New Hardness Results for (Perfect) Matching Cut and Disconnected Perfect Matching

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Abstract

The (Perfect) Matching Cut is to decide if a graph has a (perfect) matching that is also an edge cut. TheDisconnected Perfect Matching problem is to decide if a graph has a perfect matching that contains a matching cut. Both Matching Cut and Disconnected Perfect Matching are NP-complete even for graphs of girth 5, whereas Perfect Matching Cut is known to be NP-complete for graphs of arbitrarily large fixed girth. We prove the last result also for the other two problems, answering a question of Le and Le (TCS 2019) for Matching Cut. Moreover, we give three new general hardness constructions, which imply that all three problems are NP-complete for H-free graphs whenever H contains a connected component with two vertices of degree at least 3. Afterwards, we update the state-of-the-art summaries for H-free graphs and compare them with each other. Moreover, by combining our new hardness construction for Perfect Matching Cut with two existing results, we obtain a complete complexity classification of Perfect Matching Cut for H-subgraph-free graphs where H is any finite set of graphs.

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1 Introduction

In this paper, we continue an ongoing complexity study on deciding if graphs contain certain types of edge cuts. These cuts have in common that their edges must form a matching. In order to explain this, let \( G = (V,E) \) be a connected graph. A set \( M \subseteq E \) is a matching of \( G \) if no two edges in \( M \) share an end-vertex; \( M \) is perfect if every vertex of \( G \) is incident to an edge of \( M \). A set \( M \subseteq E \) is an edge cut of \( G \) if \( V \) can be partitioned into two sets \( B \) and \( R \), such that \( M \) consists of all the edges with one end-vertex in \( B \) and the other one in \( R \). We say that \( M \) is a (perfect) matching cut of \( G \) if \( M \) is a (perfect) matching that is also an edge cut. We refer to Figure 1 for some examples.

Matching cuts have applications in number theory [12], graph drawing [22], graph homomorphisms [11], edge labelings [2] and ILFI networks [9]. As such, (perfect) matching cuts are well studied in the literature. Instead of considering perfect matchings that are edge cuts, we can also consider perfect matchings in graphs that contain edge cuts. Such perfect matchings are called disconnected perfect matchings; see also Figure 1 again. Note that every perfect matching cut is a disconnected perfect matching. However, there exist graphs, like the cycle \( C_6 \) on six vertices, that have a disconnected perfect matching (and thus a matching cut) but no perfect matching cut. There also exist graphs, like the path \( P_3 \) on three vertices, that have a matching cut, but no disconnected perfect matching (and thus no perfect matching cut either).
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Figure 1 The graph $P_6$ from \[19\] with a matching cut that is not contained in a disconnected perfect matching (left), a matching cut that is properly contained in a disconnected perfect matching (middle) and a perfect matching cut (right). In each figure, thick edges denote matching cut edges.

We can now define the three corresponding decision problems which, as we argued above, are related but not the same.

Matching Cut

**Input:** a connected graph $G$.

**Question:** does $G$ have a matching cut?

Disconnected Perfect Matching

**Input:** a connected graph $G$.

**Question:** does $G$ have a disconnected perfect matching?

Perfect Matching Cut

**Input:** a connected graph $G$.

**Question:** does $G$ have a perfect matching cut?

As explained below, all three problems are NP-complete, and hence a number of complexity results for special graph classes appeared over the years. In the survey below (Section 1.1) we restrict ourselves to hereditary graph classes, that is, classes that are closed under vertex deletion. It is not difficult to observe that a class of graphs $G$ is hereditary if and only if $G$ can be characterized by a unique set of forbidden induced subgraphs $F_G$. We say that the graphs in $G$ are $F_G$-free, that is, they do not contain any graph from $F_G$ as an induced subgraph. For a systematic study, it is natural to start with letting $F_G$ consist of a single graph $H$. In that case we also say that an $F_G$-free graph is $H$-free.

Figure 2 The graphs $H^* = H_1^*$ (left) and $H_{i}^*$ (right).

1.1 Known Results

We first discuss the known complexity results for Matching Cut on hereditary graph classes, then for Disconnected Perfect Matching and finally for Perfect Matching Cut.

Matching Cuts. Chvátal \[7\] proved that Matching Cut is NP-complete even for $K_{1,r}$-free graphs (the $K_{1,r}$ is the $(r + 1)$-vertex star). In contrast, Bonsma \[3\] proved that Matching Cut is polynomial-time solvable for $K_{1,3}$-free graphs and $P_4$-free graphs; the latter result was extended to $P_5$-free graphs in \[10\] and to $P_6$-free graphs in \[20\]. It is also known that if Matching Cut is polynomial-time solvable for $H$-free graphs for some graph $H$, then
We solve their open problem and also complete the girth classification for graphs to obtain gadgets of high girth.

The girth was introduced by Bouquet and Picouleau [5]. They used a different name but we adapt the name of \emph{Perfect Matching Cuts}. The latter result also implies that \emph{Perfect Matching Cut} is \NP-complete for $C_s$-free graphs only if $s$ is odd. Bonsma [3] proved that \emph{Perfect Matching Cut} is \NP-complete for planar graphs of girth 5, and thus for $(C_3, C_4)$-free graphs. By using a trick of Moshi [21], \emph{Perfect Matching Cut} is \NP-complete for $(4P_3, P_{10})$-free graphs (strengthening a result of [10]) and, by using the aforementioned trick of Moshi [21], also for $H^*$-free graphs, where $H^*$ is the graph that looks like the letter “H”; see also Figure 2. See [4, 15, 20] for some results for non-hereditary graph classes, and see [6] for a comprehensive overview.

**Disconnected Perfect Matchings.** The \emph{Disconnected Perfect Matching} problem was introduced by Bouquet and Picouleau [5]. They used a different name but we adapt the name of Le and Telle [17] to avoid confusion with \emph{Perfect Matching Cut}. Bouquet and Picouleau [5] showed that \emph{Disconnected Perfect Matching} is, among others, polynomial-time solvable for $K_{1,3}$-free graphs and $P_5$-free graphs, but \NP-complete for $K_{1,4}$-free planar graphs, planar graphs of girth 5 and for bipartite graphs, and thus for $C_s$-free graphs for every odd $s$. In [19], it was shown that the problem is also \NP-complete for $(4P_3, P_{23})$-free graphs.

**Perfect Matching Cuts.** The \emph{Perfect Matching Cut} problem was first shown to be \NP-complete by Hegernes and Telle [13]. Afterwards, Le and Telle [17] proved that for every integer $g \geq 3$, \emph{Perfect Matching Cut} is \NP-complete even for subcubic bipartite graphs of girth at least $g$. They also proved that \emph{Perfect Matching Cut} is polynomial-time solvable for chordal graphs and for $S_{1,2,2}$-free graphs; the graph $S_{1,2,2}$ is obtained by subdividing two of the edges of the claw $K_{1,3}$ exactly once. These results were extended in [19], where it was shown that \emph{Perfect Matching Cut} is polynomial-time solvable for $P_s$-free graphs, and moreover for $(H + P_4)$-free graphs if it is polynomial-time solvable for $H$-free graphs.

