Singularities in cosmologies with interacting fluids

Spiros Cotsakis* and Georgia Kittou†

School of Natural Sciences
University of the Aegean
Karlovassi 83 200, Samos, Greece

February 8, 2012

Abstract

We study the dynamics near finite-time singularities of flat isotropic universes filled with two interacting but otherwise arbitrary perfect fluids. The overall dynamical picture reveals a variety of asymptotic solutions valid locally around the spacetime singularity. We find the attractor of all solutions with standard decay, and for ‘phantom’ matter asymptotically at early times. We give a number of special asymptotic solutions describing universes collapsing to zero size and others ending at a big rip singularity. We also find a very complicated singularity corresponding with a logarithmic branch point that resembles of cyclic universe, and give an asymptotic local series representation of the general solution in the neighborhood of infinity.

1 Introduction

Fluid matter with all its ramifications has always played a key role in discussions of cosmological singularities. In studies of the genericity of quasi-isotropic solutions [1], in studies of the structure and nature of the singularity and energy conditions [2], in the construction of the general isotropic singularity [3], in the singularity problem of inflationary cosmology [4], or in more recent attempts towards formulating the cosmological singularity in string and brane theory [5], one sees different manifestations of the ‘nature abhors a vacuum’ principle, i.e., using suitable ‘fluids’ to model the universe in its most extreme states.

*email: skot@aegean.gr
†email: gkittou@aegean.gr
In recent years there have been an increasing number of works devoted to analyzing diverse problems in situations involving more than one cosmological fluids that show a mutual interaction and the associated exchange of energy. Studies have been focused on a number of issues, for example a covariant description of the interaction [6], scaling solutions [7], perturbations [8], duality and symmetry transformations to obtain physically relevant solutions [9], detailed solutions with energy transfer [10, 11], and of course on the important current issues of cosmic acceleration, dark matter and dark energy [12].

It is therefore important to understand the nature of finite-time singularities that may develop in cosmological models with interacting fluids. Such an understanding will complement current studies of such models which focus on other issues and may also provide a demarkation of the range of dynamical possibilities of these models. As this issue has not been pursued in a systematic way so far, it is the purpose of this paper to carry out the first steps in providing the asymptotic properties of the solutions in the neighborhood of a finite-time singularity in cosmological models with two interacting fluids. In particular, we shall focus exclusively on a flat FRW model containing two such fluids and construct asymptotic solutions which have the property to blow up at a finite-time singularity.

The asymptotic analysis of the solutions of the dynamical system as the finite-time singularity is approached is carried out here using the method of asymptotic splittings, cf. [13, 14]. In this method, the vector field that defines the system is asymptotically decomposed in such a way as to reveal its most important dominant features on approach to the singularity. This leads to a detailed construction of all possible local asymptotic solutions valid in the neighborhood of the finite-time singularity. These provide in turn a most accurate picture of all possible dominant features that the field possesses as it is driven to a blow up. For previous applications of this asymptotic technique to cosmological singularities, we refer to [13, 15].

The plan of this paper is as follows. In the next Section, we write the basic equations describing a flat FRW universe filled with two interacting fluids as a dynamical system and we are lead to the asymptotic field decompositions that will yield all possible dominant features as the singularity is approached. In the following Sections, we present an analysis of the asymptotic properties of the system generally divided into power-law, oscillatory and complete solutions. We conclude with a discussion in the last Section, pointing into more general aspects of this problem.
2 Asymptotic splittings

We consider a flat FRW universe with scale factor \( a(t) \) containing two fluids with equations of state

\[
p_1 = (\Gamma - 1)\rho_1, \quad p_2 = (\gamma - 1)\rho_2. \tag{2.1}
\]

Setting \( H = \dot{a}/a \) for the Hubble expansion rate, the evolution of this system is governed by the equations

\[
3H^2 = \rho_1 + \rho_2 \\
\dot{\rho}_1 + 3H\Gamma \rho_1 = -\beta H^m \rho_1^\lambda + \alpha H^n \rho_2^\mu \tag{2.2} \\
\dot{\rho}_2 + 3H\gamma \rho_2 = \beta H^m \rho_1^\lambda - \alpha H^n \rho_2^\mu,
\]

where the exponents \( m, n, \lambda, \mu \) are rational numbers indicating an interaction between the two fluids that depends nonlinearly on their densities and the Hubble rate. Depending on the signs of the constants \( \alpha, \beta \), the fluids may ‘decay’ to each other exchanging energy.

