A COMPARISON OF CLASSES IN
THE JOHNSON COKERNELS OF
THE MAPPING CLASS GROUPS OF SURFACES

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Abstract. In [8], the first and the third authors introduced new classes in the
Johnson cokernels of the mapping class groups of surfaces by a representation
theoretic approach based on some previous results for the Johnson cokernels of
the automorphism groups of free groups. On the other hand, in [15], Kawazumi
and the second author introduced another type of classes by a topological
consideration of self-intersections of curves on a surface.

In this paper, we show that the classes found in [15] are contained in the
classes found in [8] in a stable range. Furthermore, we prove that the anti-
Morita obstructions \([1^{m+1}m]\) for \(m \geq 1\) obtained in [8] and a hook-type com-
ponent \([3,1^5]\) detected in [6] appear in their gap.

1. Introduction

Let \(\Sigma_{g,1}\) be a compact oriented surface of genus \(g\) with one boundary com-
ponent. The mapping class group \(M_{g,1}\) is the group of isotopy classes of orientation
preserving diffeomorphisms of \(\Sigma_{g,1}\) which fix the boundary component pointwise.
The Torelli group \(I_{g,1}\), which consists of mapping classes acting trivially on the first
homology \(H = H_1(\Sigma_{g,1}, \mathbb{Z})\), is an important subgroup of \(M_{g,1}\). There is a central
filtration \(I_{g,1} = M_{g,1}(1) \supset M_{g,1}(2) \supset M_{g,1}(3) \supset \cdots\) defined by the action on
the nilpotent quotients of the fundamental group of \(\Sigma_{g,1}\). The associated graded
quotient of this filtration is described by the Johnson homomorphisms
\[
\tau^M_k : gr^k(M_{g,1}) \rightarrow h_{g,1}(k), \quad k \geq 1.
\]

Here, \(gr^k(M_{g,1}) = M_{g,1}(k)/M_{g,1}(k+1)\) and \(h_{g,1}(k)\) is the kernel of the Lie bracket
\(H \otimes \mathbb{L}_2g(k+1) \rightarrow \mathbb{L}_2g(k+2)\), where \(\mathbb{L}_2g = \bigoplus_{m \geq 1} \mathbb{L}_2g(m)\) is the free Lie algebra
generated by \(H = \mathbb{L}_2g(1)\). Note that the collection \(\{\tau^M_k\}_k\) defines an injective
homomorphism of graded Lie algebras:
\[
\tau^M : gr(M_{g,1}) = \bigoplus_{k \geq 1} gr^k(M_{g,1}) \rightarrow h_{g,1} = \bigoplus_{k \geq 1} h_{g,1}(k).
\]

The space \(h_{g,1}\) is called the Lie algebra of symplectic derivations [18, 17].

The Johnson homomorphisms were introduced by Johnson [12, 13], and Morita [18] gave a refinement of the target. For recent developments in the theory
of Johnson homomorphisms, we refer to expository articles [9, 11, 16, 19, 21, 24].

A particularly important fact is that the map \(\tau^M_k\) is equivariant with respect to the action of the group \(M_{g,1}/I_{g,1} \cong Sp(2g, \mathbb{Z})\). This fact enables us to make use
of representation theory to analyze \(\tau^M_k\), in particular when we work over a field.
of characteristic zero. In what follows, putting $\mathbb{Q}$ as a subscript or a superscript means that one takes tensor product with the rationals.

As shown by Johnson [12] the first Johnson homomorphism $\tau_1^M$ is surjective. It was first observed by Morita [18] that the map $\tau_k^M$ is not surjective for higher $k$. That is, for any odd $k \geq 3$, he constructed the surjective homomorphism

$$\text{Tr}_k : h^Q_{g,1}(k) \to S^k H_\mathbb{Q},$$

where $S^k$ means the $k$th symmetric tensor product, and proved that $\text{Tr}_k \circ \tau_k^M = 0$. In other words, the map $\text{Tr}_k$ is an obstruction for the surjectivity of the $k$th Johnson homomorphism $\tau_k^M$. We call the quotient of $h^Q_{g,1}(k)$ by the image of $\tau_k^M$ the $k$th Johnson cokernel of the mapping class group $M_{g,1}$. The Sp-module structure of the Johnson cokernels becomes an interesting object of study. The Morita trace $\text{Tr}_k$ detects the unique Sp-irreducible component $S^k H_\mathbb{Q}$ in the $k$th Johnson cokernel.

In [3], the first and the third authors introduced the Sp-homomorphism

$$c_k : h^Q_{g,1}(k) \to C^Q_{2g}(k).$$

(See §3.2 for its definition.) Here, $C^Q_{2g}(k)$ is the quotient module of $H^Q_{g,1}$ with respect to the action of the cyclic group of order $k$ as cyclic permutations of the components of $H^Q_{g,1}$. By using the third author’s result in [22] that the space $C^Q_{2g}(k)$ coincides with the $k$th Johnson cokernel of the automorphism group of the free group, they proved that

$$\text{Im}(\tau_k^M) \subset \text{Ker}(c_k) \subset h^Q_{g,1}(k)$$

in a stable range. The map $c_k$ is a refinement of $\text{Tr}_k$ in the sense that $\text{Ker}(c_k) \subset \ker(\text{Tr}_k)$. Moreover, in [8] it was shown that for $k \equiv 1 \pmod{4}$ and $k \geq 5$, an Sp-irreducible component $[1^k]$ is detected in $h^Q_{g,1}(k)/\ker(c_k)$, hence in the $k$th Johnson cokernel. We call this component the anti-Morita obstruction.

There are several studies on the trace maps $c_k$ and their application to the Johnson cokernels. In [6], the first author and Hikoe Enomoto detected several series of hook-type components in $h^Q_{g,1}(k)/\ker(c_k)$. Recently, by using the hairy graph complex, Conant [3] detected new Sp-components in the Johnson cokernels which cannot be detected by the trace maps $c_k$.

At the present stage, the structure of the Johnson cokernels has not been completely determined. By using the trace map $c_k$, Morita, Sakasai and Suzuki [20] determined it up to degree 6.

In [15], Kawazumi and the second author introduced the map

$$\delta^\text{alg}_k : h^Q_{g,1}(k) \to \bigoplus_{p+q=k, p,q \geq 1} C^Q_{2g}(p) \otimes C^Q_{2g}(q)$$

(See §3.3 for its definition.) The map $\delta^\text{alg}_k$ arises from the Turaev cobracket, a topological operation which measures self-intersections of curves on a surface. They showed that

$$\text{Im}(\tau_k^M) \subset \ker(\delta^\text{alg}_k) \subset h^Q_{g,1}(k),$$

and that $\ker(\delta^\text{alg}_k) \subset \ker(\text{Tr}_k)$.

