Abstract. In this work, we study locally finite simple Lie superalgebras containing a Cartan subalgebra and equipped with an invariant nondegenerate even supersymmetric bilinear form. We call these Lie superalgebras locally finite basic classical simple Lie superalgebras, classify them and study the conjugacy classes of their Cartan subalgebras under the group of automorphisms.

0. Introduction

Following a physical interest in the context of supersymmetries, in 1977, V. Kac introduced Lie superalgebras (known as $\mathbb{Z}_2$-graded Lie algebras in physics). He classified classical Lie superalgebras which are finite dimensional simple Lie superalgebras whose even parts are reductive Lie algebras. These Lie superalgebras are a generalization of finite dimensional simple Lie algebras over an algebraically closed field of characteristic zero but classical Lie superalgebras are not necessarily equipped with nondegenerate invariant bilinear forms while Killing form on a finite dimensional simple Lie algebra over a field of characteristic zero is invariant and nondegenerate. To get a better super version of finite dimensional simple Lie algebras, one can work with those classical Lie superalgebras equipped with even nondegenerate invariant bilinear forms, called finite dimensional basic classical simple Lie superalgebras (f.d.b.c.s Lie superalgebras for short). A f.d.b.c.s Lie superalgebra has a weight space decomposition with respect to a Cartan subalgebra $\mathcal{H}$ of the even part whose root system is a union of a finite root system and a subset of the dual space $\mathcal{H}^*$ of $\mathcal{H}$ consisting of some elements which are self-orthogonal with respect to the induced form on $\mathcal{H}^*$.

The interaction of a finite dimensional simple Lie algebra with its root system is a powerful tool to study the structure of the Lie algebra via its root system. To consider such an approach on Lie superalgebra level, we first need to have an abstract definition of the root system of a Lie superalgebra apart from its connection with the Lie superalgebra. In [16], the author introduces the notion of extended affine root supersystems and systematically studies them. Roughly speaking, a spanning set $R$ of a nontrivial vector space over a field $F$ of characteristic zero, equipped with a symmetric bilinear form, is called an extended affine root supersystem if the root string property is also satisfied. Such an $R$ is called a locally finite root supersystem if the form is nondegenerate.

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Locally finite root supersystems which are finite had been studied by V. Serganova in 1996 under the name generalized root systems [12]. Almost all generalized root systems appear as the root systems of f.d.b.c.s Lie superalgebras. For a locally finite root supersystem $R$, the self-orthogonal elements are called nonsingular roots and the elements which are not self-orthogonal, are called real roots. Real roots of $R$ form a locally finite root system ([5], [10]) and if $R$ is irreducible with nonzero nonsingular part, its nonsingular roots have either one or two conjugacy classes under the Weyl group action depending on whether an arbitrary nonzero nonsingular root is conjugate to its opposite or not [14]. This helps us to know the structure of a Lie superalgebra $L$ whose root system is an irreducible locally finite root supersystem; more precisely, as real roots form a locally finite root system, we get that the derived algebra of the even part of $L$ is semisimple and considering the conjugacy classes of imaginary roots under the action of the Weyl group, we can show that the odd part of $L$ is a completely reducible module for the even part with at most two irreducible constituents; see Theorem 2.30.

Extended affine root supersystems appear as the root systems of the super version of invariant affine reflection algebras [11] called extended affine Lie superalgebras [15]. Roughly speaking, a nonzero Lie superalgebra is called an extended affine Lie superalgebra, if it is equipped with an invariant nondegenerate even supersymmetric bilinear form and that it has a weight space decomposition with respect to a toral subalgebra of its even part whose root vectors satisfy some natural conditions. Extended affine Lie superalgebras whose corresponding toral subalgebras are self-centralizing (referred to as Cartan subalgebras) are important in some sense, namely, finite dimensional basic classical simple Lie superalgebras and affine Lie superalgebras are examples of such Lie superalgebras. We study extended affine Lie superalgebras having a Cartan subalgebra and figure out the properties of their root spaces; in particular, we find the dimension of root spaces corresponding to so-called nonisotropic roots. We then focus on a special subclass of extended affine Lie superalgebras whose elements are called locally finite basic classical simple Lie superalgebras (l.f.b.c.s Lie superalgebras for short); the root system of a l.f.b.c.s Lie superalgebra is an irreducible locally finite root supersystem. Locally finite basic classical simple Lie superalgebras with zero odd part are exactly locally finite split simple Lie algebras [10]. We classify l.f.b.c.s Lie superalgebras; due to our classification, a l.f.b.c.s Lie superalgebra with nonzero odd part is either a finite dimensional basic classical simple Lie superalgebra or isomorphic to one and only one of the Lie superalgebras $\mathfrak{osp}(2I, 2J)$ ($I, J$ index sets with $|I \cup J| = \infty$, $|J| \neq 0$), $\mathfrak{osp}(2I + 1, 2J)$ ($I, J$ index sets with $|I| < \infty$, $|J| = \infty$) or $\mathfrak{sl}(I_\bar{0}, I_\bar{1})$ ($I$ an infinite superset with $|I_\bar{0}|, |I_\bar{1}| \neq 0$); see Subsection 2.1 for the definitions of these Lie superalgebras. We conclude the paper with studying the conjugacy classes of Cartan subalgebras of l.f.b.c.s Lie superalgebras under the group of automorphisms. We show that if $I$ is an infinite set and $J$ is a nonempty set, $\mathfrak{osp}(2I + 1, 2J)$ is isomorphic to $\mathfrak{osp}(2I, 2J)$ while their root systems with respect to their standard Cartan subalgebras are not isomorphic. This in particular means that these two standard Cartan subalgebras are not conjugate. We then prove that these are the only representatives for the conjugacy classes of Cartan subalgebras of $\mathfrak{osp}(2I + 1, 2J) \simeq \mathfrak{osp}(2I, 2J)$. We finally show that if $L$ is a locally finite basic classical simple Lie superalgebra which is not isomorphic to $\mathfrak{osp}(2I + 1, 2J) \simeq \mathfrak{osp}(2I, 2J)$ ($I$ an infinite set and $J$ a nonempty set), then all Cartan subalgebras of $L$ are conjugate under the automorphism group of $L$. 
1. Extended Affine Lie Superalgebras And Their Root Systems

Throughout this work, \( \mathbb{F} \) is a field of characteristic zero. Unless otherwise mentioned, all vector spaces are considered over \( \mathbb{F} \). We denote the dual space of a vector space \( V \) by \( V^* \). We denote the degree of a homogenous element \( v \) of a superspace by \( |v| \) and make a convention that if in an expression, we use \(|u|\) for an element \( u \) of a superspace, by default we have assumed \( u \) is homogeneous. We denote the group of automorphisms of an abelian group \( A \) or a Lie superalgebra \( A \) by \( Aut(A) \) and for a subset \( S \) of an abelian group \( A \), \( \langle S \rangle \), we mean the subgroup generated by \( S \). For a set \( S \), \( |S| \) we mean the cardinal number of \( S \). For a map \( f : A \to B \) and \( C \subseteq A \), \( f \mid_C \) we mean the restriction of \( f \) to \( C \). For two symbols \( i, j \), by \( \delta_{i,j} \), we mean the Kronecker delta. We also use \( \cup \) to indicate the disjoint union. We finally recall that the direct union is, by definition, the direct limit of a direct system whose morphisms are inclusion maps.

In this section, we recall the notions of extended affine Lie superalgebras and extended affine root supersystems from [15] and gather the information we need through the paper regarding them. In the sequel, by a symmetric form on an abelian group \( A \), we mean a map \((\cdot, \cdot) : A \times A \to \mathbb{F} \) satisfying

1. \((a, b) = (b, a) \) for all \( a, b \in A \),
2. \((a + b, c) = (a, c) + (b, c) \) and \((a, b + c) = (a, b) + (a, c) \) for all \( a, b, c \in A \).

In this case, we set \( A^0 := \{a \in A \mid (a, A) = \{0\} \} \) and call it the radical of the form \((\cdot, \cdot)\). The form is called nondegenerate if \( A^0 = \{0\} \). We note that if the form is nondegenerate, \( A \) is torsion free and we can identify \( A \) as a subset of \( \mathbb{Q} \otimes_\mathbb{Z} A \). Throughout the paper, if an abelian group \( A \) is equipped with a nondegenerate symmetric form, we consider \( A \) as a subset of \( \mathbb{Q} \otimes_\mathbb{Z} A \) without further explanation. Also if \( A \) is a vector space over \( \mathbb{F} \), bilinear forms are used in the usual sense.

We call a triple \((\mathcal{L}, \mathcal{H}, (\cdot, \cdot))\) a super-toral triple if

1. \( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \) is a nonzero Lie superalgebra, \( \mathcal{H} \) is a nonzero subalgebra of \( \mathcal{L}_0 \) and \((\cdot, \cdot)\) is an invariant nondegenerate even supersymmetric bilinear form \((\cdot, \cdot)\) on \( \mathcal{L} \),
2. \( \mathcal{L} \) has a weight space decomposition \( \mathcal{L} = \oplus_{\alpha \in \mathcal{H}^*} \mathcal{L}^{\alpha} \) with respect to \( \mathcal{H} \) via the adjoint representation.

We note that in this case \( \mathcal{H} \) is abelian; also as \( \mathcal{L}_0 \) as well as \( \mathcal{L}_1 \) are \( \mathcal{H} \)-submodules of \( \mathcal{L} \), we have

\[ \mathcal{L}_0 = \oplus_{\alpha \in \mathcal{H}^*} (\mathcal{L}_0)^{\alpha} \] and \( \mathcal{L}_1 = \oplus_{\alpha \in \mathcal{H}^*} (\mathcal{L}_1)^{\alpha} \) with \( (\mathcal{L}_i)^{\alpha} := \mathcal{L}_i \cap \mathcal{L}^{\alpha}, i = 0, 1 \),

3. the restriction of the form \((\cdot, \cdot)\) on \( \mathcal{H} \) is nondegenerate.

We call \( R := \{\alpha \in \mathcal{H}^* \mid \mathcal{L}^{\alpha} \neq \{0\}\} \), the root system of \( \mathcal{L} \) (with respect to \( \mathcal{H} \)). Each element of \( R \) is called a root. We refer to elements of \( R_0 := \{\alpha \in \mathcal{H}^* \mid (\mathcal{L}_0)^{\alpha} \neq \{0\}\} \) (resp. \( R_1 := \{\alpha \in \mathcal{H}^* \mid (\mathcal{L}_1)^{\alpha} \neq \{0\}\} \)) as even roots (resp. odd roots). We note that \( R = R_0 \cup R_1 \).

Suppose that \((\mathcal{L}, \mathcal{H}, (\cdot, \cdot))\) is a super-toral triple with corresponding root system \( R \). Since the form is invariant and even, we have

\[
(1.1) \quad ((\mathcal{L}_i)^{\alpha}, (\mathcal{L}_j)^{\beta}) = \{0\}, \quad \alpha, \beta \in R, \ i, j \in \{0, 1\}, \ i \neq j, \ \alpha + \beta \neq 0.
\]
This in particular implies that for \( \alpha \in R \) and \( i \in \{0,1\} \), the form restricted to \((\mathcal{L}_i)^\alpha + (\mathcal{L}_i)^{-\alpha}\) is nondegenerate. Take \( p : \mathcal{H} \to \mathcal{H}^* \) to be the function mapping \( h \in \mathcal{H} \) to \((h, \cdot)\). Since the form is nondegenerate on \( \mathcal{H} \), the map \( p \) is one to one (and so onto if \( \mathcal{H} \) is finite dimensional). So for each element \( \alpha \) of the image \( \mathcal{H}^p \) of \( \mathcal{H} \) under the map \( p \), there is a unique \( t_{\alpha} \in \mathcal{H} \) representing \( \alpha \) through the form \((\cdot, \cdot)\). Now we can transfer the form on \( \mathcal{H} \) to a form on \( \mathcal{H}^p \), denoted again by \((\cdot, \cdot)\) and defined by

\[
(\alpha, \beta) := (t_{\alpha}, t_{\beta}) \quad (\alpha, \beta \in \mathcal{H}^p).
\]

Although \( R \subseteq \mathcal{H}^* = \mathcal{H}^p \) if \( \mathcal{H} \) is finite dimensional, in general, by Lemma 3.1 of [15], \( \alpha \in R \) is an element of \( \mathcal{H}^p \) if there are \( i \in \{0,1\} \), \( x \in (\mathcal{L}_i)^\alpha \) and \( y \in (\mathcal{L}_i)^{-\alpha} \) with \( 0 \neq [x, y] \in \mathcal{H} \); moreover, if \( \alpha \in R \cap \mathcal{H}^p \), \( x \in \mathcal{L}^\alpha \) and \( y \in \mathcal{L}^{-\alpha} \) with \([x, y] \in \mathcal{H} \), using the same lemma, we have

\[
[x, y] = (x, y)t_{\alpha}.
\]

**Lemma 1.4.** Suppose that \( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \) is a Lie superalgebra equipped with an invariant nondegenerate even supersymmetric bilinear form \((\cdot, \cdot)\) such that

- \( \mathcal{L}_0 \) is a finite dimensional semisimple Lie algebra with the only simple ideals \( I_1 \) and \( I_2 \),
- for Cartan subalgebras \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) of simple Lie algebras \( I_1 \) and \( I_2 \) respectively, \((\mathcal{L}, \mathcal{H}_1 \oplus \mathcal{H}_2, (\cdot, \cdot))\) is a super-toral triple with \( \mathcal{L}^0 = \mathcal{H}_1 \),
- for each \( 0 \neq h \in \mathcal{H} \), there is \( \delta \in R_1 \) with \( \delta(h) \neq 0 \) and \( t_{\delta} \notin \mathcal{H}_1 \cup \mathcal{H}_2 \),

then \( \mathcal{L} \) is a simple Lie superalgebra in the sense that it has no nontrivial ideal (non necessarily \( \mathbb{Z}_2 \)-graded).

**Proof.** Suppose that \( I \) is a nonzero ideal of \( \mathcal{L} \). Then by [7, Pro. 2.1.1],

\[
I = \oplus_{\alpha \in R}(I \cap \mathcal{L}^\alpha).
\]

Through the following steps, we prove that \( I = \mathcal{L} \).

**Step 1.** \( I \cap \mathcal{H} \neq \{0\} \): Since \( I \) is nonzero, there is \( \alpha \in R \) with \( \mathcal{L}^\alpha \cap I \neq \{0\} \). If \( \alpha = 0 \), there is nothing to prove. Suppose that \( \alpha \neq 0 \) and \( 0 \neq x \in \mathcal{L}^\alpha \cap I \). Since the form is nondegenerate on \( \mathcal{L}^\alpha \oplus \mathcal{L}^{-\alpha} \), there is \( y \in \mathcal{L}^{-\alpha} \) with \((x, y) \neq 0 \). So [13] implies that \( t_{\alpha} \in I \cap \mathcal{H} \) and so \( I \cap \mathcal{H} \neq \{0\} \).

**Step 2.** \( I = \mathcal{L} \): Using Step 1, one finds \( 0 \neq h \in I \cap \mathcal{H} \). We take \( \delta \in R_1 \) to be such that \( \delta(h) \neq 0 \) and \( t_{\delta} \notin \mathcal{H}_1 \cup \mathcal{H}_2 \). Since \( \delta(h)x = [h, x] \) for all \( x \in \mathcal{L}^\delta \), we have \( \mathcal{L}^\delta \subseteq I \). Now since the form restricted to \( \mathcal{L}^\delta \oplus \mathcal{L}^{-\delta} \) is nondegenerate, there are \( x \in \mathcal{L}^\delta \), \( y \in \mathcal{L}^{-\delta} \) such that \((x, y) \neq 0 \). So [13] implies that \( t_{\delta} \in I \). But \( I \cap \mathcal{L}_0 \) is an ideal of the semisimple Lie algebra \( I_1 \oplus I_2 \) and \( t_{\delta} \in (I \cap \mathcal{H}) \setminus (\mathcal{H}_1 \cup \mathcal{H}_2) \), so \( I \cap \mathcal{L}_0 = I_1 \oplus I_2 = \mathcal{L}_0 \); in particular \( \mathcal{H} \subseteq I \). Now for each \( \alpha \in R \setminus \{0\} \), there is \( k \in \mathcal{H} \) with \( \alpha(k) \neq 0 \). Therefore, for each \( x \in \mathcal{L}^\alpha \), \( \alpha(k)x = [k, x] \in I \) and so \( \mathcal{L}^\alpha \subseteq I \). These altogether complete the proof.

**Definition 1.5.** A super-toral triple \((\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1, \mathcal{H}, (\cdot, \cdot))\) (or \( \mathcal{L} \) if there is no confusion), with root system \( R = R_0 \cup R_1 \), is called an extended affine Lie superalgebra if

- (1) for each \( \alpha \in R_1 \setminus \{0\} \) \((i \in \{0,1\}) \), there are \( x_\alpha \in (\mathcal{L}_i)^\alpha \) and \( x_{-\alpha} \in (\mathcal{L}_i)^{-\alpha} \) such that \( 0 \neq [x_\alpha, x_{-\alpha}] \in \mathcal{H} \),
• (2) for each \( \alpha \in R \) with \((\alpha, \alpha) \neq 0\) and \( x \in \mathcal{L}^\alpha, \ ad_x : \mathcal{L} \to \mathcal{L}, \) mapping \( y \in \mathcal{L} \) to \([x, y]\), is a locally nilpotent linear transformation.

By [15] Cor. 3.9, the root system of an extended affine Lie superalgebra is an extended affine root supersystem in the following sense:

**Definition 1.6.** Suppose that \( A \) is a nontrivial additive abelian group, \( R \) is a subset of \( A \) and \((\cdot, \cdot) : A \times A \to \mathbb{F}\) is a symmetric form. Set
\[
R^0 := R \cap A^0, \\
R^\times := R \setminus R^0, \\
R^\times_{re} := \{ \alpha \in R \mid (\alpha, \alpha) \neq 0 \}, \\
R := R^\times_{re} \cup \{0\}, \\
R^\times_{ns} := \{ \alpha \in R \setminus R^0 \mid (\alpha, \alpha) = 0 \}, \\
R_{ns} := R^\times_{ns} \cup \{0\}.
\]

We say \((A, (\cdot, \cdot), R)\) is an extended affine root supersystem if the following hold:

1. \((S1)\) \( 0 \in R \) and \((R) = A, \)
2. \((S2)\) \( R = -R, \)
3. \((S3)\) for \( \alpha \in R^\times_{re} \) and \( \beta \in R, 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}, \)
4. \((S4)\) (root string property) for \( \alpha \in R^\times_{re} \) and \( \beta \in R, \) there are nonnegative integers \( p, q \) with \( 2(\beta, \alpha)/(\alpha, \alpha) = p - q \) such that
\[
\{ \beta + k \alpha \mid k \in \mathbb{Z} \} \cap R = \{ \beta - p \alpha, \ldots, \beta + q \alpha \};
\]

we call \( \{ \beta - p \alpha, \ldots, \beta + q \alpha \} \) the \( \alpha \)-string through \( \beta, \)
5. \((S5)\) for \( \alpha \in R_{ns} \) and \( \beta \in R \) with \((\alpha, \beta) \neq 0, \) \( \{ \beta - \alpha, \beta + \alpha \} \cap R \neq \emptyset. \)

If there is no confusion, for the sake of simplicity, we say \( R \) is an extended affine root supersystem in \( A. \) Elements of \( R^0 \) are called isotropic roots, elements of \( R_{re} \) are called real roots and elements of \( R_{ns} \) are called nonsingular roots. A subset \( X \) of \( R^\times \) is called connected if each two elements \( \alpha, \beta \in X \) are connected in \( X \) in the sense that there is a chain \( \alpha_1, \ldots, \alpha_n \in X \) with \( \alpha_1 = \alpha, \alpha_n = \beta \) and \((\alpha_i, \alpha_{i+1}) \neq 0, i = 1, \ldots, n - 1. \) An extended affine root supersystem \( R \) is called irreducible if \( R_{re} \neq \{0\} \) and \( R^\times \) is connected (equivalently, \( R^\times \) cannot be written as a disjoint union of two nonempty orthogonal subsets). An extended affine root supersystem \((A, (\cdot, \cdot), R)\) is called a locally finite root supersystem if the form \((\cdot, \cdot)\) is nondegenerate and it is called an affine reflection system if \( R_{ns} = \{0\}; \) see [11].

**Lemma 1.7.** Suppose that \( A \) is a nontrivial additive abelian group, \( R \) is a subset of \( A \) and \((\cdot, \cdot) : A \times A \to \mathbb{F}\) is a nondegenerate symmetric form. If \((A, (\cdot, \cdot), R)\) satisfies \((S1)\) and \((S3) - (S5),\) then \((S2)\) is also satisfied. In particular, a subset \( S \) of a locally finite root supersystem \( R \) is a locally finite root supersystem in its \( \mathbb{Z}\)-span if

• the restriction of the form to \( \langle S \rangle \) is nondegenerate,
• \( 0 \in S, \)
• for \( \alpha \in S \cap R^\times_{re}, \beta \in S \) and \( \gamma \in S \cap R_{ns} \) with \((\beta, \gamma) \neq 0, r_\alpha(\beta) \in S \) and \( \{ \gamma - \beta, \gamma + \beta \} \cap S \neq \emptyset. \)

**Proof.** See [16] Lem. 1.4 & Rem. 1.6(ii)].

**Definition 1.8.** Suppose that \((A, (\cdot, \cdot), R)\) is a locally finite root supersystem.
• The subgroup $W$ of $\text{Aut}(A)$ generated by $r_\alpha (\alpha \in R^e_{\text{re}})$ mapping $a \in A$ to $a - \frac{2(\alpha, \alpha)}{(\alpha, \alpha)} \alpha$, is called the Weyl group of $R$.

• A subset $S$ of $R$ is called a sub-supersystem if the restriction of the form to $\langle S \rangle$ is nondegenerate, $0 \in S$, for $\alpha \in S \cap R^e_{\text{re}}, \beta \in S$ and $\gamma \in S \cap R_{\text{ns}}$ with $(\beta, \gamma) \neq 0$, $r_\alpha(\beta) \in S$ and $\{\gamma - \beta, \gamma + \beta\} \cap S \neq \emptyset$.

• A sub-supersystem $S$ of $R$ is called closed if for $\alpha, \beta \in S$ with $\alpha + \beta \in R$, we have $\alpha + \beta \in S$.

• If $(A, \langle \cdot, \cdot \rangle, R)$ is irreducible, $R$ is said to be of real type if $\text{span}_Q R_{\text{re}} = Q \otimes \mathbb{Z} A$; otherwise, we say it is of imaginary type.

• If $\{R_i \mid i \in I\}$ is a class of sub-supersystems of $R$ which are mutually orthogonal with respect the form $(\cdot, \cdot)$ and $R \setminus \{0\} = \bigcup_{i \in I}(R_i \setminus \{0\})$, we say $R$ is the direct sum of $R_i$'s and write $R = \oplus_{i \in I} R_i$.

• The locally finite root supersystem $(A, \langle \cdot, \cdot \rangle, R)$ is called a locally finite root system if $R_{\text{ns}} = \{0\}$; see [14].

• $(A, \langle \cdot, \cdot \rangle, R)$ is said to be isomorphic to another locally finite root supersystem $(B, \langle \cdot, \cdot \rangle', R)$ if there is a group isomorphism $\varphi : A \rightarrow B$ and a nonzero scalar $r \in F$ such that $\varphi(R) = S$ and $(a_1, a_2) = r(\varphi(a_1), \varphi(a_2))'$ for all $a_1, a_2 \in A$.

**Lemma 1.9.** (a) If $\{(X_i, \langle \cdot, \cdot \rangle_i, S_i) \mid i \in I\}$ is a class of locally finite root supersystems, then for $X := \bigoplus_{i \in I} X_i$ and $(\cdot, \cdot) := \bigoplus_{i \in I} \langle \cdot, \cdot \rangle_i$, $(X, \langle \cdot, \cdot \rangle, S) := \bigcup_{i \in I} S_i$ is a locally finite root supersystem.

(b) Suppose that $(A, \langle \cdot, \cdot \rangle, R)$ is a locally finite root supersystem with Weyl group $W$. Then we have the following:

(i) Connectedness is an equivalence relation on $R \setminus \{0\}$. Also if $S$ is a connected component of $R \setminus \{0\}$, then $S \cup \{0\}$ is an irreducible sub-supersystem of $R$. Moreover, $R$ is a direct sum of irreducible sub-supersystems.

(ii) For $A_{\text{re}} := \langle R_{\text{re}} \rangle$ and $(\cdot, \cdot)_{\text{re}} := \langle \cdot, \cdot \rangle|_{A_{\text{re}} \times A_{\text{re}}}$, $(A_{\text{re}}, \langle \cdot, \cdot \rangle_{\text{re}}, R_{\text{re}})$ is a locally finite root system.

(iii) If $R$ is irreducible and $R_{\text{ns}} \neq \{0\}$, then $R_{\text{ns}}^e = W\delta \cup -W\delta$ for each $\delta \in R_{\text{ns}}$.

**Proof.** See [14], §3.

**Lemma 1.10.** Suppose that $V$ is a vector space equipped with a symmetric bilinear form and $R$ is a subset of $V$ such that $(\langle R \rangle, \langle \cdot, \cdot \rangle_{R \times R}, R)$ is a locally finite root supersystem. If $\{\alpha_1, \ldots, \alpha_n\} \subseteq R_{\text{re}}$ is $\mathbb{Q}$-linearly independent and $\delta \in R_{\text{ns}} \setminus \text{span}_Q \{\alpha_1, \ldots, \alpha_n\}$, then $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\delta, \alpha_1, \ldots, \alpha_n\}$ are $\mathbb{F}$-linearly independent.

**Proof.** We just show that $\{\delta, \alpha_1, \ldots, \alpha_n\}$ is $\mathbb{F}$-linearly independent; the other statement is similarly proved. Take $\{1, x_i \mid i \in I\}$ to be a basis for $\mathbb{Q}$-vector space $\mathbb{F}$. Suppose that $r, r_1, \ldots, r_n \in \mathbb{F}$ and $r\delta + \sum_{j=1}^n r_j \alpha_j = 0$. Suppose that for $1 \leq j \leq n$, $r_j = s_j + \sum_{i \in I} s^j_i x_i$ with $\{s_j, s^j_i \mid i \in I\} \subseteq Q$. We first show $r = 0$. To the contrary, assume $r \neq 0$. Without loss of generality, we assume $r = 1$. So $0 = \delta + \sum_{j=1}^n r_j \alpha_j = \delta + \sum_{j=1}^n (s_j + \sum_{i \in I} s^j_i x_i) \alpha_j$. Now for $\alpha \in R_{\text{re}}$, we have

$$2(\delta, \alpha) = 2 \sum_{j=1}^n s_j \frac{2(\alpha_j, \alpha)}{(\alpha, \alpha)} + \sum_{j=1}^n \sum_{i \in I} s^j_i x_i \frac{2(\alpha_j, \alpha)}{(\alpha, \alpha)} = 0.$$  

This implies that for $\alpha \in R_{\text{re}}$ and $i \in I$, $\sum_{j=1}^n s^j_i x_i \frac{2(\alpha_j, \alpha)}{(\alpha, \alpha)} = 0$ and so $(\sum_{j=1}^n s^j_i \alpha_j, \alpha) = 0$. But it follows from Lemma 1.9(b)(ii) that the form on $\text{span}_Q R_{\text{re}}$ is nondegenerate, so $\sum_{j=1}^n s^j_i \alpha_j = 0$ for all $i \in I$. Now as
\{\alpha_j \mid 1 \leq j \leq n\} is \mathbb{Q}\text{-linearly independent, we have}

\begin{equation}
    s_j^i = 0 \quad (i \in I, j \in \{1, \ldots, n\}).
\end{equation}

Therefore, we get \(0 = \delta + \sum_{j=1}^{n} r_j \alpha_j = \delta + \sum_{j=1}^{n} (s_j + \sum_{i \in I} s_j^i x_i) \alpha_j = \delta + \sum_{j=1}^{n} s_j \alpha_j\). Thus we have \(\delta = -\sum_{j=1}^{n} s_j \alpha_j\) which is absurd. This shows that \(r = 0\). Now repeating the above argument, one gets that \(s_j^i = 0\) for all \(i \in I, j \in \{1, \ldots, n\}\) and that \(0 = \sum_{j=1}^{n} s_j \alpha_j\). Thus we have \(s_j = 0\) for all \(1 \leq j \leq n\). This implies that \(r_j = s_j + \sum_{i \in I} s_j^i x_i = 0\) for all \(1 \leq j \leq n\) and so we are done. \(\square\)

Using Lemma 1.13 to know the classification of irreducible locally finite root supersystems, we first need to know the classification of locally finite root systems. Suppose that \(T\) is a nonempty index set with \(|T| \geq 2\) and \(\mathcal{U} := \bigoplus_{i \in T} \mathbb{Z}\varepsilon_i\) is the free \(\mathbb{Z}\)-module over the set \(T\). Define the form

\begin{equation}
    (,\cdot) : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{F}
\end{equation}

and set

\begin{equation}
    \hat{A}_T := \{\varepsilon_i - \varepsilon_j \mid i, j \in T\},
    \hat{D}_T := \hat{A}_T \cup \{\pm(\varepsilon_i + \varepsilon_j) \mid i, j \in T, i \neq j\},
    \hat{B}_T := \hat{D}_T \cup \{\pm \varepsilon_i \mid i \in T\},
    \hat{C}_T := \hat{D}_T \cup \{\pm 2\varepsilon_i \mid i \in T\},
    \hat{B}\hat{C}_T := \hat{B}_T \cup \hat{C}_T.
\end{equation}

These are irreducible locally finite root systems in their \(\mathbb{Z}\)-span’s. Moreover, each irreducible locally finite root system is either an irreducible finite root system or a locally finite root system isomorphic to one of these locally finite root systems. We refer to locally finite root systems listed in \([1, 11]\) as type \(A, D, B, C\) and \(BC\) respectively.

