New bounds on the Lebesgue constants of Leja sequences on the unit disc and their projections $\mathcal{R}$-Leja sequences

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Abstract

Motivated by the development of non-intrusive interpolation methods for parametric PDE’s in high dimension, we have introduced in [8] a sparse multi-variate polynomial interpolation procedure based on the Smolyak formula. The evaluation points lie in an infinite grid $\otimes_{j=0}^{d} Z$ where $Z = (z_j)_{j \geq 0}$ is any infinite sequence of mutually distinct points in some compact $X$ in $\mathbb{R}$ or $\mathbb{C}$. A key aspect of the interpolation procedure is its hierarchical structure: the sampling set is progressively enriched together with the polynomial space. The Lebesgue constant that quantifies the stability of the resulting interpolation operator depends on $Z$. We have shown in [6] that the interpolation operator has Lebesgue constant with quadratic and cubic growth in the number of points when $Z$ is a Leja sequence on the unit disc initiated at the boundary and where $Z$ is an $\mathcal{R}$-Leja sequence on $X = [-1, 1]$ respectively. This result followed from the linear and quadratic growth of the Lebesgue constant for univariate polynomial interpolation with the previous type of sequences, also established in [6]. In this paper, we derive new properties of these sequences and new bounds on the growth of the Lebesgue constants which improve those obtained in [6].

1 Introduction

1.1 Interpolation on nested sets of points

This paper is concerned with the study of Leja sequences on the unit disc $U := \{|z| \leq 1\}$ initiated at the boundary $\partial U$ and $\mathcal{R}$-Leja sequences in the unit interval $[-1, 1]$, which are both known to have moderate algebraic in $k$ growth of the Lebesgue constant associated with the $k$-section $(z_0, \ldots, z_{k-1})$. Our main interest in such sequence comes from the study
of the stability of the interpolation process that we have introduced in [8] where the key motivation is the development of cheap and stable non-intrusive methods for high dimensional parametric PDE’s. We refer to the introduction and section 2 in [6] for a concise description on the construction of the interpolation process and the study of its stability.

Given $X$ a compact domain in $\mathbb{C}$ or $\mathbb{R}$, typically the complex unit disc $\mathcal{U}$ or the unit interval $[-1, 1]$, and $Z = (z_j)_{j \geq 0}$ a sequence of mutually distinct points in $X$, we denote by $I_k$ the univariate interpolation operator onto $\mathbb{P}_k$ associated with the section $Z_{k+1} := (z_0, \cdots, z_k)$. Since the sections are nested, it is convenient to express $I_k$ as the telescoping sum

$$I_k = \sum_{l=0}^{k} \Delta_l, \quad \text{where} \quad \Delta_k := I_k - I_{k-1}, \quad (1.1)$$

with the convention that $I_{-1} = 0$, which corresponds to the Newton interpolation formula of the operator $I_k$. The stability of the operator $I_k$ depends on the positions of the points of $Z_{k+1}$ on $X$, in particular through the Lebesgue constant associated with $Z_{k+1}$, defined by

$$L_k := \max_{f \in C(X) \setminus \{0\}} \frac{\|I_k f\|_{L^\infty(X)}}{\|f\|_{L^\infty(X)}} = \max_{z \in X} \lambda_{Z_k}(z), \quad (1.2)$$

where $\lambda_k$ is the so-called Lebesgue function associated with $Z_{k+1}$ defined over $X$ by

$$\lambda_k(t) := \sum_{i=0}^{k} |l_{i,k}(t)|, \quad \text{with} \quad l_{i,k}(z) := \prod_{j \neq i}^{k} \frac{z - z_j}{z_i - z_j}, \quad \text{for} \quad i = 0, \ldots, k, \quad (1.3)$$

We also introduce the notation

$$D_k := \max_{f \in C(X) \setminus \{0\}} \frac{\|\Delta_k f\|_{L^\infty(X)}}{\|f\|_{L^\infty(X)}}, \quad (1.4)$$

for the quantity that quantifies the stability of the difference operator $\Delta_k$.

The study of the stability of the high-dimensional interpolation process described in [8, 6] motivates the search for “good” univariate sequences $Z$ of points on $X$ such that $L_k$ or $D_k$ have moderate algebraic growth, in the dimension of the polynomial space $\mathbb{P}_k$, controlled for example by $(1 + k)^{\theta}$ for a small $\theta$. In the case of the unit disk or the unit interval, it is known that $L_k$ can not grow slower than $\frac{2}{\pi} \log(k)$ and it is well known that the Lebesgue constant grows like $\frac{2}{\pi} \log(k)$ for the $\{k + 1\}$-roots of unity and the $(k + 1)$ Tchebyshev or Gauss-Lobatto abscissas in $[-1,1]$, see [2]. However such sets of points are not the sections of a fixed sequence $Z$.

In [3, 4], the authors considered Leja sequences on $\mathcal{U}$ initiated at the boundary $\partial \mathcal{U}$ and $\mathcal{R}$-Leja sequences obtained by projection onto $[1,1]$ of the latter when initiated at 1. They showed that the growth of $L_k$ is controlled by $O((k + 1) \log(k + 1))$ and $O((k + 1)^3 \log(k + 1))$
respectively. In our previous works \cite{5,6}, we have improved these bounds to \(2(k+1)\) and \(8\sqrt{2}(1+k)^2\) respectively. We have also established in \cite{6}, the bound \(D_k \leq (1+k)^2\) for the difference operator, which could not be obtained directly from \(D_k \leq L_k + L_{k-1}\), and which is essential to prove that the multivariate interpolation operator using \(\Re\)-Leja sequences has a cubic Lebesgue constant.

In this paper, we consider again Leja and \(\Re\)-Leja sequence and study the growth of their Lebesgue constants. Our techniques of proof share several common points with those developed in \cite{5,6}, yet they are shorter and exploit to a considerable extent the properties of Leja sequences on the unit disk. The novelty in the present paper is the introduction of the “quadratic” Lebesgue constant

\[
\lambda_{k,2}(z) := \left(\sum_{i=0}^{k} |l_i(z)|^2\right)^{\frac{1}{2}}, \quad z \in X,
\]

which we study for Leja sequences on \(U\). We establish that the maximum of this function

\[
\mathbb{L}_{k,2} := \max_{z \in X} \lambda_{k,2}(z).
\]

satisfies for Leja sequence on \(U\),

\[
\mathbb{L}_{k,2} \leq 3\sqrt{2\sigma_1(k+1)} - 1,
\]

where \(\sigma_1(l)\) denote the number of ones in the binary expansion of \(l\). Applying Cauchy-schwartz inequality to \(\lambda_k\), this implies that

\[
\mathbb{L}_k \leq 3\sqrt{k+1} \sqrt{2\sigma_1(k+1)} - 1,
\]

for Leja sequence on \(U\), which improves the bound \(2(k+1)\) when the binary expansion of \(k+1\) is very sparse. Using also the bound on \(\mathbb{L}_{k,2}\) for Leja sequence on \(U\), we establish a bound on the growth of Lebesgue constant of \(\Re\)-Leja sequence that implies

\[
L_k \leq 6\sqrt{5}(k+1)2^{\sigma_1(k')}, \quad k' = k - 2^n
\]

where \(n\) is the integer such that \(2^n \leq k+1 < 2^{n+1}\). Again, we remark that the previous bound improves the bound \((k+1)^2\) established in \cite{6} when \(k'\) is small compared to \(2^n\) or very sparse in the sense of binary expansion.

