Asymptotic Stability for Some Types of Nonlinear Fractional Order Differential-Algebraic Control Systems

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Abstract
The aim of this paper is to study the asymptotically stable solution of nonlinear single and multi fractional differential-algebraic control systems, involving feedback control inputs, by an effective approach that depends on necessary and sufficient conditions.

Keywords: Asymptotically Stable, Feedback Control, Differential-Algebraic Equation, Fractional Order, Gronwall inequality, Mult-Fractional Order.

1. Introduction
The nonlinear fractional order differential-algebraic control systems appear in a variety of theories and applications. The theory of fractional descriptor ordinary and fractional partial differential equations, with different types of derivatives, have recently been addressed by several researchers for different problems. It is well known that descriptor systems or differential-algebraic systems are the major research fields of the control theory. During the past two decades, differential-algebraic systems attracted much attention due to the comprehensive applications in economics singular systems, not only those containing differential or difference equations as normal systems but also algebraic equations. Thus, their description is considered as being more general. Their class of systems has been widely studied, not only because of theoretical interest but also because of its extensive applications in areas such as robotics and power systems. The necessary and sufficient conditions for the solvability, positivity, and asymptotic stability and stabilization of the fractional descriptor linear systems were established [1-7, 8, 9]. Earlier works [10, 11] studied the partial eigenvalue assignment for stabilization of descriptor fractional discrete-time linear systems or by derivative state feedback. In other investigations [12, 10], the stabilization problem of singular fractional-order systems with...
fractional commensurate fractional order, via static output feedback, was studied. The stability problem of descriptor second-order systems was also considered [13]. Lyapunov equations for stability of second-order systems were established by using Lyapunov method. The robust admissibility problem in singular fractional-order continuous time systems was also studied with a static output feedback controller that is designed for the uncertain closed-loop system to be admissible [14]. Other articles studied the robust stability and stabilization of uncertain fractional-order differential-algebraic nonlinear systems [15, 13].

Our interest in this paper is to study the asymptotic stability of nonlinear fractional order differential-algebraic control systems, involving feedback input controls. Also, we aim to study single-fractional (15-16) and multi-fractional (21-22) order differential–algebraic control equations.

The following definitions and results are needed later on.

**Definition (1.1), [16]**

Let $f$ be a function such that $f: [0, \infty) \rightarrow R$. The $\alpha$ fractional order Caputo derivative is defined as

$$
(\frac{D^{\alpha}}{\Gamma(n-\alpha)} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) \, ds , \quad n - 1 < \alpha < n,
$$

where $\Gamma$ denotes the gamma function.

**Lemma (1.1), [17]**

The degree polynomial $a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0 = 0$ is asymptotically stable if it holds the following condition: $|\arg \lambda_i| > \frac{\pi}{2}$ for all zeros $\lambda_i, i = 1, \ldots, n$.

**Definition (1.1), [18]**

The following fractional order system

$$(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x(t)) = Ax(t)$$

is stable if for any $x_0$ there exists $\varepsilon > 0$ such that $\|x\| \leq \varepsilon$ for $t \geq 0$.

**Lemma (1.2), [18]**

Consider the linear fractional control system

$$(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x(t)) = Ax(t) + Bu(t)$$

where $x=(x_1, x_2, \ldots, x_n)^T$ and $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T, A \in \mathbb{R}^{n \times n}$ is

a) stable if for any $x_0$ there exists $\varepsilon > 0$ such that $\|x\| \leq \varepsilon$ for $t \geq 0$.

b) asymptotically stable if $\lim_{t \to \infty} \|x\| = 0$.

**2. Nonlinear Fractional Order Differential-Algebraic Control Systems**

The following two types of nonlinear fractional order differential-algebraic control systems are presented.

**2.1. Single-Fractional Order Differential-Algebraic Control Equations**

Consider the nonlinear single fractional order differential-algebraic control system:

$$(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x_1(t)) = \sum_{i=1}^{3} a_{1i} x_i(t) + \sum_{j=1}^{2} b_{1j} u_j(t) + f_1(x_1, x_2, x_3) g_1(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x_1, \frac{D^{\alpha}}{\Gamma(n-\alpha)} x_2, \frac{D^{\alpha}}{\Gamma(n-\alpha)} x_3)$$

$$(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x_2(t)) = \sum_{i=1}^{3} a_{2i} x_i(t) + \sum_{j=1}^{2} b_{2j} u_j(t) + f_2(x_1, x_2, x_3) g_2(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x_1, \frac{D^{\alpha}}{\Gamma(n-\alpha)} x_2, \frac{D^{\alpha}}{\Gamma(n-\alpha)} x_3)$$

$$(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x_3(t)) = \sum_{i=1}^{3} a_{3i} x_i(t) + \sum_{j=1}^{2} b_{3j} u_j(t) + f_3(x_1, x_2, x_3) g_3(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x_1, \frac{D^{\alpha}}{\Gamma(n-\alpha)} x_2, \frac{D^{\alpha}}{\Gamma(n-\alpha)} x_3)$$

where $a_{1i}, a_{2i}, a_{3i}, b_{1j}, b_{2j},$ and $b_{3j}$ are constants, $x_i \in \mathbb{R}$ are state vectors, $i=1,2,3, u_j(t) \in \mathbb{R}, j=1,2$ are control input functions, and $f_i, g_i, i=1,2,3$ are varying nonlinear time values.

The linear system of (1-3) is:

$$(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x_1(t)) = \sum_{i=1}^{3} a_{1i} x_i(t) + \sum_{j=1}^{2} b_{1j} u_j(t)$$

$$(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x_2(t)) = \sum_{i=1}^{3} a_{2i} x_i(t) + \sum_{j=1}^{2} b_{2j} u_j(t)$$

$$(\frac{D^{\alpha}}{\Gamma(n-\alpha)} x_3(t)) = \sum_{i=1}^{3} a_{3i} x_i(t) + \sum_{j=1}^{2} b_{3j} u_j(t)$$

From (6), it is given that:

$$x_3(t) = \sum_{i=1}^{3} a_{3i} x_i(t) = \sum_{i=1}^{3} \frac{a_{3i} x_i(t)}{1-a_{33}}$$

Hence, the solution is:

$$x_3(t) = \sum_{i=1}^{3} a_{3i} x_i(t) = \sum_{i=1}^{3} \frac{a_{3i} x_i(t)}{1-a_{33}}$$

$$\sum_{j=1}^{2} b_{3j} u_j(t) + \sum_{i=1}^{3} \frac{a_{3i} x_i(t)}{1-a_{33}}.$$
\((\hat{\delta} D^\alpha_t)x_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + a_{23}\left[\frac{a_{21}x_1(t)}{1-a_{33}} + \frac{a_{22}x_2(t)}{1-a_{33}} + \sum_{i=1}^{2} b_{3i}u_i(t)\right] + \sum_{j=1}^{2} b_{2j}u_j(t)\)

then,

\((\hat{\delta} D^\alpha_t)x_1(t) = \left[ a_{11} + \frac{a_{13}a_{31}}{1-a_{33}} \right] x_1(t) + \left[ a_{12} + \frac{a_{13}a_{32}}{1-a_{33}} \right] x_2(t) + \left[ a_{13} + \sum_{i=1}^{2} b_{3i}u_i(t) \right] + \sum_{i=1}^{2} b_{1i}u_i(t) \quad (7)\)

\((\hat{\delta} D^\alpha_t)x_2(t) = \left[ a_{21} + \frac{a_{23}a_{31}}{1-a_{33}} \right] x_1(t) + \left[ a_{22} + \frac{a_{23}a_{32}}{1-a_{33}} \right] x_2(t) + \left[ a_{23} + \sum_{i=1}^{2} b_{3i}u_i(t) \right] + \sum_{i=1}^{2} b_{2i}u_i(t) \quad (8)\)

For more simplicity, let \(\tilde{a}_{11} = a_{11} + \frac{a_{13}a_{31}}{1-a_{33}}\), \(\tilde{a}_{12} = a_{12} + \frac{a_{13}a_{32}}{1-a_{33}}\), \(\tilde{a}_{21} = a_{21} + \frac{a_{23}a_{31}}{1-a_{33}}\), \(\tilde{a}_{22} = a_{22} + \frac{a_{23}a_{32}}{1-a_{33}}\), \(\bar{b}_{11} = a_{13}b_{31}\), \(\bar{b}_{12} = a_{13}b_{32}\), \(\bar{b}_{21} = a_{23}b_{31}\), \(\bar{b}_{22} = a_{23}b_{32}\), and \(k_\rho\) are constants.

