Balanced metrics on the Fock-Bargmann-Hartogs domains

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Abstract The Fock-Bargmann-Hartogs domain \(D_{n,m}(\mu)\) is defined by the inequality \(\|w\|^2 < e^{-\mu}\|z\|^2\), where \((z, w) \in \mathbb{C}^n \times \mathbb{C}^m\), which is an unbounded non-hyperbolic domain in \(\mathbb{C}^{n+m}\). This paper introduces a Kähler metric \(\alpha g(\mu; \nu)\) on \(D_{n,m}(\mu)\), where \(g(\mu; \nu)\) is the Kähler metric associated with the Kähler potential \(\Phi(z, w) := \mu \nu \|z\|^2 - \ln(e^{-\mu}\|z\|^2 - \|w\|^2)\) (\(\nu > -1\)) on \(D_{n,m}(\mu)\). The purpose of this paper is twofold. Firstly, we obtain an explicit formula for the Bergman kernel of the weighted Hilbert space of square integrable holomorphic functions on \((D_{n,m}(\mu), g(\mu; \nu))\) with the weight \(\exp\{-\alpha \Phi\}\) for \(\alpha > 0\). Secondly, using the explicit expression of the Bergman kernel, we obtain the necessary and sufficient condition for the metric \(\alpha g(\mu; \nu)\) (\(\alpha > 0\)) on the domain \(D_{n,m}(\mu)\) to be a balanced metric. So we obtain the existence of balanced metrics for a class of Fock-Bargmann-Hartogs domains.

Key words: Balanced metrics · Bergman kernels · Fock-Bargmann-Hartogs domains · Kähler metrics

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1 Introduction

Assume that \(M\) is a complex manifold and \(\varphi\) is a strictly plurisubharmonic function on \(M\). Let \(g\) be a Kähler metric on \(M\) associated with the Kähler form \(\omega = \sqrt{-1} \partial \bar{\partial} \varphi\). For \(\alpha > 0\), let \(\mathcal{H}_\alpha\) be the weighted Hilbert space of square integrable holomorphic functions on \((M, g)\) with the weight \(\exp\{-\alpha \varphi\}\); that is,

\[
\mathcal{H}_\alpha := \left\{ f \in \text{Hol}(M) : \int_M |f|^2 \exp\{-\alpha \varphi\} \frac{\omega^n}{n!} < +\infty \right\},
\]

where \(\text{Hol}(M)\) is the space of holomorphic functions on \(M\). Let \(K_\alpha(z, \bar{z})\) be the Bergman kernel (namely, the reproducing kernel) of the Hilbert space \(\mathcal{H}_\alpha\) if \(\mathcal{H}_\alpha \neq \{0\}\). The Rawnsley’s \(\varepsilon\)-function on \(M\) associated with the metric \(g\) is defined by

\[
\varepsilon_{(\alpha, g)}(z) := \exp\{-\alpha \varphi(z)\} K_\alpha(z, \bar{z}), \quad z \in M.
\]

Note the Rawnsley’s \(\varepsilon\)-function depends only on the metric \(g\) and not on the choice of the Kähler potential \(\varphi\). The asymptotics of the Rawnsley’s \(\varepsilon\)-function \(\varepsilon_\alpha\) was expressed in terms of the parameter \(\alpha\) for compact manifolds by Catlin [4] and Zelditch [26] (for \(\alpha \in \mathbb{N}\)) and for non-compact manifolds by Ma-Marinescu [21, 22, 23]. In some particular case it was also proved by Englis [8, 9].

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Definition 1.1. The metric \( \alpha g \) on \( M \) is balanced if the Rawnsley’s \( \varepsilon \)-function \( \varepsilon_{(\alpha,g)}(z) \) (\( z \in M \)) is a positive constant on \( M \).

The definition of balanced metrics was originally given by Donaldson [6] in the case of a compact polarized Kähler manifold \((M,g)\) in 2001, who also established the existence of such metrics on any (compact) projective Kähler manifold with constant scalar curvature. The definition of balanced metrics was generalized in Arezzo-Loi [1] and Engliš [10] to the noncompact case. We give only the definition for those Kähler metrics which admit globally defined potentials in this paper.

Balanced metrics play a fundamental role in the quantization of a Kähler manifold (e.g., see Berezin [2], Cahen-Gutt-Rawnsley [3] and Engliš [7]). It also related to the Bergman kernel expansion (e.g., see Loi [15] and references therein). For the study of the balanced metrics, see also Feng-Tu [11, 12], Loi [16], Loi-Mossa [17], Loi-Zedda [18, 19], Loi-Zedda-Zuddas [20] and Zedda [25].

