THE EFFECT OF CAPUTO FRACTIONAL DIFFERENCE OPERATOR ON A NOVEL GAME THEORY MODEL

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Abstract. It is well-known that fractional-order discrete-time systems have a major advantage over their integer-order counterparts, because they can better describe the memory characteristics and the historical dependence of the underlying physical phenomenon. This paper presents a novel fractional-order triopoly game with bounded rationality, where three firms producing differentiated products compete over a common market. The proposed game theory model consists of three fractional-order difference equations and is characterized by eight equilibria, including the Nash fixed point. When suitable values for the fractional order are considered, the stability of the Nash equilibrium is lost via a Neimark-Sacker bifurcation or via a flip bifurcation. As a consequence, a number of chaotic attractors appear in the system dynamics, indicating that the behaviour of the economic model becomes unpredictable, independently of the actions of the considered firm. The presence of chaos is confirmed via both the computation of the maximum Lyapunov exponent and the 0-1 test. Finally, an entropy algorithm is used to measure the complexity of the proposed game theory model.

1. Introduction. Fractional calculus is a very interesting topic in mathematics with several potential applications in many fields of science and engineering [15, 13]. Referring to economics, a number of mathematical models describing different phenomena have been introduced [16, 14, 10]. Namely, since fractional operators are non local, they are suitable for constructing models characterized by memory effect. This is the reason why fractional-order differential or difference equations, when describing economic phenomena over large time periods of time, perform better with respect to integer-order continuous-time or discrete-time systems, respectively [15]. For example, in [16] the objective is to model the growth of national economies, (i.e., their gross domestic products) by means of a fractional order approach. In [14] a macroeconomic state space model for national economies described by fractional differential equations is presented. In [10] the stochastic dynamics of the stock and currency markets are described via fractional equations.

Recently, attention has been focused on the presence of chaotic phenomena in fractional systems, described by both differential or difference equations. Referring to economic systems described by continuous-time dynamics, several papers involving fractional calculus have been published to date. For example, in [7] the chaotic behaviour of a macroeconomic model with foreign capital investments is studied and controlled. In [20] a new fractional-order financial model is introduced and a method for controlling its chaotic dynamics is proposed. In [18] a 4D fractional chaotic financial system characterized by investment incentives is presented, whereas in [21] the dynamics of a fractional economic system characterized by transient chaos are studied. Unlike continuous-time systems, very few papers regarding chaotic phenomena in economic systems described by discrete-time dynamics have been published to date [11, 19, 2]. These economic systems, which usually involve concepts from game theory applied to oligopolistic markets, generate complex dynamics (described by fractional-order difference equations) that lead to the existence of bifurcations and chaos [11]. For example, in [19] the Hopf bifurcations and the chaotic attractors of the discrete-time version of the Bertrand duopoly game with fractional delay is studied. In [2] chaotic phenomena discovered in a fractional-order discrete-time Cournot duopoly game are analysed. In [3] the dynamics of Cournot duopoly game with fractional bounded rationality are analyzed. The local stability properties of the equilibrium points are studied, along with the effects of
the fractional marginal profit on the game dynamics [3]. In [17] a fractional-order discrete Cournot duopoly game model is introduced, which allows participants to make decisions while making full use of their historical information. The Nash equilibria, their local stability and the presence of chaos are deeply investigated [17]. In [5] a discrete dynamic system that describes the competition among four firms is presented. In particular, the fractional bounded rationality is taken into account and the stability of the Nash point is studied [5]. However, while the fractional duopoly game, where two firms compete over a market, has been studied [19, 2, 11], the fractional triopoly game, where the competition is among three firms, has not yet studied, to the best of the authors knowledge. Based on the above considerations, this paper presents a novel fractional-order triopoly game, where three firms producing differentiated products compete over a common market. The proposed game theory model, which exploits the fractional Caputo-like difference operator, is characterized by eight equilibria, including the Nash fixed point. When suitable values for the fractional order are considered, the stability of the Nash equilibrium is lost via a Neimark-Sacker bifurcation or via a flip bifurcation. As a consequence, a number of chaotic attractors appear in the system dynamics, indicating that the behaviour of the economic model becomes unpredictable, independently of the actions of the considered firm. The paper is organized as follows. In Section 2 the model of the Cournot triopoly game with bounded rationality and memory is introduced. In Section 3, the stability of the fixed points is studied, whereas in Section 4 bifurcations and chaos are analysed in details. In Section 5 the presence of chaos is confirmed via the 0-1 test, whereas in Section 6 an entropy algorithm is used to measure the complexity of the proposed game theory model.

