Effects of atomic diffraction on the Collective Atomic Recoil Laser

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We formulate a wave atom optics theory of the Collective Atomic Recoil Laser, where the atomic center-of-mass motion is treated quantum mechanically. By comparing the predictions of this theory with those of the ray atom optics theory, which treats the center-of-mass atomic motion classically, we show that for the case of a far off-resonant pump laser the ray optics model fails to predict the linear response of the CARL when the temperature is of the order of the recoil temperature or less. This is due to the fact that in this temperature regime one can no longer ignore the effects of matter-wave diffraction on the atomic center-of-mass motion.

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I. INTRODUCTION

The Collective Atomic Recoil Laser, or CARL, is the atomic equivalent of the Free Electron Laser [1]. Developed theoretically by Bonifacio et al [2-5], the CARL device has three main components: (1) the active medium, which consists of a gas of two-level atoms, (2) a strong pump laser which drives the two-level atomic transition, and (3) a ring cavity which supports an electromagnetic mode (the probe) counterpropagating with respect to the pump. Under suitable conditions, the operation of the CARL results in the generation of a coherent light field (the probe) due to the following mechanism. First, a weak probe field is initiated by noise, either optical in the form of spontaneously emitted light, or atomic in the form of density fluctuations in the atomic gas which backscatters the pump. Once initiated, the probe combines with the pump field to form a weak standing wave which acts as a periodic optical potential (light shift). The center-of-mass motion of the atoms on this potential results in a bunching (modulation) of their density, very much like the combined effects of the wiggler and the light field leads to electron bunching in the free-electron laser. This bunching process is then seen by the pump as the appearance of a polarization grating in the active medium, which results in stimulated backscattering into the probe field. The resulting increase in the probe strength further increases the magnitude of the standing wave field, resulting in more bunching followed by an increase in stimulated backscattering, etc. This positive feedback mechanism results in an exponential growth of both the probe intensity and the atomic bunching. This leads to the perhaps surprising result that the presence of the ring cavity turns the ordinarily stable system of an atomic gas driven by a strong pump laser into an unstable system.

The operation of the CARL was verified experimentally by Bigelow et al [6], using a hot atomic cell. Related experiments by Courtois et al [7] using cold cesium atoms, and by Lippi et al [8] using hot sodium atoms measured the recoil induced small-signal probe gain, which was interpreted in terms of coherent scattering off an induced polarization grating. However, these experiments lacked a probe feedback mechanism, which is necessary to see the long time scale instability which characterizes the CARL.

The CARL theory developed by Bonifacio et al considers the atoms either as classical point particles moving in the optical potential generated by the light fields, or, in a “hybrid” version, as particles whose center-of-mass is labeled by their classical position, but with quantum fluctuations about that position included. From an atom optics point of view, such theories can be described as “ray atom optics” treatments of the atomic field, in analogy with the ordinary ray optics treatment of electromagnetic fields.

Like ordinary ray optics, the ray atom optics description of CARL is expected to be valid provided that the characteristic wavelength of the matter-wave field remains much smaller than the characteristic length scale of any atom-optical element in the system. The characteristic wavelength of the atomic field is its de Broglie wavelength, determined by the atomic mass and the temperature $T$ of the atomic gas. The central atom-optical element of the CARL is the periodic optical potential, which acts as a diffraction grating for the atoms, and has the characteristic length scale of half the optical wavelength. Hence the classical “ray atom optics” description is intuitively expected to be valid provided that the temperature is high enough that the thermal de Broglie wavelength is much smaller than the optical wavelength. This gives the condition $T \gg T_R$, the recoil temperature of the atoms, as the domain of ray atom optics. In particular, it is certainly expected to hold under the temperature conditions of the experiments performed so far.

However, the spectacular recent progress witnessed by atomic cooling techniques makes it likely that CARL experiments using ultracold atomic samples can and will be performed in the future. In particular, subrecoil temperatures can now be achieved almost routinely. The purpose of this paper is to extend the CARL theory to this “wave atom optics” regime [9]. In this regime matter-
wave diffraction is expected to play a dominant role in the CARL dynamics, and thus it becomes important to determine to what extent it counteracts the bunching process in the CARL.

The wave optics theory of the CARL is similar to the analysis of atomic diffraction by standing waves \[1\], except that the electromagnetic field is now treated as a dynamical variable. It is also similar to the theory of recoil induced resonances \[1\], which describes the stimulated scattering of light off a standing wave induced polarization grating, but the absence of a feedback mechanism for the probe feedback in that case means that it lacks the instability necessary for lasing.

In this paper we focus on the case of a far off-resonant pump laser, thus permitting us to neglect the excited state population and therefore to ignore the effects of spontaneous emission (except as a hypothetical source of noise for probe initialization). We further concentrate on the linear regime, where both the probe field and the atomic bunching are considered as infinitesimal quantities, since it is this regime that determines whether or not the exponential instability occurs. Finally, we restrict our analysis to atomic densities low enough that collisions between atoms may be ignored, and neglect the transverse motion of the atoms, which in the absence of collisions is decoupled from the longitudinal degree of freedom along which bunching occurs.

