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Abstract: We calculate anomaly induced conductivities from a holographic gauge theory model using Kubo formulas, making a clear conceptual distinction between thermodynamic state variables such as chemical potentials and external background fields. This allows us to pinpoint ambiguities in previous holographic calculations of the chiral magnetic conductivity. We also calculate the corresponding anomalous current three-point functions in special kinematic regimes. We compare the holographic results to weak coupling calculations using both dimensional regularization and cutoff regularization. In order to reproduce the weak coupling results it is necessary to allow for singular holographic gauge field configurations when a chiral chemical potential is introduced for a chiral charge defined through a gauge invariant but non-conserved chiral density. We argue that this is appropriate for actually addressing charge separation due to the chiral magnetic effect.

Keywords: Gauge-gravity correspondence, QCD, Chiral Lagrangians.
1. Introduction

The chiral anomaly of QED is responsible for two particularly interesting effects of strong magnetic fields in dense, strongly interacting matter as found in neutron stars or heavy-ion collisions. At large quark chemical potential $\mu$, chirally restored quark matter gives rise to an axial current parallel to the magnetic field \[ J_5 = \frac{eN_c}{2\pi^2}\mu B, \] (1.1)
which may lead to observable effects in strongly magnetized neutron stars [4].

In heavy-ion collisions, one expects initial magnetic fields that momentarily exceed even those found in magnetars. It has been proposed by Kharzeev et al. [5–9] that the analogous effect [10]
\[ J = \frac{e^2N_c}{2\pi^2}\mu_5 B, \] (1.2)
where $J$ is the electromagnetic current and $\mu_5$ a chemical potential for an asymmetry in the number of right and left chiral quarks, could render observable event-by-event P and CP violations from topologically nontrivial gluon configurations. Indeed, there is recent experimental evidence for this “chiral magnetic effect” (CME) in the form of
charge separation in heavy ion collisions with respect to the reaction plane [11,12] (see however [13,14]), whose normal vector is expected to coincide with the direction of strong initial magnetic fields. For lattice studies of the effect, see for example [15,16],

The anomalous conductivities (1.1) and (1.2) have recently also been studied in holographic models of QCD by introducing chemical potentials for left and right chiral quarks as boundary values for corresponding bulk gauge fields [17,18]. However, it was pointed out by Ref. [19] that in these calculations the axial anomaly was not realized in covariant form and that the corresponding electromagnetic current was not strictly conserved. Correcting the situation by means of Bardeen’s counter-term [20,21] instead led to a vanishing result for the electromagnetic current in the holographic QCD model due to Sakai and Sugimoto [23, 24], while recovering the result (1.1) for the anomalous axial conductivity.

Indeed, the two anomalous conductivities (1.1) and (1.2) differ in that in the former case there is no difficulty with introducing a chemical potential for quark number, while a chemical potential for chirality refers to a chiral current that is either gauge invariant and anomalous or conserved but not gauge invariant. In Ref. [10], the chiral magnetic effect (1.2) was shown to be an exact result when the chiral chemical potential is conjugate to conserved chiral charges that are, however, only gauge invariant when integrated over all of space in spatially homogenous situations (this point was most recently also made in Ref. [26]).

However, charge separation in heavy-ion collisions clearly calls for inhomogeneous situations, since with $\nabla \cdot \mathbf{B} \equiv 0$ we have

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} = -\frac{e^2 N_c}{2\pi^2} \mathbf{B} \cdot \nabla \mu_5. \quad (1.3)$$

It would therefore appear important to introduce a chiral chemical potential conjugate to gauge invariant axial currents, despite them being anomalous. At least as long as the electric field is zero and as long as the chiral charge diffusion rate is suppressed (as it is in the large-$N_c$ limit [27], and the fact that we indeed find time-independent solutions in the presence of a chiral chemical potential is a reflection of this fact), it should be admissible to consider chemical potentials defined with respect to the gauge invariant chiral density in a thermodynamic description. Such a chiral chemical potential thus serves as a model parameter for the imbalance between the number of left-handed and right-handed fermions that is assumed to be induced by topologically non-trivial gluon field configurations during the out-of-equilibrium early stages of a heavy ion collision.

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1It has been argued that this is an artifact of the grand canonical ensemble and that the weak coupling result would be recovered in the canonical ensemble [22].

2In Ref. [25] a finite result was obtained in a bottom-up model that is nonzero only due to extra scalar fields.
It is however important to distinguish between thermodynamic state variables such as chemical potentials and background gauge fields (as also pointed out by Ref. [26]). The holographic dictionary instructs us to construct a functional of boundary fields and $n$-point functions are obtained by functional differentiation with respect to the boundary fields. For a gauge field, the expansion close to the boundary takes the form

$$A_\mu(x, r) = A_\mu^{(0)}(x) + \frac{A_\mu^{(2)}(x)}{r^2} + \ldots .$$

The leading term in this expansion is the source for the current $J_\mu$. The sub-leading term is often identified with the one-point function of the current. This is, however, not true in general. As has been pointed out in Ref. [19], in the presence of a bulk Chern-Simons term, the current receives also contributions from the Chern-Simons term and $A_\mu^{(2)}(x)$ can, in general, not be identified with the vev of the current. On the other hand, a constant value of $A_\mu^{(0)}$ is often identified with a chemical potential. This is, however, slightly misleading since the holographic realization of the chemical potential is given by the potential difference between the boundary and the horizon [28] and only in a gauge in which $A_\mu^{(0)}$ vanishes at the horizon such an identification can be made. Even in this case, we have to keep in mind that the boundary value of the gauge field is the source of the current whereas the potential difference between horizon and boundary is the chemical potential. We will keep this distinction explicit in this paper.

