The $D^4\mathcal{R}^4$ term in type IIB string theory on $T^2$ and U–duality

Anirban Basu

Institute for Advanced Study, Princeton, NJ 08540, USA

Abstract

We propose a manifestly U–duality invariant modular form for the $D^4\mathcal{R}^4$ interaction in type IIB string theory compactified on $T^2$. It receives perturbative contributions up to two loops, and non–perturbative contributions from D–instantons and $(p,q)$ string instantons wrapping $T^2$. We provide evidence for this modular form by showing that the coefficients at tree level and at one loop precisely match those obtained using string perturbation theory. Using duality, parts of the perturbative amplitude are also shown to match exactly the results obtained from eleven dimensional supergravity compactified on $T^3$ at one loop. Decompactifying the theory to nine dimensions, we obtain a U–duality invariant modular form, whose coefficients at tree level and at one loop agree with string perturbation theory.
1 Introduction

Understanding duality symmetries of string theory is important in order to analyze the dynamics of the theory beyond its perturbative regime. In particular, analyzing certain protected operators in toroidal compactifications of type IIB superstring theory which preserve all the thirty-two supersymmetries has proven useful in this regard. One such protected operator is the four graviton amplitude in the effective action of type IIB string theory, which involves various modular forms of the corresponding U–duality groups. These interactions which are of the form $D^{2k} R^4$ where $k$ is a non–negative integer, are expected to satisfy certain non–renormalization properties. It has been argued that (at least for low values of $k$) these interactions receive only a few perturbative contributions, as well as non–perturbative contributions. The $R^4$ interaction has been analyzed in various dimensions [1–11] (see [12, 13] for reviews). The $D^{2k} R^4$ interaction has been analyzed for some higher values of $k$ in [14, 15], while the non–renormalization properties have been discussed in [16–18].

In this paper, we shall focus on some aspects of the four graviton scattering amplitude in type IIB superstring theory compactified on $T^2$. This theory has a conjectured $SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z})$ U–duality symmetry [19, 20]. In fact using dualities, this U–duality symmetry has a natural geometric interpretation when one considers M theory compactified on $T^3$. The $SL(3, \mathbb{Z})$ factor is the modular group of $T^3$, while the Kahler structure modulus $T^M$ on $T^3$ defined by

$$T^M = C_3 + iV_3,$$

transforms as

$$T^M \rightarrow \frac{aT^M + b}{cT^M + d},$$

under the $SL(2, \mathbb{Z})$ factor, where $a, b, c, d \in \mathbb{Z}$, and $ad – bc = 1$. In (1), $C_3$ is the three form gauge potential of M theory, and $V_3$ is the volume of $T^3$ in the M theory metric.

From the eight dimensional point of view, this U–duality symmetry of type IIB string theory has a more involved interpretation. The eight dimensional theory has an $SL(2, \mathbb{Z})_\tau$ S–duality symmetry which is inherited from ten dimensions. It acts on the ten dimensional complexified coupling

$$\tau = \tau_1 + i\tau_2 = C_0 + ie^{-\phi}$$

as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d},$$

under $SL(2, \mathbb{Z})_\tau$. The eight dimensional theory has an $SL(2, \mathbb{Z})_\tau$ S–duality symmetry which is inherited from ten dimensions. It acts on the ten dimensional complexified coupling

$$\tau = \tau_1 + i\tau_2 = C_0 + ie^{-\phi}$$

as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d},$$
and on the combination $B_R + \tau B_N$ as

$$B_R + \tau B_N \rightarrow \frac{B_R + \tau B_N}{c\tau + d},$$

(5)

where $B_N(B_R)$ is the modulus from the NS–NS (R–R) two form on $T^2$. This theory also has an $SL(2,\mathbb{Z})_T$ T–duality symmetry which acts on the Kahler structure modulus of $T^2$ defined by

$$T = B_N + iV_2,$$

(6)

as

$$T \rightarrow \frac{aT + b}{cT + d},$$

(7)

where $V_2$ is the volume of $T^2$ in the string frame. It also acts on the complex scalar $\rho$ defined by

$$\rho = -B_R + i\tau_1 V_2,$$

(8)

as

$$\rho \rightarrow \frac{\rho}{c\rho + d},$$

(9)

while leaving the eight dimensional dilaton invariant. Now the $SL(2,\mathbb{Z})_\tau$ and the $SL(2,\mathbb{Z})_T$ transformations can be intertwined and embedded into the $SL(3,\mathbb{Z})$ factor of the U–duality group. Also the $SL(2,\mathbb{Z})$ factor of the U–duality group acts on the complex structure modulus $U$ of $T^2$ as

$$U \rightarrow \frac{aU + b}{cU + d}.$$  

(10)

The $\mathcal{R}^4$ interaction in type IIB string theory on $T^2$ has been analyzed in [3,4] directly in string theory, as well as from the point of view of eleven dimensional supergravity on $T^3$. In the Einstein frame, where the metric is U–duality invariant, the coefficient of the $\mathcal{R}^4$ interaction is given by a modular form of the U–duality group, that is modular invariant under $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$ transformations. An expression for this modular form has been conjectured in [3,4] which we shall mention later.

In this paper, we consider the $D^4\mathcal{R}^4$ interaction in type IIB string theory compactified on $T^2$. By this, we actually mean the interaction

$$(s^2 + t^2 + u^2)\mathcal{R}^4$$

(11)

involving the elastic scattering of two gravitons. We propose a manifestly U–duality invariant modular form that is the coefficient of this interaction in the Einstein frame. Explicitly, this modular form is given by

$$E_{5/2}(M)^{SL(3,\mathbb{Z})} - 8E_2(M^{-1})^{SL(3,\mathbb{Z})}E_2(U, \bar{U})^{SL(2,\mathbb{Z})},$$

(12)
where \( E_{5/2}(M)^{SL(3,\mathbb{Z})} \) and \( E_2(M^{-1})^{SL(3,\mathbb{Z})} \) are \( SL(3,\mathbb{Z}) \) invariant modular forms in the fundamental and the anti–fundamental representations of \( SL(3,\mathbb{Z}) \) respectively as we shall discuss below, and \( E_2(U,\bar{U})^{SL(2,\mathbb{Z})} \) is an \( SL(2,\mathbb{Z}) \) invariant modular form.

We first consider some systematics of the four graviton amplitude and briefly review the conjectured modular form for the \( R^4 \) interaction in eight dimensions. In the next section, we argue for the modular form for the \( D^4R^4 \) interaction. Our arguments are based on the known modular form for the \( D^4R^4 \) interaction in ten dimensions, U–duality invariance, and the perturbative equality of the amplitude in type IIA and type IIB string theories. Our proposed modular form satisfies certain non–renormalization properties: it receives perturbative contributions only up to two string loops, as well as an infinite number of non–perturbative contributions coming from D–instantons and \((p,q)\) string instantons wrapping \( T^2 \). To provide some evidence for the modular form, we next calculate the four graviton amplitude in eight dimensions using string perturbation theory at tree level and at one loop, and show that it exactly matches the amplitude given by the modular form. We also consider eleven dimensional supergravity compactified on \( T^3 \) at one loop, and obtain parts of the perturbative string theory amplitude, which are in precise agreement with the coefficients given by the modular form. In the next section, we decompactify the theory to nine dimensions, which has a conjectured \( SL(2,\mathbb{Z}) \times R^+ \) U–duality symmetry. The modular form we obtain for the \( D^4R^4 \) interaction in nine dimensions manifestly exhibits this U–duality symmetry. To provide further evidence, we calculate the four graviton amplitude in nine dimensions at tree level and at one loop, and obtain precise agreement with the amplitude given by the modular form. We end with some comments about the modular form for the \( D^4R^4 \) interaction for toroidal compactifications to lower dimensions.

