MELNIKOV ANALYSIS OF THE NONLOCAL NANOBEAM RESTING ON FRACTIONAL-ORDER SOFTENING NONLINEAR VISCOELASTIC FOUNDATIONS

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Abstract. In the present study, the dynamics of nanobeam resting on fractional order softening nonlinear viscoelastic pasternack foundations is studied. The Hamilton principle is used to derive the nonlinear equation of the motion. Approximate analytical solution is obtained by applying the standard averaging method. The Melnikov method is used to investigate the chaotic behaviors of device, the critical curve separating the chaotic and non-chaotic regions are found. It is shown that the distance between chaotic region and non-chaotic region in this kind of structure depends strongly on the fractional order parameter.

1. Introduction. Due to the recent and rapid advances in nanomechanics, nanobeam have become the most important structure used extensively in technology such as those nano-electromechanical systems (NEMS), opto-mechanical or nanoresonator devices. The exclusive properties of nanoscale beam are due to their size, this size plays an important role in static and in dynamic analysis. In front of the difficulties of classical continuum mechanics to take into account the size effect in modelling the behaviour of this structure kind, various size-dependent continuum theories have been developped. These theories include nonlocal continuum theory, strain gradient theory or a combinaison of both (nonlocal strain gradient theory), modifed couple stress theory, micropolar theory and the surface elasticity theory. Among these theories, Eringen’s nonlocal elasticity theory [9, 10] was utilized by a number of researchers to capture size-effects.

These kinds of structures can be modelled as a beam structure on a viscoelastic foudaton. The beam can be modelled as Timoshenko beam [16, 39], or as a Rayleigh beam [14] or as a Euler-bernouilli beam [8] and the foundation as a Winkler model [22, 13, 1] or as a Pasternak model or combinaison of both (Winkler-Pasternak model) or as a nonlinear elastic model and fractional order viscoelastic model [2].
The Winkler model is a one parameter model namely Winkler-type elastic foundation, consists of a serie of closely spaced elastic springs. Pasternak model is a two parameters model namely Pasternak-type viscoelastic foundation, consists of a Winkler-type elastic springs and transverse shear deformation. The nonlinear model is a three parameters in which the layer is indicated by linear elastic spring, shear deformation and cubic nonlinearity elastic spring. Fractional order Winkler-Pasternak [2] has been well developed, this fractional order is due to the long memory effects of some kind of viscoelastic materials. In vibration analysis of nanostructures, it is so important to evaluate the impact of surrounding medium on the dynamic of beams. Niknam and Aghdam [33] proposed an analytical approach to study dynamic of nonlocal functionally graded beam resting on nonlinear elastic support. A meshless approach for free transverse vibration of SWCNT was proposed by Kiani [18]. Eringen’s nonlocal theory and Timoshenko beam theory were used to make a buckling analysis of SWCNT on elastic medium [27, 35]. Non-conservative dynamic of nonlocal cantilever CNTs on viscoelastic medium is proposed [17]. Mikhasev [25] researched localized modes of free vibrations of SWCNT. Mustapha and Zhong [28] studied dynamic of non-prismatic SWCNT in viscoelastic medium. Lee and Chang [21] studied dynamic of a viscous–fluid conveying SWCNT. Kiani [19, 20] examined elastically restrained DWCNT and SWCNT for delivering nanoparticles, instability analysis of CNT conveying fluid is conducted [11], Yas and Samadi[44] examined CNT–reinforced composite on elastic medium , small scale effect in nonuniform CNT conveying fluid on viscoelastic medium is examined[36], Aydoglu [5] analysed nanorods on an elastic medium, dynamic analysis of nanotubes on elastic matrix is conducted by Wang [43], dynamic of curved SWCNT on a Pasternak elastic foundation is examined [23] . Aydogdu and Arda [6] researched the torsional dynamic of nonlocal DWCNTs. Necla [42] studied nonlinear vibration of a nonlocal nanobeam resting on Winkler-type foundation. The work of Anague [2] is based on dynamics of Rayleigh beams resting on fractional order viscoelastic Pasternak foundation subjected to moving loads.

Chaos theory is a branch of mathematic focusing on the behavior of dynamical system that are highly sensitive to initial conditions such as nanostructures, beam[37], plates [4, 45, 15] or membranes. Several important researchs have been done for this purpose. Nana Nbendjo and Woafo [31, 32] studied the control of chaos on a buckled beam, Euler’s beam. Siewe and Hegazy[41] determined the regions of homoclinic and heteroclinic chaos using Melnikov method in the dynamic of the micromechanical resonators. Demartini[7] analysed the chaos on the micro-electromechanical oscillator governed by the nonlinear Mathieu equation using the Melnikov method.

