The four-loop renormalization group QCD and QED $\beta$-functions in the $V$-scheme and their analogy with the Gell-Mann–Low function in QED and QCD

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(Dated: April 27, 2015)

Abstract

The semi-analytical expression for the forth coefficient of the renormalization group $\beta$-function in the $V$-scheme is obtained in the case of the $SU(N_c)$ gauge group. In the process of calculations we use the three-loop perturbative approximation for the QCD static potential, evaluated in the $\overline{\text{MS}}$-scheme. The importance of getting more detailed expressions for the $n_f$-independent three-loop contribution to the static potential, obtained at present by two groups, is emphasised. The comparison of the numerical structure of the four-loop approximations for the RG $\beta$-function of QCD in the gauge-independent $V$- and $\overline{\text{MS}}$-schemes and in the minimal MOM scheme in the Landau gauge are presented. Considering the limit of QED with $N$-types of leptons we discover that the $\beta^V$-function is starting to differ from the Gell-Mann–Low function $\Psi(\alpha_{\text{MOM}})$ at the level of the forth-order perturbative corrections, receiving the proportional to $N^2$ additional term. Taking this feature into account, we propose to consider the $\beta^V$-function as the most theoretically substantiated analog of the Gell-Man–Low function in QCD.

PACS numbers: 12.38.Bx, 12.20.-m, 11.10.Hi

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I. INTRODUCTION

One of the main quantities, which enter in the renormalization group (RG) method, developed in the classical works of Refs. [1], [2], [3], is the RG $\beta$-function. It defines the energy behaviour of the renormalized coupling constants of the renormalized quantum field models. In the case of applying gauge-invariant ultraviolet (UV) subtractions schemes, the RG $\beta$-function, which is computed by the perturbation theory (PT) method, depends on the definition of these schemes starting from the third term.

In QED the first two scheme-independent coefficients of the $\beta$-function were obtained in Ref. [3] from the performed in Ref. [4] analytical calculations of the two-loop approximation for the renormalized photon propagator.

In the momentum (MOM) scheme with the subtractions of the UV divergences of the photon vacuum polarization function in QED at the non-zero Euclidian point $\lambda^2$ the RG $\beta$-function coincides with the Gell-Man—Low function $\Psi(\alpha_{MOM})$, where $\alpha_{MOM}$ is the QED invariant charge. It is uniquely defined by the combinations of the Green functions [5]. The analytical expression for $\Psi(\alpha_{MOM})$ depends on the number of leptons $N$.

For $N=1$, i.e. in the case of consideration of the electron only, the three-loop term of $\Psi$-function was calculated analytically in Ref. [8]. This result was generalized to the case of the arbitrary number of $N$ massless leptons in Ref. [9]. In this more general form the four- and five-loop corrections to the Gell-Man—Low function were analytically calculated in Ref. [10] and Ref. [11] correspondingly. At $N=1$ the more general result of Ref. [11] coincides with the similar analytical expression, obtained in Ref. [12]. This feature can be considered as the strong argument in a favour of consistency of the complicated analytical five-loop calculations, performed in Ref. [11].

Another important scheme, which is used in QED, is the on-shell (OS) scheme. In this scheme the photon vacuum polarization function is defined by subtracting UV divergences at zero transferred momentum, while the renormalized on-shell masses of leptons are identified with their experimentally measured values. In QED coupling constant of this scheme $\alpha_{OS}$ is also called the QED fine coupling constant.

In the physical OS scheme the calculations of the $\beta(\alpha_{OS})$ were done at the three-loop level in Ref. [13] and at the four-loop level in Ref. [14]. In the case of arbitrary $N$ the five-loop contributions were evaluated in Ref. [15]. For $N=1$ it agrees with the analytical result for the same coefficient, extracted in Ref. [12] from the RG-relations between several analytical results, previously existing in the literature. The agreement with the results of direct calculations of Ref. [15] gives extra confidence in the correctness and self-consistency of the used in Ref. [12] results of the complicated computer calculations.

The third class of schemes which we will be interested in, is introduced when the dimensional regularization [16] is used. These schemes include the minimal subtractions (MS) scheme [17] and its modified variants, namely the MS-scheme [18] and the G-scheme [19]. It is possible to prove that in all these modifications of the MS-scheme the RG $\beta$-function is the same.

At $N=1$ the three-loop correction to the $\beta(\alpha_{MS})$ was evaluated Ref. [19] and Ref. [20] independently (this result had been also presented in the review of Ref. [21]). In the case of
the arbitrary $N$ the three-loop contribution to $\beta(\alpha_{\text{MS}})$-function was analytically obtained in Ref. [9]. The computation of the four-loop term was completed in Ref. [10]. The five-loop correction to the QED $\beta$-function in the MS-like schemes was calculated in Ref. [11]. At $N = 1$ this expression coincides with the result of non-direct analysis, performed in Ref. [12].

It is known that in QCD the MS-like schemes maintain the explicit gauge-independence of various RG quantities. This property clarifies why in multiloop QCD calculations the MS-like schemes are used more often. In QCD the first coefficient of the $\beta$-function was computed in Refs. [22], [23], [24] and for the number of quarks flavours $n_f \leq 6$ turned out to be negative. This feature revealed the existence of the asymptotic-freedom property in the gauge theory of strong interactions. The two-loop corrections to the QCD $\beta$-function in the MS-like schemes were analytically evaluated in Refs. [25], [26], [27] and are also negative $^a$.

In the MS-like schemes the three-loop QCD $\beta$-function was analytically calculated in Ref. [29]. This result was confirmed later in Ref. [30]. The four-loop term of the QCD $\beta$-function in the MS-like schemes was calculated in Ref. [31] and confirmed independently in Ref. [32]. For $n_f = 6$ the three-loop correction to the $\beta$-function in the MS-like schemes is positive (see results presented below). However, using the special method of resummation of the PT series the authors of Ref. [33] demonstrated, that this feature does not affect the property of asymptotic freedom in QCD.

In our work we will get the expression of the QCD $\beta$-function at the four-loop level in another gauge-independent scheme, namely in the V-scheme. This scheme is determined by perturbative high order QCD corrections to the static potential. As it will be shown in this work the extra interest to the expression of the QCD $\beta$-function in the V-scheme is supported by the fact that it can be considered as the real gauge-independent analog of the Gell-Man–Low function in QCD. This feature becomes more clear after presented in Sec.VI study of the four-loop QED approximation of the $\beta^V$-function. In Sec. IV the comparison between the four-loop approximation of the QCD $\beta$-function in the V-scheme, in the MS-like schemes and in the the widely used at present gauge-dependent minimal-MOM subtractions scheme is presented. The specific features of the structure of the PT series for the $\beta^V$-function in QCD and QED are outlined.

