Reducing (to) the Ranks.
Efficient Rank-based Büchi Automata Complementation
(Technical Report)

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Abstract. This paper provides several optimizations of the rank-based approach for complementing Büchi automata. We start with Schewe’s theoretically optimal construction and develop a set of techniques for pruning its state space that are key to obtaining small complement automata in practice. In particular, the reductions (except one) have the property that they preserve (at least some) so-called super-tight runs, which are runs whose ranking is as tight as possible. Our evaluation on a large benchmark shows that the optimizations indeed significantly help the rank-based approach and that, in a large number of cases, the obtained complement is the smallest from those produced by a large number of state-of-the-art tools for Büchi complementation.

1 Introduction

Büchi automata (BA) complementation remains an intensively studied problem since 1962, when Büchi introduced the automata model over infinite words as a fundamental stone for a decision procedure of a fragment of a second-order arithmetic [1]. Since then, efficient BA complementation became an important task from both theoretical and practical side. It is a crucial operation in some approaches for termination analysis of programs [2,3,4] as well as in decision procedures concerning reasoning about programs, such as S1S [4] or the temporal logics ETL and QPTL [5].

Büchi launched a hunt for an optimal and efficient complementation technique with his doubly exponential complementation approach [1] based on the infinite Ramsey theorem. A couple of years later, Safra proposed a complementation via deterministic Rabin automata with an \(n^{O(n)}\) upper bound. Simultaneously with finding an efficient complementation algorithm, another search for the theoretical lower bound was under way. Michel showed in [6] that the lower bound for BA complementation is \(n!\) (approx. \((0.36n)^n\)). This result was further refined by Yan to \((0.76n)^n\) in [7]. From the theoretical point of view, it seemed that the problem was already solved since Safra’s construction asymptotically matched the lower bound. From the practical point of view, however, a factor in the exponent has a great impact on the size of the complemented automaton (and, consequently, also affects the performance of real-world applications).
This gap became a topic of many works [9,10,11,12]. The efforts finally led to a construction of Schewe in [13] matching the lower bound modulo a polynomial factor of $O(n^2)$.

Although the construction of Schewe is worst-case optimal, it may in practice still generate a lot of states that are not necessary and negatively affect the size of the complemented automaton. In our previous work in [14], we employed direct and delay simulation between states of the original automaton to remove states from the complement that are not necessary for accepting a word. In the current paper, we build upon [14] and develop other optimizations for reducing the size of the complement that push the rank-based approach by a significant step further. Namely, (i) Delaying transitions: reducing the number of nondeterministic guesses of states introducing tight ranks. (ii) Successors constraints: reducing the maximum considered ranking in the tight part based on reasoning about the DFA support of an input automaton. (iii) Rank simulation: removing macrostates with incompatible rankings based on reasoning about states with an odd ranking. (iv) Max rank construction: considering only runs with the maximal ranking.

These optimizations require some additional computational cost, but from the perspective of BA complementation, their cost is still negligible and, as we have experimentally evaluated, their effect on the size of the output is often profound.

2 Preliminaries

Functions, words, and alphabets. We fix a finite nonempty alphabet $\Sigma$ and the first infinite ordinal $\omega = \{0, 1, \ldots\}$. For $n \in \omega$, by $[n]$ we denote the set $\{0, \ldots, n\}$. An (infinite) word $\alpha$ is represented as a function $\alpha : \omega \to \Sigma$ where the $i$-th symbol is denoted as $\alpha_i$. We abuse notation and sometimes also represent $\alpha$ as an infinite sequence $\alpha = \alpha_0\alpha_1\ldots$ The suffix $\alpha_i\alpha_{i+1}\ldots$ of $\alpha$ is denoted by $\alpha_{i\omega}$.

We use $\Sigma^\omega$ to denote the set of all infinite words over $\Sigma$. Furthermore, for a total function $f : X \to Y$ and a partial function $h : X \to Y$, we use $f \triangleleft h$ to denote the total function $g : X \to Y$ defined as $g(x) = h(x)$ when $h(x)$ is defined and $g(x) = f(x)$ otherwise. Moreover, we use $\text{img}(f)$ to denote the image of $f$, i.e., $\text{img}(f) = \{y \mid x \in X \land f(x) = y\}$ and for a set $C \subseteq X$ we use $f|_C$ to denote the restriction of $f$ to $C$, i.e., $f|_C = f \cap (C \times Y)$.

Büchi automata. A (nondeterministic) Büchi automaton (BA) over $\Sigma$ is a quadruple $A = (Q, \delta, I, F)$ where $Q$ is a finite set of states, $\delta$ is a transition function $\delta : Q \times \Sigma \to 2^Q$, and $I, F \subseteq Q$ are the sets of initial and accepting states respectively. We sometimes treat $\delta$ as a set of transitions $p \xrightarrow{\alpha} q$, for instance, we use $p \xrightarrow{\alpha} q \in \delta$ to denote that $q \in \delta(p, \alpha)$. Moreover, we extend $\delta$ to sets of states $P \subseteq Q$ as $\delta(P, \alpha) = \bigcup_{p \in P} \delta(p, \alpha)$. We use $\delta^{-1}(q, \alpha)$ to denote the set $\{s \in Q \mid s \xrightarrow{\alpha} q \in \delta\}$. For a set of states $S$ we define reachability from $S$ as $\text{reach}_A(S) = \mu Z. S \cup \bigcup_{a \in \Sigma} \delta(Z, a)$. A run of $A$ from $q \in Q$ on an input word $\omega$ is an infinite sequence $\rho : \omega \to Q$ that starts in $q$ and respects $\delta$, i.e., $\rho_0 = q$. 


and \( \forall i \geq 0 : \rho_i \overset{\delta}{\to} \rho_{i+1} \in \delta \). Let \( \inf(\rho) \) denote the states occurring in \( \rho \) infinitely often. We say that \( \rho \) is accepting iff \( \inf(\rho) \cap F \neq \emptyset \). A word \( \alpha \) is accepted by \( A \) from a state \( q \in Q \) if there is an accepting run \( \rho \) of \( A \) from \( q \), i.e., \( \rho_0 = q \). The set \( \mathcal{L}_A(q) = \{ \alpha \in \Sigma^\omega \mid A \text{ accepts } \alpha \text{ from } q \} \) is called the language of \( q \) (in \( A \)). Given a set of states \( R \subseteq Q \), we define the language of \( R \) as \( \mathcal{L}_A(R) = \bigcup_{q \in R} \mathcal{L}_A(q) \) and the language of \( A \) as \( \mathcal{L}(A) = \mathcal{L}_A(I) \). For a pair of states \( p \) and \( q \) in \( A \), we use \( p \sqsubseteq q \) to denote \( \mathcal{L}_A(p) \subseteq \mathcal{L}_A(q) \). Without loss of generality, in this paper, we assume \( A \) to be complete, i.e., for every state \( q \) and symbol \( a \), it holds that \( \delta(q, a) \neq \emptyset \). In this paper, we fix a complete BA \( A = (Q, \delta, I, F) \).

**Simulation.** The direct simulation on \( A \) is the relation \( \preceq_{di} \subseteq Q \times Q \) defined as the largest relation s.t. \( p \preceq_{di} q \) implies (i) \( p \in F \Rightarrow q \in F \) (ii) \( p \overset{\delta}{\to} p' \in \delta \Rightarrow \exists q' \in Q : q \overset{\delta}{\to} q' \in \delta \wedge p' \preceq_{di} q' \) for each \( a \in \Sigma \). Note that \( \preceq_{di} \) is a preorder and \( \preceq_{di} \subseteq \preceq_L \) [15].

# 3 Complementing Büchi Automata

In this section we describe the basic rank-based complementation algorithm presented by Schewe in [13] and some results related to runs with the minimal ranking.