The main purpose of this paper is to compare the two obstructions coming from $c_k$ and from $\delta^\text{alg}_k$. Our first result is as follows.

**Theorem 1.1.** For each $k \geq 1$ and $2g \geq k+2$, we have $\ker(c_k) \subset \ker(\delta^\text{alg}_k)$. 

Our proof is based on a relation between several contraction maps defined on $H^* \otimes \mathbb{Z}H_2(k+1)$; see Theorem 2.6. We remark that recently, Alekseev, Kawazumi, Kuno and Naef [1] showed that the above theorem holds for any $g$ in a completely different way.

Our second result gives explicit differences between the two obstructions.

**Theorem 1.2.** Assume that $g \geq k + 1$.

(i) For any $k \equiv 1 \mod 4$ such that $k \geq 5$, the Sp-irreducible component $[1^k]$ lies in $\ker(\delta^\text{alg}_k)/\ker(\delta^\text{alg}_0)$. Thus $\ker(\delta^\text{alg}_k) \subseteq \ker(\delta^\text{alg}_0)$.

(ii) For $k = 8$, an Sp-irreducible component $[3, 1^5]$ appears in $\ker(\delta^\text{alg}_8)/\ker(\delta^\text{alg}_8)$.

Topologically, each of the components in $\ker(\delta^\text{alg}_k)/\ker(\delta^\text{alg}_0)$ is a component of the $k$th Johnson cokernel, and cannot be detected by the usual Turaev cobracket, but by the framed version of it; see [1] and [14]. By some computer calculations, the first author and Hikoe Enomoto have checked that $[4, 1^5]$ also appears in $\ker(\delta^\text{alg}_8)/\ker(\delta^\text{alg}_0)$. They conjecture that $[3, 1^k-3]$ ($5 \leq k \equiv 0 \mod 4$) and $[4, 1^{k-4}]$ ($9 \leq k \equiv 1 \mod 4$) appear in $\ker(\delta^\text{alg}_k)/\ker(\delta^\text{alg}_0)$. These results and observations suggest that the difference of $\ker(\delta^\text{alg}_k)$ and $\ker(\delta^\text{alg}_0)$ are not so small.

**2. ANDREADAKIS-JOHNSON THEORY FOR Aut $F_n$**

In this section, we review the Andreadakis-Johnson filtration and the Johnson homomorphisms of the automorphism groups of free groups. For details, see [23] for example.

**2.1. Johnson homomorphisms of Aut $F_n$.** Let $F_n$ be a free group of rank $n \geq 2$ with basis $x_1, \ldots, x_n$ and let $\text{Aut} F_n$ be the automorphism group of $F_n$. The group $\text{Aut} F_n$ acts naturally on the abelianization $H := F_n^\text{ab} := F_n/[F_n, F_n]$ of $F_n$. The kernel of this action is called the IA-automorphism group and denoted by $\text{IA}_n$. The basis $x_1, \ldots, x_n$ induces a basis of $H$ and we can identify $\text{Aut} H$ with the general linear group $\text{GL}(n, \mathbb{Z})$. Thus we have the group extension

$$1 \to \text{IA}_n \to \text{Aut} F_n \to \text{GL}(n, \mathbb{Z}) \to 1.$$ 

Let $F_n = \Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of $F_n$. Namely it is defined by $\Gamma_n(1) := F_n$ and $\Gamma_n(k) := [\Gamma_n(k-1), F_n]$ for $k \geq 2$. It is classically known that the associated graded quotient

$$\mathcal{L}_n := \bigoplus_{k \geq 1} \mathcal{L}_n(k), \quad \text{where} \quad \mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1),$$

has the graded Lie algebra structure induced from the commutator bracket on $F_n$ and is isomorphic to the free Lie algebra generated by $H = \mathcal{L}_n(1)$. Moreover, we have the canonical embedding

$$\mathcal{L}_n(k) \hookrightarrow H^\otimes k.$$

The group $\text{Aut} F_n$ acts naturally on $F_n/\Gamma_n(k+1)$. The kernel of this action is denoted by $\mathcal{A}_n(k)$. Then the subgroups $\mathcal{A}_n(k)$ form the descending filtration $\text{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$ which we call the Andreadakis-Johnson filtration. Andreadakis proved the following theorem.

**Theorem 2.1** (Andreadakis [2]).

(i) For any $k, \ell \geq 1$, $\sigma \in \mathcal{A}_n(k)$ and $x \in \Gamma_n(\ell)$, we have $\sigma(x)x^{-1} \in \Gamma_n(k+\ell)$. 
(ii) For any \( k, \ell \geq 1 \), we have \([\mathcal{A}_n(k), \mathcal{A}_n(\ell)] \subset \mathcal{A}_n(k+\ell)\), namely the Andreadakis-Johnson filtration \( \{\mathcal{A}_n(k)\} \) is a descending central filtration of \( \text{IA}_n \).

By Theorem 2.1 (i), for any \( k \geq 1 \) we can define the homomorphism
\[
\tilde{\tau}_k : \mathcal{A}_n(k) \to \text{Hom}_\mathbb{Z}(H, \mathcal{L}_n(k + 1))
\]
by
\[
\sigma \mapsto (x \mod \Gamma_n(2) \mapsto \sigma(x)x^{-1} \mod \Gamma_n(k + 2)).
\]
The kernel of \( \tilde{\tau}_k \) coincides with \( \mathcal{A}_n(k+1) \) and we obtain the injective homomorphism
\[
\tau_k : \text{gr}^k(\mathcal{A}_n) \hookrightarrow \text{Hom}_\mathbb{Z}(H, \mathcal{L}_n(k + 1)) = H^* \otimes_\mathbb{Z} \mathcal{L}_n(k + 1),
\]
where \( \text{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k + 1) \). We call \( \tau_k \) the \( k \)-th Johnson homomorphism of \( \text{Aut} F_n \).

Next, we define a variant of the Johnson homomorphism \( \tau_k \). Let \( \text{IA}_n = \mathcal{A}'_n(1) \supset \mathcal{A}'_n(2) \supset \cdots \) be the lower central series of \( \text{IA}_n \), and set \( \text{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k + 1) \). By Theorem 2.1 (ii), we have \( \mathcal{A}'_n(k) \subset \mathcal{A}_n(k) \) for any \( k \). Thus we obtain the (not necessarily injective) homomorphism
\[
\tau'_k := \tau_k \circ i_k : \text{gr}^k(\mathcal{A}'_n) \to H^* \otimes_\mathbb{Z} \mathcal{L}_n(k + 1),
\]
where the map \( i_k : \text{gr}^k(\mathcal{A}'_n) \to \text{gr}^k(\mathcal{A}_n) \) is induced from the inclusion \( \mathcal{A}'_n(k) \hookrightarrow \mathcal{A}_n(k) \).

