THE ORIENTATION-PRESERVING DIFFEOMORPHISM GROUP OF $S^2$ DEFORMS TO $SO(3)$ SMOOTHLY

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ABSTRACT. Smale proved that the orientation-preserving diffeomorphism group of $S^2$ has a continuous strong deformation retraction to $SO(3)$. In this paper, we construct such a strong deformation retraction which is diffeologically smooth.

1. Introduction

In Smale’s 1959 paper “Diffeomorphisms of the 2-Sphere” ([8]), he shows that there is a continuous strong deformation retraction from the orientation-preserving $C^\infty$ diffeomorphism group of $S^2$ to the rotation group $SO(3)$. The topology of the former is the $C^k$ topology. In this paper, we construct such a strong deformation retraction which is diffeologically smooth. We follow the general idea of [8], but to achieve smoothness, some of the steps we use are completely different from those of [8]. The most notable differences are explained in Remark 2.3, 2.5, and Remark 3.4. We note that there is a different proof of Smale’s result in [2], but the homotopy is not shown to be smooth.

We start by defining the notion of diffeological smoothness in three special cases which are directly applicable to this paper. Note that diffeology can be defined in a much more general context and we refer the readers to [4].

Definition 1.1. Let $U$ be an arbitrary open set in a Euclidean space of arbitrary dimension.

- Suppose $\Lambda$ is a manifold with corners. A map $P : U \to \Lambda$ is a plot if $P$ is $C^\infty$.
- Suppose $X$ and $Y$ are manifolds with corners, and $\Lambda \subset C^\infty(X, Y)$. Denote by $ev$ the evaluation map $\Lambda \times X \to Y$ given by $(f, x) \mapsto f(x)$. A map $P : U \to \Lambda$ is a plot, if the map from $U \times X$ to $Y$ given by $(s, x) \mapsto ev(P(s), x)$ is $C^\infty$.
- Suppose $\Lambda$ is given by a product $\Lambda_1 \times \cdots \times \Lambda_n$, where each $\Lambda_i$ is either one of the above two cases. Denote by $\pi_i$ the projection onto each factor. A map $P : U \to \Lambda$ is a plot if each map $\pi_i \circ P : U \to \Lambda_i$ is a plot.

In the second case, one can think of a plot $P$ of $\Lambda$ as a smooth family of maps $\{f_s\}_{s \in U} \subset \Lambda$ by letting $f_s = P(s)$. 

1
Definition 1.2. Let $\Lambda$ be one of the three cases in Definition 1.1. Then $\Lambda$ equipped with the collection of all the plots is called a diffeological space. The second kind is referred to as the standard functional diffeology, and the third kind the product diffeology.

Let $\Lambda$ and $\Gamma$ be two diffeological spaces. We define the notion of diffeological smoothness of a map from $\Lambda$ to $\Gamma$ as follows.

Definition 1.3. A map $\varphi : \Lambda \to \Gamma$ is diffeologically smooth if for every plot $P : U \to \Lambda$, the map $\varphi \circ P : U \to \Gamma$ is a plot. If in addition, $\varphi$ has a diffeological smooth inverse, then it is a diffeomorphism.

Remark 1.4. Let $\Lambda$ and $\Gamma$ be manifolds with corners. Then $\varphi : \Lambda \to \Gamma$ is a diffeologically smooth map if and only if $\varphi$ is a smooth map between manifolds with corners. We refer the readers to [5] for the definition of smooth maps between manifolds with corners. This fact is non-trivial. The proof of “IV.13 Smooth real maps from half-spaces” in [4] contains the main ingredients for showing the equivalence of two notions of smoothness for manifolds with boundary. One can use Theorem 1 of [7] (take the group $G$ to be $(\mathbb{Z}/2)\mathbb{n}$) to modify this proof to show the equivalence for manifolds with corners.

Therefore, we see that diffeological smoothness is a generalization of the usual notion of smoothness. Throughout the rest of the paper, by a “smooth map” we always mean a diffeologically smooth map.

Now we state the main result of this paper. Let us denote

$$\Omega := \text{the orientation-preserving } C^\infty \text{ diffeomorphism group of } S^2.$$ 

Theorem 1.5 (main theorem). There is a smooth strong deformation retraction $P : I \times \Omega \to \Omega$ to SO(3) that is equivariant under the left action of SO(3). More precisely, for each $(t, f) \in I \times \Omega$ and $A \in \text{SO}(3)$,

1. $P_0(f) = f$,
2. $P_1(f) \in \text{SO}(3)$,
3. $P_t(A) = A$,
4. $P_t(A \circ f) = A \circ P_t(f)$.

Now we outline the construction of the above deformation retraction $P$. Let $x_0$ be the South Pole of $S^2 \subset \mathbb{R}^3$, and $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ the basis vectors of the tangent space $T_{x_0}S^2$. Denote by $\Omega_1$ the following subset of $\Omega$.

$$\Omega_1 := \{ f \in \Omega : f(x_0) = x_0 \text{ and } df|_{x_0} = \text{id}_{T_{x_0}S^2} \}.$$ 

Lemma 1.6. The map $i : \text{SO}(3) \times \Omega_1 \to \Omega$ given by $(A, f) \mapsto A^{-1} \circ f$ is a diffeomorphism with its image

$$i(\text{SO}(3) \times \Omega_1) = \{ f \in \Omega : df|_{x_0}e_1 \text{ and } df|_{x_0}e_2 \text{ are orthonormal} \}.$$ 

In the following theorem, we homotope $\Omega$ to $i(\text{SO}(3) \times \Omega_1)$. 

2
Theorem 1.7. There is a smooth homotopy $Q : I \times \Omega \to i(\text{SO}(3) \times \Omega_1)$ that fixes $\text{SO}(3)$ and is equivariant under the left action of $\text{SO}(3)$. More precisely, for each $(t, f) \in I \times \Omega$ and $A \in \text{SO}(3)$,

1. $Q_0(f) = f$,
2. $Q_1(f) \in i(\text{SO}(3) \times \Omega_1)$,
3. $Q_t(A) = A$,
4. $Q_t(A \circ f) = A \circ Q_t(f)$.

In the following theorem, we homotope $\Omega_1$ to $\{\text{id}_{S^2}\}$.

Theorem 1.8. There is a smooth strong deformation retraction $R : I \times \Omega_1 \to \Omega_1$ to $\{\text{id}_{S^2}\}$. More precisely, for each $(t, f) \in I \times \Omega_1$,

1. $R_0(f) = f$,
2. $R_1(f) = \text{id}_{S^2}$,
3. $R_t(\text{id}_{S^2}) = \text{id}_{S^2}$.

