Corase-graining dynamical triangulations: a new scheme

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Abstract

A new procedure for coarse-graining dynamical triangulations is presented. The procedure provides a meaning for the relevant value of observables when probing at large scales, e.g. the average scalar curvature. The scheme may also be useful as a starting point for a new type of renormalization procedure, suitable for dynamically triangulated quantum gravity. Random Delaunay triangulations have previously been used to produce discretizations of continuous Euclidean manifolds, and the coarse-graining scheme is an extension of this idea, using random simplicial complexes produced from a dynamical triangulation. In order for a coarse-graining process to be useful, it should preserve the properties of the original dynamical triangulation that are relevant when probing at large scales. Some general discussion of this point is given, along with some arguments in favour of the proposed scheme.

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1. Introduction

In the candidate theory of quantum gravity (QG) known as causal dynamical triangulations (CDTs) [1–4], a regulator is introduced to ‘cut out’ short distance detail from the model and render the physical quantities finite. To obtain a continuum theory the regulator has to be sent to zero. At the same time, some kind of renormalization is expected to be necessary in order to recover finite results for physical quantities. Without renormalization, for example, the average scalar curvature in the four-dimensional CDT model diverges.

Renormalization depends on being able to identify actions (or Hamiltonians) that represent the same physics at large scales, but with different cut-off scales. This can be done by identifying, and then integrating over, degrees of freedom that are irrelevant when probing the system at large scales. For example, when probing water with optical light, density fluctuations near the atomic scale are irrelevant in this sense.

In order to do this, a coarse-graining scheme is necessary: from a ‘fine-grained’ history with a certain cut-off, a coarse graining is found with a lower cut-off, but with corresponding
properties at large scales. For lattice quantum field theory (QFT), one way to achieve this is by the Kadanoff block spin transformation (see e.g. [5, 6]). By such techniques, the high-frequency modes of a field can be removed from a history without significantly affecting low-frequency modes. This accords with the idea of removing small-scale fluctuations that are not relevant when probing on large scales.

In view of this success, it would be of interest to extend these ideas to CDT quantum gravity. Previous attempts have been made to introduce coarse graining to Euclidian dynamical triangulations [7–12]. Also, in the spin-foam quantum gravity program [13, 14], some preliminary studies of the problem have been made, discussing differences between quantum field theory coarse graining and quantum gravity coarse graining, and suggesting the use of some interesting algebraic structures to extend the standard renormalization group techniques [15–17]. In the causal set approach [18, 19] there is a natural coarse-graining procedure that transforms the discreteness scale [20, 21]: some of the statistical techniques used below are similar to those employed in the causal set case.

For CDTs, the most immediate use of coarse-graining ideas would be the definition and calculation of coarse-grained observables, for comparison to simple observations and accepted theory. The theory is uniquely advanced in this regard: the expectation values of some interesting observables have already been computed, with good results (see e.g. [22, 23]). However, it has so far been difficult to find a large number of physically relevant observables to compute.

In closest analogy to the QFT case, in quantum gravity theories the idea would presumably be to take our regularized (or fundamentally discrete) histories, and produce from each one a coarse-grained version. This idea of coarse graining each history has been very useful until now in fixed-background coarse graining, and none of the generalizations required by quantum gravity suggest any new reason to abandon it (see [24], where this point is discussed). As suggested by the brief comments above, it is absolutely crucial that, on large scales, this coarse graining corresponds to the original fine-grained version on large scales. This must be the first concern of any such scheme, as without this property the coarse graining is of no physical significance.

This criterion of ‘physical aptness’ is more problematic in quantum gravity than in standard cases. In simple cases of lattice field theory, the problem is so close to trivial that it is barely discussed. There is a straightforward prescription that tells us what features of a history are relevant when probing at large scales (they are simply the low-frequency modes of the field), and many different block spin coarse-graining schemes can be found which adequately preserve these features. In lattice QCD things are less trivial; some coarse-graining schemes better encode the degrees of freedom that are relevant at larger scales. In quantum gravity, the notions of large scale and small scale themselves become subtler due to the dynamical nature of space time. Because of this it becomes necessary to consider the question ‘when do two geometries correspond at large scales?’, and provide a reasonable physical justification for the answer. For example, a candidate answer to this question, based on properties of the Lapacian operator, is a cornerstone of the successful application of renormalization group ideas to Euclidean quantum gravity [25–27]. This approach has produced evidence for the non-perturbative renormalizability of the theory in 4D.

When designing a coarse-graining scheme for quantum gravity, a number of new requirements arise, some stemming from the non-trivial nature of this physical aptness requirement. These present difficulties for any naïve blocking procedure, as discussed below in section 2. On the basis of these considerations, a new coarse-graining scheme for dynamical triangulations, suitable for use in the CDT program, is presented here. The procedure improves on previous ideas, as it arguably accords better with the physical criteria discussed below. From
this, large-scale effective versions of many observables, such as the average scalar curvature, can be defined. The class of observables also provides the basis for a scheme that can be used to define effective actions for CDT theory, at least in principle. Most of this discussion is complementary to other recent studies, which were largely concerned with the general mathematical tools necessary for coarse graining in quantum gravity.

As the proposed scheme has many novel aspects, a full explanation is made here without also including computational results, which will be left for future work. In section 2, the new problems of coarse graining for a theory of dynamical geometry are discussed. The coarse-graining scheme is then presented in section 3, along with some coarse-grained observables of particular interest. In section 4, a simplified version of the coarse-graining process that is practical to apply to CDT simulations is given, and there is some discussion on how to apply it.

2. Coarse graining and dynamical geometry

As already noted, physically relevant observables are hard to come by in lattice QG, as there are a number of difficult criteria that they must satisfy in order to be useful. These will be of relevance throughout the paper. The observables must be the following.

(a) well-defined generally covariant observables for the fundamental theory;
(b) possible to define in the discrete setting of CDTs;
(c) practical to calculate (so far this has meant by computer simulation);
(d) relevant when probing the system at large scales.

Clearly, (d) entails that the observables do not diverge to infinity in the continuum limit. Perhaps the best existing example is the spectral dimension [22], which is a measure of the effective dimension derived from the spectrum of the Laplacian operator. This will serve as an example of the kind of observable we are searching for. However, the spectral dimension was not derived from a general scheme.¹

Before attempting to define our observables and to justify them against the conditions set out above, it is necessary to consider the physical requirements, especially (d).

