Pointwise approximation by a Durrmeyer variant of Bernstein-Stancu operators

Lvxiu Dong and Dansheng Yu

Abstract
In the present paper, we introduce a kind of Durrmeyer variant of Bernstein-Stancu operators, and we obtain the direct and converse results of approximation by the operators.

MSC: 41A25; 41A35

Keywords: Bernstein-Stancu type operators; pointwise and global estimates; inverse results

1 Introduction
For any \( f \in C_{[0,1]} \), the corresponding Bernstein operators and Bernstein-Durrmeyer operators are defined by

\[
B_n(f, x) := \sum_{k=0}^{n} f \left( \frac{k}{n} \right) p_{nk}(x) \tag{1.1}
\]

and

\[
D_n(f, x) := (n + 1) \sum_{k=0}^{n} p_{nk}(x) \int_{0}^{1} f(t)p_{nk}(t) \, dt, \tag{1.2}
\]

respectively, where \( p_{nk}(x) := \binom{n}{k} x^k (1 - x)^{n-k}, \) \( k = 0, 1, \ldots, n \). Both \( B_n(f, x) \) and \( D_n(f, x) \) have played very important roles in approximation theory and computer science. There are many generalizations of the operators \( B_n(f, x) \) and \( D_n(f, x) \). Among them, Gadjiev and Ghorbanalizadeh [1] introduced the following new generalized Bernstein-Stancu type operators with shifted knots:

\[
S_{\alpha, \beta}(f, x) := \binom{n + \beta_2}{n} \sum_{k=0}^{n} f \left( \frac{k + \alpha_1}{n + \beta_1} \right) q_{nk}(x), \tag{1.3}
\]

where \( x \in A_n := \left[ \frac{\alpha_2}{n + \beta_2}, \frac{n + \alpha_2}{n + \beta_2} \right] \), and

\[
q_{nk}(x) := \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)^k \left( \frac{n + \alpha_2}{n + \beta_2} - x \right)^{n-k}, \quad k = 0, 1, \ldots, n,
\]

© The Author(s) 2017. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.
with $\alpha_k, \beta_k, k = 1, 2$ positive numbers satisfying $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$. Obviously, when $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, $S_{n,\alpha,\beta}(f, x)$ reduces to the classical Bernstein operators in (1.1), when $\alpha_2 = \beta_2 = 0$, it reduces to the so-called Bernstein-Stancu operators which were introduced by Stancu [2]:

$$B_{n,\alpha,\beta}(f, x) := \sum_{k=0}^{n} \binom{n}{k} \binom{n+\alpha}{k+\beta} \rho_n(x). \quad (1.4)$$

Some approximation properties and generalizations of the operators $S_{n,\alpha,\beta}(f, x)$ can be found in [3–5].

Motivated by (1.3), we introduce the following generalization of the operators (1.2):

$$\tilde{S}_{n,\alpha,\beta}(f, x) := \left( \frac{n+\beta_2}{n} \right) \sum_{k=0}^{n} \lambda_{nk} q_{nk}(x) \int_{A_n} q_{nk}(t)f\left( \frac{nt+\alpha_1}{n+\beta_1} \right) dt,$$

where

$$\lambda_{nk} = \int_{A_n} q_{nk}(t) dt, \quad k = 0, 1, \ldots, n,$$

and $\alpha_k, \beta_k, k = 1, 2$ positive numbers satisfying $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$.

By Lemma 1 in Section 2, we observe that $\tilde{S}_{n,\alpha,\beta}(f, x)$ can be rewritten as follows:

$$\tilde{S}_{n,\alpha,\beta}(f, x) = \left( \frac{n+\beta_2}{n} \right)^{2n+1} \sum_{k=0}^{n} q_{nk}(x)(n+1) \int_{A_n} q_{nk}(t)f\left( \frac{nt+\alpha_1}{n+\beta_1} \right) dt.$$

Especially, when $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, $\tilde{S}_{n,\alpha,\beta}(f, x)$ reduces to the classical Bernstein-Durrmeyer operators in (1.2). Many authors have studied some special cases of the operators $\tilde{S}_{n,\alpha,\beta}(f, x)$. For example, the case $\alpha_1 = \alpha_2 = \beta_1 = 0$ in [6] by Jung, Deo, and Dhamija, the case $\alpha_1 = \beta_1 = 0$ in [7] by Acar, Aral, and Gupta.

The main purpose of the present paper is to establish pointwise direct and converse approximation theorems of approximation by $\tilde{S}_{n,\alpha,\beta}(f, x)$. To state our result, we need some notations:

$$\omega^2_{\varphi}(f, t) = \sup_{0 \leq h \leq t} \sup_{x \in [a, b]} |\Delta_{bp}^2 f(x)|,$$

$$D_2^2 = \{ f \in C(A_n), f' \in A.C.loc, \|\varphi^{25} f''\| < \infty \},$$

$$K_{\varphi'}(f, t^2) = \inf_{g \in D_2^2} \|f - g\| + t^2 \|\varphi^{25} g''\|,$$

$$\overline{D}_2^2 = \{ f \in D_2^2, \|f''\| < \infty \},$$

$$\overline{K}_{\varphi'}(f, t^2) = \inf_{g \in \overline{D}_2^2} \|f - g\| + t^2 \|\varphi^{25} g''\| + t^{4(2-\lambda)} \|g''\|, \quad (1.6)$$

and $\varphi(x) = \sqrt{x - \frac{\alpha_2}{n+\beta_2}} \sqrt{\frac{n+\alpha_2}{n+\beta_2} - x}, \ 0 \leq \lambda \leq 1$. It is well known (see [8], Theorem 3.1.2) that

$$\omega^2_{\varphi}(f, t) \sim K_{\varphi'}(f, t^2) \sim \overline{K}_{\varphi'}(f, t^2), \quad (1.7)$$

where $x \sim y$ means that there exists a positive constant $c$ such that $c^{-1} y \leq x \leq cy$. 

\[ \]
Our first result can be read as follows.