In this paper, we elaborate on the simplest case where the exponents \( m, n, \lambda, \mu \) are all set equal to one. This case corresponds to the problem studied in [10, 11], and it will be interesting to compare certain of our results with theirs. However, our approach is completely different, the focus here being exclusively on the asymptotic approach to the singularities of these models. Setting all the exponents \( m, n, \lambda, \mu = 1 \) and renaming \( x = H \), the system (2.2) becomes equivalent to the dynamical system

\[
\dot{x} = y \\
\dot{y} = -Ax y - Bx^3, \tag{2.3}
\]

where \( A = \alpha + \beta + 3\gamma + 3\Gamma, \quad B = 3(\alpha\Gamma + \beta\gamma + 3\Gamma\gamma)/2. \)

This defines the vector field

\[
f(x, y) = (y, -Ax y - Bx^3). \tag{2.5}
\]

The method of asymptotic splittings developed in [13, 14] scrutinizes all possible modes that the vector field (2.5) attains on approach to the finite-time singularity located at \( t = 0 \). These modes correspond to the different ways that (2.5) splits as \( t \to 0 \). For the case we consider, the possible

\footnote{by a solution with a finite-time singularity we mean one where there is a time at which at least one of its components diverges. We note that the usual dynamical systems analysis through linearization etc is not relevant here, for in that one deals with equilibria, not singularities.}
asymptotic modes are given by the following three distinct decompositions:

\[ f^{(1)} = (y, -Ax - Bx^3), \quad \text{(all terms dominant case)} \]  
\[ f^{(2)} = (y, -Ax), \]  
\[ f^{(3)} = (y, -Bx^3). \]

Each one of the three decompositions (2.6)-(2.8) into which the vector field (2.5) splits, contains different dominant balances that describe the precise ways into which the dynamical system is driven asymptotically as we approach the singularity. These balances are in general non unique. By further analyzing the balances of each particular decomposition, we are led to the construction of a number of possible asymptotic formal series valid locally in the neighborhood of the singularity, or in the neighborhood of infinity (the latter correspond to the behaviour of the system away from singularities, describing all possible complete solutions). From a close examination of the form of these asymptotic representations, we can obtain valuable information about the genericity of the asymptotic solutions, the stability/attractor properties of dominating solutions in the developments, and other precise information most valuable to create a detailed shape of the asymptotic evolution.

We shall briefly comment on possible more general forms derived from the system (2.2) in the last Section.

3 Power-law solutions, the \( \delta \to 0 \) attractor

We start here our analysis of the possible asymptotic solutions towards the finite time singularity of the system (2.2) by searching first for power-law type solutions. The first such solution we give in this Section is the simplest and perhaps the most important of them. Let us take the second decomposition

\[ f^{(2)} = (y, -Ax), \]

and look for the possible dominant balances, by substituting in the system \((\dot{x}, \dot{y})(t) = f^{(2)}\) the forms

\[ x(t) = \theta t^p, \quad y(t) = \xi t^q, \]

where the coefficients \( \Xi \equiv (\theta, \xi) \in \mathbb{C} \), while the exponents \( p \equiv (p, q) \in \mathbb{Q} \). This leads to the unique balance

\[ B^{(2)}_1 = [\Xi, p] = [(2/A, -2/A), (-1, -2)], \quad A \neq 0. \]  

