Abstract

Recurrent neural networks are a powerful means to cope with time series. We show how autoregressive linear, i.e., linearly activated recurrent neural networks (LRNNs) can approximate any time-dependent function \( f(t) \). The approximation can effectively be learned by simply solving a linear equation system; no backpropagation or similar methods are needed. Furthermore, and this is the main contribution of this article, the size of an LRNN can be reduced significantly in one step after inspecting the spectrum of the network transition matrix, i.e., its eigenvalues, by taking only the most relevant components. Therefore, in contrast to other approaches, we do not only learn network weights but also the network architecture. LRNNs have interesting properties: They end up in ellipse trajectories in the long run and allow the prediction of further values and compact representations of functions. We demonstrate this by several experiments, among them multiple superimposed oscillators (MSO), robotic soccer (RoboCup), and stock price prediction. LRNNs outperform the previous state-of-the-art for the MSO task with a minimal number of units.

Keywords: recurrent neural network; linear activation; time-series analysis; prediction; dimensionality reduction; approximation theorem; ellipse trajectories.

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1. Introduction

Deep learning in general means a class of machine learning algorithms that use a cascade of multiple layers of nonlinear processing units for feature extraction and transformation (Deng and Yu, 2014). The tremendous success of deep learning in diverse fields of artificial intelligence, such as computer vision and natural language processing, seems to depend on a bunch of ingredients: artificial, possibly recurrent neural networks (RNNs) with nonlinearly activated neurons, convolutional layers, and iterative training methods like backpropagation (Goodfellow et al., 2016). But which of these components are really essential for machine learning tasks such as time-series analysis?

Research in time series analysis and hence modeling dynamics of complex systems has a long tradition and is still highly active due to its crucial role in many real-world applications (Lipton et al., 2015) like weather forecast, stock quotations, comprehension of trajectories of objects and agents, or solving number puzzles (Ragni and Klein, 2011; Glüge and Wendemuth, 2013). The analysis of time series allows, among others, data compression, i.e., compact representation of time series, e.g., by a function $f(t)$, and prediction of further values.

Numerous research addresses these topics by RNNs, in particular variants of networks with long short-term memory (LSTM) (Hochreiter and Schmidhuber, 1997). In the following, we consider an alternative, simple but very powerful type of RNNs: linear recurrent neural networks (LRNNs). Thus we only use linear activation, which allows us to minimize the network size in one step, namely by inspecting the eigenvalues of the network transition matrix (cf. Section 4.3). Therefore, in contrast to other approaches, we do not only learn network weights but also the network architecture.

The rest of this paper is structured as follows: First, we briefly review related works (Section 2). We then formally introduce LRNNs as a special and simple kind of RNNs together with their properties, including the general network dynamics and their long-term behavior (Section 3). Afterwards, learning LRNNs is explained (Section 4). It is a relatively straightforward procedure which allows network size reduction; no backpropagation or gradient-descent method is needed. We then discuss results and experiments (Section 5), before we end up with conclusions (Section 6).

Interestingly, many results for LRNNs we present here can be achieved by applying linear algebra and matrix analysis. Nonetheless, to the best of our knowledge, the main contributions of this article have not been presented before, namely that LRNNs can effectively be learned by a simple procedure and, even more, their
size can be reduced significantly in one step (cf. Section 4). In addition, the empirical evaluations of, in particular, the MSO benchmark and stock price prediction (cf. Sections 5.1 and 5.4) with LRNNs and other approaches, e.g., LSTM networks, demonstrate the usefulness of our approach. We therefore hope that several interested communities can profit from this work, such as the neural engineering community, the time series prediction community, and the ESN community.

2. Related Works

2.1. Echo State Networks

Echo state networks (ESNs) (Jaeger and Haas, 2004; Jaeger, 2007) play a significant role in RNN research as they provide an architecture and supervised learning principle for RNNs. They do this by driving a random, large, fixed RNN, called reservoir in this context, with the input signal which then induces in each neuron within this reservoir network a nonlinear response signal. They combine a desired output signal by a trainable linear combination of all response signals, allowing dimensionality reduction by so-called conceptors (Jaeger, 2014, 2017).

Xue et al. (2007) propose a variant of ESNs that work with several independent (decoupled) smaller networks. ESN-style (random) initialization has been shown effective for training RNNs with Hessian-free optimization (Martens and Sutskever, 2011). The latter paper addresses the problem of how to effectively train recurrent neural networks on complex and difficult sequence modeling problems which may contain long-term data dependencies. This can also be done with LRNNs (cf. MSO benchmark, Section 5.1). Tiño (2018) considers the effect of weight changes in linear symmetric ESNs on (Fisher) memory of the network. Furthermore, Couillet et al. (2016) investigate the asymptotic performance of linear ESNs from a solely theoretical point of view.

2.2. Recurrent Neural Networks

Simple RNNs are proposed by Elman (1990). By allowing them to accept sequences as inputs and outputs rather than individual observations, RNNs extend the standard feedforward multilayer perceptron networks. As shown in many sequence modeling tasks, data points such as video frames, audio snippets, and sentence segments are usually highly related in time. This results in RNNs being used as the indispensable tools for modeling such temporal dependencies. Linear RNNs and some of their properties (like short-term memory) are already investigated by White et al. (1994). Unfortunately, however, it can be a struggle to train RNNs to capture long-term dependencies (Bengio et al., 1994; Pascanu et al., 2013). This
is due to the gradients vanishing or exploding during backpropagation which in turn makes the gradient-based optimization difficult.

Nowadays, probably the most prominent and dominant type of RNNs are long short-term memory (LSTM) networks introduced by Hochreiter and Schmidhuber (1997). The expression “long short-term” refers to the fact that LSTM is a model for the short-term memory which can last for a long period of time. An LSTM is well-suited to classify, process and predict time series given time lags of unknown size. They were developed to deal with the exploding and vanishing gradient problem when training traditional RNNs. A common LSTM unit is composed of a cell, an input gate, an output gate, and a forget gate. Each unit type is activated in a different manner, whereas in this paper we consider completely linearly activated RNNs.

Ollivier et al. (2015) suggest the NoBackTrack algorithm in RNNs to train its parameters. This algorithm works in an online, memoryless setting which therefore requires no backpropagation through time. It is also scalable, thus avoiding the large computational and memory cost of maintaining the full gradient of the current state with respect to the parameters, but it still uses an iterative method, namely gradient descent. In contrast to this and other related works, in this paper we present a method working with linearly activated RNNs that does not require backpropagation or similar procedures in the learning phase.

Hu and Qi (2017) propose a novel state-frequency memory (SFM) RNN, which aims to model the frequency patterns of the temporal sequences. The key idea of the SFM is to decompose the memory states into different frequency states. In doing so, they can explicitly learn the dependencies of both the low and high frequency patterns. As we will see (cf. Section 5.1), RNNs in general can easily learn time series that have a constant frequency spectrum, which may be obtained also by Fourier analysis.

Voelker et al. (2019) propose a memory cell for recurrent neural networks that maintains information across long windows of time using relatively few resources, called Legendre memory unit (LMU). It is derived from the linear transfer function for a continuous-time history of its input signal across a sliding window, approximated by coupled differential equations, which can implicitly also be solved by LRNNs (cf. Property 1). Carta et al. (2021) propose an approach to address memorization challenges in RNNs which puts forward a way between the random encoding in the reservoir paradigm and the vanishing-gradient prone approach of fully-trained RNNs. The objective is to train memorization units to maximize their short-term memory capacity, employing a linear autoencoder for sequences. In both cases, backpropagation is employed, which is not needed for LRNNs whose
network size is reduced significantly in addition (cf. Section 4.3), thus we address the topic of architecture learning.

Architecture learning, in particular the pruning of neural networks, has been studied extensively in the literature. An early survey on pruning algorithms for neural networks is given by Reed (1993). In more recent work, Lee et al. (2019) present a method that prunes irrelevant connections between neurons for a given task prior to training and is applicable to a variety of modern neural network models, resulting in sparse networks. Furthermore, Molchanov et al. (2019) propose a method that estimates the contribution of a neuron to the final loss and iteratively removes those with smaller scores. In contrast to this, for LRNNs, the network architecture and hence its size is reduced in one step by analyzing the network transition matrix (cf. Section 4.3).

2.3. Autoregression

An autoregressive model is a representation of a type of random process (Akaike, 1969). It specifies that the output variable or a vector thereof depends linearly on its own predecessor values and on a stochastic term (white noise). In consequence, the model is in the form of a stochastic differential equation as in general (physical) dynamic systems (Colonius and Kliemann, 2014). An LRNN is also linearly activated, but its output does not only depend on its own predecessor values and possibly white noise but on the complete state of the possibly big reservoir whose dynamics is explicitly dealt with. In addition, the size of the network might be reduced in the subsequent process. We will continue the comparison between autoregression and LRNNs later (cf. Section 4.1), as we mainly consider autoregressive tasks in the following.

A popular choice in this context is the autoregressive integrated moving average (ARIMA) model (Hyndman and Khandakar, 2008; Hyndman and Athanasopoulos, 2013). A standard ARIMA($p, d, q$) model consists of autoregression AR($p$) and a moving average MA($q$). The parameter $p$ describes the history length (lag order) used to predict the time series at time $t$. We have $f(t) = c_1 f(t-1) + \cdots + c_p f(t-p) + e_t$, where $c_1, \ldots, c_p$ are (real-valued) autocorrelation coefficients and $e_t$, the residuals, are Gaussian white noise. In the moving average process MA($q$), the value $q$ specifies the number of past residuals considered for prediction. An underlying trend of the time series is modeled by using a drift, i.e., a constant that extends the term. This procedure is particularly well-suited for stationary time series, i.e., whose properties do not vary with time. Many time series, however, exhibit non-stationary behavior and thus require a transformation
to make them stationary. This is achieved by investigating the derivatives of the series, with the order of this process given by the parameter $d$.

Autoregressive frameworks are common in current machine-learning applications like language modeling, e.g., in the generative pre-trained transformer GPT-3 and their successors, by next word prediction (Brown et al., 2020). The models applied in this context, however, are very complex (175 billion parameters and even much more) and non-linear. Related to autoregression is autoencoding of sequences. For this, Pasa and Sperduti (2014) show that linear autoencoders can be used for pre-training of RNNs, while we establish completely linear RNNs here. Furthermore, Sperduti (2006) gives an exact closed-form solution for the weights of the linear autoencoder, which is related to the approximation theorem for LRNNs (cf. Property 8).

3. Linear Recurrent Neural Networks

RNNs often host several types of neurons, each activated in a different manner (Elman, 1990; Hochreiter and Schmidhuber, 1997). In contrast, we understand a homogeneous interconnected group of standard neurons simply as a (recurrent) neural network here, which may have arbitrary loops, akin to biological neuronal networks. We adopt a discrete time model, that is, the input and output are represented by a time series and are processed synchronously and stepwise by the network.

**Definition 1** (time series). A **time series** is a series of data points in $d$ dimensions $S(0), \ldots, S(n) \in \mathbb{R}^d$ with $d \geq 1$ and $n \geq 0$.

**Definition 2** (recurrent neural network). A **recurrent neural network** (RNN) is a directed graph consisting of altogether $N$ nodes, called **neurons**. $x(t)$ denotes the **activation** of the neuron $x$ at (discrete) time $t$. We may distinguish three groups of neurons (cf. Figure 1):

- $N^{\text{in}}$ **input** neurons (usually without incoming edges) whose activation is given by an external source, e.g., a time series,

- $N^{\text{out}}$ **output** neurons (usually without outgoing edges) whose activation represents some output function, and

- $N^{\text{res}}$ **reservoir** or **hidden** neurons (arbitrarily connected) that are used for auxiliary computations.
The sets of input and output neurons are not necessarily disjoint, they may even be identical (cf. Definition 3). Therefore, in the following, let $N_{\text{in out}}$ denote the overall number of neurons in the union of both sets. Obviously it holds $N = N_{\text{in out}} + N_{\text{res}}$ and $N_{\text{in out}} \leq N_{\text{in}} + N_{\text{out}}$.

