ON THE CHERN NUMBERS AND THE HILBERT POLYNOMIAL OF AN ALMOST COMPLEX MANIFOLD WITH A CIRCLE ACTION

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Abstract. Let \((M, J)\) be a compact almost complex manifold of dimension \(2n\) endowed with a \(J\)-preserving circle action with isolated fixed points, and assume that the first Chern class \(c_1\) is not a torsion element. In this note we analyse the geography problem for such manifolds, deriving equations in the Chern numbers that depend on \(k_0\), the index of \((M, J)\), namely the ‘largest integer’ dividing \(c_1\) modulo torsion. This analysis is carried out by studying the symmetries and zeros of the Hilbert polynomial associated to \((M, J)\).

We prove several formulas for the Chern numbers of \((M, J)\), in particular for \(c_n^1[M]\) and \(c_1^{-2}c_2[M]\). Moreover we prove that for \(\dim(M) \leq 8\) and \(k_0 \geq n\) all the Chern numbers can be expressed as linear combinations of integers \(N_j\) defined by the action, which correspond to the Betti numbers of \(M\) if the manifold is symplectic and the action Hamiltonian.

We apply these results to the symplectic category, giving necessary and sufficient conditions for the action to be non-Hamiltonian; the question of whether such actions exist is a long standing problem in equivariant symplectic geometry. Finally, we give an upper bound for the minimal Chern number of a symplectic manifold supporting a Hamiltonian circle action, and prove the equivariant symplectic analogue of the Kobayashi-Ochiai theorem in dimension 4.

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1. Introduction

Let \((M, J)\) be a compact almost complex manifold of dimension \(2n\) and let \(c = \sum_{i=1}^{n} c_i\) be the total Chern class of the tangent bundle of \(M\). To each partition of \(n\) one can associate an integer, called Chern number, given by \(\int_M c_{i_1} \cdots c_{i_k}\), where \(i_1 + \cdots + i_k = n\). The problem of determining which lists of integers can arise as the Chern numbers of a compact almost complex manifold \((M, J)\) of a given dimension (also known as the geography problem) has been investigated in different settings. Without additional assumptions on \((M, J)\), a theorem of Milnor (cf. [27]) implies that it is necessary and sufficient for these integers to satisfy a certain set of congruences depending on the dimension of \(M\). In [13] Geiges proved that for \(n \geq 2\) no further restrictions arise by requiring \(M\) to be connected. When \((M, \omega)\) is a compact symplectic manifold then the same conclusion follows in dimension 6 by the work of Halic [21] and in dimension 8 by that of

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\(^1\)Here the Chern classes are associated to an almost complex structure \(J\) compatible with \(\omega\)
Pasquotto [32]; the symplectic geography problem in dimension 4 is still open (see for example [49] for recent developments, and the references therein).

In this note, we are interested in exploring the geography problem for almost complex and symplectic manifolds endowed with a circle action that is compatible with the respective structures (i.e. that preserves them), where the action has isolated fixed points. (All actions are understood to be effective, unless otherwise stated.)

Henceforth, the triple $(M, J, S^1)$ will denote a compact, connected almost complex manifold acted on by a circle $S^1$ that preserves $J$, with nonempty, discrete fixed point set $M^{S^1}$, and will be referred to as an $S^1$-space.

Both in the almost complex and symplectic category, there are long standing problems concerning circle actions with isolated fixed points (see Conjectures 1.2 and 1.3).

Let $\chi(M)$ denote the Euler characteristic of $(M, J, S^1)$. Since $M^{S^1}$ is discrete, the following equalities hold\(^2\)

$$\int_M c_n = \chi(M) = |M^{S^1}|. \quad (1.1)$$

As a consequence of (1.1) we have that $\int_M c_n > 0$, which is not true in general for an almost complex manifold $(M, J)$. This easy example shows that the presence of a circle action with isolated fixed points imposes further restrictions on the Chern numbers of $(M, J)$, and the following question naturally arises:

**Question 1.1** What are all the possible values of the Chern numbers of $(M, J, S^1)$? Do they satisfy quantitative inequalities?

The last question generalises what Kosniowski [32] conjectured in 1979, namely that the number of fixed points, and hence $\int_M c_n$, grows linearly with $n$. Moreover he predicted this linear function to be $\frac{n}{2}$, thus leading to the following

**Conjecture 1.2** (Kosniowski). Let $(M, J)$ be an almost complex manifold of dimension $2n$ supporting a compatible circle action with nonempty, discrete fixed point set. Then

$$\int_M c_n \geq \left\lceil \frac{n}{2} \right\rceil$$

Even if much progress has been done to prove Kosniowski’s conjecture (see [24], and more recently [36, 37, 13, 10, 14]), a complete answer is still missing. This shows that the geography problem for an $S^1$-space $(M, J, S^1)$ is much harder.

One of the goals of this note is to derive relations among the Chern numbers of $(M, J, S^1)$ by analysing the (zeros of the) *Hilbert polynomial* of $(M, J)$ which, in turn, depends on two integers, $k_0$ and $N_0$, defined as follows. Let $c_1 \in H^2(M; \mathbb{Z})$ be the first Chern class of the tangent bundle, and assume it is not torsion in $H^2(M; \mathbb{Z})$. Consider the lattice $L$ obtained by quotienting $H^2(M; \mathbb{Z})$ by its torsion subgroup $\text{Tor}(H^2(M; \mathbb{Z}))$. We define the *index* of $(M, J)$ to be the positive integer $k_0$ such that the projection of $c_1/k_0$ to $L$ is primitive in $L$; we denote such primitive element by $\eta_0$. In other words, the index is the largest positive integer dividing the first Chern class, modulo torsion. When $M$ is simply connected and symplectic, the index coincides with the *minimal Chern number* (see Remark 1.3).

For each $p \in M^{S^1}$, consider the *weights of the $S^1$-action* at $p$ (see Section 2); then $N_0$ is the number of fixed points with 0 negative weights. More in general, $N_j$ is defined to be the number of fixed points with exactly $j$ negative weights, for each $j = 0, \ldots, n$.

Let $L_0 \to M$ be a line bundle such that the projection of the first Chern class $c_1(L_0)$ onto $L$ is $\eta_0$. Although this line bundle is not unique, the $K$-theoretic index (c.f. Sect. 2.1) of the

\(^2\)The second equality follows, for example, from applying the Atiyah-Bott-Berline-Vergne Localization theorem to the $S^1$-equivariant extension of $c_n$. 


tensor power $L_k^c$, denoted by $\text{Ind}(L_k^c)$, is well defined for each $k \in \mathbb{Z}$ (see Lemma 4.3). Let $H(z)$ be the polynomial in $\mathbb{R}[z]$ such that $H(k) = \text{Ind}(L_k^c)$ for every $k \in \mathbb{Z}$, referred to as the Hilbert polynomial of $(M, J)$ (see Section 3). If the manifold is symplectic with symplectic form $\omega$, and if the $S^1$-action extends to a toric action with $\eta_0 = [\omega]$ (hence the toric manifold is monotone), it is known that such polynomial coincides with the Ehrhart polynomial of the moment polytope (see Sect. 4.2), the study of which represents a key research topic in combinatorics. In this note we use tools originating from equivariant $K$-theory to analyse properties (in particular the zeros and the ‘symmetries’) of the Hilbert polynomial $H(z)$ of an $S^1$-space $(M, J, S^1)$. Since, by the Atiyah-Singer Index Theorem, the coefficients of $H(z)$ can be expressed as combinations of Chern numbers (see (1.3)), as a byproduct we obtain equations in the Chern numbers of $(M, J, S^1)$ which depend on $k_0$ and $N_0$. In some cases, such equations impose very strict restrictions on the Chern numbers of $(M, J, S^1)$, in particular on $f_M c_1^3$ and $f_M c_1^{n-2}c_2$; these results are summarised in Sect. 4.1.

The dependence of the Chern numbers on the index $k_0$ has already been remarked in different contexts, see for example [31], [24] and more recently [15]. In [24] Hattori assumes that $\chi(M) = n + 1$, and that the manifold possesses a quasi-ample line bundle (see Remark 4.13) to prove that if $k_0 = n + 1$ then $(M, J, S^1)$ shares many features with $\mathbb{C}P^n$, in particular it has the same Chern classes and Chern numbers as $\mathbb{C}P^n$ (see [24] Theorem 5.7, Corollaries 5.8 and 5.9). Similar investigations were carried out in the case in which $k_0 = n$ (see [24] Theorem 6.1 and Corollaries 6.2 and 6.3), where the role of $\mathbb{C}P^n$ is played by the hyperquadric $Q$, which can be identified with the Grassmannian of oriented 2-planes $Gr_2^+(\mathbb{R}^n)$. In [15], Godinho and the author proved that to each triple $(M, J, S^1)$ one can associate a family of labeled graphs, where any of these graphs can be used for computing some of the equivariant topological invariants of $(M, J, S^1)$. They prove that the classification of such labeled graphs strongly depends on $k_0$, as can be seen in [15] Sect. 5. For recent developments on Hamiltonian circle actions with $\chi(M) = n + 1$ and $k_0 = n$ or $n+1$ see also the work of Li [33].

In addition to $k_0$, in this note we also analyse the dependence of the Chern numbers on $N_0$, which is very useful for deriving results in the symplectic category. We recall that for a symplectic manifold $(M, \omega)$ with a symplectic $S^1$-action, the 1-form $\iota_\# \omega$ is necessarily closed; here $\iota_\#$ denotes the vector field generated by the circle action. Thus there is a distinction between two cases:

(H) the 1-form $\iota_\# \omega$ is exact, in which case the action is said to be Hamiltonian;

(nH) the 1-form $\iota_\# \omega$ is not exact, in which case the action is said to be non-Hamiltonian.

In the first case, if $\psi : M \rightarrow \mathbb{R}$ is an $S^1$-invariant function satisfying $\iota_\# \omega = -d\psi$, then $\psi$ is called a moment map for the $S^1$-action. In the second case, it is still not known whether there exist symplectic but non-Hamiltonian circle actions on compact symplectic manifolds with discrete fixed point set; in fact the following conjecture is still open

**Conjecture 1.3** (Frankel-McDuff). Every $S^1$-action on a compact symplectic manifold with isolated fixed points is Hamiltonian.

The topological invariants of a manifold supporting a non-Hamiltonian $S^1$-action with isolated fixed points are still somewhat mysterious. It is known however that when $n = 2$ such actions cannot exist, as proved by McDuff in [35]. (In the same paper she also proves the existence of a six-dimensional compact symplectic manifold with a non-Hamiltonian action, but the fixed point set is not discrete.) In [14] Godinho, Pelayo and the author derive lower bounds and divisibility conditions on the Euler characteristic of a class of manifolds which cannot support any Hamiltonian circle action, namely symplectic Calabi-Yau manifolds, for which $c_1 = 0$. In this note we make use of the lemma below to give a characterisation of symplectic manifolds with $c_1 \neq 0$ not supporting any Hamiltonian $S^1$-action with discrete fixed point set in terms of the vanishing of certain combinations of Chern numbers (see Corollaries 4.14 (ii') and 4.12).

\footnote{Here the role of $k_0$ is played by the constant $C$.}
Lemma 1.4 ([3]). Let $(M, \omega)$ be a compact connected symplectic manifold endowed with a symplectic $S^1$-action with isolated fixed points. Then $N_0$ can be either 0 or 1, and is 1 exactly if the action is Hamiltonian.

Finally, as a byproduct of our investigation, we prove that in low dimensions, and for big values of $k_0$, all the Chern numbers of an $S^1$-space $(M, J, S^1)$ can be expressed as linear combinations of the $N_j$, for $j = 0, \ldots , n$. The dependence of the Chern numbers on these integers is suggested by the fact that both $\int_M c_n$ and $\int_M c_1 c_{n-1}$ linearly depend on them (see (6.1) and Theorem 6.1). Note that, when the manifold is symplectic and the action is Hamiltonian, by the equivariant perfection of the moment map $\psi$ (cf. [30]) it is possible to prove that

$$b_{2j}(M) = N_j \quad \text{for every} \quad j = 0, \ldots , n,$$

where $b_{2j}(M)$ denotes the $2j$-th Betti number of $M$. Hence we obtain that when the action is Hamiltonian, for low dimensions of $M$ and large values of $k_0$ all the Chern numbers can be expressed as linear combinations of the Betti numbers of $M$.

In particular these results imply that for a compact symplectic manifold of dimension 4 endowed with a symplectic circle action, an ‘equivariant symplectic analogue’ of the Kobayashi-Ochiai Theorem [31] holds, namely: $k_0 = 3$ if and only if the manifold is equivariantly symplectomorphic to $\mathbb{C}P^2$, and $k_0 = 2$ if and only if it is equivariantly symplectomorphic to a Hirzebruch surface, with suitable actions and symplectic forms (see Proposition 6.2 (a) and (b)).

The next subsection contains a detailed description of the content and the results of the paper.

1.1. Structure of the article and summary of results. In Section 2 we introduce some notation and recall standard facts about equivariant cohomology, $K$-theory, Chern classes and line bundles. In Proposition 2.3 we study some ‘symmetry’ of the index homomorphism in equivariant cohomology and ordinary $K$-theory, which will be later applied to derive properties of the Hilbert polynomial.

The main result of Section 3 is Theorem 3.3. Here we give sufficient conditions for an equivariant line bundle $L_{S^1}$ to be rigid, namely we give conditions on its equivariant first Chern class that ensure its equivariant index to be $S^1$-invariant, hence an element of $Z \subset Z[t,t^{-1}]$; in particular we analyse when such index is zero. In the rest of the section we derive consequences of Theorem 3.3 in particular Proposition 3.9 which is the key result that will be used to prove Theorem 1.5 below.

In Section 4 we introduce the Hilbert polynomial $H(z)$ of $(M, J, S^1)$ and study its properties. In particular, in Proposition 4.15 we prove that $H(z)$ satisfies $H(0) = N_0$ and the following ‘reciprocity law’:

$$H(z) = (-1)^n H(-k_0 - z).$$

This generalises, in the sense described in Sect. 4.2, a reciprocity law known for the Ehrhart polynomial of a reflexive polytope due to Hibi [26].

The next theorem is the main result of the section:

Theorem 1.5. Let $(M, J, S^1)$ be an $S^1$ space and $N_0$ defined as before. Assume that the first Chern class $c_1$ of the tangent bundle of $M$ is not torsion. Let $k_0 \geq 1$ be the index of $(M, J)$, $H(z)$ the Hilbert polynomial and $\deg(H)$ its degree. Then

$$H(-1) = H(-2) = \cdots = H(-k_0 + 1) = 0.$$  \hspace{1cm} (1.3)

Moreover, if $H(z) \neq 0$, then

$$k_0 \leq \deg(H) + 1 \leq n + 1.$$  \hspace{1cm} (1.4)

Note that here we are not assuming the existence of a quasi-ample line bundle, which is required by Hattori [24] to prove $k_0 \leq n + 1$. In Corollary 4.17 we translate (1.3) into equations in the Chern numbers of $(M, J)$ involving Todd polynomials. We recall that the Todd polynomials $T_j \in H^{2j}(M; \mathbb{Z})$, $j = 0, \ldots , n$, are the polynomials in the Chern classes of $(M, J)$ belonging to
the power series $\frac{z}{1-e^{-z}}$. The equations in (1.3) also suggest that studying the Chern numbers of $(M, J, S^1)$ for large values of $k_0$ is easier, and a careful analysis for $k_0 \geq n - 2$ is carried out in Section 5.

The first consequence of Theorem 1.5 in the symplectic category follows from the fact that if the action is Hamiltonian $H(z)$ can never be identically zero (see Remark 1.10), and the index coincides with the minimal Chern number (see Remark 1.3), leading to the following

**Corollary 1.6.** Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$. If $(M, \omega)$ supports a Hamiltonian $S^1$-action with isolated fixed points then its minimal Chern number coincides with the index $k_0$, and the following inequalities hold

$$1 \leq k_0 \leq n + 1.$$
equation in \( \int_M c^n_1 \) and \( \int_M c^{n-2}_1 c_2 \), which depends on \( n \) and \( N_0 \), this is the content of Corollary 5.12 and Corollary 5.13. Moreover, we give necessary and sufficient conditions for the roots of \( H(z) \) to be in \( S_{k_0} \) and \( T_{k_0} \) in terms of inequalities in the Chern number \( \int_M c^n_1 \) (see Corollaries 5.9 and 5.14). As a byproduct, we prove that, for values of \( n \) sufficiently large, \( H(z) \) belongs to \( T_{k_0} \) if and only if \( U(t) \) has its roots on the unit circle, thus proving that the condition in Theorem 1.22 is also necessary.

We can summarise some of the consequences in the symplectic category of the previous investigation in the following

**Theorem 1.7 (Hamiltonian vs non-Hamiltonian symplectic \( S^1 \)-actions).** Let \((M, \omega)\) be a compact, connected symplectic manifold of dimension \( 2n \) endowed with a symplectic \( S^1 \)-action and nonempty discrete fixed point set. Then:

1. If \( k_0 > n + 1 \) the action is non-Hamiltonian and \( \int_M c^n_1 = \int_M c^{n-2}_1 c_2 = 0 \).
2. If \( k_0 = n + 1 \) then
   \[ \int_M c^n_1 \in \{0, (n+1)\} \quad \text{and} \quad \int_M c^{n-2}_1 c_2 \in \left\{0, \frac{(n+1)^{n-1}}{2}\right\} \]
3. If \( k_0 = n \) then
   \[ \int_M c^n_1 \in \{0, 2n\} \quad \text{and} \quad \int_M c^{n-2}_1 c_2 \in \left\{0, n^{n-2}(2^n - n + 1)\right\} \]

Moreover, in (II) and (III) the displayed Chern numbers are either both zero or both nonzero, and the action is Hamiltonian if and only if neither of them vanishes.

