Non-degenerate surfaces of revolution in Minkowski space that satisfy the relation $aH + bK = c$

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Abstract

In this work, we study spacelike and timelike surfaces of revolution in Minkowski space $\mathbb{E}^3_1$ that satisfy $aH + bK = c$, where $H$ and $K$ denote the mean curvature and the Gauss curvature of the surface and $a$, $b$ and $c$ are constants. The classification depends on the causal character of the axis of revolution and in all the cases, we obtain a first integral of the equation of the generating curve of the surface.

*Partially supported by MEC-FEDER grant no. MTM2007-61775 and Junta de Andalucía grant no. P06-FQM-01642.
1 Introduction

Consider the three-dimensional Minkowski space $E_3^1$, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $\langle \cdot, \cdot \rangle = (dx)^2 + (dy)^2 - (dz)^2$, where $(x, y, z)$ stand for the usual coordinates of $\mathbb{R}^3$. A vector $v \in E_3^1$ is said spacelike if $\langle v, v \rangle > 0$ or $v = 0$, timelike if $\langle v, v \rangle < 0$ and lightlike if $\langle v, v \rangle = 0$ and $v \neq 0$. A submanifold $S \subset E_3^1$ is said spacelike, timelike or lightlike if the induced metric on $S$ is a Riemannian metric (positive definite), a Lorentzian metric (a metric of index 1) or a degenerated metric, respectively. In the case that $S$ is a straight-line $L = \langle v \rangle$, this means that $v$ is spacelike, timelike or lightlike, respectively. If $S$ is a plane $P$, this is equivalent that any orthogonal vector to $P$ is timelike, spacelike or lightlike respectively.

An immersion $x : M \to E_3^1$ of a surface $M$ is said non-degenerated if the induced metric $x^*(\langle \cdot, \cdot \rangle)$ on $M$ is non-degenerate. In this setting, there is only two possibilities: if $x^*(\langle \cdot, \cdot \rangle)$ is positive definite, that is, it is a Riemannian metric and the immersion is called spacelike or $x^*(\langle \cdot, \cdot \rangle)$ is a Lorentzian metric, that is, a metric of index 1, and the immersion is called timelike. For spacelike surfaces, the tangent planes are spacelike everywhere, and for timelike surfaces, they are timelike.

We consider spacelike or timelike surfaces in $E_3^1$ that satisfy the relation

$$aH + bK = c,$$

where $H$ and $K$ are the mean curvature and the Gauss curvature of the surface, and $a$, $b$ and $c$ are constants. We say that the surface is a linear Weingarten surface of $E_3^1$. In general, a Weingarten surface is a surface that satisfies a certain smooth relation $W = W(H, K) = 0$ and our case, that is, surfaces that satisfy (1) is the simplest case of $W$, that is, that $W$ is a linear function in its variables. The family of linear Weingarten surfaces include the surfaces with constant mean curvature ($b = 0$) and the surfaces with constant Gauss curvature ($a = 0$).

In this work we study linear Weingarten surfaces that are rotational, that is, invariant by a group of motions of $E_3^1$ that pointwised fixed a straight-line. In such case, Equation (1) is a second ordinary differential equation that describes the shape of the generating curve of the surface. One can not expect to integrate this equation, because even in the trivial cases that $a = 0$ or $b = 0$, this integration is not possible. We are going to discard the cases that $H$ is constant of $K$ is constant, which are known: see for example [1, 2, 3]. We will obtain a first integration of (1). For the particular case that $a^2 - 4bc = 0$, we describe all solutions, exactly, we have

**Theorem 1.1** Let $M$ be a non-degenerate rotational surface in $E_3^1$, and take $\epsilon = 1$ if $M$ is spacelike and $\epsilon = -1$ if $M$ is timelike. Assume that $M$ is a linear Weingarten surface
such that $a^2 - 4bce = 0$. After a rigid motion of the ambient space, a parametrization $X(u,v)$ of $M$ is as follows:

