Some directional microlocal classes
defined using
wavelet transforms

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Abstract

In this short paper we discuss how the position - scale half-space of wavelet
analysis may be cut into different regions. We discuss conditions under which
they are independent in the sense that the Töplitz operators associated with
their characteristic functions commute modulo smoothing operators. This
shall be used to define microlocal classes of distributions having a well de-
ned behavior along lines in wavelet space. This allows us the description
of singular and regular directions in distributions. As an application we dis-
cuss elliptic regularity for these microlocal classes for domains with cusp-like
singularities.

Key-Words: wavelet transforms, microlocal analysis, elliptic regularity

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1 Introduction

The classical definition of local singular directions of a distribution \( \eta \) is given by the wavefront set (e.g. [Hör82]). At a given point \( x \) it is roughly speaking the cone of all directions in which the Fourier transform of the localized distribution \( \phi \cdot \eta \) does not decay rapidly where \( \phi \) is any smooth function that is supported by some neighborhood of \( x \). More precisely for fixed \( \phi \in C_0^\infty(\mathbb{R}^n) \), \( x \in \text{supp} \phi \), a direction \( \xi \in \mathbb{R}^n - \{0\} \) is regular if in some conic neighborhood \( \gamma \ni \xi \) we have

\[
 k \in \gamma \Rightarrow |(\phi \cdot \eta)^\wedge(k)| \leq c_N(1 + |k|)^{-N}, N = 0, 1, 2, \ldots,
\]

The complement of the regular directions is denoted by \( \Sigma_{x,\phi} \). The singular directions at \( x \) are then defined as

\[
 \Sigma_x = \bigcap_\phi \Sigma_{x,\phi},
\]

where the the intersection is over all \( \phi \in C_0^\infty(\mathbb{R}^n) \) with \( x \in \text{supp} \phi \). However this concept of singular and regular directions does not always fit with what one would intuitively call a regular direction in a distribution. What we mean is best illustrated by an example in \( \mathbb{R}^2 \): consider the set \( K = \{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| \leq x^2\} \) and let \( \chi \) be a function that is of very low regularity in the complement of the cusp \( K \), whereas inside the cusp it is smooth. It is plain to see that there is no direction in which the Fourier transform of the localized function \( \phi \chi \) decays rapidly and so all directions are singular. This however is in contradiction with our intuition in the sense that if we approach the singularity along any path contained in the set \( K \) no irregularity is to be noticed and one would like to call the direction of the cusp regular. Let us refine this example a little more and let us consider a functions that inside \( K \) looks like

\[
 \sin(x^{-\alpha}), \quad \alpha > 1
\]

whereas outside it is very irregular. This function although smooth inside \( K \) becomes less and less smooth as we approach the top of the cusp due to stronger and stronger oscillations. We therefore see, that the directional smoothness we want to study should have at least one more parameter which corresponds to the way how the oscillations accumulate. Therefore let us look at details of size \( a \) inside the cusp at distance \( b \) from the top. In order to see the amount of details increase—that is in order to recover the degree of
increasing non-smoothness near the top of the cusp—we must choose $b$ of the order of magnitude of $a^\alpha$. Otherwise we look at too small a scale compared to the distance, and at this scale our function seems to be uniformly smooth locally.

This is clearly only a very vague statement. To give a more precise definition we have to introduce the wavelet transform. We shall be very brief and we refer to the literature for a more detailed discussion (e.g. [Dau92], [Mey90], [CM90], [Hol95]).

Let $g \in S(\mathbb{R}^n)$, the class of Schwarz of rapidly decaying functions. In addition suppose that $g$ has all moments vanishing
\[ \int dx x^m g(x) = 0 \]
for all multi indices $m$. Then the wavelet transform of $s \in L^p(\mathbb{R}^n)$ with respect to $g$ is defined as the following convolutions
\[ W_g s(b, a) = W[g, s](b, a) = (\tilde{g}_a \ast s)(b) = \int dx \frac{1}{a^n} \overline{g} \left( \frac{x - b}{a} \right) s(x). \quad (1) \]
with $a > 0$ and $b \in \mathbb{R}^n$.

Here we have introduced the following notations, that we shall use in the sequel
\[ \tilde{g}(t) = \overline{g}(-t), \quad g_a = g(\cdot/a)/a^n, \quad g_{b, a} = g_a(\cdot - b). \]
The wavelet transform thus maps functions over the real line to functions over the open half-space $\mathbb{H}^n = \{(b, a) : b \in \mathbb{R}^n, a > 0\}$.

From the definition it is clear that the wavelet transform is a sort of mathematical microscope whose position is fixed by $b$ and whose enlargement is given by $1/a$ or to put it differently $W[g, s](b, a)$ is obtained by “looking at $s$ at position $b$ and at scale $a$”. As a general statement one can say that local regularity of $s$ is mirrored in a certain speed of decay of $W_g s$. So is for instance a uniform (in $b$) decay of $O(a^\infty)$ as $a \to 0$, of the wavelet coefficients equivalent to $C^\infty$ regularity of $s$. More quantitative information is available. So a uniform decay of $O(a^\alpha)$ with $\alpha \in (0, 1)$ is equivalent to $s \in \Lambda^\alpha$, the space of Hölder continuous functions of exponent $\alpha$.

A tentative definition of regular direction at $x$ is therefore any direction $\xi$ for which the wavelet transform decays faster than any power of $a$ if the microscope approaches the singularity along a path that is tangent to $\xi$ in $x$. 

in such a way, that it looks at a scale that is small compared to the distance to $x$. That is we say, vaguely speaking, a direction is regular if along a parabolic line we have rapid decay of the wavelet coefficients

$$W_g s(\lambda \xi, \lambda \gamma) \leq O(\lambda^{\infty}), \quad (\lambda \to 0).$$

This idea will be made more precise in section 4. In particular the definition will we modified in such a way that it becomes independent of the choice of the wavelet $g$.

## 2 The basic formulas of continuous wavelet analysis

For the convenience of the reader we shall list here the basic formulas of wavelet analysis. We limit ourselves to formal expressions. They actually have a precise meaning when we consider the wavelet analysis in $S_0(\mathbb{R}^n)$ or $S'_0(\mathbb{R}^n)$ (see below).

Let $f$ be a complex valued function over $\mathbb{R}^n$. Let $g$ be another such function. The wavelet transform of $f$ with respect to the analyzing wavelet $g$ is defined through (we write $dx$ for $n$-dimensional Lebesgue measure)

$$W_g f(b, a) = \int_{\mathbb{R}^n} dx \frac{1}{a^n} \tilde{f} \left( \frac{x - b}{a} \right) f(x), \quad b \in \mathbb{R}^n, a > 0.$$ 

We also write $W[g, f](b, a)$ instead of $W_g f(b, a)$. Here $b \in \mathbb{R}^n$ is a position parameter and $a \in \mathbb{R}_+$ is a scale parameter. The wavelet transform of a function over $\mathbb{R}^n$ is thus a function over the position scale half-space $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}_+$.

