Evading the Grossman-Nir bound with $\Delta I = 3/2$ new physics

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Abstract

Rare kaon decays with missing energy, $K \rightarrow \pi + E_{\text{miss}}$, have received considerable attention because their rates can be calculated quite precisely within the standard model (SM), where the missing energy is carried away by an undetected neutrino-antineutrino pair. Beyond the SM, clean theoretical predictions can also be made regarding these processes. One such prediction is the so-called Grossman-Nir (GN) bound, which states that the branching fractions of the $K_L$ and $K^+$ modes must satisfy the relation $B(K_L \rightarrow \pi^0 + E_{\text{miss}}) \lesssim 4.3 B(K^+ \rightarrow \pi^+ + E_{\text{miss}})$ and applies within and beyond the SM, as long as the hadronic transitions change isospin by $\Delta I = 1/2$. In this paper we extend the study of these modes to include new-physics scenarios where the missing energy is due to unobserved lepton-number-violating neutrino pairs, invisible light new scalars, or pairs of such scalars. The new interactions are assumed to arise above the electroweak scale and described by an effective field theory. We explore the possibility of violating the GN bound through $\Delta I = 3/2$ contributions to the $K \rightarrow \pi$ transitions within these scenarios and find that large violations are only possible in the case where the missing energy is due to an invisible light new scalar.

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I. INTRODUCTION

The rare kaon decays $K_L \to \pi^0\nu\bar{\nu}$ and $K^+ \to \pi^+\nu\bar{\nu}$ are known as the “golden modes” of kaon physics because they can be predicted quite precisely within the standard model (SM) and are potentially sensitive to the presence of new physics (NP) beyond it \[1, 2\]. Consequently they have received a great deal of theoretical and experimental attention. The SM predictions for their branching fractions have been known for many years \[3-10\], and their current values are \[11-13\] and have received a great deal of theoretical and experimental attention. The SM predictions for their potentially sensitive to the presence of new physics (NP) beyond it \[1, 2\]. Consequently they

In a framework of effective field theory with only SM fields, the introduction of (predominantly) CP-violating interactions which change lepton flavor/number or possess new sources of $CP$ violation also would not bring about a disruption of the GN bound \[30, 31\]. On the other hand, it has been

On the experimental side, the KOTO Collaboration at J-PARC in 2018 set an upper limit on the neutral mode based on the data collected in 2015 \[13\]: $\mathcal{B}(K_L^0 \to \pi^0\nu\bar{\nu})_{\text{KOTO15}} < 3.0 \times 10^{-9}$ at 90% confidence level (CL). For the charged mode, the combined BNL E787/949 experiments had earlier yielded $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})_{\text{E949}} = (1.73^{+1.15}_{-1.03}) \times 10^{-10}$ \[16, 17\]. Last year the NA62 Collaboration \[18\] at CERN reported the preliminary limit $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})_{\text{NA62}} < 1.85 \times 10^{-10}$ at 90% CL \[19\]. All of these results are in agreement with the SM but leave open a window for NP. This possibility is very intriguing, especially in light of the recent preliminary observation by KOTO of three candidate events in the $K_L \to \pi^0\nu\bar{\nu}$ signal region \[20\], with a single event sensitivity of $6.9 \times 10^{-10}$ having been achieved. If interpreted as signal, they imply a decay rate about two orders of magnitude higher than the SM prediction and in conflict with the experimentally established GN bound $\mathcal{B}(K_L^0 \to \pi^0\nu\bar{\nu})_{\text{GN}} < 4.3 \mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})_{\text{NA62}} = 8.0 \times 10^{-10}$ at 90% CL.

It is interesting that there are NP scenarios which can overcome this requisite. For instance, as was first pointed out in ref. \[21\], the branching fraction of $K_L \to \pi^0\chi$ with $\chi$ being an invisible particle with mass $m_\chi$ chosen to be around the pion mass, can exceed the aforementioned cap of $8.0 \times 10^{-10}$ because quests for the charged channel $K^+ \to \pi^+\chi$ do not cover the $m_\chi \sim m_\pi$ region to avoid the sizable $K^+ \to \pi^+\pi^0$ background \[17, 18\]. This kinematic loophole has been exploited in recent attempts \[22, 26\] to account for KOTO’s three anomalous events. As another example, imposing the condition $m_{K^+} - m_{\pi^+} < m_\chi < m_{K^0} - m_{\pi^0}$ renders the $K^+ \to \pi^+\chi$ channel closed (here $\chi$ can be more than one invisible particle), whereas $K_L \to \pi^0\chi$ with a big rate can still happen \[29\]. For other $m_\chi$ ranges, the stringent restriction on $K^+ \to \pi^+\chi$ can be evaded or weakened if $\chi$ is a particle with an average decay length bigger than the KOTO detector size but less than its E949 and NA62 counterparts \[22, 26\]. These cases, while still fulfilling the GN relation, only appear to contradict it by enhancing $\mathcal{B}(K_L^0 \to \pi^0\chi)$ substantially above $\mathcal{B}(K^0_L \to \pi^0\nu\bar{\nu})_{\text{GN}}^{\text{max}}$. In a framework of effective field theory with only SM fields, the introduction of (predominantly) $\Delta I = 1/2$ interactions which change lepton flavor/number or possess new sources of $CP$ violation also would not bring about a disruption of the GN bound \[30, 31\]. On the other hand, it has been
proposed that in the presence of additional light particles mediating these decays the GN bound could be violated \[32\].

In terms of the branching-fraction ratio \( r_B = \mathcal{B}(K_L \to \pi^0 \nu \bar{\nu}) / \mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu}) \), the GN bound has a theoretical model-independent maximum of \( r_B^{\text{GN}} = 4.3 \), which differs much from the central value \( r_B^{\text{SM}} = 0.36 \) of the SM prediction. A key ingredient in the derivation of the GN inequality is the assumption that the \( K \to \pi \) transitions are mediated by a two-quark \( s \leftrightarrow d \) operator, which necessarily carries isospin 1/2 and leads to a ratio of amplitudes of the neutral and charged modes given by \( A_{K^0 \to \pi^0}^\Delta I=1/2 / A_{K^+ \to \pi^+}^\Delta I=1/2 = -1/\sqrt{2} \). A true violation of the GN relation requires the \( K \to \pi \) transitions to occur via a \( \Delta I = 3/2 \) interaction as well, and this possibility has recently been investigated in refs. \[31, 33\]. A pure \( \Delta I = 3/2 \) operator would result in an amplitude ratio \( A_{K^0 \to \pi^0}^\Delta I=3/2 / A_{K^+ \to \pi^+}^\Delta I=3/2 = \sqrt{2} \) and thus translate into \( r_B^{\Delta I=3/2} \lesssim 17 \) \[31\].

In this paper we present a systematic study on breaking the GN bound with quark-level operators in the context of effective field theory (EFT). Given that the measurements of interest look for \( K \to \pi \) with \( \pi \) standing for one or more particles carrying away missing energy, we will consider for \( \pi \) several different possibilities: a neutrino-antineutrino pair \((\nu \bar{\nu})\), a pair of neutrinos \((\nu \nu)\) or antineutrinos \((\bar{\nu} \bar{\nu})\), a new invisible and light real scalar field \((S)\), and a pair of the new scalars \((SS)\). Since an operator giving rise to \( \Delta I = 3/2 \) \( K \to \pi \) transitions must contain at least four light-quark fields, the minimal mass dimension of such an operator is seven, eight, nine, and ten for \( \pi = S, SS, \nu \bar{\nu} \) or \( \nu \nu \) and \( \nu \bar{\nu} \), respectively. Being unobserved in the experiments, the neutrinos may have different flavors and can also be replaced by new invisible light fermions. We suppose that the new particles are invisible because they are sufficiently long-lived to escape detection, decay invisibly, or are stable. We will look at a complete set of operators with the lowest dimension required for each of these cases, assuming that the interactions of the new light particles with the quarks can be described by an EFT approach valid above the electroweak-symmetry breaking scale.

Before embarking in a detailed study, it is instructive to present a simple dimensional-analysis estimate for the scale \( \Lambda_{\text{NP}} \) of NP that is needed in order to produce a rate of \( K \to \pi \) that is comparable in size to the SM rate. To this end, it is useful to recall the effective Hamiltonian responsible for \( K \to \pi \nu \bar{\nu} \) in the SM,

\[
H_{\text{SM}} = \frac{G_F g^2 V_{ts}^* V_{td} X(x_t)}{\sqrt{2}} \frac{1}{16 \pi^2} S \gamma_\mu (1 - \gamma_5) d \sum_\ell \bar{\nu}_\ell \gamma^\mu (1 - \gamma_5) \nu_\ell + \text{H.c.}
\]  \( \text{(1)} \)

in the conventional notation \[13\], where \( X(x_t) \approx 1.5 \) from top-quark loops. In contrast, a generic NP operator which induces a \( \Delta I = 3/2 \) transition in the process \( K \to \pi X \) can be written as

\[
\mathcal{L}_{\text{NP}} = \frac{1}{\Lambda_{\text{NP}}^{2+n_X}} \mathcal{C} s \Gamma_1 u \bar{u} \Gamma_2 d \mathcal{X} + \text{H.c. ,}
\]  \( \text{(2)} \)

where \( n_\mathcal{X} \) tells the mass dimension of \( \mathcal{X} \) (so \( n_{\nu \bar{\nu}} = 3 \), \( n_S = 1 \), \( n_{SS} = 2 \), etc.), the coefficient \( \mathcal{C} \) is a constant, and \( \Gamma_{1,2} \) represent gamma matrices. At the amplitude level, the \( \mathcal{L}_{\text{NP}} \) contribution to
$K \to \pi\mathcal{X}$ relative to the SM top-quark contribution is then

$$\frac{A_{NP}}{A_{SM}} \sim 3.8 \times 10^5 \mathcal{C} \left( \frac{v}{\Lambda_{NP}} \right)^2 \left( \frac{m_K}{\Lambda_{NP}} \right)^{n_{\mathcal{X}}},$$

(3)

where the large numerical factor reflects the one-loop and CKM-angle suppression of the SM coefficient, $v = 2^{-1/4}G^{-1/2}_F \simeq 246$ GeV indicates the electroweak scale, and the relevant hadronic scale is taken to be the kaon mass, $m_K$. If the NP is defined to enter $\mathcal{L}_{NP}$ with $\mathcal{C}$ of order one (so any loop or mixing-angle suppression factor is absorbed into $\Lambda_{NP}$), the result in eq. (3) implies that $\Lambda_{NP} \sim 2.2$ TeV, 275 GeV, 78 GeV for $n_{\mathcal{X}} = 1, 2, 3$, respectively, corresponds to NP effects at the same level as the SM contribution.

The EFT that we employ in this paper only makes sense for $\Lambda_{NP} > v$. Otherwise, the organization of effective operators in terms of their dimensionality breaks down. This suggests that scenarios with $\mathcal{X} = S$ could amplify the rate of $K_L \to \pi^0\mathcal{X}$ to the level implied by the three KOTO events; scenarios with $\mathcal{X} = SS$ may increase the corresponding rate compared to its SM counterpart by factors of 2; and that scenarios in which $\mathcal{X} = \nu\bar{\nu}$ or $\bar{\nu}\nu$ or $\nu\bar{\nu}$ can only modify the $K \to \pi + E_{\text{miss}}$ rates marginally.

We will explore all these possibilities in detail to quantify how the underlying NP interactions influence the $K_L \to \pi^0 + E_{\text{miss}}$ decays. To handle a low-energy process involving hadrons, it is necessary to hadronize the quark-level operators at the mass scale where it takes place. For our investigation, this entails evaluating the effects of QCD renormalization group (RG) running from the EW scale to the chiral-symmetry breaking scale and subsequently matching the resulting operators to a low-energy chiral Lagrangian suitable to describe the $K \to \pi + E_{\text{miss}}$ transition. In all the cases, we present numerical results illustrating the range of values which the ratio $r_B$ can take when the NP scale is under a couple of TeV.

