Stationary Solutions of the Dirac Equation in the Gravitational Field of a Charged Black Hole

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A stationary solution of the Dirac equation in the metric of a Reissner-Nordström black hole has been found. Only one stationary regular state outside the black hole event horizon and only one stationary regular state below the Cauchy horizon are shown to exist. The normalization integral of the wave functions diverges on both horizons if the black hole is non-extremal. This means that the solution found can be only the asymptotic limit of a nonstationary solution. In contrast, in the case of an extremal black hole, the normalization integral is finite and the stationary regular solution is physically self-consistent. The existence of quantum levels below the Cauchy horizon can affect the final stage of Hawking black hole evaporation and opens up the fundamental possibility of investigating the internal structure of black holes using quantum tunneling between external and internal states.

1. INTRODUCTION

The Dirac equation in a gravitational field of general form was derived by Fock and Iwanenko in 1929 \cite{1} using the formalism of parallel spinor transport, which allowed its covariant derivative to be determined. Another method for deriving the Dirac equation in a gravitational field is based on the description of the Lorentz group in the tetrad formalism \cite{2}. The Dirac equation written in a (pseudo-)Riemannian space is

\begin{equation}
(i\gamma^\mu D_\mu - m)\psi = 0,
\end{equation}

where the Dirac matrices in a metric of general form $\gamma^\mu = e^\mu_a \gamma^a$ are expressed in terms of the standard matrices $\gamma^a$ in a Minkowski space via the tetrad $e^\mu_a$. The extended derivative is

\begin{equation}
D_\mu = \partial_\mu + iqA_\mu + \Gamma_\mu,
\end{equation}

where

\begin{equation}
\Gamma_\mu = \frac{1}{4} \gamma^a \gamma^b e^\mu_a \gamma^b e^\nu_\mu,
\end{equation}

and $A_\mu$ is the electromagnetic 4-potential.

The Dirac equation in gravitational fields of various forms was investigated in many papers (see, e.g., \textsuperscript{3} \textsuperscript{5}). The scattering of fermions by black holes and the emission of fermions in the process of quantum black hole evaporation (the Hawking effect) were studied most extensively. Most of the efforts at investigating the Dirac equation in a gravitational field were focused on problems of this type. The resonant quasistationary quantum states of scalar particles in the gravitational field of a black hole were investigated in \textsuperscript{7} \textsuperscript{9}. Similar quantum states for spinor particles were analyzed in \textsuperscript{11} \textsuperscript{12}.

The stationary states of charged particles were studied in the gravitational field of a Schwarzschild black hole \textsuperscript{13} \textsuperscript{14} and an electrically charged Reissner-Nordström black hole \textsuperscript{15} in a region outside the horizon.

The spacetime of an eternal Reissner-Nordström black hole is an infinite sequence of internal universes \textsuperscript{16}. A particle falling into an eternal Reissner-Nordström black hole can either escape into another internal universe or be left below the Cauchy horizon. The existence of stable finite orbits for classical particles below the Cauchy horizon was shown in \textsuperscript{17} \textsuperscript{21}.

Here, we consider the stationary quantum states of fermions inside and outside a black hole and show that such states and the corresponding energy levels actually exist under certain conditions. A Reissner-Nordström black hole with a charged particle at a quantum level in some respect resembles the simplest hydrogen atom. However, in the case of a charged black hole, the electron levels can be not only outside but also inside the black hole, below the Cauchy horizon. In addition, the boundary conditions for the wave functions on the black hole horizon are distinctly different, which changes qualitatively the energy level characteristics for stationary states. The existence of energy levels inside a black hole opens up the fundamental new possibility of “looking” into the black hole using quantum methods, which is impossible within the framework of classical general relativity. More specifically, if there exist energy levels inside a black hole, then the internal structure of black holes can be studied based on the spectrum of transitions between external and internal levels.

