(In)stability of quasi-static paths of some finite dimensional smooth or elastic-plastic systems

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Abstract. In this paper we discuss some mathematical issues related to the stability of quasi-
static paths of finite dimensional mechanical systems that have a smooth or an elastic-plastic
behavior. The concept of stability of quasi-static paths used here is essentially a continuity
property relatively to the size of the initial perturbations (as in Lyapunov stability) and to
the smallness of the rate of application of the external forces (which here plays the role of
the small parameter in singular perturbation problems). A related concept of attractiveness is
also proposed. Sufficient conditions for attractiveness or for instability of quasi-static paths
of smooth systems are presented. The Ziegler column and other examples illustrate these
situations. Mathematical formulations (plus existence and uniqueness results) for dynamic
and quasi-static elastic-plastic problems with linear hardening are recalled. A stability result is
proved for the quasi-static evolution of these systems.

1. Introduction
The governing equation for the dynamic evolution of mechanical systems is Newton’s law, i.e.
force equals mass times acceleration. A classical approximation for the equations that govern the
slow evolution of mechanical systems is to neglect inertia effects and take the balance equations
as static equilibrium equations, i.e. force equals zero. The slow evolutions made up of the
successive equilibrium configurations are called quasi-static evolutions.

The relationship of this issue with the theory of singular perturbations has been established
in [9], where the existence of fast (dynamic) and slow (quasi-static) time scales is recognized: a
change of variables is performed that replaces the (fast) physical time $t$ by a (slow) loading
parameter $\lambda$ whose rate of change with respect to time, $\varepsilon = d\lambda/dt$, is eventually decreased to zero.
In the finite dimensional case this change of variables leads to a system of differential equations
or inclusions defining a singular perturbation problem, i.e. a problem where some terms whose
highest order derivative with respect to $\lambda$ appear multiplied by the small parameter $\varepsilon$. In this
context we present a mathematical concept of stability of quasi-static paths that is essentially a
continuity property with respect to the smallness of the initial perturbations (as in Lyapunov
continuity).
stability) and to the smallness of the quasi-static loading rate $\varepsilon$ (that plays the role of the small parameter in singular perturbation problems).

In fact, the concept of Lyapunov stability cannot in general be directly applied to quasi-static paths with slowly varying loads, because these are not, in general, true dynamic solutions: the supposedly negligible quasi-static accelerations are not zero in general. On the other hand, most classical results of singular perturbation theory guarantee convergence of dynamic solutions to quasi-static paths by simply setting the parameter $\varepsilon$ to zero, but, in typical mechanical problems, this relies on the presence of viscous damping. These were the motivations to propose the new concept of stability of quasi-static paths discussed in this paper.

The structure of the article is the following. In Section 2, we present definitions and sufficient conditions for stability and instability of quasi-static paths of smooth systems. Some examples and numerical simulations illustrate the situations that may occur. In Section 3, existence and uniqueness of solution for dynamic and quasi-static elastic-plastic systems with hardening are recalled, and the stability of the quasi-static paths is proved.

2. (In)stability of quasi-static paths of smooth systems: definitions and sufficient conditions

The concepts of stability and attractiveness of quasi-static paths are presented in Section 2.1. Sufficient conditions for attractiveness or for instability of quasi-static paths of smooth systems are summarized in Section 2.2. Some examples and discussions of these smooth systems are presented in Section 2.3. The Ziegler column and the results of some numerical simulations are given in Section 2.4.

2.1. Definitions of stability of quasi-static paths of smooth systems

Let us consider the system:

$$\begin{cases}
x' = f(\lambda, x, y, \varepsilon), \\
\varepsilon y' = g(\lambda, x, y, \varepsilon)
\end{cases}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $f$, $g$ are vector functions of corresponding dimensions, $\varepsilon > 0$ is a small parameter, and the initial conditions are:

$$x(\lambda_1) = x_1 \text{ and } y(\lambda_1) = y_1.$$  \hspace{1cm} (2)

If we set $\varepsilon = 0$ in Equation (1) we obtain the reduced problem:

$$\begin{cases}
\bar{x}' = f(\lambda, \bar{x}, \bar{y}, 0), \\
0 = g(\lambda, \bar{x}, \bar{y}, 0)
\end{cases}$$

where the second equality in Equation (3) is an algebraic equation, so only one initial condition can be prescribed in Equation (3):

$$\bar{x}(\lambda_1) = \bar{x}_1.$$  \hspace{1cm} (4)

Let $\bar{y} = \phi(\lambda, \bar{x})$ be an isolated root of the second equation in Equation (3). Substituting $\bar{y} = \phi(\lambda, \bar{x})$ in the first equation in Equation (3), we obtain the following system:

$$\bar{x}' = f(\lambda, \bar{x}, \phi(\lambda, \bar{x}), 0),$$

with the initial condition Equation (4).

We suppose that the solution

$$\begin{cases}
x = \bar{x}(\lambda), \\
y = \phi(\lambda, \bar{x}(\lambda)) = \bar{y}(\lambda),
\end{cases}$$

$$\begin{cases}
x = \bar{x}(\lambda), \\
y = \phi(\lambda, \bar{x}(\lambda)) = \bar{y}(\lambda),
\end{cases}$$
of the reduced problem Equation (3)--Equation (4) exists on \( I = [\lambda_1, \lambda_2] \), and we call it the quasi-static solution, which, of course, depends on the initial condition \( \bar{x}_1 \). On the other hand, the solution \( x(\lambda), \ y(\lambda) \) of the original problem Equation (1)--Equation (2) will be called the dynamic solution. It depends on \( \varepsilon \) and on the initial conditions \( x_1, \ y_1 \). The dependence of the dynamic solution on \( \varepsilon \) will be frequently made explicit in the notation: \( x = x(\lambda, \varepsilon), \ y = y(\lambda, \varepsilon) \).

