A new effective potential for deuteron

Taha Koohrokhi$^{1,*}$ and Sehban Kartal$^2$

$^1$Department of Physics, Faculty of Sciences, Golestan University, Gorgan, Iran
$^2$Istanbul University, Department of Physics, 34000, Istanbul, Turkey

E-mail: t.koohrokhi@gu.ac.ir and sehban@istanbul.edu.tr

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Abstract

We calculate for the first time the static properties of the deuteron, within the framework of supersymmetric quantum mechanics, analytically. A new effective potential and its partner are derived from a superpotential so that all parameters are fitted by the experimental data. An analytical expression is obtained for the deuteron wave function and contributions of the orthogonal $^{1S_1}$ and $^{3D_1}$ states are determined, explicitly. Compared to one pion exchange, the superpotential produces an electrostatic as well as two pion exchange terms for the potential. The saddle point radius of the potential and the maximum of the wave function are linearly proportional. In comparison with other methods, the approach presented in this paper is a new and extensible symmetry-based approach that, despite its straightforward calculations and explicit analytical expressions, provides a good explanation for two-body effective interactions such as two-nucleon systems and diatomic molecules.

Keywords: superpotential, Boson exchange, supersymmetry, effective potential

(Some figures may appear in colour only in the online journal)

1. Introduction

The deuteron is the simplest nucleus consisting of two nucleons. The study of deuterons provides useful information about static nucleon–nucleon (NN) interactions. The one pion exchange potential (OPEP) is an extended version of Yukawa potential that dominates in nucleon spacings above 3 fm, and is reasonable for spacings above 2 fm. However, theoretical and experimental studies have shown that the nuclear force is not just a matter of an exchange of single pion [1]. In addition, the exact study of NN interactions requires a more fundamental theory. Nevertheless, with good approximation, at long ranges, nucleons can still be considered structureless particles and OPEP suitable for describing the NN interaction.

In general, NN interactions have been studied based on several main groups: quantum chromo dynamics (QCD) [2–4], lattice QCD [5–7], effective field theory (EFT) [8], Chiral EFT [9–11], chiral perturbation theory [12–14], boson exchange (BE) models [15–17], mean field theory [18–20] and phenomenological NN potentials [21–23]. In most of the models, potentials have quite complicated structures and are described by many parameters. Indeed an efficient theory is a theory that, while having simple calculations and reproduction of expected values, has good insights, predictions and straightforward to be developed.

Supersymmetry (SUSY) was originally conceived within the quantum field theory as a means to unify the mathematical treatment of bosons and fermions [24–26]. In this regard, supersymmetric quantum mechanics (SUSY QM) is a development of quantum mechanics that introduces new concepts such as superpotential, partner potentials, Hamiltonians hierarchy, and shape invariant potentials. The mathematical strategies in SUSY QM not only solve many problems algebraically but also classify potentials into different categories and determine criteria for them. Furthermore, approximation methods in SUSY QM provide more accurate results than those in conventional quantum mechanics [24]. The theoretical successes of SUSY have caused its applications rapidly to be extended into other branches of physics and mathematics, i.e. supersymmetric quantum chromodynamics [27] as well as nuclear physics [28], by the unification of fermionic and bosonic fields to a superfield.

This paper is the first study of static properties of deuteron by SUSY QM. Given the excellent results achieved, this approach can be developed straightforwardly to consider

$^*$ Author to whom any correspondence should be addressed.
details of the interaction. Despite the simplicity of the calculations presented in this study, the proposed superpotential not only describes the long ranges of the interaction well but also gives notable results for intermediate and short ranges. The new attitude presented in this paper has introduced a new effective potential for deuteron that is also applicable for two-body interactions such as diatomic molecules.

