Recursive formulas for the Kronecker quantum cluster algebra with principal coefficients

Ming Ding\textsuperscript{1}, Fan Xu\textsuperscript{2} & Xueqing Chen\textsuperscript{3,*}

\textsuperscript{1}School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China; \\
\textsuperscript{2}Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China; \\
\textsuperscript{3}Department of Mathematics, University of Wisconsin-Whitewater, Whitewater, WI 53190, USA

Email: dingming@gzhu.edu.cn, fanxu@mail.tsinghua.edu.cn, chenx@uww.edu

Received November 16, 2021; accepted September 16, 2022; published online March 9, 2023

Abstract We use the quantum version of Chebyshev polynomials to explicitly construct the recursive formulas for the Kronecker quantum cluster algebra with principal coefficients. As a byproduct, we obtain two bar-invariant positive $\mathbb{ZP}$-bases with one being the atomic basis.

Keywords quantum cluster algebra, cluster variable, positive basis

MSC(2020) 13F60, 16G20

Citation: Ding M, Xu F, Chen X Q. Recursive formulas for the Kronecker quantum cluster algebra with principal coefficients. Sci China Math, 2023, 66: 1933–1948, https://doi.org/10.1007/s11425-021-2038-4

1 Introduction

Cluster algebras were invented by Fomin and Zelevinsky [16], and their quantum analogues (called quantum cluster algebras) were introduced by Berenstein and Zelevinsky [2]. A (quantum) cluster algebra is generated by distinguished generators called (quantum) cluster variables which are grouped into overlapping subsets called clusters. The first important result is the Laurent phenomenon which asserts that any (quantum) cluster algebra element is a Laurent polynomial in the cluster variables in any cluster. A (quantum) cluster algebra element is said to be positive if the Laurent polynomial has non-negative coefficients.

In order to obtain more structural results in the cluster theory, one needs to construct the cluster multiplication formulas. For the classical cluster algebras of acyclic quivers, Sherman and Zelevinsky [22] firstly provided the cluster multiplication formulas in rank 2 cluster algebras of finite and affine types. This result was generalized to the rank 3 cluster algebra of affine type $A_2^{(1)}$ by Cerulli Irelli [8]. Caldero and Keller [4] constructed the cluster multiplication formulas between two generalized cluster variables for simply laced Dynkin quivers, which were generalized to affine types in [17] and to acyclic types in [23,24]. In the quantum setting, Ding and Xu [10] firstly gave the cluster multiplication formulas of the quantum cluster algebra of type $A_1^{(1)}$ without coefficients, which were generalized to type $A_2^{(1)}$ by Bai et al. [1]. Recently, Chen et al. [9] constructed the cluster multiplication formulas in the acyclic
quantum cluster algebras with arbitrary coefficients through some quotients of derived Hall algebras of acyclic valued quivers. Note that for general cases, although the structure constants deduced from the cluster multiplication formulas have the homological interpretations, it is still difficult to explicitly compute them in these formulas.

Cluster multiplication formulas play a very important role in constructing “good” bases of cluster algebras. A basis of a cluster algebra is said to be positive if its structure constants are positive, and to be atomic (or canonical) if positive elements are exactly linear combinations of basis elements with non-negative coefficients. Note that the atomic basis must be a positive basis. It attracts a lot of attention to find “good” bases such as the atomic basis or the canonical basis of a cluster algebra. Sherman and Zelevinsky [22] introduced and constructed the atomic bases in the rank 2 cluster algebras of finite and affine types, which were originally called canonical bases. Cerulli Irelli constructed the atomic basis in the rank 3 cluster algebra of affine type \(A_2^{(1)}\) in [8]. He also proved the coincidence of the atomic basis and the set of cluster monomials for cluster algebras of finite type in [7]. For non-finite type cluster algebras, the set of cluster monomials does not form a basis even if it is linearly independent. Dupont and Thomas [15] constructed the atomic bases of cluster algebras of affine type \(A\). They also provided a new proof of Cerulli Irelli’s result for cluster algebras of type \(A\).

In the quantum setting, the atomic bases of the quantum cluster algebras of types \(A_1^{(1)}\) (or \(A_{1,1}\)), \(A_2^{(2)}\) and \(A_{2n-1,1}\) without coefficients were constructed in [1, 10, 12] by using the explicit cluster multiplication formulas. However, most of quantum cluster algebras must have coefficients involved. It is natural to ask whether there exists an explicit treatment of the atomic bases for the acyclic quantum cluster algebras with coefficients by using the representation theory of quivers. Sherman and Zelevinsky [22] initiated the use of Chebyshev polynomials to construct the atomic bases of the rank 2 cluster algebras of affine type. In order to obtain the atomic bases of cluster algebras with principal coefficients, Dupont [14] introduced the deformed Chebyshev polynomials and conjectured that the deformed Chebyshev polynomials have interactions with the atomic bases or canonically the positive bases in affine cluster algebras. He proved this conjecture for the cluster algebra of affine type \(A_1^{(1)}\). Note that for the \(A_2^{(1)}\) case, Cerulli Irelli [8] already used the deformed Chebyshev polynomials to prove the existence of the atomic basis.

As the Chebyshev polynomials are used to construct the positive bases in classical cluster algebras, it is important to study them in quantized cases. In this paper, we modify the definition of Chebyshev polynomials which can be viewed as the quantum analogue of those in [8, 14]. Then we explicitly construct the recursive formulas for the Kronecker quantum cluster algebra with principal coefficients based on the representation theory of the Kronecker quiver. As a direct corollary, we obtain two bar-invariant positive \(\mathbb{Z}[P]\)-bases with one being the atomic basis. We expect to apply the results and methods used in this paper in the future studies for all the affine types.

The rest of this paper is organized as follows. In Section 2, we collect basic concepts of quantum cluster algebras and quantum cluster characters and recall some multiplication formulas. Section 3 provides detailed calculations of recursive formulas in the Kronecker quantum cluster algebra with principal coefficients. In Section 4, we obtain an explicit treatment of two bar-invariant positive \(\mathbb{Z}[P]\)-bases in terms of quiver representations.

## 2 Quantum cluster algebras and quantum cluster characters

We recall some basic features of quantum cluster algebras and quantum cluster characters.

### 2.1 Quantum cluster algebras

Let \(\Lambda : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}\) be a skew-symmetric bilinear form on \(\mathbb{Z}^m\). The natural basis of \(\mathbb{Z}^m\) is denoted by \(\{e_1, \ldots, e_m\}\). Denote by \(\mathbb{Z}[q^{\pm \frac{1}{2}}]\) the ring of integer Laurent polynomials of an indeterminate \(q\). The quantum torus \(\mathcal{T}_q\) associated with \(\Lambda\) as the \(\mathbb{Z}[q^{\pm \frac{1}{2}}]\)-algebra is freely generated by the set \(\{X^e : e \in \mathbb{Z}^m\}\) with the following twisted product:

\[X^e \cdot X^f = q^{\Lambda(e,f)/2} X^{e+f}\quad\text{for any } e, f \in \mathbb{Z}^m.\]
In what follows, we omit the twisted product “·” for the multiplication in algebras. The skew-field of fractions for \( \mathcal{T}_q \) is denoted by \( \mathcal{F}_q \). It will cause no confusion if we use the same letter \( \Lambda \) to designate the \( m \times m \) skew-symmetric integer matrix associated with the bilinear form \( \Lambda \).

Let \( \tilde{B} = (b_{ij}) \) be an \( m \times n \) integer matrix with \( n \leq m \) and \( \tilde{B}^{tr} \) be the transpose of \( \tilde{B} \). The principal part of \( \tilde{B} \) is the upper \( n \times n \) submatrix of \( \tilde{B} \) denoted by \( B \). We call the pair \((\Lambda, \tilde{B})\) compatible if \( \tilde{B}^{tr} \Lambda = (D | 0) \) for some diagonal matrix \( D \) with positive entries. An initial quantum seed \((\Lambda, \tilde{B}, X)\) of \( \mathcal{F}_q \) consists of a compatible pair \((\Lambda, \tilde{B})\) and the set \( X = \{X_1, \ldots, X_m\} \) with \( X_i := X^{e_i} \) for \( 1 \leq i \leq m \).

Let \( \mu_k \) be the quantum seed \((\Lambda, \tilde{B}, X)\) in the direction \( k \) is the new quantum seed \( \mu_k(\Lambda, \tilde{B}, X) := (\Lambda', \tilde{B}', X') \) given as follows:

1. \( \Lambda' = E^{tr} \Lambda E \), where the \( m \times m \) matrix \( E = (e_{ij}) \) is given by
   \[
   e_{ij} = \begin{cases} 
   \delta_{ij}, & \text{if } j \neq k, \\
   -1, & \text{if } i = j = k, \\
   [-b_{ik}]_+, & \text{if } i \neq j = k.
   \end{cases}
   \]

2. \( \tilde{B}' = (b'_{ij}) \) with
   \[
   b'_{ij} = \begin{cases} 
   -b_{ij}, & \text{if } i = k \text{ or } j = k, \\
   b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+, & \text{otherwise}.
   \end{cases}
   \]

3. \( X' = \{X'_1, \ldots, X'_n\} \) with
   \[
   X'_i = X_i - \sum_{1 \leq j < m} [b_{ij}]_+ e_j - e_k + \sum_{1 \leq j < m} [-b_{ik}]_+ e_j - e_k, \quad \text{if } i \neq k, \\
   X'_i = X_i, \quad \text{if } i = k,
   \]
   where \([a]_+ := \max\{0, a\}\) for \( a \in \mathbb{Z} \).

