Twist deformations in dual coordinates

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Abstract

Twist deformation $U_{\mathcal{F}}(\mathfrak{g})$ is equivalent to the quantum group $\text{Fun}_{\text{def}}(\mathfrak{g}^\#)$ and has two preferred bases: the one originating from $U(\mathfrak{g})$ and that of the coordinate functions of the dual Lie group $\mathfrak{g}^\#$. The costructure of the Hopf algebra $U_{\mathcal{F}}(\mathfrak{g}) \approx \text{Fun}_{\text{def}}(\mathfrak{g}^\#)$ is analyzed in $\mathfrak{g}^\#$-group terms. The weight diagram of the adjoint representation of the algebra $\mathfrak{g}^\#$ is constructed in terms of the root system $\Lambda(\mathfrak{g})$. The explicit form of the $\mathfrak{g} \rightarrow \mathfrak{g}^\#$ transformation can be obtained for any simple Lie algebra $\mathfrak{g}$ and the factorizable chain $\mathcal{F}$ of extended Jordanian twists. The dual group approach is used to find new solutions of the twist equations. The parametrized family of extended Jordanian deformations $U_{EJ}(\mathfrak{sl}(3))$ is constructed and studied in terms of $SL(3)^\#$. New realizations of the parabolic twist are found.

1 Introduction

Twist deformations are usually described in terms of the initial Poincare - Birkhoff-Witt basis borrowed from $U(\mathfrak{g})$. This is reasonable when the algebraic sector is of main interest. Using the $\mathfrak{g}$-basis we explicitly manifest that the algebraic relations in $U_{\mathcal{F}}(\mathfrak{g})$ remain classical. The Lie-Poisson structure is encoded in the twisting element $\mathcal{F}$. The result of the twist deformation $\mathcal{F} : U(\mathfrak{g}) \rightarrow U_{\mathcal{F}}(\mathfrak{g})$ is the Hopf algebra $U_{\mathcal{F}}(\mathfrak{g})$ with the initial multiplication, unit and counit and the deformed universal element $R_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{F}^{-1}$, costructure $\Delta_{\mathcal{F}}$ and antipode $S_{\mathcal{F}}$.

Consider the Hopf algebra $U_{\mathcal{F}}(\mathfrak{g})$ as a quantized Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^\#)$, i.e. as the deformation of $U(\mathfrak{g})$ in the direction of $\mathfrak{g}^\#$. According to the quantum duality principle [2], [4] the smooth deformation $U_{\mathcal{F}}(\mathfrak{g})$ is the quantum group $\text{Fun}_{\text{def}}(\mathfrak{g}^\#)$ of the dual Lie group $\mathfrak{g}^\#$. The group $\mathfrak{g}^\#$ is the universal covering group with the Lie algebra $\mathfrak{g}^\#$. For the twisted universal enveloping algebra
$U_F(g)$ the Lie coalgebra $(g^\#)^*$ is defined by the classical $r$-matrix

$$r = r^{lk} e_l \wedge e_k, \quad l, k = 1, \ldots, m$$ (1)

and has the composition

$$\delta : g \rightarrow g \otimes g, \quad \delta = \text{ad}_r \circ \Delta(0)$$

Here $\Delta(0)$ is the primitive coproduct $\Delta(0) (g) = g \otimes 1 + 1 \otimes g$, for $g \in g$ and $\{e_l|, l, k = 1, \ldots, m\}$ forms the basis of the carrier subalgebra of the twist $F$.

All this means that the costructure of $U_F(g)$ describes the composition law of $\text{Fun}_{\text{def}}(g^\#)$, the deformed multiplication of $\text{Fun}(g^\#)$. The deformation is due to the fact that the coordinate functions in $\text{Fun}_{\text{def}}(g^\#)$ do not commute – they are subject to the composition rules in $U(g)$.

The structure of the dual Lie algebra $g^\#$ can be described in terms of the generators of $g$ as the "second" Lie structure on the space of $g$. When the group $g^\#$ is twisted by $F$ this obviously leads not only to the deformation of the (initially cocommutative) costructure but also to the deformation of the space of coordinate functions. This can be explicitly seen in the Jordanian twist deformation $F_J = e \otimes e - H \otimes \sigma$; $F_J : U(b(2)) \rightarrow U_J(b(2))$ of $U(b(2))$. The algebra $b(2)$, with the generators $\{H, E\}$ and the relation $[H, E] = E$, is the carrier of the twist $F_J$. Thus the dual algebra $b(2)^\#$ is equivalent to $b(2)$. The classical $r$-matrix $H \wedge E$ describes the isomorphism

$$r : \begin{cases} \quad H^* \leftrightarrow E \\ \quad E^* \leftrightarrow -H \end{cases}$$

But the universal element for the Jordanian deformation $R = F_{21} F^{-1} = e^\sigma \otimes H e^{-H \otimes \sigma}$ (with $\sigma = \ln (1 + E)$) shows that in the global dualization of the algebras $U(b(2))$ and $U(b(2)^\#)$ the generator $E$ is only the first term in the expansion of $H^*$ and in fact $\sigma$ is the element dual to $H$. This is also clearly seen in the costructure of $U_J(b(2)) = \text{Fun}_{\text{def}}(B(2)^\#)$

$$\Delta_J(H) = H \otimes e^{-\sigma} + 1 \otimes H, \quad (2)$$

$$\Delta_J(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma.$$ These are the multiplication rules of the group $B(2)^\#$ (the semidirect product of the "rotation" $(e^{-\sigma})^*$ and "translation" $H^*$) with the Cartan coordinate $\sigma$ that together with $H$ forms the $g^\#$-basis here.

As we have seen in the example above the (dual) $g^\#$-coordinates are easily obtained in the case when the carrier subalgebra of the twist $g_c$ coincides with $g$.

In most of the interesting cases the carrier is a proper subalgebra of $g$, $g_c \subset g$, and we have the nontrivial decomposition of the space $V_\delta = V_{g_c} \oplus V_a$. In particular this is true when $g$ is a semisimple Lie algebra [4]. On the subspace $V_{g_c}$ and in the Hopf algebra $U_F(g_c)$ the $g^\#$-coordinates can be obtained by using the twisting element $F$ as a map $g \rightarrow g^\#$. The twist deformation on the
space $V_a$ is usually described by the set of coproducts for the basic elements of $g$ and are expressed in terms of a mixed basis containing elements from both $g$- and $g^\#$-bases. In this representation it is very difficult to study the properties of the costructure.

The deformed costructure is essential not only by itself. It is necessary to know its properties to find further deformations of $U_\mathcal{F}(g)$ and to construct new twists by enlarging the element $F$ with the additional factors, that is to find factorized solutions of the twist equations [11]. In particular the subspaces in $V_a$ with quasiprimitive costructure are of great importance [10], [11].

In this paper we shall consider universal enveloping algebras $U_\mathcal{F}(g)$ for semisimple Lie algebra $g$ twisted by chains of extended twists – the factorized 2-cocycle $\mathcal{F} = \mathcal{F}_p \mathcal{F}_{p-1} \ldots \mathcal{F}_1$ where each $\mathcal{F}_q$ is the solution of the twist equations for $U_{\mathcal{F}_{q+1} \ldots \mathcal{F}_1}(g)$. These solutions (called the twisting factors [8]) are of two types: Jordanian [3] and extension [9]. In chains of twists the Reshetikhin twisting factor is used only to "rotate" the Jordanian twist and can be always combined with the corresponding Jordanian factor. We study the group $\Phi_\mathcal{F}(g)$ and define the weight diagram for the adjoint representation of its Lie algebra $g^\#$. In Section 4 we develop the algorithm for constructing the $g^\#$-coordinates in the Hopf algebra $U_\mathcal{F}(g)$. In Section 5 the dual group approach is applied to study the properties of the Hopf algebra $U_\mathcal{F}(\mathfrak{sl}(3))$. It is demonstrated that the dual representation of the costructure can be used to find new twisted deformations.

