Abstract—Nonovershooting stabilization is a form of safe control where the setpoint chosen by the user is at the boundary of the safe set. In this article, we develop homogeneous feedback laws for fixed-time nonovershooting stabilization for nonlinear systems that are input–output linearizable with a full relative degree, i.e., for systems that are diffeomorphically equivalent to the chain of integrators. These homogeneous feedback laws can also assume the secondary role of “fixed-time safety filters,” which keep the system within the closed safe set for all time but, in the case where the user’s nominal control commands approach to the unsafe set, allow the system to reach the boundary of the safe set no later than a desired time that is independent of nominal control and independent of the value of the state at the time the nominal control begins to be overridden.

Index Terms—Control design, nonlinear control systems, robust control.

I. INTRODUCTION

NONOVERSHOOTING control is aimed at solving a tracking problem under certain state/output constrains (see, e.g., [20], [40]). Such a control can be employed in a “safety filter” framework to override a potentially unsafe nominal controller [1]. Exponential nonovershooting stabilization, including suitable extensions to systems with deterministic and stochastic disturbances, has been solved by the second author and his coauthors. Nonovershooting and safety controllers have been designed for both linear [8], [25], [28] and nonlinear plants [12], [20], [24]. Inspired by robust nonovershooting control under deterministic disturbances in [20], i.e., by stabilization to an equilibrium at the barrier along with input-to-state safety, mean-square stabilization of stochastic nonlinear systems to an equilibrium at the barrier, along with a guarantee of nonviolition of the barrier in the mean sense, is solved in [23].

In many cases, the corresponding synthesis was based on the so-called control barrier functions [3], [16], which have been extensively used in various control applications, such as automotive systems [2], [35] and multiagent robotics [36], [38]. The exponential control barrier functions were reported in [27] and allowed the use of simple linear tools for a control design. The nonovershooting prescribed-time (PT) control has been developed recently [1] for the integrator chain based on a novel time-varying backstepping procedure [19], [37]. Such a control allows safety requirements to be fulfilled for a PT interval [0, T] with T > 0. PT controllers [37], which have been developed even for PDEs [9], form a special subset of both finite-time [6] and fixed-time [32] control algorithms. The concepts of finite-time and fixed-time barrier functions were introduced in [22] and [12].

A. Homogeneous Nonovershooting Stabilization

Inspired by the mentioned results, this article deals with generalized homogeneous nonovershooting controller design. The goal is to synthesize feedback laws that ensure finite-time stabilization (FnTS) and fixed-time stabilization (FxTS) without the overshoot of the system’s output.

Homogeneity is a dilation symmetry studied in many branches of pure and applied mathematics. For instance, homogeneous differential equations and homogeneous control systems are well studied in the literature [7], [13], [14], [15], [17], [42]. Being similar (in some sense) to linear control systems, they may demonstrate faster convergence [5], better robustness [4], and smaller “peaking effect” [29, Ch. 1]. To the best of the authors’ knowledge, a homogeneous nonovershooting control has never been designed even for the integrator chain.

FnTS without an overshoot is formulated as the problem of constructing a feedback law (for example, of a homogeneous type) such that, given an initial state x0 ∈ Rn, the feedback guarantees that the trajectory of the closed-loop system initiated in x0 at the time instant t = 0 reaches the origin no later than a time instant T(x0) without overshoot in the first coordinate. If, given arbitrary T > 0, the origin is reached by the time T for any x0, such a nonovershooting feedback is referred to as fixed-time stabilizing. Fixed-time control under spatiotemporal and input constraints is designed in [11] as a solution of a state-dependent quadratic optimization problem, which has to be computed for each vector of the state space. Being computationally consuming this approach is, theoretically, applicable to a rather large class of affine-in-control systems. However, the problem of the finite/fixed-time nonovershooting stabilization of the integrator chain does not satisfy assumptions of [11].
B. Homogeneous Safety Filters

To prevent overshoots that may result from the user’s nominal control, we also design what are commonly referred to as “safety filters” and have a particularly simple structure for integrator chains [1].

a) Finite-time safety (FnTSf) and fixed-time safety (FxTSf): Since the notion of fixed-time barrier functions has already appeared in the literature [12], let us explain what FxTSf means to us. While the precise meaning of this property and the weaker FnTSf property will become clear in the statements of Theorems 3 and 2, respectively, here we only give a descriptive, intuitive definition. We consider control systems subject to a nominal control \( u_{\text{nom}} \) and equipped with a “safety filter” feedback law \( u = F(u_{\text{nom}}, x) \). The safety filter is said to ensure FxTSf with fixed time \( T \) if it ensures safety and the following holds for all initial conditions inside the safe set: whenever the applied control \( F(u_{\text{nom}}(t), x(t)) \) is different from the nominal control \( u_{\text{nom}}(t) \) over an entire time interval of duration of \( T \), the solution reaches the boundary of the safe set no later than at the end of that time interval.

Clearly, FnTSf and FxTSf impart less safety on the system than the conventional exponential and asymptotic safety properties, under which the trajectories remain away from the boundary for all finite times. This is precisely the point as FnTSf/FxTSf filters are less conservative. They allow the nominal performance to be achieved sooner, with less distortion. In the language of the “safety versus agility” tradeoff [3], the FxTSf filters are the most agile filters available.

A comparison with the prescribed-time safety (PTSf) filters in [1] is in order. Both the PTSf and FxTSf filters ensure safety while allowing convergence to the safety boundary in an amount of time that is independent of the initial condition. The difference is that FxTSf are applicable in infinite-time safety operation, whereas PTSf are ideal when a safety prohibition is only of finite duration.

b) Release time and restraint time: The time \( T \) figures in both the PTSf and FnTSf/FxTSf approaches. It is important to understand the different roles that \( T \) plays in the two approaches. In PTSf, \( T \) is the time after which the prohibition of entering the unsafe set is lifted. Hence, \( T \) should be understood in PTSf as release time (from the safety constraint). In FnTSf/FxTSf, the safety constraint applies in perpetuity—the unsafe set is prohibited for all time. So, in FnTSf/FxTSf, \( T \) is the length of a period that can commence at any moment and, if \( u_{\text{nom}} \) causes itself to be overridden by the safety filter \( F \) over the entire interval, the trajectory will necessarily be permitted to reach the boundary of the unsafe set, but not enter it. Hence in FnTSf/FxTSf, \( T \) should be understood as restraint time since the safety filter restrains the trajectory away from the boundary for no longer than \( T \). The release time is a property bestowed upon the system by a PTSf safety filter. Likewise, the restraint time is a property bestowed upon the system by the FnTSf/FxTSf safety filter \( F \). In FnTSf, the restraint time \( T \) may depend on the value of \( x \) at the moment the safety override kicks in and on the nominal control \( u_{\text{nom}} \), whereas in FxTSf the restraint time can be arbitrarily assigned with the filter and is independent of \( x \) and \( u_{\text{nom}} \).

C. Contributions and Organization

To summarize, this article’s contributions are the design and the associated stability and safety analyses for the following:

1) homogeneous finite/fixed-time nonovershooting stabilizing controllers;
2) homogeneous FnTSf/FxTSf filters.

The rest of this article is organized as follows. Section II presents the problem statement. Short preliminaries about the homogeneity are given in Section III. A homogeneous nonovershooting controller design and issues of its tuning are discussed in Section IV. Section V is devoted to homogeneous safety filters. As an example, linear, homogeneous, and PTSf filters for the double integrator are compared in Section VI. A nonovershooting homogeneous stabilizer for the triple integrator is designed in Section VII. Section VIII concludes this article. Some auxiliary results are given in the Appendix.

D. Notation

1) \( \mathbb{R} \) is the field of reals and \( \mathbb{R}_+ = \{ \alpha \in \mathbb{R} : \alpha > 0 \} \).
2) \( e_1 = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{R}^n \) is the 1th element of the canonical Euclidean basis in \( \mathbb{R}^n \).
3) \( \Sigma = \{ x \in \mathbb{R}^n : e_1^T x \leq 0 \} \) is the half-space in \( \mathbb{R}^n \) that specifies the safe set of the system.
4) \( \text{int}\Sigma \) denotes an interior of the set \( S \subset \mathbb{R}^n \).
5) \( P > 0 \) (resp. \( P < 0 \)) means that the symmetric matrix \( P \in \mathbb{R}^{n \times n} \) is positive (resp. negative) definite.
6) The matrix \( P^2 = M \) is such that \( M^2 = P \). By definition, \( P^{-\frac{1}{2}} = (P^\frac{1}{2})^{-1} \).
7) \( \lambda_{\max}(P) \) (resp. \( \lambda_{\min}(P) \)) denotes a maximal (resp. minimal) eigenvalue of the symmetric matrix \( P \).
8) \( \text{diag}\{p_1, \ldots, p_n\} \in \mathbb{R}^{n \times n} \) denotes a diagonal matrix with the elements \( p_1, p_2, \ldots, p_n \) on the main diagonal.

II. PROBLEM STATEMENT

Consider the system
\[
\dot{x} = Ax + Bu + g(t, x), \quad t > 0, \quad x(0) = x_0
\]
where \( x \in \mathbb{R}^n, u \in \mathbb{R} \)
\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1 \\
1
\end{pmatrix}
\]
and a locally bounded measurable function \( g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) models uncertainties and disturbances of the system.

Given \( x_0 \in \text{int}\Sigma, T > 0 \) and \( g = 0 \), we need to design a state feedback control \( u = u_h(x) \) such that
1) \( u_h \in C(\mathbb{R}^n \setminus \{0\}, \mathbb{R}) \) is a locally Lipschitz continuous and uniformly bounded function of the state; 2) the closed-loop system is globally uniformly finite-time stable\(^1\) with
\[
x(t, x_0) = 0 \quad \forall t \geq T
\]

\(^1\)A system \( \dot{x} = f(x), t > 0, x(0) = x_0 \) is said to be globally uniformly finite-time stable if it is Lyapunov stable and there exists a locally bounded function \( T : \mathbb{R}^n \to [0, +\infty) \) such that \( x(t, x_0) = 0, \forall t \geq T(x_0) \) for any solution \( x(\cdot, x_0) \) of the system and any \( x_0 \in \mathbb{R}^n \).
where \( x(t, x_0) \) denotes the unique solution of the closed-loop system with the initial condition \( x(0) = z \in \mathbb{R}^n \);
3) \( x(t, x_0) \in \Sigma \) for all \( t \geq 0 \).

Such a controller (if it exists) steers the state vector at zero without overshoot in the first component, i.e., \( e_1^T x(t, x_0) \leq 0 \), \( \forall t \geq 0 \). In this article, we design a homogeneous nonovershooting finite-time controller \( u_0 \), allowing also a stabilization in a fixed-time \( T > 0 \) independently of \( x_0 \). Moreover, we need to characterize a class of perturbations \( g \) which can be rejected by the designed controller such that the nonovershooting property is preserved.

Our aim also is to design a “safety filter” [1], which applies the user’s nominal control \( u_{\text{nom}} \) while the system operates in the safe set and overrides \( u_{\text{nom}} \) with the nonovershooting homogeneous control \( u_h \) in the domain where the user’s nominal control commands operation in the unsafe set.