II. PRELIMINARIES: THE HIGH-ORDER EXPRESSION OF THE STATIC POTENTIAL IN QCD IN THE MS-SCHEME

To define the four-loop approximation of the QCD $\beta$-function in the V-scheme at the beginning we will summarise the current knowledge about various perturbative QCD contributions to the static potential. This physical quantity enters various QCD applications, e.g. theoretical determinations of charm-, bottom and and top-quark masses, the studies

$^a$ The first calculation of the two-loop contribution to the QCD $\beta$-function [28] contained the bug, which resulted in the positive value of two-loop term. It was associated with the realisation of the IR-fixed point of the QCD $\beta$-function. This unexpected conclusion stimulated the independent correct calculation of this term [25]-[27], which demonstrated that the theoretical conclusion of Ref. [28] is not correct.
of the properties of bound states of the $c$ and $b$-quarks \[34],\[35],\[36] and references therein), etc.

The static potential in QCD is introduced as a potential of interaction between static quark and anti-quark at a distance $r$. It is defined by the following Fourier representation

$$V_{QCD}(\mu^2 r^2, \alpha_s(\mu^2)) = \int \frac{d^3 q}{(2\pi)^3} e^{i\vec{q}\vec{r}} V(q^2, \mu^2, \alpha_s(\mu^2))$$

$$= \int \frac{d^3 q}{(2\pi)^3} e^{i\vec{q}\vec{r}} \left(-4\pi C_F \frac{\alpha_s(q^2/\mu_V^2)}{q^2}\right)$$

where $\alpha_{s,V}(q^2/\mu_V^2)$ is the QCD coupling constant in V-scheme, $\alpha_s/4\pi = g^2/16\pi^2$, $g$ is the renormalized strong coupling constant of the QCD Lagrangian, $T^a$ is the generator of the $SU(N_c)$-group, normalized as $T^a = \lambda^a/2$ and $C_F$ is the Casimir operator, defined as $(T^a T^a)_{ij} = C_F\delta_{ij}$. In the V-scheme its coupling constant $\alpha_{s,V}(q^2/\mu_V^2)$ is related to the numerator of the momentum representation of the static potential in the $\overline{\text{MS}}$-scheme, defined in Eq. (2.1) by the following perturbative expression

$$\alpha_{s,V}(q^2/\mu_V^2) = \alpha_s(\mu^2) P(\alpha_s(\mu^2)) = \alpha_s(\mu^2) \sum_{n=0}^{\infty} P_n^{\overline{\text{MS}}}(L) \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^n. \quad (2.2)$$

The r.h.s. of Eq. (2.2) is expressed through higher order PT QCD corrections to the static potential $P_n^{\overline{\text{MS}}}(L)$ in the $\overline{\text{MS}}$-scheme which are known at present up to $O(\alpha_s^3)$-level and will be presented below.

The evolution of the $\overline{\text{MS}}$-scheme coupling constant $\alpha_s(\mu^2)$ (which depends on the $\overline{\text{MS}}$-scheme renormalization parameter $\mu^2$) is governed by the QCD $\overline{\text{MS}}$-scheme $\beta$-function. It is defined as

$$\mu^2 \frac{\partial (\alpha_s/4\pi)}{\partial \mu^2} = \beta^{\overline{\text{MS}}}(\alpha_s) = -\sum_{i=0}^{\infty} \beta_i \left(\frac{\alpha_s}{4\pi}\right)^{i+2}, \quad (2.3)$$

where the analytical expressions of the known four coefficients in $\overline{\text{MS}}$-scheme are taken from the work of Ref. [31] and have the following form

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$$

$$\beta_1 = \frac{34}{3} C_A^2 - 4C_F T_F n_f - \frac{20}{3} C_A T_F n_f$$

$$\beta_2 = \frac{2857}{54} C_A^3 + 2C_F^2 T_F n_f - \frac{205}{9} C_F C_A T_F n_f - \frac{1415}{27} C_A^2 T_F n_f +$$

$$+ \frac{44}{9} C_F T_F^2 n_f^2 + \frac{158}{27} C_A T_F^2 n_f^2$$
\[
\beta_3 = \left( \frac{150653}{486} - \frac{44}{9} \zeta(3) \right) C_A^4 + \left( -\frac{39143}{81} + \frac{136}{3} \zeta(3) \right) C_A^3 T_F n_f + \left( \frac{-4204}{27} + \frac{352}{9} \zeta(3) \right) C_A C_F^2 T_F n_f + 46 C_F^3 T_F n_f + \left( \frac{7930}{81} + \frac{224}{9} \zeta(3) \right) C_A^2 T_F^2 n_f^2 + \left( \frac{1352}{27} - \frac{704}{9} \zeta(3) \right) C_A^2 T_F^2 n_f^2 + \left( \frac{17152}{243} + \frac{448}{9} \zeta(3) \right) C_A C_F T_F^2 n_f^2 + \frac{424}{243} C_A T_F^3 n_f^2 + \frac{1232}{243} C_F T_F^3 n_f^2 + \left( \frac{-80}{9} + \frac{704}{3} \zeta(3) \right) \frac{d_{A}^{abcd} d_{A}^{abcd}}{N_A} n_f + \left( \frac{-704}{9} + \frac{512}{3} \zeta(3) \right) \frac{d_{F}^{abcd} d_{F}^{abcd}}{N_A} n_f^2 \]  

The characteristic colour structures of the group \( SU(N_c) \) are defined as in the detailed work of Ref.\[37\]. In the notations of Ref.\[37\] we have \([T^a, T^b] = i f^{abc} T^c \), where \( f^{abc} \) are antisymmetric (under permutations of any pair of indices) structure constants, which satisfy the well-known relation \( f^{aecd} f^{bced} = C_A \delta^{ab} \), \( C_A \) and \( C_F \) are the Casimir operators, \( Tr(T^a T^b) = T^c \delta^{ab} \), \( N_A \) is the number of generators of the Lie algebra of the \( SU(N_c) \), \( n_f \) is the number of quarks flavors, \( a_{F}^{abcd} = Tr(T^a T^b T^c T^d) / 6 \) is the total symmetric tensor, the notations \( (\cdot) \) are defining the procedure of symmetrisation of the generators \( T^a T^b T^c T^d \), \( a_{A}^{abcd} = Tr(C^{a} C^{b} C^{c} C^{d}) / 6 \) is the total symmetric tensor of \( (C^{a})_{bc} = -i f^{abc} \), where \( C^{a} \) are the generators of the adjoint representation of the Lie algebra of \( SU(N_c) \)-group. The expressions for the corresponding colour structures in Eqs.\( (2.4) - (2.7) \) have the form, first presented in Ref.\[31\]:

\[
C_A = N_c \quad , \quad C_F = \frac{N_c^2 - 1}{2 N_c} \quad , \quad N_A = N_c^2 - 1 \tag{2.8} \\
\frac{d_{A}^{abcd} d_{A}^{abcd}}{N_A} = \frac{N_c^2 (N_c^2 + 36)}{24} \quad , \quad \frac{d_{F}^{abcd} d_{F}^{abcd}}{N_A} = \frac{N_c (N_c^2 + 6)}{48} \tag{2.9} \\
\frac{N_c^4 - 64 N_c^2 + 18}{96 N_c^2} \tag{2.10}
\]

