Risk Bounds for Robust Deep Learning

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Abstract

It has been observed that certain loss functions can render deep-learning pipelines robust against flaws in the data. In this paper, we support these empirical findings with statistical theory. We especially show that empirical-risk minimization with unbounded, Lipschitz-continuous loss functions, such as the least-absolute deviation loss, Huber loss, Cauchy loss, and Tukey’s biweight loss, can provide efficient prediction under minimal assumptions on the data. More generally speaking, our paper provides theoretical evidence for the benefits of robust loss functions in deep learning.

Keywords: Robust deep learning; neural networks; Rademacher complexity; empirical-risk minimization; Huber loss; least-absolute deviation; weight decay.

1. Introduction

Deep learning often uses data that are rich in terms of quantity but meager in terms of quality. A well-studied problem is adversarial attacks, which means that parts of the data are corrupted by a “mean-spirited opponent.” It has been shown that adversarial attacks can make standard deep-learning pipelines fail completely (Akhtar and Mian, 2018; Yuan et al., 2019; Kurakin et al., 2016; Wang and Yu, 2019; Sharif et al., 2016; Lab, 2019; Kurakin et al., 2017), and a number of approaches to address this problem have been proposed (Madry et al., 2017; Kos and Song, 2017; Papernot et al., 2015; Tramér et al., 2017; Salman et al., 2019; Wang et al., 2018).

But statistical theory for deep-learning under adversarial attacks is scarce, and, more importantly, there are other, arguably more common, types of problems with the data. For example, data collection is often automated, and the sheer size of typical data sets makes it difficult to uphold high data quality. Moreover, data are often convenience samples, that is, the strategy for collecting data is not necessarily appropriate for the specific purpose of the analysis. Thus, we are interested in deep learning that caters to a broad spectrum of data in general. We call this topic “robust deep learning.”

Robust learning is a classical topic in statistics (Stigler, 2010). It has especially been shown that many standard estimators can be rendered robust with respect to heavy-tailed data by replacing their loss-functions, such as least-squares, by Lipschitz-continuous alternatives, such as Huber loss (Hampel et al., 2011; Huber and Ronchetti, 2009). The robustness-yielding properties of such loss functions have also been observed in a variety of deep-learning applications (Barron, 2019; Belagiannis et al., 2015; Jiang et al., 2018; Wang et al., 2016). But statistical theories for deep learning are restricted to bounded loss functions or presume (sub-)Gaussian or bounded input and output (Bartlett, 1998; Schmidt-Hieber, 2020; Taheri et al., 2020).

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In this paper, we establish a statistical theory for deep learning with Lipschitz-continuous loss functions, such as Tukey’s biweight loss, Huber loss, and absolute-deviation loss. We first establish a general risk bound that caters to empirical-risk minimizers with unbounded, Lipschitz-continuous loss functions. This result might be of independent interest. We then use the general risk bound to derive statistical guarantees for robust deep learning in a general class of feedforward neural networks. Broadly speaking, our theories suggest that robust loss function can lead to effective learning with problematic as well as with benign data.

Outline of the paper In Section 2, we establish a general risk bound that allows for Lipschitz-continuous but unbounded loss functions. In Section 3, we specify the risk bound in the case of weight decay with robust loss functions, which leads to the advertised robust guarantees. In Section 4, we give detailed proofs. In Section 5, we briefly discuss some extensions and limitations.

2. General Risk Bound

In this section, we establish a risk bound that is tailored to our needs in deep learning but might also be of independent interest. The bound is formulated in terms of the empirical risk and the Rademacher complexity and, therefore, is related to existing bounds in empirical-risk minimization. Our key innovation is that we allow for unbounded loss functions.

We first formulate the data and functions on these data. Consider i.i.d. distributed pairs \((y, x), (y_1, x_1), \ldots, (y_n, x_n) \in \mathbb{R} \times \mathbb{R}^d\) and i.i.d. Rademacher random variables \(r_1, \ldots, r_n \in \{\pm 1\}\). Also, consider a nonempty set \(F\) that consists of functions of the form \(f : \mathbb{R}^d \to \mathbb{R}\). We summarize the properties of the data and the functions in four quantities:

**Definition 1 (Complexity measures)** Given a function \(f^* \in F\), we call
\[
\begin{align*}
    s_x &:= \sqrt{\mathbb{E}_{(y,x)}[\|x\|^2_2]} \quad \text{and} \quad s_y|x := \sqrt{\mathbb{E}_{(y,x)}[\|y - f^*[x]\|^2]}
\end{align*}
\]
the expected size of the input and the expected size the noise, respectively,
\[
    w_F := \sqrt{\mathbb{E}_{(y,x)}[\sup_{f \in F}[f[x] - f^*[x]]^2]}
\]
the size of (an envelope of) \(F\), and
\[
    c_F := \mathbb{E}_{(y_1, x_1), \ldots, (y_n, x_n), r_1, \ldots, r_n}
    \left[\sup_{f \in F}\frac{1}{n} \sum_{i=1}^n r_i [f[x_i]]\right]
\]
the Rademacher complexity of \(F\).