In the remainder of the paper, we work rather with sections of length \(k\), more precisely given a sequence \(Z = (z_j)_{j \geq 0}\) of pairwise distinct points in \(X\), we study the growth of the Lebesgue constants of the \(k\)-sections \(Z_k := (z_0, \ldots, z_{k-1})\). In order to avoid confusion with the previous notations which deal rather with \(Z_{k+1} := (z_0, \ldots, z_k)\) we denote, when needed, by \(I_{Z_k}\) the interpolation operator associated with \(Z_k\), by \(\lambda_{Z_k}\) and \(\lambda_{Z_k,2}\) the Lebesgue and quadratic Lebesgue functions associated with \(Z_k\) and by \(\mathbb{L}_{Z_k}, \mathbb{L}_{Z_k,2}\) and the constants associated with \(Z_k\).
1.2 Notation

For an infinite sequence \( Z := (z_j)_{j \geq 0} \) on \( X \), we introduce the sections notation

\[
Z_k := (z_0, \ldots, z_{k-1}) \quad \text{and} \quad Z_{l,m} := (z_l, \ldots, z_{m-1}), \quad l \leq m - 1.
\] (1.10)

Given two finite sequence \( A = (a_0, \ldots, a_{k-1}) \) and \( B = (b_0, \ldots, b_{l-1}) \), we denote by \( A \land B \) the concatenation of \( A \) and \( B \), i.e.

\[
A \land B = (a_0, \ldots, a_{k-1}, b_0, \ldots, b_{l-1}).
\] (1.11)

For any finite set \( Z = (z_0, \ldots, z_{k-1}) \) of complex numbers and \( \rho \in \mathbb{C} \), we introduce the notation

\[
\rho Z := (\rho z_0, \ldots, \rho z_{k-1}), \quad \Re(Z) := (\Re(z_0), \ldots, \Re(z_{k-1})), \quad \overline{Z} := (\overline{z_0}, \ldots, \overline{z_{k-1}}).
\] (1.12)

Throughout this paper, to any finite set \( S \) of numbers, we associate the polynomial

\[
w_S(z) := \prod_{s \in S} (z - s) \quad \text{with the convention} \quad w_{\emptyset}(z) := 1
\] (1.13)

Any integer \( k \geq 1 \) can be uniquely expanded according to

\[
k = \sum_{j=0}^{n} a_j 2^j, \quad a_j \in \{0, 1\}
\] (1.14)

We denote by \( \sigma_1(k) \), \( \sigma_0(k) \) the number of ones and zeros in the binary expansion of \( k \) and by \( p_0(k) \) and \( q_0(k) \) the largest integers such that \( 2^p \) divide \( k \) and \( 2^q \) divide \( k + 1 \). For \( k = 2^n, \ldots, 2^{n+1} - 1 \) with binary expansion as above, one has

\[
\sigma_1(k) = \sum_{j=0}^{n} a_j \quad \text{and} \quad \sigma_0(k) = \sum_{j=0}^{n} (1 - a_j) = n + 1 - \sigma_1(k).
\] (1.15)

Also one has

\[
p_0(k) = \inf\{j = 0, \ldots, n : a_j = 1\} \quad \text{and} \quad q_0(k) = \inf\{j = 0, \ldots, n : a_j = 0\}
\] (1.16)

2 Leja sequences on the unit disk

Given \( X \) a compact in \( \mathbb{R} \) or \( \mathbb{C} \), a Leja sequences \( A := (a_j)_{j \geq 0} \) is defined recursively according to: \( a_0 \in X \) arbitrary and \( a_k \) satisfies

\[
|a_k - a_0| \ldots |a_k - a_{k-1}| = \max_{z \in X} |z - a_0| \ldots |z - a_{k-1}|.
\] (2.1)
Numerical evidence shows that such sequences have moderate growth of the Lebesgue constant in the case $X = [-1, 1]$, the bound $L_{A_k} \leq k$ seems valid, see [5]. However, no rigorous proof supports this evidence. It is only known that the growth of the Lebesgue constants is sub-exponential, i.e. $(L_{A_k})^{\frac{1}{k}} \to k \to \infty 0$, see [17].

Leja sequences $E = (e_j)_{j \geq 0}$ on $U$ considered in [3, 4, 5, 6] have all their initial value $e_0 \in \partial U$ the unit circle. They are defined inductively by: pick $e_0 \in \partial U$ arbitrary and for $k \geq 1$

$$e_k = \arg \max_{z \in \partial U} |z - e_{k-1}| \ldots |z - e_0|. \quad (2.2)$$

The maximum principle implies that $e_j \in \partial U$ for any $j \geq 1$. Also, the previous argmax problem might admit many solutions and $e_k$ is one of them. We call a $k$-Leja section every finite sequence $(e_0, \ldots, e_{k-1})$ obtained by the same recursive procedure. In particular, when $E := (e_j)_{j \geq 1}$ is a Leja sequence then the section $E_k = (e_0, \ldots, e_{k-1})$ is $k$-Leja section.

In contrast to the interval $[-1, 1]$ where even the first points of a Leja sequence cannot be computed explicitly, Leja sequences on $\partial U$ are much easier to compute. For instance, suppose that $e_0 = 1$, then we can immediately check that $e_1 = -1$ and $e_2 = \pm i$. Assume that $e_2 = i$ then $e_3$ maximises $|z^2 - 1||z - i|$, so that $e_3 = -i$ because $-i$ maximizes jointly $|z^2 - 1|$ and $|z - i|$. Then $e_4$ maximizes $|z^4 - 1|$, etc... We observe a binary pattern on the distribution of the first elements of $E$.

Since the element of $\partial U$ have all the same modulus 1, then an arbitrary Leja sequence $E = (e_0, e_1, \ldots)$ on $\partial U$ is merely the product by $e_0$ of a Leja sequence with initial value 1. The latter are completely determined according to the following theorem, see [11, 3, 5].

**Theorem 2.1** Let $n \geq 0$, $2^n < k \leq 2^{n+1}$ and $l = k - 2^n$. The sequence $E_k = (e_0, \ldots, e_{k-1})$, with $e_0 = 1$, is a $k$-Leja section if and only if $E_{2^n} = (e_0, \ldots, e_{2^n-1})$ and $U_l = (e_{2^n}, \ldots, e_{k-1})$ are respectively $2^n$-Leja and $l$-Leja sections and $e_{2^n}$ is any $2^n$-root of $-1$.