We note that if \(R\) is an irreducible locally finite root system as above, then \((\alpha, \alpha) \in \mathbb{N}\) for all \(\alpha \in R\). This allows us to define

\begin{align*}
    R_{sh} & := \{\alpha \in R^\times \mid (\alpha, \alpha) \leq (\beta, \beta); \text{ for all } \beta \in R\}, \\
    R_{ex} & := R \cap 2R_{sh} \quad \text{and} \quad R_{lg} := R^\times \setminus (R_{sh} \cup R_{ex}).
\end{align*}

The elements of \(R_{sh}\) (resp. \(R_{lg}, R_{ex}\)) are called short roots (resp. long roots, extra-long roots) of \(R\). We point out that following the usual notation in the literature, the locally finite root system of type \(A\) is denoted by \(\hat{A}\) instead of \(A\), as all locally finite root systems listed above are spanning sets for \(\mathbb{F} \otimes_\mathbb{Z} \mathcal{U}\) other than the one of type \(A\) which spans a subspace of codimension 1; see \([5]\) and \([16]\), Rem. 1.6(i)].

In the following two theorems, we give the classification of irreducible locally finite root supersystems.

**Theorem 1.12** ([14], Thm. 4.28]). Suppose that \(T, T'\) are index sets of cardinal numbers greater than 1 with \(|T| \neq |T'|\) if \(T, T'\) are both finite. Fix a symbol \(\alpha^*\) and pick \(t_0 \in T\) and \(p_0 \in T'\). Consider the free \(\mathbb{Z}\)-module \(X := \mathbb{Z}\alpha^* \oplus \bigoplus_{t \in T} \mathbb{Z}\varepsilon_t \oplus \bigoplus_{p \in T'} \mathbb{Z}\delta_p\) and define the symmetric form

\begin{equation}
    (,\cdot) : X \times X \longrightarrow \mathbb{F}
\end{equation}

by

\begin{align*}
    (\alpha^*, \alpha^*) & := 0, (\alpha^*, \varepsilon_{t_0}) := 1, (\alpha^*, \delta_{p_0}) := 1 \\
    (\alpha^*, \varepsilon_t) & := 0, (\alpha^*, \delta_q) := 0 \quad t \in T \setminus \{t_0\}, q \in T' \setminus \{p_0\} \\
    (\varepsilon_t, \delta_p) & := 0, (\varepsilon_t, \varepsilon_s) := \delta_{t,s}, (\delta_p, \delta_q) := -\delta_{p,q} \quad t, s \in T, p, q \in T'.
\end{align*}
Take $R$ to be $R_{re} \cup R_{ns}^+$ as in the following table:

| type      | $R_{re}$ | $R_{ns}^+$ |
|-----------|----------|------------|
| $A(0,T)$ | $\{\epsilon_t - \epsilon_s \mid t, s \in T\}$ | $\pm W\alpha^*$ |
| $C(0,T)$ | $\{\pm(\epsilon_t \pm \epsilon_s) \mid t, s \in T\}$ | $\pm W\alpha^*$ |
| $A(T,T')$ | $\{\epsilon_t - \epsilon_s - \delta_p - \delta_q \mid t, s \in T, p, q \in T'\}$ | $\pm W\alpha^*$ |

in which $W$ is the subgroup of $\text{Aut}(X)$ generated by the reflections $r_\alpha (\alpha \in R_{re} \setminus \{0\})$ mapping $\beta \in X$ to $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$, then $(A := (R), (\cdot, \cdot) |_{A \times A}, R)$ is an irreducible locally finite root supersystem of imaginary type and conversely, each irreducible locally finite root supersystem of imaginary type is isomorphic to one and only one of these root supersystems.

**Theorem 1.13 (14 Thm. 4.37).** Suppose $(X_1, (\cdot, \cdot), S_1), \ldots, (X_n, (\cdot, \cdot), S_n)$, for some $n \in \{2, 3\}$, are irreducible locally finite root systems. Set $X := X_1 + \cdots + X_n$ and $(\cdot, \cdot) := (\cdot, \cdot)_1 + \cdots + (\cdot, \cdot)_n$ and consider the locally finite root system $(X, (\cdot, \cdot), S := S_1 + \cdots + S_n)$. Take $W$ to be the Weyl group of $S$. For $1 \leq i \leq n$, we identify $X_i$ with a subset of $Q \otimes \mathbb{R} X_i$ in the usual manner. If $1 \leq i \leq n$ and $S_i$ is a finite root system of rank $\ell \geq 2$, we take $\{\omega^1_i, \ldots, \omega^\ell_i\} \subseteq Q \otimes \mathbb{R} X_i$ to be a set of fundamental weights for $S_i$ and if $S_i$ is one of infinite locally finite root systems $B_T, C_T, D_T$ or $BC_T$ as in (1.11), by $\omega^1_i$, we mean $e_1$, where 1 is a distinguished element of $T$. Also if $S_i$ is one of the finite root systems $\{0, \pm \alpha\}$ of type $A_1$ or $\{0, \pm \alpha, \pm 2\alpha\}$ of type $BC_1$, we set $\omega^1_i := \frac{1}{2} \alpha$. Consider $\delta^*$ and $R := R_{re} \cup R_{ns}^+$ as in the following table:

For $1 \leq i \leq n$, normalize the form $(\cdot, \cdot)_i$ on $X_i$ such that $(\delta^*, \delta^*) = 0$ and that for type $D(2, T)$, $(\omega^1_1, \omega^1_1)_1 = (\omega^1_1, \omega^1_2)_2$. Then $(\cdot, \cdot) |_{R \times R}, R)$ is an irreducible locally finite root supersystem of real type and conversely, if $(X, (\cdot, \cdot), R)$ is an irreducible locally finite root supersystem of real type, it is either an irreducible locally finite root system or isomorphic to one and only one of the locally finite root supersystems listed in the above table.

**Remark 1.14.** (i) We make a convention that from now on, for the types listed in column “type” of Theorems 1,12 and 1,13 we may use a finite index set $T$ and its cardinal number in place of each other, e.g., if $T$ is a nonempty finite set of cardinal number $\ell$, instead of type $B(1,T)$, we may write $B(1,\ell)$. We also mention that as we will see in Example 2,36 there is an extended affine Lie superalgebra with nonzero odd part whose root system is isomorphic to the locally finite root system of type $BC_T$. In order to distinguish the root system of that Lie superalgebra from the locally finite root system of type $BC_T$, we denote it by $B(0,T)$; more precisely, the underlying vector space of the locally finite root supersystem of type $B(0,T)$ is the same as the one of $BC_T$ and the corresponding symmetric form is the opposite of the defined form for type $BC_T$. 
(iii) Suppose \( R \) is of type \( X = C(T, T') \) or \( BC(T, T') \) with \( |T'| > 1 \). In what follows we collect some information regarding \( R \) which we will use in Lemma 2.16. Keep the same notation as in Theorem 1.13 and fix \( p, q \in T' \) with \( p \neq q \) and \( i \in T \) (\( i := 0 \) if \( |T| = 1 \)). If \( |T| = 1 \), set
\[
\epsilon := \begin{cases} \frac{1}{2} \epsilon_0 & \text{if } X = C(T, T') \\ \epsilon_0 & \text{if } X = BC(T, T'). \end{cases}
\]
Then for \( \gamma := \delta_p + \delta_q \),
\[
\theta_1 := \begin{cases} \epsilon_i + \delta_p & \text{if } |T| > 1 \\ \epsilon_i + \delta_p & \text{if } |T| = 1 \end{cases} \quad \text{and} \quad \theta_2 := \begin{cases} \epsilon_i - \delta_p & \text{if } |T| > 1 \\ \epsilon_i - \delta_p & \text{if } |T| = 1, \end{cases}
\]
we have \((\theta_1 + \theta_2, \gamma) = 0 \), \((\theta_1 - \theta_2, \theta_1 - \theta_2) = 2(\theta_1 - \theta_2, \gamma) \), \((\theta_1 - \theta_2, \gamma) = (\gamma, \gamma) \), \((\gamma, \theta_1), (\gamma, \theta_2) \neq 0 \) and the following elements are not roots:
\[
\begin{align*}
2(\theta_1 - \theta_2) - \gamma & \quad 2(\theta_1 - \theta_2) - 3\gamma \\
\theta_1 - 2\theta_2 - \gamma & \quad \theta_1 - 2\theta_2 - 2\gamma \\
2\theta_1 + \theta_2 - \gamma & \quad 2\theta_1 + \theta_2 + \gamma \\
2\theta_2 - 2\gamma & \quad 2\theta_2 + 2\gamma \\
2\theta_1 + \theta_2 & \quad \theta_1 + \gamma \\
2\theta_2 + \gamma & \quad \theta_2 - \gamma \\
\end{align*}
\]

**Lemma 1.15** (\cite{16} Lem. 2.3). Suppose that \( R \) is an irreducible locally finite root supersystem of type \( X \) in an abelian group \( A \). Then we have the following:

(i) \( A \) is a free abelian group and \( R \) contains a \( \mathbb{Z} \)-basis for \( A \).

(ii) If \( X \neq A(\ell, \ell) \), \( R \) contains a \( \mathbb{Z} \)-basis \( \Pi \) for \( A \) satisfying the partial sum property in the sense that for each \( \alpha \in R^\times \), there are \( \alpha_1, \ldots, \alpha_n \in \Pi \) (not necessarily distinct) and \( r_1, \ldots, r_n \in \{ \pm 1 \} \) with \( \alpha = r_1\alpha_1 + \cdots + r_n\alpha_n \) and \( r_1\alpha_1 + \cdots + r_t\alpha_t \in R^\times \), for all \( 1 \leq t \leq n \).

**Definition 1.16.** A subset \( \Pi \) of a locally finite root supersystem \( R \) is called an integral base for \( R \) if \( \Pi \) is a \( \mathbb{Z} \)-basis for \( A \). An integral base \( \Pi \) of \( R \) is called a base for \( R \) if it satisfies the partial sum property.

**Lemma 1.17** (\cite{16} Lem. 2.4(iii)). If \( R \) is an infinite irreducible locally finite root supersystem, then there is a base \( \Pi \) for \( R \) and a class \( \{ R_\gamma \mid \gamma \in \Gamma \} \) of finite irreducible closed sub-supersystems of \( R \) of the same type as \( R \) such that \( R \) is the direct union of \( R_\gamma \)’s and for each \( \gamma \in \Gamma , \Pi \cap R_\gamma \) is a base for \( R_\gamma \).

**Lemma 1.18** (\cite{16} Pro. 1.11(iv)). Suppose that \( (A, (\cdot, \cdot), R) \) is an extended affine root supersystem. Consider the canonical map \( - : A \to A/A^0 \) and denote the induced form on \( \widetilde{A} := A/A^0 \) by \( (\cdot, \cdot) \). Then \( (\widetilde{A}, (\cdot, \cdot), \widetilde{R}) \) is a locally finite root supersystem. Moreover, if \( R \) is irreducible, so is \( \widetilde{R} \).

**Definition 1.19.** Suppose that \( (A, (\cdot, \cdot), R) \) is an irreducible extended affine root supersystem. We define the type of \( R \) to be the type of \( \widetilde{R} \).

2. Locally Finite Basic Classical Lie Superalgebras

Throughout this section, we assume the field \( F \) is algebraically closed. Let us summarize some well-known facts regarding finite dimensional basic classical simple Lie superalgebras (f.d.b.c.s Lie superalgebras for short)
which we shall use in the sequel. For an irreducible locally finite root supersystem $R$ of type $X$, set

$$
0R := \begin{cases} 
\{ \alpha \in R_{re} \mid 2\alpha \not\in R \} \cup \{0\} & \text{if } X \neq BC(T, T') \\
R_{re} \setminus (R_{re}^2)_{sh} & \text{if } X = BC(T, T') \text{ and } R_{re} = R_{re}^1 \oplus R_{re}^2
\end{cases}
$$

and

$$
1R := R \setminus 0R.
$$

If $(\mathcal{L}, \mathcal{H}, (\cdot, \cdot))$ is an extended affine Lie superalgebra with $\mathcal{L}^0 = \mathcal{H}$ such that its corresponding root system $R$ is an irreducible locally finite root supersystem, we have $R_0 = 0R$ and $R_1 = 1R$; see [2.1], [2.6], Proposition 2.4 and Lemma 2.17. Finite dimensional basic classical simple Lie superalgebras are examples of such extended affine Lie superalgebras. A Cartan subalgebra of the even part of a f.d.b.c.s Lie superalgebra $\mathcal{L}$ is called a Cartan subalgebra of $\mathcal{L}$. It is known that Cartan subalgebras of $\mathcal{L}$ are conjugate under the group of automorphisms of $\mathcal{L}$; see [9, (3.1.2)] and [3 Cor. 16.4]. This together with the classification of f.d.b.c.s Lie superalgebras implies that the corresponding root system of a f.d.b.c.s Lie superalgebra with respect to an arbitrary Cartan subalgebra cannot be a finite root supersystem of types $BC(1, 1)$, $C(m, n)$ or $BC(m, n)$ with $n > 1$. Although, there exists no f.d.b.c.s. Lie superalgebra with root system $T := C(1, 2)$, the root system $R$ of either of $\mathfrak{osp}(3, 4)$ or $\mathfrak{osp}(5, 2)$ satisfies $R_0 = T_0$. One also knows that weight spaces of a f.d.b.c.s Lie superalgebra corresponding to nonzero roots are one dimensional other than the ones corresponding to nonzero odd roots of a f.d.b.c.s Lie superalgebra of type $A(1, 1)$ which are of dimension 2. Moreover, for a f.d.b.c.s Lie superalgebra of type $A(1, 1)$, there are nonorthogonal nonsingular roots $\delta_1, \delta_2$ such that both $\delta_1 + \delta_2$ and $\delta_1 - \delta_2$ are again roots. This is a phenomena which occurs just for type $A(1, 1)$ among all f.d.b.c.s. Lie superalgebras. We finally recall that the root system of $\mathfrak{osp}(3, 2)$ has the same even roots as $A(1, 1)$ and that it contains nonzero real odd roots while all nonzero odd roots of $A(1, 1)$ are nonsingular.

In this section, we study simple extended affine Lie superalgebras $(\mathcal{L}, \langle \cdot, \cdot \rangle, \mathcal{H})$ with $\mathcal{L}^0 = \mathcal{H}$. We show that the root system $R$ of such a Lie superalgebra $\mathcal{L}$ is an irreducible locally finite root supersystem. We show that $R$ contains a pair $(\delta_1, \delta_2)$ of nonorthogonal nonsingular roots with $\delta_1 + \delta_2, \delta_1 - \delta_2 \in R$ if and only if $R$ is of type $A(1, 1)$; in this case, the root spaces corresponding to nonzero nonsingular roots are 2 dimensional while in all other cases the root spaces corresponding to nonzero roots are 1 dimensional. We also prove that the derived subalgebra of the even part of the Lie superalgebra $\mathcal{L}$ is a semisimple Lie algebra and that $\mathcal{L}_1$ is a completely reducible $\mathcal{L}_0$-module with at most two irreducible constituents. We next conclude that $\mathcal{L}$ is a direct union of finite dimensional basic classical simple Lie subsuperalgebras.

From now on till the end of this section, we assume $(\mathcal{L}, \mathcal{H}, (\cdot, \cdot))$ is an extended affine Lie superalgebra with corresponding root system $R$ with $R^\times \neq \emptyset$. By [15] Pro. 3.10]

$$
(2.1)
R_0 \cap R_{ns} = \{0\}.
$$

Also for $\alpha \in R_i$ ($i = 0, 1$) with $(\alpha, \alpha) \neq 0$, by [15] Lem. 3.6], there are $e_\alpha \in (\mathcal{L}_i)^\alpha$ and $f_\alpha \in (\mathcal{L}_i)^{-\alpha}$ such that $(e_\alpha, f_\alpha, h_\alpha := \frac{\alpha}{(\alpha, \alpha)})$ is an $\mathfrak{sl}_2$-super-triple in the sense that

$$
(2.2)
[e_\alpha, f_\alpha] = h_\alpha, [h_\alpha, e_\alpha] = 2e_\alpha, [h_\alpha, f_\alpha] = -2f_\alpha.
$$
Moreover, the subsuperalgebra $\mathcal{G}(\alpha)$ of $\mathcal{G}$ generated by \{ $e_\alpha, f_\alpha, h_\alpha$ \} is either isomorphic to $\mathfrak{sl}_2$ or to $\mathfrak{osp}(1, 2) \simeq \mathfrak{osp}(2, 1)$; see [13, §2]. Now for $\alpha \in R_0 \setminus \{ 0 \}$ with $(\alpha, \alpha) \neq 0,$

\begin{equation}
(2.3) \quad \theta_\alpha := \exp(\text{ad}_{e_\alpha})\exp(-\text{ad}_{f_\alpha})\exp(\text{ad}_{e_\alpha})
\end{equation}

is an automorphism of $\mathcal{L}.$ Also one can easily check that $\theta_\alpha(h) = h - \alpha(h)h_\alpha$ for all $h \in \mathcal{H}.$ Now let $\beta \in R$ and $x \in \mathcal{L}^\beta.$ Suppose that $\theta_\alpha(x) = \sum_{k \in \mathbb{Z}} x_k$ in which $x_k \in \mathcal{L}^{\beta + k\alpha}.$ So for each $h \in \mathcal{H},$

$$\beta(h) \sum_{k \in \mathbb{Z}} x_k = \beta(h)\theta_\alpha(x) = \theta_\alpha[h, x]$$

$$= [\theta_\alpha(h), \theta_\alpha(x)]$$

$$= [h - \alpha(h)h_\alpha, \sum_{k \in \mathbb{Z}} x_k]$$

$$= \sum_{k \in \mathbb{Z}} (\beta + k\alpha)(h - \alpha(h)h_\alpha)x_k$$

$$= \sum_{k \in \mathbb{Z}} (\beta(h) - \alpha(h)(\beta(h) - k\alpha(h)x_k.$$  

This implies that if $x_k \neq 0$ for some $k \in \mathbb{Z}, k = -\beta(h_\alpha) = -\frac{2(\beta, \alpha)}{(\alpha, \alpha)},$ i.e.,

$$\theta_\alpha(\mathcal{L}^\beta) \subseteq \mathcal{L}^{r_\alpha(\beta)},$$

where $r_\alpha(\beta) := \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$

**Lemma 2.4.** Let $\alpha, \beta \in R,$ then $[\mathcal{L}^\alpha, \mathcal{L}^\beta] \neq \{ 0 \}$ if at least one of the following conditions is satisfied:

(i) $\alpha \in R^\times_\mathcal{L}$ and $\beta \in R$ with $\alpha + \beta \in R,$

(ii) $\alpha, \beta \in R^\times_{ns}$ with $\alpha + \beta \in R^\times.$

**Proof.** (i) Suppose that $\alpha \in R_i$ ($i \in \{ 0, 1 \}).$ Fix an $\mathfrak{sl}_2$-super triple $(y_\alpha, y_{-\alpha}, h_\alpha),$ with $y_\alpha \in (\mathcal{L}_i)^\alpha$ and $y_{-\alpha} \in (\mathcal{L}_i)^{-\alpha},$ corresponding to $\alpha.$ Take $\mathcal{G}(\alpha)$ to be the subsuperalgebra of $\mathcal{L}$ generated by \{ $y_\alpha, y_{-\alpha}, h_\alpha$ \}. Consider $\mathcal{L}$ as a $\mathcal{G}(\alpha)$-submodule and set $\mathcal{V} := \sum_{k \in \mathbb{Z}} \mathcal{L}^{\beta + k\alpha}$ which is a $\mathcal{G}(\alpha)$-submodule of $\mathcal{L}.$ For each $k \in \mathbb{Z}$ and $x \in \mathcal{L}^{\beta + k\alpha},$ take $\mathcal{V}_k(x)$ to be the submodule of $\mathcal{V}$ generated by $x.$ Then by the proof of Proposition 3.8 of [13], $\mathcal{V}_k(x)$ is finite dimensional and so $\mathcal{V}$ is completely reducible as $\mathcal{G}(\alpha)$ is a typical f.d.b.c.s Lie superalgebra. Moreover, the action of $h_\alpha$ on $\mathcal{V}$ is diagonalizable with the set of eigenvalues $\{ \beta(h_\alpha) + 2k \mid k \in \mathbb{Z}, \mathcal{L}^{\beta + k\alpha} \neq \{ 0 \} \}$ and $\mathcal{V}_{\beta(h_\alpha) + 2k} = \mathcal{L}^{\beta + k\alpha}$ for each $k \in \mathbb{Z}.$ Now as $\beta, \alpha + \beta \in R,$ we have $\beta(h_\alpha)$ and $\beta(h_\alpha) + 2$ are eigenvalues for the action of $h_\alpha$ on $\mathcal{V}.$ So considering [13 Lem. 2.4] as well as the module theory of $\mathfrak{sl}_2,$ there is a finite dimensional irreducible submodule $\mathcal{U}$ of $\mathcal{V}$ such that $\beta(h_\alpha), \beta(h_\alpha) + 2$ are eigenvalues for the action of $h_\alpha$ on $\mathcal{U},$ therefore using the $\mathfrak{sl}_2$-module theory together with [13 Cor. 2.5], we have $[y_\alpha, \mathcal{U}_{\beta(h_\alpha)}] \neq \{ 0 \}.$ But $[y_\alpha, \mathcal{U}_{\beta(h_\alpha)}] \subseteq [\mathcal{L}^\alpha, \mathcal{L}^\beta]$ and so we are done.

(ii) Set $\gamma := \alpha + \beta.$ Using Lemma [13, 15] and [16, Lem. 2.5], we have $\gamma \in R^\times_\mathcal{L}$ and $-\gamma + \alpha = -\beta \in R.$ Now part (i) implies that there are $x \in \mathcal{L}^-\gamma, y \in \mathcal{L}^\alpha$ with $0 \neq [x, y] \in \mathcal{L}^-\beta.$ Since the form on $\mathcal{L}^\beta \oplus \mathcal{L}^-\beta$ is nondegenerate, there is $z \in \mathcal{L}^\beta$ with $0 \neq ([x, y], z) = (x, [y, z])$ which implies that $0 \neq [y, z] \in [\mathcal{L}^\alpha, \mathcal{L}^\beta].$  

**Proposition 2.5.** If $\mathcal{L}^0 = \mathcal{H},$ then for $\alpha \in R^\times_{re}, \dim(\mathcal{L}^\alpha) = 1.$ Also for $\alpha \in R^\times_{re}, \alpha$ is an element of $R_0$ if and only if $2\alpha \notin R.$
Proof. We first note that using [15] Pro. 3.10(iii),

\[(2.6) \quad R^x \cap R_0 \cap R_1 = \emptyset.\]

Suppose that \(\alpha \in R^x \cap R_i\), for some \(i \in \{0,1\}\). One knows that there is an \(\mathfrak{sl}_2\)-super triple \((e,f,h)\) in which \(e \in (\mathcal{L}_i)^\alpha = \mathcal{L}^\alpha\) and \(f \in (\mathcal{L}_i)^{-\alpha} = \mathcal{L}^{-\alpha}\); see [2.0]. Take \(\mathcal{G}(\alpha)\) to be the subsuperalgebra of \(\mathcal{L}\) generated by \(\{e,f,h\}\) and set \(M := \mathcal{L}^{-2\alpha} \oplus \mathcal{L}^{-\alpha} \oplus \mathfrak{Ft}_\alpha \oplus \mathcal{L}^\alpha \oplus \mathcal{L}^{2\alpha}\). Then by (1.3) and the fact that the only scalar multiples of \(\alpha\) which can be roots are \(0, \pm 2\alpha, \pm \alpha, \pm (1/2)\alpha\), we get that \(M\) is a \(\mathcal{G}(\alpha)\)-submodule of \(\mathcal{L}\). Now using the same argument as in part (i) of Lemma [2.4] we get that \(M\) is completely reducible. Therefore, [15] Lem. 2.4] together with the \(\mathfrak{sl}_2\)-module theory implies that the number of irreducible constituents in the decomposition of \(M\) into irreducible submodules is the dimension of the 0-eigenspace. This means that \(M\) is a finite dimensional irreducible \(\mathcal{G}(\alpha)\)-module and so again using [15] Lem. 2.4] together with the \(\mathfrak{sl}_2\)-module theory, we get \(\text{dim}(\mathcal{L}^\alpha) = 1\). Now suppose \(\alpha \in R_0\), then by (2.6), \(\mathcal{L}^\pm \alpha = (\mathcal{L}_0)^\pm \alpha\) and so \([\mathcal{L}^\alpha, \mathcal{L}^\alpha] = \{0\} = [\mathcal{L}^{-\alpha}, \mathcal{L}^{-\alpha}]\) which in turn implies that \(\mathcal{L}^{-\alpha} \oplus \mathfrak{Ft}_\alpha \oplus \mathcal{L}^\alpha\) is a \(\mathcal{G}(\alpha)\)-submodule of \(M\). But \(M\) is irreducible and so \(M = \mathcal{L}^{-\alpha} \oplus \mathfrak{Ft}_\alpha \oplus \mathcal{L}^\alpha\). This together with [15] Lem. 3.6] completes the proof.

Lemma 2.7. Let \(\mathcal{L}^0 = \mathcal{H}\). We have the following statements:

(i) For \(\alpha, \beta \in R\) with \((\alpha, \beta) \neq 0\), if \(\alpha - \beta \notin R\), then for nonzero elements \(x \in \mathcal{L}^\alpha, y \in \mathcal{L}^\beta\), we have \([x,y] \neq 0\).

(ii) Suppose ((\(\langle \cdot, \cdot \rangle\), \(\cdot \)), \(\cdot\)) is a locally finite root supersystem. Assume \(S\) is a sub-supersystem of \(R\) and set \(\mathcal{H}_S := \mathcal{H} \cap \sum_{\alpha \in S} [\mathcal{L}^\alpha, \mathcal{L}^{-\alpha}]\). Take \(\pi : \mathcal{H}^* \rightarrow \mathcal{H}_S^*\) to be the function mapping \(\alpha \in \mathcal{H}^*\) to \(\alpha |_{\mathcal{H}_S}\). Then for \(\alpha, \beta \in \langle S \rangle\) with \(\alpha - \beta \in \langle S \rangle\), if \(\pi(\alpha) = \pi(\beta)\), we have \(\alpha = \beta\); in particular \(\pi\) restricted to \(\langle S \rangle\) is injective.

