Abstract. We study the specification and verification of hyperproperties for probabilistic systems represented as Markov decision processes (MDPs). Hyperproperties are system properties that describe the correctness of a system as a relation between multiple executions. Hyperproperties generalize trace properties and include information-flow security requirements, like noninterference, as well as requirements like symmetry, partial observation, robustness, and fault tolerance. We introduce the temporal logic PHL, which extends classic probabilistic logics with quantification over schedulers and traces. PHL can express a wide range of hyperproperties for probabilistic systems, including both classical applications, such as differential privacy, and novel applications in areas such as robotics and planning. While the model checking problem for PHL is in general undecidable, we provide methods both for proving and for refuting a class of probabilistic hyperproperties for MDPs.

1 Introduction

Ten years ago, Clarkson and Schneider coined the term hyperproperties [7] for the class of properties that describe the correctness of a system as a relation between multiple executions. Hyperproperties include information-flow security requirements, like noninterference [18], as well as many other types of system requirements that cannot be expressed as trace properties, including symmetry, partial observation, robustness, and fault tolerance. Over the past decade, the CAV community has developed a rich set of tools for the specification and verification of hyperproperties. HYPERLTL and HYPERCTL∗ [8] are extensions to LTL and CTL∗ that can express a wide range of hyperproperties. There are a number of algorithms and tools for hardware model checking [15,17,10], satisfiability checking [14], monitoring [6,20], and reactive synthesis [13] for hyperproperties.

The natural next step is to consider probabilistic systems. Randomization plays a key role in the design of security-critical and distributed systems. In fact, randomization is often added specifically to implement a certain hyperproperty. For example, randomized mutual exclusion protocols use a coin flip to decide which process gets access to the critical resource in order to avoid breaking the
symmetry based on the process id \[2\]. Databases employ privacy mechanisms based on randomization in order to guarantee (differential) privacy \[12\].

Previous work on probabilistic hyperproperties \[1,12,27,28\] has focused on the specification and verification of probabilistic hyperproperties for Markov chains. The temporal logic HyperPCTL \[1\] extends the standard probabilistic logic PCTL with quantification over states. For example, the HyperPCTL formula

\[
\forall \sigma \forall \sigma'. (\text{init}_\sigma \land \text{init}_{\sigma'}) \rightarrow \mathbb{P}(\Box \text{terminate}_\sigma) = \mathbb{P}(\Box \text{terminate}_{\sigma'})
\]

specifies that the probability that the system terminates is the same from all initial states. If the initial state encodes some secret, then the property guarantees that this secret is not revealed through the probability of termination.

Because Markov chains lack nondeterministic choice, they are a limited modeling tool. In an open system, the secret would likely be provided by an external environment, whose decisions need to be represented by nondeterminism. In every step of the computation, such an environment would typically set the values of some low-security and some high-security input variables. In such a case, we would like to specify that the publicly observable behaviour of our system does not depend on the infinite sequence of values of the values of the high-security input variables. Similarly, nondeterminism is needed to model the possible strategic decisions in autonomous systems, such as robots, or the content of the database in a privacy-critical system.

In this paper, we initiate the study of hyperproperties for Markov decision processes (MDPs). To formalize hyperproperties in this setting, we present PHL, a general temporal logic for probabilistic hyperproperties. The nondeterministic choices of an MDP are resolved by the scheduler; correspondingly, our logic quantifies over schedulers. For example, the PHL formula

\[
\forall \sigma \forall \sigma'. \mathbb{P}(\Box \text{terminate}_\sigma) = \mathbb{P}(\Box \text{terminate}_{\sigma'})
\]

specifies that the probability of termination is the same for all of the possible (infinite) combinations of the nondeterministic choices. If we wish to distinguish different types of inputs, for example those that are provided through a high-security variable \( h \) vs. those provided through a low-security variable \( l \), then the quantification can be restricted to those schedulers that make the same low-security choices:

\[
\forall \sigma \forall \sigma'. (\forall \pi : \sigma, \pi' : \sigma'. \Box (l_\pi \leftrightarrow l_{\pi'})) \rightarrow \mathbb{P}(\Box \text{terminate}_\sigma) = \mathbb{P}(\Box \text{terminate}_{\sigma'})
\]

The path quantifier \( \forall \pi : \sigma \) works analogously to the quantifiers in HyperCTL*, here restricted to the paths of the Markov chain induced by the scheduler assigned to variable \( \sigma \). The formula thus states that all schedulers that agree on the low-security inputs induce the same probability of termination.

As we show in the paper, PHL is a very expressive logic, thanks to the combination of scheduler quantifiers, path quantifiers and a probabilistic operator. PHL has both classical applications, such as differential privacy, as well as novel applications in areas such as robotics and planning. For example, we can quantify the interference of the plans of different agents in a multi-agent system, such
as the robots in a warehouse, or we can specify the existence of an approximately optimal policy that meets given constraints. A consequence of the generality of the logic is that it is impossible to simply reduce the model checking problem to that of a simpler temporal logic in the style of the reduction of HyperPCTL to PCTL [1]. In fact, we show that the emptiness problem for probabilistic Büchi automata (PBA) can be encoded in PHL, which implies that the model checking problem for PHL is, in general, undecidable.

We present two verification procedures that approximate the model checking problem from two sides. The first algorithm overapproximates the model checking problem by quantifying over a combined monolithic scheduler rather than a tuple of independent schedulers. Combined schedulers have access to more information than individual ones, meaning that the set of allowed schedulers is overapproximated. This means that if a universal formula is true for all combined schedulers it is also true for all tuples of independent schedulers. The second procedure is a bounded model checking algorithm that underapproximates the model checking problem by bounding the number of states of the schedulers. This algorithm is obtained as a combination of a bounded synthesis algorithm for hyperproperties, which generates the schedulers, and a model checking algorithm for Markov chains, which computes the probabilities on the Markov chains induced by the schedulers. Together, the two algorithms thus provide methods both for proving and for refuting a class of probabilistic hyperproperties for MDPs.

Related work Probabilistic noninterference originated in information-flow security [19,23] and is a security policy that requires that the probability of every low trace should be the same for every low equivalent initial state. Volpano and Smith [26] presented a type system for checking probabilistic noninterference of concurrent programs with probabilistic schedulers. Sabelfeld and Sands [25] defined a secure type system for multi-threaded programs with dynamic thread creation which improves on that of Volpano and Smith. None of these works is concerned with models combining probabilistic choice with nondeterminism, nor with general temporal logics for probabilistic hyper properties.

The specification and verification of probabilistic hyperproperties has recently attracted research attention. Abraham and Bonakdarpour [1] are the first to study a temporal logic for probabilistic hyperproperties, called HyperPCTL. The logic allows for explicit quantification over states of a Markov chain, and is capable of expressing information flow properties like probabilistic noninterference. The authors present a model checking algorithm for verifying HyperPCTL on finite-state Markov chains. Later, HyperPCTL was extended to a logic called HyperPCTL* [27] that allows nesting of temporal and probability operators, and a statistical model checking method for HyperPCTL* was proposed. Recently, an extension of Signal Temporal Logic, called hyper probabilistic signal temporal logic (HyperPSTL) [28] was introduced for specifying probabilistic hyperproperties of continuous-time Markov chains, and methods for their verification via statistical model checking were proposed. Our present work, on the other hand is concerned with the specification and model checking of probabilistic hyperproperties for system models featuring both probabilistic choice and nondeterminism,
which are beyond the scope of all previous temporal logics for probabilistic hyperproperties. Due to the presence of scheduler quantifiers in our logic we can essentially capture synthesis problems for probabilistic temporal properties.

2 Preliminaries

2.1 Markov Decision Processes

Definition 1 (Markov Decision Process (MDP)). A Markov Decision Process (MDP) is a tuple \( M = (S, \text{Act}, P, t, \text{AP}, L) \) where \( S \) is a finite set of states, \( \text{Act} \) is a finite set of actions, \( P : S \times \text{Act} \times S \to [0,1] \) is the transition probability function such that \( \sum_{s' \in S} P(s, a, s') \in [0,1] \) for every \( s \in S \) and \( a \in \text{Act} \), \( t : S \to [0,1] \) is the initial distribution such that \( \sum_{s \in S} t(s) = 1 \), \( \text{AP} \) is a finite set of atomic propositions and \( L : S \to 2^{\text{AP}} \) is a labelling function.

A finite path in an MDP \( M = (S, \text{Act}, P, t, \text{AP}, L) \) is a sequence \( s_0s_1 \ldots s_n \) in which for every \( 0 \leq i < n \) there exists \( a_i \in \text{Act} \) such that \( P(s_i, a_i, s_{i+1}) > 0 \). Infinite paths in \( M \) are defined analogously. We denote with \( \text{Paths}_{\text{fin}}(M) \) and \( \text{Paths}_{\text{inf}}(M) \) the sets of finite and infinite paths in \( M \) respectively. For an infinite path \( \rho = s_0s_1 \ldots \) and \( i \in \mathbb{N} \) we denote with \( \rho[i, \infty) \) the infinite suffix \( s_is_{i+1} \ldots \).

Given a state \( s \in S \), we define \( \text{Paths}_{\text{fin}}(M, s) = \{s_0s_1 \ldots s_n \in \text{Paths}_{\text{fin}}(M) \mid s_0 = s\} \), and similarly \( \text{Paths}_{\text{inf}}(M, s) \).

Given any state \( s \in S \) in an MDP \( M = (S, \text{Act}, P, t, \text{AP}, L) \) we denote with \( M_s = (S, \text{Act}, P, t_s, \text{AP}, L) \) the MDP obtained from \( M \) by making \( s \) the single initial state. That is, \( t_s(s) = 1 \) and \( t_s(t) = 0 \) for all \( t \neq s \).

For a set \( A \) we denote with \( \mathcal{D}(A) \) the set of probability distributions on \( A \).

Definition 2 (Scheduler). A scheduler for an MDP \( M = (S, \text{Act}, P, t, \text{AP}, L) \) is a function \( \mathfrak{S} : (S \cdot \text{Act})^* S \to \mathcal{D}(\text{Act}) \) such that for all sequences \( s_0a_0 \ldots a_{n-1}s_n \in (S \cdot \text{Act})^* S \) it holds that if \( \mathfrak{S}(s_0a_0 \ldots a_{n-1}s_n)(a) > 0 \) then \( \sum_{a \in \text{Act}} P(s_n, a, t) > 0 \), that is, each action in the support of \( \mathfrak{S}(s_0a_0 \ldots a_{n-1}s_n) \) is enabled in \( s_n \). We define \( \text{Sched}(M) \) to be the set consisting of all schedulers for an MDP \( M \).

Given an MDP \( M = (S, \text{Act}, P, t, \text{AP}, L) \) and a scheduler \( \mathfrak{S} \) for \( M \), we denote with \( M_\mathfrak{S} \) the Markov chain of \( M \) induced by \( \mathfrak{S} \), which is defined as the tuple \( M_\mathfrak{S} = ((S \cdot \text{Act})^* S, P_\mathfrak{S}, t_\mathfrak{S}, \text{AP}, L_\mathfrak{S}) \) where for every sequence \( h = s_0a_0 \ldots a_{n-1}s_n \in (S \cdot \text{Act})^* S \) it holds that \( P_\mathfrak{S}(h, h \cdot s_{n+1}) = \sum_{a \in \text{Act}} \mathfrak{S}(\pi)(a) \cdot P(s_n, a, s_{n+1}) \) and \( L_\mathfrak{S}(h) = L(s_n) \). Note that \( M_\mathfrak{S} \) is infinite even when \( M \) is finite. The different types of paths in a Markov chain are defined as for MDPs.

A scheduler is memoryless if for all \( h, h' \in (S \cdot \text{Act})^* \) and \( s \in S \) it holds that \( \mathfrak{S}(h \cdot s) = \mathfrak{S}(h' \cdot s) \). Thus, a memoryless scheduler can be represented simply as a function \( \mathfrak{S} : S \to \mathcal{D}(\text{Act}) \). It is easy to see that if a scheduler \( \mathfrak{S} \) is memoryless then the Markov chain for an MDP \( M = (S, \text{Act}, P, t, \text{AP}, L) \) induced by \( \mathfrak{S} \) is finite and \( M_\mathfrak{S} = (S, P_\mathfrak{S}, t_\mathfrak{S}, \text{AP}, L_\mathfrak{S}) \) where \( P_\mathfrak{S}(s, s') = \sum_{a \in \text{Act}} \mathfrak{S}(a) \cdot P(s, a, s') \).

