Non-local, non-convex functionals converging to Sobolev norms

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A B S T R A C T

We study the pointwise convergence and the $I'$-convergence of a family of non-local, non-convex functionals $\Lambda_\delta$ in $L^p(\Omega)$ for $p > 1$. We show that the limits are multiples of $\int_\Omega |\nabla u|^p$. This is a continuation of our previous work where the case $p = 1$ was considered.

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1. Introduction and statement of the main results

Assume that $\varphi : [0, +\infty) \to [0, +\infty)$ is defined at every point of $[0, +\infty)$, $\varphi$ is continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$ where it admits a limit from the left and from the right, and $\varphi(0) = 0$. Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) denote a domain which is either bounded and smooth, or $\Omega = \mathbb{R}^d$. Given a measurable function $u$ on $\Omega$, and a parameter $\delta > 0$, we define the following non-local functionals, for $p > 1$,

$$\Lambda(u, \Omega) := \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{p+d}} \, dx \, dy \quad \text{and} \quad \Lambda_\delta(u, \Omega) := \delta^p \Lambda(u/\delta, \Omega). \quad (1.1)$$

To simplify the notation, we will often delete $\Omega$ and write $\Lambda_\delta(u)$ instead of $\Lambda_\delta(u, \Omega)$.

As in [3], we consider the following four assumptions on $\varphi$:

$$\varphi(t) \leq at^{p+1} \text{ in } [0, 1] \text{ for some positive constant } a, \quad (1.2)$$

$$\varphi(t) \leq b \text{ in } \mathbb{R}_+ \text{ for some positive constant } b, \quad (1.3)$$

$$\varphi \text{ is non-decreasing,} \quad (1.4)$$

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and
\[ \gamma_{d,p} \int_0^\infty \varphi(t)t^{-(p+1)} \, dt = 1, \]
where \( \gamma_{d,p} := \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^p \, d\sigma \) for some \( e \in \mathbb{S}^{d-1} \).

In this paper, we study the pointwise and the \( \Gamma \)-convergence of \( A_\delta \) as \( \delta \to 0 \) for \( p > 1 \). This is a continuation of our previous work [3] where the case \( p = 1 \) was investigated in great details. Concerning the pointwise convergence of \( A_\delta \), our main result is

**Theorem 1.** Let \( d \geq 1 \) and \( p > 1 \). Assume (1.2), (1.3), and (1.5) (the monotonicity assumption (1.4) is not required here). We have

(i) There exists a positive constant \( C_{p,\Omega} \) such that
\[ A_\delta(u, \Omega) \leq C_{p,\Omega} \int_\Omega |\nabla u|^p \, dx \quad \forall u \in W^{1,p}(\Omega), \forall \delta > 0; \]
moreover,
\[ \lim_{\delta \to 0} A_\delta(u, \Omega) = \int_\Omega |\nabla u|^p \, dx \quad \forall u \in W^{1,p}(\Omega). \]

(ii) Assume in addition that \( \varphi \) satisfies (1.4). Let \( u \in L^p(\Omega) \) be such that
\[ \liminf_{\delta \to 0} A_\delta(u, \Omega) < +\infty, \]
then \( u \in W^{1,p}(\Omega) \).

**Remark 1.** Theorem 1 provides a characterization of the Sobolev space \( W^{1,p}(\Omega) \) for \( p > 1 \):
\[ W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega); \liminf_{\delta \to 0} A_\delta(u) < +\infty \right\}. \]
This fact is originally due to Bourgain and Nguyen [1,4] when \( \varphi = \hat{\varphi}_1 := c\mathbb{I}_{(1,+\infty)} \) for an appropriate constant \( c \).

There are some similarities but also striking differences between the cases \( p > 1 \) and \( p = 1 \).

(a) First note a similarity. Let \( p = 1 \) and \( \varphi \) satisfy (1.2)–(1.4), and assume that \( u \in L^1(\Omega) \) verifies
\[ \liminf_{\delta \to 0} A_\delta(u, \Omega) < +\infty, \]
then \( u \in BV(\Omega) \) (see [1,3]).