2. Model with memory. We consider a monopolistic market where three firms produce different products. The inverse demand function is given by:

\[ p_i = \alpha - q_i - \beta \sum_{j=1}^{3} q_j, \quad i = 1, 2, 3, \] (1)

in which \( q_i \) denote the outputs of products produced by the three firms, and the constants \( \alpha > 0, 0 < \beta < \sqrt{0.5} \) are the coefficients of the market demand function. Now, we assume that the cost function of these firms is proposed in the nonlinear form:

\[ C_i(q_i) = \gamma_i q_i + \delta_i \sum_{j=1, j \neq i}^{3} q_i q_j, \quad i = 1, 2, 3, \] (2)

where \( \gamma_i \) and \( \delta_i \) are two positive constants. In this setting, firms \( i \) relative profit is given by the difference between the absolute profit of the \( i \) firm and the sum of the other firms profit:

\[ \Pi_i(q_1, q_2, q_3) = \left[ q_i \left( \alpha - q_i - \beta \sum_{j \neq i}^{3} q_j \right) - \gamma_i q_i - \delta_i \sum_{j \neq i}^{3} q_i q_j \right] - \sum_{j \neq i}^{3} p_j(q_1, q_2, q_3)q_j + C_j(q_j), \quad i = 1, 2, 3. \] (3)
Through substituting Eq. (1) and Eq. (2) into Eq. (3), the relative profit functions can be given as:

\[
\begin{align*}
\Pi_1(q_1, q_2, q_3) &= \alpha(q_1 - q_2 - q_3) - q_1^2 + q_2^2 + q_3^2 + 2\beta q_2 q_3 - \gamma_1 q_1 + \gamma_2 q_2 + \gamma_3 q_3, \\
\Pi_2(q_1, q_2, q_3) &= \alpha(q_2 - q_1 - q_3) - q_1^2 + q_2^2 + q_3^2 + 2\beta q_1 q_3 - \gamma_2 q_2 + \gamma_3 q_3, \\
\Pi_3(q_1, q_2, q_3) &= \alpha(q_3 - q_1 - q_2) - q_1^2 + q_2^2 + q_3^2 + 2\beta q_1 q_2 - \gamma_3 q_3 + \gamma_1 q_1 + \gamma_2 q_2, \\
\end{align*}
\]

(4)

With this assumption, the maximizing profit is obtained by setting \( \frac{\partial \Pi_i}{\partial q_i} = 0 \). In order to construct the integer order dynamical system of this game we assume that each firm try to use information based on the marginal profit \( \frac{\partial \Pi_i}{\partial q_i} \). The mathematical model is written as follows:

\[
q_i(n + 1) = q_i(n) + \varepsilon_i q_i(n) \frac{\partial \Pi_i}{\partial q_i}, \quad i = 1, 2, 3,
\]

(5)

\( \varepsilon_i \) is a positive constant which is referring to the speed of adjustment. This game model agrees well with the model proposed by Al-Khedhairi, et al., which has been studied in [2].

Based on the dynamical system (5), we propose a new generalized model by introducing the Caputo-like difference operator on the system. (5). Specifically, our main interest is to study the dynamics of three bounded rationality firms with relative profit maximization and long memory of output decision. The definition of fractional Caputo-like difference operator will be given firstly.

**Definition 2.1.** for a given function \( u : N_a \rightarrow \mathbb{R} \), the definition of the fractional Caputo operator of order \( \nu \notin \mathbb{N} \) is:

\[
{^c} \Delta^\nu u(t) = \frac{1}{\Gamma(1-\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{-\nu} \Delta_s u(s),
\]

(6)

where the symbol \( \Gamma(.) \) represents the Euler’s gamma function and \( t \in N_{a+1-\nu} \). According to reference [6], the definition of \( \nu \)-th fractional sum for \( \Delta_s u(t) \) is expressed in the following:

\[
\Delta_{-\nu} u(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} u(s),
\]

(7)

with \( t \in N_{a+\nu} \) and \( \nu > 0 \).