### III. Preliminaries: Homogeneity

#### A. Dilation Symmetry of Functions and Vector Fields

Homogeneity is a symmetry of a set or a mapping with respect to a group of transformations called dilation (see, e.g., [4], [5], [14], [17], [29], [42]). In this article, we deal only with the following weighted dilations:

\[
d(s) = e^{sG_d}, \quad G_d = \text{diag}\{n_i + 1\}_{i=1}^n. \tag{4}
\]

In a more general case, a linear dilation [29] in \( \mathbb{R}^n \) is defined by an arbitrary anti-Hurwitz matrix \( G_d \in \mathbb{R}^{n \times n} \). Recall [17] that a vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) (a function \( h : \mathbb{R}^n \to \mathbb{R}^n \)) is said to be \( d \)-homogeneous of degree \( \mu \) if

\[
f(d(s)x) = e^{\mu s}d(s)f(x) \quad \forall x \in \mathbb{R}^n \quad \forall s \in \mathbb{R}. \tag{5}
\]

(resp. \( h(d(s)x) = e^{\mu s}h(x) \) \( \forall x \in \mathbb{R}^n \) \( \forall s \in \mathbb{R} \)).

Simple computations show that the linear vector field \( x \mapsto Ax \) is \( d \)-homogeneous of degree \(-1\)

\[
Ad(s) = e^{\mu s}d(s)A \quad \forall s \in \mathbb{R}. \tag{7}
\]

where the dilation \( d \) is given by (4) and \( A \) is defined by (2).

Homogeneous control systems are similar (in some sense) to linear control systems, but they may have additional useful properties, such as better robustness, faster convergence, and smaller overshoot [29, Ch. 1]. It is well known [5], [26], [41] that any asymptotically homogeneous system

\[
\dot{x} = f(x), \quad t > 0, \quad x(0) = x_0 \tag{8}
\]

of (negative (positive) degree is globally uniformly finite-time (nearly fixed-time)\(^2\) stable, where \( f \in C(\mathbb{R}^n \setminus \{0\}) \). To guarantee an FnTSS of system (1), (2), it is sufficient to design an asymptotically stabilizing homogeneous control \( u : \mathbb{R}^n \to \mathbb{R}^n \) such that \( u(d(s)x) = u(x), \forall x \in \mathbb{R}^n, \forall s \in \mathbb{R} \). Some schemes of homogeneous and locally homogeneous control design for the integrator chain are presented in [4], [5], [7], and [21]. This article uses the technique proposed in [32] allowing simple rules for control parameters tuning and the settling time adjustment to be applied.

\(^2\)The system is globally uniformly nearly fixed-time stable if it is globally uniformly asymptotically stable and for any \( \tau > 0 \) there exists \( T_\tau > 0 \) such that \( \|x(t, x_0)\| \leq r, \forall t \geq T_\tau, \forall x_0 \in \mathbb{R}^n \).

### B. Canonical Homogeneous Norm

Inspired by [30], let us define the so-called canonical homogeneous norm \( \| \cdot \|_d : \mathbb{R}^n \to [0, +\infty) \) as follows \( \|0\|_d = 0 \) and

\[
\|x\|_d = e^{\tau x} : \|d(-s_d)x\| = 1, \quad x \neq 0 \tag{9}
\]

where \( \|x\| = \sqrt{x^TPx} \) is the weighted Euclidean norm

\[
P = PT \in \mathbb{R}^{n \times n} : PG_d + G_d P > 0, \quad P > 0. \tag{10}
\]

The latter matrix inequalities guarantee that the linear dilation \( d \) is monotone, i.e., the function \( s \mapsto \|d(s)x\| \) is monotone for any \( x \in \mathbb{R}^n \). One can be shown [30] that \( \| \cdot \|_d \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}) \) and

1) \( \|d(s)x\|_d = e^{\tau s}\|x\|_d \) for all \( x \in \mathbb{R}^n \) and all \( s \in \mathbb{R} \);
2) \( \|x\|_d = 1 \iff \|x\|_d \leq 1 \iff \|x\|_d \leq 1 \).

Moreover, for \( x \neq 0 \) we have

\[
\frac{\partial}{\partial x} \|x\|_d = \|x\|_d \frac{2\tau d(-\ln \|x\|_d)\|d(-\ln \|x\|_d)x\|}{\|x\|_d}, \tag{11}
\]

In the following, we utilize the homogeneous norm \( \|x\|_d \) for both control design and stability analysis (as a Lyapunov function). Since the homogeneous norm is defined implicitly, then, to compute \( \|x\|_d \) for a given \( x \neq 0 \), the equation \( d(-s_d)x = 1 \) needs to be solved with respect to \( s_d \in \mathbb{R} \).

**Example 1:** For \( G_d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), by definition, \( \|x\|_d = V \) is a positive definite solution

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} V^{-2} & 0 \\ 0 & V^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \tag{12}
\]

where

\[
P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} > 0. \tag{13}
\]

For \( x \neq 0 \) the solution always exists and unique if

\[
P > 0, \quad PG_d + G_d P > 0. \tag{14}
\]

Hence, the homogeneous norm \( \|x\|_d \) is a unique positive root of the quartic equation

\[
V^4 - 2p_{22}x_2^2 V^2 - 2p_{12}x_1x_2 V - x_1^2 p_{11} = 0 \tag{15}
\]

which can be found using the Ferrari formula (see Appendix). \( \diamond \)

Therefore, for \( n = 2 \), the canonical homogeneous norm \( \|x\|_d \) can be calculated explicitly. In other cases, a special numerical algorithm may be utilized in order to compute \( \|x\|_d \) in practice, (see, e.g., [31], [32], [39] for more details).

### IV. Homogeneous Stabilization Without Overshoot

This section presents a two step procedure for a homogeneous nonovershooting control design. First, inspired by [20], we design a linear controller, which stabilizes the system exponentially at zero without overshoot. Next, inspired by [29, Ch. 9], we transform (“upgrade”) a linear controller to a homogeneous one such that an upper estimate of the settling time for the system with the given initial state \( x_0 \in int\Sigma \) can be assigned a priori.

#### A. Linear Nonovershooting Control Design

For \( \lambda > 0, i = 1, \ldots, n - 1 \), let us introduce the row vectors

\[
h_i = -e_1^T \lambda A + \lambda I \tag{16} \]

and consider the positive cone \( \Sigma \)

\[
\Omega = \{ x \in \mathbb{R}^n : h_i x \geq 0, \ i = 1, \ldots, n \} \subset \Sigma. \tag{17}
\]

A set \( D \subset \mathbb{R}^n \) is said to be a positive cone if \( x \in D \Rightarrow e^{\tau x} \in D, \forall s \in \mathbb{R} \).
The following lemma is a particular case of the results obtained in [20].

**Lemma 1:** The linear feedback

\[ u_{\text{lin}} = Kx \quad (17) \]

stabilizes exponentially invariant system (1) at \( x = 0 \) and renders the set \( \Omega \) strictly positively invariant for the closed-loop system (1), (17). Moreover, given initial value \( x_0 \in \text{int} \Sigma \), choosing \( \lambda > 0 \) such that

\[ \sum_{j=0}^{i-1} C^{-1}_{j} e^{-T_{i-j}} x_{0} \lambda^{j} \leq 0, \quad C^{-1}_{j} = C_{j}^{-1} = \left( \begin{array}{cc} i-j \end{array} \right) \eta^{j} \eta^{i-j}, \quad i = 2, \ldots, n \]

(19)

guarantees that \( x_{0} \in \Omega \).

**Proof:** Inspired by [20], let us consider the barrier functions \( \varphi_{i} : \mathbb{R}^{n} \rightarrow \mathbb{R} \) defined as

\[ \varphi_{i} = -x_{1} \quad (20) \]

\[ \varphi_{1} = -\lambda \varphi_{i-1}, \quad i = 2, \ldots, n \]

(21)

namely

\[ \varphi_{i} = h_{i-1} \dot{x} + \lambda h_{i-1} x = h_{i-1} A x + \lambda h_{i-1} x = h_{i} x \]

(22)

which is compactly represented as

\[ \Phi = \begin{bmatrix} \varphi_{1} \\ \vdots \\ \varphi_{n} \end{bmatrix} = Hx \quad (23) \]

\[ H = \begin{bmatrix} h_{1} \\ \vdots \\ h_{n} \end{bmatrix} \begin{bmatrix} -e_{1}^{T} \\ -e_{1}^{T} (A + \lambda I) \\ \vdots \\ -e_{1}^{T} (A + \lambda I)^{n-1} \end{bmatrix} \]

(24)

The positive orthant \( \{ \Phi \in \mathbb{R}^{n} : \varphi_{i} \geq 0, \quad i = 1, \ldots, n \} \) is the same as the set \( \Omega \). Furthermore, \( H \) is the observability matrix of the closed-loop system (1) and (17), which is governed by \( x = [A - e_{1} e_{1}^{T} (A + \lambda I)^{n}] x \), has the solution \( x(t) = e^{-\lambda t} H x_{0} \), and this system is exponentially stable at \( x = 0 \).

According to Lemma 1, the set \( \Omega \) is strictly positively invariant for the closed-loop system (1) and (17). Due to \( \Omega \subset \Sigma \) this means that any trajectory initiated in \( \Omega \) will never leave the safety zone. Moreover, the linear controller (17) can always be tuned in such a way that the initial state \( x_{0} \) will belong to the set \( \Omega \). The functions \( \varphi_{i}(x) = h_{i} x \), which define \( \Phi \) [see (16)], can be interpreted as barrier functions, since the inequalities \( \varphi_{i}(x) \geq 0 \) are never violated along the trajectory of the closed-loop system.

The following corollary presents a simple condition for selection of an appropriate \( \lambda > 0 \) to fulfill (19).

**Corollary 1:** Given \( x_{0} \in \text{int} \Sigma \) if \( \lambda > 0 \) satisfies the inequalities

\[ \lambda \geq 1 - \frac{\epsilon_{i} x_{0}}{e_{1}^{T} x_{0}} C_{i-1}^{-1} e_{1} x_{0}, \quad k = 1, \ldots, i - 1, \quad i = 2, \ldots, n \]

(30)

then inequalities (19) hold.

**Proof:** Indeed, for any \( i \geq 2 \) we derive

\[ \sum_{j=0}^{i-1} C_{j}^{-1} e_{1}^{T} x_{0} \lambda^{j} = C_{0}^{-1} e_{1}^{T} x_{0} \]

\[ + \lambda (C_{i-1}^{-1} e_{1}^{T} x_{0} + \lambda (C_{i-1}^{-1} e_{1}^{T} x_{0} + \lambda C_{i-1}^{-1} e_{1}^{T} x_{0}))) \]

Since \( e_{1}^{T} x_{0} < 0 \) then inequalities (30) imply that

\[ C_{i-1}^{-1} e_{1}^{T} x_{0} \leq e_{1}^{T} x_{0}, \quad k = 1, \ldots, i - 1 \]

(31)

Hence, taking into account \( C_{i-1}^{-1} = 1 \) we conclude

\[ C_{i-1}^{-1} e_{1}^{T} x_{0} \leq e_{1}^{T} x_{0} \]

\[ C_{i-3}^{-1} e_{1}^{T} x_{0} \leq C_{i-2}^{-1} e_{1}^{T} x_{0} \]

\[ C_{i-3}^{-1} e_{1}^{T} x_{0} \leq C_{i-2}^{-1} e_{1}^{T} x_{0} \]

\[ C_{i-1}^{-1} e_{1}^{T} x_{0} \leq e_{1}^{T} x_{0} \]

\[ + \lambda C_{i-1}^{-1} e_{1}^{T} x_{0}) \leq e_{1}^{T} x_{0} \]

(34)

The proof is complete.