Smoothly concatenating homotopies $Q$ and $R$ gives the desired deformation retraction $P$.

When proving Theorem 1.8, we need a proposition about the diffeomorphisms of the square $[-1,1]^2$. Let $\mathcal{F}$ be the space of those orientation-preserving diffeomorphisms of the square $[-1,1]^2$ such that for each $f \in \mathcal{F}$, there exists a neighborhood of the boundary $\partial([-1,1]^2)$ on which $f$ is the identity map.

Proposition 1.9. There is a smooth strong deformation retraction $F : I \times \mathcal{F} \to \mathcal{F}$ to $\{\text{id}_{I^2}\}$. More precisely, for each $(t, f) \in I \times \mathcal{F}$,

1. $F_0(f) = f$,
2. $F_1(f) = \text{id}_{[-1,1]^2}$,
3. $F_t(\text{id}_{[-1,1]^2}) = \text{id}_{[-1,1]^2}$.

This paper is organized as follows. In Section 2, we construct the homotopy of Theorem 1.7 and prove Theorem 1.8 while assuming Proposition 1.9. Lastly we prove the main theorem. In Section 3, we prove Proposition 1.9.

Acknowledgement. We would like to express our deepest gratitude to Yael Karshon for her time and patience involved in supervising this project. We would also like to thank Katrin Wehrheim for a helpful suggestion in the proof of Lemma 2.4.

2. Construction of homotopies

We start by surveying some commonly used properties regarding diffeological smoothness. One convenience of working with diffeology on $C^\infty$ function spaces is that verifying diffeological smoothness often reduces to checking
the usual smoothness in finite dimensions. The proofs of these properties are left as exercises.

**Remark 2.1.**

1. Let $\Lambda$, $\Gamma$, and $\Sigma$ be diffeological spaces. If $\varphi: \Lambda \to \Gamma$ and $\psi: \Gamma \to \Sigma$ are both diffeologically smooth, then $\psi \circ \varphi$ is diffeologically smooth.
2. Let $X$, $Y$ and $Z$ be manifolds with corners. Let $\Lambda$ be a subset of $C^\infty(X,Y)$, and $\Gamma$ a subset of $C^\infty(Y,Z)$. Then the map from $\Lambda \times \Gamma$ to $C^\infty(X,Z)$ given by $(f,g) \mapsto g \circ f$ is diffeologically smooth. (This fact can be proved by using the usual implicit function theorem.)
3. Let $X$ be a manifold with corners, and denote by $\text{Diff}(X)$ the diffeomorphism group of $X$. Let $\Lambda$ be a subset of $\text{Diff}(X)$. Then the map from $\Lambda$ to $\text{Diff}(X)$ given by $f \mapsto f^{-1}$ is diffeologically smooth.
4. Let $X$ and $Y$ be manifolds with corners, and $\Lambda$ a subset of $C^\infty(X,Y)$. Then the map from $\Lambda$ to $C^\infty(TX,TY)$ given by $f \mapsto Tf$ is diffeologically smooth.

Now we prove Lemma 1.6. Recall that $x_0$ is the South Pole of $S^2$, and $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ are the basis vectors of the tangent space $T_{x_0}S^2$.

**Proof of Lemma 1.6**. Denote by $\tilde{\Omega}$ the following set

$$\tilde{\Omega} := \{ f \in \Omega : df|_{x_0}e_1 \text{ and } df|_{x_0}e_2 \text{ are orthonormal} \}.$$

It follows from property (2) of Remark 2.1 that the map $i : \text{SO}(3) \times \Omega_1 \to \Omega$ given by $(A,f) \mapsto A^{-1} \circ f$ is smooth. Moreover, tangent vectors $d(A^{-1} \circ f)|_{x_0}e_1$ and $d(A^{-1} \circ f)|_{x_0}e_2$ are clearly orthonormal. Thus $i(\text{SO}(3) \times \Omega_1) \subset \tilde{\Omega}$.

Define the map $\alpha : \tilde{\Omega} \to \text{SO}(3)$ as follows. For $g \in \tilde{\Omega}$, let $\alpha(g)$ be the element in $\text{SO}(3)$ that sends the ordered basis $\{g(x_0), dg(x_0) e_1, dg(x_0) e_2\}$ to the ordered basis $\{x_0, e_1, e_2\}$. Note that $\alpha(g)$ can be written as a matrix involving the partial derivatives of $g$, and is easily seen to be smooth with respect to $g$. It is easy to see that $\alpha(g) \circ g$ is an element of $\Omega_1$.

It is straightforward to check that the map from $\tilde{\Omega}$ to $\text{SO}(3) \times \Omega_1$ given by $g \mapsto (\alpha(g), \alpha(g) \circ g)$ is the smooth inverse of the map $i$. \hfill \square

Now we construct the smooth homotopy from $\Omega$ to

$$i(\text{SO}(3) \times \Omega_1) = \{ f \in \Omega : df|_{x_0}e_1 \text{ and } df|_{x_0}e_2 \text{ are orthonormal} \}.$$

The construction involves some computation in local coordinates. Let $p : S^2 \setminus \{-x_0\} \to \mathbb{R}^2$ be the stereographic projection from the North Pole. More precisely, $p(x_1, x_2, x_3) = (\frac{2x_1}{1-x_3}, \frac{2x_2}{1-x_3})$. It is convenient to denote by $B(\delta)$ the open ball in $\mathbb{R}^2$ around $0$ with radius $\delta$, that is,

$$B(\delta) := \{ y \in \mathbb{R}^2 : |y| < \delta \}.$$
Proof of Theorem 1.4. Given \( f \in \Omega \), define tangent vectors \( u \) and \( v \) in \( T_{f(x_0)}S^2 \) by \( u = df|_{x_0}e_1 \) and \( v = df|_{x_0}e_2 \). Let \( u_1 = \frac{u}{||u||} \) be the normalization of \( u \), and \( u_2 \) the vector which makes \( \{u_1, u_2\} \) a positively oriented orthonormal basis in \( T_{f(x_0)}S^2 \).

