Frenet Frame With Respect to Conformable Derivative

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Abstract. Conformable fractional derivative is introduced by the authors Khalil at al in 2014. In this study, we investigate the frenet frame with respect to conformable fractional derivative. Curvature and torsion of a conformable curve are defined and the geometric interpretation of these two functions is studied. Also, fundamental theorem of curves is expressed for the conformable curves and an example of the curve corresponding to a fractional differential equation is given.

1. Introduction

The differential geometry of curves and surfaces has two aspects. One of them is called classical differential geometry and the other one is called global differential geometry. The classical geometry is the study of local properties of curves and surfaces. By local properties we mean those properties which depend only on the behavior of the curve or surface in the neighborhood of a point. The methods which have shown themselves the adequate in the study of such properties are the methods of differential calculus.

Recently, in [9] the authors Khalil at al. introduced a new definition of the fractional derivative called conformable fractional derivative. Moreover, in that paper, the definition of fractional integral is also defined. This paper pioneered many new studies. In [11, 13], the authors investigate Lyapunov-type inequalities in the frame of conformable derivatives. In the papers [4, 12], conformable fractional operators and semigroup operators are studied. Further studies about conformable derivatives and its applications are found in [1, 3, 5–8]. In addition to this, conformable fractional derivative for multivariable functions is given in [1, 3, 8]. These papers give a chance to introduce the notion of conformable curve which is a generalized form of a curve.

In this study, the classical differential geometry of curves is investigated with respect to conformable fractional derivative and fractional integral. In this sense, conformable curve is defined and \( \alpha \)–Frenet formulas are given. If \( \alpha = 1 \), the theory of conformable curves coincides with the classical theory of curves. Especially, the fundamental theorem of the local theory of the conformable curves is given. And, for \( \alpha \)–differentiable functions \( \kappa > 0 \) and \( \tau \), a system of fractional order differential equations is formed, then a conformable curve is obtained by the solution of this system.
2. Conformable Fractional Derivative

In this section, we give some basic definitions and properties of conformable fractional derivative introduced in [8–10].

**Definition 2.1.** [9] Given a function \( f : [0, \infty) \rightarrow \mathbb{R} \). The conformable derivative of the function \( f \) of order \( \alpha \) is defined by

\[
T_{\alpha}(f)(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}
\]

for all \( x > 0 \), \( \alpha \in (0, 1) \).

**Theorem 2.2.** [9] If a function \( f : [0, \infty) \rightarrow \mathbb{R} \) is \( \alpha \)-differentiable at \( t_0 > 0 \), \( \alpha \in (0, 1] \), then \( f \) is continuous at \( t_0 \).

**Theorem 2.3.** [9] Let \( \alpha \in (0, 1] \) and \( f, g \) be \( \alpha \)-differentiable at a point \( t > 0 \). Then

1. \( T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g) \), for all \( a, b \in \mathbb{R} \).
2. \( T_{\alpha}(p^r) = rp^{r-\alpha} \) for all \( p \in \mathbb{R} \).
3. \( T_{\alpha}(\lambda) = 0 \), for all constant functions \( f(t) = \lambda \).
4. \( T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f) \).
5. \( T_{\alpha}\left(\frac{f}{g}\right) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2} \).

6. If, in addition, \( f \) is differentiable, then \( T_{\alpha}(f)(t) = t^{1-\alpha} \frac{d}{dt} f(t) \).

**Theorem 2.4.** [10] Assume \( f, g : (0, \infty) \rightarrow \mathbb{R} \) be two \( \alpha \)-differentiable functions where \( \alpha \in (0, 1] \). Then \( g \circ f \) is \( \alpha \)-differentiable and for all \( t \) with \( t \neq 0 \) and \( f(t) \neq 0 \) we have

\[
T_{\alpha}(g \circ f)(t) = T_{\alpha}(g)(f(t))T_{\alpha}(f)(t)^{\alpha-1}.
\]

