Graphs with unique zero forcing sets and Grundy dominating sets

Boštjan Brešar\textsuperscript{a,b} Tanja Dravec\textsuperscript{a,b}

March 19, 2021

\textsuperscript{a} Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
bostjan.bresar@um.si; tanja.dravec@um.si

\textsuperscript{b} Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

Abstract

The concept of zero forcing was introduced in the context of linear algebra, and was further studied by both graph theorists and linear algebraists. It is based on the process of activating vertices of a graph $G$ starting from a set of vertices that are already active, and applying the rule that an active vertex with exactly one non-active neighbor forces that neighbor to become active. A set $S \subseteq V(G)$ is called a zero forcing set of $G$ if initially only vertices of $S$ are active and the described process enforces all vertices of $G$ to become active. The size of a minimum zero forcing set in $G$ is called the zero forcing number of $G$. While a minimum zero forcing set can only be unique in edgeless graphs, we consider the weaker uniqueness condition, notably that for every two minimum zero forcing sets in a graph $G$ there is an automorphism that maps one to the other. We characterize the class of trees that enjoy this condition by using properties of minimum path covers of trees. In addition, we investigate both variations of uniqueness for several concepts of Grundy domination, which first appeared in the context of domination games, yet they are also closely related to zero forcing. For each of the four variations of Grundy domination we characterize the graphs that have only one Grundy dominating set of the given type, and characterize those forests that enjoy the weaker (isomorphism based) condition of uniqueness. The latter characterizations lead to efficient algorithms for recognizing the corresponding classes of forests.

Keywords: Grundy total domination number; Grundy domination number; zero forcing number; tree, graph automorphism

AMS subject classification: 05C69, 05C05, 05C35, 05C60.
1 Introduction

Finding an extremal set that attains a given graph invariant is the most basic problem concerning graph invariants. Another basic question is how many extremal sets for a given invariant are there in a graph. For instance, this question was studied recently in relation with the number of minimum total dominating sets [22] and the number of minimum dominating sets [3]. Back in 1985, Hopkins and Staton [24] studied the graphs having a unique maximum independent set, and called them the unique independence graphs. The study was continued by Gunther et al. [20], while recently Jaume and Molina [25] provided an algebraic characterization of unique independence trees, which can be used for efficient recognition of such trees. In another paper of Gunther et al. [19] the trees having a unique minimum dominating set were characterized, while Haynes and Henning in [21] characterized the trees with a unique minimum total dominating set. Two recent papers considered graphs, and in particular trees, that have unique maximum (open) packings [5, 16]. In addition, it was proved in [16] that the recognition of the graphs with a unique maximum (open) packing is polynomially equivalent to the recognition of the graphs with a unique maximum independent set, and that the complexity of all three problems is not polynomial, unless P=NP [16]. In this paper, we extend this investigation to problems of zero forcing and Grundy domination, concepts initiated in [2, 11, 13]; see also [6], where a close relation between these concepts was established. The zero forcing number is a useful lower bound for the minimum rank of a graph [2], and received great attention in the last decade (one can find about 100 papers in MathSciNet concerning zero forcing in graphs).

Next, we present some domination concepts including Grundy domination.

Let \( G \) be a graph, and \( v \in V(G) \). The open (respectively closed) neighborhood of a vertex \( v \) in \( G \) is the set \( N_G(v) \) (respectively \( N_G[v] \)) that contains all neighbors of \( v \) (respectively, \( N_G[v] = N_G(v) \cup \{v\} \)). We say that a vertex \( v \) dominates vertices in \( N_G[v] \). A set \( D \subseteq V(G) \) is a dominating set of \( G \) if every vertex \( u \in V(G) \) is dominated by some \( v \in D \). A vertex \( v \) totally dominates vertices in \( N_G(v) \). A set \( D \subseteq V(G) \) is a total dominating set if every vertex \( u \in V(G) \) is totally dominated by a vertex \( v \in D \). The minimum cardinality of a (total) dominating set is the (total) domination number of \( G \), denoted by \( \gamma(G) \) (respectively, \( \gamma_t(G) \)). Now, consider domination as a process of adding vertices to a dominating set of a graph \( G \), which results in a sequence of vertices in \( G \) such that each vertex \( x \) in the sequence dominates a vertex that was not dominated by vertices that precede \( x \) in the sequence. Vertices of a longest such sequence form a Grundy dominating set of \( G \). A Grundy total dominating set is defined similarly by requiring that every vertex in a sequence totally dominates a vertex that was not totally dominated by preceding vertices. A small modification of the condition for Grundy dominating set yields the so-called Z-Grundy dominating set, which is dual to the zero forcing set (the complement of a Z-Grundy dominating set is always a zero forcing set and vice-versa [6]).

If \( G \) is a graph that has a unique Grundy dominating set, then \( G \) is a unique Grundy domination graph. A weaker version of uniqueness is defined as
follows. If for every two Grundy dominating sets $S_1$ and $S_2$ of $G$, there is an automorphism $\phi : V(G) \to V(G)$ such that $\phi(S_1) = S_2$, then $G$ is an iso-unique Grundy domination graph. In a similar way, we define the classes of unique zero forcing graphs and iso-unique zero forcing graphs, as well as unique Grundy total domination graphs and iso-unique Grundy total domination graphs. We refer to the next section for formal definitions of the mentioned concepts.

In the next section, we establish the notation, present main definitions and preliminary results on Grundy domination and zero forcing. In Section 2, we first prove that the unique zero forcing graphs are only the empty graphs, and then concentrate on the iso-unique zero forcing forests. We prove a characterization of these forests by using the concept of path cover of a forest; the result also leads to a quadratic algorithm for recognizing iso-unique zero forcing forests. Then, in Section 3 we investigate unique Grundy domination graphs and again prove that only empty graphs have this property. In addition, we characterize the iso-unique Grundy domination graphs as the graphs in which each connected component is a complete graph. The study of uniqueness with respect to Grundy total domination, in Section 4 is again more involved. The unique Grundy total domination graphs are characterized by using a characterization from [13] of the graphs whose Grundy total domination number equals their order. Moreover, we characterize the iso-unique Grundy total domination forests, and present a linear algorithm for recognition of these forests. Section 5 is concerned with the fourth version of Grundy domination, the so-called L-Grundy domination, where we again characterize all graphs that have a unique extremal set for this invariant. In addition, it turns out that all forests are iso-unique L-Grundy domination graphs.

2 Notation and preliminaries

Zero forcing is a propagation model based on the following activation rule. If all neighbors of an active vertex $u$ except one neighbor $v$ are active, then $v$ becomes active. We say that $u$ forces $v$ and write $u \to v$. A set $S \subseteq V(G)$ is a zero forcing set of $G$ if initially only vertices of $S$ are active and the propagation of activation rules enforces all vertices of $G$ to become active. The zero forcing number $Z(G)$ is the minimum cardinality of a zero forcing set of $G$. The concept was introduced in [2], motivated by its close relation with the minimum rank of a graph.

A chronological list of forcings describes an order in which the non-active vertices are forced and by which vertices they are forced. A forcing chain is a maximal sequence of vertices $v_1, \ldots, v_k$ such that $v_i \to v_{i+1}$ for all $i \in [k-1]$. Clearly, a vertex of a zero forcing set $S$ starts a forcing chain and each forcing chain yields an induced path in $G$, since one vertex of a forcing chain can force at most one vertex. Moreover, every vertex of $G$ belongs to exactly one forcing chain.

The degree, $\deg_G(v)$, of a vertex $v$ in a graph $G$ is $|N_G(v)|$. A vertex $v$ with $\deg_G(v) = 0$ is an isolated vertex. A graph with no edges is empty. In other
words, a graph is empty when all its vertices are isolated. We denote by \( i(G) \) the number of isolated vertices in a graph \( G \), and \( n(G) \) is the order of \( G \). A leaf is a vertex of degree 1. A vertex adjacent to a leaf is a support vertex. A support vertex adjacent to at least two leaves is a strong support vertex. A path between vertices \( u \) and \( v \) is a \( u, v \)-path. The distance, \( d_G(u, v) \), between vertices \( u \) and \( v \) in a graph \( G \) is the length (number of edges) of a shortest \( u, v \)-path. The eccentricity of a vertex \( v \) in a graph \( G \) is \( \text{ecc}_G(v) = \max\{d_G(v, x) : x \in V(G)\} \). The center of \( G \) is the set of all vertices in \( G \), which have minimum eccentricity. It is well known that the center of a tree is either a vertex or a pair of adjacent vertices. (We may omit the indices in the corresponding notions if the graph \( G \) is clear from the context.) Let \( [n] = \{1, \ldots, n\} \), where \( n \in \mathbb{N} \).

