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An Experimental Approach on Beating in Vibration Due to Rotational Unbalance

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Abstract: This paper proposes a study in theoretical and experimental terms focused on the vibration beating phenomenon produced in particular circumstances: the addition of vibrations generated by two rotating unbalanced shafts placed inside a lathe headstock, with a flat friction belt transmission between the shafts. The study was done on a simple computer-assisted experimental setup for absolute vibration velocity signal acquisition, signal processing and simulation. The input signal is generated by a horizontal geophone as the sensor, placed on a headstock. By numerical integration (using an original antiderivative calculus and signal correction method) a vibration velocity signal was converted into a vibration displacement signal. In this way, an absolute velocity vibration sensor was transformed into an absolute displacement vibration sensor. An important accomplishment in the evolution of the resultant vibration frequency (or combination frequency as well) of the beating vibration displacement signal was revealed by numerical simulation, which was fully confirmed by experiments. In opposition to some previously reported research results, it was discovered that the combination frequency is slightly variable (tens of millihertz variation over the full frequency range) and it has a periodic pattern. This pattern has negative or positive peaks (depending on the relationship of amplitudes and frequencies of vibrations involved in the beating) placed systematically in the nodes of the beating phenomena. Some other achievements on issues involved in the beating phenomenon description were also accomplished. A study on a simulated signal proves the high theoretical accuracy of the method used for combination frequency measurement, with less than 3 microhertz full frequency range error. Furthermore, a study on the experimental determination of the dynamic amplification factor of the combination vibration (5.824) due to the resonant behaviour of the headstock and lathe on its foundation was performed, based on computer-aided analysis (curve fitting) of the free damped response. These achievements ensure a better approach on vibration beating phenomenon and dynamic balancing conditions and requirements.

Keywords: lathe headstock; rotational unbalance; vibration; beating; signal processing

1. Introduction

Rotating unbalance is a topic frequently mentioned in the analysis of the dynamics of rotary bodies (rotordynamics [1]). The rotating unbalance occurs due to an asymmetry of mass distribution (in some different regions of the rotary body, the center of mass is not placed on the axis of rotation). Centrifugal forces occur in these unbalanced regions of the rotary body. The resultant of these centrifugal forces is transmitted through the bearings to the structure where the rotary body is placed.
In each bearing the resultant of centrifugal forces has two components at orthogonal directions. Each component acts as a harmonic excitation force against the structure, thus generating vibrations.

For some specific appliances these vibrations are desirable (e. g. in vibration shakers used also as mechanical vibration exciters [2,3], vibration alert systems in mobile phones, electronic vibrating bracelets [4], and haptic feedback devices with vibrations [5]). Generally, the vibrations due to the rotating body unbalances have highly undesirable effects (e.g., bad surface quality in the grinding process [6,7], premature bearing destruction [8,9]), and human body discomfort [10]). In order to measure [11] and to eliminate the unbalance of rotary bodies [12–14] (using additional balancing masses and additional inertia [15]), several special requirements must be met and specialized equipment must be used [16,17]. Actually, one of the best ways to balance rotary unbalanced bodies is through the use of self-balancing systems [18,19].

Sometimes two rotary unbalanced bodies having almost the same angular speed (or rotational frequency) which rotate in the same structure (e.g., in centerless grinding machines, [20]) produce a vibration beating phenomenon [10,21,22].

Each rotary unbalanced body generates a vibration. The addition of two vibrations, having slightly different frequencies, produces the aforementioned beating phenomenon. This is a resultant vibration with periodical variation of amplitude, with nodes (where the amplitude has a minimum value, the addition of the two vibrations produces destructive interference, 180 degrees out of phase between the constituents of the resultant vibration) and anti-nodes (a maximum amplitude, where the addition of the two vibrations produces constructive interference, with zero degrees shift of phase between constituents) [23]. Obviously the beating phenomenon in mechanics is not solely related to the vibrations produced by rotary unbalanced bodies, it also occurs when two vibration modes with almost the same modal frequency are excited [24,25], and it occurs as well when a system vibrates simultaneously due to forced sinusoidal excitation close to a resonant frequency and due to a free response [26–29].

Some specific appliances use the vibration beating phenomenon to monitor the condition of mechanical systems, e.g., monitoring the adhesion integrity of single lap joints [30], monitoring the structural integrity of helicopter rotor blades [31], and for seismic vibration testing [32]. A vibration beating mechanism in piezoelectric energy harvesting systems is proposed in [33].

In machine tools, in addition to a critical source of vibrations (self-excited vibrations in turning [34], milling [35] or grinding [36] processes), the vibrations produced by rotary unbalance (generated by tools [37], shafts [38] or work pieces [36]) and particularly the beating phenomenon [6] created by rotary unbalanced bodies, are important items. This paper proposes some approaches, in theoretical and experimental terms, to address the vibration beating phenomenon produced inside a Romanian lathe headstock SNA 360, by two inner unbalanced rotary shafts, rotating with very close angular speeds. The main achievements of this paper are producing results in the areas of: vibration beating monitoring, conversion of a velocity vibration signal in to a displacement signal (by antiderivative calculus), and the evolution of beating vibration signal frequency (pattern, simulation, measurement and accuracy measurement), as well as producing a study of the influence of headstock and lathe foundation dynamics on vibration amplitudes.

