DERIVATIVES CHARACTERIZATION OF BERGMAN-ORLICZ SPACES AND APPLICATIONS

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Abstract. It is well known that a function is in a Bergman space of the unit ball if and only if it satisfies some Hardy-type inequalities. We extend this fact to Bergman-Orlicz spaces. As applications, we obtain Gustavsson-Peetre interpolation of two Bergman-Orlicz spaces and we completely characterize symbols of bounded or compact Cesàro-type operators on Bergman-Orlicz spaces, extending known results for classical weighted Bergman spaces.

1. Introduction

Let \( z = (z_1, \cdots, z_n) \) and \( w = (w_1, \cdots, w_n) \) be vectors in \( \mathbb{C}^n \). We write \( \langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n \) and \( |z|^2 = \langle z, z \rangle = |z_1|^2 + \cdots + |z_n|^2 \).

Given a function \( \Phi : [0, \infty) \to [0, \infty) \), we say \( \Phi \) is a growth function if it is a continuous and non-decreasing function.

We denote by \( d\nu \) the Lebesgue measure on \( B^n \) the unit ball of \( \mathbb{C}^n \), and \( d\sigma \) the normalized measure on \( S^n = \partial B^n \) the boundary of \( B^n \). As usual, we denote by \( H(B^n) \) the space of holomorphic functions.

For \( \alpha > -1 \), we write \( d\nu_{\alpha}(z) = c_\alpha (1 - |z|^2)^\alpha d\nu(z) \), where \( c_\alpha \) is taken such that \( \nu_{\alpha}(B^n) = 1 \).

For \( \Phi \) a growth function, the Orlicz space \( L^\Phi_{\alpha}(B^n) \) is the space of functions \( f \) such that

\[
\|f\|_{\alpha, \Phi} := \int_{B^n} \Phi(|f(z)|)d\nu_{\alpha}(z) < \infty.
\]

The weighted Bergman-Orlicz space \( A^\Phi_{\alpha}(B^n) \) is the subspace of \( L^\Phi_{\alpha}(B^n) \) consisting of holomorphic functions.

We define on \( A^\Phi_{\alpha}(B^n) \) the following (quasi)-norm

\[
||f||_{l^\Phi_{\alpha, \Phi}} := \inf \{ \lambda > 0 : \int_{B^n} \Phi \left( \frac{|f(z)|}{\lambda} \right) d\nu_{\alpha}(z) \leq 1 \}
\]

which is finite for \( f \in A^\Phi_{\alpha}(B^n) \) (see [21]).

We observe that for \( \Phi(t) = t^p \), the corresponding Bergman-Orlicz space is the classical weighted Bergman spaces denoted by \( A^p_{\alpha}(B^n) \) and defined by

\[
\|f\|_{p, \alpha} := \int_{B^n} |f(z)|^p d\nu_{\alpha}(z) < \infty.
\]

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Recall that two growth functions $\Phi_1$ and $\Phi_2$ are said equivalent if there exists some constant $c$ such that
\[
\frac{1}{c} \Phi_1\left(\frac{t}{c}\right) \leq \Phi_2(t) \leq c \Phi_1(ct)
\]
and observe that two equivalent growth functions define the same Orlicz space.

We recall that given an analytic function $f$ on $\mathbb{B}^n$, the radial derivative $Rf$ of $f$ is defined by
\[
Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).
\]
We recall also that the gradient of $f \in H(\mathbb{B}^n)$ is defined by
\[
\nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right).
\]
The invariant gradient at $z$ of the analytic function $f$ is defined by
\[
\tilde{\nabla} f(z) = \nabla (f \circ \phi_z)(0)
\]
where $\phi_z$ is the automorphism of $\mathbb{B}^n$ mapping 0 to $z$.

We have the following inequalities between the above derivatives (see [29, Lemma 2.14]):
\[
(2) \quad (1 - |z|^2) |Rf(z)| \leq (1 - |z|^2) |\nabla f(z)| \leq |\tilde{\nabla} f(z)|, \quad \text{for all } z \in \mathbb{B}^n.
\]
The following derivatives characterization of classical weighted Bergman spaces is a well known fact (see [29, Theorem 2.16]).

**THEOREM 1.1.** Suppose $\alpha > -1$, $p > 0$, and $f$ is holomorphic in $\mathbb{B}^n$. Then the following conditions are equivalent.

(a) $f \in A^p_\alpha(\mathbb{B}^n)$.
(b) $|\tilde{\nabla} f(z)| \in L^p(\mathbb{B}^n, d\nu_\alpha)$
(c) $(1 - |z|^2)|\nabla f(z)| \in L^p(\mathbb{B}^n, d\nu_\alpha)$.
(d) $(1 - |z|^2)|Rf(z)| \in L^p(\mathbb{B}^n, d\nu_\alpha)$.

Our main aim in this note is to extend the above result to Bergman-Orlicz spaces. Let us recall some more definitions.

We say that a growth function $\Phi$ is of upper type $q \geq 1$ if there exists some constant $C > 0$ such that, for $s > 0$ and $t \geq 1$,
\[
\Phi(st) \leq C t^q \Phi(s).
\]
We denote by $\Phi^q$ the set of growth functions $\Phi$ of upper type $q$, (for some $q \geq 1$), such that the function $t \mapsto \frac{\Phi(t)}{t^q}$ is non-decreasing.

We say that $\Phi$ is of lower type $p > 0$ if there exists some constant $C > 0$ such that, for $s > 0$ and $0 < t \leq 1$,
\[
\Phi(st) \leq C t^p \Phi(s).
\]
We denote by $\Phi_p$ the set of growth functions $\Phi$ of lower type $p$, (for some $p \leq 1$), such that the function $t \mapsto \frac{\Phi(t)}{t^p}$ is non-increasing.
We also observe that we may always suppose that any $\Phi \in L^p$ (resp. $U^q$), is concave (resp. convex) and that $\Phi$ is a $C^1$ function with derivative $\Phi'(t) \geq \frac{\Phi(t)}{t}$.

Our main result is the following.

**Theorem 1.2.** Suppose $\alpha > -1$. Assume that $\Phi \in U^q \cup L^p$, and $f$ is holomorphic in $B^n$. Then the following conditions are equivalent.

(a) $f \in A^{\Phi}_\alpha (B^n)$.

(b) $|\nabla f(z)| \in L^\Phi(B^n, d\nu_\alpha)$

(c) $(1 - |z|^2)|\nabla f(z)| \in L^\Phi(B^n, d\nu_\alpha)$.

(d) $(1 - |z|^2)|Rf(z)| \in L^\Phi(B^n, d\nu_\alpha)$.

As applications, we characterize the Gustavsson-Peetre interpolate of two Bergman-Orlicz spaces and symbols of bounded Cesàro-type operators on Bergman-Orlicz spaces.

2. **Preliminary results**

We give in this section some useful tools needed in our presentation.