1. If the axis is timelike, $X(u,v) = (u \cos(v), u \sin(v), z(u))$, where
   \[
   z(u) = \pm \sqrt{\frac{4eb^2}{a^2} + \left(\frac{C}{a} \pm u\right)^2 + \mu}, \quad C = 2\sqrt{b\epsilon(-b + \lambda)}, \quad \mu, \lambda \in \mathbb{R}.
   \]

2. If the axis is spacelike, we have two possibilities:
   (a) The parametrization is $X(u,v) = (u, z(u) \sinh(v), z(u) \cosh(v))$, where
   \[
   z(u) = \pm \frac{C}{a} \pm \sqrt{\frac{4eb^2}{a^2} \pm (u \pm \mu)^2}, \quad C = 2\sqrt{b\epsilon\lambda}, \quad \mu, \lambda \in \mathbb{R}.
   \]
   (b) The parametrization is $X(u,v) = (u, z(u) \cosh(v), z(u) \sinh(v))$, where
   \[
   z(u) = -\frac{C}{a} \pm \sqrt{\frac{4b^2}{a^2} \pm (u \pm \mu)^2}, \quad C = 2\sqrt{b\lambda}, \quad \mu, \lambda \in \mathbb{R}.
   \]

3. If the axis is lightlike, $X(u,v) = (-2uv, z(u) + u - uv^2, z(u) - u - uv^2)$, where
   \[
   z(u) = \frac{1}{48} \left(\frac{-4ac\lambda + (cC^2 - 2a^2\lambda)u}{\epsilon c\lambda(2\lambda + cu^2)} \pm \frac{cC^2 + 2a^2\lambda}{\sqrt{-2c\lambda}} \arctanh(\sqrt{-\frac{c}{2\lambda} u})\right) \pm \mu, \quad \mu, \lambda \in \mathbb{R}.
   \]

2 Rotational surfaces in $E_1^3$

In this section we describe the surfaces of revolution of $E_1^3$ and we recall the concepts of mean curvature and Gauss curvature for a non-degenerate surface. We consider the rigid motions of the ambient space that leave a straight-line pointwised fixed, called, the axis of the surface. Let $L$ be the axis of the surface. Depending on $L$, there are three types of rotational motions. After an isometry of $E_1^3$, the expressions of rotational motions with respect to the canonical basis $\{e_1, e_2, e_3\}$ are as follows:

\[
R_v : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} \cos v & \sin v & 0 \\ -\sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]

\[
R_v : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh v & \sinh v \\ 0 & \sinh v & \cosh v \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]
\[
R_v : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -v & v \\ v & 1 - \frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1 + \frac{v^2}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]

See [4, 5] for more details.

**Definition 2.1** A surface \( M \) in \( \mathbb{E}^3_1 \) is a surface of revolution, or rotational surface, if \( M \) is invariant by some of the above three groups of rigid motions.

In particular, there exists a planar curve \( \alpha = \alpha(u) \) that generates the surface, that is, \( M \) is the set of points given by \( \{ R_v(\alpha(u)); u \in I, v \in \mathbb{R} \} \). We now describe the parametrizations of a rotational surface.

1. **Case \( L \) is a timelike axis.** Consider that \( L \) is the \( x_3 \)-axis. If \( p = (x_0, y_0, z_0) \notin L \), then \( \{ R_v(p); v \in \mathbb{R} \} \) is an Euclidean circle of radius \( \sqrt{x_0^2 + y_0^2} \) in the plane \( z = z_0 \). If \( \alpha(u) = (u, 0, z(u)) \) is a planar curve in the plane \( y = 0 \), then the surface of revolution generated by \( \alpha \) writes as
   \[
   X(u, v) = (u \cos(v), u \sin(v), z(u)), \ u \neq 0. \tag{2}
   \]