The edges of the directed graph represent the network connections. They are annotated with weights which are compiled in the transition matrix $W$ of size $N \times N$. An entry $w_{ij}$ in row $i$ and column $j$ denotes the weight of the edge from neuron $j$ to neuron $i$. If there is no connection, then $w_{ij} = 0$. The transition matrix has the form

$$W = \begin{bmatrix} W_{\text{out}} & W_{\text{in}} \\ W_{\text{in}} & W_{\text{res}} \end{bmatrix}$$

containing the following weight matrices:

- **input** weights $W_{\text{in}}$ (weights from the input and possibly the output to the reservoir, a matrix of size $N_{\text{res}} \times N_{\text{in out}}$),

- **output** weights $W_{\text{out}}$ (all weights to the output and possibly back to the input, a matrix of size $N_{\text{in out}} \times N$), and

- **reservoir** weights $W_{\text{res}}$ (weights within the reservoir, a matrix of size $N_{\text{res}} \times N_{\text{res}}$).

Let us now define the network activity: The initial configuration of the neural network is given by a column vector $s$ with $N$ components, called start vector. It
represents the network state at the start time \( t = t_0 \). Because of the discrete time model, the activation of a (non-input) neuron \( x_i \) at time \( t + \tau \) (for some time step \( \tau > 0 \)) from the activation at time \( t \) of the neurons \( x_1, \ldots, x_k \) (for some \( k \geq 0 \)) that are connected to \( x_i \) with the weights \( w_{i1}, \ldots, w_{ik} \) is computed as follows:

\[
x_i(t + \tau) = g(w_{i1} x_1(t) + \cdots + w_{ik} x_k(t))
\]  

This has to be done simultaneously for all neurons of the network. \( g \) is called activation function. Although we will not make use of it, \( g \) may be different for different parts of the network. Usually, a nonlinear, bounded, strictly increasing sigmoidal function \( g \) is used, e.g., the logistic function, the hyperbolic tangent (tanh), or the softplus function (Goodfellow et al., 2016, Sect. 3.10). In the following, we employ simply the (linear) identity function, i.e., with \( g(x) = x \) for all \( x \), and can still approximate arbitrary time-dependent functions (cf. Property 8).

**Definition 3** (linear recurrent neural network). A linear recurrent network (LRNN) is an RNN with the following properties:

1. For the start time, it holds that \( t_0 = 0 \) and \( \tau \) is constant, usually \( \tau = 1 \).
2. The initial state \( S(0) \) of the given time series constitutes the first \( d \) components of the start vector \( s \).
3. For all neurons we have linear activation, i.e., everywhere \( g \) is the identity.
4. The weights in \( W^{\text{in}} \) and \( W^{\text{res}} \) are taken randomly, independently, and identically distributed from the standard normal distribution, i.e., the Gaussian distribution with mean \( \mu = 0 \) and standard deviation \( \sigma = 1 \), and remain unchanged all the time, whereas the output weights \( W^{\text{out}} \) are learned (cf. Section 4.1).
5. The spectral radius of the reservoir weights matrix \( W^{\text{res}} \) is set to 1, i.e., \( W^{\text{res}} \) is divided by the maximal absolute value of all its eigenvalues. Note that the spectral radius of the overall transition matrix \( W \) (cf. Equation 1) may still be greater than 1 if required by the application (cf. Example 4).
6. There is no distinction between input and output but only one (joint) group of \( N^{\text{in/out}} = N^{\text{in}} = N^{\text{out}} = d \) input/output neurons. They may be arbitrarily connected like the reservoir neurons. We thus can imagine the whole network as a big reservoir because the input/output neurons are not particularly special.

LRNNs can run in one of two modes: either receiving input or generating (i.e., predicting) output. In output generating mode, the network runs autonomously,
thus without external input. In this case, Equation 2 is applied to all neurons including the input/output neurons. The output from the previous time step is copied to the input. In input receiving mode, the activation of every input/output neuron \( x \) at time \( t \) is always overwritten with the respective input value at time \( t \) given by the time series \( S \).

3.1. Examples

Example 1. The function \( f(t) = t^2 \) can be realized by an LRNN (in output generating mode) with three neurons (cf. Figure 2 a). The respective transition matrix \( W \) and start vector \( s \) are:

\[
W = \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
s = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

Consequently, starting at \( t = 0 \) with time step \( \tau = 1 \), we have:

- \( x_3(0) = 1, x_3(t+1) = x_3(t) \), and hence \( x_3(t) = 1 \) in general.
- \( x_2(0) = 0, x_2(t+1) = x_2(t) + x_3(t) = x_2(t) + 1 \), and hence \( x_2(t) = t \).
- \( x_1(0) = 0, x_1(t+1) = x_1(t) + 2x_2(t) + x_3(t) = x_1(t) + 2t + 1 \), and hence \( x_1(t) = t^2 \) because of the identity \((t + 1)^2 = t^2 + 2t + 1\) (cf. first row of the transition matrix \( W \)).

Thus, in the neuron \( x_1 \), the function \( f(t) \) is computed. It is the only input/output neuron in this case.

Example 2. The Fibonacci series \((0, 1, 1, 2, 3, 5, 8, \ldots)\) can be defined as follows:

\[
f(t) = \begin{cases} 
t, & t = 0, 1 \\
f(t-1) + f(t-2), & \text{otherwise}
\end{cases}
\]

It can be realized by an LRNN (in output generating mode) with just two neurons (cf. Figure 2 b). The respective transition matrix \( W \) and start vector \( s \) can be directly derived from the recursive definition of \( f \):

\[
W = \begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix}
\quad \text{and} \quad
s = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]
Figure 2: LRNNs for (a) \( f(t) = t^2 \) and (b + c) the Fibonacci series (0, 1, 1, 2, 3, 5, 8, \ldots) with time step \( \tau = 1 \). In each case, the input/output neuron \( x_1 \) is marked by a double circle. The initial values of the neurons at time \( t_0 = 0 \) are written in the nodes. The weights are annotated at the edges.

Again, the function \( f(t) \) is computed in the only input/output neuron \( x_1 \). In the other neuron \( x_2 \), \( f(t+1) \) is generated. There is a closed-form expression for the Fibonacci series, revealing its exponential growth, known as Binet’s formula:

\[
f(t) = \frac{\lambda_1^t - \lambda_2^t}{\sqrt{5}} \quad \text{with} \quad \lambda_1 = \frac{1 + \sqrt{5}}{2} \approx 1.6803 \quad \text{(golden ratio)} \quad \text{and} \quad \lambda_2 = 1 - \lambda_1 \tag{3}
\]

Interestingly, \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of the transition matrix \( W \). Moreover, Binet’s formula can be used to create another LRNN to calculate the Fibonacci series (cf. Figure 2 c). We will come back to this later (in Example 4).

**Property 1.** LRNNs are well suited to represent differential equations and to solve them numerically. To see this, consider the homogeneous linear differential equation

\[
\sum_{k=0}^{n} c_k x^{(k)}(t) = 0 \tag{4}
\]

where \( c_k \in \mathbb{R} \) are constant coefficients with \( c_n \neq 0 \), \( x^{(k)}(t) \) is the \( k \)-th derivative of the function \( x \) with respect to time \( t \), and \( n > 0 \). It can be solved approximately by LRNNs with start vector \( s \) satisfying Equation 4 and the following transition matrix:

\[
W = \begin{bmatrix}
1 & \tau & 0 & \cdots & 0 \\
0 & 1 & \tau & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1 & \tau \\
0 & -\tau c_0 & \cdots & -\tau c_{n-2} c_0 & 1 - \tau c_{n-1} c_0
\end{bmatrix} \tag{5}
\]

**Proof.** See Appendix A. \( \square \)
**Example 3.** The exponential function \( \exp(t) = e^t \) can be defined by the differential equation \( \dot{x}(t) = x(t) \), i.e., we have \( c_0 = 1 \) and \( c_1 = -1 \) in Equation 4. In consequence, according to Property 1 and because of \( \exp(0) = \exp(0) = 1 \), the transition matrix \( W \) and start vector \( s \) of the corresponding LRNN are:

\[
W = \begin{bmatrix} 1 & \tau \\ 0 & 1 + \tau \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

Induction over time yields immediately \( x(t) = \dot{x}(t) = (1 + \tau)^{t/\tau} \approx e^t \) for small \( \tau > 0 \) (according to Euler) as expected.

The strong relationship between RNNs and differential equations is already known (Kruse et al., 2016, Sect. 9) as well as the extraction of eigenvalues to describe dynamical systems (Strogatz, 2015, Sect. 5). Nevertheless, as we will show in the rest of this paper, the combination of both provides an effective method for network size reduction (cf. Section 4.3) and therefore seems to be worthwhile to be considered by the machine learning community in more detail.

### 3.2. Network Dynamics

An LRNN runs through network states \( f(t) \) for \( t \geq 0 \). It holds (in output generating mode)

\[
f(t) = \begin{cases} 
  s, & t = 0 \\
  W \cdot f(t - \tau), & \text{otherwise}
\end{cases}
\]

and hence simply \( f(t) = W' \cdot s \). Note that we assume \( \tau = 1 \) here (cf. Definition 3).

**Property 2.** Let \( W = V \cdot J \cdot V^{-1} \) be the Jordan decomposition of the transition matrix \( W \) where \( J \) is the direct sum of one or more Jordan blocks, i.e., a block diagonal matrix formed of Jordan blocks

\[
J_m(\lambda) = \begin{bmatrix} 
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \lambda & 1 \\
0 & \cdots & \cdots & 0 & \lambda
\end{bmatrix}
\]

in general with different sizes \( m \times m \) and different eigenvalues \( \lambda \), and \( V \) is a matrix consisting of the corresponding eigen- and principal column vectors. Then we have:

\[
f(t) = W^t \cdot s = V \cdot J^t \cdot V^{-1} \cdot s
\]
If we decompose $V$ into matrices $v$ of size $N \times m$ and the column vector $V^{-1} \cdot s$ into a stack of column vectors $w$ of size $m$, corresponding to the Jordan blocks in $J$, then $f(t)$ can be expressed as a sum of vectors $u = v \cdot J_m(\lambda)^t \cdot w$ where the Jordan block powers are upper triangular Toeplitz matrices, i.e., in which each descending diagonal from left to right is constant, with:

$$
\left( J_m(\lambda)^t \right)_{ij} = \binom{t}{j-i} \lambda^{t-(j-i)} \quad \text{(Horn and Johnson, 2013, Sect. 3.2.5)}
$$

(6)

**Remark 1.** Although the parameter $t$ is discrete, i.e., a nonnegative integer number, the values of $f(t) = W^t \cdot s$ can also be computed for $t \in \mathbb{R}$ and are always real. For this, we consider the Jordan block powers from Equation 6:

- The definition of the binomial coefficient $\binom{t}{k} = \frac{t(t-1) \cdots (t-k+1)}{k(k-1) \cdots 1}$ is applicable for real and even complex $t$ and nonnegative integer $k$. For negative $k$, we have $\binom{t}{k} = 0$.

- For real matrices $W$, there are always complex conjugate eigenvalue pairs $\lambda$ and $\overline{\lambda}$ and corresponding complex coefficients $c$ and $\overline{c}$ (resulting from the respective matrix $u$ and vector $v$ in Property 2). With $c = |c| e^{i\psi}$ and $\lambda = |\lambda| e^{i\omega}$, we get $c \lambda^t + \overline{c} \overline{\lambda}^t = |c| |\lambda|^t \cos(\omega t + \psi)$ applying Euler’s formula. This obviously is defined for all $t \in \mathbb{R}$ and always yields real-valued $f(t)$.

