Universality for the focusing nonlinear Schrödinger equation at the gradient catastrophe point:
Rational breathers and poles of the \textit{tritronquée} solution to Painlevé I

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Abstract

The semiclassical (zero-dispersion) limit of the one-dimensional focusing Nonlinear Schrödinger equation (NLS) with decaying potentials is studied in a full scaling neighborhood $D$ of the point of gradient catastrophe $(x_0, t_0)$. This neighborhood contains the region of modulated plane wave (with rapid phase oscillations), as well as the region of fast amplitude oscillations (spikes). In this paper we establish the following universal behaviors of the NLS solutions near the point of gradient catastrophe: i) each spike has the height $3|q_0(x_0, t_0, \varepsilon)|$ and uniform shape of the rational breather solution to the NLS, scaled to the size $O(\varepsilon)$; ii) the location of the spikes are determined by the poles of the \textit{tritronquée} solution of the Painlevé I (P1) equation through an explicit diffeomorphism between $D$ and a region into the Painlevé plane; iii) if $(x, t) \in D$ but lies away from the spikes, the asymptotics of the NLS solution $q(x, t, \varepsilon)$ is given by the plane wave approximation $q_0(x, t, \varepsilon)$, with the correction term being expressed in terms of the \textit{tritronquée} solution of P1. The latter result confirms the conjecture of Dubrovin, Grava and Klein [12] about the form of the leading order correction in terms of the \textit{tritronquée} solution in the non-oscillatory region around $(x_0, t_0)$. We conjecture that the P1 hierarchy occurs at higher degenerate catastrophe points and that the amplitudes of the spikes are odd multiples of the amplitude at the corresponding catastrophe point. Our technique is based on the nonlinear steepest descent method for matrix Riemann-Hilbert problems and discrete Schlesinger isomonodromic transformations.

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1 Introduction and main results

In this paper we consider the focusing Nonlinear Schrödinger (NLS) equation
\[ i\varepsilon \partial_t q + \varepsilon^2 \partial_x^2 q + 2|q|^2 q = 0, \tag{1-1} \]
where \( x \in \mathbb{R} \) and \( t \geq 0 \) are space-time variable and \( \varepsilon > 0 \). It is a basic model for self-focusing and self-modulation, for example, it governs nonlinear transmission in optical fibers; it can also be derived as a modulation equation for general nonlinear systems. It was first integrated by Zakharov and Shabat [30] who produced a Lax pair for it and used the inverse scattering procedure to describe general decaying solutions \( (\lim_{|x| \to \infty} q(x,0) = 0) \) in terms of radiation and solitons. Throughout this work, we will use the abbreviation NLS to mean “focusing Nonlinear Schrödinger equation”.

Our interest in the semiclassical (zero-dispersion) limit \( (\varepsilon \to 0) \) of NLS stems largely from its modulationally unstable behavior. As shown by Forest and Lee [14], the modulation system for NLS can be expressed as a set of nonlinear PDE with complex characteristics; thus, the system is ill posed as an initial value problem with Cauchy data. As a result, initial data
\[ q(x,0,\varepsilon) = A(x) e^{i\Phi(x)/\varepsilon}, \quad \Phi(0) = 0, \tag{1-2} \]
i.e. a plane wave with amplitude modulated by \( A(x) \) and phase modulated by \( \Phi(x) \) are expected to break immediately into some other, presumably disordered, wave form when the functions \( A(x) \) and \( \Phi(x) \) possess no special properties.

However, in the case of an analytic initial data, the NLS evolution displays some ordered structure instead of the disorder suggested by the modulational instability, as shown by Fig. 1 and Fig. 2, see [23], [28] and [8]. These figures clearly identify the regions where different types of behavior of the solution \( q(x,t,\varepsilon) \) to the NLS appear. These regions are separated by some curves in the \( x,t \) plane that are asymptotically independent of \( \varepsilon \). They are called breaking curves or nonlinear caustics. Within each region, the strong asymptotics of \( q(x,t,\varepsilon) \) can be expressed in terms of Riemann Theta-functions to within an error term of order \( O(\varepsilon) \) that is uniform on compact subsets of the region ([19] for the pure soliton case and [26] for the general case). In this context, regions of different asymptotic behavior of \( q(x,t,\varepsilon) \) corresponds to the different genera of the hyperelliptic Riemann surface \( R(x,t) \), whose Theta-functions enter in the asymptotic description. Note that \( R(x,t) \) does not depend on \( \varepsilon \). In the very first (genus zero) region (that contains \( t = 0 \)), the asymptotics of \( q(x,t,\varepsilon) \) is expressed through the branch-point \( \alpha(x,t) \) of \( R(x,t) \) (the branch-points are Schwarz-symmetrical) as
\[ q_0(x,t,\varepsilon) = A(x,t) e^{i\frac{1}{2} \Phi(x,t)} \Im \alpha(x,t) e^{-\frac{i}{2} \int_{(0,0)}^{(x,t)} \{2|\Re \alpha(\xi,t)|d\xi + [4|\Re \alpha(x,\tau)|^2 - 2(\Im \alpha(x,\tau))^2]d\tau\} + O(\varepsilon), \tag{1-3} \]
Note that this equation differs by the coefficient 2 from the NLS equation considered in previous papers [26]-[27], whose results we use here; this is the correct equation for the particular form of the time evolution of the scattering data (see (2-5)), adopted in [26]-[27] and in the present paper. Alternatively, one can keep the “old” form of the NLS, but scale by 2 the time variable while using the results of [26]-[27].

This formula was proven in [26], but, as stated in Theorem 1.1 there (with only \( d\xi \) part of the differential form), is correct only for \( t = 0 \).
Figure 1: Absolute value $|q(x, t, \varepsilon)|$ of a solution $q(x, t, \varepsilon)$ to the focusing NLS (1-1) versus $x, t$ coordinates. Here $A(x) = e^{-x^2}$, $\Phi'(x) = \tanh x$ and $\varepsilon = 0.03$.

where the error estimate is uniform on compact subsets of the genus zero region (away from the breaking curve).

A detailed study of asymptotic behavior of $q(x, t, \varepsilon)$ along the first breaking curve (a neighborhood of the tip of this curve was excepted) together with error estimates were conducted in our previous work [4]. Considering one of the pieces of the breaking curve (to the left, $x < 0$, or to the right, $x > 0$, from the tip, see Fig. 2), we introduced scaled coordinates $S(x, t) = \frac{1}{2}\vartheta(x, t) + \frac{1}{2}\kappa(x, t)\varepsilon \ln \varepsilon$, that map a neighborhood of the left (right) breaking curve onto a horizontal strip in $\mathbb{C}$, containing negative (positive) real semi-axis. In the coordinates $\vartheta, \kappa$, the oscillatory region of the left breaking curve is in the upper $S$-half-plane, and the shape of oscillations in the $S$-plane is shown on Fig. 3.

The tip-point $x_0, t_0$ of the breaking curve is called a point of gradient catastrophe, or elliptic umbilical singularity ([12]). It is evident from the numerical simulations shown on Fig. 1 and Fig. 2, as well as from the analysis of the spectral plane (presented below), that the behavior of the solution
Figure 2: \( A(x) = e^{-x^2}, \Phi'(x) = \tanh x \) and \( \varepsilon = 0.03 \).

The main goal of this paper is to analyze the leading order asymptotic behavior of the solution \( q(x, t, \varepsilon) \) on and around this special point of transition. More precisely we will examine a neighborhood \( D \) of \((x_0, t_0)\) that is shrinking at the rate \( O(\varepsilon^{\frac{4}{5}}) \) as \( \varepsilon \to 0 \). As the first step, we construct a diffeomorphism \( v = v(x, t, \varepsilon) \), that maps \( D \) onto a bounded disk \( V \subset \mathbb{C} \), where \( V \) is independent of \( \varepsilon \) and \( v(x_0, t_0, \varepsilon) = 0 \). It turns out that the leading order behavior of \( q(x, t, \varepsilon) \) in \( D \) can be conveniently described through a specific tritronquée solution (see Section 4.2.1) to the Painlevé I (P1) equation

\[
y'' = 6y^2 - v. \tag{1-4}
\]

That is why the diffeomorphism \( v(x, t, \varepsilon) \) and the complex \( v \)-plane will be referred to as the Painlevé coordinatization of \( D \) and the Painlevé plane respectively. Note that the diffeomorphism \( v(x, t, \varepsilon) \) near the point of gradient catastrophe plays similar role to the map \( S(x, t, \varepsilon) \) for the rest of the breaking curve.

The point of gradient catastrophe \((x_0, t_0)\) can be defined in terms of the physical variables \((x, t)\) as the point, where the genus zero approximation \((1-3)\) of the solution \( q(x, t, \varepsilon) \) to \((1-1),(1-2)\) develops an infinite \( x \)-derivative in \( A(x, t) \) and/or in \( \Phi_x(x, t) \) (while \( A(x, t) \) and \( \Phi_x(x, t) \) stay finite). In terms of the time evolution of the spectral data (see Fig. 4 below), the point of gradient catastrophe is the point where the birth/collapse of the new main arc (band) happens exactly at the end of an existing main arc (see Fig. 4).

As it was mentioned above, the asymptotics of \( q(x, t, \varepsilon) \) within the region of a given genus is given explicitly in terms of the Riemann Theta-functions, associated with \( \mathcal{R}(x, t) \), with the accuracy \( O(\varepsilon) \), see

\[
q(x, t, \varepsilon) \text{ at the tip of the breaking curve is very different from the behavior elsewhere on the breaking curve.}
\]
Figure 3: The graphs depict the ratio \( \frac{q(x,t,\varepsilon)}{q_0(x,t,\varepsilon)} \) in a vicinity of a breaking curve point \( x = 0.1, t = 0.1570482549 \) with \( \varepsilon = 10^{-10} \). Here \( q_0 \) is the genus zero approximate solution, \( q \) is the leading order approximation near the breaking curve, constructed in [4]. The graph is shown in the variables \( \theta, \kappa \) and the scale is not uniform: \( \kappa \) measures distances in the scale \( \varepsilon |\ln \varepsilon| \) whereas \( \theta \) in the scale \( \varepsilon \). The distance between consecutive "ranges", should be much longer than the separation of the neighboring peaks within the same range. The typical size of the hills is \( \varepsilon \) in all directions while the separation in the longitudinal direction is \( \varepsilon |\ln \varepsilon| \). This picture is consistent with a full–blown genus-two regime, where the solution is quasiperiodic at the scale \( \varepsilon \): as we progress into the genus-2 region the separation reduces to the natural scale \( \varepsilon \). If we were to plot this in the \((x,t)\) plane the only difference would be a linear change of coordinates and the “ranges” would be parallel to the breaking curve.

[26], [19]. (In the case of genus zero, see (1-3).) The leading order behavior of \( q(x,t,\varepsilon) \) in order \( O(\varepsilon \ln \varepsilon) \) strips around left and right branches of the breaking curve, found in [4], has the accuracy \( O(\sqrt{\varepsilon}) \). It was conjectured by B. Dubrovin, T. Grava and C. Klein in [12] that the error estimate near the gradient catastrophe is of order \( O(\varepsilon^{2/5}) \) and is expressible in terms of the tritronquée solution of P1 (see Remark 5.2 for more details on their conjecture).

1.1 Description of results

The results of this paper not only confirm the conjecture of Dubrovin, but in fact go beyond it, as we provide the leading order behavior together with the accuracy estimate in the whole domain \( D \) around the point of gradient catastrophe \( x_0, t_0 \), that include the oscillatory part of \( D \). The following notations are useful in describing our results.

Let \( V_p = \{ v_{p,1}, \ldots, v_{p,N} \} \) denote the set of poles of the tritronquée solution \( y(v) \) in \( V \), i.e., \( V_p \subset V \). Let \( B_{\delta,j} \) denote the disk of radius \( \delta > 0 \) centered at \( v_{p,j}, j = 1, \ldots, N \), \( B_{\delta} = \bigcup_{1}^{N} B_{\delta,j} \) and \( K_{\delta} = V \setminus B_{\delta} \). Denoting \( a + ib = \alpha(x_0,t_0) \), we prove the following facts:
1. There is a one to one correspondence between the poles of the tritronquée solution $y(v)$ within $V$ and the spikes of the NLS solution $q$ within $D$. Each spike is centered at the corresponding $(x_{p,j}, t_{p,j}) = v^{-1}(v_{p,j})$, where

$$v(x,t,\varepsilon) = \frac{e^{-ix/4}}{\sqrt[5]{2b/C}} \left[ x - x_0 + 2(2a + ib)(t - t_0) \right] \left( 1 + O(\varepsilon^{7/5}) \right)$$

(1-5) uniformly in $D$, with the nonzero constant $C$ explicitly defined by (3-44) in terms of the scattering data;

2. Each spike has the fixed height of $3|q_0(x_0,t_0)| + O(\varepsilon^{1/5})$, i.e., the height of each spike is three times the amplitude at the gradient catastrophe, see Theorems 6.2, 6.3;

3. Each spike has the universal shape of the (scaled) rational breather solution to the NLS eq. (1-1), see Fig. 10, i.e,

$$q(x,t,\varepsilon) = e^{\frac{i}{\varepsilon} \Phi(x_{p,j}, t_{p,j})} Q_{br} \left( \frac{x - x_{p,j}}{\varepsilon}, \frac{t - t_{p,j}}{\varepsilon} \right) \left( 1 + O(\varepsilon^7) \right),$$

(1-6)

where the rational breather

$$Q_{br}(\xi, \eta) = e^{-2i(a\xi + (2a^2-b^2)\eta)}b \left( 1 - 4 \frac{1 + 4ib^2\eta}{1 + 4b^2(\xi + 4a\eta)^2 + 16b^4\eta^2} \right)$$

(1-7)

satisfies the NLS eq. (1-1) with space-time variables $\xi, \eta$. This breather approximation of the spike is valid in the domain $\hat{B}_j = v^{-1}(B_{\delta,j})$ of $(x_{p,j}, t_{p,j})$, where $\delta = O(\varepsilon^{1/5})$, see Theorems 6.2) and [4]. The size of each spike in the physical plane (the size of $\hat{B}_j$) is thus $O(\varepsilon)$, which is consistent with the size of spikes along the breaking curve (away from $(x_0, t_0)$) and within the bulk of the genus two region, see above. Remarkably, the breather-like shape of spikes (see Fig. 3) is also preserved as one moves along the breaking curve away from the point of gradient catastrophe. The two zeroes (“roots”) and the maximum of each breather, shown on Fig. 10, occur at the same time (within the accuracy of our approximation).

4. In Thm. 5.2 we show that if $\delta > 0$ is a small fixed number then

$$q(x,t,\varepsilon) = \left( b - 2\varepsilon^3 \Re \left( \frac{y(v)}{\varepsilon^2} \right) + O(\varepsilon^{7/5}) \right) \times$$

$$\exp \left[ \frac{2i}{\varepsilon} \left( \frac{1}{2} \Phi(x_0,t_0) - (a(x-x_0) - (2a^2-b^2)(t-t_0)) + \varepsilon^9 \Re \left( \sqrt{\frac{2i}{\varepsilon b} H_I(v)} \right) \right) \right]$$

(1-8)

uniformly in $\hat{K}_\delta = v^{-1}(K_\delta)$, where $C$ is a nonzero constant explicitly given by (3-44), $y(v) = y(v(x,t,\varepsilon))$ is the tritronquée solution and $H_I = \frac{1}{2}(y'(v))^2 + vy(v) - 2y^3(v)$. Equation (1-8) is consistent with the conjecture of [12] (see Remark 5.2).
5. If \( \delta = O(\varepsilon^\nu) \), where \( \rho \in (0, \frac{1}{2}) \), and \( \hat{K}_\delta = v^{-1}(K_\delta) \), then equation (1-8) will be uniformly valid in \( \hat{K}_\delta \) provided that \( \varepsilon^\frac{1}{\nu} \) in the error term will be replaced by \( \varepsilon^{\frac{1}{\nu}-\nu} \), i.e.

\[
q(x, t, \varepsilon) = \left( \frac{2}{\varepsilon} \log \left( \frac{1}{\varepsilon} \right) \frac{v(t)}{C} + O(\varepsilon^{\frac{1}{\nu}-\nu}) \right) \times \exp \left\{ \frac{2\pi i}{\varepsilon} \left[ \frac{1}{2} \Phi(x_0, t_0) - \left( a(x - x_0) - (2a^2 - b^2)(t - t_0) + \varepsilon^{\frac{1}{\nu}} \Re \left( \sqrt{\frac{24}{5}H_1(v)} \right) \right) \right] \right\}.
\]

Note that since \( y(v) \) has a double pole and \( H_1(v) \) a simple pole – the term \( y(v) \) is actually of order \( \varepsilon^{-2\nu} \) and \( H_1(v) \) of order \( \varepsilon^{-\nu} \); clearly the description in terms of the tritronquée cannot be pushed “too close” to the pole/spike.

We also make the Conjecture 6.1 that the amplitudes of the spikes near any gradient catastrophe point in the genus zero phase are odd multiples of the amplitude at the point itself. In addition we can speculate that the shape of the spikes at the higher-degeneracy cases should be related to the higher rational breather recently investigated in [1].

Among other results obtained in this paper we mention the proof that the two branches of the breaking curves form a corner (wedge) at the point of gradient catastrophe \( (x_0, t_0) \) and give explicit expression, see (3-58), of the angle between the breaking curve in terms of \( C \) and \( \alpha(x_0, t_0) \). We further prove that the diffeomorphism \( v(x, t, \varepsilon) \) maps this corner in to the sector \( \frac{2\pi}{5} < \arg v < \frac{4\pi}{5} \) of the complex \( v \)-plane, see Fig. 9. This is consistent with another conjecture, stated in [12]: all the poles of the tritronquée solution \( y(v) \) are contained within the sector \( \frac{2\pi}{5} < \arg v < \frac{4\pi}{5} \) which is a longstanding question in the theory of Painlevé equations. According to our results, the set of spikes near the point of gradient catastrophe is, in fact, the visualization of the poles of the tritronquée solution \( y(v) \) to \( P1 \). For example, it was calculated in [18] that the very first pole \( v_{p,1} \) of \( y(v) \) has \( \arg v_{p,1} = \frac{3\pi}{5} \) and \( |v_{p,1}| \approx 2.38 \). Using (1-5), we calculate \( (x_{p,1}, t_{p,1}) = (0, 2.79126) \) for the NLS evolution of the initial data \( q(x, 0, \varepsilon) = \text{sech} x \). Numerical simulation of this evolution with \( \varepsilon = \frac{1}{78} \) shows the first spike at \( t_s \approx 2.8 \), which is in a very good agreement with \( t_{p,1} \), see Example 6.1 and Fig. 11.

Our results are based on the nonlinear steepest descent analysis of the inverse scattering transform of the semiclassical NLS (1-1) that was set in the form of a matrix Riemann-Hilbert problem (RHP). This RHP depends on the semiclassical parameter \( \varepsilon \) and external parameters \( x, t \). As it is now well known, the nonlinear steepest descent method reduces this RHP into what is called a model RHP, which has some piece-wise constant jump matrices on a set of arcs in \( \mathbb{C} \), known as main arcs or bands. Solution \( \Psi_0 \) to the model RHP represents the leading order term of the solution to the original RHP, and the main arcs are exactly the branch-cuts of the hyperelliptic Riemann surface \( \mathcal{R}(x, t) \), associated with the problem. Note that, due to the symmetries of the focusing NLS, the set of main arcs is always Schwarz-symmetrical. The asymptotic behavior of the solution \( q(x, t, \varepsilon) \) in a vicinity of \( (x, t) \) as \( \varepsilon \to 0 \) profoundly depends on the genus of \( \mathcal{R}(x, t) \), as seen on Fig. 2 and Fig. 1, with the change of genus of \( \mathcal{R}(x, t) \) corresponding to the transition between different asymptotic regimes of \( q(x, t, \varepsilon) \). In these terms, the point of gradient
catastrophe can be described as the point where a new main arc (a new branch-cut of $\mathcal{R}(x,t)$), together with its complex-conjugated main arc, branches out from the endpoint of the existing main arc.

While reducing the original RHP to the model RHP, the error is controlled through the use of local parametrices, which are designed to “mimic” the local behavior of the solution at the points of interest, for example, at the endpoints of main arcs. At the regular points $(x,t)$ in the physical plane, these parametrices can be constructed through Airy functions. As we show in Section 4, at the point of gradient catastrophe $(x_0,t_0)$, the parametrix $\mathcal{P}$ is constructed through the tritronquée solution $y(v)$ of the P1.

It is well known that Painlevé equations can be expressed as conditions of isomonodromic deformations for certain rational $2 \times 2$ systems of ODEs with rational coefficients [17, 15]. The parametrix $\mathcal{P}$ at the point of gradient catastrophe is built through the fundamental solution $\Psi(\xi;v)$ to the system of ODEs associated with the P1. The occurrence of parametrices build out of Painlevé associated linear systems is not unexpected here, as they often appear in various RHPs related to random matrices/orthogonal polynomials, for example:

- The case of random matrices with a soft-edge where the density vanishes to order $(z-\alpha)^{2k+1/2}$, corresponding to the even P1 hierarchy [10];
- The case of random matrices where a spectral band splits into two (P2 equation [5] and hierarchy);
- The trailing edge of the region of oscillations in the small–dispersion limit of KdV [9] (PII equation).

The novelty of our work lies in the fact that the matrix $\Psi(\xi;v)$, and thus, the parametrix $\mathcal{P}$, is not defined (has poles) at the poles $v_p \in V$ of the tritronquée solution $y(v)$. To our best knowledge, this paper contains the first example of parametrices with singularities, that were successfully used to control the errors.

Solving the RHP at or near the pole $v_p$ of $y(v)$, i.e., studying the shape of the spike, require several additional steps, which can be briefly listed as:

- Factorization $\Psi(\xi;v) = G(\xi,v)\hat{\Psi}(\xi,v)$, where $G(\xi,v)$ is a “simple” matrix with singularity at $v = v_p$, and $\hat{\Psi}(\xi,v)$ is regular at $v = v_p$. This factorization was introduced by D. Masoero in [22]. The existence of the limit of $\hat{\Psi}(\xi,v)$ as $v \to v_p$ that is uniform in a certain region of the spectral $\xi$-plane (see Appendix A), is an important part in establishing the shape of spikes;
- Construction of $\Psi_1$, solution of a modified model RHP, that is needed at the poles $v = v_p$ of the tritronquée solution $y(v)$. $\Psi_1$ was obtained by means of the discrete Schlesinger isomonodromic deformations of $\Psi_0$. This type of deformations were used, for example, in [3, 4];
- Construction of the new parametrix $\mathcal{P}_1$ for the modified model RHP, using $\hat{\Psi}(\xi,v)$. The formula $3b + O(\varepsilon^{1/5})$ for the height of spikes follows immediately from $\mathcal{P}_1$;
• Additional transformation of $\Psi_1$, called partial Schlesinger transformation, is used to obtain the shape of the spikes, see (1-6).

Finally, it is clear that our method can be modified to handle higher order (degenerate) gradient catastrophes, where $k$ main arcs, $k > 1$, simultaneously emerge from the endpoints of an existing main arc. The parametrices in these cases can be written in terms of the higher members of the P1 hierarchy, and that one should expect the height of the spikes to be $2k + 1$ times the amplitude at the point of gradient catastrophe.

2 A short review of the zero dispersion limit of the inverse scattering transform

Given an initial data (potential) $q(x, 0, \varepsilon)$ for the (1-1) that is decaying as $x \to \pm \infty$, the direct scattering transform for the NLS ([30]) produces the scattering data, namely: the reflection coefficient $r_0(z, \varepsilon)$ and the points of discrete spectrum together with their norming constants (solitons). The time evolution of the scattering data is simple and well-known. Thus, to find the evolution of a given potential at a time $t$, one needs to solve the inverse scattering problem at this time $t$. Equivalently, we can stipulate that the initial data is assigned directly through the scattering data; thus, one can produce a solution to the NLS (1-1) by choosing some scattering data and solving the inverse scattering problem for $t = 0$ (initial data) and for $t > 0$ (evolution of the initial data). The latter approach allows one to avoid solving the direct scattering problem and addressing many delicate issues associated with it (see [25] for more details). Since we are interested in studying the generic structure of solutions of the NLS near the point of gradient catastrophe, it will be convenient for us to define a solution to the NLS (1-1) by its scattering, not by its initial data.