Let \( G \) be a graph and \( S = (v_1, \ldots, v_k) \) a sequence of distinct vertices of \( G \); where integer \( k \) is the length of \( S \), and \( \hat{S} \) denotes the set of vertices from \( S \). Then, \( S \) is a closed neighborhood sequence, respectively an \( L \)-sequence, if for each \( i \in [k] \):

\[
N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset \quad (1), \text{ respectively } \quad N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset. \quad (2)
\]

Clearly, the minimum length of a closed neighborhood sequence \( S \) such that \( \hat{S} \) is a dominating set is the domination number \( \gamma(G) \) of a graph \( G \). The maximum length of a closed neighborhood sequence in \( G \) is the Grundy domination number, \( \gamma_{\text{gr}}(G) \) of \( G \), and a corresponding set \( \hat{S} \) is a Grundy dominating set. The corresponding sequence \( S \) is a Grundy dominating sequence of \( G \), or \( \gamma_{\text{gr}}(G) \)-sequence for short. Similarly, the maximum length of an \( L \)-sequence \( S \) in \( G \) is the \( L \)-Grundy domination number, \( \gamma_{L\text{gr}}(G) \), of \( G \), and \( \hat{S} \) is an \( L \)-Grundy dominating set.

In a similar way we define two additional types of sequences (usually they are defined on an isolate-free graph, although our definitions work on arbitrary graphs). A sequence \( S = (v_1, \ldots, v_k) \) of vertices of a graph \( G \) is an open neighborhood sequence, respectively a Z-sequence, if for each \( i \in [k] \):

\[
N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset \quad (3), \text{ respectively } \quad N(v_i) \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset. \quad (4)
\]

Note that if \( G \) has no isolated vertices, then the minimum length \( k \) of an open neighborhood sequence \( S \) such that \( \hat{S} \) is a total dominating set of \( G \), is the total domination number \( \gamma_t(G) \) of \( G \). The maximum length of an open neighborhood sequence in \( G \) is the Grundy total domination number, \( \gamma_{t\text{gr}}(G) \), of \( G \), and the corresponding set \( \hat{S} \) is a Grundy total dominating set, while \( S \) is a Grundy total dominating sequence of \( G \). Similarly, vertices of a Z-sequence form a Z-set, and the maximum size of a Z-set in \( G \) is the Z-Grundy domination number, \( \gamma_{Z\text{gr}}(G) \), of \( G \). A set \( \hat{S} \) of vertices that belong to a maximum Z-set \( S \) is a Z-Grundy dominating set. The corresponding sequence is a \( \gamma_{Z\text{gr}}(G) \)-sequence or a Z-Grundy dominating sequence of \( G \).

Grundy domination number has attracted considerable attention \( [7, 10, 15] \).
and so had its total version, which was introduced in [13] and studied further in [4, 8, 14, 15, 17]. The concept was motivated by domination games, and was surveyed within the recent book [12]. Brešar et al. found a close connection between the Grundy domination number and the zero forcing number of a graph [6], which motivated the introduction of the Z-Grundy domination number \( \gamma_{Z_{gr}}(G) \) of a graph \( G \), which is just the dual of the zero forcing number \( Z(G) \). Namely, in any graph the complement of any (minimum) zero forcing set is a (maximum) Z-set and vice versa. In particular, in every graph \( G \), \( \gamma_{Z_{gr}}(G) = n(G) - Z(G) \). Lin in [26] further explored the relations between all four types of Grundy domination numbers and some concepts from linear algebra.

Let \( S = (v_1, \ldots, v_k) \) be a sequence of distinct vertices of a graph \( G \). The initial segment \( (v_1, \ldots, v_i) \) of \( S \) will be denoted by \( S_i \). Given a sequence \( S' = (u_1, \ldots, u_m) \) of vertices in \( G \) such that \( \hat{S} \cap \hat{S}' = \emptyset \), \( S \oplus S' \) is the concatenation of \( S \) and \( S' \), that is, \( S \oplus S' = (v_1, \ldots, v_k, u_1, \ldots, u_m) \). Now, suppose that \( S \) is a closed neighborhood sequence, we say that for each \( i \in [k] \) vertex \( v_i \) footprints the vertices in \( N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \), and that \( v_i \) is the footprinter of any \( u \in N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \). If \( S \) is an open neighborhood sequence, respectively, L-sequence or Z-sequence, we also use the term t-footprinter, respectively, L-footprinter or Z-footprinter, meaning of which should be clear. Note that a vertex may be L-footprinted twice, once by itself, and later by one of its neighbors. On the other hand, the situation is simpler in the other three types of sequences. Namely, each vertex has a unique footprinter (respectively, Z-footprinter, t-footprinter), and so the function \( f : V(G) \to \hat{S} \), which maps a vertex to its footprinter (respectively, Z-footprinter, t-footprinter) is well defined.

Note that if \( S \) is a maximum length closed neighborhood sequence or L-sequence, then the set \( \hat{S} \) is a dominating set of \( G \) (sometimes \( S \) is also called a dominating sequence of \( G \)). The same holds for a Z-sequence provided that \( G \) has no isolated vertices. On the other hand, if \( G \) has no isolated vertices and \( S \) is a maximum length open neighborhood sequence or L-sequence, then \( \hat{S} \) is a total dominating set of \( G \).

In this paper, we study graphs in which, for each of the four Grundy domination numbers, a corresponding Grundy dominating set is unique. If \( G \) is a graph in which there is only one Grundy dominating set (respectively, Grundy total dominating set, Z-Grundy dominating set, L-Grundy dominating set), then \( G \) is a unique Grundy domination graph (respectively, unique Grundy total domination graph, unique Z-Grundy domination graph, unique L-Grundy domination graph). If \( G \) is a graph such that for every two Grundy dominating sets (respectively, Grundy total dominating sets, Z-Grundy dominating sets, L-Grundy dominating sets) \( A \) and \( B \) there exists an automorphism \( \phi : V(G) \to V(G) \) such that \( \phi(A) = B \), then \( G \) is an iso-unique Grundy domination graph (respectively, iso-unique Grundy total domination graph, iso-unique Z-Grundy domination graph, iso-unique L-Grundy domination graph). By the above observations, the class of unique zero forcing graphs (meaning of which should be clear) coincides with the class of unique Z-Grundy domination graphs, and the same holds for iso-unique variations of both concepts.
3 Unique zero forcing graphs

In the investigation of iso-unique zero forcing graphs we will use the relation between Z-Grundy domination and zero forcing described earlier. Since the complement of a (minimum) zero forcing set of \( G \) is a (maximum) Z-set of \( G \) and vice versa, a graph \( G \) is a unique zero forcing graph if and only if it is a unique Z-Grundy domination graph. Hence any of the two definitions can be used when establishing whether a graph belongs to this class of graphs.

**Proposition 1.** If \( G \) is a non-empty graph and \( x \) an arbitrary non-isolated vertex of \( G \), then there exists a Z-Grundy dominating sequence of \( G \) that contains \( x \).

**Proof.** Let \( S = (v_1, \ldots, v_k) \) be an arbitrary Z-Grundy dominating sequence of \( G \) and \( x \in V(G) \) a vertex with at least one neighbor in \( G \). Denote by \( I \) the set of isolated vertices of \( G \). Suppose that \( x \notin \hat{S} \). Since \( S \) is a Z-sequence, each vertex \( v_i \in S \) Z-footprints at least one vertex \( v'_i \in N(v_i) \). Note that since \( v'_i \) is footprinted by \( v_i \), \( v'_i \) is not adjacent to \( v_\ell \) for any \( \ell \in [i-1] \). As \( \hat{S} \) is a dominating set of \( G - I \), \( x \) has at least one neighbor in \( \hat{S} \). Let \( v_j \in S \) be the Z-footprinter of \( x \). Then \( S' = (v'_k, v'_{k-1}, \ldots, v'_1) \), where \( v'_j = x \), is a Z-sequence of \( G \), because for any \( i \in [k] \) vertex \( v'_i \) footprints \( v_i \). Since \( S' \) has length \( k = \gamma_{Zgr}(G) \) and it contains \( x \), the proof is complete.

**Corollary 2.** If \( G \) is a graph, \( I \) the set of isolated vertices of \( G \), and \( S = \{S : S \text{ is a } \gamma_{Zgr}(G)-\text{sequence of } G\} \), then

\[
\bigcup_{S \in S} \hat{S} = V(G) \setminus I.
\]

**Corollary 3.** Let \( G \) be a graph. Then \( G \) is a unique zero forcing graph if and only if \( G \) is empty.

**Proof.** If \( G \) is empty, then \( \gamma_{Zgr}(G) = 0 \) (and \( Z(G) = V(G) \)). Thus \( G \) is a unique zero forcing graph.

