LEAVITT PATH ALGEBRAS OF FINITE GELFAND-KIRILLOV DIMENSION

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Abstract. Groebner-Shirshov basis and Gelfand-Kirillov dimension of the Leavitt path algebra are derived.

1. Introduction.

Leavitt path algebras were introduced in [AA] as algebraic analogs of graph Cuntz-Krieger C*-algebras. Since then they have received significant attention from algebraists. In this paper we (i) find a Groebner-Shirshov basis of a Leavitt path algebra, (ii) determine necessary and sufficient conditions for polynomially bounded growth, and (iii) find Gelfand-Kirillov dimension.

2. Definitions and Terminologies

A (directed) graph $\Gamma = (V, E, s, r)$ consists of two sets $V$ and $E$, called vertices and edges respectively, and two maps $s, r : E \rightarrow V$. The vertices $s(e)$ and $r(e)$ are referred to as the source and the range of the edge $e$, respectively. The graph is called row-finite if for all vertices $v \in V$, $|\langle s^{-1}(v) \rangle| < \infty$. A vertex $v$ for which $(s^{-1}(v))$ is empty is called a sink. A path $p = e_1...e_n$ in a graph $\Gamma$ is a sequence of edges $e_1...e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, ..., (n-1)$. In this case we say that the path $p$ starts at the vertex $s(e_1)$ and ends at the vertex $r(e_n)$. If $s(e_1) = r(e_n)$, then the path is closed. If $p = e_1...e_n$ is a closed path and the vertices $s(e_1), ..., s(e_n)$ are distinct, then the subgraph $(s(e_1), ..., s(e_n); e_1, ..., e_n)$ of the graph $\Gamma$ is called a cycle.

Let $\Gamma$ be a row-finite graph and let $F$ be a field. The Leavitt path $F$-algebra $L(\Gamma)$ is the $F$-algebra presented by the set of generators $\{v, v \in V\}, \{e, e^* \in E\}$ and the set of relators (1) $v_i v_j = \delta_{v_i,v_j} v_i$ for all $v_i, v_j \in V$; (2) $s(e)e = er(e) = e$, $r(e)e^* = e^*s(e) = e^*$, for all $e \in E$; (3) $e^*f = \delta_{e,f}r(e)$, for all $e, f \in E$; (4) $v = \sum_{s(e) = v} ee^*$, for an arbitrary vertex $v \in V \setminus \{\text{sinks}\}$.

The mapping which sends $v$ to $v$, for $v \in V$, $e$ to $e^*$ and $e^*$ to $e$, for $e \in E$, extends to an involution of the algebra $L(\Gamma)$. If $p = e_1...e_n$ is a path, then $p^* = e_n^*...e_1^*$.

3. A Basis of $L(\Gamma)$

For an arbitrary vertex $v$ which is not a sink, choose an edge $\gamma(v)$ such that $s(\gamma(v)) = v$. We will refer to this edge as special. In other words, we fix a function $\gamma : V \setminus \{\text{sinks}\} \rightarrow E$ such that $s(\gamma(v)) = v$ for an arbitrary $v \in V \setminus \{\text{sinks}\}$.

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Theorem 1. The following elements form a basis of the Leavitt path algebra \( L(\Gamma) \):

(i) \( v, v \in V \), (ii) \( p, p^* \), where \( p \) is a path in \( \Gamma \), (iii) \( pq^* \), where \( p = e_1 \ldots e_n \), \( q = f_1 \ldots f_m \), \( e_i, f_j \in E \), are paths that end at the same vertex \( r(e_n) = r(f_m) \), with the condition that the last edges \( e_n \) and \( f_m \) are either distinct or equal, but not special.

Proof. Recall that a well-ordering on a set is a total order (that is, any two elements can be ordered) such that every non-empty subset of elements has a least element.

As a first step, we will introduce a certain well-ordering on the set of generators \( X = V \cup E \cup E^* \). Choose an arbitrary well-ordering on the set of vertices \( V \). If \( e, f \) are edges and \( s(e) < s(f) \) then \( e < f \). It remains to order edges that have the same source. Let \( v \) be a vertex which is not a sink. Let \( e_1, \ldots, e_k \) be all the edges that originate from \( v \). Suppose \( e_k = \gamma(v) \). We order the edges as follows: \( e_1 < e_2 < \ldots < e_k = \gamma(v) \). Choose an arbitrary well-ordering on the set \( E^* \). For arbitrary elements \( v, e \in E, f^* \in E^* \), we let \( v < f^* \). Thus the set \( X = V \cup E \cup E^* \) is well-ordered. Let \( X^* \) be the set of all words in the alphabet \( X \). The length-lex order (see [B, Be]) makes \( X^* \) a well-ordered set. For all \( v \in V \) and \( e \in E \), we extend the set of relators (1) - (4) by (5): \( ve = 0 \), for \( v \neq s(e) \); \( ev = 0 \), for \( v \neq r(e) \); \( ve^* = 0 \), for \( v \neq r(e) \); \( e^*v = 0 \), for \( v \neq s(e) \). The straightforward computations show that the set of relators (1) - (5) is closed with respect to compositions (see [B, BE]). By the Composition-Diamond Lemma ([B, BE]) the set of irreducible words (not containing the leading monomials of relators (1) - (5) as subwords) is a basis of \( L(\Gamma) \). This completes the proof. \( \square \)

