Ill-posedness for the stationary Navier-Stokes equations in critical Besov spaces

Jinlu Li¹, Yanhai Yu²,*and Weipeng Zhu³

¹ School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China
² School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, China
³ School of Mathematics and Big Data, Foshan University, Foshan, Guangdong 528000, China

June 2, 2022

Abstract: This paper presents some progress toward an open question which proposed by Tsurumi (Arch. Ration. Mech. Anal. 234:2, 2019): whether or not the stationary Navier-Stokes equations in \( \mathbb{R}^d \) is well-posed from \( \dot{B}^{-2}_{p,q} \) to \( \mathbb{P}\dot{B}_0^{0} \) with \( p = d \) and \( 1 \leq q \leq 2 \). In this paper, we prove that for the case \( 1 \leq q < \frac{d}{2} \) with \( d \geq 4 \) the stationary Navier-Stokes equations is ill-posed from \( \dot{B}^{-2}_{d,q} (\mathbb{R}^d) \) to \( \mathbb{P}\dot{B}_0^{0} (\mathbb{R}^d) \) by showing that a sequence of external forces is constructed to show discontinuity of the solution map at zero. Indeed in such case of \( q \), there exists a sequence of external forces which converges to zero in \( \dot{B}^{-2}_{d,q} \) and yields a sequence of solutions which does not converge to zero in \( \dot{B}_d^{0} \). In particular, we also prove that the stationary Navier-Stokes equations is well-posed from \( \dot{B}^{-2}_{d,2} (\mathbb{R}^d) \) to \( \mathbb{P}\dot{B}_0^{0} (\mathbb{R}^d) \) with \( d = 3, 4 \). Based on these two cases, we demonstrate that the above open question for the dimension \( d \geq 4 \) has been solved completely.

Keywords: Stationary Navier-Stokes equations, Besov spaces, Ill-posedness

MSC (2010): 35Q30; 35R25; 42B37

1 Introduction

In this paper, we consider the forced stationary Navier-Stokes equations describing the motion of incompressible fluid in the whole space \( \mathbb{R}^d \), \( d \geq 3 \)

\[
\begin{cases}
-\Delta u + u \cdot \nabla u + \nabla \Pi = f, \\
\text{div} u = 0,
\end{cases}
\]  

(SNS)

*E-mail: lijinlu@gnnu.edu.cn; yuyanghai214@sina.com(Corresponding author); mathzwp2010@163.com
where \( u = u(x) = (u^1(x), \ldots, u^d(x)) \) and \( \Pi = \Pi(x) \) denote the unknown velocity vector and the unknown pressure at the point \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), respectively, while \( f = f(x) = (f^1(x), \ldots, f^d(x)) \) denotes the given external force.

1.1 Known Well/Ill-posedness (WP/IP) results

There have been various studies on strong solutions \( u \) to (SNS) for given data \( f \) on the whole space \( \mathbb{R}^d \). Leray [10] and Ladyzhenskaya [9] showed the existence of strong solutions to (SNS), and Heywood [5] constructed solutions of (SNS) as a limit of solutions of the non-stationary Navier-Stokes equations:

\[
\begin{aligned}
    \partial_t u - \Delta u + u \cdot \nabla u + \nabla \Pi &= 0, & x &\in \mathbb{R}^d, \ t > 0, \\
    \text{div} u &= 0, & x &\in \mathbb{R}^d, \ t > 0, \\
    |u|_{t=0} &= u_0, & x &\in \mathbb{R}^d.
\end{aligned}
\]

(NNS)

We should mention that Koch and Tataru [7] obtained the global well-posedness of the 3D (NNS) for small initial data in the space \( BMO^{-1} = \dot{F}^{-1}_{\infty,2} \). On the other hand, Bourgain and Pavlović [2] showed the ill-posedness of (NNS) in \( B^{-1}_{\infty,q} \) (which includes \( BMO^{-1} \)). Later on, the ill-posedness in \( \dot{B}^{-1}_{\infty,q} \) with \( 1 \leq q < \infty \) was also showed by Yoneda in [15] (\( 2 < q < \infty \)) and Wang in [14] (\( 1 \leq q \leq 2 \)). These spaces play a crucial role since these are scaling invariant for the initial data \( u_0 \) in (NNS).

Chen [4] showed that for every small external force having a divergence-form \( f = \text{div} F \) with \( F \in L^{d/2} (\mathbb{R}^d) \), there exists an unique strong solution \( u \) of (SNS) in \( L^q_{t} (\mathbb{R}^d) \). Secchi [11] investigated existence and regularity of solutions to (SNS) in \( L^q \cap L^p \) with \( p > d \). As for the well-posedness of (SNS) in homogeneous Besov spaces, Kaneko-Kozono-Shimizu in [8] showed the well-posed result as follows:

**Theorem 1.1 (see [8])** Let \( d \geq 3 \). Suppose that \( 1 \leq p < d \) and \( 1 \leq q \leq \infty \). Then (SNS) is well-posed from \( E = \dot{B}^{-3+\frac{q}{p}}_{p,q} \) to \( S = \mathbb{P} \dot{B}^{-1+\frac{d}{q}}_{p,q} \).

Moreover, in the case \( p = d \), (SNS) is also well-posed from \( E = \dot{B}^{-2}_{d,q} \) to \( S = \mathbb{P} L^d \) if \( 1 \leq q \leq 2 \). These spaces \( E \) and \( S \) are scaling invariant for the external force \( f \) and the velocity \( u \) in (SNS) respectively. Precisely speaking, the corresponding scaling transform is \( \{ u, \pi, f \} \mapsto \{ u_\lambda, \pi_\lambda, f_\lambda \} \) with \( u_\lambda(x) = \lambda u(\lambda x), \pi_\lambda(x) = \lambda^2 \pi(\lambda x), f_\lambda(x) = \lambda^3 f(\lambda x) \), and we see that

\[
    \| f_\lambda \|_E = \| f \|_E, \quad \| u_\lambda \|_S = \| u \|_S, \quad \forall \lambda > 0.
\]

There are other previous results on the well-posedness in the case \( p = d \). Bjorland et al. [3] showed the well-posedness with more general space of external forces. In fact, they proved that there are constants \( \varepsilon, \delta > 0 \) such that if \( f \in S' \) satisfies \( \| (-\Delta)^{-1} f \|_{L^\infty} < \varepsilon \), then there exists a unique solution \( u \in B_{p,1} L_{1,\infty}(\delta) \) to (SNS), which belongs to \( L^d \) if and only if \( \mathbb{P} f \in \dot{H}^{-2,d} \). Phan and Phuc [16] showed the well-posedness in the largest critical space of external forces including \( \dot{H}^{-2,d} \).

A nature question to ask is: Whether or not (SNS) is well-posed from \( E = \dot{B}^{-3+\frac{q}{p}}_{p,q} \) to \( S = \mathbb{P} \dot{B}^{-1+\frac{d}{q}}_{p,q} \) when \( d \leq p \leq \infty \) and \( 1 \leq q \leq \infty \)?

Recently, Tsuruni gave a partial answer to the above problem. More precisely, Tsuruni [12] proved the ill-posedness of (SNS) in \( \mathbb{R}^d \) (see [13] for the Torus case \( T^d \)), namely,
Theorem 1.2 (see [12]) Let \( d < p \leq \infty, 1 \leq q \leq \infty \), and if \( p = d, 2 < q \leq \infty \), then (SNS) is ill-posed from \( E = \dot{B}^{-3+\frac{d}{q}}_{p,q} \) to \( S = \mathbb{P}\dot{B}^{-1+\frac{d}{q}}_{p,q} \) in the sense that the solution map \( f \in E \mapsto u \in S \) is, even if it exists, not continuous. More precisely, under such a condition, there exists a sequence \( \{f_N\}_{N \in \mathbb{N}} \) of external forces with \( f_N \to 0 \) in \( E \) such that there exists a unique solution \( u_N \in \mathbb{P}L^d \) of (SNS) for each \( f_N \), which never converges to zero in \( S \) (actually, even in \( \dot{B}^{-1}_{\infty,\infty} \)).

Obviously, Tsuruni’s result makes it clear that the well-posedness and ill-posedness can be divided between the case \( (p, q) \in [1, d) \times [1, \infty) \) (Theorem 1.1) and the case \( (p, q) \in (d, \infty] \times [1, \infty] \cup [d, 2) \times (2, \infty) \) (Theorem 1.2), respectively. However, we remark that there is a gap between the global well-posedness in Theorem 1.1 and the ill-posedness in Theorem 1.2. In other words, for the case \( p = d \) and \( 1 \leq q \leq 2 \), it is still unknown whether (SNS) in \( \mathbb{R}^d \) is well-posed or ill-posed from \( \dot{B}^{-2}_{d,q} \) to \( \mathbb{P}\dot{B}^0_{d,q} \). In this paper, we are devoted to answer the question.