The explicit expressions for the proportional to the \( n \)-th powers of \( L = \ln \left( \mu^2 / q^2 \right) \) terms \( P_{n}^{\overline{\text{MS}}} (L) \) in the polynomial \( P(\alpha_s(\mu^2)) \) of Eq.\( (2.2) \) read: \( P_{0}^{\overline{\text{MS}}} = 1 \), \( P_{1}^{\overline{\text{MS}}} (L) = a_{1}^{\overline{\text{MS}}} + \beta_0 L \), \( P_{2}^{\overline{\text{MS}}} (L) = a_{2}^{\overline{\text{MS}}} + (2 a_{1}^{\overline{\text{MS}}} \beta_0 + \beta_1) L + \beta_0^2 L^2 \), \( P_{3}^{\overline{\text{MS}}} (L) = a_{3}^{\overline{\text{MS}}} + (3 a_{2}^{\overline{\text{MS}}} \beta_0 + 2 a_{1}^{\overline{\text{MS}}} \beta_1 + \beta_2^{\overline{\text{MS}}}) L + (3 a_{1}^{\overline{\text{MS}}} \beta_0^2 + \frac{5}{2} \beta_0 \beta_1) L^2 + \beta_0^3 L^3 \). The coefficients before powers of \( L \) satisfy the RG-equation in the MS-like schemes at the three-loop level and were checked in the process of this work.

The coefficients \( a_{i}^{\overline{\text{MS}}} \) are calculated from the concrete Feynman diagrams. The first one, \( a_{1}^{\overline{\text{MS}}} \), was calculated long time ago in Refs.\[38\], \[39\] and has the following form

\[
a_{1}^{\overline{\text{MS}}} = \frac{31}{9} C_A - \frac{20}{9} T_F n_f \tag{2.11}
\]

5
The coefficient $a_2^{\overline{\text{MS}}}$ was found in the works [40, 41]. The bug in the obtained Refs. [40], [41] pure Yang-Mills contribution to $a_2^{\overline{\text{MS}}}$ was detected in Ref. [42].

The final result of these analytical calculations of Refs. [40], [42] is

$$a_2^{\overline{\text{MS}}} = \left( \frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3} \zeta(3) \right) C_A^2 - \left( \frac{1798}{81} + \frac{56}{3} \zeta(3) \right) C_A T_F n_f$$

\begin{equation}
- \left( \frac{55}{3} - 16\zeta(3) \right) C_F T_F n_f + \left( \frac{20}{9} T_F n_f \right)^2. \tag{2.12}
\end{equation}

The three-loop $\overline{\text{MS}}$-scheme perturbative contribution to the static potential can be presented as

$$a_3^{\overline{\text{MS}}} = a_3^{(3)} n_f^3 + a_3^{(2)} n_f^2 + a_3^{(1)} n_f + a_3^{(0)} \tag{2.13}$$

The $n_f$-dependent terms were computed in Ref. [43] and have the following form:

$$a^{(3)} = -\left( \frac{20}{9} \right)^3 T_F^3 \tag{2.14}$$

$$a_3^{(2)} = \left( \frac{12541}{243} + \frac{368}{3} \zeta(3) + \frac{64\pi^4}{135} \right) C_A T_F^2 + \left( \frac{14002}{81} - \frac{416}{3} \zeta(3) \right) C_F T_F^2 \tag{2.15}$$

$$a_3^{(1)} = -709.717 C_A^2 T_F + \left( -\frac{71281}{162} + 264 \zeta(3) + 80 \zeta(5) \right) C_A C_F T_F + \left( \frac{286}{9} + \frac{296}{3} \zeta(3) - 160 \zeta(5) \right) C_F^2 T_F - 56.83(1) \frac{d_{abcd} d_{abcd}}{N_A} \tag{2.16}$$

where the error of numerical calculation of $C_A^2 T_F$-coefficient in Eq. (2.16) is not indicated.

It is worth to emphasize that in the QED limit with $C_A = 0$, the analytical expressions of the terms, which are proportional to the powers of $n_f$ and $T_F=1$ in Eqs. (2.11), (2.12) and in Eqs. (2.14) - (2.16), are in agreement with the presented in the work [44] analytical $\overline{\text{MS}}$-scheme results for the constant terms of the three-loop approximation of the photon vacuum polarization function in QED. They were also confirmed in Ref. [15] in the process of computation of the four-loop approximation of this quantity. The agreement with the QED results of Refs. [44] gives extra confidence in the validity of the related analytical expressions of Ref. [43].

The numerical expressions of the $n_f$-independent contributions to Eq. (2.13) were obtained in Ref. [43] and read

$$a_3^{(0)} = 502.24(1) C_A^3 - 136.39(12) \frac{d_{abcd} d_{abcd}}{N_A} \tag{2.17}$$

These results agree with the results of the independent calculation of Ref. [46]

$$a_3^{(0)} = 502.22(12) C_A^3 - 136.33(14) \frac{d_{abcd} d_{abcd}}{N_A} \tag{2.18}$$

\(b\) The corrected result was confirmed later by the author in Ref. [46].
which, however, have greater inaccuracies. Moreover, it is not clear, whether the second coefficient in Eq. (2.18) should read as -136.3(14) or -136.30(14). In view of this it will be highly desirable to perform more detailed comparison between the results of Eq. (2.17) and Eq. (2.18). It will be even more interesting to get the analytical expressions for the numerically calculated coefficients of $a_3^{(0)}$ and $a_3^{(1)}$.

The three-loop $n_f$-independent correction to the static potential contain also the RG-controllable additional term $8 \pi^2 C_F^4 L$ [47]. It is associated with the infrared (IR) divergences, which begin to manifest themselves in the the static potential at the three-loop level [48], [49]. In the concrete physical analysis these IR-divergent $L$-terms are cancelled by the concrete UV-divergent contributions in the effective theory of heavy quarkonium — non-relativistic QCD (see e.g. [49]).

The main aim of this work is to find the four-loop approximation of the RG $\beta$-function in the V-scheme. This can be done by application of the RG-motivated effective charges (ECH) approach, developed in all orders of PT in the works of Refs. [50], [51] and independently at the next-to-leading order in Ref. [52] (see the work of Ref. [53], [54] as well). The four-loop approximation of the $\beta$-function in the V-scheme defines the evolution of $\alpha_s V$ in the region of intermediate and UV values of energy scales. It does not depend on the manifestation of IR physical effects, and the RG-uncontrollable $L$-dependent corrections to the static potential. In view of this we will not consider them in our further analysis.

