LOEWNER CHAINS WITH QUASICONFORMAL EXTENSIONS: AN APPROXIMATION APPROACH

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ABSTRACT. A new approach in Loewner Theory proposed by Bracci, Contreras, Díaz-Madrigal and Gumenyuk provides a unified treatment of the radial and the chordal versions of the Loewner equations. In this framework, a generalized Loewner chain satisfies the differential equation

$$\frac{\partial}{\partial t} f_t(z) = (z - \tau(t))(1 - \tau(t)z) \overline{p}(z, t),$$

where $\tau : [0, \infty) \to \mathbb{D}$ is measurable and $p$ is called the Herglotz function. In this paper, we will show that if there exists a $k \in [0, 1)$ such that $p$ satisfies

$$|p(z, t) - 1| \leq k|p(z, t) + 1|$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, then $f_t$ has a $k$-quasiconformal extension to the whole Riemann sphere for all $t \in [0, \infty)$. The radial case ($\tau = 0$) and the chordal case ($\tau = 1$) have been proven by Becker [J. Reine Angew. Math. 255 (1972), 23–43] and Gumenyuk and the author [arXiv:1511.08077]. In our theorem, no superfluous assumption is imposed on $\tau \in \mathbb{D}$. As a key foundation of our proof is an approximation method using the continuous dependence of evolution families.

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Date: August 25, 2016.

2010 Mathematics Subject Classification. Primary 30C62; Secondary 30C80, 34M15.

Key words and phrases. evolution family; quasiconformal mapping; Loewner chain; the Loewner differential equation.

This work was supported by JSPS KAKENHI Grant Numbers 26800053.
1. Introduction

1.1. Classical Loewner Theory. In the early 1900s, mathematicians became interested in the fine structure of the family of conformal maps. In particular, a conjecture posed by Ludwig Bieberbach in 1916 [Bie16] known as the Bieberbach conjecture attracted special attention. In 1923, Loewner introduced a method to solve a part of the conjecture [Löw23] with a differential equation that was later named after him. His approach was innovative, and provided a key ingredient in the complete proof of the conjecture by de Branges [dB85]. In 2000, Schramm introduced the celebrated Schramm-Loewner Evolution (SLE), which is a powerful tool for describing the scaling limits of various critical statistical mechanics models exhibiting conformal invariance. The system, which centers around time-parameterized conformal maps and the ordinary/partial differential equations that such maps satisfy, is now known as Loewner Theory.

Since the initial work by Loewner, a large number of studies have developed Loewner Theory. Among them, works by Pommerenke [Pom65], [Pom75, Chapter 6] focusing on the radial case and by Kufarev et al. [KSS68] focusing on the chordal case have made significant contributions. We now give a brief outline of these models.

Let $f_t(z) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n \ (t \geq 0)$ be a time-parameterized holomorphic function defined on the unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$ in the complex plane $\mathbb{C}$. $f_t$ is said to be a (classical) radial Loewner chain if $f_t$ is univalent in $D$ for each $t \in [0, \infty)$ and the inclusion relation $f_s(D) \subset f_t(D)$ holds for all $0 \leq s < t < \infty$. The key properties of radial Loewner chains are that $f_t$ is absolutely continuous on $[0, \infty)$ for each $z \in D$, which implies that $\partial_t f_t (\partial_t := \partial/\partial t)$ exists almost everywhere on $[0, \infty)$ and satisfies the partial differential equation

$$\partial_t f_t(z) = z \partial_z f_t(z) \cdot p(z, t) \quad (1.1)$$

for all $z \in D$ and almost all $t \in [0, \infty)$, where $p$ is holomorphic on $z \in D$ for each $t \in [0, \infty)$ and measurable on $t \in [0, \infty)$ for each $z \in D$ satisfying $p(0, t) = 1$ and $\text{Re} p(z, t) > 0$ for all $z \in D$ and $t \in [0, \infty)$. Loewner himself considered the case of slit domains $f_t(D) = \mathbb{C} \setminus \gamma(t)$, where $\gamma(t)$ is a certain Jordan curve which shrinks over time with one endpoint lying on $\infty$. In this case, $p$ is written as the special form $p(z, t) = (z + \kappa(t))/(z - \kappa(t))$, where $\kappa : [0, \infty) \to \partial D := \{|z| = 1\}$ is a continuous function (see e.g., [Dur83, Chapter 3], [Hay94, Chapter 7] for details).

The other significant case is the chordal case, first formulated by Kufarev. In their work, the dynamics of a growing single-slit map $f_t$ on the upper-half plane $\mathbb{H}^+ := \{ z \in \mathbb{C} : \text{Im} z > 0 \}$, under the normalization of not an internal but a boundary fixed point at the infinity limit and $\text{chordal case}$ have made significant contributions. We now give a brief outline of these models.

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$$\partial_t f_t(z) = z \partial_z f_t(z) \cdot p(z, t) \quad (1.1)$$

for all $z \in D$ and almost all $t \in [0, \infty)$. The chordal case is further studied e.g. in [Dur83, Chapter 3], [Hay94, Chapter 7] for details). A more general case where the image $f_t(\mathbb{H}^+)$ is a growing simply-connected domain is discussed in [GH]. To ensure consistency between the two models, we transfer everything from the half plane to the unit disk by a Cayley map $\zeta \mapsto (i\zeta + 1)/(i\zeta - 1) =: z \in D$. Then, $f_t$ satisfies the similar differential equation as (1.1),

$$\partial_t f_t(z) = -(1 - z)^2 \partial_z f_t(z) \cdot p(z, t) \quad (1.2)$$

for all $z \in D$ and almost all $t \in [0, \infty)$. The chordal case is further studied e.g. in [Gor87, GB92, Bau05]. In particular, many mathematicians have investigated the special case of $p(\zeta, t) = 2i/(\zeta + \lambda(t))$ where $\lambda(t) = i(1 + \kappa(t))/(1 - \kappa(t))$, which follows from the aforementioned single-slit map.

1.2. Quasiconformal extensions. In the radial case, any function $f$ belonging to the class $S$ can be embedded in a radial Loewner chain $f_t$ as $f = f_0$. Thus, a wide spectrum of properties of the class $S$ can be derived via the radial Loewner differential equations. In particular, this paper focuses on quasiconformal extensions of univalent holomorphic functions.

A sense-preserving homeomorphism $f$ of a plane domain $G \subset \mathbb{C}$ is said to be $k$-quasiconformal if $\partial_x f$ and $\partial_y f$ in the distributional sense are locally integrable on $G$ and fulfill $|\partial_x f| \leq k|\partial_y f|$ almost everywhere in $G$, where $k$ is a constant with $k \in \{0, 1\}$. For an introduction to the theory of quasiconformal mappings and related topics, see [Ahl06], [LV73] and [IT92, Chapter 4]. For a given $f \in S$, if there exists a $k$-quasiconformal mapping $F$ of $\mathbb{C}$ such that $F = f$
on \( \mathbb{D} \), then we say that \( f \) has a \( k \)-quasiconformal extension to \( \mathbb{C} \). Quasiconformal extendible univalent holomorphic functions were first treated by Bers ([Ber61]) in connection with research on Teichmüller theory (see [Leh87]). The first quasiconformal extension criterion was obtained by Ahlfors and Weill in 1962 ([AW62]).

In 1972, Becker discovered a criterion for quasiconformal extension of \( f \in \mathcal{S} \) by means of the radial Loewner chain.

**Theorem 1.A** ([Bec72], see also [Bec80]). Let \( f_t \) be a radial Loewner chain. If a function \( p \) determined by \( f_t \) in (1.1) satisfies

\[
\frac{|p(z,t) - 1|}{|p(z,t) + 1|} \leq k
\]

for all \( z \in \mathbb{D} \) and almost all \( t \geq 0 \), then \( f_t \) admits a continuous extension to \( \overline{\mathbb{D}} \) for each \( t \geq 0 \) and the map defined by

\[
F(re^{i\theta}) = \begin{cases} 
  f_0(re^{i\theta}) & \text{if } r < 1, \\
  f_0^r(e^{i\theta}) & \text{if } r \geq 1,
\end{cases}
\]

is a \( k \)-quasiconformal extension of \( f_0 \) to \( \mathbb{C} \).

Recently, Gumenyuk and the author [GH] showed, for the chordal variant of Theorem 1.A, that the same restriction as (1.3) to \( p \) determined by the chordal Loewner chain \( f_t \) in (1.2) ensures that \( f_0 \) admits a \( k \)-quasiconformal extension to \( \mathbb{C} \).

**Remark 1.1.** In the proof of the quasiconformal extension criteria for radial/chordal Loewner chains in [Bec72] and [GH], the technique of “radial limit” worked very well. For example for a radial Loewner chain \( f_t, f_t'(z) := f_t(rz)/r \ (r \in (0,1)) \) is again a radial Loewner chain whose boundary of the image \( f_t(\mathbb{D}) \) is a Jordan curve for all \( t \geq 0 \). Then, using normality of \( \{f_t'\}_{r \in (0,1)} \), one can take a limit \( \lim_{r \to 1} f_t' \) because a family of all \( k \)-quasiconformal maps on \( \overline{\mathbb{C}} \) is compact in the topology of locally uniform convergence.

### 1.3. Main results

The main aim of the paper is to discuss the quasiconformal extension problem in the framework of general Loewner Theory, a new approach for Loewner Theory proposed recently by Bracci, Contreras, Díaz-Madrigal and Gumenyuk ([BCDM12], [BCDM09] and [CDMG10]), which gives a unified treatment of the radial and chordal versions of the Loewner equations. In the theory, the following generalized Loewner chains are dealt with.

**Definition 1.B** ([CDMG10, Definition 1.2]). A family of holomorphic maps \( (f_t)_{t \geq 0} \) of the unit disk \( \mathbb{D} \) is called a **Loewner chain** if

- **LC1.** \( f_t : \mathbb{D} \to \mathbb{C} \) is univalent for each \( t \in [0, \infty) \);
- **LC2.** \( f_s(\mathbb{D}) \subset f_t(\mathbb{D}) \) for all \( 0 \leq s < t < \infty \);
- **LC3.** for any compact set \( K \subset \mathbb{D} \) and all \( T > 0 \), there exists a non-negative locally integrable function \( k_{K,T} : [0,T] \to \mathbb{R} \) such that

\[
|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\zeta)d\zeta
\]

for all \( z \in K \) and all \( 0 \leq s \leq t \leq T \).

Furthermore, a Loewner chain is said to be **normalized** if \( f_0 \in \mathcal{S} \).

It is known [CDMG10] that a Loewner chain \( (f_t) \) in the above sense satisfies the differential equation

\[
\partial_t f_t(z) = (z - \tau(t))(1 - \overline{\tau(t)}z)\partial_z f_t(z)\cdot p(z,t) \quad (z \in \mathbb{D}, \ a.e. \ t \geq 0),
\]

where \( \tau : [0, \infty) \to \overline{\mathbb{D}} \) is a measurable function and \( p \) is called the **Herglotz function** (Definition 2.F). Note that the cases when \( \tau = 0 \) and \( \tau = 1 \) match (1.1) and (1.2), respectively, and so the radial and the chordal Loewner chain are the special cases.
Conversely, for a given \( \tau \) and \( p \), the ordinary differential equation
\[
\frac{d\omega_{z,s}(t)}{dt} = (\omega_{z,s}(t) - \tau(t))(\tau(t)\omega_{z,s}(t) - 1) \cdot p(\omega_{z,s}(t), t) \quad (a.e. \ t \geq s)
\]
with the initial condition \( \omega_{z,s}(s) = z \) has a unique solution \( \omega_{z,s}(t) \) (Theorem 2.E, Theorem 2.G).