Equations (7) and (8) become,

\((\hat{\delta} D^\alpha_t)x_1(t) = \tilde{a}_{11}x_1(t) + \tilde{a}_{12}x_2(t) + \sum_{j=1}^{2} \tilde{b}_{1j}u_j(t) \quad (9)\)

\((\hat{\delta} D^\alpha_t)x_2(t) = \tilde{a}_{21}x_1(t) + \tilde{a}_{22}x_2(t) + \sum_{j=1}^{2} \tilde{b}_{2j}u_j(t) \quad (10)\)

\(x_3(t) = \tilde{a}_{31}x_1(t) + \tilde{a}_{32}x_2(t) + \sum_{j=1}^{2} \tilde{b}_{3j}u_j(t) \quad (11)\)

Now, we consider the following related linear feedback control system

\((\hat{\delta} D^\alpha_t)x_1(t) + (\hat{\delta} D^\alpha_t)x_2(t) = \bar{a}_{11}x_1(t) + \bar{a}_{12}x_2(t) + \bar{b}_{11}x_1(t) + \bar{b}_{12}x_2(t) \quad (12)\)

\((\hat{\delta} D^\alpha_t)x_2(t) + (\hat{\delta} D^\alpha_t)x_2(t) = \bar{a}_{21}x_1(t) + \bar{a}_{22}x_2(t) + \bar{b}_{21}x_1(t) + \bar{b}_{22}x_2(t) \quad (13)\)

\(u_i(t) = k_\rho x_1(t)u_i(t) = k_\rho x_2(t) \quad (14)\)

For the nonlinear multi-fractional order differential -algebraic control system in equations (10-11) with equations (12-13), we obtain:

\[
\begin{bmatrix}
\hat{\delta} D^\alpha_t x_1(t) \\
\hat{\delta} D^\alpha_t x_2(t) \\
\hat{\delta} D^\alpha_t x_1(t) \\
\hat{\delta} D^\alpha_t x_2(t)
\end{bmatrix} =
\begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11}K_\rho & \tilde{b}_{12}K_\rho \\
\tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{21}K_\rho & \tilde{b}_{22}K_\rho \\
\tilde{a}_{31}(\bar{b}_{11p} + \bar{b}_{12p}) & \tilde{a}_{32}(\bar{b}_{11p} + \bar{b}_{12p}) & \sum_{j=1}^{2} \bar{b}_{1j}p_{31}K_\rho + \bar{a}_{11p} & \sum_{j=1}^{2} \bar{b}_{1j}p_{32}K_\rho + \bar{a}_{12p} \\
\tilde{a}_{31}(\bar{b}_{21p} + \bar{b}_{22p}) & \tilde{a}_{32}(\bar{b}_{21p} + \bar{b}_{22p}) & \sum_{j=1}^{2} \bar{b}_{2j}p_{31}K_\rho + \bar{a}_{21p} & \sum_{j=1}^{2} \bar{b}_{2j}p_{32}K_\rho + \bar{a}_{22p}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_2(t)
\end{bmatrix}
\]

\[x_3(t) = \begin{bmatrix}
\tilde{a}_{31} & \tilde{a}_{32} & \tilde{b}_{31}K_\rho & \tilde{b}_{32}K_\rho
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
\]

Thus,
\[(\hat{\delta} D^\alpha_t)x(t) = Ax + F(x)G(\hat{\delta} D^\alpha_t)x \quad (17)\]

\[x(t) = Bx + f_3(x)g_3(\hat{\delta} D^\alpha_t)x \quad (18)\]

where
\[A = \begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11}K_\rho & \tilde{b}_{12}K_\rho \\
\tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{21}K_\rho & \tilde{b}_{22}K_\rho \\
\tilde{a}_{31}(\bar{b}_{11p} + \bar{b}_{12p}) & \tilde{a}_{32}(\bar{b}_{11p} + \bar{b}_{12p}) & \sum_{j=1}^{2} \bar{b}_{1j}p_{31}K_\rho + \bar{a}_{11p} & \sum_{j=1}^{2} \bar{b}_{1j}p_{32}K_\rho + \bar{a}_{12p} \\
\tilde{a}_{31}(\bar{b}_{21p} + \bar{b}_{22p}) & \tilde{a}_{32}(\bar{b}_{21p} + \bar{b}_{22p}) & \sum_{j=1}^{2} \bar{b}_{2j}p_{31}K_\rho + \bar{a}_{21p} & \sum_{j=1}^{2} \bar{b}_{2j}p_{32}K_\rho + \bar{a}_{22p}
\end{bmatrix}\]
Lemma (2.1.1), [19]
The Mittage-Leffler function $E_{\alpha\beta}(At^\alpha)$ satisfies the following
i. $E_{\alpha,1}(At^\alpha) \leq K_{E_{\alpha,1}}\|e^{At}\|, \alpha > 1$.
ii. $E_{\alpha,\alpha}(At^\alpha) \leq K_{E_{\alpha,\alpha}}\|e^{At}\|, \alpha > 1$.
where $A \in \mathbb{R}^{n \times n}$, $K_{E_{\alpha,1}}, K_{E_{\alpha,\alpha}}$ are finite real constants such that $K_{E_{\alpha,1}} > 1, K_{E_{\alpha,\alpha}} > 1$.

Lemma (2.1.2), [19]
Let $v(t)$ be a nonnegative function that is locally integrable on $[0,T)$, let $a(t)$ be a nonnegative, nondecreasing continuous function defined on $[0,T)$, and let $a(t) < M$. Suppose that $z(t)$ is nonnegative and locally integrable on $[0,T)$ with $z(t) \leq v(t)+a(t)\int_0^t(t-\tau)^{\alpha-1}z(\tau)d\tau$. If $v(t)$ is a non-decreasing function on $[0,T)$, then we have
$z(t) \leq v(t)E_\alpha(\Gamma(\alpha)a(t)t^\alpha)$.

Theorem (2.1.3)
Suppose that the following nonlinear fractional order differential-algebraic control system
(17-18) with feedback control (14) satisfies the following conditions:
1. $\text{Re}(\text{eig}(A)) < 0$ and $-\max\text{Re}(\text{eig}(A)) > \Gamma(\alpha)$
2. $\|F(x)G(\frac{\partial}{\partial t}x)\| = \|f_3(x)\| = o(\|x\|)$ as $\|x\| \to 0$.
where $F(x) = \begin{bmatrix}
f_1(x_1, x_2, x_3)B_1 \left(\frac{\partial}{\partial t}x_1, \frac{\partial}{\partial t}x_2, \frac{\partial}{\partial t}x_3\right)
f_2(x_1, x_2, x_3)B_2 \left(\frac{\partial}{\partial t}x_1, \frac{\partial}{\partial t}x_2, \frac{\partial}{\partial t}x_3\right)
\tilde{b}_{2,2,\alpha}f_3(x_1, x_2, x_3)B_3 \left(\frac{\partial}{\partial t}x_1, \frac{\partial}{\partial t}x_2, \frac{\partial}{\partial t}x_3\right)
\end{bmatrix}$
and $K_{gi}, i=1\ldots3$ are the nonnegative continuous function, and $\mu_{gi}$ are the continuous non-decreasing positive functions, such that $\|g_i(t, \frac{\partial}{\partial t}x)\| \leq K_{gi}(t)\mu_{gi}(\|\frac{\partial}{\partial t}x\|), i=1\ldots3$.
Then, the system (15) is a locally asymptotically stable.

Proof
By taking the Laplace transformation to (17), we get
$s^\alpha X(s) - s^{\alpha-1}x_0 = \tilde{A}X(s) + L \left\{F(x)G\left(\frac{\partial}{\partial t}x\right)\right\}$, thus
$X(s) = (s^\alpha - \tilde{A})^{-1} \left(-s^{\alpha-1}x_0 + L \left\{F(x)G\left(\frac{\partial}{\partial t}x\right)\right\}\right)$
(19)

By taking the Laplace inverse transformation to (19), we obtain
$x(t) = E_{\alpha,1}(\tilde{A}t^\alpha)x_0 + \int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\tilde{A}(t-\tau)^\alpha) F(x(\tau))G\left(\frac{\partial}{\partial t}x(\tau)\right)d\tau$
From lemma (2.1.1), we have
$\|x(t)\| = L_1\|e^{\tilde{A}t}\|\|x_0\| + L_2 \int_0^t(t-\tau)^{\alpha-1} \|e^{\tilde{A}(t-\tau)^\alpha}\| \|F(x(\tau))G\left(\frac{\partial}{\partial t}x(\tau)\right)\| d\tau$
From condition (1), the matrix $\tilde{A}$ is stable and there is a constant $L_4 > 0$ such that $\|e^{\tilde{A}t}\| \leq L_4e^{-\omega t}$, hence,
$\|x(t)\| \leq L_1L_4e^{-\omega t}\|x_0\| + L_3L_4 \int_0^t(t-\tau)^{\alpha-1} e^{-\omega(t-\tau)^\alpha} \|F(x(\tau))G\left(\frac{\partial}{\partial t}x(\tau)\right)\| d\tau$
From condition (2), we get
$\|F(x(\tau))G\left(\frac{\partial}{\partial t}x(\tau)\right)\|$
$= \left\|f_1(x_1, x_2, x_3)B_1 \left(\frac{\partial}{\partial t}x_1, \frac{\partial}{\partial t}x_2, \frac{\partial}{\partial t}x_3\right)\right\| + \left\|f_2(x_1, x_2, x_3)B_2 \left(\frac{\partial}{\partial t}x_1, \frac{\partial}{\partial t}x_2, \frac{\partial}{\partial t}x_3\right)\right\| + \left\|f_3(x_1, x_2, x_3)B_3 \left(\frac{\partial}{\partial t}x_1, \frac{\partial}{\partial t}x_2, \frac{\partial}{\partial t}x_3\right)\right\|$
\[ \left\| \mathbf{b}_{2,\rho} f_s(x_1, x_2, x_3) \mathbf{b}_3 \left( \mathbf{d}^\beta x_1, \mathbf{d}^\beta x_2, \mathbf{d}^\beta x_3 \right) + \mathbf{b}_{1,\rho} f_s(x_1, x_2, x_3) \mathbf{b}_3 \left( \mathbf{d}^\beta x_1, \mathbf{d}^\beta x_2, \mathbf{d}^\beta x_3 \right) \right\| \leq \| f_1(x_1, x_2, x_3) \| K_{g_1}(t) \mu_{g_1}(\| \mathbf{d}^\beta x \|) + \| f_2(x_1, x_2, x_3) \| K_{g_2}(t) \mu_{g_2}(\| \mathbf{d}^\beta x \|) + \left| \mathbf{b}_{2,\rho} \right| \| f_3(x_1, x_2, x_3) \| K_{g_3}(t) \mu_{g_3}(\| \mathbf{d}^\beta x \|) \]  