The candidate subdominant part \( f^{(2, \text{sub})} = (0, -Bx^3) \) of the vector field \( f^{(2)} \) satisfies

\[ \frac{f^{(2, \text{sub})}(\Xi tp)}{tp^{p-1}} \equiv \left(0, \frac{-8Bt^{3p}}{A^p t^{q-1}}\right) = \left(0, \frac{-8\delta}{A}\right). \]
Here we have utilized the Barrow-Clifton parameter \([10]\)

\[ \delta \equiv \frac{B}{A^2}, \quad (3.5) \]

that will play an important role in the following. There is no way for the vector field \(f^{(2,\text{sub})}\) to be subdominant asymptotically in the sense that

\[ \frac{f^{(2,\text{sub})}(t\mathbf{p})}{t^{p-1}} \to 0, \quad \text{as} \quad t \to 0, \quad (3.6) \]

unless we set

\[ \delta = 0. \quad (3.7) \]

Otherwise the decomposition \(f^{(2)}\) would not acceptable asymptotically. This means that in order to satisfy this constraint, the subdominant part has to be vanishing. We take in this case the subdominant exponent \(q\) to be equal to one, cf. \([13]\).

Next we calculate the Kovalevskaya matrix (\(K\)-matrix in short), given by

\[ K = Df^{(2)}(\Xi) - \text{diag}(p), \quad (3.8) \]

where \(Df^{(2)}(\Xi)\) is the Jacobian matrix of \(f^{(2)}\), at \(\Xi\), which in our case reads:

\[ K^{(2)} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}. \quad (3.9) \]

The next step is to calculate the \(K\)-exponents for this balance. These exponents are the eigenvalues of the \(K\) matrix and constitute its spectrum, \(\text{spec}(K^{(2)})\). The arbitrary constants of any (particular or general) solution first appear in those terms in the asymptotic solution series whose coefficients \(c_k\) have indices \(k = q\sigma\), where \(\sigma\) is a non-negative \(K\)-exponent. The number of non-negative \(K\)-exponents equals therefore the number of arbitrary constants that appear in the series expansions.

There is always the \(-1\) exponent that corresponds to an arbitrary constant, the position of the singularity (here at \(t = 0\) for notational convenience). If the balance \(\mathcal{B}_1^{(2)}\) is to correspond to a general solution, then it must possess a non-negative \(K\)-exponent (the second arbitrary constant is the position of the singularity). Here we find

\[ \text{spec}(K^{(2)}_1) = \{-1, 2\}, \quad (3.10) \]

so that \(\mathcal{B}_1\) indeed corresponds to a candidate general solution. Substituting the series expansions

\[ x = \sum_{j=0}^{\infty} c_{j1}t^{j-1}, \quad y = \sum_{j=0}^{\infty} c_{j2}t^{j-2}, \quad (3.11) \]
in the system \((2.2)\) and after some manipulations to determine the coefficients of the expansions recursively, we arrive at the following asymptotic solution around the singularity:

\[
x = \frac{2}{A} t^{-1} + c_{21} t - \frac{A}{10} c_{21}^2 t^3 \ldots, \tag{3.12}
\]

while the \(y\) expansion is obtained from the above by differentiation. Note the arbitrary constant \(c_{21}\) appearing in this expansion signifying that this representation corresponds to a general solution (we need two for this, the second is the arbitrary position of the singularity).

As a final test for admission of this solution, we use the Fredholm alternative to be satisfied by any admissible solution. This leads to the following compatibility condition for the positive eigenvalue 2 and an associated eigenvector, \(v_2 = (1, 1)\):

\[
v_2^\top \left( K - \frac{j}{s} I \right) c_j = 0, \tag{3.13}
\]

where \(I\) denotes the identity matrix, and we have to satisfy this at the \(j = 2\) level. This gives

\[
c_{21} = c_{22}, \tag{3.14}
\]

and this is indeed true as found previously in the recursive calculation. It follows from Eq. \((3.12)\) that all solutions are dominated by the \(x = H \sim \frac{2}{A} t^{-1}\) solution, that is the solution

\[
H \sim \frac{2}{A} t^{-1}, \quad \text{or} \quad a(t) \sim t^{-2/A}, \tag{3.15}
\]

is an attractor of all smoothly evolving solutions at early times, assuming the weight-homogeneous \(f^{(2)}\) decomposition asymptotically.