2 Dual Lie algebra

Consider the simple Lie algebra $g$ and the Cartan-Weil basis $\{e_i \mid i = 1, \ldots, n\}$ correlated with the decomposition $V_\theta = V_h \oplus V_{n_+} \oplus V_{n_-} = V_h \oplus (\nu \in \Lambda^+)(V_\nu \oplus V_{-\nu})$,

$$V_{n_+} \oplus V_{n_-} = \oplus (\nu \in \Lambda^+) \left( V_\nu \oplus V_{-\nu} = V_\nu \oplus (\nu \in \Lambda^+ | |(h_\nu)=1/2) \left( V_\nu \oplus V_{-\nu} \right) \oplus \left( \xi \in \Lambda^+ | |(\xi)=0 \right) V_\xi \oplus V_{-\xi} = V_1 \oplus V_{1/2} \oplus V_0 \oplus V_{-1/2} \oplus V_{-1}. $$

The carrier subalgebra $g_c$ of the chain $\mathcal{F}$ is the proper subalgebra of $g$, $g_c \subset g$. The generators $\{e_i \mid i = 1, \ldots, m\}$ of $g_c$ contain the root vectors $\{e_\phi \mid \phi \in \Lambda_c \subset \Lambda(g)\}$ for the fixed subset of roots $\Lambda_c$ [14]. Consider the subspace $V_a \subset V_\theta$ generated by $\{h^\lambda_{\nu} , e_\nu \nu \in \Lambda \setminus \Lambda_c \}$ where the Cartan elements $h^\lambda_{\nu}$ are orthogonal to any initial root $\lambda_0$ in $\mathcal{F}$. Then we have the direct sum decomposition $V_\theta = V_a \oplus V_c$. Construct the basis $\{e_i^*\}$ canonically dual to $\{e_i\}$, i.e. $(e_i^*, e_j) = \delta_{ij}$. Let $\Gamma$ be the weight diagram of the adjoint representation $ad_g$. Our aim is to define the weight diagram $\Gamma^\#$ for the adjoint representation of $ad_g^\#$. The structure of $\Gamma^\#$ is described in the following statement:

**Lemma 1** Let $g$ be the simple Lie algebra with the root system $\Lambda \subset \Gamma_{ad}(g)$ and the Cartan decomposition $V_\theta = V_h \oplus (\nu \in \Lambda^+)(V_\nu \oplus V_{-\nu})$, $\mathcal{F} = \mathcal{F}_p \mathcal{F}_{p-1} \ldots \mathcal{F}_1$ – the solution of the twist equations where each $\mathcal{F}_q$ is a Jordanian or an extension
twisting factor, \( r_F = \sum_{i,k=1,\ldots,\dim(g_c)} r_{ik} e_i \wedge e_k \in g \wedge g \) — the corresponding skew solution of the CYBE, \((g, g^\#)\) — the Lie bialgebra. Then the weight diagram \( \Gamma^\# \) of the dual Lie algebra \( g^\# \) is the union \( \Gamma^\# := \Gamma_{\text{ad}}(g^\#) = \Gamma^\#_{\text{ad}} \cup \Gamma^\#_{g^\#} \) of the weight diagram \( \Gamma^\#_{g^\#} := \Gamma(g^\#) \). The system \( \Gamma^\#_{g^\#} \) is the set of weights \( \{ -\gamma_l \} \) opposite to the weights \( \{ \gamma_l | l = 1, \ldots, \dim(g_c) \} \) of the carrier \( g_c \). For each component \( r_{\psi} \neq 0 \) in \( r \) the generator \( (e_\psi)^* \) takes the weight \( -\psi \) and \( (e_\psi)^* \) — the weight \( -\varphi \). The diagram \( \Gamma^\#_{g^\#} \) is the subset of the initial weight diagram: \( \Gamma^\#_{g^\#} = \Gamma_{\text{ad}}(g) \setminus \Gamma_{\text{ad}}(g_c) \). The weight vectors \( \{ e_\mu^* | \mu \in \Gamma^\#_{g^\#} \} \) generate the Abelian ideal \( g^\#_{a} \) and the dual algebra is the semidirect sum \( g^\# \approx g^\#_{a} \oplus g^\#_{c} \).

**Proof.** Consider \( \delta(g) \) for an arbitrary \( g \in g \):

\[
\delta(g) = [r_{kl} e_k \wedge e_l, g \otimes 1 + 1 \otimes g] = r_{kl} [e_k g] \wedge e_l + r_{kl} e_k \wedge [e_l g].
\]

Any nontrivial coproduct \( \delta(g) \) has a comultiplier belonging to \( g_c \). If \( g \in g_c \) then \( \delta(g) \) belongs to \( g_c \otimes g_c \), because \( g_c \) is an algebra. This proves that \( g^\#_{a} \) is the Abelian ideal.

The classical \( r \)-matrix performs the deformation of the Lie bialgebra \( r : (g, \text{Ab}) \longrightarrow (g, g^\#) \) [12]. Each term in \( r \) gives rise to a deforming function for the original Abelian composition of the dual algebra. Let \( r \) be the classical \( r \)-matrix corresponding to the chain of extended twists \( F \). To find the structure of the algebra \( g^\# \), it is sufficient to consider the deforming functions originating from two basic twisting factors: Jordanian and extension.

- The Jordanian BTF has the canonical form \( \mathcal{F}_J = e^{h \otimes \sigma_\mu}, \sigma_\mu = \ln (1 + e_\mu) \) [3]. It can be deformed by the previous twisting factors [11, 14] and reveal a more complicated structure. Still in all the cases one can write it in the form \( e^{h \otimes \sigma_\mu} \) with \( \sigma_\mu = \ln (1 + e_\mu + f(e_\alpha, \xi_\mu))(\xi_\mu \text{ are the deformation parameters of the previous twisting factors}) \). The corresponding \( r \)-matrix is

\[
r_J = h \wedge e_\mu,
\]

with the Cartan element \( h = h_\mu + \gamma h_\perp \subset V_h \), \( \mu(h_\perp) = 0 \) and \( [h, e_\mu] = \mu(h) e_\mu; \mu(h) = \mu(h_\mu) = 1 \). The elements \( h^* \) and \( h^*_\perp \) are canonically dual: \( (h^*, h_\perp) = 0, (h^*, h) = (h^*_\perp, h_\perp) = 1 \). On the carrier subalgebra \( g_{Jc} = \mathfrak{b}(2) \) (equivalent to its dual \( g^\#_{Jc} \)) we get

\[
\delta(h) = [r_J, \Delta^0(h)] = -h \wedge e_\mu,
\]

\[
\delta(e_\mu) = [r_J, \Delta^0(e_{\mu u}) e_\mu] = 0.
\]

This means that the relations in \( g^\#_{Jc} \) are defined by the deforming function

\[
f^\#_J (e^*_\mu, h^*) = h^*,
\]

and \( e^*_\mu \) is the Cartan element in \( g^\#_{Jc} \).
Consider \( a_\nu \in V_a \)

\[
\delta(a_\nu) = \left[ r_f, \Delta^0(a_\nu) \right] = \\
= \nu (h_\mu + \gamma h_\perp) a_\nu \wedge e_\mu + h \wedge [e_\mu, a_\nu] = \\
= \left( \frac{1}{2} + \gamma \nu (h_\perp) \right) a_\nu \wedge e_\mu + C_{\mu \nu}^p h \wedge e_p
\]

The corresponding deforming function is

\[
f^\#_f(e^*_\mu, a^*_\nu) = -\nu (h_\mu + \gamma h_\perp) a^*_\nu, \\
f^\#_f(h^*, e^*_\nu) = C_{\mu \nu}^p a^*_\nu,
\]

i.e. \( h^* \) has the weight \((-\mu)\) with respect to the original root system \( \Lambda \).

\[
\delta(a_{-\mu}) = e_\mu \wedge a_{-\mu} - 2\gamma h \wedge h_\perp.
\]

\[
f^\#_f(e^*_\mu, a^*_{-\mu}) = a^*_{-\mu}, \\
f^\#_f(h^*, h^*_{\perp}) = -2\gamma a^*_{-\mu}.
\]

Notice that \( h^* \) does not "shift" the element \( h^*_{\perp} \) if \( h \) is proportional to \( h_\mu \).