The ultimate fate of black holes evaporating in the Hawking process has not yet been clarified, in particular, because large deviations from predictions of the classical gravitation theory are probable as the black hole mass approaches the Planck mass $M_{Pl} = \sqrt{\hbar c/G} \approx 10^{-5}$ g. The various effects that could stabilize an evaporating primordial black hole near the mass $M_{Pl}$ were discussed (for an overview of the models, see \textsuperscript{22}). These remnants of evaporating black holes, called “planckions”, were pro-
posed as candidate dark matter (hidden mass) particles in the Universe [23, 24]. Black holes of such masses should have distinct quantum properties and, therefore, the quantum states of particles we discuss can play a prominent role in the properties of these black holes, in particular, they can change the probability of their quantum decay or final evaporation. Previously, Markov discussed the so-called maximons, semi-closed worlds that are particle-like charged solutions in general relativity [25]. Here, we show that a new type of systems remotely resembling maximons, charged black holes with charges on inner quantum orbits, can exist. These systems can also represent dark matter if they are stable and were born in sufficient quantities at early cosmological epochs.

Although eternal black holes with internal spaces are probably not formed through classical gravitational collapse, they can emerge in quantum processes during particle collisions on accelerators (experiments on the Large Hadron Collider have already tested some of such models) or when ultrahigh-energy cosmic ray particles interact with the atmosphere (these interactions are observed experimentally on several detectors) if the theories with extra space dimensions are realized [24]. Since such mini-black holes are formed in processes involving charged particles, it is natural to expect the birth of not neutral black holes are formed in processes involving charged particles, it is natural to expect the birth of not neutral extra space dimensions are realized [24]. Since such mini-black holes are formed in processes involving charged particles, it is natural to expect the birth of not neutral particles, it is natural to expect the birth of not neutral black holes are formed in processes involving charged particles, it is natural to expect the birth of not neutral black holes are formed in processes involving charged particles, it is natural to expect the birth of not neutral.

2. THE DIRAC EQUATION IN THE REISSNER-NORDSTROM METRIC

Let us briefly describe the method of separation of variables in the Dirac equation [3] as applied to a Reissner-Nordström black hole with the metric

$$ds^2 = f dt^2 - f^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

where $M$ is the mass of the black hole, $Q$ is its charge, and $f = 1 - 2M/r + Q^2/r^2$. We use the units of measurement in which $c = G = \hbar = 1$. In the case of $|Q| < M$, the equation $f(r) = 0$ has two roots $r = r_\pm = M \pm \sqrt{M^2 - Q^2}$, the event horizon and the Cauchy horizon (internal horizon). In the case of $|Q| > M$, metric [4] describes a naked singularity without an event horizon, while $|Q| = M$ corresponds to an extremal black hole. The electromagnetic potential for a static black hole with charge $Q$ is $A_\mu = (Q/r, 0, 0, 0)$. Using the tetrad

$$e^{(a)}_\mu = \text{diag}(f^{1/2}, f^{-1/2}, r, r \sin \theta),$$

the Dirac equation [11] takes the form

$$\begin{align*}
\left[ i\gamma^0 f^{1/2} \frac{\partial}{\partial t} + i\gamma^1 \frac{1}{r} \frac{\partial}{\partial r} + i\gamma^2 \frac{1}{r} \frac{\partial}{\partial \theta} + \right. \\
+ i\gamma^3 \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} + i\gamma^1 Q^2 - 3Mr + 2r^2 + \\
+ i\gamma^2 \frac{\cos \theta}{2r \sin \theta} - \gamma^0 \frac{qQ}{rf^{1/2}} - m \right] \psi = 0. \quad (6)
\end{align*}$$

Following [3], to simplify this equation further, we will redefine the wave function

$$\psi = \frac{e^{-iEt}}{f^{1/4}r \sin^{1/2} \theta}$$

and single out the operator

$$K = \gamma^0 \gamma^1 \left( \gamma^2 \frac{\partial}{\partial \theta} + \gamma^3 \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right), \quad (8)$$

with integer eigenvalues $K \Psi = k \Psi$, where $k = 0, \pm 1, \pm 2, \ldots$. Below, we will not be interested in the angular part $Z(\theta, \phi)$ expressed in terms of spherical harmonics, but we assume the integral of $|Z(\theta, \phi)|^2$ over the solid angle to be equal to one. After these substitutions, the system breaks up into two pairs of equivalent equations. Writing $\Psi$ as