(b) Next is a major difference. Let \( p = 1 \). There exists \( u \in W^{1,1}(\Omega) \) such that, for all \( \varphi \) satisfying (1.2)–(1.4), one has
\[ \lim_{\delta \to 0} A_\delta(u, \Omega) = +\infty \]
[3, Pathology 1]. In particular, (1.6) and (1.7) do not hold for \( p = 1 \). An example in the same spirit was originally constructed by Ponce and is presented in [4]. Other pathologies occurring in the case \( p = 1 \) can be found in [3, Section 2.2].

As we will see later, the proof of (1.6) involves the theory of maximal functions. The use of this theory was suggested independently by Nguyen [4] and Ponce and van Schaftingen (unpublished communication to the authors). The proof of (1.6) uses the same strategy as in [4].

We point out that assertion (ii) fails without the monotonicity condition (1.4) on \( \varphi \). Here is an example e.g. with \( \Omega = \mathbb{R} \). Let \( \varphi = c\mathbb{I}_{(1,2)} \) for an appropriate, positive constant \( c \). Let \( u = \mathbb{I}_{(0,1)} \). One can easily check that \( A_\delta(u) = 0 \) for \( \delta \in (0,1/2) \) and it is clear that \( u \not\in W^{1,p}(\mathbb{R}) \) for \( p > 1 \).
Concerning the $Γ$-convergence of $Λ_δ$, our main result is

**Theorem 2.** Let $d ≥ 1$ and $p > 1$. Assume (1.2)–(1.5). Then

$$Λ_δ(\cdot, Ω) Γ\text{-converges in } L^p(Ω) \text{ to } Λ_0(\cdot, Ω) := \kappa \int_{Ω} |\nabla \cdot|^p \, dx,$$

as $δ \to 0$, for some constant $κ$ which depends only on $p$ and $φ$, and verifies

$$0 < \kappa ≤ 1. \quad (1.9)$$

Theorem 2 was known earlier when $φ = φ_1 \ [5,6]$. The paper is organized as follows. Theorem 1 is proved in Section 2 and the proof of Theorem 2 is given in Section 3. Throughout the paper, we denote

$$φ_δ(t) := δ^p φ(t/δ) \text{ for } p > 1, δ > 0, t ≥ 0.$$

**2. Proof of Theorem 1**

In view of the fact that $\liminf_{t \to +∞} φ(t) > 0$, assertion (1.8) is a direct consequence of [1, Theorem 1]; note that [1, Theorem 1] is stated for $Ω = ℝ^d$ but the proof can be easily adapted to the case where $Ω$ is bounded. It could also be deduced from Theorem 2.

We now establish assertions (1.6) and (1.7). The proof consists of two steps.

**Step 1:** Proof of (1.6) and (1.7) when $Ω = ℝ^d$ and $u ∈ W^{1,p}(ℝ^d)$. Replacing $y$ by $x + z$ and using polar coordinates in the $z$ variable, we find

$$\int_{ℝ^d} dx \int_{ℝ^d} \frac{φ_δ(|u(x) − u(y)|)}{|x − y|^{d+p}} \, dy = \int_{ℝ^d} dx \int_{0}^{+∞} dh \int_{S^{d−1}} φ_δ\left(\frac{|u(x + hσ) − u(x)|}{h^{p+1}}\right) \, dσ. \quad (2.1)$$

We have

$$\int_{ℝ^d} dx \int_{0}^{+∞} dh \int_{S^{d−1}} φ_δ\left(\frac{|u(x + hσ) − u(x)|}{h^{p+1}}\right) \, dσ = \int_{ℝ^d} dx \int_{0}^{+∞} dh \int_{S^{d−1}} \frac{δ^p φ\left(|u(x + hσ) − u(x)|/δ\right)}{h^{p+1}} \, dσ. \quad (2.2)$$

Rescaling the variable $h$ gives

$$\int_{ℝ^d} dx \int_{0}^{+∞} dh \int_{S^{d−1}} \frac{δ^p φ\left(|u(x + hσ) − u(x)|/δ\right)}{h^{p+1}} \, dσ = \int_{ℝ^d} dx \int_{0}^{+∞} dh \int_{S^{d−1}} \frac{φ\left(|u(x + δhσ) − u(x)|/δ\right)}{h^{p+1}} \, dσ. \quad (2.3)$$

