THE MODIFICATION OF THE NONLINEAR GUIDING CENTER THEORY

G. Qin and L.-H. Zhang

1 State Key Laboratory of Space Weather, National Space Science Center, Chinese Academy of Sciences, P.O. Box 8701 Beijing 100190, China; gqin@spaceweather.ac.cn
2 College of Earth Sciences, Graduate University of Chinese Academy of Sciences, Beijing 100049, China; lhzhang@spaceweather.ac.cn

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1. INTRODUCTION

Knowledge of the charged energetic particles’ diffusion mechanism is necessary to study the transport and acceleration of cosmic rays. Matthaeus et al. (2003) developed the NonLinear Guiding Center (NLGC) (Note that the acronyms in this paper are briefly explained in Table 1.) theory to describe the perpendicular diffusion coefficient, which is written as (Matthaeus et al. 2003)

\[ \kappa_{xx} = \frac{a^2 v^2}{3B_0^2} \int d^3 k \frac{S_{xx}(k)}{k_{\parallel} + k_{\perp}^2 \kappa_{xx} + k_{\perp}^2 \kappa_{zz} + \gamma(k)}, \]  

where the parameter \( a^2 = 1/3 \) indicates the degree particles following the magnetic field line (MFL). In addition, the spectral amplitudes of the two-component model turbulence \( S_{xx}(k) \) is the sum of that of the two-dimensional (2D) component \( S_{xx}^{2D}(k) \) and slab component \( S_{xx}^{slab}(k) \) (e.g., Bieber et al. 1996),

\[ S_{xx}(k) = S_{xx}^{2D}(k) + S_{xx}^{slab}(k) \]

\[ = S_{xx}^{2D}(k_{\perp}) \frac{2k_{\perp}^2 \delta(k_{\parallel})}{\pi k_{\parallel}^2} + S_{xx}^{slab}(k_{\parallel}) \frac{\delta(k_{\perp})}{2\pi k_{\perp}}, \]

where the 2D component \( S_{xx}^{2D}(k_{\perp}) \) and the slab component \( S_{xx}^{slab}(k_{\parallel}) \) are written as

\[ S_{xx}^{2D}(k_{\perp}) = C(\nu) \lambda_{2D} \left( \frac{B_{2D}}{B_0} \right) \left( 1 + k_{\perp}^2 \lambda_{2D}^2 \right)^{-\nu} \]

\[ S_{xx}^{slab}(k_{\parallel}) = C(\nu) \lambda_{slab} \left( \frac{B_{slab}}{B_0} \right) \left( 1 + k_{\parallel}^2 \lambda_{slab}^2 \right)^{-\nu}, \]

and \( C(\nu) = (1/2\sqrt{\pi}(\Gamma(\nu)/\Gamma(\nu - 1/2)) \). Here, \( \lambda_{slab} \) and \( \lambda_{2D} \) are the spectral bend-over scales of the slab and 2D components of turbulence, respectively, and the parallel component parallel free path is related to the parallel diffusion coefficient as \( \lambda_{\parallel} = 3\kappa_{zz}/v \). We consider that NLGC is the first perpendicular diffusion theory that agrees well with simulations and spacecraft observations in typical solar wind conditions (Zank et al. 2004; Bieber et al. 2004). Note that in Equations (3) and (4), the spectra of the 2D and slab components, respectively, are all not varying in an energy range with lower wavenumbers. However, Matthaeus et al. (2007) suggested that in the energy range, \( k_{\perp} \ll 1/\lambda_{2D} \), the spectrum of the 2D component should not be constant. In addition, with a more general form of the 2D spectrum with an energy range spectrum index \( q \), Shalchi et al. (2010) found that the behavior of the spectrum in the energy range is important to determine the perpendicular diffusion.

Furthermore, the NLGC model is an integral equation for the perpendicular diffusion coefficient with the parallel diffusion coefficient as an input, so it is difficult to include NLGC in a numerical model to study the energetic particles transport or acceleration. Zank et al. (2004) and Shalchi et al. (2004a) derived explicit expressions for the perpendicular diffusion coefficient, where the parallel diffusion coefficient from the quasilinear theory (QLT; Jokipii 1966) can be used as input. Furthermore, Shalchi et al. (2004b) developed a nonlinear model, Weakly NonLinear Theory (WNLT), to describe the parallel and perpendicular diffusion simultaneously. However, the WNLT model is complicated and difficult to use in numerical models.