2 Some systematics of the higher derivative interactions and the \( R^4 \) interaction

First let us consider the effective action of type IIB string theory in ten dimensions. In particular, we consider the perturbative contributions to the protected \( R^4 \) and the \( D^4R^4 \) interactions along with the Einstein–Hilbert term in the string frame. Here \( R^4 \) stands for the \( t_8t_8R^4 \) interaction [21–23], and can be expressed entirely in terms of four powers of the Weyl tensor. Dropping various irrelevant numerical factors, these terms are given by

\[
S \sim \frac{1}{l_s^8} \int d^{10} x \sqrt{-g} e^{-2\phi} R + \frac{1}{l_s^8} \int d^{10} x \sqrt{-g} \left( 2\zeta(3)e^{-2\phi} + \frac{2\pi^2}{3} + \ldots \right) R^4
\]
\[ + l_s^2 \int d^{10} x \sqrt{-g} \left( 2\zeta(5)e^{-2\phi} + \frac{4\pi^4}{135}e^{2\phi} + \ldots \right) D^4 \mathcal{R}^4, \]

where \ldots are the various non-perturbative corrections coming from D-instantons. Now compactifying on \( T^2 \) of volume \( V_2 l_s^2 \) in the string frame and moving to the eight dimensional Einstein frame, we see that \( (13) \) gives us

\[ S \sim \frac{1}{\hbar_s} \int d^8 x \sqrt{-g_s} \hat{R} + \int d^8 x \sqrt{-g_s} V_2 \left( 2\zeta(3)e^{-2\phi} + \frac{2\pi^2}{3} + \ldots \right) \hat{R}^4 + \ldots \]

where the hat denotes quantities in the eight dimensional Einstein frame. Thus from \( (14) \), we see that \( (13) \) gives us

\[ V_2 \left( 2\zeta(3)e^{-2\phi} + \frac{2\pi^2}{3} + \ldots \right) \]

among other terms. In fact, an expression for this modular form has been conjectured in \([3,4]\). In order to write down the manifestly U-duality invariant modular form, we note that the part of the supergravity action involving the scalars can be written in the Einstein frame as (we are following the conventions of \([4]\))

\[ S \sim \frac{1}{\hbar_s} \int d^8 x \sqrt{-g_s} \left( \hat{R} - \frac{\partial_\mu U \hat{\partial}^\mu U}{2U_2^2} + \frac{1}{4} \text{Tr}(\partial_\mu M \hat{\partial}^\mu M^{-1}) + \ldots \right), \]

where \( M \) is a symmetric matrix with determinant one given by

\[ M = \nu^{1/3} \begin{pmatrix} 1/\tau_2 & \tau_1/\tau_2 & \text{Re}(B)/\tau_2 \\ \tau_1/\tau_2 & |\tau|^2/\tau_2 & \text{Re}(\bar{\tau}B)/\tau_2 \\ \text{Re}(B)/\tau_2 & \text{Re}(\bar{\tau}B)/\tau_2 & 1/\nu + |B|^2/\tau_2 \end{pmatrix}, \]

where \( B = B_R + \tau B_N \), and \( \nu = (\tau_2 V_2^2)^{-1} \). In \( (16) \), the matrices \( U \) and \( M \) parametrize the coset manifolds \( SL(2,\mathbb{R})/SO(2) \) and \( SL(3,\mathbb{R})/SO(3) \) respectively, and so we see that the scalar manifold is reducible and is given by \( SL(2,\mathbb{R})/SO(2) \times SL(3,\mathbb{R})/SO(3) \). The conjectured U-duality group is generated by the transformations \( U \rightarrow (aU + b)/(cU + d) \), and \( M \rightarrow \Omega_2 M \Omega_2^T \), where \( a, b, c, d \in \mathbb{Z} \) with \( ad - bc = 1 \), and \( \Omega_2 \in SL(3,\mathbb{Z}) \).

The conjectured U-duality invariant modular form for the \( \mathcal{R}^4 \) interaction is given by

\[ E_{3/2}(M)^{SL(3,\mathbb{Z})} = 2\pi \log(U_2|\eta(U)|^4), \]

where

\[ E_{3/2}(M)^{SL(3,\mathbb{Z})} = \sum_{m_i} \left( m_i M_{ij} m_j \right)^{-3/2}, \]
where \( m_i \) are integers, and the sum excludes \( \{m_1, m_2, m_3\} = \{0, 0, 0\} \). Here \( E_s(M)^{SL(3,\mathbb{Z})} \)

is the \( SL(3,\mathbb{Z}) \) invariant Eisenstein series of order \( s \) in the fundamental representation of \( SL(3,\mathbb{Z}) \), defined by \( \{1N\} \). Also the other term in \( \{1N\} \) is the \( SL(2,\mathbb{Z}) \) invariant Eisenstein series of order one \(^2\). From \( \{1N\} \), it follows that the \( R^4 \) interaction receives perturbative contributions only at tree level and at one loop, and non–perturbative contributions coming from D–instantons and \((p,q)\) string instantons wrapping \( T^2 \). This correctly reduces to the \( \tilde{R}^4 \) interaction in ten dimensions, which is given by \( E_{3/2}(\tau, \bar{\tau})^{SL(2,\mathbb{Z})} \), the \( SL(2,\mathbb{Z}) \) invariant Eisenstein series of order \( 3/2 \), where the Eisenstein series of order \( s \) is defined by \( \{94\} \) (see [24] for details).

3 The modular form for the \( D^4R^4 \) interaction

We now proceed to construct the modular form for the \( D^4R^4 \) interaction. From \( \{14\} \), we see that the U–duality invariant modular form for the \( \tilde{D}^4\tilde{R}^4 \) interaction must contain

\[
V_2^{5/3} e^{-4\phi/3} \left( 2\zeta(5)e^{-2\phi} + \frac{4\pi^4}{135}e^{2\phi} \right) + \ldots,
\]

among other terms. These lead to tree level and two loop contributions when converted to the string frame. Our aim is to propose an exact expression for this modular form.

3.1 The proposed modular form

The modular form for the \( \tilde{D}^4\tilde{R}^4 \) interaction in ten dimensions is given by \( E_{5/2}(\tau, \bar{\tau})^{SL(2,\mathbb{Z})} \), the \( SL(2,\mathbb{Z}) \) invariant Eisenstein series of order \( 5/2 \) [14]. From the structure of \( E_{5/2}(\tau, \bar{\tau}) \), it follows that the \( \tilde{D}^4\tilde{R}^4 \) interaction receives perturbative contributions only at tree level and at two loops, and an infinite number of non–perturbative contributions from D–instantons.

Thus, following the conjecture for the \( R^4 \) interaction and given the modular form for the \( D^4R^4 \) interaction in ten dimensions, it is natural to propose that a part of the full U–duality invariant modular form for the \( \tilde{D}^4\tilde{R}^4 \) interaction is given by the order \( 5/2 \) Eisenstein series for \( SL(3,\mathbb{Z}) \) defined by (see \( \{100\} \))

\[
E_{5/2}(M)^{SL(3,\mathbb{Z})} = 2(\tau_2^2V_2)^{5/3}\zeta(5) + \frac{4}{3}(\tau_2^2V_2)^{-1/3}E_2(T, \bar{T})^{SL(2,\mathbb{Z})}
+ \frac{8\pi^2}{3} \tau_2^{4/3}V_2^{5/3} \sum_{m_1 \neq 0, m_2 \neq 0} \left| \frac{m_1}{m_2} \right|^2 K_2(2\pi\tau_2|m_1m_2|)e^{2\pi im_1m_2\tau_1}
\]

\(^2\)The modular form for the \( R^4 \) interaction is actually divergent, and has to be regularized. For the case of the \( D^4R^4 \) interaction, there are no such divergences.
\[ + \frac{2\pi}{3} \tau_{2}^{-2/3} V_{2}^{-4/3} \sum_{m_{1} \neq 0, m_{3} \neq 0, m_{2}} \frac{1 + 2\pi m_{3}(m_{2} - m_{1}\tau)}{|m_{3}|^{3}} \times e^{-2\pi |m_{3}(m_{2} - m_{1}\tau)| V_{2}} + 2\pi i m_{3}(m_{1} B_{R} + m_{2} B_{N}), \] (21)

where we have used
\[ K_{3/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} (1 + x^{-1}). \] (22)

Now one can read off the various perturbative and non-perturbative contributions to the four graviton amplitude from (21). While the perturbative contributions are given by the first line of (21), the non-perturbative contributions are from D–instantons which are given by the second line of (21), as well as from \((p, q)\) string instantons wrapping \(T^{2}\) with \(q \neq 3\) which are given by the third line of (21).

