Constraints on shear stress tensor in viscous relativistic hydrodynamics

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We extend our hybrid model HydHSD by taking into account shear viscosity within the Israel-Stewart hydrodynamics. The influence of different forms of \( \pi_{\mu\nu} \) constraints on observables is analyzed. We show that the form of the corresponding condition plays an important role for the sensitivity of viscous hydrodynamics to the ratio of shear viscosity to the entropy density, \( \eta/s \). It is shown that the constraint used in the vHLLE model, results in most sensitivity of rapidity distributions and transverse momentum spectra to a change of the \( \eta/s \) ratio; however, their applicability for large values of \( \eta/s \) is doubtful. On the contrary, the strict constraints from [1] are very strong but most established. We also found that \( \eta/s \) as a function of the collision energy probably has an extremum at \( E_{lab} = 10.7 \mathrm{AGeV} \). However, we obtain that any considered condition does not allow to reproduce simultaneously pion and proton experimental data within our model.

PACS numbers: 24.10.Nz,25.75.-q, 25.75.Dw, 47.75.+f

Keywords: heavy ion collisions, viscous relativistic hydrodynamics

I. INTRODUCTION

Hydrodynamics is a powerful phenomenological tool having a variety of wonderful properties. It allows one to take easily into account collective effects and the equation of state (EoS) of studied matter which cannot be completely described by microscopic models. Application of hydrodynamics to theoretical description of high-energy nuclear collisions has been started with Landau’s original work [2]. The actual status and successful story of hydrodynamics approach in ultra-relativistic heavy-ion collision theory is reflected in review articles [3–8].

A problem of heavy ion collision modeling is that hydrodynamics applicability conditions are violated at the early and final stages of nuclear interaction. The main condition assumes that the mean free path of quasiparticles in a system has to be smaller than the system size. It is clear that this condition is not satisfied in dilute matter at the beginning and the end of a collision when medium, as a result, is far from the local equilibrium.

One way to get around the mentioned problem is to construct a hybrid model. Within hybrid models, one of which we developed in [9], the initial conditions for hydrodynamic equations, i.e. space distributions of the energy density, charge density, and velocity field, are calculated using the kinetic model.

The previous version of our HydHSD hybrid model [9] includes ideal hydrodynamics as a part. More realistic calculations need to take into account non-zero viscosity of QCD matter which is the aim of this paper.

The article is organized as follows. We start with the description of the set of viscous hydrodynamic equations in Sec. II A and how it is solved numerically, see Sec. II B. Sec. II C is devoted to obtaining the initial conditions. In Sec. II D the particlization procedure used for observable calculation is shortly formulated. Some words about the EoS can be found in Sec. II E. We continue the consideration in Sec. III where we discussed how our model depends on the parameters. A special attention is paid to the constraints on the shear stress tensor, see Sec. III C. Our final results are presented in Sec. IV. Technical details of our numerical algorithm are given in Appendices.

II. THE MODEL

A. Equations of viscous hydrodynamics

The system undergoing hydrodynamic evolution is described by the set of equations [10]

\[
\begin{align}
\partial_\mu T^{\mu\nu} &= 0, \\
\partial_\mu J^{\mu} &= 0,
\end{align}
\]

involving an energy-momentum tensor \( T^{\mu\nu} \) and a baryon current \( J^{\mu} \), and representing the conservation laws of the total energy, momentum, and baryon charge. Here and below we will assume the Cartesian coordinates. In the general case of a non-ideal fluid when dissipation processes are possible, the energy-momentum tensor and the...
baryon current can be cast in the form 10

\[ T^{\mu\nu} = T^{\mu\nu}_{\text{id}} - \Pi \Delta^{\mu\nu} + \pi^{\mu\nu}, \]
\[ J^\mu = n u^\mu + V^\mu, \tag{1} \]
\[ \tau^{\mu\nu}_{\text{id}} = \varepsilon u^\mu u^\nu - \frac{1}{2} \Delta^{\mu\nu}, \tag{2} \]
\[ \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \tag{3} \]

where \( T^{\mu\nu}_{\text{id}} \) is the ideal part of the energy-momentum tensor, \( \varepsilon, n, \) and \( P \) are the energy density, the baryon density, and the pressure in the local reference frame (LRF), respectively. The full energy-momentum tensor contains additional terms: the bulk pressure \( P \), the shear stress tensor \( \pi^{\mu\nu} \), and the baryon (charge) diffusion current \( V^\mu \). The 4-velocity \( u^\mu \) is defined here as an eigenvector of the full energy-density tensor \( T^\mu_\lambda u_\lambda = \varepsilon u^\mu \) (the Landau definition). It is normalized as \( u^\lambda u^\lambda = 1 \) and can be written as \( u^\mu = \gamma(1, \vec{v}) \) through the 3-velocity \( \vec{v} \) and \( \gamma = (1 - v^2)^{-1/2} \). From this definition of the flow velocity it follows that \( \pi^{\mu\nu} \) is a traceless symmetric tensor satisfying the orthogonality relations:

\[ u_\mu \pi^{\mu\nu} = 0, \quad \pi^{\mu\nu} = \pi^{\nu\mu}, \quad \pi^{\mu\mu} = 0. \tag{4} \]

Equations (1) have to be supplemented by the EoS \( P = P(\varepsilon, n) \). If one considers a perfect fluid and puts \( \pi^{\mu\nu} = 0 \), \( \Pi = 0 \), and \( V^\nu = 0 \), then the system of equations becomes closed and can be solved with respect to \( T^{\mu\nu}_{\text{id}} \) and \( J^0 \) taken as independent variables. In the viscous case, however, we need some additional equations for \( \pi^{\mu\nu} \), \( \Pi \), and \( V^\mu \) which are now independent dynamical variables. Below, for simplicity, we neglect the heat flux, i.e., \( V^\nu = 0 \) is assumed. Note that in this case the Landau and Eckart frames coincide.

Studies performed in 11 show that there can be infinitely many choices for the explicit form and coefficients in the equations of motion for \( \pi^{\mu\nu} \) and \( \Pi \). In viscous fluid dynamics, being applied to heavy ion collisions, works 1,14 usually takes the \( J^0 \) and \( T^{\mu\nu}_{\text{id}} \) components of the energy-momentum tensor. In this case, the reconstruction of the LRF quantities such as energy and baryon densities and the 3-velocity of the fluid cell becomes a complicated problem. Instead, we will use the components of the ideal-fluid tensor \( T^{\mu\nu}_{\text{id}} \) as independent variables, which allows us to apply relations (A1), (A2), and (A3) without a problem. Then evolution of \( T^{\mu\nu}_{\text{id}} \) is described by the equation

\[ \partial_\tau T^{\mu\nu}_{\text{id}} = -\partial_\mu \pi^{\nu\tau}, \tag{5} \]

which is just a rewriting of Eq. (4).

As follows from Eq. (5), only five components of the \( \pi^{\mu\nu} \) tensor are independent. The other can be reconstructed if the cell velocity is known. However, for some choices of this five component set, the reconstructed components can contain a singularity if an element of the vector \( \vec{v} \) vanishes 1. We select \( \pi^{00}, \pi^{zz}, \pi^{z0}, \pi^{zz}, \) and \( \pi^{yy} \) as independent ones in our implementation of the algorithm. As one can see from Eq. (A4), a singularity is absent for the such choice if \( v < 1 \).

Equations (1b) and (5) can be rewritten in the form

\[ \partial_\tau \vec{U}_{\text{cons}} + \sum_i \partial_i(n_i \vec{U}_{\text{cons}}) = \vec{S}_{\text{cons}} \tag{6} \]

where we introduced a 5-dimensional vector for generalized densities

\[ \vec{U}_{\text{cons}} = (J^0, T^{00}_{\text{id}}, T^{0z}_{\text{id}}, T^{0y}_{\text{id}}, T^{0z}_{\text{id}})^T \]

in calculations of the total entropy.

\[ s^0 = \left( s - \frac{\tau_\pi}{4T \eta} \pi^{\mu\nu} \pi_{\mu\nu} - \frac{\tau_\Pi}{2T \zeta} \Pi^2 \right) u^0 \tag{7} \]

where \( k_\eta = \text{const} \) and the entropy density \( s = s(\varepsilon, n) \) is given by the EoS.