1.2 New Results

Now we return to the equation (SNS). Let us write the \( i \)-th component of \( v \) as \( v^{(i)} \). For the vector fields \( v \) and \( u \), we define the tensor product \( v \otimes u \) as the \((i, j)\)-th component \( v^{(i)}u^{(j)} \) with \( 1 \leq i, j \leq d \). Thus, if \( v \) is divergence free (that is if \( \nabla \cdot v = 0 \) ) we have \( \nabla \cdot (v \otimes u) = (\nabla \cdot v)u \). Let us rewrite it to the generalized form so that we can apply successive approximation. First, we note that since \( \text{div}u = 0 \), there holds

\[
\nabla u = \sum_{i=1}^{n} \partial_{x_i} \left( u^{(i)} u \right) = \text{div}(u \otimes u).
\]

We next introduce the projection \( \mathbb{P} : L^p(\mathbb{R}^d) \to L^p_0(\mathbb{R}^d) \equiv \left\{ f \in C_0^\infty(\mathbb{R}^d); \text{div} f = 0 \right\}^{\| f \|_{L^p}} \). In \( \mathbb{R}^d \), \( \mathbb{P} \) can be defined by \( \mathbb{P} = \text{Id} + \nabla(-\Delta)^{-1} \text{div} \), or equivalently, \( \mathbb{P} = (\mathbb{P}_{ij})_{1 \leq i, j \leq d} \) with \( \mathbb{P}_{ij} \equiv \delta_{ij} + R_iR_j \).

Applying \( \mathbb{P} \) to (SNS), we obtain

\[
-\Delta u + \mathbb{P}\nabla \cdot (u \otimes u) = \mathbb{P}f,
\]

implied by \( \mathbb{P}u = u \) and \( \mathbb{P}(\nabla u) = 0 \) since \( \text{div}u = 0 \). Also, since \( \mathbb{P} \) commutes with \( -\Delta \), the solution \( u \) of (SNS) can be expressed as

\[
u = (-\Delta)^{-1}\mathbb{P}(u \cdot \nabla u) + (-\Delta)^{-1}\mathbb{P}f
= \mathbb{P}(-\Delta)^{-1}\text{div}(u \otimes u) + \mathbb{P}(-\Delta)^{-1}f
= \mathcal{B}(u, u) + g,
\]

here and in what follows, we shall denote the bilinear form

\[
\mathcal{B}(u, v) \equiv \mathbb{P}(-\Delta)^{-1}\text{div}(u \otimes v).
\]

Our main results now read as follows:

Theorem 1.3 Let \( d = 3, 4 \). (SNS) is well-posed from \( E = \dot{B}^{-3}_{d,2}(\mathbb{R}^d) \) to \( S = \mathbb{P}\dot{B}^0_{d,2}(\mathbb{R}^d) \).
Theorem 1.4 Let \( d \geq 4 \) and \( 1 \leq q < \frac{d}{2} \). (rSNS) is ill-posed from \( \dot{B}^{0}_{d,q}(\mathbb{R}^d) \) to \( \mathbb{P}\dot{B}^{0}_{d,q}(\mathbb{R}^d) \) in the following sense: There exist \( \{g_n\}_{n=1}^{\infty} \subset \dot{B}^{0}_{d,q}(\mathbb{R}^d) \) such that a sequence of solutions \( u_n \in \dot{B}^{0}_{d,q}(\mathbb{R}^d) \) to (rSNS) which satisfies

\[
\lim_{n \to \infty} \|g_n\|_{\dot{B}^{0}_{d,q}(\mathbb{R}^d)} = 0
\]

and for some positive constant \( \varepsilon_0 \)

\[
\lim_{n \to \infty} \|u_n\|_{\dot{B}^{0}_{d,q}(\mathbb{R}^d)} \geq \varepsilon_0.
\]

Remark 1.1 Theorem 1.4 demonstrates that if \( d \geq 4 \) and \( 1 \leq q < \frac{d}{2} \), there exists a sequence of external forces which converges to zero in \( \dot{B}^{0}_{d,q} \) and yields a sequence of solutions to (rSNS) which does not converge to zero in \( \dot{B}^{0}_{d,q} \). In other words, the (SNS) is ill-posed from \( \dot{B}^{0}_{d,q} \) to \( \mathbb{P}\dot{B}^{0}_{d,q} \) due to the discontinuity of the solution map at zero.

Remark 1.2 We should mention that we have completely solved the open question which proposed by Tsurumi [12] for the case \( d \geq 4 \). This can be seen clearly from the Table below.

| \( d \) | \( p \) | \( q \) | \( \dot{B}^{-3+d/p}_{p,q} \) | \( \mathbb{P}\dot{B}^{-1+d/p}_{p,q} \)
|---|---|---|---|---|
| \( d \geq 3 \) | \([1, d)\) | \([1, \infty]\) | WP, see Theorem 1.1 |
| \( d \geq 3 \) | \((d, \infty)\) | \([1, \infty]\) | IP, see Theorem 1.2 |
| \( d \geq 3 \) | \(d\) | \((2, \infty)\) | IP, see Theorem 1.2 |
| \( d = 3, 4 \) | \(d\) | \(2\) | WP, see Theorem 1.3 |
| \( d = 4 \) | \(d\) | \([1, 2)\) | WP, see Theorem 1.4 |
| \( d \geq 5 \) | \(d\) | \([1, 2)\) | IP, see Theorem 1.4 |
| \( d = 3 \) | \(d\) | \([1, 2)\) | Unknown |

Table 1: Well/Ill-posedness

1.3 Main Ideas

To illustrate our main idea, we introduce

\[
\begin{align*}
G & \equiv \mathcal{B}(g, g), \\
U & \equiv \mathcal{B}(u, u) - G,
\end{align*}
\]

then we have form (rSNS)

\[
u = g + G + U,
\]

Obviously,

\[
U = \mathcal{B}(U, g) + \mathcal{B}(g, U) + \mathcal{B}(U, U) + \mathcal{B}(U, G) \\
+ \mathcal{B}(G, U) + \mathcal{B}(g, G) + \mathcal{B}(G, g) + \mathcal{B}(G, G).
\]
Lemma 1.1 Let $d = 3, 4$. Then for $u, v \in \dot{B}^{0}_{d, 2}(\mathbb{R}^{d})$, we have $\mathcal{B}(u, v) \in L^{d}(\mathbb{R}^{d}) \cap \dot{B}^{0}_{d, 2}(\mathbb{R}^{d})$ with the estimate

$$\|\mathcal{B}(u, v)\|_{\dot{B}^{0}_{d, 2}(\mathbb{R}^{d})} \leq C\|u\|_{\dot{B}^{0}_{d, 2}(\mathbb{R}^{d})}\|v\|_{\dot{B}^{0}_{d, 2}(\mathbb{R}^{d})},$$

where $C$ is a positive constant.

**Proof.** Due to the embedding $L^{d}(\mathbb{R}^{d}) \hookrightarrow \dot{B}^{0}_{d, 2}(\mathbb{R}^{d}) \hookrightarrow \dot{B}^{-1}_{d, 2}(\mathbb{R}^{d})$ and $\dot{B}^{0}_{d, 2}(\mathbb{R}^{d}) \hookrightarrow L^{d}(\mathbb{R}^{d})$, we have

$$\|\mathcal{B}(u, v)\|_{\dot{B}^{0}_{d, 2}(\mathbb{R}^{d})} \leq C\|u \otimes v\|_{L^{d}(\mathbb{R}^{d})} \leq C\|u\|_{L^{d}(\mathbb{R}^{d})}\|v\|_{L^{d}(\mathbb{R}^{d})} \leq C\|u\|_{\dot{B}^{0}_{d, 2}(\mathbb{R}^{d})}\|v\|_{\dot{B}^{0}_{d, 2}(\mathbb{R}^{d})}.$$

**Remark 1.3** With Lemma 1.1 at our disposal, we can prove Theorem 1.3 by successive approximation method. Since the procedure is standard (see [8]), we shall not go into details.

Lemma 1.2 Let $d \geq 4$. Then for $u, v \in L^{d}(\mathbb{R}^{d})$, we have $\mathcal{B}(u, v) \in L^{d}(\mathbb{R}^{d}) \cap \dot{B}^{0}_{d, \frac{d}{2}}(\mathbb{R}^{d})$ with the estimate

$$\|\mathcal{B}(u, v)\|_{L^{d}(\mathbb{R}^{d})} + \|\mathcal{B}(u, v)\|_{\dot{B}^{0}_{d, \frac{d}{2}}(\mathbb{R}^{d})} \leq C\|u\|_{L^{d}(\mathbb{R}^{d})}\|v\|_{L^{d}(\mathbb{R}^{d})},$$

where $C$ is a positive constant.

**Proof.** We have

$$\|\mathcal{B}(u, v)\|_{L^{d}(\mathbb{R}^{d})} \leq C\|u \otimes v\|_{L^{d}(\mathbb{R}^{d})} \leq C\|u\|_{L^{d}(\mathbb{R}^{d})}\|v\|_{L^{d}(\mathbb{R}^{d})},$$

$$\|\mathcal{B}(u, v)\|_{\dot{B}^{0}_{d, \frac{d}{2}}(\mathbb{R}^{d})} \leq C\|u \otimes v\|_{\dot{B}^{-1}_{d, \frac{d}{2}}(\mathbb{R}^{d})} \leq C\|u\|_{L^{d}(\mathbb{R}^{d})}\|v\|_{L^{d}(\mathbb{R}^{d})} \leq C\|u\|_{\dot{B}^{0}_{d, \frac{d}{2}}(\mathbb{R}^{d})}\|v\|_{\dot{B}^{0}_{d, \frac{d}{2}}(\mathbb{R}^{d})}.$$
Corollary 1.2  Let \(d \geq 4\). For small enough \(\delta\) such that \(\|g\|_{L^d(\mathbb{R}^d)} < \delta\), then we have

\[
\|U\|_{L^d(\mathbb{R}^d)} + \|U\|_{B^0_{d,2} (\mathbb{R}^d)} \leq C\|g\|^3_{L^d(\mathbb{R}^d)},
\]

where \(C\) is a positive constant.

