Compactness of conformal metrics with constant $Q$-curvature. I

YanYan Li* and Jingang Xiong†

Abstract

We establish compactness for nonnegative solutions of the fourth order constant $Q$-curvature equations on smooth compact Riemannian manifolds of dimension $\geq 5$. If the $Q$-curvature equals $-1$, we prove that all solutions are universally bounded. If the $Q$-curvature is $1$, assuming that Paneitz operator’s kernel is trivial and its Green function is positive, we establish universal energy bounds on manifolds which are either locally conformally flat (LCF) or of dimension $\leq 9$. By assuming a positive mass type theorem for the Paneitz operator, we prove compactness in $C^4$. Positive mass type theorems have been verified recently on LCF manifolds or manifolds of dimension $\leq 7$, when the Yamabe invariant is positive. We also prove that, for dimension $\geq 8$, the Weyl tensor has to vanish at possible blow up points of a sequence of solutions. This implies the compactness result in dimension $\geq 8$ when the Weyl tensor does not vanish anywhere. To overcome difficulties stemming from fourth order elliptic equations, we develop a blow up analysis procedure via integral equations.

Contents

1 Introduction  
2 Preliminaries  
  2.1 Paneitz operator in conformal normal coordinates  
  2.2 Two Pohozaev type identities  
3 Blow up analysis for integral equations

*Supported in part by NSF grants DMS-1065971 and DMS-1203961.
†Supported in part by Beijing Municipal Commission of Education for the Supervisor of Excellent Doctoral Dissertation (20131002701).
1 Introduction

On a compact smooth Riemannian manifold \((M, g)\) of dimension \(\geq 3\), the Yamabe problem, which concerns the existence of constant scalar curvature metrics in the conformal class of \(g\), was solved through the works of Yamabe [57], Trudinger [54], Aubin [2] and Schoen [49]. Different proofs of the Yamabe problem in the case \(n \leq 5\) and in the case \((M, g)\) is locally conformally flat are given by Bahri and Brezis [4] and Bahri [3]. The problem is equivalent to solving the Yamabe equation

\[-L_g u = \text{Sign}(\lambda_1) u^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{on } M,\]

where \(L_g := \Delta_g - \frac{(n-2)}{4(n-1)} R_g\), \(\Delta_g\) is the Laplace-Beltrami operator associated with \(g\), \(R_g\) is the scalar curvature, and \(\text{Sign}(\lambda_1)\) denotes the sign of the first eigenvalue \(\lambda_1\) of the conformal Laplacian \(-L_g\). The sign of \(\lambda_1\) is conformally invariant, i.e., it is the same for every metric in the conformal class of \(g\).

If \(\lambda_1 < 0\), there exists a unique solution of (1). If \(\lambda_1 = 0\), the equation is linear and solutions are unique up to multiplication by a positive constant. If \(\lambda_1 > 0\), non-uniqueness has been established; see Schoen [50] and Pollack [46]. If \((M, g)\) is the standard unit sphere, all solutions are classified by Obata [44] and there is no uniform \(L^\infty\) bound for them. Schoen [52] established a uniform \(C^2\) bound for all solutions if \(M\) is locally conformally flat but not conformal to the sphere. The uniform \(C^2\) bound was established in dimensions \(n < 7\) by Li-Zhang [38] and Marques [43] independently. For \(n = 3, 4, 5\), see works of Li-Zhu [40], Druet [17, 18] and Li-Zhang [37]. For \(8 \leq n \leq 24\), the answer is positive provided that the positive mass theorem holds in these dimensions; see Li-Zhang [38] for \(8 \leq n \leq 11\), and Khuri-Marques-Schoen [32] for \(12 \leq n \leq 24\). On the other hand, the answer is negative in dimension \(n \geq 25\); see Brendle [6] for \(n \geq 52\), and Brendle-Marques [7] for \(25 \leq n \leq 51\).
Compactness of conformal metrics with constant $Q$-curvature

In this paper, we are interested in a fourth order analogue of the Yamabe problem. Namely, the constant $Q$-curvature problem. Let us recall the conformally invariant Paneitz operator and the corresponding $Q$-curvature, which are defined as

\begin{align}
P_g &= \Delta^2_g - \text{div}_g(a_n R_g g + b_n \text{Ric}_g) d + \frac{n-4}{2} Q_g \\
Q_g &= -\frac{1}{2(n-1)} \Delta_g R_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{2}{(n-2)^2} |\text{Ric}_g|^2,
\end{align}

where $R_g$ and $\text{Ric}_g$ denote the scalar curvature and Ricci tensor of $g$ respectively, and $a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}$, $b_n = -\frac{4}{n-2}$. The self-adjoint operator $P_g$ was discovered by Paneitz [45] in 1983, and $Q_g$ was introduced later by Branson [5]. Paneitz operator is conformally invariant in the sense that

- If $n = 4$, for any conformal metric $\hat{g} = e^{2w} g$, $w \in C^\infty(M)$, there holds
  \[ P_{\hat{g}} = e^{-4w} P_g \quad \text{and} \quad P_g w + Q_g = Q_{\hat{g}} e^{4w}. \] (4)

- If $n = 3$ or $n \geq 5$, for any conformal metric $\hat{g} = u^{\frac{4}{n-4}} g$, $0 < u \in C^\infty(M)$, there holds
  \[ P_{\hat{g}}(\phi) = u^{\frac{n+4}{n-4}} P_g(u\phi) \quad \forall \phi \in C^\infty(M). \] (5)

Hence, finding constant $Q$-curvature in the conformal class of $g$ is equivalent to solving

\[ P_g w + Q_g = \lambda e^{4w} \quad \text{on } M \] (6)

if $n = 4$, and

\[ P_g u = \lambda u^{\frac{n+4}{n-4}}, \quad u > 0 \quad \text{on } M, \] (7)

if $n = 3$ or $n \geq 5$, where $\lambda$ is a constant.

When $n = 4$, there is a Chern-Gauss-Bonnet type formula involving the $Q$-curvature; see Chang-Yang [11]. The constant $Q$-curvature problem has been studied by Chang-Yang [10], Djadli-Malchiodi [15], Li-Li-Liu [34] and references therein. Bubbling analysis and compactness for solutions have been studied by Druet-Robert [19], Malchiodi [42], Weinstein-Zhang [56] among others.

When $n \geq 5$, the constant $Q$-curvature problem is a natural extension of the Yamabe problem. However, the lack of maximum principle for fourth order elliptic equations makes the problem much harder. The first eigenvalues of fourth order self-adjoint elliptic operators are not necessarily simple and the associated eigenfunctions may change signs. We might not be

\[ 1 \text{If } n = 4, \frac{1}{2} Q_g \text{ is defined as the } Q\text{-curvature in some papers.} \]
able to divide the study of (7) into three mutually exclusive cases by linking the constant $\lambda$ to the sign of the first eigenvalue of the Paneitz operator. Up to now, the existence of solutions has been obtained with $\lambda = 1$, roughly speaking, under the following three types of assumptions. The first one is on the equation. Assuming, among others, the coefficients of the Paneitz operator are constants, Djadli-Hebey-Ledoux [14] proved some existence results, where they decompose the operator as a product of two second order elliptic operators and use the maximum principle of second order elliptic equations. This assumption is fulfilled, for instance, when the background metric is Einstein. The second one is on the geometry and topology of the manifolds. Assuming that the Poincaré exponent is less than $(n - 4)/2$, Qing-Raske [47, 48] proved the existence result on locally conformally flat manifolds of positive scalar curvature. The last one is purely geometric. Assuming that there exists a conformal metric of nonnegative scalar curvature and semi-positive $Q$-curvature, Gursky-Malchiodi [22] recently proved the existence result for $n \geq 5$. By their condition, the scalar curvature was proved to be positive. In a very recent preprint, Hang-Yang [24] replaced the positive scalar curvature condition by the positive Yamabe invariant (which is equivalent to $\lambda_1 > 0$). More precisely, (7) admits a solution with $\lambda = 1$ if

$$\lambda_1(-L_g) > 0, \quad Q_g \geq 0 \text{ and } Q_g > 0 \text{ somewhere on } M,$$

where $\lambda_1(-L_g)$ is the first eigenvalue of $-L_g$ defined above. See also Hang-Yang [23] for $n = 3$.

Each of the above three types of assumptions implies that

$$\text{Ker} P_g = \{0\} \text{ and the Green's function } G_g \text{ of } P_g \text{ is positive.}$$

In fact, $P_g$ is coercive in Djadli-Hebey-Ledoux [14], Qing-Raske [47, 48] and Gursky-Malchiodi [22]. We refer to the latest paper Gursky-Hang-Lin [21] for further discussions on these conditions. If (9) holds and $\lambda_1 > 0$, there exists a positive mass type theorem for $G_g$, provided $M$ is locally conformally flat or $n = 5, 6, 7$, but not conformal to the standard sphere; see Humbert-Raulot [28], Gursky-Malchiodi [22] and Hang-Yang [25].

Starting from this paper, we study the compactness of solutions of the constant $Q$-curvature equation for $n \geq 5$. For positive constant $Q$-curvature problem, there are non-compact examples. If $(M, g)$ is a sphere, the constant $Q$-curvature metrics are not compact in $C^4$ due to the non-compactness of the conformal diffeomorphism group of the sphere. Recently, Wei-Zhao [55] produced non-compact examples on manifolds of dimension $n \geq 25$ not conformal to the standard sphere.

**Theorem 1.1.** Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 5$, but not conformal to the standard sphere. Assume (9). For $1 < p \leq \frac{n+4}{n-4}$, let $0 < u \in C^4(M)$ be a solution of

$$P_g u = c(n) u^p \quad \text{on } M,$$
where \( c(n) = n(n + 2)(n - 2)(n - 4) \). Suppose that one of the following conditions is also satisfied:

i) \( \lambda_1(-L_g) > 0 \) and \((M, g)\) is locally conformally flat,

ii) \( \lambda_1(-L_g) > 0 \) and \( n = 5, 6, 7 \),

iii) \((M, g)\) is locally conformally flat or \( 5 \leq n \leq 9 \), and the positive mass type theorem holds for the Paneitz operator,

iv) The Weyl tensor of \( g \) does not vanish anywhere, i.e., \( |W_g|^2 > 0 \) on \( M \).

Then there exists a constant \( C > 0 \), depending only on \( M, g \), and a lower bound of \( p - 1 \), such that

\[
\|u\|_{C^4(M)} + \|1/u\|_{C^4(M)} \leq C. \tag{11}
\]

The assumption \( (9) \) in the theorem can be replaced by \( (8) \), as explained above. The positive mass type theorem for Paneitz operator in dimension 8, 9 is understood as in Remark 2.1. The case \( 5 \leq n \leq 9 \) for positive constant \( Q \)-curvature equation shows some similarity to \( 3 \leq n \leq 7 \) for the Yamabe equation with positive scalar curvature.

The following situations, included in Theorem 1.1, were proved before. If \( M \) is locally conformally flat and \( p = \frac{n+4}{n-4} \), \( (11) \) was established by Qing-Raske \([47, 48]\) with the assumptions that \( \lambda_1 > 0 \) and the Poincaré exponent is less than \( (n - 4)/2 \), and by Hebey-Robert \([26, 27]\) with \( C \) depending on the \( H^2 \) norm of \( u \), where they assumed that \( P_g \) is coercive.

Neither \( \lambda_1(-L_g) > 0 \) nor the positive mass type theorem for Paneitz operator is assumed, we have an energy bound of solutions.

**Theorem 1.2.** Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \( n \geq 5 \). Assume \( (9) \), and assume that either \( n \leq 9 \) or \((M, g)\) is locally conformally flat.

Let \( 0 < u \in C^4(M) \) be a solution of \( (10) \). Then

\[
\|u\|_{H^2(M)} \leq C,
\]

where \( C > 0 \) depends only on \( M, g \), and a lower bound of \( p - 1 \).

Next, we establish Weyl tensor vanishing results.

**Theorem 1.3.** Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \( n \geq 8 \). Assume \( (9) \). Let \( u_i \) be a sequence of positive solutions of

\[
P_gu_i = c(n)u_i^{p_i},
\]

where \( p_i \leq \frac{n+4}{n-4} \), \( p_i \rightarrow \frac{n+4}{n-4} \) as \( i \rightarrow \infty \). Suppose that there is a sequence of \( X_i \rightarrow \bar{X} \in M \) such that \( u_i(X_i) \rightarrow \infty \). Then the Weyl tensor has to vanish at \( \bar{X} \), i.e., \( W_g(\bar{X}) = 0 \).
Furthermore, if \( n = 8, 9 \), there exists \( X'_i \to \bar{X} \) such that, for all \( i \),
\[
|W_g(X'_i)|^2 \leq C \begin{cases}  
(\log u_i(X'_i))^{-1}, & \text{if } n = 8, \\
 u_i(X'_i)^{-\frac{4}{n-4}}, & \text{if } n = 9,
\end{cases}
\]
where \( C > 0 \) depends only on \( M \) and \( g \).

**Theorem 1.4.** In addition to the assumptions in Theorem 1.3 with \( n \geq 10 \), we assume that there exist a neighborhood \( \Omega \) of \( \bar{X} \) and a constant \( \bar{b} > 0 \) such that
\[
u_i(X) \leq \bar{b} \cdot \text{dist}_g(X, X_i)^{-\frac{4}{n-4}} \quad \forall X \in \Omega,  \tag{12}
\]

where \( X_i \) is a local maximum point of \( u_i \),
\[
\sup_{\Omega} u_i \leq \bar{b} u_i(X_i). \tag{13}
\]

Then, for sufficiently large \( i \),
\[
|W_g(X_i)|^2 \leq C \begin{cases}  
 u_i(X_i)^{-\frac{4}{n-4}} \log u_i(X_i), & \text{if } n = 10, \\
 u_i(X_i)^{-\frac{4}{n-4}}, & \text{if } n \geq 11,
\end{cases}
\]
where \( C > 0 \) depends only on \( M, g, \text{dist}_g(\bar{X}, \partial \Omega) \) and \( \bar{b} \).

The rates of decay of \( |W_g(X_i)| \) in Theorem 1.3 and Theorem 1.4 correspond to the Yamabe problem case \( n = 6, 7 \) and \( n \geq 8 \) respectively; see theorem 1.3 and theorem 1.2 in [38]. Condition (12) and (13) can often be reduced to, by some elementary consideration, in applications.

In a subsequent paper, we will establish compactness results analogous to those established in \( 8 \leq n \leq 24 \) for the Yamabe equation by Li-Zhang [38, 39] and Khuri-Marques-Schoen [32]. The present paper provides analysis foundations.

For the negative constant \( Q \)-curvature equation, we have

**Theorem 1.5.** Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \( n \geq 5 \). Then for any \( 1 < p < \infty \), there exists a positive constant \( C \), depending only on \( M, g \) and \( p \), such that every nonnegative \( C^4 \) solution of
\[
Pg(u) = -u^p \quad \text{on } M \tag{14}
\]
satisfies
\[
\|u\|_{C^4(M)} \leq C.
\]

The proofs of Theorems 1.1, Theorem 1.3 and Theorem 1.4 make use of important ideas for the proof of compactness of positive solutions of the Yamabe equation, which were outlined first by Schoen [50, 51, 52], as well as methods developed through the work Li [35], Li-Zhu [40], Li-Zhang [37, 38, 39], and Marques [43]. Our main difficulty now stems from the fourth order
Compactness of conformal metrics with constant $Q$-curvature

equation, which we explain in details. To understand the profile of possible blow up solutions, it is natural to scale the solutions in local coordinates centered at local maximum points. By the Liouville theorem in Lin [41], one can conclude that these solutions are close to some standard bubbles in small geodesic balls, whose sizes become smaller and smaller as solutions blowing up; see e.g., Proposition 6.1. Then we need to answer two questions:

(i) Do these blow up points accumulate?

(ii) If not, how do these solutions behave in geodesic balls with some fixed size?

For the first one, we may scale possible blow up points apart and look at them individually. It turns out that we end up with the situation of question (ii). After scaling we need to carry out local analysis. In the Yamabe case, properties of second order elliptic equations, which include the maximum principle, comparison principle, Harnack inequality and Bôcher theorem for isolated singularity, were used crucially. Now we don’t have these properties for fourth order elliptic equations. This leads to an obstruction to using fourth order equations to develop local analysis.

We observe that along scalings the bounds of Green’s function are preserved. In view of Green’s representation, we develop a blow up analysis procedure for integral equations and answer the above two questions completely in dimensions less than 10. This is inspired by our recent joint work with Jin [31] for a unified treatment of the Nirenberg problem and its generalizations, which in turn was stimulated by our previous work on a fractional Nirenberg problem [29, 30]. The approach of the latter two papers were based on the Caffarelli-Silvestre extension developed in [8]. Our analysis is very flexible and can easily be adapted to deal with higher order and fractional order conformally invariant elliptic equations. The organization of the paper is shown in the table of Contents.

Notations. Letters $x, y, z$ denote points in $\mathbb{R}^n$, and capital letters $X, Y, Z$ denote points on Riemannian manifolds. Denote by $B_r(x) \subset \mathbb{R}^n$ the ball centered at $x$ with radius $r > 0$. We may write $B_r$ in replace of $B_r(0)$ for brevity. For $X \in M$, $B_\delta(X)$ denotes the geodesic ball centered at $X$ with radius $\delta$. Throughout the paper, constants $C > 0$ in inequalities may vary from line to line and are universal, which means they depend on given quantities but not on solutions. $f = O^{(k)}(r^m)$ denotes any quantity satisfying $|\nabla^j f(r)| \leq C r^{m-j}$ for all integers $1 \leq j \leq k$, where $k$ is a positive integer and $m$ is a real number. $|S^{n-1}|$ denotes the area of the standard $n - 1$-sphere. Here are specified constants used throughout the paper:

- $c(n) = n(n + 2)(n - 2)(n - 4)$ appears in constant $Q$-curvature equation,

- $\alpha_n = \frac{1}{2(n - 2)(n - 4)|S^{n-1}|}$ appears in the expansion of Green’s functions,

- $c_n = c(n) \cdot \alpha_n = \frac{n(n + 2)}{2|S^{n-1}|}$.  

7
Added note on June 1, 2015: Theorem 1.1 was announced by the first named author in his talk at the International Conference on Local and Nonlocal Partial Differential Equations, NYU Shanghai, China, April 24-26, 2015; while the part of the theorem for general manifolds of dimension $n = 5, 6, 7$ and for locally conformally flat manifolds of dimension $n \geq 5$ was announced in his talk at the Conference on Partial Differential Equations, University of Sussex, UK, September 15-17, 2014. We noticed that two days ago an article was posted on the arXiv, [Gang Li, A compactness theorem on Branson’s $Q$-curvature equation, arXiv:1505.07692v1 [math.DG] 28 May 2015], where a compactness result in dimension $n = 5$, under the assumption that $R_g > 0$ and $Q_g \geq 0$ but not identically equal to zero, was proved independently.

Acknowledgments: J. Xiong is grateful to Professor Jiguang Bao and Professor Gang Tian for their supports.

2 Preliminaries

2.1 Paneitz operator in conformal normal coordinates

Let $(M, g)$ be a smooth Riemannian manifold (with or without boundary) of dimension $n \geq 5$, and $P_g$ be the Paneitz operator on $M$. For any point $X \in M$, it was proved in [33], together with some improvement in [9] and [20], that there exists a positive smooth function $\kappa$ (with control) on $M$ such that the conformal metric $\tilde{g} = \kappa^{-\frac{4}{n-4}} g$ satisfies, in $\tilde{g}$-normal coordinates $\{x_1, \ldots, x_n\}$ centered at $X$,

$$\det \tilde{g} = 1 \quad \text{in } B_\delta$$

for some $\delta > 0$. We refer such coordinates as conformal normal coordinates. Notice that $\det \tilde{g} = 1 + O(|x|^N)$ will be enough for our use if $N$ is sufficiently large. Since one can view $x$ as a tangent vector of $M$ at $X$, thus $\det g(x) = 1 + O(|x|^2)$. It follows that $\kappa(x) = 1 + O(|x|^2)$. In particular,

$$\kappa(0) = 1, \quad \nabla \kappa(0) = 0. \quad \quad (15)$$

In the $\tilde{g}$-normal coordinates,

$$R_{ij}(0) = 0, \quad \text{Sym}_{ijk} R_{ij,k}(0) = 0,$$

$$R_{,i}(0) = 0, \quad \Delta_\tilde{g} R(0) = -\frac{1}{6} |W_\tilde{g}(0)|^2,$$

where the Ricci tensor $R_{ij}$, scalar curvature $R$, and Weyl tensor $W$ are with respect to $\tilde{g}$. We also have

$$\Delta_\tilde{g} = \Delta + \partial_l \tilde{g}^{kl} \partial_k + (\tilde{g}^{kl} - \delta^{kl}) \partial_{kl} =: \Delta + d_l^{(1)} \partial_k + d_l^{(2)} \partial_{kl},$$
Compactness of conformal metrics with constant $Q$-curvature

and

$$\Delta^2_{\tilde{g}} = \Delta^2 + f_k^{(1)} \partial_k + f_{kl}^{(2)} \partial_{kl} + f_{klst}^{(3)} \partial_{klst} + f_{klst}^{(4)} \partial_{klst},$$

where

$$f_k^{(1)} := \Delta d_k^{(1)} + d_s^{(1)} \partial_s d_k^{(1)} + d_{st}^{(2)} \partial_{st} d_k^{(1)} = O(1),$$

$$f_{kl}^{(2)} := \partial_k d_l^{(4)} + d_{kl}^{(2)} + d_k^{(1)} d_l^{(1)} + d_s^{(1)} \partial_s d_{kl}^{(2)} + d_{st}^{(2)} \partial_{st} d_k^{(1)} + d_{st}^{(2)} \partial_{st} d_{kl}^{(2)} = O(1),$$

$$f_{klst}^{(3)} := 2d_s^{(1)} \delta_{kl} + \partial_s d_{kl}^{(2)} + 2d_s^{(1)} d_{kl}^{(2)} + d_{st}^{(2)} \partial_{st} d_{kl}^{(2)} = O(|x|),$$

$$f_{klst}^{(4)} := 2d_{kl}^{(2)} \delta^s + d_{kl}^{(2)} d_{st}^{(2)} = O(|x|^2).$$

Now the second term of the Paneitz operator $P_{\tilde{g}}$ can be expressed as

$$-\text{div}_{\tilde{g}}(a_n R_{\tilde{g}} \tilde{g} + b_n Ric_{\tilde{g}}) d = -\partial_l ((a_n R_{\tilde{g}}_{st} + b_n R_{st}) \tilde{g}^{sl} \tilde{g}^{tl} \partial_k) =: f_k^{(5)} \partial_k + f_{kl}^{(6)} \partial_{kl},$$

where

$$f_k^{(5)} := -\partial_l ((a_n R_{\tilde{g}}_{st} + b_n R_{st}) \tilde{g} - \tilde{g}^{sl} \tilde{g}^{tl}) = O(1),$$

$$f_{kl}^{(6)} := -(a_n R_{\tilde{g}}_{st} + b_n R_{st}) \tilde{g} + \tilde{g}^{sl} \tilde{g}^{tl} = O(|x|).$$

By abusing notations, we relabel $f_k^{(1)}$ as $f_k^{(1)} + f_k^{(5)}$, and $f_{kl}^{(2)}$ as $f_{kl}^{(1)} + f_{kl}^{(6)}$. Hence,

$$E(u) := P_{\tilde{g}} u - \Delta^2 u$$

$$= \frac{n - 4}{2} Q_{\tilde{g}} u + f_k^{(1)} \partial_k u + f_{kl}^{(2)} \partial_{kl} u + f_{klst}^{(3)} \partial_{klst} u + f_{klst}^{(4)} \partial_{klst} u,$$  (16)

where

$$f_k^{(1)} (x) = O(1), \quad f_{kl}^{(2)} (x) = O(1), \quad f_{klst}^{(3)} (x) = O(|x|), \quad f_{klst}^{(4)} (x) = O(|x|^2).$$  (17)

We point out that each term of $f_k^{(1)}$ takes up to three times derivatives of $\tilde{g}$ totally, each term of $f_{kl}^{(2)} (x)$ takes twice, each term of $f_{klst}^{(3)} (x)$ takes once, and no derivative of $\tilde{g}$ is taken in any term of $f_{klst}^{(4)}$. Hence, we see that

$$\|f_k^{(1)}\|_{L^\infty(B_0)} + \|f_{kl}^{(2)}\|_{L^\infty(B_0)} + \|\nabla f_{klst}^{(3)}\|_{L^\infty(B_0)} + \|\nabla^2 f_{klst}^{(4)}\|_{L^\infty(B_0)}$$

$$\leq C \sum_{k \geq 1, 2 \leq k + 1 \leq 4} \|\nabla^k \tilde{g}\|_{L^\infty(B_0)}$$  (18)
Lemma 2.1. In the $\tilde{g}$-normal coordinates, we have, for any smooth radial function $u$,

$$P_\tilde{g}u = \Delta^2 u + \frac{1}{2(n-1)}R_{ij}(0)x^i x^j(c_1^* u' + c_2^* u'') - \frac{4}{9(n-2)r^2} \sum_{kl} (W_{ijkl}(0)x^i x^j)^2 (u'' - \frac{u'}{r})$$

$$+ \frac{n-4}{24(n-1)} |\tilde{g}(0)|^2 u + \left( -\frac{\psi_5(x)}{r^2} + \psi_3(x) \right) u'' - \left( -\frac{\psi_5'(x)}{r^3} + \frac{\psi_3'(x)}{r} \right) u' + \psi_1(x) u$$

$$+ O(r^4) u'' + O(r^3) u' + O(r^2) u,$$

where $r = |x|$, $\psi_k(x), \psi_k'(x)$ are homogeneous polynomials of degree $k$, and

$$c_1^* = \frac{2(n-1)}{(n-2)} - \frac{(n-1)(n-2)}{2} + 6 - n, \quad c_2^* = -\frac{n-2}{2} - \frac{2}{n-2}.$$  \hspace{1cm} (19)

Proof. Since $\det \tilde{g} = 1$ and $u$ is radial, we have $\Delta^2 u = \Delta^2 u$. The rest of the proof is same as that of Lemma 2.8 of [22]. It suffices to expand the coefficients of lower order terms of $P_\tilde{g}$ in Taylor series to a higher order so that $(\frac{\psi_5(x)}{r^2} + \psi_3(x)) u'' - \left( -\frac{\psi_5'(x)}{r^3} + \frac{\psi_3'(x)}{r} \right) u' + \psi_1(x) u$ appears.