2. A Theoretical Approach

Assume that the rotary unbalance of each shaft (each of which rotates at angular speeds of \(\omega_1\) and \(\omega_2\), respectively) is reducible at the asymmetry of mass distribution with unbalance masses \(m_1\) and \(m_2\), respectively, placed in a single plane at distances \(r_1\) and \(r_2\), respectively, to the rotation axis. The horizontal projection of the centrifugal forces of each rotary unbalance (\(F_1 = m_1\omega_1^2r_1\) and \(F_2 = m_2\omega_2^2r_2\)) generates vibration displacements \(y_1 = kD_{a1}F_1\cos(\theta_1)\) and \(y_2 = kD_{a2}F_2\cos(\theta_2)\), where \(k\) is the stiffness of the headstock and the lathe foundation, \(D_{a1}\) and \(D_{a2}\) are the dynamic amplification factors and \(\theta_1 = \omega_1t + \varphi_1\) and \(\theta_2 = \omega_2t + \varphi_2\) are the instantaneous values of the angle of the centrifugal forces with respect to the horizontal direction (\(\varphi_1\) and \(\varphi_2\) being the instantaneous values of these angles at \(t = 0\)). With these considerations, a complete description of \(y_1\) and \(y_2\) of the vibrations is given below:
\begin{equation}
y_1 = kD_{af_1}m_1\omega_1^2r_1\cos(\omega_1 t + \varphi_1) = A_1\cos(\omega_1 t + \varphi_1)
\end{equation}

\begin{equation}
y_2 = kD_{af_2}m_2\omega_2^2r_2\cos(\omega_2 t + \varphi_2) = A_2\cos(\omega_2 t + \varphi_2)
\end{equation}

Here \( A_1 = kD_{af_1}m_1\omega_1^2r_1 \) and \( A_2 = kD_{af_2}m_2\omega_2^2r_2 \) are the vibration amplitudes of the two shafts, respectively. The headstock and the lathe vibrate as a single body on its foundation (as a mass–spring–damper system) with a vibratory motion which is the result of the addition \( y_1 + y_2 \) of these two vibrations, a periodical non-harmonic motion that presents itself as a beating phenomenon \[23\], with nodes and anti-nodes (as the simulation from Figure 1 proves). According to Figure 1, if the period of vibration \( y_1 \) is \( T_1 = 2\pi/\omega_1 \) and \( T_2 = 2\pi/\omega_2 \) is the period of vibration \( y_2 \), then the period \( T_b \) of the beating phenomenon (the beat period being the time between two anti-nodes or between two nodes, as well) and the periods \( T_1, T_2 \) (with \( T_2 < T_1 \)) should fulfill this obvious condition:

\begin{equation}
T_b = nT_1 = (n + 1)T_2
\end{equation}

with \( n \) being a natural number, defined from Equation (3) as:

\begin{equation}
n = T_2/(T_1 - T_2)
\end{equation}

In Figure 1 \( n = 7 \). If in Equations (3) and (4) the periods are replaced by frequencies (\( T_b = 1/f_b, T_1 = 1/f_1, T_2 = 1/f_2 \)), then the resulting frequency \( f_b \) of the beating phenomenon (beat frequency or the number of nodes per second, as well) is:

\begin{equation}
f_b = f_2 - f_1
\end{equation}

In Figure 1 \( f_2 = 8 \) Hz and \( f_1 = 7 \) Hz; these generate \( f_b = 1 \) Hz with \( T_b = 1s \) (\( A_1 = 10, A_2 = 8, \varphi_1 = 0, \varphi_2 = -\pi/2 \)).

According to \[39\], the resultant waveform of the vibration addition \( y_1 + y_2 \) has the frequency \( f_c = 1/T_c \) (as a combination frequency or modulation frequency, with the period \( T_c \) highlighted on Figure 1) defined as:

\begin{equation}
f_c = (f_1 + f_2)/2
\end{equation}

This paper will prove by simulations and experiments that this definition is not accurate, especially when \( A_1 \neq A_2 \).
Figure 1. A simulation of the beating phenomenon, where: $y_1$ is the vibration of shaft 1; $y_2$ is the vibration of shaft 2, $T_b$ is the beat period; and $T_c$ is the period of the resultant vibration $y_1 + y_2$.

These theoretical considerations and some other supplementary issues and procedures will be confirmed by experimental approach in this paper.

3. Experimental Setup

Figure 2a presents a lateral view of both shafts (1 and 2, placed in the headstock) involved in the beating phenomenon due to rotary unbalance.

A friction belt transmission with a flat drive belt 3, a pulley 4 (on shaft 1) and a pulley 5 (on shaft 2) synchronously rotates both shafts (the theoretical value of transmission speed ratio is 1:1). Here 6 depicts an additional mass (10.8 g, a permanent magnet) placed in different angular positions on pulley 5 and used to change the internal unbalancing of the shaft 2.

The shaft 1 is also the lathe main spindle (with the jaw chuck labelled with 7 on Figure 2b placed on the opposite side of Figure 2a). Figure 2b shows an absolute velocity vibration sensor 8 (an electrodynamic seismic geophone Geo Space GS 11D, now HGS Products HG4 as described in [40]), placed on the headstock. The geophone corner frequency (8 Hz) is smaller than the minimum frequency of the headstock vibration (17 Hz). No significant resonant amplification at the corner.

Figure 2. (a) A lateral view of the shafts involved in the beating phenomenon (with a flat belt transmission between the shafts); (b) A front view of the lathe headstock with the vibration sensor.
frequency can be identified (the open circuit damping being 34% of critical damping). The geophone sensitivity is 31.89 V/m/s.

The signal delivered by the vibration sensor (a voltage proportional to the vibration velocity of the headstock) is numerically acquired by a personal computer via a computer-assisted numerical oscilloscope PicoScope 4424 from Pico Technology, Saint Neots, UK (USB powered, four channels, 12 bits resolution, 80 M5/s maximum sampling, 32 MS memory) [41]. Due to the high sensitivity of the geophone and oscilloscope features (e.g., a numerically controlled internal amplifier) the supplementary amplification of the signal delivered by vibration sensor is not necessary.