Thus the perturbative contribution to \(E_{5/2}(M)^{SL(3, \mathbb{Z})}\) is given by
\[ E_{5/2}(M)^{SL(3, \mathbb{Z})}_{\text{pert}} = 2(\tau_{2}^{2} V_{2})^{5/3} \zeta(5) + \frac{4}{3} (\tau_{2}^{2} V_{2})^{-1/3} E_{2}(T, \bar{T})^{SL(2, \mathbb{Z})}. \] (23)

Now using the fact that we are restricting ourselves to the \(t_{8}t_{8}R^{4}\) part of the amplitude which involves only the even–even spin structure, we see that the perturbative contribution must be the same in type IIA and type IIB string theory. In going from type IIA to type IIB string theory, \(\tau_{2}^{2} V_{2}\) is invariant and \(U \leftrightarrow T\), and so we add the relevant two loop term to (23) to get a part of the perturbative piece of the whole amplitude
\[ 2(\tau_{2}^{2} V_{2})^{5/3} \zeta(5) + \frac{4}{3} (\tau_{2}^{2} V_{2})^{-1/3} \left( E_{2}(T, \bar{T})^{SL(2, \mathbb{Z})} + E_{2}(U, \bar{U})^{SL(2, \mathbb{Z})} \right) + \ldots. \] (24)

Now the first two terms in (24) are obtained from the modular form (21), and so we want to find a modular form which has
\[ \frac{4}{3} (\tau_{2}^{2} V_{2})^{-1/3} E_{2}(U, \bar{U})^{SL(2, \mathbb{Z})} \] (25)
as the two loop contribution. Now let us consider the modular form
\[ E_{-1/2}(M)^{SL(3, \mathbb{Z})} E_{2}(U, \bar{U})^{SL(2, \mathbb{Z})}, \] (26)
which has
\[ E_{-1/2}(M)^{SL(3, \mathbb{Z})}_{\text{pert}} E_{2}(U, \bar{U})^{SL(2, \mathbb{Z})} = - \left[ \frac{1}{6} (\tau_{2}^{2} V_{2})^{-1/3} + \frac{\Gamma(-1)}{2} (\tau_{2}^{2} V_{2})^{2/3} E_{-1}(T, \bar{T})^{SL(2, \mathbb{Z})} \right] E_{2}(U, \bar{U})^{SL(2, \mathbb{Z})} \] (27)

\(^{3}(1, 0)\) is the fundamental string in our conventions.
where we have used $\zeta(-1) = -1/12$. In (27), let us consider the second term which naively might seem problematic. This is because it contains $\Gamma(-1)$ which is infinite, and also because the terms in $E_{-1}(T, \bar{T})^{SL(2,\mathbb{Z})}$ which are not exponentially suppressed for large $T_2$ vanish using (94) because $\zeta(-2) = 0$. However using the relation (95), we see this is not the case and we get a finite answer for this quantity. In fact we get that

$$E_{-1/2}(M)^{SL(3,\mathbb{Z})} E_2(U, \bar{U})^{SL(2,\mathbb{Z})}$$

$$=-\left[\frac{1}{6}(\tau_2^2 V_2)^{-1/3} + \frac{1}{2\pi^3}(\tau_2^2 V_2)^{2/3} E_2(T, \bar{T})^{SL(2,\mathbb{Z})}\right] E_2(U, \bar{U})^{SL(2,\mathbb{Z})}. \quad (28)$$

Now the first term in (28) is proportional to (25) and yields a two loop contribution, while the second term contributes at one loop. Thus it is natural to guess that the modular form which yields (25) is given by

$$-8E_{-1/2}(M)^{SL(3,\mathbb{Z})} E_2(U, \bar{U})^{SL(2,\mathbb{Z})}$$

$$=-8E_2(M^{-1})^{SL(3,\mathbb{Z})} E_2(U, \bar{U})^{SL(2,\mathbb{Z})}, \quad (29)$$

where we have used (103) in going from a modular form of $SL(3,\mathbb{Z})$ in the fundamental representation to a modular form in the anti–fundamental representation. Thus in the Einstein frame, we get the manifestly U–duality invariant interaction in the type IIB effective action

$$l_s^4 \int d^8 x \sqrt{-g_8} \left[ E_{5/2}(M)^{SL(3,\mathbb{Z})} - 8E_2(M^{-1})^{SL(3,\mathbb{Z})} E_2(U, \bar{U})^{SL(2,\mathbb{Z})} \right] \hat{D}^4 \hat{R}^4. \quad (30)$$

Converting to the string frame and considering the perturbative parts, we see that (30) contributes at tree level, and at one and two loops only. More explicitly, the perturbative contributions to the effective action are given in the string frame by (upto an overall numerical factor)

$$l_s^4 \int d^8 x \sqrt{-g_8} \sum_{g=0}^{2} (V_2^{-1/2} \epsilon^4)^{2g-2} F_g(T, U, \bar{T}, \bar{U}) D^4 \mathcal{R}^4, \quad (31)$$

where

$$F_0(T, U, \bar{T}, \bar{U}) = 2\zeta(5),$$

$$F_1(T, U, \bar{T}, \bar{U}) = \frac{4}{\pi^3} E_2(T, \bar{T})^{SL(2,\mathbb{Z})} E_2(U, \bar{U})^{SL(2,\mathbb{Z})},$$

$$F_2(T, U, \bar{T}, \bar{U}) = \frac{4}{3} \left( E_2(T, \bar{T})^{SL(2,\mathbb{Z})} + E_2(U, \bar{U})^{SL(2,\mathbb{Z})} \right). \quad (32)$$
Thus while going from type IIB to type IIA string theory, which involves interchanging $U$ and $T$ while leaving $e^{-2\phi}V_2$ invariant, we see that \((31)\) is invariant, and so the perturbative contributions to the IIA and IIB theories are the same.

Thus, we propose that the U–duality invariant modular form for the $\hat{D}^4\hat{R}^4$ interaction is given by

$$E(M, U)^{SL(3,\mathbb{Z})} = E_{5/2}(M)^{SL(3,\mathbb{Z})} - 8E_2(M^{-1})^{SL(3,\mathbb{Z})}E_2(U, \bar{U})^{SL(2,\mathbb{Z})},$$

which satisfies non–renormalization properties characteristic of BPS saturated operators. As discussed before, it yields only a finite number of perturbative contributions, as well as an infinite number of non–perturbative contributions involving D–instantons and $(p, q)$ string instantons, as well as perturbative fluctuations about their backgrounds. It involves modular forms of $SL(2,\mathbb{Z})_U$ and $SL(3,\mathbb{Z})_M$ which satisfy the Laplace equations

$$\Delta_{SL(2,\mathbb{Z})}E_s(U, \bar{U})^{SL(2,\mathbb{Z})} = 4U_2^2 \frac{\partial^2}{\partial U \partial \bar{U}}E_s(U, \bar{U})^{SL(2,\mathbb{Z})} = s(s-1)E_s(U, \bar{U})^{SL(2,\mathbb{Z})},$$

and \([4]\)

$$\Delta_{SL(3,\mathbb{Z})}E_s(M)^{SL(3,\mathbb{Z})} = \left[ 4\tau_2^2 \frac{\partial^2}{\partial T \partial \bar{T}} + \frac{1}{\nu \tau_2} |\partial B_N - \tau \partial B_R|^2 + 3 \partial \nu (\nu^2 \partial \nu) \right]E_s(M)^{SL(3,\mathbb{Z})}$$

$$= \frac{2s(2s-3)}{3}E_s(M)^{SL(3,\mathbb{Z})},$$

on the fundamental domains of $SL(2,\mathbb{Z})_U$ and $SL(3,\mathbb{Z})_M$ respectively.

### 3.2 Evidence using string perturbation theory

We now provide some evidence for the modular form \((33)\) using superstring perturbation theory. We shall show that the four graviton amplitude in eight dimensions at tree level and at one loop precisely gives the values predicted by the modular form.