Of specific interest are also finite-memory schedulers, which are schedulers that can be represented as finite-state machines. Formally, a finite-memory
scheduler for $M$ is represented as a tuple $T_\mathcal{O} = (Q, \delta, q_0, \text{act})$, where $Q$ is a finite set of states, representing the memory of the scheduler, $\delta : Q \times S \times \text{Act} \rightarrow Q$ is a memory update function, $q_0$ is the initial state of the memory, and $\text{act} : Q \times S \rightarrow \mathcal{D}((\text{Act}))$ is a function that based on the current memory state and the state of the MDP returns an action of the MDP. Such a representation defines a function $\mathcal{O} : (S \cdot \text{Act})^* \rightarrow \mathcal{D}((\text{Act}))$ as follows. First, let us define the function $\delta^* : Q \times (S \cdot \text{Act})^* \rightarrow Q$ as follows: $\delta^*(q, \epsilon) = q$ for all $q \in Q$, and $\delta^*(q, s_0 a_0 \ldots s_n a_n s_{n+1} a_{n+1}) = \delta^*(q, s_0 a_0 \ldots s_n a_n), s_{n+1}, a_{n+1})$ for all $q \in Q$ and all $s_0 a_0 \ldots s_n a_n s_{n+1} a_{n+1} \in (S \cdot \text{Act})^*$. Now, we define the scheduler function represented by $T_\mathcal{O}$ by $\mathcal{O}(s_0 a_0 \ldots s_n a_n, s_{n+1}) = \text{act}(\delta^*(s_0 a_0 \ldots s_n a_n), s_{n+1})$.

As with memoryless schedulers, finite-memory schedulers induce finite Markov chains with simpler representation. A finite memory scheduler $\mathcal{O}$ represented by $T_\mathcal{O} = (Q, \delta, q_0, \text{act})$ induces the Markov chain $M_\mathcal{O} = (S \times Q, P_\mathcal{O}, \iota_\mathcal{O}, \text{AP}, L_\mathcal{O})$ where $P_\mathcal{O}((s, q), (s', q')) = \sum_{a \in \text{Act}} P(s, a, s') \cdot P_\mathcal{O}(s, q), q') = 0$, and $\iota_\mathcal{O}(s, q) = \iota(s)$ if $q = q_0$ and $\iota_\mathcal{O}(s, q) = 0$ otherwise.

A scheduler $\mathcal{O}$ is deterministic if for every $h \in (S \cdot \text{Act})^* S$ it holds that $\mathcal{O}(\pi)(a) = 1$ for exactly one $a \in \text{Act}$. By abuse of notation, a deterministic scheduler can be represented as a function $\mathcal{O} : S^+ \rightarrow \text{Act}$, that maps a finite sequence of states to the single action in the support of the corresponding distribution. Note that for deterministic schedulers we omit the actions from the history as they are uniquely determined by the sequence of states. We write $\text{DetSched}(M)$ for the set of deterministic schedulers for the MDP $M$.

### 2.2 Probability Spaces

A probability space is a triple $(\Omega, \mathcal{F}, \text{Prob})$, where $\Omega$ is a sample space, $\mathcal{F} \subseteq 2^\Omega$ is a $\sigma$-algebra and $\text{Prob} : \mathcal{F} \rightarrow [0, 1]$ is a probability measure.

Given a Markov chain $C = (S, P, \iota, \text{AP}, L)$, it is well known how to associate a probability space $(\Omega^C, \mathcal{F}^C, \text{Prob}^C)$ with $C$. The sample space $\Omega^C = \text{Paths}_{\text{inf}}(C)$ is the set of infinite paths in $C$, where the sets of finite and infinite paths for a Markov chain are defined in the same way as for MDP. The $\sigma$-algebra $\mathcal{F}^C$ is defined as the smallest $\sigma$-algebra that for each $\pi \in \text{Paths}_{\text{inf}}(C)$ contains the set $\text{Cyl}_C(\pi) = \{ \rho \in \text{Paths}_{\text{inf}}(C) \mid \exists \rho' \in \text{Paths}_{\text{inf}}(C) : \rho = \pi \cdot \rho' \}$ called the cylinder set of the finite path $\pi$. The probability measure $\text{Prob}^C$ is the unique probability measure such that for each $\pi = s_0 \ldots s_n \in \text{Paths}_{\text{fin}}(C)$ it holds that $\text{Prob}^C(\text{Cyl}(\pi)) = \iota(s_0) \cdot \prod_{i=0}^{n-1} P(s_i, s_{i+1})$.

Analogously, given any state $s \in S$ we denote with $(\Omega^C, \mathcal{F}^C, \text{Prob}^C_s)$ the probability space for paths in $C$ originating in the state $s$, i.e., the probability space associated with the Markov chain $C_s$ (where $C_s$ is defined as for MDPs). When considering a Markov chain $M_\mathcal{O}$ induced by an MDP $M$ and a scheduler $\mathcal{O}$, we write $\text{Prob}_{M_\mathcal{O}}$ and $\text{Prob}_{M_\mathcal{O}, s}$ for the sake of readability.

### 2.3 Omega-Regular Properties and Automata

$LTL$ (Linear-time Temporal Logic) is a logic commonly used for specifying omega-regular linear-time properties of reactive systems. In addition to Boolean
connectives it contains the usual temporal operators \( \text{Next} \) and \( \text{Until} \), and the derived operators Weak Until \( \text{W} \) and Eventually \( \text{E} \). LTL formulas are defined over a set of atomic propositions \( \text{AP} \) and interpreted over infinite sequences over \( 2^{\text{AP}} \) (see [24] for the semantics of LTL). We denote the satisfaction of an LTL formula \( \varphi \) by an infinite sequence \( w \in (2^{\text{AP}})^\omega \) by \( w \models \varphi \).

Following the convention of [4] we use LTL notation for describing events: for example if \( G \) is a set of states in a Markov chain \( C \) we write \( G \) for the event that \( G \) is visited infinitely often, and write \( \pi \models G \) instead of \( \pi \in G \).

A deterministic Rabin automaton over a given finite alphabet \( \Lambda \) is a tuple \( A = (Q, \Lambda, \delta, q_0, (B_j, G_j))_{j=1}^m \), where \( Q \) is finite set of states, \( \delta : Q \times \Lambda \to Q \) is the transition function, \( q_0 \in Q \) is the initial state and the pairs \( (B_j, G_j) \) of sets of states define the accepting condition of \( A \). A run of \( A \) on an infinite word \( w = \alpha_0 \alpha_1 \cdots \in \Lambda^\omega \) is an infinite sequence \( q_0 q_1 q_2 \cdots \in Q^\omega \) of states such that \( q_0 \) is the initial state, and for every \( i \in \mathbb{N} \) it holds that \( q_{i+1} = \delta(q_i, \alpha_i) \). Without loss of generality we assume that the transition function is total, and hence there exists exactly one run of \( A \) on each \( w \) since \( A \) is deterministic. A run \( q_0 q_1 q_2 \cdots \) is accepting if it satisfies the Rabin condition which requires that for some pair \( (B_j, G_j) \) the set \( B_j \) is visited finitely often and the set \( G_j \) is visited infinitely often. An infinite word \( w \) is accepted by \( A \) if there exists an accepting run of \( A \) on \( w \). We denote the set of infinite words accepted by \( A \) is, called the language of \( A \), by \( L(A) \). We define the size \( |A| \) to be the number of its states, i.e., \( |A| = |Q| \).

It is well know that for every LTL formula \( \varphi \) one can construct a deterministic Rabin Automaton \( A_\varphi \) with \( |A_\varphi| = 2^{O(|\varphi|)} \) such that \( L(A_\varphi) = \{ w \in 2^{\text{AP}} \mid w \models \varphi \} \).

If \( \pi = s_0 s_1 \ldots \) is an infinite path in an MDP or a Markov chain \( M \), and \( \varphi \) an LTL formula, we write \( \pi \models M \varphi \) meaning that \( L(s_0) L(s_1) \models \varphi \).

A safety property is a linear time property such that every word that does not satisfy the property has a finite prefix all of whose extensions violate the property. Safety properties can be represented by safety automata. A deterministic safety automaton over an alphabet \( \Lambda \) is a tuple \( A = (Q, \Lambda, \delta, q_0) \) where the transition function \( \delta \) is partial and every infinite run is accepting.

3 The Logic PHL

In this section we define the syntax and semantics of PHL, the logic which we introduce and study in this work. PHL allows for quantification over schedulers and integrates features of temporal logics for hyper properties, such as Hyper-LTL and HyperCTL* [9], and probabilistic temporal logics such as PCTL*.

3.1 Syntax

As we are concerned with hyperproperties interpreted over MDPs, our logic allows for quantification over schedulers and quantification over paths.

To this end, let \( V_{\text{sched}} \) be a countably infinite set of scheduler variables and let \( V_{\text{path}} \) be a countably infinite set of path variables. According to the semantics of our logic, quantification over path variables ranges over the paths in a Markov
chain associated with the scheduler represented by a given scheduler variable. To express this dependency we will associate path variables with the corresponding scheduler variable, writing \( \pi : \sigma \) for a path variable \( \pi \) associated with a scheduler variable \( \sigma \). The precise use and meaning of this notation will become clear below, once we define the syntax and semantics of the logic.

Given a set \( AP \) of atomic propositions, PHL formulas over \( AP \) will use atomic propositions indexed with scheduler variables, and both path and associated scheduler variables. We define the sets of atomic propositions indexed with scheduler variables as \( AP_{V_{sched}} = \{ a_\sigma \mid a \in AP, \sigma \in V_{sched} \} \) and indexed with path variables as \( AP_{V_{path}} = \{ a_\pi \mid a \in AP, \pi \in V_{path} \} \).

PHL (Probabilistic Hyper Logic) formulas are defined by the grammar

\[
\Phi ::= \forall \sigma. \Phi \mid \Phi \land \Phi \mid \neg \Phi \mid \chi \mid P \bowtie c
\]

where \( \sigma \in V_{sched} \) is a scheduler variable, \( \chi \) is a HyperCTL\(^*\) formula, \( P \) is a probabilistic expression defined below, \( \bowtie \in \{ \leq, \leq, \geq, \geq \} \), and \( c \in \mathbb{Q} \).

Formulas in HyperCTL\(^*\), introduced in [9], are constructed by the grammar

\[
\chi ::= a_\pi \mid \chi \land \chi \mid \neg \chi \mid \bigcirc \chi \mid \chi U \chi \mid \forall \pi : \sigma. \chi
\]

where \( \pi \) is a path variable associated with a scheduler variable \( \sigma \), and \( a \in AP \).

Probability expressions are defined by the grammar

\[
P ::= \mathbb{P}(\varphi) \mid P + P \mid c \cdot P
\]

where \( \mathbb{P} \) is the probability operator, \( c \in \mathbb{Q} \), and \( \varphi \) is an LTL formula [24] defined by the grammar below, where \( a \in AP \) and \( \sigma \) is a scheduler variable.

\[
\varphi ::= a_\sigma \mid \varphi \land \varphi \mid \neg \varphi \mid \bigcirc \varphi \mid \varphi U \varphi.
\]

We call formulas of the form \( P \bowtie c \) probabilistic predicates.

A PHL formula \( \Phi \) is well-formed if each path quantifier for \( \pi : \sigma \) that appears in \( \Phi \) is in the scope of a scheduler quantifier with the scheduler variable \( \sigma \).

A PHL formula is closed if all occurrences of scheduler and path variables are bound by scheduler and path quantifiers respectively.

In the following we consider only closed and well-formed PHL formulas.

Discussion Intuitively, a PHL formula is a Boolean combination of formulas consisting of a scheduler quantifier prefix followed by a formula without scheduler quantifiers constructed from probabilistic predicates and HyperCTL\(^*\) formulas via propositional operators. Thus, interleaving path quantifiers and probabilistic predicates is not allowed in PHL. This design decision is in line with the fact that probabilistic temporal logics like PCTL\(^*\) replace the path quantifiers with the probabilistic operator that can be seen as their quantitative counterpart. We further chose to not allow nesting of probabilistic predicates and temporal operators, as in all the examples that we considered we never encountered the need for nested \( P \) operators. Moreover, allowing arbitrary nesting of probabilistic and temporal operators would immediately make the model checking problem for the resulting logic undecidable, following from [5].
3.2 Self-Composition for MDPs

In order to define the semantics of PHL we first introduce the self-composition operation for MDPs, which lifts to MDPs the well-known self-composition of transition systems that is often used in the model checking of hyperproperties.

Let us fix, for the reminder of the section, an MDP \( M = (S, Act, P, i, AP, L) \).