If $\text{Ker} P_\tilde{g} = \{0\}$, then $P_\tilde{g}$ has unique Green function $G_\tilde{g}$, i.e., $P_\tilde{g} G_\tilde{g}(\cdot, \cdot) = \delta_X (\cdot)$ for every $X \in M$, where $\delta_X (\cdot)$ is the Dirac measure at $X$ on manifolds $(M, \tilde{g})$. It is easy to check that $\text{Ker} P_\tilde{g} = \{0\}$ is conformally invariant.

Proposition 2.1 ([22], [24]). Let $(M, \tilde{g})$ be a smooth compact Riemannian manifold of dimension $n \geq 5$, on which $\text{Ker} P_\tilde{g} = \{0\}$. Then there exists a small constant $\delta > 0$, depending only on $(M, \tilde{g})$, such that if $\det \tilde{g} = 1$ in the normal coordinate $\{x_1, \ldots, x_n\}$ centered at $\bar{X}$, the Green’s function $G(\bar{X}, \exp_{\bar{X}} x)$ of $P_\tilde{g}$ has the expansion, for $x \in B_{\delta}(0)$,

- If $n = 5, 6, 7$, or $M$ is flat in a neighborhood of $\bar{X}$,

$$G(\bar{X}, \exp_{\bar{X}} x) = \frac{\alpha_n}{|x|^{n-4}} + A + O^{(4)}(|x|),$$

- If $n = 8$,

$$G(\bar{X}, \exp_{\bar{X}} x) = \frac{\alpha_n}{|x|^{n-4}} - \frac{\alpha_n}{1440} |W(\bar{X})|^2 \log |x| + O^{(4)}(1),$$

- If $n \geq 9$,

$$G(\bar{X}, \exp_{\bar{X}} x) = \frac{\alpha_n}{|x|^{n-4}} \left( 1 + \psi_4(x) \right) + O^{(4)}(|x|^{9-n}),$$

where $\alpha_n = \frac{1}{2(n-2)(n-4)(n-1)}$, $A$ is a constant, $W(\bar{X})$ is the Weyl tensor at $\bar{X}$, and $\psi_4(x)$ a homogeneous polynomial of degree 4.
Compactness of conformal metrics with constant $Q$-curvature

**Corollary 2.1.** Suppose the assumptions in Proposition 2.1. Then in the normal coordinate centered at $\bar{X}$ we have

$$G(\exp_{\bar{X}} x, \exp_{\bar{X}} y) = \frac{\alpha_n (1 + O(\frac{4}{4}) + O(\frac{4}{4}))}{|x - y|^{n-4}} + \bar{a} + O(\frac{4}{4}|x - y|^{6-n}),$$

where $x, y \in B_\delta$, $x - y = (x_1 - y_1, \ldots, x_n - y_n)$, $|x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, $\bar{a}$ is a constant and $\bar{a} = 0$ if $n \geq 6$.

**Proof.** We only prove the case that $M$ is non-locally formally flat. Denote $X = \exp_{\bar{X}} x$ and $Y = \exp_{\bar{X}} y$ for $x, y \in B_\delta$, where $\delta > 0$ depends only on $(M, \bar{g})$. For $X \neq \bar{X}$, we can find $g_x = v^{-4} \bar{g}$ such that in the $g_x$-normal coordinate centered at $X$ there hold $\det g_x = 1$ and $v(Y) = 1 + O(\frac{4}{4}(\text{dist}_{g_x}(X, Y)^4)$. Let $G_{g_x}$ be the Green’s function of $P_{g_x}$. By Proposition 2.1.

$$G_{g_x}(X, Y) = \alpha_n \text{dist}_{g_x}(X, Y)^{4-n} + A + O(\frac{4}{4}(\text{dist}_{g_x}(X, Y)^{6-n}),$$

where $A$ is a constant and $A = 0$ if $n \geq 6$. By the conformal invariance of the Paneitz operator, we have the transformation law

$$G(X, Y) = G_{g_x}(X, Y)v(X)v(Y) = G_{g_x}(X, Y)v(X),$$

Since $g_x = v^{-4} \bar{g}$ and $v(Y) = 1 + O(\frac{4}{4}(\text{dist}_{g_x}(X, Y)^4)$. we obtain

$$\text{dist}_{g_x}(X, Y) = \text{dist}_{g_x}(\exp_{\bar{X}} x, \exp_{\bar{X}} y)$$

$$= (1 + O(\frac{4}{4}|x - y|^4)) \text{dist}_{\bar{g}}(\exp_{\bar{X}} x, \exp_{\bar{X}} y)$$

$$= (1 + O(\frac{4}{4}|x - y|^4))(1 + O(\frac{4}{4}|x|^2) + O(\frac{4}{4}|y|^2))|x - y|,$$

where $\bar{g}$ is viewed as a Riemannian metric on $B_\delta$ because of the exponential map $\exp_{\bar{X}}$.

Therefore, we get

$$G(\exp_{\bar{X}} x, \exp_{\bar{X}} y) = \alpha_n \frac{1 + O(\frac{4}{4}||\bar{x}|^2) + O(\frac{4}{4}||y|^2)}{|\bar{x} - y|^{n-4}} + O(\frac{4}{4}||\bar{x} - y|^{6-n}).$$

If $X = \bar{X}$, it follows Proposition 2.1. We complete the proof.

The following positive mass type theorem for Paneitz operator was proved through \[28\], [22] and [24].

**Theorem 2.1.** Let $(M, g)$ be a compact manifold of dimension $n \geq 5$, and $\bar{X} \in M$ be a point. Let $g$ be a conformal metric of $g$ such that $\det \bar{g} = 1$ in the $\bar{g}$-normal coordinate $\{x_1, \ldots, x_n\}$ centered at $\bar{X}$. Suppose also that $\lambda_1(-L_g) > 0$ and \[1\] holds. If $n = 5, 6, 7$, or $(M, g)$ is locally conformally flat, then the constant $A$ in Proposition 2.1. is nonnegative, and $A = 0$ if and only if $(M, g)$ is conformal to the standard $n$-sphere.
Remark 2.1. Suppose the assumptions in Theorem 2.1. If $W(\bar{X}) = 0$, it follows from Proposition 2.1 of [24] that, in the $\bar{g}$-normal coordinates centered at $\bar{X}$, the Green’s function $G$ of $P_{\bar{g}}$ has the expansion

$$G(\bar{X}, \exp_{\bar{X}} x) = \begin{cases} \alpha_8 |x|^{-4} + \psi(\theta) + \log |x|O^{(4)}(|x|), & n = 8, \\ \alpha_9 |x|^{-5}(1 + \frac{R_{i,j}(\bar{X})x^i x^j}{384}) + A + O^{(4)}(|x|), & n = 9, \end{cases}$$

where $x = |x|\theta$, $\psi$ is a smooth function of $\theta$, and $A$ is constant. In dimension $n = 8, 9$, we say the positive mass type theorem holds for Paneitz operator if $\int_{S^{n-1}} \psi(\theta) \, d\theta > 0$ and $A > 0$ respectively.

Let

$$U_\lambda(x) := \left(\frac{\lambda}{1 + \lambda^2 |x|^2}\right)^{\frac{n+4}{4}}, \quad \lambda > 0,$$

which is the unique positive solution of $\Delta^2 u = c(n)u^{\frac{n+4}{n-4}}$ in $\mathbb{R}^n$, $n \geq 5$, up to translations by Lin [41]. By Lemma 2.1 in the $\bar{g}$-normal coordinates we have

$$P_{\bar{g}} U_\lambda = c(n)U_\lambda^{\frac{n+4}{n-4}} + f_\lambda U_\lambda,$$  \hfill (20)

where $f_\lambda(x)$ is a smooth function satisfying that $\lambda^{-k} |\nabla^k f_\lambda(x)|$, $k = 0, 1, \ldots, 5$, is uniformly bounded in $B_\delta$ independent of $\lambda \geq 1$. Indeed, by direct computations

$$\partial_r U_\lambda = (4 - n)\lambda^\frac{n}{4}(1 + \lambda^2 r^2)^{\frac{2-n}{2}} r$$

$$\partial^2_{rr} U_\lambda = (4 - n)(2 - n)\lambda^\frac{n+4}{4}(1 + \lambda^2 r^2)^{\frac{2-n}{2}} r^2 + (4 - n)\lambda^\frac{n}{2}(1 + \lambda^2 r^2)^{\frac{2-n}{2}}.$$

Inserting them to the expression in Lemma 2.1 (20) follows.

Corollary 2.2. Let $(M, \bar{g})$ be a smooth compact Riemannian manifold of dimension $n \geq 5$, on which $\text{Ker} P_{\bar{g}} = \{0\}$. Then there exists a small constant $\delta > 0$, depending only on $(M, \bar{g})$, such that if $\det \bar{g} = 1$ in the normal coordinate $\{x_1, \ldots, x_n\}$ centered at $\bar{X}$, then

$$U_\lambda(x) = c(n) \int_{B_\delta} G(\exp_{\bar{X}} x, \exp_{\bar{X}} y)\{U_\lambda(y)^{\frac{n+4}{n-4}} + c'_\lambda(x)U_\lambda(y)\} \, dy + c''_\lambda(x),$$

where $\delta > 0$ depends only on $M, \bar{g},$ and $c'_\lambda, c''_\lambda$ are smooth functions satisfying

$$\lambda^{-k} |\nabla^k c'_\lambda(x)| \leq C, \quad |\nabla^k c''_\lambda(x)| \leq C\lambda^{\frac{k+n}{2}},$$

for $k = 0, 1, \ldots, 5$ and some $C > 0$ independent of $\lambda \geq 1$. 

Y. Y. Li & J. Xiong
Proof. Let \( \eta(x) = \eta(|x|) \) be a smooth cutoff function satisfying
\[
\eta(t) = 1 \text{ for } t < \delta/2, \quad \eta(t) = 0 \text{ for } t > \delta.
\]

By the Green’s representation formula, we have
\[
(U_\lambda \eta)(x) = \int_{B_\delta} G(\exp \bar{X} x, \exp \bar{X} y) P_\tilde{g}(U_\lambda \eta)(y) \, dy.
\]

Making use of (20) and Lemma 2.1, we see that \( c'_\lambda = f_\lambda c(n) \) and proof is finished.

\[\square\]

2.2 Two Pohozaev type identities

For \( r > 0 \), define in Euclidean space
\[
\mathcal{P}(r, u) := \int_{\partial B_r} \frac{n - 4}{2} \left( \Delta u \frac{\partial u}{\partial \nu} - u \frac{\partial (\Delta u)}{\partial \nu} \right) - \frac{r}{2} |\Delta u|^2
\]
\[\quad - x^k \partial_k u \frac{\partial}{\partial \nu} (\Delta u) + \Delta u \frac{\partial}{\partial \nu} (x^k \partial_k u) \, dS,
\]
where \( \nu = \frac{\bar{z}}{r} \) is the outward normal to \( \partial B_r \).

Proposition 2.2. Let \( 0 < u \in C^4(\bar{B}_r) \) satisfy
\[
\Delta^2 u + E(u) = K u^p \quad \text{in } B_r,
\]
where \( E : C^4(\bar{B}_r) \to C^0(\bar{B}_r) \) is an operator, \( p > 0, r > 0 \) and \( K \in C^1(\bar{B}_r) \). Then
\[
\mathcal{P}(r, u) = \int_{B_r} (x^k \partial_k u + \frac{n - 4}{2} u) E(u) \, dx + \mathcal{N}(r, u),
\]
where
\[
\mathcal{N}(r, u) := \left( \frac{n}{p + 1} - \frac{n - 4}{2} \right) \int_{B_r} Ku^{p+1} \, dx + \frac{1}{p + 1} \int_{B_r} x^k \partial_k K u^{p+1} \, dx
\]
\[\quad - \frac{r}{p + 1} \int_{\partial B_r} K u^{p+1} \, dS.
\]

Proof. A similar Pohozaev identity without \( E(u) \) was derived in [16]. We present the proof for completeness. For any \( u \in C^4(\bar{B}_r) \), by Green’s second identity we have
\[
\int_{B_r} u \Delta^2 u \, dx = \int_{B_r} (\Delta u)^2 \, dx + \int_{\partial B_r} u \frac{\partial}{\partial \nu} (\Delta u) - \frac{\partial u}{\partial \nu} \Delta u \, dS
\]
\[13\]
and
\[ \int_{B_r} x^k \partial_k u \Delta^2 u \, dx = \int_{B_r} \Delta(x^k \partial_k u) \Delta u \, dx + \int_{\partial B_r} x^k \partial_k u \frac{\partial}{\partial \nu} (\Delta u) - \frac{\partial}{\partial \nu} (x^k \partial_k u) \Delta u \, dS. \]

Using Green's first identity, we have
\[ \int_{B_r} \Delta(x^k \partial_k u) \Delta u \, dx = 2 \int_{B_r} (\Delta u)^2 \, dx + \frac{1}{2} \int_{B_r} x^k \partial_k (\Delta u)^2 \, dx \]
\[ = \frac{4 - n}{2} \int_{B_r} (\Delta u)^2 \, dx + \int_{\partial B_r} r (\Delta u)^2 \, dS. \]

Therefore, we obtain
\[ \frac{n - 4}{2} \int_{B_r} u \Delta^2 u \, dx + \int_{B_r} x^k \partial_k u \Delta^2 u \, dx \]
\[ = \frac{n - 4}{2} \int_{B_r} u \frac{\partial}{\partial \nu} (\Delta u) - \frac{\partial u}{\partial \nu} \Delta u \, dS \]
\[ + \int_{\partial B_r} r \frac{1}{2} |\Delta u|^2 + x^k \partial_k u \frac{\partial}{\partial \nu} (\Delta u) - \frac{\partial}{\partial \nu} (x^k \partial_k u) \Delta u \, dS. \]

By the equation of \( u \) we get
\[ \mathcal{P}(r, u) = \int_{B_r} (x^k \partial_k u + \frac{n - 4}{2} u) E(u) \, dx - \int_{B_r} (x^k \partial_k u + \frac{n - 4}{2} u) K w^p \, dx. \]

Since
\[ \int_{B_r} x^k \partial_k u K w^p \, dx = \frac{1}{p + 1} \int_{B_r} K x^k \partial_k u^{p+1} \, dx \]
\[ = -\frac{n}{p + 1} \int_{B_r} K w^{p+1} \, dx - \frac{1}{p + 1} \int_{B_r} x^k \partial_k K w^{p+1} \]
\[ + \frac{r}{p + 1} \int_{\partial B_r} K w^{p+1} \, dS, \]

we complete the proof. \( \square \)

**Lemma 2.2.** For \( G(x) = |x|^{4-n} + A + O^4(|x|) \), where \( A \) is constant. Then
\[ \lim_{r \to 0} \mathcal{P}(r, G) = -(n - 4)^2 (n - 2) A |S^{n-1}|. \]

The following proposition is a special case of Proposition 2.15 of [31].

14
Compactness of conformal metrics with constant $Q$-curvature

**Proposition 2.3.** For $R > 0$, let $0 \leq u \in C^1(\overline{B}_R)$ be a solution of

$$u(x) = \int_{B_R} \frac{K(y)u(y)^p}{|x - y|^{n-4}} \, dy + h_R(x),$$

where $p > 0$, and $h_R(x) \in C^1(B_R)$, $\nabla h_R \in L^1(B_R)$. Then

$$\left(\frac{n-4}{2} - \frac{n}{p+1}\right) \int_{B_R} K(x)u(x)^p \, dx - \frac{1}{p+1} \int_{B_R} x \nabla K(x)u(x)^{p+1} \, dx$$

$$= \frac{n-4}{2} \int_{B_R} K(x)u(x)^p h_R(x) \, dx + \int_{B_R} x \nabla h_R(x)K(x)u(x)^p \, dx$$

$$- \frac{R}{p+1} \int_{\partial B_R} K(x)u(x)^{p+1} \, dS.$$

### 3 Blow up analysis for integral equations

In the section, the idea of dealing with integral equation is inspired by [31], but we have to consider general integral kernels and remainder terms. We will use $A_1$, $A_2$, $A_3$ to denote positive constants, and $\{\tau_i\}_{i=1}^\infty$ to denote a sequence of nonnegative constants satisfying $\lim_{i \to \infty} \tau_i = 0$. Set

$$p_i = \frac{n+4}{n-4} - \tau_i. \quad (23)$$

Let $\{G_i(x, y)\}_{i=1}^\infty$ be a sequence of functions on $B_3 \times B_3$ satisfying

$$G_i(x, y) = G_i(y, x), \quad G_i(x, y) \geq A_1^{-1}|x - y|^{4-n},$$

$$|\nabla^l_x G_i(x, y)| \leq A_1|x - y|^{4-n-l}, \quad l = 0, 1, \ldots, 5$$

$$G_i(x, y) = c_n + O(1)(|x|^2 + |y|^2) + \tilde{a}_i + O(1)(|x - y|^{n-6}) \quad (24)$$

for all $x, y \in \overline{B}_3$, where $c_n = \frac{n(n+2)}{2(n-4)}$ is the constant given towards the end of the introduction, $f = O(1)(r^m)$ denotes any quantity satisfying $|\nabla^j f(r)| \leq A_1 r^{m-j}$ for all integers $1 \leq j \leq 4$, and $\tilde{a}_i$ is a constant and $\tilde{a}_i = 0$ if $n \geq 6$. Let $\{K_i\}_{i=1}^\infty \in C^\infty(\overline{B}_3)$ satisfy

$$\lim_{i \to \infty} K_i(0) = 1, \quad K_i \geq A_2^{-1}, \quad \|K_i\|_{C^5(B_3)} \leq A_2. \quad (25)$$

Let $\{h_i\}_{i=1}^\infty$ be a sequence of nonnegative functions in $C^\infty(B_3)$ satisfying

$$\max_{B_r(x)} h_i \leq A_2 \min_{B_r(x)} h_i,$$

$$\sum_{j=1}^5 r^j|\nabla^j h_i(x)| \leq A_2 \|h_i\|_{L^\infty(B_r(x))} \quad (26)$$
for all $x \in B_2$ and $0 < r < 1/2$.

Given $p_i, G_i, K_i$, and $h_i$ satisfying (23)-(26), let $0 \leq u_i \in L^{\frac{2n}{n-4}}(B_3)$ be a solution of

$$u_i(x) = \int_{B_3} G_i(x, y) K_i(y) u_i^p(y) \, dy + h_i(x) \quad \text{in } B_3. \quad (27)$$

It follows from [36] and Proposition A.2 that $u_i \in C^4(B_3)$. In the following we will always assume $u_i \in C^4(B_3)$.

We say that $\{u_i\}$ blows up if $\|u_i\|_{L^{\infty}(B_3)} \to \infty$ as $i \to \infty$.

**Definition 3.1.** We say a point $\bar{x} \in B_3$ is an isolated blow up point of $\{u_i\}$ if there exist $0 < \rho < \text{dist}(\bar{x}, \partial B_3)$, $C > 0$, and a sequence $x_i$ tending to $\bar{x}$, such that, $x_i$ is a local maximum of $u_i$, $u_i(x_i) \to \infty$, and $u_i(x) \leq C|x - x_i|^{-4/(p_i - 1)}$ for all $x \in B_{\rho}(x_i)$.

Let $x_i \to \bar{x}$ be an isolated blow up of $u_i$. Define

$$\overline{u}_i(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x_i)} u_i \, dS, \quad r > 0, \quad (28)$$

and

$$\overline{w}_i(r) = r^{4/(p_i - 1)} \overline{u}_i(r), \quad r > 0.$$

**Definition 3.2.** We say $x_i \to \bar{x} \in B_3$ is an isolated simple blow up point, if $x_i \to \bar{x}$ is an isolated blow up point, such that, for some $\rho > 0$ (independent of $i$) $\overline{w}_i$ has precisely one critical point in $(0, \rho)$ for large $i$.

**Lemma 3.1.** Given $p_i, G_i, K_i$ and $h_i$ satisfying (23)-(26), let $0 \leq u_i \in C^4(B_3)$ be a solution of (27). Suppose that $0$ is an isolated blow up point of $\{u_i\}$ with $\bar{r} = 2$, i.e., for some positive constant $A_3$ independent of $i$,

$$u_i(x) \leq A_3|x|^{-4/(p_i - 1)} \quad \text{for all } x \in B_2. \quad (29)$$

Then for any $0 < r < 1/3$ we have

$$\sup_{B_{2r} \setminus B_{r/2}} u_i \leq C \inf_{B_{2r} \setminus B_{r/2}} u_i,$$

where $C$ is a positive constant depending only on $n, A_1, A_2, A_3$.  