Proof. (i) Contemplating (2.6), (2.1) and (1.4), one finds \(i \in \{0,1\}\) with \(\mathcal{L}^\alpha = (\mathcal{L}_i)^\alpha, \mathcal{L}^{-\alpha} = (\mathcal{L}_i)^{-\alpha}\) and \(\mathcal{L}^\beta = (\mathcal{L}_j)^\beta\). Suppose that \(x \in \mathcal{L}^\alpha, y \in \mathcal{L}^\beta\) are nonzero elements. Using (1.3) together with the fact that the form on \(\mathcal{L}^\alpha \oplus \mathcal{L}^{-\alpha}\) is nondegenerate, we pick \(z \in \mathcal{L}^{-\alpha}\) with \([x,z] = t_\alpha\). Since \(\alpha - \beta \notin R\), we have \([z,y] = 0\). So

\[0 \neq \beta(t_\alpha)y = [t_\alpha, y] = [x,z, y] = [x, [z, y]] - (-1)^{|x||z|} [z, [x, y]] = -(-1)^{|x||z|} [z, [x, y]].\]

This shows that \([x,y] \neq 0\) and so we are done.

(ii) Suppose that \(\alpha, \beta \in \langle R \rangle\) and \(\alpha - \beta \in \langle S \rangle\). If \(\alpha - \beta \neq 0\), then since the form \((\cdot, \cdot)\) restricted to \(\langle S \rangle\) is nondegenerate, one finds \(\gamma \in S\) with \((\alpha - \beta, \gamma) \neq 0\). This means that \((\alpha - \beta)(t_\gamma) \neq 0\). But by (2.2), \(t_\gamma \in [\mathcal{L}^\gamma, \mathcal{L}^{-\gamma}] \subseteq \mathcal{H}_S\), so \(\pi(\alpha) \neq \pi(\beta)\). This completes the proof.

Lemma 2.8. Suppose that \(\mathcal{L}^0 = \mathcal{H}\). Assume \(\delta_1, \delta_2 \in R_{ns}\) with \((\delta_1, \delta_2) \neq 0\) and \(\alpha := \delta_1 + \delta_2, \beta := \delta_1 - \delta_2 \in R\). Consider (1.3) and fix \(e \in \mathcal{L}^\alpha, f \in \mathcal{L}^{-\alpha}, x \in \mathcal{L}^\beta\) and \(y \in \mathcal{L}^{-\beta}\) with \([e, f] = t_\alpha\) and \([x, y] = t_\beta\). Then we have the following:

(i) For nonzero elements \(a \in \mathcal{L}^{\delta_1}\) and \(b \in \mathcal{L}^{-\delta_1}\), we have

\[[f, a], [f, [y, a]], [x, b], [x, [e, b]] \neq 0\quad \text{and} \quad (a, [f, [y, a]]) = (b, [x, [e, b]]) = 0.\]

(ii) For \(a, c \in \mathcal{L}^{\delta_1}, b, d \in \mathcal{L}^{-\delta_1}\) with
We next note that for \(i,j\)

\[\text{Proof.} \]

\[2r \delta (2.10) 2\]

This means that \(a\) is nondegenerate.

\(r\delta \in L\)

\[\text{in particular, we have} \]

\[\text{Lemma 1.18. Therefore, (1.3) together with Proposition 2.5} \]

imply that

\[\alpha, \beta \in R_0\]

and

\[\dim L_0^\alpha = \dim L_0 = \dim L_0^{-\alpha} = \dim L^{-\alpha} = 1, \]

\[\dim L_0^\beta = \dim L_0 = \dim L_0^{-\beta} = \dim L^{-\beta} = 1;\]

in particular, we have

\[(2.9) \]

\[L_0^\alpha = Fe, L_0^{-\alpha} = Ff, L_0^\beta = Fx, L_0^{-\beta} = Fy.\]

We next note that for \(i,j\) with \(\{i,j\} = \{1, 2\}\) and \(r, s \in \{\pm 1\}\), we have

\[\frac{2(\delta_i, 2r \delta_1 + s \delta_j)}{(2r \delta_i + s \delta_j, 2r \delta_1 + s \delta_j)} = \frac{2s(\delta_i, \delta_j)}{4rs(\delta_i, \delta_j)} = 1/2r \not\in \mathbb{Z}.\]

This means that

\[(2.10) \]

\[2r \delta_i + s \delta_j \not\in R; \quad \{i,j\} = \{1, 2\} \quad \text{and} \quad r, s \in \{\pm 1\}.\]

Fix \(a \in L^{\delta_1}, b \in L^{-\delta_1}\), then by Lemma 2.7(i) and (2.10), we have

\([f, a] \neq 0, \quad [f, [y, a]] \neq 0, \quad [x, b] \neq 0, \quad [x, [e, b]] \neq 0.\]

Next set

\[M := Ft_\alpha + Ft_\beta + L^\alpha + L^{-\alpha} + L^\beta + L^{-\beta} + Fa + F[f, a] + F[y, a] + F[f, y, a]\]

and to the contrary assume \(-([f, a], [y, a]) = ([a, f], [y, a]) = (a, [f, [y, a]]) \neq 0.\) We recall that

\[L^0 = \mathcal{H} \quad \text{and} \quad \pm 2a, \pm 2\beta, \pm 2\delta_1, \pm 2\delta_2, \pm 2\delta_1 \pm \delta_2, \pm 2\delta_2 \pm \delta_1 \not\in R;\]

see (2.10), Lem. 2.1(ii) and Lemma 1.18 Therefore, (1.8) together with Proposition 2.3 implies that \(M\) is a subsuperalgebra of \(L\) with

\[M_0 = Ft_\alpha + Ft_\beta + L^\alpha + L^{-\alpha} + L^\beta + L^{-\beta} \quad \text{and} \quad M_1 = Fa + F[f, a] + F[y, a] + F[f, y, a].\]
Now contemplate the fact that the form is supersymmetric and consider the matrix

$$\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}$$

in which

$$A := \begin{pmatrix} (a, b) & (c, b) & (e, [x, b], b) & (e, [x, d], b) \\ (a, [f, [y, a]]) & (c, [f, [y, a]]) & (e, [x, b], [f, [y, a]]) & (e, [x, d], [f, [y, a]]) \\ (a, [f, [y, c]]) & (c, [f, [y, c]]) & (e, [x, b], [f, [y, c]]) & (e, [x, d], [f, [y, c]]) \\ (a, d) & (c, d) & (e, [x, b], d) & (e, [x, d], d) \end{pmatrix}$$

If $r_1 a + r_2 c + r_3 e, [x, b] + r_4 e, [x, d] + r_5 b + r_6 f, [y, a] + r_7 f, [y, c] + r_8 d$ is an element of the radical of the form on $X \times X$. Then since by (1.1), we have

$$\langle Fa + Fc + Fe, e, [x, b] + Fc, [x, d], Fa + Fc + Fe, e, [x, b] + Fc, [x, d] \rangle \subseteq \langle L^{\delta_1}, L^{\delta_1} \rangle = \{0\}$$

and

$$\langle Fb + Fd + Ff, [y, a] + Ff, [y, c], Fb + Fd + Ff, [y, a] + Ff, [y, c] \rangle \subseteq \langle L^{-\delta_1}, L^{-\delta_1} \rangle = \{0\},$$

$(r_1, \ldots, r_8)^t$ (t indicates the transposition) is a solution for $BX = 0$. But

$$\det(B) = -(\det A)^2 = - (\delta_1, \delta_2)^2(a, b)^4(c, d)^4 \neq 0,$$

so $BX = 0$ has sole solution 0. This in particular implies that the form restricted to $X \times X$ is nondegenerate.

Next suppose that

$$P := r_1[f, a] + r_2[f, c] + r_3[x, b] + r_4[x, d] + r_5[e, c] + r_6[e, d] + r_7[y, a] + r_8[y, c]$$

is an element of the radical of the form restricted to $Y \times Y$. Using (1.1), we have

$$\langle r_1[f, a] + r_2[f, c] + r_3[x, b] + r_4[x, d], Fa + Fc + Fe, e, [x, b] + Fc, [x, d] \rangle = \{0\},$$

$$\langle r_5[e, c] + r_6[e, d] + r_7[y, a] + r_8[y, c], Ff, a + Ff, c + Ff, e, e, [x, b] + Ff, [x, d] \rangle = \{0\}.$$
This together with (1.11) implies that
\[
m' := r_1a + r_2c + r_3(\delta_1, \delta_2)^{-1}[e, [x, b]] + r_4(\delta_1, \delta_2)^{-1}[e, [x, d]]
\]
and
\[
n' := r_5b + r_6d + r_7(\delta_1, \delta_2)^{-1}[f, [y, a]] + r_8(\delta_1, \delta_2)^{-1}[f, [y, c]]
\]
are elements of the radical of the form on \(X \times X\). Therefore, \(m' = n' = 0\) and so \(m = [f, m'] = 0\) and \(n = [e, n'] = 0\). Thus \(P = 0\). This completes the proof. \(\square\)

**Proposition 2.14.** Suppose that \(\mathcal{L}^0 = \mathcal{H}\). Then for \(\delta \in R^\times_{ns}\), \(\dim(\mathcal{L}^\delta) = 1\) if there is \(\eta \in R_{ns}\) with \(\delta - \eta \not\in R\) and \((\delta, \eta) \neq 0\); otherwise, \(\dim(\mathcal{L}^\delta) = 2\). In particular, if \(R\) is an irreducible locally finite root supersystem of imaginary type, \(\dim(\mathcal{L}^\delta) = 1\) for \(\delta \in R^\times_{ns}\).

**Proof.** Suppose that \(\delta \in R^\times_{ns}\). If there is \(\eta \in R_{ns}\) with \((\delta, \eta) \neq 0\) and \(\delta - \eta \not\in R\), then \(\delta + \eta \in R^\times_{re}\) and so we have \(\dim(\mathcal{L}^\delta + \eta) = 1\) by Proposition 2.3. Use (1.3) to fix \(x \in \mathcal{L}^\eta\) and \(y \in \mathcal{L}^{-\eta}\) with \([x, y] = t_\eta\) and consider the map \(\varphi : \mathcal{L}^\delta \rightarrow \mathcal{L}^{\delta + \eta}\) mapping \(a \in \mathcal{L}^\delta\) to \([x, a]\). Then by Lemma 2.7(i), the map \(\varphi\) is an injective linear map from \(\mathcal{L}^\delta\) into a one dimensional vector space and so it is onto, in particular, \(\dim(\mathcal{L}^\delta) = \dim(\mathcal{L}^{\delta + \eta}) = 1\).

Now suppose for each nonsingular root \(\eta \in R\), either \((\delta, \eta) = 0\) or \(\delta - \eta \in R\). Then using (10) Lem. 1.3(ii)], there is \(\eta \in R_{ns}\) with \(\delta + \eta, \delta - \eta \in R^\times_{re}\). Set
\[
\delta_1 := \delta, \delta_2 := \eta \in R^\times_{ns}\quad \text{and} \quad \alpha := \delta_1 + \delta_2, \beta := \delta_1 - \delta_2 \in R^{	imes}_{re}.
\]
As in the proof of Lemma 2.8 we have
\[
(2.15) \quad \alpha, \beta \in R_0.
\]
Since the form is nondegenerate on \(\mathcal{L}^{\delta \pm 1} \oplus \mathcal{L}^{-\delta \pm 1}\), we pick \(a \in \mathcal{L}^{\delta 1}\) and \(b \in \mathcal{L}^{-\delta 1}\) with \((a, b) \neq 0\). Next considering (1.3), we fix \(e \in \mathcal{L}^\alpha, f \in \mathcal{L}^{-\alpha}, x \in \mathcal{L}^{\beta}\) and \(y \in \mathcal{L}^{-\beta}\) with \([e, f] = t_{\alpha}\) and \([x, y] = t_{\beta}\) and set
\[
a_1 := a \in \mathcal{L}^{\delta 1}, \quad a_2 := [f, a] \in \mathcal{L}^{-\delta 2}, \quad a_3 := [y, a] \in \mathcal{L}^{\delta 2}, \quad a_4 := [f, [y, a]] \in \mathcal{L}^{-\delta 1},
\]
\[
b_1 := b \in \mathcal{L}^{-\delta 1}, \quad b_2 := [c, b] \in \mathcal{L}^{\delta 2}, \quad b_3 := [x, b] \in \mathcal{L}^{-\delta 2}, \quad b_4 := [e, [x, b]] \in \mathcal{L}^{\delta 1}.
\]
Then using (10) Lem. 2.1(ii)], (2.15), (2.10), (2.14) and (1.3), one can see that \(M := M_0 \oplus M_1\) with
\[
M_0 := \mathbb{F}t_\alpha + \mathbb{F}t_\beta + \mathbb{F}e + \mathbb{F}f + \mathbb{F}x + \mathbb{F}y \quad \text{and} \quad M_1 := \sum_{i=1}^4 (\mathbb{F}a_i + \mathbb{F}b_i)
\]
is a finite dimensional subsuperalgebra of \(\mathcal{L}\). It follows from Lemma 2.8(i) and (2.14) that the form restricted to \(M\) is nondegenerate and so by Lemma 1.4 \(M\) is a finite dimensional basic classical simple Lie superalgebra with \(M_0 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2\). So \(M\) is a finite dimensional basic classical simple Lie superalgebra isomorphic to \(A(1,1)\) as all odd roots are nonsingular. Therefore, its root spaces corresponding to nonzero nonsingular roots are 2-dimensional. This in particular implies that \(a_1\) and \(b_1\) are linearly independent and so \(\dim(\mathcal{L}^{\delta 1}) \geq 2\). Now to the contrary assume \(c \in \mathcal{L}^{\delta 1}\) is linearly independent from \(\{a_1, b_1\}\). Since the form is supersymmetric and invariant, we have \((a, [f, [y, a]]) = - (c, [f, [y, a]])\). Now contemplating (2.11) and Lemma 2.8(i), we set
\[
A := 
\begin{pmatrix}
(a, b) & (c, b) & ([e, [x, b]], b) \\
(a, [f, [y, a]]) & (c, [f, [y, a]]) & ([e, [x, b]], [f, [y, a]]) \\
(a, [f, [y, c]]) & (c, [f, [y, c]]) & ([e, [x, b]], [f, [y, c]])
\end{pmatrix}
\]
and
\[
= 
\begin{pmatrix}
(a, b) & (c, b) & 0 \\
0 & (c, [f, [y, a]]) & ([e, [x, b]], [f, [y, a]]) \\
(a, [f, [y, c]]) & 0 & ([e, [x, b]], [f, [y, c]])
\end{pmatrix}
\]
and
\[
= 
\begin{pmatrix}
(a, b) & (c, b) & 0 \\
0 & (c, [f, [y, c]]) & ([\delta_1, \delta_2]^2(a, b)) \\
(a, [f, [y, c]]) & 0 & ([\delta_1, \delta_2]^2(c, b))
\end{pmatrix}
\]
Since $\det(A) = 0$, the system $AX = 0$ has a nonzero solution $(r, s, k)^t$ (t indicates the transposition). Setting $c' := ra + sc + k[e, [x, b]]$, we have $c' \neq 0$ as $\{a, c, [e, [x, b]]\}$ is linearly independent. Also since $A(r, s, k)^t = 0$,

$$(c', b) = (c', [f, [y, a]]) = (c', [f, [y, c]]) = 0.$$ 

Suppose that $s = 0$, then $A(r, s, k)^t = 0$ implies that $r(a, b) = k(\delta_1, \delta_2)^2(a, b) = 0$. So we have $r = s = k = 0$ which is absurd. Thus $s \neq 0$. This implies that $\{a, c, [e, [x, b]]\}$ is linearly independent. Next considering the fact that the form is nondegenerate on $L^{\delta_1} \oplus L^{-\delta_1}$, one finds $d \in L^{-\delta_1}$ with $(c', d) \neq 0$. If $(a, d) = 0$, take $a' := a$, otherwise set $a' := -\frac{(c', d)}{(a, d)}a + c'$, then $(a', d) = 0$ and $-(a', [f, [y, c']]) = (c', [f, [y, a']]) = 0$. Now using Lemma 2.8(ii), it is not hard to see that the form is nondegenerate on $K \times K$ where $K := K_0 \oplus K_1$ with

$$K_0 := F\alpha + F\gamma + L^\alpha + L^{-\alpha} + L^\beta + L^{-\beta}$$

and

$$K_1 := F\alpha' + F\epsilon + F[e, [x, b]] + F[e, [x, d]] + Fb + Fd + F[f, [y, a']] + F[f, [y, c']] + F[f, a'] + F[f, c'] + F[x, d] + F[e, b] + F\epsilon, a, c + F\epsilon, d + F[y, a'] + F[y, c'].$$ 

So by Lemma 1.24, $K$ is a finite dimensional basic classical simple Lie superalgebra with $K_0 \simeq s_k \oplus s_2$. Since all odd roots are nonsingular, we get that $K$ is of type $A(1, 1)$ which is absurd due to its dimension. This completes the proof of the first assertion. Now suppose $R$ is an irreducible locally finite root supersystem of imaginary type and $\delta \in R_{ns}$, then by [10] Lem. 1.3(ii)], there is $\eta \in R_{ns}$ with $(\delta, \eta) \neq 0$; on the other hand, we know from the classification theorem for imaginary types that there are no two nonorthogonal nonsingular roots whose both summation and subtraction are again roots. So there is $s \in \{\pm 1\}$ with $\delta + s\eta \in R$ and $\delta - s\eta \notin R$. Now the result follows from the first assertion.

**Lemma 2.16.** Suppose that $L^0 = \mathcal{H}$. If $R$ is an irreducible extended affine root supersystem of type $X \neq A(1, 1), BC(1, 1)$, then for $\delta_1, \delta_2 \in R_{ns}$ with $(\delta_1, \delta_2) \neq 0$, there is a unique $r \in \{\pm 1\}$ with $\delta_1 + r\delta_2 \in R$.

**Proof.** Suppose that $\delta_1, \delta_2 \in R_{ns}$ with $(\delta_1, \delta_2) \neq 0$. Since $R$ is an extended affine root supersystem, there is at least one $r \in \{\pm 1\}$ with $\delta_1 + r\delta_2 \in R$. Now to the contrary, we assume $\alpha := \delta_1 + \delta_2, \beta := \delta_1 - \delta_2 \in R$ and get a contradiction. Since $X \neq A(1, 1), BC(1, 1)$ and there are two non-orthogonal nonsingular roots whose summation and subtraction are again roots, using Lemma 1.13 together with Theorems 1.12 and 1.13 we get that $R$ is of one of the types $C(1, T)$ or $C(T, T')$, $BC(T, T')$ with $|T'| > 1$. Considering Lemma 1.18 and Theorem 1.13 together with Remark 1.14(iii), there is $\gamma \in R_{re}$ such that $(\alpha, \gamma) = 0$, $(\gamma, \delta_1), (\gamma, \delta_2) \neq 0$, $(\beta, \gamma) = 2(\beta, \gamma), 2(\beta, \gamma) = 2$ and that the following elements are not elements of $R$:

$$
\begin{array}{cccc}
2(\delta_1 - \delta_2) - \gamma & 2(\delta_1 - \delta_2) - 3\gamma & \delta_1 - \delta_2 - 3\gamma & 2(\delta_1 - \delta_2) - \gamma \\
\delta_1 - 2\delta_2 - \gamma & 2\delta_1 - 2\delta_2 - 2\gamma & \delta_1 - 2\delta_2 - 2\gamma & 2\delta_1 - 2\delta_2 - 3\gamma \\
2\delta_1 + \delta_2 + \gamma & \delta_1 - \delta_2 + \gamma & \delta_1 + \delta_2 + \gamma & \delta_1 + \delta_2 - \gamma \\
2\delta_2 - 2\gamma & 2\delta_2 + 2\gamma & 2\delta_1 - 2\gamma & \delta_1 - 2\gamma \\
2\delta_1 + \delta_2 & 2\delta_1 - \delta_2 & \delta_1 + 2\delta_2 & \delta_1 - 2\delta_2 \\
2\delta_2 + \gamma & \delta_1 + \gamma & \delta_2 - \gamma & 2\delta_1 \\
2\delta_1 - \gamma & 2\delta_1 - \gamma & 2\delta_1 - \gamma & 2\delta_1 - \gamma
\end{array}
$$

Since $(\gamma, \delta_1), (\gamma, \delta_2) \neq 0$ and $\gamma + \delta_1 \notin R, \gamma - \delta_2 \notin R$, we have $\eta := \delta_1 - \gamma, \zeta := -\gamma - \delta_2 \in R$. We next note that $2(\beta, \gamma) = 2$ and that $\beta + \gamma \notin R$, so we have using the root string property that $\beta - \gamma, \beta - 2\gamma \in R_{re}$. In particular,
we have

\[ \eta, \zeta \in R_{n_a}^\alpha \text{ and } \eta + \zeta = \beta - 2\gamma, \eta - \zeta = \alpha \in R. \]

Fix \( e \in \mathcal{L}^\alpha, f \in \mathcal{L}^{-\alpha}, x \in \mathcal{L}^\beta, y \in \mathcal{L}^{-\beta}, m \in \mathcal{L}^\gamma \) and \( n \in \mathcal{L}^{-\gamma} \) with \([e, f] = t_\alpha, [x, y] = t_\beta \) and \([m, n] = t_\gamma\).

Since the form on \( \mathcal{L}^{\delta_1} \oplus \mathcal{L}^{\delta_2} \) is nondegenerate, one can find \( a \in \mathcal{L}^{\delta_1} \) and \( b \in \mathcal{L}^{-\delta_1} \) with \((a, b) \neq 0\). Take \( a_i, b_i \), \( 1 \leq i \leq 8 \), to be as in the following table:

| \( a_1 \) | \( a \in \mathcal{L}^{\delta_1} \) | \( b_1 \) | \( b \in \mathcal{L}^{-\delta_1} \) |
|---|---|---|---|
| \( a_2 \) | \( f, a \in \mathcal{L}^{-\delta_2} \) | \( b_2 \) | \( e, b \in \mathcal{L}^{\delta_2} \) |
| \( a_3 \) | \( y, a \in \mathcal{L}^{\delta_2} \) | \( b_3 \) | \( x, b \in \mathcal{L}^{-\delta_2} \) |
| \( a_4 \) | \( f, [y, a] \in \mathcal{L}^{-\delta_1} \) | \( b_4 \) | \( e, [x, b] \in \mathcal{L}^{\delta_1} \) |
| \( a_5 \) | \( n, a \in \mathcal{L}^\eta \) | \( b_5 \) | \( m, b \in \mathcal{L}^{-\eta} \) |
| \( a_6 \) | \( n, [f, a] \in \mathcal{L}^\xi \) | \( b_6 \) | \( m, [e, b] \in \mathcal{L}^{-\xi} \) |
| \( a_7 \) | \( m, [y, a] \in \mathcal{L}^{-\xi} \) | \( b_7 \) | \( n, [x, b] \in \mathcal{L}^\xi \) |
| \( a_8 \) | \( m, [f, y, a] \in \mathcal{L}^{-\eta} \) | \( b_8 \) | \( n, [e, [x, b]] \in \mathcal{L}^\eta \) |

and note that for \( 1 \leq i \leq 8 \), \((a_i, b_i) \neq 0\) as the form is invariant as well as supersymmetric and \((a, b) \neq 0\). This in turn implies that \([a_i, b_i] \neq 0\) by (1.3).

Next we note that if \( t_\beta = k t_\gamma \), then \( 2(\beta, \gamma) = (\beta, \beta) = k(\beta, \gamma) = k(\gamma, \gamma) = (\beta, \gamma) \) which is a contradiction. Also if \( t_\alpha \in \text{span}_F \{t_\beta, t_\gamma\} \), one concludes \((t_\alpha, t_\alpha) = 0\) which is absurd, so \( \{t_\alpha, t_\beta, t_\gamma\} \) is a linearly independent subset of \( H \). Now set

\[ K := \mathbb{F}t_\alpha + \mathbb{F}t_\beta + \mathbb{F}t_\gamma + \mathbb{F}t_{\beta - \gamma} + \mathbb{F}t_{\beta - 2\gamma} = \mathbb{F}t_\alpha \oplus \mathbb{F}t_\beta \oplus \mathbb{F}t_\gamma \]

and

\[ M = M_0 \oplus M_1 \]

with

\[ M_0 := K \oplus \oplus_{\theta \in \{\pm1, \pm\alpha, \pm\beta, \pm\eta, \pm(\beta - \gamma), \pm(\beta - 2\gamma)\}} \mathcal{L}^\theta, \quad M_1 := \sum_{i=1}^8 (\mathbb{F}a_i + \mathbb{F}b_i). \]

We claim that \( M \) is a subsuperalgebra of \( \mathcal{L} \). To prove our claim, we need to know the multiplication table of the elements of a basis of \( M \). Since \([\mathcal{L}_1, \mathcal{L}_1] \subseteq \mathcal{L}_0 \), we first show that for \( z, z' \in \{a_i, b_i \mid 1 \leq i \leq 8\} \), \([z, z']\) is an element of \( M_0 \). In the following tables, we denote the weight \([z, z']\) for all \( z, z' \in \{a_i, b_i \mid 1 \leq i \leq 8\} \); in fact the \((r, s)\)-th entry of each table denotes the summation of the weights of \( r \)-th entry of the first column and \( s \)-th entry of the first row; the red parts denote the elements which are not roots.

| \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_5 \) | \( a_6 \) | \( a_7 \) | \( a_8 \) |
|---|---|---|---|---|---|---|---|
| \( a_2 \) | \( \beta \) | \( -\beta \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( a_3 \) | \( 0 \) | \( \gamma \) | \( -\gamma \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( a_4 \) | \( 0 \) | \( 0 \) | \( \zeta \) | \( -\zeta \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( a_5 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( a_6 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( a_7 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( a_8 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

| \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | \( b_6 \) | \( b_7 \) | \( b_8 \) |
|---|---|---|---|---|---|---|---|
| \( b_2 \) | \( -\beta \) | \( \beta \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( b_3 \) | \( \gamma \) | \(-\gamma \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( b_4 \) | \( 0 \) | \( 0 \) | \( \zeta \) | \(-\zeta \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( b_5 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( b_6 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( b_7 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( b_8 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
Using these two tables together with \((\mathcal{M}, \mathcal{M})\), we get that \([M_1, M_3] \subseteq M_0\). Now we want to show that \([M_0, M_1] \subseteq M_1\). Considering Lemmas 2.8(i), 2.9(i) and Proposition 2.6, we have

\[
\mathcal{L}^{\beta-\gamma} = \mathcal{L}^{\delta_1-\delta_2-\gamma} = \mathcal{F}[a_2, a_5] = \mathcal{F}[b_3, b_8] = \mathcal{F}[a_1, a_6] = \mathcal{F}[b_4, b_7],
\]

\[
\mathcal{L}^{\beta+\gamma} = \mathcal{L}^{\delta_1+\delta_2+\gamma} = \mathcal{F}[a_3, a_8] = \mathcal{F}[b_2, b_5] = \mathcal{F}[b_1, b_6] = \mathcal{F}[a_4, a_7].
\]

Also contemplating Lemma 2.4 as well as Proposition 2.5 and setting

\[
u := [a_2, a_5], \quad \nu := [a_3, a_8], \quad z := [m, v], \quad w := [n, u],
\]

we get that

\[
\mathcal{L}^{2\gamma-\beta} = \mathcal{L}^{2\gamma-\delta_1+\delta_2} = F_Z \quad \text{and} \quad \mathcal{L}^{\beta-2\gamma} = \mathcal{L}^{-2\gamma+\delta_1-\delta_2} = F_w,
\]

as \((\delta_2, \eta), (\zeta, \delta_1) \neq 0\) and \(\eta + \delta_2, \delta_1 - \zeta \notin R\). We consider the following tables; as before, the \((r, s)\)-th entry of the first row denotes the summation of the weights of \(r\)-th entry of the first column and \(s\)-th entry of the first row. The red parts denote the elements which are not roots.

