Estimation and Clustering in Popularity Adjusted Stochastic Block Model

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Abstract

The paper considers the Popularity Adjusted Block model (PABM) introduced by Sengupta and Chen (2018). We argue that the main appeal of the PABM is the flexibility of the spectral properties of the graph which makes the PABM an attractive choice for modeling networks that appear in biological sciences. We expand the theory of PABM to the case of an arbitrary number of communities which possibly grows with a number of nodes in the network and is not assumed to be known. We produce the estimators of the probability matrix and the community structure and provide non-asymptotic upper bounds for the estimation and the clustering errors. We use the Sparse Subspace Clustering (SSC) approach to partition the network into communities, the approach that, to the best of our knowledge, has not been used for clustering network data. The theory is supplemented by a simulation study. In addition, we show advantages of the PABM for modeling a butterfly similarity network and a human brain functional network.

Keywords and phrases: Stochastic Block Model, Popularity Adjusted Block Model, Spectral Clustering, Sparse Subspace Clustering

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1 Introduction

Statistical network analysis has become a major field of research, with applications as diverse as sociology, biology, genetics, ecology, information technology to name a few. An overview of statistical modeling of random graphs can be found in, e.g., [16] and [21].

Consider an undirected network with \( n \) nodes and no self-loops and multiple edges. Let \( A \in \{0, 1\}^{n \times n} \) be the symmetric adjacency matrix of the network with \( A_{i,j} = 1 \) if there is a connection between nodes \( i \) and \( j \), and \( A_{i,j} = 0 \) otherwise. We assume that

\[
A_{i,j} \sim \text{Bernoulli}(P_{i,j}), \quad 1 \leq i \leq j \leq n, \tag{1.1}
\]

where \( A_{i,j} \) are conditionally independent given \( P_{i,j} \) and \( A_{i,j} = A_{j,i}, P_{i,j} = P_{j,i} \) for \( i > j \).

The block models assume that each node in the network belongs to one of \( K \) distinct blocks or communities \( \mathcal{N}_k, k = 1, \ldots, K \). Let \( c \) denote the vector of community assignment, with \( c_i = k \) if the node \( i \) belongs to the community \( k \). Then, the probability of connection between node \( i \in \mathcal{N}_k \) and node \( j \in \mathcal{N}_l \) depends on the pair of blocks \( (k, l) \) to which nodes \( (i, j) \) belong. One can also consider a corresponding membership (or clustering) matrix \( Z \in \{0, 1\}^{n \times K} \) such that \( Z_{i,k} = 1 \) iff \( i \in \mathcal{N}_k, i = 1, \ldots, n \).

A classical random graph model for networks with community structure is the Stochastic Block Model (SBM) that was studied by a number of authors (see, e.g., [1], [14] among others). Under this model, all nodes belonging to a community are considered to be stochastically equivalent, in the sense that the probability of connection between nodes is completely defined by the communities to which they belong. Specifically, under the \( K \)-block SBM, this probability is completely determined by the community assignment for nodes \( (i, j) \).
so that \( P_{i,j} = B_{c_i,c_j} \) where \( B_{k,l} \) is the probability of connection between communities \( k \) and \( l \). In particular, any nodes from the same community have the same degree distribution and the same expected degree.

Since the real-life networks usually contain a very small number of high-degree nodes while the rest of the nodes have very few connections (low degree), the SBM model fails to explain the structure of many networks that occur in practice. The Degree-Corrected Block Model (DCBM) addresses this deficiency by allowing these probabilities to be multiplied by the node-dependent weights (see, e.g., \([19], [40], [9]\) among others). Under the DCBM, the elements of matrix \( P \) are modeled as \( P_{i,j} = \theta_i B_{c_i,c_j} \theta_j \), where \( \theta_i, i = 1, \ldots, n \), are the degree parameters of the nodes, and \( B \) is the \((K \times K)\) matrix of baseline interaction between communities. Identifiability of the parameters is usually ensured by a constraint of the form \( \sum_{i \in \mathcal{N}_k} \theta_i = 1 \) for all \( k = 1, \ldots, K \) (see, e.g., \([19]\)).

A network feature that is closely associated with community structure is the popularity of nodes across communities defined as the number of edges between a specific node and a specific community. DCBM enforces node popularity to be uniformly proportional to the node degree, allows to correctly detect the communities, and accurately fits the total degree by enforcing the node-specific degree parameters. However, DCBM fails to model node popularities in a flexible and realistic way. For this reason, recently, Sengupta and Chen (2018) introduced the Popularity Adjusted Stochastic Block Model (PABM) which models the probability of a connection between nodes as a product of popularity parameters that depend on the communities to which the nodes belong as well as on the pair of nodes themselves. In particular, in PABM

\[
P_{i,j} = V_{i,c_i} V_{j,c_j},
\]

where \( V_{i,k}, 1 \leq i \leq n, 1 \leq k \leq K \), are the popularity scaling parameters and \( 0 \leq P_{i,j} \leq 1 \) for all \( i,j \). Sengupta and Chen \([31]\) introduced the notion of popularity of the node \( i \) in the community \( k \) as \( \mu_{i,k} = \sum_{j \in \mathcal{N}_k} P_{i,j} \). They noted that the ratio of popularities of the nodes \((i,j) \in \mathcal{N}_k\) in the same community \( k \) is equal to one for the SBM, in independent of community \( k \) (a function of \( i \) and \( j \) only) in DCBM but can vary between nodes and communities for the PABM, thus, allowing a more flexible modeling of connection probabilities. The authors showed that PABM generalizes both the SBM and the DCBM and suggested the quasi-maximum likelihood type procedure for estimation and clustering and demonstrated the improvement achieved through this new methodology.

The flexibility of PABM, however, is not limited to modeling the popularity parameters of the nodes. In order to understand the model better, consider a rearranged version \( P(Z, K) \) of matrix \( P \) where its first \( n_1 \) rows correspond to nodes from class 1, the next \( n_2 \) rows correspond to nodes from class 2 and the last \( n_K \) rows correspond to nodes from class \( K \). Consider vectors \( \Lambda^{(k,l)} \) with elements \( \Lambda^{(k,l)}_{i,j} = V_{i,l} \), where \( j = 1, \ldots, n_k \) and \( i,j \in \mathcal{N}_k \). Then, equation (1.2) implies that

\[
P^{(k,l)}(Z, K) = \Lambda^{(k,l)} \Lambda^{(l,k)}^T,
\]

so that \( P^{(k,l)}(Z, K) \) are rank-one matrices. Moreover, it follows from (1.2) that \( P^{(k,l)}(Z, K) = [P^{(l,k)}(Z, K)]^T \) and that each pair of blocks \((k,l)\) involves a unique combination of vectors \( \Lambda^{(l,k)} \). The latter implies that matrix \( P(Z, K) \) is formed by arbitrary rank one blocks and hence rank \( \text{rank}(P(Z, K)) = \text{rank}(P) \) can take any value between \( K \) and \( K^2 \). In comparison, all other block models restrict the rank of \( P \) to be exactly \( K \). This is true not only for the SBM and DCBM discussed above but also for their generalizations such as the Mixed Membership models (see, e.g., \([2]\) and \([10]\) and the Degree Corrected Mixed Membership (DCMM) (see, e.g., \([17]\)). Hence, the PABM allows for much more flexible spectral structure than any other block model above.

The latter makes the PABM model an attractive choice for modeling networks that appear in biological sciences. Indeed, while social networks exhibit assortative behavior due to the human tendency of forming strong associations, the biological networks tend to be more diverse. For this reason, PABM tends to be a useful tool for modeling such networks.
However, while the PABM model is extremely valuable, the statistical inference in Sengupta and Chen [31] has been incomplete. In particular, the authors considered only the case of a small finite number of communities \( K \); they provided only asymptotic consistency results as \( n \to \infty \) without any error bounds for finite \( n \); their clustering procedure is tailored to the case of a small \( K \), therefore, all simulations and real data examples in [31] only tackle the case of \( K = 2 \).

The purpose of the present paper is to address some of those deficiencies and to advance the theory of the PABM. Specifically, our paper makes the following contributions:

1. In contrast to [31], we consider the PABM with an arbitrary number of communities which possibly grows with a number of nodes in the network and is not assumed to be known.

2. We argue that the main appeal of the PABM is the flexibility of the spectral properties of the graph and replace the estimators in [31] that are based on averaging over the communities by more accurate counterparts based on low rank matrix approximations.

3. While Sengupta and Chen [31] only proved convergence of the estimation and clustering errors to zero as the number of nodes grows, we derive non-asymptotic upper bounds for those errors for an arbitrary number of classes. In particular, we produce an upper bound for the estimation error of the matrix of the connection probabilities and provide a condition that guarantees that the proportion of misclassified nodes is bounded above by a specified quantity. All results in the paper are non-asymptotic and are valid for any combination of parameters.

4. We use the accuracy of approximation of the adjacency matrix for various number of communities, to identify the number of communities in the network.

5. We suggest to use the Sparse Subspace Clustering (SSC) approach to partition the network into communities. While the SSC is widely used in computer vision, to the best of our knowledge, it has not been used for clustering network data. The advantage of the SSC procedure (in comparison with the Extreme Point algorithm applied in [31]) is that it has several well studied versions and can carry out clustering not only for the PABM but also for the SBM and DCBM.

6. Our simulation study as well as the real data examples handle various number of communities \( K \) between 2 and 6. In particular, we demonstrate the advantages of the PABM for modeling networks that appear in biological sciences.

The rest of the paper is organized as follows. Section 2 discusses estimation and clustering in PABM as a solution of a penalized optimization procedure. Section 2.1 introduces notations used throughout the paper. Section 2.2 formulates estimation and clustering as solutions of an optimization procedure. Section 2.3 derives upper bounds for estimation errors as well as sufficient conditions that the proportion of misclustered nodes is bounded above by a pre-specified quantity \( p_n \) with high probability. Section 3 deliberates about practical implementation of clustering and provides a simulation study and real data examples. In particular, Section 3.1 reviews the SSC and elaborates on what kind of SSC procedure we employ in this paper. Section 3.2 evaluates the performance of this method using synthetic networks with various values of \( K \). Furthermore, we compare the performance of the SSC with the Extreme Point algorithm applied in [31] using the simulation example presented in [31]) and show the superiority of the former, especially when the homophily factor is small. Section 3.3 brings two examples of biological networks that we model using PABM. Finally, Section 4 presents the proofs of all statements in the paper.
2 Estimation and clustering

2.1 Notation

For any two positive sequences \( \{a_n\} \) and \( \{b_n\} \), \( a_n \asymp b_n \) means that there exists a constant \( C > 0 \) independent of \( n \) such that \( C^{-1} a_n \leq b_n \leq C a_n \) for any \( n \). For any set \( \Omega \), denote cardinality of \( \Omega \) by \( |\Omega| \). For any vector \( t \in \mathbb{R}^p \), denote its \( \ell_2, \ell_1, \ell_0 \) and \( \ell_\infty \) norms by, respectively, \( \|t\|, \|t\|_1, \|t\|_0 \) and \( \|t\|_\infty \). Denote by \( m \) the \( m \)-dimensional column vector with all components equal to one.

For any matrix \( A \), denote its spectral and Frobenius norms by, respectively, \( \|A\|_{op} \) and \( \|A\|_F \). Let \( \text{vec}(A) \) be the vector obtained from matrix \( A \) by sequentially stacking its columns.

