On the relation between 2 and $\infty$ in Galois cohomology of number fields

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Abstract: We remove the assumption ‘$p \neq 2$ or $k$ is totally imaginary’ from several well-known theorems on Galois groups with restricted ramification of number fields. For example, we show that the Galois group of the maximal extension of a number field $k$ which is unramified outside 2 has finite cohomological 2-dimension (also if $k$ has real places).

Keywords: Galois cohomology, restricted ramification, real places, cohomological dimension.

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1 Introduction

Number theorist’s nightmare, the prime number 2, frequently causes technical problems and requires additional efforts. In Galois cohomology the problems with $p = 2$ are essentially due to the fact that the decomposition groups of the real places are 2-groups and so the case of a totally imaginary number field is comparatively easier to deal with.

A classical object of study in number theory is Galois groups with restricted ramification. For a number field $k$, a set $S$ of primes of $k$ and a prime number $p$, one is interested in the Galois group $G_S(p) = G(k_S(p)\mid k)$ of the maximal $p$-extension $k_S(p)$ of $k$ which is unramified outside $S$. If $S$ is empty, then $G_S(p)$ is the Galois group of the so-called $p$-class field tower of $k$ and, besides the fact that it can be infinite (Golod-Šafarevič), not much is known about this group. The situation is easier in the case that $S$ contains the set $S_p$ of primes dividing $p$, where the cohomological dimension of $G_S(p)$ is known to be less than or equal to two (cf. [2], (8.3.17), (10.4.9)). However, there is an exception: if $p = 2$ and $k$ has at least one real place. If, in this exceptional case, $S$ contains all real places, then these places become complex in $k_S(2)$ and therefore $G_S(2)$, containing involutions, has infinite cohomological dimension. Furthermore, the virtual cohomological dimension $vcd G_S(2)$ is less than or equal to two in this
case, i.e. $G_S(2)$ has an open subgroup $U$ with $\text{cd} \leq 2$. The case when not all real places are in $S$ has been open so far and is the subject of this paper.

**Theorem 1** Let $k$ be a number field and let $S$ be a set of primes of $k$ which contains all primes dividing 2. If no real prime is in $S$, then $\text{cd} G_S(2) \leq 2$. If $S$ contains real primes, then they become complex in $k_S(2)$ and $\text{cd} G_S(2) = \infty$. Hence, for $p = 2$ and $T = S \cup S_R$, where $S_R$ is the set of real places of $k$. Then $\text{cd} G_S(2) = 2$ is infinite, in particular, it is nontrivial. Hence, for $S \supset S_2$ and $S \cap S_R = \emptyset$, the group $G_S(2)$ is of cohomological dimension 1 or 2. The key for the proof of theorem 1 is the following theorem 2 in the case $p = 2$ and $T = S \cup S_R$, where $S_R$ is the number theoretical analogue of Riemann’s existence theorem and was previously known under the assumption that $p$ is odd or that $S$ contains $S_R$ (see [9], (10.5.1)).

**Theorem 2** Let $k$ be a number field, $p$ a prime number and $T \supset S \supseteq S_p$ sets of primes of $k$. Then the canonical homomorphism

$$
\chi_2 G_S(2)) = \sum_{i=0}^2 (-1)^i \text{dim}_{\mathbb{F}_2} H^i(G_S(2)) = \text{the second partial Euler characteristic and } r_2 = \text{the number of complex places of } k.
$$

The key for the proof of theorem 1 is the following theorem 2 in the case $p = 2$ and $T = S \cup S_R$, where $S_R$ is the set of real places of $k$. Then $\text{cd} G_S(2) = 2$ is infinite, in particular, it is nontrivial. Hence, for $S \supset S_2$ and $S \cap S_R = \emptyset$, the group $G_S(2)$ is of cohomological dimension 1 or 2. The key for the proof of theorem 1 is the following theorem 2 in the case $p = 2$ and $T = S \cup S_R$, where $S_R$ is the set of real places of $k$. Then $\text{cd} G_S(2) = 2$ is infinite, in particular, it is nontrivial. Hence, for $S \supset S_2$ and $S \cap S_R = \emptyset$, the group $G_S(2)$ is of cohomological dimension 1 or 2. The next theorem gives a criterion for which case occurs. In condition (3) below, $\text{Cl}_S^2(k)(2)$ denotes the 2-torsion part of the $S$-ideal class group in the narrow sense of $k$.

**Theorem 3** Assume that $S \supset S_2$ and $S \cap S_R = \emptyset$. Then $\text{cd} G_S(2) = 1$ if and only if the following conditions (1)–(3) hold.

1. $S_2 = \{p_0\}$, i.e. there exist exactly one prime dividing 2 in $k$.
2. $S = \{p_0\} \cup \{\text{complex places}\}$.
3. $\text{Cl}_S^2(k)(2) = 0$.

In this case, $G_S(2)$ is a free pro-$2$-group of rank $r_2 + 1$ and $p_0$ does not split in $k_{S \cup S_0}(2)$. In particular, if $k$ is totally real and $G_S(2)$ is free, then $k_S(2) = k_{\infty}(2)$.

Let $k$ be a number field, $p$ a prime number and $S \supset S_p$ a set of places of $k$. A (necessarily infinite) extension $K/k$ is called $p$-$S$-closed if it has no $p$-extension which is unramified outside $S$. If $p$ is odd and $K$ is $p$-$S$-closed, then the group $\text{Cl}_S(K(K_{\mu_p}))/\text{Cl}_{S_p}(K(K_{\mu_p}))$ is trivial for $j = 0, -1$, where $\mu_p$ is the group of $p$-th roots of unity, $(p)$ denotes the $p$-torsion part and $(j)$ the $j$-th Tate-twist (see [9], (10.4.7)). The corresponding result for $p = 2$ is the following
Theorem 4 Let $k$ be a number field, $S \supseteq S_2$ a set of primes of $k$ and $K$ a $2$-$S$-closed extension of $k$. Then the following holds.

(i) $\text{Cl}_S(K(\mu_4))(2) = 0$.

(ii) $\text{Cl}_S^0(K)(2) = 0$.

Remarks: 1. The triviality of $\text{Cl}(K)(2)$, and hence also that of $\text{Cl}_S(K)(2)$, follows easily from the principal ideal theorem; assertions (i) and (ii) do not.

2. In (i) one can replace $K(\mu_4)$ by any totally imaginary extension of degree 2 of $K$ in $K_S(2)$.

Finally, we consider the full extension $k_S$, i.e. the maximal extension of $k$ which is unramified outside $S$, and its Galois group $G_S = G(k_S|k)$.

Theorem 5 Let $k$ be a number field and $S$ a set of primes of $k$ containing all primes dividing 2. Then $\text{vcd}_2 G_S \leq 2$ and $\text{cd}_2 G_S \leq 2$ if and only if $S$ contains no real primes. For every discrete $G_S(2)$-module $A$ the inflation maps

$$\inf : H^i(G_S(2), A) \longrightarrow H^i(G_S, A)(2)$$

are isomorphisms for all $i \geq 1$.

Remark: If $\text{cd} G_S(K)(2) = 2$ (e.g. if $K$ contains at least two primes dividing 2) for some finite subextension $K$ of $k$ in $k_S$ then $\text{vcd}_2 G_S = 2$. This is always the case if $S \supseteq S_R$ because the class numbers of the cyclotomic fields $\mathbb{Q}(\mu_{2^n})$ are nontrivial for $n \gg 0$. But, for example, we do not know whether $\text{cd}_2 G(\mathbb{Q}_{S_R}|\mathbb{Q})$ equals 1 or 2. The answer would be ‘2’ if at least one of the real cyclotomic fields $\mathbb{Q}(\mu_{2^n})^+, n = 2, 3, \ldots$, would have a nontrivial class number. But this is unknown.

In section 5 we investigate the relation between the cohomology of the group $G_S(k)$ and the modified étale cohomology of the scheme Spec($\mathcal{O}_{k,S}$). A discrete $G_S(k)$-module $A$ induces a locally constant sheaf on Spec($\mathcal{O}_{k,S}$)$_{\text{ét},\text{mod}}$, which we will denote by the same letter. We show the following theorem which is well-known if $S$ contains all real primes (and also for odd $p$).

Theorem 6 Let $k$ be a number field and $S$ a finite set of primes of $k$ containing all primes dividing 2. Then for every 2-primary discrete $G_S(k)$-module $A$ the natural comparison maps

$$H^i(G_S(k), A) \longrightarrow H^i_{\text{ét},\text{mod}}(\text{Spec}(\mathcal{O}_{k,S}), A)$$

are isomorphisms for all $i \geq 0$.

For finite $A$ it is not difficult to show that the modified étale cohomology groups on the right hand side of the comparison map are finite and that they vanish for $i \geq 3$ if $S$ contains no real primes. Therefore one could deduce theorem 4 (with $G_S(k)(2)$ replaced by $G_S(k)$) from theorem 6. However, in
order to prove theorem 1, one needs information on the interaction between the decomposition groups of the real primes and so theorem 1 and theorem 6 are both consequences of theorem 2.