2.1. **Some properties of growth functions.** We recall that the complementary function $\Psi$ of the convex growth function $\Phi$, is the function defined from $\mathbb{R}_+$ onto itself by

$$\Psi(s) = \sup_{t \in \mathbb{R}_+} \{ ts - \Phi(t) \}.$$  

We observe that if $\Phi \in U^q$, then $\Psi$ is a growth function of lower type such that the function which $t \mapsto \frac{\Psi(t)}{t}$ is non-decreasing.

We say that $\Phi$ satisfies the $\Delta_2$-condition if there exists a constant $K > 1$ such that, for any $t \geq 0$,

$$\Phi(2t) \leq K \Phi(t).$$  

We say that the growth function $\Phi$ satisfies the $\nabla_2$-condition whenever both $\Phi$ and its complementary satisfy the $\Delta_2$-condition.

For $\Phi$ a $C^1$ growth function, the lower and the upper indices of $\Phi$ are respectively defined by

$$a_\Phi := \inf_{t > 0} \frac{t \Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi := \sup_{t > 0} \frac{t \Phi'(t)}{\Phi(t)}.$$  

We recall that when $\Phi$ is convex, then $1 \leq a_\Phi \leq b_\Phi < \infty$ and, if $\Phi$ is concave, then $0 < a_\Phi \leq b_\Phi \leq 1$. We observe with [7, Lemma 2.6] that a convex growth function satisfies the $\nabla_2$-condition if and only if $1 < a_\Phi \leq b_\Phi < \infty$.

It is easy to see that if $\Phi$ is a $C^1$ growth function. Then the functions $\frac{\Phi(t)}{t^{1/\Phi}}$ and $\frac{\Phi^{-1}(t)}{t^{1/b_\Phi}}$ are increasing. As a consequence, we have the following useful fact.

**Lemma 2.1.** Let $\Phi \in L_p$. Then the growth function $\Phi_p$, defined by $\Phi_p(t) = \Phi(t^{1/p})$ is in $U^q$ for some $q \geq 1$.

We also make the following observation.
**Proposition 2.2.** The following assertion holds:

\[ \Phi \in \mathcal{L}_p \text{ if and only if } \Phi^{-1} \in \mathcal{W}^{1/p}. \]

We observe that if \( \Phi \) is of upper type (resp. lower type) \( p_1 \), then it is of upper type (resp. lower type) \( p_2 \) for any \( \infty > p_2 > p_1 \) (resp. \( p_2 < p_1 < \infty \)). Hence, when we say \( \Phi \) is of upper type (resp. lower type) \( p \), we suppose that \( q \) (resp. \( p \)) is the smallest (resp. biggest) number \( q_1 \) (resp. \( p_1 \)) such that \( \Phi \) is of upper type \( q_1 \) (resp. lower type \( p_1 \)). We also observe that \( a_\Phi \) (resp. \( b_\Phi \)) coincides with the biggest (resp. smallest) number \( p \) such that \( \Phi \) is of lower (resp. upper) type \( p \).

### 2.2. Operators on Orlicz spaces.

**Definition 2.3.** Let \( \Phi \) be a growth function. A linear operator \( T \) defined on \( L^\Phi(\mathbb{B}^n, d\nu_\alpha) \) is said to be of mean strong type \( (\Phi, \Phi)_\alpha \) if

\[ \int_{\mathbb{B}^n} \Phi(|Tf|) d\nu_\alpha(z) \leq C \int_{\mathbb{B}^n} \Phi(|f|) d\nu_\alpha(z) \]

for any \( f \in L^\Phi(\mathbb{B}^n, d\nu_\alpha) \), and \( T \) is said to be mean weak type \( (\Phi, \Phi)_\alpha \) if

\[ \sup_{t>0} \Phi(t) \nu_\alpha(\{ z \in \mathbb{B}^n : |Tf(z)| > t \}) \leq C \int_{\mathbb{B}^n} \Phi(|f|) d\nu_\alpha(z) \]

for any \( f \in L^\Phi(\mathbb{B}^n, d\nu_\alpha) \), where \( C \) is independent of \( f \).

We observe that the mean strong type \( (t^p, t^p)_\alpha \) is the usual strong type \( (p, p) \) coincide. We also note if the operator \( T \) is of mean strong type \( (\Phi, \Phi)_\alpha \), then \( T \) is bounded on \( L^\Phi(\mathbb{B}^n, d\nu_\alpha) \).

The following result is adapted from [7, Theorem 4.3].

**Theorem 2.4.** Let \( \Phi_0, \Phi_1 \) and \( \Phi_2 \) be three convex growth functions. Suppose that their upper and lower indices satisfy the following condition

\[ 1 \leq a_{\Phi_0} \leq b_{\Phi_0} < a_{\Phi_2} \leq b_{\Phi_2} < a_{\Phi_1} \leq b_{\Phi_1} < \infty. \]

If \( T \) is of mean weak types \( (\Phi_0, \Phi_0)_\alpha \) and \( (\Phi_1, \Phi_1)_\alpha \), then it is of mean strong type \( (\Phi_2, \Phi_2)_\alpha \).

Let \( \beta > -1 \) be and consider the operator \( P_\beta \) defined for functions \( f \) on \( \mathbb{B}^n \) by

\[ P_\beta(f)(z) = \int_{\mathbb{B}^n} \frac{f(\xi)}{(1 - \langle z, \xi \rangle)^{n+1+\beta}} d\nu_\beta(\xi). \]

The operator \( P_\beta \) is the Bergman projection, that is the orthogonal projection of \( L^2(\mathbb{B}^n, d\nu_\beta) \) onto its closed subspace \( \mathcal{A}^2_\beta(\mathbb{B}^n) \). We have the following result.

**Theorem 2.5.** Let \( \alpha, \beta > -1 \). Let \( \Phi \) be a convex growth function and denote by \( a_\Phi \) its lower indice. Assume that there is \( 1 < p_0 < a_\Phi \) such that \( \alpha + 1 < p_0(\beta + 1) \). Then \( P_\beta \) is of mean strong type \( (\Phi, \Phi)_\alpha \).

**Proof.** This result is well known when \( \Phi \) is a power function (see for example [20, Theorem 2.10]). It follows in particular that \( P_\beta \) is bounded on \( L^{p_0}(\mathbb{B}^n, d\nu_\alpha) \) and on \( L^p(\mathbb{B}^n, d\nu_\alpha) \) for \( p_1 > b_\Phi \). Hence from the interpolation result Theorem 2.4 we deduce that \( P_\beta \) is of mean strong type \( (\Phi, \Phi)_\alpha \). □

In particular, we have the following.
THEOREM 2.6. Let $\alpha > -1$. Assume that $\Phi \in \mathcal{W}^q$ and satisfies the $\nabla_2$-condition. Then the Bergman projection $P_\alpha$ is bounded on $L^\Phi(B^n, d\nu_\alpha)$.

2.3. Some useful estimates and test functions. The next proposition gives pointwise estimates for functions in $A_\alpha^\Phi(B^n)$, $\Phi \in L_p \cup \mathcal{W}^q$ (see [21, 24]).

LEMMA 2.7. Let $\Phi \in L_p \cup \mathcal{W}^q$ and $\alpha > -1$. There is a constant $C > 1$ such that for any $f \in A_\alpha^\Phi(B^n)$,

\[
|f(z)| \leq C\Phi^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\alpha}}\right)\|f\|_{l_{a,\Phi}}.
\]

We refer to [28] Lemma 2.15 for the following result.