Combining (2.1), (2.2), and (2.3) yields

$$\int_{ℝ^d} dx \int_{ℝ^d} \frac{φ_δ(|u(x) − u(y)|)}{|x − y|^{d+p}} \, dy = \int_{ℝ^d} dx \int_{0}^{+∞} dh \int_{S^{d−1}} \frac{φ\left(|u(x + δhσ) − u(x)|/δ\right)}{h^{p+1}} \, dσ. \quad (2.4)$$

Note that

$$\lim_{δ \to 0} \frac{|u(x + δhσ) − u(x)|}{δ} = ⟨\nabla u(x), σ⟩ h \text{ for a.e. } (x, h, σ) ∈ ℝ^d × [0, +∞) × S^{d−1}. \quad (2.5)$$

Here and in what follows, $⟨., .⟩$ denotes the usual scalar product in $ℝ^d$. Since $φ$ is continuous at 0 and on $(0, +∞)$ except at a finite number of points, it follows that

$$\lim_{δ \to 0} \frac{1}{h^{p+1}} φ\left(|u(x + δhσ) − u(x)|/δ\right) = \frac{1}{h^{p+1}} φ\left(⟨\nabla u(x), σ⟩ h\right) \text{ for a.e. } (x, h, σ) ∈ ℝ^d × (0, +∞) × S^{d−1}. \quad (2.6)$$
Rescaling once more the variable $h$ gives
\begin{equation}
\int_0^\infty dh \int_{S^{d-1}} \frac{1}{h^{p+1}} \varphi\left(||\nabla u(x), \sigma||h\right) d\sigma = ||\nabla u(x)||^p \int_0^\infty \varphi(t) t^{-(p+1)} dt \int_{S^{d-1}} ||\langle \sigma, e \rangle||^p d\sigma;
\end{equation}
here we have also used the obvious fact that, for every $V \in \mathbb{R}^d$, and for any fixed $e \in S^{d-1}$,
\begin{align*}
\int_{S^{d-1}} |\langle V, \sigma \rangle|^p d\sigma = |V|^p \int_{S^{d-1}} |\langle e, \sigma \rangle|^p d\sigma.
\end{align*}
Thus, by the normalization condition (1.5), we obtain
\begin{equation}
\int_{\mathbb{R}^d} dx \int_0^\infty dh \int_{S^{d-1}} \frac{1}{h^{p+1}} \varphi\left(||\nabla u(x), \sigma||h\right) d\sigma = \int_{\mathbb{R}^d} |\nabla u|^p dx.
\end{equation}
Set
\begin{equation}
\tilde{\varphi}(t) = \begin{cases} 
  at^{p+1} & \text{for } t \in [0, 1), \\
  b & \text{for } t \in [1, +\infty).
\end{cases}
\end{equation}
Then
\begin{equation}
\tilde{\varphi} \text{ is non-decreasing and } \varphi \leq \tilde{\varphi}.
\end{equation}
Note that, for a.e. $(x, h, \sigma) \in \mathbb{R}^d \times (0, +\infty) \times S^{d-1}$,
\begin{equation}
\frac{|u(x + \delta h \sigma) - u(x)|}{\delta} \leq \frac{1}{\delta} \int_0^{h\delta} |\langle \nabla u(x + s\sigma), \sigma \rangle| ds \leq h M(\nabla u, \sigma)(x),
\end{equation}
where
\begin{equation*}
M(\nabla u, \sigma)(x) := \sup_{t > 0} \frac{1}{t} \int_0^t |\langle \nabla u(x + s\sigma), \sigma \rangle| ds.
\end{equation*}
Combining (2.4) and (2.10), we derive from (2.9) that
\begin{align*}
A_\delta(u) & \leq \int_{S^{d-1}} \int_{\mathbb{R}^d} \int_0^\infty \frac{\tilde{\varphi}(h|M(\nabla u, \sigma)(x)|)}{h^{p+1}} dh dx d\sigma \\
& = \int_0^\infty \tilde{\varphi}(t) t^{-(p+1)} dt \int_{S^{d-1}} \int_{\mathbb{R}^d} |M(\nabla u, \sigma)(x)|^p dx d\sigma.
\end{align*}
We claim that, for $\sigma \in S^{d-1}$,
\begin{equation}
\int_{\mathbb{R}^d} |M(\nabla u, \sigma)(x)|^p dx \leq C_p \int_{\mathbb{R}^d} |\nabla u(x)|^p dx.
\end{equation}
For notational ease, we will only consider the case $\sigma = e_1$. By the theory of maximal functions (see e.g. [7]), one has, for $g \in L^p(\mathbb{R})$,
\begin{equation*}
\int_\mathbb{R} \left| \sup_{t > 0} \int_{\xi - t}^{\xi + t} |g(s)| ds \right|^p d\xi \leq C_p \int_\mathbb{R} |g(\xi)|^p d\xi.
\end{equation*}
Using this inequality with $g(x_1) = \partial_{x_1} u(x_1, x')$ for $x' \in \mathbb{R}^{d-1}$, we obtain
\begin{equation*}
\int_\mathbb{R} |M(\nabla u, e_1)(x_1, x')|^p dx_1 \leq C_p \int_\mathbb{R} |\partial_{x_1} u(x_1, x')|^p dx_1 dx_1.
\end{equation*}
Integrating with respect to $x'$ yields
\begin{equation*}
\int_{\mathbb{R}^d} |M(\nabla u, e_1)(x)|^p dx \leq C_p \int_{\mathbb{R}^{d-1}} \int_\mathbb{R} |\partial_{x_1} u(x_1, x')|^p dx_1 dx_1 \leq C_p \int_{\mathbb{R}^d} |\nabla u(x)|^p dx,
\end{equation*}
and (2.12) follows.
Using (2.12), we deduce from (2.11) that

