Indirect Influences on Directed Manifolds

Leonardo Cano and Rafael Díaz

Abstract

We introduce a program aimed to studying problems arising from the theory of complex networks with differential geometric means. We study the propagation of influences on manifolds assuming that at each point only a finite number of propagation velocities are allowed. This leads to the computation of the volume of the moduli spaces of directed paths, i.e. paths satisfying the imposed tangential restrictions. The proposed settings provide a fertile ground for research with potential applications in geometry, mathematical physics, differential equations, and combinatorics. We establish the general framework, develop its structural properties, and consider a few basic examples of relevance. The interaction between differential geometry and complex networks is a new and promising field of study.

1 Introduction

Our aim in this work is to lay down the foundations for the study of the propagation of influences on directed manifolds. Our object of study can be approached from quite different viewpoints as indicated in the following, non-exhaustive, diagram:

\[ \text{Ind. Inf. on Graphs} \quad \Rightarrow \quad \text{Ind. Inf. on Directed Manifolds} \quad \Rightarrow \quad \text{Directed Spaces} \]

Our departure point is the theory of indirect influences for weighted directed graphs which has gradually emerged thanks to the efforts of several authors, among them Brin, Chung, Estrada, Godet, Hatano, Katz, Page, Motwani, and Winograd. Although the history of the subject is yet to be written, for our purposes we may consider the introduction of the Katz’s index [18] as an early modern approach to the problem of understanding the propagation of influences in complex networks. Fundamental developments in the
field came with the introduction of the MICMAC \cite{13}, PageRank \cite{3,4}, Communicability \cite{11}, and Heat Kernel \cite{7} methods. In 2009 the second author proposed the PWP method for computing the propagation of influences on networks \cite{8}. In a nutshell the method proceeds as follows. We assume as given a network (weighted directed graph) represented by its adjacency matrix \( D \), also called the matrix of direct influences. Then one defines the matrix \( T = T(D) \) of indirect influences whose entry \( T_{ij} \) measures the weight of the indirect influences exerted by vertex \( j \) on vertex \( i \). The matrix \( T \) is computed using the following expression:

\[
T = \frac{1}{e^\lambda - 1} \sum_{n=1}^{\infty} D^n \frac{\lambda^n}{n!},
\]

where \( \lambda \) is a positive real parameter. In words: indirect influences arise from the concatenation of direct influences, and the weight of a concatenation of length \( n \) comes from the product of \( n \) entries of \( D \) and the factor \( \frac{\lambda^n}{n!} \) ensuring convergency by attaching a rapidly decreasing weight to longer chains of direct influences. The PWP method has been applied to analyse educational programs, and to study indirect influences in international trade \cite{9}. Further extensions and applications are underway. The stability of the method with respect to changes in the matrix \( D \) and the parameter \( \lambda \) has been recently studied in \cite{10}.

Our first proposal in this work is that one may regard a differential manifold provided with a tuple of vector fields on it – we call such an object a directed manifold – as being a smooth analogue of a directed graph with numbered outgoing edges attached to each vertex. Armed with this intuition we pose the question: Is there an extension of the theory of indirect influences from the discrete to the smooth settings? We argue that the answer is in the affirmative, and that such an extension both interplays with many notions already studied in the literature, e.g. control theory \cite{1,17,23,25,28}, Feynman integrals \cite{24,12,29}, and directed topological spaces \cite{14}, and also demands the introduction of new ideas.

Constructing a smooth analogue for the PWP matrix of indirect influences – whose entries are given by sums over paths in a graph – leads directly to Feynman path integrals, understood in the general sense of integrals over spaces of paths on manifolds. Although of great interest, we follow an alternative approach in order to avoid the usual difficulties that have prevented, so far, the development of a fully rigorous general integration theory over infinite dimensional manifolds. Thus, in order to reduce our computations to finite dimensional integrals, we impose strong tangential conditions on the allowed paths in our domains of integration.
The background upon which we develop our constructions is the category of directed manifolds, introduced in Section 2, which is also a convenient category for studying geometric control theory. Our constructions bring about a new set of problems to geometric control theory – usually focus on the path reachability and path optimization problems – namely, the problem of computing integrals over the moduli spaces of directed paths. We remark again that strong tangency conditions are imposed in order to insure that the moduli spaces of directed paths – also called the spaces of indirect influences – split naturally into infinitely many finite dimensional pieces, each coming with a natural measure. Thus we have a notion of integration over each piece, which we extend additively to the whole moduli space of directed paths, leaving the convergency of these sums to a case by case analysis. Fortunately, in our examples we do obtain convergent sums. These ideas are developed in Section 3 where we also introduce the wave of influences \( u(p, t) \) which computes the total influence received by a point \( p \) in time \( t \), i.e. \( u(p, t) \) computes the volume of the moduli space of directed paths starting at an arbitrary point and ending up at \( p \) in time \( t \).

Our notion of directed manifolds is strongly related to the notion of directed spaces introduced by Grandis [14], and to some extend the former notion may be regarded as a smooth analogue of the latter. In Section 3 we make this connection precise.

In Section 4 we discuss invariant properties for directed manifolds and for the moduli spaces of directed paths on them. We also study invariant properties with respect to reordering of our given tuple of vectors fields. We propose a possible route for using our spaces of indirect influences to approach integrals with more general domains of integration, such as Feynman path integrals. Whether this approach can actually be implemented to work as a viable computational technique is left for future research. In Section 4 we study the moduli spaces of directed paths on product and quotient of directed manifolds.

In Section 6 we study the moduli spaces of directed paths arising from constant vector fields on affine spaces. We show that even in this case, the simplest one, our theory yields results worth studying where explicit computations are available. These settings give rise to fruitful constructions in combinatorics and probability theory [5].

Finally, in the closing Section 7 we indicate how our general settings for computing indirect influences, based on the computation of the volume of moduli spaces of directed paths, can be extended to the quantum context adopting a Hamiltonian viewpoint.
Notation. For $n \in \mathbb{N}$, we set $[n] = \{1, \ldots, n\}$, $[0, n] = \{0, \ldots, n\}$, and let $P[n]$ be the set of subsets of $[n]$. The amalgamated sum of closed subintervals of the real line $\mathbb{R}$ is given by

$$[a, b] \coprod_{b,c} [c, d] = [a, b + d - c].$$

We let $\delta_{ab}$ be the Kronecker’s delta function.

2 Basic Definitions

We let $\diman$ be the category of directed manifolds. A directed manifold is a tuple $(M, v_1, \ldots, v_k)$ where $M$ is a smooth manifold, and $v_1, \ldots, v_k$ are smooth vector fields on $M$, with $k \geq 1$. A morphism $(f, \alpha) : (M, v_1, \ldots, v_k) \to (N, w_1, \ldots, w_l)$ in $\diman$ is a pair $(f, \alpha)$ where $f : M \to N$ is a smooth map, $\alpha : [k] \to [l]$ is a map, and the following identity holds

$$df(v_i) = w_{\alpha(i)}, \quad \text{for } i \in [k].$$

Let $(g, \beta) : (N, w_1, \ldots, w_l) \to (K, z_1, \ldots, z_r)$ be another morphism. The composition morphism $(g, \beta) \circ (f, \alpha)$ is given by:

$$(g, \beta) \circ (f, \alpha) = (g \circ f, \beta \circ \alpha).$$

It satisfies the required property since

$$d(gf)(v_i) = dg(df(v_i)) = dg(w_{\alpha(i)}) = z_{\beta(\alpha(i))} = z_{\beta \alpha(i)}.$$

One can think of a directed manifold $(M, v_1, \ldots, v_k)$ as being a smooth analogue of a "finite directed graph with up to $k$ outgoing numbered edges at each vertex". Points in the manifold $M$ are thought as vertices in the smooth graph. The tangent vectors $v_i(p) \in T_pM$ are thought as infinitesimal edges starting at $p$. The out-degree of a vertex $p \in M$ is the number of non-zero infinitesimal edges starting at $p$, i.e. the cardinality of the set $\{i \in [k] \mid v_i(p) \neq 0\}$.

An actual edge from $p$ to $q$, points in $M$, is a smooth path $\varphi : [0, t] \to M$ with $\varphi(0) = p$, $\varphi(t) = q$, and such that the tangent vector at each point of $\varphi$ is an infinitesimal edge, i.e. $\dot{\varphi} = v_i(\varphi)$ for some $i \in [k]$, or more explicitly

$$\varphi(s) = v_i(\varphi(s)) \quad \text{for all } s \in [0, t].$$

We say that $p$ exerts a direct influence, in time $t > 0$, on the vertex $q$ through the path $\varphi$. Note that $\varphi$ is determined by the initial point $p$, and the index $i$ of vector field $v_i$, thus we are entitled to use the notation $\varphi(t) = \varphi_i(p, t)$, where $\varphi_i$ is the flow
generated by the vector field \( v_i \).

**Definition 1.** Let \((M, v_1, ..., v_k)\) be a directed manifold and \( p, q \in M \). The set of one-direction paths \( D_{p,q}(t) \) from \( p \) to \( q \) developed in time \( t > 0 \) is given by

\[
D_{p,q}(t) = \{i \in [k] \mid \varphi_i(p, t) = q\}.
\]

We also set

\[
D_{p,q}(0) = \begin{cases} 
\{p\} & \text{if } p = q, \\
\emptyset & \text{otherwise,}
\end{cases}
\]

i.e. each point of \( M \) exerts a direct influence over itself in time \( t = 0 \), and there are no \( t = 0 \) direct influences between different points of \( M \). Thus \( D_{p,q} \) defines a map \( D_{p,q} : \mathbb{R}_{\geq 0} \rightarrow P[k] \). We also say that \( D_{p,q}(t) \) is the set of direct influences from \( p \) to \( q \) exerted in time \( t > 0 \).

There might also be one-direction paths from \( p \) to \( q \) taking an infinite long interval of time to be exerted, these influences occur through a path \( \varphi : \mathbb{R} \rightarrow M \) such that

\[
\lim_{t \to -\infty} \varphi(t) = p \quad \text{and} \quad \lim_{t \to \infty} \varphi(t) = q.
\]

Semi-infinite direct influences can be similarly defined. One might also consider topological direct influences from \( p \) to \( q \) which are exerted through a path \( \varphi : \mathbb{R} \rightarrow M \) such that \( p \in \omega \lim_{t \to -\infty} \varphi(t) \) and \( q \in \omega \lim_{t \to \infty} \varphi(t) \). We will no further consider one-direction paths of these types in this work.

Next we introduce the notion of indirect influences which arise from the concatenation of direct influences. Our focus is on finding a convenient parametrization for the space of all such concatenations.

**Definition 2.** Let \((M, v_1, ..., v_k)\) be a directed manifold and \( p, q \in M \). A directed path from \( p \) to \( q \) displayed in time \( t > 0 \) through \( n \geq 0 \) changes of directions is given by a pair \((c, s)\) with the following properties:

- \( c = (c_0, c_1, ..., c_n) \) is a \((n + 1)\)-tuple with \( c_i \in [k] \) and such that \( c_i \neq c_{i+1} \). We say that \( c \) defines the pattern (of directions) of the directed path \((c, s)\). We let \( D(n, k) \) be the set of all such tuples, and \( l(c) = n + 1 \) be the length of \( c \). There are \( k(k - 1)^n \) different patterns in \( D(n, k) \). Note that we may regard a pattern \( c \) as a map \( c : [0, n] \rightarrow [k] \).

- \( s = (s_0, ..., s_n) \) is a \((n + 1)\)-tuple with \( s_i \in \mathbb{R}_{\geq 0} \) and such that \( s_0 + \cdots + s_n = t \). We say that \( s \) defines the time distribution of the directed path \((c, s)\), and let \( \Delta_n^t \) be the \( n \)-simplex of all such tuples.
The pair \((c, s)\) determines a \((n + 2)\)-tuple of points \((p_0, \ldots, p_{n+1}) \in M^{n+2}\) given by:

\[
p_0 = p \quad \text{and} \quad p_i = \varphi_{c_{i-1}}(p_{i-1}, s_{i-1}) \quad \text{for} \quad 1 \leq i \leq n + 1,
\]

where \(\varphi_{c_{i-1}}\) is the flow generated by the vector field \(v_{c_{i-1}}\). We denote the last point \(p_{n+1}\) by \(\varphi_c(p, s)\).

- The pair \((c, s)\) must be such that \(\varphi_c(p, s) = q\).
- Directed paths in time \(t = 0\) are the same as one-direction paths in time \(t = 0\).

**Remark 3.** By definition directed paths include one-direction paths as well, even for the conventional case \(t = 0\). We also say that \((c, s)\) determines an indirect influence from \(p\) to \(q\) exerted in time \(t\). The fact that our paths are displayed in non-negative time means that indirect influences propagate forward in time.