- Negative real eigenvalues, i.e., the case $\lambda < 0$, should be treated in a special way, namely by replacing $\lambda^t$ by $|\lambda|^t \cos(\pi t)$. Both terms coincide for integer $t$, but only the latter is real-valued for all $t \in \mathbb{R}$. The powers of positive real eigenvalues $\lambda$ are always positive and real and hence need no special consideration.

A Jordan decomposition exists for every square matrix $W$ (Horn and Johnson, 2013, Theorem 3.1.11). But if $W$ has $N$ distinct eigenvectors, there is a simpler decomposition, called *eigendecomposition*. The transition matrix $W$ is *diagonalizable* in this case, i.e., similar to a diagonal matrix $D$, and the network dynamics can be directly described by means of the eigenvalues and eigenvectors of $W$:

**Property 3.** Let $W = V \cdot D \cdot V^{-1}$ be the eigendecomposition of the transition matrix $W$ with column eigenvectors $v_1, \ldots, v_N$ in $V$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_N$, on the diagonal of the diagonal matrix $D$, sorted in decreasing order with respect to their absolute values. Like every column vector, we can represent the start vector $s$ as linear combination of the eigenvectors, namely as $s = x_1 v_1 + \ldots + x_N v_N$. 

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\[
\cdots + x_N v_N = V \cdot x \text{ where } x = [x_1 \cdots x_N]^\top. \text{ It follows } x = V^{-1} \cdot s. \text{ Since } W \text{ is a linear mapping and for each eigenvector } v_k \text{ with eigenvalue } \lambda_k \text{ with } 1 \leq k \leq N \text{ it holds that } W \cdot v_k = \lambda_k v_k, \text{ we have } W \cdot s = W \cdot (x_1 v_1 + \cdots + x_N v_N) = x_1 \lambda_1 v_1 + \cdots + x_N \lambda_N v_N. \]

Induction over \( t \) yields immediately:

\[
f(t) = W^t \cdot s = V \cdot D^t \cdot x = x_1 \lambda_1^t v_1 + \cdots + x_N \lambda_N^t v_N \quad \text{(7)}
\]

### 3.3. Real-Valued Transition Matrix Decomposition

For real-valued transition matrices \( W \), it is possible to define a decomposition that, in contrast to the ordinary Jordan decomposition in Property 2, solely makes use of real-valued components, adopting the so-called real Jordan canonical form (Horn and Johnson, 2013, Sect. 3.4.1) of the square matrix \( W \). For this completely real-valued decomposition, the Jordan matrix \( J \) is transformed as follows:

1. A Jordan block with real eigenvalue \( \lambda \) remains as is in \( J \).
2. For complex conjugate eigenvalue pairs \( \lambda = \lambda_\Re + i \lambda_\Im \) and \( \lambda = \lambda_\Re - i \lambda_\Im \) with \( \lambda_\Re, \lambda_\Im \in \mathbb{R} \), the direct sum of the corresponding Jordan blocks \( J_{m}(\lambda) \) and \( J_{m}(\lambda) \) is replaced by one real Jordan block:

\[
\begin{bmatrix}
M & I & O & \cdots & O \\
O & M & I & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & O \\
\vdots & & \ddots & M & I \\
O & \cdots & \cdots & O & M
\end{bmatrix}
\]

with \( M = \begin{bmatrix} \lambda_\Re & \lambda_\Im \\
-\lambda_\Im & \lambda_\Re \end{bmatrix} \), \( I = \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix} \), and \( O = \begin{bmatrix} 0 & 0 \\
0 & 0 \end{bmatrix} \).

This procedure yields the real Jordan matrix \( J \). In consequence, we have to transform \( V \) also into a completely real-valued form. For each complex conjugate eigenvalue pair \( \lambda \) and \( \lambda \), the corresponding two eigenvectors in \( V \) could be replaced by two real-valued vectors. But the subsequent theorem (Property 4) shows a more general way: The matrix \( V \) from Property 2 is transformed into a real matrix \( A \) and, what is more, the start vector \( s \) can be replaced by an arbitrary column vector \( y \) with all non-zero entries.

**Property 4.** Let \( W = V \cdot J \cdot V^{-1} \) be the (real) Jordan decomposition of the transition matrix \( W \) and \( s \) the corresponding start vector. Then for all column vectors \( y \) of
size $N$ with all non-zero entries, there exists a square matrix $A$ of size $N \times N$ such that for all $t \geq 0$ we have:

$$f(t) = W^t \cdot s = A \cdot J^t \cdot y$$

Proof. See Appendix B. \qed

### 3.4. Long-Term Behavior

Let us now investigate the long-term behavior of an LRNN (run in output generating mode) by understanding it as an (autonomous) dynamic system (Colonius and Kliemann, 2014; Strogatz, 2015). We will see (in Property 6) that the network dynamics may be reduced to a very small number of dimensions/neurons in the long run. They determine the behavior for $t \to \infty$. Nevertheless, for smaller $t$, the use of many neurons is important for computing short-term predictions.

**Property 5.** In none of the $N$ dimensions $f(t) = W^t \cdot s$ grows faster than a polynomial and only single-exponential in $t$.

Proof. See Appendix C. \qed

In fact, LRNNs can model polynomials (cf. Example 1, parabola), general single-exponential functions like the Fibonacci series (cf. Example 2), multiple superimposed oscillators (cf. Example 6), and many more (cf. Property 8). For this, the overall transition matrix $W$ may have (a) a spectral radius greater than 1 and (b) many eigenvalues (more than two) with absolute value 1. Nevertheless, it is interesting to investigate a special case, namely a pure random reservoir where both conditions do not hold:

**Property 6.** Consider an LRNN solely consisting of a random reservoir whose transition matrix $W_{\text{res}}$ (a) is completely real-valued, (b) has (according to Property 3) an eigendecomposition $W_{\text{res}} = V \cdot D \cdot V^{-1}$ with unit spectral radius, and thus (c) all eigenvalues are distinct (which is almost always, i.e., with probability close to 1, true for random matrices), together with a completely real-valued random start vector $s$ with unit norm. Then, almost all terms $x_k \lambda_k^t v_k$ in Equation 7 vanish for large $t$ because for all eigenvalues $\lambda_k$ with $|\lambda_k| < 1$ we have $\lim_{t \to \infty} \lambda_k^t = 0$. Although a general real matrix can have more than two complex eigenvalues which are on the unit circle, for a pure random reservoir as considered here, almost always only the (largest) eigenvalues $\lambda_1$ and possibly $\lambda_2$ have the absolute values 1. In consequence, we have one of the following cases:
1. \( \lambda_1 = +1 \). In this case, the network activity contracts to one point, i.e., to a singularity: \( \lim_{t \to \infty} f(t) = x_1 v_1 \)

2. \( \lambda_1 = -1 \). For large \( t \) it holds that \( f(t) \approx x_1 (-1)^t v_1 \). This means we have an oscillation in this case. The dynamic system alternates between the two points \( \pm x_1 v_1 \).

3. \( \lambda_1 \) and \( \lambda_2 \) are two (properly) complex eigenvalues with absolute value 1. Since \( W_{\text{rss}} \) is a real-valued matrix, the two eigenvalues as well as the corresponding eigenvectors \( v_1 \) and \( v_2 \) are complex conjugate with respect to each other. Thus, for large \( t \), we have an ellipse trajectory

\[
f(t) \approx x_1 \lambda_1^t v_1 + x_2 \lambda_2^t v_2 = \tilde{V} \cdot \tilde{D}^t \cdot \tilde{x}
\]

where \( \tilde{V} = [v_1 \ v_2] \), \( \tilde{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \), and \( \tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \).

We can build a matrix \( \hat{D} \), similar to \( \tilde{D} \) but completely real-valued (cf. Section 3.3) which states the ellipse rotation. Furthermore, the rotation speed can be derived from the eigenvalue \( \lambda_1 \). In each step of length \( \tau \), there is a rotation by the angle \( \omega \tau \) where \( \omega \) is the angular frequency which can be determined from the equation \( \lambda_1 = |\lambda_1| e^{i \omega \tau} \) (cf. Remark 1). The two-dimensional ellipse trajectory can be stated by two (co)sinusoids: \( f(t) = [a \cos(\omega t) \ b \sin(\omega t)]^\top \) where \( a, b > 0 \) are half the width and height of the ellipse. Applying the addition theorems of trigonometry, we get:

\[
f(t + \tau) = \begin{bmatrix} a \cos(\omega (t + \tau)) \\ b \sin(\omega (t + \tau)) \end{bmatrix} = \begin{bmatrix} a (\cos(\omega t) \cos(\omega \tau) - \sin(\omega t) \sin(\omega \tau)) \\ b (\sin(\omega t) \cos(\omega \tau) + \cos(\omega t) \sin(\omega \tau)) \end{bmatrix} = \begin{bmatrix} \cos(\omega \tau) & -a/b \sin(\omega \tau) \\ b/a \sin(\omega \tau) & \cos(\omega \tau) \end{bmatrix} \cdot f(t)
\]

\( \hat{D} \)

From this, we can read off the desired ellipse rotation matrix \( \hat{D} \) as indicated above. Due to Property 4, there exists a (two-dimensional) start vector \( y \) and a transformation matrix \( A \) such that

\[
f(t) \approx A \cdot \hat{D}^t \cdot y \tag{8}
\]

for large \( t \). Every LRNN with many neurons can thus be approximated by a simple network with at most two neurons. The output values lie on an ellipse in general,
thus in only two dimensions. Nonetheless, in the beginning, i.e., for small $t$, the
dynamics of the system is not that regular (cf. Figure 3). But although Property 6
states only the asymptotic behavior of random LRNNs with unit spectral radius,
interestingly the network dynamics converges relatively fast to the final ellipse
trajectory: The (Euclidean) distance between the actual value $f(t)$ (according to
Equation 7) and its approximation by the final ellipse trajectory (Equation 8) is
almost zero already after a few hundred steps (cf. Figure 4). Of course this depends
on the eigenvalue distribution of the transition matrix (Tao et al., 2010). So the
long-term behavior may be different for transition matrices other than pure random
reservoirs.

The long-term behavior of LRNNs is related to that of ESNs. For the latter,
usually the activation function is tanh and the spectral radius is smaller than 1.
Then reservoirs with zero input collapse because of $|\tanh(z)| \leq |z|$ for all $z \in \mathbb{R}$
but the convergence may be rather slow, nonetheless it guarantees contractiv-
ity and hence for any fixed input (not just the origin) the system converges to
a unique fixed point. This leads to the so-called echo state property (Manjunath
and Jaeger, 2013): Any random initial state of a reservoir is forgotten such that,
after a washout period, the current network state is a function of the driving in-
put. In contrast to ESNs, LRNNs have linear activation and a spectral radius of
exactly 1 (cf. Definition 3). But as we have just shown, there is a similar effect in
the long run: The network activity reduces to at most two dimensions which are
independent from the initial state of the network.

4. Learning LRNNs

Functions can be learned and approximated by LRNNs in two steps: First, as
for ESNs (Jaeger and Haas, 2004), we only learn the output weights $W^{\text{out}}$ (cf. Sec-
tion 4.1). The input weights $W^{\text{in}}$ and reservoir weights $W^{\text{res}}$ are arbitrary random
values and remain unchanged (cf. Definition 3). Nevertheless, in order to obtain
better numerical stability during the computation, they are adjusted as follows:

- Because of the linear activation, the spectral radius of the reservoir weights
  matrix $W^{\text{res}}$ is set to 1 (cf. Definition 3). Otherwise, with increasing $t$, the
  values of $f(t) = W^{t} \cdot s$ explode if the spectral radius is greater or vanish if
  the spectral radius is smaller than 1 (cf. Section 3.4). In consequence, the
  overall learning procedure behaves rather stable.