In section 2 we shall show that by distinguishing the chemical potential from the background gauge fields one can reproduce the usual result (1.2) for the chiral magnetic effect when $\mu_5$ refers to the gauge invariant chiral current\(^3\) and for a strictly conserved electromagnetic current $J$. However, this requires singular gauge field configurations in the bulk of AdS space, which appears to be the price to pay for having introduced a chemical potential for an anomalous charge. We also reproduce the uncontested result (1.1) for the axial current at finite quark chemical potential and magnetic fields, as well as a new anomalous conductivity, albeit one of perhaps mere academic interest as it refers to nonzero axial magnetic fields. Moreover, we derive results for anomalous three-point functions in certain kinematic limits. In section 3 we reproduce all these results in weak coupling calculations using gauge invariant dimensional regularization without and cutoff regularization with the need for introducing Bardeen’s counter-term.

2. Holographic Kubo formulas for anomalous conductivities

We will consider the simplest possible holographic model for one quark flavor in a

\(^3\)The chiral current is gauge invariant under the non-anomalous vectorial gauge transformations.
chirally restored deconfined phase.\textsuperscript{4} It consists of taking two gauge fields corresponding to the two chiralities for each quark flavor in a five dimensional AdS black hole background.

The action is given by two Maxwell actions for left and right gauge fields plus separate Chern Simons terms corresponding to separate ("consistent") anomalies for left and right chiral quarks. The Chern-Simons terms are, however, not unique but can be modified by adding total derivatives. A total derivative which enforces invariance under vector gauge transformations $\delta V_M = \partial_M \lambda_V$ corresponds to the so-called Bardeen counter-term \cite{20, 21}, leading to the action

\[
S = \int \sqrt{-g} \left( -\frac{1}{4g_v^2} F^V_M F^M_N - \frac{1}{4g_A^2} F^A_M F^A_N + \kappa^2 \epsilon^{MNPQR} A_M (F^A_N F^A_Q + 3 F^V_N F^V_Q) \right).
\]

Since the Chern-Simons term depends explicitly on the gauge potential $A_M$, the action is gauge invariant under $\delta A_M = \partial_M \lambda_A$ only up to a boundary term. This non-invariance is the holographic implementation of the axial $U(1)$ anomaly in covariant (Adler-Bell-Jackiw) form \cite{29, 30} when identifying the gauge fields as holographic sources for the currents of global $U(1)$ symmetries in the dual field theory. A rigorous string-theoretical realization of such a setup is provided for example by the Sakai-Sugimoto model \cite{23, 24}. As usually done in the latter, we neglect the back-reaction of the bulk gauge fields on the black hole geometry.

Before we proceed, we also want to clarify our conventions concerning the $\epsilon$-tensor. We define the $\epsilon$ tensor $\epsilon_{MNPQR} = \sqrt{-g} \epsilon(MNPQR)$. Here we distinguish between the tensor and the symbol. The symbol is $\epsilon(MNPQR)$ and normalized to $\epsilon(r0123) = 1$.

In order to compute the field equations and the boundary action, from which we shall obtain the two- and three-point functions of various currents, we expand around fixed background gauge fields to second order in fluctuations. The gauge fields are written as

\[
A_M = A_M + a_M, \quad V_M = V_M + v_M,
\]

where the calligraphic letters are the background fields and the lower case letters are the fluctuations.

\textsuperscript{4}The even simpler model considered in Ref. \cite{26} is instead closer to a single quark flavor in a chirally broken phase where right and left chiralities are living on the two boundaries of a single brane.
After a little algebra, we find to first order in the fluctuations

$$
\delta S^{(1)}_{\text{bulk}+\partial} = \int drd^4x \sqrt{-g} \left\{ a_M \left[ \frac{1}{g_A^2} \nabla_N f^{NM}_A + \frac{3\kappa}{2} \epsilon^{MNPQR} (f^A_{NP} f^A_{QR} + f^V_{NP} f^V_{QR}) \right] + v_M \left[ \frac{1}{g_V^2} \nabla_N f^V_{NM} + 3\kappa \epsilon^{MNPQR} (f^A_{NP} f^V_{QR}) \right] \right\}
$$

$$
+ \int d^4x \left\{ a_\mu \left( \frac{1}{g_A^2} \sqrt{-g} f^{\mu r}_A + 2\kappa \epsilon^{\mu\rho\lambda\rho} A_\rho f^A_{\rho\lambda} \right) + v_\mu \left( \frac{1}{g_V^2} \sqrt{-g} f^{\mu r}_{V} + 6\kappa \epsilon^{\mu\rho\lambda} A_\rho f^V_{\rho\lambda} \right) \right\}. 
$$

(2.3)

From the bulk term we get the equations of motion and from the boundary terms we can read the expressions for the currents,

$$
J^\mu = \lim_{r \to \infty} \frac{1}{g_V^2} \sqrt{-g} f^{\mu r}_V + 6\kappa \epsilon^{\mu\rho\lambda} A_\rho f^V_{\rho\lambda},
$$

$$
J^5_\mu = \lim_{r \to \infty} \frac{1}{g_A^2} \sqrt{-g} f^{\mu r}_A + 2\kappa \epsilon^{\mu\rho\lambda} A_\rho f^A_{\rho\lambda}.
$$

(2.4)

(2.5)

On-shell they obey

$$
\partial_\mu J^\mu = 0,
$$

$$
\partial_\mu J^5_\mu = -\frac{\kappa}{2} \epsilon^{\mu\rho\lambda} \left( 3F^V_{\mu\rho} F^V_{\rho\lambda} + F^A_{\mu\rho} F^A_{\rho\lambda} \right).
$$

(2.6)

As expected, the vector like current is exactly conserved. Comparing with the standard result from the one loop triangle calculation, we find $\kappa = -\frac{N_c}{24\pi}$ for a dual strongly coupled $SU(N_c)$ gauge theory for a massless Dirac fermion in the fundamental representation.