The function \(f^*\) can be an arbitrary element of \(F\), but we will later think of it as the “true” data-generating function or an approximation of it. It then makes sense to call the quantity \(y - f^*[x]\) the “noise.” The quantity \(w_F\) is the size of an envelope of \(f[x] - f^*[x]\) over \(F\) (Lederer and van de Geer, 2014, Section 2). The Rademacher complexity \(c_F\) is finally
(a) the absolute-deviation loss \( h[a] := |a| \) is convex but not differentiable, and it satisfies the Lipschitz condition with \( c_h = 1 \)

(b) the Huber loss \( h[a] := a^2/2 \) for \( a \in [-k, k] \) and \( h[a] := k|a| - k^2/2 \) otherwise, where \( k \in (0, \infty) \), is convex and differentiable, and it satisfies the Lipschitz condition with \( c_h = k \)

(c) the Cauchy loss \( h[a] := \log(1 + k^2 a^2) \), where \( k \in (0, \infty) \), is not convex but differentiable, and it satisfies the Lipschitz condition with \( c_h = k \)

Figure 1: three robust alternatives to the least-squares loss \( h[a] := a^2/2 \)

a well-known measure of the complexity of the set \( \mathcal{F} \) (Bartlett et al., 2002; Koltchinskii, 2001; Koltchinskii and Panchenko, 2002).

We then formulate the empirical-risk minimizer. Consider a function \( h : \mathbb{R} \to \mathbb{R} \) that is Lipschitz continuous: there is a constant \( c_h \in [0, \infty) \) such that

\[
|h[a] - h[b]| \leq c_h |a - b| \quad \text{for all } a, b \in \mathbb{R}.
\]

We also assume, without loss of generality, that \( h[0] = 0 \). We call \( h \) the *loss function*. The least-squares loss does not satisfy the Lipschitz condition, but many robust versions of it do, including the absolute-deviation loss, the Huber loss, the Cauchy loss, and Tukey’s biweight loss; in particular, we do not require the loss to be convex or differentiable (see Figure 1 for illustrations). The empirical-risk minimizers are then

\[
\hat{f} \in \arg \min_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} h[y_i - f(x_i)] \right\}.
\]

We give examples of these estimators in the following section.

We now equip the empirical-risk minimizer with a statistical guarantee:

**Theorem 2 (General risk bound)** For every \( f \in \mathcal{F} \) and \( t \in (0, 1) \), it holds with probability at least \( 1 - t \) that

\[
E_{y,x}[h[y - f(x)]] \leq \frac{1}{n} \sum_{i=1}^{n} h[y_i - f(x_i)] + 16c_h c_F + 236c_h \frac{w_F + s_y}{\sqrt{nt}}.
\]

This inequality bounds the risk of a function \( f \) in terms of its empirical loss and the complexity of the setting. As long as the complexity terms are small enough, and the empirical risk of the true data-generating function converges sufficiently fast to its expectation, the above-stated inequality ensures that the population risk of the empirical-risk minimizer \( \hat{f} \) is not much larger than the population risk of the true data-generating function:
Corollary 3 (General risk bound for $\hat{f}$) For every $t \in (0, 1)$, it holds with probability at least $1 - t$ that
\[
E_{(y, x)} \left[ h[y - \hat{f}[x]] \right] \leq E_{(y, x)} \left[ h[y - \hat{f}^* [x]] \right] + \frac{1}{n} \sum_{i=1}^{n} \left( h[y_i - \hat{f}^* [x_i]] - E_{(y, x)} \left[ h[y - \hat{f}^* [x]] \right] \right) + 16c_{h}c_{F} + 236c_{h}w_{F} + s_{y|x} \sqrt{nt}.
\]

We will use these results in the following section to derive risk bounds for robust deep learning.

The bound in Theorem 2 is similar to the one in Bartlett and Mendelson (2002, Theorem 8). The crucial difference is that their bound requires that the loss function has values only in $[0, 1]$, while our bound allows for loss functions that are Lipschitz continuous but unbounded. The price for this change in scope is the inclusion of the quantities $w_{F}$ and $s_{y|x}$, which are additional measures for the complexity of the statistical framework.