**Proof.** Using Lemma 1.2 yields

\[
\|U\|_{L^d(\mathbb{R}^d)} + \|U\|_{B^0_{d,2} (\mathbb{R}^d)} \leq C \left( \|U\|_{L^d(\mathbb{R}^d)} \|g\|_{L^d(\mathbb{R}^d)} + \|U\|_{L^d(\mathbb{R}^d)} \|g\|_{L^d(\mathbb{R}^d)} + \|g\|_{L^d(\mathbb{R}^d)}^2 \right)
\]

\[
\leq C \left( \|U\|_{L^d(\mathbb{R}^d)} \|g\|_{L^d(\mathbb{R}^d)} + \|U\|_{L^d(\mathbb{R}^d)} \|g\|_{L^d(\mathbb{R}^d)}^2 + \|g\|_{L^d(\mathbb{R}^d)}^3 \right),
\]

which enables us to complete the proof of Corollary 1.2.

**Remark 1.4** From (1.2) and Corollary 1.2, we expect that the primarily affect which leads to the discontinuity to the solution of (rSNS) is that the worst term from nonlinear interactions \(\mathcal{B}(g, g)\).

## 2 Preliminaries

Firstly, let us recall that for all \(f \in S\)', the Fourier transform \(\hat{f}\), is defined by

\[
(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi x} f(x) dx \quad \text{for any } \xi \in \mathbb{R}^d.
\]

The inverse Fourier transform of any \(g\) is given by

\[
(\mathcal{F}^{-1} g)(x) = \check{g}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\xi) e^{ix\xi} d\xi.
\]

Next, we will recall some facts about the Littlewood-Paley (L-P) decomposition, the homogeneous Besov spaces and their some useful properties.

**Proposition 2.1 (L-P decomposition, See [1])** Let \(\mathcal{B} := \{\xi \in \mathbb{R}^d : |\xi| \leq 4/3\}\) and \(\mathcal{C} := \{\xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3\}\). Choose a radial, non-negative, smooth function \(\chi : \mathbb{R}^d \mapsto [0, 1]\) such that it is supported in \(\mathcal{B}\) and \(\chi \equiv 1\) for \(|\xi| \leq 3/4\). Setting \(\varphi(\xi) := \chi(\xi/2) - \chi(\xi)\), then we deduce that \(\varphi\) is supported in \(\mathcal{C}\) and \(\varphi(\xi) \equiv 1\) for \(4/3 \leq |\xi| \leq 3/2\). Moreover,

\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\},
\]

\[
\frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}.
\]

For every \(u \in S'(\mathbb{R}^d)\), the homogeneous dyadic blocks \(\hat{\Delta}_j\) is defined as follows

\[
\hat{\Delta}_ju = \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F}u) = 2^{dj} \int_{\mathbb{R}^d} \varphi(2^j(x-y))u(y) dy, \quad \forall j \in \mathbb{Z},
\]

\[
\tilde{\Delta}_ju = \overline{\varphi}(2^{-j}D)u = \sum_{|k-j| \leq 1} \hat{\Delta}_ku.
\]
Moreover, the dyadic blocks $\hat{\Lambda}_j$ satisfies the property of almost orthogonality:

$$\hat{\Lambda}_j \hat{\Lambda}_k u \equiv 0 \quad \text{if} \quad |j - k| \geq 2.$$ 

In the homogeneous case, the following Littlewood-Paley decomposition makes sense

$$u = \sum_{j \in \mathbb{Z}} \hat{\Lambda}_j u \quad \text{for any} \ u \in S'_h(\mathbb{R}^d),$$

where $S'_h$ is given by

$$S'_h := \{ u \in S'(\mathbb{R}^d) : \lim_{j \to -\infty} \|\chi(2^{-j}D)u\|_{L^\infty} = 0 \}.$$

We turn to the definition of the Besov Spaces and norms which will come into play in our paper.

**Definition 2.1** ([1]) Let $s \in \mathbb{R}$ and $(p, q) \in [1, \infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ consists of all tempered distribution $f$ such that

$$\dot{B}^s_{p,q} = \{ f \in S'_h(\mathbb{R}^d) : \| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} < \infty \},$$

where

$$\| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} := \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{sjp} \| \hat{\Lambda}_j f \|_{L^p(\mathbb{R}^d)} \right)^{1/q}, & \text{if} \ 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{sj} \| \hat{\Lambda}_j f \|_{L^p(\mathbb{R}^d)}, & \text{if} \ q = \infty. \end{cases}$$

**Remark 2.1** We point out that the following properties will be used in the sequel.

- For the homogeneous Besov spaces, we have the embedding properties as follows:

$$\dot{B}^s_{p,q_1} \hookrightarrow \dot{B}^s_{p,q_2}, \ s \in \mathbb{R}, 1 \leq p \leq \infty, 1 \leq q_1 \leq q_2 \leq \infty$$

and

$$\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q}, \ -\infty < s_2 \leq s_1 < \infty, 1 \leq q \leq \infty, 1 \leq p_1 \leq p_2 \leq \infty$$

with $s_1 - d/p_1 = s_2 - d/p_2$.

- $\dot{B}^0_{p,2}$ is continuously included in $L^p$ and $L^p$ is continuously included in $\dot{B}^0_{p,p}$, namely,

$$\dot{B}^0_{p,2} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,p}, \ 2 \leq p < \infty.$$

- $\dot{B}^0_{p,p}$ is continuously included in $L^p$ and $L^p$ is continuously included in $\dot{B}^0_{p,2}$, namely,

$$\dot{B}^0_{p,p} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,2}, \ 1 < p \leq 2.$$

- The Riesz potential $(-\Delta)^{\frac{s}{2}} f \equiv \mathcal{F}^{-1} \left( |\xi|^{s} \hat{f}(\xi) \right)$ with $\alpha \in \mathbb{R}$ gives an isomorphism from $\dot{B}^{s+\alpha}_{p,q}$ onto $\dot{B}^s_{p,q}$ for any $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, which implies that

$$\| (-\Delta)^{\frac{s}{2}} f \|_{\dot{B}^s_{p,q}} \approx \| f \|_{\dot{B}^{s+\alpha}_{p,q}}.$$
3 Proof of Theorem

3.1 Construction of initial data

Letting \( n \gg 1 \), we write

\[ n \in 16\mathbb{N} = \{16, 32, 48, \cdots \} \quad \text{and} \quad \mathbb{N}(n) = \left\{ k \in 8\mathbb{N} : \frac{n}{4} \leq k \leq \frac{n}{2} \right\}, \]

Let \( 0 < \varepsilon \ll 1 \) (\( \varepsilon \) will be chosen below, see Remark 3.2). We define the matrix \( A \) whose \((i, j)\)-th component \((A)_{ij}\) with \(1 \leq i, j \leq d\) is given by

\[
(A)_{ij} = \begin{cases} 
\varepsilon, & 1 \leq i = j \leq 2, \\
1, & 3 \leq i = j \leq d, \\
0, & \text{else},
\end{cases}
\]

and \( \vec{e} = \frac{\sqrt{2}}{2}(1, 1, 0, \cdots, 0) \).

Define a scalar function \( \tilde{\theta} \in C_0^{\infty}(\mathbb{R}) \) with values in \([0, 1]\) which satisfies

\[
\tilde{\theta}(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq \frac{1}{200d}, \\
0, & \text{if } |\xi| \geq \frac{1}{100d}.
\end{cases}
\]

Let

\[
\phi(x) = \theta(x_1)\theta(x_2) \cdots \theta(x_{d-1})\theta(x_d) \sin\left(\frac{17}{24} 2^n \vec{e} \cdot x\right).
\]

To construct a sequence of external forces \( \{f_n\}_{n=1}^\infty \), motivated by Wang in [14] and Iwabuchi-Ogawa in [6], we firstly need to introduce

\[
b_n \equiv (-\Delta)^{-1} a_n = n^{\frac{1}{d}} \sum_{k \in \mathbb{N}(n)} 2^k \phi(2^k A(x - 2^{n+k} \vec{e})) \sin \left(\frac{17}{12} 2^n \vec{e} \cdot x\right), \quad (3.3)
\]

\[
c_n \equiv F^{-1} \left( \frac{\xi_2 - \xi_1}{\xi_2} \tilde{b}_n \right). \quad (3.4)
\]

Obviously, both \( b_n \) and \( c_n \) are real scalar functions. Also, it holds

\((\partial_1 - \partial_2) b_n = -\partial_2 c_n \).

In particular, we should emphasize the following important fact

\[
\text{supp } \tilde{b}_n(\xi) \subset \left\{ \xi \in \mathbb{R}^d : \frac{33}{24} 2^n \leq |\xi| \leq \frac{35}{24} 2^n \right\}. \quad (3.5)
\]

For the details of proof, see Lemma 4.1 in Appendix A.

We set

\[
\begin{cases} 
g_n^{(1)} \equiv b_n, \\
g_n^{(2)} \equiv c_n - b_n, \\
g_n^{(3)} \equiv \cdots \equiv g_n^{(d)} \equiv 0.
\end{cases} \quad (3.6)
\]
It is clearly seen that \( g_n \in \mathcal{S}(\mathbb{R}^d) \) and \( \text{div} g_n = 0 \), which obviously implies that \( \mathbb{P} g_n = g_n \).