The computer-aided processing of this signal was done in Matlab. Firstly, the signal delivered by sensor must be mathematically divided by the sensitivity of the sensor in order to obtain the vibration velocity evolution. Secondly, the velocity of vibration must be numerically integrated (by antiderivative calculus) in order to obtain the vibration displacement evolution. The description and analysis of the evolution of some of the other vibration features (e.g., the combination frequency $f_c$, the beat vibration amplitude and the free vibrations of the lathe on foundation) were also performed.

In order to measure the average value of the instantaneous angular speed (IAS) $\omega_1$ of the main spindle (shaft 1) rigorously, the technique described in our previous work [42] was used (with a two phase multi-pole AC generator placed in the jaw chuck, as an IAS sensor). The same technique (which refers only to signal processing, briefly described later on) was used for an accurate measurement of the combination frequency $f_c$.

4. Experimental Results and Discussion

4.1. A Beating Phenomenon Described in Vibration Velocity

Figure 3 presents the evolution of the headstock vibration velocity during a time interval of 200 s, described with 1 MS (or 1,000,000 samples as well) so a sampling interval of $\Delta t = 200 \mu s$, when the main spindle (shaft 1) rotates (and shaft 2, as well) in the steady-state regime, with constant IAS, with an average value of $\omega_1 = 109.2369 \text{ rad/s}$ (for $f_1 = 17.3856 \text{ Hz}$ average rotation frequency, or 1,043.1 revolutions per minute on average).

It is obvious that Figure 3 depicts a vibration beating phenomenon with nodes and anti-nodes, with a very high value of the period $T_b = 96.6 \text{ s}$ and consequently with a very small value of beat frequency $f_b = 1/T_b = 1/96.6 \text{ Hz}$. The beating phenomenon proves that the IASs $\omega_1$ and $\omega_2$ and also rotation frequencies $f_1$ and $f_2$ as well, are slightly different because the diameters of pulleys 4 and 5 involved in belt transmission are not strictly the same. With $T_1 = 1/f_1$ and the relationship between $T_1$ and $T_b$ from Equation (3) there are $n = T_b/f_1 = 1679$ periods $T_1$ between nodes (and between anti-nodes as well). This is an approximated value of $n$ because the frequency $f_1$ is not rigorously constant (as is proved, later in this paper). According to Equation (3), at each $n$ complete rotations of shaft 1, the shaft 2 makes $n+1$ or $n-1$ rotations (one rotation difference), which means that the experimentally revealed value of the belt transmission speed ratio is $\omega_2/\omega_1 = T_1/T_2 = n/(n \pm 1) = 1679/(1679 \pm 1)$. This is also the ratio between pulleys diameters: the diameter of pulley 4 divided by the diameter of pulley 5 (assuming that there is no slipping between the belt and the pulleys).

The additional mass 6 (Figure 2a) was placed in a certain angular position on pulley 5, in order to obtain a maximum value of the amplitude $A_2$, and a maximum difference between amplitudes in anti-nodes and nodes as well. It is obvious that the main spindle (shaft 1) is also unbalanced; otherwise the beating phenomenon does not occur.
Figure 3. The evolution of the velocity of headstock vibrations with a beating phenomenon due to rotary unbalanced shafts; here $T_b$ is the beat period, $A$ is a label for a future comment on signal evolution.

Figure 4 presents a zoomed-in detail in the area labelled $A$ in Figure 3. Here the dominant component ($\approx 6$ mm/s amplitude) is the sum of two vibrations created by rotary unbalances; the other low amplitude (and high frequency) components are related by vibrations generated by some other headstock rotary components.

With the values for $f_1$ and $f_b$ revealed before, the frequency $f_2$ is a result of Equation (5), with two possible values $(f_2 = f_1 - f_b = 17.3752 \text{ Hz} \text{ or } f_2 = f_1 + f_b = 17.3959 \text{ Hz})$. Because in Equation (5) the notations $f_1$ and $f_2$ are arbitrary, this equation should be reconsidered as $f_b = |f_2 - f_1|$.

As a consequence, the angular speed $\omega_2 = 2\pi f_2$ has two possible values $(\omega_2 = 109.1716 \text{ rad/s} \text{ or } \omega_2 = 109.3016 \text{ rad/s})$, as does the speed ratio ($\omega_2/\omega_1$) of the driving belt (with $\omega_1 = 109.2369 \text{ rad/s}$). To find the right value of $f_2$ (and $\omega_2$ as well) the technique described in [42] should be used (with an IAS sensor placed on shaft $2$).

The evolution from Figure 3 is an addition of vibrations velocities $(v = dy_1/dt + dy_2/dt)$ generated by both of the unbalanced shafts (1 and 2). It is expected that the beating phenomenon keeps the main characteristics (e.g., $T_b, T_c$ values or $f_b, f_c$ values, as well) if it is described using the addition of vibration...
displacements \((s = y_1 + y_2)\), except for the amplitudes in nodes and anti-nodes which significantly decreases.

