Second-order photonic topological insulator with corner states

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Higher-order topological insulators (HOTIs) which go beyond the description of conventional bulk-boundary correspondence, broaden the understanding of topological insulating phases. Being mainly focused on electronic materials, HOTIs have not been found in photonic systems yet. In this article, we propose a type of two-dimensional second-order photonic crystals with zero-dimensional corner states and one-dimensional boundary states for optical frequencies. All of these states are topologically non-trivial and can be understood based on the theory of topological polarization. Moreover, by tuning the easily-fabricated structure of the photonic crystals, we can realize different topological phases with unique topological boundary states straightforwardly. Our results can be generalized to higher dimensions and provides unprecedented venues for higher-order photonic topological insulators and semimetals.

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Introduction.—Topological insulators (TIs) and topological semimetals (TSMs) have been theoretically and experimentally studied due to their distinct edge states and transport properties [1, 2]. Normally, d-dimensional (dD) TIs have dD gapped bulk states and (d − 1)D gapless boundary states. Recently, the concept of higher-order topological insulator (HOTI) has been put forward to describe those topological insulators (TIs) which have lower-dimensional gapless boundary states [29, 33]. Generally speaking, a dD TI with (d − 1)D, (d − 2)D, ..., (d − n − 1)D gapped boundary states and (d − n)D gapless boundary states is called the nth-order TI. The HOTIs broaden the family of non-trivial topological insulating phases. Moreover, the HOTIs have unique boundary states which go beyond the conventional bulk-boundary correspondence and are characterized by novel topological invariants [29, 30, 32, 34].

However, it is not easy to realize these topological phases in electronic materials. One of the obstacles is that the Fermi levels of electronic materials are not always in the topologically non-trivial band gaps or at the gapless points. The band structures of photonic crystals (PCs) which have no relevance with Fermi level provide us with platforms to study various topological phases such as photonic topological insulators (PTIs) and photonic topological semimetals (PTSMs) without this obstacle [3, 28]. In terms of the HOTIs, the topological corner and hinge states in PCs can be used to design unprecedented optical devices. So far, the observations of HOTIs are only realized in mechanical metamaterials [35], electrical circuits [36, 37], and coupled optical fiber arrays [38] which are described predominantly by tight-binding models. The extension of the notion of HOTIs to PCs of which the photonic band structure is dominated by multiple Bragg scattering instead of tight-binding models is still lacking.

In this Letter, we propose a two-dimensional (2D) PTI which is the 2D photonic generalization of the Su-Schrieffer-Heeger (SSH) model [39]. Previous studies of the 2D SSH model is focused on the zero Berry curvature of the topological phases and gapless 1D edge states [40, 41]. Here, we demonstrate that there are coexisting edge and corner states when two topologically distinct PCs are placed together to form rectangular boundaries. We reveal that both the bulk polarization, described by the vector $P = (P_x, P_y)$, and the edge polarization, described by the nested Wilson-loop eigenvalues $p_x^y$ and $p_y^y$, are quantized by the mirror symmetries $M_x := x \rightarrow -x$ and $M_y := y \rightarrow -y$. While the edge states are determined by the bulk topological invariants, the emergence of the corner states are dominated by the edge polarizations. The theoretical predictions and analysis are supported by numerical simulations for all-dielectric PCs at optical frequencies. Our study provides an unprecedented avenue for photonic HOTIs and higher-order topological semimetals.

Second-order topological PCs.—We consider a 2D PC with mirror symmetries as shown in Fig. 1(a). There are four identical dielectric rods in each unit-cell which form isotropic (with $C_{1v}$ symmetry) or anisotropic (with $C_{2v}$ symmetry) PCs, depending on their positions. Origin of the coordinates is set as the center of the unit-cell throughout this Letter where we consider the transverse-magnetic (TM) photonic modes. Photonic band structures of distinct topological indices can be realized by tuning the distance between the four rods. Despite such a simple design, topological edge states and corner states [as illustrated in Fig. 1(b)] can appear in the photonic band gap (PBG) between the first and the second bands (denoted as PBG I).

For isotropic PCs, we found that the intra- and inter-unit-cell distances ($l_{\text{intra}}$ and $l_{\text{inter}}$) between the neigh-
The relative dielectric constant \( \epsilon \) and the coupling strength denoted as \( l \). The band inversion induced by changing \( l \) (topologically trivial PBG). (d) \( l = 0.25a \) (band gap closing). (e) \( l = a/2.8 \) (topologically nontrivial PBG).