Define the map \( \alpha : \Omega \to \text{SO}(3) \) as follows. For \( f \in \Omega \), let \( \alpha(f) \) be the unique element in \( \text{SO}(3) \) that sends the ordered basis \( \{f(x_0), u_1, u_2\} \) to the ordered basis \( \{x_0, e_1, e_2\} \). Define the map \( p_f : S^2 \setminus \{-f(x_0)\} \to \mathbb{R}^2 \) by

\[
p_f = p \circ \alpha(f).
\]

The map \( p_f \) is a coordinate chart with \( p_f(f(x_0)) = 0 \). The expression of the vector \( u_1 \) in this coordinate chart is \((1,0)\). More precisely,

\[
dp_f|_{f(x_0)}u_1 = dp|_{x_0}\alpha(f)u_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Similarly, \( dp_f|_{f(x_0)}u_2 = (0,1) \). By abuse of notation, we identify a vector \( w \in T_{f(x_0)}S^2 \) with its expression in the coordinate chart \( p_f \). Therefore,

\[
\begin{align*}
u_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\
u &= \begin{pmatrix} a \\ 0 \\ c \end{pmatrix}, \\
v &= \begin{pmatrix} b \\ c \\ 0 \end{pmatrix},
\end{align*}
\]

where \( a, b, \) and \( c \) depend smoothly on \( f \). By definition, \( a \) is positive. Since \( f \) is orientation-preserving, \( c \) is also positive. Define the matrix \( g_{f,1} \) by

\[
g_{f,1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1}.
\]

By construction, \( g_{f,1} \) sends \( u \) to \( u_1 \) and \( v \) to \( u_2 \). Moreover, \( g_{f,1} \) is upper triangular and with positive diagonal entries, and it depends smoothly on \( f \).

For each \( t \in [0,1] \), define the matrix \( g_{f,t} \) by the following linear interpolation.

\[
g_{f,t} = (1-t)\text{id} + tg_{f,1}.
\]

It is easy to see that the matrix \( g_{f,t} \) is an orientation-preserving isomorphism and depends smoothly on \((t,f)\). Thus the family of diffeomorphisms \( g_{f,t} : \mathbb{R}^2 \to \mathbb{R}^2 \) determines a time-dependent vector field \( X_{f,t} \) on \( \mathbb{R}^2 \). More precisely, the vector field \( X_{f,t} \) is defined by

\[
\frac{d}{dt}g_{f,t} = X_{f,t} \circ g_{f,t}.
\]

Let \( \rho : \mathbb{R}^2 \to [0,1] \) be a smooth function such that \( \rho(y) = 1 \) for \( y \in B(1) \) and \( 0 \) for \( y \notin B(2) \). Define the vector field \( \tilde{X}_{f,t} \) by \( \tilde{X}_{f,t} = \rho X_{f,t} \). This vector field determines a family of orientation-preserving diffeomorphisms \( \tilde{g}_{f,t} \), which is given by the solution to the following ordinary differential equation.

\[
\frac{d}{dt}\tilde{g}_{f,t} = \tilde{X}_{f,t} \circ \tilde{g}_{f,t}, \text{with the initial condition} \quad \tilde{g}_{f,0} = \text{id}.
\]

It follows from the standard regularity theory of ordinary differential equations that \( \tilde{g}_{f,t} \) is smooth with respect to \((t,f)\). It is easy to see that the restriction of \( \tilde{g}_{f,t} \) to the complement of \( B(2) \) is the identity map. Moreover, for
of \( Q \) sends \( e \) which makes \( G \) part of the image \( B \), \( \Psi \) matrix \( \mathcal{A} \). The map \( Q \) this shows that \( d, \alpha \) in the above construction, \( \tilde{\gamma} \) of \( B \).

Remark 2.3. Note that the map \( \alpha \) defined in the proof of Lemma \( 1.6 \) is the restriction of \( \alpha \) defined in the proof of Theorem \( 1.7 \).

Remark 2.2. In the above construction, \( \tilde{\gamma} \) is the diffeomorphism of \( \mathbb{R}^2 \) which agrees with the linear map \( g_{f,t} \) around the origin, and is the identity map outside a ball. This map is inspired by the diffeomorphism \( G_\nu(f) \) on page 626 of [S]. However, this diffeomorphism is not entirely accurate. Using our notation, the diffeomorphism \( G_t(f) : \mathbb{R}^2 \to \mathbb{R}^2 \) is defined by

\[
G_t(f)(x) = \gamma(f, x) g_{f,t}(x) + (1 - \gamma(f, x))x,
\]

where \( \gamma(f, x) \) is a function which is 1 on a ball \( B(\varepsilon(f)) \), and 0 outside a larger ball \( B(\varepsilon(f)) \). It is clear that the map \( G_t(f) \) agrees with \( g_{f,t} \) on the smaller ball \( B(\varepsilon(f)) \), and is the identity map outside \( B(\varepsilon(f)) \). However, part of the image \( g_{f,t}(B(\varepsilon(f))) \) could lie outside of the larger ball \( B(\varepsilon(f)) \), which makes \( G_t(f) \) not injective. Even though such problem can be fixed by choosing a different \( \gamma \), it is still not clear that \( \gamma \) can be chosen so that \( G_t(f) \)
is a diffeomorphism. Thus instead of trying to construct such a function $\gamma$, we use the vector field approach to construct the diffeomorphism $\tilde{g}_{f,t}$.

Now we prove Theorem 1.8, that is, there is a smooth strong deformation retraction from the set

$$\Omega_1 := \{ f \in \Omega : f(x_0) = x_0 \text{ and } df|_{x_0} = \text{id}_{T_{x_0}S^2} \}$$

to $\{ \text{id}_{S^2} \}$. We construct this deformation retraction in the following two steps.

1) Homotope each diffeomorphism $f$ in $\Omega_1$ to a diffeomorphism which is the identity map on a neighborhood of the South Pole $x_0$.

2) Homotope the diffeomorphism $f$ to the identity map on the complement of the neighborhood.

We start by proving a technical lemma about choosing such a neighborhood.

Let $p : S^2 \setminus \{-x_0\} \to \mathbb{R}^2$ be the stereographic projection from the North Pole, and $(y_1, y_2)$ the coordinate variables of $\mathbb{R}^2$. Using $p$ as a local coordinate chart, we denote the local expression of the map $f \in \Omega_1$ by $\tilde{f}$, which is given by

$$\tilde{f} := p \circ f \circ p^{-1}.$$ 

Note that $\tilde{f}$ is not necessarily well-defined on all of $\mathbb{R}^2$; if $f^{-1}(-x_0)$ is not equal to $-x_0$, then $\tilde{f}$ is not defined at the point $p(f^{-1}(-x_0))$. However, for $f \in \Omega_1$, the local expression $\tilde{f}$ is always defined at 0, and we have $\tilde{f}(0) = 0$. In addition, $d\tilde{f}|_0 = \text{id}$. Thus we can choose a neighborhood of 0 on which $\tilde{f}$ is well-defined and the operator $d\tilde{f} - \text{id}$ is small in the operator norm. The following technical lemma says that such choice of neighborhood can be made smoothly with respect to $f$.

