IKT$^\omega$ AND ŁUKASIEWICZ-MODELS

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Abstract. In this note we show first that the first-order logic IKT$^\omega$ is sound with regard to the models obtained continuum-valued Łukasiewicz-models for first-order languages by treating the quantifiers as infinitary strong disjunction/conjunction rather than infinitary weak disjunction/conjunction. We then proceed to show that these models cannot be used to provide a new consistency proof for the theory of truth IKT$^\omega$ obtained by expanding IKT$^\omega$ with transparent truth since the models are incompatible with transparent truth. Moreover, we also show that whether or not this inconsistency can be reproduced in the sequent calculus for IKT$^\omega$ depends on how vacuous quantification is treated.

1. Introduction

[1] presents the theory of truth IKT$^\omega$ which is obtained by expanding the logic IKT$^\omega$ with a transparent truth predicate. While the propositional fragment of IKT$^\omega$ is propositional affine logic, that is, linear logic with weakening, the quantifiers of IKT$^\omega$ are generalisations of multiplicative conjunction and disjunction as opposed to generalisations of additive conjunction and disjunction. To capture multiplicative quantifiers, [1] relies on a sequent calculus which is infinitary not only in the sense that it contains rules with infinitely many premises, but also that the sequents may contain infinitely many formulas.

The aim of this paper is to shed some light on the multiplicative quantifiers introduced by [1] by showing that the sequent calculus for the logic IKT$^\omega$ is sound with regard to models obtained from first-order continuum-valued Łukasiewicz-models by treating the quantifiers as infinitary strong disjunction/conjunction rather than infinitary weak disjunction/conjunction. While one might thus think that these models can be used to provide a new consistency proof for IKT$^\omega$ to replace that presented by [1] which was found to be erroneous by [2], the paper proceeds to show that not only are these models incompatible with transparent truth, but we can also reproduce the inconsistency in the sequent calculus for IKT$^\omega$ by taking advantage of vacuous quantification.

2. The sequent calculus $S_{IK^\omega}$

The logic IKT$^\omega$ is defined by [1] for a first-order language based on the connectives $\to$, $\neg$, $\exists$ and $\forall$ in addition to $\otimes$ as symbol for conjunction and $\oplus$ as symbol for disjunction. We shall in this paper restrict our attention to a first-order language

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2010 Mathematics Subject Classification. 03B47, 03C90, 03C75.

Key words and phrases. non-contractive truth; multiplicative quantifiers; infinitary sequents; soundness; $\omega$-inconsistency; inconsistency; vacuous quantification.

The first author would like to acknowledge the utility of the discussions with Casper Storm Hansen, Fausto Barbero and Uwe Petersen for the preparation of the material for this paper.
Following [1], $\text{IK}^\omega$ is defined with a sequent calculus based on sequents as multisets of $L$-formulas which are such that both antecedent and succedent multiset of a sequent can contain $\omega$ many formulas. Intuitively, multisets are collections of objects where their multiplicity but not their order matter. More formally, we will define a multiset of formulas as a pair $\langle X, f \rangle$ where $X$ is a set of formulas and $f$ a function from $X$ to $\omega + 1$. In the presentation of the sequent calculus for $\text{IKT}^\omega$ we use the upper-case Latin letters $A$ and $B$ as meta-linguistic variables for formulas of $L$, and upper-case Greek letters of the form $\Gamma$ and $\Delta$ as meta-linguistic variables for multisets of formulas. As usual, $\Gamma, A$ represents the multiset union of $\Gamma$ and $\{A\}$.

