Poincaré Bisectors in Hyperbolic Spaces

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Abstract

We determine explicit formulas for the Poincaré bisectors used in constructing a Dirichlet fundamental domain in hyperbolic two and three space. They are compared with the isometric spheres in the upper half space and plane models of hyperbolic space. This is used to revisit classical results on the Bianchi groups and also some recent results on groups having either a Dirichlet domain with at least two centers or a Dirichlet domain which is also a Ford domain. It is also shown that the relative position of the Poincaré bisector of an isometry and its inverse, determine whether the isometry is elliptic, parabolic or hyperbolic.

1 Introduction

Describing generators and relations of groups acting on hyperbolic spaces was started in the nineteenth century. The big difficulty one encounters is the construction of a fundamental domain. This problem was considered by Ford, Poincaré, Serre, Swan and many others. Only in the case of a Ford domain explicit formulas are known. Computer aided methods also exist. For Fuchsian groups we refer to [11, 12], for Bianchi groups we refer to [20] and [14] and for cocompact groups we refer to [5]. Another non-trivial problem is that of describing units in an order of a non-commutative non-split division algebra. In [10], we use the results of this paper to describe units in some of these orders.

Here, making use of the existing theory, we give explicit descriptions of the bisectors of the Dirichlet domains (see [17]) in $\mathbb{H}^n$, $n \in \{2, 3\}$. These formulas are first given in the ball model, where we show that the bisectors are exactly the isometric spheres of the isometries of the Ball, i.e., the spheres on which these isometries act as Euclidean isometries. Since working in the ball model is not fit for visualization, we deduce explicit formulas of the bisectors in the upper half space (plane) model as well.

These formulas at hand, we give a new and independent criterion for a result of [13] which describes those Fuchsian groups having a fundamental domain which is at the same time a Ford and a Dirichlet Domain, called a DF domain (see Theorem 4.6). Our criterion is of algebraic nature.

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and can be easily checked once a set of generators is given. Moreover our theorem also applies to Kleinian groups. In particular, we consider whether the figure-eight knot group and the Whitehead link complement group, which are groups of interest in this field, do have DF domains. We prove that the answer is negative. For the Bianchi groups, \( \text{PSL}(2, \mathcal{O}_K) \) \( K = \mathbb{Q}(\sqrt{-d}) \), \( d \) a square free positive integer and \( \mathcal{O}_K \) the ring of integers of \( K \), we describe a Dirichlet fundamental domain, together with its ideal points, and give a complete proof that these groups have finite covolume and are of the first kind. This is done in a complete way and independent of \([21]\). From our description it easily follows that \( \infty \) is the only ideal point for \( d \in \{1, 2, 3, 7, 11, 19\} \) and that the examples given in \([6]\) Chapter VII and the Bianchi group for \( d = 19 \) all have a DF domain. We also describe a Dirichlet fundamental domain of the figure-eight knot group whose sides are Poincaré biseectors.

The outline of the paper is as follows. In Section 2 we record fundamentals of hyperbolic geometry and give explicit formulas for an isometric sphere, in the upper half space (plane) model, to be a bisector and show that the relative position of the isometric spheres of an isometry \( \gamma \), say, and its inverse determine whether \( \gamma \) is elliptic, parabolic or hyperbolic. In the last section, we consider DF domains and complement some results contained in \([13]\). We also describe a Dirichlet fundamental domain of the figure-eight knot group. Finally, we handle the ideal points of the Bianchi groups and give another proof that they are of the first kind and of finite covolume.

## 2 Background

In this section we begin by recalling basic facts on hyperbolic spaces and we fix notation. Standard references are \([3, 4, 6, 8, 15, 17]\). By \( \mathbb{H}^n \) (respectively \( \mathbb{B}^n \)) we denote the upper half space (plane) (respectively the ball) model of hyperbolic \( n \)-space.

Let \( \mathbb{H}^3 = \mathbb{C} \times [0, \infty[. \) As is common, we shall often think of \( \mathbb{H}^3 \) as a subset of the classical quaternion algebra \( \mathcal{H} = \mathcal{H}(-1, -1, \mathbb{R}) \) by identifying \( \mathbb{H}^3 \) with the subset \( \{ z + rj \in \mathcal{H} : z \in \mathbb{C}, r \in \mathbb{R}^+ \} \subseteq \mathcal{H} \). The ball model \( \mathbb{B}^3 \) may be identified in the same way with \( \{ z + rj \in \mathbb{C} + \mathbb{R}j \mid |z|^2 + r^2 < 1 \} \subseteq \mathcal{H} \). Denote by \( \text{Iso}(\mathbb{H}^3) \) (respectively \( \text{Iso}(\mathbb{B}^3) \)) the group of isometries of \( \mathbb{H}^3 \) (respectively \( \mathbb{B}^3 \)). The groups of orientation preserving isometries are denoted by \( \text{Iso}^+(\mathbb{H}^3) \) and \( \text{Iso}^+(\mathbb{B}^3) \) respectively. It is well known that \( \text{Iso}^+(\mathbb{H}^3) \) and \( \text{Iso}^+(\mathbb{B}^3) \) are isomorphic with \( \text{PSL}(2, \mathbb{C}) \) and \( \text{PSL}(3, \mathbb{R}) \) respectively.

For \( \mathcal{H} = \{ a + b \mathbb{J} : a, b \in \mathbb{R}, a \geq b \} \subseteq \mathbb{H}^3 \), the ring of integers of \( \mathcal{H} \), denote \( \mathcal{H}^\cdot \) (respectively \( \mathcal{B}^\cdot \)) the group of isometries of \( \mathbb{H}^3 \) (respectively \( \mathbb{B}^3 \)).

The groups of orientation preserving isometries are denoted by \( \text{Iso}^+(\mathbb{H}^3) \) and \( \text{Iso}^+(\mathbb{B}^3) \) respectively.

Let \( u = u_0 + u_1 i + u_2 j + u_3 k \in \mathcal{H} \) and define \( \overline{u} \) to be \( u_0 - u_1 i - u_2 j - u_3 k \), the conjugate of \( u \). Moreover let \( u' = u_0 - u_1 i - u_2 j + u_3 k \) and \( u^* = u_0 + u_1 i + u_2 j - u_3 k \). Let

\[
\text{SB}_2(\mathcal{H}) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M(2, \mathcal{H}) \mid d = a', b = c', \text{a} \bar{a} - c \bar{c} = 1 \right\}.
\]
If \( f = \begin{pmatrix} a & c' \\ c & a' \end{pmatrix} \in \text{SB}_2(\mathcal{H}) \) then \( f^{-1} = \begin{pmatrix} \overline{c} & -\overline{a} \\ -a^* & \overline{c} \end{pmatrix} \). The group \( \text{SB}_2(\mathcal{H})/\{1,-1\} \) is isomorphic to \( \text{Iso}^+(\mathbb{R}^3) \) and hence \( \text{SB}_2(\mathcal{H}) \) is isomorphic to \( \text{SL}(2,\mathbb{C}) \). The action of \( \Psi(\gamma) \) on \( \mathbb{R}^3 \) is given by \( \Psi(\gamma)(P) = (AP+C')(CP+A')^{-1} \), where the latter is calculated in the classical quaternion algebra \( \mathcal{H}(-1,-1,\mathbb{R}) \). Further details on this may be found in [6, Section 1.2]. Let \( g = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix} \in \text{M}_2(\mathcal{H}) \). The map \( \Psi : \text{SL}(2,\mathbb{C}) \rightarrow \text{SB}_2(\mathcal{H}) \) given by \( \Psi(\gamma) = \overline{\gamma}g \) is a group isomorphism. Explicitly, for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we have

\[
\Psi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} |a| \overline{b} -(b-\overline{a})j & b+\overline{a}+(a-\overline{b})j \\ |c| \overline{d} +(d-\overline{c})j & \overline{c}d+(c-\overline{d})j \end{pmatrix}
\]

and hence \( \|\Psi(\gamma)\|^2 = \|\gamma\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 \). The map \( \eta_0 : \mathbb{H}^3 \rightarrow \mathbb{R}^3 \), with \( P \rightarrow (P - j)(-jP + 1)^{-1} \), is an \( \Psi \)-equivariant isometry between the hyperbolic models \( \mathbb{H}^3 \) and \( \mathbb{B}^3 \), that is \( \eta_0(M \gamma) = \psi(M) \eta_0(P) \), for \( P \in \mathbb{H}^3 \) and \( M \in \text{SL}(2,\mathbb{C}) \) (see [6, Proposition, I.2.3]).

The following lemma will be useful in the next section. The proof is easy and is therefore omitted. Part of it, is a simple consequence of the existence of an inverse element in \( \text{SB}_2(\mathcal{H}) \).

**Lemma 2.1** Let \( a, c \in \mathcal{H} \) be such that \( a\overline{a} - c\overline{c} = 1 \). Then

1. \( c \cdot \overline{a} = a' \cdot c'^* \), \( a^* c = c^* a \) and \( a\overline{a} - c\overline{c} a^* \in \mathbb{R}^k \),
2. \( \overline{a} \cdot c' = c' \cdot a^* \), \( a + a^* \in \mathbb{C} + rj \),
3. \( \overline{a} c + a \overline{a} = 2(|a|^2, |a + c|^2 = |a|^2 + |c|^2 + 2|a||c|, \) where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathbb{R}^4 \).

Recall that the hyperbolic distance \( \rho \) in \( \mathbb{H}^3 \) is determined by \( \cosh \rho(P, P') = \delta(P, P') = 1 + \frac{d(P,P')^2}{2} \), where \( d \) is the Euclidean distance and \( P = z + rj \) and \( P' = z' + r'j \) are two elements of \( \mathbb{H}^3 \).

Let \( \Gamma \) be a discrete subgroup of \( \text{ISO}^+(\mathbb{B}^3) \). The Poincaré method can be used to give a presentation of \( \Gamma \) in the following way (for details see for example [5]). Let \( \Gamma_0 \) be the stabilizer in \( \Gamma \) of \( 0 \in \mathbb{B}^3 \) and let \( \mathcal{F}_0 \) a fundamental domain for \( \Gamma_0 \). Denote by \( D_\gamma(0) = \{ u \in \mathbb{B}^3 \mid \rho(0, u) \leq \rho(0, \gamma(0)) \} \). The border \( \partial D_\gamma(0) = \{ u \in \mathbb{B}^3 \mid \rho(0, u) = \rho(0, \gamma(0)) \} \) is the hyperbolic bisector of the geodesic linking 0 to \( \gamma(0) \). We call this a Poincaré bisector. Then \( \mathcal{F} = \mathcal{F}_0 \cap ( \bigcap_{\gamma \in \Gamma_0} D_\gamma(0) ) \) is the Dirichlet fundamental domain of \( \Gamma \) with center 0. Indeed, since \( \Gamma \) is discrete, there exists \( P \in \mathbb{B}^3 \) such that \( \Gamma P \) is trivial. Let \( \gamma \) be an isometry mapping 0 to \( P \). It is now easy to see that we get the mentioned fundamental domain from the Dirichlet fundamental domain of \( \gamma^{-1} \Gamma \gamma \). Moreover it may be shown that \( \mathcal{F} \) is a polyhedron and if \( \Gamma \) is geometrically finite then a set of generators for \( \Gamma \) consists of the elements \( \gamma \in \Gamma \) so that \( \mathcal{F} \cap \gamma(\mathcal{F}) \) is a side of the polyhedron together with \( \Gamma_0 \). In our case \( \Gamma = (\Gamma_0 \gamma | \gamma(\mathcal{F}) \cap \mathcal{F} \text{ is a side }) \) (see [6, Theorem 6.8.3]).

Let \( \Gamma \) be a discrete subgroup of \( \text{SL}(2,\mathbb{C}) \) and denote by \( \Gamma_\infty \) the stabilizer in \( \Gamma \) of the point \( \infty \). Denote the fundamental domain of \( \Gamma_\infty \) by \( \mathcal{F}_\infty \). For \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \setminus \Gamma_\infty \), denote the isometric sphere of \( \gamma \) by \( \text{ISO}_\gamma \). Note that these are the points \( P \in \mathbb{H} \) such that \( \|cP + d\| = 1 \). Denote the set \( \{ P \in \mathbb{H} \mid \|cP + d\| \geq 1 \} \) by \( \text{ISO}_\gamma^c \). Then it is known that \( \mathcal{F} = \mathcal{F}_\infty \cap \bigcap_{\gamma \in \Gamma \setminus \Gamma_\infty} \text{ISO}_\gamma^c \) is a fundamental domain for \( \Gamma \) called the Ford fundamental domain of \( \Gamma \).
If $S_1$ and $S_2$ are two intersecting spheres in the extended hyperbolic space, then $(S_1, S_2)$ denotes the cosine of the angle at which they intersect, the dihedral angle. Explicit formulas can be found in [3]. Elements $x$ and $y$ of hyperbolic space are inverse points with respect to $S_1$ if $y = \sigma(x)$, where $\sigma$ is the reflection in $S_1$. In case $S_1 = \partial B^3 = S^2$, the boundary of $B^3$, then the inverse point of $x$ with respect to $S_1$ is denoted by $x^*$. This should give no confusion with the same notation used in the quaternion algebra and should be clear from the context.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{C})$ we write $a = a(\gamma), b = b(\gamma), c = c(\gamma)$ and $d = d(\gamma)$ when it is necessary to stress the dependence of the entries on the matrix $\gamma$. We also have that $\Psi(\gamma) = A \circ \sigma$, where $A$ is an orthogonal map and $\sigma$, a reflection in an Euclidean sphere, $\Sigma = S_{r_0}(p_0)$ say, ortogonal do $S^2$ (see [3]). Since $A$ is an Euclidean isometry it follows that $\Psi(\gamma)$ acts as an Euclidean isometry on $\Sigma$. This is the reason why $\Sigma$ is called the isometric sphere of $\Psi(\gamma)$. It can be shown that this is the unique sphere on which $\Psi(\gamma)$ acts as an Euclidean isometry. The fact that $\Sigma$ is ortogonal do $S^2$ is equivalent to saying that $1 + r^2 = \|p_0\|^2$.

