A COMPARISON THEOREM FOR MW-MOTIVIC COHOMOLOGY

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Abstract. Let $k$ be an infinite perfect field. We prove that $H^{n,n}_{MW}({\text{Spec}}(L),\mathbb{Z}) = K^{n}_{MW}(L)$ for any finitely generated field extension $L/k$ and any $n \in \mathbb{Z}$.

Contents

Introduction 1
1. MW-motivic cohomology 2
2. Main theorem 10
References 17

Introduction

This paper is the fourth of a series of papers ([1], [2] and [4]) devoted to the study of MW-motivic cohomology, which is a generalization of ordinary motivic cohomology. Our main purpose here is to compute the MW-motivic cohomology group of a field in bidegree $(n,n)$, namely the group $H^{n,n}_{MW}(L,\mathbb{Z})$. In [2, Theorem 4.2.3], we defined a graded ring homomorphism $\Phi : K^{*}_{MW} \rightarrow \bigoplus_{n \in \mathbb{Z}} H^{n,n}_{MW}$ where the left-hand side is the unramified Milnor-Witt $K$-theory sheaf constructed in [7, §3] and the right-hand side is the Nisnevich sheaf associated to the presheaf $U \mapsto H^{n,n}_{MW}(U,\mathbb{Z})$. The homomorphism $\Phi$ is obtained via a morphism of sheaves $\mathcal{G}^{n}_{\wedge} \rightarrow H^{n,n}_{MW}$ and the right-hand side has the property to be strictly $\mathbb{A}^{1}$-invariant [2, Proposition 1.2.11, Theorem 3.2.9]. It follows that $\Phi$ is then the universal morphism described in [7, Theorem 3.37]. In this article, we prove that $\Phi$ is an isomorphism. This can be checked on finitely generated field extensions of the base field $k$ ([7, Theorem 1.12]) and thus our main theorem takes the following form.

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Theorem. Let $L/k$ be a finitely generated field extension with $\text{char}(k) \neq 2$. Then, the homomorphism of graded rings

$$\Phi_L : \bigoplus_{n \in \mathbb{Z}} K_n^\text{MW}(L) \to \bigoplus_{n \in \mathbb{Z}} H_{\text{MW}}^{n,n}(L, \mathbb{Z}).$$

is an isomorphism.

The isomorphism in the theorem generalizes the result on (ordinary) motivic cohomology in the sense that the diagram commutes

$$\begin{align*}
\bigoplus_{n \in \mathbb{Z}} K_n^\text{MW}(L) &\xrightarrow{\Phi_L} \bigoplus_{n \in \mathbb{Z}} H_{\text{MW}}^{n,n}(L, \mathbb{Z}) \\
\bigoplus_{n \in \mathbb{N}} K_n^M(L) &\to \bigoplus_{n \in \mathbb{N}} H^{n,n}(L, \mathbb{Z})
\end{align*}$$

where the vertical homomorphisms are the “forgetful” homomorphisms and the bottom map is the isomorphism produced by Nesterenko-Suslin-Totaro. Unsurprisingly, our proof is very similar to theirs but there are some essential differences.

For instance, the complex in weight one, denoted by $\tilde{\mathbb{Z}}(1)$, admits an epimorphism to $K_1^\text{MW}$ paralleling the epimorphism $\mathbb{Z}(1) \to K_1^M$. However, we are not able to prove directly that the kernel of the epimorphism $\tilde{\mathbb{Z}}(1) \to K_1^\text{MW}$ is acyclic. We are thus forced to compute by hand its cohomology at the right spot in Proposition 2.6. This result being obtained, we then prove that $\Phi$ respects transfers for finitely generated field extensions. This is obtained in Theorem 2.8 using arguments essentially identical to [6, Lemma 5.11] or [8, Lemma 9.5].

The paper is organized as follows. In Section 1, we review the basics of MW-motivic cohomology needed in the paper, adding useful results. For instance, we prove a projection formula in Theorem 1.4 which is interesting on its own. In Section 2, we proceed with the proof of our main theorem, starting with the construction to a left inverse of $\Phi$. We then pass to the proof that $\Phi$ is an isomorphism in degree 1, which is maybe the most technical result of this work. As already mentioned above, we then conclude with the proof that $\Phi$ respects transfers, obtaining as a corollary our main result.

Conventions. The schemes are separated of finite type over some perfect field $k$ with $\text{char}(k) \neq 2$. If $X$ is a smooth connected scheme over $k$, we denote by $\Omega^n_{X/k}$ the sheaf of differentials of $X$ over $\text{Spec}(k)$ and write $\omega_{X/k} := \det \Omega^n_{X/k}$ for its canonical sheaf. In general we define $\omega_{X/k}$ connected component by connected component. We use the same notation if $X$ is the localization of a smooth scheme at any point. If $k$ is clear from the context, we omit it from the notation. If $f : X \to Y$ is a morphism of (localizations of) smooth schemes, we set $\omega_f = \omega_{X/k} \otimes f^*\omega_{Y/k}$. If $X$ is a scheme and $n \in \mathbb{N}$, we denote by $X^{(n)}$ the set of codimension $n$ points in $X$.

1. MW-motivic cohomology

The general framework of this article is the category of finite MW-correspondences as defined in [1, §4]. We briefly recall the construction of this category for the reader’s convenience. If $X$ and $Y$ are smooth connected schemes over $k$, we say that a closed subset $T \subset X \times Y$ is admissible if its irreducible components (endowed with their reduced structure) are finite and surjective over $X$. The set $\mathcal{A}(X,Y)$ of
which is characterized by the property

\[ \gamma \] is the cokernel of the morphism

\[ \gamma : \] on \( \text{Sm} \). Motivic cohomology.

1.1. \( \text{Induced by the unit in } \text{Sm} \)

be endowed with a (unique) structure of a sheaf with MW-transfers (\[ \gamma \]). However, one can show that the sheaf associated to \( \gamma \) is also a \( \text{sheaf with MW-transfers} \).

\[ \text{However, one can show that the sheaf associated to } \text{F} \text{ is a sheaf with MW-transfers, } \text{if } \gamma \text{ is a sheaf in this topology. Usually, we consider either the Zariski or the Nisnevich topology on } \text{Sm}. \text{ Interestingly, the representable presheaves } \gamma(X) \text{ are Zariski sheaves with MW-transfers (\cite{1}, Example 5.12). However, one can show that the sheaf associated to } \text{F} \text{ can be endowed with a (unique) structure of a sheaf with MW-transfers (\cite{2}, Proposition 1.2.11)}. \text{Note that it is easy to check that if } \text{F} \text{ is a } \tau\text{-sheaf with MW-transfers, then } \text{Hom}(\gamma(X), F) \text{ is also a } \tau\text{-sheaf with MW-transfers.}

1.1. Motivic cohomology. Let \( \mathbb{Z}\{1\} \) be the Zariski sheaf with MW-transfers which is the cokernel of the morphism

\[ \gamma(k) \to \gamma(\mathbb{G}_{m,k}) \]

induced by the unit in \( \mathbb{G}_{m,k} \). For any \( q \in \mathbb{Z} \), we consider next the Zariski sheaf with MW-transfer \( \mathbb{Z}\{q\} \) defined by

\[ \mathbb{Z}\{q\} = \ \begin{cases} \mathbb{Z}\{1\} \otimes q & \text{ if } q \geq 0, \\ \text{Hom}(\mathbb{Z}\{1\} \otimes q, \gamma(k)) & \text{ if } q < 0. \end{cases} \]

Let now \( \Delta^* \) be the cosimplicial object whose terms in degree \( n \) are

\[ \Delta^n = \text{Spec}(k[t_0, \ldots, t_n]/(\sum t_i - 1)) \]
and with usual face and degeneracy maps. For any presheaf \( F \in \Sh_k \), we obtain a simplicial presheaf \( \text{Hom}(\zeta(\Delta^n), F) \) whose associated complex of presheaves with MW-transfers is denoted by \( C_n^{\text{sing}}(F) \). If \( F \) is further a \( \tau \)-sheaf with MW-transfers, then \( C_n^{\text{sing}}F \) is a complex of sheaves with MW-transfers. In particular, \( \tilde{Z}(q) := C_n^{\text{sing}}\tilde{Z}(q) \) is such a complex and we have the following definition.

