Emergence of reflectionless scattering from linearizations of integrable PDEs around solitons

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Abstract
We present four examples of integrable partial differential equations (PDEs) of mathematical physics that—when linearized around a stationary soliton—exhibit scattering without reflection at all energies. Starting from the most well-known and the most empirically relevant phenomenon of the transparency of one-dimensional bright bosonic solitons to Bogoliubov excitations, we proceed to the sine-Gordon, Korteweg–de Vries, and Liouville’s equation whose stationary solitons also support our assertion. The proposed connection between integrability and reflectionless scattering seems to span at least two distinct paradigms of integrability: S-integrability in the first three cases, and C-integrability in the last one. We argue that the transparency of linearized integrable PDEs is necessary to ensure that they can support the transparency of stationary solitons in the original integrable PDEs. As contrasting cases, the analysis is further extended to cover two non-integrable systems: a sawtooth-Gordon and a $\phi^4$ model.

Keywords: solitons, reflectionless scattering, Korteweg–de Vries equation, Gross–Pitaevskii equation, sine-Gordon equation, Liouville’s equation, Bogoliubov–de Gennes equation

(Some figures may appear in colour only in the online journal)

1. Introduction

It is from the field of fiber optics that we know that small excitations of a soliton of the nonlinear Schrödinger (NLS) equation—also called the Gross–Pitaevskii (GP) equation in
other contexts—can penetrate the original soliton without any reflection [1]. An analogue of the effect, the transparency of the bosonic Bogoliubov–de Gennes (BdG) equation, was later identified [2] for solitons in Bose condensates [3, 4] as well.

Potentials that are transparent at all energies have been studied extensively since the 1933 discovery of the Pöschl–Teller potential [5]

\[ V(x) = -\frac{1}{2}n(n + 1) \text{sech}^2(x) \]  

with integer \(n\) for the time-independent Schrödinger equation. Several other classes of physically relevant linear differential equations have also been found to support reflectionless potentials [6–10] (see also book [11] for more references).

Reflectionless potentials also appear in the context of finding exact solutions to integrable nonlinear partial differential equations (PDEs) with the inverse scattering transform. The process of obtaining exact solutions to such S-integrable systems is greatly simplified if the corresponding Lax Liouvillian is transparent. Regardless of transparency, the bound states of Lax Liouvillians will also manifest as solitons in the original nonlinear PDE [12]. The reflectionless property of the solitonic BdG Liouvillian—a linearized NLS equation—is not related to a Lax Liouvillian however. This implies that an independent connection between integrability and reflectionless scattering may be at work.

We will begin our study by examining the BdG equation in detail, from which we may formulate a conjecture on the nature of this independent connection between integrability and reflectionless scattering. Three more examples will follow—involving the sine-Gordon (sG), Korteweg–de Vries (KdV), and Liouville’s equations—that confirm this connection. For a contrasting study, the analysis will also be extended to two non-integrable nonlinear PDEs: a sawtooth-Gordon equation and a \(\phi^4\) model.

2. Linearization of attractive NLS equation

Consider an attractive NLS equation (also called the GP equation in the context below) describing the mean-field dynamics of an ensemble of \(N\) one-dimensional bosons with attractive \(\delta\)-interactions

\[ i\hbar \partial_t \psi = \left( -\frac{\hbar^2}{2m} \partial_x^2 - gN |\psi|^2 \right) \psi, \quad (2) \]

where \(g\) is the coupling constant and \(m\) is the particle mass. \(\sqrt{N} \psi(x, t)\) constitutes a mean-field approximation for the bosonic quantum field \(\hat{\psi}(x, t)\), and the normalization condition is \(\int_{-\infty}^{\infty} dx |\psi(x, t)|^2 = 1\) accordingly. Equation (2) is known to support solitons [12] that have been successfully observed in experiments [3, 4].

