On the singular control of exchange rates

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Published online: 29 October 2019
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Abstract
Consider a central bank that wants to manage the exchange rate between its domestic currency and a foreign one. The central bank can purchase and sell the foreign currency, and each direct intervention on the exchange market leads to a proportional cost whose instantaneous marginal value depends on the current level of the exchange rate. The central bank aims at minimizing the total expected costs of interventions on the exchange market, plus a total expected running cost. We formulate this problem as an infinite time-horizon bounded-variation stochastic control problem. The exchange rate evolves as a general one-dimensional diffusion, and it is linearly controlled by two nondecreasing processes modeling the cumulative amount of foreign currency that has been purchased and sold by the central bank. We provide a complete solution to this problem by finding the explicit expression of the value function and a complete characterization of the optimal control. At each instant of time, the optimally controlled exchange rate is kept within a band whose size is endogenously determined as part of the solution to the problem. We also study the expected exit time from the band, and the sensitivity of the width of the band with respect to the model’s parameters in the case when the exchange rate evolves (in absence of any intervention) as an Ornstein–Uhlenbeck process, and the marginal proportional costs of controls are constant. The techniques employed in the paper are those of the theory of singular stochastic control and of one-dimensional diffusions.

Keywords  Singular stochastic control · Exchange rates · Target zones · Central bank · Variational inequality · Optimal stopping

Mathematics Subject Classification  93E20 · 60J60 · 60G40 · 91B64 · 91G30
1 Introduction

One of the main tools that a central bank has at disposal in order to maintain under control the volatility of the exchange rate is to properly purchase or sell foreign currency reserves. As a result of such interventions on the exchange market, in many cases one can observe that the exchange rate between two currencies is either constrained above/below a given floor/ceiling, or it is kept within announced margins on either side of a given value, the so-called central parity (or central rate).

An example of such a constrained regime of the exchange rate is that adopted in Switzerland in the period 2011–2015, when the exchange rate EUR/CHF was explicitly forced to remain above the floor 1.20 through an aggressive devaluation policy. The latter has been adopted by the Swiss National Bank (SNB) until the 15th of January 2015 (Economist 2015; Lloyd 2015), when SNB simply dropped its monetary policy with a very evident effect on the CHF/EUR exchange rate (see Fig. 1). Other countries pledge to keep the exchange rate within a specific band. Such an arrangement is usually referred to as target zone. The 12th of January 2017 marked the 30th anniversary of the Danish target zone (Mikkelsen 2017). Since the Danish economic crisis of the 1980s, the Danish Krone (DKK) was anchored to the German Mark, and then, since 1999, to Euro in such a way that the Krone’s central rate has been unchanged since January 12, 1987. The central rate is 7.46038 Krone per Euro, and the Krone is allowed to increase or decrease by 2.25% (even if the fluctuations have been far smaller for many years, see Fig. 2).

To end with a non-European example, as a response to the Black Saturday crisis in 1983, on October 17, 1983 the Hong Kong Dollar (HKD) has been pegged to the U.S. Dollar (USD), and since then the HKD/USD exchange rate is pegged to a central rate of 7.80 HKD/USD (see Fig. 3), with a band of ±0.05 HKD/USD.

It is not clear (nor of public knowledge) whether the width of the interval where the exchange rate is allowed to fluctuate is chosen according to some optimality criterion (e.g., maximization of social welfare or minimization of expected costs), or it is decided only on the basis of international and political agreements. In the latest years the economic and mathematical literature experienced an intensive research on target zone models. In particular, within the literature we can identify two main streams of research. On one hand, many papers develop stochastic models aiming at explaining the dynamics of exchange rates within a given target zone (see Bo et al. 2016; De Jong et al. 2001; Krugman 1991; Jørgensen and Mikkelsen 1996; Larsen and Sørensen 2007; Yang et al. 2016, among others). Target zone models have been pioneered in Krugman (1991) where it is assumed that the “fundamental” (and not observed) exchange rate is a Brownian motion, which is instantaneously reflected at exogenously given upper and lower barriers: this intrinsically defines a singular stochastic control problem, whose value function is the exchange rate really observed in the market. Although in Krugman (1991) many mathematical details are missing, in that seminal paper the author finds that the observed exchange rate is mean-reverting inside the given target zone. In the subsequent papers (see e.g. Bo et al. 2016; De Jong et al. 2001; Jørgensen and Mikkelsen 1996; Larsen and Sørensen 2007; Yang et al. 2016 and references therein), the authors assume that exchange rates fluctuate stochastically within an exogenously given interval according to a stochastic differential equation parametrized by a set of free parameters, and possibly satisfying reflecting boundary conditions, or with diffusion coefficient vanishing near the

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1 This floor was made explicit by Swiss National Bank in a press release (Swiss 2011). It might be possible that also a cap was present in their monetary policy, but its enforcing was completely useless in those times, and of course not declared.
Fig. 1 Plot EUR/CHF exchange rate from 2011 until 2015. One can notice the exchange rate’s sudden drop on January 15, 2015, when the Swiss National Bank stopped maintaining the floor EUR/CHF = 1.20

Fig. 2 Plot EUR/DKK exchange rate from 2008 until 2016 boundaries of the interval. The parameters are then calibrated in such a way that the model can fit the data on exchange rates, e.g. in the European monetary system.

On the other hand, several papers in the mathematical literature endogenize the width of the target zone by formulating the exchange rates’ optimal management problem as a stochastic optimal control problem (see Bertola et al. 2016; Cadenillas and Huamán-Aguilar 2016; Cadenillas and Zapatero 1999, 2000; Jeanblanc-Picqué 1993; Mundaca and Øksendal 1998; Perera et al. 2018, and references therein). In these papers, the central bank aims at adjusting the uncertain level of the exchange rate in order to minimize the spread between the instantaneous level of the exchange rate and a given central parity. To accomplish that, the central bank can purchase or sell foreign currency, but whenever the central bank intervenes, a cost for the intervention must be paid. In those papers such a cost has both a proportional and
a fixed component, thus leading to a mathematical formulation of the optimization problem as a two-sided stochastic impulsive control problem (possibly also with classical controls modeling the interventions on the domestic interest rate). It is shown that the optimally controlled exchange rate is kept within endogenously determined levels (the so-called free boundaries) and the interventions are of pure-jump type: at optimal times the exchange rate is pushed from a free boundary to another threshold level, which is also found endogenously as a part of the solution to the problem. In essence, it is optimal to follow a two-sided \((s, S)\)-policy, well-known optimal policy also in many inventory models.

However, a closer look, e.g., at the dynamics of the exchange rate HKD/USD since 2008 reveals no jumps, but a continuous reflection of the exchange rate at the boundaries of the interval where it is allowed to fluctuate (see Figs. 1 and 3). Such an observation suggests that the optimal management problem of exchange rates might be mathematically better formulated as a singular stochastic control problem, rather than as an impulsive one. Indeed, in singular stochastic control problems the optimal control usually prescribes a continuous reflection of the controlled state variable at endogenously determined level(s) (see, e.g., Chapter VIII in Fleming and Soner (2005) and Shreve (1988) for an introduction to singular stochastic control).

In this paper we thus introduce an infinite time-horizon, one-dimensional bounded variation singular stochastic control problem to model the exchange rates’ optimal management problem. In our model, the (logarithm of the) exchange rate is a one-dimensional Itô-diffusion satisfying a linearly controlled stochastic differential equation with suitable drift and volatility coefficients. Such general dynamics allows us to cover classical models where the exchange rate evolves as a geometric Brownian motion, as well as more realistic mean-reverting behaviors of the exchange rate’s dynamics (see Sweeney 2006; Tvedt 2012 and references therein). The cumulative amount of purchases and sales of the foreign currency are the control variables of the central bank; these are monotone processes, adapted to the underlying filtration, and satisfying proper integrability conditions. The central bank aims at choosing a (cumulative) purchasing-selling policy in order to minimize a total expected discounted cost functional. This is given by the sum of total expected running costs and costs of interventions. The instantaneous running cost of the exchange rate is measured by a general nonnegative convex function. This generalizes the quadratic cost function usually employed in the literature (cf., e.g., Bertola et al. 2016; Cadenillas and Zapatero 1999). Such a running cost can be seen as a measure of all the economic and real costs related to having a certain level \(X_t\) of exchange rate at time \(t\); e.g., it could be a penalization due the deviation from the aimed target for the exchange rate, a measure of the social cost of induced inflation/deflation, a change of...
exporting power of the country due to a lower exchange rate, or an increase of the importing power due to a higher value of its own currency. Also, we assume that the instantaneous proportional costs of the interventions on the exchange rate depend on the current level of the exchange rate, and they are sufficiently smooth real-valued functions. For the sake of mathematical tractability, and with the aim of obtaining an explicit solution, in our model we assume that purchases and sales are the only control variables of the central bank. We therefore neglect other instruments usually employed by central banks for the exchange-rates management, like interest rate policy and management of expectations. Also, we assume that central bank’s reserves are unlimited, and we therefore do not impose any bound on the cumulative purchases/sales of currencies. A detailed comment on these aspects is provided in Sect. 5.

We tackle the aforementioned problem via a classical guess-and-verify approach by carefully employing the properties of one-dimensional regular diffusions (see, e.g., Borodin and Salminen 2002), and of their excessive mappings (Alvarez 2003). We find that the optimal purchasing-selling policy of the central bank is triggered by two thresholds (free boundaries), which are the unique solution to a system of two coupled nonlinear algebraic equations. The optimal policy prescribes to purchase and sell the minimal amount of foreign currency that allows to keep the exchange rate within the free boundaries. Mathematically, the optimal control is given by the solution to a two-sided Skorokhod reflection problem.

It is worth noticing that, differently from models involving impulsive controls, where the actual optimality of a candidate value function is usually proved only via numerical methods (see Cadenillas and Zapatero 1999, 2000), here we are able to provide a complete analytical study by finding the explicit expression of the value function and of the optimal control process (up to the solution to the algebraic system for the two free boundaries). Moreover, we can provide a detailed comparative statics analysis of the free boundaries when the (log-)exchange rate (in absence of any intervention) evolves through an Ornstein–Uhlenbeck dynamics. The latter allows us to capture the mean-reverting behavior of exchange rates that has been observed in several empirical studies (see Sweeney 2006; Tvedt 2012 and references therein). In particular, by assuming that the instantaneous proportional costs of interventions are constant, we show that the more the exchange market is volatile, the more the central bank is reluctant to intervene. Also, we are able to numerically evaluate the expected exit times and exit probabilities from the target zone, and to relate our findings with the monetary policy adopted by the Danish Central Bank since 1987 (Mikkelsen 2017).

The contribution of this paper is twofold. On the one hand, we contribute to the literature from the modeling point of view. Indeed, by introducing a singular stochastic control problem to model the exchange rates’ optimal management problem faced by a central bank, we are able to mimick the continuous reflection of the exchange rate at the target zone’s boundaries which seems to happen in reality (see Fig. 3). From the mathematical point of view, we contribute by providing the explicit solution to a stationary bounded variation singular stochastic control in a very general setting with state variable evolving as a general one-dimensional diffusion, and with instantaneous marginal costs of control that are state-dependent. To the best of our knowledge, the explicit solution to a similar problem is not available in the literature yet.

The work that is perhaps closest to ours is Matomäki (2012), where a one-dimensional, bounded variation singular stochastic control problem over an infinite time-horizon has been studied. However, one can come across several major differences between our paper and Matomäki (2012). First of all, in Matomäki (2012) the instantaneous marginal costs of control are constant. Second of all, in Matomäki (2012) the state dynamics (in the notation of that paper) is $Z_t = X_t + U_t - D_t$, where $X$ is an uncontrolled one-dimensional regular diffusion,
and \((U, D)\) gives the minimal decomposition of a process of bounded variation. In our paper, instead, the dynamics of the state variable is given in differential form [see (2.3)], and, differently to Matomäki (2012), the controlled state process at time \(t \geq 0\) cannot be written as the sum of an uncontrolled one and of the cumulative bounded variation control exerted up to time \(t\). Finally, in Matomäki (2012) the optimal control is sought within the class of barrier policies, whereas we here obtain optimality in a larger class (see our Definition 2.9 below).

The rest of the paper is organized as follows. In Sect. 2.1 we set up the probabilistic setting, whereas in Sect. 2.2 we introduce the exchange rates’ optimal management problem that is the object of our study. In Sect. 3 we solve the problem by proving first a preliminary verification theorem, and then constructing the value function and the optimal control. In Sect. 4 we assume that the (log-)exchange rate is an Ornstein–Uhlenbeck process, and we provide the sensitivity of the free boundaries with respect to the model’s parameters and a study of the expected hitting time at the free boundaries. Sect. 5 collects the conclusions and possible extensions of this work. Finally, in the “Appendix” we prove some auxiliary results needed in the paper.

2 Setting and problem formulation

2.1 The probabilistic setting

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, \(B\) a one-dimensional Brownian motion, and denote by \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) a right-continuous filtration to which \(B\) is adapted. We introduce the nonempty sets

\[
S := \{\nu : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+, \mathbb{F}\text{-adapted and such that } t \mapsto \nu_t \text{ is a.s. (locally) of bounded variation, left-continuous and s.t. } \nu_0 = 0\},
\]

(2.1)

\[
U := \{\vartheta : \vartheta \in S \text{ and } t \mapsto \vartheta_t \text{ is nondecreasing}\}.
\]

(2.2)

Then, for any \(\nu \in S\), we denote by \(\xi, \eta \in U\) the two processes providing the minimal decomposition of \(\nu\); that is, such that

\[
\nu_t = \xi_t - \eta_t, \quad t \geq 0,
\]

and the increments \(\Delta \xi_t = \xi_{t+} - \xi_t\) and \(\Delta \eta_t := \eta_{t+} - \eta_t\) are supported on disjoint subsets of \(\mathbb{R}^+\). In the following, we set \(\xi_0 = \eta_0 = 0\) a.s., without loss of generality, and for frequent future use we notice that any \(\nu \in S\) satisfies

\[
\nu_t = \nu^c_t + \nu^j_t, \quad t \geq 0.
\]

Here \(\nu^c\) is the continuous part of \(\nu\), and the jump part \(\nu^j\) is such that \(\nu^j_t := \sum_{0 \leq s < t} \Delta \nu_s\), where \(\Delta \nu_t := \nu_{t+} - \nu_t, t \geq 0\).

We then consider on \((\Omega, \mathcal{F}, \mathbb{P})\) a process \(X\) satisfying the following stochastic differential equation (SDE)

\[
\begin{align*}
        dX_t &= \mu(X_t)dt + \sigma(X_t)dB_t + d\xi_t - d\eta_t, \quad X_0 = x \in I.
\end{align*}
\]

(2.3)

Here \(I = (x, \bar{x})\), with \(-\infty \leq x < \bar{x} \leq +\infty\), and \(\mu\) and \(\sigma\) are suitable drift and diffusion coefficients. The process \(X\) represents the (log-)exchange rate between two currencies. The drift coefficient \(\mu\) measures the trend of the exchange rate, whereas \(\sigma\) the fluctuations around
Moreover, under Assumption 2.1 one has that for any \( f \in C^1(\mathcal{I}) \)
and we make the following assumption.

We denote by
\[
e_{\mathcal{T}} := \inf\{ t \geq 0 \mid X_t^{\xi, \eta} \notin \mathcal{I}\}
\]
the first time when the controlled process \( X_t^{\xi, \eta} \) leaves \( \mathcal{I} \); i.e. the so-called explosion time, and we make the following assumption.

**Assumption 2.1** The coefficients \( \mu : \mathcal{I} \to \mathbb{R} \) and \( \sigma : \mathcal{I} \to (0, \infty) \) belong to \( C^1(\mathcal{I}) \). Moreover, \( \sigma' \) is locally-Lipschitz on \( \mathcal{I} \).

The previous conditions ensures that, for any \( \nu \in S \), there exists a unique strong solution to (2.3), for any \( t < e_{\mathcal{T}} \) (see Protter 1990, Theorem V.7, and the discussion after its proof). From now on, in order to stress its dependence on the initial value \( x \in \mathcal{I} \) and on the two processes \( \xi \) and \( \eta \), we refer to the (left-continuous) solution to (2.3) as \( X_{\mathcal{T}}^{\xi, \eta} \), where appropriate. Also, in the rest of the paper we use the notation \( \mathbb{E}_X[f(X_{\mathcal{T}}^{\xi, \eta})] = \mathbb{E}[f(X_{\mathcal{T}}^{\xi, \eta})] \). Here \( \mathbb{E}_X \) is the expectation under the measure \( \mathbb{P}_X(\cdot) := \mathbb{P}(\cdot | X_0^{\xi, \eta} = x) \) on \( (\Omega, \mathcal{F}) \), and \( f : \mathbb{R} \to \mathbb{R} \) is any Borel-measurable such that \( f(X_{\mathcal{T}}^{\xi, \eta}) \) is integrable.

**Remark 2.2** Notice that Assumption 2.1 is satisfied by relevant diffusions like Brownian motion, Brownian motion with drift, Ornstein–Uhlenbeck process, and CIR process, among others.

We also consider a one-dimensional diffusion evolving according to the SDE
\[
d\bar{X}_t = [\mu(\bar{X}_t) + (\sigma \sigma')(\bar{X}_t)]dt + \sigma(\bar{X}_t)d\bar{B}_t, \quad \bar{X}_0 = x \in \mathcal{I}. \tag{2.4}
\]
Notice that, under Assumption 2.1, there exists a unique strong solution to (2.4), up to a possible explosion time (see Chapter 5.5 in Karatzas and Shreve (1991), among others). Moreover, under Assumption 2.1 one has that for any \( x \in \mathcal{I} \) there exists \( \epsilon_o > 0 \) such that
\[
\int_{x-\epsilon_o}^{x+\epsilon_o} \frac{1 + |\mu(z)| + |\sigma \sigma'(z)|}{|\sigma^2(z)|} \, dz < +\infty. \tag{2.5}
\]
guaranteeing that \( \bar{X} \) is a regular diffusion. That is, starting from \( x \in \mathcal{I} \), \( \bar{X} \) reaches any other \( y \in \mathcal{I} \) in finite time with positive probability. Finally, to stress the dependence of \( \bar{X} \) on its initial value, from now on we write \( \bar{X}^x \), where needed, and we denote by \( \mathbb{P}_x \) the expectation under the measure \( \mathbb{P}_x(\cdot) \).

