DUALITY OF ONE-VARIABLE MULTIPLE POLYLOGARITHMS
AND THEIR $q$-ANALOGUES

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Abstract. The duality relation of one-variable multiple polylogarithms was proved by Hirose, Iwaki, Sato and Tasaka by means of iterated integrals. In this paper, we give a new proof using the method of connected sums, which was recently invented by Seki and the author. Interestingly, the connected sum involves the hypergeometric function in its connector. Moreover, we introduce two kinds of $q$-analogues of the one-variable multiple polylogarithms and generalize the duality to them.

1. Introduction

For a tuple of positive integers $k = (k_1, \ldots, k_r)$ with $k_r \geq 2$, let

$$\zeta(k) := \sum_{0 < m_1 < \cdots < m_r} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}$$

be the multiple zeta value of index $k$. There are many studies on the $\mathbb{Q}$-linear relations among multiple zeta values. A remarkable example of such relations is the duality relation

$$(1.1) \quad \zeta(k) = \zeta(k^\dagger).$$

Here, by representing the given index $k$ as

$$k = (1, \ldots, 1, b_1 + 1, \ldots, 1, \ldots, 1, \ldots, 1, b_s + 1)$$

with positive integers $a_1, b_1, \ldots, a_s, b_s$, its dual index $k^\dagger$ is defined by

$$k^\dagger := (1, \ldots, 1, a_s + 1, \ldots, 1, \ldots, 1, 1, 1, \ldots, 1).$$

As is well known (cf. [7, §9]), the duality relation (1.1) is an immediate consequence of the iterated integral expression of multiple zeta values. Recently, Seki and the author [6] gave a new method of proving the duality, which used certain multiple sums (called the connected sums) and made no use of the integrals. An advantage of this method
is that it can be directly generalized to obtain a $q$-analogue. In the present paper, we apply it to multiple polylogarithms and their $q$-analogues.

For an index $k = (k_1, \ldots, k_r)$, not necessarily with $k_r \geq 2$, let

$$\text{Li}_k(z_1, \ldots, z_r) := \sum_{0 < m_1 < \cdots < m_r} \frac{z_1^{m_1} \cdots z_r^{m_r - m_r - 1}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}$$

be the multiple polylogarithm. Moreover, let $I$ be a subset of $\{1, \ldots, r\}$ and assume that $k_r \neq 1$ or $r \in I$. Then we consider a function $\text{Li}_k^I(z)$ of a complex variable $z$ with $|z| < 1$ defined by

$$\text{Li}_k^I(z) := \sum_{0 = m_0 < m_1 < \cdots < m_r} \frac{z^{\sum_{i \in I} (m_i - m_{i-1})}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} (1.2)$$

$$= \text{Li}_k(z_1, \ldots, z_r) \text{ where } z_i = \begin{cases} z & (i \in I), \\ 1 & (i \notin I). \end{cases}$$

We call this function a one-variable multiple polylogarithm or one-variable MPL for short. This name suggests that it is a function of one variable $z$, and also that one or the variable $z$ is assigned to each of $r$ arguments of $\text{Li}_k(z_1, \ldots, z_r)$. Note that we recover the multiple zeta values by putting $I = \emptyset$ (the empty set):

$$\text{Li}_k^\emptyset(z) = \zeta(k).$$

Just as in the case of multiple zeta values, we are interested in $\mathbb{Q}$-linear relations among one-variable MPLs. For example, Hirose–Iwaki–Sato–Tasaka [3] obtained such relations which generalize the duality and sum formulas for multiple zeta values, by using iterated integral representation of one-variable MPLs.

Let us recall the duality formula of Hirose–Iwaki–Sato–Tasaka. Let $A := \mathbb{Q}(e_0, e_1, e_2)$ be the non-commutative polynomial algebra in three indeterminates $e_0$, $e_1$ and $e_2$, and define its linear subspace $A^0 := \mathbb{Q} + e_0 e_1 A + e_1 A e_0 + e_2 A e_0 + e_2 A e_1$. Then we define a linear map $L$ from $A^0$ to the space of holomorphic functions on the unit disc in $\mathbb{C}$ by $L(1) := 1$ and

$$L(e_z e_0^{k_1 - 1} \cdots e_z e_0^{k_r - 1}) := (-1)^r \text{Li}_k^I(z)$$

with $z_i$ as in (1.2).

