LYAPUNOV EXPONENTS FOR TRANSFER OPERATOR COCYCLES OF METASTABLE MAPS: A QUARANTINE APPROACH

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Abstract. This work investigates the Lyapunov–Oseledets spectrum of transfer operator cocycles associated to one-dimensional random paired tent maps depending on a parameter $\epsilon$, quantifying the strength of the leakage between two nearly invariant regions. We show that the system exhibits metastability, and identify the second Lyapunov exponent $\lambda_2^\epsilon$ within an error of order $\epsilon^2|\log \epsilon|$. This approximation agrees with the naive prediction provided by a time-dependent two-state Markov chain. Furthermore, it is shown that $\lambda_1^\epsilon = 0$ and $\lambda_2^\epsilon$ are simple, and the only exceptional Lyapunov exponents of magnitude greater than $-\log 2 + O(\log \log 1/\log 1/\epsilon)$.

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1. Introduction

A dynamical system is called metastable if there is a time scale over which its statistical behaviour appears to be in equilibrium, but such an equilibrium differs from the long term (asymptotic) equilibrium for the system. Metastability has been studied in connection with phase transitions in physics, the existence of isomers in chemistry and the presence of slowly mixing regions in the ocean. In mathematics, metastability has been initially studied from a probabilistic perspective; see for instance the monographs [7, 2] and references therein.

In [21], Keller and Liverani pioneered the study of metastability from a (chaotic) dynamical systems perspective. In this context, a natural way to define metastability is through spectral properties of transfer operators. When the system is ergodic, the transfer operator has a unique eigenvector of eigenvalue one, describing the system’s long term behaviour. Metastability is associated with the presence of an eigenvalue of magnitude near, but strictly less than one. Its corresponding eigenvector, sometimes called strange eigenmode, characterises the possible behaviours of the system in relatively long, yet not asymptotic,
time scales. This point of view has been developed by various authors in [10, 12, 5, 19, 1, 6].

While the spectrum captures information about iterates of a single operator, if the forcing is stationary, the spectral picture can often be replaced by the so-called Oseledets decomposition, discovered by Oseledets in his breakthrough paper [22], and adapted to the context of transfer operators by Froyland, Lloyd and Quas in [8, 9]. In this setup, the largest Lyapunov exponent is zero, corresponding to a (random) invariant measure for the system, and a natural way to define metastability is through the existence of a negative Lyapunov exponent $\lambda_2$ near zero. Then, the associated Oseledets spaces provide a coherent structure which decays at the slow exponential rate $\lambda_2$.

While the connection between autonomous and non-autonomous concepts of metastability is transparent at a conceptual level, the identification of examples of random metastability has proved more elusive. In [11], the authors investigate connections of this phenomenon with escape rates from random sets. Very recently, there has been some further progress: In [16] and [17], Horan provides bounds on the second Lyapunov exponent for a class of random interval maps using cone techniques, and in [4], Crimmins shows a result on stability for hyperbolic Oseledets splittings for quasi-compact cocycles, modelled on the famous stability result of Keller and Liverani [20].

A significant obstacle in the construction of examples of random metastable systems is that Lyapunov exponents are asymptotic quantities, which are generally difficult to rigorously bound, especially from below. In fact, it has been recently shown by the authors in [13] that small perturbations in the dynamics can result in drastic changes in the Lyapunov–Oseledets spectrum of a system of transfer operators.

In this work, we consider random compositions of a class of one-dimensional piecewise smooth expanding maps, and investigate the corresponding transfer operator cocycle in the space of functions of bounded variation. This framework was introduced by Buzzi in [3], where he showed existence of random absolutely continuous invariant measures, corresponding to the first Oseledets space with associated Lyapunov exponent equal to zero.

Here we focus on the so-called paired tent maps, defined in (1) following [16], and introduce a parameter $\epsilon$, to quantify the strength of the leakage between nearly invariant components. We show that for small $\epsilon > 0$ the second Lyapunov exponent $\lambda_2^\epsilon$ is of order $\epsilon$. In fact, we identify $\lambda_2^\epsilon$ within an error of order $\epsilon^2 |\log \epsilon|$. This approximation agrees with the naive prediction that a simple coarse graining by a time-dependent two-state Markov chain would yield, corresponding to
a two-bin Ulam approximation scheme. In addition, we show the corresponding Oseledets space is one-dimensional, and in some sense close to a step function. Therefore, it separates the two nearly invariant components of the system, in agreement with the (random version of the) Dellnitz–Froyland Ansatz. Furthermore, it is shown that, apart from 0 and $\lambda^2$, there are no other Lyapunov exponents of magnitude greater than $-\log 2 + O(\log \log 1/\log 2/\log 1)$. It follows from [9] that all exceptional Lyapunov exponents are greater than $-\log 2 + O(\epsilon)$.

The main idea of the proof is to keep track of the mass that has switched from one nearly invariant component to the other. One could think of this bookkeeping procedure as a quarantine period, during which the leaked densities are kept separate from the rest, until they have been mixed. In this way, it is possible to identify an invariant subset of densities, $C$, in some sense close to the invariant densities for the $\epsilon = 0$ map, which is shown to contain a slowly decaying coherent structure.

While our main results are stated in the context of random dynamical systems, that is, assuming there is an ergodic process $\sigma$ driving the dynamics, many of our arguments carry over to the non-stationary setting. Indeed, one may define orbit-wise Lyapunov exponents as in (2). Then, the set $C$ of §3.1 remains invariant in the non-stationary situation, and the results of §3.2 –except for the use of Birkhoff ergodic theorem– and §3.3 would yield estimates on first, second and third Lyapunov exponents of the non-stationary cocycle.

2. Notation and main results

For $0 \leq a, b \leq 1$, we define a paired tent map $T_{a,b}$ to be the piecewise linear map on $J = [-1, 1]$ sending $-1$ to $-1$, $-1/2$ to $b$ and $0^-$ to $-1$; $0^+$ to $1$; $1/2$ to $-a$ and $1$ to $1$, as introduced in [16]. That is,

$$T_{a,b}(x) = \begin{cases} 2(1+b)(x+1) - 1, & x \in [-1, -1/2], \\ -2(1+b)x - 1, & x \in [-1/2, 0), \\ 0, & x = 0, \\ -2(1+a)x + 1, & x \in (0, 1/2], \\ 2(1+a)(x-1) + 1, & x \in [1/2, 1]. \end{cases} \tag{1}$$

In the case $a = b = 0$, the system is reducible, consisting of a pair of tent maps on disjoint intervals, $[-1, 0]$ and $[0, 1]$. For small positive $a$ and $b$, there is a small amount of leakage between the two halves: points near $-1/2$ leak to the right half, while points near $1/2$ leak to the left half. We can think of $a$ and $b$ as leakage controls.
We study cocycles of paired tent maps, driven by an ergodic, invertible, probability preserving transformation $\sigma : \Omega \to \Omega$, with the leakage controls $a(\omega)$ and $b(\omega)$ scaled by a parameter $\epsilon$, and look at the asymptotics of the Lyapunov exponents of the associated Perron-Frobenius operator cocycle as $\epsilon$ approaches 0.

