THE DUALS OF ANNIHILATOR CONDITIONS FOR MODULES

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Abstract. Let \( R \) be a commutative ring with identity and let \( M \) be an \( R \)-module. The purpose of this paper is to introduce and investigate the sub-modules of an \( R \)-module \( M \) which satisfy the dual of Property \( \mathcal{A} \), the dual of strong Property \( \mathcal{A} \), and the dual of proper strong Property \( \mathcal{A} \). Moreover, a submodule \( N \) of \( M \) which satisfy Property \( \mathcal{S}_J(N) \) and Property \( \mathcal{I}_J^M(N) \) will be introduced and investigated.

1. Introduction

Throughout this paper, \( R \) will denote a commutative ring with identity and \( Z \) will denote the ring of integers.

Let \( M \) be an \( R \)-module. The set of zero divisors of \( R \) on \( M \) is \( Z_R(M) = \{ r \in R | rm = 0 \text{ for some nonzero } m \in M \} \) and the set of torsion elements of \( M \) with respect to \( R \) is \( T_R(M) = \{ m \in M | rm = 0 \text{ for some } 0 \neq r \in R \} \).

An \( R \)-module \( M \) satisfies Property \( \mathcal{A} \) (resp., Property \( \mathcal{T} \)) if for every finitely generated ideal \( I \) of \( R \) (resp., finitely generated submodule \( N \) of \( M \)) with \( I \subseteq Z_R(M) \) (resp., \( N \subseteq T_R(M) \)), there exists a nonzero \( m \in M \) (resp., \( r \in R \)) with \( Im = 0 \) (resp., \( rN = 0 \)), or equivalently \( (0 :_M I) \neq 0 \) (resp., \( \text{Ann}_R(N) \neq 0 \)) \([2]\). An \( R \)-module \( M \) satisfies strong Property \( \mathcal{A} \) (resp., strong Property \( \mathcal{T} \)) if for any \( r_1, \ldots, r_n \in Z_R(M) \) (resp., \( m_1, \ldots, m_n \in T_R(M) \)), there exists a non-zero \( m \in M \) (resp., \( r \in R \)) with \( r_1m = \cdots = r_nm = 0 \) (resp., \( rm_1 = \cdots = rm_n = 0 \)) \([2]\). An \( R \)-module \( M \) satisfies proper strong Property \( \mathcal{A} \) if for any proper finitely generated ideal \( I = \langle a_1, a_2, \ldots, a_n \rangle \) of \( R \) such that \( a_i \in Z_R(M) \) we have \( 0 :_M I \neq 0 \) \([1]\). The class of modules satisfies proper strong Property \( \mathcal{A} \) lying properly between the class of modules satisfies strong Property \( \mathcal{A} \) and Property \( \mathcal{A} \) \([1] \) Corollary 2.12.

Let \( M \) be an \( R \)-module. The subset \( W_R(M) \) of \( R \) (that is the dual notion of \( Z_R(M) \)) is defined by \( \{ r \in R | rM \neq M \} \) \([19]\). A non-zero submodule \( N \) of \( M \) is said to be secondal if \( W_R(N) \) is an ideal of \( R \). In this case, \( W_R(N) \) is a prime ideal of \( R \) \([6]\).

Recently, the annihilator conditions on modules over commutative rings have attracted the attention of several researchers. A brief history of this can be found in \([2,1]\). The purpose of this paper is to introduce and study the dual of Property \( \mathcal{A} \), the dual of strong Property \( \mathcal{A} \), and the dual of proper strong Property \( \mathcal{A} \) for modules over a commutative ring. Also, for a submodule \( N \) of an \( R \)-module \( M \) we introduce and investigate the Properties \( \mathcal{S}_J(N) \) and \( \mathcal{I}_J^M(N) \). Some of the results

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in this article are dual of the results for Property $\mathcal{A}$, strong Property $\mathcal{A}$, and proper strong Property $\mathcal{A}$ considered in [1] and [2].

2. The duals of Property $\mathcal{A}$ and strong Property $\mathcal{A}$ for modules

**Definition 2.1.** We say that an $R$-module $M$ satisfies the dual of Property $\mathcal{A}$ if for each finitely generated ideal $I$ of $R$ with $I \subseteq W_R(M)$ we have $IM \neq M$.

A proper submodule $N$ of an $R$-module $M$ is said to be completely irreducible if

$N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$ [13].

**Definition 2.2.** We say that an $R$-module $M$ satisfies the dual of strong Property $\mathcal{A}$ if for any $a_1, \ldots, a_n \in W_R(M)$, there exists a completely irreducible submodule $L$ of $M$ such that $a_i M \subseteq L \neq M$ for $i = 1, 2, \ldots, n$.

Clearly, if an $R$-module $M$ satisfies the dual of Property $\mathcal{A}$, then $M$ satisfies the dual of Property $\mathcal{A}$. Nevertheless, the following example shows that the converse is not true in general.

**Example 2.3.** The $\mathbb{Z}$-module $\mathbb{Z}$ satisfies the dual of Property $\mathcal{A}$ but does not satisfy the dual of strong Property $\mathcal{A}$.

**Remark 2.4.** Let $N$ and $K$ be two submodules of an $R$-module $M$. To prove $N \subseteq K$, it is enough to show that if $L$ is a completely irreducible submodule of $M$ such that $K \subseteq L$, then $N \subseteq L$ [6].

**Theorem 2.5.** Let $M$ be an $R$-module. Consider the conditions:

(a) $M$ satisfies the dual of strong Property $\mathcal{A}$;
(b) $M$ is a secondal $R$-module.

Then (a) $\Rightarrow$ (b). If further $R$ is a PID, then (b) $\Rightarrow$ (a).

**Proof.** (a) $\Rightarrow$ (b). Let $a, b \in W_R(M)$. Then by part (a), there exists a completely irreducible submodule $L$ of $M$ such that $aM \subseteq L \neq M$ and $bM \subseteq L \neq M$. Thus $(a-b)M \subseteq L \neq M$ and so $a-b \in W_R(M)$. This implies that $M$ is a secondal $R$-module.

(b) $\Rightarrow$ (a). Let $a_1, \ldots, a_n \in W_R(M)$. Then by part (b), $\langle a_1, \ldots, a_n \rangle$ is an ideal of $R$. As $R$ is a PID, there exists an $a \in R$ such that $\langle a_1, \ldots, a_n \rangle = Ra$. Thus $a \in W_R(M)$. Hence there exists a completely irreducible submodule $L$ of $M$ such that $aM \subseteq L \neq M$ by Remark 2.4. This implies that $a_i M \subseteq L \neq M$ for $i = 1, 2, \ldots, n$, as needed.