The Cartan-Hartogs domains and the Fock-Bargmann-Hartogs domains are two kinds of typical Hartogs domains (e.g., see Kim-Yamamori [14]). The Cartan-Hartogs domains are some Hartogs domains over bounded symmetric domains and there are many researches about the balanced metrics on the Cartan-Hartogs domains (e.g., see Feng-Tu [11, 12], Loi-Zedda [19] and Zedda [25]). The Fock-Bargmann-Hartogs domains are some Hartogs domains over \( \mathbb{C}^n \), and, however, very little seems to be known about the existence of balanced metrics on the Fock-Bargmann-Hartogs domains. In this paper we will obtain the existence of balanced metrics on a class of the Fock-Bargmann-Hartogs domains. Therefore, together with Feng-Tu [12], the result of the present paper represents an example of balanced metric on a complex domain which is not homogeneous.

For a given positive real number \( \mu \), the Fock-Bargmann-Hartogs domain \( D_{n,m}(\mu) \) is a Hartogs domain over \( \mathbb{C}^n \) defined by

\[
D_{n,m}(\mu) := \{ (z, w) \in \mathbb{C}^{n+m} : \|w\|^2 < e^{-\mu}\|z\|^2 \},
\]

where \( \|\cdot\| \) is the standard Hermitian norm. The Fock-Bargmann-Hartogs domains \( D_{n,m}(\mu) \) are strongly pseudoconvex, nonhomogeneous unbounded domains in \( \mathbb{C}^{n+m} \) with smooth real-analytic boundary. We note that each \( D_{n,m}(\mu) \) contains \( \{ (z, 0) \in \mathbb{C}^n \times \mathbb{C}^m \} \cong \mathbb{C}^n \). Thus each \( D_{n,m}(\mu) \) is not hyperbolic in the sense of Kobayashi and \( D_{n,m}(\mu) \) can not be biholomorphic to any bounded domain in \( \mathbb{C}^{n+m} \). Therefore, each Fock-Bargmann-Hartogs domain \( D_{n,m}(\mu) \) is an unbounded non-hyperbolic domain in \( \mathbb{C}^{n+m} \). For the general reference of the Fock-Bargmann-Hartogs domain, see Kim-Ninh-Yamamori [13], Tu-Wang [24] and references therein.

Now we introduce a new Kähler metric \( g(\mu; \nu) \) on \( D_{n,m}(\mu) \). Let \( \nu > -1 \), and define the strictly plurisubharmonic function \( \Phi(z,w) \) on the Fock-Bargmann-Hartogs domain \( D_{n,m}(\mu) \) as follows

\[
\Phi(z,w) := \nu \mu \|z\|^2 - \ln(e^{-\mu}\|z\|^2 - \|w\|^2).
\]

The Kähler form \( \omega(\mu; \nu) \) on \( D_{n,m}(\mu) \) is given by

\[
\omega(\mu; \nu) := \frac{1}{2\pi} \partial \overline{\partial} \Phi.
\]

Hence the Kähler metric \( g(\mu; \nu) \) on \( D_{n,m}(\mu) \) associated with \( \omega(\mu; \nu) \) can be expressed by

\[
g(\mu; \nu)_{ij} = \frac{\partial^2 \Phi}{\partial Z_i \partial Z_j} \quad (1 \leq i, j \leq n + m),
\]

where \( (Z_1, \cdots, Z_{n+m}) = (z,w) \).

In the case of \( n = 1 \) and \( \nu = 0 \), the Kähler metric \( g(\mu; \nu) \) reduces to the canonical metric in Loi-Zedda [18] with \( F(x) = \exp(-\mu x) \). But, we will see that, in the case of the Fock-Bargmann-Hartogs domain \( D_{n,m}(\mu) \), any metric \( \alpha g(\mu;0) \) \((\alpha > 0)\) is not balanced by Th. 1.1 of the present paper.

The main result of the paper is the following.
1.1. Let the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ be endowed with the Kähler metric $g(\mu; \nu)$. Then the metric $\alpha g(\mu; \nu)$ on $D_{n,m}(\mu)$ is balanced if and only if $\alpha > m + n$, $n = 1$ and $\nu = -\frac{1}{m+1}$.

The paper is organized as follows. In Section 2, we give an explicit formula for the Bergman kernel of the weighted Hilbert space of square integrable holomorphic functions on $(D_{n,m}(\mu), g(\mu; \nu))$ with the weight $\exp\{-\alpha \Phi\}$ for $\alpha > 0$, and thus obtain the explicit expression of the Rawnsley’s $\varepsilon$-function of $D_{n,m}(\mu)$ with respect to the metric $g(\mu; \nu)$. In Section 3, using the expression of the Rawnsley’s $\varepsilon$-function, we prove Theorem 1.1.

2. The Rawnsley’s $\varepsilon$-function for $D_{n,m}(\mu)$ with the metric $g(\mu; \nu)$

We firstly give the explicit description of automorphism of the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ as follows.