**Definition 2.1.** The quasi-static path \( x = \bar{x}(\lambda), \ y = \bar{y}(\lambda) \) is said to be stable at \( \lambda_1 \) if there exists \( 0 < \Delta \lambda \leq \lambda_2 - \lambda_1 \), such that, for all \( \delta > 0 \) there exists \( \bar{\rho}(\delta) > 0, \bar{\varepsilon}(\delta) > 0 \) such that for all initial conditions \( x_1, \ y_1, \bar{x}_1 \) and all \( \varepsilon > 0 \) such that
\[
\|x_1 - \bar{x}_1\| + \|y_1 - \bar{y}(\lambda_1)\| < \bar{\rho}(\delta) \quad \text{and} \quad \varepsilon < \bar{\varepsilon}(\delta),
\]
the solutions of problem Equation (1)--Equation (2) \( x = x(\lambda, \varepsilon), \ y = y(\lambda, \varepsilon) \) satisfy
\[
\|x(\lambda, \varepsilon) - \bar{x}(\lambda)\| + \|y(\lambda, \varepsilon) - \bar{y}(\lambda)\| < \delta,
\]
for all \( \lambda \in [\lambda_1, \lambda_1 + \Delta \lambda] \).

**Definition 2.2.** The quasi-static path \( x = \bar{x}(\lambda), \ y = \bar{y}(\lambda) \) is said to be attractive at \( \lambda_1 \) if it is stable and there exists \( \rho > 0 \), such that for all initial conditions \( x_1, \ y_1, \bar{x}_1 \) satisfying
\[
x_1 = \bar{x}_1 \quad \text{and} \quad \|y_1 - \bar{y}(\lambda_1)\| < \rho,
\]
the solutions of problem Equation (1)--Equation (2) \( x = x(\lambda, \varepsilon), \ y = y(\lambda, \varepsilon) \) satisfy
\[
\lim_{\varepsilon \to 0} x(\lambda, \varepsilon) = \bar{x}(\lambda), \quad \lambda_1 \leq \lambda \leq \lambda_1 + \Delta \lambda,
\]
\[
\lim_{\varepsilon \to 0} y(\lambda, \varepsilon) = \bar{y}(\lambda), \quad \lambda_1 < \lambda \leq \lambda_1 + \Delta \lambda.
\]

**2.2. Sufficient conditions for attractiveness and sufficient conditions for instability of quasi-static paths of smooth systems**

In this section, we shall discuss conditions under which the quasi-static solution of the system Equation (3)--Equation (4) is attractive.

Suppose that:

(i) The functions \( f, \ g, \ \phi \) are uniformly continuous and bounded with derivatives up to second order on all the variables in the region:
\[
\Omega = \{ \lambda \in [\lambda_1, \lambda_2], \ x \in \mathbb{R}^m, \ |y - \phi(\lambda, x)| \leq \rho_0, \ \varepsilon \in (0, \varepsilon_0) \}.
\]

(ii) The roots \( \Lambda_i(\lambda) \) of the characteristic equation \( |B(\lambda, \bar{x}) - \Lambda I| = 0 \) where \( B(\lambda, \bar{x}) = g_y(\lambda, \bar{x}, \phi(\lambda, \bar{x}), 0) \) satisfy the inequality
\[
\Re \Lambda_i(\lambda) \leq -2\gamma < 0, \ \forall \lambda \in [\lambda_1, \lambda_2].
\]

(iii) The initial point \( y_1 \) belongs to the basin of attraction of \( \phi(\lambda, x) \).

**Theorem 2.3.** If conditions Equation (i)--Equation (iii) are satisfied, then the quasi-static path \( x = \bar{x}(\lambda), \ y = \bar{y}(\lambda) \) of system Equation (1) is attractive at \( \lambda_1 \).

**Idea of the proof.** The result follows from the proof of the theorem of Tikhonov [19], applied to the system (1). \( \square \)

Let us consider again the system Equation (1) with initial conditions Equation (2). Suppose that:
(i) The functions \( f, g, \phi \) are uniformly continuous and bounded with first and second order derivatives on all the variables in the region:
\[
\Omega = \{ \lambda \in [\lambda_1, \lambda_2], x \in \mathbb{R}^m, \| y - \phi(\lambda, x) \| \leq \rho_0, \varepsilon \in (0, \varepsilon_0] \}.
\]

(ii) There exists a root \( \Lambda_1(\lambda, x) \) of the characteristic equation \( |B(\lambda, x) - \Lambda I| = 0 \), where \( B(\lambda, x) = g_y(\lambda, x, \phi(\lambda, x), 0) \), which satisfies the following inequality:
\[
\Re \Lambda_1(\lambda, x) \geq \alpha > 0.
\]

**Theorem 2.4.** If conditions Equation (i)–Equation (ii) are satisfied, then the quasi-static path \( x = \bar{x}(\lambda), y = \bar{y}(\lambda) \) is unstable at \( \lambda_1 \).

**Idea of the proof.** We consider the difference between the dynamic solution and the quasi-static one: \( \tilde{y} = y - \bar{y} \). The equation for \( \tilde{y} \) is
\[
\varepsilon \tilde{y}' = B(\lambda, \bar{x}) \tilde{y} + Y(\lambda, \bar{x}, \tilde{y}, \varepsilon)
\]
where \( Y \) is the non linear part of the Taylor’s series with respect to the third variable \( y \) of the function \( g \) in the neighborhood of the quasi-static path.

To prove the instability of the quasi-static path, it suffices to show that there exists a component of the vector \( \tilde{y} \) that becomes greater than any given \( \delta \) as \( \varepsilon \) tends to 0. For that purpose, we apply the change of variables \( \tilde{y} = S(\lambda) z \), which transforms the matrix \( B = S^{-1} B(\lambda, \bar{x}) S \) to:
\[
S^{-1} B(\lambda, \bar{x}) S = \begin{pmatrix}
\Lambda_1(\lambda, \bar{x}) & 0 \\
0 & D(\lambda, \bar{x})
\end{pmatrix},
\]
and we compute the variable \( z_1 \) that corresponds to the root \( \Lambda_1(\lambda, x) \).

