Weak Henstock-Orlicz space and inclusion properties

Hemanta Kalita\(a\) and Bipan Hazarika\(b\)

\(a\)Department of Mathematics, Assam Don Bosco University, Sonapur 782402, Assam, India.
\(b\)Department of Mathematics, Gauhati University, Guwahati 781014, Assam, India
Email: \(a\)hemanta30kalita@gmail.com; \(b\)bh_rgu@yahoo.co.in; bh_gu@gauhati.ac.in

Abstract. In this paper we discuss the structure of Henstock-Orlicz space with locally Henstock integrable functions. The weak Henstock-Orlicz spaces on \(\mathbb{R}^n\) and some basic properties of the weak Henstock-Orlicz spaces are studied. We obtain some necessary and sufficient conditions for the inclusion properties of these spaces.

Keywords and phrases: Weak-Henstock-Orlicz space; Henstock-Kurzweil integrable function; Complete space; Inclusion properties.

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1. Introduction

In 1912 Arnaud Denjoy presented a powerful integral which was able to integrate all finite derivatives and recover their preimitive functions which was not possible in the case of Lebesgue integral. The Denjoy and Perron integrals are generalizations of the Lebesgue integrals that recover a continuous function from its derivative (see \[3\]). J. Kurzweil introduced a generalized version of the Riemann integral (see \[12\]). In 1960’s, Henstock made the first systematic study of this new integral. Four years later, while unaware of the work of Kurzweil, Henstock published a paper on integration theory in which he discussed the same integral as Kurzweil. Throughout a series of papers in the sixties Henstock developed a substantial amount of properties of this integral. The definition of this integral as defined by Kurzweil \[12\] and Henstock \[5\] is quite elegant as it is highly reminiscent of the Riemann integral and since a substantial amount of its properties can be developed using Riemann sums and basic epsilon-delta proofs. For the honours of these mathematicians, now a days this integral is called Henstock-Kurzweil integral, also see \[6\]. During late nineties a lot of integration theorist have been studied, the Henstock-Kurzweil integral extensively and consequently the theory of this integral had been highly refined. It should be pointed out that this integral does not have a standard name at that time. It is also referred to as the Henstock-Kurzweil integral (in short Henstock integral), the

\(1\)Corresponding author

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generalized Riemann integral, and the gauge integral. Since the integrals discussed so far
(Riemann, Lebesgue, Denjoy, Perron) are named for a single person and since Henstock
launched the study of this integral, we are content to call it the Henstock integral (one can
see [2, 18] for related works of Henstock-Kurzweil, McShane and Pettis integrals). The
Henstock–Kurzweil integration on Euclidean spaces initiated by Yeong [24]. The Orlicz
space is the generalization of the $L^p$ space, which was initiated by Z.W. Birnbaum and
W. Orlicz. The fundamental properties of Orlicz space with Lebesgue measure found in
[11]. The theory of Orlicz space which is a more generalized version of $L^p$-space with
the help of Young functions and the underlying measure was discussed in [19] (also see
[20]). Nakai used Orlicz spaces in the application in Harmonic analysis in various ways
(see [13, 14]). Several inclusion properties of Orlicz and weak Orlicz spaces are found in
[17]. Liu Pei De et al. [15] discussed about the application of the weak Orlicz spaces in
the Harmonic analysis. By over coming difficulties as $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ but
not generally dense in $\mathcal{L}^\theta(\mathbb{R}^n)$ Thung in [23], presented a translation invariant subspace
$L^1(\mathbb{R}^n) \cap \mathcal{L}^\theta(\mathbb{R}^n)$ to be dense in Orlicz space $\mathcal{L}^\theta(\mathbb{R}^n)$. The concept of the Henstock-Orlicz
space (in brief $\mathbb{H}-\text{Orlicz space}$) was presented by Hazarika and Kalita in [7] which have
some difference thing of the Orlicz spaces such as in $\mathbb{H}-\text{Orlicz space}$ $C_0^\infty$ is dense but
not generally dense in Orlicz spaces. In [8], Kalita and Hazarika was investigated the
countable additivity of Henstock-dunford integrable functions on $\mathbb{H}-\text{Orlicz space}$. The
theory of $\mathbb{H}-\text{Orlicz spaces}$ with vector measure discussed in the conference paper [9].

OBJECTIVES OF OUR PAPER

In this paper we discuss about $\mathbb{H}-\text{Orlicz space}$ with a little different settings. In our
work we mainly focus on the weak $\mathbb{H}-\text{Orlicz space}$. Nakai in [13, 14] and Liu Ning et al.
[16] defined a class of weak Orlicz function spaces and their basic properties are discussed.
The major drawback of the weak Orlicz space is that it is not naturally complete. We
motivate to resolve this drawback of the weak Orlicz space. We introduce $\mathbb{H}-\text{Orlicz space}$
with locally Henstock integrable functions to overcome the difficulties of the weak Orlicz
space.

