Noncommutative Einstein spaces and TQFT

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Abstract. We have studied categorical properties of the noncommutative counter parts of so-called Einstein spaces (their Ricci tensor is proportional to the metric) in the framework of twisted gravity. We have computed the deformed Riemannian tensor and scalar curvature in the formalism of twisted gravity too. We could already see for some examples the remarkable property that being an Einstein space seems to be stable under deformation, using Killing vector field in the twist. The deformed Levi-Civita connection and the deformed Riemann tensor are just the undeformed ones. As a generalization, one should study star geometries, where the noncommutative vector fields are not Killing vectors. On the other hand, the main result of this talk can be summarized as a construction of extended topological quantum field theory.

1. Noncommutative geometry

1.1. Introduction

The subjects of the double bicategory, topological quantum field theory (TQFT), and noncommutative Einstein spaces have been studied in [1–13]. Let us describe some noncommutative geometric aspects of twisted deformations. Consider a Lie algebra $g$ over $\mathbb{C}$, and its associated universal enveloping algebra $Ug$. A general twist $F$ is an element $F \in Ug \otimes Ug$ in the tensor product of a Hopf algebra $(Ug, \cdot, \Delta, S, \varepsilon)$ given by

$$ F = f^\alpha \otimes f_\alpha, \quad F^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha, $$

and satisfying the conditions

$$ F_{12}((\Delta \otimes id)F) = F_{23}(id \otimes \Delta)F, \quad (\varepsilon \otimes id)F = 1 = (id \otimes \varepsilon)F, $$

where the elements $f^\alpha, f_\alpha, \bar{f}^\alpha, \bar{f}_\alpha$ belong to $Ug$, $\Delta$ denotes the coproduct and $\varepsilon$ the co-unit of the respective Hopf algebra [3, 14, 15].

Then, the universal $\mathcal{R}$ matrix is defined by

$$ \mathcal{R} = F_{23}F^{-1} = R^\alpha \otimes R_\alpha, \quad \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha. $$

Using the $\mathcal{R}$ matrix we obtain for functions $h$ and $g$

$$ h \ast g = \bar{R}^\alpha(g) \ast \bar{R}_\alpha(h). $$

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Our strategy is to deform a product $\circ$ of some objects $A$ and $B$ by replacing it with a twisted product $\circ^\star$:

$$
A \circ^\star B := \bar{f}^\alpha(A) \circ \bar{f}_\alpha(B).
$$

(6)

The universal enveloping algebra of vector fields can be deformed in two different ways:

- $U\Xi^\star$
  
  This is a Hopf algebra [3] defined by the deforming the structure functions of $U\Xi$:

  
  $$
  u \star v = \bar{f}^\alpha(u)\bar{f}_\alpha(v),
  $$

  (7)

  $$
  \Delta^\star(u) = u \otimes 1 + \bar{R}^\alpha \otimes \bar{R}_\alpha(u),
  $$

  (8)

  $$
  \epsilon^\star(u) = \epsilon(u) = 0,
  $$

  (9)

  $$
  S^\star(u) = -\bar{R}^\alpha(u)\bar{R}_\alpha,
  $$

  (10)

  where $\bar{R}^\alpha(u)$ is the usual Lie derivative of $u$ along the vector field $\bar{R}^\alpha$.

  There is a natural action of $\Xi^\star$ on the algebra of functions $A^\star$ given in terms of the usual undeformed Lie derivative,

  $$
  L^\star_u(h) := \bar{f}^\alpha(u)(\bar{f}_\alpha(h)),
  $$

  (11)

  which can be extended to $U\Xi^\star$.

The $\star$-Lie algebra of vector fields $\Xi^\star$ is generating the Hopf algebra $U\Xi^\star$.

- $U\Xi^F$

  We have the following structure maps:

  $$
  u \cdot^F v = u \cdot v,
  $$

  (12)

  $$
  S^F(u) = S(u),
  $$

  (13)

  $$
  \epsilon^F(u) = \epsilon(u),
  $$

  (14)

  $$
  \Delta^F(u) = F\Delta(u)F^{-1}.
  $$

  (15)

However, $U\Xi^\star$ and $U\Xi^F$ turn out to be isomorphic Hopf algebras.

The star-connection $\nabla^\star$ is defined to satisfy the following axioms:

$$
\nabla^\star_{u+u}z = \nabla^\star_u z + \nabla^\star_v z,
$$

(16)

$$
\nabla^\star_{h \star u}v = h \star \nabla^\star_u v,
$$

$$
\nabla^\star_u(h \star v) = L^\star_u(h) \star v + \bar{R}^\alpha(h) \star \nabla^\star_{\bar{R}_\alpha(u)}v,
$$

where $u, v$ and $z$ are vector fields. Next, we define connection coefficients by

$$
\nabla^\star_{\mu} \hat{\partial}_{\nu} := \Gamma^\sigma_{\mu \nu} \star \hat{\partial}_{\sigma},
$$

(17)

using the basis $\{\hat{\partial}_{\mu}\}$. The action of the covariant derivative on a one-form can be obtained employing the star-dual pairing of a vector field $v$ with a one-form $\omega$,

$$
\nabla^\star_u < v, w >_\star = L^\star_u < v, w >_\star = < \nabla^\star_u v, w >_\star + < \bar{R}^\alpha(v), \nabla^\star_{\bar{R}_\alpha(u)}w >_\star,
$$

(18)

which equivalently can be written as

$$
<v, \nabla^\star_u w >_\star = L_{\bar{R}_\alpha(u)} < \bar{R}_\alpha(v), w >_\star - < \nabla^\star_{\bar{R}_\alpha(u)}(\bar{R}_\alpha(v)), w >_\star,
$$

(19)
For given a metric
\[ g = g_{\mu\nu} \ast d\hat{x}^\mu \otimes d\hat{x}^{\nu}, \]
the connection that leaves it invariant is called a Levi-Civita connection:
\[ \nabla^*_\mu g = 0. \]

For general twist \( F^{-1} = f^\alpha \otimes \bar{f}_\alpha \), torsion and curvature tensors are given by [13]
\[ T(u, v) = \nabla^*_u v - \nabla^*_v R(u, v) = \nabla^*_u \nabla^*_v z - \nabla^*_v \nabla^*_u z - [u, v]_* z. \]

It is enough to calculate the tensor on a basis \( \hat{\partial}_\mu \), because of the tensorial property, i.e.,
\[ T(u, v) = u^\nu \ast T(\hat{\partial}_\nu, \hat{\partial}_\mu) \ast v^\mu. \]

In this frame, the star-connection is given by
\[ \nabla^*_z u = L^*_z(u^\nu) \ast \hat{\partial}_\nu + \bar{R}^\alpha(u^\nu) \ast R^\alpha(z) \ast \Gamma^\sigma_{\mu\nu} \ast \hat{\partial}_\sigma. \]

We will need to compute the components of the curvature tensor in this base. They can be expressed in the following way:
\[ R_{ij} = \langle R(\hat{\partial}_i, \hat{\partial}_j, \hat{\partial}_k), d\hat{x}^k \rangle_* . \]

Consequently, we have for the deformed Ricci tensor
\[ R_{ij} = R_{ijk}^l. \]

Classical Einstein spaces have a Ricci tensor proportional to the metric. In the noncommutative case, we are looking for spaces satisfying the same property:
\[ R_{ij} = cg_{ij}, \]
where \( c \) is some constant.

