Abstract

A general and computable expression for cumulants of a random variables in a semimartingale context is given, with resulting expressions for characteristic and cumulant generating functions. We have been inspired by a formal forest series for expectations of solutions of the Black-Scholes equation [AGR20]. Our proof is of remarkable simplicity and the result is likely to transcend the financial context from which it originates. A variety of examples are presented.

Keywords: cumulants, moments, continuous martingales, diamonds, Hermite polynomials, regular perturbation, KPZ type (Wild) expansion, trees, Lévy area, Wiener chaos, Heston and forward variance models; MSC2020 Class: 60G44, 60H99, 60L70.

1 Introduction and main result

Consider a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}; \mathbb{P})\), on which all martingales admit a continuous version. (Itô’s representation theorem, e.g. [RY13, Ch.V.3.], states that this holds true for Brownian filtration which covers all situations we have in mind.) Throughout, \(T \in (0, \infty]\) should be thought of as a fixed parameter. Our main result concerns the computation of cumulants and their generating functions in a (continuous) Itô calculus setting. This involves crucially

Definition 1.1. Given two continuous semimartingales \(A, B\) with integrable covariation process \(\langle A, B \rangle\), the diamond product of \(A\) and \(B\) is another continuous semimartingale given by

\[(A \diamond B)_t(T) := \mathbb{E}_t\left[\langle A, B \rangle_{t,T}\right] = \mathbb{E}_t\left[\langle A, B \rangle_T\right] - \langle A, B \rangle_t,\]

where \(\langle A, B \rangle_{t,T} = \langle A, B \rangle_T - \langle A, B \rangle_t\).

Although \(A \diamond B\) only depends on the respective martingale parts of \(A\) and \(B\), the (commutative, non-associative) diamond product is in general not a martingale in \(t \in [0, T]\).
Theorem 1.1. (i) Let $A_T$ be $\mathcal{F}_T$-measurable with $N \in \mathbb{N}$ finite moments. Then the recursion

$$K^1_T(T) := \mathbb{E}_t[A_T] \quad \text{and} \quad \forall n > 0 : \quad K^{n+1}_T(T) = \frac{1}{2} \sum_{k=1}^{n} (K^k_T \diamond K^{n+1-k}_T)(T) \quad (1.1)$$

is well-defined up to $K_N$ and, for $\xi \in \mathbb{R}$,

$$\log \mathbb{E}_t[e^{i\xi A_T}] = \sum_{n=1}^{N} (i\xi)^n K^n_T(T) + o(|\xi|^N)$$

which identifies $n! \times K^n_T(T)$ as the (time $t$-conditional) $n$th cumulant of $A_T$.

(ii) If $A_T$ has moments of all orders, we have the asymptotic expansion,

$$\log \mathbb{E}_t[e^{i\xi A_T}] \sim \sum_{n=1}^{\infty} (i\xi)^n K^n_T(t) \text{ as } \xi \to 0 . \quad (1.2)$$

(iii) If $A_T$ has exponential moments, so that its (time $t$-conditional) mgf $\mathbb{E}_t[e^{x A_T}]$ is a.s. finite for $x \in \mathbb{R}$ in some neighbourhood of zero, then there exist a maximal convergence radius $\rho = \rho_t(\omega) \in (0, \infty]$ a.s. such that for all $z \in \mathbb{C}$ with $|z| < \rho$, 

$$\log \mathbb{E}_t[e^{z A_T}] = \sum_{n=1}^{\infty} z^n K^n_T(t) . \quad (1.3)$$

The general term $K^n_T(T)$ in (1.1) is naturally written as a linear combination of binary “diamond” trees; we hence call (1.2), (1.3) the $K$-forest expansions of the characteristic function and cumulant generating function, respectively. To see this, it suffices to spell out a few terms in the recursion: writing $M$ as a short-hand for $K^1_T(T)$ we have

$$K^1 = M \equiv \circ$$
$$K^2 = \frac{1}{2}M \diamond M \equiv \frac{1}{2} \circ \circ$$
$$K^3 = \frac{1}{2}(M \diamond M) \diamond M \equiv \frac{1}{2} \circ \circ \circ$$
$$K^4 = \frac{1}{2}(M \diamond (M \diamond M) \diamond M) + \frac{1}{8}(M \diamond M)^{\diamond 2} \equiv \frac{1}{2} \circ \circ \circ \circ + \frac{1}{8} \circ \circ \circ \circ$$

... 

To gain some intuition, we first note that, as an immediate consequence of Itô’s isometry for $L^2$-martingales $M$ and $N$,

$$(M \diamond N)_t(T) = \mathbb{E}_t(M, N)_{t,T} = \mathbb{E}_t(M_{t,T} N_{t,T}) = \text{cov} [ M_{T}, N_{T} | \mathcal{F}_t ]$$

which identifies $2K^2 = M \diamond M = \circ \circ$ correctly as the conditional variance, i.e. the second cumulant.
For higher $n$, Theorem 1.1 encodes relations that are increasingly complex to derive by hand. We illustrate one more case ($n = 3$). Assuming suitable integrability, we compute

\[
(M \circ (M \circ M))_t(T) = M \circ (\mathbb{E} \langle M \rangle_T - \langle M \rangle_t) = M \circ (\mathbb{E} \langle M \rangle_T) \]

\[
= \text{cov}_t(M_T, \langle M \rangle_T) = \text{cov}_t(M_{t,T}, \langle M \rangle_{t,T}).
\]

From basic properties of Hermite polynomials (cf. Section 2.4) $H_3(M_{t,T}, \langle M \rangle_{t,T}) = M_3^{t,T} - 3M_t^{t,T} \langle M \rangle_t$ is a martingale increment, with zero ($t$-conditional) expectation. The desired relation between the third cumulant $\mathbb{E}_t(M_3^{t,T})$ and $\mathbb{E}_t(M_t^{t,T} \langle M \rangle_t)$ follows.

We now relate Theorem 1.1 to exponential martingales.

**Corollary 1.1** (Breaking exponential martingales). Let $M$ be a martingale such that $\mathbb{E}(e^{\epsilon \langle M \rangle_T})$ is finite for some $\epsilon > 0$. Then, with

\[
F^2 = \left(\frac{1}{2}a^2 + b\right) \circ \rho \quad \text{and} \quad \forall k > 2 : \ F^k = \frac{1}{2} \sum_{j=2}^{k-2} F^{k-j} \circ F^j + (a M \circ F^{k-1}),
\]

we have, for sufficiently small $a$ and $b$,

\[
\mathbb{E}_t \left[ e^{aM_t+b\langle M \rangle_T} \right] = e^{aM_t + \sum_{t \leq \tau} \rho^\tau}.
\]

For $b = -a^2/2$, this becomes the classical martingality of $\exp \left( aM_t - \frac{a^2}{2} \langle M \rangle_t \right)$.

Corollary 1.1 is an example in which $A_T$ (in Theorem 1.1) arises from some process $(A_t : 0 \leq t \leq T)$, in this case given by $A_t = aM_t + b\langle M \rangle_t$. Theorem 1.1 is a priori indifferent to this additional structure. However, if this process is a sufficiently integrable semimartingale, it can still be useful, as was seen in the example of the corollary, to decompose

\[
\mathbb{K}^1_\tau(T) = \mathbb{E}_\tau [A_T] = A_t + \mathbb{E}_\tau [A_{t,T}] =: A_t + X_t(T).
\]

Since $\mathbb{K}^1_\tau(T)$ and $A_\tau$ are both semimartingales, so must be $X = X_\tau(T)$. (If $A$ is a martingale, it must coincide with $\mathbb{K}^1$ and $X$ vanishes.) The diamond trees built by the recursion can then be further decomposed into contributions from $A$ and $X$, amounting to “diamond forests” of trees whose leaves are of different type, say “$A$” and “$X$” respectively. For instance, we encounter terms like $A \circ X$ or $((A \circ A) \circ X) \circ A$, representable as

\[
\circ \rho \circ \rho^\tau.
\]

Another reason for this decomposition is the fact that the (conditional) increment $X$ will be small as $t$ is close to $T$, as opposed to $A$. An explicit example of such a situation, other than Corollary 1.1, will be given in Section 4.6 where all involved trees are seen to be explicitly computable.
1.1 Relation to existing works and organization of paper.

Theorem 1.1 was motivated by the work of Alòs et al. [AGR20], which gives a (purely formal) forest series for expectations of solutions of the Black-Scholes equation in terms of iterated integrals of derivatives of the solution. The said series is different from our cumulant expansion, but we will see — after suitable forest reordering — that it corresponds to a special case of our $k$-expansion (cf. Sections 3 and 4.6). The proof of Theorem 1.1 is given in Section 2; Corollary 1.1 is shown in Section 3.