- The norms of the vectors in $W^{\text{in}}$ and $W^{\text{res}}$ should be balanced (Koryakin
  et al., 2012). To achieve this, we initialize the reservoir neurons such that
Figure 3: Dynamic system behavior of a pure random reservoir with unit spectral radius, with $N_{\text{res}} = 100$ neurons: (a) Eigenvalue spectrum of the reservoir matrix $W_{\text{res}}$ with complex conjugate eigenvalue pairs in the complex plane. (b) Visualization of $f(t)$ by planar projection. In the long run, we get an ellipse trajectory, thus only two dimensions (cf. Equation 8). (c) Projected to one (arbitrary) dimension, we have pure sinusoids with one single angular frequency for large $t$, sampled in large steps.
Figure 4: Asymptotic behavior of pure random reservoirs with unit spectral radius: The (Euclidean) distance between the actual value \( f(t) \) (according to Equation 7) and its approximation by the final ellipse trajectory (Equation 8) is almost zero already after a few hundred steps. The figure shows the distances for \( N_{\text{res}} = 100 \) (solid/blue), \( N_{\text{res}} = 500 \) (dashed/red), and \( N_{\text{res}} = 1000 \) (dotted/black) random reservoir neurons, starting with a random vector of unit length, averaged over 1000 trials.

The reservoir start vector \( r \) (with \( N_{\text{res}} \) components) has unit norm by setting:

\[
r = \frac{1}{\sqrt{N_{\text{res}}}} \cdot [1 \cdots 1]^T
\]

It is part of the start vector \( s = \begin{bmatrix} S(0) \\ r \end{bmatrix} \) (cf. Section 3).

- We usually employ fully connected graphs, i.e., all, especially the reservoir neurons are connected with each other because the connectivity has nearly no influence on the best reachable performance (Koryakin et al., 2012).

Second, if possible, we reduce the network size (cf. Section 4.3). This often leads to better generalization and avoids overfitting. Thus, in contrast to many other approaches, the network architecture is changed during the learning process. In contrast to other approaches, we do not do this by incremental derivation from the original network but in only one step.

### 4.1. Learning the Output Weights

To learn the output weights \( W_{\text{out}} \), we run the input values from the time series \( S(0), \ldots, S(n) \) through the network (in input receiving mode), particularly through
the reservoir. This means, we build the sequence of corresponding reservoir states \( R(0), \ldots, R(n) \) where the reservoir start vector \( r \) in principle can be chosen arbitrarily but with all non-zero entries (cf. Property 4):

\[
R(t_0) = r \quad \text{and} \quad R(t + \tau) = \begin{bmatrix} \text{\( W^{\text{in}} \)} & \text{\( W^{\text{res}} \)} \end{bmatrix} \cdot \begin{bmatrix} S(t) \\ R(t) \end{bmatrix}
\] (9)

We want to predict the next input value \( S(t + \tau) \), given the current input and reservoir states \( S(t) \) and \( R(t) \). To achieve this, we comprise all but the last input and reservoir states in one matrix \( X \) with:

\[
X = \begin{bmatrix} S(0) & \cdots & S(n-1) \\ R(0) & \cdots & R(n-1) \end{bmatrix}
\] (10)

Each output value shall correspond to the respective next input value \( S(t + \tau) \). Therefore, we compose another matrix

\[
Y^{\text{out}} = [S(1) \cdots S(n)]
\] (11)

consisting of the next values of the time series \( S \) to be predicted where the first value \( S(0) \) clearly has to be omitted because it has no predecessor value. We compute \( Y^{\text{out}}(t) = S(t + \tau) \) from \( X(t) \) by assuming a linear dependency:

\[
Y^{\text{out}} = W^{\text{out}} \cdot X
\] (12)

Its solution can easily be determined as \( W^{\text{out}} = Y^{\text{out}} / X \), where / denotes right matrix division, i.e., the operation of solving a linear equation system, possibly applying the least squares method in case of an overdetermined system, as implemented in many scientific programming languages like Matlab (Higham and Higham, 2017) or Octave (Eaton et al., 2017). Prediction of further values is now possible (in output generating mode) as follows:

\[
\begin{bmatrix} S(t + \tau) \\ R(t + \tau) \end{bmatrix} = W \cdot \begin{bmatrix} S(t) \\ R(t) \end{bmatrix} \quad \text{with} \quad W \quad \text{as in Equation 1}
\] (13)

**Property 7** (treatment of multiple sequences). It is also possible to learn from multiple sequences at once. For this, let several time series \( S_1, \ldots, S_K \) in \( d \) dimensions with (not necessarily identical) lengths \( n_1, \ldots, n_K \) be given. For each \( S_k \) with \( 1 \leq k \leq K \), we determine:

- the sequence of corresponding reservoir states \( R_k \) (according to Equation 9), taking always the same reservoir start vector \( r \),
• the corresponding input matrix $X_k$ (according to Equation 10), and
• the corresponding predicted output matrix $Y_{k}^{\text{out}}$ (according to Equation 11).

We aggregate the input and output matrices to $X = [X_1 \cdots X_K]$ and $Y_{out} = [Y_1^{out} \cdots Y_K^{out}]$ with $n_1 + \cdots + n_K$ columns each. Solving the linear matrix equation $Y_{out} = W_{out} \cdot X$ (identical with Equation 12) finally yields the output weight matrix $W_{out}$.

This first phase of the learning procedure is related to a linear autoregressive model (Akaike, 1969). However, one important difference to an autoregressive model is that for LRNNs the output does not only depend on its own previous values and possibly white noise but on the complete state of the possibly big reservoir whose dynamics is explicitly dealt with in the reservoir matrix $W_{res}$. The reservoir effectively allows us to do arbitrary auxiliary computation such that any (non-linear) function $f(t)$ can be approximated by an LRNN (cf. Property 8).

4.2. An Approximation Theorem

Property 8. From a function $f(t)$ in $d \geq 1$ dimensions, let a series of function values $f(t_0), \ldots, f(t_n)$ be given. Then there is an LRNN with the following properties:

1. It runs exactly through all given $n + 1$ function values, i.e., it approximates $f(t)$.
2. It can effectively be learned by the LRNN learning procedure (Section 4.1) employing

$$N_{res} \geq n - N_{in out}$$

reservoir neurons.

Proof. See Appendix D.

Therefore, at least in theory, any time-dependent function $f(t)$ can be interpolated, i.e., exactly approximated on the given function values and continued on input other than nonnegative integer numbers (cf. Remark 1), although clearly not every function can be implemented by LRNNs, in particular functions increasing faster than single-exponential (cf. Property 5) like $2^t$ (double-exponential) or $t!$ (factorial function). Also in practice, the LRNN learning procedure performs rather well (cf. Section 5). Nevertheless, the matrix $X$ may be ill-conditioned for long input sequences, because the reservoir state sequence as part of the matrix $X$
reduces to at most two dimensions for large $t$, independent of the number of reservoir neurons (cf. Section 3.4). Hence, the rank of the matrix $X$ may not be maximal and consequently Equation 12 may not be solvable numerically in practice (although we may have an equation system with the same number of equations and unknowns). A simple increase of the number of reservoir neurons does not help much.

However, one could learn not only the output weights $W_{\text{out}}$ as in ESNs but the complete transition matrix $W$: For this, we employ a random reservoir state sequence matrix $[R(0)\cdots R(n)]$ with $N_{\text{res}}$ reservoir neurons, considered as additional input. If all elements of this matrix are random numbers, independently and identically distributed from the standard normal distribution, its rank is almost always maximal. We then just have to solve the linear matrix equation $Y = W \cdot X$ (cf. Equation 12) with

$$Y = \begin{bmatrix} S(1) & \cdots & S(n) \\ R(1) & \cdots & R(n) \end{bmatrix}$$

and $X$ as in Equation 10. By this, the input and reservoir weights $W_{\text{in}}$ and $W_{\text{res}}$ are learned, not only the output weights $W_{\text{out}}$. But our experiments indicate that this procedure is less reliable than the one with given, i.e., predefined random input and reservoir weights and unit spectral radius for the reservoir (cf. Section 4.1).

The topic of learning all weights in the matrix $W$ is investigated by Palangi et al. (2013) for ESNs with nonlinear activation function in the reservoir. However, for LRNNs, the given input and reservoir weights $W_{\text{in}}$ and $W_{\text{res}}$ together with the learned output weights $W_{\text{out}}$ already provide the best approximation of the function $f(t)$. There is no need to learn the input and reservoir weights, simply because LRNNs are completely linearly activated RNNs (including the reservoir). If one tries to learn $W_{\text{in}}$ and $W_{\text{res}}$ taking not only the output time series $S$ but additionally the reservoir state time series $R$ into account, then exactly the given input and reservoir weights are learned if Equation 14 holds. Only with nonlinear activation there is a learning effect.

**Remark 2** (generalization to nonlinear activation). Because of the completely linear activation, LRNNs cannot compute a nonlinear function $f(s)$ from the possibly multi-dimensional (start) input $s$. Only linearly separable functions can be represented. Nevertheless, the LRNN learning procedure (cf. Section 4.1) and Property 8 can be generalized in a straightforward manner to nonlinear, invertable activation functions $g$. For this, $g$ has to be applied component-wise to Equation 9 while running the input through the reservoir and to Equation 12. The solution of the resulting equation $Y_{\text{out}} = g(W_{\text{out}} \cdot X)$ is then just $W_{\text{out}} = g^{-1}(Y_{\text{out}})/X$. Hence,
LRNN learning is as easy as learning a single-layer perceptron and allows us to approximate any (possibly nonlinear) function over time \( f(t) \) efficiently.

Property 8 is related to the universal approximation theorem for feedforward neural networks (Hornik, 1991). It states that a (non-recurrent) network with a linear output layer and at least one hidden layer activated by a nonlinear, sigmoidal function can approximate any continuous function on a closed and bounded subset of the \( \mathbb{R}^n \) from one finite-dimensional space to another with any desired non-zero amount of error, provided that the network is given enough hidden neurons (Goodfellow et al., 2016, Sect. 6.4.1). Since RNNs are more general than feedforward networks, the universal approximation theorem also holds for them (Maass et al., 2002). Any measurable function can be approximated with a (general) recurrent network arbitrarily well in probability (Hammer, 2000).

**Remark 3.** Any time series \( S(0), \ldots, S(n) \) can be generated by employing a backward shift matrix, i.e., a binary matrix with 1s on the subdiagonal and 0s elsewhere (Horn and Johnson, 2013, Sect. 0.9.7), as transition matrix \( W \) and \( s = [S(0) \cdots S(n)]^\top \) as start vector. But such a network clearly would have no ability to generalize to future data. Fortunately, this does not hold for a transition matrix \( W \) learned by the procedure in Section 4.1. Furthermore, the eigenvalue spectrum of the backward shift matrix is empty, whereas that of the learned \( W \) is not, which is important for network size reduction introduced in Section 4.3.

### 4.3. Network Size Reduction

To approximate a function exactly for sure, we need a large number \( N_{\text{res}} \) of reservoir neurons (cf. Property 8 and Equation 14). It is certainly a good idea to lower this number. One could do this by simply taking a smaller number of reservoir neurons, but then a good approximation cannot be guaranteed. In what follows, we therefore reduce the dimensionality of the transition matrix \( W \) in a more controlled way – after learning the output weights. Our procedure of dimensionality reduction leads to smaller networks with sparse connectivity. In contrast to other approaches, we do not learn the new network architecture by incremental derivation from the original network, e.g., by removing unimportant neurons or weights, but in only one step by inspecting the eigenvalues of the transition matrix.

