Normal bases of algebras and Exponential Diophantine equations in rings of positive characteristic

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Abstract

In this paper we discourse bases of representable algebras. This question lead to arithmetic problems. We prove algorithmical solvability of exponential-Diophantine equations in rings represented by matrices over fields of positive characteristic. Consider the system of exponential-Diophantine equations

$$\sum_{i=1}^{s} P_{ij}(n_1,\ldots,n_t) b_{ij0} a_{ij1}^{n_1} b_{ij1} \cdots a_{ijt}^{n_t} b_{ijt} = 0$$

where $b_{ijk}, a_{ijk}$ are constants from matrix ring of characteristic $p$, $n_i$ are indeterminates. For any solution $(n_1,\ldots,n_t)$ of the system we construct a word (over an alphabet containing $p^t$ symbols) $\overline{\alpha}_0,\ldots,\overline{\alpha}_q$ where $\overline{\alpha}_i$ is a $t$-tuple $\langle n_1^{(i)},\ldots,n_t^{(i)} \rangle$, $n^{(i)}$ is the $i$-th digit in the $p$-adic representation of $n$. The main result of this paper is as follows: the set of words corresponding in this sense to solutions of a system of exponential-Diophantine equations is a regular language (i.e. recognizable by a finite automaton). There exists an effective algorithm which calculates this language. This algorithm is constructed in the paper.

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1 Introduction

Systems of exponential Diophantine equations (EDE)

$$\sum_{i=1}^{s} P_{ij}(n_1,\ldots,n_t) c_{ij1}^{n_1} \cdots c_{ijt}^{n_t} = 0,$$  \hspace{1cm} (1)

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where $P_{ij}$ are some polynomials arise in various areas of modern mathematics, and in general case, as J. Robinson has shown, they are algorithmically undecidable. Yu. V. Matiyasevich has proved algorithmical undecidability for purely Diophantine equations

$$P(n_1, \ldots, n_t) = 0.$$  

A number of problems reduces to undecidability of some EDE. However it turns out that if $c_i$ belong to a field of positive characteristic (and even to a matrix ring), the problem of finding the set of solutions is algorithmically decidable. Questions arising in this context occur to be related with formal languages.

Investigation of bases of algebras is an inspiration for research of such equations. Suppose $a_1 \prec \ldots \prec a_t$ is an ordered set of generators for an algebra $A$. The order $\prec$ on this set induces lexicographical ordering on the set of words in $\{a_i\}$. A basis $M$ of $A$ as of a vector space is called normal if it is generated by non-decreasing (that is, not representable by a linear combination of lesser words) elements. If $A$ is a $PI$-algebra (in particular, if $A$ is representable) then due to Shirshov theorem on height there exist $h = ht(A)$ and a finite tuple $v_1, \ldots, v_s$ such that $M$ consists of elements of the form

$$v_{i_1}^{k_1} \ldots v_{i_t}^{k_t}$$

where $t \leq h$.

In this connection the question arises on the structure of the set consisting of degree vectors $\langle k_1, \ldots, k_t \rangle$, in particular for the representable case. If the algebra $A$ is representable and monomial (that is, defining relations are of the form $u_j = 0$ where $u_j$ are some words) then the problem of normal basis permits in some sense complete answer. We have the following

**Theorem (test for representability of a monomial algebra).**

A monomial algebra $A$ is representable iff $A$ has bounded height over some finite set of words $v_1, \ldots, v_t$, the set of defining relations can be divided into a finite number of series $v_1^{k_1} \ldots v_t^{k_t} = 0$ where 

$$\sum_i P_{ij}(k_1, \ldots, k_t)c_{i_1}^{k_1} \ldots c_{i_t}^{k_t} = 0$$

and each series corresponds to a specific system of EDE.

This theorem implies, in particular, existence of representable algebras whose Hilbert series is transcendental as well as algorithmic undecidability of isomorphism problem for a pair of subalgebras in a matrix algebra over a polynomial ring.

Nevertheless for positive characteristic the situation is much simpler. Although Diophantine problems arise more often in this case, their solution is simpler. The set of solutions of an EDE admits effective description in terms of $p$-adic decomposition of indeterminates $n_1, \ldots, n_t$. Since values of $P_{ij}(n_1, \ldots, n_t)$ are periodical with period $p$ in each $n_i$, it suffices to investigate equations of the form

$$\sum_{i=1}^{s} c_1^{n_1} \ldots c_t^{n_t} = 0.$$ 

Consider some solution of an EDE: $\langle n_1, \ldots, n_t \rangle$. To each its component $n_i$ attach its $p$-adic decomposition $n_i^k \ldots n_i^0$. Thus to each solution we attach the sequence of tuples of figures $\langle n_1^k, \ldots, n_t^k \rangle, \ldots, \langle n_1^0, \ldots, n_t^0 \rangle$. Interpret these tuples as letters, and our sequences
as words over the alphabet consisting of tuples. The set of all words corresponding to solutions of EDE forms a language over a finite alphabet. We are ready to formulate the main result of this paper.

**Theorem 1.** The set of words corresponding to a system of EDE is a regular language. In other words, there exists an oriented graph with arrows marked by letters corresponding to finite tuples of figures (the number of letters is $p^t$). Some vertex is declared initial, and some other vertices are declared final. There exists 1-1 correspondence between solutions of our system and words which may be read along a path consisting of arrows and going from the initial vertex to a final one. These paths are allowed to have arbitrary length, and each vertex (including initial and final ones) may be passed arbitrarily many times.

There exists an effective algorithm for constructing such a graph. Below we present its description.

# 2 Bases of Representable and $PI$-algebras

The earliest purely combinatorial result of this kind occurred to be A. I. Shirshov’s height theorem. Let $A$ be a finitely generated $PI$-algebra. Then there exists a finite set of elements $Y$ and an integer $H \in \mathbb{N}$ such that $A$ is linearly represented by (that is, is generated by linear combinations of) the set of elements of the form

$$v_1^{k_1}v_2^{k_2} \cdots v_h^{k_h} \quad \text{where } h \leq H, \ v_i \in Y.$$  

For $Y$ we may take the set of words of degree $\leq m$. Such an $Y$ is called a Shirshov basis of the algebra $A$.

The above theorem implies positive solution of Kurosh problem and of other Burnside-type problems for $PI$-rings. In fact, if $Y$ is a Shirshov basis consisting of algebraic elements then the algebra $A$ is finite-dimensional. Thus Shirshov theorem explicitly determines the set of elements whose algebraicity implies algebraicity of the whole algebra. We also have

**Corollary 2.1** If $A$ is a $PI$-algebra of degree $m$ and all words in its generators of degree $\leq m$ are algebraic then $A$ is locally finite.

Height theorem also implies

**Corollary 2.2 (Berele)** Let $A$ be a finitely generated $PI$-algebra. Then $\text{GKdim}(A) < \infty$.

$\text{GKdim}(A)$ is the Gelfand – Kirillov dimension of the algebra $A$, that is,

$$\text{GKdim}(A) = \lim_{n \to \infty} \ln V_A(n)/\ln(n)$$

where $V_A(n)$ is the growth function of $A$, that is, the dimension of the vector space generated by words of degree $\leq n$ in generators of $A$.

To prove the corollary, it suffices to observe that the number of solutions of inequality $k_1|v_1| + \cdots + k_h|v_h| \leq n$ with $h \leq H$ does not exceed $N^H$, and so $\text{GKdim}(A) \leq h(A)$.

Thus we obtain various consequences from Height theorem. A little later we discuss questions concerning conversion of these implications. To begin with, we introduce some notions and notation.
The number $m = \deg(A)$ will denote the degree of the algebra, that is, the minimal degree of an inequality satisfied by it; $n = \Pi\deg(A)$ is the complexity of $A$, that is, the maximal $k$ such that $\mathbb{M}_k$, the algebra of matrices of size $k$, belongs to the variety $\text{Var}(A)$ generated by $A$.

It is convenient to replace the notion of height by a close notion of essential height.

**Definition 2.3** An algebra $A$ has essential height $h$ over a finite set $Y$ which is called an $s$-basis if there exists a finite set $D \subset A$ such that $A$ is linearly representable by elements of the form $t_1 \cdot \ldots \cdot t_l$ where $l \leq 2h + 1$ and $\forall i(t_i \in D \vee t_i = y_i^{k_i}; y_i \in Y)$, and the set of $i$ having $t_i \notin D$ contains $\leq h$ elements.