### 1.2 Our Results

The \emph{girth} of a graph that is not a forest is the number of edges of a shortest cycle in it, while a forest has infinite girth. Hence, a graph is $C_3$-free if and only if it has girth at least 4, and a graph has girth at least $g$ for some integer $g \geq 4$ if and only if it is $(C_3, \ldots, C_{g-1})$-free.

Out of the three problems, it is only known that for every $g \geq 3$, \emph{Perfect Matching Cut} is \NP-complete for graphs of girth at least $g$ [17]. For the other two problems this is only known if $g \leq 5$ [3, 17]. In particular, in 2019, Le and Le [15] asked about determining the complexity status of \emph{Matching Cut} for graphs of girth at least $g$, for every integer $g \geq 6$. We solve their open problem and also complete the girth classification for \emph{Disconnected Perfect Matching}. That is, in Section 3 we prove that for every $g \geq 3$, \emph{Matching Cut} and \emph{Disconnected Perfect Matching} are \NP-complete for graphs of girth at least $g$. The latter result also implies that \emph{Disconnected Perfect Matching} is now also \NP-complete for $C_s$-free graphs for every $s$ as mentioned above, previously this was only known for odd $s$ [5]. To prove these results we use results from the theory of expander graphs to obtain gadgets of high girth.

For $\ell \geq 2$, let $H^*_\ell$ be the graph obtained from $H^* = H^*_1$ by subdividing the middle edge of $H^*_\ell$ exactly $\ell - 1$ times; see also Figure 2. In Section 3 we prove that for every $\ell \geq 1$, \emph{Matching Cut} and \emph{Disconnected Perfect Matching} are \NP-complete for
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(H₁, ..., Hₜ⁺)-free graphs. We obtain these two results by replacing the gadget of Moshi [21] with more advanced graph transformations. The result for MATCHING CUT extends the known hardness result for MATCHING CUT for H₁⁺-free graphs [19].

A graph is subcube if it has maximum degree at most 3. In Section 3, we also prove that for every g ≥ 1 and ℓ ≥ 1, PERFECT MATCHING CUT is NP-complete for (H₁⁺, ..., Hₜ⁺)-free subcube bipartite graphs of girth at least g, extending the aforementioned NP-completeness result of Le and Telle [17] for PERFECT MATCHING CUT restricted to subcube bipartite graphs of girth at least g (for every g ≥ 3).

1.3 Consequences for H-Free Graphs

We can now update the state-of-the-art for the complexity of MATCHING CUT, DISCONNECTED PERFECT MATCHING and PERFECT MATCHING CUT for H-free graphs and compare these three classifications with each other. All three classifications can be deduced by combining the results from [3, 5, 7, 10, 19, 20, 17, 21] listed in Section 1.1 with our new results. That is, we took the three state-of-the-art theorems in [19] and added the result for H₁⁺-free graphs to each of these theorems in [19]. For DISCONNECTED PERFECT MATCHING we also added the new result for C₆-free graphs for even s, as implied by our girth result.

In the the updated state-of-the-art theorems below, we write G' ⊆₁ G if G' is an induced subgraph of G and G' ⊇₁ G if G is an induced subgraph of G'.

**Theorem 1.** For a graph H, MATCHING CUT on H-free graphs is
- polynomial-time solvable if H ⊆₁ sP₃ + K₁,₃ or sP₃ + P₅ for some s ≥ 0, and
- NP-complete if H ⊇₁ Cᵣ for some r ≥ 3, K₁,₄, P₁₉, 4P₅ or Hᵣ⁺ for some j ≥ 1.

**Theorem 2.** For a graph H, DISCONNECTED PERFECT MATCHING on H-free graphs is
- polynomial-time solvable if H ⊆₁ K₁,₃ or P₅, and
- NP-complete if H ⊇₁ Cᵣ for some r ≥ 3, K₁,₄, P₂₃, 4P₇ or Hᵣ⁺ for some j ≥ 1.

**Theorem 3.** For a graph H, PERFECT MATCHING CUT on H-free graphs is
- polynomial-time solvable if H ⊆₁ sP₄ + S₁₂₂ or sP₄ + P₅ for some s ≥ 0, and
- NP-complete if H ⊇₁ Cᵣ for some r ≥ 3, K₁,₄ or Hᵣ⁺ for some j ≥ 1.

Let S denote the class of graphs, each connected component is either a path or a subdivided claw, that is, a graph obtained from the claw K₁,₃ by subdividing each of its edges zero or more times. From Theorem 1 it follows that MATCHING CUT is NP-complete for H-free graphs whenever H contains a cycle, a vertex of degree at least 4, or a connected component with two vertices of degree 3. Hence, the remaining open cases for MATCHING CUT on H-free graphs are all restricted to cases where H is a graph from S. The same remark holds for DISCONNECTED PERFECT MATCHING due to Theorem 2 and for PERFECT MATCHING CUT due to Theorem 3.

1.4 Consequences for H-Subgraph-Free Graphs

For a graph H, a graph G is H-subgraph-free if G does not contain H as a subgraph. Note that every H-subgraph-free is H-free, whereas the other direction only holds if H is a complete graph. For a set H of graphs, a graph G is H-subgraph-free if G is H-subgraph-free for every H ∈ H. Due to our results we can determine the complexity of PERFECT MATCHING CUT for H-subgraphs for every finite set of graphs H using a recent meta-theorem [14]. In order to explain this, we first give some terminology.
Let \( p \geq 0 \) be an integer. A \( p \)-subdivision of an edge \( uv \) in a graph replaces \( uv \) by a path from \( u \) to \( v \) of length \( p + 1 \). The \( p \)-subdivision of a graph \( G \) is the graph obtained from \( G \) by \( p \)-subdividing each edge of \( G \). For a graph class \( \mathcal{G} \), we let \( \mathcal{G}^p \) consist of all the \( p \)-subdivisions of the graphs in \( \mathcal{G} \). A graph problem \( \Pi \) is \( \text{NP}\)-complete under edge subdivision of subcubic graphs if there exists an integer \( q \geq 1 \) such that the following holds: if \( \Pi \) is \( \text{NP}\)-complete for the class \( \mathcal{G} \) of subcubic graphs, then \( \Pi \) is \( \text{NP}\)-complete for \( \mathcal{G}^{pq} \) for every \( p \geq 1 \).

Now, a graph problem \( \Pi \) is a \( C123\)-problem if (C1) \( \Pi \) is polynomial-time solvable for graphs of bounded treewidth; (C2) \( \Pi \) is \( \text{NP}\)-complete for subcubic graphs; and (C3) \( \Pi \) is \( \text{NP}\)-complete under edge subdivision of subcubic graphs.

In [14], the following was shown for every \( C123\)-problem \( \Pi \): for any finite set of graphs \( \mathcal{H} \), the problem \( \Pi \) on \( \mathcal{H}\)-subgraph-free graphs is polynomial-time solvable if \( \mathcal{H} \) contains a graph from \( \mathcal{S} \) and \( \text{NP}\)-complete otherwise. Le and Telle [17] observed that \text{Perfect Matching Cut} satisfies C1. As mentioned, they also proved C2. In Section 4, we prove C3 (with \( q = 4 \)). Hence, we obtain the following dichotomy.

\[ \blacktriangleright \text{Theorem 4}. \text{ For any finite set of graphs } \mathcal{H}, \text{ \text{Perfect Matching Cut} on } \mathcal{H}\text{-subgraph-free graphs is polynomial-time solvable if } \mathcal{H} \text{ contains a graph from } \mathcal{S} \text{ and } \text{NP}\text{-complete otherwise.} \]

2 Preliminaries

We only consider finite, undirected graphs without multiple edges and self-loops. Throughout this section, let \( G = (V, E) \) be a connected graph. For \( u \in V \), the set \( N(u) = \{ v \in V \mid uv \in E \} \) is the neighbourhood of \( u \) in \( G \), where \( |N(u)| \) is the degree of \( u \). For an integer \( d \geq 0 \), \( G \) is \( d \)-regular if every \( u \in V \) has degree \( d \). The graph \( G[S] \) is the subgraph of \( G \) induced by \( S \subseteq V \), that is, \( G[S] \) is the graph obtained from \( G \) after deleting the vertices not in \( S \).