## 3.1 Run DAGs

In this section, we recall the terminology from [13] (which is a minor modification of the terminology from [8]), which we use heavily in the paper. We fix the definition of the run DAG of \( A \) over a word \( \alpha \) to be a DAG (directed acyclic graph) \( G_\alpha = (V, E) \) of vertices \( V \) and edges \( E \) where

- \( V \subseteq Q \times \omega \) s.t. \( (q, i) \in V \) iff there is a run \( \rho \) of \( A \) from \( I \) over \( \alpha \) with \( \rho_i = q \),

- \( E \subseteq V \times V \) s.t. \( (q, i), (q', i') \in E \) iff \( i' = i + 1 \) and \( q' \in \delta(q, \alpha_i) \).

Given \( G_\alpha \) as above, we will write \( (p, i) \in G_\alpha \) to denote that \( (p, i) \in V \). We call \( (p, i) \) accepting if \( p \) is an accepting state, \( G_\alpha \) is rejecting if it contains no path with infinitely many accepting vertices. A vertex \( v \in G_\alpha \) is finite if the set of vertices reachable from \( v \) is finite, infinite if it is not finite, and endangered if \( v \) cannot reach an accepting vertex.

We assign ranks to vertices of run DAGs as follows: Let \( G_\alpha^0 = G_\alpha \) and \( j = 0 \). Repeat the following steps until the fixpoint or for at most \( 2n + 1 \) steps, where \( n \) is the number of states of \( A \).

- Set \( \text{rank}_\alpha(v) := j \) for all finite vertices \( v \) of \( G_\alpha^{i} \) and let \( G_\alpha^{i+1} \) be \( G_\alpha^{i} \) minus the vertices with the rank \( j \).
- Set \( \text{rank}_\alpha(v) := j + 1 \) for all endangered vertices \( v \) of \( G_\alpha^{j+1} \) and let \( G_\alpha^{j+2} \) be \( G_\alpha^{j+1} \) minus the vertices with the rank \( j + 1 \).
- Set \( j := j + 2 \).

For all vertices \( v \) that have not been assigned a rank yet, we assign \( \text{rank}_\alpha(v) := \omega \). (Note that since \( A \) is complete, then \( G_\alpha^n = G_\alpha^{n+1} \))

3
3.2 Complementation Algorithm

We use as the starting point the complementation procedure of Schewe [13, Section 3.1], which we denote as SCHWEKE. The procedure works with the notion of level rankings, originally proposed in [24]. Given $n = |Q|$, a (level) ranking is a function $f : Q \rightarrow [2n]$ such that $\{f(q) : q \in F\} \subseteq \{0, 2, \ldots, 2n\}$, i.e., $f$ assigns even ranks to accepting states of $\mathcal{A}$. For a ranking $f$, the rank of $f$ is defined as $\text{rank}(f) = \max\{f(q) : q \in Q\}$. For a set of states $S \subseteq Q$, we call $f$ to be $S$-tight if (i) it has an odd rank $r$, (ii) $\{f(s) : s \in S\} \supseteq \{1, 3, \ldots, r\}$, and (iii) $\{f(q) : q \notin S\} = \{0\}$. A ranking is tight if it is $Q$-tight; we use $T$ to denote the set of all tight rankings. We use $f \leq f'$ iff for every state $q \in Q$ it holds that $f(q) \leq f'(q)$ and $f < f'$ iff $f \leq f'$ and there is a state $p \in Q$ such that $f(p) < f'(p)$. The SCHWEKE procedure constructs the BA $\text{SCHWEKE}(\mathcal{A}) = (Q', \delta', I', F')$ whose components are defined as follows:

- $Q' = Q_1 \cup Q_2$ where
  - $Q_1 = 2^Q$ and
  - $Q_2 = \{(S, O, f, i) \in 2^Q \times 2^Q \times T \times \{0, 2, \ldots, 2n - 2\} | f$ is $S$-tight, $O \subseteq S \cap f^{-1}(i)\}$,
- $I' = \{I\}$,
- $\delta' = \delta_1 \cup \delta_2 \cup \delta_3$ where
  - $\delta_1 : Q_1 \times \Sigma \rightarrow 2^Q_1$ such that $\delta_1(S, a) = \{\delta(S, a)\}$,
  - $\delta_2 : Q_1 \times \Sigma \rightarrow 2^Q_2$ such that $\delta_2(S, a) = \{(S', \emptyset, f, 0) | S' = \delta(S, a)$, $f$ is $S'$-tight\}, and
  - $\delta_3 : Q_2 \times \Sigma \rightarrow 2^Q_2$ such that $(S', O', f', i') \in \delta_3((S, O, f, i), a)$ iff
    * $S' = \delta(S, a)$,
    * for every $q \in S$ and $q' \in \delta(q, a)$ it holds that $f'(q') \leq f(q)$
    * $\text{rank}(f) = \text{rank}(f')$,
    * and
      - $i' = (i + 2) \mod (\text{rank}(f') + 1)$ and $O' = f'^{-1}(i')$ if $O = \emptyset$ or
      - $i' = i$ and $O' = \delta(O, a) \cap f'^{-1}(i)$ if $O \neq \emptyset$, and
- $F' = \{\emptyset\} \cup ((2^Q \times \{\emptyset\} \times T \times \omega) \cap Q_2)$. 

The correctness of the construction is then given by the following theorem.

**Theorem 1.** ([13, Corollary 3.3]) Let $\mathcal{B} = \text{SCHWEKE}(\mathcal{A})$. Then $L(\mathcal{B}) = \overline{L(\overline{\mathcal{A}})}$.

We call the part of $\text{SCHWEKE}(\mathcal{A})$ with the states from $Q_1$ the waiting part and the part with the states in $Q_2$ the tight part (an accepting run in $\text{SCHWEKE}(\mathcal{A})$ simulates the run DAG of $\mathcal{A}$ over a word $w$ by waiting in $Q_1$ until it can generate tight rankings only; then it moves to $Q_2$). In the following, we assume that $\text{SCHWEKE}(\mathcal{A})$ contains only the states and transitions reachable from $I'$.

3.3 Super-Tight Runs

Let $\mathcal{B} = \text{SCHWEKE}(\mathcal{A})$. Each accepting run of $\mathcal{B}$ on $\alpha \in L(\mathcal{B})$ is tight, i.e., the rankings of macrostates it traverses in $Q_2$ are tight (this follows from the definition of $Q_2$). In this section, we show that there exists a super-tight run of $\mathcal{B}$
on \( \alpha \), which is, intuitively, a run that uses as little ranks as possible. The concept of super-tight runs is essential for our optimizations in Section 4.

Let \( \rho = S_0 \ldots S_m(S_{m+1}, O_{m+1}, f_{m+1}, i_{m+1})(S_{m+2}, O_{m+2}, f_{m+2}, i_{m+2}) \ldots \) be an accepting run of \( \mathcal{B} \) over a word \( \alpha \in \Sigma^\omega \). Given a macrostate \((S_k, O_k, f_k, i_k)\) for \( k > m \), we define its rank as \( \text{rank}((S_k, O_k, f_k, i_k)) = \text{rank}(f_k) \). Further, we define the rank of the run \( \rho \) as \( \text{rank}(\rho) = \min\{\text{rank}((S_k, O_k, f_k, i_k)) | \ k > m \} \).

Let \( \mathcal{G}_\alpha \) be the run DAG of \( \mathcal{A} \) over \( \alpha \) and \( \text{rank}_\alpha \) be the ranking of vertices in \( \mathcal{G}_\alpha \). We say that the run \( \rho \) is super-tight if for all \( k > m \) and all \( q \in S_k \), it holds that \( f_k(q) = \text{rank}_\alpha(q, k) \). Intuitively, super-tight runs correspond to runs whose ranking faithfully copies the ranks assigned in \( \mathcal{G}_\alpha \) (from some position \( m \) corresponding to the transition from the waiting to the tight part of \( \mathcal{B} \)).