Loosely speaking, each long word is a product of periodical parts and of “layers” having bounded length. Essential height is the number of these periodical pieces, and ordinary height depends also upon “layers”.

Height theorem gives rise to following questions:

1. To what classes of rings Height theorem may be extended?
2. For which $Y$ the algebra $A$ has bounded height?
   Since now, we consider the associative case.
3. How to evaluate height?
4. What does the degree vector $(k_1, \ldots, k_h)$ look like? First of all, which sets of its components are essential, that is, which sets of $k_i$ can be simultaneously unbounded? What is the value of essential height?
5. A question regarding finer structure of the set of degree vectors: does it have any regularity properties?
   At last, the range of questions forming the subject of this paper.
6. Which sets of words can be chosen for $\{v_i\}$?

Now we proceed to discuss the above questions.

**Non-associative generalizations.** Height theorem has been extended to certain classes of rings close to associative rings. S. V. Pchelintsev has proved it for alternative and $(-1,1)$ cases, S. P. Mishchenko has obtained an analogue of Height theorem for Lie algebras with a sparse identity. The author has proved Height theorem for a certain class of rings asymptotically close to associative rings and in particular including alternative and Jordan $PI$-algebras.

**Shirshov bases.** Let $A$ be a $PI$-algebra, and suppose a subset $M \subseteq A$ is its $s$-basis. Then if all elements of $M$ are algebraic over $K$ then $A$ is finite-dimensional (Kurosh problem). Boundedness of essential height over $Y$ implies “positive solution of Kurosh problem over $Y$”. The converse is much less trivial.

**Theorem 2.4 (A. Ya. Belov)** Suppose $A$ is a graded $PI$-algebra, $Y$ is a finite set of homogeneous elements. Then if the algebra $A/Y^{(n)}$ is nilpotent for each $n$ then $Y$ is an $s$-basis for $A$. If in this situation $Y$ generates $A$ as an algebra then $Y$ is a Shirshov basis for $A$. 

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(\(Y^{(n)}\) denotes the ideal generated by \(n\)th powers of elements from \(Y\).)

The following example demonstrates that the straightforward converse of Kurosh problem for non-graded case does not have positive solution. Suppose \(A = \mathbb{Q}[x, 1/x]\). Each projection \(\pi\) such that \(\pi(x)\) is algebraic has finite-dimensional image. Nevertheless the set \(\{x\}\) is not an \(s\)-basis for the algebra \(\mathbb{Q}[x, 1/x]\).

Thus the definition of Kurosh set is chosen as follows:

**Definition 2.5** A set \(M \subset A\) is called a Kurosh set if each projection \(\pi: A \otimes K[X] \to A'\) having image \(\pi(M)\) integral over \(\pi(K[X])\) is finite-dimensional over \(\pi(K[X])\).

We proceed to formulate a generalization of this theorem for non-homogeneous case.

**Theorem 2.6 (A. Ya. Belov)** Let \(A\) be a PI-algebra, \(M \subseteq A\) a Kurosh subset in \(A\). Then \(M\) is an \(s\)-basis for \(A\).

The following proposition shows that Theorem 2.6 is a generalization of Theorem 2.4:

**Proposition 2.7** Let \(A\) be a graded algebra, \(Y\) a set of homogeneous elements. Then if the algebra \(A/Y^{(n)}\) is locally nilpotent for all \(n\) then \(Y\) is a Kurosh set.

Thus boundedness of essential height is a non-commutative generalization of integrity.

**Remarks.**

a) Note that in the case of Lie PI-algebras, Kurosh problem has positive solution but Height theorem fails.

b) The theorem extends to some class of rings asymptotically close to associative rings (with bounded \(l\)-length, finitely generated algebra of left multiplications, and associative powers).

**Estimates of height.** The original A. I. Shirshov’s proof was purely combinatorial (based on elimination technique developed by him for Lie algebras, in particular in the proof of Freedom theorem), however it did not provide any reasonable estimates for height. Later A. T. Kolotov obtained an estimate for \(ht(A) \leq s^m (m = \deg(A), s\) is the number of generators). Subsequently, E. I. Zel’manov [5] raised the question on existing of an exponential estimate which was obtained later on by the Belov.

**Shirshov height theorem.** Suppose \(A\) is an \(l\)-generated PI-algebra of degree \(m\). Then the height of \(A\) over the set of words having degree \(\leq m\) is bounded by a function \(H(m, l)\) where \(H(m, l) < 2^{ml^{m+1}}\).

**Essential height.** Clearly, essential height is an estimate for Gelfand – Kirillov dimension and an \(s\)-basis is a Shirshov basis iff it generates \(A\) as an algebra.

In the representable case the converse is true.

**Theorem 2.8 (A. Ya. Belov [4])** Suppose \(A\) is a finitely generated representable algebra and \(H_{EssY}(A) < \infty\). Then \(H_{EssY}(A) = \text{GKdim}(A)\).

**Corollary 2.9 (V. T. Markov)** The Gelfand – Kirillov dimension of a finitely generated representable algebra is an integer.
Corollary 2.10 If $H_{\text{Ess}Y}(A) < \infty$ and an algebra $A$ is representable then $H_{\text{Ess}Y}(A)$ is independent of the $s$-basis $Y$.

Due to local representability of relatively free algebras, the Gelfand – Kirillov dimension in this case also equals the essential height.

**Structure of degree vectors.** Thus in the representable case both Gelfand – Kirillov dimension and essential height behave well. Nevertheless even in this case the set of degree vectors can have bad structure, namely, it can be the complement for the set of solutions for some system of exponential-polynomial Diophantine equations. Consequently, there exists an example of a representable algebra having transcendent Hilbert series. However in the case of relatively free algebra the Hilbert series is rational.

**Shirshov bases consisting of words.** Their description is given by the following theorem:

**Theorem 2.11 (A. Ya. Belov)** A set of words $Y$ is a Shirshov basis of an algebra $A$ iff for each word $u$ having length $\leq m = \text{Pldeg}(A)$, the complexity of $A$, the set $Y$ contains some word which is cyclically conjugate to some degree of $u$.

A. I. Shirshov himself has shown that for a Shirshov basis we may take the set of words having degree at most $\text{deg}(A)$). I. V. Liov has proved boundedness of height over the set of words having length at most $\text{deg}(A) - 1$. S. Amitsur and I. P. Shestakov conjectured that if all words having length not exceeding the complexity $\text{Pldeg}(A)$ are algebraic then the algebra is finite-dimensional. I. V. Liov reduced this statement to the following:

**Theorem 2.12** Let $A$ be a finite-dimensional subalgebra in the matrix algebra of order $n$, and let $a_1, \ldots, a_s$ be its generators. Then if all words in $a_1, \ldots, a_s$ having degree $\leq n$ are nilpotent then $A$ itself is nilpotent.

Note that $n$ is the precise estimate.

Shestakov’s conjecture was proved by V. A. Ufnarovsky and by G. P. Chekanu. Later the author [6] showed that for $\{v_i\}$, we may take the set of words from Shestakov’s conjecture. This result also was announced by G. P. Chekanu. Later on, another proof of this fact was obtained by V. Drensky.

In the sequel, we focus on the range of problems concerned to relations between Height theorem and Independence theorem.

Independence theorem may be formulated, in particular, as follows

**Theorem 2.13 (Independence theorem)** Suppose the following is true:

1. a word $W = a_{i_1} \ldots a_{i_n}$ is the minimal word in the left lexicographical ordering on the set of all nonzero products having length $\leq n$;

2. the extreme parts of $W$ are nilpotent.

Then initial subwords of $W$ are linearly independent.

1From a private letter by the latter: “we worked in the same area . . . We both have stood this friendly and creative concurrence (we did begin this deliberately, with agreement that we work in different languages)” . The proofs were based on “the spirit of independence”. Subsequent papers of these authors contained various specifications and generalizations of these theorems [10].
To deduce I. P. Shestakov’s conjecture (or, equivalently, I. V. L’vov’s statement) from this theorem, it suffices to consider a faithful representation of \( A \) by operators on \( n \)-dimensional space \( V \). Let \( v_1, \ldots, v_n \) be a basis of this space, then for some \( v_i \) we have \( m_i W \neq 0 \). Consider the auxiliary algebra generated by \( V \) and \( A \). Suppose \( V \cdot V = A \cdot V = 0 \) and the action of \( VA \) coincides with module multiplication. Reorder the generators as follows: \( v_1 \succ \cdots \succ v_n \succ a_1 \succ \cdots \succ a_s \) and apply Independence theorem. \( \square \)

Original proofs of Independence theorem were rather complicated. Application of symbolic dynamics technique involving infinite words or superwords allowed to clarify them. Technique of superwords occurred to be rather close to the lines of structure theory. Its role does not reduce to proving statements like Independence theorem. Using superwords allows to prove Height theorem, nilpotence of the Lie algebra generated by sandwiches [8], coincidence of nilradical and Jacobson radical in monomial algebras, to describe bases of algebras with extremal growth function \( V_A(n) = \frac{n(n+3)}{2} \), and also to describe weakly Noetherian, semisimple and semiprimary monomial algebras [4] and to obtain some other combinatorial results in the theories of semigroups and rings.