4.2. The Description of the Beating Phenomenon in Vibration Displacement by Numerical Integration

The vibration displacement evolution can be obtained from vibration velocity evolution by numerical integration (antiderivative calculus). Based on the approximate definition of velocity (derivative of displacement) \(v = ds/dt = \Delta s/\Delta t\), a current sample of velocity \(v_i\) is defined using two successive samples of displacement \(s_i, s_{i-1}\) (in the displacement interval \(\Delta s = s_i - s_{i-1}\)) and the values of time \(t_i, t_{i-1}\) for these samples, in the sampling interval \(\Delta t = t_i - t_{i-1}\) (usually this is a constant value) as:

\[
v_i = \frac{s_i - s_{i-1}}{\Delta t}
\]

This is an approximation of the first derivative of displacement as backward finite difference, with \(i > 1\) [43]. The current sample of displacement \(s_i\) can be simply mathematically extracted from Equation (7) as:

\[
s_i = v_i \Delta t + s_{i-1}
\]

Equation (8) describes a sample \(s_i\) of displacement related to velocity, this also being our proposal for a description of numerical integration of velocity (antiderivative calculus). According to Equation (8) the sample \(s_i\) depends on sampling interval \(\Delta t\) (here \(\Delta t\) is the time \(t_i - t_{i-1}\) between two consecutive samples of velocity \(v_i\) and \(v_{i-1}\), or two consecutive samples of displacement \(s_i\) and \(s_{i-1}\) as well), the velocity sample \(v_i\) and the previous sample of displacement \(s_{i-1}\), as the result of a previous step of numerical integration. The numerical integration from Equation (8) is available for \(i > 1\). Of course, it is mandatory to know the value of the first sample of displacement \(s_1\), this being an indefinite value because \(i > 1\). This is exactly the constant \(C\) of integration (usually an arbitrary value). Pure harmonic signals are numerically integrated, in that case, evidently \(C = 0\).

Figure 5a describes the graphical result of numerical integration of vibration elongation evolution from Figure 3 using Equation (8), with \(C = s_1 = 0\). Certainly this evolution is not strictly related to the vibration displacement from the beating phenomenon.

We found that the oscilloscope generates a very small negative constant zero offset. As the theory of integration establishes, the numerical integration of this constant zero offset produces a component with linear evolution, experimentally confirmed in Figure 5a by the evolution with negative slope. The removal of this linear component produces the result from Figure 5b (the evolution emphasized in blue).
It is evident that Figure 5b is not the expected evolution of the vibration displacement in beating phenomenon. Surely, there is not a mistake in the numerical integration proposal in Equation (8) because the numerical derivative of the evolution from Figure 5b using Equation (7) produces exactly the evolution of velocity, as Figure 6 indicates (by comparison with Figure 3).

![Figure 6](image)

**Figure 6.** The result of the numerical derivative of the evolution from Figure 5b (vibration velocity, practically similar with Figure 3).

Our first attempt to explain this deficiency in this result of numerical integration is related by the constant of integration $C$.

Intuitively it is supposed that somehow the hypothesis that $C = 0$ is wrong. Perhaps the effect of this wrong hypothesis is mirrored in the evolution from Figure 5b and its effect should be removed (as the influence of negative zero offset was removed before).

It was discovered by numerical simulation that the numerical integration of a computer-generated beating vibration velocity signal (similarly to those depicted in Figure 3) using Equation (8), with $C = 0$, produces a vertically shifted evolution with a constant nonzero value, which should be mathematically removed.

This approach assumed that in the result of numerical integration of vibration velocity depicted in Figure 5b, a supplementary low frequency component was generated and should be removed. For the time being we unfortunately do not have a consistent explanation for the appearance of this low frequency component. This low frequency component (depicted in Figure 5b in white) was detected by low-pass numerical filtering of the vibration displacement signal (the evolution emphasised in blue).

A computer-generated moving average filter [43] was used, with the first notch frequency equal to the combination frequency $f_c = (f_1 + f_2)/2$ (assuming that this definition from Equation (6) is accurate), in order to completely remove the variable component from Figure 5b having the resultant vibration frequency, and in order to obtain the low frequency component. The number of points in the average of the filter is defined as integer of the ratio $1/(f_c \Delta t)$. The removal of this low frequency component from the result of numerical integration (the vibration displacement signal) depicted in Figure 5b is shown in Figure 7. It is obvious that this evolution properly describes the resultant vibration displacement during the beating phenomenon, previously described in Figure 3, by the velocity of the resultant vibration.

There is a supplementary confirmation of this result: the numerical differentiation of the vibration displacement signal from Figure 7 (using Equation (7)) fits very well with the vibration velocity signal from Figure 3, as a very short detail (15 ms duration) of both evolutions (given in Figure 8a) from the area labelled as A (Figures 3 and 7) indicates. Thus, the absolute velocity vibration sensor together with the proposed numerical signal integration method acts as an absolute displacement vibration sensor.
Figure 7. The headstock vibration displacement evolution during the beating phenomenon, deduced by numerical integration and correction of the signal depicted in Figure 3.

Figure 8b presents a short detail of the vibration displacement evolution in the area labelled with A in Figure 7. This figure has the same size on the abscissa as Figure 4. By comparison with Figure 4, the evolution is much smoother here, as a consequence of numerical integration, which drastically reduces the amplitudes of high frequency components. The integration acts as a low-pass filter.

In Figure 7 two relationships between the vibrations amplitudes $A_1$ and $A_2$ are available (from Equations (1) and (2)) due to the constructive interference in anti-nodes ($A_1 + A_2 = 118 \, \mu m$) and destructive interference in nodes ($A_1 - A_2 = 52 \, \mu m$), so $A_1 = 85 \, \mu m$ and $A_2 = 33 \, \mu m$. For the time being $A_1$ does not necessarily refer to vibration amplitude generated by the main spindle or shaft 1.

Figure 8. (a) A detail concerning the evolution of velocity (Figure 3) overlaid on the numerical differentiation of the displacement depicted in Figure 7; and (b) a detail of area A of Figure 7 with $T_c$, the period of the resultant vibration.