Now one can easily check the following multiplication table:
Take \( r_1, \ldots, r_7, s_1, \ldots, s_6, k_1, \ldots, k_4 \in F \setminus \{0\} \) to be such that

\[
[\alpha, \beta] = [\alpha, \delta] = \delta \eta, \quad [\beta, \gamma] = \delta, \quad [\gamma, \delta] = \delta \eta, \quad [\delta, \eta] = \delta, \quad [\eta, \delta] = \delta \eta,
\]

Then we have

\[
[a_1, v] = -[v, a_1] = -[(a_3, a_8), a_1] = -[a_3, [a_8, a_1]] - [a_8, [a_3, a_1]] \\
= -[a_3, r_1 m] - [a_8, r_2 e] \\
= -r_1 [a_3, m] - r_2 [a_8, e] \\
= r_1 a_7 + r_2 (\alpha, \delta_2) a_7,
\]

\[
[a_2, v] = -[v, a_2] = -s_1 [(a_4, a_7), a_2] = -s_1 [a_4, [a_7, a_2]] - s_1 [a_7, [a_4, a_2]] \\
= -s_1 [a_4, s_2 m] - s_1 [a_7, s_3 f] \\
= s_1 (s_2 + s_3) a_8,
\]

\[
[a_3, u] = -[u, a_3] = -[(a_2, a_5), a_3] = -[a_2, [a_5, a_3]] - [a_5, [a_2, a_3]] \\
= [[a_2, a_3], a_5] \\
= r_3 (\delta_2, \eta) a_5,
\]

\[
[a_4, u] = -[u, a_4] = -[(a_2, a_5), a_4] = -[a_2, [a_5, a_4]] - [a_5, [a_2, a_4]] \\
= s_5 [a_2, u] - s_3 [a_5, f] \\
= (s_5 + s_3) a_6
\]

\[
[a_5, v] = -[v, a_5] = -[(a_3, a_8), a_5] = -[a_3, [a_8, a_5]] - [a_8, [a_3, a_5]] \\
= s_7 (\eta, \delta_2) a_3
\]

\[
[a_6, v] = -[v, a_6] = -s_1 [(a_4, a_7), a_6] = -s_1 [a_4, [a_7, a_6]] - s_1 [a_7, [a_4, a_6]] \\
= s_7 (\eta, \delta_2) a_7 - r_1 (\delta_1, \gamma) a_7 - r_2 (\alpha, \delta_2) (\delta_1, \gamma) a_7,
\]
\[ [a_6, z] = -[z, a_6] = -[[m, v], a_6] = -[m, [v, a_6]] + [v, [m, a_6]] = -s_6s_1(\zeta, \delta_1)[m, a_4] - (\gamma, \delta_2)[v, a_2] = (-s_6s_1(\zeta, \delta_1) + s_1(s_2 + s_3)(\delta_2, \gamma))a_8,\]

\[ [a_7, u] = -[u, a_7] = -r_4[[a_1, a_6], a_7] = -r_4[a_1, [a_6, a_7]] - r_4[a_6, [a_1, a_7]] = s_6r_4[t_\zeta, a_1] = s_6r_4(\zeta, \delta_1)a_1,\]

\[ [a_7, w] = -[w, a_7] = -[[u, u], a_7] = -[u, [u, a_7]] + [u, [u, a_7]] = (r_4s_6(\zeta, \delta_1) - r_3(\delta_2, \gamma)(\delta_2, \eta))a_5,\]

\[ [a_8, u] = -[u, a_8] = -[[a_2, a_5], a_8] = -[a_2, [a_5, a_8]] - [a_5, [a_2, a_8]] = s_7[t_\eta, a_2] = -s_7(\eta, \delta_2)a_2,\]

\[ [a_8, w] = -[w, a_8] = -[[u, u], a_8] = -[u, [u, a_8]] + [u, [u, a_8]] = s_7(\eta, \delta_2)[n, a_2] + [u, (\delta_1, \gamma)a_4] = (s_7(\eta, \delta_2) - (\delta_1, \gamma)(s_5 + s_3))a_6,\]

\[ [b_1, u] = -[u, b_1] = -r_5[[b_4, b_7], b_1] = -r_5[b_4, [b_7, b_1]] - r_5[b_7, [b_4, b_1]] = r_5k_1(\zeta, \delta_1)b_7,\]

\[ [b_2, u] = -[u, b_2] = -s_4[[b_3, b_8], b_2] = -s_4[b_3, [b_8, b_2]] - s_4[b_8, [b_3, b_2]] = s_4(\eta, \delta_2)k_2b_8,\]

\[ [b_3, v] = -[v, b_3] = -r_7[[b_2, b_5], b_3] = -r_7[b_2, [b_5, b_3]] - r_7[b_5, [b_2, b_3]] = -r_7k_2(\eta, \delta_2)b_5,\]
\[ [b_4, v] = -[v, b_4] = -r_6([b_1, b_6], b_4) = -r_6[b_1, [b_6, b_4]] - r_6[b_6, [b_1, b_4]] = -r_6 k_1(\zeta, \delta_1) b_6, \]

\[ [b_5, u] = -[u, b_5] = -s_4([b_3, b_8], b_5) = -s_4[b_3, [b_8, b_5]] - s_4[b_8, [b_3, b_5]] = k_3 s_4[t_\eta, b_3] = -s_4 k_3(\delta_2, \eta) b_3, \]

\[ [b_5, w] = -[w, b_5] = -[[n, u], b_5] = -[n, [u, b_5]] + [u, [n, b_5]] = -s_4 k_3(\delta_2, \eta)[n, b_3] + [u, (\gamma, \delta_1) b_1] = (-s_4 k_3(\delta_2, \eta) - (\gamma, \delta_1) k_1 r_5(\zeta, \delta_1)) b_7, \]

\[ [b_6, u] = -[u, b_6] = -r_5([b_4, b_7], b_6) = -r_5[b_4, [b_7, b_6]] - r_5[b_7, [b_4, b_6]] = r_5 k_4[t_\zeta, b_4] = r_5 k_4(\delta_1, \zeta) b_4, \]

\[ [b_6, w] = -[w, b_6] = -[[n, u], b_6] = -[n, [u, b_6]] + [u, [n, b_6]] = r_5 k_4(\delta_1, \zeta)[n, b_4] - (\delta_2, \gamma)[u, b_2] = (r_5 k_4(\delta_1, \zeta) + (\delta_2, \gamma) k_2 s_4(\eta, \delta_2)) b_8, \]

\[ [b_7, v] = -[v, b_7] = -r_6([b_1, b_6], b_7) = -r_6[b_1, [b_6, b_7]] - r_6[b_6, [b_1, b_7]] = r_6 k_4[t_\zeta, b_1] = -r_6 k_4(\delta_1, \zeta) b_1, \]

\[ [b_7, z] = -[z, b_7] = -[[m, v], b_7] = -[m, [v, b_7]] + [v, [m, b_7]] = -r_6 k_4(\delta_1, \zeta)[m, b_1] - (\delta_2, \gamma)[v, b_3] = (-r_6 k_4(\delta_1, \zeta) - (\delta_2, \gamma) r_7 k_2(\eta, \delta_2)) b_5, \]

\[ [b_8, v] = -[v, b_8] = -r_7([b_2, b_5], b_8) = -r_7[b_2, [b_5, b_8]] - r_7[b_5, [b_2, b_8]] = k_3 r_7[t_\eta, b_2] = k_3 r_7(\delta_2, \eta) b_2, \]
\[ [b_8, z] = -[m, v, b_8] = -[m, [v, b_8]] + [v, [m, b_8]] = k_3 r_7(d_2, \eta)[m, b_2] + (\gamma, \delta_1)[v, b_4] = (k_3 r_7(d_2, \eta) + r_6 k_1(\zeta, \delta_1)(\gamma, \delta_1)) b_6. \]

These altogether imply that \([M_0, M_1] \subseteq M_1\). Therefore, to complete the proof of our claim, we just need to show \([M_0, M_0] \subseteq M_0\) but it is immediate using (13.3) together with the following table; as before, the \((r, s)\)-th entry of the following table denotes the summation of the weights of \(r\)-th entry of the first column and \(s\)-th entry of the first row and the red parts denote the elements which are not roots:

| r  | s  | t, α | n   | m | w   |
|----|----|------|-----|---|-----|
| 1  | 2α| 2β   | 2α + β + γ | 2α - γ | 2α1 - 2γ |
| 1  | 2β| 2γ   | 2γ1 + α + β + γ | 2γ1 - α | 2γ1 - 2α |
| 1  | 2α + 2β | 2α + 2β + γ | 2α - γ | 2α1 - 2γ |
| 1  | 2γ | 2α1 + α + γ | 2α1 - α | 2α1 - 2γ |
| 1  | 2β | 2γ1 + β + γ | 2γ1 - β | 2γ1 - 2β |
| 1  | 2β | 2γ1 + γ | 2γ1 - γ | 2γ1 - 2γ |
| 1  | 2γ | 2γ1 + γ | 2γ1 - γ | 2γ1 - 2γ |

Now setting \(\epsilon := \alpha/2, \theta_1 := \beta/2\) and \(\theta_2 := -\gamma + \beta/2\), \(M\) has a weight space decomposition with respect to \(K\) with the set of weights
\[
S := \{0, \pm \alpha, \pm \beta, \pm \gamma, \pm (\beta - \gamma), \pm (\beta - 2\gamma), \pm \delta_1, \pm \delta_2, \pm \eta, \pm \zeta\}
\]
\[
= \{0, \pm 2\epsilon, \pm 2\theta_1, \pm 2\theta_2, \pm (\theta_1 + \theta_2), \pm (\epsilon \pm \theta_1), \pm (\epsilon \pm \theta_2)\}
\]
that is a locally finite root supersystem of type \(C(1, 2)\). It follows from Lemma 2.8(i) that for \(\theta \in \{\pm \delta_1, \pm \delta_2, \pm \eta, \pm \zeta\}\), the form on \(M^0 \oplus M^{-\theta}\) is nondegenerate. Also the corresponding matrix of the form restricted to \(K\) with respect to the ordered basis \(\{t_\alpha, t_\beta, t_\gamma\}\) is
\[
\begin{pmatrix}
(\alpha, \alpha) & (\alpha, \beta) & (\alpha, \gamma) \\
(\beta, \alpha) & (\beta, \beta) & (\beta, \gamma) \\
(\gamma, \alpha) & (\gamma, \beta) & (\gamma, \gamma)
\end{pmatrix}
\begin{pmatrix}
(\alpha, \alpha) & 0 & 0 \\
0 & 2(\beta, \gamma) & (\beta, \gamma) \\
0 & (\gamma, \beta) & (\gamma, \gamma)
\end{pmatrix}
\]
whose determinant is \((\alpha, \alpha)(\beta, \gamma)^2 \neq 0\). This implies that the form is nondegenerate on \(K\) and so the form is nondegenerate on \(M\). We note that \(\text{span}_F\{t_\theta \mid \theta \in \{\pm \delta_1, \pm \delta_2, \pm \zeta, \pm \eta\}\} = \text{span}_F\{t_\theta \mid \theta \in \{\pm \alpha, \pm \beta, \pm \gamma\}\} = K\) and the form is nondegenerate on \(K\), so for each \(0 \neq h \in K\), there is \(\theta \in \{\pm \delta_1, \pm \delta_2, \pm \zeta, \pm \eta\}\) such that \(\theta(h) = (h, t_\theta) \neq 0\). Now as \(M_0\) is a finite dimensional semisimple Lie algebra of type \(A_1 \oplus C_2\), using Lemma 1.1 \(M\) is a finite dimensional basic classical simple Lie superalgebra with \(L_0\) isomorphic to \(A_1 \oplus C_2\). Therefore, using the classification of f.d.b.c.s Lie superalgebras, \(M\) is either isomorphic to \(\text{osp}(3, 4)\) or \(\text{osp}(5, 2)\). But this makes a contradiction as there are nonzero real odd roots for each of \(\text{osp}(3, 4)\) or \(\text{osp}(5, 2)\) while odd roots of \(M\) are all nonsingular. This completes the proof. \(\square\)

**Lemma 2.17.** Suppose that \(L^0 = H\) and the corresponding root system \(R\) of \(L\) is irreducible with no isotropic roots. If there are nonsingular roots \(\delta_1, \delta_2\) with \((\delta_1, \delta_2) \neq 0\) and \(\delta_1 + \delta_2, \delta_1 - \delta_2 \in R\), then \(R\) is an irreducible locally finite root supersystem, in its \(Z\)-span, of type \(A(1, 1)\).
Proof. By Lemma 2.14, $R$ is either of type $A(1,1)$ or $BC(1,1)$. We show that $R$ cannot be of type $BC(1,1)$. To the contrary, suppose that $R$ is of type $BC(1,1)$, then by [16, Lem. 1.13], $R$ is an irreducible locally finite root supersystem. We assume that $R = \{0, \pm \epsilon_0, \pm \delta_0, \pm 2\epsilon_0, \pm 2\delta_0, \pm \epsilon_0 \pm \delta_0\}$. Then
\[
\mathcal{L} = \mathcal{H} \oplus \sum_{\alpha \in \{\pm \epsilon_0, \pm \delta_0, \pm 2\epsilon_0, \pm 2\delta_0, \pm \epsilon_0 \pm \delta_0\}} \mathcal{L}^\alpha.
\]
We note that by 2.20 and Proposition 2.15, for $\alpha \in \{\pm \epsilon_0, \pm \delta_0, \pm 2\epsilon_0, \pm 2\delta_0, \pm \epsilon_0 \pm \delta_0\}$, $L^\alpha \subseteq L_1$. Set $\delta_1 := r\epsilon_0 + s\delta_0$ and $\delta_2 := r\epsilon_0 - s\delta_0$ for some $r, s \in \{\pm 1\}$ and note that $[\mathcal{L}^{\epsilon_0} + \mathcal{L}^{-\epsilon_0} + \mathcal{L}^{\delta_0} + \mathcal{L}^{-\delta_0}, \mathcal{L}^{\delta_1} + \mathcal{L}^{-\delta_1} + \mathcal{L}^{\delta_2} + \mathcal{L}^{-\delta_2}] \subseteq L_1 \cap (\mathcal{L}^{\epsilon_0} + \mathcal{L}^{-\epsilon_0} + \mathcal{L}^{\delta_0} + \mathcal{L}^{-\delta_0})$ and $[\mathcal{L}^{\epsilon_0}, \mathcal{L}^{\delta_0}] \subseteq L_1 \cap \mathcal{L}^{\epsilon_0 \pm \delta_0}. Therefore
\[
(2.18) \quad [\mathcal{L}^{\delta_0}, \mathcal{L}^{\pm \delta_0}] = [\mathcal{L}^{\epsilon_0} + \mathcal{L}^{-\epsilon_0} + \mathcal{L}^{\delta_0} + \mathcal{L}^{-\delta_0}, \mathcal{L}^{\delta_1} + \mathcal{L}^{-\delta_1} + \mathcal{L}^{\delta_2} + \mathcal{L}^{-\delta_2}] = \{0\}.
\]
Take $\alpha, \beta, e, f, x, y, a_1, \ldots, a_4, b_1, \ldots, b_4$ to be as in the proof of Proposition 2.14, contemplating Proposition 2.15 and (2.18), as in Proposition 2.14 one can check that
\[
M = M_0 \oplus M_1 \text{ with }
\frac{M_0 := \mathbb{F}t_\alpha \oplus \mathbb{F}t_\beta \oplus \sum_{\gamma \in \{\pm 2\epsilon_0, \pm 2\delta_0\}} \mathcal{L}^\gamma}{M_1 := \mathcal{L}^{\epsilon_0} \oplus \mathcal{L}^{-\epsilon_0} \oplus \mathcal{L}^{\delta_0} \oplus \mathcal{L}^{-\delta_0} + \sum_{i=1}^4 (\mathbb{F}a_i + \mathbb{F}b_i)}
\]
is a finite dimensional basic classical simple Lie superalgebra. But the even part of $M$ is isomorphic to $\mathfrak{s}_2 \oplus \mathfrak{s}_2$ with Cartan subalgebra $\mathbb{F}t_\alpha + \mathbb{F}t_\beta$ and the root system of $M$ with respect to $\mathbb{F}t_\alpha + \mathbb{F}t_\beta$ is $BC(1,1)$. This is a contradiction using the classification of finite dimensional basic classical simple Lie superalgebras. So to complete the proof, we assume $R$ is of type $A(1,1)$ and show that $R$ is a locally finite root supersystem. Take $\mathcal{V} := \text{span}_\mathbb{R} R$ and denote the induced form on $\mathcal{V}$ again by $(\cdot, \cdot)$. Using the same argument as in [14, Lem. 3.10], one can see that $R_\alpha$ is locally finite in its $\mathbb{F}$-span in the sense that it intersects each finite dimensional subspace of $\text{span}_\mathbb{F} R_\alpha$ in a finite set. So using Lemmas 3.10, 3.12 and 3.21 of [14], we get that $\bar{R}$ is an irreducible locally finite root supersystem in its $\mathbb{Z}$-span. Also using [14, Lem. 3.5]; we get that $\bar{R}_\alpha$ is a locally finite root system and the restriction of the form $(\cdot, \cdot)$ to $\bar{V}_\alpha := \text{span}_\mathbb{F} \bar{R}_\alpha$ is nondegenerate. Therefore we have
\[
(2.19) \quad \text{the restriction of the form $(\cdot, \cdot)$ to } \bar{V}_0 := \text{span}_\mathbb{Q} \bar{R}_\alpha \text{ is nondegenerate.}
\]
Since $\bar{R}_\alpha$ is a locally finite root system, by [8, Lem. 5.1], it contains a $\mathbb{Z}$-linearly independent subset $T$ such that
\[
(2.20) \quad W_T T = (\bar{R}_\alpha)_{red}^\times = \bar{R}_\alpha \setminus \{2\vec{\alpha} | \alpha \in R_\alpha\},
\]
in which by $W_T$, we mean the subgroup of the Weyl group of $R_\alpha$ generated by $r_\alpha$ for all $\vec{\alpha} \in T$. On the other hand, we know there is a subset $\Pi$ of $R$ such that $\Pi$ is a $\mathbb{Z}$-basis for $\text{span}_\mathbb{Z} \bar{R}$; see [16, Lem. 2.3]. This allows us to define the linear isomorphism
\[
\varphi : \text{span}_{\mathbb{Q}} \bar{R} \to \mathbb{Q} \otimes_{\mathbb{Z}} \text{span}_{\mathbb{Z}} \bar{R}
\]

mapping $\vec{\alpha}$ to $1 \otimes \vec{\alpha}$ for all $\alpha \in \Pi$. Now suppose that $\bar{R}$ is of real type, then
\[
\varphi(\text{span}_{\mathbb{Q}} \bar{R}_\alpha) = \text{span}_{\mathbb{Q}}(1 \otimes \bar{R}_\alpha) = \mathbb{Q} \otimes \text{span}_{\mathbb{Z}} \bar{R} = \varphi(\text{span}_{\mathbb{Q}} \bar{R})
\]
which in turn implies that $\text{span}_{\mathbb{Q}} \bar{R} = \text{span}_{\mathbb{Q}} \bar{R}_\alpha$. Therefore, $\text{span}_{\mathbb{Q}} \bar{R} = \text{span}_{\mathbb{Q}} T$ and so $\text{span}_{\mathbb{F}} \bar{R} = \text{span}_{\mathbb{F}} T$. But $T$ is $\mathbb{Z}$-linearly independent and so it is $\mathbb{Q}$-linearly independent. We now prove that $T$ is $\mathbb{F}$-linearly independent.
Suppose that \( \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\} \subseteq T \) and \( \{r_1, \ldots, r_n\} \subseteq \mathbb{F} \) with \( \sum_{i=1}^{n} r_i \tilde{\alpha}_i = 0 \). Take \( \{a_j \mid j \in J\} \) to be a basis for \( \mathbb{Q} \)-vector space \( \mathbb{F} \). For each \( 1 \leq i \leq n \), suppose \( \{r_i^j \mid j \in J\} \subseteq \mathbb{Q} \) is such that \( r_i = \sum_{j \in J} r_i^j a_j \). Then for each \( \tilde{\alpha} \in T \), we have
\[
0 = \sum_{i=1}^{n} r_i \frac{2(\tilde{\alpha}_i, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})} = \sum_{i=1}^{n} \sum_{j \in J} r_i^j a_j \cdot \frac{2(\tilde{\alpha}_i, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})} = \sum_{j \in J} \sum_{i=1}^{n} r_i^j \frac{2(\tilde{\alpha}_i, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})} a_j.
\]
Since \( \frac{2(\tilde{\alpha}_i, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})} \in \mathbb{Z} \), we get that for each \( j \in J \) and \( \tilde{\alpha} \in T \),
\[
\left( \sum_{i=1}^{n} r_i^j \tilde{\alpha}_i, \tilde{\alpha} \right) = \sum_{i=1}^{n} r_i^j (\tilde{\alpha}_i, \tilde{\alpha}) = 0.
\]
So by (2.19), \( \sum_{i=1}^{n} r_i^j \tilde{\alpha}_i = 0 \) for all \( j \in J \). But \( T \) is \( \mathbb{Q} \)-linearly independent and so \( r_i^j = 0 \) for all \( 1 \leq i \leq n \) and \( j \in J \). This means that
\[
(2.21) \quad T \text{ is } \mathbb{F} \text{-linearly independent.}
\]
Next suppose that \( \hat{R} \) is of imaginary type and fix \( \alpha^* \in R_{ns}^\times \). Using a modified version of the above argument together with [14] Lem 3.14] (see also [13] Lem. 3.21]), we get that
\[
(2.22) \quad T \cup \{\tilde{\alpha}^*\} \text{ is } \mathbb{F} \text{-linearly independent.}
\]
For each element \( \alpha \in T \), we fix a preimage \( \hat{\alpha} \in R \) of \( \alpha \) under \( \cdot \) and set
\[
K := \begin{cases} \{\hat{\alpha} \mid \alpha \in T\} & \text{if } \hat{R} \text{ is of real type,} \\ \{\hat{\alpha} \mid \alpha \in T\} \cup \{\alpha^*\} & \text{if } \hat{R} \text{ is of imaginary type.} \end{cases}
\]
We have using [14] Pro. 3.14] together with (2.20) that \( V = \text{span}_F K \). Therefore setting \( \hat{V} := \text{span}_F K \) and using (2.21) and (2.22), we get that \( V = \hat{V} \oplus V^0 \). We set \( \hat{R} := \{\hat{\alpha} \in \hat{V} \mid \exists \sigma \in V^0, \hat{\alpha} + \sigma \in R\} \), then \( \hat{R} \) is a locally finite root supersystem in its \( \mathbb{Z} \)-span isomorphic to \( \hat{R} \). Also since \( K \subseteq R \cap \hat{R} \), \( -K \subseteq R \cap \hat{R} \). So the subgroup \( W_K \) of the Weyl group of \( R \) generated by the reflections based on real roots of \( K \), we have
\[
W_K(\pm K) \subseteq R \cap \hat{R} \quad \text{and} \quad \pm W_K K = \begin{cases} (\hat{R}_{re})^{\times} & \text{if } \hat{R} \text{ is of real type,} \\ \hat{R}_{re} & \text{if } \hat{R} \text{ is of imaginary type.} \end{cases}
\]
We finally set \( S_\hat{\alpha} := \{\sigma \in V^0 \mid \hat{\alpha} + \sigma \in R\} \) for \( \hat{\alpha} \in \hat{R} \). Then \( R = \bigcup_{\hat{\alpha} \in \hat{R}} (\hat{\alpha} + S_\hat{\alpha}) \). Now using Case 1 of the proof of Lemma 1.13 of [16], we have
\[
(2.23) \quad S_\hat{\alpha} \subseteq R^0 = \{0\}; \quad \hat{\alpha} \in \hat{R}_{re}.
\]
Suppose that \( \delta_1, \delta_2 \in R_{ns} \) with \( (\delta_1, \delta_2) \neq 0 \) and \( \delta_1 + \delta_2, \delta_1 - \delta_2 \in R \), then there are \( \hat{\delta}_1, \hat{\delta}_2 \in \hat{R}_{ns} \), \( \sigma \in S_{\delta_1}, \tau \in S_{\delta_2} \) such that \( \delta_1 = \hat{\delta}_1 + \sigma \) and \( \delta_2 = \hat{\delta}_2 + \tau \). Since \( \delta_1 \pm \delta_2 \in R \), we get using (2.23) that \( \sigma + \tau, \sigma - \tau = 0 \) and so \( \sigma = \tau = 0 \). Therefore, taking \( \hat{R} = \{0, \pm 2\delta_0, \pm 2\delta_0, \pm \delta_0 \} \), we have \( \epsilon_0 \pm \delta_0 \in R \). Now if \( r \in \{\pm 1\} \) and \( \delta \in S_{\epsilon_0 + r\delta_0} \), since \( (\epsilon_0 - r\delta_0, \epsilon_0 + r\delta_0, \pm \delta_0) \neq 0 \), either \( 2r\delta_0 + \delta \in R \) or \( 2\epsilon_0 + \delta \in R \) which together with (2.23) implies that \( \delta = 0 \). This shows \( R \subseteq \hat{R} \) and so \( V = \hat{V} \). Therefore \( R = \hat{R} \) is a locally finite root supersystem in its \( \mathbb{Z} \)-span of type \( A(1,1) \).

\[ \square \]

**Lemma 2.24.** Suppose that \( L^0 = \mathcal{H} \) and that \( L \) is simple. Then \( R \) is an irreducible locally finite root supersystem.

**Proof.** Set \( V := \text{span}_F R \) and denote the induced form on \( V \) again by \( (\cdot, \cdot) \). As in Lemma 2.17 there is a subspace \( \hat{V} \) of \( V \), a subset \( \hat{R} \) of \( \hat{V} \) and a class \( \{S_{\hat{\alpha}} \mid \hat{\alpha} \in \hat{R}\} \) of subsets of \( V^0 \), the radical of the form \( (\cdot, \cdot) \), such that
\[ V = \hat{V} \oplus V^0 \] and \( \hat{R} \) is a locally finite root supersystem in \( \hat{V} \) isomorphic to the image \( \hat{R} \) of \( R \) in \( V/V^0 \) under the canonical projection map, with \( R = \cup_{\alpha \in \hat{R}} (\hat{\alpha} + S_\alpha) \). Take \( K \) to be the subsuperalgebra of \( \mathcal{L} \) generated by \( \cup_{\alpha \in R^\times} \mathcal{L}^\alpha \). One can check that
\[
K = \sum_{\alpha \in R^\times} \mathcal{L}^\alpha + \sum_{\alpha \in R^\times} \sum_{\sigma \in S_\alpha} \sum_{\tau \in S_{-\alpha}} [\mathcal{L}^{\hat{\alpha} + \sigma}, \mathcal{L}^{-\hat{\alpha} + \tau}].
\]
Since \( K \) is a nonzero ideal of \( \mathcal{L} \), we have \( K = \mathcal{L} \). Now to the contrary, assume that \( R \) is not irreducible, then there are nonempty subsets \( A_1, A_2 \) of \( R^\times \) such that \( R^\times = A_1 \cup A_2 \) and that \( (A_1, A_2) = \{0\} \). Now if \( \alpha, \beta \in A_i \) \((i = 1, 2)\) are such that \( \alpha + \beta \in R \), then for \( j \in \{1, 2\} \) with \( j \neq i \), we have \( (\alpha + \beta, A_j) = \{0\} \), so either \( \alpha + \beta \in A_i \) or \( \alpha + \beta \in R^0 \). Therefore, either \( \alpha + \beta \in A_i \) or \( \alpha = \hat{\alpha} + \sigma \) and \( \beta = -\hat{\alpha} + \tau \) for some \( \hat{\alpha} \in \hat{R}^\times \) and \( \sigma \in S_\alpha, \tau \in S_{-\alpha} \). Also if \( \{i, j\} = \{1, 2\} \), \( \alpha \in A_i \) and \( \beta \in A_j \), then \( (\alpha + \beta, A_i) = (\alpha, A_i) \neq \{0\} \) and \( (\alpha + \beta, A_j) = (\beta, A_j) \neq \{0\} \). This implies that \( \alpha + \beta \) neither belongs to \( R^\times \) nor belongs to \( R^0 \). So \( \alpha + \beta \notin R \). Therefore, for \( A_i := \{\hat{\alpha} \in \hat{R} \mid \hat{\alpha} + S_\alpha \subseteq A_i\} \) \((i = 1, 2)\), \( \sum_{\alpha \in A_i} \mathcal{L}^\alpha + \sum_{\alpha \in A_i} \sum_{\sigma \in S_\alpha} \sum_{\tau \in S_{-\alpha}} [\mathcal{L}^{\hat{\alpha} + \sigma}, \mathcal{L}^{-\hat{\alpha} + \tau}] \) is a nontrivial ideal of \( \mathcal{L} = K \). This makes a contradiction, so \( R \) is an irreducible extended affine root supersystem.