**Definition 2.1.** Let $S_{\text{IK}^\omega}$ be a sequent calculus with sequents as multisets of $L$-formulas obtained with the initial sequents

$$A, \Gamma \Rightarrow \Delta, A$$

and the rules

| Rule | Symbol |
|------|--------|
| $\Gamma \Rightarrow \Delta, A$ | $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ (cut) |
| $\Gamma \Rightarrow \Delta, A$ | $\neg A, \Gamma \Rightarrow \Delta$ (L) |
| $A, \Gamma \Rightarrow \Delta, A$ | $\Gamma \Rightarrow \Delta, \neg A$ (R) |
| $A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ | $A, \Gamma \Rightarrow \Delta, B \Rightarrow \Delta, A \rightarrow B$ (R) |
| $A(t_0/x), \Gamma_0 \Rightarrow \Delta_0$ | $A(t_1/x), \Gamma_1 \Rightarrow \Delta_1$ | $A(t_2/x), \Gamma_2 \Rightarrow \Delta_2$ | $\ldots$ | $\exists \omega$ |
| $\exists x A$ | $\exists x A(t_0/x), A(t_1/x), A(t_2/x), \ldots$ | $\exists \omega$ |
| $\Gamma \Rightarrow \Delta, A(t_0/x), A(t_1/x), A(t_2/x), \ldots$ | $\Gamma \Rightarrow \Delta, \exists x A$ |

where $t_0, t_1, t_2, \ldots$ represents a complete enumeration of the closed terms of $L$.

The logic $\text{IK}^\omega$ is defined as follows where $\Gamma$ and $\Delta$ are multisets of formulas: $\langle \Gamma, \Delta \rangle \in \text{IK}^\omega$ if and only if $S_{\text{IK}^\omega} \vdash \Gamma \Rightarrow \Delta$.

3. **Łukasiewicz-models**

To simplify things, we shall here restrict our attention to valuations satisfying the conditions for first-order continuum-valued Łukasiewicz-models, thus avoiding a domain of quantification. The reader interested in ways to deal with a domain of quantification for continuum-valued Łukasiewicz-models may refer to [3].

**Definition 3.1.** A function $\mathcal{V}$ from the sentences of $L$ to $[1, 0]$ is a first-order continuum-valued Łukasiewicz-valuation (LQ-valuation) if and only if

- $\mathcal{V}(\neg A) = 1 - \mathcal{V}(A)$
- $\mathcal{V}(A \rightarrow B) = \min(1, 1 - \mathcal{V}(A) + \mathcal{V}(B))$
- $\mathcal{V}(\exists x A(x)) = \sup\{\mathcal{V}(A(t/x)) \mid t \in \text{Cter}\}$

where Cter is the set of closed terms of $L$. 

$L$ based on the connectives $\exists, \rightarrow$ and $\neg$ where the set of well-formed formulas is defined with the standard recursive clauses.
We shall refer to the logic defined as the set of formulas that is assigned 1 in every LQ-valuation as $L_\infty$.

It is quite natural to look at continuum-valued Lukasiewicz-models for soundness in the case of $S_{IK^\omega}$ since propositional affine logic is a sublogic of propositional Lukasiewicz-logic, see e.g. [4]. However, $S_{IK^\omega}$ is not sound with regard to the above models. To show this, we require a way to interpret infinitary multiset-sequents in continuum-valued Lukasiewicz-models. Following for example [5], we can distinguish between strong and weak disjunction and conjunction in Lukasiewicz-models based on the clauses used to define the connectives:

- Weak disjunction: $\max(V(A), V(B))$
- Strong disjunction: $\min(1, V(A) + V(B))$
- Weak conjunction: $\min(V(A), V(B))$
- Strong conjunction: $\max(0, V(A) + V(B) - 1)$

Whilst we haven’t included connectives defined with these clauses in our language, we can nonetheless utilise the clauses for strong disjunction and conjunction to interpret $S_{IK^\omega}$-sequents in our models. In fact, we can interpret $S_{IK^\omega}$-sequents directly in the models by treating the antecedent as an infinitary strong conjunction and the succedent as an infinitary strong disjunction. This is in line with [1]'s insistence on defining an infinitary multiset-multiset logic rather than a set-set logic, and it allows us to define an infinitary multiset-multiset logic on Lukasiewicz-models.