2. Potential and superpotential

2.1. OPEP

The OPEP is the potential derived from meson theory in the treatment of the NN system [29], and given by

\[
V_{\text{OPEP}}(r) = V_C(r) + S_{12} V_T(r),
\]

where \( r \) is equal to the length of the vector \( r \) connecting the two nucleons and \( S_{12} \) is tensor operator. The first term in the OPEP is the central potential

\[
V_C(r) = V_0(\tau_1 \tau_2)(\sigma_1 \sigma_2) \frac{e^{-r/R}}{r},
\]

where \( R = \frac{\langle r \rangle}{m_c} \) is the typical range of the nuclear force and \( m_c \) is the pion mass. The neutron–proton interaction involves the exchanges of both the neutral (\( n^0 \)) and charged (\( \pi^\pm \)) pions. For this reason, we employ the averaged-pion mass \( m_{\pi} = \frac{1}{3}(m_{\pi^0} + 2m_{\pi^\pm}) \) (table 1) [30]. The dot products, \( \tau_1, \tau_2 \) and \( \sigma_1, \sigma_2 \), indicate the isospin and spin dependencies of the potential, respectively. The second term in the OPEP is called a tensor potential consisting of a radial function and a tensor operator. Its radial part is

\[
V_T(r) = V_0(\tau_1 \tau_2) \left[ 1 + 3 \left( \frac{R}{r} \right) + 3 \left( \frac{R}{r} \right)^2 \right] \frac{e^{-r/R}}{r},
\]

and,

\[
V_0 = \frac{g^2 \hbar c}{3} \left( \frac{\hbar c}{2MR} \right)^2,
\]

where \( M \) is nucleon mass and \( g^2 \) is an empirical constant. We use the mean value as, \( g^2 = \frac{1}{3}(g_0^2 + 2g_\pi^2) \) (table 1) [30]. Experimental measurements for total spin-parity of deuteron give \( J^z = 1^+ \) [31]. The parity conservation and addition angular momenta rules indicate that the ground state of the deuteron wave function contained only two \( ^{13}S_1 \) (\( T = 0, S = 1, L = 0 \) and \( J = 1 \)) and \( ^{13}D_1 \) (\( T = 0, S = 1, L = 2 \) and \( J = 1 \)) states. With this consideration, the OPEP breaks the Schrödinger equation into the two coupled equations [31]

\[
\begin{align*}
\left\{ \frac{\hbar^2}{2m} \frac{d^2}{dr^2} + E_0 - V_C \right\} u(r) &= \sqrt{8} V_T(r) \omega(r), \\
\left\{ \frac{\hbar^2}{2m} \frac{d^2}{dr^2} - \frac{6}{r^2} \right\} E_0 + 2V_T(r) - V_C \omega(r) &= \sqrt{8} V_T(r) u(r).
\end{align*}
\]

(5)

where \( E_0 \) is energy ground state of deuteron, \( m \) is the reduced mass of proton-neutron system and \( u(r) \) and \( \omega(r) \) are the radial wave functions of \( ^{13}S_1 \) and \( ^{13}D_1 \) states, respectively. Because of the centrifugal potential, \( u(r) \) and \( \omega(r) \) have different asymptotic behavior at large distances. Furthermore, at short distances the same centrifugal barrier guarantees non-singular potentials \( V_C \) and \( V_T \) that \( u(r) \) is proportional to \( r \) and \( \omega(r) \) is proportional to \( r^3 \). Nevertheless, the OPEP presented here only for comparison. The main idea of the paper that starts from section 2.3 does not relate to solving this equation.

2.2. A unifying potential

Now we assume that \( u(r) \) and \( \omega(r) \) are proportional linearly [32]

\[
\omega(r) = \xi u(r).
\]

(6)

For a linear combination of \( ^{13}S_1 \) and \( ^{13}D_1 \) components, the ground state wave function of deuteron may be written as [33]

\[
\psi_0(r) = a_S \psi_S (r) + a_D \psi_D (r),
\]

(7)

with the normalization condition

\[
a_S^2 + a_D^2 = P_S + P_D = 1.
\]

(8)