A comment about the results of [10] is in order. They find that at early times the attractor solution takes the form:

\[
a_{BC}(t) \sim t^{-2/\sqrt{A^2 - 8B}}, \quad \text{as} \quad t \to 0, \tag{3.16}
\]

whereas we find the form \((3.15)\). In terms of the parameter \(\delta\) defined in \((3.5)\), their solution \((3.16)\) is given by

\[
a_{BC}(t) \sim \left( t^{-2/|A|} \right)^{1/(1-8\delta)^{1/2}}, \quad \delta \in [0, 1/8], \tag{3.17}
\]

and we see that our solution \((3.15)\) corresponds to the \(\delta = 0\) member of the one-parameter family of \(\delta\)-solutions of the form \((3.17)\). To enable the comparison, we note that irrespectively of the sign of \(A\) we have that \(t^{-2/|A|} > 0\), and so the Barrow-Clifton family is asymptotic to our solution,

\[
a_{BC}(t) \to t^{-2/|A|}, \quad \text{as} \quad \delta \to 0, \tag{3.18}
\]

(in this case, the exponent of the \(a_{BC}(t)\) solution in \((3.17)\) tends to 1). This result means that our solution \((3.15)\) represents a limit function for the Barrow-Clifton family of \(\delta\)-solutions \((3.17)\).
Since for the validity of the $f^{(2)}$ decomposition asymptotically we were forced to take $\delta = 0$, we arrive at the interesting conclusion that the Barrow-Clift on solutions (3.15) are all dominated by the solution (3.15) in this case.

4 Phantom singularities

Let us move on to the asymptotic analysis of the decomposition $f^{(3)} = (y, -Bx^3)$. There are two possible balances here but only one is of interest for the power-law solutions of this Section (we analyze the second balance together with other oscillatory solutions in Section 6). Substituting in the system $(\dot{x}, \dot{y})(t) = f^{(3)}$ the forms (3.2), we find that this balance is given by

$$B_1^{(3)} = [\Xi, p] = \left(\pm \sqrt{2/(-B)}, \mp \sqrt{2/(-B)}, (-1, -2)\right), \quad B < 0,$$

(4.1)

(the two branches give analogous results as we shall see). The candidate subdominant part $f^{(3, \text{sub})} = (0, -Axy)$ of the vector field $f^{(3)}$ satisfies

$$\frac{f^{(3, \text{sub})}(\Xi t^p)}{t^{p-1}} \equiv (0, -A\theta\xi) = (0, 0),$$

(4.2)

i.e., it vanishes only when we set $A = 0, \theta, \xi \neq 0$. We note that the balance $B_1^{(3)}$ corresponds to the limit

$$\delta \to -\infty,$$

(4.3)

we expect that it refers to different parts of the $\delta$-family of solutions than previously. With the Kovalevskaya matrix of this balance having

$$\text{spec}(K_1^{(3)}) = (-1, 4),$$

(4.4)

we find after some manipulation that the series expansion corresponding to this case is given by the form

$$x = \pm \sqrt{\frac{-2}{B}} t^{-1} + c_{41} t^3 + \frac{B}{12} c_{41}^2 \sqrt{\frac{-2}{B}} t^7 \cdots,$$

(4.5)

while the $y$ expansion is obtained from the above by differentiation.

---

2 We will comment later on the $\delta = 1/8$ limit of the $\delta$-parametric family of solutions. For the moment we note that as it is expected from (3.17), as $\delta \to 1/8$ all these power-law solutions for small $t$ will tend to zero (except of course for possible particular exact solutions, those with a smaller number of arbitrary constants than the general solution) and hence are expected to lose their significance asymptotically (this is like taking the limit $\lim_{k \to +\infty} c^k = 0$, with $c \in (0, 1)$).