**Definition 3 (n-self-composition of an MDP).** Let \( M = (S, Act, P, i, AP, L) \) be an MDP and \( n \in \mathbb{N}_{\geq 0} \) be a constant. The \( n \)-self-composition of \( M \) is the MDP \( M^n = (S^n, Act^n, P^n, i^n, AP, L^n) \) with the following components.

- \( S^n = \{ (s_1, \ldots, s_n) \mid s_i \in S \text{ for all } 1 \leq i \leq n \} \) is the set of states.
- \( Act^n = \{ (a_1, \ldots, a_n) \mid a_i \in Act \text{ for all } 1 \leq i \leq n \} \) is the set of actions.
- The transition probability function \( P^n \) is defined such that for every pair of states \( (s_1, \ldots, s_n), (s'_1, \ldots, s'_n) \in S^n \) and every action \( (a_1, \ldots, a_n) \in Act^n \):
  \[
  P^n((s_1, \ldots, s_n), (a_1, \ldots, a_n), (s'_1, \ldots, s'_n)) = \prod_{i=1}^{n} P(s_i, a_i, s'_i).
  \]
- We define the initial distribution such that \( \overline{\nu}((s_1, \ldots, s_n)) = \nu(s) \) if \( s_1 = \ldots = s_n = s \) for some \( s \in S \) and \( \overline{\nu}((s_1, \ldots, s_n)) = 0 \) otherwise.
- The labelling function \( L^n : S^n \to (2^{AP})^n \) maps the states of \( M^n \) to \( n \)-tuples of subsets of \( AP \) (in contrast to definition Definition where states are mapped to subsets of \( AP \)) and is given by \( L^n((s_1, \ldots, s_n)) = (L(s_1), \ldots, L(s_n)) \).

Naturally, a scheduler \( \hat{S} \in Sched(M^n) \) induces a Markov chain \( M^n_{\hat{S}} \).

Given schedulers \( \hat{S}_1, \ldots, \hat{S}_n \in Sched(M) \), their composition, a scheduler \( \hat{S} : (S^n \cdot Act^n)^* S^n \to \mathcal{D}(Act^n) \) for \( M^n \), is denoted \( \hat{S} = \hat{S}_1 \parallel \ldots \parallel \hat{S}_n \) and such that for every \( h = (s_{1,1}, \ldots, s_{1,n}) (a_{1,1}, \ldots, a_{1,n}) \ldots (s_{k,1}, \ldots, s_{k,n}) \in (S^n \cdot Act^n)^* S^n \) and \( \pi = (a_{k+1,1}, \ldots, a_{k+1,n}) \in Act^n \), \( \hat{S}(h)(\pi) = \prod_{i=1}^{k} \hat{S}_i(s_{1,i}, a_{1,i}, \ldots, s_{k,i}) (a_{k+1,i}) \).

### 3.3 Scheduler and Path Assignments

Let \( V_{sched} \) and \( V_{path} \) be the sets of scheduler and path variables respectively.

A scheduler assignment is a vector of pairs \( \Sigma = \bigcup_{\mathbb{N} \subseteq \mathbb{N}} V_{sched} \times Sched(M) \) that assigns schedulers to some of the scheduler variables. Given a scheduler assignment \( \Sigma = (\sigma_1, \hat{S}_1), \ldots, (\sigma_n, \hat{S}_n) \), we denote by \( |\Sigma| \) the length (number of pairs) of the vector. For a scheduler variable \( \sigma \in V_{sched} \) we define \( \Sigma(\sigma) = \hat{S}_i \) where \( i \) is the maximal index such that \( \sigma_i = \sigma \). If such an index \( i \) does not exists, \( \Sigma(\sigma) \) is undefined. For a scheduler assignment \( \Sigma = (\sigma_1, \hat{S}_1), \ldots, (\sigma_n, \hat{S}_n) \), a scheduler variable \( \sigma \in V_{sched} \), and a scheduler \( \hat{S} \in Sched(M) \) we define the scheduler assignment \( \Sigma[\sigma \mapsto \hat{S}] = (\sigma_1, \hat{S}_1), \ldots, (\sigma_n, \hat{S}_n), (\sigma, \hat{S}) \) obtained by adding the pair \( (\sigma, \hat{S}) \) to the end of the vector \( \Sigma \).

Given the MDP \( M \), let \( \Sigma = (\sigma_1, \hat{S}_1), \ldots, (\sigma_n, \hat{S}_n) \) be a scheduler assignment, and consider \( M^{[\Sigma]} \), the \( |\Sigma| \)-self-composition of \( M \). The scheduler assignment \( \Sigma \) defines a scheduler for \( M^{[\Sigma]} \), which is the product of the schedulers in
Scheduler assignment. As usual, the satisfaction relations are denoted by formulas requires additionally a path assignment. Probabilistic expressions and HyperCTL consider only closed and well-formed PHL formulas. PHL formulas are interpreted as elements of \( \mathbb{N} \). More precisely, a path assignment is a vector of pairs of path variables and paths in \( \text{Paths}_{\inf}(M) \). Analogously to scheduler assignments, for a path variable \( \pi \) and a path \( \rho \in \text{Paths}_{\inf}(M) \), we define \( \Pi(\pi) \) and \( \Pi[\pi \mapsto \rho] \). For \( \Pi = ((\pi_1, \rho_1), \ldots, (\pi_n, \rho_n)) \) and \( j \in \mathbb{N} \), we define \( \Pi[j, \infty] = ((\pi_1, \rho_1[j, \infty]), \ldots, (\pi_n, \rho_n[j, \infty])) \) to be the path assignment that assigns to each \( \pi_i \) the suffix \( \rho_i[j, \infty] \) of the path \( \rho_i \).

### 3.4 Semantics of PHL

We are now ready to define the semantics of PHL formulas. Recall that we consider only closed and well-formed PHL formulas. PHL formulas are interpreted over an MDP and a scheduler assignment. The interpretation of HyperCTL* formulas requires additionally a path assignment. Probabilistic expressions and LTL formulas are evaluated in the Markov chain for an MDP induced by a scheduler assignment. As usual, the satisfaction relations are denoted by \( \models \).

For an MDP \( M \) and a scheduler assignment \( \Sigma \) we define

\[
M, \Sigma \models \forall \sigma \Phi \iff \text{for all } \sigma \in \text{Sched}(M): M, \Sigma[\sigma \mapsto \sigma] \models \Phi;
\]

\[
M, \Sigma \models \Phi_1 \land \Phi_2 \iff M, \Sigma \models \Phi_1 \text{ and } M, \Sigma \models \Phi_2;
\]

\[
M, \Sigma \models \neg \Phi \iff M, \Sigma \not\models \Phi;
\]

\[
M, \Sigma \models \chi \iff M, \Sigma, \Pi_{\emptyset} \models \chi, \text{ where } \Pi_{\emptyset} \text{ is the empty path assignment};
\]

\[
M, \Sigma \models P \bowtie c \iff \llbracket P \rrbracket_M \bowtie c.
\]

For an MDP \( M \), scheduler assignment \( \Sigma \), and path assignment \( \Pi \) we define

\[
M, \Sigma, \Pi \models a \sigma \iff a \in L(\Pi(\pi))[0];
\]

\[
M, \Sigma, \Pi \models \chi_1 \land \chi_2 \iff M, \Sigma, \Pi \models \chi_1 \text{ and } M, \Sigma, \Pi \models \chi_2;
\]

\[
M, \Sigma, \Pi \models \neg \chi \iff M, \Sigma, \Pi \not\models \chi;
\]

\[
M, \Sigma, \Pi \models \bigcirc \chi \iff M, \Sigma, \Pi[1, \infty] \models \chi;
\]

\[
M, \Sigma, \Pi \models \chi_1 U \chi_2 \iff \text{there exists } i \geq 0 : M, \Sigma, \Pi[i, \infty] \models \chi_2 \text{ and for all } j < i : M, \Sigma, \Pi[j, \infty] \models \chi_1;
\]

\[
M, \Sigma, \Pi \models \forall \pi : \sigma. \chi \iff \text{for all } \rho \in \text{Paths}_{\inf}(C) : M, \Sigma, \Pi[\pi \mapsto \rho] \models \chi,
\]

where in the last item \( C \) is the Markov chain \( M_{\Sigma(\sigma)} \) when \( \Pi \) is the empty path assignment, and otherwise the Markov chain \( M_{\Sigma(\sigma), \Pi[\pi \mapsto \rho]} \) where \( \pi' \) is the path variable associated with scheduler variable \( \sigma \) that was most recently added to \( \Pi \).

For a Markov chain \( C \) of the form \( M_{\Sigma} \) or \( M_{\Sigma, \hat{s}} \), where \( \Sigma \) is a scheduler assignment and \( \hat{s} \) is a state in \( M_{\Sigma} \) we define the semantics \( \llbracket \cdot \rrbracket_C \) of probabilistic
expressions as follows:

\[
\llbracket \mathbb{P}(\varphi) \rrbracket_C = \text{Prob}_C(\{\rho \in \text{Paths}_{\inf}(C) \mid C, \rho \models \varphi\});
\]

\[
\llbracket P_1 + P_2 \rrbracket_C = [P_1]_C + [P_2]_C;
\]

\[
\llbracket c \cdot P \rrbracket_C = c \cdot [P]_C,
\]

where the semantics of path formulas (i.e., LTL formulas) is defined by

\[
C, \rho \models a_{\sigma} \quad \text{iff} \quad a \in \hat{L}(\rho[0])(\sigma);
\]

\[
C, \rho \models \varphi_1 \land \varphi_2 \quad \text{iff} \quad C, \rho \models \varphi_1 \quad \text{and} \quad C, \rho \models \varphi_2;
\]

\[
C, \rho \models \neg \varphi \quad \text{iff} \quad C, \rho \not\models \varphi;
\]

\[
C, \rho \models \bigcirc \varphi \quad \text{iff} \quad C, \rho[1, \infty] \models \varphi;
\]

\[
C, \rho \models \varphi_1 U \varphi_2 \quad \text{iff} \quad \text{there exists } i \geq 0: C, \rho[i, \infty] \models \varphi_2 \quad \text{and} \quad \text{for all } j < i: C, \rho[j, \infty] \models \varphi_1.
\]

Note that \( \text{Prob}_C(\{\rho \in \text{Paths}_{\inf}(C) \mid C, \rho \models \varphi\}) \) is well-defined as it is a known fact \( [4] \) that the set \( \{\rho \in \text{Paths}_{\inf}(C) \mid C, \rho \models \varphi\} \) is measurable.

We say that an MDP \( M \) satisfies a closed well-formed PHL formula \( \Phi \), denoted \( M \models \Phi \), iff \( M, \Sigma_0 \models \Phi \), where \( \Sigma_0 \) is the empty scheduler assignment.

Since PHL includes both scheduler and path quantification, the sets of deterministic and randomized schedulers are not interchangeable with respect to the PHL semantics. That is, there exists an MDP \( M \) and formula \( \Phi \) such that if quantifiers are interpreted over \( Sched(M) \), then \( M \models \Phi \), and if quantifiers are interpreted over \( DetSched(M) \) then \( M \not\models \Phi \). (See the appendix for an example.)

### 3.5 Examples of PHL Specifications

We now illustrate several types of properties that PHL is able to express.

**Example 1 (Differential privacy).** Using PHL we can express the property of differential privacy \([12]\). Intuitively, a randomized algorithm that produces responses to queries to an input database is differentially private if it behaves similarly on similar input databases. The schedulers in the MDP encode all possible unbounded length sequences of updates and queries to a database. Update actions are deterministic and modify the database, and query actions are probabilistic and encode the randomized responses to queries by the privacy mechanism.

Suppose that an MDP \( M_{dp} = (S_{dp}, Act_{dp}, P_{dp}, \iota_{dp}, AP_{dp}, L_{dp}) \) encodes some differential-privacy mechanism in the following way.

- Each state in \( S_{dp} \) encodes the current contents of the database, together with the last update or query and the last given response, if any.
- The set of actions \( Act_{dp} \) consists of two types of actions \( Act_{dp} = Act_{update} \oplus Act_{query} \), modelling the possible updates and queries respectively.
- The transition probability function \( P_{dp} \) is defined based on the privacy mechanism being modelled. The transitions for actions in \( Act_{update} \) are deterministic and encode how the database contents is modified. The transitions for actions \( Act_{query} \) are probabilistic and correspond to the randomized responses to queries as defined by the privacy mechanism.
The atomic propositions in $\mathsf{AP}_{dp} = \mathsf{AP}_{update} \uplus \mathsf{AP}_{query} \uplus \mathsf{AP}_{resp}$ contain propositions for the action labels in order to allow comparison of the sequence of updates and queries, as well as labels describing the responses to the queries. The labelling function $L_{dp}$ maps each state to the atomic propositions reflecting the last action and query response stored in the state.