\[ \Lambda_\delta(u) \leq C_{p,d} \int_{\mathbb{R}^d} |\nabla u|^p \, dx, \]

which is (1.6). From (2.6), (2.7), (2.8), and (2.10) we derive, using the dominated convergence theorem, that

\[ \lim_{\delta \to 0} \Lambda_\delta(u) = \int_{\mathbb{R}^d} |\nabla u|^p \, dx. \]

This completes Step 1.

**Step 2**: Proof of (1.6) and (1.7) when \( \Omega \) is bounded and \( u \in W^{1,p}(\Omega) \). We first claim that

\[ \lim_{\delta \to 0} \Lambda_\delta(u) = \hat{\Omega} |\nabla u|^p \, dx \quad \text{for} \quad u \in W^{1,p}(\Omega). \tag{2.13} \]

Indeed, consider an extension of \( u \) in \( \mathbb{R}^d \) which belongs to \( W^{1,p}(\mathbb{R}^d) \), and is still denoted by \( u \). By the same method as in the case \( \Omega = \mathbb{R}^d \), we have

\[ \lim_{\delta \to 0} \int_{\Omega} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{p+d}} \, dy = \int_{\Omega} |\nabla u|^p \, dx \tag{2.14} \]

and, for \( D \Subset \Omega \) and \( \varepsilon > 0 \),

\[ \lim_{\delta \to 0} \int_{D} dx \int_{B(x,\varepsilon)} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{p+d}} \, dy = \int_{D} |\nabla u|^p \, dx \tag{2.15} \]

Combining (2.14) and (2.15) yields (2.13).

We next show that

\[ A_\delta(u) \leq C_{p,\Omega} \int_{\Omega} |\nabla u|^p \, dx \quad \text{for} \quad u \in W^{1,p}(\Omega). \tag{2.16} \]

Without loss of generality, we may assume that \( \int_{\Omega} u = 0 \). Consider an extension \( U \) of \( u \) in \( \mathbb{R}^d \) such that

\[ \int_{\mathbb{R}^d} |\nabla U|^p \, dx \leq C_{p,\Omega} \int_{\Omega} |\nabla u|^p \, dx. \]

Such an extension exists since \( \Omega \) is smooth and \( \int_{\Omega} u = 0 \), see, e.g., [2, Chapter 9]. Using the fact

\[ A_\delta(u, \Omega) \leq A_\delta(U, \mathbb{R}^d) \leq C_{p,d} \int_{\mathbb{R}^d} |\nabla U|^p \, dx, \]

we get (2.16). The proof is complete. \( \square \)