**Lemma 2.4.** There exists a smooth function $\varepsilon : \Omega_1 \to (0, \frac{1}{4}]$ such that

- $\varepsilon$ is well-defined on the open ball $B(2\varepsilon(f))$,
- for $y \in B(\varepsilon(f))$, we have $\|df|_y - \text{id}\| < \frac{1}{4}$,

where $\| \cdot \|$ denotes the operator norm induced by the Euclidean norm on $\mathbb{R}^2$.

**Remark 2.5.** Later in the construction, the function $\varepsilon$ is involved in the deformation retraction from $\Omega_1$ to $\{ \text{id}_{S^2} \}$. Since the deformation retraction needs to depend smoothly on $f \in \Omega_1$, the function $\varepsilon$ needs to be smooth. Note that the topological construction of $\varepsilon$ on the top of page 625 of [8] is continuous but not smooth in general. As a result, our method of constructing $\varepsilon$ is completely different. We use the Sobolev inequality to construct the smooth function $\varepsilon$.

**Proof of Lemma 2.4.** Let $h$ be the height function $h : S^2 \to [-1,1]$ given by $h(x_1, x_2, x_3) = x_3$. Define $h_f : B(1) \to [-1,1]$ by $h_f = h \circ f \circ p^{-1}$. It is easy to see that $f \mapsto h_f$ is smooth. For $f \in \Omega_1$, the definition of $\Omega_1$ implies that $h_f(0) = h(x_0) = -1$. We also observe that the value $\tilde{f}(y)$ is
well-defined if and only if \( h_f(y) < 1 \). We need to use this criterion later in the proof.

By the Sobolev inequality (see Theorem 6, page 270 of [3]), there is a universal constant \( c \) such that

\[
\|u\|_{C^1(B(1))} \leq c \|u\|_{H^3(B(1))},
\]

for every smooth function \( u \) on the closed unit ball \( B(1) \).

Here \( \|u\|_{C^1(B(1))} \) is the \( C^1 \) norm of \( u \) on \( B(1) \), defined by

\[
\|u\|_{C^1(B(1))} := \sup_{B(1)} |u| + \sup_{B(1)} |\partial_{y_1} u| + \sup_{B(1)} |\partial_{y_2} u|.
\]

\( \|u\|_{H^3(B(1))} \) is the Sobolev-3 norm of \( u \) on \( B(1) \), defined by

\[
\|u\|_{H^3(B(1))} := (\sum_{|\alpha| \leq 3} \int_{B(1)} |\partial^\alpha u|^2)^{\frac{1}{2}},
\]

where \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1 \) and \( \alpha_2 \) non-negative integers, \( |\alpha| = \alpha_1 + \alpha_2 \),

and \( \partial^\alpha = \partial_{y_1}^{\alpha_1} \partial_{y_2}^{\alpha_2} \).

For \( f \in \Omega_1 \), define

\[
\varepsilon_1(f) = 1 + \frac{1}{c} \left[ c^2 + \sum_{|\alpha| \leq 3} \int_{B(1)} |\partial^\alpha h_f|^2 \right]^{\frac{1}{2}}.
\]

We claim that \( \varepsilon_1 \) is a smooth function from \( \Omega_1 \) to \((0,1], \) and \( \bar{f} \) is well-defined on \( B(\varepsilon_1(f)) \). It is easy to check that \( \varepsilon_1(f) > 0 \) and \( \varepsilon_1(f) \leq \frac{1}{c} c = 1 \).

The smoothness of \( \varepsilon_1 \) follows from the smoothness of \( h_f \) with respect to \( f \) and the smoothness property of integration. Moreover, for \( y \in B(\varepsilon_1(f)) \subset B(1) \), the mean value theorem implies that

\[
|h_f(y) - h_f(0)| \leq \sup_{B(1)} |d(h_f)| \cdot |y|.
\]

Thus for \( y \) such that \( |y| < \varepsilon_1(f) \),

\[
h_f(y) \leq h_f(0) + \sup_{B(1)} |d(h_f)| \cdot |y|
\]

\[
\leq -1 + \frac{1}{c} \left[ c^2 + \sum_{|\alpha| \leq 3} \int_{B(1)} |\partial^\alpha h_f|^2 \right]^{\frac{1}{2}}
\]

\[
\leq -1 + \frac{\|h_f\|_{C^1(B(1))}}{c \|h_f\|_{H^3(B(1))}} \leq 0 \leq 1,
\]

where the second inequality is by definition of \( \varepsilon_1 \) and the last line is by the Sobolev inequality \((2.1)\). Therefore by the criterion established earlier, \( \bar{f} \) is well-defined on \( B(\varepsilon_1(f)) \).
Now we use a similar method to refine the choice of $\varepsilon_1$ in order to achieve the second property in the statement of the lemma. For $f \in \Omega_1$, define $g_f : B(\varepsilon_1(f)) \to [\frac{1}{8}, +\infty)$ by
\[
g_f = \left[\frac{1}{64} + (\partial_{y_1} f_1^1 - 1)^2 + (\partial_{y_2} f_1^2)^2 + (\partial_{y_1} f_2^2)^2 + (\partial_{y_2} f_2^2 - 1)^2\right]^{\frac{1}{2}}.
\]
It is easy to check that $\|df\|_y - \|id\| < g_f(y)$ for $y \in B(\varepsilon_1(f))$, and it follows from the definition of $\Omega_1$ that $g_f(0) = \frac{1}{8}$.