The parameter \( \lambda > 0 \) specifies the feedback gain of the linear nonovershooting controller [see (17)]. It grows as \( \max_{i} \{ \frac{C_{i}^{-1} e_{1}^{T} x_{0}}{r_{i} e_{1}^{T} x_{0}} \} \) increases. This is necessary to guarantee the safety. Indeed, let us consider the double integrator \( (n = 2) \) and
denote \( y = e_1^Tx \), \( \dot{y} = e_2^Tx \). For \( y(0) = e_1^Tx_0 = 0 \) and \( \dot{y}(0) = e_2^Tx_0 > 0 \), we will always have an overshoot, since, independently of the controller, the system trajectory with such an initial condition will leave the safe set \( \Sigma \) right after the initial instant of time \( t = 0 \). If \( y(0) = e_1^Tx_0 < 0 \) is close to zero and \( \dot{y}(0) = e_2^Tx_0 > 0 \) is large then to avoid an overshoot the linear feedback must have a sufficiently large gain to guarantee that \( \dot{y}(t) \) will become nonpositive before \( y(t) \) will reach zero.

### B. Upgrading Linear Controller to a Homogeneous One

In this section, we design a finite-time nonsloshing controller by means of an “upgrade” (a transformation) of a linear “nonsloshing” feedback to a homogeneous one.

**Theorem 1:** Let \( q = 0 \) in (1). Given \( x_0 \in \Sigma \), let \( K \in \mathbb{R}^{1 \times n} \) be the gain of the linear controller (17) defined by (18). Then

1. the system of linear matrix inequalities (LMIs)
   \[
   \begin{cases}
   P(A + BK) + (A + BK)^TP < 0 \\
PQ_d + G_dP > 0, \quad P > 0
   \end{cases}
   \quad (35)
   \]
   has a solution \( P = P^T \in \mathbb{R}^{n \times n} \);

2. for any given \( T > 0 \) the homogeneous controller
   \[
   u_h(x) = \tilde{K}d(-\ln \| x/r \|_d)x, \quad \tilde{K} = \tilde{K}d(s)
   \quad (36)
   \]
   stabilizes system (1) at \( x = 0 \) in a finite time and
   \[
   \| x_0 \| \leq r \implies x(t, x_0) = 0 \quad \forall t \geq T
   \quad (38)
   \]

   where \( \tilde{d}(s) \) is given by (4)
   \[
   \rho = -\lambda_{\max}(Q^{-\frac{1}{2}}ZQ^{-\frac{1}{2}}) > 0
   \quad (39)
   \]
   \[
   Z = P(A + BK) + (A + BK)^TP < 0
   \quad (40)
   \]
   \[
   Q = PQ_d + G_dP > 0
   \quad (41)
   \]

   the homogeneous norm \( \| x \|_d \) is induced by the weighted Euclidean norm
   \[
   \| x \| = \sqrt{x^TD(s)Px}, \quad x \in \mathbb{R}^n
   \quad (42)
   \]

3. the homogeneous cone
   \[
   \Omega_r = \{ x \in \mathbb{R}^n : h_i\tilde{d}(-\ln \| x/r \|_d)x \geq 0, i = 1, \ldots, n \}
   \quad (43)
   \]
   is a positively invariant set for the closed-loop system (1), (36) and
   \[
   x_0 \in \Omega_r \implies x_0 \in \Omega_r \subset \Sigma
   \quad (44)
   \]

provided that \( \| x_0/r \|_d \leq e^\delta \), where \( \Omega \subset \Sigma \) is given by (16), \( h_i = \tilde{h}_i\tilde{d}(s) \) and the row vectors \( \tilde{h}_i \) are defined by (15).

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**Proof:** 1) Let \( P = H^TPH \), where \( H \) is defined by (24) and \( \tilde{P} = \tilde{P}^T \in \mathbb{R}^{n \times n} \) is a symmetric matrix to be defined as follows. By Corollary 6, we have \( H(A + BK) = (-\lambda I_n + A)H \) and \( HX_{\bar{d}} = (\bar{G}_d + \lambda(nI_n - G_d)A^T)H \). Since

\[
H(A + BK) + (A + BK)^TP = H^T(\tilde{P}(A - \lambda I_n) + (A - \lambda I_n)^T\tilde{P})H
\quad (45)
\]

\[
PG_d + G_dP = H^T(\tilde{P}M + M^T\tilde{P})H
\quad (46)
\]

\[
M = G_d + \lambda(nI_n - G_d)A^T
\quad (47)
\]

then, the system of LMIs (35) becomes

\[
\begin{cases}
\tilde{P}(A - \lambda I_n) + (A - \lambda I_n)^T\tilde{P} < 0 \\
\tilde{P}M + M^T\tilde{P} > 0
\end{cases}
\quad (48)
\]

If \( \tilde{P} = \text{diag}\{p_1, \ldots, p_n\} \), with \( p_i > 0 \) then \( \tilde{P} > 0 \) and the latter system of LMIs becomes (50), shown at the bottom of this page:

\[
\begin{pmatrix}
2\lambda p_1 & -p_2 & 0 & 0 & \ldots & 0 & 0 \\
-p_2 & 2\lambda p_2 & -p_3 & 0 & \ldots & 0 & 0 \\
0 & -p_3 & 2\lambda p_3 & -p_4 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 2\lambda p_{n-1} & -p_n & 0 \\
0 & 0 & 0 & \ldots & 0 & -2p_n & \lambda p_n
\end{pmatrix} > 0
\quad (49)
\]

Let us fix an arbitrary \( p_n > 0 \), e.g., \( p_n = 1 \). Since \( \lambda > 0 \) then there exists a sufficiently large \( p_{n-1} > 0 \) such that

\[
\begin{pmatrix}
2\lambda p_{n-1} & -p_n \\
-p_n & 2\lambda p_n
\end{pmatrix} > 0
\quad (51)
\]

\[
\begin{pmatrix}
4\lambda p_{n-1} & (n-1)\lambda p_n \\
(n-1)\lambda p_n & 2p_n
\end{pmatrix} > 0
\quad (52)
\]

simultaneously. Hence, using Schur complement we conclude that there exists a sufficiently large \( p_{n-2} > 0 \) such that the inequalities

\[
\begin{pmatrix}
2\lambda p_{n-2} & p_{n-1} & 0 \\
p_{n-1} & 2\lambda p_{n-1} & -p_n \\
0 & -p_n & 2\lambda p_n
\end{pmatrix} > 0
\quad (53)
\]

\[
\begin{pmatrix}
6p_{n-2} & (n-2)\lambda p_{n-1} & 0 \\
(n-2)\lambda p_{n-1} & 4p_{n-1} & (n-1)\lambda p_n \\
0 & (n-1)\lambda p_n & 2p_n
\end{pmatrix} > 0
\quad (54)
\]

are fulfilled simultaneously. Repeating the same consideration subsequently for \( p_{n-3}, \ldots, p_1 \), we conclude that the system of LMIs (35) is always feasible.

2) Let us show that the homogeneous norm \( \| x \|_d \) is a Lyapunov function of the closed-loop system (1), (36). Indeed, using the formula (11) and the identities \( \tilde{d}(s)A = e^\delta\tilde{d}(s)A, \tilde{d}(s)B =

| 2np_1 \quad \lambda p_2 \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0 |
| \lambda p_2 \quad (n-1)p_2 \quad 2\lambda p_3 \quad 0 \quad \ldots \quad 0 \quad 0 |
| 0 \quad 2\lambda p_3 \quad (n-2)p_3 \quad 3\lambda p_4 \quad \ldots \quad 0 \quad 0 |
| \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots |
| 0 \quad 0 \quad 0 \quad \ldots \quad 4p_{n-1} \quad (n-1)\lambda p_n |
| 0 \quad 0 \quad 0 \quad \ldots \quad 0 \quad (n-1)\lambda p_n |
| 2p_n |
to zero, it is sufficient to select \( r \) in (36) in a manner dependent on the initial state: \( r = ||x_0|| \) for \( x_0 \neq 0 \).

Remark 2: Controller (36) is \( \mathbf{d} \)-homogeneous of degree \(-1\) and globally uniformly bounded as follows:
\[
|u_h(x)|^2 = x^T \mathbf{d}(-\ln ||x/r||) \tilde{K}^T \tilde{K} \mathbf{d}(-\ln ||x/r||) x \leq r^2 \lambda_{\max} \left( P^{-\frac{1}{2}} K^T K P^{-\frac{1}{2}} \right) \quad \text{for all } x \in \mathbb{R}^n
\]  
where the identity
\[
x^T \mathbf{d}(-\ln ||x/r||) \mathbf{d}(\tilde{s}) P \mathbf{d}(\tilde{s}) \mathbf{d}(-\ln ||x/r||) x = r^2
\]
is utilized on the last step.

The positive cone \( \Omega \subset \Sigma \) defines a set of initial states of system (1) for which the linear control (17) stabilizes the system without overshoot in the first coordinate. In the case of the homogeneous nonovershooting control (36) such a positively invariant set is the \( \mathbf{d} \)-homogeneous cone \( \Omega_r \subset \Sigma \). Notice that \( |u_{\ln}(x)| = |u_h(x)| \) for \( ||x/r|| = d \). Moreover, linear and homogeneous controllers have the same maximum magnitude on the homogeneous ball
\[
B_r = \{ x \in \mathbb{R}^n : ||x/r|| \leq d \}
\]
i.e., \( \sup_{x \in B_r} |u_{\ln}(x)| = \sup_{x \in B_r} |u_h(x)| \). Theorem 1 implies that the positively invariant compact set \( \Omega_r \subset B_r \) of the homogeneous control system is larger than the positively invariant set \( \Omega \cap B_r \) of the linear control system despite that both controllers have the same maximal magnitude on \( B_r \).