The arrangement of the rest of the paper is as follows. In section II we study how the GN bound can be violated through $K \to \pi\nu\nu, \pi\bar{\nu}\bar{\nu}$ caused by $\Delta I = 3/2$ interactions in the EFT approach where the operators respect the SM gauge symmetry (SMEFT). In section III we extend the SMEFT to include a SM-singlet scalar $S$ and carry out a similar analysis with $K \to \pi S, \pi SS$. In section IV we draw our conclusions. In appendixes, we provide some details on the RG running for the SMEFT operators and on the low-energy chiral bosonization of the four-quark operators pertaining to the $K \to \pi + E_{\text{miss}}$ processes.

II. GN-BOUND VIOLATION FROM EFT OPERATORS FOR $K \to \pi 2\nu$

A. EFT operators for $K \to \pi 2\nu$

In this section we restrict the fields of the EFT to only those in the minimal SM. Accordingly, in the kaon decays of concern the missing energy is carried away by a pair of SM neutrinos (2$\nu$).
It may have no or nonzero lepton number depending on whether the underlying interaction is lepton-number conserving or violating, respectively. If the small contributions to $K \rightarrow \pi 2\nu$ from long-distance physics [1, 30, 34–36] are neglected, the only possible way to break the GN bound significantly in this case is through the inclusion of $\Delta I = 3/2$ operators, which must have nonleptonic parts with at least four quark fields [31, 33]. It follows that the lowest dimension of quark-neutrino operators with $\Delta I = 3/2$ components is nine.

In the SMEFT treatment the effective operators constructed must be singlets under the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, while in the low-energy effective field theory (LEFT) the operators need to be singlets only under the strong and electromagnetic gauge group $SU(3)_C \times U(1)_{em}$. Consequently, there are in general more requirements on the SMEFT operators than on the LEFT ones, which makes the number of quark operators relatively less in the former case. Since furthermore there is still no discovery of new particles beyond the SM, we will rely on the SMEFT to perform our examination.

The fundamental fields (with their SM gauge group assignments) available to construct the pertinent operators are the $U(1)_Y$ gauge field $B (1, 1, 0)$, the $SU(2)_L$ gauge field $W (1, 3, 0)$, the $SU(3)_C$ gluon field $G (8, 1, 0)$, the Higgs doublet $H (1, 2, 1/2)$, the left-handed quark doublet $Q (3, 2, 1/6)$, the right-handed quark fields $u (3, 1, 2/3)$ and $d (3, 1, -1/3)$, the left-handed lepton doublet $L (1, 2, -1/2)$, and the right-handed charged lepton field $e (1, 1, -1)$. All the quarks and leptons come in three families. In the SMEFT approach, operators with four quarks and two neutrinos are necessarily of dimension nine (dim-9) or higher.

If lepton number is conserved in the decays of concern, the responsible operator necessarily involves a pair of $L$ and $\bar{L}$, which provides the $\nu \bar{\nu}$ in $K \rightarrow \pi \nu \bar{\nu}$. One can always utilizes some appropriate Fierz relations to arrange the lepton fields such that they appear in the operators in the form $L\gamma_\mu L$ or $LD_\mu \gamma_\rho L$, with $D_\mu$ being a covariant derivative. To join $L\gamma_\mu L$ with four quark fields to form an operator that is a singlet under the SM gauge group, the quark portion needs to have a Lorentz index to contract with the one in the lepton current. Thus, the possible lowest-dimensional quark parts are $(\bar{Q}\gamma_\mu Q, \bar{q}\gamma_\mu q)(\bar{Q}q, \bar{q}Q)$, where $q$ may be $u$ or $d$ as appropriate, but they have odd numbers of $Q$ and hence are not $SU(2)_L$ singlets yet. These quark combinations can be made singlets by incorporating the Higgs field $H$, and so the full quark-lepton operators have dimension ten (dim-10). Additionally, one can insert $D_\mu$ in $(\bar{Q}q, \bar{q}Q)(\bar{Q}q, \bar{q}Q)$ to form singlets, and the resulting operators are also of dim-10. If the lepton bilinear is $LD_\mu \gamma_\rho L$ instead, the lowest-dimensional possibilities of the four-quark portion are $\bar{q}\gamma_\mu q\bar{q}\gamma_\rho q$, $g^{\mu\rho}\bar{Q}qqQ$, $\bar{Q}\sigma^{\mu\rho}qqQ$, $\bar{Q}qq\sigma^{\mu\rho}Q$, and $\bar{Q}\gamma_\mu Q\bar{Q}\gamma_\rho Q$. Again the singlet quark-lepton operators constructed are of dim-10.

1 We also ignore isospin-breaking effects due to the $u$- and $d$-quark mass difference and electroweak radiative corrections.

2 In this and the next paragraph, we suppress the family labels of the SM fermion fields. Generally these processes may change quark and/or lepton flavors.
If lepton-number violation is allowed, the situation is different, as we have $K \to \pi \nu \nu, \pi \bar{\nu} \bar{\nu}$. The lepton bilinear of the lowest dimension can be organized in the form $LL^c$ ($\bar{L} \sigma_{\mu \nu} L^c$) or its Hermitian conjugate. One can attach it to $\bar{Q} q q Q$ or $q \bar{q} Q Q$ ($\bar{Q} \sigma^{\mu \nu} q q Q$, $\bar{Q} q q \sigma^{\mu \nu} Q$, or $q \sigma^{\mu \nu} Q q Q$) to form a SM singlet. Thus, the resulting operators have dim-9, which is less than that of the lepton-number-conserving ones mentioned in the previous paragraph. Hereafter in this section, we concentrate on the dim-9 operators and later briefly comment on the dim-10 case.

It has been shown that all dim-9 SMEFT operators do not preserve lepton and/or baryon numbers [37]. We will not be interested in the baryon-number violating ones, as our aim is to study how dim-9 operators give rise to $K \to \pi \nu \nu, \pi \bar{\nu} \bar{\nu}$ and can violate the GN bound. In the following we enumerate all of those containing one strange quark $s$ and two neutrinos or antineutrinos. Upon imposing the SM gauge symmetry and applying Fierz transformations, we find the independent operators that can induce $K \to \pi \nu \nu$ to be

$$O^{opxy,\alpha\beta}_1 = \epsilon_{ij} \epsilon_{kl}(\bar{Q} \sigma^\mu \gamma^\nu Q)^j_p (\bar{d} \sigma^\mu u)_y (\bar{c} \sigma^\nu L^i_q) L^j_p \{\alpha \beta\},$$
$$O^{opxy,\alpha\beta}_1 = \epsilon_{ij} \epsilon_{kl}(\bar{Q} \sigma^\mu \gamma^\nu Q)^j_p (\bar{d} \sigma^\mu u)_y (\bar{c} \sigma^\nu L^j_p L^i_q) \{\alpha \beta\},$$
$$O^{opxy,\alpha\beta}_2 = \epsilon_{ij} \epsilon_{kl}(\bar{Q} \sigma^\mu \gamma^\nu Q)^j_p (\bar{d} \sigma^\mu u)_y (\bar{c} \sigma^\nu L^j_p L^i_q) \{\alpha \beta\},$$
$$O^{opxy,\alpha\beta}_2 = \epsilon_{ij} \epsilon_{kl}(\bar{Q} \sigma^\mu \gamma^\nu Q)^j_p (\bar{d} \sigma^\mu u)_y (\bar{c} \sigma^\nu L^j_p L^i_q) \{\alpha \beta\},$$
$$O^{opxy,\alpha\beta}_3 = \epsilon_{ik} \epsilon_{jl}(\bar{d} \sigma^\mu Q^j_p) (\bar{d} \sigma^\nu Q^i_q) L^l_p L^k_q \{\alpha \beta\},$$
$$O^{opxy,\alpha\beta}_3 = \epsilon_{ik} \epsilon_{jl}(\bar{d} \sigma^\mu Q^j_p) (\bar{d} \sigma^\nu Q^i_q) L^l_p L^k_q \{\alpha \beta\},$$
$$O^{opxy,\alpha\beta}_4 = \epsilon_{ik} \epsilon_{jl}(\bar{d} \sigma^\mu Q^j_p) (\bar{d} \sigma^\nu Q^i_q) L^l_p L^k_q \{\alpha \beta\},$$
$$O^{opxy,\alpha\beta}_4 = \epsilon_{ik} \epsilon_{jl}(\bar{d} \sigma^\mu Q^j_p) (\bar{d} \sigma^\nu Q^i_q) L^l_p L^k_q \{\alpha \beta\},$$

(4)

where $o, p, x, y (\alpha, \beta)$ denote quark (lepton) family indices, summation over the SU(2)$_L$ indices $i, j, k, l = 1, 2$ is implicit, and the leptonic scalar (tensor) currents have been arranged to be symmetric (antisymmetric) in their family indices with the convention $A_{[\alpha B\beta]} = (A_{\alpha B\beta} + A_{B\alpha B\beta})/2$ and $A_{[\alpha B\beta]} = (A_{\alpha B\beta} - A_{B\alpha B\beta})/2$. The two brackets ( , ) and [ , ] in the quark bilinears distinguish the two different ways of color contraction in the products of four quark fields to form color invariants: $q^m_i q^m_j q^n_k = (q_i q_j)(q_k q_l)$ and $q^m_i q^n_j q^n_k = (q_i q_j)[q_k q_l]$, with the color labels $m, n = 1, 2, 3$ being summed over. Each of the operators is accompanied by an unknown Wilson coefficient $C_i$, so that $O^{opxy,\alpha\beta}_1$ belongs to $O^{opxy,\alpha\beta}_1$, etc. The Hermitian conjugates of $O^{opxy,\alpha\beta}_{1,2,3,4}$ and $\tilde{O}^{opxy,\alpha\beta}_{1,2,3,4}$ contribute to $K \to \pi \nu \nu$.

As already mentioned, the SMEFT operators in eq. (4) are assumed to arise from NP above the EW scale. Consequently, to address their potential impact on $K \to \pi \nu \nu, \pi \bar{\nu} \bar{\nu}$, we will first need to take into account the effects of QCD on the evolution of the coefficients from the NP scale down to the hadronic scale, which we select to be the chiral-symmetry breaking scale $\Lambda_\chi = 4\pi F_\pi \approx 1.2$ GeV with $F_\pi$ being the pion decay constant. Subsequently, we will rely on chiral

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3 The factorization of the quark and lepton components is guaranteed by the application of Fierz transformations.
perturbation theory [38–40], in conjunction with spurion techniques [41–43], to derive the meson-neutrino operators which contribute to the amplitudes for the kaon decays. From chiral power-counting arguments in this procedure, the operators in eq. (4) with the leptonic scalar density are neutrino operators which contribute to the amplitudes for the kaon decays. From chiral power-counting theory [38–40], in conjunction with spurion techniques [41–43], we derive the meson operators (4) that the latter operators yield contributions to the amplitudes which are suppressed relative to those of the former by the factor $p_K p_\pi / \Lambda^2 \sim 0.05$. Therefore, examining the operators in eq. (4) pertaining to $K \rightarrow \pi \nu \nu, \pi \bar{\nu} \bar{\nu}$ and neglecting those with the leptonic tensor current, in what follows we focus on

$$
\begin{align*}
O_1^{usdu} &= (\bar{u}_L \gamma_\mu s_L)(\bar{d}_R \gamma_\mu u_R) J, \\
\tilde{O}_1^{usdu} &= (\bar{u}_L \gamma_\mu s_L)(\bar{d}_R \gamma_\mu u_R) J,
\end{align*}
\begin{align*}
O_1^{udsu} &= (\bar{u}_L \gamma_\mu d_L)(\bar{s}_R \gamma_\mu u_R) J, \\
\tilde{O}_1^{udsu} &= (\bar{u}_L \gamma_\mu d_L)(\bar{s}_R \gamma_\mu u_R) J,
\end{align*}
\begin{align*}
O_3^{dds} &= (\bar{d}_R d_L)(\bar{d}_R s_L) J, \\
\tilde{O}_3^{dds} &= (\bar{d}_R d_L)(\bar{d}_R s_L) J,
\end{align*}
\begin{align*}
O_3^{dss} &= (\bar{d}_R d_L)(\bar{s}_R d_L) J, \\
\tilde{O}_3^{dss} &= (\bar{d}_R d_L)(\bar{s}_R d_L) J,
\end{align*}
$$

where the neutrino part is expressed as $J = (\bar{\nu}_\alpha \nu_\beta)/(1 + \delta_{\alpha\beta})$.

