Stochastic Optimal Control for Multivariable Dynamical Systems Using Expectation Maximization

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Abstract—Trajectory optimization is a fundamental stochastic optimal control (SOC) problem. This article deals with a trajectory optimization approach for dynamical systems subject to measurement noise that can be fitted into linear time-varying stochastic models. Exact/complete solutions to these kind of control problems have been deemed analytically intractable in literature because they come under the category of partially observable Markov decision processes (MDPs). Therefore, effective solutions with reasonable approximations are widely sought for. We propose a reformulation of stochastic control in a reinforcement learning setting. This type of formulation assimilates the benefits of conventional optimal control procedure, with the advantages of maximum likelihood approaches. Finally, an iterative trajectory optimization paradigm called as SOC—expectation maximization (SOC-EM) is put forth. This trajectory optimization procedure exhibits better performance in terms of reduction in cumulative cost-to-go which is proven both theoretically and empirically. Furthermore, we also provide novel theoretical work which is related to uniqueness of control parameter estimates. Analysis of the control covariance matrix is presented, which handles stochasticity through efficiently balancing exploration and exploitation.

Index Terms—Expectation maximization (EM), maximum likelihood, optimal control, reinforcement learning, stochastic systems, trajectory optimization.

I. INTRODUCTION

In recent years, there has been a surge in research activities related to inference and control of dynamical systems in not only the systems and control but also the artificial intelligence communities. The dynamical systems that make intelligent and optimal decisions under uncertainty have been formulated in the category of Markov decision process (MDP) [1]. The trajectory optimization problem aims at designing control policies to generate trajectories for an MDP that minimizes some measure of performance. More applications have been seen in a wide variety of industrial processes and robotics with the development of computers. Researchers used stochastic optimal control (SOC) methodologies (see [2], [3]) to present a solution to an MDP. A specific type of SOC, reinforcement learning, has exhibited great performance in handling control-related tasks in noisy environment, as well as generalizing the learned policies to new behaviors through experience [4]–[6].

Reinforcement learning is widely used for solving an MDP by optimizing an objective function through dynamic programming that involves value iteration and policy iteration (see [7], [8]). It can be broadly classified into model-free and model-based categories. For instance, in a model-based setup, reward weighted regression was used to learn complex robot motor motions in [9] and a variant of differential dynamic programming to learn advanced robotic manipulation policies [4]. Model-based policy search has been used in trajectory optimization [10], analytical policy gradients [11], information-theoretic approaches [12], etc. The typical methods include iterative linear quadratic Gaussian (iLQG) approach [13], model predictive control (MPC) [14], path integral linear quadratic regulator (LQR) [15], and Bregman alternating...
direction method of multipliers (BADMM) [16]. These methods are well-known in searching optimal parameters for a stochastic control policy using a quadratic cost-to-go function and a linearized dynamic model in a closed form (after certain approximation).

Especially for the trajectory optimization problem, sophisticated model-based techniques are easy to implement without suffering from slow convergence. One can refer to [17] and [18] for more results in this line of research. On the contrary, model-free methods may suffer from slower trajectory optimization because of reduced sampling efficiency. In addition, the research in [4] and [15] exploited adaptability of model-based methods to rapid changes in the environment, which is beneficial in dealing with uncertainty. These advantages motivate the research on new model-based reinforcement learning policies in this article.

Due to the similarity between policy search and inference problems using a reinforcement learning objective, increased interests have been seen among statistical researchers who treat optimal control as maximum likelihood inference. A powerful tool known as expectation maximization (EM) has been widely used for solving maximum likelihood problems in two steps, i.e., guess of the missing data called latent variables and estimation of parameters that best describes the guess. It is an iterative process with the probability of guess increased in each iteration. The maximum likelihood technique has gained wide popularity in a broad variety of fields of applied statistics such as signal processing and dairy science [19], [20]. The EM approach has been used for robust estimation of linear dynamical systems in [21] and identification of nonlinear state space models in [22]. However, there are rare EM-based results available in the field of control, which brings another motivation of this article to study an EM algorithm for SOC problems.

It is worth mentioning that a few early attempts at leveraging the concepts of maximum likelihood for solving an MDP can be found in model-free reinforcement learning. For example, the early work in [23] provides an evidence of using likelihoods and cost for solving inference problems. Probabilistic control and decision has been studied in [24]–[26] to tackle SOC problems using the maximum entropy principle. The EM technique has been used in inference for optimal policies to maximize cumulative sum of cost for model-based and model-free learning in [27] and [28], respectively. However, these works did not substantially analyze the theoretical nature of the solutions. Other related results include the concept of likelihood used for SOC design in a binary reward model-free setting [28], [29] and the exploitation of EM to weight the reward factors in robot control trajectories [30], [31]. Nevertheless, EM has attracted widespread attention in model-free domain but not specifically in the model-based domain.

The aforementioned discussion has opened up curtain for the main technical scope of this article, that is, the development of a novel EM-based SOC algorithm in a model-based domain. The main feature lies in its powerful capacity of exploitation of searched state space and attenuation of measurement and/or environmental noise. The aforementioned approaches, e.g., iLQG, MPC, and BADMM, can be problematic in the presence of noise, which propagates through the state equations to generate a highly stochastic policy. On top of this, these existing approaches carry out exploration which immensely aggravates this issue. It demands an effective exploitation step. Some other relevant maximum likelihood strategies (see [27], [32]) may shed light on model-based EM optimization but also suffer from similar disadvantage of handling noise.

Measurement noise in an MDP results in partially observable MDP. For instance, the approach in [33] addressed the optimal control problem for partially observable MDPs with a linear Gaussian transition model and a mixture of Gaussians reward model, but it requires the action space to be discretized. The EM-based optimal control proposed in [28] considers estimation of control covariance matrix, but it does not dig deep into the analysis of covariance matrix that quantifies the tradeoff between exploration and exploitation of state space in a reinforcement learning environment. Furthermore, the technique of belief space planning in [34] aims to transform the partially observable problem into a belief space problem and then it provides an optimal belief-LQR deterministic policy by taking a major step toward effectively handling uncertainty in the system. However, belief-LQR does not deal with stochastic policies, which restricts the exploration mechanism of a reinforcement learning framework.

Based on the above discussions about the state-of-the-art SOC methodologies for MDPs, it is a promising target of this article to use the advantages of model-based trajectory-centric optimization paradigms together with probabilistic-inference-based techniques, specifically, EM, to establish an optimal policy in the presence of measurement noise. For this purpose, the main contributions of this article are summarized as follows.

1) A complete architecture of the EM-based probabilistic inference algorithm is developed for obtaining stochastic optimal policy parameters.
2) The algorithm shows the benefits of integrating a model-based optimal control procedure with the advantage of maximum likelihood to deliver an iterative trajectory optimization paradigm, called SOC-EM.
3) It is theoretically proved that update of policy parameters in an EM iteration leads to reduction in cumulative cost-to-go for an SOC problem, resulting in (approximate) optimal policy parameters.
4) The unique property of the maximizer of a surrogate likelihood function is theoretically laid out which offers a practically feasible lower dimensional approximation for each EM iteration.
5) It is exhibited that EM-SOC offers efficient exploitation of the highly uncertain exploration state space, which is analytically quantified by the convergence of control policy covariance matrices to 0 and numerically verified by improved state trajectories with reduced stochasticity in control actions.
6) The effectiveness of EM-SOC in handling measurement noise is explicitly demonstrated.

II. PRELIMINARIES AND PROBLEM FORMULATION

This section introduces the dynamic model under investigation, the problem formulation, and the proposed solvability procedure. It also elaborates the mathematical notations involved with addressing the problem that will be put forth in this article. Readers can refer to the symbols summarized in Nomenclature.

A. Mathematical Notation and Modeling

This article takes into account a stochastic dynamics that does not have a known model from first principles, in the
presence of uncertainties such as parameter variation, external disturbance, and sensor noise. The completed system is considered to be a global model, $O$, that is composed of multiple local models $\phi^l$, $l = [1, 2, \ldots ]$, and each of which follows an MDP, called a local model. We are interested in a finite horizon optimal control for a particular initial state, rather than for all the possible initial states. The procedure to handle initial states in a global space is laid out in [35] using a learning framework called guided policy search.

The partially observable MDP has a latent state $s_k \in \mathbb{R}^{n_s}$ and a control action $a_k \in \mathbb{R}^{n_a}$, at time instant $k = 1, 2, \ldots$, and the local state transition dynamic model is represented by a conditional probability density function (p.d.f.), i.e.,

$$p(s_{k+1}|s_k, a_k). \quad (1)$$

In particular, for $k = 1$, $s_1 \in \mathbb{R}^{n_s}$ is called the initial state, obeying a specified distribution. Variables $n_s$ and $n_a$ are integers which are the dimensions of state and action space. We specifically consider a finite horizon MDP in this article for $k = 1, 2, \ldots, T$, called an episode, with the time instant $T$ being the end of episode. It is worth mentioning that the p.d.f. in (1) varies with time $k$ and the time-varying nature is capable of characterizing more complicated dynamical behaviors but also brings more challenges in design control. It will be elaborated in Section III.

The entity $Y_k(s_k, a_k) \in \mathbb{R}^+$ denotes the instantaneous real-valued cost for executing action $a_k$ at state $s_k$. It has a more specific expression as follows:

$$Y_k(s_k, a_k) = (s_k - s^*)^TQ_k(s_k - s^*) + (a_k - a^*)^TQ_a(a_k - a^*) \quad (2)$$

where $s^*$ and $a^*$ are the target state and control action, respectively, and $Q_k \succ 0$ and $Q_a \succ 0$ are some specified matrices. As $s_k$ and $a_k$ are random variables, $Y_k(s_k, a_k)$ (with $Y_k$ a continuous and deterministic function) is also a random variable, shorted as $Y_k$. We develop another variable, i.e.,

$$ ek = \text{vec}(F_k) \quad (3)$$

where $ek$ and $ek$ are the variables of the dynamic model p.d.f. (3) and the control action p.d.f. (4), respectively. The expectation is taken over the measurement states which are a result of instantiations of the noise in the dynamical equation with an initial state $s_1$. To express the cost penalty to be explicitly dependent on $ek$, we rewrite $Y_k(s_k, a_k)$ as $Y_k(s_k, \phi_k)$ with slight abuse of notation. Also, we can rewrite (6) in terms of the controller parameter $\phi$, i.e.,

$$\min_{\phi} \mathbb{E}_{\phi}(S_{T+1}), \quad \text{for } V_{\phi}(S_{T+1}) = \sum_{k=1}^{T} Y_k(s_k, \phi_k) \quad (7)$$

A complete solution to the optimization problem (6) is hardly analytically tractable [34], [36]. It is more realistic to pursue effective solutions with reasonable approximations as seen in numerous references including [10], [13], [17], and [25]. It is worth mentioning that existence of a stationary policy for a partially observable MDP is NP-complete [37], [38]. Even a solution to a finite horizon partially observable MDP is shown to be PSPACE-complete for discrete states, actions, and observations [36]. Therefore, in this article we propose a new problem formulation with a procedure of solution that can be regarded as a decent alternative to the problem (6). The procedure is elaborated below in a four-step architecture and also illustrated in Fig. 1.