The infinitesimal generator of the uncontrolled diffusion \( X^{x:0,0} \) is denoted by \( \mathcal{L}_X \) and is defined as
\[
(\mathcal{L}_X f)(x) := \frac{1}{2} \sigma^2(x) f''(x) + \mu(x) f'(x), \quad f \in C^2(\mathcal{I}), \quad x \in \mathcal{I}, \tag{2.6}
\]
whereas the one of \( \bar{X} \) is denoted by \( \mathcal{L}_{\bar{X}} \) and is defined as
\[
(\mathcal{L}_{\bar{X}} f)(x) := \frac{1}{2} \sigma^2(x) f''(x) + (\mu(x) + \sigma(x) \sigma'(x)) f'(x), \quad f \in C^2(\mathcal{I}), \quad x \in \mathcal{I}. \tag{2.7}
\]
Letting \( r > 0 \) be a fixed constant, we make the following standing assumption.

**Assumption 2.3** \( r - \mu'(x) > 0 \) for \( x \in \mathcal{I} \).
In the subsequent optimization problem, the parameter \( r > 0 \) will play the role of the central bank’s discount factor [see (2.28) below].

We introduce \( \psi \) and \( \phi \) as the fundamental solutions of the ordinary differential equation (ODE) (see Ch. 2, Sec. 10 of Borodin and Salminen 2002),

\[
\mathcal{L}_X u(x) - ru(x) = 0, \quad x \in \mathcal{I},
\]

and we recall that they are strictly increasing and decreasing, respectively. For an arbitrary \( x_0 \in \mathcal{I} \) we also denote by

\[
S'(x) := \exp \left( - \int_{x_0}^x \frac{2\mu(z)}{\sigma^2(z)} \, dz \right), \quad x \in \mathcal{I},
\]

the derivative of the scale function of \((X_t^x, 0, 0)_{t \geq 0} \), and by \( W \) the constant Wronskian

\[
W := \frac{\psi'(x)\phi(x) - \phi'(x)\psi(x)}{S'(x)}, \quad x \in \mathcal{I}.
\]

Moreover, under Assumption 2.1, any solution to the ODE

\[
\mathcal{L}_\tilde{X} u(x) - (r - \mu'(x))u(x) = 0, \quad x \in \mathcal{I},
\]

can be written as a linear combination of the fundamental solutions \( \tilde{\psi} \) and \( \tilde{\phi} \), which again by Borodin and Salminen (2002, Chapter 2.10) are strictly increasing and decreasing, respectively. Finally, letting \( x_0 \in \mathcal{I} \) to be arbitrary, we denote by

\[
\tilde{S}'(x) := \exp \left( - \int_{x_0}^x \frac{2\mu(z) + 2\sigma(z)\sigma'(z)}{\sigma^2(z)} \, dz \right), \quad x \in \mathcal{I},
\]

the derivative of the scale function of \((\tilde{X}_t^x)_{t \geq 0} \), by

\[
\tilde{m}'(x) := \frac{2}{\sigma^2(x) \tilde{S}'(x)},
\]

the density of the speed measure of \((\tilde{X}_t^x)_{t \geq 0} \), and by \( w \) the Wronskian

\[
w := \frac{\tilde{\psi}'(x)\tilde{\phi}(x) - \tilde{\phi}'(x)\tilde{\psi}(x)}{\tilde{S}'(x)}, \quad x \in \mathcal{I}.
\]

**Remark 2.4** It is easy to see that the scale functions and speed measures of the two diffusions \( X^x, 0, 0 \) and \( \tilde{X} \) are related through \( \tilde{S}'(x) = S(x)/\sigma^2(x) \) and \( \tilde{m}'(x) = 2/S'(x) \) for \( x \in \mathcal{I} \).

Concerning the boundary behavior of the real-valued Itô-diffusions \( X^x, 0, 0 \) and \( \tilde{X} \), in the rest of this paper we assume that \( x \) and \( \tilde{x} \) are natural for those two processes (see Borodin and Salminen 2002 for a complete discussion of the boundary behavior of one-dimensional diffusions). This in particular means that they are unattainable in finite time and that

\[
\lim_{x \downarrow \Delta} \psi(x) = 0, \quad \lim_{x \downarrow \Delta} \phi(x) = +\infty, \quad \lim_{x \uparrow \bar{x}} \psi(x) = +\infty, \quad \lim_{x \uparrow \bar{x}} \phi(x) = 0,
\]

\[
\lim_{x \downarrow \Delta} \frac{\psi'(x)}{S'(x)} = 0, \quad \lim_{x \downarrow \Delta} \frac{\phi'(x)}{S'(x)} = -\infty, \quad \lim_{x \uparrow \bar{x}} \frac{\psi'(x)}{S'(x)} = +\infty, \quad \lim_{x \uparrow \bar{x}} \frac{\phi'(x)}{S'(x)} = 0,
\]

and

\[
\lim_{x \downarrow \Delta} \tilde{\psi}(x) = 0, \quad \lim_{x \downarrow \Delta} \tilde{\phi}(x) = +\infty, \quad \lim_{x \uparrow \bar{x}} \tilde{\psi}(x) = +\infty, \quad \lim_{x \uparrow \bar{x}} \tilde{\phi}(x) = 0,
\]

\[
\lim_{x \downarrow \Delta} \frac{\tilde{\psi}'(x)}{\tilde{S}'(x)} = 0, \quad \lim_{x \downarrow \Delta} \frac{\tilde{\phi}'(x)}{\tilde{S}'(x)} = -\infty, \quad \lim_{x \uparrow \bar{x}} \frac{\tilde{\psi}'(x)}{\tilde{S}'(x)} = +\infty, \quad \lim_{x \uparrow \bar{x}} \frac{\tilde{\phi}'(x)}{\tilde{S}'(x)} = 0.
\]
We show in Lemma A.1 in the “Appendix” that, under our conditions on the diffusions $X_{t}^{0,0}$ and $\hat{X}$, one has $\hat{\phi} = -\phi'$ and $\hat{\psi} = \psi'$ (see also the second part of the proof of Lemma 4.3 in Alvarez and Matomäki 2015).

**Remark 2.5** Recall that boundaries of the state space of the diffusions $X_{t}^{0,0}$ and $\hat{X}$ are unattainable, being natural. Defining the new measure $Q_{x}$ through the Radon–Nikodym derivative

$$Z_{t} := \frac{dQ_{x}}{dP_{x}} \bigg|_{\mathcal{F}_{t}} = \exp \left\{ \int_{0}^{t} \sigma'(X_{s}^{0,0}) dB_{s} - \frac{1}{2} \int_{0}^{t} (\sigma')^{2}(X_{s}^{0,0}) dP_{s} \right\}, \quad P_{x} \text{ a.s.,} \quad (2.17)$$

Theorem 2.1 and Corollary 2.2 in Mijatovic and Urusov (2012) (see also Carole et al. 2017) ensure that $Z$ is an exponential martingale. Then by the Girsanov theorem the process

$$\hat{B}_{t} := B_{t} - \int_{0}^{t} \sigma'(X_{s}^{0,0}) dP_{s} \quad (2.18)$$

is a standard Brownian motion under $Q_{x}$, and it is not hard to verify that $\text{Law} \left( X_{0,0}^{0,0} \bigg| Q_{x} \right) = \text{Law} \left( \hat{X} \bigg| P_{x} \right)$.

**Remark 2.6** It is worth noticing that all the assumptions that we have made regarding the diffusions $X_{t}^{x,0,0}$ and $\hat{X}$ (namely, Assumptions 2.1, 2.3) are satisfied in the relevant cases of a drifted Brownian motion (i.e. $\mu(x) = \mu > 0$ and $\sigma(x) = \sigma > 0$), or by an Ornstein–Uhlenbeck process (i.e. $\mu(x) = \theta(\mu - x)$, for some constants $\theta > 0$, $\mu \in \mathbb{R}$ and $\sigma(x) = \sigma > 0$), both defined on $\mathcal{I} = \mathbb{R}$, i.e. with $x = -\infty$, $\bar{x} = +\infty$. Also the mean-reverting exponential process in Dayanik and Karatzas (2003, Section 6.5) with $\mu(x) = \theta x(\mu - x)$, $\theta > 0$, $\mu > 0$, and $\sigma(x) = \sigma x$, $\sigma > 0$, fulfill our requirements.

For future reference, for all $x, y \in \mathcal{I}$ we introduce the Green functions associated to the diffusion $X_{t}^{x,0,0}$

$$G(x, y) := W^{-1} \cdot \begin{cases} \psi(x)\phi(y), \quad x \leq y, \\ \phi(x)\psi(y), \quad x \geq y, \end{cases} \quad (2.19)$$

and to the diffusion $\hat{X}_{t}^{x}$

$$\hat{G}(x, y) := w^{-1} \cdot \begin{cases} \hat{\psi}(x)\hat{\phi}(y), \quad x \leq y, \\ \hat{\phi}(x)\hat{\psi}(y), \quad x \geq y. \end{cases} \quad (2.20)$$

Then one has that the resolvents

$$(Rf)(x) := \mathbb{E}_{x} \left[ \int_{0}^{\infty} e^{-rs} f(X_{s}^{0,0}) ds \right], \quad x \in \mathcal{I}, \quad (2.21)$$

and

$$(\hat{R}f)(x) := \hat{\mathbb{E}}_{x} \left[ \int_{0}^{\infty} e^{-f_{0}(r-\mu'(\hat{X}_{u})) du} f(\hat{X}_{s}) ds \right], \quad x \in \mathcal{I}, \quad (2.22)$$

which are defined for any function $f$ such that the previous expectations are finite, admit the representations

$$(Rf)(x) = \mathbb{E}_{x} \left[ \int_{0}^{\infty} e^{-rs} f(X_{s}^{0,0}) ds \right] = \int_{\mathcal{I}} f(y)G(x, y)\hat{m}'(y) dy, \quad (2.23)$$

and

$$(\hat{R}f)(x) = \hat{\mathbb{E}}_{x} \left[ \int_{0}^{\infty} e^{-f_{0}(r-\mu'(\hat{X}_{u})) du} f(\hat{X}_{s}) ds \right] = \int_{\mathcal{I}} f(y)\hat{G}(x, y)\hat{m}'(y) dy, \quad (2.24)$$
for all \( x \in \mathcal{I} \). Notice that \( Rf \) and \( \hat{R}f \) solve the ODEs
\[
(L_X - r)(Rf)(x) = -f(x), \quad (L_X - (r - \mu'(x)))(\hat{R}f)(x) = -f(x),
\]
for any \( x \in \mathcal{I} \). Moreover,
\[
(Rf)'(x) = (\hat{R}f')(x), \quad x \in \mathcal{I},
\]
for any \( f \in C^1(\mathcal{I}) \) such that \( Rf \) and \( \hat{R}f' \) are well defined (a proof of relation (2.26) can be found in the “Appendix” for the sake of completeness).

Finally, the following useful equations hold for any \( x < \alpha < \beta < \infty \) (cf. par. 10, Ch. 2 of Borodin and Salminen 2002):
\[
\begin{align*}
\hat{\psi}'(\beta) \hat{S}'(\beta) - \hat{\psi}'(\alpha) \hat{S}'(\alpha) &= \int_{\alpha}^{\beta} \hat{\psi}(y)(r - \mu'(y))\hat{m}'(y)dy, \\
\hat{\phi}'(\beta) \hat{S}'(\beta) - \hat{\phi}'(\alpha) \hat{S}'(\alpha) &= \int_{\alpha}^{\beta} \hat{\phi}(y)(r - \mu'(y))\hat{m}'(y)dy.
\end{align*}
\]

### 2.2 The optimal control problem

In this section we introduce the optimization problem faced by the central bank. The central bank can adjust the level of the exchange rate only by purchasing or selling one of the two currencies (i.e. by properly exerting \( \xi \) and \( \eta \)). For the sake of mathematical tractability, we assume that the central bank’s reserves are unlimited, and that no implicit actions on the exchange rate through, e.g., an interest rate policy are considered (see Sect. 5 below for comments). A direct action on the exchange rate via a currency’s devaluation or evaluation policy is clearly costly as, e.g., it might be perceived by investors as a signal of existing volatility in the market with consequent real and financial costs. We therefore suppose that purchases/sales of currencies result into proportional costs, \( c_1 \) and \( c_2 \), that depend on the current level of the exchange rate. Also, we assume that, being \( X_t \) the level of the (log-)exchange rate at time \( t \geq 0 \), the central bank faces a running cost \( h(X_t) \). This can be seen as a measure of all the economic and real costs related to having a certain level \( X_t \) of exchange rate at time \( t \); e.g., it could be a penalization due the deviation from the aimed target for the exchange rate, or a measure of the social cost of induced inflation/deflation.

The total expected cost associated to a central bank’s policy \( \nu \in \mathcal{S} \) is therefore
\[
\mathcal{J}_x(\nu) := \mathbb{E}_x \left[ \int_0^{\sigma_I} e^{-rs} h(X^x,\xi,\eta) ds + \int_0^{\sigma_I} e^{-rs} \left( c_1(X^x,\xi,\eta) \oplus d\xi_s + c_2(X^x,\xi,\eta) \ominus d\eta_s \right) \right].
\]

In (2.28) \( r > 0 \) is a suitable discount factor of the central bank,
and
\[
\int_0^{\sigma I} e^{-rs} c_2(X_s^{\xi, \eta}) \, d\eta_s := \int_0^{\sigma I} e^{-rs} c_2(X_s^{\xi, \eta}) \, d\eta_s^c + \sum_{s < \sigma I} e^{-rs} \int_0^{\Delta\eta_s} c_2(X_s^{\xi, \eta} - z) \, dz,
\]
(2.30)
and \(\xi^c\) and \(\eta^c\) denote the continuous parts of \(\xi\) and \(\eta\), respectively. Notice that the definition of the costs of control as in (2.29) and (2.30) has been introduced in Zhu (1992), and it is now common in the singular stochastic control literature (see Lon and Zervos 2011, among many others).

Regarding the running cost \(h\) and the proportional costs \(c_i\), we suppose the following.

**Assumption 2.7**

(i) \(h : \mathbb{R} \to [0, +\infty)\) belongs to \(C^1(\mathcal{I})\);

(ii) For any \(i = 1, 2, c_i : \mathbb{R} \to \mathbb{R}\) belongs to \(C^2(\mathcal{I})\). Moreover, setting \(\hat{c}_i := (\mathcal{L}_X - (r - \mu'))c_i\), \(i = 1, 2\), we have

\[
\begin{align*}
-\hat{c}_1(x) + h'(x) &\begin{cases}
< 0, & x < \tilde{x}_1, \\
= 0, & x = \tilde{x}_1, \\
> 0, & x > \tilde{x}_1,
\end{cases}\\
\hat{c}_2(x) + h'(x) &\begin{cases}
< 0, & x < \tilde{x}_2, \\
= 0, & x = \tilde{x}_2, \\
> 0, & x > \tilde{x}_2,
\end{cases}
\end{align*}
\]

for some \(\tilde{x}_1, \tilde{x}_2\) such that \(\underline{x} < \tilde{x}_1 < \tilde{x}_2 < \overline{x}\). Furthermore,

\[
c_1(x) + c_2(x) > 0, \quad x \in \overline{\mathcal{I}},
\]

\[
\hat{c}_1(x) + \hat{c}_2(x) < 0, \quad x \in \mathcal{I},
\]

and the representation

\[
c_i(x) = -\hat{E}_x \left[ \int_0^\infty e^{-\int_0^z (r - \mu'(\hat{X}_u)) \, du} \hat{c}_i(\hat{X}_s) \, ds \right] = -\left( \hat{R} \hat{c}_i \right)(x), \quad x \in \mathcal{I},
\]
(2.31)
holds true. Finally, there exists \(K_i > 0\) and \(\gamma \geq 1\) such that

\[
|c_i(x)| \leq K_i (1 + |x|^{\gamma'}), \quad x \in \mathcal{I}.
\]

**Remark 2.8**

(1) All the results of this paper also hold for a slightly weaker regularity condition on \(c_i\), \(i = 1, 2\); namely, if \(c_i \in W^{2,\infty}_{\text{loc}}(\mathcal{I})\). The latter is equivalent by Sobolev’s embeddings (see, e.g., Cor. 9.15 in Ch. 9 of Brezis 2011) to assuming that, for any \(i = 1, 2\), \(c_i\) is continuously differentiable with second derivative which is locally bounded in \(\mathcal{I}\).

(2) It is easy to verify that, for example, \(h(x) = \frac{1}{2}(x - \theta)^2, \theta \in \mathbb{R}\), and \(c_i(x) = c_i > 0\) for all \(x \in \mathcal{I}\) satisfy Assumption 2.7.
It is worth noticing that (2.31) is in essence an integrability condition. Indeed, if the
trasversality condition
\[
\lim_{t \to +\infty} \hat{E}_{x} \left[ e^{-\int_{0}^{t} (r - \mu'(\hat{X}_{s})) ds} c_{i}(\hat{X}_{t}) \right] = 0, \quad i = 1, 2,
\]
holds true and
\[
\hat{E}_{x} \left[ \int_{0}^{\infty} e^{-\int_{s}^{t} (r - \mu'(\hat{X}_{u})) du} |c_{i}(\hat{X}_{s})| ds \right] < \infty,
\]
then an application of Dynkin’s formula (up to a standard localization argument) gives (2.31).

The following definition characterizes the class of admissible controls.