3 Indeed, this range of $\delta$ means that the fluid parameters $\Gamma, \gamma$ cannot be positive simultaneously (this is shown in detail in Ref. [10]), hence the title of this Section.
We note here that although the dominant term in this expansion is the same as in (3.12), the whole formal expansion is a different one. The arbitrary constant $c_{41}$ appearing in the series (4.5) signifies that this representation corresponds to a general solution. Indeed, this becomes true since the compatibility condition for the positive eigenvalue 4 (with an associated eigenvector say, $v_2 = (1, 3)$),

$$v_2^\top \left( K - \frac{j}{s} I \right) c_j = 0,$$

(4.6)

at the $j = 4$ level gives

$$3c_{41} = c_{42},$$

(4.7)

and this is true as it follows from the recursive calculation.

It follows from Eq. (4.5) that assuming the weight-homogeneous $f^{(3)}$ decomposition asymptotically, all solutions dominated by the balance $B^{(3)}_1$ (that is, those included in the family defined by (4.5)) are attracted on approach to the singularity by the asymptotic solution $x = H \sim 2B^{-1} t^{-1}$. That is, the dominating solution

$$H \sim \frac{2}{-B} t^{-1}, \text{ or } a(t) \sim t^{-2/B},$$

(4.8)

is an attractor of all smoothly evolving ‘phantom’ solutions at early times. Other solutions, dominated by the second balance of this decomposition, are elucidated in the next Section.

5 Decaying cosmologies and the borderline case

We now focus on the asymptotic analysis of the all-terms-dominant case, that is the decomposition $f^{(1)} = (y, -Ax - Bx^3)$. The subdominant vector field is the zero field in this case, and there are two balances:

$$B^{(1)}_1 = \begin{bmatrix} \left( \frac{A + \sqrt{A^2 - 8B}}{2B}, -A - \sqrt{A^2 - 8B} \right), & (-1, -2) \end{bmatrix},$$

(5.1)

$$B^{(1)}_2 = \begin{bmatrix} \left( \frac{A - \sqrt{A^2 - 8B}}{2B}, -A + \sqrt{A^2 - 8B} \right), & (-1, -2) \end{bmatrix}. $$

(5.2)

Our analysis closely monitors the different values $\delta$ may take and we focus in this Section exclusively on power law solutions, leaving the treatment of cyclic solutions for the next Section. Regarding the first balance of the $f^{(1)}$ decomposition, the Kovalevskaya matrix is given by

$$\mathcal{K}^{(1)} = \begin{pmatrix} 1 & 1 \\ -\mu + 6 & -\mu + 2 \end{pmatrix}, \text{ where } \mu = \frac{1 + \sqrt{1 - 8\delta}}{\delta},$$

(5.3)
and we find
\[ \text{spec}(K_1^{(1)}) = \{-1, \frac{-\mu + 8}{2}\}. \tag{5.4} \]

As in this Section we restrict attention to power law asymptotic solutions, we examine the case \( \delta = 1/8 \). Then we find
\[ \text{spec}(K_1^{(1)}) = \{-1, 0\}, \tag{5.5} \]
with corresponding eigenvector
\[ u_2^T = (1, -1). \tag{5.6} \]

The solution of the system is particular (only one arbitrary constant, cf. \([13, 14]\) for this terminology) and given by
\[ x \equiv H = \frac{4}{A} t^{-1}, \quad \text{or} \quad a \sim t^{4/A}. \tag{5.7} \]

We notice the two branches of this solution, one describing universes collapsing to zero size asymptotically \((A > 0)\), and the other ending at a big rip singularity \((A < 0)\).

For the specific case \( \delta = 1/8 \), we note the general solution found in \([16]\) is given by
\[ H^2 = a^{-\frac{8}{\delta}} (c_3 + c_4 \ln a). \tag{5.8} \]

If we set \( c_4 = 0 \), then this is the same as the 1-parameter solution \((5.7)\) found above. An exact solution identical to our solution \((5.7)\) was also found in \([10]\).