The wave function \( \psi_0(r) = R_0(r) Y_{\ell m}(\theta, \varphi) \chi(S) T(t) \) is the product of radial \( R_0(r) = \frac{\ell \gamma(r)}{r} \), angular \( Y_{\ell m}(\theta, \varphi) \), spin \( \chi(S) \), and isospin \( T(t) \) terms, respectively. The orthogonality of non-radial parts requires that [31]

\[
\begin{align*}
|a_S|^2 &= P_S = \int_0^\infty u_S^2(r) dr, \\
|a_D|^2 &= P_D = \int_0^\infty u_D^2(r) dr = \xi^2 \int_0^\infty u_S^2(r) dr,
\end{align*}
\]

(9)

where in the last term we use assumption in equation (6). As a result, if expansion coefficients are real, i.e. \( a_S^* = a_S \) and \( a_D^* = a_D \), then we have

\[
\begin{align*}
u(r) &= a_S U_0(r) \\
\omega(r) &= a_D U_0(r),
\end{align*}
\]

(10)

where \( \xi = \frac{a_D}{a_S} \).
The obtained coefficients for the potentials and superpotential for $R_m = R$.

| $a_s$ | $a_D$ | $P_f\%$ | $P_i\%$ | $A$ (fm$^{-1}$) | $L$ | $C$ | $D$ (fm) | $N$ |
|-------|-------|---------|---------|-----------------|------|-----|---------|-----|
| $\pm 0.99049$ | $\pm 0.13755$ | 98.10 | 1.89 | 0.2316 | 0.1029 | 28.09 | $-37.16$ | 0.082 |
| $\alpha$ (fm$^{-1}$) | $\beta$ | $\gamma$ (fm) | $\gamma_-$ (fm$^{-1}$) | $\lambda_-$ | $\lambda_+$ | $\chi_-$ (fm) | $\chi_+$ (fm) |
| $-1.30$ | $-4.03$ | $-5.76$ | 32.66 | $-77.08$ | 7.65 | $-6.64$ | $-81.27$ | 156.27 |

By replacing equations (10) in (5), we have

$$
\begin{aligned}
&-a_s \frac{h^2}{2m} U_0''(r) + a_D \sqrt{8} V_T(r) U_0(r) + a_s \{ V_C(r) - E_0 \} U_0(r) = 0, \\
&-a_D \frac{h^2}{2m} U_0''(r) = V_U(r),
\end{aligned}
$$

where the unifying potential is

$$
V_U(r) = \frac{h^2}{2m} \left( \frac{2m}{\hbar^2} E_0 + \frac{6b}{r^2} + \frac{\alpha e^{-r/R}}{r} + \beta \frac{e^{-r/R}}{r} + \frac{\gamma}{r^3} \right),
$$

and constant coefficients are

$$
\begin{aligned}
&b = \frac{a_D}{a_s + a_D}, \\
&\alpha = \frac{2m}{\hbar^2} [\sigma_1 \sigma_2 + (\sqrt{8} - 2b)] V_0 \tau_1 \tau_2, \\
&\beta = \frac{6m}{\hbar^2} (\sqrt{8} - 2b) V_0 \tau_1 \tau_2 R, \\
&\gamma = \frac{6m}{\hbar^2} (\sqrt{8} - 2b) V_0 \tau_1 \tau_2 R^2.
\end{aligned}
$$

The corresponding values are listed in Table 2.

### 2.3. Superpotential and partner potentials

In SUSY QM, by definition the superpotential as logarithmic derivative of ground state wave function [34]

$$
W(r) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dr} \ln U_0(r),
$$

the Schrödinger equation as a quadratic differential equation is reduced to the first-order differential equation, as follows

$$
W^2(r) + \frac{\hbar}{\sqrt{2m}} W(r) = V_\pm(r),
$$

where minus and plus signs are related to $V_-(r)$ and $V_+(r)$, respectively. This equation is known as Riccati equation and $V_-(r)$ and $V_+(r)$ which are connected by the superpotential are known as supersymmetric partner potentials.