**Lemma 2.1** (Kim-Ninh-Yamamori [13]). The automorphism group $\text{Aut}(D_{n,m}(\mu))$ is generated by all the following automorphisms of $D_{n,m}(\mu)$:

- $\varphi_U : (z, w) \rightarrow (Uz, w)$, $U \in \mathcal{U}(n)$; \hspace{1cm} (2.1)
- $\varphi_V : (z, w) \rightarrow (z, Vw)$, $V \in \mathcal{U}(m)$; \hspace{1cm} (2.2)
- $\varphi_a : (z, w) \rightarrow (z - a, e^{\mu(z,a) - \frac{\nu}{2} \|a\|^2}w)$, $a \in \mathbb{C}^n$, \hspace{1cm} (2.3)

where $\mathcal{U}(n)$ denotes the set of the $n \times n$ unitary matrices.

**Lemma 2.2.** Let $F(z, w) := (z - a, e^{\mu(z,a) - \frac{\nu}{2} \|a\|^2}w)$ $(a \in \mathbb{C}^n)$ (i.e., an automorphism of $D_{n,m}(\mu)$ of the form (2.3)). Then we have

$$\partial \Phi(F) = \partial \Phi,$$ \hspace{1cm} (2.4)

and moreover

$$\det \left( \frac{\partial F_j}{\partial Z_i} \right)(z_0, w_0) = e^{m\mu(z_0)a - \frac{\nu}{2} \|a\|^2},$$ \hspace{1cm} (2.5)

where $\Phi$ is defined by (1.3) and $(Z_1, \cdots, Z_{m+n}) = (z, w) \in D_{n,m}(\mu)$.

**Proof.** By the definition of $F$, we have

$$\Phi(F) = \mu \|z - a\|^2 - \ln e^{2\mu \text{Re}(z,a) - \mu \|a\|^2}(e^{-\mu \|z\|^2} - \|w\|^2)$$

$$= \mu \|z\|^2 - \mu(\nu + 1)(2\text{Re}(z,a) - \|a\|^2) - \ln(e^{-\mu \|z\|^2} - \|w\|^2).$$

It follows that

$$\partial \Phi(F) = \partial \Phi.$$ \hspace{1cm}

On the other hand, it is easy to see

$$\left( \frac{\partial F_j}{\partial Z_i} \right)(z_0, w_0) = \begin{pmatrix} I_{n \times n} & 0 \\ * & e^{\mu(z_0,a) - \frac{\nu}{2} \|a\|^2} I_{m \times m} \end{pmatrix},$$

where $I_{n \times n}$ and $I_{m \times m}$ denote the $n \times n$ and $m \times m$ diagonal matrices with the diagonal elements 1, respectively. It implies

$$\det \left( \frac{\partial F_j}{\partial Z_i} \right)(z_0, w_0) = e^{m\mu(z_0,a) - \frac{\nu}{2} \|a\|^2}.$$  

The proof is finished. \hfill $\square$

Balanced metrics
Lemma 2.3. Let $\Phi$ be defined by (1.3). Then we have
\[
\det \left( \frac{\partial^2 \Phi}{\partial Z_i \partial Z_j} \right)(z, w) = \frac{\mu^n \nu + (1 - \|\tilde{w}\|^2)^{-1}}{(1 - \|\tilde{w}\|^2)^{m+1}} e^{m\mu\|z\|^2},
\] (2.6)
where $(Z_1, \cdots, Z_{m+n}) = (z, w) \in D_{n,m}(\mu)$ and $\tilde{w}$ is defined by
\[
\tilde{w} := e^{\frac{\mu}{2}\|z\|^2} w.
\] (2.7)

Proof. For any $(z_0, w_0) \in D_{n,m}(\mu)$, consider the automorphism $F$ of $D_{n,m}(\mu)$ as follows:
\[
F(z, w) := (z - z_0, e^{\mu(z, z_0)} - \|\tilde{w}\|^2 w).
\]
Thus, $F(z_0, w_0) = (0, \tilde{w}_0)$, where $\tilde{w}_0 = e^{\frac{\mu}{2}\|z_0\|^2} w_0$. Due to (2.4), we get
\[
\det \left( \frac{\partial^2 \Phi(F)}{\partial Z_i \partial Z_j} \right)(z_0, w_0) = \det \left( \frac{\partial^2 \Phi}{\partial Z_i \partial Z_j} \right)(z_0, w_0).
\] (2.8)

Since
\[
\left( \frac{\partial^2 \Phi(F)}{\partial Z_i \partial Z_j} \right)(z_0, w_0) = \frac{\partial F_i}{\partial Z_i} \left( \frac{\partial^2 \Phi}{\partial Z_i \partial Z_j} \right)(F(z_0, w_0)) \frac{\partial F_j}{\partial Z_j},
\] (2.9)
by combining with (2.8), we have
\[
\det \left( \frac{\partial^2 \Phi}{\partial Z_i \partial Z_j} \right)(z_0, w_0) = \left| \det \left( \frac{\partial F_i}{\partial Z_i} \right)(z_0, w_0) \right|^2 \det \left( \frac{\partial^2 \Phi}{\partial Z_i \partial Z_j} \right)(F(z_0, w_0)).
\] (2.10)

