GROTHENDIECK—SERRE CONJECTURE FOR GROUPS OF TYPE $F_4$
WITH TRIVIAL $f_3$ INVARIANT

V. PETROV AND A. STAVROVA

Abstract. Assume that $R$ is a semi-local regular ring containing an infinite perfect field. Let $K$ be the field of fractions of $R$. Let $H$ be a simple algebraic group of type $F_4$ over $R$ such that $H_K$ is the automorphism group of a 27-dimensional Jordan algebra which is a first Tits construction. If $\text{char } K \neq 2$ this means precisely that the $f_3$ invariant of $H_K$ is trivial. We prove that the kernel of the map

$$H^1_{\text{ét}}(R, H) \to H^1_{\text{ét}}(K, H)$$

induced by the inclusion of $R$ into $K$ is trivial.

This result is a particular case of the Grothendieck—Serre conjecture on rationally trivial torsors. It continues the recent series of papers [PaSV], [Pa], [PaPS] and complements the result of Chernousov [Ch] on the Grothendieck—Serre conjecture for groups of type $F_4$ with trivial $g_3$ invariant.

1. Introduction

In the present paper we address the Grothendieck—Serre conjecture [Se p. 31, Remarque], [Gr, Remarque 1.11] on the rationally trivial torsors of reductive algebraic groups. This conjecture states that for any reductive group scheme $G$ over a regular ring $R$, any $G$-torsor that is trivial over the field of fractions $K$ of $R$ is itself trivial; in other words, the natural map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$$

has trivial kernel. It has been settled in a variety of particular cases, and we refer to [Pa] for a detailed overview. The most recent result belongs to V. Chernousov [Ch] who has proved that the Grothendieck—Serre conjecture holds for an arbitrary simple group $H$ of type $F_4$ over a local regular ring $R$ containing the field of rational numbers, given that $H_K$ has a trivial $g_3$ invariant. We prove that the Grothendieck—Serre conjecture holds for another natural class of groups $H$ of type $F_4$, those for which $H_K$ has trivial $f_3$ invariant. In fact, since our approach is characteristic-free, we establish the following slightly more general result.

Theorem 1. Let $R$ be a semi-local regular ring containing an infinite perfect field. Let $K$ be the field of fractions of $R$. Let $J$ be a 27-dimensional exceptional Jordan algebra over $R$ such that $J_K$ is a first Tits construction. Then the map

$$H^1_{\text{ét}}(R, \text{Aut } (J)) \to H^1_{\text{ét}}(K, \text{Aut } (J))$$

induced by the inclusion of $R$ into $K$ has trivial kernel.

Corollary. Let $R$ be a semi-local regular ring containing an infinite perfect field $k$ such that $\text{char } k \neq 2$. Let $K$ be the field of fractions of $R$. Let $H$ be a simple group scheme of type $F_4$ over $R$ such that $H_K$ has trivial $f_3$ invariant. Then the map

$$H^1_{\text{ét}}(R, H) \to H^1_{\text{ét}}(K, H)$$

induced by the inclusion of $R$ into $K$ has trivial kernel.
2. Isotopes of Jordan algebras

In the first two sections \( R \) is an arbitrary commutative ring.

A \textit{(unital quadratic)} Jordan algebra is a projective \( R \)-module \( J \) together with an element \( 1 \in J \) and an operation

\[
J \times J \rightarrow J \\
(x, y) \mapsto U_x y,
\]

which is quadratic in \( x \) and linear in \( y \) and satisfies the following axioms:

\begin{itemize}
  \item \( U_1 = \text{id}_J \);
  \item \( \{x, y, U_z z\} = U_x \{y, x, z\} \);
  \item \( U_{U_x y} = U_x U_y U_z \),
\end{itemize}

where \( \{x, y, z\} = U_{x+y} - U_x y - U_y z \) stands for the linearization of \( U \). It is well-known that the split simple group scheme of type \( F_4 \) can be realized as the automorphism group scheme of the split 27-dimensional exceptional Jordan algebra \( J_0 \). This implies that any other group scheme of type \( F_4 \) is the automorphism group scheme of a twisted form of \( J_0 \).