**Remark 4.** The geometric meaning of directed paths is made clear through the following construction. A pair \((c, s)\) as above determines a piece-wise smooth path

\[
\varphi_{c,s} : [0, s_0 + \cdots + s_n] \simeq [0, s_0] \bigsqcup \cdots \bigsqcup [0, s_n] \rightarrow M
\]

such that the restriction of \(\varphi_{c,s}\) to the interval \([0, s_i]\), for \(0 \leq i \leq n\), is given by

\[
\varphi_{c,s}|_{[0, s_i]}(r) = \varphi_{c_i}(p_i, r) \quad \text{for all} \quad r \in [0, s_i].
\]

We say that \(\varphi_{c,s}\) is the directed path determined by the pair \((c, s)\). Indirect influences are exerted through such directed paths. Whenever necessary we write \(\varphi_{v,c,s}\) instead of \(\varphi_{c,s}\) to make explicit that these paths do depend on the vector fields \(v = (v_1, \ldots, v_k)\). Figure 1 shows the directed path associated to a pair \((c, s)\).

![Figure 1. Directed path associated to a pair \((c, s)\).](image-url)
Note that directed paths in the sense above are examples of horizontal paths as defined in geometric control theory [1].

**Remark 5.** Our notion of indirect influences on directed manifolds may be regarded as a limit case of the propagation of disturbances in geometric optics, see Arnold [2]. In geometric optics one works with a Riemannian manifold $M$, and is given a map $v : SM \rightarrow \mathbb{R}_{\geq 0}$ from the unit sphere bundle of $M$ to the non-negative real numbers. The number $v(l)$ gives the speed allowed for the propagation of a disturbance along the direction $l$. Indirect influences on a directed manifold $(M, v_1, ..., v_k)$ correspond to the propagation of disturbances in geometric optics, if one lets $v$ be the singular map that is zero everywhere except at the directions defined by the vector fields $v_j$, and on this directions it assumes the values $|v_j|$. Note that the notion of indirect influences does not demand a Riemannian structure on $M$. Figure 2 illustrates the relation between indirect influences and geometric optics, by displaying the deformation of the indicatrix surface (the image of $v$) from a smooth ellipses to a curve concentrated on three vectors.

![Figure 2. Indirect influences as a limit case of propagations in geometric optics.](image)

**Remark 6.** Although not strictly necessary, for simplicity we usually assume that the flows generated by the vector fields $v_j$ are globally defined by smooth maps

$$\varphi_j(\cdot, s) : M \times \mathbb{R} \rightarrow M$$

yielding a one-parameter group of diffeomorphisms of $M$:

- The map $\varphi_j(\cdot, s) : M \rightarrow M$ is a diffeomorphism for all $s \in \mathbb{R}$.
- The group condition $\varphi_j(\varphi_j(p, s_1), s_2) = \varphi_j(p, s_1 + s_2)$ holds for $s_1, s_2 \in \mathbb{R}$. 


A pattern \( c \in D(n, k) \) defines an iterated flow given by the smooth map
\[
\varphi_c : M \times \mathbb{R}^{n+1} \longrightarrow M
\]
defined by recursion on the length of \( c \) as follows:
\[
\varphi_c(p, s_0, \ldots, s_n) = \varphi_{c_n}(\varphi_{c_{[0,n-1]}}(p, s_0, \ldots, s_{n-1}), s_n).
\]
Fixing a time distribution \( (s_0, \ldots, s_n) \) we obtain the diffeomorphism
\[
\varphi_c(\cdot, s_0, \ldots, s_n) : M \longrightarrow M.
\]
These constructions justify the notation \( \varphi_c(p, s) \) for the point \( p_{n+1}(c, s) \) introduced in Definition 8.

We regard the \( n \)-simplex \( \Delta^n_t \) introduced in Definition 8 as a smooth manifold with corners. There are at least three different approaches to differential geometry on manifolds with corners. First we can apply differential geometric notions on the interior of \( \Delta^n_t \). Also it is possible to introduce differential geometric objects on \( \Delta^n_t \) by considering objects that are smooth on an open neighborhood of \( \Delta^n_t \) in \( \mathbb{R}^{n+1} \). A third and more intrinsic approach for doing differential geometry on \( \Delta^n_t \) relies on deeper results in the theory of manifolds with corners. For a fresh approach the reader may consult [16]. Although this more comprehensive approach is certainly desirable, for simplicity, we will not further consider it.

**Proposition 7.** For a pattern \( c \in D(n, k) \), the map
\[
\varphi_c : M \times \Delta^n_t \longrightarrow M
\]
sending a pair \( (p, s) \in M \times \Delta^n_t \) to the point \( \varphi_c(p, s) \in M \) is a smooth map and a diffeomorphism for a fixed time distribution \( s \in \Delta^n_t \).

Next we introduce the main objects of study in this work, namely, the moduli spaces of directed path, also called the spaces of indirect influences, on directed manifolds. These spaces parametrize directed paths from a given point to another.

**Definition 8.** Let \( (M, v_1, \ldots, v_k) \) be a directed manifold and \( p, q \in M \). The moduli space \( \Gamma_{p,q}(t) \) of directed paths from \( p \) to \( q \) developed in time \( t > 0 \) is given by
\[
\Gamma_{p,q}(t) = \left\{(c, s) \mid \varphi_c(p, s) = q\right\} = \bigcap_{n=0}^{\infty} \bigcap_{c \in D(n,k)} \{s \in \Delta^n_t \mid \varphi_c(p, s) = q\} = \bigcap_{n=0}^{\infty} \bigcap_{c \in D(n,k)} \Gamma^c_{p,q}(t).
\]
In addition we set
\[ \Gamma_{p,q}(0) = \Gamma_{p,q}^0(0) = \begin{cases} \{p\} & \text{if } p = q, \\ \emptyset & \text{otherwise}. \end{cases} \]

Figure 3 shows a schematic picture of a component \( \Gamma^c_{p,q}(t) \) of the moduli space of indirect influences.

Figure 3. Moduli space of directed paths \( \Gamma^c_{p,q}(t) \).

**Remark 9.** For a fixed pattern \( c \) the continuity of the iterated flow \( \varphi_c(p, \ ) \) implies that the moduli space of directed paths \( \Gamma^c_{p,q}(t) \) is compact, as it is a closed subspace of \( \Delta^t_n \).

The moduli spaces of directed paths come equipped with the structure of a category. Indeed directed paths are pretty close of being the free category generated by one-direction path, but not quite since we have ruled out repeated directions.

**Theorem 10.** Altogether the moduli spaces of directed paths on a directed manifold form a topological category.

**Proof.** Given a directed manifold \((M, v_1, \ldots, v_k)\) we let \( \Gamma = \Gamma(M, v_1, \ldots, v_k) \) be the category of directed paths on \( M \). The objects of \( \Gamma \) are the points of \( M \). Given \( p, q \in M \), the space of morphisms in \( \Gamma \) from \( p \) to \( q \) is given by
\[ \Gamma_{p,q} = \coprod_{n \in \mathbb{N}} \coprod_{c \in D(n,k)} \Gamma^c_{p,q} \quad \text{where} \quad \Gamma^c_{p,q} = \{(s,t) \in \mathbb{R}^{n+2}_{\geq 0} \mid s \in \Delta^t_n, \ \varphi_c(p, s) = q\}. \]

In order to define continuous composition maps \( \circ : \Gamma_{p,q} \times \Gamma_{q,r} \rightarrow \Gamma_{p,r} \), it is enough to define componentwise composition maps
\[ \circ : \Gamma^c_{p,q} \times \Gamma^d_{q,r} \rightarrow \Gamma^{csd}_{p,r} \]
for given patterns \( c \) and \( d \) with \( n = l(c) \) and \( m = l(d) \). We consider two cases:
• If \( c_n \neq d_0 \), then \( c \ast d = (c, d) \) and 
\[
(s_0, \ldots, s_n) \circ (u_0, \ldots, u_m) = (s_0, \ldots, s_n, u_0, \ldots, u_m).
\]
• If \( c_n = d_0 \), then \( c \ast d = (c_0, \ldots, c_n) \ast (d_0, \ldots, d_m) = (c_0, \ldots, c_n, d_1, \ldots, d_m) \) and 
\[
(s_0, \ldots, s_n) \circ (u_0, \ldots, u_m) = (s_0, \ldots, s_n + u_0, \ldots, u_m).
\]

These compositions are well-defined continuous maps satisfying the associative property. The unique \( t = 0 \) directed path from \( p \in M \) to itself gives the identity morphism for each object \( p \in \Gamma \).

\[\square\]

**Remark 11.** The moduli spaces of directed paths \( \Gamma_{p,q}(t) \) can be extended from points to arbitrary subsets of \( M \) as follows. Given \( A, B \subseteq M \) we define the moduli space of directed paths from \( A \) to \( B \) as 
\[
\Gamma_{A,B}(t) = \left\{ (c, s) \mid p \in A, \varphi_c(p, s) \in B \right\} = 
\prod_{n=0}^{\infty} \prod_{c \in D(n,k)} \left\{ s \in \Delta^t_n \mid p \in A, \varphi_c(p, s) \in B \right\} = 
\prod_{n=0}^{\infty} \prod_{c \in D(n,k)} \Gamma^c_{A,B}(t).
\]

Restricting attention to embedded oriented submanifolds of \( M \), and following techniques from Chas and Sullivan’s string topology [6], this construction gives rise to some kind of transversal category.

We close this section introducing a few subsets of \( M \) useful for understanding the propagation of influences on \( M \). These sets are usually called the reachable sets in geometric control theory, and are natural generalizations of the corresponding graph theoretical notions. They also play a prominent role in general relativity [22]. For \( A \subseteq M \) we set:

• \( \Gamma_A(t) = \{ q \in M \mid \Gamma_{A,q}(t) \neq \emptyset \} \) is the set of points in \( M \) influenced by \( A \) in time \( t \).

• \( \Gamma_{A,\leq}(t) = \{ q \in M \mid \text{there is } 0 \leq s \leq t, \text{ such that } \Gamma_{A,q}(s) \neq \emptyset \} \) is the set of points in \( M \) influenced by \( A \) in time less or equal to \( t \).

• \( \Gamma_A = \{ q \in M \mid \Gamma_{A,q}(t) \neq \emptyset \text{ for some } t \geq 0 \} \) is the set of points in \( M \) that are influenced by \( A \).

• \( \Gamma_A^{-}(t) = \{ q \in M \mid \Gamma_{q,A}(t) \neq \emptyset \} \) is the set of points in \( M \) that influence \( A \) in time \( t \), i.e. the set of points on which \( A \) depends on time \( t \).
• $\Gamma_{A,\leq}(t) = \{ q \in M \mid \text{there is } 0 \leq s \leq t, \text{such that } \Gamma_{q,A}(s) \neq \emptyset \}$ is the set of points in $M$ that influence $A$ in time less or equal to $t$.

• $\Gamma_{A} = \{ q \in M \mid \Gamma_{q,A}(t) \neq \emptyset \text{ for some } t \geq 0 \}$ is the set of points in $M$ that influence $A$.

• $F_A(t) = \partial \Gamma_{A,\leq}(t)$ and $F_A^-(t) = \partial \Gamma_{A}^-(t)$ are called, respectively, the front of influence and the front of dependence of $A$ in time $t$.

Note that a directed manifold $M$ is naturally a pre-poset by setting

$$p \leq q \quad \text{if and only if} \quad q \in \Gamma_p.$$ 

The associated poset is the quotient space $M_{\sim}$, where the equivalence relation $\sim$ on $M$ is given by

$$p \sim q \quad \text{if and only if} \quad q \in \Gamma_p \text{ and } p \in \Gamma_q.$$ 

The space $M_{\sim}$ tell us how $M$ splits into components of co-influences, i.e. the path connected components of $M$ through directed paths.

Note that a directed manifold $(M, v_1, \ldots, v_k)$ comes equipped with a natural distribution, indeed for each point $p \in M$ we have the subspace

$$< v_1(p), \ldots, v_k(p) > \subseteq T_p M$$

generated by the vectors $v_1(p), \ldots, v_k(p)$. If this distribution is integrable, then directed paths are confined to live on the leaves. Thus to study the moduli spaces of directed paths, in the integrable case, we may as well forget about the manifold $M$ and work leaf by leaf. So the interesting cases of study are:

• $< v_1(p), \ldots, v_k(p) > = T_p M$, i.e. $M$ itself is the unique leaf.

• The distribution $< v_1(p), \ldots, v_k(p) > \subseteq T_p M$ is not integrable.

3 Measuring the Moduli Spaces of Directed Paths

Fix a directed manifold $(M, v_1, \ldots, v_k)$. In order to measure directed paths on $M$ we assume from now on that an orientation on $M$ has been chosen. To gauge the amount of indirect influences exerted, in time $t$, by a point $p \in M$ on a point $q \in M$ we need to define measures on the moduli spaces $\Gamma_{p,q}(t)$ of directed paths. From Definition [8]
we see that $\Gamma_{p,q}(t)$ is a disjoint union of pieces, one for each pattern $c \in D(n,k)$, of the form

$$\Gamma^c_{p,q}(t) = \{ s \in \Delta^t_n \mid \varphi_c(p, s) = q \}.$$ 

So, our problem reduces to imposing measures on the pieces $\Gamma^c_{p,q}(t)$.

