RESTRICTED MEAN VALUE THEOREMS AND METRIC THEORY OF RESTRICTED WEYL SUMS

CHANGHAO CHEN AND IGOR E. SHPARLINSKI

Abstract. We study an apparently new question about the behaviour of Weyl sums on a subset \( \mathcal{X} \subseteq [0,1]^d \) with a natural measure \( \mu \) on \( \mathcal{X} \). For certain measure spaces \((\mathcal{X}, \mu)\) we obtain non-trivial bounds for the mean values of the Weyl sums, and for \( \mu \)-almost all points of \( \mathcal{X} \) the Weyl sums satisfy the square root cancellation law. Moreover we characterise the size of the exceptional sets in terms of Hausdorff dimension.

Finally, we derive variants of the Vinogradov mean value theorem averaging over measure spaces \((\mathcal{X}, \mu)\). We obtain general results, which we refine for some special spaces \( \mathcal{X} \) such as spheres, moment curves and line segments.

Contents

1. Introduction
   1.1. Background
   1.2. Previous results and questions
   1.3. Average values and the metric properties of restricted Weyl sums
   1.4. Notation and conventions

2. Main results
   2.1. General sets
   2.2. Moment curves
   2.3. Segments
   3. Preliminaries
      3.1. The completion technique
      3.2. Continuity of exponential sums
      3.3. Covering the large values of exponential sums
      3.4. Bounds on exponential integrals

4. Proofs of general results
   4.1. Proof of Theorem 2.1

2010 Mathematics Subject Classification. 11L07, 11L15, 28A78.
Key words and phrases. Weyl sums, mean value theorem, Fourier decay, Hausdorff dimension.
1. Introduction

1.1. Background. For an integer $d \geq 2$, let $T_d = (\mathbb{R}/\mathbb{Z})^d$ be the $d$-dimensional unit torus.

For a vector $x = (x_1, \ldots, x_d) \in T_d$ and $N \in \mathbb{N}$, we consider the exponential sums

$$S_d(x; N) = \sum_{n=1}^{N} e \left( x_1 n + \ldots + x_d n^d \right),$$

which are commonly called Weyl sums, where throughout the paper we denote $e(x) = \exp(2\pi ix)$.

Weyl sums, introduced by Weyl [24] as a tool to investigate the distribution of fractional parts of real polynomials (see also [3]) also appear in a broad spectrum of other number theoretic problems. For example, they play a crucial role in estimating the zero-free region of the Riemann zeta-function and thus in turn obtaining a sharp form of the prime number theorem, see [21, Section 8.5], in the Waring problem, see [21, Section 20.2], in bounding short character sums modulo highly composite numbers [21, Section 12.6] and many others.

However, despite more than a century long history of estimating such sums, the behaviour of individual sums is not well understood, see [7,8].

The following best known bound is a direct implication of the current form of the Vinogradov mean value theorem from [5, 25] (see also (1.1) below) and is given in [4, Theorem 5]. Let $x = (x_1, \ldots, x_d) \in T_d$ be such that for some $\nu$ with $2 \leq \nu \leq d$ and some positive integers $a$ and
with \( \gcd(a, q) = 1 \) we have

\[
\left| x_\nu - \frac{a}{q} \right| \leq \frac{1}{q^2}.
\]

Then for any \( \varepsilon > 0 \) there exists a constant \( C(\varepsilon) \) such that

\[
|S_d(x; N)| \leq C(\varepsilon) N^{1+\varepsilon} \left( q^{-1} + N^{-1} + qN^{-\nu} \right)^{\frac{1}{(d-1)}}.
\]

On the other hand, thanks to recent striking results of Bourgain, Demeter and Guth \cite{5} (for \( d \geq 4 \)) and Wooley \cite{25} (for \( d = 3 \)) (see also \cite{28}), for any integer \( s \geq 1 \), for the \( 2s \)-power mean value of \( S_d(x; N) \) we have

\[
\int_{T_d} |S_d(x; N)|^{2s} d\mathbf{x} \leq N^{s+o(1)} + N^{2s-s(d)+o(1)}, \quad N \to \infty,
\]

where

\[
s(d) = \frac{d(d+1)}{2},
\]

which is the best possible form of the Vinogradov mean value theorem. In particular

\[
\int_{T_d} |S_d(x; N)|^{2s(d)} d\mathbf{x} \leq N^{s(d)+o(1)}, \quad N \to \infty.
\]

1.2. **Previous results and questions.** We first outline some results concerning the metric theory of Weyl sums on \( T_d \). The metric theory means that we study the properties of Weyl sums which hold for almost all points with respect to the Lebesgue or some other measures. Moreover one also characterise the size of the exceptional sets (outside of the almost all) in terms of Hausdorff dimension.

We remark that the topic here, the metric theory of Weyl sums, is not the same as the topics in the metric theory of numbers, see, for instance, \cite{17}. However they are certainly related to each other.

We say that some property holds for *almost all* \( x \in T_d \) if it holds for a set \( \mathcal{X} \subseteq T_d \) of Lebesgue measure \( \lambda(\mathcal{X}) = 1 \).

For \( d = 2 \), Fedotov and Klopp \cite{15}, Theorem 0.1 give the following optimal lower and upper bounds. Suppose that \( \{g(n)\}_{n=1}^{\infty} \) is a nondecreasing sequence of positive numbers. Then for almost all \( x \in T_2 \) one has

\[
\lim_{N \to \infty} \frac{|S_2(x; N)|}{\sqrt{Ng(ln N)}} < \infty \quad \iff \quad \sum_{n=1}^{\infty} \frac{1}{g(n)^6} < \infty.
\]

For \( d \geq 3 \), the authors \cite{10}, Appendix A] have shown that for almost all \( x \in T_d \)

\[
|S_d(x; N)| \leq N^{1/2+o(1)}, \quad N \to \infty.
\]
One may conjecture that this is the best possible bound, see [10, Conjecture 1.1].

Let $a = (a_n)_{n=1}^\infty$ be a sequence of complex weights and denote

$$S_{a,d}(x; N) = \sum_{n=1}^N a_n e(x_1 n + \ldots + x_d n^d).$$

Extending (1.2), the authors [9, Corollary 2.2] have shown that for any complex weights $a = (a_n)_{n=1}^\infty$ with $a_n = n^{\alpha(1)}$ one has that for almost all $x \in \mathbb{T}_d$,

$$|S_{a,d}(x; N)| \leq N^{1/2+\alpha(1)}, \quad N \to \infty.$$

From the almost all results in (1.2) and (1.4) one may ask how “large” are the exceptional sets. For this purpose we introduce following notation. For $0 < \alpha < 1$ and integer $d \geq 2$, we consider the set

$$E_{a,d,\alpha} = \{x \in \mathbb{T}_d : |S_{a,d}(x; N)| \geq N^\alpha \text{ for infinity many } N \in \mathbb{N}\},$$

and call it the exceptional set. If $a = e = (1)_{n=1}^\infty$ we just write

$$E_{d,\alpha} = E_{e,d,\alpha}.$$ Using this notation we may say that for any $1/2 < \alpha < 1$ the set $E_{d,\alpha}$ has zero Lebesgue measure.

For sets of Lebesgue measure zero, it is common to use the Hausdorff dimension to describe their size; for the properties of the Hausdorff dimension and its applications we refer to [14]. We recall that for $U \subseteq \mathbb{R}^d$

$$\text{diam } U = \sup\{\|u - v\| : u, v \in U\},$$

where $\|w\|$ is the Euclidean norm in $\mathbb{R}^d$.

**Definition 1.1.** The Hausdorff dimension of a set $A \subseteq \mathbb{R}^d$ is defined as

$$\dim A = \inf \left\{ s > 0 : \forall \varepsilon > 0, \exists \{U_i\}_{i=1}^\infty, U_i \subseteq \mathbb{R}^d, \right.$$ such that $A \subseteq \bigcup_{i=1}^\infty U_i$ and $\sum_{i=1}^\infty (\text{diam } U_i)^s < \varepsilon \left. \right\}.$

We note that the authors [10] have obtained a lower bound of the Hausdorff dimension of $E_{d,\alpha}$. Among other things, it is shown in [10] for any $\alpha \in (0, 1)$ one has

$$\dim E_{d,\alpha} \geq \ell(d, \alpha)$$

with some explicit function $\ell(d, \alpha) > 0.$
Furthermore, the authors [11] have given a non-trivial upper bound for $\mathcal{E}_{d,\alpha}$. More precisely, we have

\begin{equation}
\dim \mathcal{E}_{d,\alpha} \leq u(d, \alpha),
\end{equation}

where

\begin{equation}
u(d, \alpha) = \min_{k=0, \ldots, d-1} \frac{(2d^2 + 4d)(1 - \alpha) + k(k + 1)}{4 - 2\alpha + 2k}.
\end{equation}

The bound (1.7) has some interesting implications. For instance for any $\alpha \in (1/2, 1)$ we have

\[ \dim \mathcal{E}_{d,\alpha} < d. \]

Moreover, if $\alpha \to 1$ then $u(d, \alpha) \to 0$. Indeed it is expected that if $\alpha$ becomes large then the set $\mathcal{E}_{d,\alpha}$ becomes small. We refer to [11] for more details.

Furthermore, as a counterpart to (1.6), we remark that we expect $\dim \mathcal{E}_{d,\alpha} = d$ for $\alpha \in (0, 1/2]$, see also [10, 11]. On the other hand, we do not have any plausible conjecture about the exact behaviour of $\dim \mathcal{E}_{d,\alpha}$ for $\alpha \in (1/2, 1)$.

### 1.3. Average values and the metric properties of restricted Weyl sums

The goal here is to investigate the Weyl sums over some subset $\mathcal{X} \subseteq \mathbb{T}_d$ with some natural measure on $\mathcal{X}$. Restricted type Such restriction problem arise in many other areas of mathematics. These include Diophantine approximation on manifolds, see [1], Fourier restriction problems, see [23, Chapter 19], the restricted families of projections, see [23, Section 5.4], and discrete Fourier restriction, see [18–20, 22, 27].

We recall that the support $\mathrm{spt} \mu$ of a measure $\mu$ on $\mathbb{R}^d$ is the smallest closed set $\mathcal{X}$ such that $\mu(\mathbb{R}^d \setminus \mathcal{X}) = 0$.

We consider the following very general question. We remark that the below set $\mathcal{X}$ can be some fractal set.

**Question 1.2.** Let $\mu$ be a Radon measure on $\mathbb{T}_d$ with $\mathrm{spt} \mu = \mathcal{X}$. What can we say about

- **A. mean value bounds:** $\int_{\mathcal{X}} |S_{a,d}(x; N)|^\rho d\mu(x)$;
- **B. typical bounds:** $\sup \{ \alpha \in [0, 1] : \mu(\mathcal{E}_{a,d,\alpha} \cap \mathcal{X}) = 0 \}$;
- **C. exceptional sets:** $\sup \{ \alpha \in [0, 1] : \dim(\mathcal{E}_{a,d,\alpha} \cap \mathcal{X}) = \dim \mathcal{X} \}$;

provided the measure $\mu$ has some natural geometric, algebraic or combinatorial structure?
For example, the restriction results from [18–20,22,27] address some instances of Question 1.2 A in the case when \( \mathcal{X} \) is hyperplane formed by vectors \( \mathbf{x} \in \mathbb{T}_d \) with some components fixed (often to zero).