Let $\gamma : (0, 1] \times B(1) \to [0, 1]$ be a smooth function such that $\gamma(\varepsilon, y)$ is 1 for $y \in B(\frac{1}{2}\varepsilon)$, and 0 for $y \notin B(\frac{1}{2}\varepsilon)$. Then it is straightforward to show that the function $\gamma(\varepsilon_1(f), \cdot)g_f$ is well-defined on all of $B(1)$, and it is smooth with respect to $f$. Recall that $c$ is the constant that appears in the Sobolev inequality (2.1). Define
\[
\varepsilon(f) = \frac{1}{8c} \left[\frac{1}{16c^2\varepsilon_1(f)^2} + \sum_{|\alpha| \leq 3} \int_{B(1)} |\partial^\alpha (\gamma(\varepsilon_1(f), \cdot)g_f)|^2 \right]^{-\frac{1}{2}}.
\]
We claim that the function $\varepsilon$ satisfies all the requirements in the statement of the lemma. First of all, $\varepsilon(f)$ is well-defined since $\varepsilon_1(f) > 0$. Also it is easy to see that $\varepsilon(f) > 0$ and $\varepsilon(f) \leq \frac{4\varepsilon_1(f)}{8c} = \frac{\varepsilon_1(f)}{2} \leq \frac{1}{2}$. It follows that $\tilde{f}$ is well-defined on $B(2\varepsilon(f)) \subset B(\varepsilon_1(f))$. Proving smoothness of $\varepsilon$ is similar to proving smoothness of $\varepsilon_1$. For $y \in B(\varepsilon(f)) \subset B(\frac{1}{2}\varepsilon_1(f))$, we have $\gamma(\varepsilon_1(f), y) = 1$. Then it follows from the mean value theorem that
\[
|g_f(y) - g_f(0)| \leq \sup_{B(1)} |d(\gamma(\varepsilon_1(f), \cdot)g_f)| \cdot |y|.
\]
Therefore for $y$ such that $|y| < \varepsilon(f)$,
\[
\|df\|_y - \|id\| < g_f(y) \\
\leq g_f(0) + \sup_{B(1)} |d(\gamma(\varepsilon_1(f), \cdot)g_f)| \cdot |y| \\
\leq \frac{1}{8} + \frac{1}{8c} \left[\frac{1}{16c^2\varepsilon_1(f)^2} + \sum_{|\alpha| \leq 3} \int_{B(1)} |\partial^\alpha (\gamma(\varepsilon_1(f), \cdot)g_f)|^2 \right]^{-\frac{1}{2}} \\
\leq \frac{1}{8} + \frac{1}{8c} \gamma(\varepsilon_1(f), \cdot)g_f \|C^1(B(1)) \leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.
\]
This completes the proof of Lemma 2.4.

Recall that $p : S^2 \setminus \{x_0\} \to \mathbb{R}^2$ is the stereographic projection from the North Pole, and $\tilde{f} = p \circ f \circ p^{-1}$ is the local expression of the map $f$. We homotope each $f \in \Omega_1$ to a diffeomorphism whose restriction to $p^{-1}(B(\frac{1}{2}\varepsilon(f)))$ is the identity map.

**Lemma 2.6.** There exists a smooth homotopy $S : I \times \Omega_1 \to \Omega_1$ such that for each $(t, f) \in I \times \Omega_1$, the North Pole, and $\bar{\gamma}(\varepsilon_1(f), y)$ is 1 for $y \in B(\frac{1}{2}\varepsilon)$, and 0 for $y \notin B(\frac{1}{2}\varepsilon)$. Then it is straightforward to show that the function $\gamma(\varepsilon_1(f), \cdot)g_f$ is well-defined on all of $B(1)$, and it is smooth with respect to $f$. Recall that $c$ is the constant that appears in the Sobolev inequality (2.1). Define
(1) $S_0(f) = f$,
(2) $S_1(f)$ restricted to the neighborhood $p^{-1}(B(\frac{1}{2}\varepsilon(f)))$ is the identity map,
(3) $S_t(id_{S^2}) = id_{S^2}$.

Proof. Let $\gamma : (0, \frac{1}{2}] \times B(1) \to [0, 1]$ be a smooth function such that $\gamma(\varepsilon, y)$ is 1 for $y \in B(\frac{1}{2}\varepsilon)$ and 0 for $y \notin B(\varepsilon)$, and additionally $|\partial_y \gamma| < \frac{2}{\varepsilon}$ everywhere. Given $t \in [0, 1]$ and $f \in \Omega_1$, we define $S_t(f)$ by cases.

- For $y \in B(2\varepsilon(f))$, the local expression $S_t(f)$ is defined by
  $$S_t(f)(y) = (1-t)\bar{f}(y) + t \left[ \gamma(\varepsilon(f), y) y + (1 - \gamma(\varepsilon(f), y))\bar{f}(y) \right].$$
- For $x \notin p^{-1}(B(\varepsilon(f)))$, define $S_t(f)(x) = f(x)$.

We first check that each $S_t(f)$ is a well-defined smooth map. First of all, for $y$ such that $|y| < 2\varepsilon(f)$, it follows from Lemma 2.4 that $\bar{f}(y)$ is well-defined. Thus $S_t(f)$ is well-defined in the first case. The overlap of the two cases is when $\varepsilon(f) \leq |y| < 2\varepsilon(f)$. For such $y$, it follows from the choice of $\gamma$ that $\gamma(\varepsilon(f), y) = 0$ and $S_t(f)(y) = \bar{f}(y)$. This shows that the definitions agree on the overlap. In each case, it is clear that $S_t(f)$ is smooth. The agreement on the overlap implies that $S_t(f)$ is a well-defined smooth map.

Now we show that each $S_t(f)$ is an orientation-preserving local diffeomorphism. It is enough to prove that the matrix $dS_t(f)|_y$ is invertible and of positive determinant when $y \in B(\varepsilon(f))$. For such $y$, an easy computation shows that
$$dS_t(f)|_y = id + [1 - t\gamma(\varepsilon(f), y)] (d\bar{f}|_y - id) + t(y - \bar{f}(y)) \cdot \partial_y \gamma(\varepsilon(f), y).$$
To show that the matrix $dS_t(f)|_y$ is invertible, it suffices to show that the difference $dS_t(f)|_y - id$ has operator norm less than 1. The mean value theorem and Lemma 2.4 imply that
$$|\bar{f}(y) - y| \leq \sup_{B(\varepsilon(f))} \|d\bar{f} - id\| \cdot |y| < \frac{\varepsilon(f)}{4}.$$Then it follows from $|\partial_y \gamma| < \frac{2}{\varepsilon}$ that
$$\| [1 - t\gamma(\varepsilon(f), y)] (d\bar{f}|_y - id) + t(y - \bar{f}(y)) \cdot \partial_y \gamma(\varepsilon(f), y) \| < \frac{1}{4} + \frac{\varepsilon(f)}{4} \cdot \frac{3}{\varepsilon} = 1.$$Hence $dS_t(f)|_y$ is invertible. Moreover, since $f \in \Omega_1$, the matrix $dS_0(f)|_y = d\bar{f}|_y$ has positive determinant. Thus each $dS_t(f)|_y$ has positive determinant since it depends continuously on $t$.

We can use the following standard topological argument to show that $S_t(f)$ is in fact a diffeomorphism. By using the compactness of $S^2$ and the fact that $S_t(f)$ is a local diffeomorphism, we can conclude that $S_t(f)$ is a covering map. It follows from Theorem 5.1 on page 147 of [1] that $S_t(f)$ is injective. Therefore $S_t(f)$ is a diffeomorphism. It is easy to check that $S_t(f)(0) = 0$ and $dS_t(f)|_0 = id$. Thus $S_t(f) \in \Omega_1$. 