1.2. Weyl-Moyal plane \( \mathbb{R}^4_\theta \)

The metric is the usual Minkowski or Euclidean one; the twist is Abelian [3]:
\[ F = e^{-\frac{i}{2} \theta^{\mu\nu} \partial_{\mu} \otimes \partial_{\nu}}, \]
where \( \theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R} \). The covariant derivative is given by
\[ \nabla^*_z u = z^\mu \ast \partial_{\mu}(u^\nu) \ast \partial_{\nu} + z^\mu \ast u^\nu \ast \Gamma^\sigma_{\mu\nu} \ast \partial_{\sigma}. \]

In a first step, let us show that the choice \( \Gamma^\sigma_{\mu\nu} = 0 \) is a good choice and renders the affine connection to be a Levi-Civita connection. Thus, the expression for the covariant derivative (29) becomes
\[ \nabla^*_z u = z^\mu \ast \partial_{\mu}(u^\nu) \ast \partial_{\nu}. \]

Let us show that the axioms (16) are satisfied:
\[ \nabla_{u+v}^* z = (u + v)^\mu \partial_\mu (z^\nu) \nabla_\nu = \nabla_u^* z + \nabla_v^* z, \] 

\[ \nabla_{h*u}v = (h * u^\mu) \partial_\mu (v^\nu) \partial_\nu = h * (u^\mu \partial_\mu v^\nu \partial_\nu) = h * \nabla_u^* v, \] 

\[ \nabla_u^*(h * v) = u^\mu \partial_\mu (h * v^\nu) \partial_\nu = \mathcal{L}_u^*(h) * v + u^\mu h * (\partial_\mu v^\nu) \partial_\nu \]

\[ = \mathcal{L}_u^*(h) * v + \bar{R}^\alpha(h) * \bar{R}_\alpha(u^\mu) * (\partial_\mu v^\nu) \partial_\nu \]

\[ = \mathcal{L}_u^*(h) * v + \bar{R}^\alpha(h) * \nabla_u^* v. \]

In a next step, we show that the curvature and torsion vanish. The torsion is given by

\[ T(\partial_\mu, \partial_\nu) = \nabla_\nu^* \partial_\mu - \nabla_\mu^* \partial_\nu - [\partial_\mu, \partial_\nu], z = 0, \]

since the Christoffel symbols are all zero and the derivatives commute. Similarly, we see that the curvature tensor also vanishes:

\[ R(\partial_\nu, \partial_\beta, \partial_\mu) = \nabla_\nu \nabla_\beta^* \partial_\mu - \nabla_\nu^* (\partial_\mu \nabla_\nu^* \partial_\beta - \nabla_\nu^* (\partial_\beta \nabla_\nu^* \partial_\mu), z = 0. \]

At last, we consider the covariant derivative of the metric:

\[ \nabla_\mu g = \nabla_\mu^* (g_{\alpha\beta} dx^\alpha \otimes dx^\beta) = \]

\[ = \partial_\mu (g_{\alpha\beta}) dx^\alpha \otimes dx^\beta - g_{\alpha\beta} \Gamma^\alpha_{\mu\sigma} dx^\sigma \otimes dx^\beta - g_{\alpha\beta} dx^\alpha \otimes \Gamma^\alpha_{\mu\sigma} dx^\sigma = 0, \]

since we get from the star-dual pairing (19)

\[ \nabla^*_\mu dx^\alpha = -\Gamma^\alpha_{\mu\sigma} * dx^\sigma = 0. \]

Among these metrics those that are classically Einstein metrics are also shown to be noncommutative Einstein metrics.

1.2.1. Algebra \( \mathbb{R}_q^5 \). The algebra is generated by the coordinates \( \hat{x}^1, \ldots, \hat{x}^5 \) satisfying the relations [3]

\begin{align*}
\hat{x}^1 \hat{x}^2 &= q \hat{x}^2 \hat{x}^1, & \hat{x}^1 \hat{x}^4 &= q^{-1} \hat{x}^4 \hat{x}^1, \\
\hat{x}^1 \hat{x}^5 &= \hat{x}^5 \hat{x}^1, & \hat{x}^2 \hat{x}^4 &= \hat{x}^4 \hat{x}^2, \\
\hat{x}^2 \hat{x}^5 &= q \hat{x}^5 \hat{x}^2, & \hat{x}^4 \hat{x}^5 &= q^{-1} \hat{x}^5 \hat{x}^4.
\end{align*}

The coordinate \( \hat{x}^3 \) is central. Conjugation is given by

\[ \hat{x}^1 = \hat{x}^5, \quad \hat{x}^2 = \hat{x}^4, \quad \hat{x}^3 = \hat{x}^3. \]
Hence, the twist (for symmetrical ordering) reads

\[ \mathcal{F} = \exp \left( i \frac{h}{2} (\chi_1 \otimes \chi_2 - \chi_2 \otimes \chi_1) \right), \]  

where \( \chi_1 \) and \( \chi_2 \) are the following commuting vector fields:

\[ \chi_1 = x^2 \partial_2 - x^4 \partial_4, \quad \chi_2 = x^1 \partial_1 - x^5 \partial_5. \]

Thus, we have for the inverse \( \mathcal{R} \) matrix

\[ \mathcal{R}^{-1} = \tilde{R}^\alpha \otimes \tilde{R}_\alpha = f^\alpha \tilde{\beta} \otimes f_\alpha \tilde{\beta}, \]

\[ = \sum (-1)^{m+k-l} \left( \frac{h}{2} \right)^{n+k} \left( \frac{k}{n!k!} \right) \chi_1^{m-k-l} \chi_2^{n-m+l}. \]  

1.2.2. Note on Hermitian generators. Let us introduce Hermitian generators for the algebra \( \mathbb{R}^5_q \):

\[ \hat{x}_1 = \hat{z}_1 + i \hat{\bar{z}}_2, \quad \hat{x}_5 = \hat{\bar{z}}_1 - i \hat{z}_2 \]

\[ \hat{x}_2 = \hat{y}_1 + i \hat{\bar{y}}_2, \quad \hat{x}_4 = \hat{\bar{y}}_1 - i \hat{y}_2, \]  

with \( \hat{y}_i^* = \hat{\bar{y}}_i \) and \( \hat{z}_i^* = \hat{\bar{z}}_i \), \( i = 1, 2 \). Inserting these identifications into the commutation relations (37) yields the identical relations

\[ \hat{z}_1 \hat{y}_1 = q \hat{\bar{y}}_1 \hat{\bar{z}}_1, \quad \hat{z}_1 \hat{\bar{y}}_2 = q^{-1} \hat{\bar{y}}_2 \hat{\bar{z}}_1, \]

\[ \hat{\bar{z}}_1 \hat{\bar{z}}_2 = \hat{\bar{y}}_1 \hat{\bar{y}}_2 = \hat{y}_2 \hat{y}_1, \]

\[ \hat{\bar{y}}_1 \hat{\bar{y}}_2 = \hat{z}_2 \hat{\bar{y}}_1 = \hat{\bar{y}}_2 \hat{\bar{y}}_1, \]  

in the case that \( q \) is a square root of unity.

1.3. Noncommutative Einstein spaces

Again, we propose

\[ \Gamma^\mu_{\alpha \beta} = 0, \]  

and show that this definition leads to a sensible covariant derivative and geometric tensors. The covariant derivative (25) is given by

\[ \nabla^\mu_{\nu} u = \mathcal{L}^\mu_{\nu} (u^\nu) \star \hat{\partial}_\nu \]  

This satisfies the axioms for a affine connection since:

\[ \nabla^\mu_{\nu + v} z = \mathcal{L}^\mu_{\nu + v} (z^\nu) \star \hat{\partial}_\nu = \mathcal{L}^\mu_{\nu} (z^\nu) \star \hat{\partial}_\nu + \mathcal{L}^\nu_{\nu} (z^\nu) \star \hat{\partial}_\nu \]

\[ + \mathcal{L}^\mu_{\nu} (z^\nu) \star \hat{\partial}_\nu = \nabla^\mu_{\nu} z + \nabla^\mu_{\nu} z \]  

\[ \nabla^\mu_{h \ast u} v = \mathcal{L}^\mu_{h \ast u} (v^\nu) \star \hat{\partial}_\nu \]

\[ = h \ast \mathcal{L}^\mu_{u} (v^\nu) \star \hat{\partial}_\nu = h \ast \nabla^\mu_{\nu} (v) \]  

\[ = h \ast \nabla^\mu_{\nu} (v) \]  

5
The torsion $T$ is given by

$$T(u, v) = \nabla^*_u v - \nabla^*_R(v) \bar{R}_\alpha(u) - [u, v]_\star$$

Computing the torsion for frame elements, we see explicitly that

$$T(\hat{\partial}_\mu, \hat{\partial}_\nu) = 0.$$

This is due to the tensorial property and

$$[\hat{\partial}_\mu, \hat{\partial}_\nu]_\star = [\bar{f}^\alpha(\hat{\partial}_\mu), \bar{f}_\alpha(\hat{\partial}_\nu)] = 0,$$

since the Lie derivative of $\hat{\partial}_\mu$ along $\bar{f}$, and consequently also $\bar{R}$, is again a vector field with constant coefficients: $c^\mu_\alpha \hat{\partial}_\nu, c^\nu_\mu \in \mathbb{R}$.