LEMMA 2.8. Let $0 < p \leq 1$. Then there is a constant $C > 0$ such that for any $f \in A_\alpha^\Phi(B^n)$,

\[
\int_{B^n} |f(z)|(1 - |z|^2)^{(\frac{1}{p} - 1)(n+1+\alpha)} d\nu_\alpha(z) \leq C\|f\|_\alpha^p.
\]

The following gives example of functions in Bergman-Orlicz spaces. We refer to [21, 24] for a proof.

LEMMA 2.9. Let $-1 < \alpha < \infty$, $a \in \mathbb{B}^n$. Let $k > 1$. Suppose that $\Phi \in L_p \cup \mathcal{W}^q$. Then the following function is in $A_\alpha^\Phi(B^n)$

\[
f_a(z) = \Phi^{-1}\left(\frac{1}{(1 - |a|)^{n+1+\alpha}}\right) \left(\frac{1 - |a|^2}{1 - \langle z, a \rangle}\right)^{k(n+1+\alpha)}.
\]

Moreover, $\|f_a\|_{l_{a,\Phi}} \lesssim 1$.

3. Proof of Theorem 1.2

Let us start with the following result.

LEMMA 3.1. Let $\alpha > -1$. Assume that $\Phi \in \mathcal{W}^q \cup L_p$. Then there exists a constant $C > 0$ such that for any $f \in A_\alpha^\Phi(B^n)$,

\[
\int_{B^n} \Phi(|\nabla f(z)|) d\nu_\alpha(z) \leq C \int_{B^n} \Phi(|f(z) - f(0)|) d\nu_\alpha(z).
\]

Proof. We follow the proof of [29] Theorem 2.16] making some crucial modifications where needed. We start by recalling that if $\Phi \in \mathcal{W}^q$, then $A_\alpha^\Phi(B^n)$ continuously embeds into $A_\beta^1(B^n)$, and $A_\alpha^\Phi(B^n)$ continuously embeds into $A_\beta^p(B^n)$ when $\Phi \in L_p$. Let $\beta > \alpha$. Put

\[
p_\Phi = \begin{cases} 
1 & \text{if } \Phi \in \mathcal{W}^q \\
p & \text{if } \Phi \in L_p.
\end{cases}
\]

It follows from [26] Lemma 2.4] that there exists $C_1 > 0$ such that for any $g \in H(B^n)$,

\[
|\nabla g(0)|^{p_\Phi} \leq C_1 \int_{B^n} |g(w)|^{p_\Phi} d\nu_\beta(w).
\]

Put $g = f \circ \phi_z$, $z \in \mathbb{B}^n$, where $\phi_z$ is the automorphism of $\mathbb{B}^n$ such that $\phi_z(0) = z$. We obtain

\[
|\nabla f(z)|^{p_\Phi} \leq C_1 \int_{B^n} |f(w)|^{p_\Phi} \frac{(1 - |z|^2)^{n+1+\beta}}{|1 - \langle z, w \rangle|^{2(n+1+\beta)}} d\nu_\beta(w).
\]
We observe with the help of \([20, \text{Proposition 1.4.10}]\) that \(\frac{(1-|z|^2)^{n+1+\beta}}{|1-(z,w)|^{2(n+1+\beta)}}d\nu_{\beta}(w)\) is up to a constant a probability measure. It follows using the convexity of

\[
\Phi_p(t) = \begin{cases} 
\Phi(t) & \text{if } \Phi \in \mathcal{V}^q \\
\Phi(t^n) & \text{if } \Phi \in \mathcal{L}_p
\end{cases}
\]

and Jensen’s Inequality that

\[
\Phi(|\nabla f(z)|) \leq C_2 \int_{B^n} \Phi(|f(w)|) \frac{(1-|z|^2)^{n+1+\beta}}{|1-(z,w)|^{2(n+1+\beta)}}d\nu_{\beta}(w).
\]

Finally, integrating both sides of the last inequality over \(B^n\) with respect to \(d\nu_{\alpha}(z)\) and applying Fubini’s Theorem and \([20, \text{Proposition 1.4.10}]\), we obtain

\[
\int_{B^n} \Phi(|\nabla f(z)|)d\nu_{\alpha} \leq C_2 \int_{B^n} \Phi(|f(z)|)d\nu_{\alpha}(z).
\]

Replacing \(f\) by \(f - f(0)\), we have

\[
\int_{B^n} \Phi(|\nabla f(z)|)d\nu_{\alpha} \leq C_2 \int_{B^n} \Phi(|f(z) - f(0)|)d\nu_{\alpha}(z).
\]

The proof is complete. \(\Box\)

We also obtain the following.

**Lemma 3.2.** Let \(\alpha > -1\). Assume that \(\Phi \in \mathcal{V}^q\) or \(\Phi \in \mathcal{L}_p\). Then there exists a constant \(C > 0\) such that for any \(f \in H(B^n)\) such that \((1-|z|^2)\mathcal{R} f(z) \in L^p(B^n, d\nu_{\alpha})\),

\[
\int_{B^n} \Phi(|f(z) - f(0)|)d\nu_{\alpha}(z) \leq C \int_{B^n} \Phi((1-|z|^2)|\mathcal{R} f(z)|)d\nu_{\alpha}(z).
\]

**Proof.** Let \(f \in H(B^n)\) be such that \((1-|z|^2)\mathcal{R} f(z) \in L^p(B^n, d\nu_{\alpha})\). Then following the proof of \([20, \text{Theorem 2.16}]\) at page 51, we have that for \(\beta\) large enough,

\[
|f(z) - f(0)| \leq \int_{B^n} \frac{|\mathcal{R} f(w)|}{|1-(z,w)|^{n+\beta}}d\nu_{\beta}(w).
\]

Let us first consider the case of \(\Phi \in \mathcal{V}^q\). Fix \(p\) so that \(1 < p < a_{\Phi}\), and observe that (15) is equivalent to

\[
|f(z) - f(0)| \leq C \beta_{-1}((1-|z|^2)|\mathcal{R} f(z)|)(z).
\]

Taking \(\beta\) large enough so that

\[0 < \alpha + 1 < p \beta,\]

we obtain from Theorem 2.5 that

\[
\int_{B^n} \Phi(|f(z) - f(0)|)d\nu_{\alpha}(z) \leq C \int_{B^n} \Phi((1-|z|^2)|\mathcal{R} f(z)|)d\nu_{\alpha}(z).
\]

We next consider the case of \(\Phi \in \mathcal{L}_p\). We assume that \(\beta\) is large enough so that

\[\beta = \frac{n+1+\gamma}{p} - (n+1), \ \gamma > \alpha + p.\]