4. If \( k_0 = n - 1 \) and \( n \geq 2 \) then
   \[ \int_M c^{n-2}_1 c_2 - \frac{n(n-3)}{2(n-1)^2} \int_M c^n_1 \in \left\{0, 12(n-1)^{n-2}\right\} \]
5. If \( k_0 = n - 2 \) and \( n \geq 3 \) then
   \[ \int_M c^{n-2}_1 c_2 - \frac{n-3}{2(n-2)} \int_M c^n_1 \in \left\{0, 24(n-2)^{n-2}\right\} \]

Moreover, in (IV) or (V) the action is Hamiltonian if and only if the displayed combination of Chern numbers does not vanish.

**Remark 1.8** The above results are stated in terms of the Chern numbers \( \int_M c^n_1 \) and \( \int_M c^{n-2}_1 c_2 \); however similar conclusions can be obtained for \( \int_M c^n_1 T_{n-h} \) (see also Remark 5.20).

Finally, in Section 6 we investigate how the Chern numbers of \((M, J, S^1)\) depend on the integers \( N_j \) for \( j = 0, \ldots, n \), when \( \dim(M) \leq 8 \). More precisely, we prove that for \( k_0 = n \) or \( n + 1 \) (and \( n \leq 4 \)), all the Chern numbers of \((M, J, S^1)\) can be expressed as linear combinations of the \( N_j \)'s, hence in particular as linear combinations of the Betti numbers, if the manifold is symplectic and the action Hamiltonian. This is the content of Propositions 6.2 (which implies the equivariant symplectic analogue of the Kobayashi-Ochiai theorem in dimension 4), 6.4 and 6.7.

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Although I have never met him, I would like to dedicate this work to the memory of Akio Hattori who, through his articles, thought me so much.
The main purpose of this section is to recall background material, set up notation and state preliminary results needed in the forthcoming sections.

Let \((M, J)\) be a compact, connected almost complex manifold of dimension \(2n\). Thus \(J: TM \to TM\) is a complex structure on the tangent bundle of \(M\), and for such manifold we consider the Chern classes of the tangent bundle, denoted by \(c_j \in H^{2j}(M; \mathbb{Z})\) as well as the Chern numbers \(\int_M c_j \cdots c_{j_l} \in \mathbb{Z}\), for every partition \((j_1, \ldots, j_l)\) of \(n\), i.e. \(j_1 + \cdots + j_l = n\) and \(j_m \in \mathbb{N}\) for \(m = 1, \ldots, l\).

Moreover assume that \((M, J, S^1)\) is an \(S^1\)-space, i.e. \((M, J)\) is endowed with a \(J\)-preserving \(S^1\)-action with nonempty and discrete fixed point set \(M^{S^1} = \{p_0, \ldots, p_N\}\) for some \(N \in \mathbb{Z}_{>0}\).

For every \(p_i \in M^{S^1}\) we denote by \(w_{i, 1}, \ldots, w_{i, n}\) the weights of the (isotropy) action of \(S^1\) at \(p_i\), i.e. the \(S^1\)-representation induced on \(T_p M\) is given by

\[
\alpha \cdot (z_1, \ldots, z_n) = (\alpha^{w_{i, 1}} z_1, \ldots, \alpha^{w_{i, n}} z_n) \quad \text{for every} \quad \alpha \in S^1
\]

for a suitable choice of complex coordinates \((z_1, \ldots, z_n)\) on \(T_p M \cong \mathbb{C}^n\). We also denote by \(W_i\) the (multi)set of weights at \(p_i\), i.e. \(W_i = \{w_{i, 1}, \ldots, w_{i, n}\}\). Note that \(w_{i, j}\) is nonzero for every \(i = 1, \ldots, N\) and \(j = 1, \ldots, n\), since the isotropy action commutes with the action on the manifold \(M\), and \(M^{S^1}\) is discrete. Finally, we denote by \(\lambda_i\) the number of negative weights at \(p_i \in M^{S^1}\) and by \(N_j\) the number of fixed points with exactly \(j\) negative weights, for every \(j = 0, \ldots, n\). From \([21]\) Proposition 2.6 we have that

\[
N_j = N_{n-j} \quad \text{for every} \quad j = 0, \ldots, n. \tag{2.2}
\]

Let \(K(M)\) (resp. \(K_{S^1}(M)\)) be the ordinary (resp. \(S^1\)-equivariant) \(K\)-theory ring of \(M\), i.e. the abelian group associated to the semigroup of isomorphism classes of complex vector bundles (resp. complex \(S^1\)-vector bundles) over \(M\), endowed with the direct sum \(\oplus\) and tensor product \(\otimes\) operation. Thus in particular \(K(M) \cong \mathbb{Z}\) and \(K_{S^1}(M) \cong R(S^1)\), the character ring of \(S^1\). Henceforth, we identify the latter with the Laurent polynomial ring \(\mathbb{Z}[t, t^{-1}]\), where \(t\) denotes the standard \(S^1\)-representation.

Let \(H^*_S(M; \mathbb{Z})\) be the \(S^1\)-equivariant cohomology of \(M\) with \(\mathbb{Z}\) coefficients; we recall that this is defined to be the ordinary cohomology of the Borel model, i.e. \(H^*_S(M; \mathbb{Z}) := H^*(M \times_{S^1} S^\infty; \mathbb{Z})\), where \(S^\infty\) is the unit sphere in \(\mathbb{C}^\infty\). Thus in particular \(H^*_S(pt; \mathbb{Z}) = \mathbb{Z}[x]\), where \(x\) has degree 2.

Finally, let \(\text{Pic}(M)\) (resp. \(\text{Pic}_{S^1}(M)\)) be the Picard group of isomorphism classes of line bundles (resp. equivariant line bundles) over \(M\).

In the rest of the section, \(\mathcal{H}(\cdot)\) (resp. \(\mathcal{H}_{S^1}(\cdot)\)) will either denote the cohomology (resp. equivariant cohomology) ring with \(\mathbb{Z}\) coefficients, the \(K\)-theory (resp. equivariant \(K\)-theory) ring, or the Picard group (resp. equivariant Picard group).

For \(p \in M^{S^1}\) let \(i_p: \{p\} \hookrightarrow M\) and \(i: M^{S^1} \hookrightarrow M\) denote the natural inclusions; since they are equivariant we have the following induced maps:

\[
i_p^*: \mathcal{H}_{S^1}(M) \to \mathcal{H}_{S^1}(\{p\})
\]

and

\[
i^* = \bigoplus_{p \in M^{S^1}} i_p^*: \mathcal{H}_{S^1}(M) \to \mathcal{H}_{S^1}(M^{S^1}) = \bigoplus_{p \in M^{S^1}} \mathcal{H}_{S^1}(\{p\}). \tag{2.3}
\]

We denote \(i_p^*(K)\) simply by \(K(p)\), for every \(p \in M^{S^1}\) and \(K \in \mathcal{H}_{S^1}(M)\).

Observe that the unique map \(M \to \{pt\}\) induces maps

\[
\mathcal{H}_{S^1}(\{pt\}) \to \mathcal{H}_{S^1}(M) \quad \text{and} \quad \mathcal{H}([pt]) \to \mathcal{H}(M),
\]

4To avoid confusion, if in the same paragraph we also deal with Chern classes of other bundles, we will denote the Chern classes of the tangent bundle by \(c_j(M)\).
which give $\mathcal{H}_{S^1}(M)$ the structure of an $\mathcal{H}_{S^1}(\{pt\})$-module, and $\mathcal{H}(M)$ the structure of an $\mathcal{H}(\{pt\})$-module.

Finally, if $e$ denotes the identity element in $S^1$, the inclusion homomorphism $\{e\} \hookrightarrow S^1$ induces a restriction map, also called the “forgetful homomorphism”

$$r_\mathcal{H} : \mathcal{H}_{S^1}(M) \to \mathcal{H}(M).$$

(2.4)

When $M$ is a point, $r_\mathcal{H}$ coincides with the evaluation at $x = 0$ in cohomology, and with the evaluation at $t = 1$ in $K$-theory and in the Picard group. The homomorphism (2.4) will be denoted by $r_\mathcal{H}$ in cohomology, by $r_K$ in $K$-theory and by $r_{\text{Pic}}$ for the Picard group.

2.1. Indices of $K$-theory classes. Let

$$\text{Ind} : K(M) \to K(\{pt\}) \simeq \mathbb{Z}$$

(2.5)

and

$$\text{Ind}_{S^1} : K_{S^1}(M) \to K_{S^1}(\{pt\}) \simeq \mathbb{Z}[t, t^{-1}]$$

(2.6)

be the index homomorphisms (or $K$-theoretic push forwards) in ordinary and equivariant $K$-theory.

By the Atiyah-Singer formula, the index in (2.5) can be computed as

$$\text{Ind}(V) = \int_M \text{Ch}(V) \, \mathcal{T}(M), \quad \text{for every } V \in K(M),$$

(2.7)

where $\text{Ch}(\cdot)$ is the Chern character homomorphism $\text{Ch} : K(M) \to H^*(M; \mathbb{Q})$, and $\mathcal{T}(M)$ is the total Todd class of $M$, i.e. the cohomology class in $H^*(M; \mathbb{Z})$ associated to the power series $\frac{x}{1 - e^{-x}}$.

This is a rational combination of Chern classes, and that the first terms of $\mathcal{T}(M)$ are given by

$$\mathcal{T}(M) = \sum_{j \geq 0} T_j = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1c_2}{24} + \frac{-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4}{720} + \ldots$$

(2.8)

where $T_j \in H^{2j}(M; \mathbb{Z})$ for every $j$. We also recall that the Todd genus $\text{Todd}(M)$ of $M$ is given by

$$\text{Todd}(M) = \int_M \mathcal{T}(M) = \int_M T_n.$$  

By the Atiyah-Segal formula [5], the equivariant index (2.6) of a class $V \in K_{S^1}(M)$ can be computed in terms of $\text{ch}(V)$ and the $S^1$ isotropy representation on $TM|_{MS^1}$. Since $M^{S^1}$ is discrete, the Atiyah-Segal formula in this case gives

$$\text{Ind}_{S^1}(V) = \sum_{i=0}^N \frac{V(p_i)}{\prod_{j=1}^n (1 - t^{-w_{i,j}})} , \quad \text{for every } V \in K_{S^1}(M).$$

(2.9)

By (2.7), (2.9) and the commutativity of the following diagram

$$
\begin{array}{ccc}
K_{S^1}(M) & \xrightarrow{r_K} & K(M) \\
\text{Ind}_{S^1} & & \downarrow \text{Ind} \\
\mathbb{Z}[t, t^{-1}] & \xrightarrow{r_K} & \mathbb{Z}. \\
\end{array}
$$

(2.10)

it follows that for every $V \in K_{S^1}(M)$ we have

$$\left( \sum_{i=0}^N \frac{V(p_i)}{\prod_{j=1}^n (1 - t^{-w_{i,j}})} \right)_{|_{t=1}} = r_K(\text{Ind}_{S^1}(V)) = \text{Ind}(r_K(V)) = \int_M \text{Ch}(r_K(V)) \, \mathcal{T}(M).$$

(2.11)
2.2. Equivariant Chern classes and equivariant complex line bundles. Given a complex vector bundle $V \to M$, denote by $c(V) = \sum_i c_i(V) \in H^*(M; \mathbb{Z})$ the total Chern class of $V$, and if $V$ is equivariant, by $c^S(V) = \sum_i c_i^S(V) \in H^S_*(M; \mathbb{Z})$ the total equivariant Chern class, i.e. the total Chern class of the bundle $V \times_{S^1} S^\infty \to M \times_{S^1} S^\infty$. It is easy to check that when $V = TM$, if $c^S(M)$ denotes the total equivariant Chern class of the tangent bundle $TM$, then for every $p_i \in M^S$, $c^S(M)(p_i) = \prod_{j=1}^n (1 + w_{i,j}x)$, and hence $c_1^S(M)(p_i) = \sigma_j(w_{i,1}, \ldots, w_{i,n})x^j$, where $\sigma_j(x_1, \ldots, x_n)$ denotes the $j$-th elementary polynomial in $x_1, \ldots, x_n$.

If $(M, J)$ is acted on by a circle $S^1$ preserving the almost complex structure, it is a natural question to ask whether a given complex vector bundle $V$ over $M$ admits an equivariant extension, i.e. whether the $S^1$-action can be lifted to $V$, making the projection $V \to M$ equivariant. This question has been studied in different settings, and for (complex) line bundles $L$ it has been completely answered by Hattori and Yoshida [25, Theorem 1.1, Corollary 1.2] (see also [22, 41] and [19, Appendix C]); here we summarise their main result in a different language.

**Theorem 2.1** (Hattori-Yoshida). The equivariant first Chern class
\[
c_1^S : \text{Pic}_{S^1}(M) \to H^2_{S^1}(M; \mathbb{Z})
\]
is an isomorphism. As a consequence, a line bundle $L$ admits an equivariant extension if and only if its first Chern class $c_1^S(L)$ is in the image of the restriction map
\[
r_H : H^2_{S^1}(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}).
\]

The second assertion follows from the commutativity of the following diagram
\[
\begin{array}{ccc}
\text{Pic}_{S^1}(M) & \xrightarrow{c_1^S} & H^2_{S^1}(M; \mathbb{Z}) \\
\downarrow & & \downarrow r_H \\
\text{Pic}(M) & \xrightarrow{c_1} & H^2(M; \mathbb{Z})
\end{array}
\]
and the fact that the first Chern class map $c_1$ on the bottom row is an isomorphism.

Moreover, for any line bundle $L$ whose first Chern class is in the image of \((2.13)\), which will henceforth be called admissible, all the possible equivariant extensions are parametrised by $H^2(CP^\infty; \mathbb{Z}) \simeq \mathbb{Z}$. More precisely, given an admissible $L$ and two equivariant extensions $L_1^S$ and $L_2^S$, there exists $a \in \mathbb{Z}$ such that $c_1(L_1^S)(L_2^S) = a \cdot x$. In particular we have that
\[
\text{if } L \text{ is trivial, then } c_1^S(L_1^S)(p) = ax \text{ for every } p \in M^S, \text{ for some } a \in \mathbb{Z}.
\]

In \([24] \text{ Lemma 3.2}\], Hattori proves that if $L$ is admissible, and $L'$ is such that $c_1(L) = kc_1(L')$ for some nonzero integer $k$, then $L'$ is also admissible: moreover every line bundle whose first Chern class is in $\text{Tor}(H^2(M; \mathbb{Z}))$, the torsion subgroup of $H^2(M; \mathbb{Z})$, is admissible. An example of admissible line bundle is given by the determinant line bundle $\Lambda^n(TM)$. In fact it is well-known that $c_1(M)$ always admits an equivariant extension, given by the equivariant first Chern class $c_1^S(M)$. Hence $\Lambda^n(TM)$ is admissible, since $c_1(\Lambda^n(TM)) = c_1(M)$. Moreover the trivial bundle is clearly admissible.

Let $L$ be the lattice given by $H^2(M; \mathbb{Z})/\text{Tor}(H^2(M; \mathbb{Z}))$ and
\[
\pi : H^2(M; \mathbb{Z}) \to L
\]
the projection. The following lemma is an immediate consequence of \([24] \text{ Lemma 3.2}\).

**Lemma 2.2.** Let $(M, J, S^1)$ be an $S^1$-space and let $c_1$ be the first Chern class of the tangent bundle. Suppose that $c_1$ is not a torsion element, i.e. $\pi(c_1) \neq 0$, and let $\eta$ be a primitive element in $L$ such that $\pi(c_1) = k \eta$, for some $k \in \mathbb{Z} \setminus \{0\}$. Then every line bundle $L$ such that $\pi(c_1(L)) = k \eta$ is admissible, for every $k \in \mathbb{Z}$. 

In the rest of this note, we will make use of the following **convention**: let \( \tau \) be an element of \( H^2_S(M^{S^1}; \mathbb{Z}) \); thus \( \tau(p) = a_p x \in H^2_S((p); \mathbb{Z}) \), where \( a_p \in \mathbb{Z} \) and \( x \) is the generator of \( H^2_S((p); \mathbb{Z}) = H^2(\mathbb{C}P^\infty; \mathbb{Z}) \). For the sake of simplicity, we henceforth identify \( \tau \in H^2_S(M^{S^1}; \mathbb{Z}) \) with the map from \( M^{S^1} \) to \( \mathbb{Z} \) which assigns to \( p \) the integer \( a_p \).