The above investigation clearly show the special importance of the study of the chaos to control the dynamic of nanostructure. In the case of the fractional order systems, this kind of work is very rare. the aim of this work is to determine the distance between the chaotic region and non-chaotic region on the dynamics of nanobeam resting on fractional order softening nonlinear viscoelastic Pasternak foundations. Using analytical Melnikov method [24] and numerical Newton-Leipnik numerical method [34]. The critical surface separating the chaotic and non-chaotic regions are depicted. The phase portrait of the system, power density spectra and wave forms on plane are drawn, the numerical results justify the correctness and precision of theoretical study. We found that the fractional order reduce the region of chaotic behavior.
2. Preliminaries.

2.1. Fractional order viscoelasticity. Fractional calculus is a part of mathematical analysis that has found many applications in nanomechanics. The role of fractional calculus is to study arbitrary real or complex order integrals and derivatives. There are many definitions of fractional order integrals and derivatives that have been given by different authors. However, in our study, we will consider only Caputo’s definition of a fractional derivative because it is not necessary to define the fractional order initial conditions when solving differential equation. If $x(\cdot)$ is an absolutely continuous function in $[a, b]$ and $n - 1 < \alpha < n$, then:

1. The left Caputo fractional derivative of order $\alpha$ is of the form
   \[
   aD_\alpha^x t x(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{x^{(n)}(\tau)}{(t - \tau)^{n-\alpha+1}} d\tau.
   \]

2. The right Caputo fractional derivative of order $\alpha$ is of the form
   \[
   bD_\alpha^x t x(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^b \frac{x^{(n)}(\tau)}{(\tau - t)^{n-\alpha+1}} d\tau.
   \]

where $x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n}$ is the ordinary derivative of integer order $n$.

2.2. Nonlocal theory. In the nonlocal elasticity theory, the stress at a point $x$ is a function of the strains at all other points of an elastic body. The integral form of the nonlocal constitutive relation for a three-dimensional structure is

\[
\sigma_{ij}(x) = \int \chi(|x - x'|, \tau) t_{ij}(x') dV(x'), \forall x \in V,
\]

where $\sigma_{ij}$ is the nonlocal stress tensor, $t_{ij}$ is the local or classical stress tensors at a point $x'$, $\chi(|x - x'|, \tau)$ denotes the attenuation function, which incorporates non-local effects into the constitutive equation, $|x - x'|$ is the distance in the Euclidean norm and $\tau = \frac{e_0 a}{l}$ is a nonlocal parameter, where $l$ is the external characteristic length (crack length or wave length), $a$ is the internal characteristic length (lattice parameter, granular, etc.) and $e_0$ is a material constant that can be determined from molecular dynamics simulations or by using the dispersive curve of the Born–Karman model of lattice dynamics. Eringen [9] proposed a differential form of the constitutive relation with an appropriate kernel function as

\[
(1 - \tau^2 l^2 \nabla^2) \sigma_{ij} = t_{ij}.
\]

For the one-dimensional case, the local stress $t_{xx}$ at a point $x'$ can be explained according to Hooke’s law as

\[
t_{xx}(x') = E \varepsilon_{xx}(x'),
\]

where $E$ denotes the elastic modulus and $\varepsilon_{xx}$ the strain. That yields the following differential form of the nonlocal constitutive equation for a one-dimensional elastic body

\[
\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx},
\]

where $\mu = (e_0 a)^2$ is the nonlocal parameter and $\sigma_{xx}$ is the nonlocal stress.
3. Governing equation of the nanobeam resting on the fractional order softening nonlinear viscoelastic foundation. This study is carried out on the basis of the nonlocal Euler–Bernouilli nanobeam of length $L$, cross-sectional area $A$, density $\rho$ and transverse deflection $w(x,t)$ in the $z$ direction. Two types of boundary conditions, which are simple-simple and clamped-clamped, are considered in this work and are shown in Figure 1. We assume that the cross-sectional area is constant along the $x$ coordinate and that the material of the nanobeam is homogeneous. The nanobeam is resting on a fractional order softening nonlinear viscoelastic Pasternak foundation in which $k_1$, $k_2$ and $k_3$ are respectively the stiffness, the shear elastic coefficient and the softening nonlinear elastic foundation stiffness and $\mu_1$ and $\mu_2$ are the fractional damping and shear viscosity coefficients. We also consider that the nanobeam is under the influence of time varying transverse load. According to Euler–Bernouilli beam theory, the displacement fields at any point of the beam can be expressed as

$$u_x(x,z,t) = u(x,t) - z \frac{\partial w(x,t)}{\partial x}, u_y = 0, u_z = w(x,t),$$

where $u$ and $w$ are the axial and transverse displacements, respectively. By assuming the von Karman nonlinear strain displacement relation for the given displacement fields, we get

$$\varepsilon_0 = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \varepsilon_1 = -z \ddot{k}, \ddot{k} = \frac{\partial^2 w}{\partial x^2}.$$

where $\varepsilon_0$ is the nonlinear extensional strain and $\ddot{k}$ is the bending strain. The von Karman nonlinear normal strain can be expressed as

$$\varepsilon = \varepsilon_0 + \varepsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2}.$$