B. Evaluation of hadronic matrix elements at low energies

In treating the kaon decay amplitudes, the contributions of the operators generated by NP above the EW scale need to be evaluated at the low energy of interest. This entails dealing with the QCD RG running of the Wilson coefficients by resumming the large logarithms caused by the ratio of the EW scale, which we take to be the $W$-boson mass $m_W$, to the chiral-symmetry breaking scale, $\Lambda$. Thus, using the one-loop QCD running results of ref. [43], we have

$$
\frac{d}{d\mu} \left( \frac{C_{axy}^{xyu}}{\tilde{C}_{axy}^{xyu}} \right) = -\frac{\alpha_s}{2\pi} \begin{pmatrix} 3 N & 0 \\ 6C_F & 1 \end{pmatrix} \left( \frac{C_{axy}^{xyu}}{\tilde{C}_{axy}^{xyu}} \right),
\frac{d}{d\mu} \left( \frac{C_{3ddxy}^{ddxy}}{\tilde{C}_{3ddxy}^{ddxy}} \right) = -\frac{\alpha_s}{2\pi} \begin{pmatrix} 2 N + 6C_F - 4 & 2 N - 4C_F + 2 \\ 4 N - 2 & -2 N - 2C_F - 2 \end{pmatrix} \left( \frac{C_{3ddxy}^{ddxy}}{\tilde{C}_{3ddxy}^{ddxy}} \right),
$$

where in the superscripts $xy = sd$ or $ds$, the color number $N = 3$, and $C_F = (N^2 - 1)/(2N) = 4/3$ is the second Casimir invariant of the color group $SU(3)_c$. The solutions to these RG equations which connect the coefficients at the two scales are [43]

\begin{align*}
C_{1^{axyu}}(\Lambda) &= 0.88 C_{1^{axyu}}(m_W), \\
\tilde{C}_{1^{axyu}}(\Lambda) &= 2.74 \tilde{C}_{1^{axyu}}(m_W) + 0.62 C_{1^{axyu}}(m_W), \\
C_{3^{ddxy}}(\Lambda) &= 1.82 C_{3^{ddxy}}(m_W) - 0.34 C_{3^{ddxy}}(m_W), \\
\tilde{C}_{3^{ddxy}}(\Lambda) &= 0.52 \tilde{C}_{3^{ddxy}}(m_W) - 0.08 C_{3^{ddxy}}(m_W).
\end{align*}

\begin{align*}
(\Lambda) &= 0.88 C_{1^{axyu}}(m_W), \\
\tilde{C}_{1^{axyu}}(\Lambda) &= 2.74 \tilde{C}_{1^{axyu}}(m_W) + 0.62 C_{1^{axyu}}(m_W), \\
C_{3^{ddxy}}(\Lambda) &= 1.82 C_{3^{ddxy}}(m_W) - 0.34 C_{3^{ddxy}}(m_W), \\
\tilde{C}_{3^{ddxy}}(\Lambda) &= 0.52 \tilde{C}_{3^{ddxy}}(m_W) - 0.08 C_{3^{ddxy}}(m_W).
\end{align*}
In the matching to chiral perturbation theory (χPT), the neutrino bilinear in a dim-9 operator behaves as a fixed external source. Thus, we only have to work with the quark portion of the operator. Suppose the latter has been decomposed into a sum of irreducible representations of the chiral group SU(3)_L × SU(3)_R under which the quarks transform as

\[ q_{L,a} \rightarrow \hat{L}_{ap} q_{L,p}, \quad \bar{q}_{R,c} \rightarrow \bar{q}_{R,p} \hat{R}^\dagger_{pc}, \quad q_{R,a} \rightarrow \hat{R}_{ap} q_{R,p}, \quad \bar{q}_{L,c} \rightarrow \bar{q}_{L,p} \hat{L}^\dagger_{pc}, \]  

(8)

where the indices \( a, c, p = 1, 2, 3 \) refer to the flavor space, summation over \( p \) is implicit, \( \hat{L} \in SU(3)_L \), and \( \hat{R} \in SU(3)_R \). Given that an irreducible representation has the general form

\[ O_q = T_{cd,ab} (\bar{q}_{\chi_1,c} \Gamma_1 q_{\chi_2,a}) (\bar{q}_{\chi_3,d} \Gamma_2 q_{\chi_4,b}), \]  

(9)

where the \( T_{cd,ab} \) represent pure numbers which depend on the irrep under consideration, the flavor indices \( a, b, c, d = 1, 2, 3 \) are summed over, \( \chi_{1,2,3,4} = L, R \), and \( \Gamma_{1,2} \) stand for combinations of Dirac matrices, then promoting \( T \) to be a spurion field that transforms properly together with the chiral transformations of quarks would render \( O_q \) chirally invariant.

On the χPT side, we introduce the standard matrices for the lightest octet of pseudoscalar mesons,

\[ \Sigma = \xi^2, \quad \xi = \exp \left( \frac{i \Pi}{\sqrt{2} F_0} \right), \quad \Pi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & K^0 & -\sqrt{\frac{2}{3}} \eta \end{pmatrix}, \]  

(10)

where \( F_0 = F_\pi/1.0627 \approx 87 \) MeV is the meson decay constant in the chiral limit. Under chiral transformations the \( \Sigma \) and \( \xi \) matrices transform as

\[ \Sigma \rightarrow \hat{L} \Sigma \hat{R}^\dagger, \quad \xi \rightarrow \hat{L} \xi \hat{U}^\dagger = \hat{U} \xi \hat{R}^\dagger, \]  

(11)

and so \( \hat{U} \in SU(3)_V \) depends on the meson fields. From the second formula, in terms of matrix elements we have

\[ \xi_{ab} \rightarrow \hat{L}_{ap} (\xi \hat{U}^\dagger)_{pb} = (\hat{U} \xi)_{ap} (\hat{R}^\dagger)_{pb}, \quad (\xi^\dagger)_{ab} \rightarrow \hat{R}_{ap} (\xi^\dagger \hat{U}^\dagger)_{pb} = (\hat{U} \xi^\dagger)_{ap} (\hat{L}^\dagger)_{pb}. \]  

(12)

How to construct the leading-order (LO) mesonic interactions from the quark operators has been prescribed previously in the literature [41–43]. Accordingly, eq. (12) implies that the matching to χPT involves the substitutions [43]

\[ \bar{q}_{L,a} \Rightarrow \xi^\dagger_{ea}, \quad q_{L,a} \Rightarrow \xi_{ae}, \quad \bar{q}_{R,a} \Rightarrow \xi_{ea}, \quad q_{R,a} \Rightarrow \xi^\dagger_{ae}, \]  

(13)

where the free indices \( e \) are to be contracted when forming an operator with \( T_{cd,ab} \).

For a given quark operator, the first step in the matching is to decompose it according to the irreducible representations (irreps) of the chiral group SU(3)_L × SU(3)_R. One can easily see
that $\tilde{O}_{1}^{udsu}$, $O_{1}^{udsu}$, and $\tilde{O}_{1}^{udsu}$ in eq. (5) belong to the $8_{L} \times 8_{R}$ irrep of the chiral group, while $O_{3}^{dds}$, $\tilde{O}_{3}^{dds}$, $O_{3}^{dds}$, and $\tilde{O}_{3}^{dds}$ belong to the $6_{L} \times 6_{R}$. Then, after applying eq. (13), one associates with the mesonic counterpart of each irrep a low energy constant $g_{\text{irrep}}$, which encodes nonperturbative QCD effects and is to be determined usually by fitting to data or a model calculation.

For example, the LEC associated with $O_{1}^{udsu} = (\bar{u}_L \gamma_\mu s_L)(\bar{d}_R \gamma_\mu u_R)_J$, which transforms as the $8_{L} \times 8_{R}$ under the chiral group, can be called $g_{8 \times 8}$, and the leading-order chiral realization of $O_{1}^{udsu}$ is derived via the procedure

$$O_{1}^{udsu} \Rightarrow \frac{F^4_0}{4} g_{8 \times 8} \xi_1 \xi_2 \xi_3 \xi_4 \delta_\phi \delta_\phi J = \frac{F^4_0}{4} g_{8 \times 8} (\xi \xi)_32(\xi_1 \xi_1)_11 J = \frac{F^4_0}{4} g_{8 \times 8} \Sigma_3 \Sigma_1 \rightarrow \frac{g_{8 \times 8}}{4} F^4_0 \sqrt{3} \xi \xi \phi \phi \phi \phi \phi \phi \phi \phi
$$

where factor $F_0^4/4$ is a normalization factor [42], other contractions among $\varepsilon, \varrho, \zeta, \phi$ in the first line vanish due to the unitarity of $\xi$, and the ellipsis represents terms with the $\eta$ field and more than two meson fields. This result is independent of the Lorentz and color structures of $O_{1}^{udsu}$ and follows from its transformation properties as an irrep of the chiral group. Evidently, $g_{8 \times 8}$ has mass dimension two. This example can be understood from the perspective of bosonization of each quark bilinear: Fierz-transforming $Q_{1}^{udsu}$ yields $-2\bar{u}_L u_R d^m L d^m R$ which is summed over the color labels $m, n$ and has a lower-order chiral realization than $(\bar{u}_L \gamma_\mu s_L)(\bar{d}_R \gamma_\mu u_R)$ being naively taken to correspond to the higher order $(\partial^\mu \Sigma \Sigma^l)_31(\Sigma^l \partial_\mu \Sigma)_12$ due to the derivatives. If the operator comprised instead solely left-handed quarks, $(\bar{u}_L \gamma_\mu s_L)(\bar{d}_L \gamma_\mu u_L)$, its chiral realization would have to be of the form $(\partial^\mu \Sigma \Sigma^l)_{op}(\partial_\mu \Sigma \Sigma^l)_{op}$ because no scalar density can be constructed with such quarks.

In table[1] we collect the chiral realization of the quark component of each of the operators in eq. (5) according to their chiral irreps. In the last column, we display the leading-order contributions to the $K \to \pi$ transitions decomposed in terms of the definite $\Delta I = 1/2, 3/2$ combinations

$$Q_{1/2}^S = F_0^2 \left( K^+ \pi^- \frac{1}{\sqrt{2}} K^0 \pi^0 \right), \quad Q_{3/2}^S = F_0^2 \left( K^+ \pi^- \sqrt{2} K^0 \pi^0 \right),$$

respectively, or their Hermitian conjugates.\textsuperscript{4} It is worth pointing out that in the combination $5 Q_{1/2}^S - 2 Q_{3/2}^S$, which occurs in all of the lines in table[1] and hence also implicitly in eq. (14), the size of the $K^0 \pi^0$ term relative to the $K^+ \pi^-$ one is three times that in $Q_{1/2}^S$ alone. It follows that every one of these operators may potentially break the GN relation.