The $n$-simplex $\Delta^t_n$ is a smooth manifold with corners, and comes equipped with a Riemannian metric and its associated volume form. Indeed using Cartesian coordinates

$$l_1 = s_0, \quad l_2 = s_0 + s_1, \quad \ldots \ldots, \quad l_n = s_0 + \cdots + s_{n-1}$$

the $n$-simplex can be identified with the following subset of $\mathbb{R}^n$:

$$\Delta^t_n = \{ (l_1, \ldots, l_n) \in \mathbb{R}^n \mid 0 \leq l_1 \leq l_2 \leq \ldots \leq l_n \leq t \}.$$

Thus $\Delta^t_n$ inherits a Riemannian metric, an orientation, and the corresponding volume form $dl_1 \wedge \cdots \wedge dl_n$. With this measure we have that

$$\text{vol}(\Delta^t_n) = \frac{t^n}{n!} \quad \text{for} \quad n \geq 0.$$

**Definition 12.** A directed manifold $(M, v_1, \ldots, v_k)$ has smooth spaces of directed path if for any pattern $c \in D(n,k)$ and points $p, q \in M$ the space of indirect influences $\Gamma^c_{p,q}(t)$ is a smooth embedded sub-manifold of $\Delta^t_n$.

For our next result we use the implicit function theorem for manifolds [15, 26]. Let $f : N \rightarrow M$ be a smooth map between differential manifolds and fix $q \in M$. Then $f^{-1}(q)$ is a smooth sub-manifold of $N$ if for each $p \in f^{-1}(q)$ the linear map

$$d_pf : T_pN \rightarrow T_qM$$

has maximal rank, that is

$$\text{rank}(d_pf) = \text{min}\{\dim(N), \dim(M)\}.$$ 

If $\text{rank}(d_pf) = \dim(N)$, then $d_pf$ is injective, $f$ is an immersion, and $f^{-1}(q)$ is a set of isolated points. If $\text{rank}(d_pf) = \dim(M)$, then $d_pf$ is surjective, $f$ is a submersion, and $f^{-1}(q)$ is a sub-manifold of $N$ of dimension $\dim(M) - \dim(N)$.

Next we apply this result to the open part of manifolds with corners.

**Theorem 13.** Let $(M, v_1, \ldots, v_k)$ be a directed manifold. Fix a pattern $c \in D(n,k)$ with $n \geq 1$, and a point $p \in M$. If for any $(s_0, \ldots, s_n)$ in the open part of $\Gamma^c_{p,q}(t)$
there are \( \min(n, \dim(M)) \) linearly independent vectors among the vectors given for 
\( i \in [0, n - 1] \) by 
\[
\left. d^M \varphi_{(c_{i+1}, \ldots, c_n)} \right|_{(s_{i+1}, \ldots, s_n)} \left[ v_{c_i}(\varphi_{c_0, \ldots, c_i}(s_0, \ldots, s_i)) \right] - v_{c_n}(\varphi_{c}(s_0, \ldots, s_n)) \in T_{\varphi_c(p, s_0, \ldots, s_n)},
\]
then \( \Gamma_{p,q}^c(t) \) is a smooth sub-manifold of \( \Delta_n^t \).

**Proof.** Fix \( c \in D(n, k) \) with \( n \geq 1 \). Recall from Remark 6 that \( \varphi_c : M \times \mathbb{R}^{n+1} \to M \) is the iterated flow associated to \( c \). The differential of \( \varphi_c \) naturally split as:
\[
d\varphi_c = d^M \varphi_c + d\mathbb{R}^{n+1} \varphi_c.
\]
Consider the map \( \phi : \Delta_n^t \to M \) given by
\[
\phi(s) = \phi(s_0, \ldots, s_{n-1}) = \varphi_c(p, s_0, \ldots, s_{n-1}, t - s_0 - \cdots - s_{n-1}),
\]
where we are using the identification
\[
\Delta_n^t = \left\{ s = (s_0, \ldots, s_{n-1}) \in \mathbb{R}_{\geq 0} \mid |s| = s_0 + \cdots + s_{n-1} \leq t \right\}.
\]
In order to guarantee that \( \Gamma_{p,q}^c(t) = \phi^{-1}(p) \) is a smooth sub-manifold of \( \Delta_n^t \) we impose the condition that \( d_s \phi \) has maximal rank for \( s \in \phi^{-1}(p) \). Next we compute for \( i \in [0, n - 1] \) the vectors
\[
\frac{\partial \phi}{\partial s_i}(s) = d_s \phi(\frac{\partial}{\partial s_i}) \in T_{\phi(s)} M.
\]
Using the identity
\[
\frac{\partial}{\partial s_n}(\varphi_{c_0, \ldots, c_n})(p, s_0, \ldots, s_n) = v_{c_n}(\varphi_{c_1, \ldots, c_n}(p, s_0, \ldots, s_n)) = v_{c_n}(\varphi_{c_0, \ldots, c_n}(s_0, \ldots, s_n)),
\]
one can show that \( \frac{\partial \phi}{\partial s_i}(s) \) is given by
\[
d^M \varphi_{c_{i+1}, \ldots, c_n} \left|_{(s_{i+1}, \ldots, s_n)} \right. \left[ v_{c_i}(\varphi_{c_0, \ldots, c_i}(s_0, \ldots, s_i)) \right] - v_{c_n}(\varphi_{c_0, \ldots, c_n}(s_0, \ldots, s_n)),
\]
where we recall that \( s_n = t - |s| \),
\[
d^M \varphi_{c_{i+1}, \ldots, c_n} = d\varphi_{c_n}(t - |s|) \circ \cdots \circ d^M \varphi_{c_{i+1}}(s_{i+1}), \quad \text{and}
\]
\[
\varphi_{c_0, \ldots, c_i}(s_0, \ldots, s_i) = \varphi_c[\varphi_{c_0, \ldots, c_{i-1}}(s_0, \ldots, s_{i-1}), s_i] \quad \text{for} \quad i \geq 1.
\]
Thus the rank of \( d_s \phi \) is maximal at each point \( s \in \phi^{-1}(q) \) if and only if there are exactly \( \min(n, \dim(M)) \) linearly independent vectors among the vectors \( \frac{\partial \phi}{\partial s_i}(s) \) given by the expression above. We have shown the desired result. \( \square \)
Corollary 14. Under the hypothesis of Theorem 13, the interior of the moduli space \( \Gamma_{p,q}(t) \) is an oriented Riemannian sub-manifold of \( \Delta^t_n \).

Proof. We use oriented differential intersection theory as developed by Guillemin [15]. Since \( \Gamma_{p,q}(t) \) is a smooth sub-manifold of \( \Delta^t_n \) it acquires by restriction a Riemannian metric. The orientation on \( \Gamma_{c}(t) \) arises as follows. For \( s \in \Gamma_{c}(t) \) write

\[
T_s \Delta^t_n \simeq N_s \Gamma_{p,q}(t) \oplus T_s \Gamma_{p,q}(t),
\]

where \( N_s \Gamma_{p,q}(t) \simeq T_s \Delta^t_n / T_s \Gamma_{c}(t) \) is the normal bundle of \( \Gamma_{c}(t) \). Note that

\[
d_s \phi(T_s \Delta^t_n) = T_{\phi(s)} M \quad \text{and thus} \quad d_s \phi : N_s \Gamma_{p,q}(t) \to T_{\phi(s)} M \quad \text{is an isomorphism.}
\]

Since \( T_s \Delta^t_n \) is oriented, and \( N_s \Gamma_{p,q}(t) \) acquires an orientation from the isomorphism above, then \( T_s \Gamma_{c}(t) \) naturally acquires an orientation.

For a directed manifold with a smooth moduli space of directed paths each piece \( \Gamma_{p,q}(t) \subseteq \Delta^t_n \) acquires from \( \Delta^t_n \) a Riemannian metric. If in addition we assume that each piece \( \Gamma_{c}(t) \) is given an orientation, then \( \Gamma_{p,q}(t) \) acquires a volume form denoted by \( dl_c \). As we have just shown this is the situation arising from the conditions of Theorem 13.

We are ready to highlight a few functions on the moduli spaces of directed paths, for a fix a time \( t > 0 \), that one would like to integrate against these measures.

1. Volume of Moduli Space of Directed Paths.

Each component \( \Gamma_{p,q}(t) \) of the space of indirect influences is compact and thus of bounded volume. We define the volume or total measure of \( \Gamma_{p,q}(t) \), leaving convergency issues to be discussed on a case by case basis, as follows:

\[
\text{vol}(\Gamma_{p,q}(t)) = \int_{\Gamma_{p,q}(t)} 1 \, dl = \sum_{n=1}^{\infty} \sum_{c \in D(n,k)} \int_{\Gamma_{p,q}(t)} dl_c = \sum_{n=1}^{\infty} \sum_{c \in C(n,k)} \text{vol}(\Gamma_{c}(t)).
\]

2. Functions on directed paths coming from differential 1-forms on \( M \).

Let \( A \) be a differential 1-form on \( M \). We formally write

\[
\int_{\Gamma_{p,q}(t)} \hat{A} \, dl = \sum_{n=1}^{\infty} \sum_{c \in D(n,k)} \int_{\Gamma_{c}(t)} \hat{A} \, dl_c,
\]

where the map \( \hat{A} : \Gamma_{c}(t) \to \mathbb{R} \) is given by

\[
A(c,s) = \int_0^t \varphi^s_{c,s} A = \sum_{i=0}^{l(c)} \int_0^{s_i} \varphi_{c,s}[0,s_i] A,
\]
with \( \varphi_{c,s} : [0, s_0 + \cdots + s_n] \rightarrow M \) the directed path associated to \( (c, s) \in \Gamma_{p,q}(t) \).

3. Functions on directed paths from Riemannian metrics on \( M \).

Let \( g \) be a Riemannian metric on \( M \). We formally write
\[
\int_{\Gamma_{p,q}(t)} e^{-l_g} \, dl = \sum_{n=1}^{\infty} \sum_{c \in D(n,k)} \int_{\Gamma_{p,q}(t)} e^{-l_g} \, dl_c,
\]
where \( e^{-l_g} : \Gamma_{p,q}(t) \rightarrow \mathbb{R} \) is the map given by \( e^{-l_g}(c, s) = e^{-l_g(\varphi_{c,s})} \) and \( l_g(\varphi_{c,s}) \) is the length of the path \( \varphi_{c,s} \), i.e.:
\[
l_g(\varphi_{c,s}) = \sum_{i=0}^{l(c)} l_g(\varphi_{c,s}|[0,s_i]) = \sum_{i=0}^{l(c)} \int_{0}^{s_i} g(v_{c_i}(\varphi_{c_i}(p_i, u)), v_{c_i}(\varphi_{c_i}(p_i, u))) du.
\]

4. Functions on direct paths from functions on \( M \).

Given a smooth map \( f : M \rightarrow \mathbb{R} \) we formally write
\[
\int_{\Gamma_{p,q}(t)} \hat{f} \, dl = \sum_{n=1}^{\infty} \sum_{c \in D(n,k)} \int_{\Gamma_{p,q}(t)} f(p_0) \cdots f(p_n) \, dl_c,
\]
with \( p_0 = p \) and \( p_{i+1} = \varphi_{c_i}(p_i, s_i) \) for \( 0 \leq i \leq n \).

5. Functions on directed paths from Lagrangian functions on \( TM \).

Let \( L : TM \rightarrow \mathbb{R} \) be a Lagrangian map. In the applications \( L \) is usually built from a Riemannian metric \( g \) on \( M \) and a potential map \( U : M \rightarrow \mathbb{R} \) as follows:
\[
L(p, v) = g(v, v) - U(p).
\]

Given a Lagrangian \( L \) we consider the following analogue of the Feynman integrals:
\[
\int_{\Gamma_{p,q}(t)} e^{\pm S} \, dl = \sum_{n=1}^{\infty} \sum_{c \in D(n,k)} \int_{\Gamma_{p,q}(t)} e^{\pm S} \, dl_c,
\]
where we set \( e^{\pm S}(c, s) = e^{\pm S_{c,s}} \), and the action map \( S \) is given by
\[
S(c, s) = \int_{0}^{t} L(\varphi_{c,s}(u), \dot{\varphi}_{c,s}(u)) \, du = \sum_{i=0}^{l(c)} \int_{0}^{s_i} L(\varphi_{c,s}|[0,s_i](u), \dot{\varphi}_{c,s}|[0,s_i](u)) \, du.
\]
This example both reveals the relations and differences between our constructions and Feynman integrals. Whereas in the latter arbitrary paths are taken into account, with our methods only paths with speeds and directions prescribed by the vector fields $v_1, \ldots, v_k$ are allowed. Also, instead of looking for a measure on the space of all paths, we first decompose our space of paths into several pieces, and then impose a measure on each piece. Fortunately, each piece is finite dimensional and thus we have at our disposal the usual techniques coming from Riemannian geometry. Convergency of the sum of the integrals over each piece is to be studied in a case by case fashion.