The most natural construction of a Leja sequence in $\partial U$ consists then in $E := (e_j)_{j \geq 0}$ inductively defined by

$$E_1 := (e_0 = 1) \quad \text{and} \quad E_{2^n+1} := E_{2^n} \land e^{i\pi} E_{2^n}, \quad n \geq 0. \quad (2.3)$$

This recursive construction of the sequence $E$ yields an interesting distribution of its elements. Indeed, by an immediate induction, see [11], it can be shown that the elements $e_k$ are given by

$$e_k = \exp \left(i \pi \sum_{j=0}^{n} a_j 2^{-j} \right) \quad \text{for} \quad k = \sum_{j=0}^{n} a_j 2^j, \quad a_j \in \{0, 1\}. \quad (2.4)$$

The construction yields then a low-discrepancy sequence on $\partial U$ based on the bit-reversal Van der Corput enumeration. This sequence was known to be a Leja sequence over $\partial U$ in many earlier works.
As stated above, Theorem 2.1 characterizes completely Leja sequences on the unit circle. It has many implications that turn out to be very useful in the analysis of the growth of Lebesgue constants. We have

**Theorem 2.2** Let $E := (e_j)_{j \geq 0}$ be a Leja sequence on $U$ initiated at $e_0 \in \partial U$. We have:

- For any $n \geq 0$, $E_{2^n} = e_0 U_{2^n}$ in the set sense where $U_{2^n}$ is the set of $2^n$-root of unity.
- For any $k \geq 1$, $|w_{E_k}(e_k)| = \sup_{z \in \partial U} |w_{E_k}(z)| = 2^{\sigma_1(k)}$.
- For any $n \geq 0$, $E_{2^n, 2^{n+1}} := (e_{2^n}, \ldots, e_{2^{n+1}-1})$ is a $2^n$-Leja section.
- For any $n \geq 0$, $B(E_{2^n}) := (e_{2^n-1}, \ldots, e_1, e_0)$ is a $2^n$-Leja section.
- The sequence $E^2 := (e_{2^j})_{j \geq 0}$ is a Leja sequence on $\partial U$.

The proofs of these properties can be found in [3, 5, 6].

It is proved in [5] that given two $k$-Leja sections $E_k$ and $F_k$, then $F_k = \rho E_k$ in the set sense for some $\rho \in \partial U$. This means that the sequence $F_k$ can be obtained from $E_k$ by a permutation and the product by $\rho$. By inspection of quadratic Lebesgue function, we have then that

$$\lambda_{F_k, 2}(z) = \lambda_{E_k, 2}(z/\rho), \quad z \in U \quad \Longrightarrow \quad \mathbb{L}_{F_k, 2} = \mathbb{L}_{E_k, 2}. \quad (2.5)$$

In order to compute the growth of $\mathbb{L}_{E_k, 2}$ for arbitrary Leja sequences $E$, it suffices then to consider $E$ to be the simple sequence given by (2.4). Unless stated otherwise, for the rest of this section, $E$ is exclusively used for this notation. Let us note that with such sequence, we have

$$E^2 = E \quad (2.6)$$

In order to study the functions $\lambda_{E_k, 2}$, we adopt the methodology that we introduced in [5]. Namely, we study the implication of $E$ being a Leja sequence in general on the growth of $\lambda_{E_k, 2}$, then we use the implication of the particular binary distribution of Leja sequences on $U$.

**Theorem 2.3** Let $E$ be a Leja sequence on a real or complex compact $X$. For any $k \geq 1$ and any $z \in X$, it holds

$$\lambda_{E_{k+1}, 2}(z) \leq \lambda_{E_k, 2}(z) + \left(\lambda_{E_k, 2}(e_k) + 1\right). \quad (2.7)$$
**Proof:** We fix \( k \geq 1 \) and denote \( l_0, \ldots, l_{k-1} \) the Lagrange polynomials associated with the section \( E_k \) and \( L_0, \ldots, L_k \) the Lagrange polynomials associated with the section \( E_{k+1} \). By Lagrange interpolation formula, for \( j = 0, \ldots, k-1 \)

\[
l_j(z) = \sum_{i=0}^{k} l_j(e_i)L_i(z) = L_j(z) + l_j(e_k)L_k(z),
\]

Therefore \( u(z) = v(z) - L_k(z)v(e_k) \) where we have introduced the notation

\[
v(z) := (l_0(z), \ldots, l_{k-1}(z)) \in \mathbb{C}^k \quad \text{and} \quad u(z) := (L_0(z), \ldots, L_{k-1}(z)) \in \mathbb{C}^k.
\]

We have that

\[
\|u(z)\| \leq \|v(z)\| + |L_k(z)||v(e_k)|,
\]

where \( \| \cdot \| \) is the euclidean norm in \( \mathbb{C}^k \). This implies

\[
\lambda_{E_{k+1},2}(z) = \left(\|u(z)\|^2 + |L_k(z)|^2\right)^{\frac{1}{2}} \leq \|u(z)\| + |L_k(z)| \leq \|v(z)\| + \left(\|v(e_k)\| + 1\right)|L_k(z)|.
\]

Since \( \|v(z)\| = \lambda_{E_k,2}(z) \) and \( \|v(e_k)\| = \lambda_{E_k,2}(e_k) \) and by the Leja definition

\[
|L_k(z)| = \frac{|w_{E_k}(z)|}{|w_{E_k}(e_k)|} \leq 1,
\]

the proof is then complete. 

The previous result shows that the growth of \( \mathbb{L}_{E_{k+1}} \) is monitored by the growth of \( \lambda_{E_k,2}(e_k) \). In particular, it is easily checked by induction on (2.7) the that

\[
\lambda_{k,2}(e_k) = \mathcal{O}(\log(k)) \implies \mathbb{L}_{E_{k+1}} = \mathcal{O}(k \log(k)), \quad (2.8)
\]

and

\[
\lambda_{k,2}(e_k) = \mathcal{O}(k^\theta) \implies \mathbb{L}_{E_{k+1}} = \mathcal{O}(k^{\theta+1}). \quad (2.9)
\]

In the following, we show that the previous identity holds with \( \theta = 1/2 \) and we use the particular structure of Leja sequences in order to show that the exponent \( \theta = 1/2 \) is not deteriorated and is valid also for \( \mathbb{L}_{E_{k+1}} \).

First, Let us note that the binary pattern of Leja sequences on the unit disk yields the following result.

**Lemma 2.4** Let \( E \) be as in (2.4). For any \( k \geq 1 \), one has

\[
\lambda_{E_{2k},2}(z) = \lambda_{E_{k},2}(z^2), \quad z \in \partial U. \quad (2.10)
\]

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Lemma 2.5

For the Leja sequence \( E \), there is a number \( \lambda \) such that

\[
\lambda = \lim_{n \to \infty} \frac{\log |E_n|}{|E_n|}
\]

Proof: Let \( l_0, \ldots, l_{2k-1} \) the Lagrange polynomials associated with \( E_{2k} \) and \( L_0, \ldots, L_{k-1} \), the Lagrange polynomials associated with \( E_k \). Since \( e_{2j+1} = -e_{2j} \) for any \( j \geq 0 \), then we have

\[
w_{E_{2k}}(z) = w_{E_k}^2(z^2) = w_{E_k}(z^2),
\]

therefore for any \( j \geq 0 \)

\[
|w_{E_{2k}}'(e_{2j+1})| = |w_{E_{2k}}'(e_{2j})| = 2|w_{E_k}'(e_{2j})| = 2|w_{E_k}'(e_j)|,
\]

where we have used \((e_{2j+1})^2 = (e_{2j})^2 = e_j\). We have for \( j = 0, \ldots, k - 1 \)

\[
|l_{2j}(z)| = \frac{|w_{E_{2k}}(z)|}{|w_{E_{2k}}(e_{2j})||z - e_{2j}|}, \quad |l_{2j+1}(z)| = \frac{|w_{E_{2k}}(z)|}{|w_{E_{2k}}(e_{2j+1})||z - e_{2j+1}|},
\]

therefore in view of the previous equalities

\[
|l_{2j}(z)|^2 + |l_{2j+1}(z)|^2 = \frac{|w_{E_k}(z^2)|^2}{4|w_{E_k}'(e_j)|^2 \left( \frac{1}{|z - e_{2j}|^2} + \frac{1}{|z + e_{2j}|^2} \right)} = \frac{|w_{E_k}(z^2)|^2}{|w_{E_k}'(e_j)|^2} \frac{1}{|z^2 - e_j|^2} = |L_j(z^2)|,
\]

where we have used \(|a - b|^2 + |a + b|^2 = 4\) for \( a, b \in \mathbb{U} \). Summing the previous identities for the indices \( j = 0, \ldots, k - 1 \), we get the result. \( \blacksquare \)

We now turn to the growth of \( \lambda_{E_k,2}(e_k) \).