10
Furthermore, it follows from the smoothness of the function $\varepsilon$ that $S_t(f)$ is smooth with respect to $(t, f)$. It is easy to see that $S_0(f) = f$. For each $y \in B(\frac{1}{2}\varepsilon(f))$, it follows from $\gamma(\varepsilon(f), y) = 1$ that $S_t(f)(y) = y$. Lastly, it is clear that $S_t(id_{S^2}) = id_{S^2}$. \hfill $\square$

In Lemma 2.6, we homotope each $f \in \Omega_1$ to the diffeomorphism $S_1(f)$ which is the identity map on the neighborhood $p^{-1}(B(\frac{1}{2}\varepsilon(f)))$ of the South Pole. To complete the proof of Theorem 1.8 we need to homotope it to the identity map on the complement of this neighborhood.

**Proof of Theorem 1.8.** Let $\tilde{p} : S^2 \setminus \{x_0\} \to \mathbb{R}^2$ be the stereographic projection from the South Pole. We use $\tilde{p}$ as the coordinate chart throughout this proof. For each $f \in \Omega_1$, we have $S_1(f)(x_0) = x_0$. Thus the local expression

$$S_1(f) = \tilde{p} \circ S_1(f) \circ \tilde{p}^{-1}$$

is a well-defined diffeomorphism of $\mathbb{R}^2$. It follows from Lemma 2.6 that $S_1(f)$ restricted to the open set $\tilde{p} \circ p^{-1}(B(\frac{1}{2}\varepsilon(f)))$ is the identity map. It is clear that

$$\tilde{p} \circ p^{-1}\left( B\left( \frac{1}{2}\varepsilon(f) \right) \right) = \{ y \in \mathbb{R}^2 : |y| > \frac{2}{\varepsilon(f)} \}.$$  

Let $\Psi_f : \left[ -\frac{3}{\varepsilon(f)}, \frac{3}{\varepsilon(f)} \right]^2 \to [-1, 1]^2$ be the scaling map, given by $y \mapsto \frac{\varepsilon(\phi)}{3} y$. Then it is easy to see that $\Psi_f \circ S_1(f) \circ \Psi_f^{-1}$ is an orientation preserving diffeomorphism of $[-1, 1]^2$, which is the identity map on a neighborhood of the boundary. Thus it belongs to the set $\mathcal{F}$ in Proposition 1.9. Recall that $F : I \times \mathcal{F} \to \mathcal{F}$ is the smooth strong deformation retraction to $\{id_{[-1, 1]^2}\}$. Define the homotopy $T$ in terms of the local expression as follows.

$$T_t(S_1(f))(y) = \begin{cases} 
\Psi_f^{-1} \circ F_t(\Psi_f \circ S_1(f) \circ \Psi^{-1}) \circ \Psi_f(y) & \text{if } y \in \left[ -\frac{3}{\varepsilon(f)}, \frac{3}{\varepsilon(f)} \right]^2, \\
y & \text{otherwise}.
\end{cases}$$

It is clear that each $T_t(S_1(f))$ is an orientation preserving diffeomorphism of $S^2$. Moreover, properties of the homotopy $F$ imply that $T_0(S_1(f)) = S_1(f)$ and $T_1(S_1(f)) = id_{S^2}$. The smoothness property of maps $F$, $S$, and $\varepsilon$ implies that $T_t(S_1(f))$ is smooth with respect to $(t, f)$. Lastly, $T_t(id_{S^2}) = id_{S^2}$ since $F_t(id_{[-1, 1]^2}) = id_{[-1, 1]^2}$.

To finish the proof, we need to smoothly concatenate the homotopies $S$ and $T$. Let $\beta_1 : [0, \frac{1}{2}] \to [0, 1]$ be a smooth map which is 0 in a neighborhood of 0 and 1 in a neighborhood of $\frac{1}{2}$. Similarly, let $\beta_2 : [\frac{1}{2}, 1] \to [0, 1]$ be a smooth map which is 0 in a neighborhood of $\frac{1}{2}$ and 1 in a neighborhood of 1. Define the homotopy $R : I \times \Omega_1 \to \Omega_1$ by

$$R_t(f) = \begin{cases} 
S_{\beta_1(t)}(f) & t \in [0, \frac{1}{2}], \\
T_{\beta_2(t)}(S_1(f)) & t \in [\frac{1}{2}, 1].
\end{cases}$$

It is easy to see that $R$ satisfies all the required properties. \hfill $\square$
To finish the proof of the main theorem, we need to smoothly concatenate the homotopy $Q$ of Theorem 1.7 and the homotopy $R$ of Theorem 1.8.

**Proof of Theorem 1.8.** Let $\beta_1 : [0, \frac{1}{2}] \to [0, 1]$ be a smooth map which is 0 in a neighborhood of 0 and 1 in a neighborhood of $\frac{1}{2}$. Similarly, let $\beta_2 : [\frac{1}{2}, 1] \to [0, 1]$ be a smooth map which is 0 in a neighborhood of $\frac{1}{2}$ and 1 in a neighborhood of 1. Define the homotopy $P : I \times \Omega \to \Omega$ by

$$P_t(f) = \begin{cases} Q_{\beta_1(t)}(f) & t \in [0, \frac{1}{2}], \\ \alpha(f)^{-1} \circ R_{\beta_2(t)}(\alpha(f) \circ Q_1(f)) & t \in [\frac{1}{2}, 1]. \end{cases}$$

It is easy to see that $R$ satisfies all the required properties. \[\square\]

### 3. Diffeomorphisms of the Square

In this section we prove Proposition 1.9. First of all, we observe that $[-1, 1]^2$ is diffeomorphic to $I^2$ as manifolds with corners. It is more convenient to work with $I^2$ than $[-1, 1]^2$. Therefore by abuse of notation, we denote by $\mathcal{F}$ the space of those orientation-preserving diffeomorphisms of the square $I^2$ such that for each $f \in \mathcal{F}$, there exists a neighborhood of the boundary $\partial I^2$ on which $f$ is the identity map.

To prove Proposition 1.9 it is equivalent to proving

**Theorem 3.1.** There is a smooth strong deformation retraction $F : I \times \mathcal{F} \to \mathcal{F}$ to $\{\text{id}_{I^2}\}$. More precisely, for each $(t, f) \in I \times \mathcal{F}$,

1. $F_0(f) = f$,
2. $F_1(f) = \text{id}_{I^2}$,
3. $F_t(\text{id}_{I^2}) = \text{id}_{I^2}$.