4.3. The Evolution of Frequency for Resultant Vibration in Beating Phenomenon

An interesting item in the beating phenomenon is the evolution of frequency of the resultant vibration $f_c$ (also known as combination frequency or modulation frequency, $f_c = 1/T_c$, with $T_c$ highlighted in Figure 8b). In [39] this frequency is defined as the average of both frequencies ($f_1, f_2$) involved in the beating (Equation (6)).
A beating phenomenon was simulated using the sum of two harmonic vibrations displacements \( y_1(A_1,f_1) \) and \( y_2(A_2,f_2) \) — already described in Equations (1) and (2) with different values of amplitudes \( A_2 > A_1 \) and frequencies \( f_1 \) and \( f_2 \), close to those from the experiment described in Figures 3 and 7 (with \( f_1 = 17.3856 \text{ Hz} \) and \( f_2 = 17.3959 \text{ Hz} \)), for a duration equal to \( T_b \) (placed between two anti-nodes). For six different values of amplitudes \( A_1 \) increases and \( A_2 \) decreases), the evolution of the combination frequency \( f_c \) and its average value was determined on the vibration beating simulated signal, as Figure 9 indicates, using a high accuracy measurement technique from a previous work [42]. Here each peak describes the value of frequency \( f_c \) in vibration beating node. It is obvious that \( f_c \) is not constant and, in contradiction with Equation (6) and [39], \( f_c \neq (f_1 + f_2)/2 \). Here, with \( A_1 > A_2 \) the average \( f_c \) is very close to \( f_2 \) with \( f_c > f_2 \).

**Figure 9.** The evolution of the instantaneous combination frequency \( f_c \) on the simulated vibration beating during a beat period \( T_b \) (with \( A_2 > A_1 \) and \( f_2 > f_1 \)).

A similar simulation was done in the same conditions, now with \( A_1 > A_2 \), as Figure 10 indicates (here \( A_1 \) decreases and \( A_2 \) increases). Similar to the simulation given in Figure 9, it is obvious that \( f_c \) is not constant and again, in contradiction with Equation (6) and [39], \( f_c \neq (f_1 + f_2)/2 \). The average frequency \( f_c \) is very close to \( f_1 \), with \( f_c < f_1 \).

**Figure 10.** The evolution of the instantaneous combination frequency on simulated vibration beating during a beat period (the same condition as in Figure 9, except for the amplitudes relationship: \( A_1 > A_2 \)).
There are two conclusions here, in contradiction with the literature [39]:

- The combination frequency $f_c$ is not constant over a period $T_b$ (even if its variation is not significant);
- The average value of the combination frequency $f_c$ over a period $T_b$ is practically the same as the frequency of the input vibration in the beating phenomenon ($y_1(A_1,f_1)$ or $y_2(A_2,f_2)$), whose amplitudes are higher (e.g., if $A_2 > A_1$ then the average $f_c = f_2$).

Some supplementary simulations for many other values of frequencies $f_1$ and $f_2$ (and consequently $T_b$) completely confirm these conclusions.

Figure 11 presents the evolution of the instantaneous combination frequency $f_c$ during the vibration beating phenomenon (displacement of headstock) experimentally described in Figure 7.

Figure 11. The evolution of the instantaneous combination frequency $f_c$ during the vibration beating phenomenon (displacement of headstock) described in Figure 7.
Figure 12. The evolution of the low pass filtered instantaneous combination frequency $f_c$ during the vibration beating phenomenon described in Figure 7 (a low pass filtering of Figure 11).

Despite a relatively strong irregular variation of the combination frequency $f_c$ (due to the variation of experimental conditions: e.g., the small variation of rotational speeds of shafts 1 and 2, caused mainly by the variation of frequency of the supplying voltages applied to the asynchronous driving motor, around a theoretical value of 50 Hz, as Figure 13 clearly indicates), the previous simulations and conclusions are fully experimentally confirmed. Three supplementary identical experiments confirm the evolution presented in Figure 12.

Firstly, in Figure 12 there are two negative peaks (for the two nodes in Figures 3 or 7; each node produces a negative peak on $f_c$ evolution, an item already discussed in the simulation from Figure 10) at a time interval very close to the beat period value $T_b$, already defined in Figure 3 (96.73 s here, compared with 96.6 s in Figure 3).

Secondly, as shown in Figure 14, a superposition of filtered frequency $f_c$ evolution from Figure 12 (here in a conventional blue coloured description) over the experimental envelopes of vibration displacement in beating (the same as those depicted in Figure 7) indicates that the negative peaks of $f_c$ are placed, as expected, in nodes.

Figure 13. Low-pass filtered rotational frequency evolution of the main spindle and supplying voltage frequency evolution of the driving motor.
Figure 14. The position of negative peaks on the combination frequency (formally represented) relative to the position of nodes on the vibration beating phenomenon.

The small displacement to the right of the negative peaks of the filtered combination frequency evolution (as against the nodes on Figure 14) is not related to the numerical filtering. This is proved by the result of the simulation of Figure 14, as given in Figure 15 (with addition of pure harmonic signals $y_1$ and $y_2$ in vibration beating simulation).

This periodic pattern of filtered combination frequency $f_c$ evolution experimentally revealed in Figures 12 and 14 (according to the simulations from Figures 10 and 15) is strongly attenuated if the amplitude $A_2$ becomes significantly lower than $A_1$ (and vice versa).

An important question is in order here due to a very small variation of filtered frequencies revealed before (less than 50 mHz full scale evolution in Figures 9, 10, 12, 13 and 15): how accurate is this frequency measurement method [42]?

Figure 15. The result of a numerical simulation for the evolutions described in Figure 14.