For ESNs, dimensionality reduction is considered, too, namely by means of so-called *conceptors* (Jaeger, 2014, 2017; Krause et al., 2021). These are special matrices which restrict the reservoir dynamics to a linear subspace that is characteristic for a specific pattern. However, as in principal component analysis (PCA)
(Jolliffe, 2011), conceptors reduce only the spatial dimensionality of the point cloud of the given data. In contrast to this, for LRNNs, we reduce the transition matrix \(W\) and hence take also into account the temporal order of the data points in the time series. By applying insights from linear algebra, the actual network size can be reduced and not only the subspace of computation as with conceptors.

**Property 9.** By Property 4, the function \(f(t) = W' \cdot s\) can be rewritten by means of the Jordan matrix of the transition matrix \(W\) as \(A \cdot J' \cdot y\), where the start vector can be chosen as non-zero constant, e.g., \(y = [1 \cdots 1]'\). Furthermore, by Property 2, \(f(t)\) can be expressed as a sum of vectors \(u = v \cdot J_m(\lambda)' \cdot w\) where \(w\) is constant because it is part of the start vector \(y\). Then it follows from Property 6 that for large \(t\) the contribution of a Jordan component vanishes if \(\|v\| \approx 0\) and/or \(|\lambda| \ll 1\).

In consequence, we omit all Jordan components causing only small errors, until a given threshold is exceeded. The error \(E\) of a network can be estimated by the root-mean-square error (RMSE) normalized to the number of all sample components between input \(x\) and predicted output \(y\):

\[
\text{RMSE}(x, y) = \sqrt{\frac{1}{n} \sum_{t=1}^{n} \|x(t) - y(t)\|^2}
\]

We shall omit all network components corresponding to Jordan blocks \(J_m(\lambda)\) with smallest errors as long as the RMSE is below a given threshold \(\theta\). Network components that are not omitted are considered **relevant**. Thus, from \(A, J,\) and \(y\) (according to Property 4), we successively derive reduced matrices \(A\) and \(J\) and the vector \(y\) as follows:

- Reduce \(A\) to the rows corresponding to the input/output components and the columns corresponding to the relevant network components.
- Reduce \(J\) to the rows and columns corresponding to the relevant network components.
- Reduce \(y\) to the rows corresponding to the relevant network components.

Note that the dimensionality reduction does not only lead to a smaller number of reservoir neurons but also to a rather simple network structure: The transition matrix \(J\) (which comprises the reservoir weights \(W_{\text{res}}\) of the reduced network) is a sparse matrix with non-zero elements only on the main and immediately adjacent diagonals. Thus, the number of connections is in \(O(N)\), i.e., linear in the number of reservoir neurons, not quadratic – as in general.
Figure 5 summarizes the overall learning procedure for LRNNs including network size reduction. It has been implemented by the authors in Octave. Note that, although the Jordan matrix \( J \) (cf. Property 2) may contain eigenvalues with multiplicity greater than 1, Octave does not always calculate exactly identical eigenvalues then. Therefore, we cluster the computed eigenvalues as follows: If the distance in the complex plane between eigenvalues is below some given small threshold \( \delta \), they are put into the same cluster which eventually is identified with its centroid. Thus it is a kind of single linkage clustering (Gower and Ross, 1969). The complete implementation of the learning procedure together with some case studies (cf. Section 5) is available under the following link:

http://github.com/OliverObst/decorating/

**Example 4.** Let us illustrate the LRNN learning procedure (Figure 5) with the Fibonacci series (Example 2). We start with the first values of the Fibonacci series \( f(0), \ldots, f(n) \) as input \( S \) (lines 1-3) and generate a random reservoir of size \( N_{\text{res}} \) (lines 4-6). After learning the output weights \( W_{\text{out}} \) (lines 7-10) and decomposing the resulting transition matrix \( W \) (lines 11-15), the network size reduction procedure (lines 16-24) often yields minimal networks with only two reservoir neurons representing

\[
f(t) \overset{\text{Property 4}}{=} A \cdot J^t \cdot y \quad \text{with} \quad A \approx \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad J \approx \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{for} \quad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

where the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are as in Binet’s formula (Equation 3). For instance, for \( N_{\text{res}} = n = 30 \) and precision threshold \( \theta = 0.001 \), we obtain minimal networks in 32% of the cases from 100 trials. They belong to the best networks with respect to their RSME. Thus, by employing a standard validation procedure, the LRNN in Figure 2c actually can be derived numerically by the LRNN learning procedure with network size reduction.

Note that the spectral radius of the reservoir weights matrix \( W_{\text{res}} \) remains 1 all the time (cf. Definition 3). However, after learning the output weights \( W_{\text{out}} \), the spectral radius of the overall transition matrix \( W \) (according to Equation 1) and hence of the matrix \( J \) may be greater than 1 if the function \( f(t) \) to be modeled is exponentially increasing. This obviously holds for the Fibonacci series (\( \lambda_1 > 1 \)).

### 4.4. Complexity and Generalization of the Procedure

**Remark 4.** In both learning steps, it is possible to employ any of the many available fast and constructive algorithms for linear regression and eigendecomposition.
1: % d-dimensional function $f$, given sampled, as time series $S$, and start vector $s$
2: $S = [f(0) \cdots f(n)]$
3: $s = \begin{bmatrix} S(0) \\ r \end{bmatrix}$ where $r = \frac{1}{\sqrt{N_{\text{res}}}} \cdot [1 \cdots 1]^T$

4: % random initialization of input and reservoir weights
5: $W^{\text{in}} = \text{randn}(N_{\text{res}}, d)$
6: $W^{\text{res}} = \text{randn}(N_{\text{res}}, N_{\text{res}})$ normalized to unit spectral radius

7: % learn output weights by linear regression
8: $X = W^{\text{out}} \cdot s$
9: $Y^{\text{out}} = [S(1) \cdots S(n)]$
10: $W^{\text{out}} = Y^{\text{out}} / X$

11: % transition matrix and its decomposition
12: $W = \begin{bmatrix} W^{\text{out}} \\ W^{\text{in}} & W^{\text{res}} \end{bmatrix}$
13: $J = \text{jordan\_matrix}(W)$ with components sorted in decreasing order
14: with respect to $\text{Error}(J_{(1,\ldots,k-1,k+1,\ldots,K)})$
15: where $K = \# \text{Jordan components in } J$

16: % network size reduction (with binary search)
17: $L = 1$ % left index border
18: $R = K$ % right index border
19: while ($L \neq R$)
20: $M = \left\lfloor \frac{L+R}{2} \right\rfloor$
21: if $\text{Error}(J_{(1,\ldots,M)}) < \theta$
22: then $R = M$
23: else $L = M + 1$
24: return ($M$)

25: % subroutine $\text{Error}(J_I)$
26: % compute error for Jordan matrix reduced to indexed components
27: reduce $J$ to components indexed by $I$
28: $y = [1 \cdots 1]^T$
29: $Y = [J' \cdot y]_{i=0,\ldots,n}$ % run in output generating mode
30: $A = X / Y$ with rows restricted to input/output dimensions
31: return ($\text{RMSE}(S, A \cdot Y)$)

Figure 5: Pseudocode for learning LRNNs including network size reduction. A binary search algorithm is employed for determining the relevant network components with smallest errors. For this, the network components are sorted by their RMSE. The program returns the number $M$ of relevant components in the Jordan matrix $J$ (line 24). The subroutine $\text{Error}(J_I)$ (lines 25-31) computes the error of the predicted output for the Jordan matrix reduced to the components indexed by $I$. 

Therefore, the time complexity is just $O(N^3)$ for both output weights learning and network size reduction. In theory, if we assume that the basic numerical operations like $+$ and $\cdot$ can be done in constant time, the asymptotic complexity is even a bit better. In practice, however, the complexity depends on the bit length of numbers in floating point arithmetic, of course, and may be worse hence. The size of the learned network is in $O(N)$ (cf. Section 4.3).

Note that, in contrast, feedforward networks with three threshold neurons already are NP-hard to train (Blum and Rivest, 1992). This results from the fact that the universal approximation theorem for feedforward networks differs from Property 8 because the former holds for multi-dimensional functions and not only time-dependent input. In this light, the computational complexity of $O(N^3)$ for LRNNs does not look overly expensive. It dominates the overall time complexity of the whole learning procedure because it is not embedded in a time-consuming iterative learning procedure (like backpropagation) as in other state-of-the-art methods.

**Remark 5.** We observe that most of the results presented in this paper still hold if the transition matrix $W$ contains complex numbers. This means in particular that also complex functions can be learned (from complex-valued time series) and represented by LRNNs (Property 8). Nonetheless, the long-term behavior of networks with a random complex transition matrix $W$ differs from the one described in Section 3.4 because then there are no longer pairs of complex conjugate eigenvalues.

### 5. Experiments

In this section, we demonstrate evaluation results for learning and predicting time series, approximating them by a function $f(t)$ represented by an LRNN, for several tasks. We consider the following benchmarks: multiple superimposed oscillators (MSO), number puzzles, robot soccer simulation, and predicting stock prices. All experiments are performed with a program written by the authors in Octave (Eaton et al., 2017) (cf. Section 4.3). Let us start with an example that illustrates the overall method.

**Example 5.** The graphs of the functions $f_1(t) = 4t(1-t)$ (parabola) and $f_2(t) = \sin(\pi t)$ (sinusoid) look rather similar for $t \in [0, 1]$ (cf. Figure 6). Can both functions be learned and distinguished from each other by our LRNN learning procedure (cf. Section 4)?
Both can be learned and distinguished by LRNNs from the visually similar positive parts of the respective graphs, i.e., function values for \( t \in [0, 1] \). In this interval, all values of the parabola (solid/blue) are greater than or equal to those of the sinusoid (dashed/red).

To investigate this, we sample both graphs for \( t \in [0, 1] \) with time step \( \tau = 0.01 \). After that, we learn the output weights \( W_{\text{out}} \) (cf. Section 4.1), starting with a large enough reservoir consisting of up to \( N_{\text{res}} = 100 \) neurons (cf. Property 8). Finally, we reduce the size of the overall transition matrix \( W \) with precision threshold \( \theta = 0.01 \) and cluster threshold \( \delta = 0.03 \) (cf. Section 4.3). Minimal LRNNs consist of \( N_1 = 3 \) neurons for the parabola (cf. Example 1) and \( N_2 = 2 \) neurons for the sinusoid (cf. Section 3.4). The networks of minimal size are learned already with \( N_{\text{res}} = 40 \) reservoir neurons before network size reduction in about 77% (parabola) or 99% (sinusoid) of the trials (cf. Figure 7). Learning the parabola is more difficult because the corresponding transition matrix \( W \) (cf. Example 1) has no proper eigendecomposition according to Property 3 but only a Jordan decomposition according to Property 2.

5.1. **Multiple Superimposed Oscillators**

**Example 6.** Multiple superimposed oscillators (MSO) count as difficult benchmark problems for RNNs (Koryakin et al., 2012; Schmidhuber et al., 2007). The corresponding time series is generated by summing up several (pure) sinusoids. Formally it is described by

\[
S(t) = \sum_{k=1}^{K} \sin(\alpha_k t)
\]
Figure 7: For Example 5, how often LRNNs of minimal size are learned after network size reduction, i.e., with $N_1 = 3$ neurons for the parabola and $N_2 = 2$ neurons for the sinusoid? The diagram shows the success rate of the learning procedure in this regard as a function of the number of reservoir neurons $N^\text{res}$ before network size reduction (for 100 trials). Networks of minimal size are learned starting already with $N^\text{res} = 40$ reservoir neurons in about 77% (parabola, solid/blue) or 99% (sinusoid, dashed/red) of the trials.

where $K \leq 8$ denotes the number of sinusoids and $\alpha_k \in \{0.200, 0.311, 0.420, 0.510, 0.630, 0.740, 0.850, 0.970\}$ their frequencies.