For the converse, let \( G \) be a unique zero forcing graph and let \( I \) be the set of isolated vertices of \( G \). For the purpose of contradiction suppose that \( G \) is not empty. Let \( S \) be an arbitrary Z-Grundy dominating sequence of \( G \). Then \( \hat{S} \cap I = \emptyset \). Since \( G \) is not empty, \( \gamma_{Zgr}(G) \leq n(G) - i(G) - 1 \) and hence there exists \( x \in V(G) \setminus I \) that is not contained in \( \hat{S} \). By Proposition 1 there exists Z-Grundy dominating sequence \( S' \) of \( G \) that contains \( x \). Since \( S \neq S' \), we get a contradiction.

For a connected graph \( G \), Corollary 3 also follows from results proved in [4], where it was shown that no connected graph of order greater than one has a unique minimum zero forcing set.

Next, we focus on the version of unique zero forcing graphs that involves automorphisms. Although \( \gamma_{Zgr}(G) < n(G) \) holds for any graph \( G \), there are many iso-unique zero forcing graphs already in the class of trees. The simplest
A path cover of a tree $T$ is a set of vertex disjoint induced paths of $T$ that cover all vertices of $T$. A path cover $\mathcal{P}$ of $T$ is minimum if no other path cover of $T$ has fewer paths than $\mathcal{P}$, and the path cover number $P(T)$ is the number of paths in a minimum path cover. Let $\mathcal{P} = \{Q_1, \ldots, Q_k\}$ be a path cover of a tree $T$. An edge $e = xy$, where $x \in V(Q_i)$, $y \in V(Q_j)$, $i \neq j$, is a connector edge for $\mathcal{P}$ and the end-vertices $x$ and $y$ of this edge are connector vertices of $\mathcal{P}$. A connector vertex is interior if it is an interior vertex of the path of $\mathcal{P}$ in which it is contained. A path cover $\mathcal{P}$ of $T$ is interior if every connector vertex in $\mathcal{P}$ is interior. A tree $T$ is a generalized star if it contains at most one vertex of degree more than 2. A pendant generalized star of a tree $T$ that is not a generalized star is an induced subgraph $K$ of $T$ such that there is exactly one vertex $v$ of $K$ with $\deg_T(v) = k + 1 \geq 3$, $k$ connected components of $T - v$ are pendant paths and $K$ is a subgraph of $T$ induced by those $k$ pendant paths and $v$. The vertex $v$ is called the mid vertex of $K$.

A path $P$ with $V(P) = \{x_1, \ldots, x_k\}$ and $E(P) = \{x_i, x_{i+1} : i \in [k-1]\}$ will be denoted by $P : x_1, \ldots, x_k$. The path $x_k, x_{k-1}, \ldots, x_1$ will be denoted by $P^{-1}$. Furthermore, given two vertices $x, y \in V(G)$ of a path $P$, an edge $xy \in E(G)$ and two vertices $v, u \in V(G)$ of a path $R$, we will denote by $xPy, vRu$ the $x, u$-walk in $G$ that starts in $x$ and follows $P$ until $y$, continues to $v$ and then follows $R$ until $u$.

In the seminal paper for zero forcing [2] it was proved that $Z(T) = P(T)$ for any tree $T$, and consequently $Z(F) = P(F)$ for any forest $F$; see also [18]. Moreover, the set obtained from a path cover $\mathcal{P}$ of a tree $T$ by taking one end-vertex of each path $P \in \mathcal{P}$ is a zero forcing set of $T$. Conversely, if $S = \{v_1, \ldots, v_k\}$ is a minimum zero forcing set of $T$, then there exists a minimum path cover $\mathcal{P} = \{Q_1, \ldots, Q_k\}$ of $T$ such that $v_i$ is an end-vertex of $Q_i$ for all $i \in [k]$. Indeed, for any $i \in [k]$, $Q_i$ can be the forcing chain that starts in $v_i$.

The trees having a unique minimum path cover turn out to be important in the investigation of iso-unique zero forcing trees. In [23] Corollary 16, Hogben and Johnson characterized such trees as the trees having an interior path cover.

**Proposition 4.** [23] A minimum path cover $\mathcal{P}$ of a tree $T$ is the unique minimum path cover of $T$ if and only if $\mathcal{P}$ is an interior path cover.

We continue with a basic observation about minimum path covers in trees.

**Lemma 5.** If $\mathcal{P}$ is a minimum path cover of a tree $T$ and $e = xy$ is a connector edge of $\mathcal{P}$, then at least one end-vertex of $e$ is an interior connector vertex.

**Proof.** Suppose that both connector vertices $x \in V(Q_i)$ and $y \in V(Q_j)$ are end-vertices of $Q_i$ and $Q_j$, respectively. Let $P_i : x, x_2, \ldots, x_t$ and $P_j : y, y_2, \ldots, y_k$. Then $\mathcal{P}' = (\mathcal{P} \setminus \{Q_i, Q_j\}) \cup \{P\}$, where $P : y_kP_j^{-1}y, xP_ix_t$, is a path cover containing less paths than $\mathcal{P}$, a contradiction. \qed
Lemma 6. If $T$ is an iso-unique zero forcing tree and $P$ is a minimum path cover of $T$, then both end-vertices of $P \in \mathcal{P}$ are leaves of $T$.

Proof. Suppose first that $P \in \mathcal{P}$ is an $x, y$-path with $\text{deg}_T(x) = 1$ and $\text{deg}_T(y) > 1$. Since $\mathcal{P}$ is a minimum path cover, there exist minimum zero forcing sets $S$ and $S'$ such that $S \setminus S' = \{x\}$ and $S' \setminus S = \{y\}$. This is a contradiction, since there is clearly no automorphism of $T$ that maps $S$ to $S'$.

Suppose now that $P \in \mathcal{P}$ is an $x, y$-path with $\text{deg}_T(x) > 1$ and $\text{deg}_T(y) > 1$ (note that $x = y$ is also possible). Since $x$ is not a leaf of $T$, it has a neighbor $x_1 \notin V(P)$. Let $P_1 : x_1, 1, x_1, 2, \ldots, x_1, n_1$ be the path from $\mathcal{P}$ that contains $x_1$. By Lemma 5, $x_1 = x_1, j_1$ for $j_1 \in \{2, \ldots, n_1 - 1\}$. Now, we will find a sequence of $\ell \geq 1$ paths. If $x_1, 1$ is a leaf of $T$, then $\ell = 1$, otherwise $x_1, 1$ has a neighbor $x_2 \notin V(P_1)$. Let $P_2 : x_2, 1, x_2, 2, \ldots, x_2, n_2$ be the path from $\mathcal{P}$ that contains $x_2$. By Lemma 5, $x_2 = x_2, j_2$ for $j_2 \in \{2, \ldots, n_2 - 1\}$. If $x_2, 1$ is a leaf of $T$, then $\ell = 2$, otherwise we continue with this procedure until the path $P_t : x_{t, 1}, \ldots, x_{t, n_t}$, which is the path containing a neighbor $x_t$ of $x_{t-1, 1} \in V(P_{t-1})$ and $\text{deg}_T(x_{t, 1}) = 1$. Then $P' = (P \setminus \{P_1, P_2, \ldots, P_t\}) \cup \{P', P'_1, \ldots, P'_{\ell}\}$, where

$$P' : x_{t, 1}, P_t x_{t, t-1}, P_{t-1} x_{t-1, t}, x_{t-2, 1}, \ldots, x_{1, 1}, P_1 x_1, xP_y,$$

and $P'_i : x_{i, j_i}, 1, P_i x_{i, n_i}$, for all $i \in [\ell]$,

is clearly a minimum path cover of $T$. Since $\mathcal{P}$ is a minimum path cover, there is a minimum zero forcing set $S$ of $T$ with

$$S \cap (V(P) \cup V(P_1) \cup \ldots \cup V(P_t)) = \{x_1, n_1, x_2, n_2, \ldots, x_{t, n_t}, x\}.$$

As $\mathcal{P}'$ is also a minimum path cover, $S' = (S \setminus \{x\}) \cup \{x_{t, 1}\}$ is a zero forcing set. Since $S'$ has more leaves than $S$, there is no automorphism of $T$ that maps $S$ to $S'$, which is a contradiction. Thus, $\text{deg}_T(x) = \text{deg}_T(y) = 1$. \hfill \Box

Lemma 7. Let $T$ be an iso-unique zero forcing tree and let $P$ be a minimum path cover of $T$. If $e = xy$ is a connector edge for $\mathcal{P}$, then connector vertices $x$ and $y$ are interior or one of them is interior and the other is the only vertex of a path from $\mathcal{P}$.