**Definition 3.2.** A $S_{IK^\omega}$-sequent $\Gamma \Rightarrow \Delta$ is sound with regard to a LQ-valuation $V$ if and only if

$$1 - \min(1, \sum_{A \in \Gamma} (1 - V(A))) \leq \min(1, \sum_{B \in \Delta} V(B))$$

An $S_{IK^\omega}$-sequent $\Gamma \Rightarrow \Delta$ is valid if and only if $\Gamma \Rightarrow \Delta$ is sound with regard to every LQ-valuation. A $S_{IK^\omega}$-rule $R$ is sound with regard to LQ-valuations if and only if, for every LQ-valuation $V$, if every premise $\Gamma' \Rightarrow \Delta'$ of $R$ is sound with regard to $V$, then the conclusion $\Gamma'' \Rightarrow \Delta''$ of $R$ is sound with regard to $V$.

In the following, we help ourselves to the notation $V(\Gamma)$ and $V(\Delta)$ where $V(\Gamma)$ is an abbreviation of $1 - \min(1, \sum_{A \in \Gamma} (1 - V(A)))$ and $V(\Delta)$ is an abbreviation of $\min(1, \sum_{B \in \Delta} V(B))$. Importantly, $\Gamma$ and $\Delta$ are multisets, so the value of $A/B$ must be added once for each occurrence of formula $A/B$ in $\Gamma/\Delta$.

**Theorem 3.3.** The rule $\exists R^\omega$ of $S_{IK^\omega}$ is not sound with regard to LQ-valuations.

**Proof.** Assume that $V(\Gamma) \leq \min(1, \sum_{i=0}^{\infty} (V(A(t_i/x))) + V(\Delta))$ where $t_0, t_1, \ldots$ is a complete enumeration of the closed terms of $\mathcal{L}$. It doesn’t follow that $V(\Gamma) \leq \min(1, V(\exists x A x) + V(\Delta))$ because it is not guaranteed that $\sup\{V(A(t/x)) \mid t$ is a closed term} $\geq \sum_{i=0}^{\infty} (V(A(t_i/x)))$. One counterexample is a model such that $V(\Gamma) = 1$ and each $V(A(t_i/x)) = \frac{1}{2^i}$. \qed

4. Soundness for $S_{IK^\omega}$

The sup-clause for $\exists$ is not strong enough for our purposes. This shouldn’t actually be surprising because the rules $\exists L^\omega$ and $\exists R^\omega$ are generalisations of strong disjunction, not weak disjunction. Moreover, a natural proposal to overcome that problem is to replace the sup-clause for $\exists$ with the corresponding sum-clause as follows:
Definition 4.1. Let a first-order continuum-valued Łukasiewicz-valuation with multiplicative quantifier (LMQ-valuation) be like a LQ-valuation with the exception that
\[ V(\exists x A(x)) = \min(1, \sum_{i=0}^{\infty} (V(A(t_i/x)))) \]
where \( t_0, t_1, \ldots \) is a complete enumeration of the closed terms of \( \mathcal{L} \)

It is left to show that \( S_{IK-\omega} \) is sound with regard to our new models where sequents are interpreted in the same way as above.

Definition 4.2. An \( S_{IK-\omega} \)-sequent \( \Gamma \Rightarrow \Delta \) is sound with regard to a LMQ-valuation \( V \) if and only if
\[ 1 - \min(1, \sum_{A \in \Gamma} (1 - V(A))) \leq \min(1, \sum_{B \in \Delta} (V(B))) \]
An \( S_{IK-\omega} \)-sequent \( \Gamma \Rightarrow \Delta \) is valid with regard to LMQ-valuations if and only if \( \Gamma \Rightarrow \Delta \) is sound with regard to every LMQ-valuation. A \( S_{IK-\omega} \)-rule \( R \) is sound with regard to LMQ-valuations if and only if, for every LMQ-valuation \( V \), if every premise \( \Gamma' \Rightarrow \Delta' \) of \( R \) is sound with regard to \( V \), then the conclusion \( \Gamma' \Rightarrow \Delta' \) of \( R \) is sound with regard to \( V \).