**Definition 2.5.** [8] Let \( f \) be a vector valued function with \( n \) real variables such that \( f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \). Then we say that \( f \) is \( \alpha \)-differentiable at \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) where each \( a_i > 0 \), if there is a linear transformation \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that

\[
\lim_{h \to 0} \frac{\|f(a_1 + h_1a_1^{1-\alpha}, \ldots, a_n + h_na_n^{1-\alpha}) - f(a_1, \ldots, a_n) - L(h)\|}{\|h\|} = 0
\]

where \( h = (h_1, \ldots, h_n) \) and \( \alpha \in (0, 1] \). The linear transformation is denoted by \( D^\alpha f(a) \) and called the conformable derivative of \( f \) of order \( \alpha \) at \( a \).

**Theorem 2.6.** [8] Let \( f \) be a vector valued function with \( n \) variables. If \( f \) is \( \alpha \)-differentiable at \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \), each \( a_i > 0 \), then there is a unique linear transformation \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that

\[
\lim_{h \to 0} \frac{\|f(a_1 + h_1a_1^{1-\alpha}, \ldots, a_n + h_na_n^{1-\alpha}) - f(a_1, \ldots, a_n) - L(h)\|}{\|h\|} = 0.
\]

**Theorem 2.7.** [8] If a vector valued function \( f \) with \( n \) variables is \( \alpha \)-differentiable at \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \), each \( a_i > 0 \), then \( f \) is continuous at \( a \in \mathbb{R}^n \).
Theorem 2.8. [8] (Chain Rule) Let \( x \in \mathbb{R}^n, y \in \mathbb{R}^m \). If \( f(x) = (f_1(x), ..., f_m(x)) \) is \( \alpha \)-differentiable at \( a = (a_1, ..., a_n) \in \mathbb{R}^n \), each \( a_i > 0 \) such that \( \alpha \in (0, 1) \) and \( g(y) = (g_1(y), ..., g_m(y)) \) is \( \alpha \)-differentiable at \( f(a) \in \mathbb{R}^m \), all \( f_i(a) > 0 \) such that \( \alpha \in (0, 1] \). Then the composition \( g \circ f \) is \( \alpha \)-differentiable at \( a \) and

\[
D^\alpha (g \circ f)(a) = D^\alpha g(f(a)) \circ f(a)^{\alpha-1} \circ D^\alpha f(a)
\]

where \( f(a)^{\alpha-1} \) is the linear transformation from \( \mathbb{R}^m \) to \( \mathbb{R}^m \) defined by \( f(a)^{\alpha-1}(x_1, ..., x_m) = (x_1 f_1(a)^{\alpha-1}, ..., x_m f_m(a)^{\alpha-1}) \).

Theorem 2.9. [8] Let \( f \) be a vector valued function with \( n \) variables such that \( f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n)) \). Then \( f \) is \( \alpha \)-differentiable at \( a = (a_1, ..., a_n) \in \mathbb{R}^n \), each \( a_i > 0 \) if and only if each \( f_j \) is, and

\[
D^\alpha f(a) = (D^\alpha f_1(a), ..., D^\alpha f_m(a)).
\]

3. Conformable Curves

In this section, we are going to introduce conformable curves and their basic properties.

Definition 3.1. The function \( \gamma : (0, \infty) \rightarrow \mathbb{R}^3 \) is called a conformable curve in \( \mathbb{R}^3 \) if \( \gamma \) is \( \alpha \)-differentiable.

Notation: Along the work, for a conformable curve \( \gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \), we use the notation

\[
D^\alpha \gamma(t) = T_\alpha \gamma(t) = (T_\alpha \gamma_1(t), T_\alpha \gamma_2(t), T_\alpha \gamma_3(t)).
\]

Definition 3.2. Let \( \gamma : (0, \infty) \rightarrow \mathbb{R}^3 \) be a conformable curve. Velocity vector of \( \gamma \) is determined by

\[
\frac{T_\alpha \gamma(t)}{t^{1-\alpha}},
\]

for all \( t \in (0, \infty) \).