We let BV denote the Banach space of functions of bounded variation on $[-1, 1]$, where, as usual, we identify functions that disagree on a set of measure 0. For convenience later, we equip BV with a non-standard norm (although it is equivalent to the standard one), namely we define
\[
\|f\|_{BV} = \max(\text{var}_{[-1,0]} f, \|f\|_1),
\]
where $\text{var}_{[-1,0]} f$ denotes $\text{var}_{[-1,0]} f + \text{var}_{[0,1]} f$. For any $I \subset [-1, 1]$, we let $\int_I f := \int_I f dm$, where $m$ is the Lebesgue measure, and $\|f\|_1 = \int_{[-1,1]} |f| dm$. We also let $\text{sgn} = 1_{[1,0]} - 1_{[-1,0]}$ be the sign function.

The main result of this work is the following.

**Theorem 2.1.** Let $\sigma$ be an invertible ergodic measure-preserving transformation of a probability space $(\Omega, P)$. Let $a$ and $b$ be non-zero measurable functions from $\Omega$ to $[0, 1]$, and for each $\omega \in \Omega$ and $\epsilon \in (0, 1)$, let $T_{\epsilon \omega}$ denote the map $T_{\epsilon a(\omega), \epsilon b(\omega)}$. Let $L_{\epsilon \omega}$ denote the corresponding Perron-Frobenius operator.

Then the leading two Lyapunov exponents of the cocycle $L_{\epsilon (n)} (acting on BV)$ are $\lambda_1^{\epsilon} = 0$ and $\lambda_2^{\epsilon} = -\epsilon \int (a(\omega) + b(\omega)) dP(\omega) + O(\epsilon^2 |\log \epsilon|)$. Furthermore, $\lambda_1^{\epsilon}$ and $\lambda_2^{\epsilon}$ have multiplicity one and the remainder of the Lyapunov spectrum is bounded above by $-\log 2 + O(\log \log \frac{1}{\epsilon} / \log \frac{1}{\epsilon})$.

**Remark 2.2.** Horan, in his thesis (See [18, Remark 4.5.7]), discusses the fact that the known versions of operator-valued METs when applied to Perron-Frobenius operators on BV only prove the existence of (measurable) Oseledets spaces in cases where the map $\omega \mapsto T_{\omega}$ has countable range. He explains that this is not an artefact of the proof, but rather a consequence of the geometry of BV. Indeed, he points out that $\{L_{T_{\omega}, 1}\}$ is a uniformly discrete set as $(a,b)$ run over $[0,1]^2$. This means that even the map $(a,b) \mapsto L_{T_{\omega}, 1}$ is not measurable with respect to the Borel $\sigma$-algebra on BV.

In the absence of an MET for more general choices of $\omega \mapsto T_{\omega}$, a natural way to interpret the statement of Theorem 2.1 is by defining the $k$-th Lyapunov exponent $\lambda_k^{\epsilon}(\omega)$ as the supremum over $k$-dimensional subspaces $V$ of the minimal growth rate of $\|L_{\epsilon}^{(n)} f\|_{BV}$, for $f \in V \setminus \{0\}$. That is,

\[
\lambda_k^{\epsilon}(\omega) = \sup_{V \in \mathcal{U}_k(BV)} \inf_{f \in V \setminus \{0\}} \limsup_{n \to \infty} \frac{1}{n} \log \|L_{\epsilon}^{(n)} f\|_{BV},
\]
where $G_k(BV)$ is the Grassmannian of $k$-dimensional subspaces of $BV$. In this context, our arguments show that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\lambda_1^\epsilon(\omega) = 0$, $\lambda_2^\epsilon(\omega) = -\epsilon \int (a(\omega) + b(\omega)) \, d\mathbb{P}(\omega) + O(\epsilon^2|\log \epsilon|)$ and $\lambda_3^\epsilon(\omega) \leq -\log 2 + O(\log \log \frac{1}{\epsilon} / \log \frac{1}{\epsilon})$.

In Section 4, we give an alternative interpretation of Theorem 2.1 in the context of multiplicative ergodic theory. In particular, it implies measurability of the top Oseledets spaces when regarded as functions from $\Omega$ to $L^1$.

2.1. Connection with time-dependent two-state Markov chains.

Let $A^\epsilon_\omega$ denote the matrix \[
\begin{pmatrix}
1 - eb(\omega) & ea(\omega) \\
eb(\omega) & 1 - ea(\omega)
\end{pmatrix},
\]
with $a$ and $b$ as in the statement of Theorem 2.1. Let $A^{(n)}_\omega = A^\epsilon_{\sigma_{n-1}\omega} \circ \cdots \circ A^\epsilon_\omega$ be the corresponding matrix cocycle. This is the sequence of transition (Ulam) matrices one would obtain by coarsely partitioning $J = [-1, 1]$ into $J^- = [-1, 0]$ and $J^+ = [0, 1]$. The two Lyapunov exponents of the cocycle $A^{(n)}_\omega$ are $0$ and $\int \log(1 - \epsilon(a(\omega) + b(\omega))) \, d\mathbb{P}(\omega)$. To see this, notice that $(1 \ 1) A^\epsilon_\omega = (1 \ 1)$, ensuring that $0$ is a Lyapunov exponent of the dual cocycle, and hence of the primal cocycle. Also, the sum of the Lyapunov exponents is given by the average value of the log-determinant, so that the second Lyapunov exponent is $\int \log(1 - \epsilon(a(\omega) + b(\omega))) \, d\mathbb{P}(\omega)$ as claimed. By taking a Taylor expansion, we see that the second Lyapunov exponent is $-\epsilon \int (a(\omega) + b(\omega)) \, d\mathbb{P}(\omega) + O(\epsilon^2)$ so that the top two Lyapunov exponents of the paired tent map cocycle $L^\epsilon_\omega$ of Theorem 2.1 agree with those of its coarse Ulam approximation up to order $\epsilon^2|\log \epsilon|$.

3. Proof of the main theorem

In this section, we present the proof of Theorem 2.1. Let $\epsilon > 0$ be fixed and sufficiently small (with the precise conditions specified below, but where $\epsilon < \frac{1}{2000}$ suffices). Let $k$ be such that $2k\epsilon < \frac{1}{4}$ but $2^{k+1}\epsilon \geq \frac{1}{4}$.