Rewriting (15) as

\[
|f(z) - f(0)| \leq \int_{B^n} \left| \frac{|\mathcal{R} f(w)|}{1-(z,w)} \right|^{\frac{n+\gamma}{p}} \left| (1-|w|^2)^{\frac{1}{p}-1} \right|^{(n+\gamma)} d\nu(w),
\]
we obtain from Lemma 2.8 that
\[ |f(z) - f(0)|^p \leq C \int_{\mathbb{B}^n} \left| \frac{\mathcal{R}f(w)}{(1 - \langle z, w \rangle)^{\alpha + \beta}} \right|^p d\nu_\gamma(w), \]
or equivalently,
\[ |f(z) - f(0)|^p \leq C \int_{\mathbb{B}^n} \left( (1 - |w|^2) |\mathcal{R}f(w)| \right)^p \frac{d\nu_{\gamma-p}(w)}{(1 - \langle z, w \rangle)^{\alpha + 1 + (\gamma-p)}} = C P_{\gamma-p} \left( (1 - |\cdot|^2) |\mathcal{R}f(\cdot)|^p \right)(z). \]

As the growth function \( t \mapsto \Phi_p(t) = \Phi(t^{\frac{1}{p}}) \) is in \( \mathcal{U}^q \), proceeding as in the first part of this proof, we obtain
\[ \int_{\mathbb{B}^n} \Phi(|f(z) - f(0)|) d\nu_\alpha(z) \leq C \int_{\mathbb{B}^n} \Phi_p \left( (1 - |\cdot|^2) |\mathcal{R}f(\cdot)|^p \right)(z) d\nu_\alpha(z) = C \int_{\mathbb{B}^n} \Phi\left( (1 - |z|^2) |\mathcal{R}f(z)| \right) d\nu_\alpha(w). \]
The proof is complete. \( \square \)

We can now prove Theorem 1.2.

**Proof of Theorem 1.2.** That (b) implies (c) and (c) implies (d) follow from (2). That (a) implies (b) is Lemma 3.1 and that (d) implies (a) is Lemma 3.2. The proof is complete. \( \square \)

4. Applications

4.1. The Gustavsson-Peetre interpolation of two Bergman-Orlicz spaces. Our aim in this section is to give an application of Theorem 1.2 to a generalized interpolation of quasi-Banach spaces due to J. Gustavsson and J. Peetre. For this, we first introduce several definitions and results.

**Definition 4.1.** A function \( \rho : [0, \infty) \to [0, \infty) \) is said to be pseudo-concave, if it is continuous on \((0, \infty)\) and
\[ \rho(s) \leq \max(1, \frac{s}{t}) \rho(t), \text{ for all } s, t > 0. \]
We denote by \( \mathcal{T} \) the set of all pseudo-concave functions.

**Definition 4.2.** A function \( \rho \in \mathcal{T} \) is said to be in \( \mathcal{T}^+-\), if
\[ \sup_x \frac{\rho(\lambda x)}{\rho(x)} = o(\max(1, \lambda)) \text{ as } x \to 0 \text{ or } x \to \infty. \]

As example of functions in \( \mathcal{T}^+- \) we have of course concave functions. The following function was provided in [10] as a non trivial element in \( \mathcal{T}^+-\):
\[ \rho(t) = t^\theta (\log(e + t))^{\alpha}(e + \frac{1}{t})^\beta \]
where \( 0 < \theta < 1 \), and \( \alpha, \beta \) are real numbers.
DEFINITION 4.3 (J. Gustavsson and J. Peetre [10]). Let $A_0$ and $A_1$ be quasi-Banach spaces, both embedded in a Hausdorff topological space $A$. We call $\bar{A} = (A_0, A_1)$ a quasi-Banach couple. Let $\rho \in T$. We denote by $\langle \bar{A} \rangle_\rho = \langle A_0, A_1 \rangle_\rho$ the space of all elements $a \in \sum(\bar{A}) := A_0 + A_1$ such that there exists a sequence $u = \{u_\alpha\}_{\alpha \in \mathbb{Z}}$ of elements $u_\alpha \in \Delta(\bar{A}) := A_0 \cap A_1$ such that

$$a = \sum_{\alpha \in \mathbb{Z}} u_\alpha \quad \text{(convergence in } \sum(\bar{A})),$$

for every finite subset $F \subset \mathbb{Z}$ and every real sequence $\xi = \{\xi_\alpha\}_{\alpha \in F}$ with $|\xi_\alpha| \leq 1$, we have

$$\left\| \sum_F \frac{\xi_\alpha 2^{k_\alpha} u_\alpha}{\rho(2^{k_\alpha})} \right\| \leq C \quad (k = 0, 1)$$

with $C$ independent of $F$ and $\xi$.

We equip the space $\langle \bar{A} \rangle_\rho$ with the semi-norm

$$\|a\|_{\langle \bar{A} \rangle_\rho} := \inf_u C,$$

where the infimum is taken over all admissible sequences $u$ as above.

As observed in [10], if $\rho \in T^{++}$, then $\|a\|_{\langle \bar{A} \rangle_\rho}$ is a quasi-norm and $\langle \bar{A} \rangle_\rho$ is a quasi-Banach space.

Let us recall that if $T : A \to B$ is a continuous linear operator between two quasi-Banach spaces $A$ and $B$, the operator norm of $T$ denoted $\|T\|_{A \to B}$ is defined by

$$\|T\|_{A \to B} := \sup_{x \in A, x \neq 0} \frac{\|Tx\|_B}{\|x\|_A}.$$ 

PROPOSITION 4.4. ([10 Proposition 6.1.] Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two quasi-Banach couples. If $T : \bar{A} \to \bar{B}$ is a continuous linear mapping, that is the restriction $T_i = T|A_i : A_i \to B_i$, $(i = 0, 1)$ is continuous, then $T : \langle \bar{A} \rangle_\rho \to \langle \bar{B} \rangle_\rho$ is continuous and

$$\|T\|_{\langle \bar{A} \rangle_\rho \to \langle \bar{B} \rangle_\rho} \leq \max(\|T\|_{A_0 \to B_0}, \|T\|_{A_1 \to B_1}).$$

That is the functor $(A_0, A_1) \mapsto \langle \bar{A} \rangle_\rho$ is an interpolation space.

We call $\langle \bar{A} \rangle_\rho$ the Gustavsson-Peetre interpolate of $A_0$ and $A_1$. As observed in [10], when $\rho(s) = s^\theta, \ 0 < \theta < 1$, $\langle \bar{A} \rangle_\rho$ corresponds to the complex interpolation (see also [5] [14] [18]). For more on complex interpolation, we refer to the book [2].

The following is a restriction of [10 Theorem 7.3] to the class of growth functions we are interested in and our spaces.

PROPOSITION 4.5. Let $\Phi_i \in \mathcal{H}^q, \ i = 0, 1$ satisfying the $\vee_2$—condition, and let $\alpha > -1$. Assume that $\rho \in T^{++}$ and let $\Phi$ be defined by

$$\Phi^{-1} = \Phi_0^{-1} \rho \left( \frac{\Phi_0^{-1}}{\Phi_0^{-1}} \right).$$

Then

$$L^\Phi(\mathbb{R}^n, \nu_\alpha) = L^\Phi_0(\mathbb{R}^n, \nu_\alpha) = \langle L^{\Phi_0}(\mathbb{R}^n, \nu_\alpha), L^{\Phi_1}(\mathbb{R}^n, \nu_\alpha) \rangle_\rho$$
with equivalence of (quasi)-norms. In particular, \( L^\Phi_\rho(\mathbb{B}^n, d\nu_\alpha) \) is an interpolation space with respect to \( (L^\Phi_{p_0}(\mathbb{B}^n, d\nu_\alpha), L^\Phi_{p_1}(\mathbb{B}^n, d\nu_\alpha)) \).