The PHL formula $\Phi_{dp}$ given below encodes $(\varepsilon, \delta)$-differential privacy of the privacy mechanism encoded by an MDP of the above form. It states that for every two schedulers that correspond to sequences of updates that differ in at most one position, and that issue the same set of queries, the probabilities of each response to each query are "close" as defined by $\varepsilon$ and $\delta$.

$$\Phi_{dp} = \forall \sigma_1 \forall \sigma_2. (\forall \pi_1 : \sigma_1. \forall \pi_2 : \sigma_2. \varphi_{sim} \land \varphi_{queries} \rightarrow \varphi_{prob})$$

- $\varphi_{sim}$ specifies that the two sequences differ in at most one update:

$$\varphi_{sim} = \Box \left( \bigvee_{a \in \mathsf{AP}_{update}} \neg(a_{\pi_1} \leftrightarrow a_{\pi_2}) \right) \rightarrow \Box \Box \left( \bigvee_{a \in \mathsf{AP}_{update}} \neg(a_{\pi_1} \leftrightarrow a_{\pi_2}) \right).$$

- $\varphi_{queries}$ specifies that the same set of queries are being issued:

$$\varphi_{queries} = \bigwedge_{a \in \mathsf{AP}_{query}} (\Diamond a_{\pi_1} \leftrightarrow \Diamond a_{\pi_2}).$$

- $\varphi_{prob}$ specifies that for every query each response has close enough probability of occurrence in the two schedulers (i.e., the two databases):

$$\varphi_{prob} = \bigwedge_{q \in \mathsf{AP}_{query}} \bigwedge_{r \in \mathsf{AP}_{resp}} \mathbb{P}(\Box (q_{\sigma_1} \rightarrow r_{\sigma_1})) - \exp(\varepsilon) \mathbb{P}(\Box (q_{\sigma_2} \rightarrow r_{\sigma_2})) < \delta.$$ 

**Example 2 (Plan non-interference).** Another scenario that can be captured in PHL is the non-interference of plans of possibly adversarial agents in a multi-agent system. Consider two robots in a warehouse, possibly attempting to reach the same location. Our goal is to determine whether all possible plans for the first robot to move towards the goal are robust towards interferences from the plans of the other robot. That is, we want to check whether for every plan of robot 1 the probability that it reaches the goal under an arbitrary plan of robot 2 is close to that of the same plan for robot 1 executed under any other plan for robot 2. Formally, this property can be expressed in PHL as

$$\forall \sigma_1 \forall \sigma_2. (\forall \pi_1 : \sigma_1. \forall \pi_2 : \sigma_2. \Box (move1_{\pi_1} \leftrightarrow move1_{\pi_2}) \rightarrow \mathbb{P}(\Box (goal1_{\pi_1} \land \neg goal2_{\pi_2})) - \mathbb{P}(\Box (goal1_{\pi_2} \land \neg goal2_{\pi_1})) \leq \varepsilon,$$

where the proposition $move1$ corresponds to robot 1 making a move towards the goal, and the propositions $goalN$ correspond to robot $N$ being at the goal.
Example 3 (Bounding performance loss). PHL allows us to compare the performance of different schedulers in an MDP. For instance, we can specify the property that the difference in the probability of success of two schedulers that differ in certain action choices is bounded by a given constant. For instance, consider an MDP where actions come in pairs, for example one corresponding to using a more precise (but more power consuming) sensor and the other one corresponding to using a less precise (but less power consuming) sensing mechanism. The property that we want to state is that for every policy replacing some of the more precise actions with their less precise counterpart has a bounded effect on the probability of reaching some set of goal states.

This property is specified by the following formula where the propositions \(a'\) and \(a''\) correspond to the more precise and less precise actions in a pair:

\[
\forall \sigma_1 \forall \sigma_2 . \chi_{\text{actions}} \rightarrow P(\bigcirc \text{goal}_{\sigma_1}) - P(\bigcirc \text{goal}_{\sigma_2}) \leq \varepsilon,
\]

where

\[
\chi_{\text{actions}} = \forall \pi_1 : \sigma_1 . \forall \pi_2 : \sigma_2 . (\bigwedge_a (a'_{\pi_1} \land a''_{\pi_2} \lor a''_{\pi_1} \land a'_{\pi_2})) \bigwedge \neg (\in_{\pi_1} \leftrightarrow \in_{\pi_2}).
\]

Example 4 (Existence of near-optimal scheduler under hard constraint). Consider an MDP \(M = (S, \text{Act}, P, \iota, \text{AP}, L)\) and two LTL specifications \(\varphi_{\text{hard}}\) and \(\varphi_{\text{soft}}\) over AP. We can express in PHL the property that there exists scheduler \(\mathcal{S}\) for \(M\) such that \(\text{Prob}_{M,\mathcal{S}}(\{\pi \mid \varphi_{\text{hard}}\}) \geq c\) such that for every \(\mathcal{S}' \in \text{Sched}(M)\) it holds that \(\text{Prob}_{M,\mathcal{S}}(\{\pi \mid \varphi_{\text{hard}}\}) - \text{Prob}_{M,\mathcal{S}}(\{\pi \mid \varphi_{\text{soft}}\}) \leq \varepsilon\). That is, \(\mathcal{S}\) satisfies \(\varphi_{\text{hard}}\) with probability at least \(c\) and satisfies \(\varphi_{\text{soft}}\) with probability that is at most \(\varepsilon\)-away from that of the optimal scheduler. The formula is:

\[
\exists \sigma . \text{Prob}(\varphi_{\text{hard}}[\text{AP}_\sigma/\text{AP}]) \geq c \land \forall \sigma' . \text{Prob}(\varphi_{\text{soft}}[\text{AP}_{\sigma'}/\text{AP}]) - \text{Prob}(\varphi_{\text{soft}}[\text{AP}_\sigma/\text{AP}]) \leq \varepsilon,
\]

where \(\varphi_{\text{hard}}[\text{AP}_\sigma/\text{AP}]\) is the formula \(\varphi_{\text{hard}}\) in which each \(a \in \text{AP}\) is replaced by \(a_\sigma\) and similarly for the other two formulas.

4 Undecidability of PHL Model Checking

In this section we study the decidability of the model checking problem for PHL. We show that due to the fact that PHL allows quantification over both schedulers and paths, the model checking problem for PHL is undecidable.

Theorem 1. The model checking problem for PHL is undecidable.

Proof. We prove the undecidability of the model checking problem for PHL via a reduction from the emptiness problem for probabilistic Büchi automata (PBA), which is known to be undecidable [3]. More precisely, we show how for each PBA \(B\) to construct an MDP \(M\) and an PHL formula \(\Phi\) such that there exists an infinite word \(w\) over \(B\)'s alphabet such that \(\text{Prob}_B(w) > 0\) iff \(M \models \Phi\).

Let \(B = (Q, A, \delta, \mu, F)\) be a PBA, where \(Q\) is a finite set of states, \(A\) is a finite alphabet, \(\delta : Q \times A \times Q \to [0, 1]\) is the transition probability function, \(\mu\) is the initial distribution and \(F \subseteq Q\) is the set of accepting states.
We will now define an MDP in which the scheduler picks letters in \( A \) and the probabilistic choices mimic the behaviour of the PBA. The PHL formula will assert the existence of a scheduler such that in the resulting Markov chain all paths have the same scheduler choices (i.e., the scheduler is "blind") and in the resulting Markov chain the probability of visiting an accepting state infinitely often is greater than 0. This is equivalent to the scheduler generating word \( w \in A^\omega \) regardless of what states are visited by the automaton, and \( \text{Prob}_{\mathbf{B}}(w) > 0 \).

We define the MDP \( M = (S, \mathit{Act}, \mathbf{P}, \iota, \Lambda, \mathit{AP}, L) \) as follows.

- The states and actions are respectively \( S = Q \times (A \uplus \{ \bot \}) \) and \( \mathit{Act} = A \).
- The transition probability function \( \mathbf{P} \) is defined by the function \( \delta \) in \( \mathcal{B} \) and is such that for every \( (q, \alpha), (q', \alpha') \in S \) and \( \beta \in A \) we have \( \mathbf{P}((q, \alpha), \beta, (q', \alpha')) = \delta(q, \beta, q') \) if \( \alpha' = \beta \), and \( \mathbf{P}((q, \alpha), \beta, (q', \alpha')) = 0 \) otherwise.
- The initial distribution \( \iota \) is defined by the initial distribution \( \mu \) from \( \mathcal{B} \) and is such that \( \iota((q, \alpha)) = \mu(q) \) if \( \alpha = \bot \) and \( \iota((q, \alpha)) = 0 \) for \( \alpha \neq \bot \).
- The atomic propositions \( \Lambda = A \uplus \{ f \} \) correspond to the alphabet of \( \mathcal{B} \) plus a fresh proposition \( f \), and \( L \) is such that for every \( (q, \alpha) \in S \) we have that:
  - if \( \alpha = \bot \): if \( q \not\in F \), then \( L((q, \alpha)) = \emptyset \), and if \( q \in F \), \( L((q, \alpha)) = \{ f \} \);
  - if \( \alpha \in A \): if \( q \not\in F \), then \( L((q, \alpha)) = \{ \alpha \} \), otherwise \( L((q, \alpha)) = \{ \alpha, f \} \).

Let \( \Phi = \exists \sigma. (\exists \pi : \sigma \forall \pi' : \sigma. \Box_{\alpha \in A}(\alpha_{\pi} \leftrightarrow \alpha_{\pi'})) \land \Box f \).

The proof that there exists a word \( w \) such that \( \text{Prob}_{\mathbf{B}}(w) > 0 \) if and only if \( M \models \Phi \) can be found in the appendix.

In the proof of Theorem 1, we reduced the emptiness problem for PBA to the model checking of a PHL formula of the form \( \exists \sigma \cdot \forall \sigma' \cdot (\forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n, \psi) \rightarrow P \bowtie c \) which contains a single existential scheduler quantifier and a HyperCTL* formula in the \( \exists \forall \) fragment of HyperLTL. An alternative undecidability proof would be to reduce the general synthesis problem for HyperLTL to the model checking problem for PHL. Intuitively, for any HyperLTL formula \( \chi \) one can construct an MDP in which the actions correspond to the possible outputs of a strategy and the probabilistic choices encode the nondeterministic input from the environment. Since schedulers can be randomized, the PHL formula needs to encode the requirement that the implementation must be deterministic, which is easily done in HyperLTL. Thus, the undecidability of synthesis for different fragments of HyperLTL can be used to establish the undecidability of model checking of PHL formulas without probabilistic expressions. However, note that the synthesis problem \( \exists \forall \) fragment of HyperLTL is decidable, so the probabilistic predicate in the PHL formula in the above proof plays a key role in establishing the undecidability of the model checking for PHL formulas of this form.

We saw in the previous section that several examples of interest are PHL formulas of the form \( \forall \sigma_1 \ldots \forall \sigma_n. (\forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n, \psi) \rightarrow P \bowtie c \). Analogously to Theorem 1 we can show that the model checking problem for PHL formulas of the form \( \exists \sigma_1 \ldots \exists \sigma_n. (\forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n, \psi \land P \bowtie c) \) is undecidable. (The proof can be found in the appendix). The undecidability for formulas of the form \( \forall \sigma_1 \ldots \forall \sigma_n. (\forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n, \psi) \rightarrow P \bowtie c \) then follows by duality. In the next two sections we present an approximate model checking procedure and a bounded model checking procedure for PHL formulas in these two classes.
5 Approximate Model Checking

In this section we provide a sound but incomplete procedure for model checking a fragment of PHL. The fragment we consider consists of those PHL formulas that are positive Boolean combinations of formulas of the form

$$\Phi = \forall \sigma_1 \ldots \forall \sigma_n. (\chi \rightarrow c_1 \cdot P(\varphi_1) + \ldots + c_k \cdot P(\varphi_k) \bowtie c)$$

(1)

where \(\chi = \forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n. \psi\) and the formula \(\psi\) is such that:

- \(\psi\) does not contain path quantifiers,
- \(\psi\) describes an \(n\)-safety property (intuitively, a safety property on \(M^n [7]\)).

The PHL formulas in Example 1, Example 2 and Example 3 in Section 3.5 fall into this class (to see this, note that for Example 1 the conjunction of the probabilistic expressions can be taken outside of the scope of the scheduler quantifiers).