In a Markovian situation our expansion can be related to perturbative expansion of a non-linear parabolic partial differential equation of HJB type. We make this explicit in the case when $A = f(B)$, for a Brownian motion $B$ and suitable $f$, in which case the $k$’s are described by a cascade of linear PDEs, detailed in Section 4.4 indexed by trees such as 1.4, reminiscent of the “Wild expansion” used in Hairer’s KPZ analysis [Hai13]. That said, computing $\log E_t \left[ e^{\epsilon A_T} \right]$ may also be viewed as a (linear) backward SDE with “Markovian” terminal data given by $e^{\epsilon A_T} = e^{\epsilon f(B_T)}$; upon suitable exponential change of variables this becomes a quadratic BSDE as studied by Kobylanski [Kob00], Briand-Hu [BH08] and many others, in the weakly non-linear regime (BSDE driver of order $\epsilon$). Yet another point of view comes from Dupire’s functional Itô-calculus [Dup19] which would lead to similar (at least formal) computations as conducted in Section 4.4 for general $\mathcal{F}_T$-measurable $A_T$. And yet another point of view comes from the Boué–Dupuis [BD98] Formula which gives an exact variational representation of $\log E \left[ e^{A_T} \right]$ when $A_T$ is a sufficiently integrable measurable function of Brownian motion up to time $T$; here 1.3 can be viewed as an asymptotic solution to the Boué–Dupuis variational problem in the weakly non-linear regime. Theorem 1.1 should also be compared to Nourdin–Peccati [NP10] where the authors use Malliavin integration by parts to describe cumulants of certain Wiener functionals, and notably compute cumulants of elements in a fixed Wiener chaos, cf. Section 4.5. In Section 4.3 to demonstrate the computational power of the diamond product and forest expansion, we give an elementary proof of Paul Lévy’s classical closed-form formula for the cgf of the Lévy area. Finally, in Section 4.6 we apply Theorem 1.1 to establish a formula for the joint mgf of a process $X$, its quadratic variation $\langle X \rangle$, and $E_t[d\langle X \rangle_T/dT]$, quantities that play an important role in stochastic modeling (not only) in finance. The forest expansion is given explicitly for a general (infinite-dimensional affine [GKR19]) class which covers e.g. Feller square-root diffusion and the popular classical and rough Heston models.

Acknowledgement. PKF has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 683164) and the DFG Excellence Cluster MATH+ (Projekt AA4-2).

Update. After we uploaded our paper to the arXiv, Vincent Vargas kindly informed us that part (iii) of our Thm 1.1. had been obtained recently in the context of the Sine-Gordon model in QFT. Their argument (Lacoin–Rhodes–Vargas, arxiv.org/abs/1903.01394, Chapter 3) is different from our argument, but equally short. The other parts of this paper remain unaffected by this overlap.
2 Proof of Theorem 1.1

2.1 Recap on cumulants

Let \( A = A(\omega) \) be a real-valued random variable. The functions \( M : R \to (0, \infty), M(x) = \mathbb{E}[e^{xA}] \) and \( \Lambda : R \to (-\infty, +\infty], \Lambda = \log M \) are known as the moment generating function (mgf) and the cumulant generating function (cgf) of \( A \), respectively. These functions are known to be real-analytic on the open interval

\[
G = \{ x : M(x) < \infty \}^0 = \{ x : \Lambda(x) < \infty \}^0 = (a, b), \quad -\infty \leq a \leq 0 \leq b \leq +\infty ,
\]

i.e. are given as power series in a neighbourhood of any \( t \in (a, b) \). In fact, if \([ -\epsilon, \epsilon ] \subset (a, b)\), the mgf is given by the absolutely convergent series

\[
M(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathbb{E}[A^n] \quad \forall x \in [-\epsilon, \epsilon] .
\]

This and the following lemma are well-known, see e.g. [Mor84, Luk70].

**Lemma 2.1.** (i) Let \( A \) be a real-valued random variable with \( n \) moments, \( n \in \mathbb{N} \). Then the characteristic function \( \xi \mapsto \mathbb{E}[e^{i\xi A}] \) is \( n \) times differentiable at zero and, as \( \xi \to 0 \),

\[
\phi_A(\xi) = \mathbb{E}[e^{i\xi A}] = \sum_{j=1}^{n} \frac{\kappa_j^{(j)}}{j!} (i\xi)^j + o(|\xi|^n)
\]

where \( \kappa_j := i^{-j} \phi_A^{(j)}(0) \) is called j.th cumulant of \( A \).

(ii) Let \( A \) be a real-valued random variable with exponential moments by which we mean that the mgf \( M(x) = \mathbb{E}[e^{xA}] \) is finite in neighbourhood of 0. Then, for \( x \) in a (possibly smaller) neighbourhood of 0, in terms of cumulants \( \kappa_n := \Lambda^{(n)}(0) \),

\[
\log \mathbb{E}[e^{xA}] = \Lambda(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \kappa_n .
\]

This expansion is also valid for \( x \in \mathbb{R} \) replaced by (small) enough \( z \in \mathbb{C} \).

2.2 Proof of part (iii)

With \( M_t := \mathbb{E}_t[A_T] \) we have a martingale \( (M_t : 0 \leq t \leq T) \) which inherits integrability from \( A_T \). Specifically, our assumption implies that the (time \( t \)-conditional) mgf of the increment \( M_{t,T} = M_T - M_t \) is finite in some neighbourhood of zero. By Lemma 2.1 its logarithm, the (time \( t \)-conditional) cgf of the increment \( M_{t,T} \) is analytic near zero and of the form

\[
\Lambda^T_t(\epsilon) := \sum_{k=1}^{\infty} \epsilon^k \frac{\kappa_k(T)}{k!} = \epsilon \mathbb{E}_t M_{t,T} + \frac{\epsilon^2}{2!} \mathbb{V}_t M_{t,T} + ...
\]
where $\kappa_{k,t}(T)$ denotes the $k$th (time $t$-conditional) cumulant of $M_{t,T}$. Note $\mathbb{E}_t M_{t,T} = 0$ by martingality of $M$ so that with $\mathbb{K}_t^k := \kappa_{k,t}(T)/k!$,

$$\Lambda_t^T(\epsilon) = \sum_{k=2}^{\infty} \epsilon^k \mathbb{K}_t^k$$

Since $\Lambda_t^T(\epsilon) = 0$ and $\mathbb{E}_t e^{\epsilon M_{t,T}} = e^{\Lambda_t^T(\epsilon)}$, by definition of the cgf, we obtain

$$\mathbb{E}_t e^{\epsilon M_{t,T} + \Lambda_t^T(\epsilon)} = e^{\epsilon M_{t,T} + \Lambda_t^T(\epsilon)}, \quad (2.1)$$

which exhibits $e^{\epsilon M_{t,T} + \Lambda_t^T(\epsilon)} : 0 \leq t \leq T$, for fixed $T$, as (exponential) martingale. By Itô’s Formula, $L^t := \epsilon M_t + \Lambda_t^T(\epsilon)$ is hence a stochastic logarithm, meaning that $L^t + \frac{1}{2}\langle L^t \rangle$ is a martingale on $[0, T]$,

$$\mathbb{E}_t \left[ \epsilon M_{t,T} - \Lambda_t^T(\epsilon) + \frac{1}{2}(\epsilon M_t + \Lambda_t^T(\epsilon))_{t,T} \right] = 0.$$ 

Upon inserting $\Lambda_t^T(\epsilon) = \epsilon^2 \mathbb{K}_t^2 + \epsilon^3 \mathbb{K}_t^3 + ...$, it then suffices to collect terms of order $[\epsilon^k]$ and set them to zero. At order $[\epsilon^1]$ we see confirmed $\mathbb{E}_t M_{t,T} = 0$ (martingale property of $M$) without obtaining new information. At order $[\epsilon^2]$, $[\epsilon^3]$ and $[\epsilon^4]$,

$$\mathbb{K}_t^2 = \frac{1}{2} \mathbb{E}_t \langle M \rangle_{t,T}, \quad \mathbb{K}_t^3 = \mathbb{E}_t \langle M, \mathbb{K}_t^2 \rangle_{t,T}, \quad \mathbb{K}_t^4 = \mathbb{E}_t \langle M, \mathbb{K}_t^3 \rangle_{t,T} + \frac{1}{2} \mathbb{E}_t \langle \mathbb{K}_t^2, \mathbb{K}_t^2 \rangle_{t,T}.$$ 

Upon setting $\mathbb{K}_t^1 := M$, the general recursion is then easily seen to be given by, for all $n > 0$,

$$\mathbb{K}_{t,n+1}^1(T) = \frac{1}{2} \sum_{k=1}^{n} (\mathbb{K}_t^k \circ \mathbb{K}_{t,n+1-k}^1)(T),$$

as was stated in (1.1).