Proof. By Lemma 5, at least one connector vertex, say $x$, is interior. For the purpose of contradiction assume that $y$ is an end-vertex of a path $P' \in \mathcal{P}$ of length at least 2. Then $\text{deg}_{P'}(y) \geq 2$, which is a contradiction with Lemma 6. \hfill \Box

Lemma 8. Let $T$ be an iso-unique zero forcing tree and let $\mathcal{P}$ be a minimum path cover of $T$. If $P \in \mathcal{P}$ contains only one vertex $x$, then the neighbor of $x$ is the center of the path $R \in \mathcal{P}$ of length $2$, that is, $R \equiv P_3$.

Proof. By Lemma 6, $x$ is a leaf of $T$. Let $y$ be the neighbor of $x$ in $T$ and let $R : y_1, y_2, \ldots, y_k$ be the path of $\mathcal{P}$ that contains $y$. Lemma 6 implies that $y$ is an interior vertex of $R$ and hence $y = y_i$ for some $i \in \{2, \ldots, k - 1\}$. By Lemma 6, $y_1$ and $y_k$ are leaves of $T$. For the purpose of contradiction assume that $|V(R)| \geq 4$. Hence at least one of the subpaths of $R$, the $y_1, y_{i-1}$-subpath
or the $y_{i+1}, y_k$-subpath, has length at least 2. Without loss of generality assume that $i > 2$ (otherwise we change the roles of both parts of $R$). Let $P' : x, y, R y_k$ and $R' : y, R y_{k-1}$. Then $P' = (P \setminus \{P, R\}) \cup \{P', R'\}$ is also a minimum path cover of $T$. Since $P'$ contains the path $R'$ with the end-vertex $y_{i-1}$ that is not a leaf of $T$, we get a contradiction with Lemma 6. □

The following result is based on the fact that minimum zero forcing sets in a tree can be obtained from minimum path covers by taking one end-vertex from each of the paths in a path cover of $T$. Given a path cover $P$, exchanging two end-vertices of a path $P \in P$ and keeping end-vertices of other paths in $P$ fixed, we get two zero forcing sets that differ only in one vertex. This yields the existence of an automorphism of a tree, which leads to the following result.

**Lemma 9.** Let $T$ be an iso-unique zero forcing tree and let $P$ be a minimum path cover of $T$. Let $P : x_1, \ldots, x_{\ell} \in P$, and let $p = \ell - 1$ if $\ell$ is odd, and $p = \ell$ if $\ell$ is even.

(a) If $\ell$ is odd, then the connected components of $T - x_p$ that contain $x_{p-1}$ and $x_{p+1}$, respectively, are isomorphic.

(b) If $\ell$ is even, then the connected components of $T - x_p x_{p+1}$ that contain $x_p$ and $x_{p+1}$, respectively, are isomorphic.

Moreover, there is an automorphism of $G$ that maps $x_1$ to $x_\ell$ and $x_\ell$ to $x_1$.

**Proof.** Let $T$ be an iso-unique zero forcing tree and let $P$ be a minimum path cover of $T$. Adopting the notation in the formulation of the lemma, assume first that $\ell$ is odd. Let $S$ be a minimum zero forcing set of $T$, and assume without loss of generality that $x_1 \in S$ and $x_\ell \notin S$. (Note that a minimum zero forcing set of $T$ can be taken by choosing a leaf of every path in $P$.) On the other hand, $S' = (S \setminus \{x_1\}) \cup \{x_\ell\}$ is also a minimum zero forcing set of $T$, and therefore there is an automorphism $\alpha$ of $T$ that maps $S$ to $S'$. Let $C_1$ be the component of $T - x_p$ that contains $x_{p-1}$ and $C_2$ be the component of $T - x_p$ that contains $x_{p+1}$. If $\alpha$ fixes $x_{p-1}$, then $|S \cap V(C_1)| = |S' \cap V(C_1)|$, which is a contradiction, since $S \cap V(C_1) = (S' \cap V(C_1)) \cup \{x_1\}$. Therefore, $\alpha(x_{p-1}) = x_p^{(1)}$, and $x_p^{(1)}$ is a neighbor of $x_p^{(1)}$, where $\alpha(x_p) = x_p^{(2)}$ (possibility $x_p^{(1)} = x_p$ is not excluded). From the same reason, the component $C_1^{(1)}$ of $T - x_p^{(1)}$ that contains $x_p^{(1)}$ is isomorphic to $C_1$ and $\alpha(x_1) = x_1^{(1)}$, which is a leaf in $C_1^{(1)}$.

Since $S - \{x_1, x_\ell\} = S' - \{x_1, x_\ell\}$, we infer that $x_1^{(1)} \in S \cap S'$, unless $C_1^{(1)} = C_2$. Now, using the same arguments we infer that $\alpha$ maps $C_1^{(1)}$ to $C_1^{(2)}$, which is isomorphic to $C_1^{(1)}$. In particular, $\alpha(x_1^{(1)}) = x_1^{(2)}$, which is a leaf in $C_1^{(2)}$. Since $S - \{x_1, x_\ell\} = S' - \{x_1, x_\ell\}$, we infer that $x_1^{(2)} \in S \cap S'$, unless $C_1^{(2)} = C_2$. We continue with the same reasoning, and since $T$ is finite, we eventually reach the case when $\alpha$ maps $C_1^{(k)}$ to $C_1^{(k+1)}$ (for some $k \geq 0$, assuming that $C_1^{(0)} = C_1$), however, the leaf $x_1^{(k)} \in S \cap V(C_1^{(k)})$ is mapped to $x_1^{(k+1)}$, which is not in $S$. Hence, $x_1^{(k+1)} = x_\ell$. Clearly, the subgraphs $C_1 = C_1^{(0)}, C_1^{(1)}, \ldots, C_1^{(k+1)} = C_2$.
are all pairwise isomorphic. Hence, there is an automorphism that maps \( V(C_1) \) to \( V(C_2) \), and, in particular, it maps \( x_1 \) to \( x_\ell \), fixes \( x_p \), and maps \( x_\ell \) to \( x_1 \).

The case (b), when \( \ell \) is even can be proved in a similar way.

In the above lemmas we presented several necessary conditions for iso-unique zero forcing trees. In the next result we prove that all those conditions together yield a sufficient condition for \( T \) being an iso-unique zero forcing tree.

**Theorem 10.** A tree \( T \) is an iso-unique zero forcing tree if and only if the following conditions are satisfied for every minimum path cover \( \mathcal{P} \) of \( T \):

(i) Both end-vertices of a path \( P \in \mathcal{P} \) are leaves of \( T \).

(ii) If \( e \) is a connector edge of \( \mathcal{P} \), then the connector vertices of \( e \) are either both interior or one of them is interior vertex of a path in \( \mathcal{P} \) isomorphic to \( P_3 \) and the other is the only vertex of a path in \( \mathcal{P} \).

(iii) For every path \( P : x_1, \ldots, x_\ell \) of \( \mathcal{P} \) with \( p = \frac{\ell + 1}{2} \) if \( \ell \) is odd, and \( p = \frac{\ell}{2} \) if \( \ell \) is even, the following holds.

(a) If \( \ell \) is odd, then the connected components of \( T - x_p \) that contain \( x_{p-1} \) and \( x_{p+1} \) are isomorphic.

(b) If \( \ell \) is even, then the connected components of \( T - x_p x_{p+1} \) that contain \( x_p \) and \( x_{p+1} \), respectively, are isomorphic.

Moreover, there is an automorphism of \( G \) that maps \( x_1 \) to \( x_\ell \) and \( x_\ell \) to \( x_1 \).

**Proof.** By Lemmas 6, 7, 8, and 9, the conditions (i), (ii), and (iii) are necessary for a tree \( T \) to be an iso-unique zero forcing graph. So, let \( T \) be a tree such that for any minimum path cover \( \mathcal{P} \) of \( T \) conditions (i), (ii), and (iii) are satisfied.