Various publications have investigated the MSO problem with different numbers of sinusoids. We concentrate here solely on the most complex case $K = 8$ because in contrast to other approaches it is still easy to learn for LRNNs. Applying the LRNN learning procedure with precision threshold $\theta = 0.5$, we arrive at LRNNs with only $N = 16$ reservoir neurons and an RMSE less than $10^{-5}$ (cf. Figure 8), if we start with a large enough reservoir (cf. Figure 9). Since two neurons are required for each frequency (cf. Section 3.4), $2K = 16$ is the minimal reservoir size. Thus LRNNs outperform the previous state-of-the-art for the MSO task with a minimal number of units. Koryakin et al. (2012) report $N^\text{res} = 68$ as the optimal reservoir size for ESNs, but in contrast to our approach, this number is not further reduced.

For a more systematic evaluation, we generalize the MSO benchmark (Example 6) by considering 20 times $K = 8$ random frequencies $\alpha_1, \ldots, \alpha_8$ uniformly distributed on the interval $[0, 1]$. We take the first 250 time steps, i.e., $t = 1, \ldots, 250$, as training data and the subsequent 50 time steps as testing data and compare the performance of LRNNs with other approaches (cf. Section 2) with respect to their RSME on the testing data, partially using the Python library
Figure 8: The signal $S(t)$ of $K = 8$ multiple superimposed oscillators (for $1 \leq t \leq 300$ and time step $\tau = 1$) does not have a simple periodic structure (small figure). LRNN learning leads to minimal networks with only $N = 16 = 2K$ reservoir neurons, i.e., two for each frequency in the signal, with RMSE less than $10^{-5}$ (dashed line). Other methods do not perform so well on the MSO benchmark (cf. dotted lines).

Figure 9: Experimental results for the MSO benchmark ($K = 8$). The diagram shows the success rate (from 100 trials): from an initial reservoir size of $N^{\text{res}}$ neurons, how often is the minimal LRNN with size $N = 16$ learned? The two curves are for different lengths $T$ of the time series $S(t)$ used for training. Already for $T = 150$ (solid/blue), a minimal-size LRNN is learned in at least 96% of the trials if $N^{\text{res}} \geq 70$. For these minimal LRNNs, the RMSE is smaller than $10^{-5}$. As one can see, for $T = 130$ (dashed/red) the given information does not always suffice and leads to overfitting.
Darts (Herzen et al., 2022, see also http://unit8co.github.io/darts/) for time series, including a simple baseline:

- Baseline: Constantly predict the arithmetic mean of the training data.
- ARIMA: We use the AutoARIMA package from Hyndman and Khandakar (2008).
- ESN: For this, we adapt the simple sparse ESN demo by Mantas Lukoševičius from http://mantas.info/code/simple_essen/. We also use ReservoirPy (Trouvain et al., 2020).
- LSTM: We employ the respective forecasting model implemented in Darts with hyperparameter optimization taking the last 50 time steps of the training data for validation.
- LRNN: Here we adopt validation data as above and choose the best model with respect to the RMSE on the validation data from 100 trials.

Table 1 shows the evaluation results. As one can see, LRNNs outperform all other approaches on this benchmark by far. The reason for this is certainly the network size reduction procedure (cf. Section 4.3), unique in our approach, because it exactly selects the relevant components of the network: Each complex conjugate eigenvalue pair corresponds to one of the frequencies \( \alpha_1, \ldots, \alpha_8 \). In general, an LRNN with \( 2K \) neurons suffices to represent a signal, which might be a musical harmony (Stolzenburg, 2017), consisting of \( K \) sinusoids (cf. Section 3.4). It can be learned by the LRNN learning procedure with network size reduction. However, if several frequencies are close to each other (cf. example #1 in Table 1) or are rather small (cf. example #12), then LRNNs do not perform quite so well.

LRNNs are also advantageous considering the time required to train a well performing model with less than 3 s per series for training and testing on average:

| Arima (R) | Arima (Python) | ESN | LSTM | LRNN |
|-----------|----------------|-----|------|------|
| Train + test time | 0.35 s | 3.2 s | 1.24 s | 125.43 s | 2.97 s |

The ARIMA experiments with an R implementation are fastest (0.35 s per series) but often perform even worse than our baseline. Interestingly, an ARIMA implementation in Python turns out to be much slower (3.20 s per series). ESN learning takes 1.24 s per series. LRNNs more or less extend the ESN approach
and take 2.97 s per series on average with our approach, including required repeated reservoir generations. LSTM learning including hyperparameter selection takes 125.4 s per series. All experiments are run on an Intel i9-10940X, 3.3 GHz CPU, and 128 GB RAM.

5.2. Solving Number Puzzles

**Example 7.** Number series tests are a popular type of intelligence test. The function represented by a number series can often be learned also by artificial neural networks, in particular RNNs. Glüge and Wendemuth (2013) list 20 number puzzles (cf. Ragni and Klein, 2011). Among them are the series:

\[
S_8 = [28, 33, 31, 36, 34, 39, 37, 42] \quad f(t) = f(t - 2) + 3
\]

\[
S_9 = [3, 6, 12, 24, 48, 96, 192, 384] \quad f(t) = 2f(t - 1)
\]

\[
S_{15} = [6, 9, 18, 21, 42, 45, 90, 93] \quad f(t) = 2f(t - 2) + 4.5 + 1.5(-1)^{t-1}
\]

\[
S_{19} = [8, 12, 16, 20, 24, 28, 32, 36] \quad f(t) = f(t - 1) + 4
\]

We apply the LRNN learning procedure to all 20 number puzzles taking small reservoirs because the number series are short. As a side effect, this leads to learning more general functions, which seems to be fully adequate because number puzzles are usually presented to humans. The first 7 of 8 elements of each series are given as input. In each trial, we repeatedly generate LRNNs, until the RMSE is smaller than \(\theta = 0.1\). Then the last (8-th) element of the respective series is predicted (according to Equation 13) and rounded to the nearest integer because all considered number series are integer.

Table 2 lists the percentages of correct predictions of the last element for different settings. Here, the series with definitions recurring to \(f(t-2)\) but not \(f(t-1)\), e.g., \(S_8\) and \(S_{15}\), turned out to be the most difficult. If we now just add the previous values of the time series, i.e., \(f(t-2)\), as clue to the input, then the correctness of the procedure increases significantly: For 19 of 20 number puzzles, the most frequently predicted last element (simple majority) is the correct one. It is predicted in 76.5\% on average over all trials and number puzzles. Let us remark that the whole evaluation with altogether \(20 \cdot 50 \cdot 1000 = 1\,000\,000\) trials including possibly repeated network generation and network size reduction ran in only a few minutes on standard hardware.

5.3. Replaying Soccer Games

RoboCup (Kitano et al., 1997) is an international scientific robot competition in which teams of multiple robots compete against each other. Its differ-
| # | frequencies | Baseline | ARIMA | ESN | LSTM | LRNN |
|---|-------------|----------|-------|-----|------|------|
|   | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ | $\alpha_7$ | $\alpha_8$ | RMSE | RMSE | RMSE | units | RMSE | $N$ | RMSE |
| 1 | 0.334 | 0.336 | 0.399 | 0.403 | 0.412 | 0.438 | 0.442 | 0.724 | 2.05613 | 2.07496 | 0.23208 | 5 + 32 | 0.16038 | 10 | 0.04761 |
| 2 | 0.049 | 0.091 | 0.161 | 0.292 | 0.472 | 0.715 | 0.832 | 0.997 | 2.25490 | 3.06428 | 0.19716 | 10 + 2 | 0.19218 | 16 | 0.00051 |
| 3 | 0.308 | 0.521 | 0.597 | 0.607 | 0.736 | 0.766 | 0.924 | 0.957 | 1.34810 | 1.46533 | 0.13680 | 5 + 8 | 0.11722 | 16 | 0.00060 |
| 4 | 0.031 | 0.348 | 0.448 | 0.476 | 0.476 | 0.613 | 0.628 | 0.833 | 2.03748 | 2.13161 | 0.25979 | 5 + 8 | 0.19209 | 14 | 0.00003 |
| 5 | 0.059 | 0.239 | 0.324 | 0.421 | 0.437 | 0.519 | 0.747 | 0.777 | 1.68045 | 2.41015 | 0.16561 | 10 + 1 | 0.14548 | 16 | 0.00011 |
| 6 | 0.013 | 0.029 | 0.262 | 0.543 | 0.636 | 0.705 | 0.740 | 0.807 | 1.79038 | 2.91774 | 0.19343 | 5 + 4 | 0.16770 | 16 | 0.00038 |
| 7 | 0.155 | 0.226 | 0.286 | 0.512 | 0.661 | 0.692 | 0.746 | 0.930 | 1.48729 | 1.75658 | 0.13109 | 5 + 4 | 0.13761 | 16 | 0.00012 |
| 8 | 0.017 | 0.027 | 0.273 | 0.475 | 0.616 | 0.848 | 0.962 | 0.989 | 1.87712 | 2.66991 | 0.22200 | 5 + 2 | 0.16481 | 16 | 0.02033 |
| 9 | 0.092 | 0.318 | 0.335 | 0.413 | 0.593 | 0.743 | 0.747 | 0.799 | 1.84272 | 2.31084 | 0.22680 | 5 + 6 | 0.16025 | 16 | 0.00142 |
| 10 | 0.108 | 0.122 | 0.262 | 0.307 | 0.391 | 0.577 | 0.589 | 0.603 | 1.64237 | 1.66822 | 0.18361 | 10 + 16 | 0.13289 | 16 | 0.00772 |
| 11 | 0.071 | 0.264 | 0.557 | 0.609 | 0.641 | 0.719 | 0.853 | 0.964 | 1.71070 | 2.01792 | 0.13613 | 5 + 16 | 0.14964 | 16 | 0.00003 |
| 12 | 0.036 | 0.052 | 0.062 | 0.222 | 0.279 | 0.316 | 0.563 | 0.672 | 1.52224 | 1.83523 | 0.18562 | 10 + 8 | 0.15571 | 14 | 0.15984 |
| 13 | 0.036 | 0.481 | 0.571 | 0.724 | 0.750 | 0.750 | 0.864 | 0.898 | 2.51888 | 2.50217 | 0.15961 | 5 + 16 | 0.22408 | 14 | 0.00067 |
| 14 | 0.175 | 0.220 | 0.258 | 0.419 | 0.487 | 0.513 | 0.628 | 0.663 | 2.17787 | 1.90370 | 0.23698 | 5 + 8 | 0.19731 | 16 | 0.00069 |
| 15 | 0.185 | 0.300 | 0.461 | 0.751 | 0.814 | 0.833 | 0.840 | 0.992 | 2.18527 | 1.63141 | 0.15527 | 5 + 2 | 0.20659 | 14 | 0.03709 |
| 16 | 0.088 | 0.120 | 0.137 | 0.245 | 0.478 | 0.793 | 0.797 | 0.992 | 2.27108 | 2.51581 | 0.15872 | 10 + 1 | 0.19325 | 16 | 0.01439 |
| 17 | 0.002 | 0.242 | 0.348 | 0.503 | 0.734 | 0.748 | 0.759 | 0.862 | 1.47542 | 1.76223 | 0.23496 | 10 + 8 | 0.13808 | 16 | 0.00150 |
| 18 | 0.018 | 0.352 | 0.583 | 0.625 | 0.714 | 0.824 | 0.838 | 0.888 | 2.44582 | 2.71492 | 0.24585 | 5 + 32 | 0.25433 | 16 | 0.00010 |
| 19 | 0.046 | 0.105 | 0.263 | 0.351 | 0.517 | 0.556 | 0.758 | 0.807 | 1.79806 | 2.24052 | 0.18220 | 5 + 16 | 0.18124 | 16 | 0.00005 |
| 20 | 0.091 | 0.141 | 0.375 | 0.578 | 0.686 | 0.785 | 0.951 | 0.996 | 2.05839 | 2.21768 | 0.15490 | 5 + 32 | 0.17551 | 16 | 0.00001 |