If we introduce the dilation ($D_a$) and translation operators ($T_b$)

$$T_b s(x) = s(x - b), \quad D_a s(x) = s(x/a)/a^n,$$

then we may also write the wavelet transform as a family of scalar products

$$W_g f(b, a) = \langle g_{b,a} \mid f \rangle, \quad g_{b,a} = T_b D_a g,$$

or as a family of convolutions indexed by a scale parameter

$$W_g f(b, a) = (\tilde{g}_a * f)(b), \quad \tilde{g}_a = D_a \tilde{g}, \quad \tilde{g}(x) = \tilde{g}(-x).$$
The convolution product is defined as usual

\[(s \ast r)(x) = \int_{\mathbb{R}^n} dy s(x - y) r(y) = (r \ast s)(x)\]

If we introduce the Fourier transform

\[\hat{s}(k) = \int_{\mathbb{R}^n} dx e^{-ikx} s(x), \quad s(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} dk e^{ikx} \hat{s}(k)\]

then the wavelet transform may also be written as

\[\mathcal{W}_g f(b, a) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} dk \overline{\hat{g}(ak)} e^{ibk} \hat{f}(k).\]

The wavelet synthesis \(\mathcal{M}\) maps functions over the position-scale half-space to functions over \(\mathbb{R}^n\). Let \(r = r(b, a)\) be a complex valued function over \(\mathbb{H}^n\) and \(h\) a function over \(\mathbb{R}^n\). Then the wavelet synthesis of \(r\) with respect to the synthesizing wavelet \(h\) is defined as

\[\mathcal{M}_h r(x) = \int_{\mathbb{H}^n} \frac{db da}{a} r(b, a) \frac{1}{a^n} h\left(\frac{x - b}{a}\right).\]

### 2.1 Relation between \(\mathcal{W}\) and \(\mathcal{M}\)

We now list some relations between \(\mathcal{W}\) and \(\mathcal{M}\). The wavelet synthesis is the adjoint of the wavelet transform—both with respect to the same wavelet—

\[\mathcal{W}_g^* = \mathcal{M}_g\]

\[\int_{\mathbb{H}^n} \frac{db da}{a} \mathcal{W}_g s(b, a) r(b, a) = \int_{\mathbb{R}^n} dx s(x) \mathcal{M}_g r(x).\]

The combination \(\mathcal{M}_h \mathcal{W}_g\) reads in Fourier space

\[\mathcal{M}_h \mathcal{W}_g : \hat{s}(k) \mapsto m_{g,h}(k) \hat{s}(k), \quad m_{g,h}(k) = \int_0^\infty \frac{da}{a} \overline{\hat{g}(ak)} \hat{h}(ak)\]

Note that the Fourier multiplier \(m_{g,h}\) depends only on the direction of \(k\), \(m_{g,h} = m_{g,h}(k/|k|)\). This is because the measure \(da/a\) is scaling invariant. In case that \(g\) and \(h\) are such that

\[m_{g,h}(k) = \int_0^\infty \frac{da}{a} \overline{\hat{g}(ak)} \hat{h}(ak) = c_{g,h}\]
with $0 < |c_{g,h}| < \infty$, we say that $g$, $h$ are an analysis reconstruction pair, or that $h$ is a reconstruction wavelet for $g$. In this case we have

$$\mathcal{M}_h \mathcal{W}_g = c_{g,h} \mathbb{I}.$$  

We say that $g$ is strictly admissible if $g$ is its own reconstruction wavelet, or (what is the same) if

$$\forall k \in \mathbb{R}^n \setminus \{0\} : \int_0^\infty \frac{da}{a} |\hat{g}(ak)|^2 = c_g$$  

A sufficient condition for $g$ to have a reconstruction wavelet $r$ is that for some $c > 1$ we have

$$c^{-1} \leq \int_0^\infty \frac{da}{a} |\hat{g}(ak)|^2 \leq c.$$  

If this condition holds, we call $g$ admissible. In this case the following function $r$ will be a reconstruction wavelet for $g$

$$\hat{r}(k) = \hat{g}(k) / \sqrt{\int_0^\infty \frac{da}{a} |\hat{g}(ak)|^2}.$$  \hspace{1cm} (2)  

If $g$ and $h$ are an analysis reconstruction pair, then the following formula holds

$$\int_{\mathbb{H}^n} \frac{dbda}{a} \mathcal{W}_g s(b, a) \mathcal{W}_h r(b, a) = c_{g,h} \int_{\mathbb{R}^n} dx \ \overline{s(x)} r(x).$$  

In particular if $g$ is strictly admissible, then we have conservation of energy ($c_g = c_{g,g}$)

$$\int_{\mathbb{H}^n} \frac{dbda}{a} |\mathcal{W}_g s(b, a)|^2 = c_g \int_{\mathbb{R}^n} dx \ |s(x)|^2.$$  

Let $\Pi$ and $r$ be functions over the half-space $\mathbb{H}^n$. We define a (non-commutative) convolution for functions over the half-space via

$$(\Pi \ast r)(b, a) = \int_{\mathbb{H}^n} \frac{db'da'}{a'} \frac{1}{a'^n} \Pi \left( \frac{b - b'}{a'}, \frac{a}{a'} \right) r(b', a').$$  \hspace{1cm} (3)  

Formally we can write

$$(\mathcal{W}_g \mathcal{M}_h s)(b, a) = \Pi_{g,h} \ast s(b, a), \quad \Pi_{g,h} = \mathcal{W}_g h.$$
In the case where \( g \) and \( h \) are an analysis reconstruction pair with \( c_{g,h} = 1 \) we clearly have \((W_g M_h)^2 = W_g M_h\) and hence

\[
\Pi_{g,h} \ast \Pi_{g,h} = \Pi_{g,h}.
\]

The mapping \( r \mapsto \Pi_{g,h} \ast r \) is a projector into the range of the wavelet transform \( W_g \). In case that \( g \) is strictly admissible with \( c_{g,g} = 1 \), we have that \( \Pi_{g,g} \) is an orthogonal projector.

The most important formula for our work is the following. Consider a function \( s \) over \( \mathbb{R}^n \). Suppose that \( g \) is admissible. It thus has a reconstruction wavelet \( r \). An explicit formula has been given in 2. Then, since \( W_h = (W_h M_r) W_g \), the wavelet transform of \( s \) with respect to \( g \) and the one with respect to \( h \) are related via the so called cross kernel equation.

\[
W_h s = \Pi_{g \rightarrow h} \ast W_g s, \quad \Pi_{g \rightarrow h} = W_h r. \tag{4}
\]

3 Some function spaces.

All formulas that we have given so far have a well defined meaning if the wavelets are taken in some subset of the class of Schwarz as we will recall now.

3.1 The analysis of \( S_0(\mathbb{R}^n) \)

Let \( S(\mathbb{R}^n) \) denote the class of Schwarz consisting of those functions that together with their derivatives decay faster than any polynomial such that the following norms are all finite for all multi-indices \( \alpha \) and \( \beta \)

\[
\| s \|_{\alpha,\beta} = \sup_{t \in \mathbb{R}^n} |t^\alpha \partial^\beta s(t)| < \infty.
\]

They generate a locally convex topology which makes \( S(\mathbb{R}^n) \) a Fréchet space. We denoted by \( S_0(\mathbb{R}^n) \) the closed set of functions in \( S(\mathbb{R}^n) \) for which all moments vanish

\[
\forall \alpha \in \mathbb{N}^n, \int dx s(x) x^\alpha = 0 \iff \forall m > 0, \ \hat{s}(k) = o(k^m) \ (|k| \to 0).
\]
The Schwarz space of functions over the half-plane \( \mathbb{H}^n \) shall be denoted by \( S(\mathbb{H}) \). It consists of those functions \( r \) for which the following norms are all finite

\[
\|r\|_{k,l,m,n} = \sup_{(b,a) \in \mathbb{H}^n} |(a + 1/a)^k (1 + |b|)^l \partial_b^m \partial_a^n r(b,a)| < \infty.
\]

Note that this means that \( r \) together with all its derivatives decays rapidly for large \( b \) and for large or small \( a \). It can be shown that

\[
\mathcal{W} : S_0(\mathbb{R}^n) \times S_0(\mathbb{R}^n) \to S(\mathbb{H}^n), \quad (g, s) \mapsto \mathcal{W}_g s
\]
is continuous. The same holds for the wavelet synthesis defined through

\[
\mathcal{M}_h r(x) = \int_{\mathbb{H}^n} \frac{dbda}{a} r(b,a) \frac{1}{a^n} h \left( \frac{x-b}{a} \right),
\]
and we have that

\[
\mathcal{M} : S_0(\mathbb{R}^n) \times S(\mathbb{H}^n) \to S_0(\mathbb{R}), \quad (h, r) \mapsto \mathcal{M}_h r
\]
is continuous too. However that in this paper we will not discuss topologies on the microlocal classes we define. This can be done in an obvious way and we want to streamline the discussion.