Many properties of algebras are defined by monomial relations. For example, such are the conditions of Shestakov’s conjecture, namely, nilpotence of words whose degree does not exceed complexity.

This conjecture is related to the structure of the matrix algebra. Multiplication of matrix units \( E_{ij} \) is almost monomial, and the language of representations of matrix algebras clarifies “matrix” properties of semisimple components. It is no coincidence that many authors dealing with independence actively used a similar technique of matrix constructions for other problems concerning local finiteness [11].

### 3 Preliminaries

Recall now some facts from the theory of formal languages.

**Definitions.** A finite automaton is a finite oriented graph some vertex of which is declared initial, some vertices are declared final, and each edge is marked by a symbol of some finite alphabet.

A regular language is a set of words which may be read at edges of some finite automaton along a path from the initial vertex to a final one. We say that this automaton represents the given language.

A concatenation \( vu \) of two words \( u \) and \( v \) is obtained by adding \( v \) to \( u \) (in our case from the left). A concatenation of two languages \( L_1 \) and \( L_2 \) is the language \( L = \{ uv \mid u \in L_1, v \in L_2 \} \).

The closure \( L^* \) of a language \( L \) is the set of all powers of words from \( L \).

An atomary language consists of a single word consisting of a single letter.

One of the simplest instances of regular languages is the set of all words not including subwords from a fixed finite list. A description of regular languages in terms of operations over languages is provided by the following

**Theorem (Cleenee).** A language is regular iff it can be obtained from atomary languages by finite number of operations of joint, meet, concatenation and closure.

For more detail and for the proof of Cleenee’s theorem see Salomaa [1, p. 24-37].
4 Basic notation and constructions

In the sequel, we use following notation.
\( \overline{\vartheta} = (\vartheta_1, \ldots, \vartheta_r) \) is a tuple of variables.
\( \overline{n} = (n_1, \ldots, n_t) \) is a tuple of indeterminates.
\( \mathbb{N} \) is the set of natural numbers.
\( \Sigma_1 \) is the alphabet consisting of tuples of figures \( 0, 1, \ldots, p - 1 \) having length \( t \).
\( \Sigma_2 \) is the alphabet consisting of tuples of figures \( 0, 1, \ldots, p - 1 \) having length \( r \).
\( \Sigma_1^* \) is the set of all finite words over \( \Sigma_1 \).
\( \Sigma_2^* \) is the set of all finite words over \( \Sigma_2 \).
Words from \( \Sigma_1^* \), \( \Sigma_2^* \) are written from the right to the left.
\( \lambda \) is the empty word.
\( l(u) \) is the length of the word \( u \).
\( R = \mathbb{Z}_p[\vartheta_1, \ldots, \vartheta_r] \) is the ring of polynomials over \( \mathbb{Z}_p \).
\( F \) is the quotient field for \( R \).
\( A \) is the algebraic closure for \( F \).
Since to each sequence of figures with radix \( p \) there corresponds a number from \( \mathbb{N} \), the following maps are well-defined:
\( \phi : \Sigma_1^* \to \mathbb{N}^t \) which maps any word from \( \Sigma_1^* \) to a tuple consisting of \( t \) numbers written with radix \( p \),
\( \psi : \Sigma_2^* \to \mathbb{N}^r \) which maps any word from \( \Sigma_2^* \) to a tuple consisting of \( r \) numbers written with radix \( p \).
\( \phi^{(i)} : \Sigma_1^* \to \mathbb{N} \) is \( i \)th component of \( \phi \).
\( \psi^{(i)} : \Sigma_2^* \to \mathbb{N} \) is \( i \)th component of \( \psi \).
\( \overline{f} = (f_1, \ldots, f_t) \) is a tuple of polynomials.
In the sequel, we also denote by \( \overline{f}^p \) the tuple consisting of \( p \)th powers of \( f_i \): \( (f_1^p, \ldots, f_t^p) \),
and we denote by \( \overline{\vartheta}^p \) the tuple consisting of \( p \)th powers of \( \vartheta_i \): \( (\vartheta_1^p, \ldots, \vartheta_r^p) \).
Products of the form \( f_1^{\phi(1)}(u) \cdots f_t^{\phi(t)}(u) \) and \( \vartheta_1^{\psi(1)}(v) \cdots \vartheta_r^{\psi(r)}(v) \) will be denoted \( \overline{f}^{\phi(u)} \) and \( \overline{\vartheta}^{\psi(v)} \) accordingly.

5 Equations over a ring of polynomials

Let \( K \) be a matrix ring having positive characteristic \( p \). We prove now an auxiliary statement which allows to reduce the class of considered equations.

Proposition. If for each equation of the form
\[ \sum_{i=1}^{s} b_{i0}a_{i1}^{n_1}b_{i1} \cdots a_{it}^{n_t}b_{it} = 0 \]  
(2)

having coefficients from \( K \) the set of words corresponding to its solutions (as it was defined in Introduction) is a regular language then the set of words corresponding to solutions of any system of equations of the form
\[ \sum_{i=1}^{s} P_{ij}(n_1, \ldots, n_t)b_{ij0}a_{ij1}^{n_1}b_{ij1} \cdots a_{ijt}^{n_t}b_{ijt} = 0 \]  
(3)
over $K$ is a regular language.

Proof. First note that the set of solutions of a system is the meet of sets of solutions for equations of the system. So by virtue of Cleenee’s theorem, regularity of languages corresponding to single equations of the form (3) implies regularity of languages corresponding to systems of such equations.

Consider now an equation of the form (3). Let $\langle n_1, \ldots, n_t \rangle$ be a tuple of numbers $n_i = n_i^0 + p n_i'$ where $n_i^0$ is the last digit with radix $p$ in the number $n_i$. For fixed $\langle n_1', \ldots, n_t' \rangle$ we have

$$P(n_1', \ldots, n_t') = P(n_1, \ldots, n_t),$$

so $\langle n_1, \ldots, n_t \rangle$ is a solution of an equation of the form (3) iff $\langle n_1', \ldots, n_t' \rangle$ is a solution of an equation of the form (2). Regularity of the set of all $\langle n_1', \ldots, n_t' \rangle$ obviously implies regularity of the set of all $\langle n_1^0, \ldots, n_t^0 \rangle$. Finally, the complete set of solutions of the original equation is the joint of sets of solutions corresponding to distinct tuples $\langle n_1^0, \ldots, n_t^0 \rangle$. Again by Cleene’s theorem we have regularity of languages corresponding to any equations of the form (3) over $K$. The proof is complete.

Thus we have reduced investigation of solutions for some system of EDE to the case of a single equation which furthermore has no polynomial (in $n$) parts.

Consider an EDE over $\mathbb{R}$:

$$\sum_{i=1}^{s} Q_i(\overline{\vartheta})[P_{ii}^{n_i}] \cdot \cdot \cdot [P_{it}^{n_t}](\overline{\vartheta}) = 0. \quad (4)$$

Its solution is a tuple of numbers $\overline{n} = (n_1, \ldots, n_t), n_i \in \mathbb{N}$.

Definition. A word-solution of the EDE (4) is $u \in \Sigma_1^*$ such that $\phi(u) = \overline{n}$ where $\overline{n}$ is a solution for (4).

