Dynamics with choice

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Abstract

Dynamics with choice is a generalization of discrete-time dynamics where instead of the same evolution operator at every time step there is a choice of operators to transform the current state of the system. Many real-life processes studied in chemical physics, engineering, biology and medicine, from autocatalytic reaction systems to switched systems to cellular biochemical processes to malaria transmission in urban environments, exhibit the properties described by dynamics with choice. We study the long-term behaviour in dynamics with choice. We prove very general results on the existence and properties of global compact attractors in dynamics with choice. In addition, we study the dynamics with restricted choice when the allowed sequences of operators correspond to subshifts of the full shift. One of the practical consequences of our results is that when the parameters of a discrete-time system are not known exactly and/or are subject to change due to internal instability or a strategy or Nature’s intervention, the long-term behaviour of the system may not be correctly described by a system with ‘averaged’ values for the parameters. There may be a Gestalt effect.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The mathematical setting for discrete dynamics is a space X and a map $S : X \rightarrow X$. The space X is the state space, the space of all possible states of the system. The map S, the evolution operator, defines the change of states over one time step: $x \in X$ at time $t = 0$ evolves into $S(x)$ at $t = 1$, $S(S(x))$ at $t = 2$, ..., $S^n(x)$ at $t = n$, etc. If instead of one operator, S, we have a choice of evolution operators, $S_0$, $S_1$, ..., $S_{N-1}$, and at every time step
choose one of them, then we have a dynamics with choice. One way to visualize the multitude of choice through time is to generate the infinite tree of choices. This is an infinite rooted tree in which the root has $N$ children, every child has $N$ children and so on. The root corresponds to $t = 0$, its children correspond to $t = 1$, the children of the children correspond to $t = 2$, etc. At every step, the children of each node are labelled 0 through $N - 1$. Beginning at the root infinite branches (paths, strategies) represent the possible choices: for example, in figure 1 we choose the path $w$ that starts with 011... (bold edges). For this choice, the first few points in the trajectory of a point $x_0 \in X$ are $x_1 = S_0(x_0)$, $x_2 = S_1(x_1) = S_1(S_0(x_0))$, $x_3 = S_1(x_2) = S_1(S_1(S_0(x_0)))$, etc. It is natural to encode the infinite paths (beginning at the root) by one-sided infinite words (strings, sequences) on $N$ symbols. If $w$ is such a sequence, it is convenient to align it with the set of non-negative integers $\mathbb{Z}_{\geq 0}$ and denote by $w(k)$ the $(k + 1)$st letter of $w$, i.e. $w = w(0)w(1)w(2)...$. Thus, $w(0) = 0$, $w(1) = 1$, $w(2) = 1$ are the first three symbols of the path $w = 011....$

In this paper we study dynamics with choice, i.e. the dynamics of points and subsets of $X$ along all possible paths simultaneously. We will explain what this means momentarily. Here we would like to emphasize that, from the point of view of long-term behaviour, dynamics with choice, in general, is not the same as the union of trajectories along different infinite paths. We will return to this point later.

Let $\Sigma$ be the one-sided shift on $N$ symbols [15]. This means that, first, $\Sigma$ is the set of all one-sided infinite strings $w = w(0)w(1)w(2)...$, where each $w(j)$ is a symbol from the list of $N$ symbols $[0, 1, \ldots, N - 1]$, and second, there is the shift operator $\sigma : \Sigma \to \Sigma$ acting by erasing the first symbol, $\sigma(w) = w(1)w(2)w(3)\ldots$. Given the state space $X$ and operators $S_0, \ldots, S_{N-1}$, we define the corresponding dynamics with choice as the discrete dynamics on the product space $\mathcal{X} = X \times \Sigma$ generated by the following evolution operator:

$$\mathcal{G} : (x, w) \mapsto (S_{w(0)}(x), \sigma(w)).$$  \hspace{1cm} (1)

One can think of a $w \in \Sigma$ as a plan, a strategy or as Nature’s intervention. Dynamics with choice is a language to describe processes where different strategies could be applied or happen. Most mathematical models in the natural sciences and engineering are expressed in terms of differential equations. Those equations are often continuous limits of discrete equations. The continuous case is easier for qualitative analysis. However, there are situations where discrete
equations describe the processes better. Every realistic model comes with parameters. We are interested in situations where parameters may change due to, for example, internal instability or outside intervention. In an illustrative example in section 3, the coefficients $a$ and $b$ are proportional to the biting rate of mosquitoes which depends, for example, on temperature and humidity which may change from day to day and during the day.

In this paper we study long-term regimes in dynamics with choice. More specifically, we define and study global compact attractors in dynamics with choice. By a global compact attractor we mean the minimal compact set that attracts all bounded sets, see section 2.1 for definitions and references. Thinking in terms of a model with parameters, assume we know that for each admissible fixed (in time) set of parameters the system possesses a global compact attractor. What happens when the parameters switch between admissible values? Is there an attractor? How is it related to attractors corresponding to fixed parameters? Is there a Gestalt effect? These are the questions we address in this paper.

There are many real life and engineered systems that switch between different modes of operation (so-called hybrid systems). When the behaviour in each mode is modelled using continuous dynamics and the transitions are viewed as discrete-time events, such systems are called switching or switched. Analysis and especially control of switching systems is an area of intensive research, see, for example, Liberzon’s book [23] and the survey by Margaliot [25]. There is a natural affinity between switching systems and dynamics with choice (see, e.g., [18]), but we will not explore it at this time.

Readers familiar with iterated function systems (IFSs) [6, 14], may wonder if there is a connection between IFSs and dynamics with choice. Indeed there is, but we have to establish it (in section 2.3.3).

A (general) IFS can be viewed as a discrete dynamics on the space $2^X$ (of subsets of $X$). The operators $S_0, S_1, \ldots, S_{N-1}$ define evolution on $2^X$ by means of the Hutchinson–Barnsley operator:

$$F : A \mapsto F(A) := S_0(A) \cup S_1(A) \cup \cdots \cup S_{N-1}(A).$$

(2)

Following a long-standing tradition, people studying dynamics are first of all interested in fixed points. In the case of an IFS, those are the fixed points of the Hutchinson–Barnsley operator. As has been well illustrated by Barnsley, for many simple IFSs on the plane one can (use a computer to) plot their compact fixed points (sets) and obtain fascinating fractals, see [6, 7]. Generating fractals is one of the main motivations in the study of IFSs. In some papers a fractal is defined as the compact invariant set of an IFS, see [7] and references therein.

To prove that an IFS does have a fixed point, the general definition should be made more specific. One needs to specify the properties of the space $X$; the space $2^X$ should be narrowed to an appropriate class of subsets; assumptions should be made on the operators $S_0, S_1, \ldots, S_{N-1}$. As an example we state the original result of Hutchinson [14, section 3].

**Theorem (Hutchinson) 1.** Let $X$ be a complete metric space (with metric $d$). Denote by $\bar{B}(X)$ the space of all non-empty closed bounded subsets of $X$. Assume that each operator $S_0, S_1, \ldots, S_{N-1}$ is a strict contraction (i.e. there is a number $\gamma \in (0, 1)$ such that $d(S_j(x), S_j(y)) \leq \gamma d(x, y)$ for every pair $x, y \in X$ and for all $j$). Define the evolution operator $\bar{F} : \bar{B}(X) \to \bar{B}(X)$ by the formula

$$\bar{F} : A \mapsto \bar{F}(A) = S_0(A) \cup S_1(A) \cup \cdots \cup S_{N-1}(A).$$

(3)

Then there exists a unique fixed point $K \in \bar{B}(X)$ of $\bar{F}$. Viewed as a subset of $X$, the set $K$ is compact. Also, $K$ attracts every closed bounded subset of $X$ in the sense that, for

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1 We use the abbreviation IFS for single and IFSs for plural forms.
any \( C \in \bar{B}(X) \),

\[
d_H(\bar{F}^n(C), K) \to 0 \quad \text{as} \quad n \to \infty ,
\]

where \( d_H \) is the Hausdorff distance.

The IFSs with contractive operators \( S_j \) are called hyperbolic. Over the years this result has been generalized in many different directions (different assumptions on \( X \) and/or \( S_j \)), see [4] for references.

Returning to dynamics with choice, we repeat that our interest has not been motivated by fractals. We would like to understand the long-term behaviour in dynamics with choice. We assume that \( X \) is a complete metric space (with metric \( d \)), the operators \( S_0, \ldots, S_{N-1} \) are continuous, and each of the (semi)dynamical systems \((X, d, S_j)\) possesses a global compact attractor. Consider the corresponding dynamics with choice as the dynamics on the product metric space\(^2\) \( \mathcal{X} = X \times \Sigma \) generated by the operator \( \mathcal{S} \) acting according to the rule (1). From general theory (see section 2.1) we know that a system ought to enjoy certain compactness and dissipativity properties in order for it to possess the global compact attractor.