**Definition 1.1.** For any \( p, q \in \mathbb{Z} \) and any smooth scheme \( X \), we set

\[
\text{H}^{p,q}_{\text{MW}}(X, \mathcal{Z}) = \text{H}^{p}_{\text{Zar}}(X, \tilde{Z}(q)).
\]

**Remark 1.2.** In [2, §3.2.13, Definition 3.3.5], the motivic cohomology groups are defined using the complexes associated to the simplicial Nisnevich sheaves with MW-transfers constructed from the Nisnevich sheaves with transfers associated to the presheaves \( \tilde{Z}(q) \). The two definitions coincide by [4, Corollary 4.0.5].

The complexes \( C_n^{\text{sing}}\tilde{Z}(q) \) are in fact complexes of Zariski sheaves of \( K_{\text{MW}}^0(k) \)-modules ([1, §5.3]), and it follows that the MW-motivic cohomology groups are indeed \( K_{\text{MW}}^0(k) \)-modules. These modules are by construction contravariantly functorial in \( X \). Moreover, for any \( p, q \in \mathbb{Z} \), we have a homomorphism of \( K_{\text{MW}}^0(k) \)-modules

\[
\text{H}^{p,q}_{\text{MW}}(X, \mathcal{Z}) \to \text{H}^{p,q}(X, \mathcal{Z})
\]

where the latter denotes the ordinary motivic cohomology group of \( X \), with \( \text{H}^{p,q}(X, \mathcal{Z}) = 0 \) for \( q < 0 \) and the \( K_{\text{MW}}^0(k) \)-module structure on the right-hand side is obtained via the rank homomorphism \( K_{\text{MW}}^0(k) \to \mathbb{Z} \) ([1, §6.1]).

Even though MW-motivic cohomology is defined a priori only for smooth schemes, it is possible to extend the definition to limits of smooth schemes, following the usual procedure (described for instance in [1, §5.1]). In particular, we can consider MW-motivic cohomology groups \( \text{H}^{p,q}_{\text{MW}}(L, \mathcal{Z}) \) for any finitely generated field extension \( L/k \). We will use this routinely in the sequel without further comments.

1.2. **The ring structure.** The definition of MW-motivic cohomology given in [2, Definition 3.3.5] immediately yields a (bigraded) ring structure on MW-motivic cohomology

\[
\text{H}^{p,q}_{\text{MW}}(X, \mathcal{Z}) \otimes \text{H}^{p',q'}_{\text{MW}}(X, \mathcal{Z}) \to \text{H}^{p+p',q+q'}_{\text{MW}}(X, \mathcal{Z})
\]

fulfilling the following properties.

1. The product is (bi-)graded commutative in the sense that

\[
\text{H}^{p,q}_{\text{MW}}(X, \mathcal{Z}) \otimes \text{H}^{p',q'}_{\text{MW}}(X, \mathcal{Z}) \to \text{H}^{p+p',q+q'}_{\text{MW}}(X, \mathcal{Z})
\]

is \((-1)^{pp'}((-1)^{qq'})\)-commutative. In particular, \( \text{H}^{0,0}_{\text{MW}}(X, \mathcal{Z}) \) is central and the \( K_{\text{MW}}^0(k) \)-module structure is obtained via the ring homomorphism

\[
K_{\text{MW}}^0(k) = \text{H}^{0,0}_{\text{MW}}(k, \mathcal{Z}) \to \text{H}^{0,0}_{\text{MW}}(X, \mathcal{Z}).
\]

2. The homomorphism \( \text{H}^{*,*}_{\text{MW}}(X, \mathcal{Z}) \to \text{H}^{*,*}(X, \mathcal{Z}) \) is a graded ring homomorphism.
1.3. A projection formula. In this section, we prove a projection formula for finite surjective morphisms having trivial relative bundles. Let then $f : X \to Y$ be a finite surjective morphism between smooth connected schemes, and let $\chi : \mathcal{O}_X \to \omega_f = \omega_{X/k} \otimes f^*\omega_{Y/k}$ be a fixed isomorphism. Recall from [1, Example 4.17] that we have then a finite MW-correspondence $\alpha := \alpha(f, \chi) : Y \to X$ defined as the composite

$$K^\text{MW}_0(X) \simeq K^\text{MW}_0(X, \omega_f) \xrightarrow{f_*} \text{CH}^d_X(Y \times X, \omega_{Y \times X/k} \otimes \omega_{Y/k}) \simeq \text{CH}^d_X(Y \times X, \omega_X)$$

where the first isomorphism is induced by $\chi$, the second homomorphism is the push-forward along the (transpose of the) graph $\Gamma_f : X \to Y \times X$ and the third isomorphism is deduced from the isomorphisms of line bundles

$$\omega_{Y \times X/k} \otimes \omega_{Y/k} \simeq \omega_Y \otimes \omega_X \otimes \omega_{Y/k} \simeq \omega_X \otimes \omega_Y \otimes \omega_{Y/k} \simeq \omega_X$$

where the second isomorphism is $(-1)^{d_X d_Y}$ times the switch isomorphism.

We observe that $\alpha$ induces a "push-forward" homomorphism $F(X) \to F(Y)$ for any $F \in \text{PSh}_k$ through the composite

$$F(X) = \text{Hom}_{\text{PSh}_k}(\hat{c}(X), F) \xrightarrow{(-1)^\alpha} \text{Hom}_{\text{PSh}_k}(\hat{c}(Y), F) = F(Y).$$

In particular, we obtain homomorphisms

$$f_* : H^p_{\text{MW}}(X, \mathbb{Z}) \to H^p_{\text{MW}}(Y, \mathbb{Z})$$

for any $p, q \in \mathbb{Z}$, which depend on the choice of $\chi$.

On the other hand, $f$ induces a finite MW-correspondence $X \to Y$ that we still denote by $f$ and therefore a pull-back homomorphism

$$f^* : H^p_{\text{MW}}(Y, \mathbb{Z}) \to H^p_{\text{MW}}(X, \mathbb{Z}).$$

We will need the following lemma to prove the projection formula.

**Lemma 1.3.** Let $f : X \to Y$ be a finite surjective morphism between smooth connected schemes, and let $\chi : \mathcal{O}_X \to \omega_f$ be an isomorphism. Let $\Delta_X$ (resp. $\Delta_Y$) be the diagonal embedding $X \to X \times X$ (resp. $Y \to Y \times Y$). Then, the following diagram commutes

$$\begin{array}{ccc}
Y & \xrightarrow{\Delta_Y} & Y \times Y \\
\downarrow{\alpha} & & \downarrow{(1 \times \alpha)} \\
Y & \xrightarrow{\Delta_X} & X \times X
\end{array}$$

i.e. $(1 \times \alpha)\Delta_Y = (f \times 1)\Delta_X \alpha$.