Note the formula (2.5) implies
\[
\left| \det \left( \frac{\partial F_i}{\partial Z_i} \right)(z_0, w_0) \right|^2 = e^{m\mu\|z_0\|^2}.
\] (2.11)

By (1.3), we have
\[
\frac{\partial^2 \Phi}{\partial z_i \partial z_j} (z, w) = \mu \nu \delta_{ij} + \frac{1}{(1 - \|\tilde{w}\|^2)^2} \left[ \mu \delta_{ij} (1 - \|\tilde{w}\|^2) + \mu^2 z_i z_j \right] \|\tilde{w}\|^2,
\]
\[
\frac{\partial^2 \Phi}{\partial z_i \partial w_j} (z, w) = \frac{e^{\mu\|z\|^2}}{(1 - \|\tilde{w}\|^2)^2} \tilde{w}_j,
\]
\[
\frac{\partial^2 \Phi}{\partial w_i \partial w_j} (z, w) = \frac{e^{\mu\|z\|^2}}{(1 - \|\tilde{w}\|^2)^2} \left[ \delta_{ij} (1 - \|\tilde{w}\|^2) + \tilde{w}_i \tilde{w}_j \right].
\]

In particular, when we evaluate at $(0, \tilde{w}_0)$, we obtain
\[
\left( \frac{\partial^2 \Phi}{\partial Z_i \partial Z_j} \right)(0, \tilde{w}_0) = \left( \begin{array}{cc} \mu [\nu + (1 - \|\tilde{w}_0\|^2)^{-1}] & 0 \\ 0 & \frac{1}{1 - \|\tilde{w}_0\|^2} \delta_{ij} \end{array} \right) \left( \begin{array}{cc} I_{m \times n} & 0 \\ 0 & \frac{1}{1 - \|\tilde{w}_0\|^2} \tilde{w}_0 \end{array} \right),
\]
which implies
\[
\det \left( \frac{\partial^2 \Phi}{\partial Z_i \partial Z_j} \right)(0, \tilde{w}_0) = \frac{\mu^n \nu + (1 - \|\tilde{w}_0\|^2)^{-1}}{(1 - \|\tilde{w}_0\|^2)^{m+1}}
\] (2.12)

Therefore, combining (2.10), (2.11) and (2.12), we get (2.6). The proof is completed. \qed
Lemma 2.4. For $\alpha > m + k - 1$, the following multiple integration exists and
\[
\int_0^1 dx_m \cdots \int_0^{1-\sum_{i=2}^m x_i} \left(1 - \sum_{i=1}^m x_i\right)^{\alpha-m-k} \prod_{i=1}^m x_i^{q_i} \, dx_i = \frac{\prod_{i=1}^m \Gamma(q_i + 1) \Gamma(\alpha - m - k + 1)}{\Gamma(\alpha + \sum_{i=1}^m q_i - k + 1)},
\]
where $q = (q_1, \cdots, q_m) \in (\mathbb{R}_+)^m$, here $\mathbb{R}_+$ denotes the set of positive real numbers.

Proof. The proof is trivial, we omit it. \qed

Lemma 2.5. For any $p \in \mathbb{N}^n$, $q \in \mathbb{N}^m$ and $\alpha > m + n$, we have
\[
\|z^p w^q\|^2 = \frac{\prod_{i=1}^n \Gamma(p_i + 1) \prod_{i=1}^m \Gamma(q_i + 1)}{\left|\mu(\nu + 1)\alpha + |q|\right|^n \chi(\alpha, |q|)},
\]
where $w^q$, $|q|$, $\chi(\alpha, |q|)$ and $\|z^p w^q\|^2$ are given by
\[
w^q := \sum_{j=1}^m w_j^{q_j}, \quad |q| = \sum_{j=1}^m q_j,
\]
\[
\chi(\alpha, |q|) := \frac{(\nu + 1)\alpha + |q|^n}{\sum_{d=0}^n \binom{n}{d} \nu^n d \Gamma(\alpha - m - d) \Gamma(\alpha - d + |q|)}
\]
and
\[
\|z^p w^q\|^2 := \int_{D_{n,m}(\mu)} |z|^{2p} |w|^{2q} e^{-\mu((\nu + 1)\alpha - m)|z|^2} (1 - ||\tilde{w}||^2)^{\alpha - m - 1} \left[\nu + (1 - ||\tilde{w}||^2)^{-1}\right]^n \, dm(z) \, dm(w),
\]
for $w := (w_1, \ldots, w_m)$, $q := (q_1, \ldots, q_m)$.