Next, we compute the curvature tensor:

$$R(u, v, z) = \nabla^*_u \nabla^*_v z - \nabla^*_R(v) \nabla^*_{R(u)} z - \nabla^*_R(u) \nabla^*_v z - \nabla^*_R(u, v) z$$

$$= \mathcal{L}^*_u \mathcal{L}^*_v(z^\nu) + \hat{\partial}_\nu - \mathcal{L}^*_{\bar{R}^\alpha(v)}(\mathcal{L}^*_u(z^\nu)) \hat{\partial}_\nu - \mathcal{L}^*_{[u, v]} (z^\nu) \hat{\partial}_\nu - \mathcal{L}^*_{u \star v}(z^\nu) \hat{\partial}_\nu - \mathcal{L}^*_R(u, v) z - [u, v]_\star z$$

$$\bar{R}(u \star v) = \mathcal{L}^*_u \mathcal{L}^*_v(z^\nu) + \hat{\partial}_\nu - \mathcal{L}^*_{\bar{R}^\alpha(v)}(\mathcal{L}^*_u(z^\nu)) \hat{\partial}_\nu - \mathcal{L}^*_{[u, v]} (z^\nu) \hat{\partial}_\nu - \mathcal{L}^*_R(u, v) z - [u, v]_\star z.$$

The Riemann curvature tensor vanishes identically. In a next step, we show that this connection is a metric one. We have to evaluate the covariant derivative of the metric:

$$\nabla^*_\mu g = \nabla^*_\mu (g_{\alpha \beta} d\hat{x}^\alpha \otimes d\hat{x}^\beta),$$

where

$$(g_{\alpha \beta}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
1.3.1. Quantum space for $Gl_q(N)$. The quantum space for $Gl_q(N)$ [12] is defined by

$$\hat{x}^i \hat{x}^j = q \hat{x}^j \hat{x}^i, \quad i < j,$$

and therefore we have for the twist

$$F^{-1} = \exp \left( -\frac{i\hbar}{2} \sum_{i<j} (\hat{x}^j \partial_j \otimes \hat{x}^i \partial_i - \hat{x}^i \partial_i \otimes \hat{x}^j \partial_j) \right).$$

In the same way as before, we can show that the trivial connection satisfies all requirements and defines a Levi-Civita connection with vanishing curvature tensor.

1.3.2. Twisted sphere. The twisted sphere is defined by the relations (37) and the additional condition [15]

$$r^2 = 2(\hat{x}^1 \hat{x}^5 + \hat{x}^2 \hat{x}^4) + (\hat{x}^3)^2.$$  

(54)

Using stereographic coordinates $y^i$, $i = 1, 2, 4, 5$, the metric is given by

$$g^* = \frac{4r^2}{(r^2 + \kappa^2)^2} C_{ij} dy^i \otimes_* dy^j,$$

(55)

where

$$(C_{ij}) = \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}.$$  

In order to simplify the notation, we introduce the following definitions: For the vector fields, let us define

$$t_i := y^i \frac{\partial}{\partial y^i} = y^i \partial_i,$$

(56)

note that here no summation over the index $i$ is implied. Hence, we write for the twist

$$F = \exp \left( -\frac{i\hbar}{2} \varphi_{ij} t_i \otimes t_j \right),$$

(57)

with

$$\varphi_{ij} = -\varphi_{ji} = -\varphi_{ij'},$$

$$\varphi_{12} = 1, \quad \varphi_{ii} = \varphi_{ii'} = 0,$$

(58-59)

and $i' = 6 - i$. Furthermore, let us introduce $P_{ij}$ and its square,

$$P_{ij} = e^{\frac{i\hbar}{2} \varphi_{ij}}, \quad q_{ij} = P_{ij}^2.$$  

(60)

Using these definitions, we can write for the metric

$$g^* = \sum_{ij} g_{ij} dy^i \otimes_* dy^j = \frac{4r^2}{(r^2 + \kappa^2)^2} \sum_{i,j} C_{ij} P_{ij} dy^i \otimes dy^j.$$  

(61)

The Levi-Civita connection can be obtained by demanding vanishing torsion and vanishing covariant derivative of the metric. The former condition reads

$$\Gamma^k_{ij} = q_{ij} \Gamma^k_{ji}.$$  

(62)
The latter condition then leads to
\[
\Gamma^k_{ij} = \frac{1}{2} g^{lk} \left( q_{ij} \partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \right).
\] (63)

As a result, the universal connection is the same as in the undeformed case:
\[
\nabla^* = \nabla.
\] (64)

The converse is also true: Assuming (64), we obtain (63) for the connection coefficients.

Similarly, we obtain for the Riemann curvature:
\[
R^* = R, \tag{65}
\]
and in terms of components
\[
R^*_{ijkl} = \frac{1}{r^2} \left( g_{il} g_{jk} - q_{ik} g_{lj} g_{ik} \right).
\] (67)

Now let us consider a possible transformation between 5d theta-deformed plane (see section 1.2) and a 5d q-deformed one (see section 1.2.1). The theta deformed space is chosen in the following way: \([x_i, x_j] = i \theta_{ij}\) with the coordinate \(x_3\) commuting with all other coordinates and \(\theta_{ij} = \begin{pmatrix} 0 & h & -h & 0 \\ -h & 0 & -h & 0 \\ h & 0 & 0 & -h \\ 0 & -h & h & 0 \end{pmatrix} \).

Then with map: \(y_i = \exp(x_i)\) we obtain the correct commutation relations (37). But unfortunately this map does not respect the complex structure and the induced metric seems not to be the proper metric for the q-deformed plane. But another possible map is from the q-deformed sphere to a plane, via stereographic projection. Starting with the q-deformed sphere: commutation relations (37) and the constraint \(r^2 = 2(x^1 x^5 + x^2 x^4) + (x^3)^2\), one define a map to the plane in the usual way by: \(y^3 = x^3, y^i = (x^i r)/(r - x^3), i = 1, 2, 4, 5\). The induced metric is then given by (55).

2. Category theory

2.1. Basic definitions

**Definition 2.1** [20] A category is a quadruple \((\text{Ob}, \text{Hom}, \text{id}, \circ)\) consisting of:

(C1) a class \(\text{Ob}\) of objects;
(C2) for each ordered pair \((A, B)\) of objects a set \(\text{Hom}(A, B)\) of morphisms;
(C3) for each object \(A\) a morphism \(\text{id}_A \in \text{Hom}(A, A)\), the identity of \(A\);
(C4) a composition law associating to each pair of morphisms \(f \in \text{Hom}(A, B)\) and \(g \in \text{Hom}(B, C)\) a morphism \(g \circ f \in \text{Hom}(A, C)\);

which is such that:

(M1) \(h \circ (g \circ f) = (h \circ g) \circ f\) for all \(f \in \text{Hom}(A, B)\), \(g \in \text{Hom}(B, C)\) and \(h \in \text{Hom}(C, D)\);
(M2) \(\text{id}_B \circ f = f \circ \text{id}_A = f\) for all \(f \in \text{Hom}(A, B)\);
(M3) the sets \(\text{Hom}(A, B)\) are pairwise disjoint.
This last axiom is necessary so that given a morphism we can identify its domain \( A \) and codomain \( B \), however it can always be satisfied by replacing \( \Hom(A, B) \) by the set \( \Hom(A, B) \times \{A\} \times \{B\} \).