Finally, we quote the expression for the entropy. We need only zero component of the entropy 4-vector, \( s^0 \), which reads for a cell as

\[ \eta = k_\eta s, \quad \tau_\pi = \frac{5\eta}{\varepsilon + P}, \tag{8} \]
and the corresponding sources $S'_{\text{cons}}$, see Appendix A. This set of equations is solved numerically by means of the SHASTA (the SHarp and Smooth Transport Algorithm) algorithm [15, 16]. In this article, we follow the numerical scheme outlined in Section 4.2 of Ref. [1] extending it to $3 + 1$ dimensions. The corresponding formulae are collected in Appendix B.

In principle, the relaxation equations (6) can also be solved in a similar way but as shown in [15], the algorithm becomes more stable if one uses a simple centered second-order difference scheme for spatial gradients on the left-hand side of Eqs. (6). So, the shear stress tensor components, $\pi = (\pi^{xy}, \pi^{xz}, \pi^{yz}, \pi^{yy}, \pi^{zz})^T$, are propagated according to the described simple scheme.

In some cells the relaxation time $\tau_\pi$ given by Eq. (7) may become smaller than the calculation time step. Then, following the idea from Section 3.2 of Ref. [17], we evolve $\pi^{\mu\nu}$ using the formal solution of Eq. (6):

$$\pi^{\mu\nu}(t_{n+1}) = [\pi^{\mu\nu}(t_n) - \eta W^{\mu\nu}]e^{-\Delta t/(\gamma \tau_\pi) + \eta W^{\mu\nu}}.$$  

(11)

It is important that at each calculation step we have to ensure the applicability of the hydrodynamic equations. We have to be sure that viscous effect are kept only as corrections to the ideal fluid energy-momentum tensor. Therefore, at each time step in each cell we calculate the ratio

$$q = q_S = \max_{\mu,\nu} \frac{|\pi^{\mu\nu}|}{|T_{\text{id}}^{\mu\nu}|}, \quad \text{(S-cond.)}$$  

(12)

and verify the fulfillment of the condition [1]

$$q < C,$$  

(13)

where $C$ is a predefined positive constant, $C < 1$. If the opposite occurs we rescale the shear stress tensor as

$$\pi^{\mu\nu} \rightarrow \tau^{\mu\nu}_{\text{corr}} = \pi^{\mu\nu} C$$  

(14)

Such a rescaling prescription is frequently used in the literature [17, 20]; however, there are differences in how the quantity $q$ is evaluated. This aspect will be considered in detail in Section III C. The condition (13) evaluated with $q$ from Eq. (12) will be denoted as the strict (S-) condition. By default we use $C = 0.3$.

To verify our hydrodynamical code, we performed a test similar to that proposed in Ref. [1], namely, we solved numerically the $(1+1)$-dimensional Riemann problem for two states with constant pressure (or energy density since $p = \varepsilon/3$) equal to $p_0$ on one side and to 0 (vacuum) on the other side separated by a membrane located at $x = 0$. The evolution of energy density and velocity profiles is presented in Figs. 1a and 1b, respectively. We used here $\Delta x = 0.2$ fm and put very small shear viscosity, $\eta/s \approx 0.01$ to simulate numerically a viscous free flow. We see a good agreement of numerical solutions with analytical ones.

However, the price of imposing the constraint (13) by hand is that the total energy and total entropy of the system become non-monotonic functions of time, see Fig. 2. The total energy fluctuates around a constant value while the entropy oscillates about a slowly increasing average value. The reason of such behaviour is the rescaling procedure (14). Note that the total energy is a constant at the beginning, since we start with $\pi^{\mu\nu} = 0$ and no $\pi$-rescaling is needed for the first several time steps. Small initial entropy decreasing is just due to numerical inaccuracy.

We also checked our algorithm by 1D boost-invariant expansion [22].

Let us remind that in our previous work [9] we solved equations of ideal hydrodynamics by the ‘phoenical’ version of SHASTA [16] and used the operator-splitting method to treat three-dimensional operators. Our new code implements the ‘explicit’ SHASTA [16] and straightforwardly solves the 3D problem. So, as one additional test...

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1. Eq. (11) is applied before the antidiffusion step.
of our new code, we can compare proton rapidity distributions calculated within ideal hydrodynamics by old ‘splitting’ and new versions of SHASTA. The corresponding results for the Au+Au collision at \(E_{\text{lab}} = 10.7\) A\(\text{GeV}\) with the freeze-out temperature \(T_{\text{fo}} = 100\) MeV are shown in Fig. 2. One sees that if we use the default value of the so-called mask coefficient \(A_{\text{ad}} = 1\), the rapidity distributions essentially differ from each other at intermediate rapidities where the new code leads to humps. We need to note that such differences disappear with growing \(T_{\text{fo}}\) and are caused by too large antidiffusion in peripheral cells. The latter conclusion is confirmed by Fig. 3 where the humps are suppressed if \(A_{\text{ad}}\) decreases until \(A_{\text{ad}} = 0.6 - 0.5\).

Large antidiffusion in the "explicit" SHASTA was early mentioned in \cite{23} where authors proposed to set \(A_{\text{ad}}\) to be proportional to \(1/((k/\epsilon)^2 + 1)\), where \(k\) is some small constant of order \(10^{-5}\) GeV/fm\(^3\). In this way, \(A_{\text{ad}}\) goes smoothly to zero near the boundaries of the grid, i.e., we increase the amount of numerical diffusion in that region \cite{23}. However, such choice does not affect the solution while we have a more complicated problem and need to suppress humps. For our calculations, we take \(A_{\text{ad}} = 0.6\) which noticeably decrease the proton yield in the fragmentation region, see Fig. 3.

C. Initialization of hydrodynamic evolution

The differential equations of hydrodynamics must be supplemented by appropriate initial conditions. In hybrid models these conditions are usually deduced from

\[ T^\mu_\text{init}(\vec{r}) = \sum_a \frac{p_a^\mu}{p_a^0} K(\vec{r} - \vec{r}_a), \]

\[ J^\mu_\text{init}(\vec{r}) = \sum_a \frac{p_a^\mu}{p_a^0} K(\vec{r} - \vec{r}_a), \] (15)

where the bar stands for the event averaging and the sum runs over particles at the positions \(\vec{r}_a\), \(K(\vec{r})\) is the smoothing function which in our case performs just averaging in the volume element, \(\Delta V\)

\[ K(\vec{r}) = \begin{cases} 1/\Delta V, & \vec{r} \in \Delta V \\ 0, & \vec{r} \notin \Delta V \end{cases}. \] (16)

Assuming the \(T^\mu_\text{init}\) structure as for an ideal fluid, see Eq. (3) and the absence of the baryon diffusion current, \(V^\mu = 0\), we obtain from quantities (15) the initial energy density, \(\epsilon_\text{init}\), and the baryon density, \(n_\text{init}\), in a fluid cell

\[ \text{Particularly we use version 1.0 of the Parton-Hadron String Dynamics model with the switched off partonic option.} \]
and the cell velocity, \( \vec{v} \), with the help of relations \( \text{A1} \), \( \text{A2} \), and \( \text{A3} \). Now we can evaluate the initial entropy and other thermodynamical quantities in each cell using the equation of state, e.g., \( s_{\text{init}} = s(\epsilon_{\text{init}}, n_{\text{init}}) \), cf. Eq. [8]. The initial total entropy and the baryon number of the system is calculated as

\[
\begin{align*}
S_{\text{init}} &= \sum_{\text{cell}} \frac{s_{\text{init}}}{\sqrt{1 - v^2}}, \\
N_{\text{init}} &= \sum_{\text{cell}} \frac{n_{\text{init}}}{\sqrt{1 - v^2}}.
\end{align*}
\tag{17}
\]

A transition from a kinetic to a hydrodynamic regime occurs at an instant \( t_{\text{start}} \). We assume that at this moment the system is close to equilibrium and the ratio of the entropy to the baryon number \( S(t)/N(t) \) ceases changing, see Fig. 1 in Ref. [9]. At the beginning of the hydrodynamical stage all components of the shear-stress tensor are always initialized with zero values, which is found to be a useful approximation in the literature.