The table also shows that each operator with a tilde and its counterpart without it have the same chiral realization but their LECs are different. This is attributable to the fact that the chiral realization of a quark-level operator relies only on its representation under the chiral group,

\textsuperscript{4} For completeness, in appendix[1] we decompose of the quark part of each operator in eq. (5) in terms of its $\Delta I = 1/2, 3/2$ components.
TABLE I: The chiral representations and realizations of the four-quark portions of the effective operators in eq. (5). In the last column, \( Q_{1/2}^S \) and \( Q_{3/2}^S \) are the mesonic operators defined in eq. (15) and correspond to \( \Delta I = 1/2 \) and \( 3/2 \) transitions, respectively.

whereas the LEC encodes QCD effects. Numerically, we adopt the values of the LECs extracted from ref. [42] which employed \( \chi \)PT to connect the matrix elements of \( \pi^+ \to \pi^- \) transitions to kaon-mixing matrix elements for which lattice QCD results were available. Thus we have

\[
\begin{align*}
g_{8 \times 8} &= -2.9 \text{ GeV}^2, \quad \bar{g}_{8 \times 8} = -12.4 \text{ GeV}^2, \quad g_{6 \times 6} = 2.7 \text{ GeV}^2, \quad \bar{g}_{6 \times 6} = -0.91 \text{ GeV}^2. \quad (16) 
\end{align*}
\]

C. Numerical analysis

From the results of preceding subsection, we can write down the effective interactions pertinent to \( K^+ \to \pi^+ \nu \nu, \pi^+ \bar{\nu} \bar{\nu} \) and \( K_L \to \pi^0 \nu \nu, \pi^0 \bar{\nu} \bar{\nu} \), namely

\[
\begin{align*}
C_{1}^{sdu} \mathbf{0}_{sdu} + \text{H.c.} &\Rightarrow \frac{g_{8 \times 8}}{4} F_0^2 \left[ \frac{3}{2} \left( C_{1}^{sdu} J + C_{1}^{sdu} J^\dagger \right) J^\dagger \pi^- K^+ - C_{1}^{sdu} J^\dagger \pi^- K^+ \right], \\
C_{1}^{rdsu} \mathbf{0}_{sdu} + \text{H.c.} &\Rightarrow \frac{g_{6 \times 6}}{4} F_0^2 \left[ \frac{3}{2} \left( C_{1}^{rdsu} J + C_{1}^{rdsu} J^\dagger \right) J^\dagger \pi^- K^+ - C_{1}^{rdsu} J^\dagger \pi^- K^+ \right], \\
C_{3}^{dds} \mathbf{0}_{dds} + \text{H.c.} &\Rightarrow \frac{g_{6 \times 6}}{4} F_0^2 \left[ \frac{3}{2} \left( C_{3}^{dds} J + C_{3}^{dds} J^\dagger \right) J^\dagger \pi^- K^+ - C_{3}^{dds} J^\dagger \pi^- K^+ \right], \\
C_{3}^{ddsd} \mathbf{0}_{ddsd} + \text{H.c.} &\Rightarrow \frac{g_{6 \times 6}}{4} F_0^2 \left[ \frac{3}{2} \left( C_{3}^{ddsd} J + C_{3}^{ddsd} J^\dagger \right) J^\dagger \pi^- K^+ - C_{3}^{ddsd} J^\dagger \pi^- K^+ \right],
\end{align*}
\]

and analogous expressions for the tilded operators, where \( J^\dagger = (\bar{\nu}_\alpha \nu^\dagger_\beta)/(1 + \delta_{\alpha\beta}) \). With these, we arrive at the decay amplitudes

\[
\begin{align*}
\mathcal{A}_{K_L^+ \to \pi^+ \nu \nu} &= \frac{3}{8} F_0^2 (C_A + C_B^*) \bar{\nu}_\alpha \nu^\dagger_\beta, \quad \mathcal{A}_{K_L^+ \to \pi^+ \bar{\nu} \bar{\nu}} = -\frac{1}{4} F_0^2 C_B^* \bar{\nu}_\alpha \nu^\dagger_\beta, \\
\mathcal{A}_{K_L^+ \to \pi^0 \nu_\alpha \nu_\beta} &= \frac{3}{8} F_0^2 (C_A + C_B) \bar{\nu}_\alpha \nu^\dagger_\beta, \quad \mathcal{A}_{K_L^+ \to \pi^0 \bar{\nu}_\alpha \bar{\nu}_\beta} = -\frac{1}{4} F_0^2 C_A \bar{\nu}_\alpha \nu^\dagger_\beta,
\end{align*}
\]
involving the effective coupling constants

\begin{align*}
C_A &= g_{888} C_{udsu}^1(\lambda_\chi) + \tilde{g}_{888} \tilde{C}_{udsu}^1(\lambda_\chi) + g_{666} C_{ddsd}^3(\lambda_\chi) + \tilde{g}_{666} \tilde{C}_{ddsd}^3(\lambda_\chi), \\
C_B &= g_{888} C_{usdu}^1(\lambda_\chi) + \tilde{g}_{888} \tilde{C}_{usdu}^1(\lambda_\chi) + g_{666} C_{ddds}^3(\lambda_\chi) + \tilde{g}_{666} \tilde{C}_{ddds}^3(\lambda_\chi),
\end{align*}

which implicitly carry the neutrino family labels \( \alpha \) and \( \beta \). The RG running effects on the parameters between the chiral-symmetry breaking scale \( \mu = \Lambda_\chi \) and the electroweak scale \( \mu = m_W \) can be included by using the results in eq. (7). From eq. (18), we obtain the spin-summed absolute squares

\begin{align*}
\sum_{\text{spins}} |A_{K_L \rightarrow \pi^0\nu_\alpha\bar{\nu}_\beta}|^2 &= \frac{9}{32} F_0^4 |C_A + C_B|^2 \hat{s}, \\
\sum_{\text{spins}} |A_{K_L \rightarrow \pi^0\bar{\nu}_\alpha\bar{\nu}_\beta}|^2 &= \frac{9}{32} F_0^4 |C_A + C_B|^2 \hat{s}, \\
\sum_{\text{spins}} |A_{K_L \rightarrow \pi^0\bar{\nu}_\alpha\bar{\nu}_\beta}|^2 &= \frac{9}{32} F_0^4 |C_A + C_B|^2 \hat{s}, \\
\sum_{\text{spins}} |A_{K_L \rightarrow \pi^0\nu_\alpha\nu_\beta}|^2 &= \frac{9}{32} F_0^4 |C_A + C_B|^2 \hat{s},
\end{align*}

where \( \hat{s} = (p_1 + p_2)^2 \). The \( \nu_\alpha\nu_\beta \) and \( \bar{\nu}_\alpha\bar{\nu}_\beta \) channels having no interference with each other, their branching fractions add up to

\begin{align*}
\mathcal{B}(K_L \rightarrow \pi^0\nu_\alpha\nu_\beta) + \mathcal{B}(K_L \rightarrow \pi^0\bar{\nu}_\alpha\bar{\nu}_\beta) &= \frac{9}{32} F_0^4 \frac{|C_A + C_B|^2}{16} \frac{\tau_{K_L}}{2m_{K^0}} \int d\Pi_3 \hat{s} \\
&= 1.42 \times 10^{-9} \left| \hat{C}_A + \hat{C}_B \right|^2, \\
\mathcal{B}(K^+ \rightarrow \pi^+\nu_\alpha\nu_\beta) + \mathcal{B}(K^+ \rightarrow \pi^+\bar{\nu}_\alpha\bar{\nu}_\beta) &= \frac{F_0^4}{8} \frac{|C_A|^2 + |C_B|^2}{1 + \delta_{\alpha\beta}} \frac{\tau_{K^+}}{2m_{K^+}} \int d\Pi_3 \hat{s} \\
&= 6.98 \times 10^{-11} \left| \hat{C}_A \right|^2 + \left| \hat{C}_B \right|^2, \tag{21}
\end{align*}

where \( \tau_{K_L} \) and \( \tau_{K^+} \) stand for the measured \( K_L \) and \( K^+ \) lifetimes \[13\], the factor \( 1/(1 + \delta_{\alpha\beta}) \) in each equation accounts for the identical particles in the final state if \( \alpha = \beta \), and \( d\Pi_3 \) denotes the three-body phase space factor. Also, in eq. (21) we have defined the dimensionless parameters

\begin{align*}
\hat{C}_A &= (50 \text{ GeV})^5 \left[ C_{udsu}^1(m_W) + 3.3 \tilde{C}_{udsu}^1(m_W) - 0.55 C_{ddsd}^3(m_W) + 0.15 \tilde{C}_{ddsd}^3(m_W) \right], \\
\hat{C}_B &= (50 \text{ GeV})^5 \left[ C_{usdu}^1(m_W) + 3.3 \tilde{C}_{usdu}^1(m_W) - 0.55 C_{ddds}^3(m_W) + 0.15 \tilde{C}_{ddds}^3(m_W) \right],
\end{align*}

upon incorporating eqs. (7) and (16). It is instructive to notice the ratio

\begin{equation}
\hat{r}_B^{\text{NP}} \equiv \frac{\mathcal{B}(K_L \rightarrow \pi^0\nu_\alpha\nu_\beta) + \mathcal{B}(K_L \rightarrow \pi^0\bar{\nu}_\alpha\bar{\nu}_\beta)}{\mathcal{B}(K^+ \rightarrow \pi^+\nu_\alpha\nu_\beta) + \mathcal{B}(K^+ \rightarrow \pi^+\bar{\nu}_\alpha\bar{\nu}_\beta)} = \frac{20.3 \left| \hat{C}_A + \hat{C}_B \right|^2}{\left| \hat{C}_A \right|^2 + \left| \hat{C}_B \right|^2} \leq 40.6. \tag{23}
\end{equation}

Assuming that the NP operators involve only one pair of \( \alpha \) and \( \beta \neq \alpha \), since these modes do not interfere with the SM contributions, we can combine eq. (21) with their SM counterparts to find

\begin{equation}
\mathcal{B}(K_L \rightarrow \pi^0 + E_{\text{miss}}) = \mathcal{B}(K_L \rightarrow \pi^0\nu\bar{\nu})_{\text{SM}} + \mathcal{B}(K_L \rightarrow \pi^0\nu_\alpha\nu_\beta) + \mathcal{B}(K_L \rightarrow \pi^0\bar{\nu}_\alpha\bar{\nu}_\beta),
\end{equation}
\[
B(K^+ \to \pi^+ E_{\text{miss}}) = B(K^+ \to \pi^+ \nu \bar{\nu})_{\text{SM}} + B(K^+ \to \pi^+ \nu_\alpha \nu_\beta) + B(K^+ \to \pi^+ \bar{\nu}_\alpha \bar{\nu}_\beta),
\]
which lead to
\[
\begin{align*}
\frac{r_B}{B(K_L \to \pi^+ E_{\text{miss}})} &= \frac{B(K^+ \to \pi^+ E_{\text{miss}})}{B(K^+ \to \pi^+ E_{\text{miss}})} = r_B^{\text{SM}} + \frac{(r_B^{\text{NP}} - r_B^{\text{SM}}) \epsilon}{1 + \epsilon} \leq 0.36 + \frac{40.3 \epsilon}{1 + \epsilon}, \\
\epsilon &= \frac{B(K^+ \to \pi^+ \nu_\alpha \nu_\beta) + B(K^+ \to \pi^+ \bar{\nu}_\alpha \bar{\nu}_\beta)}{B(K^+ \to \pi^+ \nu \bar{\nu})_{\text{SM}}},
\end{align*}
\]
The latest preliminary NA62 result [19] translates into an upper bound for \( \epsilon \) of about 1, implying that \( r_B \) is capped to be about 20.5. This can accommodate KOTO’s anomalous events [20].

Imposing the KOTO 15 [15] and most recent NA62 [19] limits on eq. (24) and further assuming that the only nonvanishing Wilson coefficients are \( C_{u \bar{d} s u} = C_{u \bar{s} d u} = \Lambda^{-5}_{\text{NP}} \), we have
\[
\begin{align*}
B(K_L \to \pi^0 + E_{\text{miss}}) &= 3.0 \times 10^{-11} + 5.7 \times 10^{-9} \left( \frac{50 \text{ GeV}}{\Lambda_{\text{NP}}} \right)^{10} \leq 3.0 \times 10^{-9}, \\
B(K^+ \to \pi^+ + E_{\text{miss}}) &= 8.5 \times 10^{-11} + 1.4 \times 10^{-10} \left( \frac{50 \text{ GeV}}{\Lambda_{\text{NP}}} \right)^{10} \leq 1.85 \times 10^{-10},
\end{align*}
\]
where the first number in each line is the corresponding SM central value [11–13]. The stronger of the empirical limits, from KOTO in the first line, translates into \( \Lambda_{\text{NP}} \gtrsim 53 \text{ GeV} \). In figure 1, we illustrate the dependence of \( B(K_L \to \pi^0 + E_{\text{miss}}) \) and \( B(K^+ \to \pi^+ + E_{\text{miss}}) \) in eq. (27) on \( \Lambda_{\text{NP}} \). Also plotted are the corresponding SM predictions and limits from KOTO [15] and NA62 [19]. We see that the NP needs to have an effective scale \( \Lambda_{\text{NP}} = \mathcal{O}(60 \text{ GeV}) \) to be responsible for the KOTO

FIG. 1: The branching fractions of \( K \to \pi \nu \bar{\nu}, \pi \bar{\nu} \bar{\nu} \) arising from the dim-9 operators as functions of the NP scale \( \Lambda_{\text{NP}} \), as described in the text. Also displayed are the corresponding SM predictions for \( K \to \pi \nu \bar{\nu} \) (red and blue horizontal bands) and upper limits from KOTO [15] and NA62 [19] (blue and red horizontal thin lines). The light-blue region is excluded by the KOTO bound. The blue dot corresponds to KOTO’s three events.
anomaly. The preferred $\Lambda_{NP}$ is below the EW scale, which implies that for $\Lambda_{NP}$ near $v \simeq 246$ GeV the dim-9 operators would have negligible impact on $K \to \pi + E_{\text{miss}}$ and respect the GN bound.

