Abstract. While the composite fermion picture is so effective as to describe the excitation spectra including the spin wave for Laughlin’s quantum liquid, “how heavy and how strongly-interacting” remains a formidable question for the composite fermions, to which this article first addresses. The effective mass (purely interaction originated) defined from the excitation spectrum and obtained for various even- as well as odd-fractions exhibits a curious, step-like filling dependence basically determined by the number of flux quanta attached to each fermion, where the non-monotonic behaviour indicates a strong effect of gauge-field fluctuations. The excitation spectrum fits a Fermi liquid, but again a large effect of inter-composite fermion interaction appears as anomalous Landau’s parameters.

We have then moved on to see how the introduction of three-dimensionality (where the shape of the Fermi surface becomes relevant) affects the interacting electron system, and propose the magnetic-field induced SDW in three-dimensional systems. This should be a good candidate, in entirely realistic magnetic fields, for the integer QHE recently predicted by Koshino et al to occur in 3D on the fractal energy spectrum similar to Hofstadter’s. The mechanism for the field-induced phase is an effect of interaction in Landau’s quantisation on incompletely-nested (i.e., multiply-connected) Fermi surfaces, so the interplay of many-body physics and the magnetic quantisation on various Fermi surfaces may provide an interesting future avenue for 3D systems.

1 Introduction

To commemorate the two decades of the quantum Hall effect (QHE), I shall discuss two of the important problems in the QHE physics — interaction and dimensionality.

Fractional quantum Hall system is unique as a correlated electron system in two ways. First, the system is in the limit of strong electron correlation in high-$T_C$ language is infinite) in that the kinetic energy is quenched due to Landau’s quantisation. Second, spatial dimensionality of two allows Chern-Simons gauge field theoretic treatments. The composite fermion picture, one of the most fascinating concepts derived from the fractional QHE, indeed provides a good description of not only the ground-state properties, but even the excitation spectra including the spin wave and the charge mode for Laughlin’s quantum liquid.

However, we are still far from a complete understanding of the composite fermion. Specifically, as soon as we go beyond the mean field, the problem of “(i) how heavy and (ii) how strongly interacting composite fermions are” becomes a formidable question. Unlike the ordinary system, Landau’s quantisation makes the “dressing the bare mass” impossible. By the same token whether the composite fermions are nearly free or strongly interacting is entirely determined by the fluctuations beyond the mean field. Here we have first found that the effective mass of a composite fermion(CF) exhibits a curious step-function like behaviour against the Landau level filling. A most well-defined way to probe the interaction is to see whether Landau’s Fermi liquid picture holds. We have then examined this to conclude that the inter-CF interaction is of Hund’s type (negative exchange) in both spin and orbital channels, and the strength of the interaction is significant (or can even be anomalous).

The strong-correlation limit raises an interesting question of how other quantum phases such as the BCS paired state should appear, especially as compared with ordinary correlated electron systems on lattice structures. For the latter, it is becoming increasingly clear that the shape of the Fermi surface controls the occurrence of pairing, which is
anisotropic and includes the possibility of p, d, f symmetries. By contrast the FQH system is isotropic but the interaction (between the composite fermions) is controlled by the Landau index, for which we here examine the CF pseudopotential in the particle-hole channel from the viewpoint of the Fermi surface effect.

Finally we ask ourselves: can such QHE physics in 2D systems have a possible extension to three-dimensional systems in strong magnetic fields? In 3D the kinetic energy, hence the shape of the Fermi surface, should become relevant. As a starting point I shall describe our recent proposal for the magnetic-field induced SDW in 3D, where the integer QHE, which is predicted by Koshino et al to occur in 3D on the fractal energy spectrum similar to Hofstader’s, should be realised in entirely realistic magnetic fields. The mechanism for the field-induced phase is an effect of interaction in Landau’s quantisation on incompletely-nested Fermi surfaces (that become multiply-connected after the SDW formation), so the interference of the magnetic quantisation and the shape of the Fermi surface suggests an interesting future avenue for the many-body physics in 3D systems.