**Lemma 1.** Let \( \alpha \in \mathcal{L}(\mathcal{B}) \). Then there is a super-tight accepting run \( \rho \) of \( \mathcal{B} \) on \( \alpha \).

**Proof.** This follows directly from the definition of a super-tight run and the SCHWE architecture.

Let \( \rho = S_0 \ldots S_m(S_{m+1}, O_{m+1}, f_{m+1}, i_{m+1})(S_{m+2}, O_{m+2}, f_{m+2}, i_{m+2}) \ldots \) be a run and consider a macrostate \((S_k, O_k, f_k, i_k)\) for \( k > m \). We call a set \( C_k \subseteq S_k \) a tight core of a ranking \( f_k \) if \( f_k(C_k) = \{1, 3, \ldots, \text{rank}(f_k)\} \) and \( f_k|_{C_k} \) is injective (i.e., every state in the tight core has a unique odd rank). Moreover, \( C_k \) is a tight core of a macrostate \((S_k, O_k, f_k, i_k)\) if it is a tight core of \( f_k \). We say that an infinite sequence \( \tau = C_{m+1}C_{m+2} \ldots \) is a trunk of run \( \rho \) if for all \( k > m \) it holds that \( C_k \) is a tight core of \( \rho(k) \) and there is a bijection \( \theta : C_k \to C_{k+1} \) s.t. if \( \theta(q_k) = q_{k+1} \) then \( q_{k+1} \in \delta(q_k, \alpha_k) \). We will, in particular, be interested in trunks of super-tight runs. In these runs, a trunk (there can be several) represents runs of \( \mathcal{A} \) that keep the super-tight ranks of \( \rho \). The following lemma shows that every state in any tight core in a trunk of such a run has at least one successor with the same rank.

**Lemma 2.** Let \( \rho = S_0 \ldots S_m(S_{m+1}, O_{m+1}, f_{m+1}, i_{m+1}) \ldots \) be an accepting super-tight run of \( \mathcal{B} \) on \( \alpha \). Then there is a trunk \( \tau = C_{m+1}C_{m+2} \ldots \) of \( \rho \) and, moreover, for every \( k > m \) and all states \( q_k \in C_k \), it holds that there is a state \( q_{k+1} \in C_{k+1} \) such that \( f_k(q_k) = f_{k+1}(q_{k+1}) \).

**Proof.** First we show how to inductively construct a trunk \( \tau = C_{m+1}C_{m+2} \ldots \).

(i) As the base case, let \( C_{m+1} \) be an arbitrary tight core of \( f_{m+1} \). (ii) As the inductive step, consider a tight core \( C_k \) from the trunk and let us construct \( C_{k+1} \). Since \( \rho \) is a super-tight run, for each \( q \in C_k \) there is a state \( q' \in S_{k+1} \) s.t. \( f_k(q) = f_{k+1}(q') \). This follows from the run DAG ranking procedure. We put \( q' \in C_{k+1} \).

Such a constructed set is a tight core of \( f_{k+1} \) and from the construction we have the property stated in the lemma.

### 4 Optimized Complement Construction

In this section, we introduce our optimizations of SCHWE that are key to producing small automata in practice.
Showing Proof.

Schewe for every accepting super-tight run of \( \theta \) we can set the number of nondeterministic transitions between the waiting part does not close a cycle where the transition to the tight part would be infinitely often delayed. Practically, it means that when constructing \( Q_1 \), we need to check whether successors of a macrostate close a cycle in the so-far generated part of \( Q_1 \). We give the construction in Algorithm 1, and we refer to it as Delay. We use DELAY as the starting point of our other optimizations.

**Algorithm 1: The DELAY construction**

| Input: A Büchi automaton \( \mathcal{A} = (Q, I, \delta, F) \) |
| Output: A Büchi automaton \( \mathcal{C} \) s.t. \( \mathcal{L}(\mathcal{C}) = \overline{\mathcal{L}(\mathcal{A})} \) |
| 1 \( S \leftarrow \{I\}, Q_1 \leftarrow \{I\}, \theta_2 \leftarrow \emptyset, (\cdot, \delta_1 \cup \delta_2 \cup \delta_3, I', F') \leftarrow \text{SCHWEVE}(\mathcal{A}); \) |
| 2 \( \text{while } S \neq \emptyset \text{ do} \) |
| 3 \( \text{Take a state } R \text{ from } S; \) |
| 4 \( \text{foreach } a \in \Sigma \text{ do} \) |
| 5 \( \text{if } \exists T \in \delta_1(R, a) \text{ s.t. } R \xrightarrow{\alpha} T \text{ closes a cycle in } Q_1 \text{ then} \) |
| 6 \( \theta_2 \leftarrow \theta_2 \cup \{R \xrightarrow{\alpha} U \mid U \in \delta_2(R, a)\}; \) |
| 7 \( \text{foreach } T \in \delta_1(R, a) \text{ s.t. } T \notin Q_1 \text{ do} \) |
| 8 \( S \leftarrow S \cup \{T\}; \) |
| 9 \( Q_1 \leftarrow Q \cup \{T\}; \) |
| 10 \( Q_2 \leftarrow \text{reach}_{s_3}(\text{img}(\theta_2)); \) |
| 11 \( \text{return } \mathcal{C} = (Q_1 \cup Q_2, \delta_1 \cup \theta_2 \cup \delta_3, I', F' \cap Q_2); \) |

### 4.1 Delaying the Transition from Waiting to Tight

Our first optimization of the construction of the complement automaton reduces the number of nondeterministic transitions between the waiting and the tight part. This optimization is inspired by the idea of partial order reduction in model checking \cite{16,17,18}. In particular, since in each state of the waiting part, it is possible to move to the tight part, we can arbitrarily delay such a transition (but need to take it eventually) and, therefore, significantly reduce the number of transitions (and, as our experiments later show, also significantly reduced the number of reachable states in \( Q_2 \)).

Speaking in the terms of partial order reduction, when constructing the waiting part of the complement BA, given a macrostate \( S \in Q_1 \) and a symbol \( a \in \Sigma \), we can set \( \theta_2 \subseteq \delta_2 \) such that \( \theta_2(S, a) := \emptyset \) if the cycle closing condition holds and \( \theta_2(S, a) := \delta_2(S, a) \) otherwise. Informally, the cycle closing condition (often denoted as \( \mathbf{C}3 \)) holds for \( S \) and \( a \) if the successor of \( S \) over \( a \) in the waiting part does not close a cycle where the transition to the tight part would be infinitely often delayed. Practically, it means that when constructing \( Q_1 \), we need to check whether successors of a macrostate close a cycle in the so-far generated part of \( Q_1 \). We give the construction in Algorithm 1, and we refer to it as DELAY. We use DELAY as the starting point of our other optimizations.

**Lemma 3.** Let \( \mathcal{A} \) be a BA. Then \( \mathcal{L}(\text{DELAY}(\mathcal{A})) = \mathcal{L}(\text{SCHWEVE}(\mathcal{A})) \). Moreover, for every accepting super-tight run of \( \text{SCHWEVE}(\mathcal{A}) \) on \( \alpha \), there is an accepting super-tight run of \( \text{DELAY}(\mathcal{A}) \) on \( \alpha \).

