Weak type commutator and Lipschitz estimates: resolution of the Nazarov-Peller conjecture

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American Journal of Mathematics, Volume 141, Number 3, June 2019, pp. 593-610 (Article)

Published by Johns Hopkins University Press
DOI: https://doi.org/10.1353/ajm.2019.0019

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WEAK TYPE COMMUTATOR AND LIPSCHITZ ESTIMATES: RESOLUTION OF THE NAZAROV-PELLER CONJECTURE

By M. Caspers, D. Potapov, F. Sukochev, and D. Zanin

Abstract. Let $\mathcal{M}$ be a semi-finite von Neumann algebra and let $f: \mathbb{R} \to \mathbb{C}$ be a Lipschitz function. If $A, B \in \mathcal{M}$ are self-adjoint operators such that $[A, B] \in L_1(\mathcal{M})$, then

$$\|f(A)B\|_{1,\infty} \leq c_{\text{abs}} \|f'(\cdot)\|_{1,\infty} \|[A, B]\|_1,$$

where $c_{\text{abs}}$ is an absolute constant independent of $f$, $\mathcal{M}$ and $A, B$ and $\|\cdot\|_{1,\infty}$ denotes the weak $L_1$-norm. If $X, Y \in \mathcal{M}$ are self-adjoint operators such that $X - Y \in L_1(\mathcal{M})$, then

$$\|f(X) - f(Y)\|_{1,\infty} \leq c_{\text{abs}} \|f'(\cdot)\|_{1,\infty} \|X - Y\|_1.$$

This result resolves a conjecture raised by F. Nazarov and V. Peller implying a couple of existing results in perturbation theory.

1. Introduction. Let $H$ be a separable Hilbert space. We denote by $B(H)$ the algebra of all bounded linear operators on $H$. Let $L_p(H)$ be the Schatten-von Neumann ideal of $B(H)$. It consists of all compact operators for which its sequence of singular values lies in $\ell_p$. Let $F_p$ be the class of functions $f: \mathbb{R} \to \mathbb{C}$ such that $f(B) - f(C) \in L_p(H)$, for all self-adjoint $B, C$ such that $B - C \in L_p(H)$ and set

$$\|f\|_{F_p} = \sup_{B \neq C} \frac{\|f(B) - f(C)\|_p}{\|B - C\|_p}.$$

It was conjectured by M. G. Krein [19] that whenever the derivative $f' \in L_\infty(\mathbb{R})$ we have $f \in F_1$. This conjecture does not hold as was shown by Y. B. Farforovskaya in [13]. Also it was shown that the analogue of Krein’s problem fails in the case $p = \infty$ (see [11, 12]). In fact already for the absolute value function it was found by T. Kato that Krein’s problem has a negative answer [17]; and similarly in the case $p = 1$ by E. B. Davies [5].

A positive result in this direction was first obtained by M. Birman and M. Solomyak [1, Theorem 10] who proved that $C^{1+\epsilon} \subseteq F_1$ for every $\epsilon > 0$, and later improved by V. Peller [23] who showed that $B_{pq}^1 \subseteq F_1$. Here $B_{p,q}^1$ is the class of...
Besov spaces for which we refer to [14]. The Krein problem for the case $1 < p < \infty$, $p \neq 2$ remained open until [26]. In [26] it was shown by the second and third named author that $F_p$ consists exactly of all Lipschitz functions. Moreover in [3] a quantitative estimate for $\|f\|_{F_p}$ was found, namely $\|f\|_{F_p} \simeq p^2/(p-1)$. Earlier the same problem had been considered by M. de la Salle (unpublished, see [6] for related results on the problem) who was able to show already that $\|f\|_{F_p} \leq c_{abs} \cdot p^4/(p-1)^2$ with an absolute constant $c_{abs}$, which is more optimal than [26].

Other results concerning this problem have been obtained in [10, 18] and in the context of this paper we also mention [27] in which weak estimates for martingale inequalities were obtained.

Using interpolation, the above results would follow from a weak type Lipschitz estimate between $L_1$ and the weak-$L_1$ space $L_{1,\infty}$. The estimate was conjectured in a paper of F. Nazarov and V. Peller [21], and has remained the major open question in the study of Lipschitz properties of operator valued functions. Denote $L_{1,\infty}(H)$ for the weak $L_1$-space consisting of all compact operators $A$ whose sequence $\{\mu(k,A)\}_{k \geq 0}$ of singular values satisfies $\mu(k,A) = O(1/k+1)$.

CONJECTURE 1.1. Let $f : \mathbb{R} \to \mathbb{C}$ be Lipschitz. Whenever $A, B \in B(H)$ are self-adjoint operators such that $A - B \in L_1(H)$, we have that $f(A) - f(B) \in L_{1,\infty}(H)$ and

$$\|f(A) - f(B)\|_{1,\infty} \leq c_{abs} \|f'\|_{\infty} \|A - B\|_1,$$

for some absolute constant $c_{abs}$.

Nazarov and Peller [21] gave an affirmative answer under the assumption that the rank of $A - B$ equals 1. Since $\|\cdot\|_{1,\infty}$ is a quasi-norm and not a norm for $L_{1,\infty}(H)$ it is impossible to extend their result for when $A - B$ is a general trace class operator.

Another positive result to the conjecture was found by the current authors in [4] in the special case when $f$ is the absolute value map. The proof relies on the observation that the Schur multiplier of divided differences

$$\left( \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \right)_{\lambda \neq \mu}$$

can be written as a finite sum of compositions of a positive definite Schur multiplier and a triangular truncation operator. For general Lipschitz functions there is no reason that the latter fact should be true which renders the technique of [4] inapplicable.

The main result of this paper is a proof of Conjecture 1.1. The importance of this result lies in the fact that this gives the sharpest possible estimate for perturbations and commutators. In particular it retrieves $\|f\|_{F_p} \simeq p^2/(p-1)$ [3] and the Nazarov-Peller result [21]. A key ingredient in our proof is the connection with
non-commutative Calderón-Zygmund theory and in particular J. Parcet’s extension of the classical Calderón-Zygmund theorem (see Theorem 2.1 and [22]).

In the text we prove a somewhat stronger result in the terms of double operator integrals (see next section for definition), of which Conjecture 1.1 is a corollary.