**Theorem 2.6.** Let $f : R \to \hat{R}$ be a homomorphism of commutative rings and let $M$ be an $\hat{R}$-module. Consider $M$ as an $R$-module with $rm := f(r)m$ for $r \in R$ and $m \in M$.

(a) Suppose for each (finitely generated) ideal $I$ of $R$, $f(I)\hat{R} = \{f(i)\hat{r} | i \in I, \hat{r} \in \hat{R}\}$ (e.g., $f$ is surjective or $f : R \to R_N$, $f(r) = r/1$, where $N$ is a multiplicatively closed subset of $R$). Then $M$ satisfies the dual of Property $\mathcal{A}$ as an $\hat{R}$-module implies $M$ satisfies the dual of Property $\mathcal{A}$ as an $R$-module.

(b) Suppose that every (finitely generated) ideal $J$ of $\hat{R}$ has the form $J = f(I)\hat{R}$ for some (finitely generated) ideal $I$ of $R$ (e.g., $f$ is surjective or $f : R \to R_N$, $f(r) = r/1$, where $N$ is a multiplicatively closed subset of $R$). Then
Proposition 2.8. Let $M$ module of $R$. Also, $M$ is a comultiplication ring if, as an $R$-module, it satisfies the dual of Property $A$ as an $\hat{R}$-module.

Proof. (a) Suppose $M$ satisfies the dual of Property $A$ as an $R$-module. Let $I$ be an ideal of $R$ with $I \subseteq W_R(M)$. So for $i \in I$, there is a $m \in M \setminus iM = M \setminus f(i)M$. Hence, $f(i) \in W_R(M)$ and so $\hat{r}f(i) \in W_{\hat{R}}(M)$ for each $\hat{r} \in \hat{R}$. Thus $f(I)\hat{R} = \{f(i)\hat{r} \mid i \in I, \hat{r} \in \hat{R}\}$ is an ideal of $\hat{R}$ with $f(I)\hat{R} \subseteq W_{\hat{R}}(M)$. Suppose that $I$ is finitely generated. Then $f(I)\hat{R}$ is finitely generated. Hence there is a $m \in M \setminus f(I)\hat{R}M$. This implies that $m \in M \setminus IM$. Thus $M$ satisfies the dual of Property $A$ as an $R$-module.

(b) Suppose that $M$ satisfies the dual of Property $A$ as an $R$-module. Let $J$ be an ideal of $\hat{R}$ with $J \subseteq W_{\hat{R}}(M)$. Then there is an ideal $I$ of $R$ with $J = f(I)S$. For $i \in I$, $f(i) \in W_R(M)$. So, there is a $m \in M \setminus f(i)M = M \setminus iM$. So, $I \subseteq W_R(M)$. If $J$ is finitely generated, we can choose $I$ to be finitely generated. Since $M$ satisfies the dual of Property $A$ as an $R$-module, there is a $m \in M \setminus IM$. It follows that $m \in M \setminus f(I)\hat{R}M = M \setminus JM$. So, $M$ satisfies the dual of Property $A$ as an $R$-module. \hfill \Box

Corollary 2.7. Let $M$ be an $R$-module, $J \subseteq \text{Ann}_R(M)$ an ideal of $R$, and put $\hat{R} = R/J$. Then $M$ satisfies the dual of Property $A$ as an $R$-module if and only if $M$ satisfies the dual of Property $A$ as an $\hat{R}$-module. In particular, $M$ satisfies the dual of Property $A$ as an $R$-module if and only if $M$ satisfies the dual of Property $A$ as an $R/\text{Ann}_R(M)$-module.

Proof. This follows from Theorem 2.6. \hfill \Box

Recall that an $R$-module $M$ is said to be Hopfian (resp. co-Hopfian) if every surjective (resp. injective) endomorphism $f$ of $M$ is an isomorphism.

An $R$-module $M$ is said to be a multiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$ [12].

A submodule $N$ of an $R$-module $M$ is said to be idempotent if $N = (N : R M)^2 M$. Also, $M$ is said to be fully idempotent if every submodule of $M$ is idempotent [7].

An $R$-module $M$ is said to be a comultiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 : M I)$ [3]. $R$ is said to be a comultiplication ring if, as an $R$-module, $R$ is a comultiplication $R$-module [4].

A submodule $N$ of an $R$-module $M$ is said to be coidempotent if $N = (0 : M \text{Ann}_R(N))^2$. Also, an $R$-module $M$ is said to be fully coidempotent if every submodule of $M$ is coidempotent [7].

Proposition 2.8. Let $M$ be an $R$-module. Then we have the following.

(a) If $R$ is a comultiplication ring and $M$ is a faithful $R$-module, then $M$ satisfies Property $A$ and the dual of Property $A$.

(b) If $M$ is a Hopfian comultiplication (in particular, $M$ is a fully coidempotent) $R$-module and satisfies the dual of Property $A$, then $M$ satisfies Property $A$.

(c) If $M$ is a co-Hopfian multiplication (in particular, $M$ is a fully idempotent) $R$-module and satisfies Property $A$, then $M$ satisfies the dual of Property $A$.

(d) If $R$ is a principal ideal ring, then $M$ satisfies the dual of strong Property $A$. 
Proof. (a) This follows from [4, Lemma 3.11].

(b) First note that every fully coidempotent \( R \)-module is a Hopfian comultiplication \( R \)-module by [7, Theorem 3.9 and Proposition 3.5]. As \( M \) is a Hopfian comultiplication \( R \)-module, \( Z_R(M) = W_R(M) \). Now the result follows from [5, Proposition 3.1].

(c) First note that every fully idempotent \( R \)-module is a co-Hopfian multiplication \( R \)-module by [7, Proposition 2.7]. Since \( M \) is a co-Hopfian multiplication \( R \)-module, \( Z_R(M) = W_R(M) \). Now the result follows from [18, Note 1.13]. □

Lemma 2.9. Let \( S \) be a multiplicatively closed subset of \( R \), \( I \) and ideal of \( R \), and \( M \) be an \( R \)-module. Then we have the following.

(a) If \( S^{-1}I \subseteq W_{S^{-1}R}(S^{-1}M) \), then \( I \subseteq W_R(M) \).

