Comparison between Second Variation of Area and Second Variation of Energy of a Minimal Surface

Norio Ejiri
Department of Mathematics, Faculty of Science and Technology, Meijo University
1-501 Shiogamaguchi, Tempaku-ku, Nagoya-shi, Aichi 468-8502 Japan
ejiri@ccmfs.meijo-u.ac.jp

Mario Micallef
Mathematics Institute, University of Warwick, Coventry, CV4 7AL, U.K.
M.J.Micallef@warwick.ac.uk

February 1, 2008

1 Statement and discussion of the results

The conformal parameterisation of a minimal surface is harmonic and therefore, it is natural
to compare the Morse index $i_A$ of a minimal surface as a critical point of the area functional
$A$ with its Morse index $i_E$ as a critical point of the energy functional $E$. Indeed, one way by
which minimal surfaces are produced is by first finding a map which is harmonic with respect
to a fixed conformal structure on the surface and then varying the conformal structure until
we find a harmonic map whose energy is critical with respect to variations of the conformal
structure. This procedure is well known and has been used very successfully by Douglas
[6], Courant [5], Schoen and Yau [32], Sacks and Uhlenbeck [31], Tomi and Tromba [34] and
others.

We now state a theorem relating $i_A$ to $i_E$:

Theorem 1.1. Let $F : \Sigma_g \rightarrow M$ be a (possibly branched) minimal immersion of a closed
Riemann surface of genus $g$ into a Riemannian manifold $M$. Then

$$i_E \leq i_A \leq i_E + r$$

where, if $b =$ total number of branch points of $F$ counted with multiplicity then

$$r = \begin{cases}
6g - 6 - 2b & \text{if } b \leq 2g - 3, \\
4g - 2 + 2 \left\lfloor \frac{b}{2} \right\rfloor & \text{if } 2g - 2 \leq b \leq 4g - 4 \text{ and } \lfloor x \rfloor \text{ denotes the} \\
\text{largest integer less than or equal to } x, \\
0 & \text{if } b \geq 4g - 3.
\end{cases}$$

Note: if $g = 0$, $r = 0$ and if $g = 1$, then $r = 2$ if $b = 0$ and $r = 0$ if $b > 0$.

Remarks.

$^1$Recall that the index of a functional at a critical point $F$ is the number, counted with multiplicity, of
negative eigenvalues of the hessian (i.e., Jacobi operator) of the functional at the critical point. Equivalently,
the index is the dimension of a maximal subspace of the space of infinitesimal variations of $F$ on which the
hessian is negative definite.
(1) $r \leq 6g - 6 = \text{real dimension of Teichmüller space}$. This is not surprising in light of the first paragraph of this paper; see also §2.

(2) If $g = 0$, then any harmonic map is conformal (and therefore also minimal) and $i_A = i_E$. This is due to the fact that the two-sphere carries a unique conformal structure and it was proved by the second author in [22], Lemma 3.2. This paper arose out of that work.

(3) We can also compare the nullity of $F$ as a critical point of the area functional with the nullity of $F$ as a critical point of the energy functional. See Theorem 3.1 in §3 for a precise statement.

(4) Moore has also recently studied the relation between the second variations of area and energy in §5 of [24]. However, his line of investigation is different from ours. In particular, he does not compare the indices of the two functionals.

Theorem 1.1 enables us to obtain an upper bound on the index of a minimal surface in an arbitrary Riemannian manifold which depends on the area and genus of the surface, and the dimension and geometry of the ambient manifold; see Theorem 1.3. The bound does not depend on the second fundamental form of the minimal surface.

We now consider the index of complete minimal surfaces in $\mathbb{R}^d$. Let $(\Omega_i)_{i \in \mathbb{N}}$ be an exhaustion of a complete minimal surface $\Sigma$ by an increasing sequence of compact subsets. The index of $\Sigma$ is defined as $\sup_{i \to \infty} \text{index}(\Omega_i)$. In her pioneering work [12], Fischer-Colbrie showed that a complete, oriented minimal surface $\Sigma$ of finite total curvature in $\mathbb{R}^3$ has finite index; see also [16]. The proof works equally well in $\mathbb{R}^d$, $d > 3$ (see, for instance, [25]). However, no bound was given on the index in terms of the total curvature. For $d = 3$, this was first carried out by Tysk in [33] and then improved by Nayatani in [27], Theorem 4. The case $d > 3$ was treated by the first author in [8], and also by Cheng and Tysk in [1]. Among other results, they proved that there exists a constant $c_d$, depending only on the dimension $d$ of the ambient Euclidean space, such that

$$\text{index}(\Sigma) \leq c_d \int_{\Sigma} (-K) \, dA$$

where $K$ is the Gauss curvature of $\Sigma$ and $dA$ is the element of area on $\Sigma$. Unfortunately $c_d \to \infty$ as $d \to \infty$.

The method used in the proof of Theorem 1.1 can also be used to establish the following:

**Theorem 1.2.** Let $\Sigma$ be a complete, oriented, non-planar minimal surface with finitely many branch points and of finite total curvature in $\mathbb{R}^d$. Then

$$\text{index}(\Sigma) \leq \frac{1}{\pi} \int_{\Sigma} (-K) \, dA + 2g - 2 \leq \frac{3}{2\pi} \int_{\Sigma} (-K) \, dA - r + b,$$

where $g = \text{genus of } \Sigma$, $r = \text{number of ends of } \Sigma$ and $b = \text{total number of branch points counted with multiplicity}$. When $d = 3$ the above inequality may be improved to

$$\text{index}(\Sigma) \leq \frac{1}{\pi} \int_{\Sigma} (-K) \, dA + 2g - 3 \leq \frac{3}{2\pi} \int_{\Sigma} (-K) \, dA - r + b - 1.$$  \hspace{1cm} (1.3)

\[2\] We shall always mean the area index when referring to the index of a minimal surface in $\mathbb{R}^d$.

\[3\] In [15] Grigory’yan and Yau have proved an estimate of this type even for a minimal surface in $\mathbb{R}^3$ with boundary.
Remark. We can also make statements about nullity(Σ); see Theorem 3.2 in §3 for a precise statement. It suffices to state here that we will show that, when \( d = 3 \),

\[
\text{index}(\Sigma) + \text{nullity}(\Sigma) \leq \frac{1}{\pi} \int_{\Sigma} (-K)\,dA + 2g.
\]

This estimate is similar to, but worse than, the one proved by Nayatani in [27], Theorem 4:

\[
\text{If } \int_{\Sigma} (-K)\,dA \geq 8\pi, \text{ then } \text{index}(\Sigma) + \text{nullity}(\Sigma) \leq \frac{3}{4\pi} \int_{\Sigma} (-K)\,dA + 3g.
\]

(1.4)