Note that for every \( L^{S^1} \in \text{Pic}_{S^1}(M) \) and every \( p \in M^{S^1} \)

\[
L^{S^1}(p_i) = e^{a_i}, \quad \text{where } a_i \text{ is the integer given by } c_1^{S^1}(L^{S^1})(p_i). \quad (2.15)
\]

In virtue of the isomorphism (2.12), given a class \( \tau \in H^2_S(M; \mathbb{Z}) \) (resp. \( \tau' \in H^2(M; \mathbb{Z}) \)), we will denote by \( e^{2\pi i \tau} \) the isomorphism class of equivariant line bundles whose first equivariant Chern class is \( \tau \) (resp. the isomorphism class of line bundles whose first Chern class is \( \tau' \)). We conclude this section with the following

**Proposition 2.3.** Let \( (M, J, S^1) \) be an \( S^1 \)-space with \( M^{S^1} = \{ p_0, \ldots, p_N \} \). Let \( c_1 \) and \( c_1^{S^1} \) be respectively the first Chern class and the equivariant first Chern class of the tangent bundle of \( M \). Then, for every \( \tau \in H^2_S(M; \mathbb{Z}) \) we have

\[
\text{Ind}_{S^1}(e^{2\pi i \tau}) = (-1)^n \text{Ind}_{S^1}(e^{2\pi i (-\tau - c_1^{S^1})}),
\]

where \( S^1 \) is the circle \( S^1 \) with orientation reversed. Thus

\[
\text{Ind}(e^{2\pi i r_H(\tau)}) = (-1)^n \text{Ind}(e^{2\pi i (-r_H(\tau) - c_1)}).
\]

**Proof.** By (2.10) and (2.11) we have that

\[
\text{Ind}_{S^1}(e^{2\pi i \tau}) = \sum_{i=0}^N \frac{t^{\tau(p_i)}}{\prod_{j=1}^n (1 - t^{w_i,j})} = \sum_{i=0}^N (-1)^n \frac{t^{\tau(p_i) + w_{i,1} + \cdots + w_{i,n}}}{\prod_{j=1}^n (1 - t^{w_j})} = (-1)^n \text{Ind}_{S^1}(e^{2\pi i (-\tau - c_1^{S^1})}),
\]

and (2.17) follows from (2.10), (2.16) and the fact that \( r_H(c_1^{S^1}) = r_H(c_1^{S^1}) = c_1 \).

3. **Computation of equivariant indices**

In this section we analyse some properties of the equivariant index of an equivariant line bundle \( L^{S^1} \). In particular we study under which conditions \( L^{S^1} \) is ‘rigid’, namely when its equivariant index \( \text{Ind}_{S^1}(L^{S^1}) \) is \( S^1 \)-invariant, i.e. it belongs to \( \mathbb{Z} \subset \mathbb{Z}[t, t^{-1}] \), and determine what the constant is in terms of the restriction to the fixed points of its equivariant first Chern class: this is the content of Theorem 3.3. As a consequence, we derive conditions that ensure the equivariant index of an equivariant line bundle to be zero. This is a generalisation of arguments which had already been used in different ways by several authors, see for example Hattori [23, Proposition 2.6], Hirzebruch et al. [28, Section 5.7], Li [35] and Li-Liu [36, Proposition 2.5].

The rest of the section is devoted to deriving applications of Theorem 3.3 which will be used in the forthcoming sections.

For every point \( p_i \in M^{S^1} \), we order the isotropy weights \( w_{i,1}, \ldots, w_{i,n} \) at \( p_i \) in such a way that the first \( \lambda_i \) are exactly the negative weights at \( p_i \). We define \( c_1^+ \) and \( c_1^- \) in \( H^2_S(M^{S^1}; \mathbb{Z}) \) to be

\[
c_1^+(p_i) = w_{i,\lambda_i+1} + \cdots + w_{i,n} \quad \text{and} \quad c_1^-(p_i) = -(w_{i,1} + \cdots + w_{i,\lambda_i}). \quad (3.1)
\]

From the definition it follows that \( c_1^+(p_i) \geq 0 \) (resp. \( c_1^-(p_i) \geq 0 \)) and equality holds if and only if \( \lambda_i = n \) (resp. \( \lambda_i = 0 \)). Moreover, if \( c_1^{S^1} \) denotes the equivariant first Chern class of \( M \), we have that \( i^*(c_1^{S^1}) = c_1^+ - c_1^- \).

**Definition 3.1.** A class \( \tau \in H^2_S(M; \mathbb{Z}) \) is said to be dominated by \( c_1^+ \) (resp. by \( c_1^- \)) if \( \tau(p) \leq c_1^+(p) \) (resp. if \( -\tau(p) \leq c_1^-(p) \)) for every \( p \in M^{S^1} \) (resp. if \( -\tau(p) \leq c_1^-(p) \) for every \( p \in M^{S^1} \)).
Remark 3.2 It is easy to check that the classes 0 and $c_1^{S^1}$ are always dominated by both $c_1^+$ and $c_1^-$. Moreover, if $\tau \in H^2_{S^1}(M;\mathbb{Z})$ satisfies $\tau(p) \leq 0$ (resp. $\tau(p) \geq 0$) for every $p \in M^{S^1}$ then $\tau$ is dominated by $c_1^+$ (resp. $c_1^-$).

Theorem 3.3. Let $(M, J, S^1)$ be an $S^1$-space with $M^{S^1} = \{p_0, \ldots, p_N\}$. Let $\tau$ be an element of $H^2_{S^1}(M;\mathbb{Z})$ and $c_1^+$, $c_1^-$ as defined above. For every $p \in M^{S^1}$, define $\delta^+(p)$ (resp. $\delta^-(p)$) to be 1 if $\tau(p) = c_1^+(p)$ (resp. $\tau(p) = c_1^-(p)$) and zero otherwise. Then

(i) If $\tau \in H^2_{S^1}(M;\mathbb{Z})$ is dominated by $c_1^+$ then

$$\text{Ind}_{S^1}(e^{2\pi i(-\tau)}) = \sum_{j \geq 0} b_j t^j \in \mathbb{Z}[t], \quad \text{and} \quad b_0 = \sum_{i=0}^N \delta^+(p_i)(-1)^{n-\lambda_i}$$

(ii) If $\tau \in H^2_{S^1}(M;\mathbb{Z})$ is dominated by $c_1^-$ then

$$\text{Ind}_{S^1}(e^{2\pi i(-\tau)}) = \sum_{j \leq 0} b_j t^j \in \mathbb{Z}[t^{-1}], \quad \text{and} \quad b_0 = \sum_{i=0}^N \delta^-(p_i)(-1)^{\lambda_i}$$

(iii) If $\tau \in H^2_{S^1}(M;\mathbb{Z})$ is dominated by $c_1^+$ and $c_1^-$ then

$$\text{Ind}_{S^1}(e^{2\pi i(-\tau)}) = b_0 \in \mathbb{Z}$$

where

$$b_0 = \sum_{i=0}^N \delta^+(p_i)(-1)^{n-\lambda_i} = \sum_{i=0}^N \delta^-(p_i)(-1)^{\lambda_i}$$

Proof. By (2.9) and (2.15), we have that

$$\text{Ind}_{S^1}(e^{2\pi i(-\tau)}) = \sum_{i=0}^N \frac{t^{-\tau(p_i)}}{\prod_{j=1}^n (1 - t^{-w_{i,j}})}$$

For every $i = 0, \ldots, N$, let $f_i(t)$ be the rational function $\frac{t^{-\tau(p_i)}}{\prod_{j=1}^n (1 - t^{-w_{i,j}})}$, and observe that

$$\sum_{i=0}^N f_i(t) \in \mathbb{Z}[t, t^{-1}]$$

Thus, in order to prove (i), it is sufficient to prove that $\lim_{t \to 0} \sum_{i=0}^N f_i(t)$ is finite, and its value will be equal to $b_0$. Observe that by definition of $c_1^+$, $f_i(t)$ can be rewritten as

$$(-1)^{n-\lambda_i} t^{-\tau(p_i)+c_1^+(p_i)}$$

Since by assumption $i^*(\tau)$ is dominated by $c_1^+$, $\lim_{t \to 0} f_i(t)$ is finite for all $i = 0, \ldots, N$, and by definition of $\delta^+$ it follows that its value equals $\delta^+(p_i)(-1)^{n-\lambda_i}$, thus proving (i).

The proof of (ii) follows by a similar argument, by taking $\lim_{t \to \infty} \sum_{i=0}^N f_i(t)$, and by observing that $f_i(t)$ can be written as

$$(-1)^{\lambda_i} t^{-\tau(p_i)-c_1^-(p_i)}$$

Finally, (iii) follows from (i) and (ii).

Example 3.4 Consider $(CP^3, J)$ with the standard (almost) complex structure, and $S^1$-action given by

$$\lambda \cdot [z_0 : z_1 : z_2 : z_3] = [\lambda^a z_0 : \lambda^{a+b} z_2 : \lambda^{a+b+c} z_3],$$

where $a, b, c$ are pairwise coprime positive integers. This action is “standard”, in the sense that it is the restriction to a subtorus of dimension 1 of the standard toric action of the 3-dimensional torus $T^3$ on $CP^3$. The fixed point set is given by four points $p_0, p_1, p_2, p_3$, corresponding respectively to $[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0], [0 : 0 : 0 : 1]$. Let $\tau_0$ be the generator of $H^2(CP^3, \mathbb{Z})$ such
that \( c_1(\mathbb{C}P^3) = 4\tau_0 \). It can be checked that \( \tau_0 \) admits an equivariant extension\(^3\) \( \tau \in H^2_\mathbb{Z}(\mathbb{C}P^3, \mathbb{Z}) \), i.e. \( r_H(\tau) = \tau_0 \); we pick \( \tau \) so that \( \tau(p_0) = 0 \). The (multi)sets of isotropy weights at each fixed point, as well as \( \nu(\tau) \), \( c_1^+ \) and \( c_1^- \), are given in the following table:

| \( \mathbb{p}_0 \) | \( W_i \) | \( \nu(\tau) \) | \( c_1^+ \) | \( c_1^- \) |
|----------------|--------|----------------|----------|---------|
| \{a, a + b, a + b + c\} | 0 | 3a + 2b + c | 0 |
| \{-a, b, b + c\} | -a | 2b + c | a |
| \{-b, -a, -b, c\} | -a - b | c | a + 2b |
| \{-c, -b, -a - b, -c\} | -a - b - c | 0 | a + 2b + 3c |

Observe that \( \tau \) is dominated by both \( c_1^+ \) and \( c_1^- \), and by definition \( \delta^+ \equiv 0 \). Thus Theorem 3.3(iii) implies that \( \text{Ind}_{S^1}(e^{2\pi i(\nu - \tau)}) = 0 \), as it can also be checked directly from here

\[
\text{Ind}_{S^1}(e^{2\pi i(\nu - \tau)}) = \frac{1}{(1-t^{-a})(1-t^{-a+b})(1-t^{-a+b+c})} + \frac{t^a}{(1-t^a)(1-t^{-b})(1-t^{-b+c})} + \frac{t^{a+b}}{(1-t^b)(1-t^{a+b})(1-t^{-c})} + \frac{t^{a+b+c}}{(1-t^c)(1-t^{a+b+c})(1-t^{-a+b+c})} = 0
\]

**Remark 3.5** Following the discussion in Remark 3.2, by Theorem 3.3 we have that if \( \tau \in H^2_\mathbb{Z}(M; \mathbb{Z}) \) satisfies \( \tau(p) \geq 0 \) (resp. \( \tau(p) \leq 0 \)) for all \( p \in M^{S^1} \), then \( \text{Ind}_{S^1}(e^{2\pi i(\nu - \tau)}) \in \mathbb{Z}[t] \) (resp. \( \text{Ind}_{S^1}(e^{2\pi i(\nu - \tau)}) \in \mathbb{Z}[t^{-1}] \)).

As an immediate consequence of Theorem 3.3 we have the following

**Corollary 3.6.** Let \((M, J, S^1)\) be an \( S^1 \)-space with \( M^{S^1} = \{p_0, \ldots, p_N\} \). Let \( N_i \) be the number of fixed points with exactly \( i \) negative weights.

If \( 1 \in \text{Pic}_{S^1}(M) \) denotes the trivial line bundle over \( M \), where \( c_1^{S^1}(1) = 0 \), then

\[
\text{Ind}_{S^1}(1) = N_0 = N_n .
\]

If \( \tilde{L}^{S^1} \in \text{Pic}_{S^1}(M) \) denotes the determinant line bundle \( \Lambda^n(T^*M) \), where \( c_1^{S^1}(\tilde{L}^{S^1}) = c_1^{S^1}(\Lambda^n(T^*M)) = -c_1^{S^1} \), then

\[
\text{Ind}_{S^1}(\tilde{L}^{S^1}) = (-1)^n N_0 = (-1)^n N_n .
\]

**Proof.** As we have already remarked, the classes 0 and \( c_1^{S^1} \) are dominated by \( c_1^+ \) and \( c_1^- \). Thus (3.5) and (3.6) follow from Theorem 3.3(iii) and the definition of \( N_0 \) and \( N_n \). \( \square \)

Note that equation (3.5) is already known, see for example [24 Corollary 2.7] (also cf. [35 Theorem 2.3]).

Observe that (3.5) can also be obtained by noticing that since \( c_1^{S^1} \) is dominated by \( c_1^+ \) and \( c_1^- \), (3.2) implies that \( \text{Ind}_{S^1}(\tilde{L}^{S^1}) \) is an integer, thus \( \text{Ind}_{S^1}(\tilde{L}^{S^1}) = \text{Ind}(r_K(\tilde{L}^{S^1})) \), and so (3.6) follows from (2.17) in Proposition 2.3 and (3.5).

We also remark that \( \text{Ind}_{S^1}(1) \) is the Todd genus of \( M \); in fact from (2.11) we have that

\[
\text{Todd}(M) = \int_M T_n = \int_M \text{Ch}(r_K(1)) T(M) = \text{Ind}(r_K(1)) = \text{Ind}_{S^1}(1)
\]

where the second equality follows from observing that \( \text{Ch}(r_K(1)) = 1 \), and the last equality follows from (2.10) and the fact that \( \text{Ind}_{S^1}(1) \) is an integer, thus \( \text{Ind}(r_K(1)) = r_K(\text{Ind}_{S^1}(1)) = \text{Ind}_{S^1}(1) \). By combining (3.5) and (3.7) we recover the following well-known fact (see for example [24 Remark 2.10]).

\(^3\)Indeed, in this case, every class \( \gamma \in H^1(\mathbb{C}P^3, \mathbb{Z}) \) admits an equivariant extension, for every \( j \). This is due to the fact that \( \mathbb{C}P^3 \) with the above \( S^1 \)-action is *equivariantly formal* (see for example [39]).
Corollary 3.7. Let \((M, J, S^1)\) be an \(S^1\)-space, \(N_i\) the number of fixed points with exactly \(i\) negative weights, and \(\text{Todd}(M)\) the Todd genus of \(M\). Then

\[
\text{Todd}(M) = N_0 = N_n.
\]

Before giving the main application of Theorem 3.3, we prove the following easy but useful lemma.

Lemma 3.8. Let \((M, J, S^1)\) be an \(S^1\)-space, \(c_1\) the first Chern class of the tangent bundle of \(M\), \(N_i\) the number of fixed points with exactly \(i\) negative weights, and \(\text{Todd}(M)\) the Todd genus of \(M\).

(a1) If \(\eta \in \text{Tor}(H^2(M, \mathbb{Z}))\) then

\[
\text{Ind}(e^{2\pi i\eta}) = \text{Todd}(M) = N_0
\]

and

\[
\text{Ind}_{S^1}(e^{2\pi i\eta^{S^1}}) = t^a \text{Todd}(M) = t^a N_0,
\]

where \(\eta^{S^1} \in H^2_{S^1}(M, \mathbb{Z})\) denotes an equivariant extension of \(\eta\), and \(a = \eta^{S^1}(p)\) for every \(p \in M^{S^1}\).

(a2) If \(c_1 \in \text{Tor}(H^2(M, \mathbb{Z}))\) then \(N_0 = N_n = 0\) and \(\text{Todd}(M) = 0\).

Proof. (a1) First of all, observe that if \(\eta \in \text{Tor}(H^2(M, \mathbb{Z}))\) then, by the discussion in Section 2.4, it admits an equivariant extension \(\eta^{S^1} \in H^2_{S^1}(M, \mathbb{Z})\). By the commutativity of (2.10), in order to prove (3.8) it is sufficient to prove (3.9). If \(\eta\) is torsion then there exists \(k \in \mathbb{Z} \setminus \{0\}\) such that \(k\eta = 0\). Thus if we consider an equivariant extension \(\eta^{S^1}\), by (2.14) we have that \(\eta^{S^1}(p) = a\) for some \(a \in \mathbb{Z}\), for every \(p \in M^{S^1}\). Hence

\[
\text{Ind}_{S^1}(e^{2\pi i\eta^{S^1}}) = t^a \text{Ind}_{S^1}(1) = t^a \text{Todd}(M) = t^a N_0
\]

where the first equality follows from (2.14), the second from (3.8), and the last from Corollary 3.7.

(a2) By a similar argument, we have that the integer \(c_1^{S^1}(p)\) does not depend on \(p \in M^{S^1}\). However \(c_1^{S^1}(p_i) = \sum_{j=1}^n w_{ij}\), and by (2.12) we have \(N_0 = N_n\). So by definition of \(N_0\) and \(N_n\) we must have that \(N_0 = N_n = 0\), and by Corollary 3.7 that \(\text{Todd}(M) = 0\).