2.1. Geometry and scale

In the final analysis, the meaning of ‘properties relevant at large/small-scales’ could be determined by detailed physical arguments, for example involving gadanken experiments. However, usually a more practical approach is taken. In the fixed background case, the distinction is made by considering field modes of different momenta. Block-spin transformations do not exactly preserve the amplitudes of these field modes, but they have been proven to be successful by producing non-trivial, plausible results, and then by comparison to each other and to experimental data. Thus the physical aptness of a coarse graining is not usually derived from first principles, but by a process of physical insight combined with experimental (and computational) input. However, the guiding principle that large scales are associated with small field momenta plays a crucial role.

In the QG case we are just beginning to retrace these steps. Without a fixed background geometry we must generalize even the simple guiding principle mentioned above, which was

¹ However, it is true that much more information is encoded in the spectrum of the Laplacian than just the dimension. This may provide a different way to define interesting observables beyond the spectral dimension, and to compare results of CDT simulations to simple classical geometries. These questions are currently under investigation by the author and Dario Benedetti.
previously the easiest part of the argument. In this vein, it would be useful to have at least some conception of the following: a reasonable measure of the scale on which two geometries differ. The measure could be based on spectral properties as in the recent application of the renormalization group to gravity, or perhaps applying the Gromov–Hausdorff distance [28, 29] between two geometries. This would raise some difficult questions. For example, how should the kind of ‘disordered locality’ discussed in [30] be treated? Also, should the measure of closeness (and hence the coarse-graining cut-off) be frame dependent, or should it respect general covariance? Usually it is taken to be frame dependent, to the extent that a foliation of space time is used to perform the Wick rotation, and coarse graining proceeds on the Euclidean side. A truly covariant coarse graining, on the other hand, would presumably mean that small volume behaviour is cut-out, hopefully leading to high-momentum behaviour being suppressed in the frame relevant to any interaction (the causal set coarse graining, which depends only on causal and volume information, is a possible example of this [20]). Below, the standard route used for fixed-background cases is followed, i.e. to consider issues of scale after Wick rotation, but apart from that to preserve covariance as much as possible.

Even if the coarse-graining scheme itself is not covariant on the Lorentzian side (as in standard cases), care must be taken to apply it in a way that respects the symmetries of the theory—the relevant scale of coarse graining should be set at a physical scale relevant to the system under consideration, e.g. centre-of-mass energy, and therefore must never be decided by reference to any non-dynamical frame or foliation. Then the calculated observables should be approximately covariant. This is how condition (a) of section 2 is met in practice.

At this early stage, a detailed discussion of this measure of closeness of geometries can be deferred. Finding one coarse-graining scheme that gave good results for CDTs would be an advance, and could help to guide further progress in defining the conditions themselves. But these considerations do lead, at least, to some necessary conditions for a QG coarse-graining scheme to be physically apt. As part of the general problem of separating large- and small-scales properties, some new pitfalls arise in this case. These are discussed in the following subsection.

2.2. Some pitfalls for QG coarse graining

There are some key differences between the QG and standard cases of coarse graining, some of which have been noted in previous studies. First, a coarse graining for discrete QG cannot preserve the lattice structure. Second, the quantum geometry can give radical departures from the classical geometry that we are used to. Dimension, as shown in [1, 22], is now a scaling property in the CDT model. The same would probably be true of topology, if dynamical topology was allowed. Indeed, at some scales it is not clear that the geometry should closely resemble an extended smooth space time at all, to any extent beyond the fact that it is also a metric space. If the coarse-graining scheme does not reflect this, we may already have fallen foul of a classical prejudice.

Third, the coarse-graining scheme should not preserve the average scalar curvature (at least in more than 2 dimensions²). Most importantly, this is because this quantity has been seen to diverge in the case of CDTs. Also, this does not seem consistent with a coarse graining

² In two dimensions the situation is rather different, as a consequence of the Gauss–Bonnet theorem. This may be why preservation of average curvature is looked upon more favourably in [11], where a coarse-graining scheme for 2D dynamical triangulations is developed.
that preserves large-scale features of the geometry. It is not difficult to come up with examples of two geometries that differ only in many small volume, non-overlapping regions (or are otherwise intuitively ‘close’), but have arbitrarily different average scalar curvature. As well as the divergence, this speaks against preserving average curvature under coarse graining.

Forth, there is a problem that applies particularly to DTs. By decreasing the lattice spacing, any Euclidean geometry can be arbitrarily well approximated by a dynamical triangulation (a precise theorem to this effect can be found in [29]). Because of this, they are a useful way to discretize the space of geometries for quantum gravity. However, there is no equilateral triangulation of flat space above 2D, only dynamical triangulations that approach it as the lattice spacing $a \to 0$. The same is true of any space that has low curvature on large scales—exactly the type of geometry we hope to obtain from coarse-graining. This might not seem problematic; after all, in all lattice approximations, the degree of approximation to continuum configurations depends on the lattice length. But it does compromise a coarse-graining scheme that merely increases the lattice length for dynamical triangulations. In this background-independent case it is highly non-trivial to identify those DTs that are close to e.g. flat space (for example, the value of the Regge action of a DT that is close to some low-curvature geometry may not be correspondingly close to the value of the Einstein–Hilbert action for that geometry). Without a means of doing so the usefulness of the coarse-graining scheme is in doubt, since it cannot produce observables that are relevant at large scales.

Any good coarse-graining scheme for CDTs needs to avoid these problems, and most of the problems are also relevant for other approaches such as spin foams. Previously considered coarse-graining schemes were very similar to Kadanoff coarse graining. The general idea was to replace a block of many simplices with a block containing fewer simplices, which were the same except for the lattice scale. This avoids the first pitfall, but not the others. This kind of blocking does not allow the dimension to vary, or allow the coarse graining be anything other than a triangulation of a manifold. Some schemes advocated preserving average curvature. Also, schemes for dynamical triangulations have always produced dynamical triangulations as output.

Some of these problems might be overcome by a more sophisticated treatment of the continuum approximation, if it were possible to identify smooth approximations to DTs in some systematic way. However, even if this could be achieved, this is not a very natural approach. Geometries of many different dimensions would be represented by DTs with some particular fixed dimension, for example. The new coarse-graining process presented below is designed with these issues in mind.

3. The new scheme

3.1. Delaunay complexes

How can we extract such large-scale information from a given geometry? An answer to this question is a step towards a physically apt coarse-graining scheme. Fortunately, some similar ideas have been developed in other contexts, where statistical and combinatorial techniques have been used to make discrete approximations to Euclidean geometries, with controllable cut-off scales. The relevant concept is the Voronoi procedure, which produces discrete structures from a metric space. The large-scale properties of a continuous Euclidean signature space are conjectured to be encoded in these so-called ‘random Delaunay triangulations’ of that space [31]. This procedure has previously been applied to the problem of continuum approximations for spin network configurations in loop quantum gravity [31]. Outside quantum gravity, similar techniques are important for the random lattice formulation of QFT [32, 33]. Before
the application is given, the definition of the procedure will be stated and some comments made as to how the resulting structures encode the large-scale geometry.