Let \( \varphi_{s,t}(z) := w_{z,s}(t) \) for all \( 0 \leq s \leq t < \infty \) and all \( z \in \mathbb{D} \), then a family of two-parametrized holomorphic functions \( (\varphi_{s,t}(z))_{0 \leq s \leq t < \infty} \) on \( z \in \mathbb{D} \) generates a Loewner chain \( (f_t) \) that fulfills \( f_s = f_t \circ \varphi_{s,t} \) for all \( 0 \leq s \leq t < \infty \) (Section 2.3). If we further assume that \( (f_t) \) is range-normalized, i.e., \( f_0 \in \mathcal{S} \) and \( \bigcup_{t \geq 0} f_t(\mathbb{D}) \) is either \( \mathbb{C} \) or a Euclidean disk whose center is the origin, then such a chain \( (f_t) \) is determined uniquely (Theorem 2.I).

In this article, the following quasiconformal extension criterion for Loewner chains is proven. We emphasize that our theorem imposes no superfluous assumptions on \( \tau \).

**Theorem 1.2.** Let \( k \in [0, 1) \). Let \( (f_t) \) be a Loewner chain and \( p \) be the Herglotz function associated with \( (f_t) \) in (1.4). Suppose that \( p \) satisfies
\[
\left| \frac{p(z,t) - 1}{p(z,t) + 1} \right| \leq k
\]
for all \( z \in \mathbb{D} \) and almost all \( t \in [0, \infty) \). Then:

(i) for each \( t \in [0, \infty) \), \( f_t \) has a \( k \)-quasiconformal extension to \( \overline{\mathbb{C}} \);

(ii) for each \( s \in [0, \infty) \) and \( t \in [s, \infty) \), \( \varphi_{s,t} := f_t^{-1} \circ f_s \) has a \( k \)-quasiconformal extension to \( \overline{\mathbb{C}} \);

(iii) \( \bigcup_{t \geq 0} f_t(\mathbb{D}) = \mathbb{C} \).

For proving this theorem, taking the same radial limit technique mentioned in Remark 1.1 is not effective, because any normalization is not assumed to a Loewner chain in Definition 1.B. In this paper, we introduce an approximation method for Loewner chains that is discussed in Section 3 (Lemma 3.4 and Lemma 3.6). Similar results are obtained under more restrictive situations (see e.g., [Law05, Section 4.7], [JVST12, Proposition 1], [RS, Theorem 2.4] and [DMS, Theorem 1.1]).

**Theorem 1.3.** Let \( \tau_n : [0, \infty) \to \overline{\mathbb{D}} \) be a sequence of measurable functions and \( p_n \) a sequence of Herglotz functions. Suppose that \( G_n(z,t) := (z - \tau_n(t))(\tau_n(t)z - 1)p_n(z,t) \) has the following properties:

1. for all \( z \in \mathbb{D} \), \( G_n \) has a weak limit \( G \) of the form \( G(z,t) := (z - \tau(t))(\tau(t)z - 1)p(z,t) \), where \( \tau : [0, \infty) \to \overline{\mathbb{D}} \) and \( p \) are again a measurable function and a Herglotz function;

2. for almost all \( t \in [0, \infty) \), \( \{G_n(\cdot,t)\}_{n \in \mathbb{N}} \) forms a normal family.

Let \( \omega_{z,s} \) be a unique solution of the ordinary differential equation
\[
\left\{ \begin{array}{l}
\frac{d\omega_{z,s}(t)}{dt} = G(\omega_{z,s}(t),t), \quad \text{for almost all } t \geq s \\
\omega_{z,s}(s) = z, \quad t = s
\end{array} \right.
\]
and \( \omega_{z,s}^n \) a unique solution of the above ODE by \( G^n \) in the same fashion. Let \( \varphi_{s,t}(z) := \omega_{z,s}(t) \) and \( \varphi_{s,t}^n(z) := \omega_{z,s}^n(t) \) for all \( 0 \leq s \leq t < \infty \) and all \( z \in \mathbb{D} \). Then \( (\varphi_{s,t}^n)_{0 \leq s \leq t < \infty} \) converges to \( (\varphi_{s,t})_{0 \leq s \leq t < \infty} \) locally uniformly on \((z,t) \in \mathbb{D} \times [s, \infty) \) as \( n \to \infty \).

Further, let \( (f_t^n) \) and \( (f_t) \) be range-normalized Loewner chains associated uniquely with \( (\varphi_{s,t}^n) \) and \( (\varphi_{s,t}) \), respectively. Then \( (f_t^n) \) converges locally uniformly to \( (f_t) \) on \((z,t) \in \mathbb{D} \times [0, \infty) \) as \( n \to \infty \).

This paper is structured as follows: In Section 2, we collect some preliminary results from Semigroup Theory and Loewner Theory. In Section 3, we prove the quasiconformal extension theorem (Theorem 3.1) with Loewner chains and decreasing Loewner chains (Definition 2.1K) in more general setting than Theorem 1.2. The proof is divided into three steps where \( \tau \) is
a constant, a step function, and a measurable function on $\overline{\mathbb{D}}$, respectively. An approximation method for Loewner chains is also discussed in this section. In Section 4, we verify Theorem 1.2 and further results of quasiconformal extensions that are corollaries of the theorem in Section 3. We conclude Section 4 and this paper with a brief consideration of the Loewner Range $\bigcup_{t \geq 0} f_t(\mathbb{D})$.

Acknowledgement: The author would like to express his deepest gratitude to Professor Oliver Roth and Doctor Sebastian Schliefinger for their fruitful discussions and suggestions on this paper. Part of this work was done while the author was a guest researcher at the University of Würzburg.

2. Preliminaries

2.1. Semigroups of holomorphic mappings. Let $D \subset \mathbb{C}$ be a simply connected domain. We denote a family of all holomorphic functions on $D$ by $\operatorname{Hol}(D, \mathbb{C})$. If $f \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ is a self-mapping of $\mathbb{D}$, then we denote the family of such functions by $\operatorname{Hol}(\mathbb{D})$.

An easy consequence of the well-known Schwarz-Pick Lemma is that $f \in \operatorname{Hol}(\mathbb{D}) \setminus \{\text{id}\}$ may have at most one fixed point in $\mathbb{D}$. If such a point exists, then it is called the Denjoy-Wolff point of $f$. On the other hand, if $f$ does not have a fixed point in $\mathbb{D}$, then the Denjoy-Wolff theorem (see e.g. [ES10]) claims that there exists a unique boundary fixed point $\lim_{t \to -1} f(z) = \tau \in \partial \mathbb{D}$ such that the sequence of iterates $\{f^n\}_{n \in \mathbb{N}}$ converges to $\tau$ locally uniformly, where $\lim$ stands for an angular (or non-tangential) limit, and $f^n$ a $n$-th iterate of $f$, namely, $f^1 := f$ and $f^n := f^{n-1} \circ f$. In this case the boundary point $\tau$ is also called the Denjoy-Wolff point.

A family $\{\phi_t\}_{t \geq 0}$ of holomorphic self-maps of $\mathbb{D}$ is called a one-parameter (continuous) semigroup if

- $\phi_0 = \text{id}_\mathbb{D}$;
- $\phi_{s+t} = \phi_t \circ \phi_s$ for all $s, t \in [0, \infty)$;
- $\lim_{t \to s} \phi_t(z) = \phi_s(z)$ for all $s \in [0, \infty)$ and $z \in \mathbb{D}$;
- $\lim_{t \to s} \phi_t(z) = z$ locally uniformly on $\mathbb{D}$.

It is well-known that for a semigroup $\phi_t$ there exists a holomorphic function $G \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ such that $\phi_t$ is a unique solution of the ordinary differential equation

$$\frac{d\phi_t(z)}{dt} = G(\phi_t(z)) \quad (t \geq 0)$$

with the initial condition $\phi_0(z) = z$. The above function $G$ is called the infinitesimal generator of a semigroup. Various criteria which guarantee that a function $G \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ is an infinitesimal generator are known. As one of them, in 1978 Berkson and Porta [BP78] showed that a holomorphic function $G \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ is an infinitesimal generator if and only if there exists a $\tau \in \overline{\mathbb{D}}$ and a function $p \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ with $\operatorname{Re} p(z) \geq 0$ for all $z \in \mathbb{D}$ such that

$$G(z) = (\tau - z)(1 - \tau \bar{z})p(z) \quad (2.1)$$

for all $z \in \mathbb{D}$. (2.1) is called the Berkson-Porta representation. The point $\tau$ in (2.1) is the common Denjoy-Wolff of all functions $\phi_t$ which are not identity.

2.2. Evolution families and Herglotz vector fields. We introduce an evolution family, the core of general Loewner Theory.

Definition 2.A ([BCDM12, Definition 3.1]). A family of holomorphic self-maps of the unit disk $(\varphi_{s,t})$, $0 \leq s \leq t < \infty$, is an evolution family if

- EF1. $\varphi_{s,s}(z) = z$;
- EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < \infty$;
EF3. for all $z \in \mathbb{D}$ and for all $T > 0$ there exists a locally integrable function $k_{z,T} : [0, +\infty) \to [0, +\infty)$ such that
\[
|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi
\]
whenever $0 \leq s \leq u \leq t < \infty$.

**Remark 2.1.** In [BCDM12] and [CDMG10], the definitions of evolution families, Loewner chains and some other relevant notions contain an integrability order $d \in [1, +\infty]$. Since this parameter is not important for the discussions in this paper, we assume that $d = 1$ which is the most general case of the order.

We denote a family of all evolution families by $\mathcal{EF}$. Some properties of evolution families are listed below;

**Theorem 2.B** ([BCDM12, Proposition 3.7, Corollary 6.3]). Let $(\varphi_{s,t}) \in \mathcal{EF}$.

1. $\varphi_{s,t}$ is univalent in $\mathbb{D}$ for all $0 \leq s \leq t < \infty$.
2. For each $z_0 \in \mathbb{D}$ and $s_0 \in [0, \infty)$, $\varphi_{s_0,t}(z_0)$ is locally absolutely continuous on $t \in [s_0, \infty)$.
3. For each $z_0 \in \mathbb{D}$ and $t_0 \in (0, \infty)$, $\varphi_{s,t_0}(z_0)$ is absolutely continuous on $s \in [0, t_0]$.

Next, we define a Herglotz vector field.

**Definition 2.C** ([BCDM12, Definition 4.1, Definition 4.3]). A weak holomorphic vector field on the unit disk $\mathbb{D}$ is a function $G : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ with the following properties:

WV1. for all $t \in [0, \infty)$, the function $G(\cdot, t)$ is holomorphic on $\mathbb{D}$;
WV2. for all $z \in \mathbb{D}$, the function $G(z, \cdot)$ is measurable on $t \in [0, \infty)$;
WV3. for any compact set $K \subset \mathbb{D}$ and for all $T > 0$, there exists a non-negative locally integrable function $k_{K,T} : [0, T] \to \mathbb{R}$ such that
\[
|G(z,t)| \leq k_{K,T}(t)
\]
for all $z \in K$ and for almost every $t \in [0, T]$.

Furthermore, $G$ is said to be a Herglotz vector field if $G(\cdot, t)$ is an infinitesimal generator of a semigroup of holomorphic functions for almost all $t \in [0, \infty)$.

