it is only nonzero when $\tilde{F}$ commutes with the background. Unfortunately, this does not seem to give exactly the results that we would expect from string theory. Nonetheless, there is some indication that this prescription comes close to doing the right thing. To see this let us consider a special case where the field strength is anti-self-dual, corresponding to a BPS brane configuration. In this case we have $F_{21} = F_{34}$. The determinant factor in the action gives a single factor of $g_{(12)}^{-1}$. This cancels with the $g$ factors in the terms quadratic in $\tilde{F}_{0i}$, so that the kinetic terms are canonically normalized. The potential terms quadratic in $\tilde{F}$ carry two factors of $g$ so that they are multiplied by an overall factor of $g_{(12)}$ compared to the pure Yang-Mills theory.

At this point, we encounter a pleasant surprise. Applying the relation (20) to the metric leads to

$$\frac{1}{F_{21}} \int_0^{F_{21}} dF \ g_{(12)} = \frac{1}{F_{21}} \int_0^{F_{21}} dF \ \frac{1}{1 + (2\pi\alpha' F)^2} = \frac{\tan^{-1}(2\pi\alpha' F_{21})}{F_{21}} = \frac{\theta_1/2}{\tan(\theta_1/2)}$$

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This is precisely the rescaling necessary to match the field theory spectrum with the string theory result! In fact, in this case we find that the spacing of the fluctuation spectrum corresponds precisely to that predicted by string theory. Unfortunately, however, the energy of the lowest fluctuation is also affected by the term linear in $\tilde{F}$. The factors of $g$ cancel in this term so that the calculation of standard Yang-Mills theory is not modified by the symmetrized trace form of the Born-Infeld action. Thus, even in the special case where the field is anti-self-dual, we do not get exactly the spectrum predicted by string theory.

We would expect that even in the more general case where the field is not anti-self-dual, the spectra computed from string theory and from the non-abelian Born-Infeld theory should agree. In the more general case, however, we find that even the spacing of the energy levels fails to be reproduced exactly by the symmetrized trace prescription. Thus, it seems clear that this specific formulation of the non-abelian Born-Infeld theory is not sufficiently general to capture all the relevant physics of the situation we are considering here.

There are several possible explanations for why the symmetrized trace prescription does not exactly match the string theory calculation. One possibility is that the association of a single generator to each factor of the field strength should be modified. As an example of this sort of modification, we might choose to normalize the fluctuations $\delta A$ so that the kinetic terms in the action always have a canonical normalization. For cases where the field strength is not anti-self-dual this will incorporate extra factors into the potential terms. It can be shown that these extra terms guarantee that the scaling factor associated with the quadratic term in $\tilde{F}$ will give fluctuations with a level spacing precisely that predicted by string theory, as we showed above explicitly in the anti-self-dual case. Unfortunately, however, it does not seem to be possible to arrange the normalization so that the linear term in $\tilde{F}$ also scales correctly.

Another possible reason why the symmetrized trace prescription does not reproduce the string theory results correctly is that covariant derivative terms $DF$ or commutator terms $[F,F]$ may be relevant in this situation. In Tseytlin’s discussion of the non-abelian Born-Infeld action, terms of these forms were explicitly dropped. Since the fluctuations we are concerned with here are theta functions which give nontrivial spatially dependent fluxes, it is quite possible that the effects of covariant derivative terms must be incorporated into the non-abelian Born-Infeld action in order to correctly reproduce the string theory results. A discussion of a related discrepancy was given in [27], where it was shown that it was necessary to add commutator and covariant derivative terms to the Born-Infeld action already at order $F^4$ to reproduce results from string theory.

In any case, we do not have a complete understanding of why the prescription we have described corrects some, but not all, of the terms appearing in the action, and as such we have not provided a satisfactory resolution to the problem of the $\theta/2$ vs. $\tan(\theta/2)$ discrepancy. Nonetheless, the relation between $\theta/2$ and $\tan(\theta/2)$ is transcendental, and subtle details must work out just right in order for the correct transformation to emerge from this kind of analysis. The fact that one can naturally derive from the symmetrized trace prescription
The precise scaling factor needed to achieve agreement with string theory seems to provide some evidence that there is an element of truth in this description. We believe that when it is correctly defined, the full non-abelian Born-Infeld action should resolve the discrepancy we have discussed here. We therefore pose the resolution of the $\theta/2$ vs. $\tan(\theta/2)$ discrepancy as a test which should be satisfied by any candidate for the full non-abelian Born-Infeld action.