The next proposition also follows from Theorem 3.3, but it is a key result for the theorems in the next sections (see also [24, Assertion 4.10] and [36, Proposition 2.5]).

Proposition 3.9. Let \((M, J, S^1)\) be an \(S^1\)-space. Let \(c_1^{S^1}\) be the equivariant first Chern class of the tangent bundle of \(M\) and \(k\) a positive integer such that \(c_1^{S^1}(p) = k \eta^{S^1}(p) + c\) for all \(p \in M^{S^1}\), for some \(\eta^{S^1} \in H^2_{S^1}(M; \mathbb{Z})\) and \(c \in \mathbb{Z}\). Then

\[
\text{Ind}_{S^1}(e^{2\pi i(-h\eta^{S^1})}) = 0 \quad \text{for every} \quad h = 1, \ldots, k - 1.
\]

Remark 3.10 Observe that if \(c_1\) is torsion then \(r_H(\eta^{S^1})\) is also torsion, and by Lemma 3.8 it follows that

\[
\text{Ind}_{S^1}(e^{2\pi i(-h\eta^{S^1})}) = 0 \quad \text{for every} \quad h \in \mathbb{Z}
\]

Proof of Proposition 3.9. First of all, observe that it is not restrictive to assume that \(c = 0\). In fact, let \(S^1 \times M \to M\), \((\lambda, q) \to \lambda \cdot q\) be the given \(S^1\)-action on \(M\), and consider a new action given by \((\lambda, q) \to \lambda^k \cdot q\); we denote by \(\tilde{S}^1\) the new circle acting on \(M\). Note that the set of fixed points of this action coincides with the old one, and the new isotropy weights are the old ones multiplied by \(k\). Thus \(c_1^{S^1}(p)\) is divisible by \(k\), for every \(p \in M^{S^1} = M^{\tilde{S}^1}\). So there exists \(\tilde{\eta} \in H^2_{\tilde{S}^1}(M; \mathbb{Z})\) such that \(c_1^{S^1} = k\tilde{\eta}\). Moreover, if \(\text{Ind}_{\tilde{S}^1}(e^{2\pi i\tilde{\eta}}) = P(t, t^{-1})\) for some \(P \in \mathbb{Z}[x, y]\), then \(\text{Ind}_{S^1}(e^{2\pi i\eta}) = P(1, 1)\).
4. The Hilbert polynomial of \((M,J)\) and the equations in the Chern numbers

We recall from Section 2.2 that \(\mathcal{L}\) is the lattice given by \(H^2(M;\mathbb{Z})/\text{Tor}(H^2(M;\mathbb{Z}))\) and \(\pi\) the projection \(\pi: H^2(M;\mathbb{Z}) \to \mathcal{L}\). Henceforth we assume that the first Chern class \(c_1\) of the tangent bundle of \((M,J)\) is not a torsion element of \(H^2(M;\mathbb{Z})\). The next Lemma proves that in the Hamiltonian case, this condition is automatically satisfied (see also [47, Lemma 3.8]).

**Lemma 4.1.** Let \((M,\omega)\) be a compact symplectic manifold endowed with a Hamiltonian \(S^1\)-action with isolated fixed points. Then \(c_1\) is not a torsion element in \(H^2(M;\mathbb{Z})\).

**Proof.** It is sufficient to combine Lemma 4.4 with Lemma 3.8 (a2).

Since \(c_1\) is not torsion we have \(\pi(c_1) \neq 0\), so there exists a non-zero element \(\eta \in \mathcal{L}\) such that \(\pi(c_1) = k \eta\) for some \(k \in \mathbb{Z}\setminus\{0\}\). This justifies the following

**Definition 4.2.** Let \((M,J)\) be a compact almost complex manifold, and assume that \(c_1\) is not a torsion element of \(H^2(M;\mathbb{Z})\). An integer \(k_0 \geq 1\) is called the index of \((M,J)\) if

\[
\pi(c_1) = k_0 \eta_0 \quad \text{for some primitive element } \eta_0 \in \mathcal{L}.
\]

The positive integer \(k_0\) and the associated \(\eta_0 \in \mathcal{L}\), uniquely defined by (4.1), will play a crucial role in the rest of the section.

**Remark 4.3** Let \((M,\omega)\) be a compact symplectic manifold with first Chern class \(c_1\), and suppose it is not torsion. Following Definition 6.4.2 in [40], the minimal Chern number of \((M,\omega)\) is defined to be the integer \(N\) such that \(\langle c_1,\pi_2(M)\rangle = N\mathbb{Z}\). If \(M\) is simply connected then, by the Hurewicz theorem, we have \(\pi_2(M) = H_2(M,\mathbb{Z})\) which, modulo torsion, is isomorphic to \(H^2(M,\mathbb{Z})\), thus implying that the minimal Chern number agrees with the index of \((M,\omega)\). A result of Li [34] implies that if the \(S^1\)-action on \((M,\omega)\) is Hamiltonian with isolated fixed points then \(M\) is simply connected. So it follows that if \((M,\omega)\) is endowed with a Hamiltonian \(S^1\)-action with isolated fixed points, the minimal Chern number always agrees with the index \(k_0\) (which is well-defined, thanks to Lemma 4.1).

Before proceeding, we prove the following Lemma:

**Lemma 4.4.** Let \(\eta \in H^2(M;\mathbb{Z})\) and \(\tau \in \text{Tor}(H^2(M;\mathbb{Z}))\). Then

\[
\text{Ind}(e^{2\pi i(\eta + \tau)}) = \text{Ind}(e^{2\pi i \eta}).
\]

**Proof.** By (2.7) we have that
By (2.7) we obtain that
\[ \text{Ind}(e^{2\pi i(\eta + \tau)}) = \int_M \text{Ch}(e^{2\pi i(\eta + \tau)}) T(M) = \int_M \left(1 + \eta + \frac{\eta^2}{2} + \cdots \right) \left(1 + \tau + \frac{\tau^2}{2} + \cdots \right) T(M) = \int_M \left(1 + \eta + \frac{\eta^2}{2} + \cdots \right) T(M) = \text{Ind}(e^{2\pi i\eta}), \]

where the second-last equality follows from the fact that if \( \tau \) is torsion then \( \int_M \tau^k \alpha = 0 \) for all \( k > 0 \) and \( \alpha \in H^{2n-2k}(M; \mathbb{Z}) \). \( \square \)

Let \( \mathcal{L}_0 \) be the one dimensional lattice in \( \mathcal{L} \) generated by \( \eta_0 \), i.e. \( \mathcal{L}_0 = \mathbb{Z}/(\eta_0) \). In virtue of Lemma [4.4] it makes sense to consider the map which assigns to each \( \eta \), where \( \pi z \)

The first properties of \( H(z) \)

Proposition 4.5. Let \( (M,J,S^1) \) be an \( S^1 \)-space with \( N_0 \) fixed points with zero negative weights. Let \( c_1 \) be the first Chern class of the tangent bundle of \( M \) and assume that it is not torsion. Let \( k_0 \geq 1 \) be the index of \( (M,J) \), \( H(z) \) the Hilbert polynomial, and \( \deg(H) \) its degree. Then

1. \( H(0) = \text{Todd}(M) = N_0; \)
2. \( H(z) = (-1)^n H(-k_0 - z) \) for every \( z \in \mathbb{C}; \)
3. \( \deg(H) \equiv n \mod 2. \)

Remark 4.6 By Lemma [4.4] and Proposition [4.5] [1], note that if \( (M,\omega) \) is a compact symplectic manifold supporting a Hamiltonian \( S^1 \)-action with isolated fixed points, then the Hilbert polynomial \( H(z) \) can never be identically zero.

Remark 4.7 Proposition [4.5] [3] implies that if there exists \( k \) such that \( a_{n-2h} = 0 \) for every \( h = 0, \ldots, k \), then \( a_{n-2h-1} = 0 \) for every \( h = 0, \ldots, k \).
Proof. Property (1) follows from the definition of $H(z)$ and Corollary 3.7. By Lemma 2.2, every line bundle $L$ such that $\pi(c_1(L)) = k \eta_0$ is admissible. So from Proposition 2.3 we have that for all $k \in \mathbb{Z}$

$$H(k) = \text{Ind}(e^{2\pi i k \eta_0}) = (-1)^n \text{Ind}(e^{2\pi i (-k - k_0) m}) = (-1)^n \text{H}(-k_0 - k),$$

and (2) follows from observing that the polynomial given by $Q(z) = H(z) - (-1)^n \text{H}(-k_0 - z)$ is zero for all $k \in \mathbb{Z}$, hence it must be identically zero.

In order to prove (3) it is sufficient to notice that, if $H(z) = \sum_{j=0}^{m} a_n z^n$, with $m = \text{deg}(H)$, from (2) it follows that $a_m = (-1)^{m+n} a_m$.

Before proceeding with the main results of the section, we introduce some terminology that will be used in the discussion of the position of the roots of $H(z)$.

**Definition 4.8.** Fix a positive integer $k$.

1) We denote by $T_k$ the family of polynomials in $\mathbb{R}[z]$ that can be written as $C(z) \prod_{j=1}^{k-1}(z+j)$, where $C(z) \in \mathbb{R}[z]$ has all its roots on the line $l_k = \{ x + iy \in \mathbb{C} \mid x = \frac{-1}{k} \}$.

2) We define $S_k$ to be the subset of the complex plane given by

$$S_k = \{ x + iy \in \mathbb{C} \mid -k < x < 0 \}$$

and we refer to it as the canonical strip (centred at $-\frac{k}{2}$).

3) The subset of the complex plane $C_k = \{ z = x + iy \in \mathbb{C} \mid y = 0 \text{ or } x = \frac{-1}{k} \}$ is called the cross at $-\frac{1}{k}$.

The terminology in 1) and 2) is inspired respectively by [15] and [17]. Indeed, in the beautiful note [15], Rodriguez-Villegas analyses conditions which ensure a polynomial $H(z) \in \mathbb{R}[z]$ to belong to $T_k$, for some $k \in \mathbb{Z}_{>0}$. In Sect. [4.4] and Section 6 we explore connections among our results and those in [15]; we study under which conditions $H(z)$ belongs to $T_{k_0}$, for certain values of $k_0$.

In [17], Golyshin analyses the position of the roots of the Hilbert polynomial of a Fano variety and a variety of general type. In particular, after adapting his terminology to ours, he asks under which conditions all the zeros of $H(z)$ belong to the canonical strip $S_{k_0}$. In Section 5 we will study the position of the roots of $H(z)$ in terms of inequalities in the Chern numbers and of $k_0$, when $k_0 \geq n - 2$ (see Remarks 5.2, 5.3 and Corollaries 5.9 and 5.10).

The next corollary is a straightforward consequence of Proposition 4.6.

**Corollary 4.9.** Let $(M, J, S^1)$ be an $S^1$-space. Let $k_0 \geq 1$ be the index of $(M, J)$, and assume that the Hilbert polynomial $H(z)$ is of positive degree $\text{deg}(H) > 0$. If at least $\text{deg}(H) - 3$ roots of $H(z)$, counted with multiplicity, belong to $C_{k_0}$, then all the roots of $H(z)$ belong to $C_{k_0}$. In particular, if $n \leq 3$, then all the roots of $H(z)$ belong to $C_{k_0}$.

Proof. Let $h$ be the number of roots, counted with multiplicity, which belong to $C_{k_0}$; by assumption $h \geq \text{deg}(H) - 3$. Suppose that one of the remaining $\text{deg}(H) - h$ roots, $z_0 \in \mathbb{C}$, does not belong to $C_{k_0}$. Then, by Proposition 4.6 (2), we have that $z_1 = -k_0 - z_0$ is also a root, and since $H(z) \in \mathbb{R}[z]$, the complex conjugates $z_2 = \frac{-z_0}{k_0}$ and $z_3 = \frac{-k_0 - z_0}{k_0}$ are also roots. Since $z_0 \notin C_{k_0}$, it follows that $z_1 \neq z_j$ for $i \neq j$, and $z_i \notin C_{k_0}$ for $i = 0, 1, 2, 3$, implying that $H(z)$ has at least $h + 4 \geq \text{deg}(H) + 1$ roots, which is impossible since we are assuming $H(z)$ to be non identically zero.

We are now ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** Let $\eta_0 \in \mathcal{L}$ be as in (1.1), and $\tilde{\eta}_0 \in H^2(M; \mathbb{Z})$ be such that $\pi(\tilde{\eta}_0) = \eta_0$. Thus $c_1 = k_0 \tilde{\eta}_0 + \tau$, for some $\tau \in \text{Tor}(H^2(M; \mathbb{Z}))$. By Lemma 2.2 both $\tilde{\eta}_0$ and $\tau$ admit equivariant extensions $\tilde{\eta}_0^{S^1}$ and $\tau^{S^1}$ in $H^2_{S^1}(M; \mathbb{Z})$. Since $\tau^{S^1}(p)$ does not depend on $p \in M^{S^1}$ (see (2.1.1)), it follows that $c_1^{S^1}(p) = k_0 \tilde{\eta}_0^{S^1}(p) + c$ for all $p \in M^{S^1}$, for some $c \in \mathbb{Z}$. Thus by Proposition 3.9 we have that

$$\text{Ind}_{S^1}(e^{2\pi i k \tilde{\eta}_0^{S^1}}) = 0 \quad \text{for all} \quad k = -1, -2, \ldots, -k_0 + 1, \quad (4.6)$$

Corollary 4.10. Hamiltonian actions.

or not, and by Lemma 1.4, this immediately translates into properties of Hamiltonian and non-
thus if \( H(z) \neq 0 \) we must have that \( |C_0| = k_0 - 1 \leq \deg(H) \leq n \).

Note that by Proposition 1.5, \( H(z) \) has a different behaviour depending on whether \( N_0 = 0 \) or not, and by Lemma 1.4 this immediately translates into properties of Hamiltonian and non-
Hamiltonian actions.

**Corollary 4.10.** Let \((M, J, S^1), N_0, k_0 \) and \( H(z) \) be as in Theorem 1.5. Then:

(i) If \( N_0 \neq 0 \) then \( k_0 \leq \deg(H) + 1 \leq n + 1 \);

(ii) If \( N_0 = 0 \) then for all \( k_0 \leq \deg(H) \), thus in particular for all \( k_0 \geq n \), we have

\[
H(z) \equiv 0 \quad \text{and} \quad \int_M c_1^h T_{n-h} = 0 \quad \text{for every} \quad h = 0, \ldots, n.
\]

Suppose in addition that \((M, \omega)\) and the \( S^1 \)-action are symplectic. Then:

(i') If the \( S^1 \)-action is Hamiltonian, then the same claim of (i) holds;

(ii') If the \( S^1 \)-action is non-Hamiltonian, then the same claim of (ii) holds.

**Proof.** Observe that if \( N_0 \neq 0 \) then Proposition 1.5 (i) implies that \( H(z) \) is not identically zero, and (i) follows from (1.3).

On the other hand, if \( N_0 = 0 \) then by Proposition 1.5 (1) and (2), and by (1.3), we have that the set of roots of \( H(z) \) contains \( C_0' = \{0, -1, \ldots, -k_0\} \). So if \( |C_0'| = k_0 + 1 > \deg(H) \) then \( H(z) \equiv 0 \), and the second claim follows from (1.3).

In the symplectic case, the claims follow from (i), (ii) and Lemma 1.4.

**Corollary 1.10** follows from Corollary 4.10 (i') and the discussion in Remark 1.3.

**Remark 4.11.** Observe that by Corollary 1.10 (ii') it follows that if \((M, \omega)\) supports a non-Hamiltonian action and \( k_0 \geq n \), then \( \int_M c_1^n = \int_M c_1^n - 2c_2 = 0 \). In Theorem 1.7 we strengthen this fact and prove that when \( k_0 \geq n \) the vanishing of one of these Chern numbers is indeed equivalent to having a non-Hamiltonian action. Moreover, if \( k_0 = n-2 \) or \( k_0 = n-1 \), then a suitable linear combination of those Chern number is zero if and only if the action is non-
Hamiltonian.

The next Corollary is also an easy consequence of Theorem 1.5, combined with the observation in Remark 1.6 and gives a criterion to tell whether an action is non-Hamiltonian in terms of the vanishing of some (combination of) Chern numbers.

**Corollary 4.12.** Let \((M, \omega)\) be a compact symplectic manifold of dimension \( 2n \) endowed with a symplectic \( S^1 \)-action and isolated fixed points. Then:

(1) If \( n \neq k_0 \mod 2 \) then

\[
\int_M c_1^h T_{n-h} = 0 \quad \text{for all} \quad h \geq k_0 - 1 \quad \Rightarrow \quad \text{the action is non-Hamiltonian};
\]

(2) If \( n \equiv k_0 \mod 2 \) then

\[
\int_M c_1^h T_{n-h} = 0 \quad \text{for all} \quad h \geq k_0 \quad \Rightarrow \quad \text{the action is non-Hamiltonian}.
\]

**Proof.** By definition of the \( a_n \), the coefficients of the Hilbert polynomial in (1.6), we have that \( \int_M c_1^h T_{n-h} = 0 \) if and only if \( a_h = 0 \). Thus, \( a_n = 0 \) for all \( h \geq k_0 - 1 \) implies that \( \deg(H) \leq k_0 - 2 \). By Theorem 1.5, \( H(z) \) has at least \( k_0 - 1 \) zeroes, and so it must be identically zero. However, by
Remark 4.10. The Hilbert polynomial of a symplectic manifold with a Hamiltonian $S^1$-action can never be identically zero.