16
Compactness of conformal metrics with constant $Q$-curvature

**Proof.** For every $0 < r < 1/3$, set

$$w_i(x) = r^{4/(p_i-1)}u_i(rx).$$

By the equation of $u_i$, we have

$$w_i(x) = \int_{B_{3/r}} G_{i,r}(x, y) K_i(ry) w_i(y)^{p_i} \, dy + \tilde{h}_i(x) \quad x \in B_{3/r},$$

where

$$G_{i,r}(x, y) = r^{n-4} G_i(rx, ry) \quad \text{for } r > 0$$

and $\tilde{h}_i(x) := r^{4/(p_i-1)}h_i(rx)$. Since 0 is an isolated blow up point of $u_i$,

$$w_i(x) \leq A_3 |x|^{-4/(p_i-1)} \quad \text{for all } x \in B_3. \quad (30)$$

Set $\Omega_1 = B_{5/2} \setminus B_{1/4}$, $\Omega_2 = B_2 \setminus B_{1/2}$ and $V_i(y) = K_i(ry) w_i(y)^{p_i-1}$. Thus $w_i$ satisfies the linear equation

$$w_i(x) = \int_{\Omega_1} G_{i,r}(x, y) V_i(y) w_i(y) \, dy + \tilde{h}_i(x) \quad \text{for } x \in B_{5/2} \setminus B_{1/4},$$

where

$$\tilde{h}_i(x) = \bar{h}_i(x) + \int_{B_{3/r} \setminus \Omega_1} G_{i,r}(x, y) K_i(ry) w_i(y)^{p_i} \, dy.$$

By (30) and (25), $\|V_i\|_{L^\infty(\Omega_1)} \leq C(n, A_1, A_2, A_3) < \infty$. Since $K_i$ and $w_i$ are nonnegative, by (24) on $G_i$ and (26) on $h_i$ we have $\max_{\Omega_2} \tilde{h}_i \leq C(n, A_1, A_2) \min_{\Omega_2} \bar{h}_i$. Applying Proposition A.1 to $w_i$ gives

$$\max_{\Omega_2} w_i \leq C \min_{\Omega_2} w_i,$$

where $C > 0$ depends only on $n$, $A_1$, $A_2$ and $A_3$. Rescaling back to $u_i$, the lemma follows. \qed

**Proposition 3.1.** Suppose that $0 \leq u_i \in C^4(B_3)$ is a solution of (27) and all assumptions in Lemma 3.1 hold. Let $R_i \to \infty$ with $R_i^\gamma = 1 + o(1)$ and $\varepsilon_i \to 0^+$, where $o(1)$ denotes some quantity tending to 0 as $i \to \infty$. Then we have, after passing to a subsequence (still denoted as $\{u_i\}$, $\tau_i$ and etc . . . ),

$$\|m_i^{-1} u_i(m_i^{-(p_i-1)/4}) \cdot (1 + | \cdot |^2)^{(4-n)/2}} \|_{C^3(B_{2R_i}(0))} \leq \varepsilon_i,$$

$$r_i := R_i m_i^{-(p_i-1)/4} \to 0 \quad \text{as } i \to \infty,$$

where $m_i = u_i(0)$.  

17
Proof. Let
\[ \varphi_i(x) = m_i^{-1} u_i(m_i^{-(p_i-1)/4} x) \quad \text{for} \ |x| < 3m_i^{(p_i-1)/4}. \]

By the equation of \( u_i \), we have,
\[ \varphi_i(x) = \int_{B_{3m_i^{(p_i-1)/4}}} \tilde{G}_i(x, y) \tilde{K}_i(y) \varphi_i(y) p_i \, dy + \tilde{h}_i(x), \quad (31) \]
where
\[ \tilde{G}_i(x, y) = G_{i, m_i^{-(p_i-1)/4}(x, y)}, \quad \tilde{K}_i(y) = K_i(m_i^{-(p_i-1)/4} y) \quad \text{and} \quad \tilde{h}_i(x) = m_i^{-1} h_i(m_i^{-(p_i-1)/4} x). \]

First of all, \( \max_{\partial B_1} \partial h_i \leq \max_{\partial B_1} \partial u_i \leq A_3 \), by (26) we have \( \tilde{h}_i \to 0 \) in \( C^5_{\text{loc}}(\mathbb{R}^n) \) as \( i \to \infty \).

Secondly, since 0 is an isolated blow up point of \( u_i \),
\[ \varphi_i(0) = 1, \quad \nabla \varphi_i(0) = 0, \quad 0 < \varphi_i(x) \leq A_3 |x|^{-4/(p_i-1)}. \quad (33) \]

For any \( R > 0 \), we claim that
\[ \| \varphi_i \|_{C^4(B_R)} \leq C(R) \quad (34) \]
for sufficiently large \( i \).

Indeed, by Proposition A.2 and (33), it suffices to prove that \( \varphi_i \leq C \) in \( B_1 \). If \( \varphi_i(\bar{x}_i) = \sup_{B_1} \varphi_i \to \infty \), set
\[ \tilde{\varphi}_i(z) = \varphi_i(\bar{x}_i)^{-1} \varphi_i(\bar{x}_i)^{-1/4} z + \bar{x}_i \leq 1 \quad \text{for} \ |z| \leq \frac{1}{2} \varphi_i(\bar{x}_i)^{(p_i-1)/4}. \]

By (33),
\[ \tilde{\varphi}_i(z_i) = \varphi_i(\bar{x}_i)^{-1} \varphi_i(0) \to 0 \]
for \( z_i = -\varphi_i(\bar{x}_i)^{(p_i-1)/4} \bar{x}_i \). Since \( \varphi_i(\bar{x}_i) \leq A_3 |\bar{x}_i|^{-4/(p_i-1)} \), we have \( |z_i| \leq A_3^{4/(p_i-1)} \). Hence, we can find \( t > 0 \) independent of \( i \) such that such that \( z_i \in B_t \). Applying Proposition A.1 to \( \tilde{\varphi}_i \) in \( B_{2t} \) (since \( \tilde{\varphi}_i \) satisfies a similar equation to (31)), we have
\[ 1 = \tilde{\varphi}_i(0) \leq C \tilde{\varphi}_i(z_i) \to 0, \]
which is impossible. Hence, \( \varphi_i \leq C \) in \( B_1 \).

It follows from (34) that there exists a function \( \varphi \in C^4(\mathbb{R}^n) \) such that, after passing subsequence,
\[ \varphi_i(x) \to \varphi \quad \text{in} \ C^4_{\text{loc}}(\mathbb{R}^n) \quad \text{as} \ i \to \infty. \quad (35) \]
Compactness of conformal metrics with constant $Q$-curvature

Thirdly, for every $R > 0$, let
\[ g_i(R, x) := \int_{B_{3m_i^{(p_i-1)/4}} \setminus B_R} \tilde{G}_i(x, y) \tilde{K}_i(y) \varphi_i(y)^{p_i} \, dy. \]

Since $K_i$ and $\varphi_i$ are nonnegative, a simple computation using (24) gives that, for any $x \in B_{R-1}$,
\[ |\nabla^k g_i(R, x)| \leq C g_i(R, x), \quad k = 1, \ldots, 5. \]

Note that $g_i(R, x) \leq \varphi_i(x) \leq C(R)$, it follows that, after passing to a subsequence,
\[ g_i(R, x) \to g(R, x) \geq 0 \quad \text{in } C^4(B_{R-1}) \quad \text{as } i \to \infty. \tag{36} \]

By (24) and (25), we have
\[ \tilde{G}_i(x, y) \to \frac{1}{|x - y|^{n-4}} \quad \forall x \neq y \]
and $\tilde{K}_i(y) \to K_i(0) = 1$. Combining (32), (35) and (36) together, by (31) we have that for any fixed $R > 0$ and $x \in B_{R-1}$
\[ g(R, x) = \varphi(x) - c_n \int_{B_R} \frac{\varphi(y)^{n/4}}{|x - y|^{n-4}} \, dy. \tag{37} \]

By (37), $g(R, x)$ is non-increasing in $R$. For any fixed $x$ and $|y| \geq R \gg |x|$, by (24) we have
\[ \frac{G_{i,m_i^{(p_i-1)/4}}(x, y)}{G_{i,m_i^{(p_i-1)/4}}(0, y)} = \frac{G_{i,|y|m_i^{(p_i-1)/4}}(x, y)}{G_{i,|y|m_i^{(p_i-1)/4}}(0, y)} = 1 + O\left(\frac{|x|}{|y|}\right). \]

Hence, $g_i(R, x) = (1 + O\left(\frac{|x|}{R}\right))g_i(R, 0)$, which implies
\[ \lim_{R \to \infty} g(R, x) = \lim_{R \to \infty} g(R, 0) := c_0 \geq 0. \tag{38} \]

Sending $R$ to $\infty$ in (37), it follows from Lebesgue’s monotone convergence theorem that
\[ \varphi(x) = c_n \int_{\mathbb{R}^n} \frac{\varphi(y)^{n/4}}{|x - y|^{n-4}} \, dy + c_0 \quad x \in \mathbb{R}^n. \]

We claim that $c_0 = 0$. If not,
\[ \varphi(x) - c_0 = c_n \int_{\mathbb{R}^n} \frac{\varphi(y)^{n/4}}{|y|^{n-4}} \, dy > 0, \]
which implies that
\[ 1 = \varphi(0) \geq c_n \int_{\mathbb{R}^n} \frac{c_0}{|x - y|^{n-4}} = \infty. \]
This is impossible.

The use of monotonicity in the above argument is taken from [31].

It follows from the classification theorem in [12] or [36] that
\[ \varphi(x) = \left(1 + |x|^2\right)^{-\frac{n-4}{2}}, \]
where we have used that \( \varphi(0) = 1 \) and \( \nabla \varphi(0) = 0 \).

The proposition follows immediately.

Since passing to subsequences does not affect our proofs, in the rest of the paper we will always choose \( R_i \to \infty \) with \( R_{i_1}^n = 1 + o(1) \) first, and then \( \varepsilon_i \to 0^+ \) as small as we wish (depending on \( R_i \)) and then choose our subsequence \( \{u_{j_i}\} \) to work with. Since \( i \leq j_i \) and \( \lim_{i \to \infty} \tau_{j_i} = 0 \), one can ensure that \( R_{j_i}^n = 1 + o(1) \) as \( i \to \infty \). In the sequel, we will still denote the subsequences as \( u_i, \tau_i \) and etc.

**Remark 3.1.** By checking the proof of Proposition 3.1 together with the fact \( \nabla^2 (1 + |x|^2)^{-\frac{n-4}{2}} \)
is negatively definite near zero and the \( C^2 \) convergence in a fixed neighborhood of zero, the following statement holds. Let \( 0 \leq u_i \in C^4(B_3) \) be a solution of (27) and satisfy (29). Suppose that \( u_i(0) \to \infty \) as \( i \to \infty \), \( \nabla u_i(0) = 0 \) and \( \max_{B_3} u_i \leq b u_i(0) \) for some constant \( b \geq 1 \) independent of \( i \). Then, after passing to a subsequence, \( 0 \) must be a local maximum point of \( u_i \) for \( i \) large. Namely, \( 0 \) is an isolated blow up point of \( u_i \) after passing to a subsequence.

**Proposition 3.2.** Under the hypotheses of Proposition 3.1 there exists a constant \( C > 0 \), depending only on \( n, A_1, A_2 \) and \( A_3 \), such that,
\[ u_i(x) \geq C^{-1} m_i (1 + m_i^{(n-1)/2} |x|^2)^{(4-n)/2}, \quad |x| \leq 1. \]

In particular, for any \( e \in \mathbb{R}^n, |e| = 1 \), we have
\[ u_i(e) \geq C^{-1} m_i^{-1 + (n-4)/4} \tau_i. \]
Compactness of conformal metrics with constant $Q$-curvature

**Proof.** By change of variables and using Proposition 3.1, we have for $r_i \leq |x| \leq 1$,

$$u_i(x) \geq C^{-1} \int_{|y| \leq r_i} \frac{u_i(y)^{p_i}}{|x - y|^{n-4}} \, dy$$

$$\geq C^{-1} m_i \int_{|z| \leq R_i} \frac{(m_i^{-1} u_i(m_i^{-(p_i-1)/4} z))^p_i}{m_i^{(p_i-1)/4} x - z|^{n-4}} \, dz$$

$$\geq C^{-1} m_i \int_{|z| \leq R_i} \frac{U_1(z)^{p_i}}{m_i^{(p_i-1)/4} x - z|^{n-4}} \, dz$$

$$\geq \frac{1}{2} C^{-1} m_i \int_{\mathbb{R}^n} \frac{U_1(z)^{n+4}}{m_i^{(p_i-1)/4} x - z|^{n-4}} \, dz$$

$$= \frac{1}{2} C^{-1} m_i U_1(m_i^{(p_i-1)/4} x).$$

Recall that

$$U_\lambda(z) = \left( \frac{\lambda}{1 + \lambda^2 |z|^2} \right)^{(n-4)/2}, \quad \lambda > 0.$$ 

The proposition follows immediately.

**Lemma 3.2.** Suppose the hypotheses of Proposition 3.1 and in addition that $0$ is also an isolated simple blow up point with the constant $\rho > 0$. Then there exist $\delta_i > 0$, $\delta_i = O(R_i^{-4})$, such that

$$u_i(x) \leq C u_i(0)^{-\lambda_i} |x|^{4-n+\delta_i}, \quad \text{for all } r_i \leq |x| \leq 1,$n

where $\lambda_i = (n - 4 - \delta_i)(p_i - 1)/4 - 1$ and $C > 0$ depends only on $n, A_1, A_3$ and $\rho$.

**Proof.** We divide the proof into several steps.

**Step 1.** From Proposition 3.1 we see that

$$u_i(x) \leq C m_i \left( \frac{1}{1 + |m_i^{(p_i-1)/4} x|^2} \right)^{\frac{n+4}{2}}$$

$$\leq C m_i R_i^{4-n} \quad \text{for all } |x| = r_i = R_i m_i^{-(p_i-1)/4}.$$ (40)

Let $\overline{u_i}(r)$ be the average of $u_i$ over the sphere of radius $r$ centered at 0. It follows from the assumption of isolated simple blow up points and Proposition 3.1 that

$$r^{4/(p_i-1)} \overline{u_i}(r) \quad \text{is strictly decreasing for } r_i < r < \rho.$$ (41)
By Lemma 3.1, (41) and (40), we have, for all \( r_i < |x| < \rho \),
\[
|x|^{4/(p_i-1)} u_i(x) \leq C|x|^{4/(p_i-1)} \overline{u}_i(|x|) \\
\leq C R_i^{4/(p_i-1)} \overline{u}_i(r_i) \\
\leq C R_i^\frac{4-n}{2},
\]
where we used \( R_i = 1 + o(1) \). Thus,
\[
u_i(x)^{p_i-1} \leq C R_i^{-4} |x|^{-4} \quad \text{for all } r_i \leq |x| \leq \rho. \tag{42}
\]

Step 2. Let \( L_i \phi(y) := \int_{B_3} G_i(x,z) K_i(z) u_i(z)^{p_i-1} \phi(z) \, dz \).

Thus
\[
u_i = L_i u_i + h_i.
\]

Note that for \( 4 < \mu < n \) and \( 0 < |x| < 2 \),
\[
\int_{B_3} G_i(x,y) |y|^{-\mu} \, dy \leq A_1 \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-4}|y|^{4-n}} \, dy \\
= A_1 |x|^{4-n} \int_{\mathbb{R}^n} \frac{1}{|x|^{-1}x - |x|^{-1}y|^{n-4}|y|^{\mu}} \, dy \\
= A_1 |x|^{-\mu+4} \int_{\mathbb{R}^n} \frac{1}{|x|^{-1}x - z|^{n-4}|z|\mu} \, dz \\
\leq C \left( \frac{1}{n-\mu} + \frac{1}{\mu-4} \right) |x|^{-\mu+4},
\]
where we did the change of variables \( y = |x|z \). By \( (42) \), one can properly choose \( 0 < \delta_i = O(R_i^{-4}) \) such that
\[
\int_{r_i < |y| < \rho} G_i(x,y) K_i(y) u_i(y)^{p_i-1} |y|^{-\delta_i} \, dy \leq \frac{1}{4} |x|^{-\delta_i}, \tag{43}
\]

and
\[
\int_{r_i < |y| < \rho} G_i(x,y) K_i(y) u_i(y)^{p_i-1} |y|^{4-n+\delta_i} \, dy \leq \frac{1}{4} |x|^{4-n+\delta_i}, \tag{44}
\]
for all \( r_i < |x| < \rho \).

Set \( M_i := 4^n A_1^2 \max_{\partial B_{\rho}} u_i + 2 \max_{B_{\rho}} h_i \),
\[
f_i(x) := M_i \rho^\delta |x|^{-\delta_i} + A m_i^{-\lambda_1} |x|^{4-n+\delta_i},
\]
Compactness of conformal metrics with constant $Q$-curvature

and

\[ \phi_i(x) = \begin{cases} f_i(x), & r_i < |x| < \rho, \\ u_i(x), & \text{otherwise}, \end{cases} \]

where $A > 1$ will be chosen later.

By (43) and (44), we have for $r_i < |x| < \rho$.

\[ \mathcal{L}_i \phi_i(x) = \int_{B_3} G_i(x, y) K_i(y) u_i(y)^{p_i-1} \phi_i(y) \, dy 
   = \left( \int_{|y| \leq r_i} + \int_{r_i < |y| < \rho} + \int_{|y| > 3} \right) G_i(x, y) K_i(y) u_i(y)^{p_i-1} \phi_i(y) \, dy 
   \leq A_1 \int_{|y| \leq r_i} \frac{u_i(y)^{p_i}}{|x - y|^{n-4}} \, dy + \frac{f_i}{4} + \frac{M_i}{2^{n-4}}, \]

where we used, in view of (24),

\[ \int_{|y| \leq 3} G_i(x, y) K_i(y) u_i(y)^{p_i-1} \phi_i(y) \, dy 
   = \int_{|y| \leq 3} G_i(x, y) K_i(y) u_i(y)^{p_i} \, dy 
   \leq A_1^2 2^{n+4} \int_{|y| < 3} G_i\left(\frac{|x|}{|y|}\right) K_i(y) u_i(y)^{p_i} \, dy 
   \leq A_1^2 2^{n+4} \max_{\partial B_{\rho}} u_i \leq 2^{4-n} M_i. \]

By change of variables and using Proposition 3.1 we have, similar to (39),

\[ \int_{|y| \leq r_i} \frac{u_i(y)^{p_i}}{|x - y|^{n-4}} \, dy = m_i \int_{|z| \leq R_i} \frac{\left(m_i^{-1} u_i(m_i^{-1/4} z)\right)^{p_i}}{|m_i^{-1/4} x - z|^{n-4}} \, dz \]
\[ \leq 2m_i \int_{|z| \leq R_i} \frac{U_1(z)^{p_i}}{|m_i^{-1/4} x - z|^{n-4}} \, dz \]
\[ \leq C m_i \int_{\mathbb{R}^n} \frac{U_1(z)^{p_i}}{|m_i^{-1/4} x - z|^{n-4}} \, dz = C m_i U_1(m_i^{-1/4} x), \]

where we used $R^{(n-4)r_i} = 1 + o(1)$. Since $|x| > r_i$, we see

\[ m_i U_1(m_i^{(p_i-1)/4} x) \leq C m_i^{1-(p_i-1)(n-4)/4} |x|^{4-n} \leq C m_i^{-\lambda_i} |x|^{4-n+\delta_i}. \]
Therefore, we conclude that
\[ \mathcal{L}_i \phi_i(x) + h_i(x) \leq \phi_i(x) \quad \text{for all } r_i \leq |x| \leq \rho, \] (45)
provided \( A \) is large independent of \( i \).

Step 3. Note that
\[ \liminf_{|x| \to r_i^+} f_i(x) > A m_i^{-\lambda_i} R_i^{1-n+\delta_i} m_i^{(p_i-1)(n-4-\delta_i)/4} = A R_i^{1-n+\delta_i} m_i. \]

In view of (40), we may choose \( A \) large such that
\[ \liminf_{|x| \to r_i^+} (f_i(x) - u_i(x)) > 0. \] (46)

We claim that
\[ u_i(x) \leq \phi_i(x). \] (47)

Indeed, if not, let
\[ 1 < t_i := \inf\{t > 1, t \phi_i(x) \geq u_i(x) \quad \text{for all } r_i \leq |x| \leq \rho\} < \infty. \]

By (46), \( t_i > 1 \), together with \( f_i > u_i \) on \( \partial B_\rho \), we can find a sufficient small open neighborhood of \( \partial B_{r_i} \cup \partial B_\rho \) in which \( t_i \phi_i > u_i \). By the continuity there exists \( y_i \in B_\rho \setminus \overline{B}_{r_i} \) such that
\[ 0 = t_i \phi_i(y_i) - u_i(y_i) \geq \mathcal{L}_i (t_i \phi_i - u_i)(y_i) + (t_i - 1) h_i(y_i) > 0. \]

We derived a contradiction and thus (47) is valid.

Step 4. By (26), we have \( \max_{B_\rho} h_i \leq A_2 \max_{\partial B_\rho} h_i \leq A_2 \max_{\partial B_\rho} u_i \). Hence,
\[ M_i \leq C \max_{\partial B_\rho} u_i. \]

For \( r_i < \theta < \rho \),
\[
\rho^{4/(p_i-1)} M_i \leq C \rho^{4/(p_i-1)} \overline{u_i}(\rho) \\
\leq C \theta^{4/(p_i-1)} \overline{u_i}(\theta) \\
\leq C \theta^{4/(p_i-1)} \left\{ M_i \rho^{\delta_i} \theta^{-\delta_i} + A m_i^{-\lambda_i} \theta^{1-n+\delta_i} \right\}.
\]

Choose \( \theta = \theta(n, \rho, A_1, A_2, A_3) \) sufficiently small so that
\[ C \theta^{4/(p_i-1)} \rho^{\delta_i} \theta^{-\delta_i} \leq \frac{1}{2} \rho^{4/(p_i-1)}. \]

24
Compactness of conformal metrics with constant $Q$-curvature

Hence, we have

$$M_i \leq Cm_i^{-\lambda_i}.$$

It follows from (47) that

$$u_i(x) \leq \phi_i(x) \leq Cm_i^{-\lambda_i}|x|^{-\delta_i} + Am_i^{-\lambda_i}|x|^{4-n+\delta_i} \leq Cm_i^{-\lambda_i}|x|^{4-n+\delta_i}.$$

We complete the proof of the lemma.

**Lemma 3.3.** Under the assumptions in Lemma 3.2, for $k < n$ we have

$$I_k[u_i](x) \leq C \left\{ \begin{array}{ll}
\frac{n-2k+4+o(1)}{m_i^{n-4}}, & \text{if } |x| < r_i, \\
m_i^{-1+o(1)}|x|^{k-n}, & \text{if } r_i \leq |x| < 1,
\end{array} \right.$$

where

$$I_k[u_i](x) = \int_{B_1} |x-y|^{k-n}u_i(y)^p_i \, dy.$$

**Proof.** Making use of Proposition 3.1 and Lemma 3.2 we have

$$I_k[u_i](x) = \int_{B_{r_i}} \frac{u_i(y)^p_i}{|x-y|^{n-k}} \, dy + \int_{B_1 \setminus B_{r_i}} \frac{u_i(y)^p_i}{|x-y|^{n-k}} \, dy$$

$$\leq Cm_i^{-n+4+o(1)} \int_{B_{r_i}} \frac{U_1(z)^p_i}{m_i^{p_i-1/4}|x-z|^{n-k}} \, dz$$

$$+ Cm_i^{-n+4+o(1)} \int_{B_1 \setminus B_{r_i}} \frac{1}{|x-y|^{n-k}|y|^{n+4}} \, dy.$$

If $|x| < r_i$, we see that

$$\int_{B_{r_i}} \frac{U_1(z)^p_i}{m_i^{p_i-1/4}|x-z|^{n-k}} \, dz \leq C$$

by Lemma B.1, and

$$\int_{B_1 \setminus B_{r_i}} \frac{1}{|x-y|^{n-k}|y|^{n+4}} \, dy \leq C(n)R_i^{-(n-k+4)} \frac{2(n-k+4)+o(1)}{m_i^{n-4}}.$$

Hence, $I_k[u_i](x) \leq Cm_i^{-\frac{n-2k+4+o(1)}{n-4}}$. 