2. PRELIMINARIES AND AUXILIARY RESULTS

In the whole article, we consider $(\mathbb{R}^n, \Sigma_\infty, \nu_\infty)$ is an abstract measure space, where $\Sigma_\infty$
is an $\sigma$-algebra of its subsets on which a $\sigma$-additive function $\nu_\infty : \Sigma_\infty \to \mathbb{R}^+$ is given and $\nu_\infty$
is the Lebesgue measure. It is known that a measure space has the finite subset property
if for every $A \in \Sigma_\infty$ with $\nu_\infty(A) = \infty$ there exists a family of subsets $\{A_i\}_{i=1}^\infty \subset \Sigma_\infty$ with
$A_i \subset A$; $0 < \nu_\infty(A_i) < \infty$ and $\nu_\infty\left(\bigcup_{i=1}^\infty A_i\right) = \infty$. 
This gives us
\[ \nu_\infty(A) = \begin{cases} 
0, & \text{if } A = \emptyset, \\
+\infty, & \text{if } A \neq \emptyset 
\end{cases} \]

Otherwise it does not restrict the generality of our assumption. The space of all Henstock integrable functions defined on \( R^n \), is denoted by \( HK(\nu_\infty) \). \( HK(\nu_\infty) \) is a vector space under the usual operations of pointwise addition and scalar multiplication on \( R \) was studied in \( 3, 21, 22 \). In the one-dimensional case, Alexiewicz \( 1 \) has shown that the class of Henstock integrable functions, with respect to the norm
\[ \|h\|_{HK} = \sup_{t} \left| \int_{-\infty}^{t} h(s)d(s) \right|. \]
is a normed space, and it is known that \( HK(R) \) is not complete (see \( 1 \)).

### 2.1. Henstock-integral on \( R^n \)

The elements of \( R^n \) will be denoted by \( z = (z_1, z_2, \ldots, z_n) \). An interval in \( R^n \) is a set of the form \( J = [z, w] := \prod_{i=1}^{n} [z_i, w_i] \), where \( -\infty < z_i < w_i < \infty \) for \( i = 1, 2, \ldots, n \). The set \( \prod_{i=1}^{n} [z_i, w_i] \subset R^n \) is known as a degenerate interval if \( z_i = w_i \) for some \( i \in \{1, 2, \ldots, n\} \). Two intervals \( J = [z, w], I = [u, v] \) in \( R^n \) are said to be non-overlapping if \( \prod_{i=1}^{n} (z_i, w_i) \cap \prod_{i=1}^{n} (u_i, v_i) \) is empty. The union of two intervals in \( R^n \) is an interval in \( R^n \) (see Lemma 2.1.2 \( [24] \)). We know that the space \( R^n \) equipped with the maximum norm \( ||.|| \), where \( ||z|| = \max_{1\leq i \leq n} |z_i| \). With this norm, we denote the closed ball of \( R^n \) by \( B[x, r] = \{x \in R^n : ||y - x|| \leq r\} \), whose center is \( x \) with sides parallel to the co-ordinates axes of length \( 2r \). It is a closed interval for side \( i \) about \( x_i \) is in \( [z_i, w_i] \). So, let \( B[x, r] = [J, x] \), where \( J = \prod_{i=1}^{n} [z_i, w_i], J \) is closed interval in \( R^n \).

**Definition 2.1.** \( [7, 24] \) Let \( A \) is a compact ball in \( R^n \), a partition \( P \) of \( A \) is a collection \( \{\{J_i, x_i\} : x_i \in J_i, 1 \leq i \leq m\} \), where \( J_1, J_2, \ldots, J_m \) are non-overlapping closed intervals i.e., \( \nu_\infty[J_i \cap J_j] = 0, i \neq j \) and \( \bigcup_{i=1}^{m} J_i = A \).

If \( \delta \) is a positive function on \( A \) we say \( P \) is Henstock \( \delta \)-partition of \( A \) if for each \( i, J_i \subset B'(x_i, \delta(x_i)) \). The function \( \delta \) is a gauge on \( A \).

**Definition 2.2.** \( [7, 24] \) A function \( h : A \rightarrow R \) is said to be Henstock integrable on \( A \), if there exists a number \( L \) such that for any \( \varepsilon > 0 \) there exist a gauge \( \delta \) and Henstock \( \delta \)-partition on \( A \) such that
\[ \left| \sum_{i=1}^{m} h(x_i)\nu_\infty(J_i) - L \right| < \varepsilon. \]

Now we introduce the concept of locally Henstock-Kurzweil integrable function as follows:

**Definition 2.3.** A measurable function \( h : \Gamma \subset R^n \rightarrow R \) is called locally Henstock-Kurzweil integrable if \( h\chi_K \in HK(K) \) for all \( K \subset \Gamma \) compact where \( \chi_K \) is the characteristics
functions of $\mathcal{K}$. We denote the set of locally Henstock-Kurzweil integrable functions as $HK_{loc}$. Recalling that the function $h$ is Henstock integrable on a measurable set $\Gamma \subset \mathbb{R}^n$ if $h \chi_{\Gamma}$ is Henstock integrable on $\Gamma$. That is $h \in HK_{loc}(\mathbb{R}^n)$ means $h \in HK(\mathcal{K})$, where $\mathcal{K} \subseteq \Gamma$ compact. Also with easy analogous, $L^1_{loc}(\mathbb{R}^n) \subset HK_{loc}(\mathbb{R}^n)$.

**Definition 2.4.** [7, 19] A function $\theta : \mathbb{R} \rightarrow \mathbb{R}^+$ is said to be Young function, so that $\theta(x) = \theta(-x), \theta(0) = 0, \theta(x) \rightarrow \infty$ as $x \rightarrow \infty$, but $\theta(x_0) = +\infty$ for some $x_0 \in \mathbb{R}$ is permitted. We assume $\mathcal{F}$ be the class of Young’s function $\theta : \mathbb{R} \rightarrow \mathbb{R}^+$ is an increasing, bijective, continuous and concave satisfying $\theta(0) = 0; \lim_{t \rightarrow 0} \theta(t) = 0$ and $\lim_{t \rightarrow \infty} \theta(t) = \infty$. We denote $\theta_1 \Delta \theta_2$ for $\theta_1, \theta_2 \in \mathcal{F}$ if there is a constant $C > 0$ such that $\theta_1(t) \leq \theta_2(Ct)$ for all $t \geq 0$.