Moreover, because of the nonlinear effects, QLT is not very accurate compared to the simulation results (e.g., Qin et al. 2002). Based on the NLGC theory, a nonlinear model, the NonLinear Parallel (NLPA) theory of parallel diffusion coefficient, is derived as (Qin 2007)

\[ \kappa_{zz} = \left\{ \frac{6a_{xx}}{\nu} \right\} \int d^3 k \frac{S_{zz}(k)}{B_0^2} \left( \frac{\Omega}{v} \right)^2 \left[ \frac{2\pi r_L}{\lambda_{\perp}} + k_{\perp}^2 \kappa_{xx} + k_{\parallel}^2 \kappa_{zz} + \gamma(k) \right]^{-1}, \]

with the parameter

\[ a_{xx} = \frac{1}{2} \sqrt{\frac{E_s}{[\xi + (1 + \xi)(1/b) + \bar{b}/(2\xi)}}, \]

and \( \bar{r} = 2\pi r_L/\lambda_{\perp}, \bar{b} = b/B_0, \xi = \bar{r}/\bar{b}, E_s = E_{slab}/E_{total}, \) and \( E_{total} = E_{slab} + E_{2D} \). Here, the correlation length of the
Table 1

| Terms                  | Explanation                                    | Output  | Input  | Papers |
|-----------------------|------------------------------------------------|---------|--------|--------|
| NLGC                  | NonLinear Guiding Center theory                | $\kappa_\perp$ | $\kappa_\parallel$ | M03$^a$ |
| WNLT                  | Weakly NonLinear Theory                        | $\kappa_\perp, \kappa_\parallel$ | S04B$^b$ |
| NLPA                  | NonLinear Parallel diffusion theory            | $\kappa_\parallel$ | $\kappa_\perp$ | Q07$^c$ |
| NLGC-E                | Combination of NLGC and NLPA                   | $\kappa_\perp, \kappa_\parallel$ | Q07$^c$ |
| INLGC                 | Unified diffusion theory                        | $\kappa_\perp$ | $\kappa_\parallel$ | S10$^d$, TA11$^e$ |
| NLGC-N                | Modification of NLGC                           | $\kappa_\perp$ | $\kappa_\parallel$ | TP$^f$ |
| NLGCE-N               | Combination of NLGC-N and NLPA                 | $\kappa_\perp, \kappa_\parallel$ | TP$^f$ |
| NLGCE-F               | Polynomial fitting of NLGCE-N                  | $\kappa_\perp, \kappa_\parallel$ | TP$^f$ |

Notes.

$^a$ Matthaeus et al. (2003).

$^b$ Shalchi et al. (2004b).

$^c$ Qin (2007).

$^d$ Shalchi (2010).

$^e$ Term noted in Tautz & Shalchi (2011).

$^f$ This paper.

Figure 1. Top and bottom panels show perpendicular and parallel mean free paths, respectively, as a function of $E_{\text{slab}}/E_{\text{total}}$, with $r_L/\lambda_c = 0.048$, $b/B_0 = 1$ and $\lambda_{2D}/\lambda_{\text{slab}} = 0.1$. Diamonds are from the simulations in Qin (2007). Dotted, dashed, and dashed-dotted lines indicate results from NLGC-E, NLGCE-N, and NLGCE-F, respectively.

In addition, Shalchi (2010) developed a unified diffusion theory for perpendicular diffusion based on Matthaeus et al. (2003; NLGC), in which the Fokker–Planck equation is used to compute the fourth-order correlations during derivation. Furthermore, Tautz & Shalchi (2011) compared the unified diffusion theory, noted as INLGC (as well as NLGC) with simulations. It is shown that the unified diffusion theory, INLGC, can be used for three-dimensional (3D) turbulence. Moreover, INLGC automatically satisfies the subdiffusive result for slab turbulence ($\kappa_\perp = 0$), and corresponds to NLGC for two-component turbulence without any additional assumptions.

In this study, for simplicity, we modify the NLGC theory directly by replacing the spectral amplitude of the two-component model magnetic turbulence with that of the 2D component and replace the constant that indicates the amount particles following the MFL with a variable of magnetic turbulence. We also fit the numerical results of the modified model with polynomials.