**Proof.** It suffices to compute both compositions, and we start with the top one. The composite of these two finite MW-correspondences is given by the commutative
is given by the correspondence $(\Gamma_p)$. Finally, the equality $\Delta_Y$, the latter equals

where the squares are Cartesian and the non-labelled arrows are projections (vertically to the first factors and horizontally to the last factors). The composite is given by the push-forward along the projection $p: Y \times Y \times Y \times Y \to Y \times Y \times Y$ defined by $(y_1, y_2, y_3, y_4, x) \mapsto (y_1, y_4, x)$ of the product of the respective pull-backs to $Y \times Y \times Y \times Y$ of $(\Gamma_{\Delta_Y})_*(\langle 1 \rangle)$ and $(\Gamma_{(1 \times f)})_*(\langle 1 \rangle)$. Using the base change formula ([1, Proposition 3.2]), we see that it amounts to push-forward the product

Using the projection formula for Chow-Witt groups with supports ([1, Corollary 3.5]), the latter equals

and the base-change formula once again shows that we have to push-forward along $p$ the cycle

Finally, the equality $p \circ (\Gamma_{\Delta_Y} \times 1 \times 1) = \text{id}$ shows that the composite $(1 \times \alpha) \circ \Delta_Y$ is given by the correspondence $(\Gamma_{\Delta_Y} \circ f)_*(\langle 1 \rangle)$.

For the second composite, we consider the following commutative diagram

where, as before, the squares are Cartesian and the non-labelled arrows are projections (vertically to the first factors and horizontally to the last factors). Arguing as above, we find that the composite is the push-forward along the projection $q: Y \times Y \times Y \times Y \to Y \times Y \times Y$ omitting the second factor of the product

$(\Gamma'_f \times 1 \times 1)_*(\langle 1 \rangle) \cdot (1 \times (\Gamma'_{(f \times 1)\Delta_X}))_*(\langle 1 \rangle)$. 


The projection and the base-change formulas show that the latter is equal to
\[(\Gamma^f_1 \times 1 \times 1)_*, (\Gamma_1((f \times 1)_* \Delta_Y)_*)((1))\]
whose push-forward along \(q\) is \((\Gamma^f_1((\Delta_Y \circ f)_*)((1))\) as
\[q(\Gamma^f_1 \times 1 \times 1)(\Gamma_1((f \times 1)_* \Delta_Y)_*) = \Gamma^f_1((\Delta_Y \circ f)_*)^*\]
\(\square\)

**Theorem 1.4** (Projection formula). Let \(f : X \to Y\) be a finite surjective morphism between smooth connected schemes, and let \(\chi : \mathcal{O}_X \to \omega_f\) be an isomorphism. For any \(x \in H^{p,q}_{\text{MW}}(X,\mathbb{Z})\) and \(y \in H^{p',q'}_{\text{MW}}(Y,\mathbb{Z})\), we have
\[y \cdot f_*(x) = f_*(f^*y \cdot x)\]
in \(H^{p+p',q+q'}_{\text{MW}}(Y,\mathbb{Z})\).

**Proof.** Let \(	ext{DM}^\text{eff}(k)\) be the category of MW-motives ([2, §3.2]). By [2, Corollary 3.3.8], we have \(H^p_{\text{MW}}(X,\mathbb{Z}) = \text{Hom}_{\text{DM}^\text{eff}(k)}(\hat{M}(X),\hat{\mathbb{Z}}\{p-q\})\) for any \(p,q \in \mathbb{Z}\). The product structure on MW-motivic cohomology is defined via the tensor product as follows. If \(x, x'\) are respectively in \(\text{Hom}_{\text{DM}^\text{eff}(k)}(\hat{M}(X),\hat{\mathbb{Z}}\{p-q\})\) and \(\text{Hom}_{\text{DM}^\text{eff}(k)}(\hat{M}(X),\hat{\mathbb{Z}}\{p'-q'\})\), we can take their tensor product to get a morphism \(x \otimes x'\) in \(\text{Hom}_{\text{DM}^\text{eff}(k)}(\hat{M}(X) \otimes \hat{M}(X),\hat{\mathbb{Z}}\{p+q'-q'-q\})\). Now, \(\hat{M}(X) \otimes \hat{M}(X) = \hat{M}(X \times X)\) and the diagonal morphism \(\Delta_X : X \to X \times X\) induces a morphism \(\hat{M}(X) \to \hat{M}(X \times X)\). Composing the latter with \(x \otimes x'\), we obtain a morphism \(x \cdot x' \in \text{Hom}_{\text{DM}^\text{eff}(k)}(\hat{M}(X),\hat{\mathbb{Z}}\{p+q'-q'-q\})\) which represent the product of \(x\) and \(x'\) (after identification of \(\hat{\mathbb{Z}}\{q\} \otimes \hat{\mathbb{Z}}\{q'\}\) with \(\hat{\mathbb{Z}}\{q+q'\}\)).

This being said, let then \(x \in \text{Hom}_{\text{DM}^\text{eff}(k)}(\hat{M}(X),\hat{\mathbb{Z}}\{p-q\})\) and let \(y \in \text{Hom}_{\text{DM}^\text{eff}(k)}(\hat{M}(Y),\hat{\mathbb{Z}}\{p'-q'\})\). The product \(y \cdot f_*(x)\) is then of the form \((y \otimes x) \circ (1 \otimes \alpha) \circ \Delta_Y\), while \(f_*(f^*y \cdot x)\) is of the form \((y \otimes x) \circ (f \circ 1) \circ \Delta_X \circ \alpha\). The result then follows from Lemma 1.3. \(\square\)

**Remark 1.5.** It would suffice to have a fixed isomorphism \(\mathcal{L} \otimes \mathcal{L} \simeq \omega_f\) (for some line bundle \(\mathcal{L}\) on \(X\)) to get an orientation in the sense of [1, §2.2] and thus a finite MW-correspondence \(\alpha\) as above. We let the reader make the necessary modifications in the arguments of both Lemma 1.3 and Theorem 1.4.

**Remark 1.6.** It follows from [2, Theorem 3.4.3] that the same formula holds for the left module structure, i.e.
\[f_*(x) \cdot y = f_*(x \cdot f^*y).\]

**Example 1.7.** As usual, it follows from the projection formula that the composite \(f_* f^*\) is multiplication by \(f_*(1)\). Let us now compute \(f_* f_*\) in some situations that will be used later. Let us start with the general situation, i.e. \(f : X \to Y\) is a finite surjective morphism and \(\chi : \mathcal{O}_X \to \omega_f\) an isomorphism. The composite \(f_* f_*\) is given by precomposition with the correspondence \(f \circ \alpha(f,\chi)\) which we can compute
using the diagram

\[
\begin{array}{ccccccc}
X \times_Y X & \xrightarrow{(1 \times 1)} & X \times X & \xrightarrow{1 \times \Gamma_f^t} & X \\
\downarrow & & \downarrow & & \downarrow \\
X \times X & \xrightarrow{\Gamma_f \times 1} & X \times Y \times X & \xrightarrow{\Gamma_f \times 1} & Y \times X & \xrightarrow{\Gamma_f^t} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\Gamma_f} & X \times Y & \xrightarrow{\Gamma_f} & Y & \xrightarrow{\Gamma_f} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & & & & & & 
\end{array}
\]

where the non-labelled vertical arrows are projections on the first factor and the non-labelled horizontal arrows are projections on the second factor. As usual, the base change formula shows that the composite is equal to the projection on \(X \times X\) of

\[(\Gamma_f \times 1)_*(\langle 1 \rangle) \cdot (1 \times \Gamma_f^t)_*(\langle 1 \rangle),\]

In general the top left square is not transverse, and we can’t use the base-change formula to compute the above product.