The eigenmodes of a square supercell where (e)-configuration is inside a box and (c)-configuration is outside the box. There are four degenerate states localized at the four corners in the band gap.

The photonic band structure of our PC, as dominated by multiple Bragg scattering, cannot be quantitatively captured by tight-binding models. Nevertheless, the qualitative feature, particularly the topological properties, can be captured by the tight-binding model depicted in Fig. 1(a) of which the Hamiltonian is

\[
H(k) = \begin{pmatrix}
0 & h_{12} & h_{13} & 0 \\
-h_{12} & 0 & 0 & h_{24} \\
0 & 0 & h_{34} & 0 \\
0 & h_{24} & h_{34} & 0
\end{pmatrix}.
\]

Here

- \( h_{12} = t_a + t_b \exp(ik_x) \),
- \( h_{13} = t_a + t_b \exp(-ik_y) \),
- \( h_{24} = t_a + t_b \exp(-ik_y) \),
- \( h_{34} = t_a + t_b \exp(ik_x) \),

and \( k = (k_x, k_y) \).

**Bulk and edge polarizations.** The tight-binding parameters, \( t_a, t_b \), reflect the intra- and inter-unit-cell distances between the neighboring rods, respectively. It is known that the above tight-binding model has nontrivial topology as characterized by the 2D polarization \( P \) where

\[
P_i = -\frac{1}{(2\pi)^2} \int_{BZ} d^2k Tr[\hat{A}_i], \quad i = x, y
\]

where \( (\hat{A}_i)_{mn}(k) = i \langle u_m(k) | \partial_{k_i} | u_n(k) \rangle \), where \( m, n \) run over all occupied bands, \( |u_m(k)\rangle \) is the periodic Bloch function for the \( m \)th band. The 2D polarization is connected to the 2D Zak phase \( \theta_i = 2\pi P_i \) for \( i = x, y \). 2D Zak phase of the PC equals \( (\pi, \pi) \) [i.e., \( P = (\frac{1}{2}, \frac{1}{2}) \)] for \( t_a < t_b \) (i.e., \( l_{\text{intra}} > l_{\text{inter}} \)) and reveals that the PC is in topological non-trivial insulating phase, while it equals \( (0, 0) \) for \( t_a > t_b \) (i.e., \( l_{\text{intra}} < l_{\text{inter}} \)) where the PC is in the trivial insulating phase. Beyond the isotropic tight-binding model, the bulk states with nontrivial topology include \( P = (\frac{1}{2}, 0) \), \( P = (0, \frac{1}{2}) \), and \( P = (\frac{1}{4}, \frac{1}{4}) \). We will show that only the last case can lead to both topological edge and corner states, whereas the first two cases can only appear in anisotropic PCs which support nontrivial edge states but not corner states.

Now consider a box boundary between the PC with \( P = (\frac{1}{2}, \frac{1}{2}) \) and the PC with \( P = (0, 0) \) [see Fig. 2(a)]. There are boundaries perpendicular to the \( x \) and \( y \) directions which support topological edge states due to the 2D Zak phase. That is, the nontrivial Zak phase \( \theta_x = \pi \) for each \( k_y \) gives rise to the edge states on the boundaries perpendicular to the \( x \) direction for the whole edge Brillouin zone \( k_y \in [-\pi/a, \pi/a] \). Similarly, there are the edge states on the boundaries perpendicular to the \( y \) direction due to nontrivial Zak phase \( \theta_y = \pi \).