The paper is organized as follows. We discuss the modification of the NLGC theory in Section 2. The polynomial fitting of the new model is discussed in Section 3. Finally, conclusions are presented in Section 4.
2. MODIFICATION OF THE NLGC THEORY

The NLGC theory agrees with simulation results very well in general solar wind conditions. However, from simulation results in Qin (2007), especially in Figure 3 of Qin (2007), we find that the NLGC theory for perpendicular diffusion does not agree with simulation results well when the turbulence is nearly pure slab or pure 2D, so it is necessary to modify the NLGC theory. First, we consider that the slab component of the turbulent magnetic field does not directly contribute to the perpendicular diffusion (Shalchi 2006, 2010). Therefore, in the NLGC theory of the perpendicular diffusion coefficient Equation (1), the spectral amplitude of the two-component model magnetic turbulence $S_{xx}(k)$ should only include the 2D component, $S_{2D}^1(k)$ (Shalchi 2006). Second, we assume that the amount of particles following MFL varies with the conditions of magnetic turbulence, so we modify the parameter $a^2$ in Equation (1) with different forms and compare with simulation results in Qin (2007). Thus far, the best form obtained is

$$a^2 = \left( \frac{\lambda_{2D} E_{total}}{\lambda_{slab} E_{slab}} + \frac{4 E_{total}}{3 E_{2D}} \right)^{-1}$$

(7)

Therefore, by replacing $S_{xx}(k)$ and $a^2$ with $S_{2D}^1(k)$ and $a^2$, respectively, in Equation (1), we obtain a modified NLGC theory for the perpendicular diffusion coefficient,

$$\kappa_{xx} = \frac{a^2}{3} \int dk_\perp 2C(v)\lambda_{2D} \frac{(b_{2D}^2 B_0^2)}{B_0^2} (1 + k_\perp^2 \lambda_{2D}^2)^{-\nu}$$

$$\times \frac{1}{\frac{q}{2} + k_\perp^2 \kappa_{xx} + \gamma(k)}.$$  

(8)

Here, the symmetric 2D component turbulence is assumed. In addition, the 2D spectrum of Equation (3) with the energy range spectrum index $q = 0$ is used in order to compare with simulations in Qin (2007). In the future, a more general form of the 2D spectrum with $q \neq 0$ can be used. In the following, we call this modified model NLGC-N. In addition, we combine the NLGC-N and NLPA models, Equations (8) and (5), respectively, to obtain an NLGCE-N model. Next we compare the numerical results of the NLGCE-N model with that of NLGC-E and the simulation results from Qin (2007). Here, $\gamma(k)$ is chosen to be 0 for the static magnetic turbulence, and $v$ is chosen to be 5/6. In addition, we can also define a parameter, the perpendicular mean free path, as $\lambda_{\perp} = 3\kappa_{xx}/v$ for simplicity.

The top and bottom panels of Figure 1 show the perpendicular and parallel mean free paths, respectively, as a function of $E_{slab}/E_{total}$, with $r_L/\lambda_e = 0.048$, $b/B_0 = 1$, and $\lambda_{2D}/\lambda_{slab} = 0.1$. The diamonds are from the simulations in Qin (2007). The dotted, dashed, and dashed-dotted lines indicate results from NLGC-E, NLGCE-N, and NLGCE-F, respectively. Later, we will study NLGCE-F, which is the polynomial fitting of NLGCE-N. The simulation results in Figure 1 are obtained from Figure 3 of Qin (2007). As already shown in Qin (2007), the perpendicular diffusion coefficient from NLGCE-E agrees well with simulation results when $E_{slab}/E_{total} \sim 0.2$, but it does not agree well with simulations when $E_{slab}/E_{total} \ll 0.2$ or $E_{slab}/E_{total} \rightarrow 1$. However, the perpendicular diffusion coefficient from the new model, NLGCE-N, agrees very well for the entire range of $E_{slab}/E_{total}$, from $E_{slab}/E_{total} \ll 0.2$ to $E_{slab}/E_{total} \rightarrow 1$. At the same time, the parallel diffusion coefficient from NLGCE-N generally agrees well with simulations. Although NLGCE-N is relatively worse than NLGCE-E with $E_{slab}/E_{total} \ll 0.02$, the results of NLGCE-N are still acceptable.

Figure 2 is similar to Figure 1, except that the $x$-axis is $\lambda_{2D}/\lambda_{slab}$ with $E_{slab}/E_{total} = 0.2$, and the simulation results are obtained from Figure 4 of Qin (2007). From the bottom panel of Figure 2, we can see that both NLGCE-E and NLGCE-N agree well with simulations in the parallel diffusion. However, from the top panel of Figure 2, the two models are different in the perpendicular diffusion. NLGCE-E agrees well with simulations in the perpendicular diffusion coefficient with $\lambda_{2D}/\lambda_{slab} \sim 0.01$, but the agreement decreases as $\lambda_{2D}/\lambda_{slab}$ increases. On the other hand, to compare to simulations,
NLGCE-N is worse than NLGC-E in the perpendicular diffusion with \( \lambda_{2D}/\lambda_{slab} \lesssim 0.02 \), but it is better than NLGC-E with \( \lambda_{2D}/\lambda_{slab} \gtrsim 0.03 \). Generally, NLGCE-N agrees better with simulations than NLGC-E in the perpendicular diffusion.

