Jordan domains with a rectifiable arc in their boundary

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To the memory of Professor Alain Dufresnoy.

Abstract

We show that if an open arc $J$ of the boundary of a Jordan domain $\Omega$ is rectifiable, then the derivative $\Phi'$ of the Riemann map $\Phi : D \to \Omega$ from the open unit disk $D$ onto $\Omega$ behaves as an $H^1$ function when we approach the arc $\Phi^{-1}(J')$, where $J'$ is any compact subarc of $J$.

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1 Introduction

In [6] the Reflection principle has been used in order to prove that if a conformal collar, bounded by a Jordan arc $\delta$ has some nice properties, then any other conformal collar of $\delta$ on the same side has the same nice properties. We use the same method in order to generalize a well-known theorem about rectifiable Jordan curves, [3].

Theorem 1.1. Let $\tau$ be a Jordan curve and $\Phi : D \to \Omega$ be a Riemann map from the open unit disc $D$ onto the interior $\Omega$ of $\tau$. Then 1 and 2 below are equivalent:

1) $\tau$ is rectifiable.
2) The derivative $\Phi'$ belongs to the Hardy class $H^1$. 

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The generalization we obtain is that if \( \tau \) is not rectifiable, but an open arc \( J \) of it has finite length, then the derivative \( \Phi' \) behaves as an \( H^1 \) function when we approach the compact subsets of the arc \( \Phi^{-1}(J) \subset \{ z \in \mathbb{C} : |z| = 1 \} \).

In the proof we combine the statement of Theorem 1.1 with the Reflection principle, [1].

The above suggests that the Hardy spaces \( H^p \) on the disc can be generalized to larger spaces containing exactly all holomorphic functions \( f \) on the open unit disc \( D \), such that \( \sup_{0 < r < 1} \int_a^b |f(re^{it})|^p dt < +\infty \) for some fixed \( a, b \) with \( a < b < a + 2\pi \). One can investigate what is the natural topology on that new space, if it is complete and Baire’s theorem can be applied to yield some generic results as non-extendability results, and study properties of the functions, belonging to these spaces. What can be said for their zeros? All these will be investigated in future papers.

2 prelemmaries

In order to state our main result we will need some already known results and the lemma 2.2 below.

**Definition 2.1.** Let \( 0 < p < \infty \). A function \( f(z) \) analytic in the unit disk \( |z| < 1 \) is said to be of class \( H^p \) if

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta
\]

remains bounded as \( r \to 1 \).
The functions of the $H^p$ class share some useful properties such as:

(a) If $U$ is a Jordan domain with rectifiable boundary and $\Phi : D \to U$ is a Riemann map, then $\Phi' \in H^1(D)$.

(b) Let $f \in H^p$; then
\[
\int_0^{2\pi} \log|f(re^{i\theta})|d\theta \geq \log|f(0)|
\]
and \[
\int_0^{2\pi} \log|f(re^{i\theta})|d\theta > -\infty,
\]
provided that $f \neq 0$.

(c) Let $f \in H^p$. Then $f(re^{i\theta})$ has non-tangential limits almost everywhere, on the unit circle, as $r \to 1^-$.

**Lemma 2.2.** Let $\gamma$ be a Jordan curve and $J \subseteq \gamma$ a rectifiable, open arc and $J' \subseteq J$ a compact arc. Then, $J'$ can be extended to a rectifiable Jordan curve $\gamma'$ and the interior of $\gamma'$ is a subset of the interior of $\gamma$.

**Proof.** Let $I$ be a closed interval such that $\gamma(I)$ is a Jordan curve. Let $(A,B)$ be an open interval such that $J := \gamma((A,B))$ and let $[a,b]$ be a compact subset of $(A,B)$ such that $J' = \gamma([a,b])$. There exists a $t_1$ in $(A,B)$ and $\delta > 0$ such that $A < t_1 - \delta < t_1 + \delta < a$; thus, $\gamma([t_1 - \delta, t_1 + \delta]) \cap J' = \emptyset$ and
\[
\{\gamma(t_1)\} \cap \gamma(I/(t_1 - \delta, t_1 + \delta)) = \emptyset.
\]
Therefore, there exists $\eta > 0$ such that
\[
\text{dist}(\gamma(t_1), \gamma(I/(t_1 - \delta, t_1 + \delta))) = \eta > 0
\]
since $I/(t_1 - \delta, t_1 + \delta)$ is compact and $\gamma$ is continuous.