**Definition 2.9** For any \( x \in \mathcal{I} \) we say that \( \nu \in \mathcal{S} \) is an admissible control, and we write \( \nu \in A(x) \) if \( X_{x}, \xi, \eta, t \in \mathcal{I} \) for all \( t > 0 \) (i.e., \( e^{\mathcal{I}} = +\infty \) \( \mathbb{P}_{x} \)-a.s.) and the following hold true:

(a) \( E_{x} \left[ \int_{0}^{\infty} e^{-rs} |c_{1}(X_{s}^{\xi, \eta})| \oplus d\xi_{s} + \int_{0}^{\infty} e^{-rs} |c_{2}(X_{s}^{\xi, \eta})| \ominus d\eta_{s} \right] < +\infty; \)

(b) \( E_{x} \left[ \int_{0}^{\infty} e^{-rs} h(X_{s}^{\xi, \eta}) \right] < +\infty; \)

(c) \( E_{x} \left[ \sup_{t \geq 0} e^{-\frac{r}{2} t} |X_{t}^{\xi, \eta}|^{1+\gamma} \right] < +\infty \) (for \( \gamma \) as in Assumption 2.7-(ii)).

The central bank aims at picking an admissible \( \nu^{*} \) such that the total expected cost functional (2.28) is minimized; that is, it aims at solving

\[
v(x) := \inf_{\nu \in A(x)} J_{x}(\nu), \quad x \in \mathcal{I}.
\]

Problem (2.32) takes the form of a singular stochastic control problem (see, e.g., Shreve 1988 for an introduction); that is, a problem where the (random) measure on \( \mathbb{R}_{+} \) induced by a control process might be singular with respect to the Lebesgue measure.

### 3 Solving the problem

#### 3.1 A preliminary verification theorem

In this section we prove a verification theorem, which provides a set of sufficient conditions under which a candidate value function and a candidate control process are indeed optimal. To this end, we notice that according to the classical theory of singular stochastic control (see, e.g., Chapter VIII of Fleming and Soner 2005), we expect \( v \) to identify with a suitable solution to the Hamilton–Jacobi–Bellman (HJB) equation

\[
\min \left\{ (L_{x} - r) u(x) + h(x), c_{2}(x) - u'(x), u'(x) + c_{1}(x) \right\} = 0, \quad x \in \mathcal{I}.
\]  

(3.1)

In fact, the latter takes the form of a variational inequality with state-dependent gradient constraints.

**Theorem 3.1** (Verification Theorem) Suppose that Assumption 2.7 holds true and assume that the Hamilton–Jacobi–Bellman equation (3.1) admits a \( C^{2} \) solution \( u: \mathcal{I} \to \mathbb{R} \) such that

\[
|u(x)| \leq K (1 + |x|^{1+\gamma}), \quad x \in \mathcal{I},
\]

where \( K \) is a constant. Then, for any \( x \in \mathcal{I} \), there exists a control \( \nu(x) \) such that

\[
v(x) = J_{x}(\nu(x)),
\]

and the pair \( (\nu(x), u_{x}(\nu(x))) \) is optimal.
for some $K > 0$, and where $\gamma \geq 1$ is the growth coefficients of $c_i$, $i = 1, 2$ [see Assumption 2.7-(ii)]. Then one has that $u \leq v$ on $\mathcal{I}$.

Moreover, given an initial condition $x \in \mathcal{I}$, suppose also that there exists $\hat{v} \in \mathcal{A}(x)$ such that the processes $\hat{\xi}$ and $\hat{\eta}$ providing its minimal decomposition are such that

$$X_t^{x, \hat{\xi}, \hat{\eta}} \in \left\{ x \in \mathcal{I} : (\mathcal{L}_X - r)u(x) + h(x) = 0 \right\},$$

Lebesgue-a.e. $\mathbb{P}$-a.s., the process

$$\left( \int_0^t e^{-rs} \sigma(X_s^{x, \hat{\xi}, \hat{\eta}})u'(X_s^{x, \hat{\xi}, \hat{\eta}}) \, dB_s \right)_{t \geq 0}$$

is an $\mathbb{F}$-martingale,

and

$$\begin{cases}
\int_0^T (u'(X_t^{x, \hat{\xi}, \hat{\eta}}) + c_1(X_t^{x, \hat{\xi}, \hat{\eta}})) \oplus d\hat{\xi}_t = 0, \\
\int_0^T (c_2(X_t^{x, \hat{\xi}, \hat{\eta}}) - u'(X_t^{x, \hat{\xi}, \hat{\eta}})) \oplus d\hat{\eta}_t = 0,
\end{cases}$$

for all $T \geq 0$ $\mathbb{P}$-a.s. Then $u = v$ on $\mathcal{I}$ and $\hat{v}$ is optimal for (2.32).

**Proof** The proof is organized in two steps. We first prove that $u \leq v$ on $\mathcal{I}$, and then that $u \geq v$ on $\mathcal{I}$, and $\hat{v}$ is optimal for (2.32).

**Step 1** Let $x \in \mathcal{I}$ and $v \in \mathcal{A}(x)$. Since $u \in C^2(\mathcal{I})$ we can apply Itô-Meyer’s formula for semimartingales (see Meyer 1976, pp. 278–301) to the process $(e^{-rt}u(X_t^{x, \xi, \eta}))_{t \geq 0}$ on an arbitrary time interval $[0, T]$, $T > 0$. Then, recalling that $\xi^c$ and $\eta^c$ denote the continuous parts of $\xi$ and $\eta$, respectively, we have

$$u(x) = e^{-rT}u(X_T^{x, \xi, \eta}) - \int_0^T e^{-rs}(\mathcal{L}_X - r)u(X_s^{x, \xi, \eta}) \, ds - M_T^{x, \xi, \eta}$$

$$- \int_0^T e^{-rs}u'(X_s^{x, \xi, \eta}) \, d\xi^c_s + \int_0^T e^{-rs}u'(X_s^{x, \xi, \eta}) \, d\eta^c_s$$

$$- \sum_{0 \leq s < T} e^{-rs}(u(X^{x, \xi, \eta}_s) - u(X^{x, \xi, \eta}_{s+})), \quad (3.5)$$

where we have set

$$M_T^{x, \xi, \eta} := \int_0^T e^{-rs} \sigma(X_s^{x, \xi, \eta})u'(X_s^{x, \xi, \eta}) \, dB_s.$$

Since the processes $\xi$ and $\eta$ jump on disjoint subsets of $\mathbb{R}_+$ we can write

$$\sum_{0 \leq s < T} e^{-rs}(u(X^{x, \xi, \eta}_s) - u(X^{x, \xi, \eta}_{s+}))$$

$$= \sum_{0 \leq s < T} e^{-rs} \left[ \int_0^{\Delta\xi_s} u'(X_s^{x, \xi, \eta} + z) \, dz - \int_0^{\Delta\eta_s} u'(X_s^{x, \xi, \eta} - z) \, dz \right].$$
and because $(L_X - r)u \geq -h$ and $-c_1 \leq u' \leq c_2$ on $\mathcal{I}$ by (3.1), we end up from (3.5) with

$$u(x) \leq e^{-\frac{r}{2}T} u(X_T^{x;\xi,\eta}) + \int_0^T e^{-rs} h(X_s^{x;\xi,\eta}) \, ds - M_T^{x;\xi,\eta}$$

$$+ \int_0^T e^{-rs} c_1(X_s^{x;\xi,\eta}) \, d\xi_s + \int_0^T e^{-rs} c_2(X_s^{x;\xi,\eta}) \, d\eta_s,$$

(3.6)

upon recalling (2.29) and (2.30).

By assumption, for all $x \in \mathcal{I}$ one has $|u(x)| \leq K (1 + |x|^{r+1})$, and therefore we can write for some $K > 0$

$$u(x) \leq e^{-\frac{r}{2}T} K \left(1 + |X_T^{x;\xi,\eta}|^{r+1}\right) e^{-\frac{r}{2}T} T + \int_0^T e^{-rs} h(X_s^{x;\xi,\eta}) \, ds - M_T^{x;\xi,\eta}$$

$$+ \int_0^T e^{-rs} c_1(X_s^{x;\xi,\eta}) \, d\xi_s + \int_0^T e^{-rs} c_2(X_s^{x;\xi,\eta}) \, d\eta_s$$

$$\leq e^{-\frac{r}{2}T} K \left(1 + \sup_{t \geq 0} e^{-\frac{r}{2}t} |X_t^{x;\xi,\eta}|^{r+1}\right) + \int_0^T e^{-rs} H(X_s^{x;\xi,\eta}) \, ds - M_T^{x;\xi,\eta}$$

$$\quad + \int_0^T e^{-rs} c_1(X_s^{x;\xi,\eta}) \, d\xi_s + \int_0^T e^{-rs} c_2(X_s^{x;\xi,\eta}) \, d\eta_s.$$  

(3.7)

From the previous equation we have that, for all $T > 0$,

$$M_T^{x;\xi,\eta} \leq -u(x) + K \left(1 + \sup_{t \geq 0} e^{-\frac{r}{2}t} |X_t^{x;\xi,\eta}|^{r+1}\right)$$

$$+ \int_0^\infty e^{-rs} |c_1(X_s^{x;\xi,\eta})| \, d\xi_s + \int_0^\infty e^{-rs} |c_2(X_s^{x;\xi,\eta})| \, d\eta_s$$

so that $M_T^{x;\xi,\eta} \in L^1(\mathbb{P})$ by admissibility of $\nu$ [cf. Definition (2.9)]; hence, $(M_T^{x;\xi,\eta})_{T \geq 0}$ is a submartingale. Then, taking expectations in (3.7) we have

$$u(x) \leq e^{-\frac{r}{2}T} \mathbb{E}_x \left[K \left(1 + \sup_{t \geq 0} e^{-\frac{r}{2}t} |X_t^{x;\xi,\eta}|^{r+1}\right)\right]$$

$$\quad + \mathbb{E}_x \left[\int_0^T e^{-rs} h(X_s^{x;\xi,\eta}) \, ds + \int_0^T e^{-rs} \left(c_1(X_s^{x;\xi,\eta}) \, d\xi_s + c_2(X_s^{x;\xi,\eta}) \, d\eta_s\right)\right].$$

Taking limits as $T \uparrow +\infty$, and using the fact that $\nu$ is admissible [cf. Definition 2.9], by the dominated convergence theorem we get $u(x) \leq J_\nu(v)$. Since the latter holds for any $x \in \mathcal{I}$ and $v \in \mathcal{A}(x)$ we conclude that $u \leq v$ on $\mathcal{I}$.

Step 2 Let again $x \in \mathcal{I}$ be given and fixed, and take the admissible $\tilde{\nu}$ satisfying (3.2), (3.3) and (3.4). Then all the inequalities leading to (3.6) become equalities, and taking expectations we obtain

$$u(x) = \mathbb{E}_x \left[e^{-rT} u(X_T^{x;\tilde{\xi},\tilde{\eta}}) + \int_0^T e^{-rs} h(X_s^{x;\tilde{\xi},\tilde{\eta}}) \, ds \right.$$

$$\quad + \int_0^T e^{-rs} c_1(X_s^{x;\tilde{\xi},\tilde{\eta}}) \, d\tilde{\xi}_s + \int_0^T e^{-rs} c_2(X_s^{x;\tilde{\xi},\tilde{\eta}}) \, d\tilde{\eta}_s \left].

(3.8)
By assumption we have that, for any \( x \in \mathcal{I} \), \( u(x) \geq -K(1 + |x|^{1+\gamma}) \), so that we can continue from \((3.8)\) by writing

\[
u(x) \geq -e^{-\frac{t}{c}}\mathbb{E}_x \left[K \left(1 + \sup_{\nu \geq 0} e^{-\frac{t}{c} |X_{t}^{x,\hat{\xi},\hat{\eta}}|^{\gamma+1}} \right) \right] + \mathbb{E}_x \left[\int_0^T e^{-rs} h(X_s^{x,\hat{\xi},\hat{\eta}}) \, ds \right.
\]
\[
+ \int_0^T e^{-rs} c_1(X_s^{x,\hat{\xi},\hat{\eta}}) \, d\hat{\xi}_s + \int_0^T e^{-rs} c_2(X_s^{x,\hat{\xi},\hat{\eta}}) \, d\hat{\eta}_s \right]. \tag{3.9}
\]

By admissibility of \( \hat{\nu} \) (see Definition 2.9) we can take limits as \( T \uparrow \infty \), invoke the dominated convergence theorem for the second expectation in the right hand-side of \((3.9)\), and finally find that

\[
u(x) \geq \mathbb{E}_x \left[\int_0^\infty e^{-rs} h(X_s^{x,\hat{\xi},\hat{\eta}}) \, ds + \int_0^\infty e^{-rs} c_1(X_s^{x,\hat{\xi},\hat{\eta}}) \, d\hat{\xi}_s + \int_0^\infty e^{-rs} c_2(X_s^{x,\hat{\xi},\hat{\eta}}) \, d\hat{\eta}_s \right].
\]

Hence \( \nu(x) \geq J_\nu(\hat{\nu}) \geq v(x) \). Combining this inequality with the fact that \( u \leq v \) on \( \mathcal{I} \) by Step 1, we conclude that \( u = v \) on \( \mathcal{I} \) and that \( \hat{\nu} \) is optimal. \( \square \)

### 3.2 Constructing a candidate solution

We here construct a solution to the HJB equation \((3.1)\). In particular, given the structure of our problem, we conjecture that there exist two constant trigger values to be determined, say \( a \) and \( b \), such that

\[
\{ x \in \mathcal{I} : (L_x - r)u(x) + h(x) = 0 \} = (a, b),
\]

and that

\[
\{ x \in \mathcal{I} : u'(x) = -c_1(x) \} = (x, a), \quad \text{and} \quad \{ x \in \mathcal{I} : u'(x) = c_2(x) \} = [b, \overline{x}). \tag{3.10}
\]

Following this conjecture we thus start by solving the ODE

\[
(L_x - r)u(x) + h(x) = 0 \tag{3.11}
\]

in \( (a, b) \subset \mathcal{I} \), for some \( a < b \) to be found. Recalling \((2.25)\), the general solution to equation \((3.11)\) is given by

\[
u(x) = A \psi(x) + B \phi(x) + (R h)(x), \quad x \in (a, b)
\]

for some \( A, B \in \mathbb{R} \). Also, with regard to \((3.10)\) we set

\[
u(x) = A \psi(a) + B \phi(a) + (R h)(a) + \int_a^x c_1(y) \, dy
\]

for any \( x \in (x, a) \), and

\[
u(x) = A \psi(b) + B \phi(b) + (R h)(b) + \int_b^x c_2(y) \, dy
\]

for any \( x \in [b, \overline{x}) \). Notice that in this way the function \( u \) is automatically continuous at \( a \) and \( b \).
In order to determine the four unknown constants $A, B, a, \text{ and } b$, we assume that $u \in C^2(\mathcal{I})$, and, using (2.26) and (2.31), we then find the nonlinear system of four equations

\begin{align*}
A\psi'(a) + B\phi'(a) + (\hat{R}h')(a) &= (\hat{R}\hat{c}_1)(a), \\
A\psi''(a) + B\phi''(a) + (\hat{R}h')'(a) &= (\hat{R}\hat{c}_1)'(a), \\
A\psi'(b) + B\phi'(b) + (\hat{R}h')(b) &= -(\hat{R}\hat{c}_2)(b), \\
A\psi''(b) + B\phi''(b) + (\hat{R}h')'(b) &= -(\hat{R}\hat{c}_2)'(b).
\end{align*}

(Solving (3.13)–(3.14) with respect to $A$ and $B$, simple algebra and the fact that $\psi' = \hat{\psi}$ and $\phi' = -\hat{\phi}$ (cf. Lemma A.1 in the “Appendix”) give

\begin{align*}
A &= \frac{\hat{\phi}'(a)[\hat{R}(h' - \hat{c}_1)]'(a) - \hat{\phi}(a)[\hat{R}(h' - \hat{c}_1)]'(a)}{\phi'(a)\hat{\psi}'(a) - \hat{\phi}'(a)\hat{\psi}(a)}, \\
B &= \frac{\hat{\psi}'(a)[\hat{R}(h' - \hat{c}_1)]'(a) - \hat{\psi}(a)[\hat{R}(h' - \hat{c}_1)]'(a)}{\phi'(a)\hat{\psi}'(a) - \hat{\phi}'(a)\hat{\psi}(a)}.
\end{align*}

Analogous calculations starting from (3.15)–(3.16) reveal

\begin{align*}
A &= \frac{\hat{\phi}'(b)[\hat{R}(h' + \hat{c}_2)]'(b) - \hat{\phi}(b)[\hat{R}(h' + \hat{c}_2)]'(b)}{\phi'(b)\hat{\psi}'(b) - \hat{\phi}'(b)\hat{\psi}(b)}, \\
B &= \frac{\hat{\psi}'(b)[\hat{R}(h' + \hat{c}_2)]'(b) - \hat{\psi}(b)[\hat{R}(h' + \hat{c}_2)]'(b)}{\phi'(b)\hat{\psi}'(b) - \hat{\phi}'(b)\hat{\psi}(b)}.
\end{align*}

Recalling (2.12), we can then write $A = I_1(a) = I_2(b)$ and $B = J_1(a) = J_2(b)$, with

\begin{align*}
I_1(a) &:= \frac{1}{w} \left[ \frac{\hat{\phi}'(a)}{\hat{S}'(a)} [\hat{R}(h' - \hat{c}_1)](a) - \frac{\hat{\phi}(a)}{\hat{S}(a)} [\hat{R}(h' - \hat{c}_1)]'(a) \right], \\
I_2(b) &:= \frac{1}{w} \left[ \frac{\hat{\phi}'(b)}{\hat{S}'(b)} [\hat{R}(h' + \hat{c}_2)](b) - \frac{\hat{\phi}(b)}{\hat{S}(b)} [\hat{R}(h' + \hat{c}_2)]'(b) \right], \\
J_1(a) &:= \frac{1}{w} \left[ \frac{\hat{\psi}'(a)}{\hat{S}'(a)} [\hat{R}(h' - \hat{c}_1)](a) - \frac{\hat{\psi}(a)}{\hat{S}(a)} [\hat{R}(h' - \hat{c}_1)]'(a) \right], \\
J_2(b) &:= \frac{1}{w} \left[ \frac{\hat{\psi}'(b)}{\hat{S}'(b)} [\hat{R}(h' + \hat{c}_2)](b) - \frac{\hat{\psi}(b)}{\hat{S}(b)} [\hat{R}(h' + \hat{c}_2)]'(b) \right],
\end{align*}

so that the system for $a$ and $b$ reads

$$I_1(a) - I_2(b) = 0, \quad J_1(a) - J_2(b) = 0.$$

We now make the following standing assumption.