Remark 3: The simple combination of the linear and homogeneous feedbacks
\[
\begin{cases}
  u(x) = \begin{cases}
  u_h(x) & \text{if } ||x/r|| \leq d \leq e^\delta \\
  K x & \text{if } ||x/r|| > d > e^\delta
  \end{cases}
\end{cases}
\]
gives a nonovershooting control \( u \in C(\mathbb{R}^n \setminus \{0\}) \) as well. The closed-loop system (1), (64) is globally finite-time stable with the positively invariant set \( \Omega \cup B_r \). Indeed, let us consider the nonsmooth Lyapunov function
\[
V(x) = \frac{1}{2} ||x/r||^2 - \tilde{V}(x)
\]
By construction, the function \( V \) is continuous on \( \mathbb{R}^n \), locally Lipschitz continuous on \( \mathbb{R}^n \setminus \{0\} \), continuously differentiable on the set \( \{ x \in \mathbb{R}^n : ||x/r|| = e^\delta \} \setminus \{0\} \) and
\[
\tilde{V}(x) \leq \begin{cases}
  -1/T & \text{if } 0 < V(x) < e^{\delta} \\
  -\tilde{p}V(x) & \text{if } V(x) > e^\delta
  \end{cases}
\]
where \( \tilde{p} = -\lambda_{\max} (P^{-1/2} Z P^{-1/2}) > 0 \). For \( V(x) = e^{\delta} \) using the chain rule for the Clarke’s gradient, we derive \( \tilde{V}(x) \leq -\min\{1/T, \tilde{p}e^\delta\} < 0 \). The continuous barrier functions \( \varphi_i, i = 1, \ldots, n \) for this system
\[
\varphi_i(x) = \begin{cases}
  \phi_i(x) & \text{if } V(x) \leq e^\delta \\
  \phi_i(x) & \text{if } V(x) > e^\delta
  \end{cases}
\]
combine smooth linear and homogeneous barrier functions \( \varphi_i, \varphi_0 \), respectively. Since \( \tilde{V}(x) < 0 \) then the vector \( \tilde{\phi} = (\varphi_1, \ldots, \varphi_n)^T \) satisfies the switched positive system
\[
\frac{d\tilde{\phi}}{dt} = \begin{cases}
  e^\delta \Pi \tilde{\phi} & \text{if } V(x) \leq e^\delta \\
  (-\lambda I_n + A) \tilde{\phi} & \text{if } V(x) > e^\delta
  \end{cases}
\]
having a unique switching isolated in time.

The differential equations (26) and (60) for linear and homogeneous barrier functions are slightly different. In the linear case, the equation has the constant Metzler matrix \(-\lambda I_n + A\), while the tridiagonal Metzler matrix \( \Pi \) in the homogeneous
case depends on the time derivative of the Lyapunov function \( \|x/r\|_d^\circ \). This dependence complicates the analysis and design of the homogeneous safety filters.

Despite the obvious difference of positively invariant sets of linear and homogeneous controllers, their mathematical descriptions in polar homogeneous coordinates are identical provided that we do not care about tuning of the setting time of the homogeneous system, namely, if \( \bar{s} = 0 \) and \( r = 1 \). Indeed, the polar homogeneous coordinates [34] in \( \mathbb{R}^n \) are given by the polar radius \( \rho = \|x\|_d \) and the unit vector \( y = d(-\ln \|x\|_d)x \), which defines the so-called homogeneous projection on the unit sphere. In the linear case, we deal with the standard dilation \( \tilde{d}(s) := e^{s\rho} \), so the polar radius is just the Euclidean norm \( \tilde{d}(s) := \|x\| \) and the homogeneous projection is \( y := d(-\ln \|x\|_d)x = x/\|x\| \). Hence, in the polar coordinates, we have \( \Omega = \{ (\rho, y) : h_i y \geq 0, \forall i \} \) and \( \Omega_r = \{ (\rho, y) : h_i y \geq 0, \forall i \} \) provided that \( r = 1 \) and \( \bar{s} = 0 \).

The barrier functions as well as the structure of system (60) depend on the particular control law. Some interesting homogeneous control algorithms for the integrator chain are introduced in [4], [5], [7], and [21]. However, a design of barrier functions and a selection of control parameters allowing a nonovershooting FntS/fixes-time stabilization based on these algorithms are unclear yet.

C. On Robustness of the Nonovershooting Property

The homogeneous controller (36) is known to be efficient in rejecting of a certain class of perturbations [32]. The following corollary characterizes subclass of perturbations under which the nonovershooting property is preserved as well.

**Corollary 2:** Let the homogeneous controller (36) be defined as in Theorem 1. If a locally bounded measurable function \( g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) in (1) fulfills the inequality

\[
\varrho < \rho \bar{s}
\]

then system (1) with the homogeneous controller (36) is globally uniformly finite-time stable. Moreover, if, additionally

\[
\exists \delta \geq 0 : \text{tr}(\tilde{P} + \delta I_n)\phi(x) + h_i d(-\ln \|x\|_d)g(t,x) \geq 0
\]

for all \( t \geq 0 \), for all \( x \in \Omega \setminus \{0\} \) and for all \( i = 1, \ldots, n \), where \( \tilde{P} = A + (\rho \bar{s}) - \varrho \) (\( G_d + \lambda (n I_n - G_d)A^T \)) is a Metzler matrix, the function \( \phi \) is defined by (57) and the row vector \( h_i \) is given by (15), then the conclusion 3) of Theorem 1 remains valid.

**Proof:** Since the function \( g \) is locally bounded and measurable then the function \( t \to g(t,x) \) is locally Lebesgue integrable for each fixed \( x \). If \( g(t,x) \) is continuous on \( x \) then the perturbed system has Carathéodory solutions (see, for example, [10, Ch. 1, Th. 1]). In this case, inequality (55) becomes

\[
\frac{d}{dt}\|x\|_d^\circ \leq \|x\|_d^\circ \text{tr}(\tilde{P} + \delta I_n)\phi(x) + h_i d(-\ln \|x\|_d)g(t,x)
\]

that is fulfilled almost everywhere along any trajectory \( x(x_0) \) of the system as long as \( x(x_0) \neq 0 \). Hence, inequality (69) implies the finite-time stability of the closed-loop system with a perturbation \( g(t,x) \) being continuous in \( x \). Moreover, formula (60) in the perturbed case has the form

\[
\dot{\phi} = \frac{d^\circ}{\|x/r\|_d^\circ} \Pi \phi + H d(-\ln \|x/r\|_d)g(t,x)
\]

where the Metzler matrix \( \Pi \) is given by (61) with \( \gamma = -\frac{d^\circ}{\|x/r\|_d^\circ} \geq \rho \bar{s} - \varrho > 0 \) and the matrix \( H \) is given by (24).

The condition (71) guarantees that the latter system is positive on \( \Omega_r \), is a strictly positively invariant set of the system.

All abovementioned considerations remain valid for \( g(t,x) \) being discontinuous in \( x \). Indeed, in the latter case the perturbed closed-loop system has Filippov solutions [10, Ch. 2, Th. 8].

The time derivative of \( \|x/r\|_d \) along a trajectory of the system admits the estimate (73), where \( g(t,x) \) is replaced with \( \tilde{g}(t,x) \) being a selector of the set-valued mapping

\[
\tilde{g}(t,x) \in \tilde{G}(t,x) := \bigcap_{\epsilon > 0} \bigcap_{N \in \mathbb{N}} \mathbb{N} g(t, x + B(\epsilon) \setminus N)
\]

where \( \mathbb{N} \) is the closed convex hall of the set \( M \subset \mathbb{R}^n \), \( \mu(N) \) is a measure of the set \( N \subset \mathbb{R}^n \). The latter conclusion is not valid for \( \bar{s} = 0 \) then they are fulfilled for any selector \( \tilde{g} \) of the mapping \( G \) as well.

Using the Cauchy–Schwarz inequality for (69) we derive a more simple (but more conservative) restriction

\[
\sup_{t \geq 0, x \in \mathbb{R}^n} \|x\|_d^\circ \|d(-\ln \|x/r\|_d)\tilde{g}(t,x)\| \leq \varrho \rho \bar{s}
\]

where \( \varrho = 0.5 \min \{ PG_d + G_d P \} \) and the norm \( \| \cdot \| \) is defined as in Theorem 1. Taking into account the formula (4) we conclude that inequality (69) holds if for sufficiently small \( \beta_i \geq 0 \) one holds

\[
e_i^T g(t,x) \leq \beta_i \|x/r\|_d^\circ \quad \forall t \geq 0 \quad \forall x \in \mathbb{R}^n
\]

for all \( i = 1, \ldots, n \). Since for \( i = n \) we have \( \|x/r\|_d^\circ = 1 \) then the homogeneous controller preserves the finite-time stability property for sufficiently small nonvanishing matched perturbations. The mismatched components (i.e., for \( 1 \leq i \leq n-1 \)) have to be vanishing at \( x = 0 \) and proportional to a certain power of the canonical homogeneous norm of the state. The nonovershooting property can be preserved under the additional restriction (71), where by construction, the matrix \( H \) have nonpositive elements. Notice that this does not meant that \( g \) has to be nonpositive to fulfill (71).

**Remark 4:** If \( n = 2, g = Bg_0 \) then the condition (71) becomes

\[
e_i^T (\tilde{P} + \delta I_n)\phi(x) \leq g_0(t,x) \quad \forall t \geq 0 \quad \forall x \in \Omega_r
\]

Since \( \tilde{P} \phi(x) = 1, \forall x \neq 0 \) then one can be shown that

\[
\inf_{x \in \Omega_r} e_i^T (\tilde{P} + \delta I_n)\phi(x) = \omega > 0
\]

so the sign-indefinite matched perturbation may be nonvanishing on \( \Omega_r \), but the nonovershooting property of the homogeneous controller is preserved if \( g_0(t,x) \leq \omega \) for all \( t \geq 0 \) and for all \( x \in \Omega_r \), where \( g_0 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \).

The latter conclusion is not valid for \( n \geq 3 \) since \( \omega = 0 \) in this case. To fulfill (71) for a sign-indefinite \( g \), some nonnegative components of the vector \( g(t,x) \) have to vanish to zero as \( x \) tends to the boundary of \( \Omega_r \).

D. On Implementation of Homogeneous Controller

In practice, the homogeneous controller (36) can be implemented in a sampled/switching manner using a simple numerical
procedure proposed in [32]. It is important to study the nonovershooting properties of the system for this case.

Corollary 3: Let $0 = t_0 < t_1 < \ldots$ be an arbitrary sequence of time instances such that $t_i \to +\infty$ as $i \to +\infty$. Let all parameters of the homogeneous nonovershooting controller (36) be defined as in Theorem 1. For $g = 0$, system (1) with the linear switching controller

$$u(t) = \begin{cases} \hat{K}d(-\ln \kappa_j)x(t), & t \in [t_j, t_{j+1}) \\ \kappa_j = \|x(t_j)/r\|_d, & j = 0, 1, \ldots \end{cases}$$

is globally asymptotically stable and

$$x_0 \in \Omega \land \|x_0\| \leq r \implies x(t, x_0) \in \Omega \forall t \geq 0$$

where $x(t, x_0)$ is a solution of the closed-loop system (1), (80), (81) with the initial condition $x(0) = x_0 \in \mathbb{R}^n$.

Proof: The asymptotic stability follows from [32, Corollary 8], where in particular, it is shown that

$$\kappa_j = \|x(t_j, x_0)/r\|_d > \|x(t, x_0)/r\|_d, \quad t \in (t_j, t_{j+1})$$

as long as $x(t, x_0) \neq 0$. Indeed, the mentioned corollary deals with $r = 1$, but the change of variables $\hat{x} = x/r$ generalizes it to our case.