The most common way to solve this equation is to use ansatz for superpotential and to match the terms with potential via the Riccati equation [24, 35]. In the same way, we now introduce an OPEP-like superpotential as follows

$$
W(r) = \frac{\hbar}{\sqrt{2m}} \left( A - \frac{L + 1}{r} + C e^{-r/R} + D e^{-r/R} \right),
$$

As we will see in the next section, this superpotential is designed so that it can produce the main terms of potential, namely bound energy, centrifugal potential, and pion exchange terms. The ground state wave function obtained by replacing this superpotential in equation (15) is equal to

$$
U_0(r) = N \exp \{ -Ar + (L + 1) \ln(r) + \left( C - \frac{D}{R} \right) \Gamma(0, r/R) + D e^{-r/R} \},
$$

where the normalization constant $N$ is acquired by $\int_0^{\infty} |U_0(r)|^2 dr = 1$, and $A, L, C$ and $D$ are parameters to be determined by deuteron ground state properties in the next section. The superpotential $W(r)$ generates partner potentials via equation (16) as

$$
V_\pm(r) = \frac{h^2}{2m} \left( A^2 - \frac{\alpha_\pm}{r} + \frac{\beta_\pm}{r^2} + \frac{\gamma_\pm}{r^3} \right) \frac{e^{-r/R}}{r^2} + \lambda_\pm \frac{e^{-r/R}}{r^2} + \chi_\pm \frac{e^{-r/R}}{r^3} + \left( C^2 + \frac{2CD}{r} + \frac{D^2}{r^2} \right) e^{-2r/R},
$$
where its parameters are,
\[
\begin{align*}
\alpha_r &= 2A(L+1) \\
\beta_r &= L(L+1), \quad \beta_x = (L+1)(L+2) \\
\gamma_r &= \left(2A + \frac{1}{R}\right)C, \quad \gamma_x = \left(2A - \frac{1}{R}\right)C \\
\lambda_r &= 2AD - \frac{D}{R}C(2L+1), \quad \lambda_x = 2AD - \frac{D}{R} - C(2L+3) \\
\chi_r &= -2DL, \quad \chi_x = -2D(L+2)
\end{align*}
\]  

(20)

The obtained values are listed in table 2.

3. Determination of parameters

In order to determine the six constants \(a_S, a_D, L, C\), and \(D\), we need six equations provided by using the static properties of deuteron ground state, as follows:

3.1. \(A\)

According to the unbroken SUSY, the ground state energy of \(V_r(r)\) should be zero. Hence, the constant term in equation (19) is proportional to the ground state energy as

\[
A = \pm \sqrt{-\frac{2mE_0}{\hbar^2}} = \pm 0.2316,
\]

(21)

here we use experimental value for deuteron ground state energy \(E_0\) (table 1) [36]. The plus sign is a valid selection for \(A\) because the minus sign does not satisfy asymptotic condition \(\lim_{r \rightarrow \infty} U_0(r) \ll 1\) for a bound state.

3.2. \(a_S\) and \(a_D\)

The deuteron electric quadrupole moment is determined from the wave functions [31]

\[
Q = e \int_0^\infty \left\{ \frac{\sqrt{3}}{10} u(r)\omega(r) - \frac{1}{20} \omega^2(r) \right\} r^2 dr,
\]

(22)

by using equation (10), we have

\[
Q = 1.56a_D \sqrt{2} a_S - \frac{a_D}{2} e.
\]

(23)

We choose to use the empirical value for deuteron quadrupole moment (table 1). By solving simultaneous two linear equations (23) and (8) the two coefficients \(a_S\) and \(a_D\) are determined.

3.3. Effective angular momentum \(L\)

The third term in \(V_r(r)\) is the centrifugal potential that always appears when we deal with the spherical coordinate system. The expectation value of the square of the angular momentum \(\langle \hat{L}_z^2 \rangle\) with \(|\psi_0\rangle\) equation (7) is equal to \(6\hbar^2 P_D\). As a result, by the following equality

\[
\hbar^2 L(L+1) = 6\hbar^2 P_D,
\]

(24)

we find an effective angular momentum as

\[
L = \frac{1}{2}(-1 \pm \sqrt{1 + 24P_D}).
\]

(25)

The plus sign is acceptable for a real angular momentum.