By applying the Hamilton principle to the infinitesimal element of the nanobeam, the equilibrium equation can be obtained as

$$\int_{0}^{t} \delta (U - T + V) \, dt = 0,$$

where $U$, $T$ and $V$ are elastic energy, kinetic energy and the work done by the external load respectively.

$$U = \frac{1}{2} \int_{0}^{L} \int_{A} \sigma_{xx} \varepsilon_{xx} dA dx,$$
\[ U = \frac{1}{2} \int_0^L \left[ N \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - M \frac{\partial^2 w}{\partial x^2} \right] dx. \quad (12) \]

Then
\[ \delta U = \int_0^L \left[ - \frac{\partial N}{\partial x} \delta u - \left( \frac{\partial^2 M}{\partial x^2} + N \frac{\partial^2 w}{\partial x^2} \right) \delta w \right] dx. \quad (13) \]

\[ (N, M) = \int_A (\sigma_{xx}, \sigma_{xx}) \, dA, \quad (14) \]

where \( N \) and \( M \) are the axial force and the bending moment, respectively. \( \sigma_{xx} \) is a normal stress component. The kinetic energy can be express as
\[ T = \frac{1}{2} \int_0^L \int_A \rho \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] \, dAdx, \quad (15) \]

i.e
\[ T = \frac{1}{2} \int_0^L \int_A \rho \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] \, dAdx. \quad (16) \]

Then
\[ \delta T = \int_0^L \left[ I_0 \left[ \frac{\partial u}{\partial t} \delta \left( \frac{\partial u}{\partial t} \right) + \frac{\partial w}{\partial t} \delta \left( \frac{\partial w}{\partial t} \right) \right] + I \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \delta}{\partial x \partial t} \left( \frac{\partial w}{\partial t} \right) \right] dx. \quad (17) \]

\[ (I_0, I) = \int_A \rho \left( 1, z^2 \right) \, dA. \quad (18) \]

The variational external influence can be express as
\[ \delta V = \int_0^L \left( -k_1 w + k_2 \frac{\partial^2 w}{\partial x^2} + k_3 w^3 \right) \delta wdx. \quad (19) \]

Inserting (14), (19) and (21) into (10) and integrating, these two equations are obtained
\[ \frac{\partial N}{\partial x} = I_0 \frac{\partial^2 u}{\partial t^2}, \quad (20) \]
\[ \frac{\partial^2 M}{\partial x^2} + N \frac{\partial^2 w}{\partial x^2} + k_1 w - k_2 \frac{\partial^2 w}{\partial x^2} - k_3 w^3 = I_0 \frac{\partial^2 w}{\partial t^2} - I \frac{\partial^4 w}{\partial x^2 \partial t^2}. \quad (21) \]

Using the nonlocal constituve equation we can express the nonlocal axial force and bending moment as follows
\[ N - (\varepsilon_0 a)^2 \frac{\partial^2 N}{\partial x^2} = EA \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right), \quad (22) \]
\[ M - (\varepsilon_0 a)^2 \frac{\partial^2 M}{\partial x^2} = -EI \frac{\partial^2 w}{\partial x^2}. \quad (23) \]
Taking into account the transverse periodic and time-dependant forcing term, fractional damping and shear viscosity terms and combining (20-23), the fractional nonlinear equation of motion is obtained as follows

$$E I \frac{\partial^4 w}{\partial x^4} - \frac{EA}{2L} \left( \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 \, dx \right) \frac{\partial^2 w}{\partial x^2} - k_1 w + k_2 \frac{\partial^2 w}{\partial x^2} + k_3 w^3 + I_0 \frac{\partial^2 w}{\partial t^2} - I \frac{\partial^4 w}{\partial x^2 \partial t^2}$$

$$= - (\omega_0^2) \frac{\partial^2 w}{\partial x^2} \left( \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 \, dx \right) \frac{\partial^2 w}{\partial x^2} - k_1 w + k_2 \frac{\partial^2 w}{\partial x^2} + k_3 w^3 + I_0 \frac{\partial^2 w}{\partial t^2} - I \frac{\partial^4 w}{\partial x^2 \partial t^2}$$

$$= F \cos \Omega \, t - \mu_1 D^\alpha_1 \, w - \mu_2 D^\alpha_2 \left( \frac{\partial w}{\partial x} \right)^2, \quad (24)$$

where the longitudinal inertia is neglected based on the discussion about the nonlinear vibration of continuous system \([29, 30]\), then the normal force \(N\) could be represented as

$$N = \frac{EA}{2L} \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 \, dx. \quad (25)$$