We note at the outset that our theory is semiclassical in that it treats the electromagnetic field classically. While this approximation can not fully describe the statistics of the CARL output, it is sufficient to describe the small-signal gain of the system, provided that one makes the implicit assumption that small fluctuations will trigger it, an approach familiar from conventional laser theory and nonlinear optics. We also emphasize that it is not inconsistent to treat the matter waves quantum mechanically while treating the light classically, since the limits under which a quantum description is required are independent. For light, this limit is usually associated with weak intensities, while for matter waves it is normally a low temperature limit.

This rest of this paper is organized as follows: Section II briefly reviews the ray atom optics model of the CARL, establishing the notation and setting the stage for a comparison of its predictions with those of the wave atom optics theory, which is introduced in section III. Section IV discusses the collective instability leading to CARL operation, compares the ray atom optics and the wave atom optics predictions, and determines the domain of validity of the former theory. Finally section IV is a summary and outlook.

## II. RAY ATOM OPTICS MODEL

The Ray Atom Optics (RAO) model of the CARL has been developed and extensively studied by Bonifacio et al.\[2\] It begins with the classical \(N\)-particle Hamiltonian

\[
H_N = \sum_{j=1}^{N} H_1(z_j, p_j),
\]

where \(z_j\) and \(p_j\) are the classical position and momentum of the \(j\)th atom, obeying the canonical equations of motion \(d\mathbf{z}_j/dt = \partial H_N/\partial p_j\) and \(dp_j/dt = -\partial H_N/\partial z_j\). The single-particle Hamiltonian \(H_1\) is given explicitly by

\[
H_1(z_j, p_j) = \frac{p^2}{2m} + \frac{\hbar \omega_0}{2} \sigma_{zj} + i\hbar \left[ g_1 a^*_j e^{-ik_1 z_j} \sigma_{-j} + g_2 a^*_j e^{-ik_2 z_j} \sigma_{-j} - c.c. \right],
\]

where \(m\) is the atomic mass, \(\omega_0\) is the natural frequency of the atomic transition being driven by the pump and probe lasers, and \(g_1\) is the atom-probe electric dipole coupling constant. It is given by \(g_1 = \mu_1 [ck_1/(2\hbar\epsilon_0 V)]^{1/2}\), where \(\mu_1\) is the projection of the atomic dipole moment along the probe polarization, \(k_1\) is the probe wavenumber, and \(V\) is the quantization volume. The atom-pump coupling constant \(g_2\) is defined analogously to \(g_1\), but depending on \(\mu_2\), the projection of the atomic dipole moment along the pump polarization, and \(k_2\) the pump wavenumber. The normal variables \(a_1\) and \(a_2\) describe the probe and pump laser fields, respectively. They obey Maxwell’s equation

\[
\frac{d}{dt} a_i = -i \omega_i a_i + g_i \sum_{j=1}^{N} e^{-ik_i z_j} \sigma_{-j},
\]

where \(\omega_i\) is the natural frequency of the probe \((i = 1)\) or the pump \((i = 2)\) field. Note that these equations are also valid for quantized electromagnetic fields, provided that \(a_i\) are interpreted as annihilation operators, but we describe the light fields classically in this paper.

The variables \(\sigma_{-j}\) and \(\sigma_{zj}\) are the expectation values of the quantum mechanical Pauli pseudo-spin operators which describe the internal state of the \(j\)th atom. They obey the familiar optical Bloch equations, appropriately modified to include the center-of-mass motion of the atoms and with spontaneous emission neglected\

\[
\frac{d}{dt} \sigma_{-j} = -i \omega_0 \sigma_{-j} + \left[ g_1^* a_1 e^{ik_1 z_j} + g_2^* a_2 e^{ik_2 z_j} \right] \sigma_{zj},
\]

and

\[
\frac{d}{dt} \sigma_{zj} = -2 \left[ g_1 a_1^* e^{-ik_1 z_j} + g_2 a_2^* e^{-ik_2 z_j} \right] \sigma_{-j} + c.c.\]

\(^1\)Spontaneous emission is neglected in anticipation of the future approximation that the pump lasers are far-off resonant, and therefore the excited state population may be safely neglected.
It is convenient to introduce slowly varying variables via the transformations \( a_1 = a_1' e^{-i \omega t} \), \( a_2 = a_2' e^{-i \omega t} \), and \( \sigma'_{-j} = \sigma_{-j} e^{-i(\omega t - k_2 z_j)} \), where \( \omega \) is the pump frequency shifted by the frequency pulling contribution due to the induced atomic polarization. The exact value of \( \omega \) will be derived in a self-consistent manner shortly in a way similar to the approach used in conventional laser theory. These new variables obey the equations of motion

\[
\frac{d}{dt} z_j = \frac{p_j}{m},
\]

\[
\frac{d}{dt} p_j = -\hbar \left[ g_1 k_1 a_1'^* e^{-i(k_1 - k_2)z_j} + g_2 k_2 a_2'^* \right] \sigma'_{-j} + \text{c.c.},
\]

\[
\frac{d}{dt} a_1' = i(\omega - \omega_1) a_1' + g_1 \sum_{j=1}^{N} e^{-i(k_1 - k_2)z_j} \sigma'_{-j},
\]

\[
\frac{d}{dt} a_2' = i(\omega - \omega_2) a_2' + g_2 \sum_{j=1}^{N} \sigma'_{-j},
\]

\[
\frac{d}{dt} \sigma_{zj} = -2 \left[ g_1 a_1'^* e^{-i(k_1 - k_2)z_j} + g_2 a_2'^* \right] \sigma'_{-j} + \text{c.c.},
\]

and

\[
\frac{d}{dt} \sigma'_{-j} = i(\omega - \omega_0 - \frac{k_2}{m} p_j) \sigma'_{-j} + \left[ g_1^* a_1'^* e^{i(k_1 - k_2)z_j} + g_2^* a_2'^* \right] \sigma_{zj}.
\]

