Remarks on surfaces with $c_1^2 = 2\chi - 1$ having non-trivial 2-torsion

Masaaki MURAKAMI

On the occasion of 60-th birthday of Prof. Fabrizio Catanese

Abstract

We shall show that any complex minimal surface of general type with $c_1^2 = 2\chi - 1$ having non-trivial 2-torsion divisors, where $c_1^2$ and $\chi$ are the first Chern number of a surface and the Euler characteristic of the structure sheaf respectively, has the Euler characteristic $\chi$ not exceeding 4. Moreover, we shall give a complete description for the surfaces of the case $\chi = 4$, and prove that the coarse moduli space for surfaces of this case is a unirational variety of dimension 29. Using the description, we shall also prove that our surfaces of the case $\chi = 4$ have non-birational bicanonical maps and no pencil of curves of genus 2, hence being of so-called non-standard case for the non-birationality of the bicanonical maps.

1 Introduction

In classification of regular surfaces of general type, the torsion parts of the Picard groups (the torsion groups for short) sometimes play an important role. One of the reasons for this lies in variety of topological types under single values of numerical invariants, which is common especially in cases of small geometric genus; the torsion group of a regular surface, isomorphic to the first homology group with integral coefficients, carries information that the numerical invariants $c_1^2$ and $\chi$ do not.

Studies on surfaces of general type done using the torsion groups are well-known for cases of vanishing geometric genus (see, e.g., Barth-Peters-Van de Ven [3, p. 237]). In those studies, they tried to determine the structures of surfaces with given isomorphism classes of the torsion groups. There are,
however, some other cases of numerical invariants for which similar studies have been successfully developed. Consider the case $c_1^2 = 2\chi - 2$. In this case, by Ciliberto-Mendes Lopes [7], the orders of the torsion groups do not exceed 2, and the Euler characteristics $\chi$’s for the cases of non-trivial torsion do not exceed 5. Complete descriptions for the surfaces with non-trivial torsion with $\chi = 2, 3, 4,$ and 5 are given in Catanese-Debarre [5], Ciliberto-Mendes Lopes [7], Bartalesi-Catanese [2], and Ciliberto-Mendes Lopes [7] respectively. We remark that even in cases of vanishing geometric genus, complete descriptions are known only for a small number of classes.

In the present paper, we study minimal surfaces with $c_1^2 = 2\chi - 1$ having non-trivial 2-torsion divisors. Note that if $X$ is a minimal surface with $c_1^2 = 2\chi - 1$, then $X$ has vanishing irregularity, hence geometric genus $p_g = \chi - 1$. We shall prove the bound $\chi \leq 4$ for the Euler characteristics $\chi$’s (Theorem 1), describe the surfaces of the case $\chi = 4$ (Theorem 2, Remark 2), and study the moduli space for surfaces of this case (Theorem 3). By the main theorem of [17], the order of the torsion group of a minimal surface with $c_1^2 = 2\chi - 1$ is at most 3 if $\chi = 2$, and at most 2 if $\chi \geq 3$. Thus for our surfaces with $\chi \geq 2$, two conditions $\mathbb{Z}/2 \subset \text{Tors}$ and $\text{Tors} \cong \mathbb{Z}/2$ are equivalent, where $\text{Tors}$ denotes the torsion group. The case $\chi = 1$ on this line is that of the numerical Godeaux surfaces (i.e., minimal surfaces of general type with $c_1^2 = 1$ and $p_g = 0$).

Surfaces with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\text{Tors} \cong \mathbb{Z}/2$ are known to exist and can be found in [8]. In [8], Ciliberto and Mendes Lopes completely classified regular surfaces with $p_g = 3$ having non-birational bicanonical maps and without genus 2 pencils, i.e., regular surfaces with $p_g = 3$ and of non-standard case for the non-birationality of the bicanonical maps. Among their results, they showed that any regular surface of non-standard case with $c_1^2 = 7$ and $p_g = 3$ is obtained by performing a certain operation on what is known as Du Val’s ancestor with $c_1^2 = 8$ and $p_g = 4$. Since these surfaces have non-trivial 2-torsion divisors, as has been shown in [8], these are examples of our surfaces for the case $\chi = 4$. In fact, our structure theorem for surfaces with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\text{Tors} \cong \mathbb{Z}/2$ shows that although we start from the different assumption, the resulting surfaces are exactly those seen in the paper [8].

Our complete description for the surfaces with $\chi = 4$ asserts that any such surface $X$ is obtained roughly as a free quotient by $\mathbb{Z}/2$ of a double cover of the Hirzebruch surface $\Sigma_d = \mathbb{P}(\mathcal{O}_\mathbb{P}^1 \oplus \mathcal{O}_\mathbb{P}^1(d))$ ($d = 0$ or 2). We shall describe the branch divisor of the double cover, and determine the free action by $\mathbb{Z}/2$ (Theorem 2, Remark 4). The branch divisor of the double cover turns out to be a member of the quadruple anticanonical system having exactly two $[3, 3]$-points. The action by $\mathbb{Z}/2$ turns out to be a lifting of that on the
Hirzebruch surface $\Sigma_d$.

This description induces another description of our surfaces of the case $\chi = 4$ (Proposition 14), which is almost the same as a description appearing in Ciliberto-Mendes Lopes [8]. Using our descriptions, we shall show that our surfaces of the case $\chi = 4$ has non-birational bicanonical maps and no pencil of curves of genus 2 (Proposition 15), hence completely coinciding with those seen in [8] (see also Remark 6).

The coincidence of the resulting surfaces certainly implies possibility of another proof of our complete description, i.e., of a proof, like one for the case $c_1^2 = 2\chi - 2$ in Ciliberto-Mendes Lopes [7], by showing that our surfaces with $\chi = 4$ are of non-standard case for the non-birationality of the bicanonical maps. We however do not chose this way. We remark that our method has an advantage in the sense that we can show the irreducibility of the moduli space in a very explicit and elementary way.

The present paper is organized as follows. In order to show our main theorem, we follow Miyaoka [14] and Reid [19], and take the unramified double cover $Y \rightarrow X$ corresponding to a torsion divisor. We study its canonical map $\Phi_{K_Y}$ using the action by the Galois group of $Y$ over $X$. In Section 2 we state our main results and show, on the assumption $\chi \geq 4$, that we have $\deg \Phi_{K_Y} = 1$ or 2, and that $\deg \Phi_{K_Y} = 1$ implies $\chi = 4$. Note here that to obtain our main theorem, we only need to study the case $\chi \geq 4$. In Section 3 we study the case $\deg \Phi_{K_Y} = 2$. We divide this case into three according to the degree of the canonical image $Z = \Phi_{K_Y}(Y) \subset \mathbb{P}^n$: the case $\deg Z = n + 1$, the case $\deg Z = n$, and the case $\deg Z = n - 1$. We shall classify non-degenerate surfaces in $\mathbb{P}^n$ of degree $n + 1$ of which minimal desingularizations have vanishing irregularities (Proposition 3), and use this classification to study the case $\deg Z = n + 1$. In Section 4 we study the case $\deg \Phi_{K_Y} = 1$ and $\chi = 4$, and then prove Theorems 1 and 2. In the case $\deg \Phi_{K_Y} = 1$ and $\chi = 4$, the surface $Y$ has the first Chern number 14, geometric genus 7, and irregularity 0. Hence the surface $Y$ in this case is a canonical surface whose invariant lies on the Castelnuovo line. We use results given in Ashikaga-Konno [1] to exclude this case. Finally in Section 5 we study the coarse moduli space for the surfaces of the case $\chi = 4$, and prove Theorem 3. To prove the unirationality of the moduli space and the uniqueness of the deformation type, we describe our surfaces of the case $\chi = 4$ as double planes, which is almost the same as the description in Ciliberto-Mendes Lopes [8] for the surfaces of the non-standard case (see also Ciliberto-Francia-Mendes Lopes [6]). Using the two descriptions of our surfaces, we show that our surfaces of the case $\chi = 4$ in fact are of the non-standard case for the non-birationality of bicanonical maps.
Acknowledgment

The author performed the first half of the computations included in this article during his stay at Max-Planck-Institut für Mathematik in Bonn from October 2004 until March 2005. He expresses his deepest gratitude to the institute for the comfortable environment and the financial support which he received during his stay. During the final preparation of the manuscript, the author was supported by the research grant fellowship by the University of Padua, and then later by DFG Forschergruppe 790 “Classification of algebraic surfaces and compact complex manifolds”. The author is thankful to the referee, who suggested him the proof of Proposition 13 shorter than the proof in the earlier version. The author is thankful also to Prof. Kazuhiro Konno for letting him know the paper [14], to Prof. Margarida Mendes Lopes for her kind comments on the first manuscript, and to Prof. Fabrizio Catanese and Prof. Ingrid Bauer for giving him comfortable environment for the study at Bayreuth, where he performed the final revision of the manuscript.

Notation and Terminology

Let $S$ be a compact complex manifold of dimension 2. We denote by $c_1(S)$, $p_g(S)$, and $q(S)$ the first Chern class, the geometric genus, and the irregularity of $S$ respectively. The torsion group of $S$, denoted by $\text{Tors}(S)$, is the torsion part of the Picard group of $S$. If $V$ is a complex manifold, $K_V$ is a canonical divisor of $V$. For a coherent sheaf $F$ on $V$, we denote by $H^i(F)$, $h^i(F)$, and $\chi(F)$ the $i$-th cohomology group, its dimension $\dim \mathbb{C} H^i(F)$, and the Euler characteristic $\sum (-1)^i h^i(F)$ respectively. Let $f : V \to W$ be a morphism to a complex manifold $W$, and $D$, a divisor on $W$. Then $f^*(D)$ and $f^{-1}_*(D)$ denote the total transform and the strict transform respectively of $D$. The symbol $\sim$ means the linear equivalence of divisors. We denote by $\Sigma_d \to \mathbb{P}^1$ the Hirzebruch surface of degree $d$. The divisors $\Delta_0$ and $\Gamma$ are its minimal section and its fiber respectively. Let $C$ be a curve on $S$. We denote by $\text{mult}_x C$ the multiplicity of $C$ at a point $x \in S$. Let $x$ be a triple point of a reduced curve $C$ on $S$, and $S' \to S$, the blowing-up at $x$. Assume that the strict transform $C'$ of $C$ has an infinitely near triple point $x'$. Then the point $x$ is called a $[3,3]$-point of $C$, if the strict transform $C''$ of $C$ to $S''$, where $S'' \to S'$ is the blowing-up at $x'$, has at most negligible singularities on the exceptional locus of $S'' \to S$.

2 Statement of the main theorem

In [17], we obtained a bound for the orders of the torsion groups of minimal surfaces with $c_1^2 = 2\chi - 1$ and $\chi \geq 2$. In the present paper, we study the case of 2-torsion divisors, and sharpen the bound. Our goals are a bound for
the Euler characteristic \( \chi \), a complete description for the surfaces of the case of maximal \( \chi \), and the unirationality of the moduli space for surfaces of this case. The following three are the main theorems:

**Theorem 1.** Let \( X \) be a minimal surface of general type with \( c_1^2 = 2\chi - 1 \) and torsion group \( \text{Tors}(X) \cong \mathbb{Z}/2 \). Then the Euler characteristic \( \chi \) of the structure sheaf does not exceed 4.

**Theorem 2.** Let \( X \) be a minimal surface with \( c_1^2 = 2\chi - 1, \chi = 4 \), and torsion group \( \text{Tors}(X) \cong \mathbb{Z}/2 \). Then the unramified double cover \( Y \) of \( X \) admits a generically two-to-one morphism \( f \) onto the Hirzebruch surface \( \Sigma_d \) of degree \( d = 0 \) or 2 satisfying the following conditions:

i) the action by the Galois group \( G = \text{Gal}(Y/X) \cong \mathbb{Z}/2 \) of \( Y \) over \( X \) induces one on \( \Sigma_d \), of which fixed locus is a set of four points on \( \Sigma_d \);

ii) the branch divisor \( B \) of \( f \) is a member of the linear system \( |-4K_{\Sigma_d}| \) passing no fixed point of the action by \( G \);

iii) the branch divisor \( B \in |-4K_{\Sigma_d}| \) has exactly two \([3, 3]\)-points, and all other singularities, if any, are negligible ones.

**Theorem 3.** Any two minimal surfaces with \( c_1^2 = 2\chi - 1, \chi = 4 \), and \( \text{Tors} \cong \mathbb{Z}/2 \) are equivalent under deformation of complex structures. The coarse moduli space for minimal surfaces with these invariants is a unirational variety of dimension 29.

Theorem 1 sharpens the bound given in [17] into the following:

**Theorem 4.** Let \( X \) be a minimal algebraic surface with \( c_1^2 = 2\chi - 1 \). Then the following hold:

i) if \( \chi = 2 \), then \( \sharp\text{Tors}(X) \leq 3 \);

ii) if \( \chi \geq 3 \), then \( \sharp\text{Tors}(X) \leq 2 \);

iii) if \( \chi \geq 5 \), then \( \sharp\text{Tors}(X) = 1 \).

**Remark 1.** In Theorem 2 we can describe the action by \( G \) on \( \Sigma_d \) more concretely: if an involution of the Hirzebruch surface \( \Sigma_d \) has exactly four fixed points \((d: \text{even})\), then there exists an open cover \( \{U_i\}_{i=0,1} \) of \( \Sigma_d \) satisfying \( U_i = \{(u_i, (t_i : 1))\} = \mathbb{C} \times \mathbb{P}^1, u_0 = 1/u_1, \) and \( t_0 = u_1^2t_1 \), such that this involution is given by \((u_0, t_0) \mapsto (-u_0, -t_0)\). (1)

**Remark 2.** Theorem 2 asserts that any minimal surface \( X \) with \( c_1^2 = 2\chi - 1, \chi = 4 \), and \( \text{Tors}(X) \cong \mathbb{Z}/2 \) is obtained by the following procedure: 1) set \( d = 0 \) or 2; the involution \((1)\) defines an action by \( G = \mathbb{Z}/2 \) on the Hirzebruch surface \( \Sigma_d \); 2) take a reduced member \( B \in |-4K_{\Sigma_d}| \) stable under this action
that satisfies the conditions ii) and iii) in Theorem 2; 3) take the double cover of \( \Sigma_d \) branched along \( B \), and denote by \( Y \) its minimal desingulraization; there exists a unique free lifting to \( Y \) of the action by \( G \) on \( \Sigma_d \); 4) take the quotient of \( Y \) by this free action.

It is not difficult to check that this procedure in fact gives surfaces of the case \( \chi = 4 \) for sufficiently general \( B \).

Remark 3. Let \( \Sigma_d \) be the Hirzebruch surface which appears in Theorem 2. It is obvious from Remarks 1 and 2 that the fibration \( \Sigma_d \to \mathbb{P}^1 \) induces a hyperelliptic fibration \( Y \to \mathbb{P}^1 \) of genus 3 and that the divisor class of a fiber of this fibration is stable under the action by the Galois group \( G = \text{Gal}(Y/X) \). So we obtain a hyperelliptic fibration \( X = Y/G \to \mathbb{P}^1/G \) of genus 3 with two multiple fibers \( 2A_1 \) and \( 2A_2 \) corresponding to the fixed points of the action by \( G \) on \( \mathbb{P}^1 \). As is explained also in [8, p. 85], the difference \( A_1 - A_2 \) gives a non-trivial 2-torsion divisor of our surface \( X \).

In what follows, \( X \) is a minimal surface with \( c_1^2 = 2\lambda - 1, \chi = \lambda \geq 4, \) and \( \text{Tors}(X) \simeq \mathbb{Z}/2 \). We denote by \( \pi : Y \to X \) the unramimfied double cover corresponding to the torsion group \( \text{Tors}(X) \). Note that we have assumed \( \lambda \geq 4 \). The following lemma follows from the unbranched covering trick.

**Lemma 2.1.** \( K_Y^2 = 2(2\lambda - 1), p_g(Y) = 2\lambda - 1, \text{and } q(Y) = 0. \)

In order to show Theorems 1 and 2 we study the canonical map \( \Phi_{K_Y} : Y \to \mathbb{P}^n \) of \( Y \), where \( n = 2\lambda - 2 \). We denote by \( Z = \Phi_{K_Y}(Y) \) the canonical image of the surface \( Y \).

**Proposition 1.** The canonical image \( Z \) is a surface. The equality \( \deg \Phi_{K_Y} = 1 \) or 2 holds. Moreover, if \( \deg \Phi_{K_Y} = 1 \), then \( \lambda = 4 \).

Proof. Since we have assumed \( \lambda \geq 4 \), we have

\[
K_Y^2 - 3p_g(Y) = -(2\lambda - 1) \leq -7.
\]

By this together with \( q(Y) = 0 \) and [11, Theorem 1.1], we see that \( |K_Y| \) is not composite with a pencil. Thus we have

\[
\deg \Phi_{K_Y} \leq \frac{K_Y^2}{\deg Z} \leq \frac{2(n + 1)}{n - 1} = 2 + \frac{4}{n - 1} \leq 2 + \frac{4}{5},
\]

hence \( \deg \Phi_{K_Y} \leq 2 \). The second assertion follows from Castelnuovo’s inequality. \( \square \)

If \( \lambda = 4 \), then the Chern invariant of \( Y \) is on the Castelnuovo line. Thus we can use results given in [11] to study the case \( \deg \Phi_{K_Y} = 1 \).
3 The case deg $\Phi_{K_Y} = 2$

In this section, we study the case $\deg \Phi_{K_Y} = 2$. We begin with the study of the base locus of the canonical system $|K_Y|$. Let $|M|$ and $F$ be the variable part and the fixed part of the linear system $|K_Y|$. We take the shortest composite $p : \tilde{Y} \to Y$ of quadric transformations such that the variable part $|L|$ of $p^*|M|$ is free from base points, and denote by $E$ the fixed part of $p^*|M|$. Then we have $p^*|K_Y| = |L| + E + p^*F$ and

$$K_Y^2 = L^2 + LE + MF + K_YF,$$

(2)

where each term of the right hand is a non-negative integer. Note that the eigenvectors of the natural action by $G = \text{Gal}(Y/X)$ span the space of global section $H^0(\mathcal{O}_Y(K_Y))$. This implies that the linear systems $|K_Y|$, $|M|$, and $F$ are spanned by the pull-backs of divisors on $X$. Hence, for example, we have $MF \equiv 0 \mod 2$, since $\pi : Y \to X$ is of mapping degree 2. In the same way, we obtain

$$L^2 \equiv LE = -E^2 \equiv MF \equiv K_YF \equiv 0 \mod 2$$

(3)

(for the detail, see [16, Section 3]).

**Proposition 2.** Let $M$, $F$, $L$, and $E$ be divisors as above. Then one of the following holds:

1) $|K_Y| = |L|$; the canonical system $|K_Y|$ is free from base points;

2) $L^2 = K_Y^2 - 2$, $F = 0$, and $LE = 2$;

3-1) $L^2 = K_Y^2 - 4$, $F = 0$, and $LE = 4$;

3-2) $L^2 = K_Y^2 - 4$, $|L| = |M|$, $K_YF = 0$, and $F^2 = -4$.

**Proof.** First, note that we have $L^2 = K_Y^2$, $K_Y^2 - 2$, or $K_Y^2 - 4$. This follows from [3] and [9, Lemma 2]. Second, note that

$$MF \equiv 0 \mod 4.$$  

(4)

This follows from the Riemann–Roch theorem, since we have $MF = M(M + K_Y) - 2M^2 = M(M + \pi^*K_X) - 2M^2$, $\deg \pi = 2$, and $M \sim \pi^*M'$ for a certain divisor $M'$ on $X$. Then the assertion follows from (2), (4), (3), and Hodge’s index theorem.

In case 3-1), the number of the base points of $|M|$ cannot be 1, since the action by $G$ on $Y$ has no fixed point. Thus in this case, the morphism $p : \tilde{Y} \to Y$ is a composite of four quadric transformations. In the same way, we see that, in case 2), the morphism $p : \tilde{Y} \to Y$ is a blowing-up of $Y$ at two distinct points. In case 3-2), the divisor $F$ is a sum of two fundamental cycles of rational double points.
We denote by $\Phi_L : \tilde{Y} \to Z \subset \mathbb{P}^n$ the morphism associated with the linear system $|L|$. The action by $G$ on $Y$ induces one on $\tilde{Y}$. We study the morphism $\Phi_L$ using this action.

### 3.1 The case $|K_Y| = |L|$ 

Let us first exclude case 1) in Proposition [2]. In what follows, we assume $|K_Y| = |L|$. Thus we have $\deg Z = n + 1$. We shall prove the following proposition in Appendix.

**Proposition 3.** Let $n \geq 4$ be an integer, $Z$, a non-degenerate surface in $\mathbb{P}^n$ of degree $n + 1$, and $Z' \to Z$, its minimal desingularization. Assume that the morphism $Z' \to Z$ is given by a complete linear system $|D'|$ and that $q(Z') = 0$ holds. Then $n$ does not exceed 11. Further, there exist an integer $0 \leq d \leq 3$ and a blowing-up $r : Z' \to \Sigma_d$ at (possibly infinitely near) $11 - n$ points such that the equivalence $D' \sim -K_{Z'} + r^*\Gamma$ holds. Here, the divisor $\Gamma$ is a fiber of the Hirzebruch surface $\Sigma_d \to \mathbb{P}^1$.

In our case, we have $n = 2\lambda - 2$, $\lambda \geq 4$, and $q(Y) = 0$. Moreover $Z$ is the canonical image of $Y$. Thus our surface $Z = \Phi_{K_Y}(Y)$ satisfies all the conditions in the proposition above. It follows that there exist an integer $0 \leq d \leq 3$ and a blowing-up $r : Z' \to \Sigma_d$ at $11 - n$ points such that the morphism $\Phi_{D'} : Z' \to Z$, where $\Phi_{D'}$ is a morphism corresponding to the complete linear system $|D'| = |-K_{Z'} + r^*\Gamma|$, gives the minimal desingularization of $Z$.

**Proposition 4.** The canonical map $\Phi_{K_Y} : Y \to Z$ lifts to a morphism $f' : Y' \to Z'$. The branch divisor $B'$ of $f'$ is a member of the linear system $|2(2D' - r^*\Gamma)|$ having at most negligible singularities.

Proof. Let us first show the liftability of the canonical map $\Phi_{K_Y}$. Let $p' : Y' \to Y$ be the shortest composite of quadric transformations such that the morphism $\Phi_{K_Y} \circ p'$ factors through $\Phi_{D'} : Z' \to Z$. We denote by $f' : Y' \to Z'$ the unique morphism satisfying $\Phi_{K_Y} \circ p' = \Phi_{D'} \circ f'$. Then we have $K_{Y'} \sim p'^*K_Y + \eta$ for a certain effective divisor $\eta$ on $Y'$. If $f'_*\eta = 0$, then $p' : Y' \to Y$ is an isomorphism. Thus we only need to show $f'_*\eta = 0$.