3. **Proof of Theorem 2**

We first recall the meaning of \( \Gamma \)-convergence. One says that \( A_\delta(\cdot, \Omega) \overset{\Gamma}{\rightharpoonup} A_0(\cdot, \Omega) \) in \( L^p(\Omega) \) as \( \delta \to 0 \) if

(G1) For each \( g \in L^p(\Omega) \) and for every family \( (g_\delta) \subset L^p(\Omega) \) such that \( (g_\delta) \) converges to \( g \) in \( L^p(\Omega) \) as \( \delta \to 0 \), one has

\[ \liminf_{\delta \to 0} A_\delta(g_\delta, \Omega) \geq A_0(g, \Omega). \]

(G2) For each \( g \in L^p(\Omega) \), there exists a family \( (g_\delta) \subset L^p(\Omega) \) such that \( (g_\delta) \) converges to \( g \) in \( L^p(\Omega) \) as \( \delta \to 0 \), and

\[ \limsup_{\delta \to 0} A_\delta(g_\delta, \Omega) \leq A_0(g, \Omega). \]
Denote $Q$ the unit open cube, i.e., $Q = (0,1)^d$ and set

$$U(x) = d^{-1/2} \sum_{j=1}^{d} x_j \text{ in } Q,$$

so that $|\nabla U| = 1$ in $Q$.

In the following two subsections, we establish properties (G1) and (G2) where $\kappa$ is the constant defined by

$$\kappa = \inf \liminf_{\delta \to 0} \Lambda_{\delta}(v_{\delta},Q). \quad (3.1)$$

Here the infimum is taken over all families of functions $(v_{\delta}) \subset L^p(Q)$ such that $v_{\delta} \to U$ in $L^p(Q)$ as $\delta \to 0$.

### 3.1. Proof of Property (G1)

We begin with

**Lemma 1.** Let $d \geq 1$, $p > 1$, $S$ be an open bounded subset of $\mathbb{R}^d$ with Lipschitz boundary, and let $g$ be an affine function. Then

$$\inf \liminf_{\delta \to 0} \Lambda_{\delta}(g_{\delta},S) = \kappa |\nabla g|^p |S|, \quad (3.2)$$

where the infimum is taken over all families $(g_{\delta}) \subset L^p(S)$ such that $g_{\delta} \to g$ in $L^p(S)$ as $\delta \to 0$.

**Proof.** The proof of Lemma 1 is based on the definition of $\kappa$ in $(3.1)$ and a covering argument. It is identical to the one of the first part of [3, Lemma 6]. The details are omitted. □

The proof of Property (G1) for $p > 1$ relies on the following lemma with roots in [6].

**Lemma 2.** Let $d \geq 1$, $p > 1$, and $\varepsilon > 0$. There exist two positive constants $\hat{\delta}_1, \hat{\delta}_2$ such that for every open cube $\bar{Q}$ which is an image of $Q$ by a dilation, for every $a \in \mathbb{R}^d$, every $b \in \mathbb{R}$, and every $h \in L^p(\bar{Q})$ satisfying

$$\int_Q |h(x) - (\langle a,x \rangle + b)|^p \, dx \leq \hat{\delta}_1 |a|^p |\bar{Q}|^{p/d}, \quad (3.3)$$

one has

$$\Lambda_{\delta}(h,\bar{Q}) \geq (\kappa - \varepsilon) |a|^p |\bar{Q}| \text{ for } \delta \in (0, \hat{\delta}_2 |a| |\bar{Q}|^{1/d}). \quad (3.4)$$

Hereafter, as usual, we denote $f_A f = \frac{1}{|A|} \int_A f$.