2 Effective mass of the composite fermion against ν

There are several ways to define the mass of the composite fermion. The problem, related with the gauge-field fluctuations, is highly nonperturbative, where one way is to determine the mass and the interaction numerically from the (electron-hole) excitation spectra (i.e., a two-particle property) for finite systems. So we have systematically studied even and odd fractions with spin degrees of freedom included. We have done this in two steps. We first assume a free composite fermion picture to estimate the mass before moving on to Landau’s Fermi liquid picture.

For the FQH states at odd fractions, \(\nu = 2\pi n_e/(eB) = d_s p/(d_s \phi p \pm 1)\), we estimate the effective mass, \(m^*\), from an excitation gap, which is the effective cyclotron energy, \(\omega_c^* = eB^*/m^*\) for \(eB^* = eB - 2\pi \phi n_e\), in the free composite fermion picture with an effective \(\nu^* \equiv 2\pi n_e/(eB^*) = d_s p\). Here \(\phi\) is the number of flux quanta attached to each electron, \(n_e\) the number density of electrons, \(p\) a positive integer, \(d_s = 2\) the spin degeneracy (\(d_s = 1\) for the spinless case) and the natural units with \(\hbar = c = 1\) are adopted.

For even fractions, \(\nu = 1/\phi\), we can estimate \(m^*\) for a metal of composite fermions. In this case the low-lying excitation is (electron-hole pair) excitations around the “Fermi surface”, and \(m^*\) can be determined how the excitation gap vanishes for \(N_e \to \infty\), when the number of electrons is \(N_e = d_s (l_F + 1)^2\), i.e., the closed-shell case with \(l_F\) being the Fermi angular momentum.

Let us look at the effective mass thus estimated in the free CF picture for various values of Landau-level filling in Fig.2, which plots the inverse effective mass against \(\nu\) for the spinless case for the 1/r Coulomb interaction. We can immediately see that the effective mass against \(\nu\), while basically becoming heavier for \(\nu = 1/2 \to 1/4 \cdots\), exhibits a step-function-like behaviour, where each step corresponds to each number of attached flux quanta, \(\phi(= 2, 4, \cdots)\). Within each step, \(m^*\) is only weakly dependent on \(\nu\) (regardless of fractions odd or even), which implies that the effective mass is basically determined by \(\phi\), the number of attached fluxes.

This is totally unexpected, since the CF theory in a mean-field treatment predicts a smooth function (dashed line in the figure, which will discussed below) \([12]\). Within each step, \(m^*\) is nearly constant in \(\nu\), exhibits a step-function-like behaviour, where each step corresponds to each number of attached flux quanta, \(\phi(= 2, 4, \cdots)\). Within each step, \(m^*\) is only weakly dependent on \(\nu\) (regardless of fractions odd or even), which implies that the effective mass is basically determined by \(\phi\), the number of attached fluxes.

Note at the same time that the usual practice of linearly plotting \(h\omega_c^*\) versus \(B^*\) is allowed only when \(m^*\) is nearly constant in each region specified by \(\phi\). Experimentally, the effective mass from the thermal activation energy in the Shubnikov-de Haas oscillation by Leadley et al shows a difference in the \(B\)-dependence between \(\phi = 2\) and 4 \([14]\).
3 Composite fermion gas: a Fermi liquid?

Inclusion of the spin degrees of freedom turns out to vastly modify the excitation spectrum, which is a sign that the spin-spin (exchange) interaction between CF’s is not negligible. The exchange interaction between composite fermions has been estimated for the flat wave, where the spin stiffness is shown to be larger as we change \( \nu = 1/3 \rightarrow 1/5 \cdots \), which is explained by the composite-fermion picture (Fig.1; [3, 15]).

