(p, q)-string in the wrapped supermembrane on 2-torus

A classical analysis of the bosonic sector

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Abstract

We consider a wrapped supermembrane on $\mathbb{R}^9 \times T^2$. We examine a double-dimensional reduction to deduce a $(p, q)$-string in type IIB superstring theory from the wrapped supermembrane. In particular, directly from the wrapped supermembrane action, we explicitly derive the action of a string which carries the RR 2-form charge as well as the NSNS 2-form charge, and the tension of the string agrees with the $(p, q)$-string tension.
1 Introduction

M-theory includes the supermembrane in eleven dimensions [1] which is expected to play an important role to understand the fundamental degrees of freedom in M-theory. Actually, it was shown that the wrapped supermembrane on $\mathbb{R}^{10} \times S^1$ is related to type IIA superstring on $\mathbb{R}^{10}$ by means of the double dimensional reduction [2]. On the other hand, type IIB superstring is related to type IIA superstring via T-duality, or type IIA superstring on $\mathbb{R}^9 \times S^1$ leads to type IIB superstring on $\mathbb{R}^{10}$ in the shrinking limit of the $S^1$ radius. Hence, type IIB superstring in 10 dimensions is to be deduced from supermembrane on a vanishing 2-torus.

Schwarz showed an $SL(2, \mathbb{Z})$ family of string solutions of type IIB supergravity [3]. The $(p, q)$-string [3, 4] is considered to be the bound state of fundamental strings (F-strings) and D1-branes (D-strings) in type IIB superstring theory. Furthermore, it was pointed out that the supermembrane which is wrapping $p$-times around one of two compact directions and $q$-times around the other direction gives a $(p, q)$-string. However, it has not been derived directly from the supermembrane action. In this paper we consider shrinking the 2-torus to approach type IIB superstring theory. Actually we deduce $(p, q)$-strings in type IIB superstring theory from the wrapped supermembrane on $\mathbb{R}^9 \times T^2$ in the shrinking limit of the 2-torus.

The plan of this paper is as follows. In the next section, we consider the supermembrane on $\mathbb{R}^9 \times T^2$. We shall carefully rewrite the eleven-dimensional supergravity background fields to the nine dimensional ones and consider the double dimensional reduction along an oblique direction of $T^2$. In section 3, we consider the T-dual of the derived string action along another compact direction of the 2-torus to deduce a string action with the $(p, q)$-string tension. We shall see that the string carries $p$-times the unit NSNS 2-form charge and $q$-times the unit RR 2-form charge as well, which indicates that the deduced string is, in fact, a $(p, q)$-string in type IIB superstring theory. The final section contains some discussion.

2 Double dimensional reduction

The supermembrane in a eleven-dimensional supergravity background is given by [1]

$$ S = T \int d\tau \int_0^{2\pi} d\sigma d\rho \left[ \frac{1}{2} \sqrt{-\hat{\gamma}} \hat{\gamma}^{\alpha\beta} E_\alpha^A E_\beta^B \eta_{AB} - \frac{1}{2} \sqrt{-\hat{\gamma}} - \frac{1}{3!} e^{\alpha\beta\gamma} E_\alpha^A E_\beta^B E_\gamma^C \hat{A}_{C\hat{B}\hat{A}} \right], $$ (2.1)

where $T$ is the tension of the supermembrane, $\hat{\gamma}_{\alpha\beta}$ ($\alpha, \beta = 0, 1, 2$) is the worldvolume metric, $\hat{\gamma} = \det \hat{\gamma}_{\alpha\beta}$, and the target space is a supermanifold with the superspace coordinates $\hat{Z}^M = (X^M, \theta^m)$ ($M = 0, \cdots, 10$, $m = 1, \cdots, 32$). Furthermore, $\hat{A}_{MNP}(Z)$ is the super three-form and $E_\alpha^A \equiv (\partial_\alpha Z^M) E_M^A$ where $E_M^A$ is the supervielbein and $A = (A, a)$ is the tangent space index. We shall consider a dimensional reduction and in order that one can see the procedure easily we focus on the bosonic degrees of freedom hereafter. The bosonic background fields are included in the superfields as

$$ E_M^A(Z) \bigg|_{\text{fermions}=0} = e_M^A(X), \quad \hat{A}_{MNP}(Z) \bigg|_{\text{fermions}=0} = A_{MNP}(X). $$ (2.2)
Then the action (2.1) is reduced to

\[
S = T \int d\tau \int_0^{2\pi} d\sigma d\rho \left[ \frac{1}{2} \sqrt{-\hat{\gamma}} \hat{\gamma}^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN} (X) \right.
\]

\[
- \frac{1}{2} \sqrt{-\hat{\gamma}} + \frac{1}{3!} \epsilon^{\alpha\beta\gamma} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P A_{MNP} (X) \right].
\] (2.3)

Note that variation w.r.t. $\hat{\gamma}^{\alpha\beta}$ yields the induced metric,

\[
\hat{\gamma}^{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N G_{MN} (X),
\] (2.4)

and plugging it back into the original action leads to the Nambu-Goto form

\[
S = T \int d\tau \int_0^{2\pi} d\sigma d\rho \left[ \sqrt{-\hat{\gamma}} + \frac{1}{3!} \epsilon^{\alpha\beta\gamma} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P A_{MNP} (X) \right].
\] (2.5)