According to the conditions we impose above, \(\psi\) contains at most one path variable associated with each scheduler variable in \(\{\sigma_1, \ldots, \sigma_n\}\). This will allow us to use the classical self-composition approach to obtain an automaton for \(\chi\). Furthermore, requiring that \(\psi\) describes an \(n\)-safety property will enable us to construct a deterministic safety automaton for \(\chi\) which, intuitively, represents the most general scheduler in \(M^n\), such that every scheduler that refines it results in a Markov chain in which all paths satisfy the formula \(\psi\).

Since for every Markov chain \(C\) we have \(Prob^C(\{\pi \in Paths_{inf}(C) \mid \pi \models \varphi\}) = 1 - Prob^C(\{\pi \in Paths_{inf}(C) \mid \pi \models \neg \varphi\})\), it suffices to consider the case when \(\inf\) is \(\leq\) (or \(<\)) and \(c_i \geq 0\) for each \(i = 1, \ldots, k\). Otherwise, if \(\inf\) is \(\geq\) (or \(>\)) we replace each \(c_i \cdot P(\varphi_i)\) by \(c_i \cdot (1 - P(\neg \varphi_i))\), and then move the sum \(- \sum c_i \cdot P(\neg \varphi_i)\) to the other side of the inequality.

When \(\inf\) is \(\leq\) (or \(<\)), if for some \(i\) we have \(c_i < 0\) we can rewrite the probabilistic expression by replacing \(c_i \cdot P(\varphi_i)\) by \(c_i \cdot (1 - P(\neg \varphi_i))\), which is equal to \(-c_i \cdot P(\neg \varphi_i) + c_i\). The constant term \(c_i\) is then moved to the constant expression on the right of \(\inf\), and we are left with \(d_i \cdot P(\varphi_i)\) where \(d_i = -c_i > 0\).

Thus, it suffices to consider probabilistic predicates of the form \(\sum_{i=1}^{k} c_i \cdot P(\varphi_i) \leq c\), where \(c_i > 0\) for each \(i = 1, \ldots, k\). The case when \(\inf\) is \(\leq\) is \(<\) is analogous.

We now describe a sound procedure for checking whether an MDP \(M = (S, Act, P, \Pi, AP, L)\) satisfies a PHL formula \(\Phi\) of the form (1). If the answer is positive, then we are guaranteed that \(M \models \Phi\), but otherwise the result is inconclusive. The method proceeds in the following steps.

1. **Deterministic safety automaton for \(\psi\).** We begin by constructing a deterministic safety automaton \(D_\psi = (Q, S^\omega, \delta, q_0)\) for the \(n\)-hyper safety property \(\psi\). For the language of \(D_\psi\) we have that for each word \(w \in (S^n)^\omega\) it holds that \(w \in L(D_\psi)\) if and only if for some scheduler assignment \(\Sigma\) it holds that \(M, \Sigma, H_w \models \psi\), where \(H_w\) is the path assignment corresponding to the word \(w = (s_{0,1}, s_{0, 0, n}) (s_{1,1}, \ldots, s_{1,n}) \ldots\), formally defined as

\[
H_w = ((\pi_1, s_{0,1} s_{1,1} \ldots), \ldots, (\pi_n, s_{0, n} s_{1,n} \ldots)).
\]
Note that, since $\psi$ does not contain path quantifiers, and all path variables appearing in $\psi$ are assigned in the path assignment $P_\omega$, the scheduler assignment $\Sigma$ does not play any role for the satisfaction of $\psi$.

2. **Product MDP for $M^n$ and $D_\omega$.** As a second step we construct the $n$-self-composition MDP $M^n = (S^n, Act^n, \tilde{P}, \tilde{\epsilon}, AP, \tilde{L})$, and then build the product of $M^n$ with the deterministic safety automaton $D_\omega$, which is the MDP $\tilde{M}_\chi = (\tilde{S}_\chi, Act^n, \tilde{P}_{\chi}, \tilde{\epsilon}_\chi, AP, \tilde{L}_\chi)$ with the following components:

- $\tilde{S}_\chi = S^n \times Q$;
- $\tilde{P}_\chi((\hat{s}, q), a, (\hat{s}', q')) = \begin{cases} \tilde{P}((\hat{s}, a, \hat{s}')) & \text{if } q' = \delta(q, \hat{s}') \\ 0 & \text{otherwise}; \end{cases}$
- $\tilde{\epsilon}_\chi((\hat{s}, q)) = \begin{cases} \tilde{\epsilon}((\hat{s})) & \text{if } q = \delta(q_0, \hat{s}) \\ 0 & \text{otherwise}; \end{cases}$
- $\tilde{L}_\chi((\hat{s}, q)) = \bigcup_{i=1}^n \{a_{\sigma_i} \mid a \in L(\hat{s})(\sigma_i)\}$.

Intuitively, the infinite paths in $\tilde{M}_\chi$ correspond to the tuples of infinite paths in $M$ such that each such tuple satisfies the $n$-hyper safety property $\psi$.

3. **Overapproximation of $\max_{\Sigma_1,\ldots,\Sigma_n} \sum_{i=1}^k (c_i \cdot Probs_{\tilde{M}_\chi,\Sigma_{\chi}}(\varphi_i))$.**

Now, after constructing the MDP $\tilde{M}_\chi$, our goal is to check that for every scheduler assignment $\Sigma$ for the MDP $M$ such that $S = \Sigma_1 \| \ldots \| \Sigma_n \in Sched(M)$ the inequality $\sum_{i=1}^k (c_i \cdot Probs_{\tilde{M}_\chi,\Sigma_{\chi}}(\varphi_i)) \leq c$ is satisfied. That would mean, intuitively, that every scheduler assignment that satisfies $\chi$ also satisfies the above inequality, which is the property stated by $\Phi$.

Note that, if we establish that $\max_{\Sigma_1,\ldots,\Sigma_n} \sum_{i=1}^k (c_i \cdot Probs_{\tilde{M}_\chi,\Sigma_{\chi}}(\varphi_i)) \leq c$, then we have established the above property. Computing exactly the value $\max_{\Sigma_1,\ldots,\Sigma_n} \sum_{i=1}^k (c_i \cdot Probs_{\tilde{M}_\chi,\Sigma_{\chi}}(\varphi_i))$, however, is not algorithmically possible in light of the undecidability results established in the previous section.

Therefore, we will overapproximate this value by computing a value $c^* \geq \max_{\Sigma_1,\ldots,\Sigma_n} \sum_{i=1}^k (c_i \cdot Probs_{\tilde{M}_\chi,\Sigma_{\chi}}(\varphi_i))$ and if $c^* \leq c$, then we can conclude that the property holds. The value $c^*$ is computed as

$$c^* = \max_{\tilde{S} \in Sched(\tilde{M}_\chi)} \sum_{i=1}^k (c_i \cdot Probs_{\tilde{M}_\chi, \tilde{S}}(\varphi_i)).$$

Note that for the schedulers $\tilde{S}$ considered in the maximization of (2) it is not in general possible to decompose $\tilde{S}$ into schedulers $\tilde{S}_1, \ldots, \tilde{S}_n \in Sched(M)$. Therefore we have the following inequality

$$\max_{\tilde{S} \in Sched(\tilde{M}_\chi)} \sum_{i=1}^k (c_i \cdot Probs_{\tilde{M}_\chi, \tilde{S}}(\varphi_i)) \geq \max_{\tilde{S} = \tilde{S}_1 \| \ldots \| \tilde{S}_n} \sum_{i=1}^k (c_i \cdot Probs_{\tilde{M}_\chi, \tilde{S}}(\varphi_i)),$$

which implies that $c^*$ has the desired property.

To compute the value defined in (2) we proceed as follows.
(a) **Deterministic Rabin automata for \( \varphi_1, \ldots, \varphi_k \).** According to the definition of PHL, each of the formulas \( \varphi_i \) is actually an LTL formula over atomic propositions in \( \mathcal{AP}_{\text{sched}} \). Thus, for each \( \varphi_i \), we can construct a deterministic Rabin automaton \( A_i = (Q_i, 2^{\mathcal{AP}_{\text{sched}}}, \delta_i, q_i, 0, (B_{i,j}, G_{i,j})_{j=1}^m) \), such that for every \( w \) we have \( w \in L(A_i) \) if and only if \( w \models \varphi_i \).

(b) **Product MDP for \( M_\chi \) and \( A_1, \ldots, A_k \).** We now compute the product of the MDP \( M_\chi \) constructed earlier and the automata \( A_1, \ldots, A_k \) which is the MDP \( M = M_\chi \otimes A_1 \otimes \ldots \otimes A_k \) defined as given below.

\[
M = M_\chi \otimes A_1 \otimes \ldots \otimes A_k = (\tilde{S}, \text{Act}^n, \tilde{\nu}, Q_1 \times \ldots \times Q_k, \tilde{L}),
\]

where

\[
\begin{align*}
\tilde{S} &= S_\chi \times Q_1 \times \ldots \times Q_k; \\
\tilde{\nu}((\tilde{s}, q_1, \ldots, q_k), a, (\tilde{s}', q_1', \ldots, q_k')) &= \begin{cases} 
\nu_\chi(\tilde{s}, a, \tilde{s}') & \text{if } q_i = \delta_i(q_i, \tilde{L}_\chi(\tilde{s}')) \\
0 & \text{otherwise};
\end{cases}
\end{align*}
\]

\[
\tilde{L}(\tilde{s}, q_1, \ldots, q_k) = \begin{cases} 
\tilde{L}_\chi(\tilde{s}) & \text{if } q = \delta_i(q_i, \tilde{L}_\chi(\tilde{s})) \text{ for } i = 1, \ldots, k \\
0 & \text{otherwise};
\end{cases}
\]

\[
\tilde{L}(\tilde{s}, q_1, \ldots, q_k) = \{q_1, \ldots, q_k\}.
\]

(c) **Success sets for combinations of \( \varphi_1, \ldots, \varphi_k \).** Now we consider each combination of formulas in \( \{\varphi_1, \ldots, \varphi_k\} \), i.e., each subset \( I \subseteq \{1, \ldots, k\} \) such that \( I \neq \emptyset \). For each \( I \), the conjunction of the accepting conditions of the deterministic Rabin automata \( A_i \) for \( i \in I \) is

\[
\bigwedge_{i \in I} \bigvee_{1 \leq j \leq m_i} (\square \neg B_{i,j} \land \square \diamond G_{i,j}),
\]

where \( B_{i,j} = \{(s, q_1, \ldots, q_k) \in \tilde{S} | q_i \in B_{i,j}\} \) and similarly for \( G_{i,j} \).

The above property can be rewritten as

\[
\bigvee_{(j_i \in \{1, \ldots, m_i\})_{i \in I}} \left( (\square \bigwedge_{i \in I} \neg B_{i,j_i}) \land \bigwedge_{i \in I} \square \diamond G_{i,j_i} \right).
\]

With this representation, we apply the methods described in [4] to compute the so called success set \( U_I \subseteq \tilde{S} \) for the property \([3]\) for each \( I \subseteq \{1, \ldots, k\} \). Intuitively, in \( U_I \) there exists a scheduler that can enforce the conjunction of the properties whose indices are in the set \( I \).

(d) **Linear program for optimal scheduler.** Finally, in order to compute the value defined in \([3]\) we solve the following linear program.

\[
\text{minimize } \sum_{\tilde{s} \in \tilde{S}} x_{\tilde{s}} \text{ subject to: }
\]

\[
x_{\tilde{s}} \geq 0 \quad \text{for all } \tilde{s} \in \tilde{S}
\]

\[
x_{\tilde{s}} \geq \sum_{i \in I} c_i \quad \text{for all } I \subseteq \{1, \ldots, k\} \text{ and } \tilde{s} \in U_I
\]

\[
x_{\tilde{s}} \geq \sum_{\tilde{a} \in \tilde{S}} P(\tilde{s}, a, \tilde{a}) \cdot x_{\tilde{a}} \quad \text{for all } \tilde{s} \in \tilde{S} \text{ and } a \in \text{Act}^n.
\]
Let \( (x^*_s)_{s \in \tilde{S}} \) be the optimal solution of the linear program (4), and let
\[
c^* = \sum_{\tilde{s} \in \tilde{S}} \tilde{\iota}(\tilde{s}) \cdot x^*_s.
\]

4. **Checking** \( M \models \Phi \). If \( c^* \leq c \), then for all tuples of schedulers \( \mathcal{S}_1, \ldots, \mathcal{S}_n \) we have that if \( M_{\mathcal{S}_1 \parallel \ldots \parallel \mathcal{S}_n} \models \chi \), then for their product \( \mathcal{S} = \mathcal{S}_1 \parallel \ldots \parallel \mathcal{S}_n \) it holds that
\[
\sum_{i=1}^k (c_i \cdot \text{Prob}_{M, \mathcal{S}}(\varphi_i)) \leq c,
\]
and we conclude that \( M \models \Phi \).