For a fixed $\omega$ and $\epsilon$, let $H^+ = \left[ \frac{1}{2(1+ca(\omega))}, 1 - \frac{1}{2(1+ca(\omega))} \right]$, the set of points that leak from $J^+ := [0, 1]$ to $J^- := [-1, 0]$ under $T^\epsilon_\omega$ and similarly let $H^- = \left[ -1 + \frac{1}{2(1+cb(\omega))}, -\frac{1}{2(1+cb(\omega))} \right]$, the set of points leaking from $J^-$ to $J^+$. We set $H_\omega = H^+_\omega \cup H^-_\omega$.

We consider the space of $(k + 1)$-tuples of BV functions: $X = (BV[-1,1])^{k+1}$ and define for each $\omega$, an operator $\Lambda^\epsilon_\omega$ on $X$ by

\[
\Lambda^\epsilon_\omega(f_0, \ldots, f_k) = (\mathcal{L}^\epsilon_\omega(f_0 1_{H^+} + f_k), \mathcal{L}^\epsilon_\omega(1_{H^-} f_0), \mathcal{L}^\epsilon_\omega f_1, \ldots, \mathcal{L}^\epsilon_\omega f_{k-1}),
\]
where $\mathcal{L}^\epsilon_\omega = L^\epsilon_\omega$. Note that $H^\epsilon_\omega$ and $\Lambda^\epsilon_\omega$ also depend on $\epsilon$, but since $\epsilon$ will be fixed throughout the proofs, we do not write this dependence explicitly.
The above should be interpreted as follows: any two visits to the hole are separated by at least \( k \) steps. The overall density is represented by \( f_0 + \ldots + f_k \). For \( 1 \leq j \leq k \), the term \( f_j \) represents the part of the density coming from mass that leaked through one of the holes \( j \) steps ago, while \( f_0 \) is the part of the density coming from mass that has not recently passed through a hole, so that, for example, the 0-term of \( \Lambda_\omega(f_0, \ldots, f_k) \) consists of two parts: the image of the part of \( f_0 \) that did not pass through the hole, together with the image of \( f_k \), the contribution to the density that passed through the hole \( k \) steps previously.

### 3.1. Construction of an invariant set \( C \)

Let \( C \) be the collection of elements \( \tilde{f} = (f_0, \ldots, f_k) \) of \( X \) satisfying the following conditions:

- (C1) \( \varrho_\tilde{f} f_j \leq 4 \cdot (2(1 - 39\epsilon))^{-j} \| f_0 \|_1 \) for \( j = 1, \ldots, k \);
- (C2) \( \| f_j \|_1 \leq 3(1 - 39\epsilon)^{-j} \| f_0 \|_1 \) for \( j = 1, \ldots, k \);
- (C3) \( \varrho_\tilde{f} f_0 \leq 33\epsilon \| f_0 \|_1 \).

We now show \( C \) is invariant under each \( \Lambda_\omega \). Let \( \tilde{f} = (f_0, \ldots, f_k) \in C \), \( \omega \in \Omega \) and \( \tilde{g} = (g_0, \ldots, g_k) = \Lambda_\omega \tilde{f} \). We first prove that \( \| g_0 \|_1 \geq (1 - 39\epsilon) \| f_0 \|_1 \).

Since \( \varrho_\tilde{g} f_0 \leq 33\epsilon \| f_0 \|_1 \), we see that for any \( x \in J^+ \),

\[
|f_0(x) - \int_{J^+} f_0| \leq 33\epsilon \| f_0 \|_1,
\]

and similarly for any \( x \in J^- \),

\[
|f_0(x) - \int_{J^-} f_0| \leq 33\epsilon \| f_0 \|_1.
\]

If \( \int_{J^+} |f_0| \geq 33\epsilon \| f_0 \|_1 \), then \( |f_0| \) takes a value at least \( 33\epsilon \| f_0 \|_1 \) in \( J^+ \), so that (C3) implies \( f_0 \) is non-negative throughout \( J^+ \) or \( f_0 \) is non-positive throughout \( J^+ \). In either case we see that

\[
\int_{J^+} |\mathcal{L}_\omega(f_0 1_{H_\omega^0})| = \left| \int_{J^+} \mathcal{L}_\omega(f_0 1_{H_\omega^0}) \right|
= \left| \int_{J^+} f_0 1_{H_\omega^0} \right| = \int_{J^+} |f_0| - \int_{H_\omega^0} \| f_0 \|
\geq \int_{J^+} |f_0| - m(H_\omega^+) \left( \int_{J^+} |f_0| + 33\epsilon \| f_0 \|_1 \right)
\geq \int_{J^+} |f_0| - 2\epsilon \| f_0 \|_1,
\]

where we used the fact that \( m(H_\omega^+) \leq \epsilon \) and we assumed that \( \epsilon < \frac{1}{33} \).

Of course if \( \int_{J^+} |f_0| < 33\epsilon \| f_0 \|_1 \), then \( \int_{J^+} |\mathcal{L}_\omega(1_{H_\omega^0} f_0)| \geq 0 > \int_{J^+} |f_0| - 33\epsilon \| f_0 \|_1 \).
A completely symmetric argument holds for \( J^- \). Assume \( \epsilon < \frac{1}{66} \). Then, at least one of \( \int_{J^+} |f_0| \geq 33\epsilon \|f_0\|_1 \) and \( \int_{J^-} |f_0| \geq 33\epsilon \|f_0\|_1 \) is satisfied. Hence, summing the relevant inequalities, we obtain

\[
\| L_\omega f_0 1_{H^\pm} \|_1 \geq \|f_0\|_1 (1 - 35\epsilon).
\]

We then have

\[
\|g_0\|_1 = \| L_\omega (f_0 1_{H^\pm} + f_k)\|_1 \\
\geq \| L_\omega (f_0 1_{H^\pm})\|_1 - \| L_\omega f_k\|_1 \\
\geq (1 - 35\epsilon) \|f_0\|_1 - \|f_k\|_1 \\
\geq \left( 1 - (35 + 3/\epsilon) \right) \|f_0\|_1.
\]

At this point, recalling that \( k \geq \log_2(\frac{1}{\epsilon}) - 3 \), let us assume that \( \epsilon \) is sufficiently small that \( 3/(1 - 39\epsilon)^k < 4 \), so that

\[
\|g_0\|_1 \geq (1 - 39\epsilon) \|f_0\|_1.
\]

Now if \( 1 \leq j < k \), \( \text{var}_{\omega^j} f_j \leq 4/(2(1 - 39\epsilon)) \|f_0\|_1 \), then

\[
\text{var}_{\omega^j} g_{j+1} = \text{var}_{\omega^j} L_\omega f_j \\
\leq \frac{1}{2} \text{var}_{\omega^j} f_j \\
\leq 4/(2(1 - 39\epsilon)) \|f_0\|_1 \\
\leq 4/(2(1 - 39\epsilon)) \|g_0\|_1,
\]

so that \( \tilde{g} \) satisfies (C1) for \( j = 2, \ldots, k \).