Below we consider the following heavy-ion collisions: \( \text{Au+Au} \) collisions for AGS energies at \( E_{\text{lab}} = 6 \) and 10.7 \( \text{AGeV} \), and \( \text{Pb+Pb} \) collisions for SPS energies at \( E_{\text{lab}} = 40, 80, \) and 158 \( \text{AGeV} \). All calculations are performed for the impact parameter \( b = 1 \text{ fm} \). The corresponding transition times from HSD to hydrodynamics, \( t_{\text{start}} \), are shown in Table I together with the corresponding starting entropy. These times are not changed comparing with our previous work [3].

### Table I. Starting times of hydrodynamical calculations

| \( E_{\text{lab}} \) [\( \text{AGeV} \)] | \( 6 \) | \( 10.7 \) | \( 40 \) | \( 80 \) | \( 158 \) |
|-----------------|-----|-----|-----|-----|-----|
| \( t_{\text{start}} \) [\( \text{fm/c} \)] | \( 7.9 \) | \( 7.18 \) | \( 4.57 \) | \( 3.8 \) | \( 3.01 \) |
| \( S_{\text{start}} \times 10^{-3} \) | \( 3.5 \) | \( 4.4 \) | \( 7.1 \) | \( 8.7 \) | \( 10.6 \) |

D. Particilization procedure and observables

To convert fluids to particles, we realized a particilization procedure according to the Cooper-Frye formule\( \text{A}[8] \):

\[
E \frac{d^3N_a}{dp^3} = \frac{g_a}{(2\pi)^3} \int d\sigma_{\mu} p^\mu f_a(x,p),
\tag{18}
\]

where \( p^\mu = (E, \vec{p}) \) is the particle 4-momentum, \( f_a(x,p) \) represents the distribution function (Wigner function) of the particle of type ‘a’ and \( g_a \) is the corresponding spin-isospin degeneracy factor, \( d\sigma_{\mu} = n_{\mu} d^3\sigma \) is an element of the space-time freeze-out hypersurface with the normal \( n_{\mu} \). The freeze-out hypersurface, as in the previous work [6], is determined with the help of the CORNELIUS algorithm [28].

In the ideal fluid case, the particle distribution function is given by usual the Fermi/Bose distribution

\[
f_a^{(0)}(x,p) = \frac{1}{e^{\beta(p^\mu n_{\mu}(x) - \mu_a(x))} + 1},
\tag{19}
\]

where \( \beta = 1/T \) is the inverse local temperature, \( \mu_a \) is the chemical potential of the particle of type a (Recall that the Coulomb interaction is neglected and all particles within in a given isospin multiplet have the same chemical potential). The plus and minus signs correspond to fermions and bosons, respectively. For viscous fluids, one has to take into account the modification of the distribution function because of non-equilibrium viscous effects. The common way is to approximate the distribution function by the following expression [18]:

\[
f_a(x,p) = f_a^{(0)}(x,p) \times \left[ 1 + (1 \mp f_a^{(0)}(x,p)) \frac{p_T p_v \pi^{\mu \nu}}{2T^2(\varepsilon + P)} \right].
\tag{20}
\]

We use exactly the same methods of particle momentum generation [29, 30] as described in [18]. To use it, one has to convert \( \pi^{\mu \nu} \) to the LRF. Due to the orthogonality relations [17] are explicitly fulfilled in our code, we have \( \pi^{\mu \nu} = 0 \) where the asterisk refers to the LRF.

Another difference, in comparison to the ideal hydrodynamics, is that to apply the rejection procedure, one has to know the upper limit of the viscous correction factor. So one assumes that the square brackets in Eq. [20] should not be larger than 2, since the viscous term has to be only a small correction and, definitively, cannot be larger than unity.

To calculate the proton fraction among nucleons, we use isospin factor 1/2 while for pions 1/3.

After generating “thermal” contributions by Eq. [18] and [20], resonance decays are taken into account in the zero-width approximation.

E. Equation of state

The used EOS [31] includes all known hadrons with masses up to 2 GeV in the zero-width approximation. The equation of state of hadron resonance gas at finite temperature and baryon density is calculated thermodynamically taking into account a density-dependent mean field that guarantees the nuclear matter saturation.

To account for mean-field effects, an effective potential \( U = U(n) \) acting on a hadron is introduced. It depends only on the baryon density, \( n \), and does not depend on momenta of interacting baryons. Then the baryons single-particle energy can be obtained simply by adding \( U(n) \) to the kinetic energy. In this case, the partition function of the hadronic system can be calculated analytically [32]. As the result, the following expressions for thermodynamic functions of the hadron EoS can be written:

\[
P = \sum_a P_a(T, \mu^*, \mu_S) + P_f(n),
\tag{21}
\]

\[
\varepsilon = \sum_a \varepsilon_a(T, \mu^*, \mu_S) + \varepsilon_f(n),
\tag{22}
\]
where the effective baryon chemical potential, $\mu^*$, is obtained by the shift $\mu^* = \mu_B - U(n)$. The “field” contributions (marked by index ‘F’) to the energy density and pressure are found as

$$\epsilon(n) = n U(n) - P_f(n) = \int_0^n dn_1 U(n_1).$$

In this approach meson contributions are given by ideal gas expressions.

The mean-field potential is parameterized in a line with the Skyrme approach as $U(n) = \alpha n/n_0 + \beta (n/n_0)^2$, where $n_0$ is the saturation density of nuclear matter and $\alpha, \beta, \gamma = \text{const}$. In the following, we fix $\gamma = 7/6$ and choose the remaining parameters from the requirements $P(T = 0, n_0) = 0$, $\varepsilon(T = 0, n_0)/n_0 = E_b + m_N$ where the binding energy $E_b = -16$ MeV and $n_0 = 0.15$ fm$^{-3}$.

For more details on the EOS, see Ref. [31].

### III. Influence of Model Parameters on Momentum Spectra

In this section we consider how a variation of the hydrodynamic model parameters can manifest itself in rapidity ($y$) distributions and transverse momentum ($m_T$) spectra at $y = 0$ of protons and pions. To be specific, we consider Pb+Pb collisions at 40 AGeV.

#### A. Shear viscosity

First of all, let us compare proton and pion $y$- and $m_T$-distributions evaluated for viscous and ideal hydrodynamics. The results are collected in Fig. 4. One expects that calculations with a very small value of $\eta/s = 0.01$ have to be very close to ideal-hydro calculations. However, Figs. 4a and 4b demonstrate that even such a small viscosity changes visibly the proton and pion rapidity distributions. For the $m_T$ spectra the difference between calculations with $\eta/s = 0$ and $\eta/s = 0.01$ are small, as we see in panels (c) and (d) of Fig. 4. An increase in the viscosity up to $\eta/s = 0.1$ leads to sizable changes in the rapidity distributions, see the dashed lines in Figs. 4a and 4b. Particularly, the two-hump structure in the proton rapidity distribution becomes much more pronounced for the viscous case. It occurs because the shear viscosity slows the fireball longitudinal expansion and the fluid velocity, which is reflected in the form of the $y$-distribution. It narrows of the proton rapidity distributions and because of the baryon number conservation, the narrowing leads to an increase of the hump height. At the same time, the height of proton distribution at mid-rapidity ($y = 0$) is almost independent of $\eta/s$. Oppositely, the viscous corrections make the pion rapidity distributions higher than in the ideal case, see Fig. 4b, as was anticipated in Ref. [9].