Note that the mutation is an involution, i.e., \( \mu_k^2(\Lambda, \tilde{B}, X) = (\Lambda, \tilde{B}, X) \). Thus we have an equivalence relation: the quantum seed \((\Lambda, \tilde{B}, X)\) is mutation-equivalent to \((\Lambda', \tilde{B}', X')\), denoted by \((\Lambda, \tilde{B}, X) \sim (\Lambda', \tilde{B}', X')\), if they can be obtained from each other by the finite sequence of mutations. In the context of a quantum seed \((\Lambda', \tilde{B}', X')\), we call the set \( \{X'_i \mid 1 \leq i \leq n\} \) the cluster of the seed and call the elements \( X'_i \) the cluster variables. The cluster monomials are the elements \( X^{e_i} \) with \( e_i \in (\mathbb{Z}_{\geq 0})^m \) and \( e_i = 0 \) for \( n + 1 \leq i \leq m \). The elements in the set \( \mathcal{P} := \{X_i \mid n + 1 \leq i \leq m\} \) are called the coefficients.

Let \( \mathbb{Z} \mathcal{P} \) be the ring of Laurent polynomials in the elements of \( \mathcal{P} \) with coefficients in \( \mathbb{Z}[q^{1/2}] \). The quantum cluster algebra \( \mathcal{A}_q(\Lambda, \tilde{B}) \) is defined as the \( \mathbb{Z} \mathcal{P} \)-subalgebra of \( \mathcal{F}_q \) generated by all the cluster variables.

On \( \mathcal{T}_q \), we can define the \( \mathbb{Z} \)-linear bar-involution
\[
\overline{q^r X^e} = q^{-r} X^e \quad \text{for any } r \in \mathbb{Z} \text{ and } e \in \mathbb{Z}^m.
\]
It is straightforward to show that \( \overline{XY} = \overline{Y} \overline{X} \) for any \( X, Y \in \mathcal{T}_q \) and that quantum cluster monomials are bar-invariant. The bar-involution on \( \mathcal{A}_q(\Lambda, \tilde{B}) \) can be naturally induced.

The following celebrated quantum Laurent phenomenon proved by Berenstein and Zelevinsky [2] is one of the most important structural results on quantum cluster algebras, which asserts that every cluster variable can be written as a Laurent polynomial of cluster variables in any other cluster.

**Theorem 2.1** (See [2, Theorem 5.1]). The quantum cluster algebra \( \mathcal{A}_q(\Lambda, \tilde{B}) \) is a subalgebra of the ring of Laurent polynomials in the cluster variables in any cluster over \( \mathbb{Z} \mathcal{P} \).

### 2.2 Quantum cluster characters

Let \( k = \mathbb{F}_q \) be a finite field of order \( q \). Fix two positive integers \( n \leq m \). Let \( \tilde{Q} \) be an acyclic valued quiver with the vertex set \( \{1, 2, \ldots, m\} \). The full subquiver \( Q \) on the vertices \( \{1, \ldots, n\} \) is called the principal part of \( \tilde{Q} \). The elements in the subset \( \{n + 1, \ldots, m\} \) are called the frozen vertices.
Let $\mathcal{G} = kQ$ and $\mathcal{G} = k\tilde{Q}$ be the path algebras of $Q$ and $\tilde{Q}$, respectively. Let $S_i$, $P_i$ and $I_i$ be the simple, projective and injective $\mathcal{G}$-modules associated with the vertex $i$ for any $1 \leq i \leq m$, respectively. The $m \times n$ matrix $\tilde{B} = (b_{ij})$ associated with $\tilde{Q}$ is defined by

$$
b_{ij} = \dim \text{End}_{\mathcal{G}}(S_i) \text{Ext}_{\mathcal{G}}^1(S_i, S_j) - \dim \text{End}_{\mathcal{G}}(S_j) \text{Ext}_{\mathcal{G}}^1(S_j, S_i)
$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. Note that the upper $n \times n$ part of $\tilde{B}$ is $B$. The left $m \times n$ submatrix of the identity matrix of size $m \times m$ is denoted by $\tilde{I}$.

Assume that there exists some skew-symmetric integer matrix $\Lambda$ of size $m \times m$ such that

$$\Lambda(-\tilde{B}) = \begin{pmatrix} D_n & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix},$$

where $D_n = \text{diag}(d_1, \ldots, d_n)$ with $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. Note that if we enlarge the quiver $Q$ to $\tilde{Q}$, where we attach principal frozen vertices to the acyclic valued quiver $Q$, such a matrix $\Lambda$ always exists [25].

The quantum cluster algebra associated with this obtained compatible pair $(\Lambda, \tilde{B})$ is simply denoted by $A_q(Q)$.

Let $\tilde{R} = \tilde{R}_{\tilde{Q}} = (\tilde{r}_{ij})$ be the $m \times n$ matrix with $\tilde{r}_{ij} := \dim \text{End}_{\mathcal{G}}(S_i) \text{Ext}_{\mathcal{G}}^1(S_i, S_j) - \dim \text{End}_{\mathcal{G}}(S_j) \text{Ext}_{\mathcal{G}}^1(S_j, S_i)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Define $\tilde{R} := \tilde{R}_{\tilde{Q}^\text{opp}}$. Note that the principal part of $\tilde{R}$ is $R$, $\tilde{B} = \tilde{R} - R$ and $B = R - R$.

Let $C_{\tilde{Q}}$ be the cluster category of the acyclic valued quiver $\tilde{Q}$ with the shift functor [1] and the Auslander-Reiten translation functor $\tau$. For more details about the cluster category, the readers can refer to [3,21]. Note that each object $M$ of $C_{\tilde{Q}}$ can be uniquely decomposed up to isomorphism as follows:

$$M = M_0 \oplus P_M[1],$$

where $M_0$ is a $\tilde{G}$-module and $P_M$ is a projective $\tilde{G}$-module. Let $P_M = \bigoplus_{1 \leq i \leq m} m_i P_i$. The definition of the dimension vector $\underline{\dim}$ on $\tilde{G}$-modules can be extended on objects of $C_{\tilde{Q}}$ by defining

$$\underline{\dim} M = \underline{\dim} M_0 - (m_i)_{1 \leq i \leq m}.$$

In the following, we denote the dimension vector of an $\mathcal{G}$-module $X$ by the corresponding underlined lower case letter $x$ viewed as a column vector in $\mathbb{Z}^n$. For simplifying notations, we write $\text{Ext}^1(M, N) := \text{Ext}^1_{\mathcal{G}}(M, N)$ and $\text{Hom}(M, N) := \text{Hom}_G(M, N)$. The Euler form on $\tilde{G}$-modules $M$ and $N$ is given by

$$(M, N) = \dim \text{Hom}(M, N) - \dim \text{Ext}^1(M, N).$$

Let $A_q(Q)$ be the specialization of the quantum cluster algebra $A_q(Q)$ by evaluating $q = q$. For an $\mathcal{G}$-module $N$, we denote its socle and radical by $\text{Soc} N$ and $\text{rad} N$, respectively. Let $M$ be an $\mathcal{G}$-module and $I$ an injective $\tilde{G}$-module. Then the quantum cluster character of $A_q(Q)$ is defined as

$$X_M |_{I[-1]} = \sum X_M \left| \sum \left\{ \text{Gr}_x M | q^{-\frac{1}{2}(x, m - z, \tilde{I} - \tilde{r}) + \dim \text{Soc} I} \right. \right.$$

where $\dim I = i$, $\dim M = m$ and $\text{Gr}_x M$ is the Grassmannian of sub-representations of $M$ with the dimension vector $x$. Note that for any projective $\tilde{G}$-module $P$ and injective $\tilde{G}$-module $I$ with $\text{Soc} I = P/\text{rad} P$, we have

$$X_P |_{I[1]} = X_{\tau P} = X^{\dim P/\text{rad} P} = X^{\dim \text{Soc} I} = X_{I[-1]} = X_{\tau I}.$$

The main results in [20,21] show that $A_q(Q)$ as the $\mathbb{ZQ}$-subalgebra of $\mathcal{F}_q$ is generated by cluster variables

$$\{X_M | M \text{ is an indecomposable rigid } \mathcal{G} \text{-module} \} \cup \{X_{I_{i[-1]}[1]} | 1 \leq i \leq n \}.$$

Let $M$ and $N$ be two $\mathcal{G}$-modules and $\theta : N \rightarrow \tau M$ be a morphism. We can deduce an exact sequence

$$0 \rightarrow D \rightarrow N \xrightarrow{\theta} \tau M \rightarrow \tau A \oplus I \rightarrow 0,$$

where $D = \ker \theta$, $\tau A \oplus I = \text{coker} \theta$, $I$ is an injective $\tilde{G}$-module, and $A$ and $M$ have the same maximal projective summand.