In the case where the lasers are tuned far off resonance, and the atoms are initially in the ground state, the excited state population remains small and can be neglected. This is equivalent to describing the atoms as classical Lorentz atoms, and is accomplished by setting \( \sigma_{zj} = -1 \) in Eq. (11). Assuming further that the detuning \( \omega - \omega_0 \) is much larger than any other frequency in Eq. (11), allows one to adiabatically eliminate \( \sigma'_{-j} \) with

\[
\sigma'_{-j} \approx -i \frac{1}{(\omega - \omega_0)} \left[ g_1^* a_1'^* e^{i(k_1 - k_2)z_j} + g_2^* a_2'^* \right],
\]

where we have in addition neglected the Doppler shift \( k_2 p_j/m \) compared to \( \omega - \omega_0 \). This leads to the reduced set of equations

\[
\frac{d}{dt} z_j = \frac{p_j}{m},
\]

\[
\frac{d}{dt} p_j = -i \frac{2 \hbar k_0}{(\omega - \omega_0)} \left[ g_1^* g_2 a_2'^* a_1'^* e^{2i \hbar k_0 z_j} - \text{c.c.} \right],
\]

\[
\frac{d}{dt} a_1' = i \left[ \omega - \frac{N |g_1|^2}{(\omega - \omega_0) - \omega_1} \right] a_1' - i \frac{g_2^* g_1}{(\omega - \omega_0)} \sum_{j=1}^{N} e^{-i \hbar k_0 z_j},
\]

and

\[
\frac{d}{dt} a_2' = i \left[ \omega - \frac{N |g_2|^2}{(\omega - \omega_0) - \omega_2} \right] a_2',
\]

where we have introduced \( k_0 = (k_1 - k_2)/2 \).

We now introduce the undepleted pump approximation, valid in the linear regime where \( a_1' \) remains small. This is achieved by dropping the term proportional to \( a_1' \) in Eq. (10). This yields

\[
\frac{d}{dt} a_2' = i \left[ \omega - \frac{N |g_2|^2}{(\omega - \omega_0) - \omega_2} \right] a_2',
\]

which has the steady state solution \( a_2'(t) = a_2(0) \) provided that the frequency pulling condition

\[
\omega - \frac{N |g_2|^2}{(\omega - \omega_0)} = \omega_2 = 0.
\]

is satisfied. Note that this equation has two solutions, but we must choose the branch which gives the result \( \omega = \omega_2 \) when \( N = 0 \). This leads to the solution

\[
\omega = \frac{1}{2} \left[ \omega_0 + \omega_2 \pm \sqrt{(\omega_2 - \omega_0)^2 + 4 N |g_2|^2} \right],
\]

where the plus sign must be taken for positive detunings \( \omega_2 > \omega_0 \) and the minus sign for negative detunings \( \omega_2 < \omega_0 \). Expanding this relation to lowest order in \( (\omega_2 - \omega_0)^{-1} \) gives the expected result

\[
\omega \approx \omega_2 + \frac{N |g_2|^2}{(\omega_2 - \omega_0)}.
\]

To proceed analytically past this point, it is convenient to introduce the dimensionless variables \( \theta_j \equiv 2 \hbar k_0 z_j, P_j = p_j/\hbar k_0, A = g_1^* g_2 a_2(0) a_1'/[\omega(\omega - \omega_0)] \) and \( \tau = 4 \omega_1 t \), where the recoil frequency \( \omega_r \) is given by

\[
\omega_r = \hbar k_0^2/2m.
\]

These variables obey the equations of motion

\[
\frac{d}{d\tau} \theta_j = P_j,
\]

\[
\frac{d}{d\tau} P_j = -i A e^{i \theta_j} + \text{c.c.,}
\]


\[
\frac{d}{d\tau} A = i\Delta A - i\alpha \frac{1}{N} \sum_{j=1}^{N} e^{-i\theta_j}, \quad (24)
\]

where we have introduced the dimensionless control parameters

\[
\Delta = \delta_1 / 4\omega_r, \quad (25)
\]

and

\[
\alpha = N|g_1|^2|g_2|^2|a(2)|^2 / 8\omega^2_r (\omega - \omega_0)^2, \quad (26)
\]

where \(\delta_1 = \omega - \omega_1 - N|g_1|^2 / (\omega - \omega_0)\). We note that both \(\Delta\) and \(\alpha\) are real numbers, and furthermore that \(\alpha \geq 0\).