(b) If \( Z_R(M) \cap S = \emptyset \), \( W_R(M) \cap S = \emptyset \), and \( I \subseteq W_R(M) \), then \( S^{-1}I \subseteq W_{S^{-1}R}(S^{-1}M) \).

(c) If \( M \) is an Hopfian module (in particular, \( M \) is a multiplication or coidempotent module), \( W_R(M) \cap S = \emptyset \), and \( I \subseteq W_R(M) \), then \( S^{-1}I \subseteq W_{S^{-1}R}(S^{-1}M) \).

Proof. (a) This is clear.

(b) Suppose that \( S^{-1}I \not\subseteq W_{S^{-1}R}(S^{-1}M) \) and seek for a contradiction. Then \( S^{-1}(aM) = S^{-1}M \) for some \( a \in I \). As \( I \subseteq W_R(M) \), there exists \( m \in M \setminus IM \). Now we have \( stm = sam_1 \) for some \( s, t \in S \) and \( m_1 \in M \). Since \( W_R(M) \cap S = \emptyset \), \( tM = M \) and so \( m_1 = tm_2 \) for some \( m_2 \in M \). Hence, \( st(m - am_2) = 0 \). Now \( Z_R(M) \cap S = \emptyset \) implies that \( m = am_2 \), which is a contradiction.

(c) This follows from the fact that \( Z_R(M) \subseteq W_R(M) \) and part (b). □

Corollary 2.10. Let \( S \) be a multiplicatively closed subset of \( R \) and \( M \) be an \( R \)-module. Consider the conditions:

(a) \( MS \) satisfies the dual of Property \( \mathcal{A} \) as an \( R \)-module;
(b) \( MS \) satisfies the dual of Property \( \mathcal{A} \) as an \( R_S \)-module;
(c) \( M \) satisfies the dual of Property \( \mathcal{A} \) as an \( R \)-module.

Then (a) \( \iff \) (b). If further \( S \cap W_R(M) = \emptyset \) and \( S \cap Z_R(M) = \emptyset \), (a), (b) and (c) are equivalent.

Proof. The equivalence of (a) and (b) follows from Theorem 2.6. Now assume that \( S \cap W_R(M) = \emptyset \) and \( S \cap Z_R(M) = \emptyset \). Then (b) \( \iff \) (c) from Lemma 2.9 (b). □

Proposition 2.11. Let \( X \) be an indeterminate over \( R \), \( M \) be an \( R \)-module, and \( M[X] \) satisfies the dual of strong Property \( \mathcal{A} \) over \( R[X] \). Then \( M \) satisfies the dual of Property \( \mathcal{A} \).

Proof. Let \( I = \langle a_1, \ldots, a_n \rangle \) be a finitely generated ideal of \( R \) such that \( a_i \in W_R(M) \) for \( i = 1, \ldots, n \). Then \( I[X] = \langle a_1, \ldots, a_n \rangle R[X] \) is a finitely generated ideal of \( R[X] \) such that \( a_i \in W_{R[X]}(M[X]) \) for \( i = 1, \ldots, n \). Since \( M[X] \) satisfies the dual of strong Property \( \mathcal{A} \) over \( R[X] \), we get \( I[X]M[X] \neq M[X] \). This implies that \( IM \neq M \). Thus \( M \) satisfies the dual of Property \( \mathcal{A} \). □

Recall that a ring \( R \) is called \( \text{Bézout} \) if every finitely generated ideal \( I \) of \( R \) is principal.

A submodule \( N \) of an \( R \)-module \( M \) is \textit{small} if for any submodule \( X \) of \( M \), \( X + N = M \) implies that \( X = M \).
A prime ideal $P$ of $R$ is said to be a \textit{coassociated prime ideal} of an $R$-module $M$ if there exists a cocyclic homomorphic image $T$ of $M$ such that $\text{Ann}_R(T) = P$. The set of coassociated prime ideals of $M$ is denoted by $\text{Coass}(M)$ \cite{10}.

\textbf{Theorem 2.12.} \begin{enumerate}
\item[(a)] The trivial $R$-module vacuously satisfies the dual of Property $A$.
\item[(b)] Every module over a Bézout ring satisfies the dual of Property $A$.
\item[(c)] Let $R$ be a zero-dimensional commutative ring (e.g., $R$ is Artinian). Then every $R$-module satisfies the dual of Property $A$.
\item[(d)] Let $M$ be a finitely generated $R$-module. Then $M$ satisfies the dual of Property $A$.
\item[(e)] Let $M$ be an Artinian $R$-module. Then $M$ satisfies the dual of Property $A$. In fact, for any ideal $I$ of $R$ with $I \subseteq W_R(M)$, $IM \neq M$.
\item[(f)] Let $M$ and $\tilde{M}$ be $R$-modules with $W_R(M) \subseteq W_R(\tilde{M})$. If $\tilde{M}$ satisfies the dual of Property $A$ (respectively, the dual of strong Property $A$), then $M \oplus \tilde{M}$ satisfies the dual of Property $A$ (respectively, the dual of strong Property $A$).
\item[(g)] Let $N$ be a small submodule of $M$. Then $M$ satisfies the dual of Property $A$ (resp. $M$ is a secondal module) if and only if $M/N$ satisfies the dual of Property $A$ (resp. $M/N$ is a secondal module).
\end{enumerate}

\textbf{Proof.} (a) Note that $W_R(0) = \emptyset$.

(b) This is clear.

(c) Suppose $\dim R = 0$ and $M$ is an $R$-module. We can assume $M \neq 0$. Let $I$ be a finitely generated ideal of $R$ with $I \subseteq W_R(M)$. So, $I \subseteq P \subseteq W_R(M)$ for some prime ideal $P$ of $R$ by using \cite{10} Theorem 2.15]. Since $htP = 0$ and $I$ is finitely generated, $I^n_p = 0$ for some $n \geq 1$. Hence there is an $s \in R \setminus P$ with $I^n s = 0$. Since $s \in R \setminus P$, $s \not\in \text{Ann}_R(M)$. Thus $sM \neq 0$. We have $I^n s M = 0$. Suppose $I^n s M \neq 0$, but $I^{n+1} s M = 0$. Then $IM \subseteq (0 :_M I^n)$ and $sM \neq 0$. Therefore, $IM \neq M$.