From the Jordan theorem there exists a sequence $(z_n)_{n \in \mathbb{N}}$ in the interior of the Jordan curve $\gamma$ such that $z_n \to \gamma(t_1)$. Therefore, there exists a $z_0$ in the interior of $\gamma$ and in the disc $B(\gamma(t_1), \eta/100)$ with center $\gamma(t_1)$ and radius $\eta/100$ and there also exists a $t'_1$ in $I$ : $|z_0 - \gamma(t'_1)| = \min(\text{dist}(z_0, \gamma(I)))$.

We claim that, $\gamma(t'_1) \in \gamma([t_1 - \delta, t_1 + \delta])$.

Now we will prove that claim.

Let us suppose that $\gamma(t_1) \notin \gamma([t_1 - \delta, t_1 + \delta])$ to arrive to a contradiction. Then we have $|z_0 - \gamma(t'_1)| < \eta/100$ and $|\gamma(t_1) - z_0| < \eta/100$.

Therefore, $|\gamma(t'_1) - \gamma(t_1)| < 2\eta/100 < \eta$ which contradicts the fact that
\[ \text{dist}(\gamma(t_1), \gamma(I/(t_1 - \delta, t_1 + \delta))) = \eta > 0. \]

Thus, \( t'_1 \in [t_1 - \delta, t_1 + \delta] \) and \([z_0, \gamma(t'_1)] \cap \gamma(I) = \{ \gamma(t'_1) \}\). Therefore, there exists an open segment inside the interior of \( \gamma \), which joins \( z_0 \) with \( \gamma(t'_1) \). We repeat the procedure for \( b < t_1 - \delta < t_1 + \delta < B \) and will find \( \gamma(t'_2) \) and \( z_1 \) in the interior of \( G \) of \( \gamma \), such that the open segment \( (z_1, \gamma(t'_2)) \) is included in \( G \).

Therefore, there exists a polygonal line \( W \) that connects \( z_1 \) and \( z_0 \) in \( G \). It can easily be proven that this polygonal line can be chosen to be simple. The Jordan curve

\[ \gamma[t'_1, t'_2] \cup [\gamma(t'_1), z_0] \cup W \cup [z_1, \gamma(t'_2)] \]

has the desired properties.

This completes the proof of the lemma.

\[ \square \]

3 Main Result

According to a well known theorem of Osgood - Caratheodory, \([5]\), every Riemann map, from the open unit disc to the interior of the Jordan curve, extends to a homeomorphism between the closed unit disc and the closure of the Jordan domain. Our main result is the following.

**Theorem 3.1.** Let \( \gamma \) be a Jordan curve and \( J \subseteq \gamma \) a rectifiable open arc. Let also \( J' \subseteq J \) be a compact arc. Let \( G \) be the interior of \( \gamma \). Let \( \Phi : D \to G \) be a conformal mapping from the open unit disk \( D \) onto \( G \) and let \( J' = \{ \Phi(e^{it}) : a \leq t \leq b \} \). Then

\[ \int_a^b |\Phi'(r_1 e^{it}) - \Phi'(r_2 e^{it})| dt \to 0, \]

as \( r_1, r_2 \to 1^- \).

**Proof.** According to Lemma 2.2 the compact arc \( J' \) can be extended to a rectifiable Jordan curve \( \gamma' \) defining a Jordan domain \( G' \subset G \).