Actually, we consider a wrapped supermembrane action on $\mathbb{R}^9 \times T^2$. We shall take the shrinking limit of the 2-torus and deduce the $(p, q)$-string action directly from the action. We take the 10th and 9th directions to compactify on $T^2$, whose radii are $L_1$ and $L_2$, respectively. In taking the shrinking volume limit of the 2-torus, we keep the ratio of the radii finite,

\[
g_b \equiv \frac{L_1}{L_2} : \text{finite.} \quad (L_1, L_2 \to 0)
\] (2.6)

Now we consider two cycles on $T^2$ characterized by two sets of co-prime integers $(p, q)$ and $(r, s)$. We impose the following condition on the two sets of co-prime integers in order that one can adjust the spacetime coordinates to the target space,

\[
pr + qs = 0, \quad ps - qr \neq 0, \quad p, q, r, s \in \mathbb{Z}.
\] (2.7)

Considering the line-element in $(X^9, X^{10})$ surface,

\[
G_{99} (dX^9)^2 + 2G_{910} dX^9 dX^{10} + G_{1010} (dX^{10})^2
\]

\[
= \left( G_{99} - \frac{(G_{910})^2}{G_{1010}} \right) (dX^9)^2 + G_{1010} \left( dX^{10} + \frac{G_{910}}{G_{1010}} dX^9 \right)^2,
\] (2.8)

we shall represent the wrapping of the supermembrane as

\[
\sqrt{G_{1010}} X^{10} (\tau, \sigma, \rho + 2\pi) = 2\pi w_1 L_1 p + \sqrt{G_{1010}} X^{10} (\tau, \sigma, \rho),
\] (2.9)

\[
\sqrt{G_{99}} - \frac{(G_{910})^2}{G_{1010}} X^9 (\tau, \sigma, \rho + 2\pi) = 2\pi w_1 L_2 q + \sqrt{G_{99}} - \frac{(G_{910})^2}{G_{1010}} X^9 (\tau, \sigma, \rho),
\] (2.10)

\[
\sqrt{G_{1010}} X^{10} (\tau, \sigma + 2\pi, \rho) = 2\pi w_2 L_1 r + \sqrt{G_{1010}} X^{10} (\tau, \sigma, \rho),
\] (2.11)

\[
\sqrt{G_{99}} - \frac{(G_{910})^2}{G_{1010}} X^9 (\tau, \sigma + 2\pi, \rho) = 2\pi w_2 L_2 s + \sqrt{G_{99}} - \frac{(G_{910})^2}{G_{1010}} X^9 (\tau, \sigma, \rho),
\] (2.12)

1 The mass dimensions of the worldvolume parameters $(\tau, \sigma, \rho)$ and the eleven-dimensional background fields $(G_{MN}, A_{MNP})$ are 0. And the mass dimension of worldvolume metric $\hat{\gamma}^{\alpha\beta}$ is $-2$.

2 Geometrically, the first condition in eq.(2.5) indicates the orthogonality of the two vectors $(p, q)$ and $(r, s)$ and the second condition means that the area defined by the two vectors is non-zero.
or

\[
X^{10}(\xi^\alpha) = \frac{L_1 (w_1 \rho + w_2 r \sigma)}{\sqrt{G_{1010}}} + Y^1(\xi^\alpha),
\]

\[
X^9(\xi^\alpha) = \frac{L_2 (w_1 q \rho + w_2 s \sigma)}{\sqrt{G_{99} - (G_{910})^2}} + Y^2(\xi^\alpha),
\]

where \( \xi^\alpha = \tau, \sigma, \rho \) and

\[
w_n \in \mathbb{N}. \quad (n = 1, 2)
\]

Note that \( w_n \) can be negative, or \( w_n \in \mathbb{Z} \setminus \{0\} \), however, one can flip the signs of \( p, q \rightarrow -p, -q \) (for \( w_1 \)) and \( r, s \rightarrow -r, -s \) (for \( w_2 \)) to have eq.(2.15) without loss of generality. The above equations indicate that the supermembrane is wrapping \( w_1p \)-times around one of the two compact directions \( (X^{10}) \) and \( w_1q \)-times around the other direction \( (X^9) \) if one advances by \( 2\pi \) along the \( \rho \)-direction on the worldsheet. Thus, this wrapped supermembrane is expected to give \((p, q)\)-strings [3]. Actually, we shall see that the \((p, q)\)-string comes out through the double dimensional reduction and T-duality in the next section.

Now that we shall adopt the double dimensional reduction technique [2]. However, we should be careful to deduce \((p, q)\)-strings. First we determine the spacetime direction to align with one of the worldvolume coordinate, or we fix the gauge. We define \( X^y \) and \( X^z \) by an \( \text{SO}(2) \) rotation of the target space,

\[
\begin{pmatrix}
X^z \\
X^y
\end{pmatrix} = O_{(p,q)} \begin{pmatrix}
X^{10} \\
X^9
\end{pmatrix},
\]

where

\[
O_{(p,q)} = \frac{1}{c_{pq}} \begin{pmatrix}
p & q \\
-q & p
\end{pmatrix} \equiv \begin{pmatrix}
\hat{p} & \hat{q} \\
-\hat{q} & \hat{p}
\end{pmatrix} \in \text{SO}(2), \quad c_{pq} \equiv \sqrt{p^2 + q^2}.
\]

