Nonrelativistic Limit of the Scalar Chern-Simons Theory and the Aharonov-Bohm Scattering

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Abstract

We study the nonrelativistic limit of the quantum theory of a Chern-Simons field minimally coupled to a scalar field with quartic self-interaction. The renormalization of the relativistic model, in the Coulomb gauge, is discussed. We employ a procedure to calculate scattering amplitudes for low momenta that generates their $|p|/m$ expansion and separates the contributions coming from high and low energy intermediary states. The two body scattering amplitude is calculated up to order $p^2/m^2$. It is shown that the existence of a critical value of the self-interaction parameter for which the 2-particle scattering amplitude reduces to the Aharonov-Bohm one is a strictly nonrelativistic feature. The subdominant terms correspond to relativistic corrections to the Aharonov-Bohm scattering. A nonrelativistic reduction scheme and an effective nonrelativistic Lagrangian to account for the relativistic corrections are proposed.

I. INTRODUCTION

It is generally expected that nonrelativistic quantum theories, which provide very good descriptions of many physical phenomena, might be obtained from corresponding relativistic ones as appropriate limits for low momenta. In relativistic quantum mechanics, the reduction to nonrelativistic wave equations and Hamiltonians is based on the use of canonical transformations of Foldy-Wouthuysen type $^1$. For a relativistic field theory, at classical level, such a procedure can be applied to the bilinear part of the Lagrangian but the treatment of interactions is not straightforward. Another way to arrive at Schrödinger type of Lagrangian and equations of motion is to redefine the fields, by separating the factor $\exp(-imc^2t/\hbar)$, and then take the large $c$ (or large-mass) limit which eliminates rapidly oscillating terms $^2$. The nonrelativistic Chern-Simons (CS) Lagrangian, which is relevant for the physics of anyons $^3$, can be obtained from the relativistic one by performing such a limit $^4$. Due to quantum oscillations and renormalization problems, however, the nonrelativistic limit of a quantum field theory is much more subtle. In this respect, inspired in renormalization group arguments, there has been proposals of construction of effective Lagrangians by...
amending the nonrelativistic theories with other interaction terms, representing the effect of the integration of the relativistic degrees of freedom [5].

Recently [6], we have introduced a scheme of nonrelativistic approximation of a quantum field theory which uses an intermediate cutoff device and allows a systematic expansion of the scattering amplitudes in powers of $|p|/m$, permitting the identification, in the Hilbert space of intermediate states, of the origin of each contribution to the S-matrix perturbative series. The procedure was applied to $\lambda \phi^4$ theory in 2+1 dimensions, where there exist exact results in one loop order, so that it could be explicitly verified. This model has a well defined nonrelativistic counterpart which presents an interesting scale anomaly [7].

Here, we extend the discussion of nonrelativistic limit to a more involving and physically appealing gauge theory by considering the Chern-Simons field minimally coupled to a scalar field with quartic self-interaction [8]. The coupling of a matter field with a gauge field governed by a Chern-Simons action [9] has also been studied as a limiting case of the topologically massive gauge theory [10] in a covariant gauge [11], and more recently, the quantization of the fermionic model, in the Coulomb gauge, has been constructed [12]. The corresponding nonrelativistic model [13], including the quartic self-interaction [4,14], has been suggested as a field-theoretical formulation of the Aharonov-Bohm (AB) scattering [15] and as an effective theory for the fractional quantum Hall effect [16].

In the nonrelativistic theory, there exist critical values of the renormalized self-coupling parameter for which the one loop scattering vanishes and scale invariance is maintained. By choosing the positive coupling, corresponding to a repulsive contact interaction, the tree level reduces to the AB scattering [14]. The existence of this scale invariance at the critical self-interaction has been explicitly verified up to three loops using differential regularization [17] and was recently proven to hold in all orders of perturbation theory [18]. This has also been obtained for the nonrelativistic limit (in leading order) of the 1-loop particle-particle scattering of scalar self-interacting particles minimally coupled to a CS gauge field [19]. This criticality, however, ceases to exist in the relativistic case [20].

In this article we discuss the nonrelativistic approximation of the scattering amplitudes in the scalar Chern-Simons theory by using the above mentioned cutoff procedure. After computing next to leading nonrelativistic contributions to the two body scattering, we are able to construct a Galilean effective Lagrangian which extends the Aharonov-Bohm theory. We organize the article as follows. In Sec. II, we present the relativistic model and discuss its renormalization at one loop level. We then, in Sec. III, introduce the intermediate cutoff procedure we use to obtain the $|p|/m$ expansion of the quantum amplitudes. The calculation of the 1-loop particle-particle scattering amplitude up to order $p^2/m^2$ is presented in Sec. IV and in the Appendix. It is seen that the leading term of the $|p|/m$ expansion possesses the same critical self-coupling as in the nonrelativistic case. However, the subdominant parts do not vanish at the critical self-interaction values and so the aforesaid criticality is strictly nonrelativistic. The implications of this fact for the AB scattering are discussed in Sec. V where we also introduce a nonrelativistic reduction scheme for the scattering amplitudes. We consider the possibility of generating the relativistic corrections by adding nonrenormalizable interactions to the nonrelativistic Lagrangian and suggest an effective Lagrangian that accounts for the results up to order $p^2/m^2$. 


II. THE RELATIVISTIC MODEL

We consider a charged self-interacting scalar field in $2 + 1$ dimensions minimally coupled to a Chern-Simons gauge field described by the Lagrangian density

$$
L = (D_\mu \phi)^*(D^\mu \phi) - m^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2 + \frac{\Theta}{2} \epsilon_{\sigma\mu\nu} A^\sigma \partial^\mu A^\nu,
$$

(2.1)

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, $\epsilon_{\sigma\mu\nu}$ is the fully antisymmetric tensor normalized to $\epsilon_{012} = +1$, the Minkowski metric signature is $(1, -1, -1)$ and the units are such that $\hbar = c = 1$.

As it stands, the $A_\mu$ is not a dynamical variable, its equation of motion represents a constraint and so there does not exist real gauge particles [9]. However, one has to specify a gauge to work with and, by adding a gauge fixing term to the Lagrangian (2.1), one can treat the gauge field dynamically in the internal lines. For convenience in discussing the nonrelativistic limit, we choose the Coulomb gauge where

$$
L_{GF} = -\frac{\xi}{2} (\partial_i A^i)^2.
$$

(2.2)

The free propagator of the gauge field is then given by

$$
D_{\mu\nu}(k) = -\frac{i}{\xi (k^2)^2} - \frac{1}{\Theta} \epsilon_{\mu\rho\nu} \frac{\vec{k}^\rho}{k^2},
$$

(2.3)

where $\vec{k} = (0, k^1, k^2)$, which reduces, in the Landau limit ($\xi \to \infty$), to a totally antisymmetric form in the Minkowski indices with the only nonvanishing components given by

$$
D_{0i}(k) = -D_{i0}(k) = \frac{1}{\Theta} \frac{\epsilon_{ij} k^j}{k^2},
$$

(2.4)

where $\epsilon_{ij} = \epsilon_{0ij}$. This gauge fixing coincides with the choice of ref. [14] where the corresponding nonrelativistic model is discussed. Notice also that, by requiring $\xi$ in (2.2) to be dimensionless, we fix the mass dimension of $A_\mu$ as $1/2$ and so $\Theta$ as having dimension of mass. The other dimensional parameters appearing in (2.1) are $e$ and $\lambda$ which dimensions are $1/2$ and 1, respectively. In $2 + 1$ dimensions, the bosonic matter field has mass dimension equal to $1/2$ and its free propagator is given by $\Delta(p) = i (p^2 - m^2 + i\varepsilon)^{-1}$. Besides the scalar field quadrilinear self-interaction vertex, minimal gauge coupling with $A_\mu$ generates two kinds of vertices. The corresponding Feynman rules are presented in Fig. 1, where we use a wavy line to represent the gauge field propagator.