Similarly, since \( \tilde{f} \) satisfies (C2), then for \( 1 \leq j < k \)

\[
\|g_{j+1}\|_1 = \| L_\omega f_j\|_1 \\
\leq \|f_j\|_1 \leq 3(1 - 39\epsilon)^{-j} \|f_0\|_1 \\
\leq 3(1 - 39\epsilon)^{-1} \|g_0\|_1,
\]

so that \( \tilde{g} \) satisfies (C2) for \( j = 2, \ldots, k \).

To establish (C2) for \( g_1 \), recall \( g_1 = L_\omega (f_0 1_{H^\omega}) \) so that \( \|g_1\|_1 \leq \|f_0 1_{H^\omega}\|_1 \). From (3), we see \( |f_0| \) takes values at most \( |\int_{J^+} f_0| + 33\epsilon \|f_0\|_1 \) on \( H^+ \) and similarly takes values at most \( |\int_{J^-} f_0| + 33\epsilon \|f_0\|_1 \) on \( H^- \). Hence \( \|g_1\|_1 \leq m(H^\omega) (\|f_0\|_1 + 33\epsilon \|f_0\|_1) \leq 3\epsilon \|g_0\|_1 \) (where we assumed that \( \epsilon \) was sufficiently small that \( 2\epsilon(1 + 33\epsilon)/(1 - 39\epsilon) < 3\epsilon \)), thereby establishing (C2) for \( j = 1 \).

We now show (C1) for \( j = 1 \). We have \( \text{var}_{\omega^1} g_1 = \text{var}_{\omega^1} (L_\omega 1_{H^\omega} f_0) \leq \frac{1}{2} \text{var}_{H^\omega} f_0 + \|f_0(\frac{1}{2})\| + |f_0(-\frac{1}{2})| \leq \frac{33}{2} \|f_0\|_1 + (|\int_{J^+} f_0| + 33\epsilon \|f_0\|_1) + (|\int_{J^-} f_0| + 33\epsilon \|f_0\|_1) \leq (1 + \frac{105}{2} \epsilon) \|f_0\|_1 \leq 2/(1 - 39\epsilon) \|g_0\|_1 \) as required.

Finally, we establish (C3) for \( \tilde{g} \). We have \( \text{var}_{\omega^\epsilon} g_0 = \text{var}_{\omega^\epsilon} L_\omega (1_{H^\omega} f_0 + f_k) \leq \frac{1}{2} (\text{var}_{\omega^\epsilon} f_0 + \text{var}_{\omega^\epsilon} f_k) \leq \frac{33}{2} \|f_0\|_1 + 2 \cdot 2^{-k} (1 - 39\epsilon)^{-k} \|f_0\|_1 \). By the
3.2. Estimation of first and second Lyapunov exponents. Up to this point, we have written explicit bounds in terms of $\epsilon$, since the bounds are inter-dependent and this allows us to verify that they are simultaneously satisfiable. At this point, we switch to using the more compact $O(\cdot)$ notation (that is $A = O(B)$ means that there is a universal constant $C$ such that for all $\bar{f} \in \mathcal{C}$, the inequality $|A(\bar{f})| \leq C|B(\bar{f})|$ is satisfied). The rationale for using this notation from here on is that from now on, each inequality will be seen to follow from earlier estimates. Additionally, we can conveniently write expressions like $\phi(\Lambda \bar{f}) = \phi(\bar{f})(1 + c\epsilon + O(\epsilon^2 \log \epsilon))$. It would not be hard to write explicit constants in place of the $O(\cdot)$ notation if desired.

For $\bar{f} = (f_0, \ldots, f_k) \in X$, we let

$$\Phi(\bar{f}) = f_0 + \ldots + f_k \in \text{BV}.$$ 

3.2.1. Estimation of the first Lyapunov exponent. If $f \in \text{BV}$, then for a suitably large constant $c$, $f$ may be expressed as $f = g + h$ where $g = f - c$, $h = c$, and $\text{var}_0 g < 3\epsilon \|g\|_1$. Write $\bar{g} = (g, 0, \ldots, 0)$ and $\bar{h} = (h, 0, \ldots, 0)$. Then, $\bar{g}, \bar{h} \in \mathcal{C}$. It is straightforward to verify that for any $\bar{f} = (f_0, \ldots, f_k) \in X$, $\sum_{i=0}^{k} \|(\Lambda \bar{f})_i\|_1 \leq \sum_{i=0}^{k} \|f_i\|_1$. Hence, $\|(\Lambda^{(n)} \bar{g})_0\|_1$ and $\|(\Lambda^{(n)} \bar{h})_0\|_1$ are uniformly bounded above in $n$, so that by the definition of $\mathcal{C}$, $\|\Phi(\Lambda^{(n)} \bar{g})\|_{\text{BV}}$ and $\|\Phi(\Lambda^{(n)} \bar{h})\|_{\text{BV}}$ are uniformly bounded above in $n$. Since $\mathcal{L}^{(n)} f = \Phi(\Lambda^{(n)} \bar{g}) - \Phi(\Lambda^{(n)} \bar{h})$, it follows that $\|\mathcal{L}^{(n)} f\|_{\text{BV}}$ is uniformly bounded above in $n$.

On the other hand, if $\int f \neq 0$, then $\|\mathcal{L}^{(n)} f\|_{\text{BV}} \geq \|\mathcal{L}^{(n)} f\|_1 \geq |\int \mathcal{L}^{(n)} f| = |\int f| \neq 0$, so that there is a uniform lower bound also. It follows that $\lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}^{(n)} f\|_{\text{BV}} = 0$. That is, the top Lyapunov exponent is $\Lambda_1^f = 0$.