The following theorems are the generalizations of those in [11,20] to acyclic valued quivers.
**Theorem 2.2** (See [21, Theorem 4.5]). Assume that $M$ and $N$ are $\mathcal{G}$-modules with a unique (up to scalar) nontrivial extension $E \in \text{Ext}^1(M, N)$; in particular $\dim_{\text{End}(M)} \text{Ext}^1(M, N) = 1$. Let $\theta \in \text{Hom}(N, \tau M)$ be the equivalent morphism with $A, D$ and $I$ described as above. Furthermore, assume that $\text{Hom}(A \oplus D, I) = 0 = \text{Ext}^1(A, D)$. Then we have

$$X_M X_N = q^{-\frac{1}{2} \Lambda((I-R)^m) \mu(I-R)_{1}} X_E + q^{-\frac{1}{2} \Lambda((I-R)^m) \mu(I-R)_{1}} + \frac{1}{2}(M, N) - \frac{1}{2}(A, D) X_{D \oplus A \oplus I[-1]}.$$

Let $M$ be an $\mathcal{G}$-module and $I$ be an injective $\tilde{\mathcal{G}}$-module. Let $\nu$ be the Nakayama functor on $\tilde{\mathcal{G}}$-modules and write $P = \nu^{-1}(I)$. From morphisms $f : M \to I$ and $g : P \to M$, we can deduce two exact sequences

$$0 \to G \to M \xrightarrow{f} I \to I' \to 0$$

and

$$0 \to P' \to P \xrightarrow{g} M \to F \to 0,$$

where $G = \text{Ker} f$, $I' = \text{coker} f$ is an injective $\tilde{\mathcal{G}}$-module, $P' = \text{Ker} g$ is a projective $\tilde{\mathcal{G}}$-module and $F = \text{coker} g$.

**Theorem 2.3** (See [21, Theorem 4.8]). Let $M$ be an $\mathcal{G}$-module, and $I$ and $P$ be $\tilde{\mathcal{G}}$-modules defined as above. Assume that there exist unique (up to scalar) morphisms $f \in \text{Hom}(M, I)$ and $g \in \text{Hom}(P, M)$; in particular,

$$\dim_{\text{End}(I)} \text{Hom}(M, I) = \dim_{\text{End}(P)} \text{Hom}(P, M) = 1.$$

Define $F$, $G$, $I'$ and $P'$ as above and assume further that $\text{Hom}(P', F) = \text{Hom}(G, I') = 0$. Then we have

$$X_M X_{I[-1]} = q^{-\frac{1}{2} \Lambda((I-R)^m) \mu(I-R)_{1}} X_{G \oplus I[-1]} + q^{-\frac{1}{2} \Lambda((I-R)^m) \mu(I-R)_{1}} - \frac{1}{2} \dim_{\text{End}(I)} X_{F \oplus P'[1]}.$$

### 3 Recursive formulas for the Kronecker quantum cluster algebra with principal coefficients

Consider the Kronecker quiver $Q$, i.e., the quiver of type $\tilde{A}_{1,1}$:

$$\begin{array}{cccc}
1 & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\end{array}$$

It is well known in [13] that indecomposable $kQ$-modules are grouped into (up to isomorphism) three families: the indecomposable preprojective modules with the dimension vector $(l, l+1)$ for $l \in \mathbb{Z}_{\geq 0}$ denoted by $V(-l)$, the indecomposable regular modules (in particular, denote by $R(l)$) the indecomposable regular module with the dimension vector $(l, l)$ for $l \in \mathbb{Z}_{\geq 1}$ from the same homogeneous tube whose mouth has the dimension vector $(1, 1)$, and the indecomposable preinjective modules with the dimension vector $(l+1, l)$ for $l \in \mathbb{Z}_{\geq 0}$ denoted by $V(l+3)$.

The preprojective component looks like the following:

$$\begin{array}{cccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\end{array}$$

The preinjective components looks like the following:

$$\begin{array}{cccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\end{array}$$
We consider the following ice quiver $\tilde{Q}$ with frozen vertices 3 and 4:

```
1 • ➔ ➔ ➔ ➔ • 2
3 •  ↑  ↑  • 4.
```

Thus we have

\[
\begin{align*}
\tilde{R}' &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\tilde{I} - \tilde{R} &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

An easy calculation shows that the following skew-symmetric $4 \times 4$ integer matrix

\[
\Lambda = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 \end{pmatrix}
\]

satisfies

\[
\Lambda(-\tilde{B}) = \begin{pmatrix} D_2 \\ 0 \end{pmatrix},
\]

where

\[
D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The quantum cluster algebra $A_q(Q)$ associated with this pair $(\Lambda, \tilde{B})$ is called the Kronecker quantum cluster algebra with principal coefficients. We denote the coefficients by $y_1 = X_{I_1}[-1]$ and $y_2 = X_{I_2}[-1]$, and all the cluster variables by $x_l = X_{V(l)}$ for $l \in \mathbb{Z}$, where $x_1 = X_{V(1)} = X_{I_1}[-1]$ and $x_2 = X_{V(2)} = X_{I_2}[-1]$ are initial cluster variables.

By the definition of quantum cluster characters, it follows that

\[
X_{R(1)} = X^{(1,-1,1,1)} + X^{(-1,-1,1,0)} + X^{(-1,1,0,0)}.
\]

Therefore, the expression of $X_{R(1)}$ is independent of the choice of the indecomposable regular module with the dimension vector $(1,1)$ and then we denote $X_{R(1)}$ by $X_\delta$. Note that the element $X_\delta$ coincides with the quantum loop element $s_1$ defined in [6] in terms of perfect matchings associated with the Kronecker quantum cluster algebra with principal coefficients. A direct calculation shows that $X_\delta = x_0x_3 - q^2x_1x_2y_2$, and thus $X_\delta$ belongs to $A_q(Q)$ (see [6]).

The $n$-th Chebyshev polynomials of the second kind $S_n(x)$ appearing in [5] are related to the dual semi-canonical basis of the Kronecker cluster algebra. We modify the definition of $S_n(x)$ for $n \in \mathbb{Z}_{\geq 0}$ on $X_\delta$ as follows:

\[
S_0(X_\delta) = 1, \quad S_1(X_\delta) = X_\delta, \quad S_2(X_\delta) = S_1(X_\delta)S_1(X_\delta) - X^{(0,0,1,1)}, \\
S_{n+1}(X_\delta) = S_n(X_\delta)S_1(X_\delta) - X^{(0,0,1,1)}S_{n-1}(X_\delta) \quad \text{for } n \geq 2.
\]

In the following, we denote $S_n(X_\delta)$ by $S_n$ for simplicity. According to the definition of $S_n$ and the fact that $S_1 \in A_q(Q)$, it is easy to see that $S_n \in A_q(Q)$ for any $n \geq 1$.

We start with some technical lemmas.
Lemma 3.1. For any $n \geq 0$ and $m \geq 0$, we have

1. $y_1 S_n = q^{-n} S_n y_1$;
2. $y_2 S_n = q^n S_n y_2$;
3. $X^{(0,0,1,1)} S_n = S_n X^{(0,0,1,1)}$;
4. $S_m S_n = S_n S_m$.

Proof. (1) When $n = 0$, it is trivial. When $n = 1$, we have

$$y_1 S_1 = X^{(0,0,1,0)} \left( X^{(1,-1,1,1)} + X^{(-1,-1,1,0)} + X^{(-1,1,0,0)} \right)$$

$$= q^{1}(X^{(1,-1,1,1)} + X^{(-1,-1,1,0)} + X^{(-1,1,0,0)}) X^{(0,0,1,0)}$$

$$= q^{-1} S_1 y_1.$$

By induction, we have

$$y_1 S_{n+1} = y_1 (S_n S_1 - X^{(0,0,1,1)} S_{n-1})$$

$$= q^{-(n+1)} S_n S_1 y_1 - q^{(n+1)} X^{(0,0,1,1)} y_1 S_{n-1}$$

$$= q^{-(n+1)} S_n S_1 y_1 - q^{-(n+1)} X^{(0,0,1,1)} S_{n-1} y_1$$

$$= q^{-(n+1)} S_{n+1} y_1.$$

(2) The proof is similar to (1).
(3) Note that $X^{(0,0,1,1)} = q y_1 y_2$, and the proof follows from (1) and (2).
(4) According to (3) and by induction, we see that the statement follows immediately.

Lemma 3.2. For $n \geq 0$, we have

1. $S_n^{\bar{\delta}} = S_n$;
2. $X^{(0,0,1,1)} S_n = X^{(0,0,1,1)} S_n$.

Proof. (1) When $n = 0$, it is trivial. When $n = 1$, by the definition of $X_\delta$, we see that $S_1^{\bar{\delta}} = S_1$, i.e., $S_1^{\bar{\delta}} = S_1$. Then by induction and Lemma 3.1, we have

$$S_{n+1} = S_n S_1 - X^{(0,0,1,1)} S_{n-1}$$

$$= S_1 S_n - S_{n-1} X^{(0,0,1,1)}$$

$$= S_1 S_n - S_{n-1} X^{(0,0,1,1)}$$

$$= S_n S_1 - X^{(0,0,1,1)} S_{n-1}$$

$$= S_{n+1}.$$

(2) By (1) and Lemma 3.1, we have $X^{(0,0,1,1)} S_n = S_n X^{(0,0,1,1)} = X^{(0,0,1,1)} S_n$. Hence, the proof is finished.