We note here the following important fact. For admissible \( g \in S_0(\mathbb{R}^n) \), and arbitrary \( h \in S_0(\mathbb{R}^n) \) the crosskernel \([4]\) is a function in \( S(\mathbb{H}^n) \). It thus is very well localized.

### 3.2 Wavelet analysis of \( S'_0(\mathbb{R}^n) \).

We denote the space of linear continuous functionals \( \eta : S_0(\mathbb{R}^n) \to \mathbb{C} \) by \( S'_0(\mathbb{R}^n) \). We consider it together with its natural weak-* topology. The space \( S'_0(\mathbb{R}^n) \) can canonically be identified with \( S'(\mathbb{R})/P(\mathbb{R}^n) \), where \( P(\mathbb{R}^n) \) is the space of polynomials in \( n \) variables. The wavelet transform of \( \eta \in S'_0(\mathbb{R}^n) \) can now be defined pointwise as

\[
\mathcal{W}_g \eta(b,a) = \eta(\mathfrak{f}_{b,a}).
\]

This is a smooth function (e.g. \([\text{Hol}95]\)) that satisfies at

\[
|\mathcal{W}_g \eta(b,a)| \leq c(1 + |b|)^m(a + a^{-1})^m
\]

(5)
for some $m > 0$. By duality we have that the mapping (this time for fixed wavelet $g \in S_0(\mathbb{R}^n)$)

$$\mathcal{W}_g : S'_0(\mathbb{R}^n) \rightarrow S'(\mathbb{H}^n), \quad \eta \mapsto \mathcal{W}_g \eta$$

is continuous. Here $S'(\mathbb{H}^n)$ is the dual of $S(\mathbb{H}^n)$ together with the weak-* topology. Vice versa any let $r \in S'(\mathbb{H}^n)$. Then we may set for $s \in S_0(\mathbb{R}^n)$

$$(\mathcal{M}_h r)(s) = r(\mathcal{W}_h^* s)$$

Clearly again ($h \in S_0(\mathbb{R}^n)$)

$$\mathcal{M}_h : S'(\mathbb{H}^n) \rightarrow S'_0(\mathbb{R}^n), \quad r \mapsto \mathcal{M}_h r$$

is continuous. In the case of a locally integrable function $r$ of at most polynomial growth we have

$$\mathcal{M}_h r(s) = \int_{\mathbb{R}^n} \frac{dbda}{a} r(b, a) \mathcal{W}_h^* s(b, a).$$

### 3.3 More general spaces

Many other function spaces can be characterized in terms of wavelet coefficients. As a rule, the faster the wavelet coefficients decay, the more the analyzed function is regular.

We come to the details. For this consider a vector space $B(\mathbb{H}^n)$ of locally integrable functions

$$S(\mathbb{H}^n) \subset B(\mathbb{H}^n) \subset S'(\mathbb{H}^n).$$

Suppose in addition that $B(\mathbb{H}^n)$ is invariant under convolutions with highly localized kernels

$$r \in S(\mathbb{H}^n), \quad s \in B(\mathbb{H}^n) \quad \Rightarrow \quad r \ast s, \quad s \ast r \in B(\mathbb{H}^n).$$

The convolution of two functions over $\mathbb{H}^n$ is defined through 3. It then makes sense to pull back $B(\mathbb{H}^n)$ to a vector space of distributions over $\mathbb{R}^n$. We shall denote this space of distributions by $B(\mathbb{R}^n)$. It is defined through the following theorem
Theorem 1 For a distribution $\eta \in S'_0(\mathbb{R}^n)$ the following is equivalent.

- There is a wavelet $g \in S_0(\mathbb{R}^n)$ which is admissible in that it satisfies
  \[
  \int_0^{\infty} \frac{da}{a} |\hat{g}(ak)|^2 \sim 1.
  \]
  for which we have
  \[W_g \eta \in B(\mathbb{H}^n).\]

- For all $h \in S_0(\mathbb{R}^n)$ we have
  \[W_h \eta \in B(\mathbb{H}^n).\]

Proof. The passage from $W_g \eta$ to $W_h \eta$ is given by the highly localized cross
kernel. By definition, this operation leaves invariant $B(\mathbb{H}^n)$.
Therefore it makes sense to speak of the space $B(\mathbb{H}^n)$ associated with
$B(\mathbb{H}^n)$. It is precisely the space of distributions for which $W_g \eta \in B(\mathbb{H}^n)$
where $g$ is as given in the theorem.

We still shall need an additional technical assumption on the spaces $B(\mathbb{H}^n)$:
their multiplier algebra should contain the bounded functions
\[m \in L^\infty(\mathbb{H}^n), s \in B(\mathbb{H}^n) \Rightarrow m \cdot s \in B(\mathbb{H}^n).\]

This allows us to define the Töplitz operators
\[\mathcal{M}_h m W_g : B(\mathbb{R}^n) \to B(\mathbb{R}^n).\]

For the rest of the paper we refer to these spaces $B(\mathbb{H}^n)$ satisfying all stated
properties and their pulled back counter part over $\mathbb{R}^n$, $B(\mathbb{R}^n)$, as admissible
local regularity spaces.

We end this section with a remark concerning topology. Suppose that in
addition $B(\mathbb{H}^n)$ is a Banach space. Suppose in addition that it is a Banach
lattice
\[\|s\|_{B(\mathbb{H}^n)} = \|s\|_{B(\mathbb{H}^n)}.\]

Suppose further that for fixed $\Pi \in S(\mathbb{H}^n)$ we have
\[r \mapsto \Pi * r\]
is continuous. Then we can define a norm on $B(\mathbb{R}^n)$ which makes it a Banach space by setting

$$\|s\|_{B(\mathbb{R}^n)} = \|\mathcal{W}_g s\|_{B(\mathbb{H}^n)}.$$  

This is well defined, since for different wavelets satisfying the hypothesis of theorem we obtain equivalent norms. There is an easy to verify sufficient condition for $B(\mathbb{H}^n)$ to be stable under convolution with localized kernels in case that $B(\mathbb{H}^n)$ is a Banach space. It is enough to find $K$ and $c$ such that for all $s \in B(\mathbb{H}^n)$ we can estimate

$$\|s(\alpha + \beta, \alpha')\|_{B(\mathbb{H}^n)} \leq c(\alpha + 1/\alpha)^K (1 + |\beta|)^K.$$ 

Indeed, by a simple change of variables, we may write

$$\|\Pi * s\|_{B(\mathbb{H}^n)} \leq \|s\|_{B(\mathbb{H}^n)} c \int_{\mathbb{H}^n} \frac{d\beta d\alpha}{\alpha} \Pi(\beta, \alpha)(\alpha + 1/\alpha)^K (1 + |\beta|)^K.$$ 

The last integral is a finite constant. For the sake of simplicity, we shall not give a detailed discussion of possible topologies on the microlocal class we are going to define in the next chapter.

### 3.4 Some examples of local regularity spaces

Many function spaces of day to day functional analysis can be characterized with this easy concept. Most of them are contained in the following two scales of spaces. Let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\kappa : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be two tempered weight functions over $\mathbb{R}_+$ and $\mathbb{R}$ respectively. By this we understand that they satisfy at

$$\phi(aa') \leq O(1)(a + 1/a)^n \phi(a'), \quad \kappa(b + b') \leq O(1) (1 + |b|)^n \kappa(b'),$$

for some $n > 0$. Then the following expressions define norms for functions over the half-space $1 \leq p, q \leq \infty$

$$\|r\| = \left\{ \int_0^\infty \frac{da}{a} \phi(a) \left( \int_{-\infty}^{+\infty} db \kappa(b) |r(b,a)|^q \right)^{p/q} \right\}^{1/q}$$

or

$$\|r\| = \left\{ \int_{-\infty}^{+\infty} db \kappa(b) \left( \int_0^\infty \frac{da}{a} \phi(a) |r(b,a)|^q \right)^{p/q} \right\}^{1/q}.$$
The associated Banach spaces are stable under convolution with highly regular kernels and thus they may be pulled back to \( \mathbb{R}^n \) giving rise to two scales of spaces. The first scale of spaces contains the Besov spaces, whereas the second scale contains the \( L^p \)-spaces and Sobolev spaces (see e.g. [Mey90], [CM90], [Hol95], [Tri84] for more details).