Let \( v \) be an \textit{invertible} element of \( J \) (that is, \( U_v \) is invertible). An \textit{isotope} \( J^{(v)} \) of \( J \) is a new Jordan algebra whose underlying module is \( J \), while the identity and \( U \)-operator are given by the formulas

\[
1^{(v)} = v^{-1}; \\
U_x^{(v)} = U_x U_v.
\]

An \textit{isotopy} between two Jordan algebras \( J \) and \( J' \) is an isomorphism \( g: J \rightarrow J^{(v)} \); it follows that \( v = g(1)^{-1} \). We are particularly interested in \textit{autotopies} of \( J \); one can see that \( g \) is an autotopy if and only if

\[
U_{g(x)} = g U_x g^{-1} U_{g(1)}
\]

for all \( x \in J \). In particular, transformations of the form \( U_x \) are autotopies. The group scheme of all autotopies is called the \textit{structure group} of \( J \) and is denoted by \( \text{Str}(J) \). Obviously it contains \( \mathbb{G}_m \) acting on \( J \) by scalar transformations.

It is convenient to describe isotopies as isomorphisms of some algebraic structures. This was done by O. Loos who introduced the notion of a \textit{Jordan pair}. We will not need the precise definition, see \cite{Lo75} for details. It turns out that every Jordan algebra \( J \) defines a Jordan pair \( (J, J) \), and the isotopies between \( J \) and \( J' \) bijectively correspond to the isomorphisms of \( (J, J) \) and \( (J', J') \) \cite[Proposition 1.8]{Lo75}). In particular, the structure group \( \text{Str}(J) \) is isomorphic to \( \text{Aut}((J, J)) \). We use this presentation of \( \text{Str}(J) \) to show that, if \( J \) is a 27-dimensional exceptional Jordan algebra, \( \text{Str}(J) \) can be seen as a Levi subgroup of a parabolic subgroup of type \( P_7 \) (with the enumeration of roots as in \cite{BI}) in an adjoint group of type \( E_7 \). See also Garibaldi \cite{Ga}.

\textbf{Lemma 1.} \textit{Let } \( J \text{ be a 27-dimensional exceptional Jordan algebra over a commutative ring } R \). \textit{There exists an adjoint simple group } \( G \text{ of type } E_7 \text{ over } R \text{ such that } \text{Str}(J) \text{ is isomorphic to a Levi subgroup } L \text{ of a maximal parabolic subgroup } P \text{ of type } P_7 \text{ in } G \).

\textit{Proof.} By \cite{Lo75} Theorem 4.6 and Lemma 4.11] for any Jordan algebra \( J \) the group \( \text{Aut}((J, J)) \) is isomorphic to a Levi subgroup of a parabolic subgroup \( P \) of a reductive group \( \text{PG}(J) \) (not necessarily connected; the definition of a parabolic subgroup extends appropriately). Moreover, \( \text{PG}(J) \cong \text{Aut}(\text{PG}(J)/P) \). If \( J \) is a 27-dimensional exceptional Jordan algebra, i.e., an Albert algebra, the group \( \text{PG}(J) \) is of type \( E_7 \) and \( P \) is a parabolic subgroup of type \( P_7 \). Let \( G \) be the corresponding adjoint group of type \( E_7 \). Then by \cite[Théoreme 1]{Dem} we have \( \text{Aut}(\text{PG}(J)/P) \cong \text{Aut}(G) \cong G \). Hence \( \text{Aut}((J, J)) \) is isomorphic to a Levi subgroup of a parabolic subgroup \( P \) of type \( P_7 \) in \( G \). \( \square \)
3. Cubic Jordan algebras and the first Tits construction

A cubic map on a projective $R$-module $V$ consists of a function $N: V \to R$ and its partial polarization $\partial N: V \times V \to R$ such that $\partial N(x, y)$ is quadratic in $x$ and linear in $y$, and $N$ is cubic in the following sense:

- $N(tx) = t^3 N(x)$ for all $t \in R$, $x \in V$;
- $N(x + y) = N(x) + \partial N(x, y) + \partial N(y, x) + N(y)$ for all $x, y \in V$.