Since \( \| F(x) \| G(\mathbf{d}^\beta x) \| = o(\| x \|) \) as \( \| x \| \to 0 \), thus \( \lim_{\| x \| \to 0} \frac{\| F(x) \| G(\mathbf{d}^\beta x) \|}{\| x \|} = 0 \), which implies that \( \| F(x) \| G(\mathbf{d}^\beta x) \| \leq \frac{1}{L_1 L_4} \), therefore 

\[ \left\| F(x(t)) \mathbf{G}(\mathbf{d}^\beta x(t)) \right\| \leq \frac{1}{L_1 L_4} \left[ K_{g_1}(t) \mu_{g_1}(\| \mathbf{d}^\beta x \|) + \| f_2(x_1, x_2, x_3) \| K_{g_2}(t) \mu_{g_2}(\| \mathbf{d}^\beta x \|) + \left| \mathbf{b}_{2,\rho} \right| + \left| \mathbf{b}_{1,\rho} \right| K_{g_3}(t) \mu_{g_3}(\| \mathbf{d}^\beta x \|) \right\| \| x(t) \| \].

Now, we set that \( L_5 = [K_{g_1}(t) \mu_{g_1}(\| \mathbf{d}^\beta x \|) + \| f_2(x_1, x_2, x_3) \| K_{g_2}(t) \mu_{g_2}(\| \mathbf{d}^\beta x \|)] \), then 

\[ \| x(t) \| \leq c_1 L_1 L_4 \| x_0 \| e^{-\omega(t-t_0)} \| x(t) \| \text{d} t \],

thus 

\[ e^{\omega t} \| x(t) \| \leq c_1 L_1 L_4 \| x_0 \| + L_5 \int_{t_0}^{t} e^{\omega(t-t')} \| x(t') \| \text{d} t' \].

By using Gr"{o}nwall inequality and lemma(2.1.2), we have 

\[ e^{\omega t} \| x(t) \| \leq c_1 L_1 L_4 \| x_0 \| e^{-\omega(t-t_0)} \| x(t) \| \text{d} t \]  

As \( t \to \infty \), \( \| x(t) \| \to 0 \) for \( \omega > L_5 \Gamma(t) \).  

From condition (2), we have that \( x \to 0 \) as \( t \to \infty \), \( x_3(t) = Bx + f_3(x) \mathbf{b}_3 \mathbf{d}^\beta x \). We get 

\[ \| x_3(t) \| \leq \| B \| + \| f_3(x) \| \| g_3(t, \mathbf{d}^\beta x) \| \].

Now, since \( \| f_3(x) \| = o(\| x \|) \) as \( \| x \| \to 0 \), this implies that \( \lim_{\| x \| \to 0} \frac{\| f_3(x) \|}{\| x \|} = 0 \), so that \( \| f_3(x) \| < \delta_1 \), and since 

\[ \| g_3(t, \mathbf{d}^\beta x) \| \leq K_{g_3}(t) \mu_{g_3}(\| \mathbf{d}^\beta x \|) \],

therefore 

\[ \| x_3(t) \| \leq \| B \| + \delta_1 K_{g_3}(t) \mu_{g_3}(\| \mathbf{d}^\beta x \|) \| x \|.  

Hence, \( \| x_3(t) \| \to 0 \), which implies that system (17-18) is asymptotically stable.

**Example (2.1.2)**

The following single-fractional order differential-algebraic equations (19-20) with Caputo derivative are asymptotically stable, as follows 

\[ \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11} \mu_{\beta} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{12} \mu_{\beta} \\ \tilde{a}_{31}(\tilde{b}_{11} + \tilde{b}_{12}) & \tilde{a}_{32}(\tilde{b}_{11} + \tilde{b}_{12}) & \sum_{j=1}^{\gamma} \tilde{b}_{11,j} \mu_{\beta} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \sum_{j=1}^{\gamma} \tilde{b}_{11,j} \mu_{\beta} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \]  

\[ + \begin{bmatrix} f_1(x_1, x_2, x_3) \mathbf{b}_1 \left( \mathbf{d}^\beta x_1, \mathbf{d}^\beta x_2, \mathbf{d}^\beta x_3 \right) \\ f_2(x_1, x_2, x_3) \mathbf{b}_2 \left( \mathbf{d}^\beta x_1, \mathbf{d}^\beta x_2, \mathbf{d}^\beta x_3 \right) \\ \tilde{b}_{12} f_3(x_1, x_2, x_3) \mathbf{b}_3 \left( \mathbf{d}^\beta x_1, \mathbf{d}^\beta x_2, \mathbf{d}^\beta x_3 \right) \end{bmatrix} \]  

\[ + \begin{bmatrix} \tilde{b}_{22} \rho f_3(x_1, x_2, x_3) \mathbf{b}_3 \left( \mathbf{d}^\beta x_1, \mathbf{d}^\beta x_2, \mathbf{d}^\beta x_3 \right) \end{bmatrix} \]  

\[ + \begin{bmatrix} \tilde{b}_{22} \rho f_3(x_1, x_2, x_3) \mathbf{b}_3 \left( \mathbf{d}^\beta x_1, \mathbf{d}^\beta x_2, \mathbf{d}^\beta x_3 \right) \end{bmatrix} \]  

(19)
\[ x_3(t) = \begin{bmatrix} 1 & 0 & 0 \\ x_1(t) \\ x_3(t) \end{bmatrix} + f_3(x_1, x_2, x_3)g_3\left(\frac{\partial D_\alpha^\beta x_1}{\partial t} , \frac{\partial D_\alpha^\beta x_2}{\partial t} , \frac{\partial D_\alpha^\beta x_3}{\partial t}\right) \tag{20} \]

To compute condition (1):

Since \( A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \) and the \( \text{eig}(A) \) has values such as \( \lambda_{1,2,3} = -2 \) then

\( \text{Re}(\text{eig}(A)) < 0 \) and \( \omega = -\max \text{Re}(\text{eig}(A)) = 2 > \Gamma(\alpha) = 1.772. \)

To compute condition (2):

\[
\lim_{\|\mathbf{x}\| \to 0} \frac{\sqrt{(x_1 x_2 x_3)^2 K_{g_3}(t) \mu_{g_3}(\|\partial D_\alpha^\beta x\|)}}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \leq \lim_{\|\mathbf{x}\| \to 0} \sqrt{\frac{(x_1 x_2 x_3)^2 K_{g_3}(t) \mu_{g_3}(\|\partial D_\alpha^\beta x\|)}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}
\]

\[
\lim_{\|\mathbf{x}\| \to 0} \frac{\sqrt{x_1^2 x_2^2}}{\sqrt{x_1^2}} = \lim_{\|\mathbf{x}\| \to 0} \sqrt{x_1^2} = \lim_{\|\mathbf{x}\| \to 0} \sqrt{x_1^2} = 0
\]

\[
\lim_{\|\mathbf{x}\| \to 0} \frac{\sqrt{x_1^2 x_3^2}}{\sqrt{x_1^2}} = \lim_{\|\mathbf{x}\| \to 0} \sqrt{x_3^2} = \lim_{\|\mathbf{x}\| \to 0} \sqrt{x_3^2} = 0
\]

\[
\lim_{\|\mathbf{x}\| \to 0} \frac{\sqrt{x_2^2 x_3^2}}{\sqrt{x_2^2}} = \lim_{\|\mathbf{x}\| \to 0} \sqrt{x_3^2} = \lim_{\|\mathbf{x}\| \to 0} \sqrt{x_3^2} = 0
\]

and \( \|\mathbf{x}\| \to 0 \). Then, by theorem (2.1.1) we show that the zero solution of the system (19-20) is asymptotically stable.