The transverse momentum spectra of pions and protons show a very weak dependence on the $\eta/s$ value, see Figs. 4a and 4b, especially for pions. The inclusion of viscosity leads to a slight increase of slopes of the $m_T$ spectra which makes the spectra closer to the experimental data. It can be considered as an additional argument for the necessity of non-zero shear viscosity.

The slope of the calculated pion $m_T$ spectrum roughly agrees with the experimental one, while for protons the calculated spectrum is too steep and underestimates the data for $m_T - m_N > 120$ MeV.

Remarkably, in Fig. 4 we observe saturation of the viscosity effects with an increase in the $\eta/s$ ratio; indeed the lines calculated for $\eta/s = 0.2$ and 0.5 are barely distinguishable. This is because of the strict constraint on the $\pi\mu^\nu$ tensor [13] with [12], which we apply in our calculations, and the large gradients appearing in collisions at this energy. As we will see below, this saturation effect is specific for quite high energies and, for example, the sensitivity to $\eta/s$ is higher for collisions at 6 AGeV, see Fig. 9 below. However, this property together with the not too large increase of the pion rapidity distribution height leads to that the experimental data for $E_{lab} = 40$ AGeV still cannot be reproduced.

#### B. Freeze-out temperature

The influence of the freeze-out temperature, $T_{f.o.}$, on rapidity distributions and transverse momentum spectra

![Fig. 4. Rapidity distributions and transverse momentum spectra for protons and pions produced in Pb+Pb collisions at 40 AGeV in comparison with the calculations of the HydHSD model for various values of the $\eta/s$ ratio. The freeze-out temperature is fixed at $T_{f.o.} = 160$ MeV. Experimental points are taken from Refs. [32, 33].](image)
at mid-rapidity is illustrated in Fig. 5. As is seen in the figure, proton rapidity distributions become higher if

\[ dN/dy \]

the freeze-out temperature \( T_{f.o.} \) is lower. The value of \( dN/dy \) at \( y = 0 \) is moderately sensitive to \( T_{f.o.} \) as well as to \( \eta/s \), as we discussed above. This value is mainly determined by the hydrodynamics start time, \( t_{\text{start}} \). Confronting Fig. 5a and Fig. 4a, we observe a similarity in the effects caused by a decrease of \( T_{f.o.} \) and an increase of \( \eta/s \), both lead to a growth of the humps in the proton \( y \)-distribution. The reason of that is an increase of the evolution duration in both cases and a correlation of the parameters. A larger viscosity leads to a slower drop of the cell temperatures which increases the number of frozen cells in similar way as a choice of a smaller freeze-out temperature, see Fig. 11 below and the corresponding discussion. In the viscous case the two-hump structure of proton rapidity distribution at \( E_{\text{lab}} = 40 \text{AGeV} \) is clearly seen for any \( T_{f.o.} \) (compare with Fig. 4b from 4).

The width of the pion rapidity distribution is larger for smaller values of \( T_{f.o.} \), as is seen in Fig. 5b, whereas the distribution height is weakly dependent on \( T_{f.o.} \). Also we observe saturation of the height of the pion \( y \)-distribution with decreasing \( T_{f.o.} \) similar to the dependence on \( \eta/s \). As can be seen in Figs. 4b, 5a, and 5b, the height of the pion rapidity distribution saturates at the level which is significantly below the experimental data at mid-rapidity. As a result, our calculations for \( E_{\text{lab}} = 40 \text{AGeV} \) can reproduce only the proton rapidity distribution but not the pion one if we vary both \( \eta/s \) and \( T_{f.o.} \) parameters.

Irrespective of the saturation of the pion distribution height with a decrease of \( T_{f.o.} \), Fig. 4 demonstrates that there is a internal tension in attempts to describe simultaneously the proton and pion rapidity distributions in our model. Let us show how it can be explained, at hand of the results for \( E_{\text{lab}} = 40 \text{AGeV} \). The reason of this failure in reproducing both distributions is the discussed-above insensitivity of the \( y \)-distributions for \( \eta/s > 0.1 \). Therefore, after the increase in the distribution by the variation of \( \eta/s \) is exhausted, we have only one parameter \( T_{f.o.} \) to tune both proton and pion distributions. So for a freeze-out temperature, \( 140 \text{MeV} \lesssim T_{f.o.} \lesssim 160 \text{MeV} \), which is needed to fit the proton rapidity distribution, we have only the correct width of the pion distribution.

Transverse momentum spectra of protons and pions at mid-rapidity (\( y = 0 \)) are shown in Figs. 5c and 5d, respectively, for various values of freeze-out temperature. The striking feature is that the slope of the pion spectra is almost insensitive to the variation of \( T_{f.o.} \), and the proton spectra demonstrate very weak dependence, whereby the slope’s steepness decreases with a \( T_{f.o.} \). As the result, to approach experimental data for the proton spectrum we have to choose \( T_{f.o.} < 120 \text{MeV} \), whereas \( T_{f.o.} \simeq 150 \text{MeV} \) is necessary to describe the slope of the proton rapidity distribution. In contrast, the pion \( m_T \) spectrum is well described by our model.

C. Constraints on the shear stress tensor

The above results lead to two questions. Why viscous effects in our 2-stage hybrid model for pion rapidity distribution are so small (~10%) while the results of the authors [18] within the vHLL+UrQMD model demonstrate that the response is large (about 20%, see Fig. 4 in the cited article)? It cannot be explained by taking into account the electric charge conservation since this effect is included in both ideal and viscous versions of the model [18]. The second question is why our model is insensitive to the \( \eta/s \) value at \( E_{\text{lab}} = 40 \text{AGeV} \), if \( \eta/s > 0.1 \), see Fig. 4 above.

First of all, we check how the viscous response is changed if the constant \( C \) in Eq. (13) is increased. When \( C \) is larger, the viscous effects are expected to be more pronounced. As Fig. 6 shows, this is indeed the case. From Fig. 6b we see that a two-hump structure in proton rapidity distributions is more pronounced for larger values of \( C \)-parameter, while Fig. 6d demonstrates that simultaneously pion rapidity distribution is getting a bit higher but the gain at mid-rapidity is too small to improve noticeably the agreement with the experiment.

Second, we have to note that the vHLL+ model [17] uses another constraint on the \( \pi^{\mu
u} \) tensor magnitude using the criterion (13) where the quantity \( q \) is calculated
We will call it the M-condition and also consider the condition which is applied in the LLE model as the V-condition. For a further comparison with the calculations with the stricter constraint (12). We will denote the condition used in the vH-LLE model as the V-condition. It results in a weaker condition than with our definition (12). As one can see from the function QuestRevert of MUSIC code, the developers use an energy-dependent cut-off parameter $C = C(\varepsilon)$ in Eq. (13). We take just a constant value.

FIG. 6. Proton (a) and pion (b) rapidity distributions for two values of $C$-parameter in the shear-stress tensor constraint (12) at $T_{\text{lab}} = 160$ MeV and $\eta/s = 0.1$. The experimental data are the same as in Figs. 3 and 5.

FIG. 7. Rapidity distributions of protons (a) and pions (b) for Pb+Pb collisions at $E_{\text{lab}} = 40$ AGeV for different choices of $\pi^{\mu\nu}$ constraints (12), (23), and (24). Calculations are carried out for the freeze-out temperatures $T_{\text{lab}} = 160$ MeV and two values of the shear viscosity $\eta/s = 0.1$ and 0.5. The experimental data are the same as in Figs. 3 and 5.