So, perhaps the most striking feature of (1.2) is that it is valid for all \( d \geq 3 \). The translational Jacobi fields show that, for a non-planar minimal surface in \( \mathbb{R}^3 \), nullity(\(\Sigma\)) \(\geq 3\). In particular,

\[
\text{If } \int_{\Sigma} (-K)\,dA \geq 8\pi, \text{ and } d = 3, \text{ then } \text{index}(\Sigma) \leq \frac{3}{4\pi} \int_{\Sigma} (-K)\,dA + 3g - 3.
\]

(1.5)

Fischer-Colbrie showed in [12] that a complete, oriented minimal surface in \( \mathbb{R}^3 \) of finite index has finite total curvature; see also [17]. This does not hold for minimal surfaces in \( \mathbb{R}^d \), \( d \geq 4 \) because any holomorphic curve (in particular, one with infinite total curvature) in \( \mathbb{C}^2 = \mathbb{R}^4 \) is area-minimizing on compact subsets and therefore has index zero. (A partial converse to this fact was proved in [21].) The work of Fischer-Colbrie, of course, raises the question of obtaining lower bounds on the index of a minimal surface in \( \mathbb{R}^3 \) in terms of the total curvature. The first result in this direction was obtained by Fischer-Colbrie and Schoen [13] (see also [7] and [29]) and states that a complete, stable (i.e., index zero) oriented, minimal surface in \( \mathbb{R}^3 \) is a plane. Since then, several authors have obtained lower bounds, some of which we shall compare in the following remarks to the upper bounds furnished by (1.3) and (1.5). We refer to §7 of [18] for a fuller discussion of the index of minimal surfaces of finite total curvature in \( \mathbb{R}^3 \).

Remarks.

(1) If the total curvature of \( \Sigma \) in \( \mathbb{R}^3 \) is \(-4\pi\), then the Gauss map is 1-1 and therefore \( \text{index}(\Sigma) = 1 \). It will also follow from Theorem 3.2 in §3 that the nullity in this case is precisely equal to 3. The only complete immersed minimal surfaces with total curvature equal to \(-4\pi\) are the catenoid and Enneper’s surface; see [28]. However, Rosenberg and Toubiana have constructed several examples in [30] of branched minimal surfaces in \( \mathbb{R}^3 \) of total curvature \(-4\pi\). When \( \int_{\Sigma} (-K)\,dA = 4\pi \), genus of \(\Sigma\) = 0 (because the Gauss map is 1-1) and therefore, (1.3) gives \(\text{index} \leq 1\). This shows that (1.3) is sharp in this sense. We also note that, conversely, Lópéz and Ros proved in [20] that the only complete immersed minimal surfaces in \( \mathbb{R}^3 \) of index 1 are the catenoid and Enneper’s surface; see also [23].

(2) If \( \Sigma_k \) is the Jorge-Meeks \(k\)-noid of genus zero and \( k \) ends, then the estimate in (1.3) can be improved to \( \text{index}(\Sigma_k) + \text{nullity}(\Sigma_k) \leq 2k \); see §4. But in [2] it is shown that \( \text{index}(\Sigma_k) \geq 2k - 3 \). Therefore, \( \text{index}(\Sigma_k) = 2k - 3 \) and nullity(\(\Sigma_k\)) = 3. (Nayatani obtained this result by direct calculation in [26]; see also [23].) Once again the methods of this paper yield sharp results. However, the remarks below indicate that this is not so in general.

(3) If the total curvature of \( \Sigma \) in \( \mathbb{R}^3 \) is \(-8\pi\) and \( \Sigma \) has genus zero then, (1.3) and the lower bound in [2] yield \( 5 \geq \text{index}(\Sigma) \geq 3 \) whereas according to (1.5), \( \text{index}(\Sigma) \leq 3 \) and therefore, \( \text{index}(\Sigma) = 3 \). This has been proved by the first author and Kotani in [11], Corollary 4.3.
The Chen-Gackstatter surface has total curvature equal to $-8\pi$ and genus 1. Montiel and Ros showed that the index of this surface is equal to 3 in [23], Corollary 15. On the other hand, according to (1.5) the index is at most 6 and, according to (1.3) the index is at most 7. This lack of sharpness of (1.3) is not surprising as it does not take into account any special geometric properties of the minimal surface whereas Corollary 15 in [23] exploits the fact that the branching values of the Gauss map of the Chen-Gackstatter surface all lie on an equator.

(4) The first author and Kotani [11] and Montiel and Ros [23] independently proved that the index of a complete minimal surface in $\mathbb{R}^3$ of genus zero is at most $\frac{1}{2\pi} \int_{\Sigma} (-K) dA - 1$ and that generically the index is equal to this number. This shows that (1.5) (and therefore also (1.3)) is not sharp when the total curvature is less than or equal to $-\frac{1}{12}\pi$.

(5) We leave the reader to check that (1.3) and (1.5) are also not sharp for Bryant’s surface and the Hoffman-Meeks surfaces of genus $g$ and three ends.

This article is organised as follows. In §2 we show that the second variation of area for a given normal deformation $s$ is less than or equal to the second variation of energy for a deformation $v$ whose normal component is $s$. Formula (2.27) shows that the difference between the two second variations vanishes precisely when $v$ is an infinitesimal conformal deformation, which is defined by (2.5). The proofs of the main theorems, based on (2.27) and an application of the Riemann-Roch theorem, are given in §3. In the final section we prove the upper bound on the index of a minimal surface in an arbitrary Riemannian manifold mentioned above. We also obtain a smaller upper bound than that given by Theorem 1.2 for the index of a minimal surface of finite total curvature in $\mathbb{R}^3$ which has appropriate symmetry; see Theorem 4.5.

2 Infinitesimal conformal deformations and motivation for the proof of Theorem 1.1

Given a map $F: \Sigma \to M$ from a Riemann surface into a Riemannian manifold, let $z = x + iy$ be a local complex co-ordinate on $\Sigma$. Then,

the energy integrand, $e(F) := \frac{1}{2} \{|F_x|^2 + |F_y|^2\}$, (2.1)

where $F_x = F_*(\partial_x)$, $F_y = F_*(\partial_y)$, and the norm $|.|$ is taken with respect to the Riemannian metric $\langle\cdot,\cdot\rangle$ on $TM$. The

area integrand, $g(F) := \{|F_x|^2|F_y|^2 - \langle F_x, F_y \rangle^2\}^{1/2}$. (2.2)

Therefore

$e(F) \geq g(F)$ with equality if, and only if, $F$ is conformal. (2.3)

We therefore see that, a variation which decreases the energy $E := \int_{\Sigma} e(F) dxdy$ of a conformal harmonic map $F$ must also decrease the area $A := \int_{\Sigma} g(F) dxdy$ of the map, and therefore $i_A \geq i_E$. Conversely, a variation which decreases the area of a conformal harmonic map will also decrease the energy if we could reparameterise the variation so as to maintain it conformal with respect to the initial conformal structure. Of course, the obstruction to doing this comes from Teichmüller space.
We now make the above reasoning more formal. Let \( \nu \) denote the normal bundle of \( \Sigma \) and let \( s \in \Gamma(\nu) \) be such that the second variation of area in the direction of \( s \), \( (\delta^2A)(s) \), is negative. Let \( \xi \) denote the ramified tangent bundle of \( \Sigma \), i.e. \( \xi \) is the tangent bundle of \( \Sigma \) twisted at the branch points by an amount equal to the order of branching so that \( F^*(TM) = \xi \oplus \nu \). We wish to find \( \sigma_s \in \Gamma(\xi) \) such that

1. the map \( s \mapsto \sigma_s \) is linear,
2. the family of maps corresponding to the variation vector field \( s + \sigma_s \) is a family of conformal maps.