We denote by $\mathcal{HV}$ a family of all Herglotz vector fields. It is verified that $G \in \mathcal{HV}$ is locally Lipschitz continuous;

**Lemma 2.D** ([BCDM12, Lemma 4.2]). Let $G \in \mathcal{HV}$. Then for any compact set $K \subset \mathbb{D}$ and for all $T > 0$, there exists a non-negative integrable function $\hat{k}_{K,T} : [0, T] \to \mathbb{R}$ such that
\[
|G(z_1,t) - G(z_2,t)| \leq \hat{k}_{K,T}(t)|z_1 - z_2|
\]
for all $z_1, z_2 \in K$ and for almost every $t \in [0, T]$.

The next theorem shows that a Herglotz vector field is characterized by an evolution family and vice versa. In what follows, an essential unique $f(x)$ means if there exists another function $g(x)$ which satisfies the statement then $f(x) = g(x)$ for almost all $x$.

**Theorem 2.E** ([BCDM12, Theorem 5.2, Theorem 6.2]). For any $(\varphi_{s,t}) \in \mathcal{EF}$, there exists an essentially unique $G \in \mathcal{HV}$ such that
\[
\frac{d\varphi_{s,t}(z)}{dt} = G(\varphi_{s,t}(z), t)
\]
for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$. Conversely, for any $G \in \mathcal{HV}$, a family of unique solutions of (2.2) with the initial condition $\varphi_{s,s}(z) = z$ is an evolution family.

A similar mutual characterization holds between Herglotz vector fields and the Berkson-Porta datas as below.

**Definition 2.F** ([BCDM12, Definition 4.5]). A Herglotz function on the unit disk $\mathbb{D}$ is a function $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ with the following properties:

- HF1. for all fixed $z \in \mathbb{D}$, the function $p(\cdot, t)$ is locally integrable on $[0, \infty)$ for all $z \in \mathbb{D}$;
- HF2. for all fixed $t \in [0, \infty)$, the function $p(z, \cdot)$ is holomorphic on $\mathbb{D}$;
- HF3. $\Re p(z, t) \geq 0$ for all $z \in \mathbb{D}$ and $t \in [0, \infty)$.

Then, $\mathbb{H}$ stands for a family of all Herglotz functions.

**Theorem 2.G** ([BCDM12, Theorem 4.8]). Let $G \in \mathbb{H}$. Then there exists an essential unique measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$ and $p \in \mathbb{H}$ such that

$$G(z, t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z, t)$$  \hspace{1cm} (2.3)

for all $z \in \mathbb{D}$ and almost all $z \in [0, \infty)$. Conversely, for a given measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$ and $p \in \mathbb{H}$, the equation (2.3) forms a Herglotz vector field.

For our convenience, in this paper we call the above measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$ the Denjoy-Wolff function. A family of the Denjoy-Wolff functions is denoted by $\mathbb{D}W$. A pair $(p, \tau)$ of $p \in \mathbb{H}$ and $\tau \in \mathbb{D}W$ is called the Berkson-Porta data for $(\varphi_{s,t})$. We denote a set of all Berkson-Porta datas by $\mathbb{B}P$.

Hence, there is a one-to-one correspondence among evolution families $(\varphi_{s,t}) \in \mathbb{E}F$, Herglotz vector fields $G \in \mathbb{H}$ and Berkson-Porta datas $(p, \tau) \in \mathbb{B}P$. In particular, the relation of $(\varphi_{s,t})$ and $(p, \tau)$ are expressed by the ordinary differential equation

$$\frac{d\varphi_{s,t}(z)}{dt} = (\varphi_{s,t}(z) - \tau(t))(\overline{\tau(t)}\varphi_{s,t}(z) - 1)p(\varphi_{s,t}(z), t) \quad (z \in \mathbb{D}, \ a.e. \ t \geq s)$$  \hspace{1cm} (2.4)

with the initial condition $\varphi_{s,s}(z) = z$.

### 2.3. Generalized Loewner chains.**

By Definition 1.B, the notion of Loewner chains is lifted to the same framework as evolution families. A family of all Loewner chains is denoted by $\mathbb{L}C$.

A relation between Loewner chains and evolution families is given as below.

**Theorem 2.H** ([CDMG10, Theorem 1.3]). For any $(f_t) \in \mathbb{L}C$, if we define

$$\varphi_{s,t} := f_t^{-1} \circ f_s$$

for all $0 \leq s \leq t < \infty$, then $(\varphi_{s,t}) \in \mathbb{E}F$. Conversely, for any $(\varphi_{s,t}) \in \mathbb{E}F$ there exists $(f_t) \in \mathbb{L}C$ such that the equation

$$f_t \circ \varphi_{s,t} = f_s$$  \hspace{1cm} (2.5)

holds for all $0 \leq s \leq t < \infty$.

Differentiating both sides of the equation (2.5) with respect to $t$, we obtain $\partial_t f_t \cdot \partial_t \varphi_{s,t} + \partial_t f_t = 0$. Therefore combining to (2.4) we have the following generalized Loewner-Kufarev PDE

$$\partial_t f_t(z) = (z - \tau(t))(1 - \overline{\tau(t)}z)\partial_z f_t(z)p(z, t).$$  \hspace{1cm} (2.6)

In general, for a given evolution family, the equation (2.5) does not define a unique Loewner chain. That is, it is not always true that $\mathcal{L}[(\varphi_{s,t})]$, a family of all normalized Loewner chains associated with $(\varphi_{s,t}) \in \mathbb{E}F$, consists of one function. However, $\mathcal{L}[(\varphi_{s,t})]$ always includes a certain Loewner chain and in this sense $(f_t)$ is determined uniquely. Further, it is sometimes a unique member of $\mathcal{L}[(\varphi_{s,t})]$. The following theorems states such properties of uniqueness for Loewner chains. Here $\Omega[(f_t)]$ stands for the Loewner range defined by

$$\Omega[(f_t)] := \bigcup_{t \geq 0} f_t(\mathbb{D}).$$

**Theorem 2.I** ([CDMG10, Theorem 1.6 & Theorem 1.7]). Let $(\varphi_{s,t}) \in \mathbb{E}F$. Then there exists a unique $(f_t) \in \mathcal{L}[(\varphi_{s,t})]$ such that $\Omega[(f_t)]$ is $\mathbb{C}$ or an Euclidean disk in $\mathbb{C}$ whose center is the origin. Furthermore;
• The following 4 statements are equivalent;
  (1) \( \Omega[(f_t)] = \mathbb{C} \),
  (2) \( \mathcal{L}[(\varphi_{s,t})] \) consists of only one function,
  (3) \( \beta(z) = 0 \) for all \( z \in \mathbb{D} \), where
  \[
  \beta(z) := \lim_{t \to +\infty} \frac{|\varphi'_{0,t}(z)|}{1 - |\varphi_{0,t}(z)|^2},
  \]
  (4) there exists at least one point \( z_0 \in \mathbb{D} \) such that \( \beta(z_0) = 0 \).

• On the other hand, if \( \Omega[(f_t)] \neq \mathbb{C} \), then the Euclidean disk is written by
  \[
  \Omega[(f_t)] = \{ w : |w| < \frac{1}{\beta(0)} \}
  \]
  and all \( g_t \in \mathcal{L}[(\varphi_{s,t})] \) have an expression
  \[
  g_t(z) = \frac{h(\beta(0)f_t(z))}{\beta(0)} \quad (h \in \mathcal{S}).
  \]

\( (f_t) \in \mathcal{L} \) determined uniquely in the above theorem is called the range-normalized Loewner chain associated with \( (\varphi_{s,t}) \in \mathcal{E} \F \). We denote a family of range-normalized Loewner chains by \( \mathcal{L}C_0 \).

In order to construct a unique range-normalized Loewner chain from \( (\varphi_{s,t}) \in \mathcal{E} \F \), we need the following lemma which will be used later. Here, \( \mathcal{A}C((X,Y), \mathbb{D}) \) is the class of locally absolutely continuous functions \( f : X \to \mathbb{D} \) such that \( f' \) is locally integrable on \( X \).

**Lemma 2.J** ([CDMG10, Proposition 2.9]). Let \( (\varphi_{s,t}) \in \mathcal{E} \F \). Then there exist unique functions \( \alpha \in \mathcal{A}C([0, \infty), \mathbb{D}) \) and \( \beta \in \mathcal{A}C([0, \infty), \partial \mathbb{D}) \), and a unique \( (\psi_{s,t}) \in \mathcal{E} \F \) such that the following assertions hold;

1. \( \alpha(0) = 0 \) and \( \beta(0) = 1 \),
2. \( \psi_{s,t}(0) = 0 \) and \( \psi'_{s,t}(0) > 0 \) for all \( 0 \leq s \leq t \),
3. \( \varphi_{s,t} \) has an expression
   \[
   \varphi_{s,t} = M_t \circ \psi_{s,t} \circ M_s^{-1}
   \]
   for all \( 0 \leq s \leq t < +\infty \), where
   \[
   M_t(z) := \frac{\beta(t)z + \alpha(t)}{1 + \beta(t)\alpha(t)z} \quad (t \geq 0, \ z \in \mathbb{D}).
   \]

**Remark 2.2.** In Lemma 2.J, the functions \( \alpha \) and \( \beta \) are given explicitly by

\[
\alpha(t) = \varphi_{0,t}(0) \quad \text{and} \quad \beta(t) = \frac{\varphi'_{0,t}(0)}{|\varphi_{0,t}(0)|},
\]

respectively ([CDMG10, p.987]).

For later use, we recall how to construct a range-normalized Loewner chain from a given evolution family. Let \( (\varphi_{s,t}) \in \mathcal{E} \F \). Then by Lemma 2.J, there exists a unique \( (\psi_{s,t}) \in \mathcal{E} \F \) such that the assertions of the lemma are satisfied. It is known ([CDMG10]) that the function

\[
h_s(z) := \lim_{t \to +\infty} \frac{\psi_{s,t}(z)}{\psi'_{0,t}(0)}
\]

forms a range-normalized Loewner chain associated with \( (\psi_{s,t}) \), where the limit is attained uniformly on compact subsets of the unit disk \( \mathbb{D} \). Thus, setting \( f_t := h_t \circ M_t^{-1} \), we obtain \( (f_t) \in \mathcal{L}[(\varphi_{s,t})] \), where \( M_t \) is defined in (2.7).
2.4. Decreasing Loewner chains. In Section 3, we will use the decreasing setting of evolution families and Loewner chains which are discussed in [CDMG14].

**Definition 2.K** ([CDMG14, Definition 1.6]). A family \((g_t)_{t \geq 0}\) of holomorphic maps of the unit disk is called a *decreasing Loewner chain* if it satisfies the following conditions:

DLC1. \(g_t\) is univalent on \(\mathbb{D}\) for each \(t \in [0, \infty)\);

DLC2. \(g_0(z) = z\) and \(g_s(\mathbb{D}) \supset g_t(\mathbb{D})\) for all \(0 \leq s < t < \infty\);

DLC3. for any compact set \(K \subset \mathbb{D}\) and all \(T > 0\), there exists a locally integrable function \(k_{K,T} : [0, +\infty) \to [0, +\infty)\) such that

\[
|g_s(z) - g_t(z)| \leq \int_s^t k_{K,T}(\zeta)d\zeta
\]

for all \(z \in K\) and all \(0 \leq s \leq t \leq T\).

It is a generalization of an *inverse Loewner chain* introduced by Betker [Bet92].

According to decreasing Loewner chains, the decreasing counterpart of evolution families is also defined.