We will now define some useful colouring terminology for matching cuts; the notions below have been used in other papers as well (see e.g. [19]). A red-blue colouring of \( G \) colours every vertex of \( G \) either red or blue. If every vertex of some set \( S \subseteq V \) has the same colour (red or blue), then \( S \) (and also \( G[S] \)) is called monochromatic. A red-blue colouring is valid if the following holds:

1. every blue vertex has at most one red neighbour;
2. every red vertex has at most one blue neighbour; and
3. both colours red and blue are used at least once.

If a red vertex \( u \) has a blue vertex neighbour \( v \), then \( u \) and \( v \) are matched. See also Figure 1.

For a valid red-blue colouring of \( G \), we let \( R \) be the red set consisting of all vertices coloured red and \( B \) be the blue set consisting of all vertices coloured blue (so \( V = R \cup B \)). The red interface is the set \( R' \subseteq R \) consisting of all vertices in \( R \) with a (unique) blue neighbour, and the blue interface is the set \( B' \subseteq B \) consisting of all vertices in \( B \) with a (unique) red neighbour in \( R \). A red-blue colouring of \( G \) is perfect if it is valid and moreover \( R' = R \) and \( B' = B \). A red-blue colouring of a graph \( G \) is perfect-extendable if it is valid and \( G[R \setminus R'] \) and \( G[B \setminus B'] \) both contain a perfect matching. In other words, the matching defined by the edges with one end-vertex in \( R' \) and the other one in \( B' \) can be extended to a perfect matching in \( G \) or, equivalently, is contained in a perfect matching in \( G \).

We now make the following straightforward observation (see also e.g. [19]).

\[ \blacktriangleright \text{Observation 5}. \text{ Let } G \text{ be a connected graph. The following two statements hold:} \]
\[ \text{(i) } G \text{ has a matching cut if and only if } G \text{ has a valid red-blue colouring;} \]
(ii) \( G \) has a disconnected perfect matching if and only if \( G \) has a perfect-extendable red-blue colouring;

(iii) \( G \) has a perfect matching cut if and only if \( G \) has a perfect red-blue colouring.

3 Hardness for Arbitrary Given Girth

In this section, we will show that Matching Cut and Disconnected Perfect Matching remain \( \text{NP} \)-complete even for graphs of girth at least \( g \), for every \( g \geq 3 \); recall that previously both problems were known to be \( \text{NP} \)-complete for (planar) graphs of girth at least \( g \), only for \( g \leq 5 \).

We need the following notions. The edge expansion \( h(G) \) of a graph \( G \) on \( n \) vertices is defined as

\[
h(G) = \min_{0 \leq |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|},
\]

where \( \partial S := \{uv \in E(G) : u \in S, v \in V(G) \setminus S\} \). A graph is said to be immune if it admits no matching cut.

3.1 Matching Cut

The following lemma follows from known results in the theory of expander graphs.

**Lemma 6.** For every integer \( g \geq 3 \), there is an immune graph with girth at least \( g \).

**Proof.** We first observe that every graph \( G \) with \( h(G) > 1 \) is immune: for every set \( S \subseteq V \) with \( 0 \leq |S| \leq \frac{n}{2} \), the average number of neighbours the vertices of \( S \) have in \( V \setminus S \) is strictly larger than 1.

Let \( G \) be a \( d \)-regular graph. Dodziuk \cite{Dodziuk84} and, independently, Alon and Milman \cite{Alon86} showed that \( h(G) \geq \frac{1}{2}(d - \lambda_2(G)) \), where \( \lambda_2(G) \) is the second largest eigenvalue of the adjacency matrix of \( G \). We know from \cite{Kleitman84} that for every pair of integers \( d, g \geq 3 \), there exists a \( d \)-regular graph \( G \) with \( \lambda_2(G) \leq 2\sqrt{d - 1} \) and girth at least \( g \). By taking \( d \geq 9 \) and \( g \geq 3 \), we conclude that there exists a \( d \)-regular graph \( G \) with girth at least \( g \), such that \( h(G) > 1 \). In other words, \( G \) is immune.

An instance \( (X, \mathcal{C}) \) of Restricted Positive 1-in-\( k \)-SAT, with \( k \geq 3 \), consists of a set of Boolean variables \( X = \{x_1, \ldots, x_n\} \) and a collection of clauses \( \mathcal{C} = \{c_1, \ldots, c_m\} \), where each clause is a disjunction of exactly \( k \) variables, and each variable occurs at most \( k \) times. The question is to determine whether there exists a satisfying truth assignment so that exactly one variable in each clause is set to true. This problem is known to be \( \text{NP} \)-complete \cite{Chen97}. For a variable \( x \in X \) appearing in a clause \( c \in \mathcal{C} \), we let \( c(x) \) denote the number of occurrences of \( x \) in \( c \).

**Theorem 7.** For every integer \( g \geq 3 \), Matching Cut is \( \text{NP} \)-complete for graphs of girth at least \( g \).

**Proof.** The problem Matching Cut is clearly in \( \text{NP} \). Fix \( g \geq 3 \), and let \( F = (X, \mathcal{C}) \) be any instance of Restricted Positive 1-in-\( g \) SAT. We construct, in polynomial time, a graph \( G \) with girth at least \( g \) such that \( F \) is satisfiable if and only if \( G \) has a matching-cut.

Fix \( \ell = 4k \geq g \), for some positive integer \( k \). Let \( X = \{x_1, \ldots, x_n\} \) and \( \mathcal{C} = \{c_1, \ldots, c_m\} \), and let \( I(s, t) \) denote an immune graph with girth at least \( g \) and with two designated vertices \( s \) and \( t \) at distance at least \( \lfloor g/2 \rfloor \); \( I(s, t) \) exists by Lemma 6.
For each \( x_i \in X \), we build a variable gadget as depicted in Figure 3. More precisely, the variable gadget \( V(x_i) \) consists of \( V_{x_i} \cup U_{x_i} \), where

\[
V_{x_i} = \bigcup_{1 \leq j \leq \ell/2} I(v_{x_i}, v_{x_i,j}) \cup \bigcup_{1 \leq j \leq g} I(v_{x_i}, v_{x_i,j})
\]

and

\[
U_{x_i} = \bigcup_{1 \leq j \leq \ell/2} I(u_{x_i}, u_{x_i,j}) \cup \bigcup_{1 \leq j \leq g} I(u_{x_i}, u_{x_i,j}).
\]

We connect the variable gadgets as follows. For \( i \in \{1, \ldots, n-1\} \), we connect \( V(x_i) \) to \( V(x_{i+1}) \) by adding the edge \( u_{x_i}^{\ell/4} v_{x_i}^{1/4} \) and edges \( u_{x_i}^{j+\ell/4} u_{x_{i+1}, j}^{\ell/4} \), \( v_{x_i}^{j+\ell/4} u_{x_{i+1}, j}^{\ell/4} \), \( v_{x_{i+1}, j}^{j+\ell/4} v_{x_{i+1}, j}^{\ell/4} \) for \( 1 \leq j \leq \ell/4 \), thereby forming a cycle of length \( \ell \). See also Figure 4.