25
If \( r_i < |x| < 1 \), then \( |m_i^{(p_i-1)/4}|x| \geq 1 \). It follows from Lemma B.1 that
\[
\int_{B_{r_i}} \frac{U_i(z)^{p_i}}{|m_i^{(p_i-1)/4}|x-z|^{n-k}} \, dz \leq \int_{B_{r_i}} \frac{1}{|m_i^{(p_i-1)/4}|x-z|^{n-k}(1 + |z|)^{n+4+o(1)}} \, dz 
\leq C|m_i^{(p_i-1)/4}|x|^{k-n}.
\]

By change of variables \( z = m_i^{(p_i-1)/4}y \),
\[
\int_{B_{1}\backslash B_{r_i}} \frac{1}{|x-y|^{n-k}|y|^{n+4}} \, dy = m_i^{\frac{2(n-k+4)}{n-4}+o(1)} \int_{B_{m_i^{(p_i-1)/4}\backslash B_{r_i}}} \frac{1}{|m_i^{(p_i-1)/4}|x-z|^{n-k}|z|^{n+4}} \, dz 
\leq Cm_i^{\frac{2(n-k+4)}{n-4}+o(1)}|m_i^{(p_i-1)/4}|x|^{k-n}.
\]
Thus
\[
I_k[u_i](x) \leq Cm_i^{\frac{n-2k+4}{n-4}+o(1)}|m_i^{(p_i-1)/4}|x|^{k-n} = m_i^{1+o(1)}|x|^{k-n}.
\]
Therefore, the proof of the lemma is completed.

\( \square \)

**Lemma 3.4.** Under the assumptions in Lemma 3.2 we have
\[
\tau_i = O(u_i(0)^{-2/(n-4)+o(1)}).
\]

Consequently, \( m_i^{\tau_i} = 1 + o(1) \).

**Proof.** For \( x \in B_1 \), we write equation (27) as
\[
u_i(x) = c_n \int_{B_1} \frac{K_i(y)u_i(y)^{p_i}}{|x-y|^{n-4}} \, dy + b_i(x),
\]
where \( b_i(x) := Q'_1(x) + Q''_i(x) + h_i(x) \),
\[
Q'_1(x) := \int_{B_1} (G_i(x,y) - c_n|x-y|^{4-n})K_i(y)u_i(y)^{p_i} \, dy
\]
and
\[
Q''_i(x) := \int_{B_1\backslash B_{1/2}} G_i(x,y)K_i(y)u_i(y)^{p_i} \, dy.
\]

Notice that
\[
|G_i(x,y) - c_n|x-y|^{4-n}| \leq \frac{C|x|^2}{|x-y|^{n-4}} + |a_i| + C|x-y|^{6-n}.
\]
\[
|\nabla_x(G_i(x,y) - c_n|x-y|^{4-n})| \leq \frac{C|x|^2}{|x-y|^{n-3}} + \frac{C|x|}{|x-y|^{n-4}} + C|x-y|^{5-n}.
\]
Compactness of conformal metrics with constant $Q$-curvature

Hence,

$$|Q'_i(x)| \leq C(|x|^2u_i(x) + |a_i||u_i^{p_i}|_{L^1(B_1)} + I_6[u_i^{p_i}](x)),$$

$$|\nabla Q'_i(x)| \leq C(|x|^2I_3 + |x|I_4 + I_5)[u_i^{p_i}](x),$$

where $I_k[u_i^{p_i}](x) = \int_{B_1} |x - y|^{k-n}u_i(y)^{p_i} \, dy$.

By Lemma 3.2 we have $u_i(x) \leq Cm_i^{-\lambda}$ for all $x \in B_{3/2} \setminus B_{1/2}$. Hence, $Q''_i(x) + h_i(x) \leq u_i(x) \leq Cm_i^{-1+o(1)}$ for any $x \in \partial B_1$. It follows from (26) that

$$\max_{\bar{B}_2} h_i(x) \leq C \min_{\partial B_1} h_i(x) \leq Cm_i^{-1+o(1)}.$$

and

$$|\nabla h_i(x)| \leq C \max_{\bar{B}_2} h_i(x) \leq Cm_i^{-1+o(1)} \quad \text{for all } x \in B_1.$$

Since $u_i$ is nonnegative, by (24) it is easy to check that

$$|Q''_i(x)| + |\nabla Q''_i(x)| \leq Cm_i^{-1+o(1)} \quad \text{for all } x \in B_1.$$

Applying Proposition 2.3 to (48), we have

$$\tau_i \int_{B_1} u_i(x)^{p_i+1} - A_2 \int_{B_1} |x|u_i(x)^{p_i+1} \, dx$$

$$\leq C \left( \int_{B_1} (|Q'_i(x)| + |x|\nabla Q'_i(x))u_i(x)^{p_i} + m_i^{-1+o(1)} \int_{B_1} u_i^{p_i} + \int_{\partial B_1} u_i^{p_i+1} \, ds \right). \quad (50)$$

By Proposition 3.1 and change of variables,

$$\int_{B_1} u_i(x)^{p_i+1} \, dx \geq C^{-1} \int_{B_{1/4}(0)} \frac{m_i^{p_i+1}}{(1 + |m_i^{(p_i-1)/4}y|^{2(n-4)(p_i+1)/2})} \, dy$$

$$\geq C^{-1}m_i^{\tau_i(1-n/4)} \int_{R_t} \frac{1}{(1 + |z|^{2(n-4)(p_i+1)/2})} \, dz$$

$$\geq C^{-1}m_i^{\tau_i(1-n/4)},$$

By Proposition 3.1 Lemma 3.2 we have

$$\int_{B_1} u_i^{p_i} \leq Cm_i^{-1+o(1)},$$

$$\int_{B_1} |x|^s u_i^{p_i+1} \leq Cm_i^{-2s/(n-4) + o(1)}, \quad \text{for } -n < s < n,$
and
\[ \int_{\partial B_i} u_i^{p_i+1} \, ds \leq C m_i^{-2n/(n-4) + o(1)}. \]

It follows from Lemma 3.3 that
\[ \int_{B_i} (|Q_i'(x)| + |x| |\nabla Q_i'(x)|) u_i(x)^{p_i+1} \, dx \leq C m_i^{-2/(n-4) + o(1)}. \]

Therefore, we complete the proof.

**Lemma 3.5.** For \(-4 < s < 4\), we have, as \(i \to \infty\),
\[ m_i^{1+\frac{2s}{n-4}} \int_{B_{ri}} |y|^s u_i(y)^{p_i} \, dy \to \int_{\mathbb{R}^n} |z|^s (1 + |z|^2)^{-\frac{n+s}{2}} \, dz \]
and
\[ m_i^{1+\frac{2s}{n-4}} \int_{B_i \setminus B_{ri}} |y|^s u_i(y)^{p_i} \, dy \to 0. \]

**Proof.** By a change of variables \(y = m_i^{-(p_i-1)/4} z\), we have
\[ \int_{B_{ri}} |y|^s u_i(y)^{p_i} \, dy = m_i^{-(p_i-1)(s+n)/4 + p_i} \int_{B_{ri}} |z|^s (m_i^{-(p_i-1)/4} z)^{p_i} \, dz \]

By Lemma 3.4, \(m_i^{-(p_i-1)(s+n)/4 + p_i} = (1 + o(1)) m_i^{1-\frac{2s}{n-4}}\). In view of Proposition 3.1 and \(-4 < s < 4\), it follows from Lebesgue’s dominated convergence theorem that
\[ \int_{B_{ri}} |z|^s (m_i^{-1} u_i(m_i^{-(p_i-1)/4} z))^{p_i} \, dz \to \int_{\mathbb{R}^n} |z|^s (1 + |z|^2)^{-\frac{n+s}{2}} \, dz. \]

Hence, the first convergence result in the lemma follows.

By Lemma 3.2,
\[ \int_{r_i \leq |y| < 1} |y|^s u_i(y)^{p_i} \, dy \leq C m_i^{-\lambda_i p_i} \int_{r_i \leq |y| < 1} |y|^s |y|^{(4-n-\delta_i) p_i} \, dy \]
\[ \leq C m_i^{-\lambda_i p_i} \frac{m_i^{(p_i-1)(n-4-\delta_i)p_i}}{4} R_i^{n+s-(n-4-\delta_i)p_i} R_i^{n+s-(n-4-\delta_i)p_i} \]
\[ = C m_i^{-(p_i-1)(s+n)+p_i} R_i^{n+s-(n-4-\delta_i)p_i}, \]
where \(0 < \delta_i = O(R_i^{-4})\) and \(\lambda_i = (n - 4 - \delta_i)(p_i - 1)/4 - 1\). Since \(m_i^{-(p_i-1)(s+n)+p_i} = (1 + o(1)) m_i^{1-\frac{2s}{n-4}}\) and \(n + s - (n - 4 - \delta_i)p_i \to s - 4 < 0\) as \(i \to \infty\), we have the second convergence result in the lemma.

In conclusion, the lemma is proved. \(\square\)
Compactness of conformal metrics with constant \(Q\)-curvature

**Proposition 3.3.** Under the assumptions in Lemma 3.2 we have
\[
u_i(x) \leq C u_i^{-1}(0) |x|^{n} , \quad \text{for all } |x| \leq 1.
\]

**Proof.** For \(|x| \leq r_i\), the proposition follows immediately from Proposition 3.1 and Lemma 3.4.

We shall show first that\[
\sup_{|x| = 1} u_i(x) \leq C.
\] (51)

If not, then along a subsequence we have, for some unit vectors \(\{e_i\}\),\[
\lim_{i \to \infty} u_i(\rho e_i) u_i(0) = +\infty.
\]

Since \(u_i(x) \leq A_3 |x|^{-4/(p_i - 1)}\) in \(B_2\), it follows from Proposition A.1 that for any \(0 < \varepsilon < 1\) there exists a positive constant \(C(\varepsilon)\), depending only on \(n, A_1, A_2, A_3\) and \(\varepsilon\), such that\[
\sup_{B_{3/2} \setminus B_\varepsilon} u_i \leq C(\varepsilon) \inf_{B_{3/2} \setminus B_\varepsilon} u_i.
\] (52)

Let \(\varphi_i(x) = u_i(\rho e_i)^{-1} u_i(x)\). Then for \(|x| \leq 1\),\[
\varphi_i(x) = \int_{B_3} G_i(x, y) K_i(y) u_i(\rho e_i)^{p_i - 1} \varphi_i(y)^{p_i} dy + \tilde{h}_i(x),
\]
where \(\tilde{h}_i(x) = u_i(\rho e_i)^{-1} h_i(x)\). Since \(\varphi_i(\rho e_i) = 1\), by (52)\[
\|\varphi_i\|_{L^\infty(B_{3/2} \setminus B_\varepsilon)} \leq C(\varepsilon) \quad \text{for } 0 < \varepsilon < 1.
\] (53)

By (26), we have that for any \(x \in B_1\),\[
\tilde{h}_i(x) \leq A_2 \tilde{h}_i(\rho e_i) \leq A_2.
\]

Besides, by Lemma 3.2\[
u_i(\rho e_i)^{p_i - 1} \to 0
\] (54)
as \(i \to \infty\). Because of (24)-(26), by applying Proposition A.2 to \(\varphi_i\) we conclude that there exists \(\varphi \in C^3(B_1 \setminus \{0\})\) such that \(\varphi_i \to \varphi\) in \(C^3_{loc}(B_1 \setminus \{0\})\) after passing to a subsequence.

Let us write the equation of \(\varphi_i\) as\[
\varphi_i(x) = \int_{B_1} G_i(x, y) K_i(y) u_i(\rho e_i)^{p_i - 1} \varphi_i(y)^{p_i} dy + b_i(x), \label{55}
\]
where \(b_i(x) := \int_{B_3 \setminus B_1} G_i(x, y) K_i(y) u_i(\rho e_i)^{p_i - 1} \varphi_i(y)^{p_i} dy + \tilde{h}_i(x)\). By (53), there exists \(b \in C^3(B_1)\) such that\[
b_i(x) \to b(x) \geq 0 \quad \text{in } C^3_{loc}(B_1) \label{56}
\]
after passing to a subsequence. Therefore,
\[
\int_{B_1} G_i(x, y) K_i(y) u_i(\rho e_i)^{p_i-1} \varphi_i(y)^{p_i} \, dy = \varphi_i(x) - b_i(x) \to \varphi(x) - b(x)
\]
in $C^3_{\text{loc}}(B_1 \setminus \{0\})$. Denote $\Gamma(x) := \varphi(x) - b(x)$. For any $|x| > 0$ and $0 < \varepsilon < \frac{1}{2} |x|$, in view of (53) and (54) we have
\[
\Gamma(x) = \lim_{i \to \infty} \int_{B_\varepsilon} G_i(x, y) K_i(y) u_i(\rho e_i)^{p_i-1} \varphi_i(y)^{p_i} \, dy
\]
where we used Lemma 3.5 in the third identity, $a(\varepsilon)$ is a bounded nonnegative function of $\varepsilon$,
\[
G_\infty(x, 0) = c_n |x|^{4-n} + \bar{a} + O'(|x|^{6-n})
\]
by (24), $\bar{a} \geq 0$ and $\bar{a} = 0$ if $n \geq 6$. Clearly, $a(\varepsilon)$ is nondecreasing in $\varepsilon$, so $\lim_{\varepsilon \to 0} a(\varepsilon)$ exists which we denote as $a$. Sending $\varepsilon \to 0$, we obtain
\[
\Gamma(x) = a G_\infty(x, 0).
\]
Since $0$ is an isolated simple blow point of $\{u_i\}_{i=1}^\infty$, we have $r^{\frac{n-1}{2}} \varphi(r) \geq \rho^{\frac{n-1}{2}} \varphi(\rho)$ for $0 < r < \rho$. It follows that $\varphi$ is singular at $0$, and thus, $a > 0$. Hence,
\[
\lim_{i \to \infty} \int_{B_{1/8}} K_i(y) u_i(\rho e_i)^{p_i-1} \varphi_i(y)^{p_i} \, dy \geq a(\varepsilon) \geq a > 0.
\]
However,
\[
\int_{B_{1/8}} K_i(y) u_i(\rho e_i)^{p_i-1} \varphi_i(y)^{p_i} \, dy \leq C u_i(\rho e_i)^{-1} \int_{B_{1/8}} u_i(y)^{p_i} \, dy \leq \frac{C}{u_i(\rho e_i) u_i(0)} \to 0 \quad \text{as } i \to \infty,
\]
where we used Lemma 3.5 in the last inequality. This is a contradiction.
Compactness of conformal metrics with constant $Q$-curvature

Without loss of generality, we may assume that $\rho \leq 1/2$. It follows from Proposition \ref{A.1} and (51) that Proposition \ref{3.3} holds for $\rho \leq |x| \leq 1$. To establish the inequality in the Proposition for $r_i \leq |x| \leq \rho$, we only need to rescale and reduce it to the case of $|x| = 1$. Suppose the contrary that there exists a subsequence $x_i$ satisfying $|x_i| \leq \rho$ and $\lim_{i \to \infty} u_i(x_i)u_i(0)|x_i|^{n-4} = +\infty$.

Set $\tilde{r}_i := |x_i|$, $\tilde{u}_i(x) = r_i^{4/(p_i-1)}u_i(\tilde{r}_i x)$. Then $\tilde{u}_i$ satisfies
\[
\tilde{u}_i(x) = \int_{B_3} G_{\tilde{r}_i}(x,y) K_{\tilde{r}_i}(\tilde{r}_i y) \tilde{u}_i(y)^{p_i} \, dy + \tilde{h}_i(x) \quad \text{for } x \in B_2,
\]
where $\tilde{h}_i(x) = \int_{B_{3/\tilde{r}_i} \backslash B_3} G_{\tilde{r}_i}(x,y) K_{\tilde{r}_i}(\tilde{r}_i y) \tilde{u}_i(y)^{p_i} \, dy + \tilde{r}_i^{4/(p_i-1)} \tilde{h}_i(\tilde{r}_i x)$. One can easily check that $\tilde{u}_i$ and the above equation satisfy all hypotheses of Proposition \ref{3.3} for $u_i$ and its equation. It follows from (51) that
\[
\tilde{u}_i(0) \tilde{u}_i(\tilde{r}_i) \leq C.
\]
It follows (using Lemma \ref{3.4}) that
\[
\lim_{i \to \infty} u_i(x_i)u_i(0)|x_i|^{n-4} < \infty.
\]
This is again a contradiction.

Therefore, the proposition is proved. \hfill \qed

**Proposition 3.4.** Under the assumptions in Lemma \ref{3.2} we have
\[
|\nabla^k u_i(x)| \leq C u_i^{-1}(0)|x|^{4-n-k}, \quad \text{for all } r_i \leq |x| \leq 1,
\]
where $k = 1, \ldots, 4$.

**Proof.** Since 0 is an isolated blow up point in $B_2$, by Proposition \ref{A.1} we see that Proposition \ref{3.3} holds for all $|x| \leq \frac{3}{2}$. For any $r_i \leq |x| < 1$, let
\[
\varphi_i(z) = \left( \frac{|x|}{4} \right)^{\frac{4}{p_i-1}} u_i(x + \frac{|x|}{4} z).
\]
By the equation of $u_i$, we have
\[
\varphi_i(z) = \frac{1}{\{y : |x| + \frac{|x|}{4} y \leq 3\}} \frac{G_i(z,y)}{K_i(y)} \varphi_i(y)^{p_i-1} \varphi_i(y) \, dy + \tilde{h}_i(z),
\]
where $\tilde{G}_i(z,y) = (\frac{|x|}{4})^{n-4} G_i(x + \frac{|x|}{4} z, x + \frac{|x|}{4} y)$, $\tilde{K}_i(y) = K_i(x + \frac{|x|}{4} y)$, and $\tilde{h}_i(z) = (\frac{|x|}{4})^{\frac{4}{p_i-1}} h_i(x + \frac{|x|}{4} z)$. Since 0 is an isolated blow up point of $u_i$, we have $\varphi_i(z)^{p_i-1} \leq A_2^{p_i-1}$ for all $|z| \leq 1$. Since $\varphi_i, \tilde{G}_i, \tilde{K}_i$ and $\tilde{h}_i$ are nonnegative, by Proposition \ref{A.2} we have
\[
|\nabla^k \varphi_i(0)| \leq C(\|\varphi_i\|_{L^\infty(B_1)} + \|\tilde{h}_i\|_{C^1(B_1)}).
\]
This gives
\[
\left(\frac{|x|}{4}\right)^k |\nabla^k u_i(x)| \leq C\|u_i\|_{L^\infty(B_{1/4}(x))} + Cm_i^{-1} \\
\leq Cu_i(0)^{-1}|x|^{4-n}.
\]

Therefore, the proposition follows.

\[\square\]

**Corollary 3.1.** Under the hypotheses of Lemma 3.2 we have
\[
\int_{B_1} |x|^s u_i(x)^{p_i+1} \leq Cu_i(0)^{-2s/(n-4)}, \quad \text{for } -n < s < n,
\]

**Proof.** Making use of Proposition 3.1, Lemma 3.4 and Proposition 3.3, the corollary follows immediately.

By Proposition 3.3 and its proof, we have the following corollary.

**Corollary 3.2.** Under the assumptions in Lemma 3.2 if we let \( T_i(x) = T_i'(x) + T_i''(x) \), where
\[
T_i'(x) := u_i(0) \int_{B_1} G_i(x, y)K_i(y)u_i(y)^{p_i} \, dy
\]
and
\[
T_i''(x) := u_i(0) \int_{B_3 \setminus B_1} G_i(x, y)K_i(y)u_i(y)^{p_i} \, dy + u_i(0)h_i(x).
\]
then, after passing a subsequence,
\[
T_i'(x) \to aG_\infty(x, 0) \quad \text{in } C^3_{loc}(B_1 \setminus \{0\})
\]
and
\[
T_i''(x) \to h(x) \quad \text{in } C^3_{loc}(B_1)
\]
for some \( h(x) \in C^3(B_2) \), where \( G_\infty \) is the limit of a subsequence of \( G_i \) in \( C^3_{loc}(B_1 \setminus \{0\}) \),
\[
a = \int_{\mathbb{R}^n} \left( \frac{1}{1 + |y|^2} \right)^{\frac{n+4}{4}} \, dy. \tag{58}
\]

Consequently, we have
\[
u_i(0)u_i(x) \to aG_\infty(x, 0) + h(x) \quad \text{in } C^3_{loc}(B_1 \setminus \{0\}).
\]
Compactness of conformal metrics with constant $Q$-curvature

Proof. Similar to that in the proof of Proposition 3.3, we set $\varphi_i(x) = u_i(0)u_i(x)$, which satisfies

$$
\begin{align*}
\varphi_i(x) &= \int_{B_3} G_i(x, y) K_i(y)u_i(0)1^{-p_i}\varphi_i(y)p_i \, dy + u_i(0)h_i(x) \\
&=: \int_{B_3} G_i(x, y) K_i(y)u_i(0)1^{-p_i}\varphi_i(y)p_i \, dy + T_i''(x) = T_i'(x) + T_i''(x).
\end{align*}
$$

We have all the ingredients as in the proof of Proposition 3.3. Hence, we only need to evaluate the positive constant $a$. By (57) and Lemma 3.5, we have

$$
a = \lim_{\varepsilon \to 0} \lim_{i \to \infty} u_i(0) \int_{B_{\varepsilon}} K_i(y)u_i(y)p_i \, dy = \int_{\mathbb{R}^n} \left( \frac{1}{1 + |y|^2} \right)^{\frac{n+4}{2}} d y.
$$

\[ \square \]

4 Expansions of blow up solutions of integral equations

In this section, we are interested in stronger estimates than that in Proposition 3.3. To make statements closer to the main goal of the paper, we restrict our attention to a special $K_i$. Namely, given $p_i, G_i,$ and $h_i$ satisfying (23), (24) and (26) respectively, $\kappa_i$ satisfying (25) with $K_i$ replaced by $\kappa_i$, let $0 \leq u_i \in C^4(B_3)$ be a solution of

$$
u_i(x) = \int_{B_3} G_i(x, y) \kappa_i(y)^{\gamma_i} u_i^{p_i}(y) \, dy + h_i(x) \quad \text{in } B_3.
$$

(59)

We also assume

$$
\nabla \kappa_i(0) = 0.
$$

(60)

Suppose that 0 is an isolated simple blow up point of $\{u_i\}$ with $\rho = 1$, i.e.,

$$
u_i(x) \leq A_3|x|^{-4/(p_i-1)} \quad \text{for all } x \in B_2.
$$

(61)

and $r^{4/(p_i-1)}u_i(r)$ has precisely one critical point in $(0, 1)$.

Let us first introduce a non-degeneracy result.