The form of the gauge field propagator (2.4) also interferes in the power counting. For a generic graph, the degree of superficial divergence is given by $d(G) = 3 - N_\phi/2 - N_A - V_{\phi^4}$, where $N_\phi$ and $N_A$ are the numbers of external particle and gauge field lines and $V_{\phi^4}$ is the number of self-interaction vertices of the diagram. As we will see, the 1-loop order self-energy corrections of both matter and gauge fields are linearly divergent whereas the vertex correction is finite, although it is superficially divergent. The particle four-point function has also divergent contributions. In fact, the 1-loop graphs containing the seagull vertex are linearly divergent. Therefore, the inclusion of the bare $\phi^4$ vertex in the Lagrangian (2.1) is necessary to render the theory renormalizable.
Let us initially examine the 1-loop particle self-energy, the graphs of Fig. 2. Owning to the antisymmetry of the gauge field propagator \((2.4)\), the contributions that represent emission and absorption of a virtual gauge particle are all identically null, a fact that does not happen in the fermionic case \([9,12]\). The tadpole with a gauge propagator neck vanishes by charge conjugation so that the 1-loop particle self-energy reduces to the tadpole graph and a simple ultraviolet cutoff \(\Lambda_0\) introduced in the \(|k|\) integration gives

\[
\hat{\Sigma}^{(1)}(m; \Lambda_0) = -\frac{i\lambda}{2} \int_0^{\Lambda_0} \frac{d^3k}{(2\pi)^3} \frac{i}{k^2 - m^2 + ie} = \frac{i\lambda}{8\pi} m - \frac{i\lambda}{8\pi} \Lambda_0. \tag{2.5}
\]

Had we used dimensional regularization, which acts as a renormalization at one loop level in odd dimensions, we would have obtained only the first term of the above expression. We see that \(\hat{\Sigma}^{(1)}\) does not depend on the external momentum and thus there is no wave function renormalization at one loop level. The mass counterterm can be written as \(-\delta m^2 \phi^* \phi\) with \(\delta m^2 = -i\hat{\Sigma}^{(1)}\).

The one loop self-energy of the gauge field, the vacuum polarization, has two linearly divergent parcels represented by the diagrams shown in Fig. 3. It does not involve the gauge field propagator and so it is independent of the gauge fixing chosen. Dimensional renormalization generates, naturally, a gauge independent result which is given by

\[
\Pi^{(1)}_{\mu\nu}(q) = e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(2k + q)_\mu (2k + q)_\nu}{(k^2 - m^2 + ie)[(k + q)^2 - m^2 + ie]} - 2e^2 \int \frac{d^3k}{(2\pi)^3} \frac{g_{\mu\nu}}{k^2 - m^2 + ie}
\]

\[
= -\frac{i e^2}{2\pi} m \left(1 - \int_0^1 dx \sqrt{1 - q^2 x(1 - x)/m^2}\right) \frac{1}{q^2} \left[q^2 g_{\mu\nu} - q_\mu q_\nu\right]. \tag{2.6}
\]

In the low momentum regime, \(|q^2| \ll m^2\), one has

\[
\Pi^{(1)}_{\mu\nu}(q) \simeq -\frac{i e^2}{24\pi m} \left(1 + \frac{q^2}{20m^2}\right) \left[q^2 g_{\mu\nu} - q_\mu q_\nu\right]. \tag{2.7}
\]

If a gauge non invariant cutoff regularization is employed, an additional linear divergent contribution \((\frac{ie^2}{6\pi} \Lambda_0 g_{\mu\nu})\) arises and the transversality of the vacuum polarization is lost. It should be noticed that, the transverse nature of the finite part \((2.6)\) requires the inclusion of the tadpole diagram. For this reason, we found more convenient to consider the ordinary product of the operators in the interaction Lagrangian instead of the Wick ordering. One should also notice that, only a Maxwell term is generated by radiative corrections to the gauge field propagator, in contrast with the fermionic case where a Chern-Simons term is produced too \([10,21]\).

The 1-loop correction to the vectorial gauge coupling vertex, shown in Fig. 4, is finite. The other possible 1-loop contribution, which has one trilinear and one self-interaction vertex inserted in a particle loop, is null by charge conjugation. The sum of the two first parcels, the contribution involving the seagull vertex,

\[
\Gamma^{(1)\mu}_{3(8)}(p) = \frac{2ie^3}{\Theta} \int \frac{d^3k}{(2\pi)^3} \frac{(2p - k)^{\sigma} \epsilon_{\sigma \nu \rho}}{k^2 [(p - k)^2 - m^2 + ie]} - [p \leftrightarrow p'], \tag{2.8}
\]
can be exactly calculated and, for external particles legs in the mass shell, is given by
\( \Gamma^{(1)0}_{3(S)} = 0 \) and
\[
\Gamma^{(1)l}_{3(S)} = -\frac{e^3}{2\pi\Theta} \epsilon_{ij} q^l \left[ \frac{p^i \sqrt{m^2 + p^2}}{\sqrt{m^2 + m^2 + p^2}} - \frac{p'^i \sqrt{m^2 + p'^2}}{\sqrt{m^2 + m^2 + p'^2}} \right] \tag{2.9}
\]
for \( l = 1, 2 \). Although this contribution has not been considered in the calculation of the anomalous magnetic moment using the covariant Landau gauge in ref. [11], it is relevant in the present case. The triangle graph contribution, again for external momenta in the mass shell, can be expressed in terms of Feynman integrals as
\[
\Gamma^{(1)\mu}_{3(T)} = i\frac{e^3}{\Theta} \int \frac{d^3k}{(2\pi)^3} \frac{(2p - k)^\sigma}{k^2} \epsilon_{\sigma\nu\rho} \sqrt{k^\rho} (2p' - k)\nu (p + p' - 2k)^\mu \left[ \frac{A^\mu}{(Q_0^2 - yQ^2)^{3/2}} + \frac{B^\mu}{(Q_0^2 - yQ^2)^{1/2}} \right] \tag{2.10}
\]
with the numerators given by
\[
A^0 = \epsilon_{ij} p^i q^j Q^0 \left( P^0 - 2Q^0 \right) , \tag{2.11}
\]
\[
B^0 = -2\epsilon_{ij} p^i q^j , \tag{2.12}
\]
\[
A^l = \epsilon_{ij} p^i q^j Q^0 \left( P^l - 2yQ^l \right) , \tag{2.13}
\]
\[
B^l = \epsilon_{ij} \left( 4V^i + 2Q^0 q^i \right) q^l , \tag{2.14}
\]
where \( q^\mu = p^\mu - p'^\mu, P^\mu = p^\mu + p'^\mu, V^i = q^0 p^i - p^0 q^i \) and \( Q^\mu(x) = p^\mu - q^\mu (1 - x) \). The \( y \) integration in (2.10) can be easily done but the remaining \( x \) integration is very complicated. We do not push any harder to get an exact result, but rather, we shall look directly for the small momenta behavior. For sake of simplicity, we restrict ourselves to the situation with \( p^0 = p'^0 \) which implies \( p^2 = p'^2 \) for particles in the mass shell. Actually, this is the situation we shall find when dealing with the vertex insertions into the tree level particle-particle scattering in the center of mass frame (Sec. 4). In this case, \( Q_0^2 = m^2 + p^2 \) and expanding (2.9) and (2.10) for \( |p|/m \) small one obtains the total trilinear vertex correction, up to order \( p^2/m^2 \), as
\[
\Gamma^{(1)0}_{3}(p, p - q) \big|_{q^0 = 0} \simeq -\frac{e^3}{4\pi\Theta} \left[ \frac{\epsilon_{ij} p^i q^j}{m} \right] \tag{2.15}
\]
and
\[
\Gamma^{(1)l}_{3}(p, p - q) \big|_{q^0 = 0} \simeq -\frac{e^3}{4\pi\Theta} \left[ \frac{\epsilon_{ij} p^i q^j}{m} \right] \left( 2 + \frac{1}{12}(5 + \cos \theta) \frac{p^2}{m^2} \right) + \frac{e^3}{4\pi\Theta} \left[ \frac{\epsilon_{ij} p^i q^j}{m} \right] \left( \frac{p + p'}{4m} \right)^l , \tag{2.16}
\]
where \( \theta \) is the angle between the vectors \( p \) and \( p' \).