**Theorem 1.1** (Duality of one-variable MPLs, [3 Theorem 1.1]). Let $\tau : A \to A$ be the anti-automorphism of $\mathbb{Q}$-algebra defined by

$$\tau(e_0) = e_z - e_1, \quad \tau(e_1) = e_z - e_0, \quad \tau(e_z) = e_z.$$

Then we have

$$L(w) = L(\tau(w))$$

(1.3) for any $w \in A^0$.

**Remark 1.2.** In [3], $L$ is defined as a map from $A^0$ to the space of holomorphic functions on $\mathbb{C} \setminus [0, 1]$ (note that our variable $z$ corresponds to their $z^{-1}$). Analytic continuation of $\text{Li}_k^I(z)$ to $\mathbb{C} \setminus [1, \infty)$ is obtained by using an iterated integral expression.
Note that, unlike (1.1), Theorem 1.1 does not give a one-to-one identity of one-variable MPLs in general. For instance, the relation
\[ L(e_1e_0^2) = L((e_z - e_1)^2(e_z - e_0)) \] (1.3) for \( w = e_1e_0^2 \)
amounts to the linear relation
\[ \zeta(3) = \text{Li}_{1,1,1}(z, z, z) - \text{Li}_{1,1,1}(z, 1, z) - \text{Li}_{1,1,1}(1, 1, z) + \text{Li}_{1,1,1}(1, 1, 1) \]
\[ + \text{Li}_{1,2}(z, z) - \text{Li}_{1,2}(z, 1) - \text{Li}_{1,2}(1, z) + \zeta(2) \]
among nine one-variable MPLs. In fact, as is done below, it is possible to rephrase Theorem 1.1 as a one-to-one identity just like (1.1) by considering appropriate linear combinations of one-variable MPLs. This reformulation is necessary for our proof of Theorem 1.1.

We call a pair \( \tilde{k} = (k, \mu) \in \mathbb{Z}_{>0} \times \{0, 1\} \) an augmented positive integer, and a tuple \( \tilde{k} = (k_1, \ldots, k_r) \) of augmented positive integers an augmented index. Such \( \tilde{k} \) is said admissible if \( r = 0 \) (we write \( \tilde{k} = \emptyset \) in this case), or \( r > 0 \) and \( k_r \neq (1, 1) \). For an augmented index \( \tilde{k} = ((k_1, \mu_1), \ldots, (k_r, \mu_r)) \), we define \( w(\tilde{k}) \in \mathcal{A} \) by
\[ w(\tilde{k}) := y_{\mu_1}x^{k_1-1} \cdots y_{\mu_r}x^{k_r-1}, \]
where
\[ x := e_0, \quad y_0 := -e_z, \quad y_1 := e_z - e_1. \]
For \( \tilde{k} = \emptyset \), we understand \( w(\emptyset) = 1 \). Then it is easy to see that \( w(\tilde{k}) \), with \( \tilde{k} \) running over all admissible augmented indices, form a basis of \( \mathcal{A}^0 \). Moreover, for any admissible augmented index \( \tilde{k} \), there exists a unique admissible augmented index \( \tilde{k}^\dagger \), called the dual of \( \tilde{k} \), such that \( \tau(w(\tilde{k})) = w(\tilde{k}^\dagger) \).

For a non-empty admissible index \( k = ((k_1, \mu_1), \ldots, (k_r, \mu_r)) \), we define
\[ \text{Li}(\tilde{k}; z) := \sum_{0 = m_0 < m_1 < \cdots < m_r} \prod_{i=1}^r \frac{\mu_i + (-1)^{\mu_i}z^{m_i - m_{i-1}}}{m_i^{k_i}}, \]
and set \( \text{Li}(\emptyset; z) := 1 \). Then the map \( L \) satisfies (and is determined by) the formula
\[ L(w(\tilde{k})) = \text{Li}(\tilde{k}; z), \]
and hence Theorem 1.1 is equivalent to the following:

**Theorem 1.3.** For any admissible augmented index \( \tilde{k} \), we have
\[ \text{Li}(\tilde{k}; z) = \text{Li}(\tilde{k}^\dagger; z). \]

**Remark 1.4.** Kaneko and Tsumura study a version of multiple zeta values of level two defined by
\[ T(k_1, \ldots, k_r) := 2^r \sum_{0 < m_1 < \cdots < m_r \atop m_i \equiv r \mod 2} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}. \]
In our notation, this can be written as
\[ T(k_1, \ldots, k_r) = \text{Li}((k_1, 1), \ldots, (k_r, 1); -1), \]
and Theorem 1.3 is a generalization of the duality of $T$-values [4, Theorem 3.1].