We seek solutions of these equations which are perturbations about the case \(A = 0\). Thus we make the substitutions

\[
\theta_j = \theta_j(0) + P_j(0)\tau + \delta\theta_j, \quad (27)
\]

and

\[
P_j = P_j(0) + \delta P_j, \quad (28)
\]

where \(\theta_j(0)\) is randomly taken from a uniform distribution, and \(P_j(0)\) is randomly taken from the initial momentum distribution. The new variables \(\delta\theta_j\) and \(\delta P_j\) give the perturbations on the atomic center-of-mass motion due to a nonzero \(A(0)\). We introduce finally the linearized velocity group bunching parameter and its "conjugate" momentum according to

\[
B(k) = \frac{1}{N} \sum_{j=1}^{N} \delta\theta_j(0) P_j(k) (1 - i\delta\theta_j) e^{-i(\theta_j(0) + P_j(0)\tau)}, \quad (29)
\]

and

\[
\Pi(k) = \frac{1}{N} \sum_{j=1}^{N} \delta\theta_j P_j(k) \delta P_j e^{-i(\theta_j(0) + P_j(0)\tau)}
+ P(k) B(k). \quad (30)
\]

We note that

\[
\sum_k B(k) = \langle e^{-i2k_0 z} \rangle, \quad (31)
\]

and the amplitude of \(\langle \rangle\) is a measure of the degree of bunching of the atomic gas. A magnitude of zero indicates no bunching, while a magnitude of one indicates maximum bunching. This leads to the equations

\[
\frac{d}{d\tau} B(k) = -i\Pi(k), \quad (32)
\]

\[
\frac{d}{d\tau} \Pi(k) = i \left[ P^2(k) B(k) - 2P(k) \Pi(k) - \frac{N(k)}{N} A \right], \quad (33)
\]

and

\[
\frac{d}{d\tau} A = i \left[ \Delta A - \alpha \sum_k B(k) \right], \quad (34)
\]

where \(N(k)\) is the number of atoms in the velocity group with momentum \(\hbar k_B P(k)\) and we have assumed that

\[
\sum_{j=1}^{N} \delta P_j(0), P(k) e^{-i2\theta_j(0)} = 0, \quad (35)
\]

an assumption that requires that \(N(k) \gg 1\). Note that this formulation implies a discretization of the initial momentum distribution, and furthermore assumes that the atomic positions in each velocity group are initially randomly distributed along the CARL cavity. Fluctuations in the initial distributions can of course readily be included into the initial conditions of the perturbation variables.

### III. WAVE ATOM OPTICS MODEL

In order to quantize the center-of-mass motion of a gas of Bosonic atoms, one may either utilize first quantization, and replace the variables \(z_j\) and \(p_j\) in the \(N\)-particle Hamiltonian (1) with operators satisfying the canonical commutation relations \([\hat{z}_j, \hat{p}_{j'}] = i\hbar\delta_{jj'}\), or equivalently we can second-quantize the single particle Hamiltonian (2), introducing creation and annihilation operators for excited and ground state atoms of a given center-of-mass momentum. It is this second method which we will adopt in deriving the Wave Atom Optics (WAO) model. In the absence of collisions, the second-quantized Hamiltonian is simply

\[
\hat{H} = \sum_k \hat{H}(k), \quad (36)
\]

where \(\hat{H}(k)\) is given by

\[
\hat{H}(k) = \frac{\hbar^2 k^2}{2m} \hat{\epsilon}_g(k) \hat{\epsilon}_g(k) + \left( \frac{\hbar^2 k^2}{2m} + \hbar\omega_0 \right) \hat{\epsilon}_c(k) \hat{\epsilon}_c(k)
+ i\hbar \left[ g_1 a^*_1 \hat{\epsilon}_g(k + k_1) \hat{\epsilon}_c(k) + g_2 a^*_2 \hat{\epsilon}_g(k + k_2) \hat{\epsilon}_c(k)
- \hat{H}\text{c.} \right], \quad (37)
\]

where the field operator \(\hat{\epsilon}_g(k)\) annihilates a ground state atom of momentum \(\hbar k\), and \(\hat{\epsilon}_c(k)\) annihilates an excited atom of momentum \(\hbar k\). We assume that the atoms in the sample are bosonic, so that these operators obey the commutation relations

\[
[\hat{\epsilon}_g(k), \hat{\epsilon}_g^\dagger(k')] = [\hat{\epsilon}_c(k), \hat{\epsilon}_c^\dagger(k')] = \delta_{kk'}, \quad (38)
\]

all other commutators being equal to zero.

With the atomic polarization now expressed in terms of field operators, Maxwell’s equations (3) for the classical laser fields become
\[
\frac{d}{dt} a_i = -i\omega_i a_i + g_i \sum_k \langle \hat{c}_g^\dagger (k + k_i) \hat{c}_e (k) \rangle.
\] (39)

Hence, all that is required to determine the field evolution are the expectation value of bilinear combinations of atomic creation and annihilation operators. The evolution of these expectation values is easily obtained by introducing the “single-particle” atomic density operators \[\hat{\rho}_{gg}(k, k') = \hat{c}_g^\dagger (k') \hat{c}_g (k),\] (40) and
\[\hat{\rho}_{eg}(k, k') = [\hat{\rho}_{ge}(k', k)]^\dagger = \hat{c}_g^\dagger (k') \hat{c}_e (k),\] (41)
and
\[\hat{\rho}_{ee}(k, k') = \hat{c}_e^\dagger (k') \hat{c}_e (k).\] (42)

