Phaseless signal recovery from triple-window short-time Fourier measurements

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Abstract. In this paper, we present a signal recovering algorithm from the magnitudes of its short-time Fourier transform when triple window functions are utilized. Some sufficient conditions are proved on the window functions under which the magnitude of the signal vector can be recovered and the phases of the rest of coordinates can be propagated. Finally, we propose an example to show that the signal vector, up to a unimodular, can be recovered from the magnitudes of its triple-window short-time Fourier transforms.

1. Introduction

Phaseless signal recovery considers recovering the input signal from its magnitudes of the (non)linear measurements, namely phase retrieval. In many applications, only the amplitudes of the sampled input signals can be measured and we are still required to recover the input signals. For example, this often occurs in X-ray, crystallography, transmission electron microscopy and coherent diffractive imaging ([6],[12],[14],[17]). The phase retrieval problem by nature is often ill-posed due to the nonlinear nature of the measurement map A. So it presents great challenges in designing special type of input signals that can provide efficient recovering algorithms. Phase retrieval from the short-time Fourier magnitudes (i.e. finite-dimensional Gabor frames are used for the linear measurements) have been used in several signal processing applications, for example in speech and audio processing ([10],[15]). It has also been applied extensively in optics, for example in frequency resolved optical gating and psychogaphical CDI([11],[13]), which are used to characterize ultra-short laser pulses by optically producing the short-time transform magnitude of the measured pulse.

Many early methods relied on additional information about the sought signal, such as band limitation, nonzero support, and nonnegativity to successfully recover the signal. One of the popular methods for phasless signal recovery is the Gerchberg-Saxton algorithm, which utilized a priori on the support and alternates between the Fourier and inverse Fourier transforms to obtain a phase estimate from a set of Fourier magnitude measurement[8-9]. More recently phase retrieval have been treated using semi-definite relaxation, and low-rank matrix recovery ideas([5],[18]). Papers [6] and [15] considered to signal recovery from the magnitude of short-time Fourier transform over general input sets of inputs, and some necessary/sufficient conditions on stability and uniqueness in real phase retrieval problem were obtained. Recently the authors in ([7],[19-20]) considered the classical 1D phase retrieval problem from the magnitude of short-time Fourier transform for nonvanishing inputs. While their approach provides an efficient linear recovering algorithms when only one window function is used in the short-time Fourier transform, it is not clear whether the same results hold for triple window STFT. In this paper, we show that with the help of the discrete Fourier basis
representations for the magnitude and some involved vectors originated from the triple window functions. We obtain sufficient conditions on the triple window functions that can ensure the phase propagations for non-vanishing signals.

2. Triple-window short-Time Fourier measurement

In this paper, let $\mathbb{Z}$ and $\mathbb{C}$ be the set of integers and complex numbers, respectively. Given a signal $x = (x(0), x(1), \ldots, x(N-1))^T \in \mathbb{C}^N$, we consider a signal $x$ defined on $0 \leq n \leq N-1$ and assume that the triple window $\omega_r = (\omega_r(0), \omega_r(1), \ldots, \omega_r(N-1))^T$ ($r = 0, 1, 2$) are periodically extended over the boundaries.

The triple-window short-time Fourier transform of signal $x$ is given by

$$X_{\omega_r}(3m,k) = \sum_{n=0}^{N-1} x(n)\omega_r(3m-n)e^{-i\frac{2\pi kn}{N}}, \quad r = 0, 1, 2$$

where $\omega_r, r = 0, 1, 2$ is a family of windows.

3. Main results

In this section, we assume that the magnitudes $|X_{\omega_r}(3m,k)|$ of triple-window short-time Fourier measurement are given. Let $x = (x(0), x(1), \ldots, x(N-1))^T \in \mathbb{C}^N$. There exist scalars $\alpha_0, \alpha_1, \ldots, \alpha_t$ such that

$$(|x(0)|^2, |x(1)|^2, \ldots, |x(N-1)|^2)^T = \sum_{n=0}^{N-1} \alpha_0 e^{-2\pi i n/N} \sum_{n=0}^{N-1} \alpha_1 e^{-\frac{2\pi i n}{N}} \sum_{n=0}^{N-1} \alpha_t e^{-\frac{2\pi i (N-1) n}{N}}$$

For every window $\omega_r, r = 0, 1, 2$ and there exist scalars $\beta_{s}^{(r)}, s = 0, 1, \ldots, N-1, r = 0, 1, 2$ such that

$$|\omega_r(3m-n)|^2 = \sum_{s=0}^{N-1} \beta_{s}^{(r)} e^{-2\pi i (3m-n)/N}, \quad n = 0, 1, \ldots, N-1$$

**Theorem 1** Let $T = N/3$ be integer. If matrix $A$ is invertible, then the magnitude of $x$ can be recover uniquely from $|X_{\omega_r}(3m,k)|$, where

$$A = \begin{bmatrix} \beta_{0}^{(0)} & \ldots & \beta_{T}^{(0)} & 0 & \ldots & \beta_{0}^{(1)} & \ldots & \beta_{T}^{(1)} & 0 & \ldots & \beta_{0}^{(r)} & \ldots & \beta_{T}^{(r)} & \ldots & \beta_{0}^{(N-1)} & \ldots & \beta_{T}^{(N-1)} \end{bmatrix}_{N \times N}$$