$$\Psi = Z(\theta, \phi) \left[ \frac{g(r)I_2}{ih(r)I_2} \right], \quad (9)$$

where $I_2$ is the column $(1, 1)^T$, we obtain a system of equations for the radial wave functions:

$$\begin{align*}
\frac{dg}{dr} - \frac{gk}{rf^{1/2}} + \frac{h}{f^{1/2}} \left[ \frac{1}{f^{1/2}} \left( E - \frac{Q^2}{r} \right) + m \right] = 0, \quad (10) \\
\frac{dh}{dr} + \frac{hk}{rf^{1/2}} - \frac{g}{f^{1/2}} \left[ \frac{1}{f^{1/2}} \left( E - \frac{Q^2}{r} \right) - m \right] = 0. \quad (11)
\end{align*}$$

These equations can be easily derived from Eqs. (39) in [3] if we set $e^\nu = e^{-\lambda} = f$ and transform the expression for the electromagnetic potential. Because of the divergence $f(r) \to \infty$ on the black hole horizon, Eqs. (10) and (11) belong to the class of differential equations with singular points.

The zeroth component of the fermionic field probability flux $j^0 = \psi \gamma^0 \psi$, the probability density, can be normalized to unity:

$$2 \int_0^r \frac{|g|^2 + |h|^2}{f(r)} dr = 1, \quad (12)$$

where we used Eqs. (7) and (9) and performed the integration $\int f^{1/2} d^3 r$ with the spatial metric tensor $\gamma_{\alpha \beta}$. The probability flux along the radius $j^1 = 2i(g^* h - gh^*)$ becomes zero, in particular, for real $g$ and $h$ when there are no traveling waves in the solution.
3. A NON-EXTREMAL BLACK HOLE WITH $|Q| < M$

We have $f \rightarrow Q^2/r^2$ at $Q \neq 0$ near the singularity $r \rightarrow 0$ and the system of equations (10), (11) for $k = 0$ has an asymptotic solution,

$$
g = C_1 + C_2 \frac{q^2}{r^2}, \quad h = -C_1 \frac{q^2}{Q^2} + C_2,
$$

where $C_1$ and $C_2$ are constants. If $k \neq 0$, then (10) and (11) take the form $dg/dr = kg/Q$ and $dh/dr = -kh/Q$ and their asymptotic solutions are

$$
g \propto \exp(kr/Q), \quad h \propto \exp(-kr/Q).
$$

Thus, the solutions enter into the singularity with zero and finite derivatives for $k = 0$ and $k \neq 0$, respectively. Solutions (13) and (14) are valid both for non-extremal black holes and for extremal ones and naked singularities.

Let us first consider the region below the internal Cauchy horizon $r < r_-$. According to the classification by I.D. Novikov, this is the $R$-region. The local structure of the spacetime in it is the same as that outside the black hole; in particular, $t$ and $r$ have the meaning of time and radial coordinates. As was shown in [17–21], classical orbits of charged particles can exist at $r < r_-$. In this section, we study the question about the quantum orbitals of particles below the Cauchy horizon.

To investigate the wave functions when $r \rightarrow r_-$, it is convenient to introduce a new variable $y = (1-r/r_-)^{1/2}$. Equations (10) and (11), to within $O(y^2)$, will then take the form

$$
y \frac{dg}{dy} + 2kpyg - 2r_- ph \left[ \left( E - \frac{qQ}{r_-} \right) p + my \right] = 0, \quad (15)
y \frac{dh}{dy} + 2kpyh + 2r_- ph \left[ \left( E - \frac{qQ}{r_-} \right) p - my \right] = 0, \quad (16)
$$

where $p = (r_+/r_- - 1)^{-1/2}$. Let us prove that a regular solution of system (15), (16) exists for $r \rightarrow r_-$ only in the case

$$
E = \frac{qQ}{r_-}, \quad (17)
$$

We write $g$ and $h$ in the form of series,

$$
g = y^s \sum_{n=0}^{\infty} a_n y^n, \quad h = y^w \sum_{n=0}^{\infty} b_n y^n,
$$

where $a_0 \neq 0$ and $b_0 \neq 0$. We reason from the contrary. Suppose that $E \neq qQ/r_-$. If $s \neq 0$ and $w \neq 0$, then substituting (15) into (13) and (14) and writing out the coefficients at the smallest powers of $y$, we obtain $s = w$ and then

$$
a_0 s = 2r_- \frac{p^2}{b_0} \left( E - \frac{qQ}{r_-} \right), \quad b_0 s = -2r_- \frac{p^2}{a_0} \left( E - \frac{qQ}{r_-} \right), \quad (19)
$$

