ON POSITIVITY AND BASE LOCI OF VECTOR BUNDLES

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1. INTRODUCTION

The aim of this note is to shed some light on the relationships among some notions of posi-
tivity for vector bundles that arose in recent decades.

Positivity properties of line bundles have long played a major role in projective geometry; 
they have once again become a center of attention recently, mainly in relation with advances in
birational geometry, especially in the framework of the Minimal Model Program. Positivity of
line bundles has often been studied in conjunction with numerical invariants and various kinds
of asymptotic base loci (see for example [ELMNP06] and [BDPP13]).

At the same time, many positivity notions have been introduced for vector bundles of higher
rank, generalizing some of the properties that hold for line bundles. While the situation in
rank one is well-understood, at least as far as the interdependencies between the various positivity
concepts is concerned, we are quite far from an analogous state of affairs for vector bundles in
general.

In an attempt to generalize bigness for the higher rank case, some positivity properties have
been put forward by Viehweg (in the study of fibrations in curves, [Vie83]), and Miyaoka (in
the context of surfaces, [Miy83]), and are known to be different from the generalization given
by using the tautological line bundle on the projectivization of the considered vector bundle
(cf. [Laz04]). The differences between the various definitions of bigness are already present in
the works of Lang concerning the Green-Griffiths conjecture (see [Lan86]).

Our purpose is to study several of the positivity notions studied for vector bundles with
some notions of asymptotic base loci that can be defined on the variety itself, rather than on the
projectivization of the given vector bundle. We relate some of the different notions conjectured
to be equivalent with the help of these base loci, and we show that these can help simplify the
various relationships between the positivity properties present in the literature.

In particular, we define augmented and restricted base loci $\mathcal{B}_+(E)$ and $\mathcal{B}_-(E)$ of a vector
bundle $E$ on the variety $X$, as generalizations of the corresponding notions studied extensively
for line bundles. As it turns out, the asymptotic base loci defined here behave well with respect
to the natural map induced by the projectivization of the vector bundle $\mathcal{E}$, as shown in Section 3.

The relationship between these base loci with the positivity notions appearing in the literature
goes as follows:

**Theorem 1.1.** Let $X$ be a smooth projective variety and $E$ a vector bundle on $X$. Then:

1. $E$ is ample if and only if $\mathcal{B}_+(E) = \emptyset$;
2. $E$ is nef if and only if $\mathcal{B}_-(E) = \emptyset$;
3. $E$ is pseudo-effective if and only if $\mathcal{B}_-(E) \neq X$;

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(1.1.4) \( E \) is weakly positive if and only if \( \mathcal{B}_-(E) \neq X \) (see Section 5);
(1.1.5) \( E \) is \( V \)-big if and only if \( \mathcal{B}_+(E) \neq X \) (see Section 6);
(1.1.6) Assume that \( E \) is a nef vector bundle. Then \( E \) is almost everywhere ample if and only if \( \mathcal{B}_+(E) \neq X \) (cf. Section 8).

The paper is organized as follows: in sections 2 and 3 we give the definition and basic properties of the asymptotic base loci for vector bundles, and relate these loci with the ones on the projectivizations. In Section 4 we recall the various positivity properties for line bundles and their relationship with asymptotic base loci. Section 5 is devoted to a study of positivity properties of vector bundles related to the restricted base locus, while Section 6 is given over to an investigation of connection between positivity properties of vector bundles related and augmented base loci. In sections 7 and 8 we study almost everywhere ampleness and relate it to \( V \)-bigness.

2. Definitions and first properties

Convention 2.1. Throughout the paper we are working with vector bundles of finite rank, but for various reasons we find it more convenient to work with the associated sheaf of sections which is a locally free coherent \( \mathcal{O}_X \)-module. We will follow the usual abuse of terminology and while exclusively using this associated sheaf, we will still call it a vector bundle. If, rarely, we want to refer to a vector bundle and mean a vector bundle we will call it the total space of the vector bundle.

We will also work with line bundles, which of course refers to a locally free sheaf of rank 1. For a line bundle \( L \) we will denote by \( c_1(L) \) the associated Weil divisor on \( X \).

With that convention fixed we are making the following notation that we will use through the entire paper:

Notation 2.2. Let \( X \) be a smooth projective variety over the complex numbers, and \( E \) a vector bundle (i.e., according to 2.1 really a locally free sheaf) over \( X \). For a point \( x \in X \), \( E_x = E \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x} \) denotes the stalk of \( E \) at the point \( x \) and \( E(x) = E \otimes_{\mathcal{O}_X} \mathcal{O}_x \) where \( \mathcal{O}_x \) is the residue field at \( x \). Clearly, \( E(x) \) is the fiber of the total space of \( E \) over the point \( x \). In particular, \( E(x) \) is a vector space of dimension \( r = \text{rk} E \).

Definition 2.3. We define the base locus of \( E \) (over \( X \)) as the subset
\[
\text{Bs}(E) := \{ x \in X \mid H^0(X, E) \to E(x) \text{ is not surjective} \}
\]
and the stable base locus of \( E \) (over \( X \)) as
\[
\mathcal{B}(E) := \bigcap_{m > 0} \text{Bs}(\text{Sym}^m E).
\]

Remark 2.4. The assertions below follow immediately from the definition:

(2.4.1) As \( \text{Bs}(E) = \text{Bs}(\text{Im}(\bigwedge^\text{rk} E H^0(X, E) \to H^0(X, \text{det} E))) \), these loci are closed subsets, and carry a natural scheme structure.

(2.4.2) For any positive integer \( c > 0 \), \( \mathcal{B}(E) = \mathcal{B}(\text{Sym}^c E) \), and the same holds for \( \mathcal{B}_- \) and \( \mathcal{B}_+ \).

Remark 2.5. The rank of the natural linear map \( H^0(X, E) \to E(x) \) induces a stratification of \( X \) into locally closed subsets.
Definition 2.6. Let \( r = p/q \in \mathbb{Q}_{>0} \) be a positive rational number, and \( A \) a line bundle on \( X \). We will use the following notation:

\[
\mathcal{B}(E + rA) := \mathcal{B}(\text{Sym}^q E \otimes A^p), \quad \text{and} \quad \mathcal{B}(E - rA) := \mathcal{B}(\text{Sym}^q E \otimes A^{-p}).
\]

Note that if \( r = p'/q' \) is another representation of \( r \) as a fraction, then \( q'p = p'q \), hence

\[
\text{Sym}^{q'}(\text{Sym}^q E \otimes A^p) \simeq \text{Sym}^{q'q} E \otimes A^{p'} \simeq \text{Sym}^q(\text{Sym}^{q'} E \otimes A^{p'}),
\]

and therefore, by (2.4.2), \( \mathcal{B}(\text{Sym}^q E \otimes A^p) = \mathcal{B}(\text{Sym}^{q'} E \otimes A^{p'}) \) and hence \( \mathcal{B}(E + rA) \) is well-defined. A similar argument shows that \( \mathcal{B}(E - rA) \) is also well-defined.

Let \( A \) be an ample line bundle on \( X \), we define the **augmented base locus** of \( E \) as

\[
\mathcal{B}^A_+(E) := \bigcap_{r \in \mathbb{Q}_{>0}} \mathcal{B}(E - rA),
\]

and the **restricted base locus** of \( E \) as

\[
\mathcal{B}^A_-(E) := \bigcup_{r \in \mathbb{Q}_{>0}} \mathcal{B}(E + rA).
\]

Remark 2.7. The definitions above yield the following properties:

1. **(2.7.1)** The loci \( \mathcal{B}^A_+(E) \) and \( \mathcal{B}^A_-(E) \) do not depend on the choice of the ample line bundle \( A \), so we can write \( \mathcal{B}_+(E) \) and \( \mathcal{B}_-(E) \) for the augmented and restricted base locus of \( E \), respectively.