Since \( |z_1| \neq 0 \), we can consider \( w = 1/z_1 \) and prove that \( w \) tends to 0 as \( \varepsilon \) tends to 0, using Tikhonov theorem or integral inequalities. In particular, using the Bihari inequality (see [1]), we can obtain the final estimate for \( z_1 \):
\[
|z_1(\lambda)| \geq |z_1^0| \exp \left( \frac{\alpha + 2|z_1^0|}{\alpha + |z_1^0|} \left( \frac{A(\delta + \varepsilon) b(\varepsilon, \lambda_1, \lambda)}{A(\delta + \varepsilon) b(\varepsilon, \lambda_1, \lambda)} \right) \right)
\]
where \( z_1^0 \) is the initial value of \( z_1 \) and \( b(\varepsilon, \lambda_1, \lambda) = (1 - \exp \left( \frac{-\alpha(\lambda - \lambda_1)}{\varepsilon} \right)) \). This estimate yields the result in the Theorem. \( \square \)

2.3. Examples and discussion on the smooth cases

Consider the forced linear oscillator:
\[
\varepsilon^2 u'' + 2\varepsilon \zeta u' + u = f(\lambda), \ \forall \zeta \geq 0,
\]
which can be rewritten as the first order system:
\[
\begin{cases}
\varepsilon u' = v, \\
\varepsilon v' = -u - 2\zeta v + f(\lambda).
\end{cases}
\] (8)

The quasi-static path is \( \bar{v} = 0, \bar{u} = f(\lambda) \). It is easy to see that the quasi-static solution is stable but not attractive in the no-damping case \( \zeta = 0 \), and it is attractive if \( \zeta > 0 \). Note also that in the no-damping case, \( \zeta = 0 \), Theorem 2.3 does not give an answer.
Consider now the following system:

\[
\begin{align*}
\varepsilon u' &= \varepsilon u - v - u(u^2 + v^2), \\
\varepsilon v' &= u + \varepsilon v - v(u^2 + v^2).
\end{align*}
\]

(9)

The quasi-static path is \(\bar{u} = 0, \bar{v} = 0\). In order to make the system behavior clear we change to polar coordinates \(u = r \cos \phi, v = r \sin \phi\). We obtain:

\[
\begin{align*}
\varepsilon r' &= r(\varepsilon - r^2), \\
\varepsilon \phi' &= 1.
\end{align*}
\]

(10)

Any dynamic solution starting from an initial point different from \((0, 0)\) tends to a circumference of radius \(\sqrt{\varepsilon}\). The solution \((0, 0)\) is thus unstable in the sense of Lyapunov for all \(\varepsilon > 0\). But if, for arbitrary small \(\delta\), we take initial conditions inside a circumference of radius \(\bar{\rho} = \delta\) and \(\bar{\varepsilon} < \delta^2\) we obtain a solution satisfying

\[
||u(\lambda, \varepsilon) - \bar{u}(\lambda)|| + ||v(\lambda, \varepsilon) - \bar{v}(\lambda)|| < \delta,
\]

for all \(\lambda\) and all \(0 < \varepsilon < \bar{\varepsilon}\). The quasi-static path \(\bar{u} = 0, \bar{v} = 0\) is thus stable in the sense of Definition 2.1. Note that the stability picture could be reversed by appropriately changing signs in the system (10). Note also that the tangent operator

\[
g_y(\lambda, \bar{x}, \phi(\lambda, \bar{x}), \varepsilon) = \begin{pmatrix}
-\varepsilon & -1 \\
1 & -\varepsilon
\end{pmatrix},
\]

at the quasi-static and dynamic solution \((0, 0)\) has the eigenvalues \(\Lambda = \varepsilon \pm i\), which implies the Lyapunov instability of that solution for all \(\varepsilon > 0\). It is clear that \(g_y(\lambda, x, \phi(\lambda, x), \varepsilon)\) approaches \(B(\lambda, x) = g_y(\lambda, x, \phi(\lambda, x), 0)\), as \(\varepsilon\) tends to 0, and, in that limit, the real parts of those eigenvalues vanish. For this reason: (i) one cannot apply the theory of Section 2.2 to study stability or instability of the system (9) in the sense of Definition 2.1; and (ii) instability in the sense of Lyapunov (for all \(\varepsilon > 0\)) does not imply instability in the sense of Definition 2.1, namely because the strict inequalities satisfied by relevant eigenvalues of \(g_y\) may be not preserved as \(\varepsilon\) tends to 0.

It should also be clear that the tangent operator \(g_y(\lambda, x, \phi(\lambda, x), \varepsilon)\) at the quasi-static solution in general depends on \(\lambda\), so that a linearized form of the dynamic equations \(\varepsilon y' = g(y, x, y, \varepsilon)\) (1) in the neighborhood of the quasi-static path would be in general non-autonomous. Sometimes this fact has not been recognized in the mechanics literature on the stability of quasi-static paths. Other times this has been recognized ([4], [7], [14]) but, in an attempt to apply well known results on the stability of autonomous systems that use the nature of the eigenvalues of the tangent operator, the dependence of \(g_y\) on \(\lambda\) has been neglected or "frozen" by some authors ([4], [7]). A complete and sound mathematical justification for this has not been presented, but the arguments advanced by those authors have a correct idea behind them: the change of \(g_y\) with \(\lambda\) is much slower than the changes (decay or growth) of the dynamic solution that are relevant for the stability/instability analysis. Although not explicitly defined, the existence of two time scales (the slow load time scale, \(\lambda\), and the fast scale of the physical time, \(\lambda/\varepsilon\)) correctly emerges from those arguments. Following the proposals in [9], [11], the present paper gives a sound mathematical basis for these studies, in the case of smooth finite dimensional systems:

- the problem of the stability of quasi-static paths is explicitly put into its correct singular perturbation, two time scale, framework;
- an appropriate definition of stability of quasi-static paths is proposed;
- the nature of the eigenvalues of the tangent operator at the quasi-static path for \(\varepsilon = 0\) is shown to lead, under some conditions, to attractiveness (Theorem 2.3) or to instability (Theorem 2.4).
2.4. The Ziegler column

The Ziegler column is a plane double pendulum made of two hinged rods, subjected to a follower force \( P \), as shown in Figure 1 (see [22], [8]). The angles \( \theta_1 \) and \( \theta_2 \) by which the rods deviate from the vertical are the generalized coordinates of this two degree-of-freedom system. The force \( P \) is said to be follower because it makes an angle with the vertical that is proportional to the angle \( \theta_2 \) made by the second rod with the vertical (the proportionality constant is denoted by \( \chi \)). Rotational springs and dashpots (with stiffness and damping constants \( K \) and \( C \), respectively) are present at the two hinges, two masses (\( M \) and \( m \)) are located at the relevant extremities of the rods. The rods have the same length \( L \), and gravity is neglected.