Moving from bounded to unbounded loss functions also requires changing the proof techniques. For example, proofs in the bounded case can use McDiarmid’s inequality (McDiarmid, 1989)—see, for example, Bartlett and Mendelson (2002, Proof of Theorem 8) and Mohri et al. (2018, Proof of Theorem 3.3.). We instead use a concentration inequality for heavy-tailed data from Lederer and van de Geer (2014). The proof is deferred to Section 4.1.

We finally mention the fact that by applying the results of Lederer and van de Geer (2014) in a slightly different way, one can relax the assumptions on the data from a second-moment condition to a $(1 + b)$th-moment condition, $b > 0$, at the price of getting a slower rate; we omit the details to avoid digression.

3. Guarantees for Robust Deep Learning

We now use the above-stated risk bound to develop guarantees for robust deep learning. We consider layered, feedforward neural networks, that is, we consider $\mathcal{F} := \{ f_{\Theta} : \mathbb{R}^{d} \rightarrow \mathbb{R} : \Theta \in \mathcal{M} \}$ with $\mathcal{M}$ a nonempty subset of $\overline{\mathcal{M}} := \{ \Theta = (\Theta^l, \ldots, \Theta^0) : \Theta^j \in \mathbb{R}^{p^j+1 \times p^j} \}$ and
\[
f_{\Theta}[x] := \Theta^l a^l[\Theta^{l-1} \cdot \cdot \cdot a^1[\Theta^0 x]] \quad \text{for} \quad x \in \mathbb{R}^{d}. \quad (2)
\]

The functions $a^j : \mathbb{R}^{p^j} \rightarrow \mathbb{R}^{p^j}$ are called the activation functions, $l$ the depth of the network, $p^0 := d$ and $p^{l+1} := 1$ the input and output dimensions, respectively, and $w := \max\{p^1, \ldots, p^l\}$ the width of the network. To fix ideas, we assume the popular and well-established ReLU activation: $(a^j[v])_i := \max\{0, v_i\}$ (Hahnloser, 1998; Salinas and Abbott, 1996).

The empirical-risk minimizers are then the functions
\[
\hat{f} := f_{\hat{\Theta}} \quad \text{with} \quad \hat{\Theta} \in \arg \min_{\Theta \in \mathcal{M}} \left\{ \frac{1}{n} \sum_{i=1}^{n} h[y_i - f_{\Theta}[x_i]] \right\}. \quad (3)
\]

The parameter set is assumed to satisfy
\[
\mathcal{M} \subset \left\{ \Theta \in \overline{\mathcal{M}} : \max_{j \in \{0, \ldots, l\}} \|\Theta^j\|_{F} \leq b_{\mathcal{M}} \right\}
\]
for a fixed $b_M \in [0, \infty)$ and the Frobenius norm

$$
\|\Theta^j\|_F := \sqrt{\sum_{i=1}^{p_j+1} \sum_{k=1}^{p^j} |(\Theta^j)_{ik}|^2} \quad \text{for } j \in \{0, \ldots, l\}, \Theta^j \in \mathbb{R}^{p_j+1 \times p^j}.
$$

Such choices of $\mathcal{M}$ have been popular for more than three decades already and are known under the name “weight decay” (Krogh and Hertz, 1991).

A standard question is how the empirical-risk minimizers compare with an oracle. If the model is correct, the oracle is typically the true data-generating function; otherwise, the oracle is an approximation of it. We do not need to know the specifics: our theory works for every oracle $f^* := f_{\Theta^*}$ with a fixed $\Theta^* \in \mathcal{M}$. But, in any case, we can interpret $f^*$ as the “best” neural network.

Common loss functions for classification, such as the logistic sigmoid function, are bounded. Statistical guarantees for corresponding empirical-risk minimizers can then be derived based on well-established risk bounds, such as (Bartlett and Mendelson, 2002, Theorem 8). Common loss functions for regression-type tasks, in contrast, are unbounded. Particularly interesting for us are Lipschitz-continuous alternatives to the least-squares loss $h[a] := a^2$. A basis for deriving statistical guarantees is then Theorem 2. Indeed, we find the following result:

**Theorem 4 (Robust deep learning)** For every $t \in (0, 1/2)$, it holds with probability at least $1 - t$ that

$$
E_{(y,x)}[h[y - \hat{f}(x)]] \leq E_{(y,x)}[h[y - f^*(x)]] + ac_h \frac{(b_M)^{l+1}(l + 1)s_x + s_y|x}{\sqrt{nt}},
$$

where $a \in (0, \infty)$ is a numerical constant.