The following non-dimensional quantities aim to study the problem in the general form as

$$\bar{x} = \frac{x}{L}, \bar{w} = \frac{w}{L}, \bar{t} = \frac{t}{L^2 \sqrt{\frac{EI}{\rho A}}}, \eta^2 = \frac{\mu}{L^2}, K_1 = \frac{k_1 L^4}{EI}, K_2 = \frac{k_2 L^2}{EI}, K_3 = \frac{k_3 L^6}{EI}, \bar{F} = \frac{F L^2}{EI}$$

$$\bar{\Omega} = \sqrt{\frac{\rho A L^4}{EI}}, \bar{\mu} = \frac{\mu_1 L^4}{EI} \left( \frac{EI}{\rho A L^4} \right)^{\frac{1}{2}} \alpha_1 \alpha_2, \bar{\nu} = \frac{\mu_2 L^2}{EI} \left( \frac{EI}{\rho A L^4} \right)^{\frac{1}{2}} \alpha_2, \delta = \frac{1}{AL^2}. \quad (26)$$

The non-dimensional form of (24) can be expressed as

$$\frac{\partial^4 \bar{w}}{\partial \bar{t}^4} - \frac{1}{2} \left( \int_0^L \left( \frac{\partial \bar{w}}{\partial \bar{t}} \right)^2 \, d\bar{x} \right) \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - K_1 \bar{w} + K_2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + K_3 \bar{w}^3 + \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2}$$

$$- \eta^2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \left( \int_0^L \left( \frac{\partial \bar{w}}{\partial \bar{t}} \right)^2 \, d\bar{x} \right) \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - K_1 \bar{w} + K_2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + K_3 \bar{w}^3 + \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2}$$

$$= \bar{F} \cos \bar{\Omega} \, \bar{t} - \bar{\mu}_1 D^{\alpha_1}_1 \bar{w} - \bar{\mu}_2 D^{\alpha_2}_2 \left( \frac{\partial \bar{w}}{\partial \bar{x}} \right)^2, \quad (27)$$

in which \(K_1, K_2\) and \(K_3\) denote dimensionless stiffness, shear elastic and softening nonlinear elastic stiffness respectively, \(\mu_1\) and \(\mu_2\) denote the dimensionless fractional damping and the shear viscosity coefficient, \(\bar{F}\) represent the dimensionless amplitude of transverse load and \(\eta, \bar{w}\) and \(\bar{t}\) denote the nonlocal parameter, transversal displacement and time.

The non-dimensional form of boundary conditions can be expressed as

Simple-simple case

$$\bar{w}(0) = 0, \bar{w}(1) = 0, \bar{w}''(0) = 0, \bar{w}''(1) = 0; \quad (28)$$

Clamped-clamped case

$$\bar{w}(0) = 0, \bar{w}(1) = 0, \bar{w}'(0) = 0, \bar{w}'(1) = 0. \quad (29)$$
4. Dynamical analysis. In this section, approximated analytical solution of (24) is obtained by applying the standard averaging method [38, 40], the method of Andronov and Witt [3] is used to study the stability of the steady-state solutions. In order to investigate the chaotic behaviors of the system, specially the distance between chaotic and non-chaotic regions, the Melnikov method is used and some important curves are found. For better appreciate the correctness and satisfactory precision of the analytical stability analysis, numerical analysis using Newton-Leipnik definition of fractional order derivative is conducted and the effects of the variation of different parameters belonging to the application problems on the system are calculated numerically and depicted.