The one loop correction to the seagull vertex, presented in Fig. 5, is of fourth order in the charge \( e \) and so it does not participate in the particle-particle scattering at one loop level.
No essential new contribution is expected from this correction since, by gauge invariance, it must be in accordance with the trilinear vertex one. In fact, by noticing that only the second parcel in Fig. 2 has the external momentum flowing through the diagram, one can readily verify the Ward identities at one loop level

\[-\frac{1}{e} \Gamma^{(1)}_{3\mu}(p, p) = \frac{\partial \hat{\Sigma}^{(1)}_{1}}{\partial p^\mu}, \tag{2.17}\]

\[-\frac{1}{e^2} \Gamma^{(1)}_{4\mu\nu}(p, p; 0) = \frac{\partial^2 \hat{\Sigma}^{(1)}_{1}}{\partial p^\mu \partial p^\nu}. \tag{2.18}\]

Owing to the particular gauge we fixed, the above quantities are identically null. Notice also that, there exist corrections to the quadrilinear gauge vertex with one self-interaction vertex and one seagull or two trilinear vertices inserted into a particle loop which do not vanish separately due to charge conjugation. However, because of the Ward relation

\[e \frac{\partial}{\partial q^\nu} \Gamma^{(1)}_{3\mu} = \Gamma^{(1)}_{4\mu\nu} \big|_{q'=q}, \tag{2.19}\]

their sum does not contribute to the vertex correction.

There remains to discuss the correction to the self-interaction vertex which represents the 2-particle scattering in one loop order. This four-point function, which will be calculated in Sec. IV, is linearly divergent and therefore a counterterm of the form $C(\Lambda_0) (\phi^* \phi)^2$ has to be introduced in the Lagrangian. Counterterms linear in $p^\mu$, the first order Taylor subtraction in the BPHZ scheme, are absent due to the rotational invariance, a fact that can be explicitly verified. The particle-particle scattering, for low momenta, will be calculated using the approximation scheme we shall now describe.

### III. NONRELATIVISTIC APPROXIMATION

The nonrelativistic approximation scheme we shall employ to calculate particle-particle amplitudes in one loop order consists in the following steps [3]. First of all, denoting by $k$ the loop momentum, we integrate over $k^0$ without making any restriction in order to guarantee locality in time. This integration is greatly facilitated in the gauge we are working since $k^0$ appears in the integrand of the amplitudes only via the particle propagator and the trilinear vertex factors. The advantage in using Coulomb gauge to discuss the nonrelativistic limit is that it gives naturally a static interaction between the particles.

The remaining integration over the Euclidean $k$ plane is then separated into two parcels through the introduction of an intermediate cutoff $\Lambda$ in the $|k|$ integration. This auxiliary cutoff $\Lambda$ is chosen such that it satisfies the conditions

(i) $|p| \ll \Lambda \ll m$ and (ii) $\left( \frac{|p|}{\Lambda} \right)^2 \approx \left( \frac{\Lambda}{m} \right)^2 \approx \frac{|p|}{m} \approx \eta$, \hspace{1cm} (3.1)

where $|p|$ stands for the external nonrelativistic momenta, $m$ is the renormalized particle mass and $\eta$ is established as the small expansion parameter. These conditions can be easily
met in nonrelativistic condensed matter systems \[22\]. Clearly, the cutoff \( \Lambda \) splits the space of the intermediate states into two parts. The circle of radius \( \Lambda \) and center at the origin of the \( k \)-plane will be referred as the low (L) energy sector whereas the \( |k| > \Lambda \) region is the high (H) energy one.

The approximation procedure we work with rely heavily on the intrinsic nature of each of these two sectors. In the low energy one, where all the spatial momenta involved are small \((|p|/m, |k|/m \ll 1)\), we perform a \( 1/m \) expansion of the integrand. This means, for example, that the free particle energy dispersion relation can be expanded, in the L-sector, as

\[
w_k = \sqrt{m^2 + k^2} = m + \frac{k^2}{2m} - \frac{(k^2)^2}{8m^3} + \ldots \tag{3.2}
\]

The expanded integrand is then integrated, term by term, in the region \( 0 < |k| < \Lambda \) and, naturally, a \( \Lambda \)-dependent result is obtained. A further expansion in \( \Lambda/m \) may be necessary to get the \( \eta \) expansion of the L-contribution to the amplitude up to the desired order.

In the H-region, \( |k| \gg |p| \) and the integrand can be expanded in a Taylor series around \( |p| = 0 \). This permits analytical calculations of the integrals in every order in \( \eta \). Certainly, this expansion does not suppress ultraviolet divergences so, in that case, a regularization procedure is assumed. The result is also \( \Lambda \)-dependent and one must expand in \( \Lambda/m \) to obtain the \( \eta \) expansion of the H-contribution. However, as we should expect for sake of consistence of the nonrelativistic approximation, the \( \Lambda \)-dependent parcels of the L and the H contributions of each diagram cancel identically. Apparently the two sectors L and H are treated differently but, in fact, the same \( 1/m \) expansion is made and the exact cancellation of the \( \Lambda \)-dependent terms reflects just the additivity property of the integration. The complete amplitude obtained by adding the contributions of the L and H sectors is the correct \( |p|/m \) expansion up to the order we have worked.

It should be noticed that by choosing a distinct routing of the external momenta through the diagram (or by using the Feynman parametrization to simplify the integrand before making the \( k^0 \) integration) one gets different L and H contributions for the graph. The changes, however, occur only in the coefficients of the \( (|p|/\Lambda)^n \) which are not relevant for the reduction process we shall discuss later.

We can apply this scheme to the calculation of the one loop self-energy, vacuum polarization and vertex corrections and identify the origin of each of them. For the self-energy, the separation of the L- and H-contributions gives, up to the order \( \eta^2 \approx p^2/m^2 \),

\[
\hat{\Sigma}^{(1)}_L (m; \Lambda) \simeq \frac{i\lambda}{16\pi} m \left[ \frac{\Lambda^2}{m^2} + \frac{1}{4m^4} \right] \tag{3.3}
\]

and

\[
\hat{\Sigma}^{(1)}_H (m; \Lambda, \Lambda_0) \simeq \frac{i\lambda}{16\pi} m \left[ \frac{\Lambda^2}{m^2} - \frac{1}{4m^4} \right] + \frac{i\lambda}{8\pi} m - \frac{i\lambda}{8\pi} \Lambda_0. \tag{3.4}
\]

We see that the low energy intermediate states contribution is polynomial in \( \Lambda/m \) and that both finite and divergent (as \( \Lambda_0 \to \infty \)) parts of \( \Sigma_3 \) come from the high energy sector. This also happens with the vacuum polarization, for which one has (in the case of \( |q^2| \ll m^2 \))
\[ \Pi^{(1)\mu\nu}(q; \Lambda) \simeq \frac{ie^2}{6\pi} mg^{\mu\nu} \left[ \frac{\Lambda^2}{m^2} - \frac{1}{2} \frac{\Lambda^4}{m^4} - \frac{1}{4} \frac{q^2}{m^2} \left( \frac{1}{3} \frac{\Lambda^2}{m^2} - \frac{1}{4} \frac{\Lambda^4}{m^4} \right) \right] \\
+ \frac{ie^2}{16\pi} \frac{q^\mu q^\nu}{m} \left[ \frac{1}{3} \frac{\Lambda^2}{m^2} - \frac{1}{4} \frac{\Lambda^4}{m^4} \right] \tag{3.5} \]

and
\[ \Pi^{(1)\mu\nu}_H(q; \Lambda, \Lambda_0) \simeq -\frac{ie^2}{24\pi m} \left( 1 + \frac{q^2}{20m^2} \right) \left[ q^2 g^{\mu\nu} - q^\mu q^\nu \right] \\
+ \frac{ie^2}{6\pi} \Lambda_0 g^{\mu\nu} - \Pi^{(1)\mu\nu}_L(q; \Lambda) . \tag{3.6} \]

The L-contribution to the trilinear vertex correction is given by
\[ \Gamma^{(1)0}_{3L} \simeq 0 \quad \text{and} \quad \Gamma^{(1)1}_{3L} \simeq -\frac{e^3}{4\pi \Theta} \left[ \epsilon_{ij} q^i q^j \right] \left\{ \frac{\Lambda^2}{m^2} - 3 \frac{\Lambda^4}{4 m^4} \right\} , \tag{3.7} \]

and again the whole correction, equations (2.15) and (2.16), comes from the high energy intermediate states. In the above equations, and from now on, the symbol \( \simeq \) denotes that the expression which follows holds up to the order \( \eta^2 \).