### 3.4 Brane-anti-brane configurations

We have discussed the correspondence between fluctuation spectra in a variety of situations corresponding to tilted and intersecting D-branes. As mentioned in Section 3.1, the formulae for the fluctuation spectra around intersecting D-branes extend to cases where the configuration of branes breaks supersymmetry. However, in such situations the state is no longer BPS and generally admits a tachyonic instability [37]. In this subsection we discuss the simplest example of such an instability in slightly more explicit terms.

Consider a pair of 1-branes wrapped on $T^2$ with winding numbers $(1, 1)$ and $(1, -1)$. These 1-branes intersect at two points. This configuration is unstable, since a simple change in the topology at one of the crossing points turns the configuration into a single 1-brane with winding $(2, 0)$ (see Figure 7). We can understand this instability in the system from both the string theory and the Yang-Mills/Born-Infeld point of view. In the gauge theory picture, we have a pair of 2-branes on $T^2$, one carrying a unit of 0-brane charge, and the other carrying a negative 0-brane charge, corresponding to a single anti-0-brane. Thus, the tachyonic instability in this picture corresponds to the instability of a brane-anti-brane configuration.

To be specific, we will consider a $U(2)$ gauge theory on a $T^2$ with sides of length $L_1, L_2$. We will take the boundary conditions to be given by

$$\Omega_1(x_2) = e^{2\pi i (x_2/L_2) \tau_3}$$
\[ \Omega_2(x_1) = I \]

Although these seem like nontrivial boundary conditions, they define a trivial \( U(2) \) bundle over \( T^2 \) in an unusual choice of gauge; this gauge choice allows us to write the background connection in a form which manifestly corresponds to the desired T-dual D-brane configuration. The boundary conditions on the gauge fields are given by

\[
\begin{align*}
A_1(x_1 + L_1, x_2) &= e^{2\pi i (x_2/L_2) \tau_3} A_1(x_1, x_2) e^{-2\pi i (x_2/L_2) \tau_3} \\
A_1(x_1, x_2 + L_2) &= A_1(x_1, x_2) \\
A_2(x_1 + L_1, x_2) &= e^{2\pi i (x_2/L_2) \tau_3} A_2(x_1, x_2) e^{-2\pi i (x_2/L_2) \tau_3} + \left( \frac{2\pi}{L_2} \right) \tau_3 \\
A_2(x_1, x_2 + L_2) &= A_2(x_1, x_2)
\end{align*}
\]

We will be considering the constant curvature background corresponding to the connection

\[
\begin{align*}
A^0_1 &= 0 \\
A^0_2 &= \frac{2\pi}{L_1 L_2} x_1 \tau_3.
\end{align*}
\]

Clearly, through T-duality this corresponds to a pair of 1-branes whose transverse positions are given by

\[ X_2 = \pm \frac{4\pi^2 \alpha'}{L_1 L_2} x_1, \]

just as shown in figure 7.

The tachyonic instability of this system is easy to see in the D-brane language. The string excitation spectrum is essentially given by (18) where we set \( \alpha_2 = n_2 = 0 \) and where \( \alpha_1 \) is related to the angle \( \theta \) between the 1-branes by \( \alpha_1 = \theta/\pi \). As mentioned in section 3.1, this spectrum contains a tachyon.

We now discuss the instability of this system from the gauge theory point of view. As we have seen in previous sections, the Born-Infeld action is necessary to get the precise normalization of the fluctuation spectra. However, the sign of the various fluctuations are not dependent upon such a precise calculation, so we should be able to detect the instability within the simple Yang-Mills framework. To find the unstable modes, we can use essentially the same analysis as that of van Baal [17], simply setting to zero one of the two curvatures \( f_1, f_2 \). We find that indeed there are modes with negative eigenvalues corresponding to unstable directions in the system. To describe these modes more precisely, let us consider the boundary conditions on the off-diagonal modes of the system. Decomposing as above

\[ A_\mu = A^0_\mu + \left( \begin{array}{c} b^1_\mu \\ \sqrt{2} c_\mu \\ b^2_\mu \end{array} \right) \]

we find that each \( c_\mu \) satisfies the boundary conditions

\[
\begin{align*}
c_\mu(x_1 + L_1, x_2) &= e^{-4\pi i (x_2/L_2)} c_\mu(x_1, x_2) \\
c_\mu(x_1, x_2 + L_2) &= c_\mu(x_1, x_2).
\end{align*}
\] (22)
These conditions imply that \(c_\mu\) is a section of a \(U(1)\) bundle with \(C_1 = 2\). We can express \(c_\mu\) as a theta function satisfying the condition

\[
\Theta(z + q) = e^{2\pi i q \bar{z} + \pi |q|^2} \Theta(z)
\]

where \(q \in \mathbb{Z} + i\mathbb{Z}\) by transforming

\[
c_\mu(x_1, x_2) = \exp \left( \frac{2\pi i x_1 x_2}{L_1 L_2} - \pi \left( \frac{x_1^2}{L_1^2} + \frac{x_2^2}{L_2^2} \right) \right) \Theta(x_1/L_1 + ix_2/L_2).
\]