First, a note on terminology is necessary. In general, the terminology is standard to dynamical triangulations (see e.g. [29]), with the proviso that simplicial complexes are here considered in the abstract definition only. An abstract simplicial complex $C = \{S_0, S_1, \ldots, S_{D-1}\}$ is a set of vertices, or ‘0-simplices’, $S_0$ and non-empty sets $S_n$ of $(n+1)$-tuples of vertices, or ‘$n$-simplices’. $D$ is the dimension of the simplicial complex. To be an abstract simplicial complex, all faces of simplices (i.e., subsets of simplices) must also be simplices. As the alternative geometrical description of simplicial complexes will not be used in this paper, the prefix ‘abstract’ is dropped below. The $n$-skeleton of a simplicial complex $K$ is the sub-complex $K_n \subset K$ made up of all simplices in $K$ of dimension $\leq n$. For example, the 1-skeleton is made up of edges and vertices only, and as such is a graph. Where no confusion arises, the 1-skeleton will be referred to as the skeleton.

Consider a Euclidean geometry $M$ and a collection of $N_c$ points $P$ in $M$. A simplicial complex called the ‘Delaunay complex’ $C_d(P,M)$ can be associated with the points. The points are the vertices in the complex, and simplices are made up of sets of nearby vertices, according to the following prescription. Associated with each point $p \in P$ is an open region $\omega(p) \subset M$, its ‘Voronoi neighbourhood’, made up of all points in $M$ that are closer to $p$ than any other point in $P$. In the Delaunay complex, an edge is placed between two vertices $p$ and $q$ iff the closures of $\omega(p)$ and $\omega(q)$ share any points (i.e., if these regions border each other). For higher dimensional simplices, the prescription is similar: an $n$-simplex is placed between a set $v \subset P$ of $n+1$ vertices iff

$$\bigcap_{p \in v} \overline{\omega(p)} \neq \emptyset,$$

where $\overline{\omega(p)}$ signifies the closure of $\omega(p)$. If the points in $P$ are selected uniformly at random according to the volume measure on $M$, we call the resulting simplicial complex a random Delaunay complex $C_d(N_c,M)$.

If $M$ is flat or uniformly curved, then in this random process, it is with probability 0 that any point in $M$ is equidistant from more than $D+1$ points in $P$, where $D$ is the dimension of $M$ [31]. For any manifold, if the points are ‘at generic locations and sufficiently dense’ [31], the resulting complex $C_d(P,M)$ is a triangulation of $M$. This means that, if the ‘sprinkling density’ $\rho$ ($N_c$ divided by the volume of $M$) is sufficiently large with respect to the maximum curvature of $M$, then the random Delaunay complex $C_d(N_c,M)$ will be a triangulation with high probability. Below we refer to such manifolds as ‘low curvature on the sprinkling density scale’.

For higher curvatures this may not be true. As a visually accessible example, consider a 2-sphere connected to another 2-sphere by a thin neck, such that the volume of the neck is much less than that of the spheres. Below a certain density of sprinkled points, it is unlikely that the neck will contain any sprinkled points. Instead, it will probably be covered by two Voronoi neighbourhoods, one associated with a sprinkled point in each sphere. Thus, there will be an edge between these points. This will be the only edge connecting two points sprinkled into different spheres. The Delaunay complex will therefore contain an edge that is not part of any triangle. This is the reason that $C_d(P,M)$ is referred to here as a Delaunay complex, rather than the more commonly used ‘Delaunay triangulation’.

We are interested in the random Delaunay process because the complexes $C_d(P,M)$ produced are arguably insensitive to small-scale detail of $M$. However, there are some counter-examples to this involving non-trivial topology (e.g., holes). The application to the presently
considered CDT models will not be affected by this, and so the problem is not addressed in detail here. For now attention will be restricted to cases without complicated dynamical topology (although one can consider modifying the procedure to cope with microscopic topological detail, a subject that will be dealt with elsewhere).

3.2. Delaunay complex observables

A large class of observables of a Euclidean geometry $\mathcal{M}$ can be defined using the idea of the Delaunay complex of finite sets of points in $\mathcal{M}$ (we will assume $\mathcal{M}$ to be of finite volume $V(\mathcal{M})$ here). First, consider any real-valued function of (isomorphism classes of) simplicial complexes, say $f(S)$. This function holds a value for the Delaunay complex $C_d(P, \mathcal{M})$, but this value depends on the positions of the points $P$. To form an observable $O_f$, we must average over all positions of the points:

$$O_f = \int_{\mathcal{M}} d^D p_1 \sqrt{g(p_1)} \int_{\mathcal{M}} d^D p_2 \sqrt{g(p_2)} \cdots \int_{\mathcal{M}} d^D p_{N_c} \sqrt{g(p_{N_c})} f(C_d(P, \mathcal{M})).$$

(2)

where $D$ is the dimension of $\mathcal{M}$ and $P = \{p_1, p_2, \ldots, p_{N_c}\}$.

These are dubbed ‘Delaunay complex observables’. This kind of observable would clearly be extremely difficult to calculate analytically, but may be possible to approximate. The observable $O_f$ is the average value of $f(C_d(N_c, \mathcal{M}))$ for random Delaunay complexes $C_d(N_c, \mathcal{M})$, and so the random Delaunay complex procedure could in principle be used to sample the value of $f(C_d(P, \mathcal{M}))$, as a means to estimate $O_f$. If the observable is relevant in the classical regime, it should have low variance, and so this random sampling would quickly converge to the correct value for $O_f$. Indeed, there is already evidence that a single sample may be enough to accurately calculate some observables [31].

As manifestly generally covariant quantities, the observables satisfy criterion (a) given at the beginning to section 2 above (although after Wick rotation; see section 2.1). Criteria (b) and (c) will be dealt with later by giving a discrete version of the Delaunay process that is suitable for CDT computer simulations.

The conjecture is now made that a large class of Delaunay complex observables satisfy criterion (d), as discussed in section 2.1.