**Theorem 1.2.** If $A$ is a self-adjoint operator affiliated with a semifinite von Neumann algebra $\mathcal{M}$, and if $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz then

$$
\left\| T_{f[1]}^{A,A}(V) \right\|_{1,\infty} \leq c_{abs} \| f' \|_{\infty} \| V \|_1, \quad V \in (L_1 \cap L_2)(\mathcal{M}).
$$

Commutator estimate follows from the observation that the double operator integral $T_{f[1]}^{A,A}([A,B])$ equals $[f(A),B]$. As explained in the proof of Theorem 5.3, Lipschitz estimates follow from commutator ones.

**Acknowledgment.** The authors thank Javier Parcet for a detailed explanation of [22].

2. Preliminaries.

2.1. General notation. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. In this paper, we always presume that $\mathcal{M}$ is represented on a separable Hilbert space.

A (closed and densely defined) operator $x$ affiliated with $\mathcal{M}$ is called $\tau$-measurable if $\tau(E|_x(s,\infty)) < \infty$ for sufficiently large $s$. We denote the set of all $\tau$-measurable operators by $S(\mathcal{M},\tau)$. For every $x \in S(\mathcal{M},\tau)$, we define its singular value function $\mu(x)$ by setting

$$
\mu(t,x) = \inf \left\{ \| x(1-p) \|_{\infty} : \tau(p) \leq t \right\}.
$$

Equivalently, for positive operator $x \in S(\mathcal{M},\tau)$, we have

$$
n_x(s) = \tau(E_x(s,\infty)), \quad \mu(t,x) = \inf \left\{ s : n_x(s) < t \right\}.
$$

For all $x, y \in S(\mathcal{M},\tau)$, we have (see e.g., [20, Corollary 2.3.16])

$$
(2.1) \quad \mu(t+s,x+y) \leq \mu(t,x) + \mu(s,y), \quad t, s > 0.
$$

2.2. Non-commutative spaces. For $1 \leq p < \infty$ we set,

$$
L_p(\mathcal{M}) = \left\{ x \in S(\mathcal{M},\tau) : \tau(|x|^p) < \infty \right\}, \quad \| x \|_p = \left( \tau(|x|^p) \right)^{\frac{1}{p}}.
$$

The Banach spaces $(L_p(\mathcal{M}), \| \cdot \|_p)$, $1 \leq p < \infty$ are separable.

Define the space $L_{1,\infty}(\mathcal{M})$ by setting

$$
L_{1,\infty}(\mathcal{M}) = \left\{ x \in S(\mathcal{M},\tau) : \sup_{t>0} t \mu(t,x) < \infty \right\}.
$$
We equip $L_{1,\infty}(\mathcal{M})$ with the functional $\| \cdot \|_{1,\infty}$ defined by the formula

$$\|x\|_{1,\infty} = \sup_{t>0} t\mu(t,x), \quad x \in L_{1,\infty}(\mathcal{M}).$$

It follows from (2.1) that

$$\|x + y\|_{1,\infty} = \sup_{t>0} t\mu(t,x + y) \leq \sup_{t>0} t\left(\mu\left(\frac{t}{2},x\right) + \mu\left(\frac{t}{2},y\right)\right) \leq \sup_{t>0} t\mu\left(\frac{t}{2},x\right) + \sup_{t>0} t\mu\left(\frac{t}{2},y\right) = 2\|x\|_{1,\infty} + 2\|y\|_{1,\infty}.$$  

In particular, $\| \cdot \|_{1,\infty}$ is a quasi-norm. The quasi-normed space $(L_{1,\infty}(\mathcal{M}),\| \cdot \|_{1,\infty})$ is, in fact, quasi-Banach (see e.g., [16, Section 7] or [30]). In view of our main result it is important to emphasize that the quasi-norm $\| \cdot \|_{1,\infty}$ is not equivalent to any norm on $L_{1,\infty}(\mathcal{M})$ (see e.g., [16, Theorem 7.6]).

### 2.3. Weak type inequalities for Calderón-Zygmund operators.

Parcet [22] proved a non-commutative extension of Calderón-Zygmund theory. Let $K$ be a tempered distribution which we refer to as the convolution kernel. We let $W_K$ be the associated Calderón-Zygmund operator, formally given by $f \mapsto K * f$. In what follows, we only consider tempered distributions having local values (that is, which can be identified with measurable functions $K : \mathbb{R}^d \to \mathbb{C}$).

Let $\mathcal{M}$ be a semi-finite von Neumann algebra with normal, semi-finite, faithful trace $\tau$. The operator $1 \otimes W_K$ can, under suitable conditions, be defined as non-commutative Calderón-Zygmund operators by letting them act on the second tensor leg of $L_1(\mathcal{M}) \hat{\otimes} L_1(\mathbb{R}^d)$. The following theorem in particular gives a sufficient condition for such an operator to act from $L_1$ to $L_{1,\infty}$. We use $V$ to denote the gradient $(\frac{1}{i} \frac{\partial}{\partial x_1}, \ldots, \frac{1}{i} \frac{\partial}{\partial x_d})$, which is understood as unbounded self-adjoint operator on $L_2(\mathbb{R}^d)$.

**Theorem 2.1.** [22] Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a kernel satisfying the conditions

$$|K|(t) \leq \frac{\text{const}}{|t|^d}, \quad |\nabla K|(t) \leq \frac{\text{const}}{|t|^{d+1}}.$$  

Let $\mathcal{M}$ be a semi-finite von Neumann algebra. If $W_K \in B(L_2(\mathbb{R}^d))$, then the operator $1 \otimes W_K$ defines a bounded map from $L_1(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^d))$ to $L_{1,\infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^d))$.

### 2.4. Double operator integrals.

Let $A = A^*$ be an operator affiliated with $\mathcal{M}$. Symbolically, a double operator integral is defined by the formula

$$T^A_{\xi}(V) = \int_{\mathbb{R}^2} \xi(\lambda,\mu) dE_A(\lambda) V E_A(\mu), \quad V \in L_2(\mathcal{M}).$$  

In the subsequent paragraph, we provide a rigorous definition of the double operator integral.
Consider projection valued measures on $\mathbb{R}$ acting on the Hilbert space $L_2(\mathcal{M})$ by the formulae $V \rightarrow E_A(B)V$ and $V \rightarrow V E_A(B)$. These spectral measures commute and, hence (see Theorem V.2.6 in [2]), there exists a countably additive (in the strong operator topology) projection-valued measure $\nu$ on $\mathbb{R}^2$ acting on the Hilbert space $L_2(\mathcal{M})$ by the formula

\[ \nu(B_1 \otimes B_2) : V \rightarrow E_A(B_1)V E_A(B_2), \quad V \in L_2(\mathcal{M}). \]

Integrating a bounded Borel function $\xi$ on $\mathbb{R}^2$ with respect to the measure $\nu$ produces a bounded operator acting on the Hilbert space $L_2(\mathcal{M})$. In what follows, we denote the latter operator by $T_{\xi}^{A,A}$ (see also [8, Remark 3.1]).