**Step 1 (Dynamic Model Fitting):** From an initial state $s_1$ sampled from a specified distribution, the real system is operated with the controller (4) for a preselected controller parameter $\phi = \phi^0 = \phi^{(0)}, \phi^{(0)}, \ldots, \phi^{(0)}$, and the control actions $\{a_1, a_2, \ldots, a_T\}$ and the states $\{s_1, s_2, \ldots, s_{T+1}\}$ are recorded. Calculate $Y_k(s_k, \phi^0)$ and hence $ek = e^{-T}$. Then, the dynamic model (3) is identified by fitting it to the collected tuples of data $\{s_k, a_k, s_{k+1}, ek\}$, $k = 1, \ldots, T$.

**Step 2 (Generation of Cost Observation):** From an initial state $s_1$ sampled from a specified distribution, the cost observations $Y_T = \{y_1, y_2, y_3, \ldots, y_T\}$ are generated using the
dynamic model (3) (obtained from Step 1) and the controller (4) with the controller parameter $\phi = \phi^0$.

Step 3 (Optimization of Control Action): Let $p_\phi(S_{T+1}|Y_T)$ be the probability of the latent states $S_{T+1} = \{s_1, s_2, \ldots, s_{T+1}\}$ given the observation $Y_T$ (obtained from Step 2), obeying the closed-loop system composed of the dynamic model (3) (obtained from Step 1) and the controller (4) with a controller parameter $\phi$. The optimization of a local control policy is formulated as follows:

$$
\phi^* = \arg\min_\phi E_{p_\phi(S_{T+1}|Y_T)} V_\phi(S_{T+1}). \tag{8}
$$

Step 4 (Implementation and Evaluation): Run the real system with the controller (4) for the optimal parameter $\phi = \phi^*$ and evaluate the performance.

A practical approach to solve the optimization problem (8) is to use the following strategy:

$$
\widehat{\phi}^{i+1} = \arg\min_\phi E_{p_\phi(S_{T+1}|Y_T)} V_\phi(S_{T+1}) \tag{9}
$$

recursively with $\widehat{\phi}^i = \phi^*$, for $i = 0, 1, \ldots$ It is expected that $\widehat{\phi}^i$ approaches $\phi^*$ as $i$ goes to $\infty$.

Throughout this article, we use the simplified notation

$$
E_\phi(s|Y_T) \triangleq E_{p_\phi(S_{T+1}|Y_T)}(s) \tag{10}
$$

and (9) can be rewritten as

$$
\widehat{\phi}^* = \arg\min_\phi E_{\widehat{\phi}^i} \left( V_\phi(S_{T+1}) | Y_T \right). \tag{11}
$$

After each iteration $i$, one has an updated controller parameter $\widehat{\phi}^{i+1}$, and Steps 1 and 2 are repeated with $\phi = \widehat{\phi}^{i+1}$ for an updated dynamic model and updated cost observation.

Remark 1: In Steps 1 and 4, the real dynamical system is operated for data generation and performance evaluation, respectively. The state $s_k$ is physically measured, which represents the observed system state carrying measurement noise. However, in Steps 2 and 3, only theoretical computation is conducted without operating the real system, thus leveraging the latency nature of states. Here, $s_k$ represents the explored state obeying a joint probability $p_\phi(S_{T+1}|Y_T)$ conditioned on the observation $Y_T$ and parameterized with a given $\phi = \phi^i$ at each iteration. It is thus called a latent state. In both the cases, either the states carrying noise or the explored state samples are adopted. In other words, the real/true system states are not observed, with which the model (3) is treated as a partially observable MDP.

Remark 2: Sections IV and V are concerned about the optimization problem (11) which is regarded as the approximation of the original optimization problem (6). It is easy to see that (11) is equivalent to

$$
\min_{a_1, a_2, \ldots, a_T} \mathbb{E}_{p_{\mu}(S_{T+1}|Y_T)} \sum_{k=1}^T Y_k(s_k, a_k). \tag{12}
$$

It is expected that recursively solving the problem (11) or (12) will approach a solution to (6). However, the global convergence of the recursion is of great challenge and the effectiveness can only be numerically verified in this article. The gap between (6) and (12) is further discussed as follows. The optimization in (12) can be intuitively interpreted as finding a probability distribution of state trajectories whose samples contain lowest expected cost-to-go. One can refer to [10, Sec. 3.3] which describes a similar type of objective function for achieving trajectories of lowest cost. It can be a reasonable approximation of the real-time cost-to-go in (6).

In the remaining sections, we first elaborate Step 1, the dynamic model fitting procedure, in Section III. The main technical challenges in optimization of the control action (11), accounting for Steps 2 and 3, are addressed in Sections IV and V, in a novel systematic framework. Step 4 is discussed in Section VI.

### III. Dynamic Model Fitting

In this section, we elaborate the procedure formulated in Step 1 to attain linear time-varying parameter estimates of the dynamic model (3). Technically, we merge the procedure adopted in [4] with the existing variational Bayesian (VB) strategies for a finite mixture model that can be referred to in [39].

We first give a specific definition of $y_k$ as follows:

$$
y_k(s_k, a_k) = e^{-Y_k(s_k, a_k)}. \tag{13}
$$

Intuitively, $y_k(s_k, a_k) \in (0, 1]$ characterizes the likelihood of $(s_k, a_k)$ being near the optimal trajectory. When an action results in a less cost $Y_k(s_k, a_k)$, it implies a larger $y_k(s_k, a_k)$ representing a higher likelihood of being near the optimal trajectory. Such an exponential transformation has been proven successful in determining the probability of occurrence of an optimal event in optimal control (see [23], [32]).

As described in the aforementioned Step 1, we can run one experiment and collect the tuples $(s_k, a_k, s_{k+1}, a_{k+1}, y_k)$ for every episode $k = 1, \ldots, T$. In practice, the experiments can be repeated for $M$ times from the same initial conditions with a random seed value to gather sufficiently many samples, each of which is denoted by

$$
D^m = \{s_k, a_k, s_{k+1}, y_k\}_{m \text{th experiment}} \quad \text{for } m = 1, \ldots, M.
$$

Let $D_k = \{D^1, \ldots, D^M\}$ and $D = \{D_1, \ldots, D_T\}$.

Research in [4] suggests that using simple linear regression to fit the dataset $D$ requires a large amount of samples and may become problematic in high-dimensional scenarios. However, the linear Gaussian fitting approach has been proven to be effective in reducing sample complexity noting that the samples from a dynamical system in adjacent time steps are correlated. More specifically, it is assumed that the dataset $D$ is generated from a mixture of a finite number of Gaussian distributions with unknown parameters, to which one can fit a Gaussian mixture model. The procedure involves constructing normal-inverse Wishart distributions\(^1\) to act as prior

\(^1\)Wishart distribution is a generalization of gamma distribution to multiple dimension, and in statistics it is used in the context of the multivariate normal distribution where it is a conjugate prior to the precision matrix $\Sigma^{-1}$, where $\Sigma$ is the covariance matrix.
for means and covariances of Gaussian distributions involved in the mixture model. In addition, Dirichlet distributions are defined to be the prior on the weights of the Gaussian distributions which would explain the mixing proportions of Gaussians. Then, the iterative VB strategy is adopted to increase the likelihood of a joint variational distribution (see [39, Sec. 10.2]) to determine the parameters of the mixture model, i.e., the means, covariances, and weights of the Gaussians for a particular time instant \( k \). More specifically, the Gaussian distribution is of the form

\[ p(s_k, a_k, s_{k+1}, y_k) = \mathcal{N}(\mu_k, \Lambda_k) \]  

for the mean \( \mu_k \) and the covariance \( \Lambda_k \). The parameters \( \mu_k \) and \( \Lambda_k \) are the a posteriori estimates which are evaluated by a Bayesian update rule with the information of the dataset \( \mathcal{D} \) and normal inverse Wishart prior.

The Gaussian distribution (14) can then be conditioned on states and action, i.e., \( (s_k, a_k) \), using standard identities of multivariate Gaussians, which result in (3) for the following parameters:

\[
A_k' = \begin{bmatrix} A_k^d & B_k^d \\ \Lambda_k & \Lambda_k \\ \end{bmatrix}, \quad \Sigma_k'' = \begin{bmatrix} \Sigma_k^d & \Sigma_k^r \\ \Sigma_k^r & \Sigma_k^r \\ \end{bmatrix}.
\]

The dimensions of the matrices are \( A_k^d \in \mathbb{R}^{n \times n} \), \( B_k^d \in \mathbb{R}^{n \times n} \), \( \Lambda_k \in \mathbb{R}^{n \times n} \), \( \Sigma_k^d \in \mathbb{R}^{1 \times n} \), \( B_k' \in \mathbb{R}^{1 \times n} \), \( \Sigma_k^r \in \mathbb{R}^{n \times n} \), and \( \Sigma_k'^r \in \mathbb{R}^{n \times n} \). In the dynamic model (3), the term \( \Sigma_k^d \) denotes the correlation between \( s_k \) and \( y_k \). Without loss of generality, we assume that \( \Sigma_k^d \neq 0 \). Note that one can also consider \( \Sigma_k^d = 0 \) and use methods of de-correlation to carry out the entire procedure in a similar way. It is assumed that the covariance matrices are symmetric positive definite, that is, \( \Sigma_k^d > 0, \Sigma_k^r > 0, \Sigma_k'^r > 0 \) throughout this article.

We consider the dynamic model (3) for the episode \( k = 1, \ldots, T \), assuming the initial time \( k = 1 \). This kind of modeling resembles with preexisting studies in [4], [5], [13], [14], [17], and [18]. It is noted that shifting the model (3) by \( k_0 \geq 0 \) gives a model as follows, in the new episode \( k = k_0 + 1, \ldots, k_0 + T \):

\[
p\left(\begin{bmatrix} s_{k+1} \\ y_k \end{bmatrix} \mid s_k, a_k\right) = \mathcal{N}\left(\begin{bmatrix} A_k'_{k_0} s_k \\ a_k \end{bmatrix}, \Sigma_k''_{k_0}\right).
\]  

Therefore, the time-varying feature of the linear Gaussian model (3) is not absolute but relative. By relatively time-varying, we mean that the dynamical system parameters \( A_k^d \) and \( \Sigma_k^d \), in (15) do not depend on the absolute time \( k \), but on the relative time interval \( k - k_0 \). In other words, the model is independent of the initial time \( k_0 \). The time-varying nature of the model (3) is capable of characterizing the complicated dynamical behaviors studied in this article by accurately capturing the nonlinearity in a piecewise linear Gaussian manner. On the contrary, a time-invariant model with a unique Gaussian distribution in (3) for all \( k \) could be oversimplified, inaccurate, and would definitely not describe a complicated model. Nevertheless, it is possible to fit only a relatively time-varying model to the collected data by running multiple experiments at different time instants.