Denote by \( \mathcal{M}_{n,k} \) a collection of clustering matrices \( Z \in \{0,1\}^{n \times K} \) such that \( Z_{i,k} = 1 \) iff \( i \in N_k \), \( i = 1, \ldots, n \), and \( Z^T Z = \text{diag}(n_1, \ldots, n_K) \) where \( n_k = |N_k| \) is the size of community \( k \), where \( k = 1, \ldots, K \). Denote by \( \mathcal{P}_{Z,K} \in \{0,1\}^{n \times n} \) the permutation matrix corresponding to \( Z \in \mathcal{M}_{n,k} \) that rearranges any matrix \( B \in \mathbb{R}^{n \times n} \), so that its first \( n_1 \) rows correspond to nodes from class 1, the next \( n_2 \) rows correspond to nodes from class 2 and the last \( n_K \) rows correspond to nodes from class \( K \). Recall that \( \mathcal{P}_{Z,K} \) is an orthogonal matrix with \( \mathcal{P}_{Z,K}^1 = \mathcal{P}_{Z,K}^T \). For any \( \mathcal{P}_{Z,K} \) and any matrix \( B \in \mathbb{R}^{n \times n} \) denote the permuted matrix and its blocks by, respectively, \( B(Z,K) \) and \( B^{(k,l)}(Z,K) \), where \( B^{(k,l)}(Z,K) \in \mathbb{R}^{n_k \times n_l} \), \( k = 1, \ldots, K \), and \( l = 1, \ldots, K \).

\[
B(Z,K) = \mathcal{P}_{Z,K}^T B \mathcal{P}_{Z,K}, \quad B = \mathcal{P}_{Z,K} B(Z,K) \mathcal{P}_{Z,K}^T.
\]

(4.4)

Also, throughout the paper, we use the star symbol to identify the true quantities. In particular, we denote the true matrix of connection probabilities by \( P_* \), the true number of classes by \( K_* \) and the true clustering matrix that partitions \( n \) nodes into \( K_* \) communities by \( Z_* \).

2.2 Optimization procedure for estimation and clustering

In this section we consider estimation of the true probability matrix \( P_* \). Consider block \( P_*^{(k,l)}(Z_*,K_*) \) of the rearranged version \( P_*^{(Z_*,K_*)} \) of \( P_* \). Let \( \Lambda = \Lambda(Z_*,K_*) \in [0,1]^{n \times K_*} \) be a block matrix with each column \( l \) partitioned into \( K_* \) blocks \( \Lambda^{(k,l)} = \Lambda^{(k,l)}(Z_*,K_*) \in [0,1]^{n_k} \). Then, due to (1.3), \( P_*^{(k,l)}(Z_*,K_*) \) are rank-one matrices such that \( P_*^{(k,l)}(Z_*,K_*) = \mathcal{P}_{Z,K} (Z_*,K_*)^{(k,l)} \) and that each pair of blocks \( (k,l) \) involves a unique combination of vectors \( \Lambda^{(k,l)} \).

Observe that although matrices \( P_*^{(k,l)}(Z_*,K_*) \) in (1.3) are well defined, vectors \( \Lambda^{(k,l)} \) and \( \Lambda^{(l,k)} \) can be determined only up to a multiplicative constant. In particular, under the constraint

\[
\mathcal{T}_{n_k} \Lambda^{(k,l)} = \mathcal{T}_{n_l} \Lambda^{(l,k)},
\]

(2.5)
of (3.1), one obtained explicit expressions for vectors \( \Lambda^{(k,l)} \) and \( \Lambda^{(l,k)} \) in (1.3).

In reality, \( K_* \) and matrices \( Z_* \) and \( P_* \) are unknown and need to be recovered. If \( K_* \) were known, in order to estimate \( Z_* \) and \( P_* \), one could permute the rows and the columns of the adjacency matrix \( A \) using permutation matrix \( \mathcal{P}_{Z,K_*} \), obtaining matrix \( A(Z_*,K_*) = \mathcal{P}_{Z,K_*}^T A \mathcal{P}_{Z,K_*} \) and then, following assumption (1.3), minimize some divergence measure between blocks of \( A(Z_*,K_*) \) and the product \( \Lambda^{(k,l)} \mathcal{P}_{Z,K_*}^T \). One of such measures is the Bregman divergence between \( A(Z_*,K_*) \) and \( \Lambda^{(k,l)} \mathcal{P}_{Z,K_*}^T \).

The Bregman divergence between vectors \( x \) and \( y \) associated with a continuously-differentiable, strictly convex function \( F \) is defined as

\[
D_F(x,y) = F(x) - F(y) - \langle \nabla F(y), x-y \rangle
\]

where \( \nabla F(y) \) is the gradient of \( F \) with respect to \( y \). The Bregman divergence between any matrices \( X \) and \( Y \) of the same dimension can be defined as the Bregman divergence between their vectorized versions:
$D_F(X, Y) = D_F(\text{vec}(X), \text{vec}(Y))$. It is well known that $D_F(X, Y) \geq 0$ for any $X$ and $Y$ and $D_F(X, Y) = 0$ iff $X = Y$. In particular, the Poisson log-likelihood maximization used in [31] corresponds to minimizing the Bregman divergence with

$$F(x) = \sum_i (x_i \ln x_i - x_i).$$

Under the assumption (1.3) and the constraint (2.5) of [31], the latter leads to maximization over $\Lambda^{(k,l)}$ and $Z \in M_{n,K}$ of the following quantity

$$l(\Lambda|A) = -D_F(A, \Lambda) = \sum_{k,l=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{n_l} \left[ A_{i,j}^{(k,l)} \ln \left( \Lambda_i^{(k,l)} \Lambda_j^{(l,k)} \right) - \left( \Lambda_i^{(k,l)} \Lambda_j^{(l,k)} \right) \right]. \tag{2.6}$$

where $A^{(k,l)}$ stands for $A^{(k,l)}(Z, K_*)$, the $(k, l)$-th block of matrix $A(Z, K_*)$. It is easy to see that the expression (2.6) coincides with $A^{(k,l)}(Z, K_*)$, the Poisson log-likelihood up to a which is independent of $P, Z$ and $K_*$ and depends on matrix $A$ only. Maximization of (2.6) over $\Lambda$ under condition (2.5), for given $Z$ and $K_*$, leads to the estimators of $\Lambda$ obtained in [31]

$$\hat{\Lambda}^{(k,l)} = \frac{A^{(k,l)}(Z, K_*)1_{n_k}}{\sqrt{1_{n_k}^T A^{(k,l)}(Z, K_*)1_{n_k}}}; \quad \hat{\Lambda}^{(l,k)} = \frac{(A^{(k,l)}(Z, K_*)^T 1_{n_k})}{\sqrt{1_{n_k}^T A^{(k,l)}(Z, K_*)1_{n_k}}}. \tag{2.7}$$

Afterwards, Sengupta and Chen [31] plug the estimators (2.7) into (2.6), thus, obtaining the likelihood modularity function which they further maximize in order to obtain community assignments.

In the present paper, we use the Bregman divergence associated with the Euclidean distance ($F(x) = \|x\|^2$) which, for a given $K$, leads to the following optimization problem

$$(\hat{\Lambda}, \hat{Z}) \in \text{argmin}_{\Lambda, Z} \left\{ \sum_{k,l=1}^{K} \left\| A^{(k,l)}(Z, K) - \Lambda^{(k,l)}[\Lambda^{(l,k)}]^T \right\|_F^2 \right\} \quad \text{s.t.} \quad A(Z, K) = \mathcal{P}_{Z,K} A \mathcal{P}_{Z,K} \tag{2.8}$$

Note that recovery of the components $\Lambda^{(k,l)}$ and $\Lambda^{(l,k)}$ of the products above relies on an identifiability condition of the type (2.5). Since these conditions can be imposed in a variety of ways, we denote $\Theta^{(k,l)} = \Lambda^{(k,l)}[\Lambda^{(l,k)}]^T$ and recover the uniquely defined rank one matrix $\Theta^{(k,l)}$. In addition, since the number of clusters $K$ is unknown, we impose a penalty on $K$ in order to safeguard against choosing too many clusters. Hence, we need to solve the following optimization problem

$$(\hat{\Theta}, \hat{Z}, \hat{K}) \in \text{argmin}_{\Theta, Z, K} \left\{ \sum_{k,l=1}^{K} \left\| A^{(k,l)}(Z, K) - \Theta^{(k,l)} \right\|_F^2 + \text{Pen}_r(n, K) \right\} \quad \text{s.t.} \quad A(Z, K) = \mathcal{P}_{Z,K} A \mathcal{P}_{Z,K}, \quad \text{rank}(\Theta^{(k,l)}) = 1; \quad k, l = 1, 2, \ldots, K. \tag{2.8}$$

Here, $\hat{\Theta}$ is the block matrix with blocks $\hat{\Theta}^{(k,l)}, k, l = 1, \ldots, \hat{K}$ and $\tau > 0$ is a custom-chosen parameter which guarantees that the estimation and clustering errors are bounded above with the probability at least $1 - 4n^{-\tau}$.

Observe that, if $\hat{Z}$ and $\hat{K}$ were known, the best solution of problem (2.8) would be given by the rank one approximations $\hat{\Theta}^{(k,l)}$ of matrices $A^{(k,l)}(\hat{Z}, \hat{K})$

$$\hat{\Theta}^{(k,l)}(\hat{Z}, \hat{K}) = \Pi_{\hat{Z}, \hat{K}} \left( A^{(k,l)}(\hat{Z}, \hat{K}) \right) = \tilde{\sigma}_1^{(k,l)} \tilde{u}^{(k,l)}(\hat{Z}, \hat{K})(\tilde{v}^{(k,l)}(\hat{Z}, \hat{K}))^T, \tag{2.9}$$

where $\tilde{\sigma}_1^{(k,l)}$ are the largest singular values of matrices $A^{(k,l)}(\hat{Z}, \hat{K})$; $\tilde{u}^{(k,l)}(\hat{Z}, \hat{K})$, $\tilde{v}^{(k,l)}(\hat{Z}, \hat{K})$ are the corresponding singular vectors, and $\Pi_{\hat{Z}, \hat{K}} \left( A^{(k,l)}(\hat{Z}, \hat{K}) \right)$ is the rank one projection of matrix $A^{(k,l)}(\hat{Z}, \hat{K})$ (see
Lemma \ref{lemma:optimal_solution} in Section \ref{sec:optimization} (Appendix) for the exact expression. Plugging \eqref{eq:2.9} into \eqref{eq:2.8}, we rewrite optimization problem \eqref{eq:2.8} as

\begin{equation}
\hat{Z}, \hat{K} \in \arg\min_{Z, K} \left\{ \sum_{k,l=1}^{K} \left\| A^{(k,l)}(Z, K) - \Pi_{\hat{u}, \hat{v}} \left( A^{(k,l)}(Z, K) \right) \right\|^2_F + \text{Pen}_{\tau}(n, K) \right\}
\end{equation}

\text{s.t.} \quad A(Z, K) = \mathcal{P}_Z K A \mathcal{P}_Z K

In order to obtain \((\hat{Z}, \hat{K})\), one needs to solve optimization problem \eqref{eq:2.10} for every \(K\), obtaining

\begin{equation}
\hat{Z}_K \in \arg\min_{Z \in \mathcal{M}_{n, K}} \left\{ \sum_{k,l=1}^{K} \left\| A^{(k,l)}(Z, K) - \Pi_{\hat{u}, \hat{v}} \left( A^{(k,l)}(Z, K) \right) \right\|^2_F \right\}
\end{equation}

and then find \(\hat{K}\) as

\begin{equation}
\hat{K} \in \arg\min_{K} \left\{ \sum_{k,l=1}^{K} \left\| A^{(k,l)}(\hat{Z}_K, K) - \Pi_{\hat{u}, \hat{v}} \left( A^{(k,l)}(\hat{Z}_K, K) \right) \right\|^2_F + \text{Pen}_{\tau}(n, K) \right\}.
\end{equation}

Note that if the true number of clusters \(K_s\) were known, the penalty in \eqref{eq:2.10} would be unnecessary.