Note that e.g. the expectation value of the diagonal operator \(\langle \hat{\rho}_{gg}(k, k) \rangle\) gives the mean number of ground state atoms with momentum \(\hbar k\). The expectation values of these operators obey the equations of motion
\[
\frac{d}{dt} \rho_{jj'}(k, k') = \frac{i}{\hbar} \left\{ [\hat{H}, \hat{\rho}_{jj'}(k, k')] \right\}
\] (43)
where \(\rho_{jj'}(k, k') = \langle \hat{\rho}_{jj'}(k, k') \rangle\). The full form of these equations is given in the Appendix. The important point is that they are depend only on \(\rho_{jj'}(k, k')\), hence they form a closed set of equations which describe the response of the atomic field to the driving laser fields. We note that had we included collisions in our model, this would no longer be the case.

Introducing in analogy to the ray optics description the rotating variables \(a_1 = a_1' e^{-i\omega t}\), \(a_2 = a_2' e^{-i\omega t}\), and \(\rho_{eg}(k, k') = \rho_{eg}'(k - k_0, k') e^{-i\omega t}\), neglecting the excited state population, and solving adiabatically for \(\rho_{eg}'(k, k')\) yields
\[
\rho_{eg}'(k, k') \approx -\frac{i}{\omega - \omega_0} \left\{ a_1' a_1' \rho_{gg}(k + 2k_0, k') + g_2 a_2' \rho_{gg}(k, k') \right\}.
\] (44)
Substituting Eq. (44) into Maxwell’s equation (38) for the pump and making once more the undepleted pump approximation leads to the solution \(a_2'(t) = a_2(0)\) provided that \(\omega\) is given by Eq. (13). We then substitute Eq. (44) into the equation of motion for \(\rho_{gg}(k, k')\), and introduce the dimensionless wavenumber \(\kappa = k/(k_1 - k_2)\) and the mean density \(\rho(\kappa, \kappa') = \rho_{gg}(k, k')/N\), in addition to the dimensionless variables already defined in the ray atom optics model. We arrive at the wave optics equations of motion
\[
\frac{d}{d\tau} \rho(\kappa, \kappa') = -i(\kappa^2 - \kappa'^2) \rho(\kappa, \kappa') + \frac{i}{2} \left\{ A^* \left[\rho(\kappa, \kappa' + 1) - \rho(\kappa - 1, \kappa')\right] - \frac{i}{2} \left[ A\left[\rho(\kappa + 1, \kappa') - \rho(\kappa, \kappa' - 1)\right] \right],
\] (45)
and
\[
\frac{d}{d\tau} A = i\Delta A - i\alpha \sum_k \rho(\kappa, k + 1),
\] (46)
where the parameters \(\Delta\) and \(\alpha\) are given by Eqs. (23) and (24), respectively.

As in Sec. II, we seek a solution which is a perturbation about the case \(A = 0\). From Eq. (43), the unperturbed solution is readily found to be
\[
\rho(\kappa, \kappa', \tau) = \frac{N(\kappa)}{N} \delta_{\kappa, \kappa'},
\] (47)
We consider specifically an atomic sample initially in thermal equilibrium, so that Eq. (47) becomes
\[
\rho(\kappa, \kappa', \tau) = \frac{N(\kappa)}{N} \delta_{\kappa, \kappa'},
\] (48)
where \(N(\kappa)\), the number of atoms with initial wavenumber \(2k_0\kappa\), is given by a thermal distribution function. We introduce the perturbation variables \(\delta \rho(\kappa, \kappa')\) according to
\[
\rho(\kappa, \kappa') = \frac{N(\kappa)}{N} \delta_{\kappa, \kappa'} + \delta \rho(\kappa, \kappa'),
\] (49)
and observe that Maxwell’s equation (46), which becomes
\[
\frac{d}{d\tau} A = i\Delta A - i\alpha \sum_k \delta \rho(\kappa, k + 1),
\] (50)
with the linearized equation
\[
\frac{d}{d\tau} \delta \rho(\kappa, k + 1) = i(2\kappa + 1) \delta \rho(\kappa, k + 1) - i \left[ \frac{N(k + 1) - N(k)}{2N} \right] A.
\] (51)
form a closed set of equations which underlies the dynamics of the CARL in the linear regime of wave atom optics.

**IV. COLLECTIVE INSTABILITY**

The most important feature of the CARL is the appearance of a collective instability, which gives rise to exponential gain under appropriate parameter settings. This instability is characterized by an imaginary frequency component in the spectrum of the probe field \(A(\tau)\). As has been demonstrated in Ref. 3, one needs
not solve the complete set of equations derived in the previous sections in order to determine the necessary conditions for the collective instability. Instead, by taking the Laplace transform of these equations one can derive a “characteristic equation” which allows one to determine whether exponential gain occurs, and if so what the exponential growth rate is.