By using the relations between the eleven-dimensional supergravity and nine-dimensional (or \( S^1 \)-compactified) type IIB fields [5, 6], we have

\[
\sqrt{\frac{G_{99} - (G_{910})^2}{G_{1010}}} = e^{-\varphi} = g_b = \frac{L_2}{L_1},
\]

where \( \varphi \) is a type IIB dilaton background. Then we have

\[
X^z = \frac{L_1 w_1 c_{pq} \rho}{\sqrt{G_{1010}}} + \hat{p} Y^1(\xi^\alpha) + \hat{q} Y^2(\xi^\alpha),
\]

\[
X^y = \frac{L_2 (ps - qr) w_2 \sigma}{c_{pq} \sqrt{G_{99} - (G_{910})^2}} - \hat{q} Y^1(\xi^\alpha) + \hat{p} Y^2(\xi^\alpha).
\]

A suitable choice of the target-space metric is \((M, N = 0, 1, \cdots, 8, 9, 10)\)

\[
G_{MN} = \begin{pmatrix}
\frac{1}{\sqrt{G_{1010}}} g_{\hat{\mu} \hat{\nu}} + \frac{1}{G_{1010}} G_{\hat{\mu} 10} G_{\hat{\nu} 10} & G_{\hat{\mu} 10} \\
G_{\hat{\nu} 10} & G_{1010}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\sqrt{G_{1010}}} g_{\mu \nu} + \frac{1}{G_{1010}} G_{\mu 10} G_{\nu 10} & \frac{1}{\sqrt{G_{1010}}} g_{\mu 9} + \frac{1}{G_{1010}} G_{\mu 10} G_{9 10} & G_{\mu 10} \\
\frac{1}{\sqrt{G_{1010}}} g_{\nu 9} + \frac{1}{G_{1010}} G_{9 10} G_{\nu 10} & G_{9 10} & G_{9 10}
\end{pmatrix},
\]

\[
G_{\nu 10}
\]

\[
G_{9 10}
\]

\[
G_{10 10}
\]

\[
G_{10 10}
\]

\[
G_{10 10}
\]

\[
G_{10 10}
\]
where \( \hat{\mu}, \hat{\nu} = 0, 1, \cdots, 8, 9 \) and \( \mu, \nu = 0, 1, \cdots, 8 \). On the other hand, due to eq.(2.16) we may also write \((U, V = 0, 1, \cdots, 8, y, z)\)

\[
\tilde{G}_{UV} = G_{MN} \frac{\partial X^M}{\partial U} \frac{\partial X^N}{\partial V} = \left( \begin{array}{ccc}
\frac{1}{\sqrt{\tilde{g}_{zz}}} \tilde{g}_{\mu\nu} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{\nu z} & \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu y} + \frac{1}{G_{zz}} \tilde{G}_{\mu y} \tilde{G}_{z y} \tilde{G}_{\nu z} & \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu z} \tilde{G}_{\nu z} \tilde{G}_{z z} \\
\frac{1}{\sqrt{G_{zz}}} \tilde{g}_{y \nu} + \frac{1}{G_{zz}} \tilde{G}_{y z} \tilde{G}_{\nu z} & \tilde{G}_{\nu z} & \tilde{G}_{y z}
\end{array} \right),
\]

(2.22)

and hence we have

\[
\tilde{G}_{zz} = \hat{q}^2 G_{99} + 2 \hat{p} \hat{q} G_{910} + \hat{p}^2 G_{1010},
\]

(2.23)

\[
\tilde{G}_{yy} = \hat{p}^2 G_{99} - 2 \hat{p} \hat{q} G_{910} + \hat{q}^2 G_{1010},
\]

(2.24)

\[
\tilde{G}_{yz} = \hat{p} \hat{q} G_{99} + (\hat{p}^2 - \hat{q}^2) G_{910} - \hat{p} \hat{q} G_{1010}.
\]

(2.25)

Now we shall make a (partial) gauge choice of (cf. Ref.[2])

\[
X^z = \frac{L_1 w_1 c_{pq}}{\sqrt{G_{1010}}} \rho \equiv C \rho,
\]

(2.26)

or the \( z \)-direction is aligned with one of the space direction \( \rho \) of the worldvolume. Then the dimensional reduction is achieved by imposing the following conditions on the membrane-coordinates and the background fields,

\[
\partial_\rho X^y = \partial_\rho X^\mu = 0,
\]

(2.27)

and

\[
\partial_z G_{MN} = \partial_z A_{MNP} = 0.
\]

(2.28)

Thus the induced metric on the worldvolume is given by [2]

\[
\hat{\gamma}_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N G_{MN}(X) = \Phi^{-2/3} \left( \begin{array}{cc}
\gamma_{ij} + \Phi^2 A_i A_j & \Phi^2 A_i \\
\Phi^2 A_j & \Phi^2
\end{array} \right),
\]

(2.29)

where \( i, j = 0, 1 \) and

\[
\Phi^{4/3} = C^2 \tilde{G}_{zz},
\]

(2.30)

\[
\Phi^{4/3} A_i = C (\partial_i X^\mu \tilde{G}_{\mu z} + \partial_i X^y \tilde{G}_{y z}),
\]

(2.31)

\[
\gamma_{ij} = C (\partial_i X^\mu \partial_j X^\nu \tilde{g}_{\mu \nu} + 2 \partial_i X^{\mu\nu} \partial_j \tilde{g}_{\mu \nu} + \partial_i X^y \partial_j X^y \tilde{g}_{yy}).
\]

(2.32)

Note that

\[
\det \hat{\gamma}_{\alpha\beta} = \det \gamma_{ij},
\]

(2.33)

which is to be used in calculating the first term in eq.(2.5). From eqs.(2.16), (2.26) and (2.27), we have

\[
\epsilon^{\alpha\beta\gamma} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P A_{MNP}
= 3C \left\{ \epsilon^{ij} \partial_i X^\mu \partial_j X^\nu (\hat{q} A_{\mu \nu 9} + \hat{p} A_{\mu \nu 10}) + 2 \epsilon^{ij} \partial_i X^\mu \partial_j X^y A_{\mu 910} \right\}.
\]