We prove Theorem 1.3 in §2. In §3, we introduce two kinds of $q$-analogues of $\text{Li}(k; z)$, and prove for each of them a generalization of Theorem 1.3.

2. The proof of Theorem 1.3 via connected sums

Let $F\left(\frac{\alpha, \beta}{\gamma}; z\right)$ denote the hypergeometric series

$$F\left(\frac{\alpha, \beta}{\gamma}; z\right) := \sum_{n=0}^{\infty} \frac{(\alpha)_{n} (\beta)_{n} z^{n}}{(\gamma)_{n} n!}$$

where $(\alpha)_{n} := \alpha(\alpha + 1)\ldots(\alpha + n - 1)$ is the rising factorial. We use the following contiguous relations.

Lemma 2.1. The hypergeometric series satisfies the identities

\begin{align}
(2.1) & \quad F\left(\frac{\alpha, \beta}{\gamma}; z\right) = F\left(\frac{\alpha, \beta + 1}{\gamma}; z\right) - \frac{\alpha}{\gamma} F\left(\frac{\alpha + 1, \beta + 1}{\gamma + 1}; z\right), \\
(2.2) & \quad (\gamma - \alpha) F\left(\frac{\alpha, \beta}{\gamma + 1}; z\right) = \gamma F\left(\frac{\alpha, \beta}{\gamma}; z\right) - \alpha F\left(\frac{\alpha + 1, \beta}{\gamma + 1}; z\right).
\end{align}

Proof. These are well known identities, see e.g. the equations (9.2.13) and (9.2.5) in [5]. Indeed, they are shown by comparing the coefficients of $z^{n}$, that is, by checking

\begin{align}
(\alpha)_{n} (\beta)_{n} \frac{\gamma}{(\gamma + 1)_{n} n!} & = \frac{\alpha (\alpha + 1)_{n-1} (\beta + 1)_{n-1}}{(\gamma + 1)_{n-1} (n-1)!}, \\
(\gamma - \alpha) (\alpha)_{n} (\beta)_{n} \frac{\gamma}{(\gamma + 1)_{n} n!} & = \frac{\alpha (\alpha + 1)_{n} (\beta)_{n}}{(\gamma + 1)_{n} n!} - \frac{\alpha (\alpha + 1)_{n} (\beta)_{n}}{(\gamma + 1)_{n} n!},
\end{align}

respectively. \hfill \Box

From now on, we assume that $z$ is a real number with $0 \leq z < 1$, which is harmless to the proof of Theorem 1.3. The following definition is the core of our proof of Theorem 1.3.

Definition 2.2. Let $\mathbf{k} = ((k_{1}, \mu_{1}), \ldots, (k_{r}, \mu_{s}))$ and $\mathbf{l} = ((l_{1}, \nu_{1}), \ldots, (l_{s}, \nu_{s}))$ be augmented indices (admissible or not). Then we define

$$\tilde{\text{Li}}(\mathbf{k}; z) := \sum_{0=m_{0}<m_{1}<\cdots<m_{r}} \prod_{i=1}^{r} \frac{\mu_{i} + (-1)^{\mu_{i}} z^{m_{i} - m_{i-1}}}{m_{i}^{\mathbf{k}_{i}}} \times \prod_{j=1}^{s} \frac{\nu_{j} + (-1)^{\nu_{j}} z^{n_{j} - n_{j-1}}}{n_{j}^{\mathbf{l}_{j}}} C(m_{r}, n_{s}; z),$$

where

$$C(m, n; z) := \frac{m! n!}{(m + n)!} F\left(\frac{m, n}{m + n + 1}; z\right).$$