2. **Case \( L \) is a spacelike axis.** Consider that \( L \) is the \( x_1 \)-axis. If \( p = (x_0, y_0, z_0) \) does not belong to \( L \), then \( \{ R_v(p); v \in \mathbb{R} \} \) is an Euclidean hyperbola in the plane \( x = x_0 \) and with equation \( y^2 - z^2 = y_0^2 - z_0^2 \). For this kind of rotational surfaces, we have two type of surfaces:
   (a) If \( \alpha(u) = (u, 0, z(u)) \) is a planar curve in the plane \( y = 0 \), then the surface of revolution generated by \( \alpha \) writes as
       \[
       X(u, v) = (u \cos(v), z(u) \sinh(v), z(u) \cosh(v)), \ u \neq 0. \tag{3}
       \]
   (b) If \( \alpha(u) = (u, z(u), 0) \) is a planar curve in the plane \( z = 0 \), then the surface is given by
       \[
       X(u, v) = (u, z(u) \cosh(v), z(u) \sinh(v)), \ u \neq 0. \tag{4}
       \]

3. **Case \( L \) is a lightlike axis.** Consider that \( L \) is the straight-line \( v_1 =< (0, 1, 1) > \). If \( p = (x_0, y_0, z_0) \) does not belong to the plane \( < e_1, v_1 > \), the orbit \( \{ R_v(p); v \in \mathbb{R} \} \) is the curve
   \[
   \beta(v) = (x - (y - z)v, xv + y - (y - z)\frac{v^2}{2}, xv + z - (y - z)\frac{v^2}{2}).
   \]
   The curve \( \beta \) lies in the plane \( y - z = y_0 - z_0 \) and describes a parabola in this plane, namely,
   \[
   \beta(v) = (x, y, z) + v(-(y - z)e_1 + xv_1) - \frac{y - z}{2}v^2v_1.
   \]
Consider \( \alpha(u) \) a planar curve in the plane \(<0, 1, 1>, (0,1,-1)> \) given as a graph on the straight-line \(<0,1,-1)> \), that is, \( \alpha(u) = (0, u + z(u), -u + z(u)) \). The surface of revolution generated by \( \alpha \) is

\[
X(u, v) = (-2uv, z(u) + u - uv^2, z(u) - u - uv^2), \; u \neq 0.
\]

Let \( M \) be surface and \( x : M \rightarrow \mathbb{E}_1^3 \) a non-degenerate immersion and we simply say that \( M \) is non-degenerate. The surface could be not orientable, but if the immersion is spacelike, then \( M \) is necessarily orientable. This is due to the following fact. At each point \( p \in M \) there is two possible choices of a unit normal vector to the tangent plane \( T_p M \) of \( M \) at \( p \). The normal vector to \( M \) is a timelike vector, and in Minkowski space, two any timelike vectors are not orthogonal. Thus, if \( E_3 = (0,0,1) \), at each point \( p \in M \), we take that unit normal vector \( N(p) \) such that \( \langle N(p), E_3 \rangle < 0 \). This allows to define an global orientation on \( M \), proving that \( M \) is orientable. With this choice of \( N \), we say that \( N \) is future directed. In the case that the immersion is timelike, we will assume that \( M \) is orientable.

Let \( x : M \rightarrow \mathbb{E}_1^3 \) be a non-degenerate immersion of a surface \( M \) and let \( N \) be a Gauss map. Let \( U, V \) be vector fields to \( M \) and we denote by \( \nabla^0 \) and \( \nabla \) the Levi-Civitta connections of \( \mathbb{E}_1^3 \) and \( M \) respectively. The Gauss formula says \( \nabla^0_U V = \nabla_U V + II(U, V) \), where \( II \) is the second fundamental form of the immersion. The Weingarten endomorphism is \( A_p : T_p M \rightarrow T_p M \) defined as \( A_p(U) = -(\nabla^0_U N)_p = (-dN)_p(U) \). We have then \( II(U, V) = -\epsilon(II(U, V), N)N = -\epsilon(AU, V)N \), where \( \epsilon = 1 \) if \( M \) is spacelike and \( \epsilon = -1 \) if \( M \) is timelike. The mean curvature vector \( \vec{H} \) is defined as \( \vec{H} = (1/2)\text{trace}(II) \) and the Gauss curvature \( K \) as the determinant of \( II \) computed in both cases with respect to an orthonormal basis. The mean curvature \( H \) is the function given by \( H = \vec{H}, \) that is, \( H = -\epsilon(\vec{H}, N) \). If \( \{e_1, e_2\} \) is an orthonormal basis at each tangent plane, with \( \langle e_1, e_1 \rangle = 1, \langle e_2, e_2 \rangle = \epsilon \), then

\[
\vec{H} = \frac{1}{2}(II(e_1, e_1) + II(e_2, e_2)) = -\frac{1}{2}(\langle Ae_1, e_1 \rangle + \epsilon \langle Ae_2, e_2 \rangle)N = -\epsilon(\frac{1}{2}\text{trace}(A))N
\]

\[
K = -\epsilon\text{det}(A).
\]