Suppose now that \(f : X \to Y\) is finite and étale. In that case, we have a canonical isomorphism \(f^*\omega_Y \simeq \omega_X\) yielding a canonical choice for the isomorphism

\[\chi : \mathcal{O}_X \to \omega_f.\]

Moreover, \(X \times_Y X\) decomposes as \(X \times_Y X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_n\) where each term \(X_i\) is finite and étale over \(X\) with "structural" morphism \(p_i : X_i \to X\). In that case, the above top right square is transverse and we see that

\[(\Gamma_f \times 1)_*(\langle 1 \rangle) \cdot (1 \times \Gamma_f^t)_*(\langle 1 \rangle) = (\Gamma_f \times 1)_*(\Delta_\ast \sum (p_i)_*(\langle 1 \rangle)).\]

where \(\Delta : X \to X \times X\) is the diagonal map. Thus the composite \(f \circ \alpha(f, \chi)\) is equal to \(\Delta_\ast \sum (p_i)_*(\langle 1 \rangle)\). It follows immediately that we have a commutative diagram

\[
\begin{array}{cccc}
H_{MW}^{p,q}(X, \mathbb{Z}) & \xrightarrow{\sum p_i^*} & \bigoplus_i H_{MW}^{p,q}(X_i, \mathbb{Z}) \\
\downarrow f_* & & \downarrow \Sigma(p_i)_* \\
H_{MW}^{p,q}(Y, \mathbb{Z}) & \xrightarrow{f^*} & H_{MW}^{p,q}(X, \mathbb{Z})
\end{array}
\]

for any \(p, q \in \mathbb{Z}\).

Suppose next that \(\text{char}(k) = p\), that \(X \subset Y \times \mathbb{A}^1\) is the set of zeroes of \(t^p - a\) for some global section \(a \in \mathcal{O}_Y(Y)\) (we still suppose that \(X\) is smooth over \(k\)). In that case, we see that the reduced scheme of \(X \times_Y X\) is just \(X\) (but the former has nilpotent elements) and it follows that \(f \circ \alpha(f, \chi)\) is a correspondence supported on the diagonal \(\Delta(X) \subset X \times X\). It follows that there is an element \(\sigma \in K_0^{MW}(X)\)
such that the following diagram commutes

\[ \begin{array}{ccc}
K^0_{MW}(X) & \longrightarrow & \widetilde{Cor}_k(X, X) \\
\sigma \downarrow & & \downarrow f_{\text{ MW}}(f, \chi) \\
K^0_{MW}(X) & \longrightarrow & \widetilde{Cor}_k(X, X)
\end{array} \]

where the horizontal arrows are induced by the push-forward map \( \Delta_*: K^0_{MW}(X) \rightarrow \Ch^X(X \times X, \omega_X) \). Now, \( \sigma \) can be computed using the composite \( K^0_{MW}(k(X)) \rightarrow K^0_{MW}(k(Y)) \rightarrow K^0_{MW}(k(X)) \), where the first map is the push-forward (defined using \( \chi \)) and the second map the pull-back. It follows essentially from \([3, \text{ Lemme 6.4.6}]\) that \( \sigma = p \epsilon \).

1.4. **The homomorphism.** Let \( L/k \) be a finitely generated field extension. It follows from the definition of MW-motivic cohomology that \( H^{p,q}_{MW}(L, \mathbb{Z}) = 0 \) provided \( p > q \). The next step is then to identify \( H^{p,p}_{MW}(L, \mathbb{Z}) \). To this aim, we constructed in \([2, \text{ Theorem 4.2.2}]\) a graded ring homomorphism

\[ K^*_L(L) \rightarrow \bigoplus_{n \in \mathbb{Z}} H^{n,n}_{MW}(L, \mathbb{Z}) \]

which we now recall. For \( a \in L^\times \), we can consider the corresponding morphism \( a: \text{Spec}(L) \rightarrow \mathbb{G}_{m,k} \) which defines a finite MW-correspondence \( \Gamma_a \in \widetilde{Cor}_k(L, \mathbb{G}_{m,k}) \). Now, we have a surjective homomorphism \( \widetilde{Cor}_k(L, \mathbb{G}_{m,k}) \rightarrow H^{1,1}_{MW}(L, \mathbb{Z}) \) and we let \( s([a]) \) be the image of \( \Gamma_a \) under this map. Next, consider the element

\[ \eta[t] \in K^0_{MW}(\mathbb{G}_{m,L}) = \widetilde{Cor}_k(\mathbb{G}_{m,L}, k) = \widetilde{Cor}_k(\mathbb{G}_{m,k} \times L, k) = \text{Hom}(\check{c}(\mathbb{G}_{m,k}), \check{c}(k))(L). \]

We define \( s(\eta) \) to be its image under the projections

\[ \text{Hom}(\check{c}(\mathbb{G}_{m,k}), \check{c}(k))(L) \rightarrow \text{Hom}(\check{Z}(1), \check{c}(k))(L) \rightarrow H^{-1,-1}_{MW}(L, \mathbb{Z}). \]

The following theorem is proved in \([2, \text{ Theorem 4.2.2}]\) (using computations of \([4, \text{ \S 6}]\)).

**Theorem 1.8.** The associations \([a] \mapsto s([a]) \) and \( \eta \mapsto s(\eta) \) induce a homomorphism of graded rings

\[ \Phi_L: K^*_L(L) \rightarrow \bigoplus_{n \in \mathbb{Z}} H^{n,n}_{MW}(L, \mathbb{Z}). \]

By construction, the above homomorphism fits in a commutative diagram of graded rings

\[ \begin{array}{ccc}
K^*_L(L) & \rightarrow & \bigoplus_{n \in \mathbb{Z}} H^{n,n}_{MW}(L, \mathbb{Z}) \\
\Phi_L \downarrow & & \downarrow \\
K^*_L(L) & \rightarrow & \bigoplus_{n \in \mathbb{Z}} H^{n,n}(L, \mathbb{Z})
\end{array} \]

where the vertical projections are respectively the natural map from Milnor-Witt \( K \)-theory to Milnor \( K \)-theory and the ring homomorphism of the previous section, and the bottom horizontal homomorphism is the map constructed by Totaro-Nesterenko-Suslin.
2. Main theorem

2.1. A left inverse. In this section, we construct for \( q \geq 0 \) a left inverse to the homomorphism \( \Phi_L \) of Section 1.4. By definition,

\[
\tilde{c}(\mathbb{G}^q_m)(L) := \bigoplus_{x \in (\mathbb{G}^q_m, \omega_{\mathbb{G}^q_m})^L} \mathcal{H}_x^q(L, \omega_{\mathbb{G}^q_m}).
\]

Now, for any point \( x \) in \((\mathbb{G}^q_m, \omega_{\mathbb{G}^q_m})^L\) with maximal ideal \( m \), we have an exact sequence

\[
m/m^2 \to \Omega^q_{\mathbb{G}^q_m/k} \to \Omega^q_{L(x)/k} \to 0.
\]

Using the fact that \( k \) is perfect and counting dimensions, we see that this sequence is also exact on the left. We find an isomorphism

\[
\wedge^q(m/m^2)^\vee \otimes \omega_{\mathbb{G}^q_m/k} \simeq \omega_{L(x)/k}.
\]

Now, \( \omega_{\mathbb{G}^q_m/k} \simeq p_1^*\omega_{\mathbb{G}^q_m/k} \otimes p_2^*\omega_{L/k} \) and it follows that

\[
\wedge^q(m/m^2)^\vee \otimes \omega_{\mathbb{G}^q_m/k} \simeq \omega_{L(x)/k} \otimes \omega_L^\vee
\]

yielding

\[
\tilde{c}(\mathbb{G}^q_m)(L) = \bigoplus_{x \in (\mathbb{G}^q_m, \omega_{\mathbb{G}^q_m})^L} K^\text{MW}_0(L(x), \omega_{L(x)/k} \otimes \omega_L^\vee/k).
\]

Now any closed point \( x \) in \((\mathbb{G}^q_m, \omega_{\mathbb{G}^q_m})^L\) can be identified with a \( q \)-uple \((x_1, \ldots, x_q)\) of elements of \( L(x) \). For any such \( x \), we define a homomorphism

\[
f_x : K^\text{MW}_0(L(x), \omega_{L(x)/k} \otimes \omega_L^\vee/k) \to K^\text{MW}_0(L)
\]

by \( f_x(\alpha) = \text{Tr}_{L(x)/L}(\alpha \cdot [x_1, \ldots, x_q]) \). We then obtain a homomorphism

\[
f : \tilde{c}(\mathbb{G}^q_m)(L) \to K^\text{MW}_0(L)
\]

which is easily seen to factor through \((\mathbb{Z}/q)(L)\) since \([1] = 0 \in K^\text{MW}_0(L)\).