Proof. Firstly, it is well known that
\[
\left(\sum_{j=1}^n \frac{\partial \phi}{\partial q_j}\right)^{n+m} = \det \left(\frac{\partial^2 \phi}{\partial Z_i \partial Z_j}\right) \frac{\omega_0^{n+m}}{(n+m)!}.
\]
where $\omega_0 = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^{n+m} dZ_i \wedge d\overline{Z}_i$. Therefore, by Lemma 2.3, we obtain
\[
\|z^p w^q\|^2 = \mu^n \int_{D_{n,m}(\mu)} \|z\|^{2p} |w|^{2q} e^{-\mu((\nu + 1)\alpha - m)||z||^2} (1 - ||\tilde{w}||^2)^{\alpha - m - 1} \left[\nu + (1 - ||\tilde{w}||^2)^{-1}\right]^n \, dm(z) \, dm(w),
\]
where $dm$ is the Euclidean measure. Thus, by using the polar coordinates (namely, $z_j = r_j e^{i\theta_j}$, $w_l = k_l e^{i\phi_l}$), it follows
\[
\|z^p w^q\|^2 = \mu^n \int_{\left\{k^2 \leq \mu^2 \nu m \sum_{i=1}^{n+m} |z|^2 \sum_{i=1}^{n+m} r^2 \geq k^2 \right\}} r^{2p+1} k^{2q+1} e^{-\mu((\nu + 1)\alpha - m)||r||^2} (1 - ||\tilde{k}||^2)^{\alpha - m - 1} \left[\nu + (1 - ||\tilde{k}||^2)^{-1}\right]^n \, dr \, dk.
\]
where $\tilde{k} = e^{i\frac{\theta}{2} ||r||^2} k$, $r = (r_1, \cdots, r_n)$ and $k = (k_1, \cdots, k_m)$. Therefore, by setting $s_i = r_i^2$ ($1 \leq i \leq n$) and $t_j = k_j^2$ ($1 \leq j \leq m$), we have
\[
\|z^p w^q\|^2 = \mu^n \int_{\left\{s \geq 0, t \geq 0, s_i \leq \frac{1}{\nu m \sum_{i=1}^{n+m} |z|^2} \sum_{j=1}^{n+m} t_j \geq 0 \right\}} s^{p+1} e^{-\mu((\nu + 1)\alpha - m) \sum_{i=1}^n s_i} (1 - \sum_{i=1}^m t_i)^{\alpha - m - 1} \left[\nu + (1 - \sum_{i=1}^m t_i)^{-1}\right]^n \, ds \, dt,
\]
where \( \tilde{t}_i = e^{\mu \sum_{i=1}^{n} s_i} t_i \), and so it follows
\[
\| z^p w^q \|^2 = \mu^n \int_{(\mathbb{R}^+)^n} s^p e^{-\mu((\nu + 1)\alpha + |q|) \sum_{i=1}^{n} s_i} ds \sum_{i=1}^{n} \left( \frac{n}{d} \right)^{\nu - d}(1 - \sum_{i=1}^{m} \tilde{t}_i)\alpha - m - 1 - d \tilde{r}^d. 
\]

Since \( \alpha > m + n \), by Lemma 2.4, we have
\[
\| z^p w^q \|^2 = \mu^n \sum_{d=0}^{n} \left( \frac{n}{d} \right)^{\nu - d}(1 - \sum_{i=1}^{m} \tilde{t}_i)\alpha - m - 1 - d \tilde{r}^d. 
\]

By the definition of Gamma functions, we obtain
\[
\| z^p w^q \|^2 = \mu^n \sum_{d=0}^{n} \left( \frac{n}{d} \right)^{\nu - d}(1 - \sum_{i=1}^{m} \tilde{t}_i)\alpha - m - 1 - d \tilde{r}^d. 
\]

The proof is completed. □

**Theorem 2.6.** Let \( D_{n,m}(\mu,g(\mu;\nu)) \) be the Fock-Bargmann-Hartogs domain \( D_{n,m}(\mu) \) endowed with the metric \( g(\mu;\nu) \). Then, for \( \alpha > m + n \), the Bergman kernel of the weight Hilbert space \( \mathcal{H}_\alpha \) defined by
\[
\mathcal{H}_\alpha := \left\{ f \in \text{Hol}(D_{n,m}(\mu)) : \int_{D_{n,m}(\mu)} |f|^2 \exp\{-\alpha \Phi\} \frac{\omega(\mu;\nu)^{\alpha + n}}{(n + m)!} < \infty \right\}
\]
can be expressed as
\[
K_\alpha(z,w;\overline{z},\overline{w}) = (\alpha - m - n)_{m+n} e^{\mu(\nu + 1)\alpha} \sum_{q \in \mathbb{N}^m} \psi(\alpha,|q|) \frac{\Gamma(|q| + \alpha)}{\Gamma(\alpha)} \frac{\prod_{i=1}^{m} \Gamma(q_i + 1)}{\prod_{i=1}^{m} \Gamma(\alpha + q_i)} \omega(z,w)^{\alpha + n},
\]
where \( (\alpha)_k \) and \( \psi(\alpha,|q|) \) are defined by
\[
(\alpha)_k := \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \alpha(\alpha + 1) \cdots (\alpha + k - 1)
\]
and
\[
\psi(\alpha,|q|) := \frac{\Gamma(\alpha - m - n)\chi(\alpha,|q|)}{\Gamma(\alpha + |q|)}.
\]