We emphasize that only by demanding an exact conservation law for the vector current can we consistently couple it to an (external) electromagnetic field. This leaves no ambiguity in the definitions of the above currents as the ones obtained by varying the action with respect to the gauge fields and which obey (2.6). In particular, we have to keep the contributions from the Chern-Simons terms in the action, which are occasionally ignored in holographic calculations.

The second order term in the expansion of the action is

$$
S^{(2)}_{\text{bulk}+\partial} = \int \text{bulk} \left\{ a_M \left[ \frac{1}{2g_A^2} \nabla_N f^{NM}_A + \frac{3\kappa}{2} \epsilon^{MNPQR} (f^A_{NP} f^A_{QR} + f^V_{NP} f^V_{QR}) \right] + v_M \left[ \frac{1}{2g_V^2} \nabla_N f^V_{NM} + \frac{3\kappa}{2} \epsilon^{MNPQR} (f^A_{NP} f^V_{QR}) \right] \right\}
$$

$$
+ \int \partial \left[ \frac{\sqrt{-g}}{2} \left( \frac{1}{g_A^2} a_\mu f^{\mu r}_A + \frac{1}{g_V^2} v_\mu f^{\mu r}_V \right) + \kappa \epsilon^{\mu\rho\lambda} (A_\rho a_\mu f^A_{\rho\lambda} + 3v_\rho A_\mu f^V_{\rho\lambda} + 3v_\mu A_\nu f^V_{\rho\lambda} \right) \right],
$$

(2.7)
where $f_{MN}$ is the field strength of the fluctuations. Again, the action is already in the form of bulk equations of motion plus boundary term.

As gravitational background, we take the planar AdS Schwarzschild metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}(dx^2 + dy^2 + dz^2), \quad (2.8)$$

with $f = \frac{r^2}{L^2} - \frac{r_H^2}{L^2}$. The temperature is given in terms of the horizon by $r_H = L^2 \pi T$. We rescale the $r$ coordinate such that the horizon lies at $r = 1$ and we also will set the AdS scale $L = 1$. Furthermore, we also rescale time and space coordinates accordingly. To recover the physical values of frequency and momentum we thus have to do replace $(\omega, k) \rightarrow (\omega/(\pi T), k/(\pi T))$.

The background gauge fields are

$$A_0(r) = \Phi(r) = \alpha - \frac{\beta}{r^2}, \quad (2.9)$$

$$V_0(r) = \Psi(r) = \nu - \frac{\gamma}{r^2}. \quad (2.10)$$

We need to relate the integration constants $\alpha, \beta, \gamma, \nu$ to physical observables now\(^5\). It is often stated in the literature that one needs to choose a gauge in which the fields in Eqs. (2.9) and (2.10) vanish on the horizon in order to make $A_M A^M$ and $V_M V^M$ well defined there. This is, however, not a physical constraint. After all, the value of a gauge field has no intrinsic meaning.

We will, instead, define the chemical potentials of the global $U(1)$ symmetries as the potential difference between the horizon and the boundary [28]. This can be expressed as the integrated radial electric flux between horizon and boundary and is therefore a manifestly gauge invariant quantity,

$$\mu = \int_{r_H=1}^{r_H=\infty} \partial_r A_0 dr = A_0(B) - A_0(H), \quad (2.11)$$

where $A_0$ stands for a generic gauge potential. The variation of the chemical potential can be thought of as either being a variation of the gauge potential on the boundary, the horizon or an arbitrary combination thereof. However, by a gauge transformation we can always think of $\delta \mu$ to result from a variation that vanishes on the horizon. Then $\delta \mu$ is just a special case of the general gauge field variation (2.3), if we interpret $a_\mu$, $v_\mu$ to be variations of the background fields. We see, therefore, that this definition automatically reproduces $\frac{\delta S}{\delta \mu} = \langle Q \rangle$ where $Q$ is the integrated charge density $J^0$. In general, a variation of $\mu$ is different from a variation with respect to the vector field. A variation in $\mu$ changes the ground state, $\delta \mu : |Q\rangle \rightarrow |Q + \delta Q\rangle$, whereas $\delta / \delta A_0$ inserts the operator $J^0$ into correlation functions.

\(^5\)Note that these integration constants with respect to the radial integration are independent of $(t, x, y, z)!$
We can think of (2.11) as the difference of energy in the system with a unit of charge at the boundary and a unit of charge at the horizon. This is the cost of energy to add a unit charge to the system and by definition represents thus the chemical potential. By the definition (2.11), the integration constants $\beta$ and $\gamma$ are thus fixed to

$$\beta = \mu_5, \quad \gamma = \mu,$$

(2.12)

(2.13)

where $\mu$ is the chemical potential of the vector symmetry and $\mu_5$ the chemical potential of the axial $U(1)$. The constants $\alpha$ and $\nu$ we take to be arbitrary and we will eventually consider them as sources for insertions of the operators $J^0$ and $J_5^0$ at zero momentum. Due to our choice of coordinates, the physical value of the chemical potentials is recovered by $\mu \to \pi T \mu$.