Let \( f : D \to G' \) be a Riemann map. Thus, \( f' \) is of class \( H^1 \) on \( D \). We consider the function \( h : D \to D \) where \( h = \Phi^{-1} \circ f \), maps the arc \( f^{-1}(J') \subseteq \mathbb{T} \) onto the arc \( \{ e^{it} : a \leq t \leq b \} \), where \( \mathbb{T} \) is the unit circle. According to the Reflection Principle the function \( h \) is injective and holomorphic on a compact neighbourhood \( \mathbb{V} \) of the compact arc \( f^{-1}(J') \). Therefore, on \( \mathbb{V} \) the derivative \( h' \) satisfy \( 0 < \delta < |h'(z)| < M < +\infty \) and \( h \) (and all its derivatives) are uniformly continuous. We have \( \Phi = f \circ h^{-1} = f \circ g \), where \( g = h^{-1} \) maps a
compact neighbourhood $\overline{W}$ of $\{e^{it} : a \leq t \leq b\}$ biholomorphically on $V$ and $0 < \delta < \|g'(z)\| < \overline{M} < +\infty$ on $\overline{W}$ and $g$ (as well as all its derivatives) are uniformly continuous.

Therefore, $\Phi' = f' \circ g \cdot g'$.

There exists $r_0 < 1$ so that for every $t \in [a, b]$ and every $r \in [r_0, 1]$ it holds $re^{it} \in \overline{W}$. Let $r_1, r_2 \in [r_0, 1]$. Then

$$|\Phi'(re^{it}) - \Phi'(2e^{it})| = |f'(g(re^{it})) \cdot g'(re^{it}) - f'(g(2e^{it}))g'(2e^{it})|$$

$$= |f'(g(re^{it}))g'(re^{it}) - f'(g(2e^{it}))(g'(re^{it}) + f'(g(2e^{it}))g'(re^{it}) - f'(g(2e^{it}))g'(2e^{it})|$$

$$\leq |f'(g(re^{it})) - f'(g(2e^{it}))||g'(re^{it})| + |f'(g(2e^{it}))||g'(re^{it}) - g'(2e^{it})|.$$
Convergence theorem.
Let \( u \) denote the non-tangential maximal function \( u(t) = \sup \{|f'(z)| : z \in \Gamma_{t/2}, |g(e^{it}) - z| < 1/2\} \). Since \( f' \) belongs to the Hardy class \( H^1 \), according to \( 3 \), it follows that \( u \) is integrable on \([a, a + 2\pi] \supset [a, b] \). We also have \( |f'(g(r_1 e^{it})) - f'(g(r_2 e^{it}))| \leq 2u(t) \). Therefore, \( \lim_{r_1, r_2 \to 1^-} I(r_1, r_2) = 0 \). Now we prove the claim.

We have \( g(e^{i\theta}) = e^{iw(\theta)}, w(\theta) \in \mathbb{R} \). In order to prove that \( g(r e^{i\theta}) \in \Gamma_{\theta, \pi/2} \) it suffices to prove that \( |Arg[1 - g(r e^{i\theta})/g(e^{i\theta})]| < \pi/4 \). But

\[
1 - \frac{g(re^{i\theta})}{g(e^{i\theta})} = \frac{g(e^{i\theta}) - g(re^{i\theta})}{g(e^{i\theta})} = \int_{re^{i\theta}}^{e^{i\theta}} \frac{g'(y)}{g(e^{i\theta})} dy = \int_r^1 \frac{g'(te^{i\theta})e^{i\theta}}{g(e^{i\theta})} dt
\]

Since \( g(e^{i\theta}) = e^{iw(\theta)}, w(\theta) \in \mathbb{R} \) it follows that \( \frac{d}{d\theta} g(e^{i\theta}) = g'(e^{i\theta})ie^{i\theta} = e^{iw(\theta)}iw'(\theta) = g(e^{i\theta})iw'(\theta) \). Thus,

\[
\frac{g'(e^{i\theta})e^{i\theta}}{g(e^{i\theta})} = w'(\theta) \in \mathbb{R} - \{0\}
\]