It is easy to check using the definition of a pseudo-concave function, that given \( \Phi_0, \Phi_1 \in \mathcal{B}^q \), the growth function \( \Phi \) defined by (18) is of upper type \( q > 1 \).

We can now state our main result of this section.

**Theorem 4.6.** Let \( \Phi_i \in \mathcal{B}^q, i = 0, 1 \) satisfying the \( \nabla_2 \)-condition, and let \( \alpha > -1 \). Assume that \( \rho \in \mathcal{T}^+ \) and let \( \Phi \) be defined as in (18). Then

\[
A^\Phi_\alpha(\mathbb{B}^n) = (A^{\Phi_0}_\alpha(\mathbb{B}^n), A^{\Phi_1}_\alpha(\mathbb{B}^n))_\rho
\]

with equivalence of (quasi)-norms.

**Proof.** As \( \Phi_0 \) and \( \Phi_1 \) satisfy the \( \nabla_2 \)-condition, we have from Theorem 2.6 that the Bergman projection \( P_\alpha \) maps \( L^{\Phi_0}(\mathbb{B}^n, d\nu_\alpha) \) boundedly onto \( A^{\Phi_0}_\alpha(\mathbb{B}^n) \), and it maps \( L^{\Phi_1}(\mathbb{B}^n, d\nu_\alpha) \) boundedly onto \( A^{\Phi_1}_\alpha(\mathbb{B}^n) \). It follows from Proposition 4.4 that \( P_\alpha \) maps \( L^\Phi(\mathbb{B}^n, d\nu_\alpha) = (L^{\Phi_0}(\mathbb{B}^n, d\nu_\alpha), L^{\Phi_1}(\mathbb{B}^n, d\nu_\alpha))_\rho \) boundedly into \( (A^{\Phi_0}_\alpha(\mathbb{B}^n), A^{\Phi_1}_\alpha(\mathbb{B}^n))_\rho \). As \( P_\alpha (L^\Phi(\mathbb{B}^n, d\nu_\alpha)) = L^\Phi(\mathbb{B}^n) \), we conclude that

\[
A^\Phi_\alpha(\mathbb{B}^n) \subset (A^{\Phi_0}_\alpha(\mathbb{B}^n), A^{\Phi_1}_\alpha(\mathbb{B}^n))_\rho.
\]

Conversely, if we denote by \( L \) the operator defined on \( H(\mathbb{B}^n) \) by

\[
L(f)(z) := (1 - |z|^2)^\alpha \mathcal{R}f(z), \quad z \in \mathbb{B}^n,
\]

then following Theorem 1.2 and specially Lemma 3.2, we have that \( L \) maps \( A^{\Phi_0}_\alpha(\mathbb{B}^n) \) boundedly into \( L^{\Phi_0}(\mathbb{B}^n, d\nu_\alpha) \), and it maps \( A^{\Phi_1}_\alpha(\mathbb{B}^n) \) into \( L^{\Phi_1}(\mathbb{B}^n, d\nu_\alpha) \). It follows once more from Proposition 4.4 that \( L \) maps \( (A^{\Phi_0}_\alpha(\mathbb{B}^n), A^{\Phi_1}_\alpha(\mathbb{B}^n))_\rho \) into \( L^\Phi(\mathbb{B}^n, d\nu_\alpha) \). That is if \( f \in (A^{\Phi_0}_\alpha(\mathbb{B}^n), A^{\Phi_1}_\alpha(\mathbb{B}^n))_\rho \), then the function \( z \mapsto (1 - |z|^2)^\alpha \mathcal{R}f(z) \) belongs to \( L^\Phi(\mathbb{B}^n, d\nu_\alpha) \), which by Theorem 1.2 is equivalent to saying that \( f \in A^{\Phi}_\alpha(\mathbb{B}^n) \). We deduce that

\[
(A^{\Phi_0}_\alpha(\mathbb{B}^n), A^{\Phi_1}_\alpha(\mathbb{B}^n))_\rho \subset A^\Phi_\alpha(\mathbb{B}^n).
\]

From (19) and (20), we conclude that

\[
(A^{\Phi_0}_\alpha(\mathbb{B}^n), A^{\Phi_1}_\alpha(\mathbb{B}^n))_\rho = A^\Phi_\alpha(\mathbb{B}^n).
\]

The proof is complete. \( \square \)

Restricting to power functions, we deduce the following.

**Corollary 4.7.** Let \( 1 \leq p_0 < p_1 < \infty \), and let \( \alpha > -1 \). Assume that \( \rho \in \mathcal{T}^+ \) and let \( \Phi \) be defined by \( \Phi^{-1}(t) = t^{\frac{1}{p_0}}(t^{\frac{1}{p_1}})^\alpha \). Then

\[
A^\Phi_\alpha(\mathbb{B}^n) = (A^{p_0}_\alpha(\mathbb{B}^n), A^{p_1}_\alpha(\mathbb{B}^n))_\rho
\]

with equivalence of (quasi)-norms.

Note that the above corollary tells us that given two classical Bergman spaces with the same weight, their Gustavsson-Peetre interpolation space is in general a Bergman-Orlicz space while the complex interpolation of weighted Bergman spaces always gives another weighted Bergman space (see [20]).
4.2. Boundedness and compactness of weighted Cesàro-type integrals. For \( g \in \mathcal{H}(\mathbb{B}^n) \) with \( g(0) = 0 \), we consider the following integral-type operator defined on \( \mathcal{H}(\mathbb{B}^n) \) by

\[
T_g f(z) = \int_0^1 f(tz) R g(tz) \frac{dt}{t}.
\]

The operator \( T_g \) is the so-called extended Cesàro operator introduced in [11]. The boundedness and compactness of \( T_g \) on the weighted Bergman space \( \mathcal{A}_p^\alpha(\mathbb{B}^n) \) were studied by J. Xiao [27]. In [16] the same questions between different weighted Bergman spaces were studied. Note that Z. Hu also considered the boundedness and compactness of \( T_g \) between weighted Bergman spaces for a large class of weights [13].

We aim in this section to characterize symbols \( g \) such that \( T_g \) is a bounded or compact operator from a weighted Bergman-Orlicz space to itself.

We first prove an estimate for derivative of functions in Bergman-Orlicz spaces.