Let \( S \) be a minimum zero forcing of \( T \). For every \( x \in S \) there is a forcing chain (a path) that starts in \( x \). Hence, every minimum zero forcing set \( S \) yields a path cover \( \mathcal{P} \) of \( T \) such that each vertex of \( S \) is an end-vertex of (a unique) path in \( \mathcal{P} \). Since \( Z(T) = P(T) \), \( \mathcal{P} \) must be a minimum path cover of \( T \). On the other hand, it is also known and easy to see that every minimum zero forcing \( S \) of \( T \) consists only of end-vertices of the paths from the corresponding path cover \( \mathcal{P} \). By (i), all end-vertices of paths in \( \mathcal{P} \), and thus all vertices in \( S \), are leaves of \( T \). Clearly, two different zero forcings \( S \) and \( S' \) may yield the same path cover \( \mathcal{P} \) of \( T \); however, since they are different, some of the end-vertices of the paths in \( \mathcal{P} \) are different in \( S \) and \( S' \).

Let \( S \) and \( S' \) be arbitrary minimum zero forcing sets of \( T \), and first assume that both \( S \) and \( S' \) yield the same path cover \( \mathcal{P} \); that is, each of \( S \) and \( S' \) consists of end-vertices of paths from \( \mathcal{P} \), one end-vertex of each such path.

**Case 1:** \( |S \setminus S'| = 1 \). Hence there exists a path \( P : x_1, \ldots, x_k \in \mathcal{P} \) such that \( x_1 \in S \) and \( x_k \in S' \). By (iii), there exists an automorphism that exchanges \( x_1 \) to \( x_k \) and consequently maps \( S \) to \( S' \).

**Case 2:** \( |S \setminus S'| = \ell > 1 \). Let \( Q_1, \ldots, Q_\ell \) be the paths of \( \mathcal{P} \) for which \( S \cup S' \) contains both of their end-vertices. First, let \( S_1 \) be the set of vertices obtained
from $S_0 = S$ by replacing the end-vertex of $Q_1$ that is contained in $S$ with the other end-vertex of $Q_1$, that is, with the end-vertex of $Q_1$ that is contained in $S'$. Case 1 implies that there exists an automorphism $f_1 : V(T) \rightarrow V(T)$ that maps vertices of $S = S_0$ to vertices of $S_1$. We continue with the procedure so that in the $i$th step, where $i \in \{2, \ldots, \ell\}$, $S_i$ is the set of vertices obtained from $S_{i-1}$ by replacing the end-vertex of $Q_i$ that is contained in $S_{i-1}$ with the other end-vertex of $Q_i$, that is, with the end-vertex of $Q_i$ that is contained in $S'$. Case 1 implies that there exists an automorphism $f_i : V(T) \rightarrow V(T)$ that maps vertices of $S_{i-1}$ to vertices of $S_i$. Clearly, $f = f_\ell \circ f_{\ell-1} \circ \ldots \circ f_1$ is an automorphism of $T$ that maps $S$ to $S'$.

Finally, let $S$ correspond to $P$ and $S'$ correspond to $P'$, where $P'$ and $P$ are distinct minimum path covers of $T$. Let $K$ be an arbitrary pendant generalized star of $T$ with mid vertex $v$. If $\deg_K(v) \geq 3$, then if follows from (i) and (ii) that $K$ is a star. If $K$ is a path, then condition (iii) implies that $v$ is the center of $K$ and clearly $K$ must be an element of any minimum path cover of $T$. In particular, $K$ belongs to $P$ and $P'$. For each pendant generalized star $K$ of $T$ that is a star, we remove from $T$ all except two leaves of $K$ and denote the resulting tree by $T'$. Here we are assuming that when leaves that belong to either $P$ or $P'$ are removed, the resulting path cover instead adopts leaves that remained in the pendant generalized star. It follows from (ii) that any connector edge of any minimum path cover of $T'$ is interior and thus the minimum path cover is unique by Proposition 4. Hence $P$ and $P'$ when restricted to $T'$ coincide. Thus the minimum covers $P$ and $P'$ can only differ in some of the leaves of a pendant generalized star $K_{1,k}$, where $k - 2$ of these leaves are covered by one-vertex paths. In any case, any minimum zero forcing set of $T$ contains exactly $k-1$ vertices (that are leaves of $T$) of any pendant generalized star $K_{1,k}$. Clearly, there is an automorphism that maps $k - 1$ leaves to some other $k - 1$ leaves, which are all attached to the same support vertex. Combining this with the initial case when two zero forcings yielded the same path cover, we deduce that there is an automorphism that maps $S$ to $S'$.

The following corollary of Theorem 10 will be useful in the recognition algorithm for iso-unique zero forcing trees. (Note that by Proposition 4, a minimum path cover is unique if and only if it is an interior path cover.)

**Corollary 11.** If $T$ is an iso-unique zero forcing tree and $T'$ is the tree obtained from $T$ such that for every strong support vertex $v$ of $T$, which is adjacent to more than two leaves, all but two leaves adjacent to $v$ are removed, then $T'$ has the unique minimum path cover.

We are ready to present the announced algorithm for deciding whether a given tree is an iso-unique zero forcing graph. It is based on Theorem 10 and Corollary 11. The results can be directly extended from trees to forests, hence the input of the algorithm is an arbitrary forest.
Algorithm Iso-Unique Zero Forcing Forest

Input. A forest $T$.
Output. YES if $T$ is an iso-unique zero forcing forest, NO otherwise.

(1) Let $T'$ be the forest obtained from $T$ such that for every strong support vertex $v$ of $T$, which is adjacent to more than two leaves, all but two leaves adjacent to $v$ are removed.

Let $P$ be a minimum path cover of $T'$. If $P$ is not interior, then RETURN NO.

(2) Consider $P$ in $T$.

(3) For every path $P : x_1, \ldots, x_\ell$ of $P$ with $p = \frac{\ell+1}{2}$ if $\ell$ is odd, and $p = \frac{\ell}{2}$ if $\ell$ is even, check:

(a) If $\ell$ is odd, then the connected components of $T - x_p$ that contain $x_{p-1}$ and $x_{p+1}$ are isomorphic.
(b) If $\ell$ is even, then the connected components of $T-x_px_{p+1}$ that contain $x_p$ and $x_{p+1}$, respectively, are isomorphic.

Moreover, there is an automorphism of $G$ that maps $x_1$ to $x_\ell$ and $x_\ell$ to $x_1$.

If true for all paths $P \in P$, then RETURN YES, otherwise RETURN NO.

The correctness of the algorithm is a direct consequence of Theorem 10 and Corollary 11. If $T$ is an iso-unique zero forcing forest, then the minimum path cover $P$ of $T'$ is interior by Corollary 11 and thus the algorithm does not stop in step (1). By Theorem 10, the condition of step (3) is satisfied for every path $P \in P$ and hence the algorithm returns YES. For the converse, if $T$ is not an iso-unique zero forcing forest, then one condition of Theorem 10 is not satisfied. If (i) or (ii) of Theorem 10 does not hold, then the minimum path cover $P$ of $T'$ is not interior and thus the algorithm returns NO. If (iii) does not hold, then step (3) returns NO.

Clearly, one can construct $T'$ from $T$ in linear time. By an algorithm from [23], one can construct a minimum path cover of $T$ in linear time. Indeed, the mentioned algorithm is based on finding a pendant generalized star, and providing a path cover for it, and then continuing the process in the tree from which this pendant generalized star is removed. In addition, checking if the resulting path cover is interior can be done efficiently by going through all connector edges and checking if the end-vertices are interior vertices of their paths. This resolves (1). To consider $P$ in $T$ we only need to add additional one-vertex paths to $P$, which consist of vertices deleted in the previous step. This resolves step (2). For step (3), we apply an algorithm for verifying whether two trees are isomorphic. We can use the classical AHU algorithm for checking tree isomorphism [1], applied on the corresponding components of the forest. We slightly modify the algorithm by fixing the two vertices and checking whether an
algorithm maps one to the other. This algorithm is linear, and since the number of paths in a path cover is $O(n)$, we derive that entire algorithm performs in $O(n^2)$ time. We summarize these observations in the following result.

**Theorem 12.** Algorithm Iso-Unique Zero Forcing Forest verifies with time complexity $O(n^2)$ whether a given forest is an iso-unique zero forcing forest.

### 4 Unique Grundy domination graphs

**Proposition 13.** If $G$ is a graph and $x$ an arbitrary vertex of $G$, then there exists a Grundy dominating sequence of $G$ that contains $x$.