So we prove the equality above. Let $R'$ be the ramification divisor of $f'$, and $B' = f'_*R'$, its direct image. Then from $R' \sim K_{Y'} - f'^*K_{Z'} \sim f'^*(2D' - r^*\Gamma) + \eta$, we infer

$$B' \sim 2(2D' - r^*\Gamma + \alpha), \quad (5)$$

where $\alpha$ is a divisor satisfying $2\alpha \sim f'_*\eta$. We denote by $Y'' \to Z'$ the double cover branched along $B'$, and by $Y^2 \to Y''$ its canonical resolution. To show
the equality $f'_* \eta = 0$, we compute the Euler characteristic $\chi(\mathcal{O}_{Y'})$ in two ways and compare them. Note that $\dim(\Phi_{K_Y} \circ p^*)(\eta) = 0$, and that any general member of $|r^*|G|$ is a 0-curve. It follows $D' \alpha = D'f'_* \eta/2 = 0$ and $D'(r^* \Gamma) = -K_{Z'}(r^* \Gamma) = 2$. Thus by [5] and [9, Lemma 6], we obtain

$$\chi(\mathcal{O}_{Y'}) = 2 + \frac{1}{2}(2D' - r^* \Gamma + \alpha)((2D' - r^* \Gamma + \alpha) + K_{Z'}) - \beta$$

$$= 2 + \frac{1}{2}(2D' - r^* \Gamma + \alpha)(D' + \alpha) - \beta$$

$$= D'^2 + 1 - \frac{1}{4}(r^* \Gamma)(f'_* \eta) + \frac{1}{8}(f'_* \eta)^2 - \beta,$$

(6)

where $\beta$ is a term coming from essential singularities of the branch divisor $B'$. Here, we have three inequalities

$$-\frac{1}{4}(r^* \Gamma)(f'_* \eta) \leq 0, \quad \frac{1}{8}(f'_* \eta)^2 \leq 0, \quad \text{and} \quad -\beta \leq 0.$$

(7)

The first one follows from the absence of base points of $|r^*|G|$, the second one from $D'^2 > 0$ and $D'f'_* \eta = 0$, and the last one from the definition of $\beta$. Meanwhile we have $\chi(\mathcal{O}_{Y'}) = \chi(\mathcal{O}_Y) = n + 2 = D'^2 + 1$. Thus by (6) and (7), we obtain $(f'_* \eta)^2 = 0$, from which together with Hodge’s index theorem, we infer $f'_* \eta = 0$. Hence the canonical map $\Phi_{K_Y}$ lifts.

The remaining assertion easily follows from the proof above. \qed

Note that the action by $G = \text{Gal}(Y/X)$ on $Y$ induces one on $Z'$. We can verify it as follows. Since $Z$ is the canonical image of our surface $Y$, the action on $Y$ induces one on $Z$. Meanwhile the surface $Z'$ is the minimal desingularization of our surface $Z$. Thus this action on $Z$ induces one on $Z'$.

**Lemma 3.1.** The induced action by $G$ on $Z'$ is non-trivial. The fixed locus of this action has a one-dimensional irreducible component $C'_0$ satisfying $C'_0^2 \equiv 1 \mod 2$.

Proof. The first assertion is trivial, since the action on $Y$ has no fixed point. Let us show the second assertion. Let $\{z_1, \ldots, z_b\}$ be the set of isolated fixed points of the action on $Z'$, and $r'' : Z'' \to Z'$ the blowing-up at these $b$ points. We denote by $C''_i$ the $(−1)$-curve lying over $z_i$. Let $\{C'_1, \ldots, C'_a\}$ be the set of 1-dimensional irreducible components of the fixed locus of the action on $Z'$. We use the same symbol $C'_i$ for the total transform to $Z''$ of the divisor $C'_i$. Note that the divisor $\sum_{i=1}^a C'_i + \sum_{i=1}^b C''_i$ has no singularity, since we have $G \simeq Z/2$. It follows that the quotient $Z''/G$ is smooth, where the action by $G$ is the lifting of that on $Z'$. We denote by $C'_i$ and $C''_i$ the image to $Z''/G$ of the divisor $C'_i$ and that of the divisor $C''_i$, respectively. Then since
Thus we infer the assertion. This implies that the action by $G$ on the set $Z$ is linearly equivalent to twice a divisor on $Z''/G$, we have
\[ \sum_{i=1}^{a} C_i'' + \sum_{i=1}^{b} C_i'' = (\sum_{i=1}^{a} \bar{C}_i'' + \sum_{i=1}^{b} \bar{C}_i'')^2 / 2 \equiv 0 \mod 2. \]

Meanwhile, since $K_{Z''}$ is linearly equivalent to a pull-back of a divisor on $Z''/G$, we have $K_{Z''}^2 = K_{Z''} - b = n - 3 - b \equiv 0 \mod 2$, hence $b \equiv 1 \mod 2$. Thus we infer $\sum_{i=1}^{b} C_i'' \equiv 1 \mod 2$, which implies the second assertion.

**Lemma 3.2.** Let $C_0'$ be an irreducible curve as in Lemma 3.1. Then $B'C_0' \neq 0$ holds.

Proof. We derive a contradiction by assuming $B'C_0' = 0$. Assume that $B'C_0' = 0$ holds. Then by Proposition 4, we have
\[(2D' - r^*\Gamma)C_0' = (-2K_{Z'} + r^*\Gamma)C_0' = 0.\]

If $(r^*\Gamma)C_0' = 0$, then by the equality above, we obtain $K_{Z'}C_0' = 0$, which contradicts $C_0'^2 \equiv 1 \mod 2$. Thus we have $(r^*\Gamma)C_0' > 0$, hence $-2K_{Z'}C_0' = -(r^*\Gamma)C_0' < 0$. It follows $C_0'$ is a fixed component of the anti-canonical system $| - K_{Z'} |$. Then since $-K_{\Sigma_d} \sim 2\Delta_0 + (2 + d)\Gamma$, we obtain $(r^*\Gamma)C_0' \leq 2$, hence $(r^*\Gamma)C_0' = 2K_{Z'}C_0' = 2$. Thus $r_*C_0' \sim 2\Delta_0 + c\Gamma$ holds for a certain integer $c \geq 1$. Meanwhile since $0 \leq d \leq 3$, we have $h^0(O_Z(-K_{Z'})) \geq h^0(O_{\Sigma_d}(-K_{\Sigma_d})) - (11 - n) = n - 2$. Thus we obtain
\[ n - 2 \leq h^0(O_Z(-K_{Z'})) = h^0(O_Z(-K_{Z'}, - C_0')) \leq h^0(O_{\Sigma_d}(-K_{\Sigma_d} - r_*C_0')) = 3 + d - c, \]

hence $c - d \leq 5 - n < 0$. It follows $(r_*C_0')\Delta_0 = (c - d) - d < 0$, which contradicts the irreducibility of $C_0'$. Hence $B'C_0' \neq 0$ holds.

Now let us exclude case 1) in Proposition 2.

**Proposition 5.** Case 1) in Proposition 2 does not occur.

Take an irreducible curve $C_0'$ as in Lemma 3.1. Then by Lemma 3.2, we have $B' \cap C_0' \neq \emptyset$. So let us take a point $x \in B' \cap C_0'$. Then the preimage $f^{-1}(x) \subset Y$ is stable under the action by $G$ on $Y$. By Proposition 4, however, the set $f^{-1}(x)$ is either a point or a base space of the fundamental cycle of a rational double point. This implies that the action by $G$ on $f^{-1}(x)$ has a fixed point, which contradicts the definition of $\pi : Y \to X$. Thus we have the assertion.
3.2 The case $L^2 = K_Y^2 - 4$

Next we exclude cases 3-1) and 3-2) in Proposition 2. In these two cases, we have $L^2 = 2(n-1)$; hence the canonical image $Z$ is a non-degenerate surface in $\mathbb{P}^n$ of minimal degree $n - 1$. Thus from the well-known classification, it follows that our $Z$ is a image of the Hirzebruch surface $Z' = \Sigma_d$ by the morphism associated with the complete linear system $|D'| = |\Delta_0 + \frac{n-1+d}{2}E|$, where $0 \leq d \leq n - 1$ and $d \equiv n - 1 \mod 2$ (see [18] or [10, Lemma 1.2]). Let us denote this morphism by $\Phi_D : Z' \to Z \subset \mathbb{P}^n$. Then $\Phi_D$ is an embedding if $d < n - 1$, and is the contraction of $\Delta_0$ if $d = n - 1$. Note that in the later case, our $Z$ is a cone over a rational curve embedded in $\mathbb{P}^{n-1}$ by $\mathcal{O}_{\mathbb{P}^1}(n - 1)$.

For the case $d < n - 1$, the lemma below is trivial. For the case $d = n - 1$, we can give a proof by the same method as in [10, Lemma 1.5].

Lemma 3.3. The morphism $\Phi_L : \tilde{Y} \to Z$ lifts to a morphism $f' : \tilde{Y} \to Z'$.

By the same argument as in the exclusion of case 1), we see that the action by $G$ on $Y$ induces one on $Z'$.

Let us recall the morphism $p : \tilde{Y} \to Y$ and the base locus of $|K_Y|$. In case 3-1) in Proposition 2 the morphism $p$ is the blowing-up at (possibly infinitely near) four points, which we shall call $y_1, \ldots, y_4$. Let $E_i$ denote the total transform to $\tilde{Y}$ of the $(-1)$-curve corresponding to $y_i$. Then we have $E = \sum_{i=1}^4 E_i$ and $LE_i = 1$ ($1 \leq i \leq 4$). Since the action by $G$ on the set of base points of $|M|$ has no fixed point, we have only two cases: i) the case where $y_1, \ldots, y_4$ are four distinct points on $\tilde{Y}$, and ii) the case where $y_1$ and $y_2$ are distinct points on $\tilde{Y}$, and $y_{i+2}$ is infinitely near to $y_i$ for $i = 1, 2$. In the later case, the divisor $E'_i = E_i - E_{i+2}$ is a $(-2)$-curve satisfying $LE'_i = 0$.

Meanwhile in case 3-2), the morphism $p : \tilde{Y} \to Y$ is an isomorphism. Hence we may assume $\tilde{Y} = Y$. We have $|M| = |L|$ and $F = \sum_{i=1,2} F_i$, where $F_i$ is a fundamental cycle of a rational double point. Since the action on $Y$ has no fixed point, we have $F_1 \cap F_2 = \emptyset$; hence the generator of $G$ maps $F_1$ onto $F_2$. It follows $LF_1 = LF_2 = 2$.

In what follows, we put $T = 2E$ for case 3-1), and $T = F$ for case 3-2). Then we have

$$K_{\tilde{Y}} \sim L + T.$$

Lemma 3.4. Let $T$ be the divisor above. Then $\Gamma(f^*_T) \equiv 2 \mod 4$ holds.

Proof. Since $d \equiv n - 1 \equiv 1 \mod 2$, we have $d \neq 0$. Thus the action by $G$ on $Z' = \Sigma_d$ induces one on $\mathbb{P}^1$ via the natural fibration $\Sigma_d \to \mathbb{P}^1$ of the Hirzebruch surface. It follows there exists a member $\Gamma_0 \in |\Gamma|$ stable under the action by $G$. Let us take a blowing-up $\tilde{X} \to X$ such that $\tilde{Y} = \tilde{X} \times_X Y$ holds. The base change $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ is an unramified double cover satisfying
Gal(\(\hat{Y}/\hat{X}\)) \simeq \text{Gal}(Y/X)$. Then since \(f''G_0\) is stable under the action by \(G\) on \(\hat{Y}\), the divisor \(f''G_0\) is a pull-back by \(\hat{\pi}\) of a certain divisor on \(\hat{X}\). Thus from \(\hat{\pi}^*K_{\hat{X}} \simeq K_{\hat{Y}}\) and the Riemann–Roch theorem, we infer

\[
(f''G_0)^2 + (f''G_0)K_{\hat{Y}} = (f''G_0)(L + T) = 2 + (f''G_0)T \equiv 0 \mod 4.
\]

Hence we have the assertion.

**Lemma 3.5.** The morphism \(f' : \hat{Y} \to Z'\) contracts no \((-1)\)-curve on \(\hat{Y}\). Further, the following hold:

i) if \(C\) is a \((-1)\)-curve on \(\hat{Y}\) satisfying \(\text{LC} = 1\), then \(f'_*C \sim \Gamma\);

ii) if \(C\) is a \((-2)\)-curve on \(\hat{Y}\) satisfying \(\text{LC} = 0\), then \(f'\) contracts \(C\).

Proof. The first assertion trivially follows from the definition of \(p : \hat{Y} \to Y\). In order to prove i) and ii), we put \(f'_*C \sim a\Delta_0 + b\Gamma\). We denote by \(\theta\) the involution of \(\hat{Y}\) over \(Z'\). This involution exists, since \(f'\) contracts no \((-1)\)-curve.

First, let us prove the assertion i). Assume that \(C\) is a \((-1)\)-curve on \(\hat{Y}\) satisfying \(\text{LC} = 1\). Then since \(L \sim f''D'\), we have

\[(\Delta_0 + d\Gamma)f'_*C + \frac{n-1-d}{2}\Gamma f'_*C = 1,\] (8)

where each term of the left hand is a non-negative integer. Thus we obtain 
\((\Delta_0 + d\Gamma)f'_*C = 0\) or \(1\). Assume that \((\Delta_0 + d\Gamma)f'_*C = 1\). Then we have 
\(f'_*C \sim a\Delta_0 + \Gamma\) and \((\frac{n-1-d}{2})a = 0\). Thus, in this case, we only have to show \(a = 0\), which is trivial if \(n - 1 - d \neq 0\). If \(n - 1 - d = 0\), then by the irreducibility of \(C\), we have \(\Delta_0f'_*C = 1-a(n-1) \geq 0\), hence \(a = 0\). Assume next that \((\Delta_0 + d\Gamma)f'_*C = 0\). Then by (8), we obtain \(f'_*C = \Delta_0\) and 
\(d = n - 3\). We exclude this case as follows. We have \(f''\Delta_0 = C + \theta(C) + \xi\) for a certain effective divisor \(\xi\) exceptional with respect to \(f'\). It follows

\[
(f''\Delta_0)^2 = (C + \theta(C) + \xi)(C + \theta(C)) \geq C^2 + \theta(C)^2 + 2C\theta(C) \geq -4,
\]

hence \(-2(n-3) \geq -4\). This contradicts \(\lambda \geq 4\). Thus we have \((\Delta_0 + d\Gamma)f'_*C \neq 0\), which completes the proof of the assertion i).

Next, let us prove the assertion ii). Assume that \(f'(C)\) is a curve. Then since \(\Phi'_D\) contracts \(f'(C)\), we have \(d = n - 1\) and \(f'(C) = \Delta_0\). Note that we have \(f'_*C = \Delta_0\) or \(2\Delta_0\), since \(\deg f' = 2\). Assume that \(f'_*C = \Delta_0\). Then we have \(f''\Delta_0 = C + \theta(C) + \xi\) for a certain effective divisor \(\xi\) exceptional with respect to \(f'\). Then by the same method as in the proof of i), we obtain

\[-2(n - 1) = (f''\Delta_0)^2 \geq -8\]

which contradicts \(\lambda \geq 4\). Assume next that \(f'_*C = 2\Delta_0\). Then we have \(f''\Delta_0 = C + \xi\) for a certain effective divisor \(\xi\)
exceptional with respect to \( f' \). Then again by the same method, we obtain 
\(-2(n-1) \geq -2\), which contradicts \( \lambda \geq 4 \). Thus we have the assertion ii).

If our \( Y \) is of case 3-1) in Proposition 2, then by the lemma above we have 
\( f'T = 2f_*E \sim 8\Gamma \), which contradicts Lemma 3.4. Thus we have the following:

**Proposition 6.** Case 3-1) in Proposition 2 does not occur.

So in what follows, we assume that our \( Y \) is of case 3-2) in Proposition 2.

**Lemma 3.6.** Let \( C \) be an irreducible component of \( F_1 \) satisfying 
\( D'f'_*C > 0 \). Then one of the following holds:

i) \( D'f'_*C = 1 \) and \( f'_*C \sim \Gamma \);

ii) \( D'f'_*C = 2 \) and \( f'_*C \sim 2\Gamma \);

iii) \( D'f'_*C = 2 \), \( f'_*C = \Delta_0 \), and \( d = n - 5 = 1 \).

Proof. First, note that if \( f'(C) = \Delta_0 \), then we have \( C \neq \theta(C) \), where \( \theta \) is the involution of \( \tilde{Y} = Y \) over \( Z' \). We can verify this as follows. Let \( \iota \) be the generator of the Galois group \( G \), and \( \iota|_{Z'} \), the corresponding automorphism of \( Z' \). Then since \( d \neq 0 \), we have 
\( f'(\iota(C)) = \iota|_{Z'}(f'(C)) = \Delta_0 = f'(C) \).
This means \( C \neq \theta(C) = \iota(C) \), since we have \( \iota(C) \subset F_2 \) and \( F_1 \cap F_2 = \emptyset \). Next, note that \( C \) is a \((-2)\) curve satisfying 
\( 0 < D'f'_*C \leq D'f'_*F_1 = 2 \). Then we can prove the assertion by the same method as in the proof of Lemma 3.5.

By \( D'f'_*F_1 = 2 \) together with Lemmas 3.5 and 3.6, we see that either of the following holds:

a) \( f'_*F_1 = f'_*(\iota(F_2)) \sim 2\Gamma \);

b) \( f'_*F_1 = f'_*(\iota(F_2)) = \Delta_0 \), and \( d = n - 5 = 1 \),

where \( \iota \) is the generator of the Galois group of \( G \). Case a) above, however, contradicts the assertion in Lemma 3.4. Thus we have the following:

**Lemma 3.7.** \( f'_*F_1 = f'_*F_2 = \Delta_0 \) and \( d = n - 5 = 1 \).

Now let us study the morphism \( f' : \tilde{Y} = Y \rightarrow Z' = \Sigma_1 \). Let \( R' \) be the ramification divisor of \( f' \), and \( B' \) the branch divisor. Then by the lemma above we obtain

\[
R' \sim f'^*(3\Delta_0 + 6\Gamma) + \sum_{i=1,2} F_i \quad \text{and} \quad B' \sim 2(4\Delta_0 + 6\Gamma). \tag{9}
\]

We take the double cover of \( Z' \) with branch divisor \( B' \), and denote by \( Y^\sharp \) its canonical resolution. Let us recall how to obtain the canonical resolution. Set \( Z'_0 = Z' \) and \( B'_0 = B' \). We define \( Z'_i \) and \( B'_i \) inductively as follows. Choose
a singularity $z_i$, if any, of $B'_{i-1}$, and take the blowing-up $q'_i : Z'_i \to Z'_{i-1}$ at this point. We denote by $\varepsilon_i$ the $(-1)$-curve corresponding to $z_i$. Let $m_i$ be the multiplicity of $B'_{i-1}$ at $z_i$, and $[\frac{m_i}{2}]$, the largest integer not exceeding $\frac{m_i}{2}$. Then we define $B'_i$ by $B'_i = q'_i B'_{i-1} - 2[\frac{m_i}{2}] \varepsilon_i$. For a certain $s \geq 0$, the divisor $B'_s$ is non-singular. So take the double cover $f^\sharp : \tilde{Y}_s \to Z^s = Z'_s$ with branch divisor $B = B'_s$. Then this $\tilde{Y}_s$ is our canonical resolution $Y^\sharp$. Put $q' = (q'_1 \circ q'_2 \circ \cdots \circ q'_s) : Z^s \to Z'$. There exists a natural birational morphism $p^\sharp : Y^\sharp \to \tilde{Y}$ satisfying $q' \circ f^\sharp = f' \circ p^\sharp$. We use the same symbol $\varepsilon_i$ for the total transform to $Z^s$ of the $(-1)$-curve $\varepsilon_i \subset Z'_i$. Note, for our case, the action by the Galois group $G = \text{Gal}(Y/X)$ on $\tilde{Y}$ induces one on $Z^s$ and one on $Y^\sharp$. This action on $Y^\sharp$ is free.

By the same method as in [9, Section 2], we obtain the following:

**Proposition 7.** There exist $i_1$ and $i_2$ ($i_1 < i_2$) satisfying $[\frac{m_{i_1}}{2}] = [\frac{m_{i_2}}{2}] = 2$. For any $i \neq i_1, i_2$, the equality $[\frac{m_i}{2}] = 1$ holds. The morphism $p^\sharp : Y^\sharp \to \tilde{Y} = Y$ is a composite of two quadratic transformations.

Thus the branch divisor $B'$ has an essential singularity. By the proposition above, we obtain

$$K_{Y^\sharp} \sim f'^\ast (2\Delta_0 + 3\Gamma) - \varepsilon_{i_1} - \varepsilon_{i_2}). \quad (10)$$

**Lemma 3.8.** Every essential singularity of $B'$ lies on $\Delta_0$.

Proof. Since $f'$ contracts no $(-1)$-curve, $f'^\ast B' - 2R' = 2\zeta'$ holds for a certain effective divisor $\zeta'$ on $\tilde{Y}$. This $\zeta'$ satisfies

$$2\zeta' \sim 2(f'^\ast (\Delta_0) - \sum_{i=1,2} F_i), \quad (11)$$

since we have [3]. Let $\zeta' = \sum \zeta'_i$ be the decomposition into connected components. Note that $f'$ maps each $\zeta'_i$ to a point on $Z'$. Then, for any $i$ satisfying $f'(\zeta'_i) \notin \Delta_0$, we infer from (11) that $\zeta'_i = \zeta'_j = 0$, hence $\zeta'_i = 0$, which implies the assertion.

**Lemma 3.9.** Let $\eta^\sharp \sim K_{Y^\sharp} - p^\sharp \ast K_{\tilde{Y}}$ be the exceptional divisor corresponding to $p^\sharp : Y^\sharp \to \tilde{Y}$. Then the fixed part of $|K_{Y^\sharp}|$ is given by $\sum_{i=1,2} p^\sharp \ast F_i + \eta^\sharp$. Further, the linear equivalence $\sum_{i=1,2} p^\sharp \ast F_i + \eta^\sharp \sim f'^\ast (q'^\ast \Delta_0 - \varepsilon_{i_1} - \varepsilon_{i_2})$ holds, where $i_1$ and $i_2$ are integers given in Proposition [7].