**Definition 2.L** ([CDMG14, Definition 1.9]). A family \((\omega_{s,t})_{0 \leq s \leq t}\) of holomorphic self-maps of the unit disk \(\mathbb{D}\) is called a *reverse evolution family* if the following conditions are fulfilled:

REF1. \(\omega_{s,s}(z) = z\);

REF2. \(\omega_{s,t} = \omega_{s,u} \circ \omega_{u,t}\) for all \(0 \leq s \leq u \leq t < \infty\);

REF3. for all \(z \in \mathbb{D}\) and for all \(T > 0\) there exists a non-negative locally integrable function \(k_{s,t} : [0, +\infty) \to [0, +\infty)\) such that

\[
|\omega_{s,u}(z) - \omega_{s,t}(z)| \leq \int_u^t k_{s,t}(\zeta)d\zeta
\]

for all \(0 \leq s \leq u \leq t \leq T\).

DLC and REF denote families of all decreasing Loewner chains and reverse evolution families, respectively.

The following theorems state the mutual relations between decreasing Loewner chains, reverse evolution families and Herglotz vector fields.

**Theorem 2.M** ([CDMG14, Theorem 4.1]). The formula

\[
\omega_{s,t} := g_{s}^{-1} \circ g_t \quad (0 \leq s \leq t < \infty) \tag{2.9}
\]

establishes a 1-to-1 correspondence between decreasing Loewner chains \((g_t)\) and reverse evolution families \((\omega_{s,t})\). Namely, for every \((g_t) \in \text{DLC}\), the function \((\omega_{s,t})\) defined by (2.9) is a reverse evolution family. Conversely, for any \((\omega_{s,t}) \in \text{REF}\), the function \((g_t) := (\omega_{0,t})\) is a decreasing Loewner chain satisfying the equality (2.9).

**Theorem 2.N** ([CDMG14, Theorem 4.2]). The formula

\[
\frac{dw}{ds} = -G(w, s), \quad s \in [0,t], \ w(t) = s \tag{2.10}
\]

establishes an essentially 1-to-1 correspondence between reverse evolution families \((\omega_{s,t})\) and Herglotz vector fields \(G\). Namely, given \((\omega_{s,t}) \in \text{REF}\), there exists an essentially unique \(G \in \text{HV}\) such that for each \(t \geq 0\) and \(z \in \mathbb{D}\) the function \([0,t] \ni s \mapsto w(s) := \omega_{s,t}\) solves the initial value problem (2.10). Conversely, given \(G \in \text{HV}\), for every \(t > 0\) and every \(z \in \mathbb{D}\) the initial value problem (2.10) has a unique solution \(s \mapsto w := w_{z,t}(s)\) defined for all \(s \in [0,t]\) and the formula \(\omega_{s,t} := w_{z,t}(s)\) for all \(z \in \mathbb{D}\) and all \(0 \leq s \leq t < \infty\) defines a reverse evolution family \((\omega_{s,t})\).
Thus, \((\omega_{q,t}) \in \text{REF}\) and \((g_t) \in \text{DLC}\) satisfy the following differential equations
\[
\frac{d\omega_{q,t}(z)}{dt} = (\omega_{q,t}(z) - \sigma(t))(1 - \sigma(t)\omega_{q,t}(z))q(\omega_{q,t}(z), t) \quad (z \in \mathbb{D}, \text{ a.e. } t \geq s),
\]
and
\[
\frac{\partial g_t(z)}{\partial t} = (z - \sigma(t))(\sigma(t)z - 1)\frac{\partial g_t(z)}{\partial z}q(z, t) \quad (z \in \mathbb{D}, \text{ a.e. } t \geq 0),
\]
where \(q \in \text{HF}\) and \(\sigma \in \text{DW}\).

For \((g_t) \in \text{DLC}\), an intersection of all image of \(\mathbb{D}\) under \((g_t)\) for \(t \geq 0\) is denoted by \(\Lambda[(g_t)]\), i.e.,
\[
\Lambda[(g_t)] := \bigcap_{t \geq 0} g_t(\mathbb{D}).
\]
In the work by Betker \cite{Bet92}, an image of inverse Loewner chain is assumed to shrink to the origin as \(t\) tends to infinity. On the other hand, the situation is rather complicated in the case of decreasing Loewner chains. One cannot expect that \(\Lambda[(g_t)]\) is even a simply-connected domain.

Lastly, we define \(\Delta[(g_t)]\) by
\[
\Delta[(g_t)] := \left\{ \frac{1}{w} : w \in \mathbb{C} \setminus \Lambda[(g_t)] \right\}.
\]

3. General quasiconformal extension criterion

3.1. Statement of the main theorem. We will prove the following theorem.

**Theorem 3.1.** Let \(k \in [0, 1]\). Let \((f_t) \in \text{LC}\) and \((p, \tau) \in \text{BP}\) associated with \((f_t)\). We denote by \(T \in [0, \infty]\) the smallest number such that \(p(\mathbb{D}, t) \in \mathfrak{iR}\) for almost all \(t \in (T, \infty)\). Suppose that \(T > 0\) and \(p \in \text{HF}\) satisfies
\[
|p(z, t) - q(z, t)| \leq k \cdot |p(z, t) + q(z, t)|
\]
for all \(z \in \mathbb{D}\) and almost all \(t \in [0, \infty)\), where \(q \in \text{HF}\). Let \((\omega_{q,t}) \in \text{REF}\) associated with \((q, \tau) \in \text{BP}\) and \((g_t) \in \text{DLC}\) associate with \((\omega_{q,t})\). Then for each \(t \in [0, T)\), \(f_t\) and \(g_t\) have continuous extensions to \(\overline{\mathbb{D}}\). Further, \(\Phi\) defined by
\[
\Phi(z) := f_0(z), \quad z \in \mathbb{D};
\]
\[
\Phi \left( \frac{1}{g_t(e^{i\theta})} \right) := f_t(e^{i\theta}), \quad \theta \in [0, 2\pi) \text{ and } t \in [0, T),
\]
is a \(k\)-quasiconformal mapping on \(\Delta[(g_t)]\) onto \(\Omega[(f_t)]\).

**Remark 3.2.** We have assumed \(T > 0\), because if \(T = 0\), then \(f_t(\mathbb{D})\) and \(g_t(\mathbb{D})\) does not vary over time. Hence \(\Phi\) in (3.2) is conformal on \(\mathbb{D}\). In this case there is nothing to prove.

**Remark 3.3.** The inequality (3.1) implies that for a fixed \((z_0, t_0) \in \mathbb{D} \times [0, \infty)\), \(p(z_0, t_0)\) lies on a circle of Apollonius on \(\mathbb{H}\) with foci \(q(z_0, t_0)\) and \(-q(z_0, t_0)\), symmetric w.r.t. the imaginary axis. Hence, if either \(p(z, t)\) or \(q(z, t)\) gets close to the imaginary axis, then so is the other one as well.

3.2. Approximation lemmas for \((\omega_{q,t})\) and \((f_t)\). In the proof of Theorem 3.1, the following approximation methods for evolution families and Loewner chains play key roles.

The idea of the statement and the proof of the first lemma comes from a standard result in Control Theory \cite[Lemma I.37]{Rot98}. Since in our setting some arguments are simplified, we reconstruct a proof of the lemma. Here a sequence of functions \(\{f_n\}\) defined on \(D\) converges weakly to \(f\) on \(D\) means that for any compact subset \(D' \subset D\),
\[
\lim_{n \to \infty} \int_{D'} f_n(u)\phi(u)du = \int_{D'} f(u)\phi(u)du,
\]
where \(\phi \in C_0^\infty(D)\) is a test function.
Lemma 3.4. Let $\Gamma$ be a family of Herglotz vector fields such that $\{G(\cdot, t) : G \in \Gamma\}$ forms a normal family for almost every fixed $t \in [0, \infty)$. If $\{G_n\}_{n \in \mathbb{N}} \subset \Gamma$ is a sequence converging weakly to $G \in \mathbb{H}^*$, then a sequence of evolution families $(\varphi_{s,t}^n)$ associated with $G_n$ converges locally uniformly to $(\varphi_{s,t})$ associated with $G$ on $(z, t) \in \mathbb{D} \times [s, \infty)$.

The proof relies on the following lemma due to Gronwall:

Lemma 3.A (Gronwall’s inequality (see e.g. [FR75, p.198])). Let $m : [t_0, T] \to \mathbb{R}$ be a continuous function satisfying

$$0 \leq m(t) \leq h(t) + \int_{t_0}^{t} g(s)m(s)ds, \quad (t \in [t_0, T])$$

where $g(s) \geq 0$ on $[t_0, T]$, $\int_{t_0}^{T} g(s)ds < \infty$ and $h$ is bounded on $[t_0, T]$. Then

$$m(t) \leq h(t) + \int_{t_0}^{t} g(s)h(s) \exp \left\{ \int_{s}^{t} g(u)du \right\} ds$$

for all $t_0 \leq t \leq T$.

Proof of Lemma 3.4. Let $K$ be a compact subset in $\mathbb{D}$ and $T > s$. We prepare some notations.

Let $O_{K,T} := \{\varphi_{s,t}(z) : z \in K, t \in [s, T]\}$. Since $O_{K,T}$ is compact, there exist real constants $0 < b < b' < \infty$ such that

$$O_{K,T} \subset A_{K,T} \subset B_{K,T} \subset \mathbb{D},$$

where

$$A_{K,T} := \{w \in \mathbb{D} : d(w, O_{K,T}) < b\}$$

and

$$B_{K,T} := \{w \in \mathbb{D} : d(w, O_{K,T}) \leq b'\}.$$

Then, take $\varepsilon \in (0, b)$.

(a). Let

$$\alpha_n(z, t) := \int_{s}^{t} [G(\varphi_{s,u}(z), u) - G_n(\varphi_{s,u}(z), u)] du.$$

Since $G_n$ converges weakly to $G$, $\alpha_n(z, t)$ tends to 0 pointwise on $(z, t) \in K \times [s, T]$ as $n \to \infty$.

We will show that the above convergence is uniform on $K \times [s, T]$. By the normality of $\Gamma$, one can find a uniform constant $M_{K,T}$ depending only on $K$ and $T$ such that

$$|G_n(w_1, t) - G_n(w_2, t)| \leq M_{K,T}|w_1 - w_2|$$

for all $w_1, w_2 \in B_{K,T}$, almost all $t \in [s, T]$ and all $n \in \mathbb{N}$. Moreover, there exists a constant $M'_{K,T}$ such that $|G_n(w, t)| \leq M'_{K,T}$ for all $w \in B_{K,T}$, almost all $t \in [s, T]$ and $n \in \mathbb{N}$. Therefore,

$$\left| \int_{s}^{t_1} G_n(w_1, u)du - \int_{s}^{t_2} G_n(w_2, u)du \right|$$

$$\leq \int_{s}^{t_1} |G_n(w_1, u) - G_n(w_2, u)| du + \int_{t_1}^{t_2} G_n(w_2, u)du$$

$$\leq M_{K,T}|w_1 - w_2| \cdot |t_1 - s| + M'_{K,T}|t_1 - t_2|$$

for all $w_1, w_2 \in B_{K,T}$, almost all $t_1, t_2 \in [s, T]$ and all $n \in \mathbb{N}$. It implies that the family $\{\int_{s}^{t} G_n(\varphi_{s,t}(z), u)du\}_{n \in \mathbb{N}}$ is equicontinuous on $K \times [s, T]$. Hence $\alpha_n(z, t)$ converges to 0 uniformly on $K \times [s, T]$ as $n \to \infty$.