4. LEAVITT PATH ALGEBRAS OF POLYNOMIAL GROWTH

Recall some general facts on the growth of algebras. Let \( A \) be an algebra (not necessarily unital), which is generated by a finite dimensional subspace \( V \). Let \( V^k \) denote the span of all products \( v_1 \cdots v_k \), \( v_i \in V \), \( k \leq n \). Then \( V = V^1 \subset V^2 \subset \cdots \), \( A = \bigcup_{n \geq 1} V^n \) and \( g\gamma_n = \dim V^n < \infty \). Given the functions \( f, g \) from \( N = \{1, 2, \ldots\} \) to the positive real numbers \( R_+ \), we say that \( f \lesssim g \) if there exists \( c \in N \) such that \( f(n) \leq cg(cn) \) for all \( n \). If \( f \sim g \) and \( g \sim f \) then the functions \( f, g \) are said to be asymptotically equivalent, and we write \( f \sim g \). If \( W \) is another finite dimensional subspace that generates \( A \), then \( g\gamma_n \sim g\gamma(W) \). If \( g\gamma_n \) is polynomially bounded, then we define the Gelfand-Kirillov dimension of \( A \) as \( \text{GKdim } A = \limsup_{n \to \infty} \frac{\ln g\gamma_n}{\ln n} \). The definition of \( \text{GK}- \text{dimension} \) does not depend on a choice of the generating space \( V \) as long as \( \dim V < \infty \). If the growth of \( A \) is not polynomially bounded, then \( \text{GKdim } A = \infty \).

We now focus on finitely generated algebras and we will assume that the graph \( \Gamma \) is finite. Let \( C_1, C_2 \) be distinct cycles such that \( V(C_1) \cap V(C_2) \neq \phi \). Then we can renumber the vertices so that \( C_1 = (v_1, \ldots, v_m; e_1, \ldots, e_m) \), \( C_2 = (w_1, \ldots, w_n; f_1, \ldots, f_n) \), \( v_1 = w_1 \). Let \( p = e_1 \cdots e_m \), and \( q = f_1 \cdots f_n \).

Lemma 2. The elements \( p, q \) generate a free subalgebra in \( L(\Gamma) \).

Proof. By Theorem 1, different paths viewed as elements of \( L(\Gamma) \) are linearly independent. If \( u_1, u_2 \) are different words in two variables, then \( u_1(p, q) \) and \( u_2(p, q) \) are different paths. Indeed, cutting out a possible common beginning, we can assume that \( u_1, u_2 \) start with different letters, say, \( u_1(p, q) = p \cdots \), \( u_2(p, q) = q \cdots \). If
If two distinct cycles have a common vertex, then they are different. This proves the lemma. □

Corollary 3. If two distinct cycles have a common vertex, then $L(\Gamma)$ has exponential growth.

From now on we will assume that any two distinct cycles of the graph $\Gamma$ do not have a common vertex.

For two cycles $C', C''$, we write $C' \implies C''$, if there exists a path that starts in $C'$ and ends in $C''$.

Lemma 4. If $C', C''$ are two cycles such that $C' \implies C''$, and $C'' \implies C'$, then $C' = C''$.