2.2. Multi - Fractional Order Differential – Algebraic Control Equation

Consider the following nonlinear multi-fractional order differential – algebraic control system:

\[
\left(\frac{\partial D_\alpha^\beta}{\partial t}\right) x_1(t) + \left(\frac{\partial D_\alpha^\beta}{\partial t}\right) x_2(t) = \sum_{i=1}^{3} a_{1i} x_i(t) + \sum_{i=1}^{3} b_{1i} u_i(t) + f_1(x_1, x_2, x_3)g_1 \left(\frac{\partial D_\alpha^\beta x_1}{\partial t}, \frac{\partial D_\alpha^\beta x_2}{\partial t}, \frac{\partial D_\alpha^\beta x_3}{\partial t}\right) \tag{21} \]
\[(\delta D_t^{a_1})x_2(t) + (\delta D_t^{a_2})x_2(t) = \sum_{i=1}^3 a_{2i}x_i(t) + \sum_{i=1}^2 b_{2i}u_i(t) + f_2(x_1, x_2, x_3)g_2 \]

\[(\delta D_t^b)x_1(t), D_t^b x_2, D_t^b x_3) \]

\[x_3(t) = \sum_{i=1}^3 a_{3i}x_i(t) + \sum_{i=1}^2 b_{3i}u_i(t) + f_3(x_1, x_2, x_3)g_3(\delta D_t^\beta x_1, \delta D_t^\beta x_2, \delta D_t^\beta x_3) \]

where \(a_{1i}, a_{2i}, a_{3i}, b_{1i}, b_{2i}, b_{3i}\) and \(b_{3i}\) are constants, \(x_i \in R\) are state vectors, \(i=1..3\)

\(u_i(t) \in R\), \(i=1, 2\) are control input functions, and \(f_1, f_2, \ldots, f_i\) are varying nonlinear time values.

System (21-23) yields a linear dynamical system, as follows

\[(\delta D_t^{a_1})x_1(t) + (\delta D_t^{a_2})x_1(t) = \sum_{i=1}^3 a_{1i}x_i(t) + \sum_{i=1}^2 b_{1i}u_i(t) \]

\[(\delta D_t^{a_1})x_2(t) + (\delta D_t^{a_2})x_2(t) = \sum_{i=1}^3 a_{2i}x_i(t) + \sum_{i=1}^2 b_{2i}u_i(t) \]

\[x_3(t) = \sum_{i=1}^3 a_{3i}x_i(t) + \sum_{i=1}^2 b_{3i}u_i(t) \]

We get

\[(\delta D_t^{a_1})x_1(t) + (\delta D_t^{a_2})x_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + a_{13} \left[ \frac{a_{31}x_1(t)}{1-a_{33}} + \frac{a_{32}x_2(t)}{1-a_{33}} + \frac{\sum_{i=1}^2 b_{3i}u_i(t)}{1-a_{33}} \right] \]

\[(\delta D_t^{a_1})x_2(t) + (\delta D_t^{a_2})x_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + a_{23} \left[ \frac{a_{31}x_1(t)}{1-a_{33}} + \frac{a_{32}x_2(t)}{1-a_{33}} + \frac{\sum_{i=1}^2 b_{3i}u_i(t)}{1-a_{33}} \right] \]

\[x_3(t) = \frac{a_{31}x_1(t)}{1-a_{33}} + \frac{a_{32}x_2(t)}{1-a_{33}} + \frac{\sum_{i=1}^2 b_{3i}u_i(t)}{1-a_{33}} \]

Assume that

\[\tilde{a}_{11} = a_{11} + \frac{a_{31}d_{21}}{1-a_{33}}, \tilde{a}_{12} = a_{12} + \frac{a_{32}d_{21}}{1-a_{33}}, \tilde{a}_{21} = a_{21} + \frac{a_{31}d_{21}}{1-a_{33}}, \tilde{a}_{22} = a_{22} + \frac{a_{32}d_{21}}{1-a_{33}} \]

\[\tilde{a}_{31} = \frac{a_{31}}{1-a_{33}}, \tilde{a}_{32} = \frac{a_{32}}{1-a_{33}}, \tilde{b}_{11} = b_{11} + \frac{b_{31}}{1-a_{33}}, \tilde{b}_{12} = b_{12} + \frac{b_{32}}{1-a_{33}}, \tilde{b}_{21} = b_{21} + \frac{b_{31}}{1-a_{33}}, \tilde{b}_{22} = b_{22} + \frac{b_{32}}{1-a_{33}} \]

Therefore

\[(\delta D_t^{a_1})x_1(t) + (\delta D_t^{a_2})x_1(t) = \tilde{a}_{11}x_1(t) + \tilde{a}_{12}x_2(t) + \sum_{i=1}^2 \tilde{b}_{1i}u_i(t) \]

\[(\delta D_t^{a_1})x_2(t) + (\delta D_t^{a_2})x_2(t) = \tilde{a}_{21}x_1(t) + \tilde{a}_{22}x_2(t) + \sum_{i=1}^2 \tilde{b}_{2i}u_i(t) \]

\[x_3(t) = \tilde{a}_{31}x_1(t) + \tilde{a}_{32}x_2(t) + \sum_{i=1}^2 \tilde{b}_{3i}u_i(t) \]

Now, consider the following related linear feedback control system

\[(\delta D_t^{a_1})x_{1,\rho}(t) + (\delta D_t^{a_2})x_{1,\rho}(t) = \tilde{a}_{11}x_{1,\rho}(t) + \tilde{a}_{12}x_{2,\rho}(t) + \tilde{b}_{11}p_{x_1,\rho}(t) + \tilde{b}_{12}p_{x_2,\rho}(t) + \tilde{b}_{13}p_{x_3,\rho}(t) \]

\[(\delta D_t^{a_1})x_{2,\rho}(t) + (\delta D_t^{a_2})x_{2,\rho}(t) = \tilde{a}_{21}x_{1,\rho}(t) + \tilde{a}_{22}x_{2,\rho}(t) + \tilde{b}_{21}p_{x_1,\rho}(t) + \tilde{b}_{22}p_{x_2,\rho}(t) + \tilde{b}_{23}p_{x_3,\rho}(t) \]

\[u_\rho(t) = K_p\tilde{x}_{1,\rho}(t), u_\rho(t) = K_p\tilde{x}_{2,\rho}(t) \]

where \(x_{1,\rho}, x_{2,\rho} \in R\), \(\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{21}, \tilde{a}_{22}, \tilde{b}_{21}, \tilde{b}_{22}, p_{x_1,\rho}, p_{x_2,\rho}, p_{x_3,\rho}\), and \(K_p\) are constants.

We have that

\[\begin{bmatrix} (\delta D_t^{a_1})x_1(t) + (\delta D_t^{a_2})x_1(t) \\ (\delta D_t^{a_1})x_2(t) + (\delta D_t^{a_2})x_2(t) \\ (\delta D_t^{a_1})x_{1,\rho}(t) + (\delta D_t^{a_2})x_{1,\rho}(t) \\ (\delta D_t^{a_1})x_{2,\rho}(t) + (\delta D_t^{a_2})x_{2,\rho}(t) \end{bmatrix} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11}K_p & \tilde{b}_{12}K_p \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{21}K_p & \tilde{b}_{22}K_p \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{b}_{31}K_p & \tilde{b}_{32}K_p \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{b}_{31}K_p & \tilde{b}_{32}K_p \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1,\rho}(t) \\ x_{2,\rho}(t) \end{bmatrix} + \begin{bmatrix} f_1(x_1, x_2, x_3)g_1(\delta D_t^{a_1}x_1, \delta D_t^{a_2}x_2, \delta D_t^{a_2}x_3) \\ f_2(x_1, x_2, x_3)g_2(\delta D_t^{a_1}x_1, \delta D_t^{a_2}x_2, \delta D_t^{a_2}x_3) \\ b_{11}\rho f_3(x_1, x_2, x_3)g_3(\delta D_t^{a_1}x_1, \delta D_t^{a_2}x_2, \delta D_t^{a_2}x_3) \\ b_{22}\rho f_3(x_1, x_2, x_3)g_3(\delta D_t^{a_1}x_1, \delta D_t^{a_2}x_2, \delta D_t^{a_2}x_3) \end{bmatrix} \]

(24)
\[ x_3(t) = \begin{bmatrix} a_{31} & a_{32} & b_{31}K_\rho & b_{32}K_\rho \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_1p(t) \\ x_2p(t) \end{bmatrix} + f_3(x_1, x_2, x_3)g_3(t, \beta \frac{d}{dt}^\alpha x_1, \beta \frac{d}{dt}^\alpha x_2, \beta \frac{d}{dt}^\alpha x_3) \] (25)