We now check that this homomorphism vanishes on the image of \((\mathbb{Z}/q)(\mathbb{A}^1_L)\) in \((\mathbb{Z}/q)(L)\) under the boundary homomorphism. This will follow from the next lemma.

Lemma 2.1. Let \( Z \in \mathcal{A}(\mathbb{A}^1_L, \mathbb{G}^q_m) \). Let moreover \( p : \mathbb{G}^q_m \to \text{Spec}(L) \) and \( p_{\mathbb{A}^1_L} : \mathbb{A}^1_L \times \mathbb{G}^q_m \to \mathbb{A}^1_L \) be the projections and \( Z_i := p^{-1}_i(\{0\}) \cap Z \) (endowed with its reduced structure) for \( i = 0, 1 \). Let \( j_i : \text{Spec}(L) \to \mathbb{A}^1_L \) be the inclusions in \( i = 0, 1 \) and let \( g_i : \mathbb{G}^q_m \to \mathbb{A}^1_L \times \mathbb{G}^q_m \) be the induced maps. Then the homomorphisms

\[
p_*(g_i)^* : \mathcal{H}_Z^q(\mathbb{A}^1_L \times \mathbb{G}^q_m, \omega_{\mathbb{G}^q_m}) \to \mathcal{H}_Z^q(\mathbb{G}^q_m, \omega_{\mathbb{G}^q_m}) \to K^\text{MW}_0(L)
\]

are equal.

Proof. For \( i = 0, 1 \), consider the Cartesian square

\[
\begin{array}{ccc}
\mathbb{G}^q_m \times \mathbb{G}^q_m & \xrightarrow{g_i} & \mathbb{A}^1_L \times \mathbb{G}^q_m \\
p \downarrow & & \downarrow p_{\mathbb{A}^1_L} \\
\text{Spec}(L) & \xrightarrow{j_i} & \mathbb{A}^1_L
\end{array}
\]

We have \((j_0)^*(p_{\mathbb{A}^1_L})_* = p_*(g_i)^*\) by base change. The claim follows from the fact that \((j_0)^* = (j_1)^*\) by homotopy invariance. \( \square \)
Proposition 2.2. The homomorphism $f : \tilde{c}(G_m^q)(L) \to K_q^{MW}(L)$ induces a homomorphism

$$\theta_L : H^{q,q}_{MW}(L, \mathbb{Z}) \to K_q^{MW}(L)$$

for any $q \geq 1$.

Proof. Observe that the group $H^{q,q}_{MW}(L, \mathbb{Z})$ is the cokernel of the homomorphism

$$\partial_0 - \partial_1 : \tilde{Z}(q)(A_1^1) \to \tilde{Z}(q)(L)$$

It follows from [1, Example 4.16] that $\partial_i : \tilde{c}(G_m^q)(A_1^1) \to \tilde{c}(G_m^q)(L)$ is induced by $g_i^*$. We can use the above lemma to conclude. □

Corollary 2.3. The homomorphism

$$\Phi_L : \bigoplus_{n \in \mathbb{Z}} K_n^{MW}(L) \to \bigoplus_{n \in \mathbb{Z}} H^{n,n}_{MW}(L, \mathbb{Z}).$$

is split injective.

Proof. It suffices to check that $\theta_L \Phi_L = \text{id}$, which is straightforward. □

The following result will play a role in the proof of the main theorem.

Proposition 2.4. Let $n \in \mathbb{Z}$ and let $F/L$ be a finite field extension. Then, the following diagram commutes

$$
\begin{array}{ccc}
H^{n,n}_{MW}(F, \mathbb{Z}) & \xrightarrow{\theta_F} & K_n^{MW}(F) \\
\big| Tr_{F/L} \big| & & \big| Tr_{F/L} \big| \\
H^{n,n}_{MW}(L, \mathbb{Z}) & \xrightarrow{\theta_L} & K_n^{MW}(L).
\end{array}
$$

Proof. Let $X$ be a smooth connected scheme and let $\beta \in \text{Cor}_k(X, G_m^{\times n})$ be a finite MW-correspondence with support $T$ (see [1, Definition 4.7] for the notion of support). Each connected component $T_i$ of $T$ has a fraction field $k(T_i)$ which is a finite extension of $k(X)$ and, arguing as in the beginning of Section 2.1, we find that $\beta$ can be seen as an element of

$$\bigoplus_i K_0^{MW}(k(T_i), \omega_{k(T_i)/k} \otimes \omega_{k(X)/k})$$

Now, the morphism $T_i \subset X \times G_m^{\times n} \to G_m^{\times n}$ gives invertible global sections $a_1, \ldots, a_n$ and we define a map

$$\theta_X : \text{Cor}_k(X, G_m^{\times n}) \to K_n^{MW}(k(X))$$

by $\beta \mapsto \sum Tr_{k(T_i)/k(X)}(\beta_i[a_1, \ldots, a_n])$, where $\beta_i$ is the component of $\beta$ in the group $K_0^{MW}(k(T_i), \omega_{k(T_i)/k(X)})$. This map is easily seen to be a homomorphism, and its limit at $k(X)$ is the morphism defined at the beginning of Section 2.1.
Let now $X$ and $Y$ be smooth connected schemes over $k$, $f : X \to Y$ be a finite morphism and $\chi : \mathcal{O}_X \to \omega_f$ be an isomorphism inducing a finite MW-correspondence $\alpha(f, \chi) : Y \to X$ as in Section 1.3. We claim that the diagram

$$
\begin{array}{ccc}
\widetilde{\text{Cor}}_k(X, \mathbb{G}_m^\times) & \xrightarrow{\theta_X} & K^\text{MW}_n(k(X)) \\
\circ \alpha(f, \chi) & & \downarrow \text{Tr}_{k(X)/k(Y)} \\
\widetilde{\text{Cor}}_k(Y, \mathbb{G}_m^\times) & \xrightarrow{\theta_Y} & K^\text{MW}_n(k(Y)),
\end{array}
$$

where the right arrow is obtained using $\chi$, commutes. If $\beta$ is as above, we have

$$\text{Tr}_{k(X)/k(Y)}(\theta_X(\beta)) = \sum \text{Tr}_{k(T_i)/k(Y)}(\beta_i[a_1, \ldots, a_n])$$

and the latter is equal to $\sum \text{Tr}_{k(T_i)/k(Y)}(\beta_i[a_1, \ldots, a_n])$ by functoriality of the transfers. On the other hand, the isomorphism $\chi : \mathcal{O}_X \to \omega_f$ can be seen as an element in $K^\text{MW}_n(X,\omega_f)$, yielding an element of $K^\text{MW}_n(k(X),\omega_f)$ that we still denote by $\chi$. The image of $\beta \circ \alpha(f, \chi)$ can be seen as the element $\beta \cdot \chi$ of

$$\bigoplus_i K^\text{MW}_0(k(T_i),\omega_{k(T_i)/k(Y)})$$

where we have used the isomorphism

$$\omega_{k(T_i)/k(Y)} \otimes \omega_f = \omega_{k(T_i)/k(X)} \otimes \omega_{k(X)/k(Y)} \simeq \omega_{k(T_i)/k(Y)}.$$

It is now clear that $\theta_Y(\beta \circ \alpha(f, \chi)) = \sum \text{Tr}_{k(T_i)/k(Y)}(\beta_i[a_1, \ldots, a_n])$ and the result follows.