Definition 3.3. Let \( \gamma : (0, \infty) \rightarrow \mathbb{R}^3 \) be a conformable curve. If the function \( h : (0, \infty) \rightarrow (0, \infty) \) is \( \alpha \)-differentiable, the conformable curve \( \beta = \gamma(h) : (0, \infty) \rightarrow \mathbb{R}^3 \) is called the reparametrization of \( \gamma \) with \( h \).

Definition 3.4. Let \( \gamma : (0, \infty) \rightarrow \mathbb{R}^3 \) be a conformable curve. Then the velocity function \( v \) of \( \gamma \) is defined by

\[
v(t) = \frac{\|T_\alpha \gamma(t)\|}{t^{1-\alpha}}
\]

for all \( t \in (0, \infty) \).

Definition 3.5. Let \( \gamma : (0, \infty) \rightarrow \mathbb{R}^3 \) be a conformable curve. The arc length function \( s \) of \( \gamma \) is defined by

\[
s(t) = \frac{T_\alpha \|T_\alpha \gamma(t)\|}{1}
\]

for all \( t \in (0, \infty) \). If \( v(t) = 1 \) for all \( t \in (0, \infty) \), it’s said that \( \gamma \) has unit speed.

Lemma 3.6. Let the function \( f : (0, \infty) \rightarrow \mathbb{R} \) be continuous and increasing (or decreasing) on \( (0, \infty) \). If the function \( f \) is \( \alpha \)-differentiable for all \( x \in (0, \infty) \) and \( T_\alpha f(x) \neq 0 \), then the inverse function \( f^{-1} : (f(0^+), f(0^*)) \rightarrow \mathbb{R} \) (or \( f^{-1} : (f(b^-), f(b^*)) \rightarrow \mathbb{R} \)) is \( \alpha \)-differentiable at \( y = f(x) \) where \( b \in (0, \infty) \). Furthermore,

\[
(T_\alpha f^{-1})(y) = \frac{(xy)^{1-\alpha}}{T_\alpha f(x)}.
\]
Therefore, Let us show that Since \( \gamma \) and \( t \) are fixed. Consider the arc length function \( s(t) = T_{\alpha}^0 \|T_{\alpha}\gamma(t)\| \).

Since \( \gamma \) is conformable regular, \( T_\alpha s(t) = \|T_\alpha \gamma(t)\| \neq 0 \). By Lemma (3.6), \( s(t) \) has an inverse function \( t = t(s) \) and \( t = t(s) \) is also differentiable. On the other hand, \( \beta \) is a reparametrization of \( \gamma \) such that \( \beta(s) = \gamma(t(s)) \).

Let us show that \( \beta \) has unit speed: We know that \( T_\alpha \beta(s) = (T_\alpha \gamma)(t(s))T_\alpha(t(s))t(s)^{-1} \) by Theorem (2.8).

Therefore,

\[
\frac{\|T_\alpha \beta(s)\|}{s^{1-\alpha}} = \frac{\|(T_\alpha \gamma)(t(s))\|T_\alpha(t(s))t(s)^{\alpha-1}}{s^{1-\alpha}} = \frac{(T_\alpha s)(t(s))T_\alpha(t(s))t(s)^{\alpha-1}}{s^{1-\alpha}} = \frac{T_\alpha s(t(s))}{s^{2-\alpha}} = 1.
\]

This completes the proof. \( \square \)

4. \( \alpha \)-Frenet Formulas

4.1. Conformable Curves With Unit Speed

Let \( \beta \) be a unit speed conformable curve, so \( \frac{\|T_\alpha \beta(s)\|}{s^{1-\alpha}} = 1 \) for each \( s \in (0, \infty) \). Then \( E_1(s) = \frac{T_\alpha \beta(s)}{s^{1-\alpha}} \) is called the unit tangent vector field on \( \beta \). Since \( E_1 \) has constant length 1, we call \( T_\alpha E_1 \) the curvature vector field of \( \beta \). \( \alpha \)-differentiation of \( E_1 \cdot E_1 = 1 \) gives \( 2T_\alpha E_1 \cdot E_1 = 0 \), so \( T_\alpha E_1 \) is always orthogonal to \( E_1 \), that is, normal to \( \beta \).