![Figure 1. The Ziegler column.](image)

Denoting by \( \tau \) the non-dimensional time

\[
\tau = \frac{t}{L} \sqrt{\frac{K}{m}},
\]

and continuing to denote by \( \theta_\alpha, \alpha = 1, 2 \), the \( \tau \) dependent generalized coordinates, we obtain the non-dimensional form of the equations of motion

\[
m(\theta(\tau))\ddot{\theta}(\tau) + 2\zeta c\dot{\theta}(\tau) + k\theta(\tau) = f(\theta(\tau), \dot{\theta}(\tau)) + \lambda q(\theta(\tau))
\]

where

\[
\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad m = \begin{pmatrix} 1 + \rho & \cos(\theta_1 - \theta_2) \\ \cos(\theta_1 - \theta_2) & 1 \end{pmatrix}, \quad c = k = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},
\]

\[
f(\theta, \dot{\theta}) = \begin{pmatrix} -\sin(\theta_1 - \theta_2)\dot{\theta}_2^2 \\ \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 \end{pmatrix}, \quad q(\theta) = \begin{pmatrix} -\sin(\chi \theta_2 - \theta_1) \\ -\sin((\chi - 1)\theta_2) \end{pmatrix},
\]

\[
\rho = \frac{M}{m}, \quad \zeta = \frac{C}{2\sqrt{KmL}}, \quad \lambda = \frac{|P|L}{K},
\]

and \( (\cdot) \) denotes the derivative \( d(\cdot)/d\tau \).
Note that, for constant load $\lambda = \lambda_1$, the linearized form of the governing dynamic equations in the neighborhood of the static equilibrium configuration $\theta_1 = \dot{\theta}_1 = 0$ is
\begin{equation}
\mathbf{m}_1 \ddot{\theta}(\tau) + 2 \zeta c \dot{\theta}(\tau) + \mathbf{k}_1 \delta \theta(\tau) = 0,
\end{equation}
and the corresponding linearized stability eigenproblem is
\begin{equation}
[\Lambda \mathbf{m}_1 + 2 \zeta \Lambda \mathbf{c} + \mathbf{k}_1] \mathbf{V} = 0,
\end{equation}
where
\begin{equation}
\mathbf{m}_1 = \begin{pmatrix} 1 + \rho & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \mathbf{k}_1 = \mathbf{k} - \lambda_1 \frac{\partial \mathbf{q}}{\partial \theta}(\theta_1) = \begin{pmatrix} 2 - \lambda_1 & -1 + \lambda_1 \chi \\ -1 & 1 - \lambda_1(1 - \chi) \end{pmatrix}.
\end{equation}
It is the non-symmetry of the tangent stiffness matrix $\mathbf{k}_1$ in the presence of follower forces ($\chi > 0$) that may lead to flutter instability ($\Re(\Lambda) > 0$, $\Im(\Lambda) \neq 0$), for values of the load parameter smaller than the critical load $\lambda_d$ for divergence instability ($\det \mathbf{k}_1(\lambda_d) = 0$).

Considering now a load $\lambda$ slowly increasing according to $\lambda = \varepsilon \tau + \lambda_1$, $\varepsilon = d\lambda/d\tau$, $\lambda(\tau) = \lambda(\varepsilon)$, the following singular perturbation governing system is obtained
\begin{equation}
\varepsilon^2 \mathbf{m}\ddot{\theta}(\lambda) + 2 \varepsilon \zeta c \dot{\theta}(\lambda) + \mathbf{k}(\lambda) = \varepsilon^2 \mathbf{f}(\theta(\lambda), \dot{\theta}(\lambda)) + \lambda \mathbf{q}(\theta(\lambda)),
\end{equation}
for which the reduced (quasi-static) system is:
\begin{equation}
\mathbf{k}\ddot{\theta}(\lambda) - \lambda \mathbf{q}(\theta(\lambda)) = 0.
\end{equation}
Note that the governing system (17) could be obviously rewritten in first order differential equation form. For more details, the reader is referred to [11]. Note also that $\bar{\theta} \equiv 0$ is a trivial quasi-static path that solves (18) for increasing load $\lambda$, and coincides with a trivial solution of the dynamic problem (12) for any constant load $\lambda$. This coincidence facilitates the comparison of the classical concept of Lyapunov stability and the concept of “stability of quasi-static paths” proposed here.

We illustrate numerically the behavior of dynamic solutions starting in a small neighborhood of the coinciding trivial equilibria under constant loads and of trivial quasi-static path under a slowly varying load.

Numerical simulations are performed for $\rho = 2$, $\chi = 1$ and $\zeta = 0.03$. As described in [8] for similar parameter values, there is a critical load $\lambda_f = 1.4692857145$, below which the trivial dynamic solution under constant load is asymptotically stable, and above which it is unstable by flutter. The results of the simulations for $\lambda_1 = 1 < \lambda_f$ and for $\lambda_1 = 3.2 > \lambda_f$ are presented in Figures 2 and 3, respectively.

We present now the results of dynamic simulations performed with slowly increasing loads $\lambda = \lambda_1 + \varepsilon \tau$. A fast increase of the oscillation amplitude occurs immediately after the motion starts at $\lambda_1 = 2.5 > \lambda_f$ (see Figures 4 and 5). The size of the limit cycles obtained for different constant values of $\lambda$ are compared in Figures 4 and 5 with the amplitude of the oscillations obtained with slowly increasing loads and $\varepsilon = 0.01$ or $0.001$.