Proof. The first assertion follows from $|K_{Y^\sharp}| = |K_{\tilde{Y}}| + \eta^\sharp$, since $|L|$ has no base point. The second assertion follows from (10) and $\sum p^\sharp \ast F_i + \eta^\sharp \sim K_{Y^\sharp} - p^\sharp \ast L \sim K_{Y^\sharp} - p^\sharp \ast f'^\ast D'$. □
Lemma 3.10. There exists a member $\Gamma_1 \in |\Gamma|$ contained in the fixed locus of the action by $G$ on $Z' = \Sigma_1$.

Proof. The action by $G$ on $Z' = \Sigma_1$ induces one on $\mathbb{P}^1$ via the natural fibration $Z' = \Sigma_1 \to \mathbb{P}^1$ of the Hirzebruch surface. Let us show that this induced action on $\mathbb{P}^1$ is non-trivial. There exists a member $\Delta_1 \in |\Delta_0 + \Gamma|$ stable under the action by $G$ satisfying $\Delta_1 \cap \Delta_0 = \emptyset$. Assume that the induced action on $\mathbb{P}^1$ is trivial. Then this $\Delta_1$ is contained in the fixed locus of the action by $G$ on $Z'$. From this together with $B'\Delta_1 = 12$ and Lemma 3.8, it follows that $B'$ has a smooth point or a negligible singularity that is stable under the action by $G$. This, however, leads us to a contradiction by the same argument as in the proof of Proposition 5. Thus the induced action on $\mathbb{P}^1$ is non-trivial. Now take two fibers of $Z' \to \mathbb{P}^1$ that lie over the fixed points of the action on $\mathbb{P}^1$. Since $Z' = \Sigma_1$, one of these two fibers are contained in the fixed locus of the action by $G$.

Let us exclude case 3-2) in Proposition 2.

Proposition 8. Case 3-2) in Proposition 2 does not occur.

Proof. Let $\Gamma_1 \in |\Gamma|$ be the member as in Lemma 3.10. By (9), we have $B'\Gamma_1 = 8$, hence $B' \cap \Gamma_1 \neq \emptyset$. If a smooth point or a negligible singularity of $B'$ lies on $B' \cap \Gamma_1$, we can derive a contradiction by the same argument as in the proof of Proposition 5. Thus by Lemma 3.8 we see that $B' \cap \Gamma_1 = \Delta_0 \cap \Gamma_1$ and that this point is an essential singularity of $B'$. So we put $\Delta_0 \cap \Gamma_1 = \{z_1\}$, where the point $z_1$ is the center of the first blowing-up $q'_{\ast} : Z'_1 \to Z_0 = Z'$ in the procedure to obtain the canonical resolution $Y^\sharp$. Then, by Proposition 7 we have $3 \leq m_1 \leq 5$. If $m_1$ is odd, then the strict transform $\varepsilon_1^\sharp \simeq \mathbb{P}^1 \subset Z^\sharp$ of the exceptional curve $\varepsilon_1 \subset Z'_1$ is a component of $B'$ stable under the action by $G$. This, however, leads us to a contradiction, since the action by $G$ on $Y^\sharp$ is free. It follows $m_1 \equiv 0 \mod 2$, hence $m_1 = 4$. Thus we have $B'_1 = q'^{-1}_{\ast}B' - 4\varepsilon_1$ and $B'_1q'^{-1}_{\ast}(\Gamma_1) = 4$, where the divisor $q'^{-1}_{\ast}(\Gamma_1)$ is the strict transform of $\Gamma_1$ by $q'_1 : Z'_1 \to Z'$. Note that the action by $G$ on $Z'$ induces one on $Z'_1$, and that the strict transform $q'^{-1}_{\ast}(\Gamma_1)$ is contained in the fixed locus of this induced action. By the same argument as that on $\Gamma_1$ above, we see that the point $B'_1 \cap q'^{-1}_{\ast}(\Gamma_1) = \varepsilon_1 \cap q'^{-1}_{\ast}(\Gamma_1)$ is an essential singularity of $B'_1$, that we can set $\varepsilon_1 \cap q'^{-1}_{\ast}(\Gamma_1) = \{z_2\}$, where the point $z_2$ is the center of the second blowing-up $q'_2 : Z'_2 \to Z'_1$, and that $m_2 = 4$, where $m_2$ is the multiplicity of $B'_1$ at $z_2$. Thus we have $i_1 = 1$ and $i_2 = 2$, where $i_1$ and $i_2$ are the integers given in Proposition 7.

Now we derive a contradiction. Let $\Gamma_1^\sharp$ be the strict transform to $Z^\sharp$ of of the divisor $\Gamma_1$. Note that we have $z_1 \in \Gamma_1$ and $z_2 \in q'^{-1}_{\ast}(\Gamma_1)$. Thus by
Lemma 3.9 we obtain
\[ f^\#(\sum p^\# F_i + \eta^\#)\Gamma^1 = 2(\Delta_0 \Gamma + \varepsilon_1^2 + \varepsilon_2^2) = -2 < 0. \]

From this together with Lemma 3.7 we infer that the divisor \( \Gamma^1 \) is the image by \( f^\# \) of an irreducible component of \( \eta^\# \), which contradicts the equality \( \dim(q \circ f^\#)(\eta^\#) = \dim(f^\# \circ p^\#)(\eta^\#) = 0 \). Hence we have the assertion.

### 3.3 The case \( L^2 = K_Y^2 - 2 \)

Finally, we study case 2) in Proposition 2. It will turn out that \( \lambda = 4 \) in this case, and that the surfaces of this case have the structure as in the statement of Theorem 2. In what follows, we assume that our \( Y \) is of case 2) in Proposition 2, hence \( \deg Z = L^2/2 = n \). Note that in this case, the morphism \( p : \tilde{Y} \to Y \) is a blowing-up at two distinct points on \( Y \). Let \( E_1 \) and \( E_2 \) denote the \((-1)\)-curves corresponding to the centers of this blowing-up. Then we have \( p^*|K_Y| = |L| + \sum_{i=1,2} E_i \) and \( LE_1 = LE_2 = 1 \). The Galois group \( G = \text{Gal}(Y/X) \) acts transitively on the set \( \{E_1, E_2\} \). We denote by \( Z' \) the minimal desingularization of \( Z \).

**Lemma 3.11.** There exists a blowing-up \( r : Z' \to \mathbb{P}^2 \) at (possibly infinitely near) \( 9 - n \) points such that the anticanonical morphism \( Z' \to Z \subset \mathbb{P}^n \) gives the minimal desingularization of \( Z \).

**Proof.** Note that our \( Z = \Phi_{K_Y}(Y) \) is a non-degenerate surface in \( \mathbb{P}^n \) of degree \( n \). Hence our \( Z \) is one of the following (see [18] or [12, Section 3]):

i) a projection of a surface of degree \( n \) in \( \mathbb{P}^{n+1} \) from a point outside the surface;

ii) the Veronese embedding into \( \mathbb{P}^8 \) of a quadric in \( \mathbb{P}^3 \) (\( n = 8 \));

iii) the anticanonical image of \( \mathbb{P}^2 \) blown up at \( 9 - n \) points;

iv) a cone over an elliptic curve in \( \mathbb{P}^{n-1} \) of degree \( n \).

Since \( Z' \to Z \) is given by a complete linear system, case i) above is impossible for our case. Since \( q(Y) = 0 \), case iv) also is impossible. Thus it suffices to exclude case ii). In case ii), however, the divisor \( L \) is linearly equivalent to twice a divisor on \( \tilde{Y} \), which contradicts the equality \( LE_i = 1 \). Hence we have the assertion.

In what follows, we put \( D' = -K_{Z'} \) and denote by \( \Phi_{D'} : Z' \to Z \subset \mathbb{P}^n \) the anticanonical map of \( Z' \). Note that the action by \( G = \text{Gal}(Y/X) \) on \( \tilde{Y} \) induces one on \( Z' \).

**Lemma 3.12.** If the surface \( Z' \) has no \((-2)\)-curve, or if every \((-2)\)-curve on \( Z' \) is stable under the action by \( G \) on \( Z' \), then \( \Phi_L : \tilde{Y} \to Z \subset \mathbb{P}^n \) lifts to a morphism \( f' : \tilde{Y} \to Z' \).
Proof. Take the shortest composite \( p' : Y' \to \tilde{Y} \) of quadric transformations such that \( Y' \) admits a morphism \( f' : Y' \to Z' \) satisfying \( \Phi_L \circ p' = \Phi_D \circ f' \). Then the action by \( G \) on \( \tilde{Y} \) induces one on \( Y' \). Note that \( f' \) contracts no \((-1)\)-curve. This follows from \( LE_i = 1 \) and the definition of \( p' \), since the surface \( Y \) is of general type. To obtain the assertion, we only need to show that \( p' : Y' \to \tilde{Y} \) is an isomorphism. Assume that \( p' : Y' \to \tilde{Y} \) is not an isomorphism. Then there exists a \((-1)\)-curve \( C \) on \( Y' \) exceptional with respect to \( p' \). Since the anticanonical map \( \Phi_D : Z' \to Z \subset \mathbb{P}^n \) contracts \( f'(C) \) to a point, the curve \( f'(C) \) is a \((-2)\)-curve on \( Z' \), hence, by the assumption in the statement, stable under the action by \( G \) on \( Z' \). Meanwhile by the same method as in Lemma 3.5, we see that \( f'_1 C = f'(C) \) or \( 2f'(C) \), and if \( f'_2 C = f'(C) \), then \( C \) is a component of the ramification divisor of \( f' \). It follows that \( C \simeq \mathbb{P}^1 \) is stable under the action by \( G \) on \( Y' \), which implies the existence of fixed points of this action. This, however, contradicts the definition of \( \pi : Y \to X \). Thus we have the assertion.

\[ \square \]

**Lemma 3.13.** Assume that \( \Phi_L : \tilde{Y} \to Z \) lifts to a morphism \( f' : \tilde{Y} \to Z' \). Then \( f'(E_1) \) and \( f'(E_2) \) are \((-1)\)-curves on \( Z' \). Further, the following hold:

i) \( f'_i E_i = f'(E_i) \) for \( i = 1, 2 \);

ii) the ramification divisor \( R' \) of \( f' \) satisfies \( R' \sim f''(-2K_{Z'}) + 2 \sum_{i=1,2} E_i \);

iii) the branch divisor \( B' \) of \( f' \) satisfies \( B' \sim -4K_{Z'} + 2 \sum_{i=1,2} f'(E_i) \);

iv) \( f'(E_1) \) and \( f'(E_2) \) are distinct components of the branch divisor \( B' \).

Proof. The first assertion and the assertion i) follow from \( E_i L = E_i f'' D' = 1 \), which implies \( \Phi_L(E_i) \) is a line in \( \mathbb{P}^n \). The assertions ii) and iii) follow from \( D' \sim -K_{Z'} \) and the assertion i). So it suffices to prove the assertion iv). Let us prove the assertion iv). Let \( \theta \) be the involution of \( \tilde{Y} \) over \( Z' \). Since \( Y \) is of general type, the divisors \( E_1 \) and \( E_2 \) are the only \((-1)\)-curves on \( \tilde{Y} \). It follows that if \( f'(E_1) \neq f'(E_2) \), then \( \theta(E_i) = E_i \) holds for \( i = 1, 2 \). Thus we only need to show \( f'(E_1) \neq f'(E_2) \). Assume that \( f'(E_1) = f'(E_2) \). Then \( f''(f'(E_1)) = f''(f'(E_2)) = E_1 + E_2 + \xi \) holds for a certain effective divisor \( \xi \) exceptional with respect to \( f' \). Since we have \( E_1 \cap E_2 = \emptyset \), we see, by the same method as in the proof of Lemma 3.5, that \( \xi^2 = -(E_1 + E_2)\xi = 0 \), hence \( \xi = 0 \). It follows \( f''(f'(E_1)) = f''(f'(E_2)) = E_1 + E_2 \). From this together with the assertions ii) and iii), we infer \( f'' B' - 2R' = 0 \), which implies that the branch divisor \( B' \) has at most negligible singularities. Thus by [9, Lemma 6], we obtain

\[
\chi(O_Y) = 2\chi(O_Z) + \frac{1}{2}(-2K_{Z'} + \sum f'(E_i))(-K_{Z'} + \sum f'(E_i)) = n + 3,
\]

which contradicts \( \chi(O_Y) = n + 2 \). Thus we have \( f'(E_1) \neq f'(E_2) \), which completes the proof of the assertion iv).

\[ \square \]
Lemma 3.14. If the surface $Y$ is of case 2) in Proposition 2, then $\lambda = 4$.

Proof. By Lemma 3.11, we have $n = 2\lambda - 2 \leq 4$, hence $\lambda \leq 5$. Thus we only need to exclude the case $\lambda = 5$. Assume $\lambda = 5$. Then $r : Z' \to \mathbb{P}^2$ is a blowing-up at one point, hence $Z' = \Sigma_1$. Thus by Lemmas 3.12 and 3.13 we see that $\Phi : \tilde{Y} \to Z$ lifts to a morphism $f' : \tilde{Y} \to Z'$, and that $f'(E_i)$'s are $(-1)$-curves. The minimal section $\Delta_0$, however, is the unique $(−1)$-curve on the Hirzebruch surface $\Sigma_1$. Thus we have $f'(E_1) = f'(E_2) = \Delta_0$, which contradicts Lemma 3.13. Hence we have the assertion. \hfill $\Box$

Thus we only need to study the case $\lambda = 4$. In what follows we assume $\lambda = 4$, hence $n = 6$. In this case, the morphism $r : Z' \to \mathbb{P}^2$ is a blowing-up at three points.

Lemma 3.15. Assume that $\Phi : \tilde{Y} \to Z$ lifts to a morphism $f' : \tilde{Y} \to Z'$, and that $f'(E_1) \cap f'(E_2) = \emptyset$ holds. Let $r' : Z' \to W$ denote the blowing-down of the two $(-1)$-curves $f'(E_1)$ and $f'(E_2)$. Then the branch divisor $B$ of the morphism $r' \circ f' : \tilde{Y} \to W$ is a member of the linear system $|-4K_Y|$ having $[3, 3]$-points at $r'(f'(E_i))$ and $r'(f'(E_2))$. Except for these two $[3, 3]$-points, the branch divisor $B$ has at most negligible singularities. Further, the surface $Y$ gives the minimal desingularization of the double cover (of the surface $W$) with branch divisor $B$.

Proof. Note that $f'$ contracts no $(-1)$-curve. Thus the divisor $f''B' - 2R'$, linearly equivalent to $2(\sum f''(E_i)) - 2 \sum E_i)$ by Lemma 3.13, is twice a certain effective divisor $\zeta$ on $\tilde{Y}$. We have $\zeta E_j = (\sum f''(E_i)) - 2 \sum E_i)E_j = 1$, hence $\zeta(\zeta \cap E_j) = 1$ for $j = 1, 2$. So we put $\{z_j\} = f'(\zeta \cap E_j)$. Then the point $z_j \in f'(E_j)$, where $1 \leq j \leq 2$, is an essential singularity of the branch divisor $B'$. Meanwhile, by Lemma 3.13, we see that the divisor $B' - \sum f'(E_i)$ is effective, and that $(B' - \sum f'(E_i))f'(E_j) = 3$ for each $j = 1, 2$, from which we infer mult$_{z_j}(B' - \sum f'(E_i)) \leq 3$. If, moreover, we have mult$_{z_j}(B' - \sum f'(E_i)) \leq 2$, then $z_j$ is a negligible singularity of the branch divisor $B'$; the singularity $z_j$ of $B'$ decomposes into a sum of points of multiplicity at most 2 by the blowing-up at $z_j$. Thus we obtain mult$_{z_j}(B' - \sum f'(E_i)) = 3$, hence $(B' - \sum f'(E_i)) \cap f'(E_j) = \{z_j\}$ and mult$_{z_j}B' = 4$. Let $q_1 \circ q_2 : Z'_2 \to Z'$ be the blowing-up at $z_1$ and $z_2$, and $\varepsilon_j = (q_1 \circ q_2)^{-1}(z_j)$, the $(-1)$-curve corresponding to $z_j$. Then by the same method as in [9, Section 2], we infer that the divisor $B'_2 = (q_1 \circ q_2)^*(B' - 4 \sum_1^2 \varepsilon_i)$ has at most negligible singularities, and that the surface $\tilde{Y}$ gives the canonical resolution of the double cover with branch divisor $B'$. It follows that $B = r' B'$ has $[3, 3]$-points at $r'(f'(E_1))$ and $r'(f'(E_2))$, that the divisor $B$ has no essential singularity except for these two $[3, 3]$-points, and that the surface $Y$ gives the minimal desingularization of the double cover with branch divisor $B$. Now all we have
left is the linear equivalence $B \sim -4K_W$, which, however, is trivial by iii) in Lemma 3.13.

Lemma 3.16. Let $r' : Z' \rightarrow W$ be the blowing-down given in Lemma 3.15. Then the surface $W$ is the Hirzebruch surface $\Sigma_d$ of degree $d = 0$ or 2. The action by $G = \text{Gal}(Y/X)$ on $Z'$ induces one on $W$, of which fixed locus is a set of four isolated points. Further, none of these four fixed points lies on the branch divisor $B$.

Proof. The action by $G$ on $Z'$ induces one on $W$, since the divisor $f'(E_1) + f'(E_2)$ is stable under the action by $G$. Note that the anticanonical system $|-K_{Z'}|$ has no fixed component. From this together with $K_W^2 = K_Z^2 + 2 = 8$, we see that $W = \Sigma_d$ for a certain integer $0 \leq d \leq 2$.

Let us show that the class of $\Gamma$, a fiber of the Hirzebruch surface $W = \Sigma_d \rightarrow \mathbb{P}^1$, is stable under the action by $G$ on $W$. If the class of $\Gamma$ is not stable, then we see that $d = 0$ and that the generator of $G$ maps $\Gamma$ to a member of the linear system $|\Delta_0|$. It follows that there exists an irreducible member $\Delta \in |\Delta_0 + \Delta|$, contained in the fixed locus of the action by $G$ on $W$. We have $\Delta \cap B \neq \emptyset$, since $\Delta$ is a 2-curve. Meanwhile since the Galois group $G$ acts transitively on the set $\{r'(f'(E_1)), r'(f'(E_2))\}$, neither $r'(f'(E_1))$ nor $r'(f'(E_2))$ lies on $\Delta$. Thus by Lemma 3.15, every point in $\Delta \cap B$ is at most a negligible singularity of $B$. Then the same argument as in the proof of Proposition 5 leads us to a contradiction. Hence the class of $\Gamma$ is stable.

Now let us show the assertions. The argument above shows that the action by $G$ on $W$ induces one on $\mathbb{P}^1$ via the natural fibration of the Hirzebruch surface $W = \Sigma_d \rightarrow \mathbb{P}^1$. Note that if this induced action on $\mathbb{P}^1$ is trivial, then there exists an irreducible member $\Delta_1 \in |\Delta_0 + d\Gamma|$ contained in the fixed locus of the action by $G$ on $W$, which, together with the same argument as in the case of $\Delta$ above, leads us to a contradiction. Thus the induced action on $\mathbb{P}^1$ is non-trivial. It follows that $|\Gamma|$ has exactly two members stable under the action on $W$, which we shall call $\Gamma_1$ and $\Gamma_2$. The same argument as in the case of $\Delta$ above shows that the induced action on $\Gamma_i$ is non-trivial for each $i = 1, 2$. Thus we see that $d \neq 1$, and that if $d = 0$ or 2, then the fixed locus of the induced action on $W$ is a set of four isolated points. The absence of the fixed points lying on $B$ follows from the same argument as in the case of $\Delta$ above.

By Lemmas 3.15 and 3.16, we see that if $\Phi_L$ lifts to $f' : \bar{Y} \rightarrow Z'$, and if $f'(E_1) \cap f'(E_2) = \emptyset$, then our surface $X$ has the structure as in Theorem 2. Let us check that these two conditions are in fact satisfied. To do this, we study the arrangement of $(-1)$-curves and $(-2)$-curves on $Z'$, and use Lemmas 3.12 and 3.13.
Let \( r_i : Z'_i \to Z'_{i-1} \), where \(-2 \leq i \leq 0\), be the blowing-up such that \( r = (r_2 \circ r_1 \circ r_0) : Z'_0 = Z' \to Z'_{-3} = \mathbb{P}^2 \) holds. We denote by \( z_i \in Z'_{i-1} \) and \( \varepsilon_i = r_i^{-1}(z_i) \) the center of the blowing-up \( r_i \) and its corresponding \((-1)\)-curve, respectively. For each \(-2 \leq i \leq 0\), we denote by \( \varepsilon'_i \) the strict transform to \( Z' \) of the exceptional curve \( \varepsilon_i \). For the total transform to \( Z' \) of \( \varepsilon_i \), we use the same symbol \( \varepsilon_i \).

**Lemma 3.17.** Let \( m \leq 2 \) be a non-negative integer, and \( C \), a \((-m)\)-curve on \( Z' \) not exceptional with respect to \( r : Z' \to \mathbb{P}^2 \). Then \( C \) is a strict transform to \( Z' \) of a line on \( \mathbb{P}^2 \) passing exactly \( m+1 \) of the tree points \( z_i \)'s \((-2 \leq i \leq 0\).

Proof. Let \( l \) be a line on \( \mathbb{P}^2 \). Then we have \( C \sim m_0 r^*(l) - \sum_{i=-2}^{0} n_i \varepsilon_i \) for certain integers \( m_0 \geq 1 \) and \( n_i \geq 0 \)'s. Note that \( C^2 = -m \) and \(-K_{Z'} C = 2 - m\), since \( C \) is a \((-m)\)-curve. Thus we have

\[
m_0^2 - \sum_{i=-2}^{0} n_i^2 = -m, \quad 3m_0 - \sum_{i=-2}^{0} n_i = -m + 2. \quad (12)
\]

From these equalities, we infer

\[
5 \sum_{i=-2}^{0} n_i^2 + \sum_{-2 \leq i < j \leq 0} (n_i - n_j)^2 + \sum_{i=-2}^{0} (n_i + m - 2)^2 = 9m + 4(m - 2)^2 \leq 18,
\]

hence \( \sum_{i=-2}^{0} n_i^2 \leq 3 \). Thus we have \( n_i^2 = n_i \) for any \(-2 \leq i \leq 0\). By this together with the equalities (12), we obtain \( m_0 = 1 \) and \( \sum_{i=-2}^{0} n_i = m + 1 \). Thus we have the assertion.