The following proposition gives the multiplication formulas for the modified Chebyshev polynomials of the second kind on $X_\delta$.

Proposition 3.3. For $n \geq m \geq 1$, we have

$$S_m S_n = S_{n+m} + X^{(0,0,1,1)} S_{n+m-2} + \cdots + X^{(0,0,m-1,m-1)} S_{n-m+2} + X^{(0,0,m,m)} S_{n-m}.$$

Proof. When $m = 1$, the statement is just the definition. Assume that we have the equality for all positive integers $m$ less than or equal to $k$. Then by Lemma 3.1, we have

$$S_{k+1} S_n = (S_1 S_k - X^{(0,0,1,1)} S_{k-1}) S_n$$

$$= S_1 (S_{n+k} + X^{(0,0,1,1)} S_{n+k-2} + \cdots + X^{(0,0,k-1,k-1)} S_{n-k+2} + X^{(0,0,k,k)} S_{n-k})$$

$$- X^{(0,0,1,1)} (S_{n+k-1} + X^{(0,0,1,1)} S_{n+k-3} + \cdots + X^{(0,0,k-2,k-2)} S_{n-k+3} + X^{(0,0,k-1,k-1)} S_{n-k+1})$$

$$= (S_{n+k+1} + X^{(0,0,1,1)} S_{n+k-1}) + X^{(0,0,1,1)} (S_{n+k-1} + X^{(0,0,1,1)} S_{n+k-3}) + \cdots.$$
Theorem 3.6. By Theorem 2.2, we have

\[ X^{(0,0,k-1,k-1)}(S_{n-k+3}) + X^{(0,0,1,1)}S_{n-k+1} + X^{(0,0,k,k)}(S_{n-k+1}) + X^{(0,0,1,1)}S_{n-k-1} \]

\[ - X^{(0,0,1,1)}(S_{n-k-1}) + X^{(0,0,1,1)}S_{n-k-3} + \cdots \]

\[ + X^{(0,0,k-2,k-2)}S_{n-k+3} + X^{(0,0,k-1,k-1)}S_{n-k+1} \]

\[ = S_{n+k+1} + X^{(0,0,1,1)}S_{n+k-1} + \cdots + X^{(0,0,k,k)}S_{n-k+1} + X^{(0,0,k+1,k+1)}S_{n-k-1}. \]

Hence, the proof is finished. \( \square \)

Now, we give a representation-theoretic interpretation of the element \( S_n \) for any \( n \geq 1 \).

**Proposition 3.4.** For \( n \geq 1 \), we have \( S_n = X_{R(n)} \).

**Proof.** The case \( n = 1 \) is trivial. Noting that we have

\[ 0 \to R(1) \to R(n+1) \to R(n) \to 0, \]

since \( \dim \text{Ext}^1(R(n), R(1)) = \dim \text{Hom}(R(1), \tau R(n)) = 1 \), we have

\[ 0 \to R(1) \to \tau R(n) \to \tau R(n-1) \oplus I_3 \oplus I_4 \to 0. \]

By Theorem 2.2, we then have \( X_{R(n)}X_{R(1)} = X_{R(n+1)} + X_{R(n-1)} \oplus (I_3 \oplus I_4)[-1] \). A direct calculation shows that

\[ X^{(0,0,1,1)}X_{\tilde{E}z_1}(\tilde{I}-\tilde{R})(n-1,n-1)^{\tau} = X_{\tilde{E}z_1}(\tilde{I}-\tilde{R})(n-1,n-1)^{\tau} + (0,0,1,1), \]

which implies \( X_{R(n)}X_{R(1)} = X_{R(n+1)} + X^{(0,0,1,1)}X_{R(n-1)}. \)

Note that \( S_nS_1 = S_{n+1} + X^{(0,0,1,1)}S_{n-1} \). Therefore, by induction, we can obtain \( S_{n+1} = X_{R(n+1)}. \) \( \square \)

**Lemma 3.5.** In \( A_\delta(Q) \), we have

1. \( S_1x_0 = x_1 + q^{-\frac{1}{2}}X^{(1,0,0,1)} \).
2. \( S_1x_m = x_{-(m+1)} + X^{(0,0,1,1)}x_{-(m-1)} \) for \( m \geq 1 \).

**Proof.** (1) Noting that we have

\[ 0 \to V(0) \to V(-1) \to R(1) \to 0, \]

since \( \dim \text{Ext}^1(R(1), V(0)) = \dim \text{Hom}(V(0), \tau R(1)) = 1 \), we have

\[ 0 \to V(0) \to \tau R(1) \to I_1 \oplus I_4 \to 0. \]

By Theorem 2.2, we can then obtain \( S_1x_0 = X_{R(1)}X_{V(0)} = x_1 + q^{-\frac{1}{2}}X^{(1,0,0,1)}. \)

(2) Noting that \( \dim \text{Ext}^1(R(1), V(-m)) = \dim \text{Hom}(V(-m), \tau R(1)) = 1 \), we have

\[ 0 \to V(-m) \to V(-m-1) \to R(1) \to 0 \]

and

\[ 0 \to V(-m+1) \to V(-m) \to \tau R(1) \to I_3 \oplus I_4 \to 0. \]

By Theorem 2.2, we have \( S_1x_m = x_{-(m+1)} + q^{-\frac{1}{2}}X_{V(-m+1)} \oplus (I_3 \oplus I_4)[-1] \). Noting that

\[ X^{(0,0,1,1)}X_{\tilde{E}z_1}(\tilde{I}-\tilde{R})(m-1,m)^{\tau} = q^{-\frac{1}{2}}X_{\tilde{E}z_1}(\tilde{I}-\tilde{R})(m-1,m)^{\tau} + (0,0,1,1), \]

which implies \( S_1x_m = x_{-(m+1)} + X^{(0,0,1,1)}x_{-(m-1)} \) for \( m \geq 1 \). \( \square \)

We are now able to prove the following recursive formulas between modified Chebyshev polynomials of the second kind on \( X_\delta \) and cluster variables.

**Theorem 3.6.** In \( A_\delta(Q) \), we have

1. \( S_nx_0 = x_n + q^{-\frac{1}{2}}S_{n-1}X^{(1,0,0,1)} \) for \( n \geq 1 \);
2. \( S_nx_m = x_{-(m+1)} + X^{(0,0,1,1)}S_{n-1}x_{-(m-1)} \) for \( n \geq 1 \) and \( m \geq 1 \);
3. \( x_1S_n = x_{n+1} + q^{-\frac{1}{2}}x_0y_1S_{n-1} \) for \( n \geq 1 \);
4. \( x_mS_n = x_{n+m} + x_{m-1}X^{(0,0,1,1)}S_{n-1} \) for \( n \geq 1 \) and \( m \geq 2 \).
Proof. We only prove (1) and (2). The proofs of (3) and (4) are similar.

(1) When $n = 1$, by Lemma 3.5(1), the statement is true. Assume that we have the equality for all positive integers less than or equal to $k$. Then

$$S_{k+1}x_0 = (S_1S_k - X^{(0,0,1,1)}S_{k-1})x_0$$

$$= S_1(x_{-k} + q^{-\frac{1}{2}}S_{k-1}X^{(0,0,1,1)}(1^{m,0,0,0,0}) - X^{(0,0,1,1)}(x_{-(k-1)} + q^{-\frac{1}{2}}S_{k-2}X^{(1,0,0,0,0)}))$$

$$= x_{-(k-1)} + X^{(0,0,1,1)}x_{-(k-1)} + q^{-\frac{1}{2}}(S_k + X^{(0,0,1,1)}S_{k-2})X^{(1,0,0,1)}$$

$$- X^{(0,0,1,1)}(x_{-(k-1)} + q^{-\frac{1}{2}}S_{k-2}X^{(1,0,0,0,0)}))$$

$$= x_{-(k+1)} + q^{-\frac{1}{2}}S_kX^{(1,0,0,0,1)}.$$

(2) When $n = 1$, by Lemma 3.5(2), the statement is true. Assume that we have the equality for all positive integers less than or equal to $k$. Then

$$S_{k+1}x_{m-n} = (S_1S_k - X^{(0,0,1,1)}S_{k-1})x_{m-n}$$

$$= S_1(x_{-(m+k)} + X^{(0,0,1,1)}S_{k-1}x_{-(m-1)} - X^{(0,0,1,1)}(x_{-(m+k-1)} + X^{(0,0,1,1)}S_{k-2}x_{-(m-1)}))$$

$$= x_{-(m+k+1)} + X^{(0,0,1,1)}x_{-(m+k-1)} + X^{(0,0,1,1)}(S_k + X^{(0,0,1,1)}S_{k-2})x_{-(m-1)}$$

$$- X^{(0,0,1,1)}(x_{-(m+k-1)} + X^{(0,0,1,1)}S_{k-2}x_{-(m-1)}))$$

$$= x_{-(m+k+1)} + X^{(0,0,1,1)}S_kx_{-(m-1)}.$$}

Hence, the proof is finished.