We can now compute the charges present in the system from the zero components of the currents (2.4),

$$J^0 = \frac{2\gamma}{g_V^2}, \quad (2.14)$$

$$J_5^0 = \frac{2\beta}{g_A^2}. \quad (2.15)$$

Note that this is, in fact, the standard holographic definition in the grand canonical ensemble. Often the gauge choice $A_0(H) = 0$ is imposed from the outset and that fixes the integration constants $\alpha$ and $\nu$ to take the values of the chemical potentials. It is important to realize that without a Chern-Simons term, the action for a gauge field in the bulk depends only on the field strengths and is, therefore, independent of constant boundary values of the gauge field. The action does, of course, depend on the physically relevant and gauge invariant difference of the potential between the horizon and the boundary. For our particular model, the choice of the Chern-Simons term results, however, also in an explicit dependence on the integration constant $\alpha$. It is crucial to keep in mind that $\alpha$ is a priori unrelated to the chiral chemical potential but plays the role of the source for the operator $J_5^0$ at zero momentum.

For the fluctuations we choose the gauge $a_r = 0$. We take the fluctuations to be of plane wave form with frequency $\omega$ and momentum $k$ in $x$-direction. The relevant polarizations are then the $y$- and $z$-components, i.e. the transverse gauge field fluctuations. The equations of motion are

$$v'' + \left(\frac{f'}{f} + \frac{1}{r}\right)v' + \frac{(\omega^2 r^2 - f k^2)}{f^2 r^2} v + \frac{12 i \kappa g_V^2 k}{f r} \epsilon_{ij}(\Phi' v_j + \Psi' a_j) = 0, \quad (2.16)$$

$$a'' + \left(\frac{f'}{f} + \frac{1}{r}\right)a' + \frac{(\omega^2 r^2 - f k^2)}{f^2 r^2} a + \frac{12 i \kappa g_A^2 k}{f r} \epsilon_{ij}(\Phi' a_j + \Psi' v_j) = 0. \quad (2.17)$$
The indices \((i, j) \in \{y, z\}\) and a prime denotes differentiation with respect to the radial coordinate \(r\). The two-dimensional epsilon symbol is \(\epsilon_{yz} = 1\).

There is also a longitudinal sector of gauge field equations. They receive no contribution from the Chern-Simons term and so are uninteresting for our purposes.

The boundary action in Fourier space in the relevant transversal sector is

\[
S^{(2)} = \int \partial \frac{r f}{2} \left( \frac{1}{g_A} a^i_k (a^i_k)' + \frac{1}{g_V} v^i_k (v^i_k)' \right) - 2ik \kappa \epsilon_{ij} \alpha \left( a^i_k a^j_k + 3v^i_k v^i_k \right)
\]

(2.18)

As anticipated, the second order boundary action depends on the boundary value of the axial gauge field but not on the boundary value of the vector gauge field.

From this we can compute the holographic Green function. The way to do this is to compute four linearly independent solutions that satisfy in-falling boundary conditions on the horizon [31, 32]. At the AdS boundary we require that the first solution asymptotes to the vector \((v_y, v_z, a_y, a_z) = (1, 0, 0, 0)\), the second solution to the vector \((0, 1, 0, 0)\) and so on. We can therefore build up a matrix of solutions \(F_k^I J(r)\) where each column corresponds to one of these solutions [33]. Given a set of boundary fields \(a^{(0)}_i(k), v^{(0)}_i(k)\), which we collectively arrange in the vector \(\varphi^{(0)}(k)\), the bulk solution corresponding to these boundary fields is

\[
\varphi^I(k, r) = F_k^I J \varphi^{(0)}(k).
\]

Here, \(F\) is the (matrix valued) bulk-to-boundary propagator for the system of coupled differential equations.

The holographic Green function is then given by

\[
G_{IJ} = -2 \lim_{r \to \infty} (A_{IL} (F_k^L J)' + B_{IJ}).
\]

(2.20)

The matrices \(A\) and \(B\) can be read off from the boundary action as

\[
A = -\frac{1}{2} r f \left( \begin{array}{cc} \frac{1}{g_V} & 0 \\ 0 & \frac{1}{g_A} \end{array} \right), \quad B = -2i \kappa k \alpha \left( \begin{array}{cc} 3 \epsilon_{ij} & 0 \\ 0 & \epsilon_{ij} \end{array} \right),
\]

(2.21)

(notice that \(F\) becomes the unit matrix at the boundary).

We are interested here only in the zero frequency limit and the first order in an expansion in the momentum \(k\). In this limit, the differential equations can be solved explicitly. To this order the matrix bulk-to-boundary propagator is

\[
F = \left( \begin{array}{cccc}
1 & -g_A^2 \mu_s g(r) & 0 & -g_V^2 \mu_s g(r) \\
g_A^2 \mu_s g(r) & 1 & g_V^2 \mu_s g(r) & 0 \\
0 & -g_A^2 \mu_s g(r) & 1 & -g_A^2 \mu_s g(r) \\
g_V^2 \mu_s g(r) & 0 & g_A^2 \mu_s g(r) & 1
\end{array} \right),
\]

(2.22)
where \( g(r) = 6i\kappa \kappa \log(1 + 1/r^2) \). We find then the holographic current two-point functions

\[
\langle J^i J^j \rangle = -12i\kappa k (\mu_5 - \alpha) \epsilon_{ij}, \quad (2.23)
\]

\[
\langle J^i_5 J^j \rangle = -12i\kappa k \mu \epsilon_{ij}, \quad (2.24)
\]

\[
\langle J^i_5 J^j_5 \rangle = -4i\kappa k (3\mu_5 - \alpha) \epsilon_{ij}. \quad (2.25)
\]

Although \( \mu, \mu_5 \) and the boundary gauge field value \( \alpha \) enter in very similar ways in this result, we need to remember their completely different physical meaning. The chemical potentials \( \mu \) and \( \mu_5 \) are gauge invariant physical state variables whereas \( \alpha \) is the source for insertions of \( J^0_5(0) \). Had we chosen the “gauge” \( \alpha = \mu_5 \), we would have concluded (erroneously) that the two-point correlator of electric currents vanishes.