\textbf{Remark 1. Advantages of our estimation procedure.} There are several advantages of the estimator \eqref{eq:2.9} in comparison with estimators \eqref{eq:2.7} of \cite{sengupta2018community}. First, rather than obtaining estimators in \eqref{eq:2.7} by averaging, we derive the best rank one approximation of the unknown matrix of probabilities (see, e.g., \cite{albano2017community}) even when there are misclustered nodes and, therefore, the matrices \(P^{(k,l)}_\ast(\hat{Z}, \hat{K})\) are not of rank one. Indeed, the estimators obtained by averaging are suboptimal since matrix \(P_\ast\) is contaminated with errors. Second, recoveries of the matrices \(\hat{\Theta}^{(k,l)}\) do not require any identifiability conditions that can be imposed in a variety of ways. Finally, estimators \(\hat{\Lambda}^{(k,k)}\) of vectors \(\Lambda^{(k,k)}\) in \eqref{eq:2.7} require the knowledge of the diagonal elements of matrix \(A\) that are not available. On the contrary, the rank one approximation of a matrix can be achieved in the presence of missing values (see, e.g., \cite{albano2017community}).

\textbf{Remark 2. The true community assignment.} Sengupta and Chen \cite{sengupta2018community} show that, with the true matrix \(P_\ast\), the likelihood modularity is maximized at the true community assignment provided the so-called, detectability condition holds: for any two distinct communities \(N_i\) and \(N_k\) and any two nodes, \(j_1 \in N_i\) and \(j_2 \in N_k\), the set \(\{(P_\ast)_{i,j_1}/(P_\ast)_{i,j_2}\}_{i=1}^{n}\) assumes at least \(K_s + 1\) distinct values, where \(K_s\) is the true (known) number of clusters. In our case, the correct community assignment is a solution of the optimization problem \eqref{eq:2.11} if matrix \(P_\ast\) is a unique combination (up to permutations) of the \(K^2\) rank one matrices. The latter is guaranteed if subspaces \(S_k = \text{Span}(\Lambda^{(k,1)}, \ldots, \Lambda^{(k,K_s)}), k = 1, \ldots, K_s\) are independent for different values of \(k\). Here, vectors \(\Lambda^{(k,l)}\) are obtained by completing vectors \(\Lambda^{(k,l)}\) with zeros to the \(n\)-dimensional vectors. Milder conditions based on the affinity between pairs of subspaces \(S_k\) and \(S_l, k, l = 1, \ldots, K_s\), can be found in \cite{sengupta2018community}.

\subsection{2.3 The errors of estimation and clustering}

In this section we evaluate the estimation and the clustering errors. We choose the penalty which, with high probability, exceeds the random errors. In particular, we denote

\begin{align}
F_1(\tau, n, K) &= 72nK + 2K^2c \left[ (\tau + 2) \ln n + (n + 1) \ln K \right], \\
F_2(\tau, n, K) &= 2[(\tau + 1) \ln n + n \ln K],
\end{align}
Theorem 2. Let (\hat{\Theta}, \hat{Z}, \hat{K}) be a solution of optimization problem (2.8). Construct the estimator \( \hat{P} \) of \( P_* \) of the form

\[ \hat{P} \equiv \hat{P}(\hat{Z}, \hat{K}) = \hat{\mathcal{P}}_{\hat{Z}, \hat{K}} \hat{\Theta}(\hat{Z}, \hat{K}) \hat{\mathcal{P}}^T_{\hat{Z}, \hat{K}} \]

where \( \mathcal{P}_{\hat{Z}, \hat{K}} \) is the permutation matrix corresponding to \((\hat{Z}, \hat{K})\). Then, for any \( \tau > 0 \), with probability at least \( 1 - 4n^{-\tau} \), one has

\[ \frac{1}{n^2} \left\| \hat{P} - P_* \right\|_F^2 \leq \frac{\text{Pen}_\tau(n, K_*)}{(1 - \beta_1 - \beta_2)n^2} \]

Moreover, if \( \tau \geq 2 \) in (2.15), then

\[ \frac{1}{n^2} E \left\| \hat{P} - P_* \right\|_F^2 \leq \frac{2}{n} + \frac{\text{Pen}_\tau(n, K_*)}{(1 - \beta_1 - \beta_2)n^2}. \]

The value of \( \tau \) in the penalty is a custom chosen parameter. It is easy to see that the term containing \( \tau \) in the penalty (2.15) is asymptotically negligible if \( n \) is large. In practice, one can choose \( \tau = 2 \) that guarantees the upper bounds (2.17) and (2.18) with probability \( 1 - 4n^{-2} \) and in expectation.

In order to evaluate the clustering error, we assume that the true number of classes \( K = K_* \) is known. Let \( Z_* \in \mathcal{M}_{n,K_*} \) be the true clustering matrix. Then \( \hat{Z} \equiv \hat{Z}_K \) is a solution of the optimization problem (2.11). Note that if \( Z_* \) is the true clustering matrix and \( Z \) is any other clustering matrix, then the proportion of misclustered nodes can be evaluated as

\[ \text{Err}(Z, Z_*) = (2n)^{-1} \min_{\mathcal{P}_k \in \mathcal{P}_k} \| Z \mathcal{P}_k - Z_* \|_1 = (2n)^{-1} \min_{\mathcal{P}_k \in \mathcal{P}_k} \| Z \mathcal{P}_k - Z_* \|_F^2 \]

where \( \mathcal{P}_k \) is the set of permutation matrices \( \mathcal{P}_k : \{1, 2, \ldots, k\} \rightarrow \{1, 2, \ldots, k\} \). Let

\[ \Upsilon(Z_*, \rho) = \left\{ Z \in \mathcal{M}_{n,K} : (2n)^{-1} \min_{\mathcal{P}_k \in \mathcal{P}_k} \| Z \mathcal{P}_k - Z_* \|_1 \geq \rho \right\} \]

be the set of clustering matrices with the proportion of misclustered nodes being at least \( \rho \), \( \rho < 1 \). Observe that, since \( n(1 - \rho) \) nodes are clustered arbitrarily in the set \( \Upsilon(Z_*, \rho) \), the log-cardinality of the set \( \Upsilon(Z_*, \rho) \) is asymptotically equal to that of \( \mathcal{M}_{n,K} : \ln(|\Upsilon(Z_*, \rho)|) \propto n \ln K \).

The success of clustering in (2.11) relies upon the fact that matrix \( P_* \) is a collection of \( K^2 \) rank one blocks, so that the operator and the Frobenius norms of each block are the same. On the other hand, if clustering were incorrect, the ranks of the blocks would increase which would lead to the discrepancy between their operator and Frobenius norms. In particular, the following statement is true.

Theorem 2. Let \( K = K_* \) be the true number of clusters and \( Z_* \in \mathcal{M}_{n,K_*} \) be the true clustering matrix. If for some \( \alpha_1, \alpha_2 \in (0, 1) \) and \( \rho_n > 0 \), one has

\[ \| P_* \|_F^2 - \frac{1 + \alpha_2}{1 - \alpha_1} \max_{Z \in \Upsilon(Z_*, \rho_n)} \sum_{k,l=1}^{K} \left\| P_<(k,l)(Z) \right\|_o^2 \geq H_1 nK^2 \ln K + H_2 nK + H_3 K^2 \ln n, \]

then, with probability at least \( 1 - 4n^{-7} \), the proportion of the misclassified nodes is at most \( \rho_n \). Here, \( H_l \equiv H_l(\alpha_1, \alpha_2), \ l = 1, 2, 3, \) are functions of \( \alpha_1 \) and \( \alpha_2 \) only (the exact expressions for \( H_1, H_2 \) and \( H_3 \) are given in (4.60) in the Appendix).
3 Simulations and a real data example

3.1 Sparse subspace clustering

In Section 2, we obtained an estimator \( \hat{Z} \) of the true clustering matrix \( Z \) as a solution of optimization problem (2.10). Minimization in (2.10) is somewhat similar to modularity maximization in (5) or (40) in the sense that modularity maximization as well as minimization in (2.10) are NP-hard, and, hence, require some relaxation in order to obtain an implementable clustering solution.

In the case of the SBM and the DCBM, possible relaxations include semidefinite programming (see, e.g., [3] and references therein), variational methods (8) and spectral clustering and its versions (see, e.g., [13], [23] and [30] among others). Since in the case of PABM, columns of matrix \( P_\ast \) that correspond to nodes in the same class are neither identical, nor proportional, direct application of spectral clustering to matrix \( P_\ast \) does not deliver the partition of nodes. However, it is easy to see that the columns of the matrix \( P_\ast \) that correspond to nodes in the same class form a matrix with \( K \) rank-one blocks, hence, those columns lie in the subspace of the dimension at most \( K \). Therefore, matrix \( P_\ast \) is constructed of \( K \) clusters of columns (rows) that lie in the union of \( K \) distinct subspaces, each of the dimension \( K \). For this reason, the subspace clustering presents a technique for obtaining a fast and reliable solution of optimization problem (2.10).

Subspace clustering has been widely used in computer vision and, for this reason, it is a very well studied and developed technique in comparison with the Extreme Points algorithm used in [31]. Subspace clustering is designed for separation of points that lie in the union of subspaces. Let \( \{X_j \in \mathbb{R}^D\}_{j=1}^n \) be a given set of points drawn from an unknown union of \( K \geq 1 \) linear or affine subspaces \( \{S_i\}_{i=1}^K \) of unknown dimensions \( d_i = \dim(S_i), \) \( 0 < d_i < D, \) \( i = 1, ..., K. \) In the case of linear subspaces, the subspaces can be described as

\[
S_i = \{x \in \mathbb{R}^D : x = U_i y\}, \quad i = 1, ..., K
\]

where \( U_i \in \mathbb{R}^{D \times d_i} \) is a basis for subspace \( S_i \) and \( y \in \mathbb{R}^{d_i} \) is a low-dimensional representation for point \( x. \) The goal of subspace clustering is to find the number of subspaces \( K \), their dimensions \( \{d_i\}_{i=1}^K \), the subspace bases \( \{U_i\}_{i=1}^K \), and the segmentation of the points according to the subspaces.

Several methods have been developed to implement subspace clustering such as algebraic methods [6] [26], [37], iterative methods [7] [29] [34], and spectral clustering based methods [13] [24] [25] [33] [35] [12] [36]. In this paper, we shall use the latter group of techniques.

Spectral clustering algorithms rely on construction of an affinity matrix whose entries are based on some distance measures between the points. In particular, in the case of the SBM, adjacency matrix itself serves as an affinity matrix, while for DCBM, affinity matrix is obtained by normalizing rows/columns of \( A. \) In the case of the subspace clustering problem, one cannot use the typical distance-based affinity because two points could be very close to each other, but lie in different subspaces, while they could be far from each other, but lie in the same subspace. One of the solutions is to construct the affinity matrix using self-representation of the points with the expectation that a point is more likely to be presented as a linear combination of points in its own subspace rather than from a different one. A number of approaches such as Low Rank Representation (see, e.g., [24] [25]) and Sparse Subspace Clustering (see, e.g., [12] [35]) have been proposed in the past decade for the solution of this problem.