The new game model with fractional difference operator is described by:

\[
\begin{align*}
{^c} \Delta^\nu q_1(t) &= \varepsilon_1 q_1(t - 1 + \nu)(\alpha - \gamma_1 - 2\beta_1(t - 1 + \nu)} \\
&\quad + (\zeta_1 - \zeta_2) q_2(t - 1 + \nu) - (\zeta_1 - \zeta_3) q_3(t - 1 + \nu)), \\
{^c} \Delta^\nu q_2(t) &= \varepsilon_2 q_2(t - 1 + \nu)(\alpha - \gamma_2 - 2\beta_2(t - 1 + \nu)} \\
&\quad - (\zeta_2 - \zeta_1) q_1(t - 1 + \nu) - (\zeta_2 - \zeta_3) q_3(t - 1 + \nu)), \\
{^c} \Delta^\nu q_3(t) &= \varepsilon_3 q_3(t - 1 + \nu)(\alpha - \gamma_3 - 2\beta_3(t - 1 + \nu)} \\
&\quad - (\zeta_3 - \zeta_1) q_1(t - 1 + \nu) - (\zeta_3 - \zeta_2) q_2(t - 1 + \nu)),
\end{align*}
\]

(8)

where \( \nu \in (0,1) \)and \( \zeta_i = \beta + \delta_i \). To study the dynamics of the three bounded rationality with long memory we need to define the discrete version of the game model, for that we need to replace \( a \) by zero and \( q_i \) by \( x_i \). According to [4] the
equivalent discrete formula is defined by:

\[
\begin{align*}
    x_1(n) &= x_1(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\varepsilon_1 x_1(j-1)(\alpha - \gamma_1) \\
    x_2(n) &= x_2(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\varepsilon_2 x_2(j-1)(\alpha - \gamma_2) \\
    x_3(n) &= x_3(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\varepsilon_3 x_3(j-1)(\alpha - \gamma_3))
\end{align*}
\]  

(9)

where \( \theta_{ij} = \zeta_i - \zeta_j, \) \( \forall i,j = 1,2,3. \)

3. **Stability analysis.** For calculating the equilibrium points of the fractional game model (8), we assign its left hand side to zero:

\[
\begin{align*}
    \varepsilon_1 x_1(\alpha - \gamma_1) - 2x_1 - \theta_{12} x_2 - \theta_{13} x_3 &= 0, \\
    \varepsilon_2 x_2(\alpha - \gamma_2) - 2x_2 - \theta_{12} x_1 - \theta_{23} x_3 &= 0, \\
    \varepsilon_3 x_3(\alpha - \gamma_3) - 2x_3 - \theta_{13} x_1 - \theta_{23} x_2 &= 0.
\end{align*}
\]  

(10)

By algebraic computation, we obtain the following fixed points:

\[
\begin{align*}
    F_1 &= (0, 0, 0), \quad F_2 = (0, 0.5(\alpha - \gamma_2), 0), \quad F_3 = (0.5(\alpha - \gamma_1), 0, 0), \quad F_4 = (0, 0.5(\alpha - \gamma_3)), \\
    F_5 &= \left( \frac{2(\gamma_1 - \alpha) + \theta_{12}(\alpha - \gamma_2)}{\theta_{12}^2 + 4}, \frac{2(\alpha - \gamma_2) + \theta_{12}(\alpha - \gamma_1)}{\theta_{12}^2 + 4}, 0 \right), \\
    F_6 &= \left( 0, \frac{2(\alpha - \gamma_2) + \theta_{12}(\gamma_3 - \alpha)}{\theta_{23}^2 + 4}, \frac{2(\alpha - \gamma_3) + \theta_{13}(\gamma_2 - \alpha)}{\theta_{23}^2 + 4} \right), \\
    F_7 &= \left( \frac{2(\alpha - \gamma_1) + \theta_{13}(\gamma_3 - \alpha)}{\theta_{13}^2 + 4}, 0, \frac{2(\alpha - \gamma_3) + \theta_{13}(\alpha - \gamma_1)}{\theta_{13}^2 + 4} \right), \\
    F_8 &= (A_1, A_2, A_3),
\end{align*}
\]  

(11)

in which

\[
\begin{align*}
    A_1 &= \frac{(\alpha - \gamma_3)(\theta_{12} \theta_{23} - 2 \theta_{13}) - (\alpha - \gamma_2)(\theta_{13} \theta_{23} + 2 \theta_{12}) + (\alpha - \gamma_1)(\theta_{23}^2 + 4)}{2(\theta_{13}^2 + \theta_{23}^2 + \theta_{12}^2 + 4)}, \\
    A_2 &= \frac{-(\alpha - \gamma_3)(\theta_{12} \theta_{13} + 2 \theta_{23}) + (\alpha - \gamma_2)(\theta_{13}^2 + 4) + (\alpha - \gamma_1)(\theta_{13} \theta_{23} + 2 \theta_{12})}{2(\theta_{13}^2 + \theta_{23}^2 + \theta_{12}^2 + 4)}, \\
    A_3 &= \frac{(\alpha - \gamma_3)(\theta_{23}^2 + 4) + (\alpha - \gamma_2)(-\theta_{12} \theta_{13} + 2 \theta_{23}) + (\alpha - \gamma_1)(\theta_{12} \theta_{23} + 2 \theta_{13})}{2(\theta_{13}^2 + \theta_{23}^2 + \theta_{12}^2 + 4)}.
\end{align*}
\]