The sum of the contributions to the four graviton amplitude at tree level \([21, 23]\) and at one loop \([23, 25]\) in type II string theory compactified on an $n$ dimensional torus $T^n$ is proportional to \(\dfrac{\Omega}{\bar{\Omega}}\)

$$4 \int d^2 \Omega \frac{\Omega^2}{\bar{\Omega}^2} \text{Z}_{\text{lat}} F(\Omega, \bar{\Omega}),$$

where $V_n$ is the volume of $T^n$ in the string frame, and $I$ is obtained from the one loop amplitude, and is given by

$$I = \int \frac{d^2 \Omega}{\bar{\Omega}^2} \text{Z}_{\text{lat}} F(\Omega, \bar{\Omega}),$$

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\(\dfrac{\Omega}{\bar{\Omega}}\)The calculation actually yields $\mathcal{R}^4$ at the linearized level.
where $F$ is the fundamental domain of $SL(2, \mathbb{Z})$, and $d^2 \Omega = d\Omega d\bar{\Omega}/2$. The relative coefficient between the tree level and the one loop terms in (36) is fixed using unitarity [26]. In (37), the lattice factor $Z_{lat}$ which depends on the moduli is given by

$$Z_{lat} = V_n \sum_{m_i, n_i \in \mathbb{Z}} e^{-\frac{\pi}{2\Omega} \sum_{i,j}(G+B_N)_{ij}(m_i+n_i\Omega)(m_j+n_j\bar{\Omega})},$$

(38)

where $i, j = 1, \ldots, n$. Specializing to the case of $T^2$, we get that [27]

$$Z_{lat} = V_2 \sum_{m_1, m_2, n_1, n_2 \in \mathbb{Z}} e^{-\pi T_2 \Omega \frac{(m_1 \Omega + n_1 \bar{\Omega})(m_2 \Omega + n_2 \bar{\Omega})}{2}} \exp \left[ -2\pi i T (\det A) - \frac{\pi T_2}{\Omega^2} \left( \begin{array}{c} 1 \\ U_1 \\ U_2 \\ \mid U \mid \end{array} \right) \right],$$

(39)

where

$$G_{ij} = T_2 \left( \begin{array}{c} 1 \\ U_1 \\ U_2 \end{array} \right).$$

(40)

Also the dynamical factor $F(\Omega, \bar{\Omega})$ in (37) is given by

$$F(\Omega, \bar{\Omega}) = \int_\mathcal{T} \prod_{i=1}^3 d^2 \nu_i \frac{\chi_{12}^{\nu_i} \chi_{13}^{\nu_i} \chi_{14}^{\nu_i} \chi_{23}^{\nu_i}}{\Omega^2}.$$

(41)

In (41), $\nu_i$ ($i = 1, \ldots, 4$) are the positions of insertions of the four vertex operators on the toroidal worldsheet, and $\nu_4$ has been set equal to $\Omega$ using conformal invariance. Also $d^2 \nu_i = dv^R_i dv^I_i$, where $v^R_i$ ($v^I_i$) are the real (imaginary) parts of $\nu_i$. The integral over $\mathcal{T}$ is over the domain $\mathcal{T} = \{-1/2 \leq \nu^R_i < 1/2, 0 \leq \nu^I_i < \Omega_2\}$. Finally, $\ln \chi_{ij}(\nu_i - \nu_j; \Omega)$ is the scalar Green function between the points $\nu_i$ and $\nu_j$ on the toroidal worldsheet and is given by

$$\ln \chi(\nu; \Omega) = \frac{1}{4\pi} \sum_{(m,n) \neq (0,0)} \frac{\Omega^2}{|m\Omega + n|^2} e^{\pi [\rho(m\Omega + n) - \nu(m\Omega + n)]/\tau_2} + \frac{1}{2} \ln \left| (2\pi)^{1/2} \eta(\Omega) \right|^2.$$

(42)

In (42), the last term which is the zero mode does not contribute to the on-shell amplitude and hence can be dropped. In evaluating (41) to fourth order in the momenta, we use the relation [28]

$$\int_\mathcal{T} \frac{d^2 \nu_i d^2 \nu_j}{\Omega^2} |\ln \chi(\nu_i - \nu_j; \Omega)|^2 = \frac{1}{16\pi^2} \sum_{(m,n) \neq (0,0)} \frac{\Omega^2}{|m\Omega + n|^2} = \frac{1}{16\pi^2} E_2(\Omega, \bar{\Omega})^{SL(2, \mathbb{Z})},$$

(43)

which can be deduced using (42) with the zero mode term removed. Thus, expanding to fourth order in the momenta, the total contribution of the tree level term and the one loop
term in (36) gives

\[ 2\zeta(5)V_2 e^{-2\phi} + \frac{4\pi}{\Omega_2^2} \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} \mathcal{Z}_{\text{lat}} E_2(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})} \] \( l_s^4 (s^2 + t^2 + u^2) \mathcal{R}^4. \) \( (44) \)

In (44), note that the one loop contribution has been integrated over the restricted fundamental domain \( \mathcal{F}_L \) of \( SL(2,\mathbb{Z}) \), which is obtained from \( \mathcal{F} \) by restricting to \( \Omega_2 \leq L \). This is necessary to separate the analytic parts of the amplitude from the non–analytic parts (see [28] for a detailed discussion). The integral over \( \mathcal{F}_L \) gives both finite and divergent terms to the amplitude in the limit \( L \to \infty \). The terms which are finite in this limit are the analytic parts of the amplitude. The parts which diverge in this limit cancel in the whole amplitude when the contribution from the part of the moduli space \( \mathcal{F} \) with \( \Omega_2 > L \) is also included. In addition to these divergences which cancel, the contribution from \( \mathcal{F} \) with \( \Omega_2 > L \) also gives the various non–analytic terms in the amplitude. Keeping this in mind, we shall consider only the contributions which are finite in the limit \( L \to \infty \) in (44), and drop all divergent terms. In the calculations below, we shall see that the domain of integration \( \mathcal{F} \) shall often be changed to the upper half plane or a strip. Then truncating to \( \mathcal{F}_L \) to calculate the analytic terms cannot be done when the integration over \( \mathcal{F}_L \) produces divergences of the form \( \ln L \) [28]. However, from (44), using the expression for \( E_2(\tau, \bar{\tau})^{SL(2,\mathbb{Z})} \), we see that there are no logarithmic divergences, and so this is not a problem for us.

We write

\[ \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} \mathcal{Z}_{\text{lat}} E_2(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})} = I_1 + I_2 + I_3, \] \( (45) \)

where \( I_1, I_2, \) and \( I_3 \) are the contributions from the zero orbit, the non–degenerate orbits and the degenerate orbits of \( SL(2,\mathbb{Z}) \) respectively [27] (also see [29]). Now, from (44), we get that

\[ E_2(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})} = 2\zeta(4)\Omega_2^2 + \frac{\pi \zeta(3)}{\Omega_2} + 2\pi^2 \sqrt{\Omega_2} \sum_{m_1 \neq 0, m_2 \neq 0} \left| \frac{m_1}{m_2} \right|^{3/2} K_{3/2}(2\pi\Omega_2|m_1 m_2|) e^{2\pi i m_1 m_2 \Omega_1}. \] \( (46) \)

We now calculate the contributions to (45) from the various orbits. In doing the integrals, we frequently make use of the definition

\[ K_s(x) = \frac{1}{2} \left( \frac{x}{2} \right)^s \int_0^\infty \frac{dt}{t^{s+1}} e^{-t-x^2/4t}. \] \( (47) \)

(i) The contribution from the zero orbit involves setting \( A = 0 \) in (39) leading to

\[ I_1 = V_2 \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} E_2(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})} = 0, \] \( (48) \)
involves integers \( m, n \) and then finally do the \( \tau \) expressions are written, and we have also used \( k > j \) in (39), where \( \hat{m}, \hat{n} \) cover of the upper half plane. This leads to when \( \Delta \rightarrow \infty \) we do not get any term which is finite as \( L \to \infty \), and thus \( \Delta \) vanishes [28]. In fact, this is the reason why the one loop contribution to the \( D^4 R^4 \) interaction vanishes in ten dimensions.

The contributions from the non–degenerate and degenerate orbits yield finite pieces when \( L \to \infty \), and so we directly integrate over \( F \) rather than \( F_L \) in the expressions below.