**Theorem 1** Let \( f \) be a continuous function on \( A_n \), \( \lambda \in [0, 1] \) be a fixed positive number. Then there exists a positive constant \( C \) only depending on \( \lambda, \alpha_1, \alpha_2, \beta_1, \) and \( \beta_2 \) such that

\[
\left| \bar{S}_{n, \alpha, \beta}(f, x) - f(x) \right| \leq C \left( \omega_\varphi^\lambda \left( f, \frac{\delta_{\alpha_2}^{1-k}(x)}{\sqrt{n}} \right) + \omega \left( f, \frac{1}{n} \right) \right),
\]

(1.9)

where \( \delta_{\alpha}(x) = \varphi(x) + 1/\sqrt{n} \sim \max \{ \varphi(x), 1/\sqrt{n} \} \), and \( \omega(f, t) \) is the usual modulus of continuity of \( f \) on \( A_n \).

Throughout the paper, \( C \) denotes either a positive absolute constant or a positive constant that may depend on some parameters but not on \( f, x, \) and \( n \). Their values may be different at different locations.

For the converse result, we have the following.

**Theorem 2** Let \( f \) be a continuous function on \( A_n \), \( 0 < \alpha < \frac{2}{3-\lambda}, 0 \leq \lambda \leq 1 \). Then

\[
\left| \bar{S}_{n, \alpha, \beta}(f, x) - f(x) \right| = O\left( \left( n^{-1/2} \delta_{\alpha_2}^{1-k}(x) \right)^{\alpha} \right)
\]

implies that

(i) \( \omega^2_{\varphi, \alpha}(f, t) = O(t^\alpha) \); (ii) \( \omega(f, t) = O(t^{(1-\lambda)/2}) \). (1.11)

2 Auxiliary lemmas

**Lemma 1** We have

\[
\lambda_{kn} = \int_{A_n} q_{nk}(t) dt = \left( \frac{n}{n + \beta_2} \right)^{n+1} \frac{1}{n+1}, \quad k = 0, 1, \ldots, n.
\]

(2.1)

**Proof** For \( p, q = 1, 2, \ldots \), set

\[
B^*(p, q) := \int_{A_n} \left( x - \frac{\alpha_2}{n + \beta_2} \right)^{p-1} \left( \frac{n + \alpha_2}{n + \beta_2} - x \right)^{q-1} dx
\]

\[
= \int_0^{\frac{n}{n + \beta_2}} x^{p-1} \left( \frac{n}{n + \beta_2} - x \right)^{q-1} dx.
\]

Then

\[
B^*(p, q) = \frac{q-1}{p} \int_0^{\frac{n}{n + \beta_2}} x^p \left( \frac{n}{n + \beta_2} - x \right)^{q-2} dx
\]

\[
= \frac{q-1}{p} \int_0^{\frac{n}{n + \beta_2}} \left( \frac{n}{n + \beta_2} x^{p-1} - x^{p-1} \left( \frac{n}{n + \beta_2} - x \right) \right) \left( \frac{n}{n + \beta_2} - x \right)^{q-2} dx
\]

\[
= \frac{q-1}{p} \cdot \frac{n}{n + \beta_2} B^*(p, q - 1) - \frac{q-1}{p} B^*(p, q),
\]

which implies that

\[
B^*(p, q) = \frac{q-1}{p+q-1} \cdot \frac{n}{n + \beta_2} B^*(p, q - 1).
\]
Therefore,

\[
\lambda_{kn} = \left( \frac{n}{k} \right) B^* (k + 1, n - k + 1)
\]

\[
= \left( \frac{n}{n + \beta_2} \right)^{n-k} \left( \frac{n}{k} \right) \left( \frac{n - k}{(n + 1)n \cdots (k + 2)} \right) B^* (k + 1, 1)
\]

\[
= \left( \frac{n}{n + \beta_2} \right)^{n-k} \frac{k + 1}{(n + 1)} \int_0^{\frac{n}{n + \beta_2}} x^k \, dx
\]

\[
= \left( \frac{n}{n + \beta_2} \right)^{n+1} \frac{1}{n + 1}.
\]

\[\square\]

**Lemma 2** For any \( x \in A_n \), we have

\[
\tilde{S}_{n, \alpha, \beta}(t-x)^2, x \leq \frac{C}{n} \delta_n^2(x).
\] 

(2.2)

**Proof** Write

\[
\tilde{D}_{n, \alpha, \beta}(f, x) := \left( \frac{n + \beta_2}{n} \right)^{2n+1} \sum_{k=0}^{n} q_{nk}(x)(n + 1) \int_{A_n} q_{nk}(t)f(t) \, dt.
\]

Then [7]

\[
\tilde{D}_{n, \alpha, \beta}(1, x) = 1, \tilde{D}_{n, \alpha, \beta}(t, x) = \frac{n}{n + 2} x + \frac{n + 2 \alpha_2}{(n + 2)(n + \beta_2)},
\]