A morphism \( a: A \to B \) is called isomorphism if there exists a morphism \( b: B \to A \) such that \( a \circ b = i_B \) and \( b \circ a = i_A \). In this case objects \( A \) and \( B \) are called isomorphic.

Morphisms \( a: D \to A \) and \( b: D \to B \) are called isomorphic if there exists an isomorphism \( c: A \to B \) such that \( c \circ a = b \).

An object \( Z \) is called terminal object if for any object \( A \) there exists a unique morphism from \( A \) to \( Z \), which is denoted \( z_A: A \to Z \) in what follows.

A category \( D \) is called a subcategory of a category \( C \) if \( \text{Ob}(D) \subseteq \text{Ob}(C) \), \( \text{Ar}(D) \subseteq \text{Ar}(C) \), and morphism composition in \( D \) coincide with their composition in \( C \).

It is said that a category has (pairwise) products if for every pair of objects \( A \) and \( B \) there exists their product, that is, an object \( A \times B \) and a pair of morphisms \( \pi_{A,B}: A \times B \to A \) and \( \nu_{A,B}: A \times B \to B \), called projections, such that for any object \( D \) and for any pair of morphisms \( a: D \to A \) and \( b: D \to B \) there exists a unique morphism \( c: D \to A \times B \), satisfying the following conditions:

\[
\pi_{A,B} \circ c = a, \quad \nu_{A,B} \circ c = b. \tag{68}
\]

We call such morphism \( c \) the product of morphisms \( a \) and \( b \) and denote it \( a \ast b \).

It is easily seen that existence of products in a category implies the following equality:

\[
(a \ast b) \circ d = (a \circ d) \ast (b \circ d). \tag{69}
\]

In a category with products, for two arbitrary morphisms \( a: A \to C \) and \( b: B \to D \) one can define the morphism \( a \times b \):

\[
a \times b: A \times B \to C \times D, \quad a \times b \overset{\text{def}}{=} (a \circ \pi_{A,B}) \ast (b \circ \nu_{A,B}). \tag{70}
\]

This definition and (68) obviously imply that the morphism \( c = a \times b \) satisfy the following conditions:

\[
\pi_{C,D} \circ c = a \circ \pi_{A,B}, \quad \nu_{C,D} \circ c = b \circ \nu_{A,B}. \tag{71}
\]

Moreover, \( c = a \times b \) is the only morphism satisfying conditions (71).

It is also easily seen that (69) and (70) imply the following equality:

\[
(a \times b) \circ (c \ast d) = (a \circ c) \ast (b \circ d). \tag{72}
\]

Suppose \( A \times B \) and \( B \times A \) are two products of objects \( A \) and \( B \) taken in different order. By the properties of products, the objects \( A \times B \) and \( B \times A \) are isomorphic and the natural isomorphism is

\[
\sigma_{A,B}: A \times B \to B \times A, \quad \sigma_{A,B} \overset{\text{def}}{=} \nu_{A,B} \ast \pi_{A,B}. \tag{73}
\]

Moreover, for any object \( D \) and for any morphisms \( a: D \to A \) and \( b: D \to B \), the morphisms \( a \ast b \) and \( b \ast a \) are isomorphic, that is,

\[
\sigma_{A,B} \circ (a \ast b) = b \ast a. \tag{74}
\]

Similarly, by the properties of products, the objects \( (A \times B) \times C \) and \( A \times (B \times C) \) are isomorphic.

Let

\[
\alpha_{A,B,C}: (A \times B) \times C \to A \times (B \times C)
\]

be the isomorphism that satisfies

\[
\alpha_{A,B,C} (\pi_{A,B} \circ a \times c) = \pi_{A,B} \circ a \times b, \quad \alpha_{A,B,C} (\nu_{A,B} \circ a \times c) = \nu_{A,B} \circ a \times b.
\]
be the corresponding natural isomorphism. Its “explicit” form is:

\[ \alpha_{A,B,C} \overset{\text{def}}{=} (\pi_{A,B} \circ \pi_{A \times B,C}) \ast \left( (\nu_{A,B} \circ \pi_{A \times B,C}) \ast \nu_{A \times B,C} \right). \]  

(75)

Then for any object \( D \) and for any morphisms \( a: D \to A \), \( b: D \to B \), and \( c: D \to C \) we have

\[ \alpha_{A,B,C} \circ ((a \ast b) \ast c) = a \ast (b \ast c). \]

(76)

**Example 2.1** The classic example is \textbf{Sets}, the category with sets as objects and functions as morphisms, and the usual composition of functions as composition. But lots of the time in mathematics one is some category or other, e.g.:

- \textbf{Vect}_k — vector spaces over a field \( k \) as objects; \( k \)-linear maps as morphisms;
- \textbf{Group} — groups as objects, homomorphisms as morphisms;
- \textbf{Top} — topological spaces as objects, continuous functions as morphisms;
- \textbf{Diff} — smooth manifolds as objects, smooth maps as morphisms;
- \textbf{Ring} — rings as objects, ring homomorphisms as morphisms;

or in physics:

- \textbf{Symp} — symplectic manifolds as objects, symplectomorphisms as morphisms;
- \textbf{Poiss} — Poisson manifolds as objects, Poisson maps as morphisms;
- \textbf{Hilb} — Hilbert spaces as objects, unitary operators as morphisms.

**Example 2.2** The category \textbf{NC Einst}. Objects of the category \textbf{NC Einst} are noncommutative Einstein spaces \textbf{NC Einst} defined in sections 1.2 – 1.3.2 by the induced metric (55). For a morphisms \( s, t: \text{NC Einst} \to \text{NC Einst}' \) we define a map to the plane in the usual way by:

\[ y^3 = x^3, \quad y^i = (x^i r)/(r - x^3), \quad i = 1, 2, 4, 5. \]

2.2. From groups to categories

The typical way to think about symmetry is with the concept of a “group”. But to get a concept of symmetry that’s really up to the demands put on it by modern mathematics and physics, we need — at the very least — to work with a “category” of symmetries, rather than a group of symmetries.

To see this, first ask: what is a category with one object? It is a — “monoid”. The "usual" definition of a monoid is like this: a set \( M \) with an associative binary product and a unit element \( 1 \) such that \( a \ast 1 = 1 \ast a = a \) for all \( a \) in \( M \). Monoids abound in mathematics; they are in a sense the most primitive interesting algebraic structures.

To check that a category with one object is “essentially just a monoid”, note that if our category \( C \) has one object \( x \), the set \( \text{Hom}(x,x) \) of all morphisms from \( x \) to \( x \) is indeed a set with an associative binary product, namely composition, and a unit element, namely \( \text{id}_x \).

How about categories in which every morphism is invertible? We say a morphism \( f: x \to y \) in a category has inverse \( g: y \to x \) if \( f \circ g = \text{id}_y \) and \( g \circ f = \text{id}_x \). Well, a category in which every morphism is invertible is called a “groupoid”.

Finally, a group is a category with one object in which every morphism is invertible. It’s both a monoid and a groupoid!

When we use groups in physics to describe symmetry, we think of each element \( g \) of the group \( G \) as a “process”. The element \( 1 \) corresponds to the “process of doing nothing at all”. We can compose processes \( g \) and \( h \) — do \( h \) and then \( g \) — and get the product \( g \circ h \). Crucially, every process \( g \) can be “undone” using its inverse \( g^{-1} \).