Then
\[ (\beta \frac{d}{dt}^\alpha x(t) + (\beta \frac{d}{dt}^\alpha x(t) = Ax + F(x)G(t, \beta \frac{d}{dt}^\alpha x) \] (26)
\[ x_3(t) = Bx + f_3(x)g_3(t, \beta \frac{d}{dt}^\alpha x), x = (x_1, x_2, x_3) \] (27)

where
\[
A = \begin{bmatrix}
\bar{a}_{11} & \bar{a}_{12} & \bar{b}_{11}K_\rho & \bar{b}_{12}K_\rho \\
\bar{a}_{21} & \bar{a}_{22} & \bar{b}_{21}K_\rho & \bar{b}_{22}K_\rho \\
\bar{a}_{31}(\bar{b}_{11}p + \bar{b}_{12}p) & \bar{a}_{32}(\bar{b}_{11}p + \bar{b}_{12}p) & \sum_{j=1}^{2} \bar{b}_{1j}p \bar{b}_{1j}K_\rho + \bar{a}_{11}p & \sum_{j=1}^{2} \bar{b}_{1j}p \bar{b}_{32}K_\rho + \bar{a}_{12}p \\
\bar{a}_{31}(\bar{b}_{21}p + \bar{b}_{22}p) & \bar{a}_{32}(\bar{b}_{21}p + \bar{b}_{22}p) & \sum_{j=1}^{2} \bar{b}_{2j}p \bar{b}_{31}K_\rho + \bar{a}_{21}p & \sum_{j=1}^{2} \bar{b}_{2j}p \bar{b}_{32}K_\rho + \bar{a}_{22}p \\
\end{bmatrix}
\]
\[
B = [\bar{a}_{31} \quad \bar{a}_{32} \quad \bar{b}_{31}K_\rho \quad \bar{b}_{32}K_\rho], F(x)G \left( \frac{\beta}{\delta} \frac{d}{dt}^\alpha x \right) = \begin{bmatrix}
f_1(x_1, x_2, x_3)g_1(\beta \frac{d}{dt}^\alpha x_1, \beta \frac{d}{dt}^\alpha x_2, \beta \frac{d}{dt}^\alpha x_3) \\
f_2(x_1, x_2, x_3)g_2(\beta \frac{d}{dt}^\alpha x_1, \beta \frac{d}{dt}^\alpha x_2, \beta \frac{d}{dt}^\alpha x_3) \\
\vdots \\
f_n(x_1, x_2, x_3)g_n(\beta \frac{d}{dt}^\alpha x_1, \beta \frac{d}{dt}^\alpha x_2, \beta \frac{d}{dt}^\alpha x_3) \\
\end{bmatrix}
\]

\[ f_3(x)g_3(\beta \frac{d}{dt}^\alpha x) = f_3(x_1, x_2, x_3)g_3(\beta \frac{d}{dt}^\alpha x_1, \beta \frac{d}{dt}^\alpha x_2, \beta \frac{d}{dt}^\alpha x_3). \]

2.2.3 The stable equivalent system

**Theorem (2.2.1), [11]**

Consider the following system of fractional differential equations
\[
\begin{align*}
\beta \frac{d}{dt}^\alpha x_1(t) &= f_1(x, t) \\
\beta \frac{d}{dt}^\alpha x_2(t) &= f_2(x, t) \\
&\vdots \\
\beta \frac{d}{dt}^\alpha x_n(t) &= f_n(x, t)
\end{align*}
\] (28)
\[
x^{(k)}(0) = x^{(k)}_{i_0}, \quad i = 1, 2, \ldots, n; \quad k = 0, 1, \ldots, m_i - 1, \quad t \geq 0, \quad (29)
\]

\[
x = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R},
\]

Where \( f_i, i = 1, 2, \ldots, n \) are continuous functions, the fractional orders defend by \( m_i - 1 < \alpha_i < m_i \in \mathbb{Z}_+, i = 1, 2, \ldots, \) also \( n \) means the \( N \)-dimensional system of the following fractional differential equations
\[
\begin{align*}
&\frac{\partial}{\partial \tau}^\alpha_t x_{11}(t) = x_{11}(t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{1M}(t) = x_{1_{M+1}}(t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{1_{(m-1)M}}(t) = x_{1_{(m-1)_{M+1}}}(t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{1_{1M}}(t) = f_1(x, t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{22}(t) = x_{22}(t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{2_{M+1}}(t) = x_{2_{M+1}}(t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{2_{(m-1)M}}(t) = x_{2_{(m-1)_{M+1}}}(t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{2_{a2M}}(t) = f_2(x, t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{n_n}(t) = x_{n_n}(t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{n_{M+1}}(t) = x_{n_{M+1}}(t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{n_{(m-1)M}}(t) = x_{n_{(m-1)_{M+1}}}(t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{n_{a2M}}(t) = f_n(x, t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{n_{aM}}(t) = f_n(x, t), \\
&\frac{\partial}{\partial \tau}^\alpha_t x_{n_{aM}}(t) = f_n(x, t), \\
\end{align*}
\]

where \(\alpha_i, \frac{c_i}{d_i}, \text{g.c.d}(c_i, d_i)=1\), and \(i=1,2\). Also, \(M\) is a lower common multiple of \(c_i\) and \(d_i\), \(\sigma = \frac{1}{M}\), \(N=\text{g.c.d}(c_i, d_i)=1\), and \(i=1,2\). Also, \(M\) is a lower common multiple of \(c_i\) and \(d_i\), \(\sigma = \frac{1}{M}\), \(N=\text{g.c.d}(c_i, d_i)=1\), and \(i=1,2\).

\[
\begin{align*}
&x_{ij} = \begin{cases} x_{ij}^{(k)} & j = kM + 1; k = 0, 1, \ldots, m_i - 1, i=1,2,\ldots,n. \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

whenever \([x_{11}(t), x_{12}(t), \ldots, x_{1_{a1M}}(t), x_{21}(t), x_{22}(t), \ldots, x_{2_{a2M}}(t), x_{n1}(t), x_{n2}(t), \ldots, x_{n_{aM}}(t)]^T\)

is a solution of system (27-28) and \([x_{11}(t), x_{12}(t), \ldots, x_{n1}(t)]^T \in \mathbb{C}^{n1}[0,b] \times \mathbb{C}^{n2}[0,b] \times \ldots \times \mathbb{C}^{nn}[0,b]\), solved system (28-29).

Whenever \([x_{11}(t), x_{12}(t), \ldots, x_{n1}(t)]^T \in \mathbb{C}^{n1}[0,b] \times \mathbb{C}^{n2}[0,b] \times \ldots \times \mathbb{C}^{nn}[0,b]\) is a solution to system (28-29)

\[\[x_{11}(t), x_{12}(t), \ldots, x_{1_{a1M}}(t), x_{21}(t), x_{22}(t), \ldots, x_{2_{a2M}}(t), x_{n1}(t), x_{n2}(t), \ldots, x_{n_{aM}}(t)]^T = \left[x_{11}(t), \frac{\partial}{\partial \tau}^\alpha_t x_{11}(t), \ldots, \frac{\partial}{\partial \tau}^\alpha_t (x_{1_{a1M}}(t) - 1)^\sigma x_{11}(t), x_{21}(t), \frac{\partial}{\partial \tau}^\alpha_t x_{21}(t), \ldots, \frac{\partial}{\partial \tau}^\alpha_t (x_{2_{a2M}}(t) - 1)^\sigma x_{21}(t), \ldots, x_{n1}(t), \frac{\partial}{\partial \tau}^\alpha_t x_{n1}(t), \ldots, \frac{\partial}{\partial \tau}^\alpha_t (x_{n_{aM}}(t) - 1)^\sigma x_{n1}(t)\right]^T\]

satisfies system (27-28).

Consider the following multi- fractional differential equation:

\[
\frac{\partial}{\partial \tau}^\alpha_t x(t) + b_1 \frac{\partial}{\partial \tau}^\alpha_t (x(t) - 1)^\sigma x(t) + \ldots + b_{n-1} \frac{\partial}{\partial \tau}^\alpha_t x(t) + b_n = f(x(t)), \quad t > 0
\]

(32)

\[
x^{(k)}(0) = x^{(k)}_0, \quad k=0,1,\ldots,m_{n-1}.
\]

(33)

where \(x(t) \in \mathbb{R}, f: \mathbb{R} \to \mathbb{R}\) is a continues function \(D \subseteq \mathbb{R}\) and \(b_i, i=1,\ldots,n\) are constant numbers. The order \(\alpha_i, i=1,2,\ldots,n\) are rational numbers such that \(m_i - 1 < \alpha_i < m_i \in \mathbb{Z}_+\), \(i=1,2,\ldots,n\), \(\alpha_n > \alpha_{n-1} > \ldots > \alpha_1\), and \(x(t) \in \mathbb{C}[a,b]\) is a solution of system (27-28).