(2.34)
Thus, by the double dimensional reduction of eqs.(2.26)-(2.28), the supermembrane action (2.3), or (2.5), is reduced to the following equivalent one,

\[
S_{ddr} = \frac{2\pi T}{2} \int d\tau \int_0^{2\pi} d\sigma \left[ \sqrt{-\gamma} \tilde{\gamma}^{ij} (\partial_i X^\mu \partial_j X^\nu \tilde{g}_{\mu\nu} + 2\partial_i X^\mu \partial_j X^y \tilde{g}_{\mu y} + \partial_i X^y \partial_j X^y \tilde{g}_{yy}) + \{\epsilon^{ij} \partial_i X^\mu \partial_j X^y (\hat{q} A_{\mu\nu y} + \hat{p} A_{\mu\nu 10}) + 2\epsilon^{ij} \partial_i X^\mu \partial_j X^y A_{\nu 910}\} \right],
\]

(3.5)

where the first term on the r.h.s. has been rewritten in Polyakov form by introducing the worldsheet metric \(\tilde{\gamma}_{ij}\) instead of Nambu-Goto form. As is pointed out in [2], this action (3.5) has conformal invariance.

3 (p, q)-string from wrapped supermembrane

In this section, we derive the \((p, q)\)-string action from the reduced supermembrane action in eq.(2.35). We shall take T-dual along the other compactified \(X^y\)-direction (cf. eq.(2.16)). Introducing a variable \(\tilde{X}^y\), which is seen to be dual to the other compactified \(X^y\)-direction, eq.(2.35) can be rewritten by

\[
S_{ddr} = \frac{2\pi T}{2} \int d\tau \int_0^{2\pi} d\sigma C \left[ \sqrt{-\gamma} \tilde{\gamma}^{ij} (\partial_i X^\mu \partial_j X^\nu \tilde{g}_{\mu\nu} + 2\partial_i X^\mu \partial_j X^y \tilde{g}_{\mu y} + Y_i Y_j \tilde{g}_{yy}) + \{\epsilon^{ij} \partial_i X^\mu \partial_j X^y (\hat{q} A_{\mu\nu y} + \hat{p} A_{\mu\nu 10}) + 2\epsilon^{ij} \partial_i X^\mu \partial_j X^y A_{\nu 910} + 2\epsilon^{ij} \tilde{X}^y \partial_j Y_i\} \right],
\]

(3.1)

since the variation w.r.t. \(\tilde{X}^y\) leads to \(\epsilon^{ij} \partial_j Y_j = 0\) or \(Y_j = \partial_j X^y\) and hence eq.(2.35) can be reproduced.\(^3\) On the other hand, assuming that all the fields are independent of \(Y_j\) (or \(X^y\)), the variation w.r.t. \(Y_i\) leads to

\[
Y_i = -\frac{\tilde{\gamma}_{ij} \epsilon^{ik}}{\sqrt{-\gamma} \tilde{g}_{yy}} \left( \partial_k \tilde{X}^y - A_{\nu 910} \partial_k X^\nu \right) \frac{\tilde{g}_{yy}}{\tilde{g}_{yy}} - \frac{\tilde{g}_{yy}}{\tilde{g}_{yy}} \partial_i X^\mu,
\]

(3.2)

and hence we have

\[
S_{ddr} = \frac{2\pi T}{2} \int d\tau \int_0^{2\pi} d\sigma C \left[ \sqrt{-\gamma} \tilde{\gamma}^{ij} \left\{ \partial_i X^\mu \partial_j X^\nu \left( \tilde{g}_{\mu\nu} - \tilde{g}_{\mu y} \tilde{g}_{\nu y} - A_{\nu 910} A_{\mu 910} \right) \right. \\
- 2\partial_i X^\mu \partial_j \tilde{X}^y \frac{A_{\mu\nu 910}}{\tilde{g}_{yy}} + \partial_i \tilde{X}^y \partial_j \tilde{X}^y \frac{1}{\tilde{g}_{yy}} \right\} + \epsilon^{ij} \partial_i X^\mu \partial_j X^y \left( \hat{q} A_{\mu\nu y} + \hat{p} A_{\mu\nu 10} \right) \\
- \frac{2A_{\nu 910} \tilde{g}_{yy}}{\tilde{g}_{yy}} + 2\epsilon^{ij} \partial_i \tilde{X}^y \partial_j X^\mu \frac{\tilde{g}_{yy}}{\tilde{g}_{yy}} \right].
\]

(3.3)

Now that we consider T-dual for the background fields in eq.(2.35) (or eq.(3.3)). Since we regard \(X^{10}\) (not \(X^y\)) as the 11th direction, we should take T-dual along the \(X^y\)-direction to transform type IIA superstring theory to type IIB superstring theory. Then we can rewrite the background fields in terms of those of the type IIB supergravity as follows (cf. Appendix