To prove the nonovershooting property (82), let us denote

$$\phi^{[j]}(x) = h_j(d(s - \ln \kappa_j)x, \quad i = 1, \ldots, n, \quad j = 0, 1, \ldots)$$

$$\tilde{\Omega}_j = \left\{ z \in \mathbb{R}^n : \phi^{[j]}(z) \geq 0, i = 1, \ldots, n \right\}$$

$$\tilde{B}_j = \left\{ z \in \mathbb{R}^n : \|z/r\|_d \leq e^{\kappa_j \ln \kappa_j} \right\}$$

On the one hand, since $\hat{s} \geq 0$ then inequality (83) guarantees that

$$x(t, x_0) \in \tilde{B}_j, \quad \forall t \in [t_j, t_{j+1}) \land j \geq 0. \quad (87)$$

Notice also that

$$x_0 \in \Omega \land \|x_0\| \leq r \implies x_0 \in \tilde{\Omega}_0 \cap \tilde{B}_0$$

where the norm $\| \|$ is defined as in Theorem 1. Using the identities $d(s)A = e^{s}d(s)A, d(s)B = e^{s}B, \forall s \in \mathbb{R}$ we derive

$$\phi^{[j]} = \frac{\hat{\phi}^{[j]}}{\kappa_j}(-\lambda I_n + A)\phi^{[j]}, \quad t \in [t_j, t_{j+1})$$

where $\phi^{[j]} = (\phi_1^{[j]}, \ldots, \phi_n^{[j]})^T$. System (89) is positive, so

$$x(t_j, x_0) \in \tilde{\Omega}_j \implies x(t, x_0) \in \tilde{\Omega}_j \forall t \in [t_j, t_{j+1}). \quad (90)$$

Repeating the considerations of the last paragraph of the proof of Theorem 1 one can be shown

$$x(t_{j+1}, x_0) \in \Omega \cap \tilde{B}_j \implies x(t_{j+1}, x_0) \in \tilde{\Omega}_{j+1}. \quad (91)$$

Therefore, from (87) and (91), we conclude

$$x(t, x_0) \in \tilde{\Omega}_j \cap \tilde{B}_j \forall t \in [t_j, t_{j+1}) \land j \geq 0. \quad (92)$$

On the other hand, repeating the considerations of the last paragraph of the proof of Theorem 1 we conclude

$$x(t, x_0) \in \tilde{\Omega}_j \cap \tilde{B}_j \implies d(-\ln \kappa_j)x(t, x_0) \in \Omega_r. \quad (93)$$

We complete the proof taking into account that $\Omega_r$ is a d-homogeneous cone: $d(\lambda)s, s \in \Omega, \forall \lambda \in \mathbb{R}$. The proven corollary guarantees that the switching controller (80), (81) preserves the nonovershooting properties established for the homogeneous controller (36). If the dwell time $\max_j(t_{j+1} - t_j)$ tends to zero, then the trajectory of the system with the switching controller converges to the trajectory of the system with the homogeneous controller uniformly on compact intervals of time. Moreover, the numerical algorithm developed in [32] can be utilized for practical implementation of the homogeneous controller (36).

Corollary 4: All conclusions of Corollary 3 remain valid for $g \neq 0$ in (1) provided that the condition (69) is fulfilled and for any $j \geq 0$ there exists $\delta \geq 0$ such that

$$e^T(\delta + \delta I_n)\phi^{[j]}(x) + \kappa_j e^{\kappa_j \ln \kappa_j} \phi^{[j]}(x) \geq 0$$

for all $\forall i = 1, \ldots, n$, for all $t \in [t_j, t_{j+1})$ and for all $x \in \tilde{\Omega}_j \cap \tilde{B}_j$ where $\bar{\phi}_j, \tilde{\Omega}_j$, and $\tilde{B}_j$ are defined by the formulas (84), (85), (86), respectively.

Proof: In [33], it is shown that inequality (81) holds for the perturbed case as well provided that inequality (69) holds. If $g(t, x)$ is continuous in $x$

$$\phi^{[j]} = \frac{\hat{\phi}^{[j]}}{\kappa_j}(-\lambda I_n + A)\phi^{[j]} + H(d(s))d(-\ln \kappa_j)g(t, x)$$

Hence, condition (94) guarantees that $\phi^{[j]}(x(t, x_0))$ for all $t \in [t_j, t_{j+1})$ provided that the vector $\phi^{[j]}(x(t, x_0))$ is nonnegative. So, all steps of the proof of Corollary 3 can be repeated for $g \neq 0$. The case of discontinuous $g$ can be treated as in the proof of Corollary 2.

The linear switched controller (80), (81) preserves the asymptotic stability and the nonovershooting property of the closed-loop system if the perturbation $g$ in system (1) satisfies the restrictions (69), (94). Remark 4 can be generalized to this case showing that for $n = 2$, the matched perturbation may be nonvanishing and sign-indefinite.

V. HOMOGENEOUS SAFETY FILTERS

A nonovershooting controller can be implemented in practice as the so-called “safety filter” [1]

$$u = \min(u_{nom}, u_{lin})$$

where $u_{nom}$ is a nominal controller which may have overshoots and $u_{lin}$ is a nonovershooting linear controller whose gain can be time-invariant as in Lemma 1 or time-varying as in [1]. Such an implementation allows overshoots of the nominal controller $u_{nom}$ to be eliminated. For $n \leq 2$, the same safety filter is applicable in the case of the homogeneous nonovershooting controller (36). For $n \geq 3$, we show that the overshoots of the nominal controller can be eliminated by the linear controller using a modified filter.

Let us define $\Delta_r = +\infty$ for $n \leq 2$ and

$$\Delta_r(x) = \frac{\gamma_0(x)}{\gamma_n(x)} + \min_{i=2,\ldots,n-1}\frac{\gamma_i(x)}{\gamma_{i-1}(x)} \quad (97)$$

where $n \geq 3, x \in \Omega, c_i > 0$ are arbitrary constants, $\phi_i(x) = \hat{\phi}d(-\ln \|r/d\|_d)x, i = 1, \ldots, n$ are the homogeneous barrier functions and

$$\gamma_r(x) = \frac{\phi_r(x)}{\phi_r(x)P(RH)G^{-1}R\phi_r(x)} \geq 0 \quad \forall x \in \mathbb{R}^n$$

$$\gamma_u(x) = \frac{\phi_u(x)}{\phi_u(x)(P(RH)G^{-1}R\phi_u(x))} \geq 0 \quad \forall x \in \Omega$$

with $P > 0$ being a diagonal matrix defined in the proof of Theorem 1. The positive function $\Delta_r$ may have infinite values on the boundary of the set $\Omega_r$.

$$\Delta_r(x) = \lim_{z \to x} \inf \Delta_r(z) = \{0, +\infty\}, \quad z \in \Omega_r, x \in \partial\Omega_r$$

Finally, let us define $\Delta_r(x) = +\infty$ for $x \in \Omega \setminus \Omega_r$.

Theorem 2 (FnTSF Filter): Let $g = 0$ in (1) and the safety filter be defined by the formula

$$u = \max\{u_{nh}(x) - \Delta_r(x), \min\{u_{nom}, u_{nh}(x)\}\}$$

(101)
where \( x \in \mathbb{R}^n \setminus \{0\} \), \( u_{nom} \in C(\mathbb{R}) \) is a nominal control signal and \( u_h : \mathbb{R}^+ \rightarrow \mathbb{R} \) is a nonovershooting homogeneous feedback defined by Theorem 1 with \( P = H^TPH, \quad \tilde{P} = \text{diag}(p_1, \ldots, p_n) \), \( p_n = 1 \).

1. The \( \varepsilon \)-homogeneous cone \( \Omega_\varepsilon \) is a strictly positively invariant set of the closed-loop system (1), (101).

2. Let \( \varepsilon > 0 \) and \( x(\sigma, x_0) = 0 \) for some \( \sigma \geq 0 \).
   
   a) If the inequality \( u_{nom}(t) < 0 \) is fulfilled for all \( t \in (\sigma, \sigma + \varepsilon) \) then \( x(t, x_0) \in \Omega_\varepsilon \) for all \( t \in (\sigma, \sigma + \varepsilon) \).
   
   b) If the inequality \( u_{nom}(t) \geq 0 \) is fulfilled for all \( t \in [\sigma, \sigma + \varepsilon] \) then \( x(t, x_0) = 0 \) for all \( t \in [\sigma, \sigma + \varepsilon] \).

3. For any instant of time \( \tau > 0 \) for which
   
   \[ u_{nom}(t) \geq u_h(x(t, x_0)) \quad \forall t \in \mathbb{T}_\tau \]  
   
   it holds that
   
   \[ x(\tau + T_0, x_0) = 0 \]  
   
   where \( T_0 = \|x(\tau, x_0)/r\|\,dT \).

   **Proof:** 1) Repeating the derivation of the formula (60) in the proof of Theorem 1, we obtain
   
   \[ \dot{\phi} = e^{\tilde{A}(t, x)} - \frac{\varepsilon}{\varepsilon - \sigma} \cdotp e^{\tilde{A}(\sigma, x)} \frac{d}{dt} x(r) \]  
   
   as long as \( x(t, x_0) \neq 0 \), where \( \gamma = -\varepsilon \frac{d}{dt} x(r) \) and
   
   \[ H_GH^{-1} = A + \lambda_1(u_1 - G_A), \quad H \]  

   Using formula (11) and the identities \( H(B \bar{K}) = (-\lambda_1 + A)H \), \( d(s)A = e^{s}d(s)A, \quad d(s)B = e^{s}B, \forall s \in \mathbb{R} \),

   we derive
   
   \[ \gamma = \gamma_r - \gamma_u(u_h - u) \]  
   
   where \( \gamma \) is the settling time estimate of the controller (36) within the safe region. The uniqueness analysis is omitted since it goes out of the scope of this article.

   2. Since \( u_h \) is discontinuous at \( x = 0 \), then the origin may be a sliding mode of the closed-loop system. In this case, solutions are defined in sense of Filippov [10] and
   
   \[ x_n \in [u_{min}, u_{max}] \text{ for } x = 0 \]  
   
   where due to \( \lim_{z \to 0} \Delta_r(z) = +\infty \), we have
   
   \[ u_{min} = \min \left\{ u_{nom}, \liminf_{z \to 0} u_h(z) \right\} \]  
   \[ u_{max} = \min \left\{ u_{nom}, \limsup_{z \to 0} u_h(z) \right\} \]  

   If \( u_{nom} \geq 0 \) then \( u_{min} \leq u_{max} \).

   **Remark 5:** A safety filter with the mixed nonovershooting controller (64) can also be realized by the formula (101) selecting \( \Delta(x) = +\infty \) for \( \|x(r)/r\| > e^2 \).

   The key result of this theorem is (102), (104), which guarantees that if, starting from some time \( \tau \geq 0 \), the user’s nominal controller commands approach to the unsafe set then the safety filter overlaps the nominal controller such a way that the system would reach zero no later than the time instant \( \tau + T_0 \). Moreover, the system returns back into the interior of the safe set as soon as actions of the user’s defined nominal controller becomes “safe.”

   The time \( T_0 = \|x(\tau, x_0)/r\|\,dT \) is unknown a priori. It depends implicitly on the user’s nominal control \( u_{nom} \) and on the initial state \( x_0 \in \Omega_\varepsilon \). To specify the settling time estimate \( T_0 \) independently of \( u_{nom} \) and \( x_0 \), we slightly modify the homogeneous controller (36) making the parameter \( r \) dependent of the system trajectory

   \[ r = r(t) = \max \left\{ \min_{\tau \in [0, T]} \|d(-\hat{s})x(\tau)\| \right\} \]  

   where \( r_{min} > 0 \) is an arbitrary parameter and \( \hat{s} \) is defined in Theorem 1. System (1), (36), (114) becomes a functional differential equation in this case. The existence of its solution \( x(t, x_0) \) for any initial condition \( x(0) = x_0 \in \mathbb{R}^n \) follows, for example, from [18, Th. 4.1]. For \( g = 0 \), one can be shown that solutions are unique. The uniqueness analysis is omitted since it goes out of the scope of this article.