If we succeed, then \( (\delta^2E)(s + \sigma_s) = (\delta^2A)(s) < 0 \). Of course, \( \delta^2E \) is the hessian (or second variation) of the energy functional \( E \).

We will now derive a differential equation for \( \sigma_s \) that will guarantee property (2) at the infinitesimal level. Let \( I = (-\varepsilon, \varepsilon) \subset \mathbb{R}, \varepsilon > 0 \) and let \( V : I \times \Sigma \to M \) be such that \( V(0, \cdot) = F(\cdot) \) and \( (V_\varepsilon(0, \cdot))(\partial_t) = s(\cdot) \). We now want \( \varphi : I \times \Sigma \to \Sigma \) such that \( \varphi(0, \cdot) = \text{identity} \) and \( \tilde{V}(t, \cdot) := V(t, \varphi(t, \cdot)) \) is conformal with respect to the given fixed conformal structure on \( \Sigma \) for all \( t \in I \). The conformality of \( \tilde{V}(t, \cdot) \) can be expressed by

\[
\langle \tilde{V}_s(\partial_z), \tilde{V}_s(\partial_z) \rangle = 0
\]  

(2.4)

where \( z \) is a local complex co-ordinate on \( \Sigma \) and \( \langle \cdot, \cdot \rangle \) denotes the Riemannian metric on \( TM \) extended complex bilinearly to \( T_\mathbb{C}M := TM \otimes_{\mathbb{R}} \mathbb{C} \). Differentiating (2.4) with respect to \( t \) and setting \( t = 0 \) gives:

\[
\langle \nabla_{\partial_t}(s + \sigma_s), F_z \rangle = 0
\]  

(2.5)

where \( \sigma_s = F_s((\phi_s(0, \cdot))\partial_t) \in \Gamma(\xi) \) and \( \nabla \) is the Levi-Civita connection on \( M \) pulled back to \( F^*(TM) \) and extended complex linearly to \( F^*(T_\mathbb{C}M) \). For obvious reasons a vector field \( v \in \Gamma(F^*(TM)) \) which satisfies \( \langle \nabla_{\partial_z}v, F_z \rangle = 0 \) is called an infinitesimal conformal deformation.

We now recall that the complex structure on \( \Sigma \) gives rise to the splitting \( \xi_\mathbb{C} := \xi \otimes_{\mathbb{R}} \mathbb{C} = \xi^{1,0} \oplus \xi^{0,1} \) where the fibre of \( \xi^{1,0} \) (\( \xi^{0,1} \)) is locally spanned by \( F_z(\bar{F}_z) \) away from the branch points. (The holomorphicity of \( F_z \) is required to explicitly trivialize \( \xi^{1,0} \) on a neighbourhood of a branch point.) Therefore we may write \( \sigma_s = \sigma_s^{1,0} + \sigma_s^{0,1} \). Next observe that

\[
\langle \nabla_{\partial_z}\sigma_s^{1,0}, F_z \rangle = 0 \quad \text{and} \quad \langle \nabla_{\partial_z}\sigma_s^{0,1}, F_z \rangle = 0 \quad \text{by conformality of} \ F. 
\]  

(2.6)

Moreover

\[
\langle \nabla_{\partial_z}s, F_z \rangle = -\langle s, \nabla_{\partial_z}F_z \rangle = 0 \quad \text{by harmonicity of} \ F. 
\]  

(2.7)

Using (2.6) and (2.7) one sees that (2.5) may be re-written as

\[
(\nabla_{\partial_z}\sigma_s^{0,1})^\top = -(\nabla_{\partial_z}s)^\top
\]  

(2.8)

where the superscript \( ^\top \) denotes orthogonal projection onto \( \xi_\mathbb{C} \). Of course, the global form of (2.8) is

\[
D'\sigma_s^{0,1} = -(\nabla's)^\top
\]  

(2.9)

where \( \nabla' = dz \otimes \nabla_{\partial_z}, \ D' = dz \otimes D_{\partial_z} \) and \( D \) is the connection on \( \xi_\mathbb{C} \) induced by \( \nabla \) and orthogonal projection onto \( \xi_\mathbb{C} \). (2.9) is the differential equation that \( \sigma_s \) has to satisfy in order for \( v = s + \sigma_s \) to be an infinitesimal conformal deformation. Theorem 2.1 below essentially asserts the converse.

\[ ^4 \text{I shall always denote the space of smooth sections of a bundle.} \]
We now calculate the last two terms of (2.17):

\[ (\delta^2 A)(s) \leq (\delta^2 E)(s + \sigma) \text{ with equality if and only if } \sigma \text{ satisfies (2.9).} \]  

(2.10)

In (2.10), \( F \) is, of course, being regarded as a harmonic map which is conformal (away from the branch points) with respect to the conformal structure on \( \Sigma \) induced by \( F \). A more precise relationship between \( \delta^2 E \) and \( \delta^2 A \) is given by (2.27) below.

**Proof.** One could try to prove this theorem by reversing the argument that led to the derivation of (2.9) but we prefer to give a more formal proof that works unchanged also in the case of complete minimal surfaces of finite total curvature.

Let \( z = x + iy \) be a local complex co-ordinate on \( \Sigma \) and let \( v = s + \sigma \). Then

\[ (\delta^2 E)(v) = \int_{\Sigma} \left( |\nabla_{\partial_x} v|^2 + |\nabla_{\partial_y} v|^2 - \langle R(v, F_x)F_x, v \rangle - \langle R(v, F_y)F_y, v \rangle \right) \, dx dy \]  

(2.11)

where \( R \) is the Riemann curvature tensor of \( M \). Now

\[ |\nabla_{\partial_x} v|^2 + |\nabla_{\partial_y} v|^2 = 4|\nabla_{\partial_z} v|^2. \]  

(2.12)

We let \( \perp \) denote orthogonal projection onto \( \nu_{\Sigma} := \nu \otimes_{\mathbb{R}} \mathbb{C} \) and obtain

\[ \nabla_{\partial_z} v = (\nabla_{\partial_z} v)^\perp + \eta + (\nabla_{\partial_z} \sigma^{1.0})^\top \]  

(2.13)

where, as suggested by (2.5),

\[ \eta := (\nabla_{\partial_z} s)^\top + (\nabla_{\partial_z} \sigma^{0.1})^\top. \]  

(2.14)