Figure 3 is similar to Figure 1, except that the x-axis is \( r_L/\lambda_c \) with \( E_{slab}/E_{total} = 0.2 \), and the simulation results are obtained from Figure 1 of Qin (2007). From Figure 3, we can see that both NLGC-E and NLGCE-N agree well with simulations. In addition, Figure 4 is similar to Figure 1, except that the x-axis is \( r_L/\lambda_c \) with \( E_{slab}/E_{total} = 0.2 \) and \( b/B_0 = 0.2 \), and the simulation results are obtained from Figure 2 of Qin (2007). Again, from Figure 4, we can see that both NLGC-E and NLGCE-N agree well with simulations.

Therefore, we show that the new NLGCE-N model is improved compared to the NLGC-E model, especially when the magnetic turbulence is nearly pure slab or 2D.

In addition, we compare the modified model NLGCE-N with the INLGC model (Shalchi 2010; Tautz & Shalchi 2011). Since NLGC-N and INLGC are models for perpendicular diffusion with parallel diffusion coefficients as input, we use \( \kappa_{ij} \) from the simulation results in Qin (2007) as input. The left and right panels of Figure 5 are similar to the top panels of Figures 1 and 2, respectively, except that dashed and dashed-dotted lines indicate results from NLGC-N and INLGC, respectively. From the left panel of the figure, we see that when \( 0.1 < E_{slab}/E_{total} \lesssim 0.5 \), both NLGC-N and INLGC agree well with simulations. However, when \( E_{slab}/E_{total} \lesssim 0.1 \) or \( E_{slab}/E_{total} > 0.5 \), NLGCE-N agrees better with simulations than INLGC. Furthermore, from the right panel of the figure, we can see that when \( \lambda_{2D}/\lambda_{slab} \sim 0.1 \), both NLGCE-N and INLGC agree well with simulations, but when \( \lambda_{2D}/\lambda_{slab} \not\approx 0.1 \), NLGCE-N agrees better with simulations than INLGC.

### 3. POLYNOMIAL FITTING OF NLGCE-N

Although the new NLGCE-N model agrees well with simulations, numerous iteration of integrations for both equations are needed to solve it numerically. In order to simplify the numerical calculations needed using the model to study the energetic particles transport or acceleration, we fit the equations of the NLGCE-N model with polynomials in the parameters of the magnetic field and particles, i.e., \( r_L/\lambda_{slab}, E_{slab}/E_{total}, b^2/B_0^2, \) and \( \lambda_{slab}/\lambda_{2D} \), or

\[
\ln \frac{\lambda_{th}}{\lambda_{slab}} = \sum_{i=0}^{n_{s,1}} a_i^\alpha \left( \ln \frac{r_L}{\lambda_{slab}} \right)^i, 
\]

with

\[
a_i^\alpha = \sum_{j=0}^{n_{s,2}} b_{i,j}^\alpha \left( \ln \frac{E_{slab}}{E_{total}} \right)^j
\]

\[
b_{i,j}^\alpha = \sum_{k=0}^{n_{s,3}} c_{i,j,k}^\alpha \left( \ln \frac{b^2}{B_0^2} \right)^k
\]

\[
c_{i,j,k}^\alpha = \sum_{l=0}^{n_{s,4}} d_{i,j,k,l}^\alpha \left( \ln \frac{\lambda_{slab}}{\lambda_{2D}} \right)^l.
\]
where $\alpha \perp \perp \perp$. Note that this formula is not valid in pure 2D turbulence. By fitting the numerical results of NLGC-N with the polynomials in wide ranges of parameters, $r_T/\lambda_{slab}$, $E_{slab}/E_{total}$, $b_1^{\parallel}/B_1^\perp$, and $\lambda_{slab}/\lambda_{2D}$ as shown in Table 2, we obtain $\Pi_{jk,l}^{\parallel}(n_a+1)$ coefficients $c_{i,j,k,l}$ for either parallel or perpendicular diffusion. Note that with larger values of $n_a$, we can obtain a fitting formula with higher accuracy, but more fitting parameters $c_{i,j,k,l}$ are needed. Thus, we must balance between the accuracy and the simplicity of the fitting formula. We attempted to fit the formulae with different sets of $n_a$ and carefully compared the results with NLGC-N (not shown), and we found that with $n_a = 5$, $n_{a2} = 3$, $n_{a3} = 3$, and $n_{a4} = 2$ for both parallel and perpendicular diffusion, we can obtain a fitting formula with good accuracy and an acceptable calculation scale. In this way, we can directly calculate the parallel and perpendicular diffusion coefficients without the iteration of integrations, and the polynomial fitting results of parallel and perpendicular diffusion coefficients are called NLGC-F. The $\Pi_{jk,l}^{\parallel}(n_a+1) = 6 \times 4 \times 4 \times 3 = 288$ coefficients $c_{i,j,k,l}$ for parallel $(\alpha = \parallel)$ and perpendicular $(\alpha = \perp)$ diffusion in the NLGC-F model are shown in Tables 3 and 4, respectively.