If $n \equiv k_0 \mod 2$ then, by Proposition 4.3 (3) we have that $\deg(H) \leq k_0 - 1$ implies $\deg(H) \leq k_0 - 2$, and the conclusion follows.

Remark 4.13. In [24], Hattori also analyses inequalities for $k_0$ when $(M, J, S^1)$ is an $S^1$-space and it possesses a quasi-ample line bundle, defined as follows: An equivariant line bundle $\mathbb{L}^S$ is fine if the restrictions of $\mathbb{L}^S$ at the fixed points are mutually distinct $S^1$-modules, i.e., if $a_i \neq a_j$ for every $a_i \neq a_j$, where $\mathbb{L}^S(p_i) = t^{a_i}$ and $p_i \in M^{S^1}$ (see (2.15)). It is quasi-ample if it is fine and $\int_M c_1(L)^n \neq 0$, where $L = r_{Pic}(L^S)$. In [24] Theorem 5.1] the author proves that if $(M, J)$ possesses a quasi-ample line bundle $\mathbb{L}^S$ such that $c_1 = k_0 c_1(L)$, with $L = r_{Pic}(L^S)$ and $k_0 \in \mathbb{Z}_{>0}$, then $k_0 \leq n + 1 \leq \chi(M)$. Observe that in this case we must have $\int_M c_1^n \neq 0$. In Corollary 4.10 (i) the existence of a quasi-ample line bundle is not required, and it is replaced by $N_0 \neq 0$ which, by Lemma 4.3, is naturally satisfied when $(M, \omega)$ is a symplectic manifold and the $S^1$-action is Hamiltonian. Moreover, Corollary 4.10 is a straightforward consequence of Theorem 1.5, and gives better bounds on $k_0$ when $\int_M c_1^n = 0$, as remarked below.

Remark 4.14. From (4.5), Proposition 4.3 (3) and Corollary 4.10 it follows that

- If $\int_M c_1^n = 0$ and $N_0 \neq 0$, then $1 \leq k_0 \leq n - 1$;
- If $\int_M c_1^n = 0$ and $N_0 = 0$, then $H(z) \equiv 0$ for all $k_0 \geq n - 2$.

Similarly,

- If $\int_M c_1^n = \int_M c_1^{n-2} c_2 = 0$ and $N_0 \neq 0$, then $1 \leq k_0 \leq n - 3$;
- If $\int_M c_1^n = \int_M c_1^{n-2} c_2 = 0$ and $N_0 = 0$, then $H(z) \equiv 0$ for all $k_0 \geq n - 4$.

Another consequence of Theorem 1.5 is the following

Corollary 4.15. Let $(M, J, S^1)$, $N_0$, $k_0$ and $H(z)$ be as in Theorem 1.5. Then

$$\text{if } n \equiv k_0 \mod 2 \text{ then } H \left( -\frac{k_0}{2} \right) = 0. \quad (4.7)$$

Moreover, if $H(z) \neq 0$ and $n \equiv k_0 \equiv 0 \mod 2$, then the multiplicity of the root $-\frac{k_0}{2}$ is at least $2$.

Proof. In order to prove (4.7), observe that by (1.3) we have that $\tilde{H}(z) = \frac{H(z)}{\prod_{j=1}^{k_0-1}(z+j)}$ is a polynomial, and by Proposition 4.3 (2) it satisfies

$$\tilde{H}(z) = \frac{H(-k_0-z)}{\prod_{j=1}^{k_0-1}(-k_0-z+j)} = \frac{(-1)^n}{\prod_{j=1}^{k_0-1}(z+j)} H(z). \quad (4.8)$$

Hence if $n \equiv k_0 \mod 2$, from (4.8) it follows that $\tilde{H}(z) = 0$, thus proving (4.7). Finally, if $k_0$ is even then $-\frac{k_0}{2} \in \{-1, \ldots, -k_0+1\} \subset \mathbb{Z}$, hence it is a root of both $\prod_{j=1}^{k_0-1}(z+j)$ and $\tilde{H}(z)$. □

From Theorem 1.5 we also have the following refinement of Corollary 4.8 which concerns the position of the roots of $H(z)$.

Corollary 4.16. Let $(M, J, S^1)$, $N_0$, $k_0$ and $H(z)$ be as in Theorem 1.2 and assume that $\deg(H) > 0$. If $k_0 \geq n - 2$ then all the roots of $H(z)$ belong to $C_{k_0}$.

The next corollary gives useful equations in the Chern numbers depending on the index $k_0$ and the parity of $n - k_0$. 


Corollary 4.17 (Equations in the Chern numbers). Let \((M, J, S^1)\) be as in Theorem 1.5. Then
\[
\sum_{h=0}^{n} \frac{1}{h!} \left( \frac{k}{k_0} \right)^h \int_M c_1^h T_{n-h} = 0 \quad \text{for all } k \in \{-1, -2, \ldots, -k_0 + 1\}. \tag{4.9}
\]
Moreover, if \(n \equiv k_0 \mod 2\) then
\[
\sum_{h=0}^{n} \frac{(-1)^h}{2^h h!} \int_M c_1^h T_{n-h} = 0, \tag{4.10}
\]
and if \(n \equiv k_0 \equiv 0 \mod 2\) then
\[
\sum_{h=1}^{n} \frac{(-1)^{h-1}}{2^{h-1}(h-1)!} \int_M c_1^h T_{n-h} = 0. \tag{4.11}
\]

Proof. It is sufficient to notice that (4.11) is equivalent to having \(H'(-k_0) = 0\), and the proof of Corollary 4.17 is a direct consequence of Theorem 1.5, Corollary 4.15 and the definition of Hilbert polynomial (4.4).

Thus the cases in which we can derive more restrictions on the Chern numbers are when \(k_0\) is “large” (see Section 5). On the converse, when \(k_0 = 1\) and \(n\) is even, Corollary 4.17 does not give any restriction.

Before proceeding with the analysis of \(H(z)\) for different values of \(k_0\), in the next subsection we study the properties of the generating function of the sequence \(\{H(k)\}_{k \in \mathbb{N}}\).

4.1. The generating function associated to the Hilbert polynomial. We recall that the generating function of a sequence \(\{b_k\}_{k \in \mathbb{N}} \subset \mathbb{R}\) is the formal power series
\[
P(t) = \sum_{k \geq 0} b_k t^k.
\]
The following result is due to Popoviciu [44] (see also [46, Corollary 4.7]).

Proposition 4.18 (Popoviciu). Let \(H(z)\) be a polynomial of degree \(m\) and \(P(t)\) the generating function of the sequence \(\{H(k)\}_{k \in \mathbb{N}}\). Then
\[
P(t^{-1}) = (-1)^{m+1} t^{k_0} P(t) \tag{4.12}
\]
for some \(k_0 \in \mathbb{Z}_{\geq 1}\) if and only if
\[
H(-1) = H(-2) = \cdots = H(-k_0 + 1) = 0 \tag{4.13}
\]
and
\[
H(k) = (-1)^m H(-k_0 - k) \quad \text{for every } k \in \mathbb{Z}. \tag{4.14}
\]

As a consequence of the properties satisfied by \(H(z)\), we have the following

Proposition 4.19. Let \((M, J, S^1), k_0, N_0\) and \(H(z)\) be as in Theorem 1.5 and let \(\deg(H) = m\). Then the generating function \(P(t)\) of \(\{H(k)\}_{k \in \mathbb{N}}\) is given by
\[
P(t) = \frac{U(t)}{(1 - t)^{m+1}} \tag{4.15}
\]
where \(U(t)\) is a polynomial in \(\mathbb{R}[t]\) such that \(U(0) = N_0\), with
\[
P(t^{-1}) = (-1)^{m+1} t^{k_0} P(t) \tag{4.16}
\]
and
\[
U(t^{-1}) = t^{k_0 - m - 1} U(t). \tag{4.17}
\]
Moreover, if $H(z) \neq 0$, then

$$\frac{m + 1 - k_0}{2} \leq \deg(U) \leq m + 1 - k_0,$$

and $\deg(U) = m + 1 - k_0$ if and only if $N_0 \neq 0$. Here $\deg(U)$ denotes the degree of $U$.

Thus, by Lemma 4.14, if $(M, \omega)$ is a compact symplectic manifold and the $S^1$-action is Hamiltonian, then the polynomial $U(t)$ is of degree $m + 1 - k_0$.

**Proof.** It is well known that the generating function of a sequence $\{H(k)\}_{k \in \mathbb{N}}$, where $H \in \mathbb{R}[z]$ is a polynomial of degree $m$, is of the form given by (4.15), where $U(t) \in \mathbb{R}[t]$ is a polynomial of degree at most equal to $m$. In order to prove that $U(0) = N_0$, observe that

$$P(t) = \frac{U(t)}{(1 - t)^{m+1}} = U(t) \sum_{k \geq 0} \binom{m + k}{m} t^k = U(0) + tQ(t)$$

for some formal power series $Q(t) \in \mathbb{R}[[t]]$. Thus $U(0) = P(0) = H(0)$, and by Proposition 4.3, $H(0) = N_0$.

As for (4.17), observe that by Theorem 1.5 (1.3) we have that (4.13) is satisfied for $k_0 = k_0$, the index of $(M, J)$. Moreover, by Proposition 4.5 (2) and (3), we have that (4.14) is satisfied as well. Thus by Proposition 4.18 the generating function $P(t)$ of $\{H(k)\}_{k \in \mathbb{N}}$ satisfies (4.16), obtaining

$$P(t^{-1}) = (-1)^{m+1} \frac{U(t^{-1})}{(1 - t)^{m+1}} = (-1)^{m+1} \frac{k_0 U(t)}{(1 - t)^{m+1}} = (-1)^{m+1} k_0 P(t)$$

and (4.17) follows.

Let $e = \deg(U)$ and $U(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_e t^e$. By (4.17) we have that

$$\alpha_e t^{m+1-k_0} - e + \alpha_{e-1} t^{m+1-k_0} - e + \cdots + \alpha_0 t^{m+1-k_0} = \alpha_0 + \alpha_1 t + \cdots + \alpha_e t^e,$$

hence we must have $0 \leq m + 1 - k_0 - \epsilon \leq e$, and (4.18) follows. The equality in (4.19) also implies that $\alpha_0 = U(0) = N_0 \neq 0$ if and only if $m + 1 - k_0 - \epsilon = 0$.

We recall that a polynomial of degree $e$, $U(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_e t^e$, is called self-reciprocal if $t^e U(t^{-1}) = U(t)$.

(4.20)

Such a polynomial is sometimes also referred to as a palindrome, since (4.20) is equivalent to saying that the list of coefficients $\alpha_0, \alpha_1, \ldots, \alpha_e$ is a palindrome, i.e. $\alpha_i = \alpha_{e-i}$ for every $i$.

**Corollary 4.20.** With the same notation of Proposition 4.19, we have that:

(i) $U(t)$ is divisible by $t^{m+1-k_0} - e$, where $e = \deg(U)$, and the polynomial $t^e U(t)$ is self-reciprocal.

(ii) If $(M, \omega)$ is a symplectic manifold and the $S^1$-action is Hamiltonian, then $U(t)$ is self-reciprocal. Moreover if $(M, \omega)$ is monotone, i.e. $c_1 = k_0[\omega]$, then $\deg(U) = n + 1 - k_0$.

**Proof.** The claim in (i) is a consequence of Proposition 4.19 (4.17). If $\deg(U) = e = m + 1 - k_0$, which by Proposition 4.19 is equivalent to having $N_0 \neq 0$, we obtain that $U(t)$ is self-reciprocal, and the first claim in (ii) follows from Lemma 4.13. The second claim follows from observing that monotonicity implies $\int_M c_1^k \neq 0$, hence $\deg(H) = n$.

**Remark 4.21.** Observe that the polynomial $U(t)$ determines $P(t)$ which, in turns, determines $H(z)$. Thus the Hilbert polynomial, and hence all the combinations of Chern numbers $a_0, \ldots, a_n$ in (4.5), are completely determined by the coefficients of $U(t)$. Moreover, if $N_0$ is given, the coefficient of degree zero in $U(t)$ is known, since by Proposition 4.19 $U(0) = N_0$. In conclusion, from Corollary 4.20 it follows that the number of coefficients of $U(t)$ to determine is at most equal to $\left\lfloor \frac{m - k_0 - 1}{2} \right\rfloor + 1$. As we will also see in the next section, this explains why the number of conditions that completely determine the Hilbert polynomial (and hence the combinations of Chern
numbers \( a_0, \ldots, a_n \) in \((\mathbb{R}^\omega)\) is the same when \( k_0 = n + 1 - 2k \) and \( k_0 = n - 2k \), for every \( k \in \mathbb{Z} \) such that \( 0 \leq k \leq \frac{n-1}{2} \).

In our context, the main theorem in [45] can be stated in the following way:

**Theorem 4.22** (Rodriguez-Villegas [45]). Let the notation be as in Proposition 4.19 and assume that \( H(z) \neq 0 \). If all the roots of \( U(t) \) are on the unit circle, then \( H(z) \) belongs to \( T_{k_0} \).

In the next section we analyse the different expressions of \( U(t) \) for \( k_0 \in \{n-2, n-1, n, n+1\} \). As a consequence, we prove that if \( k_0 = n \) or \( k_0 = n + 1 \), then \( H(z) \) always belongs to \( T_{k_0} \) (unless \( H(z) = 0 \)). If \( k_0 = n - 2 \) or \( n - 1 \), we derive necessary and sufficient conditions on the Chern numbers that ensure \( H(z) \) to be in \( T_{k_0} \), or more in general that ensure its roots to be on the canonical strip \( S_{k_0} \) (see Corollaries 5.9 and 5.13). As a byproduct, we prove that when \( N_0 = 1 \) and \( n \) is big enough, then \( H(z) \) belongs to \( T_{k_0} \) if and only if the roots of \( U(t) \) are on the unit circle (see Corollaries 5.10 and 5.16).

### 4.2. Connection with Ehrhart polynomials

Some of the results in Section 4 can be regarded as a generalisation of what is already known for the Ehrhart polynomial of a reflexive polytope.

The link between Hilbert polynomials of \( S^1 \)-spaces and Ehrhart polynomials of reflexive polytopes is given by monotone symplectic toric manifolds.

Suppose that \((M, \omega)\) is a compact symplectic manifold of dimension \(2n\), and that the \( S^1 \)-action extends to a toric action, i.e. \( S^1 \) is a circle in an \( n \)-dimensional torus \( T^n \) which is acting effectively on \((M, \omega)\) with moment map \( \Psi : (M, \omega) \to \text{Lie}(T^n)^* \). We identify \( \text{Lie}(T^n)^* \) with \( \mathbb{R}^n \), and let the dual lattice be \( \mathbb{Z}^n \). By the Atiyah [2] and Guillemin-Sternberg [20] convexity theorem, we know that \( \Psi(M) =: \Delta \) is a convex polytope, more precisely it is the convex hull of its vertices, which coincide with the images of the fixed points of the \( T^n \) action. Suppose that \((M, \omega)\) is also monotone, i.e. \( c_1 = k_0 [\omega] \) (so \( [\omega] \) is primitive in \( H^2(M; \mathbb{Z}) \), which is torsion free in this case) and choose the moment map \( \Psi \) so that all the vertices of \( \Delta \) belong to the lattice \( \mathbb{Z}^n \): we call such polytope \( \Delta \) primitive and integral. As a consequence of a result of Danilov [11], we have that the Hilbert polynomial \( H(z) \) of \((M, \omega)\) coincides with the Ehrhart polynomial \( i_{\Delta}(z) \) of \( \Delta \). Moreover, it is well-known that there exists a (unique) \( k \in \mathbb{Z}_{\geq 0} \) such that the dilated polytope \( \Delta' = k\Delta \), suitably translated by an integer vector, is reflexive. By a result of Hibi [26], this is equivalent to saying that the Ehrhart polynomial \( i_{\Delta}(z) \) and its associated generating function \( P_{\Delta}(t) = \frac{U(t)}{(1-t)^{n+1}} \) satisfy

\[
    i_{\Delta'}(z) = (-1)^n i_{\Delta'}(-1-z) \quad \text{and} \quad P_{\Delta'}(t^{-1}) = (-1)^{n+1} t P_{\Delta}(t). \tag{4.21}
\]

The following gives a combinatorial characterisation of the index \( k_0 \) of \((M, \omega)\) (which, by Remark 4.3, coincides with the minimal Chern number):

**Lemma 4.23.** Let \((M, \omega, T, \Psi)\) be a monotone symplectic toric manifold, with symplectic form satisfying \( c_1 = k_0 [\omega] \). Consider the primitive integral moment polytope image \( \Delta' \). Then the index \( k_0 \) is the unique integer so that \( \Delta' = k_0 \Delta \) is reflexive.