If, on the other hand, we have \( c^* > c \), then the result is inconclusive.

In the appendix we establish the correctness of the model checking procedure for establishing \( M \models \Phi \) for formulas of the form (1), stated in the next theorem.

**Theorem 2 (Correctness).** Let \( \tilde{M} = \tilde{M}_\chi \otimes A_1 \otimes \ldots \otimes A_k \) be the MDP constructed above, and let \( (x^*_s)_{s \in \tilde{S}} \) be the optimal solution to the linear program (4).

If \( \vDash \in \{\leq, <\} \), then it holds that
\[
\sum_{\tilde{s} \in \tilde{S}} \tilde{\iota}(\tilde{s}) \cdot x^*_\tilde{s} \vDash c \implies M \models \forall \sigma_1 \ldots \forall \sigma_n. (\chi \rightarrow \sum_{i=1}^k c_i \cdot \text{Prob}(\varphi_i) \vDash c).
\]

The success set \( U_I \) for each of the sets \( I \subseteq \{1, \ldots, k\} \) can be computed in time polynomial in the size of \( \tilde{M} \). Since we can include in (4) for each \( \tilde{s} \in \tilde{S} \) only the inequality \( x_{\tilde{s}} \geq \sum_{i \in I} c_i \) with the largest \( \sum_{i \in I} c_i \), the size of the constraint system in (4) is polynomial in the size of \( \tilde{M} \). Thus, we have the following result.

**Theorem 3 (Complexity).** Given an MDP \( M = (S, \text{Act}, \text{P}, \iota, \text{AP}, L) \) and a PHL formula \( \Phi \) of the form (7) the model checking procedure above runs in time polynomial in the size of \( M \) and doubly exponential in the size of \( \Phi \).

Note that when \( M \models \Phi \) the result of the above procedure will be inconclusive. In the next section we present a bounded model checking procedure which can be used to search for counterexamples to PHL formulas of the form (1).

## 6 Bounded Model Checking

We present a bounded mode-checking procedure for PHL formulas of the form
\[
\Phi = \exists \sigma_1 \ldots \exists \sigma_n. (\chi \land c_1 \cdot \text{P}(\varphi_1) + \ldots + c_k \cdot \text{P}(\varphi_k) \vDash c)
\]
where \( \chi \) is a formula in the \( \forall^* \) fragment of HYPERLTL [16].

Note that by finding a scheduler assignment that is a witness for a PHL formula of the form (5) we can find counterexamples to PHL formulas of the form (1), the model checking of which we studied in the previous section.

Given an MDP \( M = (S, \text{Act}, \text{P}, \iota, \text{AP}, L) \), a bound \( b \in \mathbb{N} \), and a PHL formula \( \Phi = \exists \sigma_1 \ldots \exists \sigma_n. (\chi \land c_1 \cdot \text{P}(\varphi_1) + \ldots + c_k \cdot \text{P}(\varphi_k) \vDash c) \), the bounded model checking
problem for \( M, b \) and \( \Phi \) is to determine whether there exists a deterministic finite-memory scheduler \( \hat{\mathcal{S}} = \hat{\mathcal{S}}_1 | \cdots | \hat{\mathcal{S}}_n \) for \( M^n \) composed of deterministic finite-memory schedulers \( \mathcal{S}_i = (Q_i, \delta_i, q_{i0}, \text{act}_i) \) for \( M \) for \( i \in \{1, \ldots, n\} \), with \( |\hat{\mathcal{S}}| = b \) such that \( M^n_{\hat{\mathcal{S}}} \models \chi \land \sum_{i=1}^k (c_i \cdot \mathbb{P}(\varphi_i)) \bowtie c \).

In the following, we present a procedure for solving the bounded model checking problem that employs bounded synthesis for HyperLTL \cite{16} and model checking of Markov chains \cite{22}. The workflow diagram of our procedure is depicted in Figure 1. The procedure starts by checking whether there is a scheduler \( \hat{\mathcal{S}} \) for \( M^n \) composed of schedulers \( \mathcal{S}_1, \ldots, \mathcal{S}_n \) for \( M \) that satisfies the constraint given by the hyperproperty \( \chi \). This is done by synthesizing an implementation of size \( b \) for the HyperLTL formula \( \varphi_M^n \) composed of the formula \( \chi \), an encoding of \( M \), which ensures that the schedulers \( \mathcal{S}_1, \ldots, \mathcal{S}_n \) defining \( \hat{\mathcal{S}} \) follow the structure of \( M \), and an additional consistency constraint that requires \( \hat{\mathcal{S}} \) to be a composition of \( n \) schedulers \( \mathcal{S}_1, \ldots, \mathcal{S}_n \) for \( M \). The precise formulation of \( \varphi_M^n \) is given below. If \( \varphi_M^n \) is realizable, then the procedure proceeds by applying the synthesized scheduler \( \hat{\mathcal{S}} \) to the \( n \)-self-composition of the MDP \( M \), which results in a Markov chain \( \hat{C} = M^n_{\hat{\mathcal{S}}} \). To check whether the synthesized scheduler also satisfies the probabilistic constraint \( P \bowtie c \), we apply a probabilistic model checker to the Markov chain \( \hat{C} \) to compute for each \( \varphi_i \) the probability \( \mathbb{P} \hat{C}(\varphi_i) \), and then we evaluate the probabilistic predicate \( P \bowtie c \). If \( \hat{C} \) satisfies \( P \bowtie c \), then \( M^n_{\hat{\mathcal{S}}} \models \chi \land \sum_{i=1}^k (c_i \cdot \mathbb{P}(\varphi_i)) \bowtie c \), implying that \( M \models \Phi \). If not, we return back to the HyperLTL synthesizer to construct a new implementation. In order to exclude the scheduler \( \hat{\mathcal{S}} \) from the search for a new implementation, a new constraint \( \forall \mathcal{S} \) is added to \( \varphi_M^n \). The formula \( \varphi_M^n \) imposes the requirement that the synthesized scheduler should be different from the scheduler \( \hat{\mathcal{S}} \). This iterative procedure is repeated until a scheduler that is a witness for \( \Phi \) is found, or all candidate schedulers within the given bound \( b \) have been checked.

The formula \( \varphi_M^n \) is constructed as follows. Let \( \chi = \forall \pi_1 : \sigma_1 \cdots \forall \pi_n : \sigma_n, \psi \). Then \( \varphi_M^n = \forall \pi_1 \cdots \forall \pi_n : (\psi \land \psi_M \land \psi_{\text{consistent}}) \), where

![Workflow Diagram](https://example.com/workflow.png)
### 6.2 Model Checking Plan Non-interference

| Benchmark     | MDP Size | # Iterations | Synthesis time (s) | Model checking time (s) |
|---------------|----------|--------------|--------------------|-------------------------|
| Arena 3       | 16       | 6            | 12.04              | 2.68                    |
| Arena 4       | 36       | 5            | 17.23              | 2.19                    |
| Arena 5       | 81       | 5            | 18.49              | 2.76                    |
| Arena 7       | 256      | 5            | 19.46              | 3.01                    |
| Arena 9       | 625      | 7            | 168.27             | 4.72                    |
| 3-Robots Arena 4 | 36   | 9            | 556.02             | 4.5                     |

Table 1. Model checking plan non-interference.

| Iteration | Synthesis (s) | Model checking (s) |
|-----------|---------------|--------------------|
| 1         | 3.723         | 0.504              |
| 2         | 3.621         | 0.478              |
| 3         | 3.589         | 0.469              |
| 4         | 3.690         | 0.495              |
| 5         | 3.934         | 0.514              |
| 6         | 4.898         | 0.528              |
| 7         | 11.429        | 0.535              |
| 8         | 60.830        | 0.466              |
| 9         | 460.310       | 0.611              |

Table 2. Experimental data of 3-Robots Arena 4 benchmark

\[- \psi_M \text{ encodes the transitions of the MDP } M \text{ and is given as the formula} \]
\[
\bigwedge_{1 \leq i \leq n} \bigwedge_{s \in S} \left( (s_{\sigma_i})_{\pi_i} \rightarrow \bigwedge_{a \in \text{Act} \setminus \text{Act}_s} \neg (a_{\sigma_i})_{\pi_i} \right)
\]

where \( \text{Act}_s = \{a \in \text{Act} \mid \exists s' \in S. P(s, a, s') \neq 0\} \) for \( s \in S \).

\[- \psi_{\text{consistent}} \text{ specifies that it should be possible to decompose the synthesized scheduler to } n \text{ schedulers for } M \text{ and is given by} \]
\[
\bigwedge_{1 \leq i \leq n} \bigwedge_{a \in \text{Act}} (a_{\sigma_i})_{\pi_1} = (a_{\sigma_i})_{\pi_2} \land (s_{\sigma_i})_{\pi_1} \neq (s_{\sigma_i})_{\pi_2}.
\]

Theorem 4 (Complexity). Given an MDP \( M = (S, \text{Act}, P, \iota, AP, L) \), a bound \( b \in \mathbb{N} \), and a PHL formula \( \Phi = \exists \sigma_1 \ldots \exists \sigma_n. \chi \land c_1 P(\varphi_1) + \ldots + c_k P(\varphi_k) \preceq c \), the bounded model checking problem for \( M, b \) and \( \Phi \) can be solved in time polynomial in the size of \( M \), exponential in \( b \), and exponential in the size of \( \Phi \).

### 7 Experimental Results

We developed a proof-of-concept implementation of bounded model checking approach described in Figure 1. We used the tool BoSyHyper [16] for the scheduler synthesis and the tool PRISM [21] to model check the Markov chains resulting from applying the synthesized scheduler to the self-composition of the input MDP. For our experiments, we used a machine with 3.3 GHz dual-core Intel Core i5 and 16 GB of memory.
Table 1 shows the results of model checking the “plan non-interference” specification introduced in Section 3.5 against MDP’s representing two robots that try to reach a designated cell on grid arenas of different sizes ranging from 3 to 9 cells. In the last instance, we increased the number of robots to three to raise the number of possible schedulers. The specification thus checks whether the probability for a robot to reach the designated area changes with the movements of another robot in the arena. In the initial state, every robot is positioned on the edges of the grid, i.e., the farthest point from the designated cell.

In all instances, presented in Table 1, the specification with $\varepsilon = 0.25$ is violated. The table displays the number of iterations, i.e., the number of schedulers synthesized, until a counterexample was found. The synthesis and model checking time represent the total time for synthesizing and model checking all schedulers.

Table 1 shows the feasibility of approach, however, it also demonstrates the inherent difficulty of the synthesis problem for hyperproperties. This is further detailed in Table 2 where the time needed for synthesizing a scheduler quickly increases in the last iterations. The time for model checking the resulting Markov chains remains stable for each of the schedulers. Hence, the time needed for overall model checking approach is dominated by the time needed for synthesis.

Despite recent advances on the synthesis from hyperproperties [16,10], synthesis tools for hyperproperties are still in their infancy. PHL model checking will directly profit from future progress on this problem.

8 Conclusion

We presented a new logic, called PHL, for the specification of probabilistic temporal hyperproperties. The novel and distinguishing feature of our logic is the combination of quantification over both schedulers and paths, and a probabilistic operator. This makes PHL uniquely capable of specifying hyperproperties of MDPs. We demonstrated that PHL is capable of expressing interesting properties both from the realm of security and from the planning and synthesis domains. While, unfortunately, the expressiveness of PHL comes at a price as the model checking problem for PHL is undecidable, we show how to approximate the model checking problem from two sides by providing sound but incomplete procedures for proving and for refuting universally quantified PHL formulas. We developed a proof-of-concept implementation of the refutation procedure and demonstrated its principle feasibility on an example from planning.

We believe that this work opens up a line of research on the verification of MDPs against probabilistic hyperproperties. One direction of future work is identifying fragments of the logic or classes of models that are practically amenable to verification. Furthermore, investigating the connections between PHL and simulation notions for MDPs, as well studying the different synthesis questions expressible in PHL are both interesting avenues for future work.
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Example 5. Let $M = (\{s_0, s_1, s_2\}, \{\alpha_1, \alpha_2\}, P, \iota, \{a\}, L)$ be the MDP shown in Figure 2.