(2.3)

\[
\tilde{D}_{n, \alpha, \beta}(t^2, x) = \left( x - \frac{\alpha_2}{n + \beta_2} \right)^2 \frac{n(n-1)}{(n+2)(n+3)}
\]

\[
+ \frac{n}{n + \beta_2} \left( x - \frac{\alpha_2}{n + \beta_2} \right) \frac{4n}{(n+2)(n+3)}
\]

\[
+ \left( \frac{n}{n + \beta_2} \right)^2 \frac{2}{(n+2)(n+3)} + \frac{2n \alpha_2}{(n+2)(n+\beta_2)} \left( x - \frac{\alpha_2}{n + \beta_2} \right)
\]

\[
+ \frac{2n \alpha_2}{(n+2)(n+\beta_2)^2} + \left( \frac{\alpha_2}{n + \beta_2} \right)^2,
\]

and

\[
\tilde{D}_{n, \alpha, \beta}(t-x)^2, x \leq \frac{C}{n} \delta_n^2(x).
\]

By the facts that

\[
\tilde{S}_{n, \alpha, \beta}(1, x) = \tilde{D}_{n, \alpha, \beta}(1, x) = 1,
\]

\[
\tilde{S}_{n, \alpha, \beta}(t, x) = \frac{n}{n + \beta_1} \tilde{D}_{n, \alpha, \beta}(t, x) + \frac{\alpha_1}{n + \beta_1},
\]

(2.4)

and

\[
\tilde{S}_{n, \alpha, \beta}(t^2, x) = \frac{n^2}{(n + \beta_1)^2} \tilde{D}_{n, \alpha, \beta}(t^2, x) + \frac{2n \alpha_1}{(n + \beta_1)^2} \tilde{D}_{n, \alpha, \beta}(t, x) + \frac{\alpha_1^2}{(n + \beta_1)^2},
\]
we get

\[
\tilde{S}_{n,a,b}( (t-x)^2, x) = \frac{n^2}{(n + \beta_1)^2} \tilde{D}_{n,a,b}( (t-x)^2, x)
\]

\[
+ \left( \frac{2n^2x}{(n + \beta_1)^2} + \frac{2n\alpha_1}{n + \beta_1} \right) \tilde{D}_{n,a,b}(t, x)
\]

\[
+ \frac{\alpha_2^2}{(n + \beta_1)^2} \left( 2\alpha_1 \frac{x^2}{n + \beta_1} + x^2 - \frac{n^2}{(n + \beta_1)^2} \right)
\]

\[
= \frac{n^2}{(n + \beta_1)^2} \tilde{D}_{n,a,b}( (t-x)^2, x) + \left( \frac{\beta_1^2 + 4\beta_1}{(n + \beta_1)^2(n + 2)} \right) 2n\alpha_1(\beta_1 + \beta_2 + 2)n^2 + 2n\alpha_1(\beta_1 \beta_2 + 2\beta_1 + 2\beta_2) + 4\alpha_1\beta_1\beta_2 x
\]

\[
+ \frac{\alpha_2^2}{(n + \beta_1)^2}
\]

\[
\leq \tilde{D}_{n,a,b}( (t-x)^2, x) + \frac{C}{n^2}
\]

\[
\leq \frac{C}{n^2}(x).
\]

**Lemma 3** For any given \( \gamma \geq 0 \), we have

\[
\sum_{k=0}^{n} \left[ \frac{k + \alpha_2}{n + \beta_2} - x \right]^{\gamma} |q_{nk}(x)| \leq \frac{C}{n^{\gamma/2}} \delta_n^2(x), \quad x \in [0, 1].
\]

**Proof** It was showed in [3] that

\[
\sum_{k=0}^{n} \left[ \frac{k + \alpha_1}{n + \beta_1} - x \right]^{\gamma} |q_{nk}(x)| \leq \frac{C}{n^{\gamma/2}} \delta_n^2(x), \quad x \in [0, 1],
\]

where \( \delta_n^2(x) := \psi(x) + \frac{1}{\sqrt{n}} \) and \( \psi(x) = \sqrt{x(1-x)} \). We verify that

\[
\delta_n^2(x) \sim \delta_n(x), \quad x \in [0, 1].
\]

In fact, when \( x \in \left[ \frac{2\alpha_2 + 1}{n + \beta_2}, \frac{n - \beta_2 + 2\alpha_2}{n + \beta_2} \right] \), we have

\[
\frac{1}{2} x \leq x - \frac{\alpha_2}{n + \beta_2} \leq x,
\]

\[
\frac{1}{2} (1-x) \leq \frac{n + \alpha_2}{n + \beta_2} - x \leq 1 - x.
\]

Thus,

\[
\psi(x) \sim \psi(x),
\]

which implies (2.7) for \( x \in \left[ \frac{2\alpha_2 + 1}{n + \beta_2}, \frac{n - \beta_2 + 2\alpha_2}{n + \beta_2} \right] \). When \( x \in \left[ 0, \frac{2\alpha_2 + 1}{n + \beta_2} \right) \cup \left( \frac{n - \beta_2 + 2\alpha_2}{n + \beta_2}, 1 \right] \), we have

\[
\delta_n^2(x) \sim \delta_n(x) \sim \frac{1}{\sqrt{n}},
\]

and thus (2.7) also holds.
Now, by (2.6) and (2.7), we have
\[
\sum_{k=0}^{n} \frac{k + \alpha_2}{n + \beta_2} - x \left| q_{nk}(x) \right| \leq \sum_{k=0}^{n} \frac{k + \alpha_2}{n + \beta_2} - \frac{k + \alpha_1}{n + \beta_1} \left| q_{nk}(x) \right| + \sum_{k=0}^{n} \frac{k + \alpha_1}{n + \beta_1} - x \left| q_{nk}(x) \right|
\]
\[
\leq \frac{C}{n^\gamma} \sum_{k=0}^{n} q_{nk}(x) + C \frac{\delta_n^\gamma(x)}{n^{\gamma/2}}
\]
\[
\leq C \delta_n^\gamma(x) \frac{n}{n^{\gamma/2}}. \quad \Box
\]