2. **(2.7.2)** For any \( r_1 > r_2 > 0 \) we have \( \mathcal{B}(E + r_1 A) \subseteq \mathcal{B}(E + r_2 A) \) and \( \mathcal{B}(E - r_2 A) \subseteq \mathcal{B}(E - r_1 A) \).

3. **(2.7.3)** In particular, for any \( \varepsilon > 0 \) we have \( \mathcal{B}(E + \varepsilon A) \subseteq \mathcal{B}(E) \subseteq \mathcal{B}(E - \varepsilon A) \).

4. **(2.7.4)** Therefore we have that

\[
\mathcal{B}_+(E) := \bigcap_{q \in \mathbb{N}} \mathcal{B}(E - (1/q)A) \quad \text{and} \quad \mathcal{B}_-(E) := \bigcup_{q \in \mathbb{N}} \mathcal{B}(E + (1/q)A).
\]

5. **(2.7.5)** It follows that \( \mathcal{B}_+(E) \) is closed, but even for line bundles, the locus \( \mathcal{B}_-(E) \) is not closed in general: Lesieutre [Les12] proved that this locus can be a proper dense subset of \( X \), or a proper dense subset of a divisor of \( X \).

3. Asymptotic Invariants for Vector Bundles

In the following sections we will relate augmented and restricted base loci for vector bundles to various positivity notions found in the literature. In order to achieve a better understanding of these positivity properties and the relations between them, it is necessary to investigate the dependence of asymptotic base loci for vector bundles, and the corresponding loci of the tautological quotient line bundles on the appropriate projectivizations.

Let \( E \) be a vector bundle on a smooth projective variety \( X \), \( \pi : \mathbb{P}(E) \to X \) the projective bundle of rank one quotients of \( E \), and \( \mathcal{O}_{\mathbb{P}(E)}(1) \) the universal quotient of \( \pi^* E \) on \( \mathbb{P}(E) \). Then we immediately have

\[
\pi(\mathcal{B}(\mathcal{O}_{\mathbb{P}(E)}(1))) \subseteq \mathcal{B}(E).
\]

In fact, if the evaluation map \( H^0(X, E) \otimes \mathcal{O}_X \to E \) is surjective over a point \( x \in X \), then the map

\[
H^0(X, E) \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \mathcal{O}_{\mathbb{P}(E)} = H^0(X, E) \otimes \mathcal{O}_{\mathbb{P}(E)} \to \pi^* E \to \mathcal{O}_{\mathbb{P}(E)}(1)
\]
is surjective over any point \( y \in \mathbb{P}(E) \) such that \( \pi(y) = x \), and a similar argument applies to \( \text{Sym}^m E \).

More precisely, we have \( \pi(\text{Bs}(\mathcal{O}_{\mathbb{P}(E)}(1))) = \text{Bs}(E) \): if a point \( x \in X \) does lie in \( \text{Bs}(E) \), then the image of the map \( H^0(X, E) \to E(x) \) is contained in some hyperplane \( H \subset E(x) \), where the hyperplane \( H \) corresponds to a point \( y \in \pi^{-1}(x) \) contained in \( \text{Bs}(\mathcal{O}_{\mathbb{P}(E)}(1)) \).

It is not clear whether the inclusion \( \pi(\mathbb{B}(\mathcal{O}_{\mathbb{P}(E)}(1))) \subseteq \mathbb{B}(E) \) of stable loci is strict in general. However, as we will show right below, some useful connections rely on properties of augmented and restricted base loci, which exhibit a more predictable behavior with respect to the map \( \pi \).

**Proposition 3.1.** Let \( E \) be a vector bundle on a smooth projective variety \( X \), \( \pi : \mathbb{P}(E) \to X \) the projective bundle of one dimensional quotients of \( E \), and \( \mathcal{O}_{\mathbb{P}(E)}(1) \) the universal quotient of \( \pi^* E \) on \( \mathbb{P}(E) \). Then

\[
\pi(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))) = \mathbb{B}_-(E).
\]

**Proof.** Let us fix \( H \in \text{Pic}(X) \), a sufficiently ample line bundle such that \( A := \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* H \) is very ample on \( \mathbb{P}(E) \).

Then the line bundle \( \mathcal{O}_{\mathbb{P}(E)}(1) \otimes A \) is generated on any point \( x \), so the fibre \( \pi^{-1}(x) \) is contained in the complement of \( \mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1)) \).

The easier inclusion is \( \pi(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))) \subseteq \mathbb{B}_-(E) \). In order to show this, suppose that \( x \in X \) and that \( x \notin \mathbb{B}_-(E) \). Then for any integer \( a > 0 \) there exists a \( b > 0 \) such that the vector bundle \( \text{Sym}^{ab} E \otimes H^b \) is generated by its global sections at \( x \). Then for all \( a > 0 \) the line bundle \( \mathcal{O}(2(a - 1)b) \otimes A^{2b} = \mathcal{O}(2ab) \otimes \pi^* H^{2b} \) which is a quotient of \( \pi^*(\text{Sym}^{2ab} E \otimes H^{2b}) \) is generated by its global sections (defined over the whole space \( \mathbb{P}(E) \)) on any point of the fibre \( \pi^{-1}(x) \), so the fibre \( \pi^{-1}(x) \) is contained in the complement of \( \mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1)) \).

Let us show now that \( \pi(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))) \supseteq \mathbb{B}_-(E) \): Let \( x \in \mathbb{P}(E) \) be a point such that \( x \notin \pi(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))) \). Then for any \( a > 0 \) there exists a \( b > 0 \) such that \( \mathcal{O}_{\mathbb{P}(E)}(2(a - 1)b) \otimes A^b \) is generated on any point \( y \in \pi^{-1}(x) \) by its global sections (defined on the whole \( \mathbb{P}(E) \)).

Then the line bundle

\[
L := \mathcal{O}_{\mathbb{P}(E)}(2ab) \otimes \pi^* H^{2b} \simeq \mathcal{O}_{\mathbb{P}(E)}(2(a - 1)b) \otimes \mathcal{O}_{\mathbb{P}(E)}(b) \otimes \pi^* H^b \otimes \mathcal{O}_{\mathbb{P}(E)}(b) \otimes \pi^* H^b \\
\simeq \mathcal{O}_{\mathbb{P}(E)}(2(a - 1)b) \otimes A^b
\]

is the product of a line bundle which is generated by global sections (on \( \mathbb{P}(E) \)) at any point of the fiber \( \mathbb{P}_x := \pi^{-1}(x) = \mathbb{P}(E(x)) \) with a very ample line bundle, so its global sections (on \( \mathbb{P}(E) \)) define a closed immersion of \( \mathbb{P}_x \) into a projective space. In other words, the linear system \( H^0(\mathbb{P}(E), L) \) defines a rational map \( \varphi: \mathbb{P}(E) \dashrightarrow \mathbb{P}(H^0(\mathbb{P}(E), L)) = \mathbb{P}^N \) which is a regular immersion on \( \mathbb{P}_x \). Then in particular, for \( m \gg 0 \) the multiplication map \( \text{Sym}^m H^0(\mathbb{P}(E), L) \to H^0(\mathbb{P}(E), L) \) is surjective.