In the special case when $A$ is bounded and $\text{spec}(A) \subset \mathbb{Z}$, we have

\[ T_{\xi}^{A,A}(V) = \sum_{k,l \in \mathbb{Z}} \xi(k,l)E_A(\{k\}) V E_A(\{l\}). \]

We are mostly interested in the case $\xi = f^{[1]}$ for a Lipschitz function $f : \mathbb{R} \rightarrow \mathbb{C}$. Here,

\[ f^{[1]}(\lambda, \mu) = \begin{cases} 
\frac{f(\lambda) - f(\mu)}{\lambda - \mu}, & \lambda \neq \mu \\
0, & \lambda = \mu.
\end{cases} \]

3. Approximate intertwining properties of Fourier multipliers. We prove intertwining properties of Fourier multipliers partly inspired by K. de Leeuw’s proof of his restriction theorem for $L^p$-multipliers [7, Section 2].

In what follows,

\[ G_l(s) = \frac{1}{l\sqrt{\pi}}e^{-(\frac{s}{l})^2}, \quad s \in \mathbb{R}, \ l > 0. \]

That is, $G_l$ is a probability density function for certain Gaussian random variable. The notation $G_l^\otimes d$ stands for the function from $L_1(\mathbb{R}^d)$ given by the tensor product of $G_l$ with itself repeated $d$ times.

**Lemma 3.1.** For every $f \in L_1(\mathbb{R})$ with $\int_{-\infty}^{\infty} f(s)ds = 0$, we have $f \ast G_l \rightarrow 0$ in $L_1(\mathbb{R})$ as $l \rightarrow \infty$.

**Proof.** Suppose first that $f$ is a step function of the form $f = \sum_{k=1}^{m} \alpha_k \chi_{I_k}$, $m \geq 1$ where $I_k = [a_k, b_k]$, $1 \leq k \leq m$ are disjoint intervals and $\sum_{k=1}^{m} \alpha_k m(I_k) = 0$. 

We have
\[
(f \ast G_l)(t) = \sum_{k=1}^{m} \alpha_k \int_{a_k}^{b_k} G_l(t-s) \, ds = \sum_{k=1}^{m} \alpha_k \int_{t-a_k}^{t-b_k} G_l(u) \, du
\]
\[
= \sum_{k=1}^{m} \alpha_k \int_{t-b_k}^{t-a_k} G_1(s) \, ds = \sum_{k=1}^{m} \alpha_k \left( F\left( \frac{t-a_k}{l} \right) - F\left( \frac{t-b_k}{l} \right) \right),
\]
where \( F(t) = \int_{-\infty}^{t} G_1(s) \, ds \). To prove the assertion for our \( f \), it suffices to show that
\[
l \int_{-\infty}^{\infty} \left| \sum_{k=1}^{m} \alpha_k \left( F\left( \frac{t-a_k}{l} \right) - F\left( \frac{t-b_k}{l} \right) \right) \right| \, dt \to 0
\]
Clearly,
\[
\left| F\left( \frac{t-a_k}{l} \right) - F(t) + \frac{a_k}{l} F'(t) \right| \leq \frac{a_k^2}{2l^2} \max_{s \in [t-\frac{a_k}{l}, t]} |F''(s)|.
\]
If \( l > \max_{1 \leq k \leq m} |a_k| \) and \( l > \max_{1 \leq k \leq m} |b_k| \), then
\[
\left| \sum_{k=1}^{m} \alpha_k \left( F\left( \frac{t-a_k}{l} \right) - F\left( \frac{t-b_k}{l} \right) \right) \right| \leq \frac{1}{2l^2} \left( \max_{s \in [t-1, t+1]} |F''(s)| \right) \sum_{k=1}^{m} |\alpha_k| \left( a_k^2 + b_k^2 \right)
\]
This proves the assertion for \( f \) as above.

To prove the assertion in general, fix \( f_m \) as above (i.e., mean zero step functions) such that \( f_m \to f \) in \( L_1(\mathbb{R}) \). Since \( \|G_l\|_1 = 1 \), it follows from Young’s inequality that
\[
\|f \ast G_l\|_1 \leq \|(f-f_m) \ast G_l\|_1 + \|f_m \ast G_l\|_1 \leq \|f-f_m\|_1 + \|f_m \ast G_l\|_1.
\]
Therefore,
\[
\limsup_{l \to \infty} \|f \ast G_l\|_1 \leq \|f-f_m\|_1.
\]
Passing \( m \to \infty \), we conclude the proof. □

By Fubini Theorem, linear span of elementary tensors
\[
(f_1 \otimes \cdots \otimes f_d): (t_1, \ldots, t_d) \mapsto f_1(t_1) \cdots f_d(t_d), \quad f_1, \ldots, f_d \in L_1(\mathbb{R})
\]
is dense in \( L_1(\mathbb{R}^d) \).

**Lemma 3.2.** For every \( f \in L_1(\mathbb{R}^d) \) with \( \int_{\mathbb{R}^d} f(s) \, ds = 0 \), we have \( f \ast G_l^{\otimes d} \to 0 \) in \( L_1(\mathbb{R}^d) \) as \( l \to \infty \).
Proof. Suppose first that $f$ is a linear combination of elementary tensors. That is,

$$f = \sum_{k=1}^{m} \bigotimes_{j=1}^{d} f_{jk}, \quad f_{jk} \in L_1(\mathbb{R}).$$

(3.1)

Firstly, we consider the case when for every $k$, $1 \leq k \leq m$ there exists $j$, $1 \leq j \leq d$ such that $\int_{\mathbb{R}} f_{jk}(s) = 0$. In this case, by Lemma 3.1 we have that

$$\|f \ast G_l^{\otimes d}\|_1 \leq \sum_{k=1}^{m} \prod_{j=1}^{d} \|f_{jk} \ast G_l\|_1 \longrightarrow 0 \quad \text{as} \quad l \longrightarrow \infty.$$ 