These data allow to extend $N$ to $V_S = V \otimes_R S$ for any ring extension $S$ of $R$.

A cubic Jordan algebra is a projective module $J$ equipped with a cubic form $N$, quadratic map $\# : J \to J$ and an element $1 \in J$ such that for any extension $S/R$

- $(x^\#)^\# = N(x)x$ for all $x \in J_S$;
- $1^\# = 1; N(1) = 1$;
- $T(x^\#, y) = \partial N(x, y)$ for all $x, y \in J_S$;
- $1 \times x = T(x)1 - x$ for all $x \in J_S$,

where $\times$ is the linearization of $\#$, $T(x) = \partial N(1, x)$, $T(x, y) = T(x)T(y) - N(1, x, y)$, $N(x, y, z)$ is the linearization of $\partial N$.

There is a natural structure of a quadratic Jordan algebra on $J$ given by the formula

$$ U_{xy} = T(x, y)x - x^\# \times y. $$

Any associative algebra $A$ of degree 3 over $R$ (say, commutative étale cubic algebra or an Azumaya algebra of rank 9) can be naturally considered as a cubic Jordan algebra, with $N$ being the norm, $T$ being the trace, and $x^\#$ being the adjoint element to $x$.

Moreover, given an invertible scalar $\lambda \in R^\times$, one can equip the direct sum $A \oplus A \oplus A$ with the structure of a cubic Jordan algebra in the following way (which is called the first Tits construction):

$$ 1 = (1, 0, 0); $$

$$ N(a_0, a_1, a_2) = N(a_0) + \lambda N(a_1) + \lambda^{-1}N(a_2) - T(a_0a_1a_2); $$

$$ (a_0, a_1, a_2)^\# = (a_0^\# - a_1a_2, \lambda^{-1}a_2^\# - a_0a_1, \lambda a_1^\# - a_2a_0). $$

Now we state a transitivity result (borrowed from [PeR Proof of Theorem 4.8]) which is crucial in what follows.

**Lemma 2.** Let $E$ be a cubic étale extension of $R$, $A$ is the cubic Jordan algebra obtained by the first Tits construction from $E$, $y$ be an invertible element of $E$ considered as a subalgebra of $A$. Then $y$ lies in the orbit of $1$ under the action of subgroup of $\text{Str}(A)(R)$ generated by $G_m(R)$ and elements of the form $U_x$, $x$ is an invertible element of $A$.

**Proof.** As an element of $A$ $y$ equals $(y, 0, 0)$. Now a direct calculation shows that

$$ U_{(0, 0, 1)}U_{(0, y, 0)}y = N(y)1. $$

\[\square\]

Over a field, Jordan algebras that can be obtained by the first Tits construction can be characterized in terms of cohomological invariants. Namely, to each $J$ one associates a 3-fold Pfister form $\pi_3(J)$, and $J$ is of the first Tits construction if and only if $\pi_3(J)$ is hyperbolic (see [Pe Theorem 4.10]). Another equivalent description is that $J$ splits over a cubic extension of the base field. If the characteristic of the base field is distinct from 2, $\pi_3$ is equivalent to the cohomological $f_3$ invariant,

$$ f_3 : H^3_{\text{et}}(-, F_4) \to H^3(-, \mu_2). $$

4. Springer form

From now on $J$ is a 27-dimensional cubic Jordan algebra over $R$. 
Let $E$ be a cubic étale subalgebra of $J$. Denote by $E^\perp$ the orthogonal complement to $E$ in $J$ with respect to the bilinear form $T$ (it exists for the restriction of $T$ to $E$ is non-degenerate); it is a projective $R$-module of rank 24. It is shown in [PeR Proposition 2.1] that the operation 