**Remark 15.** In our examples we have found that the infinite sums defining the integrals above are actually convergent. Nevertheless, convergency is not a built-in property and should not be expected in general. To improve convergency properties one may look at the exponential generating series instead. For example, the vol function defined above can be replaced by the function $\text{vol}_\lambda$, with $\lambda$ a positive real parameter, defined as follows:

$$\text{vol}_\lambda(\Gamma_{p,q}(t)) = \sum_{n=1}^{\infty} \left( \sum_{c \in D(n,k)} \text{vol}(\Gamma_{p,q}^c(t)) \right) \frac{\lambda^n}{n!}.$$  

Clearly, this technique can be applied as well to the other quantities defined above. Moreover, if necessary, we may regard $\lambda$ as a formal parameter.

We have shown how to construct and integrate functions on the moduli spaces of directed paths on directed manifolds. So let us pick one such a function and call it $g$. Integrating over the moduli spaces of directed paths we obtain the kernel for the propagation of influences $k : M \times M \times \mathbb{R} \rightarrow \mathbb{R}$ which is given by

$$k(p, q, t) = \int_{\Gamma_{p,q}(t)} g \ dl.$$  

**Definition 16.** Let $(M, v_1, \ldots, v_k)$ be a directed manifold with smooth moduli space of directed paths. $M$ is given a Riemannian metric, and thus it acquires a volume form. Let $f : M \rightarrow \mathbb{R}$ be a map representing the density of influences originated at time $t = 0$. Let $g$ be a map on directed paths, and consider its associated kernel of influences $k = k_g$. The wave of influences $u : M \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is the map given by

$$u(q, t) = \int_{p \in \Gamma_{q}^{-}(t)} k(p, q, t) f(p) \, dp,$$

where we assume that $\Gamma_{q}^{-}(t)$ is a compact oriented smooth sub-manifold of $M$; thus it acquires by restriction a Riemannian metric, and comes with a volume form $dp$.  

16
Let us consider a couple of examples.

- Let \( g \) be the map constantly equal to 1, we have that
  \[
  u(q, t) = \int_{p \in \Gamma_q^M(t)} \text{vol}(\Gamma_{p,q}(t)) f(p) \, dp.
  \]

- For \( g = e^{i\pi S} \) where \( S \) is the action defined by a Lagrangian map, we have that
  \[
  u(q, t) = \int_{p \in \Gamma_q^M(t)} \int_{\Gamma_{p,q}(t)} e^{i\pi S} f(p) \, dldp.
  \]

4 Invariance, Involution, and Limit Properties

Let \((M, v_1, ..., v_k)\) be a directed manifold and \( f : M \rightarrow N \) be a diffeomorphism. Then we obtain the directed manifold

\[(N, f_*v_1, ..., f_*v_k)\]

where the push-forward vector fields \( f_*v_i \) are given for \( q \in N \) by

\[
f_*v_i(q) = d_pf(v_i(p)), \quad \text{with} \quad p = f^{-1}(q).
\]

With this notation we have the following result.

**Theorem 17.** Let \((M, v_1, ..., v_k)\) be a directed manifold and \( f : M \rightarrow N \) be a diffeomorphism. For \( p, q \in M \) the identity map gives a natural homeomorphism

\[
\Gamma^M_{p,q}(t) \simeq \Gamma^N_{f(p),f(q)}(t).
\]

Moreover, if \((M, v_1, ..., v_k)\) has a smooth moduli space of directed paths, and \( f \) is an orientation preserving diffeomorphism, then the identification above is an identity between Riemannian manifolds, and in particular we obtain that

\[
\text{vol}(\Gamma^M_{p,q}(t)) = \text{vol}(\Gamma^N_{f(p),f(q)}(t)).
\]

**Proof.** We show that \( s \in \Gamma^M_{p,q}(t) \) if and only if \( s \in \Gamma^N_{f(p),f(q)}(t) \). By construction we have that

\[
f(\varphi_{v_i}(p, t)) = \varphi_{f_*v_i}(f(p), t),
\]

and thus by induction on the length of \( c \) we have that

\[
f(\varphi_{v,c,s}(p, t)) = \varphi_{f_*v,c,s}(f(p), t),
\]
and therefore the equations
\[ \varphi_{v,c}(p, s) = q \quad \text{and} \quad \varphi_{f \cdot v,c}(f(p), s) = f(q) \]
are equivalent.

For the second part we show that the identity map \( \Gamma_{p,q}^{M,c}(t) \rightarrow \Gamma_{f(p),f(q)}^{N,c}(t) \) preserves orientation. Since the identity map preserves the splittings
\[ T_s \Delta_n \cong N_s \Gamma_{p,q}^{M,c}(t) \oplus T_s \Gamma_{p,q}^{M,c}(t) \quad \text{and} \quad T_s \Delta_n \cong N_s \Gamma_{f(p),f(q)}^{N,c}(t) \oplus T_s \Gamma_{f(p),f(q)}^{N,c}(t), \]
we just have to show that \( N_s \Gamma_{p,q}^{M,c}(t) \) and \( N_s \Gamma_{f(p),f(q)}^{N,c}(t) \) are given compatible orientations. This follows by construction, see the proof of Theorem 13, as the square
\[
\begin{array}{ccc}
N_s \Gamma_{p,q}^{M,c}(t) & \xrightarrow{d_s \varphi_{v,c}} & T_{\varphi_{v,c}(s)} M \\
1 & & \downarrow df \\
N_s \Gamma_{f(p),f(q)}^{N,c}(t) & \xrightarrow{d_s \varphi_{f \cdot v,c}} & T_{\varphi_{f \cdot v,c}(s)} N
\end{array}
\]
is a commutative diagram of orientation preserving isomorphisms, see Corollary 14.

Next result tell us how the moduli spaces of directed paths depend on the ordering on vector fields.

**Proposition 18.** Let \( (M, v_1, ..., v_k) \) be a directed manifold and \( \alpha : [k] \rightarrow [k] \) be a permutation. For the directed manifold \( (M, v_{\alpha 1}, ..., v_{\alpha k}) \) we have that
\[ \Gamma_{p,q}^{v}(t) \cong \Gamma_{p,q}^{v\alpha}(t). \]
Moreover, if \( (M, v) \) has a smooth moduli space of directed paths, then so does \( (M, v\alpha) \) and we have that
\[ \text{vol}(\Gamma_{p,q}^{v}(t)) = \text{vol}(\Gamma_{p,q}^{v\alpha}(t)). \]

**Proof.** We regard the permutation \( \alpha \) as a map
\[ \alpha_* : \Gamma_{p,q}^{v}(t) \rightarrow \Gamma_{p,q}^{v\alpha}(t) \quad \text{given by} \quad \alpha_*(c, s) = (\alpha^{-1}c, s). \]
It follows that \( \alpha \) is an homeomorphism as its restriction map
\[ \alpha_* : \Gamma_{p,q}^{v,c}(t) \rightarrow \Gamma_{p,q}^{v\alpha,c}(t) \]
is just the identity map and is a well-defined homeomorphism since
\[ \varphi_{v\alpha,c^{-1}}(p, s) = \varphi_{v,c}(p, s) = \varphi_{v,c}(p, s) = q. \]
In the case of a smooth moduli space of directed paths, the map above is clearly orientation preserving, since it is just the identity map, and we have a commutative diagram of orientation-preserving isomorphisms

\[
\begin{array}{ccc}
N_s\Gamma_{p,q}^{v,c}(t) & \xrightarrow{d_s\phi_{v,c}} & T_{\phi_{v,c}(s)}M \\
\downarrow 1 & & \downarrow 1 \\
N_s\Gamma_{p,q}^{v\alpha,\alpha-1,c}(t) & \xrightarrow{d_s\phi_{v\alpha,\alpha-1,c}} & T_{\phi_{v\alpha,\alpha-1}(s)}M
\end{array}
\]

From the Theorem 17 and Proposition 18 we see that the invariant study of directed paths on a directed oriented manifold \(M\) relies on the study, for \(k \geq 1\), of the quotient spaces

\[\chi(M)^k/\text{Diff}_+(M) \times S_k,\]

where \(\chi(M)\) is the space of vector fields on \(M\), \(S_k\) the group of permutations of \([k]\), and \(\text{Diff}(M)_+\) is the group of orientation preserving diffeomorphism of \(M\), i.e. the study of equivalence classes of tuples of vector fields under diffeomorphisms and permutations.

Next we define the direction reversion functor \(-: \text{diman} \rightarrow \text{diman}\). It sends a directed manifold \((M, v_1, ..., v_k)\) to its reversed directed manifold

\[(M, -v_1, ..., -v_k).\]

**Proposition 19.** Let \((M, v_1, ..., v_k)\) be a directed manifold and \((M, -v_1, ..., -v_k)\) its reversed directed manifold. We have canonical homeomorphisms

\[\Gamma_{v,A,B}(t) \simeq \Gamma_{-v,B,A}(t).\]

And therefore the respective reachable sets are related by:

\[\Gamma_{-v,A}(t) \simeq \Gamma_{v,A}(t), \quad \Gamma_{-v,A\leq}(t) \simeq \Gamma_{v,A\leq}(t), \quad \Gamma_{-v,A} \simeq \Gamma_{v,A}, \quad F_{-v,A}(t) \simeq \partial\Gamma_{A\leq}(t).\]

If \((M, v_1, ..., v_k)\) has a smooth moduli space of directed paths, then so does \((M, -v_1, ..., -v_k)\) and the maps above are actually diffeomorphisms. These diffeomorphisms may or may not preserve orientation.

**Proof.** We define a map \(\overline{()}: \Gamma_{v,A,B}(t) \rightarrow \Gamma_{-v,B,A}(t)\) as follows

\[\overline{(c, s)} = (c_0, ..., c_n, s_0, ..., s_n) = (\overline{c}, \overline{s}) = (c_n, ..., c_0, s_n, ..., s_0).\]
This map is an homeomorphism since the map
\[ \overline{\varphi} : D(n, k) \rightarrow D(n, k) \]
is bijective, and the map
\[ \overline{\varphi} : \Gamma_{v,A,B}(t) \rightarrow \Gamma_{\overline{v},B,A}(t) \]
is an homeomorphism as the equations
\[ \varphi_{v,c}(p,s) = q \quad \text{and} \quad \varphi_{\overline{v},c}(q,s) = p \]
are equivalent.

In quantum mechanics the proposed integration domain of a Feynman integral is usually the space of differentiable paths, with fixed endpoints, on a manifold. We think of the moduli spaces of directed paths \( \Gamma_{p,q}(t) \) as being analogues for the integration domains for Feynman path integrals, where in addition to boundary restrictions, we impose tangential restrictions on the allowed paths; these restrictions induce a partition of path-space into finite dimensional pieces. The question arises: Can we somehow approach the full Feynman domains of integration from the moduli spaces of directed paths? In other words, is it possible to relax our definition of directed paths, or perform some kind of limit procedure that allow us to approach Feynman integrals from the viewpoint of indirect influences? We left this problem open for future research, and limit ourselves to make a couple of remarks along this line of thinking.

Clearly what one should do is to allow more paths into our moduli spaces. One way to go is to replace the vector fields \( v_j \) by sections of the projective tangent bundle \( \mathbb{P}T M \), so that one fixes the directions along which our curves can move, but leave the speeds unconstrained. Although this approach may be of interest, finite dimensionality is lost. Incidentally, this approach establishes the connection with directed topological spaces \([14]\).

Instead we propose another approach. Given a directed manifold
\[ (M, v) = (M, v_1, ..., v_k) \]
we consider the tuple \( v(a, b) \) of vector fields on \( M \), for \( a, b \in \mathbb{N}_+ \), given by the lexicographically ordered set:
\[ v(a, b) = \{ \frac{i}{b} v_j | -ab \leq i \leq ab, \; j \in [k] \} \].

Indirect influences on the directed manifold \((M, v(a, b))\) are exerted through paths along the directions defined by the vector fields \( v_j \) with rather arbitrary speeds, if \( a \) and \( b \)
are large numbers. Piecewise finite dimensionality is preserved for $a$ and $b$ fix.

To relax even further the restrictions on the paths in our moduli spaces we consider directed manifolds of the form $(M, < v(a,b) >)$ where in $< v(a,b) >$ we include all vector fields that are finite sums of vector fields in $v(a,b)$. Indirect influences in $(M, < v(a,b) >)$ are exerted through paths with rather arbitrary speeds and directions; for example, if the vector fields in $v(a,b)$ at some point contain a basis of the tangent space, then essentially all directions and speeds are allowed, for $a$ and $b$ large, at that point. Piecewise finite dimensionality is preserved for $a$ and $b$ fix.

