DEFINABILITY OF RATIONAL INTEGERS
IN A CLASS OF POLYNOMIAL RINGS

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Abstract. We prove first-order definability of the ground ring of integers inside a polynomial ring with coefficients in a reduced indecomposable commutative (not necessarily Noetherian) ring. This extends a result, that has long been known to hold for integral domains, to a wider class of coefficient rings. Furthermore, we characterize indecomposable rings and reduced indecomposable rings in terms of properties of their polynomial rings. We also prove that infinite rings from the latter class have the property that polynomials inducing constant functions have necessarily degree zero.

1. Introduction

Over more than 60 years, the problem of defining rational integers inside a ring has been object of extensive investigation ([Rob51]). Much attention has been drawn onto Diophantine definability, for this would yield a counterpart result about other versions of Hilbert’s tenth problem, that is, the decidability of polynomial equations over $\mathbb{Z}$ (see [Mat70], and for details on results concerning Diophantineness, see [Shl07, Shl11] and the references therein). In a similar vein, first-order (not necessarily Diophantine) definability of integers in a ring is known to imply that the full first-order theory of such ring is undecidable. Julia Robinson showed that $\mathbb{Z}$ is first-order definable in $\mathbb{Q}$ ([Rob49]), a result that was later refined by Bjorn Poonen by reducing to two the number of universal quantifiers appearing in the defining formula ([Poo09]). Joechin Koenigsmann improved this result by using just one single universal quantifier; see [Koe16]. Concerning negative results about definability, it is well-known that definable sets in algebraically closed fields are either finite or cofinite (by elimination of quantifiers; see [Hod93, Theorem 4.4.6]), and so integers are not definable in the case of characteristic zero. Recently ([AKNS18, Lemma 4.7]) it was proved that the direct product of two infinite finitely generated rings is not bi-interpretable with $\mathbb{Z}$, which has as a consequence, for example, that $\mathbb{Z}$ is not definable in $\mathbb{Z} \times \mathbb{Z}$.

On the other hand, we have no knowledge of any attempt to extend these results outside the class of integral domains (see [Poo08] for a survey), partly due to the consistent use of field extensions of the quotient field of these rings throughout the results mentioned. In this paper, we formulate a definability criterion of $\mathbb{Z}$ inside a polynomial ring that includes a wider range of coefficient rings, which is in fact a natural class to which to extend the result, namely, the class of reduced indecomposable (commutative unital) rings (Proposition 5). In general, any Noetherian reduced ring can be written out as a finite product of such rings ([Coht03, Proposition 4.5.4]), so we may consider these rings as the basic bricks for building up an important class of objects in commutative algebra, corresponding to the notion of connected components of reduced schemes in algebraic geometry.

This work is divided as follows: in Section 2 we explore first-order definability of sets of powers, by introducing the concept of logical powers, that is, a first-order
property that coincides with the property of being a positive power of a given element of a ring, under some special conditions on both the element and the ring, mainly focusing on the case of polynomial rings in one variable. In Section 3, we investigate such special conditions, and study the class of reduced indecomposable (commutative, unital) rings, proving several of its algebraic properties; we also provide examples of such rings that are not integral domains, both Noetherian and non-Noetherian. In Section 4 we use the theory developed in Section 2 and Section 3 to construct two special definable sets of polynomials with coefficients in a reduced indecomposable ring, which are crucially used in Section 5 to define the rational integers in reduced indecomposable polynomial rings (by using explicit definitions for sets of powers of a fixed element). In Section 5 we first present a general criterion to define sets of exponents of powers of suitable elements, then we restrict this criterion to the case of polynomial rings, to obtain a specific version. We apply this specialized criterion to reduced indecomposable (commutative unital) polynomial rings, in two different versions, corresponding to two different subclasses of polynomial rings: the first and simplest one ensures integer definability upon the condition, satisfied by all polynomial rings over a field, that there exists a definable set of constants containing the integers; the second one no longer relies on this condition, but can only be proven for polynomial rings over nonfields, accounting therefore for the most general case. In this way we get a definition for \( \mathbb{Z} \) inside \( R[x] \) whenever \( R \) is a reduced indecomposable ring.

The paper ends with Section 6, a complementary collection of several results of algebraic and logical nature involving the concepts defined throughout the work, as well as specific examples of rings, where the criterion for integer definability is revisited under different viewpoints, for it applies in either of the two versions mentioned in Section 5.

All our results and proofs are developed in the framework of Zermelo–Fraenkel (ZF) set theory; in particular, our main result does not depend on AC or any choice principle. Except for Subsection 6.1, all the rings considered are nonzero, commutative and unital.

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2. A first-order approach to definability of sets of powers

Let \( S \) be a ring. For an element \( p \in S \), let \( \text{POW}(p) \) denote the set of positive powers of \( p \). As will be clearer in Section 5, the first clue for definability of \( \mathbb{Z} \) comes from the idea of “logically” identifying positive integers with the exponents of a fixed element, reducing the task to defining sets of powers of a fixed element of the ring. This has led to the search for a first-order definable notion that approximates that of “power”.

In this section we introduce an intuitive notion of positive power of an element \( p \in S \) as a multiple of \( p \) whose only divisors, up to units, are also multiples of \( p \), together with an additional property which, in the case of polynomial rings and under special conditions, also guarantees monicity (when \( p \) is monic); this condition is encapsulated by formula (2.1) below. An analogous approach is considered

\* However, some interesting issues concerning choice principles arise in Remark 11, Examples 12 and 14 and Subsection 6.7.
in [Rob51, p. 145], where it is shown that the same property is satisfied precisely by the nonnegative powers of \( p \), whenever \( p \) is a prime element and \( S \) is an integral domain (see item d of Proposition 4 for a slight generalization). We will explore our notion in a more general context where, for suitable conditions on \( p \) (Theorem 23 and Remark 24), the set \( \text{POW}(p) \) is first-order definable using \( p \) as parameter in the language of rings, with signature \((0,1,+,\cdot)\).

We adopt the standard notations of ring theory (see [Hun80] for a background); except for nilpotency, all ring-theoretic properties considered in this work are first-order definable (see [Hod93, Exercise 8.5.1] for an example of a ring whose nilradical is not definable). For the sake of brevity and notational convenience, using classical notation (e.g. "powers of ") in subsequent formulas; likewise, for two-variable formulas expressing binary relations \( \psi(\cdot,\cdot) \) which correspond to algebraic properties, we abbreviate by "logical powers".

**Definition 1.** Let \( S \) be a ring. Given \( p \in S \), we define the set \( \text{LPOW}(p) \) of **logical powers** of \( p \) as the set of elements \( f \in S \) satisfying:

- \( p \) divides \( f \);
- \( p - 1 \) divides \( f - 1 \);
- every divisor of \( f \) is either a unit or a multiple of \( p \).

Observe that \( \text{LPOW}(p) \) is defined by the one-variable formula \( \psi(\cdot,p) \), where \( \psi \) is given by

\[
\psi(f,s): \ s \mid f \land s - 1 \mid f - 1 \land \forall g \ [ g \mid f \Rightarrow (g \mid 1 \lor s \mid g)] .
\]

In what follows, we explore the similarities between \( \text{LPOW}(p) \) (a first-order definable set) and \( \text{POW}(p) \) (a set that we want to be first-order definable), in order to justify the expression "logical powers". Unfortunately, in the general case the definition of \( \text{LPOW}(p) \) fails badly in conveying the concept of "genuine powers":

**Example 2.** If \( g,h \in S \) are noninvertible and \( h \) is regular, then \( gh \notin \text{LPOW}(gh) \). In fact, we have that \( h \) divides \( gh \), yet \( h \) is not a unit, and if \( h = qgh \), then canceling \( h \) would imply that \( g \) is a unit. Therefore \( h \) cannot be a multiple of \( gh \).

Another instance in which the two definitions clash is the following: obviously we have \( 0 \in \text{POW}(p) \) if and only if \( p \) is nilpotent; on the other hand, the following result characterizes the fact of the zero element being a logical power in a wide class of rings (including the class of polynomial rings):

**Proposition 3.** Let \( S \) be a ring containing an element \( s \) such that both \( s \) and \( s + 1 \) are nonunits (for example, any polynomial ring \( R[x] \), taking \( s = x \)). Then the following are equivalent:

a. \( 0 \in \text{LPOW}(p) \);
b. Both \( p \) and \( p - 1 \) are units;
c. \( \text{LPOW}(p) = S \).

**Proof.**
a \( \Rightarrow \) b We have that \( p - 1 \) divides \( 0 - 1 = -1 \), so \( p - 1 \) is a unit. Now both \( s \) and \( s + 1 \) trivially divide \( 0 \), so they must be multiples of \( p \) (as both \( s \) and \( s + 1 \) are nonunits). Therefore \( p \) divides \( (s + 1) - s = 1 \).
b \( \Rightarrow \) c If both \( p \) and \( p - 1 \) are units, then any element \( t \in S \) obviously belongs to \( \text{LPOW}(p) \), for \( t \), as all its divisors, is a multiple of \( p \), whilst \( p - 1 \) divides \( t - 1 \).
c \( \Rightarrow \) a Obvious. \( \Box \)
Notice that the hypothesis in Proposition 3 is only used in the proof of \( a \Rightarrow b \) to prove that \( p \) is a unit, whereas \( b \Rightarrow c \Rightarrow a \Rightarrow "p − 1 is a unit" holds for any ring.

The findings above suggest that our attempt at identifying the sets \( \text{POW}(p) \) by \( \text{LPOW}(p) \) could be more successful if we avoid nilpotent and reducible elements. As a matter of fact, under certain hypotheses the two sets coincide, producing a first-order definition of the powers of some types of elements. Before proceeding in this direction, we list some general properties concerning logical powers that will be used in the sequel. At this point, one notation is worth introducing: given two elements \( f, p \) of a ring \( S \), we say that \( f \) is \textit{infinitely divisible} by \( p \) if \( f \) is a multiple of arbitrarily large powers of \( p \) (equivalently, a multiple of all positive powers of \( p \)).

**Proposition 4.** Let \( S \) be a ring, and let \( p \in S \).

\begin{itemize}
    \item[a.] Any element \( f \) of \( \text{LPOW}(p) \) is either infinitely divisible by \( p \), or an element of the\footnote{Recall that an element \( p \) is said to be \textit{irreducible} if \( p \) is nonzero, noninvertible, and it is not the product of two nonunits. The requirement that \( p \) be nonzero is actually redundant in nonzero rings, but it is convenient for our purposes to keep it explicitly in the definition.} form \( up^n \), for some \( n \geq 1 \) and some unit \( u \) satisfying \( p − 1 \mid u − 1 \). In particular, if \( u = 1 \), then \( f \in \text{POW}(p) \).
    \item[b.] If \( f \in \text{LPOW}(p) \) and \( u \) is a unit such that \( p − 1 \) divides \( u − 1 \), then \( uf \in \text{LPOW}(p) \).
    \item[c.] If \( p \) is either invertible or irreducible, then \( p \in \text{LPOW}(p) \).
    \item[d.] If \( p \) is regular and prime, then \( \text{POW}(p) \subseteq \text{LPOW}(p) \).
\end{itemize}

**Proof.**

\begin{itemize}
    \item[a.] If \( f \) is not infinitely divisible by \( p \), let \( n \geq 1 \) be the greatest exponent such that \( p^n \mid f \), so that \( f = up^n \) for some \( u \) not divisible by \( p \). Since \( u \) divides \( f \) and \( f \in \text{LPOW}(p) \), then \( u \) must be a unit. Finally, we have \( f = up^n = u(p^n − 1) + u − 1 \), and since both \( f − 1 \) and \( p^n − 1 \) are multiples of \( p − 1 \), then so is \( u − 1 \).
    \item[b.] Obviously \( p \) divides \( uf \). Since \( p − 1 \) divides both \( f − 1 \) and \( u − 1 \), then \( p − 1 \) divides \( uf(f − 1) + u − 1 = uf − 1 \). Finally, if \( g \) divides \( uf \), then \( g \) divides \( u(f − 1) = f \); since \( f \in \text{LPOW}(p) \), then \( g \) is either a unit or a multiple of \( f \), or equivalently, a multiple of \( uf \).
    \item[c.] It suffices to observe that every divisor of \( p \) would be either invertible or an associate of \( p \) (hence a multiple of \( p \)), for the other properties are trivially satisfied.
    \item[d.] Let \( n \geq 1 \). Then obviously \( p \mid p^n \) and \( p − 1 \mid p^n − 1 \), and if \( g \) is a divisor of \( p^n \), say \( p^n = gh \), then \( p^{n+1} \) cannot divide \( h \) (otherwise we would have, by canceling, that \( p \) divides \( 1 \), which contradicts the primality of \( p \)). Thus, the largest \( k \) with \( p^k \) dividing \( h \) must satisfy \( k \leq n \); after canceling we get \( p^{n−k} = gh \), with \( h \) not a multiple of \( p \). If \( k = n \) then \( g \) is invertible; otherwise \( p \) divides \( gh \), so necessarily \( p \) divides \( g \) because \( p \) is prime. \( \square \)
\end{itemize}

Now we examine the case \( S = \mathbb{R}[x] \), in order to draw some consequences from the equality \( \text{LPOW}(x) = \text{POW}(x) \):

**Proposition 5.** Let \( R \) be a ring and consider \( \mathbb{R}[x] \), the polynomial ring in one variable over \( R \). If \( x \in \text{LPOW}(x) \), then \( x \) is irreducible. If in addition one of the inclusions \( \text{LPOW}(x) \subseteq \text{POW}(x) \) or \( \text{POW}(x) \subseteq \text{LPOW}(x) \) holds, then \( R \) is reduced.

**Proof.** We always have that \( x \) is nonzero and noninvertible. Since \( x \) is regular, every divisor of it will also be regular, and so if \( x \in \text{LPOW}(x) \), then by using the contrapositive of Example 2 we can conclude that \( x \) is irreducible.
Now let \( a \in R \) with \( a^n = 0 \) for some \( n \geq 1 \). We want to prove that \( a = 0 \), obtaining in this way that \( R \) is reduced. Set \( u = 1 - a(x - 1) \). Note that \( u \) divides \( 1 - a(x - 1)^n = 1 \), that is, \( u \) is invertible, and also that \( x - 1 \) clearly divides \( u - 1 \). Consequently, by item \( b \) of Proposition 4 we have \( ux \in \text{LPow}(x) \).