For the Ray Atom Optics model, the Laplace transform of Eq. (54) yields

$$\tilde{A}_R(s) = \frac{A(0)}{R(s)},$$

(52)

where $R(s)$ is given by

$$R(s) = \left[ s - i\Delta - i\alpha \int \frac{f(k)dk}{(s + i2k)^2} \right].$$

(53)

In obtaining this result we have taken the continuum limit and assumed that $B(k)$ and $\Pi(k)$ vanish at $\tau = 0$. Here $f(k)$ is simply the normalized thermal distribution function for the dimensionless momentum $P(k) = 2k$. The roots of $R(s)$ give the characteristic exponents of the CARL. Stability requires that all roots be purely imaginary. When the collective instability occurs, however, there will be one root with a positive real part. This real part is the RAO exponential growth rate $\Gamma_R$. This result is identical to that first obtained by Bonifacio et al.

The Wave Atom Optics model, which includes the effects of atomic diffraction, yields the Laplace transform

$$\tilde{A}_W(s) = \frac{A(0)}{W(s)}.$$

(54)

$W(s)$ is given by

$$W(s) = \left[ s - i\Delta - i\alpha \int \frac{f(k)dk}{(s - i(2k - 1))(s - i(2k + 1))} \right],$$

(55)

where we have again taken the continuum limit and assumed that $\delta P(k, k + 1)$ vanishes at $\tau = 0$. If a root of $W(s)$ with a positive real part exists, that real part is the WAO exponential growth rate $\Gamma_W$. We see by comparing Eqs. (52) and (54) that the effect of atomic diffraction is to lift the degeneracy of the singularity under the integral. This expression also leads us immediately to the conclusion that if the width of the momentum distribution $f(k)$ is large compared to $2k$, then the singularity will appear as essentially degenerate, and the effects of matter waves diffraction will be negligible. Thus the RAO and WAO models should agree for large enough temperatures.

### A. Finite temperatures

In the absence of quantum degeneracies, the thermal momentum distribution is given by the Maxwell-Boltzmann distribution

$$f(k) = \frac{2\beta}{\sqrt{\pi}}e^{-\beta^2k^2},$$

(56)

where $\beta^2 = T_R/T$ and $T_R = \hbar \omega_r/k_B$ is the recoil temperature, $K_B$ being the Boltzmann constant. By substituting Eq. (56) into Eq. (53) and using the Fourier convolution theorem we find that the RAO exponential growth rate $\Gamma_R$ is determined by the equation

$$s - i\Delta - i\alpha \int_0^\infty pe^{-\frac{p^2}{4\beta^2}-ps}dp = 0,$$

(57)

which can be integrated to give the transcendental equation

$$s - i\Delta - i2\alpha\beta^2 + i2\sqrt{\pi \alpha \beta^2} e^{\beta^2s^2} \text{erfc}(\beta s) = 0.$$  (58)

In contrast, substituting Eq. (56) into Eq. (55) and again using the convolution theorem we find that the WAO exponential growth rate $\Gamma_W$ is determined by equation

$$s - i\Delta - i\alpha \int_0^\infty e^{-\frac{ps^2}{4\beta^2}-ps} \sin(p) = 0,$$

(59)

By examining Eq. (55) we see that in the case $\beta \ll 1$ we are justified in expanding $\sin(p)$ to lowest order in $p$. This exactly reproduces Eq. (57), thus showing that the WAO and RAO descriptions make indistinguishable predictions about the exponential growth rate in the limit $T \gg T_R$. However, for temperatures comparable to or less than the recoil temperature, we will see that the RAO theory fails to correctly predict the behavior of the CARL in the linear regime. Physically, this is due to the fact that it does not account for the effects of atomic diffraction, which tends to counteract the bunching process. Finally, we note that upon integration, Eq. (59) becomes the transcendental equation

$$s - i\Delta + \frac{\sqrt{\pi}}{2} \alpha e^{\beta^2(s^2-1)} \left[ e^{i2\beta^2s} \text{erfc}[\beta(s + i)] - e^{-i2\beta^2s} \text{erfc}[\beta(s - i)] \right] = 0.$$  (60)

In the next subsection we will examine in more detail the precise manner in which diffraction interferes with the bunching process for the special case of a zero temperature atomic gas. But before turning to this extreme situation, we present numerical results comparing RAO and WAO models at non-zero temperature, as determined by solving Eqs. (58) and (60).

Figures 1(b-d) compare $\Gamma_R$ with $\Gamma_W$ at $\alpha = 10$ for the three different temperature regimes, $T = T_R$, $T = 10T_R$, and $T = 100T_R$ respectively. Figures 2(b-d) shows the same comparison for $\alpha = 10^{-1}$. While we see that the behavior of $\Gamma_R$ and $\Gamma_W$ depends strongly on $\alpha$ (recall that $\alpha$ is proportional to both the pump intensity and the atomic density), the discrepancies between the two models as a function of temperature are very similar. At $T = T_R$ there are significant differences between the
predictions of the RAO and WAO models, but these differences become minimal at $T = 10T_R$, and insignificant at $T = 100T_R$. We also observe that the differences are more pronounced for lower values of $\alpha$, meaning that at lower densities and/or pump intensities, the quantum mechanical behavior becomes more apparent. The reason for this is that at high intensities the bunching process, driven by the probe field, dominates, while at low intensities the anti-bunching effects of atomic diffraction play a larger role.