Lemma 4.1. Let $v \in L^\infty(\mathbb{R}^n)$ be a solution of

$$
v(x) = c_n \frac{n + 4}{n - 4} \int_{\mathbb{R}^n} \frac{U_1(y)^{\frac{n}{n-4}} v(y)}{|x - y|^{n-4}} \, dy.
$$

Then

$$
v(z) = a_0 \left( \frac{n - 4}{2} U_1(z) + z \cdot \nabla U_1(z) \right) + \sum_{j=1}^n a_j \partial_j U_1(z),
$$

where $a_0, \ldots, a_n$ are constants.
Proof. Since \(v \in L^\infty(\mathbb{R}^n)\), by using Lemma B.1 iteratively a finite number of times we obtain
\[
|v(x)| \leq C(1 + |x|)^{4-n}.
\] (62)

Let \(F : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}, \)
\[
F(x) = \left( \frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right)
\]
denote the inverse of the inverse of the stereographic projection and \(h(F(x)) := v(x)J_F(x)^{-\frac{n-4}{2n}}\), where \(J_F = (\frac{2}{1 + |x|^2})^n\) is the Jacobian determinant of \(F\) and \(N\) is the north pole. It follows from (62) that \(h \in L^\infty(\mathbb{S}^n)\). Let \(\xi = F(x)\) and \(\eta = F(y)\). Then
\[
|\xi - \eta| = \frac{2|x - y|}{\sqrt{(1 + |x|^2)(1 + |y|^2)}} \quad \text{and} \quad d\eta = \left( \frac{2}{1 + |y|^2} \right)^n dy
\]
are respectively the distance between \(\xi\) and \(\eta\) in \(\mathbb{R}^{n+1}\) and the surface measure of \(\mathbb{S}^n\). It follows that
\[
h(\xi) = 2^{-4}n(n - 2)(n + 2)(n + 4)\alpha_n \int_{\mathbb{S}^n} |\xi - \eta|^{4-n} h(\eta) \, d\eta.
\] (63)

By the regularity theory for Riesz potentials, \(h \in C^\infty(\mathbb{S}^n)\). Note that the Paneitz operator
\[
P_{g_{\mathbb{S}^n}} = \Delta^2_{g_{\mathbb{S}^n}} - \frac{n^2 - 2n - 4}{2} \Delta_{g_{\mathbb{S}^n}} + \frac{n(n - 2)(n + 2)(n - 4)}{16}
\]
with respect to the standard metric \(g_{\mathbb{S}^n}\) on \(\mathbb{S}^n\) satisfies \(P_{g_{\mathbb{S}^n}} \phi = |J_F|^{-\frac{n+4}{2n}} \Delta^2(|J_F|^{-\frac{n+4}{2n}} \phi \circ F)\) for any \(\phi \in C^\infty(\mathbb{S}^n)\). By the integral equation of \(v\) we have
\[
P_{g_{\mathbb{S}^n}} h = 2^{-4}n(n - 2)(n + 2)(n + 4)h.
\]

Let \(Y^{(k)}\) be a spherical harmonics of degree \(k \geq 0\). We have
\[
P_{g_{\mathbb{S}^n}} Y^{(k)} = 2^{-4}(2k + n + 2)(2k + n)(2k + n - 2)(2k + n - 4)Y^{(k)}.
\]

Hence, \(h\) must be a spherical harmonics of degree one. Transforming \(h\) back, we complete the proof.

In view of Corollary 2.2, we assume in this and next section that
\[
U_\lambda(x) = \int_{B_3} G_i(x, y)\{U_\lambda(y)\frac{n+4}{n-4} + c_{\lambda,i}(y)U_\lambda(y)\} \, dy + c''_{\lambda,i}(x) \quad \forall \lambda \geq 1, \ x \in B_3
\] (64)
Compactness of conformal metrics with constant $Q$-curvature

where $c'_{\lambda,i}, c''_{\lambda,i} \in C^5(B_3)$ satisfy

$$\Theta_i := \sum_{k=0}^{5} \|\lambda^{-k} \nabla^k c'_{\lambda,i}\|_{L^\infty(B_2)} \leq A_2,$$

and $\|c''_{\lambda,i}\|_{C^5(B_2)} \leq A_2 \lambda^{-d}$, respectively.

**Lemma 4.2.** Let $0 \leq u_i \in C^4(B_3)$ be a solution of (59) and $0$ is an isolated simple blow up point of $\{u_i\}$ with some constant $\rho$, say $\rho = 1$. Suppose (64) holds and let $\Theta_i$ be defined in (65). Then we have

$$|\varphi_i(z) - U_1(z)| \leq C \begin{cases} \max\{\tau_i, m_i^{-2}\}, & \text{if } 5 \leq n \leq 7, \\ \max\{\tau_i, \Theta_i m_i^{-2} \log m_i, m_i^{-2}\}, & \text{if } n = 8, \\ \max\{\tau_i, \Theta_i m_i^{-\frac{8-n}{n-3}}, m_i^{-2}\}, & \text{if } n \geq 9, \end{cases} \quad \forall \ |z| \leq m_i^{\frac{\rho-1}{4}},$$

where $\varphi_i(z) = \frac{1}{m_i} u_i(m_i^{-\frac{\rho-1}{4}} z)$, $m_i = u_i(0)$, and $C > 0$ depends only on $n, A_1, A_2$ and $A_3$.

**Proof.** For brevity, set $\ell_i = m_i^{\frac{\rho-1}{4}}$. By the equation of $u_i$, we have

$$\varphi_i(z) = \int_{B_{\ell_i}} G_{i,\ell_i^{-1}}(z,y)\kappa_i(y)^{\tau_i} \varphi_i(y)^{p_i} \, dy + \bar{h}_i(z),$$

where $G_{i,\ell_i^{-1}}(z,y) = \ell_i^{4-n} G_i(\ell_i^{-1} x, \ell_i^{-1} y)$, $\kappa_i(z) = \kappa_i(\ell_i^{-1} z)$, and $\bar{h}_i(z) = m_i^{-1} h_i(\ell_i^{-1} z)$ with

$$\bar{h}_i(x) = \int_{B_1 \setminus B_{\ell_i}} G_i(x,y)\kappa_i(y)^{\tau_i} u_i(y)^{p_i} \, dy + h_i(x).$$

Since $0$ is an isolated simple blow up point of $u_i$, by Proposition 3.3 we have

$$u_i(x) \leq C m_i^{-1} |x|^{4-n} \quad \text{for } |x| < 1.$$

It follows that $\bar{h}_i(x) \leq C m_i^{-1} \text{ for } x \in B_1$ and $\bar{h}_i(z) \leq C m_i^{-2} \text{ for } z \in B_{\ell_i}$.

Notice that $\bar{U}_{\ell_i}(x) \leq C m_i^{-1} \text{ for } 1 \leq |x| \leq 3$. Let $z = \ell_i x$. By (64) with $\lambda = \ell_i$ we have for $|z| \leq \ell_i$

$$U_1(z) = \int_{B_{\ell_i}} G_{i,\ell_i^{-1}}(z,y)(U_1(y)^{\frac{\rho_i}{n-4}} + m_i^{-\frac{8-n}{n-3}} c'_{\lambda,i}(\ell_i^{-1} y) U_1(y)) \, dy + O(m_i^{-2})$$

$$= \int_{B_{\ell_i}} G_{i,\ell_i^{-1}}(z,y)(\kappa_i(y)^{\tau_i} U_1(y)^{p_i} + T_i(y)) \, dy + O(m_i^{-2}),$$

35
where we used $m_i^{-\alpha} = 1 + o(1)$, and

$$T_i(y) := U_1(y)^{\frac{n+4}{n-4}} - \tilde{\kappa}_i(y)^{\tau_i}U_1(y)^{p_i} + m_i^{-\frac{8}{n-4}} c_{i,\ell}^{-1}(y)U_1(y).$$

Here and throughout this section, $O(m_i^{-2})$ denotes some function $f_i$ satisfying $\|\nabla^k f_i\|_{B_1(\varepsilon \ell_i)} \leq C(\varepsilon)m_i^{-2 - \frac{2k}{n}}$ for small $\varepsilon > 0$ and $k = 0, \ldots, 5$.

In the following, we adapt some arguments from Marques [43] for the Yamabe equation; see also the proof of Proposition 2.2 of Li-Zhang [38]. Let

$$\Lambda_i = \max_{|z| \leq \ell_i} |\varphi_i - U_1|.$$

By (67), for any $0 < \varepsilon < 1$ and $\varepsilon \ell_i \leq |z| \leq \ell_i$, we have $|\varphi_i(z) - U_1(z)| \leq C(\varepsilon)m_i^{-2}$, where we used $m_i^{-\alpha} = 1 + o(1)$. Hence, we may assume that $\Lambda_i$ is achieved at some point $|z_i| \leq \frac{1}{2} \ell_i$, otherwise the proof is finished. Set

$$v_i(z) = \frac{1}{\Lambda_i}(\varphi_i(z) - U_1(z)).$$

It follows from (66) and (68) that $v_i$ satisfies

$$v_i(z) = \int_{B_{\ell_i}} G_{i,\ell_i}(z,y)(b_i(y)v_i(y) + \frac{1}{\Lambda_i}T_i(y)) \, dy + \frac{1}{\Lambda_i}O(m_i^{-2}),$$

where

$$b_i = \tilde{\kappa}_i^{\tau_i} \frac{\varphi_i^{p_i} - U_1^{p_i}}{\varphi_i - U_1}.$$

Since

$$G_{i,\ell_i}(z,y) \leq A_1 |z - y|^{4-n}$$

and

$$|T_i(y)| \leq C_\tau_i(|\log U_i| + |\log \tilde{\kappa}_i|)(1 + |y|)^{-p_i(n-4)} + \Theta_i m_i^{-\frac{8}{n-4}} (1 + |y|)^{4-n},$$

we obtain

$$\int_{B_{\ell_i}} G_{i,\ell_i}(z,y)|T_i(y)| \, dy \leq C(\tau_i + \Theta_i \alpha_i) \quad \text{for } |z| \leq \frac{\ell_i}{2},$$

where

$$\alpha_i = \begin{cases} 
  m_i^{-2}, & \text{if } 5 \leq n \leq 7, \\
  m_i^{-2} \log m_i, & \text{if } n = 8, \\
  m_i^{-\frac{8}{n-4}}, & \text{if } n \geq 9.
\end{cases}$$

(71)
Compactness of conformal metrics with constant $Q$-curvature

Since $\kappa_i(x)$ is bounded and $\varphi_i \leq CU_1$, we see

$$|b_i(y)| \leq CU_1(y)^{p_i-1} \leq C(1 + |y|)^{-7.5}, \quad y \in B_{\ell_i}.$$  \hfill (72)

Noticing that $\|v_i\|_{L^\infty(B_{\ell_i})} \leq 1$, by Lemma B.1 we have

$$\int_{B_{\ell_i}} G_i,\ell^{-1}_i(z,y)|b_i(y)v_i(y)| \, dy \leq C(1 + |z|)^{-\min\{n-4,3.5\}}.$$  

Hence, we get

$$v_i(z) \leq C((1 + |z|)^{-\min\{n-4,3.5\}} + \frac{1}{\Lambda_i}(\tau_i + \Theta_i\alpha_i + m_i^{-2})) \quad \text{for} \quad |z| \leq \frac{\ell_i}{2}$$  \hfill (73)

Suppose the contrary that $\frac{1}{\Lambda_i} \max\{\tau_i, \Theta_i\alpha_i, m_i^{-2}\} \to 0$ as $i \to \infty$. Since $v(z_i) = 1$, by (73) we see that

$$|z_i| \leq C.$$  

Differentiating the integral equation (69) up to three times, together with (70) and (72), we see that the $C^3$ norm of $v_i$ on any compact set is uniformly bounded. By Arzelà-Ascoli theorem let $v := \lim_{i \to \infty} v_i$ after passing to a subsequence. Using Lebesgue’s dominated convergence theorem, we obtain

$$v(z) = c_n \int_{\mathbb{R}^n} \frac{U_1(y)^{\frac{n}{4}} v(y)}{|z - y|^{n-4}} \, dy.$$  

It follows from Lemma 4.1 that

$$v(z) = a_0\left(\frac{n-4}{2} U_1(z) + z \cdot \nabla U_1(z)\right) + \sum_{j=1}^{n} a_j \partial_j U_1(z),$$  

where $a_0, \ldots, a_n$ are constants. Since $v(0) = 0$ and $\nabla v(0) = 0, v$ has to be zero. However, $v(z_i) = 1$. We obtain a contradiction.

Therefore, $\Lambda_i \leq C(\tau_i + \alpha_i)$ and the proof is completed. \hfill \Box

**Lemma 4.3.** Under the same assumptions in Lemma [7.2], we have

$$\tau_i \leq C \begin{cases} m_i^{-2}, & \text{if } 5 \leq n \leq 7, \\ \max\{\Theta_i m_i^{-2} \log m_i, m_i^{-2}\}, & \text{if } n = 8, \\ \max\{\Theta_i m_i^{-\frac{n}{n-4}}, m_i^{-2}\}, & \text{if } n \geq 9. \end{cases}$$  

37
Y. Y. Li & J. Xiong

**Proof.** The proof is also by contradiction. Recall the definition of \( \alpha_i \) in (71). Suppose the contrary that \( \frac{1}{\tau_i} \max\{\Theta_i \alpha_i, m_i^{-2}\} \rightarrow 0 \) as \( i \rightarrow \infty \). Set

\[
v_i(z) = \frac{\varphi_i(z) - U_1(z)}{\tau_i},
\]

It follows from Lemma 4.2 that \( |v_i(z)| \leq C \) in \( B_{\ell_i} \), where \( \ell_i = m_i^{\frac{p_i}{n-4}} \). As (69), we have

\[
v_i(z) = \int_{B_{\ell_i}} G_{i,\ell_i^{-1}}(z, y)(b_i(y)v_i(y) + \frac{1}{\tau_i} T_i(y)) \, dy + \frac{1}{\tau_i} O(m_i^{-2}),
\]

where

\[
b_i = \kappa_i^n \varphi_i^n - \varphi_i - U_1
\]

and

\[
T_i(y) := U_1(y) \frac{\kappa_i}{\tau_i} - \kappa_i(y)^{\tau_i} U_1(y)^{p_i} + m_i^{\frac{8}{n-4}} c_{i,i}(\ell_i^{-1} y) U_1(y).
\]

By the estimates (72) and (70) for \( b_i \) and \( T_i \) respectively, we conclude from the integral equation that \( \|v_i\|_{C^3} \) is uniformly bounded over any compact set. It follows that \( v_i \rightarrow v \) in \( C^2_{\text{loc}}(\mathbb{R}^n) \) for some \( v \in C^3(\mathbb{R}^n) \).

Multiplying both sides of (74) by \( b_i(z) \phi(z) \), where \( \phi(z) = \frac{\kappa_i}{2} U_1(z) + z \cdot \nabla U_1(z) \), and integrating over \( B_{\ell_i} \), we have, using the symmetry of \( G_{i,\ell_i^{-1}} \) in \( y \) and \( z \),

\[
\int_{B_{\ell_i}} b_i(z)v_i(z) \left( \phi(z) - \int_{B_{\ell_i}} G_{i,\ell_i^{-1}}(z, y)b_i(y)\phi(y) \, dy \right) \, dz = \frac{1}{\tau_i} \int_{B_{\ell_i}} T_i(z) \int_{B_{\ell_i}} G_{i,\ell_i^{-1}}(z, y)b_i(y)\phi(y) \, dy \, dz + \frac{1}{\tau_i} O(m_i^{-2}) \int_{B_{\ell_i}} b_i(z)\phi(z) \, dz.
\]

As \( i \rightarrow \infty \), we have

\[
\int_{B_{\ell_i}} G_{i,\ell_i^{-1}}(z, y)b_i(y)\phi(y) \, dy \rightarrow c_n \int_{\mathbb{R}^n} \frac{U_1(y)^{\frac{n+4}{n-4}} \phi(y)}{|z-y|^{n-4}} \, dz = \phi(z),
\]

\[
\frac{1}{\tau_i} O(m_i^{-2}) \int_{B_{\ell_i}} b_i(z)\phi(z) \, dz \rightarrow 0 \text{ by the contradiction hypothesis,}
\]

and

\[
\frac{T_i(z)}{\tau_i} \rightarrow (\log U_1(z))U_1(z)^{\frac{n+4}{n-4}}.
\]
Compactness of conformal metrics with constant $Q$-curvature

Hence, by Lebesgue’s dominated convergence theorem we obtain

\[
\lim_{i \to \infty} \frac{1}{\tau_i} \int_{B_{\ell_i}} T_i(z) \int_{B_{\ell_i}} G_i,\ell^{-1}(z, y) b_i(y) \phi(y) \, dy \, dz = \int_{\mathbb{R}^n} \phi(z)(\log U_1(z)) U_1(z)^{\frac{n+4}{n-4}} \, dz = 0.
\]

This is impossible, because

\[
\int_{\mathbb{R}^n} \phi(z)(\log U_1(z)) U_1(z)^{\frac{n+4}{n-4}} \, dz = \left(\frac{n-4}{2}\right)^2 |S^{n-1}|_4 \int_{1}^{\infty} \frac{(s^2 - 1)s^{n-1}}{(1 + s^2)^{n+1}} \log(1 + s^2) - \log s^2 \, ds > 0,
\]

where we used

\[
\int_{1}^{\frac{1}{s}} \frac{(r^2 - 1)r^{n-1}}{(1 + r^2)^{n+1}} \log(1 + r^2) \, dr = -\int_{1}^{\infty} \frac{(s^2 - 1)s^{n-1}}{(1 + s^2)^{n+1}} \log(1 + s^2) - \log s^2 \, ds
\]

by the change of variable $r = \frac{1}{s}$.

We obtain a contradiction and thus $\tau_i \leq \alpha_i$. Therefore, the lemma is proved. \(\square\)

**Proposition 4.1.** Under the hypotheses in Lemma 4.2, we have

\[
|\varphi_i(z) - U_1(z)| \leq C \begin{cases} m_i^{-2}, & \text{if } 5 \leq n \leq 7, \\
\max\{\Theta_i m_i^{-2} \log m_i, m_i^{-2}\}, & \text{if } n = 8, \\
\max\{\Theta_i m_i^{-\frac{n}{n-4}}, m_i^{-2}\}, & \text{if } n \geq 9, \end{cases}
\]

\(\forall |z| \leq m_i^{\frac{n-1}{n}}.\)

**Proof.** It follows immediately from Lemma 4.2 and Lemma 4.3. \(\square\)

**Proposition 4.2.** Under the hypotheses in Lemma 4.2, we have, for every $|z| \leq m_i^{\frac{n-1}{n}}$,

\[
|\varphi_i(z) - U_1(z)| \leq C \begin{cases} \max\{\Theta_i m_i^{-2} m_i^{\frac{2}{2(n-3)}} (1 + |z|)^{-1}, m_i^{-2}\}, & \text{if } n = 8, \\
\max\{\Theta_i m_i^{-2} m_i^{\frac{n}{n-4}} (1 + |z|)^{8-n}, m_i^{-2}\}, & \text{if } n \geq 9. \end{cases}
\]

39
Proof. Let \( \alpha_i \) be defined in (71). We may assume that \( \frac{m_i^{-2}}{\sigma_i \alpha_i} \to 0 \) as \( i \to \infty \) for \( n \geq 8 \); otherwise the proposition follows immediately from Proposition 4.1. Set

\[
\alpha_i' = \begin{cases} 
  m_i^{-2} m_i^{\frac{2}{n-8}}, & \text{if } n = 8, \\
  m_i^{-2} m_i^{\frac{n-4}{n}}, & \text{if } n \geq 9,
\end{cases}
\]

and

\[
v_i(z) = \frac{\varphi_i(z) - U_1(z)}{\Theta_i \alpha_i'}, \quad |z| \leq m_i^{\frac{p_i-1}{4}}.
\]

Since \( \frac{m_i^{-2}}{\sigma_i \alpha_i} \to 0 \), it follows from Proposition 4.1 that \( |v_i| \leq C \). Since \( 0 < \varphi_i \leq C U_1 \), we only need to prove the proposition when \( |z| \leq \frac{1}{2} \ell_i \), where \( \ell_i = m_i^{-1} \). Similar to (69), \( v_i \) now satisfies

\[
v_i(z) = \int_{B_{\ell_i}} G_{i, \ell_i^{-1}}(z, y) (b_i(y) v_i(y) + \frac{1}{\Theta_i \alpha_i'} T_i(y)) dy + \frac{1}{\Theta_i \alpha_i'} \mathcal{O}(m_i^{-2}),
\]

where

\[
b_i = \tilde{\kappa}_i \frac{\varphi_i^{p_i} - U_1^{p_i}}{\varphi_i - U_1}
\]

and

\[
T_i(y) := U_1(y)^{\frac{n+4}{n-4}} - \tilde{\kappa}_i (y)^{\tau_i} U_1(y)^{p_i} + m_i^{-\frac{8}{n-4}} c_{\ell_i, i}(\ell_i^{-1} y) U_1(y).
\]

Noticing that

\[
|T_i(y)| \leq C \tau_i \log U_1 + |\log \tilde{\kappa}_i| (1 + |y|)^{-n-4} + m_i^{-\frac{8}{n-4}} \Theta_i (1 + |y|)^{4-n},
\]

we have

\[
\frac{1}{\Theta_i \alpha_i'} \int_{B_{\ell_i}} G_{i, \ell_i^{-1}}(z, y)|T_i(y)| dy \leq C \int_{B_{\ell_i}} \frac{1}{|z - y|^{n-4}(1 + |y|)^{4-n} m_i^{\frac{2}{n-4}}} dy
\]

\[
\leq C \int_{B_{\ell_i}} \frac{1}{|z - y|^{n-4}(1 + |y|)^{5}} dy
\]

\[
\leq C (1 + |z|)^{-1},
\]

if \( n = 8 \), and

\[
\frac{1}{\Theta_i \alpha_i'} \int_{B_{\ell_i}} G_{i, \ell_i^{-1}}(z, y)|T_i(y)| dy \leq C (1 + |z|)^{8-n}
\]

if \( n \geq 9 \), where we used Lemma B.1. Thus

\[
|v_i(z)| \leq C((1 + |z|)^{-3.5} + (1 + |z|)^{-1})
\]
Compactness of conformal metrics with constant $Q$-curvature

for $n = 8$, and

$$|v_i(z)| \leq C((1 + |z|)^{-3.5} + (1 + |z|)^{8-n})$$

for $n \geq 9$. If $n = 8, 9, 10, 11$, the conclusion follows immediately from multiplying both sides of the above inequalities by $\alpha'_i$. If $n \geq 12$, the above estimate gives $|v_i(z)| \leq C(1 + |z|)^{-3.5}$. Plugging this estimate to the term $\int G_{i,\ell}^{-1}(z, y) b_i(y) v_i(y) \, dy$ yields $|v_i(z)| \leq C(1 + |z|)^{8-n}$ as long as $n \leq 14$. Repeating this process, we complete the proof.

Corollary 4.1. Under the hypotheses in Lemma 4.2, we have, for very $|z| \leq m^{-\frac{1}{4}}$,

$$|\nabla^k(\varphi_i - U_1)(z)|$$

$$\leq C(1 + |z|)^{-k} \begin{cases} m_i^{-2}, & \text{if } 5 \leq n \leq 7, \\ \max\{\Theta_i m_i^{-2} m_i^{\frac{n}{2(n-8)}} (1 + |z|)^{-1}, m_i^{-2}\}, & \text{if } n = 8, \\ \max\{\Theta_i m_i^{-2} m_i^{\frac{n}{2(n-8)}} (1 + |z|)^{8-n}, m_i^{-2}\}, & \text{if } n \geq 9. \end{cases}$$

where $k = 1, 2, 3, 4$.

Proof. Considering the integral equation of $v_i = \varphi_i - U_1$, the conclusion follows from Lemma B.1. Indeed, if $k < 4$, we can differentiate the integral equation of $v_i$ directly and then use Lemma B.1. If $k = 4$, we can use a standard technique (see the proof of Proposition A.2) for proving the higher order regularity of Riesz potential since $v_i$ and the coefficients are of $C^1$.

5 Blow up local solutions of fourth order equations

In the previous two sections, we have analyzed the blow up profiles of the blow up local solutions of integral equations. In this section, we will assume that those blow up solutions also satisfy differential equations, which is only used to check the Pohozaev identity in Proposition 2.2. It should be possible to completely avoid using differential equations after improving Corollary 2.1. This is the case on the sphere; see our joint work with Jin [31]. On the other hand, as mentioned in the Introduction, without extra information fourth order differential equations themselves are not enough to do blow up analysis for positive local solutions.

Proposition 5.1. In addition to the hypotheses in Lemma 4.2, assume that $u_i$ also satisfies

$$P_{g_i} u_i = c(n) \kappa_i^{7_p} u_i^{p_i} \text{ in } B_3,$$  \hspace{1cm} (75)

where $\det g_i = 1$, $B_3$ is a normal coordinates chart of $g_i$ at 0 and $\|g_i\|_{C^{10}(B_4)} \leq A_1$.  

41
(i) If either $n \leq 9$ or $g_i$ is flat, then

$$\liminf_{r \to 0} \mathcal{P}(r, \Gamma) \geq 0,$$

where $\Gamma$ is a limit of $u_i(0)u_i(x)$ along a subsequence.