Proof. Choose a path $p$ that starts in $C'$ and ends in $C''$. Similarly, choose a path $q$ that starts in $C''$ and finishes in $C'$. There exists also a path $p'$ on $C'$, which connects $r(p)$ with $s(q)$ and a path $q'$ on $C'$, which connects $r(q)$ with $s(p)$. Now, $p p' q q'$ is a closed path, which visits both $C'$ and $C''$. Let $t$ be a closed path with this property (visiting both $C'$ and $C''$) having a minimal length. Write $t = e_1 \cdots e_n$, $e_i \in E$. We claim that the vertices $s(e_1), \ldots, s(e_n)$ are all distinct, thus $t = (s(e_i), \ldots, s(e_n); e_1, \ldots, e_n)$ is a cycle. Assuming the contrary, let $s(e_i) = s(e_j)$, $1 \leq i < j \leq n$, and $j - i$ is minimal with this property. Then $t' = (s(e_i), s(e_{i+1}), \cdots, s(e_j); e_i, e_{i+1}, \cdots, e_{j-1})$ is a cycle. Let us "cut it out", that is, consider the path $t'' = e_1 \cdots e_{i-1} e_j \cdots e_n$. This path is shorter than $t$. Hence $t''$ can not visit both $C'$ and $C''$. Suppose that $t''$ does not visit $C'$. Then at least one of the vertices $s(e_i), \ldots, s(e_{j-1})$ lies in $C''$. Since two intersecting cycles coincide, it implies that $t'' = C''$, hence $s(e_j)$ lies in $C'$. This contradicts our assumption that $t''$ does not visit $C''$. Hence $t = C' = C''$. This proves the lemma.

A sequence of distinct cycles $C_1, \ldots, C_k$ is a chain of length $k$ if $C_1 \implies \cdots \implies C_k$. The chain is said to have an exit if the cycle $C_k$ has an exit (see [AA]), that is, if there exists an edge $e$ such that $s(e) \in V(C_k)$, but $e$ does not belong to $C_k$. Let $d_1$ be the maximal length of a chain of cycles in $\Gamma$, and let $d_2$ be the maximal length of chain of cycles with an exit. Clearly, $d_2 \leq d_1$.

Theorem 5. Let $\Gamma$ be a finite graph.

(1) The Leavitt path algebra $L(\Gamma)$ has polynomially bounded growth if and only if any two distinct cycles of $\Gamma$ do not have a common vertex;

(2) If $d_1$ is the maximal length of a chain of cycles in $\Gamma$, and $d_2$ is the maximal length of chain of cycles with an exit, then $GK \dim L(\Gamma) = \max(2d_1 - 1, 2d_2)$.

Proof. As in the proof of Theorem 1 we consider the generating set $X = V \cup E \cup E^*$ of $L(\Gamma)$. Let $E'$ be the set of edges that do not belong to any cycle. Let $P'$ be the set of all paths that are composed from edges from $E'$. Then an arbitrary path from $P'$ never arrives to the same vertex twice. Hence, $|P'| < \infty$.

By Theorem 1 the space $\text{Span}(X^n)$ is spanned by elements of the following types:

(1) a vertex,

(2) a path $p = p_1' p_1 p_2' p_2 \cdots p_k' p_{k+1}$, where $p_i$ is a path on a cycle $C_i$, $1 \leq i \leq k$,

$C_1 \implies \cdots \implies C_k$ is a chain, $p_i' \in P'$, length$(p) \leq n$,

(3) $p^*$, where $p$ is a path of the type (2),
The number of nonnegative integral solutions of the inequality

\[ pq^* \]

where \( p = p_1'p_2'...p_k' \) and \( q = q_1'q_2'...q_s'q_{s+1}' \) are paths on cycles \( C_1, D_1 \) respectively and \( C_1 \leq \cdots \leq C_k, D_1 \leq \cdots \leq D_s \) are chains; \( p_1', q_1' \in P', \) \( \text{length}(p) + \text{length}(q) \leq n \) with \( r(p) = r(q) \). We will further subdivide this case into two subcases:

1. \( r(p) \notin V(C_k) \cup V(D_s) \);
2. \( r(p) \in V(C_k) \cup V(D_s) \).

We will estimate the number of products of \( \text{length} \leq n \) in each of the above cases and then use the following elementary fact:

Let \( (a_n)_{n \in N} \) be the sum of \( s \) sequences \( (a_{in})_{n \in N}, \) \( 1 \leq i \leq s, a_{in} > 0. \) Then

\[
\limsup_{n \to \infty} \frac{\ln a_n}{\ln n} = \max(\limsup_{n \to \infty} \frac{\ln a_{in}}{\ln n}, 1 \leq i \leq s)
\]

Let us estimate the number of paths of the type (2). Fix a chain \( C_1 \Rightarrow \cdots \Rightarrow C_k \Rightarrow \cdots \Rightarrow C_{k+1} \).

Among the edges from \( V(C_1) \) to \( V(C_{k+1}) \), let the number of such paths be \( n^k \leq n^{d_1}. \)

On the other hand, let \( C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_{d_1} \) be a chain and let \( C_{d_1} \) be an exit of the cycle \( C_{d_1} \). Select paths \( p_1, p_2, ... p_{d_2}, \) where \( p_i \in P' \) and \( \text{length}(p_i) \leq m_i - 1. \) Clearly, \( m_1 + ... + m_k + k \leq n. \) This implies that the number of such paths \( \leq n^k \leq n^{d_1}. \) On the other hand, choosing a chain \( C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_{d_1} \) of length \( d_1 \), we can construct \( \sim n^{d_1} \) paths of length \( \leq n. \) The case (3) is similar to the case (2).