We first present the following lemma which is essential for establishing that the rule \( \exists L_\omega \) is sound.

Lemma 4.3. Let \( \Gamma \), \( \Delta \) and \( X \) be infinite but enumerable sets of variables assigned a value in \([0, 1]\), where \( \gamma_i, \chi_i \) and \( \delta_i \) refer to the \( i \)th variable in each set. Suppose that for each triple \( \langle \gamma_i, \chi_i, \delta_i \rangle \),
\[ 1 - \min(1, \sum_{i=0}^{\infty} (1 - \gamma_i)) \leq \min(1, \sum_{i=0}^{\infty} (1 - \chi_i)) \leq \delta_i \]
Then
\[ 1 - \min(1, \sum_{i=0}^{\infty} (1 - \gamma_i) + (1 - \chi_i)) \leq \delta_i \]

Proof. The assumptions imply the following inequalities:
(\(^*\)) \[ \chi_i \leq (1 - \gamma_i) + \delta_i \]
(\(^**\)) \[ 0 \leq (1 - \gamma_i) - \chi_i + \delta_i \]
We now split the proof into two cases.

Case 1: \( \sum_{i=0}^{\infty} (\chi_i) \geq 1 \). The desired conclusion reduces to
\[ 1 \leq \min(1, \sum_{i=0}^{\infty} (\delta_i)) + \min(1, \sum_{i=0}^{\infty} (1 - \gamma_i)) \]
We consider three subcases, \( \sum_{i=0}^{\infty} (\delta_i) \geq 1 \) or \( \sum_{i=0}^{\infty} (1 - \gamma_i) \geq 1 \), or both are strictly less than 1. It is trivial in the first two cases. In the third case, it follows from the inequality obtained by combining each instance of (\(^*\)).

Case 2: \( \sum_{i=0}^{\infty} (\chi_i) < 1 \). The desired conclusion is now
\[ 1 - \min(1, \sum_{i=0}^{\infty} (1 - \gamma_i) + (1 - \sum_{i=0}^{\infty} (\chi_i))) \leq \min(1, \sum_{i=0}^{\infty} (\delta_i)) \]
which is simplified to
\[
1 \leq \min(1, 1 + \sum_{i=0}^{\infty} (1 - \gamma_i) - \sum_{i=0}^{\infty} (\chi_i)) + \min(1, \sum_{i=0}^{\infty} (\delta_i))
\]
This is trivially true if \(1 + \sum_{i=0}^{\infty} (1 - \gamma_i) \geq 1\) or \(\sum_{i=0}^{\infty} (\delta_i) \geq 1\), so we consider the case in which both are strictly less than 1. We can then simplify the desired conclusion to \(1 \leq 1 + \sum_{i=0}^{\infty} (1 - \gamma_i) - \sum_{i=0}^{\infty} (\chi_i) + \sum_{i=0}^{\infty} (\delta_i)\)
But this is "immediate" from (**) by combining the inequalities, distributing sums and adding 1's.

With this lemma at hand, soundness is straightforward.

**Theorem 4.4.** If there is a derivation of \(\Gamma \Rightarrow \Delta\) in \(S_{IK^\omega}\) then \(\Gamma \Rightarrow \Delta\) is valid with regard to LMQ-valuations.

**Proof.** The proof proceeds by induction on the construction of \(D\) of \(\Gamma \Rightarrow \Delta\). We present the salient cases for illustration.

**Initial sequents:** With \(V(\Gamma)\) and \(V(\Delta)\) being in \([0, 1]\), we observe that \(V(\Gamma) - 1 \leq V(\Delta)\) and thus that \(V(\Delta) \leq V(\Delta) + 1\) which implies \(1 - (1 - V(\Delta) + 1 - V(\Gamma)) \leq V(\Delta) + V(\Delta) which implies \(1 - \min(1, 1 - V(\Delta) + 1 - V(\Gamma)) \leq V(\Delta) + V(\Delta).