For the second balance of \( f^{(1)} \) decomposition, power-law solutions can be found for \( \delta = 1/8 \) as well as for the standard ‘decaying fluid’ range \( 0 < \delta < 1/8 \). The Kovalevskaya matrix is given by
\[ K_2^{(1)} = \begin{pmatrix} 1 & 1 \\ \phi + 6 & \frac{\phi}{2} + 2 \end{pmatrix}, \quad \text{where} \quad \lambda = \frac{-1 + \sqrt{1 - 8\delta}}{\delta}, \tag{5.9} \]
with eigenvalues
\[ \text{spec}(K_2^{(1)}) = \{-1, \frac{\lambda + 8}{2}\}. \tag{5.10} \]

Further, we notice that the case \( \delta = 1/8 \) of this second balance has the same eigenvalues, eigenvectors and solution as the first balance of this decomposition. These are
\[ \text{spec}(K_2^{(1)}) = \{-1, 0\}, \tag{5.11} \]
with corresponding eigenvector
\[ u_2^T = (1, -1), \tag{5.12} \]
and solution
\[ x = \frac{4}{A} t^{-1}. \tag{5.13} \]
Let us now turn to the behaviour of the asymptotic solutions with standard decay, that is for $0 < \delta < \frac{1}{8}$. For definiteness, we choose the value $\delta = 1/9$. Then

$$\text{spec}(K^{(1)}_2) = \{-1, 1\}, \quad (5.14)$$

with corresponding eigenvector

$$u^T_2 = (1, 0). \quad (5.15)$$

After further manipulations, we find that in the asymptotic expansion arbitrary coefficients are expected to be in the places $c_{11}$ and $c_{12}$, but from compatibility condition we have $c_{12} = 0$, giving therefore a solution with the correct number of arbitrary constants. The final solution is a general one and reads,

$$x = \frac{3}{A}t^{-1} + c_{11} + \frac{A}{3}c_{11}^2 t + \cdots. \quad (5.16)$$

The dominant part of the solution asymptotically is given by

$$a \sim t^{3/A}, \quad (5.17)$$

in accordance with the family found in [16] (for $\delta = \frac{1}{9}$), that is

$$\left[ \frac{a}{a_0} \right]^{\frac{A}{4}} = 1 + C_5 t + C_6 t^2, \quad (5.18)$$

and it is also the member given in [10] for $\delta = \frac{1}{9}$.

6 Anti-decaying, cyclic and complete universes

This section collects together all those cases where the asymptotic solution shows a qualitatively different character than that considered so far. The $f^{(3)}$ decomposition gives imaginary solutions for $B > 0$. There are two balances:

$$B^{(3)}_1 = [(i\sqrt{2/B}, -i\sqrt{2/B}), (-1, -2)] \quad (6.1)$$

$$B^{(3)}_2 = [(-i\sqrt{2/B}, i\sqrt{2/B}), (-1, -2)]. \quad (6.2)$$

Upon considering the subdominant part, in the case $B > 0$ we find that this decomposition is asymptotically acceptable only if $A = 0$, therefore when

$$\delta \rightarrow \infty.$$

The eigenvalues of the Kovalevskaya matrix are for both balances given by

$$\text{spec}(K^{(3)}_1) = \text{spec}(K^{(3)}_2) = \{-1, 4\}, \quad (6.3)$$
with corresponding eigenvector
\[ u_2^T = (1, 3). \] (6.4)

The coefficients \( c_{41}, c_{42} \) are expected to be arbitrary and the compatibility condition fixes one of them in terms of the other, \( 3c_{41} = c_{42} \). The final solution is given by the expansion
\[ x = \pm i \sqrt{2/B} t^{-1} + c_{41} t^3 \mp i \frac{B}{12} \sqrt{2/B} c_{41}^2 t^7 + \ldots. \] (6.5)

This solution has \( \delta \to \infty \) and so it must belong to the family of antidecaying fluids considered in [10]. It is interesting that asymptotically the scale factor turns imaginary, perhaps an indication that the metric in this case becomes asymptotically Euclidean. In this case we may consider as ‘physical’ that branch of the solution that S. W. Hawking would call ‘compact’, having zero size at the singularity, cf. [17].