Furthermore, there are other types of the metric theory of Weyl sums related to Question 1.2 B. More precisely, let

\[
\{ \varphi_1(T), \varphi_2(T), \ldots, \varphi_d(T) \} = \{ T, T^2, \ldots, T^d \}.
\]

Note that here the order of \( \varphi_1, \ldots, \varphi_d \) is not specified. The works of [9,16,26] imply that for almost all \((x_1, \ldots, x_k) \in \mathbb{T}_k\) (with respect to the \(k\)-dimensional Lebesgue measure) one has

\[
\sup_{(y_1, \ldots, y_{d-k})} \left| \sum_{n=1}^{N} e \left( \sum_{j=1}^{k} x_j \varphi_j(n) + \sum_{j=k+1}^{d} y_j \varphi_j(n) \right) \right| \leq N^{1/2+\delta(d,k)+o(1)}
\]

as \( N \to \infty \), for some explicit values \( 0 < \delta(d,k) < 1 \); we refer to [9] for more details and the currently best know results in general. We note that recently special forms of such bounds, using a very different approach, have been given in [6,12], together with applications to some partial differential equations.

Here we are interested in general spaces \( \mathcal{X} \) and measures \( \mu \) and also in some special cases such as spheres (2.3), moment curves (2.5) and line segments (2.15).

1.4. Notation and conventions. Throughout the paper, the notation \( U = O(V) \), \( U \ll V \) and \( V \gg U \) are equivalent to \( |U| \leq cV \) for some positive constant \( c \), which throughout the paper may depend on the degree \( d \) and occasionally on the small real positive parameter \( \varepsilon \).

For any quantity \( V > 1 \) we write \( U = V^{o(1)} \) (as \( V \to \infty \)) to indicate a function of \( V \) which satisfies \( |U| \leq V^\varepsilon \) for any \( \varepsilon > 0 \), provided \( V \) is large enough. One additional advantage of using \( V^{o(1)} \) is that it absorbs \( \log V \) and other similar quantities without changing the whole expression.

We use \( \#S \) to denote the cardinality of a finite set \( S \).

We always identify \( \mathbb{T}_d \) with half-open unit cube \([0,1)^d\), in particular we naturally associate the Euclidean norm \( \|x\| \) with points \( x \in \mathbb{T}_d \).

We always suppose that \( d \geq 2 \).

For a measure \( \mu \) on \( \mathcal{X} \) we say that some property holds for \( \mu \)-almost all \( x \in \mathcal{X} \) if it holds for a set \( \mathcal{A} \subseteq \mathcal{X} \) such that \( \mu(\mathcal{X} \setminus \mathcal{A}) = 0 \).

For each \( q > 0 \) denote

\[(1.8) \quad s(q) = q(q+1)/2.\]
2. Main results

2.1. General sets. We consider Radon measure $\mu$ on $T_d$ which implies that $\mu$ is a Borel measure and $\mu(T_d) < \infty$, see [13, Chapter 1] for the general measure theory.

The Fourier transform of a Radon measure $\mu$ on $\mathbb{R}^d$ is defined as

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e(-x \cdot \xi) d\mu(x), \quad \xi \in \mathbb{R}^d,$$

where, as usual, the dot product $x \cdot \xi$ of vectors $x = (x_1, \ldots, x_d)$ and $\xi = (\xi_1, \ldots, \xi_d)$ is given by

$$x \cdot \xi = x_1\xi_1 + \ldots + x_d\xi_d,$$

see [23, Chapter 3] for the basic properties of the Fourier transform of measures.

We consider classes of Radon measures $\mu$ on $T_d$ such that

$$(2.1) \quad |\hat{\mu}(\xi)| \ll (1 + \|\xi\|)^{-\sigma}, \quad \forall \xi \in \mathbb{R}^d,$$

for some $\sigma > 0$.

We are mostly interested in sequences $a = (a_n)_{n=1}^\infty$ of complex weights such that

$$(2.2) \quad a_n = n^{o(1)}, \quad n \to \infty.$$

**Theorem 2.1.** Let $\mu$ be a Radon measure on $T_d$ such that (2.1) holds for some $\sigma \geq 1/d$, and let $a = (a_n)_{n=1}^\infty$ satisfy condition (2.2). Then

$$\int_{T_d} |S_{a,d}(x; N)|^2 d\mu(x) \leq N^{1+o(1)}.$$

For the case $\sigma > 1/d$ the bound in Theorem 2.1 is essentially optimal, see Remark 4.1 below. Moreover, it is interesting to know whether Theorem 2.1 still holds under the weaker condition that $\sigma > 0$.

We remark that there are many measures which satisfy the condition (2.1). These include some surface measures (for example, of spheres and paraboloids), see [23, Section 14.3]; some fractal measures (for example, natural measures on the trajectories of Brownian motion, see [23, Chapter 12] and some random Cantor measures, see [2]). Thus Theorem 2.1 claims these measures admit the square mean value theorems.

For the higher order mean value bounds we have Theorem 2.2 and 2.4 below, which depend on the rate of decay of Fourier coefficients and on boundedness of their $L^1$-norm, respectively.

For $x \in \mathbb{R}$ we define $[x]$ as the nearest integer of the number $x$ if $x - 1/2 \not\in \mathbb{Z}$ and also set $[x] = x + 1/2$ if $x - 1/2 \in \mathbb{Z}$. 
Theorem 2.2. Let \( \mu \) be a Radon measure on \( T^d \) such that (2.1) holds for some positive \( \sigma \leq d \) and let \( a = (a_n)_{n=1}^{\infty} \) satisfy the condition (2.2). Then
\[
\int_{T^d} |S_{a,d}(x;N)|^{2s(d)} d\mu(x) \leq N^{s(d)+s(\ell)(d-\sigma-\ell)+o(1)},
\]
where \( \ell = \lfloor d - \sigma + 1/2 \rfloor \) and \( s(\ell) \) is given by (1.8).

For the natural measure \( \mu_S \) on the \( d \)-dimensional sphere \( \mathbb{S}^{d-1} = \{ t = (t_1, \ldots, t_d) \in \mathbb{R}^d : (t_1 - 1/2)^2 + \ldots + (t_d - 1/2)^2 = 1/4 \} \) centred at \((1/2, \ldots, 1/2)\) and of radius \( 1/2 \), we can take \( \sigma = (d-1)/2 \) in (2.1), see [23, Equation (3.42)]. That is, we have
\[
\hat{\mu}_S(\xi) \ll (1 + \|\xi\|)^{-(d-1)/2},
\]
for any \( \xi \in \mathbb{R}^d \). Substituting in Theorem 2.2 we see the following.

Example 2.3. Let \( a_n = n^{o(1)} \). Then for the sphere we have
\[
\ell = \lfloor d - (d-1)/2 + 1/2 \rfloor = \lfloor d/2 + 1 \rfloor.
\]
Hence
\[\text{(i) if } d \text{ is even then } \ell = d/2 + 1, \text{ thus }\]
\[
\int_{\mathbb{S}^{d-1}} \left| \sum_{n=1}^{N} a_n e(t_1 n + \ldots + t_d n^d) \right|^{2s(d)} d\mu_S(t) \leq N^{s(d)+(d+2)^2/8+o(1)};
\]
\[\text{(ii) if } d \text{ is odd then } \ell = d + 3/2, \text{ thus }\]
\[
\int_{\mathbb{S}^{d-1}} \left| \sum_{n=1}^{N} a_n e(t_1 n + \ldots + t_d n^d) \right|^{2s(d)} d\mu_S(t) \leq N^{s(d)+(d+3)(d+1)/8+o(1)}.
\]

We remark that for the natural measure \( \mu_M \) on the moment curve
\[
\Gamma = \{(t, \ldots, t^d) : t \in [0,1]\},
\]
that is,
\[
\mu_M \left( \{(t, \ldots, t^d) : t \in [a, b]\} \right) = b - a,
\]
by Lemma 3.6 below we can take \( \sigma = 1/d \) in (2.1), that is,
\[
\hat{\mu}_M(\xi) \ll (1 + \|\xi\|)^{-1/d},
\]
for any $\xi \in \mathbb{R}^d$. Therefore, for all $d \geq 2$ we have $\ell = [d - 1/d + 1/2] = d$, thus Theorem 2.2 implies

\[
\int_0^1 \left| \sum_{n=1}^N e(tn + \ldots + t^d n^d) \right|^{2s(d)} dt \leq N^{2s(d) - 1 + o(1)},
\]

which is the same bound as one can instantly derive from Theorem 2.1. In fact one cannot improve the bound (2.8) as it is easy to see that for $0 \leq t \leq 0.1N$ we have

\[
\sum_{n=1}^N e(tn + \ldots + t^d n^d) \gg N
\]

and hence

\[
\int_0^1 \left| \sum_{n=1}^N e(tn + \ldots + t^d n^d) \right|^{2s(d)} dt \geq \int_0^{0.1N} \left| \sum_{n=1}^N e(tn + \ldots + t^d n^d) \right|^{2s(d)} dt \gg N^{2s(d) - 1}.
\]

However in Theorem 2.14 below we use a different argument and obtain a much stronger bound on the modification of the above integral in (2.8) over the interval $[\delta, 1]$ for any positive $\delta$. Thus for this case, we improve the bound in Theorem 2.2.

Next, we show that the proof of Theorem 2.2 implies the following result.

**Theorem 2.4.** Let $\mu$ be a Radon measure on $T_d$ such that

\[
\sum_{\xi \in \mathbb{Z}^d} |\hat{\mu}(\xi)| \ll 1,
\]

and let $a = (a_n)_{n=1}^{\infty}$ satisfy the condition (2.2). Then

\[
\int_{T_d} |S_{a,d}(x; N)|^{2s(d)} d\mu(x) \leq N^{s(d) + o(1)}.
\]

Theorem 2.5 below claims that we can derive the “almost all” individual bounds by using mean value theorems. For this purpose we now need to consider the family of sums similar to $S_{a,d}(x; N)$ in (1.3), but in the following the weights change with $N$. More precisely, let $a(N, n)$ be a “double sequence” of complex weights such that

\[
\max_{n=1,\ldots,N} |a(N, n)| \leq N^{o(1)}, \quad N \to \infty.
\]
We only consider the values $a(N, n)$ with $1 \leq n \leq N$ throughout the paper.

**Theorem 2.5.** Let $\mu$ be a Radon measure on $\mathbb{T}_d$ and let $\rho > 0$ and $0 < \vartheta < 1$ be two constants such that for any double sequence $a(N, n)$ with the condition (2.10) one has

$$
\int_{\mathbb{T}_d} \left| \sum_{n=1}^{N} a(N, n) e(x_1 n + \ldots + x_d n^d) \right|^\rho d\mu(x) \leq N^{\vartheta+o(1)},
$$

then for any complex sequence $a_n = n^{o(1)}$ and for $\mu$-almost all $x \in \text{spt } \mu$ we have

$$(2.11) \quad |S_{a,d}(x; N)| \leq N^{\vartheta+o(1)}, \quad N \to \infty.$$  

**Remark 2.6.** Let $\mu$ be a Radon measure on $\mathbb{T}_d$ with the property (2.1) for some $\sigma \geq 1/d$. By using the similar arguments as in the proof of Theorem 2.1, we derive that the measure $\mu$ satisfies the bound of Theorem 2.5 with $\rho = 2$ and $\vartheta = 1/2$. Therefore we obtain that for $\mu$-almost all $x \in \text{spt } \mu$ we have the bound (2.11).

Applying Theorem 2.5, Remark 2.6 and the bounds of (2.4) and (2.7), we obtain the square root cancellation in the following special cases of spheres and moment curves.

**Example 2.7.** Let $\mu$ be the spherical measure $\mu_S$ or the natural measure $\mu_M$ on the moment curve (2.5) and $a_n = n^{o(1)}$, then for $\mu$-almost all $x \in \text{spt } \mu$ we have

$$
|S_{a,d}(x; N)| \leq N^{1/2+o(1)}, \quad N \to \infty.
$$

Under the conditions of Theorem 2.5, we obtain that the bound (2.11) holds for $\mu$-almost all $x \in \text{spt } \mu$. Thus for any $\varepsilon > 0$ and $a_n = n^{o(1)}$ we have

$$
\mu(\mathcal{E}_{a,d,\vartheta+\varepsilon}) = 0,
$$

where the exceptional set $\mathcal{E}_{a,d,\vartheta+\varepsilon}$ is given by (1.5).