In this paper, we use Sparse Subspace Clustering (SSC) since it allows one to take advantage of the knowledge that, for a given \( K \), columns of matrix \( P_\ast \) lie in the union of \( K \) distinct subspaces, each of the dimension at most \( K. \) If matrix \( P_\ast \) were known, the weight matrix \( W \) would be based on writing every data point as a sparse linear combination of all other points by minimizing the number of nonzero coefficients

\[
\min_{W_j} \|W_j\|_0 \quad \text{s.t.} \quad (P_\ast)_j = \sum_{k \neq j} W_{kj}(P_\ast)_k
\]

(3.22)
where, for any matrix $B$, $B_j$ is its $j$-th column. The affinity matrix of the SSC is the symmetrized version of the weight matrix $W$. If the subspaces are linearly independent, then the solution to the optimization problem (3.22) is such that $W_{k,j} \neq 0$ only if points $k$ and $j$ are in the same subspace. In the case of data contaminated by noise, the SSC algorithm does not attempt to write data as an exact linear combination of other points. Instead, SSC is based on the solution of the following optimization problem

$$\hat{W}_j \in \arg\min_{W_j} \left\{ \|W_j\|_0 + \gamma \|A_j - AW_j\|_2 \right\} \text{ s.t. } W_{jj} = 0, \quad j = 1, \ldots, n,$$

where $\gamma > 0$ is a tuning parameter. Problem (3.23) can be rewritten in an equivalent form as

$$\hat{W}_j \in \arg\min_{W_j} \left\{ \|A_j - AW_j\|_2^2 \text{ s.t. } \|W_j\|_0 \leq L, \quad W_{jj} = 0 \right\}, \quad j = 1, \ldots, n,$$

where $L$ is the maximum number of nonzero elements in each column of $W$; in our case $L = K$. We solve (3.24) using the Orthogonal Matching Pursuit (OMP) algorithm [28, 39] implemented in SPAMS Matlab toolbox (see [27]). Given $\hat{W}$, the affinity matrix is defined as $|\hat{W}| + |\hat{W}^T|$ where, for any matrix $B$, matrix $|B|$ has absolute values of elements of $B$ as its entries. The class assignment (clustering matrix) $Z$ is then obtained by applying spectral clustering to $|\hat{W}| + |\hat{W}^T|$. We elaborate on the implementation of the SSC in Section 3.2.

### 3.2 Simulation on synthetic networks

In this section we evaluate the performance of our method using synthetic networks. We assume that the number of communities (clusters) $K$ is known and for simplicity consider a perfectly balanced model with $n/K$
nodes in each cluster. We generate each network from a random graph model with a symmetric probability matrix $P$ given by the PABM model with a clustering matrix $Z$ and a block matrix $\Lambda$.

Sengupta and Chen (2018), in their simulations, considered networks with $K = 2$ communities of equal sizes and matrices $\Lambda$ in (1.2) with elements $\Lambda_{i,r} = \alpha_i \sqrt{\frac{h}{1+h}}$ when node $i$ lies in class $r$, and $\Lambda_{i,r} = \beta_i \sqrt{\frac{1}{1+h}}$ otherwise, where $h$ is the homophily factor. The factors $\alpha_i$ and $\beta_i$ were set to 0.8 for half of the nodes in each class and to 0.2 for another half at random, and $h$ ranges between 1.5 and 4.0. Note that, although the data generated by the procedure above follows PABM, the probability matrix has constant blocks, for which the spectral clustering is known to deliver accurate results. In particular, the setting above leads to the SBM with four blocks. However, the spectral clustering incurs some difficulties as the probabilities of connections in every community become more diverse. In this paper, we ensure to generate networks that follow PABM with diverse probabilities of connections.

To generate a more diverse synthetic network, we start by producing a block matrix $\Lambda$ with random entries between 0 and 1. We multiply the non-diagonal blocks of $\Lambda$ by $\omega$, $0 < \omega < 1$, to ensure that most nodes in the same community have larger probability of interactions. Then matrix $P(Z, K)$ with blocks $P^{(k,l)}(Z, K) = \Lambda^{(k,l)}(\Lambda^{(l,k)})^T$, $k, l = 1, \ldots, K$, mostly has larger entries in the diagonal blocks than in the non-diagonal blocks. The parameter $\omega$ is the heterogeneity parameter. Indeed, if $\omega = 0$, the matrix $P_*$ is strictly block-diagonal, while in the case of $\omega = 1$, there is no difference between diagonal and non-diagonal blocks. Next, we generate a random clustering matrix $Z \in \mathcal{M}_{n,K}$ corresponding to the case of equal community sizes and the permutation matrix $\mathcal{P}(Z, K)$ corresponding to the clustering matrix $Z$. Subsequently, we scramble rows and columns of $P(Z, K)$ to create the probability matrix $P = \mathcal{P}_{Z,K} P(Z, K) \mathcal{P}_{Z,K}^T$. Finally we generate the lower half of the adjacency matrix $A$ as independent Bernoulli variables $A_{ij} \sim \text{Ber}(p_{ij})$, $i = 1, \ldots, n$, $j = 1, \ldots, i - 1$, and set $A_{ij} = A_{ji}$ when $j > i$. In practice, the diagonal $\text{diag}(A)$ of matrix $A$ is unavailable, so we estimate $\text{diag}(P)$ without its knowledge.

Figure 2: The clustering errors $\text{Err}(\hat{Z}, Z)$ defined in (2.19) (top) and the estimation errors $n^{-2} ||\hat{P} - P||^2_F$ (bottom) for $K = 4$ clusters. The errors are evaluated over 50 simulation runs. The number of nodes ranges from $n = 300$ to $n = 540$ with the increments of 60; $\omega = 0.5, 0.7$ and 0.9.
Figure 3: The clustering errors \( \text{Err}(\hat{Z}, Z) \) defined in (2.19) (top) and the estimation errors \( n^{-2} \| \hat{P} - P \|_F^2 \) (bottom) for \( K = 5 \) clusters. The errors are evaluated over 50 simulation runs. The number of nodes ranges from \( n = 300 \) to \( n = 540 \) with the increments of 60; \( \omega = 0.5, 0.7 \) and 0.9.

Sengupta and Chen [31] used the Extreme Points (EP) algorithm, introduced in [22], as a clustering procedure. For \( K = 2 \), the EP algorithm computes the two leading eigenvectors of the adjacency matrix \( A \), and finds the candidate assignments associated with the extreme points of the projection of the cube \([-1, 1]^n\) onto the space spanned by the two leading eigenvectors of \( A \). The technique is becoming problematic when \( K \) grows and communities are getting more diverse, hence, performances of estimation and clustering in [31] have only been studied the case of \( K = 2 \) and the choices of probability matrix \( P \) described above. As we have mentioned before, these are the settings for which the spectral clustering procedure allows to identify the communities. Considering that we are interested in studying \( K > 2 \) and the more diverse communities, we use the spectral clustering directly (SC thereafter) and compare its precision with the sparse subspace clustering (SSC) procedure.

Since the diagonal elements of matrix \( A \) are unavailable, we initially set \( A_{ii} = 0, i = 1, ..., n \). We solve optimization problem (3.24) using the Orthogonal Matching Pursuit (OMP) algorithm. After matrix \( \hat{W} \) of weights is evaluated, we obtain the clustering matrix \( \hat{Z} \) by applying spectral clustering to \( |\hat{W}| + |\hat{W}^T| \), as it is described in Section 3.1. Given \( \hat{Z} \), we generate matrix \( A(\hat{Z}) = \mathcal{P}_Z^T A \mathcal{P}_Z \) with blocks \( A^{(k,l)}(\hat{Z}), k, l = 1, \ldots, K \), and obtain \( \hat{G}^{(k,l)}(\hat{Z}, \hat{K}) \) by using the rank one approximation for each of the blocks. Finally, we estimate matrix \( P \) by \( \hat{P} = \hat{P}(\hat{Z}, \hat{K}) \) using formula (2.16) with \( \hat{K} = K \).

We compared the accuracy of SSC and SC methods in terms of the average estimation errors \( n^{-2} \| \hat{P} - P \|_F^2 \) and the average clustering errors \( \text{Err}(\hat{Z}, Z) \) defined in (2.19). Figures 1, 2, and 3 show results of these comparisons for \( K = 3, 4 \) and 5, respectively, and the number of nodes ranging from \( n = 300 \) to \( n = 540 \) with the increments of 60. Each figure consists of two plots. The top ones display the clustering errors \( \text{Err}(\hat{Z}, Z) \) while the bottom ones exhibit the estimation errors \( n^{-2} \| \hat{P} - P \|_F^2 \), as functions of the number of nodes. The errors are averaged over 50 simulation runs. Figures 1, 2, and 3 compare the errors obtained by SSC and SC with three different values of the parameter \( \omega \): \( \omega = 0.5, 0.7, \) and 0.9. All figures confirm that the SSC is...
becoming more and more accurate in comparison with SC as $\omega$ grows. The latter is due to the fact that the SSC is more suitable for handling heterogeneous connection probabilities.

Figure 4 presents the results of comparison of the clustering errors of SSC and SC in the simulations settings of Sengupta and Chen [31]. It is easy to see that, while for larger values of the homophily factor $h$ both methods perform almost equally well, the accuracy of SC deteriorates as $h$ is getting smaller, due to the fact that the differences between probabilities of connections within and between clusters become less significant. The latter shows that the SSC approach is beneficial for clustering in PABM model. Indeed, it delivers more accurate results than the SC when probabilities of connections are more diverse. On the other hand, SSC is still applicable when the PABM reduces to the SBM, although SC is more accurate in the case of the SBM since it does not require an additional step of evaluating the affinity matrix.

Remark 3. Spectral Clustering Versus Sparse Subspace Clustering. It is worth noting that when the matrix of probabilities $P_*$ is close to being block diagonal, the spectral clustering can be still used for recovering community assignments, even if $P_*$ does not follow the SBM. The latter is due to the fact that, in this situation, the graph can be well approximated by a union of distinct connected components, and, therefore, SC allows to identify the true clusters. Moreover, in such situation, SC has an advantage of not requiring an additional step of self-representation, which is computationally costly and produces additional errors. On the other hand, as we shall see from examples below, when probabilities of connections become more heterogeneous, SSC turns to be more precise than SC. In addition, since PABM has more unknown parameters than SBM, its correct fitting requires sufficient number of nodes per class (see, e.g., [33]); otherwise, its accuracy declines.

Remark 4. Unknown number of clusters. In our previous simulations we treated the true number of clusters as a known quantity. However, we can actually use $\hat{P}$ to obtain an estimator $\hat{K}$ of $K$ by solving, for every
Table 1: The relative frequencies of the estimators $\hat{K}$ of $K_*$ for $K_*$ ranging from 3 to 6, $n = 420$ and $n = 540$ and $\omega = 0.5, 0.7$ and 0.9.
suitable $K$, the optimization problem (2.12), which can be equivalently rewritten as

$$\hat{K} = \arg\min_K \{ \| \hat{P} - A \|_F^2 + \text{Pen}_r(n, K) \}. \quad (3.25)$$

The penalty $\text{Pen}_r(n, K)$ defined in (2.15) is, however, motivated by the objective of setting it above the noise level with a very high probability. In our simulations, we also study the selection of an unknown $K$ using somewhat smaller penalty

$$\text{Pen}(n, K) = \rho(A)nK\sqrt{\ln n (\ln K)^3} \quad (3.26)$$

where $\rho(A)$ is the density of matrix $A$, the proportion of nonzero entries of $A$.

In order to assess the accuracy of $\hat{K}$ as an estimator of $K$, we evaluated $\hat{K}$ as a solution of optimization problem (3.25) with the penalty (3.26) in each of the previous simulations settings over 50 simulation runs. Table 1 presents the relative frequencies of the estimators $\hat{K}$ of $K_*$ for $K_*$ ranging from 3 to 6, $n = 420$ and $n = 540$ and $\omega = 0.5, 0.7$ and 0.9. Table 1 confirms that for majority of settings, $\hat{K} = K_*$, the true number of clusters, with high probability. Moreover, the estimator of $\hat{K}$ is more reliable for higher values of $\omega$ and larger number of nodes per cluster.

### 3.3 Real data examples

In this section, we report the performance of SSC and SC in studying real life networks. The social networks usually exhibit strong assortative behavior, the phenomenon which is possibly due to the tendency of humans to form strong associations. Perhaps, for this reason, the political blogs network, the British Twitter network, and the DBLP network which have been analyzed by Sengupta and Chen (2018) have nearly block-diagonal adjacency matrices, so SC exhibits good performance in clustering of those networks (see Remark 3).