**Corollary (2.2.2)**, [11]

Suppose that \(f(x(t))\) is a real valued continuous function such that \(f(x(t)) = b_0 x(t)\) and equation (32-33) has a unique solution \(x(t) \in \mathbb{C}[a,b]\). Then, the zero solution to equation (32-33) is asymptotically stable if \(\text{arg}(\lambda) > \frac{\gamma}{2}\), where \(\lambda\) is a solution to the characteristic equation \(\det(\lambda I - A) = 0\), \(\gamma = \frac{1}{M}\), and

(34)
Theorem (2.2.3)
Consider the nonlinear multi-fractional order differential-algebraic feedback control system (26-27), such that it satisfies the following conditions:

1. $|\arg(\lambda_k)| > \frac{\gamma n}{2}$, where $\lambda$ is a solution to the characteristic equation $\det(\lambda_k I - A_k) = 0$.

$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$, and $a_{Nj} = \begin{pmatrix} b_0 & 0 & 0 & \cdots & 0 \\ -b_{n-i} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $j=1, \ldots, n-1$

$\gamma = \frac{1}{M}$ and $A_k = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_{N1} & a_{N2} & a_{N3} & a_{N4} & \cdots & a_{NN} \end{pmatrix}$, $k=1,2,3,4$

and $a_{Nj} = \begin{cases} -a_{11} & j = 1 \\ -a_{12} & j = 2 \\ -b_{11}K_p & j = 3 \\ -b_{12}K_p & j = 4 \\ 1 & j = \alpha_i M + 1, i = 1,2,\ldots, n-1 \end{cases}$

For $A_1$

$a_{Nj} = \begin{cases} \tilde{a}_{21} & j = 1 \\ \tilde{a}_{22} & j = 2 \\ \tilde{b}_{21}K_p & j = 3 \\ \tilde{b}_{22}K_p & j = 4 \\ 1 & j = \alpha_i M + 1, i = 1,2,\ldots, n-1 \end{cases}$

For $A_2$

$a_{Nj} = \begin{cases} \tilde{a}_{31}(b_{11,\rho} + \tilde{b}_{12,\rho}) & j = 1 \\ \tilde{a}_{32}(\tilde{b}_{11,\rho} + \tilde{b}_{12,\rho}) & j = 2 \\ \Sigma_{j=1}^2 \tilde{b}_{1,\rho} \tilde{b}_{3,1}K_p + \tilde{a}_{1,\rho} & j = 3 \\ \Sigma_{j=1}^2 \tilde{b}_{1,\rho} \tilde{b}_{3,2}K_p + \tilde{a}_{1,2,\rho} & j = 4 \\ 1 & j = \alpha_i M + 1, i = 1,2,\ldots, n-1 \end{cases}$

For $A_3$

$a_{Nj} = \begin{cases} \tilde{a}_{31} \tilde{b}_{2,\rho} & j = 1 \\ \tilde{a}_{32} \tilde{b}_{2,\rho} & j = 2 \\ \Sigma_{j=1}^2 \tilde{b}_{2,\rho} \tilde{b}_{3,1}K_p + \tilde{a}_{2,1,\rho} & j = 3 \\ \Sigma_{j=1}^2 \tilde{b}_{2,\rho} \tilde{b}_{3,2}K_p + \tilde{a}_{2,2,\rho} & j = 4 \\ 1 & j = \alpha_i M + 1, i = 1,2,\ldots, n-1 \end{cases}$

For $A_4$

2. $\|F(x)\| = o(\|x\|)$ as $\|x\| \to 0$.

where $F(x) = \begin{bmatrix} f_1(x_1, x_2, x_3)g_1 \left( \tilde{\xi}_1^{D^\beta_{\mathcal{P}_1}} x_1, \tilde{\xi}_2^{D^\beta_{\mathcal{P}_2}} x_2, \tilde{\xi}_3^{D^\beta_{\mathcal{P}_3}} x_3 \right) \\ f_2(x_1, x_2, x_3)g_2 \left( \tilde{\xi}_1^{D^\beta_{\mathcal{P}_1}} x_1, \tilde{\xi}_2^{D^\beta_{\mathcal{P}_2}} x_2, \tilde{\xi}_3^{D^\beta_{\mathcal{P}_3}} x_3 \right) \\ \tilde{b}_{2,\rho} f_3(x_1, x_2, x_3)g_3 \left( \tilde{\xi}_1^{D^\beta_{\mathcal{P}_1}} x_1, \tilde{\xi}_2^{D^\beta_{\mathcal{P}_2}} x_2, \tilde{\xi}_3^{D^\beta_{\mathcal{P}_3}} x_3 \right) \\ \tilde{b}_{1,\rho} f_3(x_1, x_2, x_3)g_3 \left( \tilde{\xi}_1^{D^\beta_{\mathcal{P}_1}} x_1, \tilde{\xi}_2^{D^\beta_{\mathcal{P}_2}} x_2, \tilde{\xi}_3^{D^\beta_{\mathcal{P}_3}} x_3 \right) \end{bmatrix}$ and $K_{g_i}, i=1,2,3$ are the nonnegative continuous functions and $\mu_{g_i}$ are the continuous nondecreasing positive functions, such that $\|g_i(t, \tilde{\xi}_1^{D^\beta_{\mathcal{P}_1}} x)\| \leq K_{g_i}(t) \mu_{g_i}(\|\tilde{\xi}_1^{D^\beta_{\mathcal{P}_1}} x\|)$, $i=1,2,3$. Then, the system is a locally asymptotically stable.
Proof
From theorem (2.2.1), we can transform the system (24-25) which defined as follows:
\[
\left( \frac{d}{dt}^{\alpha_1} \right) x_1(t) + \left( \frac{d}{dt}^{\alpha_2} \right) x_1(t) = \tilde{a}_{11} x_1(t) + \tilde{a}_{12} x_2(t) + \tilde{b}_{11} K_p x_{1,\rho}(t) + \tilde{b}_{12} K_p x_{2,\rho}(t)
\]

into a system of a single fractional order equations
\[
\frac{d}{dt}^{\alpha_1} x_{11}(t) = x_{12}(t),
\]
\[
\frac{d}{dt}^{\alpha_1} x_{12}(t) = x_{13}(t),
\]
\[
\vdots
\]
\[
\frac{d}{dt}^{\alpha_1} x_{1M}(t) = x_{1M+1}(t),
\]
\[
\frac{d}{dt}^{\alpha_2} x_{21}(t) = \tilde{a}_{11} x_1(t) - \tilde{a}_{12} x_2(t) - \tilde{b}_{11} K_p x_{1,\rho}(t) - \tilde{b}_{12} K_p x_{2,\rho}(t)
\]
such that
\[
A_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
\end{bmatrix}, \quad \text{and} \quad a_{Nj} = \begin{cases}
-\tilde{a}_{11} & j = 1 \\
-\tilde{a}_{12} & j = 2 \\
-\tilde{b}_{11} K_p & j = 3 \\
-\tilde{b}_{12} K_p & j = 4 \\
-1 & j = \alpha_i M + 1, i = 1, 2, \ldots, n - 1
\end{cases}
\]

Now, we transfer equation
\[
\left( \frac{d}{dt}^{\alpha_1} \right) x_1(t) + \left( \frac{d}{dt}^{\alpha_2} \right) x_1(t) = \tilde{a}_{21} x_1(t) + \tilde{a}_{22} x_2(t) + \tilde{b}_{21} K_p x_{1,\rho}(t) + \tilde{b}_{22} K_p x_{2,\rho}(t)
\]
into
\[
\frac{d}{dt}^{\alpha_1} x_{21}(t) = x_{22}(t),
\]
\[
\frac{d}{dt}^{\alpha_1} x_{22}(t) = x_{23}(t),
\]
\[
\vdots
\]
\[
\frac{d}{dt}^{\alpha_1} x_{2M}(t) = x_{2M+1}(t),
\]
\[
\frac{d}{dt}^{\alpha_2} x_{21}(t) = \tilde{a}_{21} x_1(t) + \tilde{a}_{22} x_2(t) + \tilde{b}_{21} K_p x_{1,\rho}(t) + \tilde{b}_{22} K_p x_{2,\rho}(t)
\]
such that
\[
A_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
\end{bmatrix}, \quad \text{and} \quad a_{Nj} = \begin{cases}
\tilde{a}_{21} & j = 1 \\
\tilde{a}_{22} & j = 2 \\
\tilde{b}_{21} K_p & j = 3 \\
\tilde{b}_{22} K_p & j = 4 \\
-1 & j = \alpha_i M + 1, i = 1, 2, \ldots, n - 1
\end{cases}
\]
\[ \delta D^\sigma_t x_{3,M}(t) = x_{3,M+1}(t), \]
\[ \vdots \]
\[ \delta D^\sigma_t x_{3(m_1-1)M}(t) = x_{3(m_1-1)M+1}(t), \]
\[ \delta D^\sigma_t x_{2_1,M}(t) = \bar{a}_{31}(\bar{b}_{1,1,\rho} + \bar{b}_{1,2,\rho})x_1(t) + \bar{a}_{32}(\bar{b}_{1,1,\rho} + \bar{b}_{1,2,\rho})x_2(t) + \bar{a}_{1,1,\rho}x_1(t) + (\Sigma_{j=1}^2 \bar{b}_{1,j,\rho} \bar{b}_{3,1} K_{\rho} + \bar{a}_{1,1,\rho})x_1(t) + (\Sigma_{j=1}^2 \bar{b}_{1,j,\rho} \bar{b}_{3,2} K_{\rho} + \bar{a}_{1,2,\rho})x_2(t), \]
such that
\[
A_3 = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\text{ and }
\begin{align*}
ap_{N1} & = a_{N2} a_{N3} a_{N4} \cdots a_{NN} \\
\bar{a}_{31}(\bar{b}_{1,1,\rho} + \bar{b}_{1,2,\rho}) & = j = 1 \\
\bar{a}_{32}(\bar{b}_{1,1,\rho} + \bar{b}_{1,2,\rho}) & = j = 2 \\
\Sigma_{j=1}^2 \bar{b}_{1,j,\rho} \bar{b}_{3,1} K_{\rho} + \bar{a}_{1,1,\rho} & = j = 3 \\
\Sigma_{j=1}^2 \bar{b}_{1,j,\rho} \bar{b}_{3,2} K_{\rho} + \bar{a}_{1,2,\rho} & = j = 4 \\
-1 & = \alpha_i M + 1, i = 1, 2, \ldots, n - 1
\end{align*}
\]