**Lemma 4** For any \( x \in A_n \), we have
\[
\sum_{k=0}^{n} q_{nk}(x)(n + 1) \int_{A_n} \delta_n^2(t) q_{nk}(t) dt \leq C \delta_n^2(x) \tag{2.9}
\]

and
\[
\sum_{k=0}^{n-1} q_{n-1,k}(x) n \int_{A_n} \delta_n^{-12}(t) q_{n+1,k+1}(t) dt \leq C \delta_n^{-2}(x). \tag{2.10}
\]

**Proof** By a similar calculation to that of Lemma 1, we have
\[
\int_{A_n} \phi^2(t) q_{nk}(t) dt = \left( \frac{n}{n + \beta_2} \right)^{n^3} \frac{(n - k + 1)(k + 1)}{(n + 3)(n + 2)(n + 1)}. \tag{2.11}
\]

On the other hand, we have
\[
\sum_{k=0}^{n} \left( \frac{k}{n} - \frac{k^2}{n^2} \right) q_{nk}(x) = \left( \frac{n}{n + \beta_2} \right)^{n-1} \left( x - \frac{\alpha_2}{n + \beta_2} \right) - \left( \frac{n}{n + \beta_2} \right)^{n-1} \left( x - \frac{\alpha_2}{n + \beta_2} \right) \left( \frac{n}{n + \beta_2} \right)^{n-2} \phi^2(x).
\]

Therefore,
\[
\sum_{k=0}^{n} q_{nk}(x)(n + 1) \int_{A_n} \delta_n^2(t) q_{nk}(t) dt \leq 2 \sum_{k=0}^{n} q_{nk}(x)(n + 1) \int_{A_n} \phi^2(t) + \frac{1}{n} q_{nk}(t) dt
\]
\[
\leq 2 \sum_{k=0}^{n} q_{nk}(x) \left( \frac{n}{n + \beta_2} \right)^{n^3} \frac{(n - k + 1)(k + 1)}{(n + 3)(n + 2)}
\]
\[
+ \frac{C}{n} \sum_{k=0}^{n} q_{nk}(x)
\]
\[
\leq C \sum_{k=0}^{n} q_{nk}(x) \left( \frac{(n - k)k}{n^2} + \frac{1}{n} \right) + \frac{C}{n}
\]
\[
\leq C \delta_n^2(x),
\]

which proves (2.9).
By Lemma 1, we have

\[ n \int_{A_n} \delta_n^{-2}(t)q_{n+1,k+1}(t) \, dt \leq Cn \int_{A_n} \left( \phi^{-2}(t) + n \right)q_{n+1,k+1}(t) \, dt \]

\[ \leq Cn \left( \int_{A_n} \phi^{-2}(t)q_{n+1,k+1}(t) \, dt + 1 \right) \]

\[ = Cn \left( \frac{(n + 1)n}{(k + 1)(n - k)} \int_{A_n} q_{n-1,k}(t) \, dt + 1 \right) \]

\[ \leq Cn \left( \frac{(n + 1)}{(k + 1)(n - k)} + 1 \right) \]

\[ \leq Cn. \]

Then

\[ \sum_{k=0}^{n-1} q_{n-1,k}(x)n \int_{A_n} \delta_n^{-2}(t)q_{n+1,k+1}(t) \, dt \leq Cn \sum_{k=0}^{n} q_{n-1,k}(x) \]

\[ = Cn \leq C \delta_n^{-2}(x). \]

Hence, (2.10) is proved. \( \square \)

**Lemma 5** If \( f \) is \( r \) times differentiable on \([0, 1]\), then

\[
\tilde{S}_{n,a,b}^{(r)}(f, x) = \left(\frac{n + \beta_2}{n}\right)^{2n+1} \left(\frac{n}{n + \beta_1}\right)^r \frac{(n + 1)!}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} q_{n-r,k}(x) \]

\[
\times \int_{\frac{2}{\pi n}}^{\frac{2}{\pi n}} q_{n+r,k+r}(t) f^{(r)} \left(\frac{nt + \alpha_1}{n + \beta_1}\right) \, dt. \tag{2.12}
\]

**Proof** By using Leibniz’s theorem, we have

\[
\tilde{S}_{n,a,b}^{(r)}(f, x) = \left(\frac{n + \beta_2}{n}\right)^{2n+1} \sum_{i=0}^{r} \sum_{k=i}^{n-r+i} \binom{r}{i} \frac{(-1)^{r-i}(n+1)!}{(k-i)!(n-k-r+i)!} \]

\[
\times \left( x - \frac{\alpha_2}{n + \beta_2} \right)^{k-i} \left(\frac{n + \alpha_2}{n + \beta_2} - x \right)^{n-k-r+i} \int_{\frac{2}{\pi n}}^{\frac{2}{\pi n}} q_{n,k+r}(t) f \left(\frac{nt + \alpha_1}{n + \beta_1}\right) \, dt
\]

\[
= \left(\frac{n + \beta_2}{n}\right)^{2n+1} \sum_{k=i}^{n-r+i} \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i}(n+1)!}{(n-r)!} q_{n-r,k-r+i}(x) \]

\[
\times \int_{\frac{2}{\pi n}}^{\frac{2}{\pi n}} q_{n,k+r}(t) f \left(\frac{nt + \alpha_1}{n + \beta_1}\right) \, dt
\]