The length of the curvature vector field \( T_\alpha E_1 \) gives a numerical measurement of the turning of \( \beta \). The real valued function \( \kappa \) such that \( \kappa(s) = \|T_\alpha E_1(s)\| \) for all \( s \in (0, \infty) \) is called the curvature function of \( \beta \). Let \( \kappa > 0 \).

The unit vector field \( E_2 = \frac{T_\alpha E_1}{\kappa} \) on \( \beta \) is called the principal normal vector field of \( \beta \). The unit vector field \( E_3 = E_1 \times E_2 \) on \( \beta \) is called binormal vector field of \( \beta \). Therefore the orthonormal system of the vector fields \( \{E_1, E_2, E_3\} \) on \( \beta \) is called the \( \alpha \)-frenet frame.

Remark 4.1. Now we express the terms \( T_\alpha E_1, T_\alpha E_2, T_\alpha E_3 \) in terms of \( E_1, E_2, E_3 \). Since \( E_1 = T_\alpha \beta \), we have \( T_\alpha E_1 = \kappa E_2 \). Let us consider \( T_\alpha E_3 \). Since \( E_2 \cdot E_2 = 1 \), \( T_\alpha E_3 \) is orthogonal to \( E_2 \). On the other hand, since \( E_3 \cdot E_1 = 0 \), \( T_\alpha E_3 \cdot E_1 + E_2 \cdot T_\alpha E_1 = 0 \). Hence, we have \( T_\alpha E_3 \cdot E_1 = -E_3 \cdot T_\alpha E_1 = -E_3 \cdot \kappa E_2 = 0 \), that is, \( T_\alpha E_3 \) is orthogonal to \( E_1 \). Finally, since \( T_\alpha E_3 \) is orthogonal to both \( E_1 \) and \( E_2 \), \( T_\alpha E_3 \) is, at each point, a scalar multiple of \( E_2 \). Thus, \( T_\alpha E_3 = -\tau E_2 \), where the real valued function \( \tau \) is called the torsion function of the conformable curve \( \beta \).
**Theorem 4.2.** If \( \beta \) is a unit speed conformable curve with curvature \( \kappa > 0 \) and torsion \( \tau \) then,

\[
\begin{align*}
T_aE_1 &= \kappa E_2 \\
T_aE_2 &= -\kappa E_1 + \tau E_3 \\
T_aE_3 &= -\tau E_2
\end{align*}
\]

**Proof.** By Remark (4.1) first and third formulas are essentially just the definitions of curvature and torsion. For second equation, let use orthonormal expansion to express \( T_aE_2 \) in terms of \( E_1, E_2, E_3 \):

\[
T_aE_2 = (T_aE_2 \cdot E_1)E_1 + (T_aE_2 \cdot E_2)E_2 + (T_aE_2 \cdot E_3)E_3.
\]

\( \alpha \)–differentiating of \( E_2 \cdot E_1 = 0 \) gives \( T_aE_2 \cdot E_1 = -E_2 \cdot T_aE_1 = -E_2 \cdot \kappa E_2 = -\kappa \). As usual, \( T_aE_2 \cdot E_2 = 0 \), since \( E_2 \) is a unit vector field.