Since \(1 - \min(1, 1 - V(\Delta) + 1 - V(\Gamma)) \leq 1\), we also have \(1 - \min(1, 1 - V(\Delta) + 1 - V(\Gamma)) \leq \min(1, V(\Delta) + V(\Delta))\).

\(\exists R^\omega\): Assume that \(V(\Gamma) \leq \min(1, \sum_{i=0}^{\infty} (V(A(t_i/x)) + V(\Delta)))\). Assume first that \(\sum_{i=0}^{\infty} (V(A(t_i/x)) < 1\). Then \(\sum_{i=0}^{\infty} (V(A(t_i/x))) is equal to \(\min(1, \sum_{i=0}^{\infty} (V(A(t_i/x)))\), and we’re done. Assume the contrary, that \(\sum_{i=0}^{\infty} (V(A(t_i/x))) \geq 1\). It follows then that \(\min(1, \sum_{i=0}^{\infty} (V(A(t_i/x))) + V(\Delta))\) is equal to \(\min(1, \sum_{i=0}^{\infty} (V(A(t_i/x))) + V(\Delta))\), and we’re done.

\(\exists L^\omega\): Assume that, for each closed term \(t_i\), we have a triple \((\Gamma_i, A, t_i, \Delta_i)\) such that \(1 - \min(1, (1 - V(\Gamma_i)) + (1 - V(At_i)) \leq V(\Delta_i))\). By lemma [5], we immediately obtain
\[
1 - \min(1, \sum_{i=0}^{\infty} (1 - V(\Gamma_i)) + (1 - V(\exists x A x)) \leq \min(1, \sum_{i=0}^{\infty} (V(\Delta_i)))
\]
through the clause for \(\exists\). □

5. Adding Transparent Truth?

In [1], the language \(\mathcal{L}\) is expanded with a designated one-place predicate \(T\) together with a denumerable set of constants “to serve as canonical names of all sentences in the language” [1 506] which will be referred to by either \(\gamma A^\tau\) or some lower-case letter as stipulated. Moreover, the calculus for \(IK^\omega\) is expanded with the following rules to define the calculus for \(IKT^\omega\):

\[
\frac{A, \Gamma \Rightarrow \Delta}{T^\tau A^\tau, \Gamma \Rightarrow \Delta} TL \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, T^\tau A^\tau} TR
\]

\(T\) is thus intended as a transparent truth predicate. [1] presents a cut-elimination proof for the sequent calculus for \(IKT^\omega\) from which consistency is concluded.

Now, [5] has shown that if the simple coding scheme for generating self-reference used by [1] and as sketched above is replaced with a G"odel-coding based on a theory
of arithmetic containing function symbols for certain primitive recursive functions, then the resulting theory is not only ω-inconsistent but also inconsistent. While [6] nonetheless express confidence in that the original theory is still consistent with reference to the cut-elimination proof presented by [1], [2] has shown that the reduction step employed by [1] in the cut-elimination proof for the quantifier rules is inadequate from which it follows that [1]'s claim that the theory is consistent is unsupported.

The proof presented by [6] relies on a result by [7]. [7] shows that a transparent theory of truth based on Peano Arithmetic is ω-inconsistent if it is expanded with a one-place function symbol → such that \( \lceil A \rceil \rightarrow \lceil B \rceil = \lceil A \rightarrow B \rceil \), and a two-place function symbol \( f_b \) such that \( f_b(0, x) = x \rightarrow \bot \) and \( f_b(n + 1, x) = x \rightarrow f_b(n, x) \) where + is addition, and it also satisfies the following principles, here presented in sequent calculus format for uniformity:

\[
\begin{align*}
A & \Rightarrow B \\
\exists x A & \Rightarrow \exists x B & \text{B1} \\
(A \rightarrow \exists x B) & \Rightarrow \exists x (A \rightarrow B) & \text{B2}
\end{align*}
\]

Relying on the weak diagonal lemma, the proof of ω-inconsistency in [7] employs the instance \( \mu \leftrightarrow \exists x T f_b(x, \lceil \mu \rceil) \). Importantly, the theory of truth obtained by expanding \( L_\infty \) with sufficient arithmetic to define the above function symbols and transparent truth based on a suitable Gödel-coding satisfies these conditions. Moreover, [7]'s result can be seen as a refinement of the observations by [8] and [3] in which different primitive recursive functions are used for the same purpose with regard to \( L_\infty \), namely to show that the resulting theory of truth is ω-inconsistent.