By abuse of notation, we also write $\tilde{\text{Li}}(w(\mathbf{k}); w(\mathbf{l}); z)$ for $\tilde{\text{Li}}(\mathbf{k}; \mathbf{l}; z)$. For example, $\tilde{\text{Li}}(y_{1}; y_{0} y_{1}; z)$ means $\tilde{\text{Li}}((2, 1); (1, 0), (1, 1); z)$. 
The value of the series \( \tilde{\text{Li}}(\tilde{k}; \tilde{l}; z) \) is always well defined as a non-negative real number or the positive infinity since, by the assumption \( 0 \leq z < 1 \), all terms are non-negative real numbers. As in [6], we call this series \( \tilde{\text{Li}}(\tilde{k}; \tilde{l}; z) \) the connected sum and the factor \( C(m_r, n_s; z) \) the connector: Notice that, without the connector, the sum splits into two independent sums with respect to \( m_i \)'s and \( n_j \)'s. The connected sum satisfies the obvious symmetry

\[
(2.3) \quad \tilde{\text{Li}}(\tilde{k}; \tilde{l}; z) = \tilde{\text{Li}}(\tilde{l}; \tilde{k}; z),
\]

the boundary condition

\[
(2.4) \quad \tilde{\text{Li}}(\tilde{k}; \emptyset; z) = \tilde{\text{Li}}(\tilde{l}; \emptyset; z) = \tilde{\text{Li}}(\emptyset; \tilde{k}; z),
\]

and the following transport relations:

**Proposition 2.3.** For any augmented indices \( \tilde{k} \) and \( \tilde{l} \), we have

\[
(2.5) \quad \tilde{\text{Li}}(w(\tilde{k})y_0; w(\tilde{l}); z) = \tilde{\text{Li}}(w(\tilde{l})y_0; w(\tilde{k}); z).
\]

If \( \tilde{l} \) is non-empty, we also have

\[
(2.6) \quad \tilde{\text{Li}}(w(\tilde{k})y_1; w(\tilde{l}); z) = \tilde{\text{Li}}(w(\tilde{l})x; w(\tilde{k}); z).
\]

**Proof.** In order to prove the identity (2.5), it suffices to show

\[
(2.7) \quad \sum_{a > m} z^{a-m} \frac{(a-1)! n!}{(a+n)!} F\left( a, n \atop a+n+1 \right) = \sum_{b > n} z^{b-n} \frac{a! n!}{(a+1+n)!} F\left( a+1, n+1 \atop a+1+n+1 \right)
\]

for any integers \( m, n \geq 0 \). By using (2.4), we compute the left hand side of (2.7) as follows:

\[
\sum_{a > m} z^{a-m} \frac{(a-1)! n!}{(a+n)!} F\left( a, n \atop a+n+1 \right) = \sum_{a > m} z^{a-m} \frac{(a-1)! n!}{(a+n)!} \left\{ F\left( a, n+1 \atop a+n+1 \right) - z \frac{a+1+n+1}{a+1+n+2} F\left( a+1, n+1 \atop a+1+n+2 \right) \right\}
\]

\[
= \sum_{a > m} \left\{ z^{a-m} \frac{(a-1)! n!}{(a+n)!} F\left( a, n+1 \atop a+n+1 \right) \right\} - \frac{a! n!}{(a+1+n)!} F\left( a+1, n+1 \atop a+1+n+1 \right)
\]

\[
(2.8) \quad = \frac{m! n!}{(m+n+1)!} F\left( m+1, n+1 \atop m+n+2 \right).
\]

Since the last expression is symmetric with respect to \( m \) and \( n \), we obtain (2.7).

In order to prove the identity (2.6), it suffices to show

\[
(2.9) \quad \sum_{a > m} \frac{1 - z^{a-m}}{a} C(a, n; z) = \frac{1}{n} C(m, n; z)
\]
for any integers \( m \geq 0 \) and \( n > 0 \). By using (2.2), we see that
\[
\sum_{a > m} \frac{1}{a} C(a, n; z) = \sum_{a > m} \frac{(a - 1)! n!}{(a + n)!} \left\{ (a + n) \binom{a}{n} - a \binom{a + 1}{n} \right\}
= \sum_{a > m} \frac{(a - 1)! (n - 1)!}{(a + n)!} \left\{ (a + n) \binom{a}{n} - a \binom{a + 1}{n} \right\}
= \sum_{a > m} \frac{(a - 1)! (n - 1)!}{(a + n)!} \left\{ \frac{(a + n)!}{(a - 1 + n)!} \binom{a}{n} - a \frac{(a + 1)!}{(a + n)!} \binom{a + 1}{n} \right\}
= \frac{m! (n - 1)!}{(m + n)!} \binom{m + 1}{n + 1}.
\]
This and (2.9) imply that
\[
\sum_{a > m} \frac{1}{a} C(a, n; z) = \frac{m! (n - 1)!}{(m + n)!} \binom{m + 1}{n + 1}.
\]
by (2.1) again. Thus we obtain (2.9). □