Remark 3.7. (1) In Theorem 3.6(4), let $n = 1$ and $m \geq 2$, and then we obtain $x_mS_1 = x_{m+1} + x_{m-1}X^{(0,0,1,1)}$, which is exactly the equality in [6, Lemma 4.10].

(2) In Theorem 3.6(4), let $m = 2$ and $n \geq 1$, and then we obtain $x_2S_n = x_{n+2} + x_1X^{(0,0,1,1)}S_{n-1}$. Comparing this equality with [6, Lemma 4.16(b)], we see that $S_n = s_n$. Thus, by Proposition 3.4, the elements $X_{R(n)}$ give a representation-theoretic interpretation of those elements $s_n$ which are expressed in terms of perfect matchings in [6].

We now prove the following recursive formulas between cluster variables.

Theorem 3.8. In $A_q(Q)$, we have

1. $x_{-(m+n+2)x-m} = q^{-\frac{1}{2}}S_nX^{(0,0,m,m+1)} + x_{-(n+m+1)x-(m+1)}$ for $m \geq 0$ and $n \geq 0$;

2. $x_{m+n+2}x_{m-n} = q^{-\frac{1}{2}}X^{(0,m,m+1)}X^{(n+1,n+1)}$ for $m \geq 1$ and $n \geq 0$;

3. $x_{0x_{n+1}} = q^{-\frac{1}{2}}x_{n+1}y_2 + S_n$ for $n \geq 0$;

4. $x_{m}x_1 = S_{m+1} + q^{-\frac{1}{2}}x_{-(m-1)x_0y}$ for $m \geq 1$;

5. $x_{m}x_n = qX^{(m-1)x-n-1X^{(n+1,1,1)}} + S_{m+n-2}$ for $m \geq 1$ and $n \geq 2$.

Proof. (1) Note that $\dim \text{Ext}_1^1(V(-m-2), V(-m)) = \dim \text{Hom}(V(-m), \tau V(-m-2)) = 1$. We have

$$0 \to V(-m) \to 2V(-m-1) \to V(-m-2) \to 0$$

and

$$0 \to V(-m) \to \tau V(-m-2) \to mI_4 \oplus (m+1)I_4 \to 0.$$
By induction, we then have

\[ q^{-\frac{1}{2}} S_{n+1} X^{(0,0,m,m+1)} = q^{-\frac{1}{2}} S_1 S_n X^{(0,0,m,m+1)} - q^{-\frac{1}{2}} X^{(0,0,1,1)} S_{n-1} X^{(0,0,m,m+1)} \]

\[ = S_1 (x_{-(m+n+2)}x_m - x_{-(m+n+1)}x_{-(m+1)}) \]

\[ - X^{(0,0,1,1)} (x_{-(m+n+2)}x_m - x_{-(m+n)}x_{-(m+1)}) \]

\[ = (x_{-(m+n+2)}x_m - x_{-(m+n+1)}x_{-(m+1)}) \]

\[ - X^{(0,0,1,1)} (x_{-(m+n+1)}x_m - x_{-(m+n)}x_{-(m+1)}) \]

\[ = x_{-(m+n+3)}x_m - x_{-(m+n+2)}x_{-(m+1)}. \]

(2) When \( n = 0 \), the conclusion follows from Theorems 2.2 and 2.3. When \( n = 1 \), we have

\[ x_{m+3} x_m = (S_1 x_{m+2} - X^{(0,0,1,1)} x_{m+1}) x_m \]

\[ = S_1 (x^2_{m+1} + q^{\frac{1}{2}} X^{(0,0,m,m-1)}) x_m = (x_{m+2} + X^{(0,0,1,1)} x_{m+1}) x_m + q^{\frac{1}{2}} S_1 X^{(0,0,m,m-1)} - X^{(0,0,1,1)} x_{m+1} x_m \]

\[ = x_{m+2} x_{m+1} + q^{\frac{1}{2}} S_1 X^{(0,0,m,m-1)}. \]

By induction, we then have

\[ q^{\frac{1}{2}} S_{n+1} X^{(0,0,m,m-1)} = q^{\frac{1}{2}} S_1 S_n X^{(0,0,m,m-1)} - q^{\frac{1}{2}} X^{(0,0,1,1)} S_{n-1} X^{(0,0,m,m-1)} \]

\[ = S_1 (x_{n+m+2} x_m - x_{n+m+1} x_{m+1}) \]

\[ - X^{(0,0,1,1)} (x_{n+m+1} x_m - x_{n+m} x_{m+1}) \]

\[ = (x_{n+m+3} + X^{(0,0,1,1)} x_{n+m+1}) x_m - (x_{n+m+2} + X^{(0,0,1,1)} x_{n+m}) x_{m+1} \]

\[ - X^{(0,0,1,1)} (x_{n+m+1} x_m - x_{n+m} x_{m+1}) \]

\[ = x_{n+m+3} x_m - x_{n+m+2} x_{m+1}. \]

(3) When \( n = 0 \), the conclusion follows from Theorem 2.3. When \( n = 1 \), the equality follows from the definition of \( S_1 \). According to Lemma 3.1 and by induction, we have

\[ S_{n+1} = S_n S_1 - X^{(0,0,1,1)} S_{n-1} \]

\[ = (x_0 x_{n+2} - q^{\frac{2n+1}{2}} x_1 x_{n+1} y_2) S_1 - (x_0 x_{n+1} - q^{\frac{2n-1}{2}} x_1 x_{n} y_2) X^{(0,0,1,1)} \]

\[ = x_0 (x_{n+3} + q x_{n+1} y_1 y_2) - q^{\frac{2n+1}{2}} x_1 x_{n+1} S_1 y_2 - (x_0 x_{n+1} - q^{\frac{2n-1}{2}} x_1 x_{n} y_2) q y_1 y_2 \]

\[ = x_0 x_{n+3} + q x_0 x_{n+1} y_1 y_2 - q^{\frac{2n+3}{2}} x_1 (x_{n+2} + q x_{n} y_1 y_2) y_2 - (x_0 x_{n+1} - q^{\frac{2n-1}{2}} x_1 x_{n} y_2) q y_1 y_2 \]

\[ = x_0 x_{n+3} - q^{2n+3} x_1 x_{n+2} + q^{2n+3} x_1 x_{n} y_1 y_2 + q^{2n+1} x_1 x_{n} y_2 y_2 \]

\[ = x_0 x_{n+3} - q^{2n+3} x_1 x_{n+2}. \]

(4) When \( m = 1 \), the conclusion follows from Theorem 2.3. When \( m = 2 \), we have

\[ x_{-2} = (S_1 x_{-1} - X^{(0,0,1,1)} x_0) x_1 \]

\[ = S_1 (1 + q^{\frac{1}{2}} x_0^2 y_1) - X^{(0,0,1,1)} x_0 x_1 \]

\[ = S_1 + q^{\frac{1}{2}} (x_{-1} + q^{\frac{1}{2}} x^{(1,0,0,1)} x_0 y_1) x_1 y_1 - X^{(0,0,1,1)} x_0 x_1 \]

\[ = S_1 + q^{\frac{1}{2}} x_{-1} x_0 y_1 + x_1 y_1 x_0 y_1 - X^{(0,0,1,1)} x_0 x_1 \]

\[ = S_1 + q^{\frac{1}{2}} x_{-1} x_0 y_1. \]
By induction, we have

\[ x_{-(m+1)}x_1 = (S_1x_m - X^{(0,0,1,1)}x_{-(m-1)})x_1 = S_1(S_{m-1} + qx^{(0,0,1,1)}x_{-(m-1)}y_1 - X^{(0,0,1,1)}(S_m + qx^{(0,0,1,1)}x_{-(m-2)}y_1) = S_m + X^{(0,0,1,1)}(S_m + qx^{(0,0,1,1)}x_{-(m-2)}y_1 - X^{(0,0,1,1)}S_{m-2} + qx^{(0,0,1,1)}x_{-(m-2)}y_1) = S_m + qx^{1/2}x_{m-1}y_1. \]

(5) When \( m = 1 \) and \( n \geq 2 \), we need to prove \( x_1x_n = qx_0x_{n-1}X^{(0,0,1,1)} + S_{n-1}. \) When \( n = 2 \), by Lemma 3.1, we have

\[ x_1x_2 = x_1(x_1S_1 - qx^{1/2}x_0y_1) = (qx_0x_{n-1}X^{(0,0,1,1)} + S_{n-1})S_1 - (qx_0x_{n-2}X^{(0,0,1,1)} + S_{n-2})X^{(0,0,1,1)} = qx_0(x_{n-1}S_1X^{(0,0,1,1)} + S_n + S_{n-2}X^{(0,0,1,1)} - (qx_0x_{n-2}X^{(0,0,1,1)} + S_{n-2})X^{(0,0,1,1)} = qx_0(x_n + x_{n-2}X^{(0,0,1,1)}X^{(0,0,1,1)} + S_n + S_{n-2}X^{(0,0,1,1)} = (qx_0x_{n-2}X^{(0,0,1,1)} + S_{n-2})X^{(0,0,1,1)} = qx_0x_1X^{(0,0,1,1)} + S_n. \]