An example of a two-soliton collision between a small moving soliton and a large stationary one is shown in figure 1. Assuming that the moving soliton is sufficiently small, this collision can be approximately described by a BdG linearization around the larger soliton as

\[ \psi(x, t) \approx \bar{\psi}(x, t) + \delta \psi(x, t). \]

Let the larger soliton be

\[ \bar{\psi}(x, t) = \bar{\phi}(x) \exp\left( -i\mu_0 t/\hbar \right), \]
that originates from a stationary solitonic solution
\[ \hat{\phi}(x) = (2\ell)^{-1/2} \text{sech}(x/\ell), \]
of the time-independent NLS equation
\[ \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - gN |\phi|^2 \right) \phi = \mu_0 \phi. \]
In the equations above, \( \ell = \hbar^2/(mgN) \) is the size of the soliton, and \( \mu_0 = -m(gN)^2/(8\hbar^2) \) is its chemical potential. Direct substitution into equation (2) then produces to first order the BdG equation
\[ i\hbar \delta \psi_i = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - 2gN |\bar{\psi}|^2 \right) \delta \psi_i - gN\bar{\psi}^2 \delta \psi^* i. \] (3)
As demonstrated by the simulation shown in figure 1, the BdG equation above properly supports the transparency of the larger soliton to the smaller one. This implies that the stationary scattering solutions of the equation must exhibit transparency as well.
Let us now verify the simulation in figure 1 by displaying the scattering solutions to the 
BdG equation (3) explicitly. To simplify the formulas that follow, we will use a system of 
units with \( \hbar = m = \ell = 1 \). The NLS equation thus becomes

\[
i \psi_t = \left( -\frac{1}{2} \frac{\partial^2}{\partial \xi^2} - 2 \left| \psi \right|^2 \right) \psi,
\]

and the solitonic solution is

\[
\psi(x, t) = \left( \frac{1}{\sqrt{2}} \right) \operatorname{sech} x \cdot \exp(\imath \xi/2),
\]

with \( \mu_0 = -1/2 \). Using the same linearization of

\[
\psi(x, t) \approx \tilde{\psi}(x, t) + \delta \psi(x, t),
\]

we find the time dependent BdG equation

\[
i \delta \psi_t = \left( -\frac{1}{2} \frac{\partial^2}{\partial \xi^2} - 4 \left| \tilde{\psi} \right|^2 \right) \delta \psi - 2 \tilde{\psi}^2 \delta \psi^*,
\]

that has scattering solutions of

\[
\delta \psi(x, t) = \left( u_k(x) e^{-\imath \xi t} + \left( v_k(x) \right)^* e^{\imath \xi t} \right) e^{-\imath \mu_0 t},
\]

where the two-component wavefunctions \((u_k(x), v_k(x))^T\) are the positive energy eigenstates

of a BdG Liouvillian

\[
\hat{H} = \begin{pmatrix}
-\frac{1}{2} \frac{\partial^2}{\partial \xi^2} - 2 \operatorname{sech}^2 x + \frac{1}{2} & -\operatorname{sech}^2 x \\
\operatorname{sech}^2 x & \frac{1}{2} \frac{\partial^2}{\partial \xi^2} + 2 \operatorname{sech}^2 x - \frac{1}{2}
\end{pmatrix}, \tag{4}
\]

These eigenstates have the form [1]

\[
\begin{pmatrix}
u_k(x) \\
v_k(x)
\end{pmatrix} \propto \left( 1 + \imath k^{-1} \tanh^2 x \right)^2 \exp(\imath k x), \tag{5}
\]

and satisfy the eigenvalue equation

\[
\hat{H} \begin{pmatrix}
u_k \\
v_k
\end{pmatrix} = \epsilon(k) \begin{pmatrix}
u_k \\
v_k
\end{pmatrix},
\]

\[
\epsilon(k) = \frac{1}{2} k^2 - \mu_0 = \frac{1}{2} k^2 + \frac{1}{2},
\]

and \(-\infty < k < +\infty^3\). As expected, the scattering solutions in equation (5) show no reflection

off the soliton, at all energies.