(ii) If $n \geq 8$, then $|W_{g_i}(0)|^2 \leq C^* G_i \beta_i$ with $C^* > 0$ depending only on $n, A_1, A_2, A_3$, where

$$G_i := \sum_{k \geq 1, 2 \leq k+l \leq 4} \Theta_i \| \nabla^k g_i \|_{L^\infty(B_3)} + \sum_{k \geq 1, 6 \leq k+l \leq 8} \| \nabla^k g_i \|_{L^\infty(B_3)},$$

and

$$\beta_i := \begin{cases} 
(\log m_i)^{-1}, & \text{if } n = 8, \\
-m_i^{-\frac{2}{n-4}}, & \text{if } n = 9, \\
m_i^{-\frac{1}{n-4}} \log m_i, & \text{if } n = 10, \\
m_i^{-\frac{4}{n-4}}, & \text{if } n \geq 11.
\end{cases}$$

(iii) (76) holds if $n \geq 10$ and

$$|W_{g_i}(0)|^2 > C^* G_i \beta_i.$$  \hspace{1cm} (79)

**Proof.** It follows from Corollary 3.2 that after passing a subsequence

$$\lim u_i(0)u_i(x) =: \Gamma(x),$$

where $\Gamma(x)$ is in $C^3(B_1 \setminus \{0\})$. We will still denote the subsequence as $u_i$.

Notice that for every $0 < r < 1$

$$m_i^2 x^k \nabla_k u_i \to \mathcal{P}(r, \Gamma) \quad \text{as } i \to \infty.$$  \hspace{1cm} (80)

By Proposition 2.2,

$$\mathcal{P}(r, u_i) = \int_{B_r} (x^k \partial_k u_i + \frac{n-4}{2} u_i) E(u_i) + \mathcal{N}(r, u_i),$$

where $E(u_i)$ is as in (16) with $\tilde{g}$ and $u$ replaced by $g_i$ and $u_i$ respectively, i.e.,

$$E(u_i) := P_{g_i} u_i - \Delta^2 u_i$$

$$= \frac{n-4}{2} Q_{g_i} u_i + f^{(1)}_{i,k} \partial_k u_i + f^{(2)}_{i,kl} \partial_{kl} u_i + f^{(3)}_{i,kls} \partial_{kls} u_i + f^{(4)}_{i,kl} \partial_{kl} u_i,$$

$$f^{(1)}_{i,k}(x) = O(1), \quad f^{(2)}_{i,kl}(x) = O(1), \quad f^{(3)}_{i,kls}(x) = O(|x|), \quad f^{(4)}_{i,kl}(x) = O(|x|^2).$$  \hspace{1cm} (81)
Compactness of conformal metrics with constant $Q$-curvature

and

$$\mathcal{N}(r, u_i) = \frac{c(n)\tau_i}{p_i + 1} \int_{B_r} \left( \frac{n-4}{2} \kappa_i^\tau_i + x^k \partial_k \kappa_i \kappa_i^{-1} \right) u_i^{p_i+1} - \frac{r}{p_i + 1} \int_{\partial B_r} c(n)\kappa_i^\tau_i u_i^{p_i+1}.$$  

By Proposition 3.3 for $0 < r < 1$ we have, for some $C > 0$ independent of $i$ and $r$,

$$m_i^2 \mathcal{N}(r, u_i) \geq -\frac{m_i^2 r}{p_i + 1} \int_{\partial B_r} c(n)\kappa_i^\tau_i u_i^{p_i+1} \geq -Cr^{-n}m_i^{1-p_i},$$  

where we used the facts that $\kappa_i(x) = 1 + O(|x|^2)$ and $|\nabla \kappa_i(x)| = O(|x|)$ with $O(\cdot)$ independent of $i$. Hence, we have

$$\lim_{i \to \infty} m_i^2 \mathcal{N}(r, u_i) \geq 0.$$  

(82)

Throughout this section, without otherwise stated, we use $C$ to denote some constants independent of $i$ and $r$.

If $g_i$ is flat, then we complete the proof because $E(u_i) = 0$.

Now we assume $g_i$ is not flat. By a change of variables $z = \ell_i x$ with $\ell_i = m_i^{\frac{p_i-1}{4}}$, we have

$$\mathcal{E}_i(r) : = m_i^2 \int_{B_r} \left( x^k \partial_k u_i + \frac{n-4}{2} u_i \right) E(u_i) \, dx$$

$$= m_i^2 m_i^{2+(4-n)\frac{p_i-1}{4}} \int_{B_{\ell_i r}} \left( z^k \partial_k \varphi_i + \frac{n-4}{2} \varphi_i \right) \cdot$$

$$\left( \frac{n-4}{2} \ell_i^{-4} Q_{\ell_i^{-1}}(\ell_i^{-1} z) \varphi_i + \sum_{j=1}^4 \ell_i^{-4+j} f_i^{(j)}(\ell_i^{-1} z) \nabla^j \varphi_i \right) \, dz,$$

where $\varphi_i(z) = m_i^{-1} u_i(m_i^{-\frac{p_i-1}{4}} z)$, $f_i^{(1)}(\ell_i^{-1} z) \nabla^1 \varphi_i = f_i^{(1)}(\ell_i^{-1} z) \partial_k \varphi_i$ and $f_i^{(j)}(\ell_i^{-1} z) \nabla^j \varphi_i$ is defined in the same fashion for $j \neq 1$. Define

$$\mathcal{E}_i(r) : = m_i^2 m_i^{2+(4-n)\frac{p_i-1}{4}} \int_{B_{\ell_i r}} \left( z^k \partial_k U_1 + \frac{n-4}{2} U_1 \right) \cdot$$

$$\left( \frac{n-4}{2} \ell_i^{-4} Q_{\ell_i^{-1}}(\ell_i^{-1} z) U_1 + \sum_{j=1}^4 \ell_i^{-4+j} f_i^{(j)}(\ell_i^{-1} z) \nabla^j U_1 \right) \, dz.$$

Notice that $m_i^{2+(4-n)\frac{p_i-1}{4}} = 1 + o(1)$, and $Q_g = O(1)$. By Proposition 4.1 Proposition 4.2

43
where we have used Corollary 4.1, (81) and (18), we have

\[ |E_i(r) - \hat{E}_i(r)| \leq C m^2 m_i \int_{B_{\ell}} \sum_{j=0}^{4} |\nabla^j (\varphi_i - U_1)(z)(1 + |z|)^{2-n-j}| dz \]

Hence, 

\[ r^2, \quad \max\{\Theta_i r, r^2\}, \quad \max\{\Theta_i \log(r m_i), r^2\}, \quad \max\{\Theta_i m_i^{\frac{2(n-10)}{n-18}}, r^2\}, \]

if \( n = 5, 6, 7 \), \( n = 8, 9 \), \( n = 10 \), \( n \geq 11 \).

Now we estimate \( \hat{E}_i(r) \).

If \( n = 5, 6, 7 \), we have

\[
\hat{E}_i(r) = m^2 m_i \int_{B_{\ell}} (x^k \partial_k U_{\ell_i} + \frac{n-4}{2} U_{\ell_i}) E(U_{\ell_i}) \, dx \\
= m^2 m_i \int_{B_{\ell}} (x^k \partial_k U_{\ell_i} + \frac{n-4}{2} U_{\ell_i})(P_{\ell_i} - \Delta^2)U_{\ell_i} \, dx \\
= O(1) m^2 m_i \int_{B_{\ell}} |x^k \partial_k U_{\ell_i} + \frac{n-4}{2} U_{\ell_i}|U_{\ell_i},
\]

where we have used \( (P_{\ell_i} - \Delta^2)U_{\ell_i} = O(1)U_{\ell_i} \) because of (20), and

\[ ||x^k \nabla^k U_{\ell_i}(x)|| \leq C(n, k)U_{\ell_i}(x) \text{ for } k \in \mathbb{N}. \]

Hence,

\[ |\hat{E}_i(r)| \leq Cr^{8-n} \leq Cr. \]  

Therefore, (86) follows from (82), (84) and (86) when \( n = 5, 6, 7 \).

If \( n \geq 8 \), by Lemma 2.1 we have

\[
\hat{E}_i(r) = -\frac{2}{n} \gamma_i \int_{B_{\ell_i}} (s \partial_s U_1 + \frac{n-4}{2} U_1)(c_1^2 s \partial_s U_1 + c_2^2 s^2 \partial_s U_1) \, dz \]

\[ -\frac{32(n-1)}{3(n-2)} \gamma_i \int_{B_{\ell_i}} (s \partial_s U_1 + \frac{n-4}{2} U_1)s^2 (\partial_s U_1 - \frac{\partial_s U_1}{s}) \, dz \]

\[ + (n-4) \gamma_i \int_{B_{\ell_i}} (s \partial_s U_1 + \frac{n-4}{2} U_1)U_1 \, dz + O(\alpha_i^n) \sum_{k \geq 1, 6 \leq k+l \leq 8} ||\nabla^k g_i||_{L^\infty(B_{\ell_i})}, \]

44
Compactness of conformal metrics with constant $Q$-curvature

where we used the symmetry so that those terms involving homogeneous polynomials of odd degrees are gone, $s = |z|$, $\gamma_i = \frac{2(n-s)}{m_i} \frac{2(n-4)\ell_i |W_{0,0}(s)|^2}{24(n-1)} \geq 0$,

$$\alpha''_i = \int_{B_r} |x|^2 U_{\ell_i}(x)^2 \, dx = O(1) \begin{cases} r^{10-n}, & \text{if } n = 8, 9, \\ \log r m_i, & \text{if } n = 10, \\ \frac{m_i}{2(n-10)}, & \text{if } n \geq 11. \end{cases}$$

and $c_1^*, c_2^*$ are given in (19). By direct computations,

$$r \partial_r U_1 + \frac{n - 4}{2} U_1 = \frac{n - 4}{2} \frac{1 - r^2}{(1 + r^2)^{\frac{n-2}{2}}},$$

$$c_1^* r \partial_r U_1 + c_2^* \partial_{rr} U_1 = (4 - n) \frac{(c_1^* + c_2^*) r^2}{(1 + r^2)^{\frac{n-2}{2}}} + (4 - n)(2 - n) \frac{c_2^* r^4}{(1 + r^2)^{\frac{n}{2}}}$$

$$= (4 - n) \frac{(c_1^* + c_2^*) r^2}{(1 + r^2)^{\frac{n-2}{2}}} + (c_1^* + (3-n) c_2^*) r^4,$$

$$\partial_{rr} U_1 - \frac{\partial_r U_1}{r} = (n-4)(n-2)(1 + r^2)^{-\frac{m}{2}} r^2.$$

Thus

$$\hat{\mathcal{E}}(r) = \frac{(n-4)^2}{n} \gamma_i |S^{n-1}| J_i + O(\alpha''_i) \sum_{k \geq 1, 6k + l \leq 8} \|\nabla^k g_i\|_{L^\infty(B_1)},$$

where

$$J_i := \int_0^\ell [1 - s^2] \frac{(c_1^* + c_2^* + n) s^2 + (c_1^* + (3-n) c_2^* - \frac{(n-1)(n-8)(n-6)}{3n} + \frac{n}{2}) s^4] |S^{n-1}| \, ds.$$

If $n = 8$, we have $-(c_1^* + (3-n) c_2^* + \frac{n}{2}) = (2n - 12) + \frac{14}{3} - 4 = \frac{14}{3}$. Since $\int_0^\ell \frac{s}{(1+s^2)^2} \, ds \to \infty$ as $\ell_i \to \infty$, Hence, $J_i \to \infty$ as $i \to \infty$. For $n \geq 9$, we notice that for positive integers $2 < m + 1 < 2k$,

$$\int_0^\infty \frac{t^m}{(1 + t^2)^k} \, dt = \frac{m - 1}{2k - m - 1} \int_0^\infty \frac{t^{m-2}}{(1 + t^2)^k} \, dt.$$

If $\ell_i r = \infty$, we have

$$J_i = \left\{ - \frac{2n}{n-4} - (c_1^* + c_2^* + n) \frac{8n}{(n-6)(n-4)} - (c_1^* + (3-n) c_2^* - \frac{16(n-1)}{3n} + \frac{n}{2}) \frac{12(n-2)}{(n-8)(n-6)(n-4)} \right\} \int_0^\infty \frac{s^{n-1}}{(1 + s^2)^{n-1}} \, ds.$$
We compute the coefficients of the integral,

\[
- \frac{2n}{n-4} - (c_1^* + c_2^* + n) \frac{8n}{(n-6)(n-4)} \\
- (c_1^* + (3-n)c_2^*) \frac{16(n-1) + n}{3n} \frac{2n}{(n-8)(n-6)(n-4)} \\
= \frac{2n}{n-4} \left\{ -1 + \left( \frac{n(n-2)}{2} - 8 \right) \frac{4}{n-6} + \left( \frac{3n}{2} + \frac{16(n-1)}{3n} - 12 \right) \frac{6(n+2)}{(n-8)(n-6)} \right\} \\
= \frac{2n}{n-4} \left\{ \frac{2n^2 - 5n - 26}{n-6} + \frac{9(n+2)}{n-6} + \frac{32(n-1)(n+2)}{n(n-8)(n-6)} \right\} \\
\geq \frac{4n(n^2 + 2n - 4)}{(n-4)(n-6)} > 0.
\]

Therefore, for any \(0 < r < 1\) and sufficiently large \(i\) (the largeness of \(i\) may depend on \(r\)), we have

\[
J_i \geq 1/C(n) > 0. \tag{87}
\]

In conclusion, we obtain

\[
\hat{\mathcal{E}}_i(r) \geq \begin{cases} 
\frac{1}{C} |W_g(0)|^2 \log m_i + O(\alpha_i^\prime) \sum_{k \geq 1, 6 \leq k + t \leq 8} \| \nabla^k g_i \|_{L^\infty(B_\delta)}^r, & \text{if } n = 8, \\
\frac{1}{C} |W_g(0)|^2 \left[ m_i \frac{2^{(n-8)}}{n-4} \right] + O(\alpha_i^\prime) \sum_{k \geq 1, 6 \leq k + t \leq 8} \| \nabla^k g_i \|_{L^\infty(B_\delta)}^r, & \text{if } n \geq 9.
\end{cases} \tag{88}
\]

Combing (82), (84) and (88) together, we see that, for \(n \geq 8\),

\[
m_i^2 \mathcal{P}(r, u_i) = m_i^2 \mathcal{N}(r, u_i) + (\mathcal{E}_i(r) - \hat{\mathcal{E}}_i(r)) + \hat{\mathcal{E}}_i(r) \\
\geq m_i^2 \mathcal{N}(r, u_i) + \frac{1}{2} \hat{\mathcal{E}}_i(r) - Cr
\]

where \(Cr\) can be set to zero when \(n \geq 9\). If \(n = 8, 9\), by sending \(i \to \infty\) in (89) we have \(\mathcal{P}(r, \Gamma) \geq -Cr\). Thus (76) follows and the conclusion (i) is proved.

If \(n \geq 10\) and \(|W_g(0)|^2\) satisfies (79) for large \(C^\prime > 0\), by (88) we see that \((\mathcal{E}_i(r) - \hat{\mathcal{E}}_i(r)) + \hat{\mathcal{E}}_i(r) \geq 0\). Hence, the conclusion (iii) follows.

Since \(|\mathcal{P}(r, \Gamma)| \leq C\), it follows from (89) that for large \(i\), \(\hat{\mathcal{E}}_i(r) \leq C\). In view of (88) and the definition of \(\alpha''\), the conclusion (ii) follows.

\[
\Box
\]

**Proposition 5.2.** Given \(p_i, G_i\), and \(h_i\) satisfying (23), (24) and (26) respectively, \(k_i\) satisfying (25) with \(K_i\) replaced by \(k_i\), let \(0 \leq u_i \in C^4(B_3)\) solve both (59) and (75), and assume (64) holds. Suppose that 0 is an isolated blow up point of \(\{u_i\}\) with (61) holds. Then 0 is an isolated simple blow up point, if one of the three cases happens:
Compactness of conformal metrics with constant $Q$-curvature

- $g_i$ is flat;
- $n \leq 9$;
- $n \geq 10$ and (79) holds.

Proof. By Proposition 3.1 $r^{4/(p_i-1)}\tilde{\nabla}^i(r)$ has precisely one critical point in the interval $0 < r < r_i$, where $R_i \to \infty r_i = R_iu_i(0)\frac{p_i-1}{p_i}$ as in Proposition 3.1. Suppose the contrary that 0 is not an isolated simple blow up point and let $\mu_i$ be the second critical point of $r^{4/(p_i-1)}\tilde{u}_i(r)$. Then we must have

$$\mu_i \geq r_i, \lim_{i \to \infty} \mu_i = 0. \quad (90)$$

Set

$$v_i(x) = \mu_i^{4/(p_i-1)}u_i(\mu_i x), \quad x \in B_{3/\mu_i}.$$

By the assumptions of Proposition 3.1, $v_i$ satisfies

$$v_i(x) = \int_{B_{3/\mu_i}} \tilde{G}_i(x, y)\tilde{\kappa}_i(y)^{r_i} v_i(y)^{p_i} dy + \tilde{h}_i(x)$$

$$|x|^{4/(p_i-1)}v_i(x) \leq A_3, \quad |x| < 2/\mu_i \to \infty,$$

$$\lim_{i \to \infty} v_i(0) = \infty,$$

and

$$r^{4/(p_i-1)}\tilde{\nabla}^i(r)$$

has precisely one critical point in $0 < r < 1$.

and

$$\frac{d}{dr} \left\{ r^{4/(p_i-1)}\tilde{\nabla}^i(r) \right\} \bigg|_{r=1} = 0,$$

where $\tilde{G}_i = G_{i,\mu_i}$, $\tilde{\kappa}_i(y) = \kappa_i(\mu_i y)$, $\tilde{h}_i(x) = \mu_i^{4/(p_i-1)}h_i(\mu_i x)$ and $\tilde{\nabla}^i(r) = |\partial B_r|^{-1}\int_{\partial B_r} v_i$. Therefore, 0 is an isolated simple blow up point of $\{v_i\}$.

Claim. We have

$$v_i(0)v_i(x) \to \frac{ac_\kappa}{|x|^{n-4}} + ac_\kappa \quad \text{in} \quad C^3_{\text{loc}}(\mathbb{R}^n \setminus \{0\}). \quad (91)$$

where $a > 0$ is given in (58).

First of all, by Proposition 3.3 we have $\tilde{h}_i(e) \leq v_i(e) \leq C v_i(0)^{-1}$ for any $e \in S^{n-1}$, where $C > 0$ is independent of $i$. It follows from the assumption (26) on $h_i$ that

$$v_i(0)\tilde{h}_i(x) \leq C \quad \text{for all} \quad |x| \leq 2/\mu_i$$

and

$$\|\nabla (v_i(0)\tilde{h}_i)\|_{L^\infty(B_{2/\mu_i} \setminus \{0\})} \leq \mu_i \|v_i(0)\tilde{h}_i\|_{L^\infty(B_{2/\mu_i} \setminus \{0\})} \leq C \mu_i. \quad (92)$$

47
Hence, for some constant $c_0 \geq 0$, we have, along a subsequence,
\[
\lim_{i \to \infty} \|v_i(0)^\sim \tilde{h}_i(x) - c_0\|_{L^\infty(B_t)} = 0, \quad \forall \ t > 0.
\] (93)

Secondly, by Corollary 3.2 and Proposition 3.3 we have, up to a subsequence,
\[
v_i(0) \int_{B_t} \tilde{G}_i(x, y) \tilde{k}_i(y)\tau_i v_i(y)^{p_i} \, dy \to \frac{a c_n}{|x|^{n-4}} \quad \text{in } C^3_{\text{loc}}(B_t \setminus \{0\}) \text{ for any } t > 0,
\] (94)
where we used that $\tilde{G}_i(x, 0) \to c_n |x|^{4-n}$. Notice that for any $x \in B_{t/2}$
\[
Q_i''(x) := \int_{B_{t/2} \setminus B_t} \tilde{G}_i(x, y) \tilde{k}_i(y)\tau_i v_i(y)^{p_i} \, dy \leq C(n, A_1) \max_{\partial B_t} v_i.
\]
Since $\max_{\partial B_t} v_i \leq C t^{4-n} v_i(0)^{-1}$, we have as in the proof of (35), after passing to a subsequence,
\[
v_i(0) Q_i''(x) \to q(x) \quad \text{in } C^3_{\text{loc}}(B_t) \quad \text{as } i \to \infty
\]
for any $q \in C^3(B_t)$. For any fixed large $R > t + 1$, it follows from (94) that
\[
v_i(0) \int_{t \leq |y| \leq R} \tilde{G}_i(x, y) \tilde{k}_i(y)\tau_i v_i(y)^{p_i} \, dy \to 0
\]
as $i \to \infty$, since the constant $a$ is independent of $t$. By the assumption (24) on $G_i$, for any $x \in B_t$ and $|y| > R$, we have
\[
|\nabla_x \tilde{G}_i(x, y)| \leq A_1 |x-y|^{3-n} \leq \frac{A_1}{R-t} |x-y|^{4-n} \leq \frac{A_1^2}{R-t} \tilde{G}_i(x, y).
\]
Therefore, we have $|\nabla q(x)| \leq \frac{A_1^2}{R-t} q(x)$. By sending $R \to \infty$, we have $|\nabla q(x)| \equiv 0$ for any $x \in B_t$. Thus,
\[
q(x) \equiv q(0) \quad \text{for all } x \in B_t.
\]
Since
\[
\frac{d}{dr} \left\{ r^{4/(p_i-1)} v_i(0)^\sim \tilde{v}_i(r) \right\} \bigg|_{r=1} = v_i(0) \frac{d}{dr} \left\{ r^{4/(p_i-1)} \tilde{v}_i(r) \right\} \bigg|_{r=1} = 0,
\]
we have, by choosing, for example, $t = 2$ and sending $i$ to $\infty$, that
\[
q(0) + c_0 = ac_n > 0.
\] (95)
Therefore, (93) is proved.

It follows from (95) and Lemma 2.2 that
\[
\lim_{i \to \infty} \inf v_i(0)^2 \mathcal{P}(r, v_i) = -(n-4)^2(n-2) a^2 c_n^2 |S^{n-1}| < 0 \quad \text{for all } 0 < r < 1.
\] 48
Lemma 5.1. Let
\[ g_i(z) = g_i(\mu_i z) \]
where \( g_i(z) = g_i(\mu_i z) \). It is easy to see that (64) is still correct with \( G_i \) replaced by \( \tilde{G}_i \). If \( n \leq 9 \) or \( g_i \) is flat, it follows from Proposition 5.1 that
\[ \lim_{r \to 0} \lim_{i \to \infty} v_i(0)^2 \mathcal{P}(r, v_i) \geq 0. \]  
(96)  
If \( n \geq 10 \), by (64), we have
\[ U_\lambda(x) = \int_{B_3/\mu_i} G_{i,\mu_i}(x, y) \{ U_\lambda(y)^{\frac{n+4}{n-4}} + \mu_i^4 c'_{\Lambda/\mu_i,i}(\mu_i y) U_\lambda(y) \} \, dy + \mu_i^4 c''_{\Lambda/\mu_i,i}(\mu_i x) \]
\[ = \int_{B_3} \tilde{G}_i(x, y) \{ U_\lambda(y)^{\frac{n+4}{n-4}} + \tilde{c}'_{\Lambda,i}(y) U_\lambda(y) \} \, dy + \tilde{c}''_{\Lambda,i}(x) \quad \forall \lambda \geq 1, \ x \in B_3, \]
where \( c'_{\Lambda,i}(y) := \mu_i^4 c'_{\Lambda/\mu_i,i}(\mu_i y) \) and
\[ \tilde{c}''_{\Lambda,i}(x) = \int_{B_3/\mu_i \setminus B_3} G_{i,\mu_i}(x, y) \{ U_\lambda(y)^{\frac{n+4}{n-4}} + \mu_i^4 c'_{\Lambda/\mu_i,i}(\mu_i y) U_\lambda(y) \} \, dy + \mu_i^4 c''_{\Lambda/\mu_i,i}(\mu_i x). \]
By the assumptions for \( c'_{\Lambda,i} \) and \( c''_{\Lambda,i} \), we have
\[ \tilde{\Theta}_i := \sum_{i=0}^5 \| \lambda^{-k} \nabla^k c'_{\Lambda,i} \|_{L^\infty(B_3)} \leq \mu_i^4 \Theta_i, \]
and \( \| \tilde{c}''_{\Lambda,i} \|_{C^5(B_3)} \leq C A_2 \lambda \frac{4-n}{x_2} \), where \( C > 0 \) depends only on \( n, A_1, A_2 \). Clearly, we have
\[ |W_{\tilde{g}_i}(0)|^2 = |W_{g_i}(0)|^2. \]  
Hence (79) is satisfied. By Proposition 5.1 we also have (96). We obtain a contradiction.