Consider the formula $\Phi = \exists \sigma. \exists \pi_1: \sigma. \exists \pi_2: \sigma. Q(\alpha_{\pi_1} \land \neg \alpha_{\pi_2})$. If the quantifier is interpreted over $Sched(M)$, then we have $M \models \Phi$, as witnessed by a scheduler that in state $s_0$ chooses action $\alpha_1$ with probability $\frac{1}{2}$ and $\alpha_2$ with probability $\frac{1}{2}$. If the quantifier is interpreted over $DetSched(M)$ then $M \not\models \Phi$ since for every deterministic scheduler $\mathcal{S}$, the Markov chain $M_{\mathcal{S}}$ has exactly one path.

B Further Undecidability Proofs and Results

Theorem 1. The model checking problem for PHL is undecidable.

Proof. We prove the undecidability of the model checking problem for PHL via a reduction from the emptiness problem for probabilistic Büchi automata (PBA), which is known to be undecidable [3]. More precisely, we show how for each PBA $\mathcal{B}$ to construct an MDP $M$ and an PHL formula $\Phi$ such that there exists a infinite word $w$ over $\mathcal{B}$’s alphabet such that $Prob_{\mathcal{B}}(w) > 0$ iff $M \models \Phi$.

Let $\mathcal{B} = (Q, A, \delta, \mu, F)$ be a PBA, where $Q$ is a finite set of states, $A$ is a finite alphabet, $\delta: Q \times A \times Q \to [0, 1]$ is the transition probability function, $\mu$ is the initial distribution and $F \subseteq Q$ is the set of accepting states.

We will now define an MDP in which the scheduler picks letters in $A$ and the probabilistic choices mimic the behaviour of the PBA. The PHL formula will assert the existence of a scheduler such that in the resulting Markov chain all paths have the same scheduler choices (i.e., the scheduler is ”blind”) and in the resulting Markov chain the probability of visiting an accepting state infinitely often is greater than 0. This is equivalent to the scheduler generating word $w \in A^\omega$ regardless of what states are visited by the automaton, and $Prob_{\mathcal{B}}(w) > 0$.

We define the MDP $M = (S, Act, P, \iota, AP, L)$ as follows.

- The states and actions are respectively $S = Q \times (A \cup \{\bot\})$ and $Act = A$. 

The transition probability function $P$ is defined by the function $\delta$ in $B$ and is such that for every $(q, \alpha), (q', \alpha') \in S$ and $\beta \in A$ we have $P((q, \alpha), \beta, (q', \alpha')) = \delta(q, \beta, q')$ if $\alpha' = \beta$, and $P((q, \alpha), \beta, (q', \alpha')) = 0$ otherwise.

The initial distribution $\nu$ is defined by the initial distribution $\mu$ from $B$ and is such that $\nu((q, \alpha)) = \mu(q)$ if $\alpha = \bot$ and $\nu((q, \alpha)) = 0$ for $\alpha \neq \bot$.

The atomic propositions $AP = A \cup \{ f \}$ correspond to the alphabet of $B$ plus a fresh proposition $f$, and $L$ is such that for every $(q, \alpha) \in S$ we have that:

- if $\alpha = \bot$: if $q \notin F$, then $L((q, \alpha)) = \emptyset$, and if $q \in F$, $L((q, \alpha)) = \{ f \}$;
- if $\alpha \in A$: if $q \notin F$, then $L((q, \alpha)) = \{ \alpha \}$, otherwise $L((q, \alpha)) = \{ \alpha, f \}$.

Let $\Phi = \exists \sigma. (\exists \pi : \sigma. \Box \land_{\alpha \in A} (\alpha \leftrightarrow \alpha'_\pi)) \land P(\Box \diamond f) > 0$.

The proof that for the MDP $M$ and the formula $\Phi$ defined above it holds that there exists a word $w$ such that $Prob_B(w) > 0$ if and only if $M \models \Phi$ can be found in the appendix.

$(\Rightarrow)$ Let $w = \alpha_0 \alpha_1 \ldots \in A^\omega$ be such that $Prob_B(w) > 0$.

We define the deterministic scheduler $S : (S \times Act)^* S \rightarrow Act$ such that for any $s_1 \alpha_1 \ldots s_n \alpha_n \alpha'_n \in (S \times Act)^* S$ we have $S(s_1 \alpha_1 \ldots s_n \alpha_n \alpha'_n) = \alpha_n$. That is, the scheduler $S$ blindly follows the word $w$. Thus, it is clear that all paths in $M_S$ have the same sequence of labels (which is the word $w$), and hence, the first conjunct in $\Phi$ will be satisfied. Since $P$ is defined directly following $\delta$ and $f \in L((q, \alpha))$ if and only if $q \in F$, we have that $Prob_M, S(\Box \diamond f) = Prob_B(w)$, and hence, since $Prob_B(w) > 0$ the second conjunct of $\Phi$ will be satisfied.

This concludes the proof that $M \models \Phi$.

$(\Leftarrow)$ Suppose that $M \models \Phi$. Thus, there exists a scheduler assignment $\Sigma = ((\sigma, S))$ such that $M, \Sigma \models (\exists \pi : \sigma. \forall \pi' : \sigma. \Box \land_{\alpha \in A} (\alpha \leftrightarrow \alpha'_\pi)) \land P(\Box \diamond f) > 0$.

Then, since $M_S$ satisfies the first conjunct, there exists a path assignment $\Pi$ such that $M, \Sigma, \Pi \models \forall \pi' : \sigma. \Box \land_{\alpha \in A} (\alpha \leftrightarrow \alpha'_\pi)$. Let $w \in A^\omega$ be the word obtained from the sequence of labels of the path assigned to the path variable $\pi$ in $\Pi$. By the choice of $\Pi$ we have that all paths in $M_S$ are labeled with $w$. Therefore, $Prob_M, S(\Box \diamond f) = Prob_B(w)$. Since $M_S$ satisfies the second conjunct, we have that $Prob_M, S(\Box \diamond f) > 0$, which thus implies that $Prob_B(w) > 0$.

This concludes the proof that model checking PHL is undecidable.

**Theorem 2.** The model checking problem for PHL formulas of the form

$$\exists \sigma_1 \ldots \exists \sigma_n. (\forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n. \psi) \land P \bowtie c$$

is undecidable.

**Proof.** The proof follows the lines of the proof of Theorem 1 and is again by reduction from the emptiness problem for PBA. However, the PHL formula that we will now use will assert the existence of two schedulers, one that builds up a word $w$ and another one that mimics the behaviour of the PBA on $w$. Since the schedulers are independent, this would guarantee that the word $w$ is chosen independently of the execution of the PBA. In order to allow for the two types of schedulers, the state space of the MDP that we will define will consists of two
types of states. In states of the first type the scheduler will build up the word, and in states of the second type the scheduler will mimic the PBA.

Let $B = (Q, A, \delta, \mu, F)$ be a PBA.

Let the MDP $M = (S, \text{Act}, P, \iota, \text{AP}, L)$ have the following components.

- $S = \{s_0, \hat{s}_1\} \uplus \{s_0 \mid \alpha \in A\} \uplus (Q \times (A \uplus \{\hat{\alpha}_2\}))$. Intuitively, $s_0$ is the initial state, $\{\hat{s}_1\} \uplus \{s_0 \mid \alpha \in A\}$ is the part of the states where the scheduler chooses a word $w$, and $Q \times (A \uplus \{\hat{\alpha}_2\})$ is the part of the state space where the PBA $B$ is mimicked.
- $\text{Act} = A \uplus \{\hat{\alpha}_1, \hat{\alpha}_2\}$, where the actions $\hat{\alpha}_1$ and $\hat{\alpha}_2$ serve for choosing which part of the state space will be entered.
- $P : S \times \text{Act} \times S \rightarrow [0, 1]$ is defined as follows.

$$P(s_0, \alpha, s) = \begin{cases} 1 & \text{if } \alpha = \hat{\alpha}_1 \text{ and } s = \hat{s}_1, \\ \mu(q) & \text{if } \alpha = \hat{\alpha}_2 \text{ and } s = (q, \hat{\alpha}_2), \ q \in Q, \\ 0 & \text{otherwise}; \end{cases}$$

$$P(\hat{s}_1, \alpha, s') = \begin{cases} 1 & \text{if } \alpha \in A, s' = s_0, \\ 0 & \text{otherwise}; \end{cases}$$

$$P(s_0, \alpha', s') = \begin{cases} 1 & \text{if } \alpha' \in A, s' = s_0, \\ 0 & \text{otherwise}; \end{cases}$$

$$P((q, \alpha), \alpha', s') = \begin{cases} \delta(q, \alpha', q') & \text{if } \alpha' \in A \text{ and } s' = (q', \alpha'), \ q' \in Q, \\ 0 & \text{otherwise}; \end{cases}$$

Intuitively, from state $s_0$ via one of the deterministic actions $\hat{\alpha}_1$ and $\hat{\alpha}_2$ the corresponding part of the state space is entered.

In states $\hat{s}_1$ and $s_0$ for $\alpha \in A$ the scheduler chooses the next letter and transitions via a deterministic transition to the state for that letter.

In states of the form $(q, \alpha)$ the transitions mimic the behaviour of $B$.

- $\iota(s_0) = 1$ and $\iota(s) = 0$ for $s \neq s_0$.
- $\text{AP} = \{a', a'', f\} \uplus A$. The atomic propositions $a', a''$ are used to indicate which part of the state space the state belongs to, and the proposition $f$ is used to indicate the states accepting in $B$.

- $L : S \rightarrow \text{AP}$ is defined as follows.

$$L(s) = \begin{cases} 0 & \text{if } s = s_0, \\ \{a'\} & \text{if } s = \hat{s}_1, \\ \{a''\} & \text{if } s = (q, \alpha_2), \ q \in Q \setminus F, \\ \{a'', f\} & \text{if } s = (q, \alpha_2), \ q \in F, \\ \{a\} & \text{if } s = s_0, \\ \{\alpha\} & \text{if } s = (q, \alpha), \ q \in Q \setminus F, \ \alpha \in A, \\ \{\alpha, f\} & \text{if } s = (q, \alpha), \ q \in F, \ \alpha \in A. \end{cases}$$
We define the PHL formula
\[ \Phi = \exists \sigma_1. \exists \sigma_2. (\forall \pi_1 : \sigma_1, \forall \pi_2 : \sigma_2, (\bigcirc a'_{\pi_1} \land \bigcirc a''_{\pi_2} \land \square \land_{a \in A}(\alpha_{\pi_1} \leftrightarrow \alpha_{\pi_2})) \land \mathbb{P}(\square \lozenge f_{\pi_2}) > 0). \]

The proof that for the MDP \( M \) and the formula \( \Phi \) defined above it holds that there exists a word \( w \) such that \( \text{Prob}_B(w) > 0 \) if and only if \( M \models \Phi \) is similar to that in Theorem 4.

\[ (\Rightarrow) \text{ Let } w = a_0a_1 \ldots \in A^\omega \text{ be such that } \text{Prob}_B(w) > 0. \]

We define the deterministic scheduler \( \mathcal{S}_1 : (S \times \text{Act})^* S \to \text{Act} \) such that at state \( s_0 \) it chooses \( \hat{a}_1 \) and for any longer sequence \( s_1a_1 \ldots s_n a_n s' \in (S \times \text{Act})^* S \) we have \( \mathcal{S}_1(s_1a_1 \ldots s_n a_n s') = a_{n-1} \). That is, the scheduler \( \mathcal{S}_1 \) enters the first part of the state space and then blindly follows the word \( w \). Thus, it is clear that all paths in \( M_{\mathcal{S}_1} \) have the same sequence of labels (corresponding to the word \( w \)).

We define the second deterministic scheduler \( \mathcal{S}_2 : (S \times \text{Act})^* S \to \text{Act} \) such that at state \( s_0 \) it chooses \( \hat{a}_2 \) and for any longer sequence \( s_1a_1 \ldots s_n a_n s' \in (S \times \text{Act})^* S \) we have \( \mathcal{S}_2(s_1a_1 \ldots s_n a_n s') = a_{n-1} \). That is, the scheduler \( \mathcal{S}_2 \) enters the other part of the state space and then also blindly follows the word \( w \) regardless of the outcome of the probabilistic choices. Thus, all paths in \( M_{\mathcal{S}_2} \) also have the same sequence of labels (corresponding to the word \( w \)).

Therefore, by construction of \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \), the first conjunct is satisfied.

Since \( P \) is defined following \( \delta \) and \( f \in L((q, \alpha)) \) if and only if \( q \in F \), we have that \( \text{Prob}_{M, \mathcal{S}_1}(\square \lozenge f_{\pi_2}) = \text{Prob}_B(w) \), and hence, since \( \text{Prob}_B(w) > 0 \) the second conjunct of \( \Phi \) will be satisfied.