In this work we need to compute \( H \) and \( K \) using a parametrization of the surface. Let \( X : D \subset \mathbb{R}^2 \rightarrow \mathbb{E}_1^3 \) be a parametrization of the surface, \( X = X(u, v) \). Then \( A = II(I)^{-1} \), \( I = \langle , \rangle \) and we have the known formulae ([5]):

\[
H = -\epsilon\frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}, \quad K = -\epsilon\frac{eg - f^2}{EG - F^2},
\]

where \( \{E, F, G\} \) and \( \{e, f, g\} \) are the coefficients of \( I \) and \( II \), respectively:

\[
E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle,
\]

\[
e = -\langle N_u, X_u \rangle, \quad f = -\langle N_u, X_v \rangle, \quad g = -\langle N_v, X_v \rangle,
\]
where the subscripts denote the corresponding derivatives. Here $N$ is

$$N = \frac{X_u \times X_v}{\sqrt{\epsilon(EG - F^2)}}.$$

We recall that

$$W := EG - F^2 = \epsilon |X_u \times X_v|^2 \begin{cases} \text{is positive if } M \text{ is spacelike} \\ \text{is negative if } M \text{ is timelike} \end{cases}$$

Finally, in order to the computations for $H$ and $K$, we recall that the cross-product $\times$ satisfies that for any vectors $u, v, w$ we recall that for any vectors $u, v, w$

$$\langle u \times v, w \rangle = \det(u, v, w).$$

Then the relation (1) writes as

$$H = -\frac{\epsilon G\det(X_u, X_v, X_{uu}) - 2F\det(X_u, X_v, X_{uu}) + E\det(X_u, X_v, X_{uv})}{(\epsilon(EG - F^2))^{3/2}} := \frac{H_1}{2W^{3/2}}; \quad (7)$$

$$K = -\frac{\det(X_u, X_v, X_{uu})\det(X_u, X_v, X_{uv}) - \det(X_u, X_v, X_{uv})^2}{(EG - F^2)^2} := \frac{K_1}{W^2}. \quad (8)$$

In Minkowski ambient space, the role of spheres is played by pseudohyperbolic surfaces and pseudospheres [4]. If $p_0 \in E^3_1$ and $r > 0$ the pseudohyperbolic surface centered at $p_0$ with radius $r > 0$ is $H^{2,1}(r; p_0) = \{p \in E^3_1; \langle p - p_0, p - p_0 \rangle = -r^2\}$ and the pseudosphere centered at $p_0$ and radius $r > 0$ is $S^{2,1}(r; p_0) = \{p \in E^3_1; \langle p - p_0, p - p_0 \rangle = r^2\}$. If $M$ is spacelike (resp. timelike) then $N$ is timelike (resp. spacelike) and $N : M \rightarrow H^{2,1}(1)$ (resp. $N : M \rightarrow S^{2,1}(1)$), where $H^{2,1}(1) = H^{2,1}(1; O)$ (resp. $S^{2,1}(1) = S^{2,1}(r; O)$, being $O$ the origin of coordinates of $\mathbb{R}^3$. For both kind of surfaces, we can take $N(p) = (p - p_0)/r$ and $A = -\frac{1}{r}I$. Then $H = \epsilon/r$ and $K = -\epsilon/r^2$.