### 2.3 Proof of part (i)+(ii)

We first show that the diamond recursion is well-posed to the extent that one has sufficient integrability. More precisely, we have

**Lemma 2.2.** Assume $A$ has $n$ moments, $n \in \mathbb{N}$. Then the recursion (L.L) is well-defined for $j \leq n$ and yields $(\mathbb{K}_t^j(T) : 0 \leq t \leq T)$ as a semimartingale with a $L^{n/j}$-integrable martingale part and a $L^{n/j}$-integrable bounded variation (BV) component.

**Proof.** If $A \in L^p$ then also $M := \mathbb{K}_t^1 := \mathbb{E}_t A \in L^p$. By the Burkholder–Davis–Gundy (BDG) inequality, $\sqrt{\langle M \rangle_T} \in L^p$ or $\langle M \rangle_T \in L^{p/2}$. In case $n \geq 2$,

$$2\mathbb{K}_t^2(T) = (\mathbb{K}_t^1 \circ \mathbb{K}_t^1)(T) = \mathbb{E}_t \langle M \rangle_T - \langle M \rangle_T \in L^{p/2}.$$
Call $M^{(2)}$ the martingale part of $\mathbb{K}^2$; clearly $M^{(2)}$ is a $L^{n/2}$-martingale. For $\mathbb{K}^3 = \mathbb{F}^2 \circ \mathbb{F}^1$ we first use Cauchy-Schwarz to estimate

$$|\langle \mathbb{K}^2, \mathbb{K}^1 \rangle_T| = |\langle M^{(2)}, M \rangle_T| \leq \sqrt{\langle M^{(2)} \rangle_T} \sqrt{\langle M \rangle_T}.$$ 

By BDG, the right-hand side is the product of random variables in $L^{n/2}$ and $L^{n}$ respectively. Since $\frac{1}{n^3} = \frac{1}{n^2} + \frac{1}{n}$ it follows immediately from the (generalized) Hölder inequality that $\langle \mathbb{K}^2, \mathbb{K}^1 \rangle_T \in L^{n/3}$. Assume now $n \geq 3$. Then $M^{(3)} := \mathbb{E}_\bullet \langle \mathbb{K}^2, \mathbb{K}^1 \rangle_T$ is a well-defined $L^{n/3}$-martingale which constitutes the martingale part of $\mathbb{K}^3 = M^{(3)} - \langle \mathbb{K}^2, \mathbb{K}^1 \rangle$. The general statement is that

$$\mathbb{K}^j = M^{(j)} + \text{BV}$$

with a $L^{n/j}$-martingale part and a bounded variation (BV) component, whenever $j = 1, \ldots, n$. The same reasoning gives, by induction in $n$, the general statement. □

The asymptotic expansion of part (ii) follows by definition from the validity of the expansion of part (i) for all integers $N$. We thus focus on part (i).

Given $A_T, \mathcal{F}_T$-measurable with $N \in \mathbb{N}$ finite moments – but whose mgf is not necessarily finite – we work with its two-sided truncation $(-\ell \vee A_T \wedge \ell) =: A_T^\ell$, followed by careful passage $\ell \to \infty$. Indeed, part (iii) applies to $A_T^\ell$ (bounded!) and hence shows that the $\mathbb{K}^{\ell,n}$, $n = 1, 2, \ldots$ defined by the recursion (1.1), started with $\mathbb{K}^{\ell,1} = \mathbb{E}_\bullet A_T^\ell$, are well-defined and yield (up to a factorial factor) the $n$th conditional cumulants of $A_T^\ell$. It is easy to see that the $n$th (conditional) cumulant of $A_T^\ell$, which exists by Lemma 2.2 for $n \leq N$, is the limit (a.s. and in $L^1$) of the corresponding cumulants for $A_T^\ell$, using the conditional dominated-convergence theorem. It remains to be seen that the diamond recursion is also stable under this passage to the limit. The precise integrability properties of the $\mathbb{K}$’s, obtained in Lemma 2.2 for $A_T$, are easily made uniform in the truncation parameter $\ell$; justification of taking $\ell \to \infty$ in the diamond recursion is then straightforward.

### 2.4 Remarks on the proof

**Hermite polynomials**

The reader may wonder whether or not there is a direct proof of part (i) that relates $\mathbb{K}^n$, $n = 1, 2, 3, \ldots$ with the corresponding cumulants. This is possible along the following lines. From the key identity (2.1), rewritten with $Q_T^\ell(e) := -2(\mathbb{K}^2_T(e) + e\mathbb{K}^3_T(e) + e^2\mathbb{K}^4_T(e) + \cdots)$,

$$\mathbb{E}_e e^{\epsilon \mathcal{M}_T} = \mathbb{E}_e e^{\epsilon \mathcal{M}_T - \frac{\epsilon^2}{2} Q_T^\ell(e)} = e^{e^{\epsilon \mathcal{M}_T} - \frac{\epsilon^2}{2} Q_T^\ell(e)} ,$$

we can deduce, by definition of Hermite polynomials [RY13, Ch.IV.3.] martingality of

$$e^{\epsilon \mathcal{M}_T - \frac{\epsilon^2}{2} Q_T^\ell(e)} = \sum_{n \geq 0} \epsilon^n \frac{n!}{n!} H_n(M_T, Q_T^\ell(e)).$$
By taking \((\partial/\partial \epsilon)^n|_{\epsilon=0}\) we obtain a graded family of martingales, starting with \((n=2)\)

\[ t \mapsto H_2(M_t, Q_t^0(0)) = M_t^2 - \frac{1}{2} \mathbb{K}_t^2(T) . \]

Applying Itô’s Formula over \([t,T]\) and taking \(t\)-conditional expectation then identifies \(\mathbb{K}_t^2(T)\) correctly as \(\mathbb{E}_t(M)_{t,T} = (M \circ M)_t(T)\). Using suitable relations between Hermite polynomials, this argument extends to \(n > 2\) and provides a (not so) different (but much less elegant) route to our \(\mathbb{K}\)-recursion.

Wick calculus

As already noted, the diamond product is (very) different from the Wick product. Known Wick identities, such as: \(e^{aU} = \mathbb{E}(e^{aU})^{-1}e^{aU}\) (see e.g. [GHL+93, Equ.(1.4)]) are different though vaguely reminiscent of our exponentiation result (1.3). That said, Wick calculus is an efficient tool for computations with Hermite polynomials; it is then conceivable that the (alternative) Hermite approach outlined above can be handled more efficiently using the Wick formalism.

Hopf algebras.

Hopf algebras are a classical tool in quantum field theory to organize the combinatorics of forests expansions, and more recently in the field of (branched) rough paths and regularity structures, see e.g. [FH14] and the references therein. Very recently, Ebrahimi-Fard et al. [EFPTZ18] have studied the combinatorics of Hermite vs. regular polynomials and cumulants vs. moments from a Hopf algebraic point of view. At this stage, we have not seen any advantage in reformulating our findings in a Hopf algebraic framework, although this might be useful for future extensions in a non-commutative setting.

3 Breaking the exponential martingale

Consider a martingale \((aM_t)\) with stochastic exponential \(\mathcal{E}(aM)_T = \exp \left\{ aM_T - \frac{a^2}{2} \langle M \rangle_T \right\} \).

This is a genuine martingale, for \(a\) small enough, provided \(\langle M \rangle_T\) enjoys some exponential integrability (Novikov criterion). Then

\[ \mathbb{E}_t \left[ e^{aM_T - \frac{a^2}{2} \langle M \rangle_T} \right] = e^{aM_t} \] (3.1)

with “trivial” right-hand side. This is an interesting example where the individual cumulants have more structure than their sum (given by \(aM_t\)). In fact, applied to the left-hand side above, our \(\mathbb{K}\)-recursion gives

\[ \mathbb{K}^1 = aM_t - \frac{a^2}{2} \mathbb{E}_t \left[ \langle M \rangle_{t,T} \right] , \quad \mathbb{K}^2 = \frac{1}{2} \left( aM - \frac{a^2}{2} \langle M \rangle \right)^{\circ 2} , \ldots \]
and all terms homogenous in $a^k, k \geq 2$, cancel upon summation of (finitely many) cumulants terms (namely, $\mathbb{K}^j + \cdots + \mathbb{K}^k$, where $j = \lceil k/2 \rceil$). Graphically speaking, the root cause is that (with $b = -a^2/2$)

$$\mathbb{K}^1 = aM_t + b \mathbb{E}_t(M)_{t,T} = a \circ + b \circ \rho, \quad (3.2)$$

is a linear combination of trees with different number of leaves, and this propagates to all further terms in the $\mathbb{K}$-expansion. In fact, applying the $\mathbb{K}$-recursion (1.1) with $\mathbb{K}^1$ given by (3.2), for arbitrary $a, b$, and neglecting trees with 6 or more leaves, the first few $\mathbb{K}$-forests are given by