We study the arrangement of \((-1)\)-curves and \((-2)\)-curves on \( Z' \) according to the configuration of the centers \( z_i \)'s of the blowing-up \( r : Z' \to \mathbb{P}^2 \). First, we study the case where no two of the centers \( z_{-2} \), \( z_{-1} \), and \( z_0 \) are infinitely near. This case is divided into two cases: case 2-1-1) and case 2-1-2).

Case 2-1-1): the case where the centers \( z_{-2} \), \( z_{-1} \), and \( z_0 \) are not collinear.

In this case, the surface \( Z' \) has no \((-2)\)-curve. Thus \( \Phi_L \) lifts to a morphism \( f' : \tilde{Y} \to Z' \). There exist exactly six \((-1)\)-curves: \( \varepsilon_{-2}, \varepsilon_{-1}, \varepsilon_0, r_{-1}^{-1}(l_{2,-1}), r_{-1}^{-1}(l_{-1,0}) \), and \( r_{-1}^{-1}(l_{-2,0}) \), where \( l_{i,j} \) denotes the line on \( \mathbb{P}^2 \) passing \( z_i \) and \( z_j \). Let \( (X_0 : X_1 : X_2) \) be homogeneous coordinates of \( \mathbb{P}^2 \) satisfying \( z_{-2} = (1 : 0 : 0) \), \( z_{-1} = (0 : 1 : 0) \), and \( z_0 = (0 : 0 : 1) \). For each \((a, b) \in \mathbb{C}^x \times \mathbb{C}^x\), we denote by \( \varphi_{(a, b)} \) the automorphism of \( Z' \) corresponding to the projective transformation \( (X_0 : X_1 : X_2) \mapsto (X_0 : aX_1 : bX_2) \).

Let us study the induced action by \( G \) on \( Z' \). Let \( \text{Aut}(Z') \) be the group of analytic automorphisms of the surface \( Z' \), and \( D_6 \), the dihedral group of
degree 6. Then we have a short exact sequence
\[ 0 \to \mathbb{C}^\times \times \mathbb{C}^\times \to \text{Aut}(Z') \to D_6 \to 0, \]
where the morphism $\mathbb{C}^\times \times \mathbb{C}^\times \to \text{Aut}(Z')$ is given by $(a, b) \mapsto \varphi_{(a, b)}$, and the morphism $\text{Aut}(Z') \to D_6$ corresponds to the transitions of six $(-1)$-curves on $Z'$. Let $\varphi_\sigma$ and $\varphi_\tau$ be the automorphisms of $Z'$ corresponding to the Cremona transformation $(X_0 : X_1 : X_2) \mapsto (X_2X_0 : X_0X_1 : X_1X_2)$ and the morphism $(X_0 : X_1 : X_2) \mapsto (X_0 : X_2 : X_1)$, respectively. Then we have
\[ (\varphi_\sigma)^6 = \text{id}_{Z'}, \quad (\varphi_\tau)^2 = \text{id}_{Z'}, \quad \varphi_\sigma \circ \varphi_\tau \circ \varphi_\sigma \circ \varphi_\tau = \text{id}_{Z'}. \]

Thus the short exact sequence above splits. We denote by $\sigma$ and $\tau$ the image by $\text{Aut}(Z') \to D_6$ of $\varphi_\sigma$ and $\varphi_\tau$, respectively. We have a group homomorphism $G \to \text{Aut}(Z')$ corresponding to the action by $G$ on $Z'$. Composing this homomorphism with $\text{Aut}(Z') \to D_6$, we obtain a group homomorphism $\alpha : G \to D_6$. Note that by Lemma 3.13, the morphism $\alpha$ is an injection of $G$ into $D_6$. Hence the image $\alpha(G)$ is conjugate to $\langle \tau \rangle$, $\langle \sigma \tau \rangle$, or $\langle \sigma^3 \rangle$ in $D_6$.

Assume that the image $\alpha(G)$ is conjugate to $\langle \tau \rangle$ in $D_6$. Replacing the morphism $r : Z' \to \mathbb{P}^2$ if necessary, we may assume that $\alpha(G) = \langle \tau \rangle$. Then since the Galois group $G$ acts transitively on the set $\{r'(E_1), r'(E_2)\}$, we have $\{r'(E_1), r'(E_2)\} = \{(r_\tau^{-1}(l_{-2,-1}), r_\tau^{-1}(l_{-2,0})) \text{ or } (\epsilon_{-1}, \epsilon_0)\}$, hence $r'(E_1) \cap r'(E_2) = \emptyset$. It follows that the surface $W$, where $r' : Z' \to W$ is the blowing-down of the two $(-1)$-curves $f'(E_1)$ and $f'(E_2)$, is isomorphic to the Hirzebruch surface $\Sigma_1$, which contradicts Lemma 3.16. Thus $\alpha(G)$ is not conjugate to $\langle \tau \rangle$.

Assume that the image $\alpha(G)$ is conjugate to $\langle \sigma^3 \tau \rangle$ in $D_6$. Replacing the morphism $r : Z' \to \mathbb{P}^2$ if necessary, we may assume that $\alpha(G) = \langle \sigma^3 \tau \rangle$. Then the blowing-down of the two $(-1)$-curves $\epsilon_{-2}$ and $r_\tau^{-1}(l_{-1,0})$ gives a birational morphism $r'' : Z' \to \Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ satisfying $r''(\epsilon_0) \sim r''(r_\tau^{-1}(l_{-1,0})) \sim \Delta_0$ and $r''(\epsilon_{-1}) \sim r''(r_\tau^{-1}(l_{-2,0})) \sim \Gamma$. Note that the action by $G$ on $Z'$ induces one on $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$. We take homogeneous coordinates $((\xi_0 : \xi_1), (\eta_0 : \eta_1))$ of $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ in such a way that $r''(\epsilon_{-2}) = ((1 : 0), (1 : 0))$ and $r''(r_\tau^{-1}(l_{-1,0})) = ((0 : 1), (0 : 1))$ hold, and that the automorphism of $\Sigma_0$ corresponding to the generator $\iota$ of $G$ is given by $((\xi_0 : \xi_1), (\eta_0 : \eta_1)) \mapsto ((\eta_1 : \eta_0), (\xi_1 : \xi_0))$. Since we have $-K_{Z'} \sim r''(-K_{\Sigma_0}) - \epsilon_{-2} - r_\tau^{-1}(l_{-1,0})$, the space $H^0(\mathcal{O}_{Z'}(-2K_{Z'}))$ corresponds to a certain subspace $V$ of $H^0(\mathcal{O}_{\Sigma_0}(-2K_{\Sigma_0}))$. Every element in $V$ is a homogeneous polynomial $\psi(\xi_0, \xi_1, \eta_0, \eta_1)$ of bidegree $(4, 4)$ vanishing with multiplicity at least 2 at $((1 : 0), (1 : 0))$ and $((0 : 1), (0 : 1))$.

Note that we have a natural inclusion $V \hookrightarrow H^0(\mathcal{O}_Y(2K_Y))$, since we have $L \sim f''D'$. We denote by $\phi$ the restriction to $V$ of the natural action by $G$ on
This contradicts the inequality \( \dim V \leq h^0(\mathcal{O}_X(2K_X - T_c)) \). Hence, the image \( \alpha(G) \) is not conjugate to \( \langle \sigma^3 \rangle \) in \( D_6 \).

Thus we have \( \alpha(G) = \langle \sigma^3 \rangle \). Hence, replacing \( r : Z' \to \mathbb{P}^2 \) if necessary, we may assume that \( \{ f'(E_1), f'(E_2) \} = \{ \varepsilon_{-2}, r^{-1}_s(l_{-1,0}) \} \). Then the surface \( W \) as in Lemma 3.15 obtained by blowing down the two \((-1)\)-curves \( f'(E_1) \) and \( f'(E_2) \) of \( Z' \), is isomorphic to the Hirzebruch surface \( \Sigma_0 \). Thus by Lemmas 3.15 and 3.16 our surface \( X \), in case 2-1-1), has the structure as in the case \( d = 0 \) in Theorem 2.

Case 2-1-2): the case where three points \( z_{-2}, z_{-1}, \) and \( z_0 \) are collinear. Let \( l_{-2,-1} \) be the line on \( \mathbb{P}^2 \) passing the tree points \( z_i \)'s above. Then the strict transform \( r^{-1}_s(l_{-2,-1}) \) is the unique \((-2)\)-curve on \( Z' \). Hence by Lemma 3.12 the morphism \( \Phi_L : \tilde{Y} \to Z \) lifts to \( f' : \tilde{Y} \to Z' \). Meanwhile the surface \( Z' \) has exactly three \((-1)\)-curves: \( \varepsilon_{-2}, \varepsilon_{-1}, \) and \( \varepsilon_0 \). Replacing \( r : Z' \to \mathbb{P}^2 \) if necessary, we may assume \( \{ f'(E_1), f'(E_2) \} = \{ \varepsilon_{-2}, \varepsilon_{-1} \} \) by Lemma 3.13. Let \( r' : Z' \to \mathbb{P}^2 \) be the blowing-down as in Lemma 3.15 of the two \((-1)\)-curves \( f'(E_1) \) and \( f'(E_2) \). Then we have \( W = \Sigma_1 \), \( r'_s(\varepsilon_0) = \Delta_0 \), and \( r'_s(r^{-1}_s(l_{-2,-1})) \sim \Gamma \), which contradicts Lemma 3.16. Thus case 2-1-2) does not occur.

Next, we study the case where \( z_{-2} \) and \( z_{-1} \) are distinct points on \( \mathbb{P}^2 \), and \( z_0 \) is infinitely near to \( z_{-2} \). We denote by \( l_{-2,-1} \) the unique line on \( \mathbb{P}^2 \) passing \( z_{-2} \) and \( z_{-1} \). This case is divided into two cases: case 2-2-1) and case 2-2-2).

Case 2-2-1): the case where \( z_0 \) does not lie on the strict transform \( (r_{-2} \circ r_{-1})^{-1}(l_{-2,-1}) \) of \( l_{-2,-1} \) by \( r_{-2} \circ r_{-1} \). Let \( l_{-1,0} \) be the unique line on \( \mathbb{P}^2 \) whose strict transform \( (r_{-2} \circ r_{-1})^{-1}(l_{-2,-1}) \) by \( r_{-2} \circ r_{-1} \) passes \( z_0 \). Then the surface \( Z' \) has a unique \((-2)\)-curve \( \varepsilon'_{-1} \), and exactly four \((-1)\)-curves \( \varepsilon_{-2}, \varepsilon_0, r_{-1}^{-1}(l_{-1,0}), \) and \( r_{-2}^{-1}(l_{-1,0}) \). Hence, by Lemma 3.12 the morphism \( \Phi_L : \tilde{Y} \to Z \) lifts to \( f' : \tilde{Y} \to Z' \). Note that \( \{ \varepsilon_0, r_{-1}^{-1}(l_{-2,-1}) \} \) is the set of all \((-1)\)-curves intersecting the unique \((-2)\)-curve \( \varepsilon'_{-1} \). Thus we have \( \{ f'(E_1), f'(E_2) \} = \ldots \)
We denote by $r' : Z' \to W$ the blowing-down as in Lemma 3.15 of the two ($-1$)-curves $f'(E_1)$ and $f'(E_2)$. If \( \{ f'(E_1), f'(E_2) \} = \{ \epsilon_0, r_s^{-1}(l_{-2,-1}) \} \), then we have $W = \Sigma_0$, $r'_s(\epsilon'_-1) \sim \Delta_0$ and $r'_s(\epsilon_-2) \sim r'_s(r_s^{-1}(l_{-1,0})) \sim \Gamma$. If on the other hand $\{ f'(E_1), f'(E_2) \} = \{ \epsilon_{-2}, r_s^{-1}(l_{-1,0}) \}$, then we have $W = \Sigma_2$, $r'_s(\epsilon'_-1) = \Delta_0$, and $r'_s(\epsilon_0) \sim r'_s(r_s^{-1}(l_{-2,-1})) \sim \Gamma$. Thus by lemmas 3.15 and 3.16, our surface $X$, in case 2-2-1), has the structure as in the case $d = 0$ or the case $d = 2$ in Theorem 2 according as $\{ f'(E_1), f'(E_2) \} = \{ \epsilon_0, r_s^{-1}(l_{-2,-1}) \}$ or $\{ f'(E_1), f'(E_2) \} = \{ \epsilon_{-2}, r_s^{-1}(l_{-1,0}) \}$ respectively.

Case 2-2-2): the case where $z_0$ lies on the strict transform $(r_{-2} \circ r_{-1})^{-1}(l_{-2,-1})$. In this case, the surface $Z'$ has exactly two ($-2$)-curves $\epsilon'_-1$ and $r_s^{-1}(l_{-2,-1})$, and exactly two ($-1$)-curves $\epsilon_{-2}$ and $\epsilon_0$. Note that every ($-2$)-curve on $Z'$ is stable under the action by $G$ on $Z'$; the divisor $r_s^{-1}(l_{-2,-1})$ is the unique ($-2$)-curve intersecting all ($-1$)-curves on $Z'$. Thus by Lemma 3.12 the morphism $\Phi_E : \tilde{Y} \to Z$ lifts to $f' : \tilde{Y} \to Z'$. Then it follows from Lemma 3.13 that $\{ f'(E_1), f'(E_2) \} = \{ \epsilon_{-2}, \epsilon_0 \}$. This, however, contradicts the transitivity of the action by $G$ on $\{ f'(E_1), f'(E_2) \}$, since $\epsilon_0$ is the unique ($-1$)-curve intersecting all ($-2$)-curves on $Z'$. Thus case 2-2-2) does not occur.

Finally, we study the case where $z_{-1}$ is infinitely near to $z_2$, and $z_0$ is infinitely near to $z_{-1}$. We denote by $l_{-2,-1}$ the unique line on $\mathbb{P}^2$ whose strict transform $(r_{-2} \circ r_{-1})^{-1}(l_{-2,-1})$ passes $z_{-1}$. Note that $Z'$ has no ($-3$)-curve, since the linear system $|-K_{Z'}|$ has no fixed component. Thus the point $z_0$ does not lie on the strict transform $(r_{-1})^{-1}(\epsilon_{-2})$. This case is divided into two cases: case 2-3-1) and case 2-3-2).

Case 2-3-1): the case where $z_0$ does not lie on the strict transform $(r_{-2} \circ r_{-1})^{-1}(l_{-2,-1})$. In this case, the surface $Z'$ has exactly two ($-2$)-curves $\epsilon'_-2$ and $\epsilon'_-1$, and exactly two ($-1$)-curves $\epsilon_0$ and $r_s^{-1}(l_{-2,-1})$. Since $\epsilon'_-2$ is the unique ($-2$)-curve intersecting no ($-1$)-curve on $Z'$, every ($-2$)-curve is stable under the action by $G$ on $Z'$. Thus by Lemma 3.12 the morphism $\Phi_E : \tilde{Y} \to Z$ lifts to $f' : \tilde{Y} \to Z'$. Then it follows from Lemma 3.13 that $\{ f'(E_1), f'(E_2) \} = \{ \epsilon_0, r_s^{-1}(l_{-2,-1}) \}$, hence $f'(E_1) \cap f'(E_2) = \emptyset$. Let $r' : Z' \to W$ be the blowing-down as in Lemma 3.15 of the two ($-1$)-curves $f'(E_1)$ and $f'(E_2)$. Then we have $W = \Sigma_2$, $r'_s(\epsilon'_-2) = \Delta_0$, and $r'_s(\epsilon'_-1) \sim \Gamma$. Thus by Lemmas 3.15 and 3.16, our surface $X$, in case 2-3-1), has the structure as in the case $d = 2$ in Theorem 2.

Case 2-3-2): the case where $z_0$ lies on the strict transform $(r_{-2} \circ r_{-1})^{-1}(l_{-2,-1})$. In this case, the surface $Z'$ has exactly three ($-2$)-curves $\epsilon'_-2$, $\epsilon'_-1$, and $r_s^{-1}(l_{-2,-1})$, and a unique ($-1$)-curve $\epsilon_0$. Note that $\epsilon'_-2$ is the unique ($-2$)-curve intersecting no ($-1$)-curve on $Z'$, and that $\epsilon'_-1$ is the unique ($-2$)-curve intersecting $\epsilon'_-2$. Thus every ($-2$)-curve on $Z'$ is stable under the action by $G$
on $Z'$. Thus by Lemma 3.12, the morphism $\Phi_L : \tilde{Y} \to Z$ lifts to $f' : \tilde{Y} \to Z'$. This, however, contradicts Lemma 3.13 since $\varepsilon_0$ is the unique $(-1)$-curve on $Z'$. Hence, case 2-3-2) does not occur.

Thus we have the following:

**Proposition 9.** Assume that the surface $Y$ is of case 2) in Proposition 2. Then $\lambda = 4$. Further, the surface $X$ in this case has the structure as in Theorem 2.

**4 The case $\deg \Phi_{K_Y} = 1$**

In this section, we exclude the case $\deg \Phi_{K_Y} = 1$ and give a proof for Theorems 1 and 2. In what follows, we assume that $\deg \Phi_{K_Y} = 1$. Note that by Proposition 1, we have $\lambda = 4$, hence $K_Y^2 = 14$, $p_g(Y) = 7$, and $q(Y) = 0$.

Thus our $Y$ is a canonical surface whose invariant lies on the Castelnuovo line. By [1, Lemma 1.1], the canonical system $|K_Y|$ is free from base points; hence the canonical map $\Phi_{K_Y} : Y \to \mathbb{P}^n$ is a morphism, where $n = 2\lambda - 2 = 6$.

In what follows, we frequently use results given in [1].

Let $Q \subset \mathbb{P}^n$ be the intersection of all quadrics containing the canonical image $Z = \Phi_{K_Y}(Y)$. By [1, Section 1], we obtain the following:

**Proposition 10.** Let $Q$ be the variety defined above. Then either of the following holds:

1) $Q$ is the image by $\Phi_T$ of the variety $Q' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$, where $\Phi_T$ is the morphism associated with a tautological divisor $T$ of the $\mathbb{P}^1$-bundle $\text{pr}_{Q'} : \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \to \mathbb{P}^1$;

2) $Q$ is the image by $\Phi_T$ of the variety $Q' = \mathbb{P}(\bigoplus_{i=0}^2 \mathcal{O}_{\mathbb{P}^1}(a_i))$, where $\Phi_T$ is the morphism associated with a tautological divisor $T$ of the $\mathbb{P}^2$-bundle $\text{pr}_{Q'} : \mathbb{P}(\bigoplus_{i=0}^2 \mathcal{O}_{\mathbb{P}^1}(a_i)) \to \mathbb{P}^1$, and $0 \leq a_0 \leq a_1 \leq a_2$ and $\sum_{i=0}^2 a_i = n - 2$.

First, we exclude case 1) in the proposition above.

**Proposition 11.** Case 1) in Proposition 10 does not occur.

Proof. Assume that our $Q$ is as in case 1) in Proposition 10. Then $Q$ is a cone over the Veronese surface. Let $p_0$ be the vertex of $Q$, and $A$, the linear system consisting of pull-backs by $\Phi_{K_Y}$ of all hyperplanes in $\mathbb{P}^n$ passing $p_0$. We denote by $A_0$ and $G_0$ its variable part and fixed part respectively. By [1, Proof of Claim 1], the linear system $A_0$ is free from base points and induces $\Phi_{A_0} : Y \to \mathbb{P}^{n-1}$, a morphism of mapping degree 3 onto its image. The image $\Phi_{A_0}(Y)$ is the Veronese surface, i.e., the projective plane $\mathbb{P}^2$ embedded in $\mathbb{P}^5$ by $\mathcal{O}_{\mathbb{P}^2}(2)$. Note that by the definition of $Q$, the variety $Q$ and its vertex
Proposition 12. Case 2) in Proposition 10 does not occur.

Proof. Assume that our Q is as in case 2) in Proposition 10 and that $a_0 = 0$. It follows that our Q is a cone over the Hirzebruch surface $\Sigma_{a_2-a_1}$ embedded in $\mathbb{P}^{n-1}$ by $|\Delta_0 + a_2\Gamma|$. Let $p_0$ be the vertex of Q, and A, the linear system consisting of the pull-backs by $\Phi_{K_{Y}}$ of all hyperplanes in $\mathbb{P}^n$ passing $p_0$. We denote by $A_0$ and $G_0$ the variable part and the fixed part of A respectively. By [1], the linear system $A_0$ is free from base points and induces $\Phi_{A_0} : Y \to \mathbb{P}^{n-1}$, a morphism of degree 3 onto its image. The image $\Phi_{A_0}(Y)$ is the Hirzebruch surface $\Sigma_{a_2-a_1}$ embedded in $\mathbb{P}^{n-1}$ by $|\Delta_0 + a_2\Gamma|$. By the same argument as in the proof of Proposition 11 we see that the action by $\Phi_{A_0}$ on $\Sigma_{a_2-a_1}$ are stable under this induced action on $\Sigma_{a_2-a_1}$; hence so are the class of $\Delta_0$ and $a_0$. If the variety $Q$ is as in case 2) of Proposition 10, then $a_0 = 0$.

Lemma 4.1. If the variety $Q$ is as in case 2) of Proposition 10, then $a_0 = 0$.

Proof. Assume that our variety $Q$ is as in case 2) in Proposition 10 and that $a_0 > 0$. Then $\Phi_T : Q' \to \mathbb{P}^n$ is an embedding. We identify $Q$ and $Q'$ by $\Phi_T$. By the same argument as in the proof of Proposition 11 we see that the variety $Q$ is stable under the action by $G$ on $\mathbb{P}^n$. Let $P$ be a fiber of the $\mathbb{P}^2$-bundle $\pi_Q : Q = Q' \to \mathbb{P}^1$. Then $P$ and $T$ generate the Picard group of $Q$. Using this, we see easily that if a divisor $P'$ on $Q$ satisfies $P'^3 = K_QP'^2 = 0$ and $h^0(\mathcal{O}_Q(P')) = 2$, then $P' \sim P$. Thus the class of $P$ is stable under the action by $G$ on $Q$. It follows that this action induces one on $\mathbb{P}^1$ via the projection $\pi_{Q'} : Q = Q' \to \mathbb{P}^1$, and that there exists a member $P_0 \in |P|$ stable under the action on $Q$. Now let us derive a contradiction. Since $G \simeq \mathbb{Z}/2$, the fixed locus of the action by $G$ on $P_0 = \mathbb{P}^2$ contains a line $l_0$. Hence the action on $Z$ has a fixed points. By [1, Theorem 1.5], however, the surface $Z$ has at most rational double points as its singularities. Thus, by the same argument as in the proof of Proposition 5 we infer that the action on $Y$ has fixed points, which contradicts the definition of $\pi : Y \to X$. □
and that of $\Gamma$. Thus there exist members $\Delta_1 \in |\Delta_0|$ and $\Gamma_1 \in |\Gamma|$ stable under the action on $\Sigma_{a_2-a_1}$. Then from $\Phi_{\Lambda_0}^* (\Delta_1) \Phi_{\Lambda_0}^* (\Gamma_1) = \deg \Phi_{\Lambda_0} = 3$, we derive a contradiction by the same argument as in the proof of Proposition 1. Thus we have the assertion.