IV. OPTIMIZATION OF CONTROL ACTION VIA EM

This section starts with some concepts used in the well-acknowledged EM algorithm. Basically, EM computes the maximum likelihood estimate of some parameter vector \( \phi \) (whose design is at the discretion of the user), say \( \hat{\phi}_\text{EM} \) based on an observed dataset \( \mathcal{Y}_T \). In particular, the likelihood of observing the data \( \mathcal{Y}_T \) written as \( p_\phi(\mathcal{Y}_T) \) does not decrease in an iterative manner, i.e.,

\[
\phi_\text{EM} \in \left\{ \phi \in \Phi : p_\phi(\mathcal{Y}_T) \geq p_\hat{\phi}(\mathcal{Y}_T) \right\}
\]  

(16) where \( \hat{\phi} \) is a (known) considerably good parameter estimate with which the EM approach is initialized (at the iteration labeled \( i \)).

The EM algorithm involves the observation log-likelihood

\[
L_\phi(\mathcal{Y}_T) \triangleq \log p_\phi(\mathcal{Y}_T)
\]  

(17) and an essential approximation of log of mixture likelihood of some latent variables (\( \mathcal{S}_{T+1} \)) and the observations (\( \mathcal{Y}_T \)) with a surrogate function \( \mathcal{L}(\phi, \hat{\phi}) \) defined in the following equation:

\[
\mathcal{L}(\phi, \hat{\phi}) \triangleq \mathbb{E}_\hat{\phi}\left(\log p_\phi(\mathcal{S}_{T+1}, \mathcal{Y}_T) \mid \mathcal{Y}_T\right).
\]  

(18) It is assumed that both \( L_\phi(\mathcal{Y}_T) \) and \( \mathcal{L}(\phi, \hat{\phi}) \) are differentiable in \( \phi \in \Phi \). Some lemmas used for the EM algorithm are given in the Appendix.

Next, we aim to propose an EM-based method for solving the optimal control problem (11) associated with the dynamic model (3) and the controller (4), as formulated in the aforementioned Step 3. To bridge the relationship between the optimal control problem and the EM algorithm that is originally used for maximizing the likelihood of observed data, we first recall the observation \( \mathcal{Y}_T \) in Step 2. Let \( \mathcal{S}_{T+1} \) be the latent states whose probability is denoted as \( p_\phi(\mathcal{S}_{T+1} \mid \mathcal{Y}_T) \), given the observation \( \mathcal{Y}_T \), obeying the closed-loop system composed of the dynamic model (3) and the policy (4) with a parameter \( \phi \). More specifically, one has

\[
p_\phi(\mathcal{S}_{T+1}, \mathcal{Y}_T) = p(s_1) \prod_{k=1}^{T} p_\phi(s_{k+1} \mid s_k, y_k, a_k).
\]  

(19) Hence, we can define \( L_\phi(\mathcal{Y}_T) \) and \( \mathcal{L}(\phi, \hat{\phi}) \) as in (17) and (18).

In the conventional EM, it has been revealed (see Lemma A.2) that in a recursive procedure, a new parameter \( \phi = \hat{\phi}^{i+1} \) that increases \( \mathcal{L}(\phi, \hat{\phi}) \) from \( \phi = \hat{\phi}^{i} \) also increases \( L_\phi(\mathcal{Y}_T) \). We aim to further prove that the new parameter \( \phi = \hat{\phi}^{i+1} \) also decreases \( \mathbb{E}_\phi\left(V_\phi(\mathcal{S}_{T+1} \mid \mathcal{Y}_T)\right) \) in (11), thus bridging the EM algorithm and the optimal control objective. It can be simply stated that the EM algorithm for finding \( \hat{\phi}^{i*} \) in (70) with \( \hat{\phi}^{i+1} = \hat{\phi}^{i*} \) also works for (11).

The theorem to be established in this section is based on the following assumption for the distribution of \( Y_k \).

**Assumption 1:** The p.d.f. of \( Y_k \) follows an exponential distribution with parameter \( \lambda \), i.e.,

\[
p(Y_k) = \lambda e^{-\lambda Y_k}, \quad \text{where } \lambda > 1.
\]  

(20) Remark 3: The above assumption is practically reasonable for the following two reasons. First, both \( s_k \) and \( a_k \) follow a Gaussian distribution in Section III, and therefore \( Y_k \) follows a linear combination of independent noncentral chi-squared variables with some degrees of freedom. Solving for a p.d.f. of \( Y_k \) is complicated (see [40, Appendix A.1]). All these distributions are related to a general exponential family, it is reasonable to assume that \( Y_k \) also follows an exponential distribution. Second, the justification for using an exponential distribution can also be found in relevant work. For example,
it is assumed that rewards (negative costs) are drawn from an exponential distribution in [29], and a so-called exponentiated payoff distribution is used in [41] as a link between maximum likelihood and an optimal control objective.

Then, we can give the following lemma regarding the distribution property of $Y_k$ defined in (13), which is of sole importance for establishing a theoretical relationship between the mixture likelihood function and the SOC objective.

**Lemma 1:** For $Y_k$ of the p.d.f. (20), the random variable $y_k$ in (13) has a p.d.f. of the form

$$p(y_k) = \lambda y_k^{\lambda-1}, \quad (21)$$

**Proof:** The random variable of $y_k$ has the following cumulative distribution function:

$$F_{y_k}(x) = P(y_k < x) = P(e^{-Y_k} < x) = P(Y_k > -\log(x)).$$

Further calculation implies

$$F(x) = \int_{-\log(x)}^{\infty} p(Y_k) dY_k = \int_{-\log(x)}^{\infty} \lambda e^{-\lambda y} dy_k = e^{-\lambda \log x} + \lambda x^{\lambda-1}.$$

Thus, differentiating $F(x)$ with respect to $x$ gives the p.d.f of $y_k$ as $p(x) = dF(x)/dx = \lambda x^{\lambda-1}$, which is simply denoted as (21).

Now, the main result is stated in the following theorem. Recall that $Y_k$ can be explicitly expressed by $Y_k(s_k, \phi_k)$, and accordingly, $y_k$ by $y_k(s_k, \phi_k)$, which is used in the proof of the theorem.

**Theorem 1:** Suppose the parameter $\hat{\phi}^{i+1}$ is produced such that

$$L(\hat{\phi}^{i+1}, \hat{\phi}^i) \geq L(\hat{\phi}^i, \hat{\phi}^i). \quad (22)$$

Then, the cumulative sum of expected costs defined in (11) satisfies

$$E_{\hat{\phi}}(V_{\hat{\phi}^{i+1}}(S_{T+1})|Y_T) \leq E_{\hat{\phi}}(V_{\hat{\phi}^i}(S_{T+1})|Y_T). \quad (23)$$

**Proof:** First, by Lemma A.2, (22) implies

$$L_{\hat{\phi}^{i+1}}(Y_T) - L_{\hat{\phi}^i}(Y_T) \geq 0. \quad (24)$$

Denote $\hat{\phi}^i = \{[\hat{\phi}^i_1]^T, \ldots, [\hat{\phi}^i_T]^T\}$ for $i = 0, 1, \ldots, T$. One has

$$L_{\hat{\phi}^i}(Y_T) = \log p_{\hat{\phi}^i}(Y_T) = \mathbb{E}_{p_{\hat{\phi}^i}(S_T+1|Y_T)} \left[ \log p_{\hat{\phi}^i}(Y_T) \right] = \sum_{k=1}^{T} \mathbb{E}_{p_{\hat{\phi}^i}(S_T+1|Y_T)} \left[ \log p_{\hat{\phi}^i}(y_k) \right].$$

With $p_{\hat{\phi}^i}(y_k) = p(y_k(s_k, \phi_k^i))$, the above calculation continues as follows, using the results of Lemma 1:

$$L_{\hat{\phi}^i}(Y_T) = \mathbb{E}_{p_{\hat{\phi}^i}(S_T+1|Y_T)} \left[ \sum_{k=1}^{T} \log p(y_k(s_k, \phi_k^i)) \right] = \mathbb{E}_{p_{\hat{\phi}^i}(S_T+1|Y_T)} \left[ \sum_{k=1}^{T} \log \lambda y_k(s_k, \phi_k^i)^{\lambda-1} \right] = \mathbb{E}_{p_{\hat{\phi}^i}(S_T+1|Y_T)} \left[ \sum_{k=1}^{T} (\lambda - 1)(-Y_k(s_k, \phi_k^i)) + T \log \lambda \right] = - (\lambda - 1) \mathbb{E}_{p_{\hat{\phi}^i}(S_T+1|Y_T)} \left[ \sum_{k=1}^{T} Y_k(s_k, \phi_k^i) \right] + T \log \lambda.$$  

Next, from (7), i.e., $V_{\hat{\phi}}(S_{T+i+1}) = \sum_{k=1}^{T} Y_k(s_k, \phi_k^i)$, one has

$$L_{\hat{\phi}}(Y_T) = - (\lambda - 1) \mathbb{E}_{p_{\hat{\phi}}(S_T+1|Y_T)} \left[ V_{\hat{\phi}}(S_{T+i+1}) | Y_T \right] + T \log \lambda. \quad (25)$$

As a result

$$0 \leq L_{\hat{\phi}^{i+1}}(Y_T) - L_{\hat{\phi}^i}(Y_T) = - (\lambda - 1) \left[ \mathbb{E}_{p_{\hat{\phi}}} \left[ V_{\hat{\phi}^{i+1}}(S_{T+i+1}) | Y_T \right] - \mathbb{E}_{p_{\hat{\phi}^i}} \left[ V_{\hat{\phi}^i}(S_{T+i+1}) | Y_T \right] \right].$$

It implies (23) and completes the proof.

**Remark 4:** Theorem 1 takes into account the exponential transformation according to (13) and (20) to ensure the decrease in the expected cost-to-go with increased likelihood. It suggests an effective approximate approach for the minimization objective in optimal control through pursuing the maximum likelihood objective. This approach is intuitively consistent with some results in literature. For example, the research in [32] claimed that maximum-likelihood-based inference is an approximation of the iLQG-based solution to an SOC problem. As an approximate class of inference-based techniques, a maximum likelihood method was also used in [3]. Some similar approximate relationship between a reward proportional likelihood objective and a policy gradient objective function was revealed in inference-based policy search [28].