The Jacobian matrix of the fractional order difference equations (8) at an arbitrary point \((x_1, x_2, x_3),\) is defined by:

\[
M(x_1, x_2, x_3) = \begin{pmatrix}
    D_1 - 2\varepsilon_1 x_1 & -\varepsilon_1 \theta_{12} x_1 & -\varepsilon_1 \theta_{13} x_1 \\
    \varepsilon_2 \theta_{12} x_2 & D_2 - 2\varepsilon_2 x_2 & -\varepsilon_2 \theta_{23} x_2 \\
    \varepsilon_3 \theta_{13} x_3 & \varepsilon_3 \theta_{23} x_3 & D_3 - 2\varepsilon_3 x_3
\end{pmatrix},
\]  

(12)

where

\[
\begin{align*}
    D_1 &= \varepsilon_1 (\alpha - \gamma_1 - 2x_1 - \theta_{12} x_2 - \theta_{13} x_3), \\
    D_2 &= \varepsilon_2 (\alpha - \gamma_2 - 2x_2 + \theta_{12} x_1 - \theta_{23} x_3), \\
    D_3 &= \varepsilon_3 (\alpha - \gamma_3 - 2x_3 + \theta_{13} x_1 + \theta_{23} x_2).
\end{align*}
\]

To investigate the stability of the fixed points we shall use the following theorem:
Theorem 3.1. [8] Let \( x_f \) be a fixed point of a fractional difference system \( ^C\Delta_t^\nu F(t) = F(x(t + \nu - 1)) \) where \( x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \), and \( J(x_f) = \frac{\partial F(x)}{\partial x} \bigg|_{x=x_f} \) is the Jacobian matrix at the fixed point \( x_f \). The fixed point \( x_f \) is asymptotically stable when all the eigenvalues \( (\lambda_i, i = 1, ..., n) \) of \( J \) verifies:

\[
\lambda_i \in \{ z \in \mathbb{C} : |z| < \left( \frac{2 \cos \left( \frac{\arg z - \pi}{2 - \nu} \right) \nu}{2} \right) \text{ and } |\arg z| > \frac{\nu \pi}{2} \} , \forall i = 1, ..., n. \tag{13}
\]

Now, we turn to investigate the stability of the previous fixed points.

Proposition 1. The fixed point \( F_1 = (0, 0, 0) \) is asymptotically stable if the fractional order \( \nu \) and the game parameters satisfies:

\[
\nu > \log_2|\varepsilon_1(\alpha - \gamma_i)|, \quad \text{and} \quad \alpha < \gamma_i, \quad i = 1, 2, 3. \tag{14}
\]

Proof. The Jacobian matrix (12) at the fixed point \( F_1 = (0, 0, 0) \) can be easily computed as:

\[
M_{F_1} = \begin{pmatrix}
\varepsilon_1(\alpha - \gamma_1) & 0 & 0 \\
0 & \varepsilon_2(\alpha - \gamma_2) & 0 \\
0 & 0 & \varepsilon_3(\alpha - \gamma_3)
\end{pmatrix}. \tag{15}
\]

The associated characteristic equation is defined by:

\[
(\varepsilon_1(\alpha - \gamma_1) - \lambda) \times (\varepsilon_2(\alpha - \gamma_2) - \lambda) \times (\varepsilon_3(\alpha - \gamma_3) - \lambda) = 0. \tag{16}
\]

The Eigenvalues are:

\[
\lambda_1 = \varepsilon_1(\alpha - \gamma_1), \quad \lambda_2 = \varepsilon_2(\alpha - \gamma_2), \quad \lambda_3 = \varepsilon_3(\alpha - \gamma_3).
\]

Based on Theorem 3.1, it is easy to show that the fixed point \( F_1 = (0, 0, 0) \) is always asymptotically stable when \( \nu > \log_2|\varepsilon_1(\alpha - \gamma_i)| \) and \( \alpha < \gamma_i \) \( \forall i = 1, 2, 3. \)

Proposition 2. The conditions of asymptotic stability of the fixed point \( F_2 = (0, 0.5(\alpha - \gamma_2), 0) \) are:

- If \( \alpha > \gamma_2 \) and \( \nu > \log_2|\varepsilon_2(\gamma_2 - \alpha)|. \)
- If \( \gamma_1 > \alpha + 0.5\theta_{12}(\gamma_2 - \alpha) \) and \( \nu > \log_2|\varepsilon_1(\alpha - \gamma_1 + 0.5\theta_{12}(\gamma_2 - \alpha))|. \)
- If \( \gamma_3 > \alpha + 0.5\theta_{23}(\alpha - \gamma_2) \) and \( \nu > \log_2|\varepsilon_3(\alpha - \gamma_3 + 0.5\theta_{23}(\alpha - \gamma_2))|. \)

Proof. The Jacobian matrix (12) at the fixed point \( F_2 = (0, 0.5(\alpha - \gamma_2), 0) \) can be easily computed as:

\[
M_{F_2} = \begin{pmatrix}
\varepsilon_1(\alpha - \gamma_1 - 0.5\theta_{12}(\alpha - \gamma_2)) & 0 & 0 \\
0.5\theta_{12}\varepsilon_2(\alpha - \gamma_2) & \varepsilon_2(\gamma_2 - \alpha) & 0 \\
0 & 0 & \varepsilon_3(\alpha - \gamma_3) + 0.5\theta_{23}(\alpha - \gamma_2)
\end{pmatrix}. \tag{17}
\]

The eigenvalues are:

\[
\lambda_1 = \varepsilon_1(\alpha - \gamma_1) + 0.5\varepsilon_1\theta_{12}(\gamma_2 - \alpha), \quad \lambda_2 = \varepsilon_2(\gamma_2 - \alpha), \quad \lambda_3 = \varepsilon_3(\alpha - \gamma_3) + 0.5\varepsilon_3\theta_{23}(\alpha - \gamma_2).
\]

Therefore, the eigenvalue \( \{\lambda_1, \lambda_2, \lambda_3\} \) ensure the condition (13) in Theorem (3.1).

Using the same steps, we obtain the following results.

Proposition 3. The conditions of asymptotic stability of the fixed point \( F_3 = (0.5(\alpha - \gamma_2), 0, 0) \) are:

- If \( \alpha > \gamma_1 \) and \( \nu > \log_2|\varepsilon_1(\gamma_1 - \alpha)|. \)
- If \( \gamma_2 > \alpha + 0.5\theta_{12}(\alpha - \gamma_1) \) and \( \nu > \log_2|\varepsilon_2(\alpha - \gamma_2 + 0.5\theta_{12}(\alpha - \gamma_1))|. \)
- If \( \gamma_3 > \alpha + 0.5\theta_{13}(\alpha - \gamma_1) \) and \( \nu > \log_2|\varepsilon_3(\alpha - \gamma_3 + 0.5\theta_{13}(\alpha - \gamma_1))|. \)
Proposition 4. The conditions of asymptotic stability of the fixed point $F_4 = (0, 0, 0.5(\alpha - \gamma_2))$ are:

- If $\alpha > \gamma_3$ and $\nu > \log_2 |\varepsilon_3(\gamma_3 - \alpha)|$.
- If $\gamma_1 > 0 + 0.5\theta_1(\gamma_3 - \alpha)$ and $\nu > \log_2 |\varepsilon_1(\alpha - \gamma_1 + 0.5\theta_1(\gamma_3 - \alpha))|$.
- If $\gamma_2 > 0 + 0.5\theta_2(\gamma_3 - \alpha)$ and $\nu > \log_2 |\varepsilon_2(\alpha - \gamma_2 + 0.5\theta_2(\gamma_3 - \alpha))|$.

Proposition 5. The conditions of asymptotic stability of the fixed point $F_5 = \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\alpha - \gamma_2)}{\theta_1^2 + 4}, \frac{2(\alpha - \gamma_2) + \theta_1(\alpha - \gamma_1)}{\theta_1^2 + 4}, 0\right)$ are:

- If $\left|\alpha - \gamma_3 - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right) - \theta_1 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right| < 0$ and $\nu > \log_2 |\varepsilon_3(\alpha - \gamma_1 + 0.5\theta_1(\gamma_3 - \alpha))|$.
- If $\frac{-2}{\gamma} \geq \sqrt{H}$ and $\nu > \log_2 \sqrt{\frac{H - 4H - G}{2}}$ where:

$G = \varepsilon_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right) + \varepsilon_2 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right)$.

$H = \varepsilon_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right) \times \varepsilon_2 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right)$.