In considering the semiclassical limit of (1-1), one has to consider the semiclassical limit of the corresponding scattering transform. For the case of decaying potentials of type (1-2), this limit was discussed in [27] (inverse scattering) and [25] (direct scattering), where it was shown to be a correspondence between

$$\alpha(x, t) = -\frac{1}{2} \Phi_x(x, t) + iA(x, t)$$

(2-1)
on the potential side and

$$f(z) = f_0(z) - xz - 2tz^2$$

(2-2)
on the scattering side. Here $t \geq 0$ is fixed and $f_0(z)$ has the meaning of the “scaled” logarithm of the reflection coefficient $r_0(z, \varepsilon)$ that corresponds to the initial data (1-2), that is,

$$f_0(z) = \frac{i}{2} \lim_{\varepsilon \to 0} \varepsilon r_0(z, \varepsilon).$$

(2-3)
For example, the limiting \( f_0(z) \) that corresponds to the potential \( \frac{1}{\cosh(x)} e^{-\frac{2i}{3} \ln \cosh x + i\pi} \), is given by (see [26])

\[
\begin{align*}
f_0(z) &= (1 - z) \left[ i \frac{\pi}{2} + \ln(1 - z) \right] + z \ln z + \ln 2 + \frac{\pi}{2} \varepsilon, \quad \text{when } \Re z \geq 0.
\end{align*}
\]

Moreover, if \( x(\alpha) \) denotes the inverse function to \( \alpha(x, t) \), at \( t \) fixed, then \( x(\alpha) \) and \( f_0(z) \) are connected through a certain Abel type linear integral transform, see details in [25].

Since we have taken the perspective that we are starting with the scattering data of the form \( r_0(z, \varepsilon) := e^{-\frac{2i}{3} f_0(z)} \) we shall assume that ([27]):

- \( f_0(z) \) is analytic in the upper half-plane and has a continuous limit on \( \mathbb{R} \);
- \( f_0(z) = \mathcal{O}(z) \) as \( z \to \infty \);
- there exists an interval \( (\mu_-, \mu_+) \subset \mathbb{R} \) such that \( \Re f_0(z) > 0 \) for \( z \in (\mu_-, \mu+) \) and \( \Re f_0(z) < 0 \) for \( z \in (-\infty, \mu_-) \cup (\mu_+, \infty) \);
- \( \Re f_0(z), \ z \in \mathbb{R} \), has simple zeroes at \( \mu_\pm \) and its values are separated from zero outside neighborhoods of \( \mu_\pm \).

Due to the Schwarz-symmetry of the scattering data for the NLS, \( f_0(z) \) is also a Schwarz-symmetrical function, i.e., \( f_0(\bar{z}) = \overline{f_0(z)} \); generically, \( \Re f_0(z) \) has a jump across the real axis. The above assumptions are not overly restrictive: it was shown that in the case of a solitonless potential the formula (2-3) yields such an \( f_0(z) \) after some process of analytic extension, or “folding” of certain logarithmic branch-cuts onto the real axis. In fact, for our goals it is sufficient to replace the analyticity requirement by a weaker condition: the contours of the RHP for the \( g \)-function (which will be introduced in the next section) always lie within the domain of analyticity of \( f_0(z) \) in \( \mathbb{C}^+ \). Generically, \( r_0(z) \) has a jump discontinuity along the real axis due to the discontinuity in \( \Re f_0 \).

At any time \( t \), the inverse scattering problem for (1-1) (in the solitonless case) with a fixed (not infinitesimal) \( \varepsilon \) is reducible to the following matrix RHP.

**Problem 2.1** Find a matrix \( \Gamma(z) \) analytic in \( \mathbb{C} \setminus \mathbb{R} \) such that

\[
\begin{align*}
\Gamma_+(z) &= \Gamma_-(z) \begin{bmatrix}
\frac{|r_0|^2}{r_0 e^{\frac{2i}{3} (2tz^2 + xz)}} + 1 & r_0 e^{-\frac{2i}{3} (2tz^2 + xz)} \\
r_0 e^{\frac{2i}{3} (2tz^2 + xz)} & 1
\end{bmatrix}, \quad z \in \mathbb{R},
\end{align*}
\]

\[
\Gamma(z) = 1 + \frac{1}{z} \Gamma_1 + \mathcal{O}(z^{-2}), \quad z \to \infty.
\]

(\text{In the case with solitons, there are additional jumps across small circles surrounding the points of discrete spectrum.}) Then

\[
q(x, t, \varepsilon) := -2 (\Gamma_1)_{12}
\]

is the solution of the initial value problem (1-1) for the NLS equation.
The jump matrix for the RHP admits the factorization

\[
\begin{bmatrix}
|r|^2 + 1 & r \\
r & 1
\end{bmatrix} = \begin{bmatrix}
1 & r \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
r & 1
\end{bmatrix},
\]

(2-8)

where \( r = r(z; x, t) = r_0(z)e^{\frac{2}{i\varepsilon}(2tz^2 + xz)} \).

Inspection of the RHP shows that the matrix \( \Gamma(z)^* \) (where \( ^* \) stands for the complex-conjugated, transposed matrix) solves the same RHP with the jump matrix \( M(z) \) replaced by \( M^{-1}(z) \) and hence

**Proposition 2.1** *The solution \( \Gamma(z) \) of the RHP for NLS has the symmetry*

\[
\Gamma(z)(\Gamma(z))^* \equiv 1.
\]

(2-9)

In order to study the dispersionless limit \( \varepsilon \to 0 \), the RHP (2-5)-(2-6) undergoes a sequence of transformations (that are briefly recalled in Section 2.2) along the lines of the nonlinear steepest descent method [11, 26], which reduce it to an RHP that allows for an approximation by the so-called model RHP. The latter RHP has piece-wise constant jump matrices (parametrically dependent on \( x, t, \varepsilon \)) and, in general, can be solved explicitly in terms of the Riemann Theta functions, or, in simple cases, in terms of algebraic functions. The \( g \)-function, defined below, is the key element of such a reduction.

### 2.1 The \( g \)-function

Given \( f_0(z) \), we introduce the \( g \)-function \( g(z) = g(z; x, t) \) as the solution to the following scalar RHP:

1. \( g(z) \) is analytic (in \( z \)) in \( \mathbb{C} \setminus \gamma_m \) (including analyticity at \( \infty \));

2. \( g(z) \) satisfies the jump condition

\[
g_+ + g_- = f_0 - xz - 2tz^2 \quad \text{on} \quad \gamma_m,
\]

(2-10)

for \( x \in \mathbb{R} \) and \( t \geq 0 \), and;

3. \( g(z) \) has the endpoint behavior

\[
g(z) = O(z - \alpha)^2 + \text{an analytic function in a vicinity of} \ \alpha.
\]

(2-11)

Here:

- \( \gamma_m \) is a bounded Schwarz-symmetrical contour (called the main arc) with the endpoints \( \bar{\alpha}, \alpha \), oriented from \( \bar{\alpha} \) to \( \alpha \) and intersecting \( \mathbb{R} \) only at \( \mu_+ \);

- \( g_{\pm} \) denote the values of \( g \) on the positive (left) and negative (right) sides of \( \gamma_m \);
Figure 4: The typical zero-dispersion phase diagram for a one-hump initial data. Representative at different points in the \((x,t)\)-plane are the level-curves of \(\Im(h)\). Only the upper-half spectral plane is depicted. The shape of the level curves was obtained numerically on a simple example. The plot of the amplitude of \(q(x,t,\varepsilon)\) corresponds to the initial data \(q(x,0) = \text{sech}(x), \varepsilon = \frac{1}{3}\). This is a pure-soliton case, which was used here only for the purpose of an effective illustration, as our results are valid for any generic (nondegenerate) gradient catastrophe.

- the function \(f_0 = f_0(z)\), representing the initial scattering data, is Schwarz-symmetrical and Hölder-continuous on \(\gamma_m\).

Taking into the account Schwarz symmetry, it is clear that behavior of \(g(z)\) at both endpoints \(\alpha\) and \(\bar{\alpha}\) should be the same.

Assuming \(f_0\) and \(\gamma_m\) are known, the solution \(g\) to the scalar RHP (2-10) without the endpoint condition (2-11) can be obtained by the Plemelj formula

\[
g(z) = \frac{R(z)}{2\pi i} \int_{\gamma_m} \frac{f(\zeta)}{(\zeta - z)R(\zeta)}d\zeta ,
\]

(2-12)
where \( R(z) = \sqrt{(z - \alpha)(z - \overline{\alpha})} \). We fix the branch of \( R \) by requiring that \( \lim_{z \to \infty} \frac{R(z)}{z} = 1 \). If \( f_0(z) \) is analytic in some region \( S \) that contains \( \gamma_m \setminus \{ \mu_+ \} \), the formula for \( g(z) \) can be rewritten as

\[
g(z) = \frac{R(z)}{4\pi i} \int_{\hat{\gamma}_m} \frac{f(\zeta)}{(\zeta - z)R(\zeta)} d\zeta, \tag{2-13}
\]

where \( \hat{\gamma}_m \subset S \) is a negatively oriented loop around \( \gamma_m \) (which is “pinched” to \( \gamma_m \) in \( \mu_+ \), where \( f \) is not analytic) that does not contain \( z \). Introducing function \( h = 2g - f \), we obtain

\[
h(z) = \frac{R(z)}{2\pi i} \int_{\hat{\gamma}_m} \frac{f(\zeta)}{(\zeta - z)R(\zeta)} d\zeta, \tag{2-14}
\]

where \( z \) is inside the loop \( \hat{\gamma}_m \). The endpoint condition (2-11) can now be written as

\[
h(z) = O(z - \alpha)^{3/2} \text{ as } z \to \alpha, \tag{2-15}
\]

or, equivalently,

\[
\int_{\hat{\gamma}_m} \frac{f(\zeta)}{(\zeta - z)R(\zeta)} d\zeta = 0. \tag{2-16}
\]

The latter equation is known as a modulation equation. The function \( h \) plays a prominent role in this paper. Using the fact that the Cauchy operator for the RHP (2-10), (2-11) commutes with differentiation, we have

\[
h'(z) = \frac{R(z)}{2\pi i} \int_{\hat{\gamma}_m} \frac{f'(\zeta)}{(\zeta - z)R(\zeta)} d\zeta, \tag{2-17}
\]

where \( z \) is inside the loop \( \hat{\gamma}_m \).

In order to reduce the RHP (2-5)-(2-6) to the RHP with piece-wise jump matrices, called the model RHP, the signs of \( \Im h(z) \) in the upper half-plane should satisfy the following conditions:

- \( \Im h(z) \) is negative on both sides of the contour (main arc) \( \gamma_m \);
- there exists a continuous contour \( \gamma_c \) (complementary arc) in \( \mathbb{C}_+ \) that connects \( \alpha \) and \( \mu_- \), so that \( \Im h(z) \) is positive along \( \gamma_c \). Since \( \Im h(z) > 0 \) on the interval \( (-\infty, \mu_-) \), the point \( \mu_- \) in \( \gamma_c \) can be replaced by any other point of this interval, or by \( -\infty \).

Note that the first sign requirement, together with (2-10), imply that \( \Im h(z) = 0 \) along \( \gamma_m \). Since the signs of \( \Im h(z) \) play an important role in the following discussion, we call by “sea” and “land” the regions in \( \mathbb{C}_+ \), where \( \Im h(z) \) is negative and positive respectively. In this language, the complementary arc \( \gamma_c \) goes on “land”, whereas the main arc \( \gamma_m \) is a “bridge” or a “dam”, surrounded by the sea, see Fig. 5.
2.2 Reduction to the model RHP

We start the transformation of the RHP (2-5)-(2-6) by deforming (preserving the orientation) the interval \((−∞, μ_+)\) , which is a part of its jump contour, into some contour \(γ^+\) in the upper half-plane \(\mathbb{C}_+\), such that \(μ_+ \in γ^+\). Let \(γ^-\) be the Schwarz symmetrical image of \(γ^+\). Using the factorization (2-8) the RHP (2-5)-(2-6) can be reduced to and equivalent one where:

- the right factor of (2-8) is the jump matrix on \(γ^+\);
- the left factor of (2-8) is the jump matrix on \(γ^-\);
- the jump matrix on the remaining part of \(\mathbb{R}\) is unchanged.

It will be convenient for us to change the orientation of \(γ^+\), which causes the change of sign in the off-diagonal entry of the corresponding jump matrix. On the interval \((μ_+, ∞)\) we have \(\Im f_0(z) < 0\) and it appears that the jump is exponentially close to the identity jump and hence it is possible to prove that it has no bearing on the leading order term of the solution (2-7) (as \(ε \to 0\): see [26] for the case when \(f_0\) is a one-parameter family that contains (2-4) and [27] for the general case). Therefore the leading order contribution in (2-7) comes from the contour \(γ = γ_+ ∪ γ_-\). In the genus zero case, the contour \(γ\) contains points \(α, \bar{α}\), which divide it into the main arc \(γ_m\) (contained between \(α\) and \(\bar{α}\), and the complementary arc \(γ_c = γ \setminus γ_m\). According to the sign requirements (2.1), the contour \(γ_m\) is uniquely determined as an arc of the level curve \(\Im h(z) = 0\) (bridge) that connects \(μ_+\) and \(α\), whereas \(γ_c\) can be deformed arbitrarily “on the land”. Because of the Schwarz symmetry 2.1, it is sufficient to consider \(γ\) only in the upper half-plane, i.e., it is sufficient to consider \(γ^+\).

Having found the branch-point \(α\), the \(g\)-function \(g(z)\) and the contour \(γ_m\), we introduce additional contours customarily called “lenses” that join \(α\) to \(μ_+\) on both sides of \(γ_m\) (and symmetrically down under). These lenses are to be chosen rather freely with the only condition that \(\Im h\) must be negative along them (positive in \(\mathbb{C}_-\)). This condition is guaranteed by (2.1).

The two spindle-shaped regions between \(γ_m\) and the lenses are usually called upper/lower lips (relative to the orientation of \(γ_m\). At this point one introduces the auxiliary matrix-valued function \(Y(z)\) as follows

\[
Y(z) = e^{-\frac{2i}{ε} g(∞)σ_3} \Gamma(z) \begin{cases} 
\begin{vmatrix} 
e^{-\frac{2i}{ε} g(z)σ_3} & 1 \\
e^{-\frac{2i}{ε} h(z)} & 1 \\
e^{-\frac{2i}{ε} g(z)σ_3} & 1 \\
e^{-\frac{2i}{ε} h(z)} & 1 
\end{vmatrix} & \text{outside the lips,} \\
\begin{vmatrix} 1 & -e^{-\frac{2i}{ε} h(z)} \\
0 & 1 \\
1 & e^{-\frac{2i}{ε} h(z)} \\
0 & 1 
\end{vmatrix} & \text{in the upper lip in } \mathbb{C}_+, \\
\begin{vmatrix} 1 & e^{-\frac{2i}{ε} h(z)} \\
0 & 1 \\
1 & -e^{-\frac{2i}{ε} h(z)} \\
0 & 1 
\end{vmatrix} & \text{in the lower lip in } \mathbb{C}_+. 
\end{cases}
\]

The definition of \(Y(z)\) in \(\mathbb{C}_-\) is done respecting the symmetry in Prop. 2.1, namely

\[
Y(z) = (Y(\bar{z}))^*, \ z \in \mathbb{C}_-. 
\] (2-19)

The jumps for the matrix \(Y(z)\) are reported in Fig. 5.
Figure 5: The jumps for the RHP for $Y$. The shaded region is where $\Im h < 0$ (the “sea”). The blue contour is the main arc $\gamma_m$, the black contour in $\mathbb{C}_+$ is the complementary arc $\gamma_c$ and the red contours are the lenses. The green circles show the boundaries of $\Delta_\alpha$ and $\Delta_+$. 
The model RHP. In the limit $\varepsilon \to 0$, according to the signs (2.1), the jump matrices on the complementary arc $\gamma_c$ and on the lenses are approaching the identity matrix $1$ exponentially fast. Removing these contours from the RHP for $Y(z)$, we will have only one remaining contour $\gamma_m$ with the constant jump matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ on it. This is the model RHP. Calculating the $(1,2)$ entry of the residue at infinity (see (2-7)) of the solution to the model RHP, one obtains the leading order term of the genus zero solution as follows ([26])

$$q_0(x, t, \varepsilon) = \Im \alpha(x, t)e^{i\Phi(x, t)};$$

(2-20)

where

$$\Phi(x, t) = 4g(\infty; x, t).$$

(2-21)

A direct calculation of $g(\infty; x, t)$ that uses (3-67) yields (1-3).

To justify removing contours with exponentially small jump matrices, one has to calculate the error estimates coming from neighborhoods of points $\alpha$, $\bar{\alpha}$ and $\mu_+$ (for $\alpha$ and $\mu_+$, these neighborhoods are shown as green circles in Fig. 5). This is often done through the local parametrices. We shall consider the construction of the parametrices near the point $\mu_+$ as already done and known to the reader, see [26]. The only information that we need is that these parametrices allow to approximate the exact solution to within an error term $E(z) = 1 + O(\varepsilon)$ uniformly on compact subsets of the genus zero region.

3 Analysis near the gradient catastrophe point

Let $\alpha = \alpha(x, t) \in \mathbb{C}_+$ be the branch-point in the genus zero region, where $(x, t)$ is close to the point of gradient catastrophe $(x_0, t_0)$.

For generic values of $(x, t)$ the function $h(z; x, t)$, according to (2-15), has the behavior $\frac{i}{\varepsilon}h(z; x, t) \sim O(z - \alpha)^{3/2}$; at the point $(x_0, t_0)$ of gradient catastrophe the behavior is instead $\frac{i}{\varepsilon}h(z : x_0, t_0) \sim O(z - \alpha)^{5/2}$. Thus for $(x, t)$ in the vicinity of this point we obtain

$$\frac{i}{\varepsilon}h(z; x, t) = \frac{1}{\varepsilon}(z - \alpha)^{3/2} \left(C_0 + C_1(z - \alpha) + O(z - \alpha)^2\right),$$

(3-1)

where $\alpha = \alpha(x, t)$ is the branch-point and $C_0, C_1$ are some functions of $x, t$. The gradient (umbilic) catastrophe point is the one for which $C_0(x_0, t_0) = 0$ but $C_1(0, 0) \neq 0$, this latter inequality being our standing genericity assumption.

According to (2-17) and (3-1),

$$C_0 = C_0(x, t) = \lim_{\varepsilon \to 0} \frac{i h'(z) \sqrt{z - \alpha}}{R(z)} = \frac{\sqrt{2ib}}{3\pi} \int_{\gamma_m} \frac{f'(\zeta)}{(\zeta - \alpha)R(\zeta)_+} d\zeta.$$  

(3-2)
Lemma 3.1 The value of $C_1$ at the point of gradient catastrophe is given by

$$C_1 = \frac{2\sqrt{2ib}}{15\pi} \int_{\hat{\gamma}_m} f''(\zeta) \frac{d\zeta}{(\zeta - \alpha)R(\zeta)_+}.$$  

(3-3)

Proof. To obtain $C_1$, we notice that at the point of gradient catastrophe $(x_0, t_0)$

$$h''(z) = \frac{R(z)}{2\pi i} \int_{\hat{\gamma}_m} \frac{f''(\zeta)}{(\zeta - z)R(\zeta)_+} d\zeta,$$  

(3-4)

where $z$ is inside the loop $\hat{\gamma}_m$. (This formula is not correct when $(x, t) \neq (x_0, t_0)$.) Then, similarly to (3-2),

$$C_1 = C_1(x, t) = \lim_{z \to \alpha} \frac{i h''(z) \sqrt{z - \alpha}}{R(z)} = \frac{2\sqrt{2ib}}{15\pi} \int_{\hat{\gamma}_m} \frac{f''(\zeta)}{(\zeta - \alpha)R(\zeta)_+} d\zeta.$$  

(3-5)

Q.E.D.

The goal of this section is that of introducing a suitable conformal coordinate $\zeta$ near $z = \alpha$ as in the definition below.

Definition 3.1 (Scaling coordinate) The scaling coordinate $\zeta(z) = \zeta(z; x, t, \varepsilon)$ and the exploration parameter $\tau = \tau(x, t, \varepsilon)$ are defined by

$$\frac{i}{\varepsilon} h(z; x, t) = \frac{4}{5} \zeta^{\frac{5}{2}} + \tau \zeta^{\frac{3}{2}},$$  

(3-6)

where $\zeta(\alpha) \equiv 0$.

The expression (3-6) is the normal form of the singularity defined by $h(z; x, t)$ (in the sense of singularity theory [2]).

The detailed analysis of $\tau(x, t; \varepsilon)$ on space-time will be accomplished in Sect. 3.1; for the remainder of this section we dwell a bit on the details of the construction of $\tau, \zeta$ starting from the power-series expansion of $h(z; x, t)$.

Let us denote the expansion of $h(z, x, t)$ as

$$i h(z; x, t) = C_0(x, t)(z - \alpha)^{\frac{5}{2}} + C_1(x, t)(z - \alpha)^{\frac{3}{2}} + \mathcal{O}(z - \alpha)^{\frac{7}{2}}. $$  

(3-7)

For $\Delta t = 0 = \Delta x$ we have

$$\frac{i}{\varepsilon} h(z; x_0, t_0) = C_1(z - \alpha)^{\frac{3}{2}} (1 + \mathcal{O}(z - \alpha)) $$  

(3-8)

and then the function $\zeta$ and the parameter $\tau$ are defined by the formula

$$\zeta(z) = \zeta(z; x_0, t_0, \varepsilon) := \left( \frac{5i}{4\varepsilon} h(z; x_0, t_0) \right)^{\frac{2}{3}} \Leftrightarrow \frac{i}{\varepsilon} h(z; x_0, t_0) = \frac{4}{5} \zeta^{\frac{5}{2}}, \quad \tau = 0.$$  

(3-9)
Thus, the function \( h(z; x_0, t_0) \) has a \textbf{singularity} (in the sense of \textit{singularity theory}, i.e. the study of normal forms of degeneracies of critical values) at \( z = \alpha \). For \( x \neq x_0, \ t \neq t_0 \) this function undergoes a (smooth) deformation by which the coefficient \( C_0(x, t) \) acquires a nonzero value that, consequently, is inherited by \( \tau \).

In the language of singularity theory this defines a (partial) \textbf{unfolding} of the singularity. It is a standard theorem \cite{2} that for any such deformation there is a family of changes of coordinates \( z \mapsto w \) so that

\[
\begin{align*}
\text{ih}(z; x_0, t_0) &= w^3 + T(x, t)w^2, \quad w = w(z; x, t), w(\alpha(x, t); x, t) \equiv 0, \quad (3-10)
\end{align*}
\]

where \( T(x, t) \) and \( w(z; x, t) \) have the same smoothness class as the family of the deformation.

\textbf{Remark 3.1} To be more specific, changing variable from \((z - \alpha)\) to \( q = \sqrt{z - \alpha} \) we then have a singularity for \( \text{ih} \) of type \( A_4 \), with additional symmetry

\[
\begin{align*}
\text{ih}(q) &= -\text{ih}(-q). \quad (3-11)
\end{align*}
\]

Then the theorem guarantees the existence of a conformal change \( Q = Q(q) \) such that any deformation can be recast into

\[
\begin{align*}
\text{ih}(q; x, t) &= Q^5 + T_1Q^3 + T_2Q^2 + T_1Q + T_0, \quad (3-12)
\end{align*}
\]

where \( Q \) is a local biholomorphic equivalence depending analytically on the deformations. The oddness forces \( T_0 = T_2 = 0 \) and the fact that our particular deformation for \( \text{ih} \) starts with \( q^3 \) forces \( T_1 = 0 \). Since the theorem guarantees the existence of such analytic family of change of coordinates, a computation manipulating series allows to easily set up a recursive algorithmic procedure to find this function. See the next \textbf{Remark 3.2}. The most pertinent reference is Chapter 8 in \cite{2}.