Hence, we can define the sequence of external forces \( \{f_n\}_{n=1}^{\infty} \) by \( \{g_n\}_{n=1}^{\infty} \), precisely,

\[
g_n \equiv (-\Delta)^{-1} \mathbb{P} f_n.
\]

Notice that for all \( j \in \mathbb{Z} \)

\[
\varphi(2^{-j}\xi) \equiv 1 \quad \text{for} \quad \xi \in C_j \equiv \left\{ \xi \in \mathbb{R}^d : \frac{4}{3} 2^j \leq |\xi| \leq \frac{3}{2} 2^j \right\},
\]

and

\[
\tilde{\Delta}_j b_n = \varphi(2^{-j}) \tilde{b}_n,
\]

which implies

\[
\tilde{\Delta}_j b_n = 0, \quad j \neq n.
\]

thus,

\[
\tilde{\Delta}_j (b_n) = \begin{cases} 
    b_n, & \text{if } j = n, \\
    0, & \text{otherwise}.
\end{cases}
\]

Similarly, the above also holds for \( c_n \) and \( g_n \). Based on the observation, we have the following two Lemmas involving \( b_n \) and \( c_n \).

We should remark that, here and in what follows, the positive constants \( c \) and \( C \) whose value may vary from line to line, may depend on \( \varepsilon \) and \( \phi \) but not \( n \). The positive constants \( \tilde{c} \) and \( \tilde{C} \) whose value may vary from line to line, may depend on \( \phi \) but not \( n \) and \( \varepsilon \).

**Lemma 3.1** Let \( b_n \) be defined by \eqref{3.3}. Then there holds

\[
\|b_n\|_{L^d} \leq C n^{\frac{1}{2} - \frac{1}{d}}.
\]

**Proof.** Since \( \phi \) is a Schwartz function, we have

\[
|\phi(x)| + \sum_{i=1}^{d} |\partial_{x_i} \phi(x)| \leq C (1 + |x|)^{-M}, \quad M \geq 100d.
\]

\(
(3.7)
\)

It is easy to show that

\[
\begin{align*}
\sum_{\ell \in \mathbb{N}(n)} \frac{2^{(\ell_1 + \ell_2 + \cdots + \ell_d)}}{(1 + 2^{\ell_1}|A(x + 2^{2n+\ell_1} \vec{e})|)^M (1 + 2^{\ell_2}|A(x + 2^{2n+\ell_2} \vec{e})|)^M} \\
\leq \sum_{(\ell_1, \ell_2, \ldots, \ell_d) \in \Lambda} \int_{\mathbb{R}^d} \frac{2^{2\ell}}{(1 + 2^{\ell_1}|A(x + 2^{2n+\ell_1} \vec{e})|)^M (1 + 2^{\ell_2}|A(x + 2^{2n+\ell_2} \vec{e})|)^M} \\
+ \sum_{(\ell_1, \ell_2, \ldots, \ell_d) \in \Lambda} \int_{\mathbb{R}^d} \frac{2^{2\ell_d}}{(1 + 2^{\ell_1}|A(x + 2^{2n+\ell_1} \vec{e})|)^M (1 + 2^{\ell_2}|A(x + 2^{2n+\ell_2} \vec{e})|)^M} \\
\equiv I_1 + I_2.
\end{align*}
\]

(3.8)
where the set $\Lambda$ is defined by

$$\Lambda = \{ (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d(n) \mid \exists 1 \leq k \leq d \text{ s.t. } \ell_k \neq \ell_d \}.$$  

For the term $I_1$, by direct computations, one has

$$I_1 = \sum_{\ell \in \mathbb{N}^d(n)} \int_{\mathbb{R}^d} \frac{1}{(1 + |x|)^{dM}} dx = \frac{1}{\varepsilon^2} \sum_{\ell \in \mathbb{N}^d(n)} \int_{\mathbb{R}^d} \frac{1}{(1 + |x|)^{dM}} dx \leq \frac{C}{\varepsilon^2 n}. \quad (3.9)$$

For the term $I_2$, we assume that $\ell_1 < \ell_2$ without loss of generality, then $\ell_2 - \ell_1 \geq 4$.

$$\int_{\mathbb{R}^d} \frac{1}{(1 + 2^{\ell_1}|A(x + 2^{2n+\ell_1}d)|)^M(1 + 2^{\ell_2}|A(x + 2^{2n+\ell_2}d)|)^M dx = \left( \int_{A_{\ell_1}} + \int_{A_{\ell_2}} \right) \frac{1}{(1 + 2^{\ell_1}|A(x + 2^{2n+\ell_1}d)|)^M(1 + 2^{\ell_2}|A(x + 2^{2n+\ell_2}d)|)^M dx,$$

where we defined the set $A_{\ell_1}$ by

$$A_{\ell_1} = \{ x \mid |A(x + 2^{2n+\ell_1}d)| \leq \varepsilon 2^{2n} \}.$$  

Thus

$$\int_{A_{\ell_1}} \frac{1}{(1 + 2^{\ell_1}|A(x + 2^{2n+\ell_1}d)|)^M(1 + 2^{\ell_2}|A(x + 2^{2n+\ell_2}d)|)^M dx \leq C(\varepsilon 2^{\ell_1}2^{2n})^{-M} \int_{A_{\ell_1}} \frac{1}{(1 + 2^{\ell_1}|A(x + 2^{2n+\ell_1}d)|)^M dx \leq C(\varepsilon 2^{\ell_1}2^{2n})^{-M} \varepsilon^{-2} 2^{-d\ell_1}. \quad (3.10)$$

It is easy to deduce that for $x \in A_{\ell_1}$

$$2^{\ell_1} |A(x + 2^{2n+\ell_1}d)| \geq 2^{\ell_1} |A(2^{2n+\ell_1}d)| - \varepsilon 2^{\ell_1} 2^{2n} \geq \varepsilon 2^{\ell_2} 2^{2n}.$$  

Similarly,

$$\int_{A_{\ell_1}} \frac{1}{(1 + 2^{\ell_1}|A(x + 2^{2n+\ell_1}d)|)^M(1 + 2^{\ell_2}|A(x + 2^{2n+\ell_2}d)|)^M dx \leq (\varepsilon 2^{\ell_2} 2^{2n})^{-M} \int_{A_{\ell_1}} \frac{1}{(1 + 2^{\ell_1}|A(x + 2^{2n+\ell_1}d)|)^M dx \leq C(\varepsilon 2^{\ell_2} 2^{2n})^{-M} \varepsilon^{-2} 2^{-d\ell_1}. \quad (3.11)$$

We infer from (3.10) and (3.11) that

$$I_2 \leq C \varepsilon^{-M-2} 2^{-2Mn} \sum_{(\ell_1, \ell_2, \ldots, \ell_d) \in \Lambda} (2^{-M\ell_1} 2^{-d\ell_2} + 2^{-M\ell_2} 2^{-d\ell_1}) 2^{(\ell_1 + \ell_2 + \cdots + \ell_d)} \leq C 2^{-Mn}. \quad (3.12)$$

Inserting (3.9) and (3.12) into (3.8), we have for large enough $n$

$$\|b_n\|_{L_i} \approx \|b_n\|_{L^\infty} \leq C n^{\frac{1}{d} - \frac{1}{2}}.$$  

This completes the proof of Lemma 3.1.
Lemma 3.2 Let $c_n$ be defined by (3.4). Then there holds
\[ \|c_n\|_{B_{d,1}^0} \leq C2^{-\frac{n}{2}}. \]

**Proof.** By Hausdorff-Young’s inequality, we have
\[
\|c_n\|_{L^{d'}} \leq C \left\| \frac{\xi_1 - \xi_2}{\xi_2} d_n(\xi) \right\|_{L^{\frac{d}{d-1}}} \\
\leq C \sum_{k \in \mathbb{N}(n)} 2^k \left\| \frac{\xi_1 - \xi_2}{\xi_2} \Phi_k^{\pm,\pm}(\xi) \right\|_{L^{\frac{d}{d-1}}},
\]
where $\Phi_k^{\pm,\pm}$ is given in Appendix.

Noticing that the support condition of $\Phi_k^{\pm,\pm}$ (see (4.22) in Appendix), which implies that $|\xi_1 - \xi_2| \leq 2^k$ and $|\xi_2| \approx 2^n$, thus we obtain
\[
\|c_n\|_{L^{d'}} \leq C2^{-n} \sum_{k \in \mathbb{N}(n)} 2^{2k} \left\| \Phi_k^{\pm,\pm}(\xi) \right\|_{L^{\frac{d}{d-1}}} \\
\leq C2^{-n} \sum_{k \in \mathbb{N}(n)} 2^k \\
\leq C2^{-\frac{n}{2}},
\]
where we have used the simple fact
\[
\left\| \Phi_k^{\pm,\pm}(\xi) \right\|_{L^{\frac{d}{d-1}}} \leq C2^{-k}.
\]
Due to $\|c_n\|_{B_{d,1}^0} \approx \|c_n\|_{L^{d'}}$, we obtain the desired result and finish the proof of Lemma 3.2.

Combining Lemma 3.1 and Lemma 3.2 yields

**Proposition 3.1** Let $g_n$ be defined by (3.6). Then
\[ \|g_n\|_{B_{d,1}^0} \leq Cn^{\frac{1}{d} - \frac{1}{q}}. \]

**Remark 3.1** From Proposition 3.1, we know that $g_n \in L^d(\mathbb{R}^d) \cap B_{d,1}^1(\mathbb{R}^d)$ and $\|g\|_{L^d(\mathbb{R}^d) \cap B_{d,1}^1(\mathbb{R}^d)} \leq Cn^{\frac{1}{d} - \frac{1}{q}}$. Then Corollary 1.1 tells us that the solution map $g_n \mapsto u_n \in B_{d,1}^0(\mathbb{R}^d)$.