Now, we show that the case of $f$ given by (3.1) satisfying

$$\sum_{k=1}^{m} \prod_{j=1}^{d} \int_{\mathbb{R}} f_{jk}(s) ds = 0.$$ 

(3.2)

can be reduced to the just considered case when for every $k$, $1 \leq k \leq m$ there exists $j$, $1 \leq j \leq d$ such that $\int_{\mathbb{R}} f_{jk}(s) = 0$. To this end, for every subset $\mathcal{A} \subset \{1, \ldots, d\}$, we set

$$f_{j,k,\mathcal{A}} = \begin{cases} f_{jk} - \left( \int_{\mathbb{R}} f_{jk}(s) ds \right) \chi_{(0,1)}, & j \in \mathcal{A} \\ \left( \int_{\mathbb{R}} f_{jk}(s) ds \right) \chi_{(0,1)}, & j \notin \mathcal{A}. \end{cases}$$

By the linearity, we can rewrite (3.1) as

$$f = \sum_{k=1}^{m} \left( \sum_{\mathcal{A} \subset \{1, \ldots, d\}} \sum_{j=1}^{d} \bigotimes_{j=1}^{d} f_{j,k,\mathcal{A}} \right).$$

Observing now that

$$\sum_{k=1}^{m} \bigotimes_{j=1}^{d} f_{j,k,\emptyset} = \left( \sum_{k=1}^{m} \prod_{j=1}^{d} \int_{\mathbb{R}} f_{jk}(s) ds \right) \chi^{\otimes d}_{(0,1)}$$

and appealing to (3.2), we arrive at

$$f = \sum_{k=1}^{m} \left( \sum_{\mathcal{A} \neq \emptyset} \sum_{j=1}^{d} \bigotimes_{j=1}^{d} f_{j,k,\mathcal{A}} \right).$$

(3.3)

Note that $f_{j,k,\mathcal{A}}$ is mean zero for $j \in \mathcal{A}$. Using representation (3.3) instead of (3.1) for $f$, we may assume without loss of generality that for every $k$, $1 \leq k \leq m$
there exists \( j, 1 \leq j \leq d \) such that \( \int_{\mathbb{R}} f_{jk}(s) = 0 \). This completes the proof of the lemma in the special case when \( f \) is given by (3.1) and satisfies (3.2).

To prove the general case, fix \( f \in L_1(\mathbb{R}^d) \) with \( \int_{\mathbb{R}^d} f(s)ds = 0 \), and select a sequence \( \{f_m\}_{m=1}^\infty \) of mean zero sums of elementary tensors such that \( f_m \to f \) in \( L_1(\mathbb{R}^d) \) as \( m \to \infty \). Since \( \|G_i^{\otimes d}\|_1 = 1, l \geq 1 \) it follows from Young inequality that
\[
\|f * G_i^{\otimes d}\|_1 \leq \| (f - f_m) * G_i^{\otimes d}\|_1 + \|f_m * G_i^{\otimes d}\|_1 \leq \|f - f_m\|_1 + \|f_m * G_i^{\otimes d}\|_1.
\]
Therefore,
\[
\limsup_{l \to \infty} \|f * G_i^{\otimes d}\|_1 \leq \|f - f_m\|_1.
\]
Passing \( m \to \infty \), we conclude the proof. \( \square \)

In what follows,
\[
(3.4) \quad e_k(t) := e^{i(k,t)}, \quad k, t \in \mathbb{R}^d
\]
where \( \langle k, t \rangle = \sum_{j=1}^d k_j t_j \), and \( \mathcal{F} \) stands for the Fourier transform.

**Lemma 3.3.** If \( g \in L_\infty(\mathbb{R}^d) \) is such that \( \mathcal{F}(g) \in L_1(\mathbb{R}^d) \), then for every \( k \in \mathbb{R}^d \) we have
\[
(g(\nabla))(G_i^{\otimes d} e_k) - g(k) G_i^{\otimes d} e_k \to 0
\]
in \( L_1(\mathbb{R}^d) \) as \( l \to \infty \).

**Proof.** Fix \( k \in \mathbb{R}^d \). Set \( h_1(t) := g(k) e^{-|t-k|^2} \) and \( h_0(t) := g(t) - h_1(t), t \in \mathbb{R}^d \). Observe that, for every \( t \in \mathbb{R}^d \), we have
\[
\mathcal{F}(G_i^{\otimes d})(t) = \pi^{-d/2} e^{-t^2/2}.
\]
Since \( h_1(\nabla) \) on the Fourier side is a multiplier on \( h_1 \), it follows that, for every \( t \in \mathbb{R}^d \),
\[
\mathcal{F}(G_i^{\otimes d} e_k)(t) = \pi^{-d/2} e^{-l^2/2} e^{-|t-k|^2},
\]
\[
(\mathcal{F}((h_1(\nabla))(G_i^{\otimes d} e_k)))(t) = g(k) \pi^{-d/2} e^{-l^2/2} e^{-|t-k|^2}.
\]
Applying the inverse Fourier transform to the second equality, we arrive at
\[
(h_1(\nabla))(G_i^{\otimes d} e_k) = g(k) G_i^{\otimes d} (l^2 + 1)^{1/2} e_k.
\]
A direct computation yields \( G_i^{\otimes d} (l^2 + 1)^{1/2} \to 0 \) in \( L_1(\mathbb{R}^d) \) as \( l \to \infty \). We conclude that
\[
(h_1(\nabla))(G_i^{\otimes d} e_k) - g(k) G_i^{\otimes d} e_k \to 0
\]
in $L_1(\mathbb{R}^d)$ as $l \to \infty$. It, therefore, suffices to show that
\[
(h_0(\nabla))(G^\otimes d e_k) \to 0
\]
in $L_1(\mathbb{R}^d)$ as $l \to \infty$. Define the function $f \in L_1(\mathbb{R}^d)$ by setting $f(t) = e^{i(k,t)}(\mathcal{F}h_0)(t)$, $t \in \mathbb{R}^d$. We rewrite the latter equation as $f * G^\otimes d \to 0$ as $l \to \infty$. Note that
\[
\int_{\mathbb{R}^d} f(s)ds = \int_{\mathbb{R}^d} e^{i(k,s)}(\mathcal{F}h_0)(s)ds = h_0(k) = 0.
\]
The assertion follows now from the Lemma 3.2.

\[\square\]