**Definition 2.5.** (1) A Young function $\theta$ is said to satisfy $\Delta'$ if $\lim_{k \rightarrow 0} \sup_{t > 0} \theta(kt) / \theta(t) = 0$.

(2) An $N$-function $\theta$ is said to satisfy $\Delta_2$-condition if there is a $k > 0$ such that $\theta(2x) \leq k \theta(x)$ for large values of $x$.

If $\theta$ is a convex function on $[0, \infty)$, then $\theta \in \Delta'$. In this article, we do not generally assume that $\theta$ is convex, except we mention it especially.

**Definition 2.6.** [13, Definition 2.1] Let $\theta$ be a convex function. The Orlicz space is defined as

$$L^\theta(\mathbb{R}^n) = \{ h \in L^1_{loc}(\mathbb{R}^n) : ||h||_{L^\theta(\mathbb{R}^n)} < +\infty \},$$

where

$$||h||_{L^\theta(\mathbb{R}^n)} = \inf \left\{ \alpha > 0 : (L) \int_{\mathbb{R}^n} \theta \left( \frac{|h(x)|}{\alpha} \right) d\nu_\infty(x) \leq 1 \right\} \text{ for some } \alpha > 0.$$

Also one can see [11, 14, 19, 20] for detailed on Orlicz space and it’s applications. We recalling few preliminaries of $H-$Orlicz spaces from [7].

**Definition 2.7.** [7] Let $(\mathbb{R}^n, \Sigma_\infty, \nu_\infty)$ be an arbitrary measure space. Then the space $\mathcal{H}^\theta(\nu_\infty)$ of all measurable functions $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is called $H-$Orlicz space, which defined as:

$$\mathcal{H}^\theta(\nu_\infty) = \left\{ h : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable : } \int_{\mathbb{R}^n} \theta(\alpha h) d\nu_\infty \in HK(\nu_\infty) \text{ for some } \alpha > 0 \right\}.$$

**Definition 2.8.** [7] The Luxemburg norm on $\mathcal{H}^\theta(\nu_\infty)$ as follows:

$$\mathbb{H}_\theta(h) = \inf \left\{ \alpha > 0 : HK \int_{\mathbb{R}^n} \theta \left( \frac{h}{\alpha} \right) d\nu_\infty \leq 1 \right\}.$$
It is understood that \( \inf(\theta) = +\infty \), and
\[
V = \left\{ h \text{ measurable} : HK \int_{B \subset \mathbb{R}^n} \theta(h) d\nu_\infty \leq 1 \right\},
\]
is the gauge of the set.

**Remark 2.1.** The \( \mathbb{H} \)-Orlicz spaces are Banach spaces with the Luxemburg norm defined as the definition \([2,8]\) as well as Symmetric. This spaces is separable if \( \theta \in \Delta_2 \) (see \([8]\)). The separability of \( \mathbb{H} \)-Orlicz spaces with out \( \Delta_2 \) conditions of the Young’s functions will be an aspect to work in our coming days.

3. \( \mathbb{H} \)-ORLICZ SPACES WITH LOCALLY HENSTOCK INTEGRABLE FUNCTIONS

Here we discuss about \( \mathbb{H} \)-Orlicz spaces with a little different settings. Before start the concept of \( \mathbb{H} \)-Orlicz spaces we define the \( \mathbb{H} \)-Orlicz class state as:

**Definition 3.1.** Let \( \mathcal{H}^{-\theta}(\nu_\infty) \) be the set of all \( h : K \subset \mathbb{R}^n \to \mathbb{R} \) bounded measurable with compact support for \( \Sigma_\infty \subset \mathbb{R}^n \) such that \( \int_K \theta(|f|) d\nu_\infty \) is Henstock integrable.
i.e. \( \mathcal{H}^{-\theta}(\nu_\infty) = \{ h \text{ is bounded measurable with compact support} : \int_K \theta(|h|) d\nu_\infty \in HK(\nu_\infty) \} \).

**Theorem 3.1.** The space \( \mathcal{H}^{-\theta}(\nu_\infty) \) is a linear space if and only if \( \theta \) satisfies \( \Delta_2 \)-condition.

**Proof.** The proof is similar as \([7, \text{Theorem } 2.2]\). \( \square \)

**Proposition 3.2.** For each \( h \in \mathcal{H}^{0}(\nu_\infty) \), there is an \( \alpha > 0 \) such that
\[
\mathbb{B}_\theta = \left\{ \alpha h = m \in \mathcal{H}^{-\theta}(\nu_\infty) : HK \int_{K} \theta(m) d\nu_\infty \leq 1 \right\}
\]
is a circled solid subset of \( \mathcal{H}^{-\theta}(\nu_\infty) \).