\[
= \left(\frac{n + \beta_2}{n}\right)^{2n+1} \frac{(n+1)!}{(n-r)!} \sum_{k=0}^{n-r} (-1)^{r}(n+1)! q_{n-r,k}(x) \]

\[
\times \int_{\frac{2}{\pi n}}^{\frac{2}{\pi n}} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} q_{n,k+r}(t) f \left(\frac{nt + \alpha_1}{n + \beta_1}\right) \, dt. \]
Since
\[
\frac{d^r}{dt^r}q_{n,r,k+1}(t) = \sum_{i=0}^{r} \binom{r}{i}(-1)^i \frac{(n+r)!}{n!} q_{n,k+1}(t),
\]
we have
\[
S_{n,a,b}^{(r)}(f,x) = \left(\frac{n + \beta_2}{n}\right)^{2n+1} \frac{(n + 1)!n!}{(n-r)((n+r))!} \sum_{k=0}^{n-r} q_{n-r,k}(x)
\times \int_{\frac{n-r}{n+\beta_2}}^{\frac{n+r}{n+\beta_2}} (-1)^r q_{n,r,k+1}(t) f \left(\frac{nt + \alpha_1}{n + \beta_1}\right) dt.
\]
We obtain the required result by integrating by parts \( r \) times.

Set
\[
\|f\|_0 = \sup_{x \in A_n} \{|\delta_n^{(j-1)}(x)f(x)|\};
\]
\[
C_{a,\lambda} = \{f \in C(A_n), \|f\|_0 < +\infty\};
\]
\[
\|f\|_1 = \sup_{x \in A_n} \{|\delta_n^{(j-1)}(x)f(x)|\};
\]
\[
C_{a,\lambda}^1 = \{f \in C_{a,\lambda}, \|f\|_1 < +\infty\};
\]
\[
\|f\|_2 = \sup_{x \in A_n} \{|\delta_n^{(j-1)}(x)f''(x)|\};
\]
\[
C_{a,\lambda}^2 = \{f \in C_{a,\lambda}, f'' \in A.C.loc, \|f\|_2 < +\infty\};
\]
\[
K_{a,\lambda}^1(f,t) = \inf_{g \in C_{a,\lambda}^1} \{|\|f - g\|_0 + t\|g\|_1\};
\]
\[
K_{a,\lambda}^2(f,t) = \inf_{g \in C_{a,\lambda}^2} \{|\|f - g\|_0 + t\|g\|_2\}.
\]

**Lemma 6** If \( 0 \leq \lambda \leq 1, 0 < \alpha < 2 \), then
\[
\|S_{n,a,b}^{(r)}(f)\|_1 \leq C n^{1(2-\lambda)} \|f\|_0, \quad f \in C_{a,\lambda},
\]
\[
\|S_{n,a,b}^{(r)}(f)\|_1 \leq C \|f\|_1, \quad f \in C_{a,\lambda}^1.
\]

**Proof** Firstly, we prove (2.13) by considering the following two cases.

**Case 1.** \( x \in B_n = [\frac{n + 1}{n + \beta_2}, \frac{n + 2}{n + \beta_2}]. \) In this case, we have
\[
\varphi(x) \geq \min \left( \varphi \left( \frac{\alpha_2 + 1}{n + \beta_2} \right), \varphi \left( \frac{n + \alpha_2 - 1}{n + \beta_2} \right) \right) \geq \frac{C}{\sqrt{n}},
\]
which means that
\[
\delta_n(x) \sim \varphi(x) \quad \text{for} \ x \in B_n.
\]
By simple calculations, we have

\[ q_{nk}(x) = n\varphi^{-2}(x) \left( \frac{k + \alpha_2}{n + \beta_2} - x \right) q_n(x) \quad (2.16) \]

and

\[
\delta_n \left( \frac{nt + \alpha_1}{n + \beta_1} \right) = \sqrt{\left( t - \frac{\alpha_2}{n + \beta_2} + \frac{\alpha_1 - \beta_1 t}{n + \beta_1} \right) \left( \frac{n + \alpha_2}{n + \beta_2} - t + \frac{\beta_1 t - \alpha_1}{n + \beta_1} \right) + \frac{1}{\sqrt{n}}} \\
= \sqrt{\varphi^2(t) + \frac{1}{\sqrt{n}}} + \frac{1}{\sqrt{n}} \sim \varphi(t) + \frac{1}{\sqrt{n}} = \delta_n(t). \quad (2.17)
\]

By (2.1), (2.15)-(2.17), and Hölder’s inequality, we have

\[
\left| \delta_n \left( \frac{2}{n} - \varphi^{(1-\lambda)}(x) \right) \right| \leq Cn\psi \left( \frac{2}{n} - \varphi^{(1-\lambda)}/2 \right) \left( \frac{n + \beta_2}{n} \right)^{2n+1} \times \sum_{k=0}^{n} q_{nk}(x) \left| \frac{k + \alpha_2}{n + \beta_2} - x \right| (n + 1) \left| \int_{A_n} f \left( \frac{nt + \alpha_1}{n + \beta_1} \right) q_n(t) \, dt \right| \\
\leq Cn\|f\|_0 \psi \left( \frac{2}{n} - \varphi^{(1-\lambda)}/2 \right) \sum_{k=0}^{n} q_{nk}(x) \left| \frac{k + \alpha_2}{n + \beta_2} - x \right| (n + 1) \left| \int_{A_n} \delta_n^{(1-\lambda)}(t) q_n(t) \, dt \right| \\
\leq Cn\|f\|_0 \psi \left( \frac{2}{n} - \varphi^{(1-\lambda)}/2 \right) \sum_{k=0}^{n} q_{nk}(x) \left| \frac{k + \alpha_2}{n + \beta_2} - x \right| (n + 1) \left( \int_{A_n} \delta_n(t) q_n(t) \, dt \right)^{\alpha(1-\lambda)/2} \\
\times (n + 1) \left( \int_{A_n} q_n(t) \, dt \right)^{1-\alpha(1-\lambda)/2} \\
\leq Cn\|f\|_0 \psi \left( \frac{2}{n} - \varphi^{(1-\lambda)}/2 \right) \sum_{k=0}^{n} q_{nk}(x) \left| \frac{k + \alpha_2}{n + \beta_2} - x \right| (n + 1) \left( \int_{A_n} \delta_n(t) q_n(t) \, dt \right)^{\alpha(1-\lambda)/2}.
\]