**Proof.** Since \( \left\{ \frac{z^p w^q}{\| z^p w^q \|^2} \right\} \) constitute an orthonormal basis of \( \mathcal{H}_\alpha \), we have
\[
K_\alpha(z,w;\overline{z},\overline{w}) = \sum_{p,q \in \mathbb{N}^m} \frac{z^p w^q}{\| z^p w^q \|^2} \frac{z^p w^q}{\| z^p w^q \|^2} \frac{z^p w^q}{\| z^p w^q \|^2} = \sum_{p,q \in \mathbb{N}^m} \frac{z^p w^q}{\| z^p w^q \|^2} \frac{z^p w^q}{\| z^p w^q \|^2} \frac{z^p w^q}{\| z^p w^q \|^2}
\]
by the Fock-Bargmann-Hartogs domain \( D_{n,m}(\mu) \) being a Reinhardt domain. The formula (2.13) implies that
\[
K_\alpha(z,w;\overline{z},\overline{w}) = \sum_{p,q \in \mathbb{N}^m} \frac{[\mu((\nu + 1)\alpha + |q|)]^p \chi(\alpha,|q|)}{\prod_{i=1}^{m} \Gamma(q_i + 1)} \frac{z^p w^q}{\| z^p w^q \|^2} \frac{z^p w^q}{\| z^p w^q \|^2} \frac{z^p w^q}{\| z^p w^q \|^2}
\]
and
\[
= e^{\mu((\nu + 1)\alpha + |q|)\|z\|^2} \sum_{q \in \mathbb{N}^m} \frac{\chi(\alpha,|q|)}{\prod_{i=1}^{m} \Gamma(q_i + 1)} \omega(z,w)^{\alpha + n}.
\]
where \( \tilde{w} \) is defined by (2.7). By simplifying the above formula, we have

\[
K_\alpha(z, w, \overline{z}, \overline{w}) = (\alpha - m - n)_{m+n} e^{\mu(\nu+1)\alpha\|z\|^2} \sum_{q \in \mathbb{N}^m} \frac{\Gamma(\alpha - m - n) \chi(\alpha, |q|)}{\Gamma(\alpha + |q|)} \frac{\Gamma(\alpha + |q|)}{\prod_{i=1}^{m} \Gamma(q_i + 1)} \tilde{w}^q \tilde{w}^q.
\]

So we finish the proof. \( \square \)

Now we give the explicit expression of the Rawnsley’s \( \varepsilon \)-function of the Fock-Bargmann-Hartogs domain \((D_{n,m}(\mu), g(\mu; \nu))\) as follows.

**Theorem 2.7.** Suppose \((D_{n,m}(\mu), g(\mu; \nu))\) is the Fock-Bargmann-Hartogs domain \(D_{n,m}(\mu)\) endowed with the metric \(g(\mu; \nu)\). Then the explicit expression of the Rawnsley’s \( \varepsilon \)-function of \((D_{n,m}(\mu), g(\mu; \nu))\) is given by

\[
\varepsilon_{(\alpha, g(\mu; \nu))} = (\alpha - m - n)_{m+n} e^{\alpha \cdot (1 - \|\tilde{w}\|^2)} \sum_{q \in \mathbb{N}^m} \psi(\alpha, |q|) \frac{\Gamma(|q| + \alpha)}{\prod_{i=1}^{m} \Gamma(q_i + 1)} \tilde{w}^q \tilde{w}^q.
\]

**Proof.** In fact, by the definition (1.1), we have

\[
\varepsilon_{(\alpha, g(\mu; \nu))} = e^{-\alpha \Phi(z, w)} K_\alpha(z, w, \overline{z}, \overline{w}),
\]

and by the definition (1.3), we get

\[
e^{-\alpha \Phi(z, w)} = e^{-\mu(\nu+1)\alpha\|z\|^2} (1 - \|\tilde{w}\|^2)^\alpha.
\]

Therefore, by (2.16), we obtain

\[
\varepsilon_{(\alpha, g(\mu; \nu))} = (\alpha - m - n)_{m+n} e^{\alpha \cdot (1 - \|\tilde{w}\|^2)} \sum_{q \in \mathbb{N}^m} \psi(\alpha, |q|) \frac{\Gamma(|q| + \alpha)}{\prod_{i=1}^{m} \Gamma(q_i + 1)} \tilde{w}^q \tilde{w}^q.
\]

The proof is finished. \( \square \)

### 3 The proof of main results

We at first give the following lemma.

