DETERMINANTAL VARIETY AND NORMAL EMBEDDING

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Abstract. The space $GL_n^+$ of matrices of positive determinant inherits an extrinsic metric space structure from $\mathbb{R}^{n^2}$. On the other hand, taking the infimum of the lengths of all paths connecting a pair of points in $GL_n^+$ gives an intrinsic metric. We prove bilipschitz equivalence between intrinsic and extrinsic metrics on $GL_n^+$, exploiting the conical structure of the stratification of the space of $n \times n$ matrices by rank.

1. Introduction

Consider the group $GL^+(n, \mathbb{R})$ of $n \times n$ matrices of positive determinant. It is an open submanifold of the space $\mathbb{R}^{n^2}$ of $n \times n$ matrices. Here $GL^+(n, \mathbb{R})$ carries two metrics: its extrinsic ambient Euclidean metric with distance function $d_{\text{ext}}$, and the induced intrinsic metric (i.e. least length of path) with distance function $d_{\text{int}}$. Our main result is the following.

Theorem 1.1. There is a constant $C = C(n)$ such that $d_{\text{int}} < C d_{\text{ext}}$.

Thus the determinantal variety is normally embedded; in other words, the intrinsic and extrinsic metrics are bilipschitz equivalent. We prove Theorem 1.1 first for $n = 2, 3$ and then for general $n$. The proof uses in an essential way the conical structure of the stratification of the space $\mathbb{R}^{n^2}$ of $n \times n$ matrices by rank. Indeed, the extrinsic and intrinsic metrics on a set as simple as $\{(x, y): x^2 - y^3 > 0\}$ are clearly inequivalent, due to the fact that the curve $x^2 - y^3 = 0$ has a cusp.

Bilipschitz equivalence has been studied by a number of authors; see e.g., [1], [2], [4], [7]. For a study of the Lipschitz condition in an infinitesimal context see [3], [6].

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2. Solution in Dimension 2

For $2 \times 2$ matrices with coordinates given by the entries $(a_{ij})$, the condition $a_{11}a_{22} - a_{12}a_{21} = 0$ for the determinantal variety in the space of matrices translates into

$$(a_{11} + a_{22})^2 - (a_{11} - a_{22})^2 - (a_{12} + a_{21})^2 + (a_{12} - a_{21})^2 = 0$$

or to simplify notation

$$x^2 + y^2 = z^2 + w^2. \quad (2.1)$$

**Definition 2.1.** Let $X_2 = \{ A \in \mathbb{R}^{2\times2} : \det(A) = 0 \}$ denote the determinantal variety.

**Lemma 2.2.** The intersection $X_2 \cap S^3$ with the unit sphere $S^3 \subseteq \mathbb{R}^{2\times2}$ is a flat 2-torus, namely the Clifford torus $T \subseteq S^3$.

Indeed, by (2.1) the complement $S^3 \setminus T$ is a union of two solid tori, consisting respectively of matrices of positive and negative determinant. Note that $X_2$ is a linear cone over $T \subseteq \mathbb{R}^{2\times2}$.

**Lemma 2.3.** The metrics $d_{int}$ and $d_{ext}$ on $T \subseteq \mathbb{R}^{2\times2}$ are bilipschitz equivalent.

**Proof.** By rescaling, we can assume that the furthest of the two points is at unit distance from the origin. By compactness, the only problem for a pair of points $p, q$ can arise when the points collide, i.e., $d(p, q)$ tends to 0. By the smoothness of the Clifford torus, one has $\frac{d_{int}(p, q)}{d_{ext}(p, q)} \to 1$ as $d(p, q) \to 0$. \hfill $\Box$

**Lemma 2.4.** Bilipschitz equivalence holds for the intrinsic and the extrinsic metrics on $X_2$.

**Proof.** Let $p, q \in X_2$. If the apex $O \in X_2$ is one of the points $p, q$ then the intrinsic and the Euclidean distances between them coincide by linearity of the cone.

