Signatures of many-body localization in steady states of open quantum systems

I. Vakulchyk, 1, 2 I. Yusipov, 3 M. Ivanchenko, 3 S. Flach, 1 and S. Denisov 4, 3

1 Center for Theoretical Physics of Complex Systems, IBS, Daejeon 34051, Republic of Korea
2 Basic Science Program, Korea University of Science and Technology (UST), Daejeon 34113, Republic of Korea
3 Department of Applied Mathematics, Lobachevsky University, Nizhny Novgorod, 603950, Russia
4 Institute of Physics, University of Augsburg, Universitätsstraße 1, 86159 Augsburg, Germany

Many-body localization (MBL) is a result of the balance between interference-based Anderson localization and many-body interactions in an ultra-high dimensional Fock space. It is usually expected that dissipation is blurring interference and destroying that balance so that the asymptotic state of a system with an MBL Hamiltonian does not bear localization signatures. We demonstrate, within the framework of the Lindblad formalism, that the system can be brought into a steady state with non-vanishing MBL signatures. We use a set of dissipative operators acting on pairs of connected sites (or spins), and show that the difference between ergodic and MBL Hamiltonians is encoded in the imbalance, entanglement entropy, and level spacing characteristics of the density operator. An MBL system which is exposed to the combined impact of local dephasing and pairwise dissipation evinces localization signatures hitherto absent in the dephasing-ousted steady state.

Many-body localization (MBL) is an extension of Anderson localization [1] into the world of many-body systems [2–3]. There is a spectrum of definitions/quantifiers of this multi-faceted phenomenon aimed to highlight peculiar properties of MBL systems, e.g. the absence of conductivity [3] (even in the infinite temperature limit [2]), slow logarithmic growth of the entanglement entropy after an interaction quench [4–7], the existence of an extensive set of local integrals of motion [8], and specific spectral properties of MBL Hamiltonians [9, 10]. There is a class of quantifiers which address properties of a single (eigen)state of an MBL system such as short-range correlations [11], low spatial entanglement entropy [12–14] and large spatial fluctuations of local observables [15].

Recently MBL became the subject of experiments with ultra-cold atoms [16–17]. One of the important questions concerns the impact of interactions with the environment and the fate of MBL on the large time scales. This question has been addressed recently in a series of papers [15–20], where the action of the environment was modeled with a Lindblad master equation and a set of local dephasing operators. The answer confirmed intuition: Dissipation eventually destroys localization – the steady state density operator is the normalized identity – but on the way to this state systems with MBL and non-MBL Hamiltonians behave notably differently (e.g., stretched exponential vs exponential relaxations of some observables) [21].

Can we distinguish between MBL and non-MBL (ergodic) Hamiltonians by inspecting steady states of the corresponding systems when they are subjected to some physically relevant dissipation? It was recently realized that dissipation is a full-fledged generator of evolution, no less complex and diverse than the unitary evolution generated by Hamiltonians [22–24]; e.g., dissipative mechanisms can be used to drive many-body systems into highly entangled pure states [22].

In this Letter we show that a controllable dissipation, when applied to a system with an MBL Hamiltonian, can sculpt an asymptotic state which bears detectable signatures of localization. These signatures can be revealed by using the population imbalance [18–20] (a quantity measured in experiments [16–25]), the operator spatial entanglement entropy [23–25] (a generalization of the pure state spatial entanglement entropy to open systems), and the mean spectrum gap ratio [19] of the steady state density operator.

Model. We study a conventional MBL model, an open-ended chain of $N$ (an even number) sites occupied by $N/2$ spinless fermions. The fermions interact when occupying neighboring sites and are subject to a random on-site potential $h_l$, $l = 1,\ldots,N$. The model Hamiltonian has the form

$$H = -J \sum_{l=1}^{N} \left( c_l^\dagger c_{l+1} + c_{l+1}^\dagger c_l \right) + U \sum_{l=1}^{N} n_l n_{l+1} + \sum_{l=1}^{N} h_l n_l,$$  \hspace{1cm} (1)

where $c_l^\dagger$ ($c_l$) creates (annihilates) a fermion at site $l$, and $n_l = c_l^\dagger c_l$ is the local particle number operator. Values $h_l$ are drawn from an uncorrelated uniform distribution on the interval $[-h,h]$. For $J = U = 1$ (our choice here) this system undergoes a many-body localization transition when $h > h_{MBL} \approx 3.6$ [11]. By using the Jordan-Wigner transformation, the system can be mapped onto a model of $N$ spins confined to the manifold $S^z = \sum_{i=1}^{N} s_i^z = 0$ [27]. This relation allows us to implement the time-evolving block decimation (TEBD) scheme generalized to matrix product operators [28] and propagate the model system to its steady state. As the initial state we use $\rho(0) = |\psi_0\rangle\langle\psi_0|$, $|\psi_0\rangle = |1010\ldots10\rangle$.