We next note that if \( \delta \in R^0 \setminus \{0\} \), then \( t_\delta \) is a nonzero element of the center of \( \mathcal{L} = K \) which is a contradiction. So \( R^0 = \{0\} \) and
\[
(2.26) \quad \mathcal{L} = \sum_{\alpha \in R^\times} \mathcal{L}^\alpha \oplus \sum_{\alpha \in R^\times} [\mathcal{L}^\alpha, \mathcal{L}^{-\alpha}].
\]
Now if \( R \) is not of type \( A(\ell, \ell) \), [16] Lem. 1.13 implies that \( R \) is a locally finite root supersystem in its \( \mathbb{Z}\)-span. Also if \( R \) is of type \( A(\ell, \ell) \), Lemma 2.14 and Proposition 2.26 imply that \( \mathcal{L} \) is finite dimensional, in particular, \( \mathcal{H} = \sum_{\alpha \in R^\times} [\mathcal{L}^\alpha, \mathcal{L}^{-\alpha}] = \sum_{\alpha \in R^\times} \mathbb{F} t_\alpha \) is finite dimensional. Therefore, \( \mathcal{H}^* = \text{span}_\mathbb{F} R \) and the induced form on \( \mathcal{H}^* \) is nondegenerate. But \( \mathcal{V} = \text{span}_\mathbb{F} R = \mathcal{H}^* \), and so the form on \( \mathcal{V} \) is nondegenerate. This in turn implies that the form restricted to \( \langle R \rangle \) is nondegenerate. This means that \( R \) is an irreducible locally finite root supersystem. □

**Lemma 2.27.** Suppose that \( \mathcal{L}^0 = \mathcal{H} \). If the root supersystem \( R \) of \( \mathcal{L} \) is a locally finite root supersystem in its \( \mathbb{Z}\)-span, then the ideal \( K := \sum_{\alpha \in R^\times} \mathcal{L}^\alpha + \sum_{\alpha \in R^\times} [\mathcal{L}^\alpha, \mathcal{L}^{-\alpha}] \) of \( \mathcal{L} \) is a direct sum of simple ideals.

**Proof.** Set \( A := \langle R \rangle, V := \text{span}_\mathbb{F} R \) and denote the induced form on \( V \) again by \((\cdot, \cdot)\). Since \( R \) is a locally finite root supersystem in \( A \), the form restricted to \( A \) is nondegenerate and so using the same argument as in [14] Lem. 3.21, the form on \( V \) is also nondegenerate. Suppose that \( R = \oplus_{i \in I} S_i \) is the decomposition of \( R \) into irreducible super-systems and set \( \mathcal{L}(i) := \oplus_{\alpha \in S_i^\times} \mathcal{L}^\alpha \oplus \sum_{\alpha \in S_i^\times} [\mathcal{L}^\alpha, \mathcal{L}^{-\alpha}] \). Then \( K = \sum_{i \in I} \mathcal{L}(i) \). We prove that each \( \mathcal{L}(i) \) \((i \in I)\) is a simple ideal of \( \mathcal{L} \). We note that each \( S_i \) is a closed sub-supersystem of \( R \), so \( \mathcal{L}(i) \) is an ideal of \( \mathcal{L} \). Suppose that \( i \in I \) and note that \( \mathcal{L}(i) \) has a weight space decomposition \( \mathcal{L}(i) = \oplus_{\alpha \in S_i^\times} \mathcal{L}(i)^\alpha \) with respect to \( \mathcal{H}(i) := \sum_{\alpha \in S_i^\times} [\mathcal{L}^\alpha, \mathcal{L}^{-\alpha}] = \sum_{\alpha \in S_i^\times} \mathbb{F} t_\alpha \) with \( \mathcal{L}(i)^\alpha = \mathcal{L}^\alpha \) for \( \alpha \in S_i^\times \) and \( \mathcal{L}(i)^0 = \mathcal{H}(i) \); see Lemma 2.7(i). Now suppose that \( J \) is a nonzero ideal of \( \mathcal{L}(i) \), then \( J = \oplus_{\alpha \in S_i^\times} (J \cap \mathcal{L}(i)^\alpha) \). We claim that there is \( \alpha \in S_i^\times \) with \( J \cap \mathcal{L}(i)^\alpha \neq \{0\} \). To the contrary, assume \( J = J \cap \mathcal{L}(i)^0 \) and suppose \( h := \sum_{\alpha \in S_i} r_\alpha t_\alpha \in \mathcal{L}(i)^0 \) is a nonzero element of \( J \), then since \( \sum_{\alpha \in S_i} r_\alpha \alpha \) is a nonzero element of \( \langle S_i \rangle \), one finds \( \beta \in S_i \) with \( \langle \sum_{\alpha \in S_i} r_\alpha \alpha, \beta \rangle \neq 0 \), therefore, \( [h, \mathcal{L}^\beta] = (\sum_{\alpha \in S_i} r_\alpha \alpha, \beta) \mathcal{L}^\beta \neq \{0\} \) and so \( \mathcal{L}^\beta \subseteq J \), a contradiction. Now fix \( \alpha \in S_i^\times \) with \( J \cap \mathcal{L}^\alpha = J \cap \mathcal{L}(i)^\alpha \neq \{0\} \), then \( t_\alpha \in [\mathcal{L}^{-\alpha}, \mathcal{L}^\alpha \cap J] \subseteq J \). So for each \( \beta \in S_i \) with \( (\alpha, \beta) \neq 0 \), we have \( [t_\alpha, \mathcal{L}^\beta] = (\beta, \alpha) \mathcal{L}^\beta \neq \{0\} \) which in turn
implies that $\mathcal{L}^\beta \subseteq J$. Now for an arbitrary $\beta \in S_i^\times$, as $S_i$ is irreducible, there are a chain $\alpha_0 := \alpha, \alpha_1, \ldots, \alpha_n := \beta$ of elements of $S_i^\times$ such that $(\alpha_i, \alpha_{i-1}) \neq 0$ for all $1 \leq i \leq n$, so $\mathcal{L}^{\alpha_i} \subseteq J$ for all $1 \leq i \leq n$, in particular $\mathcal{L}^\beta \subseteq J$. So $\mathcal{L}(i) \subseteq J$. This completes the proof. 

**Lemma 2.28.** Suppose that $\mathcal{L}$ is simple and $\mathcal{L}^0 = \mathcal{H}$. If $\mathcal{L}$ is finite dimensional, it is a basic classical simple Lie superalgebra and if $\mathcal{L}$ is infinite dimensional, then it is a direct union of finite dimensional basic classical simple Lie superalgebras; in particular $\mathcal{L}$ is locally finite. Moreover, up to a scalar multiple, there is a unique invariant nondegenerate even supersymmetric bilinear form on $\mathcal{L}$.

**Proof.** We first note that $R$ is an irreducible locally finite root supersystem by Lemma 2.24. We now assume $L$ is infinite dimensional. Contemplating Lemma 2.4(iii) of 10, we get that $R$ is a direct union of its irreducible closed finite sub-supersystems of the same type as $R$, say $R = \cup_{\gamma \in \Gamma} R(\gamma)$, where $\Gamma$ is a nonempty index set. For each $\gamma \in \Gamma$, set $\mathcal{L}(\gamma) := \sum_{\alpha \in R(\gamma)^\times} \mathcal{L}^\alpha + \sum_{\beta \in R(\gamma)^\times} [\mathcal{L}^\alpha, \mathcal{L}^- \beta]$. Considering Lemma 2.13 and Proposition 2.5 and using the same argument as in Lemma 2.27, one can see that $\mathcal{L}(\gamma)$ is a finite dimensional simple subsuperalgebra.

We next show that $\mathcal{L}(\gamma)$ is a finite dimensional basic classical simple Lie superalgebra. Since the form restricted to $\mathcal{L}(\gamma)$ is nonzero and even, we just need to show $\mathcal{L}(\gamma)_{\bar{0}}$ is a reductive Lie algebra or equivalently $\mathcal{L}(\gamma)_{\bar{1}}$ is a completely reducible $\mathcal{L}(\gamma)_{\bar{0}}$-module. We carry out this throughout the following two cases:

- **$R$ is of real type:** In this case, $\text{span}_{\bar{F}} \{ t_\alpha \mid \alpha \in R(\gamma) \} = \text{span}_{\bar{F}} \{ t_\alpha \mid \alpha \in R(\gamma)_{\bar{1}} \}$. This in turn implies that
  \[
  \sum_{\alpha \in R(\gamma)^\times} [\mathcal{L}^\alpha, \mathcal{L}^- \beta] = \sum_{\alpha \in R(\gamma)^\times} [\mathcal{L}^\alpha, \mathcal{L}^- \beta] = \sum_{\alpha \in R(\gamma)^\times} [\mathcal{L}^\alpha, \mathcal{L}^- \beta].
  \]

Suppose that $R(\gamma)_{\bar{0}} = \bigoplus_{i=1}^n \Phi_i$, where $n$ is a positive integer, is the decomposition of the finite root system $R(\gamma)_{\bar{0}}$ into irreducible subsystems. Using the same argument as in Lemma 2.27, we get that for each $1 \leq i \leq n$, \[
\sum_{\alpha \in \Phi_i^\times} \mathcal{L}^\alpha + \sum_{\alpha \in \Phi_i^\times} [\mathcal{L}^\alpha, \mathcal{L}^- \beta] \]
is a finite dimensional simple ideal of $\mathcal{L}(\gamma)_{\bar{0}}$ and so $\mathcal{L}(\gamma)_{\bar{0}}$ is a semisimple Lie algebra.

- **$R$ is of imaginary type:** Take $W$ to be the Weyl group of $R$ and fix $\delta^* \in R_{\text{ns}}^\times$. Without loss of generality, we assume each $R(\gamma)$ contains $\delta^*$. We recall from Lemma 14(iii) that $R_{\text{ns}}^\times = \pm W \delta^*$ and note that $-\delta^* \not\in W \delta^*$ as otherwise $-\delta^* \in \delta^* + (R_{\text{re}})$ which contradicts the fact that $R$ is of imaginary type. For $\gamma \in \Gamma$, take $W_\gamma$ to be the Weyl group of $R_\gamma$ and set $S_1 := W_\gamma \delta^*$ as well as $S_2 := -W_\gamma \delta^*$. We note that by Proposition 2.6 and 2.3, $R_1 = R_{\text{ns}}$ and claim that for $i = 1, 2$, $\sum_{\alpha \in S_i} (\mathcal{L}_i)\alpha$ is an irreducible $\mathcal{L}(\gamma)_{\bar{0}}$-module. By 14 Lem. 4.6,ices $\sum_{\alpha \in S_i} (\mathcal{L}_i)\alpha$ is a $\mathcal{L}(\gamma)_{\bar{0}}$-submodule of $\mathcal{L}_i$. Suppose that $U$ is a nonzero $\mathcal{L}(\gamma)_{\bar{0}}$-submodule of $\sum_{\alpha \in S_i} (\mathcal{L}_i)\alpha$. Then $U = \sum_{\alpha \in S_i} ((\mathcal{L}_i)\alpha \cap U)$. Since $U \neq \{0\}$, there is $\gamma \in S_i$ such that $U \cap (\mathcal{L}_i)\gamma \neq \{0\}$. So $(\mathcal{L}_i)\gamma \subseteq U$ as by Proposition 2.14 we have dim$(\mathcal{L}_i)\gamma = 1$. Suppose that $\beta \in S_i$, then there are $\gamma_1, \ldots, \gamma_n \in (R_\gamma)_{\bar{0}} \setminus \{0\}$ such that $\beta = r_{\gamma_1} \cdots r_{\gamma_n}(\gamma)$. Now considering 2.3, we have $\theta_{\gamma_1} \cdots \theta_{\gamma_n}(\mathcal{L}_i)\gamma \subseteq (\mathcal{L}_i)\beta \cap U$. This together with the fact that dim$(\mathcal{L}_i)\beta = 1$, implies that $(\mathcal{L}_i)\beta \subseteq U$. So $U = \sum_{\beta \in S_i} (\mathcal{L}_i)\beta$. Therefore $(\mathcal{L}_i)_{\bar{1}} = S_1 \oplus S_2$ is completely reducible $(\mathcal{L}_i)_{\bar{0}}$-module.

Finally, suppose that $\mathcal{L}$ is finite dimensional, using the same argument as above, if $R$ is of real type, $\mathcal{L}_{\bar{0}}$ is a finite dimensional semisimple Lie algebra and if $R$ is of imaginary type, $\mathcal{L}_i$ is a completely reducible $\mathcal{L}_{\bar{0}}$-module,
so $L$ is a finite dimensional basic classical simple Lie superalgebra. The last assertion follows easily from the facts that $L$ is a direct union of finite dimensional basic classical simple Lie superalgebras and that nonzero invariant forms on such Lie superalgebras are proportional.

**Definition 2.29.** A nonzero Lie superalgebra $L = L_0 \oplus L_1$ over an algebraically closed field is called a *locally finite basic classical simple Lie superalgebra* if

- $L$ is locally finite and simple,
- $L_0$ has a nontrivial subalgebra $H$ with respect to which $L$ has a weight space decomposition $L = \sum_{\alpha \in H^*} L^\alpha$ via the adjoint representation with corresponding root system $R$ such that $L^0 = H$ and $R^\times \neq \emptyset$,
- $L$ is equipped with an invariant nondegenerate even supersymmetric bilinear form.

**Theorem 2.30.** Suppose that $L$ is a nontrivial Lie superalgebra, then $L$ is a locally finite basic classical simple Lie superalgebra if and only if it contains a subalgebra $H$ (referred to as a Cartan subalgebra) and that it is equipped with a bilinear form $(\cdot, \cdot)$ such that $(L, H, (\cdot, \cdot))$ is a simple extended affine Lie superalgebra with corresponding root system $R$ such that $R^\times \neq \emptyset$ and $L^0 = H$. In this case, we have

(i) the root system $R$ of $L$ is an irreducible locally finite root supersystem,

(ii) $L$ is a direct union of finite dimensional basic classical simple Lie superalgebras,

(iii) $[L_0, L_0]$ is a semisimple Lie algebra,

(iv) if $L_1 \neq \{0\}$, it is a completely reducible $L_0$-module with at most two irreducible constituents.

**Proof.** Suppose that $L$ is a locally finite simple Lie superalgebra equipped with an invariant nondegenerate even supersymmetric bilinear form $(\cdot, \cdot)$ and that $L_0$ has a nontrivial subalgebra $H$ with respect to which $L$ has a weight space decomposition $L = \sum_{\alpha \in H^*} L^\alpha$ via the adjoint representation with corresponding root system $R$ such that $L^0 = H$ and $R^\times \neq \emptyset$. Use the same notation as in the text and suppose $\alpha \in R_i \setminus \{0\}$ for some $i \in \{0, 1\}$.

Since the form is nondegenerate and even, we get that the form on $(L_i)^\alpha \oplus (L_i)^{-\alpha}$ is nondegenerate and so there are $x \in (L_i)^\alpha$ and $y \in (L_i)^{-\alpha}$ such that $(x, y) \neq 0$. Now $[x, y] \in H$ and for each $h \in H$,

$$\langle h, [x, y] \rangle = \langle [h, x], y \rangle = \alpha(h)(x, y).$$

This implies that $[x, y] \neq 0$. Also as $L$ is locally finite, for each $x \in L^\alpha, y \in L^\beta$ ($\alpha \in R \setminus \{0\}, \beta \in R$), the subsuperalgebra of $L$ generated by $x, y$ is finite dimensional, so there is a positive integer $n$ such that $(ad_x)^n(y) \in L^{\alpha+\beta}$ equals to zero. This means that $ad_x$ is locally nilpotent. Altogether, we concluded that $(L, (\cdot, \cdot), H)$ is a simple extended affine Lie superalgebra. This together with Lemma 2.28 completes the proof of the first assertion.

Now suppose $L$ is a locally finite basic classical simple Lie superalgebra, $(i)$ and $(ii)$ follow from Lemmas 2.24 and 2.28. For $(iii)$, we note that if $R_0 = \oplus_{i=1}^n \Phi_i$ is the decomposition of $R_0$ into irreducible subsystems, then we have $[L_0, L_0] = \oplus_{i=1}^n L(i)$ in which $L(i) := \sum_{\alpha \in \Phi_i \setminus \{0\}} (L_0)^\alpha + \sum_{\alpha \in \Phi_i \setminus \{0\}} ([L_0]^\alpha, (L_0)^{-\alpha})$ is a simple ideal.

Finally for $(iv)$, we first suppose $R$ is of imaginary type, then by Proposition 2.25 and 2.28, $R_1 = R_{ns} \setminus \{0\}$ and
by Proposition 2.28, \( \dim L^\delta = 1 \) for \( \delta \in R_{ns}^\times \). Now one can use the same argument as in the proof of Lemma 2.28 to get that \( L_1 \) is completely reducible \( L_0 \)-module with two irreducible constituent. Next suppose \( R \) is of type \( A(\ell, \ell) \), then by Propositions 2.14 and 2.5, \( L \) is finite dimensional and so by Lemma 2.28 and the theory of finite dimensional basic classical simple Lie superalgebras, we are done. Finally, suppose \( R \) is of a real type other than type \( A(\ell, \ell) \). Then by Propositions 2.5, 2.14 and Lemma 2.17

\[(2.31) \quad \dim(L^\alpha) = 1 \quad \forall \alpha \in R^\times. \]

Also using the classification theorem for real types, we have the following:

\[(2.32) \quad \text{If } \{\beta \in R_1 \mid 2\beta \in R \} \neq \emptyset, \text{ then for } \alpha \in R_{ns}^\times, \text{ there is } \gamma \in R^\times \text{ such that } \gamma + \alpha \in R_1 \text{ and } 2(\gamma + \alpha) \in R; \text{ also for } \beta \in R_1 \text{ with } 2\beta \in R, \text{ there is } \eta \in R_{ns}^\times \text{ such that } \eta + \beta \in R_{ns}^\times. \]

We next set \( S_1 := \sum_{\alpha \in R_{ns}^\times} L^\alpha \) and \( S_2 := \sum_{\alpha \in R_{ns}^\times; 2\alpha \in R} L^\alpha \) and suppose that \( U \) is a submodule of \( L_0 \)-module \( L_1 = S_1 \oplus S_2 \), then \( U = \sum_{\alpha \in R_1}(L^\alpha \cap U) \). If \( U \neq \{0\} \), then \( U \cap L^\alpha \neq \{0\} \) for some \( \alpha \in R_{ns}^\times \) or for some \( \alpha \in R^\times \) with \( 2\alpha \in R \). Now (2.32) together with Lemma 2.4 and (2.31) imply that \( U \cap S_1 \neq \{0\} \) and if \( S_2 \neq \{0\} \), \( U \cap S_2 \neq \{0\} \). Now considering (2.31) and using the same argument as in the proof of Lemma 2.28 together with the fact that all roots \( \beta \in R_1 \) with \( 2\beta \in R \) and also all nonzero nonsingular roots are conjugate under the Weyl group action, we get that \( S_1 + S_2 \subseteq U \). So \( U = L_1 \). This completes the proof.

**Lemma 2.33.** Suppose that \((L, (\cdot, \cdot), H)\) is a locally finite basic classical simple Lie superalgebra with corresponding root system \( R \). Then \( L_0 \) is a semisimple Lie algebra if and only if \( R \) is of real type.

**Proof.** There is nothing to prove if \( L \) is a Lie algebra. So suppose that \( L_1 \neq \{0\} \). If \( R \) is of type \( B(0, T) \), take \( \delta := 0 \) and otherwise, take \( \delta \) to be a fix nonzero nonsingular root. We know that \( R_{ns} = \{0\} \cup \pm W \delta \) in which \( W \) is the Weyl group of \( R \). Also \( L = \sum_{\alpha \in R^\times} L^\alpha + \sum_{\alpha \in R^\times} [L^\alpha, L^{-\alpha}] \) as \( L \) is simple. This implies that \( H = \sum_{\alpha \in R^\times} [L^\alpha, L^{-\alpha}] = \sum_{\alpha \in R^\times} \mathbb{F}t_\alpha = \sum_{\alpha \in R_{ns}^\times} \mathbb{F}t_\alpha + \mathbb{F}t_\delta \). So

\[
L_0 = \sum_{\alpha \in R_{ns}^\times} (L^\alpha)_{\delta} \oplus H = \sum_{\alpha \in R_{ns}^\times} L^\alpha \oplus H = \sum_{\alpha \in R_{ns}^\times} L^\alpha \oplus \sum_{\alpha \in R_{ns}^\times} [L^\alpha, L^{-\alpha}] + \mathbb{F}t_\delta \]

and by (2.1) and Theorem 2.30

\[
K := [L_0, L_0] = \sum_{\alpha \in R_{ns}^\times} L^\alpha \oplus \sum_{\alpha \in R_{ns}^\times} [L^\alpha, L^{-\alpha}] = \sum_{\alpha \in R_{ns}^\times} L^\alpha \oplus \sum_{\alpha \in R_{ns}^\times} [L^\alpha, L^{-\alpha}]\]

is a semisimple ideal of \( L_0 \). Also contemplating [14, Lem. 3.5], the induced form on \( \text{span}_R R \) restricted to \( \text{span}_R R_{rc} \) is nondegenerate. Now it follows that \( t_\delta \in \text{span}_R \{t_\alpha \mid \alpha \in R_{rc}\} \) if and only if \( t_\delta \in \text{span}_Q \{t_\alpha \mid \alpha \in R_{rc}\} \); see [16, Lem. 1.8]. So we have

\[
H = \sum_{\alpha \in R_{rc}} \mathbb{F}t_\alpha \Leftrightarrow t_\delta \in \text{span}_\mathbb{F} \{t_\alpha \mid \alpha \in R_{rc}\} \Leftrightarrow t_\delta \in \text{span}_Q \{t_\alpha \mid \alpha \in R_{rc}\} \Rightarrow \delta \in \text{span}_\mathbb{Q} \{\alpha \mid \alpha \in R_{rc}\} \Rightarrow \exists n \in \mathbb{Z} \setminus \{0\} \text{ with } n\delta \in \langle R_{rc} \rangle \Rightarrow \delta \in \mathbb{Q} \otimes_{\mathbb{Z}} \langle R_{rc} \rangle \Rightarrow R \text{ is of real type}.
\]
Now if $R$ is of real type, then $\mathcal{H} = \sum_{\alpha \in R_+} F_{\alpha} = \sum_{\alpha \in R_+^0} [\mathcal{L}^\alpha, \mathcal{L}^{\alpha - \gamma}]$ and so $\mathcal{L}_0 = K$ which is a semisimple Lie algebra. Conversely, suppose $\mathcal{L}_0$ is a semisimple Lie algebra and to the contrary assume $R$ is of imaginary type. Then $\mathcal{L}_0 = K \oplus \mathbb{F}t_\alpha$. Since $\mathcal{L}_0$ is a semisimple Lie algebra and $K$ is an ideal of $\mathcal{L}_0$ of codimension 1, there is a 1-dimensional ideal $I$ of $\mathcal{L}_0$ such that $\mathcal{L}_0 = K \oplus I$. Fix $0 \neq x \in I$. We have $I = \mathbb{F}x$. If $I \subseteq \mathcal{H}$, for each $\gamma \in R_+^0$, we have $[I, (\mathcal{L}_0)^\gamma] \subseteq I \cap (\mathcal{L}_0)^\gamma \subseteq \mathcal{H} \cap (\mathcal{L}_0)^\gamma = \{0\}$; also we have $[I, \mathcal{H}] \subseteq [\mathcal{H}, \mathcal{H}] = \{0\}$. This means that $I$ is a central ideal of $\mathcal{L}_0$ which is absurd. So $x \notin \mathcal{H}$. Therefore $x = \sum_{\alpha \in R_0^+} x_\alpha + \sum_{\alpha \in R_+} r_\alpha t_\alpha + rt_\delta$ in which $r_\alpha \in \mathbb{F} (\alpha \in R_0)$, $r \in \mathbb{F} \setminus \{0\}$ and for each $\alpha \in R_0^+$, $x_\alpha \in (\mathcal{L}_0)^\alpha$ with $x_\gamma \neq 0$ for some $\gamma \in R_0 \setminus \{0\}$. Since $I = \mathbb{F}x$ is an ideal, we have

$$
[x_\gamma, (\mathcal{L}_0)^{-\gamma}] + \sum_{\alpha \in R_0^+ \setminus \{\gamma\}} [x_\alpha, (\mathcal{L}_0)^{-\gamma}] + \sum_{\alpha \in R_+} [r_\alpha t_\alpha, (\mathcal{L}_0)^{-\gamma}] + r[t_\delta, (\mathcal{L}_0)^{-\gamma}] = [x, (\mathcal{L}_0)^{-\gamma}]
$$

(2.34) \subseteq \mathbb{F}x,

but $x_\gamma \neq 0$ and the form restricted to $(\mathcal{L}_0)^\gamma \oplus (\mathcal{L}_0)^{-\gamma}$ is nondegenerate, so we have $[x_\gamma, (\mathcal{L}_0)^{-\gamma}] = \mathbb{F}x_\gamma$ by (1.13).

Now (2.34) implies that $t_\gamma \in \mathbb{F}(\sum_{\alpha \in R_+} r_\alpha t_\alpha + rt_\delta) \cap (\sum_{\alpha \in R_+} \mathbb{F}t_\alpha)$ which is a contradiction as $R$ is of imaginary type. This completes the proof. □

2.1. Examples. For a unital associative superalgebra $\mathcal{A}$ and nonempty index supersets $I, J$, by an $I \times J$-matrix with entries in $\mathcal{A}$, we mean a map $A : I \times J \to \mathcal{A}$. For $i \in I, j \in J$, we set $a_{ij} := A(i, j)$ and call it the $(i, j)$-th entry of $A$. By a convention, we denote the matrix $A$ by $(a_{ij})$. We also denote the set of all $I \times J$-matrices with entries in $\mathcal{A}$ by $\mathcal{A}^{I \times J}$. If $I = J$, we denote $\mathcal{A}^{I \times J}$ by $\mathcal{A}^I$. For $A = (a_{ij}) \in \mathcal{A}^{I \times J}$, the matrix $B = (b_{ij}) \in \mathcal{A}^{I \times I}$ with

$$
b_{ij} := \begin{cases} 
a_{ji} & |i| = |j| \\
a_{ji} & |i| = 1, |j| = 0 \\
-a_{ji} & |i| = 0, |j| = 1
\end{cases}
$$

is called the supertranspose of $A$ and denoted by $A^t$. If $A = (a_{ij}) \in \mathcal{A}^{I \times J}$ and $B = (b_{ij}) \in \mathcal{A}^{J \times K}$ are such that for all $i \in I$ and $k \in K$, at most for finitely many $j \in J$, $a_{ij}b_{jk}$’s are nonzero, we define the product $AB$ of $A$ and $B$ to be the $I \times K$-matrix $C = (c_{ik})$ with $c_{ik} := \sum_{j \in J} a_{ij}b_{jk}$ for all $i \in I, k \in K$. We note that if $A, B, C$ are three matrices such that $AB, (AB)C, BC$ and $A(BC)$ are defined, then $A(BC) = (AB)C$. We make a convention that if $I$ is a disjoint union of subsets $I_1, \ldots, I_t$ of $I$, then for an $I \times I$-matrix $A$, we write

$$
A = \begin{bmatrix}
A_{11} & \cdots & A_{1t} \\
A_{21} & \cdots & A_{2t} \\
\vdots & \ddots & \vdots \\
A_{t1} & \cdots & A_{tt}
\end{bmatrix}
$$

in which for $1 \leq r, s \leq t$, $A_{rs} = A_{r,s}$ is an $I_r \times I_s$-matrix whose $(i, j)$-th entry coincides with $(i, j)$-th entry of $A$ for all $i \in I_r, j \in I_s$. In this case, we say that $A \in \mathcal{A}^{I_1 \times \cdots \times I_t}$ and note that the defined matrix product obeys the product of block matrices. If $\{a_i : i \in I\} \subseteq \mathcal{A}$, by $\text{diag}(a_i)$, we mean an $I \times I$-matrix whose $(i, i)$-th entry is $a_i$ for all $i \in I$ and other entries are zero. If $\mathcal{A}$ is unital, we set $1_I := \text{diag}(1, I)$. A matrix $A \in \mathcal{A}^I$ is called invertible if there is a matrix $B \in \mathcal{A}^I$ such that $AB$ as well as $BA$ are defined and $AB = BA = 1_I$; such a $B$ is unique and denoted by $A^{-1}$. For $i \in I, j \in J$ and $a \in \mathcal{A}$, we define $E_{ij}(a)$ to be a matrix in $\mathcal{A}^{I \times J}$ whose $(i, j)$-th entry is $a$ and other entries are zero and if $\mathcal{A}$ is unital, we set

$$
e_{i,j} := E_{i,j}(1).
$$

Take $M_{I \times J}(\mathcal{A})$ to be the subspace of $\mathcal{A}^{I \times J}$ spanned by $\{E_{ij}(a) : i \in I, j \in J, a \in \mathcal{A}\}$. $M_{I \times J}(\mathcal{A})$ is a superspace with $M_{I \times J}(\mathcal{A})_i := \text{span}_F \{E_{r,s}(a) \mid |r| + |s| + |a| = i\}$, for $i = 0, 1$. Also with respect to the multiplication of
Lemma 2.35. (i) Suppose that $Q$ is a homogeneous element of $\mathbb{F}^I$, then $G_Q := \{ X \in \mathfrak{pl}_I(\mathbb{F}) \mid X^s Q = -(1)^{|X||Q|} X \}$ is a Lie subsuperalgebra of $\mathfrak{pl}_I(\mathbb{F})$.