The numerical results obtained in these simulations illustrate the coincidence that exists in this problem between the occurrence of stability or instability of the trivial equilibrium points (under constant loads) and the occurrence of “stability or instability of the trivial variable load path”. They also illustrate the application of Theorems 2.3 and 2.4 to study stability or instability of quasi-static paths (for $\lambda < \lambda_f$ or $\lambda > \lambda_f$, respectively).
3. Stability of quasi-static paths for finite dimensional elastic-plastic systems with hardening

The mathematical formulations for dynamic and quasi-static elastic-plastic systems with hardening are presented in Section 3.1. In Section 3.2, existence and uniqueness of solutions for the dynamic and the quasi-static problems are proved, using the theory of \textit{m-accretive} operators (see [2], [5], [10], [15], [16], [23]). In Section 3.3, \textit{a priori} estimates are obtained which enable us to prove the stability of the quasi-static path in the sense of the definition proposed in ([10], [11], [12]).

3.1. Governing equations

We consider a single degree-of-freedom elastic-plastic system with linear kinematic hardening, and we assume geometrical linearity. The governing dynamic equations can be non-dimensionalized by using the non-dimensional time (\(\tau\)) and load parameter (\(\lambda, \lambda = \lambda_1 + \varepsilon \tau\)), yielding

\[
\varepsilon^2 u'' = f_{\text{ext}}(\lambda) + f_{\text{int}}(u, r),
\]

where \(u, f_{\text{ext}}\) and \(f_{\text{int}}\) are the non-dimensional generalized displacement, external force and internal force, respectively; the latter is the force that acts on the elastic-plastic element of the
system. The non-dimensional elongation $e$ can be related to the non-dimensional generalized displacement $u$, and, in view of the presence of the elastic-plastic element, it can be decomposed into elastic, $e^e$, and plastic, $e^p$, parts:

$$e = 2u = e^e + e^p. \quad (20)$$

The internal force $f_{\text{int}}$ is equal, on one hand, to the product of the elastic part $e^e = e - e^p$ of the non-dimensional elongation with a unit elastic stiffness (Hooke’s law), and, on the other hand, it is equal to the sum of the force $2r$ in the plastic element with the force in the hardening element (the plastic elongation $e^p$ multiplied by a unit stiffness):

$$-f_{\text{int}} = e^e = e - e^p = 2u - e^p = 2r + e^p. \quad (21)$$

Therefore it follows from Equation (21) that

$$-f_{\text{int}} = u + r, \quad (22)$$

so that Equation (19) becomes

$$\varepsilon^2 u'' + u + r = f_{\text{ext}}. \quad (23)$$

The behavior of the plastic element is characterized by the non-dimensional inequalities and flow rule:

$$|r| \leq 1, \quad (e^p)' \begin{cases} 
\geq 0 \text{ if } r = +1, \\
= 0 \text{ if } -1 < r < +1, \\
\leq 0 \text{ if } r = -1.
\end{cases} \quad (24)$$

Note that the factor 2 was introduced in various points in (20) and (21) in order to have the final equation (23) in a simple form. The governing dynamic equations (23), together with the conditions (24) that characterize the behavior of the plastic element, can be put in the form of a singularly perturbed system of first order differential equations and inclusions. For that purpose, let $C$ denote the following closed convex set in $\mathbb{R}$

$$C = \{ r \in \mathbb{R} : |r| \leq 1 \}, \quad (25)$$

and let $\text{sign}^{-1}(r)$ be the normal cone to $C$ at $r \in \mathbb{R}$ defined by

if $r \notin C$ then $\text{sign}^{-1}(r) = \emptyset,$

if $r \in C$ then $\text{sign}^{-1}(r) = \{ x \in \mathbb{R} : x \geq 0, \text{ if } r = +1; x = 0, \text{ if } -1 < r < +1; x \leq 0, \text{ if } r = -1 \}.$

Then we observe that Equation (24) can be written in the differential inclusion form:

$$(e^p)' \in \text{sign}^{-1}(r). \quad (26)$$

Relations Equation (21) lead to

$$(e^p)' = u' - r'. \quad (27)$$

Substituting Equation (27) in Equation (26), we get

$$u' - r' \in \text{sign}^{-1}(r). \quad (28)$$
From Equation (23) and Equation (28) we finally obtain the governing dynamic system

\[
\begin{align*}
\epsilon u' - v &= 0, \\
\epsilon v' + u + r &= f_{\text{ext}}, \\
u' - r' &\in \text{sign}^{-1}(r),
\end{align*}
\] (29)

which must be satisfied together with some initial conditions

\[
(u(\lambda_1), v(\lambda_1), r(\lambda_1)) = (u_1, v_1, r_1) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}.
\] (30)

The corresponding quasi-static system is then (let \(\epsilon = 0\) in Equation (29))

\[
\begin{align*}
\bar{u} + \bar{r} &= f_{\text{ext}}, \\
\bar{u}' - \bar{r}' &\in \text{sign}^{-1}(\bar{r}),
\end{align*}
\] (31)

with initial condition

\[
\bar{r}(\lambda_1) = \bar{r}_1 \in \mathbb{C}.
\] (32)

Note that, consistently with the above, the quasi-static displacement rate with respect to the physical time vanishes (\(\bar{v} \equiv 0\)). Besides, if \(X\) is a space of scalar functions, the bold-face notation \(X_d\) will denote the space \(X^d\).

3.2. Existence and uniqueness of solutions for the dynamic and for the quasi-static systems

We observe that the dynamic and quasi-static systems introduced in Section 3.1 can be rewritten in a form that may be studied with the theory of \(m\)-accretive operators. Recall that existence and uniqueness to the differential inclusion problem

\[
x' + Ax \ni g(\cdot, x) \quad \text{a.e. on } (\lambda_1, \lambda_2),
\] (33a)

\[
x(\lambda_1) = x_1.
\] (33b)

follows from the following Proposition

**Proposition 3.1.** Let \(g(\cdot, x)\) be a function from \([\lambda_1, \lambda_2] \times \mathbb{R}^p\) to \(\mathbb{R}^p\) such that, for some \(\omega > 0\), the following assumptions are satisfied

\[
\forall \lambda \in [\lambda_1, \lambda_2], \forall (x, y) \in \mathbb{R}^{2p}, \quad |g(\cdot, x) - g(\cdot, y)| \leq \omega|x - y|,
\] (34a)

\[
\forall x \in \mathbb{R}^p, \quad g(\cdot, x) \in L^\infty_p(\lambda_1, \lambda_2).
\] (34b)

Assume that \(A : \mathbb{R}^p \rightarrow \mathbb{R}^p \times \mathbb{R}^p\) is an \(m\)-accretive operator. Then there exists a unique solution \(x\) of Equation (33), belonging to \(W^{1,\infty}_p(\lambda_1, \lambda_2)\).