### 3 Rotational surfaces with timelike axis

We assume that the generating curve $\alpha$ lies in the $xz$-plane and we parametrize $\alpha$ as the graph of a function $z = z(u)$, that is, $\alpha(u) = (u, 0, z(u))$, $u > 0$. Then the surface is parametrized as in (2) and $W = u^2(1 - z'^2)$. Thus $z'^2 < 1$ if the surface is spacelike and $z'^2 > 1$ if $M$ is timelike. Using (7) and (8), the expressions of $H$ and $K$ are:

$$H = -\frac{1}{2}\left(\frac{\epsilon z'}{u\sqrt{\epsilon(1 - z'^2)}} + \frac{z''}{(\epsilon(1 - z'^2))^{3/2}}\right), \quad K = -\frac{z'z''}{u(1 - z'^2)^2}.$$

Then the relation (1) writes as

$$a \left(\frac{\epsilon z'}{u\sqrt{\epsilon(1 - z'^2)}} + \frac{z''}{(\epsilon(1 - z'^2))^{3/2}}\right) + b\frac{z'z''}{u(1 - z'^2)^2} = -c.$$
Multiplying by \( u \) we obtain a first integral. Exactly, we have
\[
a \left( u \frac{e z'}{\sqrt{\epsilon(1 - z'^2)}} \right)' + b \left( \frac{1}{1 - z'^2} \right)' = -2cu.
\]
Then there exists a integration constant \( \lambda \in \mathbb{R} \) such that
\[
\epsilon \frac{auz'}{\sqrt{\epsilon(1 - z'^2)}} + \frac{b}{1 - z'^2} = -cu^2 + \lambda. \tag{9}
\]
Let
\[
\phi = \frac{z'}{\sqrt{\epsilon(1 - z'^2)}}.
\]
Then \( 1 + \epsilon \phi^2 = 1/(1 - z'^2) \) and Equation (9) writes as \( b \phi^2 + au \phi + \epsilon(b + cu^2 - \lambda) = 0 \).
Hence, we obtain \( \phi \):
\[
\frac{z'}{\sqrt{\epsilon(1 - z'^2)}} = \frac{-au \pm \sqrt{(a^2 - 4bc \epsilon)u^2 + 4b\epsilon(-b + \lambda)}}{2b}. \tag{10}
\]
We completely solve this differential equation in two particular cases:

1. Consider \( \lambda = b \). Then we have
\[
\frac{z'}{\sqrt{\epsilon(1 - z'^2)}} = \frac{-a \pm \sqrt{a^2 - 4bc \epsilon}}{2b} u = Cu, \quad C = \frac{-a \pm \sqrt{a^2 - 4bc \epsilon}}{2b}.
\]
Then
\[
z(u) = \pm \sqrt{\frac{\epsilon + C^2u^2}{C}} + \mu, \quad \mu \in \mathbb{R}.
\]
From the parametrization (2) of the surface, one concludes that \( M \) satisfies the equation \( x^2 + y^2 - (z - \mu)^2 = -\epsilon^2 \).

2. Assume \( a^2 - 4bc \epsilon = 0 \). Then
\[
\frac{z'}{\sqrt{\epsilon(1 - z'^2)}} = \frac{-au \pm C}{2b}, \quad C = 2\sqrt{bc(-b + \lambda)}.
\]
The integration of this equation is
\[
z(u) = \pm \sqrt{\frac{4\epsilon_b^2}{a^2} + \left( \frac{C}{a} \pm u \right)^2} + \mu, \quad \mu \in \mathbb{R}.
\]
4 Rotational surfaces with spacelike axis

We distinguish two cases according to the two possible parametrizations.

1. Case I. Assume that the parametrization is given by (3). The relation (1) writes as

\[
\frac{a}{2} \left( \frac{\epsilon}{z \sqrt{\epsilon (1 - z'^2)}} + \frac{z''}{(\epsilon (1 - z'^2))^{3/2}} \right) + b \frac{z''}{z (1 - z'^2)^2} = -c.
\]

Multiplying by \(zz'\), we obtain a first integral. Exactly, we have

\[
a \left( \frac{\epsilon z}{\sqrt{\epsilon (1 - z'^2)}} \right)' + b \left( \frac{1}{1 - z'^2} \right)' = -c(z^2)'.
\]

Then there exists an integration constant \(\lambda \in \mathbb{R}\) such that

\[
\epsilon \frac{az}{\sqrt{\epsilon (1 - z'^2)}} + \frac{b}{1 - z'^2} = -cz^2 + \lambda. \tag{11}
\]