There is evidence from previous studies that large-scale geometrical information is captured by such observables. In previous work on the random Delaunay complex procedure in curved spaces, Bombelli, Corichi and Winkler consider manifolds that are low curvature on the sprinkling density scale. They conjecture that, from random Delaunay complexes $C_d(N_c, \mathcal{M})$, the geometry of the original manifold $\mathcal{M}$ can be approximately reconstructed [31]. This claim is given justification in 2D. For example, they have shown that the average curvature of a 2D geometry can be calculated from the expected average valency of the vertices of a random Delaunay complex on that geometry, and is found to obey

$$R(S) = 4\pi \rho \left(1 - \frac{1}{\xi} \bar{N}_1\right),$$

(3)

where $R$ is the average curvature of a geometry $\mathcal{M}$, $\rho$ is the sprinkling density $N_c/V(\mathcal{M})$, and $\bar{N}_1$ is the mean valency of vertices in a randomly generated complex $C_d(N_c, \mathcal{M})$. Note that this curvature estimator is only a function of the skeleton of the complex. Similar results exist in 3D [34], for negative curvature. It is conjectured that the valency of vertices will also be a function of the scalar curvature in higher dimensions, for manifolds with small curvature on the sprinkling density scale as discussed above.

3 There is now good computational evidence for this conjecture as applied to spheres in 3 and 4 dimensions, which will be presented in future work.
This speaks for the preservation of large-scale information when the fine-grained geometry
is low curvature. Since the Delaunay complexes are discrete, each one contains only a small
fraction of the information of the fine-grained geometry, and so it is reasonable to conjecture
that small-scale detail is absent from the coarse graining, as required.

As in all such schemes, the final test will be in the calculation. It may not be true that
a small deformation (appropriately defined) does not affect the placement of edges in the
Delaunay complex; for example, faces of microscopic size might appear due to the exotic
geometry near the Planck regime. Hopefully, the question of whether this happens in CDT
theories can be settled by simulations, and the coarse-graining procedure could be refined if
necessary.

At large values of $N_c$, a single coarse-grained simplicial complex is conjectured to be
enough to approximately reconstruct the geometry of the manifold it was generated from.
Such a complex can be given a geometrical interpretation of its own [31]. The interpretation
of such a complex is different from the DT interpretation, which attributes the equilateral
geometry to all simplices. Here the simplicial complex corresponds to a continuum geometry
if and only if it could have come from the Random Delaunay process on that geometry with
‘relatively high probability’. This rule does not uniquely fix the geometry without further
refinement, but this is not crucial. For example, for one random Delaunay complex, the local
scalar curvature at a marked point might vary over all corresponding geometries allowed by
this definition, making it badly approximated, but this is not a quantity of observational interest
(it has no relevance at large scales). The effective average curvature of a small region will be
well approximated.

As mentioned above, dynamical triangulations with large lattice length do not approximate
well to smooth space times. It may seem odd, therefore, to use a similar structure (an
unlabelled simplicial complex) for the coarse grainings. However, using the alternative
geometrical interpretation given here, which is natural for random Delaunay complexes, avoids
the problem with DTs. With this approach, low curvature space times can be approximated,
and it is possible to read off observables of relevance at large scales from the discrete structure.
This is not the case for DTs; no criterion exists to check if a DT is close to a flat space, for
instance. It is important to note that, even when the coarse graining is a triangulation (which
is not always the case), this geometrical interpretation may differ from the interpretation as a
dynamical triangulation. There is no obvious reason for the two interpretations of unlabelled
triangulations to coincide for all observables (although it is true the expression for average
curvature is the same in 2D).

3.3. Some useful coarse-grained observables

Of the large class of observables defined above, some are of particular interest for the study
of CDT quantum gravity simulations. The effective average scalar curvature is an example.
Assuming the main conjecture of the previous section is correct, we may call the curvature
estimator $R(S)$ above, or rather its Delaunay complex observable $O_R$, the effective average
scalar curvature. Also, we can consider definitions of effective dimension. These random
Delaunay complexes have a dimension, which provides new effective dimension estimators as
Delaunay complex observables. For each vertex in a complex there is a maximum dimension
of simplex which contains it. There is also a minimum dimension of simplex that contains
the vertex but is contained in no other simplex. The average value of these numbers over
all vertices provides two fractal dimension estimators, giving alternatives to the spectral and
Hausdorff dimensions. Hopefully the new dimension estimators will agree with the other ones
for effectively manifoldlike geometries (although for other geometries they may differ without
this being problematic).
As well as familiar observables like scalar curvature, some novel observables are of interest in quantum gravity. Spacetime can be so curved that, on some scales, there may be no smooth Euclidean manifold at all which would qualify as a good coarse graining under sensible rules. In this case, more general coarse-grained observables than the above may be useful to judge whether the simulations are approaching a ‘manifoldlike’ regime at large scales. We need effective measures of manifoldlikeness.

Consider a geometry \( Y \) that is low curvature on a certain sprinkling density scale. It has been recalled that a random Delaunay complex on this geometry is with high probability a triangulation with the dimensionality of \( Y \). According to the conjecture, a manifold \( X \) that approximates \( Y \) on large scales would have this property also. Thus, the property of a random Delaunay complex being a triangulation is a measure of manifoldlikeness. It would be interesting to ask how close to a triangulation the Delaunay complexes are for a particular geometry, as a better measure of effective manifoldlikeness at large scales. This requires a measure of how like a 4D triangulation a complex is. Some observables of this type are given for 2D in appendix A which suggest generalizations to higher dimensions.

### 3.4. Effective actions

The main aim here is to define some observables that can be used to probe the large-scale features of the CDT model. We can also consider using the coarse-graining scheme to define an effective action. The procedure, and purpose, of defining an effective action in a fixed-background lattice theory is well known. A treatment in a similar context to the present one is given in \[7\]. Basically, the partition function of the model is rewritten as a sum of contributions from coarse-grained configurations\(^4\), which are in turn calculated by summing contributions from the original fine-grained configurations. The CDT theory can be written in terms of a statistical sum over geometrical configurations \( T \) weighted by \( e^{-S(T)} \) where \( S \) is our Euclideanized action. At fixed volume, the partition function may be written briefly (see e.g. \[1\]) as

\[
Z_a(V, G) = \sum_T \mu(T) e^{-S(T)},
\]

where \( Z_a(V, G) \) is the partition function which depends on \( V \) the total volume, \( G \), the bare Newton’s constant and the lattice spacing \( a \). The action \( S(T) \) and measure factor \( \mu(T) \) are defined elsewhere \[1\]. The sum is over all causal dynamical triangulations \( T \) in a certain class (for instance 4D CDTs with \( S^3 \times \mathbb{R} \) topology). The effective action takes the form

\[
e^{-S_{\text{eff}}(C)} = \frac{1}{\mu(C)} \sum_T \Pr(C, T) \mu(T) e^{-S(T)},
\]

where the sum is as above, and \( \Pr(C, T) \) is the probability of generating the simplicial complex \( C \) from the random Delaunay complex procedure on these geometries:

\[
\Pr(C, T) = \int_{M} d^{D} p_1 \sqrt{g(p_1)} \int_{M} d^{D} p_2 \sqrt{g(p_2)} \cdots \int_{M} d^{D} p_{N_c} \sqrt{g(p_{N_c})} \delta(C, C_{d}(P, M)),
\]

where \( D \) is the dimension of \( M \), \( P = \{p_1, p_2, \ldots, p_{N_c}\} \), and \( \delta(C_1, C_2) = 1 \) if graph \( C_1 \) is isomorphic to \( C_2 \), and 0 otherwise (note that \( \Pr(C, T) \) is a diffeomorphism invariant quantity