The fundamental question is whether it is possible to make any sense of the limit of the moduli spaces of directed path for the spaces $(M, < v(a,b) >)$ as $a$ and $b$ grow to infinity, a question however beyond the scope of this work.

5 Indirect Influences on Product/Quotient Manifolds

Let $(M, v_1, ..., v_k)$ and $(N, u_1, ..., u_l)$ be directed manifolds. The natural isomorphism

$$T(M \times N) \simeq \pi_M^* TM \oplus \pi_N^* TN,$$

allows us to consider

$$(M \times N, v_1, ..., v_k, u_1, ..., u_l)$$

as a directed manifold,

where one should more formally write $(v_i, 0)$ instead of $v_i$, and $(0, u_j)$ instead of $u_j$.

Let $\diman$ be the category of directed manifolds. We allow in $\diman$ manifolds with connected components of different dimensions, and assume by convention that the set with one element is a directed manifold.

**Proposition 20.** The product defined above gives $\diman$ the structure of a monoidal category with unit the set $[1]$.

Fix $A \subseteq [n]$. We say that a map $c : A \rightarrow [k]$ is a pattern if $c(i) \neq c(i+1)$ for all contiguous elements $i, i + 1 \in A$. Thus a pattern for the product manifold $M \times N$ is given by a map $c : [n] \rightarrow [k] \sqcup [l]$ such that its restrictions

$$c|_{c^{-1}[k]} : c^{-1}[k] \rightarrow [k]$$

and

$$c|_{c^{-1}[l]} : c^{-1}[l] \rightarrow [l]$$

are patterns on $c^{-1}[k]$ and $c^{-1}[l]$, respectively.
Proposition 21. Let \((p_1, p_2), (q_1, q_2) \in M \times N\), and let \(c : [n] \to [k] \sqcup [l]\) be a pattern. We have a canonical homeomorphism:

\[
\Gamma^{M \times N, c}_{(p_1, p_2), (q_1, q_2)} \simeq \Gamma^{N, c_{|c^{-1}[k]}}_{p_1, q_1} \times \Gamma^{N, c_{|c^{-1}[l]}}_{p_2, q_2}.
\]

Proof. The desired homeomorphism sends

\[
s \in \Gamma^{M \times N, c}_{(p_1, p_2), (q_1, q_2)}(t) \subseteq \Gamma^{M \times N, c}_{(p_1, p_2), (q_1, q_2)}
\]
to the pair

\[
(s|_{c^{-1}[k]}, s|_{c^{-1}[l]}) \in \Gamma^{N, c_{|c^{-1}[k]}}_{p_1, q_1}(a) \times \Gamma^{N, c_{|c^{-1}[l]}}_{p_2, q_2}(t - a),
\]

where

\[
a = \sum_{i \in c^{-1}[k]} s_i.
\]

Next we consider the moduli spaces of directed paths on quotient manifolds. Let \(M\) be a smooth manifold, \(G\) a compact Lie group acting freely on \(M\), and assume that the directed manifold \((M, v_1, \ldots, v_k)\) is invariant under the action of \(G\), i.e. the following identities hold:

\[
d_p g(v_i) = v_i(gp) \quad \text{for all} \quad p \in M, \ g \in G.
\]

Then \(M/G\) is a smooth manifold and it comes with a smooth quotient map

\[
\pi : M \to M/G,
\]

which induces a surjective map \(d\pi : TM \to T(M/G)\), and canonical isomorphisms

\[
\overline{d_p\pi} : T_p M / T_p(Gp) \to T_{\overline{p}}(M/G).
\]

Note also that we have isomorphisms

\[
T_{\overline{p}}(M/G) \simeq \left( \bigoplus_{g \in G} T_{gp} M \right) / G.
\]

Thus we obtain the directed manifold \((M/G, \overline{v}_1, \ldots, \overline{v}_k)\) with \(\overline{v}_i = d\pi(v_i)\).

Theorem 22. Let \((M, v_1, \ldots, v_k)\) be a directed manifold, invariant under the action of the compact Lie group \(G\), and let \(p, q \in M\). Then \((M/G, \overline{v}_1, \ldots, \overline{v}_k)\) with \(\overline{v}_i = d\pi(v_i)\) is a directed manifold, \(G\) acts naturally on \(\Gamma^{M, G}_{p, q}(t)\), and we have that

\[
\Gamma^{M/G, \overline{p}, \overline{q}}(t) \simeq \left( \Gamma^{M, G}_{p, q}(t) \right) / G.
\]
Proof. The result follows from the fact that there are $G$-equivariant homeomorphisms

$$
\Gamma_{p,Gq}^M(t) \rightarrow \Gamma_{\overline{p},\overline{q}}^{M/G}(t) \quad \text{and} \quad \Gamma_{p,Gq}^M(t) \rightarrow \left(\Gamma_{Gp,Gq}^M(t)\right)/G.
$$

As the vector fields $v_i$ are $G$-invariant, the corresponding flows $\varphi_i$ are also $G$-invariant:

$$
\varphi_i(gp, t) = g\varphi_i(p, t), \quad \text{and therefore} \quad \varphi_{c,s}(gp, t) = g\varphi_{c,s}(p, t)
$$

for any pattern and time distribution $(c, s)$. This shows that $G$ acts on $\Gamma_{Gp,Gq}^M(t)$, and that $\Gamma_{p,q}^M(t) \simeq \Gamma_{gp,gq}^M(t)$ for $p, q \in M$.

A pair $(c, s)$ defines a directed path from $\overline{p}$ to $\overline{q}$ in $M/G$ if and only if $\varphi_c(\overline{p}, s) = \overline{q}$. If the latter equation holds we have that

$$
\pi\varphi_c(p, s) = \overline{\varphi}_c(\overline{p}, s) = \overline{q} \quad \text{and thus} \quad \varphi_c(p, s) \in Gq.
$$

Therefore $(c, s)$ defines a directed path from $\overline{p}$ to $\overline{q}$ if and only if $(c, s)$ defines an indirect influence from $p$ to $Gq$. So we have shown that the map $\Gamma_{p,Gq}^M(t) \rightarrow \Gamma_{\overline{p},\overline{q}}^{M/G}(t)$ is a $G$-equivariant homeomorphism.

Similarly, if $a \in Gp$, then $\varphi_c(\overline{p}, s) = \overline{q}$ if and only if

$$
\varphi_c(a, s) = \varphi_c(gp, s) = g\varphi_c(p, s) \quad \text{belongs to} \quad Gq.
$$

Thus the map $\Gamma_{p,Gq}^M(t) \rightarrow \left(\Gamma_{Gp,Gq}^M(t)\right)/G$ is a $G$-equivariant homeomorphism.

\[ \square \]

6 Directed Paths for Constant Vector Fields

As a first and pretty workable example, linking the theory of indirect influences on directed manifolds with linear programming techniques, we consider constant vector fields on affine spaces. Thus we fix a directed manifold $(\mathbb{R}^d, v_1, ..., v_k)$ where the vector fields

$$
v_j = \sum_{j=1}^{d} a_{ij} \frac{\partial}{\partial x_i},
$$

have constant coefficients $a_{ij} \in \mathbb{R}$ for $i \in [d], \ j \in [k]$. 
Theorem 23. Consider the directed manifold \((\mathbb{R}^d, v_1, ..., v_k)\). Fix a pattern \(c \in D(n, k)\) and points \(p, q \in \mathbb{R}^d\). The space of directed paths \(\Gamma_{p,q}^c(t)\) is the convex polytope given on the variables \(s \in \mathbb{R}^{n+1}_{\geq 0}\) by the system of equations:

\[
a_{ic(0)}s_0 + \cdots + a_{ic(n)}s_n = q_i - p_i, \quad \text{for } i \in [d], \quad \text{and} \quad s_0 + \cdots + s_n = 1,
\]
or equivalently in matrix notation

\[
\begin{pmatrix} A_c \\ 1 \end{pmatrix} s = \begin{pmatrix} q - p \\ t \end{pmatrix},
\]

where \(A_c\) is the matrix of format \(d \times (n + 1)\) given by:

\[
(A_c)_{ij} = a_{ic(j)}, \quad 1 = (1, ..., 1) \in \mathbb{R}^{n+1},
\]

\[
s = (s_0, ..., s_n), \quad p = (p_1, ..., p_d), \quad \text{and} \quad q = (q_1, ..., q_d).
\]

Proof. The result follows from the fact that the solutions of the differential equation \(\dot{p} = v\), where \(v\) is constant and with initial condition \(a\), are of the form \(p(t) = a + tv\).

Theorem 24. Consider the directed manifold \((\mathbb{R}^d, v_1, ..., v_k)\). For \(p, q \in \mathbb{R}^d\), the volume of the space of directed paths \(\Gamma_{p,q}^c(t)\) is given by

\[
\text{vol}(\Gamma_{p,q}^c(t)) = \text{vol}(\text{Conv}(u_I)),
\]

where:

\[
\text{Conv}(u_I) \quad \text{is the convex hull of the vector } u_I \quad \text{defined by the following conditions:}
\]

- \(I \subseteq [d]\) is a subset of cardinality \(n\).
- The entries of the vector \(u_I \in \mathbb{R}^{n+1}_{\geq 0}\) vanish for indexes not in \(I\).
- For a matrix \(A\) we let \(A_I\) be its restriction to the columns with indexes in \(I\). The set \(I\) must be such that

\[
\det\begin{pmatrix} A_c \\ 1 \end{pmatrix}_I \neq 0.
\]

- \(u_I\) is the unique solution of the linear system:

\[
\begin{pmatrix} A_c \\ 1 \end{pmatrix}_I u_I = \begin{pmatrix} q - p \\ t \end{pmatrix}.
\]

Proof. Theorem 23 and standard results of linear programming [21, 27] one can show that \(\Gamma_{p,q}^c(t) = \text{Conv}(u_I)\).
6.1 Dimension One

Consider the directed manifold \((\mathbb{R}, a_1 \frac{d}{dx}, \ldots, a_k \frac{d}{dx})\) where for simplicity we assume that \(a_i \neq a_j\). Fix a pattern \(c \in D(n, k)\) and consider the space \(\Gamma_{0,x}(t)\) of directed paths from 0 to \(x\) exerted in time \(t\). The space \(\Gamma_{0,x}(t) \subseteq \mathbb{R}_{\geq 0}^{n+1}\) is the convex polytope defined by the equations

\[
a_{c(0)}s_0 + \cdots + a_{c(n)}s_n = x \quad \text{and} \quad s_0 + \cdots + s_n = t.
\]

Consider the set

\[
D = \{ (i, j) \in [n] \mid c(i) \neq c(j) \}.
\]

By Theorem \(\Gamma_{0,x}(t)\) is the convex polytope \(\text{Conv}(u_{ij})\) generated by the vectors \(u_{ij}\), given for \((i, j) \in D\) by

\[
u_{ij} = (0, \ldots, l_i, \ldots, 0, \ldots, l_j, \ldots, 0) \in \mathbb{R}_{\geq 0}^{n+1}
\]

where

\[
\begin{pmatrix}
a_{c_i} & a_{c_j} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
l_i \\
l_j
\end{pmatrix}
= \begin{pmatrix} x \\ t \end{pmatrix}.
\]

Below we use the following identity, valid for \(n, m \in \mathbb{N}\), involving the classical beta \(B\) and gamma \(\Gamma\) functions:

\[
\int_0^1 s^n(1-s)^m ds = B(n+1, m+1) = \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)} = \frac{n!m!}{(n+m+1)!}.
\]

**Theorem 25.** Consider the directed manifold \((\mathbb{R}, \frac{d}{dx}, -\frac{d}{dx})\). For \(x, y \in \mathbb{R}\) we have that \(\text{vol}(\Gamma_{0,x}(t)) = 0\) if \(|x| > t\), \(\text{vol}(\Gamma_{0,x}(t)) = 1\) if \(|x| = t\), and otherwise is given by:

\[
\sum_{n=0}^{\infty} \left[ \frac{(t+x)^n(t-x)^n}{n!^2} + 2t \frac{(t+x)^n(t-x)^n}{(n+1)!n!} \right] 2^{1-2n}.
\]

Furthermore, we have that

\[
\text{vol}(\Gamma_{x,0}(t)) = \text{vol}(\Gamma_{0,x}(t)) \quad \text{and} \quad \text{vol}(\Gamma_{x,y}(t)) = \text{vol}(\Gamma_{0,y-x}(t)).
\]

The wave of influences for \(t > 0\) is given by

\[
u(x, t) = \int_{x-t}^{x+t} \text{vol}(\Gamma_{y,x}(t)) dy \quad \text{is constant in } x \in \mathbb{R},
\]

and is given explicitly by \(\nu(x, t) = 10e^t + 6e^{-t} - 16\).
Proof. Fix \(x \in \mathbb{R}\) and a pattern \(c \in D(n, k)\). The space of directed paths \(\Gamma^c_{0,x}(t)\) is the polytope given by
\[
\sum_{i=0}^{n} (-1)^i s_i = x \quad \text{and} \quad \sum_{i=0}^{n} s_i = t.
\]

Since we have just two vector fields, a pattern \((c_0, ..., c_n)\) is determined by its initial value \(c_0\). Figure 4 shows the directed path associated to the tuple \((7, 5, 3, 7) \in \Gamma^{(1,2,1,2)}_{(0,-2)}\).