(ii) The contribution from the non–degenerate orbits involves setting

\[
A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}
\]

in (39), where \( k > j \geq 0, p \neq 0 \), and changing the domain of integration to be the double cover of the upper half plane. This leads to

\[
I_2 = 2V_1 \int_{-\infty}^{\infty} d\Omega_1 \int_{0}^{\infty} d\Omega_2 \left( \frac{d}{\Omega_2^2} \right) E_2(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})} \sum_{k \geq j > 0, p \neq 0} e^{-2\pi i T kp - \frac{\pi T_2}{\Omega_2} k|\Omega + j + p U|^2} \]

\[
= 2 \left( 2\zeta(4) U_2 + \frac{\pi \zeta(3)}{U_2} \right) \sqrt{T_2} \sum_{p \neq 0, k \neq 0} \left| \frac{p}{k} \right|^{3/2} K_{3/2}(2\pi T_2 |p|) e^{2\pi i pk T_1} \]

\[
+ 4\pi^2 \sqrt{U_2 T_2} \sum_{m \neq 0, n \neq 0, p \neq 0, q \neq 0} \left| \frac{m}{n} \right|^{3/2} K_{3/2}(2\pi |pq| U_2) K_{3/2}(2\pi \frac{mnp}{q} T_2) \times e^{2\pi i (q U_1 + mn T_1 / q)}.
\]

The three terms in (51) are obtained from the three terms in (46) in the order the expressions are written, and we have also used

\[
K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}.
\]

In evaluating the integrals in (51), we always first do the \( \tau_1 \) integral, then sum over \( j \), and then finally do the \( \tau_2 \) integral. In (51), the last term involves a restricted sum which involves integers \( m, n, \) and \( q \) such that \( (mn) / q \) is an integer. Now defining

\[
pq = \hat{p} \hat{q}, \; \frac{mnp}{q} = \hat{m} \hat{n}, \; \frac{m}{n} = \frac{\hat{m}}{\hat{n}},
\]

where \( \hat{m}, \hat{n}, \hat{p}, \) and \( \hat{q} \) are non–zero integers, we get an unrestricted sum

\[
\sum_{m \neq 0, n \neq 0, p \neq 0, q \neq 0} \left| \frac{m}{n} \right|^{3/2} K_{3/2}(2\pi |pq| U_2) K_{3/2}(2\pi \frac{mnp}{q} T_2) e^{2\pi i (q U_1 + mn T_1 / q)}
\]

\[
= \left\{ \sum_{\hat{p} \neq 0, \hat{q} \neq 0} \left| \frac{\hat{p}}{\hat{q}} \right|^{3/2} K_{3/2}(2\pi |\hat{p} \hat{q}| U_2) e^{2\pi i \hat{p} \hat{q} U_1} \right\} \left\{ \sum_{\hat{m} \neq 0, \hat{n} \neq 0} \left| \frac{\hat{m}}{\hat{n}} \right|^{3/2} K_{3/2}(2\pi |\hat{m} \hat{n}| T_2) e^{2\pi i \hat{m} \hat{n} T_1} \right\}.
\]
Thus we get that

$$ I_2 = 2\sqrt{T_2}E_2(U, \bar{U})^{SL(2,\mathbb{Z})} \sum_{p \neq 0, k \neq 0} \left| \frac{p}{k} \right|^{3/2} K_{3/2}(2\pi T_2 |p|) e^{2\pi ipkT_1} \tag{55} $$

(iii) The contribution from the degenerate orbits involves setting

$$ A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix} \tag{56} $$

in (39) such that \((j, p) \neq (0, 0)\), and changing the domain of integration to be the strip \(\Omega_2 > 0, |\Omega_1| < 1/2\), leading to

$$ I_3 = V_2 \int_{-1/2}^{1/2} d\Omega_1 \int_0^\infty d\Omega_2 \frac{E_2(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})}}{\Omega_2} \sum_{(j, p) \neq (0, 0)} e^{-\pi T_2 |j + pU|^2} = \frac{1}{\pi^2} \left( 2\zeta(4)T_2^2 + \frac{\pi \zeta(3)}{T_2} \right) E_2(U, \bar{U})^{SL(2,\mathbb{Z})}, \tag{57} $$

where have used (97) with \(s = -1\), for \(E_s(U, \bar{U})^{SL(2,\mathbb{Z})}\). Note that the contribution of the last term in (46) to (57) vanishes because of the \(\Omega_1\) integral.

Thus from (48), (55), and (57), we get that

$$ \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2} Z_{lat} E_2(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})} = \frac{1}{\pi^2} E_2(U, \bar{U})^{SL(2,\mathbb{Z})} E_2(T, \bar{T})^{SL(2,\mathbb{Z})}, \tag{58} $$

which when substituted in (14) precisely gives the coefficients \(F_0\) and \(F_1\) in (32). Thus we get a non–trivial consistency check of the proposed modular form using string perturbation theory.

We will have nothing to say about the two loop calculation of the amplitude apart from making a minor comment. Dropping numerical factors, the relevant term in the two loop amplitude involving four powers of momenta is given by [30]

$$ e^{2\phi}(s^2 + t^2 + u^2) R^4 \int_{\mathcal{M}_2} \frac{|d^2\Omega|^2}{(\text{det Im} \Omega)^3} Z_{lat}, \tag{59} $$

where \(\Omega_{AB}\) is the period matrix, \(Z_{lat}\) is the lattice factor given by

$$ Z_{lat} = V_2 \sum_{m_i, n_j} e^{\pi \sum_i (G + B_N)_{ij} (m_i A + \Omega_{AB} n_i) (\text{Im} \Omega)^A C (m_j C + \Omega_{CD} n_j)} \tag{60} $$

and the integral is over \(\mathcal{M}_2\), the fundamental domain of \(Sp(4, \mathbb{Z})\) (see [31], for example). Unlike the one loop calculation, probably one does not need to restrict the integral (59) to
a restricted fundamental domain of \( Sp(4, \mathbb{Z}) \). This is because the problematic term involves configurations where the fundamental string worldsheet has vanishing winding which gives \( Z_{lat} = V_2 \) in \(^{(60)}\). However, this term gives the volume of the fundamental domain of \( Sp(4, \mathbb{Z}) \) and is finite \(^{(32)}\). This is the reason why the two loop contribution to \( D^4 \mathcal{R}^4 \) is non–vanishing in ten dimensions. The other contributions which involve non–trivial \( Z_{lat} \) are expected to converge giving a finite answer. It would be interesting to find the two loop coefficient and see if it agrees with \( F_2 \) in \(^{(32)}\).

### 3.3 Evidence using eleven dimensional supergravity on \( T^3 \) at one loop

We now provide some evidence for \(^{(32)}\) using the four graviton amplitude in eleven dimensional supergravity compactified on \( T^3 \). It is known that the \( D^4 \mathcal{R}^4 \) interaction receives contributions only from one and two loops\(^5\). We shall consider only the one loop amplitude and show that it reproduces some terms in \(^{(32)}\).

The one loop four graviton amplitude is given by \(^{(14, 34–36)}\)

\[
A_4 = \frac{\kappa_{11}^4}{(2\pi)^{11}} \hat{K} \left[ I(S, T) + I(S, U) + I(U, T) \right],
\]

where \( \hat{K} \) involves the \( \mathcal{R}^4 \) interaction at the linearized level, and

\[
I(S, T) = \frac{2\pi^4}{l_{11}^3 V_3} \int_0^{\infty} \frac{d\sigma}{\sigma} \int_0^1 d\omega_3 \int_0^{\omega_3} d\omega_2 \int_0^{\omega_2} d\omega_1 \sum_{\{l_1, l_2, l_3\}} e^{-G^{I,J}l_{1l}^J l_{2l}l_{3l} \sigma/l_{11}^2 - Q(S, T; \omega_r) \sigma},
\]

where \( Q(S, T; \omega_r) = -S\omega_1(\omega_3 - \omega_2) - T(\omega_2 - \omega_1)(1 - \omega_3) \).

