ON ADDITIVE DIVISOR SUMS AND PARTIAL DIVISOR FUNCTIONS

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Abstract. We establish asymptotic formulae for various correlations involving general divisor functions \( d_k(n) \) and partial divisor functions \( d_l(n, A) = \sum_{q|n, q \leq n^A} d_{l-1}(q) \), where \( A \in [0, 1] \) is a parameter and \( k, l \in \mathbb{N} \) are fixed. Our results relate the parameter \( A \) to the lengths of arithmetic progressions in which \( d_k(n) \) is uniformly distributed. As applications to additive divisor sums, we establish new lower bounds and a new equivalent condition for the conjectured asymptotic. We also prove a Tauberian theorem for general additive divisor sums.

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1. Introduction

The focus of this paper is the problem of finding asymptotic formulae for ‘additive divisor sums’. That is, correlations

\[ D_{h,k,l}(x) = \sum_{n \leq x} d_k(n + h)d_l(n), \]  

where \( h, k, l \in \mathbb{N} \) are fixed and \( d_k(n) \) denotes the number of ordered ways of writing \( n \) as a product of \( k \) factors. In other words, this is the problem of counting the number of ordered solutions of the Diophantine equation

\[ h = n_1 \cdots n_k - m_1 \cdots m_l \]

where \((m_1, ..., m_l) \in \mathbb{N}^l, (n_1, ..., n_k) \in \mathbb{N}^k \) and \( n_1 \cdots n_k \leq x \).

Our results on the correlations in (1.1) are given in Section 2.2. These results are immediate corollaries of the results given in Section 2.1 which deal with correlations involving \( d_k(n) \) and partial divisor functions

\[ d_l(n, A) = \sum_{q|n:q \leq n^A} d_{l-1}(q) \quad A \in (0, 1]. \]  

As such, we emphasise the results given in Section 2.1.

1.1. Additive divisor sums. When \( k = l \), the correlations in (1.1) arise in connection with the problem of finding asymptotic formulae for the \( 2k \)th moments of the Riemann zeta function on the critical line. This connection was first exploited by Ingham [20] in the course of proving his asymptotic formula for the fourth moment of the Riemann zeta function. Ingham proved that

\[ D_{h,2,2}(x) \sim \frac{6}{\pi^2} \sigma_1(h) \log^2 x \]

where \( \sigma_z(n) = \sum_{d|n} d^z \), and subsequently Estermann [11] established the asymptotic expansion

\[ D_{h,2,2}(x) = xP_{h,2,2}(\log x) + O\left(x^{11/12+\epsilon}\right) \]

where \( P_{h,2,2} \) is a polynomial of degree 2. Estermann demonstrated that \( D_{h,2,2}(x) \) is related to the spectral theory of modular forms - his result made crucial use of a non-trivial bound for Kloosterman sums. Heath-Brown [16] subsequently used Weil’s improved bound [38] for Kloosterman sums to obtain the error term \( O(x^{5/6+\epsilon}) \) in (1.3), which was later improved by Motohashi [28] to \( O(x^{2/3+\epsilon}) \) uniformly for \( h \leq x^{20/27} \). Each of these improvements lead to corresponding improvements of the error term in the asymptotic expansion for the
fourth moment of the Riemann zeta function.

Due to the work of Hooley [19], Linnik [25], Fouvry and Tenenbaum [13], Heath-Brown [17], Drappeau [9], Motohashi [27], Deshouillers and Iwaniec [8], Bykovski and Vinogradov [3] and Topacogullari [34, 35, 36], it is now also known that for any fixed $k$ there is a $\delta > 0$ and a polynomial $P_{h,k,2}$ of degree $k$ such that

$$D_{h,k,2}(x) = xP_{h,k,2}(\log x) + O_{h,k}(x^{1-\delta}).$$

Despite these significant advances, asymptotic formulae for $D_{h,k,l}(x)$ remain elusive when both $k, l \geq 3$. The central conjecture—formulated by Conrey and Gonek [5] and Ivić [21, 22] via the ‘$\delta$-method’ of Duke, Friedlander and Iwaniec [10], and recently refined by Ng and Thom [30] and Tao [31]—is as follows.

Conjecture 1.1. If $h, k, l \in \mathbb{N}$ with $k, l$ fixed and $h = O(x^{1-\epsilon})$ for each fixed $\epsilon > 0$, then there is a $\delta > 0$ and a polynomial $P_{h,k,l}$ of degree $k + l - 2$ such that

$$D_{h,k,l}(x) = xP_{h,k,l}(\log x) + O_{\epsilon,k,l}(x^{1-\delta}).$$

The asymptotic is conjectured to be

$$\frac{D_{h,k,l}(x)}{x \log^{k+l-2} x} \sim \frac{C_{k,l}f_{k,l}(h)}{(k-1)!(l-1)!}$$

as $x \to \infty$, where

$$C_{k,l} = \prod_p \left(1 - p^{-1}\right)^{l-1} + \left(1 - p^{-1}\right)^{k-1} - \left(1 - p^{-1}\right)^{k+l-2}$$

and

$$f_{k,l}(h) = \prod_{p|h} \left(1 - p^{-1}\right) \sum_{d|p} d_{l-1}(p) \sum_{\alpha} d_k(p^\alpha) p^{-\beta} + d_k(p^\gamma) \sum_{\gamma+1} d_{l-1}(p^\alpha) p^{-\alpha} - (1 - p^{-1})^{l-1} - 1$$

(1.7)

where $h = \prod p^\gamma$.

The general form of the coefficients $C_{k,l}$ and $f_{k,l}(h)$ appearing in (1.5) were calculated by Ng and Thom [30] based on the techniques introduced by Conrey and Gonek [5], and the same prediction was made by Tao [31] based on pseudorandomness heuristics.

The asymptotic order of $D_{h,k,l}(x)$ is fairly well understood. Regarding upper bounds, it follows from the general theorem of Nair and Tenenbaum [29] that

$$D_{h,k,l}(x) = O_{h,k,l}(x \log^{k+l-2} x),$$

with uniformity in the $h$ aspect following from the work of Henriot [18]—the paper of Ng and Thom [30] discusses these matters in detail. However, when $k, l \geq 3$, it is notable that
explicit bounds on the size of the constant implied in [13] have not yet appeared in the literature. Regarding lower bounds, the best general result in the literature is due to Ng and Thom [30], who showed that for $k, l \geq 3$ there is a $B_{k,l} > 0$ such that for $h \leq \exp \left( B_{k,l} \left( \log x \log \log x \right)^{(\min(k,l)-1)/(\min(k,l)-1.99)} \right)$ we have

$$\frac{D_{h,k,l}(x)}{x \log^{k+l-2} x} \geq \left( 1 + O_{k,l} \left( \frac{\log h}{\log x} \right) \right) \frac{2^{2-k-l} C_{k,l} f_{k,l}(h)}{(k-1)! (j-1)!}.$$ (1.9)

Regarding averages over $h$, Matomaki, Radziwill and Tao [26] have recently shown that the conjectured asymptotic (1.5) holds for $k, l \geq 2$ and almost all $h \leq H$, provided that $x^{8/33+\epsilon} \leq H \leq x^{1-\epsilon}$, improving on previous work of Baier, Browning, Marasingha and Zhao [1] on the case $k = l = 3$.

1.2. Exponents of distribution. The problem of the asymptotic behaviour of additive divisor sums is closely related to the problem of improving the 'exponent of distribution' for the generalised divisor problem in arithmetic progressions. An exponent of distribution is a lower bound on the lengths of arithmetic progressions $n \equiv h \pmod{q}$, $(h, q) = g$, in which $d_k(n)$ is uniformly distributed.

Definition 1.1. A real number $0 < \theta_{g,k} \leq 1$ is an exponent of distribution for $d_k(n)$ if for every $q \leq x^{\theta_{g,k}-\epsilon}$ and each residue class $h \not\equiv 0 \pmod{q}$, we have

$$\sum_{\substack{n \leq x \\ n \equiv h \pmod{q}}} d_k(n) = \frac{1}{\phi(q/g)} \text{Res} \left( \frac{x^s}{s} \sum_{(n,q)=g} \frac{d_k(n)}{n^s}, s = 1 \right) + O_{\epsilon,\delta,k} \left( \frac{x^{1-\delta}}{\phi(q/g)} \right)$$ (1.10)

for some fixed $\delta > 0$ and $\epsilon > 0$.

An alternative way of writing (1.10) is

$$\sum_{\substack{n \leq x \\ n \equiv h \pmod{q}}} d_k(n) = \frac{1}{\phi(q/g)} \sum_{n \leq x/g} \chi_0(n) d_k(gn) + O_{\epsilon,\delta,k} \left( \frac{x^{1-\delta}}{\phi(q/g)} \right)$$ (1.11)

where $\chi_0$ is the principal Dirichlet character to the modulus $q/g$. Definition 1.1 is motivated by the fact that we expect $d_k(n)$ to be uniformly distributed even in short arithmetic progressions (i.e. with $q \leq x^{1-\epsilon}$ for every fixed $\epsilon > 0$). In other words, we expect that $\theta_{g,k} = 1$ for all $k$ provided that $g$ is not large, say $g \leq x^{1-\epsilon}$.
Currently however, we can only prove uniform distribution in sufficiently long arithmetic progressions. The best results in the literature are as follows. For \( k = 2 \), Hooley \[19\] established that we may take \( \theta_{1,2} = 2/3 \). We have \( \theta_{g,3} = 21/41 \) for all \( g \) due to Heath-Brown \[17\], \( \theta_{1,4} = 1/2 \) due to Lavrik \[24\], and \( \theta_{1,5} = 9/20, \theta_{1,6} = 5/12 \) and \( \theta_{1,k} = 8/3k \) for \( k \geq 7 \) due to Friedlander and Iwaniec \[14\]. For \( k > 2 \), the only known \( k \) for which an exponent of distribution greater than \( 1/2 \) is known is \( k = 3 \), and both proofs (including the inferior exponent \( 58/115 \) due to Friedlander and Iwaniec \[14\]) depend on Deligne’s Riemann hypothesis for algebraic varieties over finite fields. For specific moduli, further increments have also been achieved. For instance, via general estimates for sums of trace functions over finite fields twisted by Fourier coefficients of Eisenstein series, Fouvry, Kowalski and Michel \[12\] have shown that \( (1.10) \) holds for \( k = 3 \) for all primes \( q \leq x^{12/23} \) (albeit with \( x^{-\delta} \) in the error term replaced with \( \log^{-C} x \) for every \( C > 0 \)). The error term can be given explicitly in specific cases, although we will not be concerned with these details here.