Here we more closely look at the spin-dependent interaction in terms of the Fermi liquid picture for CF’s for even-fraction metals. This picture assumes that the excitation energy, \( \delta E \), is given as a functional of the deviation in the particle number from the ground state with the Landau function \( f_{l \sigma l' \sigma'} \) as a coefficient. We can then expand \( f \) to the first order in \( l \cdot l' \) (which corresponds to the spherical harmonics expansion for \( f_{pp} \) in a flat system) and \( \sigma \cdot \sigma' \). After a bit of algebra[7], we end up with

\[
\delta E = \Delta_{FL} \left[ 1 + \frac{1}{d_{\alpha}(l_F + 1)} \left( (G_0 + G_1)S \cdot S \right. \right.
\]
\[
+ \frac{1}{4} \left[ F_1 + G_1(3 - 2S \cdot S) \right] \frac{L \cdot L}{l_F(l_F + 1)} \left. \right] \right],
\]

where \( \Delta_{FL} = (l_F + 1)/(m^*_{FL} R^2) \) with \( m^*_{FL} \) being the effective mass defined in the context of the Fermi liquid theory, \( F \)'s and \( G \)'s are (dimensionless) Landau parameters, \( L = l - l' \) and \( S = (1/2)(\sigma - \sigma') \) (the total angular momentum and spin, respectively, of the excitation from a closed shell).

So we can estimate \( m^*_{FL} \) and the Landau parameters simultaneously by best-fitting the numerical results. First thing we notice is that the coefficient \( (G_0 + G_1) \) for the \( S \cdot S \) term is negative, i.e., we have a Hund’s second rule exchange interaction. This is readily seen from Fig.3(b), where \( \delta E \) for spin-flip excitations is significantly smaller (about one half, which implies that \( G_0 + G_1 \approx -1 \) for \( N_e = 8 \)). The observation is consistent with previous results.[3, 16]

We can in fact examine the whole spectrum. If we take the spinless case for simplicity and look at the low-lying excitation spectrum (solid circles in Fig.3(a)) against the angular momentum \( L \), which nuclear physicists would call an “Yrast spectrum”, the spectrum delineates the lower boundary of the Fermi liquid excitation spectrum. The free CF result (crosses) is so good as to reproduce the spectrum, including the shell structure that has to do with \( 2nk_F \) effects, where \( k_F \) is the Fermi wavenumber of the composite fermion.

To be more precise, the exact result lies significantly below the free CF result for larger \( L \). Since in the Landau’s picture the excitation energy contains a term \( \propto F_{\text{spinless}}^* L \cdot L \) with \( F_1^\text{spinless} = F_1 + 3G_1 \), this implies the Landau parameter \( F_{\text{spinless}} \) is negative, i.e., CF’s have an orbital exchange coupling of the Hund’s first rule, as consistent previous works on the \( \nu = 1/2 \) ground state.[17, 18] So the FQH system has spin- and orbital-exchange couplings both of which are Hund’s type. The Fermi-liquid result for \( m^*_{FL} \) is comparable with \( m^* \) estimated from the free CF picture.

However, we have to hasten to add that the Landau parameter, \( F_1^\text{spinless} \), is ill-behaved in that the quantity is sample-size-dependent: \( F_1 = -0.5(-1.0) \) for \( N_e = 9(16) \). Landau parameters should be scale invariant in a Fermi liquid, so the anomalous behaviour should indicate that the ordinary Fermi liquid picture cannot be directly applied. This may have to do with the relation,

\[
m^*_{FL}/m_{\text{bare}} = 1 + F_1^\text{spinless}/2
\]

(with 1/2 due to the dimensionality of two), where the infinite bare mass for the Landau level for \( N_e \rightarrow \infty \) would imply \( F_1^\text{spinless} \rightarrow -2 \), an anomalously large value.