The \( f^{(1)} \) decomposition, on the other hand, leads to a very complicated singularity for \( \delta > 1/8 \) for both remaining balances. After setting \( \delta = 1/2 \), we get
\[ \text{spec}(K^{(1)}_1) = \text{spec}(K^{(1)}_2) = \{-1, 3 \mp i \sqrt{3}\}, \] (6.6)

with corresponding eigenvectors
\[ u_2^T = (1, 2 \mp i \sqrt{3}). \] (6.7)

For both balances, the second eigenvalue of the \( K \)-matrix has positive real part. The solution of the system then reads
\[ x = \frac{1 \pm i \sqrt{3}}{A} t^{-1}, \] (6.8)

or, in terms of the scale factor we find
\[ a \sim t^{1 \pm i \sqrt{3}}. \] (6.9)

This defines a multifunction on approach to the \( t = 0 \) singularity which is obviously a logarithmic branch point admitting no Puiseux series representation. In this case the scale factor never returns to its original value no matter how many times \( t \) loops around zero.

Lastly, we give another sort of solution. When all the eigenvalues of the Kovalevskaya matrix are negative, the solution escapes away from the singularity towards infinity. For the \( f^{(1)} \) decomposition of the system, such a state appears when we examine the first balance with \( 0 < \delta < 1/8 \). If we choose \( \delta = 1/9 \), then
\[ \text{spec}(K^{(1)}_1) = \{-1, -2\}, \] (6.10)

with a corresponding eigenvector
\[ u_2^T = (1, -3). \] (6.11)
To construct a suitable expansion for this case, we need to take the multiplicative inverse \( \frac{4}{5} \) to be equal to one, \( s = -1 \), and the coefficients \( c_{21}, c_{22} \) are arbitrary. Following the method of asymptotic splittings, we are led to the compatibility condition \( 3c_{21} = -c_{22} \), and finally the asymptotic solution valid in the neighborhood of infinity:

\[
x = \frac{6}{A} t^{-1} + c_{11} t^{-2} + c_{21} t^{-3} + \cdots.
\]

(6.12)

This is a general solution valid away from any finite time singularity, showing a standard decay between the two fluids.

7 Discussion

In this paper we provided a demarcation of the singular phenomena that emerge when we consider two interacting perfect fluids in a flat FRW universe. We have examined what happens when we take this system asymptotically to a finite-time singularity. We have found a number of regimes described by different asymptotic solutions - seven different behaviours in all.

There is an asymptotic solution that acts as an attractor, a limit function to a wide family of solutions parametrized by the parameter \( \delta \). This solution is a member of a family of singular asymptotes that has the same number of arbitrary functions as the general solution, and attracts all these smoothly evolving solutions at early times in the ‘direction’ \( \delta \to 0 \). There is an analogous behaviour for the so-called ‘phantom’ regime of asymptotic solutions. There are also decaying solutions collapsing to zero size, and decaying solutions to a big rip singularity, but these are of less generality than the afore-mentioned behaviour, valid for special values of \( \delta \). The general solution towards the singularity with ‘standard decay’ (that is in the range \( \delta \in (0, 1/8) \)) was also picked by our asymptotic method, and it was constructed for a concrete parameter value. We found solutions of the ‘antidecaying’ type that approach the finite-time singularity turning purely imaginary in the parameter limit \( \delta \to \infty \), these are perhaps more amenable to a quantum cosmological description. We also gave a very peculiar solution having a log-type branch point singularity describing a ‘cyclic’ universe at ‘early’ times. Lastly, we have given the behaviour of solutions away from singularities and towards infinity.