By continuity of \( w' \) with respect to \( \theta \), we have \( w'(\theta) \in [c, k] \), for every \( \theta \in [a, b] \) or \( w'(\theta) \in [-k, -c] \) for every \( \theta \in [a, b] \), where \( 0 < c < k < +\infty \). The later case is excluded because of the following reason: the function \( g \) is a conformal equivalence between two Jordan domains \( G' \) and \( G'' \) included in \( D \) and the boundary of \( G' \) contains the arc \( \{e^{i\theta} : t \in [a, b]\} \) and \( g(e^{i\theta}) = e^{iw(\theta)}, w(\theta) \in \mathbb{R} \) for all \( \theta \in [a, b] \). Let \( z_0 \in G' \); then \( g(z_0) \in G'' \subset D \) and according to the argument principle \( Ind(g|_{G''}, g(z_0)) = 1 \). If \( w'(\theta) < 0 \) then, the homeomorphism \( g|_{G''} : \partial G' \to \partial G'' \) turns in such a sense so we should have \( Ind(g|_{G''}, g(z_0)) = -1 \neq 1 \) impossible. Therefore, \( w'(\theta) \in [c, k] \) for every \( \theta \in [a, b] \) with \( 0 < c < k < +\infty \). Thus,

\[
Arg[1 - \frac{g(re^{i\theta})}{g(e^{i\theta})}] = Arg \int_r^1 \frac{g'(te^{i\theta})e^{i\theta}}{g(e^{i\theta})} dt = Arg \frac{1}{1-r} \int_r^1 \frac{g'(te^{i\theta})e^{i\theta}}{g(e^{i\theta})} dt
\]

But \( \lim_{r \to 1^-} \frac{g'(re^{i\theta})e^{i\theta}}{g(e^{i\theta})} = \frac{g'(e^{i\theta})e^{i\theta}}{g(e^{i\theta})} = w'(\theta) \in [c, k] \) for \( 0 < c < k < +\infty \) and the limit is uniform for \( \theta \in [a, b] \). Thus, there exists \( \delta \in [r_0, 1) \) so
that for every \( r \in [\delta, 1) \) the quantity \( \frac{g'(re^{i\theta})e^{i\theta}}{g(e^{i\theta})} \) belongs to the convex angle \( \{x+iy : 0 < x, |y| \leq x\} \) which has vertex 0 and opening \( \pi/2 \) and is symmetric to the positive x-axis. Its average \( \frac{1}{1-r} \int_r^1 \frac{g'(re^{i\theta})e^{i\theta}}{g(e^{i\theta})} \) \( dr \) will belong to the same convex angle; therefore,

\[
|\text{Arg}[1 - \frac{g(re^{i\theta})}{g(e^{i\theta})}]| = |\text{Arg}\frac{1}{1-r} \int_r^1 \frac{g'(re^{i\theta})e^{i\theta}}{g(e^{i\theta})} dr| < \pi/4
\]

and the claim is verified. This completes the proof.

\[ \square \]

**Corollary 3.1.1.** For the conformal mapping \( \Phi : D \to G \) in the theorem 3.1 it holds that:

1. \( \int_a^b |\Phi'(re^{it})|dt \) is bounded for \( 0 < r < 1 \).

2. \( \Phi' \) has non-tangential limits almost everywhere on \( \{e^{it} : a < t < b\} \) which are denoted as \( \Phi'(e^{it}) \) and \( \Phi'(e^{it}) \neq 0 \) almost everywhere.

3. \( \Phi'(e^{it})|_{(a,b)} \) is integrable and \( \int_a^b |\Phi'(re^{it}) - \Phi'(e^{it})|dt \to 0 \) as \( r \to 1^- \).

4. Length of \( J' = \int_a^b |\Phi'(e^{it})|dt = \lim_{r \to 1^-} \int_a^b |\Phi'(re^{it})|dt = \lim_{r \to 1^-} \text{length of } \Phi\{re^{iu} : a \leq u \leq b\} \).