If \( \text{LPow}(x) \subseteq \text{Pow}(x) \), then \( ux = x^m \) for some \( m \geq 1 \), which forces to have \( m = 1 \) and \( u = 1 \), and so \( a = 0 \). On the other hand, \( x - a \) is not invertible and divides \( x^n - a^n = x^n \). If \( \text{Pow}(x) \subseteq \text{LPow}(x) \) (in this case the condition \( x \in \text{LPow}(x) \) is superfluous), then \( x - a \) must be a multiple of \( x \), so again \( a = 0 \). \( \square \)

Thus, for a ring \( R \), in order to have \( \text{LPow}(x) = \text{Pow}(x) \), it is necessary that \( R \) be reduced and the polynomial \( x \) be irreducible in \( R[x] \). Later we will see (Theorem 22) that these conditions are also sufficient, and in the course of the reasoning we will show (see Proposition 10) that irreducibility of the polynomial \( x \) in \( R[x] \) is equivalent to indecomposability of \( R \).

3. Reduced and indecomposable rings and some algebraic properties

In this section we study the purely algebraic properties of reduced and/or indecomposable commutative unital rings. Recall that a ring \( R \) is said to be reduced if it contains no nonzero nilpotent elements. For an element \( f \) of a polynomial ring \( R[x] \), we denote its coefficient of degree \( i \) by \( f_i \). Elements of \( R \) will be called constants.

**Lemma 6.** Let \( R \) be a ring and let \( f, g \in R[x] \).

a. If \( g \) divides \( f \), and \( m = \deg(g) > \deg(f) \), then \( f \) is annihilated by a power of \( g \). More specifically, if \( f = gh \) and \( k = \deg(h) \), then \( g^{k+1} f = 0 \). In particular, if \( R \) is reduced and \( f \) is regular, then \( g \mid f \) implies \( \deg(g) \leq \deg(f) \), and divisors of regular constant elements are itself constants.

b. If \( x^r \) divides \( gh \), then \( x^r \) divides \( g^r h \). Moreover, \( x^r = gh \) implies \( g^r = g^{r+1} h_r \).

**Proof.**

a. If \( f = gh = (g_0 x^m + \cdots + g_0)(h_k x^k + \cdots + g_0) \), then we claim that \( g^{k+1}_m \) annihilates all coefficients of \( h \); in other words, \( g^{k+1}_m \) annihilates \( h \), and consequently \( f \). In fact, as \( \deg(f) < m \leq m + k \), we have \( g_m h_k = f_{m+k} = 0 \). Again, if \( k - 1 \geq 0 \), then \( 0 = f_{m+k-1} = g_m h_{k-1} + g_{m+1} h_k \); multiplying by \( g_m \) and using the equality \( g_m h_k = 0 \) just obtained, we get \( g^2_m h_{k-1} = 0 \) and so \( g^2_m \) annihilates \( h_{k-1} \) and \( g_m \) annihilates \( h_k \), implying that \( g^2_m \) annihilates both \( h_{k-1} \) and \( h_k \). We may proceed inductively on \( i \) to prove that, if \( k - i \geq 0 \), then \( g^{i+1}_m \) annihilates \( h_k, h_{k-1}, \ldots, h_{k-i} \), which yields our claim for \( i = k \). Indeed, suppose the claim holds for \( i \) and \( k - (i + 1) \geq 0 \). Then \( 0 = f_{m+k-(i+1)} = g_m h_{k-(i+1)} + (g_{m+1} h_{k-i} + \cdots + g_{m-(i+1)} h_k) \). By induction hypothesis, the second term of this sum is annihilated by \( g^{i+1}_m \), as all coefficient of \( h \) appearing in it are. Therefore, multiplying by \( g^{i+1}_m \), one gets \( g^{i+2}_m h_{k-(i+1)} = 0 \) and \( g^{i+2}_m \) annihilates \( h_k, \ldots, h_{k-(i+1)} \), completing the induction.

For the second assertion observe that if a divisor \( g \) of \( f \) had degree higher than \( \deg(f) \), then \( f \) would be annihilated by a power of a nonzero constant (the leading coefficient of \( g \)), which is also nonzero in a reduced ring. Therefore, \( f \) would be a zero divisor, contradicting the hypothesis. The last statement follows immediately.

b. The result is obvious for \( r = 0 \). For \( r > 0 \), as \( gh \) is a multiple of \( x^r \), then all its coefficients in degrees \( 0, \ldots, r - 1 \) vanish, so we may apply a specular reasoning to that used in the previous item and get \( 0 = (gh)_0 = g_0 h_0 \) and, if \( r > 1, \) \( 0 = (gh)_1 = g_0 h_1 + g_1 h_0 \), from which \( g^2_0 h_1 = 0 \) and thus \( g^2_0 \).
Proposition 7. For a ring $R$, the following conditions are equivalent:

a. $R[x]$ is reduced;

b. $R$ is reduced;

c. $R[x]^* = R^*$.

Proof. The implication $a \Rightarrow b$ is obvious. For $b \Rightarrow c$, note that units are precisely the divisors of 1, which is a regular constant element, and apply the last assertion of Lemma 6a. Finally, if $R[x]^* = R^*$ and $f \in R[x]$ satisfies $f^m = 0$, with $m \geq 2$, then $(1 + xf^{m-1})(1 - xf^{m-1}) = 1$ implies $1 + xf^{m-1} \in R[x]^* \subseteq R$, so necessarily $f^{m-1} = 0$. Iterating this reasoning we conclude that $f = 0$, proving that $R[x]$ is reduced. \hfill $\Box$

Definition 8. A ring is said to be indecomposable if its idempotent elements are only 0 and 1.

Remark 9. Notice that the existence of an idempotent element is a first-order predicate, so that the theory of indecomposable rings is finitely axiomatizable. As a matter of fact, the same happens with reducedness, even though nilpotency of a given element cannot be expressed as a one-variable first-order formula. Indeed, observe that if $a$ is a nonzero nilpotent element of a ring and $n \geq 2$ is its index of nilpotency (i.e., the least positive integer such that $a^n = 0$), then $a^{n-1}$ is a nonzero nilpotent element with index of nilpotency 2. Therefore a ring is reduced if and only if it contains no nonzero element whose square is zero and this is obviously a first-order predicate. Consequently, the theory of reduced rings is also finitely axiomatizable.

The next result relates indecomposability of a ring $R$ to a property about its polynomial ring $R[x]$:

Proposition 10. A ring $R$ is indecomposable if and only if the polynomial $x \in R[x]$ is irreducible.

Proof. Obviously $x$ is nonzero and noninvertible. Suppose that $R$ is indecomposable, and assume $x = gh$, with $g, h \in R[x]$; we want to show that either $g$ or $h$ is a unit. Set $e := gh_1$ and $e' := g_1 h_0$. We have $e + e' = gh_1 + g_1 h_0 = (gh)_1 = (x)_1 = 1$; on the other hand, by the last part of Lemma 6b with $r = 1$ we have $g_0^2 h_1 = g_0$, so $e^2 = (g_0^2 h_1) h_1 = g_0 h_1 = e$, and therefore $e$, being idempotent, must be 0 or 1 (in a similar way one can show that $e'$ is idempotent). If $e = 1$, then $g_0 \in R$; since $g_0 h_0 = (gh)_0 = (x)_0 = 0$, it follows that $h_0 = 0$, so $x$ divides $h$, and dividing out the equality $x = gh$ by the regular element $x$, we get that $g$ is a unit. If $e = 0$, then $e' = 1$, and proceeding analogously we conclude that $h \in R[x]^*$. For the converse, observing that the only invertible idempotent in a ring is $e = e^2 e^{-1} = ee^{-1} = 1$, if $e \in R$ were an idempotent other than 0 or 1, then $x = [(1 - e) x + e] \cdot [1 - e] x + e$ would be a nontrivial factorization of $x$, since the factors on the right hand side cannot be units, as their constant terms $e$ and $1 - e$ are not. \hfill $\Box$

Remark 11. Notice that the argument above proves that, if $x$ has any nontrivial factorization, then it has one as a product of two linear polynomials. Furthermore, by putting together Propositions 7 and 10, we obtain a characterization of reduced indecomposable rings in terms of a property of the polynomials 1 and $x$ in $R[x]$; that they both be not a product of two positive degree polynomials. For those
acquainted with algebraic geometry, we recall the special meaning that indecomposability has in terms of the topology of the corresponding Zariski affine scheme: a ring \( R \) is indecomposable if and only if its prime spectrum \( \text{Spec}(R) \) is connected. The reader may feel free to check Subsection 6.3 for more equivalent definitions of indecomposability and/or reducedness.

Clearly, any integral domain is reduced and indecomposable. Below we provide some examples of reduced/indecomposable rings that are not integral domains.

**Example 12.** For a nonempty set \( X \) and a ring \( B \), let \( S = B^X \) be the set of \( B \)-valued functions on \( X \). Endowed with component-wise addition and product, \( S \) is a ring; moreover, if \( B \) is reduced, then so is any subring of \( S \). On the other hand, if \( B \) is indecomposable, then the idempotent elements of a given subring of \( S \) are precisely those functions that take only the values 0 and 1. If \( B \) is a topological ring such that its singletons are closed sets (that is, endowed with a \( T_1 \) topology), and \( X \) is a connected topological space, then \( R = C(X, B) \), the subring of \( S \) of \( B \)-valued continuous functions on \( X \), is indecomposable: for if \( f \in R \) is idempotent, then \( X = f^{-1}([0]) \cap f^{-1}([1]) \) is the disjoint union of two closed sets, so by connectedness of \( X \) we must have that \( f \) is constant. The existence in \( R \) of two continuous functions with disjoint supports, which is guaranteed in many cases (for instance, if \( B = \mathbb{R} \), this holds whenever \( X \) separates some pair of disjoint closed sets, which is the case if \( X \) is a metric space or a completely regular space or, under certain standard assumptions, whenever \( X \) is a normal space), provides examples of reduced and indecomposable rings that are not integral domains.

**Example 13.** Consider the subring \( R \) of \( \mathbb{Z} \times \mathbb{Z} \) consisting of those pairs \((m, n)\) with \( m \equiv n \pmod{2} \). Since \( \mathbb{Z} \times \mathbb{Z} \) is reduced, then \( R \) is too; moreover, the idempotents in \( \mathbb{Z} \times \mathbb{Z} \) are precisely \((0, 0), (1, 1), (1, 0) \) and \((0, 1)\); since \((1, 0), (0, 1) \notin R \), it follows that \( R \) is indecomposable.

As a consequence of our main result, we have that \( \mathbb{Z} \) is definable in the subring \( R[\mathbb{x}] \) of the ring \((\mathbb{Z} \times \mathbb{Z})[\mathbb{x}] \cong \mathbb{Z}[\mathbb{x}] \times \mathbb{Z}[\mathbb{x}] \), where \( R \) is as described in Example 13. In this line of thought, the reader may wonder whether \( \mathbb{Z} \) is definable in \((\mathbb{Z} \times \mathbb{Z})[\mathbb{x}] \). Nevertheless, as a consequence of [AKNS18, Lemma 4.7] we have that this is false (the very same result implies that \( \mathbb{Z} \) is not even definable in \( \mathbb{Z} \times \mathbb{Z}^\omega \)). In other words, the condition on the subring \( R \) in Example 13 is essential for the definability of \( \mathbb{Z} \) in \( R[\mathbb{x}] \).

**Example 14.** As a generalization of the previous example, let \( I \) be any set with more than one element. Consider the subring \( R \) of \( \mathbb{Z}^I \) consisting of those \( I \)-tuples whose entries have the same parity. Since \( \mathbb{Z}^I \) is reduced, so is \( R \), and the set of idempotents in \( \mathbb{Z}^I \) is precisely \( \{0, 1\}^I \); as \( \{0, 1\}^I \cap R = \{0_R, 1_R\} \), it follows that \( R \) is indecomposable.

Unlike the case where \( I \) is finite, the ring \( R \) is not, in general, Noetherian: indeed, if \( I \) contains a denumerable subset \( \{i_n : n \in \mathbb{N}\} \) (that is, if \( I \) is Dedekind-infinite) and \( a_i \) is the ideal in \( R \) generated by \( 2e_{i_n}, \ldots, 2e_{i_n} \) (here \( e_i \) denotes the
i-th canonical \(I\)-tuple in \(\mathbb{Z}^I\), then the ascending chain of ideals \((a_n)_{n \in \mathbb{N}}\) is not stationary\(^\blacktriangle\).

**Example 15.** We say that a ring \(R\) is local if their nonunits form an ideal\(^\blacklozenge\). It is not hard to show that this occurs precisely when \(r + 1\) is a unit for every nonunit \(r\) of \(R\). In such a case, if \(a \in R\) is idempotent, then \((a - 1)a = 0\); since one of \(a - 1\) or \(a\) is a unit, it follows that \(a = 0\) or \(a - 1 = 0\), which proves that \(R\) is indecomposable. This provides more examples of reduced indecomposable rings which are not integral domains, obtained as suitable localizations of further rings at prime ideals\(^\blacklozenge\), such as the germs of rational functions at points lying in more than one irreducible component of a (reduced) algebraic set. Notice that the rings considered in the statement of Proposition 3 happen to be exactly those that are not local.

Given an element of a ring that is zero or a unit, it trivially satisfies that some of its positive powers divides the previous corresponding power (actually, this happens with every positive power of it); for reduced and indecomposable rings, the converse holds. This basic result will be used repeatedly, and we prove it below:

**Proposition 16.** Let \(R\) be a reduced and indecomposable ring, and let \(c \in R\). Then:

a. If \(c^{m+1}\) divides \(c^m\) for some \(m \geq 0\), then \(c \in \{0\} \cup R^*\).

b. If \(c \notin \{0\} \cup R^*\), then all nonnegative powers of \(c\) are pairwise distinct.

c. If \(R\) is finite, then \(R\) is a field.

