Non-predetermined Model Theory
(short paper)

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Abstract

This article introduce a new model theory call non-predetermined model theory where functions and relations need not to be determined already and they are determined through time.

1 Introduction

A mathematical structure is set of object with a collection of distinguished functions, relations, and special elements. For example, one may consider the structure $\mathbb{N} = (\mathbb{N}, +, \cdot, <, 0, 1)$ of natural numbers where $+: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\cdot: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ are functions, $\subseteq \mathbb{N} \times \mathbb{N}$ is a relation and 0 and 1 are two distinguished elements in $\mathbb{N}$. Whenever a subject (a mathematician) wants to know the value of for example $5 + 7$, he refers to definition of the function $+$ and according to the definition he finds out that the value is 12. The function $+$ is predetermined and is independent of the subject behavior.

In this article, we introduce structures which are non-predetermined and subject-dependent. In non-predetermined structures, functions and relations are not needed to be determined already. In non-predetermined structures, for a given unary function $f$ and an object $a$ the value of $f(a)$ is not necessary determined already, and it is determined as soon as the subject intends to know (or compute) the value $f(a)$.

2 Non-predetermined Structures

In this section, we introduce non-predetermined structures.

Notation 2.1

- Let $D$ be a set. We define $D^*$ to be the set of all finite sequences (string) over $D$.

- For each string $s = \langle d_1, d_2, ..., d_k \rangle$ over $D$, and a set $A$, we let $F_s(A)$ be the set of all functions from $\{d_1, d_2, ..., d_k\}$ to $A$.

- For two strings $s = \langle d_1, d_2, ..., d_k \rangle$ and $s' = \langle d'_1, d'_2, ..., d'_n \rangle$ we let $s.s' = \langle d_1, ..., d_k, d'_1, ..., d'_n \rangle$.

- We refer to the empty string $\langle \rangle$ by $\lambda$.

- For two string $s$ and $t$, we say $s \leq t$ whenever there exists $v$ such that $t = sv$. 
Definition 2.2 A non-predetermined function from $D$ to $A$ denoted by $f : D \rightarrow A$ is a mapping $H$ where $H$ maps each string $s \in D^*$ to a function $H[s] \in F_s(A)$ with following property:

for each $s = (d_1, d_2, ..., d_k)$ in $D^*$ and each $d_{k+1} \in D$ we have for all $1 \leq i \leq k$, $H[s'](d_i) = H[s](d_i)$ where $s' = (d_1, d_2, ..., d_k, d_{k+1})$.

A first order language $L$ contains

- a finite set of predicate symbols $\mathcal{R} = \{R_i \mid i \leq m_1\}$, and a natural number $n_i$ as its ary,
- a finite set of function symbols $\mathcal{F} = \{f_i \mid i \leq m_0\}$, and a natural number $n_i$ as its ary,
- a set of constant symbols $\mathcal{C}$.

Definition 2.3 A subject-dependent $L$-structure $\mathcal{M}$ is given by following data

(i) A set $M$ called the universe or the underlying set of $\mathcal{M}$,

(ii) A collection $\mathcal{F} = M = \{f_i^M \mid i \leq m_0\}$ where each $f_i^M$ is a non-predetermined function $f_i^M : M^{n_i} \rightarrow M$ correspond to symbol function $f_i \in \mathcal{F}$,

(iii) A collection of relations $\mathcal{R} = M = \{R_i^M \mid i \leq m_1\}$ where each $R_i^M$ is a non-predetermined function $R_i^M : M^{n_i} \rightarrow \{0, 1\}$ correspond to symbol function $R_i \in \mathcal{R}$,

(iv) A collection of distinguish elements $\{c_i^M \mid i \in I_2\}$ correspond to constant symbols in $\mathcal{C}$.

A state of the structure $\mathcal{M}$ is $e = ([s_0, s_1, ..., s_{m_0}], [s'_0, s'_1, ..., s'_{m_1}])$ where $s_i, s'_i$ belong to $M^{n_i}$. Always one and only one state called the current state denoted by $(\mathcal{M}, e_c)$. Initially, we may assume that all strings in the current state are empty, that is $e_c = ([\lambda, \lambda, ..., \lambda], [\lambda, \lambda, ..., \lambda])$.

The structure $\mathcal{M}$ is like a black box for the subject.