(b). By Definition 2.C and Lemma 2.D, one can choose a constant $M''_{K,T}$ such that

$$|G(\varphi_{s,t}, t)| \leq M''_{K,T}, \quad |G_n(\varphi_{s,t}, t)| \leq M''_{K,T},$$

$$|G(\varphi_{s,t}, t) - G(\varphi_{s,t}, t)| \leq M''_{K,T}|\varphi_{s,t} - \varphi_{s,t}|$$

(3.3)
for all \(w, w' \in B_{K,T}\), almost all \(t \in [s, T]\) and all \(n \in \mathbb{N}\).

Let \(u > 0\) be a constant satisfying
\[
(\varepsilon + 2M''_{K,T}u) \exp(M''_{K,T}u) < b. \tag{3.4}
\]
We will prove that under the inequality (3.4) \(w_n \in B_{K,T}\) for all \(z \in K\) and all \(t \in [s, s + u]\). Let us fix \(n \in \mathbb{N}\) and let \(u_n\) be the largest number such that \(w_n(K, t)\) lies on \(B_{K,T}\) for all \(t \in [s, s + u_n]\).

By assumption, \(u_n\) is at least strictly greater than 0. For all \(t \in [s, s + u_n]\) we have
\[
|w(z, t) - w_n(z, t)| = \left| \int_s^t G(w(z, u), u)du - \int_s^t G_n(w_n(z, u), u)du \right|
\leq \int_s^t |G(w(z, u), u)du - G_n(w_n(z, u), u)|du + \int_s^t |G_n(w_n(z, u), u) - G_n(w_n(z, u), u)|du
\leq 2M''_{K,T}u_n + M''_{K,T} \int_s^t |w(z, u) - w_n(z, u)|du.
\]

Applying Lemma 3.A, \(|w(z, t) - w_n(z, t)| < 2M''_{K,T}u_n \exp(M''_{K,T}u_n)\) for all \(z \in K\) and all \(t \in [s, s + u_n]\). Now, suppose that \(2M''_{K,T}u_n \exp(M''_{K,T}u_n) < b\). Then \(w_n(z, t) \in A_{K,T}\) for all \(z \in K\) and all \(t \in [s, s + u_n]\). It, however, contradicts that \(u_n\) is the largest number such that \(w_n(K, t)\) lies on \(B_{K,T}\) for all \(t \in [s, s + u_n]\) (note that \(A_{K,T}\) is an open set and \(B_{K,T}\) is compact). Thus \(b \leq 2M''_{K,T}u_n \exp(M''_{K,T}u_n)\). Since \((\varepsilon + 2M''_{K,T}u) \exp(M''_{K,T}u) < b\), we obtain \(u \leq u_n\). It concludes that \(w_n(z, t) \in B_{K,T}\) for all \(z \in K\), all \(t \in [s, s + u]\) and all \(n \in \mathbb{N}\).

The above argument yields
\[
\left| \int_s^t G_n(\varphi, u(z), u)du - \int_s^t G_n(\varphi_n, u(z), u)du \right| \leq M''_{K,T} \int_s^t |\varphi_{s,t}(z) - \varphi_{s,t}(z)|ds
\]
for all \(z \in K\), all \(t \in [s, s + u]\) and all \(n \in \mathbb{N}\).

(c). By (a) and (b), we have
\[
|\varphi_{s,t}(z) - \varphi_{s,t}(z)| = \left| \int_s^t G(\varphi, u(z), u)du - \int_s^t G_n(\varphi_n, u(z), u)du \right|
\leq \int_s^t |G(\varphi, u(z), u)du - G_n(\varphi_n, u(z), u)|du + \int_s^t |G_n(\varphi_n, u(z), u)du - G_n(\varphi_n, u(z), u)|du
\leq \alpha_n(z, t) + M''_{K,T} \int_s^t |\varphi_{s,t}(z) - \varphi_{s,t}(z)|du
\]
for all \(z \in K\), all \(t \in [s, s + u]\) and all \(n \in \mathbb{N}\). Applying Lemma 3.A, \(\varphi_n(z, t)\) converges to \(\varphi_{s,t}(z)\) uniformly on \((z, t) \in K \times [s, s + u]\).

(d). We finally prove that \(\varphi_n(z, t)\) converges to \(\varphi_{s,t}(z)\) uniformly on \((z, t) \in K \times [s, T]\). Let us choose \(N \in \mathbb{N}\) such that \(|\varphi_{s,t}(z) - \varphi_n(z, t)| < \varepsilon\) for all \(z \in K\), all \(t \in [s, s + u']\) and all \(n \in \mathbb{N}\), where \(u' > 0\). Then we fix \(n > N\) and define \(u_n'\) as the largest number such that \(\varphi_n(K)\) lies on \(B_{K,T}\) for all \(t \in [s, s + u']\), as in part (b). Remark that \(u_n' \geq u'\). Then following the similar line as (b) with Lemma 3.A we have
\[
|\varphi_{s,t}(z) - \varphi_n(z, t)| \leq (|\varphi_{s,s+w}(z) - \varphi_n(z, s+w)| + 2(u_n' - u')M''_{K,T}) \exp(M''_{K,T}u_n' - u'))\]
for all \(z \in K\) and all \(t \in [s+u', s+u']\). If we suppose \((\varepsilon + 2(u_n' - u')M''_{K,T}) \exp(M''_{K,T}u_n' - u')) < b\), then \(\varphi_n(z, t) \in A_{K,T}\) for all \(z \in K\) and all \(t \in [s + u', s + u']\). So, we have \(b \leq (\varepsilon + 2(u_n' - u')M''_{K,T}) \exp(M''_{K,T}(u_n' - u'))\) and hence \(u \leq u_n' - u'\). It concludes with the last inequality in (3.3) that
\[
\int_s^t |G(\varphi, u(z), u)du - G(\varphi_n, u(z), u)|du \leq M''_{K,T} \int_s^t |\varphi,z, u(z) - \varphi_n,u(z)|du
\]
for all $z \in K$, all $t \in [s, s + u + u']$ and all $n > N$. Repeating the argument in part (c) we have

$$|\varphi_{s,t}(z) - \varphi^n_{s,t}(z)| \leq \alpha_n(z,t) + M_{K,T}^n \int_s^t |\varphi_{s,t}(z) - \varphi^n_{s,t}(z)| du$$

for all $z \in K$, all $t \in [s, s + u + u']$ and all $n > N$. Therefore $\varphi^n_{s,t}(z)$ converges to $\varphi_{s,t}(z)$ uniformly on $(z, t) \in K \times [s, s + u + u']$. Since the interval $[s, T]$ is compact, we have prove that $\varphi^n_{s,t}(z)$ converges to $\varphi_{s,t}(z)$ uniformly on $(z, t) \in K \times [s, T]$. □

**Remark 3.5.** The opposite direction of Lemma 3.4 does not hold in general. Let $(\varphi^n_{s,t})$ be a locally uniformly convergent sequence of evolution families. Then due to following Lemma 3.6, the corresponding (uniquely determined) sequence of Loewner chains $f^n_i$ (hence $\partial_i f^n_i$ also) converges locally uniformly on $\mathbb{D}$. Now observing $G_n(z, t) := -\partial_z f^n_i(z)/\partial_i f^n_i(z)$, one cannot expect a nice convergent property to $\{\partial_i f^n_i\}_n$, and therefore to $\{G_n\}$. See also [LMR10, Section 4.2] in which an explicit counterexample is presented in the choral setting.

**Lemma 3.6.** Let $(\varphi^n_{s,t})$ be a sequence of evolution families converging to $(\varphi_{s,t}) \in EF$ locally uniformly on $(z, t) \in \mathbb{D} \times [0, \infty)$. Let $(f^n_i)_i \in \mathcal{L}_0$ and $(f_i)_i \in \mathcal{L}_0$ associated with $(\varphi^n_{s,t})$ and $(\varphi_{s,t})$, respectively. Then, $(f^n_i)$ converges locally uniformly to $(f_i)$ on $(z, t) \in \mathbb{D} \times [0, \infty)$.

**Proof.** Recall that (see Theorem 2.3 and the following description)

$$f_s(z) = \lim_{t \to \infty} (L_t \circ \varphi_{s,t})(z) (3.5)$$

for a certain family $(L_t)_{t \geq 0}$ of M"{o}bius transformations of $\overline{\mathbb{C}}$ whose coefficients consist of values of $(\varphi_{0,t})$ and its derivative $(\varphi'_{0,t})$ at the origin, where the limit (3.5) is attained uniformly on compact subsets of the unit disk $\mathbb{D}$. By assumption, $L^n_i$, which appears in the relation $f^n_s(z) = \lim_{t \to \infty} (L^n_t \circ \varphi_{s,t})(z)$, converges to $L_i$ uniformly on $\mathbb{C}$. Hence

$$\lim_{n \to \infty} f^n_s(z) = \lim_{n \to \infty} \lim_{t \to \infty} (L^n_t \circ \varphi^n_{s,t})(z) = \lim_{n \to \infty} \lim_{t \to \infty} (L^n_t \circ \varphi^n_{s,t})(z) = f_s(z)$$

on a certain compact subset of $\mathbb{D}$. □

3.3. **The case when $\tau$ is constant.** We firstly show Theorem 3.1 with the additional assumption that $\tau \in DW$ is constant.

**Theorem 3.7.** Under the statement of Theorem 3.1, additionally we assume that $\tau$ is an internal fixed point in $D$. Then we obtain the same consequence as Theorem 3.1.

**Proof.** Let $\rho \in (c, 1)$ with $c > 0$ and

$$\nu[\rho](z) := \rho \left( \frac{z - \tau}{1 - \tau \rho} \right) + \tau \left( \frac{z - \tau}{1 - \tau \rho} \right) : \mathbb{D} \to \mathbb{D}$$

Define $f^n_i(z) := f_i(\nu[\rho](z))$ and $g^n_i(z) := g_i(\nu[\rho](z))$, then

$$\frac{\partial_i f^n_i(z)}{(z - \tau)(1 - \tau \rho) \partial_z f^n_i(z)} = \frac{\partial_i f_i(\nu[\rho](z))}{(\nu[\rho](z) - \tau)(1 - \tau \rho \nu[\rho](z)) \partial_z f_i(\nu[\rho](z))} = \frac{\partial_i f_i(\nu[\rho](z))}{\nu[\rho](z)} = p(\nu[\rho](z), t) =: p_{\rho}(z, t) \in \mathcal{H}$$

and

$$\frac{\partial_i g^n_i(z)}{(z - \tau)(\tau \rho - 1) \partial_z g^n_i(z)} = \frac{\partial_i g_i(\nu[\rho](z))}{(\nu[\rho](z) - \tau)(\tau \rho - 1) \partial_z g_i(\nu[\rho](z))} = \frac{\partial_i g_i(\nu[\rho](z))}{\nu[\rho](z)} = q(\nu[\rho](z), t) =: q_{\rho}(z, t) \in \mathcal{H}.$$
Therefore \((f_t^p)\) and \((g_t^p)\) satisfy all the assumptions in Theorem 3.1. Since \(f_t^p\) and \(g_t^p\) have continuous extensions to \(\overline{D}\) for all \(t \in [0, T]\), \(\Phi_p\) is defined accordingly by

\[
\begin{align*}
\Phi_p(z) &= f_0^p(z), \\
\Phi_p\left(\frac{1}{g_t^p(e^{i\theta})}\right) &= f_t^p(e^{i\theta}), \quad \theta \in [0, 2\pi) \text{ and } t \in [0, T).
\end{align*}
\]

(3.6)

Let \(I_B \subset [0, T]\) be a set such that \(p_\rho(z, t) \in i\mathbb{R}\) for all \(z \in D\) and all \(t \in I_B\) and \(I_A := [0, T) \setminus I_B\). Remark that \(I_A\) and \(I_B\) do not depend on \(\rho\).