However, PABM provides a more accurate description of more diverse networks, in particular, the networks that appear in biological sciences. Below, we consider a butterfly similarity network extracted from the Leeds Butterfly dataset described in Wang et al. [38]. Leeds Butterfly dataset contains fine-grained images of 832 butterfly species that belong to 10 different classes, with each class containing between 55 and 100 images. In this network, the nodes represent butterfly species and edges represent visual similarities between them. Visual similarities are evaluated on the basis of butterfly images and range from 0 to 1. We study a network
by extracting the four largest classes as a simple graph with 373 nodes and 20566 edges. We draw an edge between the nodes if the visual similarity between those nodes is greater than zero. We carried out clustering of the nodes using the SSC and the SC and compared the clustering assignments of both methods with the true class specifications of the species. The SSC provides 89% accuracy while SC is correct only in 64% of cases. In addition, we applied formula (3.25) with $K$ ranging from 2 to 6 and obtained the true number of clusters with 100% accuracy.

Figure 5 (left) shows the adjacency matrix of the graph (after clustering), which confirms that the network indeed follows the PABM. The latter is due to the fact that, since the phenotype of the species in the same class can vary, the SBM may not provide an adequate summary for the class similarities. Replacing the SBM by the DCBM does not solve the problem either, since it is unlikely that few butterflies are “more similar” to the others than the rest. On the other hand, the PABM allows some of the butterflies in one class to be “more similar” to species of another specific class than the other, thus, justifying application of the PABM.

As the second real network, we analyze a human brain functional network, measured using the resting-state functional MRI (fMRI). In particular we use the co-activation matrix described in Crossley et al. [11] the brain connectivity dataset. In this dataset, the brain is partitioned into 638 distinct regions and a weighted graph is used to characterize the network topology. In our analysis, we set all nonzero weights to one, obtaining the network with 18625 undirected edges. Since, for this network, the true clustering as well as the true number of clusters are unknown, we first applied formula (3.25) with $K$ ranging from 2 to 10 to find the number of clusters obtaining $\hat{K} = 6$. This agrees with the assessment in [11] where the authors partitioned the network into 6 groups (if one considers the “rich-club” communities as separate clusters). Subsequently, we applied the SSC for partitioning the network into blocks and derived the estimator $\hat{P}$ of $P_*$. Figure 5 (right) shows the adjacency matrix of the graph after clustering. The true probability matrix $P_*$ is unknown, we can only report that $n^{-2} \| \hat{P} - A \|_F^2 = 0.05$, which indicates high agreement between the two matrices.

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References

[1] Emmanuel Abbe, Community detection and stochastic block models: Recent developments, Journal of Machine Learning Research 18 (2018), no. 177, 1–86.

[2] Edoardo M. Airoldi, David M. Blei, Stephen E. Fienberg, and Eric P. Xing, Mixed membership stochastic blockmodels, J. Mach. Learn. Res. 9 (2008), 1981–2014.

[3] Arash A. Amini and Elizaveta Levina, On semidefinite relaxations for the block model, CoRR abs/1406.5647 (2014).

[4] Afonso S. Bandeira and Ramon van Handel, Sharp nonasymptotic bounds on the norm of random matrices with independent entries, The Annals of Probability 44 (2016), no. 4, 2479–2506.

[5] Peter J. Bickel and Aiyou Chen, A nonparametric view of network models and newman–girvan and other modularities, Proceedings of the National Academy of Sciences 106 (2009), no. 50, 21068–21073.

[6] Terrance Boult and Lisa Gottesfeld Brown, Factorization-based segmentation of motions, 11 1991, pp. 179 – 186.
[7] P. S. Bradley and O. L. Mangasarian, *k-plane clustering*, J. of Global Optimization 16 (2000), no. 1, 23–32.

[8] Alain Celisse, Jean-Jacques Daudin, and Laurent Pierre, *Consistency of maximum-likelihood and variational estimators in the stochastic block model*, Electron. J. Statist. 6 (2012), 1847–1899.

[9] Yudong Chen, Xiaodong Li, and Jiaming Xu, *Convexified modularity maximization for degree-corrected stochastic block models*, Ann. Statist. 46 (2018), no. 4, 1573–1602.

[10] Jie Cheng, Tianxi Li, Elizaveta Levina, and Ji Zhu, *High-dimensional mixed graphical models*, Journal of Computational and Graphical Statistics 26 (2017), no. 2, 367–378.

[11] Nicolas A Crossley, Andrea Mechelli, Petra E Vértes, Toby T Winton-Brown, Ameera X Patel, Cedric E Ginestet, Philip McGuire, and Edward T Bullmore, *Cognitive relevance of the community structure of the human brain functional coactivation network*, Proceedings of the National Academy of Sciences 110 (2013), no. 28, 11583–11588.

[12] Ehsan Elhamifar and Rene Vidal, *Sparse subspace clustering: Algorithm, theory, and applications*, IEEE Trans. Pattern Anal. Mach. Intell. 35 (2013), no. 11, 2765–2781.

[13] P. Favaro, R. Vidal, and A. Ravichandran, *A closed form solution to robust subspace estimation and clustering*, Proceedings of the 2011 IEEE Conference on Computer Vision and Pattern Recognition (Washington, DC, USA), CVPR ’11, IEEE Computer Society, 2011, pp. 1801–1807.

[14] Chao Gao, Zongming Ma, Anderson Y. Zhang, and Harrison H. Zhou, *Achieving optimal misclassification proportion in stochastic block models*, J. Mach. Learn. Res. 18 (2017), no. 1, 1980–2024.

[15] Christophe Giraud, *Introduction to high-dimensional statistics*, Chapman & Hall/CRC Monographs on Statistics & Applied Probability, CRC Press, Hoboken, NJ, 2015.

[16] Anna Goldenberg, Alice X. Zheng, Stephen E. Fienberg, and Edoardo M. Airoldi, *A survey of statistical network models*, Foundations and Trends® in Machine Learning 2 (2010), no. 2, 129–233.

[17] Jiashun Jin, Zheng Tracy Ke, and Shengming Luo, *Estimating network memberships by simplex vertex hunting*, arXiv e-prints (2017), arXiv:1708.07852.

[18] Antony Joseph and Bin Yu, *Impact of regularization on spectral clustering*, Ann. Statist. 44 (2016), no. 4, 1765–1791.

[19] Brian Karrer and Mark E. J. Newman, *Stochastic blockmodels and community structure in networks*, Physical review. E, Statistical, nonlinear, and soft matter physics 83 1 Pt 2 (2011), 016107.

[20] Olga Klopp, Karim Lounici, and Alexandre B. Tsybakov, *Robust matrix completion*, Probability Theory and Related Fields 169 (2017), no. 1, 523–564.

[21] Eric D. Kolaczyk, *Statistical analysis of network data: Methods and models*, 1st ed., Springer Publishing Company, Incorporated, 2009.

[22] Can M. Le, Elizaveta Levina, and Roman Vershynin, *Optimization via low-rank approximation for community detection in networks*, Ann. Statist. 44 (2016), no. 1, 373–400.

[23] Jing Lei and Alessandro Rinaldo, *Consistency of spectral clustering in stochastic block models*, Ann. Statist. 43 (2015), no. 1, 215–237.
[24] Guangcan Liu, Zhouchen Lin, Shuicheng Yan, Ju Sun, Yong Yu, and Yi Ma, *Robust recovery of subspace structures by low-rank representation*, IEEE Trans. Pattern Anal. Mach. Intell. 35 (2013), no. 1, 171–184.

[25] Guangcan Liu, Zhouchen Lin, and Yong Yu, *Robust subspace segmentation by low-rank representation*, ICML, 2010.

[26] Yi Ma, Allen Y. Yang, Harm Derksen, and Robert Fossum, *Estimation of subspace arrangements with applications in modeling and segmenting mixed data*, SIAM Rev. 50 (2008), no. 3, 413–458.

[27] Julien Mairal, F Bach, J Ponce, G Sapiro, R Jenatton, and G Obozinski, *Spams: A sparse modeling software, v2.3*, URL http://spams-devel.gforge.inria.fr/downloads.html (2014).

[28] S.G. Mallat and Zhifeng Zhang, *Matching pursuits with time-frequency dictionaries*, Trans. Sig. Proc. 41 (1993), no. 12, 3397–3415.

[29] N. Mustafa P. Agarwal, *k-means projective clustering*, ACM Symposium on Principles of database systems (2004).

[30] Karl Rohe, Sourav Chatterjee, Bin Yu, et al., *Spectral clustering and the high-dimensional stochastic blockmodel*, The Annals of Statistics 39 (2011), no. 4, 1878–1915.

[31] Srijan Sengupta and Yunguo Chen, *A block model for node popularity in networks with community structure*, Journal of the Royal Statistical Society Series B 80 (2018), no. 2, 365–386.

[32] Mahdi Soltanolkotabi and Emmanuel J. Candès, *A geometric analysis of subspace clustering with outliers*, Ann. Statist. 40 (2012), no. 4, 2195–2238.

[33] Mahdi Soltanolkotabi, Ehsan Elhamifar, and Emmanuel J. Candès, *Robust subspace clustering*, Ann. Statist. 42 (2014), no. 2, 669–699.

[34] Paul Tseng, *Nearest q-flat to m points*, Journal of Optimization Theory and Applications 105 (2000), no. 1, 249–252.

[35] R. Vidal and E. Elhamifar, *Sparse subspace clustering*, 2009 IEEE Conference on Computer Vision and Pattern Recognition(CVPR), vol. 00, 06 2009, pp. 2790–2797.

[36] René Vidal, *Subspace clustering*, IEEE Signal Processing Magazine 28 (2011), no. 2, 52–68.

[37] Rene Vidal, Yi Ma, and Shankar Sastry, *Generalized principal component analysis (gpca)*, IEEE transactions on pattern analysis and machine intelligence 27 (2005), no. 12, 1945–1959.

[38] Bo Wang, Armin Pourshafeie, Marinka Zitnik, Junjie Zhu, Carlos D Bustamante, Serafin Batzoglou, and Jure Leskovec, *Network enhancement: a general method to denoise weighted biological networks*, arXiv preprint arXiv:1805.03327 (2018).

[39] Sanford Weisberg, *Applied linear regression*, vol. 528, John Wiley & Sons, 2005.

[40] Yunpeng Zhao, Elizaveta Levina, Ji Zhu, et al., *Consistency of community detection in networks under degree-corrected stochastic block models*, The Annals of Statistics 40 (2012), no. 4, 2266–2292.
4 Proofs

4.1 Proof of Theorem \[1\]

Denote $\Xi = A - P_*$ and recall that, given matrix $P_*$, entries $\Xi_{i,j} = A_{i,j} - (P_*)_{i,j}$ of $\Xi$ are the independent Bernoulli errors for $1 \leq i \leq j \leq n$ and $A_{i,j} = A_{j,i}$. Then, following notation (2.4), for any $Z$ and $K$

$$\Xi(Z, K) = \mathcal{P}^T_{Z,K} \Xi \mathcal{P}_{Z,K} \quad \text{and} \quad P_*(Z, K) = \mathcal{P}^T_{Z,K} P_* \mathcal{P}_{Z,K}.$$  

Then it follows from (2.8) that

$$\left\| \mathcal{P}^T_{Z,K} A \mathcal{P}_{Z,K} - \hat{\Theta}(\hat{Z}, \hat{K}) \right\|_F^2 + \text{Pen}_r(n, \hat{K}) \leq \left\| \mathcal{P}^T_{Z_*,K_*} A \mathcal{P}_{Z_*,K_*} - \mathcal{P}^T_{Z_*,K_*} P_* \mathcal{P}_{Z_*,K_*} \right\|_F^2 + \text{Pen}_r(n, K_*)$$  

Using the fact that permutation matrices are orthogonal, we can rewrite the previous inequality as

$$\left\| A - \mathcal{P}_{Z,K} \hat{\Theta}(\hat{Z}, \hat{K}) \mathcal{P}^T_{Z,K} \right\|_F^2 + \text{Pen}_r(n, \hat{K}) \leq \left\| A - P_* \right\|_F^2 + \text{Pen}_r(n, K_*).$$  

(4.27)