Now we may write the equation in $u$

$$\sum_{i=1}^{s} Q_i(\overline{\vartheta})[P_{i}^{\phi(u)}](\overline{\vartheta}) = 0. \quad (5)$$

In the sequel, word-solutions will also be called solutions simply. The main result of this Section may be stated in the form of the following

**Theorem 1.** $L \subset \Sigma_1^*$, the set of word-solutions for the equation (5), is a regular language.

Before presenting the proof, we describe its basic idea. Let $Q(x)$ be a polynomial having coefficients from $\mathbb{Z}_p$. Let us investigate the result of removing brackets in $Q^n(x)$. Write $n$ with radix $p$:

$$n = n_0 + n_1 p + \ldots + n_k p^k + n_{k+1} p^{k+1} + \ldots + n_s p^s.$$

Put $Q_k(x) = Q^k$ where $k = 0, 1, \ldots, p - 1$. Clearly $Q_0 = 1$, $Q_1 = Q$. Then

$$Q(x)^n = Q_{n_0}(x)Q_{n_1}(x^p) \cdot \cdot \cdot Q_{n_k}(x^{p^k})Q_{n_{k+1}}(x^{p^{k+1}}) \cdot \cdot \cdot Q_{n_s}(x^{p^s}) \quad (\ast)$$

since $Q(x)^{p^k} = Q(x^{p^k})$. Consider a section of the product

$$R_k = Q_{n_0}(x) \cdot \cdot \cdot Q_{n_k}(x^{p^k}). \quad (\ast\ast)$$
Collect terms with the same remainder $\alpha$ of the degree of $x$ modulo $p^{k+1}$. In other words, represent the section as sum

$$\sum_{\alpha=0}^{p^{k+1}-1} x^\alpha R_\alpha(x^{p^{k+1}}). \quad (\ast \ast \ast)$$

Now note the following.
1. Multiplication by the rest of the product ($\ast$) (that is, by $Q_{n_k+1}(x^{p^{k+1}}) \cdots Q_{n_s}(x^{p^{s}})$) does not lead to cancellation of terms in ($\ast \ast \ast$) having distinct $\alpha$.
2. The degree of $R_k$ does not exceed $(\deg Q)(p-1)p^k$. Hence the degree of $R_\alpha$ does not exceed $(\deg Q)(p-1)$. Furthermore, since we work over a finite field, the number of distinct types of $R_\alpha$ (small types) is bounded (and does not exceed $p^{(\deg Q)(p-1)}$).
3. Due to observation 1, we need not all the information on the sum ($\ast \ast \ast$) but only the following: which polynomials $R_\alpha$ do exist for given $k$.

The set of all existing polynomials will be called a large type. Clearly the number of large types is finite (and does not exceed $2^{p(\deg Q)(p-1)}$). It is also clear that the large type for a given $k$ uniquely determines the large type for $k+1$. This implies finiteness of the space of large types for products ($\ast \ast$). If we use several monomials then we have to define the small type of the sum $S_1Q_1^{n_1} + \ldots + S_tQ_t^{n_t}$ as the tuple consisting of small types of summands, and the large type as the tuple consisting of involved small types.

Now if we consider polynomials in several variables and products of the form

$$S_iQ_{i1}^{n_{i1}} \cdots Q_{it}^{n_{it}}$$

then we have to take tuples of remainders modulo $p^{k+1}$ and to collect variables having corresponding degrees. In this case we have:

$$\sum_{\alpha_1, \ldots, \alpha_r} x_1^{\alpha_1} \cdots x_r^{\alpha_r} R_\sigma(x_1^{p^{k+1}}, \ldots, x_r^{p^{k+1}})$$

for each monomial. Then the small type of the monomial is $R_\sigma$, the small type of the system is the tuple consisting of small types of monomials with given $\alpha$, and finally the large type of the system is the tuple consisting of involved small types. It is easily seen that writing a new figure to the left from variables $n_i$ corresponds to change of large types (depending of the written figure), and vanishing of the expression

$$\sum R_iQ_{i1}^{n_{i1}} \cdots Q_{it}^{n_{it}}$$

depends only on its large types (since vanishing of its components obtained by grouping terms described above depends only on its small types). Thus we obtain a finite graph of states. Its vertices are large types, and arrows marked by tuples of figures $0, \ldots, p-1$ are transformations of large types. The initial vertex corresponds to the large type of zero, and final vertices correspond to those large types which provide cancellation of all summands.

Clearly there is a correspondence between words which may be read at arrows of the graph along a path from the initial vertex to a final, and solutions of the EDE

$$\sum R_iQ_{i1}^{n_{i1}} \cdots Q_{it}^{n_{it}} = 0.$$
Now we turn to formal details. First we introduce some important constructions. Let \( f \) be a polynomial from \( R \). Then

\[
f(\overline{y}) = \sum_{y \in \Sigma_2} f_y(\overline{y})\overline{y}^{\psi(y)},
\]

and this decomposition is unique.

**Definition.** The weeding of a polynomial \( f \) by a symbol \( y \) is the polynomial \( \varepsilon_y(f) = f_y(\overline{y}) \). The weeding of a polynomial \( f \) by a word \( v = y_k \cdots y_0 \) is the polynomial \( \varepsilon_v(f) = \varepsilon_{y_k}(\cdots(\varepsilon_{y_0}(f))\cdots) \).

**Remark.** Weeding is a way to collect, as it was mentioned above, polynomials in which degrees of variables coincide modulo \( p^k \).

It is easy to see that \( \varepsilon_y(f + g) = \varepsilon_y(f) + \varepsilon_y(g) \) and \( \deg \varepsilon_y(f) \leq \frac{1}{p}\deg f \).

**Lemma 1.** \( \varepsilon_y(f(\overline{y})g(\overline{y}')) = \varepsilon_y(f(\overline{y}))g(\overline{y}) \).

In other words, in weeding polynomials in \( \overline{y} \) are factored out looseing degree \( p \).

**Proof.** Represent \( f \) in the form \( \sum_{y \in \Sigma_2} \overline{y}^{\psi(y)}f_y(\overline{y}) \). Then

\[
f(\overline{y})g(\overline{y}') = \sum_{y \in \Sigma_2} \overline{y}^{\psi(y)}f_y(\overline{y}')g(\overline{y}') = \sum_{y \in \Sigma_2} \overline{y}^{\psi(y)}(f_y(\overline{y}')g(\overline{y}')).
\]

By definition of weeding we immediately obtain

\[
\varepsilon_y(f(\overline{y})g(\overline{y}')) = \varepsilon_y(f(\overline{y}))g(\overline{y}).
\]

**Lemma 2.** Let \( c \) be a positive integer. Then a polynomial \( f(\overline{y}) \) vanishes iff every its weeding by a word of length \( c \) vanishes.

**Proof.** Use induction in \( c \).

a) Base of induction. Suppose \( c = 1 \). Then \( f = 0 \) obviously implies \( \varepsilon_y(f) = 0 \).

Conversely, suppose \( \varepsilon_y(f) = 0 \) for any \( y \in \Sigma_2 \). Then \( f = \sum_{y \in \Sigma_2} \varepsilon_y(f)\overline{y}^{\psi(y)} = 0 \). The base of induction is proved.

b) The inductive step. Suppose the statement is valid for \( c = k \). Then \( f = 0 \) means that \( \varepsilon_y(f) = 0 \) for any \( y \in \Sigma_2 \). This in turn is equivalent to \( \varepsilon_v(\varepsilon_v(f)) = 0 \) for any \( y \in \Sigma_2 \) and any \( v' \in \Sigma_2^* \) of length \( k \). Hence \( \varepsilon_v(f) = 0 \) for any \( v \in \Sigma_2^* \) of length \( k + 1 \). The inductive step is proved.

**Special operators of an equation.**

Return to the original equation (5)

\[
\sum_{i=1}^{s} Q_i(\overline{y})[P_i^{\psi(u)}]|(\overline{y}) = 0.
\]

**Definition.** Suppose \( i \) is an integer in the interval from 1 to \( s \), \( u \) and \( v \) are two words of equal length over alphabets \( \Sigma_1 \) and \( \Sigma_2 \) accordingly. Then define the *special operator* of the equation (5) \( S_{u,v}^{(i)}(f) \) as \( \varepsilon_v(f[P_i^{\psi(u)}]) \).

**Lemma 3.** If length of words \( u_1, v_1 \) is equal, and similarly for \( u_2, v_2 \), then

\[
S_{u_1u_2,v_1v_2}^{(i)}(f) = S_{u_1,v_1}^{(i)}S_{u_2,v_2}^{(i)}(f),
\]
that is, the special operator corresponding to concatenation is the composition of special operators corresponding to its factors.

Proof. Denote by \( k \) the length of \( u_1 \) (equal to the length of \( v_1 \)). Then the composition \( S_{u_1,v_1}^{(i)} S_{u_2,v_2}^{(i)} (f) \) equals \( \varepsilon_{v_1}(\varepsilon_{v_2}(f|[P^{\phi(u_2)}_1][P^{\phi(u_1)}_1])) \). Include \([P^{\phi(u_1)}_1]\) in the weeding. We have

\[
\varepsilon_{v_1}(\varepsilon_{v_2}(f[P^{\phi(u_2)+\phi(u_1)p^k}_1])) = \varepsilon_{v_1}v_2(f[P^{\phi(u_1u_2)}_1]) = S_{u_1u_2,v_1v_2}^{(i)}(f).
\]

Lemma is proved.