The reader can find a detailed proof of Proposition 3.1 in the Appendix of [10]. By applying Proposition 3.1, we prove existence and uniqueness of solution for the dynamic system Equation (29)-Equation (30) and for the corresponding quasi-static system Equation (31)-Equation (32). Denoting \(x = (u, v, r)\), we get

\[
x' + Ax \ni g(\cdot, x)
\] (35)

where

\[
A x = \begin{pmatrix} 0 \\ 0 \\ \text{sign}^{-1}(r) \end{pmatrix} \quad \text{and} \quad g = \frac{1}{\epsilon} \begin{pmatrix} v \\ f_{\text{ext}} - u - r \end{pmatrix}.
\]

The sub-differential \(\partial \varphi = A\) is \(m\)-accretive since \(\varphi(r)\) is a convex proper and lower semi-continuous function. Observing that Equation (34) holds with \(p = 3\), Proposition 3.1 yields the following Corollary:
Corollary 3.2. Assume that $f_{\text{ext}}$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2)$ and that Equation (30) holds. Then there exists a unique solution $(u, v, r)$ of Equation (29)-Equation (30) belonging to the Sobolev space $W^{1,\infty}_3(\lambda_1, \lambda_2)$.

On the other hand, we deduce from the identity in Equation (31) that

$$\bar{u} = f_{\text{ext}} - \bar{r}. \quad (36)$$

Carrying Equation (36) into the inclusion Equation (31), we get

$$\bar{r}' + \frac{1}{2} \text{sign}^{-1}(\bar{r}) \geq \frac{1}{2} f_{\text{ext}}. \quad (37)$$

As for the dynamic system, the sub-differential $\partial \varphi(\bar{r}) = \text{sign}^{-1}(\bar{r})$ is an $m$-accretive operator since $\varphi(\bar{r})$ is a proper convex and lower semi-continuous function. Observing that Equation (34) holds with $x = \bar{r}, A = \text{sign}^{-1}/2, g = f_{\text{ext}}/2$ and $p = 1$ in Equation (33), we apply Proposition 3.1 and we obtain the following Corollary:

**Corollary 3.3.** Assume that $f_{\text{ext}}$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2)$ and $\bar{r}_1 \in C$. Then there exists a unique solution $\bar{r}$ of Equation (37) belonging to $W^{1,\infty}(\lambda_1, \lambda_2)$, and with initial conditions $\bar{r}_1 \in C$.

**Remark 3.4.** According to Corollary 3.3 and identity Equation (36), $\bar{u}$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2)$.

3.3. Stability of quasi-static paths of elastic-plastic systems

In Section 3.3.1 we adapt the definition of stability of a quasi-static path ([11], [12]) to the present elastic-plastic problem, and in Section 3.3.2 we prove a priori estimates that show that, in order to guarantee that those two solutions remain close to each other in some finite interval of load, it suffices that the dynamic solution of Equation (29) is initially close to the quasi-static solution of Equation (31) and the loading rate $\varepsilon$ is sufficiently small.

3.3.1. Definition of stability of a quasi-static path The mathematical definition of stability of a quasi-static path at an equilibrium point is presented in the context of the governing dynamic system Equation (29)-Equation (30) and the quasi-static system Equation (31)-Equation (32).

**Definition 3.5.** The quasi-static path $(\bar{u}(\lambda), \bar{r}(\lambda))$ is said to be stable at $\lambda_1$ if there exists $0 < \Delta \lambda \leq \lambda_2 - \lambda_1$, such that, for all $\delta > 0$ there exists $\rho(\delta) > 0$ and $\varepsilon(\delta) > 0$ such that for all initial conditions $u_1, v_1, r_1$, and $\bar{r}_1$, and all $\varepsilon > 0$ such that

$$|v_1|^2 + |u_1 - \bar{u}(\lambda_1)|^2 + |r_1 - \bar{r}_1|^2 \leq \rho(\delta) \text{ and } \varepsilon \leq \varepsilon(\delta),$$

the solution $(u(\lambda), v(\lambda), r(\lambda))$ of the dynamic system Equation (29)-Equation (30) satisfies

$$|v(\lambda)|^2 + |u(\lambda) - \bar{u}(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \leq \delta, \forall \lambda \in [\lambda_1, \lambda_1 + \Delta \lambda].$$

For more details, the reader is referred to [11].

3.3.2. A priori estimates and stability Let us introduce the regularized problem:

$$\begin{cases} \varepsilon^2 u'' + u + r = f_{\text{ext}}, \\ r' = \frac{1}{\mu} (r_{\mu} - \text{proj}_C r_{\mu}) = u'_{\mu}, \end{cases} \quad (38)$$

with initial conditions

$$(u_{\mu}(\lambda_1), v_{\mu}(\lambda_1), r_{\mu}(\lambda_1)) = (u_1, v_1, r_1) \in \mathbb{R} \times \mathbb{R} \times C. \quad (39)$$

Here proj$C$ denotes the projection on the convex $C$ and $v_{\mu} = \varepsilon u'_{\mu}$. The reader can find a detailed proof of the following Corollary in the Appendix of [10] or in [2]:
Corollary 3.6. Assume that \( f_{\text{ext}} \) belongs to \( W^{1,\infty}(\lambda_1, \lambda_2) \) and that Equation (39) holds. Then there exists a unique solution \((u_\mu, v_\mu, r_\mu)\) belonging to the Sobolev space \( W^{1,\infty}_3(\lambda_1, \lambda_2) \) of Equation (38) and Equation (39). Moreover, as \( \mu \) tends to zero, \((u_\mu, v_\mu, r_\mu)\) converges to the solution of Equation (29)-Equation (30).