A direct calculation of the scattering solutions is not the only way to verify that the BdG

equation (3) supports scattering without reflection however. An elegant algebraic explanation

is also provided by connecting \( \hat{H} \) to a scatterer-free Liouvillian

\[
\text{A complete solution of the time-dependent Bogoliubov–de Gennes equation, including decomposition of the initial state into a sum over the scattering solutions, and the non-scattering solutions associated with broken symmetries, is presented in [2].}
\]

4
through an intertwining relationship

$$\hat{H} \hat{A} = \hat{A} \hat{H}_0,$$  \hspace{1cm} (7)

where the intertwiner $\hat{A}$ is represented by

$$\hat{A} = \begin{pmatrix} \hat{\bar{F}} & \hat{\bar{G}} \\ \hat{\bar{G}} & \hat{\bar{F}} \end{pmatrix},$$

$$\hat{\bar{F}} = \partial_x^2 + (1 - 2 \tanh x) \partial_x^4 + (\tanh x - 1)^2 \partial_x^2$$

$$+ (1 - \text{sech}^2 x - 2 \tanh x) \partial_x + \tanh^2 x,$$

$$\hat{\bar{G}} = -\text{sech}^2 x \left( \partial_x^2 + \partial_x + 1 \right).$$  \hspace{1cm} (8)

Now we can observe that the eigenstates $w_k(x) \equiv (u_k(x), v_k(x))^T$ of $\hat{H}$ may be obtained from the eigenstates $w_{k,0}(x)$ of $\hat{H}_0$ through the map

$$w_k(x) \propto \hat{A} w_{k,0}(x).$$

The spectra of $\hat{H}$ and $\hat{H}_0$ are therefore identical up to the kernel of $\hat{A}$.

Since $\hat{H}_0$ is a differential operator with constant coefficients, and $\hat{A}$ is a differential operator with coefficients that tend to a constant value as $x \to \pm \infty$, the plane-wave eigenstates $w_{k,0}(x) = (1, 0)^T \exp(ikx)$ of $\hat{H}_0$ \(^5\) will give rise to a single-plane-wave asymptotic behavior of the eigenstates of $\hat{H}$, such that $w_k(x) \propto \exp(ikx)$ as $x \to \pm \infty$ too. This result is precisely scattering without reflection, and it is fully consistent with the explicit scattering solutions in (5).

The intertwining relationship (7) and its associated supersymmetric (SUSY) algebra have been studied extensively; a comprehensive review can be found in [13]. The presence of an intertwiner connecting a reflectionless Liouvillian $\hat{H}$ to its asymptotically constant and translationally invariant form $\hat{H}_0$ is a generic mathematical mechanism that underlies reflectionless scattering at all energies [7]. Indeed, for all four examples of integral nonlinear PDEs presented in this paper, it is possible to start from a reflectionless Liouvillian candidate $\hat{H}$, find its asymptotic form $\hat{H}_0$, solve the intertwining relation (7) directly for the intertwiner, then construct the scattering solutions of $\hat{H}$ by applying the intertwiner to the eigenstates of $\hat{H}_0$.

3. Emergence of reflectionless scattering

Through numerical simulation, explicit scattering solutions, and an intertwining relationship, the preceding section has confirmed that a linearization of the NLS equation—the BdG equation—admits scattering without reflection, but it has not resolved the question of why the

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\(^4\) The intertwiner (8) is not unique; in particular, the intertwining property will be preserved when the intertwiner $\hat{A}$ is multiplied from the left by any polynomial of $\hat{H}$ and from the right by any polynomial of $\hat{H}_0$. It appears, however, that the intertwiner (8) is the simplest member of its family.

\(^5\) The negative energy eigenstates of $\hat{H}_0$, $w_{k,0}^{-}(x) = (0, 1)^T \exp(ikx)$, lead to an identical set of scattering solutions $\delta \psi(x, t)$ for the BdG equation (3), just with the roles of $u$ and $v$ interchanged. We discard these solutions as redundant.
BdG equation should be reflectionless. Given the rarity of both integrable systems and reflectionless potentials, it would be natural to suspect a connection between the two, yet the reflectionless property of the BdG equation is not of the familiar type associated with a Lax Liouvillian. The BdG equation is merely the linearization of the NLS equation, and not the associated scattering problem that one would construct for the NLS under an inverse scattering transform.