4. Proof of Theorem 1.2 in a special case. For $s > 0$, the dilation operator $\sigma_s$ acts on the space of Lebesgue measurable functions on $\mathbb{R}$, by the formula
\[
(\sigma_s x)(t) = x(t/s),
\]
where $x$ is measurable and $\theta$ is integrable functions on $\mathbb{R}$. Let $z(t) = \begin{cases} t^{-1}, & t > 0, \\ 0, & t < 0. \end{cases}$ and let $u > 0$. For Lebesgue measurable functions $x \otimes y$ and $\theta \otimes z$ on $\mathbb{R}^2$, we have
\[
\mu(\sigma_u(x) \otimes y) = \sigma_u \mu(x \otimes y), \quad \mu(t, \theta \otimes z) = \|\theta\|_1 t^{-1}, \quad t > 0.
\]

Proof. Denoting Lebesgue measure on $\mathbb{R}^2$ by $m$, we have for every $t > 0$
\[
m(\{\sigma_u(x) \otimes y > t\}) = m\left(\left\{\left(\frac{s_1}{u}\right) y(s_2) > t\right\}\right)
= um\left(\left\{\left(\frac{s_1}{u}\right) y(s_2) > t\right\}\right)
= um\left(\{x \otimes y > t\}\right).
\]
This proves the first assertion.

Firstly, we prove the second assertion for simple function $x \in L_1(\mathbb{R})$. If $x = \sum_k a_k \chi_{B_k}$ with $B_k$ being pairwise disjoint sets, then
\[
\mu(x \otimes z) = \mu\left(\bigoplus_k (a_k \chi_{B_k} \otimes z)\right) = \mu\left(\bigoplus_k \mu(a_k \chi_{B_k} \otimes z)\right).
\]
The notation $\bigoplus_k x_k$ stands for the disjoint sum of the functions $x_k$, that is $\sum_k z_k$, where functions $z_k$ have pairwise disjoint support and $\mu(z_k) = \mu(x_k)$. We refer the reader to Definition 2.4.3 in [20] and subsequent comments.

If $B$ is a set of finite measure, then there exists a measure preserving bijection from $B$ to $(0, m(B))$ (see [15]). Therefore, we have
\[
\mu(\chi_B \otimes z) = \mu(\chi_{(0,m(B))} \otimes z) = m(B) z.
\]
Thus,

$$\mu(x \otimes z) = \mu\left( \bigoplus_k |a_k| m(B_k) z \right) = \left( \sum_k a_k m(B_k) \right) z.$$ 

The second assertion follows now by approximation. \qed

**Lemma 4.2.** For every $X \in L_{1,\infty}(M)$ and every $l > 0$, we have

$$e^{-d_x^{d_{\infty}}} \|X\|_{1,\infty} \leq \|X \otimes G_l^{\otimes d}\|_{1,\infty} \leq \|X\|_{1,\infty}.$$ 

**Proof.** For every operator $A \in S(M, \tau)$ and for every function $g \in L_{\infty}(0, \infty)$, we claim that we have

$$\mu(A \otimes g) = \mu(|A| \otimes g) = \mu(A \otimes g) = \mu(A \otimes \mu(g)). \quad (4.1)$$

Indeed, let us prove this claim. Without loss of generality, $M$ is atomless. Suppose first that $x \in M$ is $\tau$-compact. By Theorem 2.3.11 in [20], there exists a trace preserving $\ast$-isomorphism $i : L_{\infty}(0, \infty) \to M_1$ such that $i_1(\mu(A)) = |A|$. Consider the trace preserving isomorphism

$$i \otimes 1 : L_{\infty}(0, \infty) \otimes L_{\infty}(0, \infty) \to M \otimes L_{\infty}(0, \infty).$$

We have $i(\mu(A) \otimes g) = |A| \otimes g$. Since every trace preserving $\ast$-isomorphism preserves the singular value function, the claim follows for $\tau$-compact operators (for the definition of $\tau$-compact operators we refer to [20]). The general case follows by approximation, concluding (4.1).

Now let $z$ be as in Lemma 4.1. It follows from the definition $\| \cdot \|_{1,\infty}$ that

$$\mu(X) \leq \|X\|_{1,\infty} z$$

and, therefore,

$$\mu(X \otimes G_l^{\otimes d}) \overset{(4.1)}{=} \mu\left( \mu(X) \otimes G_l^{\otimes d} \right) \leq \|X\|_{1,\infty} \mu\left( z \otimes G_l^{\otimes d} \right) \overset{L_{4,1}}{=} \|X\|_{1,\infty} \mu(z).$$

This proves the right-hand side inequality.

On the other hand, $\mu(G_l) = l^{-1} \sigma_l \mu(G)$. By Lemma 4.1, we have $\mu(G_l^{\otimes d}) = l^{-d} \sigma_l \mu(G_1^{\otimes d})$. Thus,

$$\mu\left( X \otimes G_l^{\otimes d} \right) \overset{(4.1)}{=} \mu\left( X \otimes l^{-d} \sigma_l \mu(G_1^{\otimes d}) \right) \overset{L_{4,1}}{=} l^{-d} \sigma_l \mu\left( X \otimes G_1^{\otimes d} \right).$$

Therefore, we have

$$\|X \otimes G_l^{\otimes d}\|_{1,\infty} = \sup_{t > 0} \frac{t}{l d} \mu\left( \frac{t}{l d}, X \otimes G_1^{\otimes d} \right) = \sup_{s > 0} s \mu\left( s, X \otimes G_1^{\otimes d} \right) = \|X \otimes G_1^{\otimes d}\|_{1,\infty}. $$
Clearly, $\mu(G_1) \geq \frac{1}{e\sqrt{\pi}} \chi(0,1)$. It follows that

\[
\|X \otimes G_1^d\|_{1,\infty} \overset{(4.1)}{=} \|X \otimes \mu(G_1)^d\|_{1,\infty} \geq \|X \otimes \left(\frac{1}{e\sqrt{\pi}} \chi(0,1)\right)^d\|_{1,\infty}
= e^{-d\pi - \frac{d}{2}} \|X\|_{1,\infty}.
\]

This proves the left-hand side inequality. \hfill \square

The following lemma is ideologically similar to Theorem II.4.3 in [28].

**Lemma 4.3.** If $g$ is a smooth homogeneous function on $\mathbb{R}^2\setminus\{0\}$, then $\mathcal{F}g$ satisfies (possibly, after some $\delta$ distribution is subtracted) the conditions (2.2).