\(^4\) In the above scheme, a probability distribution over simplicial complexes is defined from each fine-grained configuration. This may be a rather unwieldy object to consider as a coarse graining. When \( N' \) is large, however, observables like curvature can (with high probability) be accurately estimated by taking only one sample from the random Delaunay complex process, and so one sample can be considered as ‘the coarse-grained configuration’.
for these Euclidean geometries). The random Delaunay complex procedure is defined for any metric space with volume measure, and so, in principle, it could be applied to DTs directly (although a more practical scheme is given in section 4). This is an effective action on the space of all simplicial complexes.

In this scheme, the scale is controlled by the sprinkling density $\rho = N_c / V$, where $V$ is the volume of the fine-grained geometry. Rescaling after coarse graining would be taken care of by using appropriately scaled values of $\rho$.

Note that here, due to the irregular and dynamical lattice, it is not obvious how to break $Pr(C, T)$ down into a product of terms, one for each ‘block’ (whatever that would mean), as is possible in the standard case (indeed, this feature makes it easier to find schemes that preserve large-scale properties there). Although this is a disadvantage, it is not clear if any scheme could avoid it while still meeting the criteria for a good coarse-graining scheme, as discussed above.

Now the coarse-grained histories are general simplicial complexes, to be interpreted as the results of the random Delaunay complex procedure. The fine-grained histories on the other hand are DTs. Iterating the scheme as in the fixed background case is therefore problematic. This does not destroy the usefulness of the scheme, however. The coarse graining can still be applied for different values of $\rho$ to obtain coarse grainings at different scales (similarly to the techniques used to calculate the spectral dimension). The values of observables could still be checked at different scales to search for a fixed point and/or obtain their physical values in the continuum limit, and this is the main goal of this study.

4. A practical scheme

In previous sections, a way to coarse grain dynamical triangulations has been given. It was commented that the Delaunay procedure could be applied to DTs considered as piecewise-linear manifolds. However, it is likely to be impractical to implement by computer, since it would be necessary to calculate the geometrical distances between general points in a dynamical triangulation. Because of this, it is useful to consider an approximate, discrete version of the procedure that can be applied to a dynamical triangulation, satisfying criteria (b) and (c) of section 2.

These modifications of the scheme should be compared to using random walks on the simplices to calculate the spectral dimension, rather than a continuous diffusion process [22]. It is a reasonable conjecture that the simplifying modifications of the simplicial coarse-graining procedure will not significantly effect results, particularly when the coarse-graining ratio $N_f / N_c$ is large.

4.1. Coarse graining a simplicial complex

A procedure for coarse graining a simplicial complex is now presented. The algorithm is specified to the extent that concrete implementation by computer is possible. The procedure is modelled on the random Delaunay complex process, except that it uses the combinatorial properties of the triangulation and avoids using the geometrical distance.

The fine-grained history is taken to be a simplicial complex or dynamical triangulation $C_f$. The set of vertices of $C_f$ will be referred to as $V_f$, and the number of them as $N_f$. The

\[5\] In some cases, the class of triangulations employed in dynamical triangulations is not only triangulations of manifolds, but is rather ‘pseudo-manifolds’, triangulations made from gluings [29]. These may not be simplicial complexes. However, distance along the skeleton is still defined, as is the inclusion of sub-simplices in higher dimensional simplices, and so the process about to be described is defined for these gluings.
procedure starts with the random selection of a subset \( V_c \subset V_f \) made up of \( N_c \) vertices, which are to serve as the vertices for the coarse-grained complex \( C_c(V_c, C_f) \). This amounts to restricting the coarse-grained vertices to lie on the vertices of the fine-grained triangulation. In order to approximate the uniform measure on sets of points in the continuum, the probability of selecting a particular vertex is weighted by its ‘share of volume’, which in dynamical triangulations is proportional to the number of highest dimensional simplices containing the vertex (in 2D this is equal to the valency of the vertex). As above, the coarse-grained complex can be written as a random variable \( C_c(N_c, C_f) \), which is abbreviated to \( C_c(C_f) \) when a fixed \( N_c \) is assumed. The ‘coarse-graining ratio’ \( N_f/N_c \) will also be used below.

Similarly to the continuum case, we now associate a ‘Voronoi neighbourhood’ \( \omega(v_c) \) with each coarse-grained vertex \( v_c \in V_c \). The neighbourhood \( \omega(v_c) \) is in this case a subset of the fine-grained vertices \( V_f \). Here, \( \omega(v_c) \) contains the set of fine-grained vertices that are closer to \( v_c \) than any other member of \( V_c \), in graph distance on the skeleton of \( C_f \).

However, some fine-grained vertices can be equidistant from many members of \( V_c \) with non-zero probability. A random attribution of Voronoi neighbourhood is used for these vertices. Consider a case where this occurs for a fine-grained vertex \( v_f \in V_f \). The neighbourhood \( \omega(v_c) \) is in this case a subset of the fine-grained vertices \( V_f \). Here, \( \omega(v_c) \) contains the set of fine-grained vertices that are closer to \( v_c \) than any other member of \( V_c \), in graph distance on the skeleton of \( C_f \).

Now the edges of \( C_c(C_f) \) are added. In the continuum procedure, the edges are between coarse-grained vertices whose Voronoi neighbourhoods share boundary points. In this graph procedure, the different neighbourhoods are connected by edges, which play the role of the boundaries. Analogously to the continuum case, an edge is added in the coarse-grained graph between \( v_c \) and \( w_c \) iff there is an edge in \( C_f \) between any member of \( \omega(v_c) \) and any member of \( \omega(w_c) \). This defines the skeleton of \( C_c(N_c, C_f) \).

The addition of higher order simplices is similar. In \( C_c(N_c, C_f) \), there is an \( n \)-simplex comprised of the coarse-grained vertices in set \( b \subset V_c \) if their neighbourhoods ‘share an \( n \)-simplex’, i.e. if there is an \( n \)-simplex \( s \) in \( C_f \) such that

\[
s \cap \omega(v_c) \neq \emptyset \quad \forall \; v_c \in b. \tag{7}
\]

The process is illustrated in figure 1.