Proposition 6. The conditions of asymptotic stability of the fixed point $F_6 = \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}, \frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}, \frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)$ are:

- If $\left|\alpha - \gamma_3 - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right) - \theta_1 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right| < 0$ and $\nu > \log_2 |\varepsilon_1(\alpha - \gamma_1 + 0.5\theta_1(\gamma_3 - \alpha))|$.
- If $\frac{-2}{\gamma} \geq \sqrt{J}$ and $\nu > \log_2 \sqrt{\frac{J^2 - 4J - 1}{2}}$ where:

$I = \varepsilon_2 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right) + \varepsilon_3 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right)$.

$J = \varepsilon_2 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right) \times \varepsilon_3 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right)$.

Proposition 7. The conditions of asymptotic stability of the fixed point $F_7 = \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}, 0, \frac{2(\alpha - \gamma_2) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)$ are:

- If $\left|\alpha - \gamma_3 + \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right) - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right| < 0$ and $\nu > \log_2 |\varepsilon_1(\alpha - \gamma_1 + 0.5\theta_1(\gamma_3 - \alpha))|$.
- If $\frac{-2}{\gamma} \geq \sqrt{O}$ and $\nu > \log_2 \sqrt{\frac{O^2 - 4O - L}{2}}$ where:

$L = \varepsilon_2 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right) + \varepsilon_3 \left(\alpha - \gamma_3 - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right)$.

$O = \varepsilon_2 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4} - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right) \times \varepsilon_3 \left(\alpha - \gamma_3 - \theta_1 \left(\frac{2(\gamma_1 - \alpha) + \theta_1(\gamma_3 - \alpha)}{\theta_1^2 + 4}\right)\right)$.
4. Bifurcation analysis and numerical simulation. Numerical experiments are simulated in this section to show the different route to chaos of the fractional triopoly game (8). Its phase portraits, bifurcation diagrams and maximum Lyapunov exponents (MLE) were investigated under different levels of parameters and fractional orders. As explained in [2], there is a stable closed invariant curve around the Nash fixed point $F_3 = (0.8882, 0.4283, 0.0831)$. For instance, when selecting parameters $\alpha = 2, \varepsilon_1 = 1.042811791, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4$ and initial values $x_1(0) = 0.4, x_2(0) = 0.2, x_3(0) = 0.4$, the stable closed invariant curve on $x_2 - x_3$ plane is represented in Figure 1(a). The closed invariant curve is expanded as shown in Figure 1. It verifies that the closed invariant curve is affected by the fractional order $\nu$ and a chaotic attractor is observed at $\nu = 0.81$. To further observe the dynamical behavior, the system parameters are fixed as above and the fractional order $\nu$ is varied in the range $[0.72, 1]$. Figure 2 shows the bifurcation diagram and the maximum Lyapunov exponents of the first player output $x_1(n)$. It shows that the long memory system begins from periodic states where the maximum Lyapunov exponent equal to zero, and then it exhibits chaos at 0.8257. Figures 1 and 2 shows the strong effect of the long memory on the stability of the equilibrium $F_3 = (0.8882, 0.4283, 0.0831)$. More precisely, the complexity of the model increases around the Nash fixed point and chaos appears as $\nu$ decreases.

Figure 3 shows the bifurcation diagram with respect to the adjustment parameter $\varepsilon_1$ when $\alpha = 2, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4$. Figure 3-(a) and 3-(b) are the bifurcation diagrams for $\nu = 0.985$ and $\nu = 0.972$, respectively. As one can see, the two diagrams are similar. A stable Nash fixed point is observed as $\varepsilon_1$ increases from 1 to 1.02 for $\nu = 0.985$ and when $\varepsilon_1$ increases from 1 to 1.007 for $\nu = 0.972$. We can observe that whenever we increased the value of $\varepsilon_1$ the Cournot Nash fixed point loses its stability via Neimark-Sacker bifurcation. Moreover, we observe that decreasing the value of the fractional order lead to the disappearance of the chaotic region. When $\nu = 0.985$ the model (8) exhibits chaotic behavior at $\varepsilon_1 \in [1.37, 1.443] \cup [1.461, 468]$ and for $\nu = 0.972$ the model (8) exhibits chaotic behavior for $\varepsilon_1 \in [1.38, 1.431]$. Therefore, it is possible to initially conclude that the speed of adjustment of the first player disestablish the dynamic of the market and make it unpredictable.