\(^3\)We assume that the background fields are independent of \(\tilde{X}^y\) in eq.(3.1).
On the other hand, \((p, q, r, s)\) is just the number of copies of the resulting \((1, 0)\)-string in type IIA superstring theory \([3]\) since the 11d metric \(g_{\hat{\mu}\hat{\nu}}\) is converted to the type IIA metric \(g_{\mu\nu}\) by the relation \(G_{\hat{\mu}\hat{\nu}} = g_{\mu\nu}/\sqrt{G_{1010}}\). Also, if we assume that \(l\) and \(\varphi\) are constant and hence \(e^\varphi = g_s^{\text{IB}}\), we have

\[
S_{ddr} = \frac{2\pi T}{2} \int d\tau \int_0^{2\pi} d\sigma C \sqrt{(\hat{p} + \hat{q}l)^2 + e^{-2\varphi}q^2} \sqrt{\gamma^{ij}(\partial_i X^\mu \partial_j X^\nu j_{\mu\nu} + 2 \partial_i X^\mu \partial_j \tilde{X}^y j_{9\mu} + \partial_i \tilde{X}^y \partial_j \tilde{X}^y j_{99})}
\]

\[
+ \epsilon^{ij} \partial_i X^\mu \partial_j X^\nu B_{\mu\nu}^{(pq)} + 2 \epsilon^{ij} \partial_i \tilde{X}^y \partial_j X^\mu B_{\mu\nu}^{(pq)}
\]. \hspace{1cm} (3.10)

Once we regard \(X^{10}\) as the 11th direction, the type IIA string tension \(T_s\) is given by \(2\pi L_1 T/\sqrt{G_{1010}}\) \([3]\) since the 11d metric \(G_{MN}\) is converted to the type IIA metric \(g_{\mu\nu}\) by the relation \(G_{\hat{\mu}\hat{\nu}} = g_{\mu\nu}/\sqrt{G_{1010}}\). Also, if we assume that \(l\) and \(\varphi\) are constant and hence \(e^\varphi = g_s^{\text{IB}}\), we have

\[
2\pi T C \sqrt{(\hat{p} + \hat{q}l)^2 + e^{-2\varphi}q^2} = w_1 T_s \sqrt{(p + ql)^2 + e^{-2\varphi}q^2} \equiv w_1 T_{pq}, \hspace{1cm} (3.11)
\]

where \(T_{pq}\) is the tension of a \((p, q)\)-string in type IIB superstring theory \([3]\). Actually, we see that both of the NSNS and RR antisymmetric tensors have coupled to \(X^a = (X^\mu, \tilde{X}^y)\) in eq.\((3.10)\), which implies that the reduced action \((3.10)\) is, in fact, that of \((p, q)\)-strings. Note that \(w_1\) is just the number of copies of the resulting \((p, q)\)-strings. If we allow \(q\) to be zero and take \((p, q, r, s) = (1, 0, 0, 1)\), we have the fundamental strings in type IIB superstring theory. On the other hand, \((p, q, r, s) = (0, 1, 1, 0)\) leads to the strings which couple minimally with the RR B-field, i.e., the D-strings.

4 Summary and discussion

In this paper, we have studied the double dimensional reduction of the wrapped supermembrane on \(\mathbb{R}^9 \times T^2\) and explicitly derived the bosonic sector of the \((p, q)\)-string action in
This indicates that the supermembrane actually includes a \((p, q)\)-string as an excitation mode or object. The \((1,0)\)-string (F-string) is, of course, an effective mode in a weak coupling region \(g_{\text{IIB}} \ll 1\), while the \((0,1)\)-string (D-string) in a strong coupling region \(g_{\text{IIB}} \gg 1\) for \(l = 0\). However, the valid region to treat the \((p, q)\)-string perturbatively is still obscure and is deserved to be investigated further.

The procedure of the double dimensional reduction here should be realized on the matrix-regularized wrapped supermembrane on \(\mathbb{R}^9 \times T^2\) [7], which will be reported elsewhere [8].

In this paper we have considered classically to approach the boundary of vanishing cycles of the 2-torus with the wrapped supermembrane. On the other hand, Refs.[9, 10] studied quantum mechanical justification of the double dimensional reduction in Ref.[2]. In those references, the Kaluza-Klein modes associated with the \(\rho\)-coordinate were not removed classically, but they were integrated in the path integral formulation of the wrapped supermembrane theory. Similar quantum mechanical investigation of the double dimensional reduction in this paper deserves to be investigated.

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**A Notation**

The spacetime indices:

\[
M, N, P = 0, 1, \ldots, 8, 9, 10 , \quad (A.1)
\]

\[
U, V = 0, 1, \ldots, 8, y, z , \quad (A.2)
\]

\[
\hat{\mu}, \hat{\nu} = 0, 1, \ldots, 8, 9 , \quad (A.3)
\]

\[
\mu, \nu = 0, 1, \ldots, 8 . \quad (A.4)
\]

The worldvolume and worldsheet indices:

\[
\alpha, \beta = 0, 1, 2 , \quad (A.5)
\]

\[
i, j = 0, 1 . \quad (A.6)
\]

The target-space metrics:

\[
G = 11\text{d target-space metric} , \quad (A.7)
\]

\[
\tilde{G} = 11\text{d rotated target-space metric} , \quad (A.8)
\]

\[
g = 10\text{d IIA target-space metric} , \quad (A.9)
\]

\[
\tilde{g} = 10\text{d IIA rotated target-space metric} , \quad (A.10)
\]

\[
j = 10\text{d IIB target-space metric} . \quad (A.11)
\]