(d) Let $M$ be a finitely generated $R$-module and $I$ be an ideal of $R$ with $I \subseteq W_R(M)$. Assume contrary that $IM = M$. Then $(1 + a)M = 0$ by \cite{13} Theorem 76]. As $a \in I \subseteq W_R(M)$, there exists an $m \in M \setminus aM$. Now, $(1 + a)m = 0$ implies that $m \in aM$, which is a contradiction.

(e) As $M$ is an Artinian $R$-module, $W_R(M) = \cup_{i=1}^n P_i$ by using \cite{10} Theorem 2.10 (c), Theorem 2.15, Corollary 3.2], where $P_i \in \text{Coass}(M)$. Now let $I \subseteq W_R(M)$ be an ideal. Then $I \subsetneq P_i$ for some $P_i \in \text{Coass}(M)$. Hence for some completely irreducible submodule $L$ of $M$ with $L \neq M$, we have $I \subsetneq P_i = (L :_R M)$. This implies that $IM \neq M$.

(f) Let $\tilde{M}$ satisfies the dual of Property $A$. It is easy to see that $W_R(M \oplus \tilde{M}) = W_R(M) \cup W_R(\tilde{M}) = W_R(M)$. Let $I$ be a finitely generated ideal of $R$ with $I \subseteq W_R(M) \cup W_R(\tilde{M}) = W_R(M)$. Then $IM \neq \tilde{M}$. Thus there exists an $x \in IM \setminus \tilde{M}$. This implies that $(0, x) \not\in I(M \oplus \tilde{M})$ and so $I(M \oplus M) \neq M \oplus \tilde{M}$, as needed.

(g) We always have $W_R(M/N) \subseteq W_R(M)$. As $N$ is small we get that $W_R(M) \subseteq W_R(M/N)$. Now the result is straightforward. \hfill \Box

Let $M$ be an $R$-module. The idealization $R(+)M = \{(a, m) : a \in R, m \in M\}$ of $M$ is a commutative ring whose addition is component-wise and whose multiplication is defined as $(a, m)(b, \hat{m}) = (ab, a\hat{m} + bm)$ for each $a, b \in R$, $m, \hat{m} \in M$ \cite{17}. 
Proposition 2.13. Let $M$ be an $R$-module. Then we have

$$W_{R(+)M}(R(+)M) = W_R(R(+)M).$$

Proof. First note that $W_R(M) \subseteq W_R(R)$. Let $(a, x) \in W_{R(+)M}(R(+)M)$. Then there exists $(b, y) \in R(+)M \setminus (a, x)(R(+)M)$. This implies that for each $(c, z) \in R(+)M$, $(b, y) \neq (a, x)(c, z)$. Hence, $b \neq ac$ or $y \neq a(c + z)$. If $b \neq ac$, then $a \in W_R(R)$ and we are done. If $y \neq a(c + z)$, then by setting $c = 0$, we have $y \neq az$. Thus $a \in W_R(M) \subseteq W_R(R)$ and so $W_{R(+)M}(R(+)M) \subseteq W_R(R)(+)M$. Now let $(a, x) \in W_R(R)(+)M$. Then $a \in W_R(R)$. Thus there exist $r \in R$ such that $r \in R \setminus aR$. Assume contrary that $(a, x)(R(+)M) = R(+)M$. Then $(r, 0) = (a, x)(c, y)$ for some $(c, y) \in R(+)M$. Thus $r = ac$, which is a contradiction. Hence $(a, x)(R(+)M) \neq R(+)M$, as needed. \qed

Example 2.14. Let $M = \oplus R/I$, where the sum runs over all proper finitely generated ideals of $R$. Then for a proper finitely generated ideal $I$ of $R$, $I \subseteq W_R(M)$ and $I(R/I) = 0 \neq R/I$ implies that $M$ satisfies the dual of Property $A$. As $R$ is a submodule of $M$, we have $R$ is a submodule of an $R$-module satisfying the dual of Property $A$. Let $\hat{M}$ be any $R$-module. Then $M \oplus \hat{M}$ again, satisfies the dual of Property $A$. Thus, any $R$-module is a submodule, homomorphic image, or direct factor of a module satisfying the dual of Property $A$.

3. THE DUALS OF PROPER STRONG PROPERTY $A$ FOR MODULES

Lemma 3.1. Let $M$ be an $R$-module and $S = R \setminus W_R(M)$. Then $S^{-1}R = R$ if and only if $R = U(R) \cup W_R(M)$, where $U(R)$ is the set of all invertible elements of $R$.

Proof. Assume that $S^{-1}R = R$ and $s \in S$. Then $1/s \in S^{-1}R = R$ implies that $s$ is invertible in $R$. Hence any element of $S$ is invertible in $R$. The converse is clear. \qed

Definition 3.2. We say that an $R$-module $M$ satisfies the dual of proper strong Property $A$ if for any proper finitely generated ideal $I = \langle a_1, a_2, ..., a_n \rangle$ of $R$ such that $a_i \in W_R(M)$ we have $IM \neq M$.

Theorem 3.3. Let $M$ be an $R$-module. Then the following assertions are equivalent:

(a) $M$ satisfies the dual of proper strong Property $A$;

(b) $M$ satisfies the dual of Property $A$ and $m \cap W_R(M)$ is an ideal of $R$ for each maximal ideal $m$ of $R$.

Proof. (a) $\Rightarrow$ (b) Assume that $M$ satisfies the dual of proper strong Property $A$. Clearly, $M$ satisfies the dual of Property $A$. Let $m$ be a maximal ideal of $R$. Let $a, b \in m \cap W_R(M)$ and put $I = \langle a, b \rangle$ the ideal generated by $a$ and $b$. Then $I \subseteq m$ and $a, b \in W_R(M)$. Since $M$ satisfies the dual of proper strong Property $A$, we get that $IM \neq M$. It follows that $I \subseteq m \cap W_R(M)$ and thus $m \cap W_R(M)$ is an ideal of $R$.

(b) $\Rightarrow$ (a) Let $I = \langle a_1, a_2, ..., a_n \rangle$ be a proper finitely generated ideal of $R$ such that $a_i \in W_R(M)$ for $i = 1, ..., n$. Let $m$ be a maximal ideal of $R$ with $I \subseteq m$. Then $a_1, a_2, ..., a_n \in m \cap W_R(M)$. As, by hypotheses, $m \cap W_R(M)$ is an ideal of $R$, it follows that $I \subseteq m \cap W_R(M)$. Now, since $M$ satisfies the dual of Property $A$, we get $IM \neq M$. Hence $M$ satisfies the dual of proper strong Property $A$. \qed
Theorem 3.4. Let $M$ be an $R$-module. Then

$M$ satisfies the dual of strong Property $\mathcal{A}$ ⇒
$M$ satisfies the dual of proper strong Property $\mathcal{A}$ ⇒
$M$ satisfies the dual of Property $\mathcal{A}$.