**Proof.** By assumption, $g$ is a smooth function on the circle $\{|z| = 1\}$. Thus,

\[
g(e^{i\theta}) = \sum_{k \in \mathbb{Z}} \alpha_k e^{ik\theta},
\]

where Fourier coefficients decrease faster than every power. Therefore,

\[
g = \sum_{k \in \mathbb{Z}} \alpha_k g_k, \quad g_k(z) = \frac{z^k}{|z|^k}, \quad 0 \neq z \in \mathbb{C}.
\]

For every $k \neq 0$, we have (this can be checked, e.g., by substituting $m = \Omega = g_k$ into the formula (26) in Theorem II.4.3 in [28]),

\[
\mathcal{F}g_k(z) = \frac{|k|}{2\pi i^k} \cdot g_k(z), \quad 0 \neq z \in \mathbb{C}.
\]

Hence,

\[
\mathcal{F}g(z) = \alpha_0 \delta + \frac{1}{|z|^2} h(e^{i\text{Arg}(z)})
\]

where the smooth function $h$ on the circle is defined by the formula

\[
h(e^{i\theta}) = \sum_{0 \neq k \in \mathbb{Z}} \frac{|k|}{2\pi i^k} \alpha_k e^{ik\theta}.
\]

So, $(\mathcal{F}g - \alpha_0 \delta)(z) = O(|z|^{-2})$ as $|z| \to \infty$. Furthermore, have

\[
\nabla \left( \frac{h(e^{i\text{Arg}(z)})}{|z|^2} \right) = h(e^{i\text{Arg}(z)}) \cdot \nabla \left( \frac{1}{|z|^2} \right) + \frac{1}{|z|^2} \cdot \frac{dh(e^{i\theta})}{d\theta} \big|_{\theta = \text{Arg}(z)} \cdot \nabla (\text{Arg}(z))
= O\left(\frac{1}{|z|^3}\right) \text{ as } |z| \to \infty.
\]

This completes the verification that $\mathcal{F}g - \alpha_0 \delta$ satisfies condition (2.2). \hfill \square
LEMA 4.4. There exist Schwartz functions \( \phi_m \) on \( \mathbb{R}^2 \) which vanish near 0, such that \( \phi_m(t) = 1 \) for \( |t| \in \left[ \frac{1}{m}, m \right] \) and such that \( \| \mathcal{F} \phi_m \|_1 \leq c_{abs} \) for all \( m \geq 1 \).

Proof. Let \( \psi \) be a Schwartz function on \( \mathbb{R}^2 \) such that \( \psi(t) = 1 \) for \( |t| \in [0, 1] \) and which is supported on \( t \in \mathbb{R}^2 \) with \( |t| \in [0, 2] \). Set \( \psi_m = \sigma_m \psi \cdot (1 - \sigma_{\frac{1}{m}} \psi) \). It follows that

\[
\mathcal{F}(\psi_m) = \mathcal{F}(\sigma_m \psi) \ast \mathcal{F}(1 - \sigma_{\frac{1}{m}} \psi)
= m^2 \sigma_{\frac{1}{m}}(\mathcal{F}(\psi)) - m^2 \sigma_{\frac{1}{m}}(\mathcal{F}(\psi)) \ast \frac{1}{m^2} \sigma_m(\mathcal{F}(\psi)).
\]

Applying Young’s inequality, we conclude that

\[
\| \mathcal{F}(\psi_m) \|_1 \leq \| m^2 \sigma_{\frac{1}{m}}(\mathcal{F}(\psi)) \|_1 + \| m^2 \sigma_{\frac{1}{m}}(\mathcal{F}(\psi)) \ast \frac{1}{m^2} \sigma_m(\mathcal{F}(\psi)) \|_1 
= \| \mathcal{F}(\psi) \|_1 + \| \mathcal{F}(\psi) \|_2^2.
\]

Clearly, \( \psi_m \) is 1 for all \( t \in \mathbb{R}^2 \) such that \( |t| \in \left[ \frac{2}{m}, m \right] \). Put \( \phi_m = \psi_{2m} \). Then, \( \phi_m(t) = 1 \) for \( |t| \in \left[ \frac{1}{m}, m \right] \).

THEOREM 4.5. For every \( A = A^* \in \mathcal{M} \) with \( \text{spec}(A) \subset \mathbb{Z} \) and for every Lipschitz function \( f : \mathbb{R} \to \mathbb{C} \), we have

\[
\| T^{A,A}_{f} (V) \|_{1,\infty} \leq c_{abs} \| f' \|_{\infty} \| V \|_1, \quad V \in L_1(\mathcal{M}).
\]

Proof. Fix a smooth homogeneous function \( g \) on \( \mathbb{R}^2 \) such that \( g(e^{i\theta}) = \tan(\theta) \) for \( \theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \) and for \( \theta \in (\frac{3\pi}{4}, \frac{5\pi}{4}) \). Without loss of generality, \( g \) is mean zero on the circle \( \{ |z| = 1 \} \). By Lemma 4.3, \( \mathcal{F} g \) satisfies the conditions (2.2). The operator \( g(\mathcal{V}) \in B(L_2(\mathbb{R}^2)) \) since \( g \) is bounded. Recall that \( (g(\mathcal{V}))(x) = (\mathcal{F} g) \ast x \). By Theorem 2.1, we have

\[
1 \otimes g(\mathcal{V}) : L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^2)) \longrightarrow L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^2)).
\]

By Lemma 4.4, select Schwartz functions \( \phi_m \) on \( \mathbb{R}^2 \) which vanish near 0, such that \( \phi_m(t) = 1 \) for \( |t| \in \left[ \frac{1}{m}, m \right] \) and such that \( \| \mathcal{F} \phi_m \|_1 \leq c_{abs} \) for all \( m \geq 1 \). It follows that

\[
\| 1 \otimes (g \phi_m)(\mathcal{V}) \|_{L_1 \to L_{1,\infty}} \leq \| 1 \otimes g(\mathcal{V}) \|_{L_1 \to L_{1,\infty}} \| 1 \otimes \phi_m(\mathcal{V}) \|_{L_1 \to L_1}
\leq \| 1 \otimes g(\mathcal{V}) \|_{L_1 \to L_{1,\infty}} \| \mathcal{F} \phi_m \|_1
\leq c_{abs} \| 1 \otimes g(\mathcal{V}) \|_{L_1 \to L_{1,\infty}} = c_{abs}, \quad m \geq 1.
\]

The last equality holds because \( g \) is fixed.