FIG. 8. Evolutions of the total entropy and total energy for different choices of $\pi^{\mu\nu}$ constraints for Pb+Pb collisions at $E_{\text{lab}} = 40$ AGeV with $T_{\text{lab}} = 160$ MeV for $\eta/s = 0.1$ and 0.5. As one can see from the function QuestRevert of MUSIC code, the developers use an energy-dependent cut-off parameter $C = C(\varepsilon)$ in Eq. (13). We take just a constant value.

\[ q = q_{\text{V}} \equiv \frac{\max_{\mu,\nu} |\pi^{\mu\nu}|}{\max_{\mu,\nu} |T_{\text{id}}^{\mu\nu}|} \quad (\text{V-cond.}) \]  

(23)

It results in a weaker condition than with our definition (12). We will denote the condition used in the vH-LLE model as the V-condition. For a further comparison we also consider the condition which is applied in the MUSIC model (24), where one defines

\[ q = q_{\text{M}} \equiv \sqrt{\frac{\pi^{\mu\nu} \pi_{\mu\nu}}{T_{\text{id}}^{\mu\nu} T_{\text{id}}^{\mu\nu}}} \quad (\text{M-cond.}) \]  

(24)

We will call it the M-condition.

Both V- and M-conditions can be easily realized in our code. The results of calculations for Pb+Pb collisions at

\[ E_{\text{lab}} = 40 \text{ AGeV with } T_{\text{lab}} = 160 \text{ MeV, } \eta/s = 0.1 \text{ and 0.5, and } C = 0.3 \text{ is shown in panels (a) and (b) of Fig. 7 for protons and pions, respectively. One can see that applying weaker constraints, Eqs. (23) or (24), lead to a dramatic change in the rapidity distributions. For } \eta/s = 0.1 \text{ the height of the proton humps and the maximum of the pion distribution increase sizably compared to the calculations with the stricter constraint (12). Also, Fig. 7 demonstrates that the sensitivity of the rapidity spectra to the } \eta/s \text{ value is much larger for the V- and M-conditions than for the S-condition. There appears even a three-hump structure in the proton rapidity distribution for } \eta/s = 0.5 \text{ when one applies the V- or M-conditions. We see in Fig. 7 that with the weaker constraints one can reproduce the mid-rapidity dip in the pion rapidity distribution by changing the } \eta/s \text{ parameter.}

The effect of the $\pi^{\mu\nu}$ constraint relaxing in comparison to the the strict condition (12) and (13) is qualitatively similar for the V- and M-conditions; however for the V-condition the effect is more pronounced and increases strongly for a larger value of $\eta/s$. This property of two weaker conditions is found to be valid for all considered energies.

Figure 8 demonstrates that different constraints on the $\pi^{\mu\nu}$ tensor affect also the evolution of such global quantities as the total energy and the total entropy of the system. A weaker constraint leads to longer system evolution time, especially for V-condition. For the S- and M-constraints the total energy stays constant with good precision, whereas for the V-condition the total energy starts to increase after the first 5 fm/c and continues its growth reaching a $\sim 2\%$ excess at time 12 fm/c. At the same time, the total entropy of the system decreases only slightly for the S- and M-conditions, but demonstrates a violent behaviour – first a strong decrease, then a moderate increase – for the V-condition.

To better understand how the form of the constraint on the shear stress tensor affects observables, let us consider also collisions at AGS energies. The results for Au+Au collisions at $E_{\text{lab}} = 200$ AGeV with $T_{\text{lab}} = 160$ MeV and $\eta/s = 0.1$ and 0.5.
we put cosities with \( \eta/s \) do not take into account nucleon coalescence whose effect \( \eta/s \) not necessarily related to a phase transition. Ref. [9] that a two-hump structure in proton and pion collisions at \( E_{lab} = 6 \text{ AGeV} \) at three values of \( \eta/s = 0.05, 0.1, \) and 0.5 and different conditions on the \( \pi^\tau \nu \) tensor: the S-condition [22] and the M-condition [21]. Experimental points are from [34, 37].

In Fig. 11 we illustrate the role of the viscous correction term in the Cooper-Frye formula, see Eq. (20), where we present the proton and pion rapidity distributions calculated with and without the last term in square brackets in [20]. For pions this corrections are truly perturbative leading to a slight increase of the pion number at mid-rapidity. For calculations with the M-condition the effect is stronger than for those with the S-conditions and increases with the \( \eta/s \) growth. For protons, however, in calculations with \( \eta/s = 0.5 \) the correction terms in the Cooper-Frye lead to a change in the shape of the distribution making the three-hump structure more pronounced.

Why the viscous effects promoted by the weak constraint with the V- and M-conditions lead to an increase in the pion number multiplicity? To answer this question we show in Fig. 11b the evolution of the central cell temperature for calculations done with various conditions. The viscous effects prolongs the evolution and increase the temperature. Even some reheating effect for the central cells are seen for runs with V- and M-conditions. The viscous effects prolongs the evolution and increase the temperature. Even some reheating effect.
An increase of the specific viscosity results in a further increase of the volume. The combination of higher temperatures and larger freeze-out volume leads to strong increase of the number of pions (not restricted by any conservation law) if the viscous effects are constrained by V- and M-conditions.

Three different moments of time are shown by thin lines in Fig. 12. The interval $0 < t < 1$ corresponds to the distribution of the values of $q$ and $\eta/s$ for Pb+Pb collisions at $E_{lab} = 40$ AGeV performed with the M-condition. Panel (a) shows the result for $\eta/s = 0.1$ and panel (b) for $\eta/s = 0.5$. Lines are shown for three values of $t - t_{start} = 1.6, 4.8, 8.0$ fm/c. Numbers in the square brackets show probability to find a cell with $q_m(\eta)/C < 1$.

It is interesting to try to quantify to what extend the viscous effects remains perturbative in the course of hydrodynamic evolution. With this aim we run the code for Pb+Pb collision at $E_{lab} = 40$ AGeV and calculated the distribution of the values of $q_m$ among all fluid cells with temperatures $T > 100$ MeV. The normalized (in the interval $0 < q/C < 10$) distributions obtained for three different moments of time are shown by thin lines in Fig. 12 for runs with $\eta/s = 0.1$ and $\eta/s = 0.5$. We see that initially the majority of cells, $\sim 99\%$ for $\eta/s = 0.1$ and $\sim 83\%$ for $\eta/s = 0.5$, have $q_m < 1$ but already in the middle of the fireball evolution $t - t_{start} = 4.8$ fm/c the maximum of the distribution is $q_m \sim 1$ and the rescaling of the $\pi^{\mu\nu}$ tensor must be performed in the range of about 40 to 50% of cells. At the final stage, however, more then 60% of cells are rescaled in the case of $\eta/s = 0.5$. This results are obtained for the codes running with the M-condition. The true characteristic for perturbativity of the viscous effect is however the quantity $q_m$. If $q_m > 1$ then at least one of the elements in the $\pi^{\mu\nu}$ tensor is larger than the corresponding element in $T^{\mu\nu}_{id}$, i.e. the viscous effect is non-perturbative and the applicability of the hydrodynamic equations is questionable. The distributions of values $q_m$ are shown in Fig. 12 by thin lines. We see that although the code is keeping $q_m < 1$ at each evolution step, vast majority of fluid cells have $q_m > 1$. So, already at initial steps only in 27% for $\eta/s = 0.1$ and in 14% for $\eta/s = 0.5$ of all cells the viscous effects are truly perturbative. With time passed these numbers drop further down to a very small values: at $t - t_{start} = 8$ fm/c they are 1.4 and 0.67% for $\eta/s = 0.1$ and 0.5, respectively. Thus we conclude that applying the weak M-condition we let the hydrodynamic code run, in reality, in the non-perturbative regime and the results of such calculations cannot be trustworthy.

### IV. Beam-Energy Dependence of Parameter Influence

After considering the properties of different $\pi^{\mu\nu}$ constraints, we can try to fit the rapidity distributions and transverse momentum spectra in a wide range of bombarding energies reachable at the AGS and SPS facilities. The results of this fit are presented in Fig. 13. For each energy we vary independently the parameters $T_{f.o.}$ and $\eta/s$. We also try various conditions on the shear stress tensors evaluating the quantity $q$ in the constraint $13$, keeping there $C = 0.3$, according to the strict S-condition and weaker V-, M- and M-conditions. The obtained best values of the varied parameters are collected in Table II.