3.2.2. Estimation of the second Lyapunov exponent. Let $\bar{f} = (f_0, \ldots, f_k) \in X$. We now study the evolution of the quantities $\phi^\pm(\bar{f}) = \int_{J^\pm} (f_0 + \ldots + f_k)$. We first claim that if $\bar{f} \in \mathcal{C}$ and $\epsilon$ satisfies the constraints above, then $\|f_0\|_1 = O(\max(|\phi^+(\bar{f})|, |\phi^-(\bar{f})|))$. To see this, notice that $|f_0|$ is at least $\frac{1}{2} \|f_0\|_1$ at some point $x_0 \in J^+ \cup J^-$. Without loss of generality, we assume $x_0 \in J^+$ and $f(x_0) > 0$. Now (C3) implies $|f_0(x) - f_0(x_0)| \leq 3\epsilon \|f_0\|_1$ for all $x \in J^+$, so that $f_0(x) \geq (\frac{1}{2} - 3\epsilon) \|f_0\|_1$ on $J^+$. We now have $\phi^+(\bar{f}) = \int_{J^+} (f_0 + \ldots + f_k) \geq (\frac{1}{2} - 3\epsilon - 4k\epsilon) \|f_0\|_1$
where we used (C2) to estimate $\int_{j^+} f_1, \ldots, \int_{j^+} f_k$. In particular, for all sufficiently small $\epsilon$ and all $\bar{f} \in \mathcal{C}$

$$\frac{1}{3} \|f_0\|_1 \leq \max(|\phi^+(\bar{f})|, |\phi^-(\bar{f})|) \leq \|f_0\|_1. \quad (6)$$

Writing $\bar{g} = \Lambda_\omega(\bar{f})$, we have, from the Perron-Frobenius property, that $\int_{j^+} g_0 = \int_{j^+} \Lambda_\omega f_0 + \int_{j^+} f_k$; $\int_{j^+} g_1 = \int_{H^+} f_0$ (that is, $\int_{j^+} g_1 = \int_{H^-} f_0$ and vice versa) and $\int_{j^+} g_j = \int_{j^+} f_{j-1}$ for $j = 2, \ldots, k$.

Summing, we obtain

$$\phi^\pm(\Lambda_\omega \bar{f}) = \phi^+(\bar{f}) - \int_{H^+} f_0 + \int_{H^-} f_0.$$  

(This equality is intuitively clear when one thinks of $\phi^\pm(\bar{f})$ as the total mass on the left or right side, respectively.) Adding the two quantities, we obtain $\phi^+(\Lambda_\omega \bar{f}) + \phi^-(\Lambda_\omega \bar{f}) = \phi^+(\bar{f}) + \phi^-(\bar{f})$, that is the conservation of mass. Let $\mathcal{C}_0 = \{ \bar{f} \in \mathcal{C} : \phi^+(\bar{f}) + \phi^-(\bar{f}) = 0 \}$. Thus, if $\bar{f} \in \mathcal{C}_0$ then $\Lambda_\omega \bar{f} \in \mathcal{C}_0$.

By (3) and (4),

$$\left| \int_{H^\pm} f_0 - m(H^\pm) \int_{j^+} f_0 \right| \leq 33c m(H^\pm) \|f_0\|_1 = O(\epsilon^2) \|f_0\|_1.$$  

By (C2),

$$\left| \int_{j^+} f_0 - \phi^+(\bar{f}) \right| \leq \frac{3k\epsilon}{(1 - 39\epsilon)^k} \|f_0\|_1 = O(\epsilon \log \epsilon) \|f_0\|_1.$$  

Since $m(H^+) = \epsilon a(\omega)/(1 + \epsilon a(\omega)) = \epsilon a(\omega) + O(\epsilon^2)$ and $m(H^-) = \epsilon b(\omega) + O(\epsilon^2)$, we obtain

$$\phi^+(\Lambda_\omega \bar{f}) = \phi^+(\bar{f}) - \epsilon(a(\omega)(\phi^+(\bar{f}) - b(\omega)\phi^-(\bar{f})) + O(\epsilon^2 \log \epsilon) \|f_0\|_1$$

$$\phi^-(\Lambda_\omega \bar{f}) = \phi^-(\bar{f}) + \epsilon(a(\omega)(\phi^+(\bar{f}) - b(\omega)\phi^-(\bar{f})) + O(\epsilon^2 \log \epsilon) \|f_0\|_1.$$  

In the special case $\bar{f} \in \mathcal{C}_0$.

$$\phi^+(\Lambda_\omega(\bar{f})) = \left(1 - \epsilon(a(\omega) + b(\omega)) \right) \phi^+(\bar{f}) + O(\epsilon^2 \log \epsilon) \|f_0\|_1$$

$$= \left(1 - \epsilon(a(\omega) + b(\omega)) + O(\epsilon^2 \log \epsilon) \right) \phi^+(\bar{f}).$$

It follows that if $\bar{f} \in \mathcal{C}_0$ then

$$|\phi^+(\Lambda_\omega^{(n)}(\bar{f}))| = \prod_{j=0}^{n-1} \left(1 - \epsilon(a(\sigma^j \omega) + b(\sigma^j \omega)) + O(\epsilon^2 \log \epsilon) \right) \phi^+(\bar{f}). \quad (7)$$

Suppose $f \in \text{BV}$ and $\int f = 0$. Then, choosing a sufficiently large $c$, $f$ may be expressed as $\bar{f} = g + h$ with $g(x) = f(x) - c \text{sgn}(x)$, $h(x) = c \text{sgn}(x)$, and $\text{var}_{\text{BV}} g \leq 33\epsilon \|g\|_1$. Hence $\bar{f} = (f, 0, \ldots, 0)$ can be written
as $\bar{g} + \bar{h}$ where $\bar{g} = (g, 0, \ldots, 0)$ and $\bar{h} = (h, 0, \ldots, 0)$ with $\bar{g}, \bar{h} \in \mathcal{C}_0$. Also, it is straightforward to check that $\mathcal{L}_\omega^{(n)} f = \Phi(\Lambda_\omega^{(n)}(f, 0, \ldots, 0))$.

Hence we have $\mathcal{L}_\omega^{(n)} f = \Phi(\Lambda_\omega^{(n)} \bar{g}) + \Phi(\Lambda_\omega^{(n)} \bar{h})$. Since $\bar{g}, \bar{h} \in \mathcal{C}_0$, $\phi^+ (\Lambda_\omega^{(n)} \bar{g})$ and $\phi^+ (\Lambda_\omega^{(n)} \bar{h})$ decay as in (7). Since $\Lambda_\omega^{(n)} \bar{g}$ and $\Lambda_\omega^{(n)} \bar{h}$ belong to $\mathcal{C}_0$, (6) implies that $\| (\Lambda_\omega^{(n)} \bar{g})_0 \|_1$ and $\| (\Lambda_\omega^{(n)} \bar{h})_0 \|_1$ also have this rate of decay, so by the definition of $\mathcal{C}$, $\| \Phi(\Lambda_\omega^{(n)} \bar{g}) \|_{BV}$ and $\| \Phi(\Lambda_\omega^{(n)} \bar{h}) \|_{BV}$ also decay at this rate, which ensures that

$$
\limsup_{n \to \infty} \frac{1}{n} \log \| \mathcal{L}_\omega^{(n)} f \|_{BV} 
\leq \limsup_{n \to \infty} \frac{1}{n} \log \prod_{j=0}^{n-1} \left( 1 - \epsilon(a(\sigma^j \omega) + b(\sigma^j \omega)) + O(\epsilon^2 | \log \epsilon |) \right) 
= -\epsilon \int (a(\omega) + b(\omega)) \, d\mathbb{P}(\omega) + O(\epsilon^2 | \log \epsilon |),
$$

where the Birkhoff ergodic theorem has been used in the last line, together with Taylor’s theorem.