The fact that the low energy intermediate states contributions to the basic radiative corrections are small and suppressed by part of the high ones reflects the nonrelativistic nature of the L sector. The corresponding nonrelativistic field theory, that is the theory with a Galilean invariant, Schrödinger, kinematics and the same interactions, has a distinct intrinsic nature. Particle propagation is only forward in time and interactions are instantaneous. This means that there are no dynamical corrections to the mass and to the charge of the particles which are given phenomenological parameters of the theory.

There is no objection, in principle, to extend the procedure described above to obtain the nonrelativistic approximation, that is the \( \sqrt{p} \)/\( m \) expansion, of quantum amplitudes in any order of perturbation theory. One naturally needs to introduce an intermediate cutoff for each independent loop integration and, certainly, makes use of the Feynman parametrization to symmetrize the integrand in order to perform the \( \sqrt{k^0} \) integration. An example can be found in ref. [6], where the two loop self-energy of the \( \phi^4 \) theory was calculated.

\textbf{IV. PARTICLE-PARTICLE SCATTERING}

To simplify calculations, we choose to work in the center of mass (CM) frame with external particles on the mass shell, that is \( p_1 = -p_2 = p \), \( p_1' = -p_2' = p' \) and \( p_1^0 = p_2^0 = p_1'^0 = p_2'^0 = w_p = \sqrt{m^2 + p^2} \). The tree level particle-particle amplitude has three contributions, presented in Fig. 6, and is given by

\[ A^{(0)} = -\lambda + \left\{ i e^2 (p_1 + p_1')^\mu D_{\mu\nu}(p_1 - p_1') (p_2 + p_2')_{\nu} \right\}_{\text{CM}} + \left[ p_1' \leftrightarrow p_2' \right]_{\text{CM}} \]

\[ = -\lambda - i \frac{8e^2}{\Theta} \sqrt{m^2 + p^2} \cot \theta \simeq -\lambda - i \frac{8e^2}{\Theta} m \left( 1 + \frac{p^2}{2m^2} \right) \cot \theta , \tag{4.1} \]
where \( \theta \) is the scattering angle and \( m \) is the renormalized mass of the bosonic particle. By definiteness, we take the amplitude as being \((-i)\) times the 1PI four point function. This choice is only to facilitate the comparison with the nonrelativistic case discussed in ref. [14].

We shall now calculate the 1-loop order scattering amplitude, for low external momenta and up to order \( \eta^2 \approx p^2/m^2 \). Consider firstly the amplitude that comes from the minimally subtracted vacuum polarization and gauge vertex corrections inserted into the tree level diagrams. Using equation (2.7) for the polarization tensor with \( q^0 = 0 \), one immediately gets the contribution to the amplitude due to the one loop correction to the gauge field propagator as

\[
A^{(p)} \simeq -\frac{2e^4}{\pi \Theta^2} m \left\{ \frac{1}{6} + \frac{7}{30} \frac{p^2}{m^2} \right\}.
\]  

(4.2)

Similarly, the trilinear vertex correction \( A^{(1)\mu} \), given by (2.15) and (2.16), leads to the following contribution to the one loop amplitude

\[
A^{(v)} \simeq -\frac{2e^4}{\pi \Theta^2} m \left\{ 2 + \frac{5}{3} \frac{p^2}{m^2} \right\}.
\]  

(4.3)

Diagrams, like those shown in Fig. 7, that admixes particle self-interaction and gauge field exchange, do not contribute. The first vanishes by charge conjugation, the second is null due to the antisymmetric form of the gauge field propagator whereas the chalice diagram gives

\[
A^{(ch)} = -\frac{e^2 \lambda}{8\pi \Theta} \sin \theta \frac{p^2}{m^2} + (\theta \rightarrow \pi + \theta) = 0.
\]  

(4.4)

This contribution, and also all the terms proportional to \( \sin \theta \) (or \( \cos \theta \)) appearing in the other graphs, vanishes due to due to the required symmetrization in the outgoing particles. The non admixture of the gauge coupling vertices and the self-interaction one (at least in one loop level) means, as we are going to see, that \( \lambda \phi^4 \) interaction comes about only to renormalize the particle-particle scattering due to CS gauge field exchange.

The most important one loop particle-particle scattering comes from the diagrams shown in Fig. 8 and, to calculate these contributions to the 2-particle amplitude, we shall use the approximation scheme described in the last section. The group \( (a) \) is the finite self-interaction scattering which was discussed before [3]. Adding separately the \( L \)- and the \( H \)-contributions of each graph, one obtains

\[
A_L^{(a)} \simeq \frac{\lambda^2}{32\pi m} \left\{ \left( 1 - \frac{p^2}{2m^2} \right) \left[ \ln \left( \frac{\Lambda^2}{p^2} \right) + i\pi \right] - \frac{p^2}{2\Lambda^2} - \frac{p^4}{2\Lambda^4} - \frac{3\Lambda^2}{2m^2} - \frac{21\Lambda^4}{16m^4} \right\}.
\]  

(4.5)

\[
A_H^{(a)} \simeq \frac{\lambda^2}{32\pi m} \left\{ - \left( 1 - \frac{p^2}{2m^2} \right) \ln \left( \frac{4m^2}{\Lambda^2} \right) + \frac{p^2}{\Lambda^2} + \frac{p^4}{2\Lambda^4} - \frac{3\Lambda^2}{2m^2} - \frac{21\Lambda^4}{16m^4} + 4 - \frac{p^2}{6m^2} \right\}
\]  

(4.6)

and, thus, \( A^{(a)} = A_L^{(a)} + A_H^{(a)} \) is given by

\[
A^{(a)} \simeq \frac{\lambda^2}{32\pi m} \left\{ \left( 1 - \frac{p^2}{2m^2} \right) \left[ \ln \left( \frac{4m^2}{p^2} \right) + i\pi \right] + 4 - \frac{p^2}{6m^2} \right\}.
\]  

(4.7)
As we have stressed in ref. [6], the intermediary cutoff procedure generates the correct $|p|/m$ expansion of the amplitudes in the case of $\lambda \phi^4$ theory alone. In the present case, and certainly in many other theories, where analytical results are very difficult (if not impossible) to be achieved, our procedure emerges as an useful calculational tool to generate the $|p|/m$ expansion of the amplitudes, having the exact result in the $\phi^4$ case as a basis of confidence.

The box diagrams of Fig. 8(b) are also calculated separately. Let us first concentrate in the “right” box diagram corresponding to the direct exchange of two virtual gauge particles, the first parcel of Fig. 8(b). After performing the $k^0$ and the angular integrations, by using the Cauchy-Gousart theorem and splitting the $|k|$ integration into its low and high energy contributions one is led, by adding the ($p' \leftrightarrow -p'$) companion, to

$$A^{(b)\text{dir}}_L \simeq -\frac{2e^4}{\pi \Theta^2 m} \left\{ \left(1 + \frac{P^2}{2m^2}\right) \left[\ln(2|\sin \theta|) + i\pi\right] - \frac{P^2}{m^2} \right. \\
\left. - \frac{1}{2} \cos \theta \ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) \frac{P^2}{m^2} - (1 - 2\cos^2 \theta) \frac{P^4}{\Lambda^4} \right\}$$

(4.8)

and

$$A^{(b)\text{dir}}_H \simeq -\frac{2e^4}{\pi \Theta^2 m} \left\{ (1 - 2\cos^2 \theta) \frac{P^4}{\Lambda^4} \right\}.$$  (4.9)

Adding (4.8) and (4.9), we get

$$A^{(b)\text{dir}} \simeq -\frac{2e^4}{\pi \Theta^2 m} \left\{ \left(1 + \frac{P^2}{2m^2}\right) \left[\ln(2|\sin \theta|) + i\pi\right] - \frac{P^2}{m^2} \right. \\
\left. - \frac{1}{2} \cos \theta \ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) \frac{P^2}{m^2} \right\}. \quad (4.10)$$

We see that the contribution to the amplitude coming from the direct box diagram comes entirely from the low energy, nonrelativistic, sector. The main steps of these calculations (and the others to come) are outlined in the Appendix.