We have also investigated how the situation changes as the interaction is made shorter-ranged or longer-ranged than the Coulombic, since RPA[20] and renormalisation-group results[21] suggest that the system is a marginal Fermi liquid just for the Coulombic interaction while a normal Fermi liquid recovered for shorter-ranged ones. When the functional form of the interaction is made longer-
ranged \(V(r) \propto 1/r^\alpha; \alpha > 1\), both \(m^*_\text{FL}\) and the Landau parameter \(|F_1|\) increase \((F_1 = -0.06 \rightarrow -0.8 \rightarrow -1.2\) for \(\alpha = 0.5 \rightarrow 1.0 \rightarrow 1.5\)). In the thermodynamic limit, the Landau function could possibly be singular, as considered by Stern and Halperin\[22\] by summing the diagrams in accordance with the Ward-Takahashi identity. The size-dependence of \(F_1^{\text{spinless}}\) may be related to the marginal Fermi liquid predicted with RPA in Ref.\[20\]. Note that we have deduced \(m^*\) from the excitation spectrum of the lowest-Landau-level projected model, i.e., a two-particle property in a model with an infinite bare mass. So, even when the mass defined from the pole of the one-particle Green’s function is anomalous, the excitation energy (e.g., the energy required to create a particle-hole pair) can be less anomalous. At any event, if the Fermi or marginal Fermi liquid persists in the thermodynamic limit, this will serve as an instance in which a system that has no small parameters (interaction/kinetic energy= \(\infty\), \(\hat{\phi} \sim O(1)\)) can be a Fermi liquid.

### 3.1 Effect of gauge field — Shankar-Murthy theory

Let us further discuss the mass from the viewpoint of the gauge field fluctuations. The \(1/m^* = (0.185 \pm 0.002)e^2\ell\) at \(\nu = 1/2\), where \(\ell \equiv 1/\sqrt{eB}\) is the magnetic length, obtained here from the excitation gap is slightly smaller than \(1/m^* \approx (0.2 \pm 0.02)e^2\ell\), estimated from the ground-state energy per particle\[18\]. On the other hand the present value, at \(\nu = 1/2, 1/4, 1/6\), is curiously close to an analytic estimate, \(1/m^* \approx e^2\ell/6\), obtained from the version of the composite-fermion theory due to Shankar and Murthy that incorporates the effect of the correlation hole\[23\] \[12\].

There are various versions of the composite fermion theory. While the most naive one just attaches fluxes to an electron (a singular gauge transformation), this does not say anything about why the electron-electron repulsion requires this. Then Read introduced a physically clearer version, in which the electron correlation effect is nicely incorporated as the correlation hole attached to the CF transformation. The penalty for doing that is the transformation loses its unitarity. Motivated by this, Shankar and Murthy\[23\] \[12\] introduced a Hamiltonian theory, where the unitarity is recovered, but at the cost of a complexity, in which one first expands the Hilbert space, and then restrict it to a physical one. The expanded space is called “all-\(q\)” theory, the restricted one “small-\(q\)”. The result in the latter (dashed line in Fig.3) agrees with the present result, while the former has a vastly different result.

### 3.2 Higher Landau levels and paired states

There is a growing realization that the FQH system can be very sensitively affected when we go from the lowest Landau level \((N = 0)\) to higher ones in the series \(\nu = \nu^{(N)} + 2N\). Theoretically, the interaction between CF’s strongly depends on the Landau level index. For \(\nu = 5/2(\nu^{(1)} = 1/2, N = 1)\), which sits in between a Fermi-liquid and the stripe, trial functions for paired BCS states have been proposed. Specifically, a \(p_x-ip_y\)-wave, spin-triplet paring of CF’s proposed for \(\nu = 5/2\) by Moore and Read\[24\] is supported by numerical studies by Morf\[25\] and by Rezayi and Haldane\[26\], as well as by a recent experiment by Willet et al\[27\]. However, paring mechanism has yet to be fully understood. We have seen that composite fermions are strongly interacting, and let us make two remarks here.