The group \( \text{Aut} F_n \) acts naturally on each graded quotient \( \mathcal{L}_n(k) \). Moreover, it acts on the normal subgroup \( \mathcal{A}_n(k) \) by conjugation, and hence on the graded quotients \( \text{gr}^k(\mathcal{A}_n) \) and \( \text{gr}^k(\mathcal{A}'_n) \). The action of the subgroup \( \text{IA}_n \) on these quotients is trivial, and we obtain the well-defined action of the group \( \text{GL}(n, \mathbb{Z}) = \text{Aut} F_n/\text{IA}_n \) on \( \mathcal{L}_n(k) \), \( \text{gr}^k(\mathcal{A}_n) \) and \( \text{gr}^k(\mathcal{A}'_n) \). The homomorphisms \( \tau_k \) and \( \tau'_k \) are \( \text{GL}(n, \mathbb{Z}) \)-equivariant.

In [22], the third author completely determined the structure of the cokernels of \( \tau'_k \) in a stable range. Let \( \mathcal{C}_n(k) \) be the quotient module of \( H^* \otimes^k \) by the action of the cyclic group of order \( k \). Namely,
\[
\mathcal{C}_n(k) := H^* \otimes^k/(a_1 \otimes a_2 \otimes \cdots \otimes a_k - a_2 \otimes \cdots \otimes a_k \otimes a_1 | a_i \in H).
\]
One has \( \mathcal{C}_n(0) = \mathbb{Z} \) and \( \mathcal{C}_n(1) = H \). Let
\[
\pi_k : H^* \otimes^k \to \mathcal{C}_n(k)
\]
be the natural projection, and let \( \Phi_{12} : H^* \otimes^k H^* \otimes^{k+1} \to H^* \otimes^k \) be the contraction map defined by
\[
\Phi_{12}(f \otimes a_1 \otimes a_2 \otimes \cdots a_{k+1}) = f(a_1)a_2 \otimes \cdots \otimes a_{k+1},
\]
where \( f \in H^* \) and \( a_i \in H \). For simplicity, its restriction to \( H^* \otimes \mathcal{L}_n(k + 1) \) is denoted by the same letter: thus we obtain the map
\[
\Phi_{12} : H^* \otimes \mathcal{L}_n(k + 1) \to H^* \otimes^k.
\]

**Theorem 2.2** (Satoh, [22]). Suppose \( k \geq 2 \) and \( n \geq k + 2 \).

(i) The homomorphism \( \pi_k \circ \Phi_{12} : H^* \otimes \mathcal{L}_n(k + 1) \to \mathcal{C}_n(k) \) is surjective.

(ii) We have \( \text{Im} \tau'_k = \text{Ker}(\pi_k \circ \Phi_{12}) \), namely \( \text{Coker}(\tau'_k) \cong \mathcal{C}_n(k) \).

Formulas of the \( \text{GL} \)-irreducible decompositions of \( \mathcal{C}_n(k) \) and \( \text{Im}(\tau'_k) \) are given in [7].
Remark 2.3. Recently Darné [5] showed that the natural map \( i_k : \text{gr}^k(A'_n) \to \text{gr}^k(A_n) \) is surjective for \( n \geq k + 2 \). This means that the stable \( k \)th cokernel Coker(\( \tau_k \)) coincides with Coker(\( \tau'_k \)). Namely, in the stable range, the Johnson cokernels for Aut \( F_n \) are completely determined over \( \mathbb{Z} \).

2.2. A generating set of \( \text{Im} \tau'_k \). Let \( e_1, \ldots, e_n \) be the standard basis of \( H = F^\text{ab} \) induced from the basis \( x_1, \ldots, x_n \) of \( F_n \), and \( e_1^*, \ldots, e_n^* \) the dual basis of \( H^* \). For any \( a_1, a_2, \ldots, a_k \in H \), we set

\[
[a_1, a_2, \ldots, a_k] := [\cdots[[a_1, a_2], a_3], \ldots], a_k] \in L_n(k).
\]

This is called a \( k \)-simple commutator. We have a generating set of \( \text{Im} \tau'_k \) as a \( \mathbb{Z} \)-module in a stable range.

Proposition 2.4. Suppose \( k \geq 2 \) and \( n \geq k + 2 \). Then the image of \( \tau'_k \) is generated as a \( \mathbb{Z} \)-module by the following four types of elements in \( H^* \otimes_\mathbb{Z} L_n(k + 1) \):

\[
(K_1) \ e_i^* \otimes [e_{i_1}, e_{i_2}, \ldots, e_{i_{k+1}}] \quad \text{for any } 1 \leq i, i_1, \ldots, i_{k+1} \leq n \text{ such that } i_1, \ldots, i_{k+1} \neq i,
\]

\[
(K_2) \ e_i^* \otimes [e_{i_1}, e_{i_2}, \ldots, e_{i_k}, e_i] \quad \text{for any } 1 \leq i, i_1, \ldots, i_k \leq n \text{ such that } i_1, \ldots, i_k \neq i.
\]

\[
(K_3) \ e_i^* \otimes [e_{i_1}, e_{i_2}, \ldots, e_{i_k}, e_j] - e_j^* \otimes [e_j, e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}}] \quad \text{for any } 1 \leq i, j, i_1, \ldots, i_{k-1} \leq n \text{ such that } i, j \neq i_1, \ldots, i_k. \text{ (possibly } i = j)\]

\[
(K_4) \ e_i^* \otimes [e_{i_1}, e_{i_2}, \ldots, e_{i_{k+1}}] - \sum_{j=1}^{k+1} \delta_{i,j} e_m^* \otimes [e_{i_1}, \ldots, e_{i_{j-1}}, e_m, e_{i_{j+1}}, \ldots, e_{i_k}, e_{i_{k+1}}] \quad \text{for any } 1 \leq i, m, i_1, \ldots, i_{k+1} \leq n \text{ such that } i = i_j \text{ for some } 1 \leq j \leq k + 1 \text{ and } m \neq i_1, \ldots, i_{k+1}.
\]

Proof. It is easily seen that these elements belong to Ker(\( \pi_k \circ \Phi_{12} \)). In §3.2 in [22], it was shown that these elements belong to \( \text{Im} \tau'_k \). Furthermore, by the arguments in the process of the proof of \( \text{Im} \tau'_k \supset \text{Ker}(\pi_k \circ \Phi_{12}) \), it turns out that the above elements generate \( \text{Ker}(\pi_k \circ \Phi_{12}) \) as a \( \mathbb{Z} \)-module. Since \( \text{Ker}(\pi_k \circ \Phi_{12}) = \text{Im} \tau'_k \), we obtain the required result.

