INVARIANT GIBBS DYNAMICS FOR THE TWO-DIMENSIONAL
ZAKHAROV-YUKAWA SYSTEM

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Abstract. We study the Gibbs dynamics for the Zakharov-Yukawa system on the two-
dimensional torus $\mathbb{T}^2$, namely a Schrödinger-wave system with a Zakharov-type coupling
$(-\Delta)^\gamma$. We first construct the Gibbs measure in the weakly nonlinear coupling case ($0 \leq \gamma < 1$). Combined with the non-construction of the Gibbs measure in the strongly nonlinear
coupling case ($\gamma = 1$) by Oh, Tolomeo, and the author (2020), this exhibits a phase transition
at $\gamma = 1$. We also study the dynamical problem and prove almost sure global well-posedness
of the Zakharov-Yukawa system and invariance of the Gibbs measure under the resulting
dynamics for the range $0 \leq \gamma < \frac{1}{3}$. In this dynamical part, the main step is to prove local
well-posedness. Our argument is based on the first order expansion and the operator norm
approach via the random matrix/tensor estimate from a recent work Deng, Nahmod, and
Yue (2020). In the appendix, we briefly discuss the Hilbert-Schmidt norm approach and
compare it with the operator norm approach.

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1. Introduction

1.1. Invariant Gibbs dynamics for the Zakharov-Yukawa system. In this paper, our main goal is to construct invariant Gibbs dynamics for the Schrödinger-wave systems on $T^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ with a Zakharov-type coupling:

$$
\begin{cases}
  i\partial_t u + \Delta u = uw \\
  \partial_t^2 w + (1 - \Delta)w = -\langle\nabla\rangle^{2\gamma}(|u|^2) \\
  (u, w, \partial_t w)|_{t=0} = (u_0, w_0, w_1),
\end{cases}
$$

where $0 \leq \gamma \leq 1$ and $\langle\nabla\rangle := (1 - \Delta)^{\frac{1}{2}}$. The case $\gamma = 1$ corresponds to the well-known Zakharov system, while the case $\gamma = 0$ corresponds to the Yukawa system. The Zakharov-Yukawa system (1.1) is a special case of the models introduced in [57, Section 3]

$$
\begin{cases}
  i\partial_t u + L_1 u = uw \\
  L_2 w = L_3 |u|^2,
\end{cases}
$$

where $L_1, L_2$ and $L_3$ are constant coefficient differential operators. This class of systems is referred to as Davey-Stewartson (DS) systems in the work of Zakharov-Schulman [57, Section 3]. In particular, (DS) systems is associated with a specific 2 dimensional system of the form (1.2), modeling the evolution of weakly nonlinear water waves travelling predominantly in one direction, in which the wave amplitude is modulated slowly in two horizontal directions. See, for example, [23, 24]. As for the global dynamics of (1.1), see [2].

Notice that the Zakharov-Yukawa system (1.1) is a Hamiltonian system associated with the Hamiltonian (1.3):

$$
H(u, w, \partial_t w) = \frac{1}{2} \int_{T^2} |\nabla u|^2 dx + \frac{1}{2} \int_{T^2} |u|^2 w dx + \frac{1}{4} \int_{T^2} |\langle\nabla\rangle^{1-\gamma} w|^2 dx + \frac{1}{4} \int_{T^2} |\langle\nabla\rangle^{-\gamma} \partial_t w|^2 dx.
$$

The flow of (1.1) preserves the $L^2$-norm of the Schrödinger component $u$ but not the $L^2$-norm of the wave component $w$. Hence, to avoid a problem at the zeroth frequency in the Gibbs measure construction, we work with the massive linear part $\partial_t^2 w - \Delta w + w$ and accordingly consider $\langle\nabla\rangle^{2\gamma}$ instead of $(-\Delta)^\gamma$ as a coupling to complete the Hamiltonian formulation.
Moreover, the wave energy, namely, the $L^2$-norm of the Schrödinger component
\[ M(u) = \int_{\mathbb{T}^2} |u|^2 \, dx \]
is known to be conserved. See [14]. Then, the corresponding grand-canonical (Gibbs) measure \(d\bar{\rho}_\gamma\) for the Hamiltonian system (1.1) is formally given by
\[ d\bar{\rho}_\gamma = Z^{-1} e^{-H(u, w, \partial_t w) - M(u)} \, du \otimes dw \otimes d(\partial_t w) 
= Z^{-1} e^{-\frac{1}{4} \int_{\mathbb{T}^2} |u|^2 \, dx} d(\mu_1 \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma})(u, w, \partial_t w), \] (1.4)
where for any given \(s \in \mathbb{R}\), \(d\mu_s\) denote a Gaussian field, formally defined by
\[ d\mu_s = Z_s^{-1} e^{-\frac{1}{2} \|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}^2} e^{-\frac{1}{2} (n_1^2 + |\hat{u}(n)|^2)} d\hat{u}(n), \] (1.5)
where \(\langle \cdot \rangle = (1 + | \cdot |^2)^{\frac{1}{2}}\) and \(\hat{u}(n)\) denotes the Fourier transforms of \(u\). Note that \(d\mu_s\) corresponds to the massive Gaussian free field \(d\mu_1\) when \(s = 1\) and to the white noise measure \(d\mu_0\) when \(s = 0\). Notice that the Gibbs measure \(\bar{\rho}_\gamma\) on the vector \((u, w, \partial_t w)\), formally defined in (1.4), decouples as the Gibbs measure \(\rho_\gamma\) on the vector \((u, w)\) and the Gaussian measure \(\mu_{-\gamma}\) on the third component \(\partial_t w\):
\[ \bar{\rho}_\gamma = \rho_\gamma \otimes \mu_{-\gamma}. \] (1.6)

In terms of the conservation of the Hamiltonian \(H(u, w, \partial_t w)\) and the wave energy \(M(u)\), the Gibbs measures \(d\bar{\rho}_\gamma\) in (1.4) are expected to be invariant under the Zakharov-Yukawa dynamics (1.1).

The main issue in constructing the Gibbs measure \(d\bar{\rho}_\gamma\) in (1.4) comes from the focusing nature of the potential, i.e. the interaction potential \(\int_{\mathbb{T}^2} |u|^2 \, dx\) is unbounded. In the seminal work [32], Lebowitz, Rose, and Speer initiated the study of focusing Gibbs measures in the one-dimensional setting. In this work, they constructed the one-dimensional (focusing) Gibbs measures with an \(L^2\)-cutoff
\[ d\rho_1(u, w, \partial_t w) = Z^{-1} 1_{\{u_1^2 \leq K\}} e^{-\frac{1}{4} \int_{\mathbb{T}^2} |u|^2 \, dx} d(\mu_1 \otimes \mu_0 \otimes \mu_{-1})(u, w, \partial_t w) \] (1.7)
and also the focusing \(\Phi^k_1\)-measure (Gibbs measures) in the \(L^2\)-(sub)critical setting (i.e. \(2 < k \leq 6\)) with an \(L^2\)-cutoff
\[ d\rho(u) = Z^{-1} 1_{\{u_1^2 \leq K\}} e^{\frac{1}{4} \int_{\mathbb{T}^2} |u|^2 \, dx} d\mu_1(u) \] (1.8)
where \(d\mu_1\) and \(d\mu_0\) denote the periodic Wiener measure and the white noise on \(\mathbb{T}\), respectively. The (focusing) Gibbs measures (1.7) and (1.8) were then proved to be invariant under the Zakharov system \((\gamma = 1\) in (1.1)) and cubic NLS on \(\mathbb{T}\) by Bourgain [7, 8], respectively. See Remark 1.2(i) for more explanations about the focusing \(\Phi^k_1\)-measure.

In the two-dimensional setting \(\mathbb{T}^2\), Oh, Tolomeo, and the author [23] continued the study on the (focusing) Gibbs measures (1.7) and proved that the Gibbs measure (1.7) (even with

\[ ^2 \text{A typical function}\ u \text{ in the support of } d\mu_1 \text{ (on } \mathbb{T}^2) \text{ is merely a distribution (not a function) and so a proper renormalization procedure is required. In this introduction, we keep our discussion at a formal level and do not worry about renormalizations.} \]

\[ ^3 \text{In this paper, by “focusing”, we mean “non-defocusing” (non-repulsive). Namely, the interaction potential (for example, } \int_{\mathbb{T}^2} |u|^2 w \text{ in (1.4) or } \int_{\mathbb{T}^2} |u|^k \text{ (1.8) ) is unbounded from above.} \]

\[ ^4 \text{Here, we consider the case where } k \text{ is an integer. In particular, } k \text{ is an even integer when } d\mu_1 \text{ is the complex Gaussian free field.} \]
proper renormalization on the potential energy $\int_{\mathbb{T}^2} |u|^2 w$ and on the $L^2$-cutoff) is not normalizable as a probability measure:

$$E_{\mu_1 \otimes \mu_0 \otimes \mu_{-\gamma}} \left[ 1_{\{ \int_{\mathbb{T}^2} |u|^2 : dx \leq K \}} e^{-\frac{1}{2} \int_{\mathbb{T}^2} |u|^2 : w dx} \right] = \infty$$

for any $K > 0$, where $d\mu_1$ and $d\mu_0$ denote the massive Gaussian free field and the white noise on $\mathbb{T}^2$, respectively. As for the focusing $\Phi^4_2$-measure (Gibbs measure) on $\mathbb{T}^2$, Brydges and Slade proved that the focusing $\Phi^4_2$-measure (i.e. the quartic interaction $k = 4$) (even with proper renormalization on the potential energy $\frac{1}{4} \int_{\mathbb{T}^2} |u|^4$ and on the $L^2$-cutoff) is not normalizable as a probability measure:

$$E_{\mu_1} \left[ 1_{\{ \int_{\mathbb{T}^2} |u|^2 : dx \leq K \}} e^{\frac{1}{4} \int_{\mathbb{T}^2} |u|^4 : dx} \right] = \infty$$

for any $K > 0$. An alternative proof was also given by Oh, Tolomeo, and the author. We also notice that with the cubic interaction ($k = 3$), Jaffe constructed a (renormalized) $\Phi^3_2$-measure with a Wick-ordered $L^2$-cutoff. See Remark 1.2(ii) for more explanations about the focusing $\Phi^4_2$-measure.

In this paper, our first goal is to establish the following phase transition (Theorem 1.5) at the critical value $\gamma = 1$:

(i) (weakly nonlinear coupling). Let $0 \leq \gamma < 1$. Then, we have the normalizability of the (focusing) Gibbs measure

$$E_{\mu_1 \otimes \mu_1 \otimes \mu_{-\gamma}} \left[ 1_{\{ \int_{\mathbb{T}^2} |u|^2 : dx \leq K \}} e^{-\frac{1}{2} \int_{\mathbb{T}^2} |u|^2 : w dx} \right] < \infty$$

for any $K > 0$.

(ii) (strongly nonlinear coupling). Let $\gamma = 1$. Then, we have the non-normalizability of the (focusing) Gibbs measure

$$E_{\mu_1 \otimes \mu_1 \otimes \mu_{-1}} \left[ 1_{\{ \int_{\mathbb{T}^2} |u|^2 : dx \leq K \}} e^{-\frac{1}{2} \int_{\mathbb{T}^2} |u|^2 : w dx} \right] = \infty$$

for any $K > 0$.

Therefore, the two-dimensional Zakharov system ($\gamma = 1$) turns out to be critical, exhibiting the phase transition in terms of the measure construction.

We then study the dynamical problem i.e. construct invariant Gibbs dynamics ($d\tilde{\rho}_\gamma$-almost sure global well-posedness and invariance of the Gibbs measure; see Theorems 1.8 and 1.13).

Remark 1.1. In a recent work by the author, the phase transition phenomenon in the three-dimensional setting $\mathbb{T}^3$ was explored for the Gibbs measure with $\gamma = 0$, formally expressed as

$$d\tilde{\rho}(u, w, \partial_t w) = Z^{-1} \exp \left( -\frac{\lambda}{2} \int_{\mathbb{T}^3} |u|^2 w : dx - \infty \right) 1_{\{ |f_{\mathbb{T}^3} : |u|^2 dx \leq K \}} d\tilde{\mu}_1(u, w, \partial_t w)$$

where the coupling constant $\lambda \in \mathbb{R} \setminus \{0\}$ measures the strength of the interaction potential and $\tilde{\mu}_1 = \mu_1 \otimes \mu_1 \otimes \mu_0$. In the three-dimensional scenario, the Gaussian free field $\mu_1$ is supported on a significantly rougher space, specifically $C^s(\mathbb{T}^3) \setminus C^{-\frac{1}{2}}(\mathbb{T}^3)$ for any $s < -\frac{1}{2}$, where $C^s(\mathbb{T}^d)$ denotes the Hölder-Besov space. To accommodate this difference from the two-dimensional case, an additional (non-Wick) renormalization denoted by $-\infty$ is necessary for constructing

\footnote{Since $|u|^2 w$ is not sign definite, the sign of $\lambda$ does not play any role.}
the measure. While the focusing Gibbs measure on $\mathbb{T}^2$ in (1.4) can be constructed regardless of the coupling size $|\lambda|$ (see Theorem 1.5), in [58] the author demonstrated a phase transition for the Gibbs measure (1.9) in the three-dimensional setting. Specifically, normalizability ($Z < \infty$) was established in the weak coupling regime ($0 < |\lambda| \ll 1$) for every $K > 0$, while non-normalizability ($Z = \infty$) was proven in the strong coupling case ($|\lambda| \gg 1$) for every $K > 0$. Similar phase transitions at critical exponents between strong and weak coupling regimes have been observed in other focusing models (see [39, 40]), but not with the Wick-ordered $L^2$-cutoff (i.e. the generalized grand-canonical formulation). Therefore, we point out that the size of the coupling constant $\lambda$ plays a crucial role in the analysis of the focusing Gibbs measure (1.9), particularly concerning the phase transition with respect to $|\lambda|$, marking a significant departure from focusing Gibbs measures on $\mathbb{T}^2$. In particular, in the weak coupling regime ($0 < |\lambda| \ll 1$) the Gibbs measure (1.9) is singular with respect to the base Gaussian field $\vec{\mu}_1$ while the Gibbs measure (1.4) on $\mathbb{T}^2$ is absolutely continuous with respect the base Gaussian field. This singularity of the Gibbs measure introduced additional difficulties, compared to the Gibbs measures on $\mathbb{T}^2$, studied in this paper for the measure (non-)construction part.

**Remark 1.2.**

(i) In the one-dimensional setting $\mathbb{T}$, in [32], Lebowitz, Rose, and Speer also proved non-normalizability of the focusing $\Phi^k_1$-measure

$$\mathbb{E}_{\mu_1}\left[1_{\{f_{T^2} \leq K\}} e^{\int_{T^2} |u|^k \, dx}\right] = \infty$$

in (i) the $L^2$-supercritical case ($k > 6$) for any $K > 0$ and (ii) the $L^2$-critical case ($k = 6$), provided that $K > \|Q\|_{L^2(\mathbb{R})}^2$, where $Q$ is the (unique) optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ such that $\|Q\|_{L^6(\mathbb{R})}^6 = 3\|Q\|_{L^2(\mathbb{R})}^2$. In a recent work [44], Oh, Sosoe and Tolomeo completed the focusing Gibbs measure construction program in the one-dimensional setting, including the critical case ($k = 6$) at the optimal mass threshold $K = \|Q\|_{L^2(\mathbb{R})}^2$. See [44] for more details on the (non-)construction of the focusing $\Phi^k_1$-measure in the one-dimensional setting.

(ii) In the two-dimensional setting $\mathbb{T}^2$, the non-normalizability of the focusing $\Phi^k_2$-measure also holds for the higher order interactions $k \geq 5$ (see [43] Remark 1.4)

$$\mathbb{E}_{\mu_1}\left[1_{\{f_{T^2} \leq K\}} e^{\int_{T^2} |u|^k \, dx}\right] = \infty.$$  

We point out that typical elements $u$ in the support of massive Gaussian free field $d\mu_1$ (on $\mathbb{T}^2$) is log-correlated, namely

$$\mathbb{E}_{\mu_1}[u(x)\overline{u(y)}] \sim \log |x - y|$$

for any $x, y \in \mathbb{T}^2$ with $x \neq y$. See [41, 43] for a related discussion. In particular, in [43] Oh, Tolomeo, and the author studied the non-normalizability of the Gibbs measure with log-correlated base Gaussian fields on $\mathbb{T}^d$ for any $d \geq 1$.

(iii) In the three-dimensional setting $\mathbb{T}^3$, more complicated phenomena appear. In [39], Oh, Okamoto and Tolomeo studied the (non-)construction of the focusing $\Phi^k_3$-measure. More precisely, in the weakly nonlinear regime, they proved normalizability of the $\Phi^k_3$-measure and show that it is singular with respect to the massive Gaussian free field on $\mathbb{T}^3$.

\footnote{Up to the symmetries.}
they proved that there exists a shifted measure with respect to which $\Phi^3_3$-measure is absolutely continuous. In the strongly nonlinear regime, they established non-normalizability of $\Phi^3_3$-measure.

In the case of a higher order focusing interaction ($k \geq 4$), the focusing nonlinear interaction is worse than the cubic interaction ($k = 3$) and so non-normalizability would be satisfied for the higher order interactions ($k \geq 4$). See also [40] for the non-normalizability of the focusing Hartree $\Phi^4_3$-measure.

Remark 1.3. Removing the infrared cut-off (i.e. the finite volume $T^d$) to investigate Gibbs measures on the infinite volume $\mathbb{R}^d$ poses a highly intricate challenge. The construction of these measures, initially achieved in finite volumes and subsequently extended to infinite volume, stands as a significant accomplishment in constructive quantum field theory. In the context of focusing Gibbs measures on the infinite volume, in [53, 62] it was observed that the focusing $\Phi^1_4$-measure, an invariant measure for the focusing cubic Schrödinger equation, collapses onto the unit mass on the trivial path. In other words, it converges weakly to $\delta_0$, placing unit mass on the zero path, when taking a large torus limit. Consequently, we anticipate a triviality phenomenon in the large torus limit of the Gibbs measure $\tilde{\rho}_\gamma$ (1.4), analogous to the one-dimensional focusing case, because of its focusing nature.

Remark 1.4. The equation (1.1) is also known as the Schrödinger-Klein-Gordon system with a Zakharov-type coupling. In the following, however, we simply refer to (1.1) as the Schrödinger-wave system.

We point out that for our first main result (Theorem 1.5), we need to work with the massive linear part $\partial^2_t w - \Delta w + w$ in order to avoid a problem at the zeroth frequency in the Gibbs measure construction. Hence, due to this reason, we work with the massive case in this paper.

1.2. Phase transition of the Gibbs measure. In this subsection, we explain a renormalization procedure required to give a proper meaning to the Gibbs measure $d\tilde{\rho}_\gamma$ defined in (1.4) and present our first main theorem for the phase transition i.e. (non-)construction of the focusing Gibbs measure.

The Gibbs measure for the Zakharov-Yukawa system (1.1) is formally given by

$$d\tilde{\rho}_\gamma = Z^{-1} e^{-Q(u, w)} d(\mu_1 \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma})(u, w, \partial_t w),$$

(1.10)

where the potential energy $Q(u, w)$ is given by

$$Q(u, w) = \frac{1}{2} \int_{\mathbb{T}^2} |u|^2 w \, dx.$$  

Recall that on $\mathbb{T}^2$, the Gaussian field $d\mu_s$ in (1.5) is a probability measure supported on $W^{s-1-\varepsilon} p(\mathbb{T}^2)$ for any $\varepsilon > 0$ and $1 \leq p \leq \infty$. For simplicity, we set $d\mu = d\mu_1$. We now go over the Fourier representation of functions distributed by $\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}$. We now define

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7Since the zeroth frequency is not controlled due to the lack of the conservation of the $L^2$-mass under the dynamics.

8Gaussian measure $Z^{-1} \exp \left( -\frac{1}{2} \| (u, w, \partial_t w) \|^2_{H^1 \times H^{1-\gamma} \times H^{-\gamma}} \right) du \otimes dw \otimes d(\partial_t w)$ for which $H^1(\mathbb{T}^2) \times H^{1-\gamma}(\mathbb{T}^2) \times H^{-\gamma}(\mathbb{T}^2)$ is the Cameron-Martin space.
random distributions \( u = u^\omega, w_0 = w_0^\omega \) and \( w_1 = w_1^\omega \) by the following Gaussian Fourier series:

\[
  u^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{i \langle n, x \rangle}, \quad w_0^\omega = \sum_{n \in \mathbb{Z}^2} \frac{h_n(\omega)}{\langle n \rangle - \gamma} e^{i \langle n, x \rangle}, \quad \text{and} \quad w_1^\omega = \sum_{n \in \mathbb{Z}^2} \frac{\ell_n(\omega)}{\langle n \rangle - \gamma} e^{i \langle n, x \rangle},
\]

(1.11)

where \( \{g_n, h_n, \ell_n\}_{n \in \mathbb{Z}^2} \) is a sequence of “independent standard” complex-valued Gaussian random variables on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) conditioned that \( h_{-n} = \overline{h_n} \) and \( \ell_{-n} = \overline{\ell_n} \). More precisely, with the index set \( \Lambda \) defined by

\[
  \Lambda := (\mathbb{Z} \times \mathbb{Z}_+) \cup (\mathbb{Z}_+ \times \{0\}) \cup \{(0,0)\},
\]

We define \( \{h_n, \ell_n\}_{n \in \Lambda} \) to be a sequence of independent standard complex-valued Gaussian random variables (with \( \ell_0 \) real-valued) and set \( h_{-n} = \overline{h_n} \) and \( \ell_{-n} = \overline{\ell_n} \) for \( n \in \Lambda \).

Denoting the law of a random variable \( X \) by \( \text{Law}(X) \), we then have

\[
\text{Law}((u, w_0, w_1)) = \mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}
\]

for \((u, w_0, w_1)\) in (1.11). Note that \( \text{Law}((u, w_0, w_1)) = \mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma} \) is supported on \( H^{s_1}(\mathbb{T}^2) \times H^{s_2}(\mathbb{T}^2) \times H^{s_2-1}(\mathbb{T}^2) \) for \( s_1 < 0 \) and \( s_2 < -\gamma \) but not for \( s_1 > 0 \) and \( s_2 > -\gamma \), respectively; see [10, 65, 46].

As we pointed out, the key issue in constructing the Gibbs measure \( d\tilde{\rho}_\gamma \) in (1.4) comes from the focusing nature of the potential, i.e. the potential \( Q(u, w) \) is unbounded from above. In a seminal paper [32], Lebowitz, Rose, and Speer constructed the Gibbs measure \( d\tilde{\rho}_\gamma \) when \( d = 1 \) and \( \gamma = 1 \), by inserting a cutoff in terms of the conserved wave energy \( M(u) = \|u\|_{L^2}^2 \).

Then, a natural question is to consider the construction of the Gibbs measure \( d\tilde{\rho}_\gamma \) in the two-dimensional setting \( \mathbb{T}^2 \). We point out that in [43] Oh, Tolomeo, and the author proved that the (renormalized) Gibbs measure for the Zakharov system \( (\gamma = 1) \) on \( \mathbb{T}^2 \) is not normalizable, even with a Wick-ordered \( L^2 \)-cutoff i.e. the Gibbs measure \( d\tilde{\rho}_1 \) [14] for the Zakharov system on \( \mathbb{T}^2 \) cannot be realized as a probability measure even with a Wick-ordered \( L^2 \)-cutoff on the Schrödinger component \( u \). Despite this non-normalizability result, in this paper we construct the Gibbs measure \( d\tilde{\rho}_\gamma \) for all \( \gamma < 1 \).

In view of (1.10), we can write the formal expression (1.10) for the Gibbs measure \( d\tilde{\rho}_\gamma \) as

\[
  "d\tilde{\rho}_\gamma(u, w, \partial_t w) = Z^{-1} \exp \left(-\frac{1}{2} \int_{\mathbb{T}^2} |u|^2 w \, dx \right) d(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma})(u, w, \partial_t w) " \]

(1.12)

Since \( u \) in the support of the Gaussian free field \( d\mu \) (on \( \mathbb{T}^2 \)) is not a function, the potential energy in (1.12) is not well defined and thus a proper renormalization is required to give

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9By convention, we endow \( \mathbb{T}^2 \) with the normalized Lebesgue measure \( dx_2 = (2\pi)^{-2} \, dx \).

10This means that \( h_0, \ell_0 \sim \mathcal{N}_k(0, 1) \) and \( \text{Re } g_0, \text{Im } g_0, \text{Re } g_n, \text{Im } g_n, \text{Re } h_n, \text{Im } h_n, \text{Re } \ell_n, \text{Im } \ell_n \sim \mathcal{N}_k(0, \frac{1}{2}) \) for \( n \neq 0 \).

11As for the Gaussian free field \( d\mu \) on the first component \( u \), we mean the complex Gaussian free field. On the other hand, the Gaussian fields \( d\mu_{1-\gamma}, d\mu_{-\gamma} \) on the second and third component \( w, \partial_t w \) mean the real Gaussian fields.

12Hereafter, we simply use \( Z, Z_N \), etc. to denote various normalization constants.

13A typical element \( u \) in the support of \( d\mu \) does not belong to \( L^2(\mathbb{T}^2) \).
a meaning to (1.12). In order to explain the renormalization process, we first study the regularized model. Given \( N \in \mathbb{N} \), we define the (spatial) frequency projector \( \pi_N \) by
\[
\pi_N f = \sum_{|n| \leq N} \hat{f}(n)e^{i(n.x)}.
\]
Let \( u \) be as in (1.11) and set \( u_N = \pi_N u \). Note that, for each fixed \( x \in \mathbb{T}^3 \), \( u_N(x) \) is a mean-zero real-valued Gaussian random variable with variance
\[
\sigma_N = \mathbb{E}[|u_N(x)|^2] = \sum_{|n| \leq N} \frac{1}{(n)^2} \sim \log N \to \infty,
\]
as \( N \to \infty \). We then define the Wick power :\( |u_N|^2 : \) by
\[
:|u_N|^2 : = |u_N|^2 - \sigma_N. \tag{1.14}
\]
We point out that the Wick renormalization (1.14) removes certain singularities (i.e. subtract a divergent contribution; see [10] [40]). This suggests us to consider the renormalized potential energy:
\[
Q_N(u, w) = \frac{1}{2} \int_{\mathbb{T}^2} :|u_N|^2 : w \, dx \tag{1.15}
\]
where \( u_N = \pi_N u \) as in Subsection 2.1. Thanks to the presence of \( \sigma_N \) in (1.14), we can show that \( Q_N \) converges to some limit \( Q \) in \( L^p(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}) \) if \( \gamma < 1 \). We now define the truncated renormalized Gibbs measure \( d\tilde{\rho}_{\gamma,N} \), endowed with a Wick-ordered \( L^2 \)-cutoff, by
\[
d\tilde{\rho}_{\gamma,N} = Z_N^{-1} 1_{\{|\int_{\mathbb{T}^2} |u_N|^2 \, dx| \leq K\}} e^{-Q_N(u,w)} d(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma})(u, w, \partial w). \tag{1.16}
\]
As we have already pointed out, it is well known that a typical function \( u \) in the support of \( d\mu \) (on \( \mathbb{T}^2 \)) does not belong to \( L^2(\mathbb{T}^2) \) and so we put a Wick-ordered \( L^2 \)-cutoff instead of considering a \( L^2 \)-cutoff.

In [13], Oh, Tolomeo, and the author proved the non-construction of the Gibbs measure in the strongly nonlinear coupling case (\( \gamma = 1 \)) i.e. the Gibbs measure \( d\tilde{\rho}_1 \) (even with proper renormalization on the potential energy \( \int_{\mathbb{T}^2} |u|^2 w \) and on the \( L^2 \)-cutoff) can not be defined as a probability measure. In particular, we proved
\[
\sup_{N \in \mathbb{N}} \mathbb{E}_{\mu \otimes \mu_0 \otimes \mu_{-1}} \left[ 1_{\{|\int_{\mathbb{T}^2} |u_N|^2 \, dx| \leq K\}} e^{-Q_N(u,w)} \right] = \infty. \tag{1.17}
\]
We, however, notice that once \( \gamma < 1 \) (i.e. the weakly nonlinear coupling case), we obtain the following uniform exponential integrability of the density, which allows us to construct the limiting Gibbs measure \( d\tilde{\rho}_\gamma \). We now state our first main result.

**Theorem 1.5** (Normalizability of the Gibbs measure). Let \( 0 \leq \gamma < 1 \). Then, given any finite \( p \geq 1 \), \( Q_N \) in (1.15) converges to some limit \( Q \) in \( L^p(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}) \). Moreover, there exists \( C_p > 0 \) such that
\[
\sup_{N \in \mathbb{N}} \left\| 1_{\{|\int_{\mathbb{T}^2} |u_N|^2 \, dx| \leq K\}} e^{-Q_N(u,w)} \right\|_{L^p(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma})} \leq C_p < \infty. \tag{1.18}
\]
In particular, we have
\[
\lim_{N \to \infty} 1_{\{|\int_{\mathbb{T}^2} |u_N|^2 \, dx| \leq K\}} e^{-Q_N(u,w)} = 1_{\{|\int_{\mathbb{T}^2} |u|^2 \, dx| \leq K\}} e^{-Q(u,w)} \quad \text{in } L^p(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}). \tag{1.19}
\]
As a consequence, the truncated renormalized Gibbs measure \( d\tilde{\rho}_{\gamma,N} \) defined in (1.16) converges, in the sense of (1.19), to the Gibbs measure \( d\tilde{\rho}_{\gamma} \) given by
\[
d\tilde{\rho}_{\gamma}(u, w, \partial_t w) = Z^{-1}\mathbf{1}_{\{|f_{j\ell}|: |u|^2dx|\leq K\}} e^{-Q(u,w)}d(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma})(u, w, \partial_t w).
\]
Furthermore, the resulting Gibbs measure \( d\tilde{\rho}_{\gamma} \) is absolutely continuous with respect to the Gaussian field \( d(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}) \).

Theorem 1.5 shows that a phase transition occurs: (i) the (focusing) Gibbs measure is not constructible as a probability measure for \( \gamma = 1 \) (see (1.17)) and (ii) the (focusing) Gibbs measure is constructible as a probability measure for \( \gamma < 1 \).

The main task in proving Theorem 1.5 is to show the uniform exponential integrability (1.18). We establish the bound (1.18) by applying the variational formulation of the partition function introduced by Barashkov and Gubinelli [1] in the construction of the \( \Phi_t^2 \)-measure. See also [29, 42, 39, 43, 22, 40]. We point out that the partition function \( Z_N \) comes from the expectation with respect to the product measure \( \mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma} \) i.e.
\[
Z_N = \mathbb{E}_{\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}} \left[ 1_{\{|f_{j\ell}|: |u|^2dx|\leq K\}} e^{-Q_N(u,w)} \right].
\]
Hence, it requires more careful analysis than dealing with one component measure in the variational formulation. We, however, notice that random field \( u \) under \( d\mu \) and random field \( w \) under \( d\mu_{1-\gamma} \) are independent and so we exploit some cancellation from the independence. Once the uniform bound (1.18) is established, the \( L^p \)-convergence (1.19) of the densities follows from (softer) convergence in measure of the densities. See [63, Remark 3.8].

**Remark 1.6.** The phase transition from (1.17) and Theorem 1.5 shows that the two-dimensional Zakharov system \( (\gamma = 1) \) is critical in terms of the Gibbs measure construction.