Now we are ready to prove Theorems 1 and 2.

Proof of Theorems 1 and 2

By Propositions 1, 10, 11, and 12, we have $\deg \Phi_{Y_{1,2}} = 2$. Thus Theorems 1 and 2 follow from Propostions 2, 5, 6, 8, and 9.

Remark 4. Let $X_{(1)}$ and $X_{(2)}$ be two minimal complex surfaces as in Theorem 2, $\pi_{(i)} : Y_{(i)} \rightarrow X_{(i)}$ ($i = 1, 2$), the unramified double cover corresponding to the torsion group, $f_{(i)} : Y_{(i)} \rightarrow W_{(i)} = \Sigma_{d_{(i)}}$, the generically two-to-one morphism as in Theorem 2 and $B_{(i)}$, the branch divisor of $f_{(i)}$. Then if $X_{(1)}$ and $X_{(2)}$ are isomorphic to each other, so are the triplets $(W_{(1)}, \iota|_{W_{(1)}}, B_{(1)})$ and $(W_{(2)}, \iota|_{W_{(2)}}, B_{(2)})$, where $\iota|_{W_{(i)}}$ denotes the involution of $W_{(i)}$ corresponding to the generator of the Galois group of $\pi_{(i)}$. This is verified as follows. Let $p_{(i)} : \tilde{Y}_{(i)} \rightarrow Y_{(i)}$ be the shortest composite of quadric transformations such that the variable part of $p_{(i)}^* |K_{Y_{(i)}}|$ is free from base points, and $r'_{(i)} : Z'_{(i)} \rightarrow W_{(i)} = \Sigma_{d_{(i)}}$, the blowing-up at two $[3,3]$-points, which we shall denote by $\iota|_{W_{(i)}}$ the involution of $W_{(i)}$ corresponding to the generator of the Galois group $G = \text{Gal}(Y/X)$.

Let $r' : Z' \rightarrow W$ be the blowing-up at two $[3,3]$-points, which we shall call $w_1$ and $w_2$, of the branch divisor $B$ of $f$. Then $f_W = f \circ p : \tilde{Y} \rightarrow W$ lifts to a morphism $f' : \tilde{Y} \rightarrow Z'$. We denote by $e_i = r'^{-1}(w_i)$ the

5 The moduli space for the case $\chi = 4$

In this section, we shall study the moduli space for surfaces as in Theorem 2 and give a proof for Theorem 3. For this purpose, we shall first study the explicit description of our surfaces in more detail.

Let $X$ be a minimal algebraic surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\text{Tors} \simeq \mathbb{Z}/2$. We denote by $\pi : Y \rightarrow X$ the unramified double cover corresponding to the torsion group, and by $p : \tilde{Y} \rightarrow Y$, the shortest composite of quadric transformations such that the variable part of $p^* |K_Y|$ is free from base points. Then there exist an even integer $0 \leq d \leq 2$ and a generically two-to-one morphism $f : Y \rightarrow W = \Sigma_d$ satisfying the three conditions given in Theorem 2. In what follows, we denote by $\iota|_W$ the involution of $W$ corresponding to the generator of the Galois group $G = \text{Gal}(Y/X)$.
exceptional divisor of $r'$ lying over $w_1$. Let $\tilde{r} : \tilde{Z} \to Z'$ be the blowing-up at two quadraple points, which we shall call $w'_1 \in e_1$ and $w'_2 \in e_2$, of the branch divisor of $f'$. Then $f'$ lifts to a morphism $\tilde{f} : \tilde{Y} \to \tilde{Z}$. We denote by $e'_i = \tilde{r}^{-1}(w'_i)$ the exceptional divisor of $\tilde{r}$ lying over $w'_i$. Let us use the same symbol $e_i$ for the total transform to $\tilde{Z}$ of the divisor $e_i \subset Z'$. Then there exists a reduced member $\tilde{B}_0 \in |(r' \circ \tilde{r})^*(-4K_W)| - 3 \sum e_i - 3 \sum e'_i|$ satisfying $\tilde{B}_0 \cap \tilde{r}^{-1}(e_1) = \tilde{B}_0 \cap \tilde{r}^{-1}(e_2) = \emptyset$ such that the branch divisor of $\tilde{f}$ is given by $\tilde{B}_0 + \sum \tilde{r}^{-1}_*(e_i)$. Note that the divisor $\tilde{B}_0$ has at most negligible singularities. In what follows, $\Delta_0$ and $\Gamma$ denote the minimal section and a fiber respectively of the Hirzebruch surface $W = \Sigma_d \to \mathbb{P}^1$.

**Lemma 5.1.** Let $\iota|_{Z'}$ be the involution of $Z'$ induced by the involution $\iota|_W$ of $W$. Then the configuration of the four points $w_1, w_2 = \iota|_W(w_1), w'_1,$ and $w'_2 = \iota|_{Z'}(w'_1)$ satisfies the following three conditions:

i) if $d = 2$, then $w_1 \notin \Delta_0$;

ii) if the two points $w_1$ and $w_2$ lie on one and the same member of the linear system $|\Gamma|$, then for each $i = 1, 2$, the point $w'_i$ does not lie on the strict transform to $Z'$ of this member;

iii) if $d$ equals 0, and the two points $w_1$ and $w_2$ lie on one and the same member of the linear system $|\Delta_0|$, then for each $i = 1, 2$, the point $w'_i$ does not lie on the strict transform to $Z'$ of this member.

Proof. i). Assume that $d = 2$ and $w_1 \in \Delta_0$. Then since $\Delta_0$ is stable under the action by $G$ on $W$, we have $w_2 \in \Delta_0$. Thus $r^{-1}_*(-\Delta_0)(-K_{Z'}) < 0$. It follows that $r^{-1}_*(-\Delta_0)$ is a fixed component of the linear system $|-K_{Z'}|$. This is impossible, since by the proof of our complete description the pull-back $f^*|-K_Y|$ is the variable part of $|K_Y|$. Thus we have $w_1 \notin \Delta_0$ for the case $d = 2$.

ii). Assume that $w_1$ and $w_2$ lie on one and the same member $\Gamma_0 \in |\Gamma|$. Then since $w_2 = \iota|_W(w_1)$, the member $\Gamma_0$ is stable under the action by $G$. It follows that $\Gamma_0$ passes exactly two of the fixed points of the involution $\iota|_W$. Moreover if $w'_1 \in r^{-1}_*(\Gamma_0)$, then we obtain $w'_2 \in r^{-1}_*(\Gamma_0)$, $(r' \circ \tilde{r})^{-1}_*(\Gamma_0) \sim (r' \circ \tilde{r})^*(\Gamma) - \sum e_i - \sum e'_i$, and $\tilde{B}_0((r' \circ \tilde{r})^{-1}_*(\Gamma_0)) = -4 < 0$. The last inequality implies that $\Gamma_0$ is an irreducible component of the branch divisor $B$. This however is impossible, since, by the condition in Theorem \(\square\) the branch divisor $B$ cannot pass any fixed points of the involution $\iota|_W$. Thus we have $w'_1 \notin r^{-1}_*(\Gamma_0)$.

iii). By the same argument as in the proof of ii), we can prove iii). \(\square\)

**Remark 5.** As shown in the proof of Lemma above, if the two points $w_1$ and $w_2$ lie on one and the same member $\Gamma_0 \in |\Gamma|$, then this $\Gamma_0$ is stable under the action by $G$ on $W$. There exist exactly two members of $|\Gamma|$ stable under the
For each action by $G$. In what follows, we denote by $\Gamma_1$ and $\Gamma_2$ these two members. For each $i = 1, 2$, exactly two fixed points of the action by $G$ lie on $\Gamma_i$.

Next let us show that if conversely the configuration of four points $w_i$'s and $w'_i$'s satisfies the three conditions in Lemma 5.1 then the procedure implied by our structure theorem in fact produces a minimal surface with the desired invariants. Some of the results below will be used later, in our proof of the uniqueness of the deformation type. Let $W = \Sigma_d$ be the Hirzebruch surface of degree $d = 0$ or 2, and $\iota|W$, the involution (1) given in Remark 1. Take a point $w_1 \in W$ outside the fixed locus of $\iota|W$. We denote by $r' : Z' \to W$ the blowing-up at two points $w_1$ and $w_2 = \iota|W(w_1)$, and by $e_{w_i} = r'^{-1}(w_i)$, the exceptional curve lying over $w_i$. Let $\iota|Z'$ be the involution of $Z'$ induced by $\iota|W$. Take a point $w'_1 \in e_1 \subset Z'$. We denote by $\tilde{r} : \tilde{Z} \to Z'$ the blowing-up at two points $w'_1$ and $w'_2 = \iota|Z(w'_1)$, and by $e'_{w_i} = \tilde{r}^{-1}(w'_i)$, the exceptional curve lying over $w'_i$. We use the same symbol $e_i$ for the total transform to $\tilde{Z}$ of the divisor $\iota|W$ on $Z'$. We assume that the configuration of $w_i$'s and $w'_i$'s satisfies the three conditions i), ii), and iii) in Lemma 5.1.

Let $\Gamma_1$ and $\Gamma_2$ be two distinct members of $|I|$ stable under the natural action by $G = \langle \iota|W \rangle$ on $W$ (see Remark 5). We take the minimal section $\Delta_0$ and an irreducible member $\Delta_\infty \in |\Delta_0 + dI|$ such that both are stable under the action by $G$, and $\Delta_0 \cap \Delta_\infty = \emptyset$ holds. Let $m$ be a positive integer. Since the divisor $m(\Delta_0 + \frac{d+2}{2}I_1)$ is stable under the action by $G$, we obtain a natural action on $H^0(\mathcal{O}_W(m(\Delta_0 + \frac{d+2}{2}I_1)))$ by identifying this space with that of meromorphic functions with poles at most $m(\Delta_0 + \frac{d+2}{2}I_1)$.

We put $A_m = |m(\Delta_0 + \frac{d+2}{2}I_1)|$, and denote by $A^+_m$ and $A^-_m$ the subsystems of $A_m$ corresponding to the eigenspaces of eigenvalues +1 and −1 respectively with respect to $\iota|W^*$. Moreover, for an effective divisor $C$ on $\tilde{Z}$, we put

$$A_m(C) = \{ D \in A_m; (r' \circ \tilde{r})^*(D) - C \geq 0 \} \quad \tilde{A}_m(C) = (r' \circ \tilde{r})^*A_m(C) - C$$

$$A^+_m(C) = \{ D \in A^+_m; (r' \circ \tilde{r})^*(D) - C \geq 0 \} \quad \tilde{A}^+_m(C) = (r' \circ \tilde{r})^*A^+_m(C) - C$$

$$A^-_m(C) = \{ D \in A^-_m; (r' \circ \tilde{r})^*(D) - C \geq 0 \} \quad \tilde{A}^-_m(C) = (r' \circ \tilde{r})^*A^-_m(C) - C,$$

where the symbol $\geq 0$ means effectiveness of a divisor. We abbreviate $\tilde{A}_m(0)$, $\tilde{A}^+_m(0)$, and $\tilde{A}^-_m(0)$ to $\tilde{A}_m$, $\tilde{A}^+_m$, and $\tilde{A}^-_m$ respectively. Note that if $\tilde{f} : \tilde{Y} \to \tilde{Z}$ is the generically two-to-one morphism obtained as in the beginning of this section from our structure theorem, then we have $\tilde{B}_0 \in A^+_m(3 \sum e_i + 3 \sum e'_i)$, where $\tilde{B}_0 = \sum \tilde{r}^{-1}(e_i)$ gives the branch divisor of $\tilde{f} : \tilde{Y} \to \tilde{Z}$.

**Lemma 5.2.** 1) The linear system $\tilde{A}^+_2$ has no base point.

2) The linear system $\tilde{A}^+_2(\sum e_i + \sum e'_i)$ has no base point.

3) The linear system $\tilde{A}^+_2(3 \sum e_i + 3 \sum e'_i)$ has no base point.

4) The linear systems $| - K_{2'}|$ and $| - K_{2''}|$ have no base point.

28
Proof. Since we have \( \tilde{\Lambda}_2^+ + 3\tilde{\Lambda}_2^+ (\sum e_i + \sum e'_i) \subset \tilde{\Lambda}_2^+ (3 \sum e_i + 3 \sum e'_i) \), the assertion 3) follows from the assertions 1) and 2).

Assume that we have the assertions 1) and 2). Then by \( \tilde{\Lambda}_2^+ (\sum e_i + \sum e'_i) \subset | - K_{\tilde{Z}} | \), we see that the linear system \( | - K_{\tilde{Z}} | \) has no base point. Moreover, by this together with the Riemann–Roch theorem and the vanishing theorem, we obtain \( h^0(\mathcal{O}_Z(-K_{\tilde{Z}})) = \chi(\mathcal{O}_Z) + K_{\tilde{Z}} \geq 5 \). Meanwhile, since \( r': Z' \to W \) is the blowing-up at two points \( w'_i \)'s, we have \( h^0(\mathcal{O}_{Z'}(-K_{Z'})) \geq h^0(\mathcal{O}_{W}(-K_W)) - 2 = 7 \). Thus we obtain \( h^0(\mathcal{O}_{Z'}(-K_{Z'})) - h^0(\mathcal{O}_Z(-K_{\tilde{Z}})) \geq 2 \), which implies that neither of the two points \( w'_i \)'s is a base point of \( | - K_{Z'} | \). From this, we infer that \( | - K_{Z'} | \) has no base point. So the assertion 4) also follows from the assertions 1) and 2).

Thus we only need to show the assertions 1) and 2). First, let us show the assertion 1). Let \( C_0 \) be a general member of \( | \Gamma | \). Then since the divisor \( 2\Delta_0 + \frac{d+2}{2}(C_0 + |w(C_0)|) \in \Lambda_2 \) is stable under the action by \( G \), and the divisor

\[
(2\Delta_0 + \frac{d+2}{2}(C_0 + |w(C_0)|)) - 2(\Delta_0 + \frac{d+2}{2}\Gamma_1)
\]

has no support at \( \Delta_\infty \cap \Gamma_2 \), the divisor \( 2\Delta_0 + \frac{d+2}{2}(C_0 + |w(C_0)|) \) is a member of \( \Lambda_2^+ \). Thus the base locus of \( \Lambda_2^+ \) is contained in \( \Delta_0 \). Using a similar argument, we can show that \( 2\Delta_\infty + \frac{d+2}{2}(\tilde{C}_0 + \tilde{|w(C_0)|}) \in \tilde{\Lambda}_2^+ \), so that the base locus of \( \tilde{\Lambda}_2^+ \) is contained in \( \Delta_\infty \). Thus since \( \Delta_0 \cap \Delta_\infty = \emptyset \), the linear system \( \tilde{\Lambda}_2^+ \) has no base point. Hence we have the assertion 1).

Next, let us show the assertion 2). We shall show it by dividing our situation into several cases. In what follows, for each \( i = 1, 2 \), we denote by \( \Gamma_{(i)} \) the unique member of \( | \Gamma | \) passing \( w_i \).

Case 1-1: the case where \( d = 0 \) holds, and the two points \( w_1 \) and \( w_2 \) lie neither on one and the same member of \( | \Gamma | \) nor on that of \( | \Delta_0 | \). In this case, for each \( i = 1, 2 \), we denote by \( \Delta_{(i)} \) the unique member of \( | \Delta_0 | \) passing \( w_i \). This case is divided into two subcases: case 1-1-1 and case 1-1-2.

Case 1-1-1: the subcase of case 1-1 where \( w'_1 \notin r_{s'1}(\Gamma_{(1)}) \) and \( w'_1 \notin r_{s'1}(\Delta_{(1)}) \). In this case, take global coordinates \( (s'_1, \xi'_1) \) of \( W \setminus (\Gamma_{(2)} \cup \Delta_{(2)}) \simeq \mathbb{A}^2 \) such that \( \Gamma_{(1)} \) is given by \( s'_1 = 0 \), \( \Gamma_{(2)} \) by \( s'_1 = \infty \), \( \Delta_{(1)} \) by \( \xi'_1 = 0 \), and \( \Delta_{(2)} \) by \( \xi'_1 = \infty \). Then the involution \( \iota_{W} \) is given by \( (s'_1, \xi'_1) \mapsto (1/s'_1, 1/\xi'_1) \), and the linear system \( \Lambda_s^+ \) is spanned by the five elements \( s'_1 \xi_1^m + s'_1 \xi_1^{2-l} \xi_1^{2-m} \) \((0 \leq l \leq 2, 0 \leq m \leq 2) \). Thus the linear system \( \Lambda_2^+ (\sum e_i + \sum e'_i) \) is spanned by the three elements

\[
a_0(s'_1 + s'_1 \xi_1^2) + b_0(\xi'_1 + s'_1 \xi_1^2), \quad s'_1 \xi_1, \quad s'_1^2 + \xi_1^2,
\]

where \( a_0 \neq 0 \) and \( b_0 \neq 0 \) are certain non-zero complex numbers. From this, we infer that the set \( \{w_1, w_2\} \) forms the base locus of \( \Lambda_2^+ (\sum e_i + \sum e'_i) \), and
that any general member of this linear system is smooth. By this together with \( \sum G(i) + \sum \Delta(i) \in A^2 \left( \sum e + \sum e' \right) \), we see that the linear system \( A^2 \left( \sum e + \sum e' \right) \) has no base point.

Case 1-1-2: the subcase of case 1-1 where \( w_1' \in r^{r^{-1}}(\Gamma_1) \) or \( w_1' \in r^{r^{-1}}(\Delta_1) \). Since the proof is the same, we only give a proof for the case \( w_1' \in r^{r^{-1}}(\Gamma_1) \). Assume that \( w_1' \in r^{r^{-1}}(\Gamma_1) \). Since we have \( C_0 + \iota_W(C_0) + \sum \Gamma(i) \in A^2 \sum e + \sum e' \) for any general member \( C_0 \) of \( |\Delta_0| \), the base loci of \( A^2 \left( \sum e + \sum e' \right) \) is contained in \( \left( r' \circ r \right)^{-1}(\Gamma_1) \). Meanwhile, since we have \( C_1 + \iota_W(C_1) \in A^2 \left( \sum e + \sum e' \right) \) for any general member \( C_1 \) of \( \Lambda^1(\sum e) \), the base locus of \( A^2 \left( \sum e + \sum e' \right) \) is contained in \( \left( r' \circ r \right)^{-1}(\Delta_1) \). Since \( \left( r' \circ r \right)^{-1}(\Gamma_1) \cap \left( r' \circ r \right)^{-1}(\Delta_1) \) is contained in \( 2 \left( r' \circ r \right)^{-1}(\Gamma_1) \), we see that the linear system \( A^2 \left( \sum e + \sum e' \right) \) has no base point.

Case 1-2-2: the case where \( d = 0 \) holds, and the two points \( w_1 \) and \( w_2 \) lie on one and the same member of \( |\Gamma| \) or \( |\Delta_0| \). In this case, for each \( i = 1, 2 \), we denote by \( \Delta(i) \) the unique member of \( |\Delta_0| \) passing \( w_i \). By exchanging \( \Delta_0 \) and \( \Gamma \) if necessary, we may assume that the two points \( w_1 \) and \( w_2 \) lie on one and the member \( \Gamma_0 \in |\Gamma| \). Moreover, by Remark 5 by exchanging \( \Gamma_1 \) and \( \Gamma_2 \) if necessary, we may assume that \( \Gamma_0 = \Gamma_1(= \Gamma_2) = \Gamma_1 \). Then this case is divided into two subcases: case 1-2-2 and case 1-2-1.

Case 1-2-1: the subcase of case 1-2 where \( w'_1 \notin r^{r^{-1}}(\Delta_1) \). Note that we have assumed the condition ii) of Lemma 5.1 for our configuration, so that we have \( w'_1, w'_2 \notin r^{r^{-1}}(\Gamma_1) \). For any general member \( C_0 \in |\Delta_0| \), we have \( C_0 + \iota_W(C_0) + 2 \Gamma_1 \in A^2 \left( \sum e + \sum e' \right) \) contained in \( 2 \left( r' \circ r \right)^{-1}(\Gamma_1) \). Meanwhile, for any general \( C_1 \) \((\neq \Delta_1 \sim \Gamma_1) \in A^2 \left( \sum e + \sum e' \right) \), we have \( C_1 + \iota_W(C_1) \in A^2 \left( \sum e + \sum e' \right) \) and the irreducibility and smoothness at \( w_1 \) of \( C_1 \). Thus the base loci of \( A^2 \left( \sum e + \sum e' \right) \) is contained in \( \left( r' \circ r \right)^{-1}(\Gamma_1) \) or \( \left( r' \circ r \right)^{-1}(\Delta_1) \). Since \( 2 \left( r' \circ r \right)^{-1}(\Gamma_1) + \sum \left( r' \circ r \right)^{-1}(\Delta_1) \) do not intersect each other, we see that the linear system \( A^2 \left( \sum e + \sum e' \right) \) has no base point.

Case 1-2-2: the subcase of case 1-2 where \( w_1 \in r^{r^{-1}}(\Delta_1) \). By the same argument as one given in the proof for case 1-2-1, we see that the base loci of \( A^2 \left( \sum e + \sum e' \right) \) is contained in \( \left( r' \circ r \right)^{-1}(\Gamma_1) + \sum \left( r' \circ r \right)^{-1}(\Delta_1) \). Meanwhile, for any general member \( C_1 \) \( \in |\Gamma| \), we have \( C_1 + \iota_W(C_1) + \sum \left( r' \circ r \right)^{-1}(\Delta_1) \in A^2 \left( \sum e + \sum e' \right) \). Thus the base loci of \( A^2 \left( \sum e + \sum e' \right) \) is contained in \( \left( r' \circ r \right)^{-1}(\Gamma_1) \). Since \( 2 \left( r' \circ r \right)^{-1}(\Gamma_1) + \sum \left( r' \circ r \right)^{-1}(\Delta_1) \) do not intersect each other, we see that the linear system \( A^2 \left( \sum e + \sum e' \right) \) has no base point.