$$\mathbb{K}^1 = a \circ + b \circ \rho$$
$$\mathbb{K}^2 = \frac{1}{2} (a \circ + b \circ \rho)^2 = \frac{1}{2} a^2 \circ \rho + ab \circ \rho \circ + \frac{1}{2} b^2 \circ \rho$$
$$\mathbb{K}^3 = \frac{1}{2} a^3 \circ \rho + \frac{1}{2} a^2 b \circ \rho + a^2 b \circ \rho + ab^2 \circ \rho + \frac{1}{2} ab^2 \circ \rho + \cdots$$
$$\mathbb{K}^4 = \frac{1}{2} a^4 \circ \rho + \frac{1}{2} a^4 \circ \rho + \frac{1}{2} a^3 b \circ \rho + \frac{1}{2} a^2 b \circ \rho + a^3 b \circ \rho + a^2 b \circ \rho + \frac{1}{2} a^3 b \circ \rho + \cdots$$
$$\mathbb{K}^5 = \frac{1}{2} a^5 \circ \rho + \frac{1}{2} a^{11} \circ \rho + \frac{1}{2} a^5 \circ \rho + \cdots \quad (3.3)$$

We can choose to reorder the $\mathbb{K}$-forest series into forests of trees grouped by number of leaves. Define $\mathbb{F}^\ell$ to be the (finite) linear combination of trees in the $\mathbb{K}$-expansion with $\ell \geq 1$ leaves. Since the corresponding $\mathbb{F}$-expansion will be just a reordered version of the $\mathbb{K}$ expansion, it will inherit the convergence properties of the cumulant expansion given in Lemma 2.1. Since $aM_t$ is the only tree with 1 leaf, $\mathbb{F}^1 = a M = a \circ$. Thus, by definition, in the sense of formal tree expansions,

$$\sum_{k \geq 1} \mathbb{K}^k = a \circ + \sum_{\ell \geq 2} \mathbb{F}^\ell.$$  

Reordering the $\mathbb{K}$-forests according to number of leaves, we see that the first few $\mathbb{F}$-forests are given by

$$\mathbb{F}^1 = a \circ$$
$$\mathbb{F}^2 = (\frac{1}{2} a^2 + b) \circ \rho$$
$$\mathbb{F}^3 = a (\frac{1}{2} a^2 + b) \circ \rho \circ$$
$$\mathbb{F}^4 = \frac{1}{2} a (\frac{1}{2} a^2 + b)^2 \circ \rho + a^2 (\frac{1}{2} a^2 + b) \circ \rho \circ$$
$$\mathbb{F}^5 = a (\frac{1}{2} a^2 + b)^2 \circ \rho + \frac{1}{2} a (\frac{1}{2} a^2 + b)^2 \circ \rho + a^3 (\frac{1}{2} a^2 + b) \circ \rho \circ \rho. \quad (3.5)$$

From the above, it seems clear that the $\mathbb{F}$-forests must satisfy a recursion relation. The following theorem specifies this recursion.

**Theorem 3.1.** (*$\mathbb{F}$*-recursion; proof of Corollary 1.1) With $\mathbb{F}^2 = (\frac{1}{2} a^2 + b) \circ \rho$, we have the recursion for $k \geq 2$,

$$\mathbb{F}^k = \frac{1}{2} \sum_{j=2}^{k-2} \mathbb{F}^{k-j} \circ \mathbb{F}^j + (aM \circ \mathbb{F}^{k-1}) \quad (3.6)$$
As a consequence, from (1.3), we have for sufficiently small $a$ and $b$,

$$
\mathbb{E}_t \left[ e^{aM_t + b(M)\tau} \right] = e^{aM_t + \sum_{\ell \geq 2} \mathbb{E}^{(\ell)}},
$$

(3.7)

provided $\mathbb{E}_t \left[ e^{\epsilon(M)\tau} \right]$ is finite for some $\epsilon > 0$.

Proof. Apply Theorem 1.1 with $A_T = aM_T + b(M)\tau$. The required (exponential) integrability of $A_T$ follows from the integrability assumption on $(M)_T$ by a standard argument (e.g. along the proof of Novikov’s criterion [RY13, Ch.VIII.1]). So it remains to reorder the $K$-expansion into a $F$-expansion. Note from (1.1) that

$$
K^{\ell+1} = \frac{1}{2}(K^1 \otimes K^{\ell-1} + \cdots + K^{\ell-1} \otimes K^1)
$$

(3.8)

is $(\ell + 1)$-homogenous in $a$, $b$ (but not in the number of leaves). Then

$$
\sum_{\ell \geq 1} K^\ell = \sum_{m,n \geq 0} a^m b^{n+1} =: a \circ \sum_{k \geq 2} F^k
$$

where each $F^k$ contains (by definition) exactly the trees with $k$ leaves, and each $T^{m,n}$ represents a tree with $m + n$ leaves.

Now, from the $K$-recursion (3.8), it is clear that every tree with $k \geq 2$ leaves arises precisely either from

- $\frac{1}{2} a \circ T$ or $\frac{1}{2} T \circ a$ with $T \in F^{k-1}$ (i.e. a tree with $k - 1$ leaves)

or from

- $\frac{1}{2} (T' \circ T'')$ where $T' \in F^i$ has $i \geq 2$ leaves, $T'' \in F^j$ has $j \geq 2$ leaves, and $i + j = k$.

But this says precisely that

$$
F^k = \frac{1}{2}(a \circ F^{k-1} + \frac{1}{2} F^{k-1} \circ (a \circ) + \frac{1}{2} \sum_{i,j=2}^{k-2} 1_{i+j=k} F^i \circ F^j
$$

$$
= \frac{1}{2} \sum_{i,j=2}^{k-2} 1_{i+j=k} F^i \circ F^j + (a \circ) \circ F^{k-1},
$$

which is exactly (3.6). \(\square\)

Remark 3.1. Note that for the exponential martingale case $b = -a^2/2$, the term $F^2$ vanishes (and hence so does every $F^\ell$, $\ell > 2$) and we (immediately) recover the exponential martingale identity (3.1). In the case $b \neq -a^2/2$, (3.7) can be viewed as breaking the rigid exponential martingality condition $b = -a^2/2$.

Remark 3.2. A special case of the recursion (3.6) with the identification $\tilde{F}^{\ell} = F^{\ell+2}$, $\ell \geq 0$ and $b = -\frac{1}{2} a$ appears in [AGR20].
4 Examples

4.1 Brownian motion

Example 4.1 (Brownian motion with drift). Let $A_t = \sigma B_t + \mu t$. Then
\[ \mathbb{K}_t^1(T) = \sigma B_t + \mu T = A_t + \mu(T - t), \quad \mathbb{K}_t^2(T) = \frac{1}{2}(\mathbb{K}_t^1 \circ \mathbb{K}_t^1)_t(T) = \frac{1}{2} \sigma^2(T - t). \]

and $\mathbb{K}_t^k \equiv 0$ for all $k \geq 3$. These are the cumulants of $A_T - A_t \sim N(\mu(T - t), \sigma^2(T - t))$, as predicted by Theorem 1.1 and the $\mathbb{K}$-forest expansion of the cumulant generating function (1.3) is trivially convergent (with infinite convergence radius).

Example 4.2 (Stopped Brownian motion). Consider the martingale $A = B^r$, standard Brownian motion $B$ stopped at reaching $\pm 1$. We compute
\[ \mathbb{K}_t^1(T) = \mathbb{E}_t B^r_T = B^r_t = B_{t \wedge T}, \quad \mathbb{K}_t^2(T) = \frac{1}{2} \mathbb{E}_t(B^r^2)_{t \wedge T} = \frac{1}{2} \left( \mathbb{E}_t(\tau \wedge T) - \tau \wedge t \right) \ldots . \]

By Theorem 1.1 the second quantity equals the conditional variance $\mathbb{V}_t(B^r_T) = \mathbb{E}_t(B^r_T)^2 - (B^r_t)^2$, and thus “contains” familiar identities from optional stopping. With $T = \infty$, $A_T = B_r$ takes values $\pm 1$ with equal probability. This is a bounded random variable, with globally defined and real analytic time-$t$ conditional cgf given by
\[ \Lambda_t(x) = \log \left( \frac{1}{2} [(1 + B^r_t)e^x + (1 - B^r_t)e^{-x}] \right). \]

Its convergence radius is random through the value of $B^r_T = B^r_1(\omega) \in [-1, 1]$. For instance, when $t = 0$, so that $B^r_0 = 0$, have $\Lambda_0(x) = \log \cosh(x)$ with $\mathbb{K}$-forest expansion (1.3) of finite convergence radius $\rho_0 = \pi/2$. On the other hand, on the event $E := \{ B^r_T = \pm 1 \}$, the cgf $\Lambda_t(x)$ trivially takes the value $\log e^{\pm x} = \pm x$ so that, on $E$, we have $\rho_t(\omega) = +\infty$.