So: a monoid is like a group, but the “symmetries” no longer need be invertible; a category is like a monoid, but the “symmetries” no longer need to be composable.
The operation of “evolving initial data from one spacelike slice to another” is a good example of a “partially defined” process: it only applies to initial data on that particular spacelike slice. So dynamics in special or general relativity is most naturally described using groupoids. Only after pretending that all the spacelike slices are the same can we pretend we are using a group. It is very common to pretend that groupoids are groups, since groups are more familiar, but often insight is lost in the process. Also, one can only pretend a groupoid is a group if all its objects are isomorphic. Groupoids really are more general.

In the work [21] we undertake an attempt to formulate the method of categorical extension of the theory of a group $G$ as follows:

Let $G$ be a group. Then $G$ is merely the visible part of a certain category $K$ which is invisible to the naked eye. More precisely, there exists a certain category $K$ (the train of the group $G$) such that the group itself is the automorphism group of a certain object $V$, while the semigroup $\Gamma$ is the semigroup of endomorphisms of this same object. Furthermore, each representation $\rho$ of $G^ \prime$ on a space $H$ can be extended to a representation of the category $K$. In other words, for each objects $W$ of the category $K$ we can construct a linear space $T(W)$ and for each morphism $P : W \rightarrow W^ \prime$ we can construct a linear operator $\tau(P) : T(W) \rightarrow T(W^ \prime)$ such that for any morphisms $P : W \rightarrow W^ \prime$ and $Q : W^ \prime \rightarrow W^ \prime$ we have

$$\tau(QP) = \tau(Q)\tau(P)$$

with $T(V) = H$, and for all $g \in G$ the operators $\tau(g)$ and $\rho(g)$ are the same.

We note that all the spaces $T(W)$ and all the operators $\tau(p)$ “grow out of” the one and only representation $\rho$ of $G$ and the one and only space $H$.

So: in contrast to a set, which consists of a static collection of “things”, a category consists not only of objects or “things” but also morphisms which can viewed as ”processes” transforming one thing into another. Similarly, in a 2-category, the 2-morphisms can be regarded as “processes between processes”, and so on. The eventual goal of basing mathematics upon omega-categories is thus to allow us the freedom to think of any process as the sort of thing higher-level processes can go between. By the way, it should also be very interesting to consider “$Z$-categories” (where $Z$ denotes the integers), having $j$-morphisms not only for $j = 0, 1, 2, \ldots$ but also for negative $j$. Then we may also think of any thing as a kind of process.

2.3. Functors and natural transformations

**Definition 2.2** [20] Let $X$ and $Y$ be two categories. A functor from $X$ to $Y$ is a family of functions $F$ which associates to each object $A$ in $X$ an object $FA$ in $Y$ and to each morphism $f \in \text{Hom}_X(A, B)$ a morphism $Ff \in \text{Hom}_Y(FA, FB)$, and which is such that:

- (F1) $F(g \circ f) = Fg \circ Ff$ for all $f \in \text{Hom}_X(A, B)$ and $g \in \text{Hom}_X(B, C)$;
- (F2) $Fid_A = id_{FA}$ for all $A \in \text{Ob}(X)$.

There is the definition of left and right adjoint functors. In the following we shall need two such adjoint constructions. First, in a given category the left adjoint of the diagonal functor (if it exists) is called the coproduct and the right adjoint (if it exists) is called the product: in $\text{Sets}$ the product is the Cartesian product and the coproduct is the disjoint union. Second, let the category $X$ be concrete over some category $A$ in the sense that there exists a faithful functor $U$ from $X$ to $A$, usually called the forgetful functor. The left adjoint to this functor (if it exists) is then called the free functor. A standard example is the forgetful functor from complete metric spaces to metric spaces, whose left adjoint in the completion functor. On the next higher level of abstraction the notion of a natural transformation is settled. It is a kind of a function between functors and is defined as follows.
Definition 2.3 [20] Let $F : X \to Y$ and $G : X \to Y$ be two functors. A natural transformation $\alpha : F \to G$ is given by the following data.

For every object $A$ in $X$ there is a morphism $\alpha_A : F(A) \to G(A)$ in $Y$ such that for every morphism $f : A \to B$ in $X$ the following diagram is commutative

$$
\begin{align*}
F(A) & \xrightarrow{\alpha_A} G(A) \\
F(f) & \downarrow \quad \downarrow G(f) \\
F(B) & \xrightarrow{\alpha_B} G(B).
\end{align*}
$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $G(f) \circ \alpha_A = \alpha_B \circ F(f)$.

The morphisms $\alpha_A$, $A \in \text{Ob}(A)$, are called the components of the natural transformation $\alpha$.

So, we can certainly speak, as before, of the “equality” of categories. We can also speak of the “isomorphism” of categories: an isomorphism between $C$ and $D$ is a functor $F : C \to D$ for which there is an inverse functor $G : D \to C$. I.e., $FG$ is the identity functor on $C$ and $GF$ the identity on $D$, where we define the composition of functors in the obvious way. But because we also have natural transformations, we can also define a subtler notion, the “equivalence” of categories. An equivalence is a functor $F : C \to D$ together with a functor $G : D \to C$ and natural isomorphisms $a : FG \to 1_C$ and $b : GF \to 1_D$. A “natural isomorphism” is a natural transformation which has an inverse.

As we can “relax” the notion of equality to the notion of isomorphism when we pass from sets to categories, we can relax the condition that $FG$ and $GF$ equal identity functors to the condition that they be isomorphic to identity functor when we pass from categories to the 2-category $\text{Cat}$. We need to have the natural transformations to be able to speak of functors being isomorphic, just as we needed functions to be able to speak of sets being isomorphic. In fact, with each extra level in the theory of $n$-categories, we will be able to come up with a still more refined notion of “$n$-equivalence” in this way.

2.4. Double (bi-)categories

Definition 2.4 [1] A double category $D$ consists of:

1. A category $D_0$ of objects $\text{Ob}(D_0)$ and morphisms $\text{Mor}(D_0)$ of 0-level.
2. A category $D_1$ of objects $\text{Ob}(D_1)$ of 1-level and morphisms $\text{Mor}(D_1)$ of 2-level.
3. Two functors $d, r : D_1 \to D_0$.
4. A composition functor $\ast : D_1 \times_{D_0} D_1 \to D_1$ where the bundle product is defined by commutative diagram

\[
\begin{array}{ccc}
D_1 \times_{D_0} D_1 & \xrightarrow{\pi_2} & D_1 \\
\downarrow \pi_1 & & \downarrow d \\
D_1 & \xrightarrow{r} & D_0
\end{array}
\]

5. A unit functor $ID : D_0 \to D_1$, which is a section of $d, r$.

The above data is subject to Associativity Axiom and Unit Axiom. If both of them are fulfilled only up to equivalence then the double category is called a weak double category, if they are fulfilled strictly then it is a strong double category.

Here we see that for two objects $A, B \in \text{Ob}(D_0)$ there are 0-level morphisms $D_0(A, B)$ which we note by ordinary arrows $f : A \to B$, and 1-level morphisms $D_1(A, B)$ which we note by the
arrows $\xi : A \Rightarrow B$, for $A = d(\xi)$ and $B = r(\xi)$. So with a 2-level morphism $\alpha : \xi \Rightarrow \xi'$, where $\xi : A \Rightarrow B$ and $\xi' : A' \Rightarrow B'$ we can associate the following diagram

$$
\begin{array}{c}
A \\
d(\alpha) \downarrow \\
A'
\end{array}
\begin{array}{c}
\xi \\
\downarrow r(\alpha) \\
\xi'
\end{array}
\xrightarrow{\quad \quad} 
\begin{array}{c}
B \\
\downarrow \alpha \\
B'
\end{array}
$$

and arrow $\alpha : d(\alpha) \Rightarrow r(\alpha)$

The composition on 2-level associated with the diagram

$$
\begin{array}{c}
A \\
d(\alpha) \downarrow \\
A'
\end{array}
\begin{array}{c}
\xi \\
\downarrow r(\alpha) \\
\xi'
\end{array}
\xrightarrow{\quad \quad} 
\begin{array}{c}
B \\
\downarrow \alpha \\
B'
\end{array}
$$

| Ob | Mor |
|-----|-----|
| Objects | $\bullet^x$ | $\bullet \xrightarrow{f} \bullet$ |
| Morphisms | $\bullet \xrightarrow{g} \bullet$ | $\bullet \xrightarrow{F} \bullet$ |

**Table 1.** Data of a Double Category

In a double category, thought of as an internal category in $\text{Cat}$, we have data of four sorts, as shown in Table 1. That is, a double category $\mathcal{D}$ has categories $\text{Ob}$ of objects and $\text{Mor}$ of morphisms. The first column of the table shows the data of $\text{Ob}$: its objects are the objects of $\mathcal{D}$; its morphisms are the vertical morphisms. The second column shows the data of $\text{Mor}$: its objects are the horizontal morphisms of $\mathcal{D}$; its morphisms are the squares of $\mathcal{D}$.