On using (2.6) and (2.7) in (2.13) we obtain

\[ |\nabla_{\partial_z} v|^2 = |(\nabla_{\partial_z} v)^\perp|^2 + |\eta|^2 + |(\nabla_{\partial_z} \sigma^{1.0})^\top|^2. \]  

(2.15)

Locally, and away from the branch points, we can write \( \sigma^{0.1} = f F \) for some locally defined function \( f \). Therefore

\[ (\nabla_{\partial_z} \sigma^{0.1})^\perp = 0 \text{ by harmonicity of } F. \]  

(2.16)

(2.16) allows us to re-write (2.15) as

\[ |\nabla_{\partial_z} v|^2 = |(\nabla_{\partial_z} s)^\perp|^2 + |\eta|^2 + |(\nabla_{\partial_z} \sigma^{1.0})^\top|^2 \]  

\[ + \langle (\nabla_{\partial_z} s)^\perp, \nabla_{\partial_z} \sigma^{0.1} \rangle + \langle (\nabla_{\partial_z} s)^\perp, \nabla_{\partial_z} \sigma^{1.0} \rangle. \]  

(2.17)

We now calculate the last two terms of (2.17):

\[ \langle (\nabla_{\partial_z} s)^\perp, \nabla_{\partial_z} \sigma^{0.1} \rangle = \partial_z \langle s, \nabla_{\partial_z} \sigma^{0.1} \rangle - \langle s, \nabla_{\partial_z} \nabla_{\partial_z} \sigma^{0.1} \rangle, \]  

\[ \langle (\nabla_{\partial_z} s)^\perp, \nabla_{\partial_z} \sigma^{1.0} \rangle = \partial_z \langle s, \nabla_{\partial_z} \sigma^{1.0} \rangle - \langle s, \nabla_{\partial_z} \nabla_{\partial_z} \sigma^{1.0} \rangle. \]  

(2.18)

But, from (2.16) and (2.14) we have \( \nabla_{\partial_z} \sigma^{0.1} = (\nabla_{\partial_z} \sigma^{0.1})^\top = \eta - (\nabla_{\partial_z} s)^\perp \) and therefore

\[ \langle \nabla_{\partial_z} \nabla_{\partial_z} \sigma^{0.1}, s \rangle = \langle R(F_z, F_z)\sigma^{0.1}, s \rangle + \langle \nabla_{\partial_z} (\eta - (\nabla_{\partial_z} s)^\top), s \rangle \]  

\[ = \langle R(F_z, F_z)\sigma^{0.1}, s \rangle + |(\nabla_{\partial_z} s)^\top|^2 - \langle \eta, (\nabla_{\partial_z} s)^\top \rangle. \]  

(2.19)

Similarly,

\[ \langle \nabla_{\partial_z} \nabla_{\partial_z} \sigma^{1.0}, s \rangle = \langle R(F_z, F_z)\sigma^{1.0}, s \rangle + |(\nabla_{\partial_z} s)^\top|^2 - \langle \eta, (\nabla_{\partial_z} s)^\top \rangle. \]  

(2.20)
Taking (2.18), (2.19) and (2.20) into account in (2.17), integrating and using Stokes’s theorem gives

\[ \int_{\Sigma} |\nabla_{\partial y} v|^2 \, dx \, dy = \int_{\Sigma} \left[ |(\nabla_{\partial y} s)^\perp|^2 + |\eta|^2 + |\nabla_{\partial y} \sigma^{1,0}|^2 - 2 |(\nabla_{\partial y} s)^\top|^2 \right. \\
\left. - \langle R(F_x, F_z) \sigma^{0,1}, s \rangle - \langle R(F_z, F_z) \sigma^{1,0}, s \rangle \\
+ \langle \eta, (\nabla_{\partial y} s)^\top \rangle + \langle \tilde{\eta}, (\nabla_{\partial y} s)^\top \rangle \right] \, dx \, dy. \]  

(2.21)

We now deal with the last two terms in (2.11):

\[ \langle R(v, F_x) F_x, v \rangle + \langle R(v, F_y) F_y, v \rangle = 4 \langle R(v, F_z) F_z, v \rangle \\
= 4 \left( \langle R(s, F_x) F_x, s \rangle + \langle R(\sigma^{0,1}, F_x) F_z, \sigma^{1,0} \rangle \right. \\
\left. + \langle R(s, F_z) F_x, \sigma^{1,0} \rangle + \langle R(\sigma^{0,1}, F_z) F_z, s \rangle \right). \]  

(2.22)

By the second Bianchi identity,

\[ \langle R(\sigma^{0,1}, F_z) F_z, s \rangle + \langle R(F_z, F_z) \sigma^{0,1}, s \rangle = 0, \]
\[ \langle R(\sigma^{1,0}, F_z) F_z, s \rangle + \langle R(F_z, F_z) \sigma^{1,0}, s \rangle = 0. \]  

(2.23)

Using (2.12), (2.21), (2.22) and (2.23) in (2.11) yields:

\[ (\delta^2 E)(v) = 4 \int_{\Sigma} \left[ |(\nabla_{\partial y} s)^\perp|^2 + |\nabla_{\partial y} \sigma^{1,0}|^2 + |\eta|^2 - 2 |(\nabla_{\partial y} s)^\top|^2 \right. \\
\left. - \langle R(s, F_x) F_x, s \rangle - \langle R(\sigma^{0,1}, F_z) F_z, \sigma^{1,0} \rangle \\
+ \langle \eta, (\nabla_{\partial y} s)^\top \rangle + \langle \tilde{\eta}, (\nabla_{\partial y} s)^\top \rangle \right] \, dx \, dy. \]  

(2.24)

An integration by parts shows that

\[ \int_{\Sigma} |\nabla_{\partial y} \sigma^{1,0}|^2 \, dx \, dy = \int_{\Sigma} |\nabla_{\partial y} \sigma^{1,0}|^2 \, dx \, dy + \int_{\Sigma} \langle R(F_x, F_z) \sigma^{1,0}, \sigma^{0,1} \rangle \, dx \, dy \]  

(2.25)

which, together with (2.24) and the second Bianchi identity, gives:

\[ \int_{\Sigma} |\nabla_{\partial y} \sigma^{1,0}|^2 \, dx \, dy = \int_{\Sigma} \left[ |\eta|^2 + |(\nabla_{\partial y} s)^\top|^2 - \langle \eta, (\nabla_{\partial y} s)^\top \rangle \right. \\
\left. - \langle \tilde{\eta}, (\nabla_{\partial y} s)^\top \rangle + \langle R(\sigma^{1,0}, F_z) F_z, \sigma^{0,1} \rangle \right] \, dx \, dy. \]  

(2.26)

On substituting (2.26) in (2.24) we obtain

\[ (\delta^2 E)(v) = 4 \int_{\Sigma} \left[ |(\nabla_{\partial y} s)^\perp|^2 - |(\nabla_{\partial y} s)^\top|^2 - \langle R(s, F_z) F_z, s \rangle + 2 |\eta|^2 \right] \, dx \, dy \\
= (\delta^2 A)(s) + 8 \int_{\Sigma} |\eta|^2 \, dx \, dy. \]  