The main ingredients in the proofs of theorems 1–5 are Poitou-Tate duality, the validity of the weak Leopoldt-conjecture for the cyclotomic $\mathbb{Z}_p$-extension and, most essential, the systematic use of free products of bundles of profinite groups over a topological base. The reason that the above theorems had not been proven earlier seems to be a psychological one. At least the author always thought that one has to prove theorem 1 first, before showing the other assertions. For example, theorem 2 for $p = 2$, $T = S_2 \cup S_R$ and $S = S_2$ was known if $k_{S_2}(2) = k_\infty(2)$ (see [12], §4.2 for the case $k = \mathbb{Q}$ and [13], Satz 1.4 for the general case). But now it is theorem 2 which is used in the proof of theorem 1. Finally, we should mention that theorem 1 was formulated as a conjecture in O. Neumann’s article [10].

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2 Free products of inertia groups

In this section we briefly collect some facts on free products of profinite groups and how they naturally occur in number theory. For a more detailed presentation and for proofs of the facts cited below we refer the reader to [9], chap. IV and chap. X, §1.

A profinite space is a topological space which is compact and totally disconnected. Equivalently, a profinite space is a topological inverse limit of finite discrete spaces. A profinite group is a group object in the category of profinite spaces. It can be shown that a profinite group is the inverse limit of finite groups. A full class of finite groups $c$ is a full subcategory of the category of all finite groups which is closed under taking subgroups, quotients and extensions. A pro-$c$-group is a profinite group which is the inverse limit of groups in $c$.

Let $T$ be a profinite space. A bundle of profinite groups $\mathcal{G}$ over $T$ is a group object in the category of profinite spaces over $T$. We say that $\mathcal{G}$ is a bundle of pro-$c$-groups if the fibre $\mathcal{G}_t$ of $\mathcal{G}$ over every point $t \in T$ is a pro-$c$-group. The functor “constant bundle”, which assigns to a pro-$c$-group $G$ the bundle $pr_2 : G \times T \to T$ has a left adjoint

\[
\begin{align*}
\{\text{bundles of pro-$c$-groups over } T\} & \to \{\text{pro-$c$-groups}\} \\
\mathcal{G} & \mapsto \star_T \mathcal{G}.
\end{align*}
\]

The image $\star_T \mathcal{G}$ of a bundle $\mathcal{G}$ under this functor is called its free pro-$c$-product. It satisfies a universal property which is determined by the functor adjunction. Bundles of pro-$c$-groups often arise in the following way:

Let $G$ be a pro-$c$-group and assume we are given a continuous family of closed subgroups of $G$, i.e. a family of closed subgroups $\{G_t\}_{t \in T}$ indexed by
the points of a profinite space $T$ which has the property that for every open subgroup $U \subset G$ the set $T(U) = \{ t \in T \mid G \cap U \}$ is open in $T$. Then

$$G = \{ (g,t) \in G \times T \mid G_t \subset U \}$$

is in a natural way a bundle of pro-$\mathcal{G}$-groups over $T$. We have a canonical homomorphism

$$\phi_T : G \longrightarrow G$$

and we say that $G$ is the free product of the family $\{ G_t \}_{t \in T}$ if $\phi$ is an isomorphism.

The usual free pro-$\mathcal{G}$-product of a discrete family of pro-$\mathcal{G}$-groups as defined in various places in the literature (e.g. [8]) fits into the picture as follows. For a family $\{ G_i \}_{i \in I}$ we consider the disjoint union $(\bigcup_i G_i) \cup \{ * \}$ of the $G_i$ and one external point *. Equipped with a suitable topology, this is a bundle of pro-$\mathcal{G}$-groups over the one-point compactification $\bar{I} = I \cup \{ * \}$ of $I$ and the free pro-$\mathcal{G}$-product of the family $\{ G_i \}_{i \in I}$ coincides with that of the bundle (cf. [9], chap.IV, §3, examples 2 and 4). For the free product of a discrete family of pro-$\mathcal{G}$-groups we have the following profinite version of Kurosh’s subgroup theorem (see [2] or [9], (4.2.1)).

**Theorem 2.1** Let $G = \bigast_{i \in I} G_i$ be the free pro-$\mathcal{G}$-product of the discrete family $G_i$ and let $H$ be an open subgroup of $G$. Then there exist systems $S_i$ of representatives $s_i$ of the double coset decomposition $G = \bigcup_{s_i \in S_i} H s_i G_i$ for all $i$ and a free pro-$\mathcal{G}$-group $F \subset G$ of finite rank

$$\text{rk}(F) = \sum_{i \in I} \left( \frac{[G : H] \cdot \#S_i}{G} - (G : H) + 1 \right),$$

such that the natural inclusions induce a free product decomposition

$$H = \bigast_{i,s_i} (G_i^{s_i} \cap H) \ast F,$$

where $G_i^{s_i} = s_i G_i s_i^{-1}$ denotes the conjugate subgroup.

In number theory, continuous families of pro-$\mathcal{G}$-groups occur in the following way. For a number field $k$ we denote the one-point compactification of the set of all places of $k$ by $\text{Sp}(k)$. The compactifying point will be denoted by $\eta_k$ and should be thought as the generic point of the scheme $\text{Spec}(\mathcal{O}_k)$ in the sense of algebraic geometry or as the trivial valuation of $k$ from the point of view of valuation theory. For an infinite extension $K/k$, we set

$$\text{Sp}(K) = \lim_{\leftarrow k'} \text{Sp}(k'),$$

where $k'$ runs through all finite subextensions of $k$ in $K$. The complement of the (closed and open) subset of all archimedean places of $K$ in $\text{Sp}(K)$ is naturally isomorphic to $\text{Spec}(\mathcal{O}_K)$ endowed with the constructible topology (see [3], (4.2.1)).
chap.I, §7, (7.2.11) for the definition of the constructible topology of a scheme.

Let $S$ be a set of primes of $k$ and $\bar{S}$ its closure in $\text{Sp}(k)$ ($\bar{S} = S$ if $S$ is finite, $\bar{S} = S \cup \{p\}$ if $S$ is infinite). The pre-image $\bar{S}(K)$ of $S$ under the natural projection $\text{Sp}(K) \to \text{Sp}(k)$ is the closure of the set $S(K)$ of all prolongations of primes in $S$ to $K$ in $\text{Sp}(K)$.

Now assume that $M \supset K \supset k$ are possibly infinite extensions of $k$ such that $M|K$ is Galois and $G(M|K)$ is a pro-$\mathfrak{c}$-group. The natural projection $\bar{S}(M) \to \bar{S}(K)$ has a section (in fact, there are many of them). For a fixed section $s : \bar{S}(K) \to \bar{S}(M)$ we consider the family of inertia groups $\{T_{s,p}(M|K)\}_{p \in \bar{S}(K)}$, where by convention $T_{\eta M} = \{1\}$. Since a finite extension of number fields is ramified only at finitely many primes, this is a continuous family of subgroups of $G(M|K)$ indexed by $\bar{S}(K)$. We obtain a natural homomorphism

$$\phi : \ast_{\bar{S}(K)} T_{s,p}(M|K) \longrightarrow G(M|K),$$

which we also write in the form

$$\phi : \ast_{p \in \bar{S}(K)} T_p(M|K) \longrightarrow G(M|K).$$

The cohomology groups of the free product on the left hand side with coefficients in a trivial module do not depend on the particularly chosen section $s$. The question, however, whether the homomorphism $\phi$ is an isomorphism does depend on $s$. Moreover, if $s$ is a section for which $\phi$ is an isomorphism, we always find a section $s'$ for which it is not, at least if $c$ is not the class of $p$-groups, where $p$ is a prime number. In the case of pro-$p$-groups this pathology does not occur because of the following easy and well-known

**Lemma 2.2** Let $p$ be a prime number and let $\phi : G' \longrightarrow G$ be a (continuous) homomorphism of pro-$p$-groups. Let $A$ be $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}_p/\mathbb{Z}_p$ with trivial action. Then $\phi$ is an isomorphism if and only if the induced homomorphism

$$H^i(\phi, A) : H^i(G, A) \longrightarrow H^i(G', A)$$

is an isomorphism for $i = 1$ and injective for $i = 2$.