By assumption, \( A = \sum_{j \in \mathbb{Z}} j p_j \), where \( \{ p_j \}_{j \in \mathbb{Z}} \) are pairwise orthogonal projections such that \( \sum_{j \in \mathbb{Z}} p_j = 1 \). Since \( A \) is bounded, it follows that \( p_j = 0 \) for all but
finely many $j \in \mathbb{Z}$. Hence, sums are, in fact, finite. Consider a unitary operator

$$u = \sum_{j \in \mathbb{Z}} p_j \otimes e_{(j,f(j))},$$

where $e_{(j,f(j))}$ is given in (3.4). Without loss of generality, $\|f'\|_\infty \leq 1$. For every $m \geq \|A\|_\infty$, we have $|i - j|, |f(i) - f(j)| \leq 2m$ for every $i, j \in \text{spec}(A)$. Hence,

$$(g\phi_m)(i - j, f(i) - f(j)) = g(i - j, f(i) - f(j)) = \frac{f(i) - f(j)}{i - j}, \quad i, j \in \text{spec}(A), i \neq j.$$  

It follows from the preceding paragraph and from the equality $\|G_l^{\otimes 2}\|_1 = 1$ that

$$\|(1 \otimes (g\phi_m)(\mathcal{V})) (u \otimes G_l^{\otimes 2}) u^\ast\|_{1, \infty} \leq \text{cabs} \|u \otimes G_l^{\otimes 2}) u^\ast\|_1 = \text{cabs} \|V\|_1.$$  

It is clear that

$$(1 \otimes (g\phi_m)(\mathcal{V})) (u \otimes G_l^{\otimes 2}) u^\ast = \sum_{i,j} p_i V p_j \otimes (g\phi_m(\mathcal{V}))(G_l^{\otimes 2} e_{(i - j, f(i) - f(j))}).$$  

Since there are only finitely many summands, it follows from Lemma 3.3 (as applied to the Schwartz function $g\phi_m$) that

$$(1 \otimes (g\phi_m)(\mathcal{V})) (u \otimes G_l^{\otimes 2}) u^\ast - \sum_{i \neq j} p_i V p_j \otimes \frac{f(i) - f(j)}{i - j} G_l^{\otimes 2} e_{(i - j, f(i) - f(j))} \to 0$$  

in $L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^2))$ as $l \to \infty$. It is immediate that

$$\sum_{i \neq j} p_i V p_j \otimes \frac{f(i) - f(j)}{i - j} G_l^{\otimes 2} e_{(i - j, f(i) - f(j))}$$

$$= \left(\sum_{k \in \mathbb{Z}} p_k \otimes e_{(k,f(k))}\right) \cdot \left(\sum_{i \neq j} p_i V p_j \otimes \frac{f(i) - f(j)}{i - j} G_l^{\otimes 2}\right) \cdot \left(\sum_{l \in \mathbb{Z}} p_l \otimes e_{(-l,-f(l))}\right)$$

$$\overset{(2.5)}{=} u(T_{f[l]}^{A,A}(V) \otimes G_l^{\otimes 2}) u^\ast.$$  

Therefore,

$$\|(1 \otimes (g\phi_m)(\mathcal{V})) (u \otimes G_l^{\otimes 2}) u^\ast - u(T_{f[l]}^{A,A}(V) \otimes G_l^{\otimes 2}) u^\ast\|_{1, \infty} \to 0$$  

in $L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^2))$ (and, hence, in $L_{1, \infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^2))$) as $l \to \infty$.  

Combining (4.2) and (4.3), we arrive at
\[
\limsup_{l \to \infty} \left\| u(T_{f[k]}^{A,A}(V) \otimes G^2_l)u^* \right\|_{1,\infty} \leq c_{\text{abs}} \| V \|_1.
\]

Since \( u \) is unitary, it follows that
\[
\limsup_{l \to \infty} \left\| T_{f[k]}^{A,A}(V) \otimes G^2_l \right\|_{1,\infty} \leq c_{\text{abs}} \| V \|_1.
\]

The assertion follows now from Lemma 4.2. \( \square \)

5. Proof of the main results. In this section we collect the results announced in the abstract and its corollaries. Throughout this section fix a semi-finite von Neumann algebra \( \mathcal{M} \) with normal, semi-finite, faithful trace \( \tau \).

**Lemma 5.1.** Let \( A = A^* \in \mathcal{M} \). If \( \{ \xi_n \}_{n \geq 0} \) is a uniformly bounded sequence of Borel functions on \( \mathbb{R}^2 \) such that \( \xi_n \to \xi \) everywhere, then
\[
\lim_{n \to \infty} T_{\xi_n}^{A,A}(V) = T_{\xi}^{A,A}(V), \quad V \in L_2(\mathcal{M})
\]
in \( L_2(\mathcal{M}) \) as \( n \to \infty \).

**Proof.** Let \( \nu \) be a projection valued measure on \( \mathbb{R}^2 \) considered in Subsection 2.4 (see (2.4)). Let \( \gamma : \mathbb{R} \to \mathbb{R}^2 \) be a Borel measurable bijection. Clearly, \( \nu \circ \gamma \) is a projection valued measure on \( \mathbb{R} \). Hence, there exists a self-adjoint operator \( B \) acting on the Hilbert space \( L_2(\mathcal{M}) \) such that \( E_B = \nu \circ \gamma \).

Set \( \eta_n = \xi_n \circ \gamma \) and \( \eta = \xi \circ \gamma \). We have \( \eta_n \to \eta \) everywhere on \( \mathbb{R} \). Thus,
\[
T_{\xi_n}^{A,A} = \int_{\mathbb{R}^2} \xi_n d\nu = \int_{\mathbb{R}} \eta_n(\lambda)dE_B(\lambda) = \eta_n(B) \to \eta(B) = \int_{\mathbb{R}} \eta(\lambda)dE_B(\lambda) = \int_{\mathbb{R}^2} \xi d\nu = T_{\xi}^{A,A}.
\]

Here, the convergence is understood with respect to the strong operator topology on the space \( B(L_2(\mathcal{M})) \). In particular, (5.1) follows. \( \square \)

**Proof of Theorem 1.2.**

**Step 1.** Let \( A \) is bounded. For every \( n \geq 1 \), set
\[
A_n \overset{def}{=} \sum_{k \in \mathbb{Z}} \frac{k}{n} E_A \left( \left[ \frac{k}{n}, \frac{k+1}{n} \right) \right),
\]
and
\[
\xi_n(t,s) = f^{[1]} \left( \left[ \frac{k}{n}, \frac{l+1}{n} \right), \quad t \in \left[ \frac{k}{n}, \frac{k+1}{n} \right), \quad s \in \left[ \frac{l}{n}, \frac{l+1}{n} \right).\right.
\]
It is immediate that (see e.g., Lemma 8 in [25] for much stronger assertion) for
\(V \in L^1(M)\), we have
\[ T_{\xi_n}^{A,A}(V) = T_{f_{[1]}}^{A,A}(V) = T_{(n\sigma_n f)_{[1]}}^{n,A_n,A_n}(V). \]