**Proof.** Showing \( \mathcal{L}(\text{DELAY}(\mathcal{A})) \subseteq \mathcal{L}(\text{SCHWEVE}(\mathcal{A})) \) is trivial. In order to show \( \mathcal{L}(\text{DELAY}(\mathcal{A})) \supseteq \mathcal{L}(\text{SCHWEVE}(\mathcal{A})) \), consider some \( \alpha \in \mathcal{L}(\text{SCHWEVE}(\mathcal{A})) \). Then, there is an accepting run \( \rho_m = S_0 \ldots S_m(S_{m+1}, O_{m+1}, f_{m+1}, t_{m+1}) \ldots \) on \( \alpha \) in \( \text{SCHWEVE}(\mathcal{A}) \). For each \( l > m \) there is, however, also an accepting run \( \rho_l = S_0 \ldots S_l(S_{l+1}, O_{l+1}, f_{l+1}, t_{l+1}) \ldots \) on \( \alpha \) in \( \text{SCHWEVE}(\mathcal{A}) \). Note that each \( \rho_l \) differs from \( \rho_m \) on the point where the run switched from the waiting part to the tight part.
part of \textsc{Scheewe}(A). Therefore, since the run \( \rho_m \) managed to empty \( O \) infinitely often, \( \rho_t \) will also be able to do so and, therefore, it will also be accepting.

From the \textsc{Delay} construction, we have the fact that at least one macrostate \((S_{k+1}, O_{k+1}, f_{k+1}, i_{k+1})\) where \( k > m \) is in \( Q_2 \). If this were not true, there would be a closed cycle with no state in \( Q_2 \), which is a contradiction. From the previous reasoning we have that the run \( \rho_k = S_1 \ldots S_k(S_{k+1}, O_{k+1}, f_{k+1}, i_{k+1}) \ldots \) on \( \alpha \) is present in \textsc{Delay}(A). Moreover, this run is accepting both in \textsc{Scheewe}(A) and \textsc{Delay}(A), which concludes the proof. \( \square \)

4.2 Successor Rankings

Our next optimization is used to reduce the maximum considered ranking of a macrostate in the tight part of \( B = \textsc{Scheewe}(A) \). For a given macrostate, the number of tight rankings that can occur within the macrostate rises combinatorially with the macrostate’s maximum rank (in particular, the number of tight rankings for a given set of states corresponds to the Stirling number of the second kind \cite{9}). It is hence desirable to reduce the maximum considered rank as much as possible.

The idea of our optimization, called \textsc{SuccRank} is the following. Suppose we have a macrostate \((S, O, f, i)\) from the tight part of \( B = \textsc{Scheewe}(A) \). Further, assume that the maximum size of the \( S \)-component of a macrostate that is infinitely often reachable from \((S, O, f, i)\) is \( \lceil S \rceil \). Then, we know that a super-tight accepting run that goes through \((S, O, f, i)\) will never need a rank higher than \( 2\lceil S \rceil - 1 \). Therefore, if the rank of \( f \) is higher, we can safely discard \((S, O, f, i)\) (since there will be a super-tight accepting run that goes over \((S, O', f', i')\) with \( f' < f \)). Moreover, let \( q \in S \) and let \( \lfloor \{q\} \rfloor \) be the smallest size of a set of states reachable from \( q \) over some (infinite) word infinitely often. Then, we know that those states will have a rank bounded by the rank of \( f(q) \), so there are only (at most) \( \lceil S \rceil - \lfloor \{q\} \rfloor \) states whose rank can be higher than \( f(q) \). Therefore, the rank of \( f \), which is tight, can be at most \( f(q) + 2(\lceil S \rceil - \lfloor \{q\} \rfloor) \).

We now formalize the intuition. Let us fix a BA \( A = (Q, \delta, I, F) \). Then, let us consider a BA \( R_A = (2^Q, \delta_R, \emptyset, \emptyset) \), with \( \delta_R = \{ R \mapsto S \mid S = \delta(R, a) \} \), which is tracking reachability between set of states of \( A \) (we only focus on its structure and not the language). Note that \( R_A \) is deterministic and complete. Further, given \( S \subseteq Q \), let us use \( \text{SCC}(S) \subseteq 2^Q \) to denote the set of all strongly connected components reachable from \( S \) in \( R_A \). We will use \( \text{inf-reach}(S) \) to denote the set of states \( \bigcup \text{SCC}(S) \), i.e., the set of states such that there is an infinite path in \( R_A \) starting in \( S \) that passes through a given state infinitely many times.

For \( S \subseteq Q \), we define the maximum and minimum sizes of macrostates reachable infinitely often from \( S \):

\[
[S] = \max\{|R| \mid R \in \text{inf-reach}(S)\} \quad \text{and} \quad (1)
\]
\[
[S] = \min\{|R| \mid R \in \text{inf-reach}(S)\}. \quad (2)
\]

Given a macrostate \((S, O, f, i)\), we define the condition

\[
\varphi_{\text{coarse}}((S, O, f, i)) \overset{\text{def}}{=} \text{rank}(f) \leq 2[S] - 1. \quad (3)
\]
If \((S, O, f, i)\) does not satisfy \(\varphi\text{coarse}\), we do not need to include it in the output of \(\text{SCHWEWE}(A)\) (as proved below).

Moreover, we also define the condition
\[
\varphi\text{fine}((S, O, f, i)) \overset{\text{def}}{=} \text{rank}(f) \leq \min\{f(q) + 2([S] - \lfloor \{q\}\rfloor) \mid q \in S\}. \tag{4}
\]

Again, if \((S, O, f, i)\) does not satisfy \(\varphi\text{coarse}\), it does not need to be in the result.

Putting the conditions together, we define
\[
\text{SuccRank}((S, O, f, i)) \overset{\text{def}}{=} \varphi\text{coarse}((S, O, f, i)) \land \varphi\text{fine}((S, O, f, i)) \tag{5}
\]

We abuse notation and use \(\text{SuccRank}(A)\) to denote the output of \(\text{SCHWEWE}(A) = (Q', \delta', I', F')\) where the states from the tight part of \(Q'\) are restricted to those that satisfy \(\text{SuccRank}\).

**Lemma 4.** Let \(A\) be a BA. Then \(L(\text{SuccRank}(A)) = L(\text{SCHWEWE}(A))\).

**Proof.** The inclusion \(L(\text{SuccRank}(A)) \subseteq L(\text{SCHWEWE}(A))\) is clear. Now we look at the other direction. Consider some \(\alpha \in L(\text{SCHWEWE}(A))\). Then, there is an accepting super-tight run \(\rho = S_0 \ldots S_m(O_{m+1}, f_{m+1}, i_{m+1})\ldots\) of \(\text{SCHWEWE}(A)\) over \(\alpha\). Consider \(k > m\) and a macrostate \((S_k, O_k, f_k, i_k)\). The maximum rank of this macrostate is bounded by \(2[S_k] - 1\) because \([S_k]\) is the largest size of the \(S\)-component of a macrostate reachable from \(S_k\) and, therefore, removing macrostates that do not satisfy \(\varphi\text{coarse}\) from \(\text{SCHWEWE}(A)\) will not affect this run.