**Lemma 4.8.** Let \( \Phi \in \mathcal{L}_p \cup \mathcal{U}_q \) and \( \alpha > -1 \). Then there are two positive constants \( C_1 \) and \( C_2 \) such that for any \( f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \),

\[
|\nabla f(z)| \leq \frac{C_1}{1 - |z|^2} \Phi^{-1} \left( \frac{C_2}{(1 - |z|^2)^{n+1+\alpha}} \right) \|f\|_{\text{lux}, \alpha, \Phi}, \text{ for any } z \in \mathbb{B}^n.
\]

**Proof.** Let us start by considering the case where \( \Phi \in \mathcal{U}_q \). We observe that in this case, \( \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \) continuously embeds into \( \mathcal{A}_1^\Phi(\mathbb{B}^n) \). Hence, for any \( f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \), and any \( z \in \mathbb{B}^n \),

\[
f(z) = \int_{\mathbb{B}^n} f(w) \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w).
\]

Thus for any \( j = 1, \ldots, n \),

\[
\frac{\partial f}{\partial z_j}(z) = c \int_{\mathbb{B}^n} \overline{w}_j f(w) \frac{1}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w).
\]

It follows easily that

\[
1 - |z|^2 \frac{\|f\|_{\text{lux}, \alpha, \Phi}}{\|f\|_{\text{lux}, \alpha, \Phi}} |\frac{\partial f}{\partial z_j}(z)| \leq C \int_{\mathbb{B}^n} |f(w)| \frac{1 - |z|^2}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w).
\]

It is easy to see using [20, Proposition 1.4.10] that \( \frac{1 - |z|^2}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w) \) is up to a constant a probability measure. Hence using the convexity of \( \Phi \) and Jensen’s Inequality, we obtain

\[
\Phi \left( \frac{1 - |z|^2}{\|f\|_{\text{lux}, \alpha, \Phi}} |\frac{\partial f}{\partial z_j}(z)| \right) \leq C \int_{\mathbb{B}^n} \Phi \left( \frac{|f(w)|}{\|f\|_{\text{lux}, \alpha, \Phi}} \right) \frac{1 - |z|^2}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w)
\]

\[
\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} \Phi \left( \frac{|f(w)|}{\|f\|_{\text{lux}, \alpha, \Phi}} \right) d\nu_\alpha(w)
\]

\[
\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}}.
\]
Hence
\[ \left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{1}{1 - |z|^2} \Phi^{-1} \left( \frac{C}{(1 - |z|^2)^{n+1}} \right) \|f\|_{\alpha,\Phi}^{lux} \text{, for any } z \in \mathbb{B}^n \]
from which follows (21).

We now consider the case where \( \Phi \in \mathcal{L}_p \). We recall that in this case \( \Phi \) is of lower type \( 0 < p \leq 1 \). Let \( \beta > -1 \) be large enough (this will be more precise in the next lines). As above, we have that
\[ (22) \quad \left| \frac{\partial f}{\partial z_j}(z) \right| \leq C \int_{\mathbb{B}^n} \frac{|f(w)|}{|1 - \langle z, w \rangle|^{n+2+\beta}} \, d\nu_\beta(w). \]
We assume that \( \beta = \frac{n+1+\gamma}{p} - (n+1) \) with \( \gamma > \alpha + p \). Then using Lemma \ref{lemma_25} we obtain from (22) that
\[ (23) \quad \left| \frac{\partial f}{\partial z_j}(z) \right|^p \leq C \int_{\mathbb{B}^n} \frac{f(w)}{|1 - \langle z, w \rangle|^{n+2+\beta}} \, d\nu_\gamma(w) \]
or equivalently,
\[ (24) \quad \frac{1 - |z|^2}{\|f\|_{\alpha,\Phi}^{lux}} \frac{\partial f}{\partial z_j}(z) \leq C \int_{\mathbb{B}^n} \frac{f(w)}{|1 - \langle z, w \rangle|^{n+2+\beta}} \, d\nu_\gamma(w). \]
One easily checks that \( \frac{(1 - |z|^2)^p}{|1 - \langle z, w \rangle|^{(n+2+\beta)p}} \, d\nu_\gamma(w) \) is up to a constant, a probability measure. Hence using that the function \( \Phi_p : t \mapsto \Phi_p(t) = \Phi(t^{\frac{1}{p}}) \) is convex and Jensen’s Inequality, we obtain that
\[ \Phi_p \left( \frac{1 - |z|^2}{\|f\|_{\alpha,\Phi}^{lux}} \frac{\partial f}{\partial z_j}(z) \right)^p \leq C \int_{\mathbb{B}^n} \Phi_p \left( \frac{f(w)}{|1 - \langle z, w \rangle|^{n+2+\beta}} \right)^p \, d\nu_\gamma(w). \]
Hence
\[ \Phi \left( \frac{1 - |z|^2}{\|f\|_{\alpha,\Phi}^{lux}} \frac{\partial f}{\partial z_j}(z) \right) \leq C \int_{\mathbb{B}^n} \Phi \left( \frac{f(w)}{|1 - \langle z, w \rangle|^{n+2+\beta}} \right) \frac{(1 - |z|^2)^p}{|1 - \langle z, w \rangle|^{(n+2+\beta)p}} \, d\nu_\gamma(w) = C \int_{\mathbb{B}^n} \Phi \left( \frac{f(w)}{|1 - \langle z, w \rangle|^{n+2+\beta}} \right) \frac{(1 - |z|^2)^p}{|1 - \langle z, w \rangle|^{n+1+\beta + p}} \, d\nu_\gamma(w) \leq \frac{C}{(1 - |z|^2)^{n+1}} \int_{\mathbb{B}^n} \Phi \left( \frac{f(w)}{|1 - \langle z, w \rangle|} \right) \, d\nu_\alpha(w) \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}}. \]
That is
\[ \left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{1}{1 - |z|^2} \Phi^{-1} \left( \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \right) \|f\|_{\alpha,\Phi}^{lux} \text{, for any } z \in \mathbb{B}^n \]
from which follows (21). The proof is complete. \( \square \)

We can now prove the following.
THEOREM 4.9. Let \( \Phi \in \mathcal{L}_q \cup \mathcal{U}^q \) and \( \alpha > -1 \). Assume \( g \) is a holomorphic function on \( \mathbb{B}^n \) with \( g(0) = 0 \). Then the operator \( T_g \) is bounded on \( \mathcal{A}_\alpha^p(\mathbb{B}^n) \) if and only if

\[
(25) \quad M := \sup_{z \in \mathbb{B}^n} (1 - |z|^2)|Rg(z)| < \infty.
\]

Moreover, if we denote by \( \|T_g\| \) the operator norm of \( T_g \), then

\[
\|T_g\| \sim M.
\]

Proof. Let us first assume that (25) holds. Then

\[
I := \int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2)|RT_gf(z)|}{M \|f\|_{\Phi,\alpha}^{lux}} \right) d\nu_\alpha(z)
\]

\[
= \int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2)|Rg(z)| \|f(z)\|}{M \|f\|_{\Phi,\alpha}^{lux}} \right) d\nu_\alpha(z)
\]

\[
\leq \int_{\mathbb{B}^n} \Phi \left( \frac{|f(z)|}{\|f\|_{\Phi,\alpha}^{lux}} \right) d\nu_\alpha(z) \leq 1.
\]