3.4. \(C\) and \(D\)

3.4.1. Wave function maximum. If deuteron has at least one bound state, its wave function should have a maximum inside the potential well. By assuming the maximum probability takes place at \(r = R_M\), we have

\[
W(R_M) = 0,
\]

(26)

for special case \(R_M = R\) an analytical expression obtain

\[
R_M = \frac{1}{2A}(-L + 1 - 0.368C + \sqrt{(0.368C - L - 1)^2 - 1.47AD}).
\]

(27)

However, from equations (26) and (17) the general expression \(C\) in terms of \(D\) is as follows

\[
C = R_M \exp\left(\frac{R_M}{R} \left(-A + \frac{L + 1}{R_M} \right) \frac{D}{R_M}\right).
\]

(28)

3.4.2. Structure radius. The deuteron structure \(r_{sat}\) and charge radius \(r_p\), were recently determined to use several Lamb shift transitions in muonic deuterium which in by three times more precision than previous measurements (table 1) [37, 38]. On the other hand, deuteron structure radius as a characteristic deuteron size is defined from wave function, theoretically [31]

\[
\langle r^2 \rangle_{sat} = \frac{1}{4} \int_0^\infty [u^2(r) + \omega^2(r)] r^2 dr.
\]

(29)

By putting equations (10) in (29) and using equation (8), the following equation is obtained

\[
\int_0^\infty [rU_0(r)^2] dr = 15.6 (\text{fm}^2).
\]

(30)

Figure 1 illustrates \(D\) for different values of \(R_M\) obtained by numerical integrations resulting from the replacement of equations (28) into (30). We fit an exponential function on the result as

\[
D = -3.02 \exp\left(\frac{R_M}{0.54}\right) + 4.65.
\]

(31)

By changing the \(C\) and \(D\) with \(R_M\), the normalization constant is also changed. We have performed a similar process for it, and the results of are shown in figure 2, and the fit of its
Figure 1. $D$ versus $R_M$.

Figure 2. Normalization constant versus $R_M$. 

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exponential function is as follows:

\[ N = -0.006 \exp \left( \frac{R_{M}}{1.49} \right) + 0.01. \]  

(32)

4. Results and discussions

In unbroken SUSY, the two quantum systems described by supersymmetric partner potentials (superpartners) have the same energy spectra except for the ground state of \( V_+ (r) \) [24]. The ground state energy of \( V_+ (r) \) is equal to the first excited state of \( V_- (r) \) and as a result \( V_- (r) \) has one less energy level than \( V_+ (r) \). From figure 3, it is seen that \( V_- (r) \) does not have any attractive well and thus bound states. This implies deuteron that is described by \( V_- (r) \) is a weakly bound nucleus without any excited bound states [33]. Furthermore, the superpartners are not shape invariant, thus we cannot obtain an analytical solution for \( V_- (r) \) [35].

According to equation (15), normalizability of the ground state wave function \( U_{M}(r) \) implies that \( W(r) \) must have a positive value as \( r \to \infty \) and have a negative value as \( r \to 0 \) [24]. As a result, to ensure the existence of a zero-energy state and the presence of supersymmetry, the function \( W(r) \) must have an odd number of zeros on the real axis. Using table 2, since two values \( A = 0.2316 \) and \( L + 1 - C - D = 10.1729 \) are both positive, the superpotential equation (17) satisfies essentials in two asymptotic situations, i.e. \( \lim_{r \to \infty} W(r) \approx A > 0 \), and \( \lim_{r \to 0} W(r) \approx -(L + 1 - C - D)/r < 0 \). Figure 3 also demonstrates that the superpotential \( W(r) \) has one zero on the real axis.