Lemma 4 (on decreasing the degree). There exists \( N_0 \) such that for any \( N' \geq N_0, 1 \leq i \leq s \), and for any \( u \in \Sigma^*_1, v \in \Sigma^*_2 \) of equal length \( \deg f \leq N' \) implies \( \deg S_{u,v}^{(i)}(f) \leq N' \).

In other words, rather large degrees can only decrease under the action of special operators.

Proof. The idea of proof is as follows: in a special operator, multiplying by fixed polynomials increases the degree of the original polynomial not more by a constant, and after that weeding decreases its degree not less than \( p \) times. We proceed to formalize this argument.

Denote \( \max \deg P_{ik} \) by \( M. \) Then the required \( N_0 \) equals \( \frac{prM}{p-1}. \) Indeed, \( N' = N_0 + K. \) Then if \( \deg f \leq N' \), then for all \( x \in \Sigma_1 \) we have

\[
\deg P_{i}^{\phi(x)} = \sum_{k=1}^{r}(\deg P_{ik})\phi^{(k)}(x) \leq \sum_{k=1}^{r}Mp = Mpr.
\]

Then

\[
\deg S_{u,v}^{(i)}(f) = \deg \varepsilon_{v}(f[P_{i}^{\phi(x)}]) \leq \left(\frac{\deg f + prM}{p}\right) \leq rM + \frac{N_0 + K}{p} = \frac{prM}{p-1} + \frac{K}{p} \leq N'.
\]

The assertion of Lemma now follows.

Types and their extensions.

Definitions. A small type \( T = (f_1, \ldots, f_s) \) is a string of polynomials from \( \mathbb{R} \) having degree not exceeding \( N_1 = \max \{\max \deg (Q_i), N_0\}. \)

A large type \( T \) is an arbitrary set of small types.

The extension \( \pi(u,v)\tau \) of a small type \( \tau = (f_1, \ldots, f_s) \) by a pair of words \( u \in \Sigma^*_1, v \in \Sigma^*_2 \) having equal length is the small type \( \tau' = (f'_1, \ldots, f'_s) \) where \( f'_i = S_{u,v}^{(i)}(f_i) \).

The extension \( \Pi(u)T \) of a large type \( T \) by a word \( u \in \Sigma^*_1 \) is the large type \( T' = \{\pi(u,w)\tau \mid \tau \in T, w \in \Sigma^*_2, l(w) = l(u)\}. \)

Remark. It is easy to observe that the operation of extension is defined for all small types. Indeed, if \( \deg f_1 \leq N_1 \) where \( N_1 \geq N_0 \) then \( \deg f'_1 \leq N_1 \), so \( \tau' \) is also a small type.

Moreover small types are strings of polynomials of bounded degree in \( r \) variables over a finite field, so their number is finite. The number of large types is finite as well since they are subsets of a finite set.

Definitions. The small type of a pair of words \( u \in \Sigma^*_1, v \in \Sigma^*_2 \) of equal length is \( \tau(u,v) = \pi(u,v)\tau(\lambda, \lambda) \) where \( \tau(\lambda, \lambda) = (Q_1, \ldots, Q_s) \).

Define the large type of a word \( u \in \Sigma^*_1 \) as \( T(u) = \{\tau(u,w); l(w) = l(u)\}. \)

Lemma 5. \( T(u) = \Pi(u)T(\lambda), \) that is, the large type of a word \( u \) may be obtained as an extension by \( u \) of the large type of the empty word.
Proof. By definition $T(\lambda) = \{\tau(\lambda, \lambda)\}$. Denote by $T'$ the extension of the type $T(\lambda)$ by $u$. Then $T'$ is the set of various extensions $\pi(u,v)\tau(\lambda, \lambda)$ of the type $\tau(\lambda, \lambda)$ by pairs of words $u, v$ where $v$ is an arbitrary word of the same length as $u$. Since the small type $\tau(u,v)$ is by definition $\pi(u,v)\tau(\lambda, \lambda)$ then $T'$ is the set of all $\tau(u,v)$ where $v$ is an arbitrary word of the same length as $u$, and so it coincides with $T(u)$. Lemma is proved.

Definitions. A small type $\tau = (f_1, \ldots, f_s)$ is good if $\sum_{i=1}^{s} f_i = 0$. A large type $T$ is good if all $\tau \in T$ are good.

We proceed to prove the following

Theorem 2. A large type $T(u)$ is good iff $u$ is a solution of the equation (5).

Proof. Denote the length of $u$ by $c$. A large type $T(u)$ is good iff for all $v \in \Sigma^s$ having length $c$ the small type $\pi(u,v)\tau(\lambda, \lambda)$ is good. This in turn means that for all such $v$ we have

$$\sum_{i=1}^{s} S_{u,v}(Q_i(\overline{\vartheta})) = 0$$

or equivalently

$$\sum_{i=1}^{s} \varepsilon_v(Q_i(\overline{P_i^{\phi(u)}})) = 0.$$

Furthermore due to linearity of weeding we have

$$\varepsilon_v(\sum_{i=1}^{s} Q_i(\overline{P_i^{\phi(u)}})) = 0$$

and by lemma 2

$$\sum_{i=1}^{s} Q_i(\overline{\vartheta})|\overline{P_i^{\phi(u)}}| = 0.$$

But this means that $u$ is a solution of (5).

Lemma 6. a) $\pi(u_1u_2,v_1v_2)\tau = \pi(u_1,v_1)\pi(u_2,v_2)\tau$.

b) $\Pi(u_1u_2)T = \Pi(u_1)\Pi(u_2)T$.

In other words, an extension of a type by a concatenation is a composition of extensions by factors.

Proof. a) Suppose $\tau = (f_1, \ldots, f_s)$. Then let $\tau' = (f_1', \ldots, f_s')$ denote the extension of $\tau$ by a pair of words $u_2, v_2$, and let $\tau'' = (f_1'', \ldots, f_s'')$ denote the extension of $\tau'$ by a pair of words $u_1, v_1$. We proceed to prove that $\pi(u_1u_2,v_1v_2)\tau = \tau''$. Note that $f_i' = S_{u_2,v_2}(f_i)$ and $f_i'' = S_{u_1,v_1}(f_i)$. Hence $f_i'' = S_{u_1,v_1}S_{u_2,v_2}(f_i) = S_{u_1u_2,v_1v_2}(f_i)$. Thus $\pi(u_1u_2,v_1v_2)\tau = \tau'' = \pi(u_1,v_1)\pi(u_2,v_2)\tau$. First assertion of Lemma is proved.

b) Denote $\Pi(u_2)T$ by $T'$, and $\Pi(u_1)T'$ by $T''$. We shall prove that $\Pi(u_1)\Pi(u_2)T = T'' = \Pi(u_1u_2)T$. By definition, $T'$ consists of various extensions of types from $T$ by pairs of words $u_2, v_2$ having equal length. Also by definition, $T''$ is the set of extensions of types from $T'$ by pairs of words $u_1, v_1$ having equal length. Hence $T''$ includes all small types of the form $\pi(u_1,v_1)\pi(u_2,v_2)\tau$. Using the first assertion of Lemma, we obtain that $T''$ consists of types $\pi(u_1u_2,v_1v_2)\tau$ where $\tau \in T$. Putting $v = v_1v_2$ we obtain: $T'' = \{\pi(u_1u_2,v)\tau \mid \tau \in T, l(u_1u_2) = l(v)\}$. This set is (again by definition) $\Pi(u_1u_2)T$. So Lemma is completely proved.
We proceed to return to the assertion formulated at the beginning of this Section and to prove it.

**Theorem 1.** $L \subset \Sigma^*$, the set of words-solutions for the equation \((\mathbf{5})\), is a regular language.

**Proof.** Consider the following finite automaton $G$. Its vertices are various large types. An arrow goes from $T$ to $T'$ and is marked by the symbol $x$ iff $T' = \Pi(x)T$. The initial vertex is $T(\lambda)$, and final vertices are various good large types. Some $u = x_k \cdots x_0$ is a solution iff $T(u) = \Pi(x_k) \cdots \Pi(x_0)T(\lambda)$ is a good type, and this in turn is equivalent to the assertion that $T(u)$ is a final vertex and the end of the path $x_k \cdots x_0$. Thus $u$ is a solution iff $u$ belongs to the language represented by the finite automaton $G$. This implies the assertion of Lemma.