We remark that each of \( \text{gr}^k(A'_n) \) is finitely generated since \( A_n \) is finitely generated. We should also remark that due to a recent work by Church, Ershov and Putman [4], each of \( A'_n(k) \) and \( A_n(k) \) is finitely generated in a stable range. However it seems to be still open to describe an explicit finite generating system of them.

2.3. Contractions and \( \text{Im} \tau'_k \). We generalize the contraction map \( \Phi_{12} \) in §2.1. For each \( 1 \leq \ell \leq k + 1 \), we consider the contraction map \( \Phi_{1,\ell+1} : H^* \otimes_\mathbb{Z} H^{\otimes k+1} \to H^{\otimes k} \) defined by the formula

\[
\Phi_{1,\ell+1}(f \otimes a_1 \otimes \cdots \otimes a_{k+1}) = f(a_\ell) a_1 \otimes \cdots \otimes a_{\ell-1} \otimes a_{\ell+1} \otimes \cdots \otimes a_{k+1},
\]

where \( f \in H^* \) and \( a_\ell \in H \). We denote its restriction to \( H^* \otimes_\mathbb{Z} L_n(k + 1) \) by the same letter: thus we obtain the map

\[
\Phi_{1,\ell+1} : H^* \otimes_\mathbb{Z} L_n(k + 1) \to H^{\otimes k}.
\]

For \( y \in H \) and \( e_i \) for \( 1 \leq j \leq k \), in order to describe the expansion of the simple commutator \( [y, e_{i_1}, \ldots, e_{i_k}] \) in \( H^{\otimes k} \), we introduce the following notation. For an ordered subset \( S = (i_1, i_2, \ldots, i_\ell) \) of the ordered set \( (i_1, i_2, \ldots, i_k) \), define

\[
e_{S} := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_\ell}, \quad e_{S} := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_\ell}.
\]
Let $S^c$ be the ordered complement of $S$. For example, if $S$ is the ordered subset $(2, 4, 5)$ of $(1, 2, \ldots, 6)$, we have $S^c = (1, 3, 6)$ and
\[
e_2^c = e_2 \otimes e_4 \otimes e_5, \quad e_3^c = e_3 \otimes e_4 \otimes e_2, \quad e_6^c = e_6 \otimes e_3 \otimes e_1.
\]
Then, we have
\[
[y, e_{i_1}, \ldots, e_{i_k}] = \sum_{S \subseteq (i_1, i_2, \ldots, i_k)} (-1)^{|S|} e_S \otimes y \otimes e_{S^c}
\]
where $S$ ranges over all ordered subset of $(i_1, i_2, \ldots, i_k)$, and $|S|$ denotes the number of elements in $S$. We can easily obtain the following lemma.

**Lemma 2.5.** As notation above, for any $1 \leq \ell \leq k + 1$, if $i \neq i_1, i_2, \ldots, i_k$ then we have
\[
\Phi_{1, \ell + 1}(e_i^* \otimes [e_1, e_{i_1}, \ldots, e_{i_k}]) = \sum_{S \subseteq (i_1, i_2, \ldots, i_k)} (-1)^{\ell - 1} e_S \otimes e_{S^c}.
\]

For any $1 \leq \ell \leq k + 1$, define the homomorphism
\[
\varpi_\ell : H^\otimes \kappa \rightarrow C_n(\ell - 1) \otimes C_n(k - \ell + 1)
\]
by
\[
\varpi_\ell(a_1 \otimes \cdots a_k) = \pi_{\ell - 1}(a_1 \otimes \cdots \otimes a_{\ell - 1}) \otimes \pi_{k - \ell + 1}(a_{\ell} \otimes \cdots \otimes a_k),
\]
and set
\[
\Theta_\ell := \varpi_\ell \circ \Phi_{1, \ell + 1} : H^\ast \otimes \mathbb{Z} H^\otimes k + 1 \rightarrow C_n(\ell - 1) \otimes C_n(k - \ell + 1).
\]

We denote the restriction of this map to $H^\ast \otimes \mathbb{Z} L_n(k + 1)$ by the same letter:
\[
\Theta_\ell : H^\ast \otimes \mathbb{Z} L_n(k + 1) \rightarrow C_n(\ell - 1) \otimes C_n(k - \ell + 1).
\]

**Theorem 2.6.** Suppose $k \geq 2$ and $n \geq k + 2$. For any $1 \leq \ell \leq k + 1$, we have
\[
\text{Ker}(\Theta_1) \subset \text{Ker}(\Theta_\ell)
\]
in $H^\ast \otimes \mathbb{Z} L_n(k + 1)$.

**Proof.** By Proposition 2.4 and $\text{Ker}(\Theta_1) = \text{Ker}(\pi_k \circ \Phi_{12}) = \text{Im} \tau_k$, it suffices to show that all the generators of type $K_1$, $K_2$, $K_3$ and $K_4$ of $\text{Im} \tau'_k$ belong to $\text{Ker}(\Theta_\ell)$ for any $1 \leq \ell \leq k + 1$. Clearly, generators of type $K_1$ belong to $\text{Ker}(\Theta_\ell)$. Consider a generator of type $K_2$. We have
\[
\Phi_{1, \ell + 1}(e_i^* \otimes (\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\} \otimes e_i - e_i \otimes \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\})) = \begin{cases} 0 & \text{if } \ell \neq 1, k + 1, \\ \pm[e_{i_1}, e_{i_2}, \ldots, e_{i_k}] & \text{if } \ell = 1, k + 1.
\end{cases}
\]

This shows that generators of type $K_2$ belong to $\text{Ker}(\Theta_\ell)$, since $L_n(k)$ is in the kernel of the projection $\pi_k : H^\otimes k \rightarrow C_n(k)$.