**Lemma 3.1** (see D’Angelo [5] Lemma 2). Let \( x = (x_1, \cdots, x_m) \in \mathbb{R}^m \) with \( \|x\| < 1 \) and \( s \in \mathbb{R} \) with \( s > 0 \). Then

\[
\sum_{q \in \mathbb{N}^m} \frac{\Gamma(|q| + s)}{\Gamma(s) \prod_{i=1}^{m} \Gamma(q_i + 1)} x^{2q} = \frac{1}{(1 - \|x\|^2)^s}.
\]

Now we give the proof of our main result.
Assume \( f \) to zero subspace. In our case, it is easy to see

\[ H_\alpha \neq \{0\} \iff \alpha > m + n. \]

In fact, if \( \alpha > m + n \), then by Lemma 2.5, \( H_\alpha \neq \{0\} \). Conversely, if \( H_\alpha \neq \{0\} \), we assume \( f(z, w) \in H_\alpha \setminus \{0\} \). Since \( D_{n,m}(\mu) \) is a complete Reinhardt domain, \( f(z, w) \) can be expressed by

\[ f(z, w) = \sum_\beta f_\beta(z)w^\beta, \]

where the series is uniformly convergent on any compact subset of \( D_{n,m}(\mu) \) and every \( f_\beta \) is holomorphic on \( \mathbb{C}^n \). So we have

\[
\| f \|^2 = \sum_\beta \frac{\mu^n}{\pi^{n+m}} \int_{D_{n,m}(\mu)} |f_\beta|^2 |w|^{2\beta} e^{-\mu((\nu+1)\alpha-m)||z||^2} (1 - ||\tilde{w}||^2)^{\alpha-m-1} [\nu + (1 - ||\tilde{w}||^2)^{-1}]^n dV < +\infty.
\]

Note \( ||\tilde{w}|| = ||e^\frac{\mu}{2}|| ||z||^2 < 1 \) on \( D_{n,m}(\mu) \) by definition. Thus, for each \( \beta \), we have

\[
\frac{\mu^n}{\pi^{n+m}} \int_{D_{n,m}(\mu)} |f_\beta|^2 |w|^{2\beta} e^{-\mu((\nu+1)\alpha-m)||z||^2} (1 - ||\tilde{w}||^2)^{\alpha-m-1} [\nu + (1 - ||\tilde{w}||^2)^{-1}]^n dV \leq \| f \|^2 < +\infty.
\]

Assume \( f_{\beta_0} \neq 0 \) on \( \mathbb{C}^n \). Thus, by Fubini’s Th., we have

\[
\sum_{d=1}^{n} \binom{n}{d} \nu^{n-d} \int_{||\tilde{w}||^2 < 1} |\tilde{w}|^{2\beta_0} (1 - ||\tilde{w}||^2)^{\alpha-m-1-d} d\tilde{w} < +\infty.
\]

Hence we have \( \alpha - m - 1 - d > -1 \) for all \( 1 \leq d \leq n \). Therefore \( \alpha > m + n \).

Secondly, from (2.19), we know that \( \varepsilon_{(\alpha,\beta,\mu,\nu)}(z, w) \) is independent of \( (z, w) \) if and only if there exists a constant \( \lambda \) with respect to \( (z, w) \) such that

\[ (1 - ||\tilde{w}||^2)^{-\alpha} = \lambda \sum_{q \in \mathbb{N}^m} \psi(\alpha, |q|) \frac{\Gamma(q + \alpha)}{\Gamma(\alpha) \prod_{i=1}^{m} \Gamma(q_i + 1)} \tilde{w}^{q} |w|^\alpha. \tag{3.2} \]

Thus, by Lemma 3.1, we get that \( \psi(\alpha, |q|) \) is a constant with respect to \( |q| \). From (2.14) and (2.17), we get

\[ \psi(\alpha, |q|) = \frac{[(\nu + 1)\alpha + |q|]^n}{\sum_{d=0}^{n} \binom{n}{d} \nu^{n-d} (\alpha - m - n)_{n-d} (\alpha - d + |q|)_d}, \tag{3.3} \]

and, obviously, \( \psi(\alpha, |q|) \) tends to 1 as \( |q| \to \infty \). Thus \( \varepsilon_{(\alpha,\beta,\mu,\nu)}(z, w) \) is a constant if and only if for any \( x, y \in \mathbb{R} \),

\[ [(\nu + 1)x + y]^n = \sum_{d=0}^{n} \binom{n}{d} \nu^{n-d} (x - m - n)_{n-d} (x - d + y)_d. \tag{3.4} \]

Now we claim that (3.4) holds if and only if

\[ n = 1, \quad \nu = -\frac{1}{m + 1}. \]

Indeed, if (3.4) holds, then by setting \( x + y = 1 \) in (3.4), we have

\[ (\nu x + 1)^n = \nu^n (x - m - n)_n. \tag{3.5} \]
If $\nu = 0$, it is impossible that (3.5) holds. Thus we assume $\nu \neq 0$. Then (3.5) implies
\[
\left(x + \frac{1}{\nu}\right)^n = \prod_{j=1}^{n} (x - m - j).
\] (3.6)

Since the right side of the above formula has no multiple divisor, we have $n = 1$, and so
\[
-\frac{1}{\nu} = m + 1 \Rightarrow \nu = -\frac{1}{m + 1}.
\]

On the other hand, it is easy to see that (3.4) holds when $n = 1$ and $\nu = -\frac{1}{m + 1}$. The proof is finished.