To understand the connection between integrability and reflectionless scattering for the BdG equation, we must first differentiate solitons from solitary waves in the current context. While solitons are also solitary waves—both can propagate without dispersion—solitons possess the additional ability to penetrate each other without loss of identity. This unique property enshrines solitons as a hallmark of integrable systems and distinguishes them from solitary waves that can also exist in non-integrable systems.

The ability of small solitons to penetrate large ones without reflection is thus a direct consequence of the integrability of the system. From physical intuition, we may reasonably expect this transparency to be preserved even if the underlying integrable PDE were linearized, provided that the small solitons are small enough in the appropriate sense. If a large soliton is required to be transparent to a small soliton of any size and velocity in the linearized equation, and the small soliton can be represented as a wave packet that is a superposition of scattering solutions, then we can infer that the large soliton should also be transparent to incident plane waves of any wavelength. Stated more precisely, we conjecture that the linearization of an integrable PDE around a soliton should produce scattering without reflection at all energies. Let us test this conjecture now with three integrable equations.

4. Linearization of sG equation

Linearization of the sG equation

\[ u_{xx} - u_t = \sin u \]

may at first appear ill-defined since the size of the solitons here is quantized, so they can never be small. Nonetheless, the linearized sG equation must still be transparent in order to support the ability of small moving breathers to penetrate a stationary kink-soliton in the form of \[ \bar{v}(x) = 4 \text{arctan } e^x. \]

Consider now an approximate solution that consists of the soliton and a small, time-periodic perturbation,

\[ u(x, t) \approx \bar{u}(x) + \delta u(x) e^{-i\omega t}. \]

To the first order of perturbation theory, the excitation \( \delta u(x) \) will obey

\[ \left(-\partial_x^2 - 2 \text{sech}^2 x\right) \delta u = E \delta u, \quad (9) \]

with \( E = \omega^2 - 1 \), yet this is nothing else but the celebrated Pöschl–Teller equation (1) with \( n = 1 \) whose scattering solutions correspond to pure transmission without reflection [5]. That the Pöschl–Teller Hamiltonian, \( \hat{H} \equiv -\partial_x^2 - 2 \text{sech}^2 x \), is reflectionless can easily be verified by intertwining it, in the same manner as equation (7), with its scatterer-free counterpart \( \hat{H}_0 = -\partial_x^2 \) using the intertwiner \( \hat{A} = -\partial_x + \tanh x \) [6]. The corresponding scattering solutions

\[ \delta u(x) \propto \hat{A} \exp(ikx) = (-ik + \tanh (x)) \exp(ikx) \]

with \( \omega = \pm \sqrt{k^2 + 1} \) are also well known.
5. Linearization of KdV equation

It is tempting to suggest that the equations for small excitations around solitons of integrable PDEs are candidates for new, previously unknown instances of scattering without reflection. To that end, we have found that the KdV equation

\[ u_t - 6u u_x + u_{xxx} = 0, \]

can be linearized around the stationary soliton

\[ \bar{u}(x) = -2 \text{sech}^2 x + 2/3 \]

using the approximate solution

\[ u(x, t) \approx \bar{u}(x) + \delta u(x, t) \]

to generate \( \delta u_t = \hat{H} \delta u \), where

\[ \hat{H} = -\partial_x^3 + 6\bar{u} \partial_x + 6\bar{u}^2. \]

As expected, the Liouvillian \( \hat{H} \) is reflectionless, which can be verified by intertwining it, as in equation (7) again, with a scatterer-free Liouvillian:

\[ \hat{H}_0 = -\partial_x^3 + 4\partial_x \]

using the intertwiner

\[ \hat{A} = \partial_x^2 - 4 \tanh x \partial_x^2 + \left( 4 - 8 \text{sech}^2 x \right) \partial_x^2 + 8 \text{sech}^2 x \tanh x \partial_x. \]