**Proof.** Let \( h, m \in \mathcal{H}^{0}(\nu_\infty) \). Then there exist \( \alpha_0, \beta_0 > 0 \) such that \( \alpha_0 h, \beta_0 m \in \mathcal{H}^{-\theta}(\nu_\infty) \). Let \( \mu_0 = \min(\alpha_0, \beta_0) \). Then for \( \mu_0 > 0 \) and using the known fact of convexity and monotonicity of \( \theta \), we get
\[
HK \int_{K} \theta \left( \frac{\mu_0}{2} (h + m) \right) d\nu_\infty \leq \frac{1}{2} \left[ HK \int_{K} \theta(\alpha_0 h) d\nu_\infty + HK \int_{K} \theta(\beta_0 m) d\nu_\infty \right].
\]
Clearly the right side is Henstock integrable. Since \( \frac{\mu_0}{2} > 0 \), this gives us \( h + m \in \mathcal{H}^{0}(\nu_\infty) \). Particularly, with each \( h \) in \( \mathcal{H}^{0}(\nu_\infty) \), \( 2h \in \mathcal{H}^{0}(\nu_\infty) \) and then \( nh \in \mathcal{H}^{0}(\nu_\infty) \) for all integers \( n > 1 \), so that \( \gamma_0 h \in \mathcal{H}^{0}(\nu_\infty) \) for any scalar \( \gamma_0 \). Therefore the given set is solid and circled. To hold \( \gamma_0 h \in \mathcal{H}^{0}(\nu_\infty) \) for some \( \gamma_0 > 0 \). Let \( a_n \to 0 \) be arbitrary and set \( \gamma_{a_n} = \min(\gamma_0, a_n) \).
Then \( \gamma_{a_n} \to 0 \) and \( \theta(\gamma_{a_n} h) \leq \theta(\gamma_0 h) \) and \( \theta(\gamma_{a_n} h) \to 0 \) as \( \theta \) is a continuous Young function. Now Dominated Convergence Theorem, give us \( HK \int_{K} \theta(\gamma_{a_n} h) d\nu_\infty \to 0 \) so that for some \( n_0 \), we have \( HK \int_{K} \theta(\gamma_{n_0} h) d\nu_\infty \leq 1 \). Thus \( \gamma_{n_0} h \in \mathbb{B}_\theta \). \( \square \)
To construct $\mathcal{H}^\theta(\mathbb{R}^n)$ in our setting recall the known facts that $HK(\mathbb{R}^n) \subset HK_{loc}(\mathbb{R}^n)$. For a Young function $\theta$, we can state our $H-Orlicz$ space as below:

$$\mathcal{H}^\theta(\mathbb{R}^n) = \{ h \in HK_{loc} : \| f \|_{\mathbb{H}_\theta} < +\infty \},$$

where $\| h \|_{\mathbb{H}_\theta(\mathbb{R}^n)}$ is defined as follows:

$$\| h \|_{\mathbb{H}_\theta} = \inf \left\{ \alpha > 0 : HK \int_\mathcal{K} \theta \left( \frac{h(x)}{\alpha} \right) d\nu_\infty \leq 1 \right\},$$

where $\mathcal{K}$ is a compact subset of $\mathbb{R}^n$. It is clear that this space is a normed space with respect to the norm $\mathbb{H}_\theta$.

**Theorem 3.3.** For each $h \in \mathcal{H}^\theta(\mathbb{R}^n)$, $m \in \mathcal{H}^\Phi(\mathbb{R}^n)$, if the complementary function of $\theta$ is $\phi$, then

$$HK \int_{\mathbb{R}^n} |hm|d\nu_\infty \leq \| h \|_{\mathbb{H}_\theta} \| m \|_{\mathbb{H}_\phi}.$$

**Theorem 3.4.** The classical Orlicz space is a dense subspace of $H-Orlicz$ space as continuous dense embeddings. That is, $L^\theta(\mathbb{R}^n) \hookrightarrow \mathcal{H}^\theta(\mathbb{R}^n)$ is continuous dense embeddings.

**Proof.** Let $h \in L^\theta(\mathbb{R}^n)$. Then $h \in L_{loc}$ with $\| f \|_{L^\theta(\mathbb{R}^n)} < \infty$. Then for some $\alpha > 0$, and a compact $\mathcal{K} \subset \mathbb{R}^n$ we have

$$\inf \left\{ HK \int_\mathcal{K} \theta \left( \frac{h(x)}{\alpha} \right) d\nu_\infty \right\} \leq \inf \left\{ (L) \int_{\mathbb{R}^n} \theta \left( \frac{|h(x)|}{\alpha} \right) d\nu_\infty \right\} \leq 1.$$

So, for some $\alpha > 0$, $\inf \left\{ \alpha > 0 : (L) \int_{\mathbb{R}^n} \theta \left( \frac{|h(x)|}{\alpha} \right) d\nu_\infty \leq 1 \right\}$, we get the following

$$\inf \left\{ \alpha > 0 : HK \int_\mathcal{K} \theta \left( \frac{h(x)}{\alpha} \right) d\nu_\infty \leq 1 \right\}.$$

Hence $h \in \mathcal{H}^\theta(\mathbb{R}^n)$ with $\| h \|_{\mathcal{H}^\theta(\mathbb{R}^n)} \leq \| h \|_{L^\theta(\mathbb{R}^n)}$. Hence the proof. \quad \square

**Theorem 3.5.** Suppose $\nu_\infty(\mathcal{K}) < \infty$ and $\nu_\infty$ is bounded, then $\mathcal{H}^\theta(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is continuous.

**Proof.** If $r > 0$ and $s > 0$ such that for all $p \geq 0$, we have $\theta(p) \geq rp - s$. This means $rp \leq \theta(p) + s$. Let $h \in \mathcal{H}^\theta(\mathbb{R}^n)$. Then for $\alpha > 0$ as possible small, we have

$$\alpha \int_\mathcal{K} |h|d\nu_\infty \leq \frac{1}{r} \int_\mathcal{K} [\theta(\alpha h) + s]d\nu_\infty
= \frac{1}{r} \int_\mathcal{K} \theta(\alpha h)d\nu_\infty + s \frac{\nu_\infty(\mathcal{K})}{r} < \infty$$

Therefore $h \in L^1(\mathbb{R}^n)$. Hence $\mathcal{H}^\theta(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. The inclusion is continuous follows from the similar technique that we used in the second part of the [7, Lemma 3.4]. \quad \square
Corollary 3.6. $\mathcal{H}^\theta(\mathbb{R}^n) \subset HK_{loc}(\mathbb{R}^n)$.