(2.27)

The proof of the theorem is complete.

\[ \Box \]

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1 The inequality \( i_A \geq i_E \) follows immediately from Theorem 2.1. For the other inequality, let \( S \) be a maximal subspace on which \( \delta^2 A < 0 \). By the Fredholm
alternative, we may solve (2.9) if, and only if, \((\nabla_{\partial_z}s)^\top\) is orthogonal to \(\ker D^*\), where \(D^*\) is the adjoint of \(D\). Now an integration by parts shows that \(D^* = i \ast \partial_t \Gamma(\xi^{0,1} \otimes \kappa) \rightarrow \Gamma(\xi^{0,1})\) where \(\kappa\) is the line bundle of holomorphic one-forms on \(\Sigma\) and \(\ast\) is the Hodge star operator. Therefore \(\ker D^* = H^0(\xi^{0,1} \otimes \kappa) = \text{space of holomorphic sections of } \xi^{0,1} \otimes \kappa\). Let \(h^0(\xi^{0,1} \otimes \kappa) = \text{complex dimension of } H^0(\xi^{0,1} \otimes \kappa)\). Then, we may find a subspace \(\hat{S} \subset S\) of real dimension \(\geq \dim S - 2h^0(\xi^{0,1} \otimes \kappa)\) for which (2.9) has a solution \(\sigma_s\) whenever \(s \in \hat{S}\). Moreover, we may arrange for \(\sigma_s\) to depend linearly on \(s\), since the equation for \(\sigma_s\) is linear. Let \(\hat{S} = \{ s + \sigma_s \mid s \in \hat{S} \} \subset \Gamma(F^*(TM))\). Then, by Lemma 2.1. \(\delta^2 E|_{\hat{S}} < 0\) and therefore, \(i_E \geq \dim \hat{S} = \dim \hat{S} \geq i_A - r\) where \(r = 2h^0(\xi^{0,1} \otimes \kappa)\). We now calculate \(h^0(\xi^{0,1} \otimes \kappa)\): \(c_1(\xi^{0,1}) = 2g - 2 - b\) and \(c_1(\kappa) = 2g - 2\) and therefore, by the theorem of Riemann-Roch, we have

\[
h^0(\xi^{0,1} \otimes \kappa) = 3g - 3 - b + h^0(\xi^{1,0}).
\]

If \(b \leq 2g - 3\), \(c_1(\xi^{1,0}) < 0\) and \(h^0(\xi^{1,0} \otimes \kappa) = 3g - 3 - b\). If \(b \geq 4g - 3\), \(c_1(\xi^{0,1} \otimes \kappa) < 0\) and \(h^0(\xi^{0,1} \otimes \kappa) = 0\). If \(2g - 2 \leq b \leq 4g - 4\), then \(0 \leq c_1(\xi^{0,1} \otimes \kappa) \leq 2g - 2\) and therefore, by Clifford’s theorem (see, for example, [14]), \(h^0(\xi^{0,1} \otimes \kappa) \leq \left\lfloor \frac{4g - 2 - b}{2} \right\rfloor\).

The proof of Theorem 3.1 is complete. \(\square\)

Recall that the nullity \(n\) of a functional at a critical point \(F\) is the dimension of the space of Jacobi fields of the functional at \(F\). If the index of \(F\) is \(i + n\) is equal to the dimension of a maximal subspace of the space of infinitesimal variations of \(F\) on which the hessian of the functional is negative semidefinite. A minor modification (which will be left to the reader) of the proof of Theorem 1.1 yields:

**Theorem 3.1.** Let \(F: \Sigma_g \to M\) and \(r\) be as in Theorem 1.1 and let

\[
\begin{align*}
n_A &= \text{nullity of } F \text{ as a critical point of the area functional } A, \\
n_E &= \text{nullity of } F \text{ as a critical point of the energy functional } E, \\
n_E^T &= \text{dimension of the space of purely tangential Jacobi fields of } F, \text{ as a critical point of } E.
\end{align*}
\]

Then

\[
i_E + n_E - n_E^T \leq i_A + n_A \leq i_E + n_E - n_E^T + r.
\]

The following comparison of the nullities of energy and area follows immediately from the inequalities in Theorem 1.1 and Theorem 3.1

\[
n_E - n_E^T - r \leq n_A \leq n_E - n_E^T + r.
\]

We now move on to the proof of Theorem 1.2. First, we recall that (see, for example, [19]) if a minimal surface \(\Sigma\) in \(\mathbb{R}^d\) has finite total curvature and finitely many branch points, then \(\Sigma\) is conformally diffeomorphic to a closed Riemann surface \(\Sigma\) with finitely many punctures \(\{p_1, \ldots, p_k\}\) corresponding to the ends of \(\Sigma\). Recall, too, that \(G_{2,d}\), the Grassmannian of oriented two-planes in \(\mathbb{R}^d\), may be identified with the quadric \(Q_{d-2} \subset \mathbb{C}P^{d-1}\) defined by \(\{[z] \mid z = (z_1, \ldots, z_d) \in \mathbb{C}^d \setminus \{0\}, \sum_{i=1}^d |z_i|^2 = 0\}\). Furthermore, the Gauss map \(G: \Sigma \to G_{2,d} = Q_{d-2}\) is holomorphic and extends to a holomorphic map \(G: \Sigma \to G_{2,d}\). Let \(\gamma\) be the tautological two-plane bundle over \(G_{2,d}\) and let \(\tilde{\xi} = G^*(\gamma)\). Then \(\tilde{\xi}|_{\Sigma} = \xi\), the ramified tangent bundle of \(\Sigma\) and \(c_1(\xi) = \frac{1}{2\pi} \int_{\Sigma} K \, dA\). Similarly, let \(\tilde{v}\) be the orthogonal complement of \(\xi\) in \(\Sigma \times \mathbb{R}^d\) and then \(\tilde{v}|_{\Sigma} = v\). The hessians \(\delta^2 E\) and \(\delta^2 A\) both extend to sections of \(\Sigma \times \mathbb{R}^d\) and \(\tilde{v}\) respectively and, in [12], [16] and [25] it is shown that \(i_A(\Sigma) = i_A(\Sigma)\). We also have \(i_E(\Sigma) = 0\) and \(n_E(\Sigma) = d\) because the tangent bundle of \(\mathbb{R}^d\) is trivial and the
Levi-Civita connection on $\mathbb{R}^d$ is simply the exterior derivative. Finally, the projection of a constant vector field in $\mathbb{R}^d$ onto $\bar{\nu}$ is a Jacobi field of the area functional and therefore, if $\Sigma$ is non-planar then $n_A(\Sigma) \geq d$. We shall, in fact, prove the following more precise theorem.