4.2 Diamond products of iterated stochastic integrals

At this stage, we feel the reader will be helped to see some systematic diamond computations at arbitrary level. Our example of choice is the set of iterated stochastic integrals, which play a fundamental role in stochastic numerics and rough path theory [KP92, Lyo98]. They are defined as follows. For a word $a = i_1 \ldots i_m$ of length $m$, with letters in $\mathbb{A} = \{ i : 1 \leq i \leq d \}$, write $ai$ for the word (of length $m + 1$) obtained by concatenation of $a$ with the letter $i$. Given a $d$-dimensional Brownian motion $(B^i)$, introduce the iterated Itô resp. Stratonovich integrals
\[ B^{ai} = \int_0^i B^a dB^i; \quad \hat{B}^{ai} = \int_0^i \hat{B}^a \circ dB^i; \]
set also $B^\phi = \hat{B}^\phi = 1$ when $\phi$ is the empty word. One extends these definitions by linearity to linear combination of words, which becomes a commutative algebra under the shuffle product (e.g. $12 \shuffle 3 = 312 + 132 + 123$). Then the remarkable identity
\[ B^{ai} \hat{B}^b = B^a; \quad c = a \shuffle b, \]
holds true (and reflects the validity of the usual chain rule for Stratonovich integration). In
contrast, resolving $B^i_t B^j_t$ requires quasi-shuffle (Itô Formula) which we will not introduce
here. Let us also recall Fawcett’s Formula (from e.g. Ch.3 in [FH14])

$$\mathbb{E}_0 \hat{B}_{0,1}^a = \langle e^{1/2}, a \rangle =: \sigma_a .$$

**Theorem 4.1.** Consider two (possibly empty) words $a, b$ with respective length $|a|, |b|$ and
letters $i = j \in \mathbb{A}$. Then

(Itô)

$$(B^a_i \circ B^b_j)_t(T) = B^a_i B^b_j(T - t) + \frac{T - t}{1 + (|a| + |b|)/2} (B^a \circ B^b)_t(T)$$

(Stratonovich)

$$(\hat{B}^a_i \circ \hat{B}^b_j)_t(T) = \hat{B}^a_i \hat{B}^b_j(T - t) + \hat{B}^a_i \sigma_b \frac{(T - t)^{|b| + 1}}{|b| + 1} + \hat{B}^b_j \sigma_a \frac{(T - t)^{|a| + 1}}{|a| + 1} + \frac{T - t}{1 + |a| + |b|/2} (\hat{B}^a \circ \hat{B}^b)_t(T).$$

In case $i \neq j$ both diamond products vanish.

**Proof.** (Itô) By Itô isometry, and the product rule $B^a_i B^b_j = B^a_i B^b_j + \ldots + (B^a_i, B^b_j)_{t,s}$, with
omitted martingale increment $\int_t^T (B^a dB^b + B^b dB^a),$

$$(B^a_i \circ B^b_j)_t(T) = \mathbb{E}_t(B^a_i, B^b_j)_t(T) = \delta^{ij} \int_t^T B_s^a B_s^b ds = \delta^{ij} \int_t^T (B_s^a B_s^b + (B^a \circ B^b)_t(s)) ds$$

From the scaling properties of Brownian motion, the time $t$-conditional law of $(B^a_i \circ B^b_j)_t(s)$
is equal to the law of

$$\left( \frac{s - t}{T - t} \right)^{|b|/2} (B^a \circ B^b)_t(T),$$

followed by an immediate integration over $s \in [t, T]$.

(Stratonovich) Note that

$$\hat{B}^a_i = \int \hat{B}^a dB^i + BV$$

so that, as in the Itô case (but now with non-centered dots),

$$(\hat{B}^a_i \circ \hat{B}^b_j)_t(T) = \delta^{ij} \mathbb{E}_t \int_t^T \hat{B}_s^a \hat{B}_s^b ds = \delta^{ij} \int_t^T ds (\hat{B}_s^a \hat{B}_s^b + \hat{B}_s^a \xi \hat{B}_s^b + \hat{B}_s^b \xi \hat{B}_s^a + (\hat{B}^a \circ \hat{B}^b)_t(s)).$$

Using the (known) Stratonovich expected signature of Brownian motion,

$$\mathbb{E}_t \hat{B}_{t,s}^b = (s - t)^{|b|/2} \mathbb{E}_0 \hat{B}_{0,1}^b = (s - t)^{|b|/2} \langle e^{1/2}, b \rangle =: (s - t)^{|b|/2} \sigma_b$$

we see, with $i = j$,

$$(\hat{B}^a_i \circ \hat{B}^b_j)_t(T) = \hat{B}^a_i \hat{B}^b_j(T - t) + \hat{B}^a_i \sigma_b \frac{(T - t)^{|b| + 1}}{|b| + 1} + \hat{B}^b_j \sigma_a \frac{(T - t)^{|a| + 1}}{|a| + 1} + \frac{T - t}{1 + |a| + |b|/2} (\hat{B}^a \circ \hat{B}^b)_t(T).$$

□
4.3 Lévy area

We now demonstrate the potential power of diamond calculus and the forest expansion by rederiving the following classical result.

**Theorem 4.2 (P. Lévy).** Let \( \{X, Y\} \) be 2-dimensional standard Brownian motion and stochastic (“Lévy”) area be given by

\[
\mathcal{A}_t = \int_0^t (X_s dY_s - Y_s dX_s).
\]

Then, for \( T \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \),

\[
\mathbb{E}_0[e^{\mathcal{A}_T}] = \frac{1}{\cos T}.
\]

As a warmup, we compute the first few cumulants, using the \( K \)-recursion from Theorem 1.1. By a direct computation (or using a very special case of Theorem 4.1),

\[
K^2 = \frac{1}{2} \varphi^2 = \frac{1}{2} (\mathcal{A} \circ \mathcal{A})(T) = \frac{1}{2} \int_t^T (\mathbb{E}_t [X_s^2] + \mathbb{E}_t [Y_s^2]) \, ds
\]

\[
= \frac{1}{2} (T - t)^2 + \frac{1}{2} (X_t^2 + Y_t^2) (T - t).
\]

In particular,

\[
dK^2_s = (X_s dX_s + Y_s dY_s)(T - s) + \text{BV}.
\]

Similarly, recalling that \( dK^1_s = X_s dY_s - Y_s dX_s \),

\[
K^3 = K^1 \circ K^2 = \varphi \varphi\varphi
\]

\[
= \mathbb{E}_t \left[ \int_t^T d\langle K^1, K^2 \rangle_s \right]
\]

\[
= \mathbb{E}_t \left[ \int_t^T [XY d\langle Y \rangle_s - YX d\langle X \rangle_s] (T - s) \right] = 0.
\]

It is easy to check that this pattern extends to all odd forests, i.e. they all vanish. Using again Theorem 4.1, or a direct computation (as given below),

\[
K^4 = \frac{1}{2} K^2 \circ K^2 = \frac{1}{2} \varphi \varphi \varphi \varphi
\]

\[
= \frac{1}{2} \mathbb{E}_t \left[ \int_t^T [X_s^2 d\langle X \rangle_s + Y_s^2 d\langle Y \rangle_s] (T - s)^2 \right]
\]

\[
= \frac{1}{2} \int_t^T \left( \mathbb{E}_t [X_s^2] + \mathbb{E}_t [Y_s^2] \right) (T - s)^2 ds
\]

\[
= \int_t^T (s - t) (T - s)^2 ds + \frac{1}{2} (X_t^2 + Y_t^2) \int_t^T (T - s)^2 ds
\]

\[
= \frac{1}{12} (T - t)^4 + \frac{1}{2} (X_t^2 + Y_t^2) \frac{1}{3} (T - t)^3.
\]
It is now clear how to extend this computation to all orders. Indeed we see that for each even \( n \),

\[
K_n(t(T)) = a_n I_n(t(T))
\]

for some \( a_n \in \mathbb{Q} \) where

\[
I_n(t(T)) = \frac{1}{n(n-1)} + \frac{1}{2} (X_t^2 + Y_t^2) \cdot \frac{1}{n-1} (T-t)^{n-1}.
\]

To compute the forests \( K^n \), we need the following lemma.