In the number theoretical situation above, we have the following formula for the cohomology of the free product with values in a torsion group $A$ (considered as a module with trivial action) and for $i \geq 1$:

$$H^i\left(\ast_{p \in S(K)} T_p(M|K), A\right) = \varprojlim_{k'} \bigoplus_{p \in \bar{S}(k')} H^i(T_p(M'|k'), A),$$

where $k'$ runs through all finite subextensions of $k$ in $K$ and $M'$ is the maximal pro-$\mathfrak{c}$ Galois subextension of $M|k'$ (so $M = \varinjlim M'$). The limit on the right hand side depends on $K$ and not on $k$ and we denote it by

$$\bigoplus_{p \in \bar{S}(K)} H^i(T_p(M|K), A).$$

If $K|k$ is Galois, then this limit is the maximal discrete $G(K|k)$-submodule of the product $\prod_{p \in S(K)} H^i(T_p(M|K), A)$.
3 Proof of theorem 2

Let us first remark that for \( p \in T \setminus S(k) \) the inertia group has the following structure:
- if \( p \) is nonarchimedean and \( N(p) \equiv 1 \mod p \) (i.e. if there is a primitive \( p \)-th root of unity in \( k_p \)), then \( T(k_p(p)|k_p) \) is a free pro-\( p \)-group of rank 1, i.e. isomorphic to \( \mathbb{Z}_p \).
- if \( p \) is nonarchimedean and \( N(p) \not\equiv 1 \mod p \), then \( T(k_p(p)|k_p) = \{1\} \).
- if \( p \) is real and \( p = 2 \), then \( T(k_p(p)|k_p) \cong \mathbb{Z}/2\mathbb{Z} \).
- if \( p \) is real and \( p \neq 2 \) or if \( p \) is complex, then \( T(k_p(p)|k_p) = \{1\} \).

If \( p \) is odd or if \( p = 2 \) and \( S \supset S_R \), then theorem 2 is known (see [1], (10.5.1)).

So we assume that \( p = 2 \) and \( S \not\supset S_R \). For a pro-2-group \( G \) we use the notation \( H^1(G) \) for \( H^1(G, \mathbb{Z}/2\mathbb{Z}) \).

We start with the following

Lemma 3.1 Let \( G \) and \( G' \) be pro-2-groups which are generated by involutions and assume that \( H^2(G, \mathbb{Q}_2/\mathbb{Z}_2) = 0 = H^2(G', \mathbb{Q}_2/\mathbb{Z}_2) \). Let \( \phi : G' \to G \) be a (continuous) homomorphism. Then the following assertions are equivalent.

(i) \( \phi \) is an isomorphism.
(ii) \( H^1(\phi) : H^1(G) \to H^1(G') \) is an isomorphism.
(iii) \( H^2(\phi) : H^2(G) \to H^2(G') \) is an isomorphism.

**Proof:** Clearly, (i) implies (ii) and (iii) and, by lemma 2.2, (ii) and (iii) together imply (i). So it remains to show that (ii) and (iii) are equivalent. Since \( H^2(G, \mathbb{Q}_2/\mathbb{Z}_2) = 0 \), the exact sequence \( 0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}_2/\mathbb{Z}_2 \to \mathbb{Q}_2/\mathbb{Z}_2 \to 0 \) induces the four term exact sequence

\[
0 \to H^1(G) \xrightarrow{\alpha} H^1(G, \mathbb{Q}_2/\mathbb{Z}_2) \xrightarrow{\beta} H^1(G, \mathbb{Q}_2/\mathbb{Z}_2) \xrightarrow{\gamma} H^2(G) \to 0.
\]

Since \( G \) is generated by involutions, \( \alpha \) is an isomorphism. Hence \( \beta \) is zero and \( \gamma \) is an isomorphism. The same argument also applies to \( G' \) and therefore (ii) and (iii) are both equivalent to

(iv) \( H^1(\phi, \mathbb{Q}_2/\mathbb{Z}_2) : H^1(G, \mathbb{Q}_2/\mathbb{Z}_2) \to H^1(G', \mathbb{Q}_2/\mathbb{Z}_2) \) is an isomorphism.

This concludes the proof. \( \square \)

We show theorem 2 first in the special case \( T = S_2 \cup S_R, S = S_2 \). The groups \( \star_{p \in S_2(k_{S_2}(2))} T(k_p(2)|k_p) \) and \( G(k_{S_2\cup S_2}(2)|k_{S_2}(2)) \) are both generated by involutions. Since \( H^2(T(k_p(2)|k_p), \mathbb{Q}_2/\mathbb{Z}_2) = 0 \) for every \( p \in S_2(k_{S_2}(2)) \), we have

\[
H^2(\star_{p \in S_2(k_{S_2}(2))} T(k_p(2)|k_p), \mathbb{Q}_2/\mathbb{Z}_2) = 0.
\]

By [1], (10.4.8), the inflation map

\[
H^2(G(k_{S_2\cup S_2}(2)|k_{S_2}(2)), \mathbb{Q}_2/\mathbb{Z}_2) \to H^2(G(k_{S_2\cup S_2}(2)|k_{S_2}(2)), \mathbb{Q}_2/\mathbb{Z}_2)
\]
is an isomorphism and, since \(k_{S_2}(2)\) contains the cyclotomic \(\mathbb{Z}_2\)-extension \(k_\infty(2)\) of \(k\), the validity of the weak Leopoldt-conjecture for the cyclotomic \(\mathbb{Z}_p\)-extension (see \([9], \text{(10.3.25)}\)) implies (by \([1], \text{(10.3.22)}\)) that

\[
H^2(G(k_{S_2\cup S_0}(2)|k_{S_2}(2)), \mathbb{Q}_2/\mathbb{Z}_2) = 0.
\]

By lemma \([1, \text{1.21}]\) and the calculation of the cohomology of free products (see \([9], \text{(10.4.8)}\)), this concludes the proof of theorem 2 in

\[
\text{Proposition 3.2}
\]

\[
\phi_T: T(K_p(2)|K_p) \rightarrow G(K_T(2)|K).
\]

Furthermore, we have

\[
\text{Proposition 3.2. Let } k\text{ be a number field, } p\text{ a prime number and } T \supset S \supset S_p\text{ sets of primes in } k. \text{ Let } K = p-S_p\text{-closed extension of } k. \text{ Then the following assertions are equivalent.}
\]

(i) The natural homomorphism

\[
\phi_{T,S_p} : \bigoplus_{p \in T \setminus S_p(K)} T(K_p(2)|K_p) \rightarrow G(K_T(2)|K)
\]

is an isomorphism.
(ii) The natural homomorphisms
\[
\phi_{T,S} : \bigoplus_{p \in T \setminus S(K_S(p))} T(K_p|K_p) \to G(K_T(p)|K_S(p))
\]
and
\[
\phi_{S,S_p} : \bigoplus_{p \in S \setminus S_p(K)} T(K_p|K_p) \to G(K_S(p)|K)
\]
are isomorphisms.

Here \(*\) denotes the free pro-p-product.

**Proof:** If \(\phi_{T,S_p}\) is an isomorphism, then also \(\phi_{S,S_p}\) is an isomorphism. Furthermore, a straightforward application of theorem 2.1 shows that also \(\phi_{T,S}\) is an isomorphism in this case. Let us show the converse statement. Assume that \(\phi_{T,S}\) and \(\phi_{S,S_p}\) are isomorphisms. Note that all primes in \(S \setminus S_p(K_S(p))\) split completely in \(K_T(p)|K_S(p)\). Therefore the extension of pro-p-groups
\[
1 \to G(K_T(p)|K_S(p)) \to G(K_T(p)|K) \to G(K_S(p)|K) \to 1
\]
splits. By lemma 2.2, we have to show that the induced homomorphism
\[
H^i(\phi_{T,S_p}) : H^i(G(K_T(p)|K)) \to \bigoplus_{p \in T \setminus S_p(K)} H^i(T(K_p|K_p))
\]
is an isomorphism for \(i = 1\) and injective for \(i = 2\) (coefficients \(\mathbb{Z}/p\mathbb{Z}\)). This follows easily from the Hochschild-Serre spectral sequence associated to the split exact sequence [1]:
\[
E_2^{ij} = H^i(G(K_S(p)|K), H^j(G(K_T(p)|K_S(p)))) \Rightarrow H^{i+j}(G(K_T(p)|K)).
\]
First of all, the differentials \(d_2\) are zero (\(-d_2\) is the cup-product with the extension class, see [2] (2.1.8)). Furthermore, every prime in \(T \setminus S(K)\) splits completely in \(K_S(p)|K\) because these primes are unramified in \(K_S(p)|K\) and \(K\) contains \(K_\infty(p)\). Since \(\phi_{T,S}\) is an isomorphism, the \(G(K_S(p)|K)\)-module \((j \geq 1)\)
\[
H^j(G(K_T(p)|K_S(p))) = \bigoplus_{p \in T \setminus S(K_S(p))} H^j(T(K_p|K_p))
\]
is cohomologically trivial. Therefore we obtain short exact sequences
\[
0 \to H^i(K_S(p)|K) \to H^i(K_T(p)|K) \to \bigoplus_{p \in T \setminus S(K)} H^i(T(K_p|K_p)) \to 0
\]
for \(i = 1, 2\), and the result follows from the five-lemma. \(\square\)
Now we can prove theorem 2 in the general case. It is true for odd \( p \) and for \( p = 2 \) in the special cases \( T = S_2 \cup S_R \), \( S = S_2 \) and \( T = \{ \text{all primes} \} \), \( S = S_2 \cup S_R \) and \( K = k_{S_2}(2) \), we obtain theorem 2 in the 'extremal' case \( T = \{ \text{all primes} \} \), \( S = S_2 \) and \( T = \{ \text{all primes} \} \) and \( S = S_2 \cup S_R \). Applying proposition 3.2 in the situation \( p = 2 \), \( T = \{ \text{all primes} \} \), \( S = S_2 \cup S_R \) and \( K = k_{S_2}(2) \), we obtain theorem 2 in the 'extremal' case \( T = \{ \text{all primes} \} \), \( S = S_2 \) and \( T = \{ \text{all primes} \} \) and \( S = S_2 \cup S_R \). Applying proposition 3.2 again, we obtain the case \( T = \{ \text{all primes} \} \) and \( S \) arbitrary and then the general case. This concludes the proof of theorem 2.