Now we can define for double categories **double (category) functors** and their **morphisms**, **double subcategories**, the category $\mathcal{DCat}$ of double categories, **equivalence** of double categories, **dual double categories** (changed direction of 1-level morphisms, i.e. $d, r$ are transposed), and so on [1, 22].

**Definition 2.5** [5] A **bicategory** $\mathcal{B}$ consists of the following data:

- A collection of **objects** $\text{Ob}$
- For each pair $x, y \in \text{Ob}$, a category $\text{hom}(x, y)$ whose objects are called **morphisms** of $\mathcal{B}$ and whose morphisms are called **2-morphisms** of $\mathcal{B}$
- For each object $x \in \text{Ob}$, an identity $1_x \in \text{hom}(x, x)$
- For each triple $x, y, z$ of objects, a **composition** functor $\circ : \text{hom}(x, y) \times \text{hom}(y, z) \rightarrow \text{hom}(x, z)$
- For each composable triple $f, g, h$ of morphisms, a 2-isomorphism (i.e. invertible 2-morphism) $\alpha_{f,g,h} : h \circ (g \circ f) \rightarrow (h \circ g) \circ f$ called the **associator**
• For each morphism \( f : x \to y \), left and right unitor 2-isomorphisms \( l_f : 1_y \circ f \to f \) and \( r_f : f \circ 1_x \to f \)

The associator is subject to the Pentagon identity, namely that the following diagram commutes for any 4-tuple of composable morphisms \((f, g, h, j)\):

\[
\begin{array}{c}
(f \circ g) \circ (h \circ j) \\
\downarrow a_{f,g,h,j} \\
((f \circ g) \circ h) \circ j \\
\downarrow a_{f,g,h,0j} \\
(f \circ (g \circ (h \circ j))) \\
\downarrow 1_{f \circ g, h, j} \\
(f \circ (g \circ h)) \circ j \\
\downarrow a_{f,g,0h,j} \\
\end{array}
\]

(77)

Also, the unitors and associator make the following commute for all composable \( g, f \):

\[
\begin{array}{c}
(g \circ 1_y) \circ f \xrightarrow{a_{g,1y,f}} g \circ (1 \circ f) \\
\downarrow r_{g \circ 1f} \\
g \circ f \\
\downarrow l_{g \circ f} \\
\end{array}
\]

(78)

(where \( y = t(f) = s(g) \)).

**Definition 2.6** [5] The theory of bicategories is the category (with finite limits) \( \text{Th}(\text{Bicat}) \) given by the following data:

- **Objects** \( \text{Ob}, \text{Mor}, \text{2Mor} \)
- **Morphisms** \( s, t : \text{Ob} \to \text{Mor} \) and \( s, t : \text{Mor} \to \text{2Mor} \)
- **Composition** maps \( \circ : \text{MPairs} \to \text{Mor} \) and \( \cdot : \text{BPairs} \to \mathcal{B} \), satisfying the interchange law (the requirement that this be a functor means that the **interchange law** holds):

\[
(\alpha \circ \beta) \cdot (\alpha' \circ \beta') = (\alpha \cdot \alpha') \circ (\beta \cdot \beta'),
\]

(79)

where \( \text{MPairs} = \text{Mor} \times_{\text{Ob}} \text{Mor} \) and \( \text{BPairs} = \text{2Mor} \times_{\text{Mor}} \text{2Mor} \) are equalizers of diagrams of the form:

\[
\begin{array}{c}
\text{MPairs} \\
\downarrow i \\
\text{Mor} \\
\downarrow \pi_1 \\
\text{Ob} \\
\downarrow t \\
\text{Mor} \\
\downarrow \pi_2 \\
\text{Mor} \\
\end{array}
\]

(80)

and similarly for BPairs.

- the **associator** map \( a : \text{Triples} \to \mathcal{B} \), where \( \text{Triples} = \times_{\text{Ob}} \text{Mor} \times_{\text{Ob}} \text{Mor} \) is the equalizer of a similar diagram for involving \( \text{Mor}^3 \), such that \( a \) satisfies \( s(a(f, g, h)) = (f \circ g) \circ h \) and \( t(a(f, g, h)) = f \circ (g \circ h) \)

- **unitors** \( l, r : \text{Ob} \to \text{Mor} \) with \( s \circ l = t \circ l = \text{id}_\text{Ob} \) and \( s \circ r = t \circ r = \text{id}_\text{Ob} \)
This data is subject to the conditions that the associator is subject to the Pentagon identity [22], and the unitors obey certain unitor laws

\[(g \circ 1_y) \circ f \xrightarrow{\alpha_{g,1_y}} g \circ (1 \circ f) . \]  

(81)

**Definition 2.7** [6] A double bicategory consists of:

- **bicategories** $\text{Ob}$ of objects, $\text{Mor}$ of morphisms, $\mathcal{B}$ of 2-morphisms
- **source** and **target** 2-functors
  - $s, t : \text{Mor} \to \text{Ob}$
  - $s, t : \mathcal{B} \to \text{Ob}$
  - $s, t : \mathcal{B} \to \text{Mor}$
- **Composition** 2-functors:
  - $\circ : \text{MPairs} \to \text{Mor}$
  - $\circ : \text{HPairs} \to \mathcal{B}$
  - $\cdot : \text{VPairs} \to \mathcal{B}$
  - satisfying the interchange law, where
    - $\text{MPairs} = \text{Mor} \times_{\text{Ob}} \text{Mor}$
    - $\text{HPairs} = \mathcal{B} \times_{\text{Mor}} \mathcal{B}$
    - $\text{VPairs} = \mathcal{B} \times_{\text{Ob}} \mathcal{B}$
  - are (strict) pullbacks
- **an associator** 2-functor
  - $\alpha : \text{Triples} \to 2\text{Mor}$
  - where
    - $\text{Triples} = \text{Mor} \times_{\text{Ob}} \text{Mor} \times_{\text{Ob}} \text{Mor}$
- **unitors**
  - $l, r : \text{Ob} \to \text{Mor}$

such that $\alpha$ makes the following diagram commute:

\[
\begin{array}{ccc}
\text{Pairs} & \xleftarrow{\circ \times 1} & \text{Triples} & \xrightarrow{1 \times \circ} & \text{Pairs} \\
\downarrow{\circ} & & \downarrow{\circ} & & \downarrow{\circ} \\
\text{Mor} & \xleftarrow{s} & 2\text{Mor} & \xrightarrow{t} & \text{Mor}
\end{array}
\]

and additional diagrams with the interpretation that $\alpha$ gives invertible 2-morphisms. The unitors must satisfy $s(l(x)) = t(l(x)) = x$ and $s(r(x)) = t(r(x)) = x$, and the associator should satisfy the pentagon identity (77), and the unitors should satisfy the unitor laws (81).

All the partially defined functors are defined for **composable** pairs or triples, for which source and target maps coincide in the obvious ways. The associator should satisfy the pentagon identity [22], and the unitors should satisfy the unitor laws (81).