**III. GENERAL RELATIONS BETWEEN COEFFICIENTS OF $\beta$-FUNCTIONS IN THE GAUGE-IN Variant SCHEMES AT THE FOUR-LOOP LEVEL**

Let us start this Section from writing the RG-equation for the static potential in the MS-scheme, which is defined in Eq. (2.1). In the massless limit, which is considered in this work, it has the following form

$$\left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) V(q^2, \mu^2, \alpha_s(\mu^2)) = 0$$

The study of the scheme-dependence properties of the the QCD PT series and of the QCD $\beta$-function in particular is the more delicate issue than the study of the discussed in the the Introduction problem of the scheme-dependence of the QED $\beta$-functions.

Indeed, on the contrary to the QED case, in QCD it is impossible to define the gauge-invariant analog of the MOM-scheme (see e.g. [55], [56], [57]) and thus to construct the QCD invariant charge by the unique gauge-invariant manner. In QCD the number of invariant-type charges of MOM schemes is proportional to four, namely to the number of vertexes of the QCD Lagrangian in the covariant gauges (i.e. gluon-quark-antiquark, gluon-ghost-ghost, three-gluon and four-gluon vertexes). Moreover, the definitions of these QCD invariant-type charges depend on different kinematic condition for fixing the scales of subtractions of UV divergences in the renormalized Green functions, which enter these different QCD invariant-type charges. Indeed, fixing the kinematics conditions by the different way it is possible to construct, for example, the symmetric MOM scheme [55], the variant of symmetric MOM scheme with one external zero momentum [56] and the
asymmetric MOM (AMOM) scheme \[57\]. Different gauge-dependent MOM schemes were used in the direct calculations of the massless two-loop \[58\], \[59\], \[60\], \[61\], \[62\], \[63\], \[64\] three-loop \[61\], \[62\], \[63\], \[64\] and even four-loop corrections \[62\], \[64\], \[65\] to the QCD \(\beta\)-function. These analytical calculations revealed the importance of the careful study of the dependence on gauge parameter \(c\). The classical example of the validity of this statement is the discovery, that in the AMOM the non-proper choice of the gauge in the two-loop PT correction to the QCD \(\beta\)-function can destroy the asymptotic freedom property of perturbative QCD \[59\], \[60\].

Summarising the discussions of the gauge ambiguities in the QCD analogs of the invariant charges of various MOM-schemes, we stress that in these schemes it is impossible to construct gauge-invariant QCD analog of the Gell-Man–Low function. In view of this it is of interest to study the expansions of the \(\beta\)-function in terms of physical coupling constant, which enter the effective LO approximations of the QCD static potential \(V\) \[66\].

In all these studies the ECH of Refs.\[50\], \[51\] was used. To remind the basis of this approach consider first the system of Eq.(2.1) and Eq.(2.2), which defines the expansion of the QCD coupling constant in the \(\overline{\text{MS}}\)-scheme.

At the first step, following the NLO definition of the ECH scheme of Ref.\[52\] we define the effective scale of the \(V\)-scheme as

\[
\mu^2_V = \exp[a_1^{\overline{\text{MS}}} / \beta_0] \mu^2_{\overline{\text{MS}}} \tag{3.1}
\]

where \(a_1^{\overline{\text{MS}}} = \frac{31}{9} C_A - \frac{20}{9} T_F n_f\) and \(\beta_0\) is the first coefficient of the QCD \(\beta\) function in the MS-like schemes, which is defined in Eq.(2.3). At the next step we fix \(q^2 = \mu^2_V\) in Eq.(2.2) and get the following relation between the QCD effective charge of the \(V\)-scheme and the QCD coupling constant \(\alpha_s, V\):

\[
\alpha_s(V, \mu^2_V) = \alpha_{s, \overline{\text{MS}}}(\mu^2_V) P(\alpha_s, \overline{\text{MS}}) \tag{3.2}
\]

\[
= \alpha_{s, \overline{\text{MS}}}(\mu^2_V) \left[1 + a_2^{\overline{\text{MS}}} \left(\frac{\alpha_s(V, \mu^2_V)}{4\pi}\right) + a_3^{\overline{\text{MS}}} \left(\frac{\alpha_s(V, \mu^2_V)}{4\pi}\right)^2 + O(\alpha_s^3, \overline{\text{MS}})\right].
\]

Now it is possible to define the ECH \(\beta\)-function of static potential, which corresponds to the RG \(\beta\)-function in the \(V\)-scheme:

\[
\mu^2_V \frac{\partial (\alpha_s V / 4\pi)}{\partial \mu^2_V} = \beta^V(\alpha_s, V) = - \sum_{i=0}^{\infty} \beta_i^V \left(\frac{\alpha_s(V, \mu^2_V)}{4\pi}\right)^{i+2}. \tag{3.3}
\]

The standard RG-equation relates \(\beta^V\)-function to the \(\beta\)-function in the MS-like schemes:

\[
\beta^V(\alpha_s(V, \alpha_{s, \overline{\text{MS}}}(\mu^2_V))) = \beta^{\overline{\text{MS}}}(\alpha_{s, \overline{\text{MS}}}(\mu^2_V)) \frac{d\alpha_s(V, \alpha_{s, \overline{\text{MS}}}(\mu^2_V))}{d\alpha_{s, \overline{\text{MS}}}(\mu^2_V)} \quad . \tag{3.4}
\]

\(c\) It is worth emphasise that in the Landau gauge the two-loop expressions to the QCD \(\beta\)-function in the number of MOM-schemes coincide with the MS-scheme results.
Consider now the similar to Eq.(3.4) equation between the \( \tilde{\beta} \)-functions, computed in the gauge-invariant UV subtractions schemes:

\[
\tilde{\beta}(\tilde{\alpha}_s(\alpha_s)) = \beta(\alpha_s) \frac{d\tilde{\alpha}_s(\alpha_s)}{d\alpha_s}.
\] (3.5)

where we use the similar normalization conditions for both \( \tilde{\beta} \) and \( \beta \)-functions. Wherein, the included in Eq.(3.5) \( \beta \)-functions satisfy the standard RG-equation

\[
\mu \frac{\partial(\tilde{\alpha}_s/4\pi)}{\partial \mu^2} = \tilde{\beta}_s(\tilde{\alpha}_s) = -\sum_{i=0}^{\infty} \tilde{\beta}_i \left(\frac{\alpha_s}{4\pi}\right)^{i+2}.
\] (3.6)

For these normalization conditions the coupling constant of one gauge-invariant renormalization scheme \( \tilde{\alpha}_s(\mu) \) is related to the coupling constant \( \alpha_s(\mu) \) of another gauge invariant renormalization scheme in such way that they coincide at zero order:

\[
\tilde{\alpha}_s(\mu^2) = \alpha_s(\mu^2) \left(1 + a_1 \left(\frac{\alpha_s(\mu^2)}{4\pi}\right) + a_2 \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 + a_3 \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^3 + O(\alpha^4_s)\right).
\] (3.7)