**Assumption 3.2** One has that

$$\lim_{x \to \mp} J_i(x) = 0 = \lim_{x \to \mp} I_i(x), \quad i = 1, 2.$$

By (2.13)–(2.16), the latter is essentially a requirement on the growth of $\hat{R}(h' - \hat{c}_1)$ and $\hat{R}(h' + \hat{c}_2)$, and of their derivatives. Assumption 3.2 then implies that for any $x \in \mathcal{I}$

\begin{align*}
I_i(x) &= -\int_x^\mp I_i'(z) \, dz, \quad J_i(x) = \int_x^\mp J_i'(z) \, dz, \quad i = 1, 2.
\end{align*}
Step 1  Proof sides of (3.24) and (3.25) are strictly negative and strictly positive, respectively. For there exists define the two functionals

\[ \frac{d}{dx} \left[ \frac{f'(x)}{S'(x)} \phi(x) - \frac{\phi'(x)}{S'(x)} f(x) \right] = \phi(x) \tilde{m}'(x) (L_\tilde{\gamma} - (r - \mu'(x))) f(x), \]  

(3.20)

and

\[ \frac{d}{dx} \left[ \frac{f'(x)}{S'(x)} \psi(x) - \frac{\psi'(x)}{S'(x)} f(x) \right] = \psi(x) \tilde{m}'(x) (L_\tilde{\gamma} - (r - \mu'(x))) f(x). \]  

(3.21)

As a consequence, using (3.19) we have that \( I_1(a) = I_2(b) \) is equivalent to

\[ \int_a^\infty (h'(z) - \tilde{c}_1(z)) \tilde{m}'(z) \phi(z) \, dz = \int_b^\infty (h'(z) + \tilde{c}_2(z)) \tilde{m}'(z) \phi(z) \, dz, \]  

(3.22)

whereas \( J_1(a) = J_2(b) \) is equivalent to

\[ \int_a^\infty (h'(z) - \tilde{c}_1(z)) \tilde{m}'(z) \psi(z) \, dz = \int_b^\infty (h'(z) + \tilde{c}_2(z)) \tilde{m}'(z) \psi(z) \, dz. \]  

(3.23)

Since we are looking for a solution \((a^*, b^*)\) of (3.22) and (3.23) such that \( a^* < b^* \), we can rewrite them in the form

\[ \int_a^b (h'(z) - \tilde{c}_1(z)) \tilde{m}'(z) \phi(z) \, dz = \int_b^\infty (\tilde{c}_1(z) + \tilde{c}_2(z)) \tilde{m}'(z) \phi(z) \, dz, \]  

(3.24)

\[ \int_a^b (h'(z) + \tilde{c}_2(z)) \tilde{m}'(z) \psi(z) \, dz = -\int_a^\infty (\tilde{c}_1(z) + \tilde{c}_2(z)) \tilde{m}'(z) \psi(z) \, dz. \]  

(3.25)

**Proposition 3.3** Recall \( \tilde{x}_1, \tilde{x}_2 \) as in Assumption 2.7-(ii). Then there exists a unique couple \((a^*, b^*) \in \mathcal{I} \times \mathcal{I}\) such that \( a^* < \tilde{x}_1 < \tilde{x}_2 < b^* \) that solves the system of Eqs. (3.24) and (3.25).

**Proof** Step 1 We start by proving existence. Given Assumption 2.7, note that the right-hand sides of (3.24) and (3.25) are strictly negative and strictly positive, respectively. For \( a, b \in \mathcal{I} \) define the two functionals

\[ K_1(a; b) := \int_a^b (h'(z) - \tilde{c}_1(z)) \tilde{m}'(z) \phi(z) \, dz, \]

\[ K_2(b; a) := \int_a^b (h'(z) + \tilde{c}_2(z)) \tilde{m}'(z) \psi(z) \, dz. \]

For a given and fixed \( a \in \mathcal{I} \), let \( b > a \lor \tilde{x}_2 \) and notice that by the integral mean-value theorem there exists \( \xi_2 \in (a \lor \tilde{x}_2, b) \) such that

\[ K_2(b; a) = K_2(a \lor \tilde{x}_2; a) + \int_{a \lor \tilde{x}_2}^b \frac{h'(\xi_2) + \tilde{c}_2(\xi_2)}{r - \mu'(\xi_2)} \, dz. \]

(3.26)

\[ \geq K_2(a \lor \tilde{x}_2; a) + \left( \frac{h'(\xi_2) + \tilde{c}_2(\xi_2)}{r - \mu'(\xi_2)} \right) \left[ \frac{\psi'(b)}{S'(b)} - \frac{\tilde{\psi}'(a \lor \tilde{x}_2)}{S'(a \lor \tilde{x}_2)} \right]. \]
where (2.27) has been used in the last step. Because of (2.16), and since \( h'(\xi_2) + c_2(\xi_2) > 0 \) and \( r - \mu'(\xi_2) > 0 \) by Assumption 2.3, we obtain from (3.26) that \( \lim_{b \uparrow \tau} K_2(b; a) = +\infty \), for any given \( a \in \mathcal{I} \).

On the other hand, by Assumption 2.7 one has

\[
K_2(\tilde{x}_2; a) = \int_a^{\tilde{x}_2} \left( h'(z) + c_2(z) \right) \hat{m}'(z) \hat{\psi}(z) \, dz \begin{cases}
< 0 & \text{if } a < \tilde{x}_2, \\
\leq 0 & \text{if } a \geq \tilde{x}_2.
\end{cases}
\]

Also, \( K_2(a; a) = 0 \) and \( K'_2(b; a) = \left( h'(b) + c_2(b) \right) \hat{m}'(b) \hat{\psi}(b) > 0 \) for \( b > a \vee \tilde{x}_2 \). Hence, for any given \( a \in \mathcal{I} \), by continuity and strict monotonicity of \( b \mapsto K_2(b; a) \) on \( (a \vee \tilde{x}_2, \tau) \), there exists a unique \( b(a) = y^*(a) \in (a \vee \tilde{x}_2, \tau) \) such that (3.25) is satisfied, i.e.

\[
\int_a^{y^*(a)} \left( h'(z) + c_2(z) \right) \hat{m}'(z) \hat{\psi}(z) \, dz + \int_\tilde{x}_2^{a} \left( \hat{c}_1(z) + c_2(z) \right) \hat{m}'(z) \hat{\psi}(z) \, dz = 0.
\]

Notice that, since the left-hand side of the above is \( C^1 \) both with respect to \( a \) and \( b \), and \( (h'(z) + c_2(z)) \hat{m}'(z) \hat{\psi}(z) > 0 \) for all \( z \in (\tilde{x}_2, \tau) \), \( y^* \) is \( C^1 \) by the implicit function theorem, with derivative given by

\[
y^*(a)' = \frac{h'(a) - \hat{c}_1(a)}{h'(y^*(a)) + \hat{c}_2(y^*(a))} \cdot \frac{\hat{m}'(a) \hat{\psi}(a)}{\hat{m}'(y^*(a)) \hat{\psi}(y^*(a))}.
\]

Analogously, for fixed \( b \in \mathcal{I} \), take \( a < \tilde{x}_1 \wedge b \), and for a suitable \( \xi_1 \in (a, \tilde{x}_1 \wedge b) \) one finds

\[
K_1(a; b) \leq K_1(\tilde{x}_1 \wedge b; b) + \left( \frac{h'(\xi_1) - \hat{c}_1(\xi_1)}{r - \mu'(\xi_1)} \right) \left[ \frac{\phi'(\tilde{x}_1 \wedge b)}{\tilde{S}'(\tilde{x}_1 \wedge b)} - \frac{\phi'(a)}{\tilde{S}'(a)} \right] .
\]

We thus conclude that, for any given and fixed \( b \in \mathcal{I} \), \( \lim_{a \downarrow \xi} K_1(a; b) = -\infty \), since \( x \) is natural for \( \tilde{X} \) (cf. (2.16)), \( h'(\xi_1) - \hat{c}_1(\xi_1) < 0 \), and \( r - \mu'(\xi_1) > 0 \). On the other hand, \( K_1(b; b) = 0 \),

\[
K_1(\tilde{x}_1; b) = \int_{\tilde{x}_1}^b \left( h'(z) - \hat{c}_1(z) \right) \hat{m}'(z) \hat{\phi}(z) \, dz \begin{cases}
> 0 & \text{if } b > \tilde{x}_1, \\
\geq 0 & \text{if } b \leq \tilde{x}_1,
\end{cases}
\]

and \( K'_1(b; a) = -(h'(a) - \hat{c}_1(a)) \hat{m}'(a) \hat{\phi}(a) > 0 \) for \( a < \tilde{x}_1 \wedge b \). Combining all these facts we find that for any \( b \in \mathcal{I} \) there exists a unique \( x^*(b) \in (\tilde{x}_1, \tilde{x}_1 \wedge b) \) such that (3.24) is satisfied. In analogy with \( y^* \), also \( x^* \) is differentiable by the implicit function theorem, and its derivative is

\[
(x^*)'(b) = \frac{h'(b) + \hat{c}_2(b)}{h'(x^*(b)) - \hat{c}_1(x^*(b))} \cdot \frac{\hat{m}'(b) \hat{\phi}(b)}{\hat{m}'(x^*(b)) \hat{\phi}(x^*(b))}.
\]

Since \( \tilde{x}_1 < \tilde{x}_2 \) by assumption, we clearly have that if a pair \( (a^*, b^*) \in \mathcal{I} \times \mathcal{I} \) such that \( a^* := x^*(b^*) \) and \( b^* = y^*(a^*) \) exists, then \( a^* < \tilde{x}_1 < \tilde{x}_2 < b^* \).

In order to prove that there indeed exists such a couple \((a^*, b^*)\), let \( \Theta(b) := \int_{x^*(b)}^b (h'(z) + \hat{c}_2(z)) \hat{m}'(z) \hat{\psi}(z) \, dz + \int_{\tilde{x}_2}^{x^*(b)} (\hat{c}_1(z) + \hat{c}_2(z)) \hat{m}'(z) \hat{\psi}(z) \, dz \), and notice that because \( x^*(\tilde{x}_2) < \tilde{x}_1 < \tilde{x}_2 \) one has by Assumption 2.7

\[
\Theta(\tilde{x}_2) = \int_{x^*(\tilde{x}_2)}^{\tilde{x}_2} (h'(z) + \hat{c}_2(z)) \hat{m}'(z) \hat{\psi}(z) \, dz + \int_{\tilde{x}_2}^{x^*(\tilde{x}_2)} (\hat{c}_1(z) + \hat{c}_2(z)) \hat{m}'(z) \hat{\psi}(z) \, dz < 0.
\]
On the other hand, for \( b > \tilde{x}_2 \) and for a suitable \( \xi_2 \in (\tilde{x}_2, b) \) we have by the integral mean-value theorem
\[
\Theta(b) > \int_{x^*(b)}^{\tilde{x}_2} \left( h'(z) + \hat{c}_2(z) \right) \hat{m}'(z) \hat{\psi}(z) \, dz + \int_{\tilde{x}_2}^{b} \left( h'(z) + \hat{c}_2(z) \right) \hat{m}'(z) \hat{\psi}(z) \, dz \\
+ \int_{\xi_2}^{\tilde{x}_2} \left( \hat{c}_1(z) + \hat{c}_2(z) \right) \hat{m}'(z) \hat{\psi}(z) \, dz \\
\geq \int_{\xi_2}^{\tilde{x}_2} \left( h'(\xi_2) + \hat{c}_2(\xi_2) \right) \hat{m}'(z) \hat{\psi}(z) \, dz + \int_{\tilde{x}_2}^{b} \left( \hat{c}_1(z) + \hat{c}_2(z) \right) \hat{m}'(z) \hat{\psi}(z) \, dz \\
+ \int_{\xi_2}^{\tilde{x}_2} \left( h'(\xi_2) + \hat{c}_2(\xi_2) \right) \left[ \frac{\hat{\psi}'(b)}{\hat{S}'(b)} - \frac{\hat{\psi}'(\xi_2)}{\hat{S}'(\xi_2)} \right].
\]
Here we have used that \( \hat{c}_1 + \hat{c}_2 < 0 \) on \( \mathcal{I} \) by Assumption 2.7, and Eq. (2.27). Because of (2.16), and since \( h'(\xi_2) + \hat{c}_2(\xi_2) > 0 \) and \( r - \mu'(\xi_2) > 0 \), we find from the previous equation that \( \lim_{b \uparrow \mathcal{I}} \Theta(b) = +\infty \). Also, \( b \mapsto \Theta(b) \) is continuous (since \( b \mapsto x^*(b) \) is so by the implicit function theorem), and therefore there exists \( b^* \in \mathcal{I} \) solving \( \Theta(b) = 0 \), and this value is such that \( b^* > \tilde{x}_2 \). Hence, there also exists \( a^* = x^*(b^*) \) such that \( (a^*, b^*) \in (x, \tilde{x}_1) \times (\tilde{x}_2, \mathcal{I}) \) solves system (3.24)–(3.25).

**Step 2** We now prove that the couple \( (a^*, b^*) \) is indeed unique in the domain \( (x, \tilde{x}_1) \times (\tilde{x}_2, \mathcal{I}) \). Since we already know by Step 1 that there exists a solution \( (a^*, b^*) \) belonging to \( (x, \tilde{x}_1) \times (\tilde{x}_2, \mathcal{I}) \), we can study the sign of the derivatives of the mappings \( b \mapsto x^*(b) \) and \( a \mapsto y^*(a) \) on \( (x, \tilde{x}_1) \times (\tilde{x}_2, \mathcal{I}) \). These maps have been constructed in the previous steps of this proof, and their derivatives are given by Eqs. (3.29) and (3.27). By Assumption 2.7 we have
\[
(x^*)'(b) < 0 \quad \text{for} \quad b \in (\tilde{x}_2, \mathcal{I}) \quad \text{and} \quad (y^*)'(a) < 0 \quad \text{for} \quad a \in (x, \tilde{x}_1),
\]
on recalling that \( x^*(b) < \tilde{x}_1 \) and \( y^*(a) > \tilde{x}_2 \). Moreover, at the intersection points \( b^* = y^*(a^*) \) and \( a^* = x^*(b^*) \), we have
\[
(x^*)'(b^*) = \frac{h'(b^*) + \hat{c}_2(b^*)}{h'(a^*) - \hat{c}_1(a^*)} \cdot \frac{\hat{m}'(b^*) \hat{\psi}(b^*)}{\hat{m}'(a^*) \hat{\psi}(a^*)} \cdot \frac{\hat{\psi}(a^*)}{\hat{\psi}(b^*)} \cdot \frac{\hat{\phi}(b^*)}{\hat{\phi}(a^*)} = \frac{1}{(y^*)'(a^*)} \cdot \frac{\hat{\psi}(a^*)}{\hat{\psi}(b^*)} \cdot \frac{\hat{\phi}(b^*)}{\hat{\phi}(a^*)}.
\]
Since \( (y^*)'(a^*) < 0 \) and \( \frac{\hat{\psi}(a^*)}{\hat{\psi}(b^*)} \cdot \frac{\hat{\phi}(b^*)}{\hat{\phi}(a^*)} < 1 \) (by the strict monotonicity of \( \hat{\psi} \) and \( \hat{\phi} \)), we obtain that \( (x^*)'(b^*) > \frac{1}{(y^*)'(a^*)} \), or equivalently
\[
(y^*)'(a^*) > \frac{1}{(x^*)'(b^*)} = [(x^*)^{-1}]'(a^*).\]
Together with the strict monotonicity of \( x^*(\cdot) \) and \( y^*(\cdot) \) in \( (x, \tilde{x}_1) \) and \( (\tilde{x}_2, \mathcal{I}) \), respectively, the latter shows that the intersection point is indeed unique.
Now, with \((A, B, a^*, b^*)\) as above, we define our candidate value function as

\[
    u(x) := \begin{cases} 
        A\psi(a^*) + B\phi(a^*) + (Rh)(a^*) + \int_{x}^{a^*} c_1(y) \, dy, & \text{if } x \in (x, a^*], \\
        A\psi(x) + B\phi(x) + (Rh)(x), & \text{if } x \in (a^*, b^*), \\
        A\psi(b^*) + B\phi(b^*) + (Rh)(b^*) + \int_{b^*}^{x} c_2(y) \, dy, & \text{if } x \in [b^*, \bar{x}).
    \end{cases}
\]

(3.30)

### 3.3 The value function and the optimal control

In this section we prove that the function \(u\) constructed in Sect. 3.2 [cf. (3.30) above] coincides with the value function (2.32), and we provide the optimal control \(v^*\).

**Theorem 3.4** The function \(u\) defined in (3.30) is a classical solution to the HJB equation (3.1). Moreover, there exists \(K > 0\) such that \(|u(x)| \leq K(1 + |x|^{\gamma+1})\), where \(\gamma \geq 1\) is the growth coefficient of \(c_i, i = 1, 2\) [see Assumption 2.7-(ii)].

**Proof** The proof is organized in several steps.

**Step 1** By construction, \(u \in C^2(x, \bar{x})\). Moreover, using the growth requirement on \(c_i, i = 1, 2\), of Assumption 2.7-(ii), one obtains from (3.30) that there exists \(K > 0\) such that for all \(x \in \mathcal{I}\)

\[
    |u(x)| \leq K \left( 1 + \int_{x}^{a^*} |c_1(y)| \, dy + \int_{b^*}^{x} |c_2(y)| \, dy \right) \leq K(1 + |x|^{\gamma+1}).
\]

**Step 2** We here show that \((\mathcal{L}_X - r)u(x) + h(x) \geq 0\) for all \(x \in \mathcal{I}\). This is clearly true with equality by construction for \(x \in (a^*, b^*)\), and we now prove that it also holds for \(x \in (x, a^*)\). Analogous arguments might then be employed also to show that \((\mathcal{L}_X - r)u(x) + h(x) \geq 0\) for \(x \in [b^*, \bar{x})\).