Figure 4: Directed path associated to the tuple \((7, 5, 3, 7) \in \Gamma^{(1,2,1,2)}_{(0,-2)}\).

We distinguish four cases taking into account the initial value \(c_0\) and the parity of \(n\).

Consider the pattern \((1, 2, ..., 1, 2)\) of length \(2n\), for \(n \geq 1\). Then \(\Gamma^c_{0,x}(t)\) is the polytope given by
\[
\sum_{i=0}^{2n-1} (-1)^i s_i = x \quad \text{and} \quad \sum_{i=0}^{2n-1} s_i = t.
\]

Setting
\[
\sum_{i=0}^{n-1} s_{2i} = a \quad \text{and} \quad \sum_{i=0}^{n-1} s_{2i+1} = b,
\]
the previous equations become \(a - b = x\) and \(a + b = t\), with solutions \(a = \frac{t+x}{2}\) and \(b = \frac{t-x}{2}\). By definition \(a, b \geq 0\), thus we must have \(|x| < t\) in order that \(\Gamma^c_{0,x}(t) \neq \emptyset\). For \(|x| < t\), we have that
\[
\Gamma^c_{0,x}(t) = \Delta_{n-1}(\frac{t+x}{2}) \times \Delta_{n-1}(\frac{t-x}{2}),
\]
and therefore
\[
\text{vol}(\Gamma^c_{0,x}(t)) = \frac{(t+x)^{n-1}(t-x)^{n-1}}{2^{2n-2}(n-1)!^2}.
\]

For the pattern \((1, 2, ..., 1, 2, 1)\) of length \(2n + 1\), with \(n \geq 1\), setting
\[
\sum_{i=0}^{n} s_{2i} = a \quad \text{and} \quad \sum_{i=0}^{n-1} s_{2i+1} = b \quad \text{we get that}
\]
\[
\text{vol}(\Gamma_{0,x}^c(t)) = \text{vol}\left[ \Delta_n\left(\frac{t+x}{2}\right) \times \Delta_{n-1}\left(\frac{t-x}{2}\right) \right] = \frac{(t+x)^n(t-x)^{n-1}}{2^{2n-1}n!(n-1)!}.
\]
The pattern \(c = (2,1,\cdots,2,1)\) of length \(2n\), with \(n \geq 1\), leads to
\[
\text{vol}(\Gamma_{0,x}^c(t)) = \text{vol}\left[ \Delta_{n-1}\left(\frac{t-x}{2}\right) \times \Delta_{n-1}\left(\frac{t+x}{2}\right) \right] = \frac{(t+x)^n-1(t-x)^n}{2^{2n-2}(n-1)!^2}.
\]
For the pattern \(c = (2,1,\cdots,2,1,2)\) of length \(2n+1\), with \(n \geq 1\), we get that
\[
\text{vol}(\Gamma_{0,x}^c(t)) = \text{vol}\left[ \Delta_{n}\left(\frac{t-x}{2}\right) \times \Delta_{n-1}\left(\frac{t+x}{2}\right) \right] = \frac{(t+x)^n-1(t-x)^n}{2^{2n}(n-1)!n!}.
\]
Therefore \(\text{vol}(\Gamma_{0,x}(t))\) is for \(|x| < t\) given by:
\[
\sum_{n=1}^{\infty} \frac{(t+x)^n-1(t-x)^n}{(n-1)!^2} + \frac{(t+x)^n(t-x)^n}{n!(n-1)!} + \frac{(t+x)^n-1(t-x)^n}{n!(n-1)!}2^{1-2n}
\]
yielding the desired result.

Applying Theorem 17 to translations on \(\mathbb{R}\) we obtain that:
\[
\text{vol}(\Gamma_{x,y}(t)) = \text{vol}(\Gamma_{x-x,y-x}(t)) = \text{vol}(\Gamma_{0,y-x}(t)).
\]
In particular we get that \(\text{vol}(\Gamma_{x,0}(t)) = \text{vol}(\Gamma_{0,-x}(t))\). A direct inspection of the explicit formula for \(\text{vol}(\Gamma_{x,y}(t))\) given above yields \(\text{vol}(\Gamma_{0,-x}(t)) = \text{vol}(\Gamma_{0,x}(t))\).

Next we show that the wave of influences is constant in the variable \(x\). Making the change of variables \(y - x \rightarrow y\) we get that:
\[
u(x,t) = \int_{x-t}^{x+t} \text{vol}(\Gamma_{0,x-y}(t))dy = \int_{-t}^{t} \text{vol}(\Gamma_{0,-y}(t))dy = \int_{-t}^{t} \text{vol}(\Gamma_{0,y}(t))dy = u(0,t).
\]
To compute \(u(0,t)\) we make the change of variable \(y = t(2s-1)\) in the integral
\[
\int_{-t}^{t} \sum_{n=0}^{\infty} \left[ \frac{(t+y)^n(t-y)^n}{n!^2} + 2t \frac{(t+y)^n(t-y)^n}{(n+1)!n!} \right]2^{1-2n}dy =
\]
\[
\sum_{n=0}^{\infty} 4 \frac{t^{2n+1}}{n!n!} \int_{0}^{1} s^n(1-s)^n ds + 8 \frac{t^{2n+2}}{(n+1)!n!} \int_{0}^{1} s^n(1-s)^n ds =
\]
\[
4 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} + 16 \sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!} = 4\sinh(t) + 16(cosh(t) - 1) = 10e^t + 6e^{-t} - 16.
\]
6.2 Dimension Two

Consider the directed manifold \((\mathbb{R}^2, \frac{\partial}{\partial x}, \frac{\partial}{\partial y})\), and let \(\Gamma(x, y) = \Gamma(0,0),(x,y)\) be the moduli space of directed paths from \((0, 0)\) to \((x, y)\). Note that such influences can only happen at time \(t = x + y\), and thus there is no need to include the time variable in the notation. Figure 4 shows the directed path associated to the tuple \((1, 3, 2, 1) \in \Gamma(2, 1, 2, 1)(4, 3)\).

![Diagram of directed path](image)

Figure 4. Directed path associated to \((1, 3, 2, 1) \in \Gamma(2, 1, 2, 1)(4, 3)\).

In our next results we use the following notation. For \(k \in \mathbb{N}\) we set

\[
i_k(x, y) = \sum_{n=0}^{\infty} \frac{x^n y^{n+k}}{n!(n+k)!} \quad \text{and} \quad i_{-k}(x, y) = i_k(y, x) = \sum_{n=0}^{\infty} \frac{x^{n+k} y^n}{(n+k)!n!}.
\]

The following result is easy to check.

**Lemma 26.**

- For \(l, m \in \mathbb{N}\) and \(k \in \mathbb{Z}\) we have that
  \[
  \frac{\partial^l}{\partial x^l} \frac{\partial^m}{\partial y^m} i_k(x, y) = i_{l-m+k}(x, y).
  \]

- For \(k \in \mathbb{N}\), the function \(i_k(x, y)\) is given in terms of the modified Bessel function \(I_k(z)\) by
  \[
i_k(x, y) = x^{-\frac{k}{2}} y^{\frac{k}{2}} I_k(2\sqrt{xy}),
  \]
  where we recall that
  \[
  I_v(z) = (\frac{z}{2})^v \sum_{n=0}^{\infty} \frac{(z^2/4)^n}{n! \Gamma(v + n + 1)}.
  \]

- For \(k \in \mathbb{N}\), we have that \(I_k(z) = i_{k}(\frac{z}{2}, \frac{z}{2})\).
Theorem 27. Consider the directed manifold \((\mathbb{R}^2, \frac{\partial}{\partial x}, \frac{\partial}{\partial y})\).

1. There are no directed paths from \((0,0)\) to a point \((x,y) \notin \mathbb{R}_{\geq 0}^2\).

2. \(\text{vol}(\Gamma(x,0)) = \text{vol}(\Gamma(0,x)) = 1\), for \(x \in \mathbb{R}_{>0}\).

3. For \((x,y) \in \mathbb{R}_{>0}^2\), the moduli space \(\Gamma(x,y)\) of directed paths from \((0,0)\) to \((x,y)\) has volume

\[
\text{vol}(\Gamma(x,y)) = i_{-1}(x,y) + 2i_0(x,y) + i_1(x,y) = \\
\sum_{n=0}^{\infty} \left(\frac{x^{n+1}y^n}{(n+1)!n!} + 2\frac{x^n y^n}{n!^2} + \frac{x^n y^{n+1}}{n!(n+1)!}\right).
\]

4. \(\text{vol}(\Gamma(x,y))\) is a symmetric function in \(x\) and \(y\).

5. The derivatives of the function \(\text{vol}(\Gamma) = \text{vol}(\Gamma(x,y))\) are given by:

\[
\frac{\partial^l}{\partial x^l} \frac{\partial^m}{\partial y^m} \text{vol}(\Gamma) = i_{l-m-1}(x,y) + 2i_{l-m}(x,y) + i_{l-m+1}(x,y).
\]

6. We have that \(\frac{\partial}{\partial x} \frac{\partial}{\partial y} \text{vol}(\Gamma) = \text{vol}(\Gamma)\).

7. Only points \((x,y) \in \mathbb{R}_{>0}^2\) on the segment \(x + y = t\) receive an influence from \((0,0)\) at time \(t \geq 0\). Among the points on this segment, the highest influence from \((0,0)\) is exerted on the point \((\frac{t}{2}, \frac{t}{2})\); the volume of the moduli space of directed paths from \((0,0)\) along the line of maximal influences is given by

\[
\text{vol}(\Gamma(t,t)) = 2 \sum_{n=0}^{\infty} \left(\frac{n}{\lfloor n/2 \rfloor} \right) t^n n!.
\]

8. The wave of influences \(u(x,y,t)\) is given for \(t > 0\) by

\[
u(x,y,t) = \int_0^t \text{vol}(\Gamma_{(x-s,y+s-t),(x,y)}(t)) ds = 2(e^t - 1).
\]

Proof. Item 1 is clear, and item 2 simply counts the influences that arise, respectively, from the patterns (1) and (2). Let us show 3. Since \(k = 2\), a pattern \((c_0, ..., c_n)\) is determined by its initial value \(c_0\). For \((x,y) \in \mathbb{R}_{>0}^2\) we distinguish four cases taking into account the initial value \(c_0\) and the parity of \(n\).
• Patterns \((1, 2, ..., 1, 2)\) and \((2, 1, ..., 2, 1)\) of length \(2n\), for \(n \geq 1\), have a contribution of
\[
\text{vol}(\Delta_{n-1}^x) \cdot \text{vol}(\Delta_{n-1}^y) = \frac{x^{n-1}y^{n-1}}{(n-1)!^2}
\]
to the volume of the moduli space of directed paths.

• The pattern \((1, 2, ..., 1, 2, 1)\) of length \(2n+1\), for \(n \geq 1\), have a contribution of
\[
\text{vol}(\Delta_{n-1}^x) \cdot \text{vol}(\Delta_{n-1}^y) = \frac{x^n y^{n-1}}{n!(n-1)!}
\]
to the volume of the moduli space of directed paths.

• The pattern \((2, 1, ..., 2, 1, 2)\) of length \(2n+1\), for \(n \geq 1\), have a contribution of
\[
\text{vol}(\Delta_{n-1}^x) \cdot \text{vol}(\Delta_{n}^y) = \frac{x^{n-1}y^n}{(n-1)!n!}
\]
to the volume of the moduli space of directed paths.

Putting together the three summands we obtain that
\[
\text{vol}(\Gamma(x, y)) = \sum_{n=1}^{\infty} \left( 2 \frac{x^{n-1}y^{n-1}}{(n-1)!^2} + \frac{x^n y^{n-1}}{n!(n-1)!} + \frac{x^{n-1}y^n}{(n-1)!n!} \right),
\]
an expression equivalent to our desired result after a change of variables. Clearly, \(\text{vol}(\Gamma(x, y))\) is symmetric in \(x\) and \(y\), thus item 4 follows.