\textbf{Remark 3.2} It is not hard to find –recursively– the expansion of \( w \) and \( T \) in terms of the coefficients of the series of \( \text{ih} \). If we set (for brevity we shift \( \alpha \) to the origin, without loss of generality)

\[
\begin{align*}
\text{ih}(z) &= C_0z^2 + C_1z^2 + \sum_{j=2}^{\infty} C_jz^{2j+2}, \quad w = w_1z + \sum_{j=2}^{\infty} w_jz^j \quad (3-13)
\end{align*}
\]

we can equate the series expansions of

\[
\begin{align*}
\text{ih}(z) &= w^2 + Tw^2. \quad (3-14)
\end{align*}
\]

From the coefficient of \( z^2 \) we have \( T = \frac{C_0}{w_1^2} \) and all the remaining coefficients of the expansion of \( w \) can be determined in terms of \( w_1 \) and the \( C_j \)’s. The first few are

\[
\begin{align*}
w_2 &= -2 \left( \frac{w_1^{1/2} - C_1^4}{C_0^4} \right) (w_1^2 + w_2^2C_1^4 + w_1C_1^8 + w_1^2C_1^{12} + C_1^{16})w_1, \quad (3-15)
\end{align*}
\]

\[
\begin{align*}
w_3 &= \frac{(9w_1^5 - 8C_1^2w_1^5 - 6C_2C_0 - C_1^2w_1^5)}{9C_0^2}w_1. \quad (3-16)
\end{align*}
\]
The requirement that each term in the expansion should be **analytic** at \( C_0 = 0 \) determines \( w_1 \) uniquely. For example from (3-15) we must have \( w_1 = C_1^{\frac{5}{2}} + \mathcal{O}(C_0) \); plugging this into (3-16) one sees that there can be at most a simple pole at \( C_0 = 0 \) and setting the residue to zero we determine the next coefficient in the expansion of \( w_1 \). For example we have

\[
\begin{align*}
    w_1 &= C_1^{\frac{5}{2}} + \frac{3C_0C_2}{25C_1^{\frac{3}{2}}} + \frac{C_0^2(45C_3C_1 - 72C_2^2)}{625C_1^{\frac{19}{2}}} + \mathcal{O}(C_0^3), \\
    w_1 &= C_1^{\frac{5}{2}} - \frac{6C_0C_2}{25C_1^{\frac{3}{2}}} + \mathcal{O}(C_0^2), \\
    T &= \frac{C_0}{C_1^{\frac{5}{2}}} + \mathcal{O}(C_0^3).
\end{align*}
\]

While it is clear that this recursive procedure determines a formal expansion whose coefficients are analytic at \( C_0 = 0 \), it is not clear whether the expansion should be convergent. However the above-mentioned theorem guarantees the existence of such analytic expansion and hence it must coincide with this formal manipulation.

In order to translate the normal form (3-10) into the desired one (3-6) we need to perform a simple rescaling

\[
w = \varepsilon \left( \frac{5}{4} \right)^{\frac{5}{2}} \zeta, \quad T = \varepsilon \left( \frac{4}{5} \right)^{\frac{5}{2}} \tau.
\]

The function \( \zeta \) is locally univalent in a neighborhood of \( z = \alpha \) and \( \zeta(\alpha) \equiv 0 \). The function \( \tau \) is analytic in \( C_0 \) at \( C_0 = 0 \). Their local behaviour is

\[
\begin{align*}
    \zeta &= \varepsilon \left( \frac{5}{4} \right)^{\frac{5}{2}} C_1 \left( 1 - \frac{6C_0C_2}{25C_1^{\frac{3}{2}}} + \mathcal{O}(C_0^2) \right) (z - \alpha)(1 + \mathcal{O}(z - \alpha)), \\
    \tau &= \varepsilon^{-\frac{5}{2}} C_0 \left( \frac{4}{5C_1} \right)^{\frac{5}{2}} (1 + \mathcal{O}(C_0)).
\end{align*}
\]

We point out that the determination of the root is fixed uniquely by the requirement that the image of the main arc (cut) where \( \Re h \equiv 0 \) be mapped to the **negative real** \( \zeta \)-axis.

Repeating identical considerations for the behaviour of \( h \) near \( \overline{\sigma} \) we define \( \tilde{\zeta}(z; x, t) = \zeta(\overline{\sigma}; x, t) \). Before we can proceed with the detailed asymptotic analysis of the Riemann–Hilbert Problem 2.1, we need to establish more precisely the relation between the complex parameter \( \tau(x, t, \varepsilon) \) and the \((x, t)\)-plane.

We shall consider the **scaling limit** in which \( \tau \) is uniformly bounded; this means that \( T(x, t) \) in (3-10) must tend to zero as a \( \mathcal{O}(\varepsilon^{\frac{5}{2}}) \). This means that we will be considering some shrinking neighborhood of the point of umbilic catastrophe \((x_0, t_0)\), which will be determined in more details in Sect. 3.1.

**Remark 3.3** Incidentally, one could construct critical initial data for which there is a more degenerate gradient catastrophe \( \varepsilon \frac{\partial h}{\partial t}(z; x_0, t_0) = \varepsilon (z - \alpha) \frac{2k+3}{2k+1} (C_k + \mathcal{O}(z - \alpha)) \) with \( C_k \neq 0 \). The case \( k = 0 \) correspond
to a regular (non-gradient catastrophe) point \((x_0, t_0)\), where the local parametrix is written in terms of Airy functions. The case \(k = 1\) is the one under scrutiny now and corresponds to a parametrix written in terms of Painlevé I. For \(k \geq 2\) it is easy to speculate that the PI parametrix needs to be substituted by a member of the Painlevé I hierarchy. This will be investigated elsewhere.

### 3.1 The map \((x, t) \mapsto \tau(x, t; \varepsilon)\).

The goal of this section is to determine the dependence of \(C(x, t)\), \(\alpha\), etc. on the space-time variables \((x, t)\). Here and henceforth we use the notation \(\alpha_0 = a_0 + ib_0 = \alpha(x_0, t_0)\).

#### Theorem 3.1

Near the point of gradient catastrophe \((x_0, t_0)\) the behavior of \(\alpha\) is

\[
\Delta \alpha^2 = K^2(\Delta x + 2(\alpha_0 + a_0)\Delta t) + \mathcal{O}(\Delta t^2 + \Delta x^2),
\]

with

\[
K^2 := \left( \frac{i}{4\pi} \int_{\hat{\gamma}_m} \frac{f''(\zeta)}{(\zeta - \alpha_0)R(\zeta)} d\zeta \right)^{-1}.
\]

**Proof.** The branch-point \(\alpha(x, t)\) is determined implicitly by the modulation equations (2-16) that can be written (see [26]) as

\[
\begin{align*}
\bar{F}(\alpha, \bar{\alpha}, x, t) &= \frac{1}{2i\pi} \left[ \oint_{\hat{\gamma}_m} \frac{f'(w)\delta w}{R_+(w)} , \oint_{\hat{\gamma}_m} \frac{w f'(w)\delta w}{R_+(w)} \right]^T = 0, \\
f(z; x, t) &= f_0(z) - z x - 2t z^2,
\end{align*}
\]

where \(\hat{\gamma}_m\) is a closed contour around the main arc and \(R(z)\) is chosen with the determination that behaves as \(z\) for \(z \to \infty\). The Jacobian of \(\bar{F}\) is ([27], Lemma 3.4)

\[
\frac{\partial \bar{F}}{\partial (\alpha, \bar{\alpha})} = \frac{1}{2} \begin{bmatrix} \frac{h'(z)}{R(z)} \big|_{z = \alpha} & \frac{h'(z)}{R(z)} \big|_{z = \bar{\alpha}} \\ \frac{h'(z)}{R(z)} \big|_{z = \bar{\alpha}} & \frac{h'(z)}{R(z)} \big|_{z = \alpha} \end{bmatrix}.
\]

On account of the Schwartz symmetry \(h(z) = \overline{h(\bar{z})}\), the determinant of this matrix is

\[
\det \frac{\partial \bar{F}}{\partial (\alpha, \bar{\alpha})} = \frac{ib}{4} \left| h'(z) \right|^2 |_{z = \alpha}.
\]

For \((x, t)\) away from the gradient catastrophe in the genus zero region we have \(h'(z) = C_0(z - \alpha)^{\frac{1}{2}}(1 + \mathcal{O}(z - \alpha))\) with \(C_0 = C_0(x, t) \neq 0\), so that the Jacobian (3-27) is invertible (as long as \(b = 3\alpha > 0)\) and the standard implicit function theorem yields \(\alpha(x, t)\) as a smooth function of \((x, t)\). At the point of gradient catastrophe we have \(h(z) = C_1(x_0, t_0)(z - \alpha)^{\frac{1}{2}}(1 + \mathcal{O}(z - \alpha))\) with \(C_1 = C_1(x_0, t_0) \neq 0\); therefore the matrix (3-27) is not invertible and—in fact— it is the zero matrix.

#### Remark 3.4

Note that the Jacobian matrix \(\frac{\partial \bar{F}}{\partial (x, t)}\) is

\[
\frac{\partial \bar{F}}{\partial (x, t)} = \frac{1}{2i\pi} \begin{bmatrix} \oint \frac{\delta w}{R(w)} & \oint \frac{4a \delta w}{R(w)} \\ \oint \frac{w \delta w}{R(w)} & \oint \frac{4a^2 \delta w}{R(w)} \end{bmatrix} = \begin{bmatrix} 1 & 4a \\ a & 4a^2 - 2b^2 \end{bmatrix},
\]

where \(a = a_0 + ib_0\).
where \( \alpha = a + ib \). For any \( \alpha \in \mathbb{C}_+ \), we have

\[
\det \frac{\partial \bar{F}}{\partial (x,t)} = -2b^2 \neq 0 .
\]  

(3-30)

We expand each component \( F_j \), \( j = 1, 2 \), of \( \bar{F} \) around \( \alpha_0 = \alpha(x_0, t_0) \) and its complex conjugate (we denote only the dependence on \( \alpha \) with the understanding that \( \bar{F} \) depends also on \( \bar{\alpha} \)) as

\[
F_j(\alpha; x, t) = F_j(\alpha_0; x, t) + \partial_\alpha F_j \Delta \alpha + \partial_{\bar{\alpha}} F_j \Delta \bar{\alpha} + \frac{1}{2} [\Delta \alpha, \Delta \bar{\alpha}] H_j \left[ \begin{array}{c} \Delta \alpha \\ \Delta \bar{\alpha} \end{array} \right] + \mathcal{O}(|\Delta \alpha|^3),
\]

(3-31)

where \( \Delta \alpha = \alpha - \alpha_0 \) and \( H_j \) denotes the Hessian of \( F_j \) evaluated at \( (\alpha_0, x, t) \):

\[
H_j = \frac{1}{8\pi i} \left[ \begin{array}{c} 3 \int \frac{w^j f'(w)\delta w}{(w - \alpha)^2 R_+(w)} - \int \frac{w^j f'(w)\delta w}{(w - \alpha)(w - \bar{\alpha})R_+(w)} \\ \int \frac{w^j f'(w)\delta w}{(w - \alpha)(w - \bar{\alpha})R_+(w)} \end{array} \right].
\]

(3-32)

The off-diagonal entries of \( H_j \) vanish because

\[
\frac{1}{8\pi i} \int \frac{w^j f'(w)\delta w}{(w - \alpha)(w - \bar{\alpha})R_+(w)} = -\frac{1}{16\pi b} \left[ \int \frac{w^j f'(w)\delta w}{(w - \alpha)R_+(w)} - \int \frac{w^j f'(w)\delta w}{(w - \bar{\alpha})R_+(w)} \right] = \frac{1}{16\pi b} \left[ z^j h'(z) \right]_{z=\alpha} = 0
\]

(3-33)

(3-34)

Let us denote by \( G_{j-1} \) the \((1,1)\) entry of \( H_j \) \( (j = 1, 2) \), so that

\[
G_j = \frac{3}{8\pi i} \int \frac{w^j f'(w)\delta w}{(w - \alpha)^2 R_+(w)} = \frac{3}{4\pi} \left[ z^j h'(z) \right]_{z=\alpha}, \quad j = 0, 1.
\]

(3-35)

Using the fact that \( \frac{\partial \bar{F}}{\partial (\alpha, \bar{\alpha})} = 0 \), we then have (we suppress the \((x, t)\) dependence for brevity)

\[
F_j(\alpha) = F_j(\alpha_0) + \Re(G_{j-1} \Delta \alpha^2) + \mathcal{O}(|\Delta \alpha|^3).
\]

(3-36)

Expanding also in \( x, t \) near \( x_0, t_0 \), we obtain

\[
0 = F_j(\alpha) = F_{j,x} \Delta x + F_{j,t} \Delta t + \Re \left( G_{j-1} \Delta \alpha^2 \right) + \mathcal{O}(|\Delta \alpha|^3) + \mathcal{O}(\Delta t^2 + \Delta x^2),
\]

(3-37)

where \( \Delta x = x - x_0 \), \( \Delta t = t - t_0 \). Equation (3-37) shows that \( \Delta \alpha \) is of order \( \sqrt{|\Delta x| + |\Delta t|} \). Solving equations (3-37) \( (j = 1, 2) \) for \( \Delta \alpha \) using the expressions for the \( x, t \)-derivatives in (3-29) yields

\[
(\Delta \alpha)^2 = -\frac{2}{G_0 G_1 - G_1 G_0} \left[ (\bar{G}_1 - a \bar{G}_0) \Delta x + (4\alpha_0 \bar{G}_1 - (4\alpha_0^2 - 2b_0^2) \bar{G}_0) \Delta t \right] + \mathcal{O}(\Delta t^2 + \Delta x^2)
\]

(3-38)

At \( (x_0, t_0, \alpha_0) \) we have \( G_1 = \alpha_0 G_0 \) and hence a simplification of the above equation yields

\[
\Delta \alpha^2 = -\frac{1}{G_0} [\Delta x + 2(\alpha_0 + \alpha_0) \Delta t] + \mathcal{O}(\Delta t^2 + \Delta x^2).
\]

(3-39)

From (3-35) we find \( G_0 = -\frac{a_1 \xi C_1}{8\sqrt{\pi \beta}} \) with \( C_1 \) defined by (3-3) and, thus, obtain (3-23). \textbf{Q.E.D.}
Corollary 3.1 Let us introduce the scaling variables \( x = x_0 + \varepsilon \frac{4}{5} X, \ t = t_0 + \varepsilon \frac{4}{5} T \) in the neighborhood \( D \) of \( (x_0, t_0) \). Then

\[
(\alpha_0 - \alpha)^2 = \varepsilon \frac{4}{5} K^2 (X + 2(\alpha_0 + a_0)T) (1 + \mathcal{O}(\varepsilon^{\frac{4}{5}})),
\]

\[
\tau = -\frac{4i\sqrt{2ib_0}}{3K} \left( \frac{4}{5C_1} \right)^{\frac{2}{5}} (X + 2(\alpha_0 + a_0)T)^{\frac{2}{5}} (1 + \mathcal{O}(\varepsilon^{\frac{2}{5}})),
\]

where \( K \) is given by (3-24) and \( \alpha_0 = a_0 + ib_0 \). In the above formulæ \( \sqrt{2ib_0} \) stands for \( e^{i\pi/4}\sqrt{2b_0} \) (\( b_0 > 0 \)), all the roots are principal and the argument of \( C_1 \) is determined in such a way that the direction of the main arc is

\[
\arg(z - \alpha_0) = \pi - \frac{2}{5} \arg(C_1).
\]

so that the main arc is mapped to the negative real \( \zeta \)-axis.

Moreover (3-40) can be written using (3-41, 3-24) as

\[
\Delta \alpha = \varepsilon^{\frac{2}{5}} \frac{\tau C}{2},
\]

\[
C := \left( \frac{5C_1}{4} \right)^{\frac{2}{5}} = \left( \frac{\sqrt{2ib_0}}{6\pi} \int_{\gamma_m} \frac{f''(\zeta)}{\zeta - \alpha_0} R(\zeta) d\zeta \right)^{\frac{2}{5}}.
\]

Proof. It is known from the Whitham modulation equations [27] that

\[
\partial_x h(z; x, t) = R(z), \quad \partial_t h(z; x, t) = 2(z + a)R(z),
\]

\[
R(z) := \sqrt{(z - \alpha)(z - \overline{\alpha})}, \quad \alpha = \alpha(x, t) = a + ib.
\]

Expanding in series near \( z = \alpha \) and comparing the terms we find

\[
\alpha_x C_0 = -\frac{2i\sqrt{2ib}}{3}, \quad \alpha_t C_0 = -\frac{4i\sqrt{2ib}(\alpha + a)}{3}
\]

Since \( \alpha_x = \Delta \alpha_x \), using Thm. 3.1 we have

\[
C_0 = -\frac{2i\sqrt{2ib}}{3\alpha_x} = -\varepsilon^{\frac{2}{5}} \frac{4i\sqrt{2ib}}{3\sqrt{\frac{8\sqrt{2ib_0}}{15C_1}}} \sqrt{X + 2(a + a)T} (1 + \mathcal{O}(\varepsilon^{\frac{2}{5}}))
\]

and the formula for \( \tau \) follows from the expression (3-22). Q.E.D.

Corollary 3.1 states that –in the scaling limit– the map \( (x_0 + \varepsilon^{\frac{4}{5}} X, t_0 + \varepsilon^{\frac{4}{5}} T) \rightarrow \tau^2 \) is a diffeomorphism.

For definiteness and later purposes we introduce the following definition.

Definition 3.2 The map

\[
v(x, t; \varepsilon) = \frac{3}{8} \tau^2(x, t; \varepsilon)
\]

will be called the Painlevé coordinatization.

\footnote{\textsuperscript{5}Since \( \zeta = e^{-\frac{2}{5}} C(z - \alpha_0)(1 + \ldots) \) and \( C = (4/5C_1)^{\frac{2}{5}} \), the condition for the main arc is \( \arg(\zeta) = \pi \), whence the formula (3-42).}
Figure 6: The level lines of the $\Re[ih]$ in the $\zeta$–plane and for different values of $\tau$. When $\phi := \arg(\tau) = \frac{2\pi}{5}$ or $\phi = \frac{6\pi}{5}$ the complementary arc or one of the rims of the lens is pinched and the solution enters the genus 2 region. Thus the genus 0 region corresponds to $\frac{2\pi}{5} < \arg(\tau) < \frac{6\pi}{5}$.

According to the previous analysis, the function $v(x, t; \varepsilon)$ is a local diffeomorphism in the neighborhood $D$ of the point of gradient catastrophe. To this end we formulate the following corollary.

**Corollary 3.2** In terms of the scaling coordinates $X = \frac{x - x_0}{\varepsilon^{4/5}}$, $T = \frac{t - t_0}{\varepsilon^{4/5}}$ the function $v$ reads, to the leading order as

$$v = \frac{3}{8} \tau^2 = -i \sqrt{\frac{2ib_0}{C}} (X + 2(a_0 + a_0)T)(1 + O(\varepsilon^{\frac{2}{5}})).$$

(3-50)

**Proof.** A simple manipulation from the Def. 3.2 and eq. (3-41). Q.E.D.

**Remark 3.5** The complex–valued function $v(x, t; \varepsilon)$ is a diffeomorphism that maps the neighborhood $D$ of $x_0, t_0$ of size $O(\varepsilon^{\frac{2}{5}})$ onto a neighborhood $V$ of the origin $v = 0$ uniformly bounded (in $\varepsilon$); in later sections $v$ will play the role of independent variable for the Painlevé I.
3.1.1 The image of the genus two region

We can now find the opening of the sector $S$ in the $\tau^2$-plane (and hence $v$-plane as well) that is the image of the genus two part of the neighborhood $D$ of the gradient catastrophe point.

The critical value of $\frac{1}{\varepsilon} h(z; x, t) = \frac{1}{5} \zeta^2 + \tau \zeta^2$ is given by $2 \zeta + \frac{3}{4} \tau = 0$, so the critical value $\zeta_c$ of $\zeta$ is $\zeta_c = -\frac{3}{4} \tau$. Thus

$$\frac{i}{\varepsilon} h(z_c; x, t) = \frac{4}{5} \zeta_c^2 + \tau \zeta_c^2 = \frac{4}{5} \left( -\frac{3}{4} \tau \right)^2 + \tau \left( -\frac{3}{4} \tau \right)^2.$$  \hspace{1cm} (3-51)

The breaking curves are determined implicitly by $\Re[i h(z_c; x, t)] = 0$ and hence by the condition

$$\nu(\tau) = \Re \left[ ( -3 \tau )^2 + 5 \tau ( -3 \tau )^2 \right] = -2 \Re \left[ ( -3 )^2 \tau^2 \right] = 0.$$ \hspace{1cm} (3-52)

Care must be exercised due to the presence of the fractional powers: recall that the choice of conformal parameter $\zeta$ has been made so that the main arc is mapped to $\zeta < 0$; the $\zeta$ image of the critical point, $\zeta_c$ determines whether we are in the genus zero or two region as explained presently.

The breaking curves correspond to the first directions where $\nu(\tau) = 0$ starting from the $\tau > 0$ or $\tau < 0$, which is the same--

$$\cos \left( \frac{5}{2} \arg(\zeta_c) \right) = 0 \iff \varphi = \arg(\zeta_c) = \frac{\pi}{5} + \frac{2\pi k}{5}.$$ \hspace{1cm} (3-53)

Thus the two arcs of the breaking curves correspond to two rays amongst the ones below

$$\arg(\tau) \in \left\{ \frac{2\pi k}{5}, k \in \mathbb{Z} \right\}.$$ \hspace{1cm} (3-54)

In order to explain which rays we need to choose we have to consider the topology of the level lines of $\Re[i h] = \Re \left[ \frac{4}{5} \zeta^2 + \tau \zeta^2 \right]$ for different values of $\tau$. Due to the scale invariance ($\zeta \mapsto \lambda^2 \zeta$, $\tau \mapsto \lambda^3 \tau$, $\lambda > 0$) we can restrict ourselves to studying the argument of $\tau$ only. We will consider $\tau = \frac{4}{5} e^{i \phi}$. The level lines of the real part of $i h(z)/\varepsilon$ in the $\zeta$-plane for different values of $\phi$ are plotted in Fig. 6. The transition between the genus 0 and genus 2 regions occurs when the connectivity of the complementary arc and/or the rims of the lens needs to change. This happens for $\arg \tau = \frac{2\pi}{5}$ when the complementary arc is pinched between the “sea” ($\Re[i h] < 0$) or for $\arg(\tau) = \frac{6\pi}{5}$, when the main arc is about to break into two arcs.

The above discussion about the directions of the breaking curves can be summarized in the following lemma.

**Lemma 3.2** The asymptotic image of the genus zero part of the region $D$ around the point of gradient catastrophe $(x_0, t_0)$ under the diffeomorphism $v = v(x, t, \varepsilon)$ in the limit $\varepsilon \to 0$ is the sector

$$\arg(v) \in \left[ \frac{4\pi}{5}, \frac{12\pi}{5} \right]$$ \hspace{1cm} (3-55)

in the Painlevé $v$-plane.
This is so because the argument of $v$ is twice the argument of $\tau$ and from the previous discussion.

The complementary sector of aperture $\frac{2 \pi}{5}$ is the asymptotic image of the genus–two region in the Painlevé plane; in the following section we will describe the asymptotics of $q(x,t,\varepsilon)$ in terms of the tritronquée solution, which has –conjecturally– poles only in such a sector [12].

**Remark 3.6 (The angle between the breaking curves in the $(x,,t)$–plane)** Since we know now that the breaking curves correspond to the directions $\arg(v) = \frac{2\pi}{5}, \frac{4\pi}{5}$, we can compute the angle at which the two breaking curves meet at the point of gradient catastrophe. Using (3-50), we calculate

$$\partial_X v = \kappa, \quad \partial_T v = 2\kappa(\alpha + a), \quad \kappa := -i \sqrt{\frac{2ib}{C}} = -i \left( \frac{-24\pi b^2}{\int_{\gamma_m} f''(\zeta) \frac{d\zeta}{(\zeta - \alpha_0) R(\zeta)}} \right)^{\frac{1}{4}}. \quad (3-56)$$

Now we have $v = \kappa(X + 4aT + 2ibT)(1 + O(\varepsilon^2))$.

The breaking curves correspond to $\arg(v) = \frac{2\pi}{5}, \frac{4\pi}{5}$ in the Painlevé plane. Thus, we have $\arg L_1 = \frac{2}{5} \pi - \arg \kappa$ and $\arg L_2 = \frac{4}{5} \pi - \arg \kappa$ respectively. These values of $\arg L$ define the rays

$$t = \frac{\tan(\arg L_1)}{2b - 4a \tan(\arg L_j)} x, \quad j = 1, 2 \quad (3-57)$$

on the physical plane that are tangential to the breaking curves at the point of gradient catastrophe. So, the angle $\Theta$ in the $(x,t)$–plane between the breaking curves is

$$\tan(\Theta) = \frac{\tan(\varphi_2) - \tan(\varphi_1)}{1 + \frac{\tan(\varphi_2)}{\tan(\varphi_1)} \frac{\tan(\varphi_1)}{\tan(\varphi_2)}}, \quad (3-58)$$

where $\varphi_j = \arg L_j, j = 1, 2$.