The dissipation is captured with a master equation [29],

$$\dot{\rho}(t) = \mathcal{L}_\rho(t) = -i [H, \rho(t)] + \sum_{s=1}^{M} \gamma_s \left[ A_s \rho(t) A_s^\dagger - \frac{1}{2} \{ A_s^\dagger A_s, \rho(t) \} \right],$$  \hspace{1cm} (2)
A physical interpretation of such dissipation is a chain coupled to a superfluid, which serves as a bath of Bogoliubov excitations; Raman transitions couple an antisymmetric state, by the operator $(c_l - c_{l+1})$, to the excitations which then decay into a symmetric state, through the action of $(c_l^\dagger + c_{l+1}^\dagger)\gamma_l$. With periodic boundary conditions and in the absence of disorder $\hbar = 0$ and interaction, $U = 0$, these dissipators drive the system into a uniform condensate (a dark state of all dissipators). For open boundary conditions and in the presence of the interactions and disorder, the condensate is no longer an eigenstate of the Hamiltonian so that the asymptotic state $\rho_\infty$ is not pure and not homogeneous in general.

To reveal the difference between MBL ($\hbar > h_{\text{MBL}}$) and ergodic ($\hbar < h_{\text{MBL}}$) Hamiltonians [1], we calculate three quantifiers of $\rho_\infty$. We do this either (i) by numerically finding $\rho_\infty$ as a kernel of the Lindblad generator $\mathcal{L}$ [31] ($N \leq 10$), or (ii) by propagating the matrix product representation of $\rho(t)$, until the quantifiers saturate to their asymptotic values ($10 \leq N \leq 32$) [28]. Note that our aim here is not to explore all possible regimes and parameter dependencies but to present a ‘proof of concept’. Therefore, for the following consideration we set $\gamma = 0.1$.

**Imbalance.** The imbalance is defined as

$$I(t) = \frac{N_o(t) - N_e(t)}{N/2},$$

where $N_o$ ($N_e$) is the number of fermions in odd (even) sites. This characteristics was measured in the recent experiments to quantify the MBL [10, 24] (note that due to particle loss time-dependent denominators were used).

When dissipation is non-Hermitian, the asymptotic imbalance $I = \lim_{t \to \infty} I(t)$ is a real-valued random variable $I_s$, different for different disorder realizations, $s \in \{1, 2, \ldots, M\}$. In the absence of any statistical theory of this quantity, we consider $\{I_s\}$ as a set of independent and identically distributed (iid) random variables with a probability density function (pdf) $P(I)$. In the ergodic regime $\hbar < h_{\text{MBL}}$, a configuration of site populations $n^s_i = Tr[\rho_\infty n_i^{\dagger} n_i]$ can be modeled as a random $N$-partition of the unit interval $x^s = \{x_1^s, x_2^s, \ldots, x_N^s\}$, $\sum x_i^s = 1$, uniformly distributed over the subspace $A = \{x : \forall x_k \leq 2/N\}$ ('no more than one particle per site'). The sampling results for $I_{\text{mod}}[N] = \sum_{l=1}^{N} (-1)^l x_l$.
are in a good agreement with the sampling of the model for $h = 3$ [33], see Fig. 1(a). The only notable difference is in the tail regions: While the stochastic pdf has unbounded tails, the pdf for the model (1-3) is always confined to the interval $[-1/2, 1/2]$.

Being the sum of $N$ iid random variables, $\mathcal{I}[N]$ is subject to the Central Limit Theorem [34]. Then the scaling $N^{-\beta} P(N^{\beta} \mathcal{I}[N])$ with $\beta = 0.5$ is expected. The variance of the sampled pdf $P(\mathcal{I})$ yields the exponent $\beta = 0.55$ in the ergodic regime, see inset in Fig. 1(a). For large disorder we find $\beta \approx 0.8$, Figs. 1(b-c), which indicates a transition into the MBL phase. The narrowing of the pdf can be explained by the presence of short-range anticorrelations which tie neighboring sites, a marked feature of MBL states [9].