Item 5 follows from item 3 and Lemma 26. Item 6 is a particular case of item 5. Let us show item 7. Let \(\text{vol}_n(\Gamma(x, y))\) be the \(n\)-th coefficient in the series expansion of \(\text{vol}(\Gamma(x, y))\) from item 3. The points influenced by \((0, 0)\) at time \(t\) are of the form \((s, t-s)\) with \(0 < s < t\). Thus:
\[
\text{vol}_n(\Gamma(s, t-s)) = (st - s^2)^{n-1} \left( \frac{2}{(n-1)!^2} + \frac{t}{(n-1)!n!} \right).
\]

Therefore
\[
\frac{\partial}{\partial s} \text{vol}_n(\Gamma(s, t-s)) = (n-1)(st - s^2)^{n-2} t (t-2s) \left( \frac{2}{(n-1)!^2} + \frac{t}{(n-1)!n!} \right).
\]
The sign of the expression above is determined by the sign of \((t-2s)\), as the other factors are positive. Thus the volume of the moduli space of directed paths from \((0, 0)\) exerted on time \(t\) achieves a global maximum at the point \((\frac{t}{2}, \frac{t}{2})\), and we have that
\[
\text{vol}(\Gamma(t, t)) = 2 \sum_{n=0}^{\infty} \left( \frac{t^{2n}}{n!^2} + \frac{t^{2n+1}}{(n+1)!n!} \right) =
\]
\[
2 \sum_{n=0}^{\infty} \left( \binom{2n}{n} \frac{t^{2n}}{(2n)!} + \binom{2n+1}{n} \frac{t^{2n+1}}{(2n+1)!} \right) = 2 \sum_{n=0}^{\infty} \left( \binom{n}{\lfloor n/2 \rfloor} \frac{t^n}{n!} \right).
\]

Item 8. By translation invariance the wave of influence is independent of \( x, y \). Thus we have that
\[
u(x, y, t) = u(0, 0, t) = \int_0^t \operatorname{vol}(\Gamma(-s, s-t), (0, 0))(t)ds = \int_0^t \operatorname{vol}(\Gamma(0, 0), (s, t-s))(t)ds = \int_0^t \Gamma(s, t-s)ds = \sum_{n=0}^{\infty} \int_0^t \left( 2^{n} \frac{(t-s)^n}{n!} + \frac{s^{n+1}(t-s)^n}{(n+1)!n!} + \frac{s^n(t-s)^n}{n!(n+1)!} \right)ds = 2 \sum_{n=0}^{\infty} \int_0^t \frac{s^n(t-s)^n}{n!}ds + 2 \int_0^t \frac{s^{n+1}(t-s)^n}{(n+1)!n!}ds = 2 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} + 2 \sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!} = 2(\sinh(t) + \cosh(t) - 1) = 2(e^t - 1).
\]

Next we consider the moduli spaces of directed paths on the torus \( T^2 = S^1 \times S^1 \). We use coordinates \((x, y) \in \mathbb{R}^2\) representing the point \((e^{2\pi ix}, e^{2\pi iy}) \in T^2\). Consider the vector fields on \( T^2 \) given in local coordinates by
\[
\frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y}.
\]
The moduli space of directed paths on the torus \( T^2 \) from \((1, 1)\) to \((e^{2\pi ix}, e^{2\pi iy})\) exerted in time \( t > 0 \) is denoted by \( \Gamma(e^{2\pi ix}, e^{2\pi iy}, t) \). Recall that \( D(e^{2\pi ix}, e^{2\pi iy}, t) \) is the set of one-direction paths.

**Theorem 28.** Consider the directed manifold \((T^2, \frac{\partial}{\partial x}, \frac{\partial}{\partial y})\).

1. For \( x, y \in (0, 1) \) we have that \( \operatorname{vol}(D(e^{2\pi ix}, e^{2\pi iy}, t)) = 0 \).
2. For \( x \in (0, 1] \) we have that
   \[
   \operatorname{vol}(D(e^{2\pi ix}, 1, t)) = \operatorname{vol}(D(1, e^{2\pi ix}, t)) = \sum_{m=0}^{\infty} \delta(t, x + m).
   \]
3. For \( (x, y) \in (0, 1)^2 \), the moduli space \( \Gamma(e^{2\pi ix}, e^{2\pi iy}, t) \) of directed paths from \((1, 1)\) to \((e^{2\pi ix}, e^{2\pi iy})\) is empty unless \( t = x + y + m \) for some \( m \geq 0 \), and in the latter case we have that:
   \[
   \operatorname{vol}(\Gamma(e^{2\pi ix}, e^{2\pi iy}, x + y + m)) \text{ is given by}
   \sum_{k+l=m} \sum_{n=0}^{\infty} \left( \frac{(x+k)^n(y+l)^n}{n!^2} + (x+y+k+l)\frac{(x+k)^n(y+l)^n}{(n+1)!n!} \right).
   \]
4. \( \text{vol}(\Gamma(e^{2\pi ix}, e^{2\pi iy}, x + y + m)) \) is a symmetric function in \( x \) and \( y \).

**Proof.** We can compute indirect influences on the torus as sums of indirect influences on the plane, indeed we have that
\[
\text{vol}(\Gamma(e^{2\pi ix}, e^{2\pi iy}, x + y + m)) = \sum_{k+l=m} \text{vol}(\Gamma(x+k, y+l, x+y+m)) =
\]
\[
\sum_{k+l=m} \sum_{n=0}^{\infty} \left( \frac{2(x+k)^n(y+l)^n}{n!^2} + \frac{(x+k)^{n+1}(y+l)^n}{(n+1)!n!} + \frac{(x+k)^n(y+l)^{n+1}}{n!(n+1)!} \right). 
\]

6.3 Higher Dimensions

Let us first introduce a few combinatorial notions. Given integers \( n_1, \ldots, n_k \in \mathbb{N}_{>0} \) we let \( \text{Sh}_k(n_1, \ldots, n_k) \) be the set of shuffles of \( n_1 + \cdots + n_k \) cards divided into \( k \) blocks of cardinalities \( n_1, \ldots, n_k \). Recall that a shuffle is a bijection \( \alpha \) from the set
\[
[1, n_1 + \cdots + n_k] \simeq [1, n_1] \sqcup \cdots \sqcup [1, n_k]
\]
to itself such that if \( i < j \in [1, n_s] \), then \( \alpha(i) < \alpha(j) \in [1, n_1 + \cdots + n_k] \). When we shuffle a deck of cards the idea is to intertwine the cards in the various blocks, without distorting the order in each block. We say that a shuffle is perfect if no contiguous cards within a block remain contiguous after shuffling, i.e. a shuffle \( \alpha \) is called perfect if for \( i, i+1 \in [1, n_s] \) we have that
\[
\alpha(i) + 1 < \alpha(i+1) \in [1, n_1 + \cdots + n_k].
\]

Let \( \text{PSh}_k(n_1, \ldots, n_k) \subseteq \text{Sh}_k(n_1, \ldots, n_k) \) be the set of perfect shuffles, and \( \text{psh}_k \) be the corresponding exponential generating series given by
\[
\text{psh}_k(x_1, \ldots, x_k) = \sum_{n_1, \ldots, n_k \in \mathbb{N}_{>0}} |\text{PSh}_k(n_1, \ldots, n_k)| \frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1! \cdots n_k!}.
\]

A subset \( A \subseteq [m] \) is called sparse if it does not contain consecutive elements. Let \( S_k[m] \) be the set of all sparse subsets of \( [m] \) of cardinality \( k \). Let \( p(m, k) \) count the numerical partitions of \( m \) in \( k \) positive summands.

**Lemma 29.** For \( 1 \leq k < m \in \mathbb{N} \), we have that:
\[
|S_k[m]| = p(m-k, k-1) + 2p(m-k, k) + p(m-k, k+1).
\]
Proof. If \( A \in S_k[m] \), then \(|A^c| = m - k\), and \( A^c \) comes with a naturally ordered partition with exactly \( k - 1 \) blocks if \( 1, m \in A \), \( k \) blocks if \( 1 \) or \( m \) (but not both) belong to \( A \), and \( k + 1 \) blocks if \( 1, m \notin A \). The cardinalities of the blocks of \( A^c \) provides the various kinds of numerical partitions needed to complete our result. \( \Box \)

**Lemma 30.** For \( n_1, \ldots, n_k \in \mathbb{N}_{>0} \), then \(|\text{PSh}_k(n_1, \ldots, n_k)|\) counts number of ordered partitions of \( n_1 + \ldots + n_k \) with sparse blocks of cardinalities \( n_1, \ldots, n_k \).

**Proof.** A perfect shuffle in \( \text{PSh}_k(n_1, \ldots, n_k) \) is determined by its image on each of the blocks \([1, n_s]\), which must be a sparse subsets. \( \Box \)

Let us point out the relation between patterns and perfect shuffles. Consider the map

\[ | | : C(n, k) \to \mathbb{N}^k, \]

sending a pattern \( c \in C(n, k) \) to its content multi-set given by the sequence \(|c| \in \mathbb{N}^k\) such that \(|c|_i = |c^{-1}(i)|\). The support of a pattern \( c \) is the set \( s(c) \subseteq [k] \) with \( i \in s(c) \) if and only if \(|c|_i \neq 0\).

**Lemma 31.** Fix a vector \((n_1, \ldots, n_k) \in \mathbb{N}_{>0}^k\). We have that:

\[ \left| \left\{ c \in C(n, k) \mid |c| = (n_1, \ldots, n_k) \right\} \right| = |\text{PSh}_k(n_1, \ldots, n_k)|. \]

**Proof.** The vector \((n_1, \ldots, n_k)\) gives us the content multi-set of \( c \), a shuffle on it gives us in addition the order of the vector \( c \). The perfect condition on shuffles is equivalent to the conditions \( c(i) \neq c(i + 1) \) on patterns. \( \Box \)

Consider the directed manifold \((\mathbb{R}^k, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})\). The moduli space of directed paths from \((0, \ldots, 0)\) to \((x_1, \ldots, x_k)\) is denoted by \(\Gamma(x_1, \ldots, x_k)\). Such paths can only happen at time \( t = x_1 + \cdots + x_k \).

**Theorem 32.** Consider the directed manifold \((\mathbb{R}^k, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})\).

1. There are no directed paths from \((0, \ldots, 0)\) to any point \((x_1, \ldots, x_k) \notin \mathbb{R}_{\geq 0}^k\).
2. \(\text{vol}(D(0, \ldots, 0, x, 0, \ldots, 0)) = 1\), for \( x \in \mathbb{R}_{\geq 0} \) and \( i \in [k] \).
3. For \((x_1, \ldots, x_k) \in \mathbb{R}_{\geq 0}^k\), with at least two positive entries, the moduli space \(\Gamma(x_1, \ldots, x_k)\) of directed paths from \((0, \ldots, 0)\) to \((x_1, \ldots, x_k)\) has volume

\[ \text{vol}(\Gamma(x_1, \ldots, x_k)) = \sum_{A \subseteq [k], |A| \geq 2} \frac{\partial |A|}{\partial x_A} \text{psh}_{|A|}(x_A). \]
4. \( \text{vol}(\Gamma(x_1, \ldots, x_k)) \) is a symmetric function in the variables \( x_1, \ldots, x_k \).

**Proof.** Properties 1 and 2 are clear, let us prove 3. Recall that

\[
\text{vol}(\Gamma(x_1, \ldots, x_k)) = \sum_{n=1}^{\infty} \sum_{c \in C(n, k)} \text{vol}(\Gamma^c(x_1, \ldots, x_k)),
\]

where the volume of the moduli space of directed paths with a fix pattern \( c \in C(n, k) \) is given by

\[
\text{vol}(\Gamma^c(x_1, \ldots, x_k)) = \prod_{j \in s(c)} x_j^{n_j} / (n_j! \cdot n_j). 
\]

Thus a pattern \( c \in C(n, k) \) with support \( s(c) = A \subseteq [k] \), with \( |A| \geq 2 \), contributes to the monomial

\[
\frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1! \cdots n_k!},
\]

if and only if \( |c|_i = n_i + 1 \) for \( i \in A \), and \( n_i = 0 \) for \( i \notin A \). Therefore the total contribution of the patterns with support \( A \) to this monomial is given by

\[
|\text{PSh}_{|A|}(n_A + 1)| \prod_{j \in A} \frac{x_j^{n_j}}{n_j!},
\]

where \( n_A \) is the vector obtained from the tuple \( (n_1, \ldots, n_k) \) by erasing the zero entries, and \( n_A + 1 \) is the vector obtain from \( n_A \) by adding 1 to each entry.

Summing over the \( n_j \), and setting \( x_A = (x_j)_{j \in A} \), we obtain that the total contribution of the patterns with support \( A \) to the volume of the moduli space of directed paths is given by

\[
\sum_{n_j \in \mathbb{N}; j \in A} |\text{PSh}_{|A|}(n_A + 1)| \prod_{j \in A} \frac{x_j^{n_j}}{n_j!} = \frac{\partial^{|A|}}{\partial x_A} \text{psh}_{|A|}(x_A).
\]

Adding over all possible supports \( A \subseteq [k] \), with \( |A| \geq 2 \), we obtain the desired result.