First of all we rewrite \(A\psi(a^*) + B\phi(a^*) + (Rh)(a^*)\) in a tighter form. Notice that since \(-\widetilde{\psi} = \phi'\) and \(\widetilde{\psi}' = \psi'\) by Lemma A.1 in the “Appendix”, and because \((\bar{R}h)' = (Rh)'\), we can obtain from (3.17)–(3.18)

\[
    A\psi(a^*) + B\phi(a^*) = \frac{1}{\phi''(a^*)\psi'(a^*) - \phi'(a^*)\psi''(a^*)}.\]

\[
    \cdot \left[ \psi(a^*)\phi'(a^*)(Rh)''(a^*) - \psi(a^*)\phi'(a^*)(\bar{R}\tilde{c}_1)'(a^*) - \psi(a^*)\phi''(a^*)(Rh)'(a^*) + \\
    + \psi(a^*)\phi''(a^*)(\bar{R}\tilde{c}_1)(a^*) + \phi(a^*)\psi''(a^*)(Rh)'(a^*) - \phi(a^*)\psi''(a^*)(\bar{R}\tilde{c}_1)(a^*) - \\
    - \phi(a^*)\psi'(a^*)(R_h)''(a^*) + \phi(a^*)\psi'(a^*)(\bar{R}\tilde{c}_1)'(a^*) \right].
\]

Then arranging terms and noticing that \(W S''(x) = \phi(x)\psi''(x) - \psi(x)\phi''(x), x \in \mathcal{I}\), we find from the previous equation that

\[
    A\psi(a^*) + B\phi(a^*) = \frac{W}{\phi''(a^*)\psi'(a^*) - \phi'(a^*)\psi''(a^*)}.\]

(3.31)

\[
    \cdot \left[ - S'(a^*) ((Rh)''(a^*) - (\bar{R}\tilde{c}_1)'(a^*)) + S''(a^*) ((Rh)'(a^*) - (\bar{R}\tilde{c}_1)(a^*)) \right].
\]
Then using that $S''(x) = -\frac{2\mu(x)}{\sigma^2(x)} S'(x)$, and that $f''(x) = -\frac{2\mu(x)}{\sigma^2(x)} f'(x) + \frac{2r}{\sigma^2(x)} f(x)$ for $f \in \{\phi, \psi\}$, we obtain from (3.31)
\[
A\psi(a^*) + B\phi(a^*) = -\frac{1}{r},
\]
(3.32)
\[
\frac{1}{2} \sigma^2(x)(Rh)''(a^*) + \mu(x)(Rh)'(a^*) - \left( \frac{1}{2} \sigma^2(x)(\hat{R}\eta_1)'(a^*) + \mu(a^*)(\hat{R}\eta_1)(a^*) \right).
\]
Since the resolvent satisfies [cf. (2.25)]
\[
\frac{1}{2} \sigma^2(x)(Rh)''(x) + \mu(x)(Rh)'(x) = r(Rh)(x) - h(x), \quad x \in I,
\]
and because $(\hat{R}\eta_1) = -c_1$ by (2.31), we conclude by simple algebra from (3.32) that
\[
A\psi(a^*) + B\phi(a^*) + (Rh)(a^*) = \frac{1}{r} \left[ h(a^*) - \mu(a^*)c_1(a^*) - \frac{1}{2} \sigma^2(a^*)c_1'(a^*) \right].
\]
(3.33)
Hence [cf. (3.30)]
\[
u(x) = \int_{x}^{a^*} c_1(y) \, dy + \frac{1}{r} \left[ h(a^*) - \mu(a^*)c_1(a^*) - \frac{1}{2} \sigma^2(a^*)c_1'(a^*) \right], \quad x \in (\chi, a^*].
\]
(3.34)
Thanks to (3.34) we can now easily check that $(L_X - r)u + h \geq 0$ on $(\chi, a^*].$ Indeed, since $a^* < \chi$, we have for any $x \leq a^*$ that
\[
-\int_{x}^{a^*} \left( h'(y) - (L_X - (r - \mu'(y)))c_1(y) \right) \, dy \geq 0.
\]
However, an integration by parts yields
\[
\begin{align*}
0 &\leq -\int_{x}^{a^*} \left( h'(y) - (L_X - (r - \mu'(y)))c_1(y) \right) \, dy \\
&= h(x) - h(a^*) + \frac{1}{2} \sigma^2(a^*)c_1'(a^*) - \frac{1}{2} \sigma^2(x)c_1'(x) \\
&\quad + \mu(a^*)c_1(a^*) - \mu(x)c_1(x) - r \int_{x}^{a^*} c_1(y) \, dy \\
&= h(x) - \frac{1}{2} \sigma^2(x)c_1'(x) - \mu(x)c_1(x) - r \int_{x}^{a^*} c_1(y) \, dy \\
&\quad - r \left[ \frac{1}{r} \left( h(a^*) - \frac{1}{2} \sigma^2(a^*)c_1'(a^*) - \mu(a^*)c_1(a^*) \right) \right] \\
&= (L_X - r)u(x) + h(x)
\end{align*}
\]
on $(\chi, a^*],$ upon recalling (3.34) in the last step. Hence, $(L_X - r)u(x) + h(x) \geq 0$ on $(\chi, a^*].$

**Step 3** To conclude the proof it remains to show that we have $-c_1 \leq u' \leq c_2$ on $(a^*, b^*),$ since we already know by construction that $u' = -c_1$ on $(\chi, a^*]$ and $u' = c_2$ on $[b^*, \chi).$ We here prove that $u' \geq -c_1$ on $(a^*, b^*].$ Arguments similar to those employed in the following allow to show that also $u' \leq c_2$ on $(a^*, b^*].$
By construction, the function \( u \) of (3.30) solves \((L_X - r)u = -h\) on \((a^*, b^*)\). Given the regularity of \( u \), and of \( \mu \) and \( \sigma \) (cf. Assumption 2.1), we can differentiate the previous equation with respect to \( x \) inside \((a^*, b^*)\), and find that \( u' \) solves

\[
(L_X - (r - \mu'(x)))u'(x) = -h'(x), \quad x \in (a^*, b^*),
\]

together with the boundary conditions \( u'(a^*) = -c_1(a^*) \) and \( u'(b^*) = c_2(b^*) \).

Now, take \( x \in (a^*, b^*) \), and define the two stopping times

\[
\tau_{a^*} := \inf\{t \geq 0 \mid \hat{X}_t \leq a^*\}, \quad \tau_{b^*} := \inf\{t \geq 0 \mid \hat{X}_t \geq b^*\}, \quad \hat{P}_x - \text{a.s.}
\]

Then, by the Feynman–Kac formula and the strong Markov property of \( \hat{X} \), we can write

\[
\begin{align*}
u'(x) &= \hat{E}_x \left[ e^{-\int_0^{\tau_{a^*} \wedge \tau_{b^*}} (r - \mu'(\hat{X}_u))du} w(\hat{X}_{\tau_{a^*} \wedge \tau_{b^*}}) \right] + \int_0^{\tau_{a^*} \wedge \tau_{b^*}} e^{-\int_0^t (r - \mu'(\hat{X}_u))du} h'(\hat{X}_t) \, ds \\
&= \left( \hat{R}h' \right)(x) - \hat{E}_x \left[ e^{-\int_0^{\tau_{a^*} \wedge \tau_{b^*}} (r - \mu'(\hat{X}_u))du} (\hat{R}h' + c_1(\hat{X}_{\tau_{a^*}})) \mathbf{1}_{\{\tau_{a^*} < \tau_{b^*}\}} \right] \\
&\quad - \hat{E}_x \left[ e^{-\int_0^{\tau_{b^*}} (r - \mu'(\hat{X}_u))du} (\hat{R}h' - c_2(\hat{X}_{\tau_{b^*}})) \mathbf{1}_{\{\tau_{b^*} > \tau_{a^*}\}} \right] \\
&= \left( \hat{R}h' \right)(x) - (\hat{R}h' + c_1(a^*)) \hat{E}_x \left[ e^{-\int_0^{\tau_{a^*} \wedge \tau_{b^*}} (r - \mu'(\hat{X}_u))du} \mathbf{1}_{\{\tau_{a^*} < \tau_{b^*}\}} \right] \\
&\quad - (\hat{R}h' - c_2(b^*)) \hat{E}_x \left[ e^{-\int_0^{\tau_{b^*}} (r - \mu'(\hat{X}_u))du} \mathbf{1}_{\{\tau_{b^*} > \tau_{a^*}\}} \right],
\end{align*}
\]

upon recalling (2.22). Because the function \( \hat{F}(x) := \frac{\hat{w}(x)}{\hat{\phi}(x)} \), \( x \in \mathcal{I} \), is strictly positive and strictly increasing (by strict monotonicity of \( \hat{w} \) and \( \hat{\phi} \)), we can apply Lemma 2.3 in Dayanik (2008) to evaluate the last two expectations above, and then to write that

\[
\frac{u'(x)}{\phi(x)} = \frac{\left( \hat{R}h' \right)(x)}{\phi(x)} - \frac{(\hat{R}h')(a^*) + c_1(a^*)}{\phi(a^*)} \cdot \left[ \hat{F}(b^*) - \hat{F}(a^*) \right] - \frac{(\hat{R}h')(b^*) - c_2(b^*)}{\phi(b^*)} \cdot \left[ \hat{F}(x) - \hat{F}(a^*) \right].
\]

The latter equation immediately implies that

\[
\hat{w}_1(x) := \frac{u'(x) + c_1(x)}{\phi(x)} = \frac{(\hat{R}h')(x) + c_1(x)}{\phi(x)} - \frac{(\hat{R}h')(a^*) + c_1(a^*)}{\phi(a^*)} \cdot \left[ \hat{F}(b^*) - \hat{F}(a^*) \right] - \frac{(\hat{R}h')(b^*) - c_2(b^*)}{\phi(b^*)} \cdot \left[ \hat{F}(x) - \hat{F}(a^*) \right]. \tag{3.35}
\]

We now want to show that \( \hat{w}_1 \geq 0 \) on \((a^*, b^*)\) since this is clearly equivalent to proving that \( u' \geq -c_1 \) on such interval. Clearly \( \hat{w}_1(a^*) = 0 \), and by standard differentiation one also gets that \( \hat{w}_1'(a^*) = 0 \) since \( u''(a^*) + c_1'(a^*) = 0 \). Also, by (3.35) we have that \( \hat{w}_1(b^*) = (c_1(b^*) + c_2(b^*)) / \phi(b^*) > 0 \), where the last inequality is due to Assumption 2.7 and the positivity of \( \hat{\phi} \).

If we now can prove that \( \hat{w}_1' \geq 0 \) on \((a^*, b^*)\), then we have that \( \hat{w}_1(x) \geq \hat{w}_1(a^*) = 0 \) for any \( x \in (a^*, b^*) \), and therefore that \( u' \geq -c_1 \) on \((a^*, b^*)\). Recalling (2.31), we can rewrite (3.35) as

\[
\hat{w}_1(x) := \frac{u'(x) + c_1(x)}{\phi(x)} \geq 0.
\]
\[ \hat{\nu}_1(x) = \frac{[\hat{R}(h' - \hat{c}_1)](x)}{\hat{\phi}(x)} - \frac{[\hat{R}(h' - \hat{c}_1)](a^*)}{\hat{\phi}(a^*)} \cdot \left( \frac{\hat{F}(b^*) - \hat{F}(x)}{\hat{F}(b^*) - \hat{F}(a^*)} \right) \]

Then we can employ (2.24) and (2.20) in (3.36), perform a standard differentiation with respect to \( x \), and use the fact that \( a^* \) and \( b^* \) satisfy (3.22)–(3.23) to find after some algebra

\[ \hat{\nu}_1'(x) = -\frac{1}{w} \hat{F}'(x) \int_{a^*}^{\infty} (h'(z) - \hat{c}_1(z)) \hat{\phi}(z) \hat{m}'(z) \, dz = -\frac{1}{w} \hat{F}'(x) Q(x). \]  

The function \( Q \) introduced in (3.37) is such that \( Q(a^*) = 0 \). Moreover, due to Assumption 2.7 we have that

\[ Q'(x) = (h'(x) - \hat{c}_1(x)) \hat{\phi}(x) \hat{m}'(x) \begin{cases} < 0, & x < \hat{x}_1, \\ = 0, & x = \hat{x}_1, \\ > 0, & x > \hat{x}_1, \end{cases} \]  

and \( Q(x) < 0 \) for any \( x \in (a^*, \hat{x}_1] \). We now have two cases: either (i) there exists \( \hat{x}_1 < \ell_1 < b^* \) such that \( Q(\ell_1) = 0 \) (and notice that, if it exists, such a point is unique by strict monotonicity of \( Q(\cdot) \) on \((\hat{x}_1, T)\); or (ii) \( Q(x) \leq 0 \) for any \( x \in (a^*, b^*) \).

In case (ii), we immediately conclude from (3.37) that \( \hat{\nu}_1' \geq 0 \) on \( (a^*, b^*) \), and therefore that \( u' \geq -c_1 \) on \( (a^*, b^*) \).

On the other hand, if we are in case (i), by (3.37) we see that the point \( \hat{x}_1 \) is also the unique stationary point of \( \hat{\nu}_1 \). In fact, it is a maximum of \( \hat{\nu}_1 \) since one can easily derive from (3.37) that \( \hat{\nu}_1'(\ell_1) = -w^{-1} \hat{F}'(\ell_1) Q'(\ell_1) < 0 \), where the last inequality is due to the fact that \( Q(\ell_1) = 0 \) but \( Q'(\ell_1) > 0 \). However, since we know that \( \hat{\nu}_1(a^*) = 0 \) and \( \hat{\nu}_1(b^*) > 0 \), we conclude that also in case (ii) one has that \( \hat{\nu}_1' > 0 \) on \( (a^*, b^*) \), and therefore that \( u' \geq -c_1 \) on \( (a^*, b^*) \).

**Step 4** Combining the results of the previous steps the proof is completed.

Given \( x \in \cal{I} \), let \( v^* \) be such that \( v^* = \xi^* - \eta^* \) where \((\xi^*, \eta^*) \) is the couple of nondecreasing processes that solves the following double Skorokhod reflection problem \( \text{SP}(a^*, b^*; x) \)

\[ \begin{cases} X^x_{\cdot, \xi, \eta} \in [a^*, b^*], & \mathbb{P}\text{-a.s. for } t > 0, \\ \int_0^T 1_{\{X^x_{t, \xi, \eta} > a^*\}} \, d\xi_t = 0, & \mathbb{P}\text{-a.s. for any } T > 0, \\ \int_0^T 1_{\{X^x_{t, \xi, \eta} < b^*\}} \, d\eta_t = 0, & \mathbb{P}\text{-a.s. for any } T > 0. \end{cases} \]  

Under Assumption 2.1, Problem \( \text{SP}(a^*, b^*; x) \) admits a unique pathwise solution (cf., e.g., Theorem 4.1 in Tanaka 1979). Moreover, \( t \mapsto \xi^*_t \) and \( t \mapsto \eta^*_t \) are continuous, a part possible jumps at time zero of amplitude \((a^* - x)^+ \) and \((x - b^*^+ \), respectively. It also follows that \( \text{supp}(d\xi^*_t) \cap \text{supp}(d\eta^*_t) = \emptyset \). In the following, we set \( \xi^*_0 = 0 = \eta^*_0 \) a.s., and, to simplify notation, \( X^* := X^{\xi^*, \eta^*}, \mathbb{P}_x\text{-a.s.} \)
Proof Clearly $v^* \in S$. To prove the admissibility of $v^*$, we have to verify the requirements of Definition 2.9. Since $X^*_t \in [a^*, b^*] \subset I$ for all $t > 0$, and $X^*_0 = x \in I$, we have that $|x| = +\infty \mathbb{P}_x$-a.s.; Moreover, it is easy to see that also (b) and (c) of Definition 2.9 are fulfilled.

It thus remains to show that Definition 2.9-(a) holds true, and therefore that $(\xi^*, \eta^*)$ are met due to the fact that $(\xi^*, \eta^*)$ solves $SP(a^*, b^*; x)$ we have

$$
\mathbb{E}_x \left[ \int_0^\infty e^{-rt} |c_1(X^*_t)| \, d\xi^*_t \right] = \int_0^{(a^*-x)^+} |c_1(x + z)| \, dz
$$

$$
+ |c_1(a^*)| \mathbb{E}_x \left[ \int_0^\infty e^{-rt} \xi^*_t \right],
$$

(3.40)

where we have used that $\{a^*\}$ is the support of the measure on $\mathbb{R}_+$, $d\xi^*_t$, induced by the continuous part of $\xi^*$. The continuity of $c_1$ yields

$$
\int_0^{(a^*-x)^+} |c_1(x + z)| \, dz \leq (a^*-x)^+ \max_{u \in [0,(a^*-x)^+]} |c_1(x+u)| < \infty.
$$

(3.41)

Also, arguing as in the proof of Lemma 2.1 of Shreve et al. (1984) (see in particular equations (2.16)–(2.17) therein), we have that

$$
\mathbb{E}_x \left[ \int_0^\infty e^{-rt} \xi_t \right] < \infty.
$$

(3.42)

Since analogous arguments apply to $\mathbb{E}_x \left[ \int_0^\infty e^{-rt} |c_2(X^*_t)| \, d\eta_t \right]$, by combining (3.41), (3.42) and (3.40), we conclude that the requirement of Definition 2.9-(a) holds true, and therefore that $v^* \in A(x)$. \hfill \Box

Theorem 3.6 Let $(\xi^*, \eta^*)$ solving $SP(a^*, b^*; x)$, $v^*$ such that $v^* = \xi^* - \eta^*$, and $u$ as in (3.30). Then one has that $u = v$ on $I$ and $v^*$ is optimal for (2.32).