By (2.9), (2.15) (2.5), and Hölder’s inequality again, we have

\[
\left| \delta_n \left( \frac{2}{n} - \varphi^{(1-\lambda)}(x) \right) \right| \leq Cn\|f\|_0 \psi \left( \frac{2}{n} - \varphi^{(1-\lambda)}/2 \right) \sum_{k=0}^{n} q_{nk}(x) \left| \frac{k + \alpha_2}{n + \beta_2} - x \right| \left( \int_{A_n} q_n(t) \, dt \right)^{\alpha(1-\lambda)/2} \\
\times \left( \sum_{k=0}^{n} q_{nk}(x)(n + 1) \int_{A_n} \delta_n^2(t) q_n(t) \, dt \right)^{\alpha(1-\lambda)/2} \\
\leq Cn^{1/2} \|f\|_0 \psi \left( \frac{2}{n} - \varphi^{(1-\lambda)}/2 \right) \left( \sum_{k=0}^{n} q_{nk}(x)(n + 1) \int_{A_n} \delta_n^2(t) q_n(t) \, dt \right)^{\alpha(1-\lambda)/2} \\
\leq Cn^{1/2} \|f\|_0 \psi \left( \frac{2}{n} - \varphi^{(1-\lambda)}/2 \right) \leq Cn^{1/(2-\lambda)} \|f\|_0. \quad (2.18)
\]

Case 2. \( x \in B_n^c = \left[ \frac{\alpha_2}{n + \beta_2} \right]^2 \cup \left( \frac{\alpha_2}{n + \beta_2} \right) \cup \left( \frac{\alpha_2}{n + \beta_2} \right) \cup \left( \frac{\alpha_2}{n + \beta_2} \right) \cup \left( \frac{\alpha_2}{n + \beta_2} \right). \) In this case, we have

\[ \delta_n(x) \sim \frac{1}{\sqrt{n}}, \quad x \in B_n^c. \quad (2.19) \]
Noting that
\[ q_n^*(x) = n(q_{n-1,k}(x) - q_{n-1,k}(x)) \]
with \( q_{n-1,k}(x) = q_{n-1,n}(x) = 0 \), we get
\[
\tilde{S}_{n,a,b}^*(f, x) = n \sum_{k=0}^{n-1} q_{n-1,k}(x)(n + 1) \int_{A_n} f \left( \frac{nt + \alpha_1}{n + \beta_1} \right) (q_{n,k+1}(t) - q_{n,k}(t)) \, dt.
\]

Then, by using (2.17) and Hölder’s inequality twice,
\[
\left| \delta_n^{\frac{1}{2} - \alpha(1-\lambda)}(x) \tilde{S}_{n,a,b}^*(f, x) \right| 
\leq C \delta_n^{\frac{1}{2} - \alpha(1-\lambda)}(x) \| f \|_0 \left| \sum_{k=0}^{n-1} q_{n-1,k}(x)(n + 1) \int_{A_n} \delta_n^{\alpha(1-\lambda)}(t) (q_{n,k+1}(t) + q_{n,k}(t)) \, dt \right|
\leq C \delta_n^{\frac{1}{2} - \alpha(1-\lambda)}(x) \| f \|_0 \left| \sum_{k=0}^{n-1} q_{n-1,k}(x)(n + 1) \int_{A_n} \delta_n^{2}(t) (q_{n,k+1}(t) + q_{n,k}(t)) \, dt \right|^{\frac{\alpha(1-\lambda)}{4}}
\leq C \delta_n^{\frac{1}{2} - \alpha(1-\lambda)}(x) \| f \|_0 \delta_n^{\alpha(1-\lambda)}
\leq C \delta_n^{\frac{1}{2} - \alpha(1-\lambda)} \| f \|_0,
\]

where in the fourth inequality, we used the following fact, which can be deduced exactly in the same way as (2.10):
\[
\sum_{k=0}^{n-1} q_{n-1,k}(x)(n + 1) \int_{A_n} \delta_n^{2}(t) q_{n,k+1}(t) \, dt \leq C \delta_n^{2}(x).
\]

We obtain (2.13) by combining (2.18) and (2.20).