We can obtain a corresponding lower bound: Let $\bar{f} = (\text{sgn}, 0, \ldots, 0)$ so that $\bar{f} \in \bar{\mathcal{C}}_0$. Then by (7), $\phi_+ (\Lambda_\omega^{(n)} \bar{f})$ may be expressed as an orbit product of $(1 - \epsilon(a(\sigma^j \omega) + b(\sigma^j \omega)) + O(\epsilon^2 | \log \epsilon |)$ terms, so that by the same arguments as above, the growth rate for $\bar{f}$ has a lower bound that matches the upper bound of (8).

Since any two-dimensional subspace of BV contains a non-zero function with integral 0 (and therefore with growth rate bounded as in (8)), the multiplicity of $\lambda^*_1$ is 1 and

$$
\lambda^*_2 = -\epsilon \int (a(\omega) + b(\omega)) \, d\mathbb{P}(\omega) + O(\epsilon^2 | \log \epsilon |),
$$
as required.

3.3. **Subsequent Lyapunov exponents.** We now show that all other Lyapunov exponents are at most $- \log 2 + O(\epsilon | \log \epsilon |)$.

Let $\epsilon$ be small and fixed, as before, and let $2^k \epsilon < \frac{1}{4} \leq 2^{k+1} \epsilon$, so that $2^{-k} \leq 8 \epsilon$. We record the following statement, which is a recapitulation of (7).

**Corollary 3.1.** Let $f \in BV$ satisfy $\int f = 0$ and $\text{var}_0 f \leq 33 \| f \|_1$. Then,

$$
\int_{\mathcal{J}^+} \mathcal{L}_\omega^{(n)} f = \prod_{j=0}^{n-1} \left( 1 - \epsilon(a(\sigma^j \omega) + b(\sigma^j \omega)) + O(\epsilon^2 | \log \epsilon |) \right) \int_{\mathcal{J}^+} f.
$$
Proof. The proof is by translating into and out of the space $X$. Let $f$ be as in the statement and let $\tilde{f} = (f, 0, \ldots, 0)$. Then $\tilde{f} \in C_0$, and the claimed equality is a direct restatement of (7). \qed

We recall that $\sgn = 1_{J^+} - 1_{J^-}$ denotes the sign function.

Corollary 3.2. For any $\omega \in \Omega$ and $n \geq 0$,

\[
\|L^{(n)}_\omega \sgn\|_{BV} \leq 15 \int_{J^+} L^{(n)}_\omega \sgn.
\]

Proof. Let $\tilde{f} = (\sgn, 0, \ldots, 0) \in C_0$ and consider $\tilde{g} = \Lambda^{(n)}_\omega \tilde{f}$. As already noted, $\phi^+ (\tilde{g}) = \int_{J^+} L^{(n)}_\omega \sgn$. By (6), $\|g_0\|_1 \leq 3\phi^+ (\tilde{g})$. Using (C1), (C2) and (C3), we see that for small $\epsilon$, $\|L^{(n)}_\omega \sgn\|_{BV} = \|g_0 + \ldots + g_k\|_{BV} \leq 5\|g_0\|_1 \leq 15\phi^+ (\tilde{g}) = 15 \int_{J^+} L^{(n)}_\omega \sgn$. \qed

Lemma 3.3. Let $f \in BV$ satisfy $\int_{J^+} f = \int_{J^-} f = 0$. Then $\|L^k_\omega f\|_{BV} \leq 144\epsilon \log \epsilon \|f\|_{BV}$.

Proof. Let $f \in BV$ satisfy $\|f\|_{BV} = 1$, $\int_{J^+} f = \int_{J^-} f = 0$. The definition of the norm implies

\[
\|f\|_1 \leq 1; \\
\var_{\emptyset} f \leq 1.
\]

We push $f$ forward under $L^{(k)}_\omega$ in stages, keeping track of the parts that have leaked. Let $h^{(0)} = f$ and for $j = 1, \ldots, k$, define

\[
g_j = L_{\sigma^j-1\omega} (1_{H_{\sigma^j-1\omega}} h^{(j-1)}); \quad \text{and} \quad h^{(j)} = L_{\sigma^j-1\omega} (1_{H^c_{\sigma^j-1\omega}} h^{(j-1)}),
\]

so that $L_{\sigma^j-1\omega} h^{(j-1)} = h^{(j)} + g_j$. Combining these equalities inductively, we see

\[
L^{(k)}_\omega f = L^{(k)}_\omega h^{(0)} = h^{(k)} + \sum_{j=1}^k L^{(k-j)}_{\sigma^j \omega} g_j.
\]

This should be interpreted as the parts of $f$ that did not leak through the hole during the first $k$ steps, together with the parts of $f$ that leaked through on the $j$th step, for $j$ running from 1 to $k$.

By hypothesis, we have $\var_{\emptyset} h^{(0)} \leq 1$. We see from the recursive definition (using the facts that the branches of $J^j \setminus H_{\sigma^j-1\omega}$ map fully over $J^j$ under $T_{\sigma^j-1\omega}$ and that $|T^c_{\sigma^j-1\omega}| \geq 2$) that $\var_{\emptyset} h^{(j)} \leq \frac{1}{2} \var_{\emptyset} h^{(j-1)}$ and $|\int_{J^j} h^{(j)}| \leq |\int_{J^j} h^{(j-1)}| + |\int_{J^c} g_j|$. Hence $\var_{\emptyset} h^{(j)} \leq 2^{-j}$ for
\[ j = 0, \ldots, k \text{ and} \]
\[ \left| \int_{J^+} h^{(j)} \right| \leq \int_{J^+} h^{(j-1)} + m(J^+ \cap H_{\sigma^{j-1}}) \|h^{(j-1)}\|_{\infty} J^+ \|_{\infty} \]
\[ \leq \int_{J^+} h^{(j-1)} + \epsilon \left( \int_{J^+} h^{(j-1)} + \text{var}_\omega h^{(j-1)} \right) \]
\[ = (1 + \epsilon) \int_{J^+} h^{(j-1)} + \epsilon 2^{-(j-1)}, \]
for \( j = 1, \ldots, k \); with an exactly similar inequality for \( |\int_{J^-} h^{(j)}| \). Since \( \int_{J^+} h^{(0)} = 0 \), we deduce inductively
\[ \left| \int_{J^+} h^{(j)} \right| \leq \epsilon \left( (1 + \epsilon)^{j-1} + (1 + \epsilon)^{j-2} 2^{-1} + \ldots + 2^{-(j-1)} \right). \]