For every $t \in (0, 1/2)$ and $n$ large enough, it holds with probability at least $1 - t$ that

$$
E_{(y,x)}[h[y - \hat{f}(x)]] \leq 1.1c_h s_y|x + ac_h (b_M)^{l+1}(l + 1)s_x \sqrt{\frac{\log[n]}{n}}.
$$

Broadly speaking, the first part of the theorem guarantees that the empirical-risk minimizers perform essentially as well as the best network in the class under consideration; the second part of the theorem guarantees that the expected error of the empirical-risk minimizers is essentially proportional to the variance of the noise. The key feature of the theorem is that it only requires a Lipschitz-continuous loss function and second moments of the data. Hence, the theorem confirms the empirical observations of the fact that Lipschitz-continuous alternatives to the least-squares loss can yield effective learning under very weak assumptions on the data.

The proof of Theorem 4 is based on the risk bound in Corollary 3 and on Lipschitz and Rademacher properties of neural networks (Golowich et al., 2020; Taheri et al., 2020)—see Section 4.3.

Theorem 4 is the first statistical guarantee for deep learning with unbounded, Lipschitz-continuous loss functions. Yet, the rates depend very similarly on the dimensions of the data.
and the network as the known rates for deep learning with bounded or least-squares loss (Anthony and Bartlett, 2009; Golowich et al., 2020; Schmidt-Hieber, 2020; Neyshabur et al., 2015; Taheri et al., 2020): the rates in Theorem 4 have basically a $1/\sqrt{n}$ dependence on the number of samples, no explicit dependence on the network’s input dimension and width, and an exponential dependence on the network’s depth if $b_M > 1$ and at most a linear dependence on the depth otherwise. Hence, our results support the use of robust loss functions, such as Huber loss, not only for heavily corrupted data.

But still, the most interesting case for robust loss functions is unbounded and non-Gaussian data. The specifics of the data are encapsulated in the quantities $s_x$ and $s_y|x$; broadly speaking, Theorem 4 ensures that the empirical-risk minimizer estimates the parameters effectively as long as the second moments of the input data and of the noise are reasonably small. This assumption is, of course, much weaker than the usual assumption of bounded or sub-Gaussian input data and noise (Schmidt-Hieber, 2020; Taheri et al., 2020). The following example illustrates a generic case where the weaker assumptions are crucial.

**Example 1 (Flawed input data)** A generic example where robust methods are useful is when parts of the input data are flawed. Flaws can be constructed in an adversarial manner, such as described in Moosavi-Dezfooli et al. (2017), for example, or they can stem from a nonadversarial source, such as a result of measurement errors. To fix ideas, assume that the components $x_1, \ldots, x_d$ of the input are i.i.d. and each sampled from a distribution $P_{\text{cor}}$ with probability $c$ and from a centered normal distribution with variance $\sigma^2$ otherwise. We can think of $P_{\text{cor}}$ as the type of corruption and $c \in [0, 1]$ as the level of corruption in the data.

Since we want to focus on the input data, we just assume that the second moment of the noise $y - f^*\lfloor x \rfloor$ is bounded (for example, $y - f^*\lfloor x \rfloor \sim N(0, 1)$).

Consider first $c = 1$, that is, none of the input vectors are corrupted. Then, $s_x = \sqrt{d}\sigma$, and Theorem 4 yields the rate $\sigma (b_M)^{l+1}(l+1)\sqrt{d/n}$. This rate is virtually the same as the one that follows from combining the bound for the Rademacher complexity in Golowich et al. (2020, Theorem 1) and the risk bound in Bartlett and Mendelson (2002, Theorem 8), but in contrast to those results, Theorem 4 holds for unbounded loss functions. In any case, the agreement illustrates that Theorem 4 yields good rates in the special case of few or no corrupted inputs. More broadly speaking, the agreement highlights the fact that Theorem 4 is not only useful for corrupted data but for learning with unbounded, Lipschitz-continuous loss functions, such as in regression-type settings, more generally.

Consider now $c \neq 0$, that is, about $cn$ of the total $n$ input vectors are corrupted. One can check readily that $s_x = (1 - c)\sqrt{d}\sigma + c\sqrt{dE_{\text{P}_{\text{cor}}}[(x_{\text{cor}})^2]}$, where $x_{\text{cor}} \sim P_{\text{cor}}$. Consequently, as long as $cE_{\text{P}_{\text{cor}}}[(x_{\text{cor}})^2] \lesssim \sigma^2$, Theorem 4 yields the same rate for the corrupted case as for the uncorrupted case. As a concrete example, let $P_{\text{cor}}$ be a log-normal distribution (a standard example of a heavy-tailed distribution) with parameters $(0, \gamma^2)$. Then, $s_x = (1 - c)\sqrt{d}\sigma + c\sqrt{d}\gamma^2$. Hence, as long as $\sigma$ and $\gamma$ are reasonably small, Theorem 4 ensures effective learning whatever the fraction of corrupted data is. More generally, these findings illustrate the usefulness of Theorem 4 for deep learning with corrupted input data.