**Proof.**

a. If \(c^m = c^{m+1}d\), then \((cd)^m = c^m d^m = (c^{m+1}d)d^m = (cd)^{m+1}\), hence \((cd)^m = (cd)^{m+1} = \cdots = (cd)^{2m}\). Therefore \((cd)^m\) is idempotent, hence it equals 1 or 0 (because \(R\) is indecomposable). If \((cd)^m = 1\), then \(c \in R^*\). Otherwise, since \(R\) is reduced, it follows that \(cd = 0\), which implies \(c^m = c^{m+1}d = c^m(cd) = 0\), and therefore \(c = 0\) (again by reducedness of \(R\)).

b. If two nonnegative powers of an element \(t\) coincide, say \(t^m = t^n\), with \(0 \leq m < n\), then \(t^m = t^{m+1}t^{n-m-1}\), so \(t \in \{0\} \cup R^*\) by item a.

c. If \(R\) is finite, then item b implies that \(R\) coincides with \(\{0\} \cup R^*\) and is therefore a field. \(\square\)

Next, we address the relationship between polynomials and their corresponding polynomial functions. If \(R\) is a finite ring, then the nonzero polynomial \(\prod_{r \in R}(x - r)\) is zero as a function on \(R\), so we may restrict our discussion to infinite rings. If \(D\) is an integral domain, then any nonzero polynomial \(f \in D[x]\) can only have finitely many roots; in particular, if \(D\) is infinite, then \(f\) does not vanish identically on \(R\) (as a polynomial function). For infinite reduced indecomposable rings, the set of roots of a nonzero polynomial may be infinite (take, for example, the reduced, indecomposable, characteristic zero ring \(R = \mathbb{Z}[t]/2t\mathbb{Z}[t]\), and consider the polynomial \(\tilde{f}(x^2 + x) \in R[x]\), vanishing at all integers), yet it can never be all of \(R\), as the following result shows\(^\blacklozenge\).

\(^\blacktriangle\) If \(I\) is merely infinite, then we can only prove that \(R\) has a non-finitely generated ideal, namely, that one generated by all the \(I\)-tuples \(2e_i\). See [Hod74, Section 3] for a comparison, in \(\mathbb{Z}\), of the various notions of Noetherianity.

\(^\blacklozenge\) Throughout the paper we will only work with this definition of local ring. For a deeper discussion about different notions of locality, see Subsection 6.7.

\(^\blacklozenge\) If \(A\) is a ring and \(p\) is a prime ideal in \(A\), then the localization \(R = A_p\) is local. In fact, if \(a \in A\) and \(s \in A \setminus p\) are such that \(a/s\) is not a unit in \(R\), then necessarily \(a \in p\). Now \((a/s) + 1 = (a + s)/s\), and since \(a \in p\) and \(s \notin p\), it follows that \(a + s \in A \setminus p\), which shows that \((a/s) + 1\) is invertible in \(R\).

\(^\blacklozenge\) See [Saw] for a general condition, and [vDB] for a second-order topological proof, which relies on different notions of Noetherianity (whose equivalence depends on DG) and connectedness of the prime spectrum (which depends on BPI).
Lemma 18. Let $R$ be a ring that is not a field. Then either there exists a unit $u$ with $u - 1 \not\in \{0\} \cup R^*$ or every element of $R$ is the sum of two nonunits.

Proof. If $R$ is local (see Example 15), then, as it is not a field, we may take $v \not\in \{0\} \cup R^*$ and $u = v + 1$ must be a unit, satisfying the first property. If $R$ is not local, then nonunits do not form an ideal; equivalently, they are not closed under sum. Hence, some unit $w$ must be the sum of two nonunits, say $x$ and $y$, and therefore for any $r \in R$ we have that $r = rw^{-1}w = (rw^{-1}x) + (rw^{-1}y)$ is the sum of two nonunits. □

4. LOGICAL POWERS IN REDUCED AND INDECOMPOSABLE POLYNOMIAL RINGS

In this section we study the properties of the logical powers (see Definition 1) of a polynomial for reduced and indecomposable coefficient rings.

Lemma 19. Let $R$ be a reduced ring, and let $p \in R[x]$ be nonconstant. Then no element of $\text{LPW}(p)$ can be infinitely divisible by $p$. If in addition the leading coefficient of $p$ is regular, then $\text{LPW}(p) \subseteq \text{POW}(p)$.

Proof. If $c \neq 0$ is the leading coefficient of $p$, then, since $R$ is reduced, the leading coefficient of $p^r$ is $c^r \neq 0$, for all $r \geq 1$. As $d := \text{deg}(p) > 0$, then for any given $f \in \text{LPW}(p)$ we may find $r \geq 1$ such that $\text{deg}(p^r) = rd > \text{deg}(f)$.

Now suppose by contradiction that $f$ is infinitely divisible by $p$, and hence divisible by $p^r$. Item a of Lemma 6 ensures then that $f$ is annihilated by some power of $c$, say $c^s$. Then, if we define $\ell = 1 + c^sp$, we find that $\ell$ divides $f$ (as $\ell f = f$) and that $p$ does not divide $\ell$ (otherwise it would be a nonconstant invertible polynomial, contradicting Proposition 7). Therefore, as $f \in \text{LPW}(p)$, we must have that $\ell$ is invertible. On the other hand, we have that $\ell = c^sp + 1$ is nonconstant, having coefficient $c^s + 1 \neq 0$ in degree $d > 0$, and so (again by Proposition 7) it cannot be invertible, a contradiction.
After proving that any \( f \in \text{LPOW}(p) \) cannot be infinitely divisible by \( p \), item a of Proposition 4 guarantees that \( f \) has the form \( up^n \), for some integer \( n \geq 1 \) and a unit \( u \) satisfying \( p - 1 \mid u - 1 \). As \( u \) is constant (Proposition 7) and \( p - 1 \) has positive degree and leading coefficient \( c \), then again by Lemma 6a we have that \( u - 1 \) is annihilated by a power of \( e \). Now, if \( e \) is regular, then \( u - 1 \) must be zero and \( f \in \text{POW}(p) \), proving the second assertion. \( \Box \)

**Lemma 20.** Let \( R \) be a ring that is reduced or indecomposable, and let \( p \in R[x] \).

a. If \( \text{LPOW}(p) \) contains an element that is multiple of its own square, then \( p \) must be invertible.

b. If \( p \in \text{LPOW}(p) \), then \( p \) is either invertible or irreducible.

**Proof.**

a. Let \( f \in \text{LPOW}(p) \) be a multiple of its own square, say \( f = f^2 \ell \). Then \( e = f \ell \) is idempotent and multiple of \( p \). We also have \( f = fe^n \) for all \( n \geq 1 \), that is, \( f \) is infinitely divisible by \( e \), and consequently \( f \) is infinitely divisible by \( p \); in particular, if \( R \) is reduced, then \( p \) is constant by Lemma 19. Moreover, defining \( h = 1 + (1 - e)x \) we have that \( f = fh \) because \((1 - e)f = 0 \).

If \( e = 0 \), then \( f = 0 \), and therefore \( p \) is a unit, by Proposition 3. If \( e = 1 \), then \( f \) is a unit because \( e = f \ell \), which implies that \( p \) is a unit as well (since \( p \) divides \( f \)). Since \( e = 0 \) or \( 1 \) in a indecomposable ring, this reasoning settles such case.

On the other hand, if \( R \) is reduced and \( e \neq 0,1 \), then \( h \) is not constant, hence a noninvertible divisor of \( f \) (by Proposition 7). Since \( f \in \text{LPOW}(p) \), then necessarily \( p \) divides \( h = 1 + x - ex \), so \( p \) divides \( 1 + x \) (because \( p \) divides \( e \)). As we already observed that \( p \) is constant in this case, it follows that \( p \) divides all the coefficients of \( 1 + x \), and again we conclude that \( p \) is invertible.

b. Since \( 0 \) is not a unit, it follows from Proposition 3 that \( 0 \notin \text{LPOW}(0) \), and so \( p \neq 0 \). If \( p \) is invertible, we are done; otherwise, if \( p = gh \), then \( p \) cannot divide both \( g \) and \( h \) (otherwise \( p \) would be a multiple of its square, contradicting item a above); since both \( g \) and \( h \) are divisors of \( p \) and \( p \in \text{LPOW}(p) \), it follows that one of \( g \) or \( h \) is a unit, which shows that \( p \) is irreducible. \( \Box \)

**Corollary 21.** If \( R \) is reduced and \( p \in R[x] \) is nonconstant, prime, and it has regular leading coefficient, then \( \text{LPOW}(p) = \text{POW}(p) \). In particular \( \text{LPOW}(x) = \text{POW}(x) \) when \( R \) is an integral domain.

**Proof.** The fact that \( p \) has regular leading coefficient implies that \( p \) is regular. Therefore we can apply Proposition 4d to obtain \( \text{POW}(p) \subseteq \text{LPOW}(p) \), and the reverse inclusion follows from Lemma 19. \( \Box \)

The requirement that \( \text{LPOW}(x) = \text{POW}(x) \), together with the technique shown in Theorem 28, could be at the base of a specific strategy for definability of integers in polynomial rings. However, Corollary 21 above only guarantees that \( \text{LPOW}(x) = \text{POW}(x) \) for integral domains, where the issue of definability of integers has already been worked out, in a Diophantine way ([Shl90, Theorem 5.1]). Fortunately, we now have all the tools to characterize the rings \( R \) such that, in the polynomial ring \( R[x] \), the equality \( \text{LPOW}(x) = \text{POW}(x) \) holds, obtaining in this way the converse of Proposition 5:

**Theorem 22.** Let \( R \) be a ring.

a. If \( R \) is reduced then \( \text{LPOW}(x) \subseteq \text{POW}(x) \).
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b. If \( R \) is reduced and indecomposable then \( \text{POW}(x) = \text{LPOW}(x) \).

Proof.

a. This follows immediately from Lemma 19.

b. For every \( r \geq 1 \) we have that \( x \) divides \( x^r \) and \( x - 1 \) divides \( x^r - 1 \). If \( x^r = gh \), then from Lemma 6b we obtain that \( x^r \) divides \( g_0 h \) and \( g_0' = g_0^{r+1} h \), hence \( g_0 \in \{0\} \cup R^* \) by Proposition 16a. If \( g_0 = 0 \), then \( x \) divides \( g \). Otherwise, \( x^r \) divides \( g_0 (g^r h) = h \), say \( h = x^r h \), hence \( x^r = g h = x' g \); canceling out \( x^r \), we conclude that \( g \) is invertible. This shows that \( x^r \in \text{LPOW}(x) \) for all \( r \geq 1 \), that is, \( \text{POW}(x) \subseteq \text{LPOW}(x) \), and the reverse inclusion follows from item a. \( \square \)

Next, we try to distinguish by a logical formula some elements of \( R[x] \) whose logical powers coincide with their positive powers. To this aim, it is necessary to exclude elements exhibiting logical powers infinitely divisible by them. One way of doing so, which will be presented below, relies on producing a first-order equivalent of the concept of “powers of two given elements have the same exponent”, and exploits and extends the fact that, under reasonable conditions, for polynomials \( p \) and \( q \) and \( \exp \) and \( \divides \) we have that \( p - q \) divides \( p^n - q^n \) if and only if \( m = n \).

Theorem 23. For a ring \( R \), let \( T \) be the set of all irreducible elements \( p \) in \( R[x] \) such that \( ph \in \text{LPOW}(p) \) whenever \( h \in \text{LPOW}(p) \). Moreover, let \( U \) be the set of all the elements \( p \) in \( T \) satisfying:

- For every \( q \in T \) and every \( f \in \text{LPOW}(p) \), there exists \( g \in \text{LPOW}(q) \) such that \( p - q \mid f - g \);
- If \( a \in R[x]^* \) satisfies \( p - 1 \mid a - 1 \), then \( a = 1 \).

Then the sets \( T \) and \( U \) are first-order definable, and \( \text{POW}(q) \subseteq \text{LPOW}(q) \) for every \( q \in T \). In addition, if \( R \) is reduced and indecomposable, then the following holds:

a. \( U \) is nonempty; more specifically, \( x \in U \).

b. \( \text{LPOW}(p) = \text{POW}(p) \) for every \( p \in U \).

Proof. Clearly all the clauses for membership in \( T \) are first-order definable, and the same happens to those of \( U \) (once we know that \( T \) is definable). If \( q \in T \), then \( q \) is irreducible, so \( q \in \text{LPOW}(q) \) by item c of Proposition 4, and if we assume inductively that \( m \geq 1 \) satisfies \( h = q^m \in \text{LPOW}(q) \), then \( qh = q^{m+1} \in \text{LPOW}(q) \). This shows that \( \text{POW}(q) \subseteq \text{LPOW}(q) \).

Now suppose that \( R \) is reduced and indecomposable.

a. First, we prove that \( x \in T \). Since \( R \) is indecomposable, then \( x \) is irreducible in \( R[x] \) (Proposition 10), and since \( R \) is also reduced, then \( \text{LPOW}(x) = \text{POW}(x) \) by Theorem 22b. Thus, if \( h \in \text{LPOW}(x) \), then \( h = x^k \) for some \( k \geq 1 \), hence \( xh = x^{k+1} \in \text{POW}(x) = \text{LPOW}(x) \), and \( x \in T \), as desired.

As for the remaining conditions, given \( q \in T \) and \( f \in \text{LPOW}(x) \), we want to find \( g \in \text{LPOW}(q) \) such that \( x - q \) divides \( f - g \). Since \( \text{LPOW}(x) = \text{POW}(x) \), then \( f = x^n \) for some \( n \geq 1 \). Moreover, we already know that \( q \in T \) implies \( \text{POW}(q) \subseteq \text{LPOW}(q) \), and so the element \( g := q^n \) satisfies \( g \in \text{LPOW}(q) \), and clearly \( x - q \) divides \( x^n - q^n = f - g \). Finally, if \( a \in R[x]^* \) is such that \( x - 1 \) divides \( a - 1 \), then \( a = 1 \) by Proposition 7. Writing \( a - 1 = (x - 1)1 \) and evaluating at \( x = 1 \), we obtain \( a = 1 \), and thus \( x \in U \).

b. If \( p \in U \), then \( p \in T \), and consequently \( \text{POW}(p) \subseteq \text{LPOW}(p) \). For the reverse inclusion, let \( f \in \text{LPOW}(p) \) and set \( q := x \in T \). The first condition in the definition of \( U \) guarantees the existence of an element \( g \in \text{LPOW}(q) = \text{LPOW}(x) = \text{POW}(x) \) such that \( p - x \mid f - g \), say \( g = x^n \), with \( n \geq 1 \).