Proof. The proof is clear from the definitions.

The Examples 3.5 and 3.12 show that the converse of Theorem 3.4 is not true in general.

Example 3.5. The $\mathbb{Z}$-module $\mathbb{Z}$ satisfies the dual of proper strong Property $\mathcal{A}$ but does not satisfies the dual of strong Property $\mathcal{A}$.

Let $R_i$ be a commutative ring with identity and $M_i$ be an $R_i$-module for each $i = 1, 2$. Assume that $M = M_1 \times M_2$ and $R = R_1 \times R_2$. Then $M$ is clearly an $R$-module with component-wise addition and scalar multiplication. Also, each submodule $N$ of $M$ is of the form $N = N_1 \times N_2$, where $N_i$ is a submodule of $M_i$ for each $i = 1, 2$.

Proposition 3.6. Let $R_i$ be a commutative ring with identity and $M_i$ be an $R_i$-module for each $i = 1, 2$. Then

$$W_{R_i \times R_i}(M_1 \times M_2) = (W_{R_i}(M_1) \times R_2) \cup (R_1 \times W_{R_i}(M_2)).$$

Proof. This is straightforward.

Theorem 3.7. Let $R_i$ be a commutative ring with identity and $M_i$ be an $R_i$-module for each $i = 1, 2$. Let $M = M_1 \times M_2$, $R = R_1 \times R_2$, and $S_i = R_i \backslash W_{R_i}(M_i)$. Then the following assertions are equivalent:

(a) $M$ satisfies the dual of proper strong Property $\mathcal{A}$;

(b) $M_i$ satisfies the dual of proper strong Property $\mathcal{A}$ and $S_i^{-1}R_i = R_i$ for each $i = 1, 2$.

Proof. (a) ⇒ (b) Assume that $M$ satisfies the dual of proper strong Property $\mathcal{A}$ and $I_1 = \langle a_1, a_2, ..., a_n \rangle$ is a finitely generated ideal of $R_1$ such that $I_1 \subseteq W_{R_1}(M_1)$. Set

$$I = \langle (a_1, 0), (a_2, 0), ..., (a_n, 0), (0, 1) \rangle.$$  

Then $(0, 1), (a_i, 0) \in W_{R_i \times R_i}(M_1 \times M_2)$ for $i = 1, 2, ..., n$. By part (a), $I(M_1 \times M_2) \neq M_1 \times M_2$. Thus there exists $(x_1, x_2) \in M_1 \times M_2 \setminus I(M_1 \times M_2)$. This implies that $x_1 \notin I_1M_1$. Thus $I_1M_1 \neq M_1$ and $M_1$ satisfies the dual of proper strong Property $\mathcal{A}$. Now let $r_1 \in R_1 \setminus U(R_1)$. Clearly $(r_1, 0) \in W_{R_1 \times R_2}(M_1 \times M_2)$. Set $J = \langle (r_1, 0), (0, 1) \rangle$. Thus by part (a), $J(M_1 \times M_2) \neq M_1 \times M_2$. This implies that $r_1M_1 \neq M_1$ and hence $r_1 \in W_{R_1}(M_1)$. Now by Lemma 3.1, $S_i^{-1}R_i = R_i$. Similarly, one can see that $M_2$ satisfies the dual of proper strong Property $\mathcal{A}$ and

$$S_2^{-1}R_2 = R_2.$$

(b) ⇒ (a) Let $I = \langle (a_1, b_1), (a_2, b_2), ..., (a_n, b_n) \rangle$ be a proper finitely generated ideal of $R$ such that $(a_i, b_i) \in W_{R_i \times R_i}(M_1 \times M_2)$ for each $i = 1, 2, ..., n$. Set $I_1 = \langle a_1, a_2, ..., a_n \rangle$ and $I_2 = \langle b_1, b_2, ..., b_n \rangle$. Then as $I$ is proper, $I_1$ or $I_2$ is proper. Assume that $I_1$ is proper. Then by Lemma 3.1, $I_1 \subseteq W_{R_1}(M_1)$. Thus by part (b), $I_1M_1 \neq M_1$. Hence there exists $x_1 \in M_1 \setminus I_1M_1$. Now $(x_1, 0) \in (M_1 \times M_2) \setminus I(M_1 \times M_2)$ implies that $M$ satisfies the dual of proper strong Property $\mathcal{A}$. □
Theorem 3.8. Let $M$ be an $R$-module and $S = R \setminus W_R(M)$. Then we have the following.

(a) If $S^{-1}R = R$, then $M$ satisfies the dual of proper strong Property $\mathcal{A}$ if and only if $M$ satisfies the dual of Property $\mathcal{A}$.

(b) If $S^{-1}R \neq R$, then $M$ satisfies the dual of proper strong Property $\mathcal{A}$ if and only if $M$ satisfies the dual of strong Property $\mathcal{A}$.

Proof. (a) Since $S^{-1}R = R$, we have $W_R(M) = \bigcup_{m \in \{\text{max}(R)\}} m$ by Lemma 3.1. Thus each maximal ideal $m$ of $R$, we have $m \cap W_R(M) = m$ is always an ideal of $R$. Now the result follows from Theorem 3.3. The reverse implication is clear.

(b) If $M$ satisfies the dual of strong Property $\mathcal{A}$, then clearly, $M$ satisfies the dual of proper strong Property $\mathcal{A}$. Conversely, assume that $M$ satisfies the dual of proper strong Property $\mathcal{A}$. As $S^{-1}R \neq R$, there exists $x \in R$ such that $x$ is not invertible and $x \notin W_R(M)$ by Lemma 3.1. Let $m$ be a maximal ideal of $R$ such that $x \in m$. Let $I = \langle a_1, a_2, \ldots, a_n \rangle$ be a proper ideal of $R$ such that $a_i \in W_R(M)$ for $i = 1, \ldots, n$. Then $xI = \langle xa_1, \ldots, xa_n \rangle \subseteq m$ is a proper ideal of $R$ and $xa_i \in W_R(M)$ for $i = 1, \ldots, n$. Since $M$ satisfies the dual of proper strong Property $\mathcal{A}$, $xIM \neq M$. Thus there exists a completely irreducible submodule $L$ of $M$ such that $xIM \subseteq L \neq M$ by Remark 2.4. Now, as $x \notin W_R(M)$, it follows that $IM \subseteq L$. This implies that $a_iM \subseteq L \neq M$ for $i = 1, \ldots, n$. Hence $M$ satisfies the dual of strong Property $\mathcal{A}$. □

Proposition 3.9. Let $R$ be a zero-dimensional ring. Then any faithful $R$-module $M$ satisfies the dual of proper strong Property $\mathcal{A}$. In particular, any $R$-module $M$ satisfies the dual of proper strong Property $\mathcal{A}$ over $R/\text{Ann}_RM$.