A straightforward limit process shows the following variant of theorem 2.

**Theorem 2'** Let \( k \) be a number field, \( p \) a prime number and \( T \supset S \supseteq S_p \) sets of primes of \( k \). Let \( K \) be a \( p \)-\( S \)-closed extension field of \( k \). Then the canonical homomorphism

\[
\ast_{p \in T \setminus S(K)} T(K_p(p)|K_p) \rightarrow G(K_T(p)|K)
\]

is an isomorphism.

### 4 Proofs of the remaining statements

In order to prove theorem 1, we may assume that \( S \not\supset S_R \) and we investigate the Hochschild-Serre spectral sequence

\[
E^{ij}_2 = H^i(G_S(2), H^j(G(k_{S \cup S_R}(2)|k_S(2)))) \Rightarrow H^{i+j}(G_{S \cup S_R}(2)),
\]

where the omitted coefficient are \( \mathbb{Z}/2\mathbb{Z} = \mu_2 \). By theorem 3 we have complete control over the \( G_S(2) \)-modules \( H^j(G(k_{S \cup S_R}(2)|k_S(2))) \), which are for \( j \geq 1 \) isomorphic to

\[
\text{Ind}_{G_S(2)} \bigoplus_{p \in S_R \setminus S(k)} H^j(G(\mathbb{C}|\mathbb{R})).
\]

In particular, \( E^{ij}_2 = 0 \) for \( ij \neq 0 \). Therefore the spectral sequence induces an exact sequence

\[
(2) \quad 0 \rightarrow H^1(G_S(2)) \rightarrow H^1(G_{S \cup S_R}(2)) \rightarrow \bigoplus_{p \in S_R \setminus S(k)} H^1(G(\mathbb{C}|\mathbb{R})) \rightarrow
\]

\[
H^2(G_S(2)) \rightarrow H^2(G_{S \cup S_R}(2)) \rightarrow \bigoplus_{p \in S_R \setminus S(k)} H^2(G(\mathbb{C}|\mathbb{R})) \rightarrow 0
\]

and exact sequences

\[
(3) \quad 0 \rightarrow H^i(G_S(2)) \rightarrow H^i(G_{S \cup S_R}(2)) \rightarrow \bigoplus_{p \in S_R \setminus S(k)} H^i(G(\mathbb{C}|\mathbb{R})) \rightarrow 0.
\]
for $i \geq 3$. If $S$ is finite, this shows the finiteness statement on the cohomology of $G_S(2)$ and that
\[ \chi_2(G_S(2)) = \chi_2(G_{S \cup S_R}(2)). \]
But $\chi_2(G_{S \cup S_R}(2)) = \chi_2(G_S \cup S_R) = -r_2$ (see [9], (8.6.16) and (10.4.8)).

For arbitrary $S$ and $i \geq 3$ the restriction map
\[ H^i(G_{S \cup S_R}(2)) \to \bigoplus_{p \in S \cap S_R} H^i(G(C|\mathbb{R})) \]
is an isomorphism (see [9], (8.6.13)(ii) and (10.4.8)). This together with [9] shows that the natural homomorphism
\[ H^i(G_S(2)) \to \bigoplus_{p \in S \cap S_R} H^i(G(C|\mathbb{R})) \]
is an isomorphism for $i \geq 3$. Therefore $\text{cd} G_S(2) \leq 2$ if $S \cap S_R = \emptyset$. For later use we formulate the last result as a proposition.

**Proposition 4.1** Let $k$ be a number field and $S \supset S_2$ a set of primes. Then the natural homomorphism
\[ H^i(G_S(2), \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{p \in S \cap S_R} H^i(G(C|\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \]
is an isomorphism for $i \geq 3$.

In order to conclude the proof of theorem [9] it remains to show that every real prime in $S$ ramifies in $k_S(2)$. Let $S_f$ be the subset of nonarchimedean primes in $S$. Then theorem [9] yields an isomorphism
\[ * \quad p \in S_R(k_S(2)) \quad T(k_p(2)|k_p) \cong G(k_S(2)|k_{S_f}(2)) \]
which shows the required assertion. This finishes the proof of theorem [9].

Now we prove theorem [9]. To fix conventions, we recall the following definitions. For a set $S$ of primes of $k$ the group $O_{k,S}^\times$ of $S$-units is defined as the subgroup in $k^\times$ of those elements which are units at every finite prime not in $S$ and positive at every real place not in $S$. The $S$-ideal class group $\text{Cl}_{0}^S(k)$ in the narrow sense of $k$ is the quotient of the group of fractional ideals of $k$ by the subgroup generated by the nonarchimedean primes in $S$ and the principal ideals $(a)$ with $a$ positive at every real place of $k$ not contained in $S$. In particular, $\text{Cl}_0^S(k) = \text{Cl}^S(k)$ is the ideal class group in the narrow sense and $\text{Cl}^S_{S \cup S_R}(k) = \text{Cl}_S(k)$ is the usual $S$-ideal class group. By class field theory,
\( Cl^0_S(k) \) is isomorphic to the Galois group of the maximal abelian extension of \( k \) which is unramified outside \( S \) and in which every prime in \( S \) splits completely. By Kummer theory, we can replace condition (3) of theorem \( \circ \) by the following condition
\[
(3') \quad \{ x \in k^\times \mid x \in k_p^\times \text{ and } 2 \mid v_p(x) \text{ for every finite prime } p \} = k^\times 2.
\]

**Lemma 4.2** If \( S \supseteq S_2 \) and \( cdG_{S_2}(2) = 1 \), then \( S = S_2 \).

**Proof:** By theorem \( \star \), we have an isomorphism
\[
\bigotimes_{p \in S \setminus S_2(k_{S_2}(2))} T(k_p(2)|k_p) \cong G(k_S(2)|k_{S_2}(2))
\]
Since for nonarchimedean primes \( p \notin S_2 \) the maximal unramified 2-extension of \( k_p \) is realized by \( k_{\infty}(2) \subset k_{S_2}(2) \), this shows that for \( p \in S \setminus S_2 \) the maximal 2-extension of the local field \( k_p \) is realized by \( k_S(2) \) or, in other words, the natural homomorphism
\[
G(k_p(2)|k_p) \to G_S(2)
\]
is injective. But for these primes we have \( cd G(k_p(2)|k_p) \geq 2 \) which shows that \( S \setminus S_2 = \emptyset \). \( \square \)

Now assume that \( G_{S_2}(2) \) is free. For a prime \( p \) we denote the local group \( G(k_p(2)|k_p) \) by \( G_p \) and the inertia group \( T(k_p(2)|k_p) \) by \( T_p \). By Čebotarev’s density theorem, we find a finite set of nonarchimedean primes \( T \supset S_2 \) such that the natural homomorphism
\[
H^1(G_{S_2}) \to \bigoplus_{p \in T \setminus S} H^1(G_p/T_p)
\]
is an isomorphism. It is then an easy exercise using lemma \( \star \) to show that the natural homomorphism
\[
\bigotimes_{p \in T \setminus S_2} G_p/T_p \to G_{S_2}(2)
\]
is an isomorphism. Theorem \( \star \) for \( T = S_2 \cup S \) and \( S = S_2 \) and the same arguments as in the proof of proposition \( \star \) show that the natural homomorphism
\[
\bigotimes_{p \in T \setminus S_2} G_p/T_p \to G_{S_2}(2)
\]
is an isomorphism. Then, by \( [10] \), Theorem 6) or \( [9] \), (10.7.2)), we obtain the conditions (1)–(3) and that the unique prime \( p_0 \) dividing 2 in \( k \) does not split in \( k_{S_2 \cup S} \). If, on the other hand, conditions (1)–(3) of theorem \( \star \) are satisfied, then we obtain (loc. cit.) the above isomorphism and deduce that \( G_{S_2}(2) \) is free. The statement on the rank of \( G_{S_2}(2) \) follows from \( \chi_2(G_{S_2}(2)) = -r_2 \). If \( k \) is totally real, then the homomorphism
\[
G_{S_2}(2) \to G(k_{\infty}(2)|k)
\]
is a surjection of free pro-2-groups of rank 1 and hence an isomorphism. This concludes the proof of theorem 3.