We see now that with \( \mu_5 \) introduced separately from \( \alpha \) that this is not so. We simply have obtained expressions for the correlators in the physical state described by \( \mu \) and \( \mu_5 \) in the constant external background field \( \alpha \). Due to the gauge invariance of the action under vector gauge transformations, the constant mode \( \nu \) of the corresponding source does not appear. The physical difference between the chemical potentials and the gauge field values is clear now. The susceptibilities of the two-point functions obtained by differentiating with respect to the chemical potentials are different from the three-point functions obtained by differentiating with respect to the gauge field values. Finally, we remark that the temperature dependence drops out due to the opposite scaling of \( k \) and \( \mu, \mu_5 \).

To compute the anomalous conductivities we therefore have to evaluate the two-point function for vanishing background fields \( \nu = \alpha = 0 \). We obtain, in complete agreement with the well-known weak coupling results,

\[
J^i = e^2 \sigma_{\text{CME}} B^i, \quad \sigma_{\text{CME}} = \lim_{k \to 0} \frac{i \epsilon_{ij}}{2k} \langle J^i J^j \rangle|_{\nu = \alpha = 0} = \frac{N_c}{2\pi^2} \mu_5, \quad (2.26)
\]

\[
J^i_5 = e \sigma_{\text{axial}} B^i, \quad \sigma_{\text{axial}} = \lim_{k \to 0} \frac{i \epsilon_{ij}}{2k} \langle J^i_5 J^j \rangle|_{\nu = \alpha = 0} = \frac{N_c}{2\pi^2} \mu, \quad (2.27)
\]

\[
J^i_5 = \sigma_{55} B^i, \quad \sigma_{55} = \lim_{k \to 0} \frac{i \epsilon_{ij}}{2k} \langle J^i_5 J^j_5 \rangle|_{\nu = \alpha = 0} = \frac{N_c}{2\pi^2} \mu_5. \quad (2.28)
\]

We are tempted to call all \( \sigma \)'s conductivities. This is, however, a slight misuse of language in the case of \( \sigma_{55} \). Formally, \( \sigma_{55} \) measures the response due to the presence of an axial magnetic field \( \vec{B}_5 = \nabla \times \vec{A}_5 \). Since such fields do not exist in nature, we cannot measure \( \sigma_{55} \) in the same way as \( \sigma_{\text{CME}} \) and \( \sigma_{\text{axial}} \).

Since the two-point functions in Eqs. (2.23)-(2.25) still depend on the external source \( \alpha \), we can also obtain the three point functions in a particular kinematic regime. Differentiating with respect to \( \alpha \) (and \( \nu \)) we find the three point functions
\langle J^i(k)J^j(-k)J^0(0) \rangle = 0, \quad (2.29)
\langle J^i_5(k)J^j(-k)J^0(0) \rangle = 0, \quad (2.30)
\langle J^i_5(k)J^j_5(-k)J^0(0) \rangle = 0, \quad (2.31)
\langle J^i_5(k)J^j(-k)J^0_5(0) \rangle = -ik \frac{N_c}{2\pi^2} \epsilon_{ij}, \quad (2.32)
\langle J^i_5(k)J^j(-k)J^0_5(0) \rangle = 0, \quad (2.33)
\langle J^i_5(k)J^j_5(-k)J^0_5(0) \rangle = -ik \frac{1}{3} \frac{N_c}{2\pi^2} \epsilon_{ij}. \quad (2.34)

Note the independence on chemical potentials and temperature.

Equations (2.32) and (2.34) show the sensitivity of the theory to a constant temporal component of the axial gauge field even at zero temperature and chemical potentials. If the axial $U(1)$ symmetry was exactly conserved, such a constant field value would be a gauge degree of freedom and the theory would be insensitive to it. Since this symmetry is, however, anomalous, it couples to currents through these three-point functions. The correlators (2.32) and (2.34) can therefore be understood as expressing the anomaly in the axial $U(1)$ symmetry.

In the next section we will check these results in vacuum at weak coupling by calculating the triangle diagram in the relevant kinematic regimes.

3. Weak-coupling calculations

An important property of the two- and three-point functions we just calculated is that they are independent of temperature. The three-point functions are furthermore independent of the chemical potentials. Therefore, the results for the three-point functions should coincide with correlation functions in (a chirally symmetric) vacuum. At weak coupling, all the three-point functions can be obtained from a single 1-loop Feynman integral. We only need to evaluate the diagram with two vector currents and one axial current. The diagram with three vector currents vanishes identically (due to C-parity) and the one with three axial currents can be reduced to the one with only one axial current by anti-commuting $\gamma_5$ matrices (when a regularization is applied that permits this). Similarly, it can be seen that the diagram with two axial vector currents can be reduced to the one with none, which vanishes.

When computing the three-point function, it is crucial to check the resulting anomalies. Gauge invariant regulators, like dimensional regularization, should yield the correct covariant anomaly, such that the vector currents are identically conserved. On the other hand, for example cutoff regularization breaks gauge invariance and further finite renormalizations may be needed in order to restore gauge invariance. In the following, we apply both dimensional and cutoff regularizations to compute
the three-point function and show that they give consistent results with each other and with Eqs. (2.29)-(2.34).