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References

[1] Arezzo, C., Loi, A.: Moment maps, scalar curvature and quantization of Kahler manifolds. Comm. Math. Phys. 243, 543-559 (2004)
[2] Berezin, F.A.: Quantization, Math. USSR Izvestiya. 8, 1109-1163 (1974)
[3] Cahen, M., Gutt, S., Rawnsley, J.: Quantization of Kahler manifolds. I: Geometric interpretation of Berezin’s quantization. J. Geom. Phys. 7, 45-62 (1990)
[4] Catlin, D.: The Bergman kernel and a theorem of Tian. Analysis and geometry in several complex variables (Katata, 1997), Trends Math., Birkhäuser Boston, Boston, MA, pp. 1-23 (1999)
[5] D’Angelo, J.P.: An explicit computation of the Bergman kernel function. J. Geom. Anal. 4(1), 23-34 (1994)
[6] Donaldson, S.: Scalar curvature and projective embeddings, I. J. Differential Geom. 59, 479-522 (2001)
[7] Engliš, M.: Berezin Quantization and Reproducing Kernels on Complex Domains, Trans. Amer. Math. Soc. 348, 411-479 (1996)
[8] Engliš, M.: A Forelli-Rudin construction and asymptotics of weighted Bergman kernels. J. Funct. Anal. 177, 257-281 (2000)
[9] Engliš, M.: The asymptotics of a Laplace integral on a Kähler manifold. J. Reine Angew. Math. 528, 1-39 (2000)
[10] Engliš, M.: Weighted Bergman kernels and balanced metrics. RIMS Kokyuroku 1487, 40-54(2006)
[11] Feng, Z.M., Tu, Z.H.: On canonical metrics on Cartan-Hartogs domains. Math. Z. 278, 301-320 (2014)
[12] Feng, Z.M., Tu, Z.H.: Balanced metrics on some Hartogs type domains over bounded symmetric domains. Ann. Glob. Anal. Geom. 47, 305-333 (2015)
[13] Kim, H., Ninh, V.T., Yamamori, A.: The automorphism group of a certain unbounded non-hyperbolic domain. J. Math. Anal. Appl. 409(2), 637-642 (2014)
[14] Kim, H., Yamamori, A.: An application of a Diederich-Ohsawa theorem in characterizing some Hartogs domains. Bull. Sci. Math. 139(7), 737-749 (2015)
[15] Loi, A.: The Tian-Yau-Zelditch asymptotic expansion for real analytic Kähler metrics. Int. J Geom. Methods in Modern Phys. 1, 253-263 (2004)
[16] Loi, A.: Bergman and balanced metrics on complex manifolds. Int. J. Geom. Methods Mod. Phys. 02, 553 (2005)
[17] Loi, A., Mossa, R.: Berezin quantization of homogeneous bounded domains. Geometriae Dedicata. 161(1), 119-128 (2012)
[18] Loi, A., Zedda, M.: Balanced metrics on Hartogs domains. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg. 81(1), 69-77 (2011)
[19] Loi, A., Zedda, M.: Balanced metrics on Cartan and Cartan-Hartogs domains. Math. Z. 270, 1077-1087 (2012)
[20] Loi, A., Zedda, M., Zuddas, F.: Some remarks on the Kähler geometry of the Taub-NUT metrics. Annals of Global Analysis and Geometry. 41(4), 515-533 (2012)
[21] Ma, X., Marinescu, G.: Holomorphic Morse inequalities and Bergman kernels. Progress in Mathematics, Vol. 254, Birkhäuser Boston Inc., Boston, MA (2007)
[22] Ma, X., Marinescu, G.: Generalized Bergman kernels on symplectic manifolds. Adv. Math. 217(4), 1756-1815 (2008)
[23] Ma, X., Marinescu, G.: Berezin-Toeplitz quantization on Kähler manifolds. J. reine angew. Math. 662, 1-56 (2012)
[24] Tu, Z.H., Wang, L.: Rigidity of proper holomorphic mappings between certain unbounded non-hyperbolic domains. J. Math. Anal. Appl. 419, 703-714 (2014)
[25] Zedda, M.: Canonical metrics on Cartan-Hartogs domains. International Journal of Geometric Methods in Modern Physics. 9(1), (2012)
[26] Zelditch, S.: Szegő kernels and a theorem of Tian. Internat. Math. Res. Notices. 6, 317-331 (1998)