The following proposition is crucial for the proof of the discontinuity of solutions.

**Proposition 3.2** Let $g_n$ be defined by (3.6). If $\varepsilon$ is small enough and $n$ is large enough, then there exists $c > 0$ independent of $n$ such that
\[ \|\mathcal{B}(g_n, g_n)\|_{B_{d,1}^0(\mathbb{N}(n))} \geq c. \]

**Proof.** Recalling that the definition of $\mathcal{B}(g_n, g_n)$ and $\text{div}g_n = 0$, we have
\[ \|\mathcal{B}(g_n, g_n)\|_{B_{d,1}^0(\mathbb{N}(n))} \approx \|\mathcal{P}(g_n \cdot \nabla g_n)\|_{B_{d,1}^0(\mathbb{N}(n))}. \]
Noticing that \((\partial_1 - \partial_2)b_n = -\partial_2 c_n\), we have

\[
(g_n \cdot \nabla g_n)^{(1)} = \frac{1}{2}(\partial_1 - \partial_2)[b_n^2] + c_n \partial_2 b_n \\
= (\partial_1 - \partial_2)[b_n^2] + \partial_2(b_n c_n),
\]

\[
(g_n \cdot \nabla g_n)^{(2)} = \frac{1}{2}(-\partial_1 + \partial_2)[b_n^2] + b_n \partial_1 c_n - b_n \partial_2 c_n - c_n \partial_2 b_n + c_n \partial_2 c_n \\
= (-\partial_1 + \partial_2)[b_n^2] + (\partial_1 - \partial_2)(b_n c_n) + \partial_2[c_n^2] - \partial_2(b_n c_n),
\]

\[
(g_n \cdot \nabla g_n)^{(i)} = 0, \quad i = 3, \ldots, d.
\]

Then, we can rewrite

\[
g_n \cdot \nabla g_n = E_n + F_n, \quad (3.13)
\]

where

\[
E_n^{(1)} = (\partial_1 - \partial_2)[b_n^2], \quad F_n^{(1)} = \partial_2(b_n c_n),
\]

\[
E_n^{(2)} = -(\partial_1 - \partial_2)[b_n^2], \quad F_n^{(2)} = (\partial_1 - \partial_2)(b_n c_n) + \partial_2[c_n^2] - \partial_2(b_n c_n),
\]

\[
E_n^{(i)} = F_n^{(i)} = 0, \quad i = 3, \ldots, d.
\]

Then, from (3.13), we have

\[
\|\mathbb{P}(g_n \cdot \nabla g_n)\|_{B_{d,q}^2(\mathbb{H}(n))} \geq \|\mathbb{P}(E_n)\|_{B_{d,q}^2(\mathbb{H}(n))} - \|\mathbb{P}(F_n)\|_{B_{d,q}^2(\mathbb{H}(n))} \\
\geq \|\mathbb{P}(E_n)\|_{B_{d,q}^2(\mathbb{H}(n))} - C2^{-\frac{n}{2}} \\
\geq \left\|\left(\mathbb{P}(E_n)\right)^{(1)}\right\|_{B_{d,q}^2(\mathbb{H}(n))} - C2^{-\frac{n}{2}},
\]

where we have used

\[
\|\mathbb{P}(F_n)\|_{B_{d,q}^2(\mathbb{H}(n))} \leq \left\|F_n^{(1)}\right\|_{B_{d,q}^2(\mathbb{H}(n))} + \left\|F_n^{(2)}\right\|_{B_{d,q}^2(\mathbb{H}(n))} \\
\leq C \left(\|b_n c_n\|_{B_{d,q}^{-1}(\mathbb{H}(n))} + \|c_n^2\|_{B_{d,q}^1(\mathbb{H}(n))}\right) \\
\leq C \left(\|b_n c_n\|_{B_{d,q}^0(\mathbb{H}(n))} + \|c_n^2\|_{B_{d,q}^0(\mathbb{H}(n))}\right) \\
\leq Cn^{\frac{1}{2} - \frac{1}{2}} \left(\|b_n c_n\|_{B_{d,q}^0(\mathbb{H}(n))} + \|c_n^2\|_{B_{d,q}^0(\mathbb{H}(n))}\right) \\
\leq Cn^{\frac{1}{2} - \frac{1}{2}} \left(\|b_n\|_{L^\infty} \|c_n\|_{L^2} + \|c_n^2\|_{L^2}\right) \\
\leq C2^{-\frac{n}{2}}.
\]

Notice that \(\mathbb{P} = \text{Id} + \nabla(-\Delta)^{-1}\text{div}\), then

\[
(\mathbb{P}(E_n))^{(1)} = (\partial_1 - \partial_2)[b_n^2] + \partial_1(-\Delta)^{-1}\text{div} E_n \\
= (\partial_1 - \partial_2)[b_n^2] + \partial_1(\partial_1 - \partial_2)^2(-\Delta)^{-1}[b_n^2],
\]

12
Let us introduce the set $B$. For $n \in \mathbb{N}$, direct computations gives that

$$b_n^2 = n^{-\frac{1}{2}}(H_1 + H_2),$$

where

$$H_1 \equiv \sum_{k \in \mathbb{N}(n)} 2^{2k} \phi^2(2^k A(x - 2^n \vartheta)) \sin^2 \left( \frac{17}{12} 2^n \vartheta \cdot x \right),$$

$$H_2 \equiv \sum_{k, j \in \mathbb{N}(n), k \neq j} 2^{k+j} \phi(2^k A(x - 2^n \vartheta)) \phi(2^j A(x - 2^n \vartheta)) \sin^2 \left( \frac{17}{12} 2^n \vartheta \cdot x \right).$$

We should emphasize that, the following cancelation holds

$$\|(\partial_1 - \partial_2)H_2 + \partial_1(\partial_1 - \partial_2)^2 (-\Delta)^{-1} H_2\|_{B^{-2}_{2q}(\ell^0(n))} = 0,$$

which immediately comes from the fact $\Delta_{\ell} H_2 = 0$ for $\ell \in \mathbb{N}(n)$ (for the proof see Lemma 4.2).

Thus, (3.14) reduces to

$$\left\| (\mathbb{P}(E_n))^{(1)} \right\|_{B^{-2}_{2q}(\ell^0(n))}^q = n^{-1} \|(\partial_1 - \partial_2)H_1 + \partial_1(\partial_1 - \partial_2)^2 (-\Delta)^{-1} H_1\|_{B^{-2}_{2q}(\ell^0(n))}.$$  

(3.17)

For $\ell \in \mathbb{N}(n)$, we decompose the term $H_1$ as

$$H_1 = 2^{2\ell} \phi^2(2^\ell A(x - 2^n \vartheta)) \sin^2 \left( \frac{17}{12} 2^n \vartheta \cdot x \right) + \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{2k} \phi^2(2^k A(x - 2^n \vartheta)) \sin^2 \left( \frac{17}{12} 2^n \vartheta \cdot x \right),$$

using the simple fact $\sin^2 \alpha = (1 - \cos 2\alpha)/2$, then we have

$$\Delta_{\ell} H_1 = \frac{1}{2} \Delta_{\ell} \left( 2^{2\ell} \phi^2(2^\ell A(x - 2^n \vartheta)) \right) + \frac{1}{2} \Delta_{\ell} \left( \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{2k} \phi^2(2^k A(x - 2^n \vartheta)) \right) \equiv \Delta_{\ell} H_{1,1} + \Delta_{\ell} H_{1,2}.$$

Let us introduce the set $B_{\ell}$ defined by

$$B_{\ell} \equiv \{ x : |A(x - 2^n \vartheta)| \leq 2^{-\ell} \}.$$
Then, we have
\[
\| (\partial_1 - \partial_2)H_1 + \partial_1(\partial_1 - \partial_2)^2(-\Delta)^{-1}H_1\|^{q}_{L^{2,q}(\Omega(n))} \\
\geq \sum_{\ell \in \mathbb{N}(n)} 2^{-2q\ell} \| \hat{\Delta}_\ell (\partial_1 - \partial_2)H_1 + \hat{\Delta}_\ell \partial_1(\partial_1 - \partial_2)^2(-\Delta)^{-1}H_1\|^{q}_{L^{2,q}(\mathbb{R}^d)} \\
\geq \sum_{\ell \in \mathbb{N}(n)} 2^{-2q\ell} \| \hat{\Delta}_\ell (\partial_1 - \partial_2)H_1 + \hat{\Delta}_\ell \partial_1(\partial_1 - \partial_2)^2(-\Delta)^{-1}H_1\|^{q}_{L^{2,q}(B_r)} \\
\geq \sum_{\ell \in \mathbb{N}(n)} 2^{-2q\ell} \| \hat{\Delta}_\ell (\partial_1 - \partial_2)H_{1,1}\|^{q}_{L^{2,q}(B_r)} - \sum_{\ell \in \mathbb{N}(n)} 2^{-2q\ell} \| \hat{\Delta}_\ell \partial_1(\partial_1 - \partial_2)^2(-\Delta)^{-1}H_{1,1}\|^{q}_{L^{2,q}(B_r)} \\
- \sum_{\ell \in \mathbb{N}(n)} 2^{-2q\ell} \| \hat{\Delta}_\ell (\partial_1 - \partial_2)H_{1,2}\|^{q}_{L^{2,q}(B_r)} - \sum_{\ell \in \mathbb{N}(n)} 2^{-2q\ell} \| \hat{\Delta}_\ell \partial_1(\partial_1 - \partial_2)^2(-\Delta)^{-1}H_{1,2}\|^{q}_{L^{2,q}(B_r)} \\
= J_1 - J_2 - J_3 - J_4.
\]

Now, we will show that the term \( J_1 \) contributes the main part.