Thus we can assume that $p \neq O$ and $q \neq O$. To connect a pair of points $p, q$, at different levels in the cone by a path lying in the cone, assume without loss of generality that $p$ is further than $q$ from the apex of the cone. We slide $p$ along the ray toward the apex until it reaches the level of $q$, and then connect it to $q$ by a shortest path contained in that level. The length of the combined path is clearly bilipschitz with the extrinsic distance in the ambient $\mathbb{R}^{2\times2}$. \hfill $\Box$

**Proposition 2.5.** The intrinsic and the ambient metrics on the manifold of $2 \times 2$ matrices of positive determinant are bilipschitz equivalent.
Proof. The closure of $GL^+(2, \mathbb{R})$ is a linear cone over the solid torus. Meanwhile, $GL^+(2, \mathbb{R})$ itself is the cone without the apex over the interior of the solid torus. We consider a straight line path in $\mathbb{R}^2$ connecting a pair of points in $GL^+(2, \mathbb{R})$. The length of this path is the extrinsic distance by definition.

The path does not necessarily lie entirely inside $GL^+(2, \mathbb{R})$. As the path is straight and the determinantal variety $X_2$ is algebraic, the path splits into a finite number of segments satisfying the following:

1. the interior of each segment lies fully either inside $GL^+(2, \mathbb{R})$ or inside the component $GL^-(2, \mathbb{R})$;
2. the endpoints are in $X_2$.

Applying Lemma $2.4$, we replace every segment that lies in the component $GL^-(2, \mathbb{R})$ by an arc that lies in $X_2$.

Finally, we push out the arc in $X_2$ into $GL^+(2, \mathbb{R})$, i.e., replace it by a nearby arc “just inside” $GL^+(2, \mathbb{R})$. We will explain the push-out procedure in detail since a similar argument will be used in the general case.

The vertex $O \in X_2$ is the unique singular point of the cone, i.e., the complement $X_2 \setminus \{O\}$ is a smooth manifold. Hence a path in $X_2$ disjoint from $O$ can be pushed out infinitesimally into $GL^+(2, \mathbb{R})$ by following the normal direction, without significantly affecting its length.

Suppose a path joining $P, Q \in X_2$ passes through $O$. Using the linear structure of the cone, the path can be replaced by the shorter path given by the union of the straight line segments $PO \cup OQ \subseteq X_2$. Next, $P$ and $Q$ can be replaced by nearby points $P', Q' \in GL^+(2, \mathbb{R})$. We now form a path $P'O \cup OQ'$ lying entirely within $GL^+(2, \mathbb{R})$ except for the single point $O$.

Let $p \in OP'$ be the point of intersection of the segment with a sphere $S(O, \epsilon)$ of small radius $\epsilon > 0$ centered at $O$, and similarly for point $q \in OQ'$. The intersection $S(O, \epsilon) \cap GL^+(2, \mathbb{R})$ is an open solid torus and therefore connected. Hence there is a short path $\gamma \subseteq S(O, \epsilon) \cap GL^+(2, \mathbb{R})$ joining $p$ to $q$. The resulting path $P'p \cup \gamma \cup qQ'$ is only slightly longer than the original path joining $P$ and $Q$. This completes the proof of the bilipschitz property for $GL^+(2, \mathbb{R})$. $\Box$

3. Singularities in dimension 3

For $3 \times 3$ matrices, the singular locus of the determinantal variety $X_3 = \{ A \in \mathbb{R}^{3}\times{3} : \det A = 0 \}$ consists of matrices of rank $\leq 1$. The
variety $X_3$ can be stratified as follows:

$$X_3 = X_{3,0} \cup X_{3,1} \cup X_{3,2}.$$  

Here the stratum $X_{3,i}$ consists of matrices of rank $i$. The stratum $X_{3,1}$ lies in the closure of $X_{3,2}$, while $X_{3,0}$ lies in the closure of $X_{3,1}$. Each $X_{3,i}$ is smooth. Here $X_{3,2}$ is of codimension 1 in the space of matrices, while $X_{3,1}$ is of codimension 4 in the space of matrices, and $X_{3,0}$ is a single point.

The closure $\overline{X}_{3,2}$ is a cone on a smooth manifold away from $X_{3,1}$. Thus the only potential obstruction to bilipschitz equivalence is the singularity of $X_{3,2}$ along $X_{3,1}$, which we now analyze.