The NN interaction is usually classified into three main regions [33]. At short separation distances (\( r \lesssim 1 \) fm) that is so-called hardcore, it is repulsive due to the Pauli exclusion principle of identical fermions and incompressibility of nuclear matter [39]. Similar to Van-der-Waals force in diatomic molecules, saturation property is due to particle exchange as well as strongly repulsive forces at short distances. It means that the nuclear force becomes repulsive when the nucleons try to get too close together. On other hand, at the intermediate-range (\( 1 \lesssim r \lesssim 2 \) fm), the NN potential well is attractive and causes creation bound states. Finally, the OPEP and centrifugal potential are dominated at the long-ranges \( r \gtrsim 2 \) (fm). In addition, in the asymptotic region \( r \to \infty \), the potential vanishes due to the finite range of the nuclear force between nucleons. Figure 3 shows that the potential \( V_+ (r) \) satisfies expected behavior in whole three mentioned regions. To make a comparison, the potentials \( V_+ (r) \) and superpotential \( W(r) \) are also depicted in figure 3. It can be seen that all potentials have the same asymptotic behavior \( \lim_{r \to \infty} V(r) \to 0 \) at the long ranges. Among them, only \( V_- (r) \) satisfies the intermediate and short ranges conditions.

The ground state wave function of deuteron \( U_{M}(r) \) (400X magnification) is plotted in figure 3. At the short ranges, the wave function is dropped rapidly due to the repulsive core. The peak of the wave function is located at the intermediate...
range, near the edge of the well. It is evidence of a weakly bound state within the potential well. Ultimately, the wave function decreases at the long ranges gradually.

The wave function peak radius $R_M$ is surprisingly proportional with the saddle point radius $R_S$ of the $V_-(r)$ (figure 4). We fit a line on the resulted data as

$$R_S = R_M - 0.09.$$  \(\text{(33)}\)

This means regardless of the value of $R_M$, the maximum probability of the presence of particles takes place near the well edge where the potential concavity sign is changed.

Let us now compare the new potential $V_-(r)$ (created by the superpotential) with $V_U(r)$ (created by the OPEP). It is clear that $V_-(r)$ contains two more terms than $V_U(r)$, as follows

$$-\frac{2A(L + 1)}{r} + \left(\frac{C^2}{r^2} + \frac{2CD}{r^3} + \frac{D^2}{r^4}\right)e^{-2r/R}.$$ \(\text{(34)}\)

The first one is due to Coulomb potential and the second term is related to two-pion exchange potential. It should be noted that these terms do not add to $V_-(r)$ by hand but are produced by the superpotential. Since $2A(L + 1) > 0$, the Coulomb term implies an electric dipole moment (EDM) for deuteron [40]. A permanent deuteron EDM can arise, because a CP-violating neutron–proton interaction can induce a small $^{13}P_1$ admixture in the deuteron wave function, which should be considered.

As seen from figure (3), although this fact is more prominent at the shorter distances, this term has a negligible contribution to the other components of the potential. Therefore, does not have significant impact on practical applications at low energies.

Numerous studies have shown that one boson exchange potentials (OBEP) are much easier to analyze than multi-meson ones. Therefore, in most models, the multi-pion processes have been considered as the exchange of one combined boson, rather than of multiple pions. To describe the attractive forces in the intermediate range, OBEP models need a roughly 600 MeV $0^+$ scalar boson. In fact, many OBEP models use both a 500 MeV and a 700 MeV scalar bosons. The existence of such scalar resonances has never been accepted. In this range, two-pion exchanges dominate. In such exchanges, two pions appear during the course of the interaction. The typical range is correspondingly smaller than for one-pion exchanges. Two-pion exchanges are much more difficult to crunch out than one-pion ones. However, the superpotential $W(r)$, simply produces a two-pion exchange term.

5. Conclusion

The most famous phenomenological models of the NN interaction (e.g. CD-Bonn, Reid93 and AV18) are based on the exchange of bosons and have many free parameters to be fitted with the experimental data [21–23]. In all models, the depth of the potential is well inversely proportional to the
potential width. Moreover, each channel has its specific potential. At the present unified picture, in contrast, $V_-$ is inseparable to $^{13}\Sigma_1$ and $^{13}\Delta_1$ parts. As a result, the potential $V_-$ is narrower and deeper than those obtained by the other models. In addition, the $^{13}\Delta_1$ state probability, $P_{\Delta}$ (2%-5%), is close to the lower limit. These probabilities should be recalculated due to the small $^{13}P_1$ admixture.