### 1.3. Renormalized Zakharov-Yukawa system

In this subsection, we now consider the following Zakharov-Yukawa system on \( \mathbb{T}^2 \) associated with the (renormalized) Hamiltonian
\[
\begin{aligned}
&i\partial_t u + \Delta u = uw \\
&\partial^2_t w + (1 - \Delta) w = -\langle \nabla \rangle^{2\gamma}(|u|^2 - f)|u|^2
\end{aligned}
\]
where \( f(x)dx := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x)dx \) denotes integration with respect to the normalized Lebesgue measure \( 2\pi)^{-2}dx \) on \( \mathbb{T}^2 \) and the initial data \((w^0, w_0^\gamma, w_1^\gamma)\) is distributed according to the Gibbs measure \( d\tilde{\rho}_{\gamma} \) (1.20). In view of the absolute continuity of \( d\tilde{\rho}_{\gamma} \) with respect to the Gaussian field \( d(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}) \) (Theorem 1.5), we consider the random initial data \((w^0, w_0^\gamma, w_1^\gamma)\) distributed according to \( d(\mu \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}) \) in the following discussion.

The important point is that the renormalization removes a certain singular component from the nonlinearity \( |u|^2 \) (see (1.22) below). This allows us to study well-posedness of the renormalized the Zakharov-Yukawa system (1.21) on the support of the Gibbs measure \( d\tilde{\rho}_{\gamma} \) in (1.20). In [31, 30], the Zakharov system was shown to be locally and globally well-posed in \( H^s(\mathbb{T}^2) \times H^s(\mathbb{T}^2) \times H_{-1}(\mathbb{T}^2) \) for some \( s \geq 0 \) and \( \ell \geq 0 \). It turns out, however, that Gibbs measures \( d\tilde{\rho}_{\gamma} \) in (1.20) are supported on \( H^{-\varepsilon}(\mathbb{T}^2) \times H^{-\gamma - \varepsilon}(\mathbb{T}^2) \times H_{-\gamma-\varepsilon}(\mathbb{T}^2) \) i.e. negative Sobolev spaces, which is beyond the scope of the known deterministic well-posedness results in [31, 30]. For this reason, the main part of our analysis is devoted to the probabilistic construction of local-in-time and global-in-time solutions to (1.21) on the (low regularity) support of the Gibbs measure \( d\tilde{\rho}_{\gamma} \).
Next, we interpret the nonlinearity $N(u)$:

$$N(u) = |u|^2 - \int |u|^2.$$  

Namely, define bilinear operator $N(u_1, u_2)$ by setting

$$N(u_1, u_2)(x, t) := \sum_{n \in \mathbb{Z}^2} \sum_{n_1 - n_2 = n, n_1 \neq n_2} \hat{u}_1(n_1, t) \overline{u}_2(n_2, t) e^{i(n_1 - n_2)x}$$ (1.22)

When all the arguments coincide, we simply write $N(u) = N(u, u)$. Notice that the most problematic interactions (the high-high interactions) are removed in the renormalized nonlinearity $N(u)$. Note that this renormalization of the nonlinearity in (1.21) comes from the Euclidean quantum field theory (see, for example, [60]).

This formulation first appeared in the work of Bourgain [10] for studying the invariant Gibbs measure for the defocusing cubic NLS on $\mathbb{T}^2$. See [17, 45, 26, 46] for more discussion in the context of the (usual) nonlinear Schrödinger equations.

**Remark 1.7.** We briefly look into the relation between the renormalized Zakharov-Yukawa system

$$\begin{cases}
i \partial_t u + \Delta u = uw \\
c^{-2} \partial_t^2 w + (1 - \Delta)w = - (\nabla)^{2\gamma} (|u|^2 - \int |u|^2)
\end{cases}$$ (1.23)

and the renormalized focusing Hartree NLS

$$i \partial_t u + \Delta u = - [(|u|^2 - \int |u|^2) * V] u$$

where $V$ is a convolution potential with $\tilde{V}(n) = \langle n \rangle^{-2 + 2\gamma}$. Notice that by sending the wave speed $c$ in (1.23) to $\infty$, the Zakharov-Yukawa system (Zakharov system) converges, at a formal level, to the renormalized focusing Hartree NLS (cubic NLS, respectively). As for the Zakharov system ($\gamma = 1$), see, for example, [50, 33] for rigorous convergence results on $\mathbb{R}^d$. We also refer to [54, 55] for a detailed explanation of how Gibbs measures, specifically in the context of focusing, can be microscopically derived from the perspective of many-body quantum mechanics.

**1.4. Invariant dynamics for the Zakharov-Yukawa system.** In this subsection, we establish global-in-time flow on the support of the Gibbs measure $d\tilde{\rho}_\gamma$ [33] and its invariance. The main difficulty in studying these problems, even locally in time, comes from the roughness of the support of the Gibbs measure. We first present $d\tilde{\rho}_\gamma$-almost sure local well-posedness result.

**Theorem 1.8 (Almost sure local well-posedness).** Let $0 \leq \gamma < \frac{1}{3}$. Then, the renormalized Zakharov-Yukawa system (1.21) on $\mathbb{T}^2$ is $d\tilde{\rho}_\gamma$-almost surely locally well-posed. More precisely, for any $\varepsilon > 0$, there exists a set $\Sigma \subset H^{-\varepsilon}(\mathbb{T}^2) \times H^{-\gamma-\varepsilon}(\mathbb{T}^2) \times H^{-\gamma-1-\varepsilon}(\mathbb{T}^2)$ of full $d\tilde{\rho}_\gamma$-measure such that for any $(\omega_0, \omega_0', \omega_0'' \in \Sigma$, there exists $\delta > 0$ and a solution to the Cauchy

\[\text{To be precise, it is an equivalent formulation to the Wick renormalization in handling rough Gaussian initial data.}\]
problem for (1.21) on $[-\delta, \delta]$ with data $(u_0^\omega, w_0^\omega, w_1^\omega)$, unique in the class

$$z^S,\omega + X^{s,b}_S(\delta) \subset C\left([-\delta, \delta]; H^{-\varepsilon}(T^2)\right)$$

$$z^W,\omega + X^{\ell,b}_W(\delta) \subset C\left([-\delta, \delta]; H^{-\gamma-\varepsilon}(T^2)\right) \cap C^1\left([-\delta, \delta]; H^{-\gamma-1-\varepsilon}(T^2)\right)$$

for some $b > \frac{1}{2}$ and

$$s - 1 < \ell < 1 - 2\gamma$$

with $0 < s < \frac{1}{4} - \frac{\gamma}{2}$ and $\ell > 0$, where $X^{s,b}_S(\delta)$, $X^{\ell,b}_W(\delta)$, $z^S,\omega$, and $z^W,\omega$ are defined in Subsection 2.2 and 2.1, respectively.

The proof of Theorem 1.8 is based on the first order expansion (McKean [34], Bourgain [10], Da Prato-Debussche [18, 19] type argument), which exploits the propagation of randomness under the corresponding linear flow. This method first decompose the solution by writing

$$u = \text{random linear term} + \text{smoother term} \quad (1.24)$$

i.e. the decomposition of solution as the sum of the random linear evolution terms plus a smoother remainder. Gaussian initial data in the first term of (1.21) are propagated linearly, which preserves all the independence properties of the initial data $u^\omega(0), w^\omega(0)$ in (1.21) and in the second term of (1.24) the remainder term is treated as a perturbation. The main idea is to use the propagation of the randomness in such a way that cancellations from the randomness happen and solve a fixed point problem for the remainder term (see Subsection 4.2 for more explanations about Bourgain and Da Prato-Debussche trick).

To exploit better the independene structure of the random linear term involved, in [10] Bourgain used the $TT^*$-argument with the Hilbert-Schmidt norm of the random matrices (kernel of $TT^*$ operator). In Appendix A the estimates of random matrices are based on the Hilbert-Schmidt norm with the Wiener chaos estimate (Lemma 2.11). In order to attain a stronger coupling region, we need to go beyond the Bourgain’s argument [10]. We point out that a certain room exists between the operator norm and the Hilbert-Schmidt norm as follows:

$$\|T\|_{OP}^2 = \|TT^*\|_{OP} \leq \|TT^*\|_{HS}. \quad (1.25)$$

Therefore, instead of using the Hilbert-Schmidt norm of the kernel matrix, we use the operator norm approach based on the random tensor theory. This method rely on higher order versions of Bourgain’s $TT^*$ argument introduced by Deng, Nahmod, and Yue [21]

$$\|T\|_{OP}^{2m} = \|(TT^*)^m\|_{OP},$$

which makes us exploit better the independent structure. More precisely, the essential difference between our analysis and that in [10] is that the Hilbert-Schmidt norm of kernel matrices and the Wiener chaos estimate will be replaced by the operator norm bound (Lemma 5.2) coming from the random tensor theory in [21] and the counting estimate will be also replaced by deterministic tensor estimates (see Subsection 5.2), which allows us to reach a

\[\text{In the field of stochastic PDEs, a well-posedness argument based on the decomposition (1.24) is usually referred to as the Da Prato-Debussche trick [18, 19], where the random linear solution is replaced by the solution to a linear stochastic PDE. It is worthwhile to point out that the paper [34, 10] by McKean and Bourgain precede [18, 19].}\]

\[\text{This is no longer satisfied for nonlinear solutions } u(t), w(t) \text{ as soon as } t > 0.\]
stronger coupling region (see also Remark 1.11). In [21], Deng, Nahmod, and Yue recently developed a theory of random tensors, which forms a comprehensive framework for random dispersive equations. In recent years, the random tensor theory has played a crucial role in the well-posedness study of random dispersive equations; see [21, 11, 49].

Notice that the low regularity nature of solutions in Theorem 1.5 comes from the random linear terms. We, however, point out that the random linear solutions also enjoy enhanced integrability than the smoother remainder terms thanks to the independent structure involved.

**Remark 1.9.** The decomposition (1.24) states that in the high frequency regime (i.e. at small spatial scales on the physical side), the dynamics is essentially governed by that of the random linear solution.

**Remark 1.10.** The extension of Theorem 1.8 to $\gamma < 1$ would require more sophisticated arguments. We mention recent breakthrough works (random averaging operators/random tensor) by Deng, Nahmod, and Yue [20, 21].

**Remark 1.11.** We point out that it is essential to use the operator norm approach with the random matrix/tensor theory in proving Lemma 4.6 (in particular, Subcase 2.b.(ii)), Lemma 4.7 (in particular, Subcase 2.b), and Lemma 4.8 (in particular, Subcase 2.b), where the cases cannot be proven by only using the Hilbert-Schmidt norm approach even when $\gamma = 0$ (see also Lemma 4.6’s Subcase 2.b.(i) and Appendix A to compare the methods).

**Remark 1.12.** In [10, 52, 35, 28, 39, 40], the (random) operators with kernel (random) matrices appear and the Hilbert-Schmidt norm approach was used in dealing with them as in (1.25); see also Appendix A. Hence, one can expect some improvements by using the operator norm with the random tensor theory as in this paper.

In constructing almost sure global-in-time dynamics, we exploit Bourgain’s invariant measure argument [7, 10, 15, 48] to our setting. More precisely, we use invariance of the Gibbs measure under the finite-dimensional approximation of (1.21) to obtain a uniform control on the solutions, and then apply a PDE approximation argument to extend the local solutions to (1.21) obtained from Theorem 1.8 to global ones. As a consequence, we also obtain invariance of the Gibbs measure $d\bar{\rho}_\gamma$ under the global flow of the renormalized Zakharov-Yukawa system (1.21).

**Theorem 1.13** (Almost sure global well-posedness and invariance of the Gibbs measure). Let $0 \leq \gamma < \frac{1}{3}$. Then, the Cauchy problem for (1.21) is $d\bar{\rho}_\gamma$-almost surely globally well-posed, and the Gibbs measure $d\bar{\rho}_\gamma$ is invariant under the flow. More precisely, if $\Sigma$ is as in Theorem 1.8 and if $\bar{\Phi}(t)$ denotes the flow map of (1.21) on $\Sigma$, then $d\bar{\rho}_\gamma$ is invariant under $\bar{\Phi}(t)$ in the sense that for any $d\rho_\gamma$-measurable $A \subset \Sigma$, it holds

$$\bar{\rho}_\gamma(\Phi(t)(A)) = \bar{\rho}_\gamma(A)$$

for any $t \in \mathbb{R}$.

The proof of Theorem 1.13 follows from a standard application of Bourgain’s invariant measure argument [7, 10, 15, 48], which is presented in Section 6.
1.5. **Organization of the paper.** In Section 2 we introduce some notations and preliminary (deterministic and probabilistic) lemmas. In Section 3 we present the variational formulation of the partition function and prove the phase transition (Theorem 1.5) by showing the uniform exponential integrability (1.18). In Section 4 we discuss probabilistic well-posedness with the random tensor theory. In Section 5 we provide the basic definition and some lemmas on random tensors and present the proof of random tensor estimates. In Section 6 we prove the almost sure global well-posedness and invariance of the Gibbs measure by exploiting the Bourgain’s invariant measure argument. In Appendix A we implement the Hilbert-Schmidt norm approach with the Wiener chaos estimate and compare its result with when using the operator norm bound with the random tensor theory.

2. **Notations and preliminary lemmas**

In this section, we recall and prove basic lemmas to be used in this paper.

2.1. **Notations.** If a function $f$ is random, we may use the superscript $f^\omega$ to show the dependence on $\omega \in \Omega$. Let $\eta \in C^\infty_c(\mathbb{R})$ be a smooth non-negative cutoff function supported on $[-2, 2]$ with $\eta \equiv 1$ on $[-1, 1]$ and set

$$\eta_\delta(t) = \eta(\delta^{-1}t)$$

(2.1)

for $\delta > 0$. We also denote by $\chi(t)$ another smooth non-negative cutoff function and let $\chi_\delta(t) = \chi(\delta^{-1}t)$.

Let $\mathbb{Z}_{\geq 0} := \mathbb{Z} \cap [0, \infty)$. Given a dyadic number $N \in 2\mathbb{Z}_{\geq 0}$, let $P_N$ be the (non-homogeneous) Littlewood-Paley projector onto the (spatial) frequencies $\{n \in \mathbb{Z} : |n| \sim N\}$ such that

$$f = \sum_{N \geq 1}^{\infty} P_N f.$$ 

We also denote by $P_Q$ the Fourier projector onto $Q$ where $Q$ is a spatial frequency ball of radius $N$ (not necessarily centered at the origin). Given a non-negative integer $N \in \mathbb{Z}_{\geq 0}$, we also define the Dirichlet projector $\pi_N$ onto the frequencies $\{|n| \leq N\}$ by setting

$$\pi_N f(x) = \sum_{|n| \leq N} \hat{f}(n) e^{inx}.$$ 

(2.2)

Moreover, we set

$$\pi_N = \text{Id} - \pi_N.$$ 

By convention, we also set $\pi_{-1} = \text{Id}$. By abuse of the notation, we also use the notation $P_N$ to denote the operator on functions in $(t, x)$. Also, define the operators $Q^S_L, Q^W_L$ on spacetime functions by

$$F_{t,x}(Q^S_L u)(\tau, k) := \eta_L(\tau + |k|^2) F_{t,x} u(\tau, k), \quad F_{t,x}(Q^W_L w)(\tau, k) := \eta_L(\tau \pm |k|) F_{t,x} w(\tau, k)$$

for dyadic numbers $L \geq 1$. We will write $P^S_{N,L} = P_N Q^S_L$, $P^W_{N,L} = P_N Q^W_L$ for brevity. In what follows, capital letters $N$ and $L$ are always used to denote dyadic numbers $\geq 1$. We will often use these capital letters with various subscripts, and also the notation

$$N_{\text{max}} := \max\{N_1, N_2, N\} \quad \text{and} \quad L_{\text{max}} := \max\{L_1, L_2, L\}.$$
We denote by $z^{S, \omega}$ and $z^{W, \omega}$ the linear solutions of Zakharov-Yukawa system with initial data $(u_0^\omega, w_0^\omega, w_1^\omega)$ as follows:

$$z^{S, \omega} := e^{it \Delta} u_0^\omega$$

$$z^{W, \omega} := \cos(t \langle \nabla \rangle) w_0^\omega + \frac{\sin(t \langle \nabla \rangle)}{\langle \nabla \rangle} w_1^\omega.$$

We use $c, C$ to denote various constants, usually depending only on $\alpha$ and $s$. If a constant depends on other quantities, we will make it explicit. For two quantities $A$ and $B$, we use $A \lesssim B$ to denote an estimate of the form $A \leq CB$, where $C$ is a universal constant, independent of particular realization of $A$ or $B$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$. The notation $A \ll B$ means $A \leq cB$ for some sufficiently small constant $c$. We also use the notation $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$, respectively) for arbitrarily small $\varepsilon > 0$ (this notation is often used when there is an implicit constant which diverges in the limit $\varepsilon \to 0$).

2.2. Fourier restriction norm method. In this subsection, we introduce the following Bourgain spaces for the Schrödinger and the wave equation.

**Definition 2.1** (Bourgain spaces). For $s, b \in \mathbb{R}$, define the Bourgain space for the Schrödinger equation $X^{s, b}_S$ and that for reduced wave equations $X^{s, b}_W$ by the completion of functions $C^\infty$ in space and Schwartz in time with respect to

$$\|u\|_{X^{s, b}_S(\mathbb{R} \times \mathbb{T}^2)} := \|\langle n \rangle^s (\tau - |n|^2)^b \hat{u}(\tau, n)\|_{L^2_{\tau} L^2_{n} (\mathbb{Z}^2 \times \mathbb{R})},$$

$$\|w\|_{X^{s, b}_W(\mathbb{R} \times \mathbb{T}^2)} := \|\langle n \rangle^\ell (\tau + \langle n \rangle)^b \hat{w}(\tau, n)\|_{L^2_{\tau} L^2_{n} (\mathbb{Z}^2 \times \mathbb{R}^2)}.$$

Here $\pm$ corresponds to the norm of $w_\pm$ in the system (1.2). We also define the Bourgain space for the wave equation $X^{s, b, p}_W$ by setting

$$\|w\|_{X^{s, b, p}_W} := \|\langle n \rangle^\ell (|\tau| - \langle n \rangle)^b \hat{w}(\tau, n)\|_{L^2_{\tau} L^2_{n}}$$

i.e. replacing $W_\pm$ with $W$ in the above definition of $X^{s, b}_W$. For $\delta > 0$, define the restricted space $X^{s, b}_W(\delta)$ ($\ast = S$ or $W_\pm$ or $W$) by the restrictions of distributions in $X^{s, b}_W$ to $(-\delta, \delta) \times \mathbb{T}^2$, with the norm

$$\|u\|_{X^{s, b}_W(\delta)} := \inf \{ \|v\|_{X^{s, b}_W} : v|_{(-\delta, \delta)} = u \}. \quad (2.3)$$

We note that for any $b > \frac{1}{2}$, we have $X^{s, b}_W \hookrightarrow C(\mathbb{R}; H^s(\mathbb{T}^2))$. Next, we recall the linear estimates. See [6, 25].

**Lemma 2.2.** Let $s \in \mathbb{R}$, $\ell \in \mathbb{R}$ and $0 < \delta \leq 1$.

(i) For any $b \in \mathbb{R}$, we have

$$\|e^{it \Delta} u_0\|_{X^{s, b}_S(\delta)} \leq C_b \|u_0\|_{H^s},$$

$$\|e^{it \langle \nabla \rangle} w_0\|_{X^{\ell, b}_W(\delta)} \leq C_b \|w_0\|_{H^\ell}.$$
(ii) Let $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$. Then, we have
\[
\left\| \int_0^t e^{i(t-t')\Delta} F(t') dt' \right\|_{X^{s,b}_X(\delta)} \leq C_{b,b'} \delta^{1-b+b'} \|F\|_{X^{s,b'}_X(\delta)}
\]
\[
\left\| \int_0^t e^{i(t-t')\langle\nabla\rangle} F(t') dt' \right\|_{X^{s,b'}_{W_\pm}(\delta)} \leq C_{b,b'} \delta^{1-b+b'} \|F\|_{X^{s,b'}_{W_\pm}(\delta)}
\]

By restricting the $X^{s,b}$-spaces onto a small time interval $(-\delta, \delta)$, we can gain a small power of $\delta$ (at a slight loss in the modulation).

**Lemma 2.3.** Let $s \in \mathbb{R}$ and $b < \frac{1}{2}$. Then, there exists $C = C(b) > 0$ such that
\[
\|\eta_\delta(t) \cdot u\|_{X^{s,b}} + \|\chi_\delta(t) \cdot u\|_{X^{s,b}} \leq C \delta^{\frac{1}{2} - b} \|u\|_{X^{s,b}}.
\]

For the proof of Lemma 2.3, see [17].

### 2.3. Product estimates

In this subsection, we present product estimates. We first recall the following interpolation and fractional Leibniz rule. As for the second estimate (2.4) below, see [27, Lemma 3.4].

**Lemma 2.4.** The following estimates hold.

(i) (interpolation) For $0 < s_1 < s_2$, we have
\[
\|u\|_{H^{s_1}} \lesssim \|u\|_{L^{p_j}_{2}}^{\frac{s_1}{s_2-p_j}} \|u\|_{L^{2}_{2}}^{\frac{s_2-s_1}{s_2-p_j}}.
\]

(ii) (fractional Leibniz rule) Let $0 \leq s \leq 1$. Suppose that $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. Then, we have
\[
\|\langle\nabla\rangle^s (fg)\|_{L^r(T^2)} \lesssim \left( \|f\|_{L^{p_1}(T^2)} \|\langle\nabla\rangle^s g\|_{L^{q_1}(T^2)} + \|\langle\nabla\rangle^s f\|_{L^{p_2}(T^2)} \|g\|_{L^{q_2}(T^2)} \right),
\]
where $\langle\nabla\rangle = \sqrt{1 - \Delta}$.

### 2.4. Hilbert-Schmidt norm

In this subsection, we present the Hilbert-Schmidt norm estimate of (random) matrices (kernel of $T^*T$ operator). To exploit better the independence structure of the random variable involved, we use the following lemma.

**Lemma 2.5** (Hilbert-Schmidt norm). The matrix $\mathfrak{G}$ is given by
\[
\mathfrak{G}\{b_{n_1}\} = \sum_{n_1 \in \mathbb{Z}^d} \sigma(n, n_1) b_{n_1}
\]
for any $\{b_{n_1}\} \in \ell^2$. Then, we have
\[
\|\mathfrak{G}\|_{\ell^2 \to \ell^2} = \|\mathfrak{G}^* \mathfrak{G}\|_{\ell^2 \to \ell^2} \lesssim \max_n \sum_{n_1} |\sigma(n, n_1)|^2 + \left( \sum_{n, n' : n \neq n'} \sum_{n_1} |\sigma(n_1, n') \sigma(n_1, n)|^2 \right)^{\frac{1}{2}}
\]

**Proof.** From (2.5), $\mathfrak{G}^*$ is given by
\[
\mathfrak{G}^*\{b_{n_1}\} = \sum_{n_1} \sigma(n_1, n) b_{n_1}
\]
and so
\[ G^* G \{ b_{n'} \} = \sum_{n'} \left( \sum_{n_1} \sigma(n_1, n') \sigma(n_1, n) \right) b_{n'} . \]  
(2.6)

We split (2.6) into two cases as follows
\[ G^* G \{ b_{n'} \} = \sum_{n_1} |\sigma(n_1, n)|^2 b_n + \sum_{n', n' \neq n} \left( \sum_{n_1} \sigma(n_1, n') \sigma(n_1, n) \right) b_{n'} \]  
(2.7)

Hence, from (2.7) and the Cauchy-Schwarz inequality in \( n' \), we have
\[ \| G^* G \{ b_{n'} \} \|_{L^2} \lesssim \left( \max_n \sum_{n_1} |\sigma(n_1, n)|^2 + \left( \sum_{n, n', n' \neq n} \left| \sum_{n_1} \sigma(n_1, n') \sigma(n_1, n) \right|^2 \right)^{1/2} \right) \| b_n \|_{L^2} . \]
This completes the proof of Lemma 2.5. \( \square \)

2.5. On discrete convolutions. Next, we recall the following basic lemma on a discrete convolution.

**Lemma 2.6.** (i) Let \( d \geq 1 \) and \( \alpha, \beta \in \mathbb{R} \) satisfy
\[ \alpha + \beta > d \quad \text{and} \quad \alpha, \beta < d . \]
Then, we have
\[ \sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^{\alpha} \langle n_2 \rangle^{\beta}} \lesssim \langle n \rangle^{d-\alpha-\beta} \]
for any \( n \in \mathbb{Z}^d \).

(ii) Let \( d \geq 1 \) and \( \alpha, \beta \in \mathbb{R} \) satisfy \( \alpha + \beta > d \). Then, we have
\[ \sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^{\alpha} \langle n_2 \rangle^{\beta}} \lesssim \langle n \rangle^{d-\alpha-\beta} \]
for any \( n \in \mathbb{Z}^d \).

Namely, in the resonant case (ii), we do not have the restriction \( \alpha, \beta < d \). Lemma 2.6 follows from elementary computations. See, for example, Lemmas 4.1 and 4.2 in [35] for the proof.

2.6. Strichartz estimates on \( T^2 \). In this subsection, we record the \( L^4_{t,x} \)-Strichartz estimates on \( T^2 \). We first recall (see [31])
\[ \left\| \sum_{n \in Q} a_n e^{i \langle n, x \rangle + |n|^2 t} \right\|_{L^4_{t,x}([0,1] \times T^2)} \lesssim |Q|^\varepsilon \left( \sum_{n \in Q} |a_n|^2 \right)^{1/2} \]  
(2.8)
where $Q$ is a spatial frequency ball of radius $N$ (not necessarily centered at the origin) and $|Q| = \#Q$. Then, from (2.8) and Hölder’s inequality, we have

$$
\|P_Q u\|_{L^4_t([0,1] \times \mathbb{T}^2)} \leq \int_R \left\| \sum_{n \in Q} \hat{u}(\tau + |n|^2, n) e^{i(n,x + |n|^2 t)} \right\|_{L^4_t([0,1] \times \mathbb{T}^2)} d\tau
$$

$$
\lesssim |Q|^\varepsilon \left( \sum_{n \in Q} \int \langle \tau + |n|^2 \rangle^{2b'} |\hat{u}(\tau + |n|^2, n)|^2 \right)^{\frac{1}{2}} \lesssim N^\varepsilon \|u\|_{X^{s,b}_S}
$$

(2.9)

for any $b > \frac{1}{4}$, where $P_Q$ is the Fourier projector onto $Q$. In the following lemma, we improve the $L^4_t$-Strichartz estimates (2.9) by using the Hausdorff-Young inequality and an interpolation.

**Lemma 2.7.** Let $\varepsilon > 0$. Then, we have

$$
\|P_Q u\|_{L^4_t([0,1] \times \mathbb{T}^2)} \lesssim \varepsilon N^{\varepsilon} \|P_Q u\|_{X^{0,1}_{S}} \tag{2.10}
$$

where $P_Q$ is the Fourier projector onto $Q$ and $Q$ is a spatial frequency ball of radius $N$ (not necessarily centered at the origin).

**Proof.** From the Hausdorff-Young inequality, we have

$$
\|P_Q u\|_{L^4_t} \leq \left( \sum_{n \in Q} \int \hat{u}(\tau, n) \frac{d\tau}{|n|^3} \right)^{\frac{4}{3}}
$$

$$
\leq \left( \sum_{n \in Q} \left( \int \langle \tau + |n|^2 \rangle^{2b'} |\hat{u}(\tau, n)|^2 \right)^{\frac{1}{2}} \right)^{\frac{3}{2}} \lesssim N^{\frac{1}{2}} \|P_Q u\|_{X^{s,b}_S}.
$$

(2.10)

where $N$ is the size of $Q$ and $b' > \frac{1}{4}$. By interpolating (2.9) and (2.10), we obtain the desired result.

\[ \square \]

For further use in the following sections, we also record another estimate (2.11). By interpolating the following two estimates coming from (2.9) and the definition of $X^{s,b}_S$

$$
\|P_Q u\|_{L^4_t([0,1] \times \mathbb{T}^2)} \lesssim \varepsilon N^\varepsilon \|P_Q u\|_{X^{s,b}_{\varepsilon}}
$$

$$
\|P_Q u\|_{L^4_t([0,1] \times \mathbb{T}^2)} \lesssim \|P_Q u\|_{X^{s,b}_S}.
$$

we have

$$
\|P_Q u\|_{L^2_t([0,1] \times \mathbb{T}^2)} \lesssim N^{0+} \|P_Q u\|_{X^{s,0+}_S}.
$$

(2.11)
2.7. Counting estimates for lattice points and a key multilinear estimate. In this subsection, we first record the following counting estimates.

**Lemma 2.8** (high-high interactions and low-modulation). Let

\[ 1 \ll N \lesssim N_1 \sim N_2 \quad \text{and} \quad M \ll N_1. \]

Then, for any fixed \( n \in \mathbb{Z}^2 \) with \( |n| \sim N \), we have

\[
\#\{ n_1 \in \mathbb{Z}^2 : |n_1|^2 \pm |n| - |n_2|^2 = O(M), \ n_1 + n_2 = n, \ |n_1| \sim N_1, \ \text{and} \ |n_2| \sim N_2 \}
\]

\[ \lesssim \left( \frac{M}{N} + 1 \right) N_1 \]

**Proof.** Note that \( |n_1|^2 \pm |n| - |n_2|^2 = O(M) \) and \( n_1 + n_2 = n \) imply

\[ -|n|^2 + 2\langle n, n_1 \rangle \pm |n| = O(M). \]

Hence, from \( |n| \sim N \), we have

\[ \frac{n}{|n|} \cdot n_1 = \frac{|n|}{2} \mp \frac{1}{2} + O\left( \frac{M}{N} \right) \]

i.e. the component of \( n_1 \) parallel to \( n \) is restricted in an interval of length \( O\left( \frac{M}{N} \right) \). Hence, we have

\[
\#\{ n_1 \in \mathbb{Z}^2 : |n_1|^2 \pm |n| - |n_2|^2 = O(M), \ n_1 + n_2 = n, \ |n_1| \sim N_1, \ \text{and} \ |n_2| \sim N_2 \}
\]

\[ \lesssim \left( \frac{M}{N} + 1 \right) N_1, \]

which proves the desired result.

Next, we state the following lattice point counting bound that will be used in the proof of multilinear estimates in Subsection 4.4 and 4.5. For the proof, see Lemma 4.3 in [20].

**Lemma 2.9.** Given \( 0 \neq m \in \mathbb{Z}[i], a_0, b_0 \in \mathbb{C}, \) and \( M, N > 0 \), the number of tuples \( (a, b) \in \mathbb{Z}[i]^2 \) that satisfies

\[ ab = m, \ |a - a_0| \leq M, \ |b - b_0| \leq N \]

is \( O(M^\varepsilon N^\varepsilon) \) for any small \( \varepsilon > 0 \), where the constant depends only on \( \varepsilon > 0 \).