Table 1: Evaluation results for 20 generalized MSO examples with 8 randomly generated frequencies each. The best performing approach is highlighted by bold face. For the LSTMs, the number of input and hidden units of the best performing neural network is given. For LRNNs, the network size $N$ after network size reduction is shown.
Table 2: Percentages of correct predictions of the last element for 20 number puzzles (Ragni and Klein, 2011; Glüge and Wendemuth, 2013) in 1000 trials for different settings: (a) with fixed reservoir size $N_{\text{res}} = 3, 4, 5$; (b) with network size reduction starting with $N_{\text{res}} = 7$ reservoir neurons; (c) same procedure but in addition always the previous time series value is used as clue.

| series | $N_{\text{res}} = 3$ | $N_{\text{res}} = 4$ | $N_{\text{res}} = 5$ | with reduction | plus clue |
|--------|----------------------|----------------------|----------------------|----------------|----------|
| $S_1$  | 2.2%                 | 1.3%                 | 1.3%                 | 0.8%           | 33.4%    |
| $S_2$  | 37.6%                | 42.2%                | 29.4%                | 32.0%          | 100.0%   |
| $S_3$  | 5.4%                 | 4.1%                 | 1.1%                 | 3.2%           | 100.0%   |
| $S_4$  | 23.8%                | 24.2%                | 16.8%                | 23.2%          | 99.9%    |
| $S_5$  | 56.9%                | 57.6%                | 44.2%                | 44.7%          | 99.7%    |
| $S_6$  | 31.7%                | 33.7%                | 16.1%                | 11.0%          | 100.0%   |
| $S_7$  | 72.8%                | 68.2%                | 56.2%                | 64.5%          | 100.0%   |
| $S_8$  | 5.1%                 | 3.4%                 | 1.3%                 | 1.8%           | 76.3%    |
| $S_9$  | 100.0%               | 100.0%               | 100.0%               | 100.0%         | 100.0%   |
| $S_{10}$ | 48.9%             | 71.5%                | 67.6%                | 72.6%          | 100.0%   |
| $S_{11}$ | 10.6%            | 9.0%                 | 3.4%                 | 3.7%           | 100.0%   |
| $S_{12}$ | 23.8%              | 21.1%                | 11.0%                | 8.5%           | 43.2%    |
| $S_{13}$ | 56.5%            | 58.1%                | 41.5%                | 47.2%          | 99.8%    |
| $S_{14}$ | 6.7%              | 7.4%                 | 2.1%                 | 2.7%           | 87.1%    |
| $S_{15}$ | 1.6%              | 2.6%                 | 2.5%                 | 0.3%           | 1.1%     |
| $S_{16}$ | 6.8%              | 5.9%                 | 3.4%                 | 2.9%           | 73.3%    |
| $S_{17}$ | 11.9%             | 12.0%                | 6.8%                 | 6.6%           | 41.0%    |
| $S_{18}$ | 3.1%              | 2.0%                 | 1.1%                 | 0.9%           | 18.0%    |
| $S_{19}$ | 59.6%             | 70.1%                | 72.0%                | 77.8%          | 99.8%    |
| $S_{20}$ | 1.5%              | 0.5%                 | 0.6%                 | 0.5%           | 57.2%    |

ent leagues provide many sources of robotics data that can be used for further analysis and application of machine learning. A soccer simulation game lasts 10 mins and is divided into 6000 time steps where the length of each cycle is 100 ms. Logfiles contain information about the game, in particular about the current positions of all players and the ball including velocity and orientation for each cycle. Michael et al. (2019) describe a research dataset using some of the released binaries of the RoboCup 2D soccer simulation league (Chen et al., 2003; Gabel et al., 2017) from 2016 and 2017 (Michael et al., 2018). In our experiments we evaluated ten games of the top-five teams (available from http://bitbucket.org/oliverobst/robocupsimdata), considering only the $(x, y)$-coordinates of the ball and the altogether 22 players for all time points during the so-called “play-on” mode.

For LRNN learning, we use only every 10th time step of each 6000 step game with $d = 2 + 2 \cdot 22 = 46$ input dimensions and start with a reservoir consisting of
Table 3: For ten RoboCup simulation games, an LRNN is learned with initially $N = 500 + 46 = 546$ neurons. The table shows the RMSE (1) before and (2) after dimensionality reduction where $\theta = 1$ m. The network size can be reduced significantly – 29.2% on average (last column).

$N_{res} = 500$ neurons. We repeat the learning procedure until the RMSE is smaller than 1; on average, already two attempts suffice for this. This means, if we replay the game by the learned LRNN (in output generating mode) then on average the predicted positions deviate less than 1 m from the real ones (Euclidean distance) – over the whole length of the game (cf. Figure 10). Network size reduction leads to significantly less neurons compared to the original number $N = 46 + 500 = 546$ – on average 29.2% if we concentrate on the relevant components for the ball trajectory (cf. Table 3). Note that the size of the learned network is in $O(N)$ (cf. Remark 4). Thus the LRNN model is definitely smaller than the original time series representation of a game. The complete learning procedure runs in less than a minute on standard hardware.

Property 7 shows how we can learn from multiple time series at once. This is also helpful here because by this procedure we can investigate the overall behavior of a specific robot soccer agent. As example for this, we consider the trajectories of the goalkeeper of the RoboCup simulation team FRA-UNIted during the seeding and the qualifying round of RoboCup Japan Open 2020 (see http://bit.ly/japanopen2020ssim). For learning one LRNN from this, we employ a reservoir with $N_{res} = 1000$ neurons, adopt again a maximum threshold for the RMSE of $\theta = 1$ m, and only use every 20th step of each of the 7 games. The overall trajectory of the FRA-UNIteed goalkeeper can be learned easily then (cf. Figure 11). From this, one may conclude that the goalkeeper takes up three basic positions in front of the goal, does not approach the centre line more than about 30 m and hardly leaves the centre line.
Figure 10: Ball trajectory of RoboCup 2D soccer simulation game #6 (Oxsy 0 versus Gliders 2016) on a pitch of size 105 m × 68 m. For all time steps, the original trajectory of the ball during play is shown (dotted/black). The game can be replayed by an LRNN with \( N = 500 + 46 = 546 \) neurons with high accuracy (solid/blue). The reduced network with \( N = 354 \) reservoir neurons still mimics the trajectory with only small error (dashed/red).

Figure 11: Left: Trajectory of the FRA-UNIted goalkeeper in front of the goal during games at the RoboCup Japan Open 2020. Right: Dots (in blue) mark positions that were visited more than three times (larger dots: more visits, 0.5 m resolution), information that can be derived from predictions, highlighting three larger, frequently visited regions in front of the goal.
5.4. Predicting Stock Prices

Stock price prediction is a topic that receives a considerable amount of attention in finance. Complexity of markets resulting in multiple and sudden changes in trends of stock prices make their prediction a challenge. Consequently, a number of different approaches and methods have been developed. Litz (2020) analyzes 30 different stocks by ARIMA and LRNNs using the closing stock prices 2016–2019. The stock price time series (consisting of 762 data points each) are split into training and testing data, with the first 80% of each series for training and the final 20% for evaluation. For a representative comparison, the RMSE of the predictions on every stock in the set is calculated. The average RMSE using LRNNs with $N_{\text{res}} = 600$ reservoir neurons is $E_{\text{test}} = 18.40\,€$, lower than the average RMSE using ARIMA models with seasonal patterns modeled using Fourier terms (Hyndman and Athanasopoulos, 2013, p. 321) which is $E_{\text{test}} = 24.23\,€$. For shorter term predictions of 60 steps, it is possible to slightly reduce the RMSE further to $E_{\text{test}} = 17.46\,€$ by using smaller LRNNs of $N_{\text{res}} = 200$ reservoir neurons and a smaller training set of 240 data points. With an average stock price of $286.71\,€$ of all stocks in the set, the average deviation is only 6.1%.

Apart from the good prediction results, LRNNs have the advantage that they allow the prediction of multiple stocks at the same time. An LRNN can read in 30 stocks and predict each of them concurrently. For a concurrent forecast for 60 steps, LRNNs achieve an average RMSE of $E_{\text{test}} = 30.02\,€$ with 240 training steps. Compared to ARIMA, LRNNs have also an advantage when it comes to the number of hyperparameters that have to be tuned. The LRNN model is robust when it comes to choosing the number of reservoir neurons, whereas the ARIMA model requires the adjustment of many parameters (e.g., for seasonal patterns). The compute time for ARIMA increases significantly with the number of hyperparameters. For the considered 30 stocks, LRNNs are computed about 15 times faster than the ARIMA models with the selected number of Fourier terms.

For a more systematic evaluation, we take the stocks of the German stock market index DAX (again from http://de.finance.yahoo.com/) and consider the last 250 + 50 data points until the end of 2021 as training and testing data, respectively, for each stock in the DAX existing at least 300 trading days until that time. We apply the same approaches as in Section 5.1 and compute their RMSE with respect to the testing data, each normalized (i.e., divided) by the arithmetic mean of the corresponding training data. As baseline we just constantly predict the stock price of the last trading day of the training data.

As one can see in Table 4, LRNNs outperform the other approaches in the majority of all cases, namely for 19 of 39 stocks, where the resulting network
size $N$ after size reduction often is rather small. However, the overall performance for all approaches often is not much better than the baseline (see also Figure 12). In line with that, the review by Shah et al. (2019) shows that predicting stock prices remains a challenging problem, especially for a longer timeframe, which we investigate here. Contemporary research often uses complex models, ranging from LSTM RNNs (Nelson et al., 2017; Roondiwala et al., 2017) to attention-based models that use further information about the events that drive the stock prices, e.g., news texts from social media (Liu et al., 2019). These models and also LRNNs yield rather accurate results, but mainly only in the short run.

6. Conclusions

In this paper, we have introduced LRNNs. The major innovation in this work is a closed-form approach to network size reduction (cf. Section 4.3) that learns both architecture and parameters of linearly activated RNNs. No backpropagation, gradient-descent, or other iterative procedure with several epochs is required, and the approach leads to significantly smaller, sparsely connected, and in many cases even minimal size networks.