**Lemma 4.1.**

\[
(I^{(m)} \circ I^{(n)})_t(T) = \frac{2}{(m-1)(n-1)} I^{(n+m)}_t(T).
\]

**Proof.**

\[
dI^{(n)}_s(T) = (X_s dX_s + Y_s dY_s) \int_s^T (T-r)^{n-2} dr + BV
\]

\[
= \frac{1}{n-1} (X_s dX_s + Y_s dY_s) (T-s)^{n-1} + BV.
\]

Thus

\[
(I^{(m)} \circ I^{(n)})_t(T) = \frac{1}{(m-1)(n-1)} \int_t^T \left( \mathbb{E}_t \left[ X_s^2 \right] + \mathbb{E}_t \left[ Y_s^2 \right] \right) (T-s)^{n+m-2} ds
\]

\[
= \frac{2}{(m-1)(n-1)} I^{(n+m)}_t(T).
\]

\( \square \)

Note from above that \( K^2 = I^{(2)} \) and \( K^4 = I^{(4)} \). The next nonzero forest is then computed as

\[
K^6 = I^{(4)} \circ I^{(2)} = \frac{2}{3} \frac{1}{1} I^{(6)} = \frac{(T-t)^6}{45} + \frac{2}{3} \frac{1}{2} (X_t^2 + Y_t^2) \cdot \frac{1}{5} (T-t)^5.
\]

In principle, we could go on for ever, computing forests (or cumulants) using Theorem 1.1. With not too much extra effort, we can sum all these cumulants and so recover Lévy’s theorem.

**Remark 4.1.** Levin and Wildon\([LW08]\) obtain — in a combinatorial tour de force — Lévy’s theorem from moment (rather than cumulant) considerations.

**Proof.** (Lévy’s theorem) First note that for each \( n \),

\[
I^{(n)}_0(T) = \frac{T^n}{n(n-1)}.
\]
We now drop references to $t$ and $T$ for ease of notation. Let $\mathbb{K}=a_n I^{(n)}$. From Lemma 4.1 for $n > 2$,

$$a_n I^{(n)} = \mathbb{K} = \frac{1}{2} \sum_{j=2, j \text{ even}}^{n-2} \mathbb{K}^j \circ \mathbb{K}^{n-j} = \frac{1}{2} \sum_{j=2, j \text{ even}}^{n-2} \frac{2}{(j-1)(n-j-1)} a_j a_{n-j} I^{(n)}.$$

Thus

$$a_n = \frac{1}{2} \sum_{j=2, j \text{ even}}^{n-2} \frac{2}{(j-1)(n-j-1)} a_j a_{n-j},$$

and

$$\sum_{n=2, n \text{ even}}^{\infty} a_n \frac{T^n}{n(n-1)} = \frac{T^2}{2} + \sum_{n=4, n \text{ even}}^{\infty} \frac{2}{n-4} \sum_{j=2, j \text{ even}}^{n-2} \frac{2}{(j-1)(n-j-1)} a_j a_{n-j} \frac{T^n}{n(n-1)}.$$

Differentiating twice with respect to $T$ gives

$$\sum_{n=2, n \text{ even}}^{\infty} a_n T^{n-2} = a_2 + \sum_{n=4, n \text{ even}}^{\infty} \sum_{j=2, j \text{ even}}^{n-2} \frac{a_j}{j-1} \frac{a_{n-j}}{n-j-1} T^{n-2} = a_2 + \left( \sum_{j=2, j \text{ even}}^{\infty} \frac{a_j}{j-1} T^{j-1} \right)^2.$$

Define the function

$$f(T) = \sum_{n=2, n \text{ even}}^{\infty} \frac{a_n}{n-1} T^{n-1}.$$

It satisfies

$$f''(T) = f(T)^2 + a_2 = f(T)^2 + 1$$

with $f(0) = 0$. The solution to this ODE is $f(T) = \tan(T)$. Finally,

$$\sum_{j=1}^{\infty} \mathbb{K}^j = \sum_{n=2, n \text{ even}}^{\infty} a_n \frac{T^n}{n(n-1)} = \int_0^T \tan s \, ds = -\log \cos T,$$

which together with Theorem 1.1 concludes the proof of Lévy’s theorem. □
4.4 A Markov perspective and perturbative HJB analysis

Consider $f(B)$, the image of $d$-dimensional Brownian motion $B$ under $f : \mathbb{R}^d \to \mathbb{R}$. For sufficiently nice $f$ we know that

$$\mathbb{E}_t \left[ e^{\varepsilon f(B_T)} \right] =: u^\varepsilon(t, B_t)$$

defines the unique (bounded, classical) solution to backward PDE

$$-\partial_t u^\varepsilon = \frac{1}{2} \Delta u^\varepsilon, \quad u^\varepsilon(T, \cdot) = e^{\varepsilon f}.$$

With Cole-Hopf, $u^\varepsilon = \exp(\varepsilon v^\varepsilon)$, we get the Hamilton–Jacobi–Bellman equation

$$-\partial_t v^\varepsilon = \frac{1}{2} \Delta v^\varepsilon + \frac{1}{2} \varepsilon |\nabla v^\varepsilon|^2, \quad v^\varepsilon(T, \cdot) = f.$$

Write $v(0) \equiv v^*$ for the solution of the linear problem with $\varepsilon = 0$. Similar to (the basic step) in Hairer’s KPZ analysis [Hai13, p.571] we now perform a perturbative expansion, attributed to Wild (1951), and set $v^\varepsilon(t, x) = v^*(x) + \varepsilon g^\varepsilon(t, x)$, which gives

$$-\partial_t g^\varepsilon = \frac{1}{2} \Delta g^\varepsilon + \frac{1}{2} \left| \nabla (v^* + \varepsilon g^\varepsilon) \right|^2, \quad g^\varepsilon(T, \cdot) = 0.$$

The linearization of this equation has explicit solution

$$v^\varepsilon := K * |\nabla v^*|^2$$

where $K *$ denotes (space-time) convolution with the heat-kernel. Define degree $| \bullet | = 1$ and then $|\tau| := |\tau_1| + |\tau_2|$ whenever $\tau = [\tau_1, \tau_2]$ is obtained by “root joining” of (binary) trees. For example $|v| = ||\bullet, \bullet|| = 2$. Then, noting already the resemblance with the forest expansion (1.4),

$$v^\varepsilon(t, x) = v^* + \varepsilon v^\varepsilon + \varepsilon^2 v^{\varepsilon^2} + \cdots = \sum_{\tau} \varepsilon^{|\tau|-1} v^\tau(t, x),$$

with sum over binary trees and a cascade of linear PDEs given by, for all $\tau \neq \bullet$,

$$- \partial_t v^\tau = \frac{1}{2} \Delta v^\tau + \frac{1}{2} \nabla v^{\tau_1} \cdot \nabla v^{\tau_2}, \quad v^\tau(T, \cdot) = 0. \quad (4.1)$$

We shall assume that (for a given $f$) this cascade of linear PDEs is well-posed, with $v^\tau$ given by space-time convolution of the forcing term. This is straightforward in case of $C^\infty$-bounded $f$, but under more natural assumptions (say, measurable bounded) this is not trivial and requires careful tracking of the gradients. Since this subsection only serves as an illustration, we will pursue this issue any further.\footnote{The reader must not be troubled by the fact that $u^\varepsilon$ can be written as 1D integral against a standard Gaussian. What follows would be the same for a Brownian motion on a compact Riemannian manifold where no such explicit solution is available.}

\footnote{The first author would like to thank Herbert Koch (Bonn) for a related discussion.}
Theorem 4.3. We have the formal expansion

$$\log \mathbb{E}_t \left[ e^{\varepsilon f(B_T)} \right] = e^v^*(t, B_t) + \sum_{k \geq 2} \varepsilon^k \sum_{\tau_1 | \tau | = k} v^\tau (t, B_t)$$

We can precisely match this with our $K$-expansion (which provides — en passant — conditions under which this formal series converges).