For any $\alpha > \vartheta$ we study the Hausdorff dimension of the exceptional set $\mathcal{E}_{a,d,\alpha}$ of Theorem 2.5, that is the set for which (2.11) fails. For this purpose we need impose some regularity properties on the measure $\mu$.

**Definition 2.8.** Let $\mathbb{R}_+ = (0, \infty)$ be the set of positive real numbers. For $x \in \mathbb{R}^d$ and $\zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{R}_+^d$, we define the $d$-dimensional rectangle (or box) with the centre $x$ and the side lengths $2\zeta$ by

$$
\mathcal{R}(x, \zeta) = [x_1 - \zeta_1, x_1 + \zeta_1] \times \ldots \times [x_d - \zeta_d, x_d + \zeta_d].
$$
Let $\mu$ be a Radon measure on $\mathbb{R}^d$. Suppose that there exists a function $f : \mathbb{R}_+^d \to \mathbb{R}_+$ such that for any $\zeta \in \mathbb{R}_+^d$ one has
\[ \mu(\mathcal{R}(x, \zeta)) \geq f(\zeta), \quad \forall x \in \text{spt} \mu, \]
then we say that the measure $\mu$ is $f$-regular.

To illustrate Definition 2.8, we give the following example of an $f$-regular measure. Let $L$ be a segment of $\mathbb{R}_+^d$ and $\mu$ be the natural measure on $L$. Then for any $\zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{R}_+^d$ and any $x \in L$ we have
\[ \mu(\mathcal{R}(x, \zeta)) \geq \min \{\zeta_1, \ldots, \zeta_d\}. \]
Thus the measure $\mu$ is $f$-regular with $f(\zeta) = \min \{\zeta_1, \ldots, \zeta_d\}$.

**Theorem 2.9.** Let $\mu$ be a $f$-regular Radon measure on $\mathbb{T}_d$ for some function $f$ and let $\rho > 0$, $0 < \vartheta < 1$ be two constants such that for any double sequence $a(N, n)$ with the condition (2.10) one has
\[ \int_{\mathbb{T}_d} \left| \sum_{n=1}^{N} a(N, n) e(x_1 n + \ldots + x_d n^d) \right|^\rho \, d\mu(x) \leq N^{\vartheta \rho + o(1)}. \]
Then we have
\[ \dim(\mathcal{E}_{a, d, \alpha} \cap \text{spt} \mu) \leq u_{\rho, \vartheta}(f; d, \alpha), \]
with
\[ u_{\rho, \vartheta}(f; d, \alpha) = \inf \{t > 0 : \exists \varepsilon > 0 \text{ such that} \}
\[ \sum_{i=1}^{\infty} N_i^{\rho \vartheta - \rho \alpha + s(k) + t(\alpha - k - 2) + c(d)\varepsilon} f(\zeta_i(\varepsilon))^{-1} < \infty, \]
for some $k = 0, \ldots, d - 1$, where $c(d)$ is a positive constant which depends only on $d$ and
\[ \zeta_i(\varepsilon) = (\zeta_{i,1}(\varepsilon), \ldots, \zeta_{i,d}(\varepsilon)) \]
with $\zeta_{i,j}(\varepsilon) = N_i^{\alpha - j - 1 - \varepsilon}$, $j = 1, \ldots, d$, and $N_i = 2^i$, $i \in \mathbb{N}$.

**Remark 2.10.** As we have claimed before there are many measures that satisfy the condition (2.1), and thus fulfil the conditions in Theorems 2.1 and 2.2. Moreover, applying the similar arguments as in the proofs of Theorems 2.1 and 2.2, one obtains that their conclusions still hold even when we take any double sequence $a(N, n)$ with the condition (2.10) instead of the sequence $a_n$. Thus these measures also satisfy the mean value bounds in Theorem 2.9. Furthermore, many measures also satisfy the $f$-regular condition of Definition 2.8 for some function $f$. Thus for these measures we deduce the dimension bounds for the sets.
\( \mathcal{E}_{a,d,\alpha} \cap \text{spt } \mu \). Below we give some concrete examples of applications of Theorem 2.9.

We remark that if \( \mu \) is the Lebesgue measure on \( T_d \) then for any rectangle \( R(x, \zeta) \) one has

\[
\mu(R(x, \zeta)) = f(\zeta) \quad \text{with} \quad f(\zeta) = \prod_{j=1}^{d} \zeta_j.
\]

For this special case and for \( \dim \mathcal{E}_{a,d,\alpha} \), after simple calculations we obtain the same upper bound as (1.6) for \( \mathcal{E}_{d,\alpha} \). More precisely, we have the following.

**Example 2.11.** Let \( \mu \) be the Lebesgue measure on \( T_d \), then for \( 1/2 < \alpha < 1 \) and \( a_n = n^{\alpha(1)} \) we have \( \dim \mathcal{E}_{a,d,\alpha} \leq u(d, \alpha) \), where \( u(d, \alpha) \) is given by (1.6).

Now we consider the Weyl sums on sphere \( S^{d-1} \). Observe that for any \( \zeta \in \mathbb{R}^d_+ \) with

\[
1 > \zeta_1 \geq \ldots \geq \zeta_d,
\]
we have

\[
\mu_S(R(x, \zeta) \cap S^{d-1}) \gg \prod_{j=2}^{d} \zeta_j, \quad x \in S^{d-1},
\]

hence \( \mu_S \) is \( f \)-regular if we take

(2.12) \[ f(\zeta) = c_0(d)\zeta_2 \ldots \zeta_d, \]

for some constant \( c_0(d) > 0 \) which depends only on \( d \).

Furthermore suppose that \( \mu_S \) satisfies the condition in Theorem 2.9 for some \( \rho \) and \( \vartheta \), then using (2.12) in the setting of Theorem 2.9, we see that

\[
f(\zeta_i(\varepsilon)) \gg N_i^{(d-1)(\alpha-1-\varepsilon)-s(d)+1},
\]

and we obtain that

\[
\dim(\mathcal{E}_{a,d,\alpha} \cap S^{d-1}) \leq t
\]

provided that there exists \( k = 0, 1, \ldots, d-1 \) such that

\[
\rho \vartheta - \rho \alpha + s(k) + t(\alpha - k - 2) + (1 - \alpha)(d - 1) + s(d) - 1 < 0,
\]

which is the same as

\[
t > \frac{\rho \vartheta - \rho \alpha + s(d) + s(k) + (d - 1)(1 - \alpha) - 1}{k + 2 - \alpha}.
\]

We formulate these arguments into the following.
Example 2.12. Suppose that there exists $\rho > 0$, $0 < \vartheta < 1$ such that for any double sequence $a(N, n)$ with the condition (2.10) one has
\[
\int_{T_d} \left| \sum_{n=1}^{N} a(N, n) e(x_1 n + \ldots + x_d n^d) \right|^\rho d\mu_S(x) \leq N^{\rho \vartheta + o(1)}.
\]
Then we have
\[
\dim(E_{a,d,\alpha} \cap S^{d-1}) \leq u_{S,\rho,\vartheta}(d, \alpha),
\]
where
\[
u_{S,\rho,\vartheta}(d, \alpha) = \min_{k=0,1,\ldots,d-1} \frac{\rho \vartheta - \rho \alpha + s(d) + s(k) + (d-1)(1-\alpha) - 1}{k + 2 - \alpha}.
\]

Remark 2.13. It is expected that for some specific sets we could obtain better bounds than the general one of Theorem 2.9. Below we give such results for moment curves (2.5) and line segments (2.15).

2.2. Moment curves. We start with giving an improved version of the bound (2.8) when we integrate over the interval $[\delta, 1]$.

Theorem 2.14. For any $s \geq 1$ and any $0 < \delta < 1$ and $N > 1/\delta$ we have
\[
\int_\delta^1 \left| \sum_{n=1}^{N} a_n e(tn + \ldots + t^d n^d) \right|^{2s} dt \leq \delta^{(1-d)/2} \left( N^s + N^{2s-s(d)/2} \right) N^{s(d)/2-1/2+o(1)}.
\]

Note that for a fixed $\delta$, by Theorem 2.1 and Lemma 3.6 we obtain
\[
\int_\delta^1 \left| \sum_{n=1}^{N} a_n e(tn + \ldots + t^d n^d) \right|^2 dt \leq N^{1+o(1)},
\]
and hence for any $s \geq 1$,
\[
\int_\delta^1 \left| \sum_{n=1}^{N} a_n e(tn + \ldots + t^d n^d) \right|^{2s} dt \leq N^{2s-1+o(1)}.
\]

(2.13)

Clearly, for $s \geq s(d)/2$ the bound of Theorem 2.14 takes form
\[
\int_\delta^1 \left| \sum_{n=1}^{N} a_n e(tn + \ldots + t^d n^d) \right|^{2s} dt \leq \delta^{(1-d)/2} N^{2s-d/2+o(1)}.
\]

(2.14)

Thus for a fixed $\delta$, the bound (2.14) improve the trivial bound (2.13) for any $d > 2$.

We remark that one can obtain a similar result for the integral over any interval $[\delta, L]$ from some $0 < \delta < L$. 

It is interesting to understand whether the exponent $2s - d/2$ is optimal in (2.14). However we have the following lower bound. Note that there exists a small constant $\varepsilon_0 > 0$ such that $t \in [1 - \varepsilon_0/N^d, 1]$ implies
\[
\left| \sum_{n=1}^{N} e(tn + \ldots + t^d n^d) \right| \geq \frac{1}{2} N,
\]
and hence, say for $\delta < 1/2$ and any $s > 0$,
\[
\int_{\delta}^{1} \left| \sum_{n=1}^{N} e(tn + \ldots + t^d n^d) \right|^{2s} dt \\
\geq \int_{1-\varepsilon_0/N^d}^{1} \left| \sum_{n=1}^{N} e(tn + \ldots + t^d n^d) \right|^{2s} dt \\
\geq \int_{1-\varepsilon_0/N^d}^{1} \left| \sum_{n=1}^{N} e(tn + \ldots + t^d n^d) \right|^{2s} dt \gg N^{2s-d}.
\]

For the moment curve $\Gamma$ defined by (2.5), Example 2.7 asserts that for $\mu_M$-almost all $x \in \Gamma$ one has
\[
|S_{a,d}(x; N)| \leq N^{1/2+o(1)}, \quad N \to \infty.
\]

For the exceptional sets $E_{a,d,\alpha} \cap \Gamma$ we have the following.

**Theorem 2.15.** For the moment curve $\Gamma$ defined by (2.5) and any $a_n = n^{\alpha(1)}$ we have
\[
\dim(E_{a,d,\alpha} \cap \Gamma) \leq 1 - \frac{2\alpha - 1}{d + 1 - \alpha}.
\]

We note that Theorem 2.15 gives a non-trivial bound for any $0 < \alpha < 1$. If $\alpha \to 1$ then the bounds of (1.6) gives
\[
\dim(E_{a,d,\alpha} \cap \Gamma) \leq \dim E_{a,d,\alpha} \to 0,
\]
which is better than the bound in Theorem 2.15. However, if $\alpha$ is close to $1/2$ then Theorem 2.15 implies a better bound.

It is natural to expect that if $\alpha \to 1/2$ then
\[
\dim E_{a,d,\alpha} \to d \quad \text{and} \quad \dim(E_{a,d,\alpha} \cap \Gamma) \to 1.
\]

2.3. Segments. We investigate the Weyl sums on the given segment. Among other things, our results imply that the condition (2.1) of Theorem 2.1 is not necessary.