Hence, (4.27) and (2.16) yield

$$\left\| A - \hat{P}(\hat{Z}, \hat{K}) \right\|_F^2 \leq \left\| A - P_* \right\|_F^2 + \text{Pen}_r(n, K_*) - \text{Pen}_r(n, \hat{K})$$  

(4.28)

Subtracting and adding $P_*$ in the norm of the left-hand side of (4.28), we rewrite (4.28) as

$$\left\| \hat{P}(\hat{Z}, \hat{K}) - P_* \right\|_F^2 \leq \Delta(\hat{Z}, \hat{K}) + \text{Pen}_r(n, K_*) - \text{Pen}_r(n, \hat{K}),$$  

(4.29)

where

$$\Delta(\hat{Z}, \hat{K}) = 2\text{Tr} \left[ \Xi^T (\hat{P}(\hat{Z}, \hat{K}) - P_*) \right].$$  

(4.30)

Again, using orthogonality of the permutation matrices, we can rewrite

$$\Delta(\hat{Z}, \hat{K}) = 2 \langle \Xi(\hat{Z}, \hat{K}), (\hat{\Theta}(\hat{Z}, \hat{K}) - P_*(\hat{Z}, \hat{K})) \rangle,$$

where $\langle A, B \rangle = \text{Tr}(A^T B)$. Then, in the block form, $\Delta(\hat{Z}, \hat{K})$ appears as

$$\Delta(\hat{Z}, \hat{K}) = \sum_{k,l=1}^{K} \Delta^{(k,l)}(\hat{Z}, \hat{K})$$  

(4.31)

where

$$\Delta^{(k,l)}(\hat{Z}, \hat{K}) = 2 \left\langle \Xi^{(k,l)}(\hat{Z}, \hat{K}), \Pi_{\hat{u}, \hat{v}} \left( A^{(k,l)}(\hat{Z}, \hat{K}) - P_*^{(k,l)}(\hat{Z}, \hat{K}) \right) \right\rangle$$

and $\Pi_{\hat{u}, \hat{v}}$ is defined in (4.61) of Lemma \[1\]

Let $\hat{u} = \hat{u}^{(k,l)}(\hat{Z}, \hat{K})$, $\hat{v} = \hat{v}^{(k,l)}(\hat{Z}, \hat{K})$ be the singular vectors of $P_*^{(k,l)}(\hat{Z}, \hat{K})$ corresponding to the largest singular value of $P_*^{(k,l)}(\hat{Z}, \hat{K})$. Then, according to Lemma \[1\]

$$\Pi_{\hat{u}, \hat{v}} \left( P_*^{(k,l)}(\hat{Z}, \hat{K}) \right) = \hat{u}^{(k,l)}(\hat{Z}, \hat{K}) (\hat{u}^{(k,l)}(\hat{Z}, \hat{K}))^T P_*^{(k,l)}(\hat{Z}, \hat{K}) \hat{v}^{(k,l)}(\hat{Z}, \hat{K}) (\hat{v}^{(k,l)}(\hat{Z}, \hat{K}))^T$$  

(4.32)

Recall that

$$\Pi_{\hat{u}, \hat{v}}(A^{(k,l)}(\hat{Z}, \hat{K})) = \Pi_{\hat{u}, \hat{v}} \left[ P_*^{(k,l)}(\hat{Z}, \hat{K}) + \Xi^{(k,l)}(\hat{Z}, \hat{K}) \right]$$
Then, $\Delta^{(k,l)}(\hat{Z}, \hat{K})$ can be partitioned into the sums of three components

$$
\Delta^{(k,l)}(\hat{Z}, \hat{K}) = \Delta^{(k,l)}_1(\hat{Z}, \hat{K}) + \Delta^{(k,l)}_2(\hat{Z}, \hat{K}) + \Delta^{(k,l)}_3(\hat{Z}, \hat{K}), \quad k, l = 1, 2, \ldots, K, \tag{4.33}
$$

where

$$
\Delta^{(k,l)}_1(\hat{Z}, \hat{K}) = 2\langle \Xi^{(k,l)}(\hat{Z}, \hat{K}), \Pi_{\hat{u}, \hat{v}}(\Xi^{(k,l)}(\hat{Z}, \hat{K})) \rangle \tag{4.34}
$$

$$
\Delta^{(k,l)}_2(\hat{Z}, \hat{K}) = 2\langle \Xi^{(k,l)}(\hat{Z}, \hat{K}), \Pi_{\hat{u}, \hat{v}}(P_*(\hat{Z}, \hat{K})) - P_*(\hat{Z}, \hat{K}) \rangle \tag{4.35}
$$

$$
\Delta^{(k,l)}_3(\hat{Z}, \hat{K}) = 2\langle \Xi^{(k,l)}(\hat{Z}, \hat{K}), \Pi_{\hat{u}, \hat{v}}(P_*(\hat{Z}, \hat{K})) - \Pi_{\hat{u}, \hat{v}}(P_*(\hat{Z}, \hat{K})) \rangle \tag{4.36}
$$

With some abuse of notations, for any matrix $B$, let $\Pi_{\hat{u}, \hat{v}}\left(B(\hat{Z}, \hat{K})\right)$ be the matrix with blocks $\Pi_{\hat{u}, \hat{v}}\left(B^{(k,l)}(\hat{Z}, \hat{K})\right)$, and $\Pi_{\hat{u}, \hat{v}}\left(B(\hat{Z}, \hat{K})\right)$ be the matrix with blocks $\Pi_{\hat{u}, \hat{v}}\left(B^{(k,l)}(\hat{Z}, \hat{K})\right)$, $k, l = 1, 2, \ldots, K$. Then, it follows from (4.33)–(4.36) that

$$
\Delta(\hat{Z}, \hat{K}) = \Delta_1(\hat{Z}, \hat{K}) + \Delta_2(\hat{Z}, \hat{K}) + \Delta_3(\hat{Z}, \hat{K}) \tag{4.37}
$$

where

$$
\Delta_1(\hat{Z}, \hat{K}) = 2\langle \Xi(\hat{Z}, \hat{K}), \Pi_{\hat{u}, \hat{v}}(\Xi(\hat{Z}, \hat{K})) \rangle \tag{4.38}
$$

$$
\Delta_2(\hat{Z}, \hat{K}) = 2\langle \Xi(\hat{Z}, \hat{K}), \Pi_{\hat{u}, \hat{v}}(P_*(\hat{Z}, \hat{K})) - P_*(\hat{Z}, \hat{K}) \rangle \tag{4.39}
$$

$$
\Delta_3(\hat{Z}, \hat{K}) = 2\langle \Xi(\hat{Z}, \hat{K}), \Pi_{\hat{u}, \hat{v}}(P_*(\hat{Z}, \hat{K})) - \Pi_{\hat{u}, \hat{v}}(P_*(\hat{Z}, \hat{K})) \rangle \tag{4.40}
$$

Now, we need to derive an upper bound for each component in (4.33) and (4.37).

Observe that

$$
\Delta^{(k,l)}_1(\hat{Z}, \hat{K}) = 2\langle \Xi^{(k,l)}(\hat{Z}, \hat{K}), \Pi_{\hat{u}, \hat{v}}(\Xi^{(k,l)}(\hat{Z}, \hat{K})) \rangle = 2\left\| \Pi_{\hat{u}, \hat{v}}(\Xi^{(k,l)}(\hat{Z}, \hat{K})) \right\|_F^2 = 2\| \Xi^{(k,l)}(\hat{Z}, \hat{K}) \|_{op}^2.
$$

Now, let $\Omega_{1, \tau}$ be the set where $\left\| \Xi(\hat{Z}, \hat{K}) \right\|_{op}^2 \leq F_1(\tau, n, \hat{K})$. According to Lemma \[^4\]

$$
\mathbb{P}(\Omega_{1, \tau}) \geq 1 - 2n^{-\tau}, \tag{4.41}
$$

and, for $\omega \in \Omega_{1, \tau}$, one has

$$
|\Delta_1(\hat{Z}, \hat{K})| \leq 2 \sum_{k, l=1}^{\hat{K}} \| \Xi^{(k,l)}(\hat{Z}, \hat{K}) \|_{op}^2 \leq 2F_1(\tau, n, \hat{K}) \tag{4.42}
$$

Now, consider $\Delta_2(\hat{Z}, \hat{K})$ given by (4.39). Note that

$$
|\Delta_2(\hat{Z}, \hat{K})| = 2\| \Pi_{\hat{u}, \hat{v}}(P_*(\hat{Z}, \hat{K})) - P_*(\hat{Z}, \hat{K}) \|_F \| \Xi(\hat{Z}, \hat{K}), H_{\hat{u}, \hat{v}}(\hat{Z}, \hat{K}) \| \tag{4.43}
$$

where

$$
H_{\hat{u}, \hat{v}}(\hat{Z}, \hat{K}) = \frac{\Pi_{\hat{u}, \hat{v}}(P_*(\hat{Z}, \hat{K})) - P_*(\hat{Z}, \hat{K})}{\| \Pi_{\hat{u}, \hat{v}}(P_*(\hat{Z}, \hat{K})) - P_*(\hat{Z}, \hat{K}) \|_F}
$$

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Since for any $a, b$ and $\alpha_1 > 0$, one has $2ab \leq \alpha_1 a^2 + b^2 / \alpha_1$, obtain
\[
|\Delta_2(\hat{Z}, \hat{K})| \leq \alpha_1 \|\Pi_{\hat{u}, \hat{v}}\left(P_*(\hat{Z}, \hat{K}) - P_*(\hat{Z}, \hat{K})\right)\|_F^2 + 1 / \alpha_1 \|\langle \Xi(\hat{Z}, \hat{K}), H_{\hat{u}, \hat{v}}(\hat{Z}, \hat{K})\rangle\|^2 
\tag{4.44}
\]

Observe that if $K$ and $Z \in \mathcal{M}_{n,K}$ are fixed, then $H_{\hat{u}, \hat{v}}(Z, K)$ is fixed and, for any $K$ and $Z$, one has $\|H_{\hat{u}, \hat{v}}(Z, K)\|_F = 1$. Note also that, for fixed $K$ and $Z$, permuted matrix $\Xi(Z, K) \in [0, 1]^{n \times n}$ contains independent Bernoulli errors. It is well known that if $\xi$ is a vector of independent Bernoulli errors and $h$ is any fixed vector, then, for any $x > 0$, Hoeffding’s inequality yields
\[
P(\|\hat{\xi}^T h\|^2 > x) \leq 2 \exp(-x/2)
\]

Since $\langle \Xi(Z, K), H_{\hat{u}, \hat{v}}(Z, K)\rangle = [\text{vec}(\Xi(Z, K))]^T \text{vec}(H_{\hat{u}, \hat{v}}(Z, K))$, obtain for any fixed $K$ and $Z$:
\[
P\left(\|\langle \Xi(\hat{Z}, \hat{K}), H_{\hat{u}, \hat{v}}(\hat{Z}, \hat{K})\rangle\|^2 - x > 0 \right) \leq 2 \exp(-x/2)
\]

Now, applying the union bound, derive
\[
P\left(\|\langle \Xi(\hat{Z}, \hat{K}), H_{\hat{u}, \hat{v}}(\hat{Z}, \hat{K})\rangle\|^2 - F_2(\tau, n, \hat{K}) > 0 \right) \leq 2nK^n \exp\left\{-\left(\frac{F_2(\tau, n, \hat{K})}{2}\right)^2\right\} \leq 2n^{-r}
\tag{4.45}
\]

where $F_2(\tau, n, K)$ is defined in (2.14). By Lemma 2, one has
\[
\|\Pi_{\hat{u}, \hat{v}}\left(P_*(\hat{Z}, \hat{K}) - P_*(\hat{Z}, \hat{K})\right)\|_F^2 \leq \|\Pi_{\hat{u}, \hat{v}}\left(P_*(\hat{Z}, \hat{K}) - P_*(\hat{Z}, \hat{K})\right)\|_F^2 \leq \|\hat{P}(\hat{Z}, \hat{K}) - P_\ast\|_F^2.
\]