**2.2. Proof of the main theorem.** In this section we prove our main theorem, namely that the homomorphism

$$\Phi_L : \bigoplus_{n \in \mathbb{Z}} K^\text{MW}_n(L) \to \bigoplus_{n \in \mathbb{Z}} H^\text{MW}_{n,n}(L,\mathbb{Z})$$

is an isomorphism. We first observe that $\Phi_L$ is an isomorphism in degrees $\leq 0$. In degree 0, we indeed know from [1, §6] that both sides are $K^\text{MW}_0(L)$. Next, [4, Lemma 6.0.1] yields

$$\Phi_L(\langle a \rangle) = \Phi_L(1 + \eta[a]) = 1 + s(\eta)s(a) = \langle a \rangle.$$ 

It follows $\Phi_L$ is a homomorphism of graded $K^\text{MW}_0(L)$-algebras and the result in degrees $< 0$ follows then from the fact that $H^\text{MW}_{p,p}(L,\mathbb{Z}) = W(L) = K^\text{MW}_0(L)$ by [1, §6] and [2, Proposition 4.1.2].

We now prove the result in positive degrees, starting with $n = 1$. Recall that we know from Corollary 2.3 that $\Phi_L$ is split injective, and that it therefore suffices to prove that it is surjective to conclude.

For any $d, n \geq 1$ and any field extension $L/k$ let $M^{(d)}_n(L) \subset \widetilde{\text{Cor}}_k(L, \mathbb{G}_m^\times)$ be the subgroup of correspondences whose support is a finite union of field extensions $E/L$ of degree $\leq d$ (see [1, Definition 4.7] for the notion of support of a correspondence). Let $H^\text{MW}_{n,n}(L,\mathbb{Z})^{(d)} \subset H^\text{MW}_{n,n}(L,\mathbb{Z})$ be the image of $M^{(d)}_n(L)$ under the surjective homomorphism

$$\widetilde{\text{Cor}}_k(L, \mathbb{G}_m^\times) \to H^\text{MW}_{n,n}(L,\mathbb{Z}).$$

Observe that

$$H^\text{MW}_{n,n}(L,\mathbb{Z})^{(d)} \subset H^\text{MW}_{n,n}(L,\mathbb{Z})^{(d+1)} \quad \text{and} \quad H^\text{MW}_{n,n}(L,\mathbb{Z}) = \cup_{d \in \mathbb{N}} H^\text{MW}_{n,n}(L,\mathbb{Z})^{(d)}.$$
Lemma 2.5. The subgroup $H_{MW}^{n,n}(L, \mathbb{Z})^{(1)} \subset H_{MW}^{n,n}(L, \mathbb{Z})$ is the image of the homomorphism

$$\Phi_L : K_n^{MW}(L) \to H_{MW}^{n,n}(L, \mathbb{Z}).$$

Proof. By definition, observe that the homomorphism $K_n^{MW}(L) \to H_{MW}^{n,n}(L, \mathbb{Z})$ factors through $H_{MW}^{n,n}(L, \mathbb{Z})^{(1)}$. Let then $\alpha \in H_{MW}^{n,n}(L, \mathbb{Z})^{(1)}$. We may suppose that $\alpha$ is the image under the homomorphism $\text{Cor}_k(L, \mathbb{G}_m^n) \to H_{MW}^{n,n}(L, \mathbb{Z})$ of a correspondence $a$ supported on a field extension $E/L$ of degree 1, i.e. $E = L$. It follows that $\alpha$ is determined by a form $\phi \in K_0^{MW}(L)$ and a $n$-uple $a_1, \ldots, a_n$ of elements of $L$. This is precisely the image of $\Phi_L(\phi \cdot [a_1, \ldots, a_n])$ under the homomorphism $K_n^{MW}(L) \to H_{MW}^{n,n}(L, \mathbb{Z})$. \hfill $\square$

Proposition 2.6. For any $d \geq 2$, we have $H_{MW}^{1,1}(L, \mathbb{Z})^{(d)} \subset H_{MW}^{1,1}(L, \mathbb{Z})^{(d-1)}$.

Proof. By definition, $H_{MW}^{1,1}(L, \mathbb{Z})^{(d)}$ is generated by correspondences whose supports are field extensions $E/L$ of degree at most $d$. Such correspondences are determined by an element $a \in E^\times$ given by the composite $\text{Spec}(E) \to \mathbb{G}_m, L \to \mathbb{G}_m$ together with a form $\phi \in K_0^{MW}(E, \omega_{E/L})$ given by the isomorphism

$$K_0^{MW}(E, \omega_{E/L}) \to \widetilde{CH}_1^{MW}(\text{Spec}(E), \mathbb{G}_m, L, \omega_{\mathbb{G}_m, L}).$$

We denote this correspondence by the pair $(a, \phi)$. Recall from [1, Lemma 2.4] that there is a canonical orientation $\xi$ of $\omega_{E/L}$ and thus a canonical element $\chi$ of $\text{Cor}_k(\text{Spec}(L), \text{Spec}(E))$ yielding the transfer map

$$\text{Tr}_{E/L} : \text{Cor}_k(\text{Spec}(E), \mathbb{G}_m) \to \text{Cor}_k(\text{Spec}(L), \mathbb{G}_m)$$

which is just the composition with $\chi$ ([1, Example 4.17]). Now $\phi = \psi \cdot \xi$ for $\psi \in K_0^{MW}(E)$, and it is straightforward to check that the Chow-Witt correspondence $(a, \psi)$ in $\text{Cor}_k(\text{Spec}(E), \mathbb{G}_m)$ determined by $a \in E^\times$ and $\psi \in K_0^{MW}(E)$ satisfies $\text{Tr}_{E/L}(a, \psi) = (a, \phi)$. Now $(a, \psi) \in H_{MW}^{1,1}(E, L)^{(1)}$ and therefore belongs to the image of the homomorphism $K_1^{MW}(E) \to H_{MW}^{1,1}(E, L)$. There exists thus $a_1, \ldots, a_n, b_1, \ldots, b_m \in E^\times$ (possibly equal) such that $(a, \psi) = \sum s(a_i) - \sum s(b_j)$. To prove the lemma, it suffices then to show that $\text{Tr}_{E/L}(s(b)) \in H_{MW}^{1,1}(L, \mathbb{Z})^{(d-1)}$ for any $b \in E^\times$.