The scattering solutions can be obtained from the intertwiner above; they are \( \delta u(x, t) = \delta u(x) \exp(it\omega) \), with

\[ \delta u(x) \propto \hat{A} \exp(i k x) = k \left[ 8 \text{sech}^2(x)(k + \tanh(x)) + k \left( k^2 + 4ik \tanh(x) - 4 \right) \right] \exp(ikx), \]

and \( \omega = k^3 + 4k \).

6. Linearization of Liouville’s equation

The integrability of the NLS, sG, and KdV equations studied above all originate from inverse scattering, and thus fall under the paradigm of S-integrability [12]. Let us test our conjecture now on Liouville’s equation, which is instead C-integrable—by a change of variables. Liouville’s equation is

\[ u_{xx} - u_t = e^{-u}, \]

and it can be reduced to a linear PDE through an appropriate change of variables [14]. Its most general solution has also been obtained in [15], using elementary methods. We take the stationary soliton

\[ \bar{u}(x) = -\ln\left( 2 \text{sech}^2(x) \right), \]
that belongs to a broad class of solutions [16],
\[ u(x, t) = -\ln(-8f'(x + t)g'(x - t)) \times \text{sech}^2(f(x + t) + g(x - t) + d), \]
where \( f(\xi_+) \) and \( g(\xi_-) \) are arbitrary functions of one variable, and \( d \) is an arbitrary constant.\(^6\) The stationary soliton in (11) can be recovered by choosing \( f(\xi_+) = \xi_+/2, g(\xi_-) = -\xi_-/2, \) and \( d = 0. \)

Let us perturb the soliton in (11) as
\[ u(x, t) = \bar{u}(x) + \delta u(x)e^{i\omega t}. \]
Performing this linearization leads to
\[ \left(-\partial_x^2 - 2 \text{sech}^2 x\right) \delta u = E \delta u, \]
with \( E = \omega^2 \) and \( \omega = k^2, \) which is the Pöschl–Teller equation (1) with \( n = 1, \) so there will again be no reflection of the scattering waves.

Physically, the reflectionless property of the linearized Liouville’s equation is necessary to ensure that small localized packets can penetrate the main soliton without reflection. For example, a rightward moving packet will be given by expression (12) with \( f(\xi_+) = \xi_+/2, g(\xi_-) = -\xi_-/2 + \delta g(\xi_- + t_0), \) and \( d = 0, \) where \( \delta g(\xi_-) \) is a small packet localized at \( \xi_- = 0 \) and \( t_0 \) is the moment of collision between the packet and the main soliton.

7. Integrability and transparency

The four nonlinear PDEs that have been treated so far—BdG, sG, KdV, and Liouville’s equation—are all integrable systems confirming our conjecture that the linearization of an integrable PDE around a soliton should produce scattering without reflection at all energies. Although the preceding analysis is far from a rigorous proof of the conjecture, the evidence so far does offer strong support for it.

At this point it is natural to ask if the converse statement of our conjecture is also true: Are all PDEs with a transparent linearization also integrable? Should this proposition hold true, it would point to the possibility of producing novel integrable PDEs from reflectionless equations by inverting the linearization procedure. In order to answer this question, we will examine the behavior of two non-integrable systems under the same linearization scheme that has been utilized so far.