Summing up we find that the adjoint action of the algebra \( g^\#_{f,c} \) is fixed by the relations

\[
f^\#_f(e^*_\mu, h^*) = h^*, \\
f^\#_f(h^*, a^*_\nu) = C_{\mu \nu}^p a^*_\nu, \\
f^\#_f(h^*, h^*_{\perp}) = -2\gamma a^*_{-\mu}, \\
f^\#_f(e^*_\mu, a^*_{-\mu}) = a^*_{-\mu}.
\]

In this case the deforming function \( f^\#_f \) is itself the Lie composition, it fixes the Lie algebra \( g^\# \) and \( g^\#_{f,c} \) is a subalgebra, \( g^\# \subset g^\#_{f,c} \). The adjoint action of \( g^\#_c \) on \( V^* \) can be decomposed into the direct sum: \( \text{ad} \left( g^\#_{f,c} \right) |_{V^*} = \text{ad} \left( g^\#_c \right) |_{V^*} \oplus d \left( g^\#_{f,c} \right) |_{V^*} \) (with the representation \( d \left( g^\#_{f,c} \right) \) describing the action on \( V^*_a \)).

- The extension BTF. Consider the \( r \)-matrix

\[
r_E = e_\mu \wedge e_\nu, \quad \nu \neq -\mu.
\]

(Remember that the element itself must not satisfy the CYBE. Only the full \( r \)-matrix must be the solution.)

\[
\delta(h) = [r_E, \Delta^0(h)] = \\
= -(\mu + \nu)(h) e_\mu \wedge e_\nu, \\
\delta(e_\xi) = [r_E, \Delta^0(e_\xi)] = \\
= C_{\mu \xi}^p e_\mu \wedge e_\nu + C_{\nu \xi}^p e_\mu \wedge e_\chi, \\
\delta(e_{-\mu}) = [r_E, \Delta^0(e_{-\mu})] = \\
= 2h_\mu \wedge e_\nu + C_{\nu,-\mu}^p e_\mu \wedge e_\rho,
\]

5
The deforming function $f^\#_E$ looks as follows:

$$
\begin{align*}
    f^\#_E(e^*_\nu, e^*_\nu-\xi) &= C_{\nu\xi} e^*_\xi, \\
    f^\#_E(e^*_\mu+\zeta, e^*_\nu) &= C_{\lambda\mu} e^*_\lambda, \\
    f^\#_E(e^*_\mu, e^*_\nu) &= -(\mu + \nu) (h) h^*, \\
    f^\#_E(h^*_\mu, e^*_\nu) &= 2e^*_{-\mu}.
\end{align*}
$$

If we assign to the basic vector $e^*_\tau$ the root $\tau$ then $\text{ad} (e^*_\mu)$ and $\text{ad} (e^*_\nu)$ act on $e^*_\tau$ as having the roots $-\nu$ and $-\mu$ correspondingly. When $h$ becomes proportional to $h_\mu$ (or correspondingly to $h_\nu$) the operator $\text{ad} (e^*_\mu)$ cannot shift $h^*_1$.

Combining the properties of deforming functions corresponding to different twisting factors one can check that for the integral composition $f^\# = \sum f^\#_q$ the chain $F$ of extended twists the carrier $g^\#_{\text{ext}}$ is a subalgebra in $g^\#$ its space is the direct sum $V^*_{\text{ext}} = V^*_{\text{e}} \oplus V^*_{\text{a}}$ and the dual algebra is a semidirect sum $g^\#_{\text{dual}} \simeq g^\#_{\text{ext}} \rtimes g^\#_{\text{a}}$. ■

Notice that in the general case the dual algebra $g^\#$ must not contain $g^\#_{\text{ext}}$ as a subalgebra.

When the adjoint action $\text{ad} (g^\#_{\text{ext}})$ is considered on $h^*_1$ it always behaves as "orthogonal" to $h^*$.

### 3 Second classical limit for twisted algebras

In the twisted universal enveloping algebra considered as the deformed algebra of functions $U_\mathcal{F} (\mathfrak{g}) = \text{Fun}_{\text{def}} (\mathfrak{g}^\#)$ the group $\mathfrak{g}^\#$ is the universal enveloping group realized in terms of formal power series.

We have seen in the previous Section how the relations of the dual algebra $\mathfrak{g}^\#$ are encoded in the weight diagram $\Gamma^\#$. This description refers to the basis of functionals $e^*_\mu$ canonically dual to the generators $\{ e_\mu \}$ of $\mathfrak{g}$. Thus we can construct the group compositions for $\mathfrak{g}^\#$ as the coproducts in the Hopf algebra $\text{Fun} (\mathfrak{g}^\#)$ in terms of $e_\mu$ considered as exponential coordinates for this group. Unfortunately only in a few trivial cases the transformation to dual (noncommuting) coordinates can be determined by the direct comparison of $\text{Fun} (\mathfrak{g}^\#)$ with $\text{Fun}_{\text{def}} (\mathfrak{g}^\#) = U_\mathcal{F} (\mathfrak{g})$. In the general situation the Hopf algebra $U_\mathcal{F} (\mathfrak{g})$ in its initial form is inappropriate to extract the transformation $\mathfrak{g} \rightarrow \mathfrak{g}^\#$.

In the Hopf algebra $\text{Fun}_{\text{def}} (\mathfrak{g}^\#)$ the composition law of the group $\mathfrak{g}^\#$ is deformed (by the noncommutativity of the coordinates). In quantum deformation the compositions are usually described in terms of undeformed coordinates, i.e. the proper coordinates of undeformed $\text{Fun} (\mathfrak{g}^\#)$ are sufficient to describe the multiplication of the deformed $\text{Fun}_{\text{def}} (\mathfrak{g}^\#)$. Thus we must find the Hopf algebra $\text{Fun} (\mathfrak{g}^\#)$ with the commutative multiplication law. The transition $\text{Fun}_{\text{def}} (\mathfrak{g}^\#) \rightarrow \text{Fun} (\mathfrak{g}^\#)$ is called the second classical limit [2]. It was proved [7] that for the standard quantization (with the parameter $q = e^h$) the second
classical limit can be obtained by the trivial scaling of the Lie algebra generators
by $\varepsilon$ and tending $\varepsilon, h \to 0$ provided that $\frac{h}{\varepsilon} = \text{const}$. Taking into account that
the transition to the second classical limit is a general procedure that does not
depend on the particular form of quantization (of a Lie bialgebra) we come to
the conclusion that the same algorithm is to be applied in the case of twisted
universal enveloping algebras $U_\mathcal{F}$.

We are dealing with the factorizable twists $\mathcal{F} = \mathcal{F}_p \mathcal{F}_{p-1} \ldots \mathcal{F}_1$ with Jordanian and extension twisting factors $\mathcal{F}_j$. For the sequence $\mathcal{F}$ one can attribute the fixed number of deformation parameters $\{\xi_s = 1, \ldots, l \mid l \leq p\}$, each of them corresponds to an automorphism of $\mathfrak{g}$. This number is equal to the number of independent Jordanian factors $\mathcal{F}_s$. This means that there exists the parametrized solution $r(\{\xi_s\})$ of the CYBE, and as a result we get the parametrized sets $\mathfrak{g}^\#(\{\xi_s\})$ and $\mathfrak{G}^\#(\{\xi_s\})$.