Table 2 shows the data of the bicategories $\text{Ob}$, $\text{Mor}$, and $\mathcal{B}$, each of which has objects, morphisms, and 2-cells. Note that the morphisms in the three entries in the lower right hand corner—2-cells in $\text{Mor}$, and morphisms and 2-cells in $\mathcal{B}$—are not 2-dimensional. The 2-cells
Table 2. The data of a double bicategory

in **Mor** and morphisms in **B** are the three-dimensional “filling” inside the illustrated cylinders, which each have two square faces and two bigonal faces. The 2-cells in **B** should be drawn as 4-dimensional. The picture illustrated can be thought of as taking both square faces of one cylinder **P**₁ to those of another, **P**₂, by means of two other cylinders (**S**₁ and **S**₂, say), in such a way that **P**₁ and **P**₂ share their bigonal faces. This description works whether we consider the **P**₁ to be horizontal and the **S**₂ vertical, or vice versa. These describe the “frame” of this sort of morphism: the filling is the 4-dimensional “track” taking **P**₁ to **P**₂, or equivalently, **S**₁ to **S**₂, just as a square in a double category can be read horizontally or vertically. (Not all relevant parts of the diagrams have been labeled here, for clarity.)

By analogy with “double category”, the term “double bicategory” might be expected to describe an internal bicategory in **Bicat**, the category of all bicategories.

2.5. Cobordism theory

Cobordism theory goes back to the work of René Thom [23], who showed that it is closely related to homotopy theory. Thom showed that cobordism groups, whose elements are cobordism classes of certain spaces, can be computed as homotopy groups in a certain complex.

**Definition 2.8** **2Cob** is the category with:

- Objects: one-dimensional compact oriented manifolds
- Morphisms: diffeomorphism classes of two-dimensional compact oriented cobordisms between such manifolds.

**2Cob** is generated from four generators, called the **unit**, **counit**, **multiplication**, **comultiplication**, subject to some relations. The generating cobordisms are the following:
taking the empty set to the circle (the unit); taking two circles to one circle (the multiplication); adjoints of each of these (counit and comultiplication respectively).

\[ \text{Figure 1. Generators of Cob} \]

Crane and Yetter [24] give a bicategory of such cobordisms. J.C. Morton has described [7] a "double bicategory" of cobordisms with corners, which can be reduced to a bicategory.

Jean Bénabou [25] introduced bicategories in a 1967 paper, and one broad class of examples introduced there comes from the notion of a span. Since we will want to use a similar construction later, we remark on this here:

**Definition 2.9** (Bénabou) Given any category \( C \), a **span** \( (S, \pi_1, \pi_2) \) between objects \( X_1, X_2 \in C \) is a diagram in \( C \) of the form

\[
P_1 \xleftarrow{\pi_1} S \xrightarrow{\pi_2} P_2
\]

(82)

Given two spans \( (S, s, t) \) and \( (S', s', t') \) between \( X_1 \) and \( X_2 \) a **morphism of spans** is a morphism \( g : S \to S' \) making the following diagram commute:

\[
\begin{array}{c}
\xymatrix{
X_1 \ar[r]^{\pi_1} \ar[rd]_{\pi_1'} & S' \ar[d]_g \ar[r]_{\pi_2'} & X_2 \\
& S \ar[r]_{\pi_2} & X_2
}\end{array}
\]

(83)

Composition of spans \( S \) from \( X_1 \) to \( X_2 \) and \( S' \) from \( X_2 \) to \( X_3 \) is given by pullback: that is, an object \( R \) with maps \( f_1 \) and \( f_2 \) making the following diagram commute:

\[
\begin{array}{c}
\xymatrix{
X_1 \ar[r]^{\pi_1} \ar[rd]_{\pi_1'} & S \ar[d]^{\pi_2} \ar[r]^{\pi_2'} & \ar[r]^{\pi_3'} & X_3 \\
& X_2 \ar[r]_{\pi_2} & S' \ar[r]_{\pi_3} & X_3
}\end{array}
\]

(84)

which is terminal among all such objects. That is, given any other \( Q \) with maps \( g_1 \) and \( g_2 \) which make the analogous diagram commute, these maps factor through a unique map \( Q \to R \). \( R \) becomes a span from \( X_1 \) to \( X_3 \) with the maps \( \pi_1 \circ f_1 \) and \( \pi_2 \circ f_2 \).

**Definition 2.10** [7] \( 2\text{Cosp}(C) \) is a double bicategory of **double cospans** in \( C \), consisting of the following:

- the bicategory of objects is \( \text{Ob} = \text{Cosp}(C) \)
- the bicategory of morphisms \( \text{Mor} \) has:
  - as objects, cospans in \( C \);
– as morphisms, commuting diagrams of the form

\begin{align*}
  X_1 \xrightarrow{T_1} & S \xrightarrow{M} T_2 \\
  X_2 \xleftarrow{T_2} & X_1
\end{align*}

(85)

(in subsequent diagrams we suppress the labels for clarity);
– as 2-morphisms, cospans of cospans maps, namely commuting diagrams of the following shape:

\begin{diagram}

\end{diagram}

(86)

• the bicategory of 2-morphisms has:
  – as objects, cospans maps in \( C \) as in (83)
  – as morphisms, cospans maps of cospans:

\begin{diagram}

\end{diagram}

(87)

– as 2-morphisms, cospans maps of cospans maps:

\begin{diagram}

\end{diagram}

(88)
All composition operations are by pushout; source and target operations are the same as those for cospans. The associators and unitors in the horizontal and vertical bicategories are the maps which come from the universal property of pushouts.

**Definition 2.11** [7] The bicategory $\text{nCob}_2$ is given by the following data:

- The objects of $\text{nCob}_2$ are of the form $P = \hat{P} \times I^2$ where $\hat{P}$ may be any $(n-2)$-manifolds without boundary and $I = [0,1]$.

- The morphisms of $\text{nCob}_2$ are cobordisms $P_1 \xrightarrow{i_1} S \xleftarrow{i_2} P_2$ where $S = \hat{S} \times I$ and $\hat{S}$ is an $(n-1)$-dimensional collared cobordism with corners such that: the $\hat{P}_i \times I$ are objects, the maps are injections into $S$, a manifold with boundary, such that $i_1(P_1) \cup i_2(P_2) = \partial S \times I$, $i_1(P_1) \cap i_2(P_2) = \emptyset$.

- The 2-morphisms of $\text{nCob}_2$ are generated by:
  - diffeomorphisms of the form $f \times \text{id} : T \times [0,1] \rightarrow T' \times [0,1]$ where $T$ and $T'$ have a common boundary, and $f$ is a diffeomorphism $T \rightarrow T'$ compatible with the source and target maps, i.e. fixing the collar.
  - 2-cells: diffeomorphism classes of $n$-dimensional manifolds $M$ with corners satisfying the properties of $M$ in the diagram of equation (85), where isomorphisms are diffeomorphisms preserving the boundary

where the composite of the diffeomorphisms with the 2-cells (classes of manifolds $M$) is given by composition of diffeomorphisms of the boundary cobordisms with the injection maps of the boundary $M$.

The source and target objects of any cobordism $S$ are specified by saying that the source of $S$ is the collection of components of $\partial S \times I$ for which the image of $(x,0)$ lies on the boundary for $x \in \partial S$, and the target has the image of $(x,1)$. The source and target objects are the collars, embedded in the cobordism in such a way that the source object $P = \hat{P} \times I^2$ is embedded in the cobordism $S = \hat{S} \times I$ by a map which is the identity on $I$ taking the first interval in the object to the interval for a horizontal morphism, and the second to the interval for a vertical morphism. The same condition distinguishing source and target applies as above.

Composition of 2-cells works by gluing along common boundaries.

Let $M_d$ be the category of oriented compact $d$-dimensional smooth manifolds (with boundary) and piecewise smooth maps (it may be such continuous maps $f : M \rightarrow Y$ that are smooth on a dense open subset $U_f \subset M$), let $CM_d$ be its subcategory of closed (with empty boundary) manifolds and smooth maps, $CM_d \subset M_d$.