Furthermore, those 1D edge states themselves are similar to 1D photonic SSH model with nontrivial topology protected by \( M_y(M_x) \) for edges along the \( y(x) \) direction. The topology of the edge states is characterized by the edge polarizations, \( p_{y}^{x} \) and \( p_{y}^{y} \), for edges perpendicular to the \( x \) and \( y \) directions, respectively. The topological theory of polarization \( [29][32] \) connects the polarization of the edge states to the eigenvalues of the nested Wilson-loops (Mathematical details are given in the Supplemental Materials). In simple terms, the nested Wilson-loops construct the Zak phases of the edge states by projecting the bulk Bloch states into a sector which is topologically equivalent to the edge states using the nested Bloch states, \( |w_{x,\alpha}(k)\rangle = \sum_{n \in N_{PBG}} \nu_{n}^{\alpha}(k_y) |u_{n}(k)\rangle \). Here \( n \in N_{PBG} \) stands for summation over all bands below the PBG, and \( \nu_{n}^{\alpha}(k_y) \) is the Wilson-loop eigenvector for the Wilson-loop with \( k_x \) looping from \(-\pi/a\) to \( \pi/a\) at fixed \( k_y \) where \( \alpha \) labels the eigenvalue of the Wilson-loop. With the nested Bloch states, one can calculate the polarization for the edge states perpendicular to the \( x \) direction,

\[
p_{y}^{x} = -\frac{1}{(2\pi)^2} \int_{BZ} d^2k Tr[\hat{B}_y]
\]

where

\[
(\hat{B}_y)_{\alpha,\beta} = i \langle w_{x,\alpha}(k) | \partial_{k_y} | w_{x,\beta}(k) \rangle.
\]

Remarkably, for the PBG considered in this work, there is only one photonic band below the PBG. Hence, there is only one
Wilson-loop eigenvalue and the eigenvector \( \nu_1^1(k_y) \equiv 1 \). Therefore, \( p_x = P_x = \frac{1}{2} \) and similarly, \( p_y = P_y = \frac{1}{2} \). Numerical calculations of these topological indices are presented in the Supplemental Materials.

**Bulk-edge-corner correspondence.**— The topological edge states along the \( x \) and \( y \) directions meet at the corner of the box in a square supercell [see Fig. 2(a)]. The topological corner charge is determined by the edge polarizations as [Fig. 2(b)]

\[
Q_c = p_x^e + p_y^e.
\]

This quantized corner charge gives rise to a single corner state in each of the four corners [see Fig. 2(a)].

On the other hand, the emergence of the corner modes can also be understood via the tight-binding model. Particularly, when one considers the extreme case where \( t_a = 0 \) and \( t_b \neq 0 \). In this case, the four corner sites of the tight-binding model is isolated from other parts of the lattice and become zero modes in the band gap. Since the band structure of this extreme case can be adiabatically connected to the \( t_a \neq 0 \) case, they are in the same topological class and the corner states will always exist as long as \( |t_a| < |t_b| \). In terms of the topological invariants, this corresponding to \( P = (\frac{1}{2}, \frac{1}{2}) \) phase.

The above analysis is confirmed by numerical calculation of the photonic eigenmodes for a box geometry illustrated in Fig. 2(a). The spectrum of the eigenmodes is already shown in Fig. 1(b) where four degenerate corner modes are found in the PBG. The electromagnetic fields of the corner modes are strongly localized at each of the four corners [Fig. 2(a) shows a superposition of the four corner states]. These four corner states are protected by the nontrivial topology, as characterized by the bulk and edge polarizations which are quantized by the mirror symmetries \( M_x \) and \( M_y \). Therefore, in our photonic systems, there are coexisting edge and corner states protected by the band topology, which is an unique signature of HOTIs.

**Anisotropic PCs and the topological phase diagram.**— Next, we extend our discussions to anisotropic 2D PCs. It turns out that, according to our numerical study, the topological properties of the PCs as well as the bulk-edge-corner correspondence remain valid. The bulk and edge polarizations are determined by the intra- and inter-unit-cell distance between the rods along the \( x \) and \( y \) directions, \( l_{\text{intra}}^i \) and \( l_{\text{inter}}^i \) with \( i = x, y \). For instance, \( l_{\text{intra}}^x > l_{\text{inter}}^x \) leads to nontrivial topological indices \( P_x = p_x^e = \frac{1}{2} \). There are totally 4 different combinations. In fact, we can treat each combination as the combination of two independent 1D photonic SSH models. If the configurations in two directions are different from each other and are all in the topologically non-trivial phase, there are four degenerate edge states (DESs) and the degeneracy between two sets of 1D DESs are broken with each set only contains two DESs located in the opposite boundaries. We simulate this phenomenon in PC by requiring the same parameters \( a, r \) and \( \epsilon \) as previous isotropic 2D photonic SSH model, however, with a deformed lattice structure as shown in Fig. 3(a). The result is shown in Fig. 3(b) which agrees with the theoretical predictions well. We note that there is also a single corner state for each corner for anisotropic PCs with \( P = (\frac{1}{2}, 0) \).