If \( f \) were infinitely divisible by \( p \), then \( p \) would be constant by Lemma 19. Evaluating at \( p \) and using that \( x - p \) divides \( x^n - f \), we conclude that
\[ f(p) = p^n. \] Since \( f \) is infinitely divisible by \( p \), then there is an \( h \) such that \( f = p^{n+1}h \); in particular we have \( f(p) = p^{n+1}h(p) \), so \( p^{n+1} \) divides \( p^n \). Proposition 16a would imply then that \( p \in \{0\} \cup R^* \), which is an absurd since \( p \) is irreducible.

The contradiction above, together with item a of Proposition 4, shows that \( f = ap^k \) for some \( k \geq 1 \) and some \( a \in R[x]^* \) with \( p - 1 \mid a - 1 \); the second condition of the definition of \( U \) forces \( a = 1 \) and, consequently, \( f = p^k \in \text{POW}(p) \). \(\square\)

**Remark 24.** Let \( R \) be any ring. If \( \theta \) is a ring automorphism of \( R[x] \), then \( \theta \) preserves the logical structure, and therefore the definable sets \( T \) and \( U \) of Theorem 23 are invariant under \( \theta \), that is, \( \theta(T) = T \) and \( \theta(U) = U \). If \( \nu \in R^* \) and \( r \in R \) are given, the mapping \( \theta : R[x] \to R[x] \) given by \( \theta(f) = f(x \cdot \nu + r) \) is a ring automorphism (being \( g \mapsto g([v^{-1}(x - r)] \) its inverse. In particular, if \( R \) is reduced and indecomposable, then \( x \in U \) by Theorem 23a, and therefore we have \( \nu x + r \in U \) in this case.

The following result ensures definability of sets of powers of constants, using the constants as parameters, for reduced indecomposable coefficient rings that are not fields:

**Theorem 25.** Let \( R \) be a reduced indecomposable ring that is not a field (this implies that \( R \) is infinite, by Proposition 16c), let \( a \in R \) be a constant element of \( R[x] \), and let \( f \in R[x] \). Then \( f \in \text{POW}(a) \) if, and only if, for all \( p, q \in U \) (where \( U \) is the set defined in the statement of Theorem 23), there exist \( y \in \text{POW}(p) \) and \( z \in \text{POW}(q) \), such that:

- \( p - a \mid y - f; \)
- \( q - a \mid z - f; \)
- \( p - q \mid y - z. \)

**Proof.** If \( f = a^n \), with \( n \in \mathbb{Z}^+ \), then for any \( p, q \in U \), by taking \( y = p^n \) and \( z = q^n \), one clearly has \( p - a \mid y - f, q - a \mid z - f \) and \( p - q \mid y - z. \). Conversely, let \( f \in R[x] \) satisfy the properties listed. We will prove that \( f \) is constant as a function on \( R \).

Given any two \( \rho, \sigma \in R \) and any \( \nu \in R^* \), define the polynomials \( p = x - \rho + a \) and \( q = \nu x - \sigma + a \) and observe that both \( p \) and \( q \) lie in \( U \) (see Remark 24). By the properties listed in the hypothesis, there exist elements \( y = p^m = (x - \rho + a)^m \) and \( z = q^n = (\nu x - \sigma + a)^n \), where \( m \) and \( n \) are suitable positive integers depending on \( p \) and \( q \) (and, of course, on \( a \)), satisfying:

- \( x - \rho + a - a \mid (x - \rho + a)^m - f; \)
- \( \nu x - \sigma + a - a \mid (\nu x - \sigma + a)^n - f; \)
- \( (x - \rho + a) - (\nu x - \sigma + a) \mid p^m - q^n; \)

which yields:

- \( f(\rho) = a^m; \)
- \( f(\nu^{-1}a) = a^n; \)
- \( (1 - \nu)x + (\sigma - \rho) \mid (x - \rho + a)^m - (\nu x - \sigma + a)^n. \)

In particular, for \( \rho = 0, \sigma = 0 \) and \( \nu = 1 \), we conclude that \( f(0) = a^{m_0} \), for some \( m_0 \). If \( m \neq n \), then \( (x - \rho + a)^m - (\nu x - \sigma + a)^n \) has invertible leading coefficient, being \( 1 \) or \(-\nu^n \), and therefore the last condition can only be satisfied if the leading coefficient of \((1 - \nu)x + (\sigma - \rho)\) is also invertible. If this does not happen, we must have \( m = n \) and therefore \( f(p) = a^m = a^n = f(\nu^{-1}a) \).

The above reasoning amounts to saying that, given any \( \rho, \sigma \in R \) and any \( \nu \in R^* \), if any of the following conditions holds:

- (a) \( \nu \neq 1 \) and \( \nu - 1 \not\in R^*; \)
- (b) \( \nu = 1 \) and \( \rho - \sigma \not\in R^*; \)
then \( f(\rho) = f(v^{-1} \sigma) \).

Now take any \( r \in R \): we want to prove that \( f(r) = f(0) \). By Lemma 18, either there exists a unit \( u \) with \( u - 1 \notin \{0\} \cup R^* \) or any element of \( R \) is the sum of two nonunits. In the first case, condition (a) is satisfied for \( v = u \); taking \( \rho = r \) and \( \sigma = 0 \) we conclude that \( f(r) = f(\rho) = f(v^{-1} \sigma) = f(0) \). In the second case, there are two nonunits \( s \) and \( t \) such that \( r = s + t \). Set \( v = 1 \). Considering \( \rho = r \) and \( \sigma = s \), we can use (b) to prove that \( f(r) = f(1^{-1} \cdot s) = f(s) \). Analogously, considering \( \rho = s \) and \( \sigma = 0 \), we can use (b) again to prove that \( f(s) = f(1^{-1} \cdot 0) = f(0) \). Thus, \( f(r) = f(s) = f(0) \). We have proven that, in both cases, \( f(r) = f(0) \). As \( r \) was arbitrarily taken, it follows that \( f \) is constant as a function on \( R \) and therefore, by Theorem 17 (recall that \( R \) is infinite), \( f = f(0) = a^{m_0} \) and thus it belongs to \( \text{POW}(a) \). □

Remark 26. We have that the set \( U \) is first-order definable (Theorem 23); moreover, the sets of powers of elements of \( U \) coincide with their respective sets of logical powers (Theorem 23b), and therefore they are definable, using the respective elements as parameters; see formula (2.1). Since the condition in the statement of Theorem 25 involves quantification over the elements of \( U \), we get that the set of positive powers of any constant \( a \in R \) is definable in \( R[x] \) using \( a \) as parameter. In other words, we proved the following:

Corollary 27. Let \( R \) be a reduced indecomposable ring that is not a field. Then there is a two-variable first-order formula \( \Phi(\cdot, \cdot) \) such that, for each \( a \in R \), the formula \( \Phi(\cdot, a) \) defines the set \( \text{POW}(a) \) in \( R[x] \).

5. The main result: definability of \( \mathbb{Z} \)

In this section we prove the definability of integers in \( R[x] \), where \( R \) is a reduced and indecomposable polynomial ring. Here two cases are treated separately, according to whether the ring of constants is a field or not; obviously, in both cases, it will be sufficient to define just the positive integers.

In what follows we provide a technique that, given a formula defining a set of powers of a fixed element \( p \) such that \( p - 1 \) is regular, yields a formula that defines the set of the corresponding (integer) exponents, provided the latter is already contained in a suitable definable set. Roughly said, this corresponds to a “first-order” technique for extracting exponents from sets of powers.

Theorem 28. Let \( S \) be a ring, \( p \in S \) and \( B \subseteq \text{POW}(p) \). Define \( \log_p B \) to be the set of exponents of those powers of \( p \) that belong to \( B \). Suppose that the following conditions hold:

1. \( p - 1 \) is regular;
2. There is a subset \( A \) of \( S \) containing all the positive integers, such that \( A \) is definable by a first-order formula (possibly depending on \( p \)), and such that \( p - 1 \) divides no nonzero element of \( \mathbb{Z}^* \) − \( A \) (differences between a positive integer and an element of \( A \));
3. The set \( B \) is definable by a first-order formula (possibly depending on \( p \)).

Then the set \( \log_p B \) is definable by a formula \( L_p \). Moreover, if both \( A \) and \( B \) are definable using \( p \) as parameter (that is, as \( \alpha(\cdot, p) \) and \( \psi(\cdot, p) \), respectively, being \( \alpha \) and \( \psi \) two-variable formulas), then \( \log_p B \) is definable using \( p \) as parameter as well.

Proof. Let \( A \) and \( B \) be defined, respectively, by formulas \( \alpha_p \) and \( \psi_p \) (the dependency on \( p \) may or may not occur). Consider the formula

\[
L_p(t) :\quad \alpha_p(t) \land \exists y \exists w \left[ \psi_p(y) \land y - 1 = w(p - 1) \land p - 1 \mid w - t \right].
\]

We prove that \( L_p \) defines the set \( \log_p B \); to this end, we use the fact that the element \((p^n - 1)/(p - 1)\) is congruent to \( n \) modulo \( p - 1 \), which, together with the hypotheses,
will allow us to recover the value \( n \) from the expression \( p^n - 1 \) in a definable way. Note that if \( \alpha_p = a(\cdot, p) \) and \( \psi_p = \psi(\cdot, p) \), then \( L_p = L(\cdot, p) \), for a two-variable formula \( L \), proving in this way the second assertion of the statement.

Observe that \( L_p(t) \) holds if and only if both \( t \in A \) and there exist a positive integer \( n \) and an element \( w \in S \) satisfying \( y = p^n \in B \) (recall that \( B \subseteq \text{POW}(p) \) by hypothesis), \( y - 1 = w(p - 1) \) and \( p - 1 \) divides \( w - t \).

Let \( w_n = p^{n-1} + p^{n-2} + \cdots + p + 1 \), so \( w_n(p - 1) = y - 1 \). By regularity of \( p - 1 \) (condition (1)) the only possible value for \( w \) will be \( w_n \), so the conditions on \( t \) can be restated as follows:

(a) \( t \in A \), and there is a positive integer \( n \) such that

(b) \( n \in \log_p B \);

(c) \( p - 1 \) divides \( w_n - t \).

Now \( w_1 = 1 \) and \( w_n = \left[ \sum_{k=0}^{n-2} (n - k)p^k \right](p - 1) + n \) for \( n \geq 2 \). Therefore \( p - 1 \) divides \( w_n - n \), and therefore (c) can be replaced by

(c') \( p - 1 \) divides \( (w_n - t) - (w_n - n) = n - t \).

Given any positive integer \( n \), if \( t \) satisfies (a) and (c'), then condition (2) implies \( t = n \); conversely, since \( \mathbb{Z}^+ \subseteq A \), then the element \( t = n \) trivially satisfies both (a) and (c'). This argument shows that, in the presence of (a), we can replace (c') with

(c) \( t = n \).

Consequently, given \( t \in S \), the existence of a positive integer \( n \) such that (a), (b) and (c) hold is equivalent to having \( t \in \log_p B \).

\( \square \)

**Remark 29.** With notations as in Theorem 28, assume that \( p \in S \) satisfies condition (2) for \( A = S \). Since in this case the set of differences \( \mathbb{Z}^+ - A \) is equal to \( S \) and \( p - 1 | p - 1 \), it follows from condition (2) that \( p - 1 = 0 \); but condition (1) \( p - 1 \) is regular \( \) implies \( p - 1 \neq 0 \). Thus, if \( A = S \), then conditions (1) and (2) cannot be simultaneously satisfied.

Assume then that \( A = S \), together with conditions (1) and (3), that is, assume that \( p - 1 \) is regular and that there is a definable subset \( B \) of \( \text{POW}(p) \) (its definability possibly depending on \( p \)). Then an element \( t \in S \) satisfies formula (5.1) precisely when there exist elements \( y \in B \) and \( w \in S \) such that \( y \in B \), \( y - 1 = w(p - 1) \) and \( p - 1 \) divides \( w - t \). In this case, by reasoning as in the proof of Theorem 28, we obtain that the element \( y \) of the form \( p^n \), for a positive integer \( n \) such that \( p - 1 \) divides \( n - t \). Because condition (2) does not hold in this case, we cannot conclude the equality \( t = n \). In the proof of Theorem 35 below we circumvent this issue by strengthening formula (5.1) in a suitable way.

The next step is to define subsets of the positive integers in the ambient ring \( S \), without the dependence on the parameter \( p \). In order to achieve this, we must let \( p \) vary in a definable subset of \( S \) and impose the existence of one family of suitable subsets containing \( \mathbb{Z}^+ \) and one family of subsets of the sets \( \text{POW}(p) \), both defined by two-variable formulas using \( p \) as a parameter. More specifically, we have:

**Corollary 30.** Let \( S \) be a ring, let \( P \) be a definable subset of \( S \), and let \( \alpha \) and \( \psi \) be two-variable formulas. Assume that for each \( p \in P \) the following conditions hold:

(1) \( p - 1 \) is regular.

(2) The subset \( A_p \), of \( S \) defined by \( \alpha(\cdot, p) \), contains all the positive integers; moreover, \( p - 1 \) divides no nonzero element of \( \mathbb{Z}^+ - A_p \) (differences between a positive integer and an element of \( A_p \)).

(3) The subset \( B_p \) defined by \( \psi(\cdot, p) \), satisfies \( B_p \subseteq \text{POW}(p) \).

Then the sets \( \cup_{p \in P} \log_p B_p \) and \( \cap_{p \in P} \log_p B_p \) are first-order definable without parameters.
Proof. Let \( \pi \) be a one-variable formula defining \( P \), and let \( L_p \) be as in Equation (5.1). The proof of Theorem 28 shows that \( L_p \) is indeed of the form \( L(\cdot, p) \), for a two-variable formula \( L \), and that for each \( p \in P \) the formula \( L(\cdot, p) \) defines the subset \( \log_p B_p \). Therefore the one-variable formula \( \exists s \left[ \pi(s) \land L(\cdot, s) \right] \) defines \( \cup_{p \in P} \log_p B_p \); similarly, the one-variable formula \( \forall s \left[ \pi(s) \implies L(\cdot, s) \right] \) defines \( \cap_{p \in P} \log_p B_p \). \( \square \)

As a direct application of Corollary 30, we obtain the definability of integers in \( R[x] \) when \( R \) is a reduced and indecomposable ring, provided the definability of a set between \( \mathbb{Z}^+ \) and \( R \). This improves the result of [Rob51, §2], which assumes that \( R \) is an integral domain which is itself first-order definable in the ring \( R[x] \).