The contrary happens with the twisted box diagram, the second in Fig. 8(b). The $k^0$ integration can be easily done as a contour integration but the resulting $k$ integration has a rather non trivial angular part. The alternative possibility of introducing Feynman parameters allows the evaluation of the $k$ integration but the parametric integrals that remain are intractable. By approximating the integrands as described in the last section, one simplifies the angular and the radial integrations that appear obtaining, after adding its final particles exchanged partner,

$$A^{(b)\text{twist}}_L \simeq 0 \quad \text{and} \quad A^{(b)\text{twist}} \simeq A^{(b)\text{twist}}_H \simeq -\frac{2e^4}{\pi \Theta^2 m} \left\{ \frac{P^2}{2m^2} \right\}.$$  (4.11)

As in the case of the last two graphs of Fig. 8(a), diagrams that involve propagation backward in time possess a small, $\Lambda$ dependent, contribution coming from the L sector. For the twisted box diagram this contribution is of order $\eta^3$. The total box amplitude, $A^{(b)} = A^{(b)\text{dir}} + A^{(b)\text{twist}}$, is finite and, up to order $p^2/m^2$, is given by
\[ A^{(b)} \simeq -\frac{2e^4}{\pi \Theta^2} m \left\{ \left( 1 + \frac{p^2}{2m^2} \right) \left[ \ln(2|\sin \theta|) + i\pi \right] - \frac{p^2}{2m^2} \right. \\
- \frac{1}{2} \cos \theta \ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) \frac{p^2}{m^2} \right\}. \quad (4.12) \]

The third group, the seagull scattering \((c)\), has to be treated more carefully since it carries the divergence of the four point function. One can immediately see that the \(k^0\) integration of the first diagram, the gauge bubble, would diverge if made alone. However, the two triangle diagrams, which give identical contributions, have also divergent \(k^0\) integrations that exactly compensates the divergence of the first graph. Therefore, by taking all the diagrams of group \((c)\) together, there is no need to maintain any cutoff and the \(k^0\) integration is unrestricted. The remaining \(k\) integration has two parts. One is finite and has the same angular integration as that appearing in the direct box. The other one, with a distinct but feasible angular integration, is infinite so that an ultraviolet cutoff \(\Lambda_0\) has to be introduced in the \(|k|\) integration of the \(H\) sector. Up to order \(\eta^2\), the low energy intermediate states contribution for the seagull scattering is given by

\[ A^{(c)} L \simeq -\frac{2e^4}{\pi \Theta^2} \left\{ \left( 1 + \frac{p^2}{2m^2} \right) \left[ \ln \left( \frac{\Lambda^2}{p^2} \right) - \ln(2|\sin \theta|) \right] + \frac{p^2}{m^2} \right. \\
+ \frac{1}{2} \cos \theta \ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) \frac{p^2}{m^2} + \frac{1}{2} (1 - 2 \cos^2 \theta) \frac{p^4}{\Lambda^4} + \frac{\Lambda^4}{16m^4} \right\}. \quad (4.13) \]

The \(H\) sector contribution, cutoff regulated, is given by

\[ A^{(c)} H \simeq -\frac{2e^4}{\pi \Theta^2} m \left\{ - \left( 1 + \frac{p^2}{2m^2} \right) \ln \left( \frac{\Lambda^2}{4m^2} \right) - 1 \\
- \frac{1}{2} (1 - 2 \cos^2 \theta) \frac{p^4}{\Lambda^4} - \frac{\Lambda^4}{16m^4} + \left[ \frac{\Lambda_0}{m} \right] \right\} \quad (4.14) \]

and, thus, the total \((c)\) contribution to the amplitude is

\[ A^{(c)} \simeq -\frac{2e^4}{\pi \Theta^2} m \left\{ \left( 1 + \frac{p^2}{2m^2} \right) \left[ \ln \left( \frac{4m^2}{p^2} \right) - \ln(2|\sin \theta|) \right] \\
- 1 + \frac{1}{2} \cos \theta \ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) \frac{p^2}{m^2} + \frac{p^2}{m^2} \right\} - \frac{2e^4}{\pi \Theta^2} \Lambda_0. \quad (4.15) \]

The constant divergent term above can be suppressed by a counterterm of the form \(\frac{e^4}{2\pi \Theta^2} \Lambda_0 (\phi^4)^2\) introduced in the Lagrangian density. We can also imagine that the bare self-coupling \(\lambda\) carries a divergent part that just cancel the divergence of the four point function. In any case, we take the finite part of \((4.13)\) as the 1-loop renormalized \((c)\) contribution. This would be the result if we had used dimensional renormalization.

Before summing up all contributions to get the total one loop amplitude, let us separate the total CS exchange scattering by adding the \((b)\) and \((c)\) parts. One gets

\[ A^{(CS)} \simeq -\frac{2e^4}{\pi \Theta^2} m \left\{ \left( 1 + \frac{p^2}{2m^2} \right) \left[ \ln \left( \frac{4m^2}{p^2} \right) + i\pi \right] - 1 + \frac{p^2}{2m^2} \right\}, \quad (4.16) \]
and it is noticeable that the cancellation of the $\theta$ dependent terms of the box and the seagull amplitudes happens in both dominant and subdominant orders.

The total renormalized particle-particle scattering amplitude up to 1-loop, $A^0 + A^{(a)} + A^{(CS)} + A^{(p)} + A^{(v)}$, up to order $p^2/m^2$ is given by

\begin{equation}
A^{(1)} \simeq -\lambda - i \frac{8e^2}{\Theta} m \left( 1 + \frac{p^2}{2m^2} \right) \cot \theta \\
+ m \left( \frac{\lambda^2}{32\pi m^2} - \frac{2e^4}{\pi \Theta^2} \right) \left[ \ln \left( \frac{4m^2}{p^2} \right) + i\pi \right] \\
- m \left( \frac{\lambda^2}{64\pi m^2} + \frac{e^4}{\pi \Theta^2} \right) \frac{p^2}{m^2} \left[ \ln \left( \frac{4m^2}{p^2} \right) + i\pi \right] \\
+ m \left( \frac{\lambda^2}{8\pi m^2} - \frac{7e^4}{3\pi \Theta^2} \right) - m \left( \frac{\lambda^2}{192\pi m^2} + \frac{24e^4}{5\pi \Theta^2} \right) \frac{p^2}{m^2} .
\end{equation}

The leading term of this expansion coincides with the result of ref. [19]. An important feature of the above result is that the existence of critical values of the self-interaction parameter for which the one loop scattering vanishes is restricted to the leading order. In other words, there are critical values of $\lambda$, namely

\begin{equation}
\frac{\lambda^\pm}{4m^2} = \pm \frac{2e^2}{m|\Theta|} ,
\end{equation}

for which the leading term of $A^{(1)}$ vanishes but the subdominant parcels do not. By choosing $\lambda = \lambda^+$, the tree level amplitude becomes the Aharonov-Bohm scattering for identical particles [14]. The subdominant terms then represent relativistic corrections to the Aharonov-Bohm scattering [20]. It should be pointed out that the real constant term of (4.17) could be suppressed by a finite renormalization of the coupling constant $\lambda$.

V. NONRELATIVISTIC REDUCTION AND THE AB SCATTERING

The perturbative treatment of the AB scattering has long been known to be a very delicate problem due to the singular nature of the potential [25]. In the first quantized viewpoint [24], perturbative renormalization requires the introduction of a delta function potential which is equivalent to a $\phi^4$ self-interaction in the field theoretical approach [14,25]. In the latter case [14], the nonrelativistic (NR) scattering amplitude, that is the 2-particle CM scattering amplitude calculated with the Galilean invariant Lagrangian density

\begin{equation}
L_{NR} = \psi^* \left( iD_t + \frac{D^2}{2m} \right) \psi - \frac{v_0}{4} : (\psi^* \psi)^2 : + \frac{\Theta}{2} \partial_t A \times A - \Theta A_0 \nabla \times A
\end{equation}

and with the same gauge fixing $\xi \to \infty$ $(\xi \to \infty)$, is given, at one loop level, by

\begin{equation}
A_{NR}^{(1)} = \frac{m}{8\pi} \left( v_0^2 - \frac{4e^4}{m^2 \Theta^2} \right) \left[ \ln \left( \frac{\Lambda_{NR}^2}{p^2} \right) + i\pi \right] ,
\end{equation}

\begin{equation}
V. \text{NONRELATIVISTIC REDUCTION AND THE AB SCATTERING}
\end{equation}
where $\Lambda_{NR}$ is a nonrelativistic ultraviolet cutoff. The graphs that enter in this calculation are the fish diagram (the first of Fig. 8(a)), the direct box and the two triangle diagrams with, in all of them, the particle lines representing the nonrelativistic propagator

$$\Delta_{NR}(\omega, k) = \frac{i}{\omega - k^2/2m + i\varepsilon}.$$  \hfill (5.3)

The renormalization is implemented by redefining the nonrelativistic self-coupling constant, $v_0 = v + \delta v$, so that the total renormalized nonrelativistic amplitude, obtained by adding the tree level scattering, is given, up to order $e^4$, by

$$A_{NR} = -v - i\frac{2e^2}{m\Theta} \cot \theta + \frac{m}{8\pi} \left( v^2 - \frac{4e^4}{m^2\Theta^2} \right) \left[ \ln \left( \frac{\mu^2}{p^2} \right) + i\pi \right],$$ \hfill (5.4)

where $\mu$ is an arbitrary mass scale, introduced by the renormalization, that breaks the scale invariance of the amplitude \cite{14}.