In particular, for sufficiently small \( \epsilon \), since \( k \approx \log_2 \left( \frac{1}{\epsilon} \right) \), \( |\int_{J^\pm} h^{(j)}| \leq 3\epsilon \) for \( j = 1, \ldots, k \). Combining this with the estimate for \( \text{var}_\omega h^{(j)} \), we see \( \|h^{(j)}\|_{\infty} \leq 3\epsilon + 2^{-j} \leq 2 \cdot 2^{-j} \) for each \( j = 1, \ldots, k \).

We now use this to estimate the \( L^1 \) norm and variation of \( \mathcal{L}_\omega^{(k)} f \). For \( j = 1, \ldots, k \), we have \( \|g_j\|_1 \leq 2\epsilon \|h^{(j-1)}\|_{\infty} \leq 8\epsilon \cdot 2^{-j} \) and \( \text{var}_\omega g_j \leq \frac{1}{2} \text{var}_\omega h^{(j-1)} + 2 \|h^{(j-1)}\|_{\infty} \leq 9 \cdot 2^{-j} \), so that \( \|\mathcal{L}_\omega^{(k-j)} g_j\|_1 \leq 8\epsilon \cdot 2^{-j} \) and \( \text{var}_\omega \mathcal{L}_\omega^{(k-j)} g_j \leq 9 \cdot 2^{-k} \). Using (9), we see
\[ \|\mathcal{L}_\omega^{(k)} f\|_1 \leq 2 \cdot 2^{-k} + 8\epsilon; \]
\[ \text{var}_\omega \mathcal{L}_\omega^{(k)} f \leq 2^{-k} + 9k \cdot 2^{-k}. \]

Combining this gives \( \|\mathcal{L}_\omega^{(k)} f\|_{BV} \leq 144\epsilon \log \epsilon \) as claimed. \( \square \)

**Lemma 3.4.** Let \( \omega \in \Omega \). Then the sequence of functionals \( (\psi_\omega^{(n)}) \) in \( BV^* \) given by
\[ (10) \]
\[ \psi_\omega^{(n)}(f) = \frac{\int_{J^+} (\mathcal{L}_\omega^{(n)} (f - \frac{1}{2} \int f))}{\int_{J^+} (\mathcal{L}_\omega^{(n)} \text{sgn})} \]
converges to a functional \( \psi_\omega^* \).

If \( f \in BV \) satisfies \( \int f = 0 \) and \( \psi_\omega^*(f) = 0 \), then
\[ \limsup_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} f\|_{BV} \leq -\log 2 + O(\log \log \frac{1}{\epsilon} / \log \frac{1}{\epsilon}). \]

**Proof.** We first note by Corollary 3.1 that \( \int_{J^+} \mathcal{L}_\omega^{(n)} \text{sgn} \neq 0 \) for each \( n \), so that \( \psi_\omega^{(n)}(f) \) is defined. Observe that \( \psi_\omega^{(n)}(f + a \text{sgn}) = \psi_\omega^{(n)}(f) + a \) for any \( a, n \) and \( \omega \).

Let \( \omega \in \Omega \) and fix a function \( f \in BV \) with the property that \( \int f = 0 \) and \( \|f\|_{BV} = 1 \). We let \( a_n = \psi_\omega^{(n)}(f) \) for each \( n \). Define \( g_n = \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} f\|_{BV} \leq -\log 2 + O(\log \log \frac{1}{\epsilon} / \log \frac{1}{\epsilon}). \)
\( \mathcal{L}_\omega^{(n)}(f - a_n \text{sgn}) \). By the observation above, \( \psi_\omega^{(n)}(f - a_n \text{sgn}) = 0 \), so that \( \int_{J^+} g_n = 0 \). Since \( \int g_n = 0 \), it follows that \( \int_{J^-} g_n = 0 \) as well. Hence, Lemma 3.3 applies, giving
\[
\|\mathcal{L}_\omega^{(k)} g_n\|_{BV} \leq 144\epsilon |\log \epsilon| \|g_n\|_{BV}.
\]

Notice that \( g_{n+m} = \mathcal{L}_\omega^{(m)} (g_n - (a_{n+m} - a_n)\mathcal{L}_\omega^{(n)} \text{sgn}) \), so that
\[
(11) \quad a_{n+m} - a_n = \left( \int_{J^+} \mathcal{L}_\omega^{(m)} g_n \right) / \left( \int_{J^+} \mathcal{L}_\omega^{(n+m)} \text{sgn} \right).
\]

Also,
\[
(12) \quad \left| \int_{J^+} (a_{n+k} - a_n) \mathcal{L}_\omega^{(n+k)} \text{sgn} \right| = \left| \int_{J^+} \mathcal{L}_\omega^{(k)} g_n \right| \leq \|\mathcal{L}_\omega^{(k)} g_n\|_{BV} \leq 144\epsilon |\log \epsilon| \|g_n\|_{BV}.
\]

Hence by Corollary 3.2, \( |a_{n+k} - a_n| \|\mathcal{L}_\omega^{(n+k)} \text{sgn}\|_{BV} \leq 15 \cdot 144\epsilon |\log \epsilon| \|g_n\|_{BV} \), so that
\[
(13) \quad \|g_{n+k}\|_{BV} \leq \|\mathcal{L}_\omega^{(k)} g_n\|_{BV} + \| (a_{n+k} - a_n) \mathcal{L}_\omega^{(n+k)} \text{sgn} \|_{BV} \leq 2400\epsilon |\log \epsilon| \|g_n\|_{BV}.
\]

It follows that \( \|g_{nk}\|_{BV} \leq (2400\epsilon |\log \epsilon|)^n \), while §3.2.2 implies that
\[
(14) \quad \|\mathcal{L}_\omega^{(kn)} \text{sgn}\|_{BV} \geq (1 - 2\epsilon + O(\epsilon^2 |\log \epsilon|))^k.
\]

Using (11), this implies that
\[
(15) \quad a_{(n+1)k} - a_{nk} = \left( \int_{J^+} \mathcal{L}_\omega^{(k)} g_{nk} \right) / \left( \int_{J^+} \mathcal{L}_\omega^{(n+1)k} \text{sgn} \right) \leq \frac{(2400\epsilon |\log \epsilon|)^n}{(1 - 3\epsilon)^{(n+1)k}}.
\]

Summing the geometric series, \( |a_{nk} - \lim_{n \to \infty} a_{nk}| \leq (2500\epsilon |\log \epsilon|)^n \).