**Lemma 3.1.** The 5-dimensional closure $\overline{X}_{3,1}$ is a linear cone over the smooth compact 4-manifold $(S^2 \times S^2)/\{\pm 1\}$.

**Proof.** Here $S^2 \times S^2/\{\pm 1\}$ parametrizes the intersection of $X_{3,1}$ with the unit 8-sphere in the space of matrices. The manifold $S^2 \times S^2$ can be parametrized by pairs of unit column vectors $v, w \in S^2 \subseteq \mathbb{R}^3$ producing a rank 1 matrix

$$v^t w \in X_{3,1}.$$  

The element $-1$ acts simultaneously on both factors of $S^2 \times S^2$ by the antipodal map. \qed

Fix an element $x \in X_{3,1}$. We would like to understand the bilipschitz property for $X_3$ in a neighborhood of $x$. Exploiting the action of $GL(3, \mathbb{R}) \times GL(3, \mathbb{R})$ on $X_{3,1}$ by right and left multiplication, we may assume that

$$x = 1 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$  

**Definition 3.2.** Let

$$L_x = \{1 \oplus A : A \in \mathbb{R}^{2x}\} \subseteq \mathbb{R}^{3x}$$  

be a transverse slice to $X_{3,1} \subseteq \mathbb{R}^{3x}$ at $x$, so that

$$L_x \cap \overline{X}_{3,2} = \{1 \oplus A \in L_x : \det(A) = 0\}. \quad (3.1)$$  

Note that of course transversality is weaker than orthogonality. The ambient Euclidean metric on $\mathbb{R}^9$ plays no role here.

**Lemma 3.3.** A transverse slice for $X_{3,1} \subseteq \overline{X}_{3,2}$ in local coordinates is a linear cone over a Clifford torus.

**Proof.** The defining equation (3.1) of the slice is $\det(A) = 0$. By Lemma 2.2 this is a cone over a torus. \qed
Now let $U_T \subseteq \mathbb{R}^5$ be the unit ball. Choose open sets $U_N \subseteq L_x$ and $U \subseteq \mathbb{R}^{3^2}$, each containing $x$, and a bilipschitz diffeomorphism

$$\phi: U \to U_T \times U_N$$

(3.2)

such that $\phi(U \cap X_{3,1}) = U_T \times \{x\}$ and $\phi(U \cap X_{3,2}) = U_T \times (U_N \cap X_{3,2})$. This can be done using the $GL(3, \mathbb{R}) \times GL(3, \mathbb{R})$ action, as explained in [5, p. 138]. Hence Lemmas 3.3 and 2.4 imply that the intrinsic and extrinsic metrics on $U \cap X_{3,3}$ are bilipschitz equivalent.

**Corollary 3.4.** The intrinsic and extrinsic metrics on $X_{3,2} = X_{3}$ are bilipschitz equivalent.

This follows from Lemma 3.1 by a compactness argument.

**Theorem 3.5.** The intrinsic and extrinsic metrics on $GL^+(3, \mathbb{R})$ are bilipschitz equivalent.

**Proof.** The extrinsic distance between a pair of matrices of positive determinant is the length of the straight line path in $\mathbb{R}^{3^2}$ joining them. We partition the path into finitely many segments, where the interior of each segment lies entirely in a connected component of $GL(3, \mathbb{R})$ while the endpoints are in $X_3 \subseteq \mathbb{R}^{3^2}$. Then we apply Corollary 3.4 to replace each segment belonging to the component $GL^-(3, \mathbb{R})$ by an arc in $X_3$. If the arc in $X_3$ lies in the smooth part $X_{3,2} \subseteq X_3$ then it can be pushed out into $GL^+(3, \mathbb{R})$ by a small deformation in the normal direction, as in Section 2.

Unlike the case of Proposition 2.5, the determinantal variety is not a cone on a manifold, so that an additional argument is required to push the arc out of $X_3$ and into $GL^+_3$ while retaining bilipschitz control.

If an arc in $X_3$ joining points $P, Q$ passes through the apex $O \in X_3$ then it can be replaced by the union $PO \cup OQ$ and pushed out into $GL^+(3, \mathbb{R})$ as in the proof of Proposition 2.5.