Next, we prove the correctness of removing states from \(\text{SCHWEWE}(A)\) using \(\varphi\text{fine}\). Consider a set of states \(T \subseteq Q\); we will use \(\rho_T\) to denote the run \(\rho_T = T_0 T_1 T_2 \ldots\) of \(R_A\) from \(T (= T_0)\) over the word \(\alpha_{k,\omega}\). Since \(R_A\) is deterministic and complete, there is exactly one such run. Given a state \(q \in S_k\), let \(a\) be the smallest size of a set of states that occurs infinitely often in \(\rho(q)\) and \(b\) be the largest size of a set of states that occurs infinitely often in \(\rho_{S_k}\). From the definition of \([\cdot]\) and \(\lfloor\cdot\rfloor\), it holds that
\[
[\{q\}] \leq a \leq b \leq [S_k]. \tag{6}
\]

Since we can reach \(a\) different states from \(q\), the ranks of these states need to be less or equal to \(f_k(q)\) (no successor of \(q\) can be given a rank higher than \(f_k(q)\)). Further, since we can reach at most \(b\) different states from \(S_k\), there are at some infinitely often occurring macrostate of \(\rho\) in the worst case only \(b - a\) states that can have odd rank greater than \(f_k(q)\). Due to the tightness of all macrostates in the tight part of \(\rho\), we can conclude that the maximal rank of \(f_k\) can be bounded by \(f_k(q) + 2([S_k] - \lfloor\{q\}\rfloor)\). Therefore, a macrostate where \(\varphi\text{fine}\) does not hold will not be in a super-tight run, so removing those macrostates does not affect the language of \(\text{SCHWEWE}(A)\). \qed
4.3 Rank Simulation

The next optimization is a modification of optimization PURGE$$d_i$$ from our previous work in [14]. Intuitively, PURGE$$d_i$$ is based on the fact that if a state $$p$$ is directly simulated by a state $$r$$, i.e., $$p \preceq_d r$$, then any macrostate $$(S, O, f, i)$$ where $$f(p) > f(r)$$ can be safely removed (intuitively, any run from $$p$$ can be simulated by a run from $$r$$, where the run from $$r$$ may contain more accepting states and, therefore, needs to decrease its rank more times). PURGE$$d_i$$ is compatible with SCHWE but, unfortunately, it is incompatible with the MAXRANK construction (Section 4.5) since in MAXRANK, several runs are represented by one maximal run (wrt the ranks) and removing such a run would also remove the smaller runs (see Section 4.5 for details). We, however, change the condition and obtain a new reduction, which is incomparable with PURGE$$d_i$$ but compatible with MAXRANK. We call this reduction RANKSIM.

Consider the following relation of odd-rank simulation on $$p, r \in Q$$:

$$p \prec_{ors} r \overset{def}{=} \forall \alpha \in \Sigma^\omega, \forall i \geq 0 : (\text{rank}_\alpha(p, i) \text{ is odd} \land \text{rank}_\alpha(r, i) \text{ is odd}) \Rightarrow \text{rank}_\alpha(p, i) \leq \text{rank}_\alpha(r, i). \quad (7)$$

Intuitively, if $$p \prec_{ors} r$$ holds, then we know that in any super-tight run and a macrostate $$(S, O, f, i)$$ in such a run, if $$p, r \in S$$ and both $$f(p)$$ and $$f(r)$$ are odd, then it needs to hold that $$f(p) \leq f(r)$$. Furthermore, such a reasoning can also be applied transitively (note that $$\prec_{ors}$$ is by itself not transitive): if, in addition, $$t \in S$$ and $$f(t)$$ is odd, then it also needs to hold that $$f(r) \leq f(s)$$ and $$f(p) \leq f(s)$$.

Formally, given a ranking $$f$$, let $$\prec_{ors}^f$$ be a modification of $$\prec_{ors}$$ defined as

$$p \prec_{ors}^f r \overset{def}{=} f(p) \text{ is odd} \land f(r) \text{ is odd} \land p \prec_{ors} r \quad (8)$$

and $$\preceq_{ors}^T$$ be its transitive closure. We use $$\preceq_{ors}^T$$ to define the following condition:

$$\text{RANKSIM}((S, O, f, i)) \overset{def}{=} \forall p, r \in S : p \preceq_{ors}^T r \Rightarrow f(p) \leq f(r). \quad (9)$$

Abusing the notation, we use $$\text{RANKSIM}(A)$$ to denote the output of SCHWE($$A$$) = $$(Q', \delta', I', F')$$ where states from the tight part of $$Q'$$ are restricted to those that satisfy RANKSIM.

**Lemma 5.** Let $$A$$ be a BA. Then $$\mathcal{L}(\text{RANKSIM}(A)) = \mathcal{L}($$SCHWE($$A$$))

**Proof.** The inclusion $$\mathcal{L}(\text{RANKSIM}(A)) \subseteq \mathcal{L}($$SCHWE($$A$$)) is clear. For the reverse direction, let $$\alpha \in \mathcal{L}($$SCHWE($$A$$)). Then, there is a super-tight run $$\rho = S_1 \ldots S_m(S_{m+1}, O_{m+1}, f_{m+1}, i_{m+1}) \ldots$$, i.e., for each $$k > m$$ and each $$q \in S_k$$ we have $$f_k(q) = \text{rank}_\alpha(q, k)$$. Clearly, each macrostate of $$\rho$$ satisfies RANKSIM. \( \square \)

From the definition of $$\preceq_{ors}$$, it is not immediate how to compute the relation, since it is defined over all infinite runs of $$A$$ over all infinite words. The computation of an under-approximation of $$\preceq_{ors}$$ will be the topic of the rest of this section. We first note that $$\preceq_{di} \subseteq \preceq_{ors}$$, which is a consequence of the following lemma.
Lemma 6 (Lemma 7 in [14]). Let \( p, r \in Q \) be such that \( p \preceq_d r \) and \( G_\alpha = (V, E) \) be the run DAG of \( A \) over \( \alpha \). For all \( i \geq 0 \), it holds that \( (p, i) \in V \land (r, i) \in V \implies \text{rank}_\alpha(p, i) \preceq \text{rank}_\alpha(r, i) \).

We will extend \( \preceq_d \) into a relation \( \preceq_R \), which is computed statically on \( A \), and then show that \( \preceq_R \subseteq \preceq_{\text{ors}} \). The relation \( \preceq_R \) is defined recursively as the smallest binary relation over \( Q \) such that

1. \( \preceq_d \subseteq \preceq_R \) and
2. for \( p, r \in Q \), if \( \forall a \in \Sigma : (\delta(p, a) \setminus F) \preceq_R (\delta(r, a) \setminus F) \), then \( p \preceq_R r \).

Above, \( S_1 \preceq_R S_2 \) holds iff \( \forall x \in S_1, \forall y \in S_2 : x \preceq_R y \). The relation \( \preceq_R \) can then be computed using a standard worklist algorithm, starting from \( \preceq_d \) and adding states for which condition 2 holds until a fixpoint is reached.

Lemma 7. We have \( \preceq_R \subseteq \preceq_{\text{ors}} \).

Proof. The base case \( \preceq_d \subseteq \preceq_{\text{ors}} \) follows directly from Lemma 6. For the induction step, let \( p, r \in Q \) be such that \( \forall a \in \Sigma : (\delta(p, a) \setminus F) \preceq_R (\delta(r, a) \setminus F) \). Our induction hypothesis is that for every \( a \in \Sigma, x \in (\delta(p, a) \setminus F), \) and \( y \in (\delta(r, a) \setminus F) \), it holds that for all \( \alpha \in \Sigma^\omega \) and for all \( i \geq 0 \), if \( \text{rank}_\alpha(p, i) \) is odd and \( \text{rank}_\alpha(r, i) \) is odd, then \( \text{rank}_\alpha(p, i) \leq \text{rank}_\alpha(r, i) \). Let us fix an \( a \in \Sigma \) and a word \( \alpha \in \Sigma^\omega \) that has \( a \) at its \( i \)-th position. If \( \text{rank}_\alpha(p, i) \) or \( \text{rank}_\alpha(r, i) \) are even, the condition holds trivially.

Assume now that \( \text{rank}_\alpha(p, i) \) and \( \text{rank}_\alpha(r, i) \) are odd. From the construction of the run DAG \( G_\alpha \) in Section 3.1, it follows that there exist infinite paths from \( (p, i) \) and \( (r, i) \) in \( G_\alpha \) such that all vertices on these paths are assigned the same (odd) ranks as \( (p, i) \) and \( (r, i) \), respectively. In particular, there are direct successors \( (p', i + 1) \) of \( (p, i) \) and \( (r', i + 1) \) of \( (r, i) \) whose ranks match the ranks of their predecessors. From the induction hypothesis, it holds that \( \text{rank}_\alpha(p', i + 1) \leq \text{rank}_\alpha(r', i + 1) \) and so \( \text{rank}_\alpha(p, i + 1) \leq \text{rank}_\alpha(r, i) \) and the lemma follows. (Note that in the previous reasoning, it is essential that \( (p, i) \) and \( (r, i) \) have an odd ranking; if a node has an even ranking in \( G_\alpha \), then the condition that there needs to be a successor with the same ranking does not hold in general.)