B. The $T = 0$ limit

For a typical atom, the recoil temperature is of the order of microkelvins, e.g. for sodium we have $T_R = 2.4\mu K$. However, recent advances in cooling techniques have led to measured temperatures as low as the picokelvin regime. At these extreme temperatures the condition $T \ll T_R$ is satisfied, i.e. we are effectively in the $T \to 0$ limit. In this section we study the $T = 0$ case in detail in order to gain further insight into the exact role of matter wave diffraction in the CARL system.

For the RAO model, we have a single velocity group at $k = 0$. Thus by differentiating Eq. (32) with respect to $\tau$ and using Eq. (33), we see that the bunching parameter $B \equiv B(0) = \langle \exp(-i2k_0z) \rangle$ obeys the equation of motion

$$\frac{d^2}{d\tau^2} B = -A,$$

where we have taken $P(0) = 0$ and $N(0) = N$ to indicate that all atoms are initially at rest.

In the WAO description, setting $N(\kappa)/N = \delta_{\kappa,0}$ in Eq. (31) shows that two variables are coupled to the probe field, $\delta\rho(-1,0)$, and $\delta\rho(0,1)$. They describe the recoil of atoms initially at rest as a result of their interaction with the light fields. We proceed then by introducing the new variable $B \equiv \delta\rho(-1,0) + \delta\rho(0,1)$, which has the same physical meaning as in the RAO model, namely $B = \langle \exp(-i2k_0z) \rangle$. But in contrast to that case, the time evolution of $B$ is now governed by the equation of motion

$$\frac{d^2}{d\tau^2} B = -B - A.$$

This result shows that in contrast to the predictions of classical mechanics, where the bunching parameter $B$ has dynamics similar to a free particle driven by the probe field $A$, quantum mechanically $B$ behaves as a simple harmonic oscillator of frequency $4\omega_r$ (in original time units), and subject to that same driving force. In the linear regime, $B$ is assumed to be a small perturbation about its initial value of zero, and the forces resulting from a non-zero probe field $A$ tend to cause $B$ to increase. But this mechanism is opposed by the “restoring force” due to atomic diffraction.

FIG. 1. Comparison of the exponential growth rate as a function of pump-probe detuning $\Delta$ between the RAO (solid line) and the WAO (dashed line) models, for the case $\alpha = 10$. (a) shows the results for $T = 0$ (see Sec. III.b), (b) shows the case $T = T_R$, (c) shows $T = 10T_R$, and (d) shows $T = 100T_R$. We see that the ray atom optics model gives the correct result only in the limit $T \gg T_R$. 

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In addition to opposing any increase in the magnitude of $B$, the diffraction term also modifies its phase, which may upset any phase relation between $A$ and $B$ which might be required for the collective instability to occur.

The RAO model only makes accurate predictions at $T = 0$ in the limit $\omega_r \to 0$. Therefore, if we were to increase the mass of the atoms, thus decreasing $\omega_r$, the behavior at $T = 0$ would become more and more classical. This is because heavier atoms suffer less diffraction than lighter atoms under the influence of the light fields. We also note that the correspondence principle states that quantum mechanics should agree with classical mechanics in the limit $\hbar \to 0$, which would also cause $\omega_r$ to tend to zero. These considerations can also be derived from the statement that the RAO model is valid when $T \gg T_R$, if we note that as $\omega_r \to 0$ the recoil temperature also goes to zero.

In both the RAO and WAO models, the probe field $A$ obeys the equation
\[
\frac{d}{d\tau} A = i(\Delta A - \alpha B) .
\]
(63)

From this equation, together with Eq. (61) we find that the solutions are exponentials with exponents given by the roots of the cubic equation
\[
s^3 - i\Delta s^2 - i\alpha = 0 .
\]
(64)

This is exactly the “cold-beam” cubic equation of Bonifacio et al. However with the inclusion of atomic diffraction effects, we now see that the correct “cold-beam” cubic equation, derived from Eqs. (63) and (62), is
\[
s^3 - i\Delta s^2 + s - i(\alpha + \Delta) = 0 .
\]
(65)

These equations can also be derived from the Laplace transform method of Sec. III, with the substitution $f(k) = \delta(k)$, indicating a zero temperature momentum distribution.

From these cubic equations it is possible to determine the point of transition between the stable and the unstable regimes of the CARL. For the RAO model the collective instability occurs provided that the threshold condition
\[
\alpha > \frac{4\Delta^2}{27} .
\]
(66)

is satisfied, and above threshold the exponential growth rate is given by
\[
\Gamma_R = \frac{\sqrt{3}}{2} \left( \frac{\alpha}{4} \right)^{1/3} \left| 1 + \sqrt{C} \right|^{2/3} - \left( 1 - \sqrt{C} \right)^{2/3} \right| ,
\]
(67)

where $C = 1 - 4\Delta^3/27\alpha$. For the WAO theory the threshold condition is
\[
\alpha > \frac{2}{27} \left[ (3 + \Delta^2)^{3/2} - 9\Delta + \Delta^3 \right] .
\]
(68)
and above threshold the exponential growth rate is given by

\[
\Gamma_W = \frac{\sqrt{3}}{2} \left(\frac{\alpha}{4}\right)^{1/3} \left[ (1 + \sqrt{D})^2 + \frac{4}{27\alpha^2} (1 - \Delta^2)^2 \right]^{1/3} - \left[ (1 - \sqrt{D})^2 + \frac{4}{27\alpha^2} (1 - \Delta^2)^2 \right]^{1/3},
\]