Case 2-1: the case where \( d = 2 \) holds, and the two points \( w_1 \) and \( w_2 \) do not lie on one and the same member of \( |\Gamma| \). Note that since we have assumed the condition i) in Lemma 5.1, we have \( w_1 \notin \Delta_0 \). Note also that for this case, or more generally for case \( d = 2 \), we have \( \dim \Lambda^1 = 1 \), and any
general member of this linear system is an irreducible curve stable under the action by $G$ that passes two points $\Delta_1 \cap \Gamma_1$ and $\Delta_\infty \cap \Gamma_2$. We denote by $\Delta_1$ the unique member of $\Lambda^+_1$ that passes the two points $w_1$ and $w_2$. Then this case is divided into three subcases: case 2-1-1, case 2-1-2, and case 2-1-3.

Case 2-1-1: the subcase of case 2-1 where $w_1 \notin r_{s}^{-1}(\Gamma_{(1)})$ and $w_1 \notin r_{s}^{-1}(\Delta_1)$. Since the divisor $\Delta_0 + 2\Gamma_{(1)}$ is the unique reducible member of $\Lambda_1(e_1 + e'_1)$, and we have $h^0(\mathcal{O}_W(\Delta_0 + 2\Gamma)) = 4$, any general member of $\Lambda_1(e_1 + e'_1)$ is irreducible and non-singular. By this together with $\Delta_0 + 2\Gamma_{(1)} \in \Lambda_1(e_1 + e'_1)$, we see that $\tilde{A}_1(e_1 + e'_1)$ has no base point, and $(r' \circ \tilde{r})^{-1}(C_0 + \iota|W(C_0)) \in \Lambda^+_2(\sum e_i + \sum e'_i)$ for any general member $C_0 \in \Lambda_1(e_1 + e'_1)$. Thus we see that $\Lambda^+_2(\sum e_i + \sum e'_i)$ has no base point.

Case 2-1-2: the subcase of case 2-1 where $w_1 \in r_{s}^{-1}(\Gamma_{(1)})$. Since we have $2\Delta_0 + \sum \Gamma_{(i)} + C_0 + \iota|W(C_0) \in \Lambda^+_2(\sum e_i + \sum e'_i)$ for any general member $C_0 \in |\Gamma|$, the base locus of $\Lambda^+_2(\sum e_i + \sum e'_i)$ is contained in $2(r' \circ \tilde{r})^{-1}(\Delta_0) + \sum (r' \circ \tilde{r})^{-1}(\Gamma_{(i)})$. By this together with $2\Delta_1 \in \Lambda^+_2(\sum e_i + \sum e'_i)$, we see that the linear system $\Lambda^+_2(\sum e_i + \sum e'_i)$ has no base point.

Case 2-1-3: the subcase of case 2-1 where $w_1 \notin r_{s}^{-1}(\Delta_1)$. Since we have $C_0 + \Delta_1 \in \Lambda^+_2(\sum e_i + \sum e'_i)$ for any general member $C_0 \in \Lambda^+_1$, the base locus of $\Lambda^+_2(\sum e_i + \sum e'_i)$ is contained in $\sum (r' \circ \tilde{r})^{-1}(\Delta_0) + \sum (r' \circ \tilde{r})^{-1}(\Gamma_{(i)}) + \sum r_{s}^{-1}(e_i)$. Meanwhile since we have $h^0(\mathcal{O}_W(\Delta_0 + 2\Gamma)) = 4$, we have $C_1 + \iota|W(C_1) \in \Lambda^+_2(\sum e_i + \sum e'_i)$ for an irreducible member $C_1 \in \Lambda_1(e_1 + e'_1)$. Since $2(r' \circ \tilde{r})^{-1}(\Delta_0) + 2(r' \circ \tilde{r})^{-1}(\Gamma_{(i)}) + \sum r_{s}^{-1}(e_i)$ and $(r' \circ \tilde{r})^{-1}(C_1 + \iota|W(C_1))$ do not intersect each other, we see that the linear system $\Lambda^+_2(\sum e_i + \sum e'_i)$ has no base point.

Now that we have shown the absence of base points of $\Lambda^+_2(\sum e_i + \sum e'_i)$ for all the eight cases 1-1-1, . . . , 2-2, we have the assertion 2).

Let $\tilde{B}_0$ be a reduced member of $\Lambda^+_2(3 \sum e_i + 3 \sum e'_i)$ that has at most negligible singularities, satisfies $\tilde{B}_0 \cap \sum r_{s}^{-1}(e_i) = \emptyset$, and passes no fixed point of the action by $G$ on $\tilde{Z}$. Existence of such $\tilde{B}_0$ is ensured by Lemma 5.2. Let $\tilde{Y}$ be the canonical resolution of the double cover of $\tilde{Z}$ branched along $\tilde{B}_0 + \sum r_{s}^{-1}(e_i)$, and $\tilde{f} : \tilde{Y} \to \tilde{Z}$, the natural projection. We have $\tilde{f}^*(r_{s}^{-1}(e_i)) = 2E_i$ for a $(-1)$-curve $E_i$ on $\tilde{Y}$. Let $p : \tilde{Y} \to Y$ be the blowing-
down of $E_1$ and $E_2$. Then we have $|K_Y| = (\bar{r} \circ \bar{f})^*| - K_{Z'}| + 2 \sum E_i$. Since $-K_{Z'}$ has no base point by Lemma \[5.2\] we see that $Y$ is a minimal surface with $c_Y^2 = 14$ and $\chi = 8$. By \[15\] Lemma 3.1 and 

$$(r' \circ \bar{r})(\bar{B}_0)I_1 \equiv (r' \circ \bar{r})(\bar{B}_0)I_2 \equiv (r' \circ \bar{r})(\bar{B}_0)\Delta_0 \equiv 0 \mod 4,$$

there exists a unique free lifting to $\tilde{Y}$ of the action by $G$ on $W$. Let $X = Y/G$ be the quotient of $Y$ by the induced free action by $G$ on $Y$. Then by \[17\] Theorem 1 or \[7\] (ii) in Theorem A, the surface $X$ is a minimal surface with $c_X^2 = 2\chi - 1$, $\chi = 4$, and $\text{Tors}(X) \simeq \mathbb{Z}/2$. Thus we have the following:

**Proposition 13.** Let $W = \Sigma_d$ be the Hirzebruch surface of degree $d = 0$ or 2. Let $r' : Z' \to W$ be the blowing-up at two points $w_1$ and $w_2 = \ell|W(w_1)$, where $\ell|W$ is the involution of $W$ given in Remark \[1\] and $w_1$, a point outside the fixed locus of $\ell|W$. Let $\bar{r} : \tilde{Z} \to Z'$ be the blowing-up at two points $w'_1$ and $w'_2 = \ell|Z'(w'_1)$, where $\ell|Z'$ is the induced involution of $Z'$, and $w'_1$, a point infinitely near to $w_1$. Put $e_i = r'^{-1}(w_i)$ and $e'_i = \tilde{r}^{-1}(w'_i)$ for each $i = 1, 2$, and assume that the configuration of $w_i$’s and $w'_i$’s satisfies all the three conditions in Lemma \[5.1\]. Let $\bar{B}_0$ be a reduced member of $\bar{A}_d^\perp (3 \sum e_i + 3 \sum e'_i)$ that has at most negligible singularities, satisfies $\bar{B}_0 \cap \sum r_i^{-1}(e_i) = \emptyset$, and passes no fixed point of the induced action on $\tilde{Z}$ by $G = \langle \ell|W \rangle$. Let $\tilde{Y}$ be the canonical resolution of the double cover of $\tilde{Z}$ branched along $\bar{B}_0 + \sum r_i^{-1}(e_i)$, and $\hat{f} : \tilde{Y} \to \tilde{Z}$, the natural projection. Let $p : \hat{Y} \to Y$ be the blowing-down of two $(-1)$-curves $E_1 = \hat{f}^{-1}(r_i^{-1}(e_1))$ and $E_2 = \hat{f}^{-1}(r_i^{-1}(e_2))$. Then there exists a unique free lifting to $\hat{Y}$ of the action by $G$ on $\hat{Z}$, and the quotient $Y/G$ of $Y$ by the induced free action is a minimal surface with $c_Y^2 = 2\chi - 1$, $\chi = 4$, and $\text{Tors} \simeq \mathbb{Z}/2$.

Our Theorem \[2\] together with Remark \[1\] and Lemma \[5.1\] says that all minimal surfaces with $c_Y^2 = 2\chi - 1$, $\chi = 4$, and $\text{Tors} \simeq \mathbb{Z}/2$ are obtained by the procedure as in the proposition above. We use the following lemma in order to show the uniqueness of the deformation type.

**Lemma 5.3.** Let $r' : Z' \to W$ and $\bar{r} : \tilde{Z} \to Z'$, $w_i \in W$ and $w'_i \in Z'$ for $i = 1, 2$, and $e_i = r'^{-1}(w_i)$ and $e'_i = \tilde{r}^{-1}(w'_i)$ for $i = 1, 2$ be the morphisms, points, and divisors respectively as in Proposition \[13\]. Then any general member $B_0$ of $\bar{A}_d^\perp (3 \sum e_i + 3 \sum e'_i)$ is non-singular and reduced. Further $h^i(O_{\tilde{Z}}(\bar{B}_0)) = 0$ holds for any positive integer $i > 0$.

**Proof.** The first assertion follows from 3) in Lemma \[5.2\]. The second assertion follows from 3) and 4) in Lemma \[5.2\] and the vanishing theorem.\[\square\]

Now let us show the uniqueness of the deformation type and the unirationality of the moduli space. For this purpose, we shall give another
description of our surface $X$. Let $r' : Z' \to W$, $\tilde{r} : \tilde{Z} \to Z'$, $w_i$, $w'_i$, $e_i$, and $e'_i$ be as in Proposition 13. Let $\Gamma_1$ and $\Gamma_2$ be as in Remark 5. We take the minimal section $\Delta_0$ and an irreducible member $\Delta_\infty \in |\Delta_0 + d\Gamma|$ satisfying $\Delta_0 \cap \Delta_\infty = \emptyset$ such that both are stable under the action by $G$. Note that if $d = 2$, such $\Delta_\infty$’s form a one-dimensional family.

The fixed locus of the action by $G$ on $\tilde{Z}$ is a set of four isolated points: \{(r' \circ \tilde{r})^{-1}(\Gamma_1 \cap \Delta_j)\}_{i=1,2, j=0,\infty}$. Let $\tilde{r} : \tilde{Z} \to Z$ be the blowing-up at these four points. For $i = 1, 2$ and $j = 0, \infty$, we define the divisors $J_{ij}$ on $\tilde{Z}$ as follows:

- if $d = 0$, then $J_{ij} = (r' \circ \tilde{r} \circ \tilde{r})^{-1}(\Gamma_1 \cap \Delta_j)$ for any $i = 1, 2$ and $j = 0, \infty$;
- if $d = 2$, then $J_{10} = (r' \circ \tilde{r} \circ \tilde{r})^{-1}(\Gamma_1 \cap \Delta_\infty)$, $J_{1\infty} = (r' \circ \tilde{r} \circ \tilde{r})^{-1}(\Gamma_1 \cap \Delta_0)$, and $J_{2j} = (r' \circ \tilde{r} \circ \tilde{r})^{-1}(\Gamma_2 \cap \Delta_j)$ for any $j = 0, \infty$.

Moreover for $i = 1, 2$ and $j = 0, \infty$, we define the divisors $\bar{I}_i$, $\bar{\Delta}_j$, $\bar{e}_i$, and $\bar{e}'_i$ on $\tilde{Z}$ by $\bar{I}_i = (r' \circ \tilde{r} \circ \tilde{r})^{-1}(\bar{I}_i)$, $\bar{\Delta}_j = (r' \circ \tilde{r} \circ \tilde{r})^{-1}(\bar{\Delta}_j)$, $\bar{e}_i = \tilde{r}^* (\tilde{r}^{-1}(e_i))$, and $\bar{e}'_i = \tilde{r}^* (e'_i)$. The four divisors $J_{ij}$’s form the set of all irreducible exceptional curves of $\tilde{r} : \tilde{Z} \to Z$. The action by $G$ on $\tilde{Z}$ lifts to one on $\tilde{Z}$. Note that $\sum J_{ij}$ gives the fixed locus of the induced action by $G$ on $\tilde{Z}$.

Now let $\bar{V} = \tilde{Z}/G$ be the quotient of $\tilde{Z}$ by the induced action by $G$, and $\bar{\phi} : \tilde{Z} \to \bar{V}$, the natural projection. Then $\bar{V}$ is smooth, and $\sum J_{ij}$ gives the ramification divisor of $\bar{\phi} : \tilde{Z} \to \bar{V}$. For $i = 1, 2$ and $j = 0, \infty$, we define the divisors $\bar{I}_{ij}$, $\bar{G}_i$, and $\bar{D}_j$ on $\tilde{V}$ by $\bar{I}_{ij} = \bar{\phi}(J_{ij})$, $\bar{G}_i = \bar{\phi}(\bar{I}_i)$, and $\bar{D}_j = \bar{\phi}(\bar{\Delta}_j)$. Moreover we define the divisors $\lambda$ and $\lambda'$ on $\tilde{V}$ by $\lambda = \bar{\phi}(\bar{e}_1) = \bar{\phi}(\bar{e}_2)$ and $\lambda' = \bar{\phi}(\bar{e}'_1) = \bar{\phi}(\bar{e}'_2)$. The divisors $\bar{I}_{ij}$’s are non-singular rational curves with selfintersection $\bar{I}_{ij}^2 = -2$. Note that $\sum \bar{I}_{ij}$ gives the branch divisor of $\bar{\phi} : \tilde{Z} \to \bar{V}$.

Let $\bar{\nu} : \bar{V} \to \bar{V}'$ be the blowing-down of the $(-1)$-curve $\lambda'$. For $i = 1, 2$ and $j = 0, \infty$, we define the divisor $\bar{I}_{ij}$ on $\bar{V}$ by $\bar{I}_{ij} = \bar{\nu}(I_{ij})$. Moreover we define the divisor $\lambda$ on $\bar{V}$ by $\lambda = \bar{\nu}(\bar{\lambda})$. The divisors $\bar{I}_{ij}$ and $\lambda$ are non-singular rational curves with $\bar{I}_{ij}^2 = -2$ and $\lambda^2 = -1$ respectively.

Let $\bar{\nu}' : \bar{V} \to \bar{V}'$ be the blowing-down of the $(-1)$-curve $\lambda$. For $i = 1, 2$ and $j = 0, \infty$, we define the divisors $I''_{ij}$, $G''_i$, and $D''_j$ on $\bar{V}'$ by $I''_{ij} = (\bar{\nu}' \circ \bar{\nu})_{*}(I_{ij})$, $G''_i = (\bar{\nu} \circ \bar{\nu})_{*}(\bar{G}_i)$, and $D''_j = (\bar{\nu} \circ \bar{\nu})_{*}(\bar{D}_j)$. The divisors $I''_{ij}$, $G''_i$, and $D''_j$ are non-singular rational curves with $I''_{ij}^2 = -2$, $G''_i^2 = -1$, $D''_j^2 = -(d+2)/2$, and $D''_j^2 = (d-2)/2$ respectively.

Let $\nu'' : \bar{V}' \to \bar{V}''$ be the blowing-down of the two $(-1)$-curves $G''_1$ and $G''_2$. For $i = 1, 2$ and $j = 0, \infty$, we define the divisors $I'''_{ij}$ and $D'''_j$ on $\bar{V}''$ by $I'''_{ij} = \nu'' (I''_{ij})$ and $D'''_j = \nu'' (D''_j)$. The divisors $I'''_{ij}$, $D'''_j$, and $D'''_j$ are non-singular rational curves with $I'''_{ij}^2 = -1$, $D'''_j^2 = -(d+2)/2$ and $D'''_j^2 = (d-2)/2$ respectively.
Let \( \nu'' : V'' \to V''' \) be the blowing-down of the two \((-1)\)-curves \( I'''_{1\infty} \) and \( I'''_{2\infty} \). For \( i = 1, 2 \) and \( j = 0, \infty \), we define the divisors \( I'''_{i0} \) and \( D'''_{ij} \) on \( V''' \) by \( I'''_{i0} = \nu''(I'''_{i0}) \) and \( D'''_{ij} = \nu''(D'''_{ij}) \). The divisors \( I'''_{i0}, D'''_{i0}, \) and \( D'''_{i\infty} \) are non-singular rational curves with \( I'''_{i0} = 0, D'''_{i0} = -1, \) and \( D'''_{i\infty} = 1 \). By \( K'''_{V''} = 8 \), we see easily that \( V''' \) is isomorphic to the Hirzebruch surface \( \Sigma_1 \) of degree 1, where \( D'''_{i0} \) and \( I'''_{i0} \) give the minimal section and the fiber class respectively. We define the point \( \nu''' \in V''' \) by \( \nu''' = \nu''(I'''_{1\infty}) \). Note that we have \( \nu''' \notin D'''_{00} \) if \( d = 0 \), and \( \nu''' \in D'''_{d0} \) if \( d = 2 \).

We put \( \nu = (\nu'' \circ \nu' \circ \nu \circ \nu) : \tilde{V} \to V''' \simeq \Sigma_1 \), and use the same symbols \( I'''_{i\infty}, G'_i, \) and \( \tilde{\lambda} \) for the total transforms to \( \tilde{V} \) of the divisors \( I'''_{i\infty} \subset V''' \), \( G'_i \subset V' \), and \( \tilde{\lambda} \subset \tilde{V} \) respectively. Note that the morphism \( \nu : V \to V''' \simeq \Sigma_1 \) is a blowing-up at six points some of which are infinitely near.

**Proposition 14.** The linear system \(| - 4K_{\tilde{V}} + \tilde{\lambda} + \tilde{\lambda}' | = | \nu^*(-4K_{V''}) - 4 \sum I'''_{i\infty} - 4 \sum G'_i - 3\lambda - 3\lambda' | \) has no base point. Let \( A_0 \) be a reduced member of \(| - 4K_{\tilde{V}} + \tilde{\lambda} + \tilde{\lambda}' | \) that has at most negligible singularities, and satisfies \( A_0 \cap \tilde{\lambda} = \emptyset \) and \( A_0 \cap \sum \tilde{I}_{ij} = \emptyset \). Let \( X \) be the canonical resolution of the double cover of \( V \) branched along \( \tilde{A}_0 + \tilde{\lambda} + \sum \tilde{I}_{ij} \), and \( h : \tilde{X} \to \tilde{V} \) the natural projection. Let \( \tilde{X} \to X \) be the blowing-down of five \((-1)\)-curves \( \tilde{h}^{-1}(\tilde{\lambda}) \), \( \tilde{h}^{-1}(\tilde{I}_{10}), \tilde{h}^{-1}(\tilde{I}_{20}), \tilde{h}^{-1}(\tilde{I}_{1\infty}), \) and \( \tilde{h}^{-1}(\tilde{I}_{2\infty}) \). Then \( X \) is a minimal surface with \( c_2^1 = 2\chi - 1, \chi = 4, \) and \( \text{Tors} \simeq \mathbb{Z}/2 \). Conversely, for any minimal surface \( X(1) \) with these invariants, there exist configuration of \( w_1 \) and \( w_1' \) and a reduced member \( \tilde{A}_0 \) as above such that the surface \( X \) constructed by this procedure is isomorphic to \( X(1) \).

**Proof.** Since \( K_{\tilde{X}} \sim \tilde{\phi}^*(K_{\tilde{V}}) + \sum J_{ij}, 2J_{ij} = \tilde{\phi}^*(\tilde{I}_{ij}), \tilde{\phi}^*(\tilde{\lambda}) = \sum \tilde{e}_i, \tilde{\phi}^*(\tilde{\lambda}') = \sum \tilde{e}'_i, \) and \( \tilde{\lambda} \sim \lambda + \lambda' \), we have

\[
\tilde{\phi}^*(-4K_{\tilde{V}} + \tilde{\lambda} + \tilde{\lambda}') \sim \tilde{r}^*((r' \circ \tilde{r})^*(-4K_W) - 3 \sum e_i - 3 \sum e'_i).
\]

By this together with

\[
\sum \tilde{I}_{ij} = \nu^*(\sum I'''_{i0}) - 2 \sum G'_i \sim 2(I'''_{10} - \sum G'_j),
\]

we obtain

\[
\tilde{r}^*\hat{\lambda}^+ + (3 \sum e_i + 3 \sum e'_i) = \tilde{\phi}^* - 4K_{\tilde{V}} + \tilde{\lambda} + \tilde{\lambda}',
\]

\[
\tilde{r}^*\hat{\lambda}^- + (3 \sum e_i + 3 \sum e'_i) = \tilde{\phi}^* - 4K_{\tilde{V}} + \tilde{\lambda} + \tilde{\lambda}' - (I'''_{10} - \sum G'_j) + \sum J_{ij}.
\]

Thus the absence of base points of \(| - 4K_{\tilde{V}} + \tilde{\lambda} + \tilde{\lambda}' | \) follows from 3) of Lemma 5.2. Let \( A_0 \in \| - 4K_{\tilde{V}} + \tilde{\lambda} + \tilde{\lambda}' \| \) follows from 3) of Lemma 5.2
member, the induced morphism, and the obtained surface respectively as in the statement. Then \( \tilde{B}_0 = r(\varphi(A_0)) \) satisfies all the conditions given in Proposition 13. Thus for this \( \tilde{B}_0 \), we obtain morphisms \( f : \tilde{Y} \to \tilde{Z} \) and \( p : \tilde{Y} \to Y \) as in Proposition 13 and a minimal surface \( Y/G \) with \( c_1^2 = 2\chi - 1, \chi = 4, \) and \( \text{Tors} \simeq \mathbb{Z}/2 \). Note that the preimage by \( \tilde{f} \) of the set \( \{(r' \circ \tilde{r})^{-1}(I_i \cap \Delta_j)\}_{i=1,2, j=0, \infty} \) is composed of exactly eight points. We denote by \( \tilde{Y} \to \tilde{Y} \) the blowing-up at these eight points. Then the morphism \( \tilde{f} : \tilde{Y} \to \tilde{Z} \) induces a generically two-to-one morphism \( \tilde{f} : \tilde{Y} \to \tilde{Z} \). Moreover, the natural free action by \( G = \langle \iota \rangle \) on \( \tilde{Y} \) lifts to one on \( \tilde{Y} \) that is compatible with the induced action by \( G \) on \( \tilde{Z} \). Thus \( \tilde{f} : \tilde{Y} \to \tilde{Z} \) induces a natural morphism \( Y/G \to \tilde{V} = \tilde{Z}/G \). Since the branch divisor of \( Y/G \to \tilde{V} = \tilde{Z}/G \) is \( \tilde{A}_0 + \lambda + \sum I_{ij} \), and \( \tilde{V} \) has no non-trivial torsion divisor, the morphism \( Y/G \to \tilde{V} \) coincides with \( \tilde{h} : \tilde{X} \to \tilde{V} \). Thus by Proposition 13, \( X \simeq Y/G \) is a minimal surface with \( c_1^2 = 2\chi - 1, \chi = 4, \) and \( \text{Tors} \simeq \mathbb{Z}/2 \). The final assertion follows from Theorem 2 and Lemma 5.1.