Now we take \(\phi = 1/\sqrt{\epsilon (1 - z'^2)}\). Then Equation (11) writes as

\[
b \phi^2 + az \phi + \epsilon (cz^2 - \lambda) = 0.
\]

Then

\[
\frac{1}{\sqrt{\epsilon (1 - z'^2)}} = -az \pm \sqrt{(a^2 - 4bce)z'^2 + 4be\lambda} \frac{2b}{2b}. \tag{12}
\]

We completely solve this differential equation in two particular cases:

(a) Consider \(\lambda = 0\). Then we have

\[
\frac{1}{\sqrt{\epsilon (1 - z'^2)}} = -a \pm \sqrt{a^2 - 4bce} \frac{z}{2b}, \quad Cz, \quad C = -a \pm \sqrt{a^2 - 4bce} \frac{2b}{2b}.
\]

The solution of this differential equation is

\[
z(u) = \pm \sqrt{\frac{\epsilon}{C^2}} \pm (u \pm C\mu)^2, \quad \mu \in \mathbb{R}.
\]

From the parametrization (3) of the surface, one concludes that \(M\) satisfies the equation \((x - C\mu)^2 + y^2 - z^2 = -\frac{\epsilon}{C^2}\). Thus, if we set \(p_0 = (\pm C\mu, 0, 0)\), for \(\epsilon = 1\) we obtain that \(M\) is the pseudohyperbolic surface \(H^{2,1}(1/|C|; p_0)\) and for \(\epsilon = -1\), \(M\) is the pseudosphere \(S^{2,1}(1/|C|; p_0)\).
(b) Assume \( a^2 - 4bc\epsilon = 0 \). Then

\[
\frac{1}{\sqrt{\epsilon(1 - z'^2)}} = -\frac{az \pm C}{2b}, \quad C = 2\sqrt{b\epsilon\lambda}.
\]

The integration of this equation is

\[
z(u) = \pm \frac{C}{a} \pm \sqrt{\frac{4\epsilon b^2}{a^2} \pm (u \pm \mu)^2, \mu \in \mathbb{R}}.
\]

2. Case II. The expression of the parametrization is written in (4). In this case, the surface is timelike, since \( EG - F^2 = -z^2(1 + z'^2) \). The Weingarten relation (1) is

\[
a^2 \left( -\frac{1}{z\sqrt{1 + z'^2}} + \frac{z''}{(1 + z'^2)^{3/2}} \right) - b\frac{z''}{z(1 + z'^2)} = c.
\]

Multiplying by \( zz' \) again, we have

\[
-a \left( \frac{z}{\sqrt{1 + z'^2}} \right)' + b \left( \frac{1}{1 + z'^2} \right)' = c(z^2)'.
\]

It follows the existence of an integration constant \( \lambda \in \mathbb{R} \) such that

\[
-\frac{az}{\sqrt{1 + z'^2}} + \frac{b}{1 + z'^2} = cz^2 + \lambda. \quad (13)
\]

If we set \( \phi = 1/\sqrt{1 + z'^2} \), Equation (13) is \( b\phi^2 - az\phi - cz^2 - \lambda = 0 \), obtaining

\[
\frac{1}{1 + z'^2} = \frac{az \pm \sqrt{(a^2 + 4bc)z^2 + 4b\lambda}}{2b}. \quad (14)
\]

As in the previous case, we solve this equation in the next two cases:

(a) If \( \lambda = 0 \), then

\[
\frac{1}{\sqrt{1 + z'^2}} = -\frac{a \pm \sqrt{a^2 + 4bc}}{2b} z = Cz, \quad C = \frac{a \pm \sqrt{a^2 + 4bc}}{2b}.
\]

The solution of this equation is

\[
z(u) = \pm \sqrt{\frac{1}{C^2} - (u \pm C\mu)^2}, \mu \in \mathbb{R}\}.
\]