There are the following functors:

1. Disjoint union
   \[ \cup : M_d \times M_d \rightarrow M_d : (X,Y) \mapsto X \cup Y. \]

2. Changing of the orientation of manifolds on opposite
   \[ (-) : M_d \rightarrow M_d : X \mapsto -X. \]

3. Boundary operator
   \[ \partial : M_{d+1} \rightarrow CM_d : X \mapsto \partial X. \]

4. Multiplication on the unit segment $I = [0,1]$
   \[ I \times : CM_d \rightarrow M_{d+1} : X \mapsto I \times X. \]

Now we define a double bicategory $\mathbf{C}(d)$ with
(1) \( C(d)_0 = CM_d \).

(2) 1-level morphisms \( C(d)_{(1)}(X, X') \) is a set of pairs \((Y, f)\) where \( Z \) is oriented compact \((d+1)\)-dimensional smooth manifold with the boundary \( \partial Y \) and \( f \) is an diffeomorphism

\[
f : (-X) \cup X' \to \partial Y,
\]

where \( \cup \) notes the disjoint union of \(-X\) and \(X'\). Thus we write \((Y, f) : X \Rightarrow X'\).

(3) The composition of \((Y, f) : X \Rightarrow X'\) and \((Y', f') : X' \Rightarrow X''\) is the morphism

\[
(Y \cup_{X'} Y', (f|_X) \cup (f'|_{X'})) : X \Rightarrow X'',
\]

where \((Y \cup_{X'} Y')\) denotes the union \((Y \cup Y')\) after identification of each point \(f(y) \in f(Y)\) with the point \(f'(y) \in f'(Y)\) for all \(y \in Y\) and smoothing this topological manifold.

(4) The 1-level identical morphism \( ID_X \) is \((X \times [0; 1], id_{(-X)\cup X})\), because \( \partial(X \times [0; 1]) = (-X) \cup X \).

(5) 2-level morphisms of \( C(d)_{(1)}(\xi, \xi') \) from \( \xi = (Y, f : X' \cup (-X) \to \partial Y) : X \Rightarrow X' \) to \( \xi' = (Y', f' : X'' \cup (-X') \to \partial Y'') : X' \Rightarrow X'' \) are such triples of smooth maps \((f_1, f_2, f_3)\) that the following diagram is commutative

\[
\begin{array}{ccc}
(-X) \cup X' & \overset{f}{\to} & \partial Y \subset Y \\
\downarrow f_1 \cup f_2 & & \downarrow f_3 \\
(-X') \cup X'' & \overset{f'}{\to} & \partial Y' \subset Y'
\end{array}
\]

It easy to see that functors \( \cup \) and \((-)\) may be expanded to double bicategory functors

\[
\begin{align*}
\cup : & C(d) \to C(d), \\
(-) : & C(d) \to C(d)^o
\end{align*}
\]

and \((-)\) is an equivalence of the double bicategories.

**Remark.** It is interesting the appearance the following two formulas for 1-level morphisms in algebras and cobordisms [17–19]

\[
f : A \otimes_k B^o \to End_k(N) \quad f : (-X) \cup Y \to \partial Z,
\]

where we have correspondence between the functors

\[
\begin{align*}
(-)^o & \leftrightarrow -(-), \\
\otimes_k & \leftrightarrow \cup, \\
End_k & \leftrightarrow \partial.
\end{align*}
\]

3. Topological quantum field theory

Atiyah’s axioms for an \( n \)-dimensional topological quantum field theory (TQFT) describe it as a symmetric monoidal functor

\[
Z : n\text{Cob} \to \text{Hilb}
\]  

(89)

One value of TQFTs that they provide invariants of manifolds, and in particular, for 3-manifolds (potentially with boundary). This is closely connected to the subject of knot theory, since knots are studied by their complement in some 3-manifold. In particular, in the codimension-2 case, we replace \( n\text{Cob} \) by \( n\text{Cob}_2 \), a double bicategory whose 2-morphisms are cobordisms between cobordisms. We have also passed from \( \text{Vect} \), the category of vector
spaces and linear maps, to $\textbf{2Vect}$, a double bicategory whose objects are 2-vector spaces, and whose morphisms are 2-linear maps (linear functors), and whose 2-morphisms are natural transformations.

In general, a \ldq $k$-tuply extended TQFT\rq\rq assigns higher-categorical structures called $k$-vector spaces to manifolds of codimension $k$. In particular, it is a (weak, monoidal) $k$-functor:

$$Z : \text{nCob}_k \to \text{k-Vect}$$  \hspace{1cm} (90)

Here, $\text{nCob}_k$ is a $k$-category (that is, a higher category having $j$-morphisms for $j = 1 \ldots k$) whose objects are $(n-k)$-dimensional manifolds, and whose $j$-morphisms are $(n-k+j)$-dimensional cobordisms between $(j-1)$-morphisms. That is, they are manifolds with corners, which have the source and target cobordisms as boundary components. The $k$-category $\text{k-Vect}$ is a higher-categorical analog of $\text{Vect}$ in an appropriate sense. In our case, extended topological quantum field theory is a double functor $Z$ from the double bicategory $CM(d)$ of $d$-dimensional manifolds to the double bicategory of $H$ of (usually Hermitian) finite dimensional (noncommutative) vector spaces and some axioms are satisfied \cite{3}.

Thus, extended topological quantum field theory in dimension $d$ is a functor

$$Z : \text{C}(d) \to \text{Morph}(H),$$

between double bicategories such that:

1. the disjoint union in $\text{C}(d)$ go to the tensor product

$$\biguplus \mapsto \otimes,$$

where $(-)^* : H \to H^\ast$ is dualization of vector spaces.

2. changing of orientation in $\text{C}(d)_0$ go to dualization

$$(-) \mapsto (\cdot)^*$$

Thus, as consequence of double bicategorical functorial properties, we get

1. for each compact closed oriented smooth $d$-dimensional manifold $X \in \text{Obj}(\text{C}(d)_0)$ the value of the functor $Z(X)$ is a finite dimensional (noncommutative) vector space over the field $\text{C}$ of the complex numbers (usually with Hermitian metric),

2. for each $(Y, f) : X \to X'$ from $\text{Ob}(\text{C}(d)_1)$ the value of the functor $Z(Y, f)$ is a homomorphism $Z(X) \to Z(X')$ of (Hermitian) (noncommutative) vector spaces,

and the following well known axioms of extended topological quantum field theory are satisfied:

A(1) (involutivity) $Z(-X) = Z(X)^\ast$, where $-X$ denotes the manifold with opposite orientation, and $\ast$ denotes the dual (noncommutative) vector space.

A(2) (multiplicativity) $Z(X \cup X') = Z(X) \otimes Z(X')$, where $\cup$ denotes disconnected union of manifolds.

A(3) (associativity) For the composition $(Y'', f'') = (Y, f) \ast (Y', f')$ of cobordisms must be

$$Z(Y'', f'') = Z(Y', f') \circ Z(Y, f) \in \text{Hom}_\text{C}(Z(X), Z(X')).$$

(Usually the identifications

$$Z(X' - X) \cong Z(X)^\ast \otimes Z(X') \cong \text{Hom}_\text{C}(Z(X), Z(X'))$$

allow us to identify $Z(Y, f)$ with the element $Z(Y, f) \in Z(\partial Y)$.

A(4) For the initial object $\emptyset \in \text{Ob}(\text{C}(d)_0)$

$$Z(\emptyset) = \text{C}.$$

A(5) (trivial homotopy condition) $Z(X \times [0, 1]) = id_{Z(X)}$. 


Acknowledgments
Authors could have not succeeded in pursuing this program for many years without the large number of collaborators (cf. the references). Authors also wishing to acknowledge financial support from the Austrian Academy of Sciences and the Russian Foundation for Fundamental Research which in the framework of the collaboration with the National Academy of Sciences of Ukraine co-financed this research.

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