Proof. By Theorem 2.12(c), $M$ satisfies the dual of Property $\mathcal{A}$. We have $W_R(M) \subseteq Z_R(M) = Z_R(R)$ by using the proof of [1, Corollary 2.20]. Therefore, $W_R(M) = Z_R(R)$ because the inverse inclusion is clear. Thus $S^{-1}R = R$, where $S = R \setminus W_R(M) = R \setminus Z_R(R)$. This implies that $M$ satisfies the dual of proper strong Property $\mathcal{A}$ by Theorem 3.8(a). □

Theorem 3.10. Let $R_i$ be a commutative ring with identity and $M_i$ be an $R_i$-module for each $i = 1, 2$. Let $M = M_1 \times M_2$, $R = R_1 \times R_2$, and $S_i = R_i \setminus Z_{R_i}(M_i)$. Then the following assertions are equivalent:

(a) $M$ satisfies the proper strong Property $\mathcal{A}$;

(b) $M_i$ satisfies the proper strong Property $\mathcal{A}$ and $S_i^{-1}R_i = R_i$ for each $i = 1, 2$.

Proof. (a) $\Rightarrow$ (b) Assume that $M$ satisfies the proper strong Property $\mathcal{A}$ and $I_1 = \langle a_1, a_2, \ldots, a_n \rangle$ is a finitely generated ideal of $R_1$ such that $I_1 \subseteq Z_{R_1}(M_1)$. Set

$I = \langle (a_1, 0), (a_2, 0), \ldots, (a_n, 0), (0, 1) \rangle$.

Then $(0, 1), (a_i, 0) \in Z_{R_1 \times R_2}(M_1 \times M_2)$ for $i = 1, 2, \ldots, n$. By part (a), $(0 :_{M_1 \times M_2} I) \neq 0$. Thus there exists $0 \neq (x_1, x_2) \in M_1 \times M_2$ such that $I(x_1, x_2) = 0$. This implies that $0 \neq x_1 \subseteq (0 :_{M_1} I_1)$ and $M_1$ satisfies the proper strong Property $\mathcal{A}$. Now let $r_1 \in R_1 \setminus U(R_1)$. Clearly $(r_1, 0) \in Z_{R_1 \times R_2}(M_1 \times M_2)$. Set $J = \langle (r_1, 0), (0, 1) \rangle$. Thus by part (a), $(0 :_{M_1 \times M_2} J) \neq 0$. This implies that $r_1 \in Z_{R_1}(M_1)$. Now by [1, Lemma 2.1], $S_1^{-1}R_1 = R_1$. Similarly, one can see that $M_2$ satisfies the proper strong Property $\mathcal{A}$ and $S_2^{-1}R_2 = R_2$.

(b) $\Rightarrow$ (a) Let $I = \langle (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \rangle$ be a proper finitely generated ideal of $R$ such that $(a_i, b_i) \in Z_{R_1 \times R_2}(M_1 \times M_2)$ for each $i = 1, 2, \ldots, n$. Set
\( I_1 = \langle a_1, a_2, \ldots, a_n \rangle \) and \( I_2 = \langle b_1, b_2, \ldots, b_n \rangle \). Then as \( I \) is proper, \( I_1 \) or \( I_2 \) is proper.

Assume that \( I_1 \) is proper. Then \( I_1 \subseteq \text{Z}_{R_1}(M_1) \). Thus by part (b), \( (0 :_{M_1} I_1) \neq 0 \).

Hence there exists \( 0 \neq x_1 \in M_1 \) such that \( I_1 x_1 = 0 \). Now \((0, 0) \neq (x_1, 0) \in (0 :_{M_1 \times M_2} I) \) implies that \( M \) satisfies the proper strong Property \( A \). \( \square \)

The following examples show that the concepts of proper strong Property \( A \) and the dual of proper strong Property \( A \) are different in general.

**Example 3.11.** Consider the \( \mathbb{Z} \times \mathbb{Z} \)-module \( \mathbb{Z} \times \mathbb{Z} \). As \( W_2(\mathbb{Z}) = \mathbb{Z} \setminus \{1, -1\} \), we have \( S = \mathbb{Z} \setminus W_2(\mathbb{Z}) = \{1, -1\} \) and so \( S^{-1} \mathbb{Z} = \mathbb{Z} \). Now Theorem 3.10 implies that \( \mathbb{Z} \times \mathbb{Z} \) satisfies the dual of proper strong Property \( A \). But by [11 Example 2.13 (1)], \( \mathbb{Z} \times \mathbb{Z} \) not satisfies the proper strong Property \( A \).

**Example 3.12.** Consider the \( \mathbb{Z} \times \mathbb{Z} \)-module \( \mathbb{Q} / \mathbb{Z} \times \mathbb{Q} / \mathbb{Z} \). Since \( W_2(\mathbb{Q} / \mathbb{Z}) = \{0\} \), we have \( S = \mathbb{Z} \setminus W_2(\mathbb{Q} / \mathbb{Z}) = \mathbb{Z} \setminus \{0\} \) and so \( S^{-1} \mathbb{Z} \neq \mathbb{Z} \). Now Theorem 3.10 implies that \( \mathbb{Q} / \mathbb{Z} \times \mathbb{Q} / \mathbb{Z} \) not satisfies the dual of proper strong Property \( A \). On the other hand since \( Z_2(\mathbb{Q} / \mathbb{Z}) = \mathbb{Z} \setminus \{1, -1\} \), we have \( S = \mathbb{Z} \setminus W_2(\mathbb{Q} / \mathbb{Z}) = \{1, -1\} \) and so \( S^{-1} \mathbb{Z} = \mathbb{Z} \). Clearly, the \( \mathbb{Z} \)-module \( \mathbb{Q} / \mathbb{Z} \) satisfies the proper strong Property \( A \).