Lemma 2.5 For the Leja sequence \( E \) defined in (2.11), we have for any \( k \geq 1 \),

\[
\lambda_{E_k,2}(e_k) = \sqrt{2^{x_1(k)} - 1}
\]

Proof: First, by Lemma [2.4], \( \lambda_{E_{2N},2}(e_{2N}) = \lambda_{E_{2N},2}(e_{2N}) = \lambda_{E_{2N},2}(e_N) \). Now assume \( k \) odd and write \( k = 2N + 1 \) with \( N \geq 1 \). Let \( l_0, \ldots, l_{2N} \) the Lagrange polynomials associated with \( E_k \) and \( L_0, \ldots, L_{N-1} \) the Lagrange polynomials associated with \( E_N \). For any \( j = 0, \ldots, 2N \), we have

\[
l_j(e_k) = \frac{w_{E_k}(e_k)}{(e_k - e_j)w_{E_k}'(e_j)} = \frac{w_{E_{k+1}}(e_k)}{w_{E_{k+1}}'(e_j)}.
\]

We have \( k + 1 = 2(N + 1) \), therefore in view of (2.11), for \( j = 0, \ldots, N - 1 \) we have

\[
|l_{2j}(e_k)| = \frac{2|w_{E_{N+1}}'(e_{2j}^2)|}{2|w_{E_{N+1}}'(e_{2j})|} = \frac{|w_{E_{N+1}}'(e_N)|}{|w_{E_{N+1}}'(e_j)|} = |L_j(e_N)|
\]

The same equality holds with \( 2j + 1 \) instead of \( 2j \). Now for \( 2N \), we have

\[
|l_{2N}(e_k)| = \frac{2|w_{E_{N+1}}'(e_{2N}^2)|}{2|w_{E_{N+1}}'(e_{2N})|} = \frac{|w_{E_{N+1}}'(e_N)|}{|w_{E_{N+1}}'(e_N)|} = 1.
\]
Therefore
\[ (\lambda_{E_k,2}(e_k))^2 = 1 + 2(\lambda_{E_N,2}(e_N))^2. \]
The sequence \( \alpha := (\alpha_k = (\lambda_{E_k,2}(e_k))^2))_{k \geq 1} \) satisfies the following recursion: \( \alpha_1 = 1 \) and
\[ \alpha_{2N} = \alpha_N, \quad \alpha_{2N+1} = 1 + 2\alpha_N, \quad N \geq 1 \]
We have \( \sigma_1(2N) = \sigma_1(N) \) and \( \sigma_1(2N + 1) = \sigma_1(N) + 1 \) for \( N \geq 1 \). It is then easily checked that \( (2^{\sigma_1(k)} - 1)_{k \geq 1} \) satisfies the same recursion as \( \alpha \). This shows that \( \alpha_k = 2^{\sigma_1(k)} - 1 \) for any \( k \geq 0 \) and finishes the proof.

We are now able to conclude the main result of this section

**Theorem 2.6** For the Leja sequence \( E \) defined in (2.4), we have for any \( k \geq 1 \)
\[ \mathbb{I}_{E_k,2} \leq 3\sqrt{2^{\sigma_1(k)} - 1} \]

**Proof:** We provide a proof by induction on \( k \geq 1 \). We have
\[ \mathbb{I}_{E_1,2} = 1, \quad \mathbb{I}_{E_{2},2} = \mathbb{I}_{E_1,2} = 1, \quad \mathbb{I}_{E_{3},2} = 3, \quad \mathbb{I}_{E_{4},2} = \mathbb{I}_{E_{2},2} = 1. \]
Since \( \sigma_1(1) = \sigma_1(2) = \sigma_1(4) = 1 \) and \( \sigma_1(3) = 2 \), then (2.14) is satisfied for \( k = 1, \ldots, 4 \). Let \( N \geq 1 \) and assume the inequality valid for any \( j \leq 2N - 1 \). The bound holds for \( k = 2N \) by the induction hypothesis since the quantities in (2.14) are invariant for even numbers. Now, we assume \( k = 2N + 1 \), by Theorem 2.3, we have for any \( z \in \mathcal{U} \)
\[ \lambda_{E_{2N+1},2}(z) \leq \lambda_{E_{2N},2}(z) + \lambda_{E_{2N},2}(e_{2N}) + 1 = \lambda_{E_{N},2}(z^2) + \lambda_{E_{N},2}(e_{N}) + 1. \]
Therefore with \( \alpha_N = 2^{\sigma_1(N)} - 1 \), we have by the induction hypothesis and (2.12) that
\[ \mathbb{I}_{2N+1,2} \leq \mathbb{I}_{2N,2} + 3\sqrt{\alpha_N} + 1 = 4\sqrt{\alpha_N} + 1 \]
which is smaller than \( 3\sqrt{2\alpha_N + 1} \) for any value \( \alpha_N \geq 0 \) by elementary function study. Since \( 2\alpha_N + 1 = 2^{\sigma_1(2N+1)} - 1 \), then the induction is complete.

**Remark 2.7** The previous result has an implication on the growth of Lebesgue constant associated with \( E_k \). Indeed, by Cauchy Schwartz inequality, we have \( \lambda_{E_k}(z) \leq \sqrt{k} \lambda_{E_k,2}(z) \) for any \( z \in \mathcal{U} \), therefore
\[ \mathbb{I}_{E_k} \leq \sqrt{k} \mathbb{I}_{E_k,2} \leq 3\sqrt{k} \sqrt{2^{\sigma_1(k)} - 1} \leq 3\sqrt{k2^{\sigma_1(k)}} \]
For \( k = 2^n, \ldots, 2^{n+1} - 1 \), we have \( \sigma_1(k) = n + 1 - \sigma_0(k) \), therefore
\[ \mathbb{I}_{E_k} \leq \sqrt{\frac{18}{2^{\sigma_0(k)}}} \sqrt{2^n k} \leq \sqrt{\frac{18}{2^{\sigma_0(k)}}} k. \]
This shows in particular that \( \mathbb{I}_{E_k} \leq k \) whenever \( \sigma_0(k) \geq 5 \). This last result answers partly the conjecture raised in [3] and which states that \( \mathbb{I}_{E_k} \leq k \) for any \( k \geq 1 \).
In the following, we establish a result that we make use of in the next section. Let $E = (e_j)_{j \geq 0}$ be any Leja sequence on the unit disk initiated at $e_0 = 1$. For $n \geq 0$ and $0 \leq l \leq 2^n$, we denote $H_l = E_{2^{n+1},2^{n+1}+l}$ and introduce the quantity

$$\beta_{n,l}(E) = \sum_{j=0}^{2^n+1+l-1} \frac{1}{|w_{H_l}(e_j)|^2} \tag{2.17}$$