Now, we transfer equation

\[ (\delta D^\sigma_t) x_1(t) + (\delta D^\sigma_t) x_1(t) = \bar{a}_{31} \bar{b}_{2,\rho} x_1(t) + \bar{a}_{32} \bar{b}_{2,\rho} x_2(t) + \bar{a}_{1,1,\rho} x_1(t) + (\Sigma_{j=1}^2 \bar{b}_{2,j,\rho} \bar{b}_{3,1} K_{\rho} + \bar{a}_{2,1,\rho}) x_1(t) + (\Sigma_{j=1}^2 \bar{b}_{2,j,\rho} \bar{b}_{3,2} K_{\rho} + \bar{a}_{2,2,\rho}) x_2(t), \]

into

\[ \delta D^\sigma_t x_4(t) = x_4(t), \]
\[ \vdots \]
\[ \delta D^\sigma_t x_{4(M+1)}(t) = x_{4(M+1)}(t), \]
\[ \delta D^\sigma_t x_{4(m_1-1)M}(t) = x_{4(m_1-1)M+1}(t), \]
\[ \vdots \]
\[ \delta D^\sigma_t x_{4_{\rho}}(t) = \bar{a}_{31} \bar{b}_{2,\rho} x_1(t) + \bar{a}_{32} \bar{b}_{2,\rho} x_2(t) + (\Sigma_{j=1}^2 \bar{b}_{2,j,\rho} \bar{b}_{3,1} K_{\rho} + \bar{a}_{2,1,\rho}) x_1(t) + (\Sigma_{j=1}^2 \bar{b}_{2,j,\rho} \bar{b}_{3,2} K_{\rho} + \bar{a}_{2,2,\rho}) x_2(t), \]
such that
\[
A_4 = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{align*}
ap_{N1} & = a_{N2} a_{N3} a_{N4} \cdots a_{NN} \\
\bar{a}_{31}(\bar{b}_{2,\rho}) & = 1 = j = 1 \\
\bar{a}_{32}(\bar{b}_{2,\rho}) & = 2 = j = 2 \\
(\Sigma_{j=1}^2 \bar{b}_{2,j,\rho} \bar{b}_{3,1} K_{\rho} + \bar{a}_{2,1,\rho}) & = 3 = j = 3 \\
(\Sigma_{j=1}^2 \bar{b}_{2,j,\rho} \bar{b}_{3,2} K_{\rho} + \bar{a}_{2,2,\rho}) & = 4 = j = 4 \\
-1 & = \alpha_i M + 1, i = 1, 2, \ldots, n - 1
\end{align*}
\]

From condition (1), we have that \[ \text{arg}(\lambda_k) > \frac{\nu}{2}, \text{ then } A_k, k=1, 2, 3, 4 \text{ are stable, and from condition (2), we have that } \]
\[ \left\| F(x(t)) G(\delta D^\sigma_t x(t)) \right\| = \left\| f_1(x_1, x_2, x_3) b_1 (\delta D^\sigma_t x_1, \delta D^\sigma_t x_2, \delta D^\sigma_t x_3) \right\| + \left\| f_2(x_1, x_2, x_3) b_2 (\delta D^\sigma_t x_1, \delta D^\sigma_t x_2, \delta D^\sigma_t x_3) \right\| + \left\| b_{2,\rho} f_3(x_1, x_2, x_3) b_3 (\delta D^\sigma_t x_1, \delta D^\sigma_t x_2, \delta D^\sigma_t x_3) \right\| + \left\| b_{1,2,\rho} f_3(x_1, x_2, x_3) b_3 (\delta D^\sigma_t x_1, \delta D^\sigma_t x_2, \delta D^\sigma_t x_3) \right\| \]
\[ \leq \|f_1(x_1, x_2, x_3)\|K_1(t) + \|f_2(x_1, x_2, x_3)\|K_2(t) + \|f_3(x_1, x_2, x_3)\|K_3(t) \] 

\[ \mu_g_1(\|\delta D_1^\alpha x\|) + B_{12p}\|f_3(x_1, x_2, x_3)\|K_{33}(t) + \mu_g_3(\|\delta D_1^\alpha x\|) \] 

\[ \mu_g_3(\|\delta D_1^\alpha x\|) \]

since \( \|F(x)G(\delta D_1^\alpha x)\| = 0(\|x\|) \) as \( \|x\| \to 0 \), thus \( \lim_{\|x\| \to 0} \|F(x)G(\delta D_1^\alpha x)\| = 0 \), implies that \( \|F(x)G(\delta D_1^\alpha x)\| < L_3L_4 \), where \( L_3, L_4 > 0 \). Then, we can complete the proof as in theorem (2.1.1).

Moreover, \( \|x_3(t)\| \leq (B + L_3L_4K_{33}(t) \mu_g_3(\|\delta D_1^\alpha x\|))\|x\| \), hence \( \|x_3(t)\| \to 0 \). Then, the solution of (24-25) is asymptotically stable.

**Example (2.2.4)**

Consider the following nonlinear multi-fractional order differential-algebraic system with feedback control

\[ \begin{align*}
  \begin{bmatrix}
  (\delta D_1^\alpha x_1(t) + (\delta D_1^\beta x_1(t))x_1(t) \\
  (\delta D_1^\alpha x_2(t) + (\delta D_1^\beta x_2(t))x_2(t) \\
  (\delta D_1^\alpha x_1(t) + (\delta D_1^\beta x_2(t))x_1(t) \\
  (\delta D_1^\alpha x_2(t) + (\delta D_1^\beta x_2(t))x_2(t) \\
  \end{bmatrix} = \\
  \begin{bmatrix}
  \bar{a}_{11} & \bar{a}_{12} & \bar{b}_{11}K_\rho & \bar{b}_{12}K_\rho \\
  \bar{a}_{21} & \bar{a}_{22} & \bar{b}_{21}K_\rho & \bar{b}_{22}K_\rho \\
  \bar{a}_{31}(\bar{b}_{11} + \bar{b}_{12}) & \bar{a}_{32}(\bar{b}_{11} + \bar{b}_{12}) & \sum_{j=1}^2 \bar{b}_{1j}\bar{b}_{31}K_\rho + \bar{a}_{11}K_\rho & \sum_{j=1}^2 \bar{b}_{1j}\bar{b}_{32}K_\rho + \bar{a}_{12}K_\rho \\
  \bar{a}_{31}(\bar{b}_{21} + \bar{b}_{22}) & \bar{a}_{32}(\bar{b}_{21} + \bar{b}_{22}) & \sum_{j=1}^2 \bar{b}_{2j}\bar{b}_{31}K_\rho + \bar{a}_{21}K_\rho & \sum_{j=1}^2 \bar{b}_{2j}\bar{b}_{32}K_\rho + \bar{a}_{22}K_\rho \\
  \end{bmatrix} \\
  \begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  x_1, x_2, x_3 \\
  x_2, x_2, x_3 \\
  \end{bmatrix} \\
  \end{align*} \]

\[ x_3(t) = [\bar{a}_{31} \bar{a}_{32} \bar{b}_{31}K_\rho \bar{b}_{32}K_\rho] \begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  x_1, x_2, x_3 \\
  x_2, x_2, x_3 \\
  \end{bmatrix} + f_3(x_1, x_2, x_3)g_3(t, \delta D_1^\beta x_1, \delta D_1^\beta x_2, \delta D_1^\beta x_3) \]

where \( \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{6}, \bar{a}_{11} = \bar{a}_{22} = \sum_{j=1}^2 \bar{b}_{1j}\bar{b}_{31}K_\rho + \bar{a}_{11}K_\rho = \sum_{j=1}^2 \bar{b}_{1j}\bar{b}_{32}K_\rho + \bar{a}_{12}K_\rho = -2, \)