To introduce Lemma 2.10, we define several dyadic frequency regions:

\[ \mathcal{P}_1 := \{(\tau, k) : |k| \leq 2\}, \quad \mathcal{P}_N := \{(\tau, k) : \frac{N}{2} \leq |k| \leq 2N\}, \quad N \geq 2, \]

\[ \mathcal{G}_1 := \{(\tau, k) : |\tau + |k|^2| \leq 2\}, \quad \mathcal{G}_L := \{(\tau, k) : \frac{L}{2} \leq |\tau + |k|^2| \leq 2L\}, \quad L \geq 2, \]

\[ \mathcal{M}_1^\pm := \{(\tau, k) : |\tau \pm |k|| \leq 2\}, \quad \mathcal{M}_L^\pm := \{(\tau, k) : \frac{L}{2} \leq |\tau \pm |k|| \leq 2L\}, \quad L \geq 2. \]

We now state the following multilinear estimate in [30, Proposition 3.2]. This multilinear estimate will be used in Lemmas 4.4 and 4.5. More precisely, if an interaction is the high-low interactions where one Schrödinger frequency is much greater than the other Schrödinger frequency (i.e. high-modulation cases \( L_{\text{max}} \gtrsim N_{\text{max}}^2 \)), then we can use the following lemma.
Lemma 2.10. Let \( N_j, L_j \geq 1 \) be dyadic numbers and \( f, g_1, g_2 \in L^2(\mathbb{R} \times \mathbb{Z}^2) \) be real-valued nonnegative functions with the support properties \( \text{supp} f \subset \Psi_{N_0} \cap \mathcal{W}^+_{L_0} \), \( \text{supp} g_j \subset \Psi_{N_j} \cap \mathcal{S}_{L_j} \), \( j = 1, 2 \). Moreover, assume \( N_1 \gg N_2 \) or \( N_2 \gg N_1 \). Then, we have

\[
\sum_{n,n_1:n_1+n_2=n} \int_{\tau,\tau_1,\tau_2} f(\tau, n) g_1(\tau_1, n_1) g_2(\tau_2, n_2) \lesssim L_{\text{max}}^{\frac{3}{2}} L_{\text{med}}^{\frac{3}{2}} N_{n_1}^{\frac{1}{2}} N_{n_2}^{\frac{1}{2}} \|f\|_{L_2^2} \|g_1\|_{L_2^2} \|g_2\|_{L_2^2}.
\]

2.8. Tools from stochastic analysis. In this subsection, we present the probabilistic tools. We first recall the Wiener chaos estimate (Lemma 2.11). For this purpose, we first recall basic definitions from stochastic analysis; see \[4, 59\]. Let \((H, B, \mu)\) define a polynomial chaos of order \( k \) in \( \mathcal{S}_B \) space. Namely, \( \mu\) is a Gaussian measure on a separable Banach space \( B \) with \( H \subset B \) as its Cameron-Martin space. Given a complete orthonormal system \( \{e_j\}_{j \in \mathbb{N}} \subset B^* \) of \( H^* = H \), we define a polynomial chaos of order \( k \) to be an element of the form \( \prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j \rangle) \), where \( x \in B \), \( k_j \neq 0 \) for only finitely many \( j \)'s, \( k = \sum_{j=1}^{\infty} k_j \), \( H_{k_j} \) is the Hermite polynomial of degree \( k_j \), and \( \langle \cdot, \cdot \rangle = B\langle \cdot, \cdot \rangle_B^* \) denotes the \( B-B^* \) duality pairing. We then denote the closure of polynomial chaoses of order \( k \) under \( L^2(\mathbb{R}) \) by \( H_k \). The elements in \( H_k \) are called homogeneous Wiener chaoses of order \( k \). We also set

\[
\mathcal{H}_{\leq k} = \bigoplus_{j=0}^{k} \mathcal{H}_j
\]

for \( k \in \mathbb{N} \).

Let \( L = \Delta - x \cdot \nabla \) be the Ornstein-Uhlenbeck operator.\(^{17}\) Then, it is known that any element in \( H_k \) is an eigenfunction of \( L \) with eigenvalue \(-k\). Then, as a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup \( U(t) = e^{tL} \) due to Nelson \[37\], we have the following Wiener chaos estimate \[60\] Theorem I.22].

Lemma 2.11. Let \( k \in \mathbb{N} \). Then, we have

\[
\|X\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}
\]

for any \( p \geq 2 \) and any \( X \in \mathcal{H}_{\leq k} \). As a consequence, the multilinear Gaussian expression

\[
F_k(\omega) := \sum_{n_1, \ldots, n_k} c_{n_1, \ldots, n_k} g_{n_1}(\omega) g_{n_2}(\omega) \cdots g_{n_k}(\omega)
\]

for some \( k \geq 1 \) and \( \{c_{n_1, \ldots, n_k}\} \in l^2(\mathbb{Z}^2)^k \) satisfies

\[
\|F_k\|_{L^p(\Omega)} \leq \sqrt{k+1}(p-1)^{\frac{k}{2}} \|F_k\|_{L^2(\Omega)}
\]

for any \( p \geq 2 \). Moreover, there exists \( c > 0 \) such that for any \( \lambda > 0 \), we have

\[
\mathbb{P}\{|F_k| > \lambda\} \leq \exp\left(-c\lambda^2 \|F_k\|_{L^2(\Omega)}^{-2}\right).
\]

We next present a well known fact (see also for example \[38, 17\]).

\(^{17}\) For simplicity, we write the definition of the Ornstein-Uhlenbeck operator \( L \) when \( B = \mathbb{R}^d \).
Lemma 2.12. Let \( \varepsilon > 0 \) and \( Q \) be a lattice ball of radius \( N \) in \( \mathbb{R}^2 \) (not necessarily centered at the origin). Then, given \( \beta > 0 \), there exists a constant \( c > 0 \) such that we have
\[
\mathbb{P}\left( \max_{n \in Q} |g_n| > \delta^{-\beta}(\#Q)^{\varepsilon} \right) \lesssim N^{0}e^{-\frac{1}{\delta^2}}
\]
for any \( \delta > 0 \).

**Proof.** Note that
\[
\mathbb{P}\left( \max_{n \in Q} |g_n| > \delta^{-\beta}(\#Q)^{\varepsilon} \right) \leq \sum_{n \in Q} \mathbb{P}\left( |g_n| > \delta^{-\beta}(\#Q)^{\varepsilon} \right)
\]
\[
= \sum_{n \in Q} \int_{|g_n| > \delta^{-\beta}(\#Q)^{\varepsilon}} e^{-\frac{|g_n|^2}{2}} dg_n
\]
\[
\lesssim \sum_{n \in Q} e^{-c(\#Q)^{\varepsilon}}
\]
\[
\lesssim N^2 e^{-c(\#Q)^{\varepsilon}} \lesssim N^{0}e^{-\frac{1}{\delta^2}}N^{2C(\#Q)^{\varepsilon}+2}\log N \lesssim N^{0}e^{-\frac{1}{\delta^2}}.
\]
Hence, we obtain the desired result. \( \square \)

In probabilistic well-posedness theory, a probabilistic improvement of Strichartz estimates for random linear solutions plays an important role.

Lemma 2.13. Let
\[
f^{\omega}(t, x) = \sum_{n \in \mathbb{Z}^2} c_n g_n(\omega) e^{i(n,x) - |n|^2 t} \quad \text{or} \quad \sum_{n \in \mathbb{Z}^2} c_n g_n(\omega) e^{i(n,x) + |n|^2 t}.
\]
for \( \{c_n\}_{n \in \mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2) \). Then, for \( 2 \leq p < \infty \), there exists \( \delta_0 > 0 \) and \( c > 0 \) such that
\[
\mathbb{P}\left\{ \|f^{\omega}\|_{L^p([-\delta,\delta] \times \mathbb{T}^2)} > c\|c_n\|_{\ell^2_n} \right\} \leq e^{-\frac{1}{\delta^2}}.
\]
for \( \delta \leq \delta_0 \).

One way to prove Lemma 2.13 would be to directly apply the Wiener chaos estimate 2.11. For the proof, see \cite{17}.

3. The construction of the Gibbs measure for the Zakharov-Yukawa system

In this section, we present the proof of Theorem 1.3. The main step in proving Theorem 1.3 is to prove the following lemma (uniform exponential integrability (1.18)). We establish the bound (1.18) by applying the variational formulation of the partition function by Barashkov-Gubinelli [1].

Lemma 3.1. Let \( 0 \leq \gamma < 1 \). Then, given any finite \( p \geq 1 \), \( Q_N \) in (1.18) converges to some limit \( Q \) in \( L^p(\mu \otimes \mu_{1-\gamma}) \). Moreover, there exists \( C_p > 0 \) such that
\[
\sup_{N \in \mathbb{N}} \left\| 1_{\{ |f_{r_2}| u_N |^2 dx | \leq K \} } e^{-Q_N(u,w)} \right\|_{L^p(\mu \otimes \mu_{1-\gamma})} \leq C_p < \infty.
\]
In particular, we have
\[
\lim_{N \to \infty} 1_{\{ |f_{r_2}| u_N |^2 dx | \leq K \} } e^{-Q_N(u,w)} = 1_{\{ |f_{r_2}| u_N |^2 dx | \leq K \} } e^{-Q(u,w)} \quad \text{in} \, L^p(\mu \otimes \mu_{1-\gamma}).
\]
The convergence of $Q_N$ in (1.15) follows from a standard computation and thus we omit details. See, for example, [46, Lemma 2.5] for the related result. As we pointed out, once the uniform bound (3.1) is established, the $L^p$-convergence (3.2) of the densities follows from (softer) convergence in measure of the densities. See [63, Remark 3.8].

**Remark 3.2.** We notice that the Gibbs measure $d\rho_\gamma$ on $(u, w, \partial_t w)$, formally defined in (1.4), decouples as the Gibbs measure $d\mu_\gamma$ (1.5) on the component $(u, w)$ and the Gaussian measure $d\mu_{-\gamma}$ on the component $\partial_t w$. Therefore, once the Gibbs measure $d\rho_\gamma$ (1.6) on $(u, w)$ is established, we can construct the Gibbs measure $d\rho_\gamma$ on $(u, w, \partial_t w)$ by setting

$$d\rho_\gamma(u, w, \partial_t w) = d(\rho_\gamma \otimes \mu_{-\gamma})(u, w, \partial_t w).$$

Hence, in the following, we only discuss the construction of the (renormalized) Gibbs measure $d\rho_\gamma$ on $(u, w)$, written in (1.6).

### 3.1. Stochastic variational formulation.

We use a variational formula for the partition function as in [1, 39, 43, 40, 56, 58]. The main idea is to write the partition function as a stochastic optimization problem over time-dependent processes.

We begin by representing the measure $\mu_1 \otimes \mu_{1-\gamma}$ as the distribution of a pair of cylindrical processes at the time 1. Let $\tilde{W}(t)$ be a cylindrical Brownian motion in $L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)$. Namely, we have

$$\tilde{W}(t) = (W_1(t), W_2(t)) = \left(\sum_{n \in \mathbb{Z}^2} B_1^n(t) e^{i(n,x)}, \sum_{n \in \mathbb{Z}^2} B_2^n(t) e^{i(n,x)}\right),$$

where $\{B_1^n\}_{n \in \mathbb{Z}^2}$ and $\{B_2^n\}_{n \in \mathbb{Z}^2}$ are two sequences of mutually independent complex-valued Brownian motions such that $B_j^n = B_j^{-n}$, $n \in \mathbb{Z}^2$. Then, we define a centered Gaussian process $\tilde{Y}(t) = (Y_1(t), Y_2(t))$ by

$$\tilde{Y}(t) = \left((\nabla)^{-1}W_1(t), (\nabla)^{-1+\gamma}W_2(t)\right)$$

Note that we have

$$\text{Law}(\tilde{Y}(1)) = \mu_1 \otimes \mu_{1-\gamma},$$

where $d\mu_1$ and $d\mu_{1-\gamma}$ are the (fractional) Gaussian fields in (1.5). By setting $\tilde{Y}_N = \pi_N \tilde{Y}$, we have $\text{Law}(\tilde{Y}_N(1)) = (\pi_N)_* (\mu_1 \otimes \mu_{1-\gamma})$, i.e. the pushforward of $\mu_1 \otimes \mu_{1-\gamma}$ under $\pi_N$. In particular, we have $\mathbb{E}[Y^2_{1,N}(1)] = \sigma_N$, where $\sigma_N$ is as in (1.13).

Next, let $\mathcal{H}_{\mathbb{R}}$ denote the space of drifts, which are progressively measurable processes belonging to $L^2([0, 1]; L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))$, $\mathbb{P}$-almost surely.

We now state the Boué-Dupuis variational formula [5, 64]; in particular, see Theorem 7 in [64].

---

18By convention, we normalize $B_n$ such that $\text{Var}(B_n(t)) = t$. In particular, $B_0$ is a standard real-valued Brownian motion.

19While we keep the discussion only to the real-valued setting, the results also hold in the complex-valued setting. In the complex-valued setting, we use the Laguerre polynomial to define the Wick renormalization.
Lemma 3.3. Let $\bar{Y}$ be as in [3.2]. Fix $N \in \mathbb{N}$. Suppose that $F : C^{\infty}(\mathbb{T}^d) \times C^{\infty}(\mathbb{T}^d) \to \mathbb{R}$ is measurable such that $E[|F(\pi_N \bar{Y}(1))|^p] < \infty$ and $E[|e^{-F(\pi_N \bar{Y}(1))}|^q] < \infty$ for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$-\log E\left[e^{-F(\pi_N \bar{Y}(1))}\right] = \inf_{\tilde{\theta} \in \mathbb{H}_N} E\left[F(\pi_N \bar{Y}(1) + \pi_N \tilde{I}(\tilde{\theta})(1)) + \frac{1}{2} \int_0^1 \|	ilde{\theta}(t)\|_{L^2_{\theta}}^2 dt\right],$$

where $\tilde{I}(\tilde{\theta}) = (I_1(\theta_1), I_2(\theta_2))$ is defined by

$$\tilde{I}(\tilde{\theta})(t) = (I_1(\theta_1)(t), I_2(\theta_2)(t)) = \left(\int_0^t \langle \nabla \rangle^{-1} \theta_1(t') dt', \int_0^t \langle \nabla \rangle^{-1+\gamma} \theta_2(t') dt'\right)$$

and the expectation $E = E_\mathbb{P}$ is an expectation with respect to the underlying probability measure $\mathbb{P}$.

Before proceeding to the proof of Theorem 3.8, we state a lemma on the pathwise regularity bounds of $\bar{Y}(1)$ and $\tilde{I}(\tilde{\theta})(1)$.

Lemma 3.4. (i) Let $\varepsilon > 0$. Then, given any finite $p \geq 1$, we have

$$E\left[\|\langle Y, N(1), Y, N(1)\rangle\|_{L^{p, \infty}(\mathbb{T}^d \times \mathbb{T}^d)} + \|Y_{1, N}(1)\|_{L^{p, \infty}(\mathbb{T}^d \times \mathbb{T}^d)} \right] \leq C_{\varepsilon, p} < \infty,$$

uniformly in $N \in \mathbb{N}$.

(ii) For any $\tilde{\theta} \in \mathbb{H}_N$, we have

$$\|\tilde{I}(\tilde{\theta})(1)\|_{H^{1, 1} - \gamma} \leq \int_0^1 \|\tilde{\theta}(t)\|_{L^2_{\theta}}^2 dt. \quad (3.5)$$

In particular, we have

$$\|I_1(\theta_1)(1)\|_{H^{1, 1}} \leq \int_0^1 \|\theta_1(t)\|_{L^2_{\theta}}^2 dt,$$

$$\|I_2(\theta_2)(1)\|_{H^{1, 1} - \gamma} \leq \int_0^1 \|\theta_2(t)\|_{L^2_{\theta}}^2 dt.$$

Part (i) of Lemma 3.4 follows from a standard computation and thus we omit details. See, for example, [17] Proposition 2.3 and [27] Proposition 2.1 for related results when $d = 2$. As for Part (ii), the estimate (3.5) follows from Minkowski’s and Cauchy-Schwarz’ inequalities. See the proof of Lemma 4.7 in [29].

3.2. Uniform exponential integrability. In this section, we present the proof of Lemma 3.4. Since the argument is identical for any finite $p \geq 1$, we only present details for the case $p = 1$. Note that

$$1_{\{\cdot \leq K\}}(x) \leq \exp\left(- A|x|^\alpha\right) \exp(AK^\alpha) \quad (3.6)$$

for any $K, A, \alpha > 0$. Given $N \in \mathbb{N}$, $\alpha \gg 1$, and $A \gg 1$ sufficiently large, we define

$$\mathcal{R}_N(u, w) = \int_{\mathbb{T}^2} u_N^2 : w_N d\mathcal{L} + A \int_{\mathbb{T}^2} u_N^2 : dx \right|^\alpha. \quad (3.7)$$

Then, the following uniform exponential bound (3.8) with (3.6)

$$\sup_{N \in \mathcal{N}} \left\|e^{-\mathcal{R}_N(u)}\right\|_{L^p(d\mu_\epsilon)} \leq C_{p, A, \alpha} < \infty \quad (3.8)$$

with (3.3)
implies the uniform exponential integrability (3.1). Hence, it remains to prove the uniform exponential integrability (3.8). In view of the Boué-Dupuis formula (Lemma 3.3), it suffices to establish a lower bound on

\[ W_N(\vec{\theta}) = \mathbb{E} \left[ \mathcal{R}_N(\vec{Y}(1) + \vec{I}(\vec{\theta})(1)) + \frac{1}{2} \int_0^1 \| \vec{\theta}(t) \|_{L^2_x \times L^2_x}^2 dt \right], \]  

(3.9)

uniformly in \( N \in \mathbb{N} \) and \( \vec{\theta} \in \mathbb{H}_a \). We set \( \vec{Y}_N = \pi_N \vec{Y} = \pi_N \vec{Y}(1) \) and \( \vec{\Theta}_N = \pi_N \vec{\Theta} = \pi_N \vec{I}(\vec{\theta})(1) = (\Theta_{1,N}, \Theta_{2,N}) \).

From (3.7), we have

\[
\mathcal{R}_N(\vec{Y} + \vec{\Theta}) = \int_{\mathbb{T}^2} : (Y_{1,N} + \Theta_{1,N})^2 : (Y_{2,N} + \Theta_{2,N}) dx + A \left( \int_{\mathbb{T}^2} : (Y_{1,N} + \Theta_{1,N})^2 : dx \right)^\alpha \\
= \int_{\mathbb{T}^2} \Theta_{1,N}^2 : Y_{2,N} dx + \int_{\mathbb{T}^2} \Theta_{2,N}^2 : Y_{1,N} dx + 2 \int_{\mathbb{T}^2} Y_{1,N} \Theta_{1,N} Y_{2,N} dx \\
+ 2 \int_{\mathbb{T}^2} Y_{1,N} \Theta_{1,N} \Theta_{2,N} dx + \int_{\mathbb{T}^2} \Theta_{1,N}^2 Y_{2,N} dx + \int_{\mathbb{T}^2} \Theta_{2,N}^2 Y_{1,N} dx \\
+ A \left\{ \int_{\mathbb{T}^2} \left( : Y_{1,N}^2 : + 2 Y_{1,N} \Theta_{1,N} + \Theta_{1,N}^2 \right) dx \right\}^\alpha. 
\]  

(3.10)

Hence, from (3.9) and (3.10), we have

\[
W_N(\vec{\theta}) = \mathbb{E} \left[ \int_{\mathbb{T}^2} : Y_{1,N}^2 : Y_{2,N} dx + \int_{\mathbb{T}^2} : Y_{2,N}^2 : \Theta_{1,N} \Theta_{2,N} dx + 2 \int_{\mathbb{T}^2} Y_{1,N} \Theta_{1,N} Y_{2,N} dx \\
+ 2 \int_{\mathbb{T}^2} Y_{1,N} \Theta_{1,N} \Theta_{2,N} dx + \int_{\mathbb{T}^2} \Theta_{1,N}^2 Y_{2,N} dx + \int_{\mathbb{T}^2} \Theta_{2,N}^2 Y_{1,N} dx \\
+ A \left\{ \int_{\mathbb{T}^2} \left( : Y_{1,N}^2 : + 2 Y_{1,N} \Theta_{1,N} + \Theta_{1,N}^2 \right) dx \right\}^\alpha \right] + \frac{1}{2} \int_0^1 \| \vec{\theta}(t) \|_{L^2_x \times L^2_x}^2 dt. 
\]  

(3.11)

We first state a lemma, controlling the terms appearing in (3.11). We present the proof of this lemma at the end of this section.

**Lemma 3.5.** Let \( 0 \leq \gamma < 1 \). Then, we have the following:

(i) For any \( \delta > 0 \), we have

\[
\mathbb{E} \left[ \int_{\mathbb{T}^2} : Y_{1,N}^2 : Y_{2,N} dx \right] = 0 
\]  

(3.12)

\[
\mathbb{E} \left[ \int_{\mathbb{T}^2} Y_{1,N} Y_{2,N} \Theta_{1,N} dx \right] \leq c(\delta) + \delta \mathbb{E} \left[ \| \Theta_{1,N} \|_{H^1}^2 \right] 
\]  

(3.13)

uniformly in \( N \in \mathbb{N} \).
(ii) There exists a small $\varepsilon > 0$, a constant $c > 0$ and $\alpha \gg 1$ such that for any $\delta > 0$, we have

\[
\begin{align*}
    \left| \int_{\mathbb{T}^2} Y_{1,N}^2 : \Theta_{2,N} dx \right| & \leq c(\delta) \| Y_{1,N}^2 \|_{L^2_{\varepsilon} - \infty} + \delta \| \Theta_{2,N} \|_{H^{1-\gamma}}, \\
    \left| \int_{\mathbb{T}^2} Y_{1,N} \Theta_{1,N} \Theta_{2,N} dx \right| & \leq c(\delta) \| Y_{1,N} \|_{L^2_{\varepsilon} - \infty} + \delta \left( \| \Theta_{1,N} \|_{H^1} + \| \Theta_{1,N} \|_{L^2} \right) \\
    \left| \int_{\mathbb{T}^2} \Theta_{1,N}^2 Y_{2,N} dx \right| & \leq c(\delta) \| Y_{2,N} \|_{L^2_{\gamma - \varepsilon, \infty}} + \delta \left( \| \Theta_{1,N} \|_{H^1} + \| \Theta_{1,N} \|_{L^2} \right) \\
    \left| \int_{\mathbb{T}^2} \Theta_{1,N} \Theta_{2,N} dx \right| & \leq \frac{A}{100} \| \Theta_{2,N} \|_{L^2}^2 + \delta \left( \| \Theta_{2,N} \|_{H^{1-\gamma}} + \| \Theta_{1,N} \|_{L^2} + \| \Theta_{1,N} \|_{H^1} \right)
\end{align*}
\]  

(3.14) (3.15) (3.16) (3.17)

for any sufficiently large $A > 0$, uniformly in $N \in \mathbb{N}$.

(ii) Let $A > 0$ and $\alpha > 0$. Then, there exists $c = c(A, \alpha) > 0$ such that

\[
A \left| \int_{\mathbb{T}^2} \left( Y_{1,N}^2 + 2 Y_{1,N} \Theta_{1,N} + \Theta_{2,N} \right) dx \right|^\alpha 
\geq \frac{A}{4} \| \Theta_{1,N} \|_{L^2}^{2\alpha} + \frac{1}{100} \| \Theta_{1,N} \|_{H^1} - c \left\{ \left| \int_{\mathbb{T}^2} Y_{1,N}^2 dx \right|^\alpha \right\},
\]

(3.18)

uniformly in $N \in \mathbb{N}$.

Then, as in \[1\ 29\ 42\ 39\ 43\ 40\], the main strategy is to establish a pathwise lower bound on $\mathcal{W}_N(\tilde{\theta})$ in (3.11), uniformly in $N \in \mathbb{N}$ and $\tilde{\theta} \in \mathbb{H}_a$, by making use of the positive terms:

\[
\mathcal{U}_N(\tilde{\theta}) = \mathbb{E} \left[ \frac{A}{4} \| \Theta_{1,N} \|_{L^2}^{2\alpha} + \frac{1}{2} \int_0^1 \| \tilde{\theta}(t) \|_{L^2_{\gamma - \varepsilon, \infty}}^2 dt \right].
\]

(3.19)

coming from (3.11) and (3.18). From (3.11) and (3.19) together with Lemmas 3.5 and 3.4, we obtain

\[
\inf_{N \in \mathbb{N}} \inf_{\tilde{\theta} \in \mathbb{H}_a} \mathcal{W}_N(\tilde{\theta}) \geq \inf_{N \in \mathbb{N}} \inf_{\tilde{\theta} \in \mathbb{H}_a} \left\{ - C_0 + \frac{1}{10} \mathcal{U}_N(\tilde{\theta}) \right\} \geq - C_0 > -\infty.
\]

(3.20)

Then, the uniform exponential integrability (3.1) follows from (3.20) and Lemma 3.3. This completes the proof of Lemma 3.1.

We conclude this section by presenting the proof of Lemma 3.5.

Proof of Lemma 3.5. (i) From the independence of $Y_{1,N}$ and $Y_{2,N}$, we have

\[
\mathbb{E} \left[ \int_{\mathbb{T}^2} Y_{1,N}^2 : Y_{2,N} dx \right] = 0.
\]

(3.21)

This yields (3.12).

From Hölder’s inequality and Young’s inequality, we have

\[
\left| \int_{\mathbb{T}^2} Y_{2,N} Y_{1,N} \Theta_{1,N} dx \right| \leq \| \langle \nabla \rangle^{-1} (Y_{1,N} Y_{2,N}) \|_{L^2} \| \langle \nabla \rangle \Theta_{1,N} \|_{L^2} \leq C(\delta) \| \langle \nabla \rangle^{-1} (Y_{1,N} Y_{2,N}) \|_{L^2}^2 + \delta \| \Theta_{1,N} \|_{H^1}^2.
\]
We now consider $\|\langle \nabla \rangle^{-1}(Y_{1,N}Y_{2,N})\|_{L^2}^2$. Note that

$$
\|\langle \nabla \rangle^{-1}(Y_{1,N}Y_{2,N})\|_{L^2}^2 = \sum_{n \in \mathbb{Z}^2, \ |n| \leq 2N} \frac{1}{\langle n \rangle^2} \sum_{n_1 \in \mathbb{Z}^2, \ |n_1| \leq N, \ n_1 + n_2 = n} \sum_{n_2 \in \mathbb{Z}^2, \ |n_2| \leq N} \frac{B_{n_1}^1(1) B_{n_2}^2(1)}{\langle n_1 \rangle \langle n_2 \rangle^{1-\gamma}}^2.
$$

From taking the expectation, using the independence of $\{B_{n_1}^1\}_{n \in \mathbb{Z}^2}$ and $\{B_{n_2}^2\}_{n \in \mathbb{Z}^2}$, and Lemma 2.6, we have

$$
\mathbb{E}\left[\|\langle \nabla \rangle^{-1}(Y_{1,N}Y_{2,N})\|_{L^2}^2\right] \leq \sum_{n \in \mathbb{Z}^2, \ |n| \leq 2N} \frac{1}{\langle n \rangle^2} \sum_{n_1 \in \mathbb{Z}^2, \ |n_1| \leq N, \ n_1 + n_2 = n} \frac{1}{\langle n_1 \rangle^{2-\eta} \langle n_2 \rangle^{2(1-\gamma)}} \sum_{n_2 \in \mathbb{Z}^2, \ |n_2| \leq N} \frac{1}{\langle n_2 \rangle^{2-\eta} < \infty},
$$

uniformly in $N$, where we choose $\eta > 0$ such that $2 - 2\gamma - \eta > 0$ by using the condition $\gamma < 1$. This yields (3.13).

(ii) From Young’s inequality and the condition $\gamma < 1$, we have

$$
\left| \int_{\mathbb{T}^2} :Y_{1,N}^2 : \Theta_{2,N} dx \right| \leq \|Y_{1,N} : \|_{W^{-\varepsilon,\infty}} \|\langle \nabla \rangle^\varepsilon \Theta_{2,N}\|_{L^1} \leq c(\delta) \|Y_{1,N}^2 : \|_{W^{-\varepsilon,\infty}} + \delta \|\Theta_{2,N}\|_{H^{1-\gamma}}^2.
$$

This yields (3.14).

From the fractional Leibniz rule (Lemma 2.4 (ii)) and Sobolev’s inequality, we have

$$
\left| \int_{\mathbb{T}^2} Y_{1,N} \Theta_{1,N} \Theta_{2,N} dx \right| \leq \|Y_{1,N}\|_{W^{-\varepsilon,\infty}} \|\Theta_{1,N} \Theta_{2,N}\|_{W^{\varepsilon,1}} \leq \|Y_{1,N}\|_{W^{-\varepsilon,\infty}} \left[ \|\Theta_{1,N}\|_{L^2} + \|\langle \nabla \rangle^\varepsilon \Theta_{2,N}\|_{L^2} + \|\langle \nabla \rangle^\varepsilon \Theta_{1,N}\|_{L^2} + \|\Theta_{2,N}\|_{L^2} \right] \leq \|Y_{1,N}\|_{W^{-\varepsilon,\infty}} \left[ \|\Theta_{1,N}\|_{H^\eta} \|\Theta_{2,N}\|_{H^{1-\gamma}} + \|\Theta_{1,N}\|_{H^\eta} \|\Theta_{2,N}\|_{L^2} \right] \quad (3.21)
$$

where $0 < \varepsilon < 1 - \gamma$ and $0 < \varepsilon < \eta$ for some small $\eta > 0$. From the interpolation inequality (Lemma 2.4 (i)), we have

$$
\|\Theta_{1,N}\|_{H^\eta} \lesssim \|\Theta_{1,N}\|_{H^1} \|\Theta_{1,N}\|_{L^2}^{1-\eta}. \quad (3.22)
$$

Hence, from (3.22) and Young’s inequality, we have

$$
(3.21) \leq \|Y_{1,N}\|_{W^{-\varepsilon,\infty}} \|\Theta_{2,N}\|_{H^{1-\gamma}} \|\Theta_{1,N}\|_{H^1} \|\Theta_{1,N}\|_{L^2}^{1-\eta} \leq c(\delta) \|Y_{1,N}\|_{W^{-\varepsilon,\infty}} + \delta \left( \|\Theta_{1,N}\|_{H^1}^{2α} + \|\Theta_{1,N}\|_{L^2}^{2α} + \|\Theta_{2,N}\|_{H^{1-\gamma}}^2 \right),
$$

where $\frac{1}{2} + \frac{\eta}{2α} + \frac{1-\eta}{2α} < 1$ if $α \gg 1$. This yields (3.15).
From the fractional Leibniz rule (Lemma 2.4 (ii)) (with \( \frac{1}{2+\delta_1} = \frac{1}{2} - \frac{\eta}{2} \)), Sobolev’s inequality (with \( \frac{1}{2+\delta_1} = \frac{1}{2} - \frac{\eta}{2} \)), and Young’s inequality, we have

\[
\left| \int_{T^2} Y_{2,N} \Theta^2_{1,N} dx \right| \leq \| Y_{2,N} \|_{W^{-\gamma-\varepsilon,\infty}} \| \langle \nabla \rangle^{\gamma+\varepsilon} (\Theta^2_{1,N}) \|_{L^{1+\delta_0}} \\
\leq \| Y_{2,N} \|_{W^{-\gamma-\varepsilon,\infty}} \| \langle \nabla \rangle^{\gamma+\varepsilon} \Theta_{1,N} \|_{L^{2+\delta_1}} \| \Theta_{1,N} \|_{L^{2}} \\
\leq \| Y_{2,N} \|_{W^{-\gamma-\varepsilon,\infty}} \| \Theta_{1,N} \|_{H^1} \| \Theta_{1,N} \|_{L^{2}} \\
\leq c(\delta) \| Y_{2,N} \|_{W^{-\gamma-\varepsilon,\infty}}^2 + \delta \left( \| \Theta_{1,N} \|_{H^1}^2 + \| \Theta_{1,N} \|_{L^{2}}^2 \right)
\]

for some sufficiently small \( \delta_0, \delta_1, \) and \( \eta > 0 \), where \( \frac{1}{2} + \frac{\eta}{2} < 1 \) if \( \alpha \gg 1 \). Notice that in the fourth step, we used the condition \( \gamma < 1 \). This yields (3.16).