Therefore, 0 must be an isolated simple blow up point of \( u_i \) and the proof is completed. \( \square \)

Lemma 5.1. Let \( 0 \leq u_i \in C^4(B_3) \) solve both (59) and (75) with \( n \geq 10 \), and assume (64) holds. For \( \mu_i \to 0 \), let
\[ v_i(x) = \mu_i^{4-1} u_i(\mu_i x). \]
Suppose that 0 is an isolated blow up point of \( \{v_i\} \) and (79) holds. Then 0 is also an isolated simple blow up point.

Proof. From the end of proof of Proposition 5.2 we see that the condition (79) is preserved under the scaling \( v_i(x) = \mu_i^{4-1} u_i(\mu_i x) \). Hence, the lemma follows from Proposition 5.2. \( \square \)
6 Global analysis, and proof of Theorem 1.2

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 5\). Suppose that \(\text{Ker} P_g = \{0\}\) and the Green’s function \(G_g\) of \(P_g\) is positive. Consider the equation

\[
P_g u = c(n) u^p, \quad u \geq 0 \quad \text{on } M,
\]

where \(1 < p \leq \frac{n+4}{n-4}\).

**Proposition 6.1.** Assume the above. For any given \(R > 0\) and \(0 < \varepsilon < \frac{1}{n-4}\), there exist positive constants \(C_0 = C_0(M, g, R, \varepsilon), C_1 = C_1(M, g, R, \varepsilon)\) such that, for any smooth positive solution of (97) with

\[\max_M u(X) \geq C_0,\]

then \(\frac{n+4}{n-4} - p < \varepsilon\) and there exists a set of finite distinct points

\[\mathcal{I}(u) := \{Z_1, \ldots, Z_N\} \subset M\]

such that the following statements are true.

(i) Each \(Z_i\) is a local maximum point of \(u\) and

\[\overline{B}_{\tilde{r}_i}(Z_i) \cap \overline{B}_{\tilde{r}_j}(Z_j) = \emptyset \quad \text{for } i \neq j,\]

where \(\tilde{r}_i = R u(Z_i)^{(1-p)/4}\), and \(B_{\tilde{r}_i}(Z_i)\) denotes the geodesic ball in \(B_2\) centered at \(Z_i\) with radius \(\tilde{r}_i\).

(ii) For each \(Z_i\),

\[
\left\| \frac{1}{u(Z_i)} u \left( \exp_{Z_i} \left( \frac{y}{u(Z_i)^{(p-1)/4}} \right) \right) - \left( \frac{1}{1 + |y|^2} \right)^{\frac{n+4}{2}} \right\|_{C^4(B_{2R})} < \varepsilon.
\]

(iii) \(u(X) \leq C_1 \text{dist}_g(x, \{Z_1, \ldots, Z_N\})^{-4/(p-1)}\) for all \(X \in M\).

**Proof.** The proof is standard by now. \(\square\)

**Proposition 6.2.** If either \(n \leq 9\) or \((M, g)\) is locally conformally flat, then, for \(\varepsilon > 0\), \(R > 1\) and any solution (97) with \(\max_M u > C_0\), we have

\[|Z_1 - Z_2| \geq \delta^* > 0 \quad \text{for any } Z_1, Z_2 \in \mathcal{I}(u), \ Z_1 \neq Z_2,\]

where \(\delta^*\) depends only on \(M, g\).
Proof. Suppose the contrary, then there exist a sequence $0 \leq \frac{n+4}{n-4} - p_i < \varepsilon$ and $u_i$ satisfying (97) with $p = p_i$,

$$\max_M u_i(X) > C_0,$$

and

$$\text{dist}_g(Z_{1i}, Z_{2i}) = \min_{1 \leq k, l \leq N_1, k \neq l} \text{dist}_g(Z_{ki}, Z_{li}) \to 0 \quad (98)$$

as $i \to \infty$, where $\mathcal{S}(u_i) = (Z_{1i}, \ldots, Z_{N_1i})$ be the local maximum points of $u_i$ as selected by Proposition 6.1. Without loss of generality, we may assume

$$u_i(Z_{1i}) \geq u_i(Z_{2i}).$$

Since $B_{R_u(Z_{1i})}^{-(p_i-1)/4}(Z_{1i})$ and $B_{R_u(Z_{2i})}^{-(p_i-1)/4}(Z_{2i})$ have to be disjoint, we have, because of (98), that $u_i(Z_{1i}) \to \infty$ and $u_i(Z_{2i}) \to \infty$. Let $\{x_1, \ldots, x_n\}$ be the conformal normal coordinates centered at $Z_{1i}$. We write (97) as

$$P_{g_i} \tilde{u}_i = c(n)\kappa_i^{\tau_i} \tilde{u}_i^{p_i} \quad \text{on} \ M, \quad (99)$$

where $g_i = \kappa_i^{\frac{4}{n-4}} g$, $\tilde{u}_i = \kappa_i u_i$, $\kappa_i > 0$, $\kappa_i(Z_{1i}) = 1$, $\nabla_g \kappa_i(Z_{1i}) = 0$, and $\tau_i = \frac{n+4}{n-4} - p_i$. Since $\text{dist}_g(Z_{1i}, Z_{2i}) \to 0$, for large $i$ we let $z_{2i} \in \mathbb{R}^n$ such that $\exp_{Z_{1i}} z_{2i} = Z_{2i}$, and let

$$\vartheta_i := |z_{2i}| \to 0.$$

We will sit in the conformal normal coordinates chart $B_t$ at $Z_{1i}$, where $t > 0$ is independent of $i$, and write $f(\exp_{Z_{1i}} x)$ simply as $f(x)$. Set

$$\varphi_i(x) = \vartheta_i^{4/(p_i-1)} \tilde{u}_i(\vartheta_i x) \quad \text{for} \ |x| \leq t/\vartheta_i.$$

By the equation (99), we have

$$P_{g_i} \varphi_i(x) = c(n) \tilde{\kappa}_i^{\tau_i} \varphi_i(x)^{p_i} \quad \text{in} \ B_{t/\vartheta_i}, \quad (100)$$

where $\tilde{\kappa}_i(x) = \kappa_i(\vartheta_i x)$, $\tilde{g}_i(x) = g_i(\vartheta_i x)$. Using the Green representation for (99),

$$\tilde{u}_i(x) = c(n) \int_{B_t} G_i(x, y) \kappa_i(y)^{\tau_i} \tilde{u}_i(y)^{p_i} \, dy + h_i(x),$$

where $G_i(x, y) = G_{g_i}(\exp_{Z_{1i}} x, \exp_{Z_{1i}} y)$ and

$$h_i(x) = \int_{M \setminus \exp_{Z_{1i}} B_t} G_{g_i}(\exp_{Z_{1i}} x, Y) \kappa_i(Y)^{\tau_i} u_i(Y) \, dvol_{g_i}(Y).$$

51
Hence, \( \varphi_i \) also satisfies
\[
\varphi_i(x) = \int_{B_t/\varrho_i} G_{i,\varrho_i}(x, y) K_i(\varrho_i y) \varphi_i(y) \rho_i \, dy + \tilde{h}_i(x) \quad \text{for all } x \in B_t/\varrho_i,
\]  
(101)

where \( G_{i,\varrho_i}(x, y) = \rho_i^{n-4} G(\varrho_i x, \varrho_i y) \) and \( \tilde{h}_i = \rho_i^{4/(p_i-1)} h_i(\varrho_i y) \).

By Proposition 6.1 we have
\[
\bar{u}_i(x) \leq C_1^* |x|^{-4/(p_i-1)} \quad \text{for all } |x| \leq 3\varrho_i/4,
\]
\[
\bar{u}_i(x) \leq C_1^* |x - z_{2i}|^{-4/(p_i-1)} \quad \text{for all } |x - z_{2i}| \leq 3\varrho_i/4,
\]
where \( C_1^* \) depending only on \( C_1, M \) and \( g \). Hence,
\[
\varphi_i(x) \leq C_1^* |x|^{-4/(p_i-1)} \quad \text{for all } |x| \leq 3/4,
\]
\[
\varphi_i(x) \leq C_1^* |x - \varrho_i^{-1} z_{2i}|^{-4/(p_i-1)} \quad \text{for all } |x - \varrho_i^{-1} z_{2i}| \leq 3/4.
\]  
(102)

Set \( \xi_i = \varrho_i^{-1} z_{2i} \). We claim that, after passing to a subsequence,
\[
\varphi_i(0), \varphi_i(\xi_i) \to \infty \quad \text{as } i \to \infty.
\]  
(103)

It is clear that \( \varphi_i(0) \) and \( \varphi_i(\xi_i) \) are bounded from below by some positive constant independent of \( i \). If there exists a subsequence (still denoted as \( \varphi_i \)) such that \( \lim_{i \to \infty} \varphi_i(0) = \infty \) but \( \varphi_i(\xi_i) \) stays bounded, we have that 0 is an isolated blow up point for \( \varphi_i \) in \( B_{3/4} \) when \( i \) is large; see Remark 3.1. Using equation (101) and (102), by the same proof of (34) we have \( \sup_{B_{1/2}(\xi_i)} \varphi_i < \infty \).

It follows from Proposition 5.3 and Proposition A.1 that \( \lim_{i \to \infty} \varphi_i(\xi_i) = 0 \), but this is impossible since \( \varrho_i > Ru_i(2z_{2i})^{-4(p_i-1)/4} \) and thus \( \varphi_i(\xi_i) \geq \frac{1}{C} R \) for some \( C > 0 \) depending only on \( M, g, R \) and \( \varepsilon \). On the other hand, if there exists a subsequence (still denoted as \( \varphi_i \)) such that \( \varphi_i(0) \) and \( \varphi_i(\xi_i) \) remain bounded, we know from a similar argument as above that \( \varphi_i \) is locally bounded. The same proof of Proposition 3.1 yields that after passing to a subsequence \( \varphi_i \to \varphi \) in \( C^3_{\text{loc}}(\mathbb{R}^n) \) for some \( \varphi \) satisfying
\[
\varphi(x) = c_n \int_{\mathbb{R}^n} \frac{\varphi(y) \frac{n+4}{n-4}}{|x - y|^{n-4}} \, dy \quad \text{for all } x \in \mathbb{R}^n,
\]
\( \nabla \varphi(0) = 0, \nabla \varphi(\tilde{z}) = \lim_{i \to \infty} \nabla \varphi_i(\xi_i) = \lim_{i \to \infty} \frac{\varrho_i \varphi_i(\xi_i) \nabla \varphi_i(\xi_i)}{\kappa_i(2z_{2i})} = 0 \), where \( |\tilde{z}| = 1 \) is the limit of \( \xi_i \) up to passing a subsequence. This contradicts to the Liouville theorem in [12] or Li [36]. Hence, (103) is proved.

Since \( \nabla \varphi_i(0) = 0 \), it follows from the first inequality of (102) and (103) that 0 is an isolated blow up point of \( \{ \varphi_i \} \). Since \( n \leq 9 \) or \( (M, g) \) is locally conformally flat, by Proposition 5.2 we...
Compactness of conformal metrics with constant $Q$-curvature

conclude that $0$ is an isolated simple blow up point of $\{\varphi_i\}$. It follows from Corollary 3.2 that for all $x \in B_{1/2}$

$$\varphi_i(0) \int_{B_{1/2}} G_{i,\partial_i}(x, y) \varphi_i(y)^{p_i} K_i(\partial_i y) \, dy \to a c_n |x|^{4-n} \quad (104)$$

and

$$\varphi_i(0)(Q''_i(x) + \tilde{h}_i(x)) \to h(x) \geq 0 \quad \text{in } C^3_{\text{loc}}(B_{1/2}),$$

where $a > 0$ is given in (58), $h(x) \in C^5(B_{1/2})$ and

$$Q''_i(x) = \int_{B_{1/0}\setminus B_{1/2}} G_{i,\partial_i}(x, y) \varphi_i(y)^{p_i} K_i(\partial_i y) \, dy \quad x \in B_{1/2}.$$ 

Note that

$$\varphi_i(0)Q''_i(x) \geq \frac{1}{C} \varphi_i(0) \int_{B_{1/2}(\xi_i)} \varphi_i(y)^{p_i} \, dy.$$

It follows from (102), (103) and the proof of (34) that there exists a constant $C > 0$, depending only on $M, g, R$ and $\varepsilon$ such that $\varphi_i(x) \leq C \varphi_i(\xi_i)$ for all $|x - \xi| \leq \frac{1}{2}$. It follows from the proof of Proposition 3.1 that there exist a constant $\lambda$ and an point $x_0 \in \mathbb{R}^n$ with $1 \leq \lambda \leq C$ and $|x_0| \leq C$ such that for any fixed $\bar{R} > 0$ we have

$$\lim_{i \to \infty} \left\| \frac{1}{\varphi_i(\xi_i)} \varphi_i(\xi_i) - (p_i - 1)/4 \right\|_{C^4(B_{\bar{R}})} = 0$$

By changing of variables $y = \xi_i + \varphi_i(\xi_i)^{-(p_i - 1)/4} x$, we have

$$\varphi_i(0) \int_{B_{1/2}(\xi_i)} \varphi_i(y)^{p_i} \, dy = \varphi_i(0) \varphi_i(\xi_i)^{p_i - (p_i - 1)n/4} \int_{B_{\varphi_i(\xi_i)/2}(0)} \left( \frac{1}{\varphi(\xi_i)} \varphi_i(\xi_i) - (p_i - 1)/4 \right)^{p_i} \, dx$$

$$\geq \frac{1}{C} \varphi_i(0) \varphi_i(\xi_i)^{p_i - (p_i - 1)n/4} \int_{R^n} U_\lambda(x - x_0)^{p_i} \, dx$$

$$\geq \frac{1}{C} \varphi_i(0) \varphi_i(\xi_i)^{p_i - (p_i - 1)n/4} \int_{R^n} (1 + |x|^2)^{\frac{n+4}{2}} \, dx,$$

where we used $1 \leq \lambda \leq C$ and $|x_0| \leq C$. Since $u_i(Z_{1i}) \geq u_i(Z_{2i})$, we have $\varphi_i(0) \geq \varphi_i(\xi_i)$ for some $C$ depending only on $M, g$. By Lemma 3.4, we have $\varphi_i(\xi_i)^{1+p_i - (p_i - 1)n/4} = 1 + o(1)$. Therefore, we obtain

$$\lim_{i \to \infty} \varphi_i(0)Q''_i(x) \geq \frac{1}{C} \int_{R^n} (1 + |\xi|^2)^{\frac{n+4}{2}} \, d\xi =: a_0 > 0. \quad (105)$$
In conclusion,
\[ \varphi_i(0)\varphi_i(x) \to ac_n|x|^{4-n} + h(x) \quad \text{in } C^3_{loc}(B_{1/2} \setminus \{0\}) \]
for some nonnegative bounded function in \( C^3(B_{1/2}) \) with \( h(0) \geq a_0 \). It follows from Lemma 2.2 that
\[ \liminf_{r \to 0} \liminf_{i \to \infty} \varphi_i^2 \mathcal{P}(r, \varphi_i) < -(n-4)^2(n-2)a_0c_n|S^{n-1}|. \]

On the other hand, notice that \( \varphi_i \) satisfies (99). We also have Corollary 2.2. It follows from Proposition 5.1 that
\[ \liminf_{r \to 0} \liminf_{i \to \infty} \varphi_i(0)^2 \mathcal{P}(r, \varphi_i) \geq 0. \]
We arrive at a contradiction. Therefore, (98) is not valid and the proposition follows.

Theorem 1.2 is a part of the following theorem.

**Theorem 6.1.** Let \( u_i \in C^4(M) \) be a sequences of positive solutions of \( P_g u_i = c(n)u_i^p_i \) on \( M \), where \( 0 \leq (n+4)/(n-4) - p_i \to 0 \) as \( i \to \infty \). Assume (9). If either \( n \leq 9 \) or \((M, g)\) is locally conformally flat, then
\[ \|u_i\|_{H^2(M)} \leq C, \]
where \( C > 0 \) depending only on \( M, g \). Furthermore, after passing to a subsequence, \( \{u_i\} \) is uniformly bounded or has only isolated simple blow up points and the distance between any two blow up points is bounded below by some positive constant depending only on \( M, g \).

**Remark 6.1.** On \((M, g)\), we say a point \( X \in M \) is an isolated blow up point for \( \{u_i\} \) if there exists a sequence \( X_i \in M \), where each \( X_i \) is a local maximum point for \( u_i \) and \( X_i \to X \), such that \( u_i(X_i) \to \infty \) as \( i \to \infty \) and \( u_i(X) \leq C\text{dist}_g(X, X_i)^{-\frac{4}{n-4}} \) in \( B_\delta(X_i) \) for some constants \( C, \delta > 0 \) independent of \( i \). Under the assumptions that \( u_i \) is a positive solution of \( P_g u_i = c(n)u_i^p_i \) with \( 0 \leq \frac{n+4}{n-4} - p_i \to 0 \), \( \text{Ker}P_g = \{0\} \) and that the Green’s function \( G_g \) of \( P_g \) is positive, it is easy to see that if \( X_i \to X \in M \) is an isolated blow up point of \( \{u_i\} \), then in the conformal normal coordinates centered at \( X_i \), 0 is an isolated blow up point of \( \{\bar{u}_i(\exp_{X_i}(x))\} \), where the exponential map is with respect to conformal metric \( g_i = \kappa_i^{-\frac{n-4}{n-4}} g \), \( \kappa_i > 0 \) is under control on \( M \), and \( \bar{u}_i = \kappa_i u_i \); see Remark 3.7 Since in Theorem 6.1 and the sequel those assumptions will always be assumed, the notation of isolated simple blow up points on manifolds is understood in conformal normal coordinates.

**Proof of Theorem 6.1** The last statement follows immediately from Proposition 6.1 Proposition 6.2 and Proposition 5.2. Consequently, it follows from Corollary 3.1 and Proposition 3.3 and Proposition A.1 that \( \int_M u_i^{\frac{2n}{n-4}} \text{dvol}_g \leq C \). By the Green’s representation and standard estimates for Riesz potential, we have the \( H^2 \) estimates.

Now we consider that \( n \geq 10 \).
Compactness of conformal metrics with constant $Q$-curvature

Proposition 6.3. Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 10$. Assume (9). Let $u_i$ be a sequence of positive solutions of $P_g u = c(n) u^{p_i}$, where $p_i \leq \frac{n+4}{n-4}$, $p_i \to \frac{n+4}{n-4}$ as $i \to \infty$. Suppose that there is a sequence $X_i \to \bar{X} \in M$ such that $u_i(X_i) \to \infty$. For any small $\varepsilon > 0$ and $R > 1$, let $\mathcal{S}(u_i)$ denote the set selected as in Proposition 6.7 for $u_i$. If $|W_g(\bar{X})|^2 \geq \varepsilon > 0$ on $M$ for some constant $\varepsilon > 0$, then there exists $\delta^* > 0$ depending only on $M$, $g$ and $\varepsilon_0$ such that $\mathcal{B}_{\delta^*}(\bar{X}) \cap \mathcal{S}(u_i)$ contains precisely one point.

Proof. Let $\bar{\delta} > 0$ such that $|W_g(X)|^2 \geq \varepsilon/2$ for $X \in \mathcal{B}_{\bar{\delta}}(\bar{X})$. Assume the contrary of the proposition, then for a subsequence of $\{u_i\}$ (still denoted as $\{u_i\}$) there exist distinct points $X_{1i}, \bar{X}_{1i} \in \mathcal{S}(u_i)$ such that $X_{1i}, \bar{X}_{1i} \to \bar{X}$. Define $f_i : \mathcal{S}(u_i) \to (0, \infty)$ by

$$f_i(X) := \min_{X' \in \mathcal{S}(u_i) \setminus \{X\}} \text{dist}_g(X', X).$$

Let $R_i \to \infty$ satisfying $R_i f_i(X_{1i}) \to 0$.

Claim. There exists a subsequence of $i \to \infty$ such that one can find $X'_{1i} \in \mathcal{S}(u_i) \cap \mathcal{B}_{\bar{\delta}/9}(\bar{X})$ satisfying

$$f_i(X'_{1i}) \leq (2R_i + 1)f_i(X_{1i})$$

and

$$\min_{X \in \mathcal{S}(u_i) \cap \mathcal{B}_{R_i f_i(X_{1i})}(X_{1i})} f_i(X) \geq \frac{1}{2} f_i(X'_{1i}).$$

Indeed, suppose the contrary, then there exists $I \in \mathbb{N}$ such that for any $i \geq I$, $X'_{1i}$ in the claim can not been selected. Since $f_i(X_{1i}) \leq (2R_i + 1)f_i(X_{1i})$, by the contradiction hypothesis, there must exist $X_{2i} \in \mathcal{S}(u_i) \cap \mathcal{B}_{R_i f_i(X_{1i})}(X_{1i})$ such that $f_i(X_{2i}) < \frac{1}{2} f_i(X_{1i})$. We can define $X_{li} \in \mathcal{S}(u_i), l = 3, \ldots$, satisfying $f_i(X_{li}) < \frac{1}{2} f_i(X_{l-1i})$ and $0 < \text{dist}_g(X_{li}, X_{l-1i}) < R_i f_i(X_{l-1i})$ inductively as follows. Once $X_{li}, l \geq 2$, is defined, we have, for $2 \leq m \leq l$, that

$$\text{dist}_g(X_{mi}, X_{m-1i}) < R_i f_i(X_{m-1i}) < R_i 2^{-1} f_i(X_{m-2i}) < \cdots < R_i 2^{-m} f_i(X_{1i}),$$

which implies

$$\text{dist}_g(X_{li}, X_{1i}) \leq \sum_{m=2}^{l} \text{dist}_g(X_{mi}, X_{m-1i}) < R_i f_i(X_{1i}) \sum_{m=2}^{l} 2^{-m} < 2R_i f_i(X_{1i}),$$

and

$$f_i(X_{li}) \leq \text{dist}_g(X_{li}, X_{1i}) + f_i(X_{1i}) \leq (2R_i + 1)f_i(X_{1i}).$$

so $X'_{li} := X_{li}$ satisfies $X'_{li} \in \mathcal{S}(u_i) \cap \mathcal{B}_{\delta/9}(\bar{X})$ and the first inequality of the claim. By the contradiction hypothesis, there must exist $X_{(l+1)i} \in \mathcal{S}(u_i) \cap \mathcal{B}_{R_i f_i(X_{li})}(X_{li})$ such that $f_i(X_{(l+1)i}) < \frac{1}{2} f_i(X_{li})$. But $\mathcal{S}(u_i)$ is a finite set and we can not work for all $l \geq 2$. Therefore, the claim follows.