We introduce some notation first. Let $\omega \in \mathbb{R}^d$ with $\|\omega\| = 1$ and
\[
L_\omega = \{ t\omega : \ t \in [0, 1] \}.
\]
Let \( \mu_\omega \) be the Lebesgue measure on \( L_\omega \). The orthogonal complementary space of \( \omega \) is defined as
\[
\omega^\perp = \{ x \in \mathbb{R}^d : x \cdot \omega = 0 \}.
\]
Clearly for any \( \xi \in \omega^\perp \) we have
\[
\hat{\mu}_\omega(\xi) = \int_{L_\omega} e(-x \cdot \xi) d\mu_\omega(x) = \int_0^1 e(-t \xi \cdot \omega) dt = 1.
\]
Thus the measure \( \mu_\omega \) does not have the decay property as (2.1). However, by using the van der Corput lemma (see [23, Theorem 14.2] or Lemma 3.5 below) we obtain an analogue of the result of Theorem 2.1.

**Theorem 2.16.** Using the above notation, let \( a = (a_n)_{n=1}^\infty \) satisfy the condition (2.2), then
\[
\int_{L_\omega} |S_{a,d}(x; N)|^2 d\mu_\omega(x) \leq N^{1+o(1)}.
\]

**Corollary 2.17.** Using the above notation, for any sequence \( a_n = n^{o(1)} \) and for \( \mu_\omega \)-almost all \( x \in L_\omega \) one has
\[
|S_{a,d}(x; N)| \leq N^{1/2+o(1)}.
\]

**Corollary 2.18.** Let \( a_n = n^{o(1)} \) and \( 1/2 < \alpha < 1 \) then for any
\[
(2.16) \quad \omega = (\omega_1, \ldots, \omega_k, 0, \ldots, 0) \in \mathbb{R}^d, \quad \omega_k \neq 0, \quad \|\omega\| = 1,
\]
we have
\[
\dim(E_{a,d,\alpha} \cap L_\omega) \leq 1 - \frac{2\alpha - 1}{k + 1 - \alpha}.
\]

Note that the condition (2.16) is used in the following way. For any \( x \in L_\omega \) and \( \zeta = (\zeta_1, \ldots, \zeta_d) \) with \( 0 < \zeta_j < 1, \ j = 1, \ldots, d, \) we have
\[
\zeta_k \ll \text{diam}(L_\omega \cap \mathcal{R}(x, \zeta)) \ll \zeta_k.
\]

We remark that if \( \alpha \to 1 \) then the bounds of (1.6) gives
\[
\dim(E_{a,d,\alpha} \cap \text{spt} \mu_\omega) \leq \dim E_{a,d,\alpha} \to 0,
\]
which is better than the bound in Corollary 2.18. However the following Example 2.19 shows that Corollary 2.18 can give better bounds in some cases.

**Example 2.19.** For the horizontal segment
\[
L = \{(t, 0) : t \in [0, 1]\} \subseteq \mathbb{T}^2,
\]
and for any \( 1/2 < \alpha < 1 \), Corollary 2.18 with \( d = 2 \) and \( k = 1 \) implies that
\[
\dim(E_{d,\alpha} \cap L) \leq \frac{3(1-\alpha)}{2-\alpha}.
\]
While applying (1.6) and (1.7) and using \( k = 0, 1 \) we obtain
\[
\dim(\mathcal{E}_{d, \alpha} \cap L) \leq \dim \mathcal{E}_{d, \alpha} \leq \min \left\{ \frac{8(1 - \alpha)}{2 - \alpha}, \frac{9 - 8\alpha}{3 - \alpha} \right\}.
\]
For \( 1/2 < \alpha < 1 \) elementary calculus shows that
\[
\frac{3(1 - \alpha)}{2 - \alpha} < \min \left\{ \frac{8(1 - \alpha)}{2 - \alpha}, \frac{9 - 8\alpha}{3 - \alpha} \right\}.
\]
Therefore, the bound of Example 2.19 gives a better bound for all \( 1/2 < \alpha < 1 \).

In general, the exact comparison between the bound \( u(d, \alpha) \) and that of Corollary 2.18 is not immediately obvious.

3. Preliminaries

3.1. The completion technique. We remark that the completion technique has many applications in analytic number theory. The following bound is a special case of [9, Lemma 3.2].

**Lemma 3.1.** For \( x \in \mathbf{T}_d \) and \( 1 \leq M \leq N \) we have
\[
S_{a,d}(x; M) \ll W_{a,d}(x; N),
\]
where
\[
W_{a,d}(x; N) = \sum_{h=-N}^N \frac{1}{|h| + 1} \left| \sum_{n=1}^N a_n e \left( \frac{hn}{N} + x_1n + \ldots + x_dn^d \right) \right|.
\]
Note that for any \( N \) there exists a sequence \( b_N(n) \) such that
\[
b_N(n) \ll \log N, \quad n = 1, \ldots, N,
\]
and \( W_{a,d}(x; N) \) can be written as
\[
W_{a,d}(u; N) = \sum_{n=1}^N a_n b_N(n) e(x_1n + \ldots + x_dn^d).
\]
Indeed, for each \( h \in \mathbb{Z}, N \in \mathbb{N}, x \in \mathbf{T}_d \) and the sequence \( a = (a_n)_{n=1}^\infty \) there exists some complex number \( \vartheta(h, N, x, a) \) on the unit circle such that
\[
W_{a,d}(x; N) = \sum_{h=-N}^N \frac{\vartheta(h, N, x, a)}{|h| + 1} \sum_{n=1}^N a_n e \left( \frac{hn}{N} + x_1n + \ldots + x_dn^d \right).
\]
Hence
\[
b_N(n) = \sum_{h=-N}^N \frac{\vartheta(h, N, x, a)}{|h| + 1} e(hn/N) \ll \log N.
\]
3.2. **Continuity of exponential sums.** Analogously to [9, Lemma 3.5] and [26, Lemma 2.1] we obtain:

**Lemma 3.2.** Let \(0 < \alpha < 1\) and let \(\varepsilon > 0\) be sufficiently small. If \(|W_{a,d}(x; N)| \geq N^\alpha\) for some \(x \in \mathbb{T}_d\), then \(|W_{a,d}(y; N)| \geq N^\alpha/2\) holds for any \(y \in \mathcal{R}(x, \zeta)\) provided that \(N\) is large enough and

\[
0 < \zeta_j \leq N^{\alpha-j-1-\varepsilon}, \quad j = 1, \ldots, d.
\]

**Proof.** For any integer \(h\) with \(|h| \leq N\) we have

\[
\sum_{n=1}^{N} a_n e(hn/N) \left( e \left( x_1 n + \ldots + x_d n^d \right) - e \left( y_1 n + \ldots + y_d n^d \right) \right) \ll \sum_{n=1}^{N} \sum_{j=1}^{d} a_n \zeta_j n^j \leq N^{\alpha-\varepsilon/2}.
\]

The last estimate holds for all sufficiently large \(N\). By Lemma 3.1 we obtain

\[
|W_{a,d}(x; N) - W_{a,d}(y; N)| \ll N^{\alpha-\varepsilon/2} \log N \leq N^\alpha/2,
\]

which holds for all sufficiently large \(N\) and thus the result follows. \(\square\)

3.3. **Covering the large values of exponential sums.** In analogy to [9, Lemma 3.7] we obtain the following result.

**Lemma 3.3.** Let \(\mu\) be a \(f\)-regular Radon measure as in Theorem 2.9. Let \(0 < \alpha < 1\) and \(\varepsilon > 0\) be a small parameter and let

\[
\zeta(\varepsilon) = (\zeta_1(\varepsilon), \ldots, \zeta_d(\varepsilon))
\]

where

\[
\zeta_j(\varepsilon) = N^{\alpha-j-1-\varepsilon}, \quad j = 1, \ldots, d.
\]

Then for some

\[
L \leq N^{\rho(\theta-\alpha) + o(1)} f(\zeta(\varepsilon))^{-1},
\]

there exist \(x_1, \ldots, x_L \in \text{spt } \mu\) such that

\[
\{x \in \text{spt } \mu : |W_{a,d}(x; N)| \geq N^\alpha\} \subseteq \bigcup_{\ell=1}^{L} \mathcal{R}(x_\ell, 3\zeta(\varepsilon)),
\]

and

\[
\mathcal{R}(x_i, \zeta(\varepsilon)) \cap \mathcal{R}(x_j, \zeta(\varepsilon)) = \emptyset, \quad 1 \leq i \neq j \leq L.
\]
Proof. Let \( B = \{ x \in \text{spt } \mu : |W_{a,d}(x; N)| \geq N^\alpha \} \), and let
\[
\mathcal{R}_\ell = \mathcal{R}(x_\ell, \zeta(\varepsilon)), \quad \ell = 1, \ldots, L,
\]
be a maximal collection of pair-wise disjoint rectangles from the set \( \{ \mathcal{R}(x, \zeta(\varepsilon)) : x \in B \} \). Then, we observe that the maximality of the family of the rectangles \( \mathcal{R}_\ell, \ell = 1, \ldots, L \), implies that each blow-up rectangle \( \mathcal{R}(x_\ell, 3\zeta(\varepsilon)) \) implies that
\[
B \subseteq \bigcup_{x \in B} \mathcal{R}(x, \zeta(\varepsilon)) \subseteq \bigcup_{\ell=1}^{L} \mathcal{R}(x_\ell, 3\zeta(\varepsilon)).
\]

For each \( \ell = 1, \ldots, L \), applying Lemma 3.2 we have
\[
|W_{a,d}(x; N)| \geq N^\alpha / 2, \quad \forall x \in \mathcal{R}(x_\ell, \zeta(\varepsilon)).
\]
Thus, together with the assumption that the measure \( \mu \) is a \( f \)-regular measure on \( T_d \), we arrive at
\[
\int_{T_d} |W_{a,d}(x; N)|^\rho d\mu(x) \geq \sum_{\ell=1}^{L} \int_{T_d \cap \mathcal{R}_\ell} |W_{a,d}(x; N)|^\rho d\mu(x)
\[
\geq \sum_{\ell=1}^{L} N^{\alpha \rho} \mu(T_d \cap \mathcal{R}_\ell)
\[
\geq LN^{\alpha \rho} f(\zeta(\varepsilon)).
\]
Therefore, combining with the mean value bound of Theorem 2.9, we obtain the desired bound for \( L \). \( \square \)

From Lemmas 3.1 and 3.3, we formulate the following Corollary 3.4 for the convenience of our applications on estimating the Hausdorff dimension of the set \( E_{a,d,\alpha} \cap \text{spt } \mu \). Using the same notation as in Lemma 3.3 we denote
\[
\mathcal{R}(N) = \{ \mathcal{R}(x_\ell, 3\zeta(\varepsilon)) : \ell = 1, \ldots, L \}.
\]

**Corollary 3.4.** Let \( \mu \) be a \( f \)-regular Radon measure as in Theorem 2.9. Let \( 0 < \alpha < 1 \) and let \( \varepsilon > 0 \) be a small parameter. Let \( N_i = 2^i, i \in \mathbb{N} \). Then for any \( \eta > 0 \) we have
\[
E_{a,d,\alpha+\eta} \cap \text{spt } \mu \subseteq \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} \bigcup_{R \in \mathcal{R}(N_i)} \mathcal{R},
\]
where each \( \mathcal{R} \) of \( \mathcal{R}(N_i) \) has the side length \( \zeta(\varepsilon) = (\zeta_{i,1}(\varepsilon), \ldots, \zeta_{i,d}(\varepsilon)) \) such that
\[
\zeta_{i,j}(\varepsilon) = N_i^{\alpha-j-1-\varepsilon}, \quad j = 1, \ldots, d.
\]
and furthermore
\[ \#R(N_i) \leq N_i^{d-\mu} f(\zeta_\varepsilon))^{-1}. \]

**Proof.** We continue to use the same notation as in Lemmas 3.1 and 3.3. For each \( i \in \mathbb{N} \) and \( N_i = 2^i \) let
\[ B_i = \{ x \in \text{spt} \mu : |W_{a,d}(x; N_i)| \geq N_i^\alpha \}. \]

We intend to show that for any \( \eta > 0 \) we have
\[ (3.1) \quad \mathcal{E}_{a,d,\alpha+\eta} \cap \text{spt} \mu \subseteq \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} B_i. \]

Let \( x \in \mathcal{E}_{a,d,\alpha+\eta} \cap \text{spt} \mu. \) Suppose that
\[ x \notin \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} B_i. \]

Then there exists \( i_x \) such that for all \( i \in \mathbb{N}, i \geq i_x \) implies
\[ |W_{a,d}(x; N_i)| \leq N_i^\alpha. \]

Clearly for any \( N > N_{i_x} \) there exists \( i \geq i_x \) such that
\[ N_i \leq N < N_{i+1}. \]

By Lemma 3.1 we arrive at
\[ |S_{a,d}(x; N)| \ll |W_{a,d}(x; N_{i+1})| \ll N^\alpha. \]

Thus we have a contradiction with our condition \( x \in \mathcal{E}_{a,d,\alpha+\eta} \cap \text{spt} \mu. \) Therefore, we deduce that
\[ x \in \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} B_i, \]
and thus we have \((3.1)\). Combining this with Lemma 3.3 we obtain the desired result. \( \square \)

### 3.4. Bounds on exponential integrals.
For bounding various exponential integrals (or oscillatory integrals) we use the following **lemma of van der Corput**, see [23, Theorem 14.2].