Denote the set on which (4.47) holds by $\Omega_{2,\tau}$, so that
\[
P(\Omega_{2,\tau}) \geq 1 - 2n^{-r}.
\tag{4.46}
\]

Then inequalities (4.44) and (4.45) imply that, for any $\alpha_1 > 0$, $\tau > 0$ and any $\omega \in \Omega_{2,\tau}$, one has
\[
|\Delta_2(\hat{Z}, \hat{K})| - \alpha_1 \|\hat{P}(\hat{Z}, \hat{K}) - P_\ast\|_F^2 - 1 / \alpha_1 F_2(\tau, n, \hat{K}) \leq 0. \tag{4.47}
\]

Now consider $\Delta_3(\hat{Z}, \hat{K})$ defined in (4.40) with components (4.36). Note that matrices $\Pi_{\hat{u}, \hat{v}}(P^{(k,l)}_\ast(\hat{Z}, \hat{K})) - \Pi_{\hat{u}, \hat{v}}\left(P^{(k,l)}_\ast(\hat{Z}, \hat{K})\right)$ have rank at most two. Use the fact that (see, e.g., Giraud (2014), page 123)
\[
\langle A, B \rangle \leq \|A\|_{(2,r)} \|B\|_{(2,r)} \leq 2\|A\|_{op} \|B\|_{F}, \quad r = \min\{\text{rank}(A), \text{rank}(B)\}. \tag{4.48}
\]

Here $\|A\|_{(2,q)}$ is the Ky-Fan $(2, q)$ norm
\[
\|A\|_{(2,q)}^2 = \sum_{j=1}^{q} \sigma_j^2(A) \leq \|A\|_{F}^2,
\]

where $\sigma_j(A)$ are the singular values of $A$. Applying inequality (4.48) with $r = 2$ and taking into account that for any matrix $A$ one has $\|A\|_{(2,2)} \leq 2\|A\|_{op}^2$, derive
\[
|\Delta_3^{(k,l)}(\hat{Z}, \hat{K})| \leq 4\|\Xi^{(k,l)}(\hat{Z}, \hat{K})\|_{op} \|\Pi_{\hat{u}, \hat{v}}(P^{(k,l)}_\ast(\hat{Z}, \hat{K})) - \Pi_{\hat{u}, \hat{v}}\left(P^{(k,l)}_\ast(\hat{Z}, \hat{K})\right)\|_F.
\]
Then, for any $\alpha_2 > 0$, obtain

$$\left| \Delta_3(\hat{Z}, \hat{K}) \right| \leq \sum_{k,l=1}^{\hat{K}} |\Delta_3^{(k,l)}(\hat{Z}, \hat{K})|$$

(4.49)

$$\leq \frac{2}{\alpha_2} \sum_{k,l=1}^{\hat{K}} \|\Xi^{(k,l)}(\hat{Z}, \hat{K})\|_{op}^2 + 2\alpha_2 \sum_{k,l=1}^{\hat{K}} \left\| \Pi_{\hat{n},\hat{\omega}}(P_*^{(k,l)}(\hat{Z}, \hat{K})) - \Pi_{\hat{n},\hat{\omega}}\left(P_*^{(k,l)}(\hat{Z}, \hat{K})\right) \right\|_F^2.$$  

Since, by Lemma 2,

$$\|\Pi_{\hat{n},\hat{\omega}}(P_*^{(k,l)}(\hat{Z}, \hat{K})) - \Pi_{\hat{n},\hat{\omega}}\left(P_*^{(k,l)}(\hat{Z}, \hat{K})\right) \|_F^2 \leq 2\|\Pi_{\hat{n},\hat{\omega}}(P_*^{(k,l)}(\hat{Z}, \hat{K})) - P_*^{(k,l)}(\hat{Z}, \hat{K}) \|_F^2 + 2\|\Pi_{\hat{n},\hat{\omega}}(P_*^{(k,l)}(\hat{Z}, \hat{K})) - P_*^{(k,l)}(\hat{Z}, \hat{K}) \|_F^2$$

$$\leq 4\|\Pi_{\hat{n},\hat{\omega}}(P_*^{(k,l)}(\hat{Z}, \hat{K})) - P_*^{(k,l)}(\hat{Z}, \hat{K}) \|_F^2$$

$$\leq 4\|\Pi_{\hat{n},\hat{\omega}}(A^{(k,l)}(\hat{Z}, \hat{K})) - P_*^{(k,l)}(\hat{Z}, \hat{K}) \|_F^2 = 4\left\| \Theta^{(k,l)}(\hat{Z}, \hat{K}) - P_*^{(k,l)}(\hat{Z}, \hat{K}) \right\|_F^2$$

Therefore,

$$\sum_{k,l=1}^{\hat{K}} \left\| \Pi_{\hat{n},\hat{\omega}}(P_*^{(k,l)}(\hat{Z}, \hat{K})) - \Pi_{\hat{n},\hat{\omega}}\left(P_*^{(k,l)}(\hat{Z}, \hat{K})\right) \right\|_F^2 \leq 4\left\| \hat{\Theta}(\hat{Z}, \hat{K}) - P_*^{(k,l)}(\hat{Z}, \hat{K}) \right\|_F^2 = 4\|\hat{P}(\hat{Z}, \hat{K}) - P_* \|_F^2$$

(4.50)

Combine inequalities (4.49) and (4.50) and recall that $\left\| \Xi(\hat{Z}, \hat{K}) \right\|_{op}^2 \leq F_1(\tau, n, K)$ for $\omega \in \Omega_{1,\tau}$. Then, for any $\alpha_2 > 0$, any $\tau > 0$ and any $\omega \in \Omega_{1,\tau}$, derive the following upper bound

$$\left| \Delta_3(\hat{Z}, \hat{K}) \right| \leq 8\alpha_2 \|\hat{P}(\hat{Z}, \hat{K}) - P_* \|_F^2 + 2/\alpha_2 F_1(\tau, n, K).$$

(4.51)

Now, let $\Omega_\tau = \Omega_{1,\tau} \cap \Omega_{2,\tau}$. Then, (4.41) and (4.46) imply that $\mathbb{P}(\Omega_\tau) \geq 1 - 4n^{-\tau}$ and, for $\omega \in \Omega_\tau$, inequalities (4.42), (4.47) and (4.51) simultaneously hold. Hence, by (4.37), derive that, for any $\omega \in \Omega_\tau$,

$$\left| \Delta(\hat{Z}, \hat{K}) \right| \leq (1 + 2/\alpha_2) F_1(\tau, n, K) + 1/\alpha_1 F_2(\tau, n, K) + (\alpha_1 + 8\alpha_2) \|\hat{P}(\hat{Z}, \hat{K}) - P_* \|_F^2.$$  

Combination of the last inequality and (4.29) yields that, for any $\omega \in \Omega_\tau$,

$$(1 - \alpha_1 - 8\alpha_2) \|\hat{P}(\hat{Z}, \hat{K}) - P_* \|_F^2 \leq \left(1 + \frac{2}{\alpha_2}\right) F_1(\tau, n, K) + \frac{1}{\alpha_1} F_2(\tau, n, K) + \text{Pen}_r(n, K) - \text{Pen}_r(n, \hat{K})$$

provided $\alpha_1 + 8\alpha_2 < 1$. Setting $\text{Pen}_r(n, K) = (1 + 2/\alpha_2) F_1(\tau, n, K) + 1/\alpha_1 F_2(\tau, n, K)$ and dividing by $(1 - \alpha_1 - 8\alpha_2)$, obtain

$$\|\hat{P}(\hat{Z}, \hat{K}) - P_* \|_F^2 \leq (1 - \alpha_1 - 8\alpha_2)^{-1} \text{Pen}_r(n, K).$$

(4.52)

In order to derive (2.17), set $\beta_1 = 8\alpha_2$ and $\beta_2 = \alpha_1$. Then, the upper bound (2.18) follows from the set of
inequalities below:

\[
\mathbb{E} \left\| \hat{P}(\hat{Z}, \hat{K}) - P_* \right\|_F^2 = \\
\mathbb{E} \left\{ \left\| \hat{P}(\hat{Z}, \hat{K}) - P_* \right\|_F^2 \mathbb{I}(\omega \in \Omega_r) + \left\| \hat{P}(\hat{Z}, \hat{K}) - P_* \right\|_F^2 \mathbb{I}(\omega \not\in \Omega_r) \right\} \leq \\
(1 - \beta_1 - \beta_2)^{-1} \text{Pen}_r(n, K_*) + \sqrt{\mathbb{E} \left\| \hat{P}(\hat{Z}, \hat{K}) - P_* \right\|_F^4} \sqrt{\mathbb{P}(\mathbb{I}(\omega \not\in \Omega_r))^2} \leq \\
(1 - \beta_1 - \beta_2)^{-1} \text{Pen}_r(n, K_*) + n^2 \sqrt{4n - r} \leq 2n + (1 - \beta_1 - \beta_2)^{-1} \text{Pen}_r(n, K_*)
\]

### 4.2 Proof of Theorem 2

Note that it follows from (2.11) that

\[
\sum_{k,l=1}^{K} \left\| A^{(k,l)}(\hat{Z}) - \Pi_{\hat{\omega}, \hat{v}} \left( A^{(k,l)}(\hat{Z}) \right) \right\|_F^2 \leq \sum_{k,l=1}^{K} \left\| A^{(k,l)}(Z_*) - \Pi_{\hat{\omega}, \hat{v}} \left( A^{(k,l)}(Z_*) \right) \right\|_F^2
\]

Observe that for any \( Z \in \mathcal{M}_{n,K} \), one has

\[
\sum_{k,l=1}^{K} \left\| A^{(k,l)}(Z) - \Pi_{\hat{\omega}, \hat{v}} \left( A^{(k,l)}(Z) \right) \right\|_F^2 = \sum_{k,l=1}^{K} \left\{ \left\| A^{(k,l)}(Z) \right\|_F^2 - \left\| \Pi_{\hat{\omega}, \hat{v}} \left( A^{(k,l)}(Z) \right) \right\|_F^2 \right\},
\]

so that, due to \( \sum_{k,l=1}^{K} \left\| A^{(k,l)}(Z) \right\|_F^2 = \| A \|_F^2 \), (4.53) can be re-written as

\[
\sum_{k,l=1}^{K} \left\| \Pi_{\hat{\omega}, \hat{v}} \left( A^{(k,l)}(Z) \right) \right\|_F^2 \geq \sum_{k,l=1}^{K} \left\| \Pi_{\hat{\omega}, \hat{v}} \left( A^{(k,l)}(Z_*) \right) \right\|_F^2
\]

Applying Proposition 6.2 of Giraud [15], obtain

\[
\left\| \Pi_{\hat{\omega}, \hat{v}} \left( A^{(k,l)}(Z) \right) - P^{(k,l)}_*(Z) \right\|_F^2 \leq \left( 1 + \frac{2}{\theta} \right)^2 \sum_{r=2}^{n_{\min} \{n_k, n_l\}} \sigma_r^2 P^{(k,l)}_*(Z) + 2(1 + \theta) \left( 1 + \frac{2}{\theta} \right) \left\| \Xi^{(k,l)}(Z) \right\|_{op}^2,
\]

where \( \theta > 0 \) is an arbitrary constant, \( P_* \) is the true matrix of probabilities, \( \Xi^{(k,l)}(Z) = A^{(k,l)}(Z) - P^{(k,l)}_*(Z) \), and \( \sigma_r(B) \) is the \( r \)-th largest singular value of \( B \). Since \( \text{rank}(P^{(k,l)}_*(Z_*)) = 1 \), the previous inequality yields for \( \theta = \sqrt{2} \)

\[
\left\| \Pi_{\hat{\omega}, \hat{v}} \left( A^{(k,l)}(Z_*) \right) - P^{(k,l)}_*(Z_*) \right\|_F^2 \leq 2(1 + \sqrt{2})^2 \left\| \Xi^{(k,l)}(Z_*) \right\|_{op}^2
\]