By the claim, we can follow the proof of Proposition 6.2 with $Z_{li}$ replaced by $X'_{li}$. We then derive a contradiction to Proposition 5.1. Therefore, we complete the proof.
7 Proof of Theorems 1.1, Theorem 1.3, Theorem 1.4

Proof of Theorem 1.3 For \( n = 8, 9 \), since \( u_i(X_i) \to \infty \), it follows from Proposition 6.2 and Theorem 6.1 that \( \tilde{X} \) is an isolated simple blow up points of \( \{ u_i \} \). Then Theorem 1.3 follows from item (ii) of Proposition 5.1. When \( n \geq 10 \), we may argue by contradiction. Suppose that \( |W_g(\tilde{X})|^2 > 0 \). By Proposition 5.2 and in view of the proof of (34) under (33), that there exists a sequence of \( X'_i \to \tilde{X} \) which is an isolated simple blow up point of \( \{ u_i \} \); see Remark 6.1. By item (ii) of Proposition 5.1, we have \( |W_g(X'_i)|^2 \to 0 \). It gives \( |W_g(\tilde{X})|^2 = 0 \). We obtain a contradiction. Hence, we complete the proof.

\[ \square \]

Proof of Theorem 1.4 Suppose the contrary, then, after passing to a subsequence,

\[ |W_g(X_i)|^2 > \frac{1}{|o(1)|} \begin{cases} \frac{u_i(X_i)^{-\frac{4}{n-4}} \log u_i(X_i)}{n-4}, & \text{if } n = 10, \\ u_i(X_i)^{-\frac{4}{n-4}}, & \text{if } n \geq 11. \end{cases} \tag{106} \]

For any \( \varepsilon > 0 \) and \( R > 1 \), let \( \mathcal{S}(u_i) = \{ Z_{1i}, \ldots, Z_{Ni} \} \) be the set selected as in Proposition 6.1 for \( u_i \), where \( N_i \in \mathbb{N}^+ \). Let, without loss of generality,

\[ \text{dist}_g(X_i, Z_{2i}) = \inf_{Z_{ji} \in \mathcal{S}(u_i), Z_{ji} \neq X_i} \text{dist}_g(X_i, Z_{ji}). \]

If there exists a constant \( \delta^* > 0 \) independent of \( i \) such that \( \text{dist}_g(X_i, Z_{2i}) \geq \delta^* \), then \( X_i \in \mathcal{S}(u_i) \) for large \( i \). It follows from item (iii) of Proposition 6.1 and Proposition 5.2 that \( X_i \to \tilde{X} \) has to be an isolated simple blow up point of \( \{ u_i \} \), using the fact that (106) guarantees (79). By item (ii) of Proposition 5.1, we obtain an opposite side inequality of (106). Contradiction.

If \( \text{dist}_g(X_i, Z_{2i}) \to 0 \) as \( i \to \infty \). Let \( \{ x_1, \ldots, x_n \} \) be the conformal normal coordinates centered at \( X_i \). Define \( \varphi_i \) as that in the proof of Proposition 6.2 with \( Z_{1i} \) replaced by \( X_i \). Since \( \sup \varphi_i < b u_i(X_i) \), we must have \( \varphi_i(0) \to \infty \) by the Liouville theorem in [12] or [36]; see the proof of (103). Because of (12) and (13), 0 has to be an isolated blow up point of \( \{ \varphi_i \} \); see Remark 3.1. It follows from the contradiction hypothesis (106), which guarantees (79), as in the proof of Lemma 5.1 that 0 has to be an isolated simple blow up point of \( \{ \varphi_i \} \) for \( i \) large. Then we we arrive at a contradiction by item (ii) of Proposition 5.1 again.

Therefore, we complete the proof.

\[ \square \]

Proof of Theorem 1.7 If \( n \geq 8 \) and \( |W_g|^2 > 0 \) on \( M \), Theorem 1.1 is a direct corollary of Theorem 1.3. Hence, we only need to consider \( n \leq 9 \) or \( (M, g) \) is locally conformally flat.

By Proposition 6.1, it suffices to consider that \( p \) is close to \( \frac{n+4}{n-4} \). Suppose the contrary that there exists a sequences of positive solutions \( u_i \in C^4(M) \) of \( P_g u_i = c(n) u_i^p \) on \( M \), where \( p_i \to (n+4)/(n-4) \) as \( i \to \infty \), such that \( \max_M u_i \to \infty \). By Theorem 6.1, let \( X_i \to \tilde{X} \in M \).
be an isolated simple blow up point of \( \{ u_i \} \); see Remark 6.1. It follows from Proposition 5.1 that, in the \( g_{\bar{X}} \)-normal coordinates centered at \( \bar{X} \),

\[
\liminf_{r \to 0} \mathcal{P}(r, c(n) G) \geq 0,
\]

where \( g_{\bar{X}} \) a conformal metric of \( g \) with \( \det g_{\bar{X}} = 1 \) in an open ball \( B_{\delta} \) of the \( g_{\bar{X}} \)-normal coordinates, \( G(x) = G_{g_{\bar{X}}}(\bar{X}, \exp_{\bar{X}} x) \) and \( G_{g_{\bar{X}}} \) is the Green’s function of \( P_{g_{\bar{X}}} \). On the other hand, if \( n = 5, 6, 7 \) or \( (M, g) \) is locally conformally flat, by Theorem 2.1 and Lemma 2.2 we have

\[
\mathcal{P}(r, c(n) G) < -A \quad \text{for small } r,
\]

where \( A > 0 \) depends only on \( M, g \). We obtain a contradiction.

If \( n = 8, 9 \), by Theorem 1.3 we have

\[
W_{g_{\bar{X}}}(\bar{X}) = 0.
\]

In view of Remark 2.1, we have

\[
\lim_{r \to 0} \mathcal{P}(r, G) = \begin{cases} 
-\frac{2}{p-1} \int_{S^{n-1}} \psi(\theta), & n = 8, \\
-\frac{5}{2} A, & n = 9,
\end{cases}
\]

where \( \psi(\theta) \) and \( A \) are as in Remark 2.1. If the positive mass type theorem holds for Paneitz operator in dimension \( n = 8, 9 \), we obtain \( \lim_{r \to 0} \mathcal{P}(r, G) < 0 \). Again, we derived a contradiction.

Therefore, \( u_i \) must be uniformly bounded and the proof is completed.

### 8 Proof of Theorem 1.5

This section will not use previous analysis and thus is independent. The proof of Theorem 1.5 is divided into two steps.

**Step 1.** \( L^p \) estimate. Let \( u \geq 0 \) be a solution of (14). Integrating both sides of (14) and using Hölder inequality, we have

\[
\int_M u^p d\text{vol}_g = -\int_M u P_g(1) d\text{vol}_g \leq \frac{n-4}{2} \| u \|_{L^p(M)} \| Q_g \|_{L^{p'}(M)},
\]

where \( \frac{1}{p'} + \frac{1}{p} = 1 \). It follows that \( \| u \|_{L^p(M)}^{p-1} \leq \frac{n-4}{2} \| Q_g \|_{L^{p'}(M)} \).

**Step 2.** If \( \ker P_g = \{ 0 \} \), there exist a unique Green function of \( P_g \). If the kernel of \( P_g \) is non-trivial, since the spectrum of Paneitz operator is discrete, there exists a small constant \( \varepsilon > 0 \) such that the kernel of \( P_g - \varepsilon \) is trivial. Let \( G_g \) be the Green function of the operator \( P_g - \varepsilon \), where \( \varepsilon \geq 0 \). Then there exists a constant \( \delta > 0 \), depending only on \( M, g \) and \( \varepsilon \), such that, for every \( X \in M \), we have \( G(X, Y) > 0 \) for \( Y \in B_{\delta}(X) \) and \( |G_g(X, Y)| \leq C(\delta, \varepsilon) \) for \( Y \in M \setminus B_{\delta}(X) \). Rewrite the equation of \( u \) as

\[
P_g u - \varepsilon u = -(w^p + \varepsilon u).
\]
It follows from the Green representation theorem that
\[
\begin{align*}
    u(X) &= -\int_M G_g(X,Y)(u^p + \varepsilon u)(Y)dvol_g(Y) \\
    &= -\int_{B_{\delta}(X)} G_g(X,Y)(u^p + \varepsilon u)(Y)dvol_g(Y) \\
    &\quad - \int_{M \backslash B_{\delta}(X)} G_g(X,Y)(u^p + \varepsilon u)(Y)dvol_g(Y) \\
    &\leq -\int_{M \backslash B_{\delta}(X)} G_g(X,Y)(u^p + \varepsilon u)(Y)dvol_g(Y) \\
    &\quad \leq C \max\{\|u\|_{L^p(M)}, \|u\|_{L^p(M)}\} \leq C.
\end{align*}
\]

By the arbitrary choice of \(X\), we have \(\|u\|_{L^\infty} \leq C\). The higher order estimate follows from the standard linear elliptic partial differential equation theory; see Agmon-Douglis-Nirenberg [1]. Therefore, we complete the proof.

A  Local estimates for solutions of linear integral equations

Let \(\Omega_2 \subset\subset \Omega_1\) be a bounded open set in \(\mathbb{R}^n\), \(n \geq 5\). For \(x, y \in \Omega_1\), let \(G(x, y)\) satisfy
\[
\begin{align*}
    G(x, y) &= G(y, x), \quad G(x, y) \geq A_1^{-1}|x - y|^{4-n}, \\
    |\nabla_l^i G_i(x, y)| &\leq A_1|x - y|^{4-n-l}, \quad l = 0, 1, \ldots, 5 \\
    G(x, y) &= c_n \frac{1 + O^{(4)}(|x|^2) + O^{(4)}(|y|^2)}{|x - y|^{n-4}} + O^{(4)}\left(\frac{1}{|x - y|^{n-6}}\right)
\end{align*}
\]
and let \(0 \leq h \in C^4(\Omega_1)\) satisfy
\[
\begin{align*}
    &\sup_{\Omega_2} h \leq A_2 \inf_{\Omega_2} h \quad \text{and} \quad \sum_{j=1}^4 r^j|\nabla^j h(x)| \leq A_2\|h\|_{L^\infty(B_r(x))}
\end{align*}
\]
for all \(x \in \Omega_2\) and \(0 < r < dist(\Omega_2, \partial\Omega_1)\). We recall some local estimates for solutions of the integral equation
\[
    u(x) = \int_{\Omega_1} G(x, y)V(y)u(y)\,dy + h(x) \quad \text{for } x \in \Omega_1.
\]
Compactness of conformal metrics with constant $Q$-curvature

**Proposition A.1.** Let $G, h$ satisfy (107) and (108), respectively. Let $0 \leq V \in L^\infty(\Omega_1)$, and let $0 \leq u \in C^0(\Omega_1)$ be a solution of (109). Then we have

$$\sup_{\Omega_2} u \leq C \inf_{\Omega_2} u,$$

where $C > 0$ depends only on $n, A_1, A_2, \Omega_1, \Omega_2$ and $\|V\|_{L^\infty(\Omega_1)}$.

**Proof.** It follows from some simple modification of the proof of Proposition 2.3 of [31]. In fact, the third line of (107) is not needed.

**Proposition A.2.** Suppose the hypotheses in Proposition A.1. Then $u \in C^3(B_2)$ and

$$\|u\|_{C^3(\Omega_2)} \leq C\|u\|_{L^\infty(\Omega_1)},$$

where $C > 0$ depends only on $n, A_1, A_2$, the volume of $\Omega_2$, $\text{dist}(\Omega_2, \partial\Omega_1)$ and $\|V\|_{L^\infty(\Omega_1)}$.

If $V \in C^1(\Omega_1)$, then $u \in C^4(\Omega_2)$ and for any $\Omega_3 \subset\subset \Omega_2$ we have

$$\|\nabla^4 u\|_{L^\infty(\Omega_3)} \leq C\|u\|_{L^\infty(\Omega_1)},$$

where $C > 0$ depends only on $n, A_1, A_2$, the volume of $\Omega_2$, $\text{dist}(\Omega_2, \partial\Omega_1)$, $\text{dist}(\Omega_3, \partial\Omega_2)$ and $\|V\|_{C^1(B_1)}$.

**Proof.** Let $f := Vu$. If $k < 4$, making use of (107), (108) and (109) we have

$$\nabla^k u(x) = \int_{\Omega_1} \nabla^k G(x, y)f(y) \, dy + \nabla^k h(x) \quad \text{for } x \in \Omega_2,$$

and thus

$$|\nabla^k u(x)| \leq A_1\|f\|_{L^\infty(\Omega_1)} \int_{\Omega_1} |x-y|^{n-4+k} \, dy + |\nabla^k h(x)|$$

$$\leq C(\|u\|_{L^\infty(\Omega_1)} + \|h\|_{L^\infty(\Omega_1)}).$$

Since $u$ and $V$ are nonnegative, we have $0 \leq h(x) \leq u(x)$. We proved the first conclusion.

Let $\Omega_3 \subset\subset \Omega_2$. Without loss of generality, we may assume $\partial\Omega_2 \in C^1$. If $V \in C^1(\Omega_1)$, we have $f \in C^1(\Omega_2)$. By the third line of (107), we see

$$\nabla_x G(x, y) = -\nabla_y G(x, y) + (O^3(|x|) - O^3(|y|))|x-y|^{4-n} + O^3(|x-y|^{5-n}).$$

We have for $x \in \Omega_3$ and $1 \leq j \leq n$,

$$\nabla_{x_j} \nabla^3 u(x) = \int_{\Omega_2} \nabla_{x_j} \nabla^3 G(x, y)(f(y) - f(x)) \, dy + f(x) \int_{\partial\Omega_2} \nabla_{y_j}^3 G(x, y)\nu_j \, dS(x)$$

$$+ f(x)O(\int_{\Omega_2} |x-y|^{1-n} \, dy) + \int_{\Omega_1 \setminus \Omega_2} \nabla_{x_j} \nabla^3 G(x, y)f(y) \, dy + \nabla_{x_j} \nabla^3 h(x),$$

where $\nu = (\nu_1, \ldots, \nu_n)$ denotes the outward normal to $\partial\Omega_2$. By (107) and (108), the proof follows immediately. Hence, we complete the proof. 

\[\Box\]
B Riesz potentials of some functions

Lemma B.1. For $0 < \alpha < n$, we have

$$\int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha} (1 + |y|)^\mu} \, dy$$

$$\leq C(n, \alpha, \mu) \begin{cases} 
(1 + |x|)^{\alpha - \mu}, & \text{if } \alpha < \mu < n, \\
(1 + |x|)^{\alpha - n} \log(2 + |x|), & \text{if } \mu = n, \\
(1 + |x|)^{\alpha - n}, & \text{if } \mu > n.
\end{cases}$$

References

[1] Agmon, S.; Douglis, A.; Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Comm. Pure Appl. Math. 12 (1959), 623–727.
[2] Aubin, T.: Équations différentielles nonlinéaires et problème de Yamabe concernant la courbure scalaire. J. Math. Pures Appl. 55 (1976), 269–296.
[3] Bahri, A.: Another proof of the Yamabe conjecture for locally conformally flat manifolds. Nonlinear Anal. 20 (1993), no. 10, 1261–1278.
[4] Bahri, A.; Brezis, H.: Non-linear elliptic equations on Riemannian manifolds with the Sobolev critical exponent. Topics in geometry, 1–100, Progr. Nonlinear Differential Equations Appl., 20, Birkhäuser Boston, Boston, MA, 1996.
[5] Branson, T.P.: Differential operators canonically associated to a conformal structure. Math. Scand. 57 (1985), no. 2, 295–345.
[6] Brendle, S.: Blow up phenomena for the Yamabe equation. J. Amer. Math. Soc. 21 (4) (2008), 951–979.
[7] Brendle, S.; Marques, F.C.: Blow up phenomena for the Yamabe equation. II. J. Differential Geom. 81 (2) (2009), 225–250.
[8] Caffarelli, L.A., Silvestre, L.: An extension problem related to the fractional Laplacian. Comm. Partial. Diff. Equ., 32 (2007), 1245–1260.
[9] Cao, J.: The existence of generalized isothermal coordinates for higher-dimensional Riemannian manifolds. Trans. Amer. Math. Soc. 324 (1991), 901–920.
[10] Chang, S.-Y., Yang, P.: Extremal metrics of zeta function determinants on 4-manifolds. Ann. of Math. (2) 142 (1995), no. 1, 171–212.
[11] —–: On a fourth order curvature invariant. Comp. Math. 237, Spectral Problems in Geometry and Arithmetic, Ed: T. Branson, AMS, 9–28, 1999.
[12] Chen, W.; Li, C.; Ou, B.: Classification of solutions for an integral equation. Comm. Pure Appl. Math. 59 (2006), 330–343.
[13] de Moura Almaraz, S.: A compactness theorem for scalar-flat metrics on manifolds with boundary. Calc. Var. Partial Differential Equations 41 (2011), 341–386.
Compactness of conformal metrics with constant $Q$-curvature

[14] Djadli, Z.; Hebey, E.; Michel L.: Paneitz-type operators and applications. Duke Math. J. 104 (2000), no. 1, 129–169.

[15] Djadli, Z.; Malchiodi, A.: Existence of conformal metrics with constant $Q$-curvature. Ann. of Math. (2) 168 (2008), 813–858.

[16] Djadli, Z., Malchiodi, A., Ould Ahmedou, M.: Prescribing a fourth order conformal invariant on the standard sphere. II. Blow up analysis and applications. Ann. Sc. Norm. Super. Pisa Cl. Sci. 5 (2002), 387–434.

[17] Druet, O.: From one bubble to several bubbles: the low-dimensional case. J. Differential Geom. 63 (3) (2003), 399–473.

[18] ——: Compactness for Yamabe metrics in low dimensions. Int. Math. Res. Not. 23 (2004), 1143–1191.

[19] Druet, O.; Robert, F.: Bubbling phenomena for fourth-order four-dimensional PDEs with exponential growth. Proc. Amer. Math. Soc. 134 (2006) 897–908.

[20] Günther, M.: Conformal normal coordinates. Ann. Global Anal. Geom. 11 (1993), 173–184.

[21] Gursky, M.; Hang, F.; Lin, Y.-J.: Riemannian manifolds with positive Yamabe invariant and Paneitz operator. Preprint [arXiv:1502.01050]

[22] Gursky, M.; Malchiodi, A.: A strong maximum principle for the Paneitz operator and a non-local flow for the $Q$-curvature. To appear in J. Eur. Math. Soc. (JEMS). [arXiv:1401.3216]

[23] Hang, F.; Yang, P.: $Q$-curvature on a class of 3 manifolds. To appear in Comm. Pure Appl. Math.

[24] ——: $Q$-curvature on a class of manifolds of dimension at least 5. Preprint [arXiv:1411.3926]

[25] ——: Sign of Green’s function of Paneitz operators and the $Q$-curvature. Preprint [arXiv:1411.3924]

[26] Hebey, E.; Robert, F.: Compactness and global estimates for the geometric Paneitz equation in high dimensions. Electron. Res. Announc. AMS 10 (2004), 135–141.

[27] ——: Asymptotic analysis for fourth order Paneitz equations with critical growth. Advances in the Calculus of Variations, 3 (2011), 229–276.

[28] Humbert, E.; Raulot, S.: Positive mass theorem for the Paneitz-Branson operator. Calc. Var. Partial Differential Equations 36 (2009), no. 4, 525–531.

[29] Jin, T.; Li, Y.Y.; Xiong, J.: On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions. J. Eur. Math. Soc. (JEMS), 16 (2014), no. 6, 1111–1171.

[30] Jin, T.; Li, Y.Y.; Xiong, J.: On a fractional Nirenberg problem, part II: existence of solutions. To appear in IMRN. [arXiv:1309.4666]

[31] Jin, T.; Li, Y.Y.; Xiong, J.: The Nirenberg problem and its generalizations: A unified approach. Preprint.

[32] Khuri, M.A.; Marques, F.C.; Schoen, R.: A compactness theorem for the Yamabe problem. J. Differential Geom. 81 (1) (2009), 143–196.

[33] Lee, J.; Parker, T.: The Yamabe problem. Bull. Amer. Math. Soc. (N.S.) 17 (1987), 37–91.

[34] Li, J.; Li, Y.; Liu, P.: The $Q$-curvature on a 4-dimensional Riemannian manifold $(M,g)$ with $\int_M QdV_g = 8\pi^2$. Adv. Math. 231 (2012), no. 3-4, 2194–2223.

[35] Li, Y.Y.: Prescribing scalar curvature on $S^n$ and related problems. I. J. Differential Equations 120 (1995), 319–410.

[36] ——: Remark on some conformally invariant integral equations: the method of moving spheres. J.
[37] Li, Y.Y.; Zhang, L.: A Harnack type inequality for the Yamabe equation in low dimensions. Calc. Var. Partial Differential Equations 20 (2) (2004), 133–151.

[38] —–: Compactness of solutions to the Yamabe problem II. Calc. Var. and PDEs 25 (2005), 185–237.

[39] —–: Compactness of solutions to the Yamabe problem III. J. Funct. Anal. 245 (2006), 438–474.

[40] Li, Y.Y.; Zhu, M.: Yamabe type equations on three dimensional Riemannian manifolds. Communications in Contemporary Math. 1 (1999), 1–50.

[41] Lin, C.S.: A classification of solutions of a conformally invariant fourth order equation in \( \mathbb{R}^n \). Comment. Math. Helv. 73 (1998), 206–231.

[42] Malchiodi, A.: Compactness of solutions to some geometric fourth-order equations. J. Reine Angew. Math. 594 (2006), 137–174.

[43] Marques, F.C.: A priori estimates for the Yamabe problem in the non-locally conformally flat case. J. Differential Geom. 71 (2005), 315–346.

[44] Obata, M.: The conjectures on conformal transformations of Riemannian manifolds. J. Differential Geom. 6 (1972), 247–258.

[45] Paneitz, S.: A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds. SIGMA Symmetry Integrability Geom. Methods Appl. 4 (2008), no. Paper 036.

[46] Pollack, D.: Nonuniqueness and high energy solutions for a conformally invariant scalar curvature equation. Comm. Anal. and Geom. 1 (1993), 347–414.

[47] Qing, J.; Raske, D.: Compactness for conformal metrics with constant Q curvature on locally conformally flat manifolds. Calc. Var. Partial Differential Equations 26 (2006), 343–356.

[48] —–: On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds. Int. Math. Res. Not. Art. ID 94172, 20 pp. (2006).

[49] Schoen, R.: Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differential Geom. 20 (1984), 479–495.

[50] —–: “Variational theory for the total scalar curvature functional for Riemannian metrics and related topics.” In: Giaquinta, M. (ed.) Topics in Calculus of Variations. Lecture Notes in Mathematics, Vol. 1365 120–154. Springer, Berlin Heidelberg New York 1989.

[51] —–: Courses at Stanford University, 1988, and New York University, 1989.

[52] —–: “On the number of constant scalar curvature metrics in a conformal class.” In: Lawson, H.B., Tenenblat, K. (eds.) Differential geometry: a symposium in honor of Manfredo Do Carmo, pp. 311–320. Wiley, New York 1991.

[53] Schoen, R.; Yau, S.-T.: Conformally flat manifolds, Kleinian groups and scalar curvature. Invent. Math. 92 (1988), no. 1, 47–71.

[54] Trudinger, N.: Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. Scuola Norm. Sup. Cl. Sci. (3) 22 (1968), 265–274.

[55] Wei, J.; Zhao, C.: Non-compactness of the prescribed Q-curvature problem in large dimensions. Calc. Var. 46 (2013), 123–164.

[56] Weinstein, G.; Zhang, L.: The profile of bubbling solutions of a class of fourth order geometric equations on 4-manifolds. J. Funct. Anal. 257 (2009), no. 12, 3895–3929.

[57] Yamabe, H.: On a deformation of Riemannian structures on compact manifolds. Osaka Math. J. 12
Compactness of conformal metrics with constant $Q$-curvature

(1960), 21–37.

[58] Zhu, M.: Prescribing integral curvature equation. arxiv:1407.2967

Y.Y. Li
Department of Mathematics, Rutgers University,
110 Frelinghuysen Road, Piscataway, NJ 08854, USA
Email: yyli@math.rutgers.edu

J. Xiong
School of Mathematical Sciences, Beijing Normal University
Beijing 100875, China
Email: jx@bnu.edu.cn