From Hölder’s inequality (with \( 1 = \frac{1}{2+\delta_0} + \frac{1}{2-\delta_1} \)), we have

\[
\left| \int_{T^2} \Theta_{2,N} \Theta^2_{1,N} dx \right| \leq \| \Theta_{2,N} \|_{L^{2+\delta_0}} \| \Theta^2_{1,N} \|_{L^{2-\delta_1}} \\
\leq \| \Theta_{2,N} \|_{L^{2+\delta_0}} \| \Theta_{1,N} \|_{L^{2-\delta_1}} \leq (3.23)
\]

We first consider \( \| \Theta_{2,N} \|_{L^{2+\delta_0}} \). From the Sobolev’s inequality (with \( \frac{1}{2+\delta_0} = \frac{1}{2} - \frac{\eta}{2} \)) and the interpolation inequality (Lemma 2.4 (i)), we have

\[
\| \Theta_{2,N} \|_{L^{2+\delta_0}} \lesssim \| \Theta_{2,N} \|_{H^\varepsilon} \leq \| \Theta_{2,N} \|_{H^{1-\gamma}} \| \Theta_{2,N} \|_{L^2} \leq (3.24)
\]

where \( 0 < \varepsilon < 1 - \gamma \). We next consider \( \| \Theta_{1,N} \|_{L^{4-2\delta_1}} \). From the Sobolev’s inequality (with \( \frac{1}{4-2\delta_1} = \frac{1}{2} - \frac{1-\delta_1}{2(2-\delta_1)} \)) and the interpolation inequality (Lemma 2.4 (i)), we have

\[
\| \Theta_{1,N} \|_{L^{4-2\delta_1}} \lesssim \| \Theta_{1,N} \|_{H^{1-\gamma}} \leq \| \Theta_{1,N} \|_{H^{1}} \leq (3.25)
\]

From (3.23), (3.24), (3.25), and Young’s inequality, we have

\[
\left| \int_{T^2} \Theta_{2,N} \Theta^2_{1,N} dx \right| \leq \| \Theta_{2,N} \|_{L^{2+\delta_0}} \| \Theta_{1,N} \|_{L^{4-2\delta_1}}^2 \\
\leq \| \Theta_{2,N} \|_{L^{2+\delta_0}} \| \Theta_{1,N} \|_{L^{2}} \| \Theta_{2,N} \|_{H^{1-\gamma}} \| \Theta_{1,N} \|_{L^{2}} \leq \frac{A}{100} \| \Theta_{1,N} \|_{L^{2}}^2 + \delta \left( \| \Theta_{2,N} \|_{H^{1-\gamma}}^2 + \| \Theta_{2,N} \|_{L^{2}}^2 + \| \Theta_{1,N} \|_{H^{1}}^2 \right)
\]

where \( \frac{\varepsilon}{2(1-\gamma)} + \frac{1-\gamma-\varepsilon}{2(1-\gamma)} + \frac{1-\delta_1}{2(2-\delta_1)} + \frac{1}{(2-\delta_1)\alpha} = 1 \) if \( \alpha \gg 1 \). Notice that \( \frac{1-\delta_1}{2(2-\delta_1)} < \frac{1}{2} \) if and only if \( \delta_1 > 0 \). This yields (3.17).

(iii) Note that there exists a constant \( C_\alpha > 0 \) such that

\[
|a+b+c|^\alpha \geq \frac{1}{2} |c|^\alpha - C_\gamma (|a|^\alpha + |b|^\alpha)
\] (3.26)
for any $a, b, c \in \mathbb{R}$ (see Lemma 5.8 in [39]). Then, from (3.26), we have

$$A \left| \int_{\mathbb{T}^2} \left( Y_{1,N}^2 : + 2Y_{1,N}\Theta_{1,N} + \Theta_{1,N}^2 \right) dx \right|^\alpha \geq \frac{A}{2} \left( \int_{\mathbb{T}^2} \Theta_{1,N}^2 dx \right)^\alpha - AC_\alpha \left\{ \int_{\mathbb{T}^2} Y_{1,N}^2 : dx \right\} + \left| \int_{\mathbb{T}^2} Y_{1,N}\Theta_{1,N} dx \right|^\alpha.$$  \tag{3.27}

From the interpolation inequality (Lemma 2.4 (i)) and Young’s inequality, we have

$$\left| \int_{\mathbb{T}^2} Y_{1,N}\Theta_{1,N} dx \right|^\alpha \leq \|Y_{1,N}\|_{W^{-\varepsilon, \infty}}^\alpha \|\nabla\|_{\mathcal{H}_1}^\alpha$$

$$\leq \|Y_{1,N}\|_{W^{-\varepsilon, \infty}}^\alpha \|\Theta_{1,N}\|_{\mathcal{H}_1}^{\alpha(1-\varepsilon)} \|\Theta_{1,N}\|_{\mathcal{H}_1}^{\alpha\varepsilon}$$

$$\leq c\|Y_{1,N}\|_{W^{-\varepsilon, \infty}}^{2\alpha} + \frac{1}{100C_\alpha^2} \|\Theta_{1,N}\|_{\mathcal{H}_1}^{2\alpha} + \frac{1}{100C_\alpha} \|\Theta_{1,N}\|_{\mathcal{H}_1}^{\alpha}.$$  \tag{3.28}

Notice that $\frac{1-\varepsilon}{2} + \frac{\alpha\varepsilon}{2} < 1$ if $\varepsilon$ is sufficiently small. Hence, (3.18) follows from (3.27) and (3.28).

4. Probabilistic well-posedness

In this section, we present the proof of Theorem 1.8 by assuming random tensor estimates (Lemmas 5.5, 5.6 and 5.7) in Section 5. In particular, we show that the renormalized Zakharov-Yukawa system (1.21) is $d\tilde{\mu}_\gamma$-almost surely locally well-posed.

4.1. Reduced first-order system. In this subsection, we consider reduced first-order system for the wave part. For the Zakharov-Yukawa system, there is a standard procedure to factor the wave operator in order to derive a first-order system. We first look at the renormalized Zakharov-Yukawa system:

$$\begin{cases}
  i\partial_t u + \Delta u = uw \\
  \partial_t^2 w + (1 - \Delta)w = -\langle \nabla \rangle^{2\gamma} N(u) \\
  (u, w, \partial_t w)|_{t=0} = (u^0, w_0, w_t^0),
\end{cases}$$  \tag{4.1}

where we recall $N(u) = |u|^2 - \frac{f}{2} |u|^2$ and $(u^0, w_0, w_t^0)$ is distributed according to $d\tilde{\mu}_\gamma = d\mu \otimes d\mu_{1-\gamma} \otimes d\mu_{-\gamma}$. It is convenient to reduce the renormalized Zakharov-Yukawa system (4.1) to a first-order system by setting $w_{\pm} := w \pm i\langle \nabla \rangle^{-1} \partial_t w$, where $(u, w, \partial_t w)$ is a solution to (4.1). The new system is then given by

$$\begin{cases}
  i\partial_t u + \Delta u = \frac{1}{2}(w_+ + w_-)u \\
  i\partial_t w_{\pm} + \langle \nabla \rangle w_{\pm} = \pm \langle \nabla \rangle^{-1} + 2\gamma N(u) \\
  (u, w_{\pm})|_{t=0} = (u_{0,\pm}^0, w_{\pm}^0),
\end{cases}$$  \tag{4.2}

where $(u_{0,\pm}^0, w_{\pm}^0)$ is given by

$$u_{0,\pm}^0 = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{in \cdot x} \quad \text{and} \quad w_{0,\pm}^0 = \sum_{n \in \mathbb{Z}^2} \frac{\tilde{h}_{n,\pm}(\omega)}{\langle n \rangle^{1-\gamma}} e^{in \cdot x},$$

where $\{g_n\}_{n \in \mathbb{Z}^2}$ and $\{\tilde{h}_{n,\pm} = h_n \pm i\ell_n\}_{n \in \mathbb{Z}^2}$, see (1.11). Since $w$ is real-valued, we can recover the solution of the original system (4.1) by setting $w = Re w_{\pm}$ and this consideration allows
us to claim the statement of Theorem 1.8 by proving a similar statement about the reduced system (4.2). The corresponding system can be expressed in Duhamel formulation:

\[ u(t) = e^{it\Delta}u_0 - \frac{i}{2} \int_0^t e^{i(t-t')\Delta} [(w_+ + w_-)u](t') dt' \]

\[ w_\pm(t) = e^{\mp it\langle\nabla\rangle} w_{\pm,0} \mp i \int_0^t e^{-i(t-t')\langle\nabla\rangle} \langle\nabla\rangle^{-1+2\gamma} N(u)(t') dt'. \] (4.3)

For further use, we set

\[ N_S(u_1, u_2) := u_1 u_2 \quad \text{and} \quad N_W(u_2, u_3) := \langle\nabla\rangle^{-1+2\gamma} N(u_2, u_3) \] (4.4)

### 4.2. First order expansion

Let us first go over the basic idea of the probabilistic local well-posedness, as developed for instance in \[10, 14, 61, 17, 3, 36\]; see also \[34\]. This argument is based on the following first-order expansion: random linear term + smoother term

\[ u = z^S + R^S, \]

\[ w_\pm = z^{W_\pm} + R^{W_\pm}, \]

where \( z^S = z^{S,\omega} \) and \( z^{W_\pm} = z^{W_\pm,\omega} \) denote the random linear solutions defined by

\[ z^S(t) = z^{S,\omega}(t) := e^{it\Delta} u_0^\omega, \]

\[ z^{W_\pm}(t) = z^{W_\pm,\omega}(t) := e^{\mp i t\langle\nabla\rangle} w_{\pm,0}^\omega. \] (4.5)

By rewriting (4.3) as a fixed point problem for the residual terms \( R^S := u - z^S \) and \( R^{W_\pm} := w_\pm - z^{W_\pm} \), we obtain the following perturbed renormalized Zakharov-Yukawa system:

\[ R^S(t) = -i \int_0^t e^{i(t-t')\Delta} N_S(z^S + R^S, z^{W_\pm} + R^{W_\pm})(t') dt', \]

\[ R^{W_\pm}(t) = \mp i \int_0^t e^{\mp i(t-t')\langle\nabla\rangle} N_W(z^S + R^S, z^S + R^S, z^{W_\pm} + R^{W_\pm})(t') dt'. \] (4.6)

By viewing \( (z^S, z^{W_\pm}, z^{W_\pm}) \) as a given enhanced data set, we study the fixed point problem for the smoother term \( (R^S, R^{W_\pm}) \) in \( H^s(\mathbb{T}^2) \times H^s(\mathbb{T}^2) \) for some \( s > 0 \) and \( \ell > 0 \) in Theorem 1.8. In particular, for \( 0 \leq \gamma < \frac{1}{3} \), we will show that for each small \( \delta > 0 \), there exists an event \( \Omega_\delta \subset \Omega \) with \( P(\Omega_\delta^c) < C e^{-\delta} \) such that for each \( \omega \in \Omega_\delta \), there exists a solution \( u = z^S + R^S \) and \( w_\pm = z^{W_\pm} + R^{W_\pm} \) to the perturbed renormalized Zakharov-Yukawa system (4.6) in the class:

\[ z^S + X^{s,\frac{1}{2}+}(\delta) \subset C([-\delta, \delta]; H^{-\gamma}(\mathbb{T}^2)), \]

\[ z^{W_\pm} + X^{\ell,\frac{1}{2}+}(\delta) \subset C([-\delta, \delta]; H^{-\gamma-\varepsilon}(\mathbb{T}^2)). \]

for some \( s > 0 \) and \( \ell > 0 \) in Theorem 1.8

### 4.3. The proof of Theorem 1.8

In this subsection, we prove Theorem 1.8 by assuming Lemma 4.1. We first define the following Duhamel map

\[ \Gamma^\omega(R^S, R^{W_\pm}) = \left( \int_0^t e^{i(t-t')\Delta} N_S^\omega(R^S, R^{W_\pm})(t') dt', \mp \int_0^t e^{\mp i(t-t')\langle\nabla\rangle} N_W^\omega(R^S, R^S)(t') dt' \right) \]
where
\[ \gamma^\varphi_{0}(R^S, R^{W\pm}) := \chi_{\delta} \cdot \mathcal{N}_{S}(\pi S + R^S, \pi W + R^{W\pm}) \]
\[ \gamma^\varphi_{W}(R^S, R^W) := \chi_{\delta} \cdot \mathcal{N}_{W}(\pi S + R^S, \pi S + R^W) \]
with extensions \( \pi S \) and \( \pi W \) of the truncated random linear solution \( \chi_{\delta} \cdot \pi S \) and \( \chi_{\delta} \cdot \pi W \). Given \( R^S \) and \( R^{W\pm} \) on \( T^2 \times [-\delta, \delta] \), let \( \tilde{R}^S \) and \( \tilde{R}^{W\pm} \) be extensions of \( R^S \) and \( R^{W\pm} \) onto \( T^2 \times \mathbb{R} \). By the non-homogeneous linear estimate (Lemma 2.2), we have
\[ \left\| \int_{0}^{t} e^{i(t-t')\Delta} \gamma^\varphi_{S}(R^S, R^{W\pm})(t') dt' \right\|_{X^x_{S}^0(\delta)} \lesssim \| \eta_{\delta}(t) \int_{0}^{t} e^{i(t-t')\Delta} \gamma^\varphi_{S}(R^S, R^{W\pm})(t') dt' \|_{X^x_{S}^0(\delta)} \]
and
\[ \left\| \int_{0}^{t} e^{i(t-t')\Delta} \gamma^\varphi_{W}(R^S, R^W)(t') dt' \right\|_{X^x_{W}^0(\delta)} \lesssim \| \eta_{\delta}(t) \int_{0}^{t} e^{i(t-t')\Delta} \gamma^\varphi_{W}(R^S, R^W)(t') dt' \|_{X^x_{W}^0(\delta)} \]
where \( \eta_{\delta} \) is a smooth cutoff on \( [-2\delta, 2\delta] \) as in (2.1). The main goal of next two subsections 4.4, 4.5 is to prove the following bilinear estimates by viewing \( (\pi S, \pi W, \pi S \pi W) \) as a given (enhanced) data set.

**Lemma 4.1.** Let \( 0 \leq \gamma < \frac{1}{4} \) and \( \delta > 0 \). Then, there exists an event \( \Omega_{\delta} \subset \Omega \) and \( c' > 0 \) with \( P(\Omega_{\delta}) < e^{-\frac{c'}{\delta}} \) such that
\[ \| \gamma^\varphi_{S}(\tilde{R}^S, \tilde{R}^{W\pm}) \|_{X^x_{S}^0} \lesssim \delta^{0_{\gamma_{}}}(1 + \| \tilde{R}^S \|_{X^x_{S}^{0_{\gamma}}} + \| \tilde{R}^{W\pm} \|_{X^x_{W}^{0_{\gamma}}} + \| \tilde{R}^S \|_{X^x_{S}^{0_{\gamma}}} \| \tilde{R}^{W\pm} \|_{X^x_{W}^{0_{\gamma}}}) \]
\[ \| \gamma^\varphi_{W}(\tilde{R}^S, \tilde{R}^S) \|_{X^x_{W}^{0_{\gamma}}} \lesssim \delta^{0_{\gamma_{}}}(1 + \| \tilde{R}^S \|_{X^x_{S}^{0_{\gamma}}} + \| \tilde{R}^S \|_{X^2_{S}^{0_{\gamma}}}) \]
for all \( \omega \in \Omega_{\delta} \) and any extension \( (\tilde{R}^S, \tilde{R}^W) \) of \( (R^S, R^W) \), provided \( s - 1 < \ell < 1 - 2\gamma \)
with \( 0 < s < \frac{1}{4} - \frac{\gamma}{2} \) and \( \ell > 0 \).

From the definition (2.23) of the local-in-time norm, we then conclude from (4.7) and Lemma 4.1 that
\[ \left\| \int_{0}^{t} e^{i(t-t')\Delta} \gamma^\varphi_{S}(R^S, R^{W\pm})(t') dt' \right\|_{X^x_{S}^0(\delta)} \lesssim \delta^{0_{\gamma_{}}}(1 + \| R^S \|_{X^x_{S}^{0_{\gamma}}} + \| R^{W\pm} \|_{X^x_{W}^{0_{\gamma}}} + \| R^S \|_{X^x_{S}^{0_{\gamma}}} \| R^{W\pm} \|_{X^x_{W}^{0_{\gamma}}}) \]
and
\[ \left\| \int_{0}^{t} e^{i(t-t')\Delta} \gamma^\varphi_{W}(R^S, R^W)(t') dt' \right\|_{X^x_{W}^0(\delta)} \lesssim \delta^{0_{\gamma_{}}}(1 + \| R^S \|_{X^x_{S}^{0_{\gamma}}} + \| R^S \|_{X^2_{S}^{0_{\gamma}}}). \]
By the bilinear structure of the nonlinearity Ω^S and Ω^W, a similar estimate holds for the difference of the Duhamel map \( \Gamma^\omega(R^S_1, R^S_2) - \Gamma^\omega(R^S_1, R^W_2) \), which allows us to conclude that \( \Gamma^\omega \) is a contraction on \( B_S(1) \times B_W(1) \subset X^{s_\omega, \frac{1}{2}+}(\delta) \times X^{\ell_\omega, \frac{1}{2}+}(\delta) \) for \( \omega \in \Omega_\delta \).

4.4. **Bilinear estimates for the Schrödinger part.** In this subsection, we prove (4.8) in Lemma 4.1. In view of (4.4), in order to prove (4.8), we need to carry out case-by-case analysis on

\[ \| \chi_\delta \cdot \mathcal{N}_S(u_1, w_1) \|_{X^{s_\delta, -\frac{1}{2}+}} \]  

(4.10)

where \( u_1 \) and \( w_1 \) are taken to be either of type

(I) rough random parts:

\[ u_1 = \tilde{z}^S, \text{ where } \tilde{z}^S \text{ is some extension of } \chi_\delta \cdot z^S, \]
\[ w_1 = \tilde{z}^W, \text{ where } \tilde{z}^W \text{ is some extension of } \chi_\delta \cdot z^W, \]

where \( z^S \) and \( z^W \) denote the random linear solutions defined in (4.5).

(II) smoother ‘deterministic’ remainder (nonlinear) parts:

\[ u_1 = \tilde{R}^S, \text{ where } \tilde{R}^S \text{ is any extension of } R^S, \]
\[ w_1 = \tilde{R}^W, \text{ where } \tilde{R}^W \text{ is any extension of } R^W. \]

In the following, when \( u_j \) and \( w_j \) are of type (I), we take \( \tilde{z}^S = \eta_\delta z^S \) and \( \tilde{z}^W = \eta_\delta z^W \). Thanks to the cutoff function in (4.10), we may take \( u_j = \eta_\delta \tilde{R}^S_j \) and \( w_j = \eta_\delta \tilde{R}^W_j \) in (4.10) when \( u_j \) and \( w_j \) are of type (II).

**Remark 4.2.** In the following, we drop the ± signs and work with one \( w_+ \) or \( w_- \) since there is no role of ±. Hence, we set \( w := w_+ \) and \( W := W_+ \).

**Remark 4.3.** To estimate \( \| \chi_\delta \cdot \mathcal{N}_S(u_1, w_1) \|_{X^{s_\delta, -\frac{1}{2}+}} \), we need to perform case-by-case analysis of expressions of the form:

\[ \int_\mathbb{R} \int_{\mathbb{T}^2} (\nabla)^s(u_1 w_1) v^S dx dt, \quad \text{where } \|v^S\|_{X^{s_\delta, 0+}} \leq 1. \]

In the following, for simplicity of notation, we drop the complex conjugate sign and suppress the smooth time-cutoff function \( \eta_\delta \); and thus we simply denote them by \( z^S, z^W, R^S, \) and \( R^W \), respectively when there is no confusion. Finally, we dyadically decompose \( u_1, w_1 \) and \( v^S \) such that their spatial frequency supports are \( \text{supp } \hat{u}_1 \subset \{|n_1| \sim N_1\}, \text{ supp } \hat{w}_1 \subset \{|n_2| \sim N_2\} \), and \( \text{supp } \hat{v} \subset \{|n| \sim N\} \) for some dyadic \( N_1, N_2 \) and \( N \geq 1 \).

We now prove Lemmas 4.4, 4.6, 4.7, 4.8 which will imply (4.8) in Lemma 4.1 (bilinear estimates for the Schrödinger part).

**Lemma 4.4 (R^S R^W-case).** Let \( s > 0 \) and \( \ell > 0 \). Then, we have

\[ \| \mathcal{N}_S(R^S, R^W) \|_{X^{s_\delta, -\frac{1}{2}+}} \lesssim \delta^{0+} \|R^S\|_{X^{s_\delta, \frac{1}{2}+}} \|R^W\|_{X^{\ell_\delta, \frac{1}{2}+}}. \]
Proof. We perform the case-by-case analysis:

**Case 1: \( N_1 \gg N_2 \).**
By writing \( \{|n_1| \sim N_1\} = \bigcup_{\ell_1} J_{1,\ell_1} \) and \( \{|n| \sim N\} = \bigcup_{\ell_2} J_{2,\ell_2} \), where \( J_{1,\ell_1} \) and \( J_{2,\ell_2} \) are balls of radius \( \sim N_2 \), we can decompose \( \widehat{P_{N_1}R^S} \) and \( \widehat{P_{N}v^S} \) as

\[
\widehat{P_{N_1}R^S} = \sum_{\ell_1} P_{N_1,\ell_1} R^S \quad \text{and} \quad \widehat{P_{N}v^S} = \sum_{\ell_2} P_{N,\ell_2} v^S
\]

where \( P_{N_1,\ell_1} R^S(n_1, t) = 1_{J_{1,\ell_1}}(n_1)\widehat{P_{N_1}R^S}(n_1, t) \) and \( P_{N,\ell_2} v^S(n, t) = 1_{J_{2,\ell_2}}(n)\widehat{P_{N}v^S}(n, t) \).

Given \( n_1 \in J_{1,\ell_1} \) for some \( \ell_1 \), there exists \( O(1) \) many possible values for \( \ell_2 = \ell_2(\ell_1) \) such that \( n \in J_{2,\ell_2} \) under \( n_1 + n_2 = n \). Notice that the number of possible values of \( \ell_2 \) is independent of \( \ell_1 \).

From the \( L^4 \)-Strichartz estimate (Lemma 2.7), the Cauchy-Schwarz inequality in \( \ell_1 \), and Lemma 2.3, we have \( ^{20} \)

\[
\left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} R^S P_{N_2} R^W) P_{N} v^S \, dxdt \right| \lesssim \sum_{\ell_1} \sum_{\ell_2} \int_{\mathbb{R}} \int_{T^2} P_{N_1,\ell_1} (\langle \nabla \rangle^s R^S) P_{N_2} R^W P_{N,\ell_2} v^S \, dxdt
\]

\[
\lesssim \sum_{\ell_1} \sum_{\ell_2} \| P_{N_1,\ell_1} \langle \nabla \rangle^s R^S \|_{L^4_{L_t,x}} \| P_{N,\ell_2} v^S \|_{L^4_{L_t,x}} \| P_{N_2} R^W \|_{L^2_{L_t,x}}
\]

\[
\lesssim \sum_{\ell_1} \sum_{\ell_2} N_2^s \| P_{N_1,\ell_1} R^S \|_{X^s_\epsilon^{1,1}} \| P_{N,\ell_2} v^S \|_{X^{0,\frac{1}{2}}_{\epsilon^{1,1}}} \| P_{N_2} R^W \|_{X_{\epsilon^{-\frac{1}{2}}}}
\]

\[
\lesssim \sum_{\ell_1} \sum_{\ell_2} N_2^{\ell+\epsilon} \| P_{N_1,\ell_1} R^S \|_{X^s_\epsilon^{1,1}} \| P_{N,\ell_2} v^S \|_{X^{0,\frac{1}{2}}_{\epsilon^{1,1}}} \| P_{N_2} R^W \|_{X_{\epsilon^{-\frac{1}{2}}}^{\ell+\epsilon}}
\]

\[
\lesssim \delta^{\frac{1}{2}} N_2^{-\ell+\epsilon} \| P_{N_1} R^S \|_{X^{s}_{\epsilon^{1,1}}} \| P_{N_2} R^W \|_{X^{\ell+\epsilon}_{\epsilon^{-\frac{1}{2}}}} \| P_{N} v^S \|_{X^{0,\frac{1}{2}}_{\epsilon^{1,1}}}
\]

Hence, if \( \ell > 0 \), then we can do the dyadic summation over over \( N_1 \sim N \geq N_2 \).

**Case 2: \( N_1 \ll N_2 \) (non-resonant interaction).**
This interaction includes the high-low interactions where one Schrödinger frequency is much greater than the other Schrödinger frequency. Hence, it follows from Lemma 2.11 and 2.3 that we have

\[
\| N_S(R^S, R^W) \|_{X^{s-\frac{1}{2}}_\epsilon^{1,1}} \lesssim \sum_{L, L_1, L_2} N_2^s L^{-\frac{1}{2}+\frac{1}{2}+\frac{3}{4}} \min L^\frac{3}{4} \left( N_1^2 N_2^{-1} \right) \| P_{N_1, L_1} R^S \|_{L^2_{L_t,x}} \| P_{N_2, L_2} R^W \|_{L^2_{L_t,x}} \| P_{N, L} v^S \|_{L^2_{L_t,x}}
\]

\[
\lesssim \delta^{\frac{1}{2}} N_2^{s-\ell+1} \| P_{N_1} R^S \|_{X^{s}_{\epsilon^{1,1}}} \| P_{N_2} R^W \|_{X^{\ell+1}_{\epsilon^{-\frac{1}{2}}}} \| P_{N} v^S \|_{X^{0,\frac{1}{2}}_{\epsilon^{1,1}}}
\]

Hence, if \( (s-\ell+1) + \frac{1}{2} - s = -\ell - \frac{1}{2} < 0 \) (i.e. \( \ell > -\frac{1}{2} \)), we can perform the dyadic summation over \( N_2 \sim N \geq N_1 \).

**Case 3: \( N_1 \sim N_2 \geq N \).**

---

\(^{20}\) Here, we are assuming that \( R^S, R^W, \) and \( v^S \) have non-negative Fourier coefficients since the Bourgain spaces are \( L^2 \)-based space.
From $L^4$-Strichartz estimate (Lemma 4.4) and Lemma 2.3, we have

$$\left| \int_{\mathbb{R}} \int_{\mathbb{T}^2} \langle \nabla \rangle^a (P_{N_1} R^S P_{N_2} R^W) P_N v^S dx dt \right| \lesssim \left\| \langle \nabla \rangle^a P_{N_1} R^S \right\|_{L^4_{t,x}} \left\| P_{N_2} R^W \right\|_{L^2_{t,x}} \left\| P_N v^S \right\|_{L^4_{t,x}}$$

$$\lesssim \delta \max N^{-1+\epsilon} \left\| P_{N_1} R^S \right\|_{X^{\infty, \frac{1}{2}}} \left\| P_{N_2} R^W \right\|_{X^{1, \frac{1}{2}}} \left\| P_N v^S \right\|_{X^{0, \frac{1}{2}}}.$$ 

If $\ell \geq e$ (i.e. $\ell > 0$), then we can obtain the desired result by summing over $N_1 \sim N_2 \gtrsim N$. 

\[ \square \]

**Remark 4.5.** In Case 2, if we proceed as in Case 1 (i.e. only using the $L^4$-Strichartz estimate with the orthogonality argument), then a restriction $\ell \geq s$ happens.

**Lemma 4.6 ($z^S z^W$-case).** Let $0 < s < \frac{1}{4} - \frac{\gamma}{2}$. Then, for each small $\delta > 0$, we have

$$\left\| N_S(z^S, z^W) \right\|_{X^{\infty, -\frac{1}{2}}_{S, \delta}} \lesssim \delta^0$$

outside an exceptional set of probability $< e^{-\frac{1}{s}}$.

**Proof.** We perform the case-by-case analysis:

**Case 1:** $N_1 \gg N$ or $N \gg N_1$ (non-resonant interaction).

By symmetry, we only consider the case $N_1 \gg N$. In this interaction, we have

$$L_{\max} \gtrsim \left| |n_1|^2 \pm |n_2| - |n| \right| \gtrsim N_{\max}^2 \sim N^2_2.$$

Define

$$L_1 := \langle \tau_1 - |n_1|^2 \rangle, \quad L_2 := \langle \tau_2 \pm |n_2| \rangle, \quad \text{and} \quad L := \langle \tau - |n|^2 \rangle.$$

First, suppose that $L \sim L_{\max}$. Then, from $L_{t,x}^p L_{t,x}^{2+} L_{t,x}^2$-Hölder’s inequality, Lemma 2.13 with large $p$, (4.11), and Lemma 2.3, we have

$$\left| \int_{\mathbb{R}} \int_{\mathbb{T}^2} \langle \nabla \rangle^a (P_{N_1} z^S P_{N_2} z^W) P_N v^S dx dt \right| \lesssim N_2^2 \left\| P_{N_1} z^S \right\|_{L^p \cap L^{2+}} \left\| P_{N_2} z^W \right\|_{L^{2+}} \left\| P_N v^S \right\|_{L_{t,x}^\infty}$$

$$\lesssim L_{\max}^{-\frac{1}{4}} N_2^{\frac{5}{4}+s} \left\| P_N v^S \right\|_{X^{0, \frac{1}{2}}_S}$$

$$\lesssim \delta^0 N_2^{-1+\gamma+s} \left\| P_N v^S \right\|_{X^{0, \frac{1}{2}}_S}$$

outside an exceptional set of probability $< e^{-\frac{1}{s}}$. If $s < 1 - \gamma$, then we obtain the desired result by summing over $N_1 \sim N_2 \gtrsim N$. We point out that when $N_1 \sim N_2 \sim 1$, it is possible that $\left| |n_1|^2 \pm |n_2| - |n|^2 \right| \ll 1$ but still $L_{\max} \gtrsim N_{\max}^2 \sim N_2^2$ is true.

Next, suppose that $L \ll L_{\max}$ and so $\max \{L_1, L_2\} \sim L_{\max}$. We may assume $L_1 \sim L_{\max}$. Then, we have

$$\left| \bar{\eta}_\delta (\tau_1 - |n_1|^2) \right| \lesssim \frac{1}{L_1} \sim N_{\max}^{-2} \quad \text{(4.12)}$$
since $\hat{\eta}_h(\tau) = \delta\hat{\eta}(\delta\tau)$. Then, from Hölder’s inequality with $p \gg 1$, Lemma 2.12, Young’s inequality in $\tau$, and Lemma 2.13, we have

$$
\left| \int_{\mathbb{R}} \int_{T^2} (\nabla)^s (P_{N_1} z^S P_{N_2} z^W) P_N v^S \, dt \right|
\approx \left\| P_N v^S \right\|_{\ell_n^2 L_\tau^1} \left\| \sum_{n_1, n_2 \in \mathbb{Z}^2 : n_1 + n_2 = n \atop |n_1| \sim |n_2|} \langle n \rangle^{s} g_{n_1} (\omega) \frac{h_{n_2} (\omega)}{\langle n_1 \rangle^{1-\gamma}} \int_{\tau_1 + \tau_2 = \tau} \hat{\eta}_h (\tau_1 - |n_1|^2) \hat{\eta}_h (\tau_2 \pm |n_2|) P_N v^S (n, \tau) d\tau_1 d\tau
\right|$$

outside an exceptional set of probability $< e^{-\frac{1}{p}}$. Hence, if $s < 1 - \gamma$, then we can perform the dyadic summation over $N_1 \sim N_2 \geq N$. The case $L_2 \sim L_{\max}$ follows from the same argument in the case $L_1 \sim L_{\max}$.