For each clause \( c_j = (x_{i_1} \lor \cdots \lor x_{i_g}) \in C \), we create a clause gadget on vertices \( v_{x_{i_1}, c_j(x_{i_1})}, \ldots, v_{x_{i_g}, c_j(x_{i_g})}, u_{x_{i_1}, c_j(x_{i_1})}, \ldots, u_{x_{i_g}, c_j(x_{i_g})} \) and auxiliary vertices \( v_{c_j} \) and \( a_{k}^{c_j}, b_{k}^{c_j}, u_{k}^{c_j} \) for \( k \in \{1, \ldots, g\} \). Its edge set is defined as follows:

- Add an edge joining \( v_{c_j} \) to \( v_{x_{i_k}, c_j(x_{i_k})} \) for \( k \in \{1, \ldots, g\} \);

**Figure 3** Symbolic representation of \( I(s, t) \) (left) and variable gadget corresponding to variable \( x \) (right).

**Figure 4** An illustration of the edges (in red) forming a cycle between two neighbouring variable gadgets \( V(x_i) \) and \( V(x_{i+1}) \) in the case where \( t = 12 \).
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Proof. For any all three cases, there is no cycle of length less than has length
in the second case, we do not have any cycle, and in the third case, the large cycle with a chord
In the first case, we can see by the construction that these cycles have length at least
be contained in the subgraph I of G or would go through at least two such copies. In either case, this results, by the definition
I has girth
Claim 7.1. The graph G has girth g.

Proof. For any \( v \in X \), all edges adjacent to \( v \), \( v \), and \( u \) are contained in a copy of I(s, t), and thus any cycle containing \( v \), \( v \), and \( u \) would be either entirely contained in such a copy or would go through at least two such copies. In either case, this results, by the definition of I(s, t), in a cycle length of at least \( g \). Therefore, any cycle of length less than \( g \) must be contained in the subgraph \( G' \) induced by \( V \setminus \{ v^*, v_x, u_x : x \in X \} \); moreover, it cannot contain more than one vertex of each copy of I(s, t). Thus, the only possibilities in \( G' \) are

- cycles consisting of red edges joining two consecutive variable gadgets as in Figure 4
- stars formed from clause vertices together with their neighborhood as in the left part of Figure 5 and
- cycles with at most one chord formed from clause vertices together with their neighbouring auxiliary vertices and variable vertices as in the right part of Figure 5

In the first case, we can see by the construction that these cycles have length at least \( g \). In the second case, we do not have any cycle, and in the third case, the large cycle with a chord has length \( 4g \) and the two smaller cycles have lengths \( 2g - 1 \), respectively \( 2g + 3 \). Thus, in all three cases, there is no cycle of length less than \( g \). Therefore, G has girth \( g \). ▶

Figure 5 This is an illustration of a clause gadget for \( g = 8 \). For readability, we omitted the graph I(v*, v_c).
It remains to show that $G$ has a matching cut, i.e. a valid red-blue colouring, if and only if $F$ is satisfiable. First, suppose that $G$ has a valid red-blue colouring $\varphi$. We start with some useful claims.

Claim 7.2. The set of clause vertices and, for each $x \in X$, the graphs $V_x$ and $U_x$ are each monochromatic.

Proof. Let $x \in X$. The graphs $V_x$ and $U_x$ consist each of several immune graphs which have one vertex in common. Hence, they are also immune. The clause vertices $v_{c_j}, v_{b_j}$, for $k \in \{1, \ldots, g\}$ and $j \in \{1, \ldots, m\}$, are each connected to $v^*$ by an immune graph. Since immune graphs are necessarily monochromatic, the claim follows.

Claim 7.3. For every variable $x_i \in X$, the corresponding variable gadget $V(x_i)$ is bi-chromatic.

Proof. From Claim 7.2, we know that $U_{x_i}$ and $V_{x_i}$ are each monochromatic, for every $i \in \{1, \ldots, n\}$. Suppose for a contradiction that there exists $i \in \{1, \ldots, n\}$ such that $U_{x_i}$ and $V_{x_i}$ are both coloured blue (the proof when both are coloured red follows the same arguments). Consider the variable gadgets of $x_{i-1}$ and $x_{i+1}$, if they exist (notice that at least one of them exists). We now consider the case for $x_{i-1}$, the case for $x_{i+1}$ follows analogously. The variable gadget of $x_{i-1}$ is connected to the gadget of $x_i$ by a cycle that alternates between vertices in $U_{x_i} \cup V_{x_i}$ and vertices in $U_{x_{i-1}} \cup V_{x_{i-1}}$. Now, since all vertices in $U_{x_i} \cup V_{x_i}$ are coloured blue, the vertices in $U_{x_{i-1}} \cup V_{x_{i-1}}$ also have to be coloured blue. By repeating this argument, we conclude that the vertices of all variable gadgets must all be coloured blue. For every clause $c_j \in \mathcal{C}$, this implies that all $g$ neighbours of $v_{c_j}$ are coloured blue, forcing $v_{c_j}$ to be coloured blue as well. From Claim 7.2 and the fact that $v_{c_j}$ is coloured blue, it follows that all clause vertices are coloured blue. The remaining vertices to be considered are the auxiliary vertices. Since they each have a variable vertex and a clause vertex as a neighbour, and all of those are coloured blue, the auxiliary vertices are coloured blue as well. Thus, all the vertices in $G$ are coloured blue, a contradiction since $\varphi$ is valid.

Claim 7.4. Let $c_j = (x_1 \lor \cdots \lor x_g) \in \mathcal{C}$ be a clause in $F$. Then exactly $g-1$ vertices in $\{v_{x_1,c_j(x_1)}, \ldots, v_{x_g,c_j(x_g)}\}$ have the same colour as $v_{c_j}$.

Proof. We may assume without loss of generality that $v_{c_j}$ is coloured blue. From $\varphi$ being a valid red-blue colouring, it follows that at least $g-1$ vertices in $S = \{v_{x_1,c_j(x_1)}, \ldots, v_{x_g,c_j(x_g)}\}$ are coloured blue as well. Suppose for a contradiction that all vertices in $S$ are coloured blue. By Claim 7.2 we know that all clause vertices are coloured blue (since $v_{c_j}$ is coloured blue). Furthermore, again by Claim 7.2 $V_{x_1}, \ldots, V_{x_g}$ are coloured blue and, by Claims 7.2 and 7.3, $U_{x_1}, \ldots, U_{x_g}$ are coloured red. Since $\varphi$ is a valid red-blue colouring, it follows that the two sets of auxiliary vertices $\{a_1^{c_j}, \ldots, a_g^{c_j}\}$ and $\{b_1^{c_j}, \ldots, b_g^{c_j}\}$ are each monochromatic and have different colours. But since $a_i^{c_j}$ and $b_{(g/2)^j}$ are joined by an edge we get a contradiction and thus, the claim holds.