In this measurement method (e.g., the measurement of the combination frequency $f_c$ of vibration displacement signal from Figure 7), the computer-aided detection of the time interval between each
two consecutive zero-crossing moments \((tzc\text{ and } tzc+1)\) of a periodical signal is used. This time interval defines a semi-period \(Tc/2 = tzc+1 - tzc\) as \(Tc/2 = 1/2fc\), or a value \(fc = 1/Tc\). When the result of multiplication of two successive displacements samples \(si\) and \(si-1\) (having the sampling times \(ti\) and \(ti-1\), with \(i > 1\)) is negative or zero \((si \cdot si-1 < 0 \text{ or } si \cdot si-1 = 0)\) a zero-crossing moment is detected (e.g., \(tzc\)) and calculable as the abscissa of the intersection of a line segment defined by the points of coordinates \((ti, si)\) and \((ti-1, si-1)\) on the \(t\)-axis (as \(x\)-axis in Figure 8b). The main reason for frequency measurement error \(\varepsilon f \neq 0\) is a consequence of calculation errors for two successive zero-crossing moments \(\varepsilon j \neq 0\) (for \(tzc\)) and \(\varepsilon j+1 \neq 0\)(for \(tzc+1\)). These \(\varepsilon j\) and \(\varepsilon j+1\) errors are caused by the replacement of a harmonic evolution with a linear evolution between those two successive displacement samples involved in each zero-crossing moment definition. With \(t_{zj} - ti = \Delta t\) (\(\Delta t\) being the sampling interval) the error \(\varepsilon j = 0\) only in three situations: (1) if \(si = 0\) (the end of the line segment is placed on the \(t\)-axis, with \(tj = ti\)), (2) if \(si-1 = 0\) (the start of line segment is placed on \(t\)-axis, with \(tj = ti-1\)) and (3) if \(-si = si-1\) (the middle of the line segment is placed on \(t\)-axis, with \(tj = ti-1 + \Delta t/2\)). A similar approach is available for the next two successive samples \((si+h \text{ and } si+h-1\) involved in the definition of \(tzc+1\) moment and \(\varepsilon j+1\) error (with \(h\) as the integer part of the ratio \(Tc/\Delta t\)). If simultaneously \(\varepsilon j\) and \(\varepsilon j+1 = 0\) then \(\varepsilon f = 0\). Any other definition of sampling times generates frequency measurement errors \(\varepsilon f \neq 0\).

A computer-aided calculus was performed for frequency measurement error \(\varepsilon f\) of a harmonic simulated signal with frequency 17.383 Hz (the average value of the combination frequency \(fc\)) during a semi-period. Here 10,000 different values of sampling time \(t1\) (between 0 and \(\Delta t\), with \(\Delta t = 200\mu s\), the same sampling interval as in Figures 3 and 7) and \(t2 = \Delta t - t1\) (between \(\Delta t\) and 0) for the first two successive displacement samples involved in the calculus of the first zero-crossing time \(tzc\) was used. Figure 16 describes the evolution of the frequency measurement error \(\varepsilon f(t1)\).

![Figure 16](image-url)

Figure 16. The evolution of the frequency measurement error \(\varepsilon f\) (for a simulated harmonic signal with combination frequency \(f1 = 17.383\) Hz) versus the evolution of the first sampling time \((t1 = 0 + \Delta t, \text{ or } t1 = 0 - 200\mu s)\) involved in the first zero-crossing time \(tzc\) calculus.

As Figure 16 clearly indicates, the frequency measurement error \(\varepsilon f\) of the combination frequency \(f1\) is variable and placed between \(-0.000935\) and \(+0.0014\) mHz. The result of the measured frequency of the simulated signal is \(17.383\pm0.0014\mu Hz\) as a description of the accuracy measurement. Very similar limits for the \(\varepsilon f\) error are calculated for a harmonic signal with frequency \(f1\). If the value of the frequency \(f1 = 1/Tc\) used in simulation accomplishes the condition \(Tc = h\Delta t\) (with \(h\) being an integer), then \(\varepsilon f = 0\) for any value \(t1 = 0 + \Delta t\).

4.4. The Influence of the Lathe Suspension Dynamics on Beating Vibrations Amplitude

The relative high vibration displacement amplitude of the headstock during the beating phenomenon (as shown in Figure 7) has an evident explanation: the vibration frequencies \(f1, f2\) and \(fc\).
as well, are close to the first resonant frequency (vibration mode) of the headstock and lathe on its foundation (as a single body mass–spring–damper vibratory system). This means that the dynamic amplification factors $D_{af1}$ and $D_{af2}$ (involved in Equations (1) and (2)) are significantly higher than 1 (because of resonant amplification). In order to prove that, the resources of a very simple experiment performed with the same experimental setup are available: the evolution of headstock vibration velocity after an impulse excitation produced with a rubber mallet (hammer) in the same direction with $y_1$ and $y_2$ vibrations (as Figure 17 describes).