**Example 3.1** In the case when initial data (1-2) for the NLS (1-1) is $A(x) = -\text{sech} x$ and $S'(x) = -\mu \tanh x$, where $\mu \geq 0$, the point of gradient catastrophe was calculated to be $(x_0, t_0) = \left(0, \frac{1}{2(\mu + 2)}\right)$, the corresponding value $\alpha(x_0, t_0) = i\sqrt{\mu + 2}$ and the slopes of the two breaking curves at the point of gradient catastrophe -

$$\pm m = \pm \cfrac{\cot \frac{\pi}{5}}{2\sqrt{\mu + 2}} \quad (3-59)$$

see [26]. Since the diffeomorphism $v(x,t,\varepsilon)$ asymptotically (as $\varepsilon \to 0$) maps breaking curves (of slope $\pm m$) onto the rays $\arg v = \frac{2}{5} \pi \pm \frac{2}{5} \pi$ respectively, we have (all angle equations are mod $2\pi$)

$$\arg D = \frac{\kappa}{\sqrt{1 + m^2}} |\pm 2(\alpha + a)m| = \frac{8}{5} \pi \pm \frac{4}{5} \pi, \quad (3-60)$$

the expression for $m$ is provided in Theorem 5.6, [26], however the expression for $m$ in Theorem 1.1, [26], should be replaced by its inverse.
where the vector $\vec{u} = (1, m)$ and $D_{\vec{u}}$ denote the derivative in the direction of $\vec{u}$. Thus,

$$\tan^{-1} \frac{2bm}{1 + 4am} + \arg \varpi = \frac{2\pi}{5}, \quad \tan^{-1} \frac{2bm}{4am - 1} + \arg \varpi + \pi = \frac{4\pi}{5}. \quad (3-61)$$

Equation (3-61) defines the slope of the breaking curves at the point of gradient catastrophe $(x_0, t_0)$ in terms of $\alpha(x_0, t_0)$ and $C_1(x_0, t_0)$. Let us show the slopes (3-61), found in [26], are consistent with Lemma 3.2. Substitution of the slopes $\pm m$ from (3-59) into (3-61) yields

$$\frac{\pi}{2} \pm \frac{\pi}{5} + \arg \varpi = \frac{8\pi}{5} \pm \frac{4\pi}{5}. \quad (3-62)$$

We now use (3-3) to calculate $\varpi$. Considering for simplicity the solitonless case $\mu \geq 2$, we obtain

$$C_1 = \frac{4i\sqrt{2b}}{15\pi} \int_{\mathbb{R}} \frac{3f''(\zeta)}{(\zeta - \alpha)R(\zeta)} d\zeta \quad (3-63)$$

where $f'(\zeta) = \frac{\pi}{2} \text{sign}(1 - \chi_{[-T,T]})$, $T = \sqrt{\frac{\mu^2}{4} - 1}$, was calculated in [26], Sect. 6.4. (The choice of branch of $R(\zeta)$ in (3-3) and elsewhere in this paper is opposite to those used in [26]. That is why the sign of $f''(\zeta)$ in (3-3) is opposite to the one that would have been calculated according to [26]). Direct calculation of the latter integral yields

$$C_1 = -\frac{2i\sqrt{2b}}{15} \left[ \frac{1}{(T - \alpha)R(T)} - \frac{1}{(T + \alpha)R(-T)} \right], \quad (3-64)$$

so after some algebra we get $C_1 = \frac{32\sqrt{2}}{15(\mu + 2)^2}$. Then $\arg C_1 = \frac{\pi}{2} + 2\pi k$ for some $k \in \mathbb{Z}$. Taking into account (3-43), we obtain

$$\arg \varpi = \frac{\pi}{2} + \frac{\pi}{4} - \frac{\pi}{20} = \frac{2\pi k}{5}. \quad (3-65)$$

Substitution of (3-65) into (3-62) shows that (3-62) holds with $k = -1 \pmod{5}$. Thus, slopes (3-61) from [26] are consistent with Lemma 3.2. We also conclude that

$$C_1 = \frac{32\sqrt{2}}{15(\mu + 2)^2} e^{-\frac{2\pi}{5} i}. \quad (3-66)$$

Example 3.2 Using the same example as in Ex. 3.1 with $\mu = 2$ (see Fig. 7) one can verify that the main arc has $\vartheta = \frac{3\pi}{10} = \frac{3\pi}{2} - \frac{4\pi}{5}$ and hence the nodes are in the directions $\arg(v_{\text{node}} - v_p) = \frac{\pi}{10} + k\pi$.

Looking at the orientation of the bisecant of the sector of poles of the tritronquée we see that—in this case—the nodes are aligned perpendicularly to the bisecant.

3.2 The behavior of the phase $\Phi(x, t)$ near the point of gradient catastrophe
The genus zero (Whitham) approximation $q_0(x, t; \varepsilon)$ to the semiclassical solution $q(x, t; \varepsilon)$ is the leading approximation and it is valid uniformly in the “genus zero” region; its dependence on $x, t$ is determined by the modulation (Whitham) equations [26].

These equations can actually be utilized to extend the definition of $q_0(x, t; \varepsilon)$ beyond the genus zero region where –however– the actual solution $q(x, t; \varepsilon)$ will have a different behavior (typically of oscillatory nature), see, for example, [4], where $q_0$ was extended beyond the breaking curve. It will actually turn out that $q_0(x, t; \varepsilon)$ can still be used in a neighborhood of the point of gradient catastrophe as a “reference” for describing the actual behavior of $q(x, t; \varepsilon)$. For this reason we briefly analyze $q_0(x, t; \varepsilon)$ near $(x_0, t_0)$.

In the genus zero region, the leading order approximation of the amplitude and the phase of $q(x, t; \varepsilon)$, according to (2-20), (2-21), are given by $b(x, t)$ and $4g(\infty; x, t)$ respectively, where the branchpoint $\alpha(x, t) = a(x, t) + ib(x, t)$ and the $g$-function $g$ was defined by the scalar RHP (2-10), (2-11). In ([26] Lemma 4.3, formula (4.43)) it was shown that

$$g_z(z; x, t) = \frac{1}{2} \left( \sqrt{(z - \alpha)(z - \overline{\alpha})} - z \right), \quad g_t(z; x, t) = (z + a)(z - \alpha)(z - \overline{\alpha}) - z^2,$$

(3-67)

where the determination of the square root is such that they behave like $z$ at infinity\(^7\). Hence for the phase $\Phi(x, t) = 4g(\infty; x, t)$ we have

$$\Phi_x = -2a(x, t) = -2\Re(\alpha), \quad \Phi_t = -4a(x, t)^2 + 2b(x, t)^2 = -2\Re(\alpha + a).$$

(3-68)

**Theorem 3.2** The phase $\Delta \Phi(x, t) := \Phi(x, t) - \Phi(x_0, t_0)$ near the point of gradient catastrophe $(x_0, t_0)$ has the expansion

$$\Delta \Phi(x, t) = -2\Re \left( a_0 [\Delta x + (a_0 + a_0)\Delta t] \right) - \Re \left( \frac{4K}{3} (\Delta x + 2(a_0 + a_0)\Delta t)^2 \right) + O(\Delta x^2 + \Delta t^2) \quad (3-69)$$

$$= -2a_0 \Delta x - 2(2a_0^2 - b_0^2)\Delta t - \varepsilon \Re \left( \sqrt{\frac{2i}{K}} \frac{\tau^3}{C} \right) + O(\Delta x^2 + \Delta t^2), \quad \text{(3-70)}$$

where $\Delta x = x - x_0$, $\Delta t = t - t_0$ and $K$, $C$ and $\tau$ were defined in (3-24), (3-43) and in (3-22, 3-41) respectively.

**Proof.** If we write $\alpha = a_0 + \Delta \alpha$ we have from (3-68)

$$\Phi_x = -2a_0 - 2\Delta a, \quad \Phi_t = -4a_0^2 + 2b_0^2 - 8a_0 \Delta a + 4b_0 \Delta b - 4\Delta a^2 - 2\Delta b^2.$$

(3-71)

\(^7\)Note that in [26] the determination being used is the opposite one.
In our problem $\Delta \alpha = O(\varepsilon^{\frac{3}{5}})$ and hence we can approximate

$$
\Phi_x = -2\Re a_0 - 2\Re \Delta \alpha, \quad \Phi_t = -2\Re (\alpha_0 (\alpha_0 + a_0)) - 2\Re (2(\alpha_0 + a_0)\Delta \alpha) + O(\varepsilon^{\frac{3}{5}}).
$$

(3-72)

From (3-40) and a straightforward integration we obtain (3-69). Q.E.D.

**Remark 3.7** The formulæ for $\Phi$ allow us to extend the definition of $q_0(x,t)$ within the genus-2 region using (2-20); taking the imaginary part of (3-43) we find that

$$
b(x,t) = b_0 + \frac{1}{2} \varepsilon^{\frac{3}{5}} \Im \left( \frac{\tau}{C} \right) (1 + O(\varepsilon^{\frac{3}{5}}))
$$

(3-73)

and hence

$$
q_0(x,t) = \left( b_0 + \frac{1}{2} \varepsilon^{\frac{3}{5}} \Im \left( \frac{\tau}{C} \right) \right) \times \\
\times \exp \left[ i \varepsilon \left[ \Phi_0 - 2a_0 \Delta x - 2(2a_0^2 - b_0^2)\Delta t - \varepsilon^{\frac{3}{5}} \Re \left( \sqrt{\frac{24}{C_0^2}} \frac{\tau^3}{8} \right) \right] (1 + O(\Delta x^2 + \Delta t^2)) \right]
$$

(3-74)

(3-75)

In the following we will understand that $q_0(x,t)$, $b(x,t)$, $\alpha(x,t)$ have been extended as indicated above. It is to be noticed that this extension is discontinuous due to the definition of $\tau$ (3-41) involving a square root. Such ambiguity will not be present in the final formulæ.

### 4 The Riemann–Hilbert problem for Painlevé I

The heart of the present paper is in the detailed analysis of the “local parametrix”. This will be constructed in terms of the so-called Psi-function $\Psi(\xi, v)$ of the Painlevé I Lax system, that depends on the spectral variable $\xi$ and the Painlevé variable $v$. The analysis of the Riemann–Hilbert problem for $\Psi(\xi, v)$ is contained in a number of papers and books, see, for example, [20, 13]; this analysis, however, does not cover the case when the Painlevé variable $v$ is at or is approaching a pole $v = v_p$ of the solution to P1 that is defined through $\Psi(\xi, v)$ (as the isomonodromy condition). Furthermore, it can be shown ([22]) that $\Psi(\xi, v)$ has a pole at $v = v_p$. **Analysis of the RHP for $\Psi(\xi, v)$ at or close to a pole $v = v_p$ of the tritronquée solution (transcendent) $y(v)$ to P1 is a matter of crucial**

![Figure 8: The jump matrices for the Painlevé 1 RHP: here $\vartheta := \vartheta(\xi; v) := \frac{4}{5} \xi^{\frac{3}{2}} - \frac{5}{3} \xi^{\frac{3}{2}}$.](image-url)
importance in our study of the height and the shape of the spikes. We start from the summary of the known facts about P1. Let the invertible matrix-function \( P = P(\zeta, v) \) be analytic in each sector of the complex \( \xi \)-plane shown on Fig. 8 and satisfy the multiplicative jump conditions along the oriented boundary of each sector with jump matrices shown on Fig. 8. The entries of the jump matrices satisfy the following symmetry conditions

\[
1 + \beta_0 \beta_1 = -\beta_{-2}, \\
1 + \beta_0 \beta_{-1} = -\beta_2, \\
1 + \beta_{-2} \beta_{-1} = \beta_1,
\]

so that the jump matrices in Fig. 8 depend, in fact, only on 2 complex parameters (that uniquely define a solution to P1). The matrix function \( P(\zeta, v) \) is uniquely defined by the following RHP.

**Problem 4.1 (Painlevé 1 RHP [20])** The matrix \( P(\xi; v) \) is locally bounded, admits boundary values on the rays shown in Fig. 8 and satisfies

\[
P(\xi) = \frac{\xi^{\sigma_3/4}}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \left( I + O(\xi^{-1}) \right),
\]

where the jump matrices \( M = M(\xi; v) \) are the matrices indicated on the corresponding ray in Fig. 8.

For any fixed values of the parameters \( \beta_k \), Problem 4.1 admits a unique solution for generic values of \( v \); there are isolated points in the \( v \)-plane where the solvability of the problem fails as stated and it will need to be modified.

The piecewise analytic function \( \Psi(\xi, v) = P(\xi, v)e^{i\theta_3} \), where \( \theta := \theta(\xi; v) = \frac{4}{5} \xi^2 - v \xi^2 \), solves a slightly different RHP with constant jumps on the same rays. The new jump matrices can be obtained from the old ones by replacing the exponential factor in every jump matrix by one. It then follows that it solves the ODE [20]

\[
\frac{d}{d\xi}\Psi(\xi, v) = \begin{bmatrix} y' & 2\xi^2 + 2y\xi - v + 2y^2 \\ 2\xi - 2y & -y' \end{bmatrix} \Psi(\xi, v),
\]

where \( y = y(v) \) solves the Painlevé I equation

\[
y'' = 6y^2 - v.
\]

Direct computations using the ODE 4-4 and formal algebraic manipulations of series along the lines of [29, 17, 15, 16] show that \( \Psi \) admits the following formal solution

\[
\Psi = \frac{\xi^{\sigma_3/4}}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \times \\
\times \left( I - \frac{H_1\sigma_3}{\sqrt{\xi}} + \frac{H_1^2 + y\sigma_2}{2\xi} + \frac{(v^2 - 4H_1^3 - 2y')^2}{24\xi^2} \sigma_3 + \frac{i(y^2 - 2iH_1y)^2}{4\xi^2} \sigma_1 + O(\xi^{-2}) \right) e^{i\theta_3},
\]

29
where $O(\xi^{-2})$ denotes the sum of terms with higher order powers of $\xi^{-1}$. Such an expansion has to be understood as representing the asymptotic behavior of an actual solution of the ODE (4-4) within a sector of angular width smaller than $\frac{4\pi}{5}$.

### 4.1 Failure of the Problem 4.1

The choice of the parameters $\beta_k$ is (transcendentally) equivalent to the choice of Cauchy–initial values for the ODE (4-5); it is known since the original work of Painlevé that the only (finite) singularities of Eq. (4-5) are poles and these poles coincide precisely with the set of exceptional values of $v$ for which Problem 4.1 fails to admit a solution. From the P1 equation (4-5) for $y(v)$ one can find the Laurent expansion around any such pole $v = v_p$ to be of the form

$$y(v) = \frac{1}{(v - v_p)^2} + \frac{v_p}{10}(v - v_p)^2 + \frac{1}{6}(v - v_p)^3 + \beta(v - v_p)^4 + \frac{v_p^2}{300}(v - v_p)^6 + O((v - v_p)^7).$$

We can then proceed as follows [22]: define the matrix $\hat{\Psi}(\xi; v)$ via

$$\Psi(\xi; v) := (\xi - y)^{-\sigma_3/2} \left[ \begin{array}{cc} \frac{1}{2} (y' + \frac{1}{2(\xi - y)}) & 1 \\ 1 & 0 \end{array} \right] \hat{\Psi}(\xi; v),$$

$$\hat{\Psi}(\xi; v) := G(\xi; v)\Psi(\xi; v),$$

$$G(\xi; v) := \left[ \begin{array}{cc} 0 & 1 \\ 1 & -\frac{1}{2} (y' + \frac{1}{2(\xi - y)}) \end{array} \right] (\xi - y)^{\sigma_3/2}.$$

It then satisfies the ODE

$$\frac{d}{d\xi} \hat{\Psi}(\xi; v)e^{\sigma_3} = \left[ \begin{array}{cc} 0 & 2 \\ V(\xi; v) & 0 \end{array} \right] \hat{\Psi}(\xi; v)e^{\sigma_3},$$

$$V(\xi; v) := 2\xi^3 - v\xi - 2y^3 + yv + \frac{1}{2}(y')^2 + \frac{y'}{2(\xi - y)} + \frac{3}{8(\xi - y)^2}.$$  \hspace{1cm} (4-9)

It is promptly seen from a direct computation that the function $V(\xi; v)$ admits a limit as $v \to a$

$$V(\xi; v) = 2\xi^3 - v\xi + \frac{1}{2}(y')^2 - 2y^3 + vy - \frac{y'}{2y} + \frac{y'\xi}{y(\xi - y)} + \frac{3}{8(\xi - y)^2} = 2\xi^3 - v\xi + \hat{H}_I + \frac{y'\xi}{y(\xi - y)} + \frac{3}{8(\xi - y)^2} \to 2\xi^3 - v_p\xi - 14\beta,$$

$$\hat{H}_I := H_I - \frac{y'}{2y} = -14\beta - \frac{v_p}{6}(v - v_p)^3 + O(v - v_p)^4,$$

where the convergence is uniform over compact subsets of the $\xi$ plane [22]. It was also shown ibidem that $\hat{\Psi}(\xi; v)$ tends to a finite (holomorphic) matrix $\hat{\Psi}(\xi; v_p)$ which satisfies the (essentially a scalar ODE)

$$\frac{d}{d\xi} \hat{\Psi}(\xi; v_p) = \left[ \begin{array}{cc} 0 & 2 \\ 2\xi^3 - v_p\xi - 14\beta & 0 \end{array} \right] \hat{\Psi}(\xi; v_p) =: A(\xi; v_p, \beta)\hat{\Psi}(\xi; v_p).$$ \hspace{1cm} (4-13)
Most importantly, the solutions $\hat{\Psi}(\xi; v)$ to the system (4-9) and $\hat{\Psi}(\xi; v_p)$ to the limiting system (4-13) have the same Stokes’ matrices. In fact, the Stokes’ matrices for these solutions are the same as those of $\Psi(\xi, v)$, except minor changes introduced by the obvious nontrivial monodromy of the transformation $G(\xi, v)$ in (4-8). That follows from the isomonodromic property of the equation (4-4) that defines the P1 equation and the fact that the left multiplication by $G(\xi, v)$ does not change the Stokes’ phenomenon.

Remark 4.1 The formal monodromy around $\xi = \infty$ for $\Psi$ is $-i\sigma_2$ but the one of $\hat{\Psi}$ is $i\sigma_2$ because of the additional monodromy $(-1)$ around $\xi = y$. Using the explicit expression (4-8) the reader can also verify that

$$
\hat{\Psi}(\xi, v) = \frac{\xi^{\sigma_3/4}}{\sqrt{2}} \left[ \begin{array}{cc}
1 & i \\
1 & -i
\end{array} \right] \left( I + O(\xi^{-1/2}) \right) e^{\vartheta\sigma_3}
$$

(4-14)

4.2 Analysis in a neighborhood of the pole of PI

It is essential for our application to investigate the behavior in which $v \to v_p$ at a certain rate, namely, to study how (and in which sense) the limiting expansion (4-16) is approached. It is proven in Appendix A that

$$
\hat{\Psi}(\xi, v) = \xi^{-\frac{3}{2}\sigma_3} \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3) \left( \begin{array}{cc}
1 & 0 \\
0 & i
\end{array} \right) + O \left( \xi^{-\frac{1}{2}}, y^{-4}, e^{-p_{24}|z|^{5/2}} \right) \left( \frac{\sqrt{\xi + \sqrt{y}}}{\sqrt{\xi - y}} \right)^{\sigma_3} e^{\vartheta(\xi; v)\sigma_3}
$$

(4-15)

where the term $O(\xi^{-1/2})$ is uniform w.r.t. $v$ in a finite neighborhood of the pole $v = v_p$.

The limiting case of (4-15) for $y = \infty$ (i.e. for $v = v_p$) is

$$
\hat{\Psi}(\xi; v_p) = \frac{\xi^{-\frac{3}{2}\sigma_3}}{\sqrt{2}} \left[ \begin{array}{cc}
-i & -1 \\
1 & -i
\end{array} \right] \left( 1 + 14\beta \frac{\sigma_3}{\sqrt{\xi}} + \frac{98\beta^2}{\xi} + \frac{v_p^2}{24} + \frac{1372\beta^3}{3} \right) \frac{\sigma_3}{\xi^2} + \\
+ \left( \frac{7}{12} \beta + \frac{4802}{3} \beta^4 \right) \frac{1}{\xi^2} + \frac{v_p}{8\xi^2} \sigma_2 + O(\xi^{-3/2})
$$

(4-16)

where the expressions for the various coefficients are obtained from the formal solution of the ODE (4-13) using the standard techniques in [29].

4.2.1 The tritronquée transcendent

The term tritronquée dates back to Boutroux [6, 7]. A generic solution to the ODE (4-5) has infinitely many poles that accumulate asymptotically for large $|v|$ along the rays $\text{arg}(v) = \frac{2\pi k}{5}$. Certain one-parameter families (corresponding to the vanishing of one of the Stokes’ parameters $\beta_k$ of the associated Riemann–Hilbert problem (4.1) have the properties that along one of these rays the poles eventually stop appearing as $|v| \to \infty$, or they get truncated, whence the term tronquée. If two consecutive $\beta_j$’s vanish we have 5 very special solution for which the poles truncate along three consecutive rays, whence the naming tritronquée. In fact there are –strictly speaking– several tritronquée solutions: they correspond to the vanishing of the $\beta_j$’s on two consecutive rays in Fig. 8. There are –thus– 5 such functions.
However a closer look [20] reveals that the solutions $y(v; \{\beta_j\})$ have the symmetry

$$y(v; \{\beta_k\}) = e^{\frac{4i\pi}{5}} y \left( e^{\frac{2i\pi}{5}} v; \{\beta_{k+2n}\} \right), \quad n \in \mathbb{Z}, \beta_{k+5} := \beta_k$$

and hence there is essentially only one tritronqué solution.

Such a solution is characterized by the following theorem.

**Theorem 4.1 ([20], Thm. 2.1 and Corollary 2.5 and eqs. (2.72))**

There exists a unique solution $y(v)$ corresponding to $\beta_0 = 0 = \beta_{-1}$ with the asymptotics

$$y = \sqrt{\frac{v-i\pi}{6}} + O(v^{-2}), \quad v \to \infty,$$

$$\arg(v) \in \left[ -\frac{6\pi}{5} + 0, \frac{2\pi}{5} - 0 \right].$$

(4-18)

Such a solution has no poles for $|v|$ large enough in the above sector (or –equivalently– has at most a finite number of poles within said sector).

It is conjectured in [12] that the tritronqué solution has actually no poles at all within said sector: all poles (of which it is known to be infinitely many) lie in the complementary sector, represented in the shaded area in Fig. 9. Such conjecture is so far supported by rather compelling numerical evidence and is consistent with WKB analysis [22].

Whatever the standing of the conjecture, it is quite essential for us that the poles “want to be” in the sector of the v plane that indeed corresponds to the genus-two region under the identification of the v–plane and the (scaled) X,T plane given by Def. 3.2.

We are going to see below that –in fact– each pole of the tritronqué corresponds to a “spike” in the asymptotic solution of NLS and such spikes are to be expected only in the region of paroxysmal oscillations (genus 2).

While our analysis does not rely in the least on the position of such poles, the “physical intuition” strongly suggests that indeed they will be confined to the indicated wedge.

### 5 Leading order approximation of NLS away from a spike

As we have seen in Section 3.1 (Def. 3.2), the map $v = v(x,t;\varepsilon)$ maps diffeomorphically a neighborhood $D$ of size $O(\varepsilon^\frac{4}{5})$ to the complex-v plane, namely, to the plane of the independent variable of the Painlevé I tritronqué transcendent.

---

8 Note that in [20] the independent variable $x$ coincides with our $-v$. 

---
As we have seen in the previous section, the region of the \((x,t)\)-plane that corresponds to the genus-0 region is mapped to the complement of the sector \(\arg(v) \in (2\pi/5, 4\pi/5)\). In the present section and the following Section 6 we shall consider two different limits in which a point \((x,t)\) approaches the point of gradient catastrophe \((x_0, t_0)\). These limits, express in terms of the diffeomorphism \(v(x,t; \varepsilon)\), are:

- \(v(x,t; \varepsilon)\) is in a compact subset of the “swiss–cheese” region \(K_\delta\), \(\delta > 0\) is constant\(^9\), so that \(v\) is at least on the distance \(\delta\) away from any pole of the tritronquée solution; this is the case considered in this section;
- \(v(x, t; \varepsilon) \in B_\delta\), where \(B_\delta\) is a disk of radius \(\delta = O(\varepsilon^{1/5} + \nu)\), \(\nu \geq 0\), centered at \(v = v_p\) - one of the poles of the tritronquée i.e., \(v\) can approach a pole \(v_p\) of the tritronquée at a rate \(\varepsilon^{1/5}\) or faster; this is the case considered in Section 6.

Of course, we shall consider also the case where \(v\) is exactly at a pole (Section 6), as well as the case when \(v(x,t; \varepsilon) \to v_p\) at the rate \(O(\varepsilon^{1/5} - \nu)\), where \(\nu \in (0, 1/5)\) (this section). What will transpire from the analysis is the following enticing picture:

- The spikes are in one-to-one correspondence with the poles of the tritronquée solution;
- The transversal size of the spikes is of order \(O(\varepsilon)\);
- The separation of the spikes is of order \(O(\varepsilon^{4/5})\);
- The amplitude of the spikes is three times the amplitude of \(q_0(x_0, t_0)\), i.e., the amplitude predicted by the modulation equations for the genus-zero region;
- Each spike has the shape of a rational breather.