This concludes the proof that \( M \models \Phi \).

\[ (\Leftarrow) \text{ Suppose that } M \models \Phi. \]

Thus, there exists a scheduler assignment \( \Sigma = ((\sigma_1, \mathcal{S}_1), (\sigma_2, \mathcal{S}_2)) \) such that \( M, \Sigma \models (\forall \pi_1 : \sigma_1, \forall \pi_2 : \sigma_2, (\bigcirc a'_{\pi_1} \land \bigcirc a''_{\pi_2} \land \square \land_{a \in A}(\alpha_{\pi_1} \leftrightarrow \alpha_{\pi_2})) \land \mathbb{P}(\square \lozenge f_{\pi_2}) > 0) \). By the construction of the MDP \( M \), and since \( M_\Sigma \) satisfies the first conjunct of \( \Phi \), there must exist at least one infinite path in the Markov chain \( M_{\mathcal{S}_1} \) visiting the state \( \hat{s}_1 \). Let \( w \in A^\omega \) be the word obtained from the sequence of labels on this path. Since \( M_\Sigma \) satisfies the first conjunct, we have that all paths in \( M_{\mathcal{S}_2} \) are labeled with \( w \). Therefore, \( \text{Prob}_{M, \mathcal{S}_1}(\square \lozenge f_{\pi_2}) = \text{Prob}_B(w) \). Since \( M_\Sigma \) satisfies the second conjunct, we have that \( \text{Prob}_{M, \mathcal{S}_2}(\square \lozenge f_{\pi_2}) > 0 \), which thus implies that \( \text{Prob}_B(w) > 0 \).

This concludes the proof.

\[ \square \]

**Corollary 1.** The model checking problem for PHL formulas of the form
\[ \forall \sigma_1 \ldots \forall \sigma_n. ((\forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n, \psi) \to P \triangleright\triangleright c) \]

is undecidable.

**Proof.** The result follows from the fact that for any MDP \( M \) it holds that
\[ M \models \forall \sigma_1 \ldots \forall \sigma_n. ((\forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n, \psi) \to P \triangleright\triangleright c) \text{ iff } M \models \neg \forall \sigma_1 \ldots \forall \sigma_n. ((\forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n, \psi) \to P \triangleright\triangleright c) \text{ iff } M \not\models \exists \sigma_1 \ldots \exists \sigma_n. ((\forall \pi_1 : \sigma_1 \ldots \forall \pi_n : \sigma_n, \psi) \land P \triangleright\triangleright c). \]
where $\mathfrak{R} = \begin{cases} 
abla \text{ if } \triangleright=\triangleright, \\
abla \text{ if } \triangleright=\triangleright, \\
abla \text{ if } \triangleright=\leq, \\
abla \text{ if } \triangleright=\leq. 
\end{cases}$

\[ \mathfrak{R} \]

\section{Correctness of the Procedure in Section \[5\]

Now we provide arguments for the correctness of the procedure in Section \[5\].

For a path $\tilde{\pi} \in \text{Paths}(\tilde{M})$ we denote with $\tilde{\pi}|_{\tilde{S}_x}$ the projection of $\tilde{\pi}$ on its first component. For each $i = 1, \ldots, k$ we denote by $\tilde{\pi}|_i$ the projection of $\tilde{\pi}$ on the component corresponding to $Q_i$. By the construction of $\tilde{M}$ we have that $\tilde{\pi}|_{\tilde{S}_x} \in \text{Paths}(\tilde{M}_x)$, and for each $i = 1, \ldots, k$ the sequence $\tilde{\pi}|_i$ is a run of $\mathcal{A}_i$.

Since the automata $\mathcal{A}_i$ are deterministic, for every path $\tilde{\pi} \in \text{Paths}(\tilde{M}_x)$ there exists a unique path $\hat{\pi} \in \text{Paths}(\hat{M})$ such that $\hat{\pi}|_{\tilde{S}_x} = \tilde{\pi}$.

The next proposition establishes that the paths in the MDP $\tilde{M}_x$ are precisely the paths of the $n$-self-composition $M^n$ that satisfy the $n$-safety property $\psi$.

\begin{proposition}
We have that $\hat{\pi} = (s_{0,1}, \ldots, s_{0,n}, q_0)(s_{1,1}, \ldots, s_{1,n}, q_1) \ldots \in \text{Paths}(\tilde{M}_x)$ if and only if $\pi^n = (s_{0,1}, \ldots, s_{0,n})(s_{1,1}, \ldots, s_{1,n}) \ldots \in \text{Paths}(M^n)$ and $q_0 q_1 \ldots$ is a run of $\mathcal{D}_\psi$ on $\pi^n$. Furthermore, for every scheduler assignment $\Sigma$ and path assignment $\Pi_w = ((\sigma_0, s_{0,1} s_{1,1} \ldots), (\sigma_n, s_{0,n} s_{1,n} \ldots))$ such that $M, \Sigma, \Pi_w \models \psi$ and $\pi^n = (s_{0,1}, \ldots, s_{0,n})(s_{1,1}, \ldots, s_{1,n}) \ldots \in \text{Paths}(M^n)$ there exists a unique run $q_0 q_1 \ldots$ of $\mathcal{D}_\psi$ on $\pi^n$ and $(s_{0,1}, \ldots, s_{0,n}, q_0)(s_{1,1}, \ldots, s_{1,n}, q_1) \ldots \in \text{Paths}(\tilde{M}_x)$.
\end{proposition}

Next we state the properties of the success sets $U_I$, which are a direct consequence from the construction of $U_I$ and standard results presented in \[4\].

\begin{proposition}
For every scheduler $\tilde{\mathcal{S}} \in \text{Sched}(\tilde{M})$ it holds that
\[ \text{Prob}_{\tilde{M}, \tilde{\mathcal{S}}, \tilde{\mathcal{S}}} \left( \mathbf{O} U_I \land \bigwedge_{i \in U_I} \mathbf{O} \mathbf{H} \right) = 1. \]
\end{proposition}

\begin{proposition}
(i) There exists a finite-memory scheduler $\tilde{\mathcal{S}} \in \text{Sched}(\tilde{M})$ s.t. for all $\tilde{s} \in U_I$:
\[ \text{Prob}_{\tilde{M}, \tilde{\mathcal{S}}, \tilde{s}} \left( \bigwedge_{i \in U_I} \mathbf{O} \mathbf{H} \right) = 1. \]

(ii) For every scheduler $\tilde{\mathcal{S}} \in \text{Sched}(\tilde{M})$ it holds that
\[ \text{Prob}_{\tilde{M}, \tilde{\mathcal{S}}} \left( \tilde{\pi} \models \mathbf{O} U_I \mid \forall i \in I : \tilde{\pi} \models \mathbf{O} \bigwedge_{1 \leq j \leq m_i} (\mathbf{O} \mathbf{O} \neg B_{i,j} \land \mathbf{O} \mathbf{O} \neg G_{i,j}) \right) = 1. \]
\end{proposition}
Finally, we show the relationship between the optimal solution to the linear program [(1)] and the value defined in [(2)]. The construction in the proof of the below proposition is similar to the constructions presented in [(11)].

**Proposition 3.** Let $\tilde{M} = \tilde{M}_k \otimes A_1 \otimes \ldots \otimes A_k$ be the MDP constructed above. Then, there exists a solution $(x^*_s)_{s \in \tilde{S}}$ to the linear program [(4)], and it holds that

$$\sum_{s \in \tilde{S}} \tilde{v}(s) \cdot x^*_s = \max_{\tilde{\phi} \in \text{Sched}(\tilde{M})} \sum_{i=1}^k (c_i \cdot \text{Prob}_{\tilde{M}, \tilde{\phi}}(\bigvee_{1 \leq j \leq m_i} (\Diamond \neg \tilde{B}_{i,j} \land \Box \Diamond \tilde{G}_{i,j}))).$$

**Proof.** (Sketch.) To show the relationship between the value

$$\max_{\tilde{\phi} \in \text{Sched}(\tilde{M})} \sum_{i=1}^k (c_i \cdot \text{Prob}_{\tilde{M}, \tilde{\phi}}(\bigvee_{1 \leq j \leq m_i} (\Diamond \neg \tilde{B}_{i,j} \land \Box \Diamond \tilde{G}_{i,j})))$$

and the linear program [(4)] we define an MDP $\tilde{M}'$ obtained from $\tilde{M}$ by a construction analogous to the one presented in [(11)]. We construct the new MDP $\tilde{M}' = (\tilde{S}', \tilde{Act}', \tilde{P}', \tilde{r}', Q_1 \times \ldots \times Q_k, \tilde{L}')$ with the following components:

- $\tilde{S}' = \tilde{S} \uplus \{s^*_I | I \subseteq \{1, \ldots, k\}, I \neq \emptyset\}$;
- $\tilde{Act}' = \tilde{Act}^n \uplus \{a^*_I \mid I \subseteq \{1, \ldots, k\}, I \neq \emptyset\}$;
- $\tilde{P}'(s, a, s') = \begin{cases} P(s, a, s') & \text{if } s, s' \in \tilde{S} \text{ and } a \in \tilde{Act}^n, \\ 1 & \text{if } s \in U_I, a = a^*_I \text{ and } s' = s^*_I \text{ for some } I \subseteq \{1, \ldots, k\}, \\ 1 & \text{if } s = s^*_I \text{ and } s' = s^*_I \text{ for some } I \subseteq \{1, \ldots, k\}, \\ 0 & \text{otherwise;} \end{cases}$
- $\tilde{r}'(s) = \begin{cases} \tilde{r}(s) & \text{if } s \in \tilde{S}, \\ 0 & \text{otherwise;} \end{cases}$
- $\tilde{L}'(s) = \begin{cases} \tilde{L}(s) & \text{if } s \in \tilde{S}, \\ \emptyset & \text{otherwise.} \end{cases}$

We also equip $\tilde{M}'$ with a reward function $r : \tilde{S}' \times \tilde{Act}' \times \tilde{S}' \to \mathbb{R}_{\geq 0}$ where

$$r(s, a, s') = \begin{cases} \sum_{i \in I} c_i & \text{if } s \in U_I, a = a^*_I \text{ and } s' = s^*_I, \\ 0 & \text{otherwise.} \end{cases}$$

In the MDP $\tilde{M}'$ we have added an absorbing state $s^*_I$ for every $\emptyset \neq I \subseteq \{1, \ldots, k\}$ and a transition from every state in $U_I$ to $s^*_I$. Intuitively, reaching $s^*_I$ in $\tilde{M}'$ corresponds to reaching and staying forever in some of the end components that are subsets of $U_I$. The reward function associates reward $r_I = \sum_{i \in I} c_i$ with each of the transitions entering $s^*_I$ from $U_I$, and all other transitions have reward 0. Intuitively, a path in $\tilde{M}'$ receives reward $r_I$ upon choosing to remain forever in
the set $U_I$. We consider the maximal total expected reward in $(\tilde{M}, r)$ and claim that it is equal to the quantity of interest in $\tilde{M}$. Formally, we claim that

$$\max_{\tilde{S} \in \text{Sched}(\tilde{M})} \sum_{i=1}^{k} (c_i \cdot P_{\tilde{M}, \tilde{S}}(\bigvee_{1 \leq j \leq m_i} (\square \neg \tilde{B}_{i,j} \land \square \tilde{G}_{i,j}))) =$$

$$\max_{\tilde{S} \in \text{Sched}(\tilde{M})} \mathbb{E}_{\tilde{M}, \tilde{S}}[\sum_{j=0}^{\infty} r(s_j, a_j, s_{j+1})].$$

By the definition of the linear program (4), the quantity on the right-hand side of the above equality is in fact the optimal solution to the linear program (4), which follows from standard results in stochastic dynamic programming. Thus the claim of the theorem follows from the above equality, which, in turn can be established in an analogous way to the corresponding results in [11] using the properties of the success sets $U_I$. 

Finally, combining the results above we establish the correctness of the model checking procedure for establishing $M \models \Phi$ for formulas of the form (1).

Theorem 2 (Correctness). Let $\tilde{M} = \tilde{M} \chi \otimes A_1 \otimes \ldots \otimes A_k$ be the MDP constructed above, and let $(x^*_s)_{s \in \tilde{S}}$ be the optimal solution to the linear program (4). If $\gg \in \{\leq, <\}$, then it holds that

$$\sum_{s \in \tilde{S}} \tilde{t}(s) \cdot x^*_s \gg c \text{ implies } M \models \forall \sigma_1 \ldots \forall \sigma_n. (\chi \rightarrow \sum_{i=1}^{k} c_i \cdot P(\phi_i) \gg c).$$