With the exception of Heath-Brown’s result for \( k = 3 \), the above results on exponents of distribution are stated only for \( g = 1 \). This is usually because \( g = 1 \) is the only value that is required in applications to primes in arithmetic progressions. Yet, for applications to additive divisor sums, we require exponents of distribution for all \( g \) in some range as \( x \to \infty \). In this regard, by generalising Heath-Brown’s argument, Chace \[4\] has shown that

\[
\theta_k = \max \left( \frac{1}{k}, \theta_{1,k} + (1 - k\theta_{1,k}) \limsup_{x \to \infty} \frac{\log g}{\log x} \right)
\]

is an exponent of distribution for \( d_k(n) \) for all \( g \).

### 1.3. Partial divisor functions

A central principle in this paper is that partial divisor functions

\[
d_k(n, A) = \sum_{d | n, d \leq n^A} d_{k-1}(d) \quad A \in (0, 1]
\]

provide robust approximations to \( d_k(n) \) in arithmetic progressions. This property is essential in applications to correlation problems such as \( (1.1) \). We return to this in due course.

The pointwise relationship between \( d_k(n) \) and \( d_k(n, A) \) is generally unpredictable. In this regard, Tenenbaum \[32\] showed that

\[
\lim_{n \to \infty} \frac{d_2(n, A)}{d_2(n)}
\]

do not exist for any fixed \( A \in (0, 1) \) when \( S \subseteq \mathbb{N} \) has positive measure. Furthermore, Tenenbaum \[33\] showed that for every pair \( A, B \in (0, 1) \) there is an \( S \) of positive measure in which \( d_2(n, A) = d_2(n, B) \) for every \( n \in S \). Presumably, the same conclusions hold for every \( k \geq 2 \).
On the other hand, the limit in (1.14) exists on particular sets $S$ of zero measure. For example, if $p$ is prime then we have

\[(1.15) \quad \lim_{\alpha \to \infty} \frac{d_k(p^\alpha, A)}{d_k(p^\alpha)} = A^{k-1}\]

and so, by partial summation and (1.15), it follows that

\[(1.16) \quad \sum_{\alpha \leq X} a_\alpha d_k(p^\alpha, A) \sim A^{k-1} \sum_{\alpha \leq X} a_\alpha d_k(p^\alpha)\]

whenever $(a_\alpha)$ is a sequence of non-negative real numbers such that $\sum_{\alpha \leq X} a_\alpha \to \infty$.

On average, the relationship between $d_k(n, A)$ and $d_k(n)$ is predictable. In this direction, Deshoulliers, Dress and Tenenbaum [7] proved that the mean value of $d_2(n, A)/d_2(n)$ converges to an arcsine distribution. This has been generalised by Bareikis [2], giving a beta distribution

\[(1.17) \quad \frac{1}{x} \sum_{n \leq x} \frac{d_k(n, A)}{d_k(n)} \sim \int_0^A u^{-1/k}(1-u)^{1/k-1} du / \Gamma(1/k)\Gamma(1-1/k)\]

uniformly for $0 \leq A \leq 1$ as $x \to \infty$, for any fixed $k \geq 2$.

Roughly speaking, the mean of a partial divisor sum corresponds to the logarithmic mean over the partial range. That is, if $f : \mathbb{N} \to \mathbb{C}$, then

\[(1.18) \quad \sum_{n \leq x} \left( \sum_{d \mid n \atop d \leq n^A} f(d) \right) = x \sum_{n \leq x^A} \frac{f(n)}{n} + O \left( \sum_{n \leq x^A} \frac{|f(n)|}{n^{1-1/A}} \right) \quad A \in (0, 1],\]

uniformly for $A \geq A_0 > 0$. Taking $f(n) = d_{k-1}(n)$ in (1.18) and using Perron’s formula to evaluate the r.h.s, it is easily seen that $d_k(n, A)$ approximates $A^{k-1}d_k(n)$ in the mean, that is

\[(1.19) \quad \sum_{n \leq x} d_k(n, A) = A^{k-1} \sum_{n \leq x} d_k(n) + O_A \left( x \log^{k-2} x \right).\]

Here, it is notable that the non-multiplicativity of $d_k(n, A)$ is crucial to the quality of the approximation in (1.19). Indeed, since $d_k(p^\alpha, A) = d_k(p^{[\alpha,A]})$, the mean value of $\prod_{p \mid n} d_k(p^\alpha, A)$ exists for every $A < 1$ and $k \in \mathbb{N}$, whereas the mean of $d_k(n)$ is $\sim \log^{k-1} x/(k - 1)!$. 

An elementary refinement of (1.18) is that
\[
\sum_{n \leq x \ (\text{mod } q)} \left( \sum_{d \mid n \ d \leq n^A} f(d) \right) = \frac{x}{q} \sum_{n \leq x \ (\text{mod } q)} \frac{(n, q) f(n)}{n} + O \left( \sum_{n \leq x^A} \frac{|f(n)|}{n^{1-1/A}} \right),
\]
which may be proved by interchanging the order of summation and trivially estimating the length of the resulting arithmetic progression. However, in applications to correlation problems, the error term in (1.20) is not strong enough. Typically we require an additional factor of \(1/q\), uniformly for \(q \leq x^C\) as \(x \to \infty\) for some \(C > 1 - A\). Our first theorem (Theorem 2.1) establishes the requisite refinement of (1.20) in the case \(f(n) = d_{k-1}(n)\), with a fairly strong value of \(C\). Our second theorem (Theorem 2.2) is deduced from Theorem 2.1.

2. Results

2.1. On partial divisor functions. The theorems in this section do all the work in proving the corollaries on additive divisor sums in Section 2.2. Theorems 2.1 and 2.2 are proved in Section 4.2 and Theorem 2.3 is proved in Section 4.1. In light of the structural connection to additive divisor sums, theorems of this type are potentially of further use in such applications. Moreover, in accordance with the conjecture that \(\theta_k = 1\), we expect that the ranges of \(A\), \(B\) and \(q\) for which these formulae hold may be improved significantly.

Theorem 2.1 shows that \(d_k(n, A)\) provides a robust approximation to \(A^{k-1}d_k(n)\) in arithmetic progressions.

**Theorem 2.1.** If \(h, k \in \mathbb{N}\) are fixed and \(q \leq x^{\min(\theta_k, A\theta_{k-1})-\epsilon}\), then
\[
\sum_{n \equiv h \ (\text{mod } q)} d_k(n, A) = A^{k-1} \sum_{n \equiv h \ (\text{mod } q)} d_k(n) + O_{A, \epsilon, h, k} \left( \frac{x \log^{k-2} x}{q} \right).
\]
In other words
\[
\sum_{n \equiv h \ (\text{mod } q)} d_k(n, A) = \frac{x}{q} \sum_{n \equiv h \ (\text{mod } q)} \frac{(n, q)d_{k-1}(n)}{n} + O_{A, h, k} \left( \frac{x \log^{k-2} x}{q} \right).
\]

Theorem 2.2 gives an asymptotic formula for the correlation of \(d_k(n, A)\) with \(d_l(n, B)\).

**Theorem 2.2.** If \(A \leq 1\), \(B < \min(\theta_k, A\theta_{k-1})\) and \(h, k, l \in \mathbb{N}\) are fixed, then
\[
\sum_{n \leq x} d_k(n + h, A)d_l(n, B) = A^{k-1}B^{l-1}C_{k,l}f_{k,l}(h) \frac{x \log^{k+l-2} x}{(k-1)!(l-1)!} + O_{A, B, h, k, l} \left( x \log^{k+l-3} \right).
\]
where $C_{k,l}$ and $f_{k,l}(h)$ are defined in (1.6) and (1.7).

Theorem 2.3 gives an asymptotic expansion with power saving error term for the correlation of $d_k(n)$ and $d_l(n, A)$.

**Theorem 2.3.** If $A < \theta_k$ and $k, l \in \mathbb{N}$ are fixed, then there is a $\delta > 0$ and a polynomial $P_{A,h,k,l}$ of degree $k + l - 2$ such that

$$
\sum_{n \leq x} d_k(n + h)d_l(n, A) = xP_{A,h,k,l}(\log x) + O_{A,\delta,h,k,l}(x^{1-\delta}).
$$

(2.4)

An explicit formula for $P_{A,h,k,l}$ is given in (4.20). In particular, the coefficient of the leading term is $A^lC_{k,l}f_{k,l}(h)/(k-1)!(l-1)!$.

We note that if $\theta_k > 1/2$ and $l = 2$, then $A = 1/2$ is admissible in Theorem 2.3 which thus yields an alternative proof of (1.4) in such cases. For example, in Section 4.4 we carry out this calculation in the case $k = l = 2$ to reproduce Estermann’s asymptotic expansion (1.3) explicitly.