We are now ready to show that $F$ is satisfiable. We may assume, without loss of generality, that vertex $v^*$ and thus, by Claim 7.2 all clause vertices are coloured blue. We set a variable $x \in X$ to true if and only if the vertices in $V_x$ are red. Since we know by Claim 7.3 that for a clause $c_j \in \mathcal{C}$, vertex $v_{c_j}$ has exactly one red neighbour, we get that in each clause exactly one variable is set to true and thus $F$ is satisfiable.

Conversely, suppose $F$ is satisfiable. We will obtain a valid red-blue colouring as follows. For each variable $x \in X$, we colour the vertices in $V_x$ red if $x$ is set to true and blue otherwise.
We start with the following strengthening of Lemma 6. As in the case of Lemma 6, the proof follows from known and classical results in the theory of expander graphs. By taking a construction based on Caley graphs. By taking

From Claim 7.3, we deduce that $U_x$ has to be coloured blue if $x$ is set to true and red otherwise. Since in every clause $c_j \in C$, there is exactly one true variable, $v_{c_j}$ has $g - 1$ blue neighbours and one red neighbour and thus, must be coloured blue. From Claim 7.2, it then follows that all clause vertices must be coloured blue. Notice that exactly one variable vertex from the set $\{u_{x_{ip}, c_j(x_{ip})} : x_{ip} \in c_j\}$, say $u_{x_{ik}, c_j(x_{ik})}$, is coloured blue. It remains to colour the auxiliary vertices of each clause gadget.

To do so, we describe the following algorithm; see also Figure 6 for an illustration (with $k = 5$). We start by colouring the neighbouring vertices $a_k^j$ and $b_k^j$ of $u_{x_{ik}, c_j(x_{ik})}$ blue. Then, we continue clockwise on the cycle by colouring alternately one auxiliary vertex red and one blue, until we reach the second vertex of the chord joining $a_k^j$ to $b_k^{(g/2)}$. We make sure to colour $a_k^j$ and $b_k^{(g/2)}$ alike. We then continue clockwise with alternating colours again for the auxiliary vertices by starting the colour different from the one used for the second vertex of the chord. It is easy to see that this gives us a valid red-blue colouring of our graph $G$ and thus, $G$ has a matching cut.

### 3.2 Disconnected Perfect Matching

We start with the following strengthening of Lemma 6. As in the case of Lemma 6, the proof follows from known and classical results in the theory of expander graphs.

**Lemma 8.** For every $g \geq 3$, there is an immune graph with girth $g$ that contains a perfect matching.

**Proof.** Recall that a graph $G$ is immune if $h(G) > 1$. We also recall that when $G$ is $d$-regular, Dodziuk [8] and, independently, Alon and Milman [1] showed that $h(G) \geq \frac{1}{2}(d - \lambda_2(G))$, where $\lambda_2(G)$ is the second largest eigenvalue of the adjacency matrix of $G$.

On the other hand, it follows from [18] that for every pair of integers $d, g \geq 3$, there exists a $d$-regular bipartite graph $G$ with $\lambda_2(G) \leq 2\sqrt{d - 1}$ and girth at least $g$ (see [18] for a construction based on Caley graphs). By taking $d \geq 9$, we get $h(G) > 1$, i.e. $G$ is immune.
and as $G$ is bipartite and regular, $G$ contains a perfect matching (via a classical application of Hall’s Marriage Theorem).

We define the graph $H(s, t)$ as follows. Fix $g \geq 3$, and let $H'$ be an immune graph with girth at least $2(g + 1)$ containing a perfect matching; $H'$ exists by Lemma 8. Let $s$ and $t$ be two designated vertices at distance at least $g + 1$. We fix a perfect matching $M$ of $H'$. Let $x \in N_{H'}(s)$ and $y \in N_{H'}(t)$ be the (unique) neighbours of $s$ and $t$ in $M$. Then, since $s$ and $t$ are at distance at least $g + 1$, $x$ and $y$ are at distance at least $g - 1$. We add the edge $xy$ and denote the resulting graph by $H'(s, t)$. Notice that $H'(s, t)$ has girth at least $g$ and it is also immune. The new edge $xy$ ensures that $H'(s, t) - \{s, t\}$ contains a perfect matching as well; indeed consider $(M \setminus \{sx, ty\}) \cup \{xy\}$.

Let $k$ be an integer defined by $k = \lceil g/6 \rceil$ if $g/6$ is odd and $k = \lceil g/6 \rceil + 1$ otherwise. Then, we take $k$ copies $H'(s_1, t_1), \ldots, H'(s_k, t_k)$ of $H'(s, t)$ and identify $s_{i+1}$ with $t_i$ for all $i \in \{1, \ldots, k - 1\}$. We set $s = s_1$ and $t = t_k$ and call the resulting graph $H(s, t)$ (see Figure 7).

\begin{itemize}
  \item Figure 7 a) This is an illustration of the graph $H(s, t)$. The two lower figures show how the edges in the perfect matching of $H(s, t)$ (Fig. b)) respectively $H(s, t) - \{s, t\}$ (Fig. c)) are chosen.
  \item Lemma 9. Let $H(s, t)$ be as constructed above. Then $H(s, t)$ is immune, it has girth at least $g$ and both $H(s, t)$ and $H(s, t) - \{s, t\}$ contain a perfect matching.

Proof. By taking two immune graphs and identifying two of their vertices (one from each graph), we clearly obtain another immune graph. Since the graph $H(s, t)$ is obtained by repeatedly applying this operation, we conclude that it is immune. Every graph $H'(s_i, t_i)$ for $i \in \{1, \ldots, k\}$ has girth at least $g$. By constructing $H(s, t)$ from these graphs $H'(s_i, t_i)$, we clearly do not create any new cycles. Thus, $H(s, t)$ also has girth at least $g$. To see that $H(s, t)$ and $H(s, t) - \{s, t\}$ contain both a perfect matching, we will use the fact that both $H'(s_i, t_i)$ and $H'(s_i, t_i) - \{s_i, t_i\}$ contain a perfect matching. For $H(s, t)$, we take a perfect matching of $H'(s_1, t_1), H'(s_2, t_2), H'(s_3, t_3)$ and alternate this alternation until $H'(s_k, t_k)$, see Figure 7(b). For $H(s, t) - \{s, t\}$ we alternate as well but we start and end with $H'(s_1, t_1) - \{s_1, t_1\}$ and $H'(s_k, t_k) - \{s_k, t_k\}$, see Figure 7(c).

\item Theorem 10. For every integer $g \geq 3$, Disconnected Perfect Matching is NP-complete for graphs of girth at least $g$.

Proof. Let $g \geq 3$. We reduce from Matching Cut for graphs of girth at least $g$, which is NP-complete by Theorem 7. Let $G'$ be the graph constructed in the proof of Theorem 7.
Then, we obtain our graph $G$, by simply taking two copies $G_1$ and $G_2$ of $G'$, where we connect every vertex $v \in V(G_1)$ and its copy $v' \in V(G_2)$ using the graph $H(v, v')$.