If we repeat the above steps with the dim-10 operators discussed earlier, a scaling factor of $v/\Lambda$ will accompany them. For $\Lambda_{NP} < v$, this scaling factor helps raise $\Lambda_{NP}$ slightly, but the latter is still not very close to the EW scale, $\sim v$, in order for $K \to \pi + E_{\text{miss}}$ to be within the present experimental sensitivity reaches. Hence for $\Lambda_{NP} \gtrsim v$ the dim-10 operators would also have very little influence on these modes.

### III. GN-BOUND VIOLATION FROM EFT OPERATORS FOR $K \to \pi S$ or $\pi SS$

In the preceding section, the problem of the NP scale $\Lambda_{NP}$ being too low can be ascribed to the high dimension of the SMEFT operators. As sketched in section I, any NP that can induce $K_L \to \pi^0 X$ with a rate exceeding the SM expectation without conflicting with the $K^+$ data needs to have an effective scale $\Lambda_{NP}$ which goes roughly as $(23000 \text{ TeV}^2 m_K^n)^{1/(2+n)}$, where $m_K$ is the kaon mass and $n$ the mass dimension of the field content of $X$. Therefore, one way to increase $\Lambda_{NP}$ is by reducing the dimension of the operators, which is $6+n$. If in the dim-9 operators examined above one replaces the neutrino pair, which has $n = 3$, with a scalar field $S$, which has $n = 1$, the dimension of the operators can be decreased by 2 and in turn $\Lambda_{NP}$ can be raised to the TeV level. If $X = SS$ instead, the scale will become $\Lambda_{NP} = \mathcal{O}(v)$. In this section, we explore how SMEFT four-quark operators supplemented with $S$, which we take to be real and a singlet under the SM gauge group, can give rise to $K \to \pi S$ transitions which break the GN bound. Moreover, we apply a similar treatment to the $K \to \pi SS$ case.

#### A. Operators and matching

As is clear from the last paragraph, SMEFT operators that directly give rise to $K \to \pi S$ at leading order must be of dimension seven (dim-7). Since $S$ is a SM-gauge singlet, the quark portions of these operators are none other than the SMEFT dimension-six (dim-6) four-quark operators \[44, 45\] which contribute to nonleptonic kaon decays. In the first column of table \[11\] we list these dim-6 operators in the Warsaw basis \[45\]. In the middle column, we exhibit the relevant operators with one $s$-quark field in the LEFT approach \[46\]. The third column contains the results for the Wilson coefficients at the electroweak scale from the matching of the SMEFT operators onto the LEFT operators \[46\]. For the $K \to \pi S$ and $K \to \pi SS$ transitions, we just multiply all those operators by an $S$ field and a pair of them, respectively, and the matched Wilson coefficients should be understood as new parameters associated with the operators.

Similarly to what was done in section \[11\] for each of the dim-7 operators we begin by decomposing its four-quark part in terms of the irreducible representations of the chiral group,
TABLE II: Columns 1 and 2: the SMEFT and LEFT four-quark operators contributing to $K \to \pi S(S)$. Column 3: the results for the Wilson coefficients at the electroweak scale from the matching of former operators onto the latter.

SU(3)$_L \times$SU(3)$_R$. Subsequently, for each of the irreps we derive the chiral realization, as prescribed in subsection [13B] and complement it with a low-energy constant. Finally the resulting meson operators are expressed in terms of their isospin components. The first column of table [III] list the LEFT operators in table [II] according to their irreps. In the second column we collect the meson operators. The ones corresponding to the dim-6 operators with purely left-handed or
right-handed quarks are written in terms of the $\Delta I = 1/2, 3/2$ combinations

$$Q^V_{1/2} = F_0^2 \left( \partial_\mu K^+ \partial^\mu \pi^- - \frac{1}{\sqrt{2}} \partial_\mu K^0 \partial^\mu \pi^0 \right), \quad Q^V_{3/2} = F_0^2 \left( \partial_\mu K^+ \partial^\mu \pi^- + \sqrt{2} \partial_\mu K^0 \partial^\mu \pi^0 \right),$$

respectively. The other entries in the second column involve $Q^S_{1/2}$ and $Q^S_{3/2}$ which were already defined in eq. (15). More details on the bosonization of the irreps are relegated to appendix C. In this table, we also see that there are more LECs than in table I. For $g_{1 \times 8}$ and $g_{1 \times 27}$, which are dimensionless, we adopt

$$g_{8 \times 1} = 3.65, \quad g_{27 \times 1} = 0.303$$

from ref. [47], whereas $g_{8 \times 8}$, $g_{6 \times 6}$, and $g_{6 \times 6}$ are already given in (16). Moreover, the parity invariance of the QCD suggests that we can set

$$g_{1 \times 8} = g_{8 \times 1}, \quad g_{1 \times 27} = g_{27 \times 1}, \quad g_{6 \times 6} = g_{6 \times 6}, \quad g_{3 \times 3} = g_{3 \times 3},$$

and analogous relations for the LECs with a tilde. As for $g_{3 \times 3}$, there is no estimation yet in literature, and so we can resort to the vacuum saturation approximation (VSA) which yields $g_{3 \times 3} = g_{6 \times 6} = B^2_0 \simeq 4 \text{ GeV}^2$ with $B_0 = m_\pi^2/(m_u + m_d) = m_K^2/(m_u + m_s)$ and the quark masses at 1 GeV. Evidently the VSA value of $g_{3 \times 3}$ is not too far from $g_{6 \times 6} = 3.2 \text{ GeV}^2$ in eq. (16). Additionally, we implement simple scaling to estimate $\tilde{g}_{3 \times 3} = g_{3 \times 3} g_{6 \times 6} / g_{6 \times 6} \simeq 1.4 \text{ GeV}^2$.

### B. Numerical analysis

Summing the meson operators in table III multiplied by their respective Wilson coefficients leads to the effective Lagrangian $\mathcal{L}_{K \pi S}$ responsible for $K \to \pi S$. We can express it as

$$\mathcal{L}_{K \pi S} = F_0 \left( a_1 K^+ \pi^- - \frac{b_1}{\sqrt{2}} K^0 \pi^0 \right) S + \frac{1}{F_0} \left( a_2 \partial_\mu K^+ \partial^\mu \pi^- - \frac{b_2}{\sqrt{2}} \partial_\mu K^0 \partial^\mu \pi^0 \right) S + \text{H.c.}$$

$$+ \frac{1}{F_0} \left[ a_1 K^+ \pi^- - (\text{Re} b_1) K_L \pi^0 \right] S + \frac{1}{F_0} \left[ a_2 \partial_\mu K^+ \partial^\mu \pi^- - (\text{Re} b_2) \partial_\mu K_L \partial^\mu \pi^0 \right] S,$$

where $a_{1,2}$ and $b_{1,2}$ are dimensionless constants containing linear combinations of the Wilson coefficients $C_s$, namely

$$a_1 = \frac{1}{24} F_0 \left[ 6 \left( 2C_{1,LR}^{V,1} - C_{1,LR}^{V,1} + 2C_{1,LR}^{V,1} - C_{1,LR}^{V,1} \right) g_{8 \times 8} \right.$$

$$- \left( 2C_{8,LR}^{V,1} - C_{8,LR}^{V,1} + 2C_{8,LR}^{V,1} - C_{8,LR}^{V,1} \right) \left( g_{8 \times 8} - 3 \tilde{g}_{8 \times 8} \right) \right. $$

$$- 9 \left( C_{1,LL}^{S,1} + C_{1,RR}^{S,1} + C_{1,LL}^{S,1} + C_{1,RR}^{S,1} \right) g_{6 \times 6} \right.$$

$$\left. + \frac{3}{2} \left( C_{1,RR}^{L,1} + C_{1,RR}^{S,1} + C_{1,LL}^{S,1} + C_{1,LL}^{S,1} \right) \left( g_{6 \times 6} - 3 \tilde{g}_{6 \times 6} \right) \right. $$

$$\left. + 3 \left( C_{1,LL}^{L,1} - C_{1,LL}^{S,1} + C_{1,RR}^{S,1} - C_{1,RR}^{S,1} \right) g_{3 \times 3} \right.$$
| Chiral irrep | Contributions to $K \rightarrow \pi$ |
|-------------|----------------------------------|
| $V_{LL} \, ddsd = 0_{uddd}^{V_{LL}} | 27 \times 1 + 0_{uddd}^{V_{LL}} | 8 \times 1$ | $\frac{1}{5} g_{27 \times 1} (2Q_{1/2}^{V} - 5Q_{3/2}^{V}) + \frac{1}{5} g_{8 \times 1} Q_{1/2}^{V}$ |
| $V_{LL} \, uusd = 0_{uuus}^{V_{LL}} | 27 \times 1 + 0_{uuus}^{V_{LL}} | 8 \times 1$ | $\frac{1}{5} g_{27 \times 1} (Q_{1/2}^{V} + 5Q_{3/2}^{V}) - \frac{1}{3} g_{8 \times 1} Q_{1/2}^{V}$ |
| $V_{LL} \, uusd = 0_{uusd}^{V_{LL}} | 27 \times 1 + 0_{uusd}^{V_{LL}} | 8 \times 1$ | $\frac{1}{3} g_{27 \times 1} (Q_{1/2}^{V} + 5Q_{3/2}^{V}) + \frac{11}{36} g_{8 \times 1} Q_{1/2}^{V}$ |
| $V_{RR} \, ddsd = 0_{uddd}^{V_{RR}} | 1 \times 27 + 0_{uddd}^{V_{RR}} | 1 \times 8$ | $\frac{1}{5} g_{1 \times 27} (2Q_{1/2}^{V} - 5Q_{3/2}^{V}) + \frac{1}{5} g_{1 \times 8} Q_{1/2}^{V}$ |
| $V_{RR} \, uusd = 0_{uuus}^{V_{RR}} | 1 \times 27 + 0_{uuus}^{V_{RR}} | 1 \times 8$ | $\frac{1}{5} g_{1 \times 27} (Q_{1/2}^{V} + 5Q_{3/2}^{V}) - \frac{1}{3} g_{1 \times 8} Q_{1/2}^{V}$ |
| $V_{RR} \, uusd = 0_{uusd}^{V_{RR}} | 1 \times 27 + 0_{uusd}^{V_{RR}} | 1 \times 8$ | $\frac{1}{3} g_{1 \times 27} (Q_{1/2}^{V} + 5Q_{3/2}^{V}) + \frac{11}{36} g_{1 \times 8} Q_{1/2}^{V}$ |
| $V_{LL} \, ddsd = 0_{uddd}^{V_{LL}} | 8 \times 8 + 0_{uddd}^{V_{LL}} | 8 \times 1$ | $\frac{1}{6} g_{8 \times 8} (2Q_{1/2}^{S} + Q_{3/2}^{S})$ |
| $V_{LL} \, uusd = 0_{uuus}^{V_{LL}} | 8 \times 8 + 0_{uuus}^{V_{LL}} | 8 \times 1$ | $-\frac{1}{26} g_{8 \times 8} - 3g_{8 \times 8} (2Q_{1/2}^{S} + Q_{3/2}^{S})$ |
| $V_{LL} \, uusd = 0_{uusd}^{V_{LL}} | 8 \times 8 + 0_{uusd}^{V_{LL}} | 8 \times 1$ | $-\frac{1}{26} g_{8 \times 8} (Q_{1/2}^{S} + 2Q_{3/2}^{S})$ |
| $V_{RR} \, ddsd = 0_{uddd}^{V_{RR}} | 8 \times 8 + 0_{uddd}^{V_{RR}} | 8 \times 1$ | $\frac{1}{2} g_{8 \times 8} - 3g_{8 \times 8} (Q_{1/2}^{S} + 2Q_{3/2}^{S})$ |
| $V_{RR} \, uusd = 0_{uuus}^{V_{RR}} | 8 \times 8 + 0_{uuus}^{V_{RR}} | 8 \times 1$ | $\frac{1}{2} g_{8 \times 8} - 3g_{8 \times 8} (Q_{1/2}^{S} + 2Q_{3/2}^{S})$ |
| $V_{RR} \, uusd = 0_{uusd}^{V_{RR}} | 8 \times 8 + 0_{uusd}^{V_{RR}} | 8 \times 1$ | $\frac{1}{2} g_{8 \times 8} - 3g_{8 \times 8} (Q_{1/2}^{S} + 2Q_{3/2}^{S})$ |
| $S_{1} \, ddsd = 0_{uddd}^{S_{1}} | 6 \times 6 + 0_{uddd}^{S_{1}} | 3 \times 3$ | $-\frac{1}{24} g_{6 \times 6} (5Q_{1/2}^{S} + 4Q_{3/2}^{S} + Q_{3/2}^{V}) + \frac{1}{8} g_{3 \times 3} Q_{1/2}^{V}$ |
| $S_{2} \, ddsd = 0_{uddd}^{S_{2}} | 6 \times 6 + 0_{uddd}^{S_{2}} | 3 \times 3$ | $-\frac{1}{24} g_{6 \times 6} (8Q_{1/2}^{S} + 4Q_{3/2}^{S}) - \frac{1}{8} g_{3 \times 3} Q_{1/2}^{V}$ |
| $S_{1} \, uusd = 0_{uuus}^{S_{1}} | 6 \times 6 + 0_{uuus}^{S_{1}} | 3 \times 3$ | $-\frac{1}{24} g_{6 \times 6} (5Q_{1/2}^{S} + 4Q_{3/2}^{S}) + \frac{1}{8} g_{3 \times 3} Q_{1/2}^{V}$ |
| $S_{2} \, uusd = 0_{uuus}^{S_{2}} | 6 \times 6 + 0_{uuus}^{S_{2}} | 3 \times 3$ | $-\frac{1}{24} g_{6 \times 6} (5Q_{1/2}^{S} + 4Q_{3/2}^{S}) + \frac{1}{8} g_{3 \times 3} Q_{1/2}^{V}$ |