In general, even when the individual systems \((X, d, S_j)\) do have attractors, the system \((\mathcal{X}, \text{dist}, \mathcal{S})\) will not have a global compact attractor. There are several reasons why. One counter-example we borrow from [3] (where it is used in the context of IFSs). Take \( X = \mathbb{R} \) with standard metric \( d \) and define two maps, \( S_0 \) and \( S_1 \), as follows:

\[
S_0(x) = \begin{cases} 
0, & \text{if } x \leq 0, \\
-2x, & \text{if } x > 0,
\end{cases} \quad S_1(x) = \begin{cases} 
-2x, & \text{if } x \leq 0, \\
0, & \text{if } x > 0,
\end{cases}
\]

Each of the systems \((X, d, S_j)\) has the global compact attractor, a singleton \([0]\). At the same time, the trajectory \( x_n = S_{w(n-1)} \circ S_{w(n-2)} \circ \cdots \circ S_{w(0)}(x_0) \) corresponding to the periodic string \( w = 010101 \ldots \) is unbounded for any initial point \( x_0 \neq 0 \). Hence, there is no compact attractor attracting \((x_0, w)\).

The second example is infinite dimensional. Let \( B_0 = B_0(p_0) \) and \( B_1 = B_1(p_1) \) be two disjoint closed unit balls centred at \( p_0 \) and \( p_1 \) in an infinite-dimensional Banach space. Let \( X = B_0 \cup B_1 \). Define the maps \( S_0 \) and \( S_1 \) as follows: on \( B_0 \) the map \( S_0 \) is a contraction and it maps \( B_1 \) to \( B_0 \); the map \( S_1 \) is a contraction on \( B_1 \) and maps \( B_0 \) to \( B_1 \):

\[
S_0(x) = \begin{cases} 
p_0 + \frac{1}{2} (x - p_0), & \text{if } x \in B_0, \\
p_0 + (x - p_1), & \text{if } x \in B_1,
\end{cases} \quad S_1(x) = \begin{cases} 
p_1 + \frac{1}{2} (x - p_1), & \text{if } x \in B_1, \\
p_1 + (x - p_0), & \text{if } x \in B_0.
\end{cases}
\]

The system \((X, d, S_0)\) does have the global compact attractor, \([p_0]\), and \((X, d, S_1)\) does have the global compact attractor, \([p_1]\). The corresponding dynamics with choice, \((\mathcal{X}, \text{dist}, \mathcal{S})\), does have the global closed attractor, namely, \( \mathcal{X} \), but does not have the global compact attractor.

In the first example, the maps are compact (which is good), but they do not have a joint bounded absorbing set (lack of dissipativity in \((\mathcal{X}, \text{dist}, \mathcal{S})\)). In the second example, there is a joint bounded absorbing set, \( B_0 \cup B_1 \), but there is not enough compactness (the maps \( S_j \) are not compact, not contracting, and, more generally, not condensing).

These examples show what kind of situations do not allow global compact attractors in the dynamics with choice. Thus, we make additional assumptions. First, we assume that there exists a bounded absorbing set that absorbs every bounded set regardless of the strategy. In applications, an absorbing set is usually a ball of radius that depends on the parameters of the model. Our ‘dissipativity’ assumption means that there is a common estimate on the radius for different values of the parameters.

\(^2\) \( \Sigma \) can be equipped with a metric making it a compact metric space, see section 2.3.1 for a specific choice. We denote here by \( \text{dist} \) the corresponding product metric on \( X \times \Sigma \).
**Assumption 1.** There is a closed, bounded set $B \subset X$ such that for every bounded $A \subset X$ there exists $m(A) > 0$ such that $S_{w(\infty - 1)} \circ S_{w(\infty - 2)} \circ \cdots \circ S_{w(0)}(A) \subset B$ for every word $w = w(0)w(1)\ldots w(n - 1)$ of length $n \geq m(A)$.

Our second, ‘compactness’ assumption is that each of the operators $S_j$ is condensing with respect to a common measure of noncompactness. This assumption covers practically all situations encountered in applications: contractions, compact operators and compact plus contractions. As their name suggests, measures of noncompactness measure how far a set is from being compact. There are several different measures of noncompactness in use [1]. For example, the Hausdorff measure of noncompactness of a set $A$ is the infimum of $\epsilon > 0$ such that $A$ has a finite $\epsilon$-net. In this paper we use only very general properties shared by all popular measures of noncompactness, see definition 7 in section 2.2.

Let $\psi$ be a measure of noncompactness (as in definition 7). An operator $S : X \to X$ is condensing with respect to $\psi$ iff $\psi(S(A)) < \psi(A)$ for any non-compact set $A$, and $\psi(S(A)) = \psi(A) = 0$ if $A$ is compact. Our second general assumption is this.

**Assumption 2.** Each operator $S_j$ is $\psi$-condensing.

In section 2.3 we prove the following result.

**Theorem 1.** Let $X$ be a complete metric space and let $S_0, S_1, \ldots, S_{N-1}$ be continuous, bounded (i.e. take bounded sets to bounded sets) maps $X \to X$. In addition, let assumptions 1 and 2 be satisfied. Then the system $(X, d, \mathcal{S})$ has a global compact attractor $\mathcal{M}$. (That $\mathcal{M}$ is the global compact attractor means that $\mathcal{M}$ is the smallest compact in $X$ attracting every bounded set in $X$.)

The attractor $\mathcal{M}$ has the following properties.

1. $\mathcal{M}$ is (strictly) invariant: $\mathcal{S}(\mathcal{M}) = \mathcal{M}$.
2. $\mathcal{M}$ is the union of all closed bounded sets $A \subset X$ with the property $A \subset \mathcal{S}(A)$.
3. $\mathcal{M}$ is the maximal closed set with the property $A \subset \mathcal{S}(A)$; in particular, $\mathcal{M}$ is the maximal (strictly) invariant closed set.
4. Through every point $(x, w) \in \mathcal{M}$ passes a complete trajectory. This means there exists a two-sided sequence $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ of points in $X$ and a two-sided infinite string $\ldots s(-2)s(-1)s(0)s(1)s(2)\ldots$ such that $s(0) = x$ and $s(0)s(1)s(2)\ldots = w(0)w(1)w(2)\ldots$ and such that $S_{s(n)}(x_n) = x_{n+1}$ for every integer $n$.
5. $\mathcal{M}$ is the union of all complete, bounded trajectories in $X$.

Given the state space $X$ and operators $S_0, \ldots, S_{N-1}$, there are two ways of describing dynamics generated by the corresponding IFS. First, one can follow the trajectories of bounded subsets of $X$ under the iterations of the Hutchinson–Barnsley map $\bar{F}$, see (3). We denote such a system by $(X, d, \bar{F})$. The notion of the global compact attractor as the minimal compact set that attracts all bounded sets is well defined for $(X, d, \bar{F})$. The second possibility is to choose the space of closed bounded sets, $\mathcal{B}(X)$, as the state space of the system and study the dynamics of its points under the iterations of $\bar{F}$. As a rule, $\mathcal{B}(X)$ is equipped with the Hausdorff distance $d_H$. Thus we obtain the second system, $(\mathcal{B}(X), d_H, \bar{F})$. It turns out that from the point of view of global compact attractors the dynamical system $(\mathcal{B}(X), d_H, \bar{F})$ is not very interesting (because convergence in the Hausdorff metric is too strong). It possesses an attractor (in the sense we use here) essentially only if the maps $S_j$ are contractions, so then the attractor is just one point in $\mathcal{B}(X)$. For more general $S_j$, it makes more sense to study the fixed points of $\bar{F}$.

In sections 2.3.4 and 2.3.5 we establish the following connection between the dynamics with choice and the corresponding IFS.
**Theorem 2.** Make the same assumptions on the space \( X \) and operators \( S_0, \ldots, S_{N-1} \) as in theorem 1. Then

1. The IFS \( (X, d, \bar{F}) \) does have a global compact attractor, \( K \).
2. The set \( K \) is the largest compact set in \( X \) which is invariant under the Hutchinson–Barnsley map \( \bar{F} \). \( K = \bar{F}(K) \).
3. The attractor \( M \) of the dynamics with choice has the following product structure:

\[
M = K \times \Sigma.
\]

In the extensive literature on IFSs the main question is the existence of ‘the fractal’, i.e. the maximal compact set invariant under the Hutchinson–Barnsley operator \( \bar{F} \). This corresponds to the second assertion of our theorem 2. We believe that viewing ‘the fractal’ of an IFS as the attractor of the dynamical system \( (X, d, \bar{F}) \) is beneficial to the theory of IFSs. This approach, in particular, points to the ‘right’ assumptions on the space \( X \) and the operators \( S_j \).

IFSs with compact (possibly multi-valued) operators have been considered previously, see, e.g., [3]. The statement of theorem 5.8 in [3] which establishes the existence of a compact set invariant under \( \bar{F} \), needs some additional (dissipativity) assumption such as our assumption 1, for example. The IFSs with condensing (and multi-valued, in addition) operators have been considered by Le´sniak [22] and Andres et al [4]. The assumptions of theorem 3 in [4] require that the image \( \bar{F}(X) \) of the whole space \( X \) be bounded. The word ‘minimal’ referring to the ‘fractal’ in [4, theorem 3] should probably be replaced by ‘maximal’, see also [22, theorem 3].