**Proof.** Let $S = (v_1, \ldots, v_k)$ be an arbitrary $\gamma_{gr}(G)$-sequence of $G$ and $x \in V(G)$ an arbitrary vertex of $G$. We may assume that $x \notin \hat{S}$. Since $S$ is a closed neighborhood sequence, each vertex $v_i \in S$ footprints at least one vertex $v'_i \in N[v_i]$. Note that since $v'_i$ is footprinted by $v_i$, vertex $v'_i$ is not adjacent to $v_i$ for any $\ell \in [i - 1]$. As $\hat{S}$ is a dominating set of $G$, $x$ has at least one neighbor in $\hat{S}$. Let $v_j \in S$ be the footprint of $x$. Then $S' = (v'_1, v'_{k-1}, \ldots, v'_j)$, where $v'_j = x$ is a closed neighborhood sequence of $G$ because for any $i \in [k]$, vertex $v'_i$ footprints $v_i$. Since $S'$ has length $k = \gamma_{gr}(G)$ and it contains $x$, the proof is complete.

**Corollary 14.** If $G$ is a graph, and $S = \{S : S$ is a $\gamma_{gr}(G)$-sequence of $G\}$, then

$$\bigcup_{S \in S} \hat{S} = V(G).$$

**Corollary 15.** If $G$ is a graph, then $G$ is a unique Grundy domination graph if and only if $G$ is an empty graph.

**Proof.** If $G$ is empty, then $\gamma_{gr}(G) = n(G)$ and thus $G$ is a unique Grundy domination graph.

For the converse, let $G$ be a unique Grundy domination graph. For the purpose of contradiction suppose that $G$ is not empty. Let $S$ be an arbitrary $\gamma_{gr}(G)$-sequence. Since $G$ is not empty, $\gamma_{gr}(G) \leq n(G) - 1$ and hence there exists $x \in V(G)$ that is not contained in $S$. By Proposition 13 there exists a $\gamma_{gr}(G)$-sequence $S'$ of $G$ that contains $x$. Since $S \neq S'$, we get a contradiction.

The problem of characterizing iso-unique Grundy domination graphs is less difficult than the case of iso-unique $Z$-Grundy domination graphs presented in the previous section.

**Theorem 16.** A graph $G$ is an iso-unique Grundy domination graph if and only if each connected component of $G$ is a complete graph.

**Proof.** Let $G$ be an iso-unique Grundy domination graph, and assume that there is a connected component $C$ in $G$, which is not a complete graph. Let $S = (v_1, \ldots, v_k)$ be a Grundy dominating sequence in $G$. Clearly, at least
two vertices from \( C \) will appear in \( S \), since \( C \) is not a complete graph. Let \( v_j \) be the last vertex in the sequence \( S \), which is from \( C \). Furthermore, let \( U = (N[v_1] \cup \ldots \cup N[v_{j-1}]) \cap V(C) \) and let \( U' = V(C) \setminus U \). Then \( U' \) is the set of vertices in \( C \) footprinted by \( v_j \). Since \( C \) is connected, there exists \( u \in U \) that has a neighbor \( u' \in U' \). Since \( v_j \) is the last vertex in \( S \), which is from \( C \), the subgraph of \( G \) induced by \( U' \) is a clique. Moreover, \( u \) is adjacent to all vertices from \( U' \). Since \( u' \) has no neighbors in \( \{ v_1, \ldots, v_{j-1} \} \), we have \( u \neq v_i \) for any \( i \in [j-1] \). Thus the sequences \( S' \) and \( S'' \) that are obtained from \( S \) by replacing \( v_j \) by \( u \) and \( u' \), respectively, are closed neighborhood sequences (note that \( v_j \) can be equal to \( u \) or \( u' \), and thus it is possible that either \( S = S' \) or \( S = S'' \)). Since the set \( \tilde{S} \) induces more connected components then the set \( \tilde{S}' \), it is clear that there is no automorphism of \( G \) that maps \( \tilde{S} \) to \( \tilde{S}' \). This contradiction implies that each connected component of \( G \) is a complete graph. The converse is clear. \( \square \)

5 Unique Grundy total domination graphs

**Proposition 17.** If \( G \) is a non-empty graph and \( x \) an arbitrary non-isolated vertex of \( G \), then there exists a Grundy total dominating sequence of \( G \) that contains \( x \).

**Proof.** Let \( S = (v_1, \ldots, v_k) \) be an arbitrary \( \gamma^t_{gr}(G) \)-sequence and \( x \in V(G) \) vertex with at least one neighbor in \( G \). Furthermore denote by \( I \) the set of isolated vertices of \( G \). We may assume that \( x \notin \tilde{S} \). Since \( S \) is an open neighborhood sequence, each vertex \( v_i \in S \) t-footprints at least one vertex \( v'_i \in N(v_i) \). Note that since \( v'_i \) is footprinted by \( v_i \), \( v'_i \) is not adjacent to \( v_j \) for any \( i \in [i-1] \). As \( \tilde{S} \) is a total dominating set of \( G - I \), \( x \) has at least one neighbor in \( \tilde{S} \). Let \( v_j \in S \) be the t-footprinter of \( x \). Then \( S' = (v'_k, v'_k, \ldots, v'_1) \), where \( v'_j = x \), is a \( \gamma^t_{gr}(G) \)-sequence that contains \( x \).

**Corollary 18.** If \( G \) is a graph, \( I \) the set of isolated vertices of \( G \), and \( S = \{ S : S \) is a \( \gamma^t_{gr}(G) \)-sequence of \( G \} \), then

\[
\bigcup_{S \in S} \tilde{S} = V(G) \setminus I.
\]

**Corollary 19.** A graph \( G \) is a unique Grundy total domination graph if and only if \( \gamma^t_{gr}(G) = n(G) - i(G) \).

**Proof.** If \( G \) is a graph with \( \gamma^t_{gr}(G) = n(G) - i(G) \), then the only Grundy total dominating set of \( G \) is the set of all non-isolated vertices of \( G \). Thus \( G \) is a unique Grundy total domination graph.

For the converse, let \( G \) be a unique Grundy total domination graph and let \( I \) be the set of isolated vertices of \( G \). If \( S \) is an arbitrary \( \gamma^t_{gr}(G) \)-sequence, then \( \tilde{S} \cap I = \emptyset \). If \( \gamma^t_{gr}(G) \leq n(G) - i(G) - 1 \), then there exists \( x \in V(G) \setminus I \) that is not contained in \( S \). By Proposition 17 there exists a \( \gamma^t_{gr}(G) \)-sequence \( S' \).
that contains \( x \). Since \( S \neq S' \), we get a contradiction, which implies \( \gamma_{gr}^t(G) = n(G) - i(G) \).

Restricting our attention to graphs with no isolated vertices, Corollary \cite{19} yields the graphs with \( \gamma_{gr}^t(G) = n(G) \). A characterization of these graphs was proved in the seminal paper on Grundy total domination.

**Theorem 20.** \cite{13} Theorem 4.2] If \( G \) is a graph with no isolated vertices, then
\[
\gamma_{gr}^t(G) = n(G) \text{ if and only if there exists an integer } k \text{ such that } n(G) = 2k, \text{ and the vertices of } G \text{ can be labeled } x_1, \ldots, x_k, y_1, \ldots, y_k \text{ in such a way that}
\]
\begin{itemize}
  \item \( x_i \) is adjacent to \( y_i \) for each \( i \),
  \item \( \{x_1, \ldots, x_k\} \) is an independent set, and
  \item \( y_j \) is adjacent to \( x_i \) implies \( i \geq j \).
\end{itemize}

Note that Theorem 20 restricted to forests \( T \) simplifies to \( \gamma_{gr}^t(T) = n \) if and only if \( T \) has a perfect matching.

Next, we consider iso-unique Grundy total domination graph. There are several families of such graphs. In particular, this includes the graphs \( G \) with \( \gamma_{gr}^t(G) = n(G) \), but we can also extend this family by using the following observation. Vertices \( u \) and \( v \) in a graph \( G \) are open twins if \( N_G(u) = N_G(v) \); also, a vertex \( u \) is an open twin if there exists another vertex \( v \) such that \( u \) and \( v \) are open twins. It is easy to see that \( \gamma_{gr}^t(G) = \gamma_{gr}^t(G - u) \) if \( u \) is an open twin in \( G \); see \cite{14} Proposition 3.6.

**Proposition 21.** Let \( u \) be an open twin in a graph \( G \). If \( G \) is an iso-unique Grundy total domination graph, then \( G - u \) is also an iso-unique Grundy total domination graph. In addition, if \( G \) is a forest, then \( G \) is an iso-unique Grundy total domination graph if and only if \( G - u \) is an iso-unique Grundy total domination graph.