By induction, we have

\[ x_{n+1} = x_{n-1}(x_nS_1 - qx^{1/2}x_0y_1) = (qx_0x_{n-1}X^{(0,0,1,1)} + S_{n-1})S_1 - (qx_0x_{n-2}X^{(0,0,1,1)} + S_{n-2})X^{(0,0,1,1)} = qx_0(x_{n-1}S_1X^{(0,0,1,1)} + S_n + S_{n-2}X^{(0,0,1,1)} - (qx_0x_{n-2}X^{(0,0,1,1)} + S_{n-2})X^{(0,0,1,1)} = qx_0(x_n + x_{n-2}X^{(0,0,1,1)}X^{(0,0,1,1)} + S_n + S_{n-2}X^{(0,0,1,1)} = (qx_0x_{n-2}X^{(0,0,1,1)} + S_{n-2})X^{(0,0,1,1)} = qx_0x_nX^{(0,0,1,1)} + S_n. \]

Now we proceed with the induction on \( n \). We have

\[ x_{-(m+1)}x_n = (S_1x_m - X^{(0,0,1,1)}x_{-(m-1)})x_n = S_1(qx_{-(m-1)}x_{n-1}X^{(0,0,1,1)} + S_{m+n-2}) - X^{(0,0,1,1)}(qF_{-(m-2)}x_{n-1}X^{(0,0,1,1)} + S_{m+n-3}) = q(x_{-(m-1)}x_{n-1}X^{(0,0,1,1)} + S_{m+n-1}) + X^{(0,0,1,1)}S_{m+n-3} - X^{(0,0,1,1)}(qF_{-(m-2)}x_{n-1}X^{(0,0,1,1)} + S_{m+n-3}) = qx_{-(m-1)}x_{n-1}X^{(0,0,1,1)} + S_{m+n-1}. \]

Thus, the proof is completed.

\[ \square \]

**Remark 3.9.** According to [6, Definition 4.15, Remark 4.18 and Theorem 4.20], we see that the equality in [6, Lemma 4.16(a)] is exactly \( x_{n+3}x_1 = q^{1/2}S_{n+1} + x_{n+2}x_2 \), which is a special case in Theorem 3.8(2) for \( m = 1 \).

In order to study the atomic bases in the rank 2 cluster algebras of affine types, Sherman and Zelevinsky [22] introduced the \( n \)-th Chebyshev polynomials of the first kind \( F_n(x) \). We modify the definition of \( F_n(x) \) for \( n \in \mathbb{Z}_{\geq 0} \) on \( X_\delta \) as follows:

\[ F_0(X_\delta) = 1, \quad F_1(X_\delta) = X_\delta, \quad F_2(X_\delta) = F_1(X_\delta)F_1(X_\delta) - 2X^{(0,0,1,1)}, \quad F_{n+1}(X_\delta) = F_n(X_\delta)F_1(X_\delta) - X^{(0,0,1,1)}F_{n-1}(X_\delta) \quad \text{for } n \geq 2. \]

In the following, we denote \( F_n(X_\delta) \) by \( F_n \) for simplicity. According to the definition of \( F_n \) and the fact that \( F_1 \in \mathcal{A}_q(Q) \), it is easy to see \( F_n \in \mathcal{A}_q(Q) \) for any \( n \geq 1 \).
Lemma 3.10. In $\mathcal{A}_q(Q)$, we have
(1) $F_m x_0 = x_{-m} + q \frac{2^{m-1}}{2} y_2 x_m$ for $m \geq 1$;
(2) $F_m x_{-1} = x_{-(m+1)} + q \frac{2}{2} X^{(0,0,1,2)} x_{m-1}$ for $m \geq 2$;
(3) $F_2 x_{-2} = x_{-4} + X^{(0,0,2,2)} x_0$ and $F_m x_{-2} = x_{-(m+2)} + q \frac{2^{m-1}}{2} X^{(0,0,2,3)} x_{m-2}$ for $m \geq 3$.

Proof. (1) The case $m = 1$ follows from Lemma 3.5(1). When $m = 2$, we have

$$F_2 x_0 = X_3^2 x_0 - 2 X^{(0,0,1,1)} x_0$$

$$= X_3 (x_{-1} + q \frac{2}{2} y_2 x_1) - 2 X^{(0,0,1,1)} x_0$$

$$= x_{-2} + X^{(0,0,1,1)} x_0 + q \frac{2}{2} y_2 (x_2 + q \frac{1}{2} y_1 x_0) - 2 X^{(0,0,1,1)} x_0$$

$$= x_{-2} + q \frac{2}{2} y_2 x_2.$$ 

When $m = 3$, we have

$$F_3 x_0 = (X_4 F_2 - X^{(0,0,1,1)} X_3) x_0$$

$$= X_3 (x_{-2} + q \frac{1}{2} y_2 x_2) - X^{(0,0,1,1)} (x_{-1} + q \frac{1}{2} y_2 x_1)$$

$$= x_{-3} + X^{(0,0,1,1)} x_{-1} + q \frac{2}{2} y_2 (x_3 + X^{(0,0,1,1)} x_1) - X^{(0,0,1,1)} (x_{-1} + q \frac{1}{2} y_2 x_1)$$

$$= x_{-3} + q \frac{2}{2} y_2 x_3.$$ 

Suppose that $F_m x_0 = x_{-m} + q \frac{2^{m-1}}{2} y_2 x_m$. Then we have

$$F_{m+1} x_0 = (X_4 F_m - X^{(0,0,1,1)} F_{m-1}) x_0$$

$$= X_4 (x_{-m} + q \frac{2^{m-1}}{2} y_2 x_m) - X^{(0,0,1,1)} F_{m-1} x_0$$

$$= x_{-(m+1)} + X^{(0,0,1,1)} x_{-(m+1)} + q \frac{2^{m+1}}{2} y_2 (x_{m+1} + X^{(0,0,1,1)} x_{m+1})$$

$$- X^{(0,0,1,1)} (x_{-(m+1)} + q \frac{2^{m+1}}{2} y_2 x_{m+1})$$

$$= x_{-(m+1)} + q \frac{2^{m+1}}{2} y_2 x_{m+1}.$$ 

(2) When $m = 2$, we have

$$F_2 x_{-1} = X_3^2 x_{-1} - 2 X^{(0,0,1,1)} x_{-1}$$

$$= X_3 (x_{-2} + X^{(0,0,1,1)} x_0) - 2 X^{(0,0,1,1)} x_{-1}$$

$$= x_{-3} + X^{(0,0,1,1)} x_{-1} + X^{(0,0,1,1)} (x_{-1} + q \frac{1}{2} y_2 x_1) - 2 X^{(0,0,1,1)} x_{-1}$$

$$= x_{-3} + q \frac{2}{2} X^{(0,0,1,2)} x_1.$$ 

When $m = 3$, we have

$$F_3 x_{-1} = X_4 F_2 x_{-1} - X^{(0,0,1,1)} X_3 x_{-1}$$

$$= X_3 (x_{-3} + q \frac{1}{2} X^{(0,0,1,2)} x_1) - X^{(0,0,1,1)} (x_{-2} + X^{(0,0,1,1)} x_0)$$

$$= x_{-4} + X^{(0,0,1,1)} x_{-2} + q \frac{2}{2} X^{(0,0,1,2)} (x_2 + q \frac{1}{2} y_1 x_0) - X^{(0,0,1,1)} (x_{-2} + X^{(0,0,1,1)} x_0)$$

$$= x_{-4} + q \frac{2}{2} X^{(0,0,1,2)} x_2.$$ 

Suppose that $F_m x_{-1} = x_{-(m+1)} + q \frac{2^{m-1}}{2} X^{(0,0,1,2)} x_{m-1}$. Then we have

$$F_{m+1} x_{-1} = X_4 F_m x_{-1} - X^{(0,0,1,1)} F_{m-1} x_{-1}$$

$$= X_4 (x_{-(m+1)} + q \frac{2^{m-1}}{2} X^{(0,0,1,2)} x_{m-1}) - X^{(0,0,1,1)} (x_{m-1} + q \frac{2^{m-1}}{2} X^{(0,0,1,2)} x_{m-2})$$

$$= x_{-(m+2)} + X^{(0,0,1,1)} x_{m-1} + q \frac{2^{m+1}}{2} X^{(0,0,1,2)} (x_m + X^{(0,0,1,1)} x_{m-2})$$

$$- X^{(0,0,1,1)} (x_{m-1} + q \frac{2^{m-1}}{2} X^{(0,0,1,2)} x_{m-2})$$

$$= x_{m-1} + q \frac{2^{m+1}}{2} X^{(0,0,1,2)} x_{m-2}.$$ 

Lemma 3.11. In

\begin{align*}
\text{(3) The first equality can be calculated as follows:}
F_2x_2 &= X_3^2 x_2 - 2X^{(0,0,1,1)}x_2 \\
&= X_3(x_3 + X^{(0,0,1,1)}x_3) - 2X^{(0,0,1,1)}x_2 \\
&= x_4 + X^{(0,0,1,1)}(x_2 - 2X^{(0,0,1,1)}x_0) - 2X^{(0,0,1,1)}x_2 \\
&= x_4 + X^{(0,0,2,2)}x_0.
\end{align*}