V. PRACTICAL SOLUTION TO SOC-EM

After having established the relationship between EM and optimal control, this article proceeds toward a closed-form solution of arg max$_{\phi}$ $L(\phi, \hat{\phi})$. The first step is to deliver an explicit expression of the mixture likelihood associated with the dynamic model (3) and the controller (4).

A. Explicit Expression of Mixture Likelihood

The explicit expression of the mixture likelihood $L(\phi, \hat{\phi})$ defined in (10) is given in the following lemma.

**Lemma 2:** The function $L(\phi, \hat{\phi})$ for the dynamic model (3) and the controller (4) can be expressed as follows:

$$L(\phi, \hat{\phi}) = \log p(s_1) + \sum_{k=1}^{T} \Delta_k(s_k, \phi_k, \hat{\phi}^i) \quad (26)$$

for

$$\log p(s_1) = - \frac{1}{2} \log |P_1| + (s_1 - \mu_1) \top P_1^{-1}(s_1 - \mu_1)$$

$$\Delta_k(s_k, \phi_k, \hat{\phi}^i) = - \frac{1}{2} \text{Tr} \left\{ \Sigma_k^{-1} \left( \Theta_1(s_k) - \Theta_2(s_k)A_k \top \right) \right\} - A_k^\top \Theta_3(s_k) A_k + \Theta_2(s_k)^\top A_k - \Theta_2(s_k)^\top \Sigma_k^{-1} \Theta_3(s_k) A_k^\top$$

$$- \frac{1}{2} \log |\Sigma_k|$$

where the initial state $s_1$ follows a Gaussian distribution with known mean $\mu_1$ and covariance $P_1$, and the other terms are defined by:

$$\Theta_1(s_k) = \Phi(s_k) \hat{\phi}^i \top | Y_k \} \quad (28)$$

$$\Theta_2(s_k) = \Phi(s_k) \Sigma_k^{-1} \top | Y_k \} \quad (29)$$

$$\Theta_3(s_k) = \Phi(s_k) \Sigma_k^{-1} \top | Y_k \} \quad (30)$$

for $\Phi_k = \text{col}(s_{k+1}, y_k)$ and $z_k = \text{col}(s_k, a_k)$.

N.B. The terms $\Theta_1, \Theta_2, \Theta_3$ depend on $\phi_k$ due to (4).

**Proof:** To begin with, the application of Bayes’ rule and leveraging the time-varying dynamic model (3), one can
express the mixture likelihood function as follows:

\[
\mathcal{L}(\phi, \phi') = \mathbb{E}_{p(x_t \mid Y_T)} \left( \log p(x_t \mid Y_T) \right) = \mathbb{E}_{p(x_t \mid Y_T)} \log \left( \sum_{k=1}^{T} p(x_{t+k} \mid y_k \mid s_k) \right) = \log \sum_{k=1}^{T} \mathbb{E}_{p(x_t \mid Y_T)} \log p(x_{t+k} \mid y_k \mid s_k)
\]

which is (26) with

\[
\mathcal{L}(\phi_k, \phi'_k) = \mathbb{E}_{p(x_t \mid Y_T)} \log p(x_{t+k} \mid y_k \mid s_k).
\]

The expression of \( \log p(s_t) \) given in (27) is straightforward using the log of Gaussian p.d.f. of the initial state \( s_1 \). Again, using the log of a Gaussian p.d.f. in (32) gives

\[-2 \mathcal{L}(\phi, \phi') = \log \mathbf{\Sigma}_k + \mathbb{E}_{p(x_t \mid Y_T)} \times \left( \left( \begin{array}{c} s_{t+1} \\ y_k \end{array} \right) - \mathbf{A}_k' s_k \right) \mathbf{\Sigma}_k^{-1} \left( \left( \begin{array}{c} s_{t+1} \\ y_k \end{array} \right) - \mathbf{A}_k' s_k \right) \right] + \log |\mathbf{\Sigma}_k|\]

which matches the expression given in (27). The lemma is thus proved.

More specifically, the terms \( \Theta_1(\phi_k), \Theta_2(\phi_k), \) and \( \Theta_3(\phi_k) \) can be derived in a straightforward manner. One can refer to Appendix A for more explicit details. It is noted that they are composed of elements which can be evaluated from

\[
\mathbb{E}_{\phi}(s_t \mid Y_T), \quad \mathbb{E}_{\phi}(s_{t+k} \mid s_k \mid Y_T), \quad \mathbb{E}_{\phi}(s_{t+k} \mid y_k \mid Y_T).
\]

To evaluate the above-mentioned terms, one can take advantage of the time-varying linear Kalman filter and Rauch–Tung–Striebel smoother components that are introduced below. Readers are referred to [42, pp. 201–217] for more details about the procedure. However, one cannot use the standard version of filtering and smoothing because in our case where the control action has Gaussian noise, one has to augment the state-space filter to incorporate the covariance of the control action as well. Specifically, the Kalman filter equations after augmentation are shown below where the inputs are \( y_k, x_k, e_k, \) and \( s_k, \) for \( k = 1, 2, \ldots, T, \) noting the definitions of \( y_T \) and \( \phi. \) At each iteration, it is implemented with \( \phi = \phi'. \)

For \( k = 1, 2, \ldots, T, \) the time-varying Kalman filter equations (with initialization \( s_{1|1} = s_1 \) and \( P_{1|1} = P_1 \)) are

\begin{align*}
\hat{s}_{k+1|1} &= \tilde{A}_k' s_k + B_k' e_k \\
\hat{P}_{k+1|1} &= \tilde{A}_k' \hat{P}_{k+1|k} \tilde{A}_k' + \tilde{\Sigma}_k' \\
\hat{K}_{k+1} &= \hat{P}_{k+1|k} \left( \tilde{A}_k' \hat{P}_{k+1|k} \tilde{A}_k' + \tilde{\Sigma}_k' \right)^{-1} \\
\hat{P}_{k+1|k+1} &= \hat{P}_{k+1|k} - \hat{K}_{k+1} \tilde{A}_k' \hat{P}_{k+1|k} \tilde{A}_k' \\
\hat{s}_{k+1|k+1} &= \hat{s}_{k+1|k} + \hat{K}_{k+1} (y_k - \tilde{A}_k' \hat{s}_{k+1|k} - B_k' e_k)
\end{align*}

where

\begin{align*}
\tilde{A}_k &= A_k ' + B_k' F_k \\
\tilde{\Sigma}_k' &= B_k' \Sigma_k B_k'^T + \Sigma_k' \\
\hat{A}_k' &= A_k' + B_k' F_k \\
\hat{\Sigma}_k' &= B_k' \Sigma_k B_k'^T + \Sigma_k'.
\end{align*}

The time-varying recursive smoother equations are given as follows, for \( k = T, T-1, \ldots, 1, \) with \( \hat{s}_{T+1|T} = \hat{s}_{T+1|T} \) and \( \hat{P}_{T+1|T} = \hat{P}_{T+1|T} \):

\[
\begin{align*}
J_k &= \hat{P}_{kk} \tilde{A}_k' \left( \hat{P}_{k+1|k} \right)^{-1} \\
\hat{s}_{k+1|T} &= \hat{s}_{k+1|k} + J_k (\hat{s}_{k+1|T} - \hat{s}_{k+1|k}) \\
\hat{P}_{k+1|T} &= \hat{P}_{k+1|k} + J_k (\hat{P}_{k+1|k} - \hat{P}_{k+1|k}) J_k^T.
\end{align*}
\]

One can calculate the one-lag smoothed term \( \tilde{M}_{k+1|T} \) backward with the initialization and the iteration with \( k = T; \)

\[
\tilde{M}_{k+1|T} = \left( I - \hat{K}_{T+1|T} \hat{A}_T' \right) \hat{A}_T' \hat{P}_{T+1|T}
\]

After evaluating the filtered, smoothed estimates of states, the error covariance matrices and one-lag covariance matrices for all time steps, one can evaluate the terms of (33) as follows:

\[
\begin{align*}
\mathbb{E}_{\phi}(s_t \mid Y_T) &= \tilde{s}_1 \\
\mathbb{E}_{\phi}(s_{t+k} \mid s_k \mid Y_T) &= \tilde{s}_{k+1|T} \tilde{s}_1 T + \hat{P}_{t+1|T} \triangleq \mathcal{G}_{k} \\
\mathbb{E}_{\phi}(s_{t+k} \mid y_k \mid Y_T) &= \tilde{s}_{k+1|T} \tilde{s}_1 T + \tilde{M}_{k+1|T} \triangleq \mathcal{M}_{k+1|T}.
\end{align*}
\]

It is noted that \( \hat{P}_{k+1|T} > 0 \) (see [21, Lemma C.4]).

B. Practical Algorithm for Maximization of Mixture Likelihood

The attention is now turned toward maximization of the mixture likelihood \( \mathcal{L}(\phi, \phi') \), called the M-step in the EM architecture. The proposed optimization paradigm seeks a better policy parameter \( \phi = \phi^* \) for the next iteration than \( \phi = \phi' \) in the sense of maximizing (or increasing) \( \mathcal{L}(\phi, \phi') \). From Lemma 2, one has

\[
\phi^* = \arg \max_{\phi} \mathcal{L}(\phi, \phi') = \arg \max_{\phi} \sum_{k=1}^{T} \mathcal{L}(\phi_k, \phi'_k).
\]

So, it is ideal to select \( \phi^* = \phi' \). However, it is typically difficult to compute the optimal \( \phi^* \) over the entire sequence of control action \( \{a_1, a_2, \ldots, a_T\} \). The principle of EM as optimal control reduces the maximization of \( \mathcal{L}(\phi, \phi') \) over the entire sequence of control action for each time step. In other words, one tends to maximize \( \mathcal{L}(\phi, \phi') \) for each time step according to the iterative procedure below. For \( j = 1, \ldots, T, \) we solve the local optimization problem recursively

\[
\phi^*_{j+1} = \arg \max_{\phi} \sum_{k=1}^{T} \mathcal{L}(\phi_k, \phi'_k)
\]

\[
\phi^*_{j+1} = \arg \max_{\phi} \sum_{k=1}^{T} \mathcal{L}(\phi_k, \phi'_j, \phi'_j, \ldots, \phi'_j).
\]

Then, a better policy parameter for the next iteration is selected as \( \phi^*_{j+1} = \text{col}(\phi^*_{j+1}, \ldots, \phi^*_{j+1}) \approx \phi^* \). Obviously, the dimension of the optimization problem of \( \phi_j \) in (38), for \( j = 1, \ldots, T, \) is significantly lower than that for \( \phi \) in (37). For brevity, we would refer the above optimization problem as SO2-EM I in the subsequent part of this article.