**Proof.** 1. From Theorem 3.1, the family \( t \to \Phi'(re^{it}) \) is Cauchy \( L'(a,b) \), as \( r \to 1^- \). Therefore, there exists the limit \( g \) in \( L'(a,b) \) such that

\[
\int_a^b |\Phi'(re^{it}) - g(e^{it})|dt \to 0.
\]

We have

\[
\int_a^b |\Phi'(re^{it}) - g(e^{it})|dt \geq \int_a^b |\Phi'(re^{it})|dt - \int_a^b |g(e^{it})|dt.
\]

Therefore, for every \( \epsilon > 0 \) there exists a \( r_0 > 0 \), such that for every \( r > r_0 \) it holds that

\[
|\int_a^b |\Phi'(re^{it})|dt - \int_a^b |g(e^{it})|dt| < \epsilon.
\]
Since $\int_a^b |g(e^{it})|dt < +\infty$, it follows that $\int_a^b |\Phi'(re^{it})|dt$ is bounded as $r \to 1$.
This completes the proof of 1.

2. We use the notation of Theorem 1. Then $\Phi' = f'(h)h'$, where $h : D \to D$. Since $f' \in H^1(D)$ there exists the non-tangential limit a.e. on $\partial D$ and therefore on $J'$.
On the other hand, the function $h$ is holomorphic on $D$ and can be extended holomorphically on a neighbourhood of $J'$. Therefore, $h$ and $h'$ have non-tangential limits a.e. on $\{e^{i\theta} : a < \theta < b\}$. Thus, $\Phi' = f'(h)h'$ has non-tangential limits a.e. on $\{e^{i\theta}, a < \theta < b\}$.

Now, $f'$ is in $H^1$ and $f' \neq 0$. Thus, $f'(h(e^{i\theta})) \neq 0$ a.e. on $(a,b)$. Also $h'(e^{it}) \neq 0$ for all $t \in (a,b)$ because $h$ is injective and holomorphic on a compact neighbourhood of $J'$. Thus, $\Phi'(e^{it}) \neq 0$ almost everywhere on $(a,b)$. This completes the proof of 2.

3. Since $\Phi'(re^{it})$ is Cauchy in $L_1$ as $r \to 1^-$, there exists $g(e^{it}) := \lim_{r \to 1^-} \Phi'(re^{it})$ in $L_1$. There exists a sequence $r_n$, such that $\Phi'(r_ne^{it}) \to g(e^{it})$ a.e. But $\Phi'(re^{it}) \to \Phi'(e^{it})$ a.e. on $\{e^{it}, a < t < b\}$ non-tangentially. Therefore $g = \Phi'(e^{it})$ a.e.
Since $g \in L_1$ and $g = \Phi'(e^{it})$ a.e., it follows that $\Phi' \in L_1$.
This completes the proof of 3.

4. Let $A, B$ be such that $J' = \{f(e^{it}) : A \leq t \leq B\}$. Since $f' \in H^1(D)$ we have length of $J' = \int_A^B |f'(e^{it})|dt$, [3]. But $f = \Phi \circ h$, therefore $f' = \Phi' \circ h \cdot h'$. Thus, length $J' = \int_A^B |\Phi'(h(e^{it}))|h'(e^{it})|dt$.
We do the diffeomorphic change of variable $h(e^{it}) = e^{iu}$ that is
\[ e^{it} = h^{-1}(e^{iu}) \]
which implies
\[ ie^{it}dt = (h^{-1})'(ie^{iu}) \cdot ie^{iu}du \]
and
\[ dt = |(h^{-1})'(e^{iu})|du = \frac{1}{|h'(e^{it})|}du. \]
According to [4], for this change of variable for integrable functions we find length $J' = \int_a^b |\Phi'(e^{iu})|du$. 

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Using part 3, we take
\[ \int_{a}^{b} |\Phi'(e^{iu})|du = \]
\[ = \lim_{r \to 1^-} \int_{a}^{b} |\Phi'(re^{iu})|du = \]
\[ = \lim_{r \to 1^-} \text{length}\{\Phi(re^{iu}) : a \leq u \leq b\}. \]
The result easily follows. This completes the proof of part 4 and of the whole Corollary.