We prove that \(\Phi_p\) is homeomorphism on \(\Delta([g_t^p])\). In order to show injectivity of \(\Phi_p\), we divide \(\Delta([g_t^p])\) into two domains;

\[
\begin{align*}
D_A := \mathbb{D} \cup \bigcup_{t \in I_A} \frac{1}{g_t^p(\partial\mathbb{D})}, \\
D_B := \bigcup_{t \in I_B} \frac{1}{g_t^p(\partial\mathbb{D})}.
\end{align*}
\]

Take two distinct points \(z_1, z_2 \in D_A\). If either \(z_1\) or \(z_2\) is in \(D\), it is clear that \(\Phi_p(z_1) \neq \Phi_p(z_2)\). Now, let \(z_1, z_2 \in D_A \setminus \mathbb{D}\). The equation \(1/z_1 = g_t^p(e^{i\theta_1})\) determines a unique \(t_1 \in I_A\) and \(\theta_1 \in [0, 2\pi)\) because of the following. It is easy to see that there does not exist another \(\theta_1 \in [0, 2\pi)\) such that \(1/z_1 = g_t^p(e^{i\theta_1})\), because the curve \(\{g_t^p(e^{i\lambda}) : \lambda \in [0, 2\pi)\}\) is Jordan. If there exists another \(t_1^* \in I_A\), \(t_1^* > t_1\), such that \(1/z_1 = g_{t_1}^p(e^{i\theta_1})\), then

\[
\begin{align*}
\omega_{t_1, t_1^*}(\nu[\rho](e^{i\theta_1})) &= (g_{t_1}^{-1} \circ g_{t_1^*})(\nu[\rho](e^{i\theta_1})) \\
&= (g_{t_1}^{-1} \circ g_{t_1^*})(\nu[\rho](e^{i\theta_1})) \\
&= \nu[\rho](e^{i\theta_1}).
\end{align*}
\]

(3.7)

Since \(\omega_{t_1, t_1^*}(\tau) = \tau\), we have

\[
\eta_D(\tau, \nu[\rho](e^{i\theta_1})) = \eta_D(\omega_{t_1, t_1^*}(\tau), \omega_{t_1, t_1^*}(\nu[\rho](e^{i\theta_1}))),
\]

where \(\eta_D\) is the hyperbolic metric on \(\mathbb{D}\). Hence by the Schwarz-Pick Theorem, \(\omega_{t_1, t_2}\) is a conformal automorphism of \(\mathbb{D}\). By calculation we have

\[
\omega_{t_1, t_2}(z) = \frac{e^{i\theta(t)}(\frac{z - \tau}{1 - \tau}) + \tau}{1 + \overline{e^{i\theta(t)}(\frac{z - \tau}{1 - \tau})}} \quad (t \in [t_1, t_2])
\]

(3.8)

with some real function \(\theta : [t_1, t_2] \to \mathbb{R}\) with \(\theta(t_2) = 0\). It describes a rotation of the unit disk \(\mathbb{D}\) around the point \(\tau\) in the hyperbolic sense. Hence one can verify that \(g(\mathbb{D}, t)\) lies on the imaginary axis for all \(t \in [t_1, t_2]\). It contradicts the assumption \(t \in I_A\).

Then, suppose that there exists another \(t_1^* \in I_A\), \(t_1^* \neq t_1\), and \(\theta_1^* \in [0, 2\pi)\), \(\theta_1^* \neq \theta_1\), such that \(1/z_1 = g_{t_1^*}^p(e^{i\theta_1})\). We may assume that \(t_1^* > t_1\). Since \(g_{t_1}^p(e^{i\theta_1}) = g_{t_1^*}^p(e^{i\theta_1})\), we have \(\omega_{t_1, t_1^*}(\nu[\rho](e^{i\theta_1})) = \nu[\rho](e^{i\theta_1})\) as (3.7). Hence \(\eta_D(\tau, \nu[\rho](e^{i\theta_1})) = \eta_D(\omega_{t_1, t_1^*}(\tau), \omega_{t_1, t_1^*}(\nu[\rho](e^{i\theta_1})))\). It shows that \(\omega_{t_1, t}\) is a rotation as (3.8) and \(p(\mathbb{D}, t) \in i\mathbb{R}\) for all \(t \in [t_1, t_1^*]\), which again contradicts the assumption that \(t \in I_A\).

Therefore, there exist unique \(t_1, t_2 \in I_A\) and \(\theta_1, \theta_2 \in [0, 2\pi)\) such that \(z_1 = 1/g_{t_1}^p(e^{i\theta_1})\) and \(z_2 = 1/g_{t_2}^p(e^{i\theta_2})\). We may suppose \(t_1 \leq t_2\). If \(t_1 = t_2\), then since \(f_t^p(\partial\mathbb{D})\) is Jordan, \(\Phi_p(z_1) = f_{t_1}(e^{i\theta_1}) \neq f_{t_2}(e^{i\theta_2}) = \Phi_p(z_2)\). In the case when \(t_1 < t_2\), the same argument as above with (3.7) and (3.8) then shows that \(\Phi_p(z_1) \neq \Phi_p(z_2)\). Consequently \(\Phi_p\) is injective on \(D_A\).

We show injectivity of \(\Phi_p\) on \(D_B\). Take one connected component \(I_B^p\) in \(I_B\). Since \(p_\rho(\mathbb{D}, t), q_\rho(\mathbb{D}, t) \in i\mathbb{R}\) for all \(t \in I_B\), corresponding \(\varphi_{t_0, t} \in EF\) and \(\omega_{t_0, t} \in REF\) are rotations as (3.8) on \(t \in I_B^p\). Hence \(\{f_t^p(e^{i\theta}) : \theta \in [0, 2\pi), t \in I_B\} \) and \(\{g_t^p(e^{i\theta}) : \theta \in [0, 2\pi), t \in I_B^p\}\) are curves.
For a fixed $\theta_0 \in [0, 2\pi)$, one point on the both curves are determined. Therefore, $\Phi_\rho : 1/\{g^\rho_t(e^{i\theta}), t \in I^*_B\} : \theta \in [0, 2\pi) \to \{f^\rho_t(e^{i\theta}) : \theta \in [0, 2\pi), t \in I^*_B\}$ is injective.

Since $\Phi_\rho(D_A) \cap \Phi_\rho(D_B) = \emptyset$, $\Phi_\rho$ gives an injective map on $\Delta[(g^\rho_t)]$ onto $\Omega[(f^\rho_t)]$. It concludes that $\Phi_\rho : \Delta[(g^\rho_t)] \to \Omega[(f^\rho_t)]$ is a homeomorphism.

Differentiations of (3.6) both sides with respect to $t$ and $\theta$ yield

$$
\begin{align*}
&\frac{\partial \Phi_\rho}{\partial \phi_\rho}(\frac{1}{g^\rho_t(\zeta)}) = \left(\frac{g^\rho_t(\zeta)^2}{\partial \phi_\rho g^\rho_t(\zeta)} \right) + \frac{\partial \Phi_\rho}{\partial \lambda_\rho} \left(\frac{\partial g^\rho_t(\zeta)}{g^\rho_t(\zeta)} \right) \frac{\partial f^\rho_t(\zeta)}{\partial \phi_\rho} - \frac{\partial \Phi_\rho}{\partial \lambda_\rho} \left(\frac{\partial g^\rho_t(\zeta)}{g^\rho_t(\zeta)} \right) \frac{\partial f^\rho_t(\zeta)}{\partial \lambda_\rho} + \frac{\partial \Phi_\rho}{\partial \lambda_\rho} \left(\frac{\partial g^\rho_t(\zeta)}{g^\rho_t(\zeta)} \right) \\
&= \left(\frac{g^\rho_t(\zeta)^2}{\partial \phi_\rho g^\rho_t(\zeta)} \right) + \frac{\partial \Phi_\rho}{\partial \lambda_\rho} \left(\frac{\partial g^\rho_t(\zeta)}{g^\rho_t(\zeta)} \right) \frac{\partial f^\rho_t(\zeta)}{\partial \phi_\rho} - \frac{\partial \Phi_\rho}{\partial \lambda_\rho} \left(\frac{\partial g^\rho_t(\zeta)}{g^\rho_t(\zeta)} \right) \frac{\partial f^\rho_t(\zeta)}{\partial \lambda_\rho} + \frac{\partial \Phi_\rho}{\partial \lambda_\rho} \left(\frac{\partial g^\rho_t(\zeta)}{g^\rho_t(\zeta)} \right)
\end{align*}
$$

for almost all $t \in [0, T)$ and all $\zeta \in \partial \mathbb{D}$, where

$$
\phi[\tau](z) := \frac{(z - \tau)(1 - \tau z)}{z}.
$$

It follows from the fact $\phi[\tau](\zeta) = \phi[\tau](\zeta)$ that $|\partial \Phi_\rho(z)/\partial \lambda_\rho| \leq k$ for almost all $z \in \Delta[(g^\rho_t)]$, namely, $\Phi_\rho$ is a $k$-quasiconformal mapping on $\Delta[(g^\rho_t)]$.

Take a constant $c < 1$ as close enough to 1. Since $\Phi_\rho = f^\rho_t$ is conformal on $\mathbb{D}$, one can find three distinct points $z_1, z_2, z_3$ as $d_{C}(f^\rho_0(z_i), f^\rho_0(z_j)), i, j = 1, 2, 3, i \neq j$, is greater than some constant $d > 0$ for all $\rho \in (c, 1)$, where $d_{C}$ stands for the spherical distance. Hence by normality (remark that $\Delta[(g^\rho_t)] = \mathbb{C}$ if and only if $\Delta[(g^\rho_t)] = \mathbb{C}$) $\Phi$ is $k$-quasiconformal on $\Delta[(g^\rho_t)]$.

Accordingly, $f_t$ and $g_t$ have continuous extensions to $\mathbb{D}$ for all $t \in [0, T)$. \qed

**Theorem 3.8.** Under the statement of Theorem 3.1, additionally we assume that $\tau$ is a boundary fixed point of $\mathbb{D}$. Then we obtain the same consequence as Theorem 3.1.