It follows from Theorem 4.5 that
\[ \|T_{\xi_n}^{A,A}(V)\|_1,\infty \leq c_{\text{abs}} \|n\sigma_n f\|_\infty \|V\|_1 = c_{\text{abs}} \|f'\|_\infty \|V\|_1. \]

Note that \(\xi_n \to f_{[1]}\) everywhere. It follows from Lemma 5.1 that
\[ T_{\xi_n}^{A,A}(V) \to T_{f_{[1]}}^{A,A}(V), \quad V \in L^2(M) \]
in \(L^2(M)\) (and, hence, in measure—see e.g., [8]) as \(n \to \infty\). Since the quasi-norm in \(L^1_{1,\infty}(M)\) is a Fatou quasi-norm, it follows that
\[ \|T_{f_{[1]}}^{A,A}(V)\|_1,\infty \leq c_{\text{abs}} \|f'\|_\infty \|V\|_1, \quad V \in (L^1 \cap L^2)(M). \]

**Step 2.** Let now \(A\) be an arbitrary operator affiliated with \(M\). Set \(A_n = AE_{A([-n,n])}\). By Step 1, we have for \(V \in (L^1 \cap L^2)(M)\)
\[ \|T_{f_{[1]}}^{A_n,A_n}(V)\|_1,\infty \leq c_{\text{abs}} \|f'\|_\infty \|V\|_1. \]

It follows immediately from the definition of the double operator integral that
\[ T_{f_{[1]}}^{A_n,A_n}(V) = E_A([-n,n]) \cdot T_{f_{[1]}}^{A,A}(V) \cdot E_A([-n,n]) \to T_{f_{[1]}}^{A,A}(V) \]
in \(L^2(M)\) (and, hence, in measure) as \(n \to \infty\). Since the quasi-norm in \(L^1_{1,\infty}(M)\) is a Fatou quasi-norm, the assertion follows. \(\square\)

The following lemma is ideologically similar to Theorem 7.4 in [8].

**Lemma 5.2.** If \(A, B \in M\) are such that \([A, B] \in L^2(M)\), then, for every Lipschitz function \(f\), we have
\[ T_{f_{[1]}}^{A,A}([A, B]) = [f(A), B]. \]

**Proof.** By definition of double operator integral given in Subsection 2.4, we have
\[ (5.2) \quad T_{\xi_1}^{A,A} T_{\xi_2}^{A,A} = T_{\xi_1 \xi_2}^{A,A}. \]

Let \(\xi_1 = f_{[1]}\) and let \(\xi_2(\lambda, \mu) = \lambda - \mu\) when \(|\lambda|, |\mu| \leq \|A\|_\infty\). \(\xi_2(\lambda, \mu) = 0\) when
\(|\lambda| > \|A\|_\infty\) or \(|\mu| \leq \|A\|_\infty\).

If \(p\) is a \(\tau\)-finite projection (that is \(\tau(p) < \infty\), then \(pB \in L^2(M)\) and
\[ T_{\xi_1 \xi_2}^{A,A}(pB) = f(A)pB - pB f(A), \quad T_{\xi_2}^{A,A}(pB) = ApB - pBA, \]
\[ T_{\xi_1}^{A,A}(pB) = T_{\xi_1 \xi_2}^{A,A}(pB) = T_{\xi_1}^{A,A}(T_{\xi_2}^{A,A}(pB)). \]
Applying (5.2) to the operator $pB \in L_2(\mathcal{M})$, we obtain
\[(5.3)\quad T_{f[A]}^{A,A}(ApB - pBA) = f(A)pB - pBf(A).\]

Applying Proposition 6.6 in [8] to the operator $nA$, we construct a sequence $\{p_{n,k}\}_{k \geq 0}$ of $\tau$-finite projections such that $p_{n,k} \uparrow 1$ as $k \to \infty$ and such that $\|[nA, p_{n,k}]\|_2 \leq 1$. Let $\{\eta_m\}_{m \geq 0}$ be an orthonormal basis in $L_2(\mathcal{M})$. Fix $k_n$ so large that
\[(5.4)\quad \| (1 - p_{n,k_n})\eta_m \|_2 \leq \frac{1}{n}, \quad 0 \leq m < n.\]

Set $q_n = p_{n,k_n}$. It follows from (5.4) that $q_n \to 1$ in the strong operator topology (in the left regular representation of $\mathcal{M}$). Clearly, $[A, q_n] \to 0$ in $L_2(\mathcal{M})$.

By construction,
\[Aq_nB - q_nBA = [A, q_n]B + q_n[A, B] \to [A, B], \quad n \to \infty,\]
in $L_2(\mathcal{M})$. Since $T_{f[A]}^{A,A}$ is bounded, it follows that
\[f(A)q_nB - q_nBf(A) \stackrel{(5.3)}{=} T_{f[A]}^{A,A}(Aq_nB - q_nBA) \to T_{f[A]}^{A,A}(AB - BA), \quad n \to \infty,\]
in $L_2(\mathcal{M})$. On the other hand,
\[f(A)q_nB - q_nBf(A) \to [f(A), B], \quad n \to \infty,\]
in the strong operator topology (in the left regular representation of $\mathcal{M}$). This concludes the proof. □

**Theorem 5.3.** For all self-adjoint operators $A, B \in \mathcal{M}$ such that $[A, B] \in L_1(\mathcal{M})$ and for every Lipschitz function $f$, we have
\[\|[[f(A), B]]\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \cdot \|[A, B]\|_1.\]
For all self-adjoint operators $X, Y \in \mathcal{M}$ such that $X - Y \in L_1(\mathcal{M})$ and for every Lipschitz function $f$, we have
\[\|f(X) - f(Y)\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|X - Y\|_1.\]

**Proof.** By assumption, $[A, B] \in (L_1 \cap L_2)(\mathcal{M})$. The first assertion follows by combining Lemma 5.2 and Theorem 1.2. Applying the first assertion to the operators
\[A = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\]
we obtain the second assertion. □
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