Theorem 4.4. Let $k \in \mathbb{N}$ and $K^k$ as in (1.1) with $K^1_0 (T) = \mathbb{E}_t f(B_T) = v^*_t$. Then

$$K^k_0 (T) = \sum_{\tau_1 | \tau | = k} v^\tau (t, B_t).$$

Remark 4.2. With $A_t = f(B_t)$ it is natural to decompose $K^1_0 (T) = \mathbb{E}_t f(B_T) = A_t + X_t (T)$ where $X$ is defined by this equality. At this stage, additional ($C^2$) regularity of $f$ is necessary to ensure that $A$ and $X$ are semimartingales. Note also that $X_t (T) = \mathbb{E}_t \int_t^T \frac{1}{2} \Delta f(B_s) ds$ is of order $T - t$ and has, unlike $A_t$, the interpretation as a term in a short-time (time-to-maturity) expansion. One can then track the contribution of $A = \circ$ and $X = \bullet$ through the $K$-recursion, starting with

$$2K^2 = K^1 \circ K^1 = v^\circ + v^\circ + v^\circ + v^\circ \quad \leftrightarrow \quad v.$$

Proof. Compute

$$K^2_0 (T) = \frac{1}{2} (K^1 \circ K^1)_0 (T) = \frac{1}{2} \mathbb{E}_t \left[ \langle v^\bullet (., B) \rangle_{t, T} \right] = \frac{1}{2} \mathbb{E}_t \left[ \int_t^T |\nabla v^\bullet (r, B)|^2 dr \right] =: \tilde{v}^V (t, B_t)$$

It then follows from Feynman-Kac (or Itô’s Formula) that $\tilde{v}^V$ satisfies

$$-\partial_t \tilde{v}^V = \frac{1}{2} \Delta \tilde{v}^V + \frac{1}{2} |\nabla v^\bullet |^2, \quad \tilde{v}^V (T, .) = 0,$$

which is the same backward equation, with space-time forcing $\nabla v^\bullet$ and identical (zero) terminal data, as for $v^V$. By assumption, the PDE iteration is well-posed, i.e. solutions are unique, hence $v^V$ and $\tilde{v}^V$ must coincide. Similarly,

$$K^3_0 (T) = (K^2 \circ K^1)_0 (T) = \mathbb{E}_t \left[ \langle v^\bullet (., B), v^\bullet (., B) \rangle_{t, T} \right] =: 2\tilde{v}^\tilde{Y} (t, B_t),$$

where $\tilde{v}^\tilde{Y}$ satisfies the backward equation with forcing $\frac{1}{2} \nabla v^\bullet \cdot \nabla v^\bullet$, as given by (4.1) in case of $\tau_1 = v$ and $\tau_2 = \bullet$ (switching the roles of $\tau_1$ and $\tau_2$ accounts for the factor 2 here). A formal induction proof along the same lines is then straightforward and details omitted. □

4.5 Cumulants on Wiener-Itô chaos

On the classical Wiener space $C([0, T], \mathbb{R})$, with Brownian motion $B(\omega, t) = \omega_t$, consider an arbitrary element in the second Wiener Itô chaos, written in the form

$$A_T := I_2^1 (f) := \int_0^T \int_0^V f(w, v) dB_w dB_v ,$$

17
with $f = f_A \in L^2$ on the simplex $\Delta_T = \{(s, t) : 0 \leq s \leq t \leq T\}$. Note martingality $A_t := \mathbb{E}_t A_T$ so that $\mathbb{E}_t A_{t,T} = \mathbb{E}_t A_T - A_t = 0$. Then

$$A_{t,T} = \int_t^T \int_0^v f(w, v) dB_w dB_v = \int_t^T \int_0^v f(w, v) dB_w dB_v + \int_t^T \int_0^t f(w, v) dB_w dB_v$$

and

$$\langle A \rangle_{t,T} = \int_t^T \left( \int_0^v f(w, v) dB_w \right)^2 dv = \int_t^T \left( \int_0^t f(w, v) dB_w + \int_0^v (\ldots) \right)^2 dv,$$

so that

$$(A \circ A)_i(T) = \mathbb{E}_t \langle A \rangle_{i,T} = \int_t^T \left( \int_0^t f(w, v) dB_w \right)^2 dv + 0 + \int_t^T \int_0^v f^2(v, w) dwdv.$$

We have thus computed $\mathbb{E}_i^2(T) = \frac{1}{2} (A \circ A)_i(T)$. By polarization, for $A = I_2(f_A), C = I_2(f_C)$,

$$(A \circ C)_i(T) = \int_t^T \left( \int_0^t f_A(r, v) dB_r \right) \left( \int_0^t f_C(r, v) dB_r \right) dv + \int_t^T \int_0^v f_A(w, v) f_C(w, v) dwdv.$$

To go further, we exhibit the martingale part of $A \circ C$ by writing

$$\int_0^T \left( \int_0^t f_A(r, v) dB_r \right) \left( \int_0^t f_C(r, v) dB_r \right) dv - \int_0^T (\ldots) dv + \int_0^T \int_0^v f_A(w, v) f_C(w, v) dwdv.$$

From the product rule, with $BV_t = \int_0^t f_A(r, v) f_C(r, v) dr$, we have

$$\left( \int_0^t f_A(r, v) dB_r \right) \left( \int_0^t f_C(r, v) dB_r \right) = \int_0^T \int_0^s \left( f_A(r, v) f_C(s, v) + f_C(r, v) f_A(s, v) \right) dB_r dB_s + BV_t,$$

Letting $\otimes_1$ indicate integration in one (the right-sided) variable and tilde symmetrisation,

$$f_A \tilde{\otimes}_1 f_C := \int_0^T (f_A(r, v) f_C(s, v) + f_C(r, v) f_A(s, v)) dv$$

so that

$$(A \circ C)_i(T) = \int_0^T \int_0^s (f_A \tilde{\otimes}_1 f_C) dB_r dB_s + BV_i(T)$$

with

$$(A \circ C)_0(T) = BV_0(T) = \int_0^T \int_0^v f_A(r, v) f_C(r, v) dr dv = \langle f_A, f_C \rangle_{\Delta T} =: f_A \otimes_2 f_C.$$

In particular, we see that from (1.4) that the third cumulant of $A_T = I_2(f_A)$ is given by

$$\kappa_3(A_T) = \kappa_3(I_2(f_A)) = 3(A \circ (A \circ A))_0(T) = \langle f_A, f_A \circ A \rangle = \langle f_A, (f_A \tilde{\otimes}_1 f_A) \rangle.$$

Theorem 1.1 then provides, in the present setting, an alternative to the (Malliavin calculus based) approach of Nourdin–Peccati [NP10]: by (5.22) in that paper, the $n$th cumulant of $I_2(f)$ is given by some explicit formula which reduces to (in case $n = 3$) our formula. It is not difficult to push this “diamond” computation to recover cumulants for general integer $n$. The diamond approach of course works just as well for higher Wiener-Itô chaos and $d$-dimensional Wiener space, as was already seen in Section 4.2. Note however that [NP10] also deals with Gaussian fields, whereas Theorem 1.1 is restricted to processes.
4.6 Stochastic volatility

We return to the financial mathematics context that originally gave rise to our result. Let $S$ be a strictly positive continuous martingale. Then $X := \log S$ is a semimartingale for which we assume the quadratic variation process $\langle X \rangle$ to be absolutely continuous. Then stochastic variance and forward variance are given by

$$v_t := d\langle X \rangle_t / dt, \quad \xi(T) = \mathbb{E}_t [v_T].$$

Model specification in terms of forward variance, where one specifies directly forward variances — viewed as a family of martingales indexed by their individual time horizon $T$ — is nowadays ubiquitous in (equity) financial modeling. Pricing and hedging of equity and variance options then requires knowledge of the joint law of (log)-price, total and instantaneous stochastic variance at a future time $T$, as seen from today’s time $t$. This problem is fully resolved in our framework.

**Theorem 4.5.** For $a, b, c \in \mathbb{R}$ sufficiently small we have, with $\bar{b} = b - \frac{1}{2} a$,

$$\mathbb{E}_t [e^{a X_T + b \langle X \rangle_T + c v_T}] = \exp \left\{ a X_t + \bar{b} (X \diamond X)_t(T) + c \xi_t(T) + \sum_{k=2}^{\infty} \mathbb{K}_k^t \right\},$$

where the $\mathbb{K}^k$'s are given recursively by (1.1), starting with

$$\mathbb{K}^1 = a X + \bar{b} (X \diamond X) + c \xi = a \langle s \rangle + \bar{b} \langle \sigma \rangle + c \langle \sigma \rangle,$$

and then

$$\mathbb{K}^2 = \frac{1}{2} \left( a^2 \langle \sigma \rangle + \bar{b}^2 (X \diamond X) + c^2 \langle \sigma \rangle \right) + ab \langle \sigma \rangle + ac \langle \sigma \rangle + bc \langle \sigma \rangle.$$

**Proof.** This is a direct consequence of Theorem 1.1: the time-$T$ quantity of interest is $A_T := a X_T + b \langle X \rangle_T + c v_T$ and it suffices to compute (using that $X + \langle X \rangle/2$ is martingale)

$$\mathbb{E}_t [A_T] = a X_t + (b - \frac{1}{2} a) (X \diamond X)_t(T) + c \xi_t(T).$$

Setting $b = c = 0$ in Theorem 4.5 gives the following corollary which, after reordering as in Section 3, makes rigorous the formal expansion given as Corollary 3.1 of [AGR20].