It is also easy to define part of a variant to the scheme: a similar process on the cell complex dual to the triangulation. The definition of the skeleton of the coarse graining would be entirely analogous, and that is all that is needed for some of the interesting observables, like the proposed effective average scalar curvature. It would make a good consistency check to develop two coarse-graining schemes which could be compared, and the dual version may be more practical in cases where configurations in the CDT simulations are stored as the dual complex (this is the case in the 3D simulations [35]).

Being applicable to any simplicial complex, this would be possible to apply to dynamical triangulations. So far the discussion has been appropriate to Euclidean dynamical
triangulations, but the procedure could also be applied to Wick rotated causal dynamical triangulations. After coarse graining, the special layered structure of the CDT is lost, but that has no bearing on the recovery of physically interesting observables. A further difference in the case of causal dynamical triangulations is that there are, in the 4D model, two possible edge lengths rather than just one, for spacelike and timelike edges. This complication is ignored for now (as it has been when calculating the spectral dimension) as it is unlikely to grossly affect properties of interest. But in principle it would not be hard to incorporate different edge lengths into the coarse-graining procedure.

4.2. The application to a CDT path integral

It is helpful to have a definite scheme laid out for the application to CDT simulations, which is the next step for this program.

The methods of computing the coarse-grained observables will be very similar to those used to find the spectral dimension and other observables. The discretized CDT path integral at fixed volume has been given in equation (4). Monte Carlo simulations produce instances of causal dynamical triangulations according to the weights given by the Euclideanized path integral, which can be used to estimate the expectation values of observables at different values of the cut-off. The approach is similar to that used in lattice QFT. In the CDT simulations, the number of coarse-grained vertices $N_c$, which is proportional to the dimensionful volume,
will be held fixed while the number of fine-grained simplices is moved towards infinity. The remaining dimensionless coupling constant of the model is the inverse (bare) Newton’s constant which will be held fixed. As the lattice spacing is sent to zero, large-scale effective observables should approach a fixed value, their continuum limit. Using the techniques of the previous section, certain Delaunay complex observables of section 3.2 will be estimated.

It would be sensible to start by applying the idea to the 3D model. In 3D, the only relevant solution of the Einstein equations is De Sitter, or on the Euclidean side the sphere. So far all measured observables are consistent with the conjecture that the 3D simulations produce spheres with small fluctuations in the geometry. Because of the lack of propagating degrees of freedom in 3D gravity, it is hoped that no spatially correlated geometrical fluctuations will survive in the continuum limit, i.e. that the limiting geometry will be exactly a sphere. Applying the coarse-graining scheme, the first aim would be to compare the expectation values of certain Delaunay complex observables (such as the average valency) from the CDT simulations with the values obtained from a sphere with the same number of coarse-grained vertices. In the limit, the hope would be that the CDT expectation values converge to the values for the sphere. Fortunately, Delaunay triangulations of a sphere are easily generated by computer, making Delaunay complex observables easy to compute in this case, even if they cannot be found analytically.

The observables mentioned in section 3.3 could be used for this purpose. Beyond this, it would be desirable to know a sufficient list of Delaunay observables \( \{ O_i \} \), such that the sphere (and geometries indistinguishable from it at large scales) is picked out as the only geometry approximately holding a certain set of values \( O_i = O_i^* \). In 3D perhaps this could be explored analytically, building on the existing results [34].

If successful, similar techniques would then be applied in 4D. Also, the new scaling dimensions and the measures of effective manifoldlikeness discussed in section 3.2 would be interesting to calculate for 4D CDTs, and the results compared to the other dimension estimators.

5. Conclusion

A new method for coarse-graining dynamical triangulations has been presented, and some arguments for its physical suitability have been given. As in other aspects of the theory of dynamical triangulations, the aim has been to build on ideas from standard lattice quantum field theory. Thus, as in lattice quantum field theory, coarse-grained observables can be calculated for each history in the quantum sum, and the expectation value is found as a statistical average. In the lattice field theory case, it is crucial that the coarse-grained version of an observable only depends on large-scale properties of the history, and this remains crucial for quantum gravity coarse-graining schemes, where it becomes a more involved issue. This must be a primary consideration for any such scheme.

A number of technical issues remain. There is a need for extensions of known results of [31, 34] that relate average valency of the Random Delaunay complexes to scalar curvature. Their conjecture is that all scalar curvature invariants can be calculated from simple properties of random Delaunay complexes in three and four dimensions (it is proved in 2), for manifolds of low curvature on the sprinkling scale. It will be useful for this coarse-graining program to find these expressions for the curvature invariants. Other conjectures are raised in this paper to add to those of Bombelli, Corichi and Winkler. It would be interesting to extend their results

\[ \text{Because of the topological restrictions the simulations cannot strictly produce a sphere, but it is argued that a sphere connected at two points by a cylinder of minimal diameter should play the same role as the classical solution [35].} \]
to geometries that are ‘close’ to the low curvature geometries that have so far been studied. This would amount to a justification of the central conjecture given in section 3.2 for these observables. Finding a characterization of spheres in terms of simple properties their random Delaunay triangulations would be useful for the 3D and 4D simulations.

It would also be of use to carry out some computational tests of the scheme set out above. Some of the above conjectures are amenable to this approach. For example, a 2D torus with extended dimensions and a 3D torus with one compactified dimension are uncontroversially ‘close’ geometries in the relevant sense. Tests could be performed to see if interesting Delaunay observables converged to the same values for these two types of geometries at small sprinkling density. This could also be done for the discrete coarse graining of section 4.

Adding matter to the scheme would be fairly easy in principle. For instance, for Ising matter the spin of a coarse-grained vertex $v_c$ could be taken directly from the value on the fine-grained history (similar to Kadanoff’s decimation method [5]). Alternatively, the mode spin of vertices in the discrete Voronoi neighbourhood $\omega(v_c)$ defined in section 4.1 could be taken as the spin for the coarse-grained vertex $v_c$ (similar to the majority spin method).

The coarse-graining procedure may also have uses outside of dynamical triangulations, as it can be applied to any simplicial complex (indeed, this might even suggest new ways to discretize the path integral). Unlabelled graphs were considered as encoding geometry in [31], as a stepping stone to the spin-networks of loop quantum gravity. It is possible that this scheme would be of use in this context as well, and that a similar scheme could be developed for labelled graphs. In this case however, the issue of non-local links [30] may need to be dealt with. It would be interesting to see if some modification of the scheme was eventually necessary. The application of this, or similar coarse-graining ideas, in the spin-foam program (considered as a path-integral formulation of QG as in [13]) is also a possibility.