**Proof.** By some rotation, it is enough to consider the case when $\tau = 1$. In order to make our discussion simple, we transfer everything to the right half-plane by a Cayley map $C(z) = (1 + z)/(1 - z)$ (see [GH]). For example, a family of holomorphic functions $(\phi_{s,t})_{0 \leq s \leq t < \infty}$ on the right half-plane $\mathbb{H}$ is said to be an *evolution family* if $K^{-1} \circ \phi_{s,t} \circ K \in \mathcal{E}$. A corresponding Herglotz vector field $G_\mathbb{H}$ is given by the relation $(d/dt)\phi_{s,t}(z) = G_\mathbb{H}(\phi_{s,t}, t)$. Then (2.3) is written as

$$
G_\mathbb{H}(\zeta, t) = K'(K^{-1}(\zeta))G(K^{-1}(\zeta), t) = p_\mathbb{H}(\zeta, t)
$$

for all $t \geq 0$ and all $\zeta \in \mathbb{H}$, where $p_\mathbb{H}(\zeta, t) = 2p(K^{-1}(\zeta), t)$ stands for the right half-plane model of Herglotz function. Accordingly (2.6), (2.11) and (3.2) are written as

$$
\partial \phi_\rho f_t(\zeta) = -\partial \phi_\rho f_t(\zeta) p_\mathbb{H}(\zeta, t), \quad \partial \phi_\rho g_t(\zeta) = \partial \phi_\rho g_t(\zeta) q_\mathbb{H}(\zeta, t),
$$

and

$$
\begin{align*}
\Phi(\zeta) &= f_0(\zeta), \quad \zeta \in \mathbb{H}, \\
\Phi(-g_t(\zeta)) &= f_t(\zeta), \quad y \in \mathbb{R} and t \in [0, T).
\end{align*}
$$

Let $\rho \in (0, 1)$ and define $f^\rho_t(\zeta) := f_t(\zeta + \rho)$ and $g^\rho_t(\zeta) := g_t(\zeta + \rho)$. Then $f^\rho_t \in \mathcal{L}C$ with $(p_\mathbb{H}(\zeta + \rho, t), 1) \in \mathcal{B}P$ and $g^\rho_t \in \mathcal{D}LC$ with $(q_\mathbb{H}(\zeta + \rho, t), 1) \in \mathcal{B}P$. Thus $f^\rho_t$ and $g^\rho_t$ satisfy all the
assumption of the theorem. Further, one can see that \( f_\rho^t \) and \( g_\rho^t \) have continuous extensions to \( \mathbb{H} \). Hence
\[
\begin{cases}
\Phi_\rho(\zeta) = f_\rho^y(\zeta), & \zeta \in \mathbb{H}, \\
\Phi_\rho( -g_\rho^t(iy) ) = f_\rho^y(iy), & y \in \mathbb{R} \text{ and } t \in [0, T).
\end{cases}
\] (3.10)
is well-defined.

In principle, we follow the similar lines of the proof of Theorem 3.7 using the Julia-Wolff-Carathéodory Theorem (e.g. [ES10]). We again use the notations \( I_A, I_B \subset [0, \infty) \) and \( D_A, D_B \subset \mathbb{C} \) in the proof of Theorem 3.7. In this case \( D_A \) is understood as a union of all boundary curves of \( 1/g_\rho^t(\zeta) \) for all \( t \in I_A \) and the right half-plane \( \mathbb{H} \).

What we will prove first is that injectivity of \( \Phi_\rho \) on \( D_A \). Take two points \( \zeta_1, \zeta_2 \in D_A \setminus \mathbb{H} \). We prove that the equation \( \Psi_1, t_\star(\zeta(y_1 + \rho)) = \zeta(\Psi_2, t_\star(\zeta(y_1 + \rho))) \) determines unique \( y_1, y_2 \in \mathbb{R} \). Hence by the maximum modulus principle \( \Psi_1, t_\star(\zeta) = \zeta \) for all \( \zeta \in \mathbb{H} \). Thus \( \Psi_1, t_\star(\zeta) = \zeta \) and hence \( \rho(t, \mathbb{H}, t) = 0 \) for all \( t \in [t_\star, t^*] \). It contradicts our assumption.

Then suppose that there exists another \( t^*_1 \in I_A, t^*_1 \neq t_1, \) and \( y^*_1 \in \mathbb{R}, y^*_1 \neq y_1, \) such that \( \Psi_1, t^*_1(\zeta(y_1 + \rho)) = \zeta(\Psi_1, t_\star(\zeta(y_1 + \rho))) \). Since \( g_\rho^t(iy_1) = g_\rho^t(iy_1^*) \), we have \( \Psi_1, t^*_1(\zeta) = \zeta + iy^*_1 \). By the same argument as above, we obtain \( \Psi_1, t^*_1(\zeta) = \zeta + iy_1 \) and hence \( \rho(t, \mathbb{H}, t) = 0 \) for all \( t \in [t_\star, t^*] \) which again contradicts our assumption.

Now, there exist unique \( t_1, t_2 \in I_A \) and \( y_1, y_2 \in \mathbb{R} \) such that \( \zeta_1 = g_\rho^t(iy_1) \) and \( \zeta_2 = g_\rho^t(iy_2) \). We may suppose that \( t_1 \leq t_2 \). If \( t_1 = t_2 \), then \( \rho(t_1, \mathbb{H}, t_1) = \rho(t_2, \mathbb{H}, t_2) \). Then suppose \( t_1 < t_2 \). If \( \Phi_\rho(\zeta_1) = \Phi_\rho(\zeta_2), \) then \( \Psi_1, t_1, t_2(\zeta_1 + \rho) = (f_\rho^t \circ f_\rho^t)(\zeta_1 + \rho) = \zeta_2 \). Therefore \( \Psi_1, t_1, t_2(\zeta) = \zeta + iy_2 - y_1 \), and hence \( \rho(t, \mathbb{H}, t) = 0 \) for all \( t \in [t_\star, t^*] \).

Injectivity of \( \Phi_\rho \) on \( D_B \) is obtained by the same argument as the proof of Theorem 3.7. Hence \( \Phi_\rho : \Delta([g_\rho]) \to \Omega([\bar{f}_\rho]) \) is injective. We conclude that \( \Phi_\rho \) defines a homeomorphism on \( \Delta([g_\rho]) \) onto \( \Omega([\bar{f}_\rho]) \).

Differentiating both sides of (3.10) with respect to \( t \) and \( y \) we have
\[
\begin{align*}
\frac{\partial_2 \Phi_\rho}{\partial_2 \Phi_\rho} (\zeta) &= f_\rho^y(\zeta), & \zeta \in \mathbb{H}.
\end{align*}
\]
and hence
\[
\frac{\partial_2 \Phi_\rho}{\partial_2 \Phi_\rho} ( -g_\rho^t(iy) ) = \begin{pmatrix} \frac{\partial_1 f_\rho^t(iy)}{\partial_2 g_\rho^t(iy)} & -\frac{\partial_1 g_\rho^t(iy)}{\partial_2 g_\rho^t(iy)} \\ \frac{\partial_2 f_\rho^t(iy)}{\partial_2 g_\rho^t(iy)} & -\frac{\partial_2 g_\rho^t(iy)}{\partial_2 g_\rho^t(iy)} \end{pmatrix} \begin{pmatrix} p(t, iy + \rho, t) - q(t, iy + \rho, t) \\ p(t, iy + \rho, t) + q(t, iy + \rho, t) \end{pmatrix}.
\]
Thus \( |\partial_2 \Phi_\rho(\zeta)/\partial_2 \Phi_\rho(\zeta)| \leq k \) for almost all \( \zeta \in \Delta([g_\rho]) \) and hence \( \Phi_\rho \) is \( k \)-quasiconformal there. By the same argument as in the proof of Theorem 3.7, we conclude that \( \Phi : \Delta([g_\rho]) \to \Omega([\bar{f}_\rho]) \) is \( k \)-quasiconformal. The proof is complete.
3.4. The case when $\tau$ is a step function. Next, we assume that $\tau \in \mathbb{D} \bar{w}$ is a step function, that is, $\tau$ is of the form

$$\tau(t) = \sum_{i=1}^{n} \tau_i \cdot \chi_{I_i}(t),$$

where $\tau_i \in \mathbb{D}$, $n \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, $I_i := [t_{i-1}, t_i)$ and $\chi_I$ is a characteristic function.

**Theorem 3.9.** Under the statement of Theorem 3.1, additionally we assume that $\tau$ is a step function (in the above sense). Then we obtain the same consequence as Theorem 3.1.

**Proof.** With no loss of generality, we may assume that $I_i \setminus I_B$ is not a null-set for all $i = 1, \cdots, n$.

Firstly consider the case when $t \in I_1$. Then applying the same argument as Theorem 3.7 if $\tau_1 \in \mathbb{D}$ or Theorem 3.8 if $\tau_1 \in \partial \mathbb{D}$ for $(f_t)$ and $(g_t)$ on $t \in I_1$, one can verify that the map $\Phi_1$ defined by

$$
\begin{cases} 
\Phi_1(z) = f_0(z), & z \in \mathbb{D}, \\
\Phi_1\left(1/\overline{g_t(e^{i\theta})}\right) = f_t(e^{i\theta}), & \theta \in [0, 2\pi) \text{ and } t \in I_1,
\end{cases}
$$

is a $k$-quasiconformal mapping on $\{1/\overline{\tau} : w \in \mathbb{C} \setminus g_{\tau_1}(\mathbb{D})\}$ maps to $f_{\tau_1}(\mathbb{D})$. We remark that the above mapping gives a quasiconformal extension of $g_{\tau_1}$ to the unit disk $\mathbb{D}$ (and hence $\mathbb{C}$). Therefore $g_{\tau_1}$ has a continuous injective extension to $\overline{\mathbb{D}}$.

Next, let $t \in I_2$. Then again by means of the same argument as Theorem 3.7 or Theorem 3.8, $\Phi_2$ given by

$$\Phi_2\left(1/\overline{g_t(e^{i\theta})}\right) = f_t(e^{i\theta}), \quad \theta \in [0, 2\pi) \text{ and } t \in I_2,$$

define a $k$-quasiconformal mapping on $\{1/\overline{\tau} : w \in g_{I_2}(\mathbb{D}) \setminus \overline{g_{\tau_1}(\mathbb{D})}\}$ maps to $f_{I_2}(\mathbb{D})$. Further, since $\Phi_2$ gives a quasiconformal extension of $f_{\tau_1}$ to $f_{I_2}(\mathbb{D})$, $f_{\tau_1}$ has a continuous injective extension to $\overline{\mathbb{D}}$. Hence the well-defined mapping $\Phi_1$, $\Phi_2$,

$$\Phi(z) := \begin{cases} 
\Phi_1(z), & z \in \{1/\overline{\tau} : w \in \mathbb{C} \setminus g_{\tau_1}(\mathbb{D})\}, \\
\Phi_2(z), & z \in \{1/\overline{\tau} : w \in g_{I_2}(\mathbb{D}) \setminus \overline{g_{\tau_1}(\mathbb{D})}\},
\end{cases}
$$

is a homeomorphism of $\{1/\overline{\tau} : w \in \mathbb{C} \setminus g_{I_2}(\mathbb{D})\}$ onto $f_{I_2}(\mathbb{D})$ and is $k$-quasiconformal there. Repeating this argument, our proof is complete. \qed

3.5. The case when $\tau$ is a measurable function. Now we are ready to prove Theorem 3.1.

**The proof of Theorem 3.1.** We may assume that $(f_t) \in \text{LC}_0$, i.e., range-normalized. Since $\tau$ is measurable, there exists a sequence of step functions $\{\tau_n\} \subset \mathbb{D} \bar{w}$ converging weakly to $\tau$. For each $n \in \mathbb{N}$, $(p, \tau_n) \in \text{BF}$ associates unique $G_n \in \text{HV}$ and $(\varphi_{s,t}^n) \in \text{EF}$ and $(q_t, \tau_n)$ associates unique $G_n^* \in \text{HV}$ and $(\omega_{s,t}^n) \in \text{REF}$. Let $(f_t^n) \in \text{LC}_0$ being uniquely associated with $(\varphi_{s,t}^n) \in \text{EF}$ and $(g_t^n) := (\omega_{0,t}^n) \in \text{DLC}$. By Theorem 3.9, $\Phi_n$ defined by

$$
\begin{cases} 
\Phi_n(z) = f_t^n(z), & z \in \mathbb{D}, \\
\Phi_n\left(1/\overline{g_t^n(\zeta)}\right) = f_t^n(\zeta), & \zeta \in \partial \mathbb{D} \text{ and } t \in [0, T),
\end{cases}
$$

is a $k$-quasiconformal map on $\Delta([g_t^n])$.