However, we will give a second alternative proof for part 4. Since \( \Phi(e^{it}) \) is of bounded variation on \([a, b]\), the arc measure on \( J' \) is \( |\Phi'(e^{iu})|du + dv \), where \( dv \) is a singular non negative measure; it follows that, \(^{[4]}\),
\[ \text{length} J' \geq \int_{a}^{b} |\Phi'(e^{iu})|du. \]
We notice that, combining the relation \( \Phi' = f' \circ h \cdot h' \) with the fact that \( f' \in H^1 \), we easily conclude that the non-tangential limits of \( \Phi' \) on \( \{ e^{iu} : a \leq u \leq b \} \) coincide almost everywhere with the derivative \( \frac{d\Phi(e^{iu})}{de^{iu}} \) computed for the restriction of \( \Phi \) on \( \{ e^{iu} : a \leq u \leq b \} \), which exists almost everywhere on \( \{ e^{iu} : a \leq u \leq b \} \), because \( J' \) is rectifiable and \( \Phi(e^{iu}) \) is of bounded variation on \([a, b]\). According to part 3, we have
\[ \int_{a}^{b} |\Phi'(e^{iu})|du = \]
\[ = \lim_{r \to 1^-} \int_{a}^{b} |\Phi'(re^{iu})|du = \]
\[ = \lim_{r \to 1^-} \text{length}\{\Phi(re^{iu}) : a \leq u \leq b\} \]
Since \( \Phi(re^{iu}) \to \Phi(e^{iu}) \) as \( r \to 1^- \) we have
\[ \text{length} J' = \text{length}\{\Phi(e^{iu}) : a \leq u \leq b\} \leq \]
\[ \leq \lim \inf_{r \to 1^-} \text{length}\{\Phi(e^{iu}) : a \leq u \leq b\} \]
(see Prop. 4.1 below). Now the result easily follows. The proof is complete.
4 Further results

We have seen that \( \lim_{r \to 1^-} \text{length } \Phi \{ e^{it} : a \leq t \leq b \} = \text{length of } \Phi \{ e^{it} : a \leq t \leq b \} \) provided that for some \( a', b' : a' < a < b < b' \) the length of \( \Phi \{ e^{it} : a' \leq t \leq b' \} \) is finite. Composing \( \Phi \) with an automorphism of the open unit disc \( w(z) = \frac{z - \gamma}{1 - \overline{\gamma}z}, |c| = 1, |\gamma| < 1 \) we can obtain similar results of other families of curves converging to \( \Phi \{ e^{it} : a \leq t \leq b \} \). We will not insist towards this direction. For any arc \( \{ e^{it} : A \leq t \leq B \}, A < B < A + 2\pi \), we have the following:

**Proposition 4.1.** Under the above assumptions and notation we have the following inequality.

\[
\text{length of } \Phi \{ e^{it} : A \leq t \leq B \} \leq \liminf_{r \to 1^-} \text{ of length } \Phi \{ re^{it} : A \leq t \leq B \}.
\]

*Proof.* Let \( r_n < 1, r_n \to 1 \) and \( M \) be such that length of \( \Phi \{ r_n e^{it} : A \leq t \leq B \} \leq M \) for all \( n \). Then we will show that length of \( \Phi \{ e^{it} : A \leq t \leq B \} \leq M \). It suffices to prove that

\[
\sum_{y=0}^{N-1} |\Phi(e^{iy+1}) - \Phi(e^{iy})| \leq M
\]

for any partition \( t_0 = A < t_1 < \ldots < t_{N-1} < t_N = B \).