Remark 3.1. If $h \in \mathcal{L}^\theta(\mathbb{R}^n)$, then all theorems that are true in $\mathcal{L}^\theta(\mathbb{R}^n)$ are so in $\mathcal{H}^\theta(\mathbb{R}^n)$.

4. WEAK-HENSTOCK-ORLICZ SPACES

In this division of the paper, we discuss about the weak Henstock-Orlicz spaces (In brief weak-\(\mathbb{H}\)-Orlicz space) and its basic properties. The classical weak Orlicz space is not naturally Banach spaces. We discuss the weak Henstock-Orlicz space is naturally a Banach space. We consider all functions $h$ are $\nu_\infty$ measurable now onward. $\chi_\mathcal{B}$ be the characteristic functions of a measurable $\mathcal{B} \subset \mathbb{R}^n$. For the measurable set $\mathcal{B}$, a measurable function $h$ and $t > 0$, let

$$\mu_\infty(\mathcal{B}, h, t) = \{ x \in \mathcal{B} : \nu_\infty(h(x)) > t \}.$$ 

If $\mathcal{B} = \mathbb{R}^n$, we denote as $\nu_\infty(h, t)$.

Definition 4.1. Let $h$ be $\mu_\infty$ measurable function. Then weak-\(\mathbb{H}\)-Orlicz space is defined as:

$$\mathcal{H}^\theta_w(\mathbb{R}^n) = \{ h \in HK_{loc}(\mathbb{R}^n) : ||h||_{H^\theta_w} < +\infty \},$$

where $||h||_{H^\theta_w} = \inf\left\{ \exists \alpha > 0 : \sup_{t > 0} \theta(t)\nu_\infty(\frac{h}{\alpha}, t) \leq 1 \right\}$.

We consider another $\nu_\infty$-measurable set as

$$B^\theta_w(\mathbb{R}^n) = \left\{ h \in HK_{loc}(\mathbb{R}^n) : \forall \alpha > 0 : \sup_{t > 0} \theta(t)\nu_\infty(\frac{h}{\alpha}, t) < \infty \right\}.$$

It is observe that $B^\theta_w(\mathbb{R}^n) \subset \mathcal{H}^\theta_w(\mathbb{R}^n)$ and $B^\theta_w(\mathbb{R}^n)$ is a linear space. If $h = m$ are $\nu_\infty$-a.e. in $\mathcal{H}^\theta_w(\mathbb{R}^n)$, then we call $h = m$ in $\mathcal{H}^\theta_w(\mathbb{R}^n)$. We define the modular of function $h$ as

$$\Upsilon_\theta(h) = \sup_{t > 0} \theta(t)\nu_\infty(\frac{h}{\alpha}, t).$$

Lemma 4.1. (1) If $h \leq m$ are $\nu_\infty$-a.e. and $m \in \mathcal{H}^\theta_w$, then $h \in \mathcal{H}^\theta_w$.

(2) $\sup_{t > 0} \theta\left( \frac{t}{||h||_{H^\theta_w}} \right)\nu_\infty(f, t) \leq 1$.

(3) If $h \in \Delta_2$ and $||h||_{H^\theta_w} \leq 1$ then $\Upsilon_\theta(h) \leq ||h||_{H^\theta_w}$.

Proof. (1) It is straight forward as $h \leq m$ are $\nu_\infty$-a.e. means $|h| \leq |m|$ are $\nu_\infty$-a.e.

(2) From the definition of $||h||_{H^\theta_w}$, there exists $C_k \ \sup_{t > 0} ||h||_{H^\theta_w}$ such that $\theta\left( \frac{h}{C_k} \right)\nu_\infty(h, t) \leq 1, \ \forall \ t > 0$. When $k \to \infty$, $\sup_{t > 0} \theta\left( \frac{t}{||h||_{H^\theta_w}} \right)\nu_\infty(h, t) \leq 1$. \hfill \Box

Theorem 4.2. If $\theta \in \Delta_2$, $(\mathcal{H}^\theta_w, ||\cdot||_{H^\theta_w})$ is a quasi-Banach space.

Proof. The proof of the result is similar to [15, Lemma 1.1 (4)]. \hfill \Box
Theorem 4.3. For $h, m \in H^w_\theta$, the following inequality holds

$$||h + m||_{H^w_\theta} \leq ||h||_{H^w_\theta} + ||m||_{H^w_\theta}.$$ 

Lemma 4.4. If $\theta \in \Delta_2$ then the followings are true

1. $H^w_\theta(\mathbb{R}^n) = B^\theta_w(\mathbb{R}^n)$.
2. If $h_n \in H^w_\theta(\mathbb{R}^n)$ then $||h_n - h||_{H^w_\theta(\mathbb{R}^n)} \to 0$ if $\Upsilon_\theta(h_n - h) \to 0$.
3. If $h_n \in H^w_\theta(\mathbb{R}^n)$ then $\sup\{||h_n||_{H^w_\theta(\mathbb{R}^n)}\} \leq A$ if and only if $\sup\{\Upsilon_\theta(h_n)\} \leq A$, where $A > 0$.