### 5.1 Asymptotic behaviour away from the spikes

In the genus zero region, the leading order solution to the RHP for \(Y(z)\), i.e., solution to the model RHP, is ([26])

\[
\Psi_0(z) = \frac{1}{2} \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \left( \frac{z - \alpha}{z - \bar{\alpha}} \right)^{\frac{3}{2}} \begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix},
\]

(5.1)

More than its specific form, it is important that near the point \(z = \alpha\) it has the behavior

\[
\Psi_0(\zeta) = O(z - \alpha) \frac{1}{\sqrt{2}} (z - \alpha)^{\frac{3}{2}} \begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix},
\]

(5.2)

with the jump matrix \(i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\) on the main arc. Here \(O(\zeta - \alpha)\) denotes a matrix function that is invertible and analytic in a neighborhood of \(\alpha\).

\(^9\)The definition of \(K_\delta\) is in Sect. 1.1.
We shall construct an approximation to the matrix \( Y(z; x, t, \varepsilon) \) appearing in (2-18) (and henceforth to the matrix \( \Gamma \)) in the form
\[
Y(z) = \begin{cases} 
  \mathcal{E}(z)\Psi_0(z) & \text{for } z \text{ outside of the disks } \mathbb{D}_\alpha, \mathbb{D}_\pi \\
  \mathcal{E}(z)\Psi_0(z)\mathcal{P}_\alpha(z) & \text{for } z \text{ inside of the disk } \mathbb{D}_\alpha, \\
  \mathcal{E}(z)\Psi_0(z)\mathcal{P}_\pi(z) & \text{for } z \text{ inside of the disk } \mathbb{D}_\pi,
\end{cases}
\]
where \( \Delta_\alpha, \mathbb{D}_\pi \) are small disks centered in \( \alpha, \bar{\alpha} \) respectively, see Fig. 5.

**Remark 5.1** There is an additional parametrix for \( Y \) in a disk \( \Delta_+ \) centered at the point \( z = \mu_+ \); however, as shown in [26], this parametrix has a uniform estimate \( O(\varepsilon) \), so it will not affect the accuracy of any of our expressions.

Due to the symmetry of the problem in Prop. 2.1 we must have
\[
\mathcal{P}_\pi(z) = (\mathcal{P}_\alpha(z))^{-1}
\]
and hence it suffices to consider the construction near the point \( \alpha \) only.

**5.1.1 Local parametrix**

The local parametrix \( \mathcal{P}_\alpha(z) = \mathcal{P}(z) \) (we understand and suppress the subscript \( \alpha \) ) must satisfy a certain number of properties (see Thm. 5.1), one of them being the restriction of \( \mathcal{P}(z) \)
\[
\mathcal{P}(z) \bigg|_{z \in \partial \mathbb{D}_\alpha} = 1 + o_\varepsilon(1)
\]
on the boundary of \( \mathbb{D}_\alpha \), where \( o_\varepsilon(1) \) denotes some infinitesimal of \( \varepsilon \), uniformly in \( z \in \partial \mathbb{D}_\alpha \) and in \((x, t) \in \mathcal{K} = v^{-1}(K_\delta) \) (here a small \( \delta > 0 \) is fixed).

If \( \mathcal{P} \) (and the corresponding parametrix near \( z = \pi \theta \)) can be found that satisfy those requirements then the “error matrix” \( \mathcal{E}(z) \) is seen to satisfy a small-norms RHP and be uniformly close to the identity. More precisely the matrix \( \mathcal{E} \) has jumps on: (a) the parts of the lenses and of the complementary arcs that lie outside of the disks \( \mathbb{D}_\alpha, \mathbb{D}_\pi \), and; (b) on the boundaries of the two disks \( \mathbb{D}_\alpha, \mathbb{D}_\pi \). The jumps in (a) are exponentially close to the identity jump in any \( L^p \) norm (including \( L^\infty \)) while on the boundary of the disk \( \mathbb{D}_\alpha \) we have
\[
\mathcal{E}_+(z) = \mathcal{E}_-(z)\Psi_0(z)\mathcal{P}(z)\Psi_0^{-1}(z) \bigg|_{z \in \partial \mathbb{D}_\alpha} = \mathcal{E}_-(1 + o_\varepsilon(1))
\]
Since \( \mathcal{E}(z) = 1 + \mathcal{O}(z^{-1}) \) as \( z \rightarrow \infty \) it follows [11] that \( \|\mathcal{E}(z) - 1\| \rightarrow 0 \) (uniformly on the Riemann–sphere) and that the rate of convergence is estimated as the same as the \( o_\varepsilon(1) \) that appears in (5-5) as \( \varepsilon \rightarrow 0 \).

Thus, the accuracy of the approximation (i.e. neglecting the term \( \mathcal{E}(z) \)) is directly related to the rate of convergence to the identity matrix of the local parametrix \( \mathcal{P} \) on the boundary of the disk(s).
Definition 5.1 (Local parametrix away from the spikes.) Let \( \zeta(z; \varepsilon) \) be the local conformal coordinate introduced in Def. 3.1 so that
\[
\frac{i}{\varepsilon} h(z; x, t) = \theta(\zeta; \tau) = \frac{4}{5} \zeta^2 + \tau \zeta^\frac{3}{2}.
\] (5-7)

Let \( \Psi(\zeta; v) \) denote the Psi–function of the Painlevé I problem with \( \beta_0 = 0 = \beta_{-1} \) (and \( \beta_{-2} = -1, \beta_2 = -1 \)). The parametrix \( P(z) \) is defined by
\[
P(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \zeta^{-\frac{3}{2}} \Psi \left( \zeta + \frac{\tau}{2}; \frac{3}{8} \tau^2 \right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} e^{\theta(\zeta; \tau)} \sigma_3.
\] (5-8)

Theorem 5.1 The matrix \( P \) satisfies:

1. Within \( \mathbb{D}_\alpha \), the matrix \( P(z) \) solves the exact jump conditions on the lenses and on the complementary arc;

2. On the main arc (cut) \( P(z) \) satisfies
\[
P_+(z) = \sigma_2 P_-(z) \sigma_2,
\] (5-9)

so that \( \Psi_0 P \) within \( \mathbb{D}_\alpha \) solves the exact jumps on all arcs contained therein (the left-multiplier in the jump (5-9) cancels against the jump of \( \Psi_0 \));

3. The product \( \Psi_0(z) P(z) \) (and its inverse) are –as functions of \( z \)– bounded within \( \mathbb{D}_\alpha \), namely the matrix \( P(z) \) cancels the growth of \( \Psi_0 \) at \( z = \alpha \);

4. The restriction of \( P(z) \) on the boundary of \( \mathbb{D}_\alpha \) is
\[
P(z) \bigg|_{z \in \partial \mathbb{D}_\alpha} = 1 + \left( H_I + \frac{\tau^3}{16} \right) \frac{\sigma_3}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \left[ \left( H_I + \frac{\tau^3}{16} \right)^2 + \left( y + \frac{\tau}{4} \right) \sigma_2 \right] + O(\zeta^{-\frac{3}{2}}),
\] (5-10)

where
\[
v = \frac{3}{8} \tau^2, \quad H_I = \frac{1}{2} (y')^2 + yv - 2y^3 = \int y(s) \delta s.
\] (5-11)

Proof. (1) The matrix \( \Psi(\zeta; v) \) has constant jumps of the same triangularity as the jumps indicated in Fig. 8 (with \( \beta_0 = 0 = \beta_{-1} \) and \( \beta_{-2} = \beta_2 = -1 = -\beta_1 \)). These are altogether of the opposite triangularities we need, hence the second-last (constant) matrix in (5-8). The last multiplication with \( e^{\theta(\zeta; t)} = e^{\frac{i}{\varepsilon} h(z; x, t)} \) gives the exact (non-constant) jumps on the parts of the complementary/main arcs and lenses within the disk \( \mathbb{D}_\alpha \). On the other hand, the matrix
\[
F(z) := \frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \zeta(z)^{-\frac{3}{2}}
\] (5-12)

has the jump \( F_+(z) = i \sigma_2 F_-(z) \) on the left, whence the part (2).
As for part (3), the product $\Psi_0(z)P(z)$ is a bounded function of $z$ because the singularities of $\Psi_0(z)$ are canceled by those of $F(z)$

$$\Psi_0(z)F(z) = \frac{1}{2} \left[ \begin{array}{ccc} -i & -1 & i \\ 1 & 1 & 1 \end{array} \right] (z - \alpha) \frac{\zeta^\frac{\tau}{2}}{z - \alpha} \left[ \begin{array}{ccc} i & 1 & -i \\ 1 & 1 & -1 \end{array} \right] \frac{1}{\sqrt{2}} \left[ \begin{array}{ccc} -i & i & 1 \\ 1 & 1 & 1 \end{array} \right] \zeta(z)^{-\frac{\tau}{2}} =$$

$$= \frac{1}{2} \left[ \begin{array}{ccc} -i & i & 1 \\ 1 & 1 & 1 \end{array} \right] (z - \alpha) \frac{\zeta^\frac{\tau}{2}}{z - \alpha} \zeta(z)^{-\frac{\tau}{2}} = O(1) \quad (5-13)$$

since $\zeta(z) = O(z - \alpha)$. In fact, we see that the product is actually analytic.

Finally, part (4) follows from the asymptotics of $\Psi(\zeta; v)$, for which, for $z \in \partial D_\alpha$ the conformal coordinate $\zeta(z)$ grows (homothetically) as $\varepsilon^{-\frac{2}{\tau}}$ and hence we can use the expansion (4-6) for $\Psi(\zeta; v)$ near infinity.

To see how it works let us recall the notation

$$\vartheta(\zeta; v) := \frac{4}{5}\xi\frac{\tau^2}{\varepsilon^2} - v\xi^\frac{3}{2}, \quad v = \frac{3}{8} \tau^2, \quad (5-14)$$

so that we can write

$$\mathcal{P}(z) = \frac{1}{2} \left[ \begin{array}{ccc} -i & -1 & i \\ 1 & 1 & 1 \end{array} \right] \xi^{-\frac{\tau}{2}} \left( \zeta + \frac{\tau}{2} \right)^{\sigma_3} \left[ \begin{array}{ccc} 1 & -i & i \\ i & 1 & -1 \end{array} \right] \left( 1 - \frac{H_1\sigma_3}{\sqrt{\zeta + \frac{\tau}{2}}} + \frac{H_1^2(1 + y\sigma_2)}{2\zeta + \tau} + O(\zeta^{-\frac{2}{\tau}}) \right) \times$$

$$\times e^{\vartheta(\zeta; v)\sigma_3} \left[ \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 1 \end{array} \right] e^{\vartheta(\zeta; v)\sigma_3} = (5-15)$$

$$= 1 + \frac{\tau}{8}\sigma_2 + O(\zeta^{-2})$$

$$= \left( 1 + \frac{H_1\sigma_3}{\sqrt{\zeta}} + \frac{H_1^2(1 + (y + \frac{\tau}{4})\sigma_2)}{2\zeta} + O(\zeta^{-\frac{2}{\tau}}) \right) e^{\vartheta(\zeta; v)\sigma_3} = (5-16)$$

We also have

$$\vartheta(\zeta; \tau) - \vartheta \left( \zeta + \frac{\tau}{2} : -\frac{3}{8} \tau^2 \right) = \frac{4}{5}\xi\frac{\tau^2}{\varepsilon^2} + \tau\xi^\frac{3}{2} - \frac{4}{5}\left( \zeta + \frac{\tau}{2} \right)^{\frac{3}{2}} + \frac{3}{8} \tau^2 \left( \zeta + \frac{\tau}{2} \right)^{\frac{1}{2}} = \frac{\tau^3}{16\sqrt{\zeta}} + O(\zeta^{-\frac{3}{2}}), \quad (5-17)$$

so that –continuing from (5-16)– we have

$$\mathcal{P}(z) \big|_{z \in \partial D_\alpha} = 1 + \left( H_1 + \frac{\tau^3}{16} \right) \frac{\sigma_3}{\sqrt{\zeta}} + \frac{1}{2\zeta} \left[ \left( H_1^2 + \frac{\tau^3}{16} \right) \frac{\sigma_3}{\sqrt{\zeta}} + \frac{1}{8} \left( y + \frac{\tau}{4} \right) \sigma_2 + O(\zeta^{-\frac{3}{2}}) \right] = (5-18)$$

$$= 1 + \left( H_1 + \frac{\tau^3}{16} \right) \frac{\sigma_3}{\sqrt{\zeta}} + \frac{1}{2\zeta} \left[ \left( H_1 + \frac{\tau^3}{16} \right)^2 + \frac{1}{8} \left( y + \frac{\tau}{4} \right) \sigma_2 \right] + O(\zeta^{-\frac{3}{2}}). \quad (5-19)$$

Q.E.D.

At this point we already know that the error term $E(z)$ in (5-3) is within $O(\varepsilon^{\frac{1}{2}})$ from the identity; if we simply ignore it, we would get the leading order approximation to $Y$ and –in turn– to $\Gamma$, which would produce the leading order approximation of the NLS solution $q(x, t, \varepsilon)$. Next, we will find the first subleading approximation by solving the first step in the iterative approximation of the error term itself.
5.2 Subleading correction

**Theorem 5.2** The behavior of the focusing NLS in the domain $D$ of the point of gradient catastrophe (scaled like $\varepsilon^{\frac{4}{5}}$) is given by

\[
q(x,t,\varepsilon) = b_0 \left[ 1 - 2\varepsilon^{\frac{2}{5}} \Im \left( \frac{y(v)}{C\delta_0} \right) + \mathcal{O}(\varepsilon^{\frac{4}{5}}) \right] \times \\
\times \exp \frac{i}{\varepsilon} \left[ \Phi_0 - 2 \left( a_0 \Delta x + (2a_0^2 - b_0^2) \Delta t \right) + 2\varepsilon^{\frac{6}{5}} \Re \left( \frac{2i}{C\delta_0} H_I(v) \right) \right],
\]

where $\mathcal{O}(\varepsilon^{\frac{4}{5}})$ term is uniform "away from spikes", i.e., is uniform in $(x,t) \in \tilde{K}_{\delta} = v^{-1}(K_{\delta})$ with some fixed $\delta > 0$. Here $\Phi_0 = \Phi(x_0,t_0)$, $a(x_0,t_0) = a_0 + ib_0$, $C = \left( \frac{3C_1}{4} \right) \frac{5}{2}$ given by (3.44),

\[
H_I = \frac{1}{2} (y'\nu)^2 + vy\nu - 2y^3(v)
\]
is the Hamiltonian of the Painlevé I equation, evaluated along the tritronquée solution appearing in Theorem 4.1, while $v$ can be expressed (Corollary 3.2) as

\[
v = -i\varepsilon^{-\frac{4}{5}} \sqrt{\frac{2ib_0}{C}} \left( \Delta x + 2(a_0 + a_0) \Delta t \right) (1 + \mathcal{O}(\varepsilon^{\frac{4}{5}}))
\]
unformly in $D$.

**Remark 5.2 (Comparison with the Conjecture from [12])** The approximation formula (5-20) “away from the spikes” is consistent with the conjecture from [12] about the behavior of the amplitude and the phase of the genus zero (modulated plane wave) approximation $q_0(x,t,\varepsilon)$ in the genus zero (non-oscillatory) part of the neighborhood $D$ of the point of gradient catastrophe. This conjecture can be written as

\[
U + i\sqrt{U_0} V = U_0 + i\sqrt{U_0} V_0 + \varepsilon^\frac{2}{5} K y(v) + \mathcal{O}(\varepsilon^\frac{4}{5}),
\]
where $U = |q|^2$, $V = \varepsilon \partial_x \arg(q)$, with $K$ being a (complex) constant. Using our expression (5-20), $U_0 = b_0^2$, and the fact that $H_I^\prime(v) = y(v)$ we find

\[
U = b_0^2 - 4\varepsilon^\frac{2}{5} \Im \left( \frac{b_0 y(v)}{C} \right) + \mathcal{O}(\varepsilon^\frac{4}{5}), \quad V = -2a_0 + 4\varepsilon^\frac{2}{5} \Re \left( \frac{y(v)}{C} \right),
\]
so that

\[
U + ib_0 V = b_0^2 - 4\varepsilon^\frac{2}{5} b_0 \Im \left( \frac{y(v)}{C} \right) - 2ia_0 b_0 + 4\varepsilon^\frac{2}{5} b_0 \Re \left( \frac{y(v)}{C} \right) + \mathcal{O}(\varepsilon^\frac{4}{5}) = \\
= b_0^2 - 2ia_0 b_0 + \varepsilon^\frac{2}{5} \frac{4ib_0}{C} y(v) + \mathcal{O}(\varepsilon^\frac{4}{5}).
\]

To replace the error estimate $\mathcal{O}(\varepsilon^\frac{4}{5})$ in (5-25) with the estimate $\mathcal{O}(\varepsilon^\frac{4}{5})$ from the conjecture (5-23), if correct, would require calculation of the higher order corrections to the RHP (2.18). It may be true that
the approximation of \( q \) is in powers of \( \varepsilon^{\frac{2}{5}} \) rather than in powers of \( \varepsilon^{\frac{1}{5}} \), but that, again, would require additional analysis. The situation here may resemble the analogous statements in random matrix theory in regards to the expansion of the partition function in even powers \( 1/N \). We also did not dwell into the notation of [12] to compare all the constants used. Finally, we omitted the term proportional to \((t-t_0)^2\) in (5-23) because, in our scaling, it is of order \( \varepsilon^{\frac{2}{5}} \) and hence not “visible” at this order of approximation.

**Proof of Thm. 5.2.** In a RHP of the form

\[
E(z) = 1 + O(z^{-1}), \quad z \to \infty,
\]

\[
E_+(z) = E_-(z) \left( 1 + \Delta M(z) \right)
\]

the solution (if it exists) can be written as

\[
E(z) = 1 + \frac{1}{2i\pi} \int \frac{E_-(s)\Delta M(s)ds}{s-z},
\]

where the integration extends over all contours supporting the jumps (it is a simple exercise using Sokhotskii–Plemelji formula to verify that the above singular integral equation is equivalent to the Riemann–Hilbert formulation). If –in addition– the term \( \Delta M(z) \) is sufficiently small in the appropriate norms (at least in \( L^\infty \) and \( L^2 \) of the contours) then the above formula can be used in an iterative approximation approach, where

\[
E^{(0)}(z) \equiv 1,
\]

\[
E^{(j+1)}(z) = 1 + \frac{1}{2i\pi} \int \frac{E^{(j)}(s)\Delta M(s)ds}{s-z}, \quad j = 0, 1, \ldots
\]

which can be shown to converge to the desired solution.

In the RHP for \( E(z) \), stated in Section 5.1.1, we have \( \Delta M \) exponentially small (in \( \varepsilon \)) in any \( L^p \) norm on all parts of the contour outside the disks \( \Delta_\alpha \) and \( \Delta_{\overline{\alpha}} \), and approaching zero in the \( L^\infty \) norm on the boundaries \( \partial \mathbb{D}_\alpha, \partial \mathbb{D}_{\overline{\alpha}} \). The latter estimate is valid in any \( L^p \) norm due to compactness. We shall thus find the first correction term in the above approximation procedure.

As noted in Theorem 5.1, part 4, and in (5-6), the jump of \( E \) on \( \partial \mathbb{D}_\alpha \) is (note that \( \Psi_0 \) commutes with \( \sigma_2 \))

\[
E_+(z) = E_-(z) \Psi_0(z) \mathcal{P}_\alpha^{-1}(z) \Psi_0^{-1}(z),
\]

where

\[
\Psi_0 \mathcal{P}_\alpha^{-1} \Psi_0^{-1} = 1 - \left( H_I + \frac{\tau^3}{16} \right) \left( \frac{1}{2} \sqrt{\frac{\tau}{\xi}} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} + \frac{1}{2\sqrt{\xi}} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \right)
\]

\[
+ \frac{1}{2\xi} \left( \left( H_I + \frac{\tau^3}{16} \right)^2 \mathbf{1} - \left( y + \frac{\tau}{4} \right) \sigma_2 \right) + O(\varepsilon^{\frac{2}{5}}) =
\]

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Due to the symmetry, the jump on the disk around $z = \alpha$ is

$$
\Psi_0 P^{-1}_\alpha \Psi_0^{-1} = 1 + \frac{\eta_1}{2} \left( \sqrt{\frac{p}{\zeta}} N_1 + 1 \right) + \frac{1}{2\zeta} \left( \eta_1^2 \mathbf{1} - u_2 \sigma_2 \right),
$$

(5-35)

$$
p := \frac{z - \alpha}{z - \alpha}, \quad \eta_1 := \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix},
$$

(5-36)

$$
u_1 := H_{\tau} + \frac{\eta^3}{16}, \quad u_2 := (y + \frac{\tau}{4}).
$$

(5-37)

We now proceed to the computation of the first-order correction to $E(z)$ according to the formula (5-31). In that formula the integral should extend to all the jumps of $E$, which include the lenses, the complementary arcs and the disk around $\mu_+$; the former are exponentially small and the latter is of order $O(\varepsilon)$, thus they can be neglected altogether to within this order.