Now Theorem 3.10 implies that \( \mathbb{Q} / \mathbb{Z} \times \mathbb{Q} / \mathbb{Z} \) satisfies the proper strong Property \( A \).

**Theorem 3.13.** Let \( M = R/m_1 \oplus R/m_2 \oplus \ldots \oplus R/m_n \) be an \( R \)-module, where \( m_i \in \text{max}(R) \) for \( i = 1, \ldots, n \). Then \( M \) satisfies the dual of proper strong Property \( A \) if and only if either \( \text{max}(R) = \{m_1, m_2, \ldots, m_n\} \) or \( m_1 = m_2 = \ldots = m_n \).

**Proof.** First, note that \( W_R(M) = m_1 \cup m_2 \cup \ldots \cup m_n \). So, it is easy to see that any ideal \( I \) contained in \( W_R(M) \) is contained in some maximal ideal \( m_j \). By Theorem 2.12(d), \( M \) satisfies the dual of Property \( A \) if \( \text{max}(R) = \{m_1, m_2, \ldots, m_n\} \). Then, we get \( S^{-1} R = R \). Then, using Theorem 3.8(a), \( M \) satisfies the dual of proper strong Property \( A \). On the other hand, suppose that \( m_1 = m_2 = \ldots = m_n \). Then \( W_R(M) = m \) is an ideal of \( R \) and hence \( M \) satisfies the dual of proper strong Property \( A \). It follows that \( M \) satisfies the dual of proper strong Property \( A \). Conversely, assume that \( \{m_1, m_2, \ldots, m_n\} \subset \text{max}(R) \) and that \( \text{card}(\{m_1, m_2, \ldots, m_n\}) \geq 2 \). Then, by Lemma 3.1 \( S^{-1} R \neq R \) as there exists a maximal ideal \( m \not\subseteq m_1 \cup m_2 \cup \ldots \cup m_n = W_R(M) \) and thus there exists an element \( x \in m \) which is neither invertible nor \( x \in W_R(M) \). Suppose contrary that \( W_R(M) \) is an ideal. Then as \( W_R(M) = m_1 \cup m_2 \cup \ldots \cup m_n \), there exists \( j \in \{1, \ldots, n\} \) such that \( W_R(M) = m_j \). Hence \( m_1 = m_2 = \ldots = m_n \), which is a contradiction. Therefore, \( W_R(M) \) is not an ideal and so \( M \) not satisfies the dual of strong Property \( A \) by Theorem 2.9. Therefore, by Theorem 3.8(b), \( M \) not satisfies the dual of proper strong Property \( A \), as needed. \( \square \)

4. Properties \( S_f(\mathcal{N}) \) and \( I_f^M(\mathcal{N}) \)

Let \( J \) be an ideal of \( R \) and let \( N \) be a submodule of an \( R \)-module \( M \). Set

\[ S_f(\mathcal{N}) = \{m \in M \mid rm \in N \text{ for some } r \in R - J\}. \]

When \( J \) is a prime ideal of \( R \), then \( S_f(\mathcal{N}) \) is called the saturation of \( N \) with respect to \( J \) or \( J \)-closure of \( N \). \[11, 15, 16. \]

Set

\[ I_f^M(\mathcal{N}) = \bigcap \{L \mid L \text{ is a completely irreducible submodule of } M \text{ and } \} \]
Now, by assumption, there exists a $r \in R - J$. When $J$ is a prime ideal of $R$, then $I^M_J(N)$ is called the $J$-interior of $N$ relative to $M$.

**Definition 4.1.** We say that a submodule $N$ of an $R$-module $M$ satisfies Property $\mathcal{S}_J(N)$ if for each finitely generated submodule $K$ of $M$ with $K \subseteq \mathcal{S}_J(N)$ there exists a $r \in R \setminus J$ with $rK \subseteq N$.

**Definition 4.2.** We say that a submodule $N$ of an $R$-module $M$ satisfies Property $\mathcal{I}_J^M(N)$ (that is the dual of Property $\mathcal{S}_J(N)$) if for each submodule $K$ of $M$ with $M/K$ is finitely cogenerated and $\mathcal{I}_J^M(N) \subseteq K$ there exists a $r \in R \setminus J$ with $rN \subseteq K$.

**Definition 4.3.** We say that a submodule $N$ of an $R$-module $M$ satisfies the strong Property $\mathcal{S}_J(N)$ if for any $m_1, ..., m_n \in \mathcal{S}_J(N)$ there exists a $r \in R \setminus J$ with $rm_1 \in N$, ...$rm_n \in N$.

**Example 4.4.** Let $J$ be a prime ideal of $R$ and $N$ be a submodule of an $R$-module $M$ such that $\mathcal{S}_J(N)$ is a finitely generated submodule of $M$. Then one can see that there exists a $r \in R \setminus J$ with $rN \subseteq \mathcal{S}_J(N)$. This implies that $N$ satisfies the strong Property $\mathcal{S}_J(N)$.

**Example 4.5.** Let $J$ be a prime ideal of $R$ and $N$ be a submodule of an $R$-module $M$ such that $M/\mathcal{S}_J(N)$ is a finitely cogenerated $R$-module. Then $N$ satisfies Property $\mathcal{I}_J^M(N)$ by using [10, Lemma 2.3].

If $J = 0$ and $N = 0$ in Definition 4.1 (resp. Definition 4.3), then $M$ under the name Property $T$ (strong Property $T$) was studied in [2].

**Theorem 4.6.** Let $M$ be an $R$-module. Then we have the following.

(a) A submodule $N$ of $M$ satisfies the strong Property $\mathcal{S}_J(N)$ if and only if $N$ satisfies Property $\mathcal{S}_J(N)$ and $\mathcal{S}_J(N)$ is a submodule of $M$.

(b) The zero submodule of $M$ satisfies the strong Property $\mathcal{S}_J(t)$.

(c) If a submodule $N$ of $M$ satisfies Property $\mathcal{S}_J(N)$ (respectively, strong Property $\mathcal{S}_J(N)$) and $N \subseteq K \subseteq \mathcal{S}_J(N)$, then $K$ satisfies Property $\mathcal{S}_J(K)$ (respectively, strong Property $\mathcal{S}_J(K)$).