Denoting the torus directions as 1, 2, and 3, we choose \( G_{11} = R_{11}^2 \) to be the metric along the M theory circle, thus \( R_{11} = e^{2\phi / 3} \). Though we need the \((s^2 + u^2 + t^2)\mathcal{R}^4\) term, we shall later find it useful to extract a part of the momentum independent amplitude from \(^{(61)}\) in order to fix normalizations. This is given by

\[
A_4(S = T = U = 0) = \frac{\kappa_{11}^4 \hat{K}}{(2\pi)^{11}} \cdot \frac{\pi^4}{l_{11}^3 V_3} \int_0^{\infty} \frac{d\sigma}{\sigma} \sum_{\{l_1, l_2, l_3\}} e^{-G^{I,J}l_{1l}^J l_{2l}l_{3l} \sigma/l_{11}^2} = \frac{\kappa_{11}^4 \hat{K}}{(2\pi)^{11}} \cdot \frac{\pi^4}{l_{11}^3 V_3} \int_0^{\infty} \frac{d\sigma}{\sigma^{5/2}} \sum_{\{l_1, l_2, l_3\}} e^{-\frac{G^{I,J}l_{1l}^J l_{2l}l_{3l} \sigma}{\sigma/l_{11}^2}},
\]

\(^5\)The three loop contribution has leading dependence \( D^6 \mathcal{R}^4 \) \(^{(33)}\).

\(^6\)In this section, loops refer to spacetime loops in eleven dimensional supergravity on \( T^3 \). We shall refer to the worldsheet expansion of string perturbation theory as the genus expansion.

\(^7\)Note that \( \sigma \) has dimensions of \((\text{length})^2\).
where we have done Poisson resummation using (102). Considering the \( \hat{l}_1 \neq 0, \hat{l}_2 = \hat{l}_3 = 0 \) piece, (63) gives [3]

\[
A_4(S = T = U = 0) = \frac{k_{\perp}^4 \hat{K}}{(2\pi)^3 l_{11}^3} \left[ \pi^3 \zeta(3) e^{-2\phi^4} + \ldots \right].
\] (64)

As an aside, note that the contribution of the non-analytic part of the amplitude involves setting \( l_I = 0 \) in (62) leading to (see [36] for relevant discussion)

\[
I(S, T)_{\text{non-anal}} = \frac{2\pi^4}{l_{11}^3 V_3} \int_0^\infty \frac{d\sigma}{\sigma} \int_0^1 d\omega_3 \int_0^{\omega_3} d\omega_2 \int_0^{\omega_2} d\omega_1 \left( e^{-Q(S,T,\omega_r)\sigma} - 1 \right)
\]

\[
= -\frac{2\pi^4}{l_{11}^3 V_3} \int_0^1 d\omega_3 \int_0^{\omega_3} d\omega_2 \int_0^{\omega_2} d\omega_1 \ln(-Q(S,T;\omega_r)).
\] (65)

We now consider the analytic part of (61) which involves

\[
I(S, T)_{\text{anal}} = \frac{2\pi^4}{l_{11}^3 V_3} \sum_{n=2}^{\infty} \frac{G_{ST}^n}{n!} \sum_{(l_{11},l_{12},l_{13}) \neq (0,0,0)} \int_0^\infty \frac{d\sigma}{\sigma^{1-n}} e^{-G_{IJ}l_{1I}l_{1J}/l_{11}^2}
\]

\[
= \frac{2\pi^4 l_{11}^{2n-3}}{V_3} \sum_{n=2}^{\infty} \frac{G_{ST}^n}{n} E_n(G^{-1})^{SL(3,\mathbb{Z})},
\] (66)

where

\[
G_{ST}^n = \int_0^1 d\omega_3 \int_0^{\omega_3} d\omega_2 \int_0^{\omega_2} d\omega_1 \left( -Q(S,T;\omega_r) \right)^n.
\] (67)

and we have used (101). Focussing on the \( n = 2 \) contribution, we see that

\[
I(S, T)_{\text{anal}}^{n=2} = \pi^6 G_{ST}^2 \int_0^\infty d\sigma \sigma^{-1/2} \sum_{(l_{11},l_{12},l_{13}) \neq (0,0,0)} e^{-\pi G_{IJ}l_{1I}l_{1J}/l_{11}^2}.
\] (68)

We shall be interested only in those terms in (68) that lead to the perturbative string contributions given in (32). To evaluate (68), we split the sum over \( \hat{l}_I \) into two parts: (i) \((\hat{l}_2, \hat{l}_3) = (0, 0), \hat{l}_1 \neq 0,\) and (ii) \((\hat{l}_2, \hat{l}_3) \neq (0, 0), \hat{l}_1 \) arbitrary, and call these contributions \( I(S, T)_{\text{anal}}^1 \) and \( I(S, T)_{\text{anal}}^2 \) respectively. We get that

\[
I(S, T)_{\text{anal}}^1 = \pi^7 G_{ST}^2 l_{11} e^{2\phi^4/3}.
\] (69)

To calculate \( I(S, T)_{\text{anal}}^2 \), we Poisson resum on \( \hat{l}_1 \) to go back to \( l_1 \), to get

\[
I(S, T)_{\text{anal}}^2 = \frac{\pi^6 G_{ST}^2}{l_{11} R_{11}} \sum_{(\hat{l}_2, \hat{l}_3) \neq (0,0), l_1} \int_0^\infty d\sigma \exp \left[ \frac{2\pi i l_{1}}{G_{11}} \left( G_{12} \hat{l}_2 + G_{13} \hat{l}_3 \right) - \frac{\pi l_{11}^2 \sigma}{l_{11}^2 R_{11}^2} \right]
\]

\[
- \frac{\pi l_{11}^2}{\sigma} \left[ \hat{l}_2 \left( G_{22} - \frac{G_{12}^2}{G_{11}} \right) + \hat{l}_3^2 \left( G_{33} - \frac{G_{13}^2}{G_{11}} \right) + 2 \hat{l}_2 \hat{l}_3 \left( G_{23} - \frac{G_{12} G_{13}}{G_{11}} \right) \right].
\] (70)
We next split (70) into two parts: \((\hat{l}_2, \hat{l}_3) \neq (0, 0), l_1 = 0\) which we call \(I(S, T)_{\text{anal}}^{2,0}\), and \((\hat{l}_2, \hat{l}_3) \neq (0, 0), l_1 \neq 0\) which we call \(I(S, T)_{\text{anal}}^{2,1}\). To express them in terms of quantities in type IIA string theory, we use the IIA string frame metric

\[
g_{i-1,j-1} = R_{11}(G_{ij} - \frac{G_{1i}G_{1j}}{G_{11}}),
\]

where \(i, j = 2, 3\). Also, the moduli from the R–R one form potentials along \(T^2\) in type IIA are given by

\[
A_{i-1} = \frac{G_{1i}}{G_{11}},
\]

where \(i, j = 2, 3\). Finally, the complex structure \(U\) of \(T^2\) of volume \(T^2\) is given by

\[
U = \frac{1}{g_{22}^A} (g_{23}^A + i\sqrt{\det g^A}).
\]

Thus we get that

\[
I(S, T)_{\text{anal}}^{2,0} = \frac{\pi}{4} l_{11} R_{211}^2 G_{2ST} E_2(U, \bar{U})^{SL(2,\mathbb{Z})}.
\]

Note that \(I(S, T)_{\text{anal}}^{2,1}\) gives non–perturbative contributions which are not relevant for (32), and so we shall neglect them.\(^8\)

Thus from (64), (69), and (74), we get that

\[
A_4 = \frac{\kappa_{11}^4 K}{(2\pi)^{11} l_{11}^3} \left[ \pi^3 \zeta(3) e^{-2\phi^A} + \left( \frac{\pi^4}{6!} T_2 E_2(U, \bar{U})^{SL(2,\mathbb{Z})} + \frac{\pi^7}{3 \cdot 6!} e^{2\phi^A} \right) l_4^A (s^2 + t^2 + u^2) + \ldots \right],
\]

where we have used \(l_{11} = e^{\phi^A/3} l_s\), and

\[
G_{ST}^2 + G_{SU}^2 + G_{UT}^2 = \frac{1}{6!} (s^2 + t^2 + u^2).
\]

From (76), we see that the one loop supergravity amplitude contributes only at genus one and genus two in the \(D^4 R^4\) interaction in type IIA string theory. However, given the genus zero \(R^4\) interaction in (76), we can fix the normalization of the genus zero \(D^4 R^4\) interaction using (36). The genus zero interaction in (36) is proportional to

\[
T_2 e^{-2\phi^A} \left( \zeta(3) + \frac{\zeta(5)}{2 \cdot 16} l_4^A (s^2 + t^2 + u^2) + \ldots \right) R^4,
\]