First question is how Hund’s first and second rules, shown above for the lowest Landau level, will be modified in higher Landau levels. Morf and d’Ambrumenil conjectured, from numerical results on the violation of the Hund’s rule combined with the Rezayi-Read trial wavefunction, that the compressible state becomes unstable for \(\nu^{(N)} = 1/\hat{\phi} \geq 1/2N\). Our numerical result\[3\] for finite FQH system for the total angular momentum, \(L\), and the total spin, \(S\) in the ground state also shows that the Hund first and second rules are obeyed (\(\bigcirc\)) as
Second point is a BCS trial function with $p_x - ip_y$ pairing due to Greiter, Wen, and Wilczek,[28] who studied the Moore-Read paired state in the spherical geometry with a wavefunction,

$$\Psi(u_i, v_i) = \text{Pf} \left[ \frac{1}{u_i v_j - u_j v_i} \right] \prod_{i < j} (u_i v_j - u_j v_i)^2,$$

where $u_i$ and $v_j$ are the spinor variables for $i$-th electron and Pf stands for the Pfaffian.

A numerical result for the radial distribution function, $g(r)$, for $\nu = 1/2$ in Fig.4 exhibits a significant difference between $N = 0$ and $N = 1$ Landau levels, where the latter is characterised by the fact that the inter-CF interaction in the particle-hole channel has a dip just around $k_F$ for the composite fermion.[9] This kind of instability in a Fermi liquid reminds us of a theorem for the usual (zero $B$) electron gas due to Kohn and Luttinger,[29] who showed that the normal liquid has to become, at low enough temperatures, unstable against anisotropic pairing, where the instability is associated with the Friedel oscillation in a system with a well-defined Fermi surface.

4 Quantum Hall effect in three dimensions

Having reviewed the quantum Hall physics for two-dimensional systems, one fundamental question is: while the QHE is usually conceived as specific to two-dimensional (2D) systems, can QHE occur in three dimensions (3D), and, if so, how? It has been suggested that, if there is an energy gap with the Fermi energy lying in it, integer QHE can occur in 3D accompanied by quantised Hall tensor components $\sigma_{ij}$.[30][31][32]. Usual wisdom, however, is that gaps should disappear in 3D, since the motion along the magnetic field is basically free, and gaps should be smeared out. However, we have recently shown[11][33] that, under proper conditions, we do have gapful energy spectra, which are fractal with recursive gaps as in the Hofstadter butterfly for 2D periodic systems in magnetic fields. Interesting points are:

(i) The integer QHE in 3D is by no means a remnant of the 2D butterfly, since the 3D butterfly is washed out when the third direction hopping is turned off.

(ii) The butterfly appears in 3D when plotted versus the tilting angle of the magnetic field.

(iii) The magnetic field required for the 3D butterfly is as modest as $\sim 40$ T in anisotropic 3D system when we employ higher Landau levels for the butterfly[11], which is orders of magnitude smaller than those required for the 2D butterfly.[34]

An appealing possibility, in the context of the present article, is: can we exploit the electron-electron interaction to realise the QHE in 3D? We in fact propose that the magnetic-field induced spin density wave (FISDW), considered in 3D for the first time, is promising.[10] While this is a mean-field theoretic effect unlike the FQHE that is a correlation effect, this would be a start.[35] The FISDW has originally been conceived for a 2D organic metal (TMTSF, called Bechgaard salt), where an anisotropic 2D Fermi surface is incompletely nested, and Landau’s quantisation for the pocket arising from the SDW gap formation results in a series of gaps.[36][37][38]. The Bechgaard salt has the hopping integrals $t_x : t_y : t_z \approx 1 : 0.1 : 0.003$, where the third-direction hopping $t_z$ happens to be so small that the system is almost completely 2D. So our extension to 3D must answer the question: can a 3D-specific FISDW systematically result in a 3D IQHE? Some studies[39][40] have discussed a quantisation of $\sigma_{ij}$ when there is a 3D FISDW, but it remained to be clarified whether and how FISDW phases arise in 3D. We show that, if we have an anisotropy $t_x \gg t_y \approx t_z$ (as opposed to $t_x \gg t_y \gg t_z$ in TMTSF), a 3D butterfly spectrum, accompanied by the 3D QHE, should indeed occur. The phase diagram against the tilted magnetic field $(B_y, B_z)$ is obtained by optimising the
4.1 Magnetic field induced SDW in 3D