$$E \times E^\perp \to E^\perp;$$

$$(a, x) \mapsto -a \times x$$

equips $E^\perp$ with a structure of $E$-module compatible with its $R$-module structure. Moreover, if we write 

$$x^\# = q_E(x) + r_E(x), \quad q_E(x) \in E^\perp, \quad r_E(x) \in E,$$

then $q_E$ is a quadratic form on $E^\perp$, which is nondegenerate as one can check over a covering of $R$ splitting $J$. This form is called the Springer form with respect to $E$.

The following lemma relates the Springer form and subalgebras of $J$.

**Lemma 3.** Let $v$ be an element of $E^\perp$ such that $q_E(v) = 0$ and $v$ is invertible in $J$. Then $v$ is contained in a subalgebra of $J$ obtained by the first Tits construction from $E$.

**Proof.** It is shown in [PeR Proposition 2.2] that the embedding 

$$(a_0, a_1, a_2) \mapsto a_0 - a_1 \times v - N(v)^{-1}a_2 \times v^\#$$

defines a subalgebra desired. \hfill \Box

Recall that the étale algebras of degree $n$ are classified by $H^1(R, S_n)$, where $S_n$ is the symmetric group in $n$ letters. The sign map $S_n \to S_2$ induces a map 

$$H^1(R, S_n) \to H^1(R, S_2)$$

that associates to any étale algebra $E$ a quadratic étale algebra $\delta(E)$ called the discriminant of $E$. The norm $N_{\delta(E)}$ is a quadratic form of rank 2. We will use later on the analog of the Grothendieck-Serre conjecture for quadratic étale algebras; it follows, for example, from [EGA Corollaire 6.1.14].

Over a field, the Springer form can be computed explicitly in terms of $\pi_3(J)$ and $\delta(E)$. We will need the following particular case:

**Lemma 4.** Let $J$ be a Jordan algebra over a field $K$ with $\pi_3(J) = 0$. Then 

$$q_E = N_{\delta(E)}_{|E} \perp h_E \perp h_E \perp h_E,$$

$h$ stands for the hyperbolic form of rank 2.

**Proof.** Follows from [PeR Theorem 3.2]. \hfill \Box

We will also use the following standard result.

**Lemma 5.** Let $J$ be a Jordan algebra over an algebraically closed field $F$. Then any two cubic étale subalgebras $E$ and $E'$ of $J$ are conjugate by an element of $\text{Aut}(J)(F)$.

**Proof.** Present $E$ as $F e_1 \oplus F e_2 \oplus F e_3$, where $e_i$ are idempotents whose sum is 1; do the same with $E'$. By [Lam] Theorem 17.1 there exists an element $g \in \text{Str}(J)(F)$ such that $ge_i = e_i'$. But then $g$ stabilizes 1, hence belongs to $\text{Aut}(J)(F)$. \hfill \Box

5. **Proof of Theorem**

**Proof of Theorem** Set $H = \text{Aut}(J)$. It is a simple group of type $F_4$ over $R$. We may assume that $H_K$ is not split, otherwise the result follows from [Pa Theorem 1.0.1]. Let $J$ be the Jordan algebra corresponding to $H$; we have to show that if $J'$ is a twisted form of $J$ such that $J'_K \simeq J_K$ then $J' \simeq J$. Set $L = \text{Str}(J)$; then $L$ is a Levi subgroup of a parabolic subgroup of type $P_2$ of an adjoint simple group scheme $G$ of type $E_7$ by Lemma [SGA Exp. XXVI Cor. 5.10 (i)] the map 