We have restricted ourselves to the popular ReLU activation functions, but the robustness properties of Huber loss, absolute deviation, and so forth, are not tied to this type of activation. For example, our proof extends directly to all Lipschitz-continuous activation functions that satisfy $\mathbf{a}^T[\mathbf{0}_p] = \mathbf{0}_p$ (such as leaky ReLU, for example). Relaxing the
assumptions on the activation functions further would require generalizing the results of Golowich et al. (2020) and Taheri et al. (2020) that we use in our proofs, but importantly, the risk bounds stated in Section 2 do not impose any restrictions on the functions \( f \in \mathcal{F} \) and, therefore, do not limit our choice of the activation functions.

4. Proofs

In this section, we establish very detailed proofs.

4.1 Proof of Theorem 2

We first give a proof for the risk bound established in Section 2.

**Proof** [of Theorem 2] The key idea is to direct the problem towards an empirical process whose expectation is proportional to the Rademacher complexity, and whose deviation from the expectation is controlled by a concentration inequality.

Before we start, we introduce the shorthand

\[
 z := \sup_{g \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \left( h \left[ y_i - g(x_i) \right] - E_{(y,x)} \left[ h \left[ y - g(x) \right] \right] \right) \right|.
\]

The quantity \( z \) is the above-mentioned empirical process.

**Step 1:** We first show that

\[
 E_{(y,x)} \left[ h \left[ y - f(x) \right] \right] \leq \frac{1}{n} \sum_{i=1}^{n} h\left[ y_i - f(x_i) \right] + 2E_{(y_1,x_1),\ldots,(y_n,x_n)}[z] + z - 2E_{(y_1,x_1),\ldots,(y_n,x_n)}[z].
\]

After this first step, it remains to control the expectation of the empirical process \( z \) (Step 2) and the deviation of the empirical process from its expectation (Steps 3 and 4).

The proof of the first step is based on elementary algebra. We 1. add a zero-valued term, 2. use the linearity of finite sums, 3. use the fact that \( a - b \leq a + |b| \), 4. take the supremum over \( \mathcal{F} \) in the second term, 5. invoke the definition of \( z \), and 6. add a zero-valued term to find

\[
 E_{(y,x)} \left[ h \left[ y - f(x) \right] \right]
 = \frac{1}{n} \sum_{i=1}^{n} h\left[ y_i - f(x_i) \right] - \left( \frac{1}{n} \sum_{i=1}^{n} h\left[ y_i - f(x_i) \right] - E_{(y,x)} \left[ h \left[ y - f(x) \right] \right] \right)
 = \frac{1}{n} \sum_{i=1}^{n} h\left[ y_i - f(x_i) \right] - \frac{1}{n} \sum_{i=1}^{n} \left( h\left[ y_i - f(x_i) \right] - E_{(y,x)} \left[ h \left[ y - f(x) \right] \right] \right)
 \leq \frac{1}{n} \sum_{i=1}^{n} h\left[ y_i - f(x_i) \right] + \left( \frac{1}{n} \sum_{i=1}^{n} \left( h\left[ y_i - f(x_i) \right] - E_{(y,x)} \left[ h \left[ y - f(x) \right] \right] \right) \right)
 \leq \frac{1}{n} \sum_{i=1}^{n} h\left[ y_i - f(x_i) \right] + \sup_{g \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \left( h\left[ y_i - g(x_i) \right] - E_{(y,x)} \left[ h \left[ y - g(x) \right] \right] \right)
 = \frac{1}{n} \sum_{i=1}^{n} h\left[ y_i - f(x_i) \right] + z
\]
\[= \frac{1}{n} \sum_{i=1}^{n} h[y_i - f(x_i)] + 2E(y_1, x_1, \ldots, y_n, x_n)[z] + z - 2E(y_1, x_1, \ldots, y_n, x_n)[z],\]

as desired.

**Step 2:** We now show that

\[E(y, x)[h[y - f(x)]] \leq \frac{1}{n} \sum_{i=1}^{n} h[y_i - g(x_i)] + 16c_F + \frac{8c_F s_F}{\sqrt{n}} + z - 2E(y_1, x_1, \ldots, y_n, x_n)[z].\]

This step takes care of one of the \(2E(y_1, x_1, \ldots, y_n, x_n)[z]\) in the previous bound.