Now, we begin to prove (2.14). If \( \frac{1}{2} - \alpha(1-\lambda) < 0 \), by (2.12) and using Hölder's inequality twice, we get
\[
\left| \delta_n^{\frac{1}{2} - \alpha(1-\lambda)}(x) \tilde{S}_{n,a,b}^*(f, x) \right| 
\leq C \| f \|_1 \left| \delta_n^{\frac{1}{2} - \alpha(1-\lambda)}(x) n \sum_{k=0}^{n-1} q_{n-1,k}(x) \int_{A_n} q_{n+1,k+1}(t) \delta_n^{\frac{1}{2} - \alpha(1-\lambda)}(t) \, dt \right|
\leq C \| f \|_1 \delta_n^{\frac{1}{2} - \alpha(1-\lambda)}(x) \sum_{k=0}^{n-1} q_{n-1,k}(x) \left( n \int_{A_n} q_{n+1,k+1}(t) \delta_n^{-2}(t) \, dt \right)^{\frac{1}{2}} \left( \delta_n^{\frac{1}{2} - \alpha(1-\lambda)} \right)^{\frac{1}{2}}
\times \left( n \int_{A_n} q_{n+1,k+1}(t) \, dt \right)^{\frac{1}{2}} \left( \delta_n^{\frac{1}{2} - \alpha(1-\lambda)} \right)^{\frac{1}{2}}
\leq C \| f \|_1 \delta_n^{\frac{1}{2} - \alpha(1-\lambda)}(x) \sum_{k=0}^{n-1} q_{n-1,k}(x) \left( n \int_{A_n} q_{n+1,k+1}(t) \delta_n^{-2}(t) \, dt \right) \left( \delta_n^{\frac{1}{2} - \alpha(1-\lambda)} \right)^{\frac{1}{2}}
\quad \text{(by (2.1))}
\]
\[
\leq C \|f\|_1 \left(\frac{1}{\lambda} - 1\right) \left(\sum_{k=0}^{n-1} q_{n-1,k}(x) n \int q_{n+1,k+1}(t) \delta_n^{\lambda-2}(t) \, dt\right) \frac{1}{\lambda} \left(\frac{1}{\lambda} - 1\right) \left(\frac{1}{\lambda} - 1\right)
\]

\leq C \|f\|_1,

where, in the last inequality, (2.10) is applied.

If \((\frac{2}{\lambda} - \alpha)(\lambda - 1) > 0\), by using (2.9) instead of (2.10), we also can deduce that

\[
\left|\delta_n^{\frac{2}{\lambda} - \alpha}(x) \overline{S}_{n,\alpha,\beta}(f, x)\right| \leq C \|f\|_1.
\]

\text{Lemma 7} If \(0 \leq \lambda \leq 1, 0 < \alpha < 2\), then

\[
\left\|\overline{S}_{n,\alpha,\beta}(f)\right\|_2 \leq C n \|f\|_0, \quad f \in C_{0,\lambda},
\]

\[
\left\|\overline{S}_{n,\alpha,\beta}(f)\right\|_2 \leq C \|f\|_2, \quad f \in C_{2,\lambda}.
\]

\text{Proof} It can be proved in a way similar to Lemma 6.

\text{Lemma 8} For \(0 < t < \frac{1}{2}, \frac{1}{2} \leq x \leq 1 - \frac{t}{2}, x \in [0,1], \beta < 2\), we have

\[
\int_{t/2}^{t/2} \delta_n^\beta(x + u) \, du \leq C(\beta) t \delta_n^\beta(x).
\]

\text{Lemma 9} For \(0 < t < \frac{1}{2}, t \leq x \leq 1 - t, x \in [0,1], 0 \leq \beta \leq 2, \) we have

\[
\int_{-t/2}^{t/2} \int_{-t/2}^{t/2} \delta_n^\beta(x + u + v) \, du \, dv \leq C t^2 \delta_n^\beta(x).
\]

It has been shown in [9] that Lemma 8 and Lemma 9 are valid when \(\delta_n(t)\) is replaced by \(\delta_n^\ast(t)\), which combining with (2.8) proves Lemma 8 and Lemma 9.

\section{Proofs of theorems}

\subsection{Proof of Theorem 1}

Define the auxiliary operators \(S_{n,\alpha,\beta}(f, x)\) as follows:

\[
S_{n,\alpha,\beta}(f, x) = \overline{S}_{n,\alpha,\beta}(f, x) + L_{n,\alpha,\beta}(f, x),
\]

where

\[
L_{n,\alpha,\beta}(f, x) = f(x) - f(\overline{S}_{n,\alpha,\beta}(t, x)).
\]

By (2.3) and (2.4), we have

\[
\left|\overline{S}_{n,\alpha,\beta}(t, x) - x\right| \leq \frac{C}{n},
\]

\[
S_{n,\alpha,\beta}(1, x) = 1, \quad S_{n,\alpha,\beta}(t - x, x) = 0,
\]

and

\[
\|S_{n,\alpha,\beta}\| \leq 3.
\]
It follows from (3.2) that
\[
|L_{n,a,b}[f, x]| \leq \omega(f, |S_{n,a,b}(t, x) - x|) \leq C\omega\left(f, \frac{1}{n}\right). \tag{3.5}
\]

From (1.7) and (1.8), for any fixed \(x, \lambda,\) and \(n,\) we may choose a \(g_{n,\lambda}(t) \in \bar{D}_{\lambda}^2\) such that
\[
\|f - g\| \leq C\omega^2_{\psi^2}(f, n^{-1/2}\delta_n^{1-\lambda}(x)), \tag{3.6}
\]
\[
(n^{-1/2}\delta_n^{1-\lambda}(x))^2 \|\psi^2 g\| \leq C\omega^2_{\psi^2}(f, n^{-1/2}\delta_n^{1-\lambda}(x)), \tag{3.7}
\]
\[
(n^{-1/2}\delta_n^{1-\lambda}(x))^{4/(2-\lambda)} \|g''\| \leq C\omega^2_{\psi^2}(f, n^{-1/2}\delta_n^{1-\lambda}(x)). \tag{3.8}
\]