When $z$ is outside the disks this residue computation annihilates the analytic term with $\sqrt{p/\zeta}$ in (5-35) and yields

$$
E^{(1)}(z) = 1 - \frac{u_1}{4i\pi} \int_{|z-\alpha|=\delta} \sqrt{\frac{t - \alpha}{\zeta(t)(t - \alpha)}} \frac{dt}{t - z} + \frac{\eta_1}{4i\pi} \int_{|z-\alpha|=\delta} \sqrt{\frac{t - \alpha}{\zeta(t)(t - \alpha)}} \frac{dt}{t - z} + \frac{u_1^2 \mathbf{1} - u_2 \sigma_2}{2} \int_{|z-\alpha|=\delta} \sqrt{\frac{t - \alpha}{\zeta(t)(t - \alpha)}} \frac{dt}{t - z} + \frac{\eta_1^2 + \eta_2 \sigma_2}{2} \int_{|z-\alpha|=\delta} \sqrt{\frac{t - \alpha}{\zeta(t)(t - \alpha)}} \frac{dt}{t - z} + O(\varepsilon^3),
$$

(5-40)

$$
= 1 + \frac{\varepsilon \eta_1 N_1}{(z - \alpha) C} - \frac{\varepsilon \eta_1 N_1}{(z - \alpha) C} - \frac{\varepsilon^2 \eta_1 N_1}{(z - \alpha) C} - \frac{\varepsilon^2 \eta_1 N_1}{(z - \alpha) C} - \frac{\varepsilon^2 \eta_1 N_1}{(z - \alpha) C} - \frac{\varepsilon^2 \eta_1 N_1}{(z - \alpha) C} + O(\varepsilon^3).
$$

(5-41)

So,

$$
E^{(1)}(z) = 1 + \frac{u_1}{2} \sqrt{\frac{2ib \varepsilon z}{C}} \frac{N_1}{z - \alpha} - \frac{\eta_1}{2} \sqrt{-\frac{2ib \varepsilon z}{C}} \frac{N_1^*}{z - \alpha} - \frac{\eta_1}{2} \sqrt{-\frac{2ib \varepsilon z}{C}} \frac{N_1^*}{z - \alpha} - \frac{\eta_1}{2} \sqrt{-\frac{2ib \varepsilon z}{C}} \frac{N_1^*}{z - \alpha} - \frac{\eta_1}{2} \sqrt{-\frac{2ib \varepsilon z}{C}} \frac{N_1^*}{z - \alpha} = \eta_1 \frac{2z^2 \varepsilon}{4|C|} \frac{N_1 N_1}{z - \alpha} + \frac{u_1}{2} \sqrt{\frac{2ib \varepsilon z}{C}} \frac{N_1}{z - \alpha} + \frac{\eta_1}{2} \sqrt{-\frac{2ib \varepsilon z}{C}} \frac{N_1^*}{z - \alpha} + O(\varepsilon^3),
$$

(5-42)

(5-43)

where –by definition– $C$ appears as $\zeta(z) = \varepsilon \zeta(z - \alpha)(1 + \ldots)$. We now need to use once more (5-31) with the expression (5-43) and retain only the terms up to $\varepsilon^3$:

$$
E^{(2)}(z) = E^{(1)}(z) + \frac{\varepsilon^2}{4} \left( \frac{u_1^2 N_1 N_1}{C(z - \alpha)} + \frac{\eta_1^2 N_1^* N_1}{C(z - \alpha)} \right) + \frac{|u_1|^2 \varepsilon^2}{4|C|} \left( \frac{N_1^* N_1}{z - \alpha} \right).
$$

(5-44)

Hence the approximation of the solution $Y(z)$ is

$$
Y(z) = E(z) \Psi_0(z) = E(z) \left[ \begin{array}{cc} -i & -1 \\ 1 & i \end{array} \right] \frac{z - \alpha}{z - \alpha} \left[ \begin{array}{cc} i & \eta_1 \frac{2z^2 \varepsilon}{4|C|} \frac{N_1 N_1}{z - \alpha} \right] (z - \alpha) \left[ \begin{array}{cc} i & 1 \\ -1 & -i \end{array} \right].
$$

(5-45)
Writing $\mathcal{E}(z) = 1 + \frac{\mathcal{E}_1}{z} + \mathcal{O}(z^{-2})$ near $z = \infty$, we have

$$Y = \mathcal{E}(z) \left( 1 - \frac{\alpha - \pi}{4z} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \mathcal{O}(z^{-2}) \right) = 1 + \frac{\mathcal{E}_1}{z} + \frac{1}{2z} \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} + \mathcal{O}(z^{-2}).$$ (5-46)

The correction comes from the $(1, 2)$ entry of $\mathcal{E}_1$; we use the second iteration $\mathcal{E}^{(2)}$ and

$$\left[ \frac{u_1 e^{\pm i\theta}}{2} \sqrt{\frac{2ib}{C}} N_1 - \frac{u_1 e^{\pm i\theta}}{2} \sqrt{-\frac{2ib}{C}} N_1^* - e^\pm \left( \frac{u_1^2}{2C} + \frac{u_2^2}{2C} \right) + \frac{\varepsilon}{4} \left( \frac{u_2^2}{C} + \frac{\pi_1^2}{C} \right) \right] =$$

$$= -i e^{\pm} \frac{u_1}{2} \sqrt{\frac{2ib}{C}} - i e^{\pm} \frac{u_1}{2} \sqrt{-\frac{2ib}{C}} + \varepsilon^2 \Re \left( \frac{u_2}{C} \right) + \varepsilon^2 \Im \left( \frac{u_2}{C} \right) + \frac{\varepsilon}{2} \left( \frac{u_2}{|C|} \right) =$$

$$= -i e^{\pm} \mathcal{R} \left( \sqrt{\frac{2ib}{C}} \right) + \varepsilon^2 \Re \left( \frac{2ib}{C} u_1 \right) + \varepsilon^2 \Im \left( \frac{2ib}{C} u_1 \right) =$$

$$= -i e^{\pm} \Re \left( \sqrt{\frac{2ib}{C} u_1} \right) + \varepsilon^2 \Re \left( \frac{2ib}{C} u_1 \right).$$ (5-48)

We thus have

$$q(x, t, \varepsilon) = -2 \lim_{z \to \infty} z \Gamma_{1, 2}(z) = -2 e^{\pm i\theta} g(\infty; x, t) \lim_{z \to \infty} z (\mathcal{E} \Psi_0)_{12} =$$

$$= e^{\pm i\theta} \Phi(x, t) \left( b(x, t) + 2i e^{\pm} \Re \left( \sqrt{\frac{2ib}{C} u_1} \right) - 2i e^{\pm} \Re \left( \frac{2ib}{C} u_1 \right) \right) =$$

$$= \exp \left[ \frac{i}{\varepsilon} \Phi(x, t) + 2i e^{\pm} \Re \left( \sqrt{\frac{2ib}{C} u_1} \right) \right] \left( b(x, t) - 2i e^{\pm} \Im \left( \frac{y + \tau}{C} \right) + \mathcal{O}(\varepsilon^3) \right).$$ (5-51)

Now care must be exercised before factoring $b(x, t)$ out; indeed from (3-43) we see that

$$b(x, t) = b_0 + \frac{1}{2} e^{\pm} \Im \left( \frac{\tau}{C} \right)$$

and thus the $\tau$ in (5-51) cancels out and we obtain

$$q(x, t, \varepsilon) = b_0 \exp \left[ \frac{i}{\varepsilon} \Phi(x, t) + 2i e^{\pm} \Re \left( \sqrt{\frac{2ib}{C} u_1} \right) \right] \left[ 1 - 2i e^{\pm} \Im \left( \frac{y}{C} \right) + \mathcal{O}(\varepsilon^3) \right] =$$

$$= b_0 \exp \left[ \frac{i}{\varepsilon} \Phi(x, t) + 2i e^{\pm} \Re \left( \sqrt{\frac{2ib}{C} \left( H_1 + \frac{\tau^3}{16} \right)} \right) \right] \left( 1 - 2i e^{\pm} \Im \left( \frac{y}{C} \right) + \mathcal{O}(\varepsilon^3) \right) =$$

$$= b_0 \exp \left[ \frac{i}{\varepsilon} \Phi(x, t) + i e^{\pm} \Re \left( \sqrt{\frac{2ib}{C} \left( H_1 + \frac{\tau^3}{8} \right)} \right) \right] \left( 1 - 2i e^{\pm} \Im \left( \frac{y}{C} \right) + \mathcal{O}(\varepsilon^3) \right).$$ (5-55)
Note that we could also replace the remaining occurrences of $b$ by $b_0$ in (5-55) since it would affect the result by terms of order $O(\varepsilon^5)$. Recall here the expression (3-69) for the increment of phase near the point of gradient catastrophe:

$$
\Delta \Phi = -2a_0 \Delta x - 2(2a_0^2 - b_0^2) \Delta t - \varepsilon^6 \Re \left( \sqrt{\frac{2i}{C_b}} \frac{\tau^3}{8} \right) + O(\Delta x^2 + \Delta t^2) \tag{5-56}
$$

The reader may notice that the discontinuous term containing $\tau^3$ cancels and we complete the proof. Q.E.D.

**Remark 5.3** From the expression (5-20) it is clear that the approximation cannot hold in proximity of a pole of the tritronquée $y(v)$, for the Hamiltonian $H_I$ has a simple pole there and $y$ has a double pole.

The reader could verify that the above analysis holds as long as we approach a pole $v = v_p$ but not too quickly:

$$
v - v_p = O(\varepsilon^\frac{2}{5} - \nu), \quad \frac{1}{5} > \nu > 0 \tag{5-57}
$$

In this case the formula in the above theorem is still correct but with the error term of order $O(\varepsilon^\frac{2}{5} - 3\nu)$, and then the leading correction has—in fact—order $O(\varepsilon^\frac{2}{5} - \nu)$. Indeed the term $O(\zeta^{-\frac{3}{2}})$ contains terms with triple poles (see (4-6)), and—in general—the term $\zeta^{-\frac{3}{2}}$ has a coefficient with a pole of order $k$ in the Painlevé variable $v$. Hence the estimate $O(\varepsilon^\frac{2}{5}$ in (5-34 and following would have to be replaced throughout by $O(\varepsilon^\frac{2}{5} - 3\nu)$.

It appears that something awry is occurring when we approach a pole too fast, and a different approximation parametrix need to be constructed. This is the goal for the rest of the paper.

6 Approximation near a spike/pole of the tritronquée

With the preparatory material covered in Section 4 we shall now address the approximation of $q(x, t; \varepsilon)$ near a spike or—which is essentially the same—in a (shrinking) neighborhood of a pole of the tritronquée solution.

In order to motivate the construction used below we illustrate the difficulties in constructing the leading approximation to the solution $Y(z)$ of the RHP: it should appear that the local parametrix $P$ needs to be expressed now in terms of the modified Psi function $\hat{\Psi}$ (4-8).

Looking at the asymptotic expansion for large $\xi$ of $\hat{\Psi}$ (4-15) it appears that the first modification we need to make in order to match the boundary behaviour of the $\hat{\Psi}$ with the outer parametrix is to replace the solution $\Psi_0$ to the model RHP by

$$
\Psi_1(z) := \frac{1}{2} \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \left( \frac{z - \alpha}{z - \beta} \right)^{-\frac{1}{2} \tau^3} \begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}. \tag{6-1}
$$
The difference from $Ψ_0$ (5-1) is simply in the power growth near the endpoints $α, \bar{α}$ of the main arc. The transformation that links $Ψ_0$ to $Ψ_1$ is called (discrete) Schlesinger isomonodromic transformation. In fact the two matrices are simply related one to another as seen below:

**Lemma 6.1 (Schlesinger chain)** The matrices

$$Ψ_K(z) := \frac{1}{2} \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix} \left( \frac{z - α}{z - \bar{α}} \right)^{(4-K)σ_3} \begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}, \quad K ∈ ℤ, \quad (6-2)$$

are related by a left-multiplication by a rational matrix

$$Ψ_K(z) = R_K(z)Ψ_0(z), \quad (6-3)$$

where

$$R_K(z) = \begin{bmatrix} p^K + p^{-K} & \frac{i(p^K - p^{-K})}{2} \\ -\frac{i(p^K - p^{-K})}{2} & p^K + p^{-K} \end{bmatrix}, \quad p := \frac{z - α}{z - \bar{α}}. \quad (6-4)$$

**Proof.** The expression of $R_K$ follows from straightforward computations; we only point out that the existence of such a rational left multiplier follows from the fact that all $Ψ_K$ solve the same RHP (jump conditions and normalization at infinity), but have different growth behaviors at the points $α, \bar{α}$. Q.E.D.

Mimicking the previous case of Definition 5.1, we shall state the following new definition.

**Definition 6.1 (Local parametrix near the spikes.)** Let $ζ(z; ε)$ be the local conformal coordinate in $D$, introduced in Definition 3.1, so that

$$\frac{i}{ε} h(z; x, t) = θ(ζ; τ) = \frac{4}{5} ζ^2 + τζ^2. \quad (6-5)$$

Let $Ψ(ζ; v)$ denote the modified Psi–function (4-8) of the Painlevé I problem with $β_0 = 0 = β_{-1}$ (and $β_{-2} = -1, β_2 = -1$). Then we define the parametrix

$$P_{1; α}(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} ζ^2σ_3 \left( ζ + \frac{τ}{2} - \frac{3τ^2}{8} \right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} e^{θ(ζ; τ)σ_3}, \quad (6-6)$$

and we set

$$P_{1; π}(z) := (P_{1; α}(z))^*^{-1}. \quad (6-7)$$

For brevity we will write simply $P_1 = P_{1; α}$. We can then formulate the statement corresponding to Thm. 5.1 for the new local parametrix.

**Theorem 6.1** The matrix $P_1$ satisfies:

1. Within $D_α$, the matrix $P_1(z)$ solves the exact jump conditions on the lenses and on the complementary arc;
2. On the main arc (cut) $\mathcal{P}_1(z)$ satisfies

$$
\mathcal{P}_1(z) = \sigma_2 \mathcal{P}_1^-(z) \sigma_2 , \quad (6-8)
$$

so that $\Psi_0 \mathcal{P}_1$ within $\mathbb{D}_\alpha$ solves the exact jumps on all arcs contained therein (the left-multiplier in the jump $(6-8)$ cancels against the jump of $\Psi_0$);

3. The product $\Psi_0(z) \mathcal{P}_1(z)$ (and its inverse) are -as functions of $z$- bounded within $\mathbb{D}_\alpha$, namely the matrix $\mathcal{P}_1(z)$ cancels the growth of $\Psi_0$ at $z = \alpha$;

4. The restriction of $\mathcal{P}_1(z)$ on the boundary of $\mathbb{D}_\alpha$ is

$$
\mathcal{P}_1(z) \bigg|_{z \in \partial \mathbb{D}_\alpha} = \left( 1 + \mathcal{O}(\zeta^{-\frac{1}{2}}) \right) \left( \frac{\sqrt{1 - \zeta / y}}{1 + \sqrt{\zeta / y}} \right)^{\sigma_3} \quad (6-9)
$$

where $\mathcal{O}(\zeta^{-\frac{1}{2}})$ is uniform w.r.t. $v$ in a neighborhood of a pole $v_p$ not containing any zero of $y(v)$.

Proof. For the first three points the proceeds exactly as in Thm. 5.1 and hence is omitted.

(4) Due to the asymptotic expansion (4-15) for $\hat{\Psi}$, when restricted to the boundary $\partial \Delta_\alpha$, we have (following the same computations as in Theorem 5.1)

$$
\mathcal{P}_1(z) \bigg|_{z \in \partial \mathbb{D}_\alpha} = \frac{1}{2} \left[ \begin{array}{cc} 1 & -1 \\ i & i \end{array} \right] \zeta \frac{\hat{\Psi}}{\sqrt{2}} \left( \zeta + \frac{\tau}{2} + \frac{1}{2} \right) \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \frac{\hat{\Psi}}{2} = \frac{1}{2} \left[ \begin{array}{cc} 1 & -1 \\ -i & i \end{array} \right] \left( \frac{\sqrt{\zeta / y}}{1 + \sqrt{\zeta / y}} \right) \left( \frac{\sqrt{1 - \zeta / y}}{1 + \sqrt{\zeta / y}} \right)^{\sigma_3} = (i)^{\sigma_3} \left( 1 + \mathcal{O}(\zeta^{-\frac{1}{2}}) \right) \left( \frac{\sqrt{\zeta / y - 1}}{\sqrt{1 + \sqrt{\zeta / y}}} \right)^{\sigma_3} = \left( 1 + \mathcal{O}(\zeta^{-\frac{1}{2}}) \right) \left( \frac{\sqrt{1 - \zeta / y}}{1 + \sqrt{\zeta / y}} \right)^{\sigma_3} . \quad (6-10)
$$

Q.E.D.

Corresponding to this new local parametrix, we setup the approximation of the solution as

$$
Y(z) = \left\{ \begin{array}{ll}
\mathcal{E}(z) \Psi_1(z) & \text{for } z \text{ outside of the disks } \mathbb{D}_\alpha, \mathbb{D}_\pi,
\mathcal{E}(z) \Psi_1(z) \mathcal{P}_{1,\alpha}(z) & \text{for } z \text{ inside of the disk } \mathbb{D}_\alpha,
\mathcal{E}(z) \Psi_1(z) \mathcal{P}_{1,\pi}(z) & \text{for } z \text{ inside of the disk } \mathbb{D}_\pi.
\end{array} \right. \quad (6-11)
$$

Recall that we are considering the regime $v - v_p = \mathcal{O}(\varepsilon^{-\frac{1}{2} + \nu})$ (and hence $y = \mathcal{O}(\varepsilon^{-\frac{1}{2} - 2\nu})$) where $v = \frac{\pi}{2} r^2$ and $\nu \geq 0$. The boundaries of both disks $\mathbb{D}_\alpha, \mathbb{D}_\pi$ are mapped by the conformal changes of coordinates $\zeta, \hat{\zeta}$ on some closed curves in the respective planes, that expand homothetically with a scale factor $\varepsilon^{-\frac{1}{2}}$.

Then the behavior of the jump in $(6-12)$ is determined by the behavior of the local parametrix $\mathcal{P}_1 = \mathcal{P}_{1,\alpha}$ on the boundary of $\Delta_\alpha$. The jump of $\mathcal{E}$ is

$$
\mathcal{E}_+ (z) = \mathcal{E}_- (z) \Psi_1(z) \mathcal{P}_{1,\alpha}^{-1} \Psi_{1}^{-1}(z) , \quad z \in \partial \mathbb{D}_\alpha . \quad (6-12)
$$
From eq. (6-9) it is clear that the rate of approach of $P_1$ to the identity on the boundary is seriously impeded by the last factor in (6-9), which fails to converge to 1 when $\nu = 0$, namely, when $y = O(\varepsilon^{-2})$.

More precisely we have from (6-12) (since $\zeta/y = O(\varepsilon^{-2})$)

$$E_+ + (z) = E_-(z) \left(1 + O(\varepsilon_{\min(\varepsilon,2\nu)})\right).$$

(6-13)

Before tackling the general problem $\nu = 0$, we shall see what happens when $v = v_p$ (namely $y = \infty$) and we are exactly at the “top of a spike”

6.1 The top of the spike: amplitude

When $v = v_p$ is exactly a pole of the tritronquée we can use the expansion (4-16) for the expansion of the local parametrix on the boundary of the disks $\Delta_{\alpha}, \Delta_{\beta}$. Since the first term after the identity is of order $O(\varepsilon^{-2}) = O(\varepsilon^{-\frac{4}{5}})$ when restricted on the boundary, the error term in (6-11) is then of the form

$$Y(z) = E(z)\Psi_1(z) = \left(1 + \frac{1}{z}O(\varepsilon^{-\frac{1}{5}})\right) \left(1 + \frac{3b}{2z} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + O(z^{-2})\right).$$

(6-15)

We thus have

$$q(x,t,\varepsilon) = -2 \lim_{z \to \infty} z \Gamma_{1,2}(z) = -2e^{\frac{4}{5}\varepsilon g(\infty; x, t)} \lim_{z \to \infty} z (E\Psi_1)_{12} = e^{\frac{4}{5}\varepsilon_0(x,t)} \left(3b(x,t) + O(\varepsilon^{-\frac{3}{4}})\right) = (6-16)$$

where $(x,t) = v^{-1}(v_p)$ corresponds to the top of the spike. Since the diffeomorphism $v = v(x,t,\varepsilon)$ is scaled as $\varepsilon^{-\frac{3}{4}}$, see Corollary 3.2, the corresponding to $v_p$ spike will approach the point of gradient catastrophe $(x_0,t_0)$ at $O(\varepsilon^{-\frac{3}{4}})$ rate. Here $b$ can be taken as the value at $(x_0,t_0)$ because the difference between the values at $(x_p,t_p)$ and $(x_0,t_0)$ is of order $O(\varepsilon^{-\frac{3}{4}})$ (see (3-43) or (3-40)). Thus, we have proved the following theorem;

**Theorem 6.2** The asymptotic amplitude of a spike near the gradient–catastrophe point $(x_0,t_0)$ is (up to $O(\varepsilon^{-\frac{3}{4}})$ accuracy) three times the amplitude predicted by the Whitham modulation equations at $(x_0,t_0)$.

This result is a bit unexpected in that it entails a very simple universality: the three-fold amplitude of the first spikes appears to be entirely independent of the initial data.

In fact the factor of 3 (together with the phase-shift, i.e., the minus sign) can be traced to the exponent $-\frac{3}{4}\sigma_3$ of the outer parametrix $\Psi_1$, compared with the exponent $\frac{1}{4}\sigma_3$ of $\Psi_0$. In turn, this exponent is determined by the asymptotic behavior of the modified $\Psi$-function for the P1 problem (4-16). The latter is due to the shearing transformation for the cubic potential $V(\xi;a)$ appearing in (4-13).
It seems clear that, were we to study a nongeneric gradient catastrophe (i.e., more than one new main arc emerging from the endpoint of an already existing main arc), we would have to replace the \( P_1 \) problem by a higher member of the Painlevé I hierarchy, which are characterized by exponents \( \frac{7}{2}, \frac{9}{2}, \ldots \) etc. Thus, we conjecture that the amplitude of the first spikes after a nongeneric gradient catastrophe will be \( 5, 7, 9, \ldots \) times the amplitude at the gradient catastrophe \((x_0, t_0)\), depending on the degree of degeneracy.

**Conjecture 6.1** The amplitude of the spikes in the neighborhoods of the points of gradient catastrophe are odd multiples of the amplitude at the point itself.

We reserve to verify this in a subsequent publication. The possible shape of these spikes in the case of a degenerate gradient catastrophe are discussed in Remark 6.1 below.

### 6.2 The shape of the spike

Due to the rightmost term in (6-9), the rate of approach of \( P_1 \) to the identity on the boundary of \( \zeta(\Delta_\alpha) \) in the regime

\[
v - \nu_p = O(\varepsilon^{\frac{1}{2} + \nu}) \quad \Leftrightarrow \quad y = O(\varepsilon^{-\frac{7}{2}})
\]

becomes slower as \( \nu \) approaches the critical value \( \nu = 0 \), at which point the parametrix \( P_1 \) does not tend to the identity at all. Indeed, since \( \zeta = O(\varepsilon^{-\frac{7}{2}}) \), we see that on the boundary of \( \zeta(\Delta_\alpha) \)

\[
P_1(\zeta) \sim \left(1 + O(\varepsilon^{\frac{1}{2}})\right) \begin{align*}
\left(\frac{1 - \zeta/y}{1 + \sqrt{\zeta/y}}\right)^{\sigma_a} &= 1 + O(\varepsilon^{\min\left(\frac{2\nu}{5}, \frac{1}{2}\right)\zeta/y}),
\end{align*}
\]

(6-18)

so that the jumps of \( E \) on the circles \( \Delta_\alpha, \Delta_{\bar{\alpha}} \) are

\[
E_+ (z) = E_- (z) \left(1 + O(\varepsilon^{\min\left(\frac{2\nu}{5}, \frac{1}{2}\right)\zeta/y})\right).
\]

(6-19)

From the standard approximation theorems for Riemann–Hilbert problems, it is seen that \( E \) converges (uniformly) to the identity only up to the same rate of convergence of the jumps, in this case \( O(\varepsilon^{\min\left(\frac{2\nu}{5}, \frac{1}{2}\right)\zeta/y}) \); in particular the “error” becomes worse and worse as \( \nu \to 0 \).

As we shall presently see, it is possible (and necessary) to modify the outer parametrix \( \Psi_1 \) in such a way that the troublesome factor \( Q(z) \) above is exactly taken care of. To account for the term \( Q(z) \) we shall seek an **exact** solution of the Riemann–Hilbert problem described hereafter: let \( \zeta(z), \zeta^*(z) = \bar{\zeta}(z) \) be the local conformal scaling coordinates in the neighborhoods of \( \alpha, \bar{\alpha} \) respectively of the form

\[
\zeta(z) = \varepsilon^{-\frac{7}{2}} C(z - \alpha)(1 + O(z - \alpha)), \quad \hat{\zeta}(z) = \varepsilon^{-\frac{7}{2}} \bar{C}(z - \bar{\alpha})(1 + O(z - \bar{\alpha})),
\]

(6-20)

where \( C = (\frac{7}{2} C_1)^{\frac{7}{2}} \neq 0 \). Let us assume that the circles \( \partial \Delta_\alpha \) and \( \partial \Delta_{\bar{\alpha}} \) (oriented counterclockwise) have some small radius \( r \), so that \( |\zeta(z)/y| < 1 \) and \( |\zeta^*(z)/y| < 1 \) respectively on these two circles.
**Problem 6.1** Find a piecewise analytic matrix $E(z)$ on the complement of the two circles described above and such that

$$E(z) = 1 + O(z^{-1}) \quad \text{as} \ z \to \infty,$$  

(6-21)

$$E_+(z) = E_-(z)^{-1}Q_\alpha^{-1}(z)\Psi_1^{-1}(z) = E_- F M_\alpha(z) F^{-1}, \ |z - \alpha| = r,$$  

(6-22)

$$E_+(z) = E_-(z)^{-1}Q_\pi^{-1}(z)\Psi_1^{-1}(z) = E_- F M_\pi(z) F^{-1}, \ |z - \alpha| = r,$$  

(6-23)

$$M_\alpha(z) = \frac{1}{\sqrt{1 - \zeta/y}} \begin{bmatrix} 1 & -i \sqrt{\frac{\zeta(z)(1 - \alpha)}{y(z - \alpha)^2}} \\ i \sqrt{\frac{\zeta(z)(1 - \alpha)}{y(z - \alpha)^2}} & 1 \end{bmatrix},$$  

(6-24)

$$M_\pi(z) = \frac{1}{\sqrt{1 - \zeta/y}} \begin{bmatrix} 1 & i \sqrt{\frac{\zeta(z)(1 - \alpha)}{y(z - \alpha)^2}} \\ -i \sqrt{\frac{\zeta(z)(1 - \alpha)}{y(z - \alpha)^2}} & 1 \end{bmatrix},$$  

(6-25)

$$F := \frac{1}{\sqrt{2}} \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix}. \quad (6-26)$$

Before solving Problem 6.1 we show how its solution will be used. If $E(z)$ solves Problem 6.1 then we redefine the approximation of $Y$ as

$$Y(z) = \begin{cases} E(z)E(z)^{-1}(z) & \text{for } z \text{ outside of the disks } \Delta_\alpha, \Delta_\bar{} \alpha, \\ E(z)E(z)^{-1}(z)P_\alpha & \text{for } z \text{ inside of the disks } \Delta_\alpha, \Delta_\bar{} \alpha. \end{cases} \quad (6-27)$$

Then, according to (6-18), the jump of $E(z)$ will be

$$E_+ = E_- E_+ \Psi_1 \Psi_1^{-1} E_+^{-1} = E_- E_+ \Psi_1 Q \Psi_1^{-1} E_+^{-1} = E_- (1 + O(\varepsilon^0)) E_+^{-1} = E_- \left(1 + O(\varepsilon^0)\right),$$  

(6-28)

where the last equality holds provided that $E_-(z), E_+^{-1}(z)$ are bounded uniformly in $\varepsilon$ on the boundaries (which will be the case indeed).