To construct the desired deformation retraction $F$, we consider a superset $\mathcal{E}$ of $\mathcal{F}$, and construct a deformation retraction from this superset to $\{\text{id}_{I^2}\}$. The set $\mathcal{E}$ is defined as follows.

We denote by $e_1$ the vector $(1, 0)$. Let $I_1$ be the right boundary of $I^2$. In other words, $I_1 = \{1\} \times I$. We denote by $\mathcal{E}$ the space of those orientation-preserving diffeomorphisms of the square $I^2$ such that for each $f \in \mathcal{E}$,

- there exists a neighborhood of $\partial I^2 \setminus I_1$ on which $f$ is the identity map, and
- for $x$ close enough to 1, $df|_{(x,y)} e_1 = e_1$.

By abuse of notation, we also let $e_1$ denote the constant map taking $I^2$ to the vector $e_1$. We view $e_1$ as a constant vector field on $I^2$. Then for each $f \in \mathcal{E}$, there is a corresponding non-vanishing vector field $f_* e_1$, i.e.,

$$(f_* e_1)(x, y) = df|_{f^{-1}(x,y)} e_1.$$ 

To homotope an element $f \in \mathcal{E}$ to the identity map, we first homotope its corresponding vector field $f_* e_1$ to the constant vector field $e_1$ (Lemma 3.2),
and then integrate the vector fields to recover the corresponding diffeomorphisms in $\mathcal{E}$ (Theorem 3.3). We start by considering the following space of vector fields on $I^2$.

We denote by $H$ the space of all $C^\infty$ maps from $I^2$ to $\mathbb{R}^2 \setminus \{0\}$ such that for each $h \in H$, there exists a neighborhood of $\partial I^2$ on which $h$ is equal to the constant map $e_1$.

**Lemma 3.2.** There is a smooth homotopy $\Phi : I \times \mathcal{E} \rightarrow H$, such that for each $(t,f) \in I \times \mathcal{E}$,

1. $\Phi_0(f) = f \circ e_1$,
2. $\Phi_1(f) = e_1$,
3. $\Phi_t(id_{I^2}) = e_1$.

**Proof.** The map $\exp : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$ is a $C^\infty$ covering map of $\mathbb{R}^2 \setminus \{0\}$. Fix a vector $\tilde{e}_1 \in \exp^{-1}(\{e_1\})$ in $\mathbb{R}^2$. By Theorem 4.1 on page 143 of [1], for each $h \in H$, there is a unique continuous map $\tilde{h} : I^2 \rightarrow \mathbb{R}^2$ such that

$$\exp \circ \tilde{h} = h$$

Moreover, denote by $U_h$ a connected neighborhood of $\partial I^2$ on which $h$ is equal to $e_1$. By going through the construction of $\tilde{h}$ in Theorem 4.1 of [1], one can show that $\tilde{h}(U_h) = \{\tilde{e}_1\}$. Lastly, for a small enough neighborhood $V \subset I^2$, the image $h(V)$ is contained in a basic open set $W$, on which log is defined as a multi-valued map. Then there is a unique branch of log such that $\tilde{h}|_V = \log \circ h|_V$. Thus the map $\tilde{h}$ is $C^\infty$. A similar argument shows that the map $h \mapsto \tilde{h}$ is smooth.

Let $H : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a $C^\infty$ homotopy that contracts $\mathbb{R}^2$ to $\{\tilde{e}_1\}$. Then define $\Phi : I \times \mathcal{E} \rightarrow H$ as follows. For $(t,f) \in I \times \mathcal{E}$,

$$\Phi_t(f) = \exp \circ H_t \circ \int_s f \circ e_1.$$  

By using the previously established properties, one can show that each $\Phi_t(f)$ is indeed an element of $H$, the map $\Phi$ is smooth, and conditions (1) - (3) are satisfied. $\square$

Now we show that the space $\mathcal{E}$ can be smoothly deformed to $\{id_{I^2}\}$.

**Theorem 3.3.** There is a smooth strong deformation retraction $E : I \times \mathcal{E} \rightarrow \mathcal{E}$ to $\{id_{I^2}\}$. More precisely, for each $(t,f) \in I \times \mathcal{E}$,

1. $E_0(f) = f$,
2. $E_1(f) = id_{I^2}$,
3. $E_t(id_{I^2}) = id_{I^2}$.

**Proof.** For $(t,f) \in I \times \mathcal{E}$ and $y \in I$, let $s \mapsto \Gamma_t(f)(s,y)$ be the integral curve of the vector field $\Phi_t(f)$ from Lemma 3.2 with the initial condition $\Gamma_t(f)(0,y) = (0,y)$. It follows from the standard regularity theory of ordinary differential equations that $\Gamma_t(f)(s,y)$ depends smoothly on $(t,f,s,y)$.

Firstly, for any $t \in I$, it follows from $\Phi_t(id_{I^2}) = e_1$ that $\Gamma_t(id_{I^2})(s,y) = (s,y)$. Hence $\Gamma_t(id_{I^2}) = id_{I^2}$. When $t = 1$, the fact that $\Phi_1(f) = e_1$ implies
that $\Gamma_1(f) = \text{id}_{I^2}$. When $t = 0$, by using the fact that $\Phi_0(f) = f \cdot e_1$, one can check directly that $\Gamma_0(f)(s, y) = f(s, y)$. Thus $\Gamma_t(f)$ satisfies conditions (1) - (3). However, $(t, f) \mapsto \Gamma_t(f)$ is not the desired homotopy; for $0 < t < 1$, we need to analyze $\Gamma_t(f)$ carefully.

Since each $\Phi_t(f)$ agrees with $e_1$ on a neighborhood of $\partial I^2$, we can conclude that the integral curve $\Gamma_t(f)(\cdot, y)$ either

- leaves the square $I^2$ via the right boundary $I_1$, or
- does not leave $I^2$.

However, in the second case, the integral curve would approach asymptotically to a simple closed curve, then by the Poincaré-Bendixson theorem (see page 191 of [5]), the vector field $\Phi_t(f)$ vanishes somewhere in the interior of the closed curve, which contradicts the assumption that $\Phi_t(f) \in \mathcal{H}$. Thus each integral curve $\Gamma_t(f)(\cdot, y)$ meets $I_1$ at some time $\bar{s}$, which depends on $(t, f, y)$. More precisely, $\bar{s}(t, f, y)$ is defined by the equation

$$
\Gamma_t(f)^1(\bar{s}, y) = 1.
$$

To show that $\bar{s}$ is smooth, we compute the partial derivative

$$
(3.1) \quad \frac{\partial}{\partial s} \Gamma_t(f)^1(s, y) \bigg|_{s=\bar{s}} = \Phi_t(f)^1(\Gamma_t(f)(\bar{s}, y)) = 1,
$$

where the last equality follows from $\Phi_t(f)|_{I_1} = e_1$. Therefore it follows from the implicit function theorem that $\bar{s}(t, f, y)$ depends smoothly on $(t, f, y)$.