Taking into account Eq.(3.5), the definitions for \( \tilde{\beta}(\tilde{\alpha}_s) \) in Eq.(3.6) and the relation of Eq.(3.7), it is possible to get the following links between the coefficients of \( \beta \)-functions in two gauge-invariant schemes:

\[
\tilde{\beta}_0 = \beta_0
\] (3.8)

\[
\tilde{\beta}_1 = \beta_1
\] (3.9)

\[
\tilde{\beta}_2 = \beta_2 - a_1 \beta_1 + (a_2 - a_1^2) \beta_0
\] (3.10)

\[
\tilde{\beta}_3 = \beta_3 - 2a_1 \beta_2 + a_1^2 \beta_1 + (2a_3 - 6a_1 a_2 + 4a_1^3) \beta_0
\] (3.11)

Thus, these formulae reflect the transformation law of \( \beta \)-function from one gauge-invariant renormalization scheme to another.

IV. THE FOUR-LOOP RESULTS FOR THE QCD \( \beta \)-FUNCTION IN THE \( V \)-SCHEME AND THEIR NUMERICAL EXPRESSIONS

Let us now define the four-loop approximation for the coefficients of the QCD \( \beta_V(\alpha_s,V) \)-function, related to the QCD \( \beta_{\overline{\text{MS}}} \)-function through Eq.(3.4). The expressions for its coefficients, which are gauge-independent and scheme-independent, can be obtained from Eqs.(3.8)-(3.11) and read

\[
\beta_0^V = \beta_0^{\overline{\text{MS}}} = \frac{11}{3} C_A - 4 C_F n_f
\] (4.1)

\[
\beta_1^V = \beta_1^{\overline{\text{MS}}} = \frac{34}{3} C_A^2 - 4 C_F C_A n_f - \frac{20}{3} C_A T_F n_f
\] (4.2)

\[
\beta_2^V = \beta_2^{\overline{\text{MS}}} - a_1^{\overline{\text{MS}}} \beta_1^{\overline{\text{MS}}} + (a_2^{\overline{\text{MS}}} - (a_1^{\overline{\text{MS}}}))^2 \beta_0^{\overline{\text{MS}}}
\] (4.3)

\[
\beta_3^V = \beta_3^{\overline{\text{MS}}} - 2a_1^{\overline{\text{MS}}} \beta_2^{\overline{\text{MS}}} + (a_1^{\overline{\text{MS}}})^2 \beta_1^{\overline{\text{MS}}} + \left(2a_3^{\overline{\text{MS}}} - 6a_1^{\overline{\text{MS}}} a_2^{\overline{\text{MS}}} + 4(a_1^{\overline{\text{MS}}})^3\right) \beta_0^{\overline{\text{MS}}}
\] (4.4)
Note, that the property of scheme-independence of the coefficients $\beta_i^Y$ is the consequence of application of the ECH approach to the static potential. This means, that in the class of gauge-independent schemes, where the analogs of $\beta_2^{MS}, \beta_3^{MS}$ and $g_i^{MS} (i \geq 1)$ are known, the concrete equations Eqs. (4.3), (4.4) for the scheme-invariant coefficients $\beta_2^Y$ and $\beta_3^Y$ will be the same (see e.g. [67]) and will be related with the massless gauge-independent scheme-invariants, introduced in the work of Ref. [68].

Using now the concrete expressions for the terms in Eq. (4.3) and Eq. (4.4), which are summarized in the case of the $\overline{MS}$-scheme in Sec.II, we arrive to the following expressions for $\beta_2^Y$ and $\beta_3^Y$:

$$
\beta_2^Y = \left( \frac{206}{3} + \frac{44\pi^2}{3} - \frac{11\pi^4}{12} + \frac{242}{9} \zeta(3) \right) C_A^3
$$

(4.5)

$$
\beta_3^Y = \left( \frac{5914367}{4374} + \frac{22}{3} \cdot 502,24(1) - \frac{2728\pi^2}{9} + \frac{341\pi^4}{18} - \frac{15136}{27} \zeta(3) \right) C_A^4
$$

(4.6)

$$
+ \left( \frac{4841537}{2187} - \frac{22}{3} \cdot 709,717 - \frac{8}{3} \cdot 502,24(1) + \frac{2752\pi^2}{9} - \frac{172\pi^4}{9} + \frac{18184}{9} \zeta(3) \right) C_A^2 T_F n_f
$$

$$
+ \left( \frac{15290}{9} + \frac{1952}{3} \frac{1760}{3} \frac{\zeta(3)}{\zeta(5)} \right) C_A C_T T_F n_f
$$

$$
+ \left( 572 \frac{2288}{3} \left( \frac{\zeta(3)}{\zeta(5)} - \frac{3520}{3} \frac{\zeta(3)}{\zeta(5)} \right) \right) C_A C_T^2 T_F n_f + 46 C_T^2 T_F n_f
$$

$$
+ \left( \frac{740860}{729} - \frac{8}{3} \cdot 709,717 - \frac{640\pi^2}{9} + \frac{3208\pi^4}{405} - \frac{5696}{9} \zeta(3) \right) C_T^2 T_F n_f
$$

$$
+ \left( \frac{232}{9} - \frac{1024}{3} \frac{1280}{3} \frac{\zeta(3)}{\zeta(5)} \right) C_T^2 T_F^2 n_f + \left( \frac{9328}{9} - \frac{448}{3} \zeta(3) - \frac{640}{3} \zeta(5) \right) C_A C_T^2 T_F^2 n_f
$$

$$
+ \left( \frac{9376}{81} - \frac{512\pi^4}{405} + \frac{128}{27} \frac{\zeta(3)}{\zeta(5)} \right) C_A T_F^3 n_f
$$

$$
+ \left( -128 + \frac{256}{3} \frac{\zeta(3)}{\zeta(5)} \right) C_T F^3 T_F n_f^3 + \left( \frac{80}{9} + \frac{704}{3} \frac{\zeta(3)}{\zeta(5)} \right) \frac{d_{ab}^L d_{ab}^L}{N_A}
$$

$$
+ \left( \frac{512}{9} - \frac{1664}{3} \frac{\zeta(3)}{\zeta(5)} \right) \frac{d_{a b}^L d_{a b}^L}{N_A} n_f + \left( - \frac{704}{9} + \frac{512}{3} \frac{\zeta(3)}{\zeta(5)} \right) \frac{d_{a b}^L d_{a b}^L}{N_A} n_f
$$

- \frac{22}{3} \cdot 56,83(1) C_A \frac{d_{a b}^L d_{a b}^L}{N_A} T_F n_f - \frac{22}{3} \cdot 136,39(12) C_A \frac{d_{a b}^L d_{a b}^L}{N_A} n_f
$$

$$
+ \frac{8}{3} \cdot 56,83(1) \frac{d_{ab}^L d_{ab}^L}{N_A} T_F n_f^2 + \frac{8}{3} \cdot 136,39(12) T_F \frac{d_{ab}^L d_{ab}^L}{N_A} n_f.
$$

The analytical result for Eq. (4.5) was originally obtained in Ref. [42] and agrees with the similar one of Ref. [40], with the corrected later on $C_A^3$-term.
The semi-analytical expression of Eq.(4.6) is the new result. It depends on the semi-analytical results for the coefficients of Eq.(2.16) and Eq.(2.17), which were presented in Sec.II.