Suppose that $e_c = ([s_0, s_1, ..., s_{m_0}], [s'_0, s'_1, ..., s'_{m_1}])$.

i. Whenever the subject wants to know what is the value of a function $f_i^M$ ($1 \leq i \leq m_0$) at a point $d \in M^{n_i}$, he queries $(f_i^M, d)$ to $\mathcal{M}$ then the current state of $\mathcal{M}$ changes to $e_c = ([s_0, s_1, ..., s_i, (d), ..., s_{m_0}], [s'_0, s'_1, ..., s'_{m_1}, s'_i])$, and $H_i[s_i, (d)](d)$ is return as the value of $f_i^M(d)$ (where $H_i$ is the corresponding map to $f_i^M$).

ii. Whenever the subject wants to know what is the value of a relation $R_i^M$ ($1 \leq i \leq m_1$) at a point $d \in M^{n_i}$, he queries $(R_i^M, d)$ to $\mathcal{M}$ then the current state of $\mathcal{M}$ changes to $e_c = ([s_0, s_1, ..., s_{m_0}], [s'_0, s'_1, ..., s'_i, (d), ..., s'_{m_1}])$, and $H'_i[s'_i, (d)](d)$ is return as the value of $R_i^M(d)$ (where $H'_i$ is the corresponding map to $R_i^M$).

We refer to $f^M$ and $R^M$ and $c^M$ as interpretation of symbols $f$, $R$ and $c$ in structure $\mathcal{M}$.

Remark 2.4 Note that

1) the current state of the structure $\mathcal{M}$ is determined according to behavior of the subject,
2) All functions $f^M_i$’s and relations $R^M_i$’s are not predetermined in the structure $\mathcal{M}$. As subject freely chooses a point $d \in M^n$, and a function $f^M_i$ (or a relation $R^M_i$) the current state of the structure changes and the value $f^M_i(d)$ ($R^M_i(d)$) is determined.

**Example 2.5** Consider the structure $\mathcal{M} = \langle M = \{0, 1\}, \mathcal{R}^M = \emptyset, \mathcal{R}^M = \{R_1\}, C^M = \emptyset \rangle$ where $R_1$ is a non-predetermined unary relation defined below:

- $H_{R_1}[0](0) = 1, H_{R_1}[\langle 0, 1 \rangle][1] = 0,$
- $H_{R_1}[1](1) = 1, H_{R_1}[\langle 1, 0 \rangle][0] = 0,$
- for each $s = \langle d_1, d_2, \ldots, d_k \rangle$ in $\{0, 1\}^k$ and each $d_{k+1} \in \{0, 1\}$ we have for all $1 \leq i \leq k$,
  $H_{R_1}[s'](d_i) = H_{R_1}[s](d_i)$ where $s' = \langle d_1, d_2, \ldots, d_k, d_{k+1} \rangle$.

The relation $R_1$ initially is not determined. At first, the subject is in the current state $e_c = [\lambda, [\lambda]]$. The subject choose either 0 or 1 to know that if $R_1$ holds for it. For example, suppose that the subject choose 0, then $R_1(0)$ is true and the current state changes to $e_c = [\lambda, [\lambda]]$. After this time, $R_1(0)$ is determined to be true in the structure $\mathcal{M}$. Note that if the subject chose 1 instead of 0 (when the current state was $e_c = [\lambda, [\lambda]]$) then whenever after that $R_1(0)$ is determined, it would be false.

For two states $e = ([s_0, s_1, \ldots, s_m], [s'_0, s'_1, \ldots, s'_{m'}])$ and $e' = ([l_0, l_1, \ldots, l_m], [l'_0, l'_1, \ldots, l'_{m'}])$, we say $e \leq e'$ whenever for all $i, s_i \leq l_i$ and $s'_i \leq l'_i$.

**Definition 2.6** TERM is the smallest set containing

variable symbols,
constants symbols in $C$,

for each function symbol $f_i \in F$, if $t_1, t_2, \ldots, t_n \in$ TERM then $f(t_1, t_2, \ldots, t_n)$ is a term.

**Definition 2.7** Let $e_c = ([s_0, s_1, \ldots, s_m], [s'_0, s'_1, \ldots, s'_{m'}]), f_i \in F$ be a symbol function, and $d \in M^m$. We say the interpretation of the symbol $f_i$ for $d$ is determined in state $e_c$ whenever

- $d$ is an element of the finite sequence $s_i$.

The interpretation is defined to be $f^M_{i, e_c}(d) = H_i[s_i](d)$.

We say the interpretation of the symbol $R_i$ for $d$ is determined in state $e_c$ whenever

- $d$ is an element of the finite sequence $s'_i$.

The interpretation is defined to be $R^M_{i, e_c}(d) = H_i[s'_i](d)$.