For better observation, we choose to discuss the chaos of the fractional triopoly game for $\varepsilon_1 = 1.44$, with $\nu$ varying from 0.95 to 1 by the step size $\Delta \nu = 0.6 \times 10^{-4}$. Figure 4 shows the bifurcation diagram of $x_1$ versus $\nu$ and the MLE diagram. It illustrates that the states of the long memory system are different as $\nu$ decreases. When $\nu \in [0.9562, 0.9673] \cup [0.9809, 1]$ this model is chaotic, where the maximum Lyapunov exponent is positive, and when $\nu \in [0.9562, 0.9809]$ the model is periodic. These results indicate that the long memory decrease the speed of adjustment of the first firm. The chaotic attractor when $\nu = 0.98$ is plotted as shown in Figure 5. On the other hand, the periodic behaviour of the proposed system is reported in Figure 6 for $\nu = 0.975$, whereas the chaotic attractor obtained for $\nu = 0.96$ is shown in Figure 7. Clearly, Figure 5, Figure 6 and Figure 7 confirm both the shape of the bifurcation diagram reported in Figure 4(a) and the plot of the maximum Lyapunov exponent reported in Figure 4(b).

In [2], Al-kheidari et al showed that for the parameters $\alpha = 1, \varepsilon_2 = 0.9, \varepsilon_3 = 0.9, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4$, a flip bifurcation was observed at $\varepsilon_1 = 2.305628076$ where the equilibrium $F_3 = (0.4451, 0.2384, 0.4502)$.
Figure 1. The phase portraits of game (8) with parameter values \( \alpha = 2, \varepsilon_1 = 1.042811791, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4 \) for different fractional order values: (a) \( \nu = 1 \), (b) \( \nu = 0.9 \), (c) \( \nu = 0.865 \), (d) \( \nu = 0.81 \).

Figure 2. (a) Bifurcation diagram versus \( \nu \) when \( \alpha = 2, \varepsilon_1 = 1.042811791, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4 \). (b) The maximum Lyapunov exponents with respect to \( \nu \) corresponding to (a).
loses its stability. The bifurcation diagram of the first player output \( x_1(n) \) versus \( \varepsilon_1 \) is illustrated in Figure 8. Figure 8-(a) shows that the Nash equilibrium point undergoes flip bifurcation and period doubling route to chaos. By reducing the value of \( \nu \) to 0.7635, we obtain the bifurcation diagram shown in Figure 8-(b). It was found that chaotic motion exists in the range \( \varepsilon_1 \in [2.689, 3] \) with periodic windows at 2.873. The chaotic area increases when \( \nu = 0.7635 \) as shown in Figure 8-(b). In order to confirm the shape of the bifurcation diagram reported in Figure 8-(b), we have investigated the system behaviour when \( \varepsilon_1 = 2.6 \) and \( \varepsilon_1 = 2.9 \). Namely, Figure 9 highlights the period behaviour of the system obtained for \( \varepsilon_1 = 2.6 \), whereas Figure 10 shows the chaotic attractor obtained for \( \varepsilon_1 = 2.9 \). The simulation results


Figure 5. Chaotic attractor of the proposed game with \( \nu = 0.98 \) and for \( \alpha = 2, \varepsilon_1 = 1.44, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4. \)

demonstrate that the long memory increases the speed of adjustment of the first player and the game loses its stability faster.

5. The 0-1 test for chaos. Another tool that can be used to study the influence of the order \( \nu \) on the dynamics of the market is "0-1 test". The 0-1 test is relatively new method that was proposed by Gottwald and Melbourne [9] to test the presence of chaos in a series of data which originate from deterministic systems. For the fractional game model system (9), we consider a set of data \( x(j) \) where \( j = 1 \ldots N. \)

Following [9], we transform the states of the game into \( p-q \) plots. Generally, unbounded \( p-q \) trajectories implies chaotic behavior whereas bounded trajectories implies regular behavior. Here, we applied the 0-1 test method directly to the series data \( x_1(n) \) that obtained from the first player, the results with \( \nu = 0.865 \) and \( \nu = 0.7635 \) are shown in Figure 11 and Figure 12, respectively. In particular, Figure 11 depicts bounded trajectories for \( \nu = 0.865 \), indicating that the suggested game is stable where the output \( K = 0.000827. \) On the other hand, the unbounded trajectories in Figure 12 confirms the chaotic behavior of the game for \( \nu = 0.7635 \) and the output \( K = 0.995, \) which clearly confirms the above results.
Figure 6. Periodic attractor of the proposed game with $\nu = 0.975$ and for $\alpha = 2, \varepsilon_1 = 1.44, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4$.