The quantity $\beta_{n,l}(E)$ is well defined. Indeed by the particular structure of Leja sequences, $H_l$ is a section of $E_{2^{n+1},2^{n+1}+2^n}$ hence $H_l$ is a subset of $2^n$-root of $\alpha$ with $\alpha = \pm i$, while for $j = 0, \ldots, 2^n + l - 1$, we have $(e_j)^{2^n} \in \{1, -1, -\alpha\}$. We have the following result

Lemma 2.8 For any Leja sequence $E$ initiated at $1$ and for any $n \geq 0$ and $1 \leq l \leq 2^n$, we have

$$\beta_{n,l}(E) \leq \frac{5}{2}^{2n-\sigma_1(l)-p_0(l)} \tag{2.18}$$

Proof: Since $e_0 = 1$, $e_1 = -1$, $e_2 = \pm i$, it can be checked that $\beta_{0,1}(E) = 5/4$. We now fix $n \geq 1$. If $l = 2N$, we have for any $j = 0, \ldots, 2^n + N - 1$

$$w_{H_l}(e_{2j+1}) = w_{H_l}(e_{2j}) = w_{E_{2^n,2^n+N}}(e_{2j})$$

where $E^2 := (e_{2j})_{j \geq 0}$. Pairing the indices in (2.17) as $2j$ and $2j+1$ with $j = 0, \ldots, 2^n + N - 1$, we deduce

$$\beta_{n,l}(E) = 2\beta_{n-1,N}(E^2).$$

If $l = 2N + 1$, we may write

$$\beta_{n,l}(E) = \sum_{j=0}^{2^n+1+l-1} \frac{|e_{2^n+1+l} - e_j|^2}{|w_{H_{l+1}}(e_j)|^2} \leq \sum_{j=0}^{2^n+1+(l+1)-1} \frac{|e_{2^n+1+l} - e_j|^2}{|w_{H_{l+1}}(e_j)|^2} = 4\beta_{n-1,N+1}(E^2),$$

where we have paired the indices by $2j$ and $2j+1$ and used $e_{2j+1} = -e_{2j}$ and the identity $|a + b|^2 + |a - b|^2 = 4$ for any $a, b \in \partial U$. Let us note that $E^2$ is also a Leja sequence. Therefore $\beta_{n,l}(E) \leq \frac{5}{4} \cdot a_{n,l}$ where $(a_{n,l})_{n \geq 0,1 \leq l \leq 2^n}$ is the sequence that saturates the previous inequalities and hence is defined by

$$a_{0,1} = 1, \quad \text{and} \quad a_{n,2N} = 2a_{n-1,N}, \quad a_{n,2N+1} = 4a_{n-1,N+1}, n \geq 1$$

For $n \geq 1$ and $0 \leq l \leq 2^n$, either $l = 2N$ in which case $0 \leq N \leq 2^{n-1}$ or $l = 2N+1$ in which case $0 \leq N + 1 \leq 2^{n-1}$. Moreover

$$\sigma_1(2N) + p_0(2N) = \sigma_1(N) + p_0(N) + 1$$

and

$$\sigma_1(2N + 1) + p_0(2N + 1) = \sigma_1(2N + 1) = \sigma_1(N) + 1 = \sigma_1(N + 1) + p_0(N + 1).$$

Therefore it can be checked by induction that $a_{n,l} = 2^{2n+1-\sigma_1(l)-p_0(l)}$, which finishes the proof.
3 \( \mathcal{R} \)-Leja sequences on \([-1, 1]\)

We consider a Leja sequence \( E = (e_j)_{j \geq 0} \) on the unit disk with \( e_0 = 1 \) and project it onto the real interval \([-1, 1]\) and denote by \( R = (r_j)_{j \geq 0} \) the sequence obtained. Since \( E = (1, -1, \alpha, -\alpha, \cdots) \) with \( \alpha = \pm i \), one should make sure that no point is repeated on \( R \) simply by not projecting a point \( e_j \) such that \( e_j = e_i \) for some \( i < j \). Such sequences \( R \) were named \( \mathcal{R} \)-Leja sequences in [4]. The projection rule that prevent the repetition is well understood. It was explained in [4, Theorem 2.4] and it is given by

**Theorem 3.1** Let \( E \) be a Leja sequence on \( U \) with \( e_0 = 1 \) and \( R \) the associated \( \mathcal{R} \)-Leja sequence. Then

\[
R = \mathbb{R}(\Xi), \quad \text{with} \quad \Xi = (\xi_j)_{j \geq 0} := (1, -1) \wedge \bigwedge_{j=1}^{\infty} E_{2^j, 2^{j+1}}. \tag{3.1}
\]

This mean that in order to obtain \( R \), one projects element-wise \( E \) into \([-1, 1]\) ignoring the sections \( E_{2^j, 2^{j+1}} \).

Given \( E = (e_j)_{j \geq 1} \) a Leja sequence initiated at \( e_0 = 1 \), we have \( e_1 = -1, \ e_2 = -e_3 = \pm i \) and \( e_2j = -e_{2j+1} \) for any \( j \geq 2 \), therefore the associated \( \mathcal{R} \)-Leja sequence \( R = (r_j)_{j \geq 1} \) satisfies \( r_0 = 1, \ r_1 = -1, \ r_2 = 0 \) and

\[
r_{2j-1} = -r_{2j}, \quad j \geq 2 \tag{3.2}
\]

In addition, for \( n \geq 0 \) and \( k \) such that \( 2^n \leq k - 1 < 2^{n+1} \), using the simple identity \( k = 2 + \sum_{j=1}^{n} 2^{j-1} + (k - 1 - 2^n) \), we deduce that the section \( R_k \) and the following element \( r_k \) in \( R \) are given by

\[
R_k = \mathbb{R}(\Xi_k) \quad \text{and} \quad r_k = \mathbb{R}(\xi_k) \tag{3.3}
\]

with

\[
\Xi_k = (1, -1) \wedge \bigwedge_{j=1}^{n} E_{2^j, 2^{j+1}} \wedge E_{2^{n+1}, 2^{n+k-1}} \quad \text{and} \quad \xi_k = e_{2^n+k-1}. \tag{3.4}
\]

A simple example of a Leja sequence \( R \) is obtained from the simple Leja sequence given by the bit-reversal enumeration in (2.4). For \( k \geq 2 \) such that \( 2^n \leq k - 1 < 2^{n+1} \), we have by (3.3) and (2.4)

\[
r_k = \cos \left( \frac{\pi}{2^{n+1}} + \pi \sum_{j=0}^{n-1} a_j 2^{-j} \right) \quad \text{for} \quad k - 1 = \sum_{j=0}^{n} a_j 2^j, \quad a_j \in \{0, 1\}. \tag{3.5}
\]

Let us note that this sequence can also be computed recursively. We have \( R = (\cos \phi_k)_{k \geq 0} \) where the sequence of angles \( (\phi_k)_{k \geq 0} \) is defined by \( \phi_0 = 0, \ \phi_1 = \pi, \ \phi_2 = \frac{\pi}{2} \) and

\[
\phi_{2k+1} = \frac{\phi_{k+1}}{2}, \quad \phi_{2k+2} = \phi_{2k+1} + \pi, \quad k \geq 1. \tag{3.6}
\]
This recursion provides a very fast and simple process to construct an ℜ-Leja sequence. The distribution of the first elements of \( R \) is provided in Figure 3.