Let thus $b \in E^\times$. By definition, $s(b) \in H_{MW}^{1,1}(E, \hat{\mathbb{Z}})$ is the class of the correspondence $\tilde{\gamma}(b)$ associated to the morphism of schemes $\text{Spec}(E) \to \mathbb{G}_m$ corresponding to $b$. If $F(b) \subset E$ is a proper subfield, we see that $\text{Tr}_{E/L}(s(b)) \in H_{MW}^{1,1}(F, \hat{\mathbb{Z}})^{(d-1)}$, and we may thus suppose that the minimal polynomial of $b$ over $F$ is of degree $d$. By definition, $\text{Tr}_{E/L}(s(b))$ is then represented by the correspondence associated to the pair $(b, (1) \cdot \xi)$. Consider the total residue homomorphism (twisted by the vector space $\omega_{F[t]/k} \otimes \omega_{F/k}$)

$$(1)$$

$$\partial : K_1^{MW}(F(t), \omega_{F(t)/k} \otimes \omega_{F/k}^\vee) \to \bigoplus_{x \in \mathbb{G}_m \otimes F} K_0^{MW}(F(x), (m_x/m_x^2)^\vee \otimes F[t] \omega_{F[t]/k} \otimes \omega_{F/k}^\vee)$$

where $m_x$ is the maximal ideal corresponding to $x$. Before working further with this homomorphism, we first identify $(m_x/m_x^2)^\vee \otimes F[t] \omega_{F[t]/k} \otimes \omega_{F/k}^\vee$. Consider the canonical exact sequence of $F(x)$-vector spaces

$$m_x/m_x^2 \to \Omega_{F[t]/k} \otimes F[t] F(x) \to \Omega_{F(x)/k} \to 0.$$
A comparison of the dimensions shows that the sequence is also left exact (use the fact that \( F(x) \) is the localization of a smooth scheme of dimension \( d \) over the perfect field \( k \)), and we thus get a canonical isomorphism
\[ \omega_{F[t]/k} \otimes_{F[t]} F(x) \simeq m_x/m_x^2 \otimes_{F(x)} \omega_{F(x)/k}. \]

It follows that
\[ (m_x/m_x^2)_{\omega} \otimes_{F[t]} \omega_{F[t]/k} \otimes \omega_{F/k} \simeq \omega_{F[t]/k} \otimes \omega_{F/k}. \]

We can thus rewrite the residue homomorphism (1) as a homomorphism
\[ \partial : K^\text{MW}_1(F(t), \omega_{F(t)/k} \otimes \omega_{F/k}) \to \bigoplus_{x \in (A_k^1 \setminus 0)^{(1)}} K^\text{MW}_0(F(x), \omega_{F(x)/k} \otimes \omega_{F/k}). \]

Moreover, an easy dimension count shows that the canonical exact sequence
\[ \Omega_{F/k} \otimes F[t] \to \Omega_{F[t]/k} \to \Omega_{F[t]/F} \to 0 \]

is also exact on the left, yielding a canonical isomorphism \( \omega_{F(t)/k} \simeq \omega_{F/k} \otimes \omega_{F(t)/F} \) and thus a canonical isomorphism \( \omega_{F/k} \otimes \omega_{F(t)/k} \simeq \omega_{F(t)/F} \). If \( n \) is the transcendence degree of \( F \) over \( k \), we see that the canonical isomorphism
\[ \omega_{F/k} \otimes \omega_{F(t)/k} \simeq \omega_{F(t)/k} \otimes \omega_{F/k} \]
is equal to \((-1)^{n(n+1)}\) times the switch isomorphism, i.e. is equal to the switch isomorphism. Altogether, the residue homomorphism reads as
\[ \partial : K^\text{MW}_1(F(t), \omega_{F(t)/F}) \to \bigoplus_{x \in (A_k^1 \setminus 0)^{(1)}} K^\text{MW}_0(F(x), \omega_{F(x)/k} \otimes \omega_{F/k}). \]

Let now \( p(t) \in F[t] \) be the minimal polynomial of \( b \) over \( F \). Following [7, Definition 4.26] (or [1, §2]), write \( p(t) = p_0(t^m) \) with \( p_0 \) separable and set \( \omega = p_0(t^m) \in F[t] \) if \( \text{char}(k) = l \). If \( \text{char}(k) = 0 \), set \( \omega = p'(t) \). It is easy to see that the element \( (\omega)[p] \cdot dt \) of \( K^\text{MW}_1(F(t), \omega_{F(t)/F}) \) ramifies in \( b \in \mathbb{G}^{(1)}_{m,F} \) and on (possibly) other points corresponding to field extensions of degree \( \leq d - 1 \). Moreover, the residue at \( b \) is exactly \( (1) \cdot \xi \), where \( \xi \) is the canonical orientation of \( \omega_{F(b)/F} \).

Write the minimal polynomial \( p(t) \in F[t] \) of \( b \) as \( p = \sum_{i=0}^{d} \lambda_i t^i \) with \( \lambda_d = 1 \) and \( \lambda_0 \in F^\times \), and decompose \( \omega = c \prod_{i=1}^{d} q_i^{j_i} \), where \( c \in F^\times \) and \( q_i \in F[t] \) are irreducible monic polynomials. Let \( f = (t-1)^{d-1}((-1)^d \lambda_0) \in F[t] \). Observe that \( f \) is monic and satisfies \( f(0) = \rho(0) \). Let \( F(u, t) = (1-u)p + uf \). Since \( f \) and \( p \) are monic and have the same constant terms, it follows that \( F(u, t) = t^d + \ldots + \lambda_0 \) and therefore \( F \) defines an element of \( \mathcal{A}(\mathbb{A}_k^1, \mathbb{G}_m) \). For the same reason, every \( q_i \) (seen as a polynomial in \( F[u, t] \) constant in \( u \)) defines an element in \( \mathcal{A}(\mathbb{A}_k^1, \mathbb{G}_m) \). The image of \( (\omega)[F] : dt \in K^\text{MW}_1(F(u, t), \omega_{F(u, t)/F(u)}) \) under the residue homomorphism
\[ \partial : K^\text{MW}_1(F(u, t), \omega_{F(u, t)/F(u)}) \to \bigoplus_{x \in (A_k^1 \times_k \mathbb{G}_m)^{(1)}} K^\text{MW}_0(F(x), (m_x/m_x^2)_{\omega} \otimes_{F[u]} \omega_{F[u]/F[u]} \otimes_{F[u]} \omega_{F[u]/F[u]}) \]
is supported on the vanishing locus of \( F \) and the \( g_j \), and it follows that it defines a finite Chow-Witt correspondence \( \alpha \) in \( \text{Cor}_k(\mathbb{A}_k^1, \mathbb{G}_m) \). The evaluation \( \alpha(0) \) of \( \alpha \) at \( u = 0 \) consists from \( (1) \cdot \xi \) and correspondences supported on the vanishing locus of the \( g_j \), while \( \alpha(1) \) is supported on the vanishing locus of \( f \) and the \( g_j \). The class of \( (1) \cdot \xi \) is then an element of \( H^{1,1}_{\text{MW}}(L, \mathbb{Z})^{(d-1)}. \)
Corollary 2.7. The homomorphism
\[ \Phi_L : K_1^{MW}(L) \to H_1^{MW}(F, \mathbb{Z}) \]
is an isomorphism for any finitely generated field extension \( F/k \).

Proof. We know that the homomorphism is (split) injective. The above proposition shows that \( H_1^{MW}(F, \mathbb{Z}) = H_1^{MW}(F, \mathbb{Z})^{(1)} \) and the latter is the image of \( K_1^{MW}(L) \) under \( \Phi_L \). It follows that \( \Phi_L \) is surjective. \( \square \)

We can now prove that \( \theta \) respects transfers following [6, Lemma 5.11] and [8, Lemma 9.5].