Our assumptions on the state space and the operators guarantee that, for every fixed \( j = 0, \ldots, N-1 \), the discrete dynamics generated on \( X \) by \( S_j \) does possess the global compact attractor (in \( X \)). More generally, as we show in sections 2.3.4 and 2.3.5, it makes sense to define individual attractors, \( A_w \), corresponding to every string (infinite path in the tree of choices) \( w \in \Sigma \). The attractors generated by each \( S_j \) correspond to ‘constant’ strings, \( w = jjj \ldots \). It is not hard to see that such attractors do not exhaust the attractor (fractal) \( K \).

There are situations when the union of all \( A_w \) is \( K \) (this happens, in particular, when \( S_j \) are strict contractions). However, in general, the union \( \bigcup_{w \in \Sigma} A_w \) is strictly smaller than \( K \). We give an example of this in section 2.3.5. In the cases when \( \bigcup_{w \in \Sigma} A_w \) is strictly smaller than \( K \) we say that there is a Gestalt effect, i.e. ‘the whole is greater than the sum of its parts’. This is a new phenomenon. It has not been observed in the framework of IFSs because, as we show in lemma 16, the Gestalt effect cannot occur when operators \( S_j \) are contractions.

An important generalization of dynamics with choice is dynamics with restricted choice. The name should indicate that not all strategies (sequences \( w = w(0)w(1) \ldots \in \Sigma \)) are allowed. In particular, we consider the sets in \( \Sigma \) that are closed and shift invariant, i.e. subshifts, see [15, 24]. Given a subshift \( \Lambda \subset \Sigma \), we consider the dynamics on the product-space \( X_\Lambda = X \times \Lambda \) generated by the map \( \bar{S} \) as in (1).

**Theorem 3.** Let the space \( X \) and the operators \( S_0, \ldots, S_{N-1} \) satisfy assumptions 1 and 2. Let \( \Lambda \) be a (one-sided) subshift of \( \Sigma \). Consider the dynamical system \((X_\Lambda, \text{dist}, \bar{S})\).

1. The dynamical system \((X_\Lambda, \text{dist}, \bar{S})\) does possess a global compact attractor, \( M_\Lambda \).
2. The attractor \( M_\Lambda \) is invariant in the sense that \( \bar{S}(M_\Lambda) = M_\Lambda \). In fact, \( M_\Lambda \) is the maximal invariant compact set in \( X_\Lambda \). Also, \( M_\Lambda \) is an invariant compact subset of the global attractor \( M \) of the unrestricted dynamics \((X, \text{dist}, \bar{S})\).
3. Through every point \( (x(0), w) \in M_\Lambda \) passes a complete trajectory, i.e. there exist a two-sided sequence of points \( x(-1), x(0), x(1), \ldots \) and a two-sided symbolic sequence \( w(-1)w(0)w(1) \ldots \) extending \( w \) (in the extension of the subshift \( \Lambda \)) such that \( S_{w(i)}(x(i)) = x(i+1) \) for all integers \( i \).
(4) Let $K_{\Lambda}$ denote the projection of the attractor $\mathcal{M}_\Lambda$ onto the $X$ component. The set $K_{\Lambda}$ is a compact subset of the set $K$ of theorem 2. There exist compact sets $A_0, \ldots, A_{N-1}$ such that $K_{\Lambda} = A_0 \cup A_1 \cup \ldots \cup A_{N-1}$ and

$$K_{\Lambda} = A_0 \cup A_1 \cup \ldots \cup A_{N-1} = S_0(A_0) \cup S_1(A_1) \cup \ldots \cup S_{N-1}(A_{N-1})$$

(4)

(5) In general, the attractor $\mathcal{M}_\Lambda$ is not a product. There may be infinitely many different sets among the slices $\mathcal{M}(w) = \{ x \in X : (x, w) \in \mathcal{M}_\Lambda \}$. However, if the subshift $\Lambda$ is sofic, the number of different slices is finite.

This theorem is proved in section 2.4. For information on sofic shifts see [24] and section 2.4.

Restricted dynamics of a sort has been considered previously, see [26, 27]. For example, the graph directed Markov systems of [26] describe iterations of uniformly contracting maps indexed by the edges of a directed (possibly infinite) graph. In this case there is a correspondence between the points of the limit set and the infinite walks through the graph (the coding space). Similarly, the directed IFSs discussed in [7] are defined with the help of the aforementioned correspondence, and the fractal (or attractor) $K_{\Lambda}$ is understood in terms of the map from the code space to $K$ as the image of $\Lambda$ [7, theorem 4.16.3]. The correspondence between the points of the code space, $\Sigma$, and the points of $K$ is possible because the maps are contractions (right away, or eventually). Our approach gives a new and more general view on restricted dynamics. We justify the name—attractor—and unveil attractors’ more subtle structure (assertion 5). This new approach allows us to work in a much more general setting and with transformations that are not contractions. We do not have and do not use a map from the code space into the attractor.

We should mention the paper of Andres and Fišer [2]. They use their result of [3] on the existence of the fractal (the set $K$ in our notation) for an IFS with compact operators $S_j$ to conclude that fixed time solution operators of systems of ordinary differential equations could play the role of maps generating the IFSs. As an illustration they use five two-dimensional systems of ODEs to produce five operators (incidentally, contractions, as noted in [2]) and plot the corresponding dragon-tail-like fractal set. Although their message is that IFSs and fractals can be generated by solution operators of ODEs, their examples can serve as an illustration for our dynamics with choice attractors (due to theorem 2(3)).

Our definitions of dynamics with choice and dynamics with restricted choice as skew-product semi-flows fit in with the theory of non-autonomous semidynamical systems, see [11, 16, 17, 29] and references therein. In section 2.5 we explain how our attractors are related to the forward and pullback attractors in that theory. We should mention the paper of Cheban and Mammana [10] on discrete inclusions $u_{i+1} \in F(u_i)$ where $F(u) = \bigcup_j f_j(u)$ and $\{f_j\}$ is a collection of maps. In [10], the authors view such an inclusion as a non-autonomous system and arrive at essentially the same skew-product flow as our dynamics with choice. Motivated by IFSs, they consider only contracting (exponentially, after a finite number of iterations) maps $f_j$, prove the existence of a global attractor and discuss some of its properties.

In section 3 we apply the theory to a specific example of a discrete Ross–Macdonald type model of malaria transmission. The model can be viewed as a time discretization (with time step $\Delta t$) of the ODE model, or as a pre-ODE form of the model. The reason we have chosen this model is that it is simple and we can visualize all the attractors. The state space is the unit square $X = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. We use two sets of parameters, which define two operators, $S_0$ and $S_1$. Those operators are not contractions, but they are compact, because the system is finite dimensional. The discrete dynamical system generated on $X$ by $S_0$ has two fixed points, $(0, 0)$, which is unstable, and $(11/15, 11/16)$, which is stable. The system generated by $S_1$ also has two fixed points, $(0, 0)$ (unstable) and $(7/25, 7/12)$ (stable). The attractors of the systems $(X, d, S_0)$ and $(X, d, S_1)$ are just the heteroclinic trajectories
connecting the unstable and stable fixed points. They are depicted in figure 2. The effects of freedom of choice on the dynamics are as follows. For the unrestricted dynamics with choice, we see (figures 3 and 4) that the attractor, $K$, is a rather big set with the two individual attractors corresponding to $S_0$ and $S_1$, respectively, forming parts of the boundary of $K$ (the right and left sides). The remaining part of the boundary is quite irregular when $\Delta t$ is relatively large, figure 3. For smaller $\Delta t$, this part of the boundary becomes much smoother and looks like a smooth curve, figure 4. In the limit $\Delta t \to 0$, the set $K$ retains its two-dimensional fullness. It is not an attractor of an ODE with averaged parameters. In fact, all such attractors are one-dimensional (each is a heteroclinic trajectory connecting two fixed points) and lie inside $K$. 

Figure 2. Attractors for $(X, d, S_0)$ (right) and $(X, d, S_1)$ (left).

Figure 3. The attractor slice $K; \Delta t = 0.05$. 
We also consider the dynamics with restricted choice corresponding to the golden mean subshift. We exhibit two different slices in the global attractor. Their projections on the state space do overlap, and their union is smaller than the set $K$ for the full shift.

Finally, we remark that more general compact and condensing operators will be needed in the study of dynamics with choice related to nonlinear dissipative partial differential equations, which we plan to address at a later time.

2. Theory

2.1. Attractors: general facts

We start by collecting the basic facts about attractors. There are several books such as [5, 13, 20, 21, 30] devoted to this subject. Our presentation is closer to [20]. We present only the results that we need. For the proofs of theorems 5 and 6 see the books quoted above.

Let $Y$ be a complete metric space with metric dist and let $\Phi : Y \rightarrow Y$ be a continuous map. Iterations, $\Phi^n$, of $\Phi$ define a discrete (semi)dynamical system on $(Y, \text{dist})$. It is useful to consider not only the dynamics of individual points under the action of $\Phi$, but, more generally, the dynamics of bounded sets. Denote by $B(Y)$ the collection of all bounded subsets of $Y$. We say that the set $A \in B(Y)$ attracts the set $B \in B(Y)$ if

$$\lim_{n \to \infty} \text{dist}(\Phi^n(B), A) = 0.$$
where the one-sided distance between two sets, $\text{dist}(C, A)$, is understood as $\sup_{y \in C} \text{dist}(y, A)$.