Next we show theorem 4. Let $S$ be a set of finite primes of $k$ and $\Sigma = S \cup S_R$. If $S$ is finite, then the image of the group of $\Sigma$-units of $k$ under the logarithm map $\text{Log} : \mathcal{O}_k^\times \rightarrow \bigoplus_{v \in \Sigma} \mathbb{R}$, $a \mapsto (\log |a_v|)_{v \in S}$ is a lattice of rank equal to $#S + r_1 + r_2 - 1$ (Dirichlet’s unit theorem). Complementary to this map is the signature map (which is also defined for infinite $S$)

$$
\text{Sign}_{k,S} : \mathcal{O}_k^\times \rightarrow \bigoplus_{v \in S_R} \mathbb{R}^\times / \mathbb{R}^\times 2.
$$

More or less by definition, there exists a five-term exact sequence

$$
0 \rightarrow \mathcal{O}_k^\times \rightarrow \mathcal{O}_k^\times_{\Sigma} \rightarrow \bigoplus_{v \in S_R(k)} \mathbb{R}^\times / \mathbb{R}^\times 2 \rightarrow \text{Cl}_S^0(k) \rightarrow \text{Cl}_\Sigma^0(k) \rightarrow 0,
$$

and so the cokernel of $\text{Sign}_{k,S}$ measures the difference between the usual $S$-ideal class group $\text{Cl}_S(k) = \text{Cl}_S^0(k)$ and that in the narrow sense. Of course this discussion is void if $k$ is totally imaginary. If $K$ is an infinite extension of $k$, we define the signature map

$$
\text{Sign}_{K,S} : \mathcal{O}_K^\times \rightarrow \lim_{\kappa \rightarrow K} \bigoplus_{v \in S_R(k')} \mathbb{R}^\times / \mathbb{R}^\times
$$

as the limit over the signature maps $\text{Sign}_{k',S}$, where $k'$ runs through all finite subextension $k'|k$ of $K|k$. If $K$ is 2-$S$-closed, then $\text{Cl}_S(K)(2) = 0$ and so statement (ii) of theorem 4 is equivalent to the statement that $\text{Sign}_K$ is surjective.

Now assume that $k$, $S$, $K$ are as in theorem 4. By theorem 1 all real places in $S$ become complex in $K$. By the principal ideal theorem, $\text{Cl}(K)(2) = 2$ and so statement (i) and (ii) are trivial if $K$ is totally imaginary (note that $K = K(\mu_4)$ in this case). So we may assume that $S_R(K) \neq \emptyset$ and, by theorem 1, we may suppose $S \cap S_R = \emptyset$.

Let $K' = K(\mu_4)$. Then $K'$ is totally imaginary and $G = G(K'|K)$ is cyclic of order 2. Let $\Sigma = S \cup S_R$ and let $K_\Sigma$ be the maximal (not just the pro-2) extension of $K$ which is unramified outside $\Sigma$. Inspecting the Hochschild-Serre spectral sequence associated to $K_\Sigma|K(\Sigma)(2)|K$ and using the well-known calculation of $H^1(G(K_\Sigma|K), \mathcal{O}_{K_\Sigma,\Sigma}^\times)$ (cf. [9], (10.4.8)) we see that

$$
H^1(G(K_\Sigma(2)|K), \mathcal{O}_{K_\Sigma(2),\Sigma}^\times) = \text{Cl}_S(K)(2) = 0
$$

and the same argument shows that

$$
H^1(G(K_\Sigma(2)|K'), \mathcal{O}_{K_\Sigma(2),\Sigma}^\times) \cong \text{Cl}_S(K')(2).
$$
Next we consider the Hochschild-Serre spectral sequence for the extension $K_\Sigma(2)|K'|K$ and the module $\mathcal{O}^\times_{K_\Sigma(2),\Sigma}$. By (4) and (3), we obtain an exact sequence

$$0 \to \text{Cl}_S(K')(2)^G \to H^2(G, \mathcal{O}^\times_{K',\Sigma}) \xrightarrow{\phi} H^2(G(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}).$$

Since $G$ is a 2-group, in order to prove assertion (i), it suffices to show that $\phi$ is injective. Let $c$ be a generator of the cyclic group $H^2(G, \mathbb{Z})$. For each prime $p \in S_S(K)$ (respectively for the chosen prolongation of $p$ to $K_\Sigma(2)$, cf. the discussion in section 1), the composition $T_p(K_\Sigma(2)|K) \to G(K_\Sigma(2)|K) \to G$ is an isomorphism and we denote the image of $c$ in $H^2(T_p(K_\Sigma(2)|K), \mathbb{Z})$ by $c_p$. As is well known, the cup-product with $c$ induces an isomorphism $H^0(G, \mathcal{O}^\times_{K',\Sigma}) \xrightarrow{\sim} H^2(G, \mathcal{O}^\times_{K_\Sigma(2),\Sigma})$ and the similar statement holds for each $c_p$, $p \in S_S(K)$.

The quotient $\mathcal{O}^\times_{K_\Sigma(2),\Sigma}/\mu_2$ is uniquely 2-divisible, and so we obtain a natural isomorphism

$$H^2(G(K_\Sigma(2)|K), \mu_{2\infty}) \xrightarrow{\sim} H^2(G(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}).$$

Furthermore, for each $p \in S_S \smallsetminus S$ we obtain an isomorphism

$$H^2(T_p(K_\Sigma(2)|K), \mu_{2\infty}) \xrightarrow{\sim} H^2(T_p(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}) \cong H^2(G(K_p)|K_p), \tilde{K}_p^\times).$$

Therefore, the calculation of the cohomology in dimension $i \geq 2$ of free products with values in torsion modules (see (3), Satz 4.1 or (3), (4.1.4)) and theorem 3 for the pair $\Sigma, S$ show that we have a natural isomorphism

$$H^2(G(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}) \xrightarrow{\bigoplus} H^2(G(K_p)|K_p), \tilde{K}_p^\times).$$

(Alternatively, we could have obtained this isomorphism from the calculation of the cohomology of the $\Sigma$-units, cf. (3), (8.3.10)(iii) by passing to the limit over all finite subextensions of $k$ in $K$). We obtain the following commutative diagram

$$
\begin{array}{ccc}
H^0(G, \mathcal{O}^\times_{K',\Sigma}) & \xrightarrow{\psi} & \bigoplus_{p \in S_S(K)} H^0(G(K_p)|K_p), \tilde{K}_p^\times) \\
\xrightarrow{\iota \cup c} & \phi \downarrow & \bigoplus_{p \in S_S(K)} H^2(G(K_p)|K_p), \tilde{K}_p^\times) \\
H^2(G, \mathcal{O}^\times_{K',\Sigma}) & \xrightarrow{\phi} & H^2(G(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}) \xrightarrow{\sim} \bigoplus_{p \in S_S(K)} H^2(G(K_p)|K_p), \tilde{K}_p^\times).
\end{array}
$$

Hence $\ker(\phi) \cong \ker(\psi)$ and $\coker(\phi) \cong \coker(\psi)$. Since $H^0(G, \mathcal{O}^\times_{K',\Sigma}) = \mathcal{O}^\times_{K,\Sigma}/N_{K'}[\mathcal{O}^\times_{K',\Sigma}]$, each element in $\ker(\psi)$ is represented by an $S$-unit in $K$ and we have to show that all these are norms of $\Sigma$-units in $K'$. Let $e \in \mathcal{O}^\times_{K,S}$.
Then $K(\sqrt{e})|K$ is a 2-extension which is unramified outside $S$, hence trivial. Therefore $e$ is a square in $K$ and if $f^2 = e$, then $f \in \mathcal{O}_K^\times$ and $e = N_{K'|K}(f)$. This concludes the proof of assertion (i).

To show assertion (ii), it remains to show that $\text{coker}(\text{Sign}_{K,S}) = \text{coker}(\psi) \cong \text{coker}(\phi)$ is trivial. Using the same spectral sequence as before, in order to see that $\text{coker}(\phi) = 0$, it suffices to show that the spectral terms

- $E_2^{02} = H^0(G, H^2(G(K_S(2)|K'), \mathcal{O}_{K_S(2),(2)})^{\times})$ and
- $E_2^{11} = H^1(G, \text{Cl}_S(K')(2))$

are trivial. The first assertion is easy, because $K'$ is totally imaginary and contains $k_\infty(2)$ and so $H^2(G(K_S(2)|K'), \mathcal{O}_{K_S(2),(2)})^{\times} = 0$. That the second spectral term is trivial follows from (i). This completes the proof of theorem 4.