**Proof.** By a change of variables, without loss of generality, it suffices to prove Lemma 2 in the case $\bar{Q} = Q$, $|a| = 1$, and $b = 0$. We prove this by contradiction. Suppose that this is not true. There exist $\varepsilon_0 > 0$, a sequence of measurable functions $(h_n) \subset L^p(Q)$, a sequence $(a_n) \subset \mathbb{R}^d$, and a sequence $(\delta_n)$ converging to 0 such that $|a_n| = 1$,

$$\int_Q |h_n(x) - \langle a_n,x \rangle|^p \leq \frac{1}{n}, \quad \text{and} \quad \Lambda_{\delta_n}(h_n,Q) < \kappa - \varepsilon_0.$$

Without loss of generality, we may assume that $(a_n)$ converges to $a$ for some $a \in \mathbb{R}^d$ with $|a| = 1$. It follows that $(h_n)$ converges to $\langle a, \cdot \rangle$ in $L^p(Q)$. Applying Lemma 1 with $S = Q$ and $g = \langle a, \cdot \rangle$, we obtain a contradiction. The conclusion follows. □

The second key ingredient in the proof of Property (G1) is the following useful property of functions in $W^{1,p}(\mathbb{R}^d)$.
Lemma 3. Let $d \geq 1$, $p > 1$, and $u \in W^{1,p}(\mathbb{R}^d)$. Given $\varepsilon_1 > 0$, there exist a subset $B = B(\varepsilon_1)$ of Lebesgue points of $u$ and $\nabla u$, and an integer $\ell = \ell(\varepsilon_1) \geq 1$ such that

$$
\int_{\mathbb{R}^d B} |\nabla u|^p \, dx \leq \varepsilon_1 \int_{\mathbb{R}^d} |\nabla u|^p \, dx,
$$

(3.5)

and, for every open cube $Q'$ with $|Q'|^{1/d} \leq 1/\ell$ and $Q' \cap B \neq \emptyset$, and for every $x \in Q' \cap B$,

$$
\frac{1}{|Q'|^{p}} \int_{Q'} |u(y) - u(x) - \langle \nabla u(x), y - x \rangle|^p \, dy \leq \varepsilon_1
$$

(3.6)

and

$$
|\nabla u(x)|^p \geq (1 - \varepsilon_1) \int_{Q'} |\nabla u(y)|^p \, dy.
$$

(3.7)

Proof. We first recall the following property of $W^{1,p}(\mathbb{R}^d)$ functions (see e.g., [8, Theorem 3.4.2]): for a.e. $x \in \mathbb{R}^d$,

$$
\lim_{r \to 0} \frac{1}{r^p} \int_{Q(x,r)} |u(y) - u(x) - \langle \nabla u(x), y - x \rangle|^p \, dy = 0,
$$

(3.8)

where $Q(x,r) := x + (-(r,r) \mathbb{R}^d$ for $x \in \mathbb{R}^d$ and $r > 0$.

Given $n \in \mathbb{N}$, define, for a.e. $x \in \mathbb{R}^d$,

$$
\rho_n(x) = \sup \left\{ \frac{1}{r^p} \int_{Q(x,r)} |u(y) - u(x) - \langle \nabla u(x), y - x \rangle|^p \, dy; \ r \in (0,1/n) \right\}
$$

(3.9)

and

$$
\tau_n(x) = \sup \left\{ \int_{Q(x,r)} |\nabla u(y) - \nabla u(x)|^p \, dy; \ r \in (0,1/n) \right\}.
$$

(3.10)

Note that, by (3.8), $\rho_n(x) \to 0$ for a.e. $x \in \mathbb{R}^d$ as $n \to +\infty$. We also have, $\tau_n(x) \to 0$ for a.e. $x \in \mathbb{R}^d$ as $n \to +\infty$ (and in fact at every Lebesgue point of $\nabla u$). For $m \geq 1$, set

$$
D_m = \{ x \in (-m,m)^d; x \text{ is a Lebesgue point of } u \text{ and } \nabla u, \text{ and } |\nabla u(x)| \geq 1/m \}.
$$

Since

$$
\lim_{m \to +\infty} \int_{\mathbb{R}^d \setminus D_m} |\nabla u|^p \, dx = 0,
$$

there exists $m \geq 1$ such that

$$
\int_{\mathbb{R}^d \setminus D_m} |\nabla u|^p \, dx \leq \varepsilon_1 \int_{\mathbb{R}^d} |\nabla u|^p \, dx.
$$

(3.11)

Fix such an $m$. By Egorov’s theorem, there exists a subset $B \subset D_m$ such that $(\rho_n)$ and $(\tau_n)$ converge to $0$ uniformly on $B$, and

$$
\int_{D_m \setminus B} |\nabla u|^p \, dx \leq \frac{\varepsilon_1}{2} \int_{\mathbb{R}^d} |\nabla u|^p \, dx.
$$

(3.12)

Combining (3.11) and (3.12) yields (3.5).