Hence from Theorem 1.2, we deduce that \( T_gf \in \mathcal{A}_\alpha^p(\mathbb{B}^n) \) for any \( f \in \mathcal{A}_\alpha^p(\mathbb{B}^n) \). Moreover, from Lemma 3.2, we deduce that

\[
\|T_gf\|_{\Phi,\alpha}^{lux} \leq M \|f\|_{\Phi,\alpha}^{lux}
\]

for any \( f \in \mathcal{A}_\alpha^p(\mathbb{B}^n) \). It follows that

\[
\|T_g\| \leq M.
\]

Conversely, let us assume that for \( g \in H(\mathbb{B}^n) \) with \( g(0) = 0 \), \( T_g \) is bounded on \( \mathcal{A}_\alpha^p(\mathbb{B}^n) \). Then from Lemma 1.8 we have that there is a constant \( C > 0 \) such that for any \( f \in \mathcal{A}_\alpha^p(\mathbb{B}^n) \) and any \( z \in \mathbb{B}^n \),

\[
(1 - |z|^2)|RT_gf(z)| \leq C \Phi^{-1} \left( \frac{1}{(1 - |z|^{n+1+\alpha})} \right) \|T_gf\|_{\Phi,\alpha}^{lux}
\]

which leads to

\[
(26) \quad (1 - |z|^2)|Rg(z)||f(z)| \leq C \Phi^{-1} \left( \frac{1}{(1 - |z|^{n+1+\alpha})} \right) \|T_g\| \|f\|_{\Phi,\alpha}^{lux}.
\]

Let \( a \in \mathbb{B}^n \) be fixed and consider the function \( f_a \) defined on \( \mathbb{B}^n \) by

\[
f_a(z) = \Phi^{-1} \left( \frac{1}{(1 - |a|^{n+1+\alpha})} \right) \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle} \right)^{2(n+1+\alpha)}.
\]

We recall with Lemma 2.9 that \( f_a \in \mathcal{A}_\alpha^p(\mathbb{B}^n) \) and \( \|f_a\|_{\Phi,\alpha}^{lux} \lesssim 1 \). Let us test (26) with the function \( f = f_a \), for a fixed. We obtain that for any \( z \in \mathbb{B}^n \),

\[
(1 - |z|^2)|Rg(z)|\Phi^{-1} \left( \frac{1}{(1 - |a|^2)^{n+1+\alpha}} \right) \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle} \right)^{2(n+1+\alpha)} \leq C \Phi^{-1} \left( \frac{1}{(1 - |a|^2)^{n+1+\alpha}} \right) \|T_g\|.
\]

Putting \( z = a \), we obtain that

\[
(1 - |a|^2)|Rg(a)| \leq C \|T_g\|.
\]
As $a$ was taken arbitrary in $\mathbb{B}^n$, we deduce that there is a constant $C > 0$ such
\[ M := \sup_{a \in \mathbb{B}^n} (1 - |a|^2)|Rg(a)| \leq C\|T_g\|. \]
The proof is complete. \qed

Recall that the spaces of all holomorphic functions satisfying (25) is called the Bloch space. The equivalent characterizations in Theorem 1.2 show that the Bloch space embeds continuously into $A^\Phi_\alpha(\mathbb{B}^n)$ for any $\Phi \in \mathcal{U}^q \cup \mathcal{L}^p$. We next consider compactness of the operators $T_g$. For this, we need the following compactness criteria which can be proved following the usual arguments (see [5]).

**Lemma 4.10.** Let $\Phi \in \mathcal{U}^q \cup \mathcal{L}^p$ and let $\alpha > -1$. Let $g \in H(\mathbb{B}^n)$ with $g(0) = 0$. Suppose that $T_g : A^\Phi_\alpha(\mathbb{B}^n) \to A^\Phi_\alpha(\mathbb{B}^n)$ is bounded, then
\[ T_g : A^\Phi_\alpha(\mathbb{B}^n) \to A^\Phi_\alpha(\mathbb{B}^n) \]
is compact if and only if for every sequence $(f_j)$ in the unit ball of $A^\Phi_\alpha(\mathbb{B}^n)$ which converges to 0 uniformly on compact subsets of $\mathbb{B}^n$, one has
\[ ||T_g(f_j)||_{\mathcal{L}^\Phi, \alpha} \to 0 \text{ as } j \to \infty. \]

We have the following result.

**Theorem 4.11.** Let $\Phi \in \mathcal{U}^q \cup \mathcal{L}^p$ and $\alpha > -1$. Assume $g$ is a holomorphic function on $\mathbb{B}^n$. Then the operator $T_g : A^\Phi_\alpha(\mathbb{B}^n) \to A^\Phi_\alpha(\mathbb{B}^n)$ is compact if and only if
\[ \lim_{|z| \to 1} (1 - |z|^2)|Rg(z)| = 0. \]

**Proof.** Let us first assume that $T_g$ is compact. Let $\{a_j\}_{j \in \mathbb{N}}$ be a sequence in $\mathbb{B}^n$ such that $\lim_{j \to \infty} |a_j| = 1$. Consider the sequence of holomorphic functions on $\mathbb{B}^n$ given by
\[ f_j(z) = \Phi^{-1} \left( \frac{1}{(1 - |a_j|^2)^{n+1+\alpha}} \right) \left( \frac{1 - |a_j|^2}{1 - \langle z, a_j \rangle} \right)^{k(n+1+\alpha)} \]
with $k > 1$ to be precised where needed. From Lemma 2.9, we know that the sequence $\{f_j\}_{n \in \mathbb{N}}$ is uniformly bounded in $A^\Phi_\alpha(\mathbb{B}^n)$. Also, we have that if $\Phi \in \mathcal{U}^q$, then as $\Phi^{-1}$ is concave and as $k > 1$,
\[ |f_j(z)| \leq \frac{(1 - |a_j|^2)^{(k-1)(n+1+\alpha)}}{|1 - \langle z, a_j \rangle|^{k(n+1+\alpha)}} \to 0 \]
as $j \to \infty$ on compact subsets of $\mathbb{B}^n$. If $\Phi \in \mathcal{L}^p$, then as $\Phi^{-1} \in \mathcal{L}^{\frac{1}{p}}$, taking $k > \frac{1}{p}$, we obtain using (3) that
\[ |f_j(z)| \leq C \frac{(1 - |a_j|^2)^{(k-\frac{1}{p})(n+1+\alpha)}}{|1 - \langle z, a_j \rangle|^{k(n+1+\alpha)}} \to 0 \]
as $j \to \infty$ on compact subsets of $\mathbb{B}^n$. 