Now we turn to the proof of Theorem 1.3.

**Proof of Theorem 1.3.** By the transport relations (2.5) and (2.6) combined with the symmetry (2.3), we have
\[
\tilde{L}_i(w(\tilde{k})u; w(\tilde{l}); z) = \tilde{L}_i(w(\tilde{k}); w(\tilde{l})\tau(u); z)
\]
for \( u \in \{x, y_0, y_1\} \). By using it repeatedly, we obtain
\[
\tilde{L}_i(w(\tilde{k}); 1; z) = \tilde{L}_i(1; \tau(w(\tilde{k})); z)
\]
for any admissible augmented index \( \tilde{k} \). By the boundary condition (2.4), this is exactly what we have to show. □

**Remark 2.4.** As a by-product of the above discussion, one sees that
\[
\tilde{L}_i(\tilde{k}; \tilde{l}; z) = \infty \iff \text{one of } \tilde{k} \text{ and } \tilde{l} \text{ is empty and the other is not admissible.}
\]
Indeed, the case in which \( \tilde{k} \) or \( \tilde{l} \) is empty is easy. When both \( \tilde{k} \) and \( \tilde{l} \) are non-empty, one can use transport relations to obtain \( \tilde{L}_i(\tilde{k}; \tilde{l}; z) = \tilde{L}_i(\emptyset; \tilde{l}; z) \) for some admissible \( \tilde{l}' \), which shows that \( \tilde{L}_i(\tilde{k}; \tilde{l}; z) < \infty \).
3. $q$-ANALOGUES

There are quite many versions of $q$-analogues of multiple zeta values and results on them (see \cite{S} Chap. 12 for a general account). The duality relation and its generalization to the Ohno-type sums for the Bradley–Zhao model

\[ \sum_{0 < m_1 < \cdots < m_r} q^{(k_1 - 1)m_1 + \cdots + (k_r - 1)m_r} [m_1^{k_1} \cdots [m_r^{k_r}]. \]

($[m]_q := (1 - q^m)/(1 - q)$ denotes the $q$-integer) was proved by Bradley \cite{B} Theorem 5, and its proof via connected sums was given in \cite{B}. More recently, Brindle \cite{Br} further developed the latter proof to include the duality \cite{S} Theorem 12.3.2 for the Schlesinger–Zudilin model

\[ \sum_{0 < m_1 < \cdots < m_r} q^{k_1 m_1 + \cdots + k_r m_r} [m_1^{k_1} \cdots [m_r^{k_r}. \]

In this paper, we consider the following two kinds of $q$-analogues of $\tilde{\operatorname{Li}}(\tilde{k}; z)$.

**Definition 3.1.** For a parameter $q$ with $0 < q < 1$ and an admissible augmented index $\tilde{k} = ((k_1, \mu_1), \ldots, (k_r, \mu_r))$, we define

\[
\tilde{\operatorname{Li}}_q^{(1)}(\tilde{k}; z) := \sum_{0 = m_0 < m_1 < \cdots < m_r} \prod_{i=1}^{r} \frac{q^{(k_i - 1)m_i + (-1)\mu_i q^{m_i} z^{m_i - m_i - 1}}}{[m_i]^{k_i}},
\]

\[
\tilde{\operatorname{Li}}_q^{(2)}(\tilde{k}; z) := \sum_{0 = m_0 < m_1 < \cdots < m_r} \prod_{i=1}^{r} \frac{\mu_i q^{m_i} + (-1)\mu_i z^{m_i - m_i - 1}}{[m_i]^{k_i}}.
\]

Both of these $q$-analogues satisfy the same duality relation as Theorem 1.3.