3 Poincaré bisectors and isometric spheres

The main purpose of this section is to give explicit formulas for the bisectors in the Poincaré method in two and three dimensional hyperbolic space. To do so, we first prove that, in the ball model, these bisectors are precisely the isometric spheres. Once this has been done, we use simple hyperbolic geometry to give the formulas for the bisectors in the upper half space (plane) model. Calculations are done in dimension three. However, all formulas hold for the two dimensional case. Standard facts about the theory of hyperbolic geometry will be used freely. Standard references are [3, 4, 6, 8, 10, 17].

Let 0 $\in B^3$ be the origin and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ and $\Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix}$.

Consider the Euclidean sphere $\Sigma = \Sigma_{\Psi(\gamma)} = \{P = z + rj \in \mathbb{C} + \mathbb{R}j \mid \|CP + A'\| = 1\}$. It has center $P_{\Psi(\gamma)} = -C^{-1}A'$ and radius $R_{\Psi(\gamma)} = \frac{1}{|C|}$. We will prove that this is the isometric sphere of the element $\gamma$ in $B^3$. We remind the reader that $*$ is used both for inverse point with respect to $S^2$ as well as for the classical involution on the quaternion algebra $H$.

Recall that $\gamma \in SU(2, \mathbb{C})$ if and only if $\gamma(j) = j$ or, equivalently, $\Psi(\gamma)(0) = 0$ (see [3, 6]). In the latter case, we have that $\Psi(\gamma)$ is a linear orthogonal map. Note that if $\gamma \in SU(2, \mathbb{C})$ then $\|\gamma\|^2 = 2$.

**Theorem 3.1** Let $\gamma \in SL(2, \mathbb{C})$ with $\gamma \notin SU(2, \mathbb{C})$. Then the sphere $\Sigma_{\Psi(\gamma)}$ is the isometric sphere of $\gamma$ and equals the bisector of the geodesic segment linking 0 and $\Psi(\gamma^{-1})(0)$, i.e. $\Sigma_{\Psi(\gamma)} = \{u \in B^3 \mid \rho(0, u) = \rho(u, \Psi(\gamma^{-1})(0))\}$. Moreover $1 + \frac{1}{R^2_{\Psi(\gamma)}} = |P_{\Psi(\gamma)}|^2$, $D_{\gamma}(0) = B^3 \cap \text{Exterior}(\Sigma_{\Psi(\gamma)})$ and $P_{\Psi(\gamma)}$ and $\Psi(\gamma^{-1})(0)$ are inverse points with respect to $\partial B^3$.

**Proof.** First, using Lemma 2.1 we have, in $H$, that $P^*_{\Psi(\gamma)} = P_{\Psi(\gamma)}$, i.e., $P_{\Psi(\gamma)} \in \mathbb{C} + \mathbb{R}j$. As $P_{\Psi(\gamma)} \cdot (\Psi(\gamma^{-1})(0))^{-1} = |A|^2|C|^2 \in \mathbb{R}$, it follows that $0, P_{\Psi(\gamma)}$ and $\Psi(\gamma^{-1})(0)$ are collinear points. As also $|P_{\Psi(\gamma)}|\cdot(|\Psi(\gamma^{-1})(0)|^{-1})^{-1} = |C^{-1}A'| = |C|^{-1} = 1$, we have that $P_{\Psi(\gamma)}$ and $\Psi(\gamma^{-1})(0)$ are inverse points with respect to $S^2$. Moreover $1 + R^2_{\Psi(\gamma)} = \frac{1}{|C|^2} = |C|^2 = |P_{\Psi(\gamma)}|^2$ and hence $(\Sigma, S^2) = 0$. It follows also that the reflection, $\sigma$ say, in $\Sigma$ is an isometry of $B^3$ and that $\sigma \circ \Psi(\gamma^{-1})(0) = 0$. Consequently, $A = \sigma \circ \Psi(\gamma^{-1})$ is an Euclidean linear isometry and thus $\Sigma$ is the isometric sphere of $\Psi(\gamma)$. Note that we made use of [3 Theorems 3.4.1, 3.4.2 and 3.5.1].
Now let \( r \) be the ray through \( P_{\Psi(\gamma)} \) and \( M := r \cap \Sigma_{\Psi(\gamma)} \). Clearly \( \|M\| = \frac{|A|-1}{|C|} \). Since the hyperbolic metric \( \rho \) in \( \mathbb{R}^3 \) satisfies \( \rho(0,u) = \ln(\frac{1+|u|}{1-|u|}) \) (see [3] Formula (7.2.5)) we easily see that \( \rho(0,M) = \rho(M,\Psi(\gamma^{-1})(0)) \). The ray being orthogonal to \( \Sigma \), gives us that \( \Sigma_{\Psi(\gamma)} \) is the mentioned bisector.

Another way to prove this result is noting that \( \phi := \Psi(\gamma) \) is conformal and that \( \|d\phi_u \cdot 1\| = \frac{(1-\|\phi(u)\|^2)^2}{(1-\|u\|^2)^2} \) is the scale factor of \( \phi_u \). Elementary calculations, using Lemma 2.1, show that \( \|d\phi_u \cdot 1\| = \frac{1}{|c_{\phi_u}+A\phi_u|^2} \). From this we can now read off the equation of the isometric sphere.

Consider \( \gamma \in \text{SL}(2,\mathbb{C}) \) acting on \( \mathbb{H}^3 \). We have that \( d\gamma_p \cdot 1 = \lim_{t \to 0} \frac{\gamma(P+t)-\gamma(P)}{t} \). Using that \( \cosh(\rho(P,P')) = \delta(P,P') = 1 + \frac{||P-P'||^2}{||P-P'||^2} \) (see [3] Proposition 1.6), we obtain that \( \|\gamma(P+t)-\gamma(P)\| \to \frac{1}{||c_{P+d}||} \). From this it follows that the scale factor of \( \gamma \) is \( \|d\gamma_p \cdot 1\| = \frac{1}{||c_{P+d}||} \). Hence the isometric sphere of \( \gamma \notin \text{SL}(2,\mathbb{C})_{\infty} \) acting on \( \mathbb{H}^3 \) is the Euclidean half sphere consisting of the elements \( z + rj \in \mathbb{H}^3 \) satisfying \( |cz + d|^2 + |c|^2r^2 = 1 \). We denote this set by \( \text{ISO}_{\gamma} \). Its center is denoted by \( P_{\gamma} \) and its radius by \( R_{\gamma} \). In general, we do not have that \( \text{ISO}_{\gamma} = \eta_0^{-1}(\Sigma_{\Psi(\gamma)}) \), i.e., an isometric sphere in \( \mathbb{H}^3 \) is not necessarily a Poincaré bisector. For this reason we define \( \Sigma_{\gamma} := \eta_0^{-1}(\Sigma_{\Psi(\gamma)}) \). By Theorem 3.1 and the fact that \( \eta_0 \) is an isometry between the two models, \( \Sigma_{\gamma} \) is the bisector of the geodesic linking \( \eta_0^{-1}(0) = j \) and \( \eta_0^{-1}(\Psi(\gamma^{-1}(0))) = \gamma^{-1}(j) \). This bisector may be an Euclidean sphere or a plane perpendicular to \( \partial \mathbb{H}^3 \). In case it is an Euclidean sphere, we denote its center by \( P_{\gamma} \) and its radius by \( R_{\gamma} \). The following lemma gives some information about the position of \( \text{ISO}_{\gamma} \) with respect to \( j \) and \( 0 \).

**Lemma 3.2** Let \( \gamma \in \text{SL}(2,\mathbb{C}) \), with \( c(\gamma) \neq 0 \). Then

1. \( \text{ISO}_{\gamma} \) is the isometric sphere of \( \gamma \in \text{ISO}(\mathbb{H}^3) \).
2. \( \Sigma_{\gamma} \) is the bisector of the geodesic linking \( j \) and \( \gamma^{-1}(j) \).
3. \( j \in \text{ISO}_{\gamma} \) if and only if \( |c(\gamma)|^2 + |d(\gamma)|^2 = 1 \) and \( 0 \in \text{ISO}_{\gamma} \) if and only if \( |d(\gamma)| = 1 \).

**Proof.** The first item was just proved above and the second one follows from Theorem 3.1 and using that \( \eta_0 \) is an isometry between the two models. The last item follows easily from the formula of the isometric sphere.

The following result gives concrete formulas for the Poincaré bisectors in the upper half space model.

**Proposition 3.3** Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2,\mathbb{C}) \), with \( \gamma \notin \text{SU}(2,\mathbb{C}) \).

1. \( \Sigma_{\gamma} \) is an Euclidean sphere if and only if \( |a|^2 + |c|^2 \neq 1 \). In this case, its center and its radius are respectively given by \( P_{\gamma} = -\frac{\langle \overline{a}b + \overline{c}d \rangle}{|a|^2 + |c|^2 - 1} \) and \( R_{\gamma}^2 = \frac{1 + ||P_{\gamma}||^2}{|a|^2 + |c|^2} \).
2. \( \Sigma_{\gamma} \) is a plane if and only if \( |a|^2 + |c|^2 = 1 \). In this case \( \text{Re}(\overline{v}z) + \frac{|v|^2}{2} = 0, z \in \mathbb{C} \) is a defining equation of \( \Sigma_{\gamma} \), where \( v = \overline{a}b + \overline{c}d \).
3. \(|\overline{m}b + \overline{n}d|^2 = (|a|^2 + |c|^2)(|b|^2 + |d|^2) - 1\)

**Proof.** In the proof of Theorem 3.1 it was shown that 0 and \(\Psi(\gamma^{-1}(0))\) are inverse points with respect to \(\Sigma_{\Psi(\gamma)}\). Since \(\eta_0\) is a Mobius transformation, it follows that \(j = \eta_0^{-1}(0)\) and \(\gamma^{-1}(j) = \eta_0^{-1}(\Psi(\gamma^{-1}(0)))\) are inverse points with respect to \(\Sigma_\gamma\). So, if \(\Sigma_\gamma\) is an Euclidean sphere, then \(j\), \(\gamma^{-1}(j)\) and \(P_\gamma\) are collinear points. It follows that \(P_\gamma = l \cap \partial \mathbb{H}\), where \(l\) is the line determined by \(j\) and \(\gamma^{-1}(j)\). Since \(\gamma^{-1}(j) = -\frac{(\overline{m}b + \overline{n}d)}{\langle b, c \rangle - 1} + \frac{1}{|c|^2}j\), a simple calculation gives the formula of \(P_\gamma\). To find the radius of \(\Sigma_\gamma\) one just has to notice that \(R_\gamma^2 = \|j - P_\gamma\| \cdot \|\gamma^{-1}(j) - P_\gamma\|\). This proves the first item.

The expression of \(P_\gamma\) shows that \(\Sigma_\gamma\) is a vertical plane if and only if \(|a|^2 + |c|^2 = 1\). In this case, \(\gamma^{-1}(j) = -\overline{v} + \overline{d}j\) and hence \(v = j - \gamma^{-1}(j) = \overline{ab} + \overline{c}d\) is orthogonal to \(\Sigma_\gamma\). From this one obtains the mentioned defining equation of \(\Sigma_\gamma\), hence the second item. The last item is straightforward. \(\blacksquare\)

The next proposition gives some information about the relation of \(ISO_\gamma\) and \(\Sigma_\gamma\), for some \(\gamma \in SL(2, \mathbb{C}) \setminus SU(2, \mathbb{C})\) and such that \(c(\gamma) \neq 0\). This will be useful in the study of DF domains.

**Proposition 3.4** Let \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \setminus SU(2, \mathbb{C})\).

1. Suppose \(c \neq 0\) and \(|a|^2 + |c|^2 \neq 1\). Then \(|\hat{P}_\gamma - P_\gamma| = \frac{|d - \overline{c}|}{|c|^2(|a|^2 + |c|^2) - 1}|. Moreover \(ISO_\gamma = \Sigma_\gamma\) if and only if \(d = \overline{a}\). In this case we also have that \(c = \lambda \overline{b}\), with \(\lambda \in \mathbb{R}\).

2. If \(ISO_\gamma\) and \(\Sigma_\gamma\) exist and are equal, then \(tr(\gamma) \in \mathbb{R}\).

**Proof.** Since \(\det(\gamma) = 1\) we have that \(|\hat{P}_\gamma - P_\gamma| = \frac{|d - \overline{c}|}{|c|^2(|a|^2 + |c|^2) - 1} = \frac{|d - \overline{c}|}{|c|^2(|a|^2 + |c|^2) - 1} = \frac{1}{|c|^2} - 1\). Hence if \(ISO_\gamma = \Sigma_\gamma\) then \(d = \overline{a}\) and hence \(bc = |a|^2 - 1 \in \mathbb{R}\). With the formula of \(R_\gamma\) at hand, we readily find that \(R_\gamma = \frac{1}{|c|^2}\).