2.2. On additive divisor sums. Corollaries I, II and III follow immediately from the theorems of Section 2.1.

Corollary I sharpens the lower bound (1.9) given by Ng and Thom in [30] when $h$ is fixed and $k$ is sufficiently large in comparison with $l$.

**Corollary I.** For fixed $h, k, l \in \mathbb{N}$ we have

$$
\liminf_{x \to \infty} \frac{D_{h,k,l}(x)}{x \log^{k+l-2} x} \geq \theta_k^{l-1} \frac{C_{k,l}f_{k,l}(h)}{(k-1)!(l-1)!}.
$$

(2.5)

**Proof.** Note that $d_l(n) \geq d_l(n, A)$, and use Theorem 2.3. □

For instance, given Heath-Brown’s exponent $\theta_3 = 21/41$, it follows from Corollary I that

$$
\liminf_{x \to \infty} \frac{D_{h,3,3}(x)}{x \log^4 x} \geq 0.262 \frac{C_{3,3}f_{3,3}(h)}{4}.
$$

Corollary II gives an equivalent condition for the conjectured asymptotic (1.5).

**Corollary II.** For fixed $h, k, l \in \mathbb{N}$, the asymptotic (1.5) holds if and only if

$$
\sum_{n \leq x} d_k(n + h) \left( d_l(n) - B^{1-l}d_l(n, B) \right) = o \left( x \log^{k+l-2} x \right)
$$

for some (and hence every) $B < \theta_k$.

**Proof.** Compare (1.5) with Theorem 2.3 or Theorem 2.2 with $A = 1$. □
In support of the plausibility of (2.6), we note that

**Corollary III.** If \( A < \theta_l, B < \min(\theta_k, A\theta_{k-1}) \) and \( h, k, l \in \mathbb{N} \) are fixed, then

\[
(2.7) \quad \sum_{n \leq x} d_k(n + h, A) \left( d_l(n) - B^{1-l} d_l(n, B) \right) = O_{A, B, h, k, l} \left( x \log^{k+l-3} x \right).
\]

**Proof.** This follows from Theorem 2.2 by swapping variables \( A, B \) and \( k, l \). \( \square \)

The last result of this section is Theorem 2.4. This is a Tauberian theorem and may be viewed as an analogue of the relationship between the Prime Number Theorem and the non-vanishing of \( \zeta(1 + it) \), in which we view \( D_{h,k,l}(x) \) analogously to \( \pi(x) \). Theorem 2.4 is proved in Section 4.3.

**Theorem 2.4.** Let \( h, k, l \in \mathbb{N} \) and \( 0 \leq y < \infty \) be fixed, then the function

\[
D_{h,k,l}(s, y) = \sum_{1}^{\infty} d_k(n + h) d_l \left( \frac{y}{\log n} \right) \frac{(n + h)^s}{(n + h)^s} \quad (\sigma > 1)
\]

has an analytic continuation to the complex plane except for a pole of order \( k - 1 \) at \( s = 1 \) and, if the limit

\[
\lim_{y \to \infty} D_{h,k,l}(1 + it, y)
\]

is continuous for \( t \neq 0 \), then we have

\[
\frac{D_{h,k,l}(x)}{x \log^{k+l-2} x} \sim \frac{C_{k,l} f_{k,l}(h)}{(k-1)!(l-1)!}
\]

as \( x \to \infty \).

3. Definitions

Definitions 3.1—3.4 arise in the course of the proofs.

**Definition 3.1.** For \( j \in \mathbb{N} \) and \( s \in \mathbb{C} \), we define

\[
(s - 1)^j \zeta^j(s) = \sum_{r=0}^{\infty} \frac{a_r(j)}{r!} (s - 1)^r
\]

and

\[
\frac{1}{n!} \frac{d^n}{ds^n} \left( \frac{(s - 1)^j \zeta^j(s)}{s} \right) \bigg|_{s=1} = \sum_{r=0}^{n} \frac{(-1)^{n-r} a_r(j)}{r!} = c_n(j).
\]

**Definition 3.2.** For \( h, k, l \in \mathbb{N}, h = \prod p^\gamma, \Re w > -1 - \frac{2}{l-1} \) and \( \sigma > -\Re w \), we also define

\[
C_{k,l}(s, w) = \prod_p \left( 1 - p^{-w-1} \right)^{l-1} + \frac{(1 - p^{-s})^k}{1 - p^{-1}} - \frac{(1 - p^{-s})^k(1 - p^{-w-1})^{l-1}}{1 - p^{-1}},
\]
we see that (4.1) is
\[
\begin{align*}
\text{Proof.} \\
\text{We have Theorem 2.3.}
\end{align*}
\]

(3.3)

Using Definition 1.1 to evaluate the inner summations on the r.h.s of (4.1), for 0

(4.1)

Lastly, for

(3.2)

and note that the Dirichlet series in (3.2) converge absolutely. In particular, we have

b_{h,k,l,m,n} = \sum_{i=0}^{k-1-m} \sum_{j=0}^{l-1-n-j} \frac{a_{l-1-n-j}(l-1)c_{k-1-m-i}(k)}{(l-1-n-j)!} \frac{\partial^i}{i! \partial s^i} \frac{\partial^j}{j! \partial w^j} \sum_{q=0}^{\infty} \frac{\phi_{h,k,l}(q,s)}{q^w} \bigg|_{w=0,s=1}

(3.2)

and we define

A_{h,k,l,m} = \left( -1 \right)^m \sum_{j=m-l+2}^{l-2} \binom{i}{j} \sum_{r=m-l+2}^{m} \sum_{v=0}^{r-m+l-2} \frac{(-A)^{r-j-v} l^{l-1} a_v(l-1)(v-l+1)_r}{v!}

(3.3)

\times \binom{l-v-2}{j-r} \sum_{i=j}^{k-1} \frac{c_{k-1-i}(k)}{i!} \frac{\partial^i}{\partial s^i} \frac{\partial^j}{\partial w^j} \sum_{q=0}^{\infty} \frac{\phi_{h,k,l}(q,s)}{q^w} \bigg|_{w=0,s=1}

(3.3)

4. Proofs

4.1. Theorem 2.3

Proof. We have

\[
\sum_{n \leq x} d_k(n+h)d_l(n, A) = \sum_{n \leq x} d_k(n+h) \sum_{q \mid n} \frac{d_l(q)}{q^{\alpha+1}}
\]

(4.1)

Using Definition 1.1 to evaluate the inner summations on the r.h.s of (4.1), for 0 < A < \theta_k

we see that (4.1) is

\[
\sum_{n \leq x} d_k(n+h)d_l(n, A) = \sum_{q \leq x^A} \frac{d_l(q)}{\phi(q^{\alpha+1})} \text{Res} \left( \frac{(x+h)^s}{s} \sum_{(n,q)=(h,q)} \frac{d_k(n)}{n^s}, s = 1 \right)
\]
\[ (4.2) \quad - \sum_{q \leq x^A} \frac{d_{l-1}(q)}{\phi \left( \frac{q}{(h,q)} \right)} \text{Res} \left( \frac{(q^{1/A} + h - \delta_A(q))^s}{s} \sum_{(n,q) = (h,q)} \frac{d_k(n)}{n^s}, s = 1 \right) \]

\[ + O_{A, \delta, k} \left( x^{1-\delta} \sum_{q \leq x^A} \frac{d_{l-1}(q)}{\phi \left( \frac{q}{(h,q)} \right)} \right) \]

\[ + O_{A, \delta, k} \left( x^{1-\delta} \sum_{q \leq x^A} \frac{d_{l-1}(q)(q^{1/A} + h - \delta_A(q))^{1-\delta}}{\phi \left( \frac{q}{(h,q)} \right)} \right) \]

where \( \delta_A(q) = 0 \) or \( 1 \) depending on whether \( q^{1/A} \) is an integer or not. The summations in the error terms in the third and fourth lines of (4.2) are \( O_{A,h} (\log l - 1) x \), so it remains to evaluate the first two terms.

4.1.1. Evaluation of the primary term. We begin by evaluating the first term on the r.h.s of (4.2). Let \( \chi_0 \) denote the principal character to the modulus \( q/g \), where \( q = \prod p^{\alpha} \), \( h = \prod p^{\gamma} \) and \( g = (h,q) = \prod p^{\delta} \) so \( \delta = \min(\alpha, \gamma) \). We have

\[ \sum_{(n,q) = g} \frac{d_k(n)}{n^s} = \sum_{1}^{\infty} \frac{\chi_0(n)d_k(gn)}{(gn)^s} = \prod_{p \mid g} \sum_{0}^{\infty} d_k(p^{\beta+\delta})\chi_0(p^{\beta})p^{-(\beta+\delta)s} = L^k(s, \chi_0)b_{h,k}(s, q) \]

where

\[ b_{h,k}(s, q) = \prod_{p \mid g} (1 - \chi_0(p)p^{-s})^k \sum_{\delta} d_k(p^{\beta-\delta})\chi_0(p^{\beta-\delta})p^{-\beta s} \]

is a multiplicative function of \( g \) for all \( k, s \). By Cauchy’s theorem, the first term on the r.h.s of (4.2) is

\[ = \frac{1}{(k-1)!} \left( \frac{s-1}{s} \right)^{k-1} \zeta^k(s)Z_{h,k,l} (s, x^A) (x + h)^s \right|_{s=1} \]

\[ = \frac{1}{(k-1)!} \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{d^i}{d^{i}s} Z_{h,k,l} (s, x^A) \left| \frac{\partial^{k-1-i}}{\partial^{s-1-i} s} (s-1)^{k} \zeta^k(s)(x+h)^s \right|_{s=1} \]