The existence of the singular behaviours unraveled in this paper makes the dynamics of cosmologies with two interacting fluids especially interesting on approach to their singularities, and the singularity in such models deserves to be further studied. One aspect of the problem that is

\[\text{that is the least common multiple of the set of subdominant exponents and the positive Kovalevskaya exponents, cf. [13, 14].}\]
currently under study is whether these forms of approach to the interacting fluid singularities are stable to perturbations of the \( m, n, \lambda, \mu \) exponents away from the value one we considered in this work. This may demand a reformulation of the problem using more suitable variables. Another important issue that is also under examination is precisely how the inclusion of curvature alters the behaviours found in the flat case and whether new and distinct forms are possible. We plan to return to these more involved issues in the future.

**Acknowledgements**

We thank David Wands for discussions and useful comments. The work of G.K. is supported by a PhD grant co-funded by the European Union (European Social Fund-ESF) and national resources under the framework ‘Herakleitus II: Action for the enforcement of human research potential through the realization of doctorate work’ which is gratefully acknowledged.

**References**

[1] E. M. Lifshitz, I. M. Khalatnikov, Adv. Phys. 12 (1963) 185; L. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th Rev Ed. (Pergamon, Oxford, 1975); I. M. Khalatnikov, A. Yu. Kamenshchik, A. A. Starobinski, Class. Quant. Grav. 19 (2002) 3845.

[2] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, 1973); J. Wainwright, G.F.R. Ellis, *Dynamical Systems in Cosmology* (Cambridge University Press, 1997).

[3] J. D. Barrow, Nature 272 (1978) 211; R. Penrose, in *General Relativity, An Einstein Centenary Survey* (CUP, 1979), p. 581; S. W. Goode, J. Wainwright, Class. Quant. Grav. 2 (1985) 99.

[4] A. A. Starobinski, J.E.T.P. Lett. 30 (1979) 682; A. D. Linde, *Inflation and Quantum Cosmology* (Academic Press, 1990).

[5] J-L. Lehners, Phys. Rept. 465 (2008) 223, and references therein.

[6] J-P. Uzan, Class. Quant. Grav. 15 (1998) 1063.

[7] A. Nunes, J. P. Mimoso, T. C. Charters, Phys.Rev. D63 (2001) 083506.

[8] K. A. Malik, D. Wands, C. Ungarelli, Phys. Rev. D67 (2003) 063516; D. Langlois, F. Vernizzi, JCAP 0602 (2006) 014; N.A. Koshelev, Gen. Rel. Grav. 43 (2011) 1309.
[9] L. P. Chimento, Phys. Rev. **D65** (2002) 063517, Phys. Lett. **B633** (2006) 9, Phys. Rev. **D73** (2006) 063511.

[10] J. D. Barrow and T. Clifton, Phys. Rev. **D73** (2006) 103520.

[11] T. Clifton and J. D. Barrow, Phys. Rev. **D73** (2006) 104022.

[12] N. A. Gromov, Yu. Baryshev, P. Teerikorpi, Astr. Astroph. **415** (2004) 813; Pinto-Neto, B. M. O. Fraga, Gen. Rel. Grav. **40** (2008) 1653; J. Valiviita, E. Majerotto, R. Maartens, JCAP **0807** (2008) 020; J. C. Fabris, B. Fraga, N. Pinto-Neto, W. Zimdahl, arXiv:0910.3246; S. Z. W. Lip, Phys. Rev. **D83** (2011) 023528.

[13] S. Cotsakis and J. D. Barrow, *J. Phys. Conf. Ser.* **68** (2007) 012004; arXiv:gr-qc/0608137.

[14] A. Goriely, *Integrability and Nonintegrability of Dynamical Systems*, (World Scientific, 2001).

[15] S. Cotsakis and A. Tsokaros, *Phys. Lett.* **B651** (2007) 341-344; I. Antoniadis, S. Cotsakis and I. Klaoudatou, *Class. Quant. Grav.* **27** (2010) 235018.

[16] L. P. Chimento, J. Math. Phys. **38**(1997) 2565.

[17] S. W. Hawking, Nucl. Phys. **B239** (1984) 257.