**Proof:** By Plancherel's theorem for triple window, we obtain...
\[
\sum_{k=0}^{N-1} \left| X_{\omega_j}(3m,k) \right|^2 = N \sum_{n=0}^{N-1} \left| x(n) \right|^2 = N \sum_{n=0}^{N-1} \left| \omega_j(3m,n) \right|^2 \sum_{i=0}^{2N-1} \alpha_i e^{-2\pi i n/N} \\
= N \sum_{n=0}^{N-1} \sum_{i=0}^{2N-1} \alpha_i \left| \omega_j(2m-n) \right|^2 e^{-2\pi i n/N} \\
= N \sum_{n=0}^{N-1} \sum_{i=0}^{2N-1} \alpha_i \beta_i^{(r)} e^{-2\pi i (3m-n)\gamma/N} e^{-2\pi i n/N} \\
= N^2 \sum_{n=0}^{N-1} \alpha_i \beta_i^{(r)} e^{-2\pi i (3m)\gamma/N}
\]

Write \( s = t + Tj \), where \( 0 \leq t \leq T - 1 \), then we have
\[
\sum_{k=0}^{N-1} \left| X_{s_j}(3m,k) \right|^2 = N^2 \sum_{t=0}^{T-1} \left( \alpha_{r+T} \beta_{r+T} \right)^{-1} e^{-2\pi iT\gamma/N} \\
\frac{1}{N^2} \sum_{k=0}^{N-1} \left| X_{s_j}(3m,k) \right|^2 = \left[ \begin{array}{cccc}
\alpha_0 \beta_0^{(r)} + \alpha_T \beta_T^{(r)} \\
\alpha_1 \beta_1^{(r)} + \alpha_{T+1} \beta_{T+1}^{(r)} \\
\vdots \\
\alpha_{T-1} \beta_{T-1}^{(r)} + \alpha_{N-1} \beta_{N-1}^{(r)}
\end{array} \right] \left[ \begin{array}{c}
e^{-2\pi i 0m\gamma/N} \\
e^{-2\pi i 1m\gamma/N} \\
\vdots \\
e^{-2\pi i (T-1)m\gamma/N}
\end{array} \right]
\]

Therefore,
\[
y_m^{(r)} = \alpha_m \beta_m^{(r)} + \alpha_{m+1} \beta_{m+1}^{(r)}, \quad 0 \leq m \leq T-1, \quad 0 \leq r \leq 1
\]

Let \( Y = (y_0^{(0)}, y_1^{(1)}, \ldots, y_{T-1}^{(0)}, y_{T-1}^{(1)})^T \) and \( X = (x_0, x_1, \ldots, x_{N-1})^T \). Then we obtain
\[ Y = AX \]

Since the matrix \( A \) is invertible, this means we can obtain \( |x(n)| \) from \( \left| X_{s_j}(3m,k) \right|^2 \). \( \square \)

Now we discuss how to recover the phase of \( x(n) \). The length \( S \) are defined the minimal size of triple window \( \omega_j (r = 0, 1, 2) \) such that the support of \( \omega_j (r = 0, 1, 2) \) contain in \([a,b] \). The endpoints \( a, b \) are to be interpreted modulo \( N \).

**Theorem 2** Let \( N \) and \( S-1 \) be prime. Suppose \( N \geq 2S-1 \) and \( \omega_j (r = 0, 1, 2) \) satisfy the condition in Theorem 1, then we uniquely determine \( x(n) \).

**Proof:** Let \( Y_j[k] = X_{\omega_j}(3m,k) \) and \( y_j(n) = x(n) x(3m-n) \). Define
\[
Q_j(3m,n) = \sum_{p=0}^{N-1} y_j(p) \overline{y_j(p-n)} 
\]

Then we have
\[
Q_j(3m,n) = \frac{1}{N} \sum_{k=0}^{N-1} X_{s_j}(3m,n) e^{2\pi i n/N} \\
= \sum_{p=0}^{N-1} x(p) g_j(3m-p) x(p-n) g_j(3m-p+n)
\]
By condition $N \geq 2S -1$, we have that $\omega_r (3m - p)(3m - p + S - 1)$ is nonzero if and only if $3m - p = a + r, r = 0, 1, 2$. This implies that

$$Q_r [S - 1, 3m] = x(3m - a - r)\omega_r (a + r)x(3m - a - r - S + 1)\omega_r (a + r + S - 1)$$

Since $N$ and $S - 1$ is coprime, we get

$$x(3m - a) = \frac{|x(3m - a)|}{|x(3m - a - S + 1)|} x(3m - a - S + 1)e^{\arg[x(3m - a) - \arg[x(3m - a - S + 1)]}$$

This allows us to propagate all phase for a given $x(n)$.

\[\square\]

**Example 3** Without loss of generality, suppose that $a = 0$ and $m = 0$, we have

$m = 0: \quad Q_0[2, 0] = x(0)\omega_0(0)x(7)\omega_0(2), \quad Q_1[2, 0] = x(8)\omega_1(1)x(6)\omega_1(3)$,

$m = 1: \quad Q_0[2, 3] = x(3)\omega_0(0)x(1)\omega_0(2), \quad Q_1[2, 3] = x(1)\omega_1(2)x(5)\omega_1(4)$,

$m = 2: \quad Q_0[2, 6] = x(6)\omega_0(0)x(4)\omega_0(2), \quad Q_1[2, 6] = x(4)\omega_1(2)x(2)\omega_1(4)$

Since we assume $x(n) \neq 0$, then we propagate the global phase from arbitrarily $x(0)$.

**Simulation**

We choose rectangle windows as example. Assume $N=29, \quad S=12$. The original signal $x$ with noise $\varepsilon$ can be recovered.

![Figure 1: Phase retrieval from triple-window short-time Fourier measurement](image)

**4. Conclusion**

For the given triple-windows with small supported lengths, the support of a signal could be vital for its phase retrieval from triple-window short-time Fourier transformation. The proof of sufficient conditions give a good result if we have the magnitude of nonzero components of the original signal.
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