4.1. Solution of the governing equation. The dimensionless fractional order nonlinear partial differential equation (24) describes the transversal vibration of the nanobeam resting on a fractional order nonlinear viscoelastic foundation under the influence of periodic transversal load. In order to obtain the asymptotic approximate solution, the standard averaging method will be employed. By applying the Galerkin method, we assume the asymptotic approximate solution in the following form

\[ \ddot{w}(\bar{x}, \bar{t}) = q(\bar{t}) \phi(\bar{x}), \]

in which \( q(\bar{t}) \) is the unknown time function and \( \phi(\bar{x}) \) is the linear mode shape determined from the boundary conditions. The linear mode shapes of (28) and (29) are given by

\[ \phi(\bar{x}) = c_1 \exp i\alpha_1 \bar{x} + c_2 \exp i\alpha_2 \bar{x} + c_3 \exp i\alpha_3 \bar{x} + c_4 \exp i\alpha_4 \bar{x}. \]

The boundary conditions are applied, and the constants \( c_i \) and \( \alpha_i \) can be obtained. Mode shapes of the linear first frequency are plotted in Figure 2 and Figure 3.

By introducing (30) into (24), multiplying the results by the linear mode shape function \( \phi(\bar{x}) \) and then integrating them over the length of the nanobeam, we obtain a fractional order nonlinear ordinary differential equation expressed as:

\[ q\ddot{x} + \omega_0^2 q + \frac{1}{4} \lambda q^3 + \mu_1 D_\tau^{\alpha_1} q + \mu_2 D_\tau^{\alpha_2} q = F' \cos \Omega \bar{t}, \]

where \( \omega_0 \) is the natural frequency for the linear system, \( \mu'_1 \) and \( \mu'_2 \) are normal damping ratio and shear damping ratio, \( \chi \) is the reduced nonlinear stiffness:

\[ \omega_0^2 = \frac{K_1 (a_1 - \eta^2 a_2) + K_2 (-a_2 + \eta^2 a_3) + a_3}{(a_1 - \eta^2 a_2) + \delta (-a_2 + \eta^2 a_3)}, \chi = \frac{2a_4 (-a_2 + \eta^2 a_3) - 4K_3 a_5}{(a_1 - \eta^2 a_2) + \delta (-a_2 + \eta^2 a_3)}, \]

\[ \mu'_1 = \frac{\mu_1 a_1}{(a_1 - \eta^2 a_2) + \delta (-a_2 + \eta^2 a_3)}, \mu'_2 = \frac{\mu_2 a_2}{(a_1 - \eta^2 a_2) + \delta (-a_2 + \eta^2 a_3)}, \]

\[ \{a_1, a_2, a_3, a_4, a_5\} = \int_0^L \left\{ \phi^2, \phi \phi', \phi \phi^{IV}, \phi^2 \phi', \phi^3 \right\}. \]

The expression (32) is a new form of parametrical excited Duffing differential equation due to the presence of multiple fractional order terms. In order to determine the asymptotic approximate solution with combined effects of nonlinearity, parametric excitation and fractional order damping, we will apply the standard averaging method. Let us assume that the solution can be written as

\[ q(t) = a \cos \theta, \]

\[ q(t) = -a \Omega \sin \theta, \]

\[ q(t) = \Omega \sin \theta, \]

\[ q(t) = -\Omega \sin \theta, \]

\[ q(t) = \Omega \sin \theta. \]
where $a(t)$ and $\theta(t)$ represent the amplitude and the phase which are slow-varying functions of $t$. Substituting (35) into (32), we obtain

\begin{align}
    a_t &= -\frac{1}{\Omega} \left( P_1(a, \theta) + P_2(a, \theta) + P_3(a, \theta) \right) \sin \varphi, \quad (36a) \\
    a\theta_t &= -\frac{1}{\Omega} \left( P_1(a, \theta) + P_2(a, \theta) + P_3(a, \theta) \right) \cos \varphi, \quad (36b)
\end{align}

where

\begin{align}
    P_1(a, \theta) &= F' \cos \Omega t + a \left( \Omega^2 - \omega_0^2 \right) \cos \varphi - \frac{\chi a^3}{4} \cos^3 \varphi, \quad (37a) \\
    P_2(a, \theta) &= -\mu'_1 D_{1}^{\alpha_1}(a \cos \varphi), \quad (37b) \\
    P_3(a, \theta) &= -\mu'_2 D_{2}^{\alpha_2}(a \cos \varphi). \quad (37c)
\end{align}