**6.2.1 Solution to Problem 6.1**

The problem has an explicit solution. For the sake of simpler computations we will conjugate $E(z)$ by the constant matrix $F$, so that $\hat{E}(z) := F^{-1} E(z) F$ has the matrices $M_\alpha, M_\pi$ for jumps.

It is known that the solution $\hat{E}(z)$ (if it exists) must satisfy the integral equation

$$\hat{E}(z) = 1 + \oint_{|s - \alpha| = r} \frac{\hat{E}_-(s)(M_\alpha(s) - 1)}{s - z} \frac{ds}{2i\pi} + \oint_{|s - \pi| = r} \frac{\hat{E}_-(s)(M_\pi(s) - 1)}{s - z} \frac{ds}{2i\pi},$$  

(6-29)

where $\hat{E}_-(s)$ is the (analytic continuation of the) solution from the outside of the circles $\Delta_\alpha, \Delta_\bar{} \alpha$. It is crucial that the jump matrices $M_\alpha, M_\pi$ admit a simple decomposition of the form

$$M_\alpha(z) - 1 = O_\alpha(z) + \frac{1}{z - \alpha} \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right) =: O_\alpha(z) + \frac{n_\alpha}{z - \alpha} \sigma_+,$$  

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\begin{equation}
M_{\tau}(z) - 1 = O_{\tau}(z) + \frac{-i\varepsilon^{\frac{1}{2}} \sqrt{C/\gamma(\alpha - \alpha)}}{z - \alpha} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} =: O_{\tau}(z) + \frac{n_{\tau}}{z - \alpha} \sigma_-, \tag{6-30}
\end{equation}

where \(O_{\alpha}(z)\) denote some locally analytic matrices in the respective neighborhoods (whose expression the reader can evince from the above formulæ but which has no bearings in the considerations to follow). What is essential in the following is that when evaluated at \(z = \alpha\) and \(z = \bar{\alpha}\), these local analytic matrices will be multiples of \(\sigma_+\) and \(\sigma_-\) respectively.

Consider the Ansatz
\begin{equation}
\hat{E}_{-}(z) = 1 + \frac{A}{z - \alpha} + \frac{\hat{A}}{z - \bar{\alpha}}. \tag{6-31}
\end{equation}

The expression of \(E(z)\) in the inside of the disks \(\Delta_{\alpha}, \Delta_{\bar{\alpha}}\) (i.e. \(E_{+}(z)\)) has no particular interest for us and can be simply obtained from the jump condition. According to (6-31), the integral formula (6-29) becomes
\begin{equation}
\frac{A}{z - \alpha} + \frac{\hat{A}}{z - \bar{\alpha}} = \frac{A O_{\alpha}(\alpha)}{(\alpha - z)} + \frac{\hat{A} O_{\bar{\alpha}}(\bar{\alpha})}{(\bar{\alpha} - z)} + \frac{n_{\alpha}\sigma_+}{(\alpha - z)} + \frac{\hat{n}_{\bar{\alpha}}\sigma_-}{(\bar{\alpha} - z)} + \frac{\hat{A} \sigma_+ n_{\alpha}}{(\alpha - z)(\alpha - \bar{\alpha})} + \frac{\hat{A} \sigma_- n_{\bar{\alpha}}}{(\bar{\alpha} - z)(\bar{\alpha} - \alpha)}. \tag{6-32}
\end{equation}

We thus have the system
\begin{equation} \begin{aligned}
A \sigma_+ &= 0, \\
\hat{A} \sigma_- &= 0, \\
A + \frac{n_{\alpha}}{\alpha - \bar{\alpha}} \hat{A} \sigma_+ &= n_{\alpha} \sigma_+, \\
\hat{A} + \frac{n_{\bar{\alpha}}}{\alpha - \bar{\alpha}} A \sigma_- &= n_{\bar{\alpha}} \sigma_-.
\end{aligned} \tag{6-33}
\end{equation}

The solution is given by
\begin{equation} \begin{aligned}
A &= \frac{1}{1 + \frac{n_{\alpha} n_{\bar{\alpha}}}{(\alpha - \bar{\alpha})^2}} \begin{bmatrix} 0 & n_{\alpha} \\ n_{\alpha} & 0 \end{bmatrix} = \frac{1}{1 + \frac{2ib(C)}{\varepsilon^{\frac{1}{2}} y}} \begin{bmatrix} 0 & -i \sqrt{\frac{(2ib)^3 C}{\varepsilon^{\frac{1}{2}} y}} \\ 0 & -i \sqrt{\frac{|C| 4b^2}{\varepsilon^{\frac{1}{2}} y}} \end{bmatrix}, \\
\hat{A} &= \frac{1}{1 + \frac{n_{\alpha} n_{\bar{\alpha}}}{(\alpha - \bar{\alpha})^2}} \begin{bmatrix} n_{\alpha} & 0 \\ 0 & n_{\bar{\alpha}} \end{bmatrix} = \frac{1}{1 + \frac{2ib(C)}{\varepsilon^{\frac{1}{2}} y}} \begin{bmatrix} i \sqrt{\frac{|C| 4b^2}{\varepsilon^{\frac{1}{2}} y}} & 0 \\ \varepsilon^{\frac{1}{2}} y & 0 \end{bmatrix}. \tag{6-35}
\end{aligned} \end{equation}

Thus the solution of the problem for \(\hat{E}\) has the form (6-31), (6-35) in the region outside the disks.

6.2.2 Partial Schlesinger transformation and improved leading order asymptotics: shape of the spike

**Theorem 6.3 (Shape of the spikes)** The spikes around the point of gradient catastrophe for \(q(x, t, \varepsilon)\) are in one-to-one correspondence with the poles of the tritronquée solution \(y(v)\) of the Painlevé I equation
\begin{equation}
y'' = 6y^2 - v \tag{6-36}
\end{equation}
and have a universal shape. For a given pole \( v = v_p \) of the tritronquée solution, the shape of the spike is described by

\[
\frac{q(x,t,\varepsilon)}{q_0(x_0,t_0,\varepsilon)} = 1 + \frac{(s + 2i)(\pi + 2i)}{1 + |s|^2} \left(1 + \mathcal{O}(\varepsilon^\frac{1}{4})\right) = \left|s\right|^2 - 3 + 4i\Re(s) \left(1 + \mathcal{O}(\varepsilon^\frac{1}{4})\right),
\]

where \( q_0(x,t,\varepsilon) \) is the genus zero approximation (see (2-20), (2-21), and Remark 3.7) the variable \( s \) is defined in terms of the tritronquée where

\[
s = \left(\frac{5C_1}{4}\right)^\frac{2}{5} \frac{2ib}{\varepsilon y} = -\left(\frac{5C_1}{4}\right)^\frac{2}{5} \frac{2ib}{\varepsilon y} (v - v_p)^2 (1 + \mathcal{O}(v - v_p)^2),
\]

and the diffeomorphism \( v = v(x,t,\varepsilon) \) is given in Corollary 3.2. The formula and the error term are valid uniformly for \( (x,t) \) in a \( \mathcal{O}(\varepsilon) \)-neighborhood of the center of the spike \( (x_p,t_p) = v^{-1}(v_p) \), or –which is the same– for \( v(x,t,\varepsilon) - v_p = \mathcal{O}(\varepsilon^\frac{1}{4}) \).

**Proof.** It is apparent that both \( E_-(z) \) and \( E_-(z)^{-1} \) are uniformly bounded on the boundaries of the disks \( \Delta_{\alpha}, \Delta_{\alpha} \) as long as \( y = \mathcal{O}(\varepsilon^{-\frac{2}{5} - \nu}) \) for some \( \nu \geq 0 \). Thus, according to (6-28), the matrix \( \mathcal{E}(z) \) will be \( 1 + \mathcal{O}(\varepsilon^\frac{1}{4}) \). Then the leading order approximation is thus given by

\[
\bar{\Psi}_1 = E(z)\Psi_1(z) = F\hat{E}(z) \left(\frac{z - \alpha}{z - \bar{\alpha}}\right)^{-\frac{3}{2}p_3} F_1^{-1},
\]

where

\[
F = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix}.
\]

From this expression we find the behavior of the amplitude

\[
\bar{\Psi}_1(z) = F \left(1 + \frac{A}{z - \alpha} + \frac{\hat{A}}{z - \bar{\alpha}}\right) \left(\frac{z - \alpha}{z - \bar{\alpha}}\right)^{-\frac{3}{2}p_3} F_1^{-1} = 1 + F \left(\frac{A + \hat{A}}{z} + \frac{3(\alpha - \bar{\alpha})}{4} \frac{p_3}{z}\right) F_1^{-1} =
\]

\[
= 1 - 3ib \frac{2}{2z} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \frac{1}{1 + \frac{2ibC}{\varepsilon y} \frac{1}{z}} F \left[ \frac{i|C|4b^2}{\varepsilon y} \frac{1}{|y|} \quad -ie^{-\frac{i}{4}} \sqrt{-8iCb^2/y} \\ -ie^{-\frac{i}{4}} \sqrt{8iCb^2/y} \quad -i|C|4b^2 \frac{1}{\varepsilon y} \frac{1}{|y|} \right] F_1^{-1} + \mathcal{O}(\varepsilon^\frac{3}{2})
\]

Since \( q = -2\lim_{z \to \infty} z\Gamma_{12}(z) \), where

\[
\Gamma(z) = e^{\frac{2i}{75}g(\varepsilon)\sigma_3} \mathcal{E}(z) \bar{\Psi}_1(z) e^{-\frac{2i}{75}g(\varepsilon)\sigma_3},
\]

we have

\[
q(x,t,\varepsilon) = e^{\frac{2i}{75}g(\varepsilon)} \left(-3b + 2(F(A + \hat{A})F_1^{-1})_{12} + \mathcal{O}(\varepsilon^\frac{3}{4})\right) =
\]

\[
e^{-\frac{i}{2}} \left(3 - \frac{2b|C|}{\varepsilon y} + 4i\Re \left(\sqrt{-2ibC}\right)\right)
\]

\[
= -\frac{1}{1 + \frac{2ibC}{\varepsilon y} \frac{1}{|y|}} e^{\frac{i}{2}} b \left(3 - \frac{2b|C|}{\varepsilon y} - 4i\Re \left(\sqrt{-2ibC}\right)\right).
\]
Since the variable \( v \) is a diffeomorphism on \( D \), which is a size \( \varepsilon^{\frac{4}{5}} \) neighborhood of the point of gradient catastrophe, then the variable \( s \) defined in (6-38) is a diffeomorphism on a neighborhood of size \( \mathcal{O}(\varepsilon) \) around the spike. In terms of \( s \) we can rewrite (6-45) as

\[
q(x,t,\varepsilon) = -e^{i\varepsilon \Phi_b} \frac{1}{1 + |s|^2} \left( 1 + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \right) \frac{1 + |s + 2i(\pi/2 + 2i)|}{1 + |s|^2} \left( 1 + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \right). \tag{6-46}
\]

Q.E.D.

We remark in the following corollary that from 6-37 it follows immediately that each spike has two zeroes (nodes), one on each side.

**Corollary 6.1** For each spike near the gradient catastrophe point there are two zeroes (nodes) and they occur at the time \( t = t_p \) (asymptotically as \( \varepsilon \to 0 \)) and with \( x - x_p = \pm \varepsilon^{\frac{1}{2}} \sqrt{3} \).

### 6.2.3 The spike as a rational breather

To gain a bit more insight into the formula (6-37) we start with the following observation; in terms of the coordinates \( x, t \) we have

\[
s^2 = -\frac{4b^2}{\varepsilon^{\frac{4}{5}}} \left( \frac{x - x_p}{\varepsilon^{\frac{4}{5}}} + 2(a + a \frac{t - t_p}{\varepsilon^{\frac{4}{5}}} \right)^2 \left( 1 + \mathcal{O}(\varepsilon^{\frac{1}{5}}) \right), \tag{6-47}
\]

\[
s = \pm 2ib\varepsilon^{-\frac{1}{5}} \left( x - x_p + (4a + 2ib)(t - t_p) \right) \left( 1 + \mathcal{O}(\varepsilon^{\frac{1}{5}}) \right). \tag{6-48}
\]

The phase can be expanded near the spike using (3-69) and retaining just the linear term; indeed the scale of the spike is \( \mathcal{O}(\varepsilon) \) and hence \( x - x_p = \mathcal{O}(\varepsilon) \) and \( t - t_p = \mathcal{O}(\varepsilon) \). We thus find

\[
s = 2ib \left( \frac{x - x_p}{\varepsilon} + 4a \frac{t - t_p}{\varepsilon} \right) = -4b^2 \eta + 2b(x + 4a\eta), \tag{6-49}
\]

\[
\xi := \frac{x - x_p}{\varepsilon}, \quad \eta := \frac{t - t_p}{\varepsilon}. \tag{6-50}
\]

So,

\[
q(x,t,\varepsilon) = e^{i\Phi(x_p,t_p)} Q_{br}(\xi,\eta)(1 + \mathcal{O}(\varepsilon^{\frac{1}{5}})), \tag{6-51}
\]

where

\[
Q_{br}(\xi,\eta) := e^{-2i(a\xi + (2a^2 - b^2)n)b} \left( 1 - 4 + \frac{4ib^2 \eta}{1 + 4b^2(\xi + 4a\eta)^2 + 16b^4\eta^2} \right). \tag{6-52}
\]

The expression \( Q_{br}(\xi,\eta) \) is well known in the literature [24]; it is called the **rational breather solution** for NLS. Indeed it solves exactly the NLS equation

\[
i\varepsilon \partial_t Q_{br} + \varepsilon^2 \partial_x^2 Q_{br} + 2 |Q_{br}|^2 Q_{br} = 0, \tag{6-53}
\]

where

\[
\xi = \frac{x - x_p}{\varepsilon}, \quad \eta = \frac{t - t_p}{\varepsilon}. \tag{6-54}
\]
In our case it is obtained from the “stationary” breather (depicted in Fig. 10)

\[ Q^{0}_{br}(\xi, \eta) = e^{2i\eta} \left( 1 - 4 \frac{1 + 4i\eta}{1 + 4\xi^2 + 16\eta^2} \right) \]  

(6-55)

by applying the transformations (mapping solutions into solutions)

\[ \tilde{Q}(\xi, \eta) = \lambda Q(\lambda \xi, \lambda^2 \eta), \quad \hat{Q}(\xi, \eta) = e^{i(kx - k^2 t)} Q(\xi - 2kt, t). \]  

(6-56)

The breather has the maximum equal to 3 at \( \xi = 0 = \eta \) and tends to 1 as \( |s| \to \infty \); therefore it smoothly interpolates the regime near the spike with the regime “far” from it.

Figure 10: The theoretical shape of the spike as a function of \( s = i \sqrt{\frac{2ibC}{y\epsilon}} \) as given by Eq. (6-46). This also represents the exact single-breather solution \( Q_{br} \) where \( s = 2ib(\xi + 4a\eta + 2ib\eta) \).

**Remark 6.1** While completing the preparation of this paper we came across [1], where a hierarchy of rational breathers generalizing the standard one (6-55) is investigated. Interestingly, the maximum amplitude of these breathers is also an odd integer and hence this prompts the speculation that they should play the same rôle of (6-55) in the cases of gradient catastrophes with higher degeneracy (compare with Conjecture 6.1).
Figure 11: Some simple numerics (using Fast Fourier Transform) for the initial data $q(x,0) = \frac{1}{\cosh(x)}$ and $\varepsilon = \frac{1}{33}$; note that the amplitude at the first spike (numerically determined) is $|q_{\text{peak}}| = 3.9256$ and $|q_0| = 1.3137$. This is almost exactly three times, $3|q_0| = 3.9411$. In this case $\mu = 0$ and $t_0 = \frac{1}{4}$ as per Example 3.1.

**Remark 6.2 (Consistency Check)** Suppose $y$ remains fixed and $\varepsilon \to 0$ in the formulae for $A, \hat{A}$ (6-35); we then obtain the limit

$$\Psi_1(z) = F \left( 1 + \begin{bmatrix} \frac{\pi - \alpha}{\alpha} & 0 \\ 0 & \frac{\alpha}{z - \alpha} \end{bmatrix} \left( \frac{z - \alpha}{z - \alpha} \right) \right)^{-\frac{2}{3}} F^{-1} = \Psi_0(z).$$

(6-57)

**Example 6.1** According to numerical studies by [18], the closest pole to the origin of the tritronquée is at distance of about $|v_p| \sim 2.38$ (on the bisecant of the sector containing the poles). According to our Theorem 6.3, this is the first spike after the gradient catastrophe and its location is thus determined by the diffeomorphism $v = v(x,t,\varepsilon)$.

In the case of the initial data of Example 3.1 with $\mu = 0$, the solution is symmetric ($x \mapsto -x$), hence the first spike occurs at $x = x_0 = 0$; the time of the first spike is then estimated at

$$t_{\text{spike}} = t_0 + \varepsilon \frac{5}{4} \left( \frac{5C_1}{4} \right)^{\frac{1}{2}} \frac{1}{(2b_0)^{\frac{3}{2}}} |v_p| \left( 1 + O(\varepsilon^\frac{3}{2}) \right).$$

(6-59)
For the case of the initial data of Example 3.1 (µ = 0), the exact values of \( b_0 = b(x_0, t_0) = \sqrt{2}, \) \( t_0 = \frac{1}{4} \)
and \( |C_1| = \frac{8}{15} 2^{\frac{3}{4}} \) were found in [26]; this gives

\[
t_{\text{spike}} \sim \frac{1}{4} + \varepsilon^{\frac{3}{4}} 0.4776312294 . \tag{6-60}
\]

In Fig. 11 we report on a simple numerical example with initial data \( q(x, 0) = \text{sech}(x), \varepsilon = \frac{1}{15} \); plotted are the profiles of \( |q(x, t)| \) for \( t = t_0 = 1/4 \) which is the theoretical time of gradient catastrophe; the second curve is the time where numerically the maximum amplitude is achieved at \( x = 0. \) The numerics indicates \( t_{\text{spike}} = 0.28 \) whereas the formula (6-60) predicts \( t_{\text{spike}} = 0.2791260482, \) which – considering that \( \varepsilon \) is not that small – is an amazingly close estimate.

Our numerics was produced by integrating directly the NLS equation using FFT in the \( x \)-direction; it should be noted that the initial data we have chosen correspond to a pure soliton situation, and hence one could use a better numerical approach based on exact linear algebra as explained in [21].

### A Estimate of the parametrix on a circle of large radius uniformly for large \( y \)

This section is rather technical; its goal is to prove that the sectorial analytic matrix function \( \hat{\Psi}(\xi; v) \), that is related to the solution \( P(\xi, v) \) of the RHP (4.1) through \( \hat{\Psi}(\xi; v) = G(\xi, v)P(\xi, v)e^{\theta(\xi, v)\sigma_3} \), has asymptotic behavior (4-15) as \( \xi \to \infty \) uniformly with respect to \( v - v_p = O(\varepsilon^{\frac{3}{4}}) \), where \( v_p \) is one of the poles of the tritronquée solution \( y(v) \) of P1, or, equivalently, uniformly with respect to \( \| (x, t)-(x_p, t_p) \| = O(\varepsilon). \)

In fact, we will prove a somewhat stronger result, stated in Theorem A.1 below. Corollary A.1 of this theorem has been used in the main body of the paper.

**Theorem A.1** Let \( S \) be one of the sectors shown on Figure 8 and \( \hat{\Psi}(\xi, v) = G(\xi, v)P(\xi, v)e^{\theta(\xi, v)\sigma_3} \) be a solution of the ODE (4-9), obtained from the sectorial solutions \( P(\xi, v) \) of the Problem 4.1 in the sector \( S. \) Let \( \Lambda \) be a constant nonsingular diagonal matrix and \( D_y \) be a disk of fixed radius \( r > 0 \) centered at \( \xi = y. \) Then, there exist some constants \( \xi_*, \sigma_2 > 0, p_2 > 0, \) both independent of \( y \) and of each other, such that \( \hat{\Psi}(\xi, v) \) has the representation

\[
\hat{\Psi}(\xi, v) = 2^{\sigma_2/2}(4\xi^3 - 2\xi v)^{-\sigma_2/4} \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)T\Lambda \left( I + O \left( \xi^{-\frac{3}{4}}, e^{-p_2|\frac{\xi}{v}|^{1/2}} \right) \right) e^{q(\xi, v)\sigma_3} \tag{A.1}
\]

where

\[
q(\xi, v) = \partial(\xi, v) + \frac{y'}{4y} \int_{s}\left( s - \frac{v}{s^{\frac{3}{2}}} \right) ds, \tag{A.2}
\]

\[
T = \left( 1 - \frac{i\sigma_2}{2} A_1 \right) \left( 1 - \frac{\sigma_1}{2} A_2 \right) \quad \text{with} \quad A_1 = \frac{6\xi^2 - v}{2(4\xi^3 - 2\xi v)^{3/2}}, \quad A_2 = \frac{y'\xi}{8\xi^3 y(\xi - y)} \tag{A.3}
\]

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uniformly for \( \xi \in S(\xi_\ast) = S \cap \{ \xi : |\xi| \geq \xi_\ast \} \backslash D_y \) and for all sufficiently large \( y \). Here \( \vartheta(\xi, v) = \frac{4}{5} \xi^{5/2} - v \sqrt{\xi} \) and the contour of integration in (A.2) is a ray in the direction of the bisector of \( S \). In the case when \( y \not\in S_{\xi_\ast} \), the \( O \) term in (A.1) is independent of \( \frac{|y|}{\xi_\ast} \), i.e., it is \( O(\xi^{-5/2}) \). The same \( O \) term is valid even if \( y \in S_{\xi_\ast} \) everywhere in \( S_{\xi_\ast} \) except a certain “shadow” of \( D_y \) region, see Fig. 12 and description of this region below.

Proof of this theorem constitutes the bulk of this Appendix. Let us consider sector \( S \) that is not adjacent to \( \mathbb{R}_- \). This sector is bisected by the Stokes’ ray \( \ell \) where \( \Re \xi^{5/2} = 0 \). Let us extend \( S \) by some angle \( \delta \in (0, \frac{2}{5}\pi) \) in both directions. For definiteness, we shall focus on the sector \( S = \{ \arg(\xi) \in (-\delta, \frac{2}{5}\pi + \delta) \} \), the remaining being treated similarly. It is more convenient to work with variable \( \vartheta(\xi, v) \) instead of \( \xi \); therefore, we introduce a new variable

\[
\xi = u(\xi) = -i \vartheta(\xi, v) = -i \left[ \frac{4}{5} \xi^{5/2} - v \xi^{\frac{1}{2}} \right].
\]

We will consider \( u(\xi) \) as an analytic function (with the principal determinations of the roots) on sectors of the \( \xi \)-plane of width up to \( 4\pi/5 \) and for \( |\xi| \) sufficiently large: therefore in any such region the map \( u(\xi) \) is holomorphically invertible because \( v \) will be chosen uniformly bounded (in fact- in a neighborhood of \( v_p \)), and we will denote its inverse by \( \xi(u) \). We will write for brevity \( u \) or \( \xi \) with the understanding that they may be viewed as functions of the other variable. A direct calculation shows that

\[
\xi = u(\xi) = \left( \frac{5i}{4} \right)^{2/5} u^{2/5} \left[ 1 + \frac{ve^{-i\pi/5}}{2^{5/5} 2^{1/5}} u^{-4/5} + O(u^{-8/5}) \right],
\]

We will let \( v - v_p \) be bounded above, say, by \( 2 \), and, since the branch-points of \( u(\xi) \) are at \( \pm \sqrt{v_p/2} \), the map \( u(\xi) \) is invertible in any sectorial domain \( \{ |\xi| > |\sqrt{v_p} + 1| \} \), as long as the opening of this domain is less than \( 2\pi \).

The image of \( S \) under the map (A.4) will be denoted by \( \mathcal{S} \). Then, asymptotically for large \( u \),

\[
\mathcal{S} = \{-\pi < \arg(u) < \pi \}, \quad \frac{1}{2} \pi < \phi < \pi.
\]

It will be convenient for us to consider (A.6) as the definition for \( \mathcal{S} \), whereas \( S \) is the preimage of \( \mathcal{S} \), i.e., \( S = u^{-1}(\mathcal{S}) \). By \( \mathcal{S}_{u_0} \) we denote sector \( \mathcal{S} \) shifted from the origin to some \( u_0 \in \mathbb{C} \), i.e., \( \mathcal{S}_{u_0} = \mathcal{S} + u_0 \); for convenience we shall take \( u_0 \in \mathbb{R}_+ \). Without any loss of generality we assume \( u_0 \) to be sufficiently large. Our first step in proving Theorem A.1 is to “prenormalize” the ODE (4-9), satisfied by \( \hat{\Psi} \), written with respect to the independent variable \( u \).