The optimal controller parameters of [4] and [13] are dependent on time instant and as we use a similar framework, we also exploit the time-dependent nature of control law.
While implementing our methodology into practice the optimization routine, (38) suffers from intensive nature of computational costs. Therefore, we try to increase the speed of parameter search by converting the mixture likelihood into a surrogate function which is mathematically cheaper and tractable to evaluate. The modified optimization for each optimization instance is

$$\hat{\phi}_j^* = \arg \max_{\phi_j} \sum_{k=1}^{T} \mathcal{L}_k(\phi_j, \hat{\phi}_j), \quad j = 1, \ldots, T$$  \hspace{1cm} (39)

for searching $\phi_j$ in a neighborhood of $\hat{\phi}_j$. Then, a policy parameter for the next iteration is selected as $\hat{\phi}_j^{i+1} = \text{col}(\hat{\phi}_1^*, \ldots, \hat{\phi}_j^*) \approx \hat{\phi}_j^{i*}$. We would refer to this routine as SOC-EM II in the sequel.

Remark 5: The optimization problem (37) is in the parameter space of a very high dimension of $T(n_n + n_a^2 + n_a)$, which motivates the practical approximation (38) and (39) that are carried out with respect to an individual time step rather than all $T$ steps in one go. After convergence of the optimization routine in one time step, the result is used in the immediately next one. The approximation of SOC-EM I and SOC-EM II is a heuristic approach to deal with heavy computational expense. There is a tradeoff that undoubtedly needs to be made between the complexity of the algorithm and the quality of the solutions. It has been tested on extensive experiments that SOC-EM I and SOC-EM II have similar performance, and both of them demonstrate satisfactory performance in terms of the metrics described later in Section VI. It is worth mentioning that another advantage of the approximation (39) is that it allows parallel computation with multiple processing units, which tremendously reduces the computational time.

It is worth mentioning that initializing EM with a considerably good parameter vector $\hat{\phi}$ is critical. For example, theoretical studies by [43] and [44] revealed that the convergence of the EM algorithm is highly dependent on the parameters with which it is initialized. To address the initialization issue, we use the parameters of the well-established trajectory optimization strategies. In this article, we specifically use differential dynamic-programming-based optimal control methods such as: 1) ILQG [13]; 2) MPC [14]; and 3) BADMM [5] to carry forward the optimization routine. These three techniques will be called as the “baselines” in the sequel.

Now, the overall SOC-EM algorithm is summarized in Algorithm 1. The number of recursion in the algorithm is specified a priori. It is noted that every recursion starts with line 2 for model improvement with a new controller parameter. In practice, after a certain amount of recursions, there is no more significant model improvement, and therefore “go to 2” can be replaced by “go to 3” to skip line 2. While carrying out optimization, $(\hat{\Sigma}_k^{i*})$, $j = 1, 2, \ldots, T$, the covariance matrix component of $\hat{\phi}_j^{i*}$, might (or certainly) loose its positive definiteness property, so to preserve it, we adopt the approach originally proposed by Kaith et al. [45]. In this strategy, instead of propagating $\Sigma_k$ one propagates its square root, $\hat{\Sigma}_k^{1/2}$, i.e., $\hat{\Sigma}_k = (\hat{\Sigma}_k^{1/2})^T \hat{\Sigma}_k^{1/2}$, by carrying out a Cholesky decomposition before optimization. To save computational time, we also propagated the square roots of the filtered $\hat{P}_{k+1|k}$, $\hat{P}_{k+1|k+1}$, and smoothed $\hat{P}_{k\|T}$.

C. Uniqueness of Controller Parameter Estimation

The two theorems in this section exploit the closed-form nature of the gradient and the Hessian of the mixture log-likelihood to deliver a theoretical proof of the uniqueness of solution to the optimization problem (39), i.e., SOC-EM II. It is easy to verify that the theorems still hold with $\hat{\phi}_j$ in (40) and (48) replaced by $\hat{\phi}_j^{i*}$. In this strategy, $\hat{\phi}_j^{i*}$, $j = 1, \ldots, T - 1$ and hence guarantee the uniqueness of solution to the optimization problem (38), SOC-EM I.

Theorem 2: For the function $\mathcal{L}_k(\phi_k, \hat{\phi}_j)$ defined in (27), the following equation:

$$\nabla_{\phi_j} \left\{ \sum_{k=1}^{T} \mathcal{L}_k(\phi_j, \hat{\phi}_j) \right\} = 0, \quad j = 1, \ldots, T$$  \hspace{1cm} (40)

has a unique solution for any given parameter $\hat{\phi}_j$.

Proof: Recall that the function $\mathcal{L}_k(\phi_k, \hat{\phi}_j)$ defined in (27) is expressed in terms of $\Theta(\phi_k), \Theta_2(\phi_k), \Theta_3(\phi_k)$ in (28)-(30). The terms are composed of

$$E_{\phi}(y_k | y_T), \quad E_{\phi}(a_i a_i^T | y_T), \quad E_{\phi}(y_k a_i^T | y_T)$$

where $a_i$ explicitly depends on $\phi_k$. From the detailed expression given in Appendix B, one has

$$\nabla_{\phi_j} \left\{ \sum_{k=1}^{T} \mathcal{L}_k(\phi_j, \phi_j) \right\} = \sum_{k=1}^{T} \nabla_{\phi_j} \text{Tr} \lambda_k + \Sigma^T$$  \hspace{1cm} (41)

where

$$\lambda_k = \Sigma_k^{-1} B_k^T E_{\phi}(a_i a_i^T | y_T) B_k^T$$  \hspace{1cm} (42)

and $\Sigma$ represents some constant column vector, independent of $\phi_j = \text{col}(f_i, e_j, \sigma_j)$. In particular, one has $\lambda_{1,k} = \lambda_{1,k} + \lambda_{2,k} + \lambda_{3,k} + \lambda_{4,k} + \lambda_{5,k}$ with

$$\lambda_{1,k} = (\Sigma_k)^{-1} B_k^T F_{i} F_{i}^T B_k$$

$$\lambda_{2,k} = (\Sigma_k)^{-1} B_k^T e_j e_j^T B_k$$

$$\lambda_{3,k} = (\Sigma_k)^{-1} B_k^T a_i a_i^T B_k$$

$$\lambda_{4,k} = (\Sigma_k)^{-1} B_k^T F_{i} \hat{S}_{i\|T} e_j B_k^T$$

$$\lambda_{5,k} = (\Sigma_k)^{-1} B_k^T e_j \hat{S}_{i\|T}^T F_{i} B_k^T$$.
Also, $\mathcal{O}$ is of the special structure

$$
\mathcal{O} = \begin{bmatrix}
\sum_{k=1}^{T} \vartheta_{1,k} \\
\sum_{k=1}^{T} \vartheta_{2,k} \\
0
\end{bmatrix}
$$

(43)

with the dimensions of $\vartheta_{1,k}$, $\vartheta_{2,k}$, and $0$ corresponding to those of $f_j$, $e_j$, and $\varphi_j$, respectively. The explicit expression of $\vartheta_{1,k}$ and $\vartheta_{2,k}$ can be obtained from the equations in Appendix B.

Below, we calculate the derivative of the terms in (41) with respect to $f_j$, $e_j$, and $\varphi_j$, respectively.

First, with respect to $f_j$, one has

$$
\nabla_{f_j} \text{Tr} \lambda_{1,k} = f_j^T \mathcal{Z}_k, \quad \nabla_{f_j} \text{Tr} \lambda_{2,k} = 0, \quad \nabla_{f_j} \text{Tr} \lambda_{3,k} = 0
$$

and hence

$$
\nabla_{f_j} \sum_{k=1}^{T} \text{Tr} \lambda_{4,k} = \nabla_{f_j} \sum_{k=1}^{T} \left( \left( \sigma_{T}^{d} \right)^{-1} B_k^T F_j s_{k\|T} e_j^T B_k^T \right)
$$

$$
= \nabla_{f_j} \sum_{k=1}^{T} \left( e_j^T B_k^T \right) \left( \sigma_{k}^{-1} B_k^T F_j s_{k\|T} \right)
$$

$$
= \left( B_k^T e_j \right)^T \nabla_{f_j} \left( s_{k\|T} \otimes \sigma_{k}^{-1} B_k^T \right) f_j
$$

$$
= \left[ \left( s_{k\|T} \otimes B_k^T \right) \sigma_{k}^{-1} \right] \left( I \otimes B_k^T \right) e_j^T
$$

and, similarly

$$
\nabla_{f_j} \text{Tr} \lambda_{5,k} = \left[ \left( s_{k\|T} \otimes B_k^T \right) \left( I \otimes \sigma_{k}^{-1} B_k^T \right) \right] e_j^T.
$$

Here, $\mathcal{Z}_k = \mathcal{Z}_k^{1,0} + \mathcal{Z}_k^{1,1}$ with

$$
\mathcal{Z}_k^{1,0} = 2 s_{k\|T} \left( B_k^T \right)^T \otimes B_k^T \sigma_{k}^{-1} B_k^T
$$

$$
\mathcal{Z}_k^{1,1} = 2 s_{k\|T} B_k^T \otimes B_k^T \sigma_{k}^{-1} B_k^T.
$$

Second, with respect to $e_j$, one has

$$
\nabla_{e_j} \text{Tr} \lambda_{1,k} = 0
$$

$$
\nabla_{e_j} \text{Tr} \lambda_{2,k} = e_j^T \mathcal{Z}_k
$$

$$
\nabla_{e_j} \text{Tr} \lambda_{3,k} = 0
$$

$$
\nabla_{e_j} \text{Tr} \lambda_{4,k} = f_j^T \left( I \otimes B_k^T \right) \left( s_{k\|T} \otimes \sigma_{k}^{-1} B_k^T \right)
$$

$$
\nabla_{e_j} \text{Tr} \lambda_{5,k} = f_j^T \left( I \otimes \sigma_{k}^{-1} B_k^T \right) \left( s_{k\|T} \otimes B_k^T \right)
$$

with

$$
\mathcal{Z}_k = 2 \left( I \otimes B_k^T \right) \sigma_{k}^{-1} B_k^T.
$$

Third, with respect to $\varphi_j$, the only nonzero derivative is

$$
\nabla_{\varphi_j} \text{Tr} \lambda_{3,k} = \varphi_j^T \mathcal{Z}_k
$$

(44)

using the equations in Appendix B.