In [6], it is observed that B1 and B2 also holds in IKTω, and they proceed to show that [7]'s ω-inconsistency result holds in a suitable variant of \( S_{IKT^\omega} \). Moreover, [6] then use the rule \( \exists L^\omega \) to turn the ω-inconsistency into an outright inconsistency.

The result by [6] can be strengthened in the following way. The formula introduced by [7] involves the truth-predicate, the existential quantifier, the conditional and \( \bot \) representing absurdity. As it turns out, we can with inspiration from the result in [9] replace the implication to \( \bot \) in [7]'s formula with negation and thus employ a variation of the formula utilised by [10] for a ω-inconsistency result concerning some classical theories of truth.

Let \( \hat{T} \) be a one-place function symbol defined such that \( \hat{T}t = \lceil Tt \rceil \) and let \( f \) be a two-place function symbol defined such that \( f_m(0, \lceil A \rceil) = \lceil A \rceil \) and \( f_m(n + 1, \lceil A \rceil) = \hat{T}f_m(n, \lceil A \rceil) \). Assume moreover that the weak diagonal lemma holds. In the sequent calculus obtained by expanding IKTω with a theory of arithmetic satisfying these requirements it will be the case that \( \Rightarrow \mu \leftrightarrow \exists x \lceil T f_m(x, \lceil \mu \rceil) \rceil \) is derivable. We can thus reason as follows, leaving some steps implicit for readability:
Then both \( V \omega \) derivable. While the conditional is required for with transparent truth, this is not the case with IKT

\( \text{many of them so the result must be } 1 \). It follows that \( \text{substitution instance has the same value greater than zero but th ere are infinitely} \)

\[ \exists x T \! f_m(x, \check{\mu}) \Rightarrow T \! f_m(0, \check{\mu}) \]

\[ T \! f_m(0, \check{\mu}) \Rightarrow T \! f_m(1, \check{\mu}) \]

\[ T \! f_m(1, \check{\mu}) \Rightarrow T \! f_m(2, \check{\mu}) \]

\[ \exists x T \! f_m(x, \check{\mu}) \Rightarrow T \! f_m(0, \check{\mu}), T \! f_m(1, \check{\mu}), T \! f_m(2, \check{\mu}), \ldots \]

\[ \exists x T \! f_m(x, \check{\mu}) \Rightarrow \neg \exists x T \! f_m(x, \check{\mu}) \]

Through iterated applications of the definition of \( f_m \) and \( TL \) followed by an application of \( \exists L \omega \) and \( \neg R \) we finally obtain that the sequent \( \Rightarrow \neg \exists T \! f_m(x, \check{\mu}) \) is derivable. While the conditional is required for \( \omega \)-inconsistency in the case of \( L_\omega \) with transparent truth, this is not the case with IKT\( \omega \).

While this result is clearly not ideal for the project initiated by [1] where \( S_{IK\omega} \) is presented for the purpose of being a logic for reasoning about transparent truth as it brings forth a significant limitation with the approach, one might nonetheless think that the issue is not worse for \( S_{IK\omega} \) than for \( L_\omega \) with transparent truth or the classical theory of truth FS presented by [11] and further explored by [12] which satisfies the conditions for [10]’s \( \omega \)-inconsistency result. Perhaps one could for example block the inconsistency by supplying the theory with numerals for non-standard numbers, thereby avoiding that \( \exists L \omega \) is turned into a Hilbertian \( \omega \)-rule as the language would then contain terms for numbers that are not identical to a natural number? Moreover, since \( L_\omega \) can be expanded with transparent truth as shown by [3] as long as certain function symbols are not defined, it might even be tempting to investigate whether the soundness proof from the previous section can be utilised as a consistency proof of IKT\( \omega \) under similar constraints. After all, the consistency proof for IKT\( \omega \) presented by [1] has been shown to contain an error in [2].