Remark 6. The description above for our surfaces of the case \( \chi = 4 \) is almost the same as the description in Ciliberto-Mendes Lopes [8] Section 1] of the surfaces of the non-standard case for the non-birationality of bicanonical maps (see also [6], (b) in Theorem 3.1]). We emphasize here that in the present paper we have put neither the assumption of the non-birationality of bicanonical maps nor the assumption of the absence of pencils of curves of genus 2. By the description above, it is almost clear that our surfaces coincide with those found in [8]. To be precise, however, by showing the following proposition, we shall prove that they in fact coincide.

Proposition 15. Any minimal surface \( X \) with \( c_1^2 = 2\chi - 1, \chi = 4, \) and \( \text{Tors} \simeq \mathbb{Z}/2 \) has non-birational bicanonical map. Moreover, it has no pencil of curves of genus 2.

Proof. Let \( X \) be a minimal surface with \( c_1^2 = 2\chi - 1, \chi = 4, \) and \( \text{Tors} \simeq \mathbb{Z}/2 \), and assume that \( X \) has a pencil of curves of genus 2. This pencil is rational, since \( c_1^2 \geq 2 \). Let \( \vartheta \) be a non-trivial 2-torsion divisor, and \( C \), a general member of the pencil. Then by the same method as in the proof of (i), Lemma 1.2 of [7], we see that \( |K_X + \vartheta| = |(\chi - 1)C| + D \) for a certain effective divisor \( D \) satisfying \( K_X D = 1, CD = 2, D^2 = 3 - 2\chi \), and \( \mathcal{O}_C(D) \not\cong \mathcal{O}_C(K_C) \). Since \( K_X D = 1 \), the divisor \( D \) contains an irreducible curve \( D_1 \) satisfying \( K_X D_1 = 1 \), and all other components of \( D \) are \((-2)\)-curves. Then since \( -3 \leq D_1^2 + D_1(D - D_1) = D_1 \geq K_X D_1 - (\chi - 1)CD_1 = 1 - (\chi - 1)CD_1 \), we obtain \( 0 \leq CD_1 \leq 1 \). Assume that \( CD_1 = 1 \). Then \( D - D_1 \) contains a \((-2)\)-curve \( D_2 \) such that \( CD_2 = 1 \), and so \( -2 \leq D_2^2 + D_2(D - D_2) = DD_2 \leq -2 \chi CD_2 \), hence a contradiction. Thus we have \( CD_1 = 0 \). In
this case, if we let \( D = D' + D'' \) be the decomposition of \( D \) such that \( D' \) is the sum of all the irreducible components meeting \( C \) with positive intersection number, and \( D'' \) is the sum of all the irreducible components meeting \( C \) with intersection number 0, then we have \( D_1 \subset D'' \) and \( CD' = 2 \). Since \( D' \) has at most two irreducible components and any irreducible component of \( D' \) is a \((-2)\)-curve, by using exactly the same argument as in the proof of Lemma 2.1 of [7], we obtain a contradiction. Hence the surface \( X \) has no pencil of curves of genus 2.

Thus in order to prove Proposition 15, it only remains to prove that \( X \) has non-birational bicanonical map. For this purpose, let us use the notation in Proposition 14. We have

\[
H^0(\overline{h}_*\mathcal{O}_X(2K_{\overline{X}})) = H^0(\mathcal{O}_{\overline{V}}(2(K_{\overline{V}} + \varrho))) \oplus H^0(\mathcal{O}_{\overline{V}}(2K_{\overline{V}} + \varrho))
\]

for a certain divisor \( \varrho \) with \( \overline{A}_0 + \overline{\lambda} + \sum \overline{I}_{i,j} \sim 2\varrho \).

We however have

\[
2K_{\overline{V}} + \varrho \sim \nu^*(I''_0) - \sum G_i + \tilde{\lambda},
\]

hence \( h^0(\mathcal{O}_{\overline{V}}(2K_{\overline{V}} + \varrho)) = 0 \). This implies that the bicanonical map of \( \overline{X} \) factors through the rational map of \( \overline{V} \) associated to the linear system \( |2(K_{\overline{V}} + \varrho)| \). Hence we have the assertion.

To give a proof for Theorem 3, we also need the following lemma:

**Lemma 5.4.** Let \( \overline{V} \) be the smooth surface as in Proposition 14. Then for a member \( \overline{A}_0 \in |-4K_{\overline{V}} + \overline{\lambda} + \overline{\lambda}'| \), the following equalities hold:

\[
h^0(\mathcal{O}_{\overline{V}}(\overline{A}_0)) = 29, \quad h^1(\mathcal{O}_{\overline{V}}(\overline{A}_0)) = 0, \quad h^2(\mathcal{O}_{\overline{V}}(\overline{A}_0)) = 0.
\]

**Proof.** Since we have \( \overline{\varphi}^*(\overline{A}_0) \in \overline{r^*A}_0^+ \) \((3 \sum e_i + 3 \sum e'_i)\), the equality \( h^i(\mathcal{O}_{\overline{V}}(\overline{A}_0)) = 0 \) for any \( i > 0 \) follows from Lemma 5.3. From this together with the Riemann–Roch theorem, we infer \( h^0(\mathcal{O}_{\overline{V}}(\overline{A}_0)) = 29 \).

As the first part of our proof for Theorem 3 we shall show the following:

**Lemma 5.5.** Any two minimal algebraic surfaces with \( c_1^2 = 2\chi - 1, \chi = 4, \) and \( \text{Tors} \simeq \mathbb{Z}/2 \) are equivalent under deformation of complex structures. The coarse moduli space \( \mathcal{M} \) for surfaces with these invariants is irreducible.

**Proof.** Let \( \mathcal{M} \) be the coarse moduli space for minimal surfaces \( X \)'s with \( c_1^2 = 2\chi - 1, \chi = 4, \) and \( \text{Tors} \simeq \mathbb{Z}/2 \). In what follows, for a surface \( X \) with these invariants, we denote by \([X]\) the point in \( \mathcal{M} \) corresponding to the isomorphism class of \( X \). As a reference point of \( \mathcal{M} \), let us fix a surface \( X_{(1)} \) with these invariants for which \( d = 0 \) holds, the two points \( w_1 \) and \( w_2 \) lie neither on one and the same member of \([I]\) nor on that of \([\Delta_0]\), and \( \overline{A}_0 \) is smooth. Below, we shall give an irreducible component \( \mathcal{M}_{(1)} \) of \( \mathcal{M} \)
containing \([X(1)]\), and show that for any \(X\) with these invariants we have \([X] \in \mathcal{M}(1)\) and \(X\) has the same deformation type as that of the reference surface \(X(1)\). We divide our situation into several cases according to \(d\), the configuration of \(w_1\)'s and \(w_1''\)'s, and smoothness of \(\tilde{A}_0\) for our \(X\). In what follows, \(\epsilon\) and \(\epsilon_0\) will denote positive real numbers small enough. We shall replace these numbers with smaller ones without mentioning it explicitly.

Case 1-1: the case where \(d = 0\) holds, the two points \(w_1\) and \(w_2\) lie neither on one and the same member of \(|\Gamma|\) nor on that of \(|\Delta_0|\). This case splits into two subcases: case 1-1-1 and case 1-1-2.

Case 1-1-1: the subcase of case 1-1 where \(\tilde{A}_0\) is smooth. From the point of view of description as in Proposition 13, this case corresponds to the case where \(v_i'' \notin D''_0\), \(v_i' \notin \sum I_{ij}' + \sum G_i' + \sum D_j'\), and moreover \(\tilde{A}_0\) is smooth, where we put \(v_i'' = v''(I''_0)\) and \(v_i' = \nu(\tilde{\lambda})\). Note that for all \(X\)'s of this case \(\tilde{V}\)'s have one and the same isomorphism class. Let \(\text{pr}_{\tilde{V} \times \tilde{\lambda}} : \tilde{V} \times \tilde{\lambda} \to \tilde{\lambda} \simeq \mathbb{P}^1\) be the trivial family. Let \(\text{pr}_{\tilde{V} \times \tilde{\lambda}, \tilde{V}} : \tilde{V} \times \tilde{\lambda} \to \tilde{V}\) be the first projection.

Then we can easily construct an analytic family \(\text{pr}_{\tilde{V}} : \tilde{V} \to \tilde{\lambda} \simeq \mathbb{P}^1\) together with a projection \(\text{pr}_{\tilde{V}, \tilde{V} \times \tilde{\lambda}} : \tilde{V} \to \tilde{V} \times \tilde{\lambda}\) satisfying the following condition: for each \(t \in \tilde{\lambda}\), the natural projection \(\tilde{V}_t = \text{pr}_{\tilde{V}}^{-1}(t) \to \tilde{V} = \text{pr}_{\tilde{V} \times \tilde{\lambda}}^{-1}(t)\) is the blowing-up at \(t \in \tilde{\lambda} \subset \tilde{V}\) with exceptional divisor \(\tilde{\lambda}_t\).

Let us denote by \(\tilde{\lambda}_t\) and \(\tilde{\lambda}_u\) the strict transform and the total transform by \(\tilde{V}_t = \text{pr}_{\tilde{V}}^{-1}(t) \to \tilde{V} = \text{pr}_{\tilde{V} \times \tilde{\lambda}}^{-1}(t)\) of the divisor \(\tilde{\lambda}_t\), respectively. We denote by \(\text{pr}_{\tilde{V}, \tilde{V}} : \tilde{V} \to \tilde{V}\) the composite of two projections \(\text{pr}_{\tilde{V}, \tilde{V} \times \tilde{\lambda}}\) and \(\text{pr}_{\tilde{V} \times \tilde{\lambda}, \tilde{V}}\).

Consider the divisor \(-4K_{\tilde{V}} + \text{pr}_{\tilde{V}, \tilde{V}}^*(\tilde{\lambda}) + \cup_l \tilde{\lambda}'_l\) on \(\tilde{V}\). The restriction to \(\tilde{V}_t\) of this divisor is linearly equivalent to \(-4K_{\tilde{V}_t} + \tilde{\lambda}_t + \tilde{\lambda}'_t\). Since we have \(h^1(O_{\tilde{V}_t}(-4K_{\tilde{V}_t} + \tilde{\lambda}_t + \tilde{\lambda}'_t)) = 0\) and \(h^0(O_{\tilde{V}_t}(-4K_{\tilde{V}_t} + \tilde{\lambda}_t + \tilde{\lambda}'_t)) = 29\) by Lemma 5.4, it follows that the direct image

\[ \mathcal{F}_0 = \text{pr}_{\tilde{V}, \tilde{V}}^*O_{\tilde{V}}(-4K_{\tilde{V}} + \text{pr}_{\tilde{V}, \tilde{V}}^*(\tilde{\lambda}) + \cup_l \tilde{\lambda}'_l) \]

is a locally free sheaf on \(\tilde{\lambda} \simeq \mathbb{P}^1\) of rank 29. We denote by \(\mathcal{F}_0'\) the dual sheaf of \(\mathcal{F}_0\) on \(\tilde{\lambda}\).

Let \(\text{pr}_p : \mathbb{P} = \mathbb{P}(\mathcal{F}_0') \to \tilde{\lambda}\) be the \(\mathbb{P}^{28}\)-bundle over \(\tilde{\lambda}\) associated with \(\mathcal{F}_0'\). Then \(\mathbb{P}\) is the projectivised total space of vector bundle \(\mathcal{F}_0\). We consider the Cartesian diagram

\[
\begin{array}{ccc}
\tilde{V} \times \tilde{\lambda} \times \mathbb{P} & \longrightarrow & \mathbb{P} \\
\downarrow & & \downarrow \text{pr}_p \\
\tilde{V} \times \tilde{\lambda} & \xrightarrow{\text{pr}_{\tilde{V} \times \tilde{\lambda}}} & \tilde{\lambda},
\end{array}
\]

and denote by \(\text{pr}_{\tilde{V} \times \tilde{\lambda}, \mathbb{P}} : \tilde{V} \times \tilde{\lambda} \times \mathbb{P} \to \tilde{V}, \text{pr}_{\tilde{V} \times \tilde{\lambda}, \mathbb{P}} : \tilde{V} \times \tilde{\lambda} \times \mathbb{P} \to \mathbb{P}\), and \(\text{pr}_{\tilde{V} \times \tilde{\lambda}, \mathbb{P}} :
there exists a natural non-zero global section

\[ \psi_0 \in H^0(\pr\times_{\tilde{\lambda}} \mathbb{P} \to \tilde{\lambda}) \]

on \( \tilde{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P} \) satisfying the following condition: for each open set \( U \subset \tilde{\lambda} \) such that the restriction \( \mathcal{F}_0|_U \) is trivial, the restriction \( \psi_0|_{A|_{\tilde{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P}}^{-1}(U)} \) of \( \psi_0 \) to \( \pr\times_{\tilde{\lambda}} \mathbb{P}^{-1}(U) \) is given by \( \psi_0|_{A|_{\tilde{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P}}^{-1}(U)} = \sum_{i=1}^{29} a_i \psi_i \), where \( \{\psi_1, \ldots, \psi_{29}\} \) and \( \{a_1, \ldots, a_{29}\} \) are a base of \( H^0(\mathcal{F}_0|_U) \) and its dual base respectively (note here that we have the natural isomorphism \( \pr\times_{\tilde{\lambda}} \mathbb{P}(1) \simeq \mathcal{F}_0 \)). Denote by \( \tilde{\alpha}_0 \) the divisor on \( \tilde{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P} \) defined by the section \( \psi_0 \).

For each \( u \in \mathbb{P} \), we put \( t(u) = \pr_u(u) \in \tilde{\lambda} \). Then we have the natural isomorphism \( \pr\times_{\tilde{\lambda}} \mathbb{P}^{-1}(u) \simeq V_t(u) \). Moreover, via this identification, the restriction \( \tilde{\alpha}_0|_u = \tilde{\alpha}_0|_{A|_{\tilde{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P}}^{-1}(u)} \in \mathcal{O}_{\mathbb{P}(1)}(1) \) is trivial, the restriction \( \tilde{\alpha}_0|_{A|_{\tilde{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P}}^{-1}(u)} \) is given by \( \sum_{i=1}^{29} a_i(u) \psi_i|_V(u) \).

Let \( \mathbb{P} \to \tilde{\lambda} \) be a \( \mathbb{P}^{28} \)-bundle over a non-singular rational curve \( \tilde{\lambda} \simeq \mathbb{P}^1 \), there exists a covering \( \{U'_{\mu}\}_\mu \) of \( \mathbb{P} \) by a finite number of Zariski open subsets \( U'_{\mu} \)’s satisfying the following condition: for any \( \mu \), the restriction \( \mathcal{O}_{\mathbb{P}(1)}|_{U'_{\mu}} \) is trivial, and \( U'_{\mu} \) is isomorphic to the 29-dimensional linear space \( \tilde{\mathcal{A}}^{29} \). We fix one such cover \( \{U'_{\mu}\}_\mu \), and put \( U_0' = U'_{\mu} \cap \mathbb{P}_0 \) for each \( \mu \).

Let \( \pr_{\mathcal{V},\nu} : \tilde{\mathcal{V}} \to V' \) and \( \pr_{\mathcal{V},\nu'} : \tilde{\mathcal{V}} \to V' \) be the natural projections \( \nu' \circ \nu \circ \pr_{\mathcal{V},\nu} \) and \( \tilde{\nu} \circ \pr_{\mathcal{V},\nu} \) respectively. Then since the restriction to \( \pr\times_{\tilde{\lambda}} \mathbb{P}^{-1}(U_0') \) of \( \pr\times_{\tilde{\lambda}} \mathbb{P} \mathcal{O}_{\mathbb{P}(1)}(1) \) is trivial, it follows from (13) that the restriction to \( \pr\times_{\tilde{\lambda}} \mathbb{P}^{-1}(U_0') \) of the divisor

\[ \tilde{\alpha}_0 + \pr\times_{\tilde{\lambda}} \mathbb{P} \mathcal{O}_{\mathbb{P}(1)}(1)(\sum \tilde{I}_{ij}) \]

is linearly equivalent to twice the restriction to \( \pr\times_{\tilde{\lambda}} \mathbb{P}^{-1}(U_0') \) of the divisor

\[ \pr\times_{\tilde{\lambda}} \mathbb{P} \mathcal{O}_{\mathbb{P}(1)}(1)(-2K_{V'} + \pr_{\mathcal{V},\tilde{\nu}} \widetilde{\mathcal{V}} \mathcal{O}_{\mathbb{P}(1)}(1) + \pr_{\mathcal{V},\tilde{\nu}} \widetilde{\mathcal{V}} \mathcal{O}_{\mathbb{P}(1)}(1)) \]

So let \( \tilde{\mathcal{X}}(\mu) \to \pr\times_{\tilde{\lambda}} \mathbb{P}^{-1}(U_0') \) be the double cover branched along the restriction to \( \pr\times_{\tilde{\lambda}} \mathbb{P}^{-1}(U_0') \) of the divisor (14). Composing this morphism with the projection \( \pr\times_{\tilde{\lambda}} \mathbb{P} : \tilde{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P} \to \mathbb{P} \), we obtain an analytic family.
pr}_{X^{(u)}}: \tilde{X}^{(u)} \to U^{0}_\mu. \text{ For each } u \in U^{0}_\mu, \text{ we put } \tilde{X}^{(u)} = \text{pr}_{X^{(u)}}^{-1}(u). \text{ The inverse image by } \tilde{X}^{(u)} \to \text{pr}_{\tilde{Y} \times \tilde{P}, \tilde{P}^{-1}(U^{0}_\mu)} \text{ of } \text{pr}_{\tilde{Y} \times \tilde{P}, \tilde{P}^{-1}(U^{0}_\mu)}(\cup_{t \in \tilde{A}} + \text{pr}_{\tilde{Y}, \tilde{P}^{-1}(u)}(\sum \tilde{I})^{-1}) \text{ gives a family over } U^{0}_\mu \text{ whose fiber over each } u \in U^{0}_\mu \text{ is a sum of five disjoint } (-1)-\text{curves on } \tilde{X}^{(u)}. \text{ Blowing down this family of disjoint five } (-1)-\text{curves relatively to } \text{pr}_{\tilde{X}^{(u)}}: \tilde{X}^{(u)} \to U^{0}_\mu, \text{ we obtain an analytic family } \text{pr}_{X^{(u)}} : X^{(u)} \to U^{0}_\mu. \text{ For each } u \in U^{0}_\mu, \text{ the fiber } X^{(u)} \text{ is a minimal surface with } c_1^2 = 2\chi - 1, \chi = 4, \text{ and Tors } \simeq \mathbb{Z}/2. \text{ Moreover, we infer that the corresponding point } [X^{(u)}] \text{ lies on the irreducible component of the moduli space } M. \text{ We fix one such irreducible component and denote it by } M_{(1)}. \text{ Now let } X \text{ be a minimal surface with } c_1^2 = 2\chi - 1, \chi = 4, \text{ and Tors } \simeq \mathbb{Z}/2 \text{ for which } d = 0 \text{ holds, the two points } w_1 \text{ and } w_2 \text{ lie neither on one and the same member of } |\Delta_0| \text{ on that of } |\Gamma|, \text{ and } \tilde{A}_0 \text{ is smooth. Then by Proposition 14 and the construction of } \text{pr}_{X^{(u)}} : X^{(u)} \to U^{0}_\mu \text{ above, there exist a } \mu \text{ and a } u \in U^{0}_\mu \text{ such that } X \simeq X^{(u)} \text{ holds. Since } \mathbb{P}_0 \text{ is connected, we infer that } X \text{ has the same deformation type as that of the reference surface } X_{(1)}. \text{ Moreover, we infer that the corresponding point } [X] \text{ lies on the irreducible component } M_{(1)}. \text{ Case 1-1-2: the subcase of case 1-1 where } \tilde{A}_0 \text{ is singular. Let } X \text{ be a minimal surface with } c_1^2 = 2\chi - 1, \chi = 4, \text{ and Tors } \simeq \mathbb{Z}/2 \text{ of this case. In this case, } \tilde{A}_0 \text{ has at most negligible singularities, and by Proposition 14 the linear system } |\Delta_0| \text{ has no base point. Thus by the same method as in [9] Proof of Theorem 4], we obtain an analytic family } \text{pr}_{X^{(u)}} : X \to N = \{ u \in C : |u| < \epsilon \} \text{ of minimal surfaces with } c_1^2 = 2\chi - 1, \chi = 4, \text{ and Tors } \simeq \mathbb{Z}/2 \text{ such that } X_u = \text{pr}_{X^{(u)}}^{-1}(u) \text{ is of case 1-1-1 for each } u \neq 0 \in N, \text{ and } X_0 = \text{pr}_{X^{(u)}}^{-1}(0) \simeq X. \text{ From this together with the results for case 1-1-1, we infer that } X \text{ has the same deformation type as that of the reference surface } X_{(1)}, \text{ and that the point } [X] \text{ lies on the irreducible component } M_{(1)} \text{ in the proof for case 1-1-1. Case 1-2: the case where } d = 0 \text{ holds, and the two points } w_1 \text{ and } w_2 \text{ lie on one and the same member of } |\Gamma| \text{ or } |\Delta_0|. \text{ In this case, by Remark 5, we may assume that the two points } w_1 \text{ and } w_2 \text{ lie on the member } I_1 \in |\Gamma|. \text{ Then by Lemma 5.1 we have } w_1' \notin r_{I_1}^{-1}(I_1). \text{ This case is divided into two subcases: case 1-2-1 and case 1-2-2.}
Case 1-2-1: the subcase of case 1-2 where $\tilde{A}_0$ is smooth. From the point of view of description as in Proposition 4, this case corresponds to the case where $v''_1 \not\equiv D''_0$, $v'_0 \in G'_1 \setminus \left(\sum_j I'_{ij}\right)$, and moreover $A_0$ is smooth, where we put $v'''_1 = \nu''(I'_{1\infty})$ and $v'_0 = \nu(\lambda)$. Let $X$ be a minimal surface with $c^2_1 = 2\chi - 1$, $\chi = 4$, and $\text{Tors} \simeq \mathbb{Z}/2$ of this case. Let $\epsilon$ be a positive real number small enough. We put $N = \{t \in \mathbb{C} : |t| < \epsilon\}$, and denote by $\text{pr}_{V' \times N} : V' \times N \to N$ the trivial family. Let $\text{pr}_{V^*} : V' \times N \to V'$ be the first projection.