From above, (40) is equivalent to

$$
\nabla_{\varphi_j} \left[ \sum_{k=1}^{T} \mathcal{L}_k (\varphi_j, \hat{\theta}_j) \right] = \varphi_j^T \left[ \mathcal{Z}_k \right] + \mathcal{O} = 0
$$

(45)

where $\mathcal{Z}_2 = \sum_{k=1}^{T} \mathcal{Z}_k$, $\mathcal{Z} = \sum_{k=1}^{T} \mathcal{Z}_k$, $\mathcal{K} = \left[ \mathcal{Z}_k^{1,0} \mathcal{Z}_k^{1,1} \mathcal{Z}_k \right]$

with

$$
\mathcal{Z}_k = 2 \left( I \otimes B_k^T \right) \left( s_{k\|T} \otimes \sigma_{k}^{-1} B_k^T \right).
$$

What is left is to prove the existence of a unique solution $\varphi_j$ to (45). It suffices to show that $\mathcal{Z} > 0$ and $\mathcal{Z}^2 > 0$, or $\mathcal{Z}_k > 0$ and $\mathcal{Z}_k^2 > 0$.

Since the matrix $B_k^T$ has a full column rank, $\Sigma_k > 0$ and $\tilde{P}_{k\|T} > 0$, one has $\mathcal{Z}_k > 0$ and $\mathcal{Z}_k^2 > 0$. Next, the decomposition of the matrix $\mathcal{Z}_k$ gives

$$
\mathcal{Z}_k = \left[ \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} \left[ \mathcal{I} - \mathcal{Z}_k^3 \right] \mathcal{Z}_k^{-1} \mathcal{Z}_k^T \right] \left[ \begin{bmatrix} 1 & 0 \end{bmatrix} \mathcal{I} \right]
$$

(46)

where $\mathcal{Z}_k = \mathcal{Z}_k^3 \mathcal{Z}_k^{-1}$. It is noted that $\mathcal{Z}_k^0 = \mathcal{Z}_k^3 \mathcal{Z}_k^{-1} \mathcal{Z}_k^T$, which implies

$$
\mathcal{Z}_k^1 - \mathcal{Z}_k^2 \mathcal{Z}_k^3 \mathcal{Z}_k^T = \mathcal{Z}_k^1 > 0
$$

and hence $\mathcal{Z}_k > 0$. The proof is thus completed.

Remark 6: The unique solution $\varphi_j$ to (45) is

$$
\varphi_j = -\mathcal{Z}_k^{-1} \mathcal{Z}_k^0 \mathcal{Z}_k^2
$$

(47)

Denote $\hat{\theta}_j = \text{col}(\hat{f}_j, \hat{e}_j, \hat{\varphi}_j)$, whose covariance matrix component is $\hat{\theta}_j = 0$, due to (43). However, in practical scenarios if we use an optimization routine, then it would tend to decrease toward zero in an iterative manner which can also be validated from the simulation results.

The following theorem addresses the positive definiteness property of the negative Hessian of the surrogate function of mixture log-likelihood.

Theorem 3: For the function $\mathcal{L}_k (\varphi_k, \hat{\theta}_k)$ defined in (27), the following inequality:

$$
-\nabla_{\varphi_k}^2 \mathcal{L}_k \left( \varphi_k, \hat{\theta}_k \right) > 0, \quad j = 1, \ldots, T
$$

always holds for any given parameter $\hat{\theta}_j$.

Proof: Following the proof of Theorem 2, one has

$$
-\nabla_{\varphi_k}^2 \mathcal{L}_k \left( \varphi_k, \hat{\theta}_k \right) = -\nabla_{\varphi_k}^2 \left[ \text{Tr} \lambda_k \right]
$$

$$
= \begin{bmatrix}
-\partial (.) \\
\partial f_j \\
\partial e_j \\
\partial \varphi_j \\
\partial \sigma_j \\
\partial e_j \\
\partial \sigma_j \\
\partial e_j \\
\partial \sigma_j \\
\partial e_j \\
\partial \sigma_j \\
\partial e_j \\
\partial \sigma_j \\
\partial e_j \\
\partial \sigma_j \\end{bmatrix} \left[ \begin{bmatrix} \partial \lambda_k \end{bmatrix} \right]
$$

(49)

First, we calculate the three diagonal elements of (49) below, in (50)–(52), respectively. The calculation starts from

$$
-\nabla_{\varphi_k}^2 \text{Tr} \lambda_{1,k} = \nabla_{\varphi_k} \left[ \frac{\partial \text{Tr} \lambda_{1,k}}{\partial \hat{\theta}_j} \right]^T
$$

$$
= \nabla \left[ \text{vec} \left( \nabla_{\varphi_k} \text{Tr} \lambda_{1,k} \right) \right]^T
$$

for $X = B_k^T F_j$. By invoking some basic identities, the equation continues with

$$
= \nabla \left[ \left( \Sigma_k^{-1} X G_k + \Sigma_k^{-1} X G_k^T \right) \left( I_{u_k} \otimes B_k^T \right) \right]^T
$$
that can be used to describe the convergence error from \( \hat{\phi}^i - \hat{\phi}_{EM} \) to \( \hat{\phi}^{i+1} - \hat{\phi}_{EM} \) in the following theorem.

**Theorem 4:** Let \( \hat{\phi}^i \) be a known parameter estimate and \( \hat{\phi}^{i+1} \) satisfy

\[
\frac{\partial \mathcal{L}(\phi, \hat{\phi}^i)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^{i+1}} = 0.
\]

Then, for \( \hat{\phi}_{EM} \) defined in Lemma A.4

\[
\hat{\phi}^{i+1} - \hat{\phi}_{EM} = \mathcal{J}(\hat{\phi}^i, \hat{\phi}^{i+1})(\hat{\phi}^i - \hat{\phi}_{EM}) + o(\hat{\phi}^i - \hat{\phi}_{EM})
\]

where the notation \( o \) represents higher order smallness.

**Proof:** One can use the Taylor series expansion of \( [\partial L_{\phi}(\mathcal{Y}_T)]/\partial \phi \) about \( \phi = \hat{\phi}^i \) and evaluate it at \( \phi = \hat{\phi}_{EM} \) as follows:

\[
0 = \frac{\partial L_{\phi}(\mathcal{Y}_T)}{\partial \phi} \bigg|_{\phi = \hat{\phi}_{EM}} + \frac{\partial L_{\phi}(\mathcal{Y}_T)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^i} + o(\hat{\phi}^i - \hat{\phi}_{EM}).
\]

The first equation holds because \( \hat{\phi}_{EM} \) is a stationary point of \( L_{\phi}(\mathcal{Y}_T) \) by Lemma A.4. As a result

\[
\frac{\partial L_{\phi}(\mathcal{Y}_T)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^i} + o(\hat{\phi}^i - \hat{\phi}_{EM}).
\]

Again one can apply the Taylor series expansion of \( [\partial \mathcal{L}(\phi, \hat{\phi}^i)]/\partial \phi \) about \( \phi = \hat{\phi}^i \) and evaluate it at \( \phi = \hat{\phi}^{i+1} \) as follows, noting the explicit quadratic expression of \( \mathcal{L} \):

\[
\frac{\partial \mathcal{L}(\phi, \hat{\phi}^i)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^{i+1}} = \frac{\partial \mathcal{L}(\phi, \hat{\phi}^i)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^i} + o(\hat{\phi}^i - \hat{\phi}_{EM}).
\]

which implies, due to (55),

\[
\frac{\partial \mathcal{L}(\phi, \hat{\phi}^i)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^{i+1}} = \frac{\partial \mathcal{L}(\phi, \hat{\phi}^i)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^i} + o(\hat{\phi}^i - \hat{\phi}_{EM}).
\]

Finally, by (72) of Lemma A.3, one can equate (57) and (58) as

\[
\frac{\partial \mathcal{L}(\phi, \hat{\phi}^i)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^{i+1}} = \frac{\partial \mathcal{L}(\phi, \hat{\phi}^i)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^i} + o(\hat{\phi}^i - \hat{\phi}_{EM})
\]

and hence (56).

Next, we discuss the covariance matrix component \( \hat{\sigma}^i_k \) of \( \hat{\sigma}^i_k = \text{col}(\hat{\sigma}^i_1, \hat{\sigma}^i_2, \ldots, \hat{\sigma}^i_T) \). Denote \( \hat{\sigma}^i_k = \text{col}(\hat{\sigma}^i_1, \ldots, \hat{\sigma}^i_T) \). For \( \hat{\sigma}^{i+1} = \text{col}(\hat{\sigma}^{i+1}_1, \ldots, \hat{\sigma}^{i+1}_T) \) calculated using the SOC-EM II algorithm (39) [similar analysis holds for the SOC-EM I algorithm (38)], by Theorem 2, \( \hat{\sigma}^{i+1}_k \) is the solution to (40), that is,

\[
\frac{\partial \sum_{j=1}^T \hat{\mathcal{L}}(\phi_j, \hat{\phi}^i)}{\partial \phi_j} \bigg|_{\phi = \hat{\phi}^{i+1}, \phi = \hat{\phi}^i} = 0, \quad j = 1, \ldots, T.
\]

It approximately implies that (55) is satisfied, and hence

\[
\hat{\phi}^{i+1} - \hat{\phi}_{EM} = \mathcal{J}(\hat{\phi}^i, \hat{\phi}^{i+1})(\hat{\phi}^i - \hat{\phi}_{EM})
\]

holds with the higher order smallness ignored.
By Lemma A.4, one has \( \lim_{i \to \infty} \hat{\phi}^i = \hat{\phi}_{EM} \) if \( \hat{\phi} \) recursively generated by \( \hat{\phi}^{i+1} = \hat{\phi}^{i} \) according to (70), approximated by SOC-EM II (39). It is noted that \( \hat{\sigma}_{k, EM} \) and \( \hat{\sigma}_{k, EM} \) are the covariance matrix component of \( \hat{\phi}^i \) and \( \hat{\phi}_{EM} \), respectively. As shown in Remark 6, one has \( \hat{\sigma}^i_k = 0 \) and hence \( \hat{\sigma}_{k, EM} = 0 \). Now, from (60), one has approximately

\[
\hat{\sigma}^{i+1} = J_\Sigma(\hat{\phi}^i, \hat{\phi}^{i+1}) \hat{\sigma}^i
\tag{61}
\]

for some \( J_\Sigma \). Based on Theorem 3 and its application to (54), one can approximately conclude that

\[
\mathcal{J}(\hat{\phi}^i, \hat{\phi}^{i+1}) = \text{diag} \left[ J_\Sigma(\hat{\phi}^i, \hat{\phi}^{i+1}) \right] \leq I \tag{62}
\]

where the information matrix \( \mathcal{J}(\hat{\phi}^i, \hat{\phi}^{i+1}) \) corresponds to the counterpart of the component of \( -\nabla_{\hat{\phi}^i} \{ \sum_{k=1}^p \mathcal{L}(\hat{\phi}_j, \hat{\phi}^i) \} |_{\hat{\phi} = \hat{\phi}^{i+1}} \). Furthermore, SOC-EM II is carried out for all time instants separately with no correlation between them. Therefore, one can stack them in a matrix with diagonal elements as the individual (for each optimization instant) Hessians to create a higher dimensional Hessian which essentially provides property of the policy. Similarly, the principal minor of \( \mathcal{J}(\hat{\phi}^i, \hat{\phi}^{i+1}) \) concerned with the covariance components inherits the property from (62) and follows the following inequality:

\[
0 \leq \mathcal{J}_i(\hat{\phi}_i^i, \hat{\phi}_i^{i+1}) \leq I. \tag{63}
\]