**Estimation of \( J_1 \).** Denote
\[
h(x) \equiv \frac{1}{2}(\partial_1 - \partial_2)(\phi^2)(x) \\
= \frac{1}{2}(\partial_1 - \partial_2) \left( \theta^2(x_1)\theta^2(x_2) \right) \theta^2(x_3) \cdots \theta^2(x_{d-1})\theta^2(x_d) \sin^2 \left( \frac{17}{24} x_d \right) \\
= h_1(x) + h_2(x),
\]
where
\[
h_1(x) \equiv \frac{1}{4}(\partial_1 - \partial_2) \left( \theta^2(x_1)\theta^2(x_2) \right) \theta^2(x_3) \cdots \theta^2(x_{d-1})\theta^2(x_d), \]
\[
h_2(x) \equiv \frac{1}{4}(\partial_1 - \partial_2) \left( \theta^2(x_1)\theta^2(x_2) \right) \theta^2(x_3) \cdots \theta^2(x_{d-1})\theta^2(x_d) \cos \left( \frac{17}{12} x_d \right),
\]
by direct computations, we get for \( \ell \in \mathbb{N}(n) \),
\[
\hat{\Delta}_\ell ((\partial_1 - \partial_2)H_{1,1}) = \varepsilon 2^{3\ell} h_2(2^\ell A(x - 2^{2n-\ell} \mathbf{e})).
\]
Then by change of variables, we have
\[
\| \hat{\Delta}_\ell ((\partial_1 - \partial_2)H_{1,1})\|^{q}_{L^{2,q}(B_r)} \geq \varepsilon 2^{3\ell} \| \hat{h}_2(2^\ell A(x - 2^{2n-\ell} \mathbf{e}))\|^{q}_{L^{2,q}(B_r)} \\
\geq \varepsilon^{1 - \frac{3}{2} 2^{\ell}} \| h_2(y)\|^{q}_{L^{2,q}(B_1)} \\
\equiv c \varepsilon^{1 - \frac{3}{2} 2^{\ell}},
\]
thus
\[
J_1 = \sum_{\ell \in \mathbb{N}(n)} 2^{-2q\ell} \| \Delta_\ell ((\partial_1 - \partial_2)H_{1,1})\|^{q}_{L^{2,q}(B_r)} \geq \bar{c} \varepsilon^{1 - \frac{3}{2} 2^{\ell} n}. \quad (3.18)
\]
Estimation of $J_2$. By direct computations, we get

$$\|\hat{\Delta}_\ell (\partial_1 (\partial_1 - \partial_2)^2 (-\Delta)^{-1} H_{1,1})\|_{L^q(B_\ell)}$$

$$\leq \|\hat{\Delta}_\ell (\partial_1 (\partial_1 - \partial_2)^2 (-\Delta)^{-1} H_{1,1})\|_{L^q(\mathbb{R}^d)}$$

$$\leq C 2^{-2\ell} \|\hat{\Delta}_\ell (\partial_1 (\partial_1 - \partial_2)^2 H_{1,1})\|_{L^q(\mathbb{R}^d)}$$

$$\leq C \varepsilon^3 2^{3\ell} \|\left(\partial_1 (\partial_1 - \partial_2)^2 \phi^2\right) \left(2^\ell A(y - 2^{2n+\ell} \bar{c})\right)\|_{L^q(\mathbb{R}^d)}$$

$$\leq C \varepsilon^3 2^{3\ell} \|\left(\partial_1 (\partial_1 - \partial_2)^2 \phi^2\right)\|_{L^q(\mathbb{R}^d)}$$

where we have used

$\hat{\Delta}_\ell (\partial_1 (\partial_1 - \partial_2)^2 H_{1,1}) = \varepsilon^3 2^{3\ell} \left[\left(\partial_1 (\partial_1 - \partial_2)^2 \phi^2\right) \left(2^\ell A(y - 2^{2n+\ell} \bar{c})\right)\right].$

Therefore

$$J_2 = \sum_{\ell \in \mathbb{N}(n)} 2^{-2\ell q} \left\|\hat{\Delta}_\ell \left(\partial_1 (\partial_1 - \partial_2)^2 (-\Delta)^{-1} H_{1,1}\right)\right\|_{L^q(B_\ell)} \leq C \varepsilon^{(3 - \frac{3}{2})q} n. \quad (3.19)$$

**Remark 3.2** It is worthwhile pointing out that both the constants $\bar{c}$ and $C$ do not depend on the parameter $\varepsilon$. Combining (3.18) and (3.19), we need to take $\varepsilon$ sufficiently small such that $J_2$ can be absorbed by $J_1$. This is why in the present paper we set the parameter $\varepsilon$ in the matrix $A$.

Next, we deal with the last two terms which are much less than the first one. By making fully use of the information of decreasing rapidly functions $\bar{\phi}$ and $\phi$ when $\ell \neq \ell$, we can deal with the third term.

Estimation of $J_3$. Noting the fact that for $100d < N \in \mathbb{Z}^+$

$$|\bar{\phi}(x)| \leq C (1 + |x|)^{-N},$$

then we have

$$\left\|\hat{\Delta}_\ell ((\partial_1 - \partial_2) H_{1,2})\right\|_{L^q(B_\ell)}$$

$$\leq \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{3k} 2^{d\ell} \left\|\int_{\mathbb{R}^d} \bar{\phi}(2^\ell (x - y))((\partial_1 - \partial_2)\phi^2) \left(2^\ell A(y - 2^{2n+k} \bar{c})\right)\right\|_{L^q(B_\ell)}$$

$$\leq \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{3k} 2^{d\ell} \left\|\int_{\mathbb{R}^d} \left(1 + 2^\ell |x - y|\right)^{-N} \left(1 + 2^k |A(y - 2^{2n+k} \bar{c})|\right)^{-2N}\right\|_{L^q(B_\ell)}. \quad (3.20)$$

Dividing the integral region in terms of $y$ into the following two parts to estimate:

$$\mathbb{R}^d = \left\{ y : |A(y - 2^{\ell+2n} \bar{c})| \leq \varepsilon 2^{2n}\right\} \cup \left\{ y : |A(y - 2^{\ell+2n} \bar{c})| \geq \varepsilon 2^{2n}\right\}$$

$$\equiv A_1 \cup A_2,$$
For \( x \in B_{\ell} \) and \( y \in A_1 \), we conclude that
\[
\left| A(y - 2^{k+2n} \varepsilon) \right| = \left| A(y - 2^{k+2n} \varepsilon) + A(2^{k+2n} \varepsilon - 2^{k+2n} \varepsilon) \right|
\geq \left| (2^{k+2n} - 2^{k+2n})A\varepsilon \right| - \left| A(y - 2^{k+2n} \varepsilon) \right|
\geq \varepsilon 2^{2n}.
\]

For \( x \in B_{\ell} \) and \( y \in A_2 \), it is easy to check that
\[
|x - y| \geq \left| A(y - 2^{k+2n} \varepsilon) \right| - \left| A(x - 2^{k+2n} \varepsilon) \right| \geq \varepsilon 2^{2n} - 2^{-\ell} \geq \varepsilon 2^{2n-1}.
\]

Then, we have
\[
\left\| \int_{\R^d} (1 + 2|y - y'|)^{-N} (1 + 2^k|A(y - 2^{k+2n} \varepsilon)|)^{-2N} \right\|_{L^2(B_{\varepsilon})}^2
\leq C 2^{-2(k+2n)N} \left\| \int_{A_1} (1 + 2^k|y - y'|)^{-N} \right\|_{L^2(B_{\varepsilon})}^2
+ C 2^{-(k+2n)N} \left\| \int_{A_2} (1 + 2^k|y - y'|)^{-N} \right\|_{L^2(B_{\varepsilon})}^2
\leq C \left( 2^{-d\ell} 2^{-2(k+2n)N} + 2^{-(k+2n)N} 2^{-\ell} \right) 2^{-\ell} \leq C 2^{-n}.
\]

Plugging the above into (3.20) yields
\[
\left\| \hat{\Delta}_\ell ((\partial_1 - \partial_2)H_{1,2}) \right\|_{L^q(B_{\varepsilon})} \leq C \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{3k} 2^{d\ell} \left( 2^{-d\ell} 2^{-2(k+2n)N} + 2^{-(k+2n)N} 2^{-\ell} \right) 2^{-\ell} \leq C 2^{-n}.
\]

Therefore, we have
\[
J_3 = \sum_{\ell \in \mathbb{N}(n)} 2^{-2\ell q} \left\| \hat{\Delta}_\ell ((\partial_1 - \partial_2)H_{1,2}) \right\|_{L^q(B_{\varepsilon})}^q \leq C n 2^{-qn}.
\]