To see that $G$ has girth at least $g$, we consider first $G_1$ and $G_2$. $G_1$ and $G_2$ both have girth at least $g$. Any cycle containing vertices from both copies has to pass twice through a graph $H(s, t)$. Thus, it will always have length at least $g$, and so $G$ has girth $g$.

We need to show that $G'$ admits a matching cut, if and only if $G$ admits a disconnected perfect matching. Consider some vertex $v \in V(G_1)$ and its copy $v' \in V(G_2)$. Since they are connected by the graph $H(v, v')$, which is immune, $v$ and $v'$ will always have the same colour in any valid red-blue colouring of $G$. Thus, $G_1$ and $G_2$ will be coloured the same in any valid red-blue colouring. Now it is trivial to see that if $G$ admits a perfect-extendable red-blue colouring, then $G'$ admits a valid red-blue colouring, since it suffices to colour $G'$ the same as $G_1$ (or $G_2$).

Conversely, if $G'$ admits a valid red-blue colouring, then we obtain a perfect-extendable colouring of $G$ as follows. We colour $G_1$ and $G_2$ the same as $G'$. Notice that we colour the immune graphs connecting two copies of a same vertex such that they are monochromatic. This clearly gives us a valid red-blue colouring of $G$, i.e. a matching cut $M$ in $G$. It remains to show that the matching cut is contained in a perfect matching of $G$. Since the colourings of $G_1$ and $G_2$ are the same, we have that whenever a blue vertex $v \in V(G_1)$ is matched with a red vertex $u \in V(G_1)$, i.e. $vu \in M$, then their copies $v' \in V(G_2)$ and $u' \in V(G_2)$ are matched as well, i.e. $v'u' \in M$. By Lemma 9, we know that $H(v, v') - \{v, v'\}$ and $H(u, u') - \{u, u'\}$ contain both a perfect matching which we may add to $M$. For every vertex $v \in V(G_1)$ which has no neighbour of the other colour, we know that its copy $v' \in V(G_2)$ has no neighbour of the other colour either and thus, we can use that $H(v, v')$ contains a perfect matching by Lemma 9 and add it to $M$. By repeatedly doing this, we clearly obtain a perfect matching containing $M$.

4 Hardness for Forbidden Subdivided H-Graphs

In this section we prove that MATCHING CUT, DISCONNECTED PERFECT MATCHING and PERFECT MATCHING CUT are NP-complete for $(H^*_1, \ldots, H^*_s)$-free graphs, for every $i \geq 1$.

4.1 Matching Cut

Let $uv$ be an edge of a graph $G$. We define an edge operation as displayed in Figure 8, which when applied on $uv$ will replace $uv$ in $G$ by the subgraph $T^i_{uv}$. Note that in the new graph, the only vertices from $T^i_{uv}$ that may have neighbours outside $T^i_{uv}$ are $u$ and $v$.

![Figure 8](image-url) The edge $uv$ (left) which we replace by the subgraph $T^i_{uv}$ (right).
We are now ready to prove the first result of this section.

**Theorem 11.** For every \( i \geq 1 \), MATCHING CUT is \( \mathsf{NP} \)-complete for \( (H^*_i, \ldots, H^*_i) \)-free graphs.

**Proof.** Fix an integer \( i \geq 1 \). We reduce from MATCHING CUT itself. Let \( G = (V,E) \) be a graph. We replace every edge \( uv \in E \) by the graph \( T^i_{uv} \) (see Figure 5). Let \( G' = (V',E') \) be the resulting graph.

We claim that \( G' \) is \( (H^*_i, \ldots, H^*_i) \)-free. For a contradiction, assume that \( G' \) contains an induced \( H^*_i \) for some \( 1 \leq i' \leq i \). Then \( G' \) contains two vertices \( x \) and \( y \) that are centers of an induced claw, such that \( G' \) contains an induced path from \( x \) to \( y \) of length \( i' \). All vertices in \( V' \setminus V \) are not centers of any induced claw. Hence, \( x \) and \( y \) belong to \( V \). By construction, any shortest path between two vertices of \( V \) has length at least \( i+1 \) in \( G' \), a contradiction.

**Claim 11.1.** Let \( c' \) be a valid red-blue colouring of \( V' \), let \( M' \) be a corresponding matching cut and let \( uv \in E(G) \). Then it holds that

\[(a) \text{ either } c'(u) \neq c'(v), \text{ and then } T^i_{uv} \text{ is monochromatic, or}
\]
\[(b) \text{ } c'(u) \neq c'(v), \text{ and then } c' \text{ colours } u_1, \ldots, u_{2i} \text{ with the same colour as } u, \text{ while } c' \text{ colours all vertices of } T^i_{uv} = \{u,v,u_1,\ldots,u_{2i}\} \text{ with the same colour as } v, \text{ and moreover, } uv, u_{2i}, v_{2i} \in M'.
\]

**Proof of Claim 11.1.** First assume \( c'(u) = c'(v) \), say \( c' \) colours \( u \) and \( v \) red. As any clique of size at least 3 is monochromatic, all vertices in \( T^i_{uv} \) must be coloured red, so \( T^i_{uv} \) is monochromatic.

Now assume \( c'(u) \neq c'(v) \), say \( u \) is red and \( v \) is blue. As before, we find that all vertices \( u_1, \ldots, u_{2i} \) are coloured the same as \( u \), so they are red, while all vertices \( v_1, \ldots, v_{2i} \) are coloured the same as \( v \), so they are blue. By definition, every edge \( xy \) with \( c'(x) \neq c'(y) \) belongs to \( M' \), so \( uv, u_{2i}, v_{2i} \in M' \). This proves Claim 11.1.

We claim that \( G' \) admits a matching cut \( M' \) if and only if \( G \) admits a matching cut \( M \).

First assume that \( G' \) admits a matching cut, so \( V' \) has a valid red-blue colouring \( c' \). We construct a subset \( M \subseteq E \) as follows. We add an edge \( uv \in E \) to \( M \) if and only if \( c'(u) \neq c'(v) \).

We now show that \( M \) is a matching in \( G \). Let \( u \in V \). For a contradiction, suppose that \( M \) contains edges \( uw \) and \( uv \) for \( v \neq w \). Then \( c'(u) \neq c'(v) \) and \( c'(u) \neq c'(w) \). By Claim 11.1 we find that \( M' \) matches \( u \) in \( G' \) to vertices in \( T^i_{uv} \) and \( T^i_{uw} \), contradicting our assumption that \( M' \) is a matching (cut). Hence, \( M \) is matching.

Now let \( c \) be the restriction of \( c' \) to \( V \). If \( c \) colours every vertex of \( G \) with one colour, say red, then \( c' \) would also colour every vertex of \( G' \) red by Claim 11.1 contradicting the validity of \( c' \). Hence, \( c \) uses both colours. Moreover, for every \( uv \in E \), the following holds: if \( c(u) \neq c(v) \), then \( c'(u) \neq c'(v) \) and thus \( uv \in M \). Hence, as \( M \) is a matching, \( c \) is valid, and thus \( M \) is a matching cut of \( G \).

Conversely, assume that \( G \) admits a matching cut, so \( V \) has a valid red-blue colouring \( c \). We construct a red-blue colouring \( c' \) of \( V' \) as follows.