**Definition 4.** We call a set $\mathcal{M} \subset Y$ the global compact attractor of the system $(Y, \text{dist}, \Phi)$ if
- $\mathcal{M}$ is compact,
- $\mathcal{M}$ attracts every bounded subset of $Y$,
- $\mathcal{M}$ is the minimal set with these two properties.

For a system to possess a global compact attractor, it should enjoy certain properties, namely, some form of compactness and some dissipativity. Here is the basic existence (and uniqueness) result.

**Theorem 5.** The semidynamical system $(Y, \text{dist}, \Phi)$ has a global compact attractor if and only if it enjoys the following two properties:

(i) (‘compactness’) For every bounded sequence $(y_k)$ in $Y$ and every increasing sequence of integers $n_k \to +\infty$, the sequence $\Phi^{n_k}(y_k)$ has a convergent subsequence.

(ii) (‘dissipativity’) There exists a bounded set $B \subset Y$ which absorbs every bounded set in the sense that for every $A \in B(Y)$ there exists $m(A) > 0$ such that $\Phi^m(A) \subset B$ for all $n \geq m(A)$.

Some of the basic properties of a global compact attractor are collected in the following theorem.

**Theorem 6.** Assume that $\mathcal{M}$ is the global compact attractor of the semidynamical system $(Y, \text{dist}, \Phi)$. Then

(a) $\mathcal{M}$ is the union of all possible limits of sequences of the form $\Phi^{n_k}(y_k)$, where $y_k$ is a bounded sequence in $Y$ and $n_k \to \infty$.

(b) $\mathcal{M}$ is (strictly) invariant: $\Phi(\mathcal{M}) = \mathcal{M}$.

(c) $\mathcal{M}$ is the union of all closed bounded sets $A$ with the property $A \subset \Phi(A)$.

(d) $\mathcal{M}$ is the maximal closed set with the property $A \subset \Phi(A)$; in particular, $\mathcal{M}$ is the maximal (strictly) invariant closed set.

(e) Through every point $y \in \mathcal{M}$ passes a complete trajectory, i.e. there exists a two-sided sequence $\ldots, y_{-2}, y_{-1}, y_0, y_1, \ldots$ of points in $\mathcal{M}$ such that $y_0 = y$ and $y_{m+1} = \Phi(y_m)$ for all integers $m$.

(f) $\mathcal{M}$ is the union of all complete, bounded trajectories in $Y$.

In applications, people do not verify the ‘compactness’ property of theorem 5 directly. Instead, they use one of the known sufficient conditions that imply it. Two of the most useful sufficient conditions are:

- $\Phi$ is a compact map (i.e. $\Phi : Y \to Y$ is continuous and maps bounded sets into relatively compact sets);
- in the case $Y$ is a Banach space, $\Phi$ is a sum of a compact operator and a strict contraction.

Compact $\Phi$ arise, for example, in the finite-dimensional dynamics described by differential or difference equations, or, in the infinite-dimensional case, in dynamics described by parabolic equations. The ‘compact + contraction’ $\Phi$ appear, e.g. in hyperbolic problems with damping. Each of the two sufficient conditions implies that $\Phi$ is condensing with respect to some measure(s) of noncompactness. Since measures of noncompactness and condensing operators are not widely known, below we give a brief account of the facts we need and refer to [1] for more details.
2.2. Measures of noncompactness

Measures of noncompactness assign real non-negative numbers to bounded sets with value 0 assigned exclusively to relatively compact sets. The basic examples are the Kuratowski measure of noncompactness \( \alpha \) and the Hausdorff measure of noncompactness \( \chi \). By definition, \( \alpha(A) \) is the infimum of numbers \( \epsilon > 0 \) such that \( A \) admits a finite cover by sets of diameter less than \( \epsilon \). The number \( \chi(A) \) is the infimum of those \( \epsilon > 0 \) for which \( A \) possesses a finite \( \epsilon \)-net in \( Y \). In this paper we adopt the following definition of a general measure of noncompactness (our definition differs from that in [1]).

**Definition 7.** A function \( \psi \) assigning non-negative real numbers to bounded subsets of (a complete metric) space \( Y \) will be called a measure of noncompactness iff it has the following properties:

(i) \( \psi(A) = 0 \) if and only if \( A \) is relatively compact;
(ii) If \( A_1 \subset A_2 \), then \( \psi(A_1) \leq \psi(A_2) \);
(iii) \( \psi(A_1 \cup A_2) = \max\{\psi(A_1), \psi(A_2)\} \).

Both \( \alpha \) and \( \chi \) enjoy all these properties.

**Definition 8.** A continuous bounded map \( \Phi : Y \to Y \) is called condensing with respect to the measure of noncompactness \( \psi \) (we also say \( \Phi \) is \( \psi \)-condensing) iff \( \psi(\Phi(A)) \leq \psi(A) \) for any bounded \( A \), and \( \psi(\Phi(A)) < \psi(A) \) if \( \psi(A) > 0 \) (i.e. if \( A \) is not compact).

**Theorem 9.** Consider the system \((Y, \text{dist}, \Phi)\). Assume that \( \Phi \) is condensing with respect to some measure of noncompactness \( \psi \) and that there exists a bounded set \( B \) which absorbs every bounded set. Then \((Y, \text{dist}, \Phi)\) possesses a global compact attractor.

This is a corollary of theorem 5, because a \( \psi \)-condensing map possesses the ‘compactness’ property 1 of theorem 5, as follows (essentially) from [13, lemma 2.3.5]; in the case \( \psi \) is the Kuratowski measure of noncompactness, see also [28]. In lemma 11 we establish a more general result.

If \( Y \) is a product of two complete metric spaces, \((Y_1, d_1)\) and \((Y_2, d_2)\), then we choose
\[
d(y_1', y_1''), (y_2', y_2'') = d_1((y_1', y_2')) + d_2((y_2', y_2''))
\]as a metric on \( Y \). If \( \psi_1 \) and \( \psi_2 \) are the measures of noncompactness on \( Y_1 \) and \( Y_2 \) respectively, then it is easy to see that
\[
\psi(A) = \max\{\psi_1(pr_1 A), \psi_2(pr_2 A)\},
\]where \( pr_k \) is a projection on \( Y_k \), defines a measure of noncompactness on \( Y = Y_1 \times Y_2 \).

2.3. Dynamics with choice

2.3.1. Words, strings. Fix an integer \( N > 1 \). Using the integers 0, \ldots, \( N-1 \) as the alphabet, construct strings (words) of finite length and (one-sided) strings of infinite length. Denote by \( \Sigma^* \) the set of all finite length strings (words), and denote by \( \Sigma \) the set of all (one-sided) infinite strings. The word of length 0 is the empty word. The set of non-empty words is denoted \( \Sigma^* \). Given a string \( w \in \Sigma^* \cup \Sigma \), \( w(0) \) is the first letter of \( w \), and \( w(k) \) is the \((k+1)\)-st letter of \( w \). The length of \( w \) is denoted \( |w| \). If \( w \) is a finite string and \( u \in \Sigma^* \cup \Sigma \), their concatenation is denoted \( w.u \); if \( |w| = m \), then \( (w.u)(m+k) = u(k) \) for \( k = 0, 1, \ldots \). For a \( w \in \Sigma^* \) and \( s \in \Sigma^* \cup \Sigma \), we write \( w \subseteq s \) if \( w \) is the beginning of the string \( s \), i.e. if there exists \( u \in \Sigma^* \cup \Sigma \) such that \( s = w.u \). For an infinite string, \( s \), its first \( n \) letters form a word denoted \( s[n] \), i.e. \( s[n] = s(0)s(1) \ldots s(n-1) \). The set of all words of length \( m \) will be denoted by \( \Sigma^m \).
Equip the space $\Sigma^* \cup \Sigma$ with the metric $d_{\Sigma}$, where $d_{\Sigma}(u, v) = 2^{-m}$ if $u[m-1] = v[m-1]$ but $u[m] \neq v[m]$. It is well known [7] that both $\Sigma^* \cup \Sigma$ and $\Sigma$ with metric $d_{\Sigma}$ are compact. The shift operator, $\sigma$, acts on infinite strings by deleting the first letter, i.e. $\sigma(u) = u(1)u(2)\ldots$. The shift operator maps $\Sigma$ onto itself. It is continuous; in fact, $d_{\Sigma}(\sigma(u), \sigma(v)) \leq 2 d_{\Sigma}(u, v)$.

2.3.2. The skew-product dynamics. Let $X$ be a complete metric space with metric $d$, and let $S_0, S_1, \ldots, S_{N-1}$ be continuous, bounded maps $X \rightarrow X$. Define the product metric space $\mathcal{X} = X \times \Sigma$ with metric $\text{dist}$, $\text{dist}((x, u), (y, v)) = d(x, y) + d_{\Sigma}(u, v)$. The skew-product dynamics on $\mathcal{X}$ is generated by the map $\mathcal{S} : \mathcal{X} \rightarrow \mathcal{X}$ acting according to the rule $\mathcal{S}(x, u) = (Su(0)(x), \sigma(u))$.

Assumption 1. Assume there is a closed, bounded set $B \subset \mathcal{X}$ such that for every bounded $A \subset X$ there exists $m(A) > 0$ such that $S_w(A) \subset B$ for every word $w$ of length $n \geq m(A)$.