Proof It suffices to check that the conditions of Theorem 3.1 are met. By Theorem 3.4 the function $u$ of (3.30) is a classical solution to the HJB equation (3.1), and it is such that $|u(x)| \leq K(1 + |x|^\gamma + 1)$, for some $K > 0$ and where $\gamma \geq 1$ is the growth coefficient of $c_i$, $i = 1, 2$ [see Assumption 2.7-(ii)].

Moreover, $v^*$ is such that $X^*_t \in [a^*, b^*] = \{x \in I : (L_X - r)u(x) + h(x) = 0\}$ for a.e. $t$ and a.s.; the process (3.3) is an $\mathbb{F}$-martingale by continuity of $\sigma$ and $u'$; the conditions in (3.4) are met due to the fact that $(\xi^*, \eta^*)$ solves $SP(a^*, b^*; x)$, and that $\{x \in I : u'(x) = -c_1(x)\} = (a^*, a^*)$ and $\{x \in I : u'(x) = c_2(x)\} = [b^*, \overline{a}]$. Hence Theorem 3.1 applies and this completes the proof. \hfill \Box

3.4 A link with an optimal stopping game

The following proposition provides a probabilistic representation of the derivative $v'$ of the value function. This result plays an important role in the next section where we perform a comparative statics analysis.

Proposition 3.7 Let $v$ be the value function of (2.32). Then for any $x \in I$ one has

$$
v'(x) = \inf_{\tau} \sup_{\lambda} \hat{\mathcal{J}}_x(\tau, \lambda) = \sup_{\lambda} \inf_{\tau} \hat{\mathcal{J}}_x(\tau, \lambda),
$$

(3.43)
where $\tau, \lambda, i = 1, 2$ are $\tilde{\mathbb{F}}$-stopping times, and
\[
\mathcal{F}_x(\tau, \lambda) := \tilde{\mathbb{E}}_x \left[ \int_0^{\tau \wedge \lambda} e^{-\int_0^s (r - \mu'(\tilde{X}_u)) du} h'(\tilde{X}_s) \, ds \right.
\]
\[\left. - e^{-\int_0^\tau (r - \mu'(\tilde{X}_u)) du} c_1(\tilde{X}_\tau)1_{\{\tilde{X}_\tau < \tau\}} + e^{-\int_0^\tau (r - \mu'(\tilde{X}_u)) du} c_2(\tilde{X}_\tau)1_{\{\tilde{X}_\tau > \tau\}} \right].
\] (3.44)

Moreover, for any $x \in \mathcal{I}$, the couple of $\tilde{\mathbb{F}}$-stopping times $(\tau^*, \lambda^*)$ given by
\[
\tau^* := \inf\{t \geq 0 : \tilde{X}_t \geq b^*\} \quad \text{and} \quad \lambda^* := \inf\{t \geq 0 : \tilde{X}_t \leq a^*\}, \quad \tilde{\mathbb{P}}_x \text{ a.s.} \tag{3.45}
\]
form a saddle-point; that is,
\[
\tilde{\mathcal{F}}_x(\tau^*, \lambda^*) \leq \tilde{\mathcal{F}}_x(\tau^*, \lambda^*) \leq \tilde{\mathcal{F}}_x(\tau^*, \lambda^*)
\]
for any couple of $\tilde{\mathbb{F}}$-stopping times $(\tau, \lambda)$.

**Proof** We only provide a sketch of the proof, since its arguments are quite standard. From Theorem 3.6 we know that $v = u$ on $\mathcal{I}$, with $u$ as in (3.30). Since $v \in C^2(\mathcal{I})$, then $v' \in C(\mathcal{I})$. Moreover, it is easy to check from (3.30) that $v'''$ is locally bounded on $\mathcal{I}$; that is, $v' \in W^{2, \infty}(\mathcal{I})$. Moreover,
\[
(L_{\tilde{X}} - (r - \mu'(x)))v'(x) + h'(x) = 0 \quad \text{and} \quad -c_1(x) \leq v'(x) \leq c_2(x) \quad \text{on} \quad (a^*, b^*),
\] (3.46)
\[
v'(x) = -c_1(x) < c_2(x) \quad \text{and} \quad (L_{\tilde{X}} - (r - \mu'(x)))v'(x) + h'(x) \leq 0 \quad \text{on} \quad (x, a^*],
\] (3.47)
\[
v'(x) = c_2(x) > -c_1(x) \quad \text{and} \quad (L_{\tilde{X}} - (r - \mu'(x)))v'(x) + h'(x) \geq 0 \quad \text{on} \quad [b^*, \tilde{x}).
\] (3.48)

The first equation in (3.46) easily follows by noticing that $(L_{\tilde{X}} - r)v(x) + h(x) = 0$ for any $x \in (a^*, b^*)$, and by differentiating such an equation with respect to $x$. On the other hand, the inequalities involving $(L_{\tilde{X}} - (r - \mu'))v' + h'$ in (3.47) and in (3.48) follow from Assumption 2.7-(ii), together with the fact that $(x, a^*] \subset (\tilde{x}, \tilde{x}_1)$ and $[b^*, \tilde{x}) \subset (\tilde{x}_2, \tilde{x})$.

Given an arbitrary $\tilde{\mathbb{F}}$-stopping time $\tau$, an application of a generalized version of Itô’s lemma (see, e.g., Theorem 10.4.1 in Øksendal 2003) to the process $(e^{-\int_0^\tau (r - \mu'(\tilde{X}_u)) du} v'(\tilde{X}_\tau))_{\tau \geq 0}$ on the interval $[0, \tau \wedge \lambda^*]$ yields $v'(x) \leq \tilde{\mathcal{F}}_{\tau}(x, \lambda^*)$, upon using (3.46)–(3.48). On the other hand, applying again generalized Itô’s lemma to the process $(e^{-\int_0^\tau (r - \mu'(\tilde{X}_u)) du} v'(\tilde{X}_\tau))_{\tau \geq 0}$, but now on the interval $[0, \tau^* \wedge \lambda]$ for an arbitrary $\tilde{\mathbb{F}}$-stopping time $\lambda$, and employing (3.46)–(3.48) gives $v'(x) \geq \tilde{\mathcal{F}}_{\tau}(x, \tau^*).$ Finally, by using generalized Itô’s lemma on the process $(e^{-\int_0^\tau (r - \mu'(\tilde{X}_u)) du} v'(\tilde{X}_\tau))_{\tau \geq 0}$ on the interval $[0, \tau^* \wedge \lambda^*]$ one finds that $v'(x) = \tilde{\mathcal{F}}_{\tau}(x, \lambda^*)$ by (3.46).

Hence (3.43) holds true, and the $\tilde{\mathbb{F}}$-stopping times defined in (3.45) form a saddle-point. \qed

The previous proposition shows that $v'$ equals the value function of a zero-sum game of optimal stopping (a so-called Dynkin game, cf. Dynkin 1969). Furthermore, the boundaries that trigger the optimal control, also determine a saddle-point in the game of optimal stopping. Such a finding is consistent to the known relation between bounded variation control problems and zero-sum games of optimal stopping (see, e.g., Guo and Tomecek 2008; Karatzas and Wang 2005; Taksar 1985).
4 A case study with a mean-reverting (log-)exchange rate

In this section we assume that in (2.3) one has \( \mu(x) = \rho(m - x) \), for some \( \rho > 0 \) and \( m \in \mathbb{R} \), and \( \sigma(x) = \sigma > 0 \); that is, for a given \( v = \xi - \eta \in S \), the (log-)exchange rate \( X^{\xi, \eta} \) is a linearly controlled mean-reverting process with dynamics

\[
dX_t = \rho(m - X_t) \, dt + \sigma \, dB_t + d\xi_t - d\eta_t, \quad X_0 = x \in \mathbb{R}. \tag{4.1}
\]

In absence of interventions (i.e. \( \nu \equiv 0 \)), this specification is the simplest dynamics which keeps \( X \) in a given (suitable) region with a high probability, and empirical studies (see, e.g., Bo et al. 2016; Tvedt 2012) have concluded that it well describes several exchange rates among the main world countries.

In this section we also take the instantaneous costs \( c_i, i = 1, 2 \), such that \( c_i(x) \equiv c_i \) for all \( x \in \mathbb{R} \), and we specify a quadratic running cost function of the form

\[
h(x; \theta) = \frac{1}{2} (x - \theta)^2.
\]

The parameter \( \theta > 0 \) represents a so-called reference target, and it can be also viewed as the logarithm of the central parity (introduced in Krugman 1991 for the first time). The function \( h \) penalizes any displacement of the (log-)exchange rate from such a value.

We notice that with \( X \) as in (4.1), we have \( \sigma'(x) = 0 \) for all \( x \in I \). It thus follows that the process \( \hat{X} \) of (2.4) is the unique strong solution to

\[
d\hat{X}_t = \rho(m - \hat{X}_t) \, dt + \hat{\sigma} \, d\hat{B}_t, \quad X_0 = x \in \mathbb{R}, \tag{4.2}
\]

where \( \hat{B} \) is a standard Brownian motion. Moreover, because \( r - \mu'(x) = r + \rho \), the characteristic Eq. (2.10) reads \( \frac{1}{2} \sigma^2 f'' + \rho(m - x) f' - (r + \rho) u = 0, \, r > 0 \), and it is known that it admits the two linearly independent, positive solutions (cf. Jeanblanc et al. 2009, p. 280)

\[
\hat{\phi}(x) := e^{\frac{\rho}(\sigma^2) - \frac{(x - m)^2}{2\rho}} D_{\frac{x - m}{\sqrt{2\rho}}} \left( \frac{\sigma}{\rho} \right) \tag{4.3}
\]

and

\[
\hat{\psi}(x) := e^{\frac{\rho}(\sigma^2) + \frac{(x - m)^2}{2\rho}} D_{\frac{x - m}{\sqrt{2\rho}}} \left( -\frac{\sigma}{\rho} \right), \tag{4.4}
\]

which are strictly decreasing and strictly increasing, respectively. In both (4.3) and (4.4) \( D_\alpha \) is the cylinder function of order \( \alpha \) given by (see, e.g., Bateman 1981, Chapter VIII, Section 8.3, eq. (3) at page 119)

\[
D_\alpha(x) := \frac{e^{-\frac{x^2}{\pi}}}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-\frac{x^2}{\pi t}} \, dt, \quad \text{Re}(\alpha) < 0, \tag{4.5}
\]

where \( \Gamma(\cdot) \) is the Euler’s Gamma function.

Within this setting, it is easy to see that the two equations for the free boundaries \( a^* \) and \( b^* \) [cf. (3.24) and (3.25)] read

\[
\int_a^b (z - \theta + (r + \rho)c_1) \hat{m}(z) \hat{\phi}(z) \, dz = -(r + \rho)(c_1 + c_2) \int_b^\infty \hat{m}(z) \hat{\phi}(z) \, dz, \tag{4.6}
\]

\[
\int_a^b (z - \theta - (r + \rho)c_2) \hat{m}(z) \hat{\psi}(z) \, dz = (r + \rho)(c_1 + c_2) \int_{-\infty}^a \hat{m}(z) \hat{\psi}(z) \, dz, \tag{4.7}
\]

where \( \hat{m}' \) is given by (2.11).
In the next section we study the dependency of the optimal boundaries $a^*$ and $b^*$ with respect to the model's parameters $m, \sigma, c_1, c_2,$ and $\theta$.

4.1 Comparative statics results

In the following we will often use the notation $a^*(\cdot), b^*(\cdot)$ and $v'(x; \cdot)$ to stress the dependence of $a^*, b^*$ and $v'$ with respect to a given parameter. For some of the next results (namely, Propositions 4.1, 4.3, and 4.4) an important role is played by the representation of $v'$ given in (3.43).

Proposition 4.1 The optimal intervention boundaries $a^*$ and $b^*$ are decreasing in the long-run equilibrium level $m$; that is, $m \mapsto a^*(m)$ and $m \mapsto b^*(m)$ are decreasing.

Proof Denote by $\tilde{X}^{x;m}_t$ the unique strong solution to (4.2) when the equilibrium value is $m \in \mathbb{R}$, and notice that $m \mapsto \tilde{X}^{x;m}_t$ is a.s. increasing for all $t > 0$. Then, since $x \mapsto h'(x)$ is increasing, we have for all $m_1 \geq m_2$ that $v'(x; m_1) \geq v'(x; m_2)$ by (3.43). Hence for all $m_1 \geq m_2$ we have

$$a^*(m_1) = \sup \{x \in \mathbb{R} \mid v'(x; m_1) \leq -c_1 \} \leq \sup \{x \in \mathbb{R} \mid v'(x; m_2) \leq -c_1 \} = a^*(m_2),$$

$$b^*(m_1) = \inf \{x \in \mathbb{R} \mid v'(x; m_1) \geq c_2 \} \leq \inf \{x \in \mathbb{R} \mid v'(x; m_2) \geq c_2 \} = b^*(m_2);$$

that is, $m \mapsto a^*(m)$ and $m \mapsto b^*(m)$ are decreasing. \[\Box\]

Proposition 4.2 The more the exchange market is volatile, the more the central bank is reluctant to intervene. That is, the optimal intervention boundaries $a^*$ and $b^*$ are such that $\sigma \mapsto a^*(\sigma)$ is decreasing, and $\sigma \mapsto b^*(\sigma)$ is increasing.

Proof We borrow arguments from the proof of Theorem 6.1 in Matomäki (2012) (see also Section 3 in Alvarez 2003). Let $x \in I$, and denote by $X^\sigma$ the solution to (4.1) for $v \equiv 0$ and for a volatility coefficient $\sigma > 0$. Let $\sigma_1 \geq \sigma_2$, and for $i = 1, 2$ denote by $v^{(i)}$ the value function (2.32) when the underlying controlled process solves (4.1) with volatility $\sigma_i$, by $\mathcal{L}^{X^{\sigma_i}}$, the infinitesimal generator associated to the (uncontrolled) diffusion $X^\sigma$, and by $\mathcal{L}^{X^{\sigma_i}}$ the infinitesimal generator associated to the solution to (4.2) with volatility $\sigma_i$. Also, $a_i^*$ and $b_i^*, i = 1, 2$, are the optimal control boundaries associated to the value function $v^{(i)}, i = 1, 2$.

Then, recalling that the value function equals the function given in (3.30), use that for any $i = 1, 2$ [cf. (3.33)],

$$v^{(i)}(a_i^*) = \frac{1}{r} \left[ h(a_i^*) - \mu(a_i^*)c_1 \right], \quad v^{(i)}(b_i^*) = \frac{1}{r} \left[ h(b_i^*) + \mu(b_i^*)c_2 \right],$$

to find

$$(\mathcal{L}^{X^{\sigma_1}} - r) v^{(2)}(x) + h(x) := \begin{cases} h(x) - h(a_1^*) + c_1 (r + \rho) (x - a_1^*), & x \in (x, a_2^*], \\ \frac{1}{2} (\sigma_1^2 - \sigma_2^2) \frac{d^2}{dx^2} v^{(2)}(x), & x \in (a_1^*, b_2^*], \\ h(x) - h(b_2^*) - c_2 (r + \rho) (x - b_2^*), & x \in [b_2^*, x]. \end{cases}$$

We now prove that all the terms appearing on the right hand-side of the latter equation are nonnegative. On the one hand,

$$h(x) - h(a_2^*) + c_1 (r + \rho) (x - a_2^*)$$

$$= \int_x^{a_2^*} \left[ -h'(z) + (\mathcal{L}^{X^{\sigma_1}} - (r + \rho))c_1 \right] dz \geq 0, \quad x \in (x, a_2^*].$$
and
\[
    h(x) - h(b^*_2) - c_2(r + \rho)(x - b^*_2) = \int_{b^*_2}^{x} \left[ h'(z) + (L\tilde{x}_1 - (r + \rho))c_2 \right] dz \geq 0, \quad x \in [b^*_2, \tilde{x}],
\]
where the last inequalities are due to Assumption 2.7 and the fact that \( a^*_2 < \tilde{x}_1 < \tilde{x}_2 < b^*_2 \).

On the other hand, we notice that since \( c_1(x) = c_1 \) for all \( x \in I \), the convexity of \( x \mapsto h(x; \theta) \) and the linearity of the dynamics (4.1) imply that functional \( J_\theta(v) \) is simultaneously convex in \((x, v)\), and the set of admissible controls is convex. Therefore,
\[
    v^{(2)}(\lambda x + (1 - \lambda)x') \leq \lambda J_\theta(v) + (1 - \lambda) J_\theta(v'),
\]
for all \( x, x' \in I \), \( v \in A(x), v' \in A(x') \), and \( \lambda \in [0, 1] \). Hence \( v^{(2)} \) is convex on \( I \), and this fact in turn yields
\[
    \frac{1}{2}(\sigma^2 - \sigma'^2) \frac{d^2}{dx^2} v^{(2)}(x) \geq 0, \quad x \in (a^*_2, b^*_2),
\]
since \( \sigma^2 \geq \sigma'^2 \).

It thus follows from the previous considerations that \((L\tilde{x}_1 - r)v^{(2)}(x) + h(x) \geq 0\) for all \( x \in I \). Moreover, since \( v^{(2)} \) is the value function when \( \sigma = \sigma_2 \), we also have
\[
    \frac{d}{dx} v^{(2)}(x) \leq K(1 + |x|)\mathcal{I} \quad \text{on} \quad I, \quad \text{for some} \quad K > 0, \quad \text{and for} \quad \gamma \geq 1 \quad \text{as in Assumption 2.7.}
\]
Therefore, arguing as in Step 1 of the proof of Theorem 3.1 we can show that \( v^{(2)} \leq v^{(1)} \) on \( I \).