Equation (61) is trivially true because \( \hat{\sigma}^i_{k+1} = \hat{\sigma}^i_k = 0 \) recursively in SOC-EM II. However, in real scenarios, SOC-EM II cannot be perfectly implemented, but practically in the sense of

\[
\| \hat{\phi}^{i+1} - \text{col}(\hat{\phi}_T, \ldots, \hat{\phi}_T) \| \leq \Delta \tag{64}
\]

for some error tolerance \( \Delta \). As a result, \( \hat{\sigma}^i_{k+1} = 0 \) does not hold anymore. Nevertheless, (61) can approximately claim that \( \hat{\sigma}^i_k \) converges to zero as \( i \) goes to \( \infty \). In particular, the following theorem states the conclusion in terms of the singular values of the covariance matrices \( \hat{\Sigma}^i_k = (\hat{\Sigma}^{i-1})^{1/2} \Sigma_k^{1/2} \hat{\Sigma}^{i-1/2} \) under the condition (63), where \( \hat{\Sigma}^i_k = \text{vec}(\hat{\Sigma}_k^{i/2}) \).

**Theorem 5:** Suppose (61) holds with (63). Let \( \sigma_{k,1}^i, \ldots, \sigma_{k,n_k}^i \) be the singular values of \( \hat{\Sigma}^i_k \) for \( i = i, i+1, \ldots \), then

\[
\sum_{k=1}^T \sum_{p=1}^{n_k} \sigma_{k,p}^{i+1} \leq \sum_{k=1}^T \sum_{p=1}^{n_k} \sigma_{k,p}^i. \tag{65}
\]

**Proof:** Equation (61) multiplied by its transpose gives

\[
(\hat{\sigma}^{i+1})^T \hat{\sigma}^i = (\hat{\sigma}^i)^T J_\Sigma(\hat{\phi}^i, \hat{\phi}^{i+1}) J_\Sigma(\hat{\phi}^i, \hat{\phi}^{i+1})^T \hat{\sigma}^i \leq (\hat{\sigma}^i)^T \hat{\sigma}^i \]

where the inequality holds due to (63). It is equivalent to

\[

\sum_{k=1}^T (\hat{\sigma}^{i+1}_k)^T \hat{\sigma}_k^i \leq \sum_{k=1}^T (\hat{\sigma}^i_k)^T \hat{\sigma}_k^i.
\]

Using the following property, for \( i = i, i+1 \):

\[
(\hat{\sigma}^i_k)^T \hat{\sigma}^i_k = \text{Tr} \left( (\hat{\Sigma}^{1/2}_k)^T \Sigma_k^1 \hat{\Sigma}^{1/2}_k \right) = \text{Tr} \left( \hat{\Sigma}^{1/2}_k \right) = \sum_{p=1}^{n_k} \sigma_{k,p}^i
\]

yields (65).

\[\square\]

**VI. EXPERIMENTAL RESULTS**

In this section, we investigate the empirical performance of the proposed SOC-EM algorithm that aims to use the parameter estimates of baselines as mentioned in Section V-B to deliver better controller parameters. The experiments were conducted on a Box2D framework that is a rigid polygon mass subject to gravity, linear, and angular damping [46]. We consider sensor noise \( \epsilon_k^s \) to be Gaussian in our experiments, i.e.,

\[
s_k = x_k + \epsilon_k^s, \quad \epsilon_k^s \sim \mathcal{N}(0, \sigma^2 \mathbb{I}_n)
\tag{66}
\]

Fig. 2. Cumulative sum of costs evaluated for three baselines: 1) iLQG; 2) MPC; and 3) BADMM, for different controller parameters \( \hat{\phi}_0, \hat{\phi}_1, \) and \( \hat{\phi}_2 \).

where \( x_k \) is the real state. The sensor noise \( \epsilon_k^s \) is propagated into the design of control action \( a_k \) that forms the real input to the system. The states of the system are \( x = [x, y, v_x, v_y] \) representing the position and velocity in a 2-D environment. With the control action \( a_k = [a_x, a_y] \in \mathbb{R}^2 \), the objective is driven from \([0, 5]\) to \([5, 20]\) in the Cartesian coordinate and stays there using the shortest time. Algorithm 1 was implemented in the experiments.

**A. Cost**

The mean \( \pm \text{std-dev} \) of the cumulative sum of the real costs \( \sum_{k=1}^T Y(s_k, a_k), k = 1, \ldots, T = 30 \), is depicted in Fig. 2. The datasets take into account 20 samples of cumulative sum of the costs against the time steps evaluated on three baselines. The noise parameter was set as \( \rho = 0.3 \). The plots in red, blue, and magenta represent the performance with the controllers parameterized by \( \hat{\phi}_0, \hat{\phi}_1, \) and \( \hat{\phi}_2 \), respectively. The solid, dotted, and dashed plots delineate three baselines iLQG, MPC, and BADMM, respectively. For the three baselines, it is observed that the mean \( \pm \text{std-dev} \) of the cumulative sum decreases over subsequent iterations, which verifies that the cost-to-go is reduced through iterative EM procedures.

Fig. 3 exhibits the cost-to-go comparison of the SOC-EM I
and SOC-EM II algorithms with the iLQG baseline for three specific controller parameters. It is observed that SOC-EM I works better than SOC-EM II because the former is a more accurate high-dimensional optimization when compared with the latter.

B. Trajectories

We compare the true state trajectories, i.e., \{x_1, \ldots, x_T\} produced on the real platform excited by the control actions with the parameters obtained through repeated EM iterations. In this experiment, iLQG was used as the baseline. We performed 30 experiments and each experiment ran for ten subsequent iterations. The mean ± std-dev of the trajectories \[ x_k, y_k \], \( k = 1, \ldots, 30 \), is illustrated in Fig. 4 for the control action with \( \hat{\phi}_0, \hat{\phi}_1, \) and \( \hat{\phi}_9 \). It can also be observed that all the trajectories move to the proximity of the target \([5, 20]\) quickly (in approximately eight steps) and stay there in the remaining steps. The magenta trajectory for \( \hat{\phi}_9 \) of less jittery nature in contrast to the red one for \( \hat{\phi}_0 \) demonstrates better performance achieved by EM iterations. In particular, there is a high variance in the red trajectory near the final time step as the influence of noise is accumulated temporally. The corresponding velocity trajectories \( v_x \) and \( v_y \) versus time are illustrated in Fig. 5. It is observed that the velocities increase to the maximum to drive the object to the target position quickly (again in approximately eight steps) and then decrease to zero. The advantage gained by EM iterations can be explained by the less deviation caused by noise in the magenta trajectory.

C. Control Actions

Fig. 6 shows the evolution of control actions in terms of kernel density plots of samples collected from 100 experiments. The control actions should be generated to maximum for a large velocity at the start and then reduced to zero in an ideal environment. It is evident that the control actions with the parameters \( \hat{\phi}_0 \) do not well settle down to zero. On the contrary, the control actions as a result of \( \hat{\phi}_1 \) and \( \hat{\phi}_9 \) are of significant improvement.

D. Exploitation Efficiency

The observed improvement in control actions can be well-manifested by the exploitation mechanism in the EM approach which significantly reduces the stochasticity in the control policies. In particular, the theoretical analysis in control covariance matrix in Section V-D can be verified by the plots of iterative decrease in the sum of the singular values of covariance matrices (see Fig. 7). We simulated the entire procedure of EM with different noise factors and recorded \( \sum_{\mu=1}^{n} \sigma_{k,\mu}^{2} \) which is marked as “+” for each time step \( k = 1, \ldots, 30 \) and each EM iteration \( i = 0, \ldots, 9 \). The average (equivalent to the sum divided by 30) is represented by the solid curve. The plots in log scale better shows the decrease pattern for \( \rho = 0.2 \) as expected by the theory. It can be noted that for \( \rho = 0.7 \), the pattern is violated at \( i = 8 \), which is due to the higher order smallness in (56).

E. Measurement Noise

It has been exhibited that the exploitation functionality of the EM approach can effectively reduce the stochasticity in control policies. Next, we will further highlight this effectiveness in comparison to artificially setting the covariance matrices zero. The comparison is made among the three cases: 1) the iLQG parameter \( \hat{\phi}_0 \) with \( \Sigma^0 = 0 \); 2) the EM-iLQG parameter \( \hat{\phi}_9 \); and 3) \( \hat{\phi}_9 \) with \( \Sigma^0 = 0 \). Indeed, it is observed that artificially setting the covariance matrices zero does not satisfactorily reduce the stochasticity in the control policies, as the stochasticity propagated from measurement noise is
unavoidable. The kernel density plots in Fig. 8 show that the EM approach performs better than iLQG with zero covariance matrices. It is not surprising to see that the difference between cases 2) and 3) is minor as $\phi^0$ is closely approaching 0 using SOC-EM. The corresponding state trajectories plotted in Fig. 9 support the same conclusion.

The simulations of Steps 1, 2, and 4 were performed using the 64-bit Ubuntu 16.04 operating system on Dell Alienware 15 R2 of Intel Core i7-6700HQ@2.60GHz. The simulations of Step 3 were conducted using multiple 2.6-GHz Intel Xeon Broadwell (E5-2697A v4) processors on the high-performance computing grid located at The University of Newcastle. We switched processors to leverage parallel processing of the optimization routine of (39). It took about 93 s for the policy of 200 iterations of convergence to be generated using parallel processing of multiple processors, while it took about 2790 s for a single processor.

VII. CONCLUSION

This article has proposed a new EM-based methodology for solving the SOC problem, resulting in an SOC-EM algorithm. The method effectively bridges the relationship between the optimal control problem and the EM algorithm that is originally used for maximizing the likelihood of observed data. Moreover, we have discussed a practical solution to SOC-EM and the uniqueness of controller parameter estimation. The algorithm has been applied to the Box2D framework, and the experiments support the superiority of the new technique, compared with some of the widely known and extensively used methodologies. This article has established a new research framework that has potential development in the future work. For example, nonlinear stochastic dynamics, persistently exciting property of a system as a result of parameters obtained through EM, input constraints, fitting stable linear dynamic models, etc., are interesting topics.