For the moment we only note that the space of locally integrable functions in \( \mathbb{H}^n \) for which 
\[
| r(b, a) | \leq c a^\alpha
\]
is stable under convolution with kernels in \( S(\mathbb{H}^n) \). It can thus be pulled back to a space of distributions over \( \mathbb{R}^n \). As is well known by now, these regularity spaces correspond to the Hölder—for \( \alpha > 0, \alpha \not\in \mathbb{N} \)—respectively Zygmund classes—for \( \alpha \in \mathbb{N} \). We shall denote this space by \( \Lambda^\alpha(\mathbb{R}^n) \).

4 Some more general microlocal classes.

In this section we shall be concerned with the problem of constructing new local regularity spaces out of old ones. The idea is easily explained. Consider an arbitrary set \( \Omega \subset \mathbb{H}^n \). Eventually we want to consider lines \( \Omega = \{(b = \lambda \xi, a = \lambda \gamma)\} \) in order to define regularity in the direction \( \xi \in \mathbb{R}^n \). But for the moment we stay general since it renders the discussion more easy to understand. Now fix in addition to \( \Omega \), an admissible, local regularity space \( B(\mathbb{R}^n) \). It is characterized by the fact that the wavelet transforms of its members are in some vector space \( B(\mathbb{H}^n) \). The regularity classes we want to construct are roughly speaking distributions whose wavelet coefficients have on \( \Omega \) a growth behavior governed by the class \( B(\mathbb{H}^n) \).

A slightly more general concept is obtained if we take two local regularity spaces \( B_1(\mathbb{R}^n) \) and \( B_2(\mathbb{R}^n) \), both of the type considered in section 3.2. We now want to cut the half-space \( \mathbb{H}^n \) into two parts, say \( \Omega \) and its complement \( \Omega^c \). In some sense—to be made precise below—we consider classes of distributions whose wavelet coefficients behave inside \( \Omega \) like the wavelet coefficients of functions in \( B_1(\mathbb{R}^n) \), whereas in \( \Omega^c \), these distributions have regularity governed by \( B_2(\mathbb{R}^n) \).

A naive approach might be to require that the restriction of \( \mathcal{W}_b \eta \) to \( \Omega \) satisfies \( \mathcal{W}_b \eta \in B_1(\mathbb{H}^n) \), whereas its restriction to the complement of \( \Omega \) should correspond (via \( \mathcal{M}_\beta \)) to a function in \( B_2(\mathbb{R}^n) \). However this definition might depend on the wavelets we use and thus it is not useful.
To get around this difficulty we first construct a suitable family of neighborhoods for $\Omega$. With these neighborhoods it turns out that we can define vector spaces that are independent of the wavelets $g$ and $h$.

### 4.1 A non-Euclidian distance

The first step of the construction is to introduce a non-euclidian distance function adapted to the geometry of the half-space. A suitable choice is given by dist as defined via

$$\text{dist}((b, a), (b', a')) = |a/a'| + |a'/a| + |(b - b')/a'| + |(b' - b)/a|.$$  

This clearly is not a distance in the usual sense, since $\text{dist}((b, a), (b, a)) = 2$. However a kind of multiplicative triangular inequality holds (see lemma below). Note however that the distance function is symmetric.

$$\text{dist}((b, a), (b', a')) = \text{dist}((b', a'), (b, a)).$$

Clearly the upper half-space carries a natural group structure. It is given by the following composition law

$$(b, a)(b', a') = \delta_a \tau_b (b', a') = (ab' + b, aa').$$

where $\tau_b$ and $\delta_a$ stand for the translation and dilation as left actions of $\mathbb{R}_+$ and $\mathbb{R}^n$ on $\mathbb{H}^n$ via

$$\delta_a : (b, a) \mapsto (\alpha b, \alpha a), \quad \tau_\beta : (b, a) \mapsto (b + \beta, a'),$$

(6)

The inverse element reads

$$(b, a)^{-1} = (-b/a, 1/a),$$

and the neutral element clearly is $(0, 1)$.

If we denote by $\Delta((b, a))$ the distance of a point $(b, a)$ from the point $(0, 1)$

$$\Delta((b, a)) = \text{dist}((b, a), (0, 1)) = a + 1/a + |b| (1 + 1/a)$$

then we have the following relation

$$\text{dist}((b, a), (b', a')) = \Delta((b, a)^{-1}(b', a')).$$
Note also the following identity

$$\Delta((b, a)^{-1}) = \Delta((b, a)).$$

The next lemma shows that a kind of triangular inequality holds

**Lemma 1** We have the following triangular inequalities

$$\max \left\{ \frac{\Delta((b, a))}{\Delta((b', a'))}, \frac{\Delta((b', a'))}{\Delta((b, a))} \right\} \leq \Delta((b, a)(b', a')) \leq \Delta((b, a))\Delta((b', a')).$$

**Proof.** To prove this inequality note that an elementary direct computation shows that

$$aa'\Delta((b, a)^{-1}(b', a')) = aa'\Delta((b' - b)/a, a'/a)$$

$$= a^2 + a'^2 + a|b - b'| + a'|b - b'|$$

$$\leq a^2 + a'^2 + a|b| + a|b'| + a'|b|.$$

On the other hand

$$aa'\Delta((b, a))\Delta((b', a')) = (1 + a^2 + |b| + a|b|)(1 + a'^2 + |b'| + a'|b'|).$$

Therefore the difference between the last and the previous expression is majorized by

$$(1 - a + a^2)|b'| + (1 - a' + a'^2)|b| > 0.$$

The proof of the right most inequality follows now from the identity

$$\Delta((b, a)^{-1}) = \Delta((b, a)).$$

The remaining inequality follows as usual from the previous one, namely

$$\Delta((b, a)) \leq \Delta((b, a)(b', a'))\Delta((b', a')^{-1}) = \Delta((b, a)(b', a'))\Delta((b', a'))$$

This immediately implies the following relation for the distance function

$$\text{dist}((b, a), (b', a')) = \Delta((b, a)^{-1}(b', a')) \leq \Delta((b, a))\Delta((b', a'))$$

and

$$\text{dist}((b, a), (b', a')) \geq \Delta((b, a))/\Delta((b', a'))$$
Thus the following triangular inequality holds
\[ \text{dist}((b, a), (b'', a'')) \leq \text{dist}((b, a), (b', a')) \text{dist}((b', a'), (b'', a'')) \]
and
\[ \text{dist}((b, a), (b'', a'')) \geq \text{dist}((b, a), (b', a')) / \text{dist}((b', a'), (b'', a'')). \]

### 4.2 A family of neighborhoods.

Let us introduce the closed non-euclidian balls
\[ U((b, a), r) = \{ (b', a') \in \mathbb{H} : \text{dist}((b, a), (b', a')) \leq r \} \]

Note that they all are obtained by dilations and translations of the balls around the point \((0, 1)\). More precisely, since the distance function satisfies at
\[ \text{dist}((\gamma b + \beta, \gamma a), (\gamma b' + \beta, \gamma a')) = \text{dist}((b, a), (b', a')) \]
\[ \gamma > 0, \beta \in \mathbb{R}^n, \]
we have
\[ U((b, a), r) = \tau_0 \delta_a U((0, 1), r). \]

An equivalent system of neighborhoods \(U'\) is obtained by translating and dilating the family of balls defined via the following inequalities
\[ (a - [1 + r + 1/(1 + r)]/2)^2 + |b|^2 \leq (1 + r + 1/(1 + r))^2. \]

They are euclidian balls with the "south-pole" at the point \((b = 0, a = 1/(1 + r))\) and the "north-pole" at the point \((b = 0, a = (1 + r))\). The equivalence being expressed by the fact that for some constants \(c > 1\) we have
\[ U'((0, 1), r/c) \subset U(0, 1) \subset U'((0, 1), cr). \]

We may leave the elementary calculations to the reader.