**Theorem 3.2.** For any admissible augmented index $\tilde{k}$ and $\epsilon = 1, 2$, we have

\[ \tilde{\operatorname{Li}}_q^{(\epsilon)}(\tilde{k}; z) = \tilde{\operatorname{Li}}_q^{(\epsilon)}(\tilde{k}^{\dagger}; z). \]

**Remark 3.3.** Let $\tilde{k} = ((k_1, \mu_1), \ldots, (k_r, \mu_r))$ be an admissible augmented index with $\mu_i = 1$ for all $i = 1, \ldots, r$, and $k = (k_1, \ldots, k_r)$ the corresponding usual index. Then the dual $\tilde{k}^{\dagger}$ of $\tilde{k}$ has the same property and the corresponding index is $k^{\dagger}$, the dual of $k$. Note also that

\[
\tilde{\operatorname{Li}}_q^{(1)}(\tilde{k}; 0) = \sum_{0 = m_0 < m_1 < \cdots < m_r} \prod_{i=1}^{r} \frac{q^{(k_i - 1)m_i}}{[m_i]^{k_i}} = \zeta_q^{\text{BZ}}(k),
\]

\[
\tilde{\operatorname{Li}}_q^{(2)}(\tilde{k}; 0) = \sum_{0 = m_0 < m_1 < \cdots < m_r} \prod_{i=1}^{r} \frac{q^{m_i}}{[m_i]^{k_i}} = \zeta_q^{(1, \ldots, 1)}(k),
\]

where the last expression uses Zhao’s general notation \cite{S} (11.1). Therefore, Theorem 3.2 includes the duality relation for $\zeta_q^{\text{BZ}}(k)$ and $\zeta_q^{(1, \ldots, 1)}(k)$. 


We define the connected sums

Definition 3.5.

Let \( \phi \) and \( \tilde{\phi} \) denote the \( q \)-hypergeometric series

\[
\phi_q(\alpha, \beta ; z) := \sum_{n=0}^{\infty} (\alpha; q)_n (\beta; q)_n q^n z^n,
\]

where \( (\alpha; q)_n := (1 - \alpha)(1 - q\alpha) \cdots (1 - q^{n-1}\alpha) \) is the \( q \)-shifted factorial. We need the following contiguous relations.

Lemma 3.4. The \( q \)-hypergeometric series satisfies the identities

\[
\begin{align*}
(3.1) \quad & \phi_q(\alpha, \beta ; z) = \phi_q(\alpha, q\beta ; z) - z (1 - \alpha) \frac{\beta}{1 - \gamma} \phi_q(q\alpha, \beta ; z), \\
(3.2) \quad & (\alpha - \gamma) \phi_q(\alpha, \beta ; q\gamma ; z) = (1 - \gamma) \alpha \phi_q(\alpha, \beta ; z) - (1 - \gamma) \alpha \phi_q(q\alpha, \beta ; q\gamma ; z), \\
(3.3) \quad & \phi_q(\alpha, \beta ; qz ; \gamma ; z) = \phi_q(\alpha, q\beta ; \gamma ; z) - z \frac{1 - \alpha}{1 - \gamma} \phi_q(q\alpha, q\beta ; q\gamma ; z), \\
(3.4) \quad & (\alpha - \gamma) \phi_q(\alpha, \beta ; qz ; \gamma ; z) = (1 - \gamma) \phi_q(\alpha, \beta ; z) - (1 - \gamma) \phi_q(q\alpha, \beta ; q\gamma ; z).
\end{align*}
\]

Proof. Again, they are shown by comparing the coefficients of \( z^n \).

Definition 3.5. We define the connected sums

\[
\begin{align*}
\tilde{L}_q^{(1)}(\tilde{k}; \tilde{l}; z) := & \sum_{0=m_0 < m_1 < \cdots < m_r} \prod_{i=1}^{r} q^{(k_i - 1)m_i} (\mu_i + (-1)^{\mu_i} q^{m_i - 1} z^{m_i - m_{i-1}}) \\
& \times \prod_{j=1}^{r} q^{(l_j - 1)n_j} (\nu_j + (-1)^{\nu_j} q^{n_j - 1} z^{n_j - n_{j-1}}) C_q^{(1)}(m_r, n_s; z) \\
\end{align*}
\]