$$H^1_{\text{ét}}(R, L) \to H^1_{\text{ét}}(K, G)$$

is an isomorphism. By [SGA Exp. XXVI Cor. 5.10 (ii)] the map 

$$H^1_{\text{ét}}(R, L) \to H^1_{\text{ét}}(K, G)$$

is an isomorphism.
is injective. Since $G$ is isotropic, by \textup{[Pa]} Theorem 1.0.1 the map

$$H^1_{\text{et}}(R, G) \to H^1_{\text{et}}(K, G)$$

has trivial kernel, and so does the map

$$H^1_{\text{et}}(R, L) \to H^1_{\text{et}}(K, L).$$

But $(J'_{K}, J''_{K}) \simeq (J_{K}, J_{K})$, therefore $(J', J') \simeq (J, J)$, that is $J'$ is isomorphic to $J^{(y)}$ for some invertible $y \in J$. It remains to show that $y$ lies in the orbit of 1 under the action of $\text{Str}(J)(R)$.

Present the quotient of $R$ by its Jacobson radical as a direct product of the residue fields $\prod k_i$. An argument in \textup{[PeR]} Proof of Theorem 4.8 shows that for each $i$ one can find an invertible element $v_i \in J_{k_i}$ such that the discriminant of the generic polynomial of $U_{v_i}y_{k_i}$ is nonzero. Lifting $v_i$ to an element $v \in J$ and changing $y$ to $U_{v}y$ we may assume that the generic polynomial $f(T) \in R[T]$ of $y$ has the property that $R[T]/(f(T))$ is an étale extension of $R$. In other words, we may assume that $y$ generates a cubic étale subalgebra $E$ in $J$.

Note that $E_{K}$ is a cubic field extension of $K$; otherwise $J_{K}$ is reduced, hence split, for $\pi_{3}(J_{K}) = 0$ (see \textup{[Pa]} Theorem 4.10)). Consider the form

$$q = N_{E/K} \downarrow h_{E} \downarrow h_{E} \downarrow h_{E};$$

then by Lemma 4 $q_{K} = q_{E_{K}}$. By the analog of the Grothendieck—Serre conjecture for étale quadratic algebras, $q$ and $q_{E}$ have the same discriminant. So $q_{E}$ is a twisted form of $q$ given by a cocycle $\xi \in H^{1}(E, SO(q))$. Now $\xi_{K}$ is trivial, and \textup{[Pa]} Theorem 1.0.1 imply that $\xi$ is trivial itself, that is $q_{E} = q$. In particular, $q_{E}$ is isotropic. Let us show that there is an invertible element $v$ in $J$ such that $q_{E}(v) = 0$.

The projective quadric over $E$ defined by $q_{E}$ is isotropic, hence has an open subscheme $U \simeq \mathbb{A}_{E}^{n}$. Denote by $U'$ the open subscheme of $R_{E/R}(U)$ consisting of invertible elements. It suffices to show that $U'(k_{i})$ is non-empty for each $i$, or, since the condition on $R$ implies that $k_{i}$ is infinite, that $U'(k_{i})$ is non-empty.

But $J_{k_{i}}$ splits, and, in particular, it is obtained by a first Tits construction from a split Jordan algebra of $3 \times 3$ matrices over $k_{i}$. The diagonal matrices in this matrix algebra constitute a cubic étale subalgebra of $J_{k_{i}}$. By Lemma 5 we may assume that this étale subalgebra coincides with $E_{k_{i}}$. By \textup{[PeR]} Proposition 2.2 there exists an invertible element $v_{i} \in E_{k_{i}}^{\times}$ such that $q_{E_{k_{i}}}(v_{i}) = 0$. Thus the scheme of invertible elements intersects the quadric over $k_{i}$, hence, $U'(k_{i})$ is non-empty.

Finally, Lemma 6 and Lemma 7 show that $y$ belongs to the orbit of 1 under the group generated by $G_{2D}(R)$ and elements of the form $U_{x}$. So $J' \simeq J^{(y)} \simeq J$, and the proof is completed.

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