**Corollary 4.1.** For sufficiently small $a \in \mathbb{R},$

$$\log \mathbb{E}_t [e^{a X_T}] = \sum_{k=1}^{\infty} \mathbb{K}_k^t,$$

where the $\mathbb{K}^k$'s are given by (1.1), starting with $\mathbb{K}^1 = a X - \frac{1}{2} a (X \diamond X) = a \langle s \rangle - \frac{1}{2} a \langle \sigma \rangle.$

Our final point is that diamond trees are in many cases explicitly computable. Following [GKR19] we consider forward variance models of the form

$$\frac{dS_t}{S_t} = \sqrt{v_t} dZ_t$$

$$d\xi_t(u) = \kappa(u-t) \sqrt{v_t} dW_t,$$

(4.3)
with \(L^2\)-kernel \(\kappa\) and Brownian drivers of correlation \(d\langle W, Z\rangle_t/dt = \rho\). In this class of models, which includes classical and rough Heston (see below), we now show that all diamond trees take the form of convolutions of the forward variance curve \(\xi\) and some function \(h = h(t)\).

**Lemma 4.2.** In an affine forward variance model, all diamond trees take the form

\[
\int_t^T \xi_t(u) h(t-u) \, du
\]

(4.4)

for some function \(h\).

**Proof.** With \(X = \log S\), under the affine specification (4.3), all trees are built as diamond products of \(X\) and \(\xi\) and weighted integrals thereof. Lowest order trees are given by

\[
\triangledown = (X \diamond X)(T) = \mathbb{E}_t \left[ \int_t^T d\langle X \rangle_u \right] = \int_t^T \xi_t(u) \, du
\]

\[
\triangledowndown = (X \diamond \xi)(T) = \mathbb{E}_t \left[ \int_t^T d\langle X, \xi \rangle_u \right] = \rho \int_t^T \xi_t(u) \kappa(T-u) \, du
\]

\[
\triangledowndonw = (\xi \diamond \xi)(T) = \mathbb{E}_t \left[ \int_t^T d\langle \xi \rangle_u \right] = \int_t^T \xi_t(u) \kappa(T-u)^2 \, du,
\]

all of which are of the form (4.4). Consider two arbitrary trees denoted by \(T^i(T)\) \((i = 1, 2)\). Suppose that each of these two trees takes the form

\[
T^i(T) = \int_t^T \xi_t(u) h^i(T-u) \, du
\]

Then

\[
(T^1 \diamond T^2)(T) = \mathbb{E}_t \left[ \int_t^T d\langle T^1, T^2 \rangle_u \right]
\]

\[
= \mathbb{E}_t \left[ \int_t^T \int_u^T h^1(T-s) \, ds \int_u^T h^2(T-r) \, dr \, d\langle \xi(s), \xi(r) \rangle_u \right]
\]

\[
= \mathbb{E}_t \left[ \int_t^T \xi_t(u) \, du \int_u^T h^1(T-s) \, ds \int_u^T h^2(T-r) \kappa(T-s) \kappa(T-r) \, dr \right]
\]

\[
= \int_t^T \xi_t(u) h^{12}(T-u) \, du,
\]

where

\[
h^{12}(T-u) = \int_u^T h^1(T-s) \kappa(T-s) \, ds \int_u^T h^2(T-r) \kappa(T-r) \, dr.
\]

The result then follows by induction. \(\square\)
Example 4.3 (Classical Heston). In this case, 
\[ d\xi_t(u) = \nu e^{-\lambda(u-t)} \sqrt{v_t} \, dW_t. \]

Then, for example,
\[ \xi_t = (X \diamond (X \diamond X))_t(T) = \frac{\rho \nu}{\lambda} \int_t^T \xi_t(u) \left[ 1 - e^{-\lambda(T-u)} \right] \, du. \]

Example 4.4 (Rough Heston). In this case, with \( \alpha = H + 1/2 \in (1/2, 1) \),
\[ d\xi_t(u) = \nu \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} (u-t)^{\alpha-1} \sqrt{v_t} \, dW_t. \]

Then, for example,
\[ \xi_t = (X \diamond X)_t(T) = \int_t^T \xi_t(u) \, du, \]
\[ \xi_t = ((X \diamond X) \diamond (X \diamond X))_t(T) = \frac{\nu^2}{\Gamma(1+\alpha)^2} \int_t^T \xi_t(u) \left( \int_u^T (s-u)^{\alpha-1} \, ds \right)^2 \, du \]
\[ = \frac{\nu^2}{\Gamma(1+\alpha)^2} \int_t^T \xi_t(u) (T-u)^{2\alpha} \, du. \]

For a bounded forward variance curve \( \xi \) one then sees that diamond trees with \( k \) leaves are of order \((T-t)^{1+(k-2)\alpha}\). In this case, the \( \mathcal{F} \)-expansion (forest reordering according to number of leaves) has the interpretation of a short-time expansion, the concrete powers of which depend on the roughness parameter \( \alpha = H + 1/2 \in (1/2, 1) \), cf. [CGP18, GR19].

Remark 4.3. Lemma 4.2 combined with Theorem 4.3 characterises the triple-joint mgf of \( X_T, v_T \) and \( \langle X \rangle_T \) for an affine forward variance model which should be compared to Theorem 4.3 of [JLP19] and Proposition 4.6 of [GKR19]. In fact, it is not hard to be more explicit in our computation and obtain the convolutional form
\[ \mathbb{E}_t \left[ e^{a X_T + b v_T + c \langle X \rangle_T} \right] = \exp \{ a X_t + (\xi \star g)(\tau; a, b, c), T) \}. \]

This is consistent (and generalizes) Theorem 2.6 of [GKR19] where the same convolution Riccati equation appears, but with \( g = g(\tau; a) \) instead of \( (\tau; a, b, c) \) and different boundary conditions.

References

[AGR20] Elisa Alòs, Jim Gatheral, and Radoš Radoičić. Exponentiation of conditional expectations under stochastic volatility. *Quantitative Finance*, 20(1):13–27, 2020.
[BD98] Michelle Boué and Paul Dupuis. A variational representation for certain functionals of Brownian motion. *The Annals of Probability*, 26(4):1641–1659, 1998.

[BH08] Philippe Briand and Ying Hu. Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probability Theory and Related Fields*, 141(3-4):543–567, 2008.

[CGP18] Giorgia Callegaro, Martino Grasselli, and Gilles Pagès. Rough but not so tough: Fast hybrid schemes for fractional Riccati equations. *arXiv preprint arXiv:1805.12587*, 2018.

[Dup19] Bruno Dupire. Functional Itô calculus. *Quantitative Finance*, 19(5):721–729, 2019.

[EFPTZ18] Kurusch Ebrahimi-Fard, Frédéric Patras, Nikolas Tapia, and Lorenzo Zambotti. Hopf-algebraic deformations of products and Wick polynomials. *International Mathematics Research Notices*, December 2018.

[FH14] Peter K Friz and Martin Hairer. *A course on rough paths. With an Introduction to Regularity Structures*. Springer, 2014.

[GHL+93] Håkon Gjessing, Helge Holden, Tom Lindstrøm, Bernt Øksendal, Jan Ubøe, and Tusheng Zhang. The Wick product. *Frontiers in Pure and Applied Probability*, 1:29–67, 1993.

[GKR19] Jim Gatheral and Martin Keller-Ressel. Affine forward variance models. *Finance and Stochastics*, 23(3):501–533, 2019.

[GR19] Jim Gatheral and Radoš Radoičić. Rational approximation of the rough Heston solution. *International Journal of Theoretical and Applied Finance*, 22(3):1950010, 2019.

[Hai13] Martin Hairer. Solving the KPZ equation. *Annals of Mathematics*, 178:559–664, 2013.

[Jan97] Svante Janson. *Gaussian Hilbert spaces*, volume 129. Cambridge University Press, 1997.

[JLP19] Eduardo Abi Jaber, Martin Larsson, and Sergio Pulido. Affine Volterra processes. *The Annals of Applied Probability*, 29(5):3155–3200, 2019.

[Kob00] Magdalena Koblanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *The Annals of Probability*, 28(2):558–602, 2000.

[KP92] Peter E. Kloeden and Eckhard Platen. *Numerical Solution of Stochastic Differential Equations*. Springer Berlin Heidelberg, 1992.
[Luk70] Eugene Lukacs. *Characteristic functions*. Griffin, 1970.

[LW08] Daniel Levin and Mark Wildon. A combinatorial method for calculating the moments of Lévy area. *Transactions of the American Mathematical Society*, 360(12):6695–6709, 2008.

[Lyo98] Terry Lyons. Differential equations driven by rough signals. *Revista Matemática Iberoamericana*, pages 215–310, 1998.

[Mor84] Patrick Alfred Pierce Moran. *An introduction to probability theory*. Clarendon Press, 1984.

[NP10] Ivan Nourdin and Giovanni Peccati. Cumulants on the Wiener space. *Journal of Functional Analysis*, 258(11):3775–3791, 2010.

[RY13] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293. Springer, 2013.