and

\[
\begin{align*}
\tilde{L}_q^{(2)}(\tilde{k}; \tilde{l}; z) := & \sum_{0=m_0 < m_1 < \cdots < m_r} \prod_{i=1}^{r} \mu_i q^{m_i} + (-1)^{\mu_i} q^{m_i - m_{i-1}} \\
& \times \prod_{j=1}^{r} \nu_j q^{n_j} + (-1)^{\nu_j} q^{n_j - n_{j-1}} C_q^{(2)}(m_r, n_s; z),
\end{align*}
\]

with the connectors

\[
\begin{align*}
C_q^{(1)}(m, n; z) := q^{mn} \frac{[m]! [n]!}{[m+n]!} \phi_q(\frac{q^m q^n}{q^{m+n+1}}; z) \\
C_q^{(2)}(m, n; z) := \frac{[m]! [n]!}{[m+n]!} \phi_q(\frac{q^m q^n}{q^{m+n+1}}; qz),
\end{align*}
\]

respectively. We also use the word notation

\[
\tilde{L}_q^{(r)}(w(\tilde{k}); w(\tilde{l}); z) := \tilde{L}_q^{(r)}(\tilde{k}; \tilde{l}; z).
\]
The symmetry
\begin{equation}
\tilde{\text{Li}}_q^{(c)}(\tilde{k}; \tilde{l}; z) = \tilde{\text{Li}}_q^{(c)}(\tilde{l}; \tilde{k}; z)
\end{equation}
and the boundary condition
\begin{equation}
\tilde{\text{Li}}_q^{(c)}(\tilde{k}; \emptyset; z) = \tilde{\text{Li}}_q^{(c)}(\emptyset; \tilde{k}; z) = \tilde{\text{Li}}_q^{(c)}(\emptyset; \emptyset; z)
\end{equation}
are obvious. Furthermore, we have the following transport relations.

**Proposition 3.6.** For any augmented indices \( \tilde{k} \) and \( \tilde{l} \) and \( \epsilon = 1, 2 \), we have
\begin{equation}
\tilde{\text{Li}}_q^{(c)}(w(\tilde{k})y_0; w(\tilde{l}); z) = \tilde{\text{Li}}_q^{(c)}(w(\tilde{l})y_0; w(\tilde{k}); z).
\end{equation}
If \( \tilde{l} \) is non-empty, we also have
\begin{equation}
\tilde{\text{Li}}_q^{(c)}(w(\tilde{k})y_1; w(\tilde{l}); z) = \tilde{\text{Li}}_q^{(c)}(w(\tilde{l})x; w(\tilde{k}); z).
\end{equation}

**Proof.** First let \( \epsilon = 1 \). The first relation (3.7) follows from the identity
\begin{equation}
\sum_{a>m} q^m z^{a-m} \frac{C_q^{(1)}}{[a]}(a, n; z) = z q^m q^n \frac{[m]! [n]!}{[m+n+1]!} \phi_q \left( q^{m+1}, q^{n+1}; q^{m+n+2}; z \right),
\end{equation}
which is a symmetric expression with respect to \( m \) and \( n \). We can show (3.9) by using (3.1) and making a telescoping sum. For the proof of the second relation (3.8), we use (3.2) and make a telescoping sum to see
\begin{equation}
\sum_{a>m} \frac{1}{[a]} C_q^{(1)}(a, n; z) = q^{(m+1)n} \frac{[m]! [n-1]!}{[m+n]!} \phi_q \left( q^{m+1}, q^n; q^{m+n+1}; z \right).
\end{equation}
Then, combining this with (3.9), and using (3.1) again, we obtain
\begin{equation}
\sum_{a>m} \frac{1 - q^m z^{a-m}}{[a]} C_q^{(1)}(a, n; z) = q^n \frac{[n]}{[n]} C_q^{(1)}(m, n; z),
\end{equation}
which implies (3.3). This completes the proof for \( \epsilon = 1 \).

Almost the same argument works for \( \epsilon = 2 \) if we replace the formulas (3.1) and (3.2) with (3.3) and (3.4), respectively. \( \square \)

Given the properties (3.5)–(3.8) of the connected sum, the proof of Theorem 3.2 is the same as that of Theorem 1.3.

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