**Proof.** First of all observe that, from \( \Delta' = k\Delta \) we have \( i_{\Delta}(z) = i_{\Delta'}(\frac{z}{k}) \) for every \( z \in \mathbb{C} \). Moreover, as observed before, \( H(z) = i_{\Delta}(z) \). So from (4.21) we have that

\[
    H(z) = i_{\Delta}(z) = i_{\Delta'}\left(\frac{z}{k}\right) = (-1)^n i_{\Delta'}\left(1 - \frac{z}{k}\right) = (-1)^n i_{\Delta}(-k - z) = (-1)^n H(-k - z),
\]

for every \( z \in \mathbb{C} \). By Remark 4.3 \( H(z) \) is a nonzero polynomial, so Proposition 4.3 (2) implies that \( k_0 = k \).

It is in this sense that we can regard the symmetry property of \( H(z) \) (i.e. Proposition 4.3 (2)) and the results in Proposition 4.19 as a generalisation of (4.21).

---

6An integer polytope \( P \subset \mathbb{R}^n \) of dimension \( n \) is reflexive if it contains the origin in its interior, and its dual polytope \( P^* = \{ x \in \mathbb{R}^n \mid x \cdot y \leq 1 \text{ for any } y \in P \} \) has all its vertices on the lattice \( \mathbb{Z}^n \) too.
5. Computation of \( H(z) \) and Chern numbers for some values of \( k_0 \)

In this section, we compute explicitly the Hilbert polynomial \( H(z) \) and its associated generating function for several values of \( k_0 \), deriving many properties of the Chern numbers of \((M, J)\).

Let \( \sigma_j(x_1, \ldots, x_n) \) be the \( j \)-th elementary symmetric polynomials in \( x_1, \ldots, x_n \), for \( j = 0, \ldots, n \), and let \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) be the \( \text{unsigned Stirling numbers of the first kind} \), where \( k, n \in \mathbb{N} \) and \( 1 \leq k \leq n \), satisfying

\[
(x)^{(n)} = x(x + 1) \cdots (x + n - 1) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] x^k,
\]

(5.1)

where \( (x)^{(n)} \) is the rising factorial. Thus we have the relation:

\[
\sigma_k(1, 2, \ldots, n) = \left( \begin{array}{c} n + 1 \\ n - k + 1 \end{array} \right),
\]

(5.2)

and the following well-known identities:

\[
\begin{align*}
\sigma_0(1, 2, \ldots, n) &= \left( \begin{array}{c} n + 1 \\ n + 1 \end{array} \right) = 1, \\
\sigma_1(1, 2, \ldots, n) &= \left( \begin{array}{c} n + 1 \\ n \end{array} \right) = \frac{1}{2} (n + 1), \\
\sigma_2(1, 2, \ldots, n) &= \left( \begin{array}{c} n + 1 \\ n - 1 \end{array} \right) = \frac{1}{4} (3n + 2) \left( \begin{array}{c} n + 1 \\ 3 \end{array} \right) = \frac{(3n+2)(n+1)n(n-1)}{24},
\end{align*}
\]

(5.3)

Observe that by Corollary 4.10 if \( k_0 > n + 1 \) then \( H(z) \equiv 0 \) and \( \int_M c_1^n T_{n-h} = 0 \) for every \( h = 0, \ldots, n \). So in the rest of the section we will focus on the cases in which \( k_0 \leq n + 1 \).

**Proposition 5.1** (\( k_0 = n + 1 \)). Let \((M, J, S^1)\) be as in Theorem 1.5 and assume that \( k_0 = n + 1 \). Then

\[
H(z) = \frac{N_0}{n!} \prod_{j=1}^{n} (z + j)
\]

(5.5)

and for every \( h = 0, \ldots, n \) we have

\[
\int_M c_1^n T_{n-h} = N_0 \frac{h!(n+1)^h}{n!} \left[ \begin{array}{c} n + 1 \\ h + 1 \end{array} \right]
\]

(5.6)

In particular

\[
\int_M c_1^n = N_0(n+1)^n
\]

(5.7)

and

\[
\int_M c_1^{n-2} c_2 = N_0 \frac{n(n+1)^{n-1}}{2}.
\]

(5.8)

Moreover, the generating function of \( \{H(k)\}_{k \in \mathbb{N}} \) is given by

\[
P(t) = N_0 \frac{1}{(1-t)^{n+1}}.
\]

(5.9)

**Remark 5.2** From (5.9) and Proposition 4.19 we have that in this case \( U(t) = N_0 \), and if \( N_0 \neq 0 \), the zeros of \( H(z) \) coincide with the integers greater than \(-k_0 = -(n + 1)\) and smaller than 0, thus in particular \( H(z) \) belongs to \( T_{n+1} \), and hence all its roots are on the canonical strip \( S_{n+1} \) (cf. Theorem 4.22).
Moreover, the generating function of \( \{H(k)\}_{k \in \mathbb{N}} \) is given by

\[
P(t) = \frac{N_0}{1 - t^{n+1}}
\]
Proof of Proposition 5.4. The proof is very similar to that of Proposition 5.1, but we include it here for the sake of completeness. If $N_0 = 0$ then all the claims in Proposition 5.4 follow from Corollary 4.10 (H).

Suppose that $N_0 \neq 0$. Then by Proposition 4.5 (1) we have that $H(z) \neq 0$, and from Theorem 1.5 (1.3) and Corollary 4.15 we have that $H(z) = \beta(z + \frac{2}{3}) \prod_{j=1}^{n-1} (z + j)$. In order to determine $\beta$ we can use Proposition 4.5 (1), obtaining $\beta = \frac{4N_0}{m}$, thus implying (5.11). The equations in (5.12) follow easily from observing that

$$
\sigma_{n-h}(1, 2, \ldots, n-1, \frac{n}{2}) = \sigma_{n-h}(1, 2, \ldots, n-1) + \frac{n}{2} \sigma_{n-h-1}(1, 2, \ldots, n-1) = \left[ \frac{n}{h} \right] + \frac{n}{2} \left[ \frac{n}{h+1} \right].
$$

In order to prove (5.13) it is sufficient to consider (5.12) with $h = n$ (or $h = n - 1$). To prove (5.14), first of all observe that

$$
\sigma_1(1, 2, \ldots, n-1, \frac{n}{2}) = \sigma_1(1, 2, \ldots, n-1) + \frac{n}{2} \sigma_1(1, 2, \ldots, n-1) = \frac{1}{24} n(n-1)(3n^2 - n + 2),
$$

where the last equality follows from (5.3) and (5.4). Thus if we take $h = n - 2$ in (5.12) we obtain

$$
\int_M c_1^{n-2} \left( \frac{c_1^2 + c_2}{12} \right) = N_0 \frac{2(n-2)! n^{n-2}}{n!} \sigma_2(1, 2, \ldots, n-1, \frac{n}{2}) = \frac{N_0}{12} n^{n-2}(3n^2 - n + 2),
$$

and the conclusion follows from (5.13).

In order to prove (5.15), observe by the above discussion that if $k_0 = n$ then $H(z)$ is either of degree $n$, which happens exactly if $N_0 \neq 0$, or it is identically zero. In the first case, by Proposition 4.19 and Corollary 4.20 $U(t)$ is a self-reciprocal polynomial of degree one and $U(0) = N_0$, thus implying (5.15).

From Propositions 5.1 and 5.4 we can see that the cases $k_0 = n + 1$ and $k_0 = n$ are very similar, in the sense that the Hilbert polynomial $H(z)$, as well as the combinations of Chern numbers $\int_M c_1^h T_{n-h}$, for $h = 0, \ldots, n$ and the generating function $P(t)$ of $\{H(k)\} \in \mathbb{N}$, are completely determined (cf. Remark 4.21).

Remark 5.6 In recent work Li [83] proves that if the $2n$-dimensional manifold $M$ is symplectic, the $S^1$ action Hamiltonian and $\chi(M) = n + 1$, then having $k_0 = n + 1$ (resp. $k_0 = n$) is equivalent to having the same total Chern class of $\mathbb{C}P^n$ (resp. of the Grassmannian of oriented planes in $\mathbb{R}^{n+2}$ with $n$ odd) which, in turns, is equivalent to having the same integral cohomology ring of $\mathbb{C}P^n$ (resp. the Grassmannian). Thus in particular, under the above hypotheses, all the Chern numbers are "standard", i.e. they agree with those of $\mathbb{C}P^n$ (resp. of the Grassmannian). In particular the author uses the existence of a quasi-ample line bundle (in the sense specified in Remark 4.13 which in this case is given by the pre-quantization line bundle (see also [13], Proposition 7.5 (i))).

In the following we analyse in details the cases $k_0 = n - 1$ and $k_0 = n - 2$.

Proposition 5.7 ($k_0 = n - 1$). Let $(M, J, S^1)$ be as in Theorem 7.9 and assume that $k_0 = n - 1$ and $n \geq 2$.

(a) If $N_0 \neq 0$ and $\int_M c_1^0 \neq 0$ then

$$
H(z) = \frac{4N_0}{(n-2)! (n-1)^2 - 4a^2} \left( z^2 + (n-1)z + \frac{(n-1)^2}{4} - a^2 \right) \prod_{j=1}^{n-2} (z + j),
$$

(5.16)
where \( a \) is either real or pure imaginary and \( a \neq \pm \frac{n-1}{2} \). Moreover

\[
\int_M c_1^n = \frac{4N_0 n(n-1)^{n+1}}{(n-1)^2 - 4a^2},
\]

(5.17)

and

\[
\int_M c_1^{n-2}c_2 = \frac{4N_0(n-1)^{n-2}}{[(n-1)^2 - 4a^2]} \left[ 3 - 12a^2 - 6n + \frac{9}{2}n^2 - 2n^3 + \frac{n^4}{2} \right].
\]

(5.18)

(b) If \( N_0 \neq 0 \) and \( \int_M c_1^n = 0 \) then

\[
H(z) = \frac{N_0}{(n-2)!} \prod_{j=1}^{n-2} (z + j),
\]

(5.19)

and

\[
\int_M c_1^{n-2}c_2 = 12 N_0(n-1)^{n-2}.
\]

(5.20)

Moreover, in (a) and (b), the generating function of \( \{H(k)\}_{k \in \mathbb{N}} \) is given by

\[
P(t) = N_0 \frac{1 + bt + t^2}{(1-t)^{n+1}}
\]

(5.21)

where \( b \in \mathbb{Q} \) is such that \( bN_0 \in \mathbb{Z} \) and

\[
\int_M c_1^n = 4\, N_0(n-1)^n,
\]

(5.22)

\[
\int_M c_1^{n-2}c_2 = N_0(n-1)^{n-2}[12 + \frac{(b+2)n(n-3)}{2}].
\]

(5.23)

(Thus case (b) corresponds to taking \( b = -2 \).)

(c) If \( N_0 = 0 \) then

\[
H(z) = \gamma \prod_{j=0}^{n-1} (z + j),
\]

(5.24)

where \( \gamma = \frac{1}{(n-1)^{n+1}} \int_M c_1^n \).

Moreover, the generating function of \( \{H(k)\}_{k \in \mathbb{N}} \) is given by

\[
P(t) = \gamma n! \frac{t}{(1-t)^{n+1}}.
\]

(5.25)

\textbf{Remark 5.8} Observe that the value of \( a \) in (5.17) cannot be arbitrary, since the following fraction

\[
\frac{4N_0 n(n-1)}{(n-1)^2 - 4a^2}
\]

must be an integer. This follows from the fact that, modulo torsion, \( c_1 = (n-1)\eta_0 \) for some \( \eta_0 \in H^2(M;\mathbb{Z}) \), and hence \( \int_M c_1^n \) must be an integer.

The following corollary is a straightforward consequence of Proposition 5.7.

\textbf{Corollary 5.9.} Under the same hypotheses of Proposition 5.7, we have that:

- If \( N_0 \neq 0 \) then
  
  1. The roots of \( H(z) \) belong to the canonical strip \( S_{n-1} \) if and only if \( \int_M c_1^n \geq 0 \), or equivalently if and only if \( b \geq -2 \).
  2. \( H(z) \) belongs to \( T_{n-1} \) if and only if \( 0 \leq \int_M c_1^n \leq 4N_0(n-1)^{n-1} \), or equivalently if and only if \( -2 \leq b \leq \frac{2n+1}{n-1} \).

- If \( N_0 = 0 \) then the roots of \( H(z) \) do not belong to \( S_{n-1} \).
As a result of the analysis carried out when $k_0 = n - 1$, we can strengthen Theorem 4.24.

**Corollary 5.10.** Under the same hypotheses of Proposition 5.7 assume that $N_0 = 1$ and $n > 5$. Then $H(z)$ belongs to $T_{n-1}$ if and only if $U(t)$ has its roots on the unit circle.

**Proof.** If $N_0 = 1$ then by Proposition 5.7 we know that $b$ is an integer. If $n > 5$, from Corollary 5.9 we can see that $H(z)$ belongs to $T_{n-1}$ if and only if $-2 \leq b \leq 2$. Since $b$ is an integer, for all such values of $b$ the polynomial $U(t) = 1 + bt + t^2$ has its roots on the unit circle.

**Remark 5.11.** For $2 \leq n \leq 5$, we have that $\frac{2n+1}{n-1} \geq 3$; however for $b \geq 3$, the roots of $U(t)$ are not on the unit circle. So for $2 \leq n \leq 5$, there may exist manifolds whose associated Hilbert polynomial belongs to $T_{n-1}$, but the corresponding $U(t) = 1 + bt + t^2$ does not have its roots on the unit circle: consider for example the Fano threefold $V_5$ in Example 6.5 (3), for which $b = 3$ and the corresponding Hilbert polynomial is given by $H_{V_5}(z) = \frac{1}{3}(5z^2 + 10z + 6)(z + 1)$.

**Proof of Proposition 5.7.** (a) If $N_0 \neq 0$ then, by Proposition 4.1 (1), we have that $H(z) \neq 0$. Moreover by (4.4), if $f_M^{c_1^0} \neq 0$ then $\deg(H) = n$. By Theorem 1.5 (1), $H(z)$ has roots $-1, -2, \ldots, -n+2$. By Corollary 4.9, the remaining two roots belong to $C_{n-1}$ and, by Proposition 5.3 (4), they are of the form $-\frac{n-1}{a} - \frac{n-1}{b} + a$, where $a$ is either real or pure imaginary. Moreover $a \neq \pm \frac{n-1}{2}$ since by Proposition 4.3 (1) and (2), $H(0) = N_0$, $H(1) = (-1)^n N_0$ and by assumption $N_0 \neq 0$. Thus $H(z) = a(z^2 + (n-1)z + (\frac{n-1}{2})^2) \prod_{j=1}^{n}(z + j)$, where $a \in \mathbb{R}$ can be found by imposing that $H(0) = N_0$, obtaining $\lambda_{10}$. Equations $\lambda_{17}$ and $\lambda_{18}$ come from combining (4.4) with (6.16).

(b) If $N_0 \neq 0$ and $f_M^{c_1^0} = 0$ then, by Proposition 4.1 (1) we have that $H(z) \neq 0$ and, by (4.4), $\deg(H) \leq n - 2$. By Theorem 1.5, $H(z)$ has $n - 2$ roots given by $-1, -2, \ldots, -n + 2$. Thus $H(z)$ has degree $n - 2$ and it is of the form $H(z) = \beta \prod_{j=1}^{n-2}(z + j) = \beta \sum_{h=0}^{n-2} z^h \sigma_{n-h-2}(1, 2, \ldots, n - 2)$. By Proposition 4.1 (1) we have $\beta = \frac{N_0}{n - 2}$, and (9.19) follows. Equation (9.20) can be obtained from (6.19) and (4.4) by taking $h = n - 2$.

In order to prove $\lambda_{24}$ for $f_M^{c_1^0} \neq 0$, observe that since $\deg(H) = n$, $N_0 \neq 0$ and $k_0 = n - 1$, from Proposition 4.19 and Corollary 4.20 it follows that $U(t) = N_0(1 + bt + t^2)$ for some $b \in \mathbb{R}$. Thus we have that

$$P(t) = N_0 \frac{1 + bt + t^2}{(1 - t)^{n+1}} = N_0 \sum_{k} \left[ \binom{n + k - 2}{n} b \binom{n + k - 1}{n} + \binom{n + k}{n} \right] t^k,$$

and by definition of $P(t)$ we have that $N_0(b + n + 1) = H(1)$. Since $H(1)$ is an integer, it follows that $b N_0$ must be an integer. Moreover, by (6.16) we have that $\frac{H(1)}{N_0} = \frac{4(n - 1)n + (n - 1)^2 - a^2}{(n - 1)^2 - 4a^2} = b + n + 1$, thus obtaining $b$ in terms of $a^2$, and the expressions of $f_M^{c_1^0}$ and $f_M^{c_1^{n-2}c_2}$ in terms of $b$ follow from (6.17) and (6.18).

The proof of $\lambda_{24}$ when $f_M^{c_1^0} = 0$ also follows from Proposition 4.19 and the details are left to the reader.

(c) If $N_0 = 0$, then, by Proposition 4.1 (1) and (2), and Theorem 1.5 (1), $H(z)$ has $n$ roots given by $0, -1, -2, \ldots, -n + 1$. If $f_M^{c_1^0} = 0$ then by (4.4) and (4.5) we have that $\deg(H) \leq n - 2$, hence $H(z) \equiv 0$ and $\lambda_{22}$. Otherwise $H(z) = \gamma \prod_{j=0}^{n-1}(z + j)$ where the expression for $\gamma$ can be obtained by using (4.4), imposing that $a_n = \gamma$.