Otherwise we exploit the local product structure on $X_3 \setminus \{O\}$ as in (3.2). A path in $X = X_3$ that dips into the singular locus $X_{3,1} \subseteq X$ can be handled as follows. Given a path $\gamma: [0, 1] \to X$, let $a \in [0, 1]$ be the least parameter value $t$ such that $\gamma(t)$ is contained in the singular locus $X_{3,1} \subseteq X$, and $b \in [0, 1]$ the greatest such value.

Step 1. Consider the restriction of the path $\gamma$ to $[a, b] \subseteq [0, 1]$. We replace it by a path that lies entirely in the singular locus $X_{3,1}$. Section 1 proves that $X_{3,1}$ is embedded in $X$ in a bilipschitz fashion. Hence this replacement can be performed in a bilipschitz-controlled way.

Step 2. Once the path $\gamma([a, b])$ is in the singular locus, we exploit a local trivialisation to push it in a constant direction as follows.
Choose \( \delta > 0 \) sufficiently small to be specified later. Then \( \gamma(a - \delta) \) and \( \gamma(b + \delta) \) are in the smooth part \( X_{3,2} \subseteq X \). The path can easily be shortened to a smooth one, still denoted \( \gamma \). Over a sufficiently small neighborhood of the smooth path, we can choose a smooth (and in particular bilipschitz) trivialisation of \( X \) over the path \( \gamma([a, b]) \). Let

\[
I_{0,1}(t) = I_{0,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

be a constant section of the bundle over the path. Relative to the trivialisation of the bundle we can form a new path \( \bar{\gamma}_\epsilon(t) = \gamma(t) + \epsilon I_{0,1}(t) \) which pushes \( \gamma([a, b]) \) in the constant direction \((3.3)\) for each \( t \). By construction, the new path is contained in the nonsingular part \( X_{3,2} \subseteq X \) of the determinantal variety, where \( \epsilon > 0 \) is chosen small enough so that the length of the path stays close to the original length of the path in \( X_{3,1} \).

Step 3. It remains to check that the path can be patched up with the value of the path \( \gamma \) at the parameter values \( a - \delta \) and \( b + \delta \). Since the determinantal variety in the fiber is a cone over a connected space, the two values can be connected by an arbitrarily short path, provided \( \epsilon \) and \( \delta \) are chosen small enough.

Step 4. Once the path lies in the nonsingular part \( X_{3,2} \subseteq X \) of the determinantal variety, it can be pushed out into \( GL^+(3, \mathbb{R}) \) by following the normal direction as in the case \( n = 2 \). \( \square \)

4. The general determinantal variety

Let \( X_{n,k} \subseteq \mathbb{R}^{n^2} \) be the stratum of rank \( k \) matrices. Thus \( X_n \) is the closure of \( X_{n,n-1} \) and each \( X_{n,k} \) is a smooth connected manifold of dimension \( n^2 - (n-k)^2 \). For each \( x \in X_{n,k} \) choose an open set \( U_x \subseteq \mathbb{R}^{n^2} \) containing \( x \), a metric ball \( U^T_x \subseteq \mathbb{R}^{n^2-(n-k)^2} \), an open set \( U^N_x \subseteq \mathbb{R}^{(n-k)^2} \) containing \( 0 \), and a bilipschitz diffeomorphism \( \phi = \phi_x : U_x \to U^T_x \times U^N_x \), such that

\[
\phi(U_x \cap X_{n,k}) = U^T_x \times \{0\}, \quad \phi(U_x \cap X_n) = U^T_x \times (U^N_x \cap X_{n-k}).
\]

See \[5\] p. 138 for the construction of the diffeomorphism \( \phi_x \). To ensure the bilipschitz property, it suffices to shrink slightly \( U_x, U^T_x \) and \( U^N_x \).

Assume by induction that we have proved the bilipschitz equivalence of the intrinsic and extrinsic metrics on \( X_{\ell} \) for \( \ell < n \). Generalizing Lemma \[3\] to dimension \( n \), we see \( X_{n,1} \) is a linear cone over a compact
smooth submanifold. So, the intrinsic and extrinsic metrics on $X_{n,1}$ are bilipschitz equivalent. For each $x \in X_{n,1}$, the bilipschitz diffeomorphism $\phi_x$ and the induction hypothesis for $X_{n-1}$ show the intrinsic and extrinsic metrics are bilipschitz equivalent on $U_x \cap X_n$. Then, a compactness argument shows the bilipschitz property holds for a sufficiently small open neighborhood $U_\epsilon(X_{n,1})$ of $X_{n,1}$ in $X_n$. More precisely, $U_\epsilon$ is defined to be a cone over an $\epsilon$-neighborhood in the unit sphere.