- For every edge \( uv \in E \) with \( c(u) = c(v) \), we let \( c'(x) = c(u) \) for every \( x \in V(T^i_{uv}) \).
- For every edge \( uv \in E \) with \( c(u) \neq c(v) \), we let \( c'(u) = c'(u_1) = \cdots = c'(u_{2i}) = c(u) \) and \( c'(v) = c'(v_1) = \cdots = c'(v_{2i}) = c(v) \).

As \( c \) is valid, \( c \) uses both colours and thus by construction, \( c' \) uses both colours. Let \( u \in V \). Again as \( c \) is valid, \( c(u) \neq c(v) \) holds for at most one neighbour \( v \) of \( u \) in \( G \). Hence, by construction, \( u \) belongs to at most one non-monochromatic gadget \( T^i_{uv} \). Thus, \( c' \) colours in \( G' \)
at most one neighbour of \(u\) with a different colour than \(u\). Let \(u \in V' \setminus V\). By construction, we find again that \(c'\) colours at most one neighbour of \(u\) with a different colour than \(u\). Hence, \(c'\) is valid, and so \(G'\) has a matching cut.

\[\begin{array}{c}
\text{Figure 9} \quad \text{The edge } uv \text{ (left) which we replace by the subgraph } G_{uv}^i \text{ (right).}
\end{array}\]

### 4.2 Disconnected Perfect Matching

Let \(uv\) be an edge of a graph \(G\). We define an edge operation as displayed in Figure 9, which when applied on \(uv\) replaces \(uv\) by the subgraph \(G_{uv}^i\), for some integer \(i \geq 1\). In the new graph, the only vertices from \(G_{uv}^i\), that may have neighbours outside \(G_{uv}^i\), are \(u\) and \(v\).

We are now ready to prove the second result of this section.

**Theorem 12.** For every \(i \geq 1\), DISCONNECTED PERFECT MATCHING is \(\text{NP}\)-complete for \((H_1^*, \ldots, H_i^*)\)-free graphs.

**Proof.** Fix an integer \(i \geq 1\). As the class of \((H_1^*, \ldots, H_i^*)\)-free graphs is contained in the class of \((H_1^*, \ldots, H_{i-1}^*)\)-free graphs if \(i \geq 2\), we may assume without loss of generality that \(i\) is odd (we need this assumption at a later place in our proof). We reduce from DISCONNECTED PERFECT MATCHING itself. Let \(G = (V, E)\) be a graph. We replace every edge \(uv \in E\) by the graph \(G_{uv}^i\) (see Figure 9). Let \(G' = (V', E')\) be the resulting graph.

We claim that \(G'\) is \((H_1^*, \ldots, H_i^*)\)-free. For a contradiction, assume that \(G'\) contains an induced \(H_{i'}^*\) for some \(1 \leq i' \leq i\). Then \(G'\) contains two vertices \(x\) and \(y\) that are centers of an induced claw, such that \(G'\) contains an induced path from \(x\) to \(y\) of length \(i'\). The only vertices in \(V' \setminus V\) that are the center of an induced claw are the vertices \(v_{2i+2}\) (see Figure 9).

Suppose for a contradiction that \(x = v_{2i+2}\). Since the graph \(G'[\{u, u', v_{2i+2}, v_{2i+1}, v_{i+1}\}]\) contains an induced \(C_4\), the induced path from \(x\) to \(y\) may not contain \(u\). Thus, either \(y = v\) or the path from \(x\) to \(y\) passes through \(v\). Since a shortest path from \(v_{2i+2}\) to \(v\) has length \(i + 1\) we get a contradiction. Hence, \(x\) and \(y\) belong to \(V\). By construction, any shortest path between two vertices of \(V\) has length at least \(i + 3\) in \(G'\), a contradiction.

We now prove the following claim.

**Claim 12.1.** Let \(c'\) be a perfect-extendable red-blue colouring of \(V'\), let \(M'\) be a corresponding disconnected perfect matching and let \(uv \in E(G)\). Then it holds that

(a) either \(c'(u) = c'(v)\), and then \(G_{uv}^i\) is monochromatic, or
(b) \(c'(u) \neq c'(v)\), and then \(c'\) colours \(u'\) and all neighbours of \(u\) in \(G - v\) with the same colour as \(u\), while \(c'\) colours all vertices of \(G_{uv}^i - \{u', u\}\) with the same colour as \(v\), and moreover, \(uv, v_{i+1} \in M'\) and either \(uv' \in M'\) or \(v_{i+1} \in M'\).

**Proof of Claim 12.1.** First assume \(c'(u) = c'(v)\), say \(c'\) colours \(u\) and \(v\) red. As any clique of size at least 3 is monochromatic, all vertices \(v', v_1, \ldots, v_2\) are coloured the same as \(v\), so they are red. Now \(u'\) has two red neighbours, and thus \(u'\) must be red as well. We conclude that \(G_{uv}^i\) is monochromatic.
We claim that we find that \( M \) extendable red-blue colouring matching of perfect-extendability (and thus validity) of say red, then only to
Consequently, our rule does not add any edges \( G \) to a non-monochromatic graph \( G \). By definition, every non-monochromatic edge belongs to \( M' \). Hence, \( u, v \in M' \). Recall that \( M' \) is perfect and observe that [\( V(G_{uv}^i) \)] is even. As the vertices of \( G_{uv}^i - \{u, v\} \) have no neighbours outside \( G_{uv}^i - \{u, v\} \), these two facts imply that either \( uv' \in M' \) or \( vv_{i+1} \in M' \). If \( c'(u) \neq c'(v) \) for some neighbour \( w \) in \( G - v \) then, as we just argued, \( M' \) matches \( u \) with a vertex of \( G_{uv}^i \), contradicting the fact that \( M' \) is a matching in \( G' \). This proves Claim[12.1]

We claim that \( G' \) admits a disconnected perfect matching if and only if \( G \) admits a disconnected perfect matching.

First assume that \( G' \) admits a disconnected perfect matching \( M' \), so \( V' \) has a perfect-extendable red-blue colouring \( c' \). We construct a subset \( M \subseteq E \) by the following rule: add an edge \( uv \in E \) to \( M \) if and only if either \( c'(u) = c'(v) \) and \( M' \) matches both \( u \) and \( v \) with vertices from \( G_{uv}^i \), or \( c'(u) \neq c'(v) \).

We now show that \( M \) is a perfect matching in \( G \). Let \( u \in V' \). First suppose \( u \) belongs in \( G' \) to a non-monochromatic graph \( G_{uv}^i \). By Claim[12.1] we have \( c'(u) \neq c'(v) \) and \( c'(u) = c'(w) \) for every neighbour \( w \) of \( u \) in \( G - v \), and moreover, \( M' \) matches both \( u \) and \( v \) with vertices from \( G_{uv}^i \). Hence, \( M \) matches \( u \) to \( v \). As \( c'(u) = c'(w) \) for every neighbour \( w \) of \( u \) in \( G - v \), each \( G_{uv}^i \) is monochromatic. However, as \( M' \) already matched \( u \) to some vertex from \( G_{uv}^i \), we find that \( M' \) (being a matching) does not match \( u \) to any vertex of any \( G_{uv}^i \) with \( w \neq v \). Consequently, our rule does not add any edges \( uv \) with \( w \neq v \) to \( M \). Hence, \( M \) matches \( u \) only to \( v \).