   **Theorem 3 (FtxTsF Filter):** The conclusions 2) and 3) of Theorem 2 remain valid for system (1), (101), (36), (114) with \( g = 0 \). Moreover, the following holds:

   1. \( T_0 \leq T \), where \( T > 0 \) is a parameter of the homogeneous controller (36) and \( T_0 \) is defined in Theorem 2;
   2. if \( x_0 \in \Omega_\varepsilon \) then \( x(t, x_0) \subset \Theta_r(t) \subset \subset, \forall t \geq 0 \), where
   
   \[ \Theta_r = \Omega_\varepsilon \cap \Omega_r \]  
   
   and \( \Omega_r, B_r \) are given by (43), (63), respectively.

   **Proof:** Notice that by definition of the monotone function \( r \) we have \( \|d(-\hat{s})x(\tau)/r(\tau)\| > 1, \forall \tau \geq 0 \).

   1) Since \( \Omega_r \subset \subset \) for any \( r \in (0, +\infty) \) (see, Theorem 1) then \( \Theta_r \subset \subset \) for any \( r \in (0, +\infty) \).
Let us show that $r_1 \leq r_2 \Rightarrow \Theta_{r_1} \subset \Theta_{r_2}$. Indeed, if $x \in \Omega_{r_1}$ and $r_1 \leq r_2$ then $h_d(s)(d(-\ln \|x/r_1\|)x) = h_d(-\ln \|y/r_2\|) \geq 0$, $i = 1, \ldots, n$ and $\|d(-\tilde{s})x/r_2\| \leq \|d(-\tilde{s})x/r_1\| \leq 1$. Hence, using Corollary 6, we derive

$$h_d(s)(d(-\ln \|x/r_2\|)x) = h_d \left( \ln \frac{\|d(-\tilde{s})x/r_1\|}{\|d(-\tilde{s})x/r_2\|} \right) d(s)(d(-\ln \|x/r_1\|)x) \geq 0. \quad (116)$$

Notice also $x(t,x_0) \in \Omega_{r(t)} \Rightarrow x(t,x_0) \in \Theta_{r(t)}$ since $\|x(t,x_0)/r(t)\| \leq e^s, \forall t \geq 0$. Moreover, $x_0 \in \Theta_{r(0)}$ provided that $x_0 \in \Omega$. Let us show that $x(t,x_0) \in \Theta_{r(t)}$ for all $t \geq 0$. First, we derive (105) with $\gamma = -e^{\xi} \frac{\|d\|}{\|r\|} \|d\| \|r\|$ and $\phi_i(t_0) \geq 0, i = 1, \ldots, n$. Next, using the formula (11) and the identities $H(A+BK) = (-\lambda I_n + A)H$, $d(s)A = e^s(d(s)A), d(s)B = e^sB, \forall s \in \mathbb{R}$, we obtain

$$\gamma = \gamma_r - \gamma_u(u_0 - u) + \frac{r \phi^T \phi}{\|r\|^2} P H G A H^T \phi. \quad (117)$$

Since $t \mapsto r(t)$ is a continuous decreasing function then $r$ is differentiable almost everywhere and $\dot{r} \geq 0$.

2) Repeating considerations of the proof of Theorem 2 we conclude $x_0 \in \Omega \Rightarrow x(t,x_0) \in \Theta_{r(t)}$ for all $t \geq 0$, and the conclusion 2) of Theorem 2.

3) If $0 \leq t_1 < t_2$ are such that $u_{\text{nom}}(t) \geq u_h(x(t,x_0))$ and $x(t,x_0) \neq 0$ for all $t \in [t_1, t_2]$ then repeating the derivation of (55) we obtain

$$\frac{d}{dt} \|x(t,x_0)/r(t_1)\| \|d\| \leq -\frac{1}{t} \quad \forall t \in [t_1, t]. \quad (119)$$

The settling time estimate $T > 0$ of the safety filter (101), (36), (114) is independent of the nominal control $u_{\text{nom}}$ and the initial state $x_0 \in \mathbb{R}^n$. Such an FxTsf is guaranteed by means of an adaptation of the parameter $r$ of the homogeneous controller (36) [see, the formula (114)]. The parameter $r$ depends implicitly on the user’s nominal controller $u_{\text{nom}}$ and on the initial state $x_0 \in \Omega$ of the system. However, this parameter also specifies the maximum magnitude of the homogeneous controller [see, the formula (62)] as well as a positively invariant set of the closed-loop system [see, the formula (45)]. In practice, the maximum value of $r$ should be bounded by some $r_{\text{max}} > 0$ due to physical restrictions to admissible control signals. In this case, the safety filtering with the fixed settling time estimate $T > 0$ may be ensured only in the zone $\Theta_{r_{\text{max}}} \subset \mathbb{R}^n$.

If the safety override for a duration of time $T$ recurs, on each occasion the safety boundary will be reached after $T$ time units. The adaptive parameter $r$ is different on each of those occasions.

Corollary 5: Theorems 2, 3 remain valid for $g \neq 0$ in (1) provided that inequality (69) is fulfilled and

$$\inf_{t \geq 0, x \in \Omega} \tilde{h}_d(d(-\ln \|x/r\|)g(t,x) \geq 0 \quad \forall i = 1, \ldots, n. \quad (120)$$

Proof: Let us generalize, first, Theorem 2 to the case of a perturbation $g(t,x)$ being continuous in $x$. For the perturbed system, the parameter $\gamma$ [see the formula (106)] becomes

$$\gamma = \gamma_r - \gamma_u(u_0 - u) + \tilde{\delta} \quad (121)$$

$$\tilde{\delta} = \frac{\|x\|}{\|d\|} \frac{\phi^T \phi}{\|r\|} P H G A H^T \phi. \quad (122)$$

The barrier functions $\phi_i$ satisfies

$$\phi_i = -a_i \phi_i + b_i + \tilde{h}_d(d(-\ln \|x/r\|)g(t,x) \quad (123)$$

where $a_i, b_i$ are defined by (108). Condition (120) guarantees $\inf_{t \geq 0, x \in \Omega} \tilde{\delta}(t,x) > 0$ and $b_i \geq 0$ for all $i = 1, \ldots, n$. So, the proof of Theorem 2 can be repeated for $g \neq 0$. The case of a discontinuous perturbation $g$ can be treated as in the proof of Corollary 2. Theorem 3 can be extended similarly.

According to Remark 4, for the planar case $n = 2$, the homogeneous controller preserves the safety (nonovershooting) property for some nonvanishing sign-indefinite matched perturbations. Considering the proof of the latter corollary for $n = 2$ the same conclusion can be established for the homogeneous safety filters.

VI. EXAMPLE 1: SAFETY FILTER DESIGN FOR THE DOUBLE INTEGRATOR

A. Linear Nonovershooting Controller

Let $n = 2$. According to Lemma 1, given $x_0 \in \mathbb{R}^2$, a linear nonovershooting controller can be defined as follows:

$$u_{\text{in}}(x) = K x, \quad K = h_2 A + \lambda h_2 = (-\lambda_2 - 2\lambda) \quad (124)$$

where

$$h_1 = (-1), h_2 = (-\lambda - 1)$$

$$\lambda \geq 1 \frac{h_1 A x_0}{h_1 x_0} = 1 - \frac{e^{-T x_0}}{e^{-T x_0}}. \quad (125)$$

Indeed, if $\varphi_1 = -x_1$ and $\varphi_2 = -\lambda x_1 - x_2$, where $x = (x_1, x_2) \in \mathbb{R}^2$, then

$$\dot{\varphi}_1 = -\lambda \varphi_1 + \varphi_2 \quad (126)$$

$$\dot{\varphi}_2 = -\lambda x_2 - u_{\text{in}} = -\lambda^2 x_1 - \lambda x_2 = -\lambda \varphi_2. \quad (127)$$

System (126), (127) is globally asymptotically stable and positive. Taking into account $\lambda > -\frac{e^{T x_0}}{e^{T x_0}}$ we derive $\varphi(0) > 0$ and $\varphi(0) < 0$, so the controller stabilizes the state vector $x$ at zero without overshoot in the first component and the set

$$\Omega = \{x \in \mathbb{R}^2 : h_1 x \geq 0, h_2 x \geq 0\} \quad (128)$$

is positively invariant for the closed-loop linear system.

B. Upgrading a Linear to a Homogeneous Controller

1) Homogeneous stabilization: Since for $n = 2$ one has $G_d = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$h_1 \quad h_2 \quad \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (129)$$

Hence, for $\alpha > 0$ and

$$P = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}^T \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \quad (130)$$
the system of LMIs (35) becomes
\[
\begin{pmatrix}
 h_1 \\
 h_2
\end{pmatrix}
^T
\begin{pmatrix}
 -2\lambda_2 & 1 \\
 1 & -2\lambda_2
\end{pmatrix}
\begin{pmatrix}
 h_1 \\
 h_2
\end{pmatrix} < 0,
\]
\[
\begin{pmatrix}
 h_1 \\
 h_2
\end{pmatrix}
^T
\begin{pmatrix}
 4\alpha & \lambda \\
 \lambda & 2
\end{pmatrix}
\begin{pmatrix}
 h_1 \\
 h_2
\end{pmatrix} > 0
\] (131)
or, equivalently
\[
\alpha > \max \left\{ \frac{1}{4\lambda^2}, \frac{\lambda^2}{8} \right\}.
\] (132)
Selecting $\delta = 1$, by Theorem 1, the controller
\[
u_h(x) = K \begin{pmatrix}
 1 \\
 0
\end{pmatrix}
\]
\[
= -\left( \frac{\lambda}{\|x\|_d} \right)^2 x_1 - 2 \frac{\lambda}{\|x\|_d} x_2
\] (133)
stabilizes the double integrator to zero in a finite time and
\[
\|x_0\| \leq r \implies x(t, x_0) = 0, \ t \geq T := \frac{1}{\rho}
\] (134)
where $r > 0$ is an arbitrary parameter, the homogeneous norm $\|x\|_d$ is induced by the norm $\|x\| = \sqrt{x^T P x}$ and
\[
\rho = -\lambda_{\max} \left( \begin{pmatrix}
 4\alpha & \lambda \\
 \lambda & 2
\end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix}
 -2\lambda_2 & 1 \\
 1 & -2\lambda_2
\end{pmatrix} \begin{pmatrix}
 4\alpha & \lambda \\
 \lambda & 2
\end{pmatrix}^{-\frac{1}{2}} \right).
\] (135)
The fixed-time stabilization can be guaranteed by selection $r = \max\{r_{\text{min}}, \|x_0\|\}$, where $r_{\text{min}} > 0$.