**Theorem 31.** Let \( R \) be a reduced, indecomposable ring and suppose there is a definable set \( A \) with \( \mathbb{Z}^+ \subseteq A \subseteq R \). Then \( \mathbb{Z}^+ \) is first-order definable in \( R[x] \).

**Proof.** The idea of the proof is to take the definable subsets \( A_p \) appearing in the statement of Corollary 30 all equal to \( A \); more specifically, let \( \alpha_A(\cdot) \) be a one-variable formula defining \( A \), and let \( \alpha(f, p) = \alpha_A(f) \) for each \( f, p \in R[x] \).

Let \( U \) be the definable set of Theorem 23, and let \( P \) be the set of elements \( p \in U \) such that \( p - 1 \) is regular and that, for any \( a, b \in A \), it is the case that \( p - 1 \) divides \( a - b \) only if \( a = b \). Then \( P \) is definable; moreover, \( x \in P \) because \( x \in U \), \( x - 1 \) is regular, and if \( a, b \in A \) satisfy \( x - 1 = a - b \), then \( a = b \) because \( a - b \in R \).

Finally, let \( \psi \) be as in Equation (2.1). Then for any \( p \in R[x] \) we have that the set \( B_p \) defined by \( \psi(\cdot, p) \) is precisely LPOW\( p \), and the set \( A_p \) defined by \( \alpha(\cdot, p) \) is precisely \( A \). Since LPOW\( p \) = POW\( p \) for all \( p \in P \) (by Theorem 23b, together with the fact that \( P \subseteq U \)), we are in the hypotheses of Corollary 30, and so we conclude that the set \( D = \cup_{p \in P} \log_p B_p \) is definable in \( R[x] \); since \( \log_p B_p = \mathbb{Z}^+ \) for all \( p \in P \) and \( P \) is nonempty, it follows that \( D = \mathbb{Z}^+ \), that is, \( \mathbb{Z}^+ \) is definable in \( R[x] \). \( \square \)

**Corollary 32.** If \( R \) is a reduced indecomposable ring such that every nonzero integer is invertible, then \( \mathbb{Z}^+ \) is definable in \( R[x] \); in particular, this result holds when \( R \) is a field or \( R \) has positive characteristic.

**Proof.** For the first assertion, since \( R \) is reduced, it follows that \( R[x]^+ = R^+ \) by Proposition 7, so we may take \( A \) to be the definable set \( R[x]^+ \) in Theorem 31. For the second assertion, if \( \text{char}(R) = n > 0 \), then its prime subring \( B = \mathbb{Z}/n\mathbb{Z} \) is also reduced and indecomposable. If \( n = ab \), with \( \gcd(a, b) = 1 \), then \( 1 = ra - sb \) for some integers \( r \) and \( s \), and so \( ra(ra - 1) = rsn = 0 \) in \( B \). By indecomposability we have \( ra = 0 \) or \( ra - 1 = sb = 0 \) in \( B \), that is \( n = ab \) divides \( ra \) or \( sb \), hence \( b \mid r \) or \( a \mid s \); but \( b \mid r \) implies \( b \mid ra - sb = 1 \), and similarly \( a \mid s \) implies \( a \mid 1 \). This shows that \( n \) cannot be divisible by two distinct primes, so \( n = p^r \), with \( r \geq 1 \) and \( p \) being a prime. Since \( B \) is reduced, then the nilpotent element \( p \) must be zero in \( B \), and so \( r = 1 \). Consequently \( R \) has prime characteristic, and therefore every nonzero integer is invertible, so we are in the conditions of the first assertion. \( \square \)

**Remark 33.** In any ring of positive characteristic, say \( n \), the integers are trivially definable, via the formula

\[
\gamma_n(t) : \bigvee_{i=1}^{n} \left( t = 1 + \cdots + 1 \right),
\]

which depends on \( n \) in a cumbersome way. The remarkable fact is that, in the particular case of reduced and indecomposable polynomial rings of positive characteristic, we are able to use a uniform formula (not depending on the characteristic),
namely, that given by Corollary 32, which also covers the case of polynomial rings in one variable over any field.

Now we treat the most general case, namely, that of nonfields of characteristic zero. By using definability of powers of constants with the same constants as parameters (Corollary 27), we can now slightly modify our argument from Theorem 28 and Corollary 30 in order to get round the requirement of a definable set of constants in $R[x]$ for defining rational integers.

**Theorem 35.** Let $R$ be a reduced and indecomposable ring and let $U$ be the set defined in Theorem 23. Assume that $R$ contains an element $a$ with all distinct powers (so in particular $R$ is infinite), and let $t \in R[x]$ and $k$ be a positive integer. Suppose that, for all $p \in U$ such that $p - 1$ is regular, there exist a positive integer $n$ and a polynomial $w \in R[x]$, both possibly depending on $p$, such that:

(a) $p^n - 1 = w(p - 1)$;
(b) $p - 1 | w - t$;
(c) $p - a | p^n - a^k$.

Then $t = k$.

**Proof.** We will show that $t$ defines a constant polynomial function and then use Theorem 17 to show that $t$ is a constant polynomial. Fix $r \in R$: our hypothesis applies to the polynomial $p := x - r + 1$, as $p \in U$ (by Remark 24) and $p - 1$ is regular, resulting in the existence of a positive integer $n = n_r$ and an element $w = w_r \in R[x]$ (both depending on $r$) satisfying (a)-(c). Furthermore, as $p \in U$, we have that $B := LPOW(p)$ is a definable subset of $POW(p)$ (it actually coincides with $POW(p)$ by Theorem 23b). This condition, together with (a) and (b), and the fact that $p - 1$ is regular imply, according to Remark 29, that $x - r = p - 1$ divides $n_r - t$; thus, we have $t(r) = n_r$ for each $r \in R$.

On the other hand, $p - a | p^{n_r} - a^{n_r}$, together with condition (c), implies that $p - a = x - (r + a - 1)$ divides the constant polynomial $a^{n_r} - a^k$, which, after evaluating at $r + a - 1$, forces $a^{n_r} = a^k$, which in turn implies $n_r = k$ by the initial hypothesis that powers of $a$ are distinct.

We have proven that $t(r) = n_r$ and $n_r = k$, for all $r \in R$. Since $k$, unlike $n_r$, does not depend on $r$, $t(r) = n_r = k$ means that $t$ induces a constant polynomial function on $R$. As $R$ is reduced, indecomposable and infinite we conclude, by Theorem 17, that $t = k \in \mathbb{Z}^+$. □

**Theorem 36.** Let $R$ be a reduced and indecomposable characteristic zero ring which is not a field. Then $\mathbb{Z}^+$ is first-order definable in $R[x]$.

**Proof.** We will prove, using Lemma 34, that $\mathbb{Z}^+$ is defined by the formula

$$\theta(t) : \exists z \in \text{POW}(2) \land \forall p \left( p \in P \Rightarrow \varphi(t, 2, z, p) \right),$$

where $P$ is the definable set $\{ p \in U : p - 1 \text{ is regular} \}$ (as we defined in Theorem 23) and

$$\varphi(t, a, z, p) : \exists y \forall w \left( y \in \text{LPOW}(p) \land y - 1 = w(p - 1) \land w - t \land p - a \mid y - z \right).$$

Note that $\varphi(t, a, z, p)$ can be regarded as an enhanced version of formula (5.1) for the particular case $A = R[x]$ (see Remark 29), with the additional condition $p - a \mid y - z$ (under quantification). On the other hand we have, by Corollary 27, that for any $a \in R$ the set $\text{POW}(a)$ is defined by $\Phi(\cdot, a)$, being $\Phi$ a two-variable formula. Since $\{2\}$ is definable, it follows that the set $\text{POW}(2)$ is definable (without parameters), and therefore $\theta$ is a genuine one-variable formula without parameters. As $R$ has characteristic zero, all powers of 2 are distinct, and therefore we are in the setting of Lemma 34, taking $a = 2$. 
If \( \theta(t) \) holds, then there exists a positive integer \( k \) such that, for each \( p \in P \), the formula \( \varphi(t, 2, 2^k, p) \) holds. Therefore, for each such \( p \) there exist an element \( y \in \text{LPOW}(p) = \text{POW}(p) \), say \( y = p^n \), with \( n \geq 1 \), and an element \( w \in S \), both depending on \( p \), such that \( p^n - 1 = w(p - 1) \), \( p - 1 \mid w - t \) and \( p - 2 \mid p^n - 2 \). These conditions on the integer \( k \), the element \( a = 2 \), and the triplets \((p, n, w)\), with \( p \in P \), are precisely the hypotheses of Lemma 34, and thus we may conclude that \( t = k \).

Conversely, if \( t \in \mathbb{Z}^+ \), say \( t = n \), then it is easy to see that \( \theta(t) \) holds for the choice \( z = 2^n \), namely, for each \( p \in P \) the formula \( \varphi(t, 2, z, p) \) holds for \( y = p^n \) and \( w = p^{n-1} + p^{n-2} + \cdots + p + 1 \). \( \square \)

6. Appendix: Miscellaneous Considerations

In this section we place additional findings concerning our work, which are not strictly necessary to prove the main result but have appeared as side-results, by-products and optimal generalizations and may be useful in future attempts at extending our claims to wider class of rings or in more general contexts.

6.1. A generalization of the uniform exponent-extracting technique and an application in the noncommutative context. In this subsection, we will give another version of Theorem 28 and Corollary 30 whose claim is presented in terms of maps between sets of first-order formulas, writing out complex formulas from simpler ones.

**Theorem 36.** Let \( F_1 \) (resp. \( F_2 \)) denote the set of one-variable (resp. two-variable) first-order formulas in the language of unital rings. Consider the function \( L : F_1 \times F_2 \rightarrow F_2 \) given by

\[
[L(\alpha, \psi)](t, s) = \alpha(t) \land \exists y \exists w \left( \psi(y, s) \land y - 1 = w(s - 1) \land s - 1 \mid w - t \right),
\]

where \( \mid \) denotes right divisibility. Let \( S \) be a unital (not necessarily commutative) ring and let \( P \) be a subset of \( S \) satisfying the following conditions:

1. \( p - 1 \) is a right cancellable element of \( S \) for each \( p \in P \).
2. There is a function \( A : P \rightarrow F_1 \) and a family \( (s_p)_{p \in P} \) of functions \( s_p : \mathbb{Z}^+ \rightarrow A_p \),
   where \( A_p \) is the set defined by \( A(p)(\cdot) \), satisfying the following: for each \( n \in \mathbb{Z}^+ \), the element \( s_p(n) \) is the only element in \( A_p \) which is right congruent to \( n \) modulo \( p - 1 \).
3. There is a function \( B : P \rightarrow F_2 \) such that for all \( p \in P \), one has that \( B(p)(\cdot, p) \) defines a subset \( B_p \) of \( \text{POW}(p) \).

Then, considering \( L \circ (\mathcal{A}, \mathcal{B}) : P \rightarrow F_2 \), we have that, for all \( p \in P \), the formula \( L(\mathcal{A}(p), \mathcal{B}(p))(\cdot, p) \) defines \( s_p(\log_p B_p) \), where \( \log_p(\cdot) \) is defined on subsets of \( \text{POW}(p) \) as in Theorem 28.

**Proof.** Notice that the conditions on the functions defined on \( P \) correspond, elementwise, with the conditions of Theorem 28 on its single elements, with the exception that \( s_p(n) \) does not necessarily coincide with \( n \). Also, for each \( p \in P \) fixed, replacing \( A(p) \) and \( B(p) \) by \( a_p \) and \( \varphi_p \), respectively, the formula \( L(\mathcal{A}(p), \mathcal{B}(p))(\cdot, p) \) corresponds with \( L(a_p, \psi(\cdot, p)) \) in the proof of Theorem 28, and therefore we may proceed likewise to conclude that \( t \) satisfies the formula if and only if

1. \( t \in A_p \),
2. \( n \in \log_p B_p \),
3. \( t = s_p(n) \)

and, finally, that conditions (a), (b) and (c) altogether are equivalent to saying \( t \in s_p(\log_p B_p) \), proving the claim. The reader may notice that there is no problem
replacing regularity by right-cancellability, as simplification by \( p - 1 \) is performed by the right side. \( \square \)

The result above can be interpreted as a statement on a diagram of trivial set bundles on \( S \). More specifically, if \( L \) is defined as in the theorem, \( \mathcal{P}_D(S) \) denotes the set of definable subsets of \( S \) and we define the function “truth set” from \( F_1 \) to \( \mathcal{P}_D(S) \) as \( \Theta(\alpha(s)) \coloneqq \{ s \in S : \alpha(s) \text{ is true} \} \) and \( \Psi : F_2 \times S \to \mathcal{P}_D(S) \) as \( \Psi(\psi(\cdot, \cdot), s) \coloneqq \Theta(\psi(\cdot, s)) \), then we may consider the following diagram of trivial set bundles on \( S \):

\[
\begin{array}{c}
F_1 \times F_2 \times S \xrightarrow{L \times \text{id}_S} F_2 \times S \\
\mathcal{P}_D(S) \times \mathcal{P}_D(S) \times S \xrightarrow{-\cdots-} \mathcal{P}_D(S) \times S
\end{array}
\]

where downward arrows are given by \( \Theta \times \Psi \) and \( \Psi \), respectively. Then the theorem amounts to saying that, if we restrict the diagram to \( P \) and to the section \( (\mathcal{A}, \mathcal{B}) \), that is, if we compose from the top left with \((\mathcal{A}, \mathcal{B}, \subseteq) : P \to F_1 \times F_2 \times S\) sending \( p \) to \((\mathcal{A}, \mathcal{B}, p)\), we may take the dashed arrow to be \((X, Y, p) \mapsto s_p(\log_p(Y))\), making the restricted diagram commute, provided that, for all \( p \in P, \mathcal{B}(p) \subseteq \text{POW}(p) \) and the quotient map \( S \to S/(p - 1) \) coequalizes the inclusion \( \mathbb{Z}^+ \subseteq S \) and \( j_p \circ s_p \), where \( j_p \) is the inclusion \( A_p \subseteq S \). The last condition is equivalent to saying that the lower triangle in the diagram below commutes:

\[
\begin{array}{c}
\mathcal{P}_D(S) \times \mathcal{P}_D(S) \times P \xrightarrow{(s_n(l) \circ \log_n(l)) \circ \pi_2, \pi_3} \mathcal{P}_D(S) \times P \\
\mathcal{P}_D(S) \times \mathcal{P}_D(\text{POW}(p)) \times P \\
\mathcal{P}_D(S) \times P
\end{array}
\]

Corollary 37. With notation and hypotheses as in Theorem 36, the following results hold in each of the following special situations:

1) If for a given \( p \in P \) we have that \( \mathbb{Z}^+ \subseteq A_p \) and \( p - 1 \) is right cancellable and it divides no nonzero element of \( \mathbb{Z}^+ - A_p \) (differences between a positive integer and an element of \( A_p \)), then we may replace \( s_p \) by the inclusion \( \mathbb{Z}^+ \subseteq A_p \). The set \( \log_p B_p \subseteq \mathbb{Z}^+ \) may therefore be defined by the first-order formula

\[
L(\mathcal{A}(p), \mathcal{B}(p)) \mid (\cdot, p).
\]

2) If \( \mathcal{A} \) and \( \mathcal{B} \) are constant functions and \( P \) is first-order definable, then the sets \( \sqcup_{p \in P} s_p(\log_p B_p) \) and \( \cap_{p \in P} s_p(\log_p B_p) \) are first-order definable.