Let $t$ be a term using variable $\bar{v} = (v_1, \ldots, v_m)$, and $\bar{a} = (a_1, \ldots, a_m)$ where $a_i \in M$. Let $e_c = ([s_0, s_1, \ldots, s_m], [s'_0, s'_1, \ldots, s'_{m'}])$. For a subterm $t'$ of $t$, we inductively define interpretation $t^{e_c}(\bar{a})$ as follows:

- If $t'$ is a constant symbol $c$, then $t^{e_c}(\bar{a})$ is determined in state $e_c$ and $t^{e_c}(\bar{a}) := c^M$. 
• If $t'$ is the variable $v_i$, then $t^{pe}(\bar{a})$ is determined in state $e_c$ and $t^{pe}(\bar{a}) := a_i$.

• If $t'$ is the term $f_i(t_1, ..., t_{n_i})$, then $t^{pe}(\bar{a})$ is determined in state $e_c$ whenever
  1. for $1 \leq j \leq n_i$, $t^{pe, j}(\bar{a})$ is determined in state $e_c$,
  2. $(t^{pe, 1}(\bar{a}), ..., t^{pe, n_i}(\bar{a}))$ is an element of the finite sequence $s_i$.

If $t^{pe}(\bar{a})$ is determined then $t^{pe}(\bar{a}) := f^{(M,e_c)}(t^{pe, 1}(\bar{a}), ..., t^{pe, n_i}(\bar{a}))$.

**Definition 2.8** \( FORMULA \) is the smallest set satisfying the following conditions:

1) $t_1, t_2 \in TERM$ then $t_1 = t_2 \in FORMULA$,

2) for each predicate symbol $R_i \in \mathcal{R}$, if $t_1, t_2, ..., t_{n_i} \in TERM$ then $R(t_1, t_2, ..., t_{n_i}) \in FORMULA$,

3) $\varphi, \psi \in FORMULA$ then $¬\varphi, \varphi \land \psi, \varphi \lor \psi, \varphi \rightarrow \psi, ∀y\varphi, ∃y\varphi \in FORMULA$. We call formulas defined via item 1, 2, and 3 atomic formulas.

Let $\varphi(v_1, v_2, ..., v_n)$ be a formula with free variables $v_1, v_2, ..., v_n$. Similar to definition 1.1.6 \[1\], we define what it means for $\varphi(v_1, v_2, ..., v_n)$ to hold of $(a_1, a_2, ..., a_n) \in M^n$.

**Definition 2.9** Let $\varphi$ be a formula with free variables $\bar{v} = (v_1, v_2, ..., v_n)$, and let $\bar{a} = (a_1, a_2, ..., a_n) \in M^n$. Let $e_c = ([s_0, s_1, ..., s_{m_0}], [s'_0, s'_1, ..., s'_{m_1}])$. We inductively define $(M, e_c) \models \varphi(\bar{a})$ as follows:

i. If $\varphi$ is $t_1 = t_2$ then $(M, e_c) \models \varphi(\bar{a})$ whenever for all states $e$ after $e_c$ ($e_c \leq e$) if both $t^{e, 1}_1(\bar{a})$ and $t^{e, 2}_2(\bar{a})$ are determined in state $e$ then $t^{e, 1}_1(\bar{a}) = t^{e, 2}_2(\bar{a})$.

ii. If $\varphi$ is $R_i(t_1, t_2, ..., t_{n_i})$ then $(M, e_c) \models \varphi(\bar{a})$ whenever for all states $e$ after $e_c$ ($e_c \leq e$) if

- $t^{e, 1}_1(\bar{a}), t^{e, 2}_2(\bar{a}), ..., t^{e, n_i}_n(\bar{a})$ are determined in state $e$, and
- $R^{(M,e)}_i(t^{e, 1}_1(\bar{a}), ..., t^{e, n_i}_n(\bar{a}))$ is determined in state $e$,

then $R^{(M,e)}_i(t^{e, 1}_1(\bar{a}), ..., t^{e, n_i}_n(\bar{a})) = 1$

iii. If $\varphi$ is $¬\psi$ then $(M, e_c) \models \varphi(\bar{a})$ whenever for all states $e$ after $e_c$ ($e_c \leq e$) then $(M, e) \not\models \psi(\bar{a})$.

iv. If $\varphi$ is $\psi \land \theta$ then $(M, e_c) \models \varphi(\bar{a})$ whenever