**Lemma 3.5.** Let \( \varphi \) be a smooth function on \( \mathbb{R} \) with \( |\varphi^{(k)}(x)| \geq Q > 0 \) uniformly over \( x \in [a, b] \) and for some \( k \in \{1, 2, \ldots\} \). Then we have
\[ \int_a^b e(\varphi(x))dx \ll Q^{-1/k}, \]
provided that
(i) either \( k = 1 \) and the derivative \( \varphi' \) is a monotone function over interval \([a, b]\);
(ii) or \( k \geq 2 \).

We now present the following general bound on exponential integrals with polynomial arguments which is perhaps known already. We give a proof here for the completeness.

**Lemma 3.6.** For any interval \([a, b] \subseteq [0, 1]\) we have
\[
\int_a^b e(t\xi_1 + \ldots + t^d\xi_d) dt \ll (1 + \|\xi\|)^{-1/d}, \quad \xi \in \mathbb{R}^d.
\]

**Proof.** Without loss of generality we can assume \( \|\xi\| \geq 1 \) in the following. Let \( 1 \leq k_0 \leq d \) be such that
\[
|\xi_{k_0}| = \max\{|\xi_1|, \ldots, |\xi_d|\}.
\]

We now present a case-by-case argument which depends on the value of \( k_0 \) and \( |\xi_{k_0}| \).

**Case 1.** Suppose that \( k_0 = d \). Let
\[
\varphi(t) = t\xi_1 + \ldots + t^d\xi_d,
\]
then by (3.2) we obtain
\[
|\varphi^{(d)}(t)| = d!\xi_d \gg \|\xi\|, \quad t \in [a, b].
\]
Thus by Lemma 3.5 we obtain the desired bound in this case.

**Case 2.** Suppose that \( 1 \leq k_0 < d \) and
\[
|\xi_{k_0}| \geq 2 \sum_{j=k_0+1}^{d} \left( \frac{j}{k_0} \right) |\xi_j|b^j.
\]
It follows that for any \( t \in [a, b] \) we have
\[
|\varphi^{(k_0)}(t)| = k_0! \left| \xi_{k_0} + \sum_{j=k_0+1}^{d} \left( \frac{j}{k_0} \right) \xi_j t^{j-k} \right|
\geq k_0! \left( |\xi_{k_0}| - \sum_{j=k_0+1}^{d} \left( \frac{j}{k_0} \right) |\xi_j|b^{j-k} \right)
\geq k_0! |\xi_{k_0}|/2.
\]
Applying Lemma 3.5 and (3.2) we obtain (\( \|\xi\| \geq 1 \))
\[
\int_a^b e(t\xi_1 + \ldots + t^d\xi_d) dt \ll \|\xi\|^{-1/k_0} \leq \|\xi\|^{-1/d},
\]
which concludes this case.
Case 3. Suppose that $1 \leq k_0 < d$ and
\[ |\xi_{k_0}| < 2 \sum_{j=k_0+1}^{d} \binom{j}{k_0} |\xi_j| b_j. \]
Then there exists $k_0 + 1 \leq k_1 \leq d$ such that
\[ |\xi_{k_1}| \gg \|\xi\|. \]
Now applying the same arguments as in Case 1 and Case 2 to $k_1$, then either we obtain the desired bound or there exists $k_2 \geq k_1 + 1$ such that
\[ |\xi_{k_2}| \gg \|\xi\|. \]
Iterating this argument at most $d$ times yields the desired result. □

4. Proofs of general results

4.1. Proof of Theorem 2.1. For $n, m \in \mathbb{N}$ let
\[ \xi_{n,m} = (n - m, \ldots, n^d - m^d). \]
Clearly we have
\[ |n^d - m^d| \ll \|\xi_{n,m}\| \ll |n^d - m^d|. \]
Let
\[ I = \int_{T_d} \left| \sum_{n=1}^{N} a_n \sum_{d} a_m e(x_1 n + \ldots + x_d n^d) \right|^2 d\mu(x). \]
Expanding the square and changing the order of summation, we have
\[ I = \int_{T_d} \sum_{1 \leq n, m \leq N} a_n \overline{a_m} e \left( \sum_{i=1}^{d} x_i (n^i - m^i) \right) d\mu(x) \]
\[ = \sum_{1 \leq n, m \leq N} a_n \overline{a_m} \widehat{\mu}(\xi_{n,m}). \]
Note that $\widehat{\mu}(0) = \mu(T_d) < \infty$, where the second inequality holds by the definition of Radon measure. For $n \neq m$ applying the decay condition (2.1) and (4.1) we obtain
\[ \widehat{\mu}(\xi_{n,m}) \ll (n^d - m^d)^{-\sigma}. \]
Hence, using the symmetry of $m$ and $n$ we obtain

$$I \leq N^{1+\omega(1)} + N^{o(1)} \sum_{1 \leq m \neq n \leq N} (n^d - m^d)^{-\sigma}$$

$$\leq N^{1+\omega(1)} + N^{o(1)} \sum_{m=1}^{N} \sum_{k=1}^{N} ((m+k)^d - m^d)^{-\sigma}$$

$$\leq N^{1+\omega(1)} + N^{o(1)} \sum_{m=1}^{N} \sum_{k=1}^{N} (km^{d-1})^{-\sigma}$$

$$\leq N^{1+\omega(1)} + N^{o(1)} \left( 1 + N^{1-(d-1)\sigma} \right) \left( 1 + N^{1-\sigma} \right) \leq N^{1+\omega(1)}$$

provided $\sigma \geq 1/d$.

**Remark 4.1.** For $\sigma > 1/d$ and $d \geq 2$ the proof of Theorem 2.1 implies

$$\int_{T_d} \left| \sum_{n=1}^{N} a_n \left( x_1 n + \ldots + x_d n^d \right) \right|^2 \, d\mu(x) - N \mu(T_d) \sum_{n=1}^{N} |a_n|^2$$

$$\ll N^{o(1)} \left( N^{1-\sigma} + N^{1-(d-1)\sigma} + N^{2-d\sigma} \right) = N^{1-\kappa + o(1)},$$

where $\kappa = \min\{\sigma, d\sigma - 1\} > 0$. Thus for the case $\sigma > 1/d$ we obtain an asymptotic formula with a power saving for the square mean values of $S_{a,d}(x;N)$. In particular, this implies that the bound in Theorem 2.1 is optimal when $\sigma > 1/d$.

### 4.2. Proof of Theorem 2.2.

For $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{Z}^d$ let $J_{a,N}(\boldsymbol{\xi})$ be the number of solutions to the system of equations

$$\sum_{j=1}^{s(d)} n_j^i - \sum_{j=s(d)+1}^{2s(d)} n_j^i = \xi_i, \quad i = 1, \ldots, d,$$

$$n_j = 1, \ldots, N, \quad j = 1, \ldots, 2s(d),$$

counted with the weights

$$\prod_{j=1}^{s(d)} a_{n_j} \prod_{j=s(d)+1}^{2s(d)} \alpha_{n_j} = N^{o(1)}.$$

If $a_n = 1$ for all $n \in \mathbb{N}$ then we simply denote

$$J_N(\boldsymbol{\xi}) = J_{a,N}(\boldsymbol{\xi}).$$

From the orthogonality of exponential functions, it is not hard to see that

$$J_N(\boldsymbol{\xi}) = \int_{T_d} \left| \sum_{n=1}^{N} \left( x_1 n + \ldots + x_d n^d \right) \right|^{2s(d)} \left( -x_1 \xi_1 - \ldots - x_d \xi_d \right) \, d\boldsymbol{x}. $$
Hence,
\[ J_N(\xi) \leq J_N(0), \quad \forall \xi \in \mathbb{Z}^d. \]

Recall that the Vinogradov mean value theorem (1.1) implies
\[ J_N(0) \leq N^{s(d)+o(1)}, \quad N \to \infty. \]

From (4.2) and (4.3) we obtain
\[ J_{a,N}(\xi) \leq N^{o(1)}J_N(\xi) \leq N^{s(d)+o(1)}. \]

Note that \( J_{a,N}(\xi) = 0 \) if there is \( i = 1, \ldots, d \) such that
\[ |\xi_i| \geq s(d)N^i. \]

For each \( N \) let
\[ D_N = \{ \xi \in \mathbb{Z}^d : |\xi_i| \leq s(d)N^i, \ i = 1, \ldots, d \}. \]

Expanding sums \( |S_{a,d}(x; N)|^{2s(d)} \) and changing the order of summation, and combining with (4.4) and the definition of \( D_N \), for
\[ \mathcal{I} = \int_{\mathbb{T}^d} \left| \sum_{n=1}^{N} a_n e(x_1n + \ldots + x_d n^d) \right|^{2s(d)} d\mu(x) \]
we derive
\[ \mathcal{I} \leq \sum_{\xi \in \mathbb{Z}^d} J_{a,N}(\xi) |\hat{\mu}(\xi)| \leq N^{s(d)+o(1)} \Xi_N, \]
where
\[ \Xi_N = \sum_{\xi \in \mathcal{D}_N} (1 + \|\xi\|)^{-\sigma}. \]

Taking dyadic decomposition for the range \( 1 \leq \|\xi\| \leq N^d \) we derive that there exists a positive integer \( H < 2dN^d \) such that
\[ \Xi_N \ll H^{-\sigma} \# \mathcal{L}_{N,H} \log N, \]
where
\[ \mathcal{L}_{N,H} = \{ \xi \in \mathcal{D}_N : H \leq \|\xi\| < 2H \}. \]

Assume that
\[ N^i \leq H < N^{i+1} \]
for some \( i = 0, 1, \ldots, d \).

From (4.5) and (4.8) we obtain
\[ \# \mathcal{L}_{N,H} \ll N^{s(i)}H^{d-i}, \]
where \( s(i) \) is given by (1.8). By (4.7) we arrive at
\[ \Xi_N \ll N^{s(i)}H^{d-\sigma-i} \log N. \]
We now formulate a case-to-case argument which depends on the value $i$ and $\sigma$.