6 Equations over a matrix ring

Any element of the ring $M_n(R)$ may be interpreted in two ways: as a polynomial in $\vartheta_1, \ldots, \vartheta_r$ with coefficients from $M_n(Z_p)$, and as a matrix with entries from $R$. Suppose $f(B)$ is a polynomial in a matrix $B$ with coefficients from $R$ (the matrix itself belongs to $M_n(R)$). Denote the ring consisting of such polynomials by $R[B]$. Let $\deg B, B \in M_n(R)$, be the sum of powers of $B$ as a polynomial in $\vartheta_1, \ldots, \vartheta_r$ with matrix coefficients.

**Definition.** A matrix $B \in M_n(F)$ is rational of standard form if it has the form

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & f_0 \\
1 & 0 & \cdots & 0 & f_1 \\
0 & 1 & \cdots & 0 & f_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & f_{n-1}
\end{pmatrix}
$$

where the polynomial $\xi^n - \sum_{i=0}^{n-1} f_i \xi^i$ is irreducible and separable over $F$.

In fact this the matrix of multiplication by $\xi$ in the algebraic extension of the field $F$ by a root of the above polynomial in the basis of extension consisting of powers of this root (see [2, 429-455]).

**Definition.** A matrix $B' \in M_n(R)$ is entire of standard form if it has the form

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & \varrho_0 \\
\varrho & 0 & \cdots & 0 & \varrho_1 \\
0 & \varrho & \cdots & 0 & \varrho_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \varrho & \varrho_{n-1}
\end{pmatrix}
$$

where the polynomial $\xi^n - \sum_{i=0}^{n-1} \varrho_i \xi^i$ is irreducible and separable over $F$.

**Remark.** If a matrix $B'$ is entire of standard form then there exists a unique matrix $B$, rational of standard form, such that $B' = \varrho B, \varrho \in R$.

Let $F[B]$ be the ring of polynomials having the following form:

$$
\sum_{k=0}^m f_k(\overline{\varrho})B^k(\overline{\varrho}), f_k \in F, B \in M_n(F).
$$

We have the following
Lemma 7 (on simplifying the form of an equation).
The following assertions are equivalent:
a) The set of solutions for an EDE over $M_n(A)$ is a regular language.
b) The set of solutions for an EDE over $A$ is a regular language.
c) The set of solutions for an EDE over a finite algebraic extension $F$ is a regular language.
d) The set of solutions for an EDE over a finite separable algebraic extension $F$ is a regular language.
e) The set of solutions for an EDE over $F[B]$ where $B$ is a rational matrix of standard form is a regular language.
f) The set of solutions for an EDE over $R[B']$ where $B'$ is an entire matrix of standard form is a regular language.

Proof. Observe obvious implications: a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d).
We proceed to prove c) $\Rightarrow$ b). Consider an EDE over $A$. It involves only a finite number of coefficients, and all of them are algebraic over $F$. Hence they belong to a finite algebraic extension of $F$, and the original equation is an EDE over this extension.

Now we shall prove d) $\Rightarrow$ c). Consider an EDE over a finite separable algebraic extension of $F$: 
\[ \sum_{i=1}^{s} b_i a_{i1}^{n_1} \cdots a_{it}^{n_t} = 0. \]
For some $M$ all of $b_i^{p_M}, a_{ik}^{p_M}$ are separable over $F$. Consider the equation 
\[ \sum_{i=1}^{s} (b_i)^{p_M} (a_{i1})^{p_M n_1} \cdots (a_{it})^{p_M n_t} = \left( \sum_{i=1}^{s} b_i a_{i1}^{n_1} \cdots a_{it}^{n_t} \right)^{p_M} = 0. \]
This equation is equivalent to the original one and is an EDE over a finite separable algebraic extension of $F$.

Now we postpone the proof for the most difficult implication b) $\Rightarrow$ a) and proceed to show that d) and e) are equivalent.
d) $\Rightarrow$ e). Since $B$ is rational of standard form, $F[B]$ is a finite separable algebraic extension for $F$.
e) $\Rightarrow$ d). By the primitive element theorem, every finite separable algebraic extension may be represented in the form $F[\xi]$ where $\xi$ is a root of some separable over $F$ polynomial $z^n - \sum_{i=0}^{n-1} f_i z^i$. Consider the matrix $B$:
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & f_0 \\
1 & 0 & \cdots & 0 & f_1 \\
0 & 1 & \cdots & 0 & f_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & f_{n-1}
\end{pmatrix}.
\]
It is rational of standard form, and $F[B]$ is isomorphic to $F[\xi]$. Hence every equation from d) is an equation from e).
We proceed to prove that e) and f) are equivalent.
e) $\Rightarrow$ f). Consider an EDE over $\mathbb{R}[B']$:
\[
\sum_{i=1}^{s} Q_i(B')[P_i^\phi(u)](B') = 0.
\]
Suppose $f$ is an arbitrary polynomial over $\mathbb{R}$. Since $B' = \rho B, \rho \in \mathbb{R}$, $B$ is rational of standard form, we have $f(B') = f(\rho B) \in F[B]$, hence $\mathbb{R}[B'] \subset F[B]$. Thus our equation is an EDE over $F[B]$.

f) $\Rightarrow$ e) Consider an EDE over $F[B]$:
\[
\sum_{i=1}^{s} Q_i(B)[P_i^\phi(u)](B) = 0.
\]
Find the common denominator for all $Q_i, P_{ik}$ and put $Q_i = \frac{Q'_i}{\sigma}, P_{ik} = \frac{P'_{ik}}{\sigma}$ where $Q_i, P_{ik}, \sigma$ are polynomials over $\mathbb{R}$. We have
\[
\left(\frac{1}{\sigma^{1+\phi(1)(u)+\ldots+\phi(i)(u)}}\right) \sum_{i=1}^{s} Q'_i(B)[P_i^\phi(u)](B) = 0.
\]
This equation is equivalent to the following:
\[
\sum_{i=1}^{s} Q'_i(B)[P_i^\phi(u)](B) = 0.
\]
Now use the fact that $B = \frac{1}{\rho} B'$. Hence for any polynomial $f$ over $\mathbb{R}$ we have $f(B) = \frac{f(B')}{\rho^{m}}$ where $f'$ also is a polynomial over $\mathbb{R}$, $m = \deg f$. Then the above equation may be written in the form
\[
\left(\frac{1}{\rho^{m(1+\phi(1)(u)+\ldots+\phi(i)(u))}}\right) \sum_{i=1}^{s} Q''_i(B')[P_i^\phi(u)](B') = 0.
\]
Multiplying by $\rho^{m(1+\phi(1)(u)+\ldots+\phi(i)(u))}$, we obtain an EDE over $\mathbb{R}[B']$ equivalent to the original one.

Finally we prove the last implication.
b) $\Rightarrow$ a). Consider an EDE over $M_n(A)$:
\[
\sum_{i=1}^{s} B_{i0}A_{i1}^{n_i}B_{i1}\cdots A_{it}^{n_t}B_{it} = 0
\]
where $A_{ik}, B_{it} \in M_n(A)$. Since $A$ is algebraically closed, $A_{ik}$ is representable in the form $C_{ik}A_{jk}C_{ik}^{-1}$ where $A_{jk}$ is a Jordan matrix. Then $A_{jk} = D_{ik} + R_{ik}$ where $D_{ik}$ is a diagonal matrix and $R_{ik}$ is nilpotent. Hence there exists $M$ such that for any $i, k : R_{ik}^M = 0$ and so $A_{jk}^M = (D_{ik} + R_{ik})^M = D_{ik}^M$.