Lemma 3.7. Assume that Equation (39) holds and \( f_{\text{ext}} \) belongs to \( W^{2,\infty}(\lambda_1, \lambda_2) \). Then there exists a positive constant \( c(\lambda_1, \lambda_2) \) that depends on the interval of \( \lambda \) and such that

\[
|\varepsilon v'(\lambda)|^2 \leq c(\lambda_1, \lambda_2)(|v_1|^2 + |u_1 - \bar{u}(\lambda_1)|^2 + |r_1 - \bar{r}_1|^2
\]

\[
+ \varepsilon^2 |f'_{\text{ext}}(\lambda_1)|^2 + \varepsilon^2 |f''_{\text{ext}}(\lambda_1, \lambda_2)|^2
\]

Proof. This estimate results from the application of Gronwall’s lemma to energy estimates that are obtained by differentiating the governing system Equation (38) with respect to \( \lambda \) and multiplying the result by \( \varepsilon^2 u_\mu'' \). Integrating the resulting expression over \((\lambda_1, \lambda)\), we get

\[
\int_{\lambda_1}^{\lambda} \varepsilon^4 u''_{\mu} u_{\mu''} d\xi + \int_{\lambda_1}^{\lambda} \varepsilon^2 u'_{\mu} u''_{\mu} d\xi + \int_{\lambda_1}^{\lambda} \varepsilon^2 r'_{\mu} u''_{\mu} d\xi = \int_{\lambda_1}^{\lambda} \varepsilon^2 f''_{\text{ext}} u''_{\mu} d\xi.
\]

We shall pass to the limit in Equation (41) when \( \mu \) tends to zero. Denoting \( v_\mu = \varepsilon u'_\mu \), then

\[
\int_{\lambda_1}^{\lambda} \varepsilon^4 u''_{\mu} u_{\mu''} d\xi + \int_{\lambda_1}^{\lambda} \varepsilon^2 u'_{\mu} u''_{\mu} d\xi = \frac{1}{2} |v_\mu'|^2 + |v_\mu|^2 |v_\mu'|^2
\]

According to the second identity in Equation (38), we deduce that

\[
\int_{\lambda_1}^{\lambda} \varepsilon^2 r'_{\mu} u''_{\mu} d\xi = \int_{\lambda_1}^{\lambda} \varepsilon^2 r'_{\mu} v''_{\mu} d\xi + \int_{\lambda_1}^{\lambda} \frac{\varepsilon^2}{\mu} r'_{\mu} (r_{\mu} - \text{proj}_{\varepsilon} r_{\mu}) d\xi.
\]

Since the second integral on the right hand side of Equation (43) is positive then we have

\[
\int_{\lambda_1}^{\lambda} \varepsilon^2 r'_{\mu} u''_{\mu} d\xi \geq \frac{1}{2} |\varepsilon r'_{\mu}(\lambda)|^2 - \frac{1}{2} |\varepsilon r'_{\mu}(\lambda_1)|^2.
\]

Let us observe that the second identity in the system Equation (38) and the initial conditions Equation (39) lead to

\[
|\varepsilon r'_{\mu}(\lambda_1)|^2 = |v_1|^2.
\]

On the other hand, we deduce that

\[
|\varepsilon v'(\lambda)|^2 \leq 2(|u_1 - \bar{u}(\lambda_1)|^2 + |r_1 - \bar{r}_1|^2).
\]

Introducing Equation (42) and Equation (44) in Equation (41), integrating the right hand side of Equation (41) by parts, and using Cauchy-Schwarz’s inequality, Equation (45) and Equation (46), we get

\[
|\varepsilon v'(\lambda)|^2 + |v(\lambda)|^2 + |r'(\lambda)|^2 \leq c_1(\varepsilon) + 2|\varepsilon f'_{\text{ext}}(\lambda_1) v(\lambda)| + \varepsilon^2 \int_{\lambda_1}^{\lambda} |f''_{\text{ext}}|^2 d\xi + \int_{\lambda_1}^{\lambda} |v_\mu|^2 d\xi
\]

where

\[
c_1(\varepsilon) = 3|v_1|^2 + 2(|u_1 - \bar{u}(\lambda_1)|^2 + |r_1 - \bar{r}_1|^2) + \varepsilon^2 |f'_{\text{ext}}(\lambda_1)|^2.
\]
We estimate the product $|\varepsilon f'_{\text{ext}}(\lambda)v_\mu(\lambda)|$ by $|\varepsilon f'_{\text{ext}}(\lambda)|^2 + |v_\mu(\lambda)|^2 / 4$, then the inequality Equation (47) leads to

$$|\varepsilon v'_\mu(\lambda)|^2 + \frac{1}{2} |v_\mu(\lambda)|^2 + |\varepsilon r'_{\mu}(\lambda)|^2 \leq g(\lambda, \varepsilon) + \int_\lambda^\lambda |v_\mu|^2 \, d\xi \tag{48}$$

where

$$g(\lambda, \varepsilon) = c_1(\varepsilon) + 2\varepsilon^2 \|f'_{\text{ext}}\|_{L^2(\lambda_1, \lambda_2)}^2 + \varepsilon^2 \|f''_{\text{ext}}\|_{L^2(\lambda_1, \lambda_2)}^2.$$  

By classical Gronwall’s lemma, it is clear that

$$|v_\mu(\lambda)|^2 \leq 2g(\lambda, \varepsilon) \exp (2(\lambda_2 - \lambda_1)). \tag{49}$$

Therefore the last term on the right hand side of Equation (48) can be estimated by using Hölder’s inequality and Equation (49):

$$|\varepsilon v'_\mu(\lambda)|^2 + \frac{1}{2} |v_\mu(\lambda)|^2 + |\varepsilon r'_{\mu}(\lambda)|^2 \leq g(\lambda, \varepsilon) (1 + 2(\lambda_2 - \lambda_1) \exp (2(\lambda_2 - \lambda_1))). \tag{50}$$