Proof. 1) follows immediately from Theorem 36 whereas for 2) one can proceed analogously to Corollary 30. \( \square \)

As an application of Corollary 37 we provide an example of a noncommutative ring in which rational integers are definable with parameters.
Let $D$ be an integral domain, and let $q \in D \setminus \{0\}$. The quantum plane over $D$ with parameter $q$, denoted by $S = D_q[x, y]$, is defined as the quotient of the free noncommutative $D$-algebra over two generators $x$ and $y$, by the unique relation $yx = qxy$. Alternatively, the ring $S$ is the free $D$-module generated by the monomials $x^my^n$, with $m, n \in \mathbb{N}$, so their elements are of the form $f = \sum_{(m, n) \in \mathbb{N}^2} f_{(m, n)}x^my^n$, with $f_{(m, n)} \in D$ and $f_{(m, n)} \neq 0$ for finitely many pairs $(m, n)$. Multiplication is given by $(x^m y^n)(x^r y^s) = q^{rn} x^{m+r} y^{n+s}$, and extended by $D$-linearity. (see [Kas95, Chapter IV] for details on the case in which $D$ is a field.)

We claim that $S$ is definable in $(S, 0, 1, +, \cdot, x)$, provided that every nonzero integer is invertible in $S$: for example when $D$ is a field or it has positive characteristic (because in the latter case the characteristic is a prime number, as $D = Q[t]$). To our aim, we first endow $\mathbb{N} \times \mathbb{N}$ with the so-called degree lexicographic order: $(m, n) < (m', n')$ if either $m + n < m' + n'$, or both $m + n = m' + n'$ and $m < m'$. We observe that $<$ defines a well-ordering of $\mathbb{N} \times \mathbb{N}$, and it satisfies the property

$$(\ast) \quad (m', n') + (r', s') < (m, n) + (r, s), \text{ if } (m', n') < (m, n) \text{ and } (r', s') < (r, s).$$

For $f \in S \setminus \{0\}$, let $\max(f)$ (resp. $\min(f)$) be the maximum (resp. the minimum) pair $(m, n)$, with respect to $<$, with $f_{(m, n)} \neq 0$. Given $g, h \in S \setminus \{0\}$, if $(m', n'), (r', s')$ are such that $g_{(m', n')} \neq 0$ and $h_{(r', s')} \neq 0$, then $(m', n') < \max(g)$ and $(r', s') < \max(h)$, so by $(\ast)$ we have $(m', n') + (r', s') < \max(g) + \max(h)$, and equality occurs precisely when $(m', n') = \max(g)$ and $(r', s') = \max(h)$.

The above reasoning implies that, for $\vec{b} \in \mathbb{N} \times \mathbb{N}$, all summands of

$$(gh)^\vec{b} = \sum_{(m', n') + (r', s') = \vec{b}} q^{rn'} g_{(m', n')} h_{(r', s')}$$

vanish if $\vec{b} > \max(g) + \max(h)$, and moreover, in the same way, if we set $\max(g) = (m, n)$, $\max(h) = (r, s)$, then

$$(gh)_{\max(g) + \max(h)} = q^{rn} g_{\max(g)} h_{\max(h)} \neq 0.$$
turn that we must have \( u \in D \setminus \{0\} \), and as we already showed that \( x - 1 \) does not right-divide any nonzero constant, it follows that \( u = 1 \), yielding \( f = x^k \).

With notations as in Corollary 37, let \( B_x = \text{POW}(x) \). The proof above shows that \( B_x \) is a definable subset of \( \text{POW}(x) \), using \( x \) as a parameter; moreover, the element \( x - 1 \) is right cancellable (as every nonzero element of \( S \)). Finally, since we are assuming that all nonzero integers in \( S \) are invertible, the definable set \( A_x := S^* \) satisfies \( \mathbb{Z}^* \subseteq A_x \). The reasoning of the previous paragraph shows that \( A_x \subseteq D \) and that \( x - 1 \) does not right-divide any element of \( D \setminus \{0\} \), so in particular \( x - 1 \) right-divides no nonzero difference between a positive integer and an element of \( \mathbb{Z}^* \). Applying Corollary 37 with \( p = x \), we conclude that the set \( \log_x B_x = \mathbb{Z}^* \) is definable in \((S,0,1,+,\cdot,x)\), that is, using \( x \) as the parameter (via a uniform formula; cf. Remark 33).

6.2. Further properties of logical powers. In this subsection we list theoretical
properties of logical powers of elements of a ring under special conditions on the
elements and on the ring, which will emphasize how the usefulness of the notion
of logical power is closely related to some of the properties previously mentioned
(such as irreducibility, primality, etc.), and how difficult it is to extend its use, in a
profitable way, to more general (and somehow pathological) cases.

Proposition 38. Let \( S \) be a ring, and let \( p \in S \).

a. We have \( \text{LPOW}(0) = \{0\} \) if \( S \) is a field and \( \text{LPOW}(0) = \emptyset \) otherwise.

b. If \( p \) is a unit, then \( \text{LPOW}(p) \) is the set \( \{(p-1)g+1 : g \in S\} \), which contains
every integer power of \( p \). If in addition \( p - 1 \) is regular, then \( \text{LPOW}(p) \) properly
contains the set of nonnegative powers of \( p \).

c. We have \( \text{LPOW}(p) \neq \emptyset \) if and only if \( p \in \text{LPOW}(p) \). If \( S \) is not a field and
\( \text{LPOW}(p) \neq \emptyset \), then \( p \neq 0 \).

d. If \( S = \mathbb{R}[x] \), with \( \mathbb{R} \) reduced, and \( \text{LPOW}(p) \) contains a zerodivisor, then \( p \) is
invertible.

Proof.

a. If \( f \in \text{LPOW}(0) \), then \( 0 \) divides \( f \), so necessarily \( f = 0 \), which shows that
\( \text{LPOW}(0) \subseteq \{0\} \). Moreover, we have \( 0 \in \text{LPOW}(0) \) if and only if every
divisor of \( 0 \) is a unit or a multiple of \( 0 \). Since (trivially) every element of \( S \)
divides \( 0 \), it follows that \( 0 \in \text{LPOW}(0) \) precisely when every element of \( S \) is
a unit or \( 0 \), that is when \( S \) is a field.

b. If \( p \) is a unit, then all the clauses for membership in \( \text{LPOW}(p) \) are auto-
matically fulfilled by any element \( f \), except possibly for \( p - 1 \mid f - 1 \), and
so \( f \in \text{LPOW}(p) \) precisely when \( f - 1 = (p-1)g \) for some \( g \in S \); in par-
ticular, since for all natural \( n \) we have that \( p - 1 \) divides both \( p^n - 1 \) and
\( -p^{-n}(p^n - 1) = p^{-n} - 1 \), it follows that \( \text{LPOW}(p) \) contains every integer
power of \( p \).

For the last statement, suppose that \( \text{LPOW}(p) = \text{POW}(p) \cup \{1\} \). Since
\( p^{-1} \in \text{LPOW}(p) \), it follows that \( p^{-1} = p^m \) for some \( m \geq 0 \), and therefore
\( \text{LPOW}(p) = \text{POW}(p) \cup \{1\} = \{p, p^2, \ldots, p^{m+1} = 1\} \) is finite. If \( p - 1 \) is
regular, then the mapping \( g \mapsto (p-1)g + 1 \) is injective, hence \( \text{LPOW}(p) = \{(p-1)g+1 : g \in S\} \) has the same cardinality as \( S \). Thus, \( S \) is a finite ring;
but regular elements are invertible in finite rings, and therefore so is \( p - 1 \).

By taking \( g := -(p-1)^{-1} \) we get that \( 0 = (p-1)g + 1 \in \text{LPOW}(p) \), and
clearly \( 0 \not\in \text{POW}(p) \cup \{1\} \), a contradiction.

c. One implication is clear; for the converse, let \( f \in \text{LPOW}(p) \) be fixed. We
trivially have \( p \mid p \) and \( p - 1 \mid p - 1 \), and if \( g \in S \) divides \( p \), then it also
divides \( f \) (because \( p \mid f \)), so \( g \) is either a unit or a multiple of \( p \). Thus
\( p \in \text{LPOW}(p) \), and if \( S \) is not a field, then \( p \neq 0 \) by item a.
d. Assume that \( p \) is noninvertible, and let \( f, \ell \in R[x] \) be such that \( f \in \text{LPOW}(p) \) and \( f \ell = 0 \); our objective is to show that \( \ell = 0 \). We have \( f = fh \), being \( h = 1 + px \). Since \( p \) does not divide \( h \) (otherwise \( p \) would divide \( h - px = 1 \), \( h \) is a divisor of \( f \) and \( f \in \text{LPOW}(p) \)), it follows that \( h \) must be invertible. As \( R \) is reduced, then \( h \) is constant by Proposition 7, so \( p \ell = 0 \), which in turn implies \( p = (1 + \ell x)p \). Since we are assuming \( \text{LPOW}(p) \neq \emptyset \), then necessarily \( p \in \text{LPOW}(p) \) by item c. Now by Lemma 20a we cannot have that \( p \) is multiple of \( p^2 \), and consequently the element \( 1 + \ell x \), which is a divisor of \( p \), cannot be a multiple of it. This, together with the fact that \( p \in \text{LPOW}(p) \), implies that \( 1 + \ell x \) must be invertible. Therefore \( 1 + \ell x \) is constant (again by Proposition 7), so \( \ell = 0 \), as desired. \( \square \)

We remark that the hypothesis “\( p - 1 \) is regular” cannot be dropped in the statement of Proposition 38c: take \( S = \mathbb{Z}[t]/a \), being \( a = (2(t - 1), t^2 - 1) \), and consider \( p = t \). Then \( p \) is invertible, for \( p^2 = 1 \), and this in turn implies that every element \( g \in S \) is of the form \( g = a + (p + 1)b \), with \( a, b \in \mathbb{Z} \); using that \( (p - 1)(p + 1) = 0 \) we get \( (p - 1)g = (p - 1)a \). Writing \( a = 2k + r \), with \( k \in \mathbb{Z} \) and \( r = 0 \) or \( 1 \), and using that \( 2(p - 1) = 0 \), we conclude that \( (p - 1)g + 1 = 1 + p \). Thus, \( \text{LPOW}(p) = \{1, p\} = \text{POW}(p) \cup \{1\} \) (recall that \( p^2 = 1 \)). Note that in this case \( p - 1 \neq 0 \) (there are not \( f, g \in \mathbb{Z}[t] \) such that \( t - 1 = 2(t - 1)f + (t^2 - 1)g \) but \( (p - 1)^2 = (p^2 - 1) - 2(p - 1) = 0 \), so \( p - 1 \) is a zerodivisor.

On the other hand one may wonder, following items a and b of Lemma 20, whether reducedness could be replaced by indecomposability in the hypothesis of item d above. As the counterexample below shows, this is not possible.

Let \( R \) be a local ring (see Example 15) such that the ideal \( \mathfrak{m} \) of nonunits in \( R \) is generated by a nonzero element \( p \) with \( p^2 = 0 \) (as a concrete example, take \( R = k[z]/z^2k[z] \), \( k \) being a field, and \( p = z \)). Then \( R \) is indecomposable (Example 15), and obviously \( p \) is not invertible. We claim that \( p \) is irreducible in \( R[x] \), so we have \( p \in \text{LPOW}(p) \) by Proposition 4c, and therefore the set \( \text{LPOW}(p) \), with \( p \) noninvertible, contains the zerodivisor \( p \).

In order to prove our claim, it is sufficient to show that \( p = gh \) implies that one of \( g \) or \( h \) is a unit (because clearly \( p \notin \{0\} \cup R[x] \)). If \( g = g_0 + sx \), \( h = h_0 + tx \), with \( s, t \in R[x] \), then \( p = g_0h_0 \). Since \( p \neq 0 \), then \( p^2 = 0 \) does not divide \( p \), so one of \( g_0 \) or \( h_0 \) is not a multiple of \( p \), hence it is invertible, say \( g_0 \in R^* \) and \( h_0 = g_0^{-1}p \).