The key ingredients are symmetrization and contraction arguments, the Lipschitz property of the loss function, and the concentration of sums of Rademacher random variables. We introduce \((y'_1, x'_1), \ldots, (y'_n, x'_n) \in \mathbb{R} \times \mathbb{R}^d\) as random variables that are i.i.d. copies of \((y, x)\) and independent of the rest of the data. We first render the empirical process “symmetric.” We use 1. the definition of the empirical process, 2. the i.i.d. assumption on the data, 3. the linearity of integrals and finite sums, 4. dominated convergence, 5. the i.i.d. assumption on the data and the properties of the Rademacher random variables, 6. the linearity of finite sums, the triangle inequality, and the properties of suprema, and 7. the linearity of integrals and the i.i.d. assumption on the data to find that

\[2E(y_1, x_1, \ldots, y_n, x_n)[z] \leq 2E(y_1, x_1, \ldots, y_n, x_n)\left[\sup_{g \in F} \left| \frac{1}{n} \sum_{i=1}^{n} \left( h[y_i - g(x_i)] - E(y, x)[h[y - g(x)]] \right) \right| \right] \]

We then apply a contraction argument. We use 1. the contraction principle in (Boucheron et al., 2016, second part of Theorem 11.6 on pp. 324–325) with \(x_{i, \varphi} := y_i - g(x_i)\) (with some abuse of notation), \(\varphi_i := h\) (see our assumptions for \(h\) on Page 3), and \(\Psi\) the identity function, 2. the insertion of zero-valued term, 3. the linearity of finite sums, the triangle inequality, and the
properties of suprema, 4. the linearity of integrals, and 5. Definition 1 of the Rademacher complexity \(c_\mathcal{F}\) to show that

\[
2E(y_1, x_1, \ldots, y_n, x_n)[z] \\
\leq 8c_\mathcal{F}E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ \sup_{g \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} r_i(y_i - g(x_i)) \right] \\
= 8c_\mathcal{F}E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ \sup_{g \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} r_i(y_i - f^*[x_i] + f^*[x_i] - g(x_i)) \right] \\
\leq 8c_\mathcal{F}E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ \sup_{g \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} r_i g[x_i] + \frac{1}{n} \sum_{i=1}^{n} r_i f^*[x_i] + \frac{1}{n} \sum_{i=1}^{n} r_i(y_i - f^*[x_i]) \right] \\
\leq 16c_\mathcal{F}E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ \sup_{g \in \mathcal{F}} \left( \frac{1}{n} \sum_{i=1}^{n} r_i g[x_i] \right) \right] \\
\quad + \frac{8c_\mathcal{F}}{n} E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ \sum_{i=1}^{n} r_i(y_i - f^*[x_i]) \right] \\
= 16c_\mathcal{F}E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ \sum_{i=1}^{n} r_i(y_i - f^*[x_i]) \right].
\]

We then use a contraction property of Rademacher random variables to control the second term. We use 1. the law of iterated expectations (Durrett, 2010, Display (5.1.5) on p. 228), 2. Khinchin’s inequality (Haagerup, 1981, p. 232), 3. again the law of iterated expectations, 4. Jensen’s inequality (Durrett, 2010, Theorem 1.5.1 on p. 23), 5. the linearity of integrals and the i.i.d. assumption on the data, and 6. the definition of \(s_{y|x}\) to derive that

\[
E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ \sum_{i=1}^{n} r_i(y_i - f^*[x_i]) \right] \\
= E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ \sum_{i=1}^{n} r_i(y_i - f^*[x_i]) \mid (y_1, x_1), \ldots, (y_n, x_n) \right] \right] \\
\leq E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left[ E(y_1, x_1, \ldots, y_n, x_n, r_1, \ldots, r_n) \left( \sum_{i=1}^{n} (y_i - f^*[x_i])^2 \right)^{1/2} \mid (y_1, x_1), \ldots, (y_n, x_n) \right] \\
= E(y_1, x_1, \ldots, y_n, x_n) \left( \sum_{i=1}^{n} (y_i - f^*[x_i])^2 \right)^{1/2} \\
\leq \left( E(y_1, x_1, \ldots, y_n, x_n) \left( \sum_{i=1}^{n} (y_i - f^*[x_i])^2 \right) \right)^{1/2} \\
= \left( nE(y, x) \left( y - f^*[x] \right)^2 \right)^{1/2} \\
= \sqrt{n} s_{y|x}.
\]
Combining the inequalities derived in this step yields
\[2E_{(y_1,x_1), \ldots, (y_n,x_n)}[\varepsilon] \leq 16c_h c_F + \frac{8c_h s_{y|x}}{\sqrt{n}},\]

and combining this result with the result of Step 1 then finally gives the desired statement.