Consider now the elements of length \( \leq n \) of the type \( pq^* \), \( r(p) = r(q) \); the path \( p \) passes through the cycles of the chain \( C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k \) on the way, the path \( q \) passes through the cycles of the chain \( D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_s \) on the way and so \( p = p_1'p_2'...p_k'p_{k+1}' \), each \( p_i \) is a path on the cycle \( C_i. \) Similarly, \( q = q_1'q_2'...q_s'q_{s+1}' \). Arguing as above, we see that for fixed chains \( C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k \) and \( D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_s \), the number of such paths \( \leq n^k \leq n^{d_1}. \)

Suppose that the vertex \( v = r(p) = r(q) \) does not lie in \( V(C_k) \cup V(D_s). \) Then both cycles \( C_k \) and \( D_s \) have exits. Hence the number of paths of type (4.1) is \( \leq n^{2d_2}. \) On the other hand, let \( C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_{d_2} \) be a chain and let \( e \) be an exit of the cycle \( C_{d_2} \). Select paths \( p_1', p_2', ..., p_{d_2}, \) where \( p_i' \) connects \( C_{i-1} \) to \( C_i, \) \( p_i' \in P'. \)

Select a path \( u'' \) on the cycle \( C_1 \) which connects \( r(P_{C_{d_2}}) \) to \( s(p_i' \) in \( C_1 \) which connects \( r(p_2') \) to \( s(p_2' \) in \( C_2 \) which connects \( r(p_2') \) to \( s(P_{C_{d_2}}) \) to \( s(P_{C_{d_2}}) \), and so on. The path \( u'' \) connects \( r(P_{C_{d_2}}) \) to \( s(e). \)

Among the edges from \( s^{-1}(s(e)) \) choose a special one \( \gamma(s(e)) \) different from \( e. \) Then by Theorem 1, the elements

\[
P_{C_1}^{l_1} u''_1 P_{C_2}^{l_2} u''_2 P_{C_3}^{l_3} u''_3 ... P_{C_{d_2}}^{l_{d_2}} u''_d \text{ are linearly independent. Let } m, \text{ be the total length of all elements other than } P_{C_1}^{l_1}. \]

The number of elements in the inequality (A) above is the number of nonnegative integral solutions of the inequality

\[ \sum_{i=1}^{d_2} m_i (l_i + l_{d_2} - 1) \leq n - m, \] which is \( \sim n^{2d_2}. \)

Now suppose that the vertex \( v = r(p) = r(q) \) lies in \( C_k. \) Assume at first that \( C_k \neq D_s. \) Then the chain \( D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_s \) has an exit. If \( k \leq s, \) then the number of the paths of this type is \( \leq n^{k+s} \leq n^{2d_2}. \)
If \( s < k \), then \( n^{k+s} \leq n^{2k-1} \leq n^{2d_1} - 1 \).

Next, let \( C_k = D_s \). It means that the paths \( p_{k+1}^r, q_{s+1}^r \) are empty; \( p_k \) and \( q_s \) are both paths on the cycle \( C_k \) and in this case we have,

(i) \( p_kq_s^* = u \), if \( p_k = uq_s \), is a path on \( C_k \),
(ii) \( p_kq_s^* = u^* \), if \( q_s = up_k \), is a path on \( C_k \), and
(iii) \( p_kq_s^* = 0 \), otherwise.

The number of such elements \( pq^* \) is \( \leq n^{k+s-1} \leq n^{2d_1} - 1 \).

On the other hand, let \( C_1 \Longrightarrow C_2 \Longrightarrow \ldots \Longrightarrow C_{d_1} \) be a chain of cycles. Select paths \( p'_2, \ldots, p'_{d_1} \in P^r \); \( p'_i \) connects \( C_{i-1} \) to \( C_i \); \( u'_i, u''_i \) are paths on the cycle \( C_i \) such that \( P_{C_i}u'_iP_{C_{i+1}}u''_i = P_{C_{i+1}}(P_{C_{i+1}})^{l}_{l+1}(P_{C_{i+1}})...(P_{C_{i+1}})^{l}_{l-1} \) are linearly independent provided that \( l_i \geq 1, 1 \leq i \leq 2d_1 - 1 \). The number of these elements is \( \sim n^{2d_1} \). This proves Theorem 2. \( \square \)

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