Multiplying these relations, we have

$$
s^2 = -4r_-^2 p^4 \left( E - \frac{qQ}{r_-} \right)^2. \quad (20)
$$

Condition (20) can hold only if (17) is valid and $s = 0$, which contradicts our assumptions. Let now one of the quantities $s$ and $w$ be equal to zero, for example, let us set $s = 0$. We then find from (15) that $w \geq 1$. However, this value will contradict Eq. (16), because the power of $y$ at the coefficient $(E-qQ/r_-)$ is $s = 0$, while in the first term the exponent $w \geq 1$. The case of $w = 0$ is considered similarly. We again came to a contradiction. Consequently, the only condition under which Eqs. (15) and (16) are satisfied is specified by Eq. (17). Thus, the charge below the internal horizon of a Reissner-Nordström black hole in a stationary state $\propto e^{-iEt}$ can have only one fixed energy. Despite the fact that the solution of system (15), (16) for $r \rightarrow r_-$ under condition (17) formally exists, it cannot correspond to a real physical situation due to the divergence of the normalization integral (12) on the horizon $r_-$. Indeed, exact solutions of Eqs. (15) and (16) can be easily obtained under condition (17):

$$
g = C_1 e^{\lambda y} + C_2 e^{-\lambda y}, \quad (21)
$$

where $\lambda = 2p\sqrt{k^2 + r_-^2 m^2}$ and $h(r)$ is expressed in terms of $g(r)$ from (15). For the convergence of (12), we should choose $C_1 = -C_2$, but then $h \rightarrow \text{const}$ when $r \rightarrow r_-$. Similarly, we obtain $g \rightarrow \text{const}$ const when $r \rightarrow 0$. In both cases, (12) diverges.

Just as in the case of a Schwarzschild black hole [13], system (15), (16) for $E \neq qQ/r_-$ and $y \rightarrow 0$ has an irregular solution,

$$
g = C \sin(\alpha \ln y + \delta), \quad h = C \cos(\alpha \ln y + \delta), \quad (22)
$$

where $C$ and $\delta$ are constants, $\alpha = 2r_- p^2 (E - qQ/r_-)$. However, $|y|^2 + |h|^2 = |C|^2$ for this solution and (12) diverges on the horizon.

The solution at infinity $r \rightarrow \infty$ is the same as that in the case of an ordinary hydrogen atom:

$$
g = C_1 e^{-ir\sqrt{E^2 - m^2}} + C_2 e^{ir\sqrt{E^2 - m^2}}, \quad (23)
$$

where $h = -(E + m)^{-1} dq/dr$. A localized, exponentially decreasing solution exists only for $|E| < m$, i.e., the particlecentral charge interaction should have the pattern of attraction with a negative contribution to $E$.

To investigate the solution near the horizon $r \rightarrow r_+$, let us designate $y = (r/r_+ - 1)^{1/2}$ and $p = (1 - r_-/r_+)^{-1/2}$. We analyze the corresponding equations just as (15) and (16) and find that the charge has only one energy level,

$$
E = \frac{qQ}{r_+}, \quad (24)
$$

which was specified in (15). However, (12) again diverges. The conclusion about the divergence of the normalization
As an example, we chose the parameters \( \mu_k \) circumvented by the passage to a new variable where the difficulty with the behavior on the horizon was nonstationary solutions that are localized in the course of the external solution on the horizon was also an integral of the external solution on the horizon, reached in [12].

Divergence on the horizon points to the possibility of nonstationary solutions that are localized in the course of time on the horizons, while their energy tends to [21] and [17]. The resonant quasi-stationary levels outside a black hole were investigated in a number of papers, where the difficulty with the behavior on the horizon was circumvented by the passage to a new variable \( dr^*/dr = f^{-1}(r) \), which moves the horizon to \( r^* = -\infty \) (see [10]). However, the strictly stationary levels that we discuss here do not exist because of the above divergence on the horizon.