We consider a simple orthorhombic metal with an energy dispersion \( \epsilon(k) = -t_x \cos k_x a - t_y \cos k_y b - t_z \cos k_z c \), where \( a, b, c \) are lattice constants and the transfer energies are assumed to satisfy \( t_x \gg t_y, t_z \). The dispersion along the conductive \( k_x \) can be approximated around the Fermi energy as a linear function, \( v_F(|k_x| - k_F) \), and the three-dimensionality (warping of the Fermi surface due to \( t_y \) and \( t_z \)) is considered. We apply a magnetic field \((0, B_y, B_z)\) normal to the conductive axis \( x \), and examine the SDW order parameter \( \Delta(x) \) in a mean-field equation for the wavefunction with the 3D nesting vector \( \mathbf{q} = (q_x, q_y, q_z) \), which can be written as

\[
\begin{pmatrix}
E - H_t(x) & \Delta(x) \\
\Delta^*(x) & E - H_\perp(x)
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}(x) \\
\mathbf{v}(x)
\end{pmatrix} = 0,
\]

\[
H_t(x) = -iv_F \partial_x + \epsilon_\perp (k_\perp + e \mathbf{A}_\perp(x)),
\]

\[
H_\perp(x) = +iv_F \partial_x + \epsilon_\perp (k_\perp - \mathbf{q}_\perp + e \mathbf{A}_\perp(x)),
\]

where \( k_\perp \equiv (k_y, k_z) \), \( \mathbf{A}_\perp(x) = (B_\perp x, -B_\perp x) \) is the vector potential, \( H_t(H_\perp) \) the Hamiltonian for an electron on the right Fermi surface with up-spin (or on the left Fermi surface with down-spin) with \( \mathbf{u}(\mathbf{v}) \) being corresponding wavefunctions. The SDW gap \( \Delta(x) \) can be approximated by a single-mode, \( \Delta(x) \sim \Delta e^{iq_x x} \), and we can determine \( \Delta \) and \( \mathbf{q} \) self-consistently so as to minimize the ground state energy. The Fermi energy lies in the largest gap to minimize the energy, from which we can determine the SDW nesting vector with \( q_x = 2k_F - MG_b - NG_c \) with \( G_b = eB_x b, G_c = eB_y c \).

The situation here is reminiscent of (or indeed mathematically the same as) in the 3D butterfly studied more generally in Ref. [11]. The physics is the following. In the ordinary Hofstadter’s butterfly in 2D, interference of Landau’s quantisation and Bragg’s reflection (band gap) gives the butterfly. In the butterfly in 3D, [11] two Landau quantisations (on \( x-y \) and \( z-x \) planes) in tilted magnetic fields \((B_y, B_z)\) interfere, which gives rise to the fractal energy spectrum. In the FISDW in 3D, the two components of the nesting wavenumber, \( G_b(\propto B_z), G_c(\propto B_y) \), interfere. This reasoning dictates that the spectrum plotted against \( B_z/B_y \) should have a structure similar to Hofstadter’s butterfly, which is indeed the case as shown in Fig.5. Another way of explanation is that the Fermi surface in both cases (i.e., anisotropic 3D crystals and completely nested Fermi surfaces) have multiply-connected structure, on which Landau’s quantisation takes place.