Consider, first, the results obtained with the S-condition when we fit proton rapidity distributions shown in Fig. 13 by solid lines. As we have already seen in the previous sections, we cannot simultaneously reproduce pion and proton distributions in this case. For collisions with energies from 6 AGeV to 158 AGeV, we can properly reproduce proton rapidity distributions. In agreement with our previous results for ideal hydrodynamics, where we also well reproduced proton rapidity distributions, we obtain small values of $\eta/s \simeq 0 - 0.05$. At all considered energies the experimental pion rapidity spectra are underestimated for $-1 \simeq y \simeq 1$. Different values of $T_{f.o.}$ are found for different energies. One can see that obtained $T_{f.o.}$ values are quite close to predicted by a thermal statistical model and demonstrate a saturation at higher energies. Once we have managed to fit the proton $y$-distributions, the proton $m_T$-spectra have typically too steep shapes, except the $E_{lab} = 6$ AGeV case, where both $y$- and $m_T$-distributions can be reproduced.
reasonably. Although, even in this case the number of protons at midrapidity is too low. The slopes of the pion $m_T$ spectra are typically closer to the experiment than those for proton ones, except the 10.7 AGeV case, but the lines systematically go below the data.

We would like to note that evidence for similar small values of $\eta/s \sim 0.04$ were also found in Ref. 17 at RHIC energies from an analysis of the elliptic flow of charged hadrons.

Now we try to apply the weaker M-condition 24. Here we can follow two strategies: we can insist on fitting at best either proton $y$-distributions or the pion ones. We
denote the results obtained in the first way as (p)M-fits and as (π)M-fits in the second case. It turns out that the results of (p)M-fits are close to those obtained with the S-conditions. The fit parameters are also similar, see last 2 columns in Table II.

The situation changes if we require the best possible description of pion rapidity distributions. This is possible with the M-condition since, as demonstrated in Section 11111 the code in this case does not lose its sensitivity to the viscosity parameter, at cost of the uncontrollable increase of the number of cells where the elements of the $\pi^{\mu\nu}$ tensor exceed dramatically the components of the $T_{id}^{\mu\nu}$ tensor. As is seen from Table II, in this case we obtain systematically higher freeze-out temperatures and viscosities than for the S-constraint. Obtained $T_{f.o.}$ lies very far from those predicted by statistical model [46].

Also for the M-condition we found that the parameter $\eta/s$ has a maximum at $E_{lab} = 10.7$ GeV while for the fit of proton distributions with the S-restriction, we maybe observe a minimum (due to an inaccuracy in the fitting procedure we can have a constant). Applying the M-condition, as one can see from Fig. 13, we are able nicely reproduce both $y$- and $m_T$-distributions of pions. However, if we look at the proton rapidity distribution (not shown in [18]) we see that we miserably fail: the number of protons near the mid-rapidity is too high and the distributions are too narrow (for all energies excluding 80 AGeV). These are the strong signals of the high viscosity effects, as we discussed in Section 1111111, see Figs. 7 and 9.

The same results we are also obtained if we use the V-condition instead of the M-condition, see dash-dotted lines in Fig. 13. As we see both weak conditions give very close results if one fits pion rapidity distributions. It can be easily explained since V-condition can be presented in the equivalent quadratic form as sum of squared components of $\pi^{\mu\nu}$ on l.h.s and $T_{id}^{\mu\nu}$ on the r.h.s. The resulting expressions differs from the M-condition only by some “+” signs instead “−” in the latter.

Summarising the discussion of Fig. 13 we conclude that any considered condition does not allow to reproduce simultaneously pion and proton experimental data.

In the discussed AGS-SPS energy range, a detailed comparison of experimental data with different viscous hydro approaches has been made only in a couple of papers. Great success was reached in terms of the three-fluid dynamics (3FD) model [18] applied to energies $E_{lab} \lesssim 158$ AGeV. The 3FD approximation is a minimal way to simulate the early-stage nonequilibrium in colliding nuclei. In contrast to the conventional 1-fluid hydrodynamics, the 3FD approach takes into account a finite stopping power in a counterstreaming regime of leading baryon-rich matter at an early stage of a collision, which allows one to use a constant $t_{\text{start}}$ parameter independently of $\sqrt{s}$. Different EOS are used instead of parameter variation being in the best agreement for the case of smooth cross-over phase transition. The beam-energy dependence of rapidity (not pseudorapidity !) proton spectra is in good agreement with experiment [10] at $E_{lab} \lesssim 10$ AGeV, for all EOS, but a mixed phase with the smooth crossover dominates definitely at higher energies. A similar situation is with the transverse mass spectra at the middle rapidity [50]. Effects of the EOS are getting visible in more delicate characteristics, say, energy dependence of the slopes of transverse mass spectra for identified hadrons.

The collective behavior of the nuclear fireball can also be studied using the hydrodynamics inspired phenomenological model called the blast wave model [51]. The main underlying assumption of this model is that the particles in the system produced in the collisions are locally thermalized and the system expands collectively with a common radial velocity field undergoing an instantaneous common freeze-out. While the spherically expanding source may be expected to mimic the fireball created at low energies, at higher energies a stronger longitudinal flow might lead to cylindrical geometry. For the latter case, an appropriate formalism was first developed in Ref. [52]. Using a simple functional form for the phase space density at kinetic freeze out, the authors approximated the hydrodynamical results with the boost-invariant longitudinal flow. The common assumption for all variants of the blast wave model is the underlying boost-invariant longitudinal dynamics. Although it is a reasonable assumption at RHIC and LHC energies, longitudinal boost-invariance does not hold well at AGS-SPS energies. Recently, a non boost-invariant blast wave model has been developed [53]. The model was successfully used in the AGS-SPS energy range to fit the rapidity distributions and transverse momentum spectra with only two parameters, namely, a kinetic freeze-out

### Table II: The fitted parameters for Eq. (13) (S-condition) and for Eq. (23) if proton ((p)M-condition) or pion ((π)M- and (π)V-conditions) rapidity distribution is tuned. The fit accuracy is $\Delta T = \pm 5 - 10$ MeV, $\Delta \eta/s = \pm 0.05$.

| $E_{lab}$ [AGeV] | $S$-condition | $(\pi)$M-condition | $(\pi)$V-condition | $(\pi)$M-condition |
|-----------------|---------------|---------------------|---------------------|---------------------|
| $T_{f.o.}$ [MeV] | $\eta/s$ | $T_{f.o.}$ [MeV] | $\eta/s$ | $T_{f.o.}$ [MeV] | $\eta/s$ |
| 6  | 78  | 0.03 | 120 | 0.07 | - | - | 79 | 0.01 |
| 10.7  | 125  | 0.01 | 143 | 0.24 | - | - | 125 | 0.01 |
| 40  | 155  | 0.05 | 170 | 0.08 | 175 | 0.1 | 160 | 0.01 |
| 80  | 160  | 0 | 190 | 0.13 | 190 | 0.11 | 160 | 0 |
| 158  | 160  | 0 | 205 | 0.1 | 205 | 0.08 | 160 | 0 |
temperature $T_{T.o}$ and a radial flow strength $\beta_T$. The fitted here $T_{T.o}$ are smaller than the values in our analysis (see Table II). One should note that the model \cite{53} nicely describes the shape of distributions studied but their absolute values should be separately fitted at every energy.