It follows that the map \( \pi_* : \text{Sym}^m H^0(\mathbb{P}(E), L) \otimes \mathcal{O}_{\mathbb{P}(E)} \to \pi_* L^m \) is surjective at the point \( x \in X \). As \( \pi_* L^m = \text{Sym}^{2abm} E \otimes H^{2bm} \) we may conclude that for any \( a > 0 \) and \( m \) large enough the vector bundle \( \text{Sym}^{2abm} E \otimes H^{2bm} \) is generated at \( x \) by its global sections, hence \( x \notin \mathbb{B}_-(E) \).

\( \square \)
The analogous claim holds for augmented base locus, with a similar proof.

**Proposition 3.2.** Let $E$ be a vector bundle on a smooth projective variety $X$, with the same notations as in Proposition 3.1, we have

$$\pi(\mathbb{B}_+(\mathcal{O}(E)(1))) = \mathbb{B}_+(E).$$

**Proof.** Let $H \in \text{Pic}(X)$ be a sufficiently ample line bundle such that $A := \mathcal{O}(E)(1) \otimes \pi^*H$ is very ample on $\mathbb{P}(E)$. Then

$$\mathbb{B}_+(\mathcal{O}(E)(1)) = \bigcap_{a>0} \mathbb{B}(\mathcal{O}(E)(a) \otimes A^{-1}) = \bigcap_{a>0} \left( \bigcap_{b>0} \text{Bs}(\mathcal{O}(E)(ab) \otimes A^{-b}) \right)$$

and

$$\mathbb{B}_+(E) = \bigcap_{a>0} \left( \bigcap_{b>0} \text{Bs}(\text{Sym}^{ab}E \otimes H^{-b}) \right).$$

In order to show that $\pi(\mathbb{B}_+(\mathcal{O}(E)(1))) \subseteq \mathbb{B}_+(E)$, observe that if $\text{Sym}^{ab}E \otimes H^{-b}$ is globally generated at a point $x \in X$, then $\pi^*\text{Sym}^{ab}E \otimes \pi^*H^{-b}$ is generated at all points in $\pi^{-1}(x)$, hence $\mathcal{O}(E)(ab) \otimes \pi^*H^{-b} = \mathcal{O}(E)((a+1)b) \otimes A^{-b}$ is globally generated at all points in $\pi^{-1}(x)$.

To show the other inclusion, set $U = X \setminus \pi(\mathbb{B}_+(\mathcal{O}(E)(1)))$ and observe that $(\mathcal{O}(E)(ab) \otimes A^{-b})$ is generated by its global sections at the points of $\pi^{-1}(U)$ for $a$ and $b$ sufficiently large. Let us consider $b > 0$ a sufficiently large positive integer, $a = (b-1)k > 0$ a sufficiently large multiple of $b-1$, and set $c := ((a-1)b+1)/(b-1) = kb-1 = a+k-1$. Finally, let $L$ be the line bundle

$$L := \mathcal{O}(E)(c(b-1)) \otimes \pi^*H^{-(b-1)} = \mathcal{O}(E)((a-1)b+1) \otimes \pi^*H^{-(b-1)} \simeq (\mathcal{O}(E)(ab) \otimes (\mathcal{O}(E)(-1) \otimes \pi^*H^{-1})) \otimes (\mathcal{O}(E)(1) \otimes \pi^*H) \simeq (\mathcal{O}(E)(ab) \otimes A^{-b}) \otimes A.$$

Now for $b$ and $k$ large enough $L$ is the product of the very ample line bundle $A$ with a line bundle which is generated by global sections on $\pi^{-1}(U)$, so it is very ample on the open subset $\pi^{-1}(U)$. Furthermore, we have that $\pi_*(L) = \text{Sym}^{c(b-1)}E \otimes H^{-(b-1)}$ and so we can apply the same argument as in the proof of Proposition 3.1 to finish the proof. \]

4. Positivity Properties for Line Bundles

We recall here how augmented and restricted base loci are involved with various positivity notions of line bundles, as well as loci defined by negative curves:

**Definition 4.1.** Let $L$ be a line bundle on a smooth projective variety $X$. Fix an ample line bundle $A$ and a rational number $\varepsilon > 0$. We define

1. $T^A_\varepsilon = \{ x \mid \exists C \subseteq X \text{ curve on } X \text{ s.t. } x \in C, c_1(L) \cdot C < \varepsilon \cdot c_1(A) \cdot C \}$ to be the non-AEA locus of $L$ with respect to $A$ and $\varepsilon$;
2. $T(L) := \bigcap_{\varepsilon > 0} T^A_\varepsilon$ to be the stable non-AEA locus of $L$, and
3. $T^0(L) := \{ x \mid x \in C \text{ such that } L \cdot C < 0 \}$ the negative locus of $L$.

**Proposition-Definition 4.2.** For a line bundle $L$ on the variety $X$ we have the following.

1. $T^0(L) \subseteq \mathbb{B}_-(L)$, the inequality can be strict (cf. [BDPP13, Remark 6.3]).
2. $T(L) \subseteq \mathbb{B}_+(L)$.
3. $L$ is ample iff $\mathbb{B}_+(L) = \emptyset$. 
(4.2.4) \( L \) is semiample iff \( \mathbb{B}(L) = \emptyset \).

(4.2.5) \( L \) is nef iff \( \mathbb{B}_-(L) = \emptyset \) iff \( \mathbb{T}^0(L) = \emptyset \).

(4.2.6) \( L \) is big iff \( \mathbb{B}_+(L) \neq X \).

(4.2.7) \( L \) is psef (pseudo-effective) iff \( \mathbb{B}_-(L) \neq X \).

(4.2.8) \( L \) is almost nef iff \( \mathbb{T}^0(L) \) is contained in a countable union of proper closed subsets of \( X \).

(4.2.9) \( L \) is AEA (almost everywhere ample) if \( \mathbb{T}(L) \neq X \).

(4.2.10) \( L \) is weakly positive if \( \mathbb{B}_-(L) \neq X \).

**Proof.** Points (4.2.1-6) are well-known statements. The claims (4.2.8), (4.2.9) and (4.2.10) are the definitions of respective notions according to [BDPP13], [Miy83] and [Vie83], respectively. The only statement in need of a proof is (4.2.7).

Note that a line bundle is pseudo-effective precisely if its numerical equivalence class lies in the closure of the effective cone in the real Néron-Severi group. Hence the line bundle \( L \) is psef if and only if \( \forall m > 0 \ L + (1/m)A \) is effective, or, equivalently, if \( \forall m > 0 \ \mathbb{B}(mL + A) \neq X \). Therefore \( \mathbb{B}_-(L) \neq X \) as it is contained in a countable union of proper closed subsets of \( X \). Conversely, if \( \mathbb{B}_-(L) \neq X \), then the class of \( L \) is a limit of effective classes. \( \square \)

**Remark 4.3.** Positivity properties related to asymptotic base loci are best summarized in the form of a table.

| \( B_-(L) \) | \( B_-(L) \) | \( B(L) \) | \( B_+(L) \) |
|---|---|---|---|
| \( = \emptyset \) | nef | nef | semiample |
| \( \neq X \) | pseudo-effective | weakly positive | effective | big |

**Remark 4.4.** See Section 8 and in particular Remark 8.1 for more details about non-AEA loci and their relationship with the augmented base loci.