We now are interested in when it makes sense to speak of a certain regularity in one set and an other regularity in an other set of the half-space. Consider therefore two arbitrary subsets \(\Omega, \Sigma \subset \mathbb{H}^n\). We now say that \(\Omega\) and \(\Sigma\) are well separated if the following holds. For \((b, a) \in \Omega\) consider a non-euclidian ball \(U((b, a), r)\) with center \((b, a)\) and radius \(r\). Choose \(r\) small
enough so that $U$ does not meet $\Sigma$. Well separated means for us that for some $\epsilon > 0$ we may choose $r$ such that the following estimate holds true for small $a$

$$r > a^{-\epsilon}.$$ 

In other terms we define more formally and slightly more general

**Definition 1** We say that two sets $\Omega$ and $\Sigma$ are well separated if for some $\epsilon > 0$ we have for

$$(b, a) \in \Omega \Rightarrow \text{dist}((b, a), \Sigma) > \Delta((b, a))^{\epsilon}.$$  

Here the distance between a point and a set $\Omega \subset \mathbb{H}^n$ is defined as usual

$$\text{dist}((b, a), \Omega) = \inf_{(b', a') \in \Omega} \text{dist}((b, a), (b', a')).$$

Note that although the non-euclidian distance diverges at small scale, the euclidian distance might tend to 0 as $a \to 0$. As an example, which is somehow typical, consider in $\mathbb{H}^2$ the sets

$$\Omega = \{a > |b|^\alpha \} \cap \{a < 1/2\}, \quad \Sigma = \{a < |b|\beta \} \cap \{a < 1/2\}.$$ 

They are well separated iff $\alpha > \beta$. However their euclidian distance tends always to 0 as $a \to 0$.

We have still an other useful characterization of well separatedness of two sets $\Omega$ and $\Sigma$. For this consider the sets

$$\delta_{1/a \tau - b} \Omega, \quad (b, a) \in \Sigma.$$ 

Now both sets are well separated iff each of these sets is contained in the complement of a non-Euclidian ball $U((0, 1), r(b, a))$, with $r(b, a) > \Delta((b, a))^{\epsilon}$.

$$\delta_{1/a \tau - b} \Omega \subset U((0, 1), \Delta((b, a))^{\epsilon})^c$$  

(8)

### 4.3 More about well separated sets.

Since the distance function is continuous in the euclidian topology, it is clear that the distance of a point and a set and its euclidian closure are the same. Therefore a set is well separated form an other if and only if its closures are.

The notion of well separated is inherited by subsets. If $\Omega \subset \Sigma$ and $\Sigma$ is well separated from $\Xi$, then $\Omega$ is well separated from $\Xi$ too.

The notion of well separated is symmetric and we have

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Lemma 2 If $\Omega$ is well separated from $\Sigma$ then $\Sigma$ is well separated from $\Omega$.

Proof. By hypothesis we have that $(b, a) \in \Omega$ and $(b', a') \in \Sigma$ implies that
\[
\text{dist}((b, a), (b', a')) > \Delta((b, a))^{\epsilon} \tag{9}
\]
We claim that this implies that
\[
\text{dist}((b, a), (b', a')) > \Delta((b', a'))^{\epsilon'}
\]
for $\epsilon' = \epsilon/(1 + \epsilon)$. For suppose that on the contrary for some points we have
\[
\text{dist}((b, a), (b', a')) = \Delta((b, a)^{-1}(b', a')) \leq \Delta((b', a'))^{\epsilon'}
\]
This implies, via the second triangular inequality, in particular that
\[
\Delta((b', a'))/\Delta((b, a)) \leq \Delta((b', a'))^{\epsilon'}
\]
and therefore by the choice of $\epsilon'$ that $\Delta((b', a'))^{\epsilon'} \leq \Delta((b, a))^{\epsilon'}$. This implies
\[
\text{dist}((b, a), (b', a')) \leq \Delta((b, a))^{\epsilon'},
\]
which is in contradiction with $\tag{9}$.

For every $\epsilon > 0$ let us introduce the following non-Euclidian neighborhoods of a set $\Omega \subset \mathbb{H}^n$
\[
\Gamma_\epsilon(\Omega) = \bigcup_{(b, a) \in \Omega} U((b, a), \Delta((b, a))^{\epsilon}).
\]
The system of such neighborhoods constitutes a fundamental family of neighborhoods in the following sense. We have that $\Gamma_\epsilon(\Omega)^c$ is well separated from $\Omega$. In addition, if $\Sigma$ is well separated from $\Omega$, then for some $\epsilon > 0$ we have that
\[
\Sigma \cap \Gamma_\epsilon(\Omega) = \phi.
\]
Thanks to the triangular inequalities we have the following associativity for the $\epsilon$ neighborhoods.

Lemma 3 For any set $\Omega \subset \mathbb{H}^n$ the following holds true. For $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_3 \geq \epsilon_1 + \epsilon_2(1 + \epsilon_1)$ we have
\[
\Gamma_{\epsilon_2}(\Gamma_{\epsilon_1}(\Omega)) \subset \Gamma_{\epsilon_3}(\Omega).
\]
On the other hand for $\epsilon_2$ such that $\epsilon_2/(1 - \epsilon_2) < \epsilon_1$, there is an $\epsilon_3 > 0$ such that

$$\Gamma_{\epsilon_2}(\Gamma_{\epsilon_1}(\Omega)^c) \supset \Gamma_{\epsilon_3}(\Omega).$$

More precisely it is enough that the following relation holds

$$\epsilon_1 - \epsilon_3 - \frac{\epsilon_2(1 + \epsilon_3)}{(1 - \epsilon_2)} \geq 0.$$

Proof. We show the first part. If $(b'', a'') \in \Gamma_{\epsilon_2}(\Gamma_{\epsilon_1}(\Omega))$ then for some points $(b', a') \in \mathbb{H}^n$ and $(b, a) \in \Omega$ we have

$$\text{dist}((b'', a''), (b', a')) \leq \Delta((b', a'))^{\epsilon_2}, \quad \text{dist}((b', a'), (b, a)) \leq \Delta((b, a))^{\epsilon_1}.$$

Therefore by the triangular relation we have

$$\text{dist}((b'', a''), (b, a)) \leq \Delta((b', a'))^{\epsilon_2} \Delta((b, a))^{\epsilon_1}. $$

Now as before by the reverse triangular inequality we have

$$\Delta((b', a'))/\Delta((b, a)) \leq \Delta((b, a))^{\epsilon_1}$$

and therefore finally as claimed

$$\text{dist}((b'', a''), (b, a)) \leq a^{-\epsilon_1-\epsilon_2(1+\epsilon_1)},$$

The second statement may be rephrased as follows: if for all $(b, a) \in \Omega$ we have

$$\text{dist}((b, a), (b'', a'')) \leq \Delta((b, a))^{\epsilon_3} \tag{10}$$

then $(b'', a'') \not\in \Gamma_{\epsilon_2}(\Gamma_{\epsilon_1}(\Omega)^c)$. Suppose that the contrary is true. Then for some $(b', a') \in \mathbb{H}^n$ satisfying at

$$\forall (b, a) \in \Omega \quad \text{dist}((b, a), (b', a')) > \Delta((b, a))^{\epsilon_1},$$

we have

$$\text{dist}((b'', a''), (b', a')) \leq \Delta((b, a))^{\epsilon_2}. \tag{11}$$

Therefore we have by the triangular inequality

$$\Delta((b, a))^{\epsilon_1} < \text{dist}((b, a), (b', a'))$$

$$\leq \text{dist}((b, a), (b'', a'')) \text{dist}((b'', a''), (b', a'))$$

$$\leq \Delta((b, a))^{\epsilon_3} \Delta((b', a'))^{\epsilon_2}.$$
Now, by the reverse triangular inequality, \( \Delta((b'', a''))/\Delta((b, a)) \leq \Delta((b, a))^{\epsilon_3} \).