If \(ISO_\gamma = \Sigma_\gamma\), then \(d = \overline{a}\) and thus \(tr(\gamma) = a + \overline{a} \in \mathbb{R}\). This proves the second item. \(\blacksquare\)

**Remark 3.5** Notice that if \(c = 0\) and \(\Sigma_\gamma\) is a plane, we have that \(a = 1\) and \(1 = |a|^2 + |c|^2 = |a|^2\). Hence \(c = \lambda \overline{b}\), with \(\lambda \in \mathbb{R}\). So if both the isometric sphere and the Poincaré bisector under the form of an Euclidean sphere do not exist, we get the same result as in item 1. If one of the two spheres does not exist and the other does, it makes no sense comparing both. Similarly if \(c = 0\) and \(\Sigma_\gamma\) is a plane, one may also easily proof that \(tr(\gamma) \in \mathbb{R}\), as in item 2.

We now have concrete formulas for isometric spheres and Poincaré bisectors in the upper half space model. We also have formulas for the centre and the radius of the isometric sphere (or the bisector) in the ball model. However these formulas are given in terms of \(\Psi(\gamma)\). The next proposition gives information about the isometric sphere and the bisector in the ball model in terms of \(\gamma\).

**Proposition 3.6** Let \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})\) and \(\Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix}\). Then the following properties hold.

1. \(|A|^2 = \frac{2 + \|\overline{c}\|^2}{4}\) and \(|C|^2 = \frac{\|\overline{c}\|^2 - 2}{4}\) and \(|A|^2 - |C|^2 = 1|.
2. \( P_{\Psi}(\gamma) = \frac{1}{2 + \| \gamma \|^2} \cdot \left[ -2(\pi b + r d) + [(|b|^2 + |d|^2) - (|a|^2 + |c|^2)]j \right]. \)

3. \( \Psi^{-1}(0) = \frac{1}{2 + \| \gamma \|^2} \cdot \left[ -2(\pi b + r d) + [(|b|^2 + |d|^2) - (|a|^2 + |c|^2)]j \right] \) (notation of inverse point w.r.t. \( S^2 \)).

4. \( ||P_{\Psi}(\gamma)||^2 = \frac{2 + \| \gamma \|^2}{2 + \| \gamma \|^2}. \)

5. \( R_{\Psi}^2(\gamma) = \frac{4}{-2 + \| \gamma \|^2}. \)

6. \( \Sigma_{\Psi}(\gamma) = \Sigma_{\Psi}(\gamma_1) \text{ if and only if } \gamma_1 = \gamma_0 \gamma \text{ for some } \gamma_0 \in \text{SU}(2, \mathbb{C}). \)

**Proof.** The proof of the first five items is straightforward using the explicit formulas for \( \Psi(\gamma) \), \( A \), \( C \) (see Section 2.1) and knowing that \( P_{\Psi}(\gamma) = C^{-1}A'. \)

We now prove the last item. Suppose \( \Sigma_{\Psi}(\gamma) = \Sigma_{\Psi}(\gamma_1) \). As \( \Psi(\gamma) \) and \( \Psi(\gamma_1) \) are Möbius transformations, \( \Psi(\gamma) = A \sigma \) and \( \Psi(\gamma_1) = A_1 \sigma_1 \) for \( A \) and \( A_1 \) two orthogonal maps and \( \sigma \) and \( \sigma_1 \) reflections in the spheres \( \Sigma_{\Psi}(\gamma) \) and \( \Sigma_{\Psi}(\gamma_1) \) respectively. Thus \( \sigma = \sigma_1 \) and thus \( \Psi(\gamma_1) = A_1 \sigma_1 = A_1 \sigma = A_1 A^{-1} \Psi(\gamma) \). Put \( \gamma_0 = A_1 A^{-1} \) and one implication is proved. To prove the inverse implication suppose \( \gamma_1 = \gamma_0 \gamma \) for some \( \gamma_0 \in \text{SU}(2, \mathbb{C}) \). Then \( \Psi(\gamma) = A \sigma \) and \( \Psi(\gamma_1) = A_1 \sigma_1 \) for \( A \) and \( A_1 \) two orthogonal maps and \( \sigma \) and \( \sigma_1 \) reflections in some spheres. We have that \( \gamma_1 = \gamma_0 \gamma = \gamma_0 A \sigma \) and as \( \gamma_0 A \) is an orthogonal map, \( \gamma_0 A = A_1 \) and \( \sigma = \sigma_1 \). Consequently \( \Sigma_{\Psi}(\gamma) = \Sigma_{\Psi}(\gamma_1) \). \( \blacksquare \)

If \( \Gamma \subseteq \text{iso}^+(\mathbb{B}^3) \) is a discrete subgroup of \( \text{SL}(2, \mathbb{C}) \) then, for each \( \lambda \in \mathbb{R} \), the set \( \{ \gamma \in \Gamma \mid \| \gamma \| = \lambda \} \) is finite (this is well known, but may also easily be deduced from items 3 and 4 of the previous result) and hence, by the Proposition 3.6, for any sequence \( (\gamma_n) \subseteq \Gamma \), of two by two distinct elements of \( \Gamma \), we must have that \( R_{\gamma_n} \to 0 \) as \( n \to \infty \). Also if \( \gamma_1, \gamma \in \Gamma \) have the same norm then, by items 4 and 5 of the previous Proposition, the intersection of their isometric sphere with \( S^2 \) have the same Euclidean volume. However it is not clear that this volume is a strictly decreasing function of the norm. We shall now address this problem.

Let \( \gamma \in \text{SL}(2, \mathbb{C}) \) and \( \gamma \notin \text{SU}(2, \mathbb{C}) \) and let \( r \) be the ray through the center of \( \Sigma_{\Psi}(\gamma) \). Denote by \( M \) and \( N \), respectively, the intersection of \( r \) with \( \Sigma_{\Psi}(\gamma) \) and \( S^2 \). Put \( \rho_\gamma = ||M\bar{N}|| \). Explicitly we have that \( \rho_\gamma = 1 + R_{\Psi}(\gamma) - ||P_{\Psi}(\gamma)|| \). Our next result shows that \( \rho_\gamma \) is a strictly decreasing function of \( ||\gamma|| \). Note that the Euclidean volume of the intersection of \( \Sigma_{\Psi}(\gamma) \) with \( S^2 \), mentioned above, is a function of \( \rho_\gamma \).

**Lemma 3.7** Let \( \Gamma < \text{PSL}(2, \mathbb{C}) \) be a discrete subgroup acting on \( \mathbb{B}^3 \). Then \( \rho_\gamma \) is a strictly decreasing function of \( ||\gamma|| \) on \( \Gamma \setminus \Gamma_0 \).

**Proof.** Using Proposition 3.6 one obtains that \( \rho_\gamma = 1 - (\frac{||\gamma||^2 + 2}{||\gamma||^2 - 2})^\frac{1}{2} + 2(||\gamma||^2 - 2)^{-\frac{1}{2}} \). It is well known that for any \( \gamma \in \text{GL}_2(\mathbb{C}) \), we have that \( 2 \cdot |\det(\gamma)| \leq ||\gamma||^3 \), and thus \( ||\gamma||^3 \geq 2 \) if \( \gamma \in \text{SL}_2(\mathbb{C}) \), with equality if and only if \( \gamma \in \text{SU}_2(\mathbb{C}) \). Consider now the continuous function \( f : [2, +\infty[ \to \mathbb{R} \) given by \( f(x) = 1 - (\frac{x^2 + 2}{x^2 - 2})^\frac{1}{2} + 2(x^2 - 2)^{-\frac{1}{2}} \). Then \( f'(x) = -2x(x^2 - 2)^{-3/2}(x^2 + 2)^{-1/2}[-2 + \sqrt{x^2 + 2}] \), which shows that \( f \) is a strictly decreasing function. From this the result follows. \( \blacksquare \)

Recall that a discrete group is called cocompact if it has a fundamental domain which is compact. The following result is very useful if one has to decide if a given group is cocompact or whether a certain point of \( \partial \mathbb{B}^3 \) is an ideal point. Recall that an ideal point of a fundamental domain \( \mathcal{F} \) in \( \mathbb{H}^n \) (or \( \mathbb{B}^n \) respectively) is a point \( z \in \mathcal{F} \cap \partial \mathbb{H}^n \) (or in \( \mathcal{F} \cap \partial \mathbb{B}^n \) respectively).
Lemma 3.8 Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \), with \( \gamma \notin \text{SU}(2, \mathbb{C}) \). Then

1. \( 0 \notin \Sigma_{\Psi(\gamma)} \).
2. \( j \in \Sigma_{\Psi(\gamma)} \) if and only if \( |a|^2 + |c|^2 = 1 \).
3. \( j \in \text{Interior}(\Sigma_{\Psi(\gamma)}) \) if and only if \( |a|^2 + |c|^2 < 1 \).
4. \( -j \in \Sigma_{\Psi(\gamma)} \) if and only if \( |b|^2 + |d|^2 = 1 \).
5. \( -j \in \text{Interior}(\Sigma_{\Psi(\gamma)}) \) if and only if \( |b|^2 + |d|^2 < 1 \).

Proof. All items follow readily if one uses the previous proposition and the explicit expression of \( \Psi(\gamma) \).

Corollary 3.9 Let \( \Gamma \) be a discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \), with \( \gamma \notin \text{SU}(2, \mathbb{C}) \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). Suppose that \( \{ |a|^2, |b|^2, |c|^2, |d|^2 \} \subset \mathbb{N} \). Then \( \{ -j, j \} \cap \text{Interior}(\Sigma_{\Psi(\gamma)}) = \emptyset \). In particular, if all \( \gamma \in \Gamma \) have this property then \( \Gamma \) is not cocompact.

In the upper half space model, under the assumptions of the corollary and supposing that \( \Gamma_0 = 1 \), it means that 0 and \( \infty \) are ideal points.

Our next result will be useful when dealing with ideal points of Kleinian groups, in particular the Bianchi groups. A version for Fuchsian groups can also be given. We leave this to the reader.

Lemma 3.10 Let \( \gamma \in \text{PSL}(2, \mathbb{C}) \), with \( \gamma \notin \text{SU}(2, \mathbb{C}) \), and suppose that \( j \in \text{Exterior}(\Sigma_{\Psi(\gamma)}) \cap \text{Exterior}(\Sigma_{\Psi(\gamma^{-1})}) \). Then \( \eta_0(\hat{P}_\gamma) \in \text{Interior}(\Sigma_{\Psi(\gamma)}) \).

Proof. We have that \( \Psi(\gamma^{-1})(\Sigma_{\Psi(\gamma^{-1})}) = \Sigma_{\Psi(\gamma)} \) and the latter disconnects \( \mathbb{H}^3 \) into two connected components. Since \( \Psi(\gamma)(P_{\Psi(\gamma)}) = \infty \) it follows that \( \Psi(\gamma^{-1}) \) maps \( \text{Exterior}(\Sigma_{\Psi(\gamma^{-1})}) \) onto \( \text{Interior}(\Sigma_{\Psi(\gamma)}) \). Now observe that \( \Psi(\gamma^{-1})(j) = \eta_0(\hat{P}_\gamma) \). Hence, because of the hypothesis, the conclusion follows.

Note that, in \( \mathbb{H}^3 \), \( \Sigma_\gamma \) also divides the space into two parts and the exterior of \( \Sigma_\gamma \) is defined to be the part containing \( j = \eta_0^{-1}(0) \). If we translate the hypothesis of Lemma 3.10 we obtain that \( \eta_0^{-1}(j) = \infty \) has to be in the intersection \( \text{Exterior}(\Sigma_\gamma) \cap \text{Exterior}(\Sigma_{\gamma^{-1}}) \). This means that \( j \) and \( \infty \) are on the same side of \( \Sigma_\gamma \) and \( \Sigma_{\gamma^{-1}} \), which is nothing else than that \( S^2 \) is not contained in \( \text{Interior}(\Sigma) \cap \text{Interior}(\Sigma^{-1}) \). In particular, this is true if \( R_\gamma < 1 \).

The following result is useful if one is interested in the dihedral angles. Making use of the previous results, these can now be calculated knowing only the norm of the two elements involved. Since all models of hyperbolic space of the same dimension are isometric, the formulas of our next result hold for all models.
Lemma 3.11 Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$, $\pi_0 = \partial \mathbb{H}^n$, $n = 2, 3$, and $\theta$ the angle between $\Sigma_{\Phi(\gamma)}$ and $\Sigma \in \{\Sigma_{\Phi(\gamma)}, \pi_0\}$ with $\Sigma \cap \Sigma_{\Phi(\gamma)} \neq \emptyset$. Then

1. If $\Sigma = \Sigma_{\Phi(\gamma_1)}$ then $\cos(\theta) = \frac{2|1 - (P_{\Phi(\gamma)}(P_{\Phi(\gamma_1)})|}{2R_{\Phi(\gamma)}^2 - R_{\Phi(\gamma_1)}^2}$.