Based on the averaging method [38, 40], one could select the time terminal $T$ as $T = 2\pi$ if $P_i(a, \theta)$ are periodic function, or $T = \infty$ if $P_i(a, \theta)$ are aperiodic one. Hereby we obtain

\begin{align}
    a_t &= -\frac{F \sin \theta}{2\Omega} - \frac{a}{2\Omega} (\alpha_1, \alpha_2), \quad (38a) \\
    a\theta_t &= -\frac{F \cos \theta}{2\Omega} - \frac{a \Omega^2}{2\Omega} + \frac{3\chi}{8\Omega} a^3 + \frac{a}{2\Omega} K(\alpha_1, \alpha_2), \quad (38b)
\end{align}
where
\[
\mu(\alpha_1, \alpha_2) = \frac{\mu_1 a}{2} \Omega^{(\alpha_1-1)} \sin \left(\frac{\alpha_1 \pi}{2}\right) + \frac{\mu_2 a}{2} \Omega^{(\alpha_2-1)} \sin \left(\frac{\alpha_2 \pi}{2}\right),
\]
\[
K(\alpha_1, \alpha_2) = \omega^2 + \mu_1 \Omega^{\alpha_1} \cos \left(\frac{\alpha_1 \pi}{2}\right) + \Omega^{\alpha_2} \cos \left(\frac{\alpha_2 \pi}{2}\right).
\]

Now let us study the stability of the steady-state solution using Andronov and Witt method [3]. Assuming that \(a = \bar{a} + \Delta a\) and \(\theta = \bar{\theta} + \Delta \theta\) and substituting them in (38) yields
\[
\frac{d\Delta a}{dt} = -\frac{\mu(\alpha_1, \alpha_2)}{2} \Delta a - \frac{F \cos \bar{\theta}}{2\Omega} \Delta \theta,
\]
\[
\frac{d\Delta \theta}{dt} = \left(\frac{F \cos \bar{\theta}}{2\Omega \pi^2} + \frac{3\chi \pi}{4\Omega}\right) \Delta a + \frac{F \sin \bar{\theta}}{2\Omega} \Delta \theta,
\]

based on these steady-state equations, we could eliminated \(\bar{\theta}\). The eigenvalues of the Jacobian matrix of (40) are the roots of this equation:
\[
\det \left[ \begin{array}{c} -\frac{\mu(\alpha_1, \alpha_2)}{2} - \frac{\lambda}{\Omega^2} + \frac{K(\alpha_1, \alpha_2)}{2\Omega \pi^2} - \frac{\mu(\alpha_1, \alpha_2)}{2} - \lambda \\ \frac{\Omega}{2\pi} - \frac{3\chi \pi}{2\pi} - \frac{\mu(\alpha_1, \alpha_2)}{2} \end{array} \right] = 0,
\]
\[
\lambda^2 + \mu(\alpha_1, \alpha_2) \lambda + \frac{1}{4} \mu^2(\alpha_1, \alpha_2) + \left[ -\frac{\Omega}{2} + \frac{9\chi \pi}{8\Omega} + \frac{K(\alpha_1, \alpha_2)}{2\Omega \pi^2} \right] \left[ -\frac{\Omega}{2} - \frac{3\chi \pi}{2\Omega} - \frac{K(\alpha_1, \alpha_2)}{2\Omega \pi^2} \right] = 0.
\]

Since \(\mu(\alpha_1, \alpha_2) > 0\), the instability condition for the steady-state solution is
\[
\mu^2(\alpha_1, \alpha_2) + \left[ -\frac{\Omega}{2} + \frac{9\chi \pi}{8\Omega} + \frac{K(\alpha_1, \alpha_2)}{2\Omega \pi^2} \right] \left[ -\frac{\Omega}{2} - \frac{3\chi \pi}{2\Omega} - \frac{K(\alpha_1, \alpha_2)}{2\Omega \pi^2} \right] < 0.
\]

4.2. Melnikov analysis of chaotic behaviors. In the literature, this method is used for determining the existence of chaos in a perturbed Hamiltonian system. Here, with this goal, the distance between the stable and unstable manifolds of our perturbed system is calculated and depicted. In order to validate the present analytical results, we will compare the obtained results proposed by Zhang [45] and Massoud [26]. Now let us assume that the fractional damping, shear viscosity coefficients and transverse amplitude force are small \(\mu' = \varepsilon v_1, \mu'' = \varepsilon v_2, F' = \varepsilon f\) where \(\varepsilon\) is small parameter. The expression (32) can be rewritten as a perturbed Hamiltonian system
\[
q_t = p,
\]
\[
p_t = -\omega^2 q - \chi q^3 - \varepsilon v_1 D_t^\alpha q - \varepsilon v_2 D_t^\alpha q + \varepsilon f \cos \Omega t.
\]

If \(\varepsilon = 0\), the system (44) is considered as an unperturbed system with Hamiltonian function as follows
\[
H(q, p) = \frac{1}{2} p^2 + \frac{\omega^2}{2} q^2 + \frac{\chi}{4} q^4.
\]