(ii) If $Q_1, Q_2$ are homogeneous elements of $\mathbb{F}^I$ and $T$ is an invertible homogeneous element of $\mathbb{F}^I$ of degree zero such that $Q_2 = T^s Q_1 T$, then $G_{Q_1}$, is isomorphic to $G_{Q_2}$ via the isomorphism mapping $X$ to $T^{-1} X T$.

(iii) Suppose that $I$ and $J$ are two supersets and $\eta : I \rightarrow J$ is a bijection preserving the degree. For a matrix $A = (A_{ij})$ of $\mathbb{F}^J$, define $A^n \in \mathbb{F}^I$ to be $(A_{ij})^n$ with $A_{ij}^n = A_{\eta^{-1}(i) \eta^{-1}(j)}$. If $Q$ is a homogeneous element of $\mathbb{F}^I$ and $Q' := Q^n$, then the Lie superalgebra $G_Q := \{ X \in \mathfrak{pl}_I(\mathbb{F}) \mid X^s Q = -(1)^{|X||Q|} X \}$ is isomorphic to the Lie superalgebra $G_{Q'} := \{ X \in \mathfrak{pl}_J(\mathbb{F}) \mid X^s Q' = -(1)^{|X||Q'|} Q' X \}$.

Proof. (i), (ii) It is easy to verify.

(iii) Suppose that matrices $A, B \in \mathbb{F}^I$ are such that $AB$ is defined, then for $i, j \in I$, we have

$$(A^n B^n)_{\eta(i) \eta(j)} = \sum_{t \in I} A^n_{\eta(i) \eta(t)} B^n_{\eta(t) \eta(j)} = \sum_{t \in I} A_{\eta^{-1}(i) \eta^{-1}(t)} B_{\eta^{-1}(t) \eta^{-1}(j)} = (AB)_{ij} = (A^n B^n)_{\eta(i) \eta(j)}.$$ 

This in particular implies that if $A, B, C, D \in \mathbb{F}^I$ are such that $AB$ and $CD$ are defined and $AB = CD$, then $A^n B^n = C^n D^n$. Moreover, as $\eta$ preserves the degree, we have $(A^s)^n = (A^n)^s$. Now it is easy to see that the function $\theta : G_Q \rightarrow G_{Q'}$ mapping $X$ to $X^n$ is an isomorphism. □

Example 2.36. For two disjoint index sets $I, J$ with $J \neq \emptyset$, suppose that $\{0, i, \bar{i} \mid i \in I \cup J\}$ is a superset with $|0| = |i| = |\bar{i}| = 0$ for $i \in I$ and $|j| = |\bar{j}| = 1$ for $j \in J$. We set $\hat{I} := I \cup \bar{I}$, $\hat{I}_0 := \{0\} \cup I \cup \bar{I}$ and $\hat{J} := J \cup \bar{J}$ in which

$$\hat{I} := \{\bar{i} \mid i \in I\} \quad \text{and} \quad \hat{J} := \{\bar{j} \mid j \in J\}.$$ 

For $\mathcal{I} = \hat{I} \cup \hat{J}$ or $\mathcal{I} = \hat{I}_0 \cup \hat{J}$, we set

$$Q_{\mathcal{I}} := \left( \begin{array}{cc} M_1 & 0 \\
0 & M_2 \end{array} \right)$$

in which

$$M_2 := \sum_{j \in J} (e_{j, \bar{j}} - e_{\bar{j}, j}) \quad \text{and} \quad M_1 := \left\{ \begin{array}{ll}
2e_{0,0} + \sum_{i \in I} (e_{i,\bar{i}} + e_{\bar{i}, i}) & \text{if } \mathcal{I} = \hat{I}_0 \cup \hat{J} \\
\sum_{i \in I} (e_{i,\bar{i}} + e_{\bar{i}, i}) & \text{if } \mathcal{I} \neq \emptyset, \mathcal{I} = \hat{I} \cup \hat{J}. \end{array} \right.$$ 

Now by Lemma 2.35

$$G_{\mathcal{I}} := G_{Q_{\mathcal{I}}} = \{ X \in \mathfrak{pl}_I(\mathbb{F}) \mid X^s Q_{\mathcal{I}} = -Q_{\mathcal{I}} X \}$$

is a Lie subsuperalgebra of $\mathfrak{pl}_I(\mathbb{F})$ which we refer to as $\mathfrak{osp}(2I, 2J)$ or $\mathfrak{osp}(2I + 1, 2J)$ if $\mathcal{I} = \hat{I} \cup \hat{J}$ or $\mathcal{I} = \hat{I}_0 \cup \hat{J}$ respectively. Set

$$(2.37) \quad \mathfrak{h} := \text{span}_F \{ h_t, d_k \mid t \in I, k \in J \}$$

in which for $t \in I$ and $k \in J$,

$$h_t := e_{t, \bar{t}} - e_{\bar{t}, t} \quad \text{and} \quad d_k := e_{k, \bar{k}} - e_{\bar{k}, k}$$
and for \( i \in I \) and \( j \in J \), define
\[
\begin{align*}
\epsilon_i : & \mathfrak{h} \rightarrow \mathbb{F} & h_t \mapsto \delta_{i,t}, & d_k \mapsto 0, \\
\delta_j : & \mathfrak{h} \rightarrow \mathbb{F} & h_t \mapsto 0, & d_k \mapsto \delta_{j,k},
\end{align*}
\]
\((t \in I, k \in J)\). One sees that \( \mathcal{G}_I \) has a weight space decomposition with respect to \( \mathfrak{h} \). Taking \( R(I) \) to be the corresponding set of weights, we have
\[
R(\hat{I} \cup \hat{J}) = \{ \pm \epsilon_r, \pm(\epsilon_r \pm \epsilon_s), \pm \delta_p, \pm (\delta_p \pm \delta_q), \pm (\epsilon_r \pm \delta_p) \mid r, s \in I, \ p, q \in J, \ r \neq s \},
\]
\((2.38)\)
\[R(\hat{I} \cup \hat{J}) = \{ \pm(\epsilon_r \pm \epsilon_s), \pm(\delta_p \pm \delta_q), \pm(\epsilon_r \pm \delta_p) \mid r, s \in I, \ p, q \in J, \ r \neq s \}
\]
in which \( \pm(\epsilon_r \pm \epsilon_s) \)'s are disappeared if \(|I| = 1 \) and \( \pm \epsilon_r \)'s, \( \pm(\epsilon_r \pm \delta_p) \)'s as well as \( \pm(\epsilon_r \pm \delta_p) \)'s are disappeared if \(|I| = 0 \). Moreover, for \( r, s \in I, \ p, q \in J \) with \( r \neq s \) and \( p \neq q \), we have
\[
\begin{align*}
(G_I)^{\epsilon + \epsilon} & = \text{span}_\mathbb{F}(e_{r,s} - e_{s,r}), & (G_I)^{-\epsilon - \epsilon} & = \text{span}_\mathbb{F}(e_{r,s} - e_{s,r}), \\
(G_I)^{\epsilon - \epsilon} & = \text{span}_\mathbb{F}(e_{r,s} - e_{s,r}), & (G_I)^{\delta + \delta} & = \text{span}_\mathbb{F}(e_{p,q} + e_{p,q}), \\
(G_I)^{-\delta - \delta} & = \text{span}_\mathbb{F}(e_{p,q} - e_{q,p}), & (G_I)^{\delta - \delta} & = \text{span}_\mathbb{F}(e_{p,q} - e_{q,p}), \\
(G_I)^{\epsilon + \delta} & = \text{span}_\mathbb{F}(e_{r,s} + e_{p,q}), & (G_I)^{-\epsilon - \delta} & = \text{span}_\mathbb{F}(e_{r,s} + e_{p,q}), \\
(G_I)^{-\delta + \epsilon} & = \text{span}_\mathbb{F}(e_{r,s} - e_{p,q}), & (G_I)^{-\delta - \epsilon} & = \text{span}_\mathbb{F}(e_{r,s} - e_{p,q}), \\
(G_I)^{2\delta} & = \text{span}_\mathbb{F}(e_{p,q}), & (G_I)^{-2\delta} & = \text{span}_\mathbb{F}(e_{p,q}).
\end{align*}
\]

Define
\[
(\cdot, \cdot) : \mathcal{G}_I \times \mathcal{G}_I \rightarrow \mathbb{F}; \quad (x, y) \mapsto \text{str}(xy) \quad (x, y \in \mathcal{G}_I).
\]

Then \( (\mathcal{G}_I, \mathfrak{h}, (\cdot, \cdot)) \) is a locally finite basic classical simple Lie superalgebra whose root system is an irreducible locally finite root supersystem of type \( X \) as in the following table:

| \( X \) | \(|I|, |J|\) | \( \mathcal{I} \) | \( X \) | \(|I|, |J|\) | \( \mathcal{I} \) |
|---|---|---|---|---|---|
| \( B(0, J) \) | \(|1|, |J|\) | \( I_0 \cup J \) | \( B(0, J) \) | \(|1|, |J|\) | \( I_0 \cup J \) |
| \( B(1, J) \) | \(|1|, |2, 1| \) | \( I_0 \cup J \) | \( D(2, 1, a) \) | \(|2, 1|, |J|\) | \( I_0 \cup J \) |
| \( B(1, J) \) | \(|2, 1|, |J|\) | \( I_0 \cup J \) | \( D(2, 1, a) \) | \(|2, 1|, |J|\) | \( I_0 \cup J \) |
| \( B(1, J) \) | \(|2, 1|, |J|\) | \( I_0 \cup J \) | \( D(2, 1, a) \) | \(|2, 1|, |J|\) | \( I_0 \cup J \) |
| \( A(0, 2) \) | \(|1, 1| \) | \( I \cup J \) | \( D(1, J) \) | \(|2, 1|, |J|\) | \( I \cup J \) |

We refer to \( \mathfrak{h} \) as the standard Cartan subalgebra of \( \mathcal{G}_I \). We note that \( (\mathcal{G}_I)_0 \) is centerless unless \( \mathcal{I} = \hat{I} \cup \hat{J} \) with \(|I| = 1 \); see Lemma 2.33 In this case, suppose \( I = \{1\} \), then for a fixed index \( j \in J \), \( t_{e_1 + \delta_j} - (1/2)t_{2\delta_j} \) is a nonzero central element of the even part of \( \mathcal{G}_I \).

\( \diamond \)

**Lemma 2.39.** Suppose that \( I, J \) are two nonempty index sets with \(|I| = \infty \), then \( \mathfrak{osp}(2I, 2J) \simeq \mathfrak{osp}(2I + 1, 2J) \).

**Proof.** Consider the following matrices of \( \mathbb{F}^{(0)}_{\mathcal{I} \cup \mathcal{J} \cup \mathcal{I}} \):
\[
S := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
Q_r := \begin{pmatrix}
-2 & 0 & 0 & 0 & 0 \\
0 & 2I & 0 & 0 & 0 \\
0 & 0 & -2I & 0 & 0 \\
0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & -I & 0
\end{pmatrix},
Q := \begin{pmatrix}
-2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

then we have \( S^t Q S = Q_e \). Also for matrices
\[
S' := \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
Q' := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
2I & 0 & 0 & 0 & 0 \\
0 & -2I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & -I & 0
\end{pmatrix},
Q' := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
of $\mathbb{F}^{I_0 I_0 J_0 J}$, we have $S^{i' \tau} Q' S' = Q_0$. Now by Lemma \ref{lem:mainlemma}, $G_Q \simeq G_{Q_0}$, $G_{Q'} \simeq G_{Q_0}$, and $G_{Q_0} \simeq G_{Q_0}$. This completes the proof. \hfill \Box

Example 2.40. Suppose that $J$ is a superset with $J_0, J_1 \neq \emptyset$. Set $\mathcal{G} := \mathfrak{sl}(J_0, J_1) = \{ X \in \mathfrak{p}_J(\mathbb{F}) \mid \text{str}(X) = 0 \}$ and $\mathcal{H} := \text{span}_\mathbb{F}\{e_{i,i} - e_{r,r}, e_{j,j} - e_{s,s}, e_{i,i} + e_{j,j} \mid i, r \in J_0, j, s \in J_1\}$. For $t \in J_0, k \in J_1$, define

$$
\begin{align*}
\epsilon_t : \mathcal{H} &\longrightarrow \mathbb{F}, \\
\epsilon_{i,i} - e_{r,r} &\mapsto \delta_{i,t} - \delta_{r,t}, \ e_{j,j} - e_{s,s} \mapsto 0, \ e_{i,i} + e_{j,j} \mapsto \delta_{i,t}, \\
\delta_k : \mathcal{H} &\longrightarrow \mathbb{F}, \\
\epsilon_{i,i} - e_{r,r} &\mapsto 0, \ e_{j,j} - e_{s,s} \mapsto -\delta_{j,k} + \delta_{k,s}, \ e_{i,i} + e_{j,j} \mapsto -\delta_{j,k}, \\
(i, r \in J_0, j, s \in J_1).
\end{align*}
$$

Also define

$$
(\cdot, \cdot) : \mathcal{G} \times \mathcal{G} \longrightarrow \mathbb{F}; \ (X, Y) \mapsto \text{str}(XY).
$$

If $|J| < \infty$ and $|J_0| = |J_1|$, take $K := \mathbb{F} \sum_{j \in J} e_{jj}$ and note that it is a subset of the radical of the form $(\cdot, \cdot)$. So it induces a bilinear form on $\mathcal{G}/K$ denoted again by $(\cdot, \cdot)$. Set

$$
\mathfrak{sl}_s(J_0, J_1) := \left\{ \begin{array}{ll}
\mathcal{G}/K & \text{if } |J| < \infty \text{ and } |J_0| = |J_1| \\
\mathcal{G} & \text{otherwise}. 
\end{array} \right.
$$

Then $(\mathcal{L} := \mathfrak{sl}_s(J_0, J_1), (\cdot, \cdot), \mathcal{H}/K)$ is a locally finite basic classical simple Lie superalgebra with root system

$$
\{\epsilon_i - \epsilon_j, \delta_p - \delta_q, \pm (\epsilon_i + \delta_p) \mid i, j \in J_0, p, q \in J_1\}
$$

which is an irreducible locally finite root supersystem of type $X$ as in the following table:

| $X$ | $(|J_0|, |J_1|)$ |
|-----|-----------------|
| $A(0, J_1)$ | $(1, \geq 2)$ |
| $A(0, J_0)$ | $(\geq 2, 1)$ |
| $A(J_0, J_1)$ | $(\geq 2, \geq 2)$ |
| $A(J_0, J_1)$ | $|J_0| \neq |J_1|$ if $J_0, J_1$ are both finite |
| $A(\ell, \ell)$ | $(\ell, \ell) \ (\ell \in \mathbb{Z}^\epsilon)$ |

Also if $(|J_0|, |J_1|) \neq (1, 1)$, for $i, j \in J_0$ and $p, q \in J_1$ with $i \neq j$ and $p \neq q$, we have

$$
\begin{align*}
\mathcal{L}^{\epsilon_i - \epsilon_j} &= \mathbb{F}_{e_{i,i}, e_{j,j}}, \\
\mathcal{L}^{\delta_p - \delta_q} &= \mathbb{F}_{e_{p,p},} \\
\mathcal{L}^{\epsilon_i + \delta_p} &= \mathbb{F}_{e_{i,p},} \\
\mathcal{L}^{-\epsilon_i - \delta_p} &= \mathbb{F}_{e_{p,i}}.
\end{align*}
$$

We refer to $\mathcal{H}/K$ as the standard Cartan subalgebra of $\mathcal{L} = \mathfrak{sl}_s(J_0, J_1)$. We now need to discuss the center of $\mathcal{L}_0$ for our future purpose. We recall from finite dimensional theory of Lie superalgebras that if $|J_0|, |J_1| < \infty$, $\mathcal{L}_0$ has nontrivial center if and only if $|J_0| \neq |J_1|$ and that in this case, it has a one dimensional center. Now suppose $|J_0| \cup |J_1| = \infty$, say $|J_0| = \infty$. Fix $i_0 \in J_0$ and $j_0 \in J_1$. Then $\{e_{i_0,i_0}, e_{j,0}, e_{j_0,j_0}, e_{i_0,i_0} + e_{j_0,j_0} \mid i \in J_0 \setminus \{i_0\}, j \in J_1 \setminus \{j_0\}\}$ is a basis for $\mathcal{H}$. Suppose $i_1, \ldots, i_t$ are distinct elements of $J_0 \setminus \{i_0\}$ and $j_1, \ldots, j_n$ are distinct elements of $J_1 \setminus \{j_0\}$. If $z = \sum_{t=1}^\ell r_t (e_{i_t,i_t} - e_{i_0,i_0}) + \sum_{t=1}^n s_t (e_{j_t,j_t} - e_{j_0,j_0}) + k e_{i_0,i_0} + e_{j_0,j_0}$ (where $\sum_{t=1}^\ell s_t (e_{j_t,j_t} - e_{j_0,j_0})$ is disappeared if $|J_1| = 1$) is an element of the center of $\mathcal{L}_0$, then for each $i \in J_0 \setminus \{i_0\}$, $[z, (\mathcal{L}_0)^{r_t - e_{i_t,i_t}}] = \{0\}$. Now if $i = i_s$ for some $s \in \{1, \ldots, \ell\}$, we get $r_s + (\sum_{t=1}^\ell r_t) - k = 0$ and if $i \notin \{i_1, \ldots, i_t\}$, we get $\sum_{t=1}^\ell r_t - k = 0$. Therefore we have $r_s = 0$ for all $s \in \{1, \ldots, \ell\}$ and so $k = 0$. This shows that $\mathcal{L}$ is centerless if $|J_1| = 1$. If $|J_1| > 1$, $[z, \mathcal{L}^{e_{0,0} + \delta_{j_0,j_0}}] = \{0\}$. This implies that $\sum_{t=1}^n s_t = 0$. We also have $[z, \mathcal{L}^{\delta_{j_0,j_0}}] = \{0\}$ for all $j \in J_1 \setminus \{j_0\}$. Now it follows that $s_t = 0$ for all $t \in \{1, \ldots, n\}$. This means that $z = 0$. \hfill \Diamond
Lemma 2.41. For index sets $I, J$ with $|J| \neq 0$ and a superset $T$ with $T_0, T_1 \neq \emptyset$, set

$$a_{I,J} := \text{osp}(2I, 2J)(if I \neq \emptyset), b_{I,J} := \text{osp}(2I + 1, 2J), c_T := \text{sl}_k(T_0, T_1).$$

Suppose that $G$ and $L$ are two Lie superalgebras of the class

$$\{a_{I,J}, b_{I,J}, c_T \mid I, J, T\}.$$

Then $G$ and $L$ are isomorphic if and only if (up to changing the role of $G$ and $L$) one of the following holds:

- There are index sets $I, J$, $J'$ with $|I| = |I'| \neq 0$, $|J| = |J'| \neq 0$, $G = a_{I,J}$ and $L = a_{I',J}$,
- There are index sets $I, J$, $J'$ with $|I| = |I'|$, $|J| = |J'| \neq 0$, $G = b_{I,J}$ and $L = b_{I',J}$,
- There are supersets $I, J$ with $|I_0| = |J_0| \neq 0$, $|I_1| = |J_1| \neq 0$, or $|I_0| = |J_1| \neq 0$, $|I_1| = |J_0| \neq 0$ such that $G = c_t$ and $L = c_j$,
- There are index sets $I, J$ with $|I| = |J| = 1$ and a superset $T$ with $|T_0| = 1, |T_1| = 2$ or $|T_0| = 2, |T_1| = 1$ such that $G = a_{I,J}$ and $L = c_T$,
- There are index sets $I, J$ with $J \neq \emptyset, |I| = \infty$, $G = a_{I,J}$ and $L = b_{I,J}$.

Moreover, in each of the first three cases, the mentioned isomorphism can be chosen such that the standard Cartan subalgebra of $G$ is mapped to the standard Cartan subalgebra of $L$.

Proof. We first note that for two Lie algebras $t_1$ and $t_2$ such that $[t_1, t_1]$ and $[t_2, t_2]$ are semisimple with the complete sets of simple ideals $\{t_1^1, \ldots, t_1^m\}$ and $\{t_2^1, \ldots, t_2^m\}$ respectively, if $t_1$ and $t_2$ are isomorphic, we have

$$t_1, t_2 \text{ are isomorphic,}$$

(2.42)

- $t_1$ is centerless if and only if $t_2$ is centerless,
- $m = n$ and under a permutation of indices $t_1^i$ is isomorphic to $t_2^i$ for $i \in \{1, \ldots, n\}$.

Now take $A$ to be one of the Lie superalgebras $a_{I,J}, b_{I,J}, c_T$. We have already seen that if $A$ is infinite dimensional, then the even part of $A$ is centerless if and only if $A \neq a_{I,J}$ for some infinite index set $J$ and an index set $I$ with $|I| = 1$. Next suppose that $G$ and $L$ are as in the statement and assume they are isomorphic, then we have $G \simeq L_0$. We also know that $[G_0, G_0]$ as well as $[L_0, L_0]$ are semisimple Lie algebras by Theorem 2.30. Using these together with (2.32), Lemmas 2.35 and 2.39, classification of basic classical simple Lie superalgebras and [10] Pro.'s VI4,VI6, we are done. □

3. Classification Theorem

In this section, we classify locally finite basic classical simple Lie superalgebras and study the conjugacy classes of their Cartan subalgebras under the group of automorphisms. To this end, we need to know Chevalley bases for f.d.b.c.s. Lie superalgebras. Chevalley bases for f.d.b.c.s Lie superalgebras were introduced in 2001 by I. Kenji and K. Yoshiyuki [3] using the fact that a f.d.b.c.s Lie superalgebra is a contragredient Lie superalgebra and its Cartan matrix is symmetrizable. We define Chevalley bases in a somehow different manner from the one they have defined. Throughout this section we assume the field $F$ is algebraically closed.

Lemma 3.1. Suppose that $(G_1, \langle \cdot, \cdot \rangle_1, H_1), (G_2, \langle \cdot, \cdot \rangle_2, H_2)$ are two locally finite basic classical simple Lie superalgebras with corresponding root systems $R_1, R_2$ respectively. For $i = 1, 2$, denote the induced form on span$_g R_i \subseteq$
It is immediate that \((\tilde{H},\tilde{H})\) are Chevalley bases for basic classical simple Lie superalgebras.

Then by Lemma 1.10, \(B\) is a basis for \(\text{span}_\mathbb{Q} R_1\). We define the linear transformation \(\bar{f}\) mapping \(\alpha \in B\) to \(f(\alpha)\). It is immediate that \((\bar{f}(\alpha),\bar{f}(\alpha')) = k(\alpha,\alpha')\) for \(\alpha, \alpha' \in \text{span}_\mathbb{Q} R_1\). Now fix a basis \(\{\alpha_i \mid i \in I\} \subseteq (R_1)_{re}\) for \(\text{span}_\mathbb{Q} (R_1)_{re}\) as well as a nonzero nonsingular root \(\delta\) of \(R_1\) if \(R_1\) is of imaginary type. Set

\[
B := \begin{cases} 
\{\alpha_i \mid i \in I\} & \text{if } R_1 \text{ is of real type} \\
\{\delta, \alpha_i \mid i \in I\} & \text{if } R_1 \text{ is of imaginary type.}
\end{cases}
\]

Then by Lemma 1.10 \(B\) is \(\mathbb{F}\)-linearly independent and so by Lemma 1.9(b)(iii), it is a basis for both \(\text{span}_\mathbb{Q} R_1\) and \(\text{span}_\mathbb{Q} R_2\). Similarly, \(f(B)\) is a basis for \(\text{span}_\mathbb{Q} R_2\). We define the linear transformation \(\tilde{f}\) mapping \(\alpha \in B\) to \(f(\alpha)\). It is immediate that \((\tilde{f}(\alpha),\tilde{f}(\alpha')) = k(\alpha,\alpha')\) for \(\alpha, \alpha' \in \text{span}_\mathbb{Q} R_1\).

For \(u, v \in \mathcal{V}\), we say \(0 < u\) if \(u \neq 0\) and that the first nonzero \(r_i, 1 \leq i \leq m\), is positive; next for \(u, v \in \mathcal{V}\), we say \(u < v\) if \(0 < u - v\). We set \(R^+ := R \cap \{v \in \mathcal{V} \mid 0 < v\}\) as well as \(R^- := -R^+\). Elements of \(R^+\) are called positive and elements of \(R^-\) are called negative. As usual, for \(u, v \in \mathcal{V}\), we say \(u \leq v\) if either \(u = v\) or \(u < v\). Fix an invariant nondegenerate even supersymmetric bilinear form \((\cdot,\cdot)\) on \(\mathcal{G}\). We denote the induced nondegenerate symmetric bilinear form on \(\mathcal{H}\) again by \((\cdot,\cdot)\). We recall that for \(\alpha \in \mathcal{H}\), \(t_\alpha\) indicates the unique element of \(\mathcal{H}\) representing \(\alpha\) through the form \((\cdot,\cdot)\). For \(\alpha \in \mathcal{H}\), set

\[
\sigma_\alpha := \begin{cases} 
-1 & \alpha \in R_1 \cap R^- \\
1 & \text{otherwise.}
\end{cases}
\]

Next fix \(r \in \mathbb{F} \setminus \{0\}\) and for each \(\alpha \in R^x\), set

\[
h_\alpha := rt_\alpha.
\]

One can see that

\[
(3.2) \quad \sigma_{-\alpha} = (-1)^{|\alpha|} \sigma_\alpha \quad \text{and} \quad h_\alpha = -h_{-\alpha} \quad (\alpha \in R^x).
\]

For \(\alpha \in R^+\), fix \(Y_\alpha \in \mathcal{G}^\alpha\) and \(Y_{-\alpha} \in \mathcal{G}^{-\alpha}\) such that \([Y_\alpha, Y_{-\alpha}] = h_\alpha\). We note that for \(\alpha \in R^x\), \([Y_\alpha, Y_{-\alpha}] = \sigma_\alpha h_\alpha\).

**Definition 3.3.** A set \(\{X_\alpha, h_\alpha \mid \alpha \in R^x, i = 1, \ldots, \ell\}\) is called a Chevalley basis for \(\mathcal{G}\) if

- there are a nonzero scalar \(r\) and a subset \(\{\beta_1, \ldots, \beta_\ell\}\) of \(R^x\) such that \(h_1 := h_{\beta_1}, \ldots, h_\ell := h_{\beta_\ell}\) is a basis for \(\mathcal{H}\) where for \(\alpha \in R^x\), by \(h_\alpha\), we mean \(rt_\alpha\),
- for each \(\alpha \in R^x\), \(X_\alpha \in \mathcal{G}^\alpha\),
- for each \(\alpha \in R^x\), \([X_\alpha, X_{-\alpha}] = \sigma_\alpha h_\alpha\).
Suppose that \( \{X_\alpha, h_i \mid \alpha \in \mathbb{R}^X, i = 1, \ldots, \ell \} \) is a Chevalley basis for \( G \). We know from Lemma 2.24 that if \( \alpha, \beta \in \mathbb{R}^X \) such that \( \alpha + \beta \in \mathbb{R}^X \), then \( \mathbb{G}^\alpha \mathbb{G}^\beta \neq \{0\} \). This together with the fact that \( \dim(\mathbb{G}^{\alpha+\beta}) = 1 \) implies that there is a nonzero scalar \( N_{\alpha,\beta} \) with \( [X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta} \); we also interpret \( N_{\alpha,\beta} \) as zero for \( \alpha, \beta \in \mathbb{R}^X \) with \( \alpha + \beta \not\in R \). We refer to \( \{N_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{R}^X\} \) as a set of structure constants for \( G \) with respect to \( \{X_\alpha, h_i \mid \alpha \in \mathbb{R}^X, i = 1, \ldots, \ell \} \). Using a modified argument as in [1] Pro. 7.1, one can see the following proposition but for the convenience of readers, we give its proof in details; see also [3] §3.