That is however not the case. With inspiration from the discussion about quantifiers in [13], we can show that LMQ-valuations are inconsistent with the additional stipulation that the language contains a designated one-place predicate \( T \) and a constant \( l \) such that the following hold:

\[ \forall(Tl) = \forall(\neg \exists x Tl) \]

Note the vacuous quantification in the formula \( \neg \exists x Tl \); while dubious it’s certainly permissible as per the language stipulations in [1].

Assume now that \( \forall(Tl) > 0 \). Then \( \forall(\exists x Tl) = 1 \) by the \( \exists \)-clause since each substitution instance has the same value greater than zero but there are infinitely many of them so the result must be 1. It follows that \( \forall(\neg \exists x Tl) = 0 \) and thus \( \forall(Tl) = 0 \) which contradicts our assumption. Assume instead that \( \forall(Tl) = 0 \). Then both \( \forall(\exists x Tl) = 0 \) by the \( \exists \)-clause and \( \forall(\neg \exists x Tl) = 0 \) by the condition for \( Tl \) which contradicts the \( \neg \)-clause.
As it turns out, the corresponding result is obtainable in IKT$^\omega$. The derivation looks like this:

\[
\begin{array}{c}
Tl \Rightarrow Tl \\
\exists x Tl \Rightarrow Tl, Tl, \ldots \\
\quad \Rightarrow \neg \exists x Tl, Tl, Tl, \ldots \\
\quad \Rightarrow Tl, Tl, Tl, \ldots \\
\quad \Rightarrow \exists x Tl \\
\neg \exists x Tl \Rightarrow Tl \\
\exists x Tl \Rightarrow Tl, Tl, \ldots \\
\quad \Rightarrow \neg \exists x Tl \\
\end{array}
\]

Adding a copy of $Tl$ to an infinite sequence of $Tl$'s makes no difference to the totality of copies.

Both this derivation and the above result regarding the models rely on a particular treatment of vacuous quantification, an issue that was not addressed by [1]. In the case of the sequent calculus, we actually have two options with regard to introducing $\exists x Tl$ into a derivation using the rule $\exists R^\omega$ (and thus also $\exists L^\omega$). Either we require an infinite sequence of $Tl$ as above:

\[
\Gamma \Rightarrow \Delta, Tl(t_0/x), Tl(t_1/x), Tl(t_2/x), \ldots \\
\Rightarrow \Delta, \exists x Tl \\
\]

Let us call this “the multiplicative way”. Alternatively, since every instance of $Tl(t_i/x)$ is the same formula in the case of vacuous quantification, that the rule takes only one copy of that formula as premise:

\[
\Gamma \Rightarrow \Delta, Tl \\
\Rightarrow \Delta, \exists x Tl \\
\]

The latter option, which we can call “the additive way”, comes with a contractive flavour and seems not to be in tune with the spirit of [1]. After all, the corresponding restriction on $\exists L^\omega$ permits the derivation of $\exists x Tl \Rightarrow Tl$ which actually amounts to the claim that $\bigoplus A$ implies $A$ if we consider vacuous existential quantification as corresponding to the infinitary multiplicative disjunction of a formula. The resulting logic would thus be such that the former inference is valid while the inference from $A \oplus A$ to $A$ is invalid.

In this paper then, we have seen that the logic $S_{IK\omega}$ is sound with regard to Lukasiewicz valuations that are modified by treating the existential quantifier as a generalisation of the strong disjunction instead of the weak disjunction, but that this soundness proof cannot be used as consistency proof for the theory of truth $IKT^\omega$ since not only are the valuations in question inconsistent with transparent truth but IKT$^\omega$ is itself also inconsistent as long as vacuous quantification is treated in the multiplicative way.

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