Then we can easily construct an analytic families $\text{pr}_V : \tilde{V} \to N$, $\text{pr}_{\tilde{V}} : V \to N$ together with projections $\text{pr}_{\tilde{V}, V' \times N} : \tilde{V} \to V' \times N$, $\text{pr}_{\tilde{V}, V} : \tilde{V} \to V$ satisfying the following conditions: for each $t \in N$, the projection $\tilde{V}_t = \text{pr}_{\tilde{V}}^{-1}(t) \to V'_t = \text{pr}_{V' \times N}^{-1}(t)$ is the blowing-up at $\tilde{v}'(t)$ with exceptional divisor $\lambda'_t$, where $v' : N \to V'$ is a holomorphic section of the analytic family $\text{pr}_{V' \times N} : V' \times N \to N$; for each $t \in N$, the projection $\tilde{V}_t = \text{pr}_{\tilde{V}}^{-1}(t) \to \tilde{V}_t = \text{pr}_{\tilde{V}}^{-1}(t)$ is the blowing-up at $\tilde{v}(t)$ with exceptional divisor $\lambda_t$, where $\tilde{v} : N \to V$ is a holomorphic section of the analytic family $\text{pr}_{\tilde{V}} : \tilde{V} \to N$; $\tilde{v}(0) = \tilde{v}'(0) = \tilde{v}(\lambda'_t)$ holds, and $\tilde{v}(t) \in \lambda_t$ for any $t \in N$. Note that from the conditions above, we have in particular $\tilde{V}_0 = \tilde{V}$ and $\tilde{V}_0 = \tilde{V}$. Let us denote by $\breve{\lambda}_t$ the strict transform of $\lambda_t$ by $\tilde{v}_t = \text{pr}_{\tilde{V}}^{-1}(t) \to \tilde{V}_t = \text{pr}_{\tilde{V}}^{-1}(t)$.

Consider the divisor $-4K_{\tilde{V}} + \text{pr}_{\tilde{V}, V}^*(\cup_i \lambda_t) + \cup_i \lambda'_t$ on $\tilde{V}$. The restriction to $\tilde{V}_0 = \tilde{V}$ of this divisor is linearly equivalent to $-4K_{\tilde{V}} + \lambda + \lambda'$. Since we have $h^1(O_{\tilde{V}}(-4K_{\tilde{V}} + \lambda + \lambda')) = 0$ by Lemma 5.4, there exists a non-zero global section $\Psi \in H^0(O_{\tilde{V}}(-4K_{\tilde{V}} + \text{pr}_{\tilde{V}, V}^*(\cup_i \lambda_t) + \cup_i \lambda'_t))$ on $\tilde{V}$ satisfying the following conditions: the restriction $(\Psi)|_{\tilde{V}_0}$ to $\tilde{V}_0$ coincides with $\tilde{A}_0$, where $(\Psi)$ denotes the divisor on $\tilde{V}$ defined by the global section $\Psi$; for any $t \in N$, the restriction $(\Psi)|_{\tilde{V}_t}$ to $\tilde{V}_t$ is a reduced non-singular divisor on $\tilde{V}_t$; the divisor $(\Psi)$ does not intersects $\cup_i \lambda_t + \text{pr}_{\tilde{V}, V}^*(\sum_i I'_{ij})$, where $\text{pr}_{\tilde{V}, V} : \tilde{V} \to V'$ is the composite of three projections $\text{pr}_{\tilde{V}, \tilde{V}}$, $\text{pr}_{\tilde{V}, V' \times N}$, and $\text{pr}_{V' \times N, V'}$.

Let $\tilde{X} \to \tilde{V}$ be the double cover branched along $(\Psi) + \cup_i \lambda_t + \text{pr}_{\tilde{V}, V}^*(\sum I'_{ij})$. Then composing this morphism with the projection $\text{pr}_V : \tilde{V} \to N$, we obtain an analytic family $\text{pr}_{X} : \tilde{X} \to N$. For each $t \in N$, we put $\tilde{X}_t = \text{pr}_X^{-1}(t)$. The inverse image by $\tilde{X} \to \tilde{V}$ of $\cup_i \lambda_t + \text{pr}_{\tilde{V}, V}^*(\sum I'_{ij})$ gives a family over $N$ whose fiber over each $t \in N$ is a disjoint union of five $(−1)$-curves on $\tilde{X}_t$. Blowing down this family of five $(−1)$-curves relatively to $\text{pr}_{\tilde{X}} : \tilde{X} \to N$, we obtain an analytic family $\text{pr}_{X} : \tilde{X} \to N$. Then by the construction of $\text{pr}_{X} : \tilde{X} \to N$ above, we have $X_0 = \text{pr}_X^{-1}(0) \simeq X$, and for each $t \neq 0 \in N$, the fiber $X_t = \text{pr}_X^{-1}(t)$ is a minimal surface with $c^2_1 = 2\chi - 1$, $\chi = 4$, and $\text{Tors} \simeq \mathbb{Z}/2$ of case 1-1-1. From this together with the results for case 1-1-1, we infer that $X$ has the same deformation type as that of the reference
surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 1-2-2: the subcase of case 1-2 where $\bar{A}_0$ is singular. Let $X$ be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of this case. Then using the same argument as in case 1-1-2, we infer from the results for case 1-2-1 that $X$ has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 2-1: the case where $d = 2$ holds, and the two points $w_1$ and $w_2$ do not lie on one and the same member of $[\Gamma]$. Note that in this case we have $v'_0 \notin D'_0$ by Lemma 5.1, where we put $v'_0 = \nu(\lambda)$. This case splits into two subcases: case 2-1-1 and case 2-1-2.

Case 2-1-1: the subcase of case 2-1 where $\bar{A}_0$ is smooth. From the point of view of description as in Proposition 14, this case corresponds to the case where $v''_1 \in D''_0$, $v'_0 \notin D'_0 + \sum G'_i + \sum I'_{ij}$, and moreover $\bar{A}_0$ is smooth, where we put $v''_1 = \nu''(I'_{1\infty})$ and $v'_0 = \nu(\lambda)$. Let $X$ be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathbb{Z}/2$ of this case. Let $\epsilon$ be a positive real number small enough. We put $N = \{ t \in \mathbb{C} : |t| < \epsilon \}$, and denote by $\text{pr}_{V'' \times N} : V'' \times N \to N$ the trivial family. Let $\text{pr}_{V'' \times N, V''} : V'' \times N \to V''$ be the first projection. Let us take a holomorphic section $v''_1 : N \to V'' \times N$ satisfying the following conditions: $\text{pr}_{V'' \times N, V''}(v''_1(t)) = v''_1$ $(= \nu''(I'_{1\infty}))$ holds; $\text{pr}_{V'' \times N, V''}(v''_1(t)) \in I''_0$ for any $t \in N$; $\text{pr}_{V'' \times N, V''}(v''_1(t)) = v''_1$ if and only if $t = 0$.

Recall that the configuration corresponding to Case 1-1-1 was $v''_1 \notin D''_0$, $v'_0 \notin \sum I'_{ij} + \sum G'_i + \sum D'_j$. Thus using the holomorphic section $v''_1 : N \to V'' \times N$ above, and by the same argument as in the proof for case 1-2-1, we obtain an analytic family $\text{pr}_X : X \to N = \{ t \in \mathbb{C} : |t| < \epsilon \}$ satisfying the following conditions: $X_0 = \text{pr}_X^{-1}(0) \simeq X$; and for each $t \neq 0 \in N$, the fiber $X_t = \text{pr}_X^{-1}(t)$ is a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of case 1-1-1. From this together with the results for case 1-1-1, we infer that $X$ has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 2-1-2: the case where $d = 2$ holds, the two points $w_1$ and $w_2$ lie on
one and the same member of $|\Gamma|$. Note that in this case we have $v_0' \notin D_0'$ by Lemma 5.1, where we put $v_0' = \hat{v}(\lambda)$. In this case, by Remark 5 we may assume that the two points $w_1$ and $w_2$ lie on the member $I_1 \in |\Gamma|$. Then by Lemma 5.1, we have $w_1' \notin r^{-1}_* (I_1)$. This case is divided into two subcases: case 2-2-1 and case 2-2-2.

Case 2-2-1: the subcase of case 2-2 where $\bar{A}_0$ is smooth. From the point of view of description as in Proposition [14] this case corresponds to the case where $v_1'' \in D_0''$, $v_0' \in G_1' \setminus (\sum_j I_{1j})$, and moreover $\bar{A}_0$ is smooth, where we put $v_1'' = \nu''(I_{1\infty})$ and $v_0' = \hat{v}(\lambda)$. Note that by Lemma 5.1, we have $\bar{v}_0 \notin \tilde{\nu}_*^{-1}(G_1')$, where we put $\bar{v}_0 = \hat{v}(\lambda)$. Let $X$ be a minimal surface with $c_1^2 = 2\chi - 1, \chi = 4$, and $\text{Tors} \simeq \mathbb{Z}/2$ of this case. Let $\epsilon$ be a positive real number small enough. We put $N = \{ t \in \mathbb{C} : |t| < \epsilon \}$, and denote by $\text{pr}_{V' \times N} : V' \times N \to N$ the trivial family. Let $\text{pr}_{V' \times N} \cdot V' : V' \times N \to V'$ be the first projection. Let us take a holomorphic section $v' : N \to V' \times N$ satisfying the following conditions: $	ext{pr}_{V' \times N} \cdot v'(0)) = v_0'((\hat{v}(\lambda)))$ holds; $\text{pr}_{V' \times N} \cdot v'(t)) \in D_0' + \sum G_i + \sum I_{1j}$ if and only if $t = 0$.

Recall that the configuration corresponding to case 2-1-1 was $v_1'' \in D_0''$, $v_0' \notin D_0' + \sum G_i + \sum I_{1j}$. Thus using the holomorphic section $v' : N \to V' \times N$ above, and by the same argument as in the proof for case 1-2-1, we obtain an analytic family $\text{pr}_X : X \to N = \{ t \in \mathbb{C} : |t| < \epsilon \}$ satisfying the following conditions: $X_0 = \text{pr}_X^{-1}(0) \simeq X$; for each $t \notin \bar{N}$, the fiber $X_t = \text{pr}_X^{-1}(t)$ is a minimal surface with $c_1^2 = 2\chi - 1, \chi = 4$, and $\text{Tors} \simeq \mathbb{Z}/2$ of case 2-1-1. From this together with the results for case 2-1-1, we infer that $X$ has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 2-2-2: the subcase of case 2-2 where $\bar{A}_0$ is singular. Let $X$ be a minimal surface with $c_1^2 = 2\chi - 1, \chi = 4$, and $\text{Tors} \simeq \mathbb{Z}/2$ of this case. Then using the same argument as in case 1-1-2, we infer from the results for case 2-2-1 that $X$ has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Now that we have the results for all the eight cases 1-1-1, . . . , 2-2-2, we have the assertion. \hfill \square

Note that from the proof above, we see that the morphism $\mathbb{P}_0 \to \mathcal{M}$ in the proof for case 1-1-1 is dominant.

Now let us prove Theorem 8.

**Proof of Theorem 8**

Let $\mathbb{P}_0 \to \mathcal{M}$ be the morphism $u \mapsto [X_u^{(\mu)}]$ given in the proof (for case 1-1-1) of Lemma 5.5. Recall that we have $X_u^{(\mu_1)} \simeq X_u^{(\mu_2)}$ if $u \in U_{\mu_1}^0 \cap U_{\mu_2}^0$. So in what follows, we abbreviate $X_u^{(\mu)}$ to $X_u$. Since $\mathbb{P}_0 \to \mathcal{M}$ is a dominant
admits a birational morphism $r \Sigma$. We refer the readers for details of the following argument. Let $\epsilon$ and $Z$ be the configuration of $w_i$'s those for $X$'s of case 1-1-1.

Let $\text{Aut}(W)$ be the group of analytic automorphisms of $W \simeq \Sigma_0$, and $\iota_W$, the involution of $W$ as in Proposition 13. Let $\text{Aut}(W, \iota_W, \{w_i\})$ be the subgroup of $\text{Aut}(W)$ consisting of all $\sigma \in \text{Aut}(W)$ satisfying $(\iota_W) \circ \sigma = \sigma \circ (\iota_W)$ and $\sigma\{w_{i_1}, w_{i_2}\} = \{w_{i_1}, w_{i_2}\}$. Since $\text{Aut}(W, \iota_W, \{w_i\})$ acts naturally on the sets $\{w_{i_1}, w_{i_2}\}$ and $\{\Delta_0, \Delta_\infty, \Gamma_1, \Gamma_2\}$, we have corresponding group homomorphisms $\text{Aut}(W, \iota_W, \{w_i\}) \rightarrow \mathfrak{S}_2$ and $\text{Aut}(W, \iota_W, \{w_i\}) \rightarrow D_4$, where $\mathfrak{S}_2$ and $D_4$ denote the symmetric group of degree 2 and the dihedral group of degree 4 respectively. It is easy to see that the product $\text{Aut}(W, \iota_W, \{w_i\}) \rightarrow \mathfrak{S}_2 \times D_4$ of these two morphisms is an isomorphism.

Let $Z'/G$ be the quotient of the surface $Z'$ by the natural action by the group $G = \langle \iota_W \rangle$. Then the quotient $Z'/G$ has four nodes, and the natural morphism $V \rightarrow Z'/G$ gives the minimal desingularization of $Z'/G$. Thus via the diagram $V \rightarrow Z'/G \leftarrow Z' \rightarrow W$, the action by $\text{Aut}(W, \iota_W, \{w_i\})$ on the surface $W$ induces one on the pair $(V, \lambda)$. Let $\mathfrak{S}_2 = \langle \iota_W \rangle \rightarrow \text{Aut}(W, \iota_W, \{w_i\})$ be the natural inclusion. Since $\mathfrak{S}_2$ acts trivially on $V$ via this inclusion, we obtain a natural action by $\text{Aut}(W, \iota_W, \{w_i\})/\mathfrak{S}_2 \simeq D_4$ on the pair $(V, \lambda)$. This action on $(V, \lambda)$ induces one on $\mathbb{P}_0$. Remark however implies that we have $X_{u_1} \simeq X_{u_2}$ if and only if two points $u_1 \in \mathbb{P}_0$ and $u_2 \in \mathbb{P}_0$ belong to the same orbit of the action by $\text{Aut}(W, \iota_W, \{w_i\})/\mathfrak{S}_2$ on $\mathbb{P}_0$. Thus by $\sharp D_4 = 8$, we see that for any $u_0 \in \mathbb{P}_0$ there exist at most eight $u \in \mathbb{P}_0$'s satisfying $X_u \simeq X_{u_0}$. Hence we have the assertion. \hfill \Box

6 Appendix

Proof of Proposition 3 Let us prove Proposition 3. The method we employ here is the same as the one used in [13] Proof of Lemma 4.5, to which we refer the readers for details of the following argument. Let $Z \subset \mathbb{P}^n$, where $n \geq 4$, be a non-degenerate surface satisfying the assumptions in Proposition 3 and $Z' \rightarrow Z$, its minimal desingularization. Since we have $\deg Z < 2n - 2$, and $Z' \rightarrow Z$ is given by a complete linear system $|D'|$, the surface $Z'$ is a rational surface not isomorphic to $\mathbb{P}^2$. Thus, for an integer $d$, the surface $Z'$ admits a birational morphism $r : Z' \rightarrow \Sigma_d = Z'_0$ onto the Hirzebruch surface $\Sigma_d$ of degree $d$. Let $D'_0$ be a general member of the linear system $r_*|D'|$, and $\varepsilon'_i$'s, the total transforms to $Z'$ of the $(-1)$-curves appearing at the blowings up in $Z' \rightarrow Z$. Then we have $D' \sim r^*D'_0 - \sum_{i=1}^s m_i \varepsilon'_i$, where $m_i$'s, $s \in \mathbb{Z}$.
Lemma 6.1. There exists an $r : Z' \to Z'_0$ as above such that for any $i$ s, the equality $m_i = 1$ holds.

Proof of Lemma. Note that the general member $D'$ is a non-singular irreducible curve on $Z'$. If $h^1(O_{D'}(D')) > 0$, then by Clifford’s theorem on special divisors, we have $D'^2 \geq 2(h^0(O_{D'}(D')) - 1)$, which contradicts $n \geq 4$. Thus we have $h^1(O_{D'}(D')) = 0$. From this together with the natural short exact sequence $0 \to O_{Z'} \to O_{Z'}(D') \to O_{D'}(D') \to 0$ and the Riemann–Roch theorem, we infer

$$\chi(O_{Z'}(D')) = n + 1,$$

$$D'K_{Z'} = D'^2 + 2(1-\chi(O_{Z'}(D'))) = 1-n,$$ and $$(K_{Z'}+D'/2)D' = (3-n)/2 < 0.$$ Thus by Cone Theorem, we find that if $Z'$ is not the Hirzebruch surface, then there exists a $(-1)$-curve $\varepsilon'$ on $Z'$ satisfying $(K_{Z'}+D'/2)\varepsilon' < 0$. Since $Z' \to Z$ contracts no $(-1)$-curve, we obtain $D'\varepsilon' = 1$. Let $r' : Z' \to Z''$ be the blowing-down of $\varepsilon'$. We put $D'' = r'_*(D')$. If $Z''$ is not the Hirzebruch surface, then the same argument as above ensures the existence of a $(-1)$-curve $\varepsilon''$ on $Z''$ satisfying $D''\varepsilon'' = 1$ (for the detail, see [13, Lemma 4.4]). We can repeat the same steps until we obtain the Hirzebruch surface. \[\square\]

In what follows, we assume our $r$ satisfies the condition in the lemma above, hence $D' \sim r^*D'_0 - \sum_{i=1}^s \varepsilon'_i$. We put $D'_0 \sim a\Delta_0 + b\Gamma$, where if $d = 0$, we chose $\Delta_0$ and $\Gamma$ in such a way that $b \geq a$. Then by $\chi(O_{Z'}(D')) = n + 1$ and $D'^2 = n + 1$, we obtain the following three equalities:

$$n + 1 = D'(D' - K_{Z'})/2 + 1 = (a + 1)(b - ad/2) + a - s + 1,$$

$$n + 1 = 2a(b - ad/2) - s,$$

$$0 = D'^2 - \chi(O_{Z'}(D')) = (a - 1)(b - ad/2) - (a + 1).$$

(15) (16)

Note that we have $b - ad/2 \geq a$ if $d \neq 1$, and that $b - ad/2 \geq a/2$ if $d = 1$. Thus by (15) and (16), we find $a = 2$, $b = d + 3$, and $s = 11 - n$, hence $D' \sim -K_{Z'} + r^*\Gamma$. Since $|D'_0|$ has no fixed component, we obtain $d \leq 3$. \[\square\]

References

[1] ASHIKAGA, T., KONNO, K. Algebraic surfaces of general type with $c_1^2 = 3p_g - 7$, Tohoku. Math. J., 42 (1990), no. 4, 517–536.

[2] BARTALESI, A., CATANESI, F. Surfaces with $K^2 = 6$, $\chi = 4$, and with torsion. Conference on algebraic varieties of small dimension (Turin, 1985), Rend. Sem. Mat. Univ. Politec. Torino, 1986, Special Issue (1987), 91–110.
[3] Barth, W., Peters, C., Van de Ven, A. Compact Complex Surfaces, *Springer-Verlag*, (1984).

[4] Brieskorn, E. Die Auflösung der rationalen Singularitäten holomorpher Abbildungen, *Math. Ann.*, 178 (1968), 255–270.

[5] Catanese, F., Debarre, O. Surfaces with $K^2 = 2$, $p_g = 1$, $q = 0$, *J. Reine. Angew. Math.*, 395 (1989), 1–55.

[6] Ciliberto, C., Francia, P., Mendes Lopes, M. Remarks on the bicanonical map for surfaces of general type, *Math. Z.*, 224 (1997), 137–166.

[7] Ciliberto, C., Mendes Lopes, M. On surfaces of general type with $K^2 = 2\chi - 2$ and non-trivial torsion, *Geom. Dedicata*, 66 (1997), no. 3, 313–329.

[8] Ciliberto, C., Mendes Lopes, M. On regular surfaces of general type with $p_g = 3$ and non-birational bicanonical map, *J. Math. Kyoto Univ.*, 40 (2000), no. 1, 79–117.

[9] Horikawa, E. On deformations of quintic surfaces, *Invent. Math.*, 31 (1975), 43–85.

[10] Horikawa, E. Algebraic surfaces of general type with small $c_1^2$. I, *Ann. of Math.*, 104 (1976), no. 2, 358–387.

[11] Horikawa, E. Algebraic surfaces of general type with small $c_1^2$. III, *Invent. Math.*, 47 (1978), no. 3, 209–248.

[12] Horikawa, E. Algebraic surfaces of general type with small $c_1^2$. IV, *Invent. Math.*, 50 (1978/79), no. 2, 103–128.

[13] Konno, K. Even canonical surfaces with small $K^2$, I, *Nagoya Math. J.*, 129 (1993), 115–146.

[14] Miyaoka, Y. Tricanonical Maps of Numerical Godeaux Surfaces, *Invent. Math.*, 34 (1976), no. 2, 99–111.

[15] Murakami, M. The torsion group of a certain numerical Godeaux surface, *J. Math. Kyoto Univ.*, 41 (2001), 323–333.

[16] Murakami, M. Minimal algebraic surfaces of general type with $c_1^2 = 3$, $p_g = 1$ and $q = 0$, which have non-trivial 3-torsion divisors, *J. Math. Kyoto Univ.*, 43 (2003), no. 1, 203–215.
[17] MURAKAMI, M. A bound for the orders of the torsion groups of surfaces with $c_1^2 = 2 \chi - 1$, *Math. Z.*, **253** (2006), no. 2, 251–262.

[18] NAGATA, M. On rational surfaces I, *Mem. Coll. Sci. Univ. Kyoto, Ser. A*, **32** (1960), 351–370.

[19] REID, M. Surfaces with $p_g = 0$, $K^2 = 1$, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **25** (1978), no. 1, 75–92.

Masaaki Murakami  
University of Bayreuth, Lehrstuhl Mathematik VIII  
Universitaetsstrasse 30, D-95447 Bayreuth, Germany  
Masaaki.Murakami@uni-bayreuth.de