Consider now the real QCD case, which is based on the $SU(N_c=3)$ colour group. The analytical expressions for its colour structures have the following form: $C_A = 3$, $C_F = 4/3$, $T_F = 1/2$, $N_A = 8$, $d_{A}^{abcd}d_{A}^{abcd} = 135$, $d_{F}^{abcd}d_{A}^{abcd} = 15/2$ and $d_{F}^{abcd}d_{F}^{abcd} = -53/12$. Transforming now the analytical expressions for the coefficients of the QCD $\beta$-function in the V-scheme into the numerical form, we get the well-known expressions for the coefficients $\beta_0$ and $\beta_1$

$$\beta_0 = 11 - 0.666666n_f$$
$$\beta_1 = 102 - 12.66666n_f \quad (4.7)$$

and the numerical expressions for the third and fourth coefficients of the QCD $\beta$-function in the V-scheme

$$\beta_2^V = 4224.181 - 746.0062n_f + 20.87191n_f^2$$
$$\beta_3^V = 43175.06(6.43) - 12951.700(390)n_f + 706.9658(6)n_f^2 - 4.87214n_f^3 \quad (4.8)$$

The numerical errors of the first three coefficients in Eq.(1.9) are defined as the mean square error $\sigma = \sqrt{\sum_i^k \sigma_i^2}$, where $\sigma_i$ are the numerical errors that arise from the multiplication $2\beta_0$ by the computed errors in the coefficients $a_3^{(1)}$, $a_3^{(0)}$ of Eq.(2.16) and Eq.(2.17), evaluated in the $\overline{\text{MS}}$-scheme.

V. THE NUMERICAL FORTH-ORDER APPROXIMATIONS OF THE QCD $\beta$-FUNCTION IN THE V AND $\overline{\text{MS}}$-SCHEMES VS MINIMAL MOM SCHEME

In the last few years the interest to study of perturbative approximations of the QCD $\beta$-function in gauge-independent and gauge-dependent schemes increased due to theoretical investigations whether it is possible that the IR fixed points are manifesting themselves in the $\beta$-functions of the concrete non-abelian groups with fermions (see e.g.[69],[70],[71]).

Other, more phenomenologically oriented applications of different MOM-schemes are related to careful analysis of the minimization of gauge-dependence of the four-loop approximation of the $e^+e^-\text{-annihilation} R$-ratio in the different MOM-schemes [72]. These studies are based on transformation to the MOM-schemes of the results for the four-loop approximation of $R$-ratio, which are known from analytical evaluations, performed in Refs. [75],[76] within the $\overline{\text{MS}}$-scheme. In the process of MOM-schemes analysis of Ref. [72] the effects of three-loop $\overline{\text{MS}}$-scheme QCD corrections to $R$-ratio, completely evaluated in Ref.[72], were also taken into account.

Note, that the first study of the gauge-dependence of three-loop corrections to $R$-ratio was made within AMOM-scheme in Ref.[77]. However, this work was based on the analysis of the the gauge-dependence of the AMOM version of the erroneous $O(\alpha_s^3)$ approximation for $R$-ratio, which was obtained in the $\overline{\text{MS}}$-scheme in Ref.[78]. As it is well-known, this $\overline{\text{MS}}$-scheme result was corrected later in Ref.[73] and confirmed in Ref.[74].
In view of this it may be interesting to clarify the status of the gauge dependence of the four-loop approximation of R-ratio in the AMOM-scheme using the evaluated in Ref.\[73\] three-loop \(\overline{\text{MS}}\)-scheme QCD corrections and the obtained recently in Refs.\[75\],\[76\] \(O(\alpha_s^4)\) contributions.

In this section we will limit ourselves by the comparison of the numerical expressions of the third and fourth coefficients of the QCD \(\beta\)-function in the V-scheme (see Eqs.(4.8), (4.9)) with the similar results, obtained in the \(\overline{\text{MS}}\)-scheme and introduced in Ref.\[64\] minimal-MOM (mMOM) scheme, which was recently used in theoretical studies of Refs.\[65\],\[69\],\[71\] and in the phenomenological considerations of Ref.\[72\].

Let us briefly remind how the mMOM-scheme is defined. Using the standard notations for the renormalization constants of QCD in an arbitrary linear covariant gauge namely

\[
\psi_0 = \sqrt{Z_\psi}, \quad A_0^{a\mu} = \sqrt{Z_A} A^{a\mu}, \quad c_0^a = \sqrt{Z_c} c^a, \quad g_0 = Z_g g, \quad \lambda_0 = Z_\lambda Z^{-1}_A \lambda
\]

where \(\psi, A_0^a, c^a\) are the quarks, gluons and ghosts fields correspondingly, \(g\) is the constant of the strong interaction, \(\lambda\) is the gauge parameter, which is included in the Lagrangian QCD as \((\partial_\mu A_\mu^a)^2/2\lambda\). Therefore, the non-renormalized gluon propagator in momentum space will be written

\[
D_{\mu\nu} = i\delta_{ab} \left( -g_{\mu\nu} + (1 - \lambda) \frac{p_\mu p_\nu}{p^2 + i\varepsilon} \right) \quad (5.2)
\]

The form of the QCD Lagrangian dictates how to relate the different renormalization constants. For example, renormalization constant of the gluon-ghost-ghost vertex has the following form

\[
Z_{ccg} = Z_A Z_c^{1/2} \quad (5.3)
\]

The definition of mMOM-scheme is based on the consideration of this relation \[64\]. Taking into account Eq.(5.3) one can write down the expression for the QCD coupling constant of the mMOM-scheme \(\alpha_s^{\text{mMOM}}\) as

\[
\alpha_s^{\text{mMOM}}(\mu^2) = \frac{Z_A^{\text{mMOM}}(\mu^2) (Z_c^{\text{mMOM}}(\mu^2))^2}{(Z_{ccg}^{\text{mMOM}}(\mu^2))^2} \alpha_s^0
\]

(5.4)