Finally, we prove theorem 5. The statement on $\text{cd}_2 G_S$ and $\text{vcd}_2 G_S$ follows by choosing a 2-Sylow subgroup $H \subset G_S$ and applying theorem 4 to all finite subextensions of $k$ in $(k_S)^H$. It remains to show the statement on the inflation map. It is equivalent to the statement that

$$\inf \otimes Z(2) : H^i(G_S(2), A) \otimes Z(2) \rightarrow H^i(G_S, A) \otimes Z(2)$$

is an isomorphism for every discrete $G_S(2)$-module $A$ and all $i \geq 0$, where $Z(2)$ denotes the localization of $Z$ at the prime ideal (2).

Since cohomology commutes with inductive limits, we may assume that $A$ is finitely generated (as a $Z$-module). Using the exact sequences

$$0 \rightarrow \text{tor}(A) \rightarrow A \rightarrow A/\text{tor}(A) \rightarrow 0,$$

$$0 \rightarrow A/\text{tor}(A) \rightarrow (A/\text{tor}(A)) \otimes \mathbb{Q} \rightarrow (A/\text{tor}(A)) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

and using the limit argument for $(A/\text{tor}(A)) \otimes \mathbb{Q}/\mathbb{Z}$ again, we are reduced to the case that $A$ is finite. Every finite $G_S(2)$-module is the direct sum of its 2-part and its prime-to-2-part. The statement is obvious for the prime-to-2-part and every finite 2-primary $G_S(2)$-module has a composition series whose quotients are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Therefore we are reduced to showing the statement on the inflation map for $A = \mathbb{Z}/2\mathbb{Z}$. But it is more convenient to work with $A = \mathbb{Q}_2/\mathbb{Z}_2$ (with trivial action) which is possible by the exact sequence

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Q}_2/\mathbb{Z}_2 \rightarrow \mathbb{Q}_2/\mathbb{Z}_2 \rightarrow 0.$$

Using the Hochschild-Serre spectral sequence for the extensions $kS|k_S(2)|k$, we thus have to show that

$$H^i(G(k_S|k_S(2)), \mathbb{Q}_2/\mathbb{Z}_2) = 0$$
for \( i \geq 1 \). The case \( i = 1 \) is obvious by the definition of the field \( k_S(2) \).
By theorem 4, every real prime in \( S \) becomes complex in \( k_S(2) \) and therefore \( \text{cd}_2 G(k_S|k_S(2)) \leq 2 \). It remains to show that \( H^2(G(k_S|k_S(2)), \mathbb{Q}_2/\mathbb{Z}_2) = 0 \). Therefore the next proposition implies the remaining statement of theorem 3.

**Proposition 4.3** Let \( k \) be a number field, \( S \supseteq S_2 \) a set of primes in \( k \) and \( K \supseteq k_{\infty}(2) \) an extension of \( K \) in \( k_S \). Then
\[
H^2(G(k_S|K), \mathbb{Q}_2/\mathbb{Z}_2) = 0.
\]

**Proof:** Let \( H \) be a 2-Sylow subgroup in \( G(k_S|K) \) and \( L = (k_S)^H \). Then the restriction map
\[
H^2(G(k_S|K), \mathbb{Q}_2/\mathbb{Z}_2) \rightarrow H^2(G(k_S|L), \mathbb{Q}_2/\mathbb{Z}_2)
\]
is injective and so, replacing \( K \) by \( L \), we may suppose that \( k_S = K_S(2) \). Applying theorem 2' to the 2-S-closed field \( K_S(2) \), we obtain an isomorphism
\[
G(K_{S,S_2}(2)|K_S(2)) \cong \bigoplus_{p \in S_2(K_S(2))} T_p(K_p(2)|K_p).
\]
Hence we have complete control over the Hochschild-Serre spectral sequence associated to \( K_{S,S_2}(2)|K_S(2)|K \). Furthermore, the weak Leopoldt conjecture holds for the cyclotomic \( \mathbb{Z}_2 \)-extension and \( K \supseteq k_{\infty}(2) \), which implies that \( H^2(G(K_{S,S_2}(2)|K), \mathbb{Q}_2/\mathbb{Z}_2) = 0 \). The exact sequence 3 of 4 applied to all finite subextensions \( k'|k \) of \( K|k \) yields a surjection
\[
\bigoplus_{p \in S \setminus S(K)} H^1(T(K_p(2)|K_p), \mathbb{Q}_2/\mathbb{Z}_2) \twoheadrightarrow H^2(G(K_S(2)|K), \mathbb{Q}_2/\mathbb{Z}_2),
\]
and therefore, in order to prove the proposition, it suffices to show that the group \( H^2(G(K_S(2)|K), \mathbb{Q}_2/\mathbb{Z}_2) \) is 2-divisible. This is trivial if \( S \cap S(K) = \emptyset \) because then \( \text{cd} G(K_S(2)|K) \leq 2 \). Otherwise, this follows from the commutative diagram
\[
\begin{array}{ccc}
H^2(G(K_S(2)|K), \mathbb{Q}_2/\mathbb{Z}_2)/2 & \rightarrow & H^3(G(K_S(2)|K), \mathbb{Z}/2\mathbb{Z}) \\
\bigoplus_{p \in S \cap S(K)} H^2(T(K_p(2)|K_p), \mathbb{Q}_2/\mathbb{Z}_2)/2 & \rightarrow & \bigoplus_{p \in S \cap S(K)} H^3(T(K_p(2)|K_p), \mathbb{Z}/2\mathbb{Z}).
\end{array}
\]
The right hand vertical arrow is an isomorphism by proposition 4.1. But \( H^2(T(K_p(2)|K_p), \mathbb{Q}_2/\mathbb{Z}_2) = 0 \) for all \( p \in S \cap S(K) \) and therefore the object in the lower left corner is zero. \( \Box \)

5 Relation to étale cohomology

Let \( k \) be a number field and \( S \) a finite set of places of \( k \). We think of \( \text{Spec}(\mathcal{O}_{k,S}) \) as \( \{ \text{scheme-theoretic points of } \text{Spec}(\mathcal{O}_{k,S}) \} \cup \{ \text{real places of } k \text{ not in } S \} \). Essentially following Zink [17], we introduce the site \( \text{Spec}(\mathcal{O}_{k,S})_{\text{et,mod}} \).
Objects of the category are pairs \( \bar{U} = (U, U_{\text{real}}) \), where \( U \) is a scheme together with an \( \text{\'{e}tale} \) structural morphism \( \phi_U : U \to \text{Spec}(O_{k,S}) \) and \( U_{\text{real}} \) is a subset of the set of real valued points \( U(\mathbb{R}) = \text{Mor}_{\text{Schemes}}(\text{Spec}(\mathbb{R}), U) \) of \( U \) such that \( \phi_U(U_{\text{real}}) \subset S_{\mathbb{R}}(k) \setminus S \).

Morphisms are scheme morphisms \( f : U \to V \) over \( \text{Spec}(O_{k,S}) \) satisfying \( f(U_{\text{real}}) \subset V_{\text{real}} \).

Coverings are families \( \{ \pi_i : \bar{U}_i \to \bar{U} \}_{i \in I} \) such that \( \{ \pi_i : U_i \to U \}_{i \in I} \) is an \( \text{\'{e}tale} \) covering in the usual sense and \( \bigcup_{i \in I} \pi_i(U_{i,\text{real}}) = U_{\text{real}} \).

There exists an obvious morphism of sites \( \text{Spec}(O_{k,S})_{\text{et}} \to \text{Spec}(O_{k,S})_{\text{et}, \text{mod}} \) and both sites coincide if \( S \) contains all real places of \( k \). The pair \( \bar{X} = (X, X_{\text{real}}) \) with \( X = \text{Spec}(O_{k,S}) \) and \( X_{\text{real}} = S_{\mathbb{R}}(k) \setminus S \) is the terminal object of the category and the profinite group \( G_S(k) \) is nothing else but the fundamental group of \( \bar{X} \) with respect to this site. Let \( \eta \) denote the generic point of \( X \). For a sheaf \( A \) of abelian groups on \( \text{Spec}(O_{k,S})_{\text{et}, \text{mod}} \) and for any point \( v \) of \( \bar{X} \) we have a specialization homomorphisms \( s_v : A_v \to A_\eta \) from the stalk \( A_v \) of \( A \) in \( v \) to that in \( \eta \). For each point \( v \in X_{\text{real}} \) we consider the local cohomology \( H^i_v(\bar{X}, A) \) with support in \( v \). There is a long exact localization sequence (see \cite{[17]})

\[
\cdots \to \bigoplus_{v \in X_{\text{real}}} H^i_v(\bar{X}, A) \to H^i_{\text{et}, \text{mod}}(\bar{X}, A) \to H^i_{\text{et}}(X, A) \to \cdots
\]

and the local cohomology with support in real points is calculated as follows:

**Lemma 5.1** For \( v \in X_{\text{real}} \) the following holds.

\[
\begin{align*}
H^0_v(\bar{X}, A) &= \ker(s_v : A_v \to A_\eta) \\
H^1_v(\bar{X}, A) &= \text{coker}(s_v : A_v \to A_\eta) \\
H^i_v(\bar{X}, A) &= H^{i-1}(k_v, A_v) & \text{for } i \geq 2
\end{align*}
\]

Here the right hand side of the last isomorphism is the Galois cohomology of the field \( k_v \).