3.1 Triangle diagram with one axial current

The triangle diagram, shown in Fig. 1, with one axial current and two vector currents, is given by

\[
\Gamma_{\mu\nu\rho}(p, q) = (-1)(ie)^2(ig)(i)^3 \int \frac{d^d l}{(2\pi)^d} \text{tr} \left( \gamma_5 \frac{l - p}{(l - p)^2} \gamma^\mu \frac{l + q}{(l + q)^2} \gamma^\nu \gamma^\rho \right) \\
+ (\mu \leftrightarrow \nu, p \leftrightarrow q).
\]

(3.1)

The factors are a \((-1)\) from the fermion loop, the couplings to vector and axial gauge fields and \(i\) for each fermion propagator. We will simply set the electric and axial couplings \(e\) and \(g\) to one. Evaluation of the integral with dimensional and cutoff regularizations is presented in some detail in Appendix A.

The anomalies of the various currents coupled to the triangle diagram are obtained by contracting the three-point function above by the corresponding momenta. Applying dimensional regularization, we get immediately

\[
p_\mu \Gamma_{\mu\nu\rho}^{\text{DR}}(p, q) = 0,
\]

(3.2)

\[
q_\nu \Gamma_{\mu\nu\rho}^{\text{DR}}(p, q) = 0,
\]

(3.3)

\[
(p + q)_\rho \Gamma_{\mu\nu\rho}^{\text{DR}}(p, q) = \frac{i}{2\pi^2} p_\alpha q_\beta \epsilon^{\alpha\beta\mu\nu},
\]

(3.4)

yielding the correct Adler-Bell-Jackiw anomaly. In terms of cutoff regularization, we, however, find

\[
p_\mu \Gamma_{\mu\nu\rho}^{\text{CO}}(p, q) = -\frac{i}{6\pi^2} p_\alpha q_\beta \epsilon^{\alpha\beta\mu\rho},
\]

(3.5)

\[
q_\nu \Gamma_{\mu\nu\rho}^{\text{CO}}(p, q) = \frac{i}{6\pi^2} p_\alpha q_\beta \epsilon^{\alpha\beta\mu\rho},
\]

(3.6)

\[
(p + q)_\rho \Gamma_{\mu\nu\rho}^{\text{CO}}(p, q) = \frac{i}{6\pi^2} p_\alpha q_\beta \epsilon^{\alpha\beta\mu\nu}.
\]

(3.7)
In order to cancel the anomalies in the vector current, we must perform an additional finite renormalization by adding the Bardeen counter-term,

\[ \Gamma^{c.t.} = c \int d^4x \, \epsilon^{\mu\nu\rho\lambda} \, V_\mu \, A_\nu^5 \, F^V_{\rho\lambda}, \]

(3.8)

where \( F^V_{\rho\lambda} = \partial_\rho V_\lambda - \partial_\lambda V_\rho \). This vertex brings an additional contribution to the three-point function, and the full result reads

\[ \Gamma^{\mu\nu\rho} = \Gamma^{\mu\nu\rho}_{CO}(p, q) + 2ic(p_\lambda - q_\lambda)\epsilon^{\lambda\mu\nu}. \]

(3.9)

Choosing the coefficient \( c \) of the Bardeen counter-term appropriately, \( c = \frac{1}{12\pi^2} \), we find the anomaly equations

\[ p_\mu \Gamma^{\mu\nu\rho}(p, q) = 0, \]

(3.10)

\[ q_\mu \Gamma^{\mu\nu\rho}(p, q) = 0, \]

(3.11)

\[ (p + q)_\rho \Gamma^{\mu\nu\rho}(p, q) = \frac{i}{2\pi^2} p_\alpha q_\beta \epsilon^{\alpha\beta\mu\nu}, \]

(3.12)

in full agreement with the covariant anomaly and the result from dimensional regularization.

We next want to evaluate the triangle diagram in the special kinematic regimes of Eqs. (2.29)-(2.34). Taking \( q = -p \), corresponding to the three-point function in Eq. (2.32), only the integrands \( A \) and \( B \) in Eqs. (A.12)-(A.23) contribute and take the values \( 1/2 \) and \( -1/2 \) in dimensional regularization and \( 1/6 \) and \( -1/6 \) in cutoff, respectively. The three-point function is then

\[ \Gamma^{\mu\nu\rho}(p, -p) = \frac{i}{2\pi^2} \epsilon^{\alpha\mu\nu\rho} p_\alpha, \]

(3.13)

in agreement with Eq. (2.32). Note that with cutoff regularization, \( \frac{1}{3} \) of this result comes from the loop diagram and \( \frac{2}{3} \) comes from the counter-term.

Let us next take \( p = 0 \), i.e. we put zero momentum on one of the vector currents. The corresponding loop integral vanishes in dimensional regularization, while the loop contribution in cutoff regularization is precisely canceled by the contribution from the counter-term,

\[ \Gamma^{0\nu\rho}(0, -q) = 0. \]

(3.14)

This result is in agreement with Eq. (2.30).

### 3.2 Triangle diagram with three axial currents

From the same one loop integral we can also compute the correlator of three axial currents\(^6\). Since we can anti-commute the \( \gamma_5 \) and use \( \gamma_5^2 = -1 \), we can reduce this

\(^6\)However, as this requires commuting the \( \gamma_5 \) with the rest of the \( \gamma \) matrices, only cutoff regularization can be applied.
diagram to the one in Fig. (3.1). The Bardeen counter-term, however, does not contribute this time, and we therefore find

\[ \Gamma_5^{\mu\nu} = \frac{1}{3} \frac{i}{2\pi^2} \epsilon_{\alpha\mu\nu} p_\alpha, \]  

(3.15)

just as in Eq. (2.34). The factor \( \frac{1}{3} \) is fixed by demanding Bose symmetry on the external legs.