**Operator-space entanglement entropy (OSSE).** This quantity was introduced by Prosen and Pizorn [35] as an operator generalization of the spatial entanglement entropy (defined for pure states). OSSE was implemented for the density operator in order to monitor the relaxation of an open MBL system to the infinite temperature state [20]. To calculate this quantity, one should split the chain into two (equal in our case) parts and calculate the Schmidt decomposition of the density operator, $\varrho = \sum_{k} \sqrt{\mu_{k}} C_{k} \otimes D_{k}$, where the operators $C_{k}$ ($D_{k}$) act non-trivially on the left (right) half only and form a complete Hilbert-Schmidt basis in the corresponding subspace. The normalized coefficients $\tilde{\mu}_{k}$ define the entropy value $S^{k} = -\sum_{k} \tilde{\mu}_{k} \log_{2} \tilde{\mu}_{k}$. When the state is pure, $S^{k}$ is twice the standard entanglement entropy [36].

In the ergodic phase $h = 3$ we find that for $N \geq 10$ the averaged (over the disorder) OSSE $\bar{S}^{k}(t)$ saturates to $S^{k}(1_{\text{HF}})$, which is the entropy corresponding to the state maximally mixed over the half-filled subspace $\mathcal{H}_{L}$ [37], Fig. 2(a). This implies an effective thermalization of the system: At variance to the case of local dephasing [20], the individual realization entropy values are not all identical to $S^{k}(1_{\text{HF}})$ but distributed around it, see inset in Fig. 2(a) [as we show in the next section, a single ergodic steady state is far from being maximally mixed]. The initial short-time evolution of the entropy follows the Hamiltonian path. It is a linear growth, which in the absence of the dissipation will saturate to the Page value $S_{\text{Page}}^{k} \approx N - 1$, corresponding to the entropy of a typical random pure state uniformly ‘smeared’ over $\mathcal{H}_{L}$. After time $t \gtrsim \gamma^{-1}$ the contribution of the dissipative part of the generator $\mathcal{L}$ starts to the dynamics becomes tangible and eventually brings the entropy down to an asymptotic value near $S^{k}(1_{\text{HF}}) \ll S_{\text{Page}}^{k}$.

In the MBL phase, the averaged OSSE saturates to values below $S^{k}(1_{\text{HF}})$, see Figs. 2(b-c). This can be explained by generalizing the argument used in Ref. [11] for the Hamiltonian case. While in the ergodic phase all - even distant - sites (spins) are ‘tied’ by the conservation of the total particle number (total spin), in the MBL phase the correlations are short-ranged and restricted by the localization length. Therefore, the entanglement entropy is lower in the MBL phase. The relaxation dynamics of the OSSE in the strong localization limit is marked by a logarithmic growth, $S^{k}(t) \simeq g \log_{2}(t)$, a feature found before with local dephasing [18,20]. The prefactor $g$, as conjectured in Ref. [20], is related to the scaling of the spectral gap of the generator $\mathcal{L}$ with the size of the system. If the gap scales as $N^{\nu}$, then $g \simeq \frac{1}{\nu}$, see Fig. 2(c). This gives the value of the scaling exponent $\nu \simeq 2.5$. Whether this is a universal value belonging to one of the universality classes, discussed in Ref. [37], is an interesting question which demands a more detailed analysis.

**Ratio of consecutive level spacing for the steady state**

![Figure 2](image-url)

**FIG. 2.** Averaged operator-space entanglement entropy $\bar{S}^{k}(t)$ of the density operator $\varrho(t)$. Dashed lines are the values of the entropy for the maximally mixed (over the half-filled subspace) states [37]. The dotted line on panel (c) is $\frac{1}{2} \log_{2}(t) + \text{const.}$ Inset: The probability density function of the entropy of individual disorder realizations, for $h = 3$ and $N = 12$. Other parameters are as in Fig. 1.
density operator. There is a genetic link between changes in the spectral statistics of many-body Hamiltonians and ergodic-MBL transitions [9][10]. According to the quantum chaos theory, Poisson and Wigner-Dyson distributions of the energy spacing \( \delta_j = E_{j+1} - E_j \) correspond to regular (integrable) and chaotic (non-integrable) quantum systems [39]. Similarly, we can expect Poisson and Wigner-Dyson distributions for MBL and ergodic Hamiltonians, respectively [10]. However, these indicators assume the uniform level density which is rarely the case with physical Hamiltonians [40]. To circumvent this problem, Oganesyan and Huse considered the distribution of the ratios \( r_j = \min[\lambda_j, \lambda_j^{-1}] \), \( \lambda_j = \delta_j/\delta_{j-1} \), which do not depend on the local density of states [9]. It follows that spectral averages of \( r \) yield \( r_{\text{Poission}} \approx 0.386 \) for Poisson random variables, \( r_{\text{GOE}} \approx 0.536 \) for Gaussian orthogonal (GOE), and \( r_{\text{GUE}} \approx 0.603 \) for Gaussian unitary (GUE) ensembles [41]. The ergodic-to-MBL transitions correspond to the passage from \( r_{\text{poission}} \) to \( r_{\text{GOE}} \).