**Proof** See \cite{[17]}, Lemma 2.3. \( \square \)

**Remark:** Suppose that \( S \) contains all primes dividing 2 and no real primes. Let \( A \) be a locally constant constructible sheaf on \( \text{Spec}(O_{k,S})_{\text{et}} \) which is annihilated by a power of 2. We denote the push-forward of \( A \) to \( \text{Spec}(O_{k,S})_{\text{et}, \text{mod}} \) by the same letter. By Poitou-Tate duality, the boundary map of the long exact localization sequence

\[
H^i_{\text{et}}(X, A) \to \bigoplus_{v \in X_{\text{real}}} H^{i+1}_v(\bar{X}, A) = \bigoplus_{v \in \text{arch.}} H^i(k_v, A_v)
\]

is an isomorphisms for \( i \geq 3 \) and surjective for \( i = 2 \). Therefore, we obtain the vanishing of \( H^i_{\text{et}, \text{mod}}(\text{Spec}(O_{k,S}), A) \) for \( i \geq 3 \). In this situation the
modified étale cohomology is connected to the “positive étale cohomology” $H^2_\text{ét}(\text{Spec}(\mathcal{O}_k, S), A_+)$ defined in §3 in the following way. There exists a natural exact sequence

$$0 \to H^0_{\text{ét, mod}}(\text{Spec}(\mathcal{O}_k, S), A) \to \bigoplus_{v \text{ arch.}} H^0(k_v, A_v) \to H^1_\text{ét}(\text{Spec}(\mathcal{O}_k, S), A) \to H^1_{\text{ét, mod}}(\text{Spec}(\mathcal{O}_k, S), A) \to 0.$$  

and isomorphisms

$$H^i_\text{é}(\text{Spec}(\mathcal{O}_k, S), A_+) \cong H^{i+1}_{\text{ét, mod}}(\text{Spec}(\mathcal{O}_k, S), A)$$

for $i \geq 1$. This can be easily deduced from the long exact localization sequence, lemma 5.1 and the long exact sequence (2.4) of §3.

Now let $A$ be a discrete $G_S(k)$-module. The module $A$ induces locally constant sheaves on $\text{Spec}(\mathcal{O}_k, S)_{\text{ét, mod}}$ and $\text{Spec}(\mathcal{O}_k, S)_{\text{ét}}$, which we will denote by the same letter. According to lemma 5.1, we obtain for every $v \in X_{\text{real}}$

$$H^i(X, A) = 0 \quad \text{for } i = 0, 1.$$  

Let $\tilde{X} = (\text{Spec}(\mathcal{O}_k, S), S_\mathbb{R}(k_S) \setminus S(k_S)$ be the universal covering of $X$. The Hochschild-Serre spectral sequence

$$E^{ij}_2 = H^i(G_S(k), H^j_{\text{ét, mod}}(\tilde{X}, A)) \Rightarrow H^{i+j}_{\text{ét, mod}}(\tilde{X}, A)$$

induces natural comparison homomorphisms

$$H^i(G_S(k), A) \to H^i_{\text{ét, mod}}(\tilde{X}, A)$$

for all $i \geq 0$. It follows immediately from the spectral sequence that these homomorphisms are isomorphisms if

$$H^j_{\text{ét, mod}}(\tilde{X}, A) = 0$$

for all $j \geq 1$.

Next we are going to prove theorem §6 of the introduction. Assume that $S$ contains all primes dividing 2 and that $A$ is 2-torsion. Both sides of the comparison homomorphism commute with direct limits, and so, in order to prove theorem §6, we may suppose that $A$ is finite. Since $A$ is constant on $\tilde{X}$, we can easily reduce to the case $A = \mathbb{Z}/2\mathbb{Z}$, in order to show $H^j_{\text{ét, mod}}(\tilde{X}, A) = 0$ for $j \geq 1$. Furthermore, the assertion is trivial for $j = 1$. The theorem is well-known if $S$ contains all real primes (see [17], prop. 3.3.1 or [4], II, 2.9) and so, passing to the limit over all finite subextensions of $k$ in $k_S$, we obtain natural isomorphisms for all $j \geq 0$.

$$H^j(G_{S_\mathbb{R}}(k_S), \mathbb{Z}/2\mathbb{Z}) \cong H^j_{\text{ét}}(\mathbb{X} \setminus S_\mathbb{R}(k_S), \mathbb{Z}/2\mathbb{Z}).$$  

18
On the other hand, theorem 2 for $T = S \cup S_R$, $S = S$ applied to all finite subextensions of $k$ in $k_S$ in conjunction with theorem 5 induces isomorphisms for all $j \geq 1$.

$$H^j(G_{S \cup S_R}(k_S), \mathbb{Z}/2\mathbb{Z}) \sim \bigoplus_{v \in S_R \setminus S(k_S)} H^j(k_v, \mathbb{Z}/2\mathbb{Z}).$$

These two isomorphisms together with the long exact localization sequence show that

$$H^j_{\text{et,mod}}(\tilde{X}, \mathbb{Z}/2\mathbb{Z}) = 0$$

for $j \geq 1$. This completes the proof of theorem 6.

Theorem 6 is best understood in the context of étale homotopy, namely as a vanishing statement on the 2-parts of higher homotopy groups. For a scheme $X$ we denote by $X_{et}$ its étale homotopy type, i.e. a pro-simplicial set. The étale homotopy groups of $X$ are by definition the homotopy groups of $X_{et}$ and, as is well known, these pro-groups are pro-finite, whenever the scheme $X$ is noetherian, connected and geometrically unibranch (Theorem 11.1). If we consider the modified étale site Spec$(O_{k,S})_{\text{et,mod}}$ as above, we obtain in exactly the same manner as for the usual étale site a pro-finite simplicial set $\tilde{X}_{et,\text{mod}}$. We denote the universal covering of $\tilde{X}_{et,\text{mod}}$ by $\tilde{X}_{et,\text{mod}}$. If $p$ is a prime number and $Y$ is a pro-simplicial set, we denote the pro-$p$ completion of $Y$ by $Y^{\wedge p}$. Furthermore, we write $G(p)$ for the maximal pro-$p$ factor group of a pro-group $G$.

**Lemma 5.2** Assume that $Y$ is a simply connected (i.e. $\pi_1(Y) = 0$) pro-simplicial set such that $\pi_i(Y)$ is pro-finite for all $i \geq 2$. Then we have isomorphisms for all $i$:

$$\pi_i(Y^{\wedge p}) \to \pi_i(Y)^{(p)}.$$

**Proof:** See [13], prop. 13. □

For a pro-group $G$ we denote by $K(G, 1)$ the Eilenberg-MacLane pro-simplicial set associated with $G$ (cf. [3], (2.6)). If $S$ contains all real primes of $k$ the following theorem was proved in [13], prop. 14.

**Theorem 5.3** Let $k$ be a number field and $S$ a finite set of primes of $k$ containing all primes dividing 2. Let $\tilde{X}$ be the pair $(X, X_{\text{real}})$ with $X = \text{Spec}(O_{k,S})$ and $X_{\text{real}} = S_{\mathbb{R}}(k) \setminus S$ endowed with the modified étale topology. Then the higher homotopy groups of $\tilde{X}_{et,\text{mod}}$ have no 2-part, i.e.

$$\pi_i(\tilde{X}_{et,\text{mod}})^{(2)} = 0 \quad \text{for } i \geq 2.$$

Furthermore, the canonical morphism

$$(\tilde{X}_{et,\text{mod}})^{\wedge 2} \to K(G_S(k)(2), 1)$$

is a weak homotopy equivalence.
Proof: Since $G_S(k)$ is the fundamental group of $\tilde{X}_{et,mod}$, theorem 5.2 implies that the universal covering $\tilde{X}_{et,mod}$ of $X_{et,mod}$ has no cohomology with values in 2-primary coefficient groups. By the Hurewicz theorem ([1], (4.5)), the pro-2 completion of $\tilde{X}_{et,mod}$ is weakly contractible. Therefore lemma 5.2 implies
\[ \pi_i(\tilde{X}_{et,mod})(2) \cong \pi_i((\tilde{X}_{et,mod})^{\wedge}) = 0 \]
for $i \geq 2$, which shows the first statement of the theorem. By theorem [3], for every finite 2-primary $G_S(k)(2)$-torsion module $A$ the inflation homomorphism $H^i(G_S(k)(2), A) \rightarrow H^i(G_S(k), A)$ is an isomorphism for all $i$. The same arguments as above show that the universal covering of $(\tilde{X}_{et,mod})^{\wedge}$ is weakly contractible. This proves the second statement. □

6 Closing Remarks

1. Dualizing modules

Unfortunately, we do not have (despite semi-tautological reformulations of the definition) a good description of the $p$-dualizing module $I$ of the group $G_S$, where $S$ is a finite set of finite primes containing $S_p$. If $k$ is totally imaginary, then $I$ is determined by the exact sequence
\[ 0 \rightarrow \mu_p \rightarrow \bigoplus_{p \in S(k_S)} \mu_p \rightarrow I \rightarrow 0 \]
(see [9], (10.2.1)) and the group $G_S$ is a duality group at $p$ of dimension 2 (see [13], th.4 or [9], (10.9.1)). The general case remains unsolved (also for odd $p$).