Now suppose that \( u \) only belongs to monochromatic graphs \( G_{uv}^i \). By Claim[12.1] we have \( c'(u) = c'(v) \) for every neighbour \( v \) of \( u \) in \( G \). As \( M' \) is a perfect matching, \( u \) has exactly one neighbour \( v \) in \( G \), such that \( M' \) matches \( u \) with a vertex of \( G_{uv}^i \). As \( |V(G_{uv}^i)| \) is even and the vertices of \( G_{uv}^i - \{u, v\} \) have no neighbours outside \( G_{uv}^i - \{u, v\} \), this means that \( M' \) also matches \( v \) with a vertex from \( G_{uv}^i \). Hence, \( M \) matches \( u \) to \( v \) and to no other vertex of \( G \). We conclude that \( M \) is indeed a perfect matching in \( G \).

Finally, let \( c \) be the restriction of \( c' \) to \( V \). If \( c \) colours every vertex of \( G \) with one colour, say red, then \( c' \) would also colour every vertex of \( G' \) red by Claim[12.1] contradicting the perfect-extendability (and thus validity) of \( c' \). Hence, \( c \) uses both colours. Moreover, for every \( uv \in E \), the following holds: if \( c(u) \neq c(v) \), then \( c'(u) \neq c'(v) \) and thus \( uv \in M \). Hence, as \( M \) is a (perfect) matching, \( c \) is perfect-extendable, and thus \( M \) is a disconnected perfect matching of \( G \).

Conversely, assume that \( G \) admits a disconnected perfect matching \( M \), so \( V \) has a perfect-extendable red-blue colouring \( c \). We construct a red-blue colouring \( c' \) of \( V' \) and a matching \( M' \) in \( G' \) as follows.

- For every edge \( uv \in E \) with \( c(u) = c(v) \), we let \( c'(x) = c(u) \) for every \( x \in V(G_{uv}^i). \) If \( uv \in M \), then we add \( u, v, v_{i+1}, \ldots, v_2, v_1, u \) to \( M' \); see also Figure[10](a). If \( uv \notin M \), then we add \( v_{i}, v_{i+1}, \ldots, v_2 - 2, v_{2i-1}, v_{2i}, u \) to \( M' \) (recall that \( i \) is odd, so this is possible); see also Figure[10](b).
- For every edge \( uv \in E \) with \( c(u) \neq c(v) \), we let \( c'(u') = c(u) \) and \( c'(v') = c(v) \) for every \( x \in V(G_{uv}^i) \). We add \( u, v, v_{i+1}, \ldots, v_2 - 1 \) to \( M' \) as well as \( v_2, v_{i+1}, \ldots, v_2 - 1 \); see also Figure[10](c).
Again we find that \( c \) uses both colours and thus by construction, \( c' \) uses both colours. Let \( u \in V \). Again as \( c \) is valid, \( c(u) \neq c(v) \) holds for at most one neighbour \( v \) of \( u \) in \( G \). Hence, by construction, \( u \) belongs to at most one non-monochromatic gadget \( G_{vw}' \). Thus, \( c' \) colours in \( G' \) at most one neighbour of \( u \) with a different colour than \( u \). Let \( u \in V' \setminus V \). By construction, we find again that \( c' \) colours at most one neighbour of \( u \) with a different colour than \( u \). Hence, \( c' \) is valid. Again by construction, \( M' \) is a perfect matching containing all edges \( xy \) of \( G' \) with \( c'(x) \neq c'(y) \), and thus, \( M' \) is a disconnected perfect matching of \( G' \).

4.3 Perfect Matching Cut

In any perfect red-blue colouring of a graph \( G = (V, E) \), a vertex \( v \in V \) of degree 2 has exactly one neighbour coloured the same as itself and exactly one neighbour coloured differently than itself. We use this observation to prove the following lemma.

Lemma 13. Let \( G = (V', E') \) be the graph obtained from a graph \( G \) by 4-subdividing an edge \( e \) of \( G \). It holds that \( G \) admits a perfect matching cut if and only if \( G' \) admits a perfect matching cut.

Proof. Let \( e = uv \) and denote the resulting path in \( G' \) by \( uu_1u_2u_3u_4v \).

First suppose that \( G \) has a perfect matching cut, so \( G \) has a perfect red-blue colouring \( c \). If \( u \) and \( v \) are coloured alike, say \( u \) and \( v \) are red, then we colour \( u_1 \) red, \( u_2 \) blue, \( u_3 \) blue and \( u_4 \) red. Else we may assume that \( u \) is red and \( v \) is blue. In that case we colour \( u_1 \) blue, \( u_2 \) blue, \( u_3 \) red and \( u_4 \) red. In both cases we obtain a perfect red-blue colouring \( c' \) of \( G' \). So \( G' \) has a perfect matching cut.

Now suppose that \( G' \) has a perfect matching cut, so \( G' \) has a perfect red-blue colouring \( c' \). If \( u \) and \( v \) are coloured alike, say \( u \) and \( v \) are red, then \( u_1 \) must be red, \( u_2 \) blue, \( u_3 \) blue and \( u_4 \) red. Hence, the restriction of \( c' \) to \( V \) is a perfect red-blue colouring of \( G \). Else we may assume that \( u \) is red and \( v \) is blue. In that case \( u_1 \) must be blue, \( u_2 \) blue, \( u_3 \) red and \( u_4 \) red. Again, the restriction of \( c' \) to \( V \) is a perfect red-blue colouring of \( G \). Hence, in both cases we find that \( G \) has a perfect matching cut.

We now find that Perfect Matching Cut satisfies condition C3 from Section 1.4 take \( q = 4 \) and apply Lemma 12 on every edge of the input graph. Moreover, if we apply Lemma 13 sufficiently times on every edge of a graph and combine this with the \( \text{NP} \)-completeness of Perfect Matching Cut for subcubic bipartite graphs of girth at least \( g \) (for every \( g \geq 3 \)), then we also obtain our hereditary graph class result announced in Section 1.2.

Note that for this result we get the girth property also from Lemma 13.

Theorem 14. For every \( i \geq 1 \) and \( g \geq 3 \), Perfect Matching Cut is \( \text{NP} \)-complete for \((H_1^i, \ldots, H_k^i)\)-free subcubic bipartite graphs of girth at least \( g \).
5 Conclusions

We proved new NP-completeness results for Matching Cut, Disconnected Perfect Matching and Perfect Matching Cut. This answered, amongst others, an open problem of Le and Le [15] for Matching Cut for graphs of high girth. It also enabled us to update the state-of-art theorems for all three problems restricted to $H$-free graphs. As a consequence, for all three problems, the only open cases are when $H$ is a graph from $S$; recall that $S$ is the set of graphs, each component of which is either a path or subdivided claw.

We recall from Theorems 1 and 2 that Matching Cut and Disconnected Perfect Matching are NP-complete for $P_{19}$-free graphs and $P_{23}$-free graphs, respectively. The following open problem from [10] is still relevant:

**Open Problem 1.** Does there exist an integer $t$ such that Perfect Matching Cut is NP-complete for $P_t$-free graphs?

We also note that the transformations for Matching Cut and Disconnected Matching Cut from Section 4 do not decrease the girth and result in graphs that contain many cycles of varying length as subgraphs. Hence, we also pose the following open problem:

**Open Problem 2.** Classify the complexity of Matching Cut and Disconnected Matching Cut for $H$-subgraph-free graphs.

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