2. If $\Sigma = \pi_0$ then $\cos(\theta) = \frac{1}{\sqrt{\gamma^2 - 2}}$.

**Proof.** This follows from the known formulas for these angles (see for example [3, Section 3.2]) and the results of this section. 

Lemma 3.12 Let $\gamma, \gamma_1 \in \text{SL}(2, \mathbb{C})$ and $\gamma_2 \in \text{SU}(2, \mathbb{C})$. Then

1. $\tilde{P}_{\gamma_2 \gamma_1 \gamma} = \tilde{P}_{\gamma_1 \gamma} = \gamma_1^{-1}(\tilde{P}_\gamma)$.

2. $P_{\gamma_2 \gamma_1 \gamma} = P_{\gamma_1 \gamma} = \Psi(\gamma_1^{-1})(P_{\gamma_1})$.

**Proof.** Note that $P_{\gamma_1} = \Psi(\gamma_1^{-1})(\infty)$ and $\tilde{P}_\gamma = \gamma^{-1}(\infty)$. Two isometries have identical isometric spheres if and only if they differ by an element of $\text{SU}(2, \mathbb{C})$. These two observations prove the lemma. 

In what follows the transformation $\gamma$ is an element of a Kleinian group. Similar results can be obtained in the Fuchsian case. We establish some results which reveal the relative position of the isometric spheres of $\gamma$ and $\gamma^{-1}$.

Lemma 3.13 Let $\gamma \in \text{PSL}(2, \mathbb{C})$ be a parabolic element and $z_0 \in \mathbb{C}$ the fixed point of $\gamma$. Then $z_0 = \frac{a-d}{2c} \in \mathbb{Q}(a(\gamma), c(\gamma), d(\gamma))$ and $\Sigma_\gamma \cap \Sigma_{\gamma^{-1}} = \{z_0\}$, i.e., $\Sigma_\gamma$ and $\Sigma_{\gamma^{-1}}$ are tangent at $z_0$.

**Proof.** Let $a = a(\gamma), b = b(\gamma), c = c(\gamma)$ and $d = d(\gamma)$ and note that $c \neq 0$. Since $\gamma(z_0) = z_0$ we have that $z_0(cz_0 + d) = a z_0 + b$ and hence $cz_0^2 + (d-a)z_0 - b = 0$. The discriminant of this quadratic equation is equal to $(a-d)^2 + 4bc = (a+d)^2 - 4 = \text{tr}(\gamma)^2 - 4 = 0$ and hence $z_0 = \frac{a-d}{2c}$.

To prove the second part, let $p = \eta_0(z_0)$ and choose an orientation preserving orthogonal map sending $p$ to $j$. Such a map is of the form $\Psi(\gamma_0)$, with $\gamma_0 \in \text{SU}(2, \mathbb{C})$. Let $\gamma_1 = \gamma_0 \gamma_0^{-1}$ and note that $\Psi(\gamma_1)(j) = j$. By Proposition 3.6 and the definition of the Poincaré bisector, we have that $\Sigma_{\Psi(\gamma_1 z_0^{-1})} = \Sigma_{\Psi(\gamma_0^{-1})} = \Psi(\gamma_0)(\Sigma_{\Psi(\gamma)})$. In $\mathbb{H}^3$ we have that $\gamma_1(\infty) = \infty$ and hence $c(\gamma_1) = 0$ and $a(\gamma_1)d(\gamma_1) = 1$. Using Remark 3.5 we obtain that $\Sigma_{\gamma_1}$ and $\Sigma_{\gamma_1^{-1}}$ are parallel planes and hence they intersect only at $\infty$. Since $\Psi(\gamma_0)$ is an orthogonal map it follows that $\Sigma_{\Psi(\gamma)}$ and $\Sigma_{\Psi(\gamma^{-1})}$ are parallel bisectors and thus $\Sigma_\gamma \cap \Sigma_{\gamma^{-1}} = \{z_0\}$. 

Lemma 3.14 Let $\gamma \in \text{PSL}(2, \mathbb{C})$ be a hyperbolic or loxodromic element and $z_0, z_1 \in \partial \mathbb{H}^3$ the two fixed points of $\gamma$. Then the following properties hold.

1. $\{z_0, z_1\} \subset \text{Interior}(\Sigma_{\Phi(\gamma)}) \cup \text{Interior}(\Sigma_{\Phi(\gamma^{-1})})$.

2. If $\gamma$ is hyperbolic then $z_0 \in \text{Interior}(\Sigma_{\Phi(\gamma)})$, $z_1 \in \text{Interior}(\Sigma_{\Phi(\gamma^{-1})})$ and $\Sigma_{\Phi(\gamma)} \cap \Sigma_{\Phi(\gamma^{-1})} = \emptyset$. 

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3. If \( \gamma \) is loxodromic then there exists \( n \in \mathbb{N} \) such that \( \Sigma_{\phi(\gamma^n)} \cap \Sigma_{\phi(\gamma^{-n})} = \emptyset \).

Proof. Choose \( \gamma_0 \in SU(2, \mathbb{C}) \) such that \( \Psi(\gamma_0)(j) = z_0 \) and set \( \gamma_1 = \gamma_0 \gamma_0^{-1} \). Let \( a = a(\gamma_1) \), 
\( b = b(\gamma_1) \), \( c = c(\gamma_1) \) and \( d = d(\gamma_1) \). Then \( c(\gamma_1) = 0 \) and \( |a(\gamma_1)| \neq 1 \neq |d(\gamma_1)| \) (otherwise the trace would not be bigger than 4). Switching to \( \gamma_{-1} \) if necessary, we may suppose that \( |a(\gamma_1)| < 1 \) and hence, by Lemma 3.8, \( j \in \text{Interior}(\Sigma_{\phi(\gamma_1)}) \). Since \( \Sigma_{\phi(\gamma_0 \gamma_0^{-1})} = \Psi(\gamma_0)(\Sigma_{\phi(\gamma_1)}) \), it follows that
\( z_0 \in \text{Interior}(\Sigma_{\phi(\gamma_1)}) \).

Working in \( \mathbb{H}^3 \), we have that \( \infty \) and \( \frac{b}{a} \pm \frac{\sqrt{a^2 + b^2}}{a} \) are the fixed points of \( \gamma_1 \). By Proposition 3.3, \( P_{\gamma_1} = \frac{b}{1 - |a|^2}, \, P_{\gamma_1}^{-1} = \frac{a}{a^2 - 1} P_{\gamma_1}, \, R_{\gamma_1} = \frac{1 - |a|^2}{|a|^2} |a|^2 R_{\gamma_1} \), and \( R_{\gamma_1}^{-1} = |a|^2 R_{\gamma_1} \). From this it follows that \( R_{\gamma_1} + R_{\gamma_1}^{-1} - (P_{\gamma_1} - P_{\gamma_1}^{-1}) = (1 - |a|^2) R_{\gamma_1} - \frac{|a|^2}{a^2} |a|^2 \). The condition we look for is that \( 0 < (1 - |a|^2) R_{\gamma_1} - \frac{|a|^2}{a^2} |a|^2 \). Using the expressions obtained, it follows that \( |P_{\gamma_1}|^2 \left[ \frac{4m^2(a^2) - (1 - |a|^2)^2}{|a|^2} \right] < \frac{1 - |a|^2}{|a|^2} \). If \( \gamma \) is hyperbolic then, since \( c(\gamma) = 0 \) we have that \( a \in \mathbb{R} \), the left hand side is negative and so the bisectors are disjoint. Also, still in the hyperbolic case, \( P_{\gamma_1} = z_1 \). Noting that \( a(\gamma^n) = (a(\gamma))^n \), it follows that, in the loxodromic case, we may choose \( n \) such that the left hand side is negative.

Working in \( \mathbb{H}^3 \) it is easy to see that the axis of \( \gamma \) is perpendicular to both \( \Sigma_{\gamma} \) and \( \Sigma_{\gamma^{-1}} \) if and only if \( z_0 \) and \( z_1 \) are antipodal points in \( \mathbb{B}^3 \). The following corollary is [17] Corollary 1 of Theorem 12.3.4 in case the element \( \gamma \) is hyperbolic.

Corollary 3.15 Let \( \Gamma \) be a discrete group acting on \( \mathbb{B}^3 \), \( \mathcal{F} \) a fundamental domain of \( \Gamma \), \( \gamma \in \Gamma \) and \( z_0 \) a fixed point of \( \gamma \). If \( \gamma \) is hyperbolic or loxodromic then \( z_0 \notin \partial \mathcal{F} \).

Let \( \gamma \) be an elliptic element such that \( c = c(\gamma) = 0 \). The fixed points of \( \gamma \) in \( \mathbb{H}^3 \) are \( \infty \) and \( z_1 = \frac{b}{2Im(a)} \). It follows by Proposition 3.3 that \( \Sigma_{\gamma} \) and \( \Sigma_{\gamma^{-1}} \) are vertical planes that intersect in the vertical line \( l \), say, through \( z_1 \). Making use of Lemma 2.1 and Proposition 3.3 we find that the angle of rotation of \( \gamma \) around \( l \) is \( \frac{\theta(a^2)}{|a|^2} \). This proves the following result.

Lemma 3.16 Let \( \gamma \) be an elliptic element of \( \text{PSL}(2, \mathbb{C}) \). Then \( \Sigma_{\phi(\gamma)} \cap \Sigma_{\phi(\gamma^{-1})} \) is a circle, containing both fixed point of \( \gamma \).

4 Applications

In this section all groups \( \Gamma < \text{PSL}(2, \mathbb{C}) \) are discrete groups and are either Fuchsian or Kleinian. The following theorem is well known, but can be proved easily using the lemmas from section 3.

Theorem 4.1 If \( \Gamma \) is cocompact, then it contains no parabolic elements.

Proof. If \( \gamma_0 \in \Gamma \) is parabolic then, working with a conjugate of \( \Gamma \) if necessary, we may suppose that \( \gamma_0(z) = z + 1 \). Hence, by Shimizu’s Lemma (see [3] Theorem II.3.1), for any \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma, \gamma \notin \Gamma_\infty \), we have that \( |c| \geq 1 \). It follows that \( |a|^2 + |c|^2 \geq 1 \) and thus by Lemma 3.8 \( j \notin \text{Interior}(\Sigma_{\phi(\gamma)}). \) If \( \gamma \in \Gamma_\infty \), then \( |c| = 0 \) and by [3] Theorem II.1.8, \( |a| = 1 \) and hence, again by Lemma 3.8, \( j \in \Sigma_{\phi(\gamma)} \).

It follows that \( j \) is an ideal point of \( \Gamma \), which contradicts the cocompactness.

The next lemma gives more information about the link between discontinuous groups, discrete groups and sequences of distinct elements in those groups.
Lemma 4.2  Let $\Gamma < \text{PSL}(2,\mathbb{C})$ act on $\mathbb{H}^3$. Then the following are equivalent.

1. $\Gamma$ is discontinuous.

2. If $(\gamma_n)$ is a sequence of distinct elements in $\Gamma$, then $(\Psi(\gamma_n^{-1})(0))$ has no limit point in $\mathbb{B}^3$.

3. If $(\gamma_n)$ is a sequence of distinct elements in $\Gamma$, then $\lim_{n \to \infty} \|\gamma_n\| = \infty$.

4. $\Gamma$ is discrete.

Proof. We first prove that (1) and (2) are equivalent. Suppose that $\Gamma$ is discontinuous and let $(\gamma_n) \subseteq \Gamma$ be a sequence of distinct elements. Then, by definition of discontinuity, $(\Psi(\gamma_n^{-1})(0))$ has no limit point in $\mathbb{B}^3$. Suppose now that $(\gamma_n) \subseteq \Gamma$ is a sequence of distinct elements, such that $(\Psi(\gamma_n^{-1})(0))$ has no limit point in $\mathbb{B}^3$. By Proposition 3.6, it follows that $\|\Psi(\gamma_n^{-1})(0)\| \to 1$ and hence $\lim_{n \to \infty} \|\gamma_n\| = \infty$. Choose $P \in \mathbb{B}^3$ and let $n_0 \in \mathbb{N}$ be such that $P \in \text{Exterior}(\Sigma_{\Psi(\gamma_n)})$, $n \geq n_0$. Then $\gamma_n(P) \in \text{Interior}(\Sigma_{\Psi(\gamma_n)})$ and since $R_{\Psi(\gamma_n)} \to 0$, it follows that $(\gamma_n(P))$ has no limit point in $\mathbb{B}^3$. Note that we made use repeatedly of Proposition 3.6.

The fact that item (2) and (3) are equivalent follows from Proposition 3.6. Suppose now that $\Gamma$ is discrete and let $(\gamma_n) \subseteq \Gamma$ be a sequence of distinct elements. Suppose by contradiction, and using a subsequence if necessary, that $\|\gamma_n\| < L$, for all $n \in \mathbb{N}$, for some constant $L$. From this we have that $(a(\gamma_n))$, $(b(\gamma_n))$, $(c(\gamma_n))$ and $(d(\gamma_n))$, are all bounded sequences with at least one of them non-constant, contradicting the discreteness of $\Gamma$. Hence (4) implies (3). Suppose that (3), and hence also (2), holds. If $\Gamma$ were not discrete then there would exist a sequence of distinct elements $(\gamma_n) \subseteq \Gamma$ and $\gamma \in \Gamma$ such that $\lim_{n \to \infty} \gamma_n = \gamma$. But this would contradict the fact that $\lim_{n \to \infty} \|\gamma_n\| = \infty$. This completes the proof.