The difference between atoms with a set of levels and hydrogen-like atoms is attributable to the change of the boundary conditions for the Dirac equation. Quantization in the hydrogen atom follows from the finiteness of the wave function or the normalization probability integral. The presence of an event horizon changes the form of the boundary conditions and equations. As a consequence, only one energy level exists for a strictly stationary solution.

4. THE SOLUTION FOR AN EXTREMAL BLACK HOLE

In the case of an extremal black hole, \( M = |Q| \), both horizons coincide: \( r_+ = r_+ = r_h = M \). This changes the boundary condition and, as we will now show, creates conditions for the existence of a physically acceptable solution. The solution at the center when \( r \to 0 \) has the same asymptotics [13] and [14]. When \( r \to r_- \) from the inside, we introduce a new variable \( y = f^{1/2} = (r_h/r - 1) \).

Equations (10) and (11) will then take the form

\[
y'^2(1+y)^2\frac{dy}{dy} - hky(1+y) + grh \left[ E - \frac{qQ(1+y)}{r_h} - my \right] = 0.
\]

(25)

\[
y'^2(1+y)^2\frac{dy}{dy} + gky(1+y) - hrh \left[ E - \frac{qQ(1+y)}{r_h} + my \right] = 0.
\]

(26)

The combinations of quantities in (25) and (26) can be rewritten in standard physical units via dimensionless parameters as

\[
myr_h \to \mu = \frac{mM}{M^2}, \quad qQ \to \nu = \frac{qQ}{hc}.
\]

(27)

We can also write \( \mu = R_q/(2l_C) \), where \( R_q \) is the Schwarzschild radius and \( l_C \) is the Compton wavelength of the particle [13]. Similar to the previous case, it is proven that for a regular solution

\[
E = \frac{qQ}{r_h}.
\]

(28)

Equations (25) and (26) when \( y \to 0 \) then have asymptotic solutions,

\[
g = C_1 y^\kappa + C_2 y^{-\kappa}, \quad \text{where} \quad \kappa = \sqrt{k^2 + \mu^2 - \nu^2}.
\]

(29)

The solutions with \( C_2 = 0 \) are physical, because the part of the solution at \( C_2 \neq 0 \) makes a diverging contribution to (12), similar to the case with non-extremal black holes considered above. Then,

\[
h = C_1 y^{\kappa}\frac{k + \kappa}{\mu - \nu}.
\]

(30)

The contribution from (30) to (12) is finite under the condition \( 2\kappa - 2 > -1 \), which can be rewritten as

\[
k^2 + \mu^2 - \nu^2 > \frac{1}{4}.
\]

(31)

Hence it follows that the state with \( k = 0 \) is forbidden if \( 1/4 + \nu^2 - \mu^2 \geq 0 \). In view of the Pauli exclusion principle, each of the quantum levels found can be filled with two identical fermions. A numerical solution of Eqs. (10) and (11) for the internal region \( r < r_h \) is shown in the figure [11].

The irregular solution for \( E \neq qQ/r_h \) and \( y \to 0 \) in this case is

\[
g = -C \sin(\alpha/y + \delta), \quad h = C \cos(\alpha/y + \delta),
\]

(32)

where \( \alpha = r_h(E - qQ/r_h) \). For this solution, \(|g|^2 + |h|^2 = |C|^2 \) and (12) diverges.

To investigate the case of \( r \to r_h \) from the outside, we will set \( y = f^{1/2} = (1 - r_h/r) \). The corresponding equations are self-consistent for the same value of (28) and under condition (31) and have a physically acceptable (with a finite integral (12)) solution,

\[
g = C_1 y^\kappa, \quad h = C_1 y^\kappa\frac{k + \kappa}{\mu + \nu}.
\]

(33)
Solutions (30) and (33) lose their meaning at \( \mu = \pm \nu \), respectively. Since the black hole is extremal, with \( M = |Q| \), the conditions \( \mu = \pm \nu \) imply that the particle itself is extremal, \( m = |q| \). For ordinary particles, such as the proton or electron, this equality, of course, does not hold. For them, \( m \ll |q| \), but the concept of extremality itself loses its meaning due to the contribution of quantum effects. The solution of the Dirac equation in the metric of a classical naked Reissner-Nordström singularity with \( |Q| > M \) was found numerically in [13].