![Diagram of the first elements of the Leja sequence E and the associated ℜ-Leja sequence.](image)

Figure 3.1: Distribution of the first elements of the Leja sequence \( E \) defined in (2.4) and \( R \) the associated ℜ-Leja sequence.

The particular structure of Leja sequences \( E \) on the unit disk initiated at 1 conveys also a particular structure and useful properties for ℜ-Leja sequences. First, in view of the first property in Theorem 2.2, since for any Leja sequence on \( \mathcal{U} \) starting at 1, \( E_{2^{n+1}} = \mathcal{U}_{2^{n+1}} \) in the set sense then the projection on \([-1, 1]\) gives,

\[
R_{2^{n+1}} = \left\{ \cos \left( \frac{j\pi}{2^n} \right) : j = 0, \ldots, 2^n \right\}, \quad n \geq 0 \tag{3.7}
\]

in the set sense. Therefore \( R_{2^{n+1}} \) coincides as a set with the Gauss-Lobatto abscissas. Also, by the fifth property of Theorem 2.2, \( E^2 \) is also a Leja sequence initiated at 1, then projecting \( E^2 \) onto \([-1, 1]\) yields also an ℜ-Leja sequence. This implies in particular the following result, given in [6, Lemma 3.4],

**Lemma 3.2** Let \( R := (r_j)_{j \geq 0} \) be an ℜ-Leja sequence. The sequence

\[
R^2 := (2r^2_{2j} - 1)_{j \geq 0} \tag{3.8}
\]

is also an ℜ-Leja sequence.

The previous lemma has certain implications on the polynomials \( w_{R_k} \) and we have used it in [6] along with its implications in order to prove that \( D_k(R) \) the norm of the difference operator \( \Delta_k \) grows at worse quadratically.
In the following, we establish new bounds on the Lebesgue constants \( L_{R_k} \) and the norms of the difference operators \( D_{R_k} \) for \( \Re \)-Leja sequences. Our approach use our method of [6] where we relate, using simple calculatory arguments, the analysis of the Lebesgue constants associated with the sections of \( R \) to that of Lebesgue constants associated with the sections of \( E \).

As stated above in (3.7), for any \( \Re \)-Leja sequence \( R \), the sections \( R_{2^n+1} \) coincide in the set sense with the Gauss-Lobatto abscissas. This type of abscissas are known to have optimal Lebesgue constant in the sense \( L_{R_{2^n+1}} \sim 2 \pi \log(2^n + 1) \). More precisely, we have the bound

\[
L_{R_{2^n+1}} \leq 1 + \frac{2}{\pi} \log(2^n),
\]

see [14, Formulas 5 and 13]. The previous bound is then sharp and we are only interested in bounds for \( L_{R_k} \) when \( k - 1 \) is not power of 2. For the remainder of this section, we use the notation

\[
n \geq 0, \quad 2^n + 1 < k < 2^{n+1} + 1, \quad \text{and} \quad k' := k - (2^n + 1) \in \{1, \ldots, 2^n - 1\}
\]

In view of (3.4), we have

\[
R_k = R_{2^n+1} \land R_{2^n+1,k} = \Re(\Xi_{2^n+1}) \land \Re(\Xi_{2^n+1,k})
\]

with

\[
\Xi_{2^n+1} = (1, -1) \land \bigwedge_{j=1}^{n} E_{2j,2j+2j-1}, \quad \Xi_{2^n+1,k} = E_{2^n+1,2^n+k-1}
\]

The previous identities shows that \( R_k \) is obtained by projection of the section \( E_{2^n+k-1} \). We introduce the notation

\[
G_k = E_{2^n+k-1} = E_{2^n+1+k'}, \quad F_k = E_{2^n+1,2^n+k-1} = E_{2^n+1,2^n+1+k'}.
\]

We observe that \( G_k \) and \( F_k \) considered as sets are obtained from the sequence \( \Xi = (\xi_j)_{j \geq 0} \) by

\[
G_k = \{\xi_0, \xi_1\} \cup \{\xi_2, \overline{\xi_2}, \ldots, \xi_{2^n}, \overline{\xi_{2^n}}\} \cup F_k, \quad F_k = \{\xi_{2^n+1}, \ldots, \xi_{k-1}\}.
\]

Our approach in [6] for analysing \( L_{R_k} \) consisted in bounding the Lebesgue function \( \lambda_{R_k} \) using the Lebesgue function \( \lambda_{G_k} \) and a function associated with \( F_k \). We recall in the following this approach. In order to clarify our notation, we find it convenient to work with normalized versions of the polynomials \( w_{R_k} \) that we define by

\[
W_{R_k}(x) := 2^k w_{R_k}(x), \quad x \in [-1, 1].
\]

Using the elementary identity

\[
2|\Re(z) - \Re(z')| = |z - z'| |z - \overline{z'}|, \quad z, z' \in \partial U,
\]

we have established the following Lemma, see [6, Lemma 4.3],
Lemma 3.3 With the previous notations, for any \( z \in \partial \mathcal{U} \) and \( x = \Re(z) \)

\[
|W_{R_k}(x)| = |z^2 - 1| |w_{G_k}(z)||w_{\overline{F_k}}(z)|. \tag{3.16}
\]

Consequently, for any \( j = 0, \ldots, k - 1 \)

\[
|W'_{R_k}(r_j)| = 2\alpha_j |w'_{G_k}(\xi_j)||w_{\overline{F_k}}(\xi_j)|, \tag{3.17}
\]

where \( \alpha_j = 1 \) for every \( j \) except for \( j = 0 \) and \( j = 1 \) for which it is equal to 2.

Remark 3.4 The previous lemma stays also valid in the case \( k = 2^n + 1 \), in which case \( k' = 0 \). In such case, one has \( G_k = E_{2^{n+1}} \) and \( F_k = \emptyset \) and use the convention \( w_0 = 1 \).

The previous lemma allows to express Lagrange polynomials associated with \( R_k \) in the complex variable. If we denote by \( l_0, \ldots, l_{k-1} \) these polynomials, we have

\[
l_j(x) := \frac{W_{R_k}(x)}{W'_{R_k}(r_j)(x - r_j)}, \quad x \in [-1, 1], \tag{3.18}
\]

This combined with the previous theorem, the identity (3.15) and \( r_j = \Re(\xi_j) \), we have for \( z \in \partial \mathcal{U} \) such that \( x = \Re(z) \)

\[
|l_j(x)| = \frac{1}{\alpha_j} \left| \frac{z^2 - 1}{(z - \xi_j)(z - \overline{\xi_j})} \right| \frac{|w_{G_k}(z)|}{|w'_{G_k}(\xi_j)||w_{\overline{F_k}}(\xi_j)|} \tag{3.19}
\]

We denote by \( L_0, L_1, L_{(2,1)}, L_{(2,2)}, \ldots, L_{(2^n,1)}, L_{(2^n,2)}, L_{2^{n+1}}, \ldots, L_{k-1} \), the Lagrange polynomials associated with the set \( G_k \) following the order given in (3.13) and introduce the quotient notation

\[
q_k(z, \xi) := \frac{|w_{\overline{F_k}}(\overline{z})|}{|w_{\overline{F_k}}(\overline{\xi})|}, \quad z \in \partial \mathcal{U}, \quad \xi \in \partial \mathcal{U} \setminus F_k. \tag{3.20}
\]