Lemma 4.2 extends a classical result due to Poincaré (see for example [6, Theorem II.1.2]). In particular, to check discreteness, one has to check discontinuity only at one point.

The proof of the lemma also shows that if $\Gamma < \text{PSL}(2,\mathbb{C})$ is discrete and $K \subset \mathbb{B}^3$ (respectively $\mathbb{B}^2$) is compact and $(\gamma_n)$ is a sequence of distinct elements of $\Gamma$ then there exists $n_0 \in \mathbb{N}$ such that $\|\gamma_n\| \leq \|\gamma\|$ for all $n \geq n_0$ implies that $K \cap \gamma_n(K) = \emptyset$. Since $\Gamma$ is countable we have that $\{\gamma \in \Gamma \mid \gamma(K) \cap K\}$, is a finite subset of $\Gamma$.

We finish this subsection by some discussion about the volume of a fundamental domain. Suppose that $\Gamma_0 = 1$ and put $r = r(\Gamma) = \min\{1 - \rho_{\gamma} \mid \gamma \in \Gamma\}$, where $\rho_{\gamma}$ is defined as in Lemma 3.7. Note that $r(\Gamma) = 1 - \rho_{\gamma_1}$, where $\|\gamma_1\| \leq \|\gamma\|$ for all $\gamma \in \Gamma \setminus \{1\}$. By Lemma 3.7 the Euclidean ball $B_r(0)$ is contained in the fundamental domain of $\Gamma$ with center 0. In particular, if $\Gamma$ has finite covolume (respectively coarea) then $\text{vol}(\Gamma) \geq \pi(\text{sinh}(2\rho(0,r)) - 2\rho(0,r))$ (respectively $\text{Area}(\Gamma) \geq 4\pi \text{sinh}^2(\frac{1}{2}(\rho(0,r)))$, where $\rho$ denotes the hyperbolic distance (for a reference of the hyperbolic volume of a sphere, see [17, Exercises 3.4]). This can be used to give an estimate of $\text{vol}(\Gamma)$ (respectively $\text{Area}(\Gamma)$) for the examples in this paper. With the notation of Lemma 3.7 and using [6, Theorem II.5.4], there exists $\kappa \in [0,1]$ such that $\|\gamma\| \geq f^{-1}(1-\kappa)$, for all $\gamma \in \Gamma$ and any discrete group. If $\Gamma_0 \neq 1$ we only have to consider a portion of the balls.

If $\Gamma$ is cocompact, let $t = t(\Gamma) = 1 - \rho_{\gamma_2}$, with $\gamma_2$ having the biggest norm such that its isometric sphere is part of the boundary of the fundamental domain of $\Gamma$ with center 0. Then $\pi(\text{sinh}(2\rho(0,r)) - 2\rho(0,r)) \leq \text{vol}(\Gamma) \leq \pi(\text{sinh}(2\rho(t,0)) - 2\rho(t,0))$.  

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Consider the quaternion algebra $\mathcal{H}(K)$, with $K = \mathbb{Q}(\sqrt{-d})$, with $d$ a positive square free integer, and put $\Gamma = \text{SL}_1(\mathcal{H}(-1, -1, \mathbb{Z}[\frac{1+\sqrt{-d}}{2}]))$. One can show that $r = r(\Gamma) = \sqrt{\frac{2+d}{d}} - \sqrt{\frac{2}{d}}$ (see [10]). Since $\Gamma_0 \neq 1$, we get that $\text{vol}(\Gamma) \geq \frac{\pi}{4}(\sinh(2\rho(0,r)) - 2\rho(0,r))$. It follows that $\lim_{d \to \infty} \text{vol}(\Gamma) = \infty$.

Based on the results proved in this paper an algorithm called DAFC was developed which has as output a Poincaré fundamental domain. Some symmetries of the fundamental domains of some quaternion algebras were also described in [10, Proposition 3.7]. The interested reader can obtain more information in [10]. In particular one can very briefly obtain generators for the group $\Gamma = \text{SL}_1(\mathcal{H}(-1, -1, \mathbb{Z}[\frac{1+\sqrt{-d}}{2}]))$, which simplifies a lot the calculations done in [5].

4.1 DF Domains and Double Dirichlet Domains

In this subsection we concentrate on discrete groups having a Dirichlet fundamental domain which is also a Ford domain, called a DF domain, or which has two distinct points as center, called a double Dirichlet Domain. One may check that a lot of the examples found in [17] have one of these properties. Here we revisit the interesting paper of [13] and complement its results. In particular we give an algebraic criterion that is easy to be checked. For the symmetries appearing in this section, we recall a result from [10]. Details and proof may be found in [10].

**Proposition 4.3** [10, Proposition 3.7] Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C})$ with $|a|^2 + |c|^2 \neq 1$ (so $\Sigma_\gamma$ is an Euclidean sphere by Proposition 3.3). Denote by $\sigma$ the conjugation by the matrix $\begin{pmatrix} \sqrt{i} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$, by $\delta$ the conjugation by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $\tau(\gamma) = \overline{\gamma}$ denote complex conjugation of the entries of $\gamma$ and define $\phi = \sigma^2 \circ \delta \circ \tau$. Then in $\mathbb{H}^3$.

1. $P_{\phi(\gamma)}$ is the reflection of $P_\gamma$ in $S^2$,
2. $\tau$ induces a reflection in the plane spanned by 1 and $j$, i.e. $P_{\tau(\gamma)} = \overline{P_\gamma}$ and $R_{\tau(\gamma)} = R_\gamma$,
3. $\sigma^2$ induces a reflection in the origin, i.e. $P_{\sigma^2(\gamma)} = -P_\gamma$ and $R_{\sigma^2(\gamma)} = R_\gamma$,
4. $\sigma$ restricted to $\partial \mathbb{H}^3 = \{ z \in \mathbb{C} \}$ induces a rotation of ninety degrees around the point of origin, i.e. $P_{\sigma(\gamma)} = iP_\gamma$ and $R_{\sigma(\gamma)} = R_\gamma$.

The proposition above may also be translated into $\mathbb{H}^2$. In the following lemma we use this restriction. Abusing notations, in the following lemma $\sigma$ denotes the linear operator represented by the matrix $\begin{pmatrix} \sqrt{i} & 0 \\ 0 & \sqrt{-i} \end{pmatrix}$ and $\tau$ denotes the map sending $z \in \mathbb{H}^2$ to $\bar{z}$.

**Lemma 4.4** Let $\gamma \in \text{PSL}(2, \mathbb{R})$. Then the following are equivalent.

1. $a(\gamma) = d(\gamma)$.
2. $\gamma = \sigma^2 \circ \tau \circ \sigma_\gamma$, where $\sigma_\gamma$ is the reflection in $\Sigma_\gamma$.
3. $\Sigma_\gamma$ is the bisector of the geodesic linking $ti$ and $\gamma^{-1}(ti)$, for all $t > 0$.
4. There exists $t_0 \neq 1$ such that $\Sigma_\gamma$ is the bisector of the geodesic segment linking $t_0 i$ and $\gamma^{-1}(t_0 i)$.

**Proof.** From the definitions of the linear operators $\sigma$ and $\tau$, one may easily deduce that $\sigma^2 \circ \tau$ denotes the reflection in the imaginary axes, i.e., $\sigma^2 \circ \tau(z) = -\overline{z}$. We first prove that the two first items are equivalent.

Suppose that $a(\gamma) = d(\gamma)$. Then, by Proposition 3.4 we have that $\Sigma_\gamma = \text{ISO}_\gamma$. Hence the reflection $\sigma_\gamma$ in $\Sigma_\gamma$ is given by $\sigma_\gamma(z) = P_\gamma - (|c|^2 \sigma^2 \circ \tau(z - P_\gamma))^{-1} = \sigma^2 \circ \tau(\gamma(z))$, if $c(\gamma) \neq 0$. If $c(\gamma) = 0$, then we may take $a(\gamma) = d(\gamma) = 1$ and thus $\sigma_\gamma(z) = \sigma^2 \circ \tau(z - b(\gamma)) = \sigma^2 \circ \tau(\gamma(z))$. Hence in either case we have that $\gamma = \sigma^2 \circ \tau \circ \sigma_\gamma$. Now suppose that $\gamma = \sigma^2 \circ \tau \circ \sigma_\gamma$. We first suppose that $P_\gamma$ exists, i.e., $\Sigma_\gamma$ is an Euclidean sphere. In this case we have that $\gamma(z) = \sigma^2 \circ \tau \circ \sigma_\gamma(z) = \frac{-P_\gamma + P_\gamma^2 - R^2}{z - P_\gamma}$, from which it follows that $a(\gamma) = d(\gamma)$. If $\Sigma_\gamma$ is a vertical line, $x = x_0$ say, then $\sigma_\gamma(z) = -\overline{z} + 2x_0$. Hence $\gamma(z) = \sigma^2 \circ \tau \circ \sigma_\gamma(z) = z - 2x_0$ and hence $a(\gamma) = 1 = d(\gamma)$.

Suppose now that $\gamma = \sigma^2 \circ \tau \circ \sigma_\gamma$ and let $u \in \Sigma_\gamma$. Then $\rho(\gamma^{-1}(ti), u) = \rho(\sigma_\gamma \circ \sigma^2 \circ \tau \circ \sigma_\gamma, \sigma_\gamma(u)) = \rho(\tau, \sigma_\gamma(u)) = \rho(\sigma_\gamma(t), u) = \rho(t, \sigma_\gamma(u)) = \rho(t, u)$ and hence $\Sigma_\gamma$ is the bisector of the geodesic linking $t$ and $\gamma^{-1}(ti)$. This proves that the second item implies the third. Obviously the third item implies the fourth.

We now prove that the fourth item implies the first. Let $u \in \Sigma_\gamma$. Then we have that $\rho(t_0 i, u) = \rho(u, \gamma^{-1}(t_0 i))$ and hence $\rho(t_0 i, u) = \rho(t_0 i, \gamma(u))$. Since $\gamma$ is a Möbius transformation we have that $\text{Im}(\gamma(z)) = |\gamma'(z)||\text{Im}(z)|$. Using this and the explicit formula of the hyperbolic distance in the upper half space model (see Section 2.3), we obtain that $|\gamma'(z)||t_0 i - u|^2 = |t_0 i - \gamma(u)|^2$. It follows that $\Re(u^2 |\gamma'(u)| - \Re(\gamma(u))^2 = (|\gamma'(u)| - 1)t_0^2 + (1 - |\gamma'(u)|)\gamma'(u)|\gamma'(u)|\text{Im}(u)^2$. We may write this as an equation of the type $\alpha t^2 = \beta$ having $t = t_0$ as a solution. However as $u \in \Sigma_\gamma$, by definition $\rho(u, i) = \rho(u, \gamma^{-1}(i))$ and hence also $t = 1$ is also solution of the given equation. Thus we have that $\alpha = \beta$ and $\alpha(t_0^2 - 1) = 0$. It follows that $\alpha = 0$ and thus $|\gamma'(u)| = 1$, for all $u \in \Sigma_\gamma$, i.e. $\Sigma_\gamma = \text{ISO}_\gamma$. Applying Proposition 3.4 we obtain that $a(\gamma) = d(\gamma)$. \[\square\]

The following corollary is useful to decide if a set of matrices generates a discrete group. For example it can be used to prove that the group $\Gamma$ of [13] Section VI is discrete. Recall that an angle $\alpha$ is a submultiple of an angle $\beta$ if and only if either there is a positive integer $n$ such that $n\alpha = \beta$ or $\alpha = 0$.

**Corollary 4.5** Let $\{\gamma_1, \ldots, \gamma_n\} \subset \text{PSL}(2, \mathbb{R})$ and suppose that $a(\gamma_k) = d(\gamma_k)$ for all $1 \leq k \leq n$. Then $\langle \gamma_1, \ldots, \gamma_n \rangle$ is Fuchsian if and only if all dihedral angles of intersecting isometric spheres of the $\gamma_k$’s are submultiples of $\pi$. In particular, $\langle \gamma_1, \ldots, \gamma_n \rangle$ is the subgroup of the orientation preserving isometries of a Fuchsian reflection group.

**Proof.** First note that, as $a(\gamma_k) = d(\gamma_k)$, we have by Proposition 3.4 that $\Sigma_\gamma = \text{ISO}_\gamma$ and more precisely $\Sigma_{\gamma_k}$ has the same radius as $\Sigma_{\gamma_k}$ and their centres are the same in absolute value, but have opposite sign. Hence if $\Sigma_{\gamma_k}$ intersects the imaginary axis $\Sigma$, say, then so does $\Sigma_{\gamma_k}$ and $(\Sigma, \Sigma_{\gamma_k}) = \frac{1}{2}(\Sigma_{\gamma_k}, \Sigma_{\gamma_k})$. We first prove the sufficiency. Consider the polyhedron $P$, say, whose sides are $\Sigma$ and the $\Sigma_{\gamma_k}$’s with $P_{\gamma_k} \geq 0$. By Lemma 4.4 the group $\hat{\Gamma} = \langle \sigma^2 \circ \tau, \sigma_{\gamma_k} \mid P_{\gamma_k} \geq 0 \rangle$ is a discrete reflection group with respect to $P$ and hence by [17] Theorem 7.1.2, all the dihedral angles of $P$ are submultiples of $\pi$. By the construction of $P$ this implies that all dihedral angles of intersecting isometric spheres of the $\gamma_k$’s are submultiples of $\pi$.