**Proposition 3.4.** Keep the same notation as above; we have the following:

(i) If \( \alpha, \beta \in \mathbb{R}^X \), then \( N_{\alpha,\beta} = -(-1)^{||\alpha||\beta}N_{\beta,\alpha} \).

(ii) If \( \alpha, \beta \in \mathbb{R}^X \) with \( \alpha + \beta \in \mathbb{R}^X \), then for \( s_{\alpha,\beta} := \sigma_\alpha \sigma_{\alpha+\beta} \), we have

\[
N_{\alpha,\beta} = s_{\alpha,\beta}N_{\beta,-\alpha-\beta} = \sigma_\alpha \sigma_{\alpha+\beta}N_{\beta,-\alpha-\beta}.
\]

(iii) Suppose that \( \alpha, \beta \in \mathbb{R}^X \) with \( \alpha + \beta \in \mathbb{R}^X \), then

\[
N_{\alpha,\beta}N_{-\alpha,-\beta} = r_{\alpha,\beta} := \sigma_\beta \sigma_{\beta+\alpha} \sigma_\alpha (-1)^{||\beta||\alpha} \sum_{i=0}^{p} (-1)^{i||\alpha||}(\beta - i\alpha)(h_\alpha);
\]

where \( p = 0 \) if \( \alpha, \beta \in R_{ns} \) and otherwise, \( p \) is the largest nonnegative integer such that \( \beta - px \in R \).

(iv) If \( \alpha, \beta, \gamma, \delta \in \mathbb{R}^X \) with \( \alpha + \beta + \gamma + \delta = 0 \) such that each pair is not the opposite of the one another, then

\[
(-1)^{||\alpha||\gamma} \sigma_{\alpha+\beta}N_{\alpha,\beta}N_{\gamma,\delta} + (-1)^{||\beta||\gamma} \sigma_{\beta+\gamma}N_{\beta,\gamma}N_{\alpha,\delta} + (-1)^{||\gamma||\delta} \sigma_{\gamma+\delta}N_{\gamma,\delta}N_{\alpha,\beta} = 0.
\]

**Proof.** (i) We have

\[
N_{\alpha,\beta}X_{\alpha+\beta} = [X_\alpha, X_\beta] = -(-1)^{||X_\alpha||X_\beta} [X_\beta, X_\alpha] = -(-1)^{||\alpha||\beta}N_{\beta,\alpha}X_{\alpha+\beta}.
\]

This completes the proof.

(ii) Set \( \gamma := -\alpha - \beta \) and consider the Jacobi superidentity for \( X_\alpha, X_\beta, X_\gamma \), then we have

\[
0 = (-1)^{||\gamma||\beta}[X_\alpha, X_\alpha] + (-1)^{||A||\beta}[X_\gamma, X_\alpha] + (-1)^{||\delta||\beta}[X_\beta, X_\alpha] = (-1)^{||\gamma||\beta}[N_{\alpha,\beta}X_{\alpha+\beta}, X_\alpha] + (-1)^{||\beta||\alpha}[N_{\beta,\gamma}X_{\beta+\gamma}, X_\alpha] = (-1)^{||\gamma||\beta}N_{\alpha,\beta}X_{\alpha+\beta} - (-1)^{||\beta||\alpha}N_{\beta,\gamma}X_{\beta+\gamma} + (-1)^{||\gamma||\beta}N_{\alpha,\beta}h_{\gamma-\alpha} + (-1)^{||\beta||\alpha}N_{\beta,\gamma}h_{\alpha} = (-1)^{||\gamma||\beta}N_{\alpha,\beta}(-1)^{||\gamma||\beta}N_{\alpha,\beta}h_{\gamma-\alpha} - (-1)^{||\beta||\alpha}N_{\beta,\gamma}(-1)^{||\beta||\alpha}N_{\beta,\gamma}h_{\alpha} = (-1)^{||\gamma||\beta}N_{\alpha,\beta}(-1)^{||\gamma||\beta}N_{\alpha,\beta}h_{\gamma-\alpha} + (-1)^{||\beta||\alpha}N_{\beta,\gamma}(-1)^{||\beta||\alpha}N_{\beta,\gamma}h_{\alpha}.
\]

So as \( |\alpha| + |\beta| = |\gamma| \), we have

\[
(-1)^{||\beta||\gamma}N_{\alpha,\beta}N_{\beta,\gamma} = (-1)^{||\alpha||\beta}N_{\gamma,\alpha}N_{\beta,\gamma} + (-1)^{||\gamma||\alpha}N_{\beta,\gamma}N_{\alpha,\beta}.
\]

Now if \( \alpha, \beta \) are linearly independent, \( \{h_\alpha, h_\beta\} \) is linearly independent and so we get

\[
(-1)^{||\beta||\gamma}N_{\alpha,\beta}N_{\beta,\gamma} = (-1)^{||\alpha||\beta}N_{\gamma,\alpha}N_{\beta,\gamma} + (-1)^{||\gamma||\alpha}N_{\beta,\gamma}N_{\alpha,\beta}.
\]

Therefore, as \( |\alpha| + |\beta| = |\gamma| \), we have

\[
N_{\alpha,\beta}N_{\beta,\gamma} = (-1)^{||\gamma||\alpha}N_{\beta,\gamma}N_{\alpha,\beta}.
\]

If \( \alpha, \beta \) are linearly dependent, then using [16] Lem. 2.1(ii)] and [15] Pro. 3.8(i)], one of the following cases can happen: \( \alpha = \pm \beta, \alpha = \pm 2\beta \) or \( \beta = \pm 2\alpha \) but since \( \alpha + \beta \in R \setminus \{0\} \), we just have \( \alpha = \beta, \beta = -2\alpha \) or \( \alpha = -2\beta \). In the first case, since \( 2\alpha = \alpha + \beta \in R \setminus \{0\} \), we have \( |\alpha| = |\beta| = 1 \) and \( |\gamma| = 0 \). So 5.5 together with part (i) implies that

\[
N_{\alpha,\beta}N_{2\beta} = -N_{\gamma,\alpha}N_{\beta,\gamma}h_{\beta} + N_{\beta,\gamma}N_{2\alpha}h_{\beta} = 2N_{\beta,\gamma}N_{\beta,\gamma}h_{\beta}.
\]
so we get

\[ N_{\alpha, \beta} \sigma_{\gamma} = (-1)^{|\gamma|} \sigma_{\alpha} N_{\beta, \gamma}. \]

In the second case, \(|\alpha| = |\gamma| = 1\) and \(|\beta| = 0\) and so the result is immediate using part (i). In the last case, \(|\beta| = |\gamma| = 1\) and \(|\alpha| = 0\). Therefore (iii) together with part (i) implies that

\[ -2N_{\gamma, \alpha} \sigma_{\gamma} - N_{\gamma, \alpha} \sigma_{\beta} = N_{\alpha, \beta} \sigma_{\gamma} = N_{\gamma, \alpha} \sigma_{\beta} = -2N_{\beta, \gamma} \sigma_{\alpha}. \]

This completes the proof.

(iii) We first assume \(\alpha, \beta \in \mathbb{K}_{rre}\). Let \(0 \leq i \leq p\) and set \(\beta' := \beta - i\alpha\). We make a convention that

for \(i = p\), we set \(N_{\alpha, \beta' ; -\alpha} = N_{\alpha, \alpha ; -\beta'} = N_{\alpha, \alpha + \beta'} = N_{\alpha, \beta' ; -\alpha} := 0\)

and consider the Jacobi superidentity for \(X_{\alpha, X_{-\alpha}} X_{\beta'}\). Then we have

\[
0 = [[X_{\alpha, X_{-\alpha}}], X_{\beta'}] = [X_{\alpha}, [X_{-\alpha}, X_{\beta'}]] + (-1)^{|\alpha|} [X_{-\alpha}, [X_{\alpha}, X_{\beta'}]]
\]

\[
= [\sigma_{\alpha} h_{\alpha}, X_{\beta'}] - N_{\alpha, \beta'} N_{\alpha, -\alpha + \beta'} X_{\beta'} + (-1)^{|\alpha|} N_{\alpha, \alpha + \beta'} N_{\alpha, \beta'} X_{\beta'}.
\]

This implies that

\[ \sigma_{\alpha} \beta' (h_{\alpha}) = N_{\alpha, \beta'} N_{\alpha, -\alpha + \beta'} + (-1)^{|\alpha|} N_{\alpha, \alpha + \beta'} N_{\alpha, \beta'}. \]

Using part (ii) respectively for \((-\beta', -\alpha)\) and \((\alpha - \beta', -\alpha)\), we get

\[
\sigma_{\beta' + \alpha} N_{\beta' ; -\alpha} = (-1)^{|\beta' + \alpha|} \sigma_{\beta' + \alpha} N_{\alpha, \alpha + \beta'} \quad \text{and} \quad \sigma_{\beta'} N_{\alpha ; -\beta} = (-1)^{|\beta'|} \sigma_{\alpha - \beta} N_{\alpha, \beta'}.
\]

Thus we get

\[
N_{\alpha, \beta'} N_{\alpha, \alpha + \beta'} = (-1)^{|\beta' + \alpha|} \sigma_{\beta'} \sigma_{\alpha + \alpha} N_{\alpha, \beta'} N_{\alpha, -\beta'} = -(-1)^{|\alpha| + |\beta'|} (-1)^{|\beta'|} (-1)^{|\alpha|} \sigma_{\beta'} \sigma_{\beta' + \alpha} N_{\alpha, \beta'} N_{\alpha, -\beta'}
\]

\[
= -(-1)^{|\alpha|} (-1)^{|\beta'|} \sigma_{\beta'} \sigma_{\beta' + \alpha} N_{\alpha, \beta'} N_{\alpha, -\beta'}
\]

and

\[
N_{\alpha, \beta'} N_{\alpha, -\alpha + \beta'} = (-1)^{|\beta'|} \sigma_{\alpha - \beta} \sigma_{\beta'} N_{\alpha, -\alpha + \beta'} = -(-1)^{|\beta'|} (-1)^{|\alpha|} \sigma_{\beta'} \sigma_{\beta' - \alpha} N_{\alpha, -\alpha + \beta'} N_{\alpha, -\alpha - \beta'}.
\]

So we have

\[
(-1)^{|\alpha| + |\beta'|} \sigma_{\alpha} \beta' (h_{\alpha}) = (-1)^{|\alpha| + |\beta'|} N_{\alpha, \beta'} N_{\alpha, -\alpha + \beta'} - (-1)^{|\alpha|} (-1)^{|\beta'|} N_{\alpha, \alpha + \beta'} N_{\alpha, \beta'}
\]

\[
= \sigma_{\beta'} \sigma_{\beta' + \alpha} N_{\alpha, \beta'} N_{\alpha, -\beta'} - \sigma_{\beta'} \sigma_{\beta' - \alpha} N_{\alpha, -\alpha + \beta'} N_{\alpha, -\alpha - \beta'}.
\]
Therefore, we have
\[
\sigma_\alpha \sum_{i=0}^{p} (-1)^{|\beta-i\alpha||\alpha|}(\beta - i\alpha)(h_\alpha) = \sum_{i=0}^{p} \sigma_{\beta-i\alpha}\sigma_{\beta-(i-1)\alpha}N_{\alpha,\beta-i\alpha}N_{\alpha,-(\beta-ia)} - \sum_{i=0}^{p} \sigma_{\beta-i\alpha}\sigma_{\beta-(i+1)\alpha}N_{\alpha,\beta-(i+1)\alpha}N_{\alpha,-(\beta+i+1)\alpha} + \sigma_\beta \sigma_{\beta+\alpha}N_{\alpha,\beta}N_{\alpha,-\beta}
\]
\[
= \sum_{i=1}^{p} \sigma_{\beta-i\alpha}\sigma_{\beta-(i-1)\alpha}N_{\alpha,\beta-(i-1)\alpha}N_{\alpha,-(\beta-ia)} - \sum_{i=0}^{p-1} \sigma_{\beta-i\alpha}\sigma_{\beta-(i+1)\alpha}N_{\alpha,\beta-(i+1)\alpha}N_{\alpha,-(\beta+i+1)\alpha} + \sigma_\beta \sigma_{\beta+\alpha}N_{\alpha,\beta}N_{\alpha,-\beta}
\]
\[
= \sigma_\beta \sigma_{\beta+\alpha}N_{\alpha,\beta}N_{\alpha,-\beta}.
\]
This means that
\[
N_{\alpha,\beta}N_{\alpha,-\beta} = \sigma_\beta \sigma_{\beta+\alpha}(\beta - i\alpha)(h_\alpha) \sum_{i=0}^{p} (-1)^{|\beta||\alpha|} \sum_{i=0}^{p} (-1)^{|\alpha|}(\beta - i\alpha)(h_\alpha)
\]
as we desired. We next assume $\alpha, \beta \in R_{\alpha\beta}$, then since $\alpha + \beta \in R$, we have $\alpha - \beta \notin R$; see Proposition 2.16. Then the Jacobi superidentity for $X_\alpha, X_{-\alpha}, X_{\beta}$ together with parts (i), (ii) turns into
\[
0 = [\sigma_\alpha h_\alpha, X_\beta] + (-1)^{|\alpha|}N_{-\alpha,\alpha+\beta}N_{\alpha,\beta}X_\beta
\]
\[
= \sigma_\alpha \beta(h_\alpha)X_\beta + (-1)^{|\alpha|}N_{-\alpha,\alpha+\beta}N_{\alpha,\beta}X_\beta
\]
\[
= \sigma_\alpha \beta(h_\alpha)X_\beta + (-1)^{|\alpha|}(-1)^{|\beta+\alpha|}\sigma_{\beta+\alpha}N_{-\beta,-\alpha}N_{\alpha,\beta}X_\beta
\]
\[
= \sigma_\alpha \beta(h_\alpha)X_\beta + (-1)^{|\alpha|}(-1)^{|\beta|}(-1)^{|\beta+\alpha|}\sigma_{\beta+\alpha}N_{-\beta,-\alpha}N_{\alpha,\beta}X_\beta
\]
\[
= \sigma_\alpha \beta(h_\alpha)X_\beta - (-1)^{|\alpha|}(-1)^{|\beta|}(-1)^{|\beta+\alpha|}(-1)^{|\alpha|}\sigma_{\beta+\alpha}N_{-\alpha,-\beta}N_{\alpha,\beta}X_\beta.
\]
This implies that
\[
N_{-\alpha,-\beta}N_{\alpha,\beta} = (-1)^{|\alpha||\beta|}\sigma_\alpha \sigma_{\beta+\alpha} \sigma_{\beta\beta}(h_\alpha)
\]
and so we are done.

(iv) In what follows if $\eta_1, \eta_2 \in \mathcal{H}^*$ are such that either $\eta_1$ or $\eta_2$ is a non-zero root, we define $N_{\eta_1, \eta_2}$ to be zero. Suppose that $\alpha, \beta, \gamma, \eta$ are as in the statement. Considering the Jacobi superidentity for $X_\alpha, X_{\beta}$ and $X_{\gamma}$, we have
\[
N_{\alpha,\beta}N_{\alpha+\beta,\gamma}X_{\alpha+\beta+\gamma} = [[X_\alpha, X_\beta], X_\gamma]
\]
\[
= [X_\alpha, [X_\beta, X_\gamma]] - (-1)^{|\alpha||\beta|}[X_\beta, [X_\alpha, X_\gamma]]
\]
\[
= N_{\beta,\gamma}N_{\alpha,\beta+\gamma}X_{\alpha+\beta+\gamma} - (-1)^{|\alpha||\beta|}N_{\alpha,\gamma}N_{\beta,\alpha+\gamma}X_{\alpha+\beta+\gamma}.
\]
Since \((\alpha + \beta) + \gamma + \delta = 0\), part (ii) implies that

\[ N_{\alpha + \beta, \gamma} = (-1)^{|\delta|} \sigma_\delta \sigma_{\alpha + \beta} N_{\gamma, \delta}. \]

Note that if \(\alpha + \beta\) is not a root, the above equality is nothing but the natural identity \(0 = 0\). Similarly, we have

\[ N_{\beta + \gamma, \alpha} = (-1)^{\delta} \sigma_\delta \sigma_{\beta + \gamma} N_{\alpha, \delta}\]

and

\[ N_{\alpha + \gamma, \beta} = (-1)^{\delta} \sigma_\delta \sigma_{\alpha + \gamma} N_{\beta, \delta}. \]

Therefore, we have

\[ (-1)^{|\delta|} \sigma_\delta \sigma_{\alpha + \beta} N_{\alpha, \beta} N_{\gamma, \delta} = N_{\alpha, \beta} N_{\alpha + \beta, \gamma} \]

\[ = N_{\beta, \gamma} N_{\alpha, \beta + \gamma} - (-1)^{|\alpha|+|\beta|} N_{\alpha, \gamma} N_{\beta, \alpha + \gamma} \]

\[ = -(-1)^{|\delta|} (-1)^{|\alpha|+|\beta|+|\gamma|} \sigma_\delta \sigma_{\beta + \gamma} N_{\beta, \gamma} N_{\alpha, \delta} \]

\[ + (-1)^{|\alpha|+|\beta|+|\gamma|} \sigma_\delta \sigma_{\alpha + \gamma} N_{\alpha, \gamma} N_{\beta, \delta} \]

and so we get

\[ \sigma_{\alpha + \beta} N_{\alpha, \beta} N_{\gamma, \delta} = (-1)^{|\alpha|+|\beta|+|\gamma|} \sigma_{\beta + \gamma} N_{\beta, \gamma} N_{\alpha, \delta} + (-1)^{|\gamma|} \sigma_{\alpha + \gamma} N_{\alpha, \gamma} N_{\beta, \delta}. \]

Thus we have

\[ (-1)^{|\alpha|+|\gamma|} \sigma_{\alpha + \beta} N_{\alpha, \beta} N_{\gamma, \delta} + (-1)^{|\alpha|+|\beta|} \sigma_{\beta + \gamma} N_{\beta, \gamma} N_{\alpha, \delta} + (-1)^{|\gamma|} \sigma_{\alpha + \gamma} N_{\alpha, \gamma} N_{\beta, \delta} = 0. \]

This completes the proof. \(\square\)

We know that there are roots \(\alpha, \gamma\) such that \(\alpha \neq \pm \gamma\) and \((\alpha, \gamma) \neq 0\). So either \(\alpha + \gamma \in R^+\) or \(\alpha - \gamma \in R^+\). Replacing \(\gamma\) with \(-\gamma\) if necessary, we assume \(\eta := -(\alpha + \gamma) \in R^+\). Since \(\alpha + \gamma + \eta = 0\), either two of \(\alpha, \gamma, \eta\) are positive or two of \(-\alpha, -\gamma, -\eta\) are positive. Selecting this pair of positive roots in an appropriate order, we get a pair \((\eta_1, \eta_2)\) among the 12 pairs

\[(\alpha, \gamma), (\alpha, \eta), (\gamma, \eta), (\gamma, \alpha), (\eta, \alpha), (\eta, \gamma),\]

\[(-\alpha, -\gamma), (-\alpha, -\eta), (-\gamma, -\eta), (-\gamma, -\alpha), (-\eta, -\alpha), (-\eta, -\gamma)\]

such that \(0 < \eta_1 \leq \eta_2\); following [1], we call such a pair a special pair. More precisely, a pair \((\alpha, \beta)\) of elements of \(R^+\) is called a special pair if \(0 < \alpha \leq \beta\) and \(\alpha + \beta \in R\). A special pair \((\alpha, \beta)\) is called an extraspecial pair if for each special pair \((\delta, \gamma)\) with \(\alpha + \beta = \delta + \gamma\), we get \(\alpha \leq \delta\).

**Lemma 3.6.** Suppose that \(\mathcal{A}\) is the set of all extraspecial pairs \((\alpha, \beta)\) of \(R^+\) and \(\{N_{\alpha, \beta} \mid (\alpha, \beta) \in \mathcal{A}\}\) is an arbitrary set of nonzero scalars. Then there is \(\{e_\alpha \in G^\alpha \setminus \{0\} \mid \alpha \in R^+\}\) such that \([e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha + \beta}\) for all \((\alpha, \beta) \in \mathcal{A}\).

**Proof.** Suppose that \(R^+ = \{\alpha_1, \ldots, \alpha_n\}\) with \(\alpha_1 < \ldots < \alpha_n\) and take \(t\) to be the smallest index such that \(\alpha_t\) is the summation of the components of an extraspecial pair. We choose arbitrary elements \(e_{\alpha_i} \in G^{\alpha_i}\), for \(1 \leq i \leq t - 1\). We know that there is a unique extraspecial pair \((\alpha, \beta)\) with \(\alpha_t = \alpha + \beta\), so there is a unique pair \((i, j)\) with \(i \leq j < t\) such that \(\alpha_t = \alpha_i + \alpha_j\) and define \(e_{\alpha_t} = N_{\alpha_i, \alpha_j}^{-1}[e_{\alpha_i}, e_{\alpha_j}]\). Now using an induction process, we can complete the proof; indeed, suppose that \(t < r \leq n\) and that \(\{e_{\alpha_s} \mid 1 \leq s \leq r - 1\}\) with the desired property has been chosen. If \(\alpha_r\) is not the summation of the components of an extraspecial pair, we choose \(e_{\alpha_r}\),
arbitrary, but otherwise we pick the unique pair \((i', j')\) with \(i' \leq j' < r - 1\) such that \(\alpha_r = \alpha_{i'} + \alpha_{j'}\). Now we define \(e_{\alpha} = N_{\alpha_{i'}, \alpha_{j'}}^{-1}[e_{\alpha_{i'}}, e_{\alpha_{j'}}]\). This completes the proof. 

**Theorem 3.7.** Suppose that \(G\) and \(L\) are two finite dimensional basic classical simple Lie superalgebras with Cartan subalgebras \(H\) and \(T\) and corresponding root systems \(R = R_0 \cup R_1\) and \(S = S_0 \cup S_1\) respectively which are not of type \(A(1,1)\). Suppose that \((\cdot, \cdot)\) (resp. \((\cdot, \cdot)\)') is an invariant nondegenerate even supersymmetric bilinear form on \(G\) (resp. \(L\)) and denote the induced forms on \(H^*\) and \(T^*\) again by \((\cdot, \cdot)\) and \((\cdot, \cdot)'\) respectively. Suppose that \(((R),(\cdot, \cdot), R)\) and \(((S),(\cdot, \cdot)', S)\) are isomorphic finite root supersystems, say via \(f : (R) \to (S)\). Then we have the following:

\[(i)\] There are Chevalley bases \(\{h_i, e_\alpha | \alpha \in R^X, 1 \leq i \leq \ell\}\) and \(\{t_i, x_\beta | \beta \in S^X, 1 \leq i \leq \ell\}\) for \(G\) and \(L\) with corresponding sets of structure constants \(\{N_{\alpha, \beta} | \alpha, \beta \in R^X\}\) and \(\{M_{\gamma, \eta} | \gamma, \eta \in S^X\}\) respectively such that \(N_{\alpha, \beta} = M_{f(\alpha), f(\beta)}\) for all \(\alpha, \beta \in R^X\).

\[(ii)\] \(\{N_{\alpha, \beta} | \alpha, \beta \in R^X\}\) is completely determined in terms of \(N_{\alpha, \beta}\)'s for extraspecial pairs \((\alpha, \beta)\).

\[(iii)\] There is an isomorphism from \(G\) to \(L\) mapping \(H\) to \(T\) and \(e_\alpha\) to \(x_{f(\alpha)}\) for all \(\alpha \in R \setminus \{0\}\).

**Proof.** (i), (ii) Suppose that \(k \in F \setminus \{0\}\) is such that \((f(\alpha), f(\beta))' = k(\alpha, \beta)\) for \(\alpha, \beta \in R\). Fix \(r, s \in F \setminus \{0\}\) such that \(r = sk\). This implies that \(r(\alpha, \beta) = sk(\alpha, \beta) = s(f(\alpha), f(\beta))'\) for all \(\alpha, \beta \in R\). Use Lemma 3.1 to extend the map \(f\) to a linear isomorphism, denoted again by \(f\), from \(H^* = \text{span}_F R\) to \(T^* = \text{span}_F S\) with

\[(3.8)\]

\[r(\alpha, \beta) = sk(\alpha, \beta) = s(f(\alpha), f(\beta))' \quad (\alpha, \beta \in H^*).\]

For \(\alpha \in H^*\), take \(t_\alpha\) to be the unique element of \(H\) representing \(\alpha\) through \((\cdot, \cdot)\) and for \(\beta \in T^*\), take \(t'_\beta\) to be the unique element of \(T\) representing \(\beta\) through \((\cdot, \cdot)'\). Next set

\[(3.9)\]

\[h_\alpha := rt_\alpha \quad \text{and} \quad h'_\beta := st'_\beta \quad (\alpha \in R, \beta \in S).\]

Fix a total ordering \(\preceq\) on \(\text{span}_F R\) as at the beginning of this subsection and transfer it through \(f\) to a total ordering, denoted again by \(\preceq\), on \(\text{span}_F S\). For \(\alpha \in H^*\) and \(\beta \in T^*\), set

\[(3.10)\]

\[\sigma_\alpha := \begin{cases} -1 & \text{if } \alpha \in R^- \cap R_1 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma'_\beta := \begin{cases} -1 & \text{if } \beta \in S^- \cap S_1 \\ 1 & \text{otherwise}. \end{cases}\]

Suppose that \(A\) is the set of all extraspecial pairs of \(R\), then

\[\{(f(\alpha), f(\beta)) | (\alpha, \beta) \in A\} = \{(\eta, \gamma) | (\eta, \gamma) \text{ is an extraspecial pair of } S\}.\]

Fix a subset \(\{N_{\alpha, \beta} | (\alpha, \beta) \in A\}\) of nonzero scalars and set \(M_{f(\alpha), f(\beta)} := N_{\alpha, \beta}\), for all \((\alpha, \beta) \in A\). Using Lemma 3.6 one can find \(e_\alpha \in G^- \setminus \{0\} | \alpha \in R^+\) and \(x_\beta \in L^- \setminus \{0\} | \beta \in S^+\) such that

\[e_\alpha, e_\beta = N_{\alpha, \beta}^{} e_{\alpha + \beta} \quad \text{and} \quad [x_{f(\alpha)}, x_{f(\beta)}] = M_{f(\alpha), f(\beta)} x_{f(\alpha) + f(\beta)} \quad (\alpha, \beta) \in A.\]

Now for each \(\alpha \in R^+\) and \(\gamma \in S^+\), choose \(e_{-\alpha} \in G^- \\alpha\) and \(x_{-\gamma} \in L^- \gamma\) such that

\[e_\alpha, e_{-\alpha} = h_\alpha \quad \text{and} \quad [x_{\gamma}, x_{-\gamma}] = h'_\gamma \quad (\alpha \in R^+, \gamma \in S^+)\]

and note that we have

\[e_\alpha, e_{-\alpha} = \sigma_\alpha h_\alpha \quad \text{and} \quad [x_{\gamma}, x_{-\gamma}] = \sigma'_\gamma h'_\gamma \quad (\alpha \in R^X, \gamma \in S^X).\]
Now for each pair \((\alpha, \beta)\) of \(R^x\) with \(\alpha + \beta \in R^x\) and \((\alpha, \beta) \notin A\), take \(N_{\alpha, \beta}\) to be the unique nonzero element of \(F\) with \([e_{\alpha}, e_{\beta}] = N_{\alpha, \beta}e_{\alpha+\beta}\); also for each pair \((\gamma, \eta)\) of \(S^x\) with \(\gamma + \eta \in S^x\) such that \((\gamma, \eta)\) is not an extraspecial pair, take \(M_{\gamma, \eta}\) to be the unique nonzero element of \(F\) with \([x_{\gamma}, x_{\eta}] = M_{\gamma, \eta}e_{\gamma+\eta}\). Fix \(\{\beta_1, \ldots, \beta_\ell\}\) such that \(\{h_i := h_{\beta_i} \mid 1 \leq i \leq \ell\}\) is a basis for \(H\) and set \(t_i := h'_{f(\beta_i)}\). Then \(\{h_i, e_\alpha \mid \alpha \in R^x, 1 \leq i \leq \ell\}\) and \(\{t_i, x_\beta \mid \beta \in S^x, 1 \leq i \leq \ell\}\) are Chevalley bases for \(G\) and \(L\) respectively. We complete the proof through the following two steps. As usual we also set \(N_{\alpha, \beta} := 0\) and \(M_{\gamma, \eta} := 0\) if \(\alpha, \beta \in R^x\) and \(\gamma, \eta \in S^x\) with \(\alpha + \beta \notin R^x\) and \(\gamma + \eta \notin S^x\). By a conventional notation in these cases, we set \(N_{\xi, \alpha+\beta} := 0\) and \(M_{\xi, \eta+\gamma} := 0\) for \(\xi \in R^x\) and \(\xi \in S^x\).