Furthermore, as we mentioned earlier, Lesieutre [Les12] proved that there exist line bundles which are pseudo-effective but not weakly positive. In particular, \( \mathbb{B}_-(L) \) is not necessarily closed.

**Proposition 4.5.** A line bundle \( L \) is almost nef if and only if it is pseudo-effective.

**Proof.** One implication is obvious by (4.2.1). The other implication follows from [BDPP13], as if \( L \) is not pseudo-effective then there exists a reduced irreducible curve \( C \subseteq X \), such that \( c_1(L) \cdot C < 0 \) and \( C \) moves in a family covering all \( X \), so \( \mathbb{T}^0(L) \) cannot be contained in a countable union of proper (Zariski) closed subsets. \( \square \)

The following theorem will be proved in Section 7:

**Theorem 4.6.** A line bundle \( L \) is big if and only if it is AEA.

A recent result of Lehmann [Leh11] gives a characterization of the relationship between the non-AEA locus and the diminished base locus. We will use the following when describing all the relationships.

**Definition 4.7.** Let \( X \) be a smooth projective variety over \( \mathbb{C} \) and let \( D \) be a pseudo-effective \( \mathbb{R} \)-divisor on \( X \). Suppose that \( \phi : Y \rightarrow X \) is a proper birational map from a smooth variety \( Y \). The movable transform of \( L \) on \( Y \) is defined to be

\[
\phi_{\text{mov}}^{-1}(D) := \phi^* D - \sum_{E \phi - \text{exc}} \sigma_E(\phi^* D) \cdot E.
\]
For a pseudo-effective line bundle $L$ define $\phi^{-1}_{\text{mov}}(L)$ as the line bundle associated to $\phi^{-1}_{\text{mov}}(c_1(L))$ on $Y$.

Note that the movable transform is not linear and is only defined for pseudo-effective divisors.

**Remark 4.8.** In the above, $\sigma_E$ is the asymptotic multiplicity function introduced by Nakayama [Nak04, Section III.1]. If $X$ is a smooth projective variety, $L$ a pseudo-effective $\mathbb{R}$-divisor, $E$ a prime divisor on $X$, then

$$\sigma_E(L) := \lim_{\epsilon \to 0^+} \inf \{ \text{mult}_E L' \mid L' \geq 0 \text{ and } L' \sim L + \epsilon A \},$$

where $A$ is an arbitrary but fixed ample divisor.

**Remark 4.9.** Following [Leh11, Definition 1.2], we call an irreducible curve $C$ on $X$ to be a mov$^1$-curve, if it deforms to cover a codimension one subset of $X$.

**Theorem 4.10** [Leh11]. Let $X$ be a smooth projective variety over $\mathbb{C}$ and $D$ a pseudo-effective $\mathbb{R}$-divisor. $D$ is not movable if and only if there is a mov$^1$- curve $C$ on $X$ and a proper birational morphism $\phi : Y \to X$ from a smooth variety $Y$ such that

$$\phi^{-1}_{\text{mov}}(D) \cdot \bar{C} < 0,$$

where $\bar{C}$ is the strict transform of a generic deformation of $C$.

The following reformulation is easy to see.

**Theorem 4.11.** Let $X$ be a smooth projective variety and $D$ a pseudo-effective $\mathbb{R}$-divisor. Suppose that $V$ is an irreducible subvariety of $X$ contained in $\mathbb{B}_-(L)$ and let $\psi : X' \to X$ be a smooth birational model resolving the ideal sheaf of $V$. Then there is a birational morphism $\phi : Y \to X'$ from a smooth variety $Y$ and an irreducible curve $\bar{C}$ on $Y$ such that $\phi^{-1}_{\text{mov}}(\psi^*D) \cdot \bar{C} < 0$ and $\psi \circ \phi(\bar{C})$ deforms to cover $V$.

**Remark 4.12.** Let $L$ be any line bundle on $X$ smooth projective, then

$$\mathbb{B}_-(L) = \bigcup_{f : Y \to X} f(\mathbb{T}^0(f^{-1}_{\text{mov}}(L))).$$

There are several examples for which the loci $\mathbb{T}^0(L)$ and $\mathbb{B}_-(L)$ do not coincide, and in some cases the difference is divisorial.

**Question 4.13.** Is it true that $\mathbb{B}_-(L)$ is contained in a proper closed subset of $X$ (i.e., $L$ is weakly positive) if and only if the same holds for $\mathbb{T}^0(L)$?

**Remark 4.14.** In [BDPP13, Question 7.5] the authors ask if for a vector bundle $E$, $\mathbb{B}_-(\mathcal{O}_E(1))$ doesn’t dominate $X$ if and only if neither does $\mathbb{T}^0(\mathcal{O}_E(1))$.

5. **RESTRICTED BASE LOCI FOR VECTOR BUNDLES**

Here we explore the connections between the positivity properties of a vector bundle and the associated asymptotic base loci. We will start recalling some classical definitions for vector bundles. Note that these definitions do sometimes appear slightly differently in the literature, but we will try to follow and indicate specific selected references each time.

**Definition 5.1.** Let $E$ be a vector bundle on a smooth projective variety $X$, $\pi : \mathbb{P}(E) \to X$ the projective bundle of one dimensional quotients of $E$, and $\mathcal{O}_{\mathbb{P}(E)}(1)$ the universal quotient of $\pi^* E$ on $\mathbb{P}(E)$. We say that $E$ is
(5.1.1) nef if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle, i.e., if $\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1)) = \emptyset$;
(5.1.2) almost nef if $\pi(T^0(\mathcal{O}_{\mathbb{P}(E)}(1)))$ is contained in a countable union of proper closed subsets of $X$ (cf. [BDPP13]);
(5.1.3) pseudo-effective if $\mathbb{B}_-(E) \neq X$ (cf. [BDPP13]);
(5.1.4) weakly positive if $\mathbb{B}_-(E) \neq X$ (cf. [Vie83]);

**Proposition 5.2.** A vector bundle $E$ is nef if and only if $\mathbb{B}_-(E) = \emptyset$.

*Proof.* This follows directly from Proposition 3.1. \qed

**Proposition 5.3.** A vector bundle $E$ is pseudo-effective if and only if $\mathbb{B}_-(E) = \emptyset$.

*Proof.* This follows immediately from Proposition 3.1. \qed

**Remark 5.4.** The proposition above is the same as [BDPP13, Proposition 7.2]. Observe that the locus $L_{\text{nonnef}}$ in [BDPP13] is what we call $\mathbb{B}_-(L)$ here (we are in the smooth projective case).

The following proposition is immediate.

**Proposition 5.5.** A vector bundle $E$ is almost nef if and only if there exists a countable union $T = \bigcup_{i \in \mathbb{N}} T_i$ of proper closed subsets of $X$, such that for any curve $C \subseteq X$ not contained in $T$ the restriction $E|_C$ is a nef vector bundle.

**Remark 5.6.** It follows from the definitions and propositions above that $E$ weakly positive $\Rightarrow$ $E$ pseudo-effective $\Rightarrow$ $E$ almost nef.

We have seen that the first implication is not an equivalence, while it is an open question whether, for vector bundles, being almost nef is equivalent to being pseudo-effective, as in the line bundle case cf. [BDPP13, Question 7.5].

**Question 5.7.** Does $E$ being almost nef imply that $E$ is pseudo-effective?

If $E$ is almost nef, then the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is almost nef, hence pseudo-effective. In order to have that $E$ is pseudo-effective, one needs to show that $\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))$ does not dominate $X$.