Represent all $n_i$ in the form $n_i' + p^M n_i''$ where all $n_i' \leq p^M$. Denote $D_{ik}^{p^M}$ by $D_{ik}'$. In the new notation, the original equation takes the form
\[
\sum_{i=1}^{s} (B_{i0}C_{i1}A_{i1}^{n'_i})D_{i1}^{n'_i}((C_{i1}^{-1}B_{i1}C_{i2}A_{i2}^{n'_i})D_{i2}^{n'_2} \cdots D_{it}^{n'_t}(C_{it}^{-1}B_{it}) = 0
\]
or for fixed $n'_1, \ldots, n'_t$:

$$\sum_{i=1}^{s} B'_{i0} D_{i1}' B'_{i1} \cdots D_{it}' B'_{it} = 0.$$  

Denote the $kl$-th entry of $B'_{ij}$ by $\beta_{ij,kl} \in A$, and $k$-th entry of the diagonal matrix $D'_{ij}$ by $\lambda_{ij,k} \in A$. Let $\sigma(x, y, \overline{n'})$ denote the expression

$$\sum_{1 \leq z_1, \ldots, z_t \leq n, 1 \leq i \leq s} \beta_{i0,z_1,1} \beta_{i1,z_2,1} \cdots \beta_{it,z_t,1} \lambda_{i1,z_1} \lambda_{i2,z_2} \cdots \lambda_{it,z_t}$$

for various values of $x, y$ from 1 to $n$. Then the above equation is equivalent to the following system (depending on $\overline{n'} = \langle n'_1, \ldots, n'_t \rangle$):

$$\sigma(x, y, \overline{n'}) = 0.$$  

This is a system of $n^2$ EDE over $A$ and so by b)

$$L(\overline{n'}, x, y) = \{ u \in \Sigma_* : \sigma(x, y, \phi(u)) = 0 \}$$

is a regular language. Then

$$L(\overline{n'}) = \bigcap_{1 \leq x, y \leq n} L(\overline{n'}, x, y)$$

also is a regular language, and hence the set of solutions for the original EDE

$$L = \bigcup_{0 \leq n'_1, \ldots, n'_t < p^M} \{ \overline{n'} \} \ast L(\overline{n'})$$

is a regular language (here we apply Cleene’s theorem).

Lemma is completely proved.

Arguing for the ring of polynomials over a field, we have essentially applied the identity $\{ f(\overline{v}) \}^p = f(\overline{v}^p)$. For the ring of polynomials over a non-commutative ring, in particular over a matrix ring, this identity fails. But it turns that our constructions can be extended to the case of the ring $R[B]$ considered in assertion f) of the above lemma, by means of the following statement:

**Lemma 8 (on conjugation).**

a) Suppose $B(\overline{v})$ is a rational matrix of standard form. Then $B^p(\overline{v}) = C(\overline{v})B(\overline{v})C^{-1}(\overline{v})$ where $C \in M_n(F)$.

b) Suppose $B'(\overline{v})$ is an entire matrix of standard form. Then $B'^p(\overline{v}) = C'(\overline{v})B'(\overline{v})C'^{-1}(\overline{v})$ where $C' \in M_n(R)$.

**Proof.**

$$B = \begin{pmatrix}
0 & 0 & \cdots & 0 & f_0 \\
1 & 0 & \cdots & 0 & f_1 \\
0 & 1 & \cdots & 0 & f_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & f_{n-1}
\end{pmatrix}.$$
Suppose $\xi^n - \sum_{i=0}^{n-1} f_i \xi^i = 0$. Then $B(\overline{\vartheta})$ is the matrix of the operator $x \mapsto \xi x$ in the basis $1, \xi, \ldots, \xi^{n-1}$. The matrix $B^p(\overline{\vartheta})$ corresponds to the operator $x \mapsto \xi^p x$ in the same basis. The matrix $B(\overline{\vartheta}^p)$ corresponds to the operator $x \mapsto \xi^p x$ in the basis $1, \xi^p, \ldots, \xi^{p(n-1)}$. Due to well-known theorem of linear algebra, these matrices are conjugate.

b) The matrix $B'$ is entire of standard form. It is known that $B' = \rho B$ where $B$ is rational of standard form, $\rho \in R$. Then $B'(\overline{\vartheta}^p) = \rho^p B(\overline{\vartheta}) = \rho^p C(\overline{\vartheta}) B(\overline{\vartheta}) C^{-1}(\overline{\vartheta}) C(\overline{\vartheta}) = \frac{1}{\sigma(\overline{\vartheta})} C'(\overline{\vartheta}) C' \in M_n(R), \sigma(\overline{\vartheta})$ is the greatest common divisor for denominators of all entries of $C'(\overline{\vartheta})$. Then $C^{-1}(\overline{\vartheta}) = \sigma(\overline{\vartheta}) C'^{-1}(\overline{\vartheta})$, and so $B'^p(\overline{\vartheta}) = \rho^p C'(\overline{\vartheta}) B(\overline{\vartheta}) C'^{-1}(\overline{\vartheta}) = C'(\overline{\vartheta}) B(\overline{\vartheta}) C'^{-1}(\overline{\vartheta})$.

Consider an EDE over $R[B]$ where $B$ is an entire matrix of standard form:

$$\sum_{i=1}^{s} Q_i(B)[P_i^{\phi(u)}](B) = 0. \quad (6)$$

**Remark.** Suppose $f(\xi)$ is an arbitrary polynomial from $R[\xi]$. Then

$$f^p(B(\overline{\vartheta})) = C(\overline{\vartheta}) f(B(\overline{\vartheta}^p)) C^{-1}(\overline{\vartheta}).$$

Now put $u = x_k \cdots x_0$.

Transforming the right side of (6), we subsequently have

$$0 = \sum_{i=1}^{s} Q_i(B(\overline{\vartheta})[P_i^{\phi(x_0)}](B(\overline{\vartheta}))[P_i^{\phi(x_1)p}](B(\overline{\vartheta})) \cdots [P_i^{\phi(x_k)p}](B(\overline{\vartheta})) =$$

$$= \sum_{i=1}^{s} Q_i(B(\overline{\vartheta})[P_i^{\phi(x_0)}](B(\overline{\vartheta})) C(\overline{\vartheta})[P_i^{\phi(x_1)}](B(\overline{\vartheta}^p)) C^{-1}(\overline{\vartheta}) \cdots (C(\overline{\vartheta}) C(\overline{\vartheta}^p) \cdots$$

$$\cdots C(\overline{\vartheta}^k))(P_i^{\phi(x_k)})(B(\overline{\vartheta}^p))(C^{-1}(\overline{\vartheta}) \cdots C^{-1}(\overline{\vartheta})) =$$

$$= \sum_{i=1}^{s} Q_i(B(\overline{\vartheta})[P_i^{\phi(x_0)}](B(\overline{\vartheta})) C(\overline{\vartheta}) \cdots [P_i^{\phi(x_k)}](B(\overline{\vartheta}^k))(C(\overline{\vartheta}^p)) C^{-1}(\overline{\vartheta}) \cdots C^{-1}(\overline{\vartheta})$$

Now multiply the expression on the right side by an invertible matrix $C(\overline{\vartheta}) C(\overline{\vartheta}^p) \cdots C(\overline{\vartheta}^k)$. The resulting equation is equivalent to the original one:

$$\sum_{i=1}^{s} Q_i(B)[P_i^{\phi(x_0)}](B(\overline{\vartheta})) \cdots [P_i^{\phi(x_k)}](B(\overline{\vartheta}^k))(C(\overline{\vartheta}^k)) = 0. \quad (7)$$

We proceed to generalize constructions of the first step to the matrix case. Suppose $f \in M_n(R)$. Then as before $f(\overline{\vartheta}) = \sum_{y \in \Sigma_2} f_y(\overline{\vartheta}) \overline{\vartheta}^y$.

**Definitions.**

a) *The weeding* by a symbol $y \in \Sigma_2$ is $\varepsilon_y(f) = f_y(\overline{\vartheta})$.

b) *The weeding* by a word $v = y_k \cdots y_0$ is $\varepsilon_v(f) = \varepsilon_{y_k} \cdots (\varepsilon_{y_0}(f)) \cdots$, that is, the composition of weedings by letters of the word.
Lemma 9 (properties of the weeding operator).

a) $\varepsilon_y(f + g) = \varepsilon_y(f) + \varepsilon_y(g)$.
b) $\deg \varepsilon_y(f) \leq \frac{1}{p} \deg f$.
c) $\varepsilon_y(f(\overline{v})g(\overline{v})) = \varepsilon_y(f(\overline{v}))g(\overline{v})$.
d) If $c$ is a constant then $f = 0$ iff for any $v \in \Sigma^*_2$ having length $c$ we have $\varepsilon_v(f) = 0$.

The proofs of these properties are similar to those given at the first step.

Special operators are defined for the matrix case as follows:

Definition.

a) Suppose $x \in \Sigma_1$, $y \in \Sigma_2$. The special operator $S_{x,y}^{(i)}(f) = \varepsilon_y(f[\overline{P}^{\phi(x)}_i](B)C)$ where $C$ is the matrix from Lemma 8.
b) Suppose $u = x_k \cdots x_0$, $v = y_k \cdots y_0$ are words of equal length from $\Sigma_1^*$ and $\Sigma_2^*$ accordingly. Then $S_{u,v}^{(i)}(f) = S_{x_k,y_k}^{(i)} \cdots S_{x_0,y_0}^{(i)}(f)$.