When \( m = 3 \), we have

\begin{align*}
F_3x_2 &= X_3 F_2 x_2 - X^{(0,0,1,1)} X_3 x_2 \\
&= X_3(x_4 + X^{(0,0,2,2)}x_0) - X^{(0,0,1,1)}(x_3 + X^{(0,0,1,1)}x_1) \\
&= x_5 + X^{(0,0,1,1)}x_3 + X^{(0,0,2,2)}(x_1 - 2X^{(0,0,1,1)}x_1) - X^{(0,0,1,1)}(x_3 + X^{(0,0,1,1)}x_1) \\
&= x_5 + q^{-\frac{3}{2}} X^{(0,0,2,3)}x_1.
\end{align*}

When \( m = 4 \), we have

\begin{align*}
F_4x_2 &= X_4 F_3 x_2 - X^{(0,0,1,1)} F_2 x_2 \\
&= X_4(x_5 + q^{-\frac{3}{2}} X^{(0,0,2,3)}x_3) - X^{(0,0,1,1)}(x_4 + X^{(0,0,2,2)}x_0) \\
&= x_6 + X^{(0,0,1,1)}x_4 + q^{-\frac{3}{2}} X^{(0,0,2,3)}(x_2 + q^{\frac{3}{2}} x_1) - X^{(0,0,1,1)}(x_4 + X^{(0,0,2,2)}x_0) \\
&= x_6 + q^{-\frac{7}{2}} X^{(0,0,2,3)}x_2.
\end{align*}

Suppose that \( F_m x_2 = x_{-(m+2)} + q^{-\frac{2m+1}{2}} X^{(0,0,2,3)}x_{m-2} \). Then we have

\begin{align*}
F_{m+1} x_2 &= X_3 F_m x_2 - X^{(0,0,1,1)} F_{m-1} x_2 \\
&= X_3(x_{-(m+3)} + q^{-\frac{2m+1}{2}} X^{(0,0,2,3)}x_{m-3}) - X^{(0,0,1,1)}(x_{-(m+1)} + q^{-\frac{2m+3}{2}} X^{(0,0,2,3)}x_{m-3}) \\
&= X^{(0,0,1,1)}(x_{-(m+3)} + q^{-\frac{2m+1}{2}} X^{(0,0,2,3)}x_{m-3}) \\
&= x_{-(m+3)} + q^{-\frac{2m+1}{2}} X^{(0,0,2,3)}x_{m-3}.
\end{align*}

Hence, the proof is finished. \( \square \)

Similarly, we have the following results.

Lemma 3.11. In \( A_q(Q) \), we have

\begin{enumerate}
\item \( x_1 F_m = x_{m+1} + q^{-\frac{2m-1}{2}} x_{-(m-1)}y_1 \) for \( m \geq 1 \);
\item \( x_2 F_m = x_{m+2} + q^{-\frac{2m-1}{2}} x_{-(m-2)}X^{(0,0,2,1)} \) for \( m \geq 2 \).
\end{enumerate}

We can now prove the multiplication formulas between the modified Chebyshev polynomials of the first kind on \( X_q \) and cluster variables.

Theorem 3.12. In \( A_q(Q) \), we have

\begin{enumerate}
\item \( F_m x_n = \begin{cases} 
  x_{-(n+m)} + X^{(0,0,m,m)} x_{-(n-m)} & \text{for } 1 \leq m \leq n, \\
  x_{-(n+m)} + q^{-\frac{2m-1}{2}} X^{(0,0,n,m+1)} x_{m-n} & \text{for } n \geq 0, \ n+1 \leq m; 
\end{cases} \)
\item \( x_n F_m = \begin{cases} 
  x_{n+m} + x_{m-m} X^{(0,0,m,m)} & \text{for } 1 \leq m < n, \\
  x_{n+m} + q^{-\frac{2m-1}{2}} x_{-(m-n)} X^{(0,0,n,n-1)} & \text{for } 1 \leq n \leq m. 
\end{cases} \)
\end{enumerate}
Proof. (1) The cases where \( n = 0, n = 1 \) and \( n = 2 \) follow from Lemmas 3.5 and 3.10. Assume that \( n \geq 2 \). Then we have \( X_0 x_{-n} = x_{-(n+2)} + X^{(0,0,1,1)} x_{-n} \) and

\[
F_2 x_{-(n+1)} = X_0^2 x_{-(n+1)} - 2 X^{(0,0,1,1)} x_{-(n+1)}
= X_0 (x_{-(n+2)} + X^{(0,0,1,1)} x_{-n}) - 2 X^{(0,0,1,1)} x_{-(n+1)}
\]

\[
= x_{-(n+3)} + X^{(0,0,1,1)} x_{-(n+1)} + X^{(0,0,1,1)} x_{-(n+1)} - 2 X^{(0,0,1,1)} x_{-(n+1)}
= x_{-(n+3)} + X^{(0,0,2,2)} x_{-(n-1)}.
\]

Suppose that \( F_k x_{-(n+1)} = x_{-(n+1+k)} + X^{(0,0,k,k)} x_{-(n+1-k)} \) for \( k \leq n \). Then we have

\[
F_{k+1} x_{-(n+1)} = (X_0 F_k - X^{(0,0,1,1)} F_{k-1}) x_{-(n+1)}
= X_0 (x_{-(n+1+k)} + X^{(0,0,k,k)} x_{-(n+1-k)}) - X^{(0,0,1,1)} (x_{-(n+k)} + X^{(0,0,k-1,k-1)} x_{-(n+2-k)})
\]

\[
= x_{-(n+2+k)} + X^{(0,0,1,1)} x_{-(n+k)} + X^{(0,0,k,k)} (x_{-(n+2-k)} + X^{(0,0,1,1)} x_{-(n-k)})
- X^{(0,0,1,1)} (x_{-(n+k)} + X^{(0,0,k-1,k-1)} x_{-(n+2-k)})
= x_{-(n+2+k)} + X^{(0,0,k+1,k+1)} x_{-(n-k)}.
\]

Thus,

\[
F_{n+2} x_{-(n+1)} = (X_0 F_n - X^{(0,0,1,1)} F_{n-1}) x_{-(n+1)}
= X_0 (x_{-(2n+2)} + X^{(0,0,n+1,n+1)} x_{0}) - X^{(0,0,1,1)} (x_{-(2n+1)} + X^{(0,0,n,n)} x_{-1})
\]

\[
= x_{-(2n+3)} + X^{(0,0,1,1)} x_{-(2n+1)} + X^{(0,0,n,n)} x_{-1}
- X^{(0,0,1,1)} (x_{-(2n+1)} + X^{(0,0,n,n)} x_{-1})
= x_{-(2n+3)} + q^{-\frac{2n+3}{2}} X^{(0,0,n+1,n+2)} x_{1}.
\]

Similarly, we have

\[
F_{n+3} x_{-(n+1)} = x_{-(2n+4)} + q^{\frac{2n+3}{2}} X^{(0,0,n+1,n+2)} x_{2}
\]

and

\[
F_{n+4} x_{-(n+1)} = x_{-(2n+5)} + q^{\frac{2n+3}{2}} X^{(0,0,n+1,n+2)} x_{3}.
\]

Now, for \( m \geq n+2 \), suppose that

\[
F_m x_{-(n+1)} = x_{-(m+n+1)} + q^{\frac{2m-1}{2}} X^{(0,0,n+1,n+2)} x_{m-n-1}.
\]

Then we have

\[
F_{m+1} x_{-(n+1)} = (X_0 F_m - X^{(0,0,1,1)} F_{m-1}) x_{-(n+1)}
= X_0 (x_{-(m+n+1)} + q^{\frac{2m-1}{2}} X^{(0,0,n+1,n+2)} x_{m-n-1})
- X^{(0,0,1,1)} (x_{-(m+n)} + q^{\frac{2m-1}{2}} X^{(0,0,n+1,n+2)} x_{m-n-2})
\]

\[
= x_{-(m+n+2)} + X^{(0,0,1,1)} x_{-(m+n)} + q^{\frac{2m+1}{2}} X^{(0,0,n+1,n+2)} (x_{m-n} + X^{(0,0,1,1)} x_{m-n-2})
- X^{(0,0,1,1)} (x_{-(m+n)} + q^{\frac{2m-1}{2}} X^{(0,0,n+1,n+2)} x_{m-n-2})
= x_{-(m+n+2)} + q^{\frac{2m+3}{2}} X^{(0,0,n+1,n+2)} x_{m-n}.
\]

(2) The proof is similar to (1), where we need to use Lemma 3.11 and Theorem 3.6. \( \square \)

**Remark 3.13.** Since all the coefficients appearing in the multiplication formulas are Laurent polynomials of \( q \) which are independent of the choice of the finite field \( \mathbb{F}_q \), generically all the multiplication formulas also hold in \( A_q(Q) \) if we replace \( q \) by the indeterminate \( q \).
4 Bar-invariant positive $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-bases of $\mathcal{A}_q(Q)$

In this section, we use the multiplication formulas established in Section 3 to construct two positive $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-bases.

**Definition 4.1.** An element in $\mathcal{A}_q(Q)$ is called positive if the coefficients of its Laurent expansion with any cluster belong to $\mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$.