Actually, the model has been presented in this paper is a phenomenological model obtained by the superpotential approach. This new attitude has introduced a new superpotential and a corresponding effective potential ($V_-(r)$) for deuteron, as follows

$$W(r) = \frac{\hbar^2}{2m} \left( A - \frac{L + 1}{r} + C e^{-r/r_0} + D e^{-r/r_1} \right)$$

$$V_{\alpha}(r) = \frac{\hbar^2}{2m} \left( A' + \frac{2A(L + 1)}{r} + \frac{L(L + 1)}{r^2} + \frac{L + \lambda}{r} + \frac{\lambda}{r^2} \right) e^{-r/r_0}$$

$$+ \left( \frac{C'}{r} + D' \right) e^{-r/r_1}.$$  

These expressions satisfy different static properties of force between nucleons, such as repulsive core at the short-range, attractive at the intermediate range, finite range, spin and isospin dependencies, central and tensor parts, EDM, and one- and two-pion exchanges. Also, some static properties of deuteron are satisfied by the superpotential including unitary and normalization, wave functions, probabilities, binding energy, charge radius, quadrupole moment, and the existence of a weakly bound state as well as the lack of any excited states. Nevertheless, many questions about NN interaction are still unanswered, such as magnetic moment, aspect ratio, scattering length, effective range, phase shifts, locality and non-locality properties, energy and momentum dependencies of NN interaction, etc.

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ORCID iDs

Taha Koohrokhi  @  https://orcid.org/0000-0003-0517-4432

References

[1] Bertulani C A 2007 Nuclear Physics in a Nutshell 1st edn (Princeton, NJ: Princeton University Press)
[2] Myhrer F and Wroldsen J 1988 The nucleon–nucleon force and the quark degrees of freedom Rev. Mod. Phys. 60 629
[3] Ping J L, Huang H X, Pang H R, Wang F and Wong C W 2009 Quark models of dibaryon resonances in nucleon–nucleon scattering Phys. Rev. C 79 024001
[4] Huang F and Wang W L 2018 Nucleon–nucleon interaction in a chiral SU(3) quark model revisited Phys. Rev. D 98 047018
[5] Beane S R, Bedaque P F, Orginos K and Savage M J 2006 Nucleon–nucleon scattering from fully dynamical lattice QCD Phys. Rev. Lett. 97 012001
[6] Ratti C 2018 Lattice QCD and heavy ion collisions: a review of recent progress Rep. Prog. Phys. 81 084301
[7] Somá V 2018 From the liquid drop model to lattice QCD Eur. Phys. J. Plus 133 434
[8] Epelbaum E, Hammer H-W and Meißner U-G 2009 Modern theory of nuclear forces Rev. Mod. Phys. 81 1773
[9] Machleidt R and Entem D R 2011 Chiral effective field theory and nuclear forces Phys. Rep. 503 1
[10] Epelbaum E, Krebs H and Meißner U-G 2015 Precision nucleon–nucleon potential at fifth order in the chiral expansion Phys. Rev. Lett. 115 122301
[11] Wu S and Long B 2019 Perturbative NN scattering in chiral effective field theory Phys. Rev. C 99 024003
[12] Entem D and Machleidt R 2002 Accurate nucleon–nucleon potential based upon chiral perturbation theory Phys. Lett. B 524 93
[13] Entem D R, Kaiser N, Machleidt R and Nüsly Y 2015 Peripheral nucleon-nucleon scattering at fifth order of chiral perturbation theory Phys. Rev. C 91 014002
[14] Xiao Y, Geng L-S and Ren X-L 2019 Covariant nucleon-nucleon contact Lagrangian up to order $O(q^3)$ Phys. Rev. C 99 024004
[15] Schierholz G 1972 A relativistic one-boson-exchange model of nucleon-nucleon interaction Nucl. Phys. B 40 335
[16] Peláez J R 2016 From controversy to precision on the sigma meson: a review on the status of the non-ordinary f0(500) resonance Phys. Rep. 658 1
[17] Reuber A, Holinde K, Kim H-C and Speth J 1996 Correlated ππ and KK exchange in the baryon-baryon interaction Nucl. Phys. A 608 243
[18] Serra M, Otsuka T, Akashi Y, Ring P and Hirose S 2005 Relativistic mean field models and nucleon-nucleon interactions Prog. Theor. Phys. 113 1009
[19] Naghdí M 2014 Comparing some nucleon–nucleon potentials Phys. Part. Nucl. Lett. 11 410
[20] Naghdí M 2014 Nucleon–nucleon interaction: a typical/ concise review Phys. Part. Nucl. 45 924
[21] Machleidt R 2001 High-precision, charge-dependent Bonn nucleon–nucleon potential Phys. Rev. C 63 024001
[22] Stoks V G J, Klomp R A M, Terheggen C P F and de Swart J J 1994 Construction of high-quality NN potential models Phys. Rev. C 49 2950
[23] Wiringa R B, Stoks V G J and Schiavilla R 1995 Accurate nucleon–nucleon potential with charge-independence breaking Phys. Rev. C 51 38
[24] Cooper F, Khare A and Sukhatme U 1995 Supersymmetry and quantum mechanics Phys. Rep. 251 267
[25] Witten E 1981 Dynamical breaking of supersymmetry Nucl. Phys. B 188 513
[26] Cooper F and Freedman B 1983 Aspects of supersymmetric quantum mechanics Ann. Phys. 146 262
[27] Shifman M and Yung A 2018 Hadrons of $\mathcal{N} = 2$ supersymmetric QCD in four dimensions from little string theory Phys. Rev. D 98 085013
[28] Liang H Z 2016 Pseudospin symmetry in nuclear structure and its supersymmetric representation Phys. Scr. 91 083005
[29] Iwadare J, Otsuki S, Tamagaki R and Watari W 1956 Two-nucleon problem with pion theoretical potential. I': determination of coupling constant and deuteron problem *Prog. Theor. Phys. 16* 455