Lemma 10 (on decreasing the degree).

There exists $N_0$ such that for any $N' \geq N_0$, $1 \leq i \leq s$, $x \in \Sigma_1$, $y \in \Sigma_2$ we have $\deg f \leq N' \Rightarrow \deg S_{x,y}^{(i)}(f) \leq N'$. In other words, rather high degrees of polynomials can be only decreased by special operators.

Proof.

Denote $\max \deg P_{ik}$ by $M$, and $\frac{prM + \deg C}{p-1}$ by $N_1$. Then $N_1$ is the desired $N_0$. We proceed to prove this. Suppose $N' = N_1 + K$. Then $\deg f \leq N'$ implies

$$\deg fP^{\phi(x)}_i C \leq N' + Mpr + \deg C = (prM + \deg C) \left( \frac{p}{p-1} \right) + K.$$ 

Furthermore

$$\deg S_{x,y}^{(i)}(f) \leq \frac{1}{p} \deg fP^{\phi(x)}_i C \leq \frac{prM + \deg C}{p-1} + \frac{K}{p} \leq N'.$$

Lemma is proved.

Definitions.

a) A small type $T = (f_1, \ldots, f_s)$ is a string of matrices from $M_n(\mathbb{R})$ such that $\deg f_i \leq N_2$, $N_2 = \max\{\max \deg (P_{i,k}), N_0\}$.
b) Let $x, y$ be symbols from alphabets $\Sigma_1$ and $\Sigma_2$ accordingly. The extension $\pi(x, y)\tau$ of a small type $\tau = (f_1, \ldots, f_s)$ by these symbols is the small type $\tau' = (f'_1, \ldots, f'_s)$, $f'_i = S_{x,y}^{(i)}(f_i)$.
c) Suppose $u = x_k \cdots x_0$, $v = y_k \cdots y_0$ are words of length from $\Sigma_1^*$ and $\Sigma_2^*$ accordingly. Then the extension $\pi(u, v)\tau$ of a small type $\tau$ by this pair of words is the composition of its extensions by pairs of symbols $\pi(x_k, y_k) \cdots \pi(x_0, y_0)\tau$.

Remark. If $\tau = (f_1, \ldots, f_s)$ then $\pi(u, v)\tau = (f'_1, \ldots, f'_s)$ where $f'_i = S_{u,v}^{(i)}(f_i)$. This follows immediately from definitions of $\pi(u, v)$ and $S_{u,v}$.

Definition.

a) A large type $T$ is an arbitrary set of small types.
b) Suppose $u \in \Sigma_2^*$, $u$ is a large type $\Pi(u)T = \{\pi(u, v)\tau \mid \tau \in T, v \in \Sigma_2^*, l(v) = l(u)\}$.

Lemma 11. $\Pi(u_1u_2)T = \Pi(u_1)\Pi(u_2)T$, that is, an extension of a large type by a concatenation of two words is composition of extensions of this type by the given words.
This implies that for all $\Pi(u_2)T$ by $T'$, and $\Pi(u_1)T'$ by $T''$. By definition of extension of a large type, $T' = \{ \pi(u_2, v_2) \, \tau \mid \tau \in T, l(v_2) = l(u_2) \}$. In turn, $T''$ is (again by definition) $\{ \pi(u_1, v_1) \, \tau' \mid \tau' \in T', l(v_1) = l(u_1) \}$ and, by virtue of formula for $T'$, equals $\{ \pi(u_1, v_1) \pi(u_2, v_2) \, \tau \mid \tau \in T, l(v_1) = l(u_1), l(v_2) = l(u_2) \}$, or $\{ \pi(u_1 u_2, v) \, \tau \mid \tau \in T, l(v) = l(u_1 u_2) \}$. This precisely coincides with $\Pi(u_1 u_2)T$. Lemma is proved.

**Definition.** Let $u, v$ be words of equal length from $\Sigma^*_1$ and $\Sigma^*_2$ accordingly.

a) The small type of the pair of words $\tau(u, v)$ is $\pi(u, v) \tau(\lambda, \lambda)$ where $\tau(\lambda, \lambda) = (Q_1(B), \ldots, Q_s(B))$.

b) The large type of the word $T(u)$ is $\{ \tau(u, w) \mid l(w) = l(u) \}$.

**Lemma 12.** $T(u) = \Pi(u)T(\lambda)$, that is, the large type of the word $u$ is the extension by this word of the large type of the empty word.

**Proof.** By definition, $T(\lambda) = \{ \tau(\lambda, \lambda) \}$. So, using only definitions for extensions of large and small types, we easily obtain

$$\Pi(u)T(\lambda) = \{ \pi(u, v) \tau(\lambda, \lambda) \mid l(v) = l(u) \} = \{ \tau(u, v) \mid l(v) = l(u) \} = T(u).$$

Lemma is proved.

**Definition.** a) A small type $\tau = (f_1, \ldots, f_s)$ is good if $\sum_{i=1}^s f_i = 0$.

b) A large type $T$ is good if all $\tau \in T$ are good.

**Theorem 3.** Suppose $u$ is an arbitrary word from $\Sigma^*_1$. Then $T(u)$ is a good type iff $u$ is a solution for the original EDE.

**Proof.** Define the length of $u$ by $c$. By definition of $T(u)$, it is a good type if for any $v \in \Sigma^*_2$ of length $c$ the type $\tau(u, v) = \pi(u, v) \tau(\lambda, \lambda)$ is good. This means in turn that for any $v \in \Sigma^*_2$ of length $c$ we have

$$\sum_{i=1}^s S_{u,v}^{(i)}(Q_i(B(\overline{v}))) = 0.$$

This implies that for all $v \in \Sigma^*_2$ of length $c$ we have

$$\sum_{i=1}^s \varepsilon_v(Q_i(B) | P_i^{\phi(x_0)} | B(\overline{v}))(B(\overline{v})C(\overline{v}) \cdots [P_i^{\phi(x_c)}](B(\overline{v}^c))C(\overline{v}^c)) = 0,$$

or for any $v \in \Sigma^*_2$ of length $c$

$$\varepsilon_v \left( \sum_{i=1}^s Q_i(B) | P_i^{\phi(x_0)} | B(\overline{v})C(\overline{v}) \cdots [P_i^{\phi(x_c)}](B(\overline{v}^c))C(\overline{v}^c) \right) = 0.$$

Using ae property of weeding (assertion d of Lemma 9), we have now

$$\sum_{i=1}^s Q_i(B) | P_i^{\phi(x_0)} | B(\overline{v})C(\overline{v}) \cdots [P_i^{\phi(x_c)}](B(\overline{v}^c))C(\overline{v}^c) = 0,$$
that is, \( u \) is a solution of (7), and so \( u \) is a solution of (6). Theorem is proved.
Now it remains to construct the desired finite automaton.

Remark. The number of large and small types is finite. Small types are matrices of order \( n \), their entries are polynomials of bounded degree in \( r \) variables over \( \mathbb{Z}_p \). Large types are subsets of some finite sets.

Theorem 4. The set of solutions for an EDE over the ring \( \mathbb{R}[B] \) where \( B \) is an entire matrix of standard form, is a regular language.

Construction of the finite automaton is completely similar to the case of the ring of polynomials. Again vertices are large types, and an arrow marked by \( x \) goes from \( T_1 \) to \( T_2 \) iff \( T_2 = \Pi(x)T_1 \). The initial vertex is \( T(\lambda) \), and final vertices are all of good large types.
The proof is similar to the one given in the preceding section.

Theorem 4 and Lemma 7 immediately imply the following

Theorem 5. Suppose \( F \) is a field, \( \text{char } F = p \), then the set of solutions for an EDE over \( \mathbb{M}_n(F) \) is a regular language.

Proof. An EDE includes a finite number of matrix entries, so all of them belong to some finite extension of \( \mathbb{Z}_p \). Any such extension may be included in \( A \) if \( r \) is its transcendence degree. Hence the original equation is an EDE over \( \mathbb{M}_n(A) \).

By Lemma 7 and Theorem 4 we obtain that the set of its solutions is a regular language.

Corollary. If \( R \) is a ring representable by matrices over a field \( F \), \( \text{char } F = p \), then the set of solutions for an EDE over \( R \) is a regular language.

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