**Theorem 3.2.** Let $\Sigma$ be a complete, oriented, non-planar minimal surface in $\mathbb{R}^d$ with finitely many branch points and of finite total curvature. Define $g$, $b$ and $r$ as in Theorem 1.2. Then

$$\text{index}(\Sigma) + \text{nullity}(\Sigma) \leq \frac{1}{\pi} \int_{\Sigma} (-K) \, dA + 2g - 2 + d \leq \frac{3}{2\pi} \int_{\Sigma} (-K) \, dA - r + b + d. \quad (3.1)$$

Furthermore, if $d = 3$ then $(3.1)$ can be improved to

$$\text{index}(\Sigma) + \text{nullity}(\Sigma) \leq \frac{1}{\pi} \int_{\Sigma} (-K) \, dA + 2 \leq \frac{3}{2\pi} \int_{\Sigma} (-K) \, dA - r + b + 2. \quad (3.2)$$

**Remark.** Inequalities $(1.2)$ and $(1.3)$ in Theorem 1.2 result from using, in $(3.1)$ and $(3.2)$, the observation that $\text{nullity}(\Sigma) \geq d$ for a non-planar minimal surface.

**Proof.** It is clear that the proof of Theorem 2.1 also works for $\delta^2 E$ and $\delta^2 A$ extended to $\bar{\Sigma}$. Furthermore,

$$c_1(\xi^{1,0}) = c_1(\xi) = \frac{1}{2\pi} \int_{\Sigma} K \, dA < 0.$$

Therefore, by the Riemann-Roch theorem we have

$$h^0(\xi^{0,1} \otimes \kappa) = -c_1(\xi) + c_1(\kappa) + 1 - g = \frac{1}{2\pi} \int_{\Sigma} (-K) \, dA + g - 1.$$

The proof of the first inequality in $(3.1)$ can now be completed by an argument identical to that of the proof of Theorem 3.1. The inequality

$$\frac{1}{2\pi} \int_{\Sigma} (-K) \, dA \geq 2g - 2 + r - b$$

is due to Osserman; see for instance, [19].

The proof of $(3.2)$ when $d = 3$ requires the following lemma.

**Lemma 3.3.** Let $F : \Sigma \to \mathbb{R}^3$ be a generalised minimal immersion of a Riemann surface $\Sigma$. Let $\bar{\nu}$ be a smooth unit normal vector field on $\Sigma$. (Such a section of $\nu$ exists because $\Sigma$ is orientable.) Then $(\partial \bar{\nu})^\top \in H^0(\xi^{0,1} \otimes \kappa)$.

**Proof.** By $(2.7)$ we have that

$$(\partial \bar{\nu})^\top = \langle \partial_z \bar{\nu}, F_z \rangle \frac{F_z}{|F_z|^2} \otimes dz = -\langle \bar{\nu}, F_{zz} \rangle \frac{F_z}{|F_z|^2} \otimes dz.$$

The conformality and harmonicity of $F$ then show that $D^\nu((\partial \bar{\nu})^\top) = 0$, where $D^\nu := d\bar{z} \otimes D_{\partial_z}$, i.e., $(\partial \bar{\nu})^\top$ is a holomorphic section of $\xi^{0,1} \otimes \kappa$, as claimed.

We now return to the proof of $(3.2)$. Denote by $\mathcal{M}$ the space of meromorphic functions on $\bar{\Sigma}$. By Lemma 3.3

$$H^0(\xi^{0,1} \otimes \kappa) = \{g(\partial \bar{\nu})^\top \mid g \in \mathcal{M}, \, [g] + [(\partial \bar{\nu})^\top] \geq 0\},$$

where $[\cdot]$ denotes divisor. It is convenient to define

$$\mathcal{M}_L := \{g \in \mathcal{M} \mid [g] + [(\partial \bar{\nu})^\top] \geq 0\}, \quad (3.3)$$
where, of course, $L := \xi^{0,1} \otimes \kappa$. If $s \in \Gamma(\nu)$, then $s = f\hat{\nu}$ for some $f \in C^\infty(\Sigma)$. Therefore, the Fredholm alternative for the solvability of (2.9) can now be stated as the following condition on $f$:

$$\int_\Sigma f\bar{g}K \, dA = 0 \forall g \in \mathcal{M}_L,$$

where we have used $|\nabla \hat{\nu}|^2 = -K$. Since the constant function 1 belongs to $\mathcal{M}_L$, we see that the $\mathbb{R}$-codimension of the space of real valued smooth functions $f$ satisfying (3.4) is $2h^0(\xi^{0,1} \otimes \kappa) - 1 = \frac{1}{2} \int_\Sigma (-K) \, dA + 2g - 3$.

4 More area index estimates

Theorem 1.1 provides an upper bound on $i_A$ whenever $i_E$ may be estimated, and it is often easier to estimate $i_E$ than $i_A$ because $\delta^2 E$ does not involve the second fundamental form. For instance, if $M$ has nonpositive sectional curvature then $i_E$ is easily seen to be zero from (2.11), thereby yielding:

**Corollary 4.1.** Let $\Sigma$ be a closed minimal surface in a Riemannian manifold $M$ of nonpositive sectional curvature. Then $i_A \leq r \leq 6g - 6$ where $r$ is as in Theorem 1.1.

In particular, the index of a closed minimal surface of genus $g$ in a flat torus is at most $6g - 6$. Corollary 4.1 has already been noted in [10] by considering an energy functional on Teichmüller space.

Theorem 1.1 can be refined when the ambient space is 3-dimensional in a manner similar to that in Theorem 1.2 for minimal surfaces of finite total curvature in $\mathbb{R}^3$. More precisely,

**Theorem 4.2.** Let $F: \Sigma_g \to M^3$ be a (possibly branched) minimal immersion of a closed Riemann surface of genus $g$ into a three-dimensional space-form $M^3$. If $F$ is not totally geodesic then

$$i_E \leq i_A \leq i_E + r - 1 \quad (4.1)$$

where $r$ is as in Theorem 1.1.

The proof consists in observing that Lemma 3.3 is still valid in this setting. The argument then proceeds as in the proof of (3.2).

The Clifford torus in $\mathbb{R}P^3$ is stable as a harmonic map. (In [9], the first author has determined all harmonic tori in $\mathbb{R}P^3$ that minimize energy in their homotopy class.) However, the Clifford torus is unstable as a minimal surface and it is not hard to show that its index is 1. Furthermore $r = 2$. This is a situation in which $i_A \leq i_E$ and yet (4.1) is still sharp.

For a general minimal immersion we can prove

**Theorem 4.3.** Let $F: \Sigma_g \to M^d$ be a (possibly branched) minimal immersion of a closed Riemann surface of genus $g$ into a Riemannian manifold $M$. Then

$$i_E + n_E \leq d C(M) \text{Area}(F(\Sigma_g)) \quad (4.2)$$

and therefore,

$$i_A + n_A \leq d C(M) \text{Area}(F(\Sigma_g)) + r \quad (4.3)$$

where $r$ is as in Theorem 1.1 and $C(M)$ is a constant which depends on the second fundamental form of an isometric embedding of $M$ into Euclidean space.