To prove the necessity, note that, as $(\Sigma, \Sigma_{\gamma_k}) = \frac{1}{2}(\Sigma_{\gamma_k}, \Sigma_{\gamma_k})$, $(\Sigma, \Sigma_{\gamma_k})$ is a submultiple of $\pi$ and in the elliptic case, we even have that $(\Sigma, \Sigma_{\gamma_k}) = \frac{\pi}{2}$. Consider again the polyhedron $P$. By
Theorem 7.1.3], the group \( \hat{\Gamma} \) generated by the reflections of \( \mathbb{H}^2 \) in the sides of \( P \) is a discrete reflection group with respect to \( P \). By Lemma 4.4 we have that \( \hat{\Gamma} = \langle \sigma^2 \circ \tau, \sigma_{\gamma_k} | P_{\gamma_k} \geq 0 \rangle \) is a reflection group containing \( \langle \gamma_1, \ldots, \gamma_n \rangle \). Since \( \langle \gamma_1, \ldots, \gamma_n \rangle = \langle \gamma_k | P_{\gamma_k} \geq 0 \rangle \), we have that \( \langle \gamma_1, \ldots, \gamma_n \rangle \) is of index two in the reflection group \( \hat{\Gamma} \). In particular, \( \langle \gamma_1, \ldots, \gamma_n \rangle \) is a Fuchsian group.

In [13] Theorem 5.3], conditions are given for a Fuchsian group \( \Gamma \) to have a fundamental domain \( F \) which is both a Ford domain and a Dirichlet domain (a DF domain): A finitely generated, finite coarea Fuchsian group \( \Gamma \) admits a DF domain \( F \), if and only if \( \Gamma \) is an index 2 subgroup of a reflection group. It is also proved that in the Kleinian case \( \Gamma \) has a generating set consisting of elements whose traces are real (\[13\] Theorem 6.3].) Our next theorem complements this nice result.

**Theorem 4.6** Let \( \Gamma < \text{PSL}(2, \mathbb{C}) \) be a finitely generated discrete group, acting on \( \mathbb{H}^n, n \in \{2, 3\} \), and \( P_0 \in \{i, j\} \), according to \( \Gamma \) being Fuchsian or Kleinian. Then \( \Gamma \) admits a DF domain with center \( P_0 \) if and only if for every side-pairing transformation \( \gamma \) of \( \Gamma \) we have that \( d(\gamma) = \overline{a(\gamma)} \). In particular, \( \gamma(\tau(\gamma)) \in \mathbb{R} \) for all these elements. Moreover, if \( \Gamma \) is Fuchsian, then \( \hat{\Gamma} = \langle \sigma^2 \circ \tau, \Gamma \rangle \) is a reflection group, \( \hat{\Gamma} = \langle \sigma^2, \Gamma \rangle \) is a Coxeter Kleinian group and both contain \( \Gamma \) as a subgroup of index two.

**Proof.** Let \( F \) be a DF domain, in \( \mathbb{H}^3 \), for \( \Gamma \) with center \( P_0 \in \{i, j\} \). Let \( \Phi_0 \) be a set of sidepairing transformations, i.e., \( \Phi_0 \) consist of those elements of \( \Gamma \) whose isometric circles (respectively isometric spheres) and vertical lines (respectively vertical planes) form the boundary of \( F \).

Then there exists a bijection \( f : \Phi_0 \to \Phi_0 \) such that if \( \gamma \in \Phi_0 \), \( \gamma \notin \Gamma_\infty \), then \( \text{ISO}_\gamma = \Sigma_f(\gamma) \). Since \( F \) is a Ford domain we have that \( F \cap \gamma^{-1}(F) = \text{ISO}_\gamma \) and hence \( \gamma(\tau(\gamma)) \in \Phi_0 \). Since \( F \) is also a Dirichlet domain we have that \( F \cap \gamma^{-1}(F) = \Sigma_\gamma \). Consequently \( \text{ISO}_\gamma = \Sigma_\gamma \) and thus, by Proposition 3.4, \( d(\gamma) = \overline{a(\gamma)} \). It also follows that \( \Sigma_\gamma = \Sigma_f(\gamma) \) and hence, by item 7 in Proposition 3.6, \( f(\gamma) = g\gamma \), with \( g \in \Gamma_{P_0} \).

If \( \Sigma_\gamma \) is an Euclidean line or plane then \( \Sigma_{\gamma^{-1}} \) is also a line or a plane. Indeed suppose that \( \Sigma_{\gamma^{-1}} \) is not an Euclidean line or plane. Then \( \Sigma_{\gamma^{-1}} = \text{ISO}_{\gamma^{-1}} \) and \( \gamma^{-1} \notin \Gamma_\infty \), which implies that \( \gamma \notin \Gamma_\infty \) and hence \( \Sigma_\gamma = \text{ISO}_\gamma \) is an Euclidean sphere, a contradiction. From this it follows that \( |a(\gamma)| = |d(\gamma)| = 1 = \overline{a(\gamma)}d(\gamma) \) and thus \( d(\gamma) = \overline{a(\gamma)} \) and \( c(\gamma) = 0 \). Hence, in the Kleinian case, \( \text{tr}(\gamma) \in \mathbb{R} \), for all \( \gamma \in \Phi \).

To prove the converse, one just has to use Proposition 3.4 to obtain that \( \text{ISO}_\gamma = \Sigma_\gamma \), for all side pairing transformation \( \gamma \) whose bisector is not an Euclidean line or plane. For those \( \gamma \) such that \( \Sigma_\gamma \) is an Euclidean line or plane, one uses Remark 3.5] and hence we have a DF domain.

Suppose now that \( \Gamma \) is Fuchsian. In this case, the set of elements of \( \Phi_0 \) without an isometric circle is \( \Phi_0 \cap \Gamma_\infty \) and consists of only two elements, \( \gamma_0 \) and \( \gamma_0^{-1} \). Letting \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \gamma = f(\gamma_0) \), we have that \( \Sigma_\gamma \) is an Euclidean line and \( |a|^2 + |c|^2 = 1 \). Moreover, \( x = -\frac{ab + cd}{2} \) is a defining equation of \( \Sigma_\gamma \). Since \( \Gamma \) has finite coarea, \( \gamma_0 \) is parabolic and hence \( \gamma_0 = \left( \begin{array}{cc} 1 & -b_0 \\ 0 & 1 \end{array} \right) \). Clearly \( f(\gamma_0^{-1}) = f(\gamma_0)^{-1} \), hence \( |d|^2 + |c|^2 = 1 \), and thus \( |a|^2 = |d|^2 \). Suppose that \( c \neq 0 \); we have that \( \infty \in \Sigma_\gamma \) and \( \gamma(\infty) = \frac{a}{c} \). Since \( \gamma(\Sigma_\gamma) = \Sigma_{\gamma^{-1}} \), it follows that \( \frac{a}{c} = \frac{ab + cd}{2} \). If \( a = 0 \) then \( \gamma_0 \in \Gamma_1 \). If \( a \neq 0 \) then \( d = -a \). We obtain that \( a^2 + bc = -1 \) and hence \( \gamma_0^2 = -1 \). Hence \( o(\gamma) = 2 \) and thus \( \Sigma_\gamma = \Sigma_{\gamma^{-1}} \), a contradiction. So we must have that \( c = 0 \), hence \( a^2 = d^2 = 1 = ad \), \( a = d = 1 \) and \( b = b_0 \). This proves that for any \( \gamma \in \Phi \) we have that \( d(\gamma) = a(\gamma) \).
By Corollary 4.5 we have that $\tilde{\Gamma} = \langle \sigma^2 \circ \tau, \Phi \rangle$ is a reflection group containing $\Gamma$ as a subgroup of index 2. Since $o(\sigma^2 \gamma) = 2$, $tr(\sigma^2 \gamma) = 0$, for all $\gamma \in \Phi$ it follows that $\tilde{\Gamma} := \langle \sigma^2, \Phi \rangle$ is a Coxeter group with $[\tilde{\Gamma} : \Gamma] = 2$.

Note that a presentation of $\tilde{\Gamma}$ and $\hat{\Gamma}$ can be obtained using [6, Theorem II.7.5]. In the Fuchsian case it easily follows that the orbifold of $\Gamma$ is a punctured sphere (see [13]).

**Corollary 4.7** Let $\Gamma$ be a Fuchsian group. Then the following are equivalent.

1. $\Gamma$ is the subgroup of orientation preserving isometries of a Fuchsian reflection group.
2. $\Gamma$ has a DF domain.
3. $\Gamma$ has a Dirichlet fundamental domain $F$ such that $a(\gamma) = d(\gamma)$ for every side-pairing transformation $\gamma$.
4. $\Gamma$ has a Dirichlet fundamental domain $F$ with two distinct points as center.
5. $\Gamma$ has a Dirichlet fundamental domain $F$ with a geodesic ray all whose points are centers of $F$.

**Proof.** Theorem 4.6 shows that the second item implies the third. The equivalence of items 3, 4 and 5 is given by Lemma 4.4. Moreover by Corollary 1.5 the third item implies the first and so we just have to prove that the first item implies the second. Fix a polyhedron $P$ for the reflection group such that one of the sides is the imaginary axis $\Sigma$, say. Then $\sigma^2 \circ \tau$ is the reflection in $\Sigma$. Let $\Sigma_i$ be a side of $P$, denote by $\sigma_{\gamma_i}$ the reflection in $\Sigma_i$ and let $\gamma_i = \sigma^2 \circ \tau \circ \sigma_{\gamma_i}$. Then clearly $\Sigma_i = \Sigma_{\gamma_i}$ and the rest follows by Lemma 4.4 and Theorem 4.6.

Our proof permits that $\Gamma_j$ is non-trivial. In this case $S^n \cap \mathbb{H}$ is the isometric sphere of $\gamma$ and this can be part of the boundary of a fundamental domain of $\Gamma_{f_0}$. It follows that all examples of [6, Section VII.4] are groups whose Ford domain is also a Dirichlet domain. Note that this does not follow from the results of [13].

Another interesting example is the group inducing the unique hyperbolic structure of the tricuspunctured sphere. This is the free group on two generators $\Gamma(2) = \langle z + 2, \frac{z}{z + 2} \rangle$, the congruence group at level 2. Its orbifold is the tricuspunctured sphere (see for example [17, Theorem 9.8.8]).

By the previous theorem it has a DF domain. The same can be deduced for $\text{PSL}(2, \mathbb{Z})$. Hence we see that $\langle \Gamma(2), \sigma^2 \circ \tau \rangle < \langle \text{PSL}(2, \mathbb{Z}), \sigma^2 \circ \tau \rangle$ are both reflection groups and hence $\langle \Gamma(2), \sigma^2 \circ \tau \rangle$ is not a maximal reflection group. In this case this is due to one more symmetry in the fundamental domain of $\Gamma(2)$. In [13, Section VI] a maximal reflection group, constructed in this way, is exhibited thus answering a question of [1]. The fundamental domain does not have the symmetric property exhibited by the fundamental domain of $\Gamma(2)$. Note that only visual Euclidean symmetry is not enough.

### 4.2 The Bianchi Groups

In this section we treat some aspects of the fundamental domains of the Bianchi Groups, which will be deduced from the lemmas and theorems we have proven in this paper. These groups are of the form $\text{PSL}(2, O_d)$, where $O_d$ denotes the ring of integers in $\mathbb{Q}(\sqrt{-d})$ and $d$ is a square-free positive integer. As seen in Section 3, the totally geodesic surfaces used in the construction of the
Ford domains are exactly the isometric spheres and the planes determined by the generators of $\Gamma_{\infty}$. Here, we describe a Dirichlet fundamental domain for all $d$. We also give a complete proof that the Bianchi groups have finite covolume and are of the first kind and describe their ideal points. This makes this section independent of [6] Chapter VII of and [21]. As we already saw, for some values of $d$ a Ford domain is also a Dirichlet domain. We shall see one more example of this situation. Note that the Bianchi groups can also be handled as groups commensurable with the unit group of an order in the split quaternion algebra $\mathcal{H}(K)$, $K = \mathbb{Q}(\sqrt{-d})$ and $d \equiv 1, 2 \mod 4$ or $d \equiv 3 \mod 4$ and $\mathcal{H}(K)$ not a division ring. All this can be handled as in [10, Section 4.1] (the division assumption in that section was only used to guarantee that the groups were cocompact and hence of finite covolume).