Taking images in the integral domain \((R/m)[x] \)\* we get \( 0 = (g_0 + sx)t \). Since \( g_0 \neq 0 \), then \( (g_0 + sx)x \neq 0 \), which implies \( t = 0 \), that is \( t = p \ell \) for some \( \ell \in R[x] \), and consequently \( p = p(g_0 + sx)(g_0^{-1} + t \ell) \). If \( s = s_nx^n + \cdots + s_0 \), with \( n \geq 0 \), then by item a of Lemma 6 we have \( s_n^2p = 0 \) for some \( k \geq 1 \). As \( p \neq 0 \), it follows that \( s_n \) cannot be invertible, hence it is a multiple of \( p \), and in particular \( ps_n = 0 \) (recall that \( p^2 = 0 \)). If \( \hat{s} = s - s_nx^n \), then \( p\hat{s} = ps - ps_nx^n = ps \), and so we have \( p = p(g_0 + \hat{s}x)(g_0^{-1} + t \ell) \). Iterating this argument we conclude that \( p \) divides every coefficient of \( s \), that is, \( p \) divides \( s \), so \( s^2 = 0 \), and therefore \( (g_0 + sx)(g_0 - sx) = 0 \) is a unit, which shows that \( g \) is a unit.

6.3. Algebraic equivalences for reducedness/indecomposability.

**Proposition 39.** For a ring \( R \) the following are equivalent:

a. \( R \) contains an idempotent element other than 0 and 1.

b. \( R \) is isomorphic to the direct product of two nonzero rings.

\* If \( a, b \in R \) are such that \( ab \) is a nonunit, then \( a \) or \( b \) is a nonunit; since the set \( m \) of nonunits is already an ideal, it follows that \( m \) is indeed a prime ideal, so the quotient ring \( R/m \) is an integral domain.
c. The polynomial \( x \) in \( R[x] \) is a product of two noninvertible polynomials of degree 1.

d. The polynomial \( x \) in \( R[x] \) is a product of two noninvertible polynomials of positive degree.

e. The polynomial \( x \) in \( R[x] \) is a product of two noninvertible polynomials. Equivalently, \( x \) is reducible in \( R[x] \).

Proof.

a \( \Rightarrow \) b If \( e \in R \) is idempotent then \( f = 1 - e \) is too; moreover the ideal \( R_1 = Re \) (respectively \( R_2 = Rf \)) has \( e \) (respectively \( f \)) as a multiplicative unit. Therefore both \( R_1 \) and \( R_2 \) are unital rings, with \( 1_{R_1} = e \) and \( 1_{R_2} = f \), and the mapping \( R \to S := R_1 \times R_2 \) given by \( r \mapsto (re, rf) \) is a ring homomorphism (it respects sums, products and sends \( 1_R \) to \( (e, f) = 1_S \)), and its inverse is given by \( (ae, bf) \mapsto ae + bf \). This shows that \( R \cong S \). If \( e \neq 0, 1 \) then both \( R_1 \) and \( R_2 \) are nonzero, which proves the implication.

b \( \Rightarrow \) a If \( R_1 \) and \( R_2 \) are nonzero rings, then the element \((1,0)\) is a nontrivial idempotent in the ring \( R_1 \times R_2 \).

c \( \Rightarrow \) d \( \Rightarrow \) e Obvious.

e \( \Rightarrow \) a \( \Rightarrow \) c See the proof of Proposition 10. \( \Box \)

We may observe that, like integral domains, which are characterized by the property that \( x \) is a prime element in \( R[x] \), the class of indecomposable rings also corresponds to a specific property of the algebra generator \( x \), namely, the polynomial \( x \) is irreducible in \( R[x] \) (by Proposition 10). In the case of reduced rings, since all positive degree polynomials are noninvertible by Proposition 7, this characterization can be specialized in the following form:

**Proposition 40.** A reduced ring \( R \) is indecomposable if and only if the polynomial \( x \) in \( R[x] \) is not a product of two polynomials of positive degree.

Finally, in order to express the class of rings \( R \) we are interested in, in terms of properties of \( R[x] \), we may synthesize as follows:

**Proposition 41.**

a. A ring \( R \) is reduced if and only if the polynomial 1 in \( R[x] \) is not a product of two polynomials of positive degree.

b. A ring \( R \) is reduced and indecomposable if and only if the polynomials 1 and \( x \) in \( R[x] \) are not a product of two polynomials of positive degree.

Proof.

a. If \( R \) is reduced, then invertible elements of \( R[x] \) are constant by Proposition 7. For the converse, if \( a \in R \) and \( n \geq 2 \) satisfy \( a^n = 0 \) and \( a^{n-1} \neq 0 \), then \( 1 = (1 + a^{-1}x)(1 - a^{-1}x) \).

b. Once item a above is given, this follows from condition d of Proposition 39, as the requirement of noninvertibility of nonconstant elements becomes redundant in a reduced ring by Proposition 7. \( \Box \)

We remark that the result of Proposition 16a actually characterizes reduced and indecomposable rings: in fact, let \( R \) be a ring such that whenever \( c^{n+1} \) divides \( c^n \), then \( c \) is either zero or a unit. If \( c \in R \) is idempotent, then obviously \( c^2 \) divides \( c \), and therefore \( c = 0 \) or \( c \) is a unit, and in the latter case we have \( 1 = c c^{-1} = c^2 c^{-1} = c \), which shows that \( R \) is indecomposable. On the other hand, if \( a \in R \) is nilpotent, say \( a^n = 0 \), with \( n \geq 1 \), then obviously \( a \) cannot be a unit (recall that \( R \) is a nonzero ring), and since \( a^{n+1} = 0 \) trivially divides \( a^n = 0 \), it follows that \( a = 0 \), and consequently \( R \) is reduced.
From the very definition of polynomials and their multiplication, it follows that 0 is the only polynomial infinitely divisible by \( x \). This will be used in the proof of the following result, which shares the same spirit of Proposition 7, but concerning indecomposability:

**Proposition 42.** For any ring \( R \), a polynomial \( e \in R[x] \) is idempotent if and only if \( e \) is constant and idempotent in \( R \). In particular, \( R \) is indecomposable if and only if \( R[x] \) is indecomposable.

**Proof.** Let \( e \in R[x] \) be idempotent. Writing \( e = e_0 + px \), with \( p \in R[x] \), the equality \( e = e^2 \) becomes \( e_0 + px = e_0^2 + 2e_0px + p^2x^2 \), yielding \( e_0 = e_0^2 \), and in particular \((1 - 2e_0)px = (px)^2\). Since \((1 - 2e_0)^2 = 1\), then \((1 - 2e_0)px = [(1 - 2e_0)px]^2\). Thus \((1 - 2e_0)px = [(1 - 2e_0)p]^n x^n \) for all \( n \geq 1 \), that is, \((1 - 2e_0)px \) is infinitely divisible by \( x \), and so necessarily \((1 - 2e_0)px = 0\). Since \((1 - 2e_0)x \) is regular, it follows that \( p = 0 \), so \( e = e_0 \) is idempotent in \( R \). \( \square \)

From Propositions 7 and 42 we obtain the following characterization of reduced/indecomposability for polynomial rings in an arbitrary set of indeterminates:

**Proposition 43.** Let \( R \) be a ring and let \( I \) be a nonempty set. Then the polynomial ring \( S = R[\{x_i\}_{i \in I}] \) is reduced (resp. indecomposable) if and only if \( R \) is reduced (resp. indecomposable).

**Proof.** Obviously \( S \) reduced (resp. indecomposable) implies \( R \) reduced (resp. indecomposable). Conversely, assume that \( R \) is reduced (resp. indecomposable). Given \( f \in S \), then exists a finite subset \( I_0 \) of \( I \) such that \( f \in S_0 := R[\{x_i\}_{i \in I_0}] \). Propositions 7 and 42, together with induction, show that \( S_0 \) is reduced (resp. indecomposable) as well, and therefore \( f \) nilpotent (resp. idempotent) implies \( f = 0 \) (resp. \( f = 0 \) or 1), which shows that \( S \) is reduced (resp. indecomposable). \( \square \)

Thus, a polynomial ring (in a set of variables) with reduced/indecomposable coefficient ring is the same thing as a reduced/indecomposable polynomial ring, and therefore Corollary 32 and Theorem 35 are valid for these rings. In other words, the set of integers is first-order definable in any reduced and indecomposable polynomial ring (in any set of variables).

### 6.4 More about constant polynomial functions.

Let \( R \) be a ring such that the only polynomials in \( R[x] \) inducing constant polynomial functions on \( R \) are the constant polynomials; for example, we can take \( R \) infinite, reduced and indecomposable, by Theorem 17. If \( R \) is reduced and \( g \in R[x] \) takes finitely many values, then \( g \in R \) (and, a posteriori, \( g \) takes only one value). Indeed, assuming the contrary, let \( n \geq 2 \) be minimal such that there exists a polynomial \( g \) taking \( n \) values. If \( a \) and \( b \) are two such (distinct) values, then the polynomial \( f = (a + b - g)g \) takes the value \( ab \) when \( g \) takes the values \( a \) or \( b \), and therefore \( f \) takes at most \( n - 1 \) values. By minimality of \( n \), we necessarily have that \( f \) is constant as a polynomial function, so \( f \in R \) by the initial hypothesis. On the other hand, if \( c \) is the leading coefficient of the nonconstant polynomial \( g \), then \( a + b - g \) also has positive degree and its leading coefficient equals \(-c\). Since \( R \) is reduced, we have \(-c^2 \neq 0\), and consequently \( f = (a + b - g)g \) has positive degree, a contradiction.

Now we show, through examples, that neither reducedness nor indecomposability can be removed from the hypotheses of Theorem 17.

**Example 44.** If \( R \) is a Boolean ring (that is, \( a^2 = a \) for all \( a \in R \)), then \( R \) is decomposable unless \( R = \mathbb{F}_2 \), the field with two elements. On the other hand, Remark 9
implies immediately that \( R \) is reduced. Finally, by definition the nonconstant polynomial \( f = x^2 - x \in R[x] \) vanishes on all of \( R \). As a concrete example of infinite Boolean ring we may take \( R \) as the direct product \( F_2^\mathbb{N} \).

**Example 4.5.** Let \( S \) be the ring of polynomials in infinitely many variables \( T_1, T_2, \ldots \) over the field \( F_2 \). Let \( a \) be the ideal in \( S \) generated by the products \( T_i T_j \), with \( 1 \leq i \leq j \), and consider the factor ring \( R = S/a \). Denoting the class of \( T_i \) modulo \( a \) by \( t_i \), we have that every element of \( R \) is of the form \( p = a_0 + \sum_{i=1}^{\infty} a_i t_i \), with \( a_i = 0 \) or \( 1 \) for all \( i \geq 0 \), and \( a_i \neq 0 \) for finitely many \( i \). Moreover, \( p = 0 \) if and only if \( a_i = 0 \) for all \( i \); in particular, all the elements \( t_i \) are pairwise distinct, so \( R \) is infinite. Since \( R \) has characteristic 0, then \( p^2 = a_0^2 + \sum_{i=1}^{\infty} a_i^2 t_i^2 = a_0^2 = a_0 = 0 \) or 1, so the nonconstant polynomial \( x^2(x^2 - 1) \in R[x] \) vanishes on all of \( R \), and moreover \( p^2 - p = \sum_{i=1}^{\infty} a_i^2 t_i^2 - a_i t_i \). Thus, \( p^2 = p \) implies \( a_i = 0 \) for all \( i \geq 1 \), that is \( p = a_0 \), which shows that \( R \) is indecomposable. Obviously \( R \) is not reduced, as \( t_i \neq 0 \) for each \( i \) but \( t_i^2 = 0 \).

6.5. **Comparing powers with logical powers.** Consider the definable subsets \( T \) and \( U \) of \( R[x] \) appearing in the statement of Theorem 23. Then every element \( p \in T \) satisfies \( \text{POW}(p) \subseteq \text{LPOW}(p) \); similarly, every element \( p \in U \) satisfies \( \text{POW}(p) = \text{LPOW}(p) \), under the additional assumption that \( R \) is reduced and indecomposable.

Note that both \( T \) and \( U \) can be defined in any ring \( S \). If we replace "\( p \) is irreducible" by "\( p \in \text{LPOW}(p) \)" in the definition of \( T \), it remains true that \( \text{POW}(p) \subseteq \text{LPOW}(p) \) for each \( p \in T \). The converse is almost true: it is easy to show that if \( p \in S \) satisfies \( \text{POW}(p) = \text{LPOW}(p) \), then \( p \in T \); moreover, for any unit \( p \) we have, by Proposition 38b, that \( \text{LPOW}(p) = \{ (p - 1)g + 1 : g \in S \} \), and from this it follows that the inclusion \( S' \subseteq T \) also holds.

On the other hand, if \( p \in U \) is a unit, then by taking \( s = p \) in the second condition of the definition of \( U \) we obtain \( p = 1 \). Consequently, if we impose the additional condition \( p \neq 1 \) on the definition of \( U \), we get that \( U \) consists entirely of nonunits. As we want all elements \( p \) in \( U \) to satisfy \( \text{LPOW}(p) = \text{POW}(p) \), this restriction will be harmless, because for a unit \( p \in S \) we have, in most cases, that \( \text{LPOW}(p) \) strictly contains \( \text{POW}(p) \): namely, when \( p - 1 \) is regular (Proposition 38b). Note that this regularity condition is essential for the proofs of Theorems 28, 31 and 35 to work.

Even with the extra requirement \( p \neq 1 \) in the definition of \( U \), we are still able to prove that \( \text{LPOW}(p) = \text{POW}(p) \) for each \( p \in U \), whenever \( S = R[x] \), with \( R \) reduced and indecomposable: the proof is almost identical to that of Theorem 23b (using Proposition 16a together with Lemma 20b).

Finally, if \( S = R[x] \), with \( R \) reduced (not necessarily indecomposable), and if \( p \in S \) is nonconstant with regular leading coefficient, and satisfies \( \text{LPOW}(p) = \text{POW}(p) \), then \( p \in U \). The proof is entirely similar to that of Theorem 23a for the special case \( p = x \). Note that Corollary 21 provides examples of nonlinear polynomials satisfying the requirements above (the classic example being \( R = \mathbb{R} \) and \( p = x^2 + 1 \)). This is in contrast with Remark 24, which merely guarantees that linear polynomials with invertible leading coefficient belong to the set \( U \) (in the case \( R \) reduced and indecomposable).