Finally, \( T_aE_2 \cdot E_3 = -E_2 \cdot T_aE_3 = -E_2 \cdot -\tau E_2 = \tau \). \( \square \)

**Remark 4.3.** The equations obtained in Theorem 4.2 are the same as the equations in Theorem 3.2 in [2].

### 4.2. Conformable Curves With Arbitrary Speed

**Remark 4.4.** Let \( \gamma \) be a conformable regular curve with arbitrary speed and \( \beta \) be the unit speed reparametrization of \( \gamma \). If \( s \) is an arc length function for \( \gamma \) as in Theorem (3.8), then \( \gamma(t) = \beta(s(t)) \) for all \( t \in (0, \infty) \). If \( \kappa\| \neq 0, \tau\|, \)

\( E_1, E_2, E_3 \) are defined for \( \beta \) as in Subsection 4.1, we define for \( \gamma \) the curvature function as \( \kappa = \kappa\|\( \beta \)(s) \), torsion function as \( \tau = \tau\|\( \beta \)(s) \), unit tangent vector field as \( E_1 = E_1\|\( \beta \)(s) \), principal normal vector field as \( E_2 = E_2\|\( \beta \)(s) \) and binormal vector field as \( E_3 = E_3\|\( \beta \)(s) \).

**Lemma 4.5.** If \( \gamma \) is a conformable regular curve in \( \mathbb{R}^3 \) with \( \kappa > 0 \) then

\[
\begin{align*}
T_aE_1 &= \kappa \nu \lambda^{1-\alpha} E_1 \\
T_aE_2 &= -\kappa \nu \lambda^{1-\alpha} E_1 + \tau \nu \lambda^{1-\alpha} E_3 \\
T_aE_3 &= -\tau \nu \lambda^{1-\alpha} E_2
\end{align*}
\]

where \( \lambda = \frac{t}{s(t)} \).

**Proof.** Let \( \beta \) be a unit speed reparametrization of \( \gamma \), then by Remark 4.4 \( E_1 = E_1\|\( \beta \)(s) \), where \( s \) is an arc length function for \( \gamma \). The \( \alpha \)–chain rule as applied to differentiation of vector fields gives

\[
T_aE_1(t) = T_a(E_1\|\( \beta \)(s(t)))) = (T_aE_1\|\( \beta \))(s(t))T_a(s(t))s(t)^{a-1}
= \kappa\|\( \beta \)(s(t))E_3\|\( \beta \)(s(t))\nu t^{1-\alpha} s(t)^{a-1}
= \kappa\|\( \beta \)(s(t))t^{1-\alpha} s(t)^{a-1} = \kappa \nu \lambda^{1-\alpha} E_2
\]

where \( \lambda = \frac{t}{s(t)} \).

The formulas for \( T_aE_2 \) and \( T_aE_3 \) are derived in the same way. \( \square \)
Theorem 4.6. Let \( \gamma \) be a conformable curve in \( \mathbb{R}^3 \). Then

\[
E_1 = \frac{T_\alpha' \gamma}{\|T_\alpha' \gamma\|}
\]
\[
E_3 = \frac{T_\alpha' \gamma \times T_\alpha'' \gamma}{\|T_\alpha' \gamma \times T_\alpha'' \gamma\|}
\]
\[
E_2 = E_3 \times E_1
\]
\[
\kappa = (\frac{t}{\lambda})^{1-a} \frac{\|T_\alpha' \gamma \times T_\alpha'' \gamma\|}{\|T_\alpha' \gamma\|^{3}}
\]
\[
\tau = (\frac{t}{\lambda})^{1-a} \frac{(T_\alpha' \gamma \times T_\alpha'' \gamma) \cdot T_\alpha'' \gamma}{\|T_\alpha' \gamma \times T_\alpha'' \gamma\|^2}
\]

Proof. Since \( T_\alpha' \gamma = vt^{1-a}E_1 \), we have \( E_1 = \frac{T_\alpha' \gamma}{vt^{1-a}} \). We know \( v = \frac{\|T_\alpha' \gamma\|}{t^{1-a}} \), then \( E_1 = \frac{T_\alpha' \gamma}{\|T_\alpha' \gamma\|} \).

\( T_\alpha'' \gamma(t) = T_\alpha(vt^{1-a}E_1) = (t^{1-a}T_\alpha \gamma + (1 - \alpha)t^{1-2a}v)E_1 + \kappa v^2(t\lambda)^{1-a}E_2 \).