By calculation, we have

$$|G(z, t) - G_n(z, t)| = |(z - \tau(t)) \overline{(\tau(t)z - 1)} - (z - \tau_n(t)) \overline{(\tau_n(t)z - 1)}| \cdot |p(z, t)| \leq 4|\tau(t) - \tau_n(t)| \cdot |p(z, t)|,$$

for almost all $t \in [0, \infty)$. Hence $\{G_n\}$ is normal and converges weakly to $G$ on $(z, t) \in \mathbb{D} \times [0, \infty)$. Applying Lemma 3.4, we have $(\varphi_{s,t}^n) \to (\varphi_{s,t})$ and $(\omega_{s,t}^n) \to (\omega_{s,t})$ locally uniformly on $(z, t) \in$
\( \mathbb{D} \times [s, \infty) \). In particular, \((g^n_t) \to (g_t)\) locally uniformly on \((z, t) \in \mathbb{D} \times [0, T)\). By Lemma 3.6, \((f^n_t) \to (f_t)\) locally uniformly on \(z \in \mathbb{D}\) for each \(t \in [0, T)\).

Take an arbitrary \(t \in [0, T)\) and fixed. Let \(D^n_t := \{1/w : w \in \mathbb{C} \setminus g^n_t(\mathbb{D})\} \) and consider \(\Psi_n := \Phi_n|D^n_t\). Since \(g^n_t \to g_t\) locally uniformly and \(g_t\) is conformal on \(\mathbb{D}\), there exists \(N > 0\) and an open set \(K\) such that for all \(n \geq N\), \(K \subset g^n_t(\mathbb{D})\) and \(K \subset g_t(\mathbb{D})\). Let \(\{h_n\}\) be a sequence of conformal maps on \(\mathbb{D}\) onto \(D^n_t\). Since \(D^n_t\) does not intersect with \(\{1/z : z \in K\}\) for all \(n \geq N\), by Montel’s theorem \(\{\Psi_n \circ h_n\}_n\) forms a normal family. Consequently \(\Phi : \{1/w : w \in \mathbb{C} \setminus g_t(\mathbb{D})\} \to f_t(\mathbb{D})\) is \(k\)-quasiconformal. Since \(t\) is arbitrary, the first assertion is proved. It also concludes that \(f_t\) and \(g_t\) have continuous extensions to \(\mathbb{D}\) for all \(t \in [0, T)\). \(\square\)

4. Proof of the main theorem and further results

4.1. Proof of Theorem 1.2. Various corollaries are deduced from the results and the arguments in Section 3. In order to present them, we will need the next lemma.

**Lemma 4.1.** We have the followings:

1. For \((f_t) \in \text{LC}\) which is associated with \((1, \tau) \in \text{BP}\), \(\Omega((f_t)) = \mathbb{C}\).
2. For \((g_t) \in \text{DLC}\) which is associated with \((1, \tau) \in \text{BP}\), \(\Lambda((g_t))\) consists of one point in \(\overline{\mathbb{D}}\).

**Proof.** We may assume that \((f_t) \in \text{LC}_0\). Let \(\{\tau_n\}_{n \in \mathbb{N}}\) be a sequence of step functions converging weakly to \(\tau\). Let \((f^n_t) \in \text{LC}_0\) and \((g^n_t) \in \text{DLC}\) associated with \((1, \tau_n) \in \text{BP}\). In this case \(\Omega((f^n_t)) = \mathbb{C}\) (by Lemma 4.4 below) and \(\Lambda((g^n_t))\) consists of one point in \(\overline{\mathbb{D}}\) for all \(n \in \mathbb{N}\). By Theorem 3.9, one can deduce that \(\Phi_n\), a map defined by \((f^n_t)\) and \((g^n_t)\) as in Section 3.4, is a \(0\)-quasiconformal automorphism of \(\mathbb{C}\), i.e., a Möbius transformation. Since \(f_0 \in \mathcal{S}\), \(\Phi^n = \text{id}_\mathbb{C}\) for all \(n \in \mathbb{N}\). Hence a local uniform limit \(\Phi\) of \(\{\Phi_n\}\) is \(\text{id}_\mathbb{C}\). This proves that \(f_t(\mathbb{D})\) tends to \(\mathbb{C}\) as \(t \to \infty\). Accordingly, \(\Lambda((g_t))\) consists of one point in \(\overline{\mathbb{D}}\). \(\square\)

**The proof of Theorem 1.2.** Proof of (i): Consider the decreasing Loewner chain \((g_t)\) as the one associated with the Berkson-Porta data \((1, \tau) \in \text{BP}\). Applying Theorem 3.1, we immediately obtain the first assertion of the theorem.

Proof of (ii): We employ the same idea as the proof of [GH, Theorem 3.5]. Let us fix \(s \geq 0\) and \(t \geq s\). Then one defines \((\tilde{f}_a)_{a \geq 0}\) by

\[
\tilde{f}_t(z) := \begin{cases} \varphi_{s+a,t}(z), & a \in [0, t), \\
(a-t, e) & a \in [t, \infty).
\end{cases}
\]

By the definition, \((\tilde{f}_a)_{a \geq 0}\) is a Loewner chain. Since \(\varphi_{s+a,t} = f^{-1}_t \circ f_{s+a}\), \(\tilde{f}_t\) satisfies

\[
\frac{\partial \tilde{f}_a(z)}{\partial a} = \frac{\partial \tilde{f}_a(z)}{\partial z} ((z - \tau(s + a))(1 - \overline{(s + a)z})p(z, s + a)
\]

of all \(z \in \mathbb{D}\) and almost all \(a \in [0, t)\), a Herglotz function \(\tilde{p}\) associated with \((\tilde{f}_a)\) is given by

\[
\tilde{p}(z, a) = \begin{cases} p(z, a + s), & a \in [0, t), \\
1 & a \in [t, \infty).
\end{cases}
\]

By the first assertion of this theorem, we conclude that \(\varphi_{s,t}\) has a \(k\)-quasiconformal extension to \(\mathbb{C}\) for each \(s \geq 0\) and \(t \geq s\).

Proof of (iii): By Lemma 4.1, \(\Delta([g_t]) = \overline{\mathbb{C}} \setminus \{a\}\) where \(\{a\} := \Lambda((g_t))\). Since \(\Phi : \Delta([g_t]) \to \Omega((f_t))\) in (3.2) is a homeomorphism, it follows that \(\Omega((f_t)) = \mathbb{C}\). \(\square\)

If \(p(z, t) = q(z, t)\) for all \(z \in \mathbb{D}\) and almost all \(t \in [0, \infty)\), then the inequality (3.1) represents a sector domain;
Corollary 4.2. Let $k \in [0, 1)$. Let $(f_t) \in \text{LC}$, $(p, \tau) \in \text{BP}$ associated with $(f_t)$, and $(g_t) \in \text{DLC}$ associated with $(p, \tau)$. If $p$ is a function which is not essentially equal to 0 and
\[ p(z, t) \in \left\{ z : \arg z < \frac{k\pi}{2} \right\} \]
for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, then $f_t$ has a $\sin(k\pi/2)$-quasiconformal extension to $\Delta[(g_t)]$ onto $\Omega[(f_t)]$ for each $t \in [0, T)$, where $(g_t) \in \text{DLC}$ associated with $(p, \tau) \in \text{BP}$ and $T \in (0, \infty)$ is the smallest number such that $p(\mathbb{D}, t) \neq 0$ for almost all $t \in (T, \infty)$.

Choosing $(f_t)$ as $p \equiv 1$, we obtain the following.

Corollary 4.3. Let $(g_t) \in \text{DLC}$ and $(q, \tau) \in \text{BP}$ associated with $(g_t)$. If $q$ satisfies
\[ \left| \frac{q(z, t) - 1}{q(z, t) + 1} \right| \leq k \]
for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$. Then:
(i) for each $t \in [0, \infty)$, $g_t$ has a $k$-quasiconformal extension to $\overline{\mathbb{C}}$;
(ii) for each $s \in [0, \infty)$ and $t \in [s, \infty)$, $\omega_{s,t} := g_t^{-1} \circ g_s$ has a $k$-quasiconformal extension to $\overline{\mathbb{C}}$;
(iii) $\Lambda[(g_t)]$ consists of one point in $\overline{\mathbb{D}}$.

4.2. Loewner Range. We close the paper with some considerations on the Loewner range $\Omega[(f_t)]$.

Under some restriction on $\tau$, one can obtain the same conclusion as Theorem 1.2 (iii) with a weaker assumption on the Herglotz function $p$.

Lemma 4.4. Let $(f_t) \in \text{LC}$ and $(p, \tau) \in \text{BP}$ associated with $(f_t)$. Suppose that $\tau \in \overline{\mathbb{D}}$ is a constant, and there exist uniform constants $C_1, C_2 > 0$ such that
\[ C_1 < \text{Re} p(z, t) < C_2 \]
for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$. Then $\Omega[(f_t)] = \mathbb{C}$.

Proof. Suppose firstly $\tau \in \text{DW}$ is an internal fixed point of $\mathbb{D}$. In this case we may assume that $\tau = 0$. Let $(\varphi_{s,t}) \in \text{EF}$ associated with $(f_t)$. Then in view of $f_t'(0) = 1/\varphi_{0,t}'(0)$, Koebe’s 1/4-Theorem shows that $f_t(\mathbb{D})$ contains a disk radius $1/(4|\varphi_{0,t}'(0)|)$. Hence what we need to prove is $\lim_{t \to \infty} |\varphi_{0,t}'(0)| = 0$. Since $\tau = 0$, $(\varphi_{s,t})$ satisfies
\[ \dot{\varphi}_{s,t}(z) = -\varphi_{s,t}(z)p(\varphi_{s,t}(z), t). \]
Then calculations show that
\[ \frac{\dot{\varphi}_{0,t}(z)}{\varphi_{0,t}(z)} = -p(\varphi_{0,t}(z), t) \]
\[ \implies \log \frac{\varphi_{0,t}(z)}{z} = -\int_0^t p(\varphi_{0,u}(z), u)du \]
\[ \implies \Re \log \varphi_{0,t}'(0) = -\int_0^t \Re p(0, u)du < -C_1 t. \]
It shows that $|\varphi_{0,t}'(0)| \to 0$ as $t \to \infty$.

The case when $\tau \in \text{DW}$ is a boundary fixed point of $\mathbb{D}$ is verified in [GH].

A direct corollary of Lemma 4.4 is that the same conclusion holds if $\tau$ is a step function in $\overline{\mathbb{D}}$. On the other hand, the following example tells us that one cannot take the approximation method as in the proof of Theorem 3.1 to obtain $\Omega[f_t] = \mathbb{C}$ in general;
\[ f_t^n(z) := e^{t/n}z \quad (n \in \mathbb{N}) \]
whose Loewner Range $\Omega[\{f^n\}]$ is $\mathbb{C}$ for any fixed $n$ but not so is the locally uniform limit $\lim_{n \to \infty} f^n(z) = z$. Hence, to prove the following, we need an alternative method.

**Problem 4.5.** Let $(f_j) \in \text{LC}$ and $(p, \tau) \in \text{BP}$ associated with $(f_j)$. If there exist uniform constants $C_1, C_2 > 0$ such that $C_1 < \text{Re}(z, t) < C_2$ for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$, then $\Omega[\{f^n\}] = \mathbb{C}$.

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