2) Homogeneous barrier functions: The time-derivatives of the functions
\[
\dot{\phi}_1 = -\frac{1}{\|x\|_d^2} x_1, \quad \dot{\phi}_2 = -\frac{\lambda}{\|x\|_d^2} x_1 - \frac{\lambda}{\|x\|_d} x_2
\] (136)
along the trajectories of the closed-loop homogeneous system are
\[
\dot{\phi}_1 = 2 \frac{\phi_1}{\|x\|_d} x_1 - \frac{x_1}{\|x\|_d} \phi_1
\]
\[
= -\frac{\lambda + \frac{\lambda}{\|x\|_d}}{\|x\|_d} \phi_1 + \frac{1}{\|x\|_d} \phi_2
\] (137)
\[
\dot{\phi}_2 = 2 \frac{\phi_2}{\|x\|_d} x_1 - \frac{\lambda + \frac{\lambda}{\|x\|_d}}{\|x\|_d} x_2 - \frac{1}{\|x\|_d} \phi_1
\]
\[
= 2 \frac{\phi_2}{\|x\|_d} x_1 + \frac{\phi_2}{\|x\|_d} x_2 - \frac{\lambda + \frac{\lambda}{\|x\|_d}}{\|x\|_d} \phi_1
\]
\[
= \frac{\phi_2}{\|x\|_d} \phi_1 - \frac{\lambda + \frac{\lambda}{\|x\|_d}}{\|x\|_d} \phi_2.
\] (138)
Since $\gamma = -\frac{\lambda}{\|x\|_d} > 0$ and $\|x\|_d > 0$ as long as $x \neq 0$ then the system
\[
\begin{pmatrix}
 \phi_1 \\
 \phi_2
\end{pmatrix}
= \frac{1}{\|x\|_d} \begin{pmatrix}
 -\lambda + 2\gamma \\
 -\lambda + \gamma
\end{pmatrix} \begin{pmatrix}
 \phi_1 \\
 \phi_2
\end{pmatrix}
\] (139)
is positive and the set
\[
\Omega_r = \{ x \in \mathbb{R}^2 : \phi_1(x) \geq 0, \phi_2(x) \geq 0 \}
\] (140)
is a strictly positively invariant set of the closed loop homogeneous system.

3) Tuning of the settling time: Let us optimize the settling time estimate (134) maximizing $\rho$. From (135) we conclude that $\rho = -\max\{\eta_1, \eta_2\}$, where $\eta_1$ and $\eta_2$ are real negative roots of the algebraic equation
\[
\det \left( \begin{pmatrix}
 -2\lambda_2 & 1 \\
 1 & -2\lambda_2
\end{pmatrix} - \eta \begin{pmatrix}
 4\alpha & \lambda \\
 \lambda & 2
\end{pmatrix} \right) = 0
\] (141)
which can be rewritten as follows:
\[(8\alpha - \lambda^2)^2 + 2(3 + 4\alpha)\lambda \eta + 4\lambda^2 - 1 = 0.\] (142)
Since
\[D = (3 + 4\alpha)^2\lambda^2 - (8\alpha - \lambda^2)(4\lambda^2 - 1) > 0 \quad \text{for} \quad \alpha > \frac{\lambda}{2},
\] then
\[
\rho = \frac{(3 + 4\alpha)\lambda_1 - \sqrt{(3 + 4\alpha)^2\lambda^2 - (8\alpha - \lambda^2)(4\lambda^2 - 1)}}{(8\alpha - \lambda^2)}
\] (143)
or, equivalently
\[
\rho = \frac{1}{4\lambda^2 - 1} \sqrt{(3 + 4\alpha)^2\lambda^2 - (8\alpha - \lambda^2)(4\lambda^2 - 1)}.
\] (145)
Hence, to maximize $\rho$ the parameter $\alpha$ should be minimized. For $\lambda \geq \sqrt{2}$, we have $\alpha > \max\{\frac{1}{4\lambda^2}, \frac{\lambda^2}{8}\}$ and the smallest upper estimate of the settling time $x(t, x_0) = 0, \forall t \geq T^*$ corresponds to $\alpha^* = \frac{\lambda^2}{8}$ and
\[
T^* = \frac{1}{\rho} = \frac{6 + \lambda^2}{4\lambda^2 - 1}.
\] (146)
By Theorem 1, any settling time estimate $T > 0$ can be assigned by means of the modification of the feedback gain $K = K(d\delta, \delta \geq \ln \max\{1/(\rho T), 1\}$. Without such a rescaling of $K$, the upper estimate of the settling time cannot be less than $T^*$, since the parameter $\alpha > 0$ must satisfy the inequality $\alpha > \frac{\lambda}{2}$ to guarantee the feasibility of LMIs (35).

C. Numerical Simulations and the Use of the Homogeneous Controller as a “safety Filter”

For $x_0 = (-4, 2)$, we select
\[
\lambda = 2 > 1 - \frac{e_1^T x_0}{e_1^T x_0} \quad \text{and} \quad K = (-4 \quad -4).
\]
Then
\[
h_1 = (-1 \quad 0), \quad h_2 = (-2 \quad -1).
\] (147)
Selecting $\alpha = \frac{\lambda^2 + 0.01}{8} = 0.50125$, we derive $\nu \approx 0.7495$, so the closed-loop system with the homogeneous nonovershooting controller (36) has the following settling time estimate: $T = \frac{1}{\nu} \approx 1.3342$, i.e., $\|x(t)\| \leq r \implies x(t, x_0) = 0, \forall t \geq T$.

A nonovershooting controller for the chain of integrators can assume the role of a “safety filter” of a nominal controller $u_{\text{nom}}$ by being implemented as $[1]$
\[
u = \min\{u_{\text{nom}}, u_s\}
\] (148)
where the nominal control $u_{\text{nom}}$ may demand overshoots, while the override of such an “unsafe” nominal control is performed using a “safe” control $u_s$, which may be either of the linear kind, denoted as $u_s = u_{\text{lin}}$, or of the homogeneous kind, denoted as $u_s = u_{\text{hh}}$.

To illustrate such “safety filtering” through a simulation, we take a nominal control given by
\[
u_{\text{nom}} = -4 \left( x_1 + \sin \left( \frac{\pi t}{2} \right) + 0.8 \right) - 4 \left( x_2 + \frac{\pi}{2} \cos \left( \frac{\pi t}{2} \right) \right)
\] (149)
which periodically and persistently attempts to violate the safety condition $x_1 \leq 0$, while also periodically retreating from such an attempt. In this simulation, we compare three safety filters: a linear one based on using [20] with (148), a homogeneous one based on the results of this article, and a PT safety filter introduced in [1]. The simulation results are shown in Figs. 1 and 2.
Fig. 1. Trajectories in \((t, x_1)\)-plane without disturbance.

Fig. 2. Trajectories in \((x_1, x_2)\)-plane without disturbance.

Fig. 3. Positively invariant sets \(\Omega\) and \(\Omega_r\) for linear and homogeneous nonovershooting controllers, respectively.

But let us first examine Fig. 3, which presents the positively invariant sets \(\Omega\) and \(\Omega_r\) for, respectively, the linear nonovershooting controller and homogeneous nonovershooting controller with \(r = ||x_0|| = \sqrt{x_0^T P x_0} \approx 6.6348\). The sets define the “safety zones” outside of which the corresponding controllers cannot guarantee the absence of overshoots. In the case of the FxTSf filter this zone is adaptive and depends of \(\max_{\tau \in [0, t]} ||r(\tau)||\). The larger a zone, the less conservative the override of the nominal controller. Obviously, the homogeneous nonovershooting controller has a larger positively invariant set than the linear controller (at least close to the origin). The PT controller is a time-varying linear feedback \([1]\), so its positively invariant set is not defined.

The numerical simulations in Figs. 1 and 2 show that, in comparison with the linear safety filter, both the homogeneous and the PT safety filters perform highly unconservative overrides of the nominal controller. The main difference between the PT and homogeneous safety filters is that the operation time of the PT safety filter terminates after the prohibition on \(x_1\) being positive ends, which is at \(T = 4\) seconds in our case, whereas the homogeneous safety filter continues to restrict the operation to \(x_1 \leq 0\) for all time.

However, it is remarkable, and evident from Fig. 1 that the homogeneous safety filter keeps \(x_1(t)\) strictly negative for no longer than the fixed time \(T = 4\) s (and, in fact, for little more than 1 s), in response to the attempts of the nominal control to make \(x_1\) positive, whereas the more conservative linear safety filter keeps \(x_1(t)\) negative all the time, greatly distorting the resulting (dashed) solution \(x_1(t)\) compared to the nominal solution (red).

It is evident from Fig. 1 that the safety override with the homogeneous safety filter recurs (twice during the time shown; note the two flat tops of the blue curve). This is the result of the fact that the periodic nominal control keeps attempting to violate the safety boundary and keeps retreating.

### D. System With Disturbances

It is easy to check that all considered filters guarantee safety in the case of nonpositive disturbance \(g \leq 0\). In the planar case, the homogeneous safety filter is proven to be robust with respect to matched positive disturbances as well. The numerical simulations illustrate this on Figs. 4 and 5, where the evolution of the system in the case of the constant (“worse case”) matched disturbance \(g = \begin{pmatrix} 0 \\ 3/2 \end{pmatrix}\) is presented. The nominal controller was simulated for the nominal system (without perturbations). The linear and PT filters cannot guarantee safety in the perturbed case, while the motion of the system with the homogeneous filter remains safe.
The linear nonovershooting controller is given by (17). According to (30), for \( n = 3 \), the parameter \( \lambda > 0 \) of the linear controller can be selected as follows:

\[
\lambda > \max \left\{ 1 - \frac{c_7 x_0}{e_1^7 x_0}, 1 - \frac{c_7 x_0}{e_1^7 x_0}, 1 - 2 \frac{c_7 x_0}{e_1^7 x_0} \right\}.
\] (150)

The gain \( K \) of the linear controller (17) is given by formula (18). The vectors \( h_i, i = 1, 2, 3 \) are defined by the same \( \lambda > 0 \).

Given \( x_0 = (-4, 2, 1)^T \), we select \( \lambda = 2 \) to fulfill (150). Given \( T = 1 \), we define the parameters of the homogeneous nonovershooting controller (36) by Theorem 1

\[
P = \begin{pmatrix} 49 & 24 & 4 \\ 24 & 20 & 4 \\ 4 & 1 & 1 \end{pmatrix}, \quad \delta = 1.6983, \quad r = \|x_0\| = 21.5639.
\] (151)

The closed-loop system has been simulated in MATLAB using the explicit Euler method with the sampling step \( 10^{-3} \).

The homogeneous controller (36) has been realized using the algorithm of the practical implementation developed in [32]. The simulations results of the homogeneous nonovershooting controllers (36) for the triple integrator are depicted in Fig. 6. The triple integrator is stabilized to zero in a finite time (no later than the time instant \( t = 1 \)) without overshoot in the first component.

**VIII. CONCLUSION**

This article proposes a two-step procedure for a nonovershooting homogeneous control design for the integrator chain. First, we construct a linear nonovershooting controller using a backstepping procedure [20]. Next, we transform (“upgrade”), the linear feedback law to a generalized homogeneous one. The obtained homogeneous controller is globally uniformly bounded by a number dependent on control parameters \( T \) and \( r \), which defines a convergence time of the system initiated in the ball of radius \( r \). The positively invariant set of the homogeneous control system is larger than the positively invariant set of the linear control system utilized for the “upgrade.” This allows a better (less conservative) overriding of a potentially unsafe nominal controller for the safety filter design.