So we have

\[
T_\alpha' \gamma \times T_\alpha'' \gamma = vt^{1-a}E_1 \times [(t^{1-a}T_\alpha \gamma + (1 - \alpha)t^{1-2a}v)E_1 + \kappa v^2(t\lambda)^{1-a}E_2] 
\]
\[
= vt^{1-a}T_\alpha \gamma E_1 + (1 - \alpha)t^{1-2a}vE_1 \times E_1 + \kappa v^2(t\lambda)^{1-a}E_1 \times E_2 
\]
\[
= \kappa v^3(\lambda t^2)^{1-a}E_3.
\]

If we take norms, we get \( \|T_\alpha' \gamma \times T_\alpha'' \gamma\| = \kappa v^3(\lambda t^2)^{1-a} \).

Then,

\[
\frac{T_\alpha' \gamma \times T_\alpha'' \gamma}{\|T_\alpha' \gamma \times T_\alpha'' \gamma\|}
\]

On the other hand,

\[
\kappa = \frac{\|T_\alpha' \gamma \times T_\alpha'' \gamma\|}{vt^{3}(\lambda t^2)^{1-a}} = (\frac{t}{\lambda})^{1-a} \frac{\|T_\alpha' \gamma \times T_\alpha'' \gamma\|}{(vt^{1-a})^3(\frac{\lambda}{t})^{1-a}}.
\]

Now we need only to find \( E_3 \) component of \( T_\alpha'' \gamma \). Then,

\( T_\alpha'' \gamma(t) = \kappa v^3(\lambda t^2)^{1-a}E_3 + ... \)

Consequently,

\[
(T_\alpha' \gamma \times T_\alpha'' \gamma) \cdot T_\alpha'' \gamma = \kappa^2 v^6(\lambda t)^{3(1-a)}\tau.
\]

Thus,

\[
\tau = \frac{(T_\alpha' \gamma \times T_\alpha'' \gamma) \cdot T_\alpha'' \gamma}{\kappa^2 v^6(\lambda t)^{3(1-a)}} = (\frac{t}{\lambda})^{1-a} \frac{(T_\alpha' \gamma \times T_\alpha'' \gamma) \cdot T_\alpha'' \gamma}{\|T_\alpha' \gamma \times T_\alpha'' \gamma\|^2}.
\]

\[\square\]

Remark 4.7. Although the equations in the case of unit speed conformable curves are structurally identical with usual Frenet equations, arbitrary speed curves are different equations from usual case.
Example 4.8. We compute the $\alpha$–Frenet apparatus of the conformable curve
\[ \gamma(t) = \left(3 \cos\left(\frac{t^\alpha}{\alpha}\right), 3 \sin\left(\frac{t^\alpha}{\alpha}\right), 4 \frac{t^\alpha}{\alpha}\right). \]

The $\alpha$–derivatives are
\begin{align*}
T_1\gamma(t) & = \left(-3 \sin\left(\frac{t^\alpha}{\alpha}\right), 3 \cos\left(\frac{t^\alpha}{\alpha}\right), 0\right), \\
T_2\gamma(t) & = \left(3 \cos\left(\frac{t^\alpha}{\alpha}\right), -3 \sin\left(\frac{t^\alpha}{\alpha}\right), 0\right), \\
T_3\gamma(t) & = \left(3 \sin\left(\frac{t^\alpha}{\alpha}\right), -3 \cos\left(\frac{t^\alpha}{\alpha}\right), 0\right).
\end{align*}

Then, $\|T_3\gamma(t)\| = 5$. Applying the definition of cross product yields
\[ T_1\gamma(t) \times T_2\gamma(t) = \left(12 \sin\left(\frac{t^\alpha}{\alpha}\right), -12 \cos\left(\frac{t^\alpha}{\alpha}\right), 0\right). \]

Hence, $\|T_1\gamma(t) \times T_2\gamma(t)\| = 15$. The expression above for $T_1\gamma(t) \times T_2\gamma(t)$ and $T_3\gamma(t)$ yield
\[ (T_1\gamma(t) \times T_2\gamma(t)) \cdot T_3\gamma(t) = 6. \]