Following the proposals of Ref.\[64\] the renormalization expressions for the gluon and ghost propagators are defined by using the requirements that at \(p^2 = \mu^2\) their residues are equal to unity, namely

\[
D(p^2, \alpha_s^{\text{mMOM}}(\mu^2))|_{p^2=\mu^2} = 1 \quad , \quad G(p^2, \alpha_s^{\text{mMOM}}(\mu^2))|_{p^2=\mu^2} = 1
\]

(5.5)

Then the renormalized expression for the gluon propagator, defined in the Landau gauge \(\lambda=0\), will take the following form

\[
D_{ab}^{\mu\nu} = i\delta_{ab} \left( g_{\mu\nu} - \frac{p^\mu p^\nu}{p^2 + i\varepsilon} \right) \frac{D(p^2, \alpha_s^{\text{mMOM}}(\mu^2))}{p^2 + i\varepsilon}
\]

(5.6)

(5.7)

and for ghost propagator:

\[
D_{ab}^c = i\delta_{ab} \frac{G(p^2, \alpha_s^{\text{mMOM}}(\mu^2))}{p^2 + i\varepsilon}
\]

(5.8)
The most important additional requirements of the mMOM scheme \cite{64,65} are the special definitions of the renormalization constant of the gluon-ghost-ghost vertex and of the renormalization constant of the gauge parameter, namely

$$Z_{ccg}^{\text{mMOM}}(\alpha_s^{\text{mMOM}}) = Z_{ccg}^{\overline{\text{MS}}} (\alpha_s^{\overline{\text{MS}}}) \quad , \quad Z_{\lambda}^{\text{mMOM}}(\alpha_s^{\text{mMOM}}) = Z_{\lambda}^{\overline{\text{MS}}} (\alpha_s^{\overline{\text{MS}}})$$  \hspace{1cm} (5.9)

Taking into account the definition of the QCD coupling constant in the $\overline{\text{MS}}$-scheme through the same vertex

$$\alpha_s^{\overline{\text{MS}}} (\mu^2) = \frac{Z_{\lambda}^{\overline{\text{MS}}} (\mu^2) (Z_{ccg}^{\overline{\text{MS}}} (\mu^2))^2}{(Z_{ccg}^{\overline{\text{MS}}} (\mu^2))^2} \alpha_s^0$$  \hspace{1cm} (5.10)

and Eq.(5.3), Eq.(5.9) one can get the useful relations between the renormalization constants of the mMOM and $\overline{\text{MS}}$-schemes

$$Z_g^{\text{mMOM}} \sqrt{Z_A^{\text{mMOM}}} Z_c^{\text{mMOM}} = Z_g^{\overline{\text{MS}}} \sqrt{Z_A^{\overline{\text{MS}}} Z_c^{\overline{\text{MS}}}} \quad .$$  \hspace{1cm} (5.11)

and the relation between the renormalized QCD coupling constants of these scheme

$$\alpha_s^{\text{mMOM}} (\mu^2) = \frac{Z_A^{\text{mMOM}}}{Z_A^{\overline{\text{MS}}}} \left( \frac{Z_c^{\text{mMOM}}}{Z_c^{\overline{\text{MS}}}} \right)^2 \alpha_s^{\overline{\text{MS}}} (\mu^2)$$  \hspace{1cm} (5.12)

All the above mentioned formulas are valid for any linear covariant gauge and in particular for the Landau gauge. In this work we will use the Landau gauge $\lambda=0$. Its choice is primarily due to the simplification of the final results we will be interested in. Moreover, the choice of this gauge in Ref. \cite{64} is supported by simplification of the definite lattice-based studies (see e.g.\cite{79}).

In the mMOM scheme the analytical expressions for the three-loop and four-loop coefficients of the QCD $\beta$-function in the general covariant gauge were obtained in Ref. \cite{64} applying the results of analytical MOM-schemes calculations, performed some time ago in Ref. \cite{61}. The results of Ref. \cite{64} were recently confirmed in Ref. \cite{65} by direct symbolical three- and four-loop computations. In the Landau gauge it has the following numerical form

$$\beta_2^{\text{mMOM}} = 3040.482 - 625.3867n_f + 19.38330n_f^2$$ \hspace{1cm} (5.13)

$$\beta_3^{\text{mMOM}} = 100541.05 - 24423.330n_f + 1625.40222n_f^2 - 27.49263n_f^3$$ \hspace{1cm} (5.14)

It is interesting to compare these results with the numerical expressions of the gauge-invariant coefficients $\beta_2^V$ and $\beta_3^V$ (see Eq.(4.8) and Eq.(4.9)) of the QCD $\beta$-function in the V-scheme, which pretend to the role of the QCD analog of the Gell-Man–Low $\Psi$-function, and with the numerical values of the similar coefficients in the QCD $\beta$-function in the gauge-invariant $\overline{\text{MS}}$-scheme. These numerical expressions can be obtained from the results of analytical calculations of Refs. \cite{29} and Ref. \cite{31}, which were presented above in Eq.(2.6) and Eq.(2.7) and have the following form

$$\beta_2^{\overline{\text{MS}}} = 1428.500 - 279.6111n_f + 6.01851n_f^2$$ \hspace{1cm} (5.15)

$$\beta_3^{\overline{\text{MS}}} = 29242.96 - 6946.289n_f + 405.0890n_f^2 + 1.49931n_f^3$$ \hspace{1cm} (5.16)
One can note that the numerical values of these scheme-dependent coefficients in the \( \overline{\text{MS}} \)-scheme are smaller than the values of the similar V-scheme coefficients of Eq.\((4.8)\) and Eq.\((4.9)\).

In order to clarify the relations between the numerical values of the third and forth coefficients for the QCD \( \beta \)-function in three considered schemes we present in Table below their numerical values for the concrete numbers of quarks flavours \(1 \leq n_f \leq 6\).

| \( n_f \) | \( \beta_2^V \) | \( \beta_3^V \) | \( \beta_2^\overline{\text{MS}} \) | \( \beta_3^\overline{\text{MS}} \) | \( \beta_2^\text{mMOM} \) | \( \beta_3^\text{mMOM} \) |
|---|---|---|---|---|---|---|
| 1 | 3499.047 | 30925.46 ± 6.44 | 1154.907 | 22703.26 | 2434.478 | 77715.63 |
| 2 | 2815.656 | 20060.55 ± 6.48 | 893.351 | 16982.73 | 849.068 | 27094.64 |
| 3 | 2174.010 | 10551.11 ± 6.54 | 643.833 | 12090.37 | 1338.771 | 41157.38 |
| 4 | 1574.107 | 2367.90 ± 6.62 | 406.351 | 8035.18 | 1867.242 | 57976.06 |
| 5 | 1015.948 | −4518.30 ± 6.72 | 893.351 | 16982.73 | 1338.771 | 41157.38 |
| 6 | 499.533 | −10136.74 ± 6.84 | −32.500 | 2472.28 | −14.038 | 6577.14 |

We conclude that in spite of the fact that for all \( n_f \) the absolute values of \( \beta_2^\overline{\text{MS}} \)-scheme are smaller than the ones, calculated in the V-scheme and mMOM-scheme in the Landau gauge, while \( \beta_3^V \)-results are smaller than the ones, computed in the Landau-gauge version of the mMOM-scheme, but they are larger than the similar numbers, obtained in the gauge-independent \( \overline{\text{MS}} \)-scheme. This feature demonstrates that the asymptotic structure of the PT series for the effective \( \beta \)-function in the V-scheme has non-regular behaviour and differs from the asymptotic PT for the \( \beta \)-function in the \( \overline{\text{MS}} \)-scheme, which was obtained in Ref.\[80\] using the approach, developed in Ref.\[81\]. Note, however, that as it will be shown below the \( \beta_V \)-function can be considered as the real QCD analog of the Gell-Mann–Low function.