**Definition 4.2.** A basis of $\mathcal{A}_q(Q)$ is called a positive $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-basis if its structure constants belong to $\mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$.

Define $S = \{\text{cluster monomials}\} \cup \{S_n \mid n \geq 1\}$.

**Theorem 4.3.** The set $S$ is a bar-invariant positive $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-basis of $\mathcal{A}_q(Q)$.

**Proof.** According to Proposition 3.3, Theorems 3.6 and 3.8 and Lemmas 3.1 and 3.2, we can deduce that $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-bases of $\mathcal{A}_q(Q)$, $\mathcal{A}_q(Q)$.

In order to prove the $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-independency of these elements, we define a partial order $\leq$ on $\mathbb{Z}^2$ as follows: $(r_1, r_2) \leq (s_1, s_2)$ if $r_i \leq s_i$ for $1 \leq i \leq 2$. Moreover if $r_i < s_i$ for some $i$, we write $(r_1, r_2) < (s_1, s_2)$. According to Proposition 3.3, every element $s_i$ has a minimal non-zero term $a_m X^{-d_m R(n)}$, where $a_m \in \mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$. According to Theorem 3.6, we see that every element $x_m$ has a minimal non-zero term $b_n X^{-d_m v(m)}$, where $b_n \in \mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$. It follows from [22, Proposition 3.1] that there exists a bijection between the set of all the minimal non-zero terms in the elements in $S$ and $\mathbb{Z}^2$, which implies that the elements in $S$ are $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-independent.

By Lemma 3.2 and the fact that cluster monomials are bar-invariant, we know that every element in $S$ is bar-invariant. Again from Proposition 3.3, Theorems 3.6 and 3.8 and Lemmas 3.1 and 3.2, we can deduce that the structure constants belong to $\mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$. \hfill $\Box$

**Remark 4.4.** Using the same arguments as [18, Corollary 3.3.10] and [19, Corollary 8.3.3], we see that all the elements in $S$ are positive.

**Definition 4.5.** A basis of $\mathcal{A}_q(Q)$ is called the atomic $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-basis if positive elements in $\mathcal{A}_q(Q)$ are exactly $\mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-combinations of these basis elements.

It follows from the definition that the atomic basis, if it exists, must be a positive basis.

Define $B = \{\text{cluster monomials}\} \cup \{F_n \mid n \geq 1\}$.

**Theorem 4.6.** The set $B$ is the bar-invariant atomic $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-basis of $\mathcal{A}_q(Q)$.

**Proof.** By Theorem 4.3 and the relation between $S_i$ and $F_n$, we find that $B$ is a $\mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-basis. By applying similar discussions as in Theorem 4.3, we can show that the elements in $B$ are bar-invariant. Note that cluster variables are positive (see also Remark 4.4), and then by Theorem 3.12, we see that $F_n$ is a positive element for $n \geq 1$. Thus, all the elements in $B$ are positive.

Let $Y$ be a positive element in $\mathcal{A}_q(Q)$. $Y = \sum_{b \in B} \lambda_b b$ for some $\lambda_b \in \mathbb{Z}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$. We need to prove $\lambda_b \in \mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$. Firstly, we consider the case where the element $b$ is a cluster monomial. Without loss of generality, suppose that $b$ is a cluster monomial in the cluster $\{x_m, x_{m+1}\}$. If $b$ appears in the $\mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$-expansion of some other basis element $b'$ of $B$ associated with the cluster $\{x_m, x_{m+1}\}$, then $b|_{q=1, y_1=1, y_2=1}$ appears in the $\mathbb{Z}_{\geq 0}$-expansion of some other basis element $b'$ $|_{q=1, y_1=1, y_2=1}$ associated with the cluster $\{x_m, x_{m+1}\}$ due to the positivity of the elements in $B$. However, this cannot happen in the corresponding classical cluster algebra (see, for more details, [15, Section 4, Proof of (B3) in Theorem 1.2] or [22, Section 5, Proof of (5.10)]). Thus by the assumption that $Y$ is positive associated with the cluster $\{x_m, x_{m+1}\}$, we have $\lambda_b \in \mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$. Now, we consider the case $b = F_l$ for some $l \geq 1$. By using the same discussions and references as above, we can find a Laurent monomial $Y_l$ in a certain cluster expansion of $F_l$ with coefficients $q^{\pm \frac{1}{2}}y_1^l y_2^l$ for some $l_0, l_1, l_2 \in \mathbb{Z}$, but $Y_l$ does not appear in this cluster expansion of any other basis element in the above sum terms. Thus we have $\lambda_l q^{\pm \frac{1}{2}}y_1^l y_2^l \in \mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$ by using the assumption that $Y_l$ is positive associated with this cluster. It follows that $\lambda_l \in \mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}][y_1^\pm, y_2^\pm]$. The proof is completed. \hfill $\Box$
Remark 4.7. By specializing $q = 1$ in Theorems 4.3 and 4.6, we obtain a positive integral basis and the atomic basis of the classical cluster algebra associated with the Kronecker quiver with principal coefficients. Note that Dupont [14, Theorems 7.3 and 7.4] constructed these bases in the classical Kronecker cluster algebra with opposite principal coefficients.

Acknowledgements Ming Ding was supported by National Natural Science Foundation of China (Grant No. 11771217). Fan Xu was supported by National Natural Science Foundation of China (Grant No. 12031007). We thank the referees for their very helpful comments and suggestions.

References

1. Bai L Q, Chen X Q, Ding M, et al. Cluster multiplication theorem in the quantum cluster algebra of type $\tilde{A}_2^{(2)}$ and the triangular basis. J Algebra, 2019, 533: 106–141
2. Berenstein A, Zelevinsky A. Quantum cluster algebras. Adv Math, 2005, 195: 405–455
3. Buan A B, Marsh B R, Reineke M, et al. Tilting theory and cluster combinatorics. Adv Math, 2006, 204: 572–618
4. Caldero P, Keller B. From triangulated categories to cluster algebras. Invent Math, 2008, 172: 169–211
5. Caldero P, Zelevinsky A. Laurent expansions in cluster algebras via quiver representations. Mosc Math J, 2006, 6: 411–429
6. Canaki I, Lampe P. An expansion formula for type A and Kronecker quantum cluster algebras. J Combin Theory Ser A, 2020, 171: 105132
7. Cerulli Irelli G. Positivity in skew-symmetric cluster algebras of finite type. arXiv:1102.3050, 2011
8. Cerulli Irelli G. Cluster algebras of type $\tilde{A}^{(1)}_2$. Algebr Represent Theory, 2012, 15: 977–1021
9. Chen X Q, Ding M, Zhang H C. The cluster multiplication theorem for acyclic quantum cluster algebras. arXiv:2108.03558, 2021
10. Ding M, Xu F. Bases of the quantum cluster algebra of the Kronecker quiver. Acta Math Sin (Engl Ser), 2012, 28: 1169–1178
11. Ding M, Xu F. A quantum analogue of generic bases for affine cluster algebras. Sci China Math, 2012, 55: 2045–2066
12. Ding M, Xu F, Chen X Q. Atomic bases of quantum cluster algebras of type $\tilde{A}_{2n-1,1}$. J Algebra, 2022, 590: 1–25
13. Dlab V, Ringel C M. Indecomposable Representations of Graphs and Algebras. Memoirs of the American Mathematical Society, vol. 6, no. 173. Providence: Amer Math Soc, 1976
14. Dupont G. Quantized Chebyshev polynomials and cluster characters with coefficients. J Algebraic Combin, 2010, 31: 501–532
15. Dupont G, Thomas H. Atomic bases of cluster algebras of types A and $\tilde{A}$. Proc Lond Math Soc (3), 2013, 107: 825–850
16. Fomin S, Zelevinsky A. Cluster algebras I: Foundations. J Amer Math Soc, 2002, 15: 497–529
17. Hubery A. Acyclic cluster algebras via Ringel-Hall algebras. Http://citeseerx.ist.psu.edu/doc/10.1.1.182.6939, 2005
18. Kimura Y, Qin F. Graded quiver varieties, quantum cluster algebras and dual canonical basis. Adv Math, 2014, 262: 261–312
19. Qin F. $t$-analog of $q$-characters, bases of quantum cluster algebras, and a correction technique. Int Math Res Not IMRN, 2014, 2014: 6175–6232
20. Qin F, Keller B. Quantum cluster variables via Serre polynomials. J Reine Angew Math, 2012, 668: 149–190
21. Rupel D. Quantum cluster characters for valued quivers. Trans Amer Math Soc, 2015, 367: 7061–7102
22. Sherman P, Zelevinsky A. Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. Mosc Math J, 2004, 4: 947–974
23. Xiao J, Xu F. Green’s formula with $C^*$-action and Caldero-Keller’s formula for cluster algebras. In: Representation Theory of Algebraic Groups and Quantum Groups. Progress in Mathematics, vol. 284. Boston: Birkhäuser, 2010, 313–348
24. Xu F. On the cluster multiplication theorem for acyclic cluster algebras. Trans Amer Math Soc, 2010, 362: 753–776
25. Zelevinsky A. Quantum cluster algebras: Oberwolfach talk, February 2005. arXiv:math/0502256, 2005