APPENDIX

A. Derivation of the Terms in Lemma 2

The terms of $\Theta_1(\phi_k)$, $\Theta_2(\phi_k)$, and $\Theta_3(\phi_k)$ after expansion are shown below. First

$$\Theta_1(\phi_k) = E_{\phi_k} \left( \zeta_k z_k^T \right) = \begin{bmatrix} G_{k+1} & \gamma_1 & \gamma_2 \end{bmatrix}$$

where

$$\gamma_1 = \sum_k E_{\phi_k} (s_{k+1} s_k^T \mid y_{\tau}) + B_{k} \sum_k E_{\phi_k} (a_{k+1} a_k^T \mid y_{\tau})$$

$$\gamma_2 = A_k^T \sum_k E_{\phi_k} (s_{k+1} a_k^T \mid y_{\tau}) B_k^T$$

Similarly, the matrix $\Theta_2(\phi_k)$ can be expanded as

$$\Theta_2(\phi_k) = E_{\phi_k} \left( \zeta_k z_k^T \right) = \begin{bmatrix} \sum_k E_{\phi_k} (s_{k+1} a_k^T \mid y_{\tau}) \end{bmatrix}$$

where

$$\gamma_3 = \sum_k E_{\phi_k} (a_{k+1} a_k^T \mid y_{\tau})$$

$$\gamma_4 = \sum_k E_{\phi_k} (a_{k+1} a_k^T \mid y_{\tau})$$

The matrix $A_k^T \Theta_3(\phi_k) A_k^T$ has the expression

$$A_k^T \Theta_3(\phi_k) A_k^T = \begin{bmatrix} \gamma_8 & \gamma_9 \\ \gamma_{10} & \gamma_{11} \end{bmatrix}$$

where

$$\gamma_0 = G_k F_k^T + \sum_k E_{\phi_k} (a_{k+1} a_k^T \mid y_{\tau})$$

$$\gamma_7 = E_{\phi_k} (a_{k+1} a_k^T \mid y_{\tau})$$

$$\gamma_8 = A_k^T \left( G_k A_k^T + \gamma_0 B_k^T \right) + B_k^T \left( \gamma_0 A_k^T + \gamma_0 B_k^T \right)$$

$$\gamma_9 = A_k^T \left( G_k A_k^T + \gamma_0 B_k^T \right) + B_k^T \left( \gamma_0 A_k^T + \gamma_0 B_k^T \right)$$

$$\gamma_{10} = A_k^T \left( G_k A_k^T + \gamma_0 B_k^T \right) + B_k^T \left( \gamma_0 A_k^T + \gamma_0 B_k^T \right)$$

$$\gamma_{11} = A_k^T \left( G_k A_k^T + \gamma_0 B_k^T \right) + B_k^T \left( \gamma_0 A_k^T + \gamma_0 B_k^T \right)$$

B. Gradient of Mixture Likelihood

The gradient of the mixture log-likelihood is evaluated using properties of multivariable calculus. In particular, the gradients with respect to different parameters in $\phi_k$ are shown from the equations below, where $\gamma_{\tau}$ can be referred to in Appendix A. The equations regarding $\Theta_1(\phi_k)$ are

$$\nabla_{\phi_k} \text{Tr} \left( \Sigma_k^{-1} \Theta_1(\phi_k) \right) = 2B_k^T \otimes \Sigma_k^{-1} A_k^T G_k + \nabla_{\phi_k} B_k^T \gamma B_k^T$$

$$\nabla_{\phi_k} \text{Tr} \left( \Sigma_k^{-1} \Theta_1(\phi_k) \right) = 2B_k^T \otimes \Sigma_k^{-1} A_k^T a_k + \nabla_{\phi_k} B_k^T \gamma B_k^T$$

$$\nabla_{\phi_k} \text{Tr} \left( \Sigma_k^{-1} \Theta_1(\phi_k) \right) = \nabla_{\phi_k} B_k^T \gamma B_k^T$$

those for $\Theta_2(\phi_k)$

$$\nabla_{\phi_k} \text{Tr} \left( \Sigma_k^{-1} \Theta_2(\phi_k) A_k^T \right) = \text{vec} \left( M_{k+1} \Sigma_k^{-1} B_k^T \right)$$

$$+ 2B_k^T \otimes \Sigma_k^{-1} A_k^T G_k + \nabla_{\phi_k} B_k^T \gamma B_k^T$$
\[
\n\nabla_k \text{Tr}\left( \Sigma_k^{-1} \Theta_2 (\phi_k) A_k^T \right) \\
= \text{vec}\left( \tilde{\Sigma}_k^{-1} \Sigma_k^{-1} B_k^T \right) \\
+ 2 B_k^T \otimes \Sigma_k^{-1} A_k \tilde{\Sigma}_k^{-1} B_k^T + \nabla_k \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
\n\nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} \Theta_2 (\phi_k) A_k^T \right) \\
= \nabla_{\sigma_k} B_k \gamma B_k^T \\
\text{and those for } \Theta_3 (\phi_k) \\
\n\nabla_k \text{Tr}\left( \Sigma_k^{-1} \Theta_3 (\phi_k) A_k^T \right) \\
= 2 \text{vec}\left( G_k A_k^T \Sigma_k^{-1} B_k^T \right) \\
+ \nabla_k \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) + \nabla_k \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
+ 2 B_k^T \otimes \Sigma_k^{-1} \Theta_3 G_k \\
\n\nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} \Theta_3 (\phi_k) A_k^T \right) \\
= \nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) + \nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
+ 2 B_k^T \otimes \Sigma_k^{-1} \Theta_3 \tilde{\Sigma}_k^{-1} B_k^T. \\
\]

C. Hessian of Mixture Likelihoods

The components of the Hessian of mixture log of mixture likelihood expression can be expanded and verified with equations shown below. The equations regarding \( \Theta_1 (\phi_k) \) are

\[
\nabla_k \text{Tr}\left( \Sigma_k^{-1} \Theta_1 (\phi_k) \right) = \nabla_k \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
\nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} \Theta_1 (\phi_k) \right) = \nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
\]

those for \( \Theta_2 (\phi_k) \)

\[
\nabla_k \text{Tr}\left( \Sigma_k^{-1} \Theta_2 (\phi_k) A_k^T \right) = \nabla_k \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
\nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} \Theta_2 (\phi_k) A_k^T \right) = \nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
\]

and those for \( \Theta_3 (\phi_k) \)

\[
\nabla_k \text{Tr}\left( \Sigma_k^{-1} \Theta_3 (\phi_k) A_k^T \right) = \nabla_k \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
+ \nabla_k \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
\nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} \Theta_3 (\phi_k) A_k^T \right) = \nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
+ \nabla_{\sigma_k} \text{Tr}\left( \Sigma_k^{-1} B_k \gamma B_k^T \right) \\
\n\]

D. Some EM Lemmas

Lemma A.1 explains the lower bound maximization strategy in EM and also delineates the two main steps involved (see [47]).

**Lemma A.1:** Consider \( L_\phi (Y_T) \) and \( \mathcal{L} (\phi, \hat{\phi}) \) defined in (17) and (18), respectively, with a known parameter estimate \( \hat{\phi} \). Let

\[
\n l(\phi, \tilde{\phi}) = \mathbb{E}_{\bar{p}(\mathcal{S}_T+1)} \log \frac{p_\phi (\bar{S}_T+1, Y_T)}{\tilde{p}(\mathcal{S}_T+1)} \\
\]

for any distribution \( \tilde{p}(\mathcal{S}_T+1) \). One has

\[
\n L_\phi (Y_T) \geq l(\phi, \tilde{p}(\mathcal{S}_T+1)) \]

that is, \( l(\phi, \tilde{p}(\mathcal{S}_T+1)) \) is a lower bound of \( L_\phi (Y_T) \). Moreover, let

\[
\n \tilde{p}(\mathcal{S}_T+1) = p_{\hat{\phi}} (\mathcal{S}_T+1 | Y_T) \\
\]

and denote

\[
\n l(\phi, \hat{\phi}) = l(\phi, \hat{p}_i) (\mathcal{S}_T+1 | Y_T). \\
\]

One has

\[
\n \hat{\phi}^* = \arg \max \phi \ l(\phi, \hat{\phi}^*) = \arg \max \phi \mathcal{L} (\phi, \hat{\phi}^*). \\
\]

Furthermore, Lemma A.2 shows that in a recursive procedure, any new parameter \( \phi = \hat{\phi}^* \) that increases \( \mathcal{L} (\phi, \hat{\phi}^*) \) from \( \phi = \hat{\phi}^* \) also increase \( L_\phi (Y_T) \).

**Lemma A.2:** Suppose the parameter vector \( \hat{\phi}^* \) is produced in an iteration, one has

\[
\n L_\phi (Y_T) - L_\phi (Y_T) \geq \mathcal{L} (\phi, \hat{\phi}^*) - \mathcal{L} (\phi, \hat{\phi}^*), \\
\]

where the equality holds if \( p_{\hat{\phi}_i} (\mathcal{S}_T+1 | Y_T) = p_{\hat{\phi}} (\mathcal{S}_T+1 | Y_T) \).

Lemma A.3 provides a relationship between the gradients of \( L_\phi (Y_T) \) and \( \mathcal{L} (\phi, \hat{\phi}^*) \) evaluated at \( \phi = \hat{\phi}^* \), called Fisher’s identity [48].

**Lemma A.3:** Consider \( L_\phi (Y_T) \) and \( \mathcal{L} (\phi, \hat{\phi}^*) \) defined in (17) and (18), respectively, with a known parameter estimate \( \hat{\phi}^* \). Then,

\[
\n \frac{\partial L_\phi (Y_T)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^*} = \frac{\partial \mathcal{L} (\phi, \hat{\phi}^*)}{\partial \phi} \bigg|_{\phi = \hat{\phi}^*}. \\
\]

The property of monotonic convergence of EM undisputedly holds (see [21], [49]). The result is summarized in the following lemma (see [49, Th. 2]).

**Lemma A.4:** Let \( \hat{\phi}^* \in \Phi \), \( i = 1, 2, \ldots \), be the policy parameter estimates recursively generated by \( \hat{\phi}_i = \hat{\phi}^* \) according to (70). Then the limit point \( \lim_{i \to \infty} \hat{\phi}_i \) \( = \hat{\phi}_{\text{EM}} \) exists and is a stationary point of \( L_\phi (Y_T) \). Also, \( L_\phi (Y_T) \) converges monotonically to \( L_{\phi_{\text{EM}}} (Y_T) \) as \( i \) goes to \( \infty \).

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