\[ = \frac{1}{(k-1)!} \sum_{q \leq x^A} \frac{d_{l-1}(q)}{\phi \left( \frac{q}{(h,q)} \right)} \prod_{p \mid (r,q)} (1 - p^{-s})^k b_{h,k}(s, q), \]

\[ (4.4) \quad Z_{h,k,l} (s, x^A) = \sum_{q \leq x^A} \frac{d_{l-1}(q)}{\phi \left( \frac{q}{(h,q)} \right)} \prod_{p \mid (r,q)} (1 - p^{-s})^k b_{h,k}(s, q), \]
and so our first task is to find asymptotic formulae for $\frac{\partial^r}{\partial s^r}Z_{h,k,l}(s, Q)$ at $s = 1$ as $Q \to \infty$. To proceed, we note that the factor

$$\frac{d_i(q)}{\phi(q)} \prod_{p \mid q(h)} \left(1 - p^{-s}\right)^k$$

of the summand in (4.4) is a multiplicative function of $q$ for all $h, k, s$, and we shall now show that $b_{h,k}(s, q)$ is also. From (4.3) we have

$$b_{h,k}(s, q) = \prod_{p \mid (q,h)} \frac{d_k(p^\delta(q))}{p^{\delta(q)s}} \prod_{p \mid (q,h), p \nmid (s, q)} \left(1 - p^{-s}\right)^k \sum_{\delta(q)} d_k(p^\beta)p^{-\beta s}.$$  

If $q = rt$ with $(r, t) = 1$, we have $(rt, h) = (r, h)(t, h)$ and $\delta(rt) = \delta(r) + \delta(t)$ for every $p$ so the inclusion $p|rt, h)$ in (4.5) is multiplicative, that is

$$b_{h,k}(s, rt) = \prod_{p \mid (r, h)} \frac{d_k(p^\delta(r))}{p^{\delta(r)s}} \prod_{p \mid (r, h), p \nmid (t, h)} \left(1 - p^{-s}\right)^k \sum_{\delta(r)} d_k(p^\beta)p^{-\beta s}$$

$$\times \prod_{p \mid (t, h)} \frac{d_k(p^\delta(t))}{p^{\delta(t)s}} \prod_{p \mid (t, h), p \nmid (t, h)} \left(1 - p^{-s}\right)^k \sum_{\delta(t)} d_k(p^\beta)p^{-\beta s}.$$  

(4.6)

Since $p|r$ implies $p \nmid t$, the intersection of the sets $p|(r, h)$ and $p|t/(t, h)$ in (4.6) is already empty. It follows that if $p|(r, h)$ then the inclusion $p|t/(t, h)$ and exclusion $p \nmid t/(t, h)$ is superfluous, and vice-versa. Therefore

$$b_{h,k}(s, rt) = \prod_{p \mid (r, h)} \frac{d_k(p^\delta(r))}{p^{\delta(r)s}} \prod_{p \mid (r, h), p \nmid (t, h)} \left(1 - p^{-s}\right)^k \sum_{\delta(r)} d_k(p^\beta)p^{-\beta s}$$

$$\times \prod_{p \mid (t, h)} \frac{d_k(p^\delta(t))}{p^{\delta(t)s}} \prod_{p \mid (t, h), p \nmid (t, h)} \left(1 - p^{-s}\right)^k \sum_{\delta(t)} d_k(p^\beta)p^{-\beta s}$$

$$= b_{h,k}(s, r)b_{h,k}(s, t).$$

Thus

$$Z_{h,k,l}(s, x^A) = \sum_{q \leq x^A} \phi_{h,k,l}(s, q)$$
where the summand

$$\phi_{h,k,l}(s, q) = \frac{d_{l-1}(q)}{\phi\left(\frac{q}{(q,h)}\right)} \prod_{p|q,(q,h)} (1 - p^{-s})^k \prod_{p|(q,h)} \frac{d_k(p^{\delta(q)})}{p^{\delta(q)s}} \prod_{p|(q,h)} (1 - p^{-s})^k \sum_{\delta(q)} d_k(p^\beta)p^{-\beta s}$$

is a multiplicative function of $q$ for all $h, k, l, s$.

Since $\phi_{h,k,l}(s, q)$ is multiplicative, we define the Euler product

$$\Phi_{h,k,l}(s, w) = \prod_p \sum_{0}^{\infty} \phi_{h,k,l}(s, p^{\alpha}) p^{-\alpha w}$$

for values of $w \in \mathbb{C}$ for which the r.h.s converges absolutely. If $p \nmid h$ then $\phi_{h,k,l}(s, p^{\alpha}) = \phi_{1,k,l}(s, p^{\alpha})$ which, after some routine algebra, gives

$$\Phi_{h,k,l}(s, w) = C_{k,l}(s, w)f_{h,k,l}(s, w)\zeta^{l-1}(w + 1),$$

where $f_{h,k,l}(s, w)$ is defined in (3.1) and the Euler product

$$C_{k,l}(s, w) = \prod_p \left(1 - p^{-w-1}\right)^{l-1} + \frac{(1 - p^{-s})^k (1 - (1 - p^{-w-1})^{l-1})}{1 - p^{-1}}$$

converges absolutely for $\Re w > -1 - \frac{\sigma-1}{l-1}$ and $\sigma > -\Re w$ (this follows from the fact that the largest power of $p$ appearing in each factor of the Euler product has real part strictly less than $-1$ when $s, w$ are in this range). Consequently, $C_{k,l}(s, w)$ is analytic and bounded on compact subsets of the half planes $\Re w > -1 - \frac{\sigma-1}{l-1}$ and $\sigma > -\Re w$. It follows that for fixed $h, i, k, l$ the Dirichlet series

$$\frac{\partial^{i}}{\partial s^{i}}C_{k,l}(s, w)f_{h,k,l}(s, w) = \sum_{1}^{\infty} \frac{\partial^{i}}{\partial s^{i}} \varphi_{h,k,l}(q, s) \frac{q^{w}}{q^{d}}$$

is absolutely convergent and bounded for such values of $s, w$. Thus, using the relation

$$\phi_{h,k,l}(s, q) = \sum_{d|q} \varphi_{h,k,l}(d, s) \frac{d_{l-1}(q/d)}{q/d},$$

(4.9)
we have

\[
\frac{\partial^i}{\partial s^i} Z_{h,k,l}(s, Q) = \frac{\partial^i}{\partial s^i} \sum_{q \leq Q} \varphi_{h,k,l}(d, s) \frac{d_{l-1}(q/d)}{q/d}
\]

\[
= \frac{\partial^i}{\partial s^i} \sum_{d \leq Q} \varphi_{h,k,l}(d, s) \sum_{q \leq Q/d} \frac{d_{l-1}(q)}{q}
\]

(4.10)

\[
= \frac{\partial^i}{\partial s^i} \sum_{d \leq Q} \varphi_{h,k,l}(d, s) \left( \sum_{j=0}^{l-1} \frac{a_{l-1-j}(l-1) \log^j (Q/d)}{(l-1-j)!j!} \right)
\]

\[
+ \frac{1}{2\pi i} \int_{(c-2/l)} \zeta^{l-1} (w+1) \frac{(Q/d)^w}{w} dw
\]

\[
= \sum_{j=0}^{l-1} \frac{a_{l-1-j}(l-1)}{j!(l-1-j)!} \frac{\partial^i}{\partial s^i} \sum_{d \leq Q} \varphi_{h,k,l}(d, s) \log^j (Q/d)
\]

\[
+ O \left( Q^{c-2/l} \sum_{d \leq Q} \left| \frac{\partial^i}{\partial s^i} \varphi_{h,k,l}(d, s) \right| d^{2/l-\epsilon} \right)
\]

where the notation \( \int_{(c)} \) in the fourth line of (4.10) denotes integration along a vertical line from \( c-i\infty \) to \( c+i\infty \). That this integral is \( O \left( (Q/d)^{c-2/l} \right) \) follows from classical results on the error term in the generalised Dirichlet divisor problem (see Titchmarsh [37], for instance). Expanding \( \log^j (Q/d) \) as a polynomial in \( \log Q \), (4.10) is

\[
= \sum_{j=0}^{l-1} \frac{a_{l-1-j}(l-1)}{j!(l-1-j)!} \sum_{n=0}^{j} \binom{j}{n} \log^n Q \frac{\partial^i}{\partial s^i} \sum_{d \leq Q} \varphi_{h,k,l}(d, s) (-\log d)^{j-n} + O \left( Q^{c-2/l} \right)
\]

\[
= \sum_{n=0}^{l-1} \frac{\log^n Q}{n!} \sum_{j=0}^{l-1-n} \frac{a_{l-1-n-j}(l-1)}{j!(l-1-n-j)!} \frac{\partial^i}{\partial s^i} \frac{\partial^j}{\partial w^j} \sum_{d \leq Q} \varphi_{h,k,l}(d, s) \left| \frac{\partial^i}{\partial s^i} \frac{\partial^j}{\partial w^j} \right| w=0 + O \left( Q^{c-2/l} \right).
\]

(4.11)

We also have
\[
\frac{\partial^{k-1-i}}{\partial s^{k-1-i}} \left( (s-1)^k \zeta^k(s)(x+h)^s \right) \bigg|_{s=1} = (x+h)(k-1-i)! \sum_{r=0}^{k-1-i} \frac{a_r(k)}{r!} \sum_{m=0}^{k-1-i} \frac{(-1)^{k-1-i-r} \log^m(x+h)}{m!} (s-1)^{k-1-i} \zeta^k(s) \left( x^r + h \right) s \\
\frac{k-1-i}{s} \left( x^r + h \right) s \bigg|_{s=1} = (x+h)(k-1-i)! \sum_{m=0}^{k-1-i} \frac{\log^{k-1-i-m}(x+h)}{(k-1-i-m)!} c_m(k)
\]

(4.12)

Setting \( Q = x^A \) in (4.11) and using (4.12), we conclude that (4.3) is

\[
= (x+h) \sum_{m=0}^{k-1-i} \frac{A^n b_{h,k,l,m,n}}{m! n!} \log^m(x+h) \log^n x
\]

\[
+ O \left( (x+h) x^{e-2A/l} \sum_{i=0}^{k-1-i} \frac{\log^{k-1-i-m}(x+h)c_{m-i}(k)}{(k-1-m)!} \sum_{d \leq A} \left| \frac{\partial^i}{\partial s^i} \varphi_{h,k,l}(d, s) \bigg|_{s=1} \right| d^{2/l-\epsilon} \right),
\]

\[
= x \sum_{m=0}^{k-1-i} \frac{A^n b_{h,k,l,m,n}}{m! n!} \log^{m+n} x + O_{h,k,l} \left( x^{1-2A/l+\epsilon} \right),
\]

(4.13)

where the coefficients \( b_{h,k,l,m,n} \) are defined in Section 3.