**Case 1.** Suppose that $i \leq d - \sigma$. From (4.8) and (4.9) we obtain
$$
\Xi_N \ll N^{s(i) + (i+1)(d-\sigma - i)}.
$$
Let $f(t) = t(t+1)/2 + (t+1)(d - \sigma - t)$ then $f$ attains the maximal value at
$$
t_0 = d - \sigma - 1/2.
$$
and since $f$ is quadratic its largest value at an integer argument is attained at
$$
i_0 = \lfloor t_0 \rfloor = \lfloor d - \sigma - 1/2 \rfloor.
$$
It follows that for each $i = 0, 1, \ldots, d$ we have
$$
\Xi_N \ll N^{s(i) + (i+1)(d-\sigma - i)} \log N \leq N^{s(i_0) + (i_0+1)(d-\sigma - i_0)} \log N.
$$
It remains to observe that the definition of the function $\lfloor x \rfloor$ implies $\lfloor x - 1/2 \rfloor \leq x$, thus we obtain $i_0 \leq d - \sigma$ is within the range under consideration.

**Case 2.** Suppose that $i > d - \sigma$. Then from (4.8) and (4.9) we obtain
$$
\Xi_N \ll N^{s(i) + (i+1)(d-\sigma - i)}.
$$
Let $g(t) = t(t+1)/2 + t(d - \sigma - t)$ then $g$ attains the maximal value at
$$
u_0 = d - \sigma + 1/2.
$$
As before, it now follows that for each $i = 0, 1, \ldots, d$ we have
$$
\Xi_N \ll N^{s(i) + (d-\sigma - i)} \leq N^{s(j_0) + j_0(d - \sigma - j_0)},
$$
where
$$
j_0 = \lfloor u_0 \rfloor = \lfloor d - \sigma + 1/2 \rfloor.
$$
Again the definition of $\lfloor x \rfloor$ gives $\lfloor x + 1/2 \rfloor > x$, thus we see that $j_0 > d - \sigma$ is an admissible value.

Combining (4.11) and (4.12), we derive that
$$
\Xi_N \ll N^{s(i_0) + (i_0+1)(d-\sigma - i_0)} + N^{s(j_0) + j_0(d - \sigma - j_0)},
$$
where $i_0, j_0$ are given at (4.10) and (4.13), respectively.

Observe that the definition of $\lfloor x \rfloor$ implies that
$$
\lfloor x + 1/2 \rfloor = \lfloor x - 1/2 \rfloor + 1,
$$
and thus $j_0 = i_0 + 1$ and one now verifies that
$$
s(i_0) + (i_0+1)(d - \sigma - i_0) = s(j_0) + j_0(d - \sigma - j_0),$$
hence the two terms in (4.14) are equal. Therefore, we have
\[ \Xi_N \ll N^{s(j_0) + j_0(d - \sigma - j_0)}, \]
where \( j_0 \) is given by (4.13). Thus by (4.6) we obtain the desired bound.

4.3. Proof of Theorem 2.4. Applying a similar chain of arguments as in the proof of Theorem 2.2, substituting the bound (4.4) in the inequality
\[ \int_{T_d} \left| \sum_{n=1}^{N} a_n e(x_1 n + \ldots + x_d n^d) \right|^{2s(d)} d\mu(x) \leq \sum_{\xi \in \mathbb{Z}^d} J_{a,N}(\xi) |\hat{\mu}(\xi)|, \]
see (4.6) and recalling the condition (2.9) we obtain the desired bound.

4.4. Proof of Theorem 2.5. Let \( \alpha > \vartheta \) and set
\[ N_i = 2^i, \quad i = 1, 2, \ldots. \]
Recall that the set \( W_{a,d}(x; N) \) is given by Lemma (3.1). We now consider the set
\[ B_i = \{ x \in \text{spt} \mu : |W_{a,d}(x; N_i)| \geq N_i^\alpha \}. \]

By the Markov inequality, the definition of \( W_{a,d}(x; N_i) \) and the mean value bound of Theorem 2.5, we derive
\[ \mu(B_i) \leq N_i^{-\rho \alpha} \int_{T_d} |W_{a,d}(x; N_i)|^{\rho} d\mu(x) \leq N_i^{\rho \vartheta - \rho \alpha + o(1)}. \]
Combining with \( \alpha > \vartheta \) and \( N_i = 2^i \), we have
\[ \sum_{i=1}^{\infty} \mu(B_i) < \infty. \]
Thus the Borel–Cantelli lemma implies
\[ \mu \left( \bigcap_{i=1}^{\infty} \bigcup_{q=i}^{\infty} B_i \right) = 0. \]
It follows that for \( \mu \)-almost all \( x \in \text{spt} \mu \) there exists \( i_x \) such that for any \( i \geq i_x \) one has
\[ |W_{a,d}(x; N_i)| \leq N_i^{\alpha}. \]
We now fix this \( x \) in the following argument.
For any \( N \geq N_{i_x} \) there exists \( i \) such that
\[ N_{i-1} \leq N < N_i. \]
By Lemma 3.1 and (4.15) we have
\[ S_{a,d}(x; N) \ll W_{a,d}(x; N_i) \ll N^\alpha. \]
Since $\alpha > \vartheta$ is arbitrary, this gives the desired result.

4.5. **Proof of Theorem 2.9.** We start from some auxiliary results. We adapt the definition of the *singular value function* from [14, Chapter 9] to the following.

**Definition 4.2.** Let $\mathcal{R} \subseteq \mathbb{R}^d$ be a rectangle with side lengths

$$r_1 \geq \ldots \geq r_d.$$  

For $0 < t \leq d$ we set

$$\varphi_{0,t}(\mathcal{R}) = r_1^t,$$

and for $k = 1, \ldots, d - 1$ we define

$$\varphi_{k,t}(\mathcal{R}) = r_1 \ldots r_k r_{k+1}^{t-k}.$$

Note that for a rectangle $\mathcal{R} \subseteq \mathbb{R}^2$ with the side length $r_1 \geq r_2$ we have

$$\varphi_{k,t}(\mathcal{R}) = \begin{cases} r_1^t & \text{for } k = 0, \\ r_1 r_2^{t-1} & \text{for } k = 1. \end{cases}$$

**Remark 4.3.** The notation $\varphi_{k,t}(\mathcal{R})$ roughly means that we can cover the rectangle $\mathcal{R}$ by about (up to a constant factor)

$$\frac{r_1}{r_{k+1}} \ldots \frac{r_k}{r_{k+1}}$$

balls of radius $r_{k+1}$, and hence this leads to the term

$$\varphi_{k,t}(\mathcal{R}) = \frac{r_1}{r_{k+1}} \ldots \frac{r_k}{r_{k+1}} r_{k+1}^t$$

in the expression for the Hausdorff measure with the parameter $t$ (again up to a constant factor which does not affect our results).

We now turn to the proof of Theorem 2.9. From the definition of the Hausdorff dimension, using the above notation and applying Corollary 3.4 we obtain

$$\dim(\mathcal{E}_{a,d,\alpha + \eta} \cap \text{spt } \mu) \leq \inf \left\{ t > 0 : \sum_{i=1}^{\infty} \sum_{\mathcal{R} \in \mathcal{B}(N_i)} \varphi_{k,t}(\mathcal{R}) < \infty, \right. \left. \text{for some } k = 0, \ldots, d - 1 \right\}.$$  

(4.16)
Furthermore, for \( k = 1, \ldots, d - 1 \) and \( 0 < t < d \) we have

\[
\sum_{\mathcal{R} \in \mathcal{R}(N_i)} \varphi_{k,t}(\mathcal{R}) = \# \mathcal{R}(N_i) \zeta_{i,k+1}(\varepsilon)^{t-k} \prod_{j=1}^{k} \zeta_{i,j}(\varepsilon)
\]

\[
\leq N_i^{\rho \alpha - \rho \alpha} f(\zeta_i(\varepsilon))^{-1} \zeta_{i,k+1}(\varepsilon)^{t-k} \prod_{j=1}^{k} \zeta_{i,j}(\varepsilon)
\]

\[
\leq N_i^{\rho \alpha + s(k) + t(\alpha - k - 2) + C(d)\varepsilon} f(\zeta_i(\varepsilon))^{-1},
\]

where \( C(d) \) is a positive constant which depends only \( d \). We remark that (4.17) also holds for the case \( k = 0 \), in which we have \( s(k) = 0 \). To be precise for \( k = 0 \) we have

\[
\sum_{\mathcal{R} \in \mathcal{R}(N_i)} \varphi_{0,t}(\mathcal{R}) \leq N_i^{\rho \alpha + t(\alpha - 2) + C(d)\varepsilon} f(\zeta_i(\varepsilon))^{-1}.
\]

Applying (4.16) we derive that

\[
\dim(\mathcal{E}_{a,d,\alpha+\eta} \cap \text{spt } \mu) \leq t
\]

provided that the parameters \( \alpha, \rho, k, t, \vartheta \) satisfy the following condition

\[
\sum_{i=1}^{\infty} N_i^{\rho \alpha + s(k) + t(\alpha - k - 2) + C(d)\varepsilon} f(\zeta_i(\varepsilon))^{-1} < \infty,
\]

where \( c(d) \) is a positive constant that depends only on \( d \). By the arbitrary choice of \( \eta > 0 \) we finish the proof.

5. Proofs of results for special sets

5.1. Proof of Theorem 2.14. We start with recalling the following well-known estimate, see, for example, [21, Equation (8.6)].

**Lemma 5.1.** For any \( t \in [-1/2, 1/2] \) we have (for convenience we set \( 1/0 = \infty \))

\[
\sum_{n=1}^{N} e(nt) \ll \min \left\{ N, \frac{1}{t} \right\}.
\]

For interval \([a, b] \subseteq [0, 1] \) let \( \mu_{[a,b]} \) be the natural measure on the moment curve over the interval \([a, b] \), see (2.6).

We also need the following \( L^2 \)-type estimate which could be of independent interest.

**Lemma 5.2.** For any \( 0 < \delta < 1/2 \) and \( N \in \mathbb{N} \) such that \( N\delta > 1 \) we have

\[
\sum_{\xi \in D_N} |\widehat{\mu_{[\delta,1/2]}(\xi)}|^2 \ll \delta^{1-d} N^{\alpha(d)-d},
\]
where the notation $D_N$ is given by \((4.5)\).

Proof. Observe that

$$\sum_{\xi \in D_N} \left| \hat{\mu}_{[\delta, 1/2]}(\xi) \right|^2 = \sum_{\xi \in D_N} \left| \int_{\delta}^{1/2} e(\xi_1 t + \ldots + \xi_d t^d) dt \right|^2$$

$$= \sum_{\xi \in D_N} \int_{\delta}^{1/2} \int_{\delta}^{1/2} e(\xi_1(u - v) + \ldots + \xi_d(u^d - v^d)) dudv$$

$$= \int_{\delta}^{1/2} \int_{\delta}^{1/2} \prod_{i=1}^{d} \sum_{|\xi_i| \leq 2s(d)N^i} e(\xi_i(u^i - v^i)) dudv.$$

For each $i = 1, \ldots, d$ the Lagrange mean value theorem implies that

$$i\delta^{i-1}|u - v| \leq |u^i - v^i| \leq 1/2.$$

Combining with Lemma 5.1 we arrive at

$$\sum_{\xi \in D_N} \left| \hat{\mu}_{[\delta, 1/2]}(\xi) \right|^2$$

\[(5.1)\]

$$\ll \int_{\delta}^{1/2} \int_{\delta}^{1/2} \prod_{i=1}^{d} \min \left\{ N^i, \frac{1}{|u^i - v^i|} \right\} dudv$$

$$\ll \delta^{d-s(d)} \int_{\delta}^{1/2} \int_{\delta}^{1/2} \prod_{i=1}^{d} \min \left\{ N^i\delta^{i-1}, \frac{1}{|u - v|} \right\} dudv.$$

Now we decompose the set $S = [\delta, 1/2] \times [\delta, 1/2]$ into finite “strips” and estimate the above integral over each strip. Precisely, for each $i = 1, \ldots, d - 1$, denote

$$S_i = \left\{ (u, v) \in S : N^i\delta^{i-1} \leq \frac{1}{|u - v|} < N^{i+1}\delta^i \right\}.$$

Furthermore, for $i = 0$ and $i = d$ denote

$$S_0 = \left\{ (u, v) \in S : \frac{1}{|u - v|} < N \right\},$$

$$S_d = \left\{ (u, v) \in S : \frac{1}{|u - v|} \geq N^{d}\delta^{d-1} \right\}.$$

Integration over $S_0$. Since $N\delta > 1$, any $(u, v) \in S_0$ implies

$$\frac{1}{|u - v|} < N^i\delta^{i-1}, \quad i = 1, \ldots, d.$$
Thus we obtain

\[
\int_{S_0} \prod_{i=1}^d \min \left\{ N_i \delta_i^{-1}, \frac{1}{|u-v|} \right\} \, dudv \leq \int_{S_0} \frac{1}{|u-v|^d} \, dudv \leq \log N \sum_{k=1}^{\log N} 2^{kd} \lambda \left( \{(u,v) \in S_0 : 2^{k-1} < |u-v| \leq 2^k\} \right) \ll N^{d-1},
\]

(5.2)

where \( \lambda \) is the 2-dimensional Lebesgue measure.