Figure 6(a) shows the phase diagram, plotted against \( B_y \) and \( B_z \), where two integers represent the QHE integers. The quasi-particle spectrum plotted against \( B_z/B_y \) in Fig.6(b) has indeed a fractal structure. An important difference, however, from the butterfly in the non-interacting case is that the SDW phase, being interaction-originated, adjust itself in such a way that the largest gap in the butterfly has the Fermi level in it. The set of integers \((M, N)\) that give the largest gap vary in a complicated sequence as the field is tilted, where the \((M, N)\) have an important physical meaning — the Hall conductivity. Following Yakovenko’s formulation for 2D, Sun and Maki [11] have predicted that the Hall conductivities in the FISDW with \((M, N)\) are given by \( (\sigma_{xy}, \sigma_{xx}) = -\left(e^2/h\right)(M/c, N/b) \) (per spin). In [11] we have obtained the QHE integers for the 3D butterfly using the general Widom-Streda argument [12] due to Halperin, Kohmoto and...
Wu, where these integers are identified as topological invariants assigned to each gap in the butterfly.

4.2 Experimental feasibility

The magnetic field required for the 3D FISDW is dramatically reduced to $B \sim 10$ T (for the magnitude of transfer integrals typically found in organic metals), so this should be entirely within experimental feasibility. Another novel candidate should be doped zeolites. It has been established that guest atoms such as potassium can be incorporated into the nanometer-sized cages in zeolites, where the electronic structure is shown to be surprisingly simple. So an application of magnetic field to zeolites (preferably anisotropic ones such as ZSM-5) will be interesting. We can also apply two external modulations (such as the acoustic waves) to the otherwise uniform system to realise the long periodicity.

4.3 Wrapping current in the QHE in 3D

Having looked at the bulk property, let us consider the surface states in 3D QHE, since the edge states are an important issue in the 2D QHE. Koshino, Halperin and the present author have shown that the 3D QHE in a finite sample should accompany a wrapping current that winds around the faces of a 3D sample.

Curiously, the current direction on each facet does not coincide with the plane normal to the magnetic field $B$, but is dictated by the 3D topological (Chern) integers, which are just the quantised Hall tensor components, $(\sigma_{yz}, \sigma_{zx}, \sigma_{xy})$ in 3D, since the Hall current $j = - (\sigma_{yz}, \sigma_{zx}, \sigma_{xy}) \times E$ in an external electric field $E$. So this is a hallmark of the 3D-specific nature. We can also show that the 3D Hall conductivity when all the currents are assumed to be carried by the wrapping current exactly coincides with that given by the bulk Hall conductivity. This is shown again by Widom-Štěrda formula combined with thermodynamic Maxwell’s relation. In 2D the Hall current carried by the edge current coincides exactly with one calculated with the Kubo formula for the 2D sample, as has been shown by Hatsugai by identifying the connection between the topological (Chern) integers for the bulk and the edge states. So this property remarkably extends to 3D. We can in fact give an intuitive way to understand why surface or bulk does not really matter.

We can also propose an experiment to detect the 3D integer quantum Hall effect through the wrapping current. To observe the currents we have to attach some electrodes, and how to measure the conductivity tensor experimentally in the 3D QHE becomes much less trivial than in 2D. In analogy with the 2D Hall bar experiment we can attach two pairs of electrodes as shown in Fig.7, for which $\sigma_{\text{surface}} = \sigma_{\text{bulk}}$ can again be demonstrated.

One interesting observation is the following. In the 2D Hall bar geometry it has long been recognized that there are two “hot spots” where the chemical potential has to drastically drop dissipatively. In our 3D geometry, the hot spots extend into two “hot lines” as shown in Fig.7. In 2D cyclotron emission has been observed around the hot spots, so we may be able to extend this to the hot lines.

As a final comment, formation of the plateaus, which has been explained in terms of the localisation due to disorder by Aoki and Ando, is an interesting problem for the FQHE in 2D and QHE in 3D. For the FQHE this is a challenge for the composite-fermion picture. For the field-induced SDW, the system adjust itself in such a way that $E_F$ always sticks to the gap, which should act to widen the plateaus.