Thanks to the last inequality we can now prove that \( a^*_2 \geq a^*_1 \). We follow a contradiction scheme, and we suppose that \( a^*_2 < a^*_1 \). Then noticing that \( a^*_2 < a^*_1 < \tilde{x}_1 \), and using (4.8) and Assumption 2.7-(ii) we have that (recall that here \( \mu(x) = \rho(m - x), x \in \mathbb{R} \))
\[
    v^{(2)}(a^*_2) = \frac{1}{r}
    [h(a^*_2) - \mu(a^*_2)c_1]
    = \frac{1}{r}
    [h(a^*_2) + (ra^*_2 - \mu(a^*_2))c_1]
    - c_1 a^*_2
    > \frac{1}{r}
    [h(a^*_1) + (ra^*_1 - \mu(a^*_1))c_1]
    - c_1 a^*_2
    = v^{(1)}(a^*_1) - c_1(a^*_2 - a^*_1) = v^{(1)}(a^*_2).
\]
The latter inequality contradicts that \( v^{(2)} \leq v^{(1)} \) on \( I \), and therefore shows that \( a^*_2 \geq a^*_1 \).

\begin{proposition}
The optimal intervention boundaries \( a^* \) and \( b^* \) are such that \( c_1 \mapsto a^*(c_1) \) is decreasing, and \( c_1 \mapsto b^*(c_1) \) is increasing. Also, \( c_2 \mapsto a^*(c_2) \) is decreasing and \( c_2 \mapsto b^*(c_2) \) is increasing.
\end{proposition}

\begin{proof}
From (3.43) and (3.44) it is easy to see that
\[
    c_1 \mapsto v'(x; c_1) + c_1 \text{ is increasing,} \quad c_2 \mapsto v'(x; c_2) - c_2 \text{ is decreasing}
\]
\[
    c_1 \mapsto v'(x; c_1) \text{ is decreasing,} \quad c_2 \mapsto v'(x; c_2) \text{ is increasing.}
\]
Take now \( \overline{c} > c_1 \). Then
\[
    a^*(\overline{c}) = \sup \{ x \in \mathbb{R} | v'(x; \overline{c}) + \overline{c} \leq 0 \} \leq \sup \{ x \in \mathbb{R} | v'(x; c_1) + c_1 \leq 0 \} = a^*(c_1),
\]
\[
    b^*(\overline{c}) = \inf \{ x \in \mathbb{R} | v'(x; \overline{c}) \geq c_2 \} \geq \inf \{ x \in \mathbb{R} | v'(x; c_1) \geq c_2 \} = b^*(c_1).
\]
Hence \( c_1 \mapsto a^*(c_1) \) is decreasing and \( c_1 \mapsto b^*(c_1) \) is increasing.
\end{proof}
Analogously, taking now \( \bar{c} > c_2 \) we have

\[
\begin{align*}
    a^*(\bar{c}) &= \sup \{ x \in \mathbb{R} \mid v'(x; \bar{c}) \leq -c_1 \} \leq \sup \{ x \in \mathbb{R} \mid v'(x; c_2) \leq -c_1 \} = a^*(c_2), \\
    b^*(\bar{c}) &= \inf \{ x \in \mathbb{R} \mid v'(x; \bar{c}) - \bar{c} \geq 0 \} \geq \inf \{ x \in \mathbb{R} \mid v'(x; c_2) - c_2 \geq 0 \} = b^*(c_2);
\end{align*}
\]

i.e., \( c_2 \mapsto a^*(c_2) \) is decreasing and \( c_2 \mapsto b^*(c_2) \) is increasing.

**Proposition 4.4** The optimal intervention boundaries \( a^* \) and \( b^* \) are such that \( \theta \mapsto a^*(\theta) \) and \( \theta \mapsto b^*(\theta) \) are increasing.

**Proof** We notice that \( \theta \mapsto h'(x; \theta) \) is decreasing. It follows from (3.43) that \( \theta \mapsto v'(x; \theta) \) is decreasing as well, and therefore for all \( \theta_2 > \theta_1 \) we have

\[
\begin{align*}
    a^*(\theta_1) &= \sup \{ x \in \mathbb{R} \mid v'(x; \theta_1) \leq -c_1 \} \leq \sup \{ x \in \mathbb{R} \mid v'(x; \theta_2) \leq -c_1 \} = a^*(\theta_2), \\
    b^*(\theta_1) &= \inf \{ x \in \mathbb{R} \mid v'(x; \theta_1) \geq c_2 \} \leq \inf \{ x \in \mathbb{R} \mid v'(x; \theta_2) \geq c_2 \} = b^*(\theta_2).
\end{align*}
\]

Thus, \( \theta \mapsto a^*(\theta) \) and \( \theta \mapsto b^*(\theta) \) are both increasing, so that when the target level \( \theta \) increases, the no-intervention region \( (a^*, b^*) \) is displaced towards higher values.

**Remark 4.5** Notice that the previous monotonicity results can be easily generalized to the case of a more general diffusion. For example, assuming that a comparison principle à la Yamada-Watanabe (see, e.g., Proposition 5.2.18 in Karatzas and Shreve 1991) holds true for the diffusion (2.4), that the killing rate \( r - \mu'(\cdot) \) is decreasing (i.e. \( x \mapsto \mu(x) \) is convex), and that the holding cost function is convex, one can show that the boundaries are monotonically decreasing with respect to the drift coefficient. Also, arguing as in the proof of Theorem 6.1 in Matomäki (2012), one can prove the monotonicity of the boundaries with respect to a general state-dependent volatility coefficient. However, we decided to formulate the study of this section for an Ornstein–Uhlenbeck process because it is perhaps the simplest diffusion that captures the mean-reverting behavior of exchange rates empirically observed in some economies (see Sweeney 2006; Tvedt 2012; Yang et al. 2016, and references therein).

### 4.2 Expected exit time from the target zone

One of the great advantages of the Ornstein–Uhlenbeck model above is that many quantities about exit times and probabilities are known in closed form. We base our analysis on the results contained in Cadenillas et al. (2007, “Appendix B”). Recalling the optimally controlled process \( X^* \), define the exit time from \((a^*, b^*)\) as

\[
\tau_{(a^*, b^*)} := \inf \{ t > 0 : X_t^* \notin (a^*, b^*) \},
\]

and notice that \( \mathbb{P}_x \{ \tau_{(a^*, b^*)} < \infty \} = 1 \) for all \( x \in \mathbb{R} \). Indeed, if \( x \notin (a^*, b^*) \) then clearly \( \tau_{(a^*, b^*)} = 0 \) \( \mathbb{P}_x \)-a.s. On the other hand, if \( x \in (a^*, b^*) \) then the optimal control \( v^* \) is such that \( v_t^* \equiv 0 \) for any \( t \leq \tau_{(a^*, b^*)} \), and the (uncontrolled) Ornstein–Uhlenbeck process is positively recurrent. Also, we have that...
Furthermore, we know that the function \( q(x) := \mathbb{E}_x[\tau_{(a^*, b^*)}], x \in (a^*, b^*) \), satisfies the boundary value differential problem

\[
\mathcal{L}_X q + 1 = 0, \quad q(a^*) = q(b^*) = 0,
\]

whose solution is

\[
q(x) = A_1 + B_1 \int_{\sqrt{\frac{\sigma}{\rho}}(a^*-m)} \int_{\sqrt{\frac{\sigma}{\rho}}(b^*-m)} e^{\frac{1}{2}w^2} dw - \frac{1}{\rho} \int_{\sqrt{\frac{\sigma}{\rho}}(a^*-m)} \int_{\sqrt{\frac{\sigma}{\rho}}(b^*-m)} e^{\frac{1}{2}w^2} dw \int_{w} e^{-\frac{1}{2}u^2} du dw, \tag{4.11}
\]

with the constants \( A_1 \) and \( B_1 \) given by

\[
A_1 = \frac{1}{\rho} \int_{\sqrt{\frac{\sigma}{\rho}}(b^*-m)} \int_{\sqrt{\frac{\sigma}{\rho}}(b^*-m)} e^{\frac{1}{2}w^2} e^{-\frac{1}{2}u^2} dw du, \quad B_1 = \frac{-A_1}{\int_{\sqrt{\frac{\sigma}{\rho}}(a^*-m)} \int_{\sqrt{\frac{\sigma}{\rho}}(a^*-m)} e^{\frac{1}{2}w^2} dw}.
\]

Thanks to the previous results we can numerically compute the mean time until the exchange rate leaves the target zone, i.e. the mean time until the next central bank’s intervention. This will be done in the next section.

### 4.3 Numerical results

We now present a possible implementation of the previous model, tailored to mimick the DKK/EUR exchange rate. Since it seems that in 30 years there was no need to intervene from the Danish Central Bank, we can safely assume that the long-run mean corresponds to the logarithm of the central parity fixed to 7.46038 DKK/EUR. Remembering that the Ornstein-Uhlenbeck process in Eq. (4.1) represents the logarithm of the exchange rate, we thus let \( m = \theta = \log 7.46038 = 2.00961 \sim 2.01; \) other plausible parameters for the Ornstein-Uhlenbeck dynamics could be \( \rho = 0.001 \) and \( \sigma = 0.015 \). Given the interest rates in the current economy, a plausible value for \( r \) could be \( r = 0.5\% = 0.005 \). The values above are characteristic of the Danish and European economies, and still do not reflect the Danish Central Bank’s policy, which is instead implemented in the three parameters \( \theta, c_1 \) and \( c_2 \).

We collect the parameters up to now in Table 1.

In order to find the known intervention thresholds of \( \pm 2.25\% \) from the central parity, we must implement the following inverse problem: find \( c_1, c_2 \) such that, with the parameters above, the optimal \( a^* \) and \( b^* \) are

\[
a^* = \log 7.46038(1 - 0.0225) = 1.98685, \quad b^* = \log 7.46038(1 + 0.0225) = 2.03186
\]
Table 1 Parameters’ values for the numerical example

|   | r   | σ   | ρ   | θ   | m   |
|---|-----|-----|-----|-----|-----|
|   | 0.005 | 0.015 | 0.001 | 2.01 | 2.01 |

Table 2 Optimal continuation regions (i.e., target zones) for various fixed costs \( c = c_1 = c_2 \)

| c   | \( a^* \) | \( b^* \) | \( a^* - m \) | \( b^* - m \) |
|-----|-----|-----|-----|-----|
| 1   | 1.93729 | 2.08193 | −0.07232 | 0.07232 |
| 0.5 | 1.95302 | 2.0662 | −0.0565905 | 0.0565905 |
| 0.1 | 1.97703 | 2.04218 | −0.0325786 | 0.0325786 |
| 0.05 | 1.98383 | 2.03539 | −0.0257803 | 0.0257803 |
| 0.04 | 1.98569 | 2.03352 | −0.0239155 | 0.0239155 |
| 0.035 | 1.98674 | 2.03247 | −0.0228658 | 0.0228658 |
| 0.034 | 1.98696 | 2.03225 | −0.0226442 | 0.0226442 |
| **0.0335** | **1.98707** | **2.03214** | **−0.0225317** | **0.0225317** |
| 0.033 | 1.98719 | 2.03202 | −0.0224182 | 0.0224182 |
| 0.03 | 1.98789 | 2.03132 | −0.021712 | 0.021712 |

Given the (approximate) symmetry of our problem\(^2\), we search for \( c_1 \) and \( c_2 \) such that \( c_1 = c_2 =: c \). From the monotonicity result of Proposition 4.3 we know that, by increasing (decreasing) the common proportional cost \( c \), the continuation region \((a^*, b^*)\) will enlarge (shrink): this is a positive sign that our inverse problem can have a unique solution.

With this in mind, we search for \( c = c_1 = c_2 \) such that \([a^*, b^*] \simeq [m − 0.0225, m + 0.0225]\). We start checking for \( c = 1 \), and we continue by decreasing the value of \( c \) until we find our zone: the results are reported in Table 2.

Hence, given the parameters’ values of Table 1, if we let \( c_1 = c_2 = 0.0335 \), we find (see bold text in Table 2) that the optimal \( a^* \) and \( b^* \) well approximate the boundaries of the target zone that the Central Bank of Denmark is adopting since January 12, 1987 (Mikkelsen 2017).

By using the results in Sect. 4.2, we can also compute the expected exit time of the exchange rate from the target zone. In fact, by taking \((a^*, b^*) = (1.98707, 2.03214)\), Eq. (4.11) can be plotted as a function of initial (log-)exchange rate \( x \). The plot is in Fig. 4 below.

We can see that the maximal expected time is obtained (as expected) when the deviation from central parity is null, i.e., for \( x = \log 7.46038 \simeq 2.01 \), and decreases as the exchange rate nears the target zone’s boundaries. This maximum expected time is around 31.11 years, which is also the expected time before an intervention by the central bank is triggered. This finding is perfectly in line with the observed phenomenon that the Danish Central Bank did not need to intervene to keep the DKK/EUR exchange rate within the target zone since the last 30 years (Mikkelsen 2017).

5 Model’s extensions and concluding remarks

5.1 Future outlooks

There are several possible directions towards our study on exchange rates’ control can be extended, and two relevant ones are discussed in the following.

\(^2\) Since \( \log(1 + 0.0225) = 0.02225 \simeq 0.0225 \simeq −0.02276 = − \log(1 − 0.0225) \).
5.1.1 Optimal control of exchange rates via direct interventions and interest rate management.

A first extension is related to the possibility for the central bank to intervene on the exchange rate through the management of the level of interest rate. This aspect has been already considered in the literature in Bertola et al. (2016) and Cadenillas and Zapatero (2000). In these papers the actions of the central bank on the foreign exchange market are modeled through purely discontinuous controls (i.e. via impulse controls), while the interest rate management is performed with a time-dependent speed (i.e. via regular controls).

We believe that a more natural way to model this problem is as a mixed singular-impulse two-dimensional stochastic control problem, as we now briefly explain. Let us fix a complete probability space \((\Omega, \mathcal{F}, P)\), and on it a one-dimensional Brownian motion \(B\) generating a right-continuous filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\).

Then, for any \(\xi - \eta := \nu \in S\) [cf. (2.1)], and \(x \in \mathcal{I} = (\underline{x}, \overline{x})\), with \(-\infty \leq \underline{x} < \overline{x} \leq +\infty\), we suppose that the exchange rate between two currencies evolves as

\[
dX_t = \mu(X_t, r_t)dt + \sigma(X_t)dB_t + d\xi_t - d\eta_t, \quad t > 0, \quad X_0 = x \in \mathcal{I},
\]

(5.1)

for suitable \(\mu\) and \(\sigma\) ensuring the existence of a strong solution to the previous stochastic differential equation. As in the model solved in the previous sections, \(\xi\) and \(\eta\) represent the cumulative amounts of direct interventions on the exchange rate via the sales and purchases of the currencies. In (5.1), the expected trend of the exchange rate is affected by the current level of an interest rate \(r\), which can be fixed by the central bank (e.g. the official discount rate). In Bertola et al. (2016) and Cadenillas and Zapatero (2000), \(r\) is assumed to be a control which is continuously adjustable by the central bank, and this results in a deterministic function of \(X\). However, we deem it quite unrealistic that a central bank can adjust its official discount rate in this fashion, given also that in reality these rates are pathwise constant in time, and every adjustment is seen by the public opinion as a major move of the central bank. We thus assume that \(r\) can be directly adjusted by the central bank, with dynamics given by

\[
r_t = r + \sum_{\tau_k < t} \xi_k - \sum_{\lambda_k < t} \beta_k, \quad t \geq 0,
\]

(5.2)
for some initial level $r \in \mathbb{R}$. Here, $(\zeta_k)_{k \in \mathbb{N}}$ (resp., $(\beta_k)_{k \in \mathbb{N}}$) are the sizes of the impulses exerted by the central bank at the stopping times $(\tau_k)_{k \in \mathbb{N}}$ (resp., $(\lambda_k)_{k \in \mathbb{N}}$) in order to increase (resp., decrease) the current level of interest rate. We may assume that the central bank’s interest rate management policy is such that keeps $r_t$ in a given bounded region, for any $t \geq 0$ a.s.

The central bank aims at choosing a suitable purchasing/selling strategy $v^* := \xi^* - \eta^* \in S$, and an opportune intervention policy on the interest rate $\Phi := (\xi^*_k, \tau^*_k, \beta^*_k, \lambda^*_k)_{k \in \mathbb{N}}$ so to minimize the total expected cost functional

$$J_{(x,r)}(v, \Phi) := \mathbb{E}_{(x,r)} \left[ \int_0^\infty e^{-r s} h(X_s, R_s) \, ds + \int_0^\infty e^{-r s} \left( c_1(X_s) \oplus d \xi_s + c_2(X_s) \oplus d \eta_s \right) \right] + \sum_{k \geq 1} e^{-r \tau_k} \left( \alpha_+ \xi_k + \gamma_+ \right) + \sum_{k \geq 1} e^{-r \lambda_k} \left( \alpha_- \beta_k + \gamma_- \right),$$

for a suitable running cost function $h$, proportional cost functions $c_1$ and $c_2$, and for nonnegative numbers $\alpha_\pm$ and $\gamma_\pm$.

The presence of the fixed costs of interventions $\gamma_+$ and $\gamma_-$ makes the latter a two-dimensional singular-impulse stochastic control problem (see Aid et al. (2019) and references therein). To the best of our knowledge, a stochastic control problem as the one described before has never been solved in the literature. Its analysis requires the use of mathematical techniques different to those employed in this papers, and it is clearly outside the scopes of the current work.

### 5.1.2 Optimal control of exchange rates with limited reserves.

Another aspect that could be incorporated in an extension of our model is related to the fact that the reserves at disposal to the central bank in order to sell/purchase currencies are clearly limited. This could be modeled as follows.

Let again fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and on it a one-dimensional Brownian motion $B$ generating a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. Then, for any $\mu := v - \mu \in S$ (cf. (2.1)), the dynamics of the exchange rate is still be given by (2.3), and the central bank’s reserves’ level has dynamics

$$R_t = R + \xi_t - \eta_t, \quad t \geq 0,$$

and such that $R_t \in [0, \bar{R}]$ a.s. for any $t \geq 0$, for some given and fixed $\bar{R} \in (0, \infty)$. The latter constraint poses limits to the direct interventions of the central bank on the exchange market, as it effectively limits the possible (cumulative) amounts of sales and purchases of the two currencies.