**Case 2:** $N_2 \lesssim N_1 \sim N$ (resonant interaction).

We split the case into the high and low modulation cases.

**Subcase 2.a:** $L_{\max} \gtrsim N_1^{2s+2\gamma}$ (high modulation case).

First, suppose that $L \sim L_{\max}$. Then, from $L^p_{l_{\tau},x} L^2_{l_{\tau},x} L^2_{l_{\tau},x}$ Hölder’s inequality, Lemma 2.13 with large $p$, and Lemma 2.3, we have

$$
\left| \int_{\mathbb{R}} \int_{T^2} (\nabla)^s (P_{N_1} z^S P_{N_2} z^W) P_N v^S \, dt \right|
\lesssim N_1^{s} \left| P_{N_1} z^S \right|_{L^p_{l_{\tau},x}} \left| P_{N_2} z^W \right|_{L^2_{l_{\tau},x}} \left| P_N v^S \right|_{L^2_{l_{\tau},x}}
\lesssim N_1^{s+\gamma} \left| P_N v^S \right|_{L^2_{l_{\tau},x}}
\lesssim \delta^{\frac{1}{2}} - N_1^{0-} \left| P_N v^S \right|_{X^{0, \frac{1}{2}}_S}
$$

outside an exceptional set of measure $< e^{-\frac{1}{p}}$. Hence, we can perform the dyadic summation over $N_1 \sim N \geq N_2$.

Next, suppose that $L \ll L_{\max}$ and so $\max \{ L_1, L_2 \} \sim L_{\max}$. We may assume $L_1 \sim L_{\max}$. Then, we have

$$
\left| \hat{\eta}_h (\tau_1 - |n_1|^2) \right| \lesssim \frac{1}{L_1} \sim N_{\max}^{-2s-2\gamma}.
$$

We note that

$$
\left| \int_{\mathbb{R}} \int_{T^2} (\nabla)^s (P_{N_1} z^S P_{N_2} z^W) P_N v^S \, dt \right|
\approx \left| \int_{\tau \in \mathbb{R}} \sum_{n \in \mathbb{Z}^2 : |n| \sim N} \langle n \rangle^{s} P_N v^S (n, \tau) \left( \sum_{n_1, n_2 \in \mathbb{Z}^2 : n_1 + n_2 = n \atop |n_1| \sim |n_2|} a_{n_1, n_2} (\tau) g_{n_1} (\omega) h_{n_2} (\omega) \right) d\tau \right|
$$

(4.14)
where

\[ a_{n_1,n_2,n}(\tau) = 1_{\{n_1+n_2=n\}} \langle n_1 \rangle^{-1} \langle n_2 \rangle^{-1+\gamma} \int_{\tau_1+\tau_2=\tau} \hat{\eta}_\delta(\tau_1 - |n_1|^2)\hat{\eta}_\delta(\tau_2 \pm |n_2|)d\tau. \]

Then, from Lemma 2.11 Minkowski’s inequality in \( \tau \) (with \( p \gg 1 \)), (4.13), and Young’s inequality, we have

\[
\left\| \sum_{n_2 \in \mathbb{Z}^2: \ n_1+n_2=n, \ |n_1|\sim N_1, |n_2|\sim N_2} a_{n_1,n_2,n}(\tau) \cdot \hat{g}_{n_1}(\omega)h_{n_2}(\omega) \right\|_{L_p^{\gamma}} \lesssim \delta^0 - N_1^{-1} \lesssim \delta^0 - \left( \sum_{n_2 \in \mathbb{Z}^2: \ n_1+n_2=n, \ |n_1|\sim N_1, |n_2|\sim N_2} \langle a_{n_1,n_2,n}(\tau) \rangle^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \delta^0 - N_1^{-1} + N_2^{-1+\gamma} \times \left( \sum_{n_2 \in \mathbb{Z}^2: \ |n_2|\sim N_2} \langle a_{n_1,n_2,n}(\tau) \rangle^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \delta^0 - N_1^{-1} + N_2^{-1+\gamma} N_1^{-2s-2\gamma+2} N_2. \tag{4.15}
\]

In (4.15), we need to make sure that the probability \( e^{-\delta^0 N_1^{\omega}} \) of the exceptional sets corresponding to different dyadic blocks and different values of \( n_2 \) should be summable and bounded by \( e^{-\delta^0} \) i.e. (4.15) holds outside an exceptional set of measure:

\[
\sum_{N_1} N_2^2 e^{-\frac{\delta^0 N_1^{\omega}}{s}} \lesssim e^{-\delta^0}.
\]

From (4.14), Hölder’s inequality in \( \tau \), (4.15), and Cauchy-Schwarz inequality in \( n \), we have

\[
\text{LHS of (4.14)} \lesssim \sum_{n \in \mathbb{Z}^2: \ |n|\sim N} N^s \| P_Nv^S \|_{L_p^{\gamma}} \left( \sum_{n_2 \in \mathbb{Z}^2: \ n_1+n_2=n, \ |n_1|\sim N_1, |n_2|\sim N_2} \| a_{n_1,n_2,n}(\tau) \|_{L_p^{\gamma}}^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \delta^{\frac{1-s}{p}} - N_1^{-1} + N_2^{-1+\gamma} N^s N_1^{-2s-2\gamma+2} N_2 \| P_Nv^S \|_{X_S^{0,\frac{1}{2}}}
\]

\[
\lesssim \delta^{\frac{1-s}{p}} - N_1^{-s-\gamma} \| P_Nv^S \|_{X_S^{0,\frac{1}{2}}}.
\]

Hence, we can perform the dyadic summation over \( N_1 \sim N \geq N_2 \) if \( s+\gamma > 0 \), which holds when \( s > 0 \).

**Subcase 2.b:** \( L_{\max} \lesssim N_1^{2s+2\gamma} \) (low modulation case).

We split the case into \( N_1 \sim N \sim N_2 \) and \( N_1 \sim N \gg N_2 \).

**Subsubcase 2.b.(i):** \( N_1 \sim N \sim N_2 \).

We note that

\[
\left| \int_{\mathbb{R}} \int_{\mathbb{T}^2} (\nabla)^s (P_{N_1} z^S P_{N_2} z^W) P_N v^S dx \right| \tag{4.16}
\]

\[
= \left| \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}^2: \ |n|\sim N} \langle n \rangle^s \widehat{P_Nv^S}(n,\tau) \left( \sum_{n_2 \in \mathbb{Z}^2: \ n_1+n_2=n, \ |n_1|\sim N_1, |n_2|\sim N_2} a_{n_1,n_2,n}(\tau) \hat{g}_{n_1}(\omega)h_{n_2}(\omega) \right) d\tau \right|
\]
where
\[ a_{n_1,n_2,n}(\tau) = 1_{\{n_1+n_2=n\}} (n_1)^{-1}(n_2)^{-1+\gamma} \int_{\tau_1+\tau_2=\tau} (\hat{\eta}_{\delta}(\tau_1 - |n_1|^2)\hat{\eta}_{\delta}(\tau_2 \pm |n_2|)) d\tau. \]

Then, from Lemma 2.11 Minkowski’s inequality in \( \tau \) (with \( p \gg 1 \)), and Young’s inequality, we have
\[
\| \sum_{n_1,n_2=\pm n, |n_1|^2+|n_2|^2=O(N_1^{2\gamma+2\gamma^+}), \n_1+\n_2=N_2} a_{n_1,n_2,n}(\tau) \cdot g_{n_1}(\omega) h_{n_2}(\omega) \|_{L_p^p} \leq \delta^0 - N_1^{0+} \left( \sum_{n_1,n_2=\pm n, |n_1|^2+|n_2|^2=O(N_1^{2\gamma+2\gamma^+}), \n_1+\n_2=N_2} \|a_{n_1,n_2,n}(\tau)\|_{L_p^p}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \delta^0 - N_1^{1-\gamma} N_2^{-\gamma} \left( \sum_{n_1,n_2=\pm n, |n_1|^2+|n_2|^2=O(N_1^{2\gamma+2\gamma^+}), \n_1+\n_2=N_2} \|\hat{\eta}_{\delta}\|_{L_2^2}^2 \|\hat{\eta}_{\delta}\|_{L_1^2}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \delta^0 - N_1^{1-\gamma} N_2^{-\gamma} \left( \sum_{n_1,n_2=\pm n, |n_1|^2+|n_2|^2=O(N_1^{2\gamma+2\gamma^+}), \n_1+\n_2=N_2} \|\hat{\eta}_{\delta}\|_{L_2^2}^2 \|\hat{\eta}_{\delta}\|_{L_1^2}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \delta^0 - N_1^{1-\gamma} N_2^{-\gamma} \left( \sum_{n_1,n_2=\pm n, |n_1|^2+|n_2|^2=O(N_1^{2\gamma+2\gamma^+}), \n_1+\n_2=N_2} \|\hat{\eta}_{\delta}\|_{L_2^2}^2 \|\hat{\eta}_{\delta}\|_{L_1^2}^2 \right)^{\frac{1}{2}}
\]
outside an exceptional set of probability \( < e^{-c N_1^{\gamma^+}} \), where
\[ S_n := \left\{ n_1 \in \mathbb{Z}^2 : |n_1|^2 + |n-n_1| - |n|^2 = O(N_1^{2\gamma+2\gamma^+}), |n_1| \sim N_1, |n-n_1| \sim N_2, \text{ and } |n| \sim N \right\}. \]

Notice that in (4.17) we need to make sure that the probability \( e^{-c N_1^{\gamma^+}} \) of the exceptional sets corresponding to different dyadic blocks and different values of \( n_2 \) should be summable and bounded by \( e^{-\frac{1}{N_1}} \), i.e. (4.17) holds outside an exceptional set of measure:
\[
\sum_{N_1} N_2^2 e^{-c N_1^{\gamma^+}} \lesssim e^{-\frac{1}{N_1}}.
\]

Let \( n_1 \in S_n \). Then, we have
\[ |n_1|^2 - |n|^2 = O(N_1^{2\gamma+2\gamma^+} + N_1) = O(N_1) \]
since \( |n-n_2| \sim N_2 \) and \( N_2 \sim N_1 \). Therefore, we have
\[ \|n_1| - |n| \| \lesssim 1 \]
since we are in the case \( |n_1| \sim N_1, |n| \sim N, N_1 \sim N, \text{ and } s + \gamma < \frac{1}{2} \). Therefore, \( |n_1| \in (|n| - c, |n| + c) \) for some constants \( c \). Let \( |n_1| = \sqrt{m}, \) where \( m \geq 0 \). Then, \( m \in (|n|^2 - 2c|n| + c^2, |n|^2 + 2c|n| + c^2) \) and so the possible number of \( m \) is given by \( |n| \sim N \). Hence, we have
\[
\sup_{n} \#S_n \lesssim N_1^{\gamma^+} N \quad (4.18)
\]
since if \((x, y) = n_1\), where \( x, y \in \mathbb{Z} \), then thanks to Lemma 2.9 the number of lattice points on a circle is given by
\[ \left| \{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = m \} \right| \lesssim N_1^{\gamma^+} \]
From (4.16), Hölder’s inequality in $\tau$, (4.17), Cauchy-Schwarz inequality in $n$, and (4.18), we have

\[
\text{LHS of (4.16)} \lesssim \delta^{\frac{\gamma - 1}{2}} N_1^{-1+\gamma} \| P_N v^S \|_{X_0^0, \frac{1}{2}} \left( \sum_{n_1,n_2} \| a_{n_1,n_2,n} \|_{L_q^p}^2 \right)^{\frac{1}{2}} 
\]

\[
\lesssim \delta^{\frac{1}{2}} N_1^{-1+\gamma} \| P_N v^S \|_{X_0^0, \frac{1}{2}} \left( \sum_{n_1,n_2} \| a_{n_1,n_2,n} \|_{L_q^p}^2 \right)^{\frac{1}{2}} 
\]

\[
\lesssim \delta^{\frac{1}{2}} N_1^{-1+\gamma} \| P_N v^S \|_{X_0^0, \frac{1}{2}} \left( \sum_{n_1,n_2} \| a_{n_1,n_2,n} \|_{L_q^p}^2 \right)^{\frac{1}{2}} 
\]

Hence, if $s < \frac{1}{2} - \gamma$, then we can perform the dyadic summation over $N_1 \sim N \gg N_2$.

**Subcase 2.b.(ii):** $N_1 \sim N \gg N_2.$

In this case, if we proceed as in (4.19) which is based on the Hilbert-Schmidt norm approach, we can no longer use the $N_2^{-1+\gamma}$ to perform the dyadic summation over $N_1 \sim N$. Therefore, the summation loss $N_2$ in $n$ (the third line of (4.19) i.e. the summation in $n$) is a big obstacle to performing the dyadic summation over $N_1 \sim N \gg N_2$. However, by using the operator norm approach and random matrix estimates, we can overcome the summation loss.

By taking the Fourier transform, we have

\[
\mathcal{F}_x P_{N_1} (P_{N} \nabla^s v^S)(n_2, t) = \sum_{n \in \mathbb{Z}^2: n + n_1 = n_2} \langle n \rangle^s \hat{P}_N v^S(n,t) \eta_\delta(t) H(n, n_2, t),
\]

where $\eta_\delta(t) = \eta(\delta^{-1}t)$ is from our notation (2.1), the random matrix $H(n, n_2, t)$ is defined by

\[
H(n, n_2, t) = \sum_{n_1 \in \mathbb{Z}^2} e^{-it |n_1|^2} g_{n_1}(\omega) \langle n_1 \rangle \chi_{\{n_1 = n_2 - n\}} \chi_{\{\phi(n_1,n_2,n) = O(N_1^{2+2\gamma})\}} \chi_{\{|n| \sim N\}} \prod_{j=1}^2 \chi_{\{|n_j| \sim N_j\}}
\]

and the phase function $\phi: (\mathbb{Z}^2)^3 \rightarrow \mathbb{R}$ is defined by

\[
\phi(n_1, n_2, n) = |n_1|^2 \pm |n_2| - |n|^2. \tag{4.20}
\]

Then, from Cauchy-Schwarz inequality, Lemma 2.12 and taking the operator norm, we have

\[
\left| \int_{\mathbb{T}^2 \times \mathbb{R}} (\nabla)^s \left[ N_S(P_{N_1} z^S, P_{N_2} z^W) \right] P_N v^S dx dt \right| 
\]

\[
\lesssim \| P_{N_2} z^W \|_{L_2^q, t} \left\| \sum_n \langle n \rangle^s \hat{P}_N v^S(n, t) \eta_\delta(t) H(n, n_2, t) \right\|_{L_2^q, t} \tag{4.21}
\]

\[
\lesssim N_2^{\gamma + \frac{\gamma}{2}} \| H(n, n_2, t) \|_{\ell^q_n \rightarrow \ell^q_t} \left\| \langle n \rangle^s \hat{P}_N v^S(n, t) \eta_\delta(t) \right\|_{L_2^q, t}
\]

\[
\lesssim N_2^{\gamma + \frac{\gamma}{2}} \| H(n, n_2, t) \|_{\ell^q_n \rightarrow \ell^q_t} \left\| \langle n \rangle^s \hat{P}_N v^S \|_{L_2^q, t} \right\|
\]
which holds outside an exceptional set of measure $e^{-\frac{1}{p}}$. The random matrix $H(n, n_2, t)$ can be written with the random tensor $h(n, n_1, n_2, t)$ as follows:

$$H(n, n_2, t) = \sum_{n_1 \in \mathbb{Z}^2} \frac{e^{-it|n_1|^2} g_{n_1}(\omega)}{\langle n_1 \rangle} 1_{\{n_1 = n_2 - n\}} 1_{\{\varphi(n_1, n_2, n) = O(N_1^{2+2\gamma})\}} 1_{\{n \sim N\}} \prod_{j=1}^2 1_{\{n_j \sim N_j\}}$$

$$= \sum_{n_1 \in \mathbb{Z}^2} h(n, n_1, n_2, t) g_{n_1}(\omega),$$

where

$$h(n, n_1, n_2, t) = e^{-it|n_1|^2} 1_{\{n_1 = n_2 - n\}} 1_{\{\varphi(n_1, n_2, n) = O(N_1^{2+2\gamma})\}} 1_{\{n \sim N\}} \prod_{j=1}^2 1_{\{n_j \sim N_j\}}.$$ (4.22)

Then, from Lemma 5.2 we have

$$\sup_{t \in \mathbb{R}} \|H(n, n_2, t)\|_{L^2_{n} \to L^2_{n_2}} \lesssim N_1^{s+\gamma} N_2^{-\frac{1}{2}} + N_1^{-\frac{1}{2}}.$$ (4.23)

Hence, from (4.23) and Lemma 5.3, we have

$$\sup_{t \in \mathbb{R}} \|H(n, n_2, t)\|_{L^2_{n} \to L^2_{n_2}} \lesssim N_1^{s+\gamma} N_2^{-\frac{1}{2}} + N_1^{-\frac{1}{2}}.$$ (4.24)

Hence, by combining (4.21) and (4.24), we have

$$\text{LHS of } (4.21) \lesssim \delta^{-\frac{1}{2}} (N_1^{2+\gamma} N_2^{-\frac{1}{2}} + N_1^{-\frac{1}{2} + s + \gamma}).$$

Therefore, if $s < \frac{1}{2} - \frac{\gamma}{2}$ and $\gamma < \frac{1}{2}$, then we can perform the dyadic summation over $N_1 \sim N \gg N_2$.

\[ \Box \]

**Lemma 4.7** ($z^S R^W$-case ). Let $0 < s < \min\{\frac{1}{2}, \ell + 1\}$ and $\ell > 0$. Then, for each small $\delta > 0$, we have

$$\|N_S(z^S, R^W)\|_{X^{s+\frac{3}{4}}_{\delta} W^{1,4}} \lesssim \delta^0 \|R^W\|_{X^{\ell+\delta} W^{1,4}}$$

outside an exceptional set of probability $e^{-\frac{1}{p}}$.

**Proof.** We perform the case-by-case analysis:

**Case 1:** $N_1 \gg N$ or $N \gg N_1$ (non-resonant interaction).

We may assume $N_1 \gg N$ by symmetry. In this case, we have

$$L_{\max} \gtrsim \|n_1\|^2 + |n_2| - |n_1|^2 \gtrsim N_1^{2+2\gamma} \sim N_2^2.$$ (4.25)

First, suppose that $\max(L_2, L) \sim L_{\max}$. Then, from $L_{t,x}^{p} L_{t,x}^{2} L_{t,x}^{2+}$ Hölder’s inequality, Lemma 2.13 with large $p$, (4.25), and Lemma 2.23 we have

$$\int_{\mathbb{R}} \int_{T^2} (\nabla)^s (P_N z^S P_N R^W) P_N v^S dx dt \lesssim N_2^s \|P_N z^S\|_{L^{p}_t L^{p}_x} \|P_N R^W\|_{L^{2+}_t L^{2+}_x} \|P_N v^S\|_{L^{2+}_t L^{2+}_x}$$

$$\lesssim N_2^{s-\ell} \|P_N R^W\|_{X^{\ell+\delta}_W} \|P_N v^S\|_{X^{\delta}_{\ell+\delta}}$$

$$\lesssim \delta^{-\frac{1}{4}} (N_1^{2+\gamma} N_2^{-\frac{1}{2}} + N_1^{-\frac{1}{2} + s + \gamma}).$$
outside an exceptional set of measure \(< e^{-\frac{1}{3}}\). Hence, if \(s < \ell + 1\), then we can perform the dyadic summation over \(N_1 \sim N_2 \geq N\).

Next, suppose that \(\max(L_2, L) \ll L_{\text{max}}\) and hence \(L_1 \sim L_{\text{max}}\). Then, we have

\[
|\tilde{\eta}_0(\tau_1 - |n_1|^2)| \lesssim \frac{1}{L_1} \sim N_{\text{max}}^{-2}
\]

(4.26)

since \(\tilde{\eta}_0(\tau) = \delta \hat{\eta}(\delta \tau)\). Then, from Hölder’s inequality with \(p \gg 1\), (4.26), Young’s inequality in \(\tau\), and Lemma 2.12 we have

\[
\left| \int_\mathbb{R} \int_{\mathbb{T}^2} (\nabla)^s (P_{N_1} z^S P_{N_2} R^W) P_N v^S dx \right| \lesssim \sum_{n_1, n \in \mathbb{Z}^2; n_1 + n_2 = n} \sum_{n_1 \sim 1} \frac{|\hat{g}_{n_1}(\omega)|}{(n_1)} \int_{\tau_1 + \tau_2 = \tau} \tilde{\eta}_0(\tau_1 - |n_1|^2) P_{N_2} R^W(\tau_2, n_2) P_{N_{\tau}}(n, \tau) d\tau_1 d\tau_2
\]

\[
\lesssim \|P_N v^S\|_{\ell^{2,1} L^{\frac{3}{2}}} \sum_{n_1, n \in \mathbb{Z}^2; n_1 + n_2 = n} \sum_{n_1 \sim 1} \frac{|\hat{g}_{n_1}(\omega)|}{(n_1)} N_2^{-2(1-\varepsilon)} \|\tilde{\eta}_0\|_{L^p} \|P_{N_2} R^W\|_{L^p} \|P_{N} R^W\|_{L^p} \|P_{N} v^S\|_{X^{0,\frac{1}{2}}}
\]

\[
\lesssim \delta^{s-\frac{1}{p}} N_1^{s-1} N_2^{s-1/2} N_2^{s-1} \|P_{N_2} R^W\|_{X^{s+\frac{1}{2}}} \|P_{N} v^S\|_{X^{0,\frac{1}{2}}}
\]

outside an exceptional set of probability \(< e^{-\frac{1}{3}}\). Hence, if \(s < \ell + 1\), then we can perform the dyadic summation over \(N_1 \sim N_2 \geq N\).

**Case 2:** \(N_2 \lesssim N_1 \sim N\) (resonant interaction).

We split the case into the high and low modulation cases.

**Subcase 2.a:** \(L_{\text{max}} \gtrsim N^{s+\frac{3}{4}} N_2^{\frac{3}{2}}\) (high modulation case).

First, suppose that \(\max(L_2, L) \sim L_{\text{max}}\). Then, from \(L^p_{t, x} L^2_{t, x} L^{2^\ast}_{t, x}\)-Hölder’s inequality, Lemma 2.13 with large \(p\), and Lemma 2.23 we have

\[
\left| \int_\mathbb{R} \int_{\mathbb{T}^2} (\nabla)^s (P_{N_1} z^S P_{N_2} R^W) P_N v^S dx \right| \lesssim N^s \|P_{N_1} z^S\|_{L^p_{t, x}} \|P_{N_2} R^W\|_{L^2_{t, x}} \|P_{N} v^S\|_{L^{2^\ast}_{t, x}}
\]

\[
\lesssim N^{s+} \|P_{N_2} R^W\|_{L^p_{t, x}} \|P_{N} v^S\|_{L^{2^\ast}_{t, x}}
\]

\[
\lesssim \delta^{s-\frac{3}{4}} N_2^{s-\frac{1}{2}} N_2^{-\ell-\frac{4}{8}} \|P_{N_2} R^W\|_{X^{s+\frac{1}{2}}} \|P_{N} v^S\|_{X^{0,\frac{1}{2}}}
\]

outside an exceptional set of probability \(< e^{-\frac{1}{3}}\). Hence, if \(s < \frac{1}{4}\), we obtain the desired result by summing over \(N_1 \sim N \geq N_2\).
Next, suppose that \( \max(L_2, L) \ll L_{\max} \) and so \( L_1 \sim L_{\max} \). Then, from Hölder’s inequality with \( p \gg 1 \), Young’s inequality in \( \tau \), and Lemma 2.12, we have

\[
\left| \int_{\mathbb{R}} \int_{\mathbb{T}^2} (\nabla)^s (P_{N_1} z^S P_{N_2} R^W) P_{N_1} v^S dx dt \right|
\]

\[
\leq \sum_{n_1, n_2} \left\| P_{N_1} v^S \right\|_{L^p_{n_1}} \left\| P_{N_2} v^S \right\|_{L^p_{n_2}} \left\| \left( n_1 \right) \frac{|g_{n_1}(\omega)|}{(n_1)} \left( n_2 \right) \right\|_{L^p_{n_2}} \left\| \eta_\delta(\tau_1 - |n_1|^2) P_{N_2} R^W(n_2, \tau_2) P_{N_1} v^S(n, \tau) d\tau_1 d\tau_2 \right\|
\]

\[
\lesssim \delta^{s-\frac{1}{2}} N^{s} N^{-1} \left| n_1 \right|^{\frac{3}{2}} \left| n_2 \right|^{\frac{3}{2}} \left\| P_{N_2} R^W \right\|_{X_W^{\ell, \frac{1}{2} +}} \left\| P_{N_1} v^S \right\|_{X_S^{\ell, \frac{1}{2} -}}
\]

outside an exceptional set of probability \( \ll e^{-\frac{1}{2} \tau} \). Hence, if \( \ell > 0 \), we can perform the dyadic summation over \( N_1 \sim N \geq N_2 \).

**Subcase 2.b:** \( L_{\max} \ll N^{s+\frac{1}{2} +} N_2^{\frac{3}{2}} \) (low modulation case).

We first rewrite \( \mathcal{N}_S(P_{N_1} z^S, P_{N_2} R^W) \) as a random operator. By taking the Fourier transform, we have

\[
\mathcal{F}_x \mathcal{N}_S(P_{N_1} z^S, P_{N_2} R^W)(n, t) = \sum_{n_2} P_{N_2} R^W(n_2, t) \eta_\delta(t) H(n, n_2, t),
\]

where \( \eta_\delta(t) = \eta(\delta^{-1} t) \) is from our notation (2.1), the random matrix \( H(n, n_2, t) \) is defined by

\[
H(n, n_2, t) = \sum_{n_1} \frac{e^{-it|n_1|^2} g_{n_1}(\omega)}{(n_1)} 1_{\{n_1 = n_2\}} 1_{\{\varphi(n_1, n_2, n) = O(N^{s+\frac{1}{2} +} N_2^{\frac{3}{2}})\}} 1_{\{|n| \sim N\}} \prod_{j=1}^2 1_{\{|n_j| \sim N_j\}},
\]

and the phase function \( \varphi : (\mathbb{Z}^2)^3 \to \mathbb{R} \) is defined by

\[
\varphi(n_1, n_2, n) = |n_1|^2 \pm |n_2| - |n|^2.
\]

Then, from Cauchy-Schwarz inequality and taking the operator norm, we have

\[
\left| \int_{\mathbb{T}^2 \times \mathbb{R}} (\nabla)^s \left( \mathcal{N}_S(P_{N_1} z^S, P_{N_2} R^W) \right) P_{N_1} v^S dx dt \right|
\]

\[
\lesssim \left\| (\nabla)^s P_{N_1} v^S \right\|_{L^2_{n_1}} \left\| \sum_{n_2} P_{N_2} R^W(n_2, t) \eta_\delta(t) H(n, n_2, t) \right\|_{L^2_{n_2}}
\]

\[
\lesssim N^s \left\| P_{N_1} v^S \right\|_{L^2_{n_1}} \left\| H(n, n_2, t) \right\|_{L^2_{n_2}} \left\| P_{N_2} R^W(n_2, t) \right\|_{L^2_{n_2}}
\]

\[
\lesssim N^s N_2^{\frac{3}{2}} \sup_{t \in \mathbb{R}} \left\| H(n, n_2, t) \right\|_{L^2_{n_2}} \left\| P_{N_2} R^W \right\|_{X_W^{\ell, \frac{1}{2} +}} \left\| P_{N_1} v^S \right\|_{L^2_{n_1}}.
\]
The random matrix $H(n, n_2, t)$ can be written with the random tensor $h(n, n_1, n_2, t)$ as follows:

$$H(n, n_2, t) = \sum_{n_1 \in \mathbb{Z}^2} e^{-it|n_1|^2} g_{n_1}(\omega) \frac{1}{\langle n_1 \rangle} 1_{\{n_1 = n - n_2\}} \{v(n_1, n_2, n) = O(N^{s+\frac{1}{2}+\frac{4}{2}})\} \frac{1}{\langle n \rangle} \prod_{j=1}^{2} 1_{\{n_j \sim N_j\}}$$

$$= \sum_{n_1 \in \mathbb{Z}^2} h(n, n_1, n_2, t) g_{n_1}(\omega),$$

where

$$h(n, n_1, n_2, t) = e^{-it|n_1|^2} 1_{\{n_1 = n - n_2\}} \frac{1}{\langle n_1 \rangle} 1_{\{v(n_1, n_2, n) = O(N^{s+\frac{1}{2}+\frac{4}{2}})\}} \frac{1}{\langle n \rangle} \prod_{j=1}^{2} 1_{\{n_j \sim N_j\}}. \quad (4.29)$$

Then, from Lemma 5.2, we have

$$\sup_{t \in \mathbb{R}} \|H(n, n_2, t)\|_{\ell_n^2 \rightarrow \ell_n^2} \lesssim N_1^s \sup_{t \in \mathbb{R}} \max(\|h(t)\|_{n_1 \rightarrow n_1}, \|h(t)\|_{n_2 \rightarrow n_1}). \quad (4.30)$$

Hence, from (4.30) and Lemma 5.6, we have

$$\sup_{t \in \mathbb{R}} \|H(n, n_2, t)\|_{\ell_n^2 \rightarrow \ell_n^2} \lesssim N_1^s - \frac{3}{8} + N_2^{-\frac{1}{8}} + N_1^{-\frac{1}{8}}. \quad (4.31)$$

Hence, by combining (4.28) and (4.31), we have

$$\text{LHS of } (4.28) \lesssim N_1^{-\frac{3}{8} - \frac{3}{8}} + N_2^{-\frac{1}{8} - \ell} + N_1^{-\frac{1}{8}} + N_2^{-\ell}.$$

Therefore, if $s < \frac{1}{8}$ and $\ell > 0$, then we can perform the dyadic summation over $N_1 \sim N \gtrsim N_2$. \hfill \Box

**Lemma 4.8 (RSzW-case).** Let $\gamma < \frac{1}{3}$ and $0 < s < 1 - \gamma$. Then, for each small $\delta > 0$, we have

$$\|N_S(RS, zW)\|_{X_{S, \delta}^{s, -\frac{1}{8}}} \lesssim \delta^{0+} \|RS\|_{X_{S, \delta}^{s, -\frac{1}{8}}} \quad (4.32)$$

outside an exceptional set of probability $< e^{-\frac{1}{8}}$.