The automorphism that injects the deformation parameters together with
the mentioned above scaling leads to the substitution

$$e_j \rightarrow \varepsilon \xi^{(s)}_s \frac{\varepsilon}{\varepsilon} e_j.$$  (4)

(Here $\theta_{(s)}$ is the natural grading corresponding to the s-th link of the chain $\mathcal{F}$.)

The classical limit algorithm prescribes that the parameters $\varepsilon$ and $\xi_s$ are to be
turned to zero while their relations are fixed

$$\varepsilon, \xi_s \to 0,$$

$$\frac{\xi_s}{\varepsilon} = \zeta_s.$$

The result is the algebra of functions over the group $\mathfrak{G}^\#(\{\zeta_s\})$,

$$\lim_{\varepsilon, \xi_s \to 0, \xi_s/\varepsilon = \zeta_s} U_{\mathcal{F}_s(\xi_s)}(\mathfrak{g}(\varepsilon)) = \text{Fun}(\mathfrak{G}^\#(\{\zeta_s\})).$$  (5)

In particular one can check that when $\mathcal{F}$ is a canonically parametrized chain
of extended twists in the limit (5) the structure constants of $U_{\mathcal{F}_s(\xi_s)}(\mathfrak{g}(\varepsilon))$
remains finite, the multiplication law becomes Abelian and the Lie algebra of
the obtained group coincides with $\mathfrak{g}^\#(\{\zeta_s\})$. The corresponding calculations
are performed in Appendix.

4 Group coordinates in terms of dual algebra coordinates

In the case of the factorizable twist $\mathcal{F} = \mathcal{F}_p \mathcal{F}_{p-1} \ldots \mathcal{F}_1$ the dual group $\mathfrak{G}^\#$ is
solvable. Thus the exponential coordinates are the most suitable to describe $\text{Fun}_{\text{def}}(\mathfrak{G}^\#) = U_\mathcal{F}(\mathfrak{g})$. In these coordinates the Hopf algebra $(\text{Fun}_{\text{def}}(\mathfrak{G}^\#))^*$
dual to $\text{Fun}_{\text{def}}(\mathfrak{G}^\#)$ takes the form of the universal enveloping algebra $U(\mathfrak{g}_{(\mathfrak{g})})$
obtained in terms of (the initial) \( g \)

Example. Twisted algebra \( U \) is dual to the exponential coordinates for the group \( G \)
universal enveloping \( U \) that describe \( g \) PBW coordinates in \( g \) action of the dual Lie algebra become highly transparent demonstrating the exponential map of the adjoint
Fun
\( U \) the Cartan elements are \( I \), where the ideals \( I \) encode the Lie multiplications. It is sufficient to compare the ideals \( I (g) \) and \( I (g) \). This gives the finite set of relations

\[
e_i^{\#} = \phi_i (\{ e_j \})
\]

that describe \( g \)-basic elements in terms of \( g \)-basis and thus introduces the \( g \*-PBW \) coordinates in \( (\text{Fun} (g^\#))^* \). Constructing the relations dual to (6) we return to \( \text{Fun} (g^\#) \) and find the desired transformation

\[
e_i^* = \phi_i (\{ e_j \})
\]

Finally this transformation is to be applied to the deformed algebra \( U_F (g) = \text{Fun}_{\text{def}} (g^\#) \). This gives the deformed group \( g^\# \) in its exponential coordinates. In this presentation the commultiplication of functions on the solvable group \( g^\# \) become highly transparent demonstrating the exponential map of the adjoint action of the dual Lie algebra \( g^\# \).

5 Example. Twisted algebra \( U_F (sl (3)) \) in dual group coordinates

Let \( g = sl (3) \) and consider the \( r \)-elements: \( r_J (\gamma) = h \wedge e_{13} \) and \( r_E = e_{12} \wedge e_{23} \), where \( h = h_{13} + \gamma h_\perp \). Here the generators \( e_{ab} \) are the ordinary matrix units and the Cartan elements are \( h_{aa} = 1/2 (e_{aa} - e_{bb}) \), \( h_\perp = 1/2 (e_{11} - 2 e_{22} + e_{33}) \). The sum \( r_{EJ} (\gamma) = r_J (\gamma) + r_E \) is the set of solutions of the CYBE. Or equivalently, the sum of deforming functions \( f_J^\# (\gamma) + f_E^\# \) defines the Lie multiplication [\( [\cdot, \cdot]^\# := (f_J^\# (\gamma) + f_E^\# ) (,.) \]. This bracket is the Lie composition of the dual algebra \( s(l (3))^* \) on the space \( V_6^* \). The basic commutation relations are

\[
\begin{align*}
[e_{13}^*, h^*] &= h^*, \\
[e_{13}^*, e_{21}^*] &= + \frac{1}{2} (3 \gamma + 1) e_{21}^*, \\
[e_{13}^*, e_{12}^*] &= - \frac{1}{2} (3 \gamma - 1) e_{12}^*, \\
[e_{13}^*, e_{31}] &= [e_{21}^*, e_{32}] = e_{23}^*, \\
[e_{12}^*, e_{21}^*] &= e_{31}^*, \\
[h_\perp^*, e_{23}^*] &= (1 - \gamma) e_{21}^*, \\
[h_\perp^*, e_{12}^*] &= (1 + \gamma) e_{32}^*, \\
[h^*, e_{12}^*] &= [h^*, e_{23}^*] = 0, \\
[h^*, h_\perp^*] &= -2 \gamma e_{31}^*, \\
[e_{13}^*, e_{32}^*] &= - \frac{1}{2} (3 \gamma - 1) e_{32}^*, \\
[e_{13}^*, e_{23}^*] &= + \frac{1}{2} (3 \gamma + 1) e_{23}^*. 
\end{align*}
\]
This algebra has the four-dimensional subalgebra (equivalent to the subalgebra \( \mathfrak{g}_c \)), the carrier of the \( r \)-matrix \( r_{EJ} (\gamma) \)) with the weight system \( \Gamma^\#_c = \{0, \varphi_{31}, \varphi_{32}, \varphi_{21}\} \) and the four-dimensional Abelian ideal generated by the set \( \{h_1^*, e_{31}^\gamma, e_{32}^\gamma, e_{21}^\gamma\} \) with the weights \( \Gamma^\#_a = \{0, \rho_{31}, \rho_{32}, \rho_{21}\} \). Here \( \{\varphi_{ab}\} \) and \( \{\rho_{ab}\} \) are the copies of those vectors from \( \Lambda (\mathfrak{sl}(3)) \) that correspond to the generators \( e_{ab} \). In this particular case both subsystems contain the same sets of vectors. The action of \( \text{ad} (\hat{\mathfrak{g}}_c^\#) \) on \( h^*_1 \) shows that \( h^*_1 \) behaves as "orthogonal" to \( h^* \).

The following twisting element

\[
\mathcal{F}(\gamma) = \mathcal{F}_E(\gamma) \mathcal{F}_J(\gamma) = \exp \left( \xi e_{12} \otimes e_{23} e^{\frac{1}{3} (3\gamma - 1)} \sigma (\xi) \right) \exp (h \otimes \sigma (\xi))
\]

is the solution of the twist equations \( [1] \) corresponding to the \( r \)-matrix \( r_{EJ} (\gamma) \). Here \( \sigma (\xi) = \ln (1 + \xi e_{13}) \).