[30] Babenko V A 2017 Relation between the charged and neutral pion–nucleon coupling constants in the Yukawa model *Phys. Part. Nucl. Lett. 14* 58

[31] Garçon M and Orden J W V 2001 The deuteron: structure and form factors *Advances in Nuclear Physics Advances in the Physics of Particles and Nuclei* ed J W Negele and E W Vogt vol 26 2nd edn (Boston, MA: Springer)

[32] Nicholson A F 1962 Simple S and D deuteron ground state wavefunctions assuming central and r² tensor potentials *Aust. J. Phys. 15* 169

[33] Wong S S M 1998 *Introductory Nuclear Physics* 2nd edn (New York: Wiley)

[34] Gangopadhyaya A, Mallow J and Rasinariu C 2017 *Supersymmetric Quantum Mechanics: An Introduction* 2nd edn (Singapore: World Scientific)

[35] Koohrokhi T, Izadpanah A and Gerayloo M 2001 A unified scheme of shape invariant potentials with central symmetry in 3-dimensions arXiv:2001.02068

[36] Mohr P J, Taylor B N and Newell D B 2012 CODATA recommended values of the fundamental physical constants: 2010 *Rev. Mod. Phys. 84* 1527

[37] Pohl R 2016 Laser spectroscopy of muonic deuterium *Science 353* 669

[38] Hernandez O J, Ekström A, Dinur N N, Ji C, Bacca S and Barnea N 2018 The deuteron-radius puzzle is alive: a new analysis of nuclear structure uncertainties *Phys. Lett. B 778* 377

[39] Wang Y, Guo C, Li Q, Le Fèvre A, Leifels Y and Trautmann W 2018 Determination of the nuclear incompressibility from the rapidity-dependent elliptic flow in heavy-ion collisions at beam energies 0.4–1.0A GeV *Phys. Lett. B 778* 207

[40] Bartolini L, Bolognesi S and Gudnason S B 2020 Deuteron electric dipole moment from holographic QCD *Phys. Rev. D 101* 086009