**Proof.** We perform the case-by-case analysis:

**Case 1:** $N_1 \gg N$ or $N \gg N_1$ (non-resonant interaction).

We first consider the (worse) case $N \gg N_1$. In this case, we have

$$L_{\max} \gtrsim \|n_1|^2 + |n_2| - |n_2|^2 \gtrsim N_{\max}^2 \sim N_2^2. \quad (4.33)$$

First suppose that $\max(L_1, L) \sim L_{\max}$. Then, from $L_{t,x}^{l,p} L_{t,x}^{l,p} L_{t,x}^{l,p}$-Hölder’s inequality with $p$ large, Lemma 2.33, (2.11), (4.33), and Lemma 2.3, we have

$$\int_{\mathbb{T}^2} \int_{\mathbb{R}} \langle \nabla \rangle^s (P_{N_1} R_S P_{N_2} z W) P_N v S dx dt \lesssim N_2^s \|P_{N_1} R_S\|_{L_{t,x}^{l,p}} \|P_{N_2} z W\|_{L_{t,x}^{l,p}} \|P_N v S\|_{L_{t,x}^{l,p}} \lesssim \delta^{0+} N_2^{-\frac{1}{8} + s + \gamma} \|P_{N_1} R_S\|_{X_{S, \delta}^{s, -\frac{1}{8}}} \|P_N v S\|_{X_{S, \delta}^{s, -\frac{1}{8}}} \lesssim \delta^{0+} N_2^{-\frac{1}{8} + s + \gamma} N_1^{-s} \|P_{N_1} R_S\|_{X_{S, \delta}^{s, -\frac{1}{8}}} \|P_N v S\|_{X_{S, \delta}^{s, -\frac{1}{8}}},$$

for an exceptional set of measure $< e^{-\frac{1}{8}}$. Hence, if $s + \gamma < 1$, then we can perform the dyadic summation over $N \sim N_2 \geq N_1$. Notice that in the case $N_1 \gg N$, it suffices to assume $\gamma < 1$. 
Next, suppose that $\max(L_1, L) \ll L_{\text{max}}$ and so $L_2 \sim L_{\text{max}}$. Then, from Hölder’s inequality with $p \gg 1$, Young’s inequality in $\tau$, and Lemma 2.12, we have
\[
\left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} R^S P_{N_2} z^W) P_N v^S dx dt \right|
\leq \sum_{n_1, n_2 \in \mathbb{Z}^2 : n_1 + n_2 = n \atop |n| \sim N_1, n_2 \sim N_2} \langle n \rangle^s \frac{\hat{h}_{n_2}(\omega)}{\langle n_2 \rangle^{1-\gamma}} \int_{\tau_1 + \tau_2 = \tau} \widehat{P_{N_1} R^S}(\tau_1, n_1) \hat{\eta}_0(\tau_2 \pm |n_2|) \widehat{P_N v^S}(n, \tau) d\tau_1 d\tau_2
\leq \|P_N v^S\|_{\ell^2(L^1)} \sum_{n_1, n_2 \in \mathbb{Z}^2 : n_1 + n_2 = n \atop |n| \sim N_1, n_2 \sim N_2, n_2 \sim N} \langle n \rangle^s \frac{\hat{h}_{n_2}(\omega)}{\langle n_2 \rangle^{1-\gamma}} N_2^{-2(1-\varepsilon)} \|\hat{\eta}_0\|_L^p \|P_{N_1} R^S\|_{L^p(L^1)} \|P_N v^S\|_{L^p(L^\infty)}
\leq \delta^{2+} N_{s-1}^{1+\gamma} + N_{s-2}^{1} \|P_{N_1} R^S\|^0_{X^0_s} + \|P_N v^S\|^0_{X^0_s}
\leq \delta^{2+} N_{s-1}^{1+\gamma} + N_{s-2}^{1} \|P_{N_1} R^S\|^0_{X^0_s} + \|P_N v^S\|^0_{X^0_s}
\end{equation}
outside an exceptional set of probability $< e^{-\frac{1}{\delta}}$. Hence, if $s < 1 - \gamma$, we can perform the dyadic summation over $N \sim N_2 \geq N_1$. Notice that in the case $N_1 \gg N$, it suffices to assume $\gamma < 1$.

Case 2: $N_2 \lesssim N_1 \sim N$ (resonant interaction).

We split the case into the high and low modulation cases.

Subcase 2.a: $L_{\text{max}} \gtrsim N_2^{1+\gamma}$ (high modulation case).

When $N_1 \gg N_2$, we first write $\{|n| \sim N_1\} = \bigcup_{\ell_1} J_{1, \ell_1}$ and $\{|n| \sim N\} = \bigcup_{\ell_2} J_{2, \ell_2}$, where $J_{1, \ell_1}$ and $J_{2, \ell_2}$ are balls of radius $\sim N_2$, we can decompose $P_{N_1} R^S$ and $P_{N_2} v^S$ as
\[
P_{N_1} R^S = \sum_{\ell_1} P_{N_1, \ell_1} R^S \quad \text{and} \quad P_{N_2} v^S = \sum_{\ell_2} P_{N_2, \ell_2} v^S
\]
where $P_{N_1, \ell_1} R^S(n_1, t) = 1_{J_{1, \ell_1}}(n_1) \widehat{P_{N_1} R^S}(n_1, t)$ and $P_{N_2, \ell_2} v^S(n, t) = 1_{J_{2, \ell_2}}(n) \widehat{P_{N_2} v^S}(n, t)$.

Given $n_1 \in J_{1, \ell_1}$ for some $\ell_1$, there exists $O(1)$ many possible values for $\ell_2 = \ell_2(\ell_1)$ such that $n \in J_{2\ell_2}$ under $n_1 + n_2 = n$. Notice that the number of possible values of $\ell_2$ is independent of $\ell_1$.

First suppose that $\max(L_1, L) \sim L_{\text{max}}$. Then, from $L^2_{t,x} + L^p_{t,x} L^2_{t,x}$-Hölder’s inequality with $p$ large, Lemma 2.13, 2.11, and Lemma 2.3, we have
\[
\left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1, \ell_1} R^S P_{N_2} z^W) P_{N_2, \ell_2} v^S dx dt \right|
\leq \langle n \rangle^s \frac{\hat{h}_{n_2}(\omega)}{\langle n_2 \rangle^{1-\gamma}} \int_{\tau_1 + \tau_2 = \tau} \widehat{P_{N_1, \ell_1} R^S}(\tau_1, n_1) \hat{\eta}_0(\tau_2 \pm |n_2|) \widehat{P_{N_2} z^W}(n, \tau) d\tau_1 d\tau_2
\leq \delta^{2+} N_{s-1}^{1+\gamma} + N_{s-2}^{1} \|P_{N_1, \ell_1} R^S\|^0_{X^0_s} + \|P_{N_2, \ell_2} v^S\|^0_{X^0_s}
\leq \delta^{2+} N_{s-1}^{1+\gamma} + N_{s-2}^{1} \|P_{N_1, \ell_1} R^S\|^0_{X^0_s} + \|P_{N_2, \ell_2} v^S\|^0_{X^0_s}
\end{equation}
for an exceptional set of measure $< e^{-\frac{1}{\delta}}$. Hence, we can perform the the Cauchy-Schwarz inequality in $\ell_1$ and the dyadic summation over $N_1 \sim N \gtrsim N_2$ if $\gamma < 1$.

Next, suppose that $\max(L_1, L) \ll L_{\text{max}}$ and so $L_2 \sim L_{\text{max}}$. Then, from Hölder’s inequality with $p \gg 1$, Young’s inequality in $\tau$, Lemma 2.12 and Cauchy-Schwarz inequality in $n_1 \in J_{1, \ell_1}$,
we have

\[
\left| \int_\mathbb{T} \int_\mathbb{R} \langle \nabla \rangle^s (P_{N_1, \ell_1} R^S P_{N_2} z^W) P_{N_1, \ell_1} v^S \, dx \, dt \right|
\]

\[
= \left| \sum_{n_1, n_2 \in \mathbb{Z}^2, n_1 + n_2 = n} \langle n \rangle^s \frac{h_{n_2}(\omega)}{\langle n_2 \rangle^{1-\gamma}} \int_{\tau_1 + \tau_2 = \tau} P_{N_1, \ell_1} \hat{R}^S(n_1) \hat{\eta}_0(\tau_2 \pm |n_2|) P_{N_1, \ell_1} \hat{v}^S(n, \tau) \, d\tau_1 \, d\tau \right|
\]

\[
\lesssim \| P_{N_1, \ell_1} \hat{v}^S \|_{\ell^2_2 L^1_x} \left( \sum_{n_1 \in \mathbb{Z}^2, n_2 = n} \langle n \rangle^s \frac{|h_{n_2}(\omega)|}{\langle n_2 \rangle^{1-\gamma}} N_2^{-1+\gamma+\eta} N_2^{-1-\gamma-2\eta} \right) \| P_{N_1, \ell_1} \hat{R}^S \|_{L^p_x} \| P_{N_1, \ell_1} \hat{v}^S \|_{X^{0, \frac{1}{2}+}_S}
\]

\[
\lesssim \delta^{\frac{1}{p}} N_2^{-1+\gamma+\eta} N_2^{-1-\gamma-2\eta} \| P_{N_1, \ell_1} \hat{R}^S \|_{X^{0, \frac{1}{2}+}_S} \| P_{N_1, \ell_1} \hat{v}^S \|_{X^{0, \frac{1}{2}-}_S}
\]

outside an exceptional set of probability \( < e^{-\frac{1}{p}} \). Hence, we can perform the Cauchy-Schwarz inequality in \( \ell_1 \) and the dyadic summation over \( N \sim N_1 \geq N_2 \).

**Subcase 2.b:** \( L_{\text{max}} \lesssim N_2^{1+\gamma} \) (low modulation case).

We rewrite \( N_S(P_{N_1, \ell_1} R^S, P_{N_2} z^W) \) as a random operator. By taking the Fourier transform, we have

\[
\mathcal{F}_x N_S(P_{N_1, \ell_1} R^S, P_{N_2} z^W)(n, t) = \sum_{n_1} P_{N_1, \ell_1} \hat{R}^S(n_1, t) \eta_0(t) H(n, n_1, t),
\]

where \( \eta_0(t) = \eta(\delta^{-1} t) \) is from our notation (2.1), the random matrix \( H(n, n_1, t) \) is defined by

\[
H(n, n_1, t) := \sum_{n_2 \in \mathbb{Z}^2} e^{-i|n_2|} h_{n_2}(\omega) \mathbf{1}_{\{n_2 = n-n_1\}} \mathbf{1}_{\{\varphi(n_1, n_2, n) = O(N_2^{1+\gamma})\}} \mathbf{1}_{\{n_1 \in J_{\ell_1}\}} \mathbf{1}_{\{n_2 \sim N_2\}},
\]

and the phase function \( \varphi : (\mathbb{Z}^2)^3 \rightarrow \mathbb{R} \) is defined by

\[
\varphi(n_1, n_2, n) = |n_1|^2 \pm |n_2|^2 - |n|^2.
\]

Then, from Cauchy-Schwarz inequality, Minkowski’s inequality in \( \tau \) and taking the operator norm, we have

\[
\left| \int_{\mathbb{T}^2 \times \mathbb{R}} \langle \nabla \rangle^s [N_S(P_{N_1, \ell_1} R^S, P_{N_2} z^W)] P_{N_1, \ell_1} v^S \, dx \, dt \right|
\]

\[
\lesssim \left\| \langle \nabla \rangle^s P_{N_1, \ell_1} v^S \right\|_{L^2_{x,t}} \left( \sum_{n_1} \| P_{N_1, \ell_1} \hat{R}^S(n_1, t) \eta_0(t) H(n, n_1) \|_{L^2_{x,t}} \right)
\]

\[
\lesssim N^s \| P_{N_1, \ell_1} v^S \|_{L^2_{x,t}} \| H(n, n_1, t) \|_{\ell^2_{n_1} \rightarrow \ell^2_{n_1}} \| P_{N_1, \ell_1} \hat{R}^S(n_1, t) \|_{\ell^2_{n_1}} \| (H(n, n_1, t)) \|_{L^1_{x,t}}
\]

\[
\lesssim \sup_{t \in \mathbb{R}} \| H(n, n_1, t) \|_{\ell^2_{n_1} \rightarrow \ell^2_{n_1}} \| P_{N_1, \ell_1} v^S \|_{X^{0, \frac{1}{2}-}_S} \| P_{N_1, \ell_1} R^S \|_{X^{0, \frac{1}{2}+}_S}
\]
The random matrix $H(n, n_1, t)$ in (4.34) can be written with the random tensor $h(n, n_1, n_2, t)$ as follows:

\[ H(n, n_1, t) = \sum_{n_2 \in \mathbb{Z}^2} h(n, n_1, n_2, t) h_{n_2}(\omega), \]

where

\[ h(n, n_1, n_2, t) = e^{-it|n_2|} \mathbf{1}_{\{n_2 = n-n_1, |n_2| \sim N_2\}} \mathbf{1}_{\{\phi(n_1, n_2, n) = O(N_2^{1+\gamma})\}} \mathbf{1}_{\{n_1 \in J_1\}} \mathbf{1}_{\{n \in J_2\}} (n_2)^{-1+\gamma}. \]

(4.37)

Then, from Lemma 5.2 and 5.3, we have

\[ \|H(n, n_1, t)\|_{\ell^2_{n_1} \rightarrow \ell^2_{n_2}} \lesssim N^2 \varepsilon_{\text{max}} \|h(t)\|_{n_1n_2 \rightarrow n} \|h(t)\|_{n_1 \rightarrow n_2}. \]

(4.38)

Hence, from (4.38) and Lemma 5.7, we have

\[ \sup_{t \in \mathbb{R}} \|H(n, n_1, t)\|_{\ell^2_{n_1} \rightarrow \ell^2_{n_2}} \lesssim N^{-\frac{1}{2}+\frac{3}{2}\gamma}. \]

(4.39)

From (4.36), (4.39), and the Cauchy-Schwarz inequality in $\ell^1$, we have

\[ \text{LHS of (4.36)} \lesssim N_2^{-\frac{1}{2}+\frac{3}{2}\gamma} \|P_{N_2} v^S\|_{X^{\frac{1}{2}+\gamma}_S} \|P_{N_1} R^\ell\|_{X^{\frac{1}{2}+\gamma}_S}, \]

where we used the fact that the number of possible value of $\ell_2 = \ell_2(\ell_1)$ is independent of $\ell_1$. Therefore, if $\gamma < \frac{1}{3}$, we can perform the dyadic summation over $N_1 \sim N \gg N_2$.

When $N_1 \sim N_2$, the above argument used in the case $N_1 \gg N_2$ is itself applicable without using the orthogonality argument.

\[ \square \]

4.5. **Bilinear estimates for the wave part.** In this subsection, we prove (4.9) in Lemma 4.1. In view of (4.4), in order to prove (4.9), we need to carry out case-by-case analysis on

\[ \|\chi_\delta \cdot N_W(u_1, u_2)\|_{X^{\ell_{-\frac{1}{2}}}_{W_{-\frac{1}{2}}}} \]

(4.40)

where $u_j$ is taken to be either of type

(I) rough random part:

\[ u_j = \tilde{z}^S, \text{ where } \tilde{z}^S \text{ is some extension of } \chi_\delta \cdot z^S \]

where $z^S$ denotes the random linear solution defined in (4.5),

(II) smoother ‘deterministic’ remainder (nonlinear) part:

\[ u_j = \tilde{R}^S, \text{ where } \tilde{R}^S \text{ is any extension of } R^S, \]

In the following, when $u_j$ is of type (I), we take $\tilde{z}^S = \eta_\delta z^S$. Thanks to the cutoff function in (4.40), we may take $u_j = \eta_\delta \cdot \tilde{R}^S$ in (4.40) when $u_j$ is of type (II).

**Remark 4.9.** In the following, we drop the $\pm$ signs and work with one $w_+$ or $w_-$ since there is no role of $\pm$. Hence, we set $w := w_+$ and $W := W_+$. 
**Remark 4.10.** To estimate \( \| \chi_t \cdot \mathcal{N}_W (u_1, u_2) \|_{X_{^{\frac{1}{2}}}} \), we need to perform case-by-case analysis of expressions of the form:

\[
\int_R \int_{\mathbb{T}^2} \langle \nabla \rangle^{\ell - 1 + 2\gamma} \mathcal{N}_W (u_1, u_2) v^W \ dx \ dt,
\]

where \( \| v^W \|_{X_{^{\frac{1}{2}}}} \leq 1 \). As in Remark 1.3, for simplicity of notation, we drop the complex conjugate sign and suppress the smooth time-cutoff function \( \eta_t \) and thus simply denote them by \( Z^S \) and \( R^S \), respectively when there is no confusion. We dyadically decompose \( u_1 \) and \( u_2 \) such that their spatial frequency support are \( \text{supp} \tilde{u}_1 \subset \{ |n| \sim N_1 \} \), \( \text{supp} \tilde{u}_2 \subset \{ |n| \sim N_2 \} \), and \( \text{supp} v^W \subset \{ |n| \sim N \} \) for some dyadic \( N_1, N_2 \) and \( N \geq 1 \). Lastly, we point out that \( n_1 \neq n_2 \) thanks to the renormalization (see (1.22)).

We now prove Lemmas 4.11, 4.12, and 4.13 which will imply Lemma 4.1’s second part (4.9) (bilinear estimates for the wave part).

**Lemma 4.11** (\( R^S R^S \)-case). Let \( \ell < 1 - 2\gamma + s \) and \( s > 0 \). Then, we have

\[
\| \mathcal{N}_W (R^S, R^S) \|_{X_{^{\frac{1}{2}}}} \lesssim \delta^{\frac{1}{2} -} \| R^S \|_{X_{s, 1}} \| R^S \|_{X_{s, 1}}.
\]

**Proof.** We may assume \( N_1 \gg N_2 \) by symmetry.

**Case 1:** \( N_1 \gg N_2 \)

Let \( P_{=0} \) be the projection onto non-zero frequencies: \( P_{=0}f := f - \int_{\mathbb{T}^2} f \). Then, from the boundedness of \( P_{=0} \) on \( L^p (\mathbb{T}^2) \), \( L^4_{t,x} L^4_{t,x} L^2_{t,x} \)-Hölder’s inequality, the \( L^4 \)-Strichartz estimate (Lemma 2.7), and Lemma 2.3, we have

\[
\left| \int_R \int_{\mathbb{T}^2} \langle \nabla \rangle^{\ell - 1 + 2\gamma} P_{=0} (P_{N_1} R^S P_{N_2} R^S) P_N v^W \ dx \ dt \right|
\lesssim N^{\ell - 1 + 2\gamma} N_1^{-s} N_2^{-s} \| P_{N_1} R^S \|_{X_{s, 1}} \| P_{N_2} R^S \|_{X_{s, 1}} \| P_N v^W \|_{X_0^{0,0}}
\lesssim \delta^{\frac{1}{2} -} N^{\ell - 1 + 2\gamma - s} N_2^{-s} \| P_{N_1} R^S \|_{X_{s, 1}} \| P_{N_2} R^S \|_{X_{s, 1}} \| P_N v^W \|_{X_0^{0,0}}.
\]

Hence, if \( \ell < 1 - 2\gamma + s \) and \( s > 0 \), we can perform the dyadic summation over \( N_1 \sim N \gg N_2 \).

**Case 2:** \( N_1 \sim N_2 \gg N \)

By \( L^4_{t,x} L^4_{t,x} L^2_{t,x} \)-Hölder’s inequality, the \( L^4 \)-Strichartz estimate (Lemma 2.7), and Lemma 2.3, we have

\[
\left| \int_R \int_{\mathbb{T}^2} \langle \nabla \rangle^{\ell - 1 + 2\gamma} P_{=0} (P_{N_1} R^S P_{N_2} R^S) P_N v^W \ dx \ dt \right|
\lesssim \delta^{\frac{1}{2} -} N^{\ell - 1 + 2\gamma} N_1^{-2s} \| P_{N_1} R^S \|_{X_{s, 1}} \| P_{N_2} R^S \|_{X_{s, 1}} \| P_N v^W \|_{X_0^{0,0}}.
\]

Hence, if \( \ell < 1 - 2\gamma + 2s \) and \( s > 0 \), we can perform the dyadic summation over \( N_1 \sim N_2 \gg N \). \( \square \)

---

\( P_{=0} \) is clearly bounded on \( L^p (\mathbb{T}^2) \), \( 1 \leq p \leq \infty \).
Lemma 4.12 ($z^S z^S$-case). Let $\ell < 1 - 2\gamma$. Then, for each small $\delta > 0$, we have

$$\|N_W(z^S, z^S)\|_{X^\ell, \frac{1}{2+}} \lesssim \delta^{0+}$$

outside an exceptional set of probability $< e^{-\frac{1}{\delta^2}}$.

Proof. We may assume $N_1 \geq N_2$ by symmetry.

Case 1: $N_1 \gg N_2$ (non-resonant interaction).

In this case, we have

$$L_{\text{max}} \gtrsim \|n_1|^2 - |n_2|^2 \pm |n| \gtrsim N_{\text{max}}^2 \sim N_1^2.$$  \hfill (4.41)

First suppose that $L \sim L_{\text{max}}$. Then, from the boundedness of $P \neq 0$ on $L^p(T^2)$, $L^p_{t,x} L^2_{t,x}$, Hölder’s inequality with $p$ large, Lemma 2.13 (4.41), and Lemma 2.3, we have

$$\left| \int_{T^2} \langle \nabla \rangle^{\ell-1+2\gamma} P \neq 0 (P_{N_1} z^S P_{N_2} z^S) P_N v^W dxdt \right| \lesssim N_1^{\ell-1+2\gamma} \|P_{N_1} z^S\|_{L^p} \|P_{N_2} z^S\|_{L^2} \|P_N v^W\|_{L^2_{t,x}}$$

$$\lesssim \delta^{0+} L_{\text{max}} N_1^{\ell-1+2\gamma} \|P_{N_1} v^W\|_{X^{0, \frac{1}{2}}}$$

$$\lesssim \delta^{0+} N_1^{\ell-2+2\gamma} \|P_{N_1} v^W\|_{X^{0, \frac{1}{2}}}$$

outside an exceptional set of measure $< e^{-\frac{1}{\delta^2}}$. Hence, if $\ell < 2 - 2\gamma$, we can perform the dyadic summation over $N_1 \sim N \geq N_2$.

Next, suppose that $L \ll L_{\text{max}}$ and so $\max(L_1, L_2) \sim L_{\text{max}}$. Then, from Hölder’s inequality with $p \gg 1$, Young’s inequality in $\tau$, and Lemma 2.12, we have

$$\left| \int_{T^2} \langle \nabla \rangle^{\ell-1+2\gamma} P \neq 0 (P_{N_1} z^S P_{N_2} z^S) P_N v^W dxdt \right|$$

$$= \left| \sum_{n_2 \in \mathbb{Z}^2; n_1 - n_2 = n} \frac{\langle n \rangle^{\ell-1+2\gamma} g_{n_1}(\omega) g_{n_2}(\omega)}{\langle n_1 \rangle \langle n_2 \rangle} \int_{\tau_1 \tau_2 = \tau} \tilde{\eta}_\delta(\tau_1 - |n_1|^2) \tilde{\eta}_\delta(\tau_2 - |n_2|^2) P_N v^W(n, \tau) d\tau_1 d\tau_2 \right|$$

$$\lesssim \|P_N v^W\|_{L^2_{t,x}} \left| \sum_{n_2 \in \mathbb{Z}^2; n_1 - n_2 = n} \frac{\langle n \rangle^{\ell-1+2\gamma} g_{n_1}(\omega) g_{n_2}(\omega)}{\langle n_1 \rangle \langle n_2 \rangle} N_1^{-2(1-\varepsilon)} \|\tilde{\eta}_\delta\|_{L^p_{t,x}} \|\tilde{\eta}_\delta\|_{L^1_{t,x}} \right|$$

$$\lesssim \delta^{0+} N_1^{\ell-1+2\gamma} N_1^{-1} N_2^{-1} + N_1^{-2} N_2 N \|P_N v^W\|_{X_{S}^{0, \frac{1}{2}}}$$

$$\lesssim \delta^{0+} N_1^{\ell-2+2\gamma} \|P_N v^W\|_{X_{S}^{0, \frac{1}{2}}}.$$

Hence, if $\ell < 2 - 2\gamma$, we can perform the dyadic summation over $N_1 \sim N \gg N_2$.

Case 2: $N_1 \sim N_2 \gg N$ (resonant interaction).
We note that
\[
\left| \int_\mathbb{R} \int_\mathbb{T} (\nabla \cdot (P_{N_1} \rho S^P N_2 \rho S^R) P_N v^W \, dx \, dt) \right|
\]
\[
= \int_\mathbb{R} \sum_{n \in \mathbb{Z}^2; |n| \sim N} \langle n \rangle^{\ell - 1 + 2\gamma} \rho_{n_1 \rho n_2} (n, \tau) \left( \sum_{n_1 \neq n_2} a_{n_1 n_2, n}(\tau) f_{n_1}(\omega) \overline{f_{n_2}(\omega)} \right) d\tau
\]
\[
(4.41)
\]
where
\[
a_{n_1 n_2, n}(\tau) = \langle n_1 \rangle^{-1} \langle n_2 \rangle^{-1} \int_{\tau_1 + \tau_2 = \tau} \tilde{\gamma}_{n_1}(\tau_1 - |n_1|^2) \tilde{\gamma}(\tau_2 - |n_2|^2) d\tau_1.
\]
Then, from Lemma 2.11, Minkowski’s inequality in \(\tau\) (with \(p \gg 1\)), (4.13), Young’s inequality, and no pairing condition \(n_1 \neq n_2\), we have
\[
\left\| \sum_{n_1 \neq n_2; n_1 \neq n_2, n_1 - n_2 = n, |n_1| \sim N_1, |n_2| \sim N_2, n_1 \neq n_2} a_{n_1 n_2, n}(\tau) f_{n_1}(\omega) \overline{f_{n_2}(\omega)} \right\|_{L^p} \lesssim \delta^{0 - N_1^{\gamma} + \frac{1}{2}} \left( \sum_{n_2 \in \mathbb{Z}^2; |n_2| \sim N_2, n_1 - n_2 = n, |n_1| \sim N_1, |n_2| \sim N_2} \|a_{n_1 n_2, n}(\tau)\|_{L^p}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \delta^{0 - N_1^{\gamma} + \frac{1}{2}} N_2^{-1 - N_1^{\gamma}} N_2 \left( \sum_{n_2 \in \mathbb{Z}^2; |n_2| \sim N_2} \|\tilde{\gamma}_{n_2}\|_{L^2}^2 \right)^{\frac{1}{2}},
\]
\[
\lesssim \delta^{0 - N_1^{\gamma} + \frac{1}{2}} N_2^{-1 - N_1^{\gamma}} N_2.
\]
(4.42)
In (4.43), we need to make sure that the probability \(e^{-\varepsilon \frac{N_2^4}{\delta_1}}\) of the exceptional sets corresponding to different dyadic blocks and different values of \(n_2\) should be summable and bounded by \(e^{-\varepsilon}\) i.e. (4.43) holds outside an exceptional set of measure:
\[
\sum_{N_1} N_2^2 e^{-\varepsilon \frac{N_2^4}{\delta_1}} \lesssim e^{-\varepsilon}.
\]
From (4.12), Hölder’s inequality in \(\tau\), (4.13), and Cauchy-Schwarz inequality in \(n\), we have
\[
\text{LHS of (4.42)} \lesssim \sum_{n \in \mathbb{Z}^2; |n| \sim N} N_1^{\ell - 1 + 2\gamma} \|P_N v^W\|_{L^p} \left( \sum_{n_1 \neq n_2, n_1 \neq n_2; n_1 - n_2 = n, |n_1| \sim N_1, |n_2| \sim N_2} \|a_{n_1 n_2, n}(\tau)\|_{L^p}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \delta^{1 - \frac{1}{p} - \frac{1}{2} - N_1^{\gamma} - N_1 - N_2^{-1}} N_2 N \|P_N v^W\|_{X^{0, \frac{1}{2}}_{W, \delta}}
\]
\[
\lesssim \delta^{1 - \frac{1}{p} - N_1^{\gamma} - 2\gamma} \|P_N v^W\|_{X^{0, \frac{1}{2}}_{W, \delta}}.
\]
Hence, if \(\ell < 1 - 2\gamma\), we can perform the dyadic summation over \(N_1 \sim N_2 \gtrsim \mathcal{N}\).  

**Lemma 4.13** \((s^S R^S\text{-case})\). Let \(\ell < 1 - 2\gamma\) and \(s > 0\). Then, for each small \(\delta > 0\), we have
\[
\|N_W (z^S, R^S)\|_{X^{0, \frac{1}{2}}_{W, \delta}} \lesssim \delta^{0 + \frac{1}{2}} \|R^S\|_{X^{0, \frac{1}{2}}_{S, \delta}}
\]
outside an exceptional set of probability \(< e^{-\varepsilon}\).
Proof. We perform the case-by-case analysis:

**Case 1:** \( N_1 \gg N_2 \) (non-resonant interaction).

In this case, we have

\[
L_{\text{max}} \gtrsim \left| |n_1|^2 - |n_2|^2 \pm n \right| \gtrsim N_{\text{max}}^2 \sim N_1^2.
\]

First suppose that \( \max(L_2, L) \sim L_{\text{max}} \). Then, from the boundedness of \( P_{\neq 0} \) on \( L^p(\mathbb{T}^2) \), \( L_{t,x}^p \mathbb{T}_{t,x}^{2+} L_{t,x}^2 \)-Hölder’s inequality with \( p \) large, Lemma 2.13, 2.11, and Lemma 2.3, we have

\[
\left| \int \int_{\mathbb{T}^2} \langle \nabla \rangle^{\ell - 1 + 2\gamma} P_{\neq 0}(P_{N_1} z^S P_{N_2} R^S) P_N v^W dxdt \right|
\]

outside an exceptional set of measure \( < e^{-\frac{\delta}{2}} \). Hence, if \( \ell < 2 - 2\gamma \) and \( s > 0 \), then we can perform the dyadic summation over \( N_1 \sim N \geq N_2 \).

Next, suppose that \( \max(L_2, L) \ll L_{\text{max}} \) and so \( L_1 \sim L_{\text{max}} \). Then, from Hölder’s inequality with \( p \gg 1 \), Young’s inequality in \( \tau \), Lemma 2.12, and Cauchy-Schwarz inequality in \( n_2 \), we have

\[
\left| \int \int_{\mathbb{T}^2} \langle \nabla \rangle^{\ell - 1 + 2\gamma} P_{\neq 0}(P_{N_1} z^S P_{N_2} R^S) P_N v^W dxdt \right|
\]

\[
= \left| \sum_{n_2, n \in \mathbb{Z}^2 : n_1 - n_2 = n \atop |n_1| \sim N_1, |n_2| \sim N_2, n_1 \neq n_2} (n)^s g_{n_1}(\omega) \int_{\tau_1 + \tau_2 = \tau} \tilde{\omega}_\delta(\tau_1 - |n_1|^2) P_{N_2} R^S(n_2, \tau_2 - |n_2|^2) P_N v^W(n, \tau) d\tau_1 d\tau \right|
\]

\[
\lesssim \left| \left| P_N v^W \right|_{L^p_t L^{s+}_x} \right|^\ell \left| \left| \sum_{n_2, n \in \mathbb{Z}^2 : n_1 - n_2 = n \atop |n_1| \sim N_1, |n_2| \sim N_2, n_1 \neq n_2} (n)^{\ell - 1 + 2\gamma} \frac{g_{n_1}(\omega)}{(n_1)} N_1^{-2(1 - \epsilon)} \left| \left| \tilde{\omega}_\delta^\epsilon \right|_{L_\mu} \left| \left| P_{N_2} R^S \right|_{L^{s+}_x} \right| \right|^\ell \right|
\]

\[
\lesssim \delta^{\frac{\ell - 1 + 2\gamma}{2}} N_1^{-1 + 1} N_2^{-1 + 2} N_2 N \left| \left| P_{N_2} R^S \right|_{X^{\frac{1}{2} +}_x} \right| \left| \left| P_N v^W \right|_{X^{\frac{1}{2} -}_x} \right|
\]

\[
\lesssim \delta^{\frac{\ell - 1 + 2\gamma}{2}} N_1^{-1 + 1} N_2^{-1 + 2} N_2 N \left| \left| P_{N_2} R^S \right|_{X^{\frac{1}{2} +}_x} \right| \left| \left| P_N v^W \right|_{X^{\frac{1}{2} -}_x} \right|
\]

Hence, if \( \ell < 2 - 2\gamma \) and \( s > 0 \), then we can perform the dyadic summation over \( N_1 \sim N \gg N_2 \).