**Proof.** Let \( u \) and \( v \) be open twins in a graph \( G \). It is easy to see that at most one of these two vertices belongs to an open neighborhood sequence, and also they are both t-footprinted by the same vertex in any such sequence. Now, there is a natural automorphism \( \phi_{u \leftrightarrow v} \) that exchanges \( u \) and \( v \) and fixes all other vertices of \( G \). Note that \( \gamma_{gr}^t(G - u) = \gamma_{gr}^t(G) \), and \( S \) is a Grundy total dominating sequence in \( G \) if and only \( S' \) is a Grundy total dominating sequence in \( G - u \), where \( S' \) is obtained from \( S \) by replacing \( u \) with \( v \) if necessary (or, otherwise, if \( u \) is not in \( S \), then \( S' = S \) ). By Proposition 17 \( v \) belongs to a \( \gamma_{gr}^t(G) \)-set, \( u \) belongs to a \( \gamma_{gr}^t(G - u) \)-set.

Suppose that \( G \) is an iso-unique Grundy total domination graph. The family \( \mathcal{F} \) of \( \gamma_{gr}^t(G - u) \)-sets is a subfamily of the family \( \mathcal{F}' \) of \( \gamma_{gr}^t(G) \)-sets, where \( \mathcal{F}' \setminus \mathcal{F} \) consists of exactly those \( \gamma_{gr}^t(G) \)-sets that contain \( u \). Since in \( \mathcal{F}' \) every two sets are exchangeable by an automorphism, the same holds in \( \mathcal{F} \), which consists of \( \gamma_{gr}^t(G) \)-sets that do not contain \( u \). Hence \( G - u \) is an iso-unique Grundy total domination graph.

For the second statement of the proposition, when \( G \) is a forest, we only need to prove the reversed direction. In this case, an open twin \( u \) is necessarily
a leaf, adjacent to a vertex $w$, and let $U$ be the set of leaves adjacent to $w$ in $G$. Let $G - u$ be an iso-unique Grundy total domination graph, and let $S$ be an arbitrary $\gamma_{gr}^t(G - u)$-sequence. We claim that $w$ is $t$-footprinted with respect to $S$ by a leaf $v \in U$. Suppose that $w$ is $t$-footprinted with respect to $S$ by $z$, which is not a leaf. Then no vertex from $U$ belongs to $S$. Note that the sequence $S'$ obtained from $S$ by replacing $z$ with $v \in U$ is an open neighborhood sequence in $G - u$, hence $S'$ is a $\gamma_{gr}^t(G - u)$-sequence. This is a contradiction with $G - u$ being an iso-unique Grundy total domination graph, since $S'$ has more leaves than $S$ and so there is no automorphism of $G - u$ that maps $S$ to $S'$. We infer that $w$ is indeed $t$-footprinted by a leaf $v \in U$ in any Grundy total dominating sequence in $G - u$. Now, we claim that the same holds in $G$. Notably, if there exists a $\gamma_{gr}^t(G)$-sequence in which $w$ is footprinted by $z$, which is not a leaf, then the same sequence is an open neighborhood sequence in $G - u$, and so it is a $\gamma_{gr}^t(G - u)$-sequence, since $\gamma_{gr}^t(G) = \gamma_{gr}^t(G - u)$. This is a contradiction, which implies that $w$ is $t$-footprinted by a leaf in any $\gamma_{gr}^t(G)$-sequence in $G$. Let $S$ and $T$ be $\gamma_{gr}^t(G)$-sets. If any of them, say $S$, contains $u$, then $\phi_{u+v}$ maps $S$ to a $\gamma_{gr}^t(G)$-set $\phi_{u+v}(S)$, which is at the same time a $\gamma_{gr}^t(G - u)$-set. Since there exists an automorphism of $G - u$ that maps $\phi_{u+v}(S)$ to the $\gamma_{gr}^t(G - u)$-set $T$ (or $\phi_{u+v}(T)$, if $u$ is contained in $T$), there exists an automorphism of $G$ that maps $S$ to $T$. Hence $G$ is an iso-unique Grundy total domination graph.

The second statement of Proposition 21 does not necessarily hold if $G$ is not a forest. To see this consider the graph $H$, which is obtained from $C_5$ by adding an open twin to any vertex of the cycle. See Figure 1, where two copies of $H$ are depicted. Notice that two different $\gamma_{gr}^t(H)$-sets are depicted in these two copies of $H$, where vertices of a $\gamma_{gr}^t(H)$-set in each copy are black. It is thus clear that $H$ is not an iso-unique Grundy total domination graph. However, $H - u$ is isomorphic to $C_5$, and it is known \cite{13} Proposition 6.1 that $\gamma_{gr}^t(C_n) = n - 1$ if $n \geq 3$ is odd, and by symmetry of $C_n$ it follows that every odd cycle is an iso-unique Grundy total domination graph.

![Figure 1: Graph $H$ with two $\gamma_{gr}^t(H)$-sets marked by black vertices.](image)

From Proposition 21 we derive that when dealing with iso-unique Grundy total domination forests we may restrict our attention to forests with no open twins. In addition, from any such iso-unique Grundy total domination forest $T$ we can build infinite families of examples of iso-unique Grundy total domination forests by attaching an arbitrary number of leaves to support vertices. In the next result, we characterize all iso-unique Grundy total domination forests with
Theorem 22. If $T$ is a forest with no isolated vertices and no open twins, then $T$ is an iso-unique Grundy total domination graph if and only if $\gamma_{gr}^i(T) = n(T)$.

Proof. We start with an observation about bipartite graphs with partition $V(G) = A + B$. If $S$ is an open neighborhood sequence in $G$, then the two subsequences $S_A$ and $S_B$ of $S$, which are obtained by taking only vertices in $A$ (respectively, $B$) in the same order as they appear in $S$ are clearly both open neighborhood sequences. More importantly, the subsequences are independent of each other, and so $S_A \oplus S_B$ and $S_B \oplus S_A$ are also open neighborhood sequences (with the same length as $S$).

Now, let $T$ be an iso-unique Grundy total domination forest with no isolated vertices and no open twins, and let $S$ be a Grundy total dominating sequence of $T$. Assume that on the contrary, $\gamma_{gr}^i(T) < n(T)$, and let $v \in V(T)$ be a vertex, which is not in $S$. Given the bipartition $V(T) = A + B$, we may assume that $v \in B$. In addition, by the observation above, we may assume without loss of generality that $S = S_A \oplus S_B$, where $S_A$ (respectively $S_B$) is the subsequence of $S$ of the vertices in $S$ that belong to $A$ (respectively $B$). Let $S_B = (u_1, \ldots, u_k)$. Since $T$ has no isolated vertices, $v$ has at least one neighbor, and let $w$ be the neighbor of $v$, which is $t$-footprinted as the latest with respect to $S$ among all vertices in $N(v)$. Let $u_j$ be the $t$-footprinter of $w$. Clearly, $u_j \in B$. Now, it is easy to see that the sequence $S'$ obtained from $S$ by replacing $u_j$ with $v$ is also an open neighborhood sequence. Indeed, the initial segment $S_A \oplus (u_1, \ldots, u_{j-1})$ is the same in both sequences, $v$ footprints $w$, and since $\cup_{i=1}^{j-1} N(u_i) \cup \{v\} \subseteq \cup_{i=1}^{j-1} N(u_i)$, the remainder of $S'$ is also an open neighborhood sequence.

Let $C$ be the center of the component $T'$ of $T$, which contains $v$. Note that $C$ consists of either a vertex $c$ or two adjacent vertices $c$ and $c'$, and we consider the following measure for a set $U \subseteq V(T')$:

$$m(U) = \sum_{v \in U} \min\{d(v,c), d(v,c')\},$$

which represents the sum of distances of vertices in $U$ from the center. (In the case when $C = \{c\}$, the above calculation simplifies, but formally, we may let $c' = c$ and use the same formula.) It is clear that for every automorphism of $T'$, which maps a subset $U$ onto a subset $U'$, the equality $m(U) = m(U')$ holds. Therefore, since there is an automorphism that maps the $\gamma_{gr}^i(T)$-set $S$ to $S'$, we infer that $v$ and $u_j$ must be at the same distance from the center $C$, while their common neighbor $w$ is closer by 1 to the center than each of $v$ and $u_j$. It is also clear that $v$ and $u_j$ are not leaves, since $T$ has no open twins.