In another context, Prosen and Žnidarič proposed to quantify the non-equilibrium steady state density operators in terms of their level spacing distributions [42]. They found that the transition from integrability to non-integrability [43] corresponds to the Poisson-to-GUE transition in the distribution of the level spacing of the density operator. Here we follow this idea, but implement the averaged ratio of consecutive level spacing \( \bar{r} \) instead.

We find that in the ergodic phase the spectrum of the steady state density operator displays \( \bar{r} \) values close to \( r_{\text{GUE}}, \) while in the limit of strong localization its value approaches \( r_{\text{Poission}}, \) see Fig. 3a). This correspondence improves with increasing \( N \). The structure of the density matrices \( \varrho_\infty \) is notably different in the ergodic and strong localization regimes, see Figs. 3b,c): While in the ergodic phase matrices exhibit a well-developed off-diagonal structure and thus a relatively high purity and interference, in the deep MBL regime they have near diagonal structure, with a few ‘hot spots’ (a similar structure was found before in the context of dissipative single-particle localization [45]).

**Pairwise dissipation on top of local dephasing.** Consider a Lindblad generator \( \mathcal{L} = \mathcal{L}_{\text{MBL}} + \gamma_{\text{deph}} \mathcal{L}_{\text{deph}} + \gamma_{\text{pair}} \mathcal{L}_{\text{pair}} \), where \( \mathcal{L}_{\text{MBL}} = -i[H_{\text{MBL}},\cdot] \) and the two next terms are dissipative Liouvillians corresponding to local dephasing and pairwise dissipation respectively. In the limit \( \gamma_{\text{pair}} = 0, \) any whatever small but finite dephasing \( \gamma_{\text{deph}} \neq 0 \) will eventually bring the system into the maximally mixed state with no MBL signatures. Assume now that \( \gamma_{\text{deph}} \|\mathcal{L}_{\text{deph}}\| \ll \|\mathcal{L}_{\text{MBL}}\| \), where \( \| \cdot \| \) is a suitable operator norm [46] defined on the set of, e.g., matrix product operator (MPO) states [47], which serve a proper basis for weakly-entangled mixed states. By adding pairwise dissipation \( \gamma_{\text{deph}} \|\mathcal{L}_{\text{deph}}\| < \gamma_{\text{pair}} \|\mathcal{L}_{\text{pair}}\| \ll \|\mathcal{L}_{\text{MBL}}\| \), it is possible to create a new steady state, with the corresponding density operator bears the signatures of the MBL (though to the degree dependent on the relative values of \( \gamma_{\text{deph}} \) and \( \gamma_{\text{pair}} \)) [48]. This conjecture is based on the stability of many-body dissipative systems with no faster than linear (in time) growth of the support of initially localized operators [49].

**Discussion.** We proposed three quantitative identifiers of MBL in open systems. The imbalance statistics is accessible in experiments [16][24], but requires studying systems of different sizes. The operator-space entanglement entropy indicates differences between phases both in the asymptotic limit and during the relaxation towards it. The level spacing of the asymptotic density operator \( \varrho_\infty \) bridges MBL and quantum chaos theory [39][42]. The operator provides complete information on properties of the system in the asymptotic limit (including values of
all three identifiers); however, its numerical resolution is possible for relatively small systems, \( N \leq 10 \). The TEBD propagation is useful in case we want to explore the relaxation of the system to its steady state. To address the steady state directly, it is more advantageous to use recently developed variational methods \([54]\). MBL steady states naturally fulfill the conditions imposed on the matrix-product operators (MPOs) so that the MPO ansatz \([47]\) should work well in this case.

The considered regular pairwise dissipation is perhaps not the best choice to create an MBL steady state. Such dissipation tries to build a long-range entanglement in the system \([22]\), and in this sense it does not favor localization. The states we observed are the result of the antagonistic competition between the unitary MBL dynamics and dissipation. However, this is the only physically reasonable \([51]\) type of non-Hermitian dissipation, preserving the number of particles, which we found in the recent literature. Future studies could consider the incorporation of the disorder into local rates \( \gamma_l \). This idea leads to an intriguing question of creation MBL states by dissipative means solely, without Hamiltonian disorder. Disordered pairwise dissipation acquires relevance in the context of recent experiments with dissipatively coupled exciton-polariton condensate arrays \([52]\).

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