2. Free profinite product decompositions

In this paper we used free pro-$p$-product decompositions of Galois groups of pro-$p$-extensions of global fields into Galois groups of local pro-$p$-extensions in an essential way. One might ask whether, for sets of places $T \supset S$, the natural homomorphism
\[ \phi : \bigast_{p \in T \setminus S(k_S)} T(k_p|k) \rightarrow G(k_T|k_S) \]
is an isomorphism, where the free product on the left hand side is the free product of profinite groups. More precisely, one has to ask, whether there exists a continuous section to the natural projection $T \setminus S(k_T) \rightarrow T \setminus S(k_S)$ such that the above map is an isomorphism (cf. the discussion in section 2). We do not know the answers to this question in general. It is ‘yes’ if $S$ contains all but finitely many primes of $k$ (see below). But it seems likely that $\phi$ is never an isomorphism if $T$ and $S$ are finite. The present level of knowledge on this question is rather low. For example, we do not know whether there are infinitely many prime numbers $p$ such that $p^\infty$ divides the order of $G_T$. The best result
known in this direction is that if $T$ contains all real places and all primes dividing one prime number $p$, then there exist infinitely many prime numbers $\ell$ dividing the order of $G_T$ (see [4], cor.3 or [9], (10.9.4)).

In the case that $S$ contains all but finitely many primes of $k$, we can deduce the above statement by applying the following slightly more general result to the complement of $S$:

For a finite set $S$ of primes of $k$, let $k^S$ be the maximal extension of $k$ in which all primes in $S$ are totally decomposed. Then there exists a continuous section to the natural projection $S(\bar{k}) \to S(k^S)$ such that the natural map

$$\ast \quad \left. \prod_{p \in S(k^S)} G(\bar{k}_p|k) \to G(\bar{k}|k^S) \right.$$ 

is an isomorphism. This had been proved first in the special case $S = S_\mathbb{R}$ by Fried-Haran-Völklein [6] and then by Pop [11] for arbitrary finite $S$.

3. Leopoldt’s conjecture

The Leopoldt conjecture for $k$ and a prime number $p$ holds if and only if the group

$$H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p)$$

is trivial for one (all) finite set(s) of primes $S \supseteq S_p$. The weak Leopoldt conjecture is true for $k$, $p$ and a $\mathbb{Z}_p$-extension $k_\infty|k$ if and only if

$$H^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$$

is trivial for one (all) finite set(s) of primes $S \supseteq S_p$ (of $k$). This is well known for odd $p$ and for $p = 2$ it can be easily deduced from the above results.

4. Iwasawa theory

Let $k$ be a number field, $S \supseteq S_2$ a finite set of primes of $k$ and $k_\infty|k$ the cyclotomic $\mathbb{Z}_2$-extension of $k$. Let $\Gamma = G(k_\infty|k) \cong \mathbb{Z}_2$ and let $\Lambda = \mathbb{Z}_2[\Gamma] \cong \mathbb{Z}_2[[T]]$ be the Iwasawa algebra. We consider the compact $\Lambda$-module

$$X_S = G(k_S(2)|k_\infty)^{ab}.$$ 

Then the following holds

(i) $X_S$ is a finitely generated $\Lambda$-module.

(ii) $\text{rank}_\Lambda X_S = r_2$ (the number of complex places of $k$).

(iii) $X_S$ does not contain any nontrivial finite $\Lambda$-submodule.

(iv) the $\mu$-invariant of $X_S$ is greater than or equal to $\# S \cap S_\mathbb{R}(k)$.

Properties (i)-(iii) follow in a purely formal way (see [2], (5.6.15)) from the facts that: (a) $\chi_2(G_S(2)) = -r_2$, (b) $H^2(G_S(k_\infty)(2), \mathbb{Q}_2/\mathbb{Z}_2) = 0$ and (c)
$H^2(G_S(2), \mathbb{Q}_2/\mathbb{Z}_2)$ is 2-divisible. Assertion (iv) is trivial if $S$ contains no real places and in the general case it follows from the exact sequence

$$0 \to (A/2)\oplus S\times(k) \to X_S \to X_{S\setminus S_k} \to 0.$$ 

Now let $k^+$ be a totally real number field, $k = k^+(\mu_4)$, $k^+_{\infty}$ the cyclotomic $\mathbb{Z}_2$-extension of $k^+$ and $k_{\infty} = k^+_{\infty}(\mu_4) = k(\mu_{2n})$. Let $k_n$ be the unique subextension of degree $2^n$ in $k_{\infty}$ and let $J$ be the complex conjugation. We set $A_n = Cl(k_n)(2)$ and

$$A_n^- := \{a \in A_n \mid aJ(a) = 1\}.$$ 

Furthermore, let $A_{\infty}^- = \lim_{\to} A_n^-$, $X^+ = X_{S_2}(k^+)$, let $\vee$ denote the Pontryagin dual and $(-1)$ the Tate-twist by $-1$. Then there exists a natural homomorphism

$$\phi : (A_{\infty})^\vee \to X^+(-1)$$

whose kernel and cokernel are annihilated by 2. If the Iwasawa $\mu$-invariant of $k$ is zero (this is known if $k/\mathbb{Q}$ is abelian), then $\phi$ is a pseudo-isomorphism, i.e. $\phi$ has finite kernel and cokernel. This can be seen by a slight modification of the arguments given in [4], §2.

Let $M^+$ be the maximal abelian 2-extension of $k^+_{\infty}$ which is unramified outside $S_2$, in particular, $M^+$ is totally real. Kummer theory shows that, for an $\alpha \in k^+_{\infty}$, the field $k_{\infty}(\sqrt[2^n]{\alpha})$ is contained in $M^+k_{\infty}$ if and only if: (a) $\alpha \in k^+_{\infty}\setminus p$ for all $p \notin S(k_{\infty})$ and (b) $\alphaJ(\alpha) = \beta^{2^n}$ for a totally positive element $\beta \in k^+_{\infty}$. Let $R_n$ be the subgroup in $k^+_{\infty}/k^+_{\infty}2^n$ generated by elements satisfying (a) and (b) and let

$$\mathfrak{M}^- := \lim_{\to} R_n \subset k^+_{\infty} \otimes \mathbb{Q}_2/\mathbb{Z}_2.$$ 

Then we have a perfect Kummer pairing $X^+ \times \mathfrak{M}^- \to \mu_{2^n}$. Since all primes dividing 2 are infinitely ramified in $k_{\infty}|k$, for $\alpha \otimes 2^n \in \mathfrak{M}^-$ there exists a unique ideal $a$ in $k_{\infty}$ with $a^{2^n} = (\alpha)$ and the class $[a]$ is contained in $A_{\infty}^-$. This yields a homomorphism

$$\phi^\vee : \mathfrak{M}^- \to A_{\infty}^-.$$ 

A straightforward computation shows that $\text{im}(\phi^\vee) \supseteq (A_{\infty})^2$ and that $\text{ker}(\phi^\vee)$ is the image of $O^X_{k^+_{\infty},\varnothing}/O^X_{k^+_{\infty},S_k}$ in $\mathfrak{M}^-$ (notational conventions as in [4]). Thus, if the Iwasawa $\mu$-invariant of $k$ is zero, then the cokernel of $\phi^\vee$ is finite and it remains to show the same for its kernel. Since $\mu = 0$, the $\mathbb{F}_2$-ranks of $2\text{Cl}^0(k^+_{\infty})$ (the subgroup of elements annihilated by 2 in the ideal class groups in the narrow sense) are bounded independently of $n$. Thus also the $\mathbb{F}_2$-ranks of the kernels of the signature maps

$$O^X_{k^+_{\infty},\varnothing}/O^X_{k^+_{\infty},S_k} \to \bigoplus_{v \in S_k(k^+_{\infty})} \mathbb{R}^\times/\mathbb{R}^\times$$

are bounded independently of $n$. But the direct limit over $n$ of these kernels is just the group in question. Finally, we obtain the result by taking Pontryagin duals.
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