For a generator
\[
X = e_i^* \otimes [e_{i_1}, e_{i_1}, \ldots, e_{i_k}] - e_j^* \otimes [e_{j}, e_{i_k}, e_{i_1}, \ldots, e_{i_{k-1}}]
\]
of type $K_3$. By Lemma 2.5, we have
\[
\Phi_{1, \ell + 1}(X) = (-1)^{\ell - 1} \left[ \sum_{S \subseteq (i_1, i_2, \ldots, i_k)} e_S \otimes e_{S^c} - \sum_{T \subseteq (i_{k+1}, i_1, \ldots, i_{k-1})} e_T \otimes e_{T^c} \right].
\]
Here the first sum is written as
\[ \sum_{i_k \in S \mid |S| = \ell - 1} e^{S_i} \otimes e^{S_j} + \sum_{i_k \notin S \mid |S| = \ell - 1} e^{S_i} \otimes e^{S_j}, \]
and the second sum is written as
\[ \sum_{i_k \in T \mid |T| = \ell - 1} e^{T_i} \otimes e^{T_j} + \sum_{i_k \notin T \mid |T| = \ell - 1} e^{T_i} \otimes e^{T_j}. \]
Then we have
\[ \sum_{i_k \in S \mid |S| = \ell - 1} e^{S_i} \otimes e^{S_j} - \sum_{i_k \in T \mid |T| = \ell - 1} e^{T_i} \otimes e^{T_j} = \sum_{S_0 \subseteq (t_1, \ldots, t_{\ell - 1}) \mid |S_0| = \ell - 2} e_{i_k} \otimes e^{S_0} \otimes e^{S_j} - \sum_{T_0 \subseteq (t_1, \ldots, t_{\ell - 1}) \mid |T_0| = \ell - 2} e^{T_0} \otimes e_{i_k} \otimes e^{T_j} = 0 \]
since \( \pi_{\ell - 1}(e_{i_k} \otimes e^{S_0}) = \pi_{\ell - 1}(e^{T_0} \otimes e_{i_k}) \). Similarly, the remaining terms
\[ \sum_{i_k \in S \mid |S| = \ell - 1} e^{S_i} \otimes e^{S_j} - \sum_{i_k \notin T \mid |T| = \ell - 1} e^{T_i} \otimes e^{T_j} \]
are annihilated by \( \varpi_\ell \). This shows that \( \Phi_{1, \ell + 1}(X) \) is in the kernel of \( \varpi_\ell \) and thus generators of type \( K_31 \) belong to \( \text{Ker}(\Theta_\ell) \) for any \( 1 \leq \ell \leq k + 1 \).

Finally, consider a generator
\[ X = e^*_i \otimes \{i_1, i_2, \ldots, i_{k + 1}\} - \sum_{j=1}^{k+1} \delta_{i, i_j} e^*_m \otimes \{i_1, \ldots, i_{j-1}, e_m, i_{j+1}, \ldots, i_k, i_{k+1}\} \]
of type \( K_4 \). Assume \( i_{j_1} = \cdots = i_{j_s} = i \). For any \( 1 \leq \ell \leq k + 1 \), we can calculate \( \Phi_{1, \ell + 1}(e^*_i \otimes \{i_1, i_2, \ldots, i_{k+1}\}) \) by taking all contractions between \( e^*_i \) and \( e_{i_{j_s}} \) for \( 1 \leq s \leq t \). In particular, the contribution of the contraction between \( e^*_i \) and \( e_{i_{j_s}} \) for a fixed \( s \) is equal to that of
\[ \Phi_{1, \ell + 1}(e^*_m \otimes \{i_1, \ldots, i_{j_s-1}, e_m, i_{j_s+1}, \ldots, i_k, i_{k+1}\}). \]
Therefore we see that \( \Phi_{1, \ell + 1}(X) = 0 \). This completes the proof of Theorem 2.6.

3. Structures of the Johnson Cokernels of \( \mathcal{M}_{g,1} \)

In this section, we turn our attention to the mapping class group \( \mathcal{M}_{g,1} \) and prove Theorem 3.1 in Introduction.

3.1. Johnson homomorphisms for \( \mathcal{M}_{g,1} \). We review the Johnson homomorphisms and their cokernels of \( \mathcal{M}_{g,1} \), following [S]. Given a base point on the boundary, the fundamental group \( \pi_1(\Sigma_{g,1}) \) of the surface \( \Sigma_{g,1} \) is a free group \( \mathcal{F}_{2g} \) of rank \( 2g \). Take a basis \( x_1, x_2, \ldots, x_{2g} \) of \( \pi_1(\Sigma_{g,1}) \) such that the product \( \prod_{i=1}^{2g}[x_i, x_{i+g}] \) is parallel to the boundary component. The homology classes \( e_1, \ldots, e_{2g} \) of \( x_1, \ldots, x_{2g} \)
be the lower central series of $I$ where $J$ group.
For any $H$ and commutative diagram.
The kernel of $\pi$ embedding $M$ ral action of the mapping class group $\text{Aut}(\Sigma)$ induced from the Dehn-Nielsen embedding and the Johnson homomorphisms of $\tau$.
Recall from seminal work of Hain [10] shows that the rational images of $\tau$ classes in the Johnson cokernels. These classes are defined by the $\text{Sp}(2, \mathbb{Z})$-homomorphism $g, k$. The space $H$ can identify $k \subset M$ and be $\tau$.

Theorem 3.1 (Hain [10]). We have $\text{Im} \tau^{M}_{k, Q} = \text{Im} \tau^{M}_{k, Q}$ in $H_{Q} \otimes \mathbb{Q} L_{2g}^{Q}(k + 1)$. The space $H^{*}$ is canonically isomorphic to $H$ by the Poicaré duality and we can identify $H^{*} \otimes L_{2g}(k + 1)$ with $H \otimes L_{2g}(k + 1)$. In [18], Morita proved that $\text{Im} \tau^{M}_{k} \subset \mathfrak{h}_{g, 1}(k)$, where $\mathfrak{h}_{g, 1}(k)$ is the kernel of the left bracketing homomorphism $H \otimes L_{2g}(k + 1) \to L_{2g}(k + 2), \ X \otimes u \mapsto [X, u]$.

3.2. Enomoto-Satoh’s obstructions. In [8], Enomoto and Satoh introduced new classes in the Johnson cokernels. These classes are defined by the $\text{Sp}$-homomorphism $c_{k} : \mathfrak{h}_{g, 1}(k) \to H_{Q} \otimes \mathbb{Q} L_{2g}^{Q}(k + 1) \cong H_{Q}^{*} \otimes \mathbb{Q} L_{2g}^{Q}(k + 1) \otimes L_{2g}(k)$.
where $\Theta_1$ has been introduced in \cite{2.3}. The following commutative diagram holds:

\[
\text{Im}(\tau_{k,Q}^M) \xrightarrow{\text{Thm 3.1}} \text{Im}(\tau_{k,Q}^M) \subset \text{Im}(\tau_{k,Q}^M) \subset h_{g,1}(k) \xrightarrow{\Theta_k} C_{2g}(k) \xrightarrow{c_k} \text{Im}(\tau_{k,Q}^M) \subset \text{Ker}(c_k) \subset h_{g,1}(k).
\]

By using Theorem 2.2 and Theorem 3.1 they \cite{8} proved that $\text{Im}(\tau_{k,Q}^M) \subset \text{Ker}(c_k) \subset h_{g,1}(k)$.