In this new model, the cost functional to be minimized might then be

$$J_{(x,R)}(v) := \mathbb{E}_{(x,R)} \left[ \int_0^\infty e^{-r s} \left( h(X_s, R_s) \, ds + c_1(X_s, R_s) \oplus d \xi_s + c_2(X_s, R_s) \oplus d \eta_s \right) \right],$$

for real-valued running cost $h$ and proportional cost functions $c_1$ and $c_2$.

The resulting stochastic optimal control problem now takes the form of a finite-fuel bounded-variation singular stochastic control problem (see the recent paper Ferrari and Koch 2019 and references therein). To the best of our knowledge, a problem with the structure outlined before has never been solved in the literature before, and it definitely represents an interesting question to be addressed in a future work. In particular, since the process $R$ is purely controlled, we expect that it might be possible—but very challenging—to obtain
explicit solutions. This should involve two free boundaries, depending on the current level of the reserves, that trigger the action of the central bank on the exchange rate market.

Clearly, combining the modeling aspects discussed in the two problems above would lead to a very interesting—but extremely complex and difficult—problem, for which we see no hopes to obtain explicit solutions.

5.2 Conclusions

In this paper we have studied the optimal management problem of exchange rates faced by a central bank. We have formulated it as an infinite time-horizon singular stochastic control problem for a one-dimensional diffusion that is linearly controlled through a process of bounded variation. We have provided the explicit expression of the value function, as well as the complete characterization of the optimal control. At each instant of time, the optimally controlled exchange rate is kept within an optimal band (continuation region), whose boundaries (the so-called free boundaries) are endogenously determined as part of the solution to the problem.

A detailed comparative statics analysis of the free boundaries is provided when the (log-)exchange rate (in absence of any intervention) evolves as an Ornstein–Uhlenbeck process. This dynamics captures the mean-reverting behavior of exchange rates that has been observed in several empirical studies (see Sweeney 2006; Tvedt 2012 and references therein). Moreover, it allows the central bank to have aims, both in its cost function \( h \) as well as in its intervention costs \( c_i \), which possibly contrast with this foreign exchange dynamics. This does not happen if, for example, the minimum of \( h \) is very near to the long-term mean of the exchange dynamics: in this case, the exchange rate stays naturally with a high probability in the continuation region. This fact can be interpreted as the “target zone” introduced in Krugman (1991), and it applies, for example, to the Danish and Hong Kong currencies (Mikkelsen 2017, Hong Kong Monetary Authority).

Given the general cost structure and state dynamics, the control problem studied in this paper might be a reasonable model also in other context, as, e.g., for problems of partially reversible capacity expansion (see De Angelis and Ferrari 2014; Guo and Pham 2005, among others), for the optimal management of an inventory (see Harrison and Taksar 1983 for an early work), for the automotive cruise control of an aircraft under an uncertain wind condition (Chow et al. 1985), or for the optimal management of stabilization funds (Huamán-Aguilar 2015).

Acknowledgements

Financial support by the German Research Foundation (DFG) through the Collaborative Research Centre 1283 “Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications” is gratefully acknowledged by the first author. This work has been initiated, and then revised, while the first author was visiting the Department of Mathematics of the University of Padova thanks to the fundings provided by the “ACRI Young Investigator Training Program” (YITP-QFW2017) and the program “Visiting Scientist 2019” of the University of Padova. We also thank anonymous referees and the corresponding editors for interesting comments that helped to improve a first version of this work.

Appendix A

Lemma A.1 One has that \( \psi' = \hat{\psi} \) and \( -\phi' = \hat{\phi} \), where \( \hat{\psi} \) and \( \hat{\phi} \) are the strictly increasing and strictly decreasing fundamental solutions of the ODE \( (L_{\hat{X}} - (r - \mu')) f = 0 \) for \( \hat{X} \) killed at rate \( r - \mu'(\cdot) \).
Proof We simply repeat the arguments in the second part of the proof of Lemma 4.3 in Alvarez and Matomäki (2015) (see also Theorem 9 in Alvarez 2001). Under Assumption 2.1 standard differentiation reveals that \( \psi' \) and \( \phi' \) solve the ODE
\[
(L\hat{\psi} - (r - \mu')) f = 0. \tag{A-1}
\]
Also, for any \( x \in \mathcal{I} \) one has \( \phi''(x)\psi'(x) - \phi'(x)\psi''(x) = 2rW \hat{S}'(x) \neq 0 \), and so any solution \( f \) to the previous ODE has to be of the form \( f(x) = c_1 \psi'(x) + c_2 \phi'(x) \). Furthermore, note that under Assumption 2.1 and 2.3, Corollary 1 of Alvarez (2003) can be applied yielding that \( \phi \) and \( \psi \) are strictly convex.

We thus find that for all \( \ell_1 < \ell_2 \) and for all \( x \in (\ell_1, \ell_2) \subset \mathcal{I} \) we can write
\[
\mathbb{E}_x [ e^{-\int_0^t (r - \mu'(\hat{S})) ds} ] = \frac{f_1(x)}{f_1(\ell_1)} + \frac{f_2(x)}{f_2(\ell_2)},
\]
where \( \mathcal{T} = \inf \{ t \geq 0 : \hat{\mathcal{X}}_t \notin (\ell_1, \ell_2) \} \), \( \mathbb{P}_x \)-a.s., and \( f_1(x) := \frac{\phi'(\ell_2)}{\phi'(\ell_1)} \psi'(x) - \phi'(x) \) and \( f_2(x) := \psi'(x) - \frac{\psi'(\ell_1)}{\phi'(\ell_1)} \phi'(x) \) are the fundamental solutions of (A-1) when \( \mathcal{X} \) is killed at \( \ell_1 \) and \( \ell_2 \).

Noticing that \( \lim_{\ell_1 \downarrow \ell_2} \psi'(\ell_1)/\phi'(\ell_1) = 0 \) and \( \lim_{\ell_2 \uparrow \mathcal{T}} \phi'(\ell_2)/\psi'(\ell_2) = 0 \) by the required boundary behavior of \( X \), one has that
\[
\lim_{\ell_1 \downarrow \ell_2} \mathbb{E}_x \left[ e^{-\int_0^t (r - \mu'(\hat{S})) ds} \right] = \frac{\psi'(x)}{\psi'(\ell_2)},
\]
and
\[
\lim_{\ell_2 \uparrow \mathcal{T}} \mathbb{E}_x \left[ e^{-\int_0^t (r - \mu'(\hat{S})) ds} \right] = \frac{\phi'(x)}{\phi'(\ell_1)}.
\]
Hence, \( \psi' \) and \( -\phi' \) are the fundamental solutions of (A-1) for \( \hat{\mathcal{X}} \) killed at rate \( r - \mu'(\cdot) \). That is, \( \psi' = \hat{\psi} \) and \( -\phi' = \hat{\phi} \).

\[ \square \]

Proof of Equation (2.26) Assumption 2.1 guarantees that the flow \( x \mapsto X_t^{x;0,0} \) is a.s. continuous, increasing and differentiable for any \( t \geq 0 \) (see, e.g., Protter 1990, Ch. V.7). Defining the process \( Y \) such that \( Y_t = \partial X_t^{x;0,0}/\partial x, \ t \geq 0 \), by ordinary differentiation we find that \( Y \) satisfies
\[
dY_t = \mu'(X_t^{x;0,0}) Y_t dt + \sigma'(X_t^{x;0,0}) Y_t dB_t, \quad Y_0 = 1,
\]
and therefore that
\[
Y_t = e^{\int_0^t \mu'(X_s^{x;0,0}) ds} Z_t,
\]

where the exponential martingale \( (Z_t)_{t \geq 0} \) has been defined in Eq. (2.17).

As in Remark 2.5, consider the dynamics of \( X^{x,0} \) under the measure \( \mathbb{P}_x \), and the dynamics of \( \mathcal{X} \) under the measure \( \mathbb{P}_\mathcal{X} \). Define also a new measure \( \mathbb{Q}_x \) through the Radon–Nikodym derivative \( Z_t := \frac{d\mathbb{Q}_x}{d\mathbb{P}_x} |_{\mathcal{F}_t} \) and notice that the Girsanov theorem implies that the process
\[
\hat{B}_t := B_t - \int_0^t \sigma'(X_s^{x;0,0}) ds
\]
is a standard Brownian motion under \( \mathbb{Q}_x \).
Take now $f \in C^1(I)$, and such that $Rf$ and $\hat{R}f'$ are well defined. Then, by differentiating (2.21) we obtain
\[
(Rf)'(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-rt} Y_t f'(X^0,0)_t dt \right] = \mathbb{E}_x \left[ \int_0^\infty e^{-\int_0^t (r-\mu'(X^0,0)_s) ds} f'(X^0) dt \right].
\]
We therefore conclude by observing that Law $(X^0,0|Q_x) = \text{Law}(\hat{X}|\hat{P}_x)$ and recalling (2.22).

References

Aïd, R., Basei, M., Callegaro, G., Campi, L., & Varjoulu, T. (2019). Nonzero-sum stochastic differential games with impulse controls: A verification theorem with applications. Mathematics of Operations Research. https://doi.org/10.1287/moor.2019.0989

Alvarez, L.H.R., & Matomäki, P. (2015). Expected supremum representation of the value of a singular stochastic control problem. Preprint. arXiv:1508.02854.

Alvarez, L. H. R. (2001). Singular stochastic control, linear diffusions and optimal stopping: A class of solvable problems. SIAM Journal on Control and Optimization, 39(6), 1697–1710.

Alvarez, L. H. R. (2003). On the properties of a class of $r$-excessive mappings for a class of diffusions. The Annals of Applied Probability, 13(4), 1517–1533.

Bateman, H. (1981). Higher transcendent functions (Vol. II). New York: McGraw-Hill Book Company.

Bertola, G., Runggaldier, W. J., & Yasuda, K. (2016). On classical and restricted impulse stochastic control for the exchange rate. Applied Mathematics and Optimization, 74(2), 423–454.

Bo, L., Li, D., Ren, G. Wang, Y., & Yang, X. (2016). Modeling the exchange rates in a target zone by reflected Ornstein–Uhlenbeck process. Preprint. Available at SSRN: https://ssrn.com/abstract=2107686 or https://doi.org/10.2139/ssrn.2107686

Borodin, W. H., & Salminen, P. (2002). Handbook of Brownian motion-facts and formulae (2nd ed.). Basel: Birkhäuser.

Brezis, H. (2011). Functional analysis, Sobolev spaces and partial differential equations. Springer: Universitext.

Cadenillas, A., & Huamán-Aguilar, (2016). Explicit formula for the optimal government debt ceiling. Annals of Operations Research, 247(2), 415–449.

Cadenillas, A., Sarkar, S., & Zapatero, F. (2007). Optimal dividend policy with mean-reverting cash reservoir. Mathematical Finance, 17(1), 81–109.

Cadenillas, A., & Zapatero, F. (1999). Optimal central bank intervention in the foreign exchange market. Journal of Economic Theory, 87, 218–242.

Cadenillas, A., & Zapatero, F. (2000). Classical and impulse stochastic control of the exchange rate using interest rates and reserves. Mathematical Finance, 10, 141–156.

Carole, B., Cui, Z., & McLeish, D. (2017). On the martingale property in stochastic volatility models based on time-homogeneous diffusions. Mathematical Finance, 27(1), 194–223.

Chow, P. L., Menaldi, J. L., & Robin, M. (1985). Additive control of stochastic linear systems with finite horizon. SIAM Journal on Control and Optimization, 23(6), 858–899.

Dayanik, S. (2008). Optimal stopping of linear diffusions with random discounting. Mathematics of Operations Research, 33(3), 645–661.

Dayanik, S., & Karatzas, I. (2003). On the optimal stopping problem for one-dimensional diffusions. Stochastic Processes and Their Applications, 107(2), 173–212.

De Angelis, T., & Ferrari, G. (2014). A stochastic partially reversible investment problem on a finite-time horizon: free-boundary analysis. Stochastic Processes and Their Applications, 124, 4080–4119.

De Jong, F., Drost, F. C., & Werker, B. J. M. (2001). A jump-diffusion model for exchange rates in a target zone. Statistica Neerlandica, 55(3), 270–300.

Dynkin, E. B. (1969). Game variant of a problem on optimal stopping. Soviet Mathematics-Doklady, 10, 270–274.

Ferrari, G., & Koch, T. (2019). An optimal extraction problem with price impact. Applied Mathematics and Optimization. https://doi.org/10.1007/s00245-019-09615-9

Fleming, W. H., & Soner, H. M. (2005). Controlled Markov processes and viscosity solutions (2nd ed.). Berlin: Springer.
Guo, X., & Pham, H. (2005). Optimal partially reversible investment with entry decision and general production function. *Stochastic Processes and Their Applications, 115*, 705–736.

Guo, X., & Tomecek, P. (2008). Connections between singular control and optimal switching. *SIAM Journal on Control and Optimization, 47*(1), 421–443.

Harrison, M., & Taksar, M. I. (1983). Instantaneous control of Brownian motion. *Mathematics of Operations Research, 8*(3), 439–453.

Hong Kong Monetary Authority. *Linked exchange rate system*. http://www.hkma.gov.hk/eng/key-functions/monetary-stability/linked-exchange-rate-system.shtml

Huamán-Aguilar, R. (2015). *Stochastic control for optimal government debt management*. Ph.D. Thesis, University of Alberta.

Jeanblanc, M., Yor, M., & Chesney, M. (2009). *Mathematical methods for financial markets*. Berlin: Springer.

Jeanblanc-Picqué, M. (1993). Impulse control method and exchange rate. *Mathematical Finance, 3*, 161–177.

Jørgensen, B., & Mikkelsen, H. O. (1996). An arbitrage free trilateral target zone model. *Journal of International Money and Finance, 15*(1), 117–134.

Karatzas, I., & Shreve, S. E. (1991). *Brownian motion and stochastic calculus* (2nd ed.)., Graduate Texts in Mathematics 113 New York: Springer.

Karatzas, I., & Wang, H. (2005). Connections between bounded-variation control and Dynkin games. In J. L. Menaldi, A. Sulem, & E. Rofman (Eds.), *Optimal control and partial differential equations* (pp. 353–362), Volume in honor of Professor Alain Bensoussan’s 60th birthday Amsterdam: IOS Press.

Krugman, P. R. (1991). Target zones and exchange rate dynamics. *The Quarterly Journal of Economics, 106*(3), 669–682.

Larsen, K. S., & Sørensen, M. (2007). Diffusion models for exchange rates in a target zone. *Mathematical Finance, 17*(2), 285–306.

Lloyd, C. (2015). *On the end of the EUR CHF peg*. SNBCHF.com, February 6, 2015 https://snbchf.com/chf/colin-lloyd-end-eur-chf-peg/

Lon, P. C., & Zervos, M. (2011). A model for optimally advertising and launching a product. *Mathematics of Operations Research, 36*, 363–376.

Matomäki, P. (2012). On solvability of a two-sided singular control problem. *Mathematical Methods of Operations Research, 76*, 239–271.

Meyer, P. A. (1976). *Lecture notes in mathematics 511*. Séminaire de Probabilités X, Université de Strasbourg, New York: Springer.

Mijatovic, A., & Urusov, M. (2012). On the martingale property of certain local martingales. *Probability Theory and Related Fields, 152*(1), 1–30.

Mikkelsen, O. (2017). *Denmark’s fixed exchange rate policy: 30th anniversary of unchanged central rate*. News—Danmarks Nationalbank, January 2017 no. 1. http://www.nationalbanken.dk/en/publications/Pages/2017/01/Denmark’s-fixed-exchange-rate-policy-30th-anniversary-of-unchanged-central-rate.aspx

Mundaca, G., & Øksendal, B. (1998). Optimal stochastic intervention control with application to the exchange rate. *Journal of Mathematical Economics, 29*, 223–241.

Øksendal, B. (2003). *Stochastic differential equations* (6th ed.), Berlin: Springer.

Perera, S., Buckley, W., & Long, H. (2018). Market-reaction-adjusted optimal central bank intervention policy in a forex market with jumps. *Annals of Operations Research, 262*(1), 213–238.

Protter, P. (1990). *Stochastic integration and differential equations*. Berlin: Springer.

Shreve, S.E. (1988). An introduction to singular stochastic control. In W. Fleming, P.L. Lions (Eds.), *Stochastic differential systems, stochastic control theory and applications,IMA* (Vol. 10). New York: Springer.

Shreve, S. E., Lehoczky, J. P., & Gaver, D. P. (1984). Optimal consumption for general diffusions with absorbing and reflecting barriers. *SIAM Journal on Control and Optimization, 22*(1), 55–75.

Sweeney, R. J. (2006). Mean reversion in G-10 nominal exchange rates. *Journal of Financial and Quantitative Analysis, 41*(3), 685–708.

Swiss Central Bank Acts to Put a Cap on Franc’s Rise. The New York Times, September 6, 2011. http://www.nytimes.com/2011/09/07/business/global/swiss-franc.html

Taksar, M. I. (1985). Average optimal singular control and a related stopping problem. *Mathematics of Operations Research, 10*(1), 63–81.

Tanaka, H. (1979). Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Mathematical Journal, 9*, 163–177.

The Economist explains: Why the Swiss unpegged the franc. The Economist, January 18, 2015. http://www.economist.com/blogs/economist-explains/2015/01/economist-explains-13

Tvedt, J. (2012). Small open economies and mean reverting nominal exchange rates. *Australian Economic Papers, 51*(2), 85–95.
Yang, X., Ren, G., Wang, Y., Bo, L., & Li, D. (2016). *Modeling the exchange rates in a target zone by reflected Ornstein–Uhlenbeck process*. Perprint. Available at SSRN, https://ssrn.com/abstract=2107686 or https://doi.org/10.2139/ssrn.2107686.

Zhu, H. (1992). Generalized solution in singular stochastic control: The nondegenerate problem. *Applied Mathematics and Optimization*, 25(3), 225–245.

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