**Case 2:** \( N_1 \ll N_2 \) (non-resonant interaction).

In this case, we have

\[
L_{\text{max}} \gtrsim \left| |n_1|^2 - |n_2|^2 \pm n \right| \gtrsim N_{\text{max}}^2 \sim N_2^2.
\]

We point out that in Case 2, it suffices to assume \( \ell < 2 - 2\gamma + s \) by proceeding with the proof of Case 1 \( (N_1 \gg N_2) \).

**Case 3:** \( N_1 \sim N_2 \gg N \) (resonant interaction).
From $L^p_{t,x} L^{2+}_{t,x} L^2_{t,x}$-Hölder’s inequality with $p$ large, Lemma 2.13, (2.11), and Lemma 2.3, we have

\[
\int_{\mathbb{R}} \int_{\mathbb{T}^2} \left| \mathcal{G}^{-1} \frac{\partial}{\partial \tau} \right| \mathcal{P}_{\neq 0}(P_{N_{1}} z^S P_{N_{2}} R^S) P_{N} v^w \, dx \, dt \\
\lesssim N^{\frac{1}{2} - 2 \gamma} \| P_{N_{1}} z^S \|_{L^p_{t,x}} \| P_{N_{2}} R^S \|_{L^{2+}_{t,x}} \| P_{N} v^w \|_{L^2_{t,x}} \\
\lesssim \delta^{\frac{1}{2}} - 2 \gamma N_{2}^{-s} \| P_{N_{2}} R^S \|_{X_{S}^{\frac{1}{2}}} \| P_{N} v^w \|_{X_{W}^{\frac{1}{2}}}.
\]

outside an exceptional set of measure $< e^{-\frac{1}{10}}$. Hence, if $\ell < 1 - 2 \gamma$ and $s > 0$, we can perform the dyadic summation over $N_{1} \sim N_{2} \geq N$.

\[\square\]

5. RANDOM TENSOR THEORY

In this section, we present the proof of random tensor estimates which were used in the proof of Lemmas 4.7, 4.8, and 4.13.

5.1. Random tensors. In this subsection, we provide the basic definition and some lemmas on (random) tensors from [21, 11, 49, 12]. See [21, Sections 2 and 4] and [11, Section 4] for further discussion.

**Definition 5.1.** Let $A$ be a finite index set. We denote by $n_A$ the tuple $(n_j : j \in A)$. A tensor $h = h_{n_A}$ is a function: $(\mathbb{Z}^2)^A \to \mathbb{C}$ with the input variables $n_A$. Note that the tensor $h$ may also depend on $\omega \in \Omega$. The support of a tensor $h$ is the set of $n_A$ such that $h_{n_A} \neq 0$.

Given a finite index set $A$, let $(B, C)$ be a partition of $A$. We define the norms $\| \cdot \|_{n_A}$ and $\| \cdot \|_{n_B \to n_C}$ by

\[
\| h \|_{n_A} = \| h \|_{\ell^2_{n_A}} = \left( \sum_{n_A} |h_{n_A}|^2 \right)^{\frac{1}{2}}
\]

and

\[
\| h \|_{n_B \to n_C} = \sup \left\{ \sum_{n_C} \left| \sum_{n_B} h_{n_A} f_{n_B} \right|^2 : \| f \|_{\ell^2_{n_B}} = 1 \right\},
\]

(5.1)

where we used the short-hand notation $\sum_{n_Z}$ for $\sum_{n_Z \in (\mathbb{Z}^2)^Z}$ for a finite index set $Z$. Note that, by duality, we have $\| h \|_{n_B \to n_C} = \| h \|_{n_C \to n_B} = \| h \|_{n_B \to n_C}$ for any tensor $h = h_{n_A}$. If $B = \emptyset$ or $C = \emptyset$, then we have $\| h \|_{n_B \to n_C} = \| h \|_{n_A}$.

For example, when $A = \{1, 2\}$, the norm $\| h \|_{n_1 \to n_2}$ denotes the usual operator norm $\| h \|_{\ell^2_{n_1} \to \ell^2_{n_2}}$ for an infinite dimensional matrix operator $\{h_{n_1 n_2}\}_{n_1, n_2 \in \mathbb{Z}^2}$. By bounding the matrix operator norm by the Hilbert-Schmidt norm (= the Frobenius norm), we have

\[
\| h \|_{\ell^2_{n_1} \to \ell^2_{n_2}} \leq \| h \|_{\ell^2_{n_1 \to n_2}}.
\]

Let $(B, C)$ be a partition of $A$. Then, by duality, we can write (5.1) as

\[
\| h \|_{n_B \to n_C} = \sup \left\{ \left| \sum_{n_B, n_C} h_{n_A} f_{n_B} g_{n_C} \right| : \| f \|_{\ell^2_{n_B}} = \| g \|_{\ell^2_{n_C}} = 1 \right\},
\]

from which we obtain

\[
\sup_{n_A} |h_{n_A}| = \sup_{n_B, n_C} |h_{n_B n_C}| \leq \| h \|_{n_B \to n_C}.
\]
Before we state the main lemma in this subsection (Lemma 5.2), we first present the following notations. For a complex number \( z \), we define \( z^+ = z \) and \( z^- = \overline{z} \). Let \( A \) be a finite index set. For each \( j \in A \), we associate \( j \) with a sign \( \zeta_j \in \{ \pm \} \). For \( j_1, j_2 \in A \), we say that \( (n_{j_1}, n_{j_2}) \) is a pairing if \( n_{j_1} = n_{j_2} \) and \( \zeta_{j_1} = -\zeta_{j_2} \). Let \( \{ g_n \}_{n \in \mathbb{Z}^2} \) be a set of independent standard complex-valued Gaussian random variables. We write in polar coordinates

\[
g_k(\omega) = \rho_k(\omega) \eta_k(\omega)
\]

where \( \rho_k = |g_k| \) and \( \eta_k = \rho_k^{-1} g_k \). Then all the \( \rho_k \) and \( \eta_k \) are independent, and each \( \eta_k \) is uniformly distributed on the unit circle of \( \mathbb{C} \).

We now present the following random tensor estimate. For the proof, see Proposition 4.14 in [21].

**Lemma 5.2** (Proposition 4.14 in [21]). Let \( \delta > 0 \), \( A \) be a finite set and \( h_{bcnA} = h_{bcnA}(\omega) \) be a random tensor, where each \( n_j \in \mathbb{Z}^2 \) and \( (b,c) \in (\mathbb{Z}^2)^q \) for some integer \( q \geq 2 \). Given signs \( \zeta_j \in \{ \pm \} \), we also assume that \( \langle b \rangle, \langle c \rangle \leq M \) and \( \langle n_j \rangle \leq M \) for all \( j \in A \), where \( M \) is a dyadic number, and that in the support of \( h_{bcnA} \), there is no pairing in \( n_A \). Define the tensor

\[
H_{bc} = \sum_{n_A} h_{bcnA} \prod_{j \in A} \eta_j^{\zeta_j}.
\]

where we assume that \( \{ h_{bcnA} \} \) is independent with \( \{ \eta_n \}_{n \in \mathbb{Z}^2} \). Then, there exists constants \( C, c > 0 \) such that we have

\[
\| H_{bc} \|_{b \rightarrow c} \lesssim \delta^{-\theta} M^\theta \cdot \max_{(B,C)} \| h \|_{bnB \rightarrow cnC},
\]

outside an exceptional set of probability \( \leq C \exp(-\frac{cM^\theta}{\delta}) \) with \( \theta > 0 \), where \( (B,C) \) runs over all partitions of \( A \).

For example, under the independence assumption\(^{22}\) with high probability we have

\[
\| H_{bd} \|_{b \rightarrow d} \lesssim \max(\| h \|_{abc \rightarrow ad}, \| h \|_{ab \rightarrow cd}, \| h \|_{bc \rightarrow ad}, \| h \|_{b \rightarrow acd}), \quad \text{where} \quad H_{bd} = \sum_{a,c} h_{bdac} y^a_c y^b_c.
\]

We also present the following variant of Lemma 5.2. For the proof, see Proposition 4.15 in [21].

**Lemma 5.3** (Proposition 4.15 in [21]). Consider the same setting as in Lemma 5.2 with the following differences:

1. We only restrict \( \langle n_j \rangle \leq M \) for all \( j \in A \) but do not impose any condition on \( \langle b \rangle \) or \( \langle c \rangle \).
2. We assume that \( b, c \in \mathbb{Z}^2 \) and that in the support of the random tensor \( h_{bcnA} \) we have \( |b - \zeta c| \leq M \) where \( \zeta \in \{ \pm \} \).
3. The random tensor \( h_{bcnA} \) only depends on \( b - \zeta c, |b|^2 - \zeta |c|^2 \), and \( n_A \), and is supported in the set where \( \| b - \zeta c |^2 \| \leq M^5 \).

\(^{22}\)Lemma 5.2 requires the independence of \( \{ h_{bcnA} \} \) and \( \{ \eta_n \} \). Since \( \rho_n = |g_n| \) and \( \eta_n = \rho_n^{-1} g_n \) are independent, we know that \( \{ h_{bdac} \rho_n \rho_c \} \) and \( \{ \eta_n \} \) are independent, which satisfies the assumption in Lemma 5.2.
Then, there exists constants $C, c > 0$ such that we have
\[ \|H_{bc}\|_{b\rightarrow c} \lesssim \delta^{-\theta} M^\theta \cdot \max_{(B,C)} \|h\|_{bnB\rightarrow cnC}, \]
outside an exceptional set of probability $\leq C \exp(-\frac{cM}{\delta^\theta})$ with $\theta > 0$, where $(B, C)$ runs over all partitions of $A$.

5.2. Deterministic tensor estimates. In this subsection, we present the deterministic tensor estimates (Lemma 5.5, 5.6, and 5.7).

Remark 5.4. Lemma 5.5, 5.6, and 5.7 play an important role in proving Lemma 4.6 (in particular, Subsubcase 2.b(ii)), Lemma 4.7 (in particular, Subcase 2.b), and Lemma 4.8 (in particular, Subcase 2.b), respectively.

Lemma 5.5 (First deterministic tensor estimate). Let $h_{n,n_1,n_2}(t) := h(n, n_1, n_2, t)$ be the random tensor defined in (4.22):
\[ h(n, n_1, n_2, t) = e^{-it|n|^2}1_{\{n_1 = n_2 - n\}}1_{\{\varphi(n_1, n_2, n) = O(N_1^{2s+2\gamma^+})\}}\langle n_1 \rangle^{-1}1_{\{|n| \sim N\}} \prod_{j=1}^2 1_{\{|n_j| \sim N_j\}}, \]
where the phase function $\varphi(n_1, n_2, n)$ is given in (4.20) and $N_1 \sim N \gg N_2$. Then, we have
\[ \sup_{t \in \mathbb{R}} \|h(t)\|_{n \rightarrow n_2} \lesssim N_1^{s+\gamma-\frac{1}{2}+}N_2^{-\frac{s}{2}} + N_1^{-\frac{s}{2}+}, \quad (5.3) \]
\[ \sup_{t \in \mathbb{R}} \|h(t)\|_{n_1 \rightarrow n_2} \lesssim N_1^{-\frac{s}{2}+}. \quad (5.4) \]

Proof. We first prove (5.3). By the Schur’s test, we have that
\[ \|h(t)\|_{n_1 \rightarrow n_2}^2 \lesssim \left( \sup_{n_2} \sum_{n_1, n} |h_{n_1,n_2,n}(t)| \right) \left( \sup_{n_1, n} \sum_{n_2} |h_{n_1,n_2,n}(t)| \right) \]
\[ \lesssim N_1^{-2} \sup_{|n_2| \sim N_2} \left\{ \{(n_1, n) : n_2 = n_1 + n, \varphi(n_1, n_2, n) = O(N_1^{2s+2\gamma^+}), |n_1| \sim N_1, |n| \sim N\} \right\} \]
\[ \times \sup_{n_1, n} \left\{ n_2 : n_2 = n_1 + n, \varphi(n_1, n_2, n) = O(N_1^{2s+2\gamma^+}) \right\} \]
\[ \leq \left( \sup_{|n_2| \sim N_2} \left\{ n_2 : n_2 = n_1 + n, \varphi(n_1, n_2, n) = O(N_1^{2s+2\gamma^+}) \right\} \right) \]
\[ \approx \sup_{|n_2| \sim N_2} \left\{ n_1 : |n_1|^2 + |n_2| - |n_2 - n_1|^2 = O(N_1^{2s+2\gamma^+}) \right\} \]
\[ = \sup_{|n_2| \sim N_2} \#S_{n_2}. \quad (5.5) \]
Let $n_1 \in S_{n_2}$. Then, we have $|n_1|^2 + |n_2| - |n_2 - n_1|^2 = O(N_1^{2s+2\gamma^+})$ and so $|n_2| \sim N_2$ implies that

$$\frac{n_2}{|n_2|} \cdot n_1 = -\frac{|n_2|}{2} \pm \frac{1}{2} + O\left(\frac{N_2^{2s+2\gamma^+}}{N_2}\right)$$

i.e. the component of $n_1$ parallel to $n_2$ is restricted in an interval of length $O\left(\frac{N_2^{2s+2\gamma^+}}{N_2}\right)$. Since $|n_1| \sim N_1$, we have

$$\sup_{n_2 \sim N_2} \#S_{n_2} \lesssim \left(\frac{N_1^{2s+2\gamma^+}}{N_2} + 1\right) N_1 \lesssim N_1^{2s+2\gamma^+ + 1} N_2^{-1} + N_1$$

(5.7)

Therefore, from (5.5), (5.6), and (5.7), we have

$$\|h\|_{n_1, n_2 \to n_2}^2 \lesssim N_1^{-2} \left( N_1^{2s+2\gamma^+ + 1} N_2^{-1} + N_1 \right) \lesssim N_1^{2s+2\gamma^+ - 1} N_2^{-1} + N_1^{-1}$$

which proves (5.3).

We now prove (5.10). By the Schur’s test, we have that

$$\|h(t)\|_{n \to n_1 n_2}^2 \lesssim \left( \sup_{n_1, n_2} \sum_{n_1, n_2} |h_{n_1 n_2 n}(t)| \right) \left( \sup_{n_1, n_2} \sum_n |h_{n_1 n_2 n}(t)| \right)$$

$$\lesssim N_1^{-2} \sup_{|n| \sim N} \left\{ (n_1, n_2) : n_2 = n_1 + n, \varphi(n_1, n_2, n) = O(N_1^{2s+2\gamma^+}) \wedge |n_1| \sim N_1, |n_2| \sim N_2 \right\}$$

$$\times \sup_{n_1, n_2} \left\{ n : n_2 = n_1 + n, \varphi(n_1, n_2, n) = O(N_1^{2s+2\gamma^+}) \right\}$$

(5.8)

We first consider the last factor in (5.8). Since $n$ is uniquely determined by $n_1$ and $n_2$, the last factor in (5.8) can be bounded by 1:

$$\sup_{n_1, n_2} \left\{ n : n_2 = n_1 + n, \varphi(n_1, n_2, n) = O(N_1^{2s+2\gamma^+}) \right\} \lesssim 1.$$  

(5.9)

As for the remaining part, we have

$$\sup_{|n| \sim N} \left\{ (n_1, n_2) : n_2 = n_1 + n, \varphi(n_1, n_2, n) = O(N_1^{2s+2\gamma^+}) \wedge |n_1| \sim N_1, \text{ and } |n_2| \sim N_2 \right\}$$

$$\lesssim \sup_{|n| \sim N} \left\{ n_1 : |n_1|^2 + |n_1 + n|^2 = O(N_1^{2s+2\gamma^+}) \wedge |n_1| \sim N_1, \text{ and } |n_1 + n| \sim N_2 \right\}$$

$$= \sup_{|n| \sim N} \#S_n.$$  

(5.10)

Let $n_1 \in S_n$. Then, we have

$$|n_1|^2 - |n|^2 = O(N_1^{2s+2\gamma^+} + N_1) = O(N_1)$$

since $|n_1 + n| \sim N_2 \lesssim N_1$, and $2s + 2\gamma < 1$. Therefore, we have

$$|n_1| - |n| \lesssim 1,$$

which implies that $|n_1| \in (|n| - c, |n| + c)$ for some constants $c$. Let $|n_1| = \sqrt{m}$, where $m \geq 0$. Then, $m \in (|n|^2 - 2c|n| + c^2, |n|^2 + 2c|n| + c^2)$ and so the possible number of $m$ is given by
\[ |n| \sim N. \text{ Hence, we have} \]
\[
\sup_{|n| \sim N} \# S_n \lesssim N_1^2 N, \quad (5.11)
\]
since if \((x, y) = n_1\), where \(x, y \in \mathbb{Z}\), then thanks to Lemma 2.9 the number of lattice points on a circle is given by
\[
\left| \{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = m\} \right| \lesssim N_1^\varepsilon.
\]
Hence, from (5.8), (5.9), (5.10), and (5.11), we have
\[
\|h\|^2_{n \to n_2 n_1} \lesssim N_1^{-2} N_1^{\varepsilon} N \lesssim N_1^{-1+\varepsilon},
\]
which proves (5.4).

**Lemma 5.6** (Second deterministic tensor estimate). Let \(h_{n, n_1, n_2}(t) := h(n, n_1, n_2, t)\) be the random tensor defined in (4.29):
\[
h(n, n_1, n_2, t) = e^{-it|n|^2} \sum_{n_1 = n - n_2}^1 \left\{ \varphi(n_1, n_2, n) = O\left(N^{s+\frac{1}{2}} + N_2^\frac{3}{2}\right) \right\}
\]
where the phase function \(\varphi(n_1, n_2, n)\) is given in (4.27) and \(N_1 \sim N \gtrsim N_2\). Then, we have
\[
\sup_{t \in \mathbb{R}} \|h(t)\|_{n_1 n_2 \to n} \lesssim N_1^{\frac{s}{2} - \frac{1}{2}},
\]
\[
\sup_{t \in \mathbb{R}} \|h(t)\|_{n_2 \to n_1 n} \lesssim N_1^{\frac{s}{2} - \varepsilon} N_2^{-\frac{s}{2}} + N_1^{-\frac{s}{2}}. \quad (5.13)
\]

**Proof.** We first prove (5.12). By the Schur’s test, we have that
\[
\|h(t)\|^2_{n, n_2 \to n} \lesssim \left( \sup_{n_1, n_2} \sum_n |h_{n_1 n_2 n}(t)| \right)^2 \left( \sup_{n_1, n_2} \sum_n |h_{n_1 n_2 n}(t)| \right)
\]
\[
\lesssim N_1^{-2} \sup_{n_1, n_2} \left| \left\{ n : n = n_1 + n_2, |\varphi(n_1, n_2, n)| = O\left(N^{s+\frac{1}{2}} + N_2^\frac{3}{2}\right) \right\} \right|
\]
\[
\times \sup_{|n| \sim N} \left| \left\{ (n_1, n_2) : n = n_1 + n_2, |\varphi(n_1, n_2, n)| = O\left(N^{s+1}\right), |n_1| \sim N_1, |n_2| \sim N_2 \right\} \right| \quad (5.14)
\]
We first consider the first factor in (5.14). Since \(n\) is uniquely determined by \(n_1\) and \(n_2\), the first factor in (5.14) can be bounded by 1:
\[
\sup_{n_1, n_2} \left| \left\{ n : n = n_1 + n_2, |\varphi(n_1, n_2, n)| = O\left(N^{s+1}\right) \right\} \right| \lesssim 1. \quad (5.15)
\]
As for the remaining part, we have
\[
\sup_{|n| \sim N} \left| \left\{ (n_1, n_2) : n = n_1 + n_2, |\varphi(n_1, n_2, n)| = O\left(N^{s+1}\right), |n_1| \sim N_1, \text{ and } |n_2| \sim N_2 \right\} \right|
\]
\[
\lesssim \sup_{|n| \sim N} \left| \left\{ n_1 : |n_1|^2 + |n - n_1| - |n|^2 = O\left(N^{s+1}\right), |n_1| \sim N_1, \text{ and } |n - n_1| \sim N_2 \right\} \right|
\]
\[
= \sup_{|n| \sim N} \# S_n. \quad (5.16)
\]
Let $n_1 \in S_n$. Then, we have
\[ |n_1|^2 - |n|^2 = O(N^{s+1} + N_2) \]
Therefore, we have
\[ |n| - cN^{s+} \leq |n_1| \leq |n| + cN^{s+} \]
for some constants $c > 0$. Since $|n| \sim N$ and $N \gg cN^{s+}$ with $s < 1$, $n_1$ is contained in an annulus of thickness $\sim N^{s+}$ and a ball of radius $\sim N_1$. Hence, we have
\[ \sup_{|n| \sim N} \# S_n \lesssim N^{s+} N_1 \sim N^{s+1} \]  
Hence, from (5.14), (5.15), (5.16), and (5.17), we have
\[ \sup_{t \in \mathbb{R}} ||h(t)||^2_{n_1 n_2 \to n} \lesssim N_1^{-2} N^{s+1} \lesssim N_1^{s+1}, \]  
which proves (5.12).

As for (5.13), by the Schur’s test, we have that
\[
||h(t)||^2_{n_2 \to n} \lesssim \left( \sup_{n_2} \sum_{n_1,n} |h_{n_1,n_2}(t)| \right) \left( \sup_{n_1,n} \sum_{n_2} |h_{n_1,n_2}(t)| \right) 
\lesssim N_1^{-2} \sup_{n_1,n} \left\{ n_2 : n = n_1 + n_2, \varphi(n_1,n_2,n) = O(N^{s+1} N_2^{\frac{3}{2}}) \right\} \]
\[
\times \sup_{n_2} \sup_{|n_2| \sim N_2} \left\{ (n_1,n) : n = n_1 + n_2, \varphi(n_1,n_2,n) = O(N^{s+1} N_2^{\frac{3}{2}}), |n_1| \sim N_1, |n| \sim N \right\}.
\]
(5.18)

Since $n_2$ is uniquely determined by $n_1$ and $n$, the first factor can be bounded by 1. As for the remaining part, we have
\[
\sup_{n_2 : |n_2| \sim N_2} \left\{ (n_1,n) : n = n_1 + n_2, |\varphi(n_1,n_2,n)| = O(N^{s+1} N_2^{\frac{3}{2}}), |n_1| \sim N_1, and |n| \sim N \right\} 
\lesssim \sup_{|n_2| \sim N_2} \left\{ n_1 : |n_1|^2 \pm |n_2| - |n_1 + n_2|^2 = O(N^{s+1} N_2^{\frac{3}{2}}), |n_1| \sim N_1, and |n_1 + n_2| \sim N \right\} 
= \sup_{|n_2| \sim N_2} \# S_{n_2}.
\]
(5.19)

Let $n_1 \in S_{n_2}$. Then, we have $|n_1|^2 \pm |n_2| - |n_1 + n_2|^2 = O(N^{s+1} N_2^{\frac{3}{2}})$ and so $|n_2| \sim N_2$ implies that
\[
\frac{n_2}{|n_2|} \cdot n_1 = -\frac{|n_2|}{2} \pm \frac{1}{2} + O(N^{s+1} N_2^{-\frac{3}{2}})
\]  
i.e. the component of $n_1$ parallel to $n_2$ is restricted in an interval of length $O(N^{s+1} N_2^{-\frac{3}{2}})$. Since $|n_1| \sim N_1$, we have
\[
\sup_{|n_2| \sim N_2} \# S_{n_2} \lesssim (N^{s+1} N_2^{-\frac{3}{2}} + 1) N_1 \lesssim N^{s+1} N_2^{-\frac{3}{2}} + N_1 \]  
(5.20)
Therefore, from \((5.18), (5.19),\) and \((5.20)\), we have
\[
\sup_{t \in \mathbb{R}} \| h(t) \|_{n_2 \rightarrow n_1}^2 \lesssim N_1^{-2} \left( N^{s + \frac{1}{2}} + N_2^{-\frac{1}{2}} + N_1 \right) \lesssim N^{s - \frac{1}{2}} N_2^{-\frac{1}{2}} + N_1^{-1},
\]
which proves \((5.13)\).

\[\square\]

**Lemma 5.7** (Third deterministic tensor estimate). Let \(h_{n_1, n_2}(t) := h(n, n_1, n_2, t)\) be the random tensor defined in \((4.37)\):
\[
h(n, n_1, n_2, t) = e^{-it|n_2|} \mathbb{1}_{\{n_2 = -n_1, |n_2| \sim N_2\}} \mathbb{1}_{\{n \in J_{1+\gamma}\}} \mathbb{1}_{\{n_1 \in J_{2}\}} \langle n_2 \rangle^{1+\gamma},
\]
where the phase function \(\varphi(n_1, n_2, n)\) is given in \((4.35)\) and \(N_1 \sim N \gtrsim N_2\). Then, we have
\[
\begin{align*}
\sup_{t \in \mathbb{R}} \| h(t) \|_{n_1 \rightarrow n_2} & \lesssim N_2^{-\frac{1}{2} + \frac{1}{2} + \gamma}, \\
\sup_{t \in \mathbb{R}} \| h(t) \|_{n_1 \rightarrow n_2} & \lesssim N_2^{-\frac{1}{2} + \frac{1}{2} + \gamma}.
\end{align*}
\]

**Proof.** We first prove \((5.21)\). By the Schur’s test, we have that
\[
\| h(t) \|_{n_1 \rightarrow n_2} \lesssim \left( \sup_n \sum_{n_1, n_2} |h_{n_1, n_2}(t)| \right) \left( \sup_n \sum_{n_1, n_2} |h_{n_1, n_2}(t)| \right)
\]
\[
\lesssim \sup_{n \in J_{2\gamma}} \left\{ \{ (n_1, n_2) : n = n_1 + n_2, \ |\varphi(n_1, n_2, n)\| = O(N_2^{1+\gamma}), n_1 \in J_{1\ell_1}, |n_2| \sim N_2 \} \right\}
\times N_2^{-2+2\gamma} \sup_{n_1, n_2} \left\{ n : n = n_1 + n_2, \ |\varphi(n_1, n_2, n)\| = O(N_2^{1+\gamma}) \} \right\}.
\]

\[\text{(5.23)}\]

We first consider the last factor in \((5.23)\). Since \(n\) is uniquely determined by \(n_1\) and \(n_2\), the last factor in \((5.23)\) can be bounded by \(1\):
\[
\sup_{n_1, n_2} \{|n : n = n_1 + n_2, \ |\varphi(n_1, n_2, n)\| = O(N_2^{1+\gamma})\} \lesssim 1.
\]

\[\text{(5.24)}\]

As for the remaining part, we have
\[
\begin{align*}
\sup_{n \in J_{2\gamma}} \left\{ \{ (n_1, n_2) : n = n_1 + n_2, \ |\varphi(n_1, n_2, n)\| = O(N_2^{1+\gamma}), n_1 \in J_{1\ell_1}, |n_2| \sim N_2 \} \right\}
\lesssim \sup_{n \in J_{2\gamma}} \left\{ n_1 : |n_1|^2 \pm |n - n_1| - |n|^2 = O(N_2^{1+\gamma}), n_1 \in J_{1\ell_1}, |n - n_1| \sim N_2 \right\}
\end{align*}
\]
\[\text{(5.25)}\]

Let \(n_1 \in S_n\). Then, we have
\[
|n_1|^2 - |n|^2 = O(N_2^{1+\gamma} + N_2) = O(N_2^{1+\gamma})
\]
Therefore, we have
\[
|n_1| - |n| \lesssim N_2^{1+\gamma} N_1^{-1}.
\]

\[\text{(5.26)}\]

We now split the case into \(N_2^{1+\gamma} \ll N_1\) and \(N_2^{1+\gamma} \gg N_1\).

**Case 1:** \(N_2^{1+\gamma} \ll N_1\)
From (5.26), in this case we have
\[ |n| - \varepsilon \leq |n_1| \leq |n| + \varepsilon \]
for some \(0 < \varepsilon \ll 1\). Hence, \(n_1\) is contained in an annulus of thickness \(\sim \varepsilon\) and a ball of radius \(\sim N_2\), which means that we have
\[
\sup_{n \in J_{2\ell_2}} \# S_n \lesssim \left| \{ n_1 : n_1 \in J_{1\ell_1} \} \cap \{ n_1 : |n| - \varepsilon \leq |n_1| \leq |n| + \varepsilon \} \right|
\lesssim \varepsilon N_2.
\] (5.27)

**Case 2:** \(N_2^{1+\gamma} \geq N_1\)

From (5.26), in this case we have
\[ |n| - cN_2^{\gamma^+} \leq |n_1| \leq |n| + cN_2^{\gamma^+} \]
for some constants \(c > 0\). Since \(|n| \sim N\) and \(N \gg cN_2^{\gamma^+}\) with \(\gamma < 1\), \(n_1\) is contained in an annulus of thickness \(\sim N_2^{\gamma^+}\) and a ball of radius \(\sim N_2\). Hence, we have
\[
\sup_{n \in J_{2\ell_2}} \# S_n \lesssim \left| \{ n_1 : n_1 \in J_{1\ell_1} \} \cap \{ n_1 : |n| - N_2^{\gamma^+} \leq |n_1| \leq |n| + N_2^{\gamma^+} \} \right|
\lesssim N_2^{1+\gamma^+}.
\] (5.28)

Hence, from (5.23), (5.24), (5.25), (5.27), and (5.28), we have
\[
\sup_{t \in \mathbb{R}} \| h(t) \|_{n_1n_2 \to n}^2 \lesssim N_2^{-2+2\gamma} N_2^{1+\gamma^+} \lesssim N_2^{1+3\gamma^+},
\]
which proves (5.22).

As for (5.22), by the Schur’s test, we have that
\[
\| h(t) \|_{n_1 \to n_{n_2}} \lesssim \left( \sup_{n,n_2} \sum_{n_1} |h_{n_1n_2n}(t)| \right) \left( \sup_{n,n_2} \sum_{n_1} |h_{n_1n_2n}(t)| \right)
\lesssim N_2^{-2+2\gamma} \sup_{n,n_2} \left| \left\{ n_1 : n = n_1 + n_2, \varphi(n_1, n_2, n) = O(N_2^{1+\gamma^+}) \right\} \right|
\times \sup_{n_1 \in J_{1\ell_1}} \left| \left\{ (n, n_2) : n = n_1 + n_2, \varphi(n_1, n_2, n) = O(N_2^{1+\gamma^+}), n \in J_{2\ell_2}, |n_2| \sim N_2 \right\} \right|
\]
Since \(n_1\) is uniquely determined by \(n\) and \(n_2\), the first factor can be bounded by 1. As for the remaining part, we have
\[
\sup_{n_1 \in J_{1\ell_1}} \left| \left\{ (n, n_2) : n = n_1 + n_2, |\varphi(n_1, n_2, n)| = O(N_2^{1+\gamma^+}), n \in J_{2\ell_2}, |n_2| \sim N_2 \right\} \right|
\lesssim \sup_{n_1 \in J_{1\ell_1}} \left| \left\{ n : |n_1|^2 + |n - n_1| - |n|^2 = O(N_2^{1+\gamma^+}), n \in J_{2\ell_2}, |n - n_1| \sim N_2 \right\} \right|
= \sup_{n_1 \in J_{1\ell_1}} \# S_{n_1}.
\]
As for the counting estimate \(\# S_{n_1}\), thanks to the symmetry, we can proceed as in the case (5.25), which implies
\[
\sup_{t \in \mathbb{R}} \| h(t) \|_{n_1 \to n_{n_2}}^2 \lesssim N_2^{-2+2\gamma} N_2^{1+\gamma^+} \lesssim N_2^{1+3\gamma^+}.
\]
This proves (5.22).
6. Global well-posedness and invariance of the Gibbs measure

In this section, we extend the local solutions constructed in Theorem 1.8 to global solutions and prove invariance of the Gibbs measure (1.20) under the flow of the renormalized Zakharov-Yukawa system (1.21). We exploit the Bourgain’s invariant measure argument [7, 10, 15, 48].