The parameter \( \gamma \) describes the Reshetikhin rotation of the Jordanian factor while \( \xi \) is the deformation parameter. The latter corresponds to the automorphism

\[
e_\tau \rightarrow \xi^{\gamma(h_{13})} e_\tau,
\]

where \( \tau \in \Gamma (\mathfrak{sl}(3)) \) are the weights and the element \( h_{13} \) performs the gradation in \( \Gamma, h_{13} : \Gamma \rightarrow R^1 \). Together with the scaling this gives the parametrization appropriate to perform the second classical limit

\[
e_\tau \rightarrow \xi^{\gamma(h_{13})} e_\tau, \quad \xi = \zeta.
\]

The twisting is the transformation of the comultiplication in \( U (\mathfrak{g}) \) performed by the operator

\[
\prod_{q=p}^1 e^{\text{ad} (\ln \mathcal{F}_q)} = e^{\frac{1}{3} \text{ad} (\text{BCH} \{\Psi_{(\mathcal{F})q}(\{e_i; \zeta_i\}); \zeta_i\})}
\]

Here BCH denotes the Baker-Campbell-Hausdorff series and \( \Psi_{(\mathcal{F})q}(\{e_i; \zeta_i\}) \) are the logarithms of the twisting factors \( \mathcal{F}_j \):

\[
\Psi_{(\mathcal{F})q}(\{e_i; \zeta_i\}) = \varepsilon \ln \mathcal{F}_q, \quad \Psi \in U \otimes U; \quad q = 1, \ldots, p; i = 1, \ldots, n.
\]

For the twisting element \( \mathcal{F}_1 \) these are

\[
\Psi_{(\mathcal{F})1}(\{e_i; \zeta_i\}) = \hat{\mathcal{h}} \otimes \hat{\sigma} (\zeta),
\]

\[
\Psi_{(\mathcal{F})2}(\{e_i; \zeta_i\}) = \zeta e_{12} \otimes e_{23} e^{\frac{1}{3} (3\gamma - 1)} \hat{\sigma} (\zeta).
\]

In the second classical limit only the zero power terms in the BCH-series remain and the compositions of the dual group \( \mathfrak{G}^\# = (\mathfrak{SO}(3)) \) in \( \mathfrak{g} = \mathfrak{sl}(3) \)-coordinates, i.e. the coproducts in \( \text{Fun} (\mathfrak{G}^\#) \) in \( \mathfrak{g} \)-basis, can be obtained by the formula:

\[
\Delta_{\mathcal{F}_1(\zeta)} (e_j) = \lim_{\varepsilon \rightarrow 0} \left( e^{\frac{1}{3} \text{ad} (\text{BCH} \{\Psi_{(\mathcal{F})1}(\{e_i; \zeta_i\}); \zeta_i\}) \circ (e_j \otimes 1 + 1 \otimes e_j)} \right) = \lim_{\varepsilon \rightarrow 0} \left( e^{\frac{1}{3} \text{ad} (\Psi_{(\mathcal{F})1}(\{e_i; \zeta_i\}) + \Psi_{(\mathcal{F})2}(\{e_i; \zeta_i\})) \circ (e_j \otimes 1 + 1 \otimes e_j)} \right)
\]
In particular the scaled $g$-basic elements (and $\hat{\sigma}(\zeta)$) have the following coproducts in $\text{Fun}(\mathfrak{g}^\#)$:

\[
\begin{align*}
\Delta^\text{lim}_F(\hat{h}(\gamma)) &= \hat{h}(\gamma) \otimes e^{-\hat{\sigma}(\zeta)} + 1 \otimes \hat{h}(\gamma) - \xi \hat{e}_{12} \otimes \hat{e}_{23} e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)}; \\
\Delta^\text{lim}_F(\hat{h}_\perp) &= \hat{h}_\perp \otimes 1 + 1 \otimes \hat{h}_\perp; \\
\Delta^\text{lim}_F(\hat{e}_{12}) &= \hat{e}_{12} \otimes e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)} + 1 \otimes \hat{e}_{12}; \\
\Delta^\text{lim}_F(e_{23}) &= e_{23} \otimes e^{-\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)} + e^{\hat{\sigma}(\zeta)} \otimes e_{23}; \\
\Delta^\text{lim}_F(\hat{\sigma}(\zeta)) &= \hat{\sigma}(\zeta) \otimes 1 + 1 \otimes \hat{\sigma}(\zeta); \\
\Delta^\text{lim}_F(\hat{e}_{21}) &= \hat{e}_{21} \otimes e^{-\frac{i}{2}(\gamma + 1)\hat{\sigma}(\zeta)} + 1 \otimes \hat{e}_{21} + \zeta(1 - \gamma) \hat{h}_\perp \otimes \hat{e}_{23} e^{-\hat{\sigma}(\zeta)}; \\
\Delta^\text{lim}_F(e_{32}) &= e_{32} \otimes e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)} + 1 \otimes e_{32} + \zeta \hat{h}(\gamma) \otimes e_{12} e^{-\hat{\sigma}(\zeta)} + \\
&\quad + \zeta^2 \hat{e}_{12} \otimes \left(\hat{h}(\gamma) - (\gamma + 1) \hat{h}_\perp\right) e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)} - \\
&\quad - \zeta^2 \hat{h}(\gamma) \hat{e}_{12} \otimes \left(e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)} - e^{-\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)}\right) - \\
&\quad - \zeta^2 \hat{e}_{12} \otimes \hat{e}_{23} e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)} + \zeta^2 \hat{e}_{12} \otimes \hat{e}_{23} e^{\frac{i}{2}(\gamma - 2)\hat{\sigma}(\zeta)}; \\
\Delta^\text{lim}_F(\hat{e}_{31}) &= \hat{e}_{31} \otimes e^{-\hat{\sigma}(\zeta)} + 1 \otimes \hat{e}_{31} + \\
&\quad + 2\zeta \hat{h}(\gamma) \otimes \left(\hat{h}(\gamma) - \gamma \hat{h}_\perp\right) e^{-\hat{\sigma}(\zeta)} - \zeta \hat{h}(\gamma) \hat{h}(\gamma) - 2\gamma \hat{h}_\perp \otimes \left(e^{-\hat{\sigma}(\zeta)} - e^{-2\hat{\sigma}(\zeta)}\right) - \\
&\quad + \zeta^2 \hat{h}(\gamma) \hat{e}_{12} \otimes \hat{e}_{23} e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)} - 2\zeta^2 e_{12} e^{\frac{i}{2}(\gamma - 3)\hat{\sigma}(\zeta)} + \\
&\quad + \zeta^2 \hat{e}_{12} \otimes \hat{e}_{23} e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)} - \zeta^2 \hat{e}_{12} \otimes \hat{e}_{23} e^{\frac{i}{2}(\gamma - 2)\hat{\sigma}(\zeta)} - \\
&\quad - 2\zeta^2 \hat{e}_{12} \otimes \left(\hat{h}(\gamma) - \gamma \hat{h}_\perp\right) e_{23} e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)} + \\
&\quad + \zeta^2 \hat{e}_{12}^2 \otimes e_{23}^2 e^{\frac{i}{2}(\gamma - 1)\hat{\sigma}(\zeta)};
\end{align*}
\]

(11)