All other current three-point functions can be related to the triangle with three vector currents which is known to vanish. Therefore, we have indeed reproduced the holographic results in Eqs. (2.29)-(2.34).

### 3.3 Two-point correlator of axial currents

To conclude the weak coupling considerations, we also compute the two-point function of two axial currents in the background of an axial chemical potential. We simply can follow the analogous calculation of the chiral magnetic conductivity in [9]. The relevant two-point function of axial currents is

\[ G_{55}(P) = \frac{T}{2} \sum_{n} \int \frac{d^3q}{(2\pi)^3} \epsilon^{ijk} \text{tr} \left[ \gamma^k \gamma_5 S(Q) \gamma^j \gamma_5 S(P + Q) \right], \]  

(3.16)

where \( S(Q) \) is the fermion propagator at finite temperature and density,

\[ S(Q) = \frac{1}{i\gamma^0(\tilde{\omega}_n - i\mu - i\mu_5\gamma_5) - \gamma\tilde{q}}, \]  

(3.17)

with \( \omega_n = (2n + 1)\pi T \). Using \( \gamma_5 S(Q) = -S(Q)\gamma_5 \) we can square the \( \gamma_5 \) matrices to one and end up with the same integral as for the chiral magnetic conductivity in [9]. We therefore find at weak coupling that \( \sigma_{55} = \sigma_{CME} \), coinciding with the result in (2.28).

### 4. Discussion and Conclusion

We have computed two- and three-point functions of currents at finite density using holographic methods for a simple holographic model incorporating the axial anomaly of the standard model. We were able to reproduce the known weak-coupling results concerning the chiral magnetic effect and also found a new type of “conductivity” in the axial sector alone, \( \sigma_{55} \). Although it can not be probed by switching on external fields, as a two-point function it is as well defined as \( \sigma_{CME} \). It would be interesting to find a way of relating this anomalous conductivity to experimentally accessible observables.

Previous calculations of anomalous conductivities have been able to reproduce the weak-coupling result for \( \sigma_{\text{axial}} \propto \mu \) but not \( \sigma_{CME} \propto \mu_5 \) unless the contributions
from the Chern-Simons term to the chiral currents were dropped. In our calculation we have used the complete expressions for the currents, but of key importance was a clear distinction between the physical state variable, the chemical potential, and the external background field. The latter we viewed exclusively as a source that couples to an operator, whereas the chemical potential should correspond in the most elementary way to the cost of energy for adding a unit of charge to the system.

It is useful to remember how a chemical potential can be introduced in field theory. One possible way is by deforming the Hamiltonian according to $H \rightarrow H - \mu Q$. A second, usually equivalent, way is by imposing boundary conditions $\phi(t - i\beta) = \pm \exp(\mu\beta)\phi(t)$ on the fields along the imaginary time direction \cite{34, 35}. These methods are equivalent as long as $Q$ is a non-anomalous charge. Similarly, in holography we can introduce the chemical potential either through a boundary value of the temporal component of the gauge field or through the potential difference between boundary and horizon. Thus, for non-anomalous symmetries, the boundary value of the temporal gauge field can be identified with the chemical potential. Due to the exact gauge invariance of the action, a constant boundary value never enters in correlation functions. In the presence of a Chern-Simons term, however, the gauge symmetry is partially lost and even a constant boundary gauge field becomes observable. This can be seen explicitly from the three-point functions (2.32) and (2.34). Therefore, we should set the axial vector field to zero after having used it as a source for axial current. By defining the corresponding chemical potential as the potential difference between the horizon of the AdS black hole and the holographic boundary we are able to do so. However, the prize we have to pay is to accept singular gauge field configurations at the horizon.

The fact that a gauge field that does not vanish on the horizon is not well defined is most easily seen in Kruskal coordinates,

$$ U V = - \exp(4\pi T r_*) , \quad V / U = - \exp(4\pi T t) , $$

where $dr_* = dr/f$. We note that close to the horizon, $r - r_H \approx -UV$. The time component of a gauge field in Kruskal coordinates at the horizon is therefore

$$ A_0 dt = A_0(r_H)(dV/V - dU/U) - A'_0(r_H)(U dV - V dU) + \cdots $$

This is not a well-defined one-form unless $A_0(r_H)$ vanishes. Although the gauge field is singular at the horizon, we do not believe that well defined physical observables are effected by this. Local gauge invariant observables, i.e. the field strengths, are certainly well behaved.

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A. Evaluation of the triangle diagram

We wish to compute the integral corresponding to the triangle diagram in Fig. 1,

\[ \Gamma_{\mu\nu\rho}(p, q) = (-1)(ie)^2(iq)(i)^3 \int \frac{d^4l}{(2\pi)^d} \text{tr} \left( \gamma_\xi \frac{l - q}{(l - p)^2} \gamma^\mu \frac{l}{l^2} \gamma^\nu \frac{l + q}{(l + q)^2} \gamma^\rho \right) \]

\[ + (\mu \leftrightarrow \nu, p \leftrightarrow q). \quad (A.1) \]