The worldvolume and worldsheet metrics:

\[
\hat{\gamma} = \text{membrane worldvolume metric} , \quad (A.12)
\]

\[
\gamma = \text{string worldsheet metric} . \quad (A.13)
\]

\[\text{Of course, a BPS saturated classical solution of the } (p, q)\text{-string action } (3.10) \text{ is valid irrespective of the value of the string coupling } g_{\text{IIB}}.\]
(Anti-)symmetrization r.w.t. indices:

\[ A_{[\mu B_\nu]} = \frac{1}{2} (A_\mu B_\nu - A_\nu B_\mu) , \quad (A.14) \]

\[ A_{[\mu B_\nu C_\rho]} = \frac{1}{3!} (A_\mu B_\nu C_\rho + A_\nu B_\rho C_\mu + A_\rho B_\mu C_\nu - A_\mu B_\rho C_\nu - A_\rho B_\nu C_\mu - A_\nu B_\mu C_\rho) , \quad (A.15) \]

\[ A_{[\mu B_\nu | C_\rho]} = \frac{1}{2} (A_\mu B_\nu C_\rho - A_\rho B_\nu C_\mu) , \quad (A.16) \]

\[ A_{\{\mu B_\nu\}} = \frac{1}{2} (A_\mu B_\nu + A_\nu B_\mu) , \quad \text{etc.} \quad (A.17) \]

B 11d vs. 10d background fields

The 11-dimensional metric can be written by

\[
G_{MN} = \begin{pmatrix}
\frac{1}{\sqrt{G_{1010}}} g_{\mu\nu} + \frac{1}{G_{1010}} G_{\mu\nu} G_{10} & G_{\mu10} \\
G_{10\mu} & G_{1010}
\end{pmatrix}
\equiv e^{-\frac{\phi}{3}} \begin{pmatrix}
g_{\mu\nu} + e^{2\phi} A_{\mu \nu} & e^{2\phi} A_{\mu} \\
e^{2\phi} A_{\mu} & e^{2\phi} A_{\mu}
\end{pmatrix}
\]

\[
\equiv \begin{pmatrix}
\frac{1}{\sqrt{G_{1010}}} g_{\mu\nu} + \frac{1}{G_{1010}} G_{\mu\nu} G_{10} & G_{\mu9} + \frac{1}{G_{1010}} G_{\mu10} G_{910} & G_{\mu10} \\
G_{10\mu} & G_{1010} & G_{910} & G_{910}
\end{pmatrix}, \quad (B.1)
\]

and the third-rank antisymmetric tensor \( A_{MNP} \) is decomposed as

\[
A_{MNP} = (A_{\mu\nu\rho}, A_{\mu\nu10}, A_{\mu\nu9}, A_{\mu910})
= (C_{\mu\nu\rho}, B_{\mu\nu}, C_{\mu\nu9}, B_{\mu9}). \quad (B.2)
\]

Those fields are related to those of IIB as

\[
g_{\mu\nu} = J_{\mu\nu} - \frac{J_{\mu9} J_{\nu9} - B_{9\mu}^{(1)} B_{9\nu}^{(1)}}{J_{99}}, \quad (B.3)
\]

\[
g_{9\mu} = \frac{B_{9\mu}^{(1)}}{J_{99}}, \quad (B.4)
\]

\[
g_{99} = \frac{1}{J_{99}}, \quad (B.5)
\]

\[
C_{\mu\nu9} = \frac{B_{9\mu}^{(2)}}{J_{99}} + \frac{2 B_{9\mu}^{(2)}}{J_{99}}, \quad (B.6)
\]

\[
C_{\mu\nu\rho} = \frac{D_{9\mu\nu\rho} + \frac{3}{2} \epsilon_{\mu\nu\rho} B_{9\mu}^{(1)} B_{9\nu}^{(1)} + \frac{3}{2} \epsilon_{\mu\nu\rho} B_{9\mu}^{(1)} B_{9\nu}^{(1)}}{J_{99}}, \quad (B.7)
\]

\[
B_{\mu\nu} = \frac{B_{\mu\nu}^{(1)}}{J_{99}} + \frac{B_{9\mu}^{(1)} J_{\nu9} - B_{9\nu}^{(1)} J_{\mu9}}{J_{99}}, \quad (B.8)
\]
\[ B_{9\mu} = \frac{j_{9\mu}}{j_{99}}, \quad (B.9) \]
\[ A_{\mu} = -B_{9(2)} + lB_{9(1)}, \quad (B.10) \]
\[ A_9 = l, \quad (B.11) \]
\[ \phi = \varphi - \frac{1}{2} \ln j_{99}. \quad (B.12) \]

On the other hand, the 9-10 rotated metric is given by \((U, V = 0, 1, \cdots, 8, y, z)\)

\[
\tilde{G}_{UV} = G_{MN} \frac{\partial X^M}{\partial X^U} \frac{\partial X^N}{\partial X^V} = \begin{pmatrix}
\frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu\nu} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{\nu z} & \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu y} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{y z} & \tilde{G}_{\mu z} \\
\frac{1}{G_{zz}} \tilde{g}_{y \nu} + \frac{1}{G_{zz}} \tilde{G}_{y z} \tilde{G}_{\nu z} & \frac{1}{G_{zz}} \tilde{g}_{y y} + \frac{1}{G_{zz}} \tilde{G}_{y z} \tilde{G}_{y z} & \tilde{G}_{y z} \\
\tilde{G}_{\nu z} & \tilde{G}_{y z} & \tilde{G}_{zz} 
\end{pmatrix}. \quad (B.13) \]