The standard dualization transforms this costructure into the multiplication for the Hopf algebra $\left(U_x(\mathfrak{sl}(3))^\ast\right)^\ast = \left(\text{Fun}(\mathfrak{gl}(3)^\#)^\ast\right)^\ast = U\left((\mathfrak{sl}(3))^\#\right)$. The latter is thus obtained in terms of $g$-coordinates $\left\{e^\ast_i\right\}$. The same Hopf algebra in terms of $g^\#$-coordinates $\left\{e^\#_i\right\}$ is defined by the relations that generate the necessary ideal in the tensor algebra over $V_{g^\#}$. The deformation parameter $\zeta$ is introduced by the scaling $\xi$. Comparing the associative multiplications in $\left(U\left((\mathfrak{sl}(3))^\#\right)^\ast\right)$ and $\left(U\left((\mathfrak{sl}(3))^\#\right)^\ast\right)$ we find two types of nontrivial relations. The relations of the first type are

\[
\widehat{\sigma}(\zeta) \ast \hat{h}(\gamma) = h(\gamma) \ast \; \ldots \; e^\#_{13}(\zeta) \ast h(\gamma)^\# = h(\gamma)^\# \ast \; \ldots
\]

and

\[
\widehat{\sigma}(\zeta) \ast (\hat{e}_{23} e^{-\hat{\sigma}(\zeta)}) = \frac{i}{2} e_{13} e^{-\hat{\sigma}(\zeta)} \ast \; \ldots \; e^\#_{13}(\zeta) \ast e_{23}^\# = \frac{i}{2} e_{23}^\#.
\]

They signify that the transition to $g^\#$-generators includes the substitutions

\[
e^\#_{13}(\zeta) = \ln (1 + \xi e_{13}),
\]

\[
e^\#_{23} = e_{23} e^{-\hat{\sigma}(\zeta)}.
\]
These transformations are defined in the space $U (g^\#)$ of the carrier subalgebra. They coincide with those induced by the twisting element morphism $F : g \rightarrow g^\#$.

Much more important are the relations on the space $V_a$ complimentary to $V_{g^\#}$. They cannot be extracted from the twisting element itself and depend on the subrepresentation $\text{ad} g^\#|_{V_a}$. In our case there are only two relations of this type obtained from (11) and (7):

$$
\hat{h} (\gamma)^* \cdot \hat{e}_{12} = \left( \hat{h} (\gamma) \cdot \hat{e}_{12} \right)^* + \zeta \hat{e}_{12}^*; \quad h (\gamma)^# \cdot e_{12}^* = \left( h (\gamma)^# \cdot e_{12}^* \right)^*;
$$

$$
\hat{h} (\gamma)^* \cdot \hat{h} (\gamma) = \left( \hat{h} (\gamma) \cdot \hat{h} (\gamma) \right)^*; \quad h (\gamma)^# \cdot h (\gamma)^# = \left( h (\gamma)^# \cdot h (\gamma)^# \right)^* = 2 \left( \left( h (\gamma)^# \right)^2 \right)^* + 2 \zeta \hat{e}_{31}^*;
$$

This gives two nontrivial relations between $(g)^*$- and $(g^\#)^*$- generators

$$
\left( \hat{h} (\gamma) \cdot \hat{e}_{12} \right)^* (\zeta) = \left( h (\gamma)^# \cdot e_{12}^* \right)^* - \zeta e_{32}^* ,
$$

$$
\left( \hat{h} (\gamma) \cdot \hat{h} (\gamma) \right)^* (\zeta) = \frac{1}{2} \left( h (\gamma)^# \cdot h (\gamma)^# \right)^* - \zeta e_{31}^* ,
$$

and correspondingly between $g$- and $g^\#$- generators

$$
e_{32}^# (\zeta) = \hat{e}_{32} - \zeta \hat{h} (\gamma) \hat{e}_{12} ,
$$

$$
e_{31}^# (\zeta) = \hat{e}_{31} - \zeta \hat{h} (\gamma) .
$$

As we have already mentioned the basic elements \( \{ e_\tau^\# \} \) constructed above in terms of the undeformed group coordinates \( \hat{e}_\tau \) are appropriate also for the twisted Hopf algebra. Summing up we get the \( g \rightarrow g^\# \) transformation:

$$
e_{13}^# (\zeta) = \sigma (\zeta) ,
$$

$$
e_{23}^# (\zeta) = e_{23} e^{-\sigma (\zeta)} ,
$$

$$
e_{32}^# (\zeta) = e_{32} - \zeta h (\gamma) e_{12} ,
$$

$$
e_{31}^# (\zeta) = e_{31} - \zeta h (\gamma)^2 .
$$

In terms of the original $g$-coordinates (and the function $\sigma (\xi) = \ln (1 + \xi e_{13})$) the costructure in the twisted algebra $U_F (\mathfrak{sl}(3))$ is defined by the following
Applying to these coproducts the transformation \( (12) \) we get the same costruc-
tions \([9], [13]\):

\[
\Delta_F(h(\gamma)) = h(\gamma) \otimes e^{-\sigma(\xi)} + 1 \otimes h(\gamma) - \xi e_{12} \otimes e_{23} e^{\frac{1}{2}(\gamma-1)e(\gamma)}; \\
\Delta_F(h_{\perp}) = h_{\perp} \otimes 1 + 1 \otimes h_{\perp}; \\
\Delta_F(e_{12}) = e_{12} \otimes e^{\frac{1}{2}(3\gamma-1)e(\gamma)} + 1 \otimes e_{12}; \\
\Delta_F(e_{23}) = e_{23} \otimes e^{-\frac{1}{2}(3\gamma-1)e(\gamma)} + e^{e(\gamma)} \otimes e_{23}; \\
\Delta_F(\sigma(\xi)) = \sigma(\xi) \otimes 1 + 1 \otimes \sigma(\xi); \\
\Delta_F(e_{21}) = e_{21} \otimes e^{-\frac{1}{2}(3\gamma+1)e(\gamma)} + 1 \otimes e_{21} + \xi (1 - \gamma) h_{\perp} \otimes e_{23} e^{-\sigma(\xi)}; \\
\Delta_F(e_{32}) = e_{32} \otimes e^{\frac{1}{2}(3\gamma-1)e(\gamma)} + 1 \otimes e_{32} + \xi h(\gamma) \otimes e_{12} e^{-\sigma(\xi)} + \\
+ \xi e_{12} \otimes (h(\gamma) - (\gamma + 1) h_{\perp}) e^{\frac{1}{2}(3\gamma-1)e(\gamma)} - \\
- \xi h(\gamma) e_{12} \otimes (e^{\frac{1}{2}(3\gamma-1)e(\gamma)} - e^{-\frac{1}{2}(3\gamma-1)e(\gamma)}) - \\
- \xi e_{12} \otimes e_{23} e^{\frac{1}{2}(3\gamma-1)e(\gamma)} - \xi^2 e_{12} \otimes e_{23} e^{(3\gamma-2)e(\gamma)}; \\
\Delta_F(e_{31}) = e_{31} \otimes e^{-\sigma(\xi)} + 1 \otimes e_{31} + \\
+ 2\xi h(\gamma) \otimes (h(\gamma) - \gamma h_{\perp}) e^{-\sigma(\xi)} + \\
+ \xi (h(\gamma) - h(\gamma)^2) \otimes (e^{-\sigma(\xi)} - e^{-\gamma(\gamma)e(\gamma)} + \\
+ \xi e_{12} \otimes e_{23} e^{\frac{1}{2}(3\gamma-1)e(\gamma)} - \\
- \xi e_{12} \otimes e_{23} e^{\frac{1}{2}(3\gamma-1)e(\gamma)} + \\
+ 2\xi e_{12} \otimes e_{23} e^{\frac{1}{2}(3\gamma-5)e(\gamma)} - \xi^2 e_{12} \otimes e_{23} e^{\frac{1}{2}(3\gamma-1)e(\gamma)} - \\
- 2\xi^2 e_{12} \otimes (h(\gamma) - \gamma h_{\perp}) e_{23} e^{\frac{3}{2}(3\gamma-1)e(\gamma)} - \\
- 2\xi^2 h(\gamma) e_{12} \otimes e_{23} e^{\frac{1}{2}(3\gamma-5)e(\gamma)} + \xi^3 e_{12} \otimes e_{23} e^{\frac{1}{2}(3\gamma-1)e(\gamma)}(12)
\]