By Theorem 4.6,
\begin{align*}
E_1(t) & = \left(-\frac{3}{\alpha} \sin\left(\frac{t^\alpha}{\alpha}\right), \frac{3}{\alpha} \cos\left(\frac{t^\alpha}{\alpha}\right), 0\right), \\
E_2(t) & = \left(-\cos\left(\frac{t^\alpha}{\alpha}\right), -\sin\left(\frac{t^\alpha}{\alpha}\right), 0\right), \\
E_3(t) & = \left(\frac{12}{\alpha} \sin\left(\frac{t^\alpha}{\alpha}\right), -\frac{12}{\alpha} \cos\left(\frac{t^\alpha}{\alpha}\right), \frac{9}{\alpha}\right), \\
\kappa(t) & = \frac{3}{25}\left(\frac{1}{\lambda}\right)^{\frac{1}{\alpha}-\frac{1}{3}}, \\
\tau(t) & = \frac{4}{25}\left(\frac{1}{\lambda}\right)^{\frac{1}{\alpha}-\frac{1}{2}}.
\end{align*}

5. Fundamental Theorem of the Local Theory of Conformable Curves

Theorem 5.1. Given $\alpha$–differentiable functions $\kappa(s) > 0$ and $\tau(s)$, $s \in (0, \infty)$, there exists a conformable regular parametrized curve $\gamma : (0, \infty) \to \mathbb{R}^3$ such that $s$ is the arc length, $\kappa(s)$ is the curvature and $\tau(s)$ is the torsion of $\gamma$.

Proof. The starting point is to observe that $\alpha$–Frenet equations
\begin{align*}
T_1\gamma & = \kappa E_2 \\
T_2\gamma & = -\kappa E_1 + \tau E_3 \\
T_3\gamma & = -\tau E_2
\end{align*}
\begin{equation}
\tag{8}
\end{equation}

may be considered as a system of fractional differential equations in $(0, \infty) \times \mathbb{R}^3$,
\begin{equation}
\tag{9}
\end{equation}

where $(\rho_1, \rho_2, \rho_3) = (E_1, \rho_4, \rho_5) = (E_2, \rho_7, \rho_8, \rho_9) = (E_3, \rho_{10}, \ldots, \rho_9)$ and $f_i, i = 1, \ldots, 9$ are linear functions (with coefficients that depend on $s$) of the coordinates $\rho_i$. 

Given initial conditions $s_0 \in (0, \infty)$, $\rho_1(s_0), ..., \rho_6(s_0)$, there exists an open interval $I \subset (0, \infty)$ containing $s_0$ and a unique $\alpha$–differentiable mapping $\gamma : I \to \mathbb{R}^3$, with $\gamma(s_0) = (\rho_1(s_0), ..., \rho_6(s_0))$ and $T_\alpha \gamma(s) = (f_1, ..., f_6)$, where each $f_i, i = 1, ..., 9$ is calculated in $(s, \gamma(s)) \in I \times \mathbb{R}^3$. Furthermore, if the system is linear, $I = (0, \infty)$.

It follows that given an orthonormal, positively oriented trihedron $[E_1(s_0), E_2(s_0), E_3(s_0)]$ in $\mathbb{R}^3$ and a value $s_0 \in (0, \infty)$, there exists a family of trihedrons $[E_1(s), E_2(s), E_3(s)]$, $s \in (0, \infty)$. We shall first show that the family $[E_1(s), E_2(s), E_3(s)]$ thus obtained remains orthonormal for every $s \in (0, \infty)$. By using the system (8) to express the $\alpha$–derivatives relative to $s$ of the six quantities

$$E_1 \cdot E_2, \quad E_1 \cdot E_3, \quad E_2 \cdot E_3, \quad E_1 \cdot E_1, \quad E_2 \cdot E_2, \quad E_3 \cdot E_3$$