We have shown that despite its simplicity of using only linear activation in the recurrent layer, LRNNs are a powerful approach to model time-dependent func-
| name                  | DAX member | Baseline | ARIMA | ESN | LSTM | LRNN |
|-----------------------|------------|----------|-------|-----|------|------|
| Adidas                | ADS.DE     | 0.0554   | 0.0554| 0.0784 | 0.1162 | 2 | 0.0569 |
| Airbus                | AIR.DE     | 0.0618   | 0.1070| 0.0583 | 0.2293 | 2 | 0.0500 |
| Allianz               | ALV.DE     | 0.0252   | 0.0241| 0.0199 | 0.0916 | 2 | 0.0503 |
| BASF                  | BAS.DE     | 0.0438   | 0.0262| 0.0739 | 0.1218 | 2 | 0.0433 |
| Bayer                 | BAYN.DE    | 0.0414   | 0.0409| 0.0626 | 0.1027 | 1 | 0.0350 |
| BMW St                | BMW.DE     | 0.0706   | 0.0706| 0.0744 | 0.1509 | 11| 0.0787 |
| Brenntag              | BNR.DE     | 0.0544   | 0.0938| 0.0664 | 0.1162 | 2 | 0.0969 |
| Continental           | CON.DE     | 0.0518   | 0.0518| 0.0803 | 0.2540 | 2 | 0.1918 |
| Covestro              | COV1.DE    | 0.0488   | 0.0488| 0.0564 | 0.1612 | 1 | 0.0609 |
| Deutsche Börse        | DB1.DE     | 0.0307   | 0.0307| 0.0249 | 0.0814 | 1 | 0.0347 |
| Deutsche Bank         | DBK.DE     | 0.0442   | 0.0442| 0.0229 | 0.1794 | 1 | 0.0599 |
| Delivery Hero         | DHER.DE    | 0.1119   | 0.1040| 0.1148 | 0.2226 | 3 | 0.0798 |
| Deutsche Post         | DPW.DE     | 0.0487   | 0.0379| 0.0369 | 0.1822 | 1 | 0.0549 |
| Deutsche Telekom      | DTE.DE     | 0.0279   | 0.0279| 0.0619 | 0.1209 | 2 | 0.0224 |
| Deutsche Wohnen SE     | DWN1.DE    | 0.2410   | 0.2410| 0.2468 | 0.1706 | 1 | 0.2589 |
| Siemens Energy        | ENR.DE     | 0.0457   | 0.0457| 0.0364 | 0.2212 | 9 | 0.0349 |
| E.ON                  | EOAN.DE    | 0.0646   | 0.0646| 0.0666 | 0.4041 | 1 | 0.0389 |
| Fresenius Medical Care| FME.DE     | 0.0831   | 0.0842| 0.0916 | 0.1279 | 1 | 0.0738 |
| Fresenius             | FRE.DE     | 0.1256   | 0.1256| 0.1560 | 0.1399 | 2 | 0.1228 |
| HeidelbergCement      | HEL.DE     | 0.0504   | 0.0259| 0.0708 | 0.1171 | 2 | 0.0912 |
| Henkel Vz             | HEN3.DE    | 0.0476   | 0.0476| 0.0328 | 0.1479 | 2 | 0.0237 |
| HelloFresh            | HFG.DE     | 0.1236   | 0.1236| 0.1241 | 0.3065 | 2 | 0.1220 |
| Infineon              | IFX.DE     | 0.1035   | 0.1029| 0.1623 | 0.3574 | 1 | 0.0705 |
| Linde PLC             | LIN.DE     | 0.1077   | 0.0710| 0.1134 | 0.1803 | 1 | 0.0909 |
| Merck KGaA            | MRK.DE     | 0.1293   | 0.0826| 0.0332 | 0.2156 | 1 | 0.0361 |
| MTU Aero Engines      | MTX.DE     | 0.0598   | 0.0600| 0.1085 | 0.2046 | 2 | 0.0652 |
| Münchener Rück        | MUV2.DE    | 0.0257   | 0.0256| 0.0430 | 0.1276 | 1 | 0.0421 |
| Porsche Vz            | PAH3.DE    | 0.0690   | 0.1116| 0.0920 | 0.1710 | 2 | 0.1188 |
| Puma                  | PUM.DE     | 0.0894   | 0.0887| 0.0783 | 0.1737 | 1 | 0.0531 |
| Qiagen                | QIA.DE     | 0.0660   | 0.0660| 0.0810 | 0.1311 | 2 | 0.0225 |
| RWE                   | RWE.DE     | 0.0460   | 0.0460| 0.0805 | 0.1202 | 2 | 0.0276 |
| SAP                   | SAP.DE     | 0.0398   | 0.0398| 0.0459 | 0.1200 | 2 | 0.0442 |
| Siemens Healthineers  | SHL.DE     | 0.1057   | 0.0561| 0.0975 | 0.1898 | 1 | 0.0328 |
| Siemens               | SIE.DE     | 0.0561   | 0.0354| 0.0758 | 0.1276 | 2 | 0.0996 |
| Sartorius Vz          | SRT3.DE    | 0.0676   | 0.0676| 0.0591 | 0.2400 | 1 | 0.0463 |
| Symrise               | SY1.DE     | 0.1032   | 0.1032| 0.0893 | 0.1055 | 1 | 0.0755 |
| Vonovia               | VNA.DE     | 0.0714   | 0.0801| 0.0662 | 0.1222 | 1 | 0.0593 |
| Volkswagen Vz         | VOW3.DE    | 0.0615   | 0.0615| 0.1207 | 0.1521 | 2 | 0.1181 |
| Zalando               | ZAL.DE     | 0.0656   | 0.0656| 0.1139 | 0.2111 | 2 | 0.0600 |

Table 4: Evaluation results for all stocks in the DAX existing at least 300 trading days until the end of 2021. The best performing approach is highlighted by bold face. For LRNNs, the network size $N$ after size reduction is shown.
tions. The training procedure only uses standard matrix operations and is thus quite fast. In contrast to ESNs, also, no washout period is required. Any function can be approximated directly from its first step, with an arbitrary start vector (cf. Property 4). Experiments with reasonably large example and network sizes can be performed successfully within seconds on standard hardware. However, if thousands of reservoir neurons are employed, the procedure may become numerically unstable, at least our Octave implementation. The likelihood of almost identical eigenvectors and eigenvalues with absolute values greater than 1 in the learned transition matrix $W$ is increased then. Nonetheless, the underlying major problem here seems to be that existing scientific programming libraries do not calculate the eigenvalues of large matrices accurately enough. This point needs further investigation.

A particularly interesting application of our approach reducing the network size is in hardware implementations of neural networks, e.g., for neuromorphic or reservoir computing (Mead, 1990; Indiveri et al., 2011; Liao and Li, 2017). Neuromorphic computing refers to new hardware that mimics the functioning of the human brain, and neuromorphic hardware results from the exploration of unconventional physical substrates and nonlinear phenomena. Future work shall include improving predictive and memory capacity of LRNNs, analyzed for small networks by Marzen (2017) and to some extent also by Couillet et al. (2016). Last but not least, other machine learning tasks besides prediction shall be addressed in more detail, including classification and reinforcement learning (Sutton and Barto, 2018; Pong et al., 2017).

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Appendix A. Proof of Property 1

For a function $x$ and its first derivative $\dot{x}$ with respect to time $t$, we have

$$\dot{x}(t) = \lim_{\tau \to 0} \frac{x(t + \tau) - x(t)}{\tau}$$

and hence $x(t + \tau) \approx x(t) + \tau \dot{x}(t)$

for small time steps $\tau > 0$. We can apply this to $x^{(k)}(t)$ for $k \geq 0$ in the difference of Equation 4 between the times $t + \tau$ and $t$ divided by $\tau$ and obtain:

$$0 = \sum_{k=0}^{n} c_k x^{(k)}(t + \tau) - \sum_{k=0}^{n} c_k x^{(k)}(t)$$

for $0 \equiv \frac{\tau}{\tau}$

$$= \sum_{k=0}^{n-1} c_k \frac{x^{(k)}(t + \tau) - x^{(k)}(t)}{\tau} + c_n \frac{x^{(n)}(t + \tau) - x^{(n)}(t)}{\tau}$$

$$\approx \sum_{k=0}^{n-1} c_k x^{(k+1)}(t) + \frac{c_n}{\tau} (x^{(n)}(t + \tau) - x^{(n)}(t))$$

This is equivalent to:

$$x^{(n)}(t + \tau) \approx x^{(n)}(t) - \frac{\tau}{c_n} \sum_{k=0}^{n-1} c_k x^{(k+1)}(t)$$

From this, we can read off the desired transition matrix $W$ of size $(n + 1) \times (n + 1)$ from Equation 5. Together with a start vector $s$ satisfying Equation 4, we can thus solve differential equations approximately by LRNNs.

Appendix B. Proof of Property 4

We first prove the case where the Jordan matrix $J$ only contains ordinary Jordan blocks as in Property 2, i.e., possibly with complex eigenvalues on the diagonal. Since $J$ is a direct sum of Jordan blocks, it suffices to consider the case where $J$ is a single Jordan block because, as the Jordan matrix $J$, the matrices $A$ and also $B$ (see below) can be obtained as direct sums, too.
In the following, we use the column vectors $y = [y_1 \cdots y_N]^T$ with all non-zero entries, $x = [x_1 \cdots x_N]^T$ with $x = V^{-1} \cdot s$ (cf. Property 3), and $b = [b_1 \cdots b_N]^T$. From $b$, we construct the following upper triangular Toeplitz matrix

$$B = \begin{bmatrix} b_N & \cdots & b_2 & b_1 \\ 0 & \ddots & b_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_N \end{bmatrix}$$

which commutes with the Jordan block $J$ (Horn and Johnson, 2013, Sect. 3.2.4), i.e., it holds that (a) $J \cdot B = B \cdot J$. We define $B$ and hence $b$ by the equation (b) $x = B \cdot y$ which is equivalent to:

$$\begin{bmatrix} y_N & \cdots & y_2 & y_1 \\ 0 & \ddots & y_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & y_N \end{bmatrix} \cdot b = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

Since the main diagonal of the left matrix contains no 0s because $y_N \neq 0$ by precondition, there always exists a solution for $b$ (Horn and Johnson, 2013, Sect. 0.9.3). Then $A = V \cdot B$ does the job:

$$f(t) = W' \cdot s \overset{\text{Property 2}}{=} V \cdot J' \cdot V^{-1} \cdot s = V \cdot J' \cdot x \overset{(b)}{=}} V \cdot J' \cdot B \cdot y \overset{(a)}{=} V \cdot B \cdot J' \cdot y = A \cdot J' \cdot y$$

The generalization to the real Jordan decomposition is straightforward by applying the fact that for complex conjugate eigenvalue pairs $\lambda$ and $\bar{\lambda}$ the matrix $M$ in a real Jordan block (cf. Section 3.3) is similar to the diagonal matrix $D = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$ via $U = \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix}$ (Horn and Johnson, 2013, Sect. 3.4.1), i.e., $M = U \cdot D \cdot U^{-1}$. The above-mentioned commutation property (a) analogously holds for real Jordan blocks. This completes the proof.

**Appendix C. Proof of Property 5**

Let $f_k(t)$ denote the value of the $k$-th dimension of $f(t)$, $\lambda$ be the eigenvalue of $W$ with maximal absolute value and $m$ be the maximal (geometric) multiplicity of
the eigenvalues of the transition matrix $W$. Then, from Property 2, we can easily deduce

$$|f_k(t)| = O(t^n |\lambda|^t)$$

as asymptotic behavior for large $t$.

**Appendix D. Proof of Property 8**

First, we take the series of function values $f(t_0), \ldots, f(t_n)$ and identify them with the time series $S(0), \ldots, S(n)$. After applying the LRNN learning procedure, the LRNN runs through all given values, because by construction the upper part $[S(0) \cdots S(n-1)]$ of the matrix $X$ (cf. Equation 10) and the matrix $Y^{out}$ (cf. Equation 11) consist of the series of function values of $f$, provided that the linear matrix equation $Y^{out} = W^{out} \cdot X$ (Equation 12) has at least one solution.

Since Equation 12 is equivalent to simultaneously solving the equations $y_k = w_k \cdot X$, for $1 \leq k \leq N$ where $y_1, \ldots, y_N$ and $w_1, \ldots, w_N$ denote the row vectors of the matrices $Y^{out}$ and $W^{out}$, respectively, the latter is the case if the rank of the coefficient matrix $X$ is equal to the rank of the augmented matrix $M_k = [X \ y_k]^T$ for every $k$. This leads to the equation $\text{rank}(X) = \min(N^{in\ out} + N^{res}, n) = \text{rank}(M_k) = \min(N^{in\ out} + N^{res} + 1, n)$. From this, it follows that, as desired, $N^{res} \geq n - N^{in\ out}$ reservoir neurons have to be employed to guarantee at least one solution for the $w_k$, provided that the rank of the matrix $X$ is maximal. For the latter, we consider two cases:

- If the rank of the upper part of the matrix $X$ (see above) is not maximal, then this does not cause any problems. We only have to replace $N^{in\ out}$ by the actual rank of the upper part of the matrix $X$ in Equation 14.

- The rank of the lower part $[R(0) \cdots R(n-1)]$ of the matrix $X$ almost always has maximal rank, because we employ a random reservoir (cf. Definition 3). Thus, a suitable reservoir does the job, which completes the proof.