Applying to these coproducts the transformation \( (12) \) we get the same costruc-
ture in \( \mathfrak{g}^\# \)-coordinates:

\[
\Delta_F(h^\#(\gamma)) = h^\#(\gamma) \otimes e^{-\gamma^\#} + 1 \otimes h^\#(\gamma) - \zeta e_{12} \otimes e_{23} e^{\frac{1}{2}(\gamma-1)e(\gamma)^\#}; \\
\Delta_F(e_{13}^\#) = e_{13}^\# \otimes 1 + 1 \otimes e_{13}^\#; \\
\Delta_F(e_{12}^\#) = e_{12}^\# \otimes e^{\frac{1}{2}(3\gamma-1)e(\gamma)^\#} + 1 \otimes e_{12}^\#; \\
\Delta_F(e_{23}^\#) = e_{23}^\# \otimes e^{-\frac{1}{2}(3\gamma+1)e(\gamma)^\#} + 1 \otimes e_{23}^\#; \\
\Delta_F(h_{\perp}^\#) = h_{\perp}^\# \otimes 1 + 1 \otimes h_{\perp}^\#; \\
\Delta_F(e_{21}^\#) = e_{21}^\# \otimes e^{\frac{1}{2}(3\gamma+1)e(\gamma)^\#} + 1 \otimes e_{21}^\# + \zeta (1 - \gamma) h_{\perp}^\# \otimes e_{23}^\#; \\
\Delta_F(e_{32}^\#) = e_{32}^\# \otimes e^{\frac{1}{2}(3\gamma-1)e(\gamma)^\#} + 1 \otimes e_{32}^\# - \zeta (\gamma + 1) e_{12}^\# \otimes h_{\perp}^\# e^{\frac{1}{2}(\gamma-1)e(\gamma)^\#}; \\
\Delta_F(e_{31}^\#) = e_{31}^\# \otimes e^{-\gamma^\#} + 1 \otimes e_{31}^\# + \\
+ \zeta e_{12}^\# \otimes e_{21}^\# e^{\frac{1}{2}(3\gamma-1)e(\gamma)^\#} - \zeta e_{32}^\# \otimes e_{23}^\# e^{\frac{1}{2}(\gamma-1)e(\gamma)^\#} + \\
- 2\gamma h^\#(\gamma) \otimes h_{\perp}^\# e^{-\gamma^\#} + 2\zeta^2 h^\#(\gamma) e_{12}^\# \otimes h_{\perp}^\# e_{23}^\# e^{\frac{1}{2}(\gamma-1)e(\gamma)^\#} + (13)
\]

This presentation reveals the \( \mathfrak{g}^\# \)-group law in \( U_F(\mathfrak{sl}(3)) \) described in terms of the exponential coordinates. The first four relations correspond to the group multiplication in \( \mathfrak{g}^\# \); the other four expose the adjoint action of \( \mathfrak{g}^\# \) on the 4-dimensional space \( V_a^\# \). In the standard orthonormal basis \( \{e_i\} \) the weights of the diagram

\[
\Gamma^\#_c \cup \Gamma^\#_a = \{0, \varphi_{31}, \varphi_{32}, \varphi_{21}\} \cup \{0, \rho_{31}, \rho_{32}, \rho_{21}\}
\]
have the form:
\[ \varphi_{ab} = \rho_{ab} = e_a - e_b. \]
Consider, for example, the coproduct \( \Delta_F(e^#_{31}) \). The first two expressions are due to the trivial property of the unity and the adjoint action
\[ \left[ e^#_{13}, e^#_{31} \right] = \ldots + e^#_{31} + \ldots \]

The remaining four sets of terms are directly correlated with the weights shifts:
\[ \varphi_{32} \circ \rho_{21} \\
\varphi_{21} \circ \rho_{32} \\
\varphi_{31} \circ \rho \left( h^#_{1} \right) \\
\varphi_{32} \circ \varphi_{21} \circ \rho \left( h^#_{1} \right) \right\} = \rho_{31}. \]

The redeveloped costructure expose explicitly the properties that can be used to find new twist cocycles. First of all it is clearly seen that the parametrized set of dual groups \( G^#(\gamma) \) has tree irregular points, \( \gamma = 0, \pm 1 \).
At the point \( \gamma = 0 \) the coproducts for \( e^#_{31} \) has no terms containing \( h^#_{1} \). The shifts \( \varphi_{31} \circ \rho \left( h^#_{1} \right) \), and \( \varphi_{32} \circ \varphi_{21} \circ \rho \left( h^#_{1} \right) \) result in zeros because in this case \( h^# \) is orthogonal to the weight of \( h^#_{1} \).
The points \( \gamma = +1 (\gamma = -1) \) are especially interesting. In them the coproduct \( \Delta_F(e^#_{32}) \) (correspondingly \( \Delta_F(e^#_{21}) \)) becomes quasiprimitive. Together with the primitivity of \( \Delta_F(h^#) \) this means that in these points the twist equations for the Hopf algebra \( U_F(\mathfrak{sl}(3)) \) have the additional solutions:
\[ F_{\gamma^+} = \exp \left( -\frac{2}{3} h^# \otimes \ln \left( \left( 1 + e^#_{21} \right) e^{2e^#_{13}} \right) \right) \quad \text{for} \quad \gamma = +1, \]
\[ F_{\gamma^+} = \exp \left( +\frac{2}{3} h^# \otimes \ln \left( \left( 1 + e^#_{32} \right) e^{2e^#_{13}} \right) \right) \]
\[ = \exp \left( +\frac{2}{3} h^# \otimes \ln \left( \left( 1 + e^#_{32} - \zeta he_{12} \right) e^{2\sigma_{13}(\zeta)} \right) \right) \quad \text{for} \quad \gamma = -1. \]
Correspondingly for \( U(\mathfrak{sl}(3)) \) we have two parabolic twists:
\[ F_{\psi^+} = F_{\gamma^+} F_\xi \quad \text{for} \quad \gamma = +1, \]
\[ F_{\psi^-} = F_{\gamma^-} F_\xi \quad \text{for} \quad \gamma = -1. \]
This construction generalizes the elementary parabolic twist first presented in [15].
Both twists \( F_{\gamma^+} \) and \( F_{\gamma^-} \) are the deformed Jordanian factors and the canonical form \( F_\gamma = e^{H \otimes \sigma} \) is reobtained when \( \zeta = 0 \). This means that when the preceding extended Jordanian twist is trivialized, \( F_\xi F_\gamma |_{\zeta = 0} = 1 \otimes 1 \), the jordanian deformation \( F_\gamma = e^{\pm \frac{2}{3} h^# \otimes \sigma_{(21)}(32)} \) remains possible with the ordinary function \( \sigma_{(21)}(32) = \ln \left( 1 + e^#_{32} \right) \).
6 Conclusions

We have demonstrated that the dual group coordinates provide the natural basis for the costructure of the twisted universal enveloping algebras. The $g \rightarrow g^#$-transition applied to the twisted algebra $U_F(g)$ simplifies the problem of finding solutions to the twist equations. In the forthcoming publication we shall demonstrate how new classes of solutions are obtained in terms of $g^#$-coordinates.

The dual group approach provides the new insight in the effect of the deformed carrier spaces [11],[14]. In the weight system $\Gamma^F$ the weights $\lambda_{\perp}$ orthogonal to the initial root $\lambda_0$ (of the full extended twist $F$) cannot be reached from the points of $\Gamma^F$ by the shifts corresponding to weights in $\Gamma^F$. The reason is that the system $\Gamma^F$ is located in the negative sector while $\Gamma_{\perp,\lambda_0}$ has the zero level, $\lambda_{\perp}(h_{\lambda_0}) = 0$. For the canonical extended twists (without the Reshetikhin "rotation") this means that the coproducts $\Delta_F(e_{\lambda_{\perp}})$ are primitive. This primitivity is realized only in $g^#$-coordinates. Thus to find the additional twisting factors with the carrier subalgebras in $V_{\perp}$ one must redefine $V_{\perp}$ in terms of $g^#$-basis.