Using Lemma 4.8, we obtain that there is a constant $C > 0$ such that each $j \in \mathbb{N}$, and for any $z \in B^n$,

$$(1 - |z|^2)|\mathcal{R}T_g f_j(z)| \leq C \Phi^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right) ||T_g(f_j)||_{\Phi,\alpha}^{(ux)}$$

or equivalently,

$$(1 - |z|^2)\Phi^{-1} \left( \frac{1}{(1 - |a_j|^2)^{n+1+\alpha}} \right) \frac{1 - |a_j|^2}{1 - \langle z, a_j \rangle} |\mathcal{R}g(z)| \leq C \Phi^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right) ||T_g(f_j)||_{\Phi,\alpha}^{(ux)}.$$

Putting in particular $z = a_j$, we obtain

$$(1 - |a_j|^2)|\mathcal{R}g(a_j)| \leq C ||T_g(f_j)||_{\Phi,\alpha}^{(ux)}.$$

As $||T_g(f_j)||_{\Phi,\alpha}^{(ux)} \to 0$ as $j \to \infty$, we deduce that

$$\lim_{j \to \infty} (1 - |a_j|^2)|\mathcal{R}g(a_j)| = 0$$

which leads to

$$\lim_{|z| \to 1} (1 - |z|^2)|\mathcal{R}g(z)| = 0.$$

Conversely, let us assume that the holomorphic function $g$ satisfies (27). Note that this implies that $g \in A^\Phi_\alpha(B^n)$ and that for any $\varepsilon > 0$, there exists $\eta$ such that

$$(28) \quad (1 - |z|^2)|\mathcal{R}g(z)| < \varepsilon$$

for any $z \in B^n$ such that $\eta < |z| < 1$.

Let us start by proving that $T_g$ is bounded on $A^\Phi_\alpha(B^n)$. Let

$$K = \max\{1, C \Phi^{-1} \left( \frac{1}{(1 - \eta^2)^{n+1+\alpha}} \right), ||g||_{\Phi,\alpha}^{(ux)}\}$$

where $C$ is (11). We have at first that for any $f \in A^\Phi_\alpha(B^n)$,

$$L := \int_{B^n} \Phi \left( \frac{(1 - |z|^2)|\mathcal{R}T_g f(z)|}{K^2 ||f||_{\Phi,\alpha}^{(ux)}} \right) d\nu_\alpha(z)$$

$$= \int_{B^n} \Phi \left( \frac{(1 - |z|^2)|\mathcal{R}g(z)||f(z)|}{K^2 ||f||_{\Phi,\alpha}^{(ux)}} \right) d\nu_\alpha(z)$$

$$\leq \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2)|\mathcal{R}g(z)||f(z)|}{K^2 ||f||_{\Phi,\alpha}^{(ux)}} \right) d\nu_\alpha(z) + \int_{|z| > \eta} \Phi \left( \frac{(1 - |z|^2)|\mathcal{R}g(z)||f(z)|}{K^2 ||f||_{\Phi,\alpha}^{(ux)}} \right) d\nu_\alpha(z).$$
Using the pointwise estimate (11) and the definition of the constant $K$, we obtain

$$
L_1 := \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)||f(z)|}{K^2 \|f\|_{\text{lux} \Phi, \alpha}} \right) \, d\nu_\alpha(z)
$$

$$
\leq \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)|C\Phi^{-1}}{K^2} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right) \right) \, d\nu_\alpha(z)
$$

$$
\leq \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)|C\Phi^{-1}}{K^2} \right) \, d\nu_\alpha(z).
$$

It follows from the equivalent characterization in Theorem 1.2 that

$$
L_1 \leq \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)||f(z)|}{\|g\|_{\text{lux} \Phi, \alpha}} \right) \, d\nu_\alpha(z)
$$

$$
\leq \int_{\mathbb{B}^n} \Phi \left( \frac{|g(z)|}{\|g\|_{\text{lux} \Phi, \alpha}} \right) \, d\nu_\alpha(z) \leq 1.
$$

That is

$$
(29) \quad \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)||f(z)|}{K^2 \|f\|_{\text{lux} \Phi, \alpha}} \right) \, d\nu_\alpha(z) \lesssim 1.
$$

Using the estimate (28), we obtain

$$
L_2 := \int_{|z| > \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)||f(z)|}{K^2 \|f\|_{\text{lux} \Phi, \alpha}} \right) \, d\nu_\alpha(z)
$$

$$
\leq \int_{|z| > \eta} \Phi \left( \frac{\varepsilon |f(z)|}{K^2 \|f\|_{\text{lux} \Phi, \alpha}} \right) \, d\nu_\alpha(z)
$$

$$
\leq \int_{\mathbb{B}^n} \Phi \left( \frac{|f(z)|}{\|f\|_{\text{lux} \Phi, \alpha}} \right) \, d\nu_\alpha(z)
$$

$$
\leq 1.
$$

That is

$$
(30) \quad \int_{|z| > \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)||f(z)|}{K^2 \|f\|_{\text{lux} \Phi, \alpha}} \right) \, d\nu_\alpha(z) \leq 1.
$$

From (29), (30) and the equivalent characterizations in Theorem 1.2 we deduce that $T_g$ is bounded on $A^\Phi_{\alpha}(\mathbb{B}^n)$.

Now, let $\{f_j\}_{j \in \mathbb{N}}$ be a sequence in the unit ball of $A^\Phi_{\alpha}(\mathbb{B}^n)$ which converges to 0 uniformly on compact subsets of $\mathbb{B}^n$. Then there exists an integer $j_0 > 0$ such that for any $j > j_0$,

$$
\sup_{0 < |z| \leq \eta} |f_j(z)| < \varepsilon.
$$
Let $M := \max\{1, \|g\|_{\Phi, \alpha}^{1/p}\}$. Then we obtain for any $j > j_0$,

$$L := \int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2)|RT_j f_j(z)|}{M} \right) d\nu_\alpha(z) = \int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2)|Rg(z)||f_j(z)|}{M} \right) d\nu_\alpha(z) \leq \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)||f_j(z)|}{M} \right) d\nu_\alpha(z) + \int_{|z| > \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)||f_j(z)|}{M} \right) d\nu_\alpha(z).$$

Using the convexity of $\Phi$ if $\Phi \in \mathcal{W}^q$ and condition $\mathcal{H}$ if $\Phi \in \mathcal{L}_p$ and the definition of $p_\Phi$ in $\mathcal{H}$, it follows that

$$L \lesssim \varepsilon^{p_\Phi} \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2)|Rg(z)|}{M} \right) d\nu_\alpha(z) + \varepsilon^{p_\Phi} \int_{|z| > \eta} \Phi \left( \frac{|f_j(z)|}{M} \right) d\nu_\alpha(z) \lesssim \varepsilon^{p_\Phi} \int_{\mathbb{B}^n} \Phi \left( \frac{|g(z)|}{M} \right) d\nu_\alpha(z) + \varepsilon^{p_\Phi} \int_{\mathbb{B}^n} \Phi \left( \frac{|f_j(z)|}{M} \right) d\nu_\alpha(z) \leq 2\varepsilon^{p_\Phi}.$$ 

It follows that $\int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2)|RT_j f_j(z)|}{M} \right) d\nu_\alpha(z) \to 0$ as $j \to \infty$. Hence that $\int_{\mathbb{B}^n} \Phi \left( \frac{|T_j f_j(z)|}{M} \right) d\nu_\alpha(z) \to 0$ as $j \to \infty$. This implies that $\|T_j f_j\|_{\Phi, \alpha} \to 0$ as $j \to \infty$. Thus $T_g$ is a compact operator on $A_\Phi^\alpha(\mathbb{B}^n)$. The proof is complete. 

\[\Box\]

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