By the triangle inequality, we have, for every non-empty, open cube $Q'$ and a.e. $x \in \mathbb{R}^d$ (in particular for $x \in Q' \cap B$),

$$
\left(\int_{Q'} |\nabla u(y)|^p \, dy \right)^{1/p} \leq \left(\int_{Q'} |\nabla u(y) - \nabla u(x)|^p \, dy \right)^{1/p} + |\nabla u(x)| \leq \frac{|\nabla u(x)|}{(1 - \varepsilon_1)^{1/p}},
$$

(3.13)

provided

$$
\left(\int_{Q'} |\nabla u(y) - \nabla u(x)|^p \, dy \right)^{1/p} \leq \left(\frac{1}{(1 - \varepsilon_1)^{1/p}} - 1 \right) \frac{1}{m} \text{ and } |\nabla u(x)| \geq 1/m.
$$
Since \((\rho_n)\) and \((\tau_n)\) converge to 0 uniformly on \(B\) and \(|\nabla u(x)| \geq 1/m\) for \(x \in B\), it follows from (3.13) that there exists an \(\ell \geq 1\) such that (3.6) and (3.7) hold when \(|Q'|^{1/d} \leq 1/\ell\) and \(Q' \cap B \neq \emptyset\), and \(x \in Q' \cap B\). The proof is complete. \(\square\)

We are ready to give the

**Proof of Property (G1).** We only consider the case \(\Omega = \mathbb{R}^d\). The other case can be handled as in [3] and is left to the reader. We follow the same strategy as in [6].

In order to establish Property (G1), it suffices to prove that

\[
\liminf_{k \to +\infty} A_{\delta_k}(g_k, \mathbb{R}^d) \geq \kappa \int_{\mathbb{R}^d} |\nabla g|^p \, dx
\]  

(3.14)

for every \(g \in L^p(\mathbb{R}^d)\), \((\delta_k) \subset \mathbb{R}_+\) and \((g_k) \subset L^p(\mathbb{R}^d)\) such that \(\delta_k \to 0\) and \(g_k \to g\) in \(L^p(\mathbb{R}^d)\).

Without loss of generality, we may assume that \(\liminf_{k \to +\infty} A_{\delta_k}(g_k, \mathbb{R}^d) < +\infty\). It follows from [6] that \(g \in W^{1,p}(\mathbb{R}^d)\). Fix \(\varepsilon > 0\) (arbitrary) and let \(\hat{\delta}_1\) be the positive constant in Lemma 2. Set, for \(m \geq 1\),

\[A_m = \left\{ x \in \mathbb{R}^d ; \ x \text{ is a Lebesgue point of } g \text{ and } \nabla g, \text{ and } |\nabla g(x)| \leq 1/m \right\}.
\]

Since

\[
\lim_{m \to +\infty} \int_{A_m} |\nabla g|^p \, dx = 0,
\]

there exists \(m \geq 1\) such that

\[
\int_{A_m} |\nabla g|^p \, dx \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla g|^p \, dx.
\]  

(3.15)

Fix such an integer \(m\). By Lemma 3 applied to \(u = g\) and \(\varepsilon_1 = \min\{\varepsilon/2, \delta_1/(2m)^p\}\), there exist a subset \(B\) of Lebesgue points of \(g\) and \(\nabla g\), and a positive integer \(\ell\) such that

\[
\int_{\mathbb{R}^d \setminus B} |\nabla g|^p \, dx \leq \varepsilon_1 \int_{\mathbb{R}^d} |\nabla g|^p \, dx \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla g|^p \, dx,
\]  

(3.16)

and for every open cube \(Q'\) with \(|Q'|^{1/d} \leq 1/\ell\) and \(Q' \cap B \neq \emptyset\), and, for every \(x \in Q' \cap B\),

\[
\frac{1}{|Q'|^{p/d}} \int_{Q'} |g(y) - g(x) - \nabla g(x) \cdot (y - x)|^p \, dy \leq \varepsilon_1 \leq \hat{\delta}_1/(2m)^p
\]  