Now again by the reverse triangular inequality \([\square]\) implies
\[\Delta((b', a'))/\Delta((b'', a'')) \leq \Delta((b', a'))^{\epsilon_2}.\]

It follows that
\[\Delta((b', a')) \geq \Delta((b, a))^{(1+\epsilon_3)/(1-\epsilon_2)}.\]

Therefore
\[1 < \Delta((b, a))^{\epsilon_3 - \epsilon_1 + \epsilon_2 (1+\epsilon_3)/(1-\epsilon_2)},\]
which is impossible by the choice of \(\epsilon_3\) and since \(\Delta((b, a)) \geq 2\). \(\square\)

This immediately implies the following corollary that we shall use in the next section.

**Lemma 4** Let \(\Sigma \supset \Omega\) be such that \(\Sigma^c\) is well separated from \(\Omega\). Then there is a set \(\Xi\), \(\Sigma \supset \Xi \supset \Omega\) such that \(\Xi\) is well separated from \(\Sigma^c\) and \(\Omega\) is well separated from \(\Xi^c\).

**Proof.** Some \(\Gamma_\epsilon(\Omega)\) with \(\epsilon\) small enough will do. \(\square\)

### 4.4 Cutting the half-space.

Let us come back to our original goal of dividing the half-space into two set of different regularity. As we said already, it is not possible to speak of regularity \(B(\mathbb{H}^n)\) inside a given set \(\Omega \subset \mathbb{H}^n\), since notion is not independent under highly regular Calderón Zygmund operators, or to put it simpler, it might depend on the given wavelet we use for the definition.

However if we require regularity \(B(\mathbb{H}^n)\) in a region that that is slightly larger than \(\Omega\), it then follows that the same regularity holds true in \(\Omega\) for any wavelet.

We denote by abuse of notation \(\Sigma\), (respectively \(\Omega\)) the operator that restricts functions over \(\mathbb{H}^n\) to the set \(\Sigma\) (respectively \(\Omega\)). That is we have
\[\Sigma : r \mapsto \chi_{\Sigma} r,\]
where \(\chi_{\Sigma}\) is the characteristic function of \(\Sigma\).
Theorem 2 Consider two sets $\Sigma$ and $\Omega$ and suppose that $\Sigma \supset \Omega$ in such a way that $\Sigma^c$ and $\Omega$ are well separated. Let $g, g', h, h' \in S_0(\mathbb{R}^n)$ satisfy for some $c > 1$ $(s = g, g', h, h')$,

$$c^{-1} < \int_0^\infty \frac{da}{a} |\hat{s}(ak)|^2 < c.$$  

Suppose that $\eta \in S'(\mathbb{R}^n)$ satisfies at

$$M_h \Sigma W_g \eta \in B(\mathbb{R}^n)$$

Then

$$M_{h'} \Omega W_{g'} \eta \in B(\mathbb{R}^n).$$

Therefore it makes sense to separate $\mathbb{H}^n$ into regions of different regularity provided, the regions are well separated.

The proof is based on the following lemma. It estimates the influence under convolution operators over the half-plane of a nasty function inside some region $\Xi$ on $\Xi'$ when both are well separated.

Lemma 5 Suppose $\Xi'$ and $\Xi$ are well separated. Let $r$ be a locally integrable function over $\mathbb{H}^n$ that is equal to 0 except on $\Xi'$, where it satisfies for some $M > 0$ and some $c > 0$ $(b, a) \in \Xi' \Rightarrow |r(b, a)| \leq c \Delta((b, a))^M$.

Then for $\Pi \in S(\mathbb{H}^n)$ we have that $\tilde{r} = \Pi * r$ satisfies at

$$(b, a) \in \Xi \Rightarrow |\tilde{r}(b, a)| \leq c_k \Delta((b, a))^{-k}$$

for all $k \in \mathbb{N}$.

Proof. We have to estimate the localization of $\Pi * r(b, a)$ for $(b, a) \in \Xi$. By definition $\Pi * r(b, a)$ equals

$$\int_{\Xi'} \frac{db'da'}{a'} a^m \Pi \left( \frac{b-b'}{a'}, \frac{a}{a'} \right) r(b', a').$$

By dilation and translation we also may write using the action (6) of dilation and translation on $\mathbb{H}^n$

$$\int_{\delta_{1/a} \tau - b \Xi'} \frac{db'da'}{a'} \Pi(b', a') r(a'b' + b, aa').$$
Now by hypothesis on \( r \) we may write with some \( K \) and some \( c > 0 \)
\[
|r(a'b + b, \alpha a')| \leq c \Delta((b, a))^K \Delta((b', a'))^K.
\]

Plugging this estimation into the previous expression we have to estimate for \((b, a) \in \Xi\)
\[
\Delta((b, a))^K \int_{\delta_{1/a} - b, a' \Xi} \frac{db'da'}{a'} |\Pi'(b', a')|\]
with \( \Pi'(b', a') = \Delta((b', a'))^K \Pi(b', a') \). Together with \( \Pi \) we have that \( \Pi' \) is highly localized. For \( \lambda \geq 0 \) let us look at the following integral running over the complement of a non-euclidian ball centered at \((0, 1)\)
\[
F(\lambda) = \int_{\Delta((b', a')) > \lambda} \frac{db'da'}{a'} |\Pi'(b', a')|.
\]

Thanks to the high localization of \( \Pi' \), this function is faster decaying than any power of \( \lambda \) as \( \lambda \to 0 \). Now, since \( \Xi \) and \( \Xi' \) are well separated we may use characterization 8 to conclude that the integral in 12 is estimated by \( F(\Delta((b, a))^{\epsilon}) \) for some \( \epsilon > 0 \). But this function is again rapidly decaying as \( \Delta((b, a)) \to \infty \) and the proof is finished. \( \square \)

Note that the lemma we went to prove may be rephrased as follows: for all \( \Pi \in S(\mathbb{H}^n) \) we have that
\[
s \mapsto \Xi'(\Pi \ast (\Xi s))
\]
is infinitely smoothing in the sense that it maps functions of polynomial growth into rapidly decaying functions over the half-space.

**Proof.** (of theorem 2) These previous considerations imply the following: if we have that \( \Sigma \mathcal{W}_g \eta \in B(\mathbb{H}^n) \), with \( g \) admissible, then for all \( g' \in S_0(\mathbb{R}) \) we have that \( \Omega \mathcal{W}_{g'} \eta \in B(\mathbb{H}^n) \). To show this note the passage from \( \mathcal{W}_g \eta \) to \( \mathcal{W}_{g'} \eta \) is done by convolving with a highly localized kernel \( \Pi \). Now we may write
\[
\Omega \mathcal{W}_{g'} \eta = \Omega(\Pi \ast (\Sigma \mathcal{W}_g \eta)) + \Omega(\Pi \ast (\Sigma' \mathcal{W}_g \eta)).
\]

Since by hypothesis \( B(\mathbb{H}^n) \) is invariant under multiplications with bounded functions and convolutions with \( \Pi \) the first term is again in \( B(\mathbb{H}^n) \), whereas the second term is arbitrary smooth.
A slightly more complicated situation occurs in our theorem, since we
can not conclude from $M_h r \in B(\mathbb{R}^n)$ that $r \in B(\mathbb{H}^n)$ since the wavelet
synthesis is not injective.