In this section, we consider the case of \(\chi < 0\). The equilibrium points of the system (44) are \((0, 0), \left(\pm \sqrt{-\frac{\omega^2}{2}}, 0\right)\) where the first one is a center and others one are saddle points. We derive the heteroclinic orbits connecting the saddle points as
follows

\[ q_{het} (t) = \pm \sqrt{-\frac{\omega_0^2}{\chi} \tanh \left( \frac{\omega_0 t}{\sqrt{2}} \right)} , \quad (46a) \]

\[ p_{het} (t) = \pm \omega_0^2 \sqrt{-\frac{1}{2\chi}} \text{sech} \left( \frac{\omega_0 t}{\sqrt{2}} \right) . \quad (46b) \]

The Melnikov function of the system (44) along the heteroclinic orbits (46) is defined as follows

\[ M (t_0) = \int_{-\infty}^{+\infty} \left( -\nu_1 p (t) D_t^{\alpha_1} q (t) - \nu_2 p (t) D_t^{\alpha_2} q (t) + f p (t) \cos (\Omega (t + t_0)) \right) dt , \quad (47) \]

\[ M (t_0) = -\frac{\nu_1 \omega_0^3}{\chi \sqrt{2}} J_{\alpha_1} - \frac{\nu_2 \omega_0^3}{\chi \sqrt{2}} J_{\alpha_2} \pm \frac{f \omega_0^2}{\sqrt{-2\chi}} J , \quad (48) \]

where

\[ J = \int_{-\infty}^{+\infty} \text{sech}^2 \left( \frac{\omega_0 t}{\sqrt{2}} \right) \cos (\Omega (t + t_0)) dt = \frac{2\pi \Omega}{\omega_0^2 \sinh \left( \frac{\Omega \pi}{\omega_0 \sqrt{2}} \right)} \cos \Omega t_0 , \quad (49a) \]

\[ J_{\alpha_1} = \int_{-\infty}^{+\infty} \text{sech}^2 \left( \frac{\omega_0 t}{\sqrt{2}} \right) D_t^{\alpha_1} \left( \tanh \left( \frac{\omega_0 t}{\sqrt{2}} \right) \right) dt , \quad (49b) \]

\[ J_{\alpha_2} = \int_{-\infty}^{+\infty} \text{sech}^2 \left( \frac{\omega_0 t}{\sqrt{2}} \right) D_t^{\alpha_2} \left( \tanh \left( \frac{\omega_0 t}{\sqrt{2}} \right) \right) dt . \quad (49c) \]

The first integral \( J \) is evaluated by the residus theorem and the numerical Caputo’s definition of fractional derivative is used to integrate \( J_{\alpha_1} \) and \( J_{\alpha_2} \). According to the Melnikov method \[12\] the heteroclinic bifurcations are obtained if we can find an initial angle \( t_0 \) such that

\[ M (t_0) = 0 \text{ and } M_t (t_0) \neq 0. \]

Applying the above conditions to the expression (48) of the Melnikov’s function, the condition of appearance of chaotic behavior is given by

\[ \left| J \left( \alpha_1, \alpha_2 \right) \right| \leq \frac{2\pi \Omega}{\omega_0^2 \sinh \left( \frac{\pi \Omega / \omega_0}{\sqrt{2}} \right)} , \quad (50) \]

where \( J \left( \alpha_1, \alpha_2 \right) \) is a two-order fractional integral that depends linearly on viscosity terms. The above condition allows us to concretely evaluate the distance between chaotic and non-chaotic regions. The graphs obtained in Figure 4-5 show the variations of the size of the chaotic and non-chaotic regions as a function of the various parameters of the system, the regions below the critical curves or surfaces are considered to be the regions of the occurrence of chaos and others one are non-chaotics.

We observe in Figure 4 that the size of the chaotic area varies considerably with the fractional parameter, this area decreases when the fractional parameter increases. At the level of Figure 5 we could appreciate the variation of the size of the chaotic zone as a function of others parameters such as the viscosity, the nonlinearity and the different frequencies terms.
4.3. **Validation study.** In order to validate the present analytical results, we compared the obtained results proposed by Zhang [45] and Mir [26]. Now let us assume that the fractional order $\alpha_1 = \alpha_2 = 1$ and $\nu_1 = \nu_2 = \nu$, we can rewrite (48) as follows

$$M(t_0) = -\frac{\nu_0^3}{\chi \sqrt{2}} J_{\alpha=1} \pm \frac{f \omega_0^2}{\sqrt{-2\chi}} J,$$

(51)

where

$$J_{\alpha=1} = \int_{-\infty}^{+\infty} \sec h^2 \left( \frac{\omega_0}{\sqrt{2}t} \right) \frac{d}{dt} \left( \tanh \left( \frac{\omega_0}{\sqrt{2}t} \right) \right) dt = \frac{4}{3},$$