4.1.2. Evaluation of the secondary term. We now evaluate the second term on the r.h.s of (4.2). By Cauchy’s theorem, this is

\[
= \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial s^{k-1}} \left( s-1 \right)^k \zeta^k(s) W_{h,k,l}(s, x^A) \bigg|_{s=1} = \frac{1}{(k-1)!} \sum_{i=0}^{k-1} \left( \begin{array}{c} k-1 \\ i \end{array} \right) \frac{\partial^i}{\partial s^i} W_{h,k,l}(s, x^A) \bigg|_{s=1} \frac{\partial^{k-1-i}}{\partial s^{k-1-i}} \left( s-1 \right)^k \zeta^k(s) \bigg|_{s=1}
\]

(4.14)

\[
= \sum_{i=0}^{k-1} \frac{1}{i!} \frac{\partial^i}{\partial s^i} W_{h,k,l}(s, x^A) \bigg|_{s=1} c_{k-1-i}(k),
\]

where

\[
W_{A,h,k,l}(s, Q) = \sum_{q \leq Q} \phi_{h,k,l}(s, q)(q^{1/A} + h - \delta_A(q))^s.
\]

By (4.9) we have
\[
\frac{\partial^i}{\partial s^i} W_{A,h,k,l}(s, Q) \bigg|_{s=1} = \frac{\partial^i}{\partial s^i} \sum_{d \leq Q} \varphi_{h,k,l}(d, s) \left( \sum_{q \leq Q/d} \frac{d_{l-1}(q)(q(d)^{1/A} + h - \delta_A(q))^s}{q} \right) \bigg|_{s=1}
\]

(4.15)

\[
\sum_{j=0}^{i} \binom{i}{j} \sum_{d \leq Q} \frac{\partial^{i-j}}{\partial s^{i-j}} \varphi_{h,k,l}(d, s) \frac{\partial^j}{\partial s^j} V_{A,h,l,Q}(d, s) \bigg|_{s=1},
\]

where

\[
V_{A,h,l,Q}(d, s) = \sum_{q \leq Q/d} \frac{d_{l-1}(q)(q(d)^{1/A} + h - \delta_A(q))^s}{q}
\]

and

\[
\frac{\partial^j}{\partial s^j} V_{A,h,l,Q}(d, s) \bigg|_{s=1} = (-1)^j \sum_{q \leq Q/d} \frac{d_{l-1}(q)(q(d)^{1/A} + h - \delta_A(q)) \log^j((q(d)^{1/A} + h - \delta_A(q))}{q}
\]

\[
= (-A)^{-j} d^{1/A} \sum_{q \leq Q/d} \frac{d_{l-1}(q) \log^j(qd)}{q^{1-1/A}} + O_{A,h,j,l}(Q/d^j)
\]

(4.16)

\[
= (-A)^{-j} d^{1/A} \sum_{j=0}^{j} \binom{j}{m} \log^{j-m} d \sum_{q \leq Q/d} \frac{d_{l-1}(q) \log^m q}{q^{1-1/A}} + O_{A,h,j,l}(Q/d^j).
\]

The inner summation on the r.h.s of (4.16) may be written as

\[
\sum_{q \leq Q/d} \frac{d_{l-1}(q) \log^m q}{q^{1-1/A}} = \frac{(-1)^m}{2\pi i} \int_{(c)} \frac{d^m}{dw^m} \zeta^{l-1}(w + 1) \left( \frac{(Q/d)^{w+1/A} dw}{w + 1/A} \right),
\]

which may be evaluated using Cauchy’s Theorem and classical results on the error term in the generalised Dirichlet divisor problem (see Titchmarsh [37]). The error term is \(O_{A,h,l}(Q/d^{1/A-2l+\epsilon})\), and the residue at the pole at \(w = 0\) is

\[
\frac{(-1)^m}{(m + l - 2)!} \sum_{v=0}^{m+l-2} \binom{m + l - 2}{v} \frac{\partial^v}{\partial w^v} \frac{d^m}{dw^m} \zeta^{l-1}(w + 1) \bigg|_{w=0} \frac{\partial^{m+l-2-v}}{\partial w^{m+l-2-v}} \left( \frac{(Q/d)^{w+1/A}}{w + 1/A} \right)_{w=0}
\]

\[
= (-1)^{m-1}(Q/d)^{1/A} \sum_{v=0}^{l+m-2} \frac{a_v(l - 1)(v - l + 1)m}{v!} \sum_{r=0}^{l+m-2-v} \frac{(-A)^{l+m-1-v-r} \log^r(Q/d)}{r!}
\]

\[
= (-1)^{m-1}(Q/d)^{1/A} \sum_{r=0}^{l+m-2} \frac{\log^r(Q/d)}{r!} \sum_{v=0}^{l+m-2-r} \frac{(-A)^{l+m-1-v-r} a_v(l - 1)(v - l + 1)m}{v!}.
\]
As such, (4.16) is

\[
(4.15) \quad j
\]

and so, expanding \( \log^j (Q/d) \) as a polynomial in \( \log Q \) on the r.h.s of (4.17), it follows that (4.15) is

\[
= -Q^{1/A} \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) (-1)^j \sum_{m=0}^{j} \left( \begin{array}{c} j \\ m \end{array} \right) \sum_{r=0}^{l+m-2} \sum_{u=0}^{r} \sum_{v=0}^{l+m-2-r} \frac{\log^u Q}{v!} \frac{\log^{v-r} (Q/d)}{u! (r-u)!} \frac{Q^{1/A}}{v!} \\
\times \frac{(-A)^{l+m-1-v-r} a_v (l-1)(v-l+1)_m}{v!} \partial^{i-j} \partial^{j+1-r-u-2} \frac{\varphi_{h,k,l}(d,s)}{d^{s Q}} \bigg|_{s=1} \\
+ O \left( Q^{1/A+2/l} \right)
\]

\[
\]

\[
= -Q^{1/A} \sum_{u=2-l}^{i} \frac{\log^{l+u-2} Q}{(l + u - 2)!} \sum_{j=u}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) (-1)^j \sum_{m=u}^{j} \left( \begin{array}{c} j \\ m \end{array} \right) \sum_{r=0}^{m-u} \sum_{v=0}^{r} \frac{\log^u Q}{v!} \frac{\log^{v-r} (Q/d)}{u! (r-u)!} \frac{Q^{1/A}}{v!} \\
\times \frac{(-A)^{r-j-v+1} a_v (l-1)(v-l+1)_m}{(m - r - u)!v!} \partial^{i-j} \partial^{j+l-r-u-2} \frac{\varphi_{h,k,l}(d,s)}{d^{s Q}} \bigg|_{s=1,v=0} \\
+ O \left( Q^{1/A+2/l} \right)
\]

\[
= -Q^{1/A} \sum_{u=2-l}^{i} \frac{\log^{l+u-2} Q}{(l + u - 2)!} \sum_{j=u}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) (-1)^j \sum_{m=u}^{j} \left( \begin{array}{c} j \\ m \end{array} \right) \sum_{r=0}^{m-u} \sum_{v=0}^{r} \frac{\log^u Q}{v!} \frac{\log^{v-r} (Q/d)}{u! (r-u)!} \frac{Q^{1/A}}{v!} \\
\times \frac{(-A)^{r-j-u+1} a_v (l-1)(v-l+1)_m}{(m - r)!v!} \partial^{i-j} \partial^{j+l-r-2} \frac{\varphi_{h,k,l}(d,s)}{d^{s Q}} \bigg|_{s=1,v=0} \\
+ O \left( Q^{1/A+2/l} \right)
\]
\[
Q^{1/A} \sum_{u=2-l}^{i} \sum_{j=0}^{i} \left( \frac{\log^{l+u-2} Q}{(l+u-2)!} \right) \sum_{j=0}^{i} (-1)^j \sum_{r=0}^{j} \frac{(-A)^{r-j-u-v+1} a_v(l-1)}{v!}
\times \left( \sum_{m=r}^{j} \binom{j}{m} \frac{(v-l+1)m}{(m-r)!} \right) \frac{\partial^{j-i} \partial^{j+l-r-2} \varphi_{h,k,l}(d,s)}{\partial^{s^{i-j}} \partial^{w^{j+l-r-2}}} \sum_{d \leq Q} \frac{\varphi_{h,k,l}(d,s)}{d^w} \bigg|_{s=1, v=0} + O(Q^{1/A+\epsilon-2/l})
\]

so (4.14) is

\[
= \frac{-Q^{1/A}}{u!} \sum_{u=0}^{k-l-3} \log^{l+u-2} Q \sum_{j=0}^{i} \sum_{r=0}^{j} \frac{(-A)^{r-j-u-v+1} a_v(l-1)(v-l+1)}{v!}
\times \left( \frac{l-v-2}{j-r} \right) \sum_{i=j}^{k-1} \frac{c_{k-1-i}(k)}{i!} \frac{\partial^{j-i} \partial^{j+l-r-2} \varphi_{h,k,l}(d,s)}{\partial^{s^{i-j}} \partial^{w^{j+l-r-2}}} \sum_{d \leq Q} \frac{\varphi_{h,k,l}(d,s)}{d^w} \bigg|_{s=1, v=0} + O(Q^{1/A+\epsilon-2/l})
\]