Prior to examine the similarities between the results of the relativistic and the nonrelativistic models, the normalization of states has to be properly adjusted. In the relativistic case one takes $\langle p'|p \rangle = 2w_p \delta(p' - p)$ while the usual normalization in a nonrelativistic theory does not have the $2w_p$ factor. Besides that, the cross-sections involve the relative velocity and thus, for the purpose of comparison, the CM amplitudes calculated in the last section must be multiplied by

$$f \left( \frac{1}{\sqrt{2w_p}} \right)^4 = \frac{1}{4m^2} \left[ 1 - \frac{3p^2}{4m^2} + \ldots \right],$$ \hfill (5.5)

where $f^2 = w_p/m$ is the ratio between the nonrelativistic and relativistic velocities.

Let us now compare the renormalized nonrelativistic amplitude (5.4) with the leading term of the $|p|/m$ expansion of the minimally subtracted relativistic scattering (4.17) given by

$$A_{lead} = -\frac{\lambda}{4m^2} - i\frac{2e^2}{m\Theta} \cot \theta + \frac{m}{8\pi} \left( \frac{\lambda^2}{16m^4} - \frac{4e^4}{m^2\Theta^2} \right) \left[ \ln \left( \frac{4m^2}{p^2} \right) + i\pi \right],$$ \hfill (5.6)

where the calligraphic $A$ means that the amplitude already incorporates the nonrelativistic normalization factor (5.3).

Confronting the tree levels, one sees that the self-interaction parameters are related by

$$v = \frac{\lambda}{4m^2}. \hfill (5.7)$$

It is also immediately seen that, by fixing the nonrelativistic renormalization point such that $\mu^2 = 4m^2$, the two expressions coincide, but this choice is completely arbitrary. What in fact coincides, when comparing the 1-loop NR scattering with the leading order (in $|p|/m$) of the relativistic result, are the critical values for which the one loop scattering vanishes, that is $v_c^{\pm} = \lambda^{\pm}/4m^2 = \pm 2e^2/m|\Theta|$. For these values of the self-coupling, which can be reached by a finite renormalization, one regains the scale invariance of the NR amplitude and by choosing the $v_c^+$ value, corresponding to a repulsive contact interaction, the amplitude reduces to the Aharonov-Bohm amplitude for identical particles which is given by \cite{14}.
\[ A_{AB} = -i \frac{4\pi}{m} \alpha [\cot \theta - \text{sgn}(\alpha)] + \mathcal{O}(\alpha^3) , \] (5.8)

where \( \alpha = e^2 / 2\pi \Theta \) is the Aharonov-Bohm parameter.

The subdominant terms that survive after fixing \( \lambda = \lambda^\ast \) can be seen as relativistic field theoretical corrections to the Aharonov-Bohm scattering \([20]\) and are given by

\[
A_{\text{sub}} = \pi \frac{m}{\alpha} [3 \text{sgn}(\alpha) + i \cot \theta] \frac{p^2}{m^2} - \frac{2\pi}{m} \alpha^2 \frac{p^2}{m^2} \left[ \ln \left( \frac{4m^2}{p^2} \right) + i\pi \right] + \frac{17}{3} \frac{\pi}{m} \alpha^2 - \frac{563}{60} \frac{\pi}{m} \alpha^2 \frac{p^2}{m^2} .
\] (5.9)

Part of the correction of the tree level (\( \sim \alpha \)) is due to the normalization of states and the relative velocity factor and so has a pure kinematical origin, but not all of it since the scattering amplitude corresponding to the exchange of one virtual gauge particle (4.1) depends on the CM energy as a consequence of the minimal coupling. The other corrections come from the 1-loop (\( e^4 \)) contribution to the perturbative expansion. Terms proportional to \( \alpha^2 \) do not exist in nonrelativistic AB scattering (which exact result is function of \( \sin \pi \alpha \)) and may be detected in experiments with fast particles.

Outside the critical values, the different ultraviolet structures of the relativistic and the nonrelativistic models fully manifest. The nonrelativistic triangle graph is logarithmically divergent while the relativistic one is linearly divergent. The nonrelativistic fish diagram is also logarithmically divergent whereas the relativistic (channel s) graph is finite \([24]\). The distinct nature of the divergences in the NR theory is due to its own kinematics reflected in the form of the NR propagator \([5.3]\). The \( 1/m \) expansion we have used to calculate the low energy intermediate states contributions mimics this aspect and one naturally expects that the L-sector amplitudes could be mapped into a nonrelativistic framework. Such a reduction process can be implemented and, certainly, requires a reinterpretation of the intermediate cutoff \( \Lambda \).

Let us initially concentrate on the leading order in \( |p|/m \) which should reproduce the NR case. As we saw, contributions coming from the radiative corrections to propagators and vertices are subdominant, come entirely from H sector (the L part is polynomial in \( \Lambda^2/m^2 \approx \eta \)) and so are neglected in agreement with what is expected in the nonrelativistic theory, where the parameters \( m, e \) and \( \Theta \) are fixed at their phenomenological values. Now, by comparing the L-sector, leading order, contribution to the 1-loop scattering,

\[
A_L^{(1)} = \frac{m}{8\pi} \left( \frac{\Lambda^2}{16m^4} - \frac{4e^4}{m^2\Theta^2} \right) \left[ \ln \left( \frac{\Lambda^2}{p^2} \right) + i\pi \right] .
\] (5.10)

with the NR amplitude \([5.2]\), taking into account \([5.7]\), one sees that they coincide if the intermediate cutoff is reinterpreted as a genuine nonrelativistic ultraviolet cutoff. In such a mapping the L-sector contribution is identified with the NR scattering amplitude while the H part can be seen as the needed counterterm to render the nonrelativistic theory finite, renormalized. In this way, the renormalization of the nonrelativistic model of ref. \([14]\) can be better understood.

This identification can be extended to a more general context by considering a \( d \)-dimensional space–time. For example, the contribution of the L sector to the s channel amplitude (the first graph of Fig. 8(a)) and the NR fish diagram are given by
\[ A_L^{(s)} = \frac{\lambda^2 m^{-1}}{2^{d+2} \pi^{(d-1)/2}} \frac{1}{\Gamma \left( \frac{d-1}{2} \right)} \times \int_0^{\Lambda^2} d(k^2) \frac{(k^2)^{(d-3)/2}}{k^2 - p^2 - i\epsilon} \left[ 1 - \frac{k^2}{2m^2} + \frac{3k^4}{8m^4} - \cdots \right] \] (5.11)

and

\[ A_{NR}^{fish} = \frac{v_0^2 m}{2^d \pi^{(d-1)/2}} \frac{1}{\Gamma \left( \frac{d-1}{2} \right)} \int_0^{\Lambda_{NR}^2} d(k^2) \frac{(k^2)^{(d-3)/2}}{k^2 - p^2 - i\epsilon} . \] (5.12)

For \( d = 1 + 1 \), the nonrelativistic scattering is finite and coincides with the leading term of the \( \eta \) expansion (in fact a \( \eta^{1/2} \) expansion for \( d = 1 + 1 \)) of the \( L \) contribution to the \( s \) channel. In this case, rigorous results for the \( \phi^4 \) theory can be found in ref. [27]. For \( d > 2 + 1 \), one sees that the identification \( \Lambda \leftrightarrow \Lambda_{NR} \) can also be done and produces the same expressions. This mapping can figuratively be pictured as a NR lens which magnifies \( \Lambda \) to \( \Lambda_{NR} (\rightarrow \infty) \) making kinematics be nonrelativistic. From a pragmatic viewpoint, one sees that the nonrelativistic limit can be obtained by calculating the \( L \) contribution to the scattering amplitude using the cutoff procedure described in Sec. III.