Note that $\Gamma_t(f)$ is a diffeomorphism from the set $\bigcup_y \{y\} \times [0, \bar{s}(t, f, y)]$ to the square $I^2$. However $\bar{s}$ is not necessarily equal to 1, so we cannot take $\Gamma_t(f)$ as the desired homotopy. We need to reparametrize the $s$ variable in $\Gamma_t(f)(s, y)$ to obtain an element of $\mathcal{E}$.

Let $\chi : [0, 1] \times (0, +\infty) \to [0, +\infty)$ be a smooth function, such that for each $r \in (0, +\infty)$,

- the function $\chi(\cdot, r)$ maps the interval $[0, 1]$ diffeomorphically to $[0, r]$, with $\chi(0, r) = 0$ and $\chi(1, r) = r$,
- there exists a neighborhood $U_0$ of 0 and a neighborhood $U_1$ of 1, such that for $x \in U_0 \cup U_1$, we have $\frac{\partial}{\partial x} \chi(x, r) = 1$,
- for $x \in [0, 1]$, we have $\chi(x, 1) = x$.

We use the diffeomorphism $x \mapsto s = \chi(x, \bar{s})$ to reparametrize $\Gamma_t(f)(s, y)$. We claim that

$$
E_t(f)(x, y) := \Gamma_t(f)(\chi(x, \bar{s}(t, f, y)), y)
$$

is the right deformation retraction.

We first show that each map $E_t(f) : I^2 \to I^2$ has a smooth inverse. Fix $(t, f) \in I \times \mathcal{E}$. Let $s \mapsto \gamma_t(f)(s, x', y')$ be the integral curve of the vector field $\Phi_t(f)$, with the initial condition $\gamma_t(f)(0, x', y') = (x', y')$. Let us denote by $-\tau(x', y')$ the time when the integral curve $\gamma_t(f)(\cdot, x', y')$ meets the left boundary. More precisely, $-\tau(x', y')$ is defined by the equation

$$
\gamma_t(f)^1(-\tau, x', y') = 0.
$$
Define $y(x', y')$ by $y = \gamma_{t}(f)^{2}(-\tau, x', y')$. It is easy to see that $\Gamma_{t}(f)(\tau, y) = (x', y')$. Now it suffices to find $x(x', y')$. Recall that by the first property of $\chi$, the function $\chi(\cdot, \bar{s}(y)) : [0, 1] \to [0, \bar{s}(y)]$ has a smooth inverse function, which we denote by $\sigma(y)$. Define $x(x', y')$ by $x = \sigma(y)(\tau)$. Then it is easy to check that the inverse of $E_{t}(f)$ is given by $$(E_{t}(f))^{-1}(x', y') = (x(x', y'), y(x', y')).$$

To show that $(E_{t}(f))^{-1}$ is smooth, it suffices to show the function $\tau$ is smooth. One can carry out a similar computation as equation (3.1), and then the smoothness of $\tau$ follows from the implicit function theorem.

One can check that each $E_{t}(f)$ is in $E$ by using the second property of the function $\chi$ and the property of $\Phi_{t}(f)$ near the boundary. Lastly, the fact that $\Gamma_{t}(f)$ satisfies conditions (1) - (3) and the third property of the function $\chi$ implies that $E : I \times E \to E$ is the desired deformation retraction. This completes the proof of Theorem 3.3. □

**Remark 3.4.** On the top of page 623 of [8], Smale uses an argument in general topology to reparametrize the variable $s$, which yields a continuous but not necessarily smooth homotopy. Here we use the smooth function $\chi$ to reparametrize, and the resulting homotopy is smooth.

To finish the construction of the homotopy from $F$ to \{id$_{I^{2}}$\}, we first notice that we cannot define the homotopy $F : I \times F \to F$ to be the restriction of the homotopy $E$. This is because for $f \in F$, the diffeomorphism $E_{t}(f)$ does not necessarily lie in the set $F$. To solve this problem, we first construct a retraction $p : E \to F$, and we show that $F_{t}(f)$ given by $(p \circ E_{t})(f)$ is the right homotopy. To construct this retraction, we first consider the following set of diffeomorphisms of the interval $I$.

Let $G$ be the space of those orientation-preserving diffeomorphisms of $I$ such that for each $g \in G$, there exists a neighborhood of 0 and a neighborhood of 1 on which $g$ is the identity map.

**Lemma 3.5.** There is a smooth strong deformation retraction $G : I \times G \to G$ to \{id$_{I}$\}. More precisely, for each $(t, g) \in I \times G$,

1. $G_{0}(g) = g$,
2. $G_{1}(g) = \text{id}_{I}$,
3. $G_{t}(\text{id}_I) = \text{id}_I$.

**Proof.** Define the homotopy $G : I \times G \to G$ as follows. For $x \in I$,

$$G_{t}(g)(x) = (1 - t)g(x) + tx.$$

It is easy to check that $G$ is smooth and conditions (1) - (3) are satisfied. □

**Proof of Theorem 3.1.** For each $f \in E$, one can check that for $x$ close enough to 1, the value $f(x, y)$ is given by $(x, g_{f}(y))$, where $g_{f}(y) = f^{2}(1, y)$. It is easy to see that $g_{f}$ is an element of $G$ and it depends smoothly on $f$. 

15
Let $\beta : I \to I$ be a smooth function that is 1 in a neighborhood of 0, and 0 in a neighborhood of 1. Now define $\Psi_f : I^2 \to I^2$ by

$$\Psi_f(x, y) = \left( x, \left[ G_{\beta(x)} \left( g_f^{-1} \right) \right](y) \right).$$

It is easy to check that the inverse of $\Psi_f$ is given by the map $(x, y) \mapsto \left( x, \left[ G_{\beta(x)} \left( g_f^{-1} \right) \right]^{-1}(y) \right)$. Thus $\Psi_f$ is a diffeomorphism of the square.

One can show that the map $p : \mathcal{E} \to \mathcal{F}$ defined by

$$p(f) = \Psi_f \circ f$$

is a smooth retraction. In other words, $p$ is smooth and $p|_\mathcal{F} = \text{id}$. It follows that the homotopy $F : I \times \mathcal{F} \to \mathcal{F}$ defined by

$$F_t(f) = (p \circ E_t)(f)$$

is the desired smooth deformation retraction to $\{\text{id}_{I^2}\}$. □

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