\(^8\)This contribution is given by

\[
I(S, T)_{\text{anal}}^{2,0} = 2\pi^6 G_{ST}^2 \frac{\l_1}{\sqrt{K_{11}}} T_2 U_2 \sum_{(l_2, l_3) \neq (0,0), l_1 \neq 0} \frac{|\hat{l}_2 + \hat{l}_3 U|}{|l_1|} K_1 \left( 2\pi e^{-\phi^A} \sqrt{\frac{T_2}{U_2}} |\hat{l}_2 + \hat{l}_3 U||l_1| \right) e^{2\pi i l_1 A_i}.
\]
thus leading to

$$A^\text{total}_4 = \frac{\kappa_4^4 K}{(2\pi)^{11} l_{11}^3} \left[ \frac{\pi^3}{32} \zeta(5)e^{-2\phi^A} + \frac{\pi^4}{6!} T_2 E_2(U, \bar{U})^{SL(2,\mathbb{Z})} + \frac{\pi^7}{3 \cdot 6!} e^{2\phi^A} + \ldots \right] l_s^4 (s^2 + t^2 + u^2).$$

(79)

Thus, we see that (79) leads to terms in the type IIB effective action given by

$$l_s^4 \int d^8 x \sqrt{-g_8} \left[ (e^{-2\phi^A} V_2)^2 \zeta(5) + \frac{8\zeta(4)}{\pi^3} E_2(T, \bar{T})^{SL(2,\mathbb{Z})} U_2^2 + (e^{-2\phi^A})^{-1} \frac{8\zeta(4)}{3} U_2^2 \right],$$

(80)

where we have used $\zeta(4) = \pi^4/90$. To see that it reproduces some of the terms in (31), we keep the leading terms in $U_2$ in $F_0$, $F_1$, and $F_2$ in (32) giving us

$$F_0 = 2\zeta(5), \quad F_1 = \frac{8\zeta(4)}{\pi^3} E_2(T, \bar{T})^{SL(2,\mathbb{Z})} U_2^2 + \ldots, \quad F_2 = \frac{8\zeta(4)}{3} U_2^2 + \ldots,$$

(81)

which precisely matches (80). Thus the supergravity analysis provides some more evidence for the proposed modular form.

4 Decompressifying to nine dimensions

We now decompressify the $D^4\mathcal{R}^4$ interaction to nine dimensions to see what structure it gives, and also to make some further consistency checks. We define

$$T_2 = r_\infty r_B, \quad U_2 = \frac{r_\infty}{r_B},$$

(82)

where $r_\infty$ is the direction that is being decompressified. Here $r_\infty$ and $r_B$ are the radii of $T^2$ in the string frame. Now let us take the limit $r_\infty \to \infty$, so that $T_2, U_2 \to \infty$. From (30), using (94) and (100), we see that the non–vanishing terms in nine dimensions in the string frame are given by

$$l_s^3 \int d^9 x \sqrt{-g_9} \left[ \frac{r_B}{\sqrt{r_2}} \frac{E_5/2(\tau, \bar{\tau})^{SL(2,\mathbb{Z})}}{\pi^2 r_2^3/2 r_B^3} + \frac{4\zeta(4)}{\pi^2 r_2^3/2 r_B^3} E_3/2(\tau, \bar{\tau})^{SL(2,\mathbb{Z})} \right]$$

$$+ \frac{8}{\pi^2} \zeta(3) \zeta(4) r_B^3 + \frac{16}{\pi^3} \zeta(4) r_\infty^3 \right] D^4\mathcal{R}^4,$$

(83)

where we have set $l_s \int d^8 x \sqrt{-g_8 r_\infty} = \int d^9 x \sqrt{-g_9}$. Note that the last term in (83) is divergent in the limit $r_\infty \to \infty$. However the existence of this kind of term is crucial for the consistency of the theory. The full effective action of type IIB string theory on $T^2$ contains terms analytic (like the $(s^2 + t^2 + u^2)\mathcal{R}^4$ term we have discussed) as well as non–analytic in the external momenta of the gravitons. In taking the decompressification limit to go to nine

---

9 We drop the $E_2(T, \bar{T})^{SL(2,\mathbb{Z})}$ term in $F_2$. 

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dimensions, a part of the analytic terms diverges ([83] contains only one such term among an infinite number of such diverging terms coming from the infinite number of analytic terms). These diverging terms as well as the non–analytic terms must add up to give the massless square root threshold singularity in nine dimensions. A detailed analysis of the corresponding divergence in ten dimensions obtained by taking \( r_B \to \infty \) has been done in [14], where it was shown that the diverging term involving \( r_B^3 \) in (83) adds up with other such terms, as well as the non–analytic terms to give the logarithmic threshold singularity in ten dimensions. So the term in (83) that diverges as \( r_\infty \to \infty \) is not a part of the \( D^4\mathcal{R}^4 \) interaction in nine dimensions (just like the \( r_B^3 \) term that diverges in the ten dimensional limit is not a part of the \( D^4\mathcal{R}^4 \) interaction in ten dimensions ), and so we drop it from our analysis from now on.

Keeping only the terms that survive in the large \( r_B \) limit, note that (83) reduces to

\[
I_s^3 \int d^3 x \sqrt{-g} r_B \left[ e^{\phi_B/2} E_{5/2}(\tau, \bar{\tau})^{SL(2,\mathbb{Z})} + \frac{8}{\pi^2} \zeta(3) \zeta(4) r_B^2 \right] D^4\mathcal{R}^4, \tag{84}
\]

which is precisely what has been obtained in [14], which is a consistency check of our modular form. Also from (83), we see that the perturbative contribution to the scattering amplitude is given by (upto an irrelevant numerical factor)

\[
\left[ 2\zeta(5)(r_B e^{-2\phi_B}) + \frac{8}{\pi^2} \zeta(3) \zeta(4) \left( r_B^3 + \frac{1}{r_B^3} \right) + \frac{8}{3} \zeta(4) (r_B e^{-2\phi_B})^{-1} \left( r_B^2 + \frac{1}{r_B^2} \right) \right] (s^2 + t^2 + u^2) \mathcal{R}^4, \tag{85}
\]

where the three terms give tree level, one loop and two loop contributions respectively. Now using the relations

\[
r_B = r_A^{-1}, \quad e^{-\phi_B} = r_A e^{-\phi_A}, \tag{86}
\]

to go to type IIA string theory, from (85) we see that the perturbative contributions are the same in either theory, which should be the case. As another consistency check, we now show that the tree level and one loop contributions in (85) match the result using string perturbation theory.

From (36), for \( n = 1 \), we see that the amplitude in nine dimensions is given by

\[
\left[ 2\zeta(5) r_B e^{-2\phi_B} + \hat{I}_1 \right] I_s^4 (s^2 + t^2 + u^2) \mathcal{R}^4, \tag{87}
\]

where the one loop contribution \( \hat{I}_1 \) is given by

\[
\hat{I}_1 = \frac{4r_B}{\pi} \sum_{m,n \in \mathbb{Z}} \int_{\mathcal{F}_L} \frac{d^2 \Omega}{\Omega_2^2} e^{-\pi r_B^2 |m+n\Omega|^2/\tau_2} E_2(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})}, \tag{88}
\]
where \( \mathcal{F}_L \) is the restricted fundamental domain of \( SL(2, \mathbb{Z}) \) as before. This integral can be simplified leading to \([37–39]\)

\[
\hat{I}_1 = \frac{4r_B}{\pi} \left[ \int_{\mathcal{F}_L} \frac{d^2 \Omega}{\Omega_2^2} + \sum_{m \in \mathbb{Z}, m \neq 0} \int_{-1/2}^{1/2} d\Omega_1 \int_0^\infty \frac{d\Omega_2}{\Omega_2^2} e^{-\pi r_B^2 m^2/\Omega_2} \right] E_2(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})}. \tag{89}
\]

Now the first term in (89) is proportional to \( I_1 \) in (48), and thus vanishes. Using (46) and doing the other integral, we get

\[
\hat{I}_1 = \frac{8}{\pi^2} \zeta(3) \zeta(4) \left( r_B^3 + \frac{1}{r_B^3} \right), \tag{90}
\]

thus giving us (85). In the nine dimensional Einstein frame, we see that (83) equals

\[
l_3^3 \int d^9 x \sqrt{-\hat{g}_9} \left[ \xi^{5/2} E_{5/2}(\tau, \bar{\tau})^{SL(2,\mathbb{Z})} + \frac{4\zeta(4)}{\pi^2} \xi^{-9} E_{3/2}(\tau, \bar{\tau})^{SL(2,\mathbb{Z})} + \frac{8}{\pi^2} \zeta(3) \zeta(4) \xi^{12} \right] \hat{D}^4 \hat{R}^4, \tag{91}
\]

where the hatted indices signify quantities in the Einstein frame, and \( \xi^7 = r_B^2 \sqrt{r_2} \). Thus from (91), the \( SL(2, \mathbb{Z}) \times \mathbb{R}^+ \) U–duality symmetry of the \( \hat{D}^4 \hat{R}^4 \) interaction is nine dimensions is manifest\(^{10}\).