Now we can find a set $\Xi$ between $\Sigma$ and $\Omega$
$$\Omega \subset \Xi \subset \Sigma,$$
such that $\Xi$ is well separated from the complement of $\Sigma$ and $\Omega$ is well separated
from the complement of $\Xi$. This follows from lemma [4]. We may conclude
that
$$\Xi \mathcal{W}_g M_h \Sigma \mathcal{W}_g \eta = \Xi(\Pi_1 *(\Sigma \mathcal{W}_g \eta)) \in B(\mathbb{H}^n).$$
Where $\Pi_1 = \mathcal{W}_f h$ for any admissible $f \in S_0(\mathbb{R}^n)$. In particular we may
choose $f$ to be a reconstruction wavelet for $g$ and thus it follows that $\Pi_1 \ast
\mathcal{W}_g \eta = \mathcal{W}_g \eta$. Now writing (as characteristic functions!) $\Sigma = 1 - \Sigma^c$ the last
expression equals
$$\Xi(\Pi_1 \ast \mathcal{W}_g \eta) - \Xi(\Pi_1 \ast (\Sigma^c \mathcal{W}_g \eta)).$$
The set $\Xi$ is well separated from $\Sigma^c$ and thus the second term has rapid
decay as $\Delta((b, a))$ gets large. Let us call this function $u$. Then since $u$ is well
localized we have
$$r \ast u \in S(\mathbb{H}^n)$$
for all $r \in S(\mathbb{H}^n)$. We therefore obtain, up to a function of rapid decay
$$\Xi \mathcal{W}_g \eta \in B(\mathbb{H}^n).$$
Now $\mathcal{W}_g \eta = \Pi \ast \mathcal{W}_g \eta$ for some $\Pi \in S(\mathbb{H}^n)$. Therefore, since $\Omega \subset \Xi$ is well
separated from the complement of $\Xi$ we have as at the beginning of the proof
that $\Omega \mathcal{W}_g \eta \in B(\mathbb{H}^n)$ up to the well localized function $\Omega u$. But then clearly
$\mathcal{M}_h \Omega \mathcal{W}_g \eta \in B(\mathbb{R}^n)$. \hfill \Box

Let $\Sigma \supset \Omega$ be open and let again $\Omega$ be well separated from the complement
of $\Sigma$ as before. Consider the two Töplitz operators
$$T_{\Sigma} = \mathcal{M}_h \Sigma \mathcal{W}_g, \quad T_{\Omega} = \mathcal{M}_h \Omega \mathcal{W}_g.$$ We then have proved the following

**Corollary 1** We have that
$$[T_{\Sigma}, T_{\Omega}] = T_{\Sigma} T_{\Omega} - T_{\Omega} T_{\Sigma}$$
is infinitely smoothing in the sense that it maps the tempered distributions in
$S'_0(\mathbb{R}^n)$ into smooth function in $S_0(\mathbb{R}^n)$.
4.5 Some microlocal classes.

The theorems of the previous section may be used to define some very general micro-local classes. Suppose we are given two regularity spaces $A(\mathbb{R}^n)$ and $B(\mathbb{R}^n)$ and Suppose in addition that $B(\mathbb{R}^n) \subset A(\mathbb{R}^n)$. Consider a set $\Omega \subset \mathbb{H}^n$. Since we are only interested in local properties we may suppose that $\Omega$ is bounded in the euclidian norm. In order to avoid technicalities we suppose $\Omega$ is closed.

The first type of local regularity classes corresponds to the idea the globally, a distribution has a regularity described by $A(\mathbb{H}^n)$ whereas locally, in $\Omega$ we have some higher regularity of type $B(\mathbb{H}^n)$.

A dual idea would be to have the wavelet coefficients concentrated on the subset $\Omega$. That is that outside of $\Omega$, the wavelet coefficients are small, hence correspond to the higher regularity $B(\mathbb{H}^n)$, whereas inside $\Omega$, the coefficients are in $A(\mathbb{H}^n)$.

We now want to make these statements more precise. Consider first the case of higher local regularity. Suppose that there is a sequence of closed sets $\{\Omega_k\}$, $k = 1, 2, \ldots$ with

$$\Omega_1 \subset \ldots \subset \Omega_k \subset \Omega_{k+1} \ldots \subset \Omega.$$ 

We suppose that $\Omega_k$ converges to $\Omega$ in the sense that

$$\Omega = \bigcup_k \Omega_k.$$ 

Suppose that $\Omega_k^c$ and $\Omega_{k+1}$ are well separated for each $k$. Then clearly $\Omega_k^c$ and $\Omega_l$ are well separated for $k < l$. We then say that $\eta \in S_0'(\mathbb{R}^n)$ belongs to the microlocal class $\Omega_{A,B}$ iff for some admissible $g$ and all $k$ we have

$$\eta \in A(\mathbb{R}^n), \quad \text{and} \quad \mathcal{M}_g \Omega_k \mathcal{W}_g \eta \in B(\mathbb{R}^n).$$

By the results of the previous theorem it is clear that the definition does not depend on the specific wavelets nor on the family of approximating sets $\Omega_k$. Indeed, by lemma 3 we may take the family $\Gamma_{1/k}(\Omega)$ as universal family of approximating sets.

Note however that for arbitrary $\Omega$, the previous class might coincide with $A(\mathbb{R}^n)$. Indeed, in order to have an approximating sequence from the interior, of mutually well separated sets the “smoother” region can not be arbitrary thin. It must contain at least some non-euclidian neighborhood of some set.
Frequently one takes $A = S_0'(\mathbb{R}^n)$ in which case one is only interested in the behavior of the wavelet coefficients around $\Omega$. In the next section we shall use this kind of classes to define directional regularity in distributions.

Consider now the dual approach, where we want to formalize the idea of wavelet coefficients concentrated on $\Omega$. Suppose now that a sequence of open sets $\{\Omega_k\}, k = 1, 2, \ldots$ with

$$\Omega_1 \supset \ldots \supset \Omega_k \supset \Omega_{k+1} \ldots \supset \Omega,$$

converges to $\Omega$ in the sense that

$$\Omega = \bigcap_k \Omega_k.$$  

Again we require that $\Omega_k^c$ and $\Omega_{k+1}^c$ are well separated for each $k$. We then say that $\eta \in S_0'(\mathbb{R}^n)$ belongs to the microlocal class $\Omega^{A,B}$ iff for some admissible $g$ and all $k$ we have

$$\eta \in A(\mathbb{R}^n), \quad \text{and} \quad \mathcal{M}_g \Omega_k^c \mathcal{W}_g \eta \in B(\mathbb{R}^n).$$

Note that in the case where $B(\mathbb{R}^n) = S_0(\mathbb{R}^n)$, this corresponds to the idea having the wavelet coefficients concentrated on the set $\Omega$, where they satisfy the less restrictive regularity estimate given by $A(\mathbb{H}^n)$.

Note again that the region that contains the wavelet coefficients corresponding to the smoother behavior can not be arbitrary thin. However the set $\Omega$ on which the wavelet coefficients are concentrated is arbitrary.

5 Some directional microlocal classes.

We now propose to look at more specific examples of regularity classes. In particular to those we mentioned in the beginning of the paper, that is to classes related to the notion of singular or regular directions in distributions.