4. For a permutation \( \sigma \in S_n \) we have that \( \text{vol}(\Gamma(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \) is given by

\[
\sum_{A \subseteq [k]} \frac{\partial^{|A|}}{\partial x_{\sigma(A)}} \text{psh}_{|A|}(x_{\sigma(A)}) = \sum_{A \subseteq [k]} \frac{\partial^{|A|}}{\partial x_A} \text{psh}_{|A|}(x_A) = \text{vol}(\Gamma(x_1, \ldots, x_k)).
\]
Next we consider directed paths on the $k$-dimensional torus $T^k = S^1 \times \cdots \times S^1$. We use coordinates $(x_1, \ldots, x_k) \in \mathbb{R}^k$ representing the point $(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k}) \in T^k$. Consider the constant vector fields on $T^k$ given in local coordinates by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$.

The moduli space of directed paths on $T^k$ from $(1, \ldots, 1)$ to $(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k})$ exerted in time $t > 0$ is denoted by $\Gamma(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k}, t)$. Recall that the set of one-direction paths is denoted by $D(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k}, t)$.

**Theorem 33.** Consider the directed manifold $(T^k, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})$.

1. For $x_1, \ldots, x_k \in (0, 1)$, with at least two entries in $(0, 1)$, we have that \[
\text{vol}(D(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k}, t)) = 0.
\]
2. For $x \in (0, 1]$ we have that:
\[
\text{vol}(D(1, \ldots, e^{2\pi ix_1}, \ldots, 1, t)) = \sum_{m=0}^{\infty} \delta(t, x_i + m).
\]
3. For $x_1, \ldots, x_k \in (0, 1)$, with at least two entries in $(0, 1)$, the moduli space $\Gamma(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k}, t)$ of directed paths from $(1, \ldots, 1)$ to $(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k})$ is empty unless $t = x_1 + \cdots + x_k + m$ for some $m \geq 0$, and in the latter case we have that:
\[
\text{vol}(\Gamma(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k}, 1 + \cdots + x_k + m)) = \sum_{m_1 + \cdots + m_k = m} \sum_{A \subseteq \{d\}} \sum_{|A| \geq 2} \frac{\partial|A|}{\partial x_A} \text{psh}_{|A|}(x_A + m_A).
\]
4. \[
\text{vol}(\Gamma(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k}, x_1 + \cdots + x_k + m)) \text{ is a symmetric function on } x_1, \ldots, x_k.
\]

7 Quantum Indirect Influences

In this closing section we briefly describe how to extend the theory of indirect influences to the quantum settings. We first consider indirect influences on Poisson manifolds [2] from two different viewpoints.

Let $(M, \{ \cdot, \cdot \})$ be a Poisson manifold and $f_1, \ldots, f_k : M \to \mathbb{R}$ be $k$ smooth functions on $M$. We obtain the directed manifold
\[
(M, \{ f_1, \}, \ldots, \{ f_k, \}),
\]
where \( \{f_j, \} \), the Hamiltonian vector field on \( M \) generated by \( f_j \), is given in local coordinates by

\[
\{f_j, \} = \sum_{kl} \{x_k, x_l\} \frac{\partial f_j}{\partial x_k} \frac{\partial}{\partial x_l}.
\]

Next let \( C^\infty(M) \) be the infinite dimensional vector space of smooth functions on \( M \). We obtain the infinite dimensional directed manifold

\[
(C^\infty(M), \{f_1, \}, \ldots, \{f_k, \})
\]

where now we regard \( \{f_j, \} \) as the vector field on \( C^\infty(M) \) assigning to \( f \in C^\infty(M) \) the vector

\[
\{f_j, f\} \in T_f C^\infty(M) = C^\infty(M).
\]

Given functions \( f, g \in C^\infty(M) \) and a pattern \( c \in D(n, k) \) the moduli space of directed paths from \( f \) to \( g \) exerted in time \( t > 0 \) is given by

\[
\Gamma^c_{f,g}(t) = \{ s \in \Delta^t_n \mid \varphi_c(f, s) = g \}.
\]

The flow generated by \( \{f_j, \} \) on \( M \), and the flow generated by \( \{f_j, \} \) on \( C^\infty(M) \) (allow us to use the same notation for vector fields in different spaces) are related by the identity

\[
\varphi_j(f, s)(x) = f(\varphi_j(x, s)).
\]

Since the vector fields \( \{f_j, \} \) are linear operators on \( C^\infty(M) \), the flows generated by them – assuming suitable convergency properties – can be written as

\[
\varphi_j(f, s) = e^{\{f_j, \} s f}.
\]

Expanding the exponentials functions the iterated flow \( \varphi_c(f, s) \) can be written as

\[
\varphi_c(f, s) = \sum_{k_0, \ldots, k_n \in \mathbb{N}} \{f_{c(n)}, \ldots, f_{c(0)}, f\} k_0, \ldots, k_n \frac{s^{k_0} \ldots s^{k_n}}{k_0! \ldots k_n!},
\]

where the symbol \( \{g_n, \ldots, g_1, f\}_{k_1, \ldots, k_n} \) is defined recursively as follows:

\[
\{g_1, f\}_0 = f, \quad \{g, f\}_{k+1} = \{g, \{g, f\}_k\},
\]

\[
\{g_n, \ldots, g_1, f\}_{k_1, \ldots, k_n} = \{g_n, \{g_{n-1} \ldots, g_1, f\}_{k_1, \ldots, k_{n-1}}\}{k_n}.
\]

From this viewpoint it is clear how to extend the theory of indirect influences to the quantum context [12, 24]. Let \( \mathcal{H} \) be a Hilbert space and \( A_1, \ldots, A_k \) be bounded
Hermitian operators on $\mathcal{H}$.

In the Heisenberg picture we consider the (possibly infinite dimensional) directed manifold

$$(\mathcal{B}(\mathcal{H}), \frac{i}{\hbar}[A_1, \ldots, \frac{i}{\hbar}[A_k, ]]$$

where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on $\mathcal{H}$, $[\cdot, \cdot]$ is the commutator of bounded operators, and $\frac{i}{\hbar}[A_1, \cdot]$ is regarded as the vector field on $\mathcal{B}(\mathcal{H})$ assigning to $B \in \mathcal{B}(\mathcal{H})$ the vector

$$\frac{i}{\hbar}[A_1, B] \in T_B\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}).$$

Given operators $B, C \in \mathcal{B}(\mathcal{H})$ and a pattern $c \in D(n, k)$, the moduli space of directed paths from $B$ to $C$ exerted in time $t > 0$ is given by

$$\Gamma_{B,C}^c(t) = \{ s \in \Delta_n^t \mid \varphi_c(B, s) = C \},$$

where the iterated flow $\varphi_c(B, s)$ is given by

$$\varphi_c(B, s) = e^{\frac{i}{\hbar}A_{c(n)}s_n} \cdots e^{\frac{i}{\hbar}A_{c(1)}s_1} e^{\frac{i}{\hbar}A_{c(0)}s_0} Be^{-\frac{i}{\hbar}A_{c(1)}s_1} e^{-\frac{i}{\hbar}A_{c(0)}s_0} e^{-\frac{i}{\hbar}A_{c(n)}s_n} =$$

$$\sum_{k_0, \ldots, k_n} \left(\frac{i}{\hbar}\right)^{k_0 + \cdots + k_n} [A_{c(n)}, \ldots, A_{c(0)}, B]_{k_0, \ldots, k_n} s_0^{k_0} \cdots s_n^{k_n}$$

where the symbols

$$[A_{c(n)}, \ldots, A_{c(0)}, B]_{k_0, \ldots, k_n}$$

are defined as in the Poisson case replacing brackets $\{ \cdot, \cdot \}$ by commutators $[\cdot, \cdot]$. Clearly, we can apply this constructions in the context of deformation quantization as well [19].

In the Schrödinger picture we consider the (possibly infinite dimensional) directed manifold

$$(\mathcal{H}, -\frac{i}{\hbar}A_1, \ldots, -\frac{i}{\hbar}A_k),$$

where $-\frac{i}{\hbar}A_j$ is regarded as the vector field assigning to $v \in \mathcal{H}$ the vector

$$-\frac{i}{\hbar}A_j(v) \in T_v\mathcal{H} = \mathcal{H}.$$

Given $v, w \in \mathcal{H}$ and a pattern $c \in D(n, k)$, the moduli space of directed paths from $v$ to $w$ exerted in time $t > 0$ is given by

$$\Gamma_{v,w}^c(t) = \{ s \in \Delta_n^t \mid \varphi_c(v, s) = w \}.$$

The iterated flow $\varphi_c(v, s)$ is given by

$$\varphi_c(v, s) = \sum_{k_0, \ldots, k_n} \left(\frac{i}{\hbar}\right)^{k_0 + \cdots + k_n} \left(A_{c(n)}^{k_n} \cdots A_{c(0)}^{k_0} v\right) s_0^{k_0} \cdots s_n^{k_n}. $$

37
Acknowledgement

Our thanks to Tom Koornwinder whose comments and suggestions help us to make substantial improvements on an early version of this work.

References

[1] A. Agrachev, Y. Sachkov, Control theory from the geometric viewpoint, Springer-Verlag, Berlin 2004.

[2] V. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York 1989.

[3] S. Brin, L. Page, R. Motwani, T. Winograd, The Anatomy of a Large-Scale Hypertextual Web Search Engine, Comp. Netw. ISDN Sys. 30 (1998) 107-117.

[4] S. Brin, L. Page, R. Motwani, T. Winograd, The PageRank citation ranking: Bringing order to the Web, Technical Report, Stanford Digital Library Technologies Project 1998.

[5] L. Cano, R. Díaz, Continuous Analogues for the Binomial Coefficients and the Catalan Numbers, preprint, arXiv:1602.09132.

[6] M. Chas, D. Sullivan, String Topology, preprint, arXiv:math/9911159.

[7] F. Chung, The heat kernel as the pagerank of a graph, Proc. Natl. Acad. Sci. U.S.A. 104 (2007) 19735-19740.

[8] R. Díaz, Indirect Influences, Adv. Stud. Contemp. Math. 23 (2013) 29-41.

[9] R. Díaz, L. Gómez, Indirect Influences in International Trade, Netw. Heterog. Media 10 (2015) 149-165.

[10] R. Díaz, A. Vargas, On the Stability of the PWP method, preprint, arXiv:1504.03033.

[11] E. Estrada, N. Hatano, Communicability in complex networks, Phys. Rev. E 77 (2008) 036111.

[12] J. Glim, A. Jaffe, Quantum Physics, Springer-Verlag, New York 1981.

[13] M. Godet, De l’Anticipation à l’Action, Dunod, Paris 1992.
[14] M. Grandis, Directed Algebraic Topology, Models of non-reversible worlds, Camb. Univ. Press, Cambridge 2009.

[15] V. Guillemin, A. Pollack, Differential topology, Springer-Verlag, New York 1983.

[16] D. Joyce, On manifolds with corners, in S. Janeczko, J. Li, D. Phong (Eds.), Advances in Geometric Analysis, International Press, Boston (2012) 225-258.

[17] V. Jurdjevic, Geometric control theory, Cambridge University Press, Cambridge 1997.

[18] L. Katz, A new status index derived from sociometric analysis, Psychometrika 18 (1953) 39-43.

[19] M. Kontsevich, Deformation Quantization of Poisson Manifolds, Lett. Math. Phys. 66 (2003) 157-216.

[20] A. Langville, C. Meyer, Deeper Inside PageRank, Internet Math. 1 (2004) 335-400.

[21] J. Matoušek, B. Gaertner, Understanding and Using Linear Programming, Springer-Verlag, Berlin 2007.

[22] R. Penrose, Techniques of Differential Topology in Relativity, Soc. Industrial Appl. Math., Bristol 2014.

[23] L. Rifford, Sub-Riemannian geometry and optimal transport, Springer 2014.

[24] B. Simon, Functional integration and quantum physics, Amer. Math. Soc, Providence 2005.

[25] E. Sontag, Mathematical control theory, Springer-Verlag, New York 1998.

[26] F. Warner, Foundations of differentiable manifolds and Lie groups, Springer-Verlag, New York 1983.

[27] X. Yang, Introduction to mathematical optimization, Cambridge Int. Science Publishing, Cambridge 2008.

[28] J. Zabczyk, Mathematical control theory: an introduction, Birkhäuser, Boston 1992.

[29] J. Zinn-Justin, Path integrals in quantum mechanics, Oxford Univ. Press, Oxford 2010.
lnrdcano@gmail.com
Departamento de Matemáticas, Universidad Sergio Arboleda, Bogotá, Colombia

ragadiaz@gmail.com
Departamento de Matemáticas, Pontificia Universidad Javeriana, Bogotá, Colombia