(3.17)

and

\[
|\nabla g(x)|^p |Q'| \geq (1 - \varepsilon_1) \int_{Q'} |\nabla g|^p \, dy \geq (1 - \varepsilon) \int_{Q'} |\nabla g|^p \, dy.
\]  

(3.18)

Fix such a set \(B\) and such an integer \(\ell\). Set

\[B_m := B \setminus A_m.
\]

Since \(\mathbb{R}^d \setminus (B \setminus A_m) \subset (\mathbb{R}^d \setminus B) \cup A_m\), it follows that

\[
\int_{\mathbb{R}^d \setminus B_m} |\nabla g|^p \, dx = \int_{\mathbb{R}^d \setminus (B \setminus A_m)} |\nabla g|^p \, dx \leq \int_{\mathbb{R}^d \setminus B} |\nabla g|^p \, dx + \int_{A_m} |\nabla g|^p \, dx.
\]

We deduce from (3.15) and (3.16) that

\[
\int_{\mathbb{R}^d \setminus B_m} |\nabla g|^p \, dx \leq \varepsilon \int_{\mathbb{R}^d} |\nabla g|^p \, dx.
\]  

(3.19)
Set $P_\ell = \frac{1}{\ell} \mathbb{Z}^d$. Let $\Omega_\ell$ be the collection of all open cubes with side length $1/\ell$ whose vertices belong to $P_\ell$ and denote $J_\ell = \{Q' \in \Omega_\ell; Q' \cap B_m \neq \emptyset\}$.

Take $Q' \in J_\ell$ and $x \in Q' \setminus B_m$. Since $g_k \to g$ in $L^p(Q')$, from (3.17), we obtain, for large $k$,

$$\frac{1}{|Q'|^{p/d}} \int_{Q'} |g_k(y) - g(x) - \langle \nabla g(x), y - x \rangle|^p \, dy < \frac{\delta_1}{m^p} \leq \frac{\delta_1}{\nabla g(x)}|Q'|^p,$$

since $|\nabla g(x)| \geq 1/m$ for $x \in B_m \subset \mathbb{R}^d \setminus A_m$. Next, we apply Lemma 2 with $\hat{Q} = Q'$, $h = g_k$, $a = \nabla g(x)$, $b = g(x)$, and large $k$; we have

$$A_\delta(g_k, Q') \geq (\kappa - \varepsilon)|\nabla g(x)|^p|Q'|$$

for $\delta \in (0, \frac{\delta_1}{\nabla g(x)}|Q'|^{1/d})$, which implies, by (3.18),

$$\liminf_{k \to +\infty} A_\delta(g_k, Q') \geq (\kappa - \varepsilon)(1 - \varepsilon) \int_{Q'} |\nabla g|^p \, dy.$$  \hfill (3.20)

Since

$$\liminf_{k \to +\infty} A_\delta(g_k, \mathbb{R}^d) \geq \sum_{Q' \in J_\ell} \liminf_{k \to +\infty} A_\delta(g_k, Q'),$$

it follows from (3.20) that

$$\liminf_{k \to +\infty} A_\delta(g_k, \mathbb{R}^d) \geq (\kappa - \varepsilon)(1 - \varepsilon) \sum_{Q' \in J_\ell} \int_{Q'} |\nabla g|^p \, dx \geq (\kappa - \varepsilon)(1 - \varepsilon)^2 \int_{\mathbb{R}^d} |\nabla g|^p \, dx;$$

in the second inequality, we have used the fact $B_m$ is contained in $\bigcup_{Q' \in J_\ell} Q'$ up to a null set. Since $\varepsilon > 0$ is arbitrary, one has

$$\liminf_{k \to +\infty} A_\delta(g_k, \mathbb{R}^d) \geq \kappa \int_{\mathbb{R}^d} |\nabla g|^p \, dx.$$  \hfill (3.19)

The proof is complete. \hfill \Box

3.2. Proof of Property (G2)

The proof of Property (G2) for $p > 1$ is the same as the one for $p = 1$ given in [3]. The details are omitted.

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