(Applications $B$ is usually a closed ball of radius that depends on the parameters of the model. Showing that for different values of the parameters there is a common estimate on the radius is enough to verify assumption 1.)

Let $\psi$ be a measure of noncompactness as in definition 7.

Assumption 2. Assume that each operator $S_j$ is $\psi$-condensing.

We are going to apply theorem 5 to prove the existence of the attractor. For this we need the following fact (compare with [9, lemma 3.7]).

Lemma 10. The map $\mathcal{S}$ is $\psi_{\mathcal{X}}$-condensing, where $\psi_{\mathcal{X}}$ is the measure of noncompactness on $\mathcal{X} = X \times \Sigma$ defined on the product according to the recipe of section 2.2 with $\psi$ as the measure of noncompactness on $X$ and any measure of noncompactness, $\psi_{\Sigma}$, on $\Sigma$.

Proof. Note, that $\psi_{\mathcal{X}}(F) = 0$ for any subset $F \subset \Sigma$. Given a set $\mathcal{C} \subset \mathcal{X}$, we have $\mathcal{S}(\mathcal{C}) \subset \bigcup_j S_j(pr_X \mathcal{C}) \times \sigma(pr_{\Sigma} \mathcal{C})$. Therefore,

$$\psi_{\mathcal{X}}(\mathcal{S}(\mathcal{C})) \leq \psi_X \left( \bigcup_j S_j(pr_X \mathcal{C}) \times \sigma(pr_{\Sigma} \mathcal{C}) \right) = \psi \left( \bigcup_j S_j(pr_X \mathcal{C}) \right)$$

$$= \max_j \psi (S_j(pr_X \mathcal{C})) = \psi (S_n(pr_X \mathcal{C})) \leq \psi (pr_X \mathcal{C}) = \psi_{\mathcal{X}}(\mathcal{C})$$

If $\psi_{\mathcal{X}}(\mathcal{C}) > 0$, i.e. $\psi (pr_X \mathcal{C}) > 0$, then the inequality in the second line would be strict. The lemma is proved.

Applying now theorems 5 and 6, we immediately obtain theorem 1. We next proceed to the proof of theorem 2.
2.3.3. *Proof of theorem 2.* We start with a corollary of lemma 10.

**Lemma 11.** Under assumptions 1 and 2, let $w_n$ be a sequence of finite words of increasing lengths $|w_n| \to +\infty$. For any bounded sequence $x_n$ the sequence $S_{w_n}(x_n)$ has a convergent subsequence.

**Proof.** Let $s_n$ be any sequence of infinite strings such that $w_n \subseteq s_n$. The sequence $(x_n, s_n)$ is bounded in $X$. Since the operator $\mathcal{S}$ is condensing, the sequence $\mathcal{S}^{[w_n]}(x_n, s_n)$ has a convergent subsequence in $X$, as follows from [13, lemma 2.3.5]. The $X$-component of such a subsequence converges in $X$. The lemma is proved.

Consider the IFS dynamics $(X, d, \bar{F})$. This means that we follow the dynamics of bounded sets under the iterations of $\bar{F}$. Although the map $\bar{F}$ is inherently multi-valued, the same definition 4 of a global compact attractor makes sense. Due to assumption 1, there is an absorbing set $B$. In addition, for any bounded sequence $(s_n) \subset X$ and for any sequence of integers $m_n \to +\infty$, any sequence $y_n$, $y_n \in \bar{F}^{m_n}(x_n)$, contains a convergent subsequence, as follows from lemma 11. This is all that is needed in the proof of theorem 5. Thus, the existence of the global compact attractor for the IFS is established. The global compact attractor, $K$, comes with all the properties listed in theorem 6. In particular, $K$ is the maximal compact set invariant under $\bar{F}$.

To prove that $\mathcal{M} = K \times \Sigma$ we start by showing that the slices of the attractor corresponding to different strings are all the same, i.e. the set $\{x \in X : (x, s) \in \mathcal{M}\}$ does not depend on $s$.

**All slices are equal.** Recall that every point $(x, s)$ in $\mathcal{M}$ is a limit of some sequence $\mathcal{S}^{m_n}(x_k, s_k)$ with bounded $(x_k) \subset X$ and $\sigma^{m_n}(s_k)$ converging to $s$. As we argued above, we can write

$$\mathcal{S}^{m_n}(x_k, s_k) = (S_{w_k}(x_k), \sigma^{m_n}(s_k)),$$

where $w_k$ is a prefix of length $|w_k| = n_k$ of the string $s_k$, i.e. $s_k = w_k \cdot \sigma^{n_k}(s_k)$. The sequence $S_{w_k}(x_k)$ converges to $x$ and $\sigma^{m_n}(s_k)$ converges to $s$. The limit of the pair will not change if we replace $s_k$ by $w_k \cdot s$. Clearly, for any string $u \in \Sigma$, we have

$$\lim (S_{w_k}(x_k), \sigma^{m_n}(w_k \cdot u)) = (x, u).$$

This proves that $\mathcal{M} = A \times \Sigma$, with a compact set $A \subset X$.

Since $\Sigma = 0 \cdot \Sigma \cup 1 \cdot \Sigma \cup \cdots \cup (N - 1) \cdot \Sigma$ and since $\mathcal{S}(\mathcal{M}) = \mathcal{M}$, we get $\mathcal{S}(A \times \Sigma) = (S_0(A) \cup S_1(A) \cup \cdots \cup S_{N-1}(A)) \times \Sigma = A \times \Sigma$. In other words, $A = S_0(A) \cup S_1(A) \cup \cdots \cup S_{N-1}(A)$. Because $K$ is the maximal compact in $X$ with this property, we have $A \subset K$. On the other hand, $\mathcal{S}(K \times \Sigma) = K \times \Sigma$. Since $A \times \Sigma$ is the maximal compact in $X$ with this property, we have $K \subset A$, and hence, $A = K$. This completes the proof of theorem 2.

2.3.4. **Individual attractors.** Every fixed strategy also generates a dynamics on $X$: if $w \in \Sigma$ is the (fixed) strategy, then an $x \in X$ moves to $S_{w(0)}(x)$, then to $S_{w(1)}(S_{w(0)}(x))$, then to $S_{w(2)}(S_{w(1)}(S_{w(0)}(x)))$, etc. Denote this dynamics by $(X, d, w)$. This is not a (semi)dynamical system, but we should not worry about names. Certain important notions related to the long-term behaviour with natural adjustments still make sense. For example, the individual, i.e. corresponding to an individual strategy $w$, trajectory of a set $B$ is the union $B \cup S_{w(1)}(B) \cup S_{w(2)}(B) \cup \cdots$.

We define the individual $\omega$-limit set of a bounded set $B$ as

$$\omega(B, w) = \{y \in X : y = \lim S_{w[n_k]}(y_k) \text{ for some sequence } (y_k) \text{ in } B\}.$$
By analogy with definition 4, we say that a set $A$ is the global compact attractor of system $(X, d, w)$ if it is the minimal set with the following two properties: $A$ is compact and $A$ attracts every bounded set under the strategy $w$, i.e. for any bounded $B$, we have $\lim_{n \to \infty} \text{dist} (S_{w[n]}(B), A) = 0$.

The next theorem establishes the existence of individual compact attractors, $\mathcal{A}_w$, of systems $(X, d, w)$. Along the way we establish various properties of the $\omega$-limiting sets $\omega(B, w)$.

**Theorem 12.** Under assumptions 1 and 2, every system $(X, d, w)$ has the global compact attractor, which we denote by $\mathcal{A}_w$. This attractor is the intersection of the closures of the tails of the trajectory of the absorbing set $B$,

$$\mathcal{A}_w = \bigcap_{n \geq 1} \bigcup_{k \geq n} S_{w[k]}(B).$$

The attractor, $\mathcal{A}_w$, is the union of all $\omega(B, w)$ with bounded $B$.

**Proof.** Due to assumption 1, for every bounded set $B_0$, we have $S_{w[m]}(B_0) \subset B$ for all sufficiently large $m$. The set $\omega(B, w)$ is not empty by lemma 11. It is standard (and easy) to show that $\omega(B, w) = \bigcap_{n \geq 1} \bigcup_{k \geq n} S_{w[k]}(B)$. Hence, $\omega(B, w)$ is closed. Next,

$$\omega(B, w) \subset \bigcup_{j=0}^{N-1} S_j (\omega(B, w)). \quad (5)$$

Indeed, every point in $\omega(B, w)$ is a limit of some sequence $S_{w[n_k]}(y_k)$. In the sequence of words $w[n_k]$, infinitely many words end with the same symbol $j_0$, say. Restricting to this subsequence, we have $y = \lim S_{j_0} (S_{w[n_k]}(y_k))$. It remains to note that $S_{w[n_k-1]}(y_k)$ has a convergent subsequence with limit in $\omega(B, w)$. From (5) it follows that $\omega(B, w)$ is compact, because if it is not, we have a contradiction:

$$\psi(\omega(B, w)) \leq \psi \left( \bigcup_{j=0}^{N-1} S_j (\omega(B, w)) \right) = \max \{ \psi (S_j (\omega(B, w))) \} < \psi(\omega(B, w)).$$

It is an easy exercise to show that $\omega(B, w)$ attracts $B$. Also, it is clear that $\mathcal{A}_w = \omega(B, w) = \bigcup_{\text{bounded }B} \omega(B, w)$. The theorem is proved.