A general discussion of the theta functions of this type which are eigenfunctions of the Laplace-Beltrami operator is given in [36]. In the particular case of interest here, the description of the states in terms of the boundary conditions (22) is fairly straightforward. We are looking for eigenfunctions of the operator

\[
M_n = -\partial_1^2 + \left( \frac{4\pi x_1}{L_1 L_2} \right)^2 - \frac{8\pi}{L_1 L_2}
\]

with the minimal eigenvalue, subject to the condition that the eigenfunctions must satisfy the boundary conditions (22). There are precisely two such eigenfunctions with the minimum eigenvalue \(-4\pi/(L_1 L_2)\). These functions are given by

\[
c(x_1, x_2) = \sum_{n, m \in \mathbb{Z}} \exp \left( -\pi \frac{x_1^2}{L_1^2} + (x_2 - L_2 m)^2 \right) \Theta(x_1/L_1 + ix_2/L_2),
\]

and

\[
c'(x_1, x_2) = \sum_{\hat{n}, \hat{m} \in \mathbb{Z} + 1/2} \exp \left( \pi i (\hat{n} + \hat{m}) - \frac{\pi}{L_1 L_2} [(x_1 - L_1 \hat{n})^2 + (x_2 - L_2 \hat{m})^2] \right)
\]

\[
+ 2\pi i \left( \frac{x_1}{L_1} \hat{m} - \frac{x_2}{L_2} \hat{n} \right) - 2\pi i \frac{x_1 x_2}{L_1 L_2}).
\]

These solutions are given by superimposing copies of the gaussian harmonic oscillator ground state centered around points on a lattice. As \(L_2\) becomes small, these functions become localized around the values \(x_1 = 0\) and \(x_1 = L_1/2\), which are precisely the \(x_1\) values of the two points of intersection for the dual D-branes. In the corresponding D-brane picture, as \(L_2\) becomes small the dual radius becomes large and the angle between the branes increases, which as discussed in [9] should localize the string modes in the vicinity of the intersection points. Furthermore, since the unstable modes are associated with the \(c\) entries of the \(U(2)\) matrix, which affect the commutativity between the two brane positions, it is natural to interpret these modes geometrically in terms of a change in topology localized to the brane intersection points. This corresponds perfectly with what we would expect from the geometrical picture shown in figure 7. How this explicit description of the unstable mode relates to the D-brane instability is difficult to make completely precise, however, due to the nonlocality of the T-duality transformation in the \(x_2\) direction. A better understanding of this relationship might lead to interesting results about brane-anti-brane interactions which could be understood from a simple field theory perspective.
4 Conclusions

In this paper, we examined the spectra of low-energy fluctuations around various D-brane configurations in string theory and the T-dual spectra of fluctuations around constant background fields in gauge theory.

Background fields in the central $U(1)$ of a $U(N)$ gauge theory correspond to tilted branes on the dual torus. By taking the field theory limit while keeping the tilt angle constant, we encountered a discrepancy between the field theory spectrum and the string theory spectrum. The discrepancy was resolved by computing the fluctuation spectrum of the full Born-Infeld action instead of the Yang-Mills action. This is a natural result in light of the fact that the scaling limit which keeps the tilt constant corresponds to scaling up the background field strength keeping $2\pi\alpha'F$ constant. This forces the terms ordinarily subleading in $\alpha'$ to play a relevant role.

We also examined the spectrum of excitations around constant background fields from reducible connections, corresponding to branes intersecting at an angle on the dual torus. We found that the results of [9] can be extended to non-supersymmetric configurations and that the dynamics of tachyons can be studied naturally in the context of gauge theories. In the small angle limit, the D-brane and Yang-Mills spectra are in agreement. The correspondence breaks down, however, when the angle of intersection is fixed and finite. We investigated the possibility of making the correspondence exact even at finite angles by considering the full Born-Infeld action. Here, a non-abelian version of the Born-Infeld action is necessary to make the appropriate correspondence. We analyzed the gauge theory spectrum using Tseytlin’s recent proposal for the symmetrized non-abelian Born-Infeld action [18]. Although we were unable to resolve the discrepancy at finite angle in full, we found a strong hint in the general structure of the action suggesting how this discrepancy might be resolved. It seems rather remarkable that the subtle transcendental relations required to reproduce the correct spectrum of strings attached to the D-branes are encoded in the symmetrized Born-Infeld action. Perhaps one could turn this type of argument around and use the correspondence with string theory through T-duality as a guiding principle for resolving various difficulties which plague the non-abelian Born-Infeld action.