**Step 3:** We now show that
\[E_{(y_i,x_i)} \left[ \sup_{g \in F} \left( h[y_i - g[x_i]] - E_{(y,x)} [h[y - g[x]]] \right)^2 \right] \leq (3c_h w_F + 3c_h s_{y|x})^2\]

for all \(i \in \{1, \ldots, n\}\). This bound will be essential for applying a concentration inequality in the following step.

We use elementary tools to connect the left-hand side with the complexity measures in Definition 1. Specifically, we 1. invoke the i.i.d. assumption for the data and the linearity of integrals, 2. use the fact that \(a \leq |a|\), 3. invoke the Lipschitz condition (1) for the loss \(h\), 4. add a zero-valued term, 5. use the triangle inequality and the linearity of integrals, 6. apply dominated convergence and Jensen’s inequality, 7. use \((a+b+c+d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)\) according to Lemma 5 in Section 4.2, the properties of suprema, and the linearity of integrals, 8. use the linearity of integrals and the i.i.d. assumption on the data, 9. invoke Definition 1 of \(w_F\) and \(s_{y|x}\), and finally 10. \(a^2 + b^2 \leq (a+b)^2\) for nonnegative \(a, b\) to find
\[
E_{(y_i,x_i)} \left[ \sup_{g \in F} \left( h[y_i - g[x_i]] - E_{(y,x)} [h[y - g[x]]] \right)^2 \right] \\
= E_{(y_i,x_i)} \left[ \sup_{g \in F} \left( E_{(y,x)} [h[y_i - g[x_i]]] - h[y - g[x]] \right)^2 \right] \\
\leq E_{(y_i,x_i)} \left[ \sup_{g \in F} \left( E_{(y,x)} [h[y_i - g[x_i]]] - h[y - g[x]] \right)^2 \right] \\
\leq E_{(y_i,x_i)} \left[ \sup_{g \in F} \left( E_{(y,x)} [c_h |y_i - g[x_i]| - y + g[x]] \right)^2 \right] \\
= E_{(y_i,x_i)} \left[ \sup_{g \in F} \left( E_{(y,x)} [c_h |y_i - g[x_i]| - f^*[x_i] - f^*[x_i] - f^*[x] - y + g[x]] \right)^2 \right] \\
\leq (c_h)^2 E_{(y_i,x_i)} \left[ \sup_{g \in F} \left( E_{(y,x)} \left[ |g[x_i] - f^*[x_i]| + |g[x] - f^*[x]| + |y_i - f^*[x_i]| + |y - f^*[x]| \right] \right)^2 \right] \\
\leq (c_h)^2 E_{(y,x),(y_i,x_i)} \left[ \sup_{g \in F} \left( |g[x_i] - f^*[x_i]| + |g[x] - f^*[x]| + |y_i - f^*[x_i]| + |y - f^*[x]| \right)^2 \right] \\
\leq 4 (c_h)^2 E_{(y,x),(y_i,x_i)} \left[ \sup_{g \in F} |g[x_i] - f^*[x_i]|^2 + \sup_{g \in F} |g[x] - f^*[x]|^2 + |y_i - f^*[x_i]|^2 + |y - f^*[x]|^2 \right] \\
= 8 (c_h)^2 E_{(y,x)} \left[ \sup_{g \in F} |g[x] - f^*[x]|^2 \right] + 8 (c_h)^2 E_{(y,x)} \left[ |y - f^*[x]|^2 \right] \\
= 8 (c_h)^2 (w_F)^2 + 8 (c_h)^2 (s_{y|x})^2
\]
\[ \leq (3c_h w_F + 3c_h s_{y|x})^2, \]
as desired.

Step 4: We now show that
\[ P_{(y_1, x_1), \ldots, (y_n, x_n)} \left\{ z - 2E_{(y_1, x_1), \ldots, (y_n, x_n)}[z] \geq \frac{228c_h w_F + 228c_h s_{y|x}}{\sqrt{n}t} \right\} \leq t. \]

This deviation inequality controls the remaining term in our bound.