### 3.3. Kawazumi-Kuno’s obstructions.

In \cite{15}, Kawazumi and Kuno introduced another type of classes in the Johnson cokernels by using some topological consideration on self-intersections of loops on the surface $\Sigma_{g,1}$. In more detail, they considered an operation called the Turaev cobracket, and showed that its graded version $\delta$ gives rise to an obstruction for the Johnson image. (For more details, see \cite{15} and \cite{16}.)

The map $\delta$ is homogeneous of degree $(-2)$ and the degree $k$ part

\[
\delta_k^{a_k} : H_{\mathbb{Q}}^{\otimes k+2} \rightarrow \bigoplus_{p,q \geq 1} C_{2g}^{\otimes}(p) \otimes C_{2g}^{\otimes}(q)
\]

sends $a_1 \otimes \cdots \otimes a_{k+2}$ to

\[
\sum_{1 \leq i < j \leq k+2, \atop 1 \leq j < i \leq k+1} a_i^* \delta_j \left\{ \begin{array}{c}
\pi(a_{i+1} \otimes \cdots \otimes a_{j-1}) \otimes \pi(a_{j+1} \otimes \cdots \otimes a_{k+2} \otimes a_1 \otimes \cdots \otimes a_{i-1}) \\
-\pi(a_{j+1} \otimes \cdots \otimes a_{k+2} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \pi(a_{i+1} \otimes \cdots \otimes a_{j-1})
\end{array} \right. \}
\]

Here, $a_i^* \in H_{\mathbb{Q}}^*$ is the element corresponding to $a_i \in H_{\mathbb{Q}}$ through the Poincaré duality $H_{\mathbb{Q}}^* = H_{\mathbb{Q}}$, and $\pi$ denotes the projection $\pi : H_{\mathbb{Q}}^{\otimes l} \rightarrow C_{2g}^{\otimes}(l)$ when it is applied to $H_{\mathbb{Q}}^{\otimes l}$. By restriction (and using the same letter), we obtain the map

\[
\delta_k^{a_k} : h_{g,1}(k) \rightarrow \bigoplus_{p,q \geq 1} C_{2g}^{\otimes}(p) \otimes C_{2g}^{\otimes}(q).
\]

In \cite{15}, it was shown that

\[
\text{Im}(\tau_{k,Q}^M) \subset \text{Ker}(\delta_k^{a_k}) \subset h_{g,1}(k).
\]

### 3.4. Proof of Theorem 1.1

Here we give a proof of Theorem 1.1. Recall from \cite{2.3} the homomorphism $\Theta_\ell : H_{\mathbb{Q}}^{\otimes k+1} \rightarrow C_{2g}^{\otimes}(\ell - 1) \otimes C_{2g}^{\otimes}(k - \ell + 1)$. We can regard it as a map from $H_{\mathbb{Q}}^{\otimes k+2} = H_{\mathbb{Q}}^{\otimes k+1}$ by the Poincaré duality.

**Proof of Theorem 1.1** Let $\zeta$ be the cyclic permutation of the components of $H_{\mathbb{Q}}^{\otimes k+2}$ given by $\zeta(a_1 \otimes a_2 \otimes \cdots \otimes a_{k+2}) := a_2 \otimes \cdots \otimes a_{k+2} \otimes a_1$ and set $\zeta_{k+2} := \sum_{i=0}^{k+1} \zeta^i \in \text{End}(H_{\mathbb{Q}}^{\otimes k+2})$. Then, we see that

\[
\delta_k^{a_k} = (\Theta_2 + \cdots + \Theta_k)\zeta_{k+2}
\]

on $H_{\mathbb{Q}}^{\otimes k+2}$. Since any element of $h_{g,1}^Q(k)$ is $\zeta$-invariant in $H_{\mathbb{Q}}^{\otimes k+2}$ (for instance, see \cite[Proposition 5.2]{8}), one has $\delta_k^{a_k} = (k+2)(\Theta_2 + \cdots + \Theta_k)$ on $h_{g,1}^Q(k)$.
The homomorphism $\Theta_1$ is nothing but the trace map $c_k$, and hence $\ker c_k = \ker \Theta_1$. By Theorem 2.6, $\ker \Theta_1 \subset \ker \Theta_\ell$ for any $\ell \geq 2$. Therefore, $\ker c_k \subset \ker \delta_k^{\text{alg}}$ on $\mathfrak{h}_{g,1}(k)$.

**Remark 3.2.** There is a refinement of $\delta_k^{\text{alg}}$ which uses the same formula but we allow $j - i$ to be 1 or $k + 1$ so that $p$ and $q$ can be zero in the target. This map comes from a framed version of the Turaev cobracket and does actually have the same information as $c_k$. For more detail, see [1] and [14].

### 4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We consider polynomial representations of $\text{GL}(2g, \mathbb{Q})$ and rational representations of $\text{Sp}(2g, \mathbb{Q})$. The isomorphism classes of $\text{GL}$-irreducible polynomial representations are parametrized by partitions $\lambda$ such that their lengths $\ell(\lambda)$ are at most $2g$. We denote by $(\lambda)$ the $\text{GL}$-irreducible polynomial representation corresponding to a partition $\lambda$. The isomorphism classes of $\text{Sp}$-irreducible rational representations are parametrized by partitions $\lambda$ such that their lengths $\ell(\lambda)$ are at most $g$. We denote by $[\lambda]$ the $\text{Sp}$-irreducible rational representation corresponding to a partition $\lambda$.

#### 4.1. Anti-Morita obstruction $[1^k]$.

In this subsection, we prove Theorem 1.2(i).

First, we recall the anti-Morita obstruction. In [8], we have the following result.

**Theorem 4.1** ([Enomoto and Satoh [8], Theorem 1]). Suppose $g \geq k + 1$ and $k \equiv 1 \pmod{4}$ and $k \geq 5$. The multiplicities of $\text{Sp}$-irreducible representations $[1^k]$ are exactly one in $\mathfrak{h}_{g,1}(k)/\ker(c_k)$.