\( \bar{a}_{12} = \bar{b}_{11}K_\rho = \bar{b}_{12}K_\rho = \bar{a}_{21} = \bar{b}_{21}K_\rho = \bar{b}_{22}K_\rho = \bar{a}_{31}(\bar{b}_{11} + \bar{b}_{12}) = \bar{a}_{32}(\bar{b}_{21} + \bar{b}_{22}) = \sum_{j=1}^2 \bar{b}_{2j}\bar{b}_{31}K_\rho + \bar{a}_{21}K_\rho = 0, \)

\( \bar{a}_{31} = \bar{a}_{32} = \bar{b}_{31}K_\rho = 0, \bar{b}_{32}K_\rho = \frac{1}{2}. \)

\[ \begin{align*}
  f_1(x_1, x_2, x_3)g_1(\delta D_1^\beta x_1, \delta D_1^\beta x_2, \delta D_1^\beta x_3) \\
  f_2(x_1, x_2, x_3)g_2(\delta D_1^\beta x_1, \delta D_1^\beta x_2, \delta D_1^\beta x_3) \\
  \bar{b}_{12}K_\rho f_3(x_1, x_2, x_3)g_3(\delta D_1^\beta x_1, \delta D_1^\beta x_2, \delta D_1^\beta x_3) \\
  \bar{b}_{22}K_\rho f_3(x_1, x_2, x_3)g_3(\delta D_1^\beta x_1, \delta D_1^\beta x_2, \delta D_1^\beta x_3) \\
  \end{align*} \]

We have that

\[ \begin{bmatrix}
  (\delta D_1^\beta x_1(t) + (\delta D_1^\beta x_2(t))x_1(t) \\
  (\delta D_1^\beta x_2(t) + (\delta D_1^\beta x_2(t))x_2(t) \\
  (\delta D_1^\beta x_1(t) + (\delta D_1^\beta x_2(t))x_1(t) \\
  (\delta D_1^\beta x_2(t) + (\delta D_1^\beta x_2(t))x_2(t) \\
  \end{bmatrix} = \\
  \begin{bmatrix}
  -9/64 & 0 & 0 & 0 & 0 \\
  0 & -9/64 & 0 & 0 & 0 \\
  0 & 0 & -9/64 & 0 & 0 \\
  0 & 0 & 0 & -9/64 & 0 \\
  \end{bmatrix} \begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  x_1, x_2, x_3 \\
  x_2, x_2, x_3 \\
  \end{bmatrix} \]
\[
\begin{align*}
\hspace{10pt} x_3(t) &= \begin{bmatrix} 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1\rho}(t) \\ x_{2\rho}(t) \end{bmatrix} + x_2x_3K_1(t)\mu_1(\|\delta D^\alpha x\|) \\
\end{align*}
\]

Suppose that
\[
\bar{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1\rho}(t) \\ x_{2\rho}(t) \end{bmatrix}, \hspace{10pt} \frac{1}{\delta D_t^{1/3}}\bar{x} + \frac{1}{\delta D_t^{1/6}}\bar{x} = \begin{bmatrix} \frac{1}{\delta D_t^{1/3}}x_1(t) + \frac{1}{\delta D_t^{1/6}}x_1(t) \\ \frac{1}{\delta D_t^{1/3}}x_2(t) + \frac{1}{\delta D_t^{1/6}}x_2(t) \\ \frac{1}{\delta D_t^{1/3}}x_{1\rho}(t) + \frac{1}{\delta D_t^{1/6}}x_{1\rho}(t) \\ \frac{1}{\delta D_t^{1/3}}x_{2\rho}(t) + \frac{1}{\delta D_t^{1/6}}x_{2\rho}(t) \end{bmatrix}.
\]

\[
\bar{A} = \begin{bmatrix} -\frac{9}{64} & 0 & 0 & 0 \\ 0 & -\frac{9}{64} & 0 & 0 \\ 0 & 0 & -\frac{9}{64} & 0 \\ 0 & 0 & 0 & -\frac{9}{64} \end{bmatrix}
\]

\[
\bar{f}_1(\bar{x}) = \begin{bmatrix} x_1x_2K_1(t)\mu_1(\|\delta D^\alpha x\|)|x| \\ x_1x_3K_1(t)\mu_2(\|\delta D^\alpha x\|)|x| \\ b_{12\rho}x_2x_3K_1(t)\mu_3(\|\delta D^\alpha x\|)|x| \\ b_{22\rho}x_2x_3K_1(t)\mu_3(\|\delta D^\alpha x\|)|x| \end{bmatrix}
\]

\[
\bar{A}_{2,1} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\
\end{bmatrix}, \hspace{10pt} \bar{f}_2(\bar{x}) = x_2x_3K_1(t)\mu_3(\|\delta D^\alpha x\|)
\]

\[
\left(\frac{1}{\delta D_t^{1/3}}\right)x_1(t) + \left(\frac{1}{\delta D_t^{1/6}}\right)x_1(t) = -\frac{9}{64}x_1(t) + f_1(x_1,x_2,x_3)g_1(\delta D^\beta x_1,\delta D^\beta x_2,\delta D^\beta x_3)
\]

From theorem (2.2.1), equation (36) becomes
\[
\frac{1}{\delta D_t^{1/3}}x_1(t) = x_2(t),
\]
\[
\frac{1}{\delta D_t^{1/6}}x_2(t) = x_3(t),
\]

\[
\delta D^\alpha x_3(t) = f_1(x_1,x_2,x_3)g_1(\delta D^\beta x_1,\delta D^\beta x_2,\delta D^\beta x_3) - x_3(t) - 18x_1(t)
\]

\[
\begin{bmatrix} \frac{1}{\delta D_t^{1/3}}x_1(t) \\ \frac{1}{\delta D_t^{1/6}}x_2(t) \\ -\frac{9}{64}x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{9}{64} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_1(x_1,x_2,x_3)g_1(\delta D^\beta x_1,\delta D^\beta x_2,\delta D^\beta x_3) \end{bmatrix}
\]

The coefficient matrix of system (38) can be written as
\[
A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{9}{64} & 0 & -1 \end{bmatrix}
\]

To obtain the eigenvalues of $A_1$,
\[
\det(A_1 - \lambda I) = 0,
\]
\[
\det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{9}{64} & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0,
\]
\[
\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -\frac{9}{64} & 0 & -1 - \lambda \end{bmatrix}\right) = 0,
\]
\[ \lambda^2 (-1 - \lambda) - \frac{9}{64} = 0 \]
\[ \lambda^3 + \lambda^2 + \frac{9}{64} = 0 \]
\[ \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = -\frac{3}{4} \]
\[ \Rightarrow \arg(\lambda_1) = \frac{\pi}{6}, \quad \arg(\lambda_2) = -\frac{\pi}{6} \]
\[ \Rightarrow |\arg(\lambda_i)| = \frac{\pi}{6}, \quad i = 1, 2, 3 \]

and \( \frac{\gamma \pi}{2} = \frac{\pi}{12} = 0.26167 \)
\[ \Rightarrow |\arg(\lambda_i)| > \frac{\gamma \pi}{2}, \quad i = 1, 2, 3 \]

Then, from corollary (2.2.2), we have that \( A_1 \) is stable.

It is easy to demonstrate that \( f_1(x_1, x_2, x_3) \mid_{\mathbb{B}_1} \left( \uo_{D_t^\alpha} x_1, \uo_{D_t^\alpha} x_2, \uo_{D_t^\alpha} x_3 \right) = x_1 x_2 K_{g_1}(t) \mu_{g_1}(\|D_t^\alpha x\|) \) satisfies the following
\[
\lim_{\|x\| \to 0} \left\| f_1(x_1, x_2, x_3) \mid_{\mathbb{B}_1} \left( \uo_{D_t^\alpha} x_1, \uo_{D_t^\alpha} x_2, \uo_{D_t^\alpha} x_3 \right) \right\| = \lim_{\|x\| \to 0} \frac{\sqrt{x_1^2 x_2^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \lim_{\|x\| \to 0} \frac{\sqrt{x_2^2}}{\sqrt{x_2^2}} = 0
\]

that is \( \left\| f_1(x_1, x_2, x_3) \mid_{\mathbb{B}_1} \left( \uo_{D_t^\alpha} x_1, \uo_{D_t^\alpha} x_2, \uo_{D_t^\alpha} x_3 \right) \right\| = 0 \|x(t)\| \) as \( \|x\| \to 0 \).

By continuing in this way with other equations of (3), and
\[ \|x_3(t)\| \leq \left\| \left[ \begin{array}{ccc} 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} \\ \end{array} \right] \right\| K_{g_3}(t) \mu_{g_3}(\|D_t^\alpha x\|) \|x\|, \text{ hence } \|x_3(t)\| \to 0 \text{ as } \|x\| \to 0 \]

By using theorem (2.2.3), then the zero solution of equation (34-35) is asymptotically stable.

**Conclusions**

We studied the asymptotic stability for the proposed multi-fractional differential-algebraic control systems, involving multi control inputs, which needed to be transformed to single-fractional differential systems, using sufficient and necessary conditions.

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