Here the blue curve partially depicts the free damped vibration velocity $v_{fd}$ (acquired with the geophone sensor); the red coloured one depicts the best fitting curve of a part of the free response (with 25,000 samples and 500 ns sampling time). The curve fitting [44] was done in Matlab, with an adequate computer program specially designed for this paper, based on a known theoretical model of free viscous damped vibration velocity response [45]:

$$v_{fd}(t) = a \cdot e^{-bt} \sin (p_1 t + \alpha)$$  \hspace{1cm} (9)

![Figure 17. Some experimental results on signal processing related to free damped vibration of headstock after an impulse excitation (with a rubber mallet).](image)

The best fitting curve (in red in Figure 17) is described with $a = 1.170 \cdot 10^{-3}$ m/s, $b = 5.899$ s$^{-1}$ (as damping constant), $p_1 = 117.952$ rad/s (as angular frequency of damped harmonic vibration) and $\alpha = 4.986$ rad (as phase angle at the origin of time $t_0$ on Figure 17). The angular natural frequency ($p = \sqrt{p_1^2 + n^2} = 118.099$ rad/s) and the damping constant $b$ are useful in the definition [45] of dimensionless dynamic amplification factor $D_{af}$ from forced vibrations of harmonic excitation (as happens during the beating phenomenon, assuming that the combination frequency is approximately constant):

$$D_{af} = \frac{1}{\sqrt{(1 - (\frac{\omega}{p})^2)^2 + (2 \frac{\omega}{p} \frac{b}{p})^2}}$$  \hspace{1cm} (10)

Here $\omega = 2\pi f$ is the angular frequency of harmonic excitation on frequency $f$. Based on previous experimental results of curve fitting (with $b$ and $p$ values in Equation (10)) Figure 18 presents the simulated evolution of $D_{af}$ related to the frequency of excitation (1 ÷ 35 Hz range). Because of a low damping constant $b$, the system presents resonant amplification, with a maximum value $D_{af} = 10.01$ on $f = 18.749$ Hz frequency.

Based on the previous experimentally determined frequencies $f_1$ and $f_2$, with $f = f_1 = 17.3856$ Hz gives the result $D_{af1} = 5.831$ and with $f = f_2 = 17.3752$ Hz (or $f = f_2 = 17.3959$ Hz) the result is $D_{af2} = 5.803$
(or $D_{af} = 5.859$). For $f = \bar{f}_1 = 17.383$ Hz (Figure 12) the result is $D_{af} = 5.824$ (the coordinates of point A on Figure 18). This means that, because of mechanical resonance, the vibration amplitude generated by the beating phenomenon of the headstock and the lathe on its foundation (already revealed in Figure 7) is amplified on average by 5.824 times.

Besides the amplification of the vibration, the resonant behaviour also introduces a significant shift of phase $\gamma$ between the excitation (unbalancing) force and the vibration displacement, theoretically described [45] as depending on $\omega$ and excitation frequency $f$ with the equation:

$$\gamma = \arctan\left[\frac{2 \frac{b}{p} \frac{\omega}{p}}{1 - \left(\frac{\omega}{p}\right)^2}\right]$$

(11)

With the $b$ and $p$ values previously determined, the values of shift of phase calculated for each frequency are: $\gamma_1 = 0.5691$ rad for $f = f_1 = 17.3856$ Hz and $\gamma_2 = 0.5656$ rad (or $\gamma_2 = 0.5725$ rad) for $f = f_2 = 17.3752$ Hz (or $f_2 = 17.3959$ Hz). For $f = \bar{f}_c = 17.383$ Hz the result is $\gamma_c = 0.5682$ rad.

![Figure 18. The evolution of the dynamic amplification factor $D_{af}$ generated by the headstock foundation in the resonance area, based on Equation (10) and experimental free damped response analysis.](image-url)

The knowledge of both of these resonant characteristics (the value of the dynamic amplification factor $D_{af}$ and especially the phase shift $\gamma$) is important for a next approach of the dynamic balancing of these two shafts placed inside the headstock.

As a general comment, we should mention that the resonance behaviour of this low damped vibratory system—as previously mentioned—is a consequence of the disponibility of this system to absorb modal mechanical energy. The system works as a narrow-band modal energy absorber [46].

5. Conclusions and Future Work

Some specific features of the beating vibration phenomenon discovered on a headstock lathe have been revealed in this paper.

An experimental description (with theoretical approaches based on simulations) of this beating vibration phenomenon with very low beat frequency (1/96.6 Hz) was performed. The beating phenomenon occurs due to the addition of vibrations produced by two unbalanced shafts, rotating with very close instantaneous angular speeds (rotating frequencies), with constructive interference in anti-nodes and destructive interference in nodes.
The absolute velocity signal of vibration beating (delivered by a vibration electro-dynamic sensor placed on the headstock) was converted into a displacement signal. For this purpose, a fully confirmed method of numerical integration (antiderivative calculus), with theoretical and experimental approaches was applied. This method is deduced from the approximation of the formula for the first derivative of displacement, as a backward finite difference [43]). An appropriate technique of correction of this numerical antiderivative calculus method was also introduced (mainly by removing the low frequency displacement signal component generated by numerical integration). Thus, an absolute velocity vibration sensor together with a numerical integration procedure plays the role of an absolute vibration displacement sensor.

A consistent part of the research was focused on the resultant vibration displacement signal, mainly on the evolution of frequency (or the combination frequency \( f_c \)) related to the nodes and anti-nodes position. It was theoretically discovered (by simulation) and was experimentally proved that, in opposition to the literature reports, the combination frequency is not constant, and the definition of its average value is wrong. The evolution of the combination frequency has a specific periodic pattern (having the same frequency as the beat frequency) with small variation (tens of millihertz) and negative or positive peaks placed in beating nodes. The appearance of these peaks (negative or positive) depends on the relationship between amplitudes and frequencies of vibrations involved in the beating phenomenon. The small variation of frequency inside the pattern and the correlation between the frequencies of different experimental signals (the combination frequency \( f_c \), the rotation frequency \( f_1 \) of the main spindle, and the supply voltage frequency of the driving motor) have been correctly described as a result of a high accuracy procedure of frequency measurement, developed in a previous work [42] and successfully applied here. It was proved on a simulated signal (having a frequency of 17.383 Hz, equal to the average value of the combination frequency in vibration beating phenomenon) that this procedure has less than ±1.5 μHz measurement error.