6. AUGMENTED BASE LOCI FOR VECTOR BUNDLES

**Definition 6.1.** Let $E$ be a vector bundle on a smooth projective variety $X$, $\pi: \mathbb{P}(E) \to X$ the projective bundle of one dimensional quotients of $E$, and $\mathcal{O}_{\mathbb{P}(E)}(1)$ the universal quotient of $\pi^*E$ on $\mathbb{P}(E)$.

We say that

(6.1.1) $E$ is ample if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample on $\mathbb{P}(E)$;
(6.1.2) $E$ is $L$-big if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big on $\mathbb{P}(E)$; and
(6.1.3) $E$ is $V$-big (or Viehweg-big) if there exists an ample line bundle $A$ and a positive integer $c > 0$ such that $\text{Sym}^c E \otimes A^{-1}$ is weakly positive, i.e., such that $\mathbb{B}_-(\text{Sym}^c E \otimes A^{-1}) \subsetneq X$ (cf. (5.1.4)).

**Proposition 6.2.** A vector bundle $E$ is ample if and only if $\mathbb{B}_+(E) = \emptyset$.

*Proof.* Follows directly from Proposition 3.2. \qed
Remark 6.3. It is well-known that a line bundle $L$ is big if and only if $\mathbb{B}_+(L) \neq X$, equivalently, if there exist $A$ ample and a positive integer $c > 0$ such that $L^\otimes c \otimes A^{-1}$ is pseudo-effective. Next we will prove that the same equivalences hold for V-bigness for vector bundles of arbitrary rank.

Theorem 6.4. Let $E$ be a vector bundle on a smooth projective variety $X$. Then the following properties are equivalent:

\begin{enumerate}[(6.4.1)]
\item $E$ is V-big.
\item There exist an ample line bundle $A$ and a positive integer $c > 0$ such that $\text{Sym}^c E \otimes A^{-1}$ is pseudo-effective.
\item $\mathbb{B}_+(E) \neq X$.
\end{enumerate}

Proof. The implication (6.4.1) $\Rightarrow$ (6.4.2) is clear; let us consider (6.4.2) $\Rightarrow$ (6.4.3). Suppose there exist $A$ ample and $c > 0$ such that $\text{Sym}^c E \otimes A^{-1}$ is pseudo-effective, i.e.,

$$\mathbb{B}_-(\text{Sym}^c E \otimes A^{-1}) \neq X.$$ 

Then

$$\bigcup_{q > 0} \mathbb{B}(E - (1/c)A + (1/q)A) = \bigcup_{q > 0} \mathbb{B}(E - \frac{q-c}{qc}A) \neq X.$$ 

Now for $q \gg 0$ and $\frac{q-c}{qc} > \frac{1}{2c}$, one has

$$\mathbb{B}_+(E) \subseteq \mathbb{B}(E - \frac{1}{2c}A) \subseteq \mathbb{B}(E - \frac{q-c}{qc}A) \subsetneq X,$$

hence the validity of the desired implication.

Next we verify that (6.4.3) $\Rightarrow$ (6.4.1). If $E$ satisfies $\mathbb{B}_+(E) \neq X$, then there exists $q > 0$ such that $\mathbb{B}(E - \frac{q}{q}A) \subsetneq X$ is a closed proper subset. Consequently,

$$\mathbb{B}_-(\text{Sym}^q E \otimes A^{-1}) \subsetneq \mathbb{B}(\text{Sym}^q E \otimes A^{-1}) \subsetneq X.$$

Corollary 6.5. If $E$ is V-big, then it is L-big as well.

Proof. Theorem 6.4 yields $\mathbb{B}_+(E) \neq X$, therefore $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1)) \neq \mathbb{P}(E)$ via Proposition 3.2. As a consequence $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big on $\mathbb{P}(E)$, equivalently, $E$ is L-big.

Remark 6.6. L-big vector bundles are not necessarily V-big, as the example of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$ shows (see [Jab07, p.24]).

The key difference between V-big and L-big vector bundles is that being L-big means that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample with respect to an open set $V \subseteq \mathbb{P}$ whereas $E$ is V-big if we can take $V$ to be of the form $V = \pi^{-1}(U), U \subseteq X$ open.

Remark 6.7. A vector bundle $E$ on a variety $X$ satisfying $\mathbb{B}_+(E) \neq X$ is also called ample with respect to an open subset (cf. [Jab07, Chapter 3]).

Remark 6.8. In the case where $E = \Omega_X$ is the cotangent sheaf of a variety $X$, the definitions and Proposition 3.2 imply the following inclusion $\mathbb{B}_+(\Omega_X) \supseteq DS(X, T_X)$, where $DS(X, T_X)$ is the Demailly-Semple locus.

The work of Diverio and Rousseau [DRT13] therefore provides examples of complex projective varieties of general type $X$ where $\Omega_X$ is a semistable L-big vector bundle with a big determinant, which is nevertheless not V-big.
The notion of almost everywhere ampleness was first defined by Miyaoka in the context of his work on vector bundles on surfaces; the definition goes through in all dimensions verbatim.

**Definition 7.1** [Miy83]. Let \( X \) be a smooth projective variety, \( E \) a rank \( r \) vector bundle on \( X \). Consider the projectivized bundle \( \mathbb{P} = \mathbb{P}(E) \) with projection morphism \( \pi : \mathbb{P} \to X \) and tautological bundle \( \mathcal{O}_{\mathbb{P}}(1) \). We say that \( E \) is almost everywhere ample (AEA for short), if there exists an ample line bundle \( A \) on \( X \), a Zariski closed subset \( T \subset \mathbb{P} \), whose projection \( \pi(T) \) onto \( X \) satisfies \( \pi(T) \neq X \), and a positive number \( \varepsilon > 0 \) such that

\[
c_1(\mathcal{O}_{\mathbb{P}}(1)) \cdot C \geq \varepsilon \cdot \pi^*(c_1(A)) \cdot C
\]

for all curves \( C \subset \mathbb{P} \) that are not contained in \( T \).

For line bundles, this notion coincides with bigness:

**Proposition 7.2.** For a line bundle \( L \) on a smooth projective variety \( X \), the following are equivalent:

1. \( L \) is AEA, i.e., there is an ample line bundle \( A \) on \( X \), a number \( \varepsilon > 0 \), and a proper Zariski closed subset \( T \subset \mathbb{P} \) such that

\[
c_1(L) \cdot C \geq \varepsilon \cdot c_1(A) \cdot C
\]

for all curves \( C \subset \mathbb{P} \) not contained in \( T \).

2. For every ample line bundle \( A \) on \( X \), there is an \( \varepsilon > 0 \) and a proper Zariski closed subset \( T \subset X \) such that

\[
c_1(L) \cdot C \geq \varepsilon \cdot c_1(A) \cdot C
\]

for all curves \( C \subset X \) not contained in \( T \).

3. \( L \) is big.

**Proof.** Assume (7.2.3), and let \( A \) be any ample line bundle. Then, by Kodaira’s lemma, there is a positive integer \( m \) such that we can write

\[
mc_1(L) = c_1(A) + F,
\]

where \( F \) is an effective divisor. Taking \( T \) to be the support of \( F \), it follows for every curve \( C \subset X \) not contained in \( T \) that

\[
mc_1(L) \cdot C = c_1(A) \cdot C + F \cdot C \geq c_1(A) \cdot C
\]

and this implies (2) with \( \varepsilon := 1/m \).