We have the following result, which is established in the proof of [6] Lemma 4.4]

Lemma 3.5 With the previous notations

\[
|l_0(x)| \leq q_k(z, \xi_0)|L_0(z)| \quad \text{and} \quad |l_1(x)| \leq q_k(z, \xi_1)|L_1(z)|. \tag{3.21}
\]

For the indices \( j = 2, \ldots, 2^n \), we have

\[
|l_j(x)| \leq q_k(z, \xi_j)|L_{(j,1)}(z)| + q_k(z, \xi_j)|L_{(j,2)}(z)|. \tag{3.22}
\]

Finally for the indices \( j = 2^n + 1, \ldots, k - 1 \), we have

\[
|l_j(x)| \leq q_k(z, \xi_j)|L_j(z)| + q_k(\overline{\xi_j})|L_j(\overline{\xi})|. \tag{3.23}
\]
Using now rather the convenient notation $L_0, \ldots , L_{2^n+1+k'-1}$ for the Lagrange polynomials associated with $G_k$ seen as a section, i.e. $G_k = E_{2^n+1+k'} = (e_0, \ldots , e_{2^n+1+k'-1})$, and using the fact that $F_k = E_{2^n+1,2^n+1+k'}$ is a $k'$-Leja section, we write in view of the second point of Theorem 2.2 and the previous lemma that for $z \in \partial \mathcal{U}$ and $x = \Re(z)$

$$\lambda_{R_k}(x) \leq 2^{\sigma_1(k')} \left( \gamma_{n,k'}(z) + \gamma_{n,k'}(\overline{z}) \right), \quad \gamma_{n,k}(z) := \sum_{j=0}^{2^{n+1+k'-1}-1} \frac{|L_j(z)|}{|w_{F_k}(e_j)|},$$

(3.24)

We now can give the new bound for $L_{R_k}$ of the present paper which improve the bound 8$\sqrt{2}k^2$ established in [6, Therem 4.1].

**Theorem 3.6** Let $R$ an $\Re$-Leja sequence and $n, k$ and $k'$ as in (3.10). We have

$$L_{R_k} \leq 6\sqrt{5} \ 2^{n+\sigma_1(k')-\frac{p_0(k')}{2}}$$

(3.25)

**Proof:** By Cauchy Schwartz inequality, for any $z \in \partial \mathcal{U}$

$$\gamma_{n,k'}(z) \leq \left( \sum_{j=0}^{2^{n+1+k'-1}-1} |L_j(z)|^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{2^{n+1+k'-1}-1} \frac{1}{|w_{F_k}(e_j)|^2} \right)^{\frac{1}{2}} = \lambda_{G_k,2}(z) \beta_{n,k'}(E),$$

where $\lambda_{G_k,2}$ is the quadratic Lebesgue function associated with $G_k$ and $\beta_{n,k'}(E)$ as defined in (2.17). In view of Theorem 2.6 and Lemma 2.8, for any $z \in \partial \mathcal{U}$

$$\gamma_{n,k'}(z) \leq 3\sqrt{2^{\sigma_1(2^{n+1+k'}) - 1}} \left( \frac{5}{2} 2^{2n-\sigma_1(k')-p_0(k')} \right)^{\frac{1}{2}} \leq 3\sqrt{5} \ 2^{n-\frac{p_0(k')}{2}}$$

where we have used that $\sigma_1(2^{n+1}+k') = 1 + \sigma_1(k')$ in the last inequality. The same bound holds for $\gamma_{n,k'}(\overline{z})$, which in view of (3.24) finishes the proof.

We observe that if $k'$ is small compared to $2^n$, for example $k' \leq 2^{n/3-1}$, then $\sigma_1(k') \leq n/3$, so that

$$L_{R_k} \leq 6\sqrt{5} \ 2^{4n/3} \leq 6\sqrt{5} \ k^{4/3}$$

(3.26)

which improves the quadratic bound established in [6]. We note also that if $k'$ is such that $2\sigma_1(k') \leq p_0(k')$, then we get a linear bound.

**Remark 3.7** In the previous theorem, if $0 < k' < 2^n$ is the integer with the most number of ones in the binary expansion, i.e. $\sigma_1(k') = n$, in other words $k' = 2^n - 1$ and $k = 2^{n+1}$, we merely get the quadratic bound

$$L_{R_k} \leq 6\sqrt{5} \ 2^{2n} = \frac{3\sqrt{5}}{2} k^2.$$  

(3.27)
In [4], section 3.4, it is shown that for the values \( k = 2^n \), in other words \( R_k \) is the set of Gauss-Lobatto abscissas \((3.7)\) missing one abscissa, we have \( \mathbb{L}_{R_k} \geq k - 1 \). Indeed, for \( j = 0, \ldots, k - 1 \), one has in view of \((3.18)\)

\[
l_j(r_k) = \frac{W_{R_k}(r_k)}{W'_{R_{k+1}}(r_j)} = \frac{W'_{R_{k+1}}(r_k)}{W'_{R_{k+1}}(r_j)} \quad (3.28)
\]

Since \( k + 1 = 2^n + 1 \) then \( G_{k+1} = E_{2^{n+1}} = \mathcal{U}_{2^{n+1}} \) so that by Remark\((3.4)\), \( W'_{R_{k+1}}(r_j) = 2\alpha_j 2^{n+1} \) for any \( j = 0, \ldots, k \). This shows that for \( k = 2^n \geq 4 \)

\[
\lambda_{R_k}(r_k) = \frac{1}{2} + \frac{1}{2} + \sum_{j=2}^{k-1} 1 = k - 1 \quad (3.29)
\]

The growth of \( \mathbb{L}_{R_k} \) for \( k \geq 1 \) can not be slower than \( k \). In the final remark of this paper, we show that we have the bound \( \mathbb{L}_{R_k} \leq 2k \) even for this worst scenario of values \( k = 2^n \). This added to the observed growth of \( \mathbb{L}_{R_k} \) for values \( k \leq 128 \) suggests that the bound

\[
\mathbb{L}_{R_k} \leq 2k, \quad k \geq 1 \quad (3.30)
\]

might be valid for any \( \mathfrak{R} \)-Leja sequence \( R \). We conjecture its validity.

\section{4 Growth of the norms of the difference operators}

In this section, we discuss the growth of the norms of the difference operators \( \Delta_0 = I_0 \) and \( \Delta_k = I_k - I_{k-1} \) for \( k \geq 1 \), associated with interpolation on Leja or \( \mathfrak{R} \)-Leja sequences. We are interested in estimating their norm

\[
\mathbb{D}_k := \sup_{f \in C(X) - \{0\}} \frac{\|\Delta_k f\|_{L^\infty(X)}}{\|f\|_{L^\infty(X)}}. \quad (4.1)
\]

We recall that given \( Z \) a sequence of pair-wise distinct point in \( X \) a compact in \( \mathbb{R} \) or \( \mathbb{C} \), we have denoted by \( I_k \) the interpolation operator associated with \( Z_{k+1} = (z_0, \ldots, z_k) \). Elementary arguments, see [6], shows that

\[
\mathbb{D}_k(Z) = \left(1 + \lambda_{Z_k}(z_k)\right) \sup_{\substack{z \in X}} \frac{|w_{Z_k}(z)|}{|w_{Z_k}(z_k)|}. \quad (4.2)
\]