Using inequality (4.68) of Lemma 2 and setting \( x = (\tau + 2) \ln n \), derive

\[
\mathbb{P} \left\{ \sum_{k,l=1}^{K} \left\| \Xi^{(k,l)}(Z_*) \right\|_{op}^2 \leq 72nK + 2\bar{c}K^2(\tau + 2) \ln n \right\} \geq 1 - 2n^{-\tau}.
\]
Also, since \( \text{card}(\mathcal{M}_{n,K}) = K^n \), setting \( x = n \ln K + (\tau + 2) \ln n \) in inequality (4.68) of Lemma 4, obtain
\[
\mathbb{P} \left\{ \max_{Z \in \mathcal{M}_{n,K}} \sum_{k,l=1}^{K} \left\| \Xi^{(k,l)}(Z) \right\|_{op}^2 \leq 72nK + 2K^2 \tilde{c} \left[ (\tau + 2) \ln n + n \ln K \right] \right\} \geq 1 - 2n^{-\tau} \tag{4.57}
\]

Note that for any \( \alpha_1 \in (0, 1) \),
\[
\left\| \Pi_{\tilde{\alpha}, \tilde{\tau}} \left( A^{(k,l)}(Z) \right) \right\|_F^2 \geq \left( 1 - \alpha_1 \right) \left\| P_s \right\|_F^2 - \left( 1 + \sqrt{2} \right)^2 \left( \frac{1}{\alpha_1} - 1 \right) \sum_{k,l=1}^{K} \left\| \Xi^{(k,l)}(Z) \right\|_{op}^2. \tag{4.58}
\]
where we used the fact that \( \left\| P_s(Z) \right\|_F = \left\| P_s \right\|_F \). On the other hand, for any \( Z \in \mathcal{M}_{n,K} \) and any \( \alpha_2 > 0 \),
\[
\left\| \Pi_{\tilde{\alpha}, \tilde{\tau}} \left( A^{(k,l)}(Z) \right) \right\|_F^2 \leq \left( 1 + \alpha_2 \right) \left\| \Pi_{\tilde{\alpha}, \tilde{\tau}} \left( P_s^{(k,l)}(Z) \right) \right\|_F^2 + \left( 1 + \alpha_2^{-1} \right) \left\| \Pi_{\tilde{\alpha}, \tilde{\tau}} \left( \Xi^{(k,l)}(Z) \right) \right\|_F^2, \tag{4.59}
\]
so that
\[
\left\| \Pi_{\tilde{\alpha}, \tilde{\tau}} \left( A^{(k,l)}(Z) \right) \right\|_F^2 \leq (1 + \alpha_2) \left\| P_s^{(k,l)}(Z) \right\|_{op}^2 + \left( 1 + \alpha_2^{-1} \right) \left\| \Xi^{(k,l)}(Z) \right\|_{op}^2.
\]

Now, we prove the theorem by contradiction. Assume that \( \hat{Z} \in \mathcal{T}(Z_s, \rho_n) \) is the solution of optimization problem (2.11). Then, inequality (4.54) holds. Combining (4.54), (4.58) and (4.59), obtain that
\[
(1 - \alpha_1) \left\| P_s \right\|_F^2 - (1 + \alpha_2) \sum_{k,l=1}^{K} \left\| \Xi^{(k,l)}(Z) \right\|_{op}^2 \leq (1 - \alpha_1) \left[ H_1 K^2 \ln K + H_2 nK + H_3 K^2 \ln n \right],
\]
where
\[
H_1 \equiv H_1(\alpha_1, \alpha_2) = 2\tilde{c} \left[ \frac{1 + \alpha_2}{\alpha_2(1 - \alpha_1)} \right],
H_2 \equiv H_2(\alpha_1, \alpha_2) = 72 \left[ \frac{2(1 + \sqrt{2})^2}{\alpha_1} + \frac{1 + \alpha_2}{\alpha_2(1 - \alpha_1)} \right],
H_3 \equiv H_3(\alpha_1, \alpha_2) = 2\tilde{c}(\tau + 2) \left[ \frac{2(1 + \sqrt{2})^2}{\alpha_1} + \frac{1 + \alpha_2}{\alpha_2(1 - \alpha_1)} \right].
\tag{4.60}
\]

The latter contradicts (2.21), since \( \hat{Z} \in \mathcal{T}(Z_s, \rho_n) \). This contradiction completes the proof.
4.3 Supplementary statements and their proofs

**Lemma 1.** Let $u$ and $v$ be the singular vectors of matrix $A$ corresponding to its largest singular value $\sigma$. Then, the best rank one approximation of $A$ is given by $\Pi_{u,v}(A)$ where

$$\Pi_{u,v}(A) = \sigma uv^T = (uu^T)A(vv^T)$$

and $\langle \Pi_{u,v}(A), A - \Pi_{u,v}(A) \rangle = 0$. \hspace{1cm} (4.61)

**Lemma 2.** Let $A = P + \Xi$. Denote by $(\hat{u}, \hat{v})$ and $(u, v)$ the pairs of singular vectors of matrices $A$ and $P$, respectively, corresponding to the largest singular values. Then,

$$\|\Pi_{u,v}(P) - P\|_F \leq \|\hat{\Pi}_{u,v}(P) - P\|_F \leq \|\hat{\Pi}_{u,v}(A) - P\|_F$$ \hspace{1cm} (4.62)

where $\Pi_{u,v}(\cdot)$ is defined in (4.61).

**Proof.** The first inequality in (4.62) is true because $\Pi_{u,v}(P)$ is the best rank one approximation of $P$. Validity of the second inequality in (4.62) follows from

$$\|\Pi_{\hat{u},\hat{v}}(A) - P\|_F = \|\Pi_{\hat{u},\hat{v}}(P) - P + \Pi_{\hat{u},\hat{v}}(\Xi)\|_F = \|\Pi_{\hat{u},\hat{v}}(P) - P\|_F^2 + \|\Pi_{\hat{u},\hat{v}}(\Xi)\|_F^2$$

**Lemma 3.** Let $A : n_1 \times n_2$ be such that $A_{ij}$ are independent Bernoulli variables $A_{ij} \sim \text{Bernoulli}(P_{ij})$, and $\Xi = A - P$. Then, there exists an absolute constant $\bar{c}$ such that, for any $x > 0$

$$\mathbb{P} \left\{ \|\Xi\|_{op}^2 \geq 36(n_1 + n_2) + 2\bar{c}x \right\} \leq (n_1 + n_2) e^{-x}$$ \hspace{1cm} (4.63)

If $n_1 = n_2 = n$ and $A^T = A$, i.e $A_{ij} = A_{ji} \sim \text{Bernoulli}(P_{ij})$ with $P_{ij} = P_{ji}$, $1 \leq j \leq i \leq n$, then for any $x > 0$

$$\mathbb{P} \left\{ \|\Xi\|_{op}^2 \geq 36n + 2\bar{c}x \right\} \leq n \exp\{-x\}$$ \hspace{1cm} (4.64)

where $\bar{c}$ is an absolute constant.

**Proof.** First, consider the case of $n_1 = n_2 = n$. Apply Corollary 3.12 and Remark 3.13 of Bandeira and Handel \cite{4} with $\rho = \|P\|_\infty \leq 1$, $\bar{\sigma} \leq \sqrt{m_2} \leq \sqrt{n}$, and $\bar{\sigma}_s \leq 1$.

Let

$$\rho = \|P\|_\infty = \max_{i,j} |P_{ij}| \leq 1$$

Then in Corollary 3.12,

$$\bar{\sigma} \leq \sqrt{m_2} \leq \sqrt{n}, \quad \bar{\sigma}_s \leq 1$$

Since $\Xi_{ij} = 1 - P_{ij}$ with probability $P_{ij}$, setting $\varepsilon = 1/2$, obtain from Remark 3.13 that

$$\mathbb{P} \left\{ \|\Xi\|_{op} \geq 3\sqrt{2}\sqrt{n} + t \right\} \leq n \exp\{-t^2/\bar{c}\}$$

where $\bar{c}$ is an absolute constant. Then, setting $t^2/\bar{c} = x$, due to $(a + b)^2 \leq 2a^2 + 2b^2$, obtain

$$\mathbb{P} \left\{ \|\Xi\|_{op} \geq 36n + 2\bar{c}x \right\} \leq n \exp\{-x\}$$ \hspace{1cm} (4.65)

If $n_1 \neq n_2$, then replacing $\Xi$ by

$$\tilde{\Xi} = \left( \begin{array}{cc} 0 & \Xi \\ \Xi^T & 0 \end{array} \right)$$

and noting that $\|\Xi\|_{op} = \|\tilde{\Xi}\|_{op}$ and $n = n_1 + n_2$, obtain

$$\mathbb{P} \left\{ \|\Xi\|_{op} \geq 36(n_1 + n_2) + 2\bar{c}x \right\} \leq (n_1 + n_2) \exp\{-x\}$$ \hspace{1cm} (4.66)
Lemma 4. Let $F_1(\tau, n, K)$ be defined in (2.13). Then, for any $\tau > 0$,

$$
\mathbb{P}\left\{ \sum_{k,l=1}^{\hat{K}} \|\Xi^{(k,l)}(\hat{Z}, \hat{K})\|_{op}^2 - F_1(\tau, n, \hat{K}) \leq 0 \right\} \geq 1 - 2n^{-\tau}.
$$

(4.67)

Proof. Using Lemma 3, for any fixed $K$ and $Z \in \mathcal{M}_{n,K}$, obtain

$$
\mathbb{P} \left\{ \left\| \Xi^{(k,l)}(Z, K) \right\|_{op}^2 \geq 36(n_k + n_l) + 2\tilde{c}x \right\} \leq (n_k + n_l)e^{-x}.
$$

Application of the union bound yields

$$
\mathbb{P} \left\{ \max_{1 \leq k,l \leq K} \left\| \Xi^{(k,l)}(Z, K) \right\|_{op}^2 - 36(n_k + n_l) - 2\tilde{c}x \right\} \geq 0 \right\} \leq \sum_{k,l=1}^{K} (n_k + n_l)e^{-x} = 2nKe^{-x}
$$

The latter implies

$$
\mathbb{P} \left\{ \sum_{k,l=1}^{K} \left( \Xi^{(k,l)}(Z, K) \right\|_{op}^2 - 36(n_k + n_l) - 2\tilde{c}x \right\} \geq 0 \right\} \leq 2nKe^{-x}
$$

(4.68)

Now, setting $x = (\tau + 2) \ln n + (n + 1) \ln K$, applying union bound over $K = 1, \cdots, n$ and over sets $\mathcal{M}_{n,K}$, taking into account that the cardinality of $\mathcal{M}_{n,K}$ is at most $K^n$, derive

$$
\mathbb{P} \left\{ \sum_{k,l=1}^{K} \left\| \Xi^{(k,l)}(\hat{Z}, \hat{K})\|_{op}^2 - F_1(\tau, n, \hat{K}) \right\} \geq 0 \right\} \leq \mathbb{P} \left\{ \max_{1 \leq K \leq n} \max_{Z \in \mathcal{M}_{n,K}} \left( \sum_{k,l=1}^{K} \left\| \Xi^{(k,l)}(Z, K)\right\|_{op}^2 - F_1(\tau, n, K) \right) \geq 0 \right\}
$$

$$
\leq \sum_{k=1}^{n} \sum_{Z \in \mathcal{M}_{n,K}} \mathbb{P} \left\{ \sum_{k,l=1}^{K} \left\| \Xi^{(k,l)}(Z, K)\right\|_{op}^2 - F_1(\tau, n, K) \geq 0 \right\}
$$

$$
\leq 2n^2K \exp\{-x\} K^n \leq 2n^{-\tau}
$$

which completes the proof.