Next we consider the stratum $X_{n,2}$. The complement $X_{n,2} \setminus U_\epsilon(X_{n,1})$ is a cone on a compact smooth manifold with boundary. For each point $x \in X_{n,2} \setminus U_\epsilon(X_{n,1})$, the bilipschitz diffeomorphism $\phi_x$ and the induction hypothesis for $X_{n-2}$ show the intrinsic and extrinsic metrics are bilipschitz equivalent on $U_x \cap X_n$. Arguing by compactness as before, we obtain the bilipschitz property for a sufficiently small neighborhood $U_\epsilon(X_{n,2})$ of $X_{n,2}$ in $X_n$. Here $\epsilon$ may have to be chosen smaller than the one chosen for $X_{n,1}$.

We proceed in this way until we obtain the bilipschitz property for a neighborhood of $X_{n,n-2}$ in the determinantal variety $X_n$. By compactness, the bilipschitz property holds for $X_n$ itself.

5. Bilipschitz property for the set of matrices of positive determinant

We prove Theorem 1.1 by pushing a path in the determinantal variety out into the component $GL^+(n, \mathbb{R})$, and mimicking the proof of Theorem 3.5.

If the path $\gamma$ passes via the apex $O \in X$, it can be replaced by a pair of straight line segments and pushed out into $C$ as in Section 2.

Otherwise let $k \geq 1$ be the least rank of a matrix $\gamma(t)$ along the path, and let $a_k \leq b_k \in [0, 1]$ be respectively the first and last occurrences of a matrix of rank $k$. As in Section 4, we push the path $\gamma([a_k, b_k])$ into $X_{n,k} \subseteq X$, by applying the results of Section 4. We then push it out into $X_{n,k+1}$ by following a constant direction, and patch it up at the endpoints as in Step 3 of the proof of Theorem 3.5.

The new path is in $X_{n,k+1}$. We now choose the corresponding parameter values $a_{k+1}, b_{k+1} \in [0, 1]$ and proceed inductively. Thus the path can be pushed out into $X_{n,n-1}$. Finally we follow the normal direction to push the path out into $GL^+(n, \mathbb{R})$.

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References

[1] Birbrair, L.; Fernandes, A.; Lê, D.; Sampaio, J. “Lipschitz regular complex algebraic sets are smooth.” Proc. Amer. Math. Soc. 144 (2016), no. 3, 983–987.
[2] Birbrair, L.; Mostowski, T. “Normal embeddings of semialgebraic sets.” Michigan Math. J. 47 (2000), no. 1, 125–132.
[3] Kanovei, V.; Katz, K.; Katz, M.; Nowik, T. “Small oscillations of the pendulum, Euler’s method, and adequacy.” Quantum Studies: Mathematics and Foundations 3 (2016), no. 3, 231–236. See http://dx.doi.org/10.1007/s40509-016-0074-x and http://arxiv.org/abs/1604.06663
[4] Katz, K.; Katz, M. “Bi-Lipschitz approximation by finite-dimensional imbeddings.” Geom. Dedicata 150 (2011), 131–136.
[5] MacPherson, R.; Procesi, C. “Making conical compactifications wonderful.” Selecta Math. (N.S.) 4 (1998), no. 1, 125–139.
[6] Nowik, T., Katz, M. “Differential geometry via infinitesimal displacements.” Journal of Logic and Analysis 7:5 (2015), 1–44. See http://dx.doi.org/10.4115/jla.2015.7.5 and http://arxiv.org/abs/1405.0984
[7] Pedersen, H.; Ruas, M. “Lipschitz Normal Embeddings and Determinantal Singularities.” See http://arxiv.org/abs/1607.07746

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\(^1\)See http://mathoverflow.net/questions/222162
\(^2\)See http://mathoverflow.net/questions/230668