\textsuperscript{5} as in the proof of Theorem 1.1
Remarks. The main interest in (4.3) is that no assumption is made on the second fundamental form of \( F \). Colding and Minicozzi showed (Theorem 1.1 in [3]) that, given \( A > 0 \) and a positive integer \( g \), there are at most finitely many closed embedded minimal surfaces of genus \( g \) and with area at most \( A \) in a closed orientable 3-manifold with a bumpy metric. In particular, the Morse index of an embedded minimal surface in a 3-manifold with a bumpy metric is bounded by its genus and its area but no explicit bound like (4.3) is given in [3].

Proof. Let \( h(t) \) be the trace of the heat kernel of \( \Sigma_g \) with the metric induced by \( F \). Then, since \( F \) is an isometric harmonic map, Proposition 2.2 in [35] asserts that

\[
i_E + n_E \leq d \inf_{t>0} e^{2\mu t} h(t) \tag{4.4}
\]

where \( \mu \) is an upper bound of the sectional curvatures of \( M^d \).

Now consider \( M \) to be isometrically embedded in some Euclidean space. The method of proof of Theorem 5 on pages 991-993 in [1] can now be employed to obtain

\[
h(t) \leq \frac{\alpha_1}{(1 - e^{-\alpha_2 t})^2} \text{Area}(F(\Sigma_g)) \tag{4.5}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are constants which depend on the second fundamental form of the isometric embedding of \( M \) into Euclidean space. The bound (4.2) now follows from (4.4) and (4.5). \( \square \)

The indexes of some minimal surfaces of finite total curvature have already been mentioned in the remarks at the end of \( \S1 \). Many known examples of minimal surfaces in \( \mathbb{R}^3 \) which are embedded or, at least, whose ends are embedded, have symmetries which allow refinements to Theorem 1.2. The next proposition makes precise the type of symmetry that the minimal surface is required to have. It includes the notion of strong symmetry with respect to a plane introduced by Cosín and Ros in [4], Definition 1; see also Lemma 4 of the same article.

**Proposition 4.4.** Let \( \Sigma \) be a Riemann surface and let \( F: \Sigma \to \mathbb{R}^3 \) be a generalised minimal immersion which is complete and of finite total curvature. Suppose there exists an isometry \( \Theta: \mathbb{R}^3 \to \mathbb{R}^3 \) and a diffeomorphism \( \theta: \Sigma \to \Sigma \) such that \( \Theta \circ F = F \circ \theta \). Let \( \mathcal{M}_L \) be defined by (3.3). If \( \theta \) is anti-holomorphic and \( g \in \mathcal{M}_L \) then \( g \circ \theta \in \mathcal{M}_L \). If \( \theta \) is holomorphic and \( g \in \mathcal{M}_L \) then \( g \circ \theta \in \mathcal{M}_L \).

Proof. We shall only give the proof when \( \theta \) is anti-holomorphic; the proof when \( \theta \) is holomorphic is similar and, in fact, much easier.

From \( F(\theta(z)) = \Theta(F(z)) \) and \( \partial \theta = 0 \) we obtain:

\[
\frac{\partial F}{\partial w}(\theta(z)) \frac{\partial \bar{\theta}}{\partial z} = \Theta_0 \left( \frac{\partial F}{\partial z}(z) \right)
\]

where \( \Theta_0 \in SO(3) \) is the non-translational part of \( \Theta \); of course, \( w \) is a local complex coordinate defined on a neighbourhood of \( \theta(z) \). It follows that

\[
\hat{\nu}(\theta(z)) = \begin{cases} \Theta_0(\hat{\nu}(z)), & \text{if } \det \Theta_0 = -1, \\ -\Theta_0(\hat{\nu}(z)), & \text{if } \det \Theta_0 = 1. \end{cases} \tag{4.6}
\]

Differentiating (4.6) yields:

\[
\frac{\partial \hat{\nu}}{\partial w}(\theta(z)) \frac{\partial \bar{\theta}}{\partial z}(z) = \pm \Theta_0 \left( \frac{\partial \hat{\nu}}{\partial z}(z) \right), \text{ i.e., } \theta^*((\partial \hat{\nu})^\top) = \pm \Theta_0((\partial \hat{\nu})^\top).
\]

Therefore, \( (\partial \hat{\nu})^\top \) has a zero of order \( Q \) at \( z \) if, and only if, it also has a zero of order \( Q \) at \( \theta(z) \). The proposition follows immediately. \( \square \)
Theorem 4.5. Let $\Sigma$ be a Riemann surface and let $F: \Sigma \to \mathbb{R}^3$ be a generalised minimal immersion which is complete and of finite total curvature. Suppose there exists an isometry $\Theta: \mathbb{R}^3 \to \mathbb{R}^3$ and an anti-holomorphic involution $\theta: \Sigma \to \Sigma$ such that $\Theta \circ F = F \circ \theta$. Then

$$\text{index}(\Sigma) + \text{nullity}(\Sigma) \leq \frac{1}{2\pi} \int_{\Sigma} (-K) \, dA + g + 2. \quad (4.7)$$

In particular, since $\text{nullity}(\Sigma) \geq 3$, we have

$$\text{index}(\Sigma) \leq \frac{1}{2\pi} \int_{\Sigma} (-K) \, dA + g - 1. \quad (4.8)$$

Proof. Proposition 4.4 enables us to define $\rho: \mathcal{M}_L \to \mathcal{M}_L$ by $\rho(g) := g \circ \theta$, where $\mathcal{M}_L$ is defined by (3.3). Then $\rho$ is linear and $\rho^2 = \text{identity}$. Therefore, $\mathcal{M}_L = \mathcal{M}_L^+ \oplus \mathcal{M}_L^-$ where $\mathcal{M}_L^+$ and $\mathcal{M}_L^-$ are respectively the $+1$ and $-1$ eigenspaces of $\rho$. Similarly, we can write $C^\infty(\Sigma) = C^\infty_+(\Sigma) \oplus C^\infty_-(\Sigma)$.

Next observe that if $f \in C^\infty(\Sigma)$ then $(\delta^2 A)(f \nu) = (\delta^2 A)(f_+ \nu) + (\delta^2 A)(f_- \nu)$, where $f_{\pm} := \frac{1}{2} (f \pm f \circ \theta)$. Now let $S_+$ be a maximal subspace of $C^\infty_+(\Sigma)$ on which $\delta^2 A \leq 0$ and define $S_-$ similarly. A moment’s thought will reveal that $S := S_+ \oplus S_-$ is then a maximal subspace of $C^\infty(\Sigma)$ on which $\delta^2 A \leq 0$.