Figure 7. Chaotic attractor of the proposed game with $\nu = 0.96$ and for $\alpha = 2, \varepsilon_1 = 1.44, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4$. 
2.2 Bifurcation diagram versus $\varepsilon_1$ with order $\nu = 1$ when $\alpha = 1, \varepsilon_2 = 0.9, \varepsilon_3 = 0.9, \zeta_1 = 0.4, \zeta_2 = 0.8, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4$ (b) Bifurcation diagram versus $\varepsilon_1$ with order $\nu = 0.7635$.

Figure 8. (a) Bifurcation diagram versus $\varepsilon_1$ with order $\nu = 1$

Figure 9. Periodic attractor of the proposed game with $\nu = 0.7635$ for $\varepsilon_1 = 2.6$ and $\alpha = 1, \varepsilon_2 = 0.9, \varepsilon_3 = 0.9, \zeta_1 = 0.4, \zeta_2 = 0.8, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4$.

6. Entropy. Approximate entropy (ApEn) [12] is the measurement of the degree of complexity of a series of data from a multi-dimensional perspective. This method estimates the regularity where it assigns a nonnegative number where higher values indicate higher complexity. The detailed calculation steps for detecting ApEn are as follows. Following Pincus [12] we consider a set of points $x(1), x(2), ..., x(n)$ that are obtained from the fractional triopoly game (8). The value of the approximate
Figure 10. Chaotic attractor of the proposed game with $\nu = 0.7635$ for $\varepsilon_1 = 2.9$ and $\alpha = 1, \varepsilon_2 = 0.9, \varepsilon_3 = 0.9, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4$.

Figure 11. 0-1 test: regular dynamics of the translation components $(p, q)$ of the Cournot game (8) for $\alpha = 2, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4$ with fractional order $\nu = 0.865$. 
Figure 12. 0-1 test: regular dynamics of the translation components \((p, q)\) of the Cournot game (8) for \(\alpha = 2, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4\) with fractional order \(\nu = 0.7635\).

entropy depends on a two important parameters \(m\) and \(r\). The input \(r\) is the similar tolerance and \(m\) is the embedding dimension. Now, we reconstruct a subsequence of \(x\) such that \(\{X(i) = [x(i), ..., x(i + m - 1)]\}\), where \(m\) presents the points from \(x(i)\) to \(x(i + m - 1)\). Let \(K\) the number of \(X(i)\) such that the maximum absolute difference of two vectors \(X(i)\) and \(X(j)\) is lower or equal to the tolerance \(r\). The relative frequency of \(X(i)\) being similar to \(X(j)\) is: \(C_i^m(r) = \frac{K}{N - m + 1}\). From \(C_i^m\) calculate the logarithm and then define the average for all \(i\) as follows:

\[
\phi^m(r) = \frac{1}{N - m - 1} \sum_{i=1}^{N-m+1} \log C_i^m(r). \quad (18)
\]

The approximate entropy of order \(m\) is setting as:

\[
ApEn = \phi^m(r) - \phi^{m+1}(r). \quad (19)
\]

Here, we applied the ApEn directly to the series of data \(x_3(n)\) that was obtained from the third firm. Figure 13 shows the approximate entropy of the proposed game when \(\alpha = 2, \varepsilon_1 = 1.042811791, \varepsilon_2 = 1.1, \varepsilon_3 = 1.1, \zeta_1 = 0.4, \zeta_2 = 0.8, \zeta_3 = 0.1, \gamma_1 = 0.07, \gamma_2 = 0.03, \gamma_3 = 0.4\), and for different fractional order values. It is shown in Figure 13, that the approximate entropy results agree well with the corresponding bifurcation diagram and MLE in Figure 2. It is also shown that the smaller the fractional order \(\nu\) is, the more complex the game model is. Therefore, we must be aware of the selected fractional order in the game model (8) in order to have a relatively high structural complexity.

7. Conclusion. Since fractional operators are nonlocal, they are suitable for constructing models for long series, characterized by a memory effect. This is the reason
why fractional difference equations possess a large advantage over their integer-order counterparts, and are suitable for describing economic systems based on the game theory. This paper has presented a novel fractional-order triopoly game with bounded rationality. By considering suitable values of the fractional order, the conducted analysis has shown that the stability of the Nash equilibrium is lost via a Neimark-Sacker bifurcation or via a flip bifurcation. Chaotic attractors have been displayed, indicating that the behaviour of the economic model becomes unpredictable, independently of the actions of the considered firm. The presence of chaos has been confirmed via both the computation of the maximum Lyapunov exponent and the 0-1 test. Finally, it is worth noting that the introduction of more realistic models of game theory, such as the considered triopoly game based on fractional calculus, might lead to a better understanding of the economic implications in industrial organization, international trade and business cycles.

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