**Integration over \( S_d \).** Again since \( N\delta > 1 \), any \((u,v) \in S_d\) implies

\[
\frac{1}{|u-v|} \geq N_i \delta_i^{-1}, \quad i = 1, \ldots, d.
\]

Thus we have

\[
\int_{S_d} \prod_{i=1}^d \min \left\{ N_i \delta_i^{-1}, \frac{1}{|u-v|} \right\} \, dudv \leq \int_{S_d} N^{s(d)} \delta^{s(d)-d} \, dudv \leq N^{s(d)-d} \delta^{s(d)-2d+1}.
\]

(5.3)

**Integration over \( S_i \).** For \( i = 1, \ldots, d-1 \) the definition of \( S_i \) implies

\[
\prod_{i=1}^d \min \left\{ N_i \delta_i^{-1}, \frac{1}{|u-v|} \right\} \leq N^{s(i)} \delta^{s(i)-i} \frac{1}{|u-v|^{d-i}}.
\]

It follows that

\[
\int_{S_i} \prod_{i=1}^d \min \left\{ N_i \delta_i^{-1}, \frac{1}{|u-v|} \right\} \, dudv \leq N^{s(i)} \delta^{s(i)-i} \int_{S_i} \frac{1}{|u-v|^{d-i}} \, dudv.
\]

(5.4)

Taking dyadic decomposition over the range

\[
N^i \delta_i^{-1} \leq 1/|u-v| < N^{i+1} \delta_i,
\]

that is for each \( k \in \mathbb{N} \) let

\[
S_{i,k} = \{(u,v) \in S_i : 2^{k-1} N^i \delta_i^{-1} < 1/|u-v| < 2^k N^i \delta_i^{-1}\}.
\]

Then for the Lebesgue measure of \( S_{i,k} \) we have

\[
\lambda (S_{i,k}) \ll 2^{-k} N^{-i} \delta^{1-i}.
\]
Thus we derive
\[
\int_{S_i} \frac{1}{|u-v|^{d-i}} dudv \ll \sum_{k=1}^{\log N\delta} (2^k N^i \delta^{i-1})^{d-i} \lambda(S_{i,k})
\]
\[
\ll \sum_{k=1}^{\log N\delta} (2^k N^i \delta^{i-1})^{d-i} 2^{-k} N^{-i} \delta^{1-i}
\]
\[
= N^{(d-i-1)(i+1)} \delta^{i(d-i-1)}.
\]
Combining with (5.4) we obtain
\[
\int_{S_i} \prod_{i=1}^{d} \min \left\{ N^i \delta^{i-1}, \frac{1}{|u-v|} \right\} dudv
\]
\[
\ll N^{(i+1)(d-1-i/2)} \delta^{(d-1)i-s(i)}.
\]

The function \( f(t) = (t+1)(d-1-t/2) \) attains its maximal value at \( t_0 = d - 3/2 \).

Note that the function \( g(t) = (d-1)t - s(t) \) attains its minimal value at \( t_0 = d - 3/2 \).

Let \( i_0 = d - 1 \) (or \( i_0 = d - 2 \) by symmetry). Substituting in (5.5) we obtain
\[
\int_{S_i} \prod_{i=1}^{d} \min \left\{ N^i \delta^{i-1}, \frac{1}{|u-v|} \right\} dudv \ll N^{s(d)-d} \delta^{s(d)-2d+1}.
\]

Combining (5.1) with the estimates (5.2), (5.3) and (5.6) we arrive at
\[
\sum_{\xi \in \mathcal{D}_N} \left| \hat{\mu}_{[b,1/2]}(\xi) \right|^2 \ll \delta^{d-s(d)} N^{d-1} + \delta^{1-d} N^{s(d)-d} \ll \delta^{1-d} N^{s(d)-d},
\]
since \( N\delta > 1 \) and \( d \geq 2 \).

The proof of Lemma 5.2 implies the following result. Recall that for an interval \([a, b] \subseteq [0, 1] \) the measure \( \mu_{[a,b]} \) is the natural measure on the moment curve over the interval \([a, b] \), see (2.6).

**Lemma 5.3.** For any interval \([a, b] \subseteq [1/2, 1] \) with \( b - a \leq 1/2d \) we have
\[
\sum_{\xi \in \mathcal{D}_N} \left| \hat{\mu}_{[a,b]}(\xi) \right|^2 \ll N^{s(d)-d}.
\]

**Proof.** For each \( i = 1, \ldots, d \) and any \( u, v \in [a, b] \) the Lagrange mean value theorem implies that
\[
|u - v| \ll |u^i - v^i| \leq 1/2.
\]
Thus applying the argument as in the proof of Lemma 5.2, taking $\delta$ to be some positive constant, we obtain the desired bound.

We now turn to the proof of Theorem 2.14. Without loss of generality we assume that $0 < \delta < 1/2$. First of all let

\begin{equation}
(5.7) \quad [\delta, 1] \subseteq \bigcup_{j=0}^{2d} \mathcal{I}_j,
\end{equation}

where $\mathcal{I}_0 = [\delta, 1/2]$ and for $j = 1, \ldots, 2d$

$$
\mathcal{I}_j = [1/2 + (j-1)/2d, 1/2 + j/2d].
$$

Note that for each interval $\mathcal{I} \subseteq [0, 1]$, applying the argument as in the proof of Theorem 2.2, similarly to (4.6) we obtain

\begin{equation}
(5.8) \quad \int_{\mathcal{I}} \left| \sum_{n=1}^{N} e(tn + \ldots + t^d n^d) \right|^{2s} dt \leq \sum_{\xi \in \mathcal{D}_N} I_s(N; \xi) |\hat{\mu}_\mathcal{I}(\xi)|,
\end{equation}

where $I_s(N; \xi)$ is defined similarly to $J_{a,N}(\xi)$ as the number of solutions to the system of equations

$$
\sum_{j=1}^{s} n_j^i - \sum_{j=s+1}^{2s} n_j^i = \xi_i, \quad i = 1, \ldots, d;
$$

$$
n_j = 1, \ldots, N, \quad j = 1, \ldots, 2s.
$$

Combining with (5.7) and (5.8) we obtain

\begin{equation}
(5.9) \quad \int_{\delta}^{1} \left| \sum_{n=1}^{N} e(tn + \ldots + t^d n^d) \right|^{2s} dt
\end{equation}

$$
\leq \sum_{j=0}^{2d} \int_{\mathcal{I}_j} \left| \sum_{n=1}^{N} e(tn + \ldots + t^d n^d) \right|^{2s} dt
\leq \sum_{j=0}^{2d} \sum_{\xi \in \mathcal{D}_N} I_s(N; \xi) |\hat{\mu}_{\mathcal{I}_j}(\xi)|.
$$

By the Cauchy-Schwarz inequality we have

\begin{equation}
(5.10) \quad \sum_{\xi \in \mathcal{D}_N} I_s(N; \xi)|\hat{\mu}_{\mathcal{I}_j}(\xi)| \ll \left( \sum_{\xi \in \mathcal{D}_N} I_s(N; \xi)^2 \sum_{\xi \in \mathcal{D}_N} |\hat{\mu}_{\mathcal{I}_j}(\xi)|^2 \right)^{1/2}.
\end{equation}
We now note that by (1.1) we obtain
\begin{equation}
\sum_{\xi \in \mathcal{D}_N} I_s(N; \xi)^2 \leq I_{2s}(N;0) = \int_{\Gamma_d} |S_d(x;N)|^{4s} dx 
\leq N^{2s+o(1)} + N^{4s-s(d)+o(1)}.
\end{equation}

For \( j = 0 \), applying (5.10), (5.11) and Lemma 5.2, we arrive at
\begin{equation}
\sum_{\xi \in \mathcal{D}_N} I_s(N; \xi) |\hat{\mu}_{[d,1/2]}(\xi)| 
\ll \delta^{(1-d)/2} \left( N^s + N^{2s-s(d)/2} \right) N^{s(d)/2-d/2+o(1)}.
\end{equation}

Similarly, for each \( j = 1, \ldots, 2d \), we use Lemma 5.3 instead of Lemma 5.2 and derive
\begin{equation}
\sum_{\xi \in \mathcal{D}_N} I_s(N; \xi) |\hat{\mu}_{I_j}(\xi)| 
\ll \left( N^s + N^{2s-s(d)/2} \right) N^{s(d)/2-d/2+o(1)}.
\end{equation}

Combining this with (5.9) and (5.12) we obtain the desired bound.

5.2. **Proof of Theorem 2.15.** Let \( \delta > 0 \) and \( \Gamma_{\delta} = \Gamma \setminus \mathcal{B}(0, \delta) \). Moreover let \( \mu_{M,\delta} \) be the natural measure on \( \Gamma_{\delta} \), see (2.6). For any \( \zeta = (\zeta_1, \ldots, \zeta_d) \) with \( 0 < \zeta_j < 1, \ j = 1, \ldots, d \), we have
\begin{equation}
\zeta_d \ll \text{diam}(\mathcal{R}(x, \zeta) \cap \Gamma_{\delta}) \ll \zeta_d, \ \ x \in \Gamma_{\delta},
\end{equation}
where the implied constant may depend on \( \delta \). It follows that the measure \( \mu_{M,\delta} \) is \( f \)-regular if we take
\begin{equation}
f(\zeta) = c_d \zeta_d
\end{equation}
for some positive constant \( c_d \).

Note that Lemma 3.6 implies that
\[
\hat{\mu}_{M,\delta}(\xi) \ll (1 + \|\xi\|)^{-1/d}, \quad \xi \in \mathbb{R}^d.
\]
Thus by Remark 2.6 we derive that the measure \( \mu_{M,\delta} \) satisfies the condition of Theorem 2.9 with \( \rho = 2 \) and \( \vartheta = 1/2 \).