(d) Let $\psi = \{N_\lambda\}_{\lambda \in \Lambda}$ be a chain of submodules of $M$ with $N \subseteq N_\lambda \subseteq \mathcal{S}_J(N)$ for each $\lambda \in \Lambda$. Then $\cup_{\lambda \in \Lambda}N_\lambda$ satisfies Property $\mathcal{S}_J(\cup_{\lambda \in \Lambda}N_\lambda)$ (respectively, strong Property $\mathcal{S}_J(\cup_{\lambda \in \Lambda}N_\lambda)$) if and only if each $N_\lambda$ satisfies Property $\mathcal{S}_J(N_\lambda)$ (respectively, strong Property $\mathcal{S}_J(N_\lambda)$).

(e) If for a submodule $N$ of $M$ we have $\text{Ann}_R(M/N) \nsubseteq J$, or more generally, $\text{Ann}_R(\mathcal{S}_J(N)/N) \nsubseteq J$, then $N$ satisfies Property $\mathcal{S}_J(N)$.

(f) If $J$ is an irreducible ideal of $R$ (e.g., $R/J$ is an integral domain), then every submodule $N$ of $M$ satisfies strong Property $\mathcal{S}_J(N)$.

(g) If a submodule $N$ of $M$ is a Bezout module (respectively, chained module), then $N$ satisfies Property $\mathcal{S}_J(N)$ (respectively, strong Property $\mathcal{S}_J(N)$).

**Proof.** (a), (b), (c), and (g) are straightforward.

(c) Let $T$ be a finitely generated submodule of $M$ with $T \subseteq \mathcal{S}_J(K)$. Then $T \subseteq \mathcal{S}_J(K) \subseteq \mathcal{S}_J(\mathcal{S}_J(N))$. Clearly, $\mathcal{S}_J(\mathcal{S}_J(N)) = \mathcal{S}_J(N)$. Hence $T \subseteq \mathcal{S}_J(N)$. Now, by assumption, there exists a $r \in R \setminus J$ with $rT \subseteq N$ and so $rT \subseteq K$.

(d) First note that, if $\cup_{\lambda \in \Lambda}N_\lambda$ satisfies Property $\mathcal{S}_J(\cup_{\lambda \in \Lambda}N_\lambda)$ (respectively, strong Property $\mathcal{S}_J(\cup_{\lambda \in \Lambda}N_\lambda)$), then each $N_\lambda$ satisfies Property $\mathcal{S}_J(N_\lambda)$ (respectively, strong Property $\mathcal{S}_J(N_\lambda)$). Conversely, suppose that each $N_\lambda$ satisfies Property $\mathcal{S}_J(N_\lambda)$ (respectively, strong Property $\mathcal{S}_J(N_\lambda)$) and $K$ is a finitely generated...
submodule of \( S_f(\bigcup_{\alpha\in A}N_\alpha) \). Then \( K \) is a submodule of \( S_f(N_\alpha) \) for some \( \alpha \in A \) and hence \( (N_\alpha :_RG) \not\subseteq I \) and so \( (\bigcup_{\alpha\in A}N_\alpha :_RG) \not\subseteq I \), as desired.

(f) Let \( J \) be an irreducible ideal of \( R \) and \( m_1, \ldots, m_n \in S_f(N) \), where \( r_i m_i \in N \) with \( r_i \in R \setminus J \) for \( i = 1, 2, \ldots, n \). Since \( J \) is irreducible, \( Rr_1 \cap \ldots \cap Rr_n \neq J \). Hence there exists \( r \in (Rr_1 \cap \ldots \cap Rr_n) \setminus J \). Thus \( rm_i \in N \) for \( i = 1, 2, \ldots, n \), as needed. \( \Box \)

**Theorem 4.7.** Let \( M \) be an \( R \)-module. Then we have the following.

(a) Let \( J \) be a prime ideal of \( R \). A submodule \( N \) of \( M \) satisfies Property \( \mathcal{I}^M(J) \) if and only if for any completely irreducible submodules \( I_1, \ldots, I_n \) of \( M \) with \( \mathcal{I}^M(J) \subseteq I_1, \ldots, \mathcal{I}^M(J) \subseteq I_n \) there exists \( a \in R \setminus J \) with \( aN \subseteq I_1, \ldots, aN \subseteq I_n \).

(b) \( M \) satisfies Property \( \mathcal{I}^M_0(M) \) if and only if every submodule \( K \) of \( M \) with \( M/K \) is finitely cogenerated satisfies Property \( \mathcal{I}^M_0(M/K) \).

(c) If for a submodule \( N \) of \( M \) we have \( \text{Ann}_R(N) \not\subseteq J \), or more generally, \( \text{Ann}_R(N/\mathcal{I}^M(J)) \not\subseteq J \), then \( N \) satisfies Property \( \mathcal{I}^M(J) \).

(d) If \( J \) is an irreducible ideal of \( R \) (e.g., \( R/J \) is an integral domain), then every submodule \( N \) of \( M \) satisfies Property \( \mathcal{I}^M(J) \).

**Proof.** (a) The necessity is clear. For the sufficiency assume that for a submodule \( K \) of \( M \) with \( M/K \) is finitely cogenerated we have \( \mathcal{I}^M(J) \subseteq K \). As \( M/K \) is finitely cogenerated, there exist completely irreducible submodules \( L_1, \ldots, L_n \) of \( M \) such that \( K = \bigcap_{i=1}^n L_i \). Now by assumption, there exist \( r_1, \ldots, r_n \in R \setminus J \) such that \( r_i N \subseteq L_i \) for \( i = 1, 2, \ldots, n \). Set \( r = r_1 r_2 \ldots r_n \). As \( J \) is prime, \( r \in R \setminus J \). Now \( rN \subseteq K \), as needed.

(b) and (c) are straightforward.

(d) Let \( J \) be an irreducible ideal of \( R \) and \( \mathcal{I}^M(J) \subseteq L_i \), where \( L_i \) is a completely irreducible submodule of \( M \), \( r_i N \subseteq L_i \), and \( r_i \in R \setminus J \) for \( i = 1, 2, \ldots, n \). As \( J \) is irreducible, \( Rr_1 \cap \ldots \cap Rr_n \neq J \). Hence there exists \( r \in (Rr_1 \cap \ldots \cap Rr_n) \setminus J \). Thus \( rN \subseteq L_i \) for \( i = 1, 2, \ldots, n \), as needed. \( \Box \)

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