### 2.3.5. Interplay between individual attractors

Recall that (with assumptions 1 and 2) the global attractor $\mathfrak{A}$ of $(X, \text{dist}, \Sigma)$ is a product $\mathfrak{A} = K \times \Sigma$.

We start with a few simple observations.

**Lemma 13.** $\mathcal{A}_w \subset F(\mathcal{A}_w) \subset K$, where $F$ is the Hutchinson–Barnsley operator.

**Proof.** Pick a point, $x$, in $\mathcal{A}_w$. Then $x = \lim S_{w[n_k]}(x_k)$ for some bounded sequence $(x_k)$ in $X$ and $n_k \to \infty$. Among the last letters of the words $w[n_k]$ there is at least one, say, $j$, that repeats infinitely many times. Sparse the sequence so that every $w[n_k]$ has the last letter $j$. Then,

$$S_{w[n_k]}(x_k) = S_j (S_{w[n_k-1]}(x_k)).$$

The sequence $S_{w[n_k-1]}(x_k)$ has a convergent subsequence by lemma 11, and the limit is in $\mathcal{A}_w$. Thus, $x \in S_j (\mathcal{A}_w)$. The lemma is proved.

**Lemma 14.** $\mathcal{A}_w \subset F(\mathcal{A}_w)$. 


Proof. Again, if \( x \in A_w \), then \( x = \lim S_{u[n_k]}(x_k) \). Clearly, \( S_{u[n_k]}(x_k) = S_{\sigma(w)[n_k]-1}(S_{w(0)}(x_k)) \). The sequence \( (S_{w(0)}(x_k)) \) is bounded and \( \sigma(w)[n_k - 1] \to \sigma(w) \). The lemma is proved.

Corollary 15. If the string \( w \) is periodic, then \( A_w = A_{\sigma(w)} \).

The union of individual attractors \( A_w \) lies inside \( K \),

\[
\bigcup_{w \in \Sigma} A_w \subseteq K.
\] (6)

There are many important cases when this union equals \( K \).

Lemma 16. We have \( \bigcup_{w \in \Sigma} A_w = K \) in each of the following cases:

(a) Operators \( \{S_j\} \) are eventually strict contractions, i.e. there exist a \( 0 < \gamma < 1 \) and an integer \( M \geq 1 \) such that for any finite word \( w^* \) of length \( \geq M \) the operator \( S_{w^*} \) is a contraction with factor \( \gamma \). (This condition is automatically satisfied if each \( S_j \) is a strict contraction.)

(b) \( S_j^{-1}(K) \supseteq K \) for \( j = 0, \ldots, N - 1 \).

(c) Each operator \( S_j \) is invertible on \( K \).

Proof. The inclusion (6) is obvious. To prove the equality in the special cases (a) and (b), pick an \( x \in K \). There exists a sequence of points \( \{x_k\} \subseteq K \), and a sequence \( u_{n_k} \) of lengths \( n_k \) increasing to infinity such that \( x = \lim_{k \to \infty} S_{u_{n_k}}(x_k) \). We claim that \( x \in A_u \), where \( u = u_{n_1}, u_{n_2}, \ldots, u_{n_k} \ldots \). Denote \( u[m_k] = u_{n_1}, u_{n_2}, \ldots, u_{n_k} \). The lengths of the words \( u[m_k] \) go to infinity.

In case (a), for every \( k \) and any \( y \in K \) we have

\[
d(S_{w_{n_k}}(x_k), S_{u[m_k]}(y)) = d(S_{w_{n_k}}(x_k), S_{w_{n_k}} S_{u[m_{k-1}]}(y)) = d(S_{w_{n_k}}(x_k), S_{w_{n_k}}(z_k)),
\]

where \( z_k = S_{u[m_{k-1}]}(y) \). Then, \( d(S_{w_{n_k}}(x_k), S_{w_{n_k}}(z_k)) \leq \gamma^k \times d(x_k, z_k) \leq \gamma^k \times \text{diam}(K) \), where \( l_k \) is the round down of \( n_k / M \). Therefore, \( d(S_{w_{n_k}}(x_k), S_{u[m_k]}(y)) \to 0 \), as \( k \to \infty \). Since \( \lim_{k \to \infty} S_{u[m_k]}(y) \in A_u \), and \( \lim_{k \to \infty} S_{u[m_k]}(y) = \lim_{k \to \infty} S_{u_{n_k}}(x_k) = x \), it follows that \( x \in A_u \) and the inclusion \( K \subseteq \bigcup_{u \in \Sigma} A_u \) is proved.

In the second case, since \( S_j^{-1}(K) \supseteq K \), for every \( y \in K \) there exist \( z_j \in K \) with \( y = S_j(z_j) \), \( j \in \{0, 1, \ldots, N - 1\} \). Therefore, for every \( k \), we can find \( y_k \in K \) such that \( S_{u[m_{k-1}]}(y_k) = x_k \). Then, \( S_{u[m_k]}(y_k) = S_{u_{n_k}} S_{u[m_{k-1}]}(y_k) = S_{u_{n_k}}(x_k) \). It follows that \( x \in A_u \).

Finally, (c) is a special case of (b). This concludes the proof.

Remark 17. Case (c) may seem too restrictive. However, there are many situations where the operators \( S_j \) are not invertible on \( X \) but are invertible on the attractor \( K \). This was first observed by Ladyzhenskaya in the case of Navier–Stokes equations [19]. The fact is due to the invariance of \( K \) and, what is called, the backwards uniqueness property of certain parabolic-like equations.

Although \( K \) equals the union of individual attractors in many cases, there are situations when \( K \) is strictly larger than that union. This is what we call a Gestalt effect. This is a new phenomenon. As we have shown in lemma 16, the Gestalt effect cannot occur when operators \( S_j \) are contractions.
Example of a Gestalt effect. In this example the state space \( X \) will be the space \( \Sigma_2 \) of one-sided infinite strings of 0s and 1s. There will be two operators, \( S_0 \) and \( S_1 \), defined as follows:

\[
S_0(v) = v(2) \cdot v, \quad S_1(v) = v(1) \cdot v
\]

for all \( v = v(0)v(1)v(2)v(3) \ldots \in X \). The conditions of theorem 1 are satisfied, so let \( \mathcal{M} = K \times \Sigma_2 \) be the global compact attractor of the corresponding dynamics with choice. (Note that the global compact attractor of the system generated by \( S_0 \) is the set of all strings with period 3, and the attractor of the system generated by \( S_1 \) is the set of all strings with period 2.)

We claim that the sequence \( u = 000\overline{1}00 \) is in \( K \) but not in \( A_w \) for any \( w \in \Sigma_2 \). Let \( v = 001.\sigma^3(v) \), i.e. the first three symbols of \( v \) are 001, and let \( w_k = 000...001 \) with 3k zeros before 1. Then, for every \( k \), \( S_{w_k}(v) = 0.001001...001 \) with 001 repeating \( k \) times. Therefore, \( S_{w_k}(v) \to u \) as \( k \to \infty \), i.e. \( u \in K \). To show that \( u \) does not belong to the union \( \bigcup_{w \in \Sigma} A_w \), we argue by contradiction. If \( u \in A_w \), then there exists a sequence \( v_k \in \Sigma_2 \) such that \( \lim_{k \to \infty} S_{[w_k]}(v_k) = u \), where \( n_k \to \infty \). Therefore, we can find \( l \) such that \( S_{[w_k]}(v_k), S_{[w_{k-l}]}(v_{k-l}), \ldots, S_{[w_0]}(v_0) \), all begin with \( 000100100... \). Since \( S_{[w_k]}(v_k) = S_{[w_{k-l}]}(v_{k-l}) = S_{[w_0]}(v_0) = 000100100... \), and the action of operators \( S_0 \) and \( S_1 \) depends only on the first three symbols in the strings, it follows that \( v_l[3] \neq v_{l+1}[3] \), because if \( v_l[3] = v_{l+1}[3] \), then \( S_{[w_0]}(v_0) \) starts with at least 4 zeros, i.e. \( 0000100100... \), which is impossible. Similarly, \( v_{l+j}[3] \neq v_{l+j}[3] \) for \( j, k = 0, \ldots, 8, j \neq k \). But there can be only 8 different three-letter words in 2 symbols—a contradiction. Hence, \( u \) does not belong to the \( \bigcup_{w \in \Sigma} A_w \).

2.4. Dynamics with restricted choice

As in section 2.3.1, \( \Sigma \) denotes the space of one-sided infinite strings on \( N \) symbols, and \( d_\Sigma \) is the metric on \( \Sigma \). Let \( \Lambda \) be a subshift of \( \Sigma \), i.e. \( \Lambda \) is a closed subset of \( \Sigma \) and \( \sigma(\Lambda) = \Lambda \). Dynamics with restricted choice is defined on the space \( \mathcal{X}_\Lambda = X \times \Lambda \) by the operator \( \Theta : (x, w) \mapsto (S_{[w]}(x), \sigma(w)) \), where the strings \( w \) are now taken from \( \Lambda \) only.