V. CONCLUSIONS

In this work we have extended the HydHSD model developed in \cite{9} by inclusion of shear viscosity within the Israel-Stewart hydrodynamics. Using the updated version of our hybrid model, we considered proton and pion rapidity distributions and transverse momentum spectra for $E_{lab} \leq 160$A GeV. As in other viscous hydrodynamic calculations, genuine inaccuracy of a numerical implementation leads to an uncontrollable increase of the shear stress tensor $\pi^{\mu\nu}$ that contradict to a perturbative character of the viscous corrections to ideal hydrodynamics. To timid the problem, a regularization scheme was suggested in the literature, which assumes a rescaling of the $\eta/s$ parameter; cf. Eq (3), which is violated. We use the strict (S-) condition (12) proposed in \cite{1}, which guarantees that each element of the $\pi^{\mu\nu}$ tensor remains not more than 30% (specified by the $C$ value in (13)) of the corresponding element of the $T^{\mu\nu}_{id}$ tensor. Also we analyzed other conditions used in the literatures: the V-condition (23) used in the vHLLE code \cite{17, 18} and the M-condition (24) used in the MUSIC and iEBE-VISHNU codes \cite{19, 20}. The V- and M-conditions are weaker than S-condition.

It is proven that the form of $\pi^{\mu\nu}$-constraints plays a crucial role in sensitivity to the $\eta/s$ value. Our results demonstrate that the form of the proton humps is mainly determined by the $\eta/s$ ratio which has to be not greater than 0.1 for $E_{lab} = 40$A GeV, see Fig. 2.

A numerical algorithm with weaker V- and M-conditions is more responsive and leads to higher pion rapidity and humps in proton distribution than with the stricter S-condition for the same freeze-out parameters, see Figs. 4 and 11. Our results demonstrate that the V-condition leads to the longest evolution and, as a result, is most sensitive to a change of the $\eta/s$ parameter; however, they cannot be applied to large values of the $\eta/s$.

Such sensitivity of weaker conditions allows for a good quality fits of pion rapidity distributions. The reason for this is that the larger viscosity effects, going beyond perturbative nature of the original hydrodynamic equations, lead to higher temperatures of fluids and consequently to a higher freeze-out volume contributing to the pion yield, see Fig. 11.

However, any considered condition does not allow to reproduce simultaneously pion and proton experimental data within our model. Moreover, a particular parameter fitting of experimental distributions at every colliding energy $E_{lab}$ does not guarantee excellent agreement. We expect that quantitative improvement of the description could be reached by choosing a better equation of state and by taking into account fluctuating (event-by-event) initial conditions and the bulk viscosity. Note that for the considered moderate beam-energy range there is no systematic comparison of hydro-predictions with experimental data though for separate observable good agreement with experiment may be reached.

ACKNOWLEDGMENTS

We thank E. Bratkovskaya and W. Cassing for providing the HSD code and consultations. We appreciate very much extensive discussions with Iu. Karpenko and Yu.B. Ivanov and constructive remarks by G. Sandukovskaya. The work is supported by Slovak grant VEGA-1/0348/18 and by THOR the COST Action CA15213. A.S.K and E.E.K. acknowledge the support by the Plenipotentiary of the Slovak Government at JINR, Dubna.

Appendix A: Reconstruction of local quantities

The hydrodynamics code evolves the components of the energy-stress tensor and baryon current. The equation of state is formulated in the local system where the energy density and the particle number should be defined. To Lorentz-transform from the laboratory frame in the local rest frame one also needs to define a 4-velocity of the fluid element. If we know the components of the ideal stress tensor, $T^{\mu\nu}_{id} = T^{\mu\nu} - \pi^{\mu\nu}$ and current $J^{\mu} = n u^{\mu}$, other quantities can be recovered as follows:

$$n = J^0 \sqrt{1 - v^2}, \quad \epsilon = T^{00}_{id} - M v,$$

$$M^2 = T^{00}_{id} T^{0x}_{id} + T^{0y}_{id} T^{0y}_{id} + T^{0z}_{id} T^{0z}_{id}.$$  \hspace{2cm} (A1)

The modulus of the fluid velocity can be found as a root of the equation

$$v = \frac{M}{T^{00}_{id} - P(T^{0i}_{id} - M v, J^0 \sqrt{1 - v^2})}$$ \hspace{2cm} (A2)

and, therefore, depends on the chosen equation of state $P = P(\epsilon, n)$. The direction of the fluid velocity is determined as

$$v^i = \frac{v}{M} T^{0i}_{id}.$$ \hspace{2cm} (A3)

We use in the code $\pi^{xy}, \pi^{xz}, \pi^{yz}, \pi^{yy}$, and $\pi^{zz}$ as independent variables. Other components can be recovered
The velocity components are taken at the lous diffusion with the help of the following expressions

\[
\begin{align*}
\pi^{00} &= \frac{1}{1 - v_z^2} \left[ 2(\pi^{xy}v_x v_y + \pi^{xz}v_x v_z + \pi^{yz}v_y v_z) \\
&\quad - \pi^{yy}(v_x^2 - v_y^2) - \pi^{xz}(v_x^2 - v_z^2) \right], \\
\pi^{0x} &= \frac{1}{1 - v_z^2} \left[ (\pi^{xy}v_x + \pi^{xz}v_x)(1 + v_x^2) + 2\pi^{yz}v_x v_y v_z \\
&\quad - \pi^{yy}v_x(1 - v_y^2) - \pi^{xz}v_x(1 - v_z^2) \right], \\
\pi^{0y} &= \pi^{xy}v_x + \pi^{yy}v_y + \pi^{yz}v_z, \\
\pi^{0z} &= \pi^{xz}v_x + \pi^{yz}v_y + \pi^{z2}v_z, \\
\pi^{xx} &= \frac{1}{1 - v_z^2} \left[ 2(\pi^{xy}v_x v_y + \pi^{xz}v_x v_z + \pi^{yz}v_y v_z) \\
&\quad - \pi^{yy}(1 - v_y^2) - \pi^{xz}(1 - v_z^2) \right].
\end{align*}
\]  

(A4)

We emphasize that these expressions do not develop anomalously large values for the case of small fluid velocities.

Appendix B: 3+1 implementation of SHASTA algorithm

For completeness we provide the complete set of formulas for the 3+1 implementation of the SHASTA algorithm extending expressions provided in Ref [1]. The r.h.s. of Eq. (10) looks like

\[
\tilde{S}_{\text{cons}} = \begin{bmatrix}
0 \\
-\partial_t \pi^{00} - \text{div}(\vec{v}P) - (\partial_x \pi^{0x} + \partial_y \pi^{0y} + \partial_z \pi^{0z}) \\
-\partial_t \pi^{0x} - \partial_y P - (\partial_x \pi^{xx} + \partial_y \pi^{xy} + \partial_z \pi^{xz}) \\
-\partial_t \pi^{0y} - \partial_y P - (\partial_x \pi^{xy} + \partial_y \pi^{yy} + \partial_z \pi^{yz}) \\
-\partial_t \pi^{0z} - \partial_z P - (\partial_x \pi^{xz} + \partial_y \pi^{yz} + \partial_z \pi^{z2})
\end{bmatrix}.
\]

(B1)

For lattice realization of quantities \(U(x, y, z, t)\) we will use notations \(U_{ijk}^{[n]}\), where index \(n\) stands for temporal steps and \(i, j, k\) for spatial lattice cells in \(x, y,\) and \(z\) directions respectively.