(4.18)

Taking \( Q = x^A \) in (4.18) yields

\[
\frac{1}{(k-1)!} \frac{\partial^{k-1} \left( s-1 \right)^k e^k(s) W_{h,k,l} (s, x^A) \bigg|_{s=1}}{\partial s^{k-1}} = -x \sum_{u=0}^{k-l-3} a_{A,h,k,l,u} \log^u x \bigg|_{s=1} + O(x^{1+\epsilon-2A/l}).
\]

(4.19)

From (4.2), (4.13) and (4.19), for \( A < \theta_k \) we have

\[
\sum_{n \leq x} d_k(n + h)d_l(n, A) = x \sum_{m=0}^{k-l-3} \frac{A^m b_{h,k,l,m,n}}{m!} \log^{m+n} x + x \sum_{m=0}^{k-l-3} \frac{a_{A,h,k,l,m,n} \log^m x}{m!}
\]

(4.20)

\[
+ O_{A,h,k} \left( x^{1+\epsilon-2A/l} \right) + O_{A,h,k} \left( x^{1-\delta} \right)
\]
where the coefficients $b_{h,k,l,m,n}$ and $a_{A,h,k,l,u}$ are defined in Section 3. This concludes the proof of Theorem 2.3.

\[ \square \]

4.2. Theorems 2.1 and 2.2

**Proof of Theorem 2.1** We begin by writing

\[ d_k(n, A) = d_k(n) - \sum_{d|n \atop d \leq x^{1-A}} d_k \left( \frac{n}{d} \right) + \sum_{d|n \atop nA^{-1} < d \leq n^A} d_k-1(n), \]

so that

\[ \sum_{n \equiv h \pmod{q} \atop n \leq x} d_k(n, A) = \sum_{n \equiv h \pmod{q} \atop n \leq x} d_k(n) - \sum_{d < x^{1-A} \atop d \equiv h \pmod{q} \atop n \leq A} \sum_{n \equiv h \pmod{q} \atop \frac{1}{x} \leq d \leq \frac{1}{A}} d_k-1(n) \]

\[ + \sum_{d \leq x^A} d_k-1(d) \sum_{d \equiv h \pmod{q} \atop \frac{1}{A} \leq d < x^{1-A}} 1. \]  

(4.22)

Firstly, using Definition 1.1 in the form (1.11), the first term on the r.h.s of (4.22) is

\[ \frac{1}{\phi(q/(h,q))} \sum_{n \leq x/(h,q)} \chi_0(n)d_k((h,q)n) + O_{\epsilon,\delta,k} \left( \frac{x^{1-\delta}}{\phi(q/(h,q))} \right) \]

for $q \leq x^{\theta_k - \epsilon}$, where $\chi_0$ is the principal character to the modulus $q/(q,h)$. Secondly, the third term on the r.h.s of (4.22) is

\[ \leq \sum_{n < x^{1-A} \atop d \leq x^A} d_k-1(d) \]

\[ \leq \sum_{n \equiv h \pmod{(n,q)/h} \atop d \equiv h \pmod{(n,q)/h}} \sum_{d \leq x^A} d_k-1(d). \]  

(4.24)

Using Definition 1.1 with $q \leq x^{\theta_{k-1} - \epsilon}$ and the fact that $((n/(n,q))(h/(h,q)), q/(n,q)) = (h/(n,q), q/(n,q)) = (h,q)/(n,q)$ when $(n,q)|h$, (4.24) is

\[ = O_{A,\epsilon,k} \left( \frac{\phi^{k-2}(q/(h,q)) x^A \log^{k-2} x}{(q/(h,q))^{k-1}} \sum_{n < x^{1-A} \atop (n,q)|h} 1 \right) = O_{A,\epsilon,h,k} \left( \frac{x \log^{k-2} x}{q} \right). \]

(4.25)
As such, (4.22) is

\[
= \frac{1}{\phi(q/(h,q))} \sum_{n \leq x/(h,q)} \chi_0(n) d_k((h,q)n) \\
- \sum_{d < x^{1-A}} \sum_{n \equiv h \pmod{q}, \frac{n}{d} \leq x/d} d_{k-1}(n) + O_{A,\epsilon,h,k} \left( \frac{x \log^{k-2} x}{q} \right) \\
= \frac{1}{\phi(q/(h,q))} \sum_{n \leq x/(h,q)} \chi_0(n) d_k((h,q)n) \\
- \sum_{d < x^{1-A}} \sum_{n \leq x/(h,q))/(d/(d,q))} \chi_0(n) d_{k-1} \left( \frac{(h,q)n}{(d,q)} \right) + O_{A,\epsilon,h,k} \left( \frac{x \log^{k-2} x}{q} \right)
\]

(4.26) for \( q \leq x^{\min(\theta_k,A\theta_k-1) - \epsilon} \), where we have used Definition 1.1 again to write the second term on the r.h.s of (4.26) as a character sum. We now write the second term as

\[
= - \sum_{m \leq x/(h,q)} \sum_{d < x^{1-A}} \chi_0 \left( \frac{(d,q)m}{d} \right) d_{k-1} \left( \frac{(h,q)m}{d} \right)
\]

(4.27)

where \( r = d/(d,q) \). Since \((d,q)|h\) we have \((d,q)|(h,q)\) so \(q/(h,q)|q/(d,q)\), therefore the condition \((r,q/(d,q)) = 1\) may be replaced with \((r,q/(h,q)) = 1\) and so (4.27) is

\[
= - \sum_{m \leq x/(h,q)} \sum_{d < x^{1-A}} \chi_0 \left( \frac{m}{r} \right) d_{k-1} \left( \frac{(h,q)m}{(d,q)r} \right)
\]
\[= - \sum_{m \leq x/(h,q)} \chi_0(m) \sum_{d \leq x^{1-A} \atop d/(d,q) \mid m} \left( \frac{(h,q)m}{d} \right) \]

\[= - \sum_{m \leq x/(h,q)} \chi_0(m) \sum_{d/(d,q) \mid m} \frac{d_{k-1}}{(d,q)h} \left( \frac{(h,q)m}{d} \right) \]

\[+ \sum_{m \leq x/(h,q)} \chi_0(m) \sum_{x^{1-A} \leq d \leq x/(h,q) \atop d/(d,q) \mid m} \frac{d_{k-1}}{(d,q)h} \left( \frac{(h,q)m}{d} \right) \]

\[= - \sum_{m \leq x/(h,q)} \chi_0(m) d_{k-1} ((h,q)m) \]

(4.28)

\[+ \sum_{m \leq x/(h,q)} \chi_0(m) \sum_{x^{1-A} \leq d \leq x/(h,q) \atop d/(d,q) \mid m} \frac{d_{k-1}}{(d,q)h} \left( \frac{(h,q)m}{d} \right) , \]

where in the fourth line of (4.28) we have used the identity

\[d_k((h,q)m) = \sum_{d/(d,q)|m} \frac{d_{k-1}}{(d,q)h} \left( \frac{(h,q)m}{d} \right) .\]

Therefore, by (4.26) and (4.28), the main term in (4.22) is

\[= \frac{1}{\phi(q/(h,q))} \sum_{m \leq x/(h,q)} \chi_0(m) \sum_{x^{1-A} \leq d \leq x/(h,q) \atop d/(d,q) \mid m} \frac{d_{k-1}}{(d,q)h} \left( \frac{(h,q)m}{d} \right) \]

(4.30)

and the error term is \(O_{A,\epsilon,h,k}(x \log^{k-2} x/q)\). To evaluate the main term, we note that (4.30) is

\[= \frac{1}{\phi(q/(h,q))} \sum_{x^{1-A} \leq d \leq x/(h,q) \atop (d,q) \mid h} \sum_{m \leq x/(h,q)/(d/(d,q))} \chi_0 \left( \frac{dm}{(d,q)} \right) d_{k-1} \left( \frac{(h,q)m}{d} \right) \]

\[= \frac{1}{\phi(q/(h,q))} \sum_{x^{1-A} \leq d \leq x/(h,q) \atop (d,q) \mid h} \chi_0(d) \sum_{n \leq x/d \atop (h,q) \mid n} \chi_0 \left( \frac{n}{(h,q)} \right) d_{k-1} (n) \]
\[ = \frac{1}{\phi(q/(h, q))} \sum_{n \leq x} \chi_0 \left( \frac{n}{(h, q)} \right) d_{k-1} (n) \sum_{x^{1-A} \leq d \leq x/(h, q)} \chi_0(d), \]  

(4.31)

where the condition \((d, q)|h\) is removed because \(\chi_0(d) = 0\) if \((d, q) > 1\). This is

\[ = \frac{1}{\phi(q/(h, q))} \sum_{n \leq x^A} \chi_0(n) d_{k-1} ((h, q)n) \sum_{x^{1-A} \leq d \leq x/(h, q)} \chi_0(d), \]

\[ = \frac{1}{\phi(q/(h, q))} \sum_{n \leq x^A} \chi_0(n) d_{k-1} ((h, q)n) \sum_{d \leq x/(h, q)} \chi_0(d) + O_{A,h,k} \left( \frac{x \log^{k-2} x}{q} \right). \]