Theorem 2.8. Let \( n \in \mathbb{N} \) and let \( F/L \) be a finite field extension. Then the following diagram commutes

\[ \begin{array}{ccc}
K_n^{MW}(F) & \xrightarrow{\Phi_F} & H_n^{MW}(F, \mathbb{Z}) \\
\downarrow{\text{Tr}_{F/L}} & & \downarrow{\text{Tr}_{F/L}} \\
K_n^{MW}(L) & \xrightarrow{\Phi_L} & H_n^{MW}(L, \mathbb{Z}).
\end{array} \]

Proof. First, we know from Proposition 2.4 that the diagram

\[ \begin{array}{ccc}
H_n^{MW}(F, \mathbb{Z}) & \xrightarrow{\theta_F} & K_n^{MW}(F) \\
\downarrow{\text{Tr}_{F/L}} & & \downarrow{\text{Tr}_{F/L}} \\
H_n^{MW}(L, \mathbb{Z}) & \xrightarrow{\theta_L} & K_n^{MW}(L).
\end{array} \]

commutes. If \( \Phi_F \) and \( \Phi_L \) are isomorphisms, it follows from Corollary 2.3 that \( \theta_F \) and \( \theta_L \) are their inverses and thus that the diagram

\[ \begin{array}{ccc}
K_n^{MW}(F) & \xrightarrow{\Phi_F} & H_n^{MW}(F, \mathbb{Z}) \\
\downarrow{\text{Tr}_{F/L}} & & \downarrow{\text{Tr}_{F/L}} \\
K_n^{MW}(L) & \xrightarrow{\Phi_L} & H_n^{MW}(L, \mathbb{Z}).
\end{array} \]

also commutes. We may then suppose, using Corollary 2.7 that \( n \geq 2 \). Additionally, we may suppose that \( [F : L] = p \) for some prime number \( p \). Following [6, Lemma 5.11], we first assume that \( L \) has no field extensions of degree prime to \( p \). In that case, it follows from [7, Lemma 3.25] that \( K_n^{MW}(F) \) is generated by elements of the form \( \eta^m[a_1, a_2, \ldots, a_{n+m}] \) with \( a_i \in F^\times \) and \( a_i \in L^\times \) for \( i \geq 2 \). We conclude from the projection formula 1.4, its analogue in Milnor-Witt K-theory and the fact that \( \Phi \) is a ring homomorphism that the result holds in that case.

Let’s now go back to the general case, i.e. [\( F : L \) = \( p \)] without further assumptions. Let \( L' \) be the maximal prime-to-\( p \) field extension of \( L \). Let \( \alpha \in H_n^{MW}(L, \mathbb{Z}) \) be such that its pull-back to \( H_n^{MW}(L', \mathbb{Z}) \) vanishes. It follows then that there exists a finite field extension \( E/L \) of degree \( m \) prime to \( p \) such that the pull-back of \( \alpha \) to \( H_n^{MW}(E, \mathbb{Z}) \) is trivial. Let \( f : \text{Spec}(E) \to \text{Spec}(L) \) be the corresponding morphism. For any unit \( b \in E^\times \), we have \( (b) \cdot f^*(\alpha) = 0 \) and it follows from the projection formula once again that \( f_*(b) \cdot f^*(\alpha) = f_*(b) \cdot \alpha = 0 \). We claim that there is a unit \( b \in E^\times \) such that \( f_*(b) = m_\epsilon \). Indeed, we can consider the factorization
$L \subset F^{sep} \subset E$ where $F^{sep}$ is the separable closure of $L$ in $E$ and the extension $F^{sep} \subset E$ is purely inseparable. If the claim holds for each extension, then it holds for $L \subset E$. We may thus suppose that the extension is either separable or purely inseparable. In the first case, the claim follows from [9, Lemme 2] while the second case follows from [3, Théorème 6.4.13]. Thus, for any $\alpha \in H^{n,n}_{MW}(L, \mathbb{Z})$ vanishing in $H^{n,n}_{MW}(L', \mathbb{Z})$ there exists $m$ prime to $l$ such that $m \cdot \alpha = 0$.

Let now $\alpha \in K^{MW}_n(F)$ and $t(\alpha) = (\text{Tr}_{F/L} \circ \Phi_F - \Phi_L \circ \text{Tr}_{F/L})(\alpha) \in H^{n,n}_{MW}(L, \mathbb{Z})$. Pulling back to $L'$ and using the previous case, we find that $m \cdot t(\alpha) = 0$. On the other hand, the above arguments show that if the pull-back of $t(\alpha)$ to $F$ is trivial, then $p \cdot t(\alpha) = 0$ and thus $t(\alpha) = 0$ as $(p, m) = 1$. Thus, we are reduced to show that $f^*(t(\alpha)) = 0$ where $f : \text{Spec}(F) \to \text{Spec}(L)$ is the morphism corresponding to $L \subset F$.

Suppose first that $F/L$ is purely inseparable. In that case, we know from Example 1.7 that $f^* f_* : H^{n,n}_{MW}(F, \mathbb{Z}) \to H^{n,n}_{MW}(F, \mathbb{Z})$ is multiplication by $p$. The same property holds for Milnor-Witt $K$-theory. This is easily checked using the definition of the transfer, or alternatively using Proposition 2.4, the fact that $\theta_F$ is surjective and Example 1.7. Altogether, we see that $f^* t(\alpha) = p \cdot \Phi_F(\alpha) - \Phi_F(p \cdot \alpha)$ and therefore $f^* t(\alpha) = 0$ since $\Phi_F$ is $K^{MW}_n(F)$-linear.

Suppose next that $F/L$ is separable. In that case, we have $F \otimes_L F = \prod_i F_i$ for field extensions $F_i/F$ of degree $\leq p - 1$. We claim that the diagrams

$$
\begin{array}{ccc}
K^{MW}_n(F) & \longrightarrow & \oplus_i K^{MW}_n(F_i) \\
\text{Tr}_{F/L} & & \downarrow \sum \text{Tr}_{F_i/F} \\
K^{MW}_n(L) & \longrightarrow & K^{MW}_n(F)
\end{array}
$$

and

$$
\begin{array}{ccc}
H^{n,n}_{MW}(F, \mathbb{Z}) & \longrightarrow & \oplus_i H^{n,n}_{MW}(F_i, \mathbb{Z}) \\
\text{Tr}_{F/L} & & \downarrow \sum \text{Tr}_{F_i/F} \\
H^{n,n}_{MW}(L, \mathbb{Z}) & \longrightarrow & H^{n,n}_{MW}(F, \mathbb{Z})
\end{array}
$$

commute. The second one follows from Example 1.7 and the first one from [8, Lemma 9.4] (or alternatively from Proposition 2.4, the fact that $\theta_F$ is surjective and Example 1.7). By induction, each of the diagrams

$$
\begin{array}{ccc}
K^{MW}_n(F_i) & \xrightarrow{\Phi_{F_i}} & H^{n,n}_{MW}(F_i, \mathbb{Z}) \\
\text{Tr}_{F_i/F} & & \downarrow \text{Tr}_{F_i/F} \\
K^{MW}_n(F) & \xrightarrow{\Phi_F} & H^{n,n}_{MW}(F, \mathbb{Z}).
\end{array}
$$

commute, and it follows that $f^*(t(\alpha)) = 0$. □

**Theorem 2.9.** The homomorphism

$$
\Phi_L : K^{MW}_n(L) \to H^{n,n}_{MW}(L, \mathbb{Z})
$$

is an isomorphism for any $n \in \mathbb{Z}$ and any finitely generated field extension $L/k$.

**Proof.** As in degree 1, it suffices to prove that $\Phi_L$ is surjective. Let then $\alpha \in \text{Cor}_k(L, G^*_n)$ be a finite Chow-Witt correspondence supported on $\text{Spec}(F) \subset (\mathbb{A}^1_L)^n$. 


Such a correspondence is determined by an \( n \)-uple \((a_1, \ldots, a_n) \in (F^\times)^n\) together with a bilinear form \( \phi \in \text{GW}(F, \omega_{F/L}) \). Arguing as in Proposition 2.6, we see that such a finite MW-correspondence is of the form \( \text{Tr}_{F/L}(\Phi F(\beta)) \) for some \( \beta \in K_n^{\text{MW}}(F) \). The result now follows from Theorem 2.8. □

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