We also recall the $\text{GL}$-irreducible decomposition of $C_{2g}^Q(k)$ obtained by [7].

**Lemma 4.2** ([7], Corollary 4.2(2))). Suppose $2g \geq k$. The multiplicity $[C_{2g}^Q(k) : (1^k)]$ of $(1^k)$ in $C_{2g}^Q(k)$ is equal to 1 if $k$ is odd, and 0 if $k$ is even.

**Proof of Theorem 1.2(i).** Note that $g \geq k + 1$ implies $2g \geq k$. Assume $k \equiv 1 \pmod{4}$ and $k \geq 5$.

To prove that the $\text{Sp}$-homomorphism $\delta_k^{\text{alg}} : \mathfrak{h}_{g,1}(k) \to \bigoplus_{p,q \geq 1} C_{2g}^Q(p) \otimes C_{2g}^Q(q)$ annihilates the $\text{Sp}$-irreducible component $[1^k]$ in $\mathfrak{h}_{g,1}(k)/\ker(c_k)$, it is sufficient to show that $[1^k]$ does not appear in all $C_{2g}^Q(p) \otimes C_{2g}^Q(q)$ for $p, q \geq 1$ and $p + q = k$. By the $\text{GL}$-$\text{Sp}$ branching rule, it is enough to show that there is no $\text{GL}(2g, \mathbb{Q})$-irreducible representation $(1^k)$ in $C_{2g}^Q(p) \otimes C_{2g}^Q(q)$ for $p + q = k$ and $p, q \geq 1$.

For partitions $\mu$ and $\nu$ of $p$ and $q$ respectively, suppose $(\mu) \otimes (\nu)$ has the $\text{GL}$-irreducible representation $(1^k)$. If $\ell(\mu) < p$ or $\ell(\nu) < q$, we have $\ell(\mu) + \ell(\nu) < k$. Then by the Littlewood-Richardson rule, there is no $\text{GL}$-irreducible representation $(1^k)$ in $\mu \otimes \nu$. Hence, we consider $\ell(\mu) = p$ and $\ell(\nu) = q$. This case is nothing but $\mu = (1^p)$ and $\nu = (1^q)$. Since $p + q = k \equiv 1 \pmod{4}$, the signatures of $p$ and $q$ are different. By Lemma 4.2, there is no component $(1^p) \otimes (1^q)$ in $C_{2g}^Q(p) \otimes C_{2g}^Q(q)$. This is a contradiction. □

**Remark 4.3.** Especially, for $5 \leq k \equiv 1 \pmod{4}$ and $g \geq k + 1$, an $\text{Sp}$-irreducible component $[1^k]$ appears in $\ker(\Theta_2)/\ker(c_k)$, thus $\ker(c_k) \neq \ker(\Theta_2)$. 


4.2. A hook-type component [3,1⁵]. In this subsection, we prove Theorem 1.2 (ii).

First, in [6, Theorem 1.1], several series of hook-type Sp-irreducible components \([r+1,1^{k-r-1}]\) are detected in \(k\)th Johnson cokernel \(\mathfrak{h}_{g,1}(k)/\ker(c_k)\). An Sp-irreducible representation \([3,1⁵]\) for \(k = 8\) and \(r = 2\) is one of such components.

**Proposition 4.4.** For \(g \geq 9\), an Sp-irreducible component \([3,1⁵]\) appears in \(\mathfrak{h}_{g,1}(8)/\ker(c_8)\).

Note that the multiplicity of \([3,1⁵]\) is larger than or equal to 1 in each \(C^n_2(p) \otimes C^n_2(8-p)\) for \(1 \leq p \leq 7\). Therefore, to prove that \([3,1⁵]\) lies in \(\ker(\delta^k_8)/\ker(c_8)\), we need to use a different way from the previous subsection. We consider a maximal vector which gives a component \([3,1⁵]\) in \(\mathfrak{h}_{g,1}(8)/\ker(c_8)\) and prove that it lies in \(\ker(\Theta_8)\) for \(2 \leq \ell \leq 8\).

As in [3,1] we fix a symplectic basis \([e_1, \ldots, e_g, e_{g+1}, \ldots, e_{2g}]\) of \(H_Q\). Set \(i' := 2g - i + 1\) for each integer \(1 \leq i \leq 2g\). We see that

\[
\langle e_i, e_j \rangle = 0 = \langle e_{i'}, e_{j'} \rangle, \quad \langle e_i, e_{j'} \rangle = \delta_{ij} = -\langle e_{i'}, e_i \rangle, \quad (1 \leq i, j \leq g).
\]

For each integer \(1 \leq i \leq 2g\), we define \(e_i^* = \begin{cases} e_{i'}, & (1 \leq i \leq g), \\ -e_{i'}, & (g + 1 \leq i \leq 2g). \end{cases}\) Then

\[
\langle e_i, e_j^* \rangle = \delta_{ij} \quad \text{for any} \ i, j.
\]

Set \(\omega = \sum_{i=1}^{2g} e_i \otimes e_i^* \in H^2_Q\). We identify \(H_Q\) with \(H^*_Q\) by \(v \mapsto \langle v, \bullet \rangle\). Note that \(\langle e_{i'}, e_r \rangle e_{i'}^* = e_r\) for \(1 \leq r \leq 2g\).

We define

\[
v_{[3,1⁵]} := \omega \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6) \otimes e_1 \otimes e_1 \in H^\otimes Q
\]

where \(e_1 \wedge e_2 \wedge \cdots \wedge e_6\) is the anti-symmetrizer \(\sum_{\sigma \in \mathfrak{S}_6} \text{sgn}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \cdots \otimes e_{\sigma(6)} \in H^\otimes Q\).

Let \(s_i\) be the permutation of \(i\) and \(i + 1\). By the Brauer-Schur-Weyl duality, the set of elements \([v_{[3,1⁵]} : \tau \cdot \theta \cdot \zeta_{10} (\tau \in \mathfrak{S}_{10})]\) generates the space of Sp-maximal vectors corresponding to Sp-irreducible components \([3,1⁵]\) in \(\mathfrak{h}_{g,1}(8)\), where \(\theta = (1 \rightarrow s_2)(1 \rightarrow s_3)(1 \rightarrow s_8)(1 \rightarrow s_2)\) is the Dynkin-Specht-Weyl idempotent and \(\zeta_{10} \in \text{End}(H^\otimes Q)\) is defined in the proof of Theorem 1.1.

In [6], a component \([3,1⁵]\) is detected in \(\mathfrak{h}_{g,1}(8)/\ker(c_8)\) by proving the following claim.