We shall now investigate the possibility of constructing an effective nonrelativistic Lagrangian that reproduces the scattering amplitudes incorporating the relativistic corrections up to order \( p^2/m^2 \). To account for the next-to-leading contribution to the amplitudes in the nonrelativistic context one has to add counterterms and new interactions to the usual nonrelativistic Lagrangian (5.1). As a general strategy, we consider the \( H \)-sector contributions to the relativistic scattering amplitudes as counterterms in the nonrelativistic theory and search for Galilean invariant effective interactions, preserving gauge invariance, to generate nonrelativistic scattering processes which compare with the \( L \)-sector amplitude. From beginning, we expect these terms to be nonrenormalizable interactions, like happen in effective field theories [\textcircled{3}].

The subdominant, nonpolynomial, part of the \( L \)-sector contribution to the two-body scattering is given, neglecting terms that are powers of \( \eta \) involving \( \Lambda \), by

\[ A_L^{(1) \text{subdom}} \approx \frac{- \left( \frac{5\lambda^2}{64m^4} - \frac{e^4}{m^2\Theta^2} \right)}{8\pi} \frac{p^2}{m^2} \ln \left( \frac{\Lambda^2}{p^2} \right) + i\pi \] . (5.13)

Notice that the subdominant polynomial part is cutoff-dependent and this freedom can be used to eliminate it. The term inside the square bracket in this expression appears in both fish and triangle NR diagrams. This suggests, in order to account for the \( p^2 \) factor, that the effective interactions which might reproduce these one loop contributions should contain second order spatial derivatives. Among the operators of dimension higher than four, the simplest possibility that gives the correct to the contact scattering, the \( \lambda^2 \) term in (5.13), can be borrowed from our earlier calculation of the nonrelativistic limit of \( \phi^4 \) theory [\textcircled{3}]. In fact, the addition of the interaction Lagrangian \( \frac{v_1}{m^2} \left[ \bar{\psi}(\nabla^2 \psi) \psi^2 - (\nabla \psi)^2 \bar{\psi}^2 \right] \) produces, for the one-loop CM scattering, a term proportional to \( p^2 \ln(\Lambda^2/p^2) + i\pi \) so that the first parcel of (5.13) can be obtained by properly adjusting the value of the parameter \( v_1 \). Similarly, one can easily sees that a seagull derivative interaction of the kind \( \frac{c_2}{8m^2} A^2 \psi \bar{\psi} \nabla^2 \psi \) generates the
A parcel of (5.13) proportional to $e^4$ by adequately choosing the value of $c_1$. These matchings furnish $v_1 = -5\lambda/64m^2$ and $c_1 = -1$.

These additional interactions should, of course, be modified by taking covariant derivatives instead of ordinary ones, in order to guarantee gauge invariance. One is thus lead to suggest the following effective interaction

$$L_{int}^{NR} = -\frac{5\lambda}{256m^4} \left[ \psi^* D^2 \psi^* + (D \psi^*)^2 \right] \psi^2 - \psi^* \frac{D^4}{8m^3} \psi .$$  (5.14)

Notice that the new terms that appear by considering covariant derivatives do not contribute up to the order we have worked, since they involve powers higher than two of either $|p|$ or $e$.

The last term in the above expression can be seen as originated from the expansion of the energy dispersion relation like that which happens in the Foldy-Wouthuysen procedure. There are other dimension six operators which could also have been considered. For example, by extending the covariant derivative in a non-minimal way through the introduction of a magnetic term, that is by defining $D_j = D_j + e \epsilon_{j\rho\sigma} \partial^\rho A^\sigma$, the $-\frac{1}{2m} (D\psi)^* \cdot (D\psi)$ term in the Lagrangian introduces (besides the usual $|D\psi|^2$) parcels involving $E \times (\psi^* \nabla \psi)$, $E \times A \psi^* \psi$ and $E^2 \psi^* \psi$, which might contribute to the scattering amplitude at order $p^2/m^2$. However, to take into account appropriately all the possibilities and to derive a more complete effective Lagrangian, one has to consider other sectors of the theory. This analysis is left for future work.

VI. APPENDIX

We summarize here the calculation of the various contributions to the CS 1-loop particle-particle scattering in the CM frame, using the approximation described in Sec. III. The momentum assignment is that of Fig. 8.

(i) Direct box graph.

Following the Feynman rules, one has

$$A^{(b)dir} = -ie^4 \int \frac{d^4k}{(2\pi)^4} \left\{ \left( p_1 + k \right)^{\mu} D_{\mu\nu}(k - p_1) \left( 2p_2 + p_1 - k \right)^{\nu} \Delta(k) \right\}$$

$$\Delta(p_1 + p_2 - k) \left( -k + p_1 + p_2 + p_2' \right)^{\mu} D_{\mu\nu}(k - p_1') (k + p_1')^{\nu} \right\} + [p_1' \leftrightarrow p_2']$$

$$= -\frac{4e^4}{\pi^2\Theta^2} \int d^2k \left( \frac{w_p}{w_k} \right) \frac{1}{p^2 - k^2 + i\epsilon} \left[ \frac{(k \times p) \cdot (k \times p')}{(k - p)^2(k - p')^2} \right] + [p' \leftrightarrow -p'] ,$$  (6.1)

where the $k^0$ integration of the CM amplitude was done as a contour integral. The angular integration, which involves only the third factor of the integrand in the last line above, can be cast in the form $\frac{1}{2} \left[ \cos \theta I_0 - I_2 \right]$ where

$$I_n = \int_0^{2\pi} d\varphi \frac{\cos(n\varphi)}{2 \cos(\varphi - \theta/2) - \beta} \left[ 2 \cos(\varphi + \theta/2) - \beta \right] \left[ 2 \cos(\varphi - \theta/2) - \beta \right]$$  (6.2)

and $\beta = (k^2 + p^2)/(|k||p|)$. Making $z = \exp(i\varphi)$ and using the residue theorem, one finds

$$I_n = \frac{\pi}{B\sqrt{\beta^2 - 1}} \left( \vartheta^{n-1} + \vartheta^{-n+1} \right) \sin \left[ (n + 1)\theta/2 \right] - \left( \vartheta^{n+1} + \vartheta^{-n-1} \right) \sin \left[ (n - 1)\theta/2 \right] \sin(\theta/2) ,$$  (6.3)
where \( B = \beta^2 - 2(1 + \cos \theta) \) and \( \theta = \frac{1}{2} \left[ \beta - \sqrt{\beta^2 - 4} \right] \). Using this formula, one can write

\[
A^{(b)\text{dir}} = -\frac{e^4}{\pi \Theta^2} \int d(k^2) \left( \frac{w_p^2}{w_k} \right) \frac{1}{p^2 - k^2 + i\varepsilon} \left[ \frac{|k^2 - p^2| (k^2 + p^2)}{(k^2 + (p^2)^2 - 2k^2p^2 \cos \theta)} - 1 \right] + (\theta \leftrightarrow \pi + \theta) . \tag{6.4}
\]

The remaining \( k^2 \) integration is then divided into two pieces, from 0 to \( \Lambda^2 \) (L region) and from \( \Lambda^2 \) to \( \Lambda_0^2 \to \infty \) (H sector). In the L part, using

\[
\frac{w_p^2}{w_k} = m \left( 1 + \frac{p^2}{m^2} \right) \left[ 1 - \frac{k^2}{2m^2} + \frac{3(k^2)^2}{8m^4} + \ldots \right] \tag{6.5}
\]

and keeping terms up to order \( \eta^2 \), one obtains

\[
A^{(b)\text{dir}}_L \simeq -\frac{e^4}{\pi \Theta^2} m \left\{ \left( 1 + \frac{p^2}{2m^2} \right) [\ln(2[1 - \cos \theta]) + i\pi] - (1 - \cos \theta - 2\cos^2 \theta) \frac{p^4}{\Lambda^4} - \frac{p^2}{m^2} \right.
\]

\[
-\cos \theta \left[ \ln(2[1 - \cos \theta]) + \ln \left( \frac{p^2}{\Lambda^2} \right) \right] \frac{p^2}{m^2} + 2\cos \theta \frac{p^2}{\Lambda^2} \left] + (\theta \leftrightarrow \pi + \theta) . \tag{6.6}
\]