4.4 Ranking Restriction

Another optimization, called \text{RANKRestr}, restricts ranks of successors of states with an odd rank. In particular, in a super-tight run, every odd-ranked state has a successor with the same rank (this follows from the construction of the run DAG). Let \( A \) be a BA and \( B = \text{SCHWE}(A) = (Q, \delta_1 \cup \delta_2 \cup \delta_3, I, F) \). Then, we define the following restriction on transitions:

\[
\text{RANKRestr}((S, O, f, i) \xrightarrow{a} (S', O', f', i')) \equiv \\
\forall q \in S : f(q) \text{ is odd } \Rightarrow (\exists q' \in \delta(q, a) : f'(q') = f(q))
\]

(10)

We abuse notation and use \text{RANKRestr}(A) to denote \( B \) with transitions from \( \delta_3 \) restricted to those that satisfy \text{RANKRestr}. 

Lemma 8. Let \( A \) be a BA. Then \( \mathcal{L}(\text{RankRestr}(A)) = \mathcal{L}(\text{Schewe}(A)) \).

Proof. The inclusion \( \mathcal{L}(\text{RankRestr}(A)) \subseteq \mathcal{L}(\text{Schewe}(A)) \) is clear. We now look at the reverse direction. Consider a word \( \alpha \in \mathcal{L}(\text{Schewe}(A)) \). Let \( \rho = S_0 \ldots S_m(S_{m+1}, O_{m+1}, f_{m+1}, i_{m+1}) \ldots \) be an accepting super-tight run on \( \alpha \). Now consider a macrostate \( (S_j, O_j, f_j, i_j) \) where \( j > m \) and some state \( q \in S_j \). Since \( \rho \) is super-tight, i.e., represents the run DAG of \( A \) over \( \alpha \), it holds that if \( f_j(q) \) is odd, then there is a state \( q' \in S_{j+1} \) s.t. \( f_{j+1}(q') = f_j(q) \), which satisfies RankRestr. \( \square \)

4.5 Maximum Rank Construction

Our final optimization, named MAXRANK, is the one with the biggest practical effect. We introduce it as the last one because it depends on our previous optimizations (in particular SuccRank and RankSim). It is a corrected version of Schewe’s “Reduced Average Outdegree” construction \(^{13}\) Section 4], named ScheweREDAVOOUT, which contains a small but fatal bug (we discuss the particular issue later).

The main idea of MAXRANK is that a set of runs of \( B = \text{Schewe}(A) \) (including super-tight runs) that assign different ranks to non-trunk states are represented by a single, “maximal,” not necessarily super-tight, run in \( C = \text{MAXRANK}(A) \). More concretely, when moving from the waiting to the tight part, \( C \) needs to correctly guess a rank that is needed on an accepting run and the first tight core of a trunk of the run. The ranks of the rest of states are made maximal. Then, the tight part of \( C \) contains for each macrostate and symbol at \( \rho \) the maximal successor ranking \( \eta \) over macrostates.

Before we give the construction, let us first provide some needed notation. We further use \( (S, O, f, i) \leq (S, O, g, i) \) to denote that \( f \leq g \) and similarly for \(<\) (note that non-ranking components of the macrostates need to match). In the definitions, given a set of macrostates \( R \) from \( Q_2 \), we use \( \max_f \{(S, O, f, i) \in R \} \) to denote the set of maximal elements of the partial order \( \leq \) on macrostates.

The construction is then formally defined as \( \text{MAXRANK}(A) = (Q_1 \cup Q_2, \eta, \delta_1, \delta_2) \) with \( \eta = \delta_1 \cup \eta_2 \cup \eta_3 \cup \eta_4 \) such that \( Q_1, Q_2, \delta_1, \delta_2, \delta_3 \) are the same as in Schewe.

Let \( B = \text{DELAY}(A) = (\cdot, \delta_1 \cup \theta_2 \cup \delta_3, \cdot) \) where \( \delta_1, \theta_2, \) and \( \delta_3 \) are defined as in \( \text{DELAY} \). We define an auxiliary transition function that keeps macrostates satisfying RankSim or SuccRank as follows:

\[
\Delta^*(q, a) = \{ q' \mid q' \in \theta_2(q, a) \land \text{RankSim}(q') \land \text{SuccRank}(q') \}.
\]  

(We note that \( q \) is from the waiting and \( q' \) is from the tight part of \( B \).) Given a macrostate \( (S, O, f, i) \) and \( a \in \Sigma \), we define the maximal successor ranking \( f'_{\text{max}} = \max\text{-rank}(S, O, f, i, a) \) as follows. Consider \( q' \in \delta(S, a) \) and the rank \( r = \min\{ f(s) \mid s \in \delta^{-1}(q, a) \cap S \} \). Then
− \( f'_{\text{max}}(q') := r - 1 \) if \( r \) is odd and \( q' \in F \)
− \( f'_{\text{max}}(q') := r \) otherwise.

Let \( \delta_{\mathcal{A}} \) be the transition function of the tight part of \( \text{Schewe}(\mathcal{A}) \). We can now proceed to the definition of the missing components of \( \text{MaxRank}(\mathcal{A}) \):

− \( \eta_2(S, a) := \max_{f'} \{ (S', \emptyset, f', 0) \in \Delta^* (S, a) \} \)
− \( \eta_3((S, O, f, i), a) \): Let \( f'_{\text{max}} = \text{max-rank}(S, O, f, i, a) \). Then, we set
  • \( \eta_3((S, O, f, i), a) := \{(S', O', f'_{\text{max}}, i') \} \) when \((S', O', f'_{\text{max}}, i') \in \delta_{\mathcal{A}}((S, O, f, i), a)\)
    (i.e., if \( f'_{\text{max}} \) is tight) and
  • \( \eta_3((S, O, f, i), a) := \emptyset \) otherwise.
− \( \eta_4((S, O, f, i), a) \): Let \( \eta_4((S, O, f, i), a) = \{(S', P', h', i') \} \) such that
  • \( f' = h' < \{u \rightarrow h'(u) - 1 \mid u \in P' \setminus F\} \) and
  • \( O' = P' \cap f'^{-1}(i') \).

Then, if \( i' \neq 0 \), we set \( \eta_4((S, O, f, i), a) := \{(S', O', f', i') \} \). Otherwise, we set \( \eta_4((S, O, f, i), a) := \emptyset \).

Note that \( \eta_3 \) and \( \eta_4 \) are deterministic, so we will sometimes use the notation \((S', O', f', i') = \eta_3((S, O, f, i), a)\).

\( \text{MaxRank} \) differs from \( \text{Schewe}\text{\textsc{RedAVGOut}} \) in the definition of \( \eta_4 \). In particular, in \( \text{Schewe}\text{\textsc{RedAVGOut}} \), the condition that only non-accepting states decrease rank \((u \in P' \setminus F)\) is omitted. Instead, the rank of all states in \( P' \) is decreased, which may cause the resulting automaton make a run over a word that was in \( \mathcal{A} \), which is incorrect. Fixing this makes the proof of the theorem significantly more involved.

**Theorem 2.** Let \( \mathcal{A} \) be a BA and \( \mathcal{C} = \text{MaxRank}(\mathcal{A}) \). Then \( \mathcal{L}(\mathcal{C}) = \overline{\mathcal{L}(\mathcal{A})} \).