But \( \sum_{y=0}^{N-1} |\Phi(r_n e^{iy+1}) - \Phi(r_n e^{iy})| \leq \text{length } \Phi \{ r_n e^{it} : A \leq t \leq B \} \leq M \). Since \( \Phi(r_n e^{it}) \to \Phi(e^{it}), n \to +\infty \), passing to the limit we obtain \( \sum_{y=0}^{N-1} |\Phi(e^{iy+1}) - \Phi(e^{iy})| \leq M \). The result easily follows. \( \square \)

**Corollary 4.1.1.** Under the above assumptions and notations we have the following:

1. If length of \( \Phi(\{ e^{it} : A \leq t \leq B \}) = +\infty \), then length of \( \Phi(\{ e^{it} : A \leq t \leq B \}) = \lim_{r \to 1^-} \text{length of } \Phi(\{ re^{it} : A \leq t \leq B \}) \).

2. If there exists \( A', B', A' < A < B < B' \) such that length of \( \Phi \{ e^{it} : A' \leq t \leq B' \} < +\infty \), then length of \( \Phi \{ e^{it} : A \leq t \leq B \} = \lim_{r \to 1^-} \text{length } \Phi \{ re^{it} : A \leq t \leq B \} \).
The proof of the corollary 4.1.1 follows easily from the previous results.

We believe that it is possible to have:

\[
\text{length of } \Phi \{e^{it} : A \leq t \leq B \} < +\infty
\]

and

\[
\text{length of } \Phi \{e^{it} : A \leq t \leq B \} < \liminf_{r \to 1^-} \text{length of } \Phi \{re^{it} : A \leq t \leq B \} < \limsup_{r \to 1^-} \text{length of } \Phi \{re^{it} : A \leq t \leq B \}
\]

but we do not have an example. A candidate for such an example is the Jordan domain

\[
\Omega = \{x+iy : -5 < y < x \cos(1/x); 0 < x < 1 \} \cup \{x+iy : -5 < y < 0, -1 < x \leq 0 \}.
\]

Although \( \int_0^{2\pi} |f'(re^{it})|dt \) is increasing with respect to \( r \in (0, 1) \), we believe that this is no longer true for \( \int_a^b |\Phi'(re^{it})|dt \) and a candidate for a counter example is any convex polygonal domain \( \Omega \).

Finally, we have the following:

**Theorem 4.2.** Let \( \Omega \) be a Jordan domain and \( \Phi : D \to \Omega \) a Riemann map from the open unit disc \( D \) onto \( \Omega \). Let \( A < B < A + 2\pi \), then the following are equivalent.

1. For every \( a, b \) such that \( A < a < b < B \) the arc \( \{ \Phi(e^{it}) : a \leq t \leq b \} \) is rectifiable.

2. For every \( a, b \) such that \( A < a < b < B \) we have

\[
\sup_{0 < r < 1} \int_a^b |\Phi'(re^{it})|dt = M_{a,b} < \infty
\]

3. For every \( a, b \) such that \( A < a < b < B \) there exist curves \( \gamma_r : [a, b] \to \mathbb{C}, 0 < r < 1 \) such that \( \lim_{r \to 1^-} \gamma_r(t) = \Phi(e^{it}) \) for all \( t \in [a, b] \) and such
that the lengths of $\gamma_r$ are uniformly bounded as $r \to 1^-$, by a constant $C_{a,b} < \infty$.

Proof. We have already seen that 1. $\Rightarrow$ 2. In order to see that 2. $\Rightarrow$ 3. it suffices to set $\gamma_r(t) = \Phi(re^{it})$. Finally, to prove that 3. $\Rightarrow$ 1., it suffices to prove that

$$\sum_{j=0}^{n-1} |\Phi(e^{it_{j+1}}) - \Phi(e^{it_j})| \leq C_{a,b}$$

for all partitions $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$. But

$$\sum_{j=0}^{n-1} |\gamma_r(t_{j+1}) - \gamma_r(t_j)| \leq \leq \text{length} \gamma_r \leq C_{a,b}$$

and $\lim_{r \to 1^-} \sum_{j=0}^{n-1} |\gamma_r(t_{j+1}) - \gamma_r(t_j)| = \sum_{j=0}^{n-1} |\Phi(e^{it_{j+1}}) - \Phi(e^{it_j})|$ and the proof is completed.

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