Proof. (1) Let $\theta \in \Delta_2$ and $h \in H^w_\theta(\mathbb{R}^n)$. Then there exists $\alpha > 0$ such that

$$\sup_{t>0} \left( \frac{t}{\alpha} \right) \nu_\infty(h, t) \leq 1.$$ 

Hence $\theta(\frac{t}{\alpha}) \leq \theta(\frac{t}{\alpha})$ when $a \geq \alpha$. As $\theta \in \Delta_2$, so $\theta(\frac{t}{\alpha}) \leq C\theta(\frac{t}{\alpha})$ when $a < \alpha$. So,

$$\sup_{t>0} \left( \frac{t}{\alpha} \right) \nu_\infty(h, t) \leq 1 \text{ for all } a > 0.$$ 

Hence $h \in B^\theta_w(\mathbb{R}^n)$.

The proof of part (2) follows from the similar technique as [15, Corollary 2.1] and the proof of part (3) follows from [15, Theorem 2.1 (4)].

The weak Orlicz space is not naturally complete. The weak Orlicz space is complete if the Young function $\theta \in \Delta_2$. We observe the weak $H-$Orlicz is complete if the Young function without $\theta \in \Delta_2$. We discuss now that the completeness of weak $H-$Orlicz space.

Lemma 4.5. Let $(h_n) \in H^w_\theta(\mathbb{R}^n)$. Then the followings are true

1. If $||h_n - h||_{H^w_\theta} \to 0$, then $h_n \to h$ (convergence in measure).
2. If $0 \leq \inf(h_n) \to h$ is $\nu_\infty-a.e.$ then $||h_n||_{H^w_\theta} \to ||h||_{H^w_\theta}.$

Proof. (1) The definition of norm of the weak $H-$Orlicz space, we have

$$||h_n - h||_{H^w_\theta(\mathbb{R}^n)} = \inf\left\{ \alpha > 0 : \sup_{t>0} \theta \left( \frac{h_n - h}{\alpha} \right) \nu_\infty(h_n - h, t) \right\} \to 0$$

gives, $\theta \left( \frac{h_n - h}{\alpha} \right) \nu_\infty(h_n - h, t) \to 0$. This implies $\nu_\infty(h_n - h) \to 0$. Therefore $h_n \to h$ in the measure $\nu_\infty$.

(2) Let $\inf \nu_\infty(h_n) = \nu_\infty(h) \nu_\infty-a.e.$ Then clearly $h_n \to h$ in the measure $\nu_\infty$. This gives for all $\varepsilon > 0$ there exists a number $n_0 \in \mathbb{N}$ such that

$$\nu_\infty(h_n - h) \to 0 \text{ for } n > n_0.$$
Using the property of Young function $\theta$, we can write
\[
\theta \left( \frac{h_n-h}{\alpha} \right) \nu_\infty(h_n - h, t) \to 0
\]
\[
\Rightarrow \sup \theta \left( \frac{h_n-h}{\alpha} \right) \nu_\infty(h_n - h, t) \to 0.
\]
That is, $\inf \{ \alpha > 0 : \sup \theta \left( \frac{h_n-h}{\alpha} \right) \nu_\infty(h_n - h, t) \} < \epsilon$. As $\epsilon$ is an arbitrary so

\[
||h_n - h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} \to 0.
\]

\[\Box\]

**Theorem 4.6.** $(\mathcal{H}_w^\theta(\mathbb{R}^n), \|\cdot\|_{\mathcal{H}_w^\theta(\mathbb{R}^n)})$ is a Banach space.

**Proof.** Let $(h_n) \in \mathcal{H}_w^\theta(\mathbb{R}^n)$ be a Cauchy sequence. Then there exists a natural number $n_0$ such that $\lim_{n,e \to \infty} ||h_n - h_e||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} = 0$ for all $n, e \geq n_0$. We know $\lim_{n,e \to n_0} \nu_\infty(h_n - h_e, t) = 0$ for all $t > 0$, implies $h_n \to h$ as $\nu_\infty$ measurable. By Riesz’s Theorem, there is a subsequence $h_{n_k} \to h$, $\nu_\infty$-a.e and $\nu_\infty(h_{n_k} - h_{n_s}) \to \nu_\infty(h_{n_k} - h)$ as $\nu_\infty$-a.e. Let $n_k \geq n_0$ and $h_{n_s} \to \infty$, then by [4] Eq 1.1.15 of Rem 1.1.8 for $\epsilon > 0$, 
\[
\theta(\frac{t}{\epsilon}) \nu_\infty(h_{n_k} - h, t) \leq \lim_{n_s \to \infty} \theta(\frac{t}{\epsilon}) \nu_\infty(h_{n_k} - h, t)
\]
\[
\leq 1.
\]
Using (2) of the Lemma [4.5] $||h_{n_k} - h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} < \epsilon$ and $\lim_{n_k \to \infty} ||h_{n_k} - h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} = 0$. Now from Theorem 4.3 we can find the following
\[
||h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} \leq ||h_{n_k} - h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} + ||h_{n_k}||_{\mathcal{H}_w^\theta(\mathbb{R}^n)}.
\]
This implies $||h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} \leq ||h_{n_k}||_{\mathcal{H}_w^\theta(\mathbb{R}^n)}$. Using the first part of the Lemma 4.1 we can conclude that $h \in \mathcal{H}_w^\theta(\mathbb{R}^n)$. So, $h_n \to h$ in $\mathcal{H}_w^\theta(\mathbb{R}^n)$. \[\Box\]

**Theorem 4.7.** $L_w^\theta(\mathbb{R}^n) \hookrightarrow \mathcal{H}_w^\theta(\mathbb{R}^n)$ is a continuous dense embeddings.