5 Final remarks

To summarise, we have discussed (i) how the 2D continuous system in magnetic fields has interesting quantum phases arising from the interaction that is controlled by the Landau index, while (ii) in 3D systems the effect of the shape of the Fermi surface enters as a novel ingredient in the physics in magnetic fields.
fields. If we combine (i) and (ii), even richer physics may be expected. Incidentally, in the context of the superconductivity and magnetism in heavy fermion compounds, a magnetic field induced triplet superconductivity has been proposed.\[53\] Also, it has long been known and intensively studied that there is a rich phase diagram for liquid $^3$He that includes nonunitary pairing superfluid phases, so the combination of (i) and (ii) as conceived here may be related with, or possibly even go beyond, these. There are thus a wealth of open questions to be unraveled in the physics of quantum Hall effect even after the two decades of its discovery.

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Fig. 1 (a) The excitation spectrum for a finite FQH system, where the low-lying mode is the spin wave ($\circ$: exact result, $+$: composite-fermion result) for $\nu = 1/3, 1/5$. (b) How the Coulomb pseudopotential changes as we attach the fluxes to electrons to convert them into CF. (after [3]).

Fig. 2 Inverse effective mass estimated from the size scaling of the excitation energy in the spinless system (●). The dashed (dotted) line is the small-$q$ (all-$q$) result in the Shankar-Murthy CF mean-field. (after [6, 7]).

Fig. 3 (a) Low-lying excitation spectra for $N_e = 16$ in the exact diagonalisation (●), free composite fermion gas model (+) and the Fermi liquid theory (square) at $\nu = 1/2$. (after [4]). Arrows indicate the Hund’s coupling. (b) Excitation gap ($\propto$ inverse effective mass) estimated from the size scaling of the energy for spinflip ($\triangle$) or no-spinflip ($\circ$) excitations. (after [3]).

Fig. 4 Radial distribution function $g(r)$ against the great-circle distance $r$ for $N_e^{(1)} = 9$ system with $N_{\phi}^{(1)} = 16$. Results for $N = 1, N = 0$ Landau levels, and that for a deformed pseudopotential to remove the dip at $k_F$ are shown. (after [5]).

Fig. 5 (a) the energy spectrum of an anisotropic 3D system against the angle $\theta$ of the applied magnetic field. Pairs of numbers for each gap represent the Hall conductivity ($\sigma_{xy}, \sigma_{zx}$) in units of $-e^2/ah$. (b) The Hofstadter butterfly in 2D. Bottom insets depict sample geometries.

Fig. 6 (a) Phase diagram for the FISDW in 3D is shown against ($B_y, B_z$). The phases are labelled by the quantum Hall integers ($\sigma_{xy}, \sigma_{zx}$) in units of ($h/e^2$). The 3D-natured phases (which vanish for $t_z \to 0$) are shaded. (b) The quasi-particle energy spectrum against $B_z$ with a fixed $B_y$. Vertical lines indicate boundaries between different FISDW phases labelled by ($\sigma_{xy}, \sigma_{zx}$). (c) An incompletely nested Fermi surface, which resembles, after the SDW gap formation, the multiply-connected Fermi surface for anisotropic 3D systems.

Fig. 7 Right: Wrapping current (thin arrows) in the QHE in 3D (b), where the experimental setup of the electrodes to detect 3D QHE wrapping current is indicated. The corresponding picture in 2D is shown in (a). The “hot lines (spots)” are indicated by blurred lines (spots).
$N_e = 16, \nu = 1/2$
\[ v^{(1)} = \frac{1}{2} \quad (N_e^{(1)} = 9, \quad N_{\phi}^{(1)} = 16) \]

\[ v^{(1)} = \frac{1}{2} \quad (N_e^{(1)} = 9, \quad N_{\phi}^{(1)} = 16) \]

\[ V_K^{(1)} (N = 1 \text{ pseudopotential}) \]

\[ V_K^{(0)} (N = 0 \text{ pseudopotential}) \]