(4.32)

Since \((n/(h, q), q/(h, q)) = 1\) is equivalent to \((n, q) = (h, q)\) which implies that \((n, q)|h\), we conclude that (4.22) is

\[ \frac{x}{q} \sum_{n \leq x^A} \frac{(n,q)d_{k-1}(n)}{n} + O_{A,h,k} \left( \frac{x \log^{k-2} x}{q} \right). \]

(4.33)

By partial summation or otherwise, the remainder of the proof is trivial. \(\square\)

**Proof of Theorem 2.4.** This is a straightforward consequence of Theorem 2.1 and the method of proof of Theorem 2.3. We have

\[ \sum_{n \leq x} d_k(n + h, A)d_l(n, B) = \sum_{q \leq x^B} d_{l-1}(q) \sum_{n \equiv h \pmod{q}} d_k(n, A) \]

\[ = A^{k-1} \sum_{q \leq x^B} d_{l-1}(q) \sum_{n \equiv h \pmod{q}} d_k(n) \]

\[ + O_{A,B,h,k} \left( \frac{x \log^{k-2} x}{q} \sum_{q \leq x^B} \frac{d_{l-1}(q)}{q} \right) \]

provided that \(B < \min(\theta_k, A\theta_{k-1})\), by Theorem 2.1. The first term on the r.h.s. of (4.34) is identical to (4.11), and the summation in the error term is \(O(\log^{l-1} x)\). \(\square\)

**4.3. Theorem 2.4.**

**Proof.** We begin by establishing the analytic continuation of \(D_{h,k,l}(s, Q)\). We have
\[ D_{h,k,l}(s, Q) = \sum_{n=1}^{\infty} \frac{d_k(n + h) d_l \left( n, \frac{Q}{\log n} \right)}{(n + h)^s} \]

\[ = \sum_{n=1}^{\infty} \frac{d_k(n + h)}{(n + h)^s} \sum_{q \leq Q} d_{l-1}(q) \]

\[ = \sum_{q \leq Q} d_{l-1}(q) \sum_{n \equiv h \pmod{q}} \frac{d_k(n)}{n^s} \sum_{n > h} \frac{d_k(n)}{n^s} - \frac{d_k(h)}{h^s} \sum_{q \leq Q} d_{l-1}(q). \]

We have

\[ \sum_{n \equiv h \pmod{q}} \frac{d_k(n)}{n^s} = \frac{1}{\phi \left( \frac{2}{q} \right)} \sum_{n \equiv h \pmod{q}} \chi \left( \frac{h}{g} \right) \sum_{1}^{\infty} \frac{\chi(n) d_k(gn)}{(gn)^s} \]

where

\[ \sum_{1}^{\infty} \frac{\chi(n) d_k(gn)}{(gn)^s} = \prod_{p} \sum_{0}^{\infty} d_k(p^{\beta + \delta}) \chi(p^{\beta}) p^{-\beta - \delta} \]

\[ = L^k(s, \chi) b_k(s, \chi, g) \]

is a meromorphic function of \( s \) for all \( h, k, l \).

This shows that \( D_{h,k,l}(s, Q) \) is a meromorphic function of \( s \) for all \( h, k, l, Q \), and we observe that

\[ D_{h,k,l}(s, Q) = \zeta^k(s) Z_{h,k,l}(s, Q) + B_{h,k,l}(s, Q) \]

say, where \( Z_{h,k,l}(s, Q) \) is defined in (4.4) and \( B_{h,k,l}(s, Q) \) is an analytic function of \( s \) for all fixed \( h, k, l, Q \). Writing

\[ D_{h,k,l}(x, Q) = \sum_{n \leq x} d_k(n + h) d_l \left( n, \frac{Q}{\log n} \right), \]

we have

\[ D_{h,k,l}(s, Q) = s \int_{1}^{\infty} D_{h,k,l}(x, Q) \frac{dx}{(x + h)^{s+1}} \]

and, by (4.36), we have
We now use Definitions 3.3 and 3.4 to calculate the coefficients in (4.39). We use Estermann’s notation in the same way as in the proof of Theorem 2.3, we have
\[
\sum_{h,k,l} \frac{Z_{h,k,l}(s, Q)}{(s-1)^k} + C_{h,k,l}(s, Q)
\]
for \(\sigma > 1\), where \(C_{h,k,l}(s, Q) = O_{h,k,l,Q}(s - 1)^{1-k}\) as \(s \to 1\). By (4.36) we know that \(D_{h,k,l}(1 + it, Q)\) is continuous for \(t \neq 0\) (in fact it is analytic in a neighbourhood of the line). As such, the Delange-Ikehara Tauberian theorem \([6]\) applies, i.e.
\[
\lim_{x \to \infty} \frac{D_{h,k,l}(x, Q)}{x \log^{k-1} x} = Z_{h,k,l}(1, Q).
\]
Arguing in the same way as in the proof of Theorem 2.3, we have
\[
\frac{Z_{h,k,l}(1, Q)}{\log^{t-1} Q} = \frac{C_{k,1,h,l}(h)}{(k-1)!(l-1)!} + O_{h,k,l} \left( \frac{1}{\log Q} \right).
\]
Now, if \(\lim_{Q \to \infty} D_{h,k,l}(1 + it, Q)\) is continuous when \(t \neq 0\), then the restriction that \(Q\) is fixed in (4.37) can be removed and the Delange-Ikehara Tauberian theorem still applies. Taking \(Q = x\) we have \(D_{h,k,l}(x, Q) = D_{h,k,l}(x)\), which completes the proof. \(\square\)

4.4. The coefficients in the case \(k = l = 2\). We conclude this paper with a demonstration that Theorem 2.3 recovers Estermann’s asymptotic expansion for \(D_{h,2,2}(x)\) precisely. We take \(k = l = 2\) in (4.20), so that
\[
\sum_{n \leq x} d_2(n+h)d_2(n, A) = x \sum_{m=0}^{1} \sum_{n=0}^{1} \frac{A^n b_{h,2,2,m,n}}{m!n!} \log^{m+n} x + x \sum_{m=0}^{1} \frac{a_{A,h,2,2,m} \log^m x}{m!} + O_{A,h} \left( x^{1-\delta} \right)
\]
\[
= A b_{h,2,2,1,1} x \log^2 x + (b_{h,2,2,1,0} + A b_{h,2,2,0,1} + a_{A,h,2,2,1}) x \log x + (b_{h,2,2,0,0} + a_{A,h,2,2,0}) x + O_{A,h} \left( x^{1-\delta} \right).
\]
Thus, putting \(A = 1/2\) and using the symmetry of the divisors of \(n\) about \(n^{1/2}\) in (4.38), we obtain
\[
D_{h,2,2}(x) = b_{h,2,2,1,1} x \log^2 x + (2 b_{h,2,2,1,0} + b_{h,2,2,0,1} + 2 a_{1/2,h,2,2,1}) x \log x + 2 (b_{h,2,2,0,0} + a_{1/2,h,2,2,0}) x + O_{h} \left( x^{1-\delta} \right).
\]
We now use Definitions 3.3 and 3.4 to calculate the coefficients in (4.39). We use Estermann’s notation (4.11) notation
\[
\sigma'_{-1}(h) = \sum_{d|h} \frac{\log d}{d}, \quad \sigma''_{-1}(h) = \sum_{d|h} \frac{\log^2 d}{d}
\]
and
\[
a' = -\sum_{2}^{\infty} \frac{\mu(n) \log n}{n^2}, \quad a'' = \sum_{2}^{\infty} \frac{\mu(n) \log^2 n}{n^2}.
\]
Firstly, for the coefficient of $x \log^2 x$ we have

$$b_{h,2,2,1,1} = C_{2,2}(1,0) f_{h,2,2}(1,0) = \frac{6}{\pi^2} \sigma_1(h).$$

Secondly, for the coefficient of $x \log x$, we have

\begin{align*}
2b_{h,2,2,0,1} + b_{h,2,2,0,1} + & 2a_{1/2,h,2,2,1} \\
= & 2a_1(1)c_0(2)C_{2,2}(1,0) f_{h,2,2}(1,0) + 2a_0(1)c_0(2) \frac{\partial}{\partial w} C_{2,2}(1, w) f_{h,2,2}(1, w) \bigg|_{w=0} \\
+ & a_0(1)c_1(2)C_{2,2}(1,0) f_{h,2,2}(1,0) + a_0(1)c_0(2) \frac{\partial}{\partial s} C_{2,2}(s,0) f_{h,2,2}(s,0) \bigg|_{s=0} \\
- & a_0(1)c_0(2)C_{2,2}(1,0) f_{h,2,2}(1,0) \\
= & 2\gamma C_{2,2}(1,0) f_{h,2,2}(1,0) + 2 \frac{\partial}{\partial w} C_{2,2}(1, w) f_{h,2,2}(1, w) \bigg|_{w=0} \\
+ & (2\gamma - 1) C_{2,2}(1,0) f_{h,2,2}(1,0) + \frac{\partial}{\partial s} C_{2,2}(s,0) f_{h,2,2}(s,0) \bigg|_{s=0} \\
- & C_{2,2}(1,0) f_{h,2,2}(1,0) \\
= & \frac{12}{\pi^2} (2\gamma - 1) \sigma_1(h) + 2 \frac{\partial}{\partial w} C_{2,2}(1, w) f_{h,2,2}(1, w) \bigg|_{w=0} \\
+ & \frac{\partial}{\partial s} C_{2,2}(s,0) f_{h,2,2}(s,0) \bigg|_{s=0} \\
= & \left(\frac{12}{\pi^2} (2\gamma - 1) + \frac{6}{\pi^2} \left(2 \frac{\partial}{\partial w} C_{2,2}(1, w) \bigg|_{w=0} + \frac{\partial}{\partial s} f_{h,2,2}(s,0) \bigg|_{s=0}\right)\right) \sigma_1(h) \\
- & \frac{24}{\pi^2} \sigma_1(h).
\end{align*}