It would be interesting to prove or disprove the modular form for the \( \hat{D}^4 \hat{R}^4 \) interaction we have proposed, as well as to construct U–duality invariant modular forms for the four graviton amplitude in toroidal compactifications of type IIB string theory to lower dimensions. It might be possible to construct the modular form for the \( \hat{D}^6 \hat{R}^4 \) interaction in eight dimensions along the lines in this paper, although the analysis will get more complicated because the ten dimensional modular form satisfies a Poisson equation on the fundamental domain of \( SL(2, \mathbb{Z}) \) \([15]\). In trying to construct these modular forms, it might be useful to consider eleven dimensional supergravity compactified on torii, and consider

\(^{10}\)Let us make some comments about the possible modular form for the \( \hat{D}^4 \hat{R}^4 \) interaction in lower dimensions. In dimensions lower than eight, the coset manifold \( \mathcal{M} = G/H \) which parametrizes the scalars in the supergravity action is irreducible (see \([40, 41]\), for example). Here \( G \) is a non–compact group, and \( H \) is its maximal compact subgroup. The conjectured U–duality group is \( \hat{G} \), the discrete version of \( G \). Thus in the Einstein frame the relevant term in the supergravity action is given by

\[
S \sim \frac{1}{l_s^{8-d}} \int d^{10-d} x \sqrt{-g_{10-d}} \text{Tr}(\partial_\mu M \hat{\partial}^\mu M^{-1}), \tag{92}
\]

where \( M \) parametrizes \( \mathcal{M} \). Based on the \( \hat{D}^4 \hat{R}^4 \) interaction in ten dimensions as well as the modular form we propose, it is conceivable that the U–duality invariant modular form in lower dimensions is given by

\[
E_{5/2}(M) \hat{G} = \sum_{m_i} \left( m_i M_{ij} m_j \right)^{-5/2}. \tag{93}
\]
four graviton scattering in this background. Though this will not account for the various non–perturbative contributions like membrane instantons for M theory on $T^3$, it might give hints about the various U–duality invariant modular forms. In general, understanding the role of modular forms in toroidal compactifications of M theory that preserve all the thirty two supersymmetries is useful. At least some aspects of constructing them might not depend on a precise definition of the microscopic degrees of freedom of M theory, and might be completely determined based on the constraints of supersymmetry and U–duality invariance. Thus constructing them might shed some light on the fundamental degrees of freedom of M theory.

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5 Appendix

In the two appendices below, we write down explicit expressions for the Eisenstein series of $SL(2, \mathbb{Z})$ and $SL(3, \mathbb{Z})$ that are useful in the main text.

A The Eisenstein series for $SL(2, \mathbb{Z})$

The Eisenstein series of order $s$ for $SL(2, \mathbb{Z})$ is defined by

$$E_s(T, \bar{T})^{SL(2, \mathbb{Z})} = \sum_{(p,q) \neq (0,0)} \frac{T^s_p}{|p+qT|^{2s}}$$

$$= 2\zeta(2s)T^s_2 + 2\sqrt{\pi}T^{1-s}_2 \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1)$$

$$+ \frac{2\pi^s \sqrt{T_2}}{\Gamma(s)} \sum_{m_1 \neq 0, m_2 \neq 0} \left|\frac{m_1}{m_2}\right|^{s-1/2} K_{s-1/2}(2\pi T_2|m_1m_2|) e^{2\pi im_1m_2T_1}. \quad (94)$$

Using the relations

$$\zeta(2s-1)\Gamma(s-\frac{1}{2}) = \pi^{2s-3/2}\zeta(2-2s)\Gamma(1-s), \quad (95)$$

and

$$K_s(x) = K_{-s}(x), \quad (96)$$



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we see that
\[ \Gamma(s)E_\pi(T, \hat{T})^{SL(2, \mathbb{Z})} = \pi^{2s-1}\Gamma(1-s)E_{1-s}(T, \hat{T})^{SL(2, \mathbb{Z})}. \]  

(97)

**B The Eisenstein series for SL(3, \mathbb{Z})**

The Eisenstein series of order \( s \) for \( SL(3, \mathbb{Z}) \) in the fundamental representation is defined by

\[ E_s(M)^{SL(3, \mathbb{Z})} = \sum_{m_i} \left( m_iM_{ij}m_j \right)^{-s} \]

\[ = \sum_{m_i} \nu^{-s/3} \left( \frac{|m_1 + m_2\tau + m_3B|^2}{\tau_2} + \frac{m_3^2}{\nu} \right)^{-s}, \]

(98)

where \( m_i \) are integers, and the sum excludes \( \{m_1, m_2, m_3\} = \{0, 0, 0\} \). The integers \( m_i \) transform in the anti–fundamental representation of \( SL(3, \mathbb{Z}) \). The matrix \( M_{ij} \) has entries given by \((17)\).

Using the integral representation

\[ E_s(M)^{SL(3, \mathbb{Z})} = \frac{\nu^{-s/3}\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{s+1}} \sum_{m_i} e^{-\pi|m_1 + m_2\tau + m_3B|^2/\tau_2 + m_3^2/\nu} t, \]

(99)

we can evaluate \((99)\) to get that

\[ E_s(M)^{SL(3, \mathbb{Z})} = 2(\tau_2^2V_2)^{2s/3}\zeta(2s) + \frac{\sqrt{\pi}\Gamma(s - 1/2)}{\Gamma(s)}(\tau_2^2V_2)^{1/2 - s/3}E_{s-1/2}(T, \hat{T})^{SL(2, \mathbb{Z})} \]

\[ + \frac{2\pi^s}{\Gamma(s)}\tau_2^{s+1/2}V_2^{2s/3} \sum_{m_1\neq0, m_2\neq0} m_1^{s-1/2} K_{s-1/2}(2\pi\tau_2|m_1m_2|)e^{2\pi im_1m_2\tau_1} \]

\[ + \frac{2\pi^s}{\Gamma(s)}\tau_2^{1-2s/3}V_2^{1-s/3} \sum_{m_1\neq0, m_3\neq0, m_2} \left| \frac{m_2 - m_1\tau}{m_3} \right|^{s-1} K_{s-1}(2\pi|m_3(m_2 - m_1\tau)|V_2) \]

\[ \times e^{2\pi im_3(m_1BR + m_2BN)}. \]

(100)

We can also define the Eisenstein series of order \( s \) in the anti–fundamental representation by

\[ E_s(M^{-1})^{SL(3, \mathbb{Z})} = \sum_{\hat{m}_i} \left( \hat{m}_iM^{ij}\hat{m}_j \right)^{-s}, \]

(101)

where \( \hat{m}_i \) transforms in the fundamental representation of \( SL(3, \mathbb{Z}) \). Now using the result

\[ \sum_{l_i} e^{-\pi\sigma G^iil_i} = \sigma^{-3/2} \sqrt{\det G} \sum_{l_i} e^{-\pi G^iil_i/\sigma} \]

(102)
for invertible matrices, which can be derived using Poisson resummation, we get that

$$E_s(M^{-1})^{SL(3,\mathbb{Z})} = E_{3/2-s}^{SL(3,\mathbb{Z})}. \quad (103)$$

Thus there is a simple relationship between the Eisenstein series for the fundamental and the anti–fundamental representations.

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