As an example, to show the NLGC-F's acceptable accuracy with a controllable calculation scale, in Figure 6 we show a comparison between the NLGC-F and the results of a new fitting formula with $n_{el} = 6$, $n_{a2} = 4$, $n_{a3} = 4$, and $n_{a4} = 3$, which is called NLGC-F2. Figure 6 is similar to Figure 3, except that solid, dotted, and dashed lines indicate results from NLGC-N, NLGC-F, and NLGC-
F2, respectively. From the top panel of the figure, we can see that, for perpendicular diffusion, when $r_{\perp}/\lambda_{c} < 0.3$, NLGCE-F2 agrees better with NLGCE-N than NLGCE-F, but when $r_{\perp}/\lambda_{c} > 0.3$, NLGCE-F agrees better with NLGCE-N than NLGCE-F2. However, from the bottom panel of the figure, we can see that, for parallel diffusion, NLGCE-F and NLGCE-F2 agree very well with each other in the range 0.001 $\lesssim r_{\parallel}/\lambda_{c} \lesssim 0.3$. With comparisons including other variable ranges (not shown) we found that, generally speaking, relative to NLGCE-F2, NLGCE-F is acceptable in agreement with NLGCE-N. However, with NLGCE-F2, the number of coefficients $d_{i,j,k,l}^{p}$ for either the parallel or perpendicular diffusion is $7 \times 5 \times 5 \times 4 = 700$. In order to show the agreement between the model NLGCE-N and its polynomial fitting NLGCE-F, in Figures 1–4, we plot the results of NLGCE-N with dashed-dotted lines. From the figures, we can see that NLGCE-F agrees relatively well with NLGCE-N.

4. CONCLUSIONS

In this paper, we modified the NLGC model, Equation (1), which determines the particles perpendicular diffusion, by replacing the spectral amplitude of the two-component model magnetic turbulence $S_{d}(k)$ with that of the 2D component, $S_{d2}(k)$ (Shalchi 2006), and replace the parameter $\alpha^{2}$ with $\alpha^{2}$, which is a function of $E_{slab}/E_{total}$ and $\lambda_{d2}/\lambda_{slab}$, to obtain a new model NLGCE-N for the parallel diffusion. To combine NLGCE-N with NLPA, the model for parallel diffusion, we obtain
the model NLGCE-N, which can be solved simultaneously to describe both perpendicular and parallel diffusion. In addition, we show that NLGCE-N agrees better with simulations than NLGC-E, which is the combination of NLGC and NLPA. Furthermore, we fit the numerical results of NLGCE-N with the polynomials in wide ranges of parameters $r_L/\lambda_{slab}$, $E_{slab}/E_{total}$, $b^2/B_0^2$, and $\lambda_{slab}/\lambda_{2D}$, to obtain a new model, NLGCE-F, in order to directly calculate parallel and perpendicular diffusion coefficients simultaneously without the iteration of integrations. Therefore, this process eliminates the need to perform many numerical calculations to study diffusion coefficients.

We also note that when $E_{slab}/E_{total} \lesssim 0.1$ or $\lambda_{2D}/\lambda_{slab} \neq 0.1$, the modified model NLGC-N agrees better with simulations than the unified diffusion theory (INLGC). In addition, the modified model NLGC-N is very similar to the two-component limit of the unified diffusion theory, with a major difference, i.e., in INLGC, a constant parameter $a^2 = 1/3$ is used. Thus, we suggest that the unified diffusion theory can also adopt the similar modification of parameter $a^2$, Equation (7), as in this paper.

For future study, we plan to compare our models with simulations and with general forms of the 2D component with an energy range spectrum index $q \neq 0$. In addition, we will use the model NLGCE-F to study the transport of energetic particles in the solar wind, including solar energetic particles, anomalous cosmic rays, or galactic cosmic rays (e.g., Qin et al. 2006; Zhao et al. 2014). Furthermore, FORTRAN code used for NLGCE-F and the data of the coefficients $d_{\alpha,i,j,k,l}$ can be found online at http://www.qingang.org.cn/code/NLGCE-F to be freely downloaded and used.

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