Another interesting situation is that the anisotropic PC has topologically non-trivial configuration in one direction and topologically trivial configuration in the other direction. Without loss of generality, we assume \( x \)-direction is the topologically non-trivial direction, i.e., \( P = (\frac{1}{2}, 0) \). In this case, there are only two 1D edge states which locate at the left and right boundaries, since only the Zak phase along the \( x \) direction is nontrivial [see Fig. 3(c)]. Moreover, the polarization of the edge states is trivial, because \( p_y^e = P_y = 0 \). Hence, the corner charge is \( Q_c = 0 \) and there is no corner state for the two phases with \( P = (\frac{1}{2}, 0) \) and \( (0, \frac{1}{2}) \).

According to the above discussions, the whole phase diagram for the mirror-symmetric 2D PCs can be classified into four different phases, as depicted in Fig. 4: the topologically trivial phase with \( l_{\text{intra}}^i < l_{\text{inter}}^i \) for \( i = x \) and \( y \) of which \( P = (0, 0) \) and \( Q_c = 0 \); the two phases with nontrivial bulk topology but zero corner charge, i.e., \( P = (\frac{1}{2}, 0) \) or \( (0, \frac{1}{2}) \), and \( Q_c = 0 \) where \( l_{\text{intra}}^x < l_{\text{inter}}^x \) or \( l_{\text{intra}}^y < l_{\text{inter}}^y \); the phase with both nontrivial bulk topology and corner charge, i.e., \( P = (\frac{1}{2}, \frac{1}{2}) \) and \( Q_c = 1 \) where \( l_{\text{intra}}^i > l_{\text{inter}}^i \) for \( i = x \) and \( y \). Therefore, by simply changing the relative distances of nearest rods in anisotropic 2D photonic SSH model, we can achieve various topological phases with different 1D DESs as well as corner states.

**Discussions.**—(i). Our study predicts a simple realization of the second-order topological insulator in all-
FIG. 3: Anisotropic 2D photonic SSH model. (a.) The unit cell of the PC which has the same $a$, $r$, and $\epsilon$ as the isotropic 2D photonic SSH model but with deformed lattice structure characterized by $l_x$ and $l_y$. Here we choose perfect electric conduction (PEC) as the boundary condition. The eigenstates are solved by COMSOL. (b). $l_x = 0.37a$ and $l_y = 0.35a$. Non-trivial phase in both $x$- and $y$-directions. Two sets of DESs with different frequencies. For the DESs in the upper figure, the frequency is 112.2THz and for the DESs in the lower figure, the frequency is 114.4THz. (c). $l_x = 0.35a$ and $l_y = 0.15a$. $x$-direction in topologically non-trivial phase and $y$-direction in topologically trivial phase. There are only 1D DESs in the left and right sides. The frequency is 105.1THz.

dielectric photonic crystals (PC) with corner states. The exotic corner states can be regarded as the 0D boundary states of the 1D edge of the 2D PC. Moreover, it is topologically protected by mirror symmetries and characterized by the bulk and edge topological polarizations which are coincidentally the same in our PCs. The conventional bulk-edge correspondence is extended to bulk-edge-corner correspondence in this system.

(ii). Besides, we study the anisotropic 2D photonic SSH model and find that the 1D DESs arise due to the 1D structures existing in 2D photonic SSH model. By adjusting the distances between the nearby rods in the $x$- and $y$-directions, the emergence of the edge states and corner states can be controlled straightforwardly. These topologically protected edge and corner states may be valuable for robust waveguides, optical couplers, and lasers.

(iii). If our theory is generalized to 3D photonic SSH model, there can be second-order and third-order topological insulating phases where topological hinge states and corner states can emerge respectively. 3D second-order topological semimetals may be achieved by stacking the 2D photonic SSH model along the $z$ direction.

The 3D second-order topological semimetals have rich bulk and boundary properties which are yet to be explored [22].

Note added.— A recent paper showing 0D topological bound state in a 2D PC due to the presence of dislocation (a topological defect) has appeared [43], demonstrating another mechanism for lower-dimensional topological light-trapping.

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