**TABLE III:** The chiral representations and realizations of the LEFT four-quark operators contributing to $K \rightarrow \pi S(S)$. In the second column, $Q_{1/2}^{V}$ and $Q_{3/2}^{S}$ are the mesonic operators defined in eqs. (15) and (25) and correspond, respectively, to $\Delta I = 1/2$ and $3/2$ transitions. For $O_{V_{LL} uusd}^{V_{LL}}$ and $O_{V_{RR} uusd}^{V_{RR}}$, in rows 7-14, the contributions of the $8 \times 1$ and $1 \times 8$ terms are chirally subleading compared to their $8 \times 8$ counterparts and therefore dropped from the second column.

\[
\begin{align*}
- \frac{1}{2} \left( c_{uddd}^{S_{2},LL} - c_{uuus}^{S_{2},LL} + c_{uuus}^{S_{2},RR} - c_{uddd}^{S_{2},RR} \right) (g_{3 \times 3} - 3g_{3 \times 3}) \\
+ 3 \left( c_{uddd}^{S_{2},LL} + c_{uuus}^{S_{2},LL} + c_{uuus}^{S_{1},RR} + c_{uddd}^{S_{1},RR} \right) g_{6 \times 6}
\end{align*}
\]

\[
b_{1} = \frac{1}{24} F_{0} \left[ 6 \left( c_{uddd}^{V_{LL} uusd} + c_{uddd}^{V_{RR} uusd} \right) g_{8 \times 8} - \left( c_{uuus}^{V_{LL} uusd} + c_{uuus}^{V_{RR} uusd} \right) (g_{8 \times 8} - 3g_{8 \times 8}) \\
+ 3 \left( c_{uddd}^{S_{1},LL} + c_{uuus}^{S_{1},LL} + c_{uuus}^{S_{1},RR} + c_{uddd}^{S_{1},RR} \right) g_{6 \times 6} \right]
\]
\[-\frac{1}{2} \left( c_{uusd}^{S8,LL} + c_{uusd}^{S8,RR} + c_{uusd}^{S8,RR} \right) \left( g_{6\times6} - 3\bar{g}_{6\times6} \right) + 3 \left( c_{uusd}^{S1,LL} - c_{uusd}^{S1,LL} + c_{uusd}^{S1,RR} \right) \left( g_{3\times3} \right) - \frac{1}{2} \left( c_{uusd}^{S8,LL} - c_{uusd}^{S8,RR} + c_{uusd}^{S8,RR} \right) \left( g_{3\times3} \right) \right]_{\Lambda_{\chi}},

(33)

a_2 = \frac{1}{36} F_0^3 \left[ 6 \left( c_{dds}^{V,LL} + c_{dds}^{V,RR} - 2c_{dds}^{V,LL} - 2c_{dds}^{V,RR} \right) \left( g_{8\times1} - g_{27\times1} \right) + \left( c_{ddsd}^{V,LL} + c_{ddsd}^{V,RR} \right) \left( 11g_{8\times1} + 4g_{27\times1} \right) \right]_{\Lambda_{\chi}},

(34)

b_2 = \frac{1}{36} F_0^3 \left[ 6 \left( c_{dds}^{V,LL} + c_{dds}^{V,RR} \right) \left( g_{8\times1} + 4g_{27\times1} \right) - 6 \left( c_{dds}^{L,LL} + c_{dds}^{L,RR} \right) \left( 2g_{8\times1} + 3g_{27\times1} \right) + \left( c_{ddsd}^{V,LL} + c_{ddsd}^{V,RR} \right) \left( 11g_{8\times1} - 6g_{27\times1} \right) \right]_{\Lambda_{\chi}},

(35)

the subscript $\Lambda_{\chi}$ indicating that the $c$s on the right-hand sides are evaluated at $\mu = \Lambda_{\chi}$. These coefficients scale as $\Lambda_{\chi}^{-3}$.

For $K \rightarrow \pi SS$, the interaction Lagrangian $\mathcal{L}_{K\pi SS}$ has an expression similar to $S\mathcal{L}_{K\pi S}$, namely

$$\mathcal{L}_{K\pi SS} = \left( \hat{a}_1 K^+ \pi^- - \frac{\hat{b}_1}{\sqrt{2}} K^0 \pi^0 \right) S^2 + \left( \hat{a}_2 \partial_{\mu} K^+ \partial^\mu \pi^- - \frac{\hat{b}_2}{\sqrt{2}} \partial_{\mu} K^0 \partial^\mu \pi^0 \right) \frac{S^2}{F_0^2} + \text{H.c.}$$

(36)

The dimensionless parameters $\hat{a}_{1,2}$ and $\hat{b}_{1,2}$ are the same in form as $a_{1,2} F_0$ and $b_{1,2} F_0$, respectively, but with the Wilson coefficients now denoted $\hat{c}$s, which scale as $\Lambda_{\chi}^{-4}$ because the underlying quark-level operators are of dimension eight.

C. $K \rightarrow \pi S$

From $\mathcal{L}_{K\pi S}$, we obtain the amplitudes for $K \rightarrow \pi S$ to be

$$A_{K^+\rightarrow \pi S} = a_1 F_0 + a_2 \frac{m_{K^+}^2 + m_{\pi^+}^2 - m_S^2}{2F_0},$$

$$A_{K_L\rightarrow \pi S} = -\text{Re} b_1 F_0 - \text{Re} b_2 \frac{m_{K^0}^2 + m_{\pi^0}^2 - m_S^2}{2F_0},$$

(37)

and hence the branching fractions

$$\mathcal{B}(K^+ \rightarrow \pi^+ S) = \tau_{K^+} \frac{\sqrt{(m_{K^+}^2 - m_{\pi^+}^2)^2 - (2m_{K^+}^2 + 2m_{\pi^+}^2 - m_S^2)m_S^2}}{16\pi m_{K^+}^3} \times \left| \frac{a_1 + a_2 \frac{m_{K^+}^2 + m_{\pi^+}^2 - m_S^2}{2F_0}^2}{2F_0^2} \right| F_0^2,$$

(38)

$$\mathcal{B}(K_L \rightarrow \pi^0 S) = \tau_{K_L} \frac{\sqrt{(m_{K^0}^2 - m_{\pi^0}^2)^2 - (2m_{K^0}^2 + 2m_{\pi^0}^2 - m_S^2)m_S^2}}{16\pi m_{K^0}^3}$$

18
lead to \( \tilde{b} \) than 10 times. For this example, in figure 2 we depict upper limits on \( K \) for the latter requirement, it follows that the blue and black curves in each of these graphs are related.

\[
\times \left| \text{Re} b_1 + \text{Re} b_2 \frac{m_{K_0}^2 + m_{\pi^0}^2 - m_S^2}{2F_0^2} \right|^2 F_0^2 ,
\]

(39)

To account for the KOTO anomaly, one could consider various possibilities. For illustration, we pick a scenario in which the only contributing operators are those with purely right-handed quarks, \( Q_{ddsd}^{V,RR} \) and \( Q_{uusd}^{(V_1,V_8),RR} \), implying that \( a_1 = b_1 = 0 \) and

\[
\begin{align*}
a_2 &= \frac{F_0^3}{6} \left[ \left( c_{ddsd}^{V,RR} - 2c_{uusd}^{V,RR} + \frac{11}{6} c_{uusd}^{V,8,RR} \right) g_{8 \times 1} - \left( c_{ddsd}^{V,RR} - 2c_{uusd}^{V,RR} - \frac{2}{3} c_{uusd}^{V,8,RR} \right) g_{27 \times 1} \right] \chi, \\
b_2 &= \frac{F_0^3}{6} \left[ \left( c_{ddsd}^{V,RR} - 2c_{uusd}^{V,RR} + \frac{11}{6} c_{uusd}^{V,8,RR} \right) g_{8 \times 1} + \left( 4c_{ddsd}^{V,RR} - 3c_{uusd}^{V,1,RR} - c_{uusd}^{V,8,RR} \right) g_{27 \times 1} \right] \chi .
\end{align*}
\]

(40)

Moreover, selecting the \( g_{8 \times 1} \) terms to vanish changes eq. (40) to

\[
\begin{align*}
a_2 &= \frac{5F_0^3}{12} g_{27 \times 1} c_{uusd}^{V,8,RR} (\Lambda_\chi), \\
b_2 &= \frac{25F_0^3}{18} g_{27 \times 1} \left( \frac{5}{3} c_{uusd}^{V,1,RR} (\Lambda_\chi) - c_{uusd}^{V,8,RR} (\Lambda_\chi) \right) ,
\end{align*}
\]

(41)

with which we arrive at

\[
\tilde{r}_B^{NP} = \frac{B(K_L \to \pi^0 S)}{B(K^+ \to + S)} = 4.13 \left( \frac{\text{Re}^2 (b_1 + 17.6 b_2)}{|a_1 + 17.4 a_2|^2} \right) = 47 \left( \frac{\text{Re} [c_{uusd}^{V,8,RR} (\Lambda_\chi) - 0.6 c_{uusd}^{V,1,RR} (\Lambda_\chi)]}{c_{uusd}^{V,8,RR} (\Lambda_\chi)} \right)^2 .
\]

(42)

It is worth noting that the potential enlargement of \( \tilde{r}_B^{NP} \) in this equation can be expected from the fact that it arises from the quark operators \( Q_{uusd}^{V_1,V_8,RR} \) which, as rows 5-6 in table III show, generate the combination \( Q_{uusd}^{V_1,V_8,RR} + 5Q_{uusd}^{V_3/2} \) of the mesonic operators defined in eq. (28) and hence each contain a significant \( \Delta I = 3/2 \) component. Amplifying \( \tilde{r}_B^{NP} \) can be easily realized by appropriately tuning the parameters in eq. (42).