Using Feynman parametrization the integral can be written as

\[ \Gamma_{\mu\nu\rho}(p, q) = I_{\alpha\beta\gamma} \left[ \text{tr} \left( \gamma_\xi \gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu \gamma^\rho \right) - \text{tr} \left( \gamma_\xi \gamma^\nu \gamma^\beta \gamma^\mu \gamma^\alpha \gamma^\rho \right) \right], \quad (A.2) \]

\[ I_{\alpha\beta\gamma} = -2 \int_0^1 dx dy \Theta(1 - x - y) \int \frac{d^4l}{(2\pi)^d (l^2 + D)^3}, \quad (A.3) \]

where

\[ D = x(1 - x)p^2 + 2xpq + y(1 - y)q^2, \quad (A.4) \]

\[ r_\mu = xp_\mu - yq_\mu, \quad (A.5) \]

\[ N_{\alpha\beta\gamma} = (r - p)_\alpha r_\beta (r + q)_\gamma + \frac{l^2}{d} \left[ \delta_\alpha\beta (r + q)_\gamma + \delta_\alpha\gamma r_\beta + \delta_\beta\gamma (r - p)_\alpha \right]. \quad (A.6) \]

Here we have already taken into account that with both dimensional and cutoff regularizations, the integral with odd powers of \( l \) in the numerator of the integrand vanishes, and the remaining tensor structure is dictated by the rotational symmetry of a momentum shell at fixed \(|l|\).

Using

\[ \int_0^\Lambda \frac{l^3 dl}{(l^2 + D)^3} = \frac{1}{4D} + \mathcal{O}\left( \frac{1}{\Lambda^2} \right), \quad (A.7) \]

\[ \int_0^\Lambda \frac{l^5 dl}{(l^2 + D)^3} = \frac{1}{2} \left[ \log \left( \frac{\Lambda^2}{D} \right) - \frac{3}{2} \right] + \mathcal{O}\left( \frac{1}{\Lambda^2} \right), \quad (A.8) \]

in the cutoff regularization \((d = 4)\), and

\[ \left( \frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon} (l^2 + D)^3} = \frac{\Gamma(\epsilon)}{16\pi^2} \left( e^\gamma \mu^2 \right)^\epsilon \frac{1}{2D^{1+\epsilon}}, \quad (A.9) \]

\[ \left( \frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon} (l^2 + D)^3} = \frac{\Gamma(\epsilon)}{16\pi^2} \left( e^\gamma \mu^2 \right)^\epsilon \left( 1 - \frac{\epsilon}{2} \right) \frac{1}{D^\epsilon}, \quad (A.10) \]
in the dimensional regularization \((d = 4 - 2\epsilon)\) with MS scheme, we find

\[
\Gamma_{\mu\nu\rho}^{\text{reg}}(p, q) = \frac{i}{2\pi^2} \int_0^1 dx \, dz \, \Theta(1 - x - z) \left[ (A_p^{\text{reg}} + B_q^{\text{reg}})\epsilon^\alpha_{\mu\nu\rho} + (C_p^{\text{reg}} + D_q^{\text{reg}})\epsilon^\alpha_{\beta\mu\nu} + (C_p^{\text{reg}} + D_q^{\text{reg}})\epsilon^\alpha_{\lambda\mu\nu} \right]
\]

(A.11)

with \(\text{reg} \in \{\text{CO}, \text{DR}\}\). The coefficients are given by

\[
A_{\text{CO}} = \frac{(x - 1)r^2 + yq^2}{D} + \left[ \log \left( \frac{\Lambda^2}{D} \right) - \frac{3}{2} \right] (3x - 1),
\]

(A.12)

\[
B_{\text{CO}} = \frac{(1 - y)r^2 - xp^2}{D} + \left[ \log \left( \frac{\Lambda^2}{D} \right) - \frac{3}{2} \right] (1 - 3y),
\]

(A.13)

\[
C_{1\text{CO}} = \frac{2x(x - 1)}{D},
\]

(A.14)

\[
C_{2\text{CO}} = \frac{2xy}{D},
\]

(A.15)

\[
D_{1\text{CO}} = -\frac{2xy}{D},
\]

(A.16)

\[
D_{2\text{CO}} = \frac{2y(1 - y)}{D},
\]

(A.17)

in the cutoff regularization, and

\[
A_{\text{DR}} = \left[ \frac{(x - 1)(r^2 - D) + yq^2}{D} \right] \frac{\Gamma(\epsilon)}{D^\epsilon} \left( e^{\gamma_E \bar{\mu}^2} \right)^\epsilon,
\]

(A.18)

\[
B_{\text{DR}} = \left[ \frac{(1 - y)(r^2 - D) - xp^2}{D} \right] \frac{\Gamma(\epsilon)}{D^\epsilon} \left( e^{\gamma_E \bar{\mu}^2} \right)^\epsilon,
\]

(A.19)

\[
C_{1\text{DR}} = \frac{2x(x - 1)}{D^{1+\epsilon}} \frac{\Gamma(\epsilon)}{\Gamma(\epsilon)} \left( e^{\gamma_E \bar{\mu}^2} \right)^\epsilon,
\]

(A.20)

\[
C_{2\text{DR}} = \frac{2xy}{D^{1+\epsilon}} \frac{\Gamma(\epsilon)}{\Gamma(\epsilon)} \left( e^{\gamma_E \bar{\mu}^2} \right)^\epsilon,
\]

(A.21)

\[
D_{1\text{DR}} = -\frac{2xy}{D^{1+\epsilon}} \frac{\Gamma(\epsilon)}{\Gamma(\epsilon)} \left( e^{\gamma_E \bar{\mu}^2} \right)^\epsilon,
\]

(A.22)

\[
D_{2\text{DR}} = -\frac{2y(y - 1)}{D^{1+\epsilon}} \frac{\Gamma(\epsilon)}{\Gamma(\epsilon)} \left( e^{\gamma_E \bar{\mu}^2} \right)^\epsilon,
\]

(A.23)

in the dimensional regularization.

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