Then

\[
\tilde{G}_{zz} = \hat{q}^2 G_{99} + 2\hat{p}\hat{q} G_{910} + \hat{p}^2 G_{1010} \\
= \hat{q}^2 \left( \frac{g_{99}}{G_{1010}} + G_{1010} A_9^2 \right) + 2\hat{p}\hat{q} G_{1010} A_9 + \hat{p}^2 G_{1010} \\
= G_{1010} (\hat{p} + \hat{q} A_9)^2 + \hat{q}^2 \frac{g_{99}}{G_{1010}} = e^{4\varphi/3} j_{99}^{-2/3} \left\{ (\hat{p} + \hat{q} l)^2 + e^{-2\varphi} \hat{q}^2 \right\} , \quad (B.14) \]

\[
\tilde{G}_{yy} = \hat{p}^2 G_{99} - 2\hat{p}\hat{q} G_{910} + \hat{q}^2 G_{1010} \\
= \hat{p}^2 \left( \frac{g_{99}}{G_{1010}} + G_{1010} A_9^2 \right) - 2\hat{p}\hat{q} G_{1010} A_9 + \hat{q}^2 G_{1010} \\
= G_{1010} (\hat{q} - \hat{p} A_9)^2 + \hat{p}^2 \frac{g_{99}}{G_{1010}} = e^{4\varphi/3} j_{99}^{-2/3} \left\{ (\hat{q} - \hat{p} l)^2 + \hat{p}^2 e^{-2\varphi} \right\} , \quad (B.15) \]

\[
\tilde{G}_{yz} = \hat{p}\hat{q} (G_{99} + (\hat{p}^2 - \hat{q}^2) G_{910} - \hat{p}\hat{q} G_{1010} \\
= \hat{p}\hat{q} \left( \frac{g_{99}}{G_{1010}} + G_{1010} A_9^2 \right) + (\hat{p}^2 - \hat{q}^2) G_{1010} A_9 - \hat{p}\hat{q} G_{1010} \\
= G_{1010} (\hat{p} A_9 - \hat{q}) (\hat{q} A_9 + \hat{p}) + \hat{p}\hat{q} \frac{g_{99}}{G_{1010}} \\
= e^{4\varphi/3} j_{99}^{-2/3} \left\{ (\hat{p} l - \hat{q}) (\hat{q} l + \hat{p}) + \hat{p}\hat{q} e^{-2\varphi} \right\} , \quad (B.16) \]

Furthermore,

\[
\tilde{G}_{\mu \nu} = \hat{p} G_{\mu 9} - \hat{q} G_{\mu 10} = \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu y} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{y z}, \quad (B.17) \]

and hence

\[
\tilde{g}_{\mu y} = -\frac{1}{\sqrt{G_{zz}}} \left( \tilde{G}_{\mu y} \tilde{G}_{zz} - \tilde{G}_{\mu z} \tilde{G}_{y z} \right) \\
= -\frac{1}{\sqrt{G_{zz}}} \left\{ (\hat{p} G_{\mu 9} - \hat{q} G_{\mu 10}) (\hat{q}^2 G_{99} + 2\hat{p}\hat{q} G_{910} + \hat{p}^2 G_{1010}) \\
- (\hat{q} G_{\mu 9} + \hat{p} G_{\mu 10}) (\hat{p}\hat{q} G_{99} + (\hat{p}^2 - \hat{q}^2) G_{910} - \hat{p}\hat{q} G_{1010}) \right\} 
\]
\[
\begin{align*}
\ &= \frac{G_{\mu9} (\hat{p} G_{1010} + \hat{q} G_{109}) - G_{\mu10} (\hat{q} G_{99} + \hat{p} G_{910})}{\sqrt{G_{zz}}} \\
\ &= \sqrt{\frac{G_{1010}}{G_{zz}}} \left\{ (\hat{p} + \hat{q} A_9) g_{\mu9} - \hat{q} g_{99} A_{\mu} \right\} \\
\ &= \sqrt{\frac{G_{1010}}{G_{zz}}} \left\{ (\hat{p} + \hat{q} l) \frac{B_{\nu\mu}^{(1)}}{j_{99}} - \hat{q} B_{\nu\mu}^{(2)} + l B_{9\nu}^{(1)} \right\} = \frac{\hat{p} B_{9\mu}^{(1)} + \hat{q} B_{9\nu}^{(2)}}{j_{99} \sqrt{(\hat{p} + \hat{q} l)^2 + e^{-2q^2}}} \quad (B.18)
\end{align*}
\]

We shall calculate \( \tilde{g}_{\mu\nu} \), \( \tilde{g}_{yy} \) as follows. The equation,

\[
\tilde{g}_{yy} = \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{yy} + \frac{1}{G_{zz}} \tilde{G}_{yz} \tilde{G}_{yz}, \quad (B.19)
\]

leads to

\[
\begin{align*}
\tilde{g}_{yy} &\ = \frac{1}{\sqrt{G_{zz}}} \left( \tilde{g}_{yy} \tilde{G}_{zz} - \tilde{G}_{yz} \tilde{G}_{yz} \right) \\
&\ = \frac{1}{\sqrt{G_{zz}}} \left[ \left\{ G_{1010} (\hat{q} - \hat{p} A_9)^2 + \hat{p}^2 \frac{g_{99}}{\sqrt{G_{1010}}} \right\} \left\{ G_{1010} (\hat{p} + \hat{q} A_9)^2 + \hat{q}^2 \frac{g_{99}}{\sqrt{G_{1010}}} \right\} \\
&\quad - \left\{ G_{1010} (\hat{p} A_9 - \hat{q}) (\hat{q} A_9 + \hat{p}) + \hat{p} \hat{q} \frac{g_{99}}{\sqrt{G_{1010}}} \right\}^2 \right] \\
&\ = \sqrt{\frac{G_{1010}}{G_{zz}}} g_{99} \left\{ \hat{p}^2 (\hat{p} + \hat{q} A_9)^2 + \hat{q}^2 (\hat{q} - \hat{p} A_9)^2 - 2 \hat{p} \hat{q} (\hat{q} A_9 - \hat{q}) (\hat{q} A_9 + \hat{p}) \right\} \\
&\ = \sqrt{\frac{G_{1010}}{G_{zz}}} g_{99} = \frac{1}{j_{99} \sqrt{(\hat{p} + \hat{q} l)^2 + e^{-2q^2}}} \quad (B.20)
\end{align*}
\]