5. DISCUSSION

We showed that the stationary regular solution for a fermion has the energy level \( E = qQ/r_− \) below the internal Cauchy horizon, where \( q \) and \( q \) are the particle and black hole charges, respectively, and \( r_− \) is the radius of the Cauchy horizon. The existence of only one level is attributable to the properties of the wave equation when the Cauchy horizon is approached from the inside. There is also only one regular level with energy \( E = qQ/r_+ \) outside the event horizon and the solution at infinity corresponds to that for the hydrogen atom. However, the probability integral turned out to diverge on the horizons \( r_\pm \) and, therefore, the wave functions cannot be normalized to unity. This may imply that the real solution tends to the stationary solutions found only asymptotically, while the particle tends to be localized on the horizons. If quantum tunneling is disregarded, then, as is well known, the falling of a particle below the horizon, according to the clocks of a remote observer, will take an infinitely long time. Accordingly, the particle localization on the horizon will occur only asymptotically on long time scales. In the extremal case of \( |Q| = M \), the energies of the internal and external levels take on identical values of \( E = qQ/r_h \), where \( r_h \) is the radius of the extremal black hole horizon. In contrast to the case of a non-extremal black hole, at \( |Q| = M \) a physically acceptable solution with normalized wave functions exists both inside (below the Cauchy horizon) and outside the black hole.

The stationary solutions of the Dirac equation we considered show that black holes with charges on inner quantum orbits can exist. An alternative possibility is a naked singularity around which charge is distribution in the stationary case [13]. If \( q = −Q \), where \( q \) is the total charge of the particles at quantum levels, then the external metric of such a system is the Schwarzschild one. For an external observer, the system is neutral and interacts weakly with the surrounding matter. If, in addition, such systems are stable (which requires an additional study) and were born effectively in the early Universe, then they could be candidates for dark matter particles. Vronsky [14] considered a Dirac particle outside the horizon of a neutral Schwarzschild black hole and also came up with the idea of such systems as dark matter particles. However, if the black hole is neutral and the particle is charged, then the system carries a nonzero total electric charge and will interact via this charge with the surrounding matter. In the case of a Reissner-Nordström black hole with \( q = −Q \), the corresponding candidate dark matter particles are electrically neutral and interact very weakly.

If the black hole has internal and external energy levels, then quantum transitions between these levels are possible. This allows the charge falling into a Reissner-Nordström black hole to linger at a quantum level below the Cauchy horizon and probably to tunnel outward or into the internal universe. The quantum transition of a particle from the external stationary energy level (24) to the corresponding internal level (17) may be considered as an analog of the classical fall of a particle into a black hole. Since the energies at the external and internal levels for an extremal black hole are equal, no energy is released during the quantum transition. In the case of a non-extremal black hole, energy can be released during the transitions between quasi-stationary levels:

\[
\Delta E = E_− − E_+ = qQ \left( \frac{1}{r_−} − \frac{1}{r_+} \right) = \frac{2Mc_q}{Q} \sqrt{1 − \frac{Q^2}{GM^2}},
\]

where we restored the physical dimensions of the quantities in the last equality. We see that much energy can be radiated during such transitions for \( q^2 ∼ Q^2 \ll GM^2 \). If the systems under consideration are dark matter highly energetic particles that may contribute to the ultrahigh energy cosmic rays. There will be no such energy release for an extremal black hole, because \( \Delta E = 0 \). These transitions may turn out to be fundamentally important at the final stage of Hawking black hole evaporation, because they affect the phase volumes of the final and intermediate states of the particles produced during evaporation. The existence of the quantum levels we detected is a necessary condition for these quantum transitions, but the possibility of such transitions itself, of course, requires an additional study and justification. The transitions between levels, including those through the horizon, can occur via quantum tunneling with a probability proportional to \( \exp(−2\text{Im} S) \), where the action \( S = \int dp dr \) is calculated along a semiclassical trajectory [26], [27]. Note also the fundamental possibility of investigating the internal structure of black holes using quantum tunneling between external and internal states.

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