An intriguing aspect of the correspondence described here is related to the spatial dependence of the fluctuations in the gauge theory and string theory pictures for intersecting branes. In the gauge theory description, the off-diagonal fluctuations correspond to sections of nontrivial $U(1)$ bundles over the torus, and are described in [17] in terms of theta functions. These theta functions were described in [30] in terms of linear superpositions along the lattice of harmonic oscillator eigenfunctions. This description essentially coincides with the characterization of states in a constant magnetic field on the torus in terms of Landau levels. On the other hand, as argued by Berkooz, Douglas and Leigh in [9], the strings connecting intersecting D-branes are essentially living in a harmonic oscillator potential well and are localized to the intersection of the branes. It is natural to expect that the spatial dependence of the second quantized string excitations in the intersecting D-brane picture will correspond...
to gaussian and higher excited harmonic oscillator eigenfunctions. This agrees nicely with the theta function description in the gauge theory language; however, there is a subtlety in this relationship because a T-duality transformation has been performed in some of the directions on the torus. Making this correspondence more precise might lead to a better understanding of the action of T-duality as a map between the Hilbert spaces of quantized Born-Infeld theory and second quantized string theory.

Although in this paper we have concentrated on stationary backgrounds, the configurations of tilted and intersecting branes which we have considered here are closely related to boosted branes and systems of branes scattering from one another. For these systems, a convenient dynamical quantity to compute is the phase shift due to scattering [38, 39]. A discrepancy in the relativistic corrections to D-brane scattering processes as computed in a quantum mechanics model derived from Yang-Mills theory was discussed in [40]. This discrepancy seems similar in nature to the $\theta/2$ vs. $\tan(\theta/2)$ discrepancy we have described here. The authors of [40] also consider the possibility of resolving this discrepancy by considering the full Born-Infeld action, but encounter similar difficulties in making the exact correspondence at subleading order in backgrounds. In particular, they report a discrepancy between the phase shift computed from the relativistic 0-brane quantum mechanics (equation (63) of [40])

$$2\delta_{00}^{QM}(v) \sim \left(\frac{v}{1 - v^2}\right)^3$$

and from the semiclassical phase shift in the Eikonal approximation (equation (64) of [40])

$$2\delta_{00}^{SC}(v) \sim \frac{(1 - \sqrt{1 - v^2})^2}{v\sqrt{1 - v^2}}.$$ 

It is interesting to note that $2\delta_{00}^{QM}$ and $2\delta_{00}^{SC}$ are related by a relation similar to (20)

$$2\delta_{00}^{SC}(V) = \frac{1}{V} \int_0^V dv \left(1 - v\right)^{3/2} 2\delta_{00}^{QM}(v).$$

Although it is not completely clear why the integral prescription (20) would apply to the phase shift in resolving the discrepancy between $2\delta_{00}^{QM}$ and $2\delta_{00}^{SC}$, this observation provides an indication that by analyzing the combinatorics of Tseytlin’s symmetrized trace prescription [18], relativistic quantum mechanics of 0-branes might be made to agree with the semiclassical calculation.

Another aspect of the D-brane/gauge theory correspondence which we have not pursued here, but which is an interesting direction for further study, is the relationship between the moduli spaces of vacua in the two pictures and the corresponding 0-modes of the system. It was pointed out by van Baal in [17] that the physical 0-modes in the gauge theory picture can be determined explicitly, and that their number corresponds to the expected dimension of the moduli space of instantons. It has been argued [22, 23, 11] that the moduli space of vacua for the intersecting D-brane system is precisely given by the appropriate instanton
moduli space. The approach we have discussed here may provide a concrete framework in which this connection can be explored in more detail.

The web of relations between string theories and gauge theories has become an active area of investigation in light of our increased understanding of the role D-branes play in the non-perturbative dynamics of string theory and in reformulating string theory itself though M(atrix) theory. Backgrounds with constant fluxes have played an important role in recent studies of extended objects in M(atrix) theory [5, 21, 42, 43]. Although there is clearly more remaining to be said regarding the exact definition of the non-abelian Born-Infeld theory, it seems that when this theory is properly defined it should provide a mechanism for calculating some rather nontrivial results for systems of interacting D-branes within a field theory context.

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