(52a)
The condition of occurrence of chaos in this case is as follows
\[ \left| \frac{\nu \omega_0 f}{J} \right| \geq \left( \frac{\nu \omega_0 f}{J} \right)_{cr} = \frac{3\pi \Omega/\omega_0 \sqrt{2}}{\sqrt{2} \sinh(\pi \Omega/\omega_0 \sqrt{2})}. \] (53)

Supposing \( \pi \Omega/\omega_0 \sqrt{2} \) is X-axis and \( \nu \omega_0 f \) is Y-axis as in the work of Zhang [45] and Mir [26], we can represent the curve separating the chaotic region and other one Figure 6, we can also see that our analytical study is in perfect agreement with the work done by Zhang [45] and Mir [26].

4.4. Numerical simulation. In order to verify the correctness and satisfactory precision of analytical stability analysis, we used the Newton-Leipnik algorithm [34] by considering the following definition of fractional order derivative
\[ D_t^\alpha (x(t_n)) \approx h^\alpha \sum_{j=0}^{n} C_j^\alpha x(t_{n-j}), \] (54)
\[ C_0^\alpha = 1, \quad C_j^\alpha = \left( 1 - \frac{1 + \alpha}{j} \right) C_{j-1}^\alpha. \] (55)

where \( t_n = nh \) is the time sample point, \( h \) is the sample step, \( C_j^\alpha \) is the fractional binomial coefficient. Based on (54-55), we obtained the following numerical iterative
algorithm of (34)

\[ Z_1(t_k) = Z_2(t_{k-1}) h + Z_1(t_{k-1}) \]
\[ Z_2(t_k) = (-\omega_0^2 Z_1(t_k) - \chi Z_1^2(t_k) - \upsilon_1 Z_3(t_{k-1}) - \upsilon_2 Z_4(t_{k-1}) + f \cos \Omega t_k) h - Z_2(t_{k-1}) \]
\[ Z_3(t_k) = Z_2(t_k) h^{1-\alpha_1} - \sum_{j=1}^{k} C_j^{1-\alpha_1} Z_3(t_{k-j}) \]
\[ Z_4(t_k) = Z_2(t_k) h^{1-\alpha_2} - \sum_{j=1}^{k} C_j^{1-\alpha_2} Z_4(t_{k-j}) \]

where \( Z_1 \) is the displacement, \( Z_2 \) is the velocity, \( Z_3 \) and \( Z_4 \) are the fractional order derivative. The phase portraits, the Poincare sections and waveforms on system are depicted below. With the choice of the following parameters of the system \( \omega_0 = 5, \upsilon_1 = \upsilon_2 = 0.1, \Omega = 2, \chi = -10, \alpha_1 = \alpha_2 = 0.9, f = 80 \), we could check that the condition (50) of appearance of chaos is not verified, we obtained in Figure 7-8 the periodic behavior of the system. However, with the choice of the following parameters of the system \( \omega_0 = 5, \upsilon_1 = \upsilon_2 = 0.1, \Omega = 2, \chi = -10, \alpha_1 = \alpha_2 = 0.9, f = 400 \), the condition of appearance of the chaos is verified and this can be observed in Figure 9-10.

Figure 7. Regular motion when \( f = 80 \): (a) the phase portrait (b) the Poincare section for the initial conditions \((q_0, p_0) = (0.1, 0.1)\).

Figure 8. Regular motion when \( f = 80 \): (c) the waveform \((t, q(t))\) (d) the waveform \((t, p(t))\) for the initial conditions \((q_0, p_0) = (0.1, 0.1)\).
Figure 9. Chaotic motion when $f = 400$: (a) the phase portrait (b) the Poincare section for the initial conditions $(q_0, p_0) = (0.1, 0.1)$.

Figure 10. Chaotic motion when $f = 400$: (c) the waveform $(t, q(t))$ (d) the waveform $(t, p(t))$ for the initial conditions $(q_0, p_0) = (0.1, 0.1)$.

5. Conclusion. The stability analysis of dynamics of nanobeam resting on fractional-order softening nonlinear viscoelastic Pasternack foundation was studied using the Andronov and Witt’s method and the Melnikov’s method. Critical curves and surfaces separating the chaotic and non-chaotic regions are depicted, the phases portraits, the Poincare section and waveforms of system with different choices of parameters system in chaotic and non-chaotic regions are drawn. In the view of the graphs obtained above, we noticed that the numerical results justify the correctness and precision of analytical analysis, Melnikov’s method thus appears to be an important and accurate method in controlling the stability of nanotechnological structures.

Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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