**Proof.** Let $h \in L_w^\theta(\mathbb{R}^n)$. Then $h \in L^1_{loc}(\mathbb{R}^n)$ with $||h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} < \infty$. This is very obvious that $h \in HK_{loc}$. Now we find $||h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} < \infty$. From the fact that for some $\alpha > 0$,
\[
||h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} = \inf \left\{ \alpha > 0 : \sup_{t>0} \theta(t) \nu_\infty \left( \frac{h}{\alpha}, t \right) \leq 1 \right\}.
\]
We get $\sup_{t>0} \theta(t) \nu_\infty \left( \frac{h}{\alpha}, t \right) \leq 1$ also true when $h \in L^1_{loc}(\mathbb{R}^n)$. That is, $||h||_{\mathcal{H}_w^\theta(\mathbb{R}^n)} \leq ||h||_{L_w^\theta(\mathbb{R}^n)}$. Hence $L_w^\theta(\mathbb{R}^n) \hookrightarrow \mathcal{H}_w^\theta(\mathbb{R}^n)$ is a continuous dense embeddings. \[\Box\]

**Remark 4.1.** The $\Delta_2$ condition of the Young function $\theta$ is not necessary to proof the Bounded Convergence theorem, Control convergence theorem, Fatou-type convergence theorem, Levi-type convergence theorem, Vitali-type convergence theorem that are proved in [16] Section 3], can also be proved in the weak $\mathbb{H}$–Orlicz space.
5. **Inclusion Property of Weak $H^\theta$–Orlicz Spaces**

In this section we discuss inclusion properties of weak $H^\theta$–Orlicz spaces. Before that we find inclusion relations between $H^\theta$–Orlicz space and weak $H^\theta$–Orlicz space in the following theorem as follows:

**Theorem 5.1.** Let $\theta$ be a Young function. Then $\mathcal{H}^\theta(\mathbb{R}^n) \subset \mathcal{H}_{w}^\theta(\mathbb{R}^n)$ for every $h \in \mathcal{H}^\theta(\mathbb{R}^n)$ with $\|h\|_{H_{w}^\theta(\mathbb{R}^n)} \leq \|h\|_{H^\theta(\mathbb{R}^n)}$.

**Proof.** Let $h \in \mathcal{H}^\theta(\mathbb{R}^n)$. We need to prove $h \in \mathcal{H}_{w}^\theta(\mathbb{R}^n)$. Let

\[
\mathfrak{A}_{\theta,w} = \left\{ \alpha > 0 : \sup_{t > 0} \theta(t) \nu_\infty \left( \frac{h}{\alpha}, t \right) \leq 1 \right\}
\]

and

\[
\mathfrak{B}_{\theta,w} = \left\{ \alpha > 0 : HK \int_{\mathcal{K}} \theta \left( \frac{h}{\alpha} \right) d\mu_\infty \leq 1 \right\}.
\]

Clearly, $\|h\|_{H_{w}^\theta(\mathbb{R}^n)} = \inf \mathfrak{A}_{\theta,w}$ and $\|h\|_{H^\theta(\mathbb{R}^n)} = \inf \mathfrak{B}_{\theta,w}$. Now for any $\beta \in \mathfrak{B}_{\theta,w}$ and $t > 0$, we have

\[
\theta(t) \nu_\infty \left( \frac{h}{\beta}, t \right) \leq HK \int_{\mathcal{X} \in \mathbb{R}^n : \nu_\infty \left( \frac{h}{\beta}, t \right)} \theta \left( \frac{h}{\beta} \right) d\nu_\infty
\]

\[
\leq HK \int_{\mathcal{K}} \theta \left( \frac{h}{\beta} \right) d\nu_\infty
\]

\[
\leq 1.
\]

As $t > 0$ is arbitrary, $\sup_{t > 0} \theta(t) \nu_\infty \left( \frac{h}{\beta}, t \right) \leq 1$ and $\mathfrak{B}_{\theta,w} \leq \mathfrak{A}_{\theta,w}$. Hence $f \in \mathcal{H}_{w}^\theta(\mathbb{R}^n)$ with $\|h\|_{H_{w}^\theta(\mathbb{R}^n)} \leq \|h\|_{H^\theta(\mathbb{R}^n)}$. \hfill \Box

**Lemma 5.2.** Let $\theta$ be a Young function, $a \in \mathbb{R}^n$ and $r > 0$ be arbitrary. Then

\[
\|\chi_{B(a,r)}\|_{H_{w}^\theta(\mathbb{R}^n)} = \frac{1}{\theta^{-1} \left( \frac{1}{\nu_\infty(B(a,r))} \right)},
\]

where $\nu_\infty(B(a,r))$ is the volume of open ball $B(a,r)$. 


Proof. Theorem 5.1 gives $||h||_{H^\theta_w(\mathbb{R}^n)} \leq ||h||_{H^\phi_w(\mathbb{R}^n)}$. Now,

$$||x_{B(a,r)}||_{H^\theta_w(\mathbb{R}^n)} = HK \int_{\mathbb{R}^n} \theta \left( \chi_{B(a,r)} \theta^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right) \right) d\nu_{\infty}$$

$$= HK \int_{B(a,r)} \theta \left( \chi_{B(a,r)} \theta^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right) \right) d\nu_{\infty}$$

$$\leq HK \int_{B(a,r)} \frac{1}{\nu_{\infty}(B(a,r))} d\nu_{\infty}$$

$$= \frac{1}{\nu_{\infty}(B(a,r))} HK \int_{B(a,r)} d\nu_{\infty}$$

$$= 1.$$ 

By the definition of $||\cdot||_{H^\theta(\mathbb{R}^n)}$, we get

$$||x_{B(a,r)}||_{H^\theta_w(\mathbb{R}^n)} \leq \frac{1}{\theta^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right)}.$$ 