Similarly

\[
G_{\mu\nu} = \frac{1}{\sqrt{G_{1010}}} g_{\mu\nu} + \frac{1}{G_{1010}} g_{\mu10} G_{\nu10} = \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu\nu} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{\nu z}, \quad (B.21)
\]

leads to

\[
\begin{align*}
\tilde{g}_{\mu\nu} &\ = \frac{1}{\sqrt{G_{zz}}} \left( G_{\mu\nu} \tilde{G}_{zz} - \tilde{G}_{\mu z} \tilde{G}_{\nu z} \right) \\
&\ = \frac{1}{\sqrt{G_{zz}}} \left\{ \left( \frac{1}{\sqrt{G_{1010}}} g_{\mu\nu} + \frac{1}{G_{1010}} g_{\mu10} G_{\nu10} \right) \tilde{G}_{zz} - (\hat{q} G_{\mu9} + \hat{p} G_{\mu10} )(\hat{q} G_{\nu9} + \hat{p} G_{\nu10}) \right\} \\
&\ = \sqrt{\frac{\tilde{G}_{zz}}{G_{1010}} g_{\mu\nu} + \tilde{G}_{zz} G_{\mu10} G_{\nu10} - G_{1010} (\hat{q} G_{\mu9} + \hat{p} G_{\mu10} )(\hat{q} G_{\nu9} + \hat{p} G_{\nu10}) \\
&\quad \sqrt{G_{zz}} \left\{ g_{99} A_\mu A_\nu - \frac{g_{\mu9} g_{\nu9}}{(G_{1010})^{3/2}} - A_9 (A_\mu g_{\nu9} + A_\nu g_{\mu9}) \right\} \\
&\quad - \hat{p} \hat{q} \sqrt{\frac{G_{1010}}{G_{zz}}} (A_\mu g_{\nu9} + A_\nu g_{\mu9}) \\
&\ = \sqrt{\frac{\tilde{G}_{zz}}{G_{1010}} g_{\mu\nu} + \hat{p} \hat{q} \sqrt{\frac{G_{1010}}{G_{zz}}} \left\{ (-B_{9\mu}^{(2)} + l B_{9\mu}^{(1)}) (-B_{9\nu}^{(2)} + l B_{9\nu}^{(1)}) \frac{1}{j_{99}} - B_{9\nu}^{(1)} B_{9\mu}^{(1)} \right\} \frac{B_{9\mu}^{(1)} B_{9\nu}^{(1)}}{e^{2q^2 j_{99}}} \\
\end{align*}
\]
\[ -\hat{p}q \sqrt{\frac{G_{1010}^{zz}}{G_{zz}}} \left( \frac{(-B_{9\mu}^{(2)} + l B_{9\mu}^{(1)}) B_{9\nu}^{(1)} + (-B_{9\nu}^{(2)} + l B_{9\nu}^{(1)}) B_{9\mu}^{(1)}}{j_{99}} \right) \]

\[ = \sqrt{\frac{\tilde{G}_{zz}}{G_{1010}}} g_{\mu \nu} + \frac{1}{j_{99}} \sqrt{\frac{G_{1010}^{zz}}{G_{zz}}} \left[ (\hat{p} B_{9\mu}^{(1)} + \hat{q} B_{9\nu}^{(2)}) (\hat{p} B_{9\nu}^{(1)} + \hat{q} B_{9\mu}^{(2)}) \right. \]

\[ - \left\{ (\hat{p} + \hat{q} l)^2 + \hat{q}^2 e^{-2\varphi} \right\} B_{9\mu}^{(1)} B_{9\nu}^{(1)} \]

\[ = \sqrt{(\hat{p} + \hat{q} l)^2 + e^{-2\varphi} \hat{q}^2} \left( g_{\mu \nu} - \frac{B_{9\mu}^{(1)} B_{9\nu}^{(1)}}{j_{99}} \right) + \left( \frac{\hat{p} B_{9\mu}^{(1)} + \hat{q} B_{9\mu}^{(2)}}{j_{99}} \right) \left( \frac{\hat{p} B_{9\nu}^{(1)} + \hat{q} B_{9\nu}^{(2)}}{j_{99}} \right) \frac{(\hat{p} + \hat{q} l)^2 + e^{-2\varphi} \hat{q}^2}{j_{99}} \right] \]

\[ = \sqrt{(\hat{p} + \hat{q} l)^2 + e^{-2\varphi} \hat{q}^2} \left( j_{99} \right) \left( \frac{\hat{p} B_{9\mu}^{(1)} + \hat{q} B_{9\mu}^{(2)}}{j_{99}} \right) \left( \frac{\hat{p} B_{9\nu}^{(1)} + \hat{q} B_{9\nu}^{(2)}}{j_{99}} \right) \frac((\hat{p} + \hat{q} l)^2 + e^{-2\varphi} \hat{q}^2}{j_{99}} \right] \].

(B.22)

Note that

\[ \sqrt{\frac{\tilde{G}_{zz}}{G_{1010}}} = \sqrt{(\hat{p} + \hat{q} l)^2 + e^{-2\varphi} \hat{q}^2} \]  

(B.23)

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