To be more precise in our numerical treatment, we again need to take into account the QCD RG running of the coefficients from the EW scale, which we choose to be the \( W \)-boson mass \( m_W \) as before, down to the chiral-symmetry breaking scale \( \Lambda_\chi \). The pertinent one-loop RG equations are available in ref. [48], from which we collect the formulas in appendix B. We use them to get

\[
\begin{align*}
c_{uusd}^{V,1,RR} (\Lambda_\chi) &= 1.07 c_{uusd}^{V,1,RR} (m_W) - 0.19 c_{uusd}^{V,8,RR} (m_W) , \\
c_{uusd}^{V,8,RR} (\Lambda_\chi) &= 1.31 c_{uusd}^{V,8,RR} (m_W) - 0.86 c_{uusd}^{V,1,RR} (m_W) - 0.16 c_{ddsd}^{V,RR} (m_W) ,
\end{align*}
\]

(43)

To simplify things further, we set \( c_{uusd}^{V,1,RR} (m_W) = c_{ddsd}^{V,1,RR} (m_W) = 0 \) and \( c_{uusd}^{V,8,RR} (m_W) = \Lambda_{NP}^{-3} \), which lead to \( \tilde{r}_B^{NP} = 51 \) in eq. (42). This exceeds the maximum \( r_B^{GN} = 4.3 \) of the GN bound by more than 10 times. For this example, in figure 2 we depict \( B(K \to \pi S) \) as functions of the \( S \) mass \( m_S \) with \( \Lambda_{NP} = 1 \) TeV (left panel) and 800 GeV (right panel). In the figure, we also display the upper limits on \( K_L \to \pi^0 S \) and \( K^+ \to + S \) at 90\% CL from KOTO 2015 [15] and BNL [17], respectively, along with the standard GN constraint on \( K_L \to \pi^0 S \) from the BNL result. From the latter requirement, it follows that the blue and black curves in each of these graphs are related.
by $B(K_L \to \pi^0 S)_{\text{GN}} = 4.3\, B(K^+ \to \pi^+ S)_{\text{BNL}}$. The left panel reveals that the GN inequality is not respected in the $m_S \lesssim 110$ MeV region, with the current experimental limits being satisfied. If $\Lambda_{\text{NP}}$ is smaller, around 800 GeV as specified in the right panel, it is also possible to break the GN bound in the range $170$ MeV $\lesssim m_S \lesssim 240$ MeV. We conclude that it can be violated by dim-7 EFT operators with a NP scale $\Lambda_{\text{NP}} = \mathcal{O}(1\text{ TeV})$.

D. $K \to \pi SS$

For the three-body decays $K \to \pi SS$, the amplitudes are

$$A_{K^+ \to \pi^+ SS} = 2\hat{a}_1 + \hat{a}_2 \frac{m_{K^+}^2 + m_{\pi^+}^2 - \hat{s}}{F_0^2},$$
$$A_{K_L \to \pi^0 SS} = -2\Re \hat{b}_1 F_0 - \Re \hat{b}_2 \frac{m_{K^0}^2 + m_{\pi^0}^2 - \hat{s}}{F_0^2}, \quad (44)$$

where $\hat{s}$ is the invariant mass squared of the $SS$ pair. Since the second terms in eq. (44) are suppressed compared to the first by $\Lambda_\chi^2/\Lambda_{\text{NP}}^2$ with $\Lambda_{\text{NP}} \geq v$, to assess the $\Lambda_{\text{NP}}$ value needed to explain the KOTO anomaly, we can safely drop the second terms. Then the branching fractions become

$$B(K^+ \to \pi^+ SS) = \frac{\tau_{K^+}}{2^9\pi^3 m_{K^+}^3} \int \Pi_3 |\mathcal{M}_{K^+ \to \pi^+ SS}|^2 = \frac{\tau_{K^+} |\hat{a}_1|^2}{2^7\pi^3 m_{K^+}^3} \int \Pi_3 = 7 \times 10^{11}|\hat{a}_1|^2,$$
$$B(K_L \to \pi^0 SS) = \frac{\tau_{K_L}}{2^9\pi^3 m_{K^0}^3} \int \Pi_3 |\mathcal{M}_{K_L \to \pi^0 SS}|^2 = \frac{\tau_{K_L} (\Re \hat{b}_1)^2}{2^7\pi^3 m_{K^0}^3} \int \Pi_3 = 3 \times 10^{12}(\Re \hat{b}_1)^2. \quad (45)$$

As before, we have many options regarding the parameters which can yield a violation of the GN bound. For instance, if we assume that $\hat{a}_1$ and $\hat{b}_1$ receive nonzero contributions from only the $g_{8 \times 8}$
terms in eqs. (32) and (33) and additionally pick $C_{sdd}^{V1,LR} = C_{ddsd}^{V1,LR} = C_{uusd}^{V1,LR} = 2C_{sduu}^{V1,LR} = 2/\Lambda_{NP}^4$, we obtain
\[ \hat{a}_1 = \frac{F_0^2 g_{8\times8}}{4 \Lambda_{NP}^4}, \quad \hat{b}_1 = \frac{F_0^2 g_{8\times8}}{4 \Lambda_{NP}^4}. \] (46)

In figure 3 we plot the resulting branching fractions of $K \to \pi SS$ as functions of $\Lambda_{NP}$. Evidently, to amplify $B(K_L \to \pi^0 SS)$ to a level within KOTO’s current sensitivity reach would require a NP scale $\Lambda_{NP}$ not more than roughly 200 GeV in this particular instance. For $\Lambda_{NP}$ above this value, the GN bound is no longer violated.

FIG. 3: The branching fractions of $K \to \pi SS$ induced by the dim-8 operators as functions of the NP scale $\Lambda_{NP}$, as described in the text. Also displayed are the corresponding SM predictions for $K \to \pi \nu \bar{\nu}$ (red and blue horizontal bands) and upper limits from KOTO [15] and NA62 [19] (blue and red horizontal thin lines). The light-blue region is excluded by the KOTO bound. The blue dot corresponds to KOTO’s three events.

IV. SUMMARY AND CONCLUSIONS

Motivated by the recent preliminary observation of 3 anomalous events of $K_L \to \pi^0 \nu \bar{\nu}$ by the KOTO Collaboration, we study in detail the possibility of having new physics responsible for enhancing the $K \to \pi + E_{\text{miss}}$ modes over their SM expectations. We consider two types of scenarios:

- NP above the EW scale represented by quark-neutrino interactions which do not preserve lepton flavor/number.

- NP above the EW scale with new scalar particles that are sufficiently light to be produced in $K \to \pi + E_{\text{miss}}$ decays.
The NP is described with an effective Lagrangian above the EW scale that respects the gauge symmetries of the SM. In all cases we specifically look for true violations of the Grossman-Nir bound through four-quark $\Delta I = 3/2$ interactions.

The NP effects are classified by the mass dimensionality of the required operators. To this end, we catalogue all operators that can be responsible for the reaction $K \to \pi \mathcal{X}$ with $\mathcal{X}$ standing for one or more particles carrying away the missing energy. As itemized above, we allow for $\mathcal{X}$ to comprise: a neutrino-antineutrino pair ($\nu\bar{\nu}$), a pair of neutrinos ($\nu\nu$) or antineutrinos ($\bar{\nu}\bar{\nu}$), a new invisible and light real scalar field ($S$), and a pair of the new scalars ($SS$). These cases require a minimal dimensionality of ten, nine, seven, and eight, respectively. On general grounds, we argue that the scenarios with new scalars (dim-7 or -8 operators) are consistent with large enhancements in the $K_L \to \pi^0 + E_{\text{miss}}$ rates for NP scales above the EW scale.

We construct the effective Lagrangian for all the cases and, after identifying the $\Delta I = 3/2$ component of the operators, we discuss the renormalization group running of the couplings down to a hadronic scale followed by a matching onto chiral perturbation theory. We present numerical results illustrating the scale of NP required to enhance $K_L \to \pi^0 + E_{\text{miss}}$ above the GN bound obtained from the measurements of $K^+ \to \pi^+ + E_{\text{miss}}$.

We find that the production of a single new light scalar via $\Delta I = 3/2$ interactions allows enhancements in the $K_L \to \pi^0 + E_{\text{miss}}$ rate that are large enough to appear in the KOTO experiment and we illustrate this in figure 2. Our results are shown for stable new scalars, but long lived ones would also work as they have weaker constraints [17].

The production of two light scalars could result in substantial enhancements over the SM but not above the GN bound. We illustrate this in figure 4 where the blue area illustrates that enhancements over the SM by factors of four are possible while keeping $\Lambda_{\text{NP}} \geq v$. With a different choice of parameters, the charged mode could also be enhanced by a similar amount.

The production of two neutrinos, on the other hand, suffers from much larger $\Lambda_{\text{NP}}$ suppression. Restricting $\Lambda_{\text{NP}} \geq v$ results in very small enhancements over the SM, completely within the uncertainty of the SM predictions and thus unobservable.

We conclude that continued improvement of the KOTO upper bound on $K_L \to \pi^0 + E_{\text{miss}}$, even at current levels which are much above the GN bound, provide relevant constraints on possible new physics scenarios.

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FIG. 4: The branching fractions of $K \to \pi + E_{\text{miss}}$: in red the 90% CL SM predictions; in green the 1σ BNL 787/949 result; in brown the 90% NA62 exclusion; in grey the GN bound; and in blue a region accessible with $K \to \pi SS$ for parameters chosen to enhance mostly the neutral mode with a NP scale $\Lambda_{\text{NP}} \geq v$.

Appendix A: Isospin decomposition of quark parts of dim-9 operators

In this appendix, for completeness we decompose the quark portion of each of the dim-9 operators in eq. (5) into its $\Delta I = 1/2, 3/2$ components. This will allow us to see clearly the difference between them. Since additionally each operator also causes a definite change $\Delta I_3$ in the third isospin component, we can first group them according to their $\Delta I_3$ values and then express them as linear combinations of their $\Delta I$ terms. Inspecting the operators, we find that $O_{1,usdu}^{udsu}$, $O_{1,usdu}^{udsu}$, $O_{3,ddds}^{dss}$, and $O_{3,ddds}^{dss}$ have $\Delta I_3 = 1/2$, whereas $\tilde{O}_{1,usdu}^{udsu}$, $\tilde{O}_{1,usdu}^{udsu}$, $\tilde{O}_{3,ddds}^{dss}$, and $\tilde{O}_{3,ddds}^{dss}$ have $\Delta I_3 = -1/2$. Employing the Clebsch-Gordan decomposition rule, we then get the following results:

- The $\Delta I_3 = 1/2$ operators:

  \[
  O_{1,usdu}^{udsu} = -\frac{1}{3} O_{1,\Delta I=1/2}^{usdu} + \frac{1}{3} O_{1,\Delta I=3/2}^{usdu}, \quad O_{3,ddds}^{dss} = \frac{1}{3} O_{3,\Delta I=1/2}^{ddds} - \frac{1}{3} O_{3,\Delta I=3/2}^{ddds}, \quad (A1)
  \]

  with their components of definite $\Delta I$ being given by

  \[
  O_{1,\Delta I=1/2}^{usdu} = \left[ (d_L\gamma_{\mu}s_L)(u_R\gamma^\mu u_R) - 2(u_L\gamma_{\mu}s_L)(d_R\gamma^\mu d_R) \right] J, \\
  O_{1,\Delta I=3/2}^{usdu} = \left[ (d_L\gamma_{\mu}s_L)(u_R\gamma^\mu u_R) + (u_L\gamma_{\mu}s_L)(d_R\gamma^\mu d_R) - 2(d_L\gamma_{\mu}s_L)(d_R\gamma^\mu d_R) \right] J, \\
  O_{3,\Delta I=1/2}^{ddds} = \left[ (u_R u_L)(d_R s_L) + (d_R u_L)(u_R s_L) + 2(d_R d_L)(u_R s_L) \right] J, \\
  O_{3,\Delta I=3/2}^{ddds} = \left[ (u_R u_L)(d_R s_L) + (d_R u_L)(u_R s_L) - (d_R d_L)(d_R s_L) \right] J, \quad (A2)
  \]

  and similarly $\tilde{O}_{1,usdu}^{udsu}$ and $\tilde{O}_{3,ddds}^{dss}$.
• The $\Delta I_3 = -1/2$ operators:

$$0_{1,\text{udsu}}^{\text{udsu}} = -\frac{1}{3} 0_{1,\Delta I=1/2}^{\text{udsu}} + \frac{1}{3} 0_{1,\Delta I=3/2}^{\text{udsu}}, \quad 0_{3,\text{ddds}}^{\text{ddds}} = \frac{1}{3} 0_{3,\Delta I=1/2}^{\text{ddds}} - \frac{1}{3} 0_{3,\Delta I=3/2}^{\text{ddds}}, \quad (A3)$$

with their components of definite $\Delta I$ being given by

$$0_{1,\Delta I=1/2}^{\text{udsu}} = \left[ (u_L^{\gamma_\mu} u_L) (\bar{s}_R^{\gamma_\mu} d_R) - 2 (u_L^{\gamma_\mu} d_L) (\bar{s}_R^{\gamma_\mu} u_R) - (d_L^{\gamma_\mu} d_L) (\bar{s}_R^{\gamma_\mu} d_R) \right] J,$$

$$0_{1,\Delta I=3/2}^{\text{udsu}} = \left[ (u_L^{\gamma_\mu} u_L) (\bar{s}_R^{\gamma_\mu} d_R) + (u_L^{\gamma_\mu} d_L) (\bar{s}_R^{\gamma_\mu} u_R) - (d_L^{\gamma_\mu} d_L) (\bar{s}_R^{\gamma_\mu} d_R) \right] J,$$

$$0_{3,\Delta I=1/2}^{\text{ddds}} = \left[ (u_R u_L) (\bar{s}_R^d d_L) + (u_R d_L) (\bar{s}_R^u u_L) + 2 (d_R d_L) (\bar{s}_R^d d_L) \right] J,$$

$$0_{3,\Delta I=3/2}^{\text{ddds}} = \left[ (u_R u_L) (\bar{s}_R^d d_L) + (u_R d_L) (\bar{s}_R^u u_L) - (d_R d_L) (\bar{s}_R^d d_L) \right] J, \quad (A4)$$

and similarly their tilded counterparts.

**Appendix B: RG running of dim-6 LEFT operators for $K \to \pi$ transitions**

The 1-loop QCD RG equations of the Wilson coefficients of the LEFT dim-6 quark operators relevant to the $K \to \pi S(S)$ transitions are given by [18]

$$\frac{d}{d\mu} \left( \begin{array}{c} C_{V,LL}^{ddsd} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \end{array} \right) = - \frac{\alpha_s}{2\pi} \left( \begin{array}{cccccccc} -\frac{20}{9} & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -\frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{3} & -6 & \frac{5}{3} & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{3} & 0 & -1 & 0 & 2 & 6 & -\frac{22}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{3} & 0 & -1 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ -\frac{4}{3} & 0 & -1 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \end{array} \right) \left( \begin{array}{c} C_{V,LL}^{ddsd} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \\ C_{uusd}^{V,LL} \end{array} \right), \quad (B1)$$

$$\frac{d}{d\mu} \left( \begin{array}{c} C_{S,LL}^{uusd} \\ C_{S,LL}^{uusd} \\ C_{S,LL}^{uusd} \\ C_{S,LL}^{uusd} \end{array} \right) = - \frac{\alpha_s}{2\pi} \left( \begin{array}{cccc} 8 & -\frac{8}{9} & -\frac{32}{9} & \frac{56}{27} \\ -\frac{8}{9} & 3 & -\frac{22}{9} & \frac{9}{9} \\ -\frac{32}{9} & -\frac{56}{27} & 8 & -\frac{8}{9} \\ \frac{8}{3} & -\frac{22}{9} & -4 & -\frac{8}{3} \end{array} \right) \left( \begin{array}{c} C_{S,LL}^{uusd} \\ C_{S,LL}^{uusd} \\ C_{S,LL}^{uusd} \end{array} \right). \quad (B2)$$

The solutions of these equations between the electroweak scale, which we take to be $\mu = m_W$, and the chiral symmetry breaking $\mu = \Lambda_\chi = 4\pi F_\pi \simeq 1.2$ GeV are

$$\left( \begin{array}{c} C_{V,LL}^{ddsd} \\ C_{V,LL}^{uusd} \\ C_{V,LL}^{uusd} \\ C_{V,LL}^{uusd} \\ C_{V,LL}^{uusd} \\ C_{V,LL}^{uusd} \end{array} \right)_{\mu=\Lambda_\chi} = \left( \begin{array}{c} 0.76 & 0.00 & -0.01 & -0.00 & -0.00 & -0.00 \\ 0.01 & 1.07 & -0.19 & 0.00 & 0.00 & 0.00 \\ -0.16 & -0.86 & 1.31 & -0.01 & -0.03 & -0.01 \\ -0.01 & 0.00 & -0.00 & 1.05 & 0.11 & -0.00 \\ -0.09 & 0.01 & -0.03 & 0.51 & 0.43 & -0.01 \\ -0.01 & 0.00 & -0.00 & -0.00 & 1.05 & 0.11 \end{array} \right) \left( \begin{array}{c} C_{V,LL}^{ddsd} \\ C_{V,LL}^{uusd} \\ C_{V,LL}^{uusd} \\ C_{V,LL}^{uusd} \\ C_{V,LL}^{uusd} \end{array} \right)_{\mu=m_W}, \quad (B3)$$

24
\[
\left( \begin{array}{c}
C_{uusd}^{S1,LL} \\
C_{uusd}^{S8,LL} \\
C_{uusd}^{S1,LL} \\
C_{uusd}^{S8,LL}
\end{array} \right)_{\mu = \Lambda_{\chi}} = \left( \begin{array}{cccc}
2.97 & -0.03 & -1.17 & -0.36 \\
-1.01 & 0.71 & 0.84 & -0.16 \\
-1.17 & -0.36 & 2.97 & -0.03 \\
0.84 & -0.16 & -1.01 & 0.71
\end{array} \right) \left( \begin{array}{c}
C_{uusd}^{S1,LL} \\
C_{uusd}^{S8,LL} \\
C_{uusd}^{S1,LL} \\
C_{uusd}^{S8,LL}
\end{array} \right)_{\mu = m_{W}}. 
\]

(B4)

All of these formulas are also valid for the chirality-flipped counterparts of the operators.

**Appendix C: Chiral structure and hadronization of dim-6 operators for \( K \rightarrow \pi \) transitions**

Here we collect the SU(3)_L × SU(3)_R irreducible representations of the dim-6 four-quark operators examined in section III and the corresponding mesonic operators decomposed into their \( \Delta I = 1/2, 3/2 \) components. Adopting the normalization convention of ref. [42] for the chiral realization of each of the operators\(^5\) we have

\[
0_{uusd}^{V,LL} \big|_{27 \times 1} = \frac{1}{5} \left[ (q_{L}^{\gamma} d_{L}^{\mu} - \bar{s}_{L}^{\gamma} s_{L}^{\mu})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L}) - (u_{L}^{\gamma} d_{L}^{\mu})(\bar{s}_{L}^{\gamma} \gamma_{\mu} u_{L}) \right] - \frac{1}{5} (q_{L}^{\gamma} q_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L})
\]

\[
\Rightarrow \frac{1}{12} g_{27 \times 1} F_{0}^{4} \left[ 4 L_{\mu 22} L_{23}^{\mu} - L_{\mu 33} L_{23}^{\mu} - L_{\mu 21} L_{13}^{\mu} \right] \supset \frac{1}{18} g_{27 \times 1} \left( 2 Q_{1/2}^{V} - 5 Q_{3/2}^{V} \right),
\]

\[
0_{uusd}^{V,LL} \big|_{8 \times 1} = \frac{1}{5} \left[ (u_{L}^{\gamma} d_{L}^{\mu})(\bar{s}_{L}^{\gamma} \gamma_{\mu} u_{L}) - (u_{L}^{\gamma} u_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L}) \right] + \frac{2}{5} (q_{L}^{\gamma} q_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L})
\]

\[
\Rightarrow \frac{1}{12} g_{8 \times 1} F_{0}^{4} \left[ L_{\mu 21} L_{13}^{\mu} - L_{\mu 11} L_{23}^{\mu} \right] \supset \frac{1}{6} g_{8 \times 1} Q_{1/2}^{V},
\]

\[
0_{uusd}^{V,LL} \big|_{8 \times 1} = \frac{1}{5} \left[ (u_{L}^{\gamma} d_{L}^{\mu})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L}) - (u_{L}^{\gamma} u_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} u_{L}) \right] - \frac{1}{5} (q_{L}^{\gamma} q_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L})
\]

\[
\Rightarrow \frac{1}{12} g_{8 \times 1} F_{0}^{4} \left[ 3 L_{\mu 11} L_{23}^{\mu} + 2 L_{\mu 21} L_{13}^{\mu} \right] \supset \frac{1}{18} g_{27 \times 1} \left( Q_{1/2}^{V} + 5 Q_{3/2}^{V} \right),
\]

\[
0_{uusd}^{V,LL} \big|_{8 \times 1} = \frac{2}{5} \left[ (u_{L}^{\gamma} u_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L}) - (u_{L}^{\gamma} d_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} u_{L}) \right] + \frac{2}{5} (q_{L}^{\gamma} q_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L})
\]

\[
\Rightarrow \frac{1}{6} g_{8 \times 1} F_{0}^{4} \left[ L_{\mu 11} L_{23}^{\mu} - L_{\mu 21} L_{13}^{\mu} \right] \supset -\frac{1}{3} g_{8 \times 1} Q_{1/2}^{V} \in \Delta I = \frac{1}{2},
\]

\[
0_{uusd}^{V,LL} \big|_{27 \times 1} = \frac{1}{3} 0_{uusd}^{V,LL} \big|_{27 \times 1},
\]

\[
0_{uusd}^{V,LL} \big|_{8 \times 1} = \frac{11}{30} \left[ (u_{L}^{\gamma} d_{L}^{\mu})(\bar{s}_{L}^{\gamma} \gamma_{\mu} u_{L}) - (u_{L}^{\gamma} u_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L}) \right] + \frac{1}{15} (q_{L}^{\gamma} q_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L})
\]

\[
\Rightarrow \frac{11}{72} g_{8 \times 1} F_{0}^{4} \left[ L_{\mu 21} L_{13}^{\mu} - L_{\mu 11} L_{23}^{\mu} \right] \supset \frac{11}{36} g_{8 \times 1} Q_{1/2}^{V} \in \Delta I = \frac{1}{2},
\]

\[
0_{uusd}^{V,LL} \big|_{8 \times 1} = \frac{2}{5} \left[ (u_{L}^{\gamma} u_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L}) \right] \left[ (u_{R}^{\gamma} u_{R}) - \frac{1}{3} (q_{R}^{\gamma} q_{R}) \right]
\]

\[
\Rightarrow \frac{F_{0}^{4}}{4} g_{8 \times 8} \Sigma_{21}^{l} \Sigma_{13}^{l} \supset \frac{1}{6} g_{8 \times 8} \left( 2 Q_{1/2}^{S} + Q_{3/2}^{S} \right),
\]

\(^5\) Particularly, \((\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L}) \Rightarrow (5/12) g_{27 \times 1} F_{0}^{4} L_{\mu 23} L_{23}^{\mu}\) and \((\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{L})(\bar{s}_{L}^{\gamma} \gamma_{\mu} d_{R}) \Rightarrow (1/4) g_{8 \times 8} F_{0}^{4} \Sigma_{23} \Sigma_{23}^{l}\) for operators with purely left-handed quarks and quarks of mixed chirality, respectively.
where \( q^T = (u, d, s) \), and \( P^\mu_{ij} = (\Sigma^\dagger \partial^\mu \Sigma)_{ij} \) and \( L^\mu_{ij} = (\Sigma \partial^\mu \Sigma^\dagger)_{ij} \). For the operators with \( L \) and \( R \) being exchanged, their irreducible components and chiral realizations can be obtained from the above results by the exchange of \( L \leftrightarrow R \) and \( \Sigma \leftrightarrow \Sigma^\dagger \) respectively.

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