The exponential basis used in this paper can not be considered as universal. When the dual group is not solvable one must use other dual group bases (for example, the matrix coordinates are to be used for the parabolic dual group $G^P$ that contains the simple subgroup in $\mathfrak{S}(n)$).

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7 Appendix

We can check that the limit \( \lim_{\varepsilon, \xi_s \to 0; \xi_s/\varepsilon = \zeta_s} U_{\mathcal{F}(\{\xi_s\})} (g(\varepsilon)) \) exists and corresponds to the Hopf algebra \( \text{Fun}(\mathfrak{g}^\#(\{\zeta_s\})) \). It must be taken into account that the twisting factors \( F_q \) can be deformed by the previous twists. Their form can differ from the canonical one, \( F_J = \exp (h \otimes \sigma) \) for the Jordanian BTF and \( F_E = \exp (\epsilon_{\lambda} \otimes e_{\nu} f(\sigma)) \) for the extension. The expressions \( h \otimes \sigma \) and \( \epsilon_{\lambda} \otimes e_{\nu} f(\sigma) \) are the zero order terms in the expansion of the deformed \( \ln F_{\text{def}} \) and \( \ln F_{\xi_s}^{\text{def}} \) with respect to the deformation parameters of the preceding twists.

Let the parameters \( \xi_s \) be proportional to \( \varepsilon \)

\[ \xi_s = \varepsilon \zeta_s, \]

We shall assume that the logarithms \( \ln F_q \) of the twisting factors \( F_q \) behave as follows

\[ \ln F_q = \frac{1}{\varepsilon} \Psi_q \left( \{\xi_j; \zeta_s\} \right), \quad \Psi \in U \otimes U; \quad s = 1, \ldots, l; q = 1, \ldots, p. \quad (14) \]

It can be directly checked that in the type of deformation that we consider here this condition always holds.
Lemma 2 In the Hopf algebra $U_{\mathcal{F}}(\mathfrak{g})$ twisted by the factorizable twist $\mathcal{F} = F_p F_{p-1} \cdots F_1$ the coproducts $\xi \Delta F^\varepsilon (e_j)$ have the finite limit

$$\lim_{\varepsilon, \varepsilon \rightarrow 0; \varepsilon / \varepsilon = \varepsilon_o} \left( \frac{\xi}{\xi} \Delta F^\varepsilon (e_j) \right) = \Delta_{lim}^\varepsilon F_{i, j} (\xi_j).$$

Proof. Any operator $\text{ad} (\Psi_q (\{ \xi_j; \xi_s \}))$ with respect to $\varepsilon$ acts as a multiplication by a polynomial with strictly positive powers (in particular it multiplies by $\varepsilon$ any tensor belonging to $V_\varepsilon \otimes U(\mathfrak{g})$ or $U(\mathfrak{g}) \otimes V_\varepsilon$). In the limit the twisted coproducts of the generators $\Delta_F (e_j)$ are defined by the action of the BCH $\{ \Psi_q (\{ \xi_j; \xi_s \}); \varepsilon \}$ series that has only positive powers of $\varepsilon$. Finally for any $e_i \in \mathfrak{g}$ we get

$$\lim_{\varepsilon, \varepsilon \rightarrow 0; \varepsilon / \varepsilon = \varepsilon_o} \left( \frac{\xi}{\xi} \Delta F^\varepsilon (e_j) \right) = \lim_{\varepsilon, \varepsilon \rightarrow 0; \varepsilon / \varepsilon = \varepsilon_o} \left( \frac{\xi}{\xi} \Delta F^\varepsilon (e_j) \right) = \lim_{\varepsilon, \varepsilon \rightarrow 0; \varepsilon / \varepsilon = \varepsilon_o} \left( \frac{\xi}{\xi} \Delta F^\varepsilon (e_j) \right) = \Delta_{lim}^\varepsilon F_{i, j} (\xi_i).$$

Corollary 3 Notice that in the limit all terms arising due to rearrangement of the monomials in the PBW basis fade away. Only the zero power term of $\text{BCH}_F (\{ \Psi_q (\{ \xi_j; \xi_s \}); \varepsilon \})$ will give the contribution to the limit value $\Delta_{lim}^\varepsilon F_{i, j} (\xi_i)$, and antisymmetrize the first power terms in the coproducts

$$\delta (\xi_i) = \left. \frac{d}{d\xi} \left( \Delta_{lim}^\varepsilon F_{i, j} (\xi_i) - \tau \circ \Delta_{lim}^\varepsilon F_{i, j} (\xi_i) \right) \right|_{\xi = 0} = \left. \frac{d}{d\xi} \left( \left( \frac{\xi}{\xi} \Delta_{lim}^\varepsilon F_{i, j} (\xi_i) \right) \right) \right|_{\xi = 0} = \left. \frac{d}{d\xi} \left( \left( \frac{\xi}{\xi} \Delta_{lim}^\varepsilon F_{i, j} (\xi_i) \right) \right) \right|_{\xi = 0} = \left( \xi \otimes 1 + 1 \otimes \xi \right).$$

Let the group $\Phi^H (\{ \xi_s \})$ be defined by the commutative Hopf algebra $H$ with the costructure $\left\{ \Delta_{lim}^\varepsilon F_{i, j} (\xi_i) \right\}$. This is the second classical limit of $U_{\mathcal{F}} (\mathfrak{g})$,

$$H \left( \text{Ab}, \Delta_{lim}^\varepsilon F_{i, j} (\xi_i) \right) \approx \text{Fun} (\Phi^H (\{ \xi_s \})).$$

It is sufficient to check that the corresponding Lie algebras are equivalent. The Lie coalgebra of the group $\Phi^H (\{ \xi_s \})$ is constructed in an ordinary way: we put all the parameters proportional to $\xi$

$$\xi_s = \xi_s \xi,$$

and antisymmetrize the first power terms in the coproducts
The functions $\Psi_{(F_q)}(\{\hat{e}_j; \xi_s\xi\})$ corresponding to the twisting factors $F_q$ in $\mathcal{F}$ obey the rule:

$$\frac{d}{d\xi}|_{\xi=0} \sum_q \left( \Psi_{(F_q)}(\{\hat{e}_j; \xi_s\xi\}) - \tau \circ \Psi_{(F_q)}(\{\hat{e}_j; \xi_s\xi\}) \right) = r_{\mathcal{F}}(\{\xi_s\}).$$

Thus the Lie coalgebra of the group $\mathfrak{G}^H(\{\xi_s\})$ described by $H \left( \text{Ab}, \Delta_{\lim}^{\mathcal{F},(\xi_s)} \right)$ is fixed by the same relations as those that define $\mathfrak{g}^\#(\{\xi_s\})$,

$$\delta(\xi_i) = \left[ r_{\mathcal{F}}(\{\xi_s\}), \Delta^{(0)}(\xi_i) \right].$$

The groups $\mathfrak{G}^H(\{\xi_s\})$ and $\mathfrak{g}^\#(\{\xi_s\})$ are simply connected and have the same Lie algebra $\mathfrak{g}^\#(\{\xi_s\})$.

**Remark 4** It is essential to check the algebra $\mathfrak{g}^H(\{\xi_s\})$. Varying the behaviour of the deformation parameters we can involve the additional limit procedures and obtain various contracted groups $\lim \mathfrak{g}^\#(\{\xi_s\})$ corresponding to different boundaries of the initial parametrized set. This is the uniform character of the parameters with respect to $\xi$ that guarantees the isomorphism $\mathfrak{G}^H(\{\xi_s\}) \approx \mathfrak{g}^\#(\{\xi_s\})$. 