Let $\omega = \sqrt{-d}$ if $d \equiv 1, 2 \mod 4$, $\omega = \frac{1+\sqrt{d}}{2}$ if $d \equiv 3 \mod 4$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \text{PSL}(2, \mathbb{Z}[\omega])$. Note that $|a|^2, |b|^2, |c|^2, |d|^2 \in \mathbb{N}$ and hence by Corollary 3.9, $\Gamma$ is not cocompact. Thus the fundamental domain of $\Gamma$ has at least one ideal point. During the whole section we suppose that $j$ is an ideal point in the ball model $\mathbb{B}^3$ (this is possible by conjugation). In the ball model $\Gamma_0 = \Gamma \cap \text{SU}(2, \mathbb{C})$ and so $|a|^2 + |c|^2 = 1$ and thus $a \cdot c = 0$. This implies that $a$ or $c$ is a root of unity, being non-trivial only when $d = 1, 3$. If $d = 1, 3$ then $\Gamma_0$ is isomorphic to $C_2 \times C_2$ or $S_3$, respectively and in all other cases $\Gamma_0 = \langle \Psi(\gamma_0) \rangle$, where $\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $\gamma_0 \in \Gamma_0$, for all $d$, we have that a fundamental domain $F_0$ of $\Gamma$ is contained in $\{(x, y, z) \in \mathbb{B}^3 \mid z \geq 0\}$. Since $j$ is an ideal point of $\Gamma$, we have to find the elements $\gamma$ such that $j \in \Sigma_{\Psi(\gamma)}$. For these $\gamma$ we have, by Lemma 3.8, that $|a|^2 + |c|^2 = 1$. If $c(\gamma) \neq 0$ then $a(\gamma_0 \gamma) \neq 0$ and by, Proposition 3.6, $\Sigma_{\Psi(\gamma_0 \gamma)} = \Sigma_{\Psi(\gamma)}$. Consequently, we may suppose that $c(\gamma) = 0$ and thus $\gamma \in \Gamma_\infty$. In the ball model $\Psi(\gamma)$ is an element of $G_j$, the stabilizer of $j \in S_2$. Denoting by $F_j$ a fundamental domain of $\Gamma_j$ acting on $\mathbb{B}^3$, we have that $F \subseteq F_0 \cap F_j \cap \{(x, y, z) \in \mathbb{B}^3 \mid z \geq 0\}$.

Using $\eta_0 : \mathbb{H}^3 \to \mathbb{B}^3$ we transfer this information to $\mathbb{H}^3$. In this model $\Gamma_\infty = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad = 1, a, b, d \in \mathbb{Z}[\omega] \}$. If $d \notin \{1, 3\}$ then $\Gamma_\infty$ is isomorphic to a full lattice in $\mathbb{C}$. In any case it is a crystallographic group acting on $\mathbb{C}$. Let $F_\infty$ be a fundamental domain of $\Gamma_\infty$ acting on $\mathbb{C}$. Then, for all $d$, $F_\infty = \{ z + rj \in \mathbb{H}^3 \mid z \in F_j \}$ is a fundamental domain of $\Gamma_\infty$ in $\mathbb{H}^3$. Note that $F_\infty$ is the projection on $\partial\mathbb{H}^3$ of $\eta^{-1}(L)$, where $L$ is a link of $\Gamma$ at $j \in \mathbb{B}^3$. The following lemma describes $F_\infty$ for all $d$. It may be easily proved by the previous remarks.

**Lemma 4.8**

1. If $d \equiv 1, 2 \mod 4$ then $F_\infty = \{ z + rj \in \mathbb{H}^3 \mid -\frac{1}{2} \leq Re(z) \leq \frac{1}{2}, -\frac{\sqrt{d}}{2} \leq Im(z) \leq \frac{\sqrt{d}}{2} \}$.
2. If $3 < d \equiv 3 \mod 4$ then $F_\infty = \{ z + rj \in \mathbb{H}^3 \mid -\frac{1}{2} \leq Re(z) \leq \frac{1}{2}, -\frac{1+d}{4} \leq Re(z) + \sqrt{d}Im(z) \leq \frac{1+d}{4}, -\frac{1+d}{4} \leq Re(z) - \sqrt{d}Im(z) \leq \frac{1+d}{4} \}$.
3. If $d = 3$, then $F_\infty = \{ z + rj \in \mathbb{H}^3 \mid 0 \leq Re(z) \leq \frac{1}{2}, 0 \leq Re(z) + \sqrt{3}Im(z) \leq 1 \}$.
4. If $d = 1$ then $F_\infty = \{ z + rj \in \mathbb{H}^3 \mid -\frac{1}{2} \leq Re(z) \leq \frac{1}{2}, 0 \leq Im(z) \leq \frac{1}{2} \}$.

The next theorem is based on the remarks made above, Lemma 4.8 and basics about Dirichlet fundamental domains.

**Theorem 4.9** Let $X = \{ \gamma \in \Gamma \mid \Sigma_\gamma \cap F_\infty \neq 0, \Sigma_\gamma \not\subset \mathbb{B}^3 \}$. Then $F = (\mathbb{B}^3)^c \cap F_\infty \cap \bigcap_{\gamma \in X} \text{Exterior}(\Sigma_\gamma)$ is a Dirichlet fundamental domain for $\Gamma$. Since $\Gamma$ is of finite covolume, $X$ can be chosen to be finite.
For $d \equiv 1, 2 \bmod 4$, $\hat{F}_\infty$ is a rectangle with vertices $\pm \frac{1}{2} \pm \frac{\sqrt{d}}{2}i$ and for $d \equiv 3 \bmod 4$ it is a hexagon with vertices $\pm \frac{(d+1)\sqrt{d}}{4d}i$ and $\pm \frac{(d-1)\sqrt{d}}{4d}i$. For $d \neq 3$, all vertices of this hexagon lie on the circle centered at 0 with radius $\frac{(d+1)\sqrt{d}}{4d}$. Hence, for $d \in \{1, 2, 3, 7, 11\}$, $\partial \hat{F}_\infty \subseteq S_1(0)$ and hence $\infty$ is the only ideal point of $\Gamma$.

Based on the previous section, one may also analyse when the Bianchi groups have a DF domain. The next corollary gives a starting point to this problem. It may be proved by analysing the Dirichlet fundamental domain of these groups and using Theorem [14, Proposition 2.3] and [24].

**Corollary 4.10** Let $d \in \{1, 2, 3, 5, 7, 11, 19\}$. Then the Bianchi group $PSL(2, \mathbb{Z}[w])$ has a DF domain.

Note that the values appearing in the corollary, with the exception of $d = 5$, also appear in others contexts. See for example [2, Proposition 2.3] and [24]. This suggests that having a DF domain is equivalent to other properties. We do not know if these are the only values of $d$ for a DF domain to exist.

We now consider the cusps $C_\Gamma$ and ideal points of a Bianchi group $\Gamma$. As in [6, Definition II.1.10]), a cusp is point $z \in \mathbb{P}^1(\mathbb{C})$ such that $\Gamma z$ is a free abelian group of rank 2. First we reprove that $\Gamma$ is of finite covolume and hence geometrically finite. In what follows $K = \mathbb{Q}(\sqrt{-d})$, $\mathcal{O}_K$ is the ring of integers of $K$ and $h_K$ denotes the class number of $\mathcal{O}_K$. Recall that a discrete group $\Gamma$, acting on $\mathbb{B}^n$, is of the first kind if its limit set, i.e. the points $a \in S^{n-1}$ such that there exists a point $x \in \mathbb{B}^n$ and a sequence $(\gamma_i)_{i=1}^\infty$ in $\Gamma$ such that $(\gamma_i(x))_{i=1}^\infty$ converges to $a$, equals $\partial(\mathbb{B}^n)$.

**Theorem 4.11** Let $\Gamma$ be a Bianchi group and $\mathcal{F}$ a Dirichlet fundamental domain of $\Gamma$. Then the following hold.

1. $\Gamma$ is of the first kind.
2. The ideal points of $\mathcal{F}$ are contained in $\mathbb{P}^1(K)$.
3. The fundamental domain $\mathcal{F}$ has $h_K$ ideal points.
4. If $z = \frac{a}{b} \in K$ with $\alpha, \beta \in \mathcal{O}_K$ is an ideal point, then $\langle \alpha, \beta \rangle$ is a proper ideal of $\mathcal{O}_K$.

**Proof.** The element $\gamma \in \Gamma$, with $a(\gamma) = c(\gamma) = d(\gamma) = 1$ and $b(\gamma) = 0$, is parabolic and fixes 0. Let $\alpha, \beta \in \mathcal{O}_K$ be such that the ideal $\langle \alpha, \beta \rangle = \mathcal{O}_K$. Consider $\gamma_1 \in \Gamma$ such that $b(\gamma_1) = \alpha$ and $d(\gamma_1) = \beta$. Then $\frac{a}{b}$ is the fixed point of $\gamma_1 \gamma_1^{-1}$ and hence, by [6, Proposition VII.2.7], the set of fixed points of parabolic elements of $\Gamma$ is dense in $\mathbb{P}^1(\mathbb{C})$. Applying [17, Theorem 12.2.4] we get that $L(\Gamma) = \mathbb{P}^1(\mathbb{C})$, where $L(\Gamma)$ denotes the limit set of $\Gamma$, i.e. the set of all limit points of $\Gamma$. So $\Gamma$ is of the first kind.

To prove the second item, we follow the method of [14] and [24]. Let $\{ (c_i, d_i) | 1 \leq i \leq h_K \}$ be a set of representatives for the the ideal classes of $K$ and set $c_1 = 0$ and $d_1 = 1$. Let $a_i$ and $c_i$ be chosen in $K$ such that

$$\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad \text{det}(\gamma_i) = 1,$$

with $\gamma_1 = I$. Note that $a_i, b_i \in \mathbb{Z}[w]$ only for $a_1, b_1$. Let

$$L = \bigcup_{i=1}^{h_K} \gamma_i \Gamma.$$
Consider the groups $\Gamma^i = \gamma_i \Gamma \gamma_i^{-1}$ for $1 \leq i \leq h_K$. By [24, Lemma 4, page 41], there exists a constant $\Delta > 0$ only depending on $K$ such that for every $z + rj \in \mathbb{H}^3$, there exists $\gamma_0 \in L$ such that $\gamma_0(z + rj) = z' + r'j'$ with $r' \geq \Delta$. Choose $\epsilon > 0$ such that $\Delta - \epsilon > 0$. Then by [24, Lemma 1, page 37], the group $\Gamma^\infty$ maps

$$\{z + rj \in \mathbb{H}^3 | r > \Delta - \epsilon\}$$

onto itself. Let $F^i$ be the Dirichlet fundamental domain with center $\gamma_i(j)$ for $\Gamma^i$ (note that as $\Gamma^\infty$ is an elementary group of parabolic type, no element of $\Gamma^\infty$ fixes $\gamma_i(j)$ and hence the Dirichlet fundamental domain with center $\gamma_i(j)$ is well defined.) Denote the intersection of $F^3 \cap F^0$ with the set defined in [2] by $F^i$. Consider the union

$$\tilde{F} = \bigcup_{i=1}^{h_K} \gamma_i^{-1}(F^i).$$

Note that every region of the form $\gamma_i^{-1}(F^i)$ is bounded by bisectors relative to $j$ and hence $\tilde{F}$ is bounded by bisectors. We claim that $\tilde{F}$ contains a point or a boundary point of every orbit of $\Gamma$. Indeed, let $z + rj \in \mathbb{H}^3$. Then, as stated above, there exists $\gamma_0 = \gamma_k \gamma$ for some $\gamma \in \Gamma$ and some $1 \leq k \leq h_K$ such that $\gamma_0(z + rj) = z + r_1j$ with $r_1 > \Delta - \epsilon$. Let $z_2 + r_2j$ be such that $z_2 + r_2j = T_k(z_1 + r_1j)$ with $T_k \in \Gamma_k^h$ and $z_2 + r_2j \in F^k$. Then $z_3 + r_3j = \gamma_k^{-1}T_k \gamma \gamma(z + rj)$ is in $\gamma_i^{-1}(F^i)$. Moreover $\gamma_k^{-1}T_k \gamma \gamma \in \Gamma$ and hence the claim is proven. Thus the fundamental domain $F$ of $\Gamma$ is included in $\tilde{F}$. We consider now the ideal points of $\tilde{F}$. Every region $F_i$ has only $\infty$ as ideal point. Hence for $1 \leq i \leq h_K$, $\gamma_i^{-1}(F^i)$ has $\gamma_i^{-1}(\infty) = -\frac{d_k}{c_i}$ as only ideal point. Thus $\tilde{F}$ has exactly $h_K$ ideal points and they are of the form $-\frac{d_k}{c_i}$ for $1 \leq i \leq h_K$. As $F \subseteq \tilde{F}$, the ideal points of $F$ are included in the ideal points of $\tilde{F}$ and that proves the second item.

To prove the third item, notice that by the previous $F$ has at most $h_K$ ideal points. Thus we just have to show that those ideal points are not equivalent under the action of $\Gamma$. Let

$$F^{i, \lambda} = \{z + rj \in F^i | r \geq \lambda > 0\},$$

and let

$$\tilde{F}^{\lambda} = \bigcup_{i=1}^{h_K} \gamma_i^{-1}(F^{i, \lambda}).$$

Suppose $\lambda > \Delta$ and let $P_1 \in \tilde{F}^{\lambda}$ and $P_2 \in \tilde{F}$. Suppose there exists $\gamma_0 \in \Gamma$ such that $\gamma_0(P_1) = P_2$. Applying the same proof as in [14, Hilfsatz 4], $P_1$ and $P_2$ are at the border of $\gamma_i^{-1}(F^{k, \lambda})$ for some $1 \leq k \leq h_K$. Thus either $P_1$ is equal to $P_2$ or they are not ideal points. This proves the third item.

Suppose that $\langle \alpha, \beta \rangle = \mathcal{O}_K$ and set $\gamma = \begin{pmatrix} x & -y \\ -\beta & \alpha \end{pmatrix} \in \Gamma$. Then, by Corollary 3.9 and Lemma 3.10, $z = P_\gamma \in \text{Interior}(\Sigma_\gamma)$ and this proves the last item.