Now, consider the sequence $S$ again, and let $T_{vw}$ be the (sub)tree of $T'$, which coincides with the component of $T - vw$ that contains $v$. Similarly, let $T_{uw}$ be the (sub)tree that coincides with the component of $T - vw$, which contains $w$ (and contains also $u_j$). Let $S'_B$ be the subsequence of $S_B$ of those vertices that
belong to $T_{vw}$, and $S_B'$ be the subsequence of $S_B$ of those vertices that belong to $T_{vw}$. Note that $v$ does not belong to $S_B$ (as it does not belong to $S$), hence the subsequences $S_B'$ and $S_B''$ do not affect one another, because the distance between a vertex in one subsequence and a vertex in the other subsequence is at least 4. We thus infer that $S_A \oplus S_B' \oplus S_B''$ is also an open neighborhood sequence, which we denote by $S_1$. Clearly, $S_1$ is a Grundy total dominating sequence. However, the neighbor of $v$, which is $t$-footprinted as the latest with respect to $S_1$ among all vertices in $N(v)$, is a neighbor $z$ of $v$, which lies in $T_{vw}$ (so it is not $w$ as in $S$). Let $u_k \in S_B''$ be the vertex that footprints $z$. Clearly, $u_k \in V(T_{vw})$, and
\[ d(u_k, c) = d(v, c) + 2 = d(u_j, c) + 2. \]

Now, we replace $u_k$ with $v$ in $S_1$ and call the resulting sequence $S_2$. In the same way as earlier we derive that $S_2$ is also a Grundy total dominating sequence in $G$. However, due to the distances from the center of $v$ and of $u_k$, as shown above, we infer that $m(S_2) < m(S)$. This implies that there is no automorphism that maps $S$ onto $S_2$, which is a contradiction with $T$ being an iso-unique Grundy total domination graph. Hence, $\gamma_{gr}^t(T') = n(T)$.

The reverse direction of the statement of the theorem is trivial. □

By combining Proposition 21 and Theorem 22 we get a characterization of all iso-unique Grundy total domination forests. The result can be best described by the following algorithm for recognition of such forests.

**Algorithm Iso-Unique Grundy Total Domination Forest**

**Input.** A forest $T$.

**Output.** YES if $T$ is an iso-unique Grundy total domination forest tree, NO otherwise.

1. Let $T'$ be the forest obtained from $T$ such that for every strong support vertex $v$ of $T$, all but one leaf adjacent to $v$ are removed. (Note that $T'$ has no open twins.)

2. If $T'$ has a perfect matching, then RETURN YES, otherwise RETURN NO.

Note that a forest with no strong support vertices has no open twins. Hence, the forest $T'$ obtained in step (1) has no open twins. Hence, by Theorem 22 $T'$ is an iso-unique Grundy total domination graph if and only if $\gamma_{gr}^t(T') = n(T')$. By Theorem 20 restricted to forests, $\gamma_{gr}^t(T') = n(T)$ if and only if $T'$ has a perfect matching. Finally, by Proposition 21 $T$ is an iso-unique Grundy total domination forest if and only if $T'$ is an iso-unique Grundy total domination forest. This proves the correctness of the algorithm. Clearly, each of the steps (1) and (2) can be performed in linear time.

**Theorem 23.** Algorithm Iso-Unique Grundy Total domination Forest verifies in linear time whether a given forest is an iso-unique Grundy total domination forest.
6 Unique L-Grundy domination graphs

Recall that a vertex may L-footprint itself, in which case it has two different footprinters. This fact makes the proof of the next result slightly more involved than the proofs of similar results for other versions of Grundy domination.

**Proposition 24.** If $G$ is a graph and $x$ an arbitrary vertex of $G$, then there exists a $\gamma^L_{gr}(G)$-sequence that contains $x$.

**Proof.** Let $x$ be an arbitrary vertex of $G$ and let $S = (v_1, \ldots, v_k)$ be a $\gamma^L_{gr}(G)$-sequence. For any $i \in [k]$ we denote by $v_i'$ an arbitrary vertex that is L-footprinted by $v_i$. If $k = \gamma^L_{gr}(G) = n(G)$ or if $x \in S$, then the statement follows. Thus we may assume that $k \leq n(G) - 1$ and that $x \notin S$. Since $\hat{S}$ is a (total) dominating set, $x$ has at least one neighbor in $\hat{S}$. Let $v_i \in S$ be the L-footprinter of $x$, with respect to $S$. Since $x \notin \hat{S}$, $v_i \neq x$. If $N(x) \subseteq N(v_1) \cup \cdots \cup N(v_i)$, then $S' = (v_1, \ldots, v_{i-1}, x, v_i, \ldots, v_k)$ is an L-sequence, as $x$ L-footprints itself, $v_i$ L-footprints $x$, and for any $j > i$, $v_j$ L-footprints $v_j'$. Since $|S'| = k + 1 > \gamma^L_{gr}(G)$ we get a contradiction. Thus $N(x) \nsubseteq N(v_1) \cup \cdots \cup N(v_i)$. Let $v_j \in \{v_{i+1}, v_{i+2}, \ldots, v_k\}$ be the last vertex from $S$ that L-footprints a vertex from $N(x)$. Let $a \in N(x)$ be a vertex L-footprinted by $v_j$. First note that $a \neq v_j$. Indeed, if $a = v_j$, then $v_j$ footprints itself, and thus it is also footprinted by $v_j$ for $\ell > j$, which contradicts the choice of $j$. As $a \neq v_j$, $S' = (v_1, \ldots, v_{j-1}, x, v_{j+1}, \ldots, v_k)$ is an L-sequence ($x$ L-footprints $a$ and for any $\ell > j$, $v_\ell$ L-footprints $v_\ell'$). Since $x \in S'$ and $|S'| = \gamma^L_{gr}(G)$, the proof is complete. \qed

**Corollary 25.** If $G$ is a graph, and $S = \{S : S$ is a $\gamma^L_{gr}(G)$-sequence of $G\}$, then

$$\bigcup_{S \in \mathcal{S}} \hat{S} = V(G).$$

**Corollary 26.** A graph $G$ is a unique L-Grundy domination graph if and only if $\gamma^L_{gr}(G) = n(G)$.

There are several graph families that enjoy $\gamma^L_{gr}(G) = n(G)$. In particular, every forest $T$ with no isolated vertices enjoys $\gamma^L_{gr}(T) = n(T)$; see [9, Theorem 5.1]. A complete characterization of such graphs is yet to be found.

It is clear that a graph $G$ with $\gamma^L_{gr}(G) = n(G)$ is an iso-unique L-Grundy domination graph and hence all forests are iso-unique L-Grundy domination graphs. There are also (many) iso-unique L-Grundy domination graphs with $\gamma^L_{gr}(G) \leq n(G) - 1$. Some simple examples are complete graphs of order at least 3, cycles, and complete bipartite graphs $K_{m,n}$ with $m, n \geq 2$.

7 Concluding remarks

In this paper, we presented characterizations of graphs that have unique Grundy dominating sets for all four different types of Grundy domination. All charac-
Characterizations yield very special graphs. In the cases of unique Grundy total domination graphs and unique L-Grundy domination graphs these are the graphs in which the corresponding Grundy dominating set coincides with the vertex set of the graph (minus isolated vertices in the former case). While the former graphs have been characterized (Theorem 20), the structure of the latter graphs is largely unknown, hence we propose the following

Problem 27. Characterize the graphs $G$ with $\gamma_{\text{gr}}^L(G) = V(G)$.

When uniqueness condition is weakened by involvement of automorphisms, the situation is completely resolved for Grundy domination; notably the iso-unique Grundy domination graphs are precisely the graphs in which connected components are cliques (Theorem 10). In the other three cases, we could only give characterizations of forests that enjoy the iso-uniqueness condition. The case of iso-unique L-Grundy domination forests is in a sense trivial, since it is known that all forests $T$ enjoy $\gamma_{\text{gr}}^L(T) = V(T)$, which implies that all forests are iso-unique L-Grundy domination graphs. The cases of Grundy total domination and Z-Grundy domination are much more involved, but we could provide characterizations that yield efficient algorithms for the recognition of these two classes of forests. The natural open question remains whether one can extend the efficient recognition algorithms from iso-unique Grundy total domination forests, and iso-unique zero forcing forests, respectively, to larger classes of graphs. In addition, since the recognition of iso-unique Grundy domination graphs is polynomial, it would be interesting to see if the same holds for the other three iso-unique classes of graphs.

Problem 28. Is there a polynomial time algorithm to recognize the class of iso-unique Grundy total domination graphs, iso-unique L-Grundy domination graphs, or iso-unique zero forcing graphs, respectively?

If the answer to some of the above questions is negative (which we suspect), one could restrict it to special families of graphs that contain forests. In particular, what happens with the complexity of the above three problems in chordal graphs?

Acknowledgement

The authors acknowledge the financial support from the Slovenian Research Agency (research core funding No. P1-0297 and project grants J1-9109, J1-1693, N1-0095 and J1-2452).

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