Differentiating the first identity in the system Equation (38) with respect to $\lambda$ and integrating the resulting expression over $(\eta_1, \eta_2)$, $\eta_1$ and $\eta_2$ belonging to $(\lambda_1, \lambda_2)$, we get, since $v_\mu = \varepsilon u'_{\mu}$,

$$\varepsilon v'_\mu(\eta_2) - \varepsilon v'_\mu(\eta_1) = \int_{\eta_1}^{\eta_2} (f'_{\text{ext}} - u'_{\mu} - r'_{\mu}) \, d\xi,$$

which implies, thanks to Equation (50) and Cauchy-Schwarz’s inequality, that

$$|\varepsilon v'_\mu(\eta_2) - \varepsilon v'_\mu(\eta_1)| \leq g(\lambda, \varepsilon) (1 + 2(\lambda_2 - \lambda_1) \exp (2(\lambda_2 - \lambda_1)))|\eta_2 - \eta_1|. \tag{51}$$

As a consequence, Equation (51) and Equation (50) show that the sequence $\varepsilon v'_\mu$ is equicontinuous and bounded in $C^0(\lambda_1, \lambda_2)$. Therefore, according to Ascoli’s theorem, there exists $z$ belonging to $C^0(\lambda_1, \lambda_2)$ and a subsequence, still denoted by $\varepsilon v'_\mu$, such that

$$\varepsilon v'_\mu \to z \text{ in } C^0(\lambda_1, \lambda_2). \tag{52}$$

By uniqueness of the limit, $z = \varepsilon v'$ and, thanks to Equation (50), we obtain the result in the Lemma.  

\textbf{Remark 3.8.} The inclusions in Equation (29) and Equation (31) can be written in slightly different but equivalent forms: for all $r^*$ belonging to $C$ and for all $\lambda$ belonging to $[\lambda_1, \lambda_2]$, we have

$$\int_{\lambda_1}^{\lambda} (u' - r') (r - r^*) \, d\xi \geq 0, \tag{53}$$

and

$$\int_{\lambda_1}^{\lambda} (\bar{u}' - r') (\bar{r} - r^*) \, d\xi \geq 0. \tag{54}$$

\textbf{Proposition 3.9.} (Stability). Assume that Equation (30) and Equation (32) hold and that $f_{\text{ext}}$ belongs to $W^{2,\infty}(\lambda_1, \lambda_2)$. Then there exist $\gamma_i > 0$, $i = 1, 2$, such that

$$|v(\lambda)|^2 + |u(\lambda) - \bar{u}(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \leq \gamma_1 \left(|v_1|^2 + |u_1 - \bar{u}(\lambda_1)|^2 + |r_1 - \bar{r}_1|^2 \right) + \varepsilon \gamma_2. \tag{55}$$
Proof. We subtract the equality in the quasi-static system Equation (31) to Equation (23), then we multiply the resulting expression by \((u' - \bar{u}')\) and we integrate over \((\lambda_1, \lambda_2)\). On the other hand, we choose \(r^* = \bar{r}\) in Equation (53) and \(r^* = r\) in Equation (54), and we add Equation (53) to Equation (54). We get the following system

\[
\begin{bmatrix}
\int_{\lambda_1}^{\lambda} \varepsilon^2 u'' u' \, d\xi + \int_{\lambda_1}^{\lambda} (u - \bar{u})(u' - \bar{u}') \, d\xi + \int_{\lambda_1}^{\lambda} (r - \bar{r})(u' - \bar{u}') \, d\xi = \int_{\lambda_1}^{\lambda} \varepsilon^2 u'' u' \, d\xi,
\int_{\lambda_1}^{\lambda} (\alpha - (r' - \bar{r}'))(r - \bar{r}) \, d\xi \geq 0.
\end{bmatrix}
\]  

(56)

Since \(v = \varepsilon u'\) then

\[
\int_{\lambda_1}^{\lambda} \varepsilon^2 u'' u' \, d\xi + \int_{\lambda_1}^{\lambda} (u - \bar{u})(u' - \bar{u}') \, d\xi = \frac{1}{2} \left[ |v|^2 + |u - \bar{u}|^2 \right]_{\lambda_1}^\lambda.
\]  

(57)

Notice that the system Equation (56) leads to

\[
\int_{\lambda_1}^{\lambda} (r' - \bar{r}')(r - \bar{r}) \, d\xi \leq \int_{\lambda_1}^{\lambda} (u' - \bar{u}')(r - \bar{r}) \, d\xi
\]

which immediately implies

\[
\frac{1}{2} |r(\lambda) - \bar{r}(\lambda)|^2 - \frac{1}{2} |r_1 - \bar{r}_1|^2 \leq \int_{\lambda_1}^{\lambda} (u' - \bar{u}')(r - \bar{r}) \, d\xi.
\]  

(58)

Define now

\[
h(\xi) = |v(\xi)|^2 + |u(\xi) - \bar{u}(\xi)|^2 + |r(\xi) - \bar{r}(\xi)|^2.
\]  

(59)

Carrying Equation (57) and Equation (58) into the identity in Equation (56) and using the Cauchy-Schwarz inequality and the notation Equation (59), we obtain the following inequality

\[
h(\lambda) \leq h(\lambda_1) + 2 \left( \int_{\lambda_1}^{\lambda} |v'|^2 \, d\xi \right)^{1/2} \left( \int_{\lambda_1}^{\lambda} |u'|^2 \, d\xi \right)^{1/2}.
\]  

(60)

We conclude from Equation (60) that there exists \(\alpha > 0\) such that

\[
|v(\lambda)|^2 + |u(\lambda) - \bar{u}(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \leq \alpha h(\lambda_1) + \alpha \left( \int_{\lambda_1}^{\lambda} |v'|^2 \, d\xi \right)^{1/2} \left( \int_{\lambda_1}^{\lambda} |u'|^2 \, d\xi \right)^{1/2}.
\]

The conclusion follows then from Lemma 3.7.

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