Since \( \cos(\pi + \theta) = -\cos \theta \), one recovers equation (4.3). In the H region, the integrand is replaced by its Taylor expansion around \( p^2 = 0 \) which is given by

\[
\left[ -\frac{2m^2 \cos \theta}{(k^2)^2 \sqrt{k^2 + m^2}} \right] p^2 + \left[ \frac{2m^2(1 - 2\cos^2 \theta)}{(k^2^3 \sqrt{k^2 + m^2} - 2\cos \theta \sqrt{k^2 + m^2})} \right] \frac{p^4}{m^2} + O(p^6).
\]

Performing the \( k^2 \) integrations one obtains, up to order \( \eta^2 \),

\[
A^{(b)\text{dir}}_H \simeq -\frac{e^4}{\pi \Theta^2} m \left\{ -\cos \theta \ln \left( \frac{\Lambda^2}{4m^2} \right) \frac{p^2}{m^2} - 2\cos \theta \frac{p^2}{\Lambda^2} \right.
\]

\[
+ (1 - \cos \theta - 2\cos^2 \theta) \frac{p^4}{\Lambda^4} - \cos \theta \frac{p^2}{m^2} \left] + (\theta \leftrightarrow \pi + \theta) , \tag{6.7}
\]

which is the same as (4.9).

\[(ii) \textbf{Twisted box diagram.}\]

After performing the \( k^0 \) integration, one gets

\[
A^{(b)\text{twist}} = -\frac{e^4}{2\pi^2 \Theta^2} \int d^2k \left( \frac{w_k w_{k-s} - w_p^2}{w_k w_{k-s}(w_k + w_{k-s})} \right) \left[ \frac{(k \times s)^2 - (p \times p')^2}{(k - p)^2(k - p')^2} \right] + [p' \leftrightarrow -p'] \tag{6.8}
\]

where \( s = p + p' \) and we recall that \( w_{k-s} = \sqrt{(k - s)^2 + m^2} \). The angular integration above is very complicated and, therefore, we separate the L and H sectors before make it. The \( 1/m \) expansion of the first factor in the integrand of (6.8) is
\[
\frac{1}{2} \left( k^2 + \cos \theta p^2 - \sqrt{2(1 + \cos \theta)|k| \cos \varphi} \right) \frac{1}{m^3} + \mathcal{O}(1/m^5).
\]

One immediately suspect that there will be no contribution from the L sector up to order \( \eta^2 \). Pushing further, one can do the angular integration, which is expressible in terms of \( I_n \) \( (n = 0, \ldots, 3) \), and the \( k^2 \) integration, to get up to order \( \eta^3 \),

\[
A_{L}^{(b)\text{twist}} \approx -\frac{e^4}{4\pi \Theta^2} \frac{m(1 + \cos \theta) p^2 \Lambda^2}{m^4} + (\theta \leftrightarrow \pi + \theta) \quad (6.9)
\]

In the H region, the whole integrand is expanded around \(|p| = 0\) resulting in

\[
\left( \frac{(1 + \cos \theta) \sin^2 \varphi}{(k^2 + m^2)^{3/2}} \right) p^2 + \mathcal{O}(|p|^3),
\]

and, thus, the integrations over \( \varphi \) and \( k^2 \) lead to

\[
A_{H}^{(b)\text{twist}} \approx -\frac{e^4}{2\pi \Theta^2} m \left\{ (1 + \cos \theta) \frac{p^2 m}{m^2} - \frac{1}{2} (1 + \cos \theta) \frac{p^2 \Lambda^2}{m^4} \right\} + (\theta \leftrightarrow \pi + \theta) \quad (6.10)
\]

which reproduces (6.11).

(iii) **Seagull scattering.**

The gauge bubble contribution to the amplitude in the CM frame is given by

\[
-\frac{4e^4}{\Theta^2} \int \frac{d^3k}{(2\pi)^3} \frac{(k - p) \cdot (k - p')}{(k - p)^2 (k - p')^2}.
\]

The linear divergence of this expression manifests itself at once in the \( k^0 \) integration. However, the sum of the equal triangle diagrams has a parcel with a divergent \( k^0 \) integration which exactly cancel the preceding one so that one can write

\[
A^{(c)} = -\frac{e^4}{2\pi \Theta^2} \left\{ \int d^2k \left( \frac{w_p^2 + w_k^2}{w_k} \right) \frac{(k - p) \cdot (k - p')}{(k - p)^2 (k - p')^2} \right. \\
- \left. \int d^2k \left( \frac{4}{w_k} \right) \left[ \frac{(k \times p) (k \times p')}{(k - p)^2 (k - p')^2} \right] \right\} + (\theta \leftrightarrow \pi + \theta) \quad (6.11)
\]

The angular integration of the second term above is the same as that of the direct box diagram while that of the first parcel can be written as a linear combination of \( I_0 \) and \( I_1 \), and doing so, one gets

\[
A^{(c)} = -\frac{e^4}{2\pi \Theta^2} \left\{ \int d(k^2) \left( \frac{w_p^2 + w_k^2}{w_k} \right) \frac{\text{sgn}(k^2 - p^2)}{(k^2)^2 + (p^2)^2 - 2k^2p^2 \cos \theta} \frac{(k^2 - \cos \theta p^2)}{(k^2)^2 + (p^2)^2 - 2k^2p^2 \cos \theta} \right\} + (\theta \leftrightarrow \pi + \theta), \quad (6.12)
\]

where \( \text{sgn} \) denotes the signal function. Using (B.2) and (B.3) and performing the integrations, the L sector contribution is obtained, up to order \( \eta^2 \), as
\[ A_L^{(c)} \simeq -\frac{e^4}{2\pi\Theta^2} m \left\{ \left(2 + [1 - 2 \cos \theta] \frac{p^2}{m^2} \right) \left[ \ln \left( \frac{\Lambda^2}{p^2} \right) - \ln(2 [1 - \cos \theta]) \right] \right. \\
\left. + 2 \frac{p^2}{m^2} - 2 \cos \theta \frac{p^2}{\Lambda^2} + (1 - 2 \cos^2 \theta) \frac{p^4}{\Lambda^4} + \frac{\Lambda^4}{8m^4} \right\} + (\theta \leftrightarrow \pi + \theta), \quad (6.13) \]

which coincides with (4.13). The H part, cutoff regulated, is obtained by expanding the integrand up to \( p^4 \) and integrating from \( \Lambda^2 \) to \( \Lambda_0^2 \). It is found

\[ A_H^{(c)} \simeq -\frac{e^4}{2\pi\Theta^2} m \left\{ -\left( 2 + [1 - 2 \cos \theta] \frac{p^2}{m^2} \right) \ln \left( \frac{\Lambda^2}{4m^2} \right) - 2 + 2 \cos \theta \frac{p^2}{\Lambda^2} \\
+ \cos \theta \frac{p^2}{m^2} - (1 - 2 \cos^2 \theta) \frac{p^4}{\Lambda^4} - \frac{\Lambda^4}{8m^4} \right\} - \frac{e^4}{\pi\Theta^2} \Lambda_0 + (\theta \leftrightarrow \pi + \theta), \quad (6.14) \]

the same as (4.14).

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Figure captions

Fig. 1 - Feynman rules for the interaction vertices.

Fig. 2 - Particle self-energy insertion in 1-loop order.

Fig. 3 - Vacuum polarization correction in one loop order.

Fig. 4 - One loop correction to the trilinear vertex.

Fig. 5 - One loop correction to the seagull vertex.

Fig. 6 - Tree level scattering.

Fig. 7 - Basic mixing interactions diagrams.

Fig. 8 - One loop order particle-particle scattering. In the momenta assignment shown, which is used in the calculations, $s = p_1 + p_2$, $q = p_1 - p'_1$ and $u = p'_1 - p_2$. 