**Estimation of J_4.** Notice that
\[
\phi^2(x) = \prod_{i=1}^d \theta^2(x_i) + \frac{1}{2} \prod_{i=1}^d \theta^2(x_i) \cos \left( \frac{17}{12} x_d \right)
\equiv \Theta_1(x) + \Theta_2(x),
\]
then we have for \( k \neq \ell \),
\[
\hat{\Delta}_\ell \left( \Theta_2(2^k A(x - 2^{2n+k} \varepsilon)) \right) = 0,
\]
which implies
\[
\hat{\Delta}_\ell \left( (\partial_1 - \partial_2) (\hat{\Delta}_\ell \left( \Theta_2(2^k A(x - 2^{2n+k} \varepsilon)) \right) \right) = \hat{\Delta}_\ell \left( \hat{\Delta}_\ell \left( \Theta_1(2^k A(x - 2^{2n+k} \varepsilon)) \right) \right)
\Rightarrow \hat{\Delta}_\ell \Psi(x)
\]

16
where we denote
\[ \Psi(x) \equiv U(2^k A(x - 2^{2n+k} \partial)) \]
and
\[ U(x) = \varepsilon^3 2^k \partial_1 (\partial_1 - \partial_2)^2 (-\varepsilon^2 \partial_1^2 - \varepsilon^2 \partial_2^2 - \cdots - \partial_d^2)^{-1} (\Theta_1)(x), \]
we can show that
\[
\tilde{\Delta}_x \Psi(x) = 2\ell^d \int_{\mathbb{R}^d} \hat{h}(2^\ell (x - y)) U(2^k A(y - 2^{2n+k} \partial)) dy
\]
\[ = 2\ell^d \int_{\mathbb{R}^d} \hat{h}(2^\ell (x - z - 2^{2n+k} \partial)) U(2^k A z) dz
\]
\[ = \varepsilon^{-2} 2^{d(\ell - k)} \int_{\mathbb{R}^d} \hat{h}(2^\ell (x - 2^{-k} A^{-1} y - 2^{2n+k} \partial)) U(y) dy
\]
\[ = \varepsilon^{-2} 2^{d(\ell - k)} \int_{\mathbb{R}^d} \hat{h}(2^{\ell - k} A^{-1} (2^k A [x - 2^{2n+k} \partial] - y)) U(y) dy
\]
\[ = (P_\ell * U)(2^k A(x - 2^{2n+k} \partial)), \]
where
\[ P_\ell * U = \varepsilon^{-2} 2^{d(\ell - k)} \int_{\mathbb{R}^d} \hat{h}(2^{\ell - k} A^{-1}(x - y)) U(y) dy. \]

Notice that
\[ \mathcal{F}[\hat{h}(2^{\ell - k} A^{-1} \cdot)] = \varepsilon^2 2^{-d(\ell - k)} \hat{\varphi}(2^{k-\ell} A \xi), \]
we have
\[ (P_\ell * U)(x) = (2\pi)^{-d} \varepsilon^3 2^k \int_{\mathbb{R}^d} e^{ix \xi} \cdot \frac{-i \xi_1 (\xi_1 - \xi_2)^2}{\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2} \hat{\Theta}_1(\xi) \hat{\varphi}(2^{k-\ell} A \xi) d\xi. \]

Since \( \hat{\varphi}(x) = \varphi(\frac{1}{2} x) + \varphi(x) + \varphi(2x) \), then we have \( \hat{\varphi}(x) \) is supported in \( C := \{ \xi \in \mathbb{R}^d : 3/8 \leq |\xi| \leq 16/3 \} \). Therefore, if \( |\xi| \leq \frac{1}{8} 2^{\ell - k} \), we can deduce that \( \hat{\varphi}(2^{k-\ell} A \xi) = 0 \). Then, using
\[
(1 + |x|^2)^N (P_\ell * U)(x) = (2\pi)^{-d} \varepsilon^3 2^k \int_{\mathbb{R}^d} (1 - \Delta_\xi)^N (e^{ix \xi} \cdot \frac{-i \xi_1 (\xi_1 - \xi_2)^2}{\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2} \hat{\Theta}_1(\xi) \hat{\varphi}(2^{k-\ell} A \xi) d\xi
\]
\[ = (2\pi)^{-d} \varepsilon^3 2^k \int_{\mathbb{R}^d} e^{ix \xi} (1 - \Delta_\xi)^N \left( \frac{-i \xi_1 (\xi_1 - \xi_2)^2}{\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2} \hat{\Theta}_1(\xi) \hat{\varphi}(2^{k-\ell} A \xi) \right) d\xi, \]
we have
\[ |(1 + |x|^2)^N (P_\ell * U)(x)| \leq C 2^k 2^{(k-\ell)2N}, \]
which implies
\[ |\tilde{\Delta}_x \Psi(x)| \leq C 2^k 2^{(k-\ell)2N} (1 + 2^k |A(x - 2^{2n+k} \partial)|)^{-2N}. \]
Then, we have
\[
\left\| \hat{A}_t \left( \partial_1 (\partial_1 - \partial_2)^2 (-\Delta)^{-1} H_{1,2} \right) \right\|_{L^q(B_r)} \\
\leq \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{2k} 2^{d\ell} \left\| \int_{\mathbb{R}^d} \tilde{\varphi}(2^\ell (x - y)) (\Delta_1 (\Psi(y))) dy \right\|_{L^q(B_r)} \\
\leq \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{3k} 2^{2(\ell - 2)2N} \left\| \int_{\mathbb{R}^d} \left( 1 + 2^{\ell} |x - y| \right)^{-2N} \left( 1 + 2^k |A(y - 2^{n+k} \partial) \right)^{-2N} dy \right\|_{L^q(B_r)} \\
\leq C \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{3k} 2^{2(\ell - 2)2N} \left( 2^{-d\ell} 2^{-2(\ell-2)N} + 2^{-(\ell+2)N} 2^{-dn} \right) 2^{-\frac{\ell}{2}} \\
\leq C 2^{-n},
\]
this gives that
\[
J_4 \leq C n 2^{-qn}.
\]
Combining the above estimates yields
\[
\left\| \left( \mathbb{P}(E_n) \right)^{(1)} \right\|_{\tilde{B}^q_{2,2}(\tilde{\Omega}(n))} \geq (\tilde{c} - \tilde{C} \epsilon^2) e^{(1 - \frac{2}{q})n} - C 2^{-qn},
\]
which implies
\[
\| \mathcal{B}(g_n, g_n) \|_{\tilde{B}^0_{2,2}(\tilde{\Omega}(n))} \geq c. \tag{3.21}
\]

Thus, we complete the proof of Proposition 3.2.

Combining Propositions 3.1 and 3.2, we shall prove Theorem 1.4.

**Proof of Theorem 1.4.** Letting \( u_n = g_n + \mathcal{B}(g_n, g_n) + U_n \), then we deduce from Proposition 3.1 that
\[
\| g_n \|_{\tilde{B}^0_{d,1}(\tilde{\Omega}(n))} \leq C \| g_n \|_{\tilde{B}^0_{d,1}} \leq C n^{\frac{1}{2} - \frac{\ell}{4}}
\]
and Corollary 1.2 that
\[
\| U_n \|_{\tilde{B}^0_{d,1}(\tilde{\Omega}(n))} \leq n^{\frac{1}{2} - \frac{\ell}{4}} \| U_n \|_{\tilde{B}^0_{d,2}(\tilde{\Omega}(n))} \leq C n^{\frac{1}{2} - \frac{\ell}{4}} \| g_n \|_{L^d}^3 \leq C n^{\frac{1}{2} - \frac{\ell}{4}} n^{\frac{1}{2} - \frac{\ell}{4}} \leq C n^{\frac{1}{2} - \frac{\ell}{4}}.
\]
Therefore, we obtain for large \( n \) enough
\[
\| u_n \|_{\tilde{B}^0_{d,1}} \geq \| u_n \|_{\tilde{B}^0_{d,2}(\tilde{\Omega}(n))} \\
\geq \| \mathcal{B}(g_n, g_n) \|_{\tilde{B}^0_{d,2}(\tilde{\Omega}(n))} - \| g_n \|_{\tilde{B}^0_{d,2}(\tilde{\Omega}(n))} - \| U_n \|_{\tilde{B}^0_{d,2}(\tilde{\Omega}(n))} \\
\geq c - C n^{\frac{1}{2} - \frac{\ell}{4}} \geq \frac{c}{2}.
\]
Thus, we have completed the proof of Theorem 1.4.


4 Appendix

For the sake of convenience, here we present more details in the computations.