We assume that \( X \) and \( S_0, \ldots, S_{N-1} \) satisfy our assumptions 1 and 2. The existence of the global compact attractor, \( \mathcal{M}_\Lambda \), then follows from the abstract result, theorem 5. The assertions 2 and 3 of theorem 3 are among the general properties of global compact attractors, see theorem 6. Denote by \( K_\Lambda \) the projection of \( \mathcal{M}_\Lambda \) onto the \( X \) component. Clearly, \( K_\Lambda \) is compact. Also, \( K_\Lambda \) is a subset of the slice \( K \) corresponding to the full shift \( \Sigma \), as in theorem 2. Because of the invariance property of \( \mathcal{M}_\Lambda \), for every point \( y \in K_\Lambda \) there is a \( j \), one of the symbols 0, \ldots, \( N-1 \), and a point \( x \in K \) such that \( y = S_j(x) \). Define the sets \( A_j = \{ x \in K_\Lambda : S_j(x) \in K \} \). It is easy to see that each \( A_j \) is compact and \( K_\Lambda = A_0 \cup A_1 \cup \cdots \cup A_{N-1} \). By construction, we have \( A_0 \cup A_1 \cup \cdots \cup A_{N-1} = S_0(A_0) \cup S_1(A_1) \cup \cdots \cup S_{N-1}(A_{N-1}) \). To analyse the slices \( \mathcal{M}_\Lambda(s) = \{ x \in X : (x, s) \in \mathcal{M}_\Lambda \} \), we follow the argument of the corresponding part of section 2.3.3. Every point \( (x, s) \in \mathcal{M}_\Lambda \) is the limit of the form

\[
(x, s) = \lim_{n_k \to \infty} (S_{w_k}(s_{n_k}), \sigma^{n_k}(s_{n_k}))
\]

where \( (x_n) \) is a bounded sequence in \( X \), \( (s_n) \) is a bounded sequence in \( \Lambda \), and \( w_k \) is the prefix of \( s_{n_k} \), \( s_{n_k} = w_k \cdot \sigma^{n_k}(s_{n_k}) \). Because \( \mathcal{M}_\Lambda \) is invariant under \( \Theta \) and we know that the unrestricted dynamics has the global compact attractor \( \mathcal{M} = K \times \Sigma \), the sequence \( (x_n) \) can be taken from the compact \( K \), and we may assume that \( x_{n_k} \to x \in K \). Also, we may assume that the words \( w_k \) converge (to some infinite string \( w_k \in \Lambda \)). The strings \( \sigma^{n_k}(s_{n_k}) \) converge to \( s \). Consider all strings \( u \in \Lambda \) such that \( w_k.u \) is a string in \( \Lambda \) for infinitely many \( k \). For every such \( u \) we will have \( x \in \mathcal{M}_\Lambda(u) \).
We see that the number of different slices of the attractor $\mathcal{M}_\Lambda$ may depend on the sequence $x_{n_k}$, but more importantly, it depends on what strings can be attached to convergent sequences of finite words in $\Lambda$.

With every sequence $(w_k)$ of finite words in $\Lambda$ we associate the set $s((w_k))$ of one-sided infinite strings $u \in \Lambda$ such that $w_k u \in \Lambda$ for some subsequence $w_k$. In order to prove the third assertion of theorem 3 we will show that, if $\Lambda$ is a sofic shift, the number of different sets among all $s((w_k))$ is finite. The argument will be similar to the proof of theorem 3.2.10 in [24].

Recall that $\Lambda$ is a sofic shift if it has a presentation by a finite labelled graph, see [24]. This means that there is a directed graph, $G = (V, E)$, with a finite number of vertices, $V$, and edges, $E$; the edges are labelled by the symbols $0, 1, \ldots, N - 1$; from every vertex begins at least one infinite directed path; the labels of the edges in the infinite directed paths form infinite one-sided strings that exhaust exactly all strings in $\Lambda$.

**Lemma 18.** If $\Lambda$ is a one-sided sofic subshift of $\Sigma$, then the number of different sets among all $s((w_k))$ is finite.

**Proof.** Let $G = (V, E)$ be a labelled graph presenting $\Lambda$. Let $(w_k)$ be a sequence of finite words allowed in $\Lambda$. For each word $w_k$, pick a finite directed path in $G$ presenting it. We can find a subsequence, $(w_{k_\ell})$, such that all the words $w_{k_\ell}$ have the same terminal vertex in their presentation. If $T$ is such a vertex, then $w_{k_\ell} u \in \Lambda$ for all infinite paths $u$ starting at $T$. Because the number of vertices is finite, we are done. □

**Remark 19.** Even if the number of different sets among all $s((w_k))$ is $> 1$, the attractor $\mathcal{M}_\Lambda$ may be a product, $\mathcal{M}_\Lambda = K_\Lambda \times \Lambda$, with the same slice for every string in $\Lambda$.

Indeed, let $N = 2$ and let $\Lambda$ consist of the periodic string $u = 100100 \ldots$ and its shifts $\sigma(u) = 0100 \ldots$ and $\sigma^2(u) = 0010 \ldots$. If $(w_k)$ consists of words ending in $00$, then the only string that can be attached to $w_k$ is $u$. If $(w_k)$ consists of words ending in $1$, then the only string is $\sigma(u)$, and for words ending in $10$ the only string is $\sigma^2(u)$. Thus, we have three different sets of the form $s((w_k))$. At the same time, the individual attractors $\mathcal{A}_u$, $\mathcal{A}_{\sigma(u)}$ and $\mathcal{A}_{\sigma^2(u)}$ are all equal, as we argue in corollary 15.

One may ask whether $\mathcal{M}_\Lambda$ is always a product. The answer is no, as the following example shows. Let $\Lambda$ be the intersection of the one-sided golden mean shift with the even shift. In other words, $\Lambda$ consists of all sequences of $0$s and $1$s such that between any two $1$s there are two or a larger even number of $0$s. A graph presenting $\Lambda$ is given in figure 5. We will animate this graph to define the dynamics. First, identify the nodes with three distinct points $A$, $B$ and $C$ in $\mathbb{R}^2$, see figure 5 left, and define $X = \{A, B, C\}$. Second, define the maps $S_0$ and $S_1$ acting on points as shown by the directed edges labelled correspondingly; for example, $S_0(A) = B$, $S_1(A) = A$, and $S_0(C) = B$.

Now consider the set $\Lambda^*$ of non-empty finite words (blocks) of $\Lambda$. We divide $\Lambda^*$ into three classes and correspondingly divide the strings in $\Lambda$ into three classes. The first class of...
words in \( \Lambda^+ \) consists of the words ending in 1. Such words can serve as prefixes of strings starting with an even (or infinite) number of 0s. Denote these classes by \( \Lambda^+_\Lambda \) and \( \Lambda^+_A \). The second class of finite words consists of the words ending in an odd number of 0s. The strings for which such words can serve as prefixes are the strings starting with an odd number of 0s. These classes are denoted by \( \Lambda^+_B \) and \( \Lambda_A \). The last class in \( \Lambda^+ \) consists of words ending in an even number of 0s. The corresponding strings are those starting with 1 or with an even number of 0s. These are denoted by \( \Lambda^+_C \). By looking at the picture of the animated shift, it is easy to identify the possible limits of sequences \( S \) when \( w_k \) belong to a particular class, while \( x_k \in \{A, B, C\} \). We see that if \( w_k \in \Lambda^+_\Lambda \), then the limit set is \( \{A, B\} \). If \( w_k \in \Lambda^+_B \), then the limit set is \( \{B, C\} \). Finally, if \( w_k \in \Lambda^+_C \), then the limit set is again \( \{B, C\} \). Thus, there are two different slices in the attractor \( \mathcal{M}_A \). One slice is \( \{A, B\} \), and the other is \( \{B, C\} \). We have \( \mathcal{M}_A(u) = \{A, B\} \) if \( u \in A \Lambda \), and \( \mathcal{M}_A(u) = \{B, C\} \) if \( u \in B \Lambda \cup C \Lambda \). The global attractor \( \mathcal{M}_A \) is a union of the sets \( \{A, B\} \times A \Lambda \), \( \{B, C\} \times B \Lambda \), and \( \{B, C\} \times C \Lambda \).

Another example of different slices appears in numerical results reported in the next section.

2.5. Non-autonomous systems and dynamics with choice

It has been known for a long time that non-autonomous dynamical systems can be viewed as autonomous dynamical systems in a larger (state) space, and there are many ways of achieving this. However, for the purposes of the analysis of the long-term behaviour of solutions, the most beneficial approach was suggested by Sell [29]. The modern abstract definition of a discrete non-autonomous semidynamical system consists of a state space, \( X \), a base (or parameter) space, \( P \), a map \( \theta : P \to P \) that defines a dynamics on \( P \), and a cocycle map \( \varphi : \mathbb{Z}_{\geq 0} \times X \times P \to X \). The cocycle map \( \varphi \) has the properties

(i) \( \varphi(0, x, p) = x \)

(ii) \( \varphi(n + 1, x, p) = \varphi(n, \varphi(1, x, p), \theta(p)) \)

The skew-product dynamics is then understood as an autonomous dynamics on the product \( X \times P \) generated by the map

\[
\pi(x, p) = (\varphi(1, x, p), \theta(p)).
\]

In what follows, \( X \) and \( P \) are assumed to be complete, metric spaces, and the maps \( \theta \) and \( \varphi(1, \cdot, \cdot) \) are assumed to be continuous and bounded. Also, we assume that \( P \) is compact and \( \theta(P) = P \). Of course, our definition of dynamics with choice agrees with this construction. Our \( \Sigma \) is \( P \), our \( \sigma \) is \( \theta \), and \( \varphi(1, x, p) = S_{\delta_{P\theta}(x)} \).