- for all states $e$ after $e_c$ ($e_c \leq e$), if atomic formulas which are subformula of $\psi(\bar{a})$ are determined in state $e$ then $(M, e) \models \psi(\bar{a})$ and
- for all states $e$ after $e_c$ ($e_c \leq e$) if atomic formulas which are subformula of $\theta(\bar{a})$ are determined in state $e$ then $(M, e) \models \theta(\bar{a})$. 
v. If $\varphi$ is $\psi \lor \theta$ then $(M, e_c) \models \varphi(\bar{a})$ whenever

- either for all states $e$ after $e_c$ ($e_c \leq e$), if atomic formulas which are subformula of $\psi(\bar{a})$ are determined in state $e$ then $(M, e) \models \psi(\bar{a})$, or
- for all states $e$ after $e_c$ ($e_c \leq e$), if atomic formulas which are subformula of $\theta(\bar{a})$ are determined in state $e$ then $(M, e) \models \theta(\bar{a})$.

vi. If $\varphi$ is $\exists x \psi(\bar{v}, x)$ then $(M, e_c) \models \varphi(\bar{a})$ whenever there exists $b \in M$ such that for all states $e$ after $e_c$ ($e_c \leq e$), if atomic formulas which are subformula of $\psi(\bar{a}, b)$ are determined in state $e$ then $(M, e) \models \psi(\bar{a}, b)$.

vii. If $\varphi$ is $\forall x \psi(\bar{v}, x)$ then $(M, e_c) \models \varphi(\bar{a})$ whenever for all $b \in M$ for all states $e$ after $e_c$ ($e_c \leq e$), if atomic formulas which are subformula of $\psi(\bar{a})$ are determined in state $e$, then if $(M, e) \models \psi(\bar{a})$ then $(M, e) \models \theta(\bar{a})$.

viii. If $\varphi$ is $\psi \rightarrow \theta$ then $(M, e_c) \models \varphi(\bar{a})$ whenever for all states $e$ after $e_c$ ($e_c \leq e$), if atomic formulas which are subformula of $\psi(\bar{a})$ are determined in state $e$, and atomic formulas which are subformula of $\theta(\bar{a})$ are determined in state $e$, then if $(M, e) \models \psi(\bar{a})$ then $(M, e) \models \theta(\bar{a})$.

The main difference between Kripke structures and non-predetermined structures is that the actual state in Kripke structure is fixed, but the actual state of non-predetermined structures (called current state) changes throughout time up to subject.

**Theorem 2.10** For all $\phi \in \text{Formula}$, and a subject-dependent structure $M$, for every states $e$ and $e'$ we have:

$$e \leq e' \text{ and } (M, e) \models \phi \text{ then } (M, e') \models \phi.$$  

**Proof.** It is straightforward. $\dashv$

We prove that our proposed model theory is sound with respect of intuitionistic deduction system (see page 40 of [2]). For a set of formula $\Gamma$, and a formula $\phi$, we write $\Gamma \vdash_i \phi$ to say that $\phi$ is derivable form $\Gamma$ using intuitionistic deduction system.

**Theorem 2.11** Let $\Gamma \vdash_i \phi$ and $M$ be a structure. Assume for all formula $\psi \in \Gamma$, $(M, e) \models \psi$ (abbreviated by $(M, e) \models \Gamma$). Then $(M, e) \models \phi$.

**Proof.** The proof is similar to the proof of theorem 5.10, page 82 of [2]. It is done by induction on the derivations in intuitionistic deduction system. Let a derivation terminates with a rule

$$\Gamma_1 \vdash_i \psi_1 \quad \Gamma_2 \vdash_i \psi_2 \quad \ldots \\ \vdash \Gamma_i \psi_i.$$ 

By induction we assume that if $(M, e) \models \Gamma_i$ then $(M, e) \models \psi_i$, then using the assumption we prove that if $(M, e) \models \Gamma$ then $(M, e) \models \phi$.  

$\dashv$
3 Conclusion

We introduced structures which functions and relations are not necessary predetermined and the value of them is eventually recognized by the way that the subject interacts with the structure. One may ask that what is the use of non-predetermined structures. Suppose that $\Gamma$ is a set of formula, $\phi$ is a formula, and we want to show that $\Gamma \not\vdash \phi$. One way to show this is to find a model $M$, and prove that $M \models \Gamma$ and $M \not\models \phi$. If constructing a predetermined model is difficult then we may try to construct non-predetermined model.

References

[1] D. Marker, Model Theory: An introduction, Springer, 2002.

[2] A. S. Troelstra, D. van Dalen, Constructivism in Mathematics, An introduction, Vol. 1, North-Holland, 1988.