From (11), we see that for each \( 0 \leq j < k \), \( a_{nk+j} - a_{nk} \) also decreases exponentially in \( n \), so that the full sequence \( (a_n) \) converges to the same limit as \( (a_{nk}) \) and hence \( \psi_\omega^{(n)}(f) \) is convergent as required. Since the estimates in (13) and (14) are uniform in \( \omega \), it follows that \( \psi_\omega^{(n)}(f) \leq c \|f\|_{BV} \) for all \( f \in BV \).

Finally, let \( f \in BV \) satisfy \( \int f = 0 \) and \( \psi_\omega^{(n)}(f) = 0 \). Let \( (g_n) \) be as above, so that \( g_n = \mathcal{L}_\omega^{(n)} (f - \psi_\omega^{(n)}(f) \text{sgn}) \). We established that \( \|g_n\|_{BV} = O((2400\epsilon |\log \epsilon|)^{n/k}) \) and \( a_n = a_n - \psi_\omega^{(n)}(f) = O((2500\epsilon |\log \epsilon|)^{n/k}) \), so that
\[
\|\mathcal{L}_\omega^{(n)} f\|_{BV} \leq \|g_n\|_{BV} + |a_n| \|\mathcal{L}_\omega^{n} \text{sgn}\|_{BV} = O((2500\epsilon |\log \epsilon|)^{n/k}).
\]
Taking \( n \)th roots, we see

\[
\limsup_{n \to \infty} \| L_\omega^{(n)} f \|_{1/n}^{1} \leq (2500 \varepsilon \log \varepsilon)^{1/k}.
\]

Taking logarithms, the growth rate is at most \(- \log 2 + O(\log \log \frac{1}{\varepsilon} / \log \frac{1}{\varepsilon})\).

An immediate consequence of Lemma 3.4 is the following.

**Corollary 3.5.** If \( V \subset BV \) is any three-dimensional space, there exists \( f \in V \setminus \{0\} \) satisfying \( \int f = 0 \) and \( \psi^*_\omega (f) = 0 \), so that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \| L_\omega^{(n)} f \|_{BV} \leq - \log 2 + O(\log \log \frac{1}{\varepsilon} / \log \frac{1}{\varepsilon}).
\]

In summary, 0 and \( \lambda_2^\varepsilon \) are the only two Lyapunov exponents greater than \(- \log 2 + O(\log \log \frac{1}{\varepsilon} / \log \frac{1}{\varepsilon})\), counted with multiplicity. This concludes the proof of Theorem 2.1.

## 4. A Multiplicative Ergodic Theory (MET) Framework for Theorem 2.1

Besides BV, there are other function spaces which are well suited for the investigation of transfer operators of piecewise smooth interval maps, including paired tent maps. An example examined in [23] is given by fractional Sobolev spaces \( \mathcal{H}^t_p \), and there are versions of the MET, [14, 15], which apply to transfer operators on such spaces. Furthermore [14, Lemma 3.16] implies that for any \( 0 < \delta < \log 2 \) one can select \( p \) and \( t \) near one \( (t < \frac{1}{p} < 1) \), so that the MET applies and yields an \( \mathcal{H}^t_p \) Oseledets splitting in the setting of Theorem 2.1, with index of compactness \( \kappa^\varepsilon \) bounded above by \(- \log 2 + \delta \).

The following result gives a way to interpret Theorem 2.1 in a multiplicative ergodic theory context when \( a \) and/or \( b \) have uncountable range. In particular, it implies measurability of the first and second Oseledets spaces, for instance when regarded as functions from \( \Omega \) to \( L^1 \).

**Theorem 4.1.** Let \( 0 < \delta < \log 2 \) and assume the hypotheses of Theorem 2.1 hold. Then, for \( \varepsilon > 0 \) sufficiently small, 0 and \( \lambda_2^\varepsilon \) are the only exceptional Lyapunov exponents of the \( \mathcal{H}^t_p \) cocycle \( L_\omega^{(n)} \) which are greater than \(- \log 2 + \delta \). Furthermore, they have multiplicity one.

**Proof.** Let \( \mu_1^\varepsilon \geq \mu_2^\varepsilon \geq \cdots \geq \mu_j^\varepsilon \) be the exceptional Lyapunov exponents of the \( \mathcal{H}^t_p \) cocycle \( L_\omega^{(n)} \) greater than \(- \log 2 + \delta \). The fact that \( \mu_1^\varepsilon = 0 \) follows as in §3.2.1. Since smooth functions are dense in both \( \mathcal{H}^t_p \) and BV, the \( \mathcal{H}^t_p \)-MET implies that one is able to observe the largest \( \mathcal{H}^t_p \)
growth rate by considering BV functions with integral zero. Also, since \( \mathcal{H}_p^t \subset L^1 \) is continuously embedded, it follows from [11, Theorem 3.3] that the growth rate of functions in the Oseledets spaces associated to exceptional Lyapunov exponents measured with respect to \( \mathcal{H}_p^t \) and \( L^1 \) norms coincide. Therefore, since BV is stronger than \( L^1 \), \( \mu_2^s \leq \lambda_2 \).

In §3.2.2, we have shown that \( \mathcal{L}_\omega^{(n)} \) sgn has growth rate \( \lambda_2 \) in both BV and \( L^1 \) norms. Since \( \text{sgn} \in \mathcal{H}_p^t \) and \( \mathcal{H}_p^t \) is stronger than \( L^1 \), this implies \( \mu_2^s \geq \lambda_2 \). Combining with the previous paragraph, \( \mu_2^s = \lambda_2 \).

Recall that Corollary 3.5 ensures that in any three dimensional subspace of BV functions there exists a non-zero function with decay rate bounded above by \( -\log 2 + O(\log \log \frac{1}{\epsilon}/\log \frac{1}{\epsilon}) \) in BV, and hence also in \( L^1 \). Using once again the density argument, and the fact that Lyapunov exponents measured with respect to \( \mathcal{H}_p^t \) and \( L^1 \) norms coincide on Oseledets spaces, we conclude that if \( \epsilon \) is sufficiently small, then, in \( \mathcal{H}_p^t \), there are no further exceptional Lyapunov exponents greater than \( -\log 2 + \delta \), apart from 0 and \( \lambda_2 \). Furthermore, these have multiplicity one.

\[ \square \]

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