Obviously (2) implies (7.2.1), so let us assume condition (7.2.1) and show that it implies (7.2.3). A curve \( C \subset X \) such that \( c_1(L) \cdot C < \varepsilon \cdot c_1(A) \cdot C \) cannot be a movable curve (in the sense of [Laz04, Sect. 11.4.C]), since these cover all of \( X \) (by [Laz04, Lemma 11.4.18]), whereas \( T \neq X \). So \( L \) must have positive intersection with all movable curves. This implies that \( L \) lies in the dual of the cone of movable curves \( \text{Mov}(X) \), which by the theorem of Boucksom-Demailly-Paun-Peternell [BDPP13] is the pseudo-effective cone \( \text{Eff}(X) \). In order to conclude that \( L \) is big – and thus to complete the proof – it is therefore enough to show that \( L \) lies in the interior of that cone.

The assumption that \( L \) be AEA says that

\[
(c_1(L) - \varepsilon c_1(A))C \geq 0 \quad \text{for all } C \not\subset T.
\]
Therefore, writing \( c_1(L) - \varepsilon c_1(A) = (c_1(L) - \varepsilon c_1(A)) - \varepsilon c_1(A) \), we see that \( c_1(L) - \frac{\varepsilon}{2} c_1(A) \) is AEA as well. Moreover, every class in the open set
\[
c_1(L) - \varepsilon \frac{c_1(A)}{2} + \text{Amp}(X)
\]
is AEA, and \( c_1(L) \) lies in this open set. \( \square \)

**Proposition 7.3.** Let \( E \) be a vector bundle on a smooth projective variety \( X \), let \( \mathbb{P} = \mathbb{P}(E) \). If \( E \) is AEA on \( X \), then so is \( \mathcal{O}_\mathbb{P}(1) \) on \( \mathbb{P} \).

**Proof.** For \( E \) to be AEA means that for every ample line bundle \( A \) on \( X \), there exists a Zariski-closed subset \( T \subseteq X \), and \( \varepsilon > 0 \) such that
\[
c_1(\mathcal{O}_\mathbb{P}(1)) \cdot C \geq \varepsilon (c_1(A) \cdot C)
\]
for all irreducible curves not contained in \( T \).

Since \( \mathcal{O}_\mathbb{P}(1) \) is \( \pi \)-ample, the line bundle \( \pi^* A \otimes \mathcal{O}_\mathbb{P}(m) \) is ample for all \( m \geq m_0 \gg 0 \) by \[\text{Laz04, Proposition 1.7.10}\]. According to Proposition 7.2, \( \mathcal{O}_\mathbb{P}(1) \) is AEA if and only if it is big, therefore it suffices to prove the AEA property for \( \mathcal{O}_\mathbb{P}(k) \) for some large \( k \). This means in particular, that we are allowed to work with \( \mathbb{Q} \)-divisors as well.

Let \( m_0 \) be, as above, a positive integer such that \( \pi^* A \otimes \mathcal{O}_\mathbb{P}(m_0) \) is ample. We will prove that \( \mathcal{O}_\mathbb{P}(1) \) is AEA on \( \mathbb{P} \) with closed subset \( T \subseteq \mathbb{P} \), and a suitable \( \varepsilon' > 0 \). We need that
\[
c_1(\mathcal{O}_\mathbb{P}(1)) \cdot C \geq \varepsilon' (c_1(\pi^* A \otimes \mathcal{O}_\mathbb{P}(m_0)) \cdot C)
\]
or equivalently,
\[
c_1(\mathcal{O}_\mathbb{P}(1)) \cdot C \geq \frac{\varepsilon'}{1 - \varepsilon' m_0} (c_1(A) \cdot C)
\]
for all curves not contained in \( T \). By our assumption on \( E \), this holds whenever
\[
\varepsilon' < \frac{\varepsilon}{1 + \varepsilon m_0}.
\]
\( \square \)

**Corollary 7.4.** If \( E \) is AEA, then it is \( L \)-big.

**Proof.** Immediate from Proposition 7.2 and Proposition 7.3 \( \square \)

### 8. The Bad AEA Locus in the Line Bundle Case

Consider a line bundle \( L \), and fix an ample line bundle \( A \) and a number \( \varepsilon > 0 \). We defined the non-AEA locus of \( L \) with respect to \( A \) and \( \varepsilon \) as the subvariety
\[
T^A_\varepsilon = \text{closure}(\bigcup \{ C \mid C \text{ curve on } X \text{ with } c_1(L) \cdot C < \varepsilon \cdot c_1(A) \cdot C \})
\]
The AEA assumption on \( L \) simply means that there exists an \( \varepsilon > 0 \) such that \( T^A_\varepsilon \neq X \). For \( \varepsilon < \delta \) we have \( T^A_\varepsilon \subseteq T^A_\delta \), so that we can express the AEA condition equivalently as saying that the intersection \( \mathbb{T}(L) := \bigcap_{\varepsilon > 0} T^A_\varepsilon \) is not all of \( X \).

**Remark 8.1.** It is immediate that
\[
\mathbb{T}(L) \subseteq \mathbb{B}_+(L)
\]
In fact, by the noetherian property there are positive real numbers \( \varepsilon_0 \) and \( \delta_0 \) such that
\[
\mathbb{T}(L) = T^A_\varepsilon \forall \varepsilon \leq \varepsilon_0 \text{ and } \mathbb{B}_+(L) = \mathbb{B}(L - \delta A) \forall \delta \leq \delta_0.
\]
Now choose $\varepsilon < \min(\varepsilon_0, \delta_0)$, then
\[
T(L) = T_\varepsilon = \{ x \mid x \in C \text{ curve on } X \text{ s.t. } c_1(L) \cdot C < \varepsilon \cdot c_1(A) \cdot C \}.
\]
If $C$ is a curve such that $(L - \varepsilon A)C < 0$ then $C \subseteq B(L - \varepsilon A) = B_+(L)$ which is a closed set.

**Remark 8.2.** A line bundle $L$ is ample if and only if $T(L) = \emptyset$. In general the inclusion
\[
\{ x \mid x \in C \text{ curve on } X \text{ s.t. } c_1(L) \cdot C \leq 0 \} \subseteq T(L)
\]
is strict, as shown by a strictly nef non ample line bundle $L$, where the first set is empty but the second one is not. Examples of line bundles that are strictly nef (and even big) and non ample have been first given by Mumford (cf. [Har70]), and a complete description can be found in [Urb07].

**Remark 8.3.** A few words on the relationship between $T(L)$ and $B_\pm(L)$. We’ll show here that $T(L) \neq B_+(L)$ in general. A bit more precisely, we will try to understand the relationship of $T(L)$ to the augmented and restricted base loci of $L$ when $\dim X = 2$. Recall that
\[
B_-(L) = \bigcup_{m=1}^{\infty} B(L + \frac{1}{m}A)
\]
for any integral ample divisor $A$ on $X$.