(69)

where

\[ D = 1 - \frac{4\Delta}{3\alpha} \left(1 - \frac{\Delta^2}{9}\right) - \frac{4}{27\alpha^2} (1 - \Delta^2)^2. \]

(70)

In Figure 3(a) we examine the CARL operating regime, defined as the region in parameter space where the exponential instability occurs, at \( T = 0 \) as it would be if Ray Atom Optics were valid. We contrast this with Figure 3(b) which shows the actual CARL operating regime at \( T = 0 \), as calculated using Wave Atom Optics. From this figure we see that the operating regime of the CARL is drastically reduced at low pump intensities and/or atomic densities when the effects of atomic diffraction are included.

\[
\begin{align*}
\text{FIG. 3.} & \quad \text{The CARL operating regime (shaded region) as predicted by the RAO model (a), and the actual operating regime (b), as given by the WAO model.}
\end{align*}
\]

Figure 1(a) compares \( \Gamma_R \) with \( \Gamma_W \) for the case \( \alpha = 10 \) at \( T = 0 \), and Fig. 2(a) shows the same comparison for \( \alpha = 10^{-1} \). We see that atomic diffraction leads to the appearance of a second threshold below which the collective instability does not occur. From Fig. 2(a) we see that this second threshold may even be above \( \Delta = 0 \) for low intensities and/or densities. In fact, the threshold crosses \( \Delta = 0 \) at precisely \( \alpha = 2/3\sqrt{3} \).

Figure 2(a) shows that in the limit of weak pump intensities and/or atomic densities the peak gain for the WAO model tends to \( \Delta = 1 \), while that of the RAO model is at \( \Delta = 0 \). This result can actually be understood quite simply: The atomic center-of-mass dispersion curve tells us that the absorption of a pump photon and the emission of a probe photon by an atom initially at rest creates an energy defect of \( 4\omega_r \) due to atomic recoil. This defect can be compensated by a detuning between the pump and probe, which in dimensionless units occurs at \( \Delta = 1 \). Therefore, the fact that \( \Gamma_W \) is a sharply peaked function around \( \Delta = 1 \) is simply an expression of energy-momentum conservation. If we are to take the Ray Atom Optics model seriously at \( T = 0 \), then we must concede that we are in the limit where \( \omega_r \rightarrow 0 \), therefore, energy-momentum conservation would predict the maximum of \( \Gamma_R \) to occur at \( \Delta = 0 \). In other words, in that limit the center-of-mass dispersion curve is flat over the range of a few photon momenta.

V. DISCUSSION AND OUTLOOK

The main result of this paper is that at low temperatures the behavior of the CARL is strongly influenced by matter-wave diffraction, which tends to counteract the atomic bunching and reduces the instability range of the system. In this temperature range, the CARL presents an experimentally realizable example of dynamically coupled Schrödinger and Maxwell fields. The present theory quantizes the matter wave, but not the electromagnetic field. It will be of considerable interest to extend it to regimes where both fields need to be quantized. An analysis of the density regime where quantum degeneracy becomes important will also be a fascinating extension, in particular when two-body collisions are included. This study will allow one to investigate to which extent a Bose-Einstein condensate can be manipulated and modified in a far off-resonant CARL configuration. An intriguing possibility would be to generate in this fashion a coupled laser-"atom laser" system. The study of the coherence properties of this system will be the object of future investigations. Finally, a comparison between bosonic and fermionic CARL systems in the quantum degenerate regime should also be considered.

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APPENDIX: HEISENBERG EQUATIONS OF MOTION FOR DENSITY OPERATORS

The full equations of motion for the expectation values of the density operators are

\[
\frac{d}{dt} \rho_{gg}(k, k') = -\frac{i\hbar}{2m}(k^2 - k'^2)\rho_{gg}(k, k') \\
+ g_1^* a_1^* \rho_{eg}(k - k_1, k') + g_2 a_2^* \rho_{eg}(k - k_2, k') \\
+ g_1^* a_1 \rho_{ge}(k, k' - k_1) + g_2 a_2 \rho_{ge}(k, k' - k_2),
\]
\[ (A1) \]

\[
\frac{d}{dt} \rho_{eg}(k, k') = -\frac{i\hbar}{2m}(k^2 - k'^2 + \omega_0)\rho_{eg}(k, k') \\
+ g_1^* a_1 [\rho_{ee}(k, k' - k_1) - \rho_{gg}(k + k_1, k')] \\
+ g_2^* a_2 [\rho_{ee}(k, k' - k_2) - \rho_{gg}(k + k_2, k')],
\]
\[ (A2) \]

and

\[
\frac{d}{dt} \rho_{ee}(k, k') = -\frac{i\hbar}{2m}(k^2 - k'^2)\rho_{ee}(k, k') \\
- g_1^* a_1 \rho_{eg}(k, k' + k_1) - g_2 a_2^* \rho_{eg}(k, k' + k_2) \\
- g_1 a_1 \rho_{ge}(k + k_1, k') - g_2 a_2 \rho_{ge}(k + k_2, k').
\]
\[ (A3) \]