At the first stage of the SHASTA algorithm one calculates, at each subsequent \((n+1)\)th step, the so-called transport-diffused solution

\[
\tilde{U}_{ijk}^{[n+1]} = \tilde{U}_{ijk}^x + \tilde{U}_{ijk}^y + \tilde{U}_{ijk}^z - 2U_{ijk}^{[n]} + \Delta t S_{ijk},
\]

(B2)

where \(U_{ijk}^{[n]}\) is the full solution at the previous time step and auxillary quantities \(\tilde{U}_{ijk}^{x,y,z}\) are defined as

\[
\begin{align*}
\tilde{U}_{ijk}^x &= \frac{1}{2} \left[ (Q_{ijk}^{x+})^2 (U_{i+1,j,k}^{[n]} - U_{ij}^{[n]} - U_{i,j+1,k}^{[n]} - U_{i,j,k}^{[n]}) + (Q_{ijk}^{x-})^2 (U_{i,j,k}^{[n]} - U_{i,j-1,k}^{[n]}) \right] + (Q_{ijk}^{x+} + Q_{ijk}^{x-}) U_{ijk}^{[n]}, \\
\tilde{U}_{ijk}^y &= \frac{1}{2} \left[ (Q_{ijk}^{y+})^2 (U_{i,j+1,k}^{[n]} - U_{ij}^{[n]} - U_{i,j-1,k}^{[n]} - U_{i,j,k}^{[n]}) + (Q_{ijk}^{y-})^2 (U_{i,j,k}^{[n]} - U_{i,j,k-1}^{[n]}) \right] + (Q_{ijk}^{y+} + Q_{ijk}^{y-}) U_{ijk}^{[n]}, \\
\tilde{U}_{ijk}^z &= \frac{1}{2} \left[ (Q_{ijk}^{z+})^2 (U_{i,j,k+1}^{[n]} - U_{ij}^{[n]} - U_{i,j,k-1}^{[n]} - U_{i,j,k}^{[n]}) + (Q_{ijk}^{z+} + Q_{ijk}^{z-}) U_{ijk}^{[n]} \right].
\end{align*}
\]

(B3) \(\text{to} \) (B5)

with

\[
Q_{ijk}^\pm = \frac{1/2 \pm \lambda (v_x)^{[n]}_{ijk}}{1 \pm \lambda (v_x)^{[n]}_{ijk}}, \quad Q_{ijk}^\pm = \frac{1/2 \pm \lambda (v_y)^{[n]}_{ijk}}{1 \pm \lambda (v_y)^{[n]}_{ijk}}, \quad Q_{ijk}^\pm = \frac{1/2 \pm \lambda (v_z)^{[n]}_{ijk}}{1 \pm \lambda (v_z)^{[n]}_{ijk}}.
\]

(B6)

The velocity components are taken at the \(n\)th time step.

Further, using the transport-diffused solution one calculates an antidiffusion flux that takes into account an anomalous diffusion

\[
A_{\text{ad}}^{x,y,z} = \frac{1}{8} A_{\text{ad}}^{x,y,z} \tilde{\Delta}_{ijk}^{x,y,z}, \quad \tilde{\Delta}_{ijk}^x = \tilde{U}_{i+1,j,k}^x - \tilde{U}_{i,j,k}^x, \quad \tilde{\Delta}_{ijk}^y = \tilde{U}_{i,j+1,k}^y - \tilde{U}_{i,j,k}^y, \quad \tilde{\Delta}_{ijk}^z = \tilde{U}_{i,j,k+1}^z - \tilde{U}_{i,j,k}^z,
\]

(B7)

where \(A_{\text{ad}}^{x,y,z}\) are the antidiffusive mask coefficients. For simplicity, one takes them to be equal for all spatial directions and set \(A_{\text{ad}} = 1\) as the default value. Next, we calculate the limited antidiffusion fluxes

\[
\begin{align*}
\tilde{A}_{ijk}^x &= \sigma_{ijk}^x \text{max} \left[ 0, \text{min} \left( \sigma_{ijk}^x \tilde{\Delta}_{i+1,j,k}^x, |A_{ijk}^x|, \sigma_{ijk}^x \tilde{\Delta}_{i-1,j,k}^x \right) \right], \\
\tilde{A}_{ijk}^y &= \sigma_{ijk}^y \text{max} \left[ 0, \text{min} \left( \sigma_{ijk}^y \tilde{\Delta}_{i,j+1,k}^y, |A_{ijk}^y|, \sigma_{ijk}^y \tilde{\Delta}_{i,j-1,k}^y \right) \right], \\
\tilde{A}_{ijk}^z &= \sigma_{ijk}^z \text{max} \left[ 0, \text{min} \left( \sigma_{ijk}^z \tilde{\Delta}_{i,j,k+1}^z, |A_{ijk}^z|, \sigma_{ijk}^z \tilde{\Delta}_{i,j,k-1}^z \right) \right].
\end{align*}
\]

(B8)
The total incoming and outgoing antidiffusive fluxes in the cell are calculated as
\[ A_{ij}^{\text{in}} = \max \left(0, A_{i-1,j,k}^x - \min \left(0, A_{i,j-1,k}^y + \max \left(0, A_{i-1,j,k}^z - \min \left(0, A_{i,j,k-1}^x\right)\right), A_{i,j,k}^x\right), A_{i,j,k}^y\right), A_{i,j,k}^z\right) \]  \quad \text{(B9)}
\[ A_{ij}^{\text{out}} = \max \left(0, A_{ij}^x\right) - \min \left(0, A_{i-1,j,k}^x\right) + \max \left(0, A_{ij-1,k}^y\right) - \min \left(0, A_{ij,k-1}^y\right) + \max \left(0, A_{i,j,k}^z\right) - \min \left(0, A_{i,j,k-1}^z\right). \]  \quad \text{(B10)}

The maximal and minimal values of the transport-diffused solution \( U_{ij}^{[n+1]} \) after the antidiffusion stage are between
\[ \tilde{U}_{ij}^{\text{min}} = \min \left(\tilde{U}_{ij-1,k-1}^{[n+1]}, \tilde{U}_{ij-1,k}^{[n+1]}, \tilde{U}_{ij,k-1}^{[n+1]}, \tilde{U}_{ij,k}^{[n+1]}\right), \]  \quad \text{(B11)}
\[ \tilde{U}_{ij}^{\text{max}} = \max \left(\tilde{U}_{ij-1,k}^{[n+1]}, \tilde{U}_{ij-1,k}^{[n+1]}, \tilde{U}_{ij,k}^{[n+1]}\right). \]  \quad \text{(B12)}

This information is then used to determine the fractions of the incoming and outgoing fluxes,
\[ F_{ij}^{\text{in}} = \frac{1}{A_{ij}^{\text{in}}} (\tilde{U}_{ij}^{\text{max}} - \tilde{U}_{ij}^{[n+1]}), \quad F_{ij}^{\text{out}} = \frac{1}{A_{ij}^{\text{out}}} (\tilde{U}_{ij}^{[n+1]} - \tilde{U}_{ij}^{\text{min}}). \]  \quad \text{(B13)}

The final antidiffusion fluxes are calculated as
\[ A_{ij}^x = A_{ij}^x \left[\min \left(1, F_{ij}^{\text{in}, i+1,k} + F_{ij}^{\text{out}, i+1,k}\right)\Theta(A_{ij}^x) + \min \left(1, F_{ij}^{\text{in}, i+1,k} + F_{ij}^{\text{out}, i+1,k}\right)\Theta(-A_{ij}^x)\right], \]  \quad \text{(B14)}
\[ A_{ij}^y = A_{ij}^y \left[\min \left(1, F_{ij}^{\text{in}, i+1,k} + F_{ij}^{\text{out}, i+1,k}\right)\Theta(A_{ij}^y) + \min \left(1, F_{ij}^{\text{in}, i+1,k} + F_{ij}^{\text{out}, i+1,k}\right)\Theta(-A_{ij}^y)\right], \]  \quad \text{(B15)}
\[ A_{ij}^z = A_{ij}^z \left[\min \left(1, F_{ij}^{\text{in}, i+1,k} + F_{ij}^{\text{out}, i+1,k}\right)\Theta(A_{ij}^z) + \min \left(1, F_{ij}^{\text{in}, i+1,k} + F_{ij}^{\text{out}, i+1,k}\right)\Theta(-A_{ij}^z)\right]. \]  \quad \text{(B16)}

Finally, the full solution for \( n + 1 \) time step is given by
\[ U_{ij}^{[n+1]} = \tilde{U}_{ij}^{[n+1]} + (A_{i-1,j,k}^x - A_{ij}^x) + (A_{ij-1,k}^y - A_{ij}^y) + (A_{ij,k-1}^z - A_{ij}^z). \]  \quad \text{(B17)}

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