Let $D$ be a big divisor on a smooth projective surface $X$ with Zariski decomposition $D = P_D + N_D$. Then [ELMNP06, Examples 1.11 and 1.17] tell us that
\[
B_+(\mathcal{O}_X(D)) = \text{Null}(P_D) = \bigcup C \subseteq X \text{ irreducible } \mid P_D \cdot C = 0, \quad B_-(\mathcal{O}_X(D)) = \text{Supp } N_D.
\]

**Example 8.4.** Here we present an example where $T(L) \neq B_\pm(L)$. Let $X$ be a surface that carries a big divisor $D$ and an irreducible curve $C \subseteq X$ satisfying $C \subseteq \text{Supp } N_D$ and $D \cdot C > 0$. Then $C \not\subseteq T(L)$, but $C \subseteq B_-(\mathcal{O}_X(D)) \subseteq B_+(\mathcal{O}_X(D))$. In this case we have
\[
T(\mathcal{O}_X(D)) \neq B_-(\mathcal{O}_X(D)), B_+(\mathcal{O}_X(D)).
\]

Surfaces carrying such $D$ and $C$ exist by [BF12]: Consider a K3 surface $X$, on which the Zariski chamber decomposition does not coincide with the Weyl chamber decomposition. The latter is by [BF12, Theorem 1] the case if and only if there are $(-2)$-curves on $X$ having intersection number 1. For a concrete example one can, as done in [BF12, Section 3], take a smooth quartic surface $X \subseteq \mathbb{P}^3$ that has a hyperplane section of the form $H = L_1 + L_2 + Q$, where $L_1$ and $L_2$ are lines and $Q$ is a smooth conic. Then the divisors of the form
\[
D = H + a_1 L_1 + a_2 L_2
\]
with $a_1 \geq 1$ and $a_2 \geq 1$ have $L_1$ and $L_2$ in the support of the negative part of their Zariski decomposition, but one can find $a_1, a_2$ such that $D \cdot L_1 > 0$ and $D \cdot L_2 < 0$ (for instance $a_1 = 2, a_2 = 4$). (In the notation of [BF12], $D$ lies in the Zariski chamber $Z_{\{L_1, L_2\}}$, but in the Weyl chamber $W_{\{L_2\}}$.)

**Example 8.5.** In general $T(\mathcal{O}_X(D))$ is not contained in $B_-(\mathcal{O}_X(D))$ either, where $L = \mathcal{O}_X(D)$ for a suitable integral Cartier divisor $D$. To see this, take a surface where all negative curves
have self-intersection $-1$. Then the intersection form of the negative part of the Zariski decomposition of any big divisor is $-\text{Id}$, in other words, no two curves in it can intersect. Consequently,

$$T(\mathcal{O}_X(D)) = B_{+}(\mathcal{O}_X(D)).$$

This can be seen as follows: let $C \subseteq B_{+}(\mathcal{O}_X(D)) = \text{Null}(P_D)$ be an irreducible curve. Since $P_D$ is big and nef, the intersection form on $P_D^\perp$ is negative definite, which under the given circumstances means that $(C \cdot C') = 0$ for every irreducible curve $C \neq C'$ coming up in $N_D$. Therefore

$$D \cdot C = P_D \cdot C + N_D \cdot C = 0 + (\leq 0) = (\leq 0).$$

Consequently, $C \subseteq T(\mathcal{O}_X(D))$.

Take a non-stable (in the sense if [ELMNP06, Definition 1.22] big divisor $D$ on $X$, then $B_{-}(\mathcal{O}_X(D)) \subsetneq B_{+}(\mathcal{O}_X(D)) = T(\mathcal{O}_X(D))$.

The following lemma is well-known to experts working in the area, but for lack of an adequate reference we include it here.

**Lemma 8.6.** Let $D$ be a big divisor on a smooth projective surface, $C \subseteq X$ irreducible curve, $D \cdot C = 0$. Then $(C^2) < 0$.

**Proof.** Let $D = P_D + N_D$ denote the Zariski decomposition of $D$. If $C \subseteq \text{Supp } N_D$, then it must have negative self-intersection, since the intersection form on $N_D$ is negative definite. Assume $C$ is not in $N_D$. Then

$$0 = D \cdot C = P_D \cdot C + N_D \cdot C = (\geq 0) + (\geq 0),$$

since $P_D$ is nef, $C$ is effective, $N_D$ is effective with no common components with $C$. This can only happen if

$$P_D \cdot C = N_D \cdot C = 0.$$

Therefore, $C$ is orthogonal to the big and nef divisor $P_D$, hence we must have $(C^2) < 0$. □

9. **V-big vs. AEA**

Let $X$ be a smooth projective variety and $E$ a vector bundle on $X$. There exist two non-equivalent definitions for bigness in the literature: V-big and L-big vector bundles. It is known that V-bigness implies L-bigness and that the converse does not hold if $\text{rk } E \geq 2$ (cf. Remark 6.8).

Throughout this section we will point out some differences (for example a different Kodaira’s lemma) between L-big and V-big vector bundles, and compare V-bigness and almost everywhere amplenessq. In particular we will show that these positivity properties coincide for nef vector bundles. V-big vector bundles are also called *ample with respect to an open set* [Jab07, Chapter 3].

We have seen that if $E$ is V-big, then $E$ is also AEA cf. Remark 8.1.

**Question 9.1.** Does $E$ being AEA imply that $E$ is V-big?

We will see that this is the case if $E$ is nef.

**Remark 9.2.** As pointed out in [Jab07, Lemma 3.44], a vector bundle on a projective curve is ample with respect to an open set exactly if it is ample.

Next we will show that a strong form of Kodaira’s lemma is valid for vector bundles that are ample with respect to an open set.
Lemma 9.3 (Kodaira’s lemma for vector bundles). Let $X$ be a smooth projective variety, $E$ a vector bundle, and $A$ an ample line bundle on $X$. Then the following are equivalent.

1. $E$ is ample with respect to an open subset.
2. $\text{Sym}^m E$ contains an ample vector bundle of the same rank for some $m > 0$.
3. There exists $m > 0$ and an injective morphism

$$\bigoplus \text{rk Sym}^m E A \hookrightarrow \text{Sym}^m E,$$

which is an isomorphism over an open subset.

Proof. The equivalence of (9.3.1) and (9.3.3) is the content of [Jab07, Lemma 3.42]; (9.3.1) obviously implies (9.3.2), and (9.3.2) implies (9.3.1) holds if being ample with respect to an open set is scale-invariant. □

There is a useful characterization of ampleness with respect to an open subset in terms of $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Lemma 9.4. (cf. Proposition 9.2) With notation as above, $E$ is ample with respect to the dense open set $U \subseteq X$ precisely if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample with respect to $\pi^{-1}(U) \subseteq \mathbb{P}(E)$.

Here we have the following weaker version of the Kodaira lemma.

Lemma 9.5. Let $E$ be a vector bundle using the notation above.

1. Assume that $H^0(X, \text{Sym}^m E) \neq 0$ for some $m > 0$. Then for any ample line bundle $A$ on $X$ and any $k > 0$, $\text{Sym}^k E \otimes A$ is $L$-big.
2. Assume that for some $m > 0$ and some $x \in X$ the vector bundle $\text{Sym}^m E$ is generated at $x$ by its global sections $H^0(X, \text{Sym}^m E)$. Then for any ample line bundle $A$ on $X$ and any $k > 0$, $\text{Sym}^k E \otimes A$ is $V$-big.
3. Conversely, assume that $E$ is $L$-big. Then for any line bundle $L$ on $X$,

$$H^0(X, \text{Sym}^m E \otimes L) \neq 0$$

for all $m \gg 0$.
4. Assume that $E$ is $V$-big. Then for any line bundle $L$ on $X$, $\text{Sym}^m E \otimes L$ is generically generated by its global sections for all $m \gg 0$.