6.1. Invariance of the Gibbs measure under the truncated Zakharov-Yukawa system. In this subsection, we present frequency-truncated systems and invariance of the Gibbs measure along the flow. For fixed $\varepsilon > 0$, $\mu_1 \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}$ is a measure on

$$H^{-\varepsilon}(\mathbb{T}^2) \times \tilde{H}^{-\varepsilon}(\mathbb{T}^2)$$

where $\tilde{H}^{-\varepsilon}(\mathbb{T}^2) = H^{-\gamma-\varepsilon}(\mathbb{T}^2) \times H^{-1-\gamma-\varepsilon}(\mathbb{T}^2)$. Given $N \in \mathbb{N}$, we also define the finite-dimensional Gaussian measures $\tilde{\mu}_{\gamma,N} = (\tilde{\pi}_N)(\mu_1 \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma})$ on $\tilde{E}_N = \text{span}\{e^{in_1x}, e^{in_2x}, e^{in_3x}: |n_j| \leq N, j = 1, 2, 3\}$ as the pushforward of $\mu_1 \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}$ under $\tilde{\pi}_N$, where $\tilde{\pi}_N$ is the Dirichlet projector onto the frequencies $\{(n_1, n_2, n_3): |n_j| \leq N, j = 1, 2, 3\}$.

Consider the frequency-truncated system of the renormalized Zakharov-Yukawa system (1.21):

$$\begin{cases}
i \partial_t u^N + \Delta u^N = \pi_N N_S(u^N, w^N) \\
\partial_t^2 w^N + (1 - \Delta)w^N = \pi_N N_W(u^N, w^N) \\
(u^N, w^N, \partial_t w^N)|_{t=0} = (\pi_N u_0, \pi_N w_0, \pi_N v_0),
\end{cases}$$

(6.1)

where $N_S$ and $N_W$ denote the nonlinearity defined in (4.4) and (4.1) and $(\pi_N u_0, \pi_N w_0, \pi_N v_0) \in \tilde{E}_N$. The truncated system (6.1) is the finite-dimensional system of nonlinear ODEs on the Fourier coefficients of $(u_N, w_N, \partial_t w_N)$. Therefore, we can conclude by the Cauchy-Lipschitz theorem that the system of ODEs is locally well-posed. Furthermore, we can extend these solutions globally-in-time since $\|(u^N(t), w^N(t), \partial_t w^N(t))\|_{H^1(\mathbb{T}^2) \times H^{1-\gamma}(\mathbb{T}^2)}$ can be controlled uniformly in time (but not in $N$) by using the conservation of Hamiltonian and $L^2$-mass of the Schrödinger component: for any fixed $N \geq 1$, we have

$$\|(u^N(t), w^N(t), \partial_t w^N(t))\|^2_{H^1(\mathbb{T}^2) \times H^{1-\gamma}(\mathbb{T}^2)} \lesssim H(\pi_N u_0, \pi_N w_0, \pi_N v_0) + N^2 M(\pi_N u_0)$$

uniformly in $t \in \mathbb{R}$, where we used the Sobolev and Young’s inequality. In this section, one of our goals is to establish the uniform (in $N$) control of solutions in the support of the Gibbs measure.

Let $\tilde{\Phi}_N(t)$ denote the flow map for (6.1). Let $\tilde{d}\rho_{N,\gamma}$ denote the finite dimensional Gibbs measure associated with the density:

$$\tilde{d}\rho_{N,\gamma} = Z_N^{-1} e^{-Q_N(u,w)}1\{|f_{22}:|u^N|^2+dx|\leq K\}d\tilde{\mu}_{\gamma,N}(u, w, \partial_t w)$$

where $Q_N(u, w)$ is the interaction potential defined in (1.15) and $d\tilde{\mu}_{\gamma,N} = d(\mu_1, \mu_{1-\gamma}, \mu_{-\gamma})$ with $\mu_{s,N} := (\pi_N)^s \mu_s$. From the conservation of Hamiltonian, $L^2$-mass of $u^N$ and the Liouville theorem, we can know that the truncated Gibbs measure $\tilde{d}\rho_{N,\gamma}$ is an invariant

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23 Because of the focusing nature of the potential energy in the Hamiltonian, we cannot directly obtain the uniform (in $N$) estimate in the energy class $H^1(\mathbb{T}^2) \times H^{1-\gamma}(\mathbb{T}^2)$ by only using the energy conservation. We also point out that the $L^2$-norm is only preserved for the component $u_N$, which means that there is no a priori uniform (in both $N$ and $t$) control of solutions $(u^N(t), w^N(t), \partial_t w^N(t))$ at the $L^2$-level.
measure under the truncated system \( \tilde{\Phi}_N(t) \). We also consider the extension of (6.1) to infinite dimensions, where the higher modes evolve according to linear dynamics:

\[
\begin{align*}
\left\{ 
\begin{array}{l}
i \partial_t u^N + \Delta u^N = \pi_N N_S(\pi_N w^N, \pi_N w^N) \\
\partial_t w^N + (1 - \Delta) w^N = \pi_N N_W(\pi_N w^N, \pi_N u^N) \\
(u^N, w^N, \partial_t w^N)|_{t=0} = (u_0, w_0, v_0),
\end{array}
\right.
\end{align*}
\] (6.2)

where \((u_0, w_0, v_0) \in H^{-\varepsilon}(\mathbb{T}^2) \times \tilde{H}^{-\gamma-\varepsilon}(\mathbb{T}^2)\). In other words, (6.2) allows us to discuss the two decoupled flows, where the high frequency part evolves linearly and the low frequency part corresponds to the finite-dimensional system (6.1) of nonlinear ODEs. Let \( \tilde{\Phi}_N(t) \) be the flow map for (6.2). Then, we have

\[
\tilde{\Phi}_N(t) = \tilde{\Phi}_N(t)\pi_N + \tilde{S}(t)\pi_N^\perp
\]

where \( \pi_N^\perp := \text{Id} - \pi_N \). We denote by \( \tilde{E}_N^\perp \) the orthogonal complement of \( \tilde{E}_N \) in \( H^{-\varepsilon}(\mathbb{T}^2) \times \tilde{H}^{-\gamma-\varepsilon}(\mathbb{T}^2) \). Let \( \tilde{\mu}_N^\perp \) be the Gaussian field on \( \tilde{E}_N^\perp \) i.e. \( \tilde{\mu}_N^\perp = (\pi_N^\perp)_*(\mu_1 \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma}) \) on \( \tilde{E}_N^\perp \) as the pushforward of \( \mu_1 \otimes \mu_{1-\gamma} \otimes \mu_{-\gamma} \) under \( \pi_N^\perp \). We define the truncated Gibbs measure \( d\tilde{\rho}_{\gamma,N} \) as follows:

\[
d\tilde{\rho}_{\gamma,N} = d\tilde{\rho}_{\gamma,N}^\perp \otimes d\mu_N^\perp.
\]

From the invariance of \( d\tilde{\rho}_{\gamma,N} \) under the flow \( \tilde{\Phi}_N(t) \) and the invariance of the Gaussian measures \( d\tilde{\mu}_N^\perp \) under rotations \( \tilde{S}(t) \), we conclude the following invariance of \( d\tilde{\rho}_{\gamma,N} \) under the truncated system \( \tilde{\Phi}_N(t) \).

**Lemma 6.1.** For each \( t \in \mathbb{R} \), the Gibbs measure \( d\tilde{\rho}_{\gamma,N} \) is invariant under the flow map \( \tilde{\Phi}_N(t) \) on \( H^{-\varepsilon}(\mathbb{T}^2) \times \tilde{H}^{-\gamma-\varepsilon}(\mathbb{T}^2) \).

6.2. Almost sure global well-posedness. In this subsection, by using the invariance of the Gibbs measure for (6.2) (Lemma 6.1) and a standard PDE approximation argument, we obtain the almost sure global well-posedness.

**Lemma 6.2.** There exist small \( 0 < \varepsilon < \varepsilon_1 \ll 1 \) and \( \beta > 0 \) such that given any small \( \kappa > 0 \) and \( T > 0 \), there exists a measurable set \( \Sigma_{\kappa,T} \subset H^{-\varepsilon}(\mathbb{T}^2) \times \tilde{H}^{-\gamma-\varepsilon}(\mathbb{T}^2) \) such that (i) \( \tilde{\rho}_{\gamma}(\Sigma_{\kappa,T}) < \kappa \) and (ii) for any \((u_0, w_0, v_0) \in \Sigma_{\kappa,T}\), there exists a (unique) solution

\[
\begin{align*}
z^S_{t,\omega} + X^S_{t,\omega}(T) \subset C([-T, T]; H^{-\varepsilon}(\mathbb{T}^2)) \\
z^W_{t,\omega} + X^W_{t,\omega}(T) \subset C([-T, T]; H^{-\gamma-\varepsilon}(\mathbb{T}^2)) \cap C^1([-T, T]; H^{-\gamma-1-\varepsilon}(\mathbb{T}^2))
\end{align*}
\]

to the renormalized Zakharov-Yukawa system (1.21) with \((u, w, \partial_t w)|_{t=0} = (u_0, w_0, v_0)\) for some \( s > 0 \) and \( \ell > 0 \) in Theorem 1.8. Furthermore, given any large \( N \gg 1 \), we have

\[
\left\| (u(t), w(t), \partial_t w(t)) - \tilde{\Phi}_N(t)(u_0, w_0, v_0) \right\|_{C([-T, T]; H^{-\varepsilon_1}(\mathbb{T}^2) \times \tilde{H}^{-\gamma-\varepsilon_1}(\mathbb{T}^2))} \lesssim C(\kappa, T)N^{-\beta},
\]

where \( \tilde{\Phi}_N(t) \) denotes the flow map for (6.2).

**Proof.** Once we have almost sure local well-posedness (Theorem 1.8), the proof of Lemma 6.2 is by now standard. In the following, we only sketch key parts of the argument and refer to [7, 10, 15, 51, 52] for further details.
It is convenient to reduce the Zakharov-Yukawa system (4.1) to a first-order system by setting \( w_\pm := w \pm i(\nabla)^{-1}\partial_tw \) as in the proof of Theorem 1.8 see also Subsection 4.1. An observation is that convergence properties of \( w_\pm \) can be directly converted to convergence properties of \( w \) by taking the real part; \( w = \text{Re} \, w_\pm \). In the following, we drop the \( \pm \) signs from \( w_\pm \) and work with one \( w_+ \) or \( w_- \) since there is no role of \( \pm \).

For \( M \geq N \geq 1 \), we can write

\[
(u^M, w^M) - (u^N, w^N) = (\pi_M u^M - \pi_N u^N, \pi_M w^M - \pi_N w^N) + (\pi_{1/2} u^M, \pi_{1/2} w^M) - (\pi_{1/2} u^N, \pi_{1/2} w^N).
\]

(6.3)

The convergence of

\[
(\pi_M u^M - \pi_N u^N, \pi_M w^M - \pi_N w^N) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty
\]

in \( C\left(\left[-\delta, \delta\right]; H^{-\epsilon_1}(\mathbb{T}^2) \times H^{-\gamma}(\mathbb{T}^2)\right) \) since we also have

\[
(\pi_M e^{it\Delta} u_0^\omega - \pi_N e^{it\Delta} u_0^\omega, \pi_M e^{it(\nabla)} u_0^\omega - \pi_N e^{it(\nabla)} u_0^\omega) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty
\]

in \( C\left(\left[-\delta, \delta\right]; H^{-\epsilon_1}(\mathbb{T}^2) \times H^{-\gamma-\epsilon_1}(\mathbb{T}^2)\right) \).

On the other hand, the second and third terms in (6.3) decay like \( N^{-\beta} \) for some \( \beta > 0 \) thanks to the high frequency projections. The remaining part of the argument leading to the proof of Lemma 6.2 is contained in \([7, 10, 15, 51, 52]\). In particular, see the proof of Proposition 3.5 in \([51]\) for details in a setting analogous to our work.

□

Once we obtain Lemma 6.2 the almost sure global well-posedness follows from the Borel-Cantelli lemma. Given \( \kappa > 0 \), let \( T_j = 2^j \) and \( \kappa_j = \frac{\kappa}{2^j}, j \in \mathbb{N} \). By exploiting Lemma 6.2 we can construct a set \( \Sigma_{\kappa_j, T_j} \) and set

\[
\Sigma_\kappa := \bigcap_{j=1}^\infty \Sigma_{\kappa_j, T_j}.
\]

(6.4)

Then, we have \( \bar{\rho}_\gamma(\Sigma_\kappa^c) < \kappa \) and for any \((u_0, w_0, v_0) \in \Sigma_\kappa\), there exists a unique global-in-time solution to the renormalized Zakharov-Yukawa system (1.21) with \( (u, w, \partial_tw)|_{t=0} = (u_0, w_0, v_0) \). Finally, we set

\[
\Sigma := \bigcup_{n=1}^\infty \Sigma_{\frac{1}{\pi} n}.
\]

(6.5)

Then, we have \( \bar{\rho}_\gamma(\Sigma^c) = 0 \) and for any \((u_0, w_0, v_0) \in \Sigma\), there exists a unique global-in-time solution to the renormalized Zakharov-Yukawa system (1.21) with \( (u, w, \partial_tw)|_{t=0} = (u_0, w_0, v_0) \), which means that we prove almost sure global well-posedness.
6.3. **Invariance of the Gibbs measure.** Let \( \tilde{\Phi}(t) \) be the flow map for the renormalized Zakharov-Yukawa system \([121]\) defined on the set \( \Sigma \) of full probability constructed in \([6.5]\).

The main goal of this subsection is to establish the invariance of the Gibbs measure along the flow \( \tilde{\Phi}(t) \):

\[
\int_{\Sigma} F(\tilde{\Phi}(t)(u, w, v)) d\tilde{\rho}_\gamma(u, w, v) = \int_{\Sigma} F(u, w, v) d\tilde{\rho}_\gamma(u, w, v)
\]

for any \( F \in L^1(H^{-\varepsilon}(\mathbb{T}^2) \times \tilde{H}^{-\gamma-\varepsilon}(\mathbb{T}^2), d\tilde{\rho}_\gamma) \) and any \( t \in \mathbb{R} \). From a density argument, it is enough to show \([6.6]\) for continuous and bounded \( F \). Fix \( t \in \mathbb{R} \). Note that

\[
\left| \int_{\Sigma} F(\tilde{\Phi}(t)(u, w, v)) d\tilde{\rho}_\gamma(u, w, v) - \int_{\Sigma} F(u, w, v) d\tilde{\rho}_\gamma(u, w, v) \right|
\]

\[
\leq \left| \int_{\Sigma} F(\tilde{\Phi}(t)(u, w, v)) d\tilde{\rho}_\gamma(u, w, v) - \int_{\Sigma} F(\Phi(t)(u, w, v)) d\tilde{\rho}_{\gamma,N}(u, w, v) \right|
\]

\[
+ \left| \int_{\Sigma} F(\Phi(t)(u, w, v)) d\tilde{\rho}_{\gamma,N}(u, w, v) - \int_{\Sigma} F(\Phi_N(t)(u, w, v)) d\tilde{\rho}_{\gamma,N}(u, w, v) \right|
\]

\[
+ \left| \int_{\Sigma} F(\Phi_N(t)(u, w, v)) d\tilde{\rho}_{\gamma,N}(u, w, v) - \int_{\Sigma} F(u, w, v) d\tilde{\rho}_{\gamma,N}(u, w, v) \right|
\]

\[
= I_N + II_N + III_N + IV_N.
\]

From Lemma \([6.1]\) we have

\[
\int_{\Sigma} F(\tilde{\Phi}_N(t)(u, w, v)) d\tilde{\rho}_{\gamma,N}(u, w, v) = \int_{\Sigma} F(u, w, v) d\tilde{\rho}_{\gamma,N}(u, w, v),
\]

which implies

\[
III_N = 0.
\]

Thanks to Theorem \([1.5]\) (especially, \([1.19]\)), we have \(^{24}\) that \( d\tilde{\rho}_{\gamma,N} \) converges weakly to \( d\tilde{\rho}_\gamma \), which implies

\[
I_N + IV_N \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.
\]

Let \( \delta > 0 \). Then, the boundedness of \( F \) and the total variation convergence of \( d\tilde{\rho}_{\gamma,N} \) to \( d\tilde{\rho}_\gamma \) imply that for any sufficiently small \( \kappa > 0 \) and sufficiently large \( N \gg 1 \), we have

\[
\left| \int_{\Sigma_{\kappa}} F(\tilde{\Phi}(t)(u, w, v)) d\tilde{\rho}_{\gamma,N}(u, w, v) \right| + \left| \int_{\Sigma_{\kappa}} F(\tilde{\Phi}_N(t)(u, w, v)) d\tilde{\rho}_{\gamma,N}(u, w, v) \right| < \delta,
\]

where \( \Sigma_{\kappa} \) is defined in \([6.4]\). Let us fix one such \( \kappa > 0 \). Then, from \([6.7]\) and \([6.9]\), we have

\[
II_N \leq \delta + \int_{\Sigma_{\kappa}} \left| F(\tilde{\Phi}(t)(u, w, v)) - F(\tilde{\Phi}_N(t)(u, w, v)) \right| d(\tilde{\rho}_{\gamma,N} - \tilde{\rho}_\gamma)(u, w, v)
\]

\[
+ \int_{\Sigma_{\kappa}} \left| F(\tilde{\Phi}(t)(u, w, v)) - F(\tilde{\Phi}_N(t)(u, w, v)) \right| d\tilde{\rho}_{\gamma}(u, w, v).
\]

\(^{24}\)In fact, we have the total variation convergence of measures.
Thanks to the boundedness of $F$ and the total variation convergence of $d\tilde{\rho}_{\gamma,N}$ to $d\tilde{\rho}_\gamma$, we have

$$\int_{\Sigma_\kappa} |F(\bar{\Phi}(t)(u,w,v)) - F(\bar{\Phi}_N(t)(u,w,v))| d(\tilde{\rho}_{\gamma,N} - \tilde{\rho}_\gamma)(u,w,v) \to 0 \quad (6.11)$$

as $N \to \infty$. From Lemma (6.2), we have

$$\|\bar{\Phi}(t)(u,w,v) - \bar{\Phi}_N(t)(u,w,v)\|_{H^{-\epsilon}(T^2) \times \bar{H}^{-\gamma-\epsilon}(T^2)} \leq C(\kappa, t) N^{-\beta} \quad (6.12)$$

for any $(u, w, v) \in \Sigma_\kappa$ and sufficiently large $N \gg 1$. Hence, from (6.12), the continuity of $F$ and the dominated convergence theorem, we have

$$\int_{\Sigma_\kappa} |F(\bar{\Phi}(t)(u,w,v)) - F(\bar{\Phi}_N(t)(u,w,v))| d\tilde{\rho}_\gamma(u,w,v) \to 0 \quad (6.13)$$

as $N \to \infty$. From (6.9), (6.10), (6.11), (6.13), and taking $\delta \to 0$, we have

$$II_N \to 0 \quad \text{as} \quad N \to \infty,$$

which proves (6.6).

**Appendix A. Hilbert-Schmidt norm approach**

In this appendix, we give a brief discussion on the Hilbert-Schmidt norm approach and compare its result with when using the operator norm bound with the random tensor theory. In particular, we handle the kernel (random) matrix by using the Hilbert-Schmidt norm with the Wiener chaos estimate (Lemma 2.11). In the following, we look into Subcase 2.b in Lemma 4.8 which gave the restriction $\gamma < \frac{1}{3}$ in the operator norm approach with the random tensor estimates. We will see that by using the Hilbert-Schmidt norm approach, Lemma 4.8 is no longer satisfied for any $\gamma \geq 0$, which shows that the approach with the random tensor theory is essential to obtain the main result. Let us first recall Subcase 2.b in Lemma 4.8:

**Subcase 2.b in Lemma 4.8:**

$L_{\max} \lesssim N_2^{1+\gamma^+} \text{ (low modulation)}$ and $N_1 \sim N \gtrsim N_2 \text{ (resonant interaction)}$.

This time we try to prove the bilinear estimate (4.32) with the Hilbert-Schmidt norm approach. When $N_1 \gg N_2$, we first write $\{|n_1| \sim N_1\} = \bigcup_{\ell_1} J_{1,\ell_1}$ and $\{|n| \sim N\} = \bigcup_{\ell_2} J_{2,\ell_2}$, where $J_{1,\ell_1}$ and $J_{2,\ell_2}$ are balls of radius $\sim N_2$, we can decompose $\hat{P}_{N_1} R^S$ and $\hat{P}_{N} v^S$ as

$$\hat{P}_{N_1} R^S = \sum_{\ell_1} \hat{P}_{N_1,\ell_1} R^S \quad \text{and} \quad \hat{P}_{N} v^S = \sum_{\ell_2} \hat{P}_{N,\ell_2} v^S$$

where $\hat{P}_{N_1,\ell_1} R^S(n_1, t) = 1_{J_{1,\ell_1}}(n_1) \hat{P}_{N_1} R^S(n_1, t)$ and $\hat{P}_{N,\ell_2} v^S(n, t) = 1_{J_{2,\ell_2}}(n) \hat{P}_{N} v^S(n, t)$. Given $n_1 \in J_{1,\ell_1}$ for some $\ell_1$, there exists $O(1)$ many possible values for $\ell_2 = \ell_2(\ell_1)$ such that $n \in J_{2,\ell_2}$ under $n_1 + n_2 = n$. Notice that the number of possible values of $\ell_2$ is independent of $\ell_1$.
From the Cauchy-Schwarz inequality, we have
\[
\left| \int_{\mathbb{T}^2} \langle \nabla \rangle (P_{N_1, t_{1}} R^S P_{N_2} W) P_{N_1, t_{2}} v^S dx dt \right|
= \left| \sum_{n \in J_{2t_2}} \int_{\mathbb{R}} \langle n \rangle^S P_{N_1, t_{1}} v^S(n, t) \sum_{n_1 \in J_{t_1}} P_{N_1, t_{1}} R^S(n_1, t)(n_2) e^{-it|n_2|} h_{n_2}(\omega) \eta_{\delta}(t) dt \right|
\leq \left( \sum_{n \in J_{2t_2}} \left| \langle n \rangle^S P_{N_1, t_{1}} v^S(n, t) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{n_1 \in J_{t_1}} \sum_{n_1 + n_2 = n} \sigma^\omega(n, n_1, t) \eta_{\delta}(t) P_{N_1, t_{1}} R^S(n_1, t) \right)^{\frac{1}{2}}
\sim N^0 \| P_{N_1, t_{1}} v^S \|_{N^0_S} \left( \sum_{n_1 \in J_{t_1}} \sum_{n_1 + n_2 = n} \sigma^\omega(n, n_1, t) \eta_{\delta}(t) P_{N_1, t_{1}} R^S(n_1, t) \right)^{\frac{1}{2}},
\tag{A.1}
\]
where the random matrix \( \sigma^\omega(n, n_1, t) \) is given by
\[
\sigma^\omega(n, n_1, t) = \begin{cases} 
\frac{e^{-it|n_2|} h_{n_2}(\omega)}{\langle n_2 \rangle^2 + 1}, & \text{if } |n_1|^2 + |n - n_1| - |n|^2 = O(N_2^{1+\gamma}), \ n_1 + n_2 = n, \ |n_2| \sim N_2 \\
0, & \text{otherwise}
\end{cases}
\]
Then, from Lemma 2.5 we have
\[
\sum_{n \in J_{2t_2}} \left| \sum_{n_1 \in J_{t_1}, n_1 + n_2 = n} \sigma^\omega(n, n_1, t) \eta_{\delta}(t) P_{N_1, t_{1}} R^S(n_1, t) \right|^2 \lesssim \sup_{n \in \mathbb{R}} \max_{n_1 \in S_n} \sum_{n_1} |\sigma^\omega(n, n_1, t)|^2 + \left( \sum_{n_1 \neq n_1'} \left| \sum_{n_2 \in \mathbb{Z}^2} \sigma^\omega(n, n_1, t) \sigma^\omega(n, n_1', t) \right| \right)^{\frac{1}{2}} \| P_{N_1, t_{1}} R^S \|_{N^0_S}^2 \tag{A.2}
\]
where
\[
S_n = \{ n_1 \in J_{t_1} : |n_1|^2 + |n - n_1| - |n|^2 = O(N_2^{1+\gamma}), \ n_1 + n_1' \sim N_2 \} \tag{A.3}
\]
We now split the case into \( N_2^{1+\gamma} \ll N_1 \) and \( N_2^{1+\gamma} \gg N_1 \).

**Case 1:** \( N_2^{1+\gamma} \ll N_1 \)

In this case, from (A.3) we have
\[
|n| - \varepsilon \leq |n| \leq |n| + \varepsilon
\]
for some \( 0 < \varepsilon \ll 1 \). Hence, \( n_1 \) is contained in an annulus of thickness \( \sim \varepsilon \) and a ball of radius \( \sim N_2 \). Hence, we have
\[
\sup_{n \in J_{2t_2}} \# S_n \lesssim \sup_{n \in J_{2t_2}} \left| \{ n_1 : n_1 \in J_{t_1} \} \cap \{ n_1 : |n| - \varepsilon \leq |n| \leq |n| + \varepsilon \} \right|
\lesssim \varepsilon N_2. \tag{A.4}
\]

**Case 2:** \( N_2^{1+\gamma} \gg N_1 \)

In this case, from (A.3) we have
\[
|n| - cN_2^{\gamma} \leq |n| \leq |n| + cN_2^{\gamma}
\]
for some constants \( c > 0 \). Since \(|n| \sim N\) and \( N \gg cN_2^{\gamma^+}\) with \( \gamma < 1 \), \( n_1\) is contained in an annulus of thickness \( \sim N_2^{\gamma^+}\) and a ball of radius \( \sim N_2\). Hence, we have

\[
\sup_{n \in J_{2\ell_2}} \#S_n \leq \sup_{n \in J_{2\ell_2}} \left| \{ n_1 : n_1 \in J_{2\ell_2} \} \cap \{ n_1 : |n| - N_2^{\gamma^+} \leq |n_1| \leq |n| + N_2^{\gamma^+} \} \right|
\lesssim N_2^{1+\gamma^+}.
\]  

(A.5)

By counting the number of lattice points as in (A.4) and (A.5), the diagonal term can be estimated as follows:

\[
\sup_{n \in J_{2\ell_2}} \sum_{n \in S_n} |\sigma^\omega(n, n_1, t)|^2 \lesssim N_2^{-2+2\gamma^+} \sup_{n \in J_{2\ell_2}} \#S_n
\lesssim N_2^{-2+2\gamma^+} N_2^{1+\gamma^+} = N_2^{-1+3\gamma^+}
\]

(A.6)

outside an exceptional set of probability \( < e^{-\frac{1}{12}} \), where we used Lemma 2.12 in the first step of (A.6).

Next, we consider the non-diagonal term. Note that

\[
\left( \sum_{n_1 \neq n_1'} \left| \sum_{n \in \mathbb{Z}^2} \sigma^\omega(n, n_1, t)\sigma^\omega(n, n_1', t) \right|^2 \right)^{\frac{1}{2}}
= \left( \sum_{n_1 \neq n_1'} \sum_{n \in \mathbb{Z}^2 : [n_1'^2 + |n-n_1'| - |n|] \leq O(N_2^{2\gamma^+})} a_{n,n_1,n_1'} \cdot h_{n-n_1}(\omega)\bar{h}_{n-n_1'}(\omega) \right)^{\frac{1}{2}}
\]  

(A.7)

where

\[
a_{n,n_1,n_1'} = \langle n - n_1 \rangle^{-1+\gamma} \langle n - n_1' \rangle^{-1+\gamma} 1_{\{|n-n_1| \sim N_2, |n-n_1'| \sim N_2\}}
\]

From Lemma 2.11 and using the independence of \( \{h_n\} \) with the condition \( n_1 \neq n_1' \), we have

\[
\left| \sum_{n \in \mathbb{Z}^2 : [n_1'^2 + |n-n_1'| - |n|] \leq O(N_2^{2\gamma^+})} a_{n,n_1,n_1'} \cdot h_{n-n_1}(\omega)\bar{h}_{n-n_1'}(\omega) \right|
\lesssim \delta^{0-} N_2^{0+} \left( \mathbb{E}_P \left| \sum_{n \in \mathbb{Z}^2 : [n_1'^2 + |n-n_1'| - |n|] \leq O(N_2^{2\gamma^+})} a_{n,n_1,n_1'} \cdot h_{n-n_1}(\omega)\bar{h}_{n-n_1'}(\omega) \right|^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \delta^{0-} N_2^{0+} \left( \sum_{n \in \mathbb{Z}^2 : [n_1'^2 + |n-n_1'| - |n|] \leq O(N_2^{2\gamma^+})} |a_{n,n_1,n_1'}|^2 \right)^{\frac{1}{2}}.
\]

(A.8)
Hence, from (A.7), (A.8), and counting the number of lattice points as in (A.4) and (A.5), we have
\[
\sum_{n_1 \neq n_1'} \left| \sum_{n \in \mathbb{Z}^2} \sigma^\omega(n, n_1, t) \bar{\sigma}^\omega(n, n_1', t) \right|^2 \\
\lesssim N_2^{-4+4\gamma} \# \left\{ (n_1, n_1', n) : |n_1|^2 \pm |n - n_1| - |n|^2 = O(N_2^{1+\gamma^+}), \ n_1 \in J_1 \ell_1, \ n \in J_2 \ell_2, \ |n - n_1| \sim N_2 \right\} \\
\lesssim N_2^{-4+4\gamma} \sum_{n_1 \in J_1 \ell_1} \sum_{n_1' \in J_1 \ell_1} \sum_{n \in J_2 \ell_2} |n_1|^2 - |n_1'|^2 = O(N_2^{1+\gamma^+}) \ |n_1|^2 - |n|^2 = O(N_2^{1+\gamma^+}) \\
\lesssim N_2^{-4+4\gamma} N_2^2 N_2^{1+\gamma} N_2^{1+\gamma} \sim N_2^{6\gamma^+}. \quad (A.9)
\]
Therefore, by combining (A.1), (A.2), (A.6), and (A.9), we have
\[
\left| \int_\mathbb{R} \int_{T^2} \langle \nabla \rangle^s (P_{N_1 \ell_1} R^S P_{N_2 \ell_2} v^S) P_{N_1 \ell_1} v^S dx dt \right| \\
\leq \sup_{t \in \mathbb{R}} \left[ \max_{n \in J_2 \ell_2} \sum_{n_1 \in S_n} |\sigma^\omega(n, n_1, t)|^2 + \left( \sum_{n_1 \neq n_1'} \left| \sum_{n \in \mathbb{Z}^2} \sigma^\omega(n, n_1, t) \bar{\sigma}^\omega(n, n_1', t) \right|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \\
\times \| P_{N_1 \ell_1} R^S \|_{X^{e, 0}_0} \| P_{N_1 \ell_1} v^S \|_{X^{0, 0}_0} \\
\lesssim (N_2^{-\frac{1}{2}+\frac{3}{2}\gamma^+} + N_2^{3\gamma^+}) \| P_{N_1 \ell_1} R^S \|_{X^{e, 0}_0} \| P_{N_1 \ell_1} v^S \|_{X^{0, 0}_0}.
\]
Therefore, even if $\gamma = 0$, we cannot perform the dyadic summation over $N_1 \sim N \gtrsim N_2$, which shows that Lemma 4.8 cannot be proven by using the Hilbert-Schmidt norm approach with the lattice counting method used in this paper for any $\gamma \geq 0$. In other words, the operator norm approach with the random tensor theory is essential to obtain the main result.

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