The proof of $\lambda_{25}$ follows easily from Proposition 4.19 and the details are left to the reader. 

Proposition 5.7 implies that, when $k_0 = n - 1$, the Chern numbers $f_M^{c_1^0}$ and $f_M^{c_1^{n-2}c_2}$ are related by the following formula.
Corollary 5.12. Under the same hypotheses of Proposition 5.7 we have that
\[
\int_M c_i^{n-2}c_2 - \frac{n(n-3)}{2(n-1)^2} \int_M c_i^n = 12N_0(n-1)^{n-2}
\]

Proof. When \(N_0 \neq 0\) the claim follows from (5.22) and (5.23).

If \(N_0 = 0\) and \(\int_M c_i^n = 0\) then from (5.24) we have \(H(z) \equiv 0\), which, by (4.5) implies that
\[
a_{n-2} = \frac{1}{12(n-1)^{n-2}(n-2)!} \left( \int_M c_i^n + c_i^{n-2}c_2 \right) = 0,
\]
thus implying \(\int_M c_i^{n-2}c_2 = 0\), and the claim follows.

Otherwise, if \(N_0 \neq 0\) and \(\int_M c_i^n \neq 0\), from (5.24) and (5.10) we have that \(a_{n-2}\) is
\[
a_{n-2} = \gamma \left[ \frac{n}{n-2} \right] = \frac{(3n-1)n(n-1)(n-2)}{24},
\]
where \(\gamma = \frac{1}{(n-1)n} \int_M c_i^n\), and the claim follows from comparing the general expression of \(a_{n-2}\) with (5.26). \(\square\)

Proposition 5.13 \((k_0 = n - 2)\). Let \((M, J, S^1)\) be as in Theorem 4.5 and assume that \(k_0 = n - 2\) and \(n \geq 3\).

(a) If \(N_0 \neq 0\) and \(\int_M c_i^n \neq 0\) then
\[
H(z) = \frac{4N_0}{(n-2)![(n-2)^2 - 4a^2]}(2z + n - 2) \left( z^2 + (n-2)z + \frac{(n-2)^2 + a^2}{4} \right)^{n-3} \prod_{j=1}^{n-1} (z + j),
\]
where \(a\) is either real of pure imaginary and \(a \neq \pm \frac{n-2}{2}\). Moreover
\[
\int_M c_i^n = \frac{8N_0 n(n-1)(n-2)^n}{(n-2)^2 - 4a^2},
\]
and
\[
\int_M c_i^{n-2}c_2 = \frac{4N_0(n-2)^{n-2}(24 - 24a^2 - 30n + 17n^2 - 6n^3 + n^4)}{(n-2)^2 - 4a^2}.
\]

(b) If \(N_0 \neq 0\) and \(\int_M c_i^n = 0\) then
\[
H(z) = \frac{N_0}{(n-2)!}(2z + n - 2) \prod_{j=1}^{n-3} (z + j),
\]
and
\[
\int_M c_i^{n-2}c_2 = 24N_0(n-2)^{n-2}.
\]

Moreover, in (a) and (b), the generating function of \(\{H(k)\}_{k \in \mathbb{N}}\) is given by
\[
P(t) = N_0 \frac{1 + bt + bt^2 + t^3}{(1-t)^{n+1}}
\]
where \(b\) is such that \(bN_0\) is an integer and
\[
\int_M c_i^n = 2N_0(b+1)(n-2)^n,
\]
\[
\int_M c_i^{n-2}c_2 = N_0(n-2)^{n-2}[24 + (b+1)(n-2)(n-3)],
\]
and case (b) corresponds to taking \(b = -1\).
(c) If $N_0 = 0$ then

$$H(z) = \gamma \left(z + \frac{n-2}{2}\right)^{n-2} \prod_{j=0}^{n-2} (z + j), \quad (5.35)$$

where $\gamma = \frac{1}{(n-2)!^2} \int_M c_1^2$. Moreover, the generating function of $\{H(k)\}_{k \in \mathbb{N}}$ is given by

$$P(t) = \frac{2n!}{t + t^2} \left(1 - \frac{1}{1-t}\right)^{n+1}. \quad (5.36)$$

Remark 5.14 The same comment in Remark 5.8 applies here: the value of $a$ cannot be arbitrary, since the following fraction

$$\frac{8N_0 n(n-1)}{(n-2)^2 - 4a^2}$$

must be an integer.

The following corollary is very similar to Corollary 5.9 and is a straightforward consequence of Proposition 5.13.

**Corollary 5.15.** Under the same hypotheses of Proposition 5.13, we have that:

- If $N_0 \neq 0$ then
  
  1. The roots of $H(z)$ belong to the canonical strip $S_{n-2}$ if and only if $\int_M c_1^2 \geq 0$, or equivalently if and only if $b \geq -1$.
  2. $H(z)$ belongs to $T_{n-2}$ if and only if $0 \leq \int_M c_1^2 \leq 8N_0 n(n-1)(n-2)^{n-2}$, or equivalently if and only if $-1 \leq b \leq \frac{3n^2 - 4}{(n-2)^2}$.

- If $N_0 = 0$ then the roots of $H(z)$ do not belong to $S_{n-2}$.

In analogy with Corollary 5.10 we have the following:

**Corollary 5.16.** Under the same hypotheses of Proposition 5.13, assume that $N_0 = 1$ and $n > 14$. Then $H(z)$ belongs to $T_{n-2}$ if and only if $U(t)$ has its roots on the unit circle.

**Proof.** If $N_0 = 1$ then by Proposition 5.13 we know that $b$ is an integer. If $n > 14$, from Corollary 5.13 we can see that $H(z)$ belong to $T_{n-2}$ if and only if $-1 \leq b \leq 3$. Since $b$ is an integer, for all such values of $b$ the polynomial $U(t) = 1 + bt + bt^2 + t^3$ has its roots on the unit circle. \(\square\)

Remark 5.17 For $3 \leq n \leq 14$, we have that $\frac{3n^2 - 4}{(n-2)^2} \geq 4$; however for $b \geq 4$, the roots of $U(t) = 1 + bt + bt^2 + t^3$ are not on the unit circle. In conclusion, we can say that for $3 \leq n \leq 14$, there may exist manifolds whose associated Hilbert polynomial belongs to $T_{n-2}$, but the corresponding $U(t)$ does not have its roots on the unit circle: consider for example the Fano threefold $V_{22}$ in Example 6.6 (2), for which $b = 10$ and the corresponding Hilbert polynomial is given by $H_{V_{22}}(z) = \frac{1}{(11z^2 + 11z + 6)}(2z + 1)$.

**Proof of Proposition 5.13.** The proof of this Proposition is very similar to that of Proposition 5.7 and here we only sketch the first part. (a) If $N_0 \neq 0$ then, by Proposition 5.3 (1) we have that $H(z) \neq 0$. Moreover by (1.3), if $\int_M c_1^2 \neq 0$ then $\deg(H) = n$. By Theorem 1.5 (1.3), $H(z)$ has roots $-1, -2, \ldots, -n + 3$. By Corollary 1.15 one of the remaining three roots is $-\frac{n-2}{2}$. By Corollary 1.9 the remaining two roots are on $S_{n-2}$, and by Proposition 1.3 (2) they are of the form $-\frac{n-2}{2} - a$, $-\frac{n-2}{2} + a$, where $a$ is either real or pure imaginary. Moreover $a \neq \pm \frac{n-2}{2}$ since by Proposition 1.5 (1) and (2), $H(0) = N_0$, $H(-n + 2) = (-1)^n N_0$ and by assumption $N_0 \neq 0$. It follows that the
Hilbert polynomial is of the form
\[
H(z) = \alpha \left( 2z + n - 2 \right) \left( z^2 + (n - 2)z + \frac{(n - 2)^2}{4} - a^2 \right) \prod_{j=1}^{n-3} (z + j),
\]
where \( \alpha \) can be found by imposing \( H(0) = N_0 \), thus obtaining (5.37). The rest of the proof is left to the reader.

Similarly to the case \( k_0 = n - 1 \), Proposition 5.13 implies that the Chern numbers \( \int_M c_1^{n-2}c_2 \) are related by the following formula.

**Corollary 5.18.** Under the same hypotheses of Proposition 5.13 we have that
\[
\int_M c_1^{n-2}c_2 - \frac{n - 3}{2(n - 2)} \int_M c_1^n = 24N_0(n - 2)^{n-2}
\]

**Proof.** The proof of this Corollary is very similar to that of Corollary 5.12 and the details are left to the reader. \( \square \)

As a consequence of the analysis of \( H(z) \) when the index \( k_0 \) is \( n - 2 \) or \( n \) we have the following

**Corollary 5.19.** Let \((M, J, S^1)\) be as in Theorem 1.3 and assume that \( N_0 \neq 0 \), and \( k_0 = n \) or \( k_0 = n - 2 \) and \( n \geq 3 \). Then the Chern numbers \( \int_M c_1^n \) and \( \int_M c_1^{n-2}c_2 \) are always even.

**Proof.** When \( k_0 = n \) the claim follows from Proposition 5.13(5.13) and (5.14), and when \( k_0 = n - 2 \) it follows from Proposition 5.13(5.13) and (5.14). \( \square \)

The case in which \( k_0 = n - 3 \), where \( n \geq 4 \), is not analysed in details here. However we would like to make some remarks about it when \( N_0 \neq 0 \) and \( \deg(H) = n \), i.e. \( \int_M c_1^n \neq 0 \). First of all, observe that this is the first case in which the roots of \( H(z) \) may not belong to \( C_{k_0} \) (see Corollary 4.16). From Theorem 4.13(4.13), the roots of \( H(z) \) are \(-1, -2, \ldots, -n + 4\), plus four additional roots \( z_1, z_2, z_3, z_4 \). If the remaining four roots don’t belong to \( C_{k_0} \), from the properties of \( H(z) \) they must be of the form \(-\frac{n-3}{2} \pm a \pm i b\), for some \( a, b \in \mathbb{R} \setminus \{0\} \), thus obtaining that
\[
H(z) = \alpha \prod_{j=1}^{n-4} \left( z + \frac{n-3}{2} \pm a \pm i b \right) \prod_{j=1}^{n-3} (z + j).
\]

(5.37)

From the expression of \( a_n \) in (4.34) and Proposition 4.5(4.5) it follows that
\[
\left[ \left( \frac{n-3}{2} - a \right)^2 + b^2 \right] \left[ \left( \frac{n-3}{2} + a \right)^2 + b^2 \right] = \frac{N_0 n! (n - 3)^n}{(n - 4)!} \int_M c_1^n,
\]
which implies that \( \int_M c_1^n > 0 \). Moreover, for a fixed value of \( \int_M c_1^n \), the four roots \( z_1, \ldots, z_4 \) belong to the **Cassini oval** of equation
\[
d(p, 0) d(p, -n + 3) = \sqrt{\frac{N_0 n! (n - 3)^n}{(n - 4)!} \int_M c_1^n},
\]
where \( d(p, q) \) denotes the Euclidean distance from \( p \) to \( q \), with \( p, q \in \mathbb{R}^2 \cong \mathbb{C} \), and the foci of this oval are the points 0 and \(-n + 3\) (see Figure 5.1).

---

7We recall that a **Cassini oval** is a quartic plane curve given by the locus of points in \( \mathbb{R}^2 \cong \mathbb{C} \) satisfying the equation
\[
d(p, q_1) d(p, q_2) = d^2,
\]
where \( d \neq 0 \). The points \( q_1 \) and \( q_2 \) are called the **foci** of the Cassini oval.
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Out = Minus 6 Minus 4 Minus 2 0 2

Figure 5.1. Examples of Cassini ovals of equation \(d(p, q_1) d(p, q_2) = d^2\) with foci \(q_1 = (0, 0)\) and \(q_2 = (-4, 0)\) for different values of \(d\). The curve passing through the origin is called the lemniscate of Bernoulli, and is obtained for \(d = 4\).

5.1. Conclusions on Hamiltonian and non-Hamiltonian actions. As an application of the results obtained before, we conclude the section with the proof of Theorem 1.7, which, for \(k_0 \geq n - 2\), gives necessary and sufficient conditions for a symplectic action with nonempty, discrete fixed point set to be non-Hamiltonian. Observe that for finding such actions we can assume \(n \geq 3\). In fact, for \(n = 1\) and \(n = 2\) there do not exist symplectic non-Hamiltonian circle actions with nonempty discrete fixed point sets: for \(n = 1\) the only surface admitting such a symplectic circle action is a sphere, hence the action is Hamiltonian; for \(n = 2\) the assertion was proved by McDuff in [38, Proposition 2].

Proof of Theorem 1.7. By Lemma 1.4, \(N_0 = 0\) if and only if the action is non-Hamiltonian. Thus the claims in (I) follow from Corollary 4.10 (I'), those in (II) and (III) from Propositions 5.1 and 5.4, and those in (IV) and (V) from Corollaries 5.12 and 5.18. □

Remark 5.20 Observe that by Propositions 5.1 and 5.4, when \(k_0 = n + 1\) or \(k_0 = n\) the action is Hamiltonian if and only if all the combinations of Chern numbers \(\int_M c_1^h T_{n-h}\) do not vanish, for \(h = 0, \ldots, n\).

6. Examples: low dimensions of \((M, J)\)

In this section, we study some consequences of the results previously obtained when \(\dim(M) \leq 8\). In particular we prove that when \(k_0 = n\) or \(n + 1\) then all the Chern numbers of \((M, J, S^1)\) can be expressed as a linear combination of the \(N_j\)'s, where \(N_j\) denotes the number of fixed points with exactly \(j\) negative weights. In the Hamiltonian category, this amounts to saying that all the Chern numbers of \((M, \omega)\) can be expressed as linear combinations of the Betti numbers of \(M\) (see (12)).
The most obvious Chern number that can always be written in terms of the $N_j$s is $\int_M c_n$. In fact, by (1.1) and by definition of the $N_j$s we have
\[
\int_M c_n = \sum_{j=0}^{n} N_j.
\] (6.1)

In [15], Godinho and the author prove that the Chern number $\int_M c_1 c_{n-1}$ can also be expressed in terms of the $N_j$s. We recall its explicit expression in the following

**Theorem 6.1** ([15] Theorem 1.2). Let $(M, J, S^1)$ and $N_j$ be as above. Then
\[
\int_M c_1 c_{n-1} = \sum_{j=0}^{n} N_j \left[6j(j-1) + \frac{5n-3n^2}{2}\right].
\] (6.2)

In particular, if $N_j = 1$ for every $j = 0, \ldots, n$ then
\[
\int_M c_1 c_{n-1} = \frac{1}{2} n(n+1)^2.
\] (6.3)

Observe that Theorem 6.1 implies that $k_0$ and the $N_j$s are related. In fact, from the definition of $k_0$ it is straightforward to see that it must divide the right-hand side of (6.2). For example, if the action satisfies $N_j = 1$ for every $j = 0, \ldots, n$, then $k_0$ must divide $\frac{1}{2} n(n+1)^2$ (this happens when we have a Hamiltonian $S^1$-action with minimal number of fixed points, see [15, Section 2.1.1]).

The case in which $(M, J)$ is of (real) dimension 2 is trivial, since an almost complex surface with positive Euler characteristic (corresponding to the number of fixed points) is a sphere. Hence $k_0 = 2$, $H(z) = 1 + z$ and $\int_{S^2} c_1(S^2) = 2$.

6.1. dim$(M) = 4$. First of all, observe that by (2.2) and (6.1) we have
\[
\int_M c_2 = 2N_0 + N_1.
\] (6.4)

Moreover, by (2.2) and Theorem 6.1 (6.2) it follows that for $n = 2$ we have
\[
\int_M c_1^2 = 10N_0 - N_1.
\] (6.5)

Thus in dimension 4 all the Chern numbers can be expressed as a linear combination of the $N_j$s (independently on $k_0$). In the next Proposition we strengthen this result for $k_0 = 2$ and 3. In fact, by combining (6.3) with Propositions 5.1 and 5.4 in the next Proposition we prove that when $k_0 = 2$ or 3 the $N_j$s are related to each other, and when $M$ is symplectic we prove what we refer to as the 'symplectic analogue of the Kobayashi-Ochiai Theorem'.

**Proposition 6.2** (dim$(M) = 4$, $k_0 = 2, 3$). Let $(M, J, S^1)$ be an $S^1$-space, and let $k_0$, $H(z)$ and the $N_j$s be defined as before. Then $N_0 \neq 0$, the first Chern class $c_1$ is not a torsion element in $H^2(M; \mathbb{Z})$, and $k_0 \in \{1, 2, 3\}$. Moreover
\begin{enumerate}
    \item[(a)] If $k_0 = 3$ then
    \[N_0 = N_1 = N_2, \quad \int_M c_1^2 = 9N_0 \quad \text{and} \quad H(z) = \frac{N_0}{2}(z+1)(z+2).\] (6.6)
\item[(b)] If $k_0 = 2$ then
    \[2N_0 = N_1 = 2N_2, \quad \int_M c_1^2 = 8N_0 \quad \text{and} \quad H(z) = N_0(z+1)^2.\] (6.7)
\end{enumerate}

If $(M, \omega)$ is a connected symplectic manifold, and the $S^1$-action is symplectic, then
\begin{enumerate}
    \item[(a')] $k_0 = 3$ if and only if $(M, \omega)$ is equivariantly symplectomorphic to $(\mathbb{C}P^2, \tilde{\omega})$ for a suitable $S^1$-action on $\mathbb{C}P^2$, where $\tilde{\omega}$ is the standard symplectic structure on $\mathbb{C}P^2$, suitably rescaled.
(b') \( k_0 = 2 \) if and only if \((M, \omega)\) is equivariantly symplectomorphic to \((\mathcal{H}, \tilde{\omega}')\), where \(\mathcal{H}\) is a Hirzebruch surface endowed with a suitable \(S^1\)-action and symplectic form \(\tilde{\omega}'\).