This paper is intended as the first in a series. The next step is the application of the coarse-graining procedure, as outlined in section 4.2. The final worth of any coarse-graining scheme depends on the success of the calculations in producing good, consistent results. On the point of comparison with results, further adaptation of the scheme may become necessary, similarly to other successful lattice coarse-graining programs, as mentioned in section 2.1. From simulations so far, indications are that the program should be practical, in terms of available computational power. In this way, hopefully, evidence for the convergence of renormalized physical quantities such as average scalar curvature could be found.

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Appendix A. Observables for manifoldlikeness in 2D

It is claimed in section 3.3 that, if a geometry $X$ is effectively manifoldlike at a certain scale, then a random Delaunay complex on $X$, taken at the appropriate sprinkling density, will be a triangulation of a manifold. Effective manifoldlikeness is a very interesting observable for quantum gravity.
However, this easily stated condition of manifoldlikeness has no tolerance. The addition or subtraction of one edge can alter the result. In simulations, small deviations are bound to arise at some scales, and so we will need to quantify how close a graph is to being a triangulation. One such quantification for a graph $G$ is the size of the largest induced subcomplex in $G$ that is also an induced sub-complex of some triangulation. Another quantity would be the number of vertices whose ‘star’ (defined below) is topologically a ball of given dimension. Below, similar quantifications are introduced that are easy to calculate.

At a practical level, it is preferable to deal only with the skeleton of the complex where possible, to avoid having to compute the higher dimensional simplices. Also, it is possible that measures of manifoldlikeness depending only on the skeleton would be more robust against small deviations of the type discussed above. With this in mind, we would like to distinguish graphs which are the skeleton graph of some triangulation of a manifold. These questions are considered in this appendix for the simple case of 2D triangulations.

First, it is necessary to be more specific about the kinds of triangulation under consideration. An abstract simplicial complex has been defined in section 3.1, and we continue to drop the term ‘abstract’ and treat it as implicit in the following. A 1D simplicial complex is a graph $G = \{S_0, S_1\}$, where $S_0$ is the set of vertices and $S_1$ is the set of edges. In 2D, we have $S = \{S_0, S_1, S_2\}$ where $S_0$, $S_1$ and $S_2$ are the vertices, edges and triangles. Where definitions below differ from those in [29], it is a consequence of the use of the abstract description of simplicial complexes.

A triangulation of a manifold is called a simplicial manifold, which is a type of the simplicial complex. To characterize them further we need the following definitions. In a simplicial complex, the star of a simplex $\sigma$, $\text{star}(\sigma)$, is the set of all simplices of which $\sigma$ is a face, and their faces. The star is also a simplicial complex by this definition. The link of $\sigma$, $\text{link}(\sigma)$, is the set of all simplices in the star of $\sigma$ for which $\sigma \cap \sigma_f = \emptyset$. The following is an immediate consequence theorem 1 of [29].

**Theorem 1.** A 2D simplicial complex $S$ is a simplicial manifold iff the link of every vertex in $S$ is a cycle of edges.

To rephrase the original problem, we want to be able to distinguish those graphs $G = \{S_0, S_1\}$ for which there exists a simplicial manifold $S(G) = \{S_0, S_1, S_2\}$. The task will involve identifying $S_2$, an assignment of triangles to the graph, if one exists.

First, we must examine what theorem 1 means for the skeleton of the simplicial manifold.

A wheel is a graph of order $n$ which contains a cycle of order $n - 1$, and for which every graph vertex in the cycle is connected to one other graph vertex, which is known as the hub (see figure 2). A full wheel of a graph $G$ is a subgraph of $G$ that is a wheel and contains all the vertices of $G$; a graph $G$ will be said to have a full wheel if it contains a full wheel of itself. A few easily proved lemmas are necessary for the final results.

![Wheel graphs, of order 4, 5 and 6.](image-url)
Lemma 1. For any vertex $v$ in a 2D simplicial manifold, the skeleton of $\text{star}(v)$ is a wheel of which $v$ is the hub.

Proof. Consider $v$, a vertex in a 2D simplicial manifold. Clearly, the vertices in $\text{star}(v)$ are exactly those in $\text{link}(v)$ and $v$. The skeleton of $\text{link}(v)$ is a cycle, from theorem 1, and so the skeleton of $\text{star}(v)$ contains a cycle containing all vertices but $v$. Every vertex in the cycle $\text{link}(v)$ shares an edge with $v$, so the skeleton of $\text{star}(v)$ is a wheel with $v$ as the hub. □

Now we have characterized the skeleton of $\text{star}(v)$ for $v$ being a vertex in a simplicial manifold. But the definition of $\text{star}(v)$ depends on more than the skeleton $G$ of the simplicial manifold. To use lemma 1 we must be able to identify the skeleton of $\text{star}(v)$ in $G$ without reference to any triangles. The concept of the graph neighbourhood is useful here. Let the graph neighbourhood, $\text{gn}(v)$, of a vertex $v$ in a graph $G$ be the subgraph induced by $v$ and all of those vertices that share an edge with $v$.

Lemma 2. For any vertex $v$ in a 2D simplicial manifold, $\text{gn}(v)$ has a full wheel, of which $v$ is the hub.

Proof. Vertices in $\text{star}(v)$ either share an edge with $v$, or are $v$. Conversely all vertices that share an edge with $v$ are in $\text{star}(v)$. Therefore the skeleton of $\text{star}(v)$ is contained in $\text{gn}(v)$ and $\text{gn}(v)$ contains no other vertices. The lemma then follows from lemma 1. □

On its own, this provides a necessary condition for a graph $G$ to be the skeleton of a simplicial manifold. It is also possible to find a sufficient condition. Let us take a graph $G$ such that the condition of lemma 2 is satisfied, and such that there is one and only one full wheel in $\text{gn}(v)$ for all $v$. Each vertex in $G$ then has a unique associated wheel graph. We may then associate a unique set of triangles $t(v)$ with each vertex: the set of 3-cycles in this wheel.

Theorem 2. Let $G = \{S_0, S_1\}$ be a graph such that, for every $v$, $\text{gn}(v)$ has one and only one full wheel subgraph of which $v$ is the hub. Let $t(v)$ be the set of 3-cycles in this wheel, and let $S(G) = \{S_0, S_1, S_2\}$ be the simplicial complex such that $S_2$ is the union of $t(v)$ over all $v \in G$. If $t(v)$ are the only triangles containing $v$ for all $v$, then $S(G)$ is a 2D simplicial manifold.