6.6. **Revisiting examples.** It is possible to prove the definability of the integers in \( R[x] \), for \( R \) as in Examples 13 to 15, and in some instances of Example 12, by constructing a definable set \( A \) of \( R \) containing \( \mathbb{Z} \), and applying Theorem 31. Note that the rings in Examples 13 and 14 are always nonfields of characteristic zero, as
well as some particular cases of the rings in Examples 12 and 15, and thus these rings are also covered by Theorem 35.

Example 46 (Example 15, revisited). If $R$ is a local and reduced ring, then the set $A = \{ f \in R[x] : f \in R[x^+] \text{ or } f + 1 \in R[x] \}$ is definable, and it satisfies $A \subseteq R$ by Proposition 7. The reverse inclusion follows from the definition of local ring, and thus we may take $A = R$ in Theorem 31.

Example 47 (Example 12, revisited). Let $R$ as in Example 12. If every nonzero integer in $B$ is invertible, then the same happens to each nonzero integer constant function from $X$ to $B$, so we may apply Corollary 32 in this case.

Example 48 (Example 13, revisited). Consider the ring $R[x]$, where $R$ is defined as in Example 13. Note that an element $(m, n) \in \mathcal{R}$ is regular if and only if $m, n \neq 0$.

Let $p = (5, 1), q = (1, 5) \in R[x]$. Then the set $B = \{ p, q \}$ can be defined by the formula

$$\beta(t) : \exists r (r \mid 1 \land r \neq 1 \land r \neq -1 \land t = 3 + 2r).$$

Now we claim that $\text{LPow}(p) = \text{POW}(p)$ and $\text{LPow}(q) = \text{POW}(q)$. It is easy to see that $p$ is prime in $R$, so it remains prime in $R[x]$. Since $p$, it is also regular, then $\text{POW}(p) \subseteq \text{LPow}(p)$ by Proposition 4d. Conversely, let $h \in \text{LPow}(p)$, and let us denote $h_m$ by $(f_m, g_m) \in R$. Now $p - 1 = (4, 0)$ divides $h - 1$, so $4$ divides $f_0 - 1$, and thus $f_0$ is odd. If $p^k$ divides $h$ in $R$, then $5^k$ must divide $f_0 \neq 0$ in $\mathbb{Z}$, and so $h$ cannot be infinitely divisible by $p$. Therefore, by Proposition 4a, we can write $h = up^n$, with $n \geq 1$ and $u \in R[x^+] = R^*$ (see Proposition 7) such that $p - 1 = (4, 0)$ divides $u - 1$ in $R$. We have $u - 1 = (0, 0), (-2, -2), (0, -2)$ or $(-2, 0)$; since $-2$ is not multiple of 0 or 4, it follows that $u - 1$ must be equal to $(0, 0)$, so $h \in \text{POW}(p)$. The proof of $\text{LPow}(q) = \text{POW}(q)$ is analogous.

As a consequence of these two equalities right above, we obtain that the set $C = \{ 5^m - 1, 5^m - 1 \mid m, n \geq 1 \}$ is definable by the formula

$$\gamma(t) : \exists r \exists s \exists v \exists w \left[ \beta(r) \land r + s = 6 \land v \in \text{LPow}(r) \right. \land v \in \text{LPow}(s) \land t = v + w - 2 \left. \right].$$

If $D \subseteq R[x]$ is the set of divisors of elements in $C$, then obviously $D$ is also definable; moreover, since $R$ is reduced and $C \subseteq R$ consist entirely on regular elements, it follows from item a of Lemma 6 that $D \subseteq R$. If $\phi$ denotes the Euler’s totient function, then it is well-known that, for any positive integer $a$ not a multiple of 5, we have that $5^{\phi(a)} - 1$ is divisible by both $a$ and $-a$. Therefore $D$ contains all the elements $(d, d)$, with $d \in \mathbb{Z}$ not a multiple of 5 as a rational integer. Consequently, the set $A$ of elements $t$ such that $t \in D$ or $t + 1 \in D$ is definable, and it satisfies $\mathbb{Z} \subseteq A \subseteq R$.

Example 49 (Example 14, revisited). Let $R$ be as in Example 14 and set $S = R[x]$. We may also think of elements of $S$ as $I$-tuples of integers polynomials whose

\* In general, the ring $R = C(X, B)$ contains a isomorphic copy of $B$, namely, the subring of constant functions. We claim that $R$ is a field if and only if $B$ is a field and $R = B$. In fact, if $B$ is not a field and $b \in B$ satisfies $b \notin \{0\} \cup B^*$, then the constant function with value $b$ has no inverse in $R$; on the other hand, if $R$ properly contains $B$, then some $f \in R$ take two distinct values, say $a \neq b$ in $R$, and therefore the function $f - a$ is nonzero and noninvertible (as it takes the value 0). In either case we conclude that $R$ is not a field, and the converse is obvious.

\* It is not possible to tell apart $p$ from $q$ by using a first-order formula.

\* It is well-known that if $a$ is an ideal in a ring $R$, then $R[x]/a$, the ideal in $R[x]$ generated by $a$, is precisely the set of $a[x]$ of polynomials with coefficients in $a$, and $R[x]/a[x] \cong (R/a)[x]$. If $p \in R$ is prime and $a = pR$, then $R[x]/p = a[x]$, so $R[x]/R[x]/p = (R/a)[x]$, which is an integral domain, and this shows that $p$ remains prime in $R[x]$. 

\*\*
coefficients in any fixed degree have the same parity. Let \( D = \{ d \in S^*: 2d + 3 \text{ is irreducible} \} \), \( C = \{ 2d + 3 : d \in D \} \) and \( E = \{ d + 1 : d \in D \} \). It is easy to check that \( C \), \( D \) and \( E \) are definable sets, that \( D \) is the set of \( I \)-tuples with one entry equal to 1 and all other entries equal to \(-1\), that \( C \) consists precisely of the \( I \)-tuples, all irreducible, with one entry equal to 5 and all other entries equal to 1, and finally, that \( E \) consists of those \( I \)-tuples with one entry equal to 2 and all other entries equal to zero.

We claim that, for any \( c \in C \), one has \( \text{LPOW}(c) = \text{POW}(c) \). Indeed, for \( c \in C \), there must be \( j \in I \) such that \( c_j = 5 \) and all other entries of \( c \) are 1. If \( f \in \text{POW}(c) \), then \( c \mid f \) and \( c - 1 \mid f - 1 \) are obviously satisfied and, if \( g \mid f \), then all but one entry of \( g \) are \( \pm 1 \) and the other, \( g_j \), divides a power of 5. Thus \( g_j \) must be a constant, by Lemma 6a, and consequently it is either \( \pm 1 \) or a multiple of 5. Therefore either \( g \) is invertible or a multiple of \( c \). Conversely, if \( f \in \text{LPOW}(c) \), then \( c - 1 \mid f - 1 \) forces all but the \( j \)-th component of \( f - 1 \) to vanish and \( 4 \mid f_j - 1 \); in particular \( f_j \neq 0 \). Since \( c \mid f \), then \( f_j = m \cdot 5^n \), with \( n > 0 \) and \( m \equiv 1 \) (mod 4) not a multiple of 5. Furthermore, if \( m \) were not invertible, the element \( f \) with \( f_j = m \) and all other entries equal to 1, not divisible by \( c \), would be a noninvertible divisor of \( f \in \text{LPOW}(c) \), a contradiction. Therefore \( m = \pm 1 \). Since \( m \equiv 1 \) (mod 4), then \( m = 1 \) and, therefore, \( f \in \text{POW}(c) \). Now consider the following formula:

\[
\alpha(t) := \forall e \in E \Rightarrow \exists c \exists y \left( c \in C \land y \in \text{LPOW}(c) \land \left[ t \cdot e \mid y - 1 \lor (t + 1) \cdot e \mid y - 1 \right]\right).
\]

The formula holds whenever multiplication of \( t \) or \( t + 1 \) by any element of \( E \) divides \( y - 1 \), for some logical power \( y \) of a suitable \( c \in C \). We claim that \( r \in R \) precisely when \( \alpha(r) \) holds. Indeed, let \( r \in R \) be a constant element, and let \( e \in E \). Then there exists \( j \in I \) such that \( e \) has all entries equal to zero but its \( j \)-th entry, which is equal to \( 2 \); then \( r \cdot e \) and \( (r + 1) \cdot e \) have one constant integer entry, namely \( 2r_j \) and \( 2(r_j + 1) \), respectively, and all other entries equal to zero. By Euler’s theorem, any rational integer not divisible by \( 5 \) divides some element of the form \( 5^n - 1 \). In view of this, since \( 2r_j \) and \( 2(r_j + 1) \) cannot both be a multiple of \( 5 \), and using \( \text{LPOW}(c) = \text{POW}(c) \), for all \( c \in C \), then either \( r \cdot e \) or \( (r + 1) \cdot e \) divides \( y - 1 \) for some logical power \( y \) of the element \( c \in C \) with \( 5 \) in the \( j \)-th entry and 1 in all other entries, and so \( \alpha(r) \) holds.

Conversely, let \( s \in R[x] \) be nonconstant, say \( \deg(s_j) \geq 1 \) and consider the element \( e \in E \) such that \( e_j = 2 \) and all other entries of \( e \) are zero. Suppose \( s \cdot e \) or \((s + 1) \cdot e \) divides \( y - 1 \), for some \( y \in \text{LPOW}(c) \) and some \( c \in C \). As \( y - 1 \) has only one nonzero entry, which is a constant, and \( s \cdot e \) has all but the \( j \)-th entry equal to zero, we must have that \( (y - 1)_j \neq 0 \) is a constant and \( (y - 1)_i = 0 \) for all \( i \neq j \). But both \( (s \cdot e)_j = 2s_j \) and \((s + 1) \cdot e)_j = 2(s_j + 1) \) have positive degree and therefore they cannot divide, in the reduced ring \( \mathbb{Z}[x] \), the nonzero constant element \((y - 1)_j \), (by Lemma 6a), proving that \( \alpha(s) \) is false. Therefore \( R \) is definable and we can just take \( A = R \).

6.7. Further discussion concerning local rings and AC. It is customary to define a local ring in an alternative way to that given in Example 15, namely, as a ring with a unique maximal ideal. This property is a straightforward consequence of the definition given in Example 15 (nonunits form an ideal), and it is well-known that, in the presence of the axiom of choice (AC), these definitions are equivalent (we provide below proofs of these facts).
We would like to stress (and the reader may check this) that the only definition of “local ring” we use throughout the paper to prove the main result (concretely, in the proof of Lemma 18) is that given in Example 15: neither the alternative definition nor its relation with the definition appearing in Example 15 are needed for our arguments to work. In particular, our main result holds unconditionally on ZF and does not require assuming AC. Interestingly enough, the interchangeability between the two notions of locality is not just a consequence of AC, but is indeed equivalent to it (to the best of our knowledge, this is a new condition equivalent to the axiom of choice).

Let \( R \) be a ring such that the set \( m = R \setminus R^* \) of nonunits of \( R \) forms an ideal. Then \( m \) is maximal, for any ideal strictly containing it must contain a unit, and therefore it coincides with \( R \). On the other hand, if \( n \) is a maximal ideal in \( R \), then \( n \) contains no unit, so \( n \subseteq m \), and therefore \( n = m \) by maximality.

The argument above shows that every local ring in the sense of Example 15 has a unique maximal ideal (namely, its set of nonunits), which is the commonly accepted definition of “local ring”. The converse is not true in ZF; in fact, we contend that the assertion “In every ring with a unique maximal ideal the set of nonunits forms an ideal” is equivalent to the claim that every (nonzero commutative unital) ring has a maximal ideal. This condition, in turn, is known to be equivalent to the axiom of choice ([Hod79]).

To prove our claim, suppose that every nonzero ring has a maximal ideal. By working on quotient rings, we get that every nonunit in a ring belongs to a maximal ideal, and therefore in a ring with a unique maximal ideal all nonunits must belong to that maximal ideal, which in turn consists entirely of nonunits, and this proves that the set of nonunits of the ring forms an ideal. On the other hand, if \( A \) is a nonzero ring without maximal ideals, then the ring \( R = \mathbb{Q} \times A \) has \( 0 \times A \) as its unique maximal ideal. As we already saw, if the set of nonunits of \( R \) were an ideal, then it would be equal to the unique maximal ideal, and so \( R \setminus R^* = 0 \times A \); but this equality is impossible, because \((1,0) \in R \setminus R^* \) but \((1,0) \notin 0 \times A \), and this shows that nonunits in the ring \( R \) do not form an ideal.

Note that, incidentally, the ring \( R \) above is not indecomposable (Proposition 39), so we cannot change “local” by “the ring has a unique maximal ideal” in Example 15.

6.8. Diagram of implications. In the diagram below we show the implications between the conditions of reducedness/indecomposability of a ring \( R \), and properties of the subsets \( \text{POW}(x) \) and \( \text{LPOW}(x) \) in \( R[x] \). The converse of implication \((m)\) will be denoted by \((m)'\):
Theorem 22b.

(1)' (3)+(7), together with (3)+(5)+(6)+(4)'.

Obvious.

(2)' Counterexample: \( R = \mathbb{Z}/4\mathbb{Z} \).}

Obvious.

(3)' (7)+(11).

Obvious.

(4) Proposition 10.

(4)' Proposition 38c.

Proposition 5.

(5)' (4)+(6)'+(3)' and (1)' are true.

(6) Proposition 5.

(6)' Proposition 4c.

Proposition 5.

(7)' (7)+(3)+1) is false (counterexample: \( R = \mathbb{Z} \times \mathbb{Z} \)), but (3) and (1) are true.

(8) Proposition 5.

(8)' (4)+(6)'+(8)'+(10) imply (2)', which is false, but (4), (6)', (10) are true.

(9) Proposition 38c.

(9)' (7)+(6)+(4) is false (counterexample: \( R = \mathbb{Z} \times \mathbb{Z} \)), but (6), and (4) are true.

Theorem 22a.

(10)' Any decomposable and nonreduced ring \( R \), such as \( R \times \mathbb{Z} / 4\mathbb{Z} \), for \( R \) decomposable implies \( \text{LPOW}(x) = \emptyset \), by (9)+(6)+(4)', and so trivially we have \( \text{LPOW}(x) \subseteq \text{POW}(x) \).

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