**Remark 6.3** The ‘suitable’ \(S^1\)-actions mentioned in (a’) and (b’) are determined by the weights of the \(S^1\)-action at the fixed point set.

**Proof.** Let \( p \) be a fixed point, and \( e^{S^1}(p) \in H^2_{S^1}(\{p\}; \mathbb{Z}) = \mathbb{Z}[x] \) the equivariant Euler class of the normal bundle at \( p \), which is simply given by \( w_{1p}w_{2p}x \), where \( w_{1p} \) and \( w_{2p} \) are the weights of the isotropy \(S^1\)-action at \( p \). By the Atiyah-Bott-Berline-Vergne Localization formula \([4, 9]\) we must have

\[
\sum_{p \in M^{S^1}} \frac{1}{e^{S^1}(p)} = \int_M 1 = 0.
\]  

(6.8)

So it follows that \(M^{S^1}\) must contain points whose product of the corresponding weights is positive, as well as those for which it is negative. Thus \( N_0 + N_2 \neq 0 \) (and \( N_1 \neq 0 \)) which, together with \((2.2)\), implies that \( N_0 \neq 0 \). From Lemma 6.3 (a2) it follows that \( c_1 \) is not a torsion element in \( H^2(M; \mathbb{Z}) \), and by Corollary 4.10 (i) that \( k_0 \in \{1, 2, 3\} \).

If \( k_0 = 3 \), by Proposition 5.1 5.7 we have \( \int_M c_1^2 = 9N_0 \) which, together with \((6.3)\) and \((2.2)\), implies \( N_0 = N_1 = N_2 \). The expression for the Hilbert polynomial follows immediately from Proposition 5.1. The claims in (b) follow similarly by using Proposition 5.4.

If \((M, \omega)\) and the \(S^1\)-action are symplectic, by Lemma 1.4 and the fact that \( N_0 \neq 0 \) we have that the action is Hamiltonian and \( N_0 = 1 \). Also, observe that a standard argument in equivariant Morse theory shows that \( N_j = b_{2j}(M) \), the \(2j\)-th Betti number of \( M \). Following the work of [1] and [7], compact symplectic manifolds of dimension 4 endowed with a Hamiltonian \( S^1 \)-action were completely classified by Karshon [29]. If \( k_0 = 3 \), from (a) we have that \( b_0(M) = b_2(M) = b_4(M) = 1 \), and from the classification in [29] the manifold must be diffeomorphic to \( \mathbb{C}P^2 \). In order to prove the existence of an equivariant symplectomorphism with \((\mathbb{C}P^2, \tilde{\omega})\), by [29] Theorem 4.1 it is sufficient to endow the latter with an \( S^1 \)-action such that the weights at the three fixed points agree with those of \((M, \omega)\), and suitably rescale the standard symplectic form on \( \mathbb{C}P^2 \). On the other hand, if \((M, \omega)\) is symplectomorphic to \( \mathbb{C}P^2 \), then clearly \( k_0 = 3 \), thus proving the claims in (a').

If \( k_0 = 2 \), from (b) we have that \( b_0(M) = b_4(M) = 1 \) and \( b_2(M) = 2 \), and the claim in (b') follows from [29], the details are left to the reader. \( \square \)

If \( k_0 = 1 \) then the Hilbert polynomial is not determined, and by Proposition 5.4 (a) and (b) it depends on the value of \( \int_M c_1^2 \). It is interesting to study the position of the roots of \( H(z) \) in terms of \( \beta = \frac{N_0}{N_0} \). Observe that, by the proof of Proposition 6.2, \( \beta > 0 \) and if the action is Hamiltonian (and the manifold is connected) then \( \beta = b_2(M) \). From the definition of Hilbert polynomial of \((M, J)\) and \((6.3)\) (cf. Proposition 5.7), it is immediate to see that

\[
H(z) = \frac{N_0}{2} [10 + (10 - \beta)z^2 + (10 - \beta)z + 2].
\]  

(6.9)

Thus for \( \beta \neq 10 \) the roots, which are of the form \(-\frac{1}{2} \pm a \) with \( a \) either real or pure imaginary, have the following position:

- for \( 0 < \beta < 2 \) or \( \beta > 10 \) they are real and distinct;
- for \( \beta = 2 \) they are real and coincide;
- for \( 2 < \beta < 10 \) they live on the axis \(-\frac{1}{2} + iy\), for \( y \in \mathbb{R} \setminus \{0\} \).

Moreover when \( \| \int_M c_1^2 \| \to +\infty \), or equivalently when \( \beta \to +\infty \), the roots cluster around the “foci” 0 and −1.

Observe that by Proposition 6.2 in the symplectic case it is impossible to have \( k_0 = 1 \) and \( \beta = b_2(M) \leq 2 \). Moreover, we can have manifolds with \( b_2(M) \) arbitrarily large; it is sufficient to blow-up \( \mathbb{C}P^2 \) as many times as we want.
6.2. dim(M) = 6. Now suppose that dim(M) = 6. Then, as a consequence of (6.2) and (6.1) we have that
\[ \int_M c_3(M) = 2(N_0 + N_1), \] (6.10)
and, as a direct consequence of Theorem 6.1, that
\[ \int_M c_1 c_2 = 24 N_0. \] (6.11)
Moreover, the following proposition follows immediately from Propositions 5.1, 5.4 and Lemma 1.4:

**Proposition 6.4** (dim(M) = 6, k_0 = 3, 4). Let (M, J, S^1) be an S^1-space of dimension 6, and let k_0, H(z) and the N_j's be defined as before.

(a) If k_0 = 4 then
\[ \int_M c_1^3 = 64 N_0 \] and \[ H(z) = \frac{N_0}{6}(z + 1)(z + 2)(z + 3). \]
(b) If k_0 = 3 then
\[ \int_M c_1^3 = 54 N_0 \] and \[ H(z) = \frac{N_0}{6}(z + 3)(z + 1)(z + 2). \]

If (M, ω) is a connected symplectic manifold, and the S^1-action is symplectic, then:

(a') If k_0 = 4 and the action is Hamiltonian then
\[ \int_M c_1^3 = 64 \] and \[ \int_M c_1 c_2 = 24. \]
If the action is non-Hamiltonian then
\[ \int_M c_1^3 = \int_M c_1 c_2 = 0. \]

(b') If k_0 = 3 and the action is Hamiltonian then
\[ \int_M c_1^3 = 54 \] and \[ \int_M c_1 c_2 = 24. \]
If the action is non-Hamiltonian then
\[ \int_M c_1^3 = \int_M c_1 c_2 = 0. \]

When k_0 < 3, the Chern number \( \int_M c_1^3 \) and the Hilbert polynomial H(z) are not determined. For example, if k_0 = 2 then from Proposition 5.7 it follows that for N_0 ≠ 0 and \( \int_M c_1^3 ≠ 0 \) we have
\[ \int_M c_1^3 = \frac{48 N_0}{1 - a^2} \] and \[ H(z) = \frac{N_0}{1 - a^2}(z^2 + 2z + 1 - a^2)(z + 1) \] (6.12)
where \( a^2 ≠ 1 \) and \( a \) is either real or pure imaginary. Thus the roots of \( H(z)/(z + 1) \), given by \(-1 ± a\), are real exactly if \( \int_M c_1^3 ≥ 48 N_0 \) or \( \int_M c_1^3 < 0 \). Moreover they cluster around the "foci" 0 and \(-2 \) exactly if \( \| \int_M c_1^3 \| \to +∞ \).

**Example 6.5** In the following we give examples of manifolds of dimension 6 with k_0 = 2, together with their associated Hilbert polynomials.

1. The flag variety \( Fl(\mathbb{C}^3) := F \). The variety of complete flags in \( \mathbb{C}^3 \) is a compact symplectic (indeed Kähler) manifold of dimension 6 which can be endowed with a Hamiltonian S^1-action with exactly 6 fixed points; for details about the action see [15 Example 5.5] and the discussion preceding it. The reader can verify that the definition of k_0 given here
where

\( z \in \mathbb{R} \) exactly if

\( \int_{c_1} \zeta = 1 \) then from Proposition 5.13 it follows that for \( N_0 \neq 0 \) and \( \int_M c_1^3 \neq 0 \) we have

\[
\int_M c_1^3 = \frac{48N_0}{1 - 4a^2} \quad \text{and} \quad H(z) = \frac{N_0}{1 - 4a^2} \left[ 4z^2 + 4z + 1 - 4a^2 \right] (2z + 1) \tag{6.13}
\]

where \( a^2 \neq \frac{1}{4} \) and \( a \) is either real or pure imaginary. Thus the roots of \( H(z)/(2z + 1) \), given by

\[
-\frac{1}{2} \pm a, \quad \text{are real exactly if} \quad \int_M c_1^3 \geq 48N_0 \quad \text{or} \quad \int_M c_1^3 < 0. \quad \text{Moreover they cluster around the “foci”}
\]

If \( k_0 = 1 \) then from Proposition 6.13 it follows that for \( N_0 \neq 0 \) and \( \int_M c_1^3 \neq 0 \) we have

\[
\int_M c_1^3 = \frac{48N_0}{1 - 4a^2} \quad \text{and} \quad H(z) = \frac{N_0}{1 - 4a^2} \left[ 4z^2 + 4z + 1 - 4a^2 \right] (2z + 1) \tag{6.13}
\]

where \( a^2 \neq \frac{1}{4} \) and \( a \) is either real or pure imaginary. Thus the roots of \( H(z)/(2z + 1) \), given by

\[
-\frac{1}{2} \pm a, \quad \text{are real exactly if} \quad \int_M c_1^3 \geq 48N_0 \quad \text{or} \quad \int_M c_1^3 < 0. \quad \text{Moreover they cluster around the “foci”}
\]

**Example 6.6** In the following we give examples of manifolds of dimension 6 with \( k_0 = 1 \), together with their associated Hilbert polynomials.

\begin{enumerate}
\item \( \mathbb{CP}^1 \times \mathbb{CP}^2 =: C \). This is a compact symplectic (indeed Kähler) manifold which can be endowed with a Hamiltonian \( S^1 \)-action with 6 fixed points. Moreover \( \int_C c_1(C)^3 = 54 \), and the Hilbert polynomial is \( H_C(z) = \frac{1}{4} [9z^2 + 9z + 2] (2z + 1) \).
\item \( \text{The Fano threefold } V_{22} \) (for details see \[39\] or \[15\] Example 6.14). Similarly to Example 6.5 (3), this is a Fano manifold which can be endowed with a Hamiltonian \( S^1 \)-action with exactly 4 fixed points. The cohomology ring is given by \( \mathbb{Z}[x,y]/(x^2 - 22y, y^2) \) (where \( x \) has degree 2, and \( y \) degree 4), and \( k_0 = 1 \). Thus \( \int_{V_{22}} c_1(V_{22})^3 = 22 \), and the Hilbert polynomial is \( H_{V_{22}}(z) = \frac{1}{4} [11z^2 + 11z + 6] (2z + 1) \).
\end{enumerate}

**6.3.** \( \dim(M) = 8 \). When \( \dim(M) = 8 \), from \[22\] and \[31\] we have that

\[
\int_M c_4 = 2N_0 + 2N_1 + N_2, \tag{6.14}
\]

and from Theorem 6.1

\[
\int_M c_1c_3(M) = 44N_0 + 8N_1 - 2N_2. \tag{6.15}
\]

As for the remaining Chern numbers, we can use Propositions 5.1 and 5.4 to prove the following

**Proposition 6.7** (\( \dim(M) = 8 \), \( k_0 = 4, 5 \)). Let \( (M, J, S^1) \) be an \( S^1 \)-space of dimension 8, and let \( k_0, H(z) \) and the \( N_j \)'s be defined as before.
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(a) If $k_0 = 5$ then
\[
\int_M c_1^4 = 625 N_0, \quad \int_M c_1^2 c_2 = 250 N_0, \quad \int_M c_2^2 = 101 N_0 - 2 N_1 + N_2, \tag{6.16}
\]
and $H(z) = \frac{N_0}{24} \prod_{j=1}^{4} (z + j)$.

(b) If $k_0 = 4$ then
\[
\int_M c_1^4 = 512 N_0, \quad \int_M c_1^2 c_2 = 224 N_0, \quad \int_M c_2^2 = 98 N_0 - 2 N_1 + N_2, \tag{6.17}
\]
and $H(z) = \frac{N_0}{12} (z + 2) \prod_{j=1}^{3} (z + j)$.

Moreover, if $(M, \omega)$ is a connected symplectic manifold and the $S^1$-action is Hamiltonian then

(a') if $k_0 = 5$ we have
\[
\int_M c_1^4 = 625 N_0, \quad \int_M c_1^2 c_2 = 250 N_0, \quad \int_M c_2^2 = 101 N_0 - 2 N_1 + N_2, \quad \text{and} \quad H(z) = \frac{N_0}{24} \prod_{j=1}^{4} (z + j). \tag{6.18}
\]

(b') if $k_0 = 4$ we have
\[
\int_M c_1^4 = 512 N_0, \quad \int_M c_1^2 c_2 = 224 N_0, \quad \int_M c_2^2 = 98 N_0 - 2 N_1 + N_2, \quad \text{and} \quad H(z) = \frac{N_0}{12} (z + 2) \prod_{j=1}^{3} (z + j). \tag{6.19}
\]

Proof. The only claims in (6.16) and (6.17) which do not follow directly from Propositions 5.1 and 5.4 are the expressions of $\int_M c_1^2$ in terms of the $N_j$s. In order to obtain them, it is sufficient to use the expression of the Todd genus given in Corollary 3.7, which for $n = 4$ gives
\[
\int_M -c_1^4 + 4 c_1^2 c_2 + 3 c_2^2 + c_1 c_3 - c_4 = 720 N_0. \tag{6.20}
\]

By combining (6.20) with (5.7), (5.8), (5.13) and (5.14) we obtain the desired claims. In the symplectic case, all the claims follow from Lemma 1.4, the fact that in the Hamiltonian case $N_j = b_2(M)$, Corollary 4.10 and (6.21).

When $k_0 = 3$ or $k_0 = 2$, from Proposition 6.7 and 6.13 we can see that the coefficients of the Hilbert polynomial $a_h$, $h = 0, \ldots, 4$, are not completely determined, but they depend on the value of $\int_M c_1^4$. The following proposition exhibits the relation between $\int_M c_1^2 c_2$ and $\int_M c_2^2$ with $\int_M c_1^4$.

Proposition 6.8 ($\dim(M) = 8$, $k_0 = 2, 3$). Let $(M, J, S^1)$ be an $S^1$-space of dimension 8, and let $k_0$, $H(z)$ and the $N_j$s be defined as before. Then

(a) $k_0 = 3$ implies that
\[
\int_M c_1^2 c_2 = 108 N_0 + \frac{2}{9} \int_M c_1^4, \tag{6.22}
\]
and
\[
\int_M c_2^2 = 82 N_0 - 2 N_1 + N_2 + \frac{1}{27} \int_M c_1^4. \tag{6.23}
\]
(b) \( k_0 = 2 \) implies that
\[
\int_M c_1^2 c_2 = 96 N_0 + \frac{1}{4} \int_M c_1^4 ,
\]
and
\[
\int_M c_2^2 = 98 N_0 - 2 N_1 + N_2 .
\]

Proof. (a) In order to prove (6.22), it is sufficient to use Corollary 5.12, and equation (6.23) can be obtained by combining (6.21) with (6.14), (6.15) and (6.22).

(b) Equation (6.24) follows from Corollary 5.18, and (6.25) can be obtained by combining (6.21) with (6.14), (6.15) and (6.24).

\[\square\]

Remark 6.9 From (6.17) and (6.25) we obtain that for \( k_0 = 2 \) and \( k_0 = 4 \) the expression of \( \int_M c_2^2 \) in terms of the \( N_j \)s is the same.

Remark 6.10 It can be checked that (6.24) is equivalent to equation (7.22) in [15]; here \( M \) is an 8-dimensional compact symplectic manifold, with a Hamiltonian \( S^1 \)-action and exactly 5 fixed points. Equation (7.22) in [15] is obtained by applying some results of Hattori (see [24], and Corollary 7.7, Theorem 7.11 in [15]) which, however, only hold whenever \((M,J)\) possesses a fine line bundle. Moreover, the derivation of (7.22) from such results is rather complicated, as it can be seen from the proof of [15, Theorem 7.11]. Here we do not need to assume the existence of a fine line bundle, and (6.24) is an immediate consequence of Corollary 5.18.

When \( k_0 = 1 \) we do not obtain any restrictions on the Chern numbers (see Corollary 4.17, as well as the discussion on the case \( k_0 = n - 3 \) at the end of Section 5).

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