Now we need to prove

$$||x_{B(a,r)}||_{H^\theta_w(\mathbb{R}^n)} \geq \frac{1}{\theta^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right)}.$$ 

If possible, let $||x_{B(a,r)}||_{H^\theta_w(\mathbb{R}^n)} < \frac{1}{\theta^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right)}$. Then by the definition of $||\cdot||_{H^\theta_w(\mathbb{R}^n)}$, we find $||x_{B(a,r)}||_{H^\theta_w(\mathbb{R}^n)} \leq 1$. This contradicts our assumption and hence

$$||x_{B(a,r)}||_{H^\theta_w(\mathbb{R}^n)} = \frac{1}{\theta^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right)}.$$ 

\[\square\]

**Theorem 5.3.** Let $\theta$, $\phi$ be Young functions then the statements below are equivalent:

1. $\theta(t) \leq \phi(Ct)$ for every $t > 0$.
2. $H^\phi_w(\mathbb{R}^n) \subseteq H^\theta_w(\mathbb{R}^n)$.
3. For every $h \in H^\theta_w(\mathbb{R}^n)$, implies $||h||_{H^\theta_w(\mathbb{R}^n)} \leq C||h||_{H^\phi_w(\mathbb{R}^n)}$.

**Proof.** The proof is similar to the [17], Theorem 3.3. \[\square\]

**Corollary 5.4.** Let $\theta$, $\phi$ be Young functions with $\theta(t) \leq \phi(Ct)$ for every $t > 0$. If $h \in H^\theta_w(\mathbb{R}^n)$ then $||h||_{H^\phi_w(\mathbb{R}^n)} \leq ||h||_{H^\theta_w(\mathbb{R}^n)}$.

Now we state a necessary and sufficient condition for the inclusion properties of weak $\mathbb{H}$–Orlicz spaces generated by concave function presented in the following theorem.

**Theorem 5.5.** Let $\theta_1, \theta_2 \in \mathfrak{F}$. The followings are equivalent:

1. $\theta_1 \Delta \theta_2$.
2. $H^{\theta_1}_w(\mathbb{R}^n) \subseteq H^{\theta_2}_w(\mathbb{R}^n)$.
There exists a constant $C > 0$ such that $\|h\|_{H^{\theta_2}_{w}(\mathbb{R}^n)} \leq C\|h\|_{H^{\theta_1}_{w}(\mathbb{R}^n)}$ for every $h \in H^{\theta_2}_{w}(\mathbb{R}^n)$.

Proof. $(1) \Rightarrow (2)$ Let $h \in H^{\theta_2}_{w}(\mathbb{R}^n)$. We set

$$A_{\theta_1} = \left\{ \alpha > 0 : \sup_{t > 0} \theta_1(t) \nu_{\infty} \left( \frac{h}{\alpha}, t \right) \leq 1 \right\}$$

and

$$A_{\theta_2} = \left\{ \alpha > 0 : \sup_{t > 0} \theta_2(Ct) \nu_{\infty} \left( \frac{h}{\alpha}, t \right) \leq 1 \right\}$$

$$= \left\{ \alpha > 0 : \sup_{t > 0} \theta_2(p) \nu_{\infty} \left( \frac{Ch}{\alpha}, p \right) \leq 1 \right\}$$

for $p = Ct$. If $t > 0$ and $\alpha \in A_{\theta_2}$, then

$$\theta_1(t) \nu_{\infty} \left( \frac{h}{\alpha}, t \right) \leq \theta_2(Ct) \nu_{\infty} \left( \frac{h}{\alpha}, t \right)$$

$$= \theta_2(p) \nu_{\infty} \left( \frac{Ch}{\alpha}, p \right) \leq 1.$$

So,

$$\|h\|_{H^{\theta_1}_{w}(\mathbb{R}^n)} = \inf A_{\theta_1} \leq \inf A_{\theta_2} \leq C\|h\|_{H^{\theta_2}_{w}(\mathbb{R}^n)}.$$

Therefore $H^{\theta_2}_{w}(\mathbb{R}^n) \subseteq H^{\theta_1}_{w}(\mathbb{R}^n)$.

$(2) \Rightarrow (3)$ Since $(H^{\theta_2}_{w}(\mathbb{R}^n), H^{\theta_1}_{w}(\mathbb{R}^n))$ are pair of Banach spaces then by [10] Lemma 3.3], we get the conclusion.

$(3) \Rightarrow (1)$ From the Lemma 5.2

$$\frac{1}{\theta_1^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right)} = \|\chi_{B(a,r)}\|_{H^{\theta_1}_{w}(\mathbb{R}^n)}$$

$$\leq C\|\chi_{B(a,r)}\|_{H^{\theta_2}_{w}(\mathbb{R}^n)}$$

$$= C \frac{1}{\theta_2^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right)}.$$

Therefore for any $a \in \mathbb{R}^n$ and $r > 0$ we find

$$C\theta_1^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right) \geq \theta_2^{-1} \left( \frac{1}{\nu_{\infty}(B(a,r))} \right).$$
Using the Lemma 1.1(4), we get \( \theta_1(\frac{1}{\nu_\infty(B(a,r))}) \leq \theta_2(\frac{C}{\nu_\infty(B(a,r))}) \).

As \( r > 0 \) is an arbitrary, assuming \( t = \frac{1}{\nu_\infty(B(a,r))} \), we get the conclusion \( \theta_1(t) \leq \theta_2(Ct) \) and hence \( \theta_1 \Delta \theta_2 \). \( \square \)

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