In particular if \( Z \) is a Leja sequence on \( X \), then

\[
\mathbb{D}_k(Z) = 1 + \lambda_{Z_k}(z_k). \quad (4.3)
\]

In [5], we have established that \( \lambda_{E_k}(e_k) \leq k \) if \( E \) is a Leja sequence on \( \mathcal{U} \) initiated at \( \partial \mathcal{U} \), which implies \( \mathbb{D}_k(E) \leq 1 + k \). Here, we improve slightly this result.
Lemma 4.1 Let $E$ be a Leja sequence on the unit disk initiated at $e_0 \in \partial U$, we have $D_0 = 1$ and

$$D_k(E) \leq 1 + \sqrt{k(2\sigma_1(k) - 1)}, \quad k \geq 1.$$  \hspace{1cm} (4.4)

Proof: By Cauchy Schwartz inequality, we have $\lambda_{E_k}(e_k) \leq \sqrt{k} \lambda_{E_{k+1}}(e_k)$. Lemma 2.5 which is proved for the simple Leja sequence defined in (2.4) can also be proved for any Leja sequence $E$ and it implies the result. \hfill \blacksquare

For $\mathcal{R}$-Leja sequences $R$ on $[-1, 1]$, we have shown in [6] using a recursion argument based on the fact that $R^2$, defined in (3.8), is also an $\mathcal{R}$-Leja sequence that

$$D_k(R) \leq (1 + k)^2.$$  \hspace{1cm} (4.5)

In view of the new bounds obtained in this paper for Lebesgue constant of $\mathcal{R}$-Leja sections, the previous bound is not sharp. Indeed, we have $D_k \leq L_k + L_{k-1} \leq 12\sqrt{\mathcal{R}} k^{4/3}$ for $k$ falling in the range of values implying the inequality (3.26). We give here a sharper bound for $D_k$ which is not implied by $D_k \leq L_k + L_{k-1}$. We recall that up to a rearrangement in the formula (4.2), see [6] for justification, we may express the quantities $D_k(R)$ in a more convenient form for $\mathcal{R}$-Leja sequences,

$$D_k(R) = 2\beta_k(R) \sup_{x \in [-1,1]} |W_{R_k}(x)|$$  \hspace{1cm} (4.6)

where

$$\beta_k(R) := \frac{1}{2|W_{R_k}(r_k)|} \left( 1 + \lambda_{R_k}(r_k) \right) = \sum_{j=0}^{k} \frac{1}{|W_{R_{k+1}}(r_j)|}$$  \hspace{1cm} (4.7)

As in [6], we treat separately the quantities $\beta_k(R)$ and $\sup_{x \in [-1,1]} |W_{R_k}(x)|$, then conclude. For $\beta_k(R)$, we recall our result [6 Lemma 5.3], where we have proved that for any $k \geq 1$

$$\beta_k(R) \leq \frac{2\sigma_0(k) - p_0(k)}{4}.$$  \hspace{1cm} (4.8)

As for $\sup_{x \in [-1,1]} |W_{R_k}(x)|$, we show by a tricky argument that the bound $2^{2\sigma_1(k) + 2p_0(k) - 2}$ established in [6 Lemma 5.4] can be improved by $2^{-p_0(k)}$.

Lemma 4.2 Let $R$ be an $\mathcal{R}$-Leja sequence. We have for any $k \geq 2$,

$$\sup_{x \in [-1,1]} |W_{R_k}(x)| \leq 2^{2\sigma_1(k) + p_0(k)}.$$  \hspace{1cm} (4.9)
Proof: First, if \( k = 2^n + 1 \), then by Remark 3.4, for \( x \in [-1, 1] \) and \( z \in \partial \mathcal{U} \) such that \( \Re(z) = x \),
\[
|W_{R_k}(x)| = |z^2 - 1||w_{E_{2^n+1}}(z)| = |z^2 - 1||z^{2+1} - 1| \leq 4.
\]
We let now \( n \geq 0 \) and assume that \( 2^n + 1 < k < 2^{n+1} + 1 \) and define \( k' := k - (2^n + 1) \). By Lemma 3.3, for \( x \in [-1, 1] \) and \( z \in \partial \mathcal{U} \) such that \( \Re(z) = x \), we have
\[
|W_{R_k}(x)| = |z^2 - 1||w_{G_k}(z)||w_{\tilde{F}_k}(z)|
\]
We have that \( |z^2 - 1| = |z - \overline{z}| \leq |z - e_{2^{n+1+k'}}| + |z - e_{2^n+1+k'}| \), so that
\[
|W_{R_k}(x)| \leq |w_{G_{k+1}}(z)||w_{\tilde{F}_{k}}(z)| + |w_{G_{k}}(z)||w_{\tilde{F}_{k+1}}(z)| \leq 2^{2+\sigma_1(k') + \sigma_1(k'+1)},
\]
where we have introduced \( G_{k+1} = E_{2^{n+1+k'+1}}, F_{k+1} = E_{2^{n+1+2n+1+k'+1}} \), and used that \( G_k, G_{k+1}, F_k \) and \( F_{k+1} \) are all Leja sections with length \( 2^{n+1+k'}, 2^{n+1+k'+1}, k' \) and \( k'+1 \) respectively.

We have \( k = 2^n + 1 + k' \), so that
\[
\sigma_1(k) = 1 + \sigma_1(1+k') \quad \text{if} \quad k' < 2^n + 1 \quad \text{and} \quad \sigma_1(k) = 1 = \sigma_1(1+k') \quad \text{if} \quad k' = 2^n - 1.
\]
Moreover
\[
\sigma_1(k) - 1 + p_0(k) = \sigma_1(k - 1) = \sigma_1(2^n + k') = 1 + \sigma_1(k')
\]
Summing the two previous identities implies
\[
2 + \sigma_1(k') + \sigma_1(1+k') \leq 2\sigma_1(k) + p_0(k),
\]
which concludes the proof. \( \blacksquare \)

In view of the previous lemma and (1.8), we have the following result

**Theorem 4.3** Let \( R \) be an \( \mathcal{R} \)-Leja sequence in \([-1, 1] \). The norms of the difference operators associated with \( R \) satisfy, \( \mathcal{D}_0 = 1 \) and for \( 2^n \leq k < 2^{n+1} \)
\[
\mathcal{D}_k \leq 2^{\sigma_1(k)} 2^n \tag{4.10}
\]
**Proof:** We have
\[
\mathcal{D}_k \leq 2^{\frac{2\sigma_0(k) - p_0(k)}{4}} 2^{2\sigma_1(k) + p_0(k)} \leq 2^{n + \sigma_1(k)},
\]
where we have used \( \sigma_1(k) + \sigma_0(k) = n + 1 \) for \( 2^n \leq k < 2^{n+1} \). \( \blacksquare \)
Remark 4.4 We have by the triangular inequality that for any $k \geq 1$,

$$\mathbb{L}_{R_k} \leq \mathbb{L}_{R_{k+1}} + D_k(R) \quad (4.11)$$

In particular if $k = 2^n$, we have

$$\mathbb{L}_{R_k} \leq 1 + \frac{2}{\pi} \log(2^n) + 2^n \leq 2k \quad (4.12)$$

This shows that the particular case $k = 2^n$ which corresponds to $R_k$ being the set of Gauss-Lobatto abscissas (3.7) missing one abscissas has a linear bound. This improves the quadratic bound given provided by Theorem 3.6.

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