Proof. Consider a vertex $v$ in a simplicial complex $S(G)$ as described in the theorem. If $t(v)$ are the only triangles containing $v$ in $S(G)$, then they are the only triangles in $\text{star}(v)$ in $S(G)$. □
It is easy to see that $\text{link}(v)$ is therefore a cycle. By theorem 1, if this is true for all vertices in $S(G)$, then $S(G)$ is a 2D simplicial manifold.

Now we have a sufficient condition for a graph $G$ to be the skeleton of some simplicial manifold. In a simulation, therefore, only a positive result would be conclusive. To make a necessary condition, we would have to allow $gn(v)$ to contain more than one full wheel subgraph, and look for an assignment of $t(v)$ such that $S(G)$ is a 2D simplicial manifold. But we will now show that the class of simplicial manifolds which do not satisfy the condition of theorem 2 is not important for our purposes.

**Lemma 3.** The graph neighbourhood $gn(v)$ of every vertex $v$ in the triangulation of a sphere is a planar graph, which has a full wheel subgraph of which $v$ is the hub.

**Proof.** It is well known and easy to see that the skeleton $G$ of a triangulation of a sphere is a planar graph in which every region is bounded by three edges. The graph neighbourhood of a vertex in $G$ is a subgraph of $G$, and is therefore also planar. The lemma then follows from lemma 2.

**Lemma 4.** If $gn(v)$ in a graph $G$ is planar and has a full wheel subgraph of which $v$ is the hub, then it has only one full wheel subgraph of which $v$ is the hub.

**Proof.** In a graph $G$, consider a graph neighbourhood $gn(v)$ of a vertex $v$ that is planar and has a full wheel subgraph, of which vertex $v$ is the hub. The lemma is trivial when $gn(v)$ contains four vertices, as the full wheel subgraph is then the complete graph on four vertices.

Consider the case in which $gn(v)$ is of order 5. Let us label the vertices $\{v, 1, 2, 3, 4\}$. The graph $gn(v)$ contains edges from $v$ to all other vertices. It also contains a cycle from the full wheel graph which we can label $\{1, 2, 3, 4, 5\}$ or schematically $1-2-3-4-1$. If $gn(v)$ in order to contain more than one full wheel with $v$ as hub, it must also contain another 4-vertex cycle in the subset $\{1, 2, 3, 4\}$, and therefore would contain the edges $\{2, 4\}$ and $\{1, 3\}$. In this case, $gn(v)$ is the complete graph on five vertices, $K_5$. This is not a planar graph, and so this violates our assumptions.

For larger order $gn(v)$, it is not hard to see that any other $gn(v)$ with two full wheel subsets (with the same hub) can be reached from $K_5$ by graph expansion. Similarly to the order 5 case, let us label the vertices $\{v, 1, 2, 3, 4, \ldots, n\}$ and let one of the cycles $c$ be $1-2-3-4-\ldots-n-1$. There is another order $n$ cycle in $gn(v)$, $\tilde{c}$ containing $\{1, 2, 3, 4, \ldots, n\}$. $p_1-p_2-p_3-\cdots-p_n-p_1$, where $\{p_i\}$ is a non-cyclic permutation of $\{1, 2, 3, 4, \ldots, n\}$. From the properties of permutations, there exists some pair of edges $\{p_a, p_b\}$ and $\{p_c, p_d\}$ in $\tilde{c}$ such that $p_a < p_c < p_b < p_d$ or $p_a < p_c < p_d < p_b$. Therefore, all pairs of vertices in the set $\{v, p_a, p_b, p_c, p_d\}$ are connected by paths that do not include any other vertex in the set $\{v, p_a, p_b, p_c, p_d\}$ (this is because $v$ shares an edge with all others, $\{p_a, p_b\}$ and $\{p_c, p_d\}$ are edges, and the rest are connected by the cycle $c$). It follows from this that $K_5$ is a graph minor of $gn(v)$ (i.e. related to $gn(v)$ by edge deletion and/or edge contraction). Therefore such $gn(v)$ are not planar by Kuratowski's reduction theorem, and violate our assumptions for the same reason.

**Corollary 1.** The graph neighbourhood $gn(v)$ of every vertex $v$ in the triangulation of a sphere has one and only one full wheel subgraph of which $v$ is the hub.

This corollary is the main result of this appendix. It shows that the graph neighbourhoods in the triangulation of a sphere satisfy the conditions in theorem 2. This condition is therefore sufficient for a graph $G$ to be the skeleton of a simplicial manifold, and necessary for it to be the skeleton of a triangulation of a sphere.
If a graph neighbourhood $gn(v)$ in a graph $G$ does contain more than one full wheel graph, then it is not isomorphic to a graph neighbourhood in any triangulation of a sphere; $G$ contains a 'small handle' at $v$. If this $G$ arose from the random Delaunay process on a manifold $X$, this can be counted as short scale detail in $X$, and so it is justified to say that $X$ is not manifoldlike at that scale—this topological detail should be removed by decreasing the sprinkling density. Also, space time topology is fixed in the CDT models that are presently being used in simulations, and so we do not expect that such handles will arise there. With this in mind, we expect that the condition of theorem 2 is not too strict to be of use in CDT simulations.

This condition also suggests two measures of how close a graph is to being a triangulation. The first, $\text{Tri}_1(G)$, is the fraction of vertices $v$ such that $gn(v)$ is a planar and contains a full wheel of which $v$ is the hub. The second, $\text{Tri}_2(G)$, is the fraction of vertices $v$ in $G$ that obey this condition, plus the extra condition that the $t(v)$ are the only triangles containing $v$. Only if $\text{Tri}_2(G) = 1$ do these measures show that $G$ is the skeleton of a simplicial manifold, by theorem 2. For a dynamical triangulations simulation, if the average value of $\text{Tri}_2$ approached 1 as the lattice spacing $a$ was taken to zero, this would be evidence of manifoldlike behaviour on the sprinkling scale, according to the conjectures of section 3.2. We can add to this list the more strict measure $\text{Tri}_3(G)$ which is 1 if $G$ satisfies the conditions of theorem 2 and 0 otherwise.

These quantities have been found to be practically computable for large simplicial complexes generated by the Delaunay procedure on a 2D sphere. For higher dimension, the conditions will inevitably be more complicated, but may still be practical to use. In this case, a triangulation of a sphere is not uniquely determined by its skeleton. The alternative, of computing the link given the higher dimensional simplices, may then be preferable. This choice will be affected by the confidence in different aspects of the coarse-graining scheme, and this can only be decided as the scheme is further examined.

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