8. Linearization of a sawtooth-Gordon equation

The first non-integrable system to be examined is a sawtooth-Gordon system, defined by
\[ u_{xx} - u_{tt} = F(u), \]
\[ F(u) = \frac{1}{4} u - \left[ \frac{1}{4} (u + 2) \right]. \]

\(^6\) The csch-function featured in [16] can be recovered by choosing \( d = c + i\pi/2, \) where \( c \) is another arbitrary constant. The conventional text-book solution in [12] is a particular case of (12), obtained in the limit where \( f(\xi^+), g(\xi^-), \) and \( c \) simultaneously tend to zero.
where \([\ldots]\) denotes the floor function. This equation admits a stationary solution of

\[
\tilde{u}(x) = \begin{cases} 
2e^{x/2} & \text{for } x < 0, \\
4 - 2e^{-x/2} & \text{for } x \geq 0.
\end{cases}
\]

Moving kink and anti-kink solutions can be generated from the stationary solution above through Lorentz transformations. Numerical simulations of collisions between a moving kink and an anti-kink solution result in the annihilation of both entities and the production of radiative ripples that confirm this sawtooth-Gordon equation is indeed non-integrable. Using a linearization of

\[
u(x, t) \approx \tilde{u}(x) + \delta \phi(x)e^{\omega t},
\]

we find that the perturbation will obey

\[
\left(-\partial_x^2 - \delta(x)\right)\delta \phi = E \delta \phi,
\]

with \(E = \omega^2 - \frac{1}{4}\), which is just a generic scattering problem with a \(\delta\)-function potential and is well known to produce reflections at all finite energies. This result is consistent with the converse of our conjecture, but let us consider another non-integrable PDE before drawing a conclusion.

9. Linearization of a \(\phi^4\) model

The second non-integrable system we will examine is a \(\phi^4\) model with the equation of motion

\[
\phi_{xx} - \phi_{tt} = 2\phi^3 - 2\phi,
\]

that admits the stationary solution

\[
\tilde{\phi}(x) = \tanh(x).
\]

Lorentz transformations of the stationary solution above can produce moving kink and anti-kink solitary wave solutions. The behavior of kink and anti-kink collisions in the \(\phi^4\) model is rich with detail \([17]\). For the current study however, we only confirm the non-integrability of the system by numerically simulating a kink and anti-kink collision. With the kink and anti-kink each moving at half the speed of light, their collision results in the reflection of both entities and the creation of outward radiating ripples. Continuing now with a linearization of

\[
\phi(x, t) \approx \tilde{\phi}(x) + \delta \phi(x)e^{\omega t},
\]

we find that the perturbation will obey

\[
\left(-\partial_x^2 - 6 \sech^2 x\right)\delta \phi = E \delta \phi,
\]

with \(E = \omega^2 - 4\). This is a Pöschl–Teller potential with \(n = 2\), and is thus reflectionless.

The existence of a reflectionless equation that originates from the linearization of a non-integrable PDE clearly contradicts the converse of our conjecture. This greatly diminishes the prospect of generating integrable PDEs from reflectionless equations by inverting the linearization procedure.
10. Summary and outlook

Based on a close study of the BdG equation—a linearization of an attractive NLS equation—we conjecture that the linearization of physically relevant integrable PDEs around stationary solitons should produce linear problems that demonstrate scattering without reflection at all energies. This prediction is verified in the sG, KdV, and Liouville’s equations. The last case indicates that this phenomenon spans two paradigms of integrability: S-integrability for the NLS, sG, and Kortweg-de Vries equations, and C-integrability for Liouville’s equation. The applicability of our conjecture towards both S-integrable and C-integrable systems suggests that the transparency of linearized integrable PDEs arises from integrability itself.

We propose that the nature of this connection is as follows: the transparency of linearized PDEs may be necessary to ensure that they correctly predict the transparency of large solitons to small solitons, small breathers, or small packets (for Liouville’s equation) at the level of the original integrable nonlinear PDEs. It remains to be verified that the observed transparency persists for multi-soliton solutions and for other integrable PDEs in general.

By extending our analysis to two non-integrable systems, where linearizations around stationary solutions produced both reflective and reflectionless equations, we found that the converse of our conjecture did not hold. As reflectionless equations can arise from the linearization of both integrable and non-integrable PDEs, it is unlikely that an inverse linearization procedure could be used to reliably generate integrable PDEs. While such a procedure may yet produce interesting candidates for integrable PDEs, this line of thought will have to await further study.

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