Particular useful examples arise when we consider parabolic regions or lines in wavelet space. As measure of regularity it is useful to consider the Hölder-Zygmund scale $\Lambda^\alpha$ of spaces defined in wavelet space via

$$\|s\|_\alpha = \sup_{(b,a) \in \mathbb{H}^n} |a^{-\alpha} s(b,a)|$$
Now fix a vector $\xi \in \mathbb{R}^n$, $|\xi| > 0$ and consider the set
\[ \Xi = \Xi(\xi, \gamma) = \{(b = \lambda \xi, a = \lambda^\gamma) : 1/2 > \lambda > 0\} \]
for some $\gamma > 1$. We now say that $\eta \in \Lambda^\alpha(\mathbb{R}^n)$ is locally of type $(\alpha, \xi, \gamma)$ if it belongs to the microlocal class $\Omega^{A,B}$ with $\Omega = \Gamma_\epsilon(\Xi(\xi, \gamma))$ for some $\epsilon > 0$, $A = S(\mathbb{R}^n)$ and $B = \Lambda^\alpha(\mathbb{R}^n)$. Explicitly, this means—let us recall it once more—that the wavelet transform of $s$ satisfies for some $\epsilon > 0$ at
\[ (b, a) \in \Gamma_\epsilon(\Xi(\xi, \gamma)) \Rightarrow |W g \eta(b, a)| \leq ca^\alpha. \]
and
\[ (b, a) \notin \Gamma_\epsilon(\Xi(\xi, \gamma)) \Rightarrow |W g \eta(b, a)| \leq c(a + 1/a)^k(1 + |b|)^k. \]
This corresponds to looking at behavior of the wavelet coefficients under the following non-homogeneous dilations
\[ W g \eta(\lambda \xi, \lambda^\gamma), \quad \lambda > 0. \]
Here $c$ depends on $\epsilon$ only. Actually we may choose $\xi$ such that $|\xi| = 1$. Indeed suppose $\xi' = \beta \xi$ with $\beta > 0$ and denote by $\Xi'$ the corresponding line. Then if $(b, a) \in \Xi$ it follows that $(b, \beta^{-\alpha}a) \in \Xi'$. Therefore the non-euclidean distance between the two points is uniformly bounded by $\beta^\alpha + \beta^{-\alpha}$. Therefore, they define the same micro-local classes.

Let us explain in which sense these classes are linked to singular and regular directions. For this replace for the moment the wavelet at position $b$ and scale $a$ by the characteristic function of the euclidian ball centered at $b$ and of radius $a$. As $(b, a)$ tends to $(0, 0)$, while always $(b, a) \in (\Omega(\xi, \gamma))$, the support of the wavelets is contained in a cusp-like region, around the line in direction $\xi$. This shows, that the micro-local class $(\alpha, \xi, \gamma)$ quantifies the regularity of $s$ in direction $\xi$.

### 5.1 Some elliptic regularity

We now want to apply the classes introduced above to a problem of elliptic regularity. For the sake of simplicity, we only discuss the Laplace equation and leave more general elliptic operators for subsequent papers. We say an open domain $\Omega \subset \mathbb{R}^n$ satisfies the cusp condition of degree $\delta > 0$ at $x \in \partial \Omega$ in direction $\xi$, $\xi \in \mathbb{R}^n - \{0\}$ if there is some $c > 0$ such that
\[ \{y : |y - x| \leq c |(y - x \cdot \xi)|^\delta, |y - x| \leq c\} \subset \Omega \]
Theorem 3  Let $\Omega$ satisfy a cusp condition at 0 of type $\delta$ in direction $\xi$. Suppose that $\eta$ is a tempered distribution that satisfies inside $\Omega$ at

$$\Delta \eta = f$$

for some distribution $f$ supported by $\overline{\Omega}$. Now if $f$ is of type $(\xi, \gamma, \alpha)$, with $\gamma > \delta$, then it follows that $\eta$ is of type $(\xi, \gamma, \alpha + 2)$.

This theorem is a special case of a more general theorem. Let $B(\mathbb{R}^n)$ be a local regularity space of the type we have considered before with $B(\mathbb{H}^n)$ the associated space of functions over $\mathbb{H}^n$. It is plain to see that together with $B(\mathbb{H}^n)$, the space of functions which consists the functions $a^\gamma r(b, a)$ with $r \in B(\mathbb{H}^n)$ is again an admissible regularity space. It shall be denoted by $(a^\gamma B)(\mathbb{H}^n)$ respectively $(a^\gamma B)(\mathbb{R}^n)$.

For a set $\Omega \subset \mathbb{R}^n$ we consider the set

$$\bigcup_{b \in \Omega} K(b),$$

where $K(b) \subset \mathbb{H}^n$ is the cone of opening angle 1 with top in $b$

$$K(b) = \{ (\beta, \alpha) : |\beta - b| \leq \alpha \}.$$

We call this set the influence region of $\Omega$ in the upper half-space.

The general theorem can now be stated as follows.

Theorem 4  Let $\Omega \subset \mathbb{R}^n$ be open. Suppose that $\Xi \subset \mathbb{H}^n$ is well separated from the influence region of $\Omega$. Suppose that $\Xi' \subset \Xi$ such that $\Xi'$ and $\Xi^c$ are well separated. Suppose that $\eta$ is a tempered distribution that satisfies inside $\Omega$ at

$$\Delta \eta = f$$

for some distribution $f \in S(\mathbb{R}^n)$ supported by $\overline{\Omega}$. If now $f$ satisfies at

$$\mathcal{M}_g \Xi \mathcal{W}_g f \in B(\mathbb{R}^n),$$

with some admissible wavelet $g \in S_0(\mathbb{R}^n)$, then it follows that

$$\mathcal{M}_g \Xi' \mathcal{W}_g \eta \in (a^2 B)(\mathbb{R}^n),$$
Proof. We may suppose that \( g \in S_0(\mathbb{R}^n) \) be spherically symmetric, \( g = g(|x|) \). Then with \( h = \Delta g \) we may write

\[
W_g \Delta \eta = -a^{-2} W_h \eta.
\]

Now \( g \) and \( h \) are both admissible in the sense that

\[
\int_0^\infty \frac{da}{a} |\hat{g}(ak)|^2 \sim \int_0^\infty \frac{da}{a} |\hat{h}(ak)|^2 \sim 1
\]

From this it follows immediately that \( s \in S'(\mathbb{R}^n) \) and \( \Delta s \) satisfying at

\[
\mathcal{M}_g \Xi W_g \Delta s \in B(\mathbb{R}^n),
\]

implies that \( s \) satisfies at

\[
\mathcal{M}_g \Xi' W_g s \in (a^2 B)(\mathbb{R}^n).
\]

The theorem is therefore proved if we can show that any distribution \( f' \) that coincides with \( f \) inside \( \Omega \) satisfies again at

\[
\mathcal{M}_g \Xi W_g \Delta f \in B(\mathbb{R}^n).
\]

But this is follows from the next lemma, which justifies the name influence region for \( \Omega \).

**Lemma 6** Let \( \Xi \subset \mathbb{H}^n \) be well separated from the influence region of \( \Omega \). Then for all \( \rho \in S'(\mathbb{R}^n) \) with support in \( \Omega \) we have that

\[
\mathcal{M}_g \Xi W_g \rho \in S_0(\mathbb{R}^n).
\]

Proof. By hypothesis there is an \( \epsilon > 0 \) such that for \((b, a) \in \Gamma_\epsilon(\Xi)\) there is a Euclidian ball of radius

\[
\Delta((b, a))^{\epsilon}
\]

around \( b \) that is contained in the complement of the influence region of \( \Omega \). Denote by \( \phi \) a \( C^\infty \) function which is identically 1 on the complement of the unit ball of radius 1 and which is supported on the complement of a slightly smaller ball. Then denote by \( \phi_{b,a} \) the family of translates and dilates of \( \phi \). Therefore if \( \rho \) is supported by \( \Omega \), it follows that

\[
(b, a) \in \Xi \Rightarrow W_h \rho(b, a) = W_g(\phi_{b,\Delta((b,a))^{\epsilon}} \rho)(b, a) = \left\langle g_{b,a} \phi_{b,\Delta((b,a))^{\epsilon}} | \rho \right\rangle
\]
But now for every $\epsilon > 0$ we have that $g_{b,a} \phi_{b,\Delta((b,a))}^\epsilon$ tends to 0 as $\Delta((b,a)) \to \infty$ in $S(\mathbb{R}^n)$ in such a way that for all semi-norms in $S_0(\mathbb{R}^n)$ and all $M > 0$ we have

$$\|g_{b,a} \phi_{b,\Delta((b,a))}^\epsilon\|_{l,m} \leq c_{l,m,M} \Delta((b,a))^{-M}$$

This proofs the lemma

The theorem is proved.

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