This surface is the pseudosphere \( S^{2,1}(1/|C|; p_0) \), with \( p_0 = (\pm C\mu, 0, 0) \) since by the expression of the parametrization (4), the coordinates of \( M \) satisfies \( (x \pm C\mu)^2 + y^2 - z^2 = 1/C^2 \).
(b) If \( a^2 + 4bc = 0 \), then

\[
\frac{1}{\sqrt{1 + z'^2}} = \frac{az \pm C}{2b}, \quad C = 2\sqrt{b\lambda}.
\]

The solution of this equation is

\[
z(u) = -\frac{C}{a} \pm \sqrt{\frac{4b^2}{a^2} \pm (u \pm \mu)^2}, \quad \mu \in \mathbb{R}.
\]

5 Rotational surfaces with lightlike axis

Consider the parametrization given in (5). Then \( EG - F^2 = 16u^2z' \) and the relation (1) writes as

\[
a \left( \frac{1}{2u\sqrt{\epsilon z'}} - \frac{\epsilon z''}{4(\epsilon z')^{3/2}} \right) + b \frac{z''}{8uz'^2} = c.
\]

Multiplying by \( u \) we obtain a first integral. Exactly, we have

\[
a \left( \frac{u}{2\sqrt{\epsilon z'}} \right)' - b \left( \frac{1}{8z'} \right)' = cu.
\]

Then there exists an integration constant \( \lambda \in \mathbb{R} \) such that

\[
a \frac{u}{\sqrt{\epsilon z'}} - b \frac{1}{8z'} = \frac{c}{2} u^2 + \lambda. \quad (15)
\]

From (15), we obtain the value of \( \sqrt{\epsilon z'} \):

\[
\sqrt{\epsilon z'} = \frac{aeu \pm \sqrt{(a^2 - 4b\epsilon c)u^2 - 8b\epsilon \lambda}}{4\epsilon (cu^2 + 2\lambda)}.
\]

As in the two previous cases, we distinguish two special cases:

1. If \( \lambda = 0 \), then

\[
\sqrt{\epsilon z'} = \frac{a \pm \epsilon \sqrt{a^2 - 4b\epsilon c}}{4c} \frac{1}{u} := \frac{C}{u}, \quad C = \frac{a \pm \epsilon \sqrt{a^2 - 4b\epsilon c}}{4c}.
\]

We solve this equation obtaining

\[
z(u) = -\frac{C^2}{u} + \mu, \quad \mu \in \mathbb{R}.
\]

From the parametrization (5), we see that \( M \) satisfies the equation \( x^2 + y^2 - (z - \mu)^2 = -4\epsilon C^2 \). Thus, if \( p_0 = (0,0,\mu) \), we have that \( M = H^{2,1}(2C; p_0) \) if \( \epsilon = 1 \), and \( M = S^{2,1}(2C; p_0) \) if \( \epsilon = -1 \).
2. Assume $a^2 - 4b\epsilon \lambda = 0$. Then

$$\sqrt{\epsilon z'} = \frac{a\epsilon u \pm C}{4\epsilon (cu^2 + 2\lambda)}, \quad C = \sqrt{-8b\epsilon \lambda}.$$ 

We point out that $-8b\epsilon \lambda > 0$ and that combining with $a^2 - 4b\epsilon \lambda = 0$, we have $c\lambda \leq 0$. The solution is

$$z(u) = \frac{1}{64} \left( \frac{-4aC\lambda \pm \epsilon (cC^2 - 2a^2 \lambda) u + cC^2 + 2a^2 \lambda}{\epsilon c\lambda (2\lambda + cu^2)} \right) \arctanh \left( \frac{-c}{2\lambda} u \right) + \mu, \quad \mu, \lambda \in \mathbb{R}.$$ 

Figure 1: Rotational surfaces with timelike axis, for $a = 2$, $b = \epsilon$ and $\mu = 0$: The surface is spacelike with $\lambda = 2$ (left). The surface is timelike with $\lambda = 0$ (right).

Figure 2: Rotational surfaces with spacelike axis, for $a = 2$, $b = \epsilon$, $\lambda = 1$ and $\mu = 0$: The surface is spacelike (left). The surface is timelike (right).
Figure 3: Rotational surfaces with lightlike axis, for $a = 2, b = -\epsilon, \lambda = 1$ and $\mu = 0$: The surface is spacelike (left). The surface is timelike (right).

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