The main advantage of the proposed scheme is the simplicity of the control design and parameter tuning. Moreover, the proposed procedure allows a simple upgrade of an existing linear nonovershooting controller to a homogeneous one. The main disadvantage is the necessity to use a special computational procedure (see, [32], [39]) to implement the homogeneous controller in practice for high-order systems \( n \geq 3 \). However, Corollary 3 guarantees that the algorithm of practical implementation of the homogeneous control developed in [32] preserves the nonovershooting properties of the closed-loop system. Moreover, Corollary 2 characterizes a class of perturbations, which can be rejected by the homogeneous nonovershooting controller. It is demonstrated that, in the planar case \( n = 2 \), the safe behavior can be preserved even when the matched disturbances do not vanish as the state tends to the boundary of the safe set. A further robustness analysis of nonovershooting controllers and safety filters is the interesting problem for future research.

**APPENDIX**

**A. Cubic Equation (Cardano Formula)**

Consider the cubic equation

\[
z^3 + pz^2 + qz + r = 0, \quad p, q, r \in \mathbb{R}.
\] (152)

Introduce the following numbers:

\[
C_1 = \sqrt[3]{\frac{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0}}{2}}, \quad C_2 = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0}}{2}}
\] (153)

where \( \Delta_0 = p^2 - 3q, \Delta_1 = 2p^3 - 9pq + 27r. \)

If \( \Delta_1^2 - 4\Delta_0 \geq 0 \) then

\[
z_0 = -\frac{p+C_1+C_2}{3}
\] (154)

is a real root of the cubic equation.

**B. Quartic Equation (Ferrari Formula)**

Let us consider the quartic equation

\[
V^4 + aV^2 + bV + c = 0, \quad a, b, c \in \mathbb{R}
\] (155)

and the adjoint cubic equation

\[
z^3 + 2az^2 + (a^2 - 4e)z - b^2 = 0.
\] (156)
If \( z_0 \in \mathbb{R} \) is a real root of the cubic equation then the roots of the quartic one are

\[
\begin{align*}
V_1 &= -\sqrt{z_0} + \frac{\sqrt{z_0} - 2a + 2b/\sqrt{z_0}}{2}, \\
V_2 &= -\sqrt{z_0} - \frac{\sqrt{z_0} - 2a + 2b/\sqrt{z_0}}{2}, \\
V_3 &= \sqrt{z_0} + \frac{\sqrt{z_0} - 2a - 2b/\sqrt{z_0}}{2}, \\
V_4 &= \sqrt{z_0} - \frac{\sqrt{z_0} - 2a - 2b/\sqrt{z_0}}{2}.
\end{align*}
\]

(157)

C. Auxiliary Results

The following lemma studies some properties of the vectors \( h_i \), which are utilized below for a nonovershooting control design.

**Lemma 2**: If the diagonal matrix \( D_i \in \mathbb{R}^{n \times n} \) is given by

\[
D_i = D_{i-1} + \begin{pmatrix} I_{i-1} & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} i-1 & 0 & \ldots & \ldots & 0 \\ 0 & i-2 & \ldots & \ldots & 0 \\ \ldots & \ldots & 1 & 0 & \ldots \\ 0 & \ldots & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}
\]

\( i = 2, \ldots, n, \quad D_1 = 0 \)

(158)

then for \( i = 2, \ldots, n \) one has

1. \( D_{i-1}A = AD_i \);
2. \( Ae^{D_i s} = e^{D_{i-1}s} A, \forall s \in \mathbb{R} \);
3. \( h_i \begin{pmatrix} I_i \\ 0 \\ 0 \end{pmatrix} = h_i \) and \( h_i e^{s I_{i-1}} \begin{pmatrix} I_i \\ 0 \\ 0 \end{pmatrix} = e^{sh_i}, \quad \forall s \in \mathbb{R}; \)
4. \( h_i D_i = (i-1)\lambda h_{i-1} \);
5. \( h_i D_n = (n-i)h_i + (i-1)\lambda h_{i-1} \).

**Proof**: 1) Simple calculations show

\[
AD_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \\ i-1 & 0 & \ldots & \ldots & 0 \\ 0 & i-2 & \ldots & \ldots & 0 \\ \ldots & \ldots & 1 & 0 & \ldots \\ 0 & \ldots & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}
\]

\[
\times \begin{pmatrix} 0 & i-2 & 0 & \ldots & 0 \\ 0 & 0 & i-3 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \end{pmatrix} = D_{i-1}A. \quad (159)
\]

2) Using \( D_{i-1}A = AD_i \) we obtain

\[
A e^{D_i s} = \sum_{k=0}^{\infty} \frac{AD_i^s}{k!} = \sum_{k=0}^{\infty} \frac{D_{i-1}A D_{i-1}^{s-1}}{k!} = \sum_{k=0}^{\infty} \frac{D_{i-1}^{s-1} A}{k!} = e^{D_{i-1}s} A. \quad (160)
\]

3) By construction [see, the formula (15)], only first \( i \) components of the vector \( h_i \) are nonzero. The latter means that \( h_i \begin{pmatrix} I_i \\ 0 \\ 0 \end{pmatrix} = h_i \) and

\[
h_i e^{s I_{i-1}} \begin{pmatrix} I_i \\ 0 \\ 0 \end{pmatrix} = \sum_{k=0}^{\infty} \frac{h_i}{k!} \begin{pmatrix} I_i \\ 0 \\ 0 \end{pmatrix} = \sum_{k=0}^{\infty} \frac{h_i e^{sh_i}}{k!}.
\]

4) Since \( h_2 D_2 = (-\lambda, 0, \ldots, 0) = (i-1)\lambda h_i \) for \( i = 2 \) then, by induction, for \( i \geq 3 \), we derive

\[
h_i D_i = h_{i-1} A D_i + \lambda h_{i-1} D_i = h_{i-1} D_{i-1} A + \lambda h_{i-1} D_{i-1} + (i-2)\lambda h_{i-1} A + \lambda h_{i-1} = (i-2)\lambda h_{i-1} + (i-1)\lambda h_{i-1} - (i-1)\lambda h_{i-1}.
\]

(162)

5) Since only first \( i \) elements of \( h_i \) are nonzero then

\[
h_i D_n = h_i (I_{n-1} + D_{n-1}) = h_i + h_i D_{n-1}
\]

\[
= \cdots = (n-i)h_i + h_i D_i
\]

(163)

where \( h_i D_i = (i-1)\lambda h_{i-1} \) as shown above.

Other useful properties of the vectors \( h_i \) and the matrices \( D_i \) are given by the following corollary.

**Corollary 6**: 1) If \( \lambda > 0 \) then

\[
h_i d(s) \geq 0 \quad \forall s \geq 0 \quad (164)
\]

provided that \( h_i x \geq 0 \) for all \( i = 1, \ldots, n; \)

2) \( H(A + BK) = (A - \lambda I_n)H \), where \( K \) is given by (18)

\[
= \begin{pmatrix} 1 \\ \vdots \\ h_n \end{pmatrix} \in \mathbb{R}^{n \times n}
\]

(165)

iii) \( G_d = I_n + D_n \) and

\[
HG_d = (G_d + (nI_n - G_d)A^T) H.
\]

(166)

**Proof**: i) Let us consider the functions \( g_i : \mathbb{R}_+ \to \mathbb{R} \) defined as follows:

\[
g_i(s) = h_i e^{sh_i} x, \quad s \geq 0.
\]

(167)

Hence, we have \( g_i(0) \geq h_i x \). Let us show, by induction that these functions are nondecreasing and, consequently, nonnegative on \( \mathbb{R}_+ \). Indeed, since \( D_1 = 0 \) then \( g_1(s) = h_1 e^{sh_1} x = h_1 x \).
is nondecreasing and nonnegative. For $i \geq 2$, we derive
\[
\frac{d}{dt} q_i(s) = h_i D_i e^{s D_i} x = (i-1) \lambda h_{i-1} e^{s D_i} x
\]
where the identities $h_i D_i = (i-1) \lambda h_{i-1} D_i$ and $h_{i-1} e^{s D_i} (s_i, 0, 0) = e^s h_{i-1}$ (see Lemma 2) are utilized. Since $\lambda > 0$ then $\frac{d}{dt} q_i(s) \geq 0$ and, consequently, $q_i(s) \geq q_i(0), \forall s \geq 0$ provided that $q_i(s) \geq 0, \forall s \geq 0$. Hence, taking into account $h_i a(s) = e^{(n-i+1)s} h_i D_i e^{s D_i} x$ we conclude that $h_i d(s) \geq 0$ for all $s \geq 0$ provided that $h_i x \geq 0$ for all $i = 1, 2, \ldots, n$.

ii). The identity $H(A + BK) H^{-1} = -\lambda H_n + A$ is proven in Lemma 1 by means of the coordinate transformation $\varphi = H x$.

iii). Since $G_d = D_n + I_n$ then
\[
H G_d = H + \left( \begin{array}{c} h_1 D_n \\ \vdots \\ h_n D_n \end{array} \right) = H + \left( \begin{array}{c} (n-1) h_1 \\ (n-2) h_2 + \lambda h_1 \\ (n-3) h_3 + 2\lambda h_2 \\ \vdots \\ (n-1) \lambda h_{n-1} \end{array} \right)
\]
\[
= H + D_n H + \lambda \left( \begin{array}{c} 0 \\ h_1 \\ \vdots \\ (n-1) h_{n-1} \end{array} \right) H
\]
\[
= G_d H + \lambda (n I_n - G_d) A^T H.
\]
\[
= G_d H + \lambda (n I_n - G_d) A^T H.\]
\[
(170)
\]
\[
(171)
\]
\[
(172)
\]
[35] Y. Rahman, M. Jankovic, and M. Santill, “Driver intent prediction with barrier functions,” in Proc. Amer. Control Conf., 2021, pp. 224–230.
[36] M. Santillo and M. Jankovic, “Collision free navigation with interacting, non-communicating obstacles,” in Proc. Amer. Control Conf., 2021, pp. 1637–1643.
[37] Y. Song, Y. Wang, J. Holloway, and M. Krstic, “Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time,” Automatica, vol. 83, pp. 243–251, 2017.
[38] L. Wang, A. D. Ames, and M. Egerstedt, “Safety barrier certificates for collisions-free multirobot systems,” IEEE Trans. Robot., vol. 33, no. 3, pp. 661–674, Jun. 2017.
[39] S. Wang, A. Polyakov, and G. Zheng, “On generalized homogenization of linear quadrotor controller,” in Proc. IEEE Int. Conf. Robot. Autom., 2020, pp. 6190–6195.
[40] P. Wieland and F. Allgöwer, “Constructive safety using control barrier functions,” IFAC Proc. Volumes, vol. 40, pp. 462–467, 2007.
[41] V. I. Zubov, Methods of A. M. Lyapunov and Their Applications. (Translated from Russian: V. I. Zubov, Metody Lyapunova i ih primenenie, Leningrad: LGU, 1957), Leiden, The Netherlands: Noordhoff, 1964.
[42] V. I. Zubov, “On systems of ordinary differential equations with generalized homogenous right-hand sides,” Izvestia vuzov. Mathematica (in Russian), vol. 1, pp. 80–88, 1958.

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