**Proof.** Let \( \mathcal{B} = \text{Schewe}(\mathcal{A}) \). Showing \( \mathcal{L}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{B}) \) is easy (the transitions of \( \mathcal{C} \) are contained in the transitions of \( \mathcal{B} \)). Next, we show that \( \mathcal{L}(\mathcal{C}) \supseteq \mathcal{L}(\mathcal{B}) \).

Let \( \alpha \in \mathcal{L}(\mathcal{B}) \) and \( \rho \) be a super-tight run (from Lemma \[1\] we know that a super-tight run exists) of \( \mathcal{B} \) over \( \alpha = \alpha_0 \alpha_1 \alpha_2 \ldots \) s.t.

\[
\rho = S_0 \ldots S_m(S_{m+1}, O_{m+1}, f_{m+1}, i_{m+1})(S_{m+2}, O_{m+2}, f_{m+2}, i_{m+2}) \ldots
\]

Let \( \tau = C_{m+1}C_{m+2} \ldots \) be a trunk of \( \rho \). We will construct the run

\[
\rho' = S_0 \ldots S_m(S_{m+1}, O'_{m+1}, f'_{m+1}, i'_{m+1})(S_{m+2}, O'_{m+2}, f'_{m+2}, i'_{m+2}) \ldots
\]

of \( \mathcal{C} \) on \( \alpha \) in the following way (note that the \( S \)-components of the macrostates traversed by \( \rho \) and \( \rho' \) are the same):

1. For the transition from the waiting to the tight part, we set \( O'_{m+1} := \emptyset \) and \( i'_{m+1} := 0 \). The ranking \( f'_{m+1} \) is set as follows. Let \( r \) be the rank of \( \rho \) (remember that \( \rho \) is super-tight). We first construct an auxiliary (tight) ranking

\[
g = f_{m+1} < \{u \mapsto \max\{r - 1, f_{m+1}(u)\} \mid u \in S_{m+1} \setminus C_{m+1}\}.
\]

Note that \( C_{m+1} \) is also a tight core of \( g \). There are now two possible cases:
(a) \((S_{m+1}, \emptyset, g, 0) \in \eta_2(S_m, \alpha_m)\): If this holds, we set \(f'_{m+1} := g\).

(b) Otherwise, \(\eta_2(S_m, \alpha_m)\) contains at least one macrostate with a ranking \(h\) s.t. \(g < h\). We pick an arbitrary such ranking \(h\) from \(\eta_2(S_m, \alpha_m)\) and set \(f'_{m+1} := h\). Note that \(C_{m+1}\) is also a tight core of \(h\).

Note that \(\eta_2(S_m, \alpha_m)\) contains at least one macrostate \((S_{m+1}, \emptyset, h, 0)\) with the rank \(r\) such that \(g \leq h\). This follows from the fact that the reductions \textsc{SuccRank} and \textsc{RankSim} only remove macrostates that do not occur on super-tight runs and that \((S_{m+1}, \emptyset, g, 0)\) is not removed using the reductions. The latter follows from (iii) and (iii).

2. Let \(k > m\) and \(i_\ast\) be such that \((s, f_\ast, i_\ast) = \eta_k((S_k, O_k, f_k', i_k'), \alpha_k)\). Then,

- we set \((S_{k+1}, O'_{k+1}, f'_{k+1}, i'_{k+1}) := \eta_3((S_k, O_k, f_k', i_k'), \alpha_k)\) if \(O_{k+1} = \emptyset\), \(i_\ast = i_{k+1}\), and \(f_\ast \geq f_{k+1}\),
- otherwise, we set \((S_{k+1}, O'_{k+1}, f''_{k+1}, i''_{k+1}) := \eta_3((S_k, O_k, f_k', i_k'), \alpha_k)\).

Intuitively, \(\rho'\) simulates the super-tight run \(\rho\) of \(B\) with the difference that (i) the transition from the waiting to the tight part sets the ranks of all non-core states to \(r - 1\), (ii) in the tight part, \(\rho'\) keeps taking the maximizing \(\eta_3\) transitions until it happens that \(\rho'\) is stuck with emptying some \(O\), in which case, the ranks of all non-accepting states in \(O\) are decreased (the \(\eta_4\) transition).

First, we prove that the run \(\rho'\) constructed according to the procedure above is infinite. Intuitively, there are two possibilities how the construction of \(\rho'\) can break: (i) the macrostate \((S_{m+1}, \emptyset, f'_{m+1}, 0)\) is not in \(\eta_2(S_m, \alpha_m)\), (ii) \(\eta_3((S_m, O_m, f_m, i_m), \alpha_m) = \emptyset\), or (iii) \(\eta_4((S_m, O_m, f_m, i_m), \alpha_m) = \emptyset\).

**Claim 1:** For every \(k > m\) the following conditions hold:

(i) the macrostate \(\rho'(k)\) is well defined,
(ii) \(f'_k \geq f_k\), and
(iii) \(C_k\) is a tight core of \(\rho'(k)\).

**Proof:** By induction on the position \(k > m\) in \(\rho'\).

- \(k = m + 1:\)
  (i) Proving \((S_{m+1}, \emptyset, f'_{m+1}, 0) \in \eta_2(S_m, \alpha_m)\): This easily follows from the construction of \((S_{m+1}, O_{m+1}, f_{m+1}, i_{m+1})\) given above.
  (ii) Proving \(f'_{m+1} \geq f_{m+1}\): This, again, easily follows from the construction of \((S_{m+1}, O_{m+1}, f_{m+1}, i_{m+1})\). In particular, the \(g\) constructed in (ii) satisfies the property \(g \geq f_{m+1}\) and the \(f_{m+1}\) constructed from it satisfies \(f'_{m+1} \geq g\).
  (iii) We have that \(C_{m+1}\) is a tight core of \((S_{m+1}, O_{m+1}, f_{m+1}, i_{m+1})\). From the definition of \(f'_{m+1}\) we directly obtain that \(C_{m+1}\) is also a tight core of \((S_{m+1}, O'_{m+1}, f''_{m+1}, i''_{m+1})\).
- \(k + 1:\) Suppose the claim holds for \(k\).
  (i) (and (iii)) For proving \(\rho'(k + 1)\) is well-defined, from the construction, we need to prove the following:
• If \( \rho'(k + 1) \) is the \( \eta_3 \)-successor of \( \rho'(k) \), we need to show that 
\[
\eta_3((S_k, O'_k, f'_k, i'_k, \alpha_k) \neq \emptyset.
\]
This condition can be proved by showing that the ranking \( f'_{\text{max}} = \max\text{-rank}(S_k, O'_k, f'_k, i'_k, \alpha_k) \) is tight. From the induction hypotheses ("\( f'_k \geq f_k \)" and "\( C_k \) is a tight core of \( f''_k \)"), we know that \( f_k \) and \( f'_k \) coincide on states from \( C_k \). Further, from Lemma 2 it holds that for every state \( q_k \in C_k \) there is a state \( q_{k+1} \in C_{k+1} \) such that \( f_k(q_k) = f_{k+1}(q_{k+1}) \). From the construction of \( f'_{\text{max}} \), we can conclude that it also holds that \( f'_k(q_k) = f'_{\text{max}}(q_{k+1}) \) (which proves (iii)). Using induction hypothesis one more time ("\( f'_k \) is tight"), we can conclude that \( f'_{\text{max}} \) is also tight.