Let \( 1/2 < \alpha < 1 \) and let \( \varepsilon > 0 \) be a small parameter. In the following we use the notation from Corollary 3.4. Let \( N_i = 2^i, \ i \in \mathbb{N} \). Then for any \( \eta > 0 \) we have
\[
\mathcal{E}_{a,d,\alpha+\eta} \cap \Gamma_{\delta} \subseteq \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} \bigcup_{\mathcal{R} \in \mathcal{R}(N_i)} (\mathcal{R} \cap \Gamma_{\delta}).
\]
Moreover, by Corollary 3.4 the centre of each \( \mathcal{R} \in \mathcal{R}(N_i) \) is in \( \Gamma_{\delta} \). For \( t > 0 \), applying Corollary 3.4, the upper bound in (5.13), and (5.14),
we obtain
\[ \sum_{i=1}^{\infty} \#R(N_i) \text{diam}(R \cap \Gamma_\delta)' \leq N_i^{1-2\alpha + o(1)} \zeta_{i,d}^{-1+\sigma} \]
\[ \leq N_i^{d+2-3\alpha + t(\alpha-d-1) + C(d)\varepsilon + o(1)}. \]

where \( C(d) > 0 \) is a constant that depends only on \( d \). Applying (4.16), we conclude that
\[ \dim(\mathcal{E}_{a,d,\alpha+\eta} \cap \Gamma_\delta) \leq t \]
provided that the parameters satisfy the following further condition
\[ d + 2 - 3\alpha + t(\alpha - d - 1) + C(d)\varepsilon < 0. \]

By the arbitrary choice of \( \varepsilon > 0 \) it is sufficient to have
\[ d + 2 - 3\alpha + t(\alpha - d - 1) < 0. \]

Thus we conclude that
\[ (5.15) \quad \dim(\mathcal{E}_{a,d,\alpha+\eta} \cap \Gamma_\delta) \leq \frac{d + 2 - 3\alpha}{d + 1 - \alpha}. \]

Note that Hausdorff dimension has the following countable stability (see [14, Section 2.2]): for \( A_i \subseteq \mathbb{R}^d, \ i \in \mathbb{N} \) we have
\[ (5.16) \quad \dim \bigcup_{i=1}^{\infty} A_i = \sup_{i \in \mathbb{N}} \dim A_i. \]

Clearly we have
\[ \Gamma = \bigcup_{i=1}^{\infty} \Gamma_{1/i} \cup \{0\}. \]

Therefore, combining (5.15) and (5.16) we derive that
\[ \dim(\mathcal{E}_{a,d,\alpha+\eta} \cap \Gamma) \leq \frac{d + 2 - 3\alpha}{d + 1 - \alpha}. \]

By the arbitrary choice of \( \eta \) we obtain the desired bound.

5.3. **Proof of Theorem 2.16.** Let \( \omega = (\omega_1, \ldots, \omega_d) \) and \( \xi_{n,m} = (n - m, \ldots, n^d - m^d) \).

Let
\[ \mathcal{J} = \int_{L_\omega} |S_{a,d}(x)|^2 d\mu_\omega(x) = \int_{0}^{1} \left| \sum_{n=1}^{N} a_n e(t\omega_1n + \ldots t\omega_dn^d) \right|^2 dt. \]

Expanding the square, we obtain
\[ \mathcal{J} = \int_{0}^{1} \sum_{1 \leq n,m \leq N} a_n \overline{a_m} e(t\omega \cdot \xi_{n-m}) dt. \]
Observe that there exists a positive constant $C_\omega$ such that if
\[
\max\{n, m\} \geq C_\omega,
\]
and $n \neq m$, then we have
\[
|\omega \cdot \xi_{n-m}| \gg |n-m|.
\]
Indeed, for $\omega$ let $1 \leq k \leq d$ be the maximal number such that $\omega_k \neq 0$. If $k = 1$ then clearly we have (5.17). For the case $k > 1$, for each $1 \leq i < k$ and $n \neq m$, we have
\[
\left|\frac{w_i(n^i - m^i)}{w_k(n^k - m^k)}\right| \ll \min\left\{\frac{1}{n}, \frac{1}{m}\right\},
\]
where the implied constant may depend on $\omega$. Thus by choosing $C_\omega$ large enough, and $n \neq m$ we obtain
\[
|\omega \cdot \xi_{n-m}| \gg |w_k(n^k - m^k)| \gg |n-m|,
\]
which gives (5.17).

Applying Lemma 3.5 with $k = 1$ and the estimate (5.17), we deduce
\[
J \leq \int_0^1 \sum_{1 \leq n, m \leq N} a_n\overline{a_m}e(t\omega \cdot \xi_{n-m})dt
\]
\[
\ll \sum_{1 \leq n = m \leq N} a_n\overline{a_m} + N^{o(1)} \sum_{1 \leq m \neq n \leq N} 1/|n-m|
\]
\[
\leq N^{1+o(1)},
\]
which gives the desired result.

5.4. Proof of Corollary 2.17. Let $a(N,n)$ be a double sequence with the condition (2.10). Taking $a(N,n)$ instead of $a_n$ in the proof of Theorem 2.16, we obtain
\[
\int_0^1 \left|\sum_{n=1}^N a(n, N) e(t\omega_1n + \ldots t\omega_d n^d)\right|^2 dt \leq N^{1+o(1)}.
\]
Using Theorem 2.5 we obtain the desired result.

5.5. Proof of Corollary 2.18. Let $1/2 < \alpha < 1$ and let $\varepsilon > 0$ be a small parameter. In the following we use the notation from Corollary 3.4. Let $N_i = 2^i$, $i \in \mathbb{N}$. Then for any $\eta > 0$ we have
\[
\mathcal{E}_{a,d,\alpha+\eta} \cap L_\omega \subseteq \bigcap_{q=1}^\infty \bigcup_{i=q}^\infty \bigcup_{\mathcal{R} \in \mathcal{R}(N_i)} (\mathcal{R} \cap L_\omega).
\]
Note that the centre of each \( R \in \mathfrak{R}(N_i) \) is in \( L_\omega \). Thus the assumption (2.16) implies that
\[
\zeta_{i,k} \ll \text{diam}(R \cap L_\omega) \ll \zeta_{i,k}. \tag{5.18}
\]

By applying arguments similar to that in the proof of Theorem 2.15, taking \( \zeta_{i,k} \) instead of \( \zeta_{i,d} \), we obtain the desired bound.

**Remark 5.4.** Note that for segments we have the uniform bound (5.18), thus there is not need to use the decomposition argument as in the proof of Theorem 2.15.

6. Comments

Certainly the method of the proof of Theorem 2.14 works for many other polynomial curves and rationally parametrised varieties, that is, for exponential sums with polynomials
\[
f_t(X) = g_1(t)X + \ldots + g_d(t)X^d, \quad t = (t_1, \ldots, t_m) \in \mathbb{R}^m
\]
where \( g_i(T) \in \mathbb{R}[T] \), \( i = 1, \ldots, d \), are polynomials in \( m \) variables, although the specific estimate in Lemma 5.2 depends on the specific form of the moment curve (2.5).

However we do not see any approach to improving the general bound of Theorem 2.1 and Theorem 2.2 for the parameter \( x \) which runs through some general algebraic variety \( \mathcal{V} \), that is, for the integrals
\[
I_s(\mathcal{V}) = \int_{\mathcal{V}} \left| \sum_{n=1}^{N} a_n e \left( x_1 n + \ldots + x_d n^d \right) \right|^s d\mu_{\mathcal{V}}(x),
\]
where \( \mu_{\mathcal{V}} \) is some natural measure on \( \mathcal{V} \).

Note that Example 2.3 gives upper bounds on \( I_{2s(d)}(S^{d-1}) \), which follow directly from Theorem 2.2. We are however interested in stronger results utilising some specific properties of \( \mathcal{V} \), which we pose as an open question.

**Acknowledgement**

The authors are grateful to Bryce Kerr for his encouragement and many helpful discussions.

This work was supported by ARC Grant DP170100786.

**References**

[1] V. Beresnevich, R. Vaughan, S. Velani and E. Zorin, ‘Diophantine approximation on manifolds and the distribution of rationals: contributions to the convergence theory’, *Intern. Math. Res. Notices* 10 (2017), 2885–2908.

[2] C. Bluhm, ‘Random recursive construction of Salem sets’, *Ark. Mat.*, 34 (1996), 51–63.
[3] R. C. Baker, ‘Metric number theory and the large sieve’, *J. London Math. Soc.*, 24 (1981), 34–40.
[4] J. Bourgain, ‘On the Vinogradov mean value’, *Proc. Steklov Math. Inst.*, 296 (2017), 30–40.
[5] J. Bourgain, C. Demeter and L. Guth, ‘Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three’, *Ann. Math.*, 184 (2016), 633–682.
[6] J. Brandes, S. T. Parsell, K. Poulias, G. Shakan and R. C. Vaughan, ‘On generating functions in additive number theory, II: Lower-order terms and applications to PDEs’, *Preprint*, 2020, available at https://arxiv.org/abs/2001.05629.
[7] J. Brüdern, ‘Approximations to Weyl sums’, *Acta Arith.*, 184 (2018), 287–296.
[8] J. Brüdern and D. Daemen, ‘Imperfect mimesis of Weyl sums’, *Intern. Math. Res. Notices*, 2009 (2009), 3112–3126.
[9] C. Chen and I. E. Shparlinski, ‘New bounds of Weyl sums’, *Intern. Math. Research Notices*, (to appear).
[10] C. Chen and I. E. Shparlinski, ‘Hausdorff dimension of the large values of Weyl sums’, *Preprint*, 2019, available at https://arxiv.org/abs/1901.01551.
[11] C. Chen and I. E. Shparlinski, ‘Hausdorff dimension of the large values of Weyl sums’, *Preprint*, 2019, available at https://arxiv.org/abs/1904.04457.
[12] M. B. Erdoğan and G. Shakan, ‘Fractal solutions of dispersive partial differential equations on the torus’, *Selecta Math.* 25 (2019), Art. 11.
[13] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Boca Raton, FL: CRC, 1992.
[14] K. J. Falconer, *Fractal geometry: Mathematical foundations and applications*, John Wiley, 2nd Ed., 2003.
[15] A. Fedotov and F. Klopp, ‘An exact renormalization formula for Gaussian exponential sums and applications’, *Amer. J. Math.*, 134 (2012), 711–748.
[16] L. Flaminio and G. Forni, ‘On effective equidistribution for higher step nilflows’, *Preprint*, 2014, available at https://arxiv.org/abs/1407.3640.
[17] G. Harman, *Metric number theory*, London Math. Soc. Monographs. New Ser., vol. 18, The Clarendon Press, Oxford Univ. Press, New York, 1998.
[18] K. Henriot and K. Hughes, ‘Discrete restriction estimates of ε-removal type for k-th powers and paraboloids’, *Math. Ann.*, 372 (2018), 963–998.
[19] K. Henriot and K. Hughes, ‘On restriction estimates for discrete quadratic surfaces’, *Intern. Math. Res. Notices*, (to appear).
[20] K. Hughes and T. D. Wooley, ‘Discrete restriction for (x, x^3) and related topics’, *Preprint*, 2019, available at https://arxiv.org/abs/1911.12262.
[21] H. Iwaniec and E. Kowalski, *Analytic number theory*, Amer. Math. Soc., Providence, RI, 2004.
[22] X. Lai and Y. Ding, ‘A note on the discrete Fourier restriction problem’, *Proc. Amer. Math. Soc.*, 146 (2018), 3839–3846.
[23] P. Mattila, *Fourier analysis and Hausdorff dimension*, Cambridge Studies in Advanced Math., vol. 50, Cambridge Univ. Press, 2015.
[24] H. Weyl, ‘Über die Gleichverteilung von Zahlen mod Eins’, *Math. Ann.*, 77 (1916), 313–352.
[25] T. D. Wooley, ‘The cubic case of the main conjecture in Vinogradov’s mean 
value theorem’, *Adv. in Math.*, 294 (2016), 532–561.

[26] T. D. Wooley, ‘Perturbations of Weyl sums’, *Intern. Math. Res. Notices*, 2016 
(2016), 2632–2646.

[27] T. D. Wooley, ‘Discrete Fourier restriction via efficient congruencing’, *Intern. 
Math. Res. Notices*, 2017 (2017), 1342–1389.

[28] T. D. Wooley, ‘Nested efficient congruencing and relatives of Vinogradov’s 
mean value theorem’, *Proc. London Math. Soc.*, 118 (2019), 942–1016.

Department of Pure Mathematics, University of New South Wales, 
Sydney, NSW 2052, Australia

E-mail address: changhao.chenm@gmail.com

Department of Pure Mathematics, University of New South Wales, 
Sydney, NSW 2052, Australia

E-mail address: igor.shparlinski@unsw.edu.au