Proof. To prove (9.5.1), assume that $H^0(X, \text{Sym}^m E) \neq 0$ for some $m > 0$. This means that $H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \simeq H^0(X, \text{Sym}^m E) \neq 0$, hence $\mathcal{O}_{\mathbb{P}(E)}(1)$ is $\mathbb{Q}$-effective. Then

$$\mathcal{O}_{\mathbb{P}(E \otimes A)}(1) \simeq \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* A.$$

By [Laz04, Proposition 1.7.10], the $\mathbb{Q}$-divisor $ac_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \pi^* c_1(L)$ is ample for $0 < a \ll 1$. This implies that

$$c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \pi^* c_1(L) = (1-a)c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + (ac_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \pi^* c_1(L))$$

can be written as the sum of an effective and an ample divisor, hence it is big.

To prove (9.5.2), assume that for some $m > 0$ and some $x \in X$ the vector bundle $\text{Sym}^m E$ is generated at $x$ by its global sections. Then $\mathbb{B}(E) \neq X$ and hence $\mathbb{B}_-(E) \subset \mathbb{B}(E) \neq X$. Therefore $E$ is weakly positive and for all $H$ ample

$$\mathbb{B}_+(E + (1/m)H) = \bigcap \mathbb{B}(E + (1/m)H - (1/n)H) \subseteq \mathbb{B}(E) \neq X.$$
(9.5) is a reformulation of the Kodaira lemma on $\mathbb{P}(E)$ (see [Laz04, Lemma 2.2.6]).

Finally, for (9.5), assume that $E$ is $V$-big, $H$ an ample line bundle, and $L$ any line bundle on $X$. Using notations of the first chapters we have that for $m$ sufficiently large $H - (1/m)L$ is ample, and as $E$ is $V$-big we have

$$\mathbb{B}(\text{Sym}^k E + L) = \mathbb{B}(\text{Sym}^k E + (1/m)L) \subseteq \mathbb{B}(\text{Sym}^k E - (H - (1/m)L)) \neq X$$

for $k$ big enough.

**Remark 9.6.** A line bundle is $(L)$-big if and only if it is ample with respect to an open set. We have seen in 6.6 that there exist simple vector bundles that are $L$-big, but not $V$-big, and hence not ample with respect to an open set.

**Proposition 9.7.** Let $X$ be a smooth projective variety, $E$ a vector bundle on $X$ containing an ample vector bundle $A$ of the same rank. Then

(9.7.1) $E$ is almost everywhere ample with respect to the closed subset $T = \text{Supp}(E/A)$.

(9.7.2) $E$ is ample with respect to the dense open set $X \setminus T$.

**Proof.** Let $H$ be an ample line bundle on $X$, and $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ and $C \subseteq \mathbb{P}(E)$ an irreducible curve. If $C$ is contained in a fibre that maps to a point away from $T$, then

$$L \cdot C = \pi^* A \cdot C ,$$

which is good. If $C$ is contained in a fibre mapping to $T$, then we do not care about the intersections numbers at all.

We may now assume that $C$ is not contained in a fibre of $\pi$. Let $B \overset{\text{def}}{=} \pi(C) \subseteq X$.

By restricting everything to $B$ via base change along $B \hookrightarrow X$, we may assume that $X$ is a curve, and $\pi|_C : C \to X$ is a dominant morphism. Consider the short exact sequence of sheaves

$$0 \to A \to E \to \mathcal{O} \to 0 ,$$

where $\mathcal{O} \overset{\text{def}}{=} E/A$ is a torsion sheaf on $X$ with support $T$.

The vector bundle map $\pi^* E \to \mathcal{O}_{\mathbb{P}(E)}(1)$ is surjective, hence $\pi^* A$ maps surjectively onto a sub-line-bundle $\mathcal{B}$ of $\mathcal{O}_{\mathbb{P}(E)}(1)$. Since $\pi^* A$ is ample, so is the quotient $\mathcal{B}$. □

**Remark 9.8.** In the case of a line bundle $L$, the largest open subset over which the evaluation map

$$H^0(X, L) \otimes \mathcal{O}_X \to L$$

is surjective is the complement of the stable base locus $X \setminus \mathbb{B}(L)$.

An $L$-big vector bundle $E$ is $V$-big if $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1))$ is contained in a union of fibres over a proper Zariski closed subset of $X$.

When $L$ is a line bundle on a surface, then it is immediate from the intersection-theoretic characterizations that

$$\mathbb{B}_+(L) = \mathbb{T}(P_D) .$$

Since we would need something along these lines on $\mathbb{P}(E)$, which in interesting cases has dimension at least three, the above observations can only serve as a pointer what kind of statements we would like to prove in higher dimensions.

**Lemma 9.9.** Let $X$ be an irreducible projective variety, $L$ a nef line bundle on $X$. Then

$$\mathbb{B}_+(L) = \mathbb{T}(L) .$$
Proof. We have already observed in Remark 8.1 that $T(L) \subseteq B_+(L)$ in general. To prove the other direction in the case where $L$ is nef, we use the main result of [Nak00] (see also [ELMNP09 Corollary 5.6]):

$$B_+(L) = \bigcup_{V \subseteq X, (L^dV) = 0} V. $$

We have to show that if $V \subseteq X$ is an irreducible subvariety for which $L|_V$ is not big, then $V$ is contained in $T(L)$. We will show that $V$ is covered by curves $C$ satisfying $L \cdot C < \varepsilon A \cdot C$ for $\varepsilon$ small enough.

Assume first that $V$ is smooth, and apply [BDPP13] (the pseudo-effective cone is the dual of the cone of moving curves). Then $L|_V$ not big implies that it is not in the interior of the pseudo-effective cone, hence there must exist a real 1-cycle $0 \neq C \in N_1(V)$ limiting moving curves $C_n \subseteq V$ with $L|_V \cdot C = 0$. As $A|_V$ is ample on $V$, then $A|_V \cdot C > 0$, so $\lim \frac{L|_V \cdot C_n}{A|_V \cdot C_n} = 0$. Thus, for $n$ sufficiently large, $L \cdot C_n < \varepsilon A \cdot C_n$. The class of $C_n$ however covers $V$, which implies $V \subseteq T(L)$.

If $V$ is not smooth, then let $\mu : V' \to V \subseteq X$ be a resolution of singularities. Since $L|_V$ was pseudo-effective and not big to begin with, the same applies to $\mu^*(L|_V)$. Using the argument as above, there exist moving curves $C_n$ on $V'$ such that their limit is a non zero 1-cycle $C \in N_1(V')$ such that $\mu^*(L|_V) \cdot C = 0$, now let $A$ be an ample divisor on $X$, then $\mu^*(A)$ is big on $V'$, so $\mu^*(A) \cdot C > 0$. Then as above $\lim \frac{\mu^*(L|_V) \cdot C_n}{\mu^*(A) \cdot C_n} = 0$. Thus, for $n$ sufficiently large, $\mu^*(L|_V) \cdot C_n < \varepsilon \mu^*(A) \cdot C_n$, hence by projection formula $L \cdot \mu_+(C_n) < \varepsilon A \cdot \mu_+(C_n)$. And the class of $\mu_+(C_n)$ covers $V$, which implies $V \subseteq T(L)$. \qed

Proposition 9.10. Let $E$ be a nef vector bundle on an irreducible projective variety $X$. Then $E$ is AEA if and only if it is $V$-big.

Proof. Both AEA and $V$-bigness imply that $L \overset{\text{def}}{=} \mathcal{O}_X(1)$ is a big and nef line bundle. By the previous lemma, $B_+(L) = T(L)$, hence

$qE$ is $V$-big $\iff \pi(B_+(L)) \neq X \iff \pi(T(L)) \neq X \iff E$ is AEA. \qed

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