Let $p = \frac{a}{b} \in K$ correspond to a non-trivial element of the ideal class group of $\mathcal{O}_K$ and, based on the construction in [6, Definition VII.3.5], define $\gamma_+ = \begin{pmatrix} 1 + \alpha \beta & -\alpha^2 \\ \beta^2 & 1 - \alpha \beta \end{pmatrix}$, $\gamma_- = \begin{pmatrix} 1 + \omega \alpha \beta & -\omega \alpha^2 \\ \omega \beta^2 & 1 - \omega \alpha \beta \end{pmatrix}$ and $\gamma = \begin{pmatrix} \alpha \beta & 0 \\ \beta^2 & 1 \end{pmatrix}$, where $\omega$ is defined as $\frac{d_K + \sqrt{d_K}}{2}$ with $d_K$ being the discriminant of $K$. Then $\langle \gamma_+, \gamma_- \rangle < \Gamma_p$, $\gamma_+^{-1} \gamma_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma_-^{-1} \gamma_- = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$ and hence $\langle \gamma_+, \gamma_- \rangle$ is isomorphic to $\mathbb{Z}^2$. His gives rise to the following corollary.
Corollary 4.12 The ideal points of $F$ are in a one-to-one correspondence with the cusps contained in $F$.

Proof. From the discussion above, it follows that every ideal point of $F$ is a cusp. Moreover by [5, Theorem VII.2.4] there are exactly $h_K$ non-equivalent cusps under the action of $\Gamma$. Thus the result follows by the above proposition. 

The intersection of the union of the interiors of the bisectors $\Sigma_{\gamma^2 \pm 1}$ and $\Sigma_{\gamma^2 \pm 1}$ with $\partial \mathbb{H}^3$, contains a punctured disk of $\partial \mathbb{H}^3$ centered in $p$. This is easily seen using Lemma 3.13. From this it follows easily that a cusp region has finite covolume. We now prove that all ideal points of $\Gamma$, except for $\infty$, arise in this way.

Theorem 4.13 Let $\Gamma$ be a Bianchi group. Then the following hold.

1. $\Gamma$ has finite covolume and is geometrically finite.
2. If $z \in \mathbb{P}^1(\mathbb{C})$ is an ideal point of $\Gamma$, it corresponds to a non-trivial element of the ideal class group of $O_K$ and $1 \leq |z|$.

Proof. By the observation before the theorem, each cusp region has finite covolume from which it easily follows that $vol(\Gamma)$ is finite. For every ideal point $z_i$, for $1 \leq i \leq h_K$, let $V_i$ be the interior of the projection on $\partial \mathbb{H}^3$ of a horoball centered at $z_i$. Let $V = \hat{\mathcal{F}}_\infty - \bigcup_i V_i$. Then $V$ is compact and Theorem 4.11 shows that it can be covered by a finite number of bisectors. Hence $\Gamma$ is geometrically finite. (This also follows by a result of Garlan and Raghunathan, [6, Theorem II.2.7].)

By Theorem 4.11, an ideal point is of the form $z = \frac{a}{\overline{a}}$ with $n \in \mathbb{N}$, $\alpha \in \mathbb{Z}[w]$ and with $I = \langle n, \alpha \rangle$ a proper non-zero ideal of $\mathbb{Z}[w]$. Suppose it were principal, i.e., $\langle n, \alpha \rangle = I = \langle \beta \rangle$. We may write $\beta = nx + ya$, $n = u\beta$ and $\alpha = v\beta$, with $x, y, u, v \in \mathbb{Z}[w]$. From this it follows that $ux + yv = 1$ and hence $z = \frac{x}{y}$ with $\langle u, v \rangle = \mathbb{Z}[w]$, contradicting Theorem 4.11. The fact that $1 \leq |z|$ follows trivially from the fact that $F \subseteq \{ (x, y, r) \in \mathbb{B}^3 | r > 0 \}$.

The proofs of Theorem 4.11 and Theorem 4.13 suggests that a Bianchi group is virtually generated by parabolic elements. In case $d = 3$ a subgroup of finite index generated by parabolic elements is the figure-eight knot group.

We give more information about an ideal point $z \in \mathbb{C}$ of the previous corollary which will allow the determination of all the ideal points. Write $t = \gcd(n, |\alpha|^2 - 1)$, $n = ts$ and $|\alpha|^2 - 1 = tm$. If we set $\gamma_1 = \left( \begin{array}{cc} \alpha & -m \\ -t & \alpha \end{array} \right)$, then $sz = P_{\gamma_1}$ and $\Sigma_{\gamma_1} = \text{ISO}_{\gamma_1}$. Hence, since $z$ corresponds to an ideal point of the fundamental domain $F$ of $\Gamma$ (given in Theorem 4.9), we have that $s \neq 1$, i.e., $n$ is not a divisor of $|\alpha|^2 - 1$. By [6, Theorem IV.4.24], $I = \langle n, \alpha \rangle$ can be represented as $a\mathbb{Z} + (b + cw)\mathbb{Z}$, where $a, b, c \in \mathbb{Z}$, where $a > 0$, $c > 0$, $0 \leq b < a$, $c | b, c | a$ and $ac \mid |b + cw|^2$. Hence we may write $b = cr$ and $a = cq$, with $r, q \in \mathbb{Z}$. From this we conclude that $z = \frac{a}{m} = \frac{b + cw}{a} = \frac{r + w}{q}$, with $r, q \in \mathbb{Z}$, $q > 0$, $0 \leq r < q$ and $q \mid |r + w|^2$. In particular, $z$ corresponds to the element $[q, r + w] \in \mathbb{P}^1(\mathbb{C})$. Since $z \in F_\infty$, it follows that $|\text{Re}(r + w)| \leq \frac{q}{2}$, $q \leq |r + w|$ and $\frac{2q^2}{4} \leq |\text{Im}(r + w)|^2 \leq \frac{2r^2}{4}$. By Theorem 4.13 $\Gamma$ has $h_K$ non $\Gamma$-equivalent ideal points. Hence we have described all the ideal points of $F$ and thus we have proved the following theorem which is due to Swan (see [21, Corollary 7.6]) in the context of a Ford domain. In case $d \in \{1, 2, 3, 7, 11, 19\}$, then the only ideal point of $\Gamma$ is $\infty$. 19
Theorem 4.14 The ideal points of the fundamental domain $F$, given in Theorem 4.9, of the Bianchi group $\Gamma = \text{PSL}(2, \mathbb{Z}[\omega])$ are $\infty$ and the points $g(z), z = \frac{r+w}{q} \in F_\infty$, with $g \in \langle \sigma^2, \tau \rangle$, $r, q \in \mathbb{Z}$, $q > 0$, $0 \leq r < q$, $0 \leq 2|\text{Re}(r+w)| \leq q$, $q \leq |r+w|$ and $|\text{Im}(r+w)|^2 \leq \frac{dq^2}{4}$. More precisely we have:

1. For $d \equiv 1, 2 \mod 4$, we have that $4 \leq q^2 \leq \frac{4d}{3}, 0 \leq r \leq \frac{q}{2}$ and $q \mid |r+w|^2$.

2. For $d \equiv 3 \mod 4$, we have that $1 < q \leq \frac{2+\sqrt{4+3d}}{3}, 0 \leq r \leq \frac{q-1}{2}$ and $q \mid |r+w|^2$.

We now consider a subgroup of finite index of the Bianchi group with $d = 3$. It is well known that the figure-eight knot group $\Gamma_8 = \langle \gamma_1, \gamma_2 \rangle$ is generated by $\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$, where $w = \frac{-1+\sqrt{-3}}{2}$ (see [6, 22]). A matrix presentation for $\Gamma_8$ was first described in [18] where also a fundamental domain was given (see also [7, 16]). Looking at the generating set, and having in mind the criterium of Theorem 4.6, one might think that $\Gamma_8$ has a DF domain. However we show that this is not the case as it is also not the case for the group defining the hyperbolic structure on the complement of the Whitehead Link. To see this we will use the results of Section 3 and [17, Figures 10.3.2 and 10.3.12] which show the gluing patterns for the figure-eight knot complement and the Whitehead Link complement. We stick to the notation used in the figures of [17]. In the first case face $B$ is a vertical plane which is mapped to $B'$ an Euclidean sphere and face $D$ is an Euclidean sphere which is mapped to a vertical plane. In the case of the Whitehead Link face $A$ is a vertical plane which is mapped to $A'$ an Euclidean sphere. In fact, in this case all vertical planes are mapped to Euclidean spheres and vice versa. This violates a basic property of a DF domain:

*If $F$ is a fundamental domain for $\Gamma$ and $\gamma$ is a side-pairing transformation such that $\infty \in \Sigma_\gamma$ then $\gamma \in \Gamma_\infty$. This follows from the proof of Theorem 4.6.*

We now describe a Dirichlet fundamental domain $F$ for $\Gamma_8$. We shall make use of the matrix presentation given in [18]. We could use the Cayley graph and apply DAFC but, in this case, it is very easy to get a set of generators needed to construct a Dirichlet fundamental domain.

Indeed, if we set $\gamma_3 = \begin{pmatrix} 0 & w \\ -w^2 & 1-w \end{pmatrix}$, $\gamma_4 = \begin{pmatrix} 0 & -w \\ w^2 & 1-w \end{pmatrix}$ and $\gamma_5 = \begin{pmatrix} 1 & -1 \\ -w & 1+w \end{pmatrix}$ then, using the results of Section 3, one can obtain, by hand, the Poincaré bisectors of these elements and of their inverses. Note that $\Sigma_{\gamma_3}$ is a vertical plane but $\Sigma_{\gamma_3^{-1}}$ is an Euclidean sphere. This is also the case for $\gamma_4$. As explained, this can be deduced from the gluing patterns and the results of Section 3. The pictures below show the projection of the link and the fundamental domain.
Finally, we use the results of Section 3 to obtain a matrix representation of the group $\Gamma$ defining the hyperbolic structure on the Whitehead Link complement. We will use the regular octahedron $T$ of [17, Figure 10.3.11], the gluing pattern [17, Figure 10.3.12] and the side-pairing transformations explained on [17, Page 455]. The notation of [17, Page 455] is maintained. We will now work in both $B_3$ and $H_3$ and use freely the results of Section 3. Set $\Psi(\gamma_1) = g_A$, $\Psi(\gamma_2) = g_B$, $\Psi(\gamma_3) = g_C$ and $\Psi(\gamma_4) = g_D$. Clearly $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle$. We have that $\Sigma_{\Psi(\gamma_1)} = A$, $\Sigma_{\Psi(\gamma_2)} = B$, $\Sigma_{\Psi(\gamma_3)} = C$ and $\Sigma_{\Psi(\gamma_4)} = D$. Note that $\eta_0^{-1}$ fixes pointwise the unit circle $B_3 \cap \partial H_3$ (in fact it is its isometric sphere) and $F = \eta_0^{-1}(T)$ is a Dirichlet fundamental domain in $H_3$ of $\Gamma$. We can now easily describe explicitly all the $\gamma_k$'s. Since they are Möbius transformations we only have to give them at three points. We use ordered triples to write them. Using the gluing pattern we easily deduce that $\gamma_1 : (i, 1, 0) \mapsto (-i, \infty, 1)$, $\gamma_2 : (i, -1, 0) \mapsto (i, \infty, 1)$, $\gamma_3 : (-1, -i, 0) \mapsto (\infty, i, -1)$ and $\gamma_4 : (-i, 1, 0) \mapsto (-i, \infty, -1)$. Using cross-ratio, we find that $\gamma_1 = \lambda \left( \begin{array}{cc} -2 - i & i \\ -i & i \end{array} \right)$, $\gamma_2 = \lambda \left( \begin{array}{cc} 2 - i & -i \\ -i & -i \end{array} \right)$, $\gamma_3 = \lambda \left( \begin{array}{cc} -2 - i & -i \\ -i & i \end{array} \right)$ and $\gamma_4 = \lambda \left( \begin{array}{cc} 2 - i & i \\ -i & -i \end{array} \right)$, where $\lambda = \frac{\sqrt{2i}}{2}$. Note that $\gamma_3 = \sigma^2(\gamma_1)$ is elliptic, $\gamma_4 = \sigma^2(\gamma_2)$ is parabolic and $\phi(F) = F$. One can check, using the formulas of Section 3 that the $\gamma_k$'s are indeed the side-pairing transformations of $F$. Clearly $\langle \gamma_1, \gamma_2, \sigma^2 \rangle$ is a Kleinian group containing $\Gamma$ as a subgroup of index two, i.e., the Whitehead link complement is the double cover of an orbifold. This is also known for the figure-eight knot complement group as $\langle \Gamma_8, \delta \rangle = \text{PSL}(2, \mathbb{Z}[w])$, with $d = 3$. Remember that $\sigma, \delta$ and $\phi$ are the symmetries induced by algebra isomorphisms (see [10]) given in Proposition 4.3.

Finally note that it is easy to deduce from the gluing pattern given in [17, Figure 10.3.20], that...
the group giving the hyperbolic structure of the Borromean rings complement has a fundamental domain in which the basic property of a DF domain is satisfied. Using the presentation given in [19] and the results of Section 3, one can construct a Dirichlet fundamental domain of the Borromean rings complement group.

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