• If \( O_{k+1} = \emptyset, i_{k+1} = i'_{k+1} \), and \( f'_{k+1} \geq f_{k+1} \) hold at the same time (i.e., \( \rho'(k + 1) \) is the \( \eta_3 \)-successor of \( \rho'(k) \)), we need to show that 
\[
\eta_4((S_k, O'_k, f'_k, i'_k, \alpha_k) \neq \emptyset.
\]
Above, we have already shown that \( \eta_3((S_k, O'_k, f'_k, i'_k, \alpha_k) \neq \emptyset. 
\]
From the definition, in order for \( \eta_4((S_k, O'_k, f'_k, i'_k, \alpha_k) \emptyset, \) it would need to hold that \( i'_{k+1} = 0 \). Since our assumption is that \( \mathcal{A} \) is complete and we know that \( \rho \) is accepting, it needs to hold that at every position \( j > m \), we have \( f_j(q) > 0 \) for any state \( q \in S' \) (otherwise, if \( q \) appeared in the \( O \)-component of some macrostate in \( \rho \), it would never disappear and so \( \rho \) could not be accepting).

The proof of (iii) easily follows from the previous step for \( \eta_3 \), since the ranking function of the result of \( \eta_4 \) differs from the one for \( \eta_3 \) only on states from the \( O \)-component, which are even and therefore, by definition, not in a tight core.

(ii) Proving \( f'_{k+1} \geq f_{k+1} \) assuming the induction hypothesis \( f'_k \geq f_k \):

• If \( \rho'(k + 1) \) is the \( \eta_4 \)-successor of \( \rho'(k) \), \( f'_{k+1} \geq f_{k+1} \) follows immediately from the fact that the \( \eta_3 \) transition function yields the maximal successor ranking.

• If \( O_{k+1} = \emptyset, i_{k+1} = i'_{k+1} \), and \( f'_{k+1} \geq f_{k+1} \) hold at the same time (i.e., \( \rho'(k + 1) \) is the \( \eta_4 \)-successor of \( \rho'(k) \)), \( f'_{k+1} \geq f_{k+1} \) is already a condition for \( \eta_4 \) to be taken.

Next, we will show that \( \rho' \) is accepting, i.e., that \( O \)-component of macrostates in \( \rho' \) is emptied infinitely many times. For the sake of contradiction, assume that \( \rho' \) is not accepting, i.e., for some \( \ell > m \), it happens that for all \( k \geq \ell \) it holds that \( O'_k \neq \emptyset \) and \( i'_k = i'_\ell \) (the run is “stuck” at some \( i' \) and cannot empty \( O' \)). We will show that if \( \rho' \) is “stuck” at some \( i' \), it will contain infinitely many macrostates obtained using an \( \eta_4 \) transition. An \( \eta_4 \) transition decreases ranks of all non-final states in \( O' \) and, as a consequence, removes such tracked runs from \( O' \). Therefore, if the rank of some run in \( O' \) is infinitely often not decreased by \( \eta_4 \), there needs to be a corresponding run of \( \mathcal{A} \) with infinitely many occurrences of an accepting state, so it would need to hold that \( \alpha \in \mathcal{A} \), leading to a contradiction.

Let us now prove the previous reasoning more formally. First, we show that \( \rho' \) needs to contain infinitely many occurrences of \( \eta_4 \)-obtained macrostates. Since \( \rho \) is accepting, it satisfies infinitely often the condition that \( O \) is empty and \( i = i'_\ell \).
To satisfy the condition for executing an \( \eta_4 \) transition, we need to show that for infinitely many \( k \) it in addition holds that \( f'_k = f'_\text{max} < \{ q \mapsto f'_\text{max}(q) - 1 \mid u \in O'_k \setminus F \} \geq f_k \) where \( f'_\text{max} \) is as in the definition of \( \eta_3 \). In particular, we will show that for infinitely many \( k \), we will have \( i_k = i'_\ell \), \( O_k = \emptyset \), and \( \forall q \in O'_k \setminus F : f'_\text{max}(q) > f(q) \) (from Claim 1 we already know that \( f'_\text{max} \geq f_k \)).

Let \( p > \ell \) be a position such that \( i_{p-1} \neq i'_\ell \), \( O_{p-1} = \emptyset \), and \( i_p = i'_\ell \), i.e., a position at which run \( \rho \) started emptying runs with rank \( i'_\ell \). Because \( \rho \) is accepting, there is a position \( k \geq p \) such that \( \rho(k) = (S_k, \emptyset, f_k, i'_\ell) \), therefore, the ranks of all runs tracked in \( O_p \) were decreased (otherwise, \( O_k \) could not be empty). Consider the following claim.

**Claim 2:** \( \forall q \in O'_k : f_k(q) < f'_k(q) \)

**Proof:** The weaker property \( f_k \leq f'_k \) follows from Claim 1. We prove the strict inequality for states in \( O'_k \) by contradiction. Assume that \( f_k(q) = f'_k(q) \) for some \( q \in O'_k \). Then there needs to be a predecessor \( s \) of \( q \) in \( S_p \) such that \( f_p(s) = i'_\ell \) and so \( s \in O_p \). But since \( q \notin O_k \), then somewhere between \( p \) and \( k \), the rank of the run in \( A \) must have been decreased. Therefore, \( f_k(q) < f'_k(q) \).

From Claim 2 and the fact that \( f'_k(q) \leq f'_\text{max} \), it follows that \( \forall q \in O'_k \setminus F : f'_\text{max}(q) > f(q) \), so an \( \eta_4 \) transition was taken infinitely often in \( \rho' \).

The last thing to show is that when \( \eta_4 \) is taken infinitely often, \( O' \) will be eventually empty. The condition does not hold only in the case when the rank of a run of \( A \) tracked in \( O' \) is infinitely often not decreased because it is represented in \( O' \) by a final state \( q \in O' \cap F \). But then \( A \) contains a run over \( \alpha \) that touches an accepting state infinitely often, so \( \alpha \in \mathcal{L}(A) \), which is a contradiction. \( \square \)

### 4.6 Backing Off

Our final optimization, called **BackOff**, is a strategy for guessing when our optimized rank-based construction is likely (despite the optimizations) to generate too many states and when it might be helpful to give up and use a different complementation procedure instead. We evaluate this afer the initial phase of **DELAY**, constructing \( \theta_2 \) (or \( \eta_2 \) in **MAXRANK**; we will just use \( \theta_2 \) in what follows) finishes. In particular, we provide a set of pairs \( \{(\text{StateSize}_i, \text{RankMax}_i)\}_{i \in \mathcal{I}} \) for some index set \( \mathcal{I} \). We then check (after \( \theta_2 \) is constructed) that for all \( (S, O, f, i) \in \text{img}(\theta_2) \) and all \( i \in \mathcal{I} \) it holds that either \( |S| < \text{StateSize}_i \) or rank \( (f) < \text{RankMax}_i \). If for some \( (S, O, f, i) \) and \( i \) the condition does not hold, we terminate the construction and execute a different procedure.

### 5 Experimental Evaluation

Experimental evaluation has been omitted in this version of the technical report.
6 Related Work

The problem of BA complementation has attracted researchers since Büchi’s seminal work [1]. Since then, there have appeared several branches of approaches for BA complementation. Ramsey-based complementation using the original argument of Büchi, decomposing the language accepted by an automaton into a finite number of equivalence classes, was later improved in [19]. Determinization-based complementation was introduced by Safra in [20] and later improved by Piterman in [21]. Determinization-based approaches convert an input BA into an equivalent intermediate deterministic automaton with different accepting condition (e.g. Rabin automaton) that can be easily complemented, and then converted back into a BA. Rank-based complementation, studied in [8,9,13], extends the subset construction for determinizing finite automata, with some additional information kept in each macrostate to track the acceptance condition of all runs of the input automaton. Rank-based construction can be optimized using simulation relations as shown in [14]. Here the simulation relations can be used to prune macrostates that are redundant for accepting of a word. Slice-based complementation uses a reduced abstraction on a run tree to track the acceptance condition [10,11]. A learning-based approach was presented in [22]. Except of complementation of standard nondeterministic Büchi automata, there are specific approaches for complementation of special types of BAs, such as deterministic [23], semi-deterministic [24], or unambiguous [25].

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