Step 1. For a special pair \((\alpha, \beta)\), we have \(N_{\alpha, \beta} = M_{f(\alpha), f(\beta)}\) and \(N_{\alpha, \beta}\) is determined in terms of \(N_{\alpha', \beta'}\)'s for extraspecial pairs \((\alpha', \beta')\) : Suppose that \((\alpha, \beta)\) is a special pair which is not an extraspecial pair. So there is a unique extraspecial pair \((\gamma, \delta)\) with \(\alpha + \beta = \gamma + \delta\). We have \(\alpha + \beta + (-\gamma) + (-\delta) = 0\). Therefore by Proposition 3.4, we have

\[
(1)^{|\alpha||\gamma|} \sigma_{\alpha+\beta} N_{\alpha, \beta} N_{-\gamma, -\delta} + (1)^{|\alpha||\delta|} \sigma_{\beta-\gamma} N_{\beta, -\gamma} N_{\alpha, -\delta} + (1)^{|\beta||\gamma|} \sigma_{\alpha-\gamma} N_{-\gamma, \alpha} N_{\beta, -\delta} = 0.
\]

So,

\[
N_{\alpha, \beta} = - (1)^{|\alpha||\beta|+|\alpha||\gamma|} \sigma_{\alpha+\beta} \sigma_{\beta-\gamma} N_{-\gamma, -\delta} N_{\beta, -\gamma} N_{\alpha, -\delta} - (1)^{|\beta||\gamma|+|\alpha||\gamma|} \sigma_{\alpha+\beta} \sigma_{\alpha-\gamma} N_{-\gamma, \alpha} N_{\beta, -\delta} N_{\alpha, -\delta} - (1)^{|\beta||\gamma|+|\alpha||\gamma|} \sigma_{\alpha+\beta} \sigma_{\alpha-\gamma} N_{-\gamma, \alpha} N_{\beta, -\delta} N_{\alpha, -\delta} - (1)^{|\beta||\alpha|+|\alpha||\gamma|} \sigma_{\alpha+\beta} \sigma_{\alpha-\gamma} N_{-\gamma, \alpha} N_{\beta, -\delta} N_{\alpha, -\delta} - (1)^{|\beta||\alpha|+|\alpha||\gamma|} \sigma_{\alpha+\beta} \sigma_{\alpha-\gamma} N_{-\gamma, \alpha} N_{\beta, -\delta} N_{\alpha, -\delta} = 0.
\]

Since \(N_{\alpha, \beta} \neq 0\), either \(N_{-\gamma, -\delta} N_{\beta, -\gamma} N_{\alpha, -\delta} \neq 0\) or \(N_{-\gamma, \alpha} N_{\beta, -\delta} N_{\beta, -\gamma} \neq 0\). So

\[
N_{\alpha, \beta} = \begin{cases} (1)^{|\alpha||\beta|+|\alpha||\gamma|} \sigma_{\alpha+\beta} \sigma_{\beta-\gamma} N_{-\gamma, -\delta} N_{\beta, -\gamma} N_{\alpha, -\delta} N_{\gamma, -\alpha} N_{\delta, -\beta} = 0 & N_{-\gamma, \alpha} N_{\beta, -\delta} N_{\beta, -\gamma} = 0 \\ (1)^{|\beta||\gamma|+|\alpha||\gamma|} \sigma_{\alpha+\beta} \sigma_{\alpha-\gamma} N_{-\gamma, \alpha} N_{\beta, -\delta} N_{\alpha, -\delta} N_{\gamma, -\alpha} N_{\delta, -\beta} = 0 & N_{-\gamma, \alpha} N_{\beta, -\delta} N_{\beta, -\gamma} = 0 \\ (1)^{|\beta||\alpha|+|\alpha||\gamma|} \sigma_{\alpha+\beta} \sigma_{\alpha-\gamma} N_{-\gamma, \alpha} N_{\beta, -\delta} N_{\alpha, -\delta} N_{\gamma, -\alpha} N_{\delta, -\beta} = 0 & otherwise. \end{cases}
\]

But \(\gamma < \alpha \leq \beta < \delta\), so if each of pairs \((\gamma, \beta - \gamma), (\alpha, \delta - \alpha), (\gamma, \alpha - \gamma), (\beta, \delta - \beta)\) is a pair of roots, it is a pair of positive roots whose sum of its components is less than \(\alpha + \beta = \gamma + \delta\) with respect to the ordering \(\leq\).

Now we use induction on \(\alpha + \beta\) with respect to the ordering \(\leq\) to complete the proof. If \((\alpha, \beta)\) is a special pair such that \(\alpha + \beta\) is as small as possible with respect to \(\leq\), it follows from the above argument that \((\alpha, \beta)\) is an extraspecial pair and so there is nothing to prove. Next suppose \((\alpha, \beta)\) is a special pair and that we have the result for \(N_{\xi_1, \xi_2}\), where \((\xi_1, \xi_2)\) is a special pair satisfying \(\xi_1 + \xi_2 < \alpha + \beta\). If \((\alpha, \beta)\) is extraspecial, there is nothing to prove, otherwise we are done using (3.11) together with (3.8) and the induction hypothesis.

Step 2. For \(\alpha, \beta \in R^x\) with \(\alpha + \beta \in R^x\), we have \(N_{\alpha, \beta} = M_{f(\alpha), f(\beta)}\) and \(N_{\alpha, \beta}\) is completely determined in terms of \(N_{\alpha', \beta'}\)'s for extraspecial pairs \((\alpha', \beta')\) : Suppose that \(\alpha, \beta \in R^x\) and \(\alpha + \beta \in R^x\), then for \(\gamma := -(\alpha + \beta)\), we have \(\alpha + \beta + \gamma = 0\), so as we have seen before, there is a special pair \((\eta_1, \eta_2)\) among the 12 pairs

\[
(\alpha, \beta), (\alpha, \gamma), (\beta, \gamma), (\beta, \alpha), (\gamma, \alpha), (\gamma, \beta), \\
(-\alpha, -\beta), (-\alpha, -\gamma), (-\beta, -\gamma), (-\beta, -\alpha), (-\gamma, -\alpha), (-\gamma, -\beta).
\]

Now using Proposition 3.3, for each pair \((\gamma_1, \gamma_2)\) of these 12 pairs, \(N_{\gamma_1, \gamma_2} = M_{f(\gamma_1), f(\gamma_2)}\) are uniquely determined in terms of \(N_{\eta_1, \eta_2} = M_{f(\eta_1), f(\eta_2)}\). Now we get the result using Step 1 together with (3.8).

(iii) Use the same notation as above. Define \(\theta : G \to L\) mapping \(h_i := h_{\beta_i}\) to \(t_i := h'_{f(\beta_i)}\) and \(e_\alpha\) to \(x_{f(\alpha)}\) for all \(\alpha \in R^x\) and \(1 \leq i \leq \ell\). We claim that \(\theta\) is a Lie superalgebra isomorphism. We first note that by Proposition 2.6 2.0 and 2.1, \(f(R_0) = S_0\) and \(f(R_1) = S_1\). Therefore, we have \(\theta(G_1) \subseteq L_i\) for \(i = 0, 1\). Now we need to
show $\theta[x, y] = [\theta(x), \theta(y)]$ for all $x, y \in G$. If $x = h_{\beta_i}$ and $y = e_\alpha$, for some $1 \leq i \leq \ell$ and $\alpha \in R^\times$, by (3.3), we have

$$\theta[h_{\beta_i}, e_\alpha] = \theta(\alpha(h_{\beta_i}) e_\alpha) = \alpha(h_{\beta_i}) \theta(e_\alpha) = f(\alpha)(h_{f(\beta_i)}) x f(\alpha) = [h_{f(\beta_i)}, x f(\alpha)]$$

$$= [\theta(h_{\beta_i}), \theta(e_\alpha)].$$

Next suppose $\alpha, \beta \in R^\times$. If $\alpha + \beta \notin R$, then $f(\alpha) + f(\beta) \notin S$ and so $[e_\alpha, e_\beta] = 0$, also if $\alpha + \beta \in R^\times$, then by part (i),

$$\theta[e_\alpha, e_\beta] = \theta(N_{\alpha, \beta} e_{\alpha+\beta}) = N_{\alpha, \beta} \theta(e_{\alpha+\beta}) = M_{f(\alpha), f(\beta)} x f(\alpha+\beta) = [x f(\alpha), x f(\beta)]$$

$$= [\theta(e_\alpha), \theta(e_\beta)].$$

Finally, for $\alpha \in R^\times$, if $h_\alpha = \sum_{i=1}^\ell r_i h_{\beta_i}$ for some $r_i \in F$ ($1 \leq i \leq \ell$), we get $\alpha = \sum_{i=1}^\ell r_i h_{\beta_i}$ and so $f(\alpha) = \sum_{i=1}^\ell r_i h_{f(\beta_i)}$ which in turn implies that

$$h_{f(\alpha)} = \sum_{i=1}^\ell r_i h_{f(\beta_i)} = \sum_{i=1}^\ell r_i \theta(h_{\beta_i}) = \theta(h_\alpha).$$

Therefore, we have

$$\theta[e_\alpha, e_{-\alpha}] = \theta(\sigma h_\alpha) = \sigma_\alpha h_{f(\alpha)} = \sigma f(\alpha) h_{f(\alpha)} = [x f(\alpha), x_{-f(\alpha)}] = [\theta(e_\alpha), \theta(e_{-\alpha})].$$

This completes the proof. $\square$

Suppose that $G$ is a finite dimensional basic classical simple Lie superalgebra with a Cartan subalgebra $H$ and corresponding root system $R$. For a group homomorphism $\phi : \langle R \rangle \rightarrow F \setminus \{0\}$, the linear transformation $\tilde{\phi} : G \rightarrow G$ mapping $x \in G^\alpha$ ($\alpha \in R$) to $\phi(\alpha) x$ is a superalgebra automorphism referred to as a diagonal automorphism.

**Lemma 3.13.** Keep the same notations and assumptions as in Theorem [3.7] and its proof. Suppose that $\Pi$ is an integral base for $R$ and fix nonzero elements $f_\alpha \in G^\alpha$ and $y_\alpha \in L^f(\alpha)$ for all $\alpha \in \Pi$. Then there is an isomorphism from $G$ to $L$ mapping $f_\alpha$ to $y_\alpha$ and $h_\alpha$ to $h_{f(\alpha)}$ for all $\alpha \in \Pi$. Moreover, if $\Pi$ is a base, then such an isomorphism is unique.

**Proof.** Consider the Chevalley bases $\{e_\alpha, h_1 \mid \alpha \in R^\times, 1 \leq i \leq \ell\}$ and $\{x_\beta, t_1 \mid \beta \in S^\times, 1 \leq i \leq \ell\}$ as well as the isomorphism $\theta : G \rightarrow L$ as in Theorem [3.4]. Since $f_\alpha \in G^\alpha = F e_\alpha$, there is $k_\alpha \in F \setminus \{0\}$ such that $f_\alpha = k_\alpha e_\alpha$. Similarly, there is a nonzero scalar $k'_\alpha$ such that $y_\alpha = k'_\alpha x f(\alpha)$. Define $\phi : \langle R \rangle \rightarrow F \setminus \{0\}$ mapping $\alpha \in \Pi$ to $k^{-1}_\alpha$ and $\phi : \langle S \rangle \rightarrow F \setminus \{0\}$ mapping $f(\alpha) \in f(\Pi)$ to $k'_\alpha$. Now using (3.12), one can see that isomorphism $\psi := \tilde{\phi}' \circ \theta \circ \tilde{\phi}$ has the desired properties. Next suppose $\Pi$ is a base for $R$ and $\psi$ and $\psi'$ are two isomorphisms from $G$ to $L$ mapping $f_\alpha$ to $y_\alpha$ and $h_\alpha$ to $h_{f(\alpha)}$ for all $\alpha \in \Pi$. Then $\varphi := \psi^{-1} \circ \psi'$ is an automorphism of $G$ mapping $f_\alpha$ to $f_\alpha$ and $h_\alpha$ to $h_\alpha$, for $\alpha \in \Pi$. Since for $\alpha \in \Pi$, $\varphi(f_\alpha) = f_\alpha$, we have $\varphi(e_\alpha) = e_\alpha$. On the other hand, as $\varphi$ is identity on $H$, $\varphi$ preserves the root spaces. This together with the fact that $[e_\alpha, \varphi(e_{-\alpha})] = [\varphi(e_\alpha), \varphi(e_{-\alpha})] = \varphi(h_\alpha) = h_\alpha$ for all $\alpha \in \Pi$, implies that $\varphi(e_{-\alpha}) = e_{-\alpha}$. Now suppose that $\alpha \in R^\times$, since $\Pi$ is a base, there are $r_1, \ldots, r_n \in \{\pm 1\}$ and $\alpha_{i_1}, \ldots, \alpha_{i_n} \in \Pi$ such that $\alpha = r_1 \alpha_{i_1} + \cdots + r_n \alpha_{i_n}$ and that $r_1 \alpha_{i_1} + \cdots + r_n \alpha_{i_n} \in R^\times$ for all $1 \leq t \leq n$. This
Lemma 3.13. This allows us to define the isomorphism of $G$ with $(\Pi)$ and $T$ is a Lie superalgebra with Cartan subalgebra $R$. Then as in the proof of Lemma 2.28, $f, k$ is a scalar. Using Lemma 3.1, we extend $f$ to a linear isomorphism, denoted again by $f$, from $\text{span}_R R$ to $\text{span}_S S$ with $(f(\alpha), f(\beta))' = k(\alpha, \beta)$ for $\alpha, \beta \in \text{span}_R R$. Fix $r, s \in \mathbb{F} \setminus \{0\}$ such that $r = sk$. Therefore, we have

$$r(\alpha, \beta) = sk(\alpha, \beta) = s(f(\alpha), f(\beta))' \quad (\alpha, \beta \in \text{span}_R R).$$

For $\alpha \in \text{span}_R R$, take $t_\alpha$ to be the unique element of $H$ representing $\alpha$ through $(\cdot, \cdot)$ and for $\beta \in \text{span}_S S$, take $t'_\beta$ to be the unique element of $T$ representing $\beta$ through $(\cdot, \cdot)'$ and set

$$h_\alpha := rt_\alpha \quad \text{and} \quad h'_\beta := st'_\beta \quad (\alpha \in R, \beta \in S).$$

By Lemma 3.13, there is a base $\Pi$ for $R$ and a class $\{R_\gamma \mid \gamma \in \Gamma\}$ of finite irreducible closed sub-supersystems of $R$ of the same type as $R$ such that $R$ is the direct union of $R_\gamma$’s and for each $\gamma \in \Gamma$, $\Pi \cap R_\gamma$ is a base for $R_\gamma$. Now $\Pi' := f(\Pi)$ is a base for $S$ and $S = \cup_{\gamma \in \Gamma} S_\gamma$ in which $S_\gamma := f(R_\gamma)$ is a finite irreducible closed sub-supersystem of $S$ and $\Pi'_\gamma := \Pi' \cap S_\gamma$ is a base for $S_\gamma$. For each $\gamma \in \Gamma$, set

$$G(\gamma) := \sum_{\alpha \in R^\gamma_\gamma} G^{\alpha} \oplus \sum_{\alpha \in R^\gamma_\beta} [G^{\alpha}, G^{-\alpha}] \quad \text{and} \quad H(\gamma) := \sum_{\alpha \in R^\gamma_\gamma} [G^{\alpha}, G^{-\alpha}].$$

and

$$L(\gamma) := \sum_{\alpha \in S^\gamma_\gamma} L^{\alpha} \oplus \sum_{\alpha \in S^\gamma_\beta} [L^{\alpha}, L^{-\alpha}] \quad \text{and} \quad T(\gamma) := \sum_{\alpha \in S^\gamma_\gamma} [L^{\alpha}, L^{-\alpha}].$$

Then as in the proof of Lemma 2.28, $G(\gamma)$ is a finite dimensional basic classical simple Lie superalgebra with Cartan subalgebra $H(\gamma)$ and corresponding root system $R_\gamma$ and $L(\gamma)$ is a finite dimensional basic classical simple Lie superalgebra with Cartan subalgebra $T(\gamma)$ and corresponding root system $S_\gamma$. Now fix $\{f_\alpha \in G^{\alpha} \mid \alpha \in \Pi\}$ and $\{y_\alpha \in L^{f(\alpha)} \mid \alpha \in \Pi\}$. By Lemma 3.13, for each $\gamma \in \Gamma$, there is a unique isomorphism $\theta_\gamma$ from $G(\gamma)$ to $L(\gamma)$ mapping $f_\alpha$ to $y_\alpha$ and $h_\alpha$ to $h'_f(\alpha)$ for $\alpha \in \Pi_\gamma$. Now for $\gamma_1, \gamma_2 \in \Gamma$ with $R_{\gamma_1} \subseteq R_{\gamma_2}$, $\theta_{\gamma_1}, \theta_{\gamma_2}$ are isomorphisms from $G(\gamma_1)$ to $L(\gamma_1)$ mapping $f_\alpha$ to $y_\alpha$ and $h_\alpha$ to $h'_f(\alpha)$ for all $\alpha \in \Pi_{\gamma_2}$; therefore, we have $\theta_{\gamma_1} = \theta_{\gamma_2} \mid G(\gamma_1)$ by Lemma 3.13. This allows us to define the isomorphism $\theta : G \to L$ by $\theta(x) = \theta_\gamma(x)$ if $x \in G(\gamma)$. The isomorphism $\theta$ maps $H$ onto $T$. This completes the proof. □
Corollary 3.15. Suppose that \( \mathcal{L} \) is a locally finite basic classical simple Lie superalgebra. Assume \( \mathcal{H} \) and \( \mathcal{T} \) are two Cartan subalgebras of \( \mathcal{L} \) with corresponding root systems \( R \) and \( S \) respectively. Then \( \mathcal{H} \) and \( \mathcal{T} \) are conjugate under \( \text{Aut}(\mathcal{L}) \) if and only if \( R \) and \( S \) are isomorphic locally finite root supersystems.

Proof. Assume \( \phi : \mathcal{L} \rightarrow \mathcal{L} \) is a Lie superalgebra automorphism such that \( T = \phi(H) \). Define the bilinear form \((\cdot,\cdot)' : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{F} \) with \((x,y) := (\phi^{-1}(x),\phi^{-1}(y))\) for all \( x,y \in \mathcal{L} \). The linear isomorphism \( \phi \mid_{\mathcal{H}} : \mathcal{H} \rightarrow T \) induces a linear isomorphism \( \hat{\phi} : \mathcal{H}^* \rightarrow T^* \) mapping \( \alpha \in \mathcal{H}^* \) to \( \alpha \circ (\phi|_{\mathcal{T}})^{-1} \). For each \( h \in \mathcal{H} \), \( \alpha \in R \) and \( x \in \mathcal{L}_\alpha \), we have \( [\hat{\phi}(h),\hat{\phi}(x)] = \hat{\phi}(h) \hat{\phi}(x) - \hat{\phi}(\alpha)(\hat{\phi}(h)) \hat{\phi}(x) \). Now it follows that \( (\mathcal{L},(\cdot,\cdot)',T) \) is an extended affine Lie superalgebra, \( \hat{\phi}(\mathcal{L}_\alpha) = \mathcal{L}_\hat{\phi}(\alpha) \) and that \( S = \hat{\phi}(R) \). For \( \beta \in \text{span}_{\mathbb{F}} S \), take \( t'_\beta \) to be the unique element of \( T \) representing \( \beta \) through \((\cdot,\cdot)\)' and for \( \alpha \in \text{span}_{\mathbb{F}} R \), take \( t_\alpha \) to be the unique element of \( \mathcal{H} \) representing \( \alpha \) through \((\cdot,\cdot)\). One can easily check that \( \hat{\phi}(t_\alpha) = t'_\hat{\phi}(\alpha) \) and \( (t_\alpha,t_\beta) = (t'_\hat{\phi}(\alpha),t'_\hat{\phi}(\beta))' \) for all \( \alpha,\beta \in R \). These all together imply that \( \hat{\phi} \mid_{\mathcal{T}} \) defines an isomorphism from \( R \) to \( S \). The reverse part follows from Proposition 3.14 [11 Cor. IV.5] and finite dimensional theory of Lie superalgebras.

Using [11] Thm. IV.6, one knows the classification of locally finite split simple Lie algebras, i.e., locally finite basic classical simple Lie superalgebras with zero odd part. In what follows using Theorem 3.16 Examples 2.36 2.40 Proposition 3.14 and Lemma 2.41 we give the classification of locally finite basic classical simple Lie superalgebras with nonzero odd part:

Theorem 3.16. Each locally finite basic classical simple Lie superalgebra with nonzero odd part is either a finite dimensional basic classical simple Lie superalgebra or isomorphic to one and only one of the Lie superalgebras \( \mathfrak{osp}(2I,2J) \) \((I,J \text{ index sets with } |I \cup J| = \infty, |J| \neq 0)\), \( \mathfrak{osp}(2I+1,2J) \) \((I,J \text{ index sets with } |I| < \infty, |J| = \infty)\) or \( \mathfrak{sl}(I_0,I_1) \) \((I \text{ an infinite superset with } I_0,I_1 \neq \emptyset)\).

Proposition 3.17. Suppose that \( \mathcal{L} \) is an infinite dimensional locally finite basic classical simple Lie superalgebra with nonzero odd part, then if for an infinite index set \( I \) and a nonempty index set \( J \), \( \mathcal{L} \simeq \mathfrak{osp}(2I+1,2J) \simeq \mathfrak{osp}(2I,2J) \), there are two conjugacy classes for Cartan subalgebras of \( \mathcal{L} \) under \( \text{Aut}(\mathcal{L}) \); otherwise all Cartan subalgebras of \( \mathcal{L} \) are conjugate under \( \text{Aut}(\mathcal{L}) \), i.e., there is just one conjugacy class for Cartan subalgebras of \( \mathcal{L} \) under \( \text{Aut}(\mathcal{L}) \).

Proof. We first assume \( I \) is an infinite index set, \( J \) a nonempty index set and \( \mathcal{L} \simeq \mathfrak{osp}(2I+1,2J) \simeq \mathfrak{osp}(2I,2J) \). We know form Example 2.36 that there are Cartan subalgebras \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) for \( \mathcal{L} \) with corresponding root systems \( R_1 \) of type \( B(I,J) \) and \( R_2 \) of type \( D(I,J) \) respectively; in particular thanks to Corollary 3.15 there are at least two conjugacy classes for Cartan subalgebras of \( \mathcal{L} \) under \( \text{Aut}(\mathcal{L}) \). We next note that there is a decomposition \( \mathcal{L}_0 = \mathcal{G}_1 \oplus \mathcal{G}_2 \) for \( \mathcal{L}_0 \) into simple ideals in which \( \mathcal{G}_1 \) is isomorphic to \( \mathfrak{osp}(2I+1,\mathbb{F}) \simeq \mathfrak{osp}(2I,\mathbb{F}) \) and \( \mathcal{G}_2 \) is isomorphic to \( \mathfrak{osp}(J,\mathbb{F}) \); see [11] for the notations. By [11] Cor. VI.8 and finite dimensional theory of Lie algebras, there are two conjugacy classes for Cartan subalgebras of \( \mathcal{G}_1 \) under \( \text{Aut}(\mathcal{G}_1) \) and there is just one conjugacy class for Cartan subalgebras of \( \mathcal{G}_2 \) under \( \text{Aut}(\mathcal{G}_2) \). Therefore, up to \( \text{Aut}(\mathcal{G}_1) \)-conjugacy, \( \mathcal{H}_1 \cap \mathcal{G}_1, \mathcal{H}_2 \cap \mathcal{G}_1 \) are the only non-conjugate Cartan subalgebras of \( \mathcal{G}_1 \); also \( \mathcal{H}_1 \cap \mathcal{G}_2, \mathcal{H}_2 \cap \mathcal{G}_2 \) are \( \text{Aut}(\mathcal{G}_2) \)-conjugate Cartan subalgebras of \( \mathcal{G}_2 \).
and in fact up to $\text{Aut}(G^2)$-conjugacy, $\mathcal{H}_1 \cap G^2$ is the only Cartan subalgebra of $G^2$. Now suppose that $T$ is a Cartan subalgebra of $L$ with corresponding root system $S = S_0 \cup S_1$. We want to show that $T$ is either conjugate to $H_1$ or to $H_2$. Since $T \cap G^1$ is a Cartan subalgebra of $G^1$ and $T \cap G^2$ is a Cartan subalgebra of $G^2$, there are $i \in \{1, 2\}$ and $\phi_1 \in \text{Aut}(G^1), \phi_2 \in \text{Aut}(G^2)$ such that $\phi_1(T \cap G^1) = H_i \cap G^1$ and $\phi_2(T \cap G^2) = H_i \cap G^2$. So $\phi_1 \circ \phi_2$ is an automorphism of $L_0$ mapping $T = (T \cap G^1) \oplus (T \cap G^2)$ to $H_i = (H_i \cap G^1) \oplus (H_i \cap G^2)$. This implies that $(R_i)_0$ is isomorphic to $S_0$. So using the classification tables of locally finite root supersystems, Proposition 2.5 Lemma 2.17 and the fact that $|R_i|, |S| = \infty$, $R_i$ is isomorphic to $S$. Therefore there is an automorphism of $L$ mapping $T$ to $H_i$ by proposition 3.14. This implies that there are exactly two conjugacy classes for Cartan subalgebras of $L$ under $\text{Aut}(L)$.

Next suppose that $L$ is one of the Lie superalgebras $\mathfrak{osp}(2I, 2J), \mathfrak{osp}(2I + 1, 2J)$ where $I, J$ are index sets with $0 \neq |I| < \infty, |J| \neq 0$ or $\mathfrak{s}(I_0, I_1)$ where $I$ is an infinite superset with $|I| = \infty$ and $I_0, I_1 \neq \emptyset$. Take $h$ to be the standard Cartan subalgebra of $L$ introduced in Examples 2.10 and 2.11 and consider its corresponding root system $R$. Next suppose that $T$ is another Cartan subalgebra of $L$ and take $S$ to be the corresponding root system of $L$ with respect to $T$. Then $S$ is an irreducible locally finite root supersystem with $|S| = \infty$. From the classification tables of locally finite root supersystems and Lemmas 2.10 and 2.11 $S$ is isomorphic to the root system of one of the Lie superalgebras $a_{I', J'}, b_{I', J'}, c_{I'}$ introduced in Lemma 2.44. Call this Lie superalgebra $G$ and take $H$ to be its standard Cartan subalgebra, so by Proposition 3.14, there is an isomorphism $\phi: \mathcal{L} \rightarrow G$ mapping $T$ to $H$. Now since $\mathcal{L} \simeq G$, using Lemma 2.44, there is an isomorphism $\psi$ form $\mathcal{L}$ to $G$ mapping $h$ to $H$. Therefore, $\psi^{-1} \circ \phi$ is an automorphism of $\mathcal{L}$ mapping $T$ to $h$. This completes the proof.

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