By (3.4) and (3.6), we have
\[
|S_{n,a,b}(f, x) - f| \leq |S_{n,a,b}(f, g, x)| + |f(x) - g(x)| + |S_{n,a,b}(g, x) - g(x)|
\leq 4\|f - g\| + |S_{n,a,b}(g, x) - g(x)|
\leq C\omega^2_{\psi^2}(f, n^{-1/2}\delta_n^{1-\lambda}(x)) + |S_{n,a,b}(g, x) - g(x)|. \tag{3.9}
\]

Noting that \(\psi^2(x)\) and \(\delta_n^{\beta}(x)\) are concave functions on \([0, 1],\) for any \(t, x \in [0, 1],\) and \(u\) between \(x\) and \(t,\) say \(u = \theta x + (1 - \theta)t,\) \(0 \leq \theta \leq 1,\) we have
\[
\frac{|t - u|}{\psi^2(u)} = \frac{\theta |t - x|}{\psi^2((\theta x + (1 - \theta)t)} \leq \frac{\theta |t - x|}{\psi^2(x)} \leq \frac{|t - x|}{\psi^2(x)}, \tag{3.10}
\]
\[
\frac{|t - u|}{\delta_n^{\beta}(u)} \leq \frac{|t - x|}{\delta_n^{\beta}(x)}. \tag{3.11}
\]

By using Taylor’s expansion
\[
g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u)g''(u)\,du,
\]
(3.3), and (3.11),
\[
|S_{n,a,b}(g, x) - g(x)| = |S_{n,a,b}^\prime\left(\int_x^t (t - u)g''(u)\,du, x\right)|
\leq |S_{n,a,b}^\prime\left(\int_x^t (t - u)g''(u)\,du, x\right)|
+ \int_x^t \left|\tilde{S}_{n,a,b}(t, x) - u\right|g''(u)\,du.
\]

When \(x \in B_n,\) by (2.15), (3.10), (3.2), and (2.2), we have
\[
|S_{n,a,b}(g, x) - g(x)| \leq C\|\psi^2 g\|\|\tilde{S}_{n,a,b}\left(\frac{(t - x)^2}{\psi^2(x)}, x\right) + \psi^2(x)\|\psi^2 g''\|\left(\tilde{S}_{n,a,b}(t, x) - x\right)^2
\leq Cn^{-1}\delta^{2-\beta}_n(x)\|\psi^2 g''\|
\leq C\omega^2_{\psi^2}(f, n^{-1/2}\delta_n^{1-\lambda}(x)), \tag{3.12}
\]

where in the last inequality, (3.7) is applied.
When $x \in B_n$, by (2.19), (3.10), (3.2), and (2.2), we have

$$
\left| S_{n,\alpha,\beta}(g, x) - g(x) \right| \leq C \left\| \delta_n^{2\alpha} g'' \right\| \left( \left( t - x \right)^2 \delta_n^{2\alpha}(x) + \delta_n^{-2\alpha}(x) \right) \left\| \delta_n^{2\alpha}(t-x)^2 \right\| \left( \left( S_{n,\alpha,\beta}(t,x) - x \right)^2 \right)
\leq C n^{-1} \delta_n^{2\alpha}(x) \left( \left\| \varphi^{2\alpha} g'' \right\| + \frac{1}{n^\alpha} \left\| g'' \right\| \right)
\leq C n^{-1} \delta_n^{2\alpha}(x) \left\| \varphi^{2\alpha} g'' \right\| + C \left( n^{-1/2} \delta_n^{1/2}(x) \right)^{4/(2-\lambda)} \left\| g'' \right\|
\leq C n^{-2\alpha} \left( f_n^{-1/2} \delta_n^{1/2}(x) \right),
$$

where in the last inequality, we used (3.7) and (3.8).

We complete the proof of Theorem 1 by combining (3.1), (3.5), (3.9), (3.12), and (3.13).

3.2 Proof of Theorem 2

With Lemma 6-Lemma 9, the proof of Theorem 2 can be found exactly in the same way as that of [9]. We omit the details here.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
The authors contributed equally to this work. All authors read and approved the final manuscript

Received: 19 August 2016 Accepted: 27 December 2016 Published online: 27 January 2017

References
1. Gadjiev, AD, Ghorbanalizae, AM: Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables. Appl. Math. Comput. 216, 890-901 (2010)
2. Stancu, DD: Approximation of functions by a new class of linear polynomial operators. Rev. Roum. Math. Pures Appl. 13, 1173-1194 (1968)
3. Wang, ML, Yu, DS, Zhou, P: On the approximation by operators of Bernstein-Stancu types. Appl. Math. Comput. 246, 79-87 (2014)
4. Içöz, G: A Kantorovich variant of a new type Bernstein-Stancu polynomials. Appl. Math. Comput. 218, 8552-8560 (2012)
5. Taşdelen, F, Başcanbaz-Tunca, G, Erençin, A: On a new type Bernstein-Stancu operators. Fasc. Math. 48, 119-128 (2012)
6. Jung, HS, Deo, N, Dhamija, M: Pointwise approximation by Bernstein type operators in mobile interval. Appl. Math. Comput. 244, 683-694 (2014)
7. Acar, T, Aral, A, Gupta, V: On approximation properties of a new type Bernstein-Durrmeyer operators. Math. Slovaca 65, 1107-1122 (2015)
8. Ditzian, Z, Totik, V: Moduli of Smoothness. Springer, New York (1987)
9. Guo, S, Liu, L, Liu, X: The pointwise estimate for modified Bernstein operators. Studia Sci. Math. Hung. 37(1), 69-81 (2001)