There is a considerable literature devoted to attractors of non-autonomous systems, see [8, 9, 11, 16, 17] and references therein. Several authors (e.g. [9]) view the product \( X \times P \) as a fibre bundle over the base \( P \), and then it makes sense to define the attractors fibred over \( P \). The following definitions are compiled from [9, 17].

Let \( \hat{M} \) be a collection of compact subsets \( M(p) \subset X \) parametrized by the points of \( P \).

**Definition 20.** \( \hat{M} = \{M(p)\}_{p \in P} \) is a forward attractor of the non-autonomous system \((X, \varphi, (P, \mathbb{Z}_{\geq 0}, \theta))\) iff

- \( \bigcup_{q \in M} \varphi(1, M(q), q) = M(p) \);
- The set \( \bigcup_{p \in P} M(p) \) is compact;
- \( \lim_{n \to +\infty} \text{dist}_X(\varphi(n, B, p), M(\theta^n(p))) = 0 \), for every bounded \( B \subset X \) and \( p \in P \).
The following result follows from [17, section 5]

**Lemma 21.** Assume that the skew-product semidynamical system \((X \times P, \pi)\) corresponding to the non-autonomous system \((X, \varphi, (P, \mathbb{Z}_{\geq 0}, \theta))\) possesses a global compact attractor \(M\). Then the slices \(M(p) = \{x \in X : (x, p) \in M\}\) form the forward attractor.

In the case of the full dynamics with choice, the global attractor of the skew-product dynamics is \(\mathcal{M} = K \times \Sigma\) and the forward attractor has \(K\) assigned to every string \(w \in \Sigma\). In the case of the restricted dynamics with choice, the global attractor, \(M/\Lambda\), is not a product, in general. The collection of slices \(M/\Lambda(w) = \{x \in X : (x, w) \in M/\Lambda\}\) forms the forward attractor. Lemma 18 shows that among those slices there are only finitely many different sets.

In the theory of non-autonomous systems, serious attention is paid to the notion of a pullback attractor because of its role in random dynamical systems [12]. Usually this notion is introduced under the assumption that the map \(\theta\) is a homeomorphism on \(P\). In our case, the shift on the one-sided strings is not invertible, so the usual definition is not applicable. However, the definition can be extended to the case of non-invertible \(\theta\), as shown in [17].

**Definition 22.** \(\hat{M} = \{M(p)\}_{p \in P}\) is a pullback attractor of the non-autonomous system \((X, \varphi, (P, \mathbb{Z}_{\geq 0}, \theta))\) iff

- \(\bigcup_{q : \theta(q) = p} \varphi(1, M(q), q) = M(p)\);
- The set \(\bigcup_{p \in P} M(p)\) is compact;
- \(\lim_{n \to +\infty} \text{dist}_X (\varphi(n, B, \theta^{-n}(p)), M(p)) = 0\), for every bounded \(B \subset X\) and \(p \in P\).

In this definition, \(\theta^{-n}(p)\) is the set of all \(q \in P\) such that \(\theta^n(q) = p\). The slices of the attractors \(\mathcal{M}\) and \(\mathcal{M}/\Lambda\) in dynamics with choice compose both forward and pullback attractors of the corresponding non-autonomous systems.

### 3. Example

The simplest mathematical model of malaria transmission goes back to Ross and Macdonald. The state of the human–mosquito interaction system is described by the portion of infected humans, \(x\), and the portion of infected mosquitoes, \(y\). The change in time is described by the following simple system of ordinary differential equations:

\[
\begin{align*}
\dot{x} &= ay(1-x) - rx, \\
\dot{y} &= bx(1-y) - my.
\end{align*}
\]

(7)

The nature of the positive coefficients \(a, b, r\) and \(m\) is discussed in [31]. In particular, the coefficients \(a\) and \(b\) are proportional to the biting rate and the transmission efficiencies (infected human to mosquito and infected mosquito to human), \(r\) is the recovery rate (in humans) and \(1/m\) is the average mosquito life-span. In practice, it is hard to measure these parameters. Also, there are many factors that affect their values, see [31], page 8, and the values may change in time.

The state space for the model (7) is the closed square \(X = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}\). For initial conditions in \(X\) the solution stays in \(X\) for all \(t\). If the quantity \(R_0 = \frac{ab}{rm}\) is \(\leq 1\), all trajectories starting in \(X\) converge to the origin, and the global compact attractor consists of a single point, \(P_1 = (0, 0)\). If \(R_0 > 1\), the equilibrium \(P_1\) becomes unstable and there emerges the second fixed point, \(P_2 = (x_*, y_*)\), inside the square \(X\),

\[
\begin{align*}
x_* &= \frac{ab - rm}{b(a + r)}, \\
y_* &= \frac{ab - rm}{a(b + m)}.
\end{align*}
\]

(8)
This second equilibrium is stable, and the global compact attractor of the system consists of the two equilibria, \( P_1 \) and \( P_2 \), and of the heteroclinic trajectory connecting them (and staying entirely inside \( X \)). The number \( R_0 \), known as the basic reproductive number, detects the emergence of epidemics: when \( R_0 > 1 \) there is a stable portion of infected population.

We consider a discrete version of equations (7):

\[
\begin{align*}
x(t + \Delta t) &= x(t) + \Delta t \left( ay(t) (1 - x(t)) - rx(t) \right), \\
y(t + \Delta t) &= y(t) + \Delta t \left( bx(t) (1 - y(t)) - my(t) \right).
\end{align*}
\]

(9)

The time step map \((x(t), y(t)) \mapsto (x(t + \Delta t), y(t + \Delta t))\) maps \( X \) into itself provided

\[
\Delta t < \min \left\{ \frac{1}{a + r}, \frac{1}{b + m} \right\}.
\]

(10)

The fixed points for (9) are the same as for (7). As in the continuous case, if \( ab > rm \) and the time step satisfies (10), the global attractor for (9) consists of the two fixed points, \( P_1 \) and \( P_2 \), and the heteroclinic trajectory connecting them.

We choose two sets of parameters, \( \text{pset}_0 = \{ a = 4, b = 6, r = 1, m = 2 \} \) and \( \text{pset}_1 = \{ a = 2, b = 10, r = 3, m = 2 \} \), and denote the corresponding time step maps.
by $S_0$ and $S_1$. These sets of parameters are not related to any real-life situation but rather chosen to better visualize the attractors. The fixed point $P_2$ for pset$_0$ is $(11/15, 11/16)$ and for pset$_1$ it is $(7/25, 7/12)$. Figures 2–9 show the results of numerical computation. The results depend on the size of the time step $\Delta t$. In figures 6 and 7, the left line (the heteroclinic trajectory) is the (global compact) attractor for the discrete system $(X, S_1)$, and the right line is the attractor of $(X, S_0)$. The two lines between them form the individual attractor $A_w$ corresponding to the periodic string $w = 1010...$ (in figure 7 the two lines are very close). For our example of dynamics with choice, $\Sigma$ is the space of one-sided infinite strings of symbols 0 and 1. According to theorem 2, the global compact attractor for $(\mathcal{X}, \Sigma)$ has one slice,
i.e. $\mathcal{M} = K \times \Sigma$. The set $K$ for $\Delta t = 0.05$ and for $\Delta t = 0.005$ are depicted in figures 3 and 4, respectively. We have also looked at the dynamical systems corresponding to convex combinations of the parameter sets $\text{pset}_0$ and $\text{pset}_1$ and plotted their global attractors. The result is different from $K$, see figure 8 where the ‘convex combination’ is superimposed onto the set $K$.

When $\Delta t \to 0$, the upper part of the boundary of $K$ becomes smooth. Note that the limit set is not an attractor of any system (9) with a fixed, averaged set of parameters $a$, $b$, $r$ and $m$. It would be interesting to understand whether the limit set can be obtained as a union of the attractors of the systems $(\mathcal{X}, S_t)$, where the operator $S_t$ corresponds to a certain parameter set $\text{pset}_t$ for some curve connecting $\text{pset}_0$ with $\text{pset}_1$ in the space of parameters.

Next, we consider restricted dynamics associated with the golden mean subshift $\Lambda$ (made of one-sided strings of 0s and 1s such that each 1 is necessarily followed by 0). The graph representing the golden mean shift is shown in figure 10.

Our analysis in section 2.4 shows that the global attractor of the restricted dynamics, $(\mathcal{X}, \Lambda)$ may have at most two different slices: one corresponding to sequences of words ending in 1 (the red slice), and the other one corresponding to sequences of words ending in 0 (the blue slice). Our computation shows that the attractor of the restricted dynamics $(\mathcal{X}, \Lambda)$ indeed has two slices. The slices are shown in figures 11 and 12.

As point sets on the plane, the slices overlap. Their union is plotted in figure 9.
Dynamics with choice

Figure 12. The blue slice; $\Delta t = 0.05$.

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