**Proposition 4.5** ([6 Proposition 3.8]). \(c_8(v_{[3,1⁵]} \theta \zeta_{10}) \neq 0\).

Recall from [3,4] that, up to scalar, \(\delta^k_8\) is equal to \(\Theta_2 + \cdots + \Theta_8\). Therefore the following theorem implies Theorem 1.2 (ii).

**Theorem 4.6.** For \(2 \leq \ell \leq 8\), we have \(\Theta_\ell(v_{[3,1⁵]} \theta \zeta_{10}) = 0\).

**Proof.** Note that it is sufficient to prove the claim for \(\ell = 2, 3, 4, 5\). We use the following notations. The \((i, j)\)-expansion operator \(D_{ij} : H^\otimes_Q \rightarrow H^\otimes_{Q+2}\) is given by

\[
(v_1 \otimes \cdots \otimes v_k) D_{ij} = \sum_{r=1}^{2g} v_1 \otimes \cdots \otimes v_{i-1} \otimes e_r \otimes v_i \otimes \cdots \otimes v_{j-2} \otimes e_r^* \otimes v_{j-1} \otimes \cdots \otimes v_k.
\]
The element \( \Lambda_{a,b} \in H^\otimes_Q \) is given by
\[
\sum_{\sigma \in S_6} \text{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(6)}.
\]

In [6, Proposition 3.3], we have
\[
v[3,15]^{\omega} = (e_1 \wedge \cdots \wedge e_6) \otimes e_1^{\otimes 2} \cdot (D_{12} - 3D_{14} + 3D_{16} - D_{18}) + e_1 \otimes (e_1 \wedge \cdots \wedge e_6) \otimes e_1 \cdot (-2D_{13} + 6D_{15} - 6D_{17} + 2D_{19}) + e_1^{\otimes 2} \otimes (e_1 \wedge \cdots \wedge e_6) \cdot (D_{14} - 3D_{16} + 3D_{18} - D_{11,9}).
\]

Let us denote the three terms in the right hand side by \( v_1, v_2 \) and \( v_3 \).
For the 13-contraction operator \( \Phi_{13} \), we obtain
\[
\Phi_{13}(v_1) = 2\Lambda_{1,8} + 2\Lambda_{6,7} - 2\Lambda_{2,3} + 3\Lambda_{1,4} - 3\Lambda_{1,6},
\]
\[
\Phi_{13}(v_2) = (-4g - 2)\Lambda_{1,8} + (-4g - 2)\Lambda_{1,2} - 4\Lambda_{6,8} + 4\Lambda_{2,4},
\]
\[
\Phi_{13}(v_3) = 2\Lambda_{1,2} - 2\Lambda_{3,4} + 2\Lambda_{7,8} - 3\Lambda_{1,4} + 3\Lambda_{1,6}.
\]

Then we have
\[
\Phi_{13}(v[3,15]^{\omega})= -(4g)(\Lambda_{1,2} + \Lambda_{1,8}) + 2(\Lambda_{6,7} - \Lambda_{2,3}) + 4(\Lambda_{2,4} - \Lambda_{6,8}) + 2(\Lambda_{7,8} - \Lambda_{3,4}).
\]

The first term is in the kernel of \( \varpi_2 : H^\otimes_Q \to C_{2g}(1) \otimes C_{2g}(7) \) because \( \Lambda_{1,2} \) and \( \Lambda_{1,8} \) are of the form \( e_1 \otimes \) (a maximal vector with weight \( (2,1^5) \) in \( H^\otimes_Q \)), and \( (2,1^5) \) does not appear in \( C_{2g}(7) \) ([6 Corollary 4.2]). The remaining three terms are also in the kernel of \( \varpi_2 \) because they cancel each other in \( C_{2g}(1) \otimes C_{2g}(7) \). Hence we obtain \( v[3,15]^{\omega} \in \text{Ker}(\Theta_2) \).

For the 14-contraction operator \( \Phi_{14} \), we have
\[
\Phi_{14}(v[3,15]^{\omega}) = (4g)\Lambda_{1,2} + (-6g + 1)(\Lambda_{3,4} + \Lambda_{7,8}) - 2\Lambda_{3,5} - 2\Lambda_{4,5} + 6\Lambda_{4,6} - 6\Lambda_{5,6} + 6\Lambda_{5,7} - 2\Lambda_{6,7} - 2\Lambda_{6,8}.
\]

The first term is in the kernel of \( \varpi_3 \) because \([1^6] \) does not appear in \( C_{2g}(6) \). All the other terms are in the kernel of \( \varpi_3 \) because each term is contained in \( e_i \otimes e_j \otimes v - e_j \otimes e_i \otimes v \in H^\otimes_Q \). Hence we obtain \( v[3,15]^{\omega} \in \text{Ker}(\Theta_3) \).

For the 15-contraction operator \( \Phi_{15} \), we have
\[
\Phi_{15}(v[3,15]^{\omega}) = 12g(\Lambda_{1,8} + \Lambda_{3,4}) + 4(\Lambda_{4,5} - \Lambda_{7,8}) + 4(\Lambda_{5,6} - \Lambda_{6,7}) + 8(\Lambda_{6,8} - \Lambda_{4,6}).
\]

In the first term, by dividing \( \sum_{\sigma \in S_6} \sum_{\sigma(1)=1}^{\sigma(2)=1} + \sum_{\sigma(3)=1}^{\sigma(5)=1} \) they are in the kernel of \( \varpi_4 \). The remaining three terms are also in the kernel of \( \varpi_4 \) because they cancel each other in \( C_{2g}(3) \otimes C_{2g}(5) \). Thus we obtain \( v[3,15]^{\omega} \in \text{Ker}(\Theta_4) \).

For the 16-contraction operator \( \Phi_{16} \), we have
\[
\Phi_{16}(v[3,15]^{\omega}) = -(6g + 1)(\Lambda_{1,2} + \Lambda_{3,4} - \Lambda_{5,6} - \Lambda_{7,8}) + 2(\Lambda_{6,7} - \Lambda_{2,3} + \Lambda_{2,4} - \Lambda_{6,8} + \Lambda_{1,3} - \Lambda_{5,7}).
\]

Since the projection \( H^\otimes_Q \to C_{2g}(4) \) annihilates the elements \( e_i \wedge e_j \wedge e_k \wedge e_l \), all the terms are in the kernel of \( \varpi_5 \). Therefore we obtain \( v[3,15]^{\omega} \in \text{Ker}(\Theta_5) \). \( \square \)
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