VI. THE FOUR-LOOP QED RESULT IN THE V-SCHEMEN

Replacing the \( SU(N_c) \) group weights in Eqs.\((4.1)\), Eq.\((4.2)\), Eq.\((4.5)\), Eq.\((4.6)\) by the Abelian \( U(1) \)-group weights as \( C_A = 0, \ C_F = 1, \ T_F = 1, \ d_A^{abcd} = 0, \ d_F^{abcd} = 1, \ N_A = 1 \) and \( n_f = N \), we obtain the following four-loop semi-analytical expression for the \( \beta_V \)-function in QED

\[
\beta^V_{\text{QED}}(a_V) = \frac{4}{3} N a_V^2 + 4 N a_V^3 + \left(-2N + \left(\frac{64}{3} \zeta(3) - \frac{184}{9} \right) N^2\right) a_V^4 + \left(-46N + \left(104 + \frac{512}{3} \zeta(3) - \frac{1280}{3} \zeta(5) - \frac{8}{3} \cdot 56.83(1)\right) N^2 + \left(128 - \frac{256}{3} \zeta(3)\right) N^3\right) a_V^5 + O(a_V^6)
\]

where \( a_V = \alpha_V/4\pi \), and \( N \) is the number of leptons.

In the numerical form the scheme-dependent coefficients of \( \beta^V_{\text{QED}} \)-function read:

\[
\beta_2^V = 2N - 5.19943N^2
\]

\[
\beta_3^V = 46N - 284.818(26)N^2 + 25.42447N^3
\]
The analogous expressions for the three- and four-loop coefficients of the QED $\beta$-function in the $\overline{\text{MS}}$-scheme follow from the analytical results of Ref.[10] and can be presented as

\begin{align}
\beta_2^{\overline{\text{MS}}} &= 2N - 4.88888N^2 \\
\beta_3^{\overline{\text{MS}}} &= 46N - 82.9753N^2 - 5.06995N^3
\end{align}

(6.5)  (6.6)

The numerical expressions for the analogous coefficients of the Gell-Man–Low $\Psi$-function (or the QED $\beta$-function in MOM-scheme), which we obtain from the same work of Ref.[10], are

\begin{align}
\Psi_2 &= 2N - 5.19943N^2 \\
\Psi_3 &= 46N - 133.2714N^2 + 25.42447N^3
\end{align}

(6.7)  (6.8)

One should note that coefficients $\Psi_2$ and $\beta_2^V$ at the three-loop level are equal each other and differ from the three-loop coefficient of the QED $\beta$-function in the $\overline{\text{MS}}$-scheme [82]. Thus $\beta_2^V \equiv \Psi_2 \neq \beta_2^{\overline{\text{MS}}}$. Moreover coefficients that stand in front of $N$ and $N^3$ in $\beta_3^V$ coincide with the corresponding coefficients in $\Psi_3$-function. The only difference between $O(\alpha^5)$ approximation of the $\beta^V$- and $\Psi$-functions is the appearance of extra numerical contribution in the $N^2$ term of $\beta_3^V$-coefficient only. Using these results, we conclude that it is reasonable to consider the $\beta^V$-function as an analog of the Gell-Man–Low $\Psi$-function.

At the $O(\alpha^5)$-level the relation between these functions has the following form

\[
\Psi(\alpha_{\text{MOM}}) = \beta_{QED}^V(\alpha_V) + \frac{8}{3} \cdot 56.83(1)N^2 \left(\frac{\alpha_V}{4\pi}\right)^5 + O(\alpha_V^6)
\]

(6.9)

Note, that the $\beta_{QED}^V$-function differs from the Gell-Man–Low function in additional makeweight $-8 \cdot 56.83(1)N^2a_5^V/3$. It follows from the proportional to the new colour factors $d_F^{abcd}d_F^{abcd}$ of the SU($N_c$)-group in the $O(\alpha_5^4)$ approximation of the QCD RG $\beta^V$-function (see Eq.(4.6)).

For the completeness we present the QED expressions for the $O(\alpha^5)$ approximations for the $\Psi$ and $\beta_{QED}^V$-functions in the case of $N=1$ :

\begin{align}
\Psi &= 1.3333a^2 + 4a^3 + 3.1994a^4 - 153.8469a^5 + O(a^6) \\
\beta_{QED}^V &= 1.3333a_V^2 + 4a_V^3 + 3.1994a_V^4 - 305.3936(266)a_V^5 + O(a_V^6)
\end{align}

(6.10)  (6.11)

VII. CONCLUSION

In this work we consider the definition of the gauge-independent RG QCD $\beta$-function in the $V$-scheme. Using higher-order corrections to the static potential of the quark-antiquark interaction and $\beta$-function in $\overline{\text{MS}}$-scheme, we compute the forth term of the PT expression for the $\beta$-function in V-scheme of SU($N_c$) group analog of QCD. The comparison of the numerical expressions of the scheme-dependent coefficients of the $\beta^V$-function of QCD with the similar coefficients of the QCD $\beta$-function in the $\overline{\text{MS}}$ and mMOM-scheme in the Landau gauge are presented. The indication that the structure of the PT series for the effective $\beta$-function in the V-scheme has non-regular asymptotic behaviour and differs from the asymptotic PT for the $\beta$-function in the $\overline{\text{MS}}$-scheme are presented.
Considering the QED limit of the SU(Nc)-group $\beta^V$-function we observe that its perturbative expression is starting to differ from the perturbative expression for the Gell-Mann–Low $\Psi$-function at the $O(\alpha^5_V)$-level only. In view of this we propose to consider the $\beta^V$-function as the most theoretically substantiated analog of the Gell-Man–Low function in QCD.

Acknowledgements

This work is done within the scientific program of the Russian Foundation for Basic Research, Grant N 14-01-00647 and the grant for the Leading Scientific Schools NS-2835.2014.2. The part of this work was done when one of us (ALK) was visiting CERN in November-December of 2014. He is grateful to colleagues from Theory Unit of CERN Physics Department for the warm hospitality.

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