as functions of these same quantities, we obtain that the system of $\alpha$–differential equations

$$T_\alpha E_1(s) = \kappa(s)E_2(s) + \tau(s)E_3(s),$$

$$T_\alpha E_2(s) = \kappa(s)E_1(s) - \tau(s)E_3(s),$$

$$T_\alpha E_3(s) = -\kappa(s)E_1(s) + \tau(s)E_2(s).$$

It is easily checked that

$$E_1 \cdot E_2 = 0, \quad E_1 \cdot E_3 = 0, \quad E_2 \cdot E_3 = 0, \quad E_1 \cdot E_1 = 1, \quad E_2 \cdot E_2 = 1, \quad E_3 \cdot E_3 = 1$$

is a solution of above system with initial conditions $0, 0, 0, 1, 1, 1$. By uniqueness, the family $[E_1(s), E_2(s), E_3(s)]$ is orthonormal for every $s \in (0, \infty)$, as we claimed.

From the family $[E_1(s), E_2(s), E_3(s)]$ it is possible to obtain a conformable curve by setting

$$\gamma(s) = I_\alpha E_1(s).$$

Example 5.2. Let $\alpha$–differentiable functions $\kappa(s) = 3$ and $\tau(s) = 4$. Then the associated fractional differential equations are

$$T_\alpha E_1 = 3E_2,$$

$$T_\alpha E_2 = -3E_1 + 4E_3,$$

$$T_\alpha E_3 = -4E_2.$$
and given initial conditions

\[ E_1(\frac{\pi^2}{400}) = (1, 0, 0) \]
\[ E_2(\frac{\pi^2}{400}) = (0, 1, 0) \]
\[ E_3(\frac{\pi^2}{400}) = (0, 0, 1). \]

Then the solutions of the system of \( \alpha \)-differential equations (10) are

\[ E_1(s) = (\frac{9}{25} \sin 10 \sqrt{s} + \frac{16}{25}, -\frac{3}{5} \cos 10 \sqrt{s} - \frac{12}{25} \sin 10 \sqrt{s} + \frac{12}{25}) \]
\[ E_2(s) = (\frac{3}{5} \cos 10 \sqrt{s} \sin 10 \sqrt{s}, -\frac{4}{5} \cos 10 \sqrt{s}) \]
\[ E_3(s) = (-\frac{12}{25} \sin 10 \sqrt{s} + \frac{12}{25}, \frac{4}{5} \cos 10 \sqrt{s} - \frac{16}{25} \sin 10 \sqrt{s} + \frac{9}{25}). \]

Finally, the conformable curve corresponding to fractional system (10) with initial conditions (11) is the curve

\[ \gamma(s) = \frac{1}{125} \left( 160 \sqrt{s} - 9 \cos 10 \sqrt{s} - 8\pi, -15 \sin 10 \sqrt{s} + 15, 120 \sqrt{s} + 12 \cos 10 \sqrt{s} - 6\pi \right). \]

6. Conclusions

In our study, the classical differential geometry of curves is investigated with respect to conformable fractional derivative and conformable fractional integral. In this sense, a conformable curve is defined and \( \alpha \)-Frenet formulas are given. If \( \alpha = 1 \), the theory of conformable curves coincides with the classical theory of curves. A conformable curve is a natural generalization of a classical curve. Studying this problem, we use conformable derivatives because it provides useful properties such as the product rule and chain rule compared to other fractional derivatives including Caputo and Riemann-Liouville. Thus the fundamental theorem of the local theory of the conformable curves is given. And, for \( \alpha \)-differentiable functions \( \kappa > 0 \) and \( \tau \), a system of fractional order differential equations is formed, then a conformable curve is obtained by the solution of this system.
Acknowledgements

The authors would like to thank the anonymous reviewers for their helpful and constructive comments that greatly contributed to improving the final version of the paper. They would also like to thank the Editors for their generous comments and support during the review process. Finally, the first author would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during his doctorate studies.

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