The influence of the behaviour of the headstock and lathe foundation dynamics (as a rigid body placed on a spring–damper system) on the vibration induced by unbalanced rotors and the beating phenomenon was also investigated. Based on computer-aided analysis of free damped viscous response (by curve fitting), the characteristics of foundation dynamics were experimentally revealed (mainly the values of natural angular frequency and the damping constant). Considering these values, the dynamic amplification factors of vibrations (mainly of the resultant vibration) and phase shift between centrifugal forces (as excitation forces produced by unbalanced rotary shafts) and the vibrations generated by these forces were calculated.

For each experiment, numerical simulation and signal processing procedures, several computer programs written in Matlab were successfully used.

In the future, the theoretical and experimental approaches will be focused on the influence of dynamic unbalancing and vibration beating on the active and instantaneous electrical power absorbed by the driving motor of the headstock. There is a logical reasoning for these approaches: the headstock vibration motion (especially during the resonant amplification behaviour revealed in Figure 18) should be mechanically powered. Of course, the instantaneous and active mechanical power (difficult to measure) is delivered by the driving motor as an equivalent of instantaneous and active electric power absorbed from the electrical supply network (easier to measure).

Several theoretical and experimental studies on computer-aided balancing of each rotary shaft inside the lathe headstock will be performed (using two absolute velocity sensors and an appropriate method of computer-assisted experimental balancing). A study on the vibration beating phenomenon produced by more than two unbalanced rotary bodies will be done.

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**Nomenclature**

| Symbol | Definition |
|--------|------------|
| $A_1, A_2$ | The amplitudes of vibrations $y_1, y_2$ [m] |
| $ae^+$ | The envelope of free viscous damped vibration velocity response [m/s] |
| $b$ | The damping constant [s$^{-1}$] |
| $C$ | The constant of velocity signal integration [m] |
| $D_d$ | Theoretical dynamic amplification factors of vibrations [ ] |
| $D_{01, D_{02}}$ | Dynamic amplification factors of vibrations $y_1, y_2$ produced by shafts 1, 2 [ ] |
| $D_{d0}$ | Dynamic amplification factor of resultant vibration $y_1 + y_2$ at average frequency $f_0$ [ ] |
| $dy_1/dt, dy_2/dt$ | The derivative of vibration displacements $y_1, y_2$ (vibration velocities) [m/s] |
| $f$ | The frequency of harmonic excitation of the lathe headstock [Hz] |
| $F_1, F_2$ | The horizontal projection of the rotary unbalance forces generated by shafts 1 and 2 [N] |
| $f_b, f_1, f_2$ | The frequency of vibrations $y_1, y_2$ [Hz] |
| $f_b$ | The beat frequency [Hz] |
| $f_c$ | The frequency of the resultant vibration $y_1 + y_2$, or combination frequency [Hz] |
| $IAS$ | Instantaneous angular speed [rad/s] |
| $k$ | The stiffness of headstock and lathe foundation [N/m] |
| $m_1, m_2$ | Unbalance mass on rotary shafts 1, 2 [Kg] |
| $n$ | A natural number involved in the definition of the beat period $T_b$ |
| $p$ | The natural angular frequency [rad/s] |
| $p_n$ | The angular frequency of damped harmonic vibration [rad/s] |
| $r_1, r_2$ | The distance between the center of the unbalance mass and the rotation axis on shafts 1, 2 [m] |
| $s$ | The addition of vibration displacements $s = y_1 + y_2$ [m] |
| $s_{o1}, s_{o+1}$ | Two successive displacement samples of vibration [m] |
| $s_{o2}, s_{+2}$ | Two successive displacement samples of vibration [m] |
| $t$ | Time [s] |
| $t_0$ | The origin of time for the theoretical model of free damped vibration velocity [s] |
| $t_{o1}, t_{o+1}$ | Two successive zero-crossing moments of the displacement vibration signal involved in frequency measurement [s] |
| $\Delta t$ | Sampling interval for a numerically described signal [s] |
| $T_1, T_2$ | The periods of vibrations $y_1, y_2$ [s] |
| $T_b$ | The beat period, with $T_b = 1/f_b$ [s] |
| $T_c$ | The period of the resultant vibration $y_1 + y_2$, with $T_c = 1/f_c$ [s] |
| $v$ | The velocity of the resultant vibration in beating [m/s] |
| $v_s$ | The vibration velocity of the headstock during a free damped response [m/s] |
| $v_1$ | A sample of the vibration velocity [m/s] |
| $y_1, y_2$ | The vibration displacement generated by shafts 1, 2 [m] |
| $y_{v1}, y_{v2}$ | Simulated vibration displacement signals [m] |
| $\alpha$ | The phase angle at the origin of time $t_0$ for a theoretical model of free damped vibration velocity [rad/s] |
| $\varepsilon_f$ | The error in the frequency measurement [Hz] |
| $\varepsilon_{o1}, \varepsilon_{o+1}$ | The calculus errors for two successive zero-crossing moments [s] |
| $\gamma$ | The shift of phase between the excitation force and the vibration displacement in the free damped response [rad] |
| $\theta_1, \theta_2$ | The instantaneous value of the angle of centrifugal forces to the horizontal direction [rad] |
| $\psi_1, \psi_2$ | The values of $\theta_1$ and $\theta_2$ at the origin of time, $t = 0$ [rad] |
| $\omega$ | The angular frequency of harmonic excitation of the lathe headstock [rad/s] |
| $\omega_n, \omega_{2n}$ | The instantaneous angular speed of the rotary shafts 1, 2 [rad/s] |
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