Lastly, for the coefficient of $x$ we have

\begin{align*}
2b_{h,2,2,0,0} & + 2a_{1/2,h,2,2,0} \\
= & 2a_1(1)c_1(2)C_{2,2}(1,0) f_{h,2,2}(1,0) + 2a_1(1)C_{2,2}(1,0) \frac{\partial}{\partial s} f_{h,2,2}(s,0) \bigg|_{s=1} \\
+ & 2a_1(1) f_{h,2,2}(1,0) \frac{\partial}{\partial s} C_{2,2}(s,0) \bigg|_{s=1} + 2a_0(1)c_1(2)C_{2,2}(1,0) \frac{\partial}{\partial w} f_{h,2,2}(1, w) \bigg|_{w=0} \\
+ & 2c_1(2) f_{h,2,2}(1,0) \frac{\partial}{\partial w} C_{2,2}(1, w) \bigg|_{w=0} + 2C_{2,2}(1,0) \frac{\partial}{\partial s} \frac{\partial}{\partial w} f_{h,2,2}(s, w) \bigg|_{w=0,s=1} \\
+ & 2 \frac{\partial}{\partial w} C_{2,2}(s, w) \bigg|_{w=0} \frac{\partial}{\partial s} f_{h,2,2}(s, w) \bigg|_{s=1} + 2 \frac{\partial}{\partial s} C_{2,2}(s, w) \bigg|_{s=1} \frac{\partial}{\partial w} f_{h,2,2}(s, w) \bigg|_{w=0}.
\end{align*}
\[
+2f_{h,2,2}(1,0) \frac{\partial}{\partial w} \frac{\partial}{\partial s} C_{2,2}(s, w) \bigg|_{w=0, s=1} \\
- c_1(2)C_{2,2}(1,0)f_{h,2,2}(1,0) - f_{h,2,2}(1,0) \frac{\partial}{\partial s} C_{2,2}(s, 0) \bigg|_{s=1} - C_{2,2}(1,0) \frac{\partial}{\partial s} f_{h,2,2}(s, 0) \bigg|_{s=1} \\
+ C_{2,2}(1,0)f_{h,2,2}(1,0) \\
= \frac{12\gamma}{\pi^2}(2\gamma - 1)\sigma_{-1}(h) - \frac{24\gamma}{\pi^2} \sigma'_{-1}(h) + 4\gamma a' \sigma_{-1}(h) - \frac{12}{\pi^2}(2\gamma - 1)\sigma'_{-1}(h) + 2(2\gamma - 1)a' \sigma_{-1}(h) \\
+ \frac{24}{\pi^2} \sigma''_{-1}(h) - 8a' \sigma''_{-1}(h) + 4a'' \sigma_{-1}(h) - \frac{6}{\pi^2}(2\gamma - 1)\sigma_{-1}(h) + \frac{12}{\pi^2} \sigma'_{-1}(h) - 2a' \sigma_{-1}(h) + \frac{6}{\pi^2} \sigma_{-1}(h) \\
= \left(\frac{6}{\pi^2}(2\gamma - 1)^2 + \frac{6}{\pi^2} + 4a'(2\gamma - 1) + 4a'' \right) \sigma_{-1}(h) - \left(\frac{24}{\pi^2}(2\gamma - 1) + 8a' \right) \sigma'_{-1}(h) + \frac{24}{\pi^2} \sigma''_{-1}(h).
\]

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**References**

[1] S. Baier, T. D. Browning, G. Marasingha, L. Zhao, *Averages of shifted convolutions of \( d_3(n) \)*, Proc. Edinb. Math. Soc. (2) 55 (2012), no. 3, 551–576.

[2] G. Bareikis. *On the DDT theorem*, Acta Arith. 126 (2) 155–168 (2007).

[3] A. Bykovski, A. I. Vinogradov. *Inhomogeneous convolutions* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 160 (1987), no. Anal. Teor. Chisel i Teor.Funktsii. 8, 1630, 296; translation in J. Soviet Math. 52 (1990), no. 3, 3004–3016.

[4] C. E. Chace. *The divisor problem for arithmetic progressions with small modulus*, Acta Arithmetica 61.1 (1992) 35–50.

[5] J.B. Conrey, S.M. Gonek. *High moments of the Riemann zeta-function*, Duke Math. J. 107 (2001), 577–604.

[6] H. Delange. Généralisation du théorème de Ikehara, Ann. Sci. Ec. Norm. Sup., 71 (1954), 213–242.

[7] J.-M. Deshouilliers, F. Dress, G. Tenenbaum. *Lois de répartition des diviseurs, I*, Acta Arith. 2, 147-51 (1979).

[8] J.-M. Deshouillers, H. Iwaniec. *An additive divisor problem*, J. London Math. Soc. 26 (1982), 1–14.

[9] S. Drappeau. *Sums of Kloosterman sums in arithmetic progressions, and the error term in the dispersion method*, arxiv.org/abs/1504.05549.

[10] W. Duke, J.B. Friedlander, and H. Iwaniec. *A quadratic divisor problem*, Invent. Math. 115 (1994), no. 2, 209–217.
[11] T. Estérenn. *Über die Darstellung einer Zahl als Differenz von zwei Produkten*, J. Reine Angew. Math. 164 (1931), 173–182.

[12] É. Fouvry, E. Kowalski, P. Michel. *On the exponent of distribution of the Ternary divisor function*. Mathematika, 61(1), 121–144 (2015).

[13] É. Fouvry and G. Tenenbaum. *Sur la corrélation des fonctions de Piltz*, Rev. Mat. Iberoamericana 1 (1985), no. 3, 43–54.

[14] J. B. Friedlander and H. Iwaniec. *The divisor problem for arithmetic progressions*, Acta Arith. 45 (1985), 273–277.

[15] D. A. Goldston and S. M. Gonek. *Mean value theorems for long Dirichlet polynomials and tails of Dirichlet series*, Acta Arith. 84 (1998), 155–192.

[16] D.R. Heath-Brown. *The fourth power moment of the Riemann zeta function*, Proc. London Math. Soc. (3) 38 (1979), no. 3, 385–422.

[17] D. R. Heath-Brown. *The divisor function $d_3(n)$ in arithmetic progressions*, Acta Arith. 47(1) (1986) 29–56.

[18] K. Henriot. *Nair-Tenenbaum bounds uniform with respect to the discriminant*, Mathematical Proceedings of the Cambridge Philosophical Society (2012), 152(3), 405–424.

[19] C. Hooley. *An asymptotic formula in the theory of numbers*, Proc. London Math. Soc. (3) 7: 396–413, (1957).

[20] A. Ingham. *Some asymptotic formulae in the theory of numbers*, J. London Math. Soc. 2 (1927), 202–208.

[21] A. Ivić. *The general additive divisor problem and moments of the zeta-function*, New trends in probability and statistics, Vol. 4 (Palanga, 1996), 69-89, VSP, Utrecht, 1997.

[22] A. Ivić. *On the ternary additive divisor problem and the sixth moment of the zeta-function*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), 205–243, London Math. Soc. Lecture Note Ser., 237, Cambridge Univ. Press, Cambridge, 1997.

[23] N.V. Kuznetsov. *Petersson’s conjecture for forms of weight zero and Linnik’s conjecture*, Sums of Kloosterman sums, Mathematics of the USSR-Sbornik 39(3), (1981).

[24] A. F. Lavrik. *On the principal term in the divisor problem and the power series of the Riemann zeta-function in a neighborhood of its pole*, English transl. in Proc. Steklov Inst. Math. 1979, no. 3, 175–183.

[25] Y. Motohashi. *On some additive divisor problems*, J. Math. Soc. Japan 28 (1976), 772–784.

[26] Y. Motohashi. *The binary additive divisor problem*, Ann. Sci École Norm. Sup. (4)27 (1994) 529–572.

[27] M. Nair, G. Tenenbaum. *Short sums of certain arithmetic functions*, Acta Math. 180 (1998), no. 1, 119–144.

[28] N. Ng, M. Thom. *Bounds and conjectures for additive divisor sums*, arXiv:1609.01411v1 (2016).

[29] T. Tao. *Heuristic computation of correlations of higher order divisor functions*, https://terrytao.wordpress.com (2016).

[30] G. Tenenbaum. *Lois de répartition des diviseurs, 4*, Ann. Inst. Fourier 29, 1–15 (1979).

[31] G. Tenenbaum. *Lois de répartition des diviseurs, 2*, Acta Arith. 38, 1-36 (1980).

[32] B. Topacogullari. *Shifted convolution of divisor sums*, arxiv.org/abs/1506.02608 (2015).

[33] B. Topacogullari. *On a certain additive divisor sum*, arxiv.org/abs/1512.05770 (2015).

[34] B. Topacogullari. *The shifted convolution of generalized divisor functions*, http://arxiv.org/abs/1605.02364 (2016).

[35] E.C. Titchmarsh. *The theory of the Riemann zeta-function*, 2nd ed. Oxford Univ. Press, New York, (1986).

[36] A. Weil. *On some exponential sums*, Proc. Nat. Acad. Sci. U. S. A. 34 (1948), 204–207.