Let $\{f_1, \ldots, f_p\}$ and $\{f_{p+1}, \ldots, f_q\}$ be bases of $S_+$ and $S_-$ respectively and let $\{g_1, \ldots, g_p\}$ and $\{g_{p+1}, \ldots, g_q\}$ be bases of $\mathcal{M}_L^+$ and $\mathcal{M}_L^-$ respectively. Then, for $j \in \{1, \ldots, q\}$ and $\alpha \in \{1, \ldots, \nu\}$ we have:

$$\int_{\Sigma} f_j \tilde{g}_\alpha K \, dA = \int_{\Sigma} (f_j \circ \theta)(\tilde{g}_\alpha \circ \theta)\theta^*(K \, dA)$$

$$= \pm \int_{\Sigma} f_j \tilde{g}_\alpha K \, dA.$$ 

Therefore, $\int_{\Sigma} f_j \tilde{g}_\alpha K \, dA$ is either real or purely imaginary. It follows that the $\mathbb{R}$-codimension of the space of real valued smooth functions $f$ satisfying (3.4) is $h^0(\xi^{0,1} \otimes \kappa) = \frac{1}{2\pi} \int_{\Sigma} (-K) \, dA + g - 1$. (4.7) and (4.8) follow immediately.

The Jorge-Meeks $k$-roid $\Sigma_k$ of genus zero and $k$ ends has total curvature equal to $4\pi(k-1)$ and is strongly symmetric in the sense of Cosín and Ros in [4]. Therefore, we may apply Theorem 4.5 to conclude, as asserted in a remark in §1, that index($\Sigma_k$) + nullity($\Sigma_k$) $\leq 2k$.

References

[1] Cheng, S.-Y. and Tysk, J., *Schrödinger operators and index bounds for minimal submanifolds*, Rocky Mountain J. Math. 24 (1994) 977-996

[2] Choe, J., *Index, vision number and stability of complete minimal surfaces*, Arch. Rational Mech. Anal. 109 (1990) 195-212

[3] Colding, T. and Minicozzi II, W., *Embedded minimal surfaces without area bounds in 3-manifolds* Proceedings of Conference on Geometry and Topology (Aarhus, 1998), Contemp. Math., 258, Amer. Math. Soc., Providence, RI (2000) 107-120

[4] Cosín, C. and Ros, A., *A Plateau problem at infinity for properly immersed minimal surfaces with finite total curvature*, Indiana Univ. Math. J. 50 (2001) 847-879

[5] Courant, R., *Dirichlet’s principle, conformal mapping, and minimal surfaces*, Interscience Publishers, New York (1950)

12
[6] Douglas, J., *Minimal surfaces of higher topological structure*, Ann. of Math. (2) **40** (1939) 205-298

[7] do Carmo, M. and Peng, C. K., *Stable complete minimal surfaces in $\mathbb{R}^3$ are planes*, Bull. Amer. Math. Soc. (N.S.) **1** (1979) 903-906

[8] Ejiri, N., *Two applications of the unit normal bundle of a minimal surface in $\mathbb{R}^N$*, Pacific J. Math. **147** (1991) 291-300

[9] Ejiri, N., *Homotopically energy-minimizing harmonic maps of tori into $\mathbb{R}P^3$*, Tokyo J. Math. **23** (2000) 503-518

[10] Ejiri, N., *A differential-geometric Schottky problem, and minimal surfaces in tori*, Differential geometry and integrable systems (Tokyo, 2000), Contemp. Math., **308**, Amer. Math. Soc., Providence, RI (2002) 101-144

[11] Ejiri, N. and Kotani, M., *Index and flat ends of minimal surfaces*, Tokyo J. Math. **16** (1993) 37-48

[12] Fischer-Colbrie, D., *On complete minimal surfaces with finite Morse index in three-manifolds*, Invent. Math. **82** (1985) 121-132

[13] Fischer-Colbrie, D. and Schoen, R., *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980) 199-211

[14] Griffiths, P. and Harris, J., *Principles of Algebraic Geometry*, John Wiley and Sons, New York (1978)

[15] Grigor’yan, A. and Yau, S.-T., *Isoperimetric properties of higher eigenvalues of elliptic operators*, Amer. J. Math. **125** (2003) 893-940

[16] Gulliver, R., *Index and total curvature of complete minimal surfaces*, Proc. Sympos. Pure Math. **44** (1986) 207-211

[17] Gulliver, R. and Lawson, H. Blaine, Jr., *The structure of stable minimal hypersurfaces near a singularity*, Proc. Sympos. Pure Math. **44** (1986) 213-237

[18] Hoffman, D. and Karcher, H., *Complete embedded minimal surfaces of finite total curvature*, Encyclopaedia Math. Sci. **90**, Geometry V, Springer, Berlin (1997) 5-93

[19] Lawson, H. Blaine, Jr., *Lectures on minimal submanifolds, Vol. I*, Mathematics Lecture Series **9** Publish or Perish, Inc. (1980)

[20] López, F.J. and Ros, A., *Complete minimal surfaces with index one and stable constant mean curvature surfaces*, Comment. Math. Helv. **64** (1989) 34-43

[21] Micallef, M., *Stable minimal surfaces in Euclidean space*, J. Differential Geom. **19** (1984) 57-84

[22] Micallef, M., *On the topology of positively curved manifolds*, Research Report, Centre for Mathematical Analysis, Australian National University **18** (1986) 39 pp

[23] Montiel, S. and Ros, A., *Schrödinger operators associated to a holomorphic map*, Global differential geometry and global analysis (Berlin, 1990), Lecture Notes in Math. **1481** Springer, Berlin (1991) 147-174
[24] Moore, J.D., *Second variation of energy for minimal surfaces in Riemannian manifolds*, preprint 20pp

[25] Nayatani, S., *On the Morse index of complete minimal surfaces in Euclidean space*, Osaka J. Math. 27 (1990) 441-451

[26] Nayatani, S., *Lower bounds for the Morse index of complete minimal surfaces in Euclidean 3-space*, Osaka J. Math. 27 (1990) 453-464

[27] Nayatani, S., *Morse index and Gauss maps of complete minimal surfaces in Euclidean 3-space*, Comment. Math. Helv. 68 (1993) 511-537

[28] Osserman, R., *A survey of minimal surfaces*, 2nd edition, Dover Publications, New York (1986)

[29] Pogorelov, A. V., *On the stability of minimal surfaces*, Dokl. Akad. Nauk SSSR 260 (1981) 293-295; English transl.: Soviet Math. Dokl. 24 (1981) 274-276

[30] Rosenberg, H. and Toubiana, É., *Complete minimal surfaces and minimal herissons*, J. Differential Geom. 28 (1988) 115-132

[31] Sacks, J. and Uhlenbeck, K., *The existence of minimal immersions of 2-spheres*, Ann. of Math. (2) 113 (1981) 1-24

[32] Schoen, R. and Yau, S.-T., *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) 110 (1979) 127-142

[33] Tysk, J., *Eigenvalue estimates with applications to minimal surfaces*, Pacific J. Math. 128 (1987) 361-366

[34] Tomi, F., Tromba, A.J., *Existence theorems for minimal surfaces of non-zero genus spanning a contour*, Mem. Amer. Math. Soc. 71 (1988) Number 382

[35] Urakawa, H., *Stability of harmonic maps and eigenvalues of the Laplacian*, Trans. Amer. Math. Soc. 301 (1987) 557-589