**Lemma A.1** Let \( W(u) \) be defined by

\[
\hat{\Psi}(\xi) = 2^{\sigma_1/2} (4\xi^3 - 2\xi v)^{-\sigma_3/4} \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3) T(\xi) W(\xi)
\]

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where the matrix $T$ is defined by (A.3). Then $W$ satisfies the ODE in the variable $u$

$$W_u = \left[ \left( i + \frac{i y' \xi}{8 \xi^3 y (\xi - y)} \right) \sigma_3 + B(u) \right] W,$$

(A.8)

where $B(u) = O(u^{-6/5})$ uniformly with respect to sufficiently large values of $y$ in $u \in R_{u_0, y} = \mathcal{S}_{u_0} \setminus \Delta$. Here $\Delta = u(D_y)$.

**Proof.** The transformation

$$\hat{\Psi} = 2^{\sigma_3 / 2} (4 \xi^3 - 2 \xi y) \sigma_3 \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3) X$$

(A.9)

reduces the system (4-9) to

$$X_{\xi} = \left[ \sqrt{4 \xi^3 - 2 \xi y} \sigma_3 + \frac{6 \xi^2 - v}{2(4 \xi^3 - 2 \xi v)} \sigma_1 + \frac{M}{2 \sqrt{4 \xi^3 - 2 \xi v}} (\sigma_3 + i \sigma_2) \right] X,$$

(A.10)

where

$$M = 2 \hat{H}_1 + M_2 + M_3 = \left( 2H - \frac{y'}{y} \right) + \frac{y' \xi}{y(\xi - y)} + \frac{3}{4(\xi - y)^2}.$$  

(A.11)

The three terms in $M$ have uniform bounds in $S_{\xi_0, y} = u^{-1}(R_{u_0, y})$, where $u_0 = u(\xi_0)$:

- Since $\xi \notin D$ we have $|M_3| < \frac{1}{\xi^2}$;
- According to (4-7) and the fact that $H'(v) = y(v)$, we have
  $$2 \hat{H}_1 = 2H(v) - \frac{y'}{y(v)} = 28 \beta + O(v - v_p);$$
  (A.12)
- According to (4-7), $(\ln y)' = -2 \sqrt{y}(1 + O(v - v_p)^4)$ for all sufficiently large $y$, so we now need to estimate $\frac{y' \xi}{\xi - y}$. Note that $\frac{\sqrt{\xi}}{\xi - y} \leq \frac{1}{d_0}$ outside the domain $|\xi - y| < d_0 |\xi|$. Thus, $M_2 = O(\xi^{1/2})$ outside this domain. Inside the latter domain but outside $D_y$ we have $\frac{y' \xi}{\xi - y} = O(\xi^{1/2})$, so that
  $$M_2 = O(\xi^{1/2})$$

(A.13)

for all $\xi \in S_{\xi_0, y}$ and uniformly in all sufficiently large $y$.

Rewriting (A.10) in the variable $u$ we obtain

$$X_u = \left[ \frac{2 \sqrt{\xi}}{4 \xi^2 - v} \left[ \sqrt{4 \xi^3 - 2 \xi y} \sigma_3 + \frac{6 \xi^2 - v}{2(4 \xi^3 - 2 \xi v)} \sigma_1 + \frac{M}{2 \sqrt{4 \xi^3 - 2 \xi v}} (\sigma_3 + i \sigma_2) \right] X = \left[ \sigma_3 + \frac{6 \xi^2 - v}{2(4 \xi^3 - 2 \xi v)^{3/2}} \sigma_1 + \frac{M}{2(4 \xi^3 - 2 \xi v)} (\sigma_3 + i \sigma_2) + O(u^{-8/5}) \right] X.$$

(A.14)
Dropping all the terms in $M$ except $M_2$, the previous equation becomes (see the bulleted list above)

$$X_u = i \left[ \sigma_3 + \frac{6\xi^2 - v}{2(4\xi^3 - 2\xi v)^{3/2}} \sigma_1 + \frac{y'\xi(\sigma_3 + i\sigma_2)}{2y(\xi - y)(4\xi^3 - 2\xi v)} + O(u^{-6/5}) \right] X. \quad (A.15)$$

Using (A.13) we have

$$\frac{y'\xi}{2y(\xi - y)(4\xi^3 - 2\xi v)} = \frac{y'}{8y\xi^2(\xi - y)}(1 + O(\xi^{-2})) = \frac{y'}{8y\xi^2(\xi - y)} + O(u^{-7}) \quad (A.16)$$

and hence

$$X_u = i \left[ \sigma_3 + \frac{6\xi^2 - v}{2(4\xi^3 - 2\xi v)^{3/2}} \sigma_1 + \frac{y'\xi(\sigma_3 + i\sigma_2)}{8\xi^2y(\xi - y)} + O(u^{-6/5}) \right] X \quad (A.17)$$

for $u \in \mathbb{R}_{u_0}$ uniformly in $y$ (for sufficiently large $y$). Setting $X = TW$ with $T$ as indicated in (A.3, A.7), simplifying and keeping track of the orders already estimated, yields the equation (A.8). Q.E.D.

We will use the ODE (A.8) in order to setup a recursive approximation scheme within each sector; the key is that the estimate for $B(u)$ in (A.8) is uniform for $v$ in a neighborhood of the pole $v_p$ (hence, for large values of $y$) and $|\xi - y|$ bounded below.

The domain $\Delta = u(D_y)$ in the $u$ plane is of size $O(|u(y)|^{\frac{2}{5}})$. For simplicity (and without real loss of generality) we modify $D$ to be the preimage of the exact disk $\Delta$ in the $u$-plane centered at $u(y)$ of the radius $d_0|u(y)|^{\frac{2}{5}}$.

**Lemma A.2** Any solution $W(u)$ to (A.8) can be written in the form $W = e^{q(u)}\sigma_3$ where

$$\frac{d}{du} q(u) = \left( i + \frac{iy'\xi}{8\xi^3y(\xi - y)} \right) \quad (A.18)$$

and $\Phi(u)$ solves

$$\Phi' = q'[\sigma_3, \Phi] + B \Phi \quad (A.19)$$

with $B = B(u)$ as in Lemma A.1 (eq. (A.8)).

**Proof.** A direct substitution of the proposed expression into (A.8). Q.E.D.

The differential equation (A.8) is equivalent to the following Volterra integral equation

$$\Phi(u) = \Lambda + e^{q(u)} \left( \int_{\Omega(u)} e^{-q(\eta)} B\Phi e^{q(\eta)} d\eta \right) e^{-q(u)}\sigma_3 = \Lambda + \mathcal{I}\Phi(u), \quad (A.20)$$

where $q$ from (A.18) can be written as

$$q(u) := iu + \frac{iy'}{8y} \int_u^\infty \frac{du}{(\xi(u) - y)\xi^2(u)}. \quad (A.21)$$

The symbol $\int_{\Omega(u)}$ denotes the integration along a set of contours originating at $u$ and extending to $u = \infty$, with a different direction of contour $\Omega_{i,j}(u)$ for different entries of the matrix, see Fig. 12 for the choice of
Figure 12: Region $R_{u_0,y}$ is the complement of $\Delta$ and the union of $\Delta^0$ (darker) and $R^0_{u_0,y}$ (lighter). Shown in the picture are the contours of integration $\Omega_{i,j}(u)$, $u \in R^0_{u_0,y}$, for the $(i,j)$ entry of the Volterra operator (A.20).

the contours (the exponential of the integrand should decrease in the direction of the contour, traversed from $u$ to $\infty$). It is promptly seen from the fundamental theorem of calculus is equivalent to (A.18). The matrix $\Lambda$ is a constant of integration, which, at this point, we choose to be a diagonal and nonsingular matrix, but otherwise undetermined.

Changing variable $u = u(\xi)$ in the integral (A.21) and using $\frac{du}{d\xi} = -2i\xi^{3/2} + \frac{iv}{\sqrt{\xi}}$ and

$$\int_{\infty}^{\xi} \frac{dt}{\sqrt{t}(t-y)} = \frac{1}{\sqrt{y}} \ln \frac{\sqrt{\xi} - \sqrt{y}}{\sqrt{\xi} + \sqrt{y}},$$

we find

$$\mathbf{q}(u) = iu + \frac{y'}{4y^{3/2}} \ln \frac{\xi - y}{(\sqrt{\xi} + \sqrt{y})^2} + O(u^{-3/5})$$

in the region $R_{u_0,y}$ uniformly in $y$. Note that

$$\frac{y'}{4y^{3/2}} = -\frac{1}{2} + \frac{a}{8} (v - v_p)^4 + O((v - v_p)^5) = -\frac{1}{2} + O(y^{-2})$$

and hence

$$e^{\mathbf{q}(u)\sigma_3} = e^{i(\frac{4}{3} \xi^3 - v\xi^2)} \left(\frac{\sqrt{\xi} + \sqrt{y}}{\sqrt{\xi} - \sqrt{y}}\right)^{\sigma_3} (I + O(u^{-3/5}, y^{-2}))$$

in the region $R_{u_0,y}$ uniformly for large $y$; the cut of the logarithm is taken from $y$ to $\xi\infty$ parallel to the Stokes line (bisectant to the sector).
The strategy is to show that the integral operator $I$, defined in (A.20) is contractive in a suitable space, so that a solution can be sought through an iteration scheme; the analysis depends on the region $u$ belongs to. For practical reasons we state the following lemma.

**Lemma A.3** The function \( \left( \frac{\sqrt{\xi + y} + \sqrt{\xi - y}}{\sqrt{\xi - y}} \right) \) and its inverse satisfy the inequalities

\[
\left| \frac{\sqrt{\xi + y} + \sqrt{\xi - y}}{\sqrt{\xi - y}} \right| \leq K_0 |\xi|^\frac{1}{2}
\]

outside the circle $|\xi - y| \geq r$, with $K_0$ a constant independent of $y$ (but depending on $r$).

**Proof.** The function \( \left( \frac{\sqrt{\xi + y} + \sqrt{\xi - y}}{\sqrt{\xi - y}} \right) \) is continuous (in fact smooth except at $\xi = 0$ and $\xi = y$) on $\mathbb{C} \setminus \{y\}$ and has limit 1 at $\xi = \infty$, being unbounded in any neighborhood of $\xi = y$. It can be also seen by calculus that it has no critical values in $\mathbb{C} \setminus \{0, y\}$ and hence the maximum is attained on a disk around $y$. The value there is easily estimated as in (A.26). Similar arguments work for the inverse function. Q.E.D.

### A.1 Convergence of iterations in $R^0_{u_0,y}$

The region $R^0_{u_0,y}$ is the lighter shaded region in Fig. 12; it is obtained by excising from $R_{u_0,y}$ the “cone of shadow” of the disk $\Delta$ with the angle $\phi_2 \in (\phi, \pi)$ indicated in Fig. 12 chosen arbitrarily and fixed once and for all. The region $\Delta^0$ will be such cone of shadow. For a given $u \in R^0_{u_0,y}$, the contours of integration $\Omega_{i,j}(u)$ are also shown in Fig. 12. It is important that the collection of contours $\Omega(u) \subset R^0_{u_0,y}$ for any $u \in R^0_{u_0,y}$.

**Lemma A.4** If a matrix-function $\chi(u)$ satisfies $\|\chi(u)\| \leq c|u|^{-m}$ in $R^0_{u_0,y}$ for some $c > 0$ and some $m \geq 0$ then $\|I_1 \chi(u)\| \leq cK_1 |u|^{-m-1/5}$, where $I_1$ denoted the Volterra integral operator (A.20) with the contours $\Omega(u)$ specified above. The constant $K_1$ does not depend on $\chi$, $u_0 \in \mathbb{R}^+$ and $y \in \mathbb{C}$, but depend on $m$ (it was assumed above that $u_0$ and $|y|$ are large).

**Proof.** The diagonal entries of $B\chi$ are unaffected by the conjugation by $e^{\alpha z^3}$ and are of order $u^{-m - \frac{3}{2}}$; after integration they become of order $u^{-m - \frac{1}{2}}$. Regarding the off-diagonal entries of $B\chi$, let $F(u)$ denote one of them; then, according to (A.20), $|F(u)| \leq cL |u|^{-m - \frac{3}{2}}$ on $R_{u_0,y}$, where the constants $L > 0$ and $m \geq 0$ do not depend on $y$ and we have (from (A.25))

\[
e^{\pm 2i\alpha} F(u) = e^{\pm 2i\alpha} F(u) \left( \frac{\sqrt{\xi + y} + \sqrt{\xi - y}}{\sqrt{\xi - y}} \right)^2 \left( I + O(u^{-3/5}, y^{-2}) \right),
\]

so that

\[
|F(u) \left( \frac{\sqrt{\xi + y} + \sqrt{\xi - y}}{\sqrt{\xi - y}} \right) (I + O(u^{-3/5}, y^{-2}))| \leq cLK_0 \sqrt{|\xi| |u|^{-m}} = cLK_0 |u|^{-m-1}.
\]

Due to the choice of the contours $\Omega(u)$, the integration along the corresponding $\Omega_{i,j}(u)$ can change the previous estimate only by a constant that does not depend on $u_0$ and $y$ (but depends on $\phi_2$). Q.E.D.
Let us consider successive iterations $\Phi_n = I_1 \Phi_{n-1}$, $n \geq 1$, of equation (A.20), where $\Phi_0 = 0$. Let $\Delta \Phi_n = \Phi_n - \Phi_{n-1}$. Then $\Phi_1 = \Delta \Phi_1 = \Lambda$ and $\Delta \Phi_n = I_1 \Delta \Phi_{n-1}$. We are now going to prove uniform in $y$ convergence of the series $\sum_1^\infty \Delta \Phi_n$ in different subregions of $R_{u_0,y}$, provided that $u_0$, which is independent of $y$, is large enough. In all the analysis below $y$ is assumed to be large.

Let $\|\Lambda\| = c$. Then, according to Lemma A.4, $\|\Delta \Phi_2\| \leq cK_1 |u|^{-1/5}$ in $R_{u_0,y}$ uniformly in $u_0$ and in $y$. We can choose $u_0 \in \mathbb{R}^+$ so large that $K_1 |u|^{-1/5} < \frac{1}{2}$ for all $u \in R_{u_0,y}^0$. This choice of $u_0$ guarantees convergence of the series $\sum_1^\infty \Delta \Phi_n$ in $R_{u_0,y}^0$ uniformly in $y$. Moreover, we obtain the estimate

$$\Phi(u) = \Lambda[I + O(u^{-1/5})]$$

(A.29)

in $R_{u_0,y}^0$ uniformly in $y$.

Let $\mathcal{R}_{\xi_0,y}$ denote the preimage of $R_{u_0,y}^0$ under the map $u = u(\xi)$, where $u_0 = u(\xi_0)$. Going back to the system (4.9), we obtain the following result.

**Lemma A.5** The solution $\hat{\Psi}(\xi,v)$ to the system (4.9) has behavior

$$\hat{\Psi}(\xi,v) = 2^{\sigma_3/2}(4\xi^3 - 2\xi v)^{-\sigma_3/4} \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3) T \Lambda(I + O(u^{-1/5})) e^{q(u(\xi))}$$

(A.30)

in $\mathcal{R}_{\xi_0,y}$ uniformly in $y$ provided $\xi_0$ is large enough. Moreover, for a fixed diagonal $\Lambda$, condition (A.30) uniquely determines the solution $\hat{\Psi}(\xi,v)$ of (4.9).

Comparison of (A.30) and (4.8) together with the asymptotics of $\Psi$ (4-6) and (4-14) yields

$$\Lambda = \text{diag}(1,i).$$

(A.31)

**A.2 Convergence of iterations in $\Delta^0$**

If $y$ does not belong to the sector $S$ we are considering, we can always arrange that $\mathcal{D}_y \cap S = \emptyset$, so that $\mathcal{R}_{\xi_0,y}$ coincides with the preimage of $\mathcal{S}_{u_0}$ and, thus, the estimate of Lemma A.5 holds throughout that sector. We want to extend the statement of Lemma A.5 into the preimage $\mathcal{D}^0$ of the region $\Delta^0$ for the case that $y \in S$. Because of the construction of $S$ we can assume, without any loss of generality, that $y \in \hat{S}$, where $\hat{S}$ is any proper subsector of $S$.

Consider the tangent line $\lambda$ to the disk $\Delta$ (which is centered at $u(y)$) that is parallel to $e^{-i\phi_1}$ and located above $\Delta$, i.e., if $u_*$ is the point of tangency, then $\arg u_* > \arg u(y)$. Here $\phi_1 \in (\phi, \phi_2)$. It is clear that $\lambda$ divides $\mathcal{S}_{u_0}$ into two regions. Let $R_{u_0,y}^1$ denote one of these regions, namely, the one that does not contain $\Delta$. We want to extend the statement of Lemma A.5 into the region $R_{u_0,y}^1$. To this end, we construct a solution $\hat{\Phi}$ of the integral equation (A.20) in $R_{u_0,y}^1$ by successive iterations. Let $u_1$ denote the point of intersection of $\partial \mathcal{S}_{u_0}$ and $\lambda$. The collection of contours of integration $\Omega(u)$ in the case of the region $R_{u_0,y}^1$ is similar to the case of the region $R_{u_0,y}^0$, considered above, except that the contour of
integration for the entry (1, 2) is the segment \([u_1, u]\), provided that \([u_1, u] \subset R_{u_0, y}^1\). For \(u \in R_{u_0, y}^1\) that do not satisfy the latter condition, the contour of integration is the union of \([u_1, u_0] \cup [u_0, u]\).

Let \(\mathcal{I}_2\) denote the integral operator in the Volterra equation (A.20) in the region \(R_{u_0, y}^1\) with the contours defined above. We can repeat the previous estimates of integrals to extend the statement of Lemma A.4 to the integral operator \(\mathcal{I}_2\) in the region \(R_{u_0, y}^1\); the only difference is the finite contour of integration, where the desired estimate comes from Lemma 14.2. [29]. The solution \(\tilde{\Phi}\) to (A.20) thus satisfies the same estimate of Lemma A.5. We also have

\[
\tilde{\Phi}(u) = \Lambda[I + O(u^{-1/5})]
\]

in \(R_{u_0, y}^1\) uniformly in \(y\). Here \(\Lambda\) is given by (A.31).

Let \(u_0 \in \mathbb{R}_+\) be fixed: for any sufficiently large \(|u(y)|\) the set \(R_{u_0, y}^0 \cap R_{u_0, y}^1\) consists of two disjoint components \(\Upsilon_{u_0, y}^{1, 2}\) shown in Fig. 13 (otherwise, we can extend \(\Delta\) in such a way that it will intersect the boundary of \(\Upsilon_{u_0}\) and the corresponding \(\Delta^0\) disappear). Let \(S(y)\) denote the Stokes matrix connecting solutions of (A.8) \(\Phi e^{q y^3}\) and the previously constructed one \(\Phi e^{q y^3}\). Then

\[
\Phi(u, y) = \tilde{\Phi}(u, y) e^{q(u, y) y^3} S(y) e^{-q(u, y) y^3}, \quad S(y) = \begin{pmatrix} 1 & s(y) \\ 0 & 1 \end{pmatrix},
\]

where the triangularity of \(S(y)\) follows from the fact that \(\Phi, \tilde{\Phi} \to \Lambda\) as \(u \to \infty\), \(u \in \Upsilon_{u_0, y}^2\). Writing \(q = q(u, y)\) in (A.33), we emphasize that \(q\) depends on \(y\).

**Estimate for \(s(y)\).** We now want to obtain an estimate for \(s(y)\) by comparing \(\Phi\) and \(\tilde{\Phi}\) in the region \(\Upsilon_{u_0, y}^1\). Let us denote by \(\hat{\Phi}\) the analytic continuation of \(\Phi\) from \(\Upsilon_{u_0, y}^2\) throughout \(R_{u_0, y}^1\) (throughout the dark shaded region) into \(\Upsilon_{u_0, y}^1\). According to (4-8), \(\hat{\Phi}(\xi, y)\) has monodromy \(-1\) as \(\xi\) goes around \(\xi = y\). Because of (A.7), \(W = \Phi e^{q y^3}\) has the same monodromy. It follows from (A.23) that \(e^{q y^3}\) has monodromy \(M m_y := \exp \left[ \frac{2i\pi y^3}{4y^3} \right]\) around \(\xi = y\). Therefore we have

\[
\hat{\Phi}(u; y) = -\Phi(u; y) m_y^{-1}, \quad u \in \Upsilon_{u_0, y}^1.
\]

The matrix \(m_y\) (which is constant in \(u!\)) has the behavior \(m_y = -1 + O(y^{-2})\) on account of (A.24). According to (A.29) and the fact that \(m_y\) is uniformly bounded (for all large \(y\)), we have

\[
||\hat{\Phi} - \Lambda|| \leq K_2|u|^{-\frac{3}{2}}, \quad ||\Phi - \Lambda|| \leq K_2|u|^{-\frac{3}{2}} \quad \text{in} \quad \Upsilon_{u_0, y}^1,
\]

with a constant \(K_2\) independent of \(y\). But, according to (A.32), \(\hat{\Phi}\) too satisfies the same estimate. Substituting these estimates into (A.33), we obtain

\[
s(y) e^{2q(u)} = O(u^{-\frac{3}{2}})
\]

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Figure 13: The region $R^1_{u_0,y}$ (shaded) and its two subregions $\Upsilon^1_{u_0,y}$, $\Upsilon^2_{u_0,y}$ (lighter shading). The goal is to extend the estimate (A.30) into the remaining third (darker shading) subregion of $R^1_{u_0,y}$.

in $\Upsilon^1_{u_0,y}$ uniformly in $y$. In particular, according to (A.23), $|s(y)| \leq K_3|u_1|^{\frac{1}{5}}e^{2\Im u_1}$ for some $K_3 > 0$ that is independent of $y$. Therefore, in the region $\Delta^0 \cup R^1_{u_0,y}$ (darker shade on Fig. 13), we have

$$|s(y)e^{q(u)}| \leq K_3|u_1|^{\frac{1}{5}}e^{2\Im (u_1-u)} \tag{A.37}$$

Given the geometry of regions and the fact that $y \in \hat{S}$, we have

$$\Im (u - u_1) \geq K_4|u(y) - u_1| \geq K_5|u(y) - u_0|, \quad \forall u \in \Delta^0, \tag{A.38}$$

where $K_4, K_5$ denote some positive constants independent of $u_0, y$. Thus, the matrix $e^{q(u,y)\sigma_3}S(y)e^{-q(u,y)\sigma_3}$ in (A.33) is exponentially close to 1 as $y - \xi_0 \to \infty$, where $u(\xi_0) = u_0$. So, according to (A.33), the estimate of Lemma A.5 (with a modified $O$ term) can be extended to $\Delta^0$ (more precisely, to the shaded part of it $\Delta^0 \cap R^1_{u_0,y}$), provided $y - \xi_0 \to \infty$. Because of the construction of $\hat{S}$, there exists $\mu > 0$, such that $\hat{S}$ contains $S(\mu|\xi_0|)$, where by $S$ now we mean the original sector of the RHP (8), see Theorem (A.1). We denote $\xi_* = \mu|\xi_0|$. Existence of the positive constant $p_2$ in the estimate (A.1) follows from (A.33) (A.37) and (A.38). The case of sectors $S$ adjacent to the jump contour $\mathbb{R}^-$ on Fig. 8 can be considered similarly since we can rotate this contour in both directions by angle up to $\frac{\pi}{2}$.

The proof of Theorem A.1 is completed.

According to (A.3) and (A.13), the matrix $T$ from (A.1) can be absorbed into $1 + O(\xi^{-\frac{1}{2}})$ term. To within the same estimate we can replace also $\sqrt{4\xi^3 - v\xi}$ by $2\xi^{\frac{3}{2}}$ and recast the theorem into
Corollary A.1 Under the same assumptions and in the same notations of Theorem A.1 we have
\[
\hat{\Psi}(\xi, v) = \xi^{-\frac{3}{4}} \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3) \left( A + O \left( \xi^{-\frac{1}{4}}, y^{-4}, e^{-p_{2|\xi|^{\frac{3}{2}}}} \right) \right) \left( \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - y} \right) \sigma_3 \epsilon^{\theta(\xi, v)\sigma_3}
\] (A.39)

Theorem A.1 together with the convergence of iterations imply the following corollary.

Corollary A.2 For any sufficiently large \(\xi\) we have \(\lim_{v \to v_p} \hat{\Psi}(\xi, v) = \hat{\Psi}(\xi, v_p)\).

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