Lemma 4.1 Let $b_n$ be defined by (3.3). Then there holds

$$\text{supp } b_n(\xi) \subset \left\{ \xi \in \mathbb{R}^d : \frac{33}{24} 2^n \leq |\xi| \leq \frac{35}{24} 2^n \right\}.$$ 

Proof. For the sake of simplicity, we denote

$$\Phi_k := \phi \left( 2^k A(x - 2^{n+k} \hat{e}) \right) \sin (\lambda_n \hat{e} \cdot x) \quad \text{with} \quad \lambda_n := \frac{17}{12} 2^n.$$ 

Using the fact $\sin \alpha = \frac{1}{i} (e^{-i\alpha} - e^{i\alpha})$ from Euler's formula, we deduce easily that

$$\mathcal{F} (\Phi_k) = \int_{\mathbb{R}^d} e^{-i2^k \xi \cdot \hat{e}} \phi \left( 2^k A(x - 2^{n+k} \hat{e}) \right) \sin (\lambda_n \hat{e} \cdot x) \, dx \quad \text{for} \quad k \geq 0.$$ 

$$= \frac{i}{2} \int_{\mathbb{R}^d} e^{-i2^k \xi \cdot \hat{e}} \phi \left( 2^k A(x - 2^{n+k} \hat{e}) \right) \phi \left( 2^k A(x - 2^{n+k} \hat{e}) \right) \, dx$$ 

$$= \frac{i}{2} (\Phi_k^1 - \Phi_k^2),$$

where

$$\Phi_k^1 := \int_{\mathbb{R}^d} e^{-i2^k \xi \cdot \hat{e}} \phi \left( 2^k A x \right) \, dx,$$ 

$$\Phi_k^2 := \int_{\mathbb{R}^d} e^{-i2^k \xi \cdot \hat{e}} \phi \left( 2^k A x \right) \, dx.$$ 

Let $\lambda_n := \frac{\lambda_n}{2^k}$. Then by change of variables, we have

$$\Phi_k^1 = \frac{1}{\xi_2^k 2^k} e^{-i2^{k+1} \xi \cdot \hat{e}} \int_{\mathbb{R}^d} e^{-i\xi_1 \cdot e^{-1/2^k} (\xi_1 + \lambda_n)} \theta(x_1) \, dx_1 \int_{\mathbb{R}^d} e^{-i\xi_2 \cdot e^{-1/2^k} (\xi_2 + \lambda_n)} \theta(x_2) \, dx_2$$ 

$$\times \int_{\mathbb{R}^d} e^{-i\xi_3 \cdot e^{-1/2^k} \xi_3} \theta(x_3) \, dx_3 \cdots \int_{\mathbb{R}^d} e^{-i\xi_d \cdot e^{-1/2^k} \xi_d} \theta(x_d) \, dx_d$$ 

$$= \frac{i}{\xi_2^k 2^k} e^{-i2^{k+1} \xi \cdot \hat{e}} \theta \left( \xi_1 + \lambda_n \right) \theta \left( \xi_2 + \lambda_n \right) \theta \left( \xi_3 \right) \cdots \theta \left( \xi_d \right) \left[ \theta \left( \frac{\xi_d}{2^k} + \frac{17}{24} \right) - \theta \left( \frac{\xi_d}{2^k} - \frac{17}{24} \right) \right]$$ 

$$= \Phi_k^{1+} - \Phi_k^{1-}.$$ 

Similarly,

$$\Phi_k^2 = \frac{i}{\xi_2^k 2^k} e^{-i2^{k+1} \xi \cdot \hat{e}} \theta \left( \xi_1 - \lambda_n \right) \theta \left( \xi_2 - \lambda_n \right) \theta \left( \xi_3 \right) \cdots \theta \left( \xi_d \right) \left[ \theta \left( \frac{\xi_d}{2^k} + \frac{17}{24} \right) - \theta \left( \frac{\xi_d}{2^k} - \frac{17}{24} \right) \right]$$ 

$$= \Phi_k^{2+} - \Phi_k^{2-}.$$ 

19
Recalling that the support condition of \( \hat{\theta} \), we have
\[
\text{supp } \Phi^+_{k, j} \subset \left\{ \xi : \left| \xi_i \pm \frac{17\sqrt{2}}{24}2^n \right| \leq \frac{\varepsilon 2^k}{100d}, \ i = 1, 2, \right. \\
|\xi_j| \leq \frac{2^k}{100d}, \ j = 3, \cdots, d - 1, \left. \left| \xi_d \pm \frac{17\sqrt{2}}{24}2^n \right| \leq \frac{2^k}{100d} \right\}.
\] (4.22)

Without loss of generality, we assume that \( \text{supp } \hat{\Phi}_k \subset \text{supp } \Phi^+_{k, j} \). Then for all \( k \in \mathbb{N}(n) \), we have
\[
\text{supp } \hat{\Phi}_k \subset \left\{ \xi : \frac{17\sqrt{2}}{24}2^n - \frac{\varepsilon 2^k}{100d} \leq |\xi| \leq \frac{17\sqrt{2}}{24}2^n + \frac{\varepsilon 2^k}{100d}, \ i = 1, 2, \right. \\
|\xi_j| \leq \frac{2^k}{100d}, \ j = 3, \cdots, d - 1, \left. \frac{2^k}{3} \leq |\xi_d| \leq \frac{3\sqrt{2}}{4}2^k \right\},
\]
which implies that
\[
\text{supp } \hat{\Phi}_k \subset \left\{ \xi : \frac{33}{24}2^n \leq |\xi| \leq \frac{35}{24}2^n \right\}.
\]

Thus, we finish the proof of Lemma 4.1.

**Lemma 4.2** Let \( H_2 \) be defined by (3.16). Then there holds for \( \ell \in \mathbb{N}(n) \)
\[
\hat{\Delta}_\ell H_2 = 0.
\] (4.23)

**Proof.** Obviously, one has
\[
H_2 = \frac{1}{2} \sum_{k, j \in \mathbb{N}(n), k \neq j} 2^{k+j} \Phi_{k, j}(x) + \frac{1}{2} \sum_{k, j \in \mathbb{N}(n), k \neq j} 2^{k+j} \Phi_{k, j}(x) \cos \left( \frac{17}{12}2^{n+1} \cdot x \right)
\]
where
\[
\Phi_{k, j}(x) := \phi(2^k A(x - 2^{n+k} \bar{\varepsilon})) \phi(2^j A(x - 2^{n+j} \bar{\varepsilon})).
\]

Notice that the definition of \( \phi \), we deduce that for \( j < k \)
\[
\text{supp } \Phi_{k, j} \subset \left\{ \xi \in \mathbb{R}^d : \frac{33}{48}2^k \leq |\xi| \leq \frac{35}{48}2^k \right\},
\]
which in turn gives that for \( j < k \)
\[
\text{supp } \mathcal{F} \left( \Phi_{k, j} \cos \left( \frac{17}{12}2^{n+1} \cdot x \right) \right) \subset \left\{ \xi \in \mathbb{R}^d : \frac{33}{24}2^{n+1} \leq |\xi| \leq \frac{35}{24}2^{n+1} \right\}.
\]
Then, for \( j < k \), (4.23) holds. Similarly, (4.23) also holds for \( j > k \).

Thus, we finish the proof of Lemma 4.2.

**Acknowledgments**

J. Li is supported by the National Natural Science Foundation of China (11801090 and 12161004) and Jiangxi Provincial Natural Science Foundation (20212BAB211004). Y. Yu is supported by the National Natural Science Foundation of China (12101011) and Natural Science Foundation of Anhui Province (1908085QA05). W. Zhu is supported by the Guangdong Basic and Applied Basic Research Foundation (2021A1515111018).
Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier analysis and nonlinear partial differential equations, Grundlehren der Mathematischen Wissenschaften, 343, Springer-Verlag, Berlin, Heidelberg, 2011.

[2] J. Bourgain, N. Pavlović, Ill-posedness of the Navier–Stokes equations in a critical space in 3D, J. Funct. Anal., 255 (2008) 2233–2247.

[3] C. Bjorland, L. Brandolese, D. Iftimie, M.E. Schonbek, $L^p$-solutions of the steady-state Navier-Stokes equations with rough external forces. Commun. Partial Differ. Equations 36, No. 1-3(2011) 216-246.

[4] Z. Chen, $L^n$ solutions of the stationary and nonstationary Navier-Stokes equations in $\mathbb{R}^n$, Pac. J. Math. 158 (1993) 293-303.

[5] J.G. Heywood, On stationary solutions of the Navier-Stokes equations as limits of non-stationary solutions, Arch. Ration. Mech. Anal. 37 (1970) 48-60.

[6] T. Iwabuchi, T. Ogawa, Ill-posedness for the compressible Navier-Stokes equations under the barotropic condition in the limiting Besov spaces. J. Math. Soc. Jpn. (2021) doi: 10.2969/jmsj/81598159

[7] H. Koch, D. Tataru, Well-posedness for the Navier-Stokes equations, Adv. Math. 157 (2001), 22-35.

[8] K. Kaneko, H. Kozono, S. Shimizu, Stationary solution to the Navier-Stokes equations in the scaling invariant Besov space and its regularity. Indiana Univ. Math. 68:3 (2019) 857-880.

[9] O.A. Ladyzhenskaya, Investigation of the Navier-Stokes equation for stationary motion of an incompressible fluid, Uspekhi Mat. Nauk 14 (1959) 57-97.

[10] J. Leray, Etude de diverses équations intégrales non linéaires et de quelques proléms que pose l’hydrodynamique, J. Math. Pures Appl. 12 (1933) 1-82.

[11] P. Secchi, On the stationary and nonstationary Navier–Stokes equations in $\mathbb{R}^n$, Ann. Mat. Pure Appl. 153 (1988) 293-305.

[12] H. Tsurumi, Well-posedness and ill-posedness problems of the stationary Navier-Stokes equations in scaling invariant Besov spaces, Arch. Ration. Mech. Anal. 234:2 (2019) 911-923.

[13] H. Tsurumi, Well-posedness and ill-posedness of the stationary Navier-Stokes equations in toroidal Besov spaces. Nonlinearity 32 (2019) 3798-3819.

[14] B. Wang, Ill-posedness for the Navier-Stokes equations in critical Besov spaces $\dot{B}^{-1}_{\infty,q}$, Adv. Math. 268 (2015) 350-372.
[15] T. Yoneda, Ill-posedness of the 3D-Navier-Stokes equations in a generalized Besov space near $\text{BMO}^{-1}$, J. Funct. Anal. 258 (2010) 3376-3387.

[16] T.V. Phan, N.C. Phuc, Stationary Navier-Stokes equations with critically singular external forces: existence and stability results. Adv. Math. 241 (2013) 137-161.