The proof is based on Step 3 and a concentration result by Lederer and van de Geer (2014). The coordinates of the random vectors in Lederer and van de Geer (2014, Section 2) are in our case (with some abuse of notation) \( Z_i[y] := h[y_i - g[x_i]] - E_{(y,x)}[h[y - g[x]]] \). As coordinates of the envelope, we simply take \( E_i := \sup_{g \in F} |h[y_i - g[x_i]] - E_{(y,x)}[h[y - g[x]]]| \). According to Step 3, it holds that \( \sigma \leq M \leq 3c_h w_F + 3c_h s_{y|x} \) for \( p = 2 \)—see their Equation (4).

Hence, Lederer and van de Geer (2014, Corollary 3) yields (with \( \epsilon := 1, l := 1, \) and \( p := 2 \))
\[ P_{(y_1, x_1), \ldots, (y_n, x_n)} \left\{ z - 2E_{(y_1, x_1), \ldots, (y_n, x_n)}[z] \geq v \right\} \leq \frac{72(3c_h w_F + 3c_h s_{y|x})}{\sqrt{n}v} + \frac{4(3c_h w_F + 3c_h s_{y|x})}{\sqrt{n}v} \]
\[ = \frac{228c_h w_F + 228c_h s_{y|x}}{\sqrt{n}v} \]
for all \( v \in (0, \infty) \). Setting \( v := (228c_h w_F + 228c_h s_{y|x})/(\sqrt{n}t) \) then gives the desired result.

Combining Steps 2 and 4 and using that \( t < 1 \) finally yields the bound stated in the theorem.

4.2 An Auxiliary Result

We now state an simple auxilliary result that was used in the above-stated proof of Theorem 2. The result is very standard, but for the sake of completeness, we prove it nevertheless.

Lemma 5 (Binomial) For every \( a, b, c, d \in \mathbb{R} \), it holds that
\[ (a + b + c + d)^2 \leq 4a^2 + 4b^2 + 4c^2 + 4d^2. \]

Proof [of Lemma 5.] Noting that
\[ 2uv = -(u - v)^2 + u^2 + v^2 \leq u^2 + v^2 \]
for all \( u, v \in \mathbb{R} \), we find
\[ (a + b + c + d)^2 \]
\[ = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd \]
\[ \leq a^2 + b^2 + c^2 + d^2 + a^2 + b^2 + a^2 + c^2 + a^2 + d^2 + b^2 + b^2 + d^2 + c^2 + d^2 \]
\[ = 4a^2 + 4b^2 + 4c^2 + 4d^2, \]
as desired.
4.3 Proof of Theorem 4

We finally give a proof for the robust guarantee established in Section 3.

**Proof** [of Theorem 4] We need to control the terms of the right-hand side of the inequality in Corollary 3. Two results that are especially important in our derivations are a Lipschitz property of neural networks developed in Taheri et al. (2020) and a bound for the Rademacher complexity of neural networks developed in Golowich et al. (2020).

**Step 1:** We first show that with probability at least 1 − 2t, it holds that

\[
E_{(y,x)}[h[y - \hat{f}(x)]] \leq E_{(y,x)}[h[y - f^*[x]]] + 16c_h c_F + 237 c_h \frac{w_F + s_{y|x}}{\sqrt{n}t}.
\]

This first step takes care of the empirical loss in the bound of Corollary 3.

The proof of the first step is based on Corollary 3 and Markov’s inequality. We use 1. the definition of \( \hat{f} \) as a risk minimizer in (3), 2. a rearrangement of the terms and the linearity of finite sums, 3. Markov’s inequality (Durrett, 2010, Display (1.6.1) on p. 29), 4. the i.i.d. assumption on the data and the linearity of integrals, 5. a consolidation of the factors, 6. the fact that \( E[(v - E[v])^2] \leq E[v^2] \) and the i.i.d. assumption on the data, 7. the assumption \( h[0] = 0 \) on Page 3, 8. the Lipschitz assumption (1) on the loss \( h \), 9. the linearity of integrals and a consolidation, and 10. Definition 1 of \( s_{y|x} \) and the fact that \( t \in (0,1) \) to find

\[
P_{(y_1,x_1),\ldots,(y_n,x_n)} \left\{ \frac{1}{n} \sum_{i=1}^{n} h[y_i - \hat{f}(x_i)] \right\} \geq E_{(y,x)}[h[y - f^*[x]]] + \frac{c_h s_{y|x}}{\sqrt{n}t}
\]

\[
\leq P_{(y_1,x_1),\ldots,(y_n,x_n)} \left\{ \frac{1}{n} \sum_{i=1}^{n} h[y_i - f^*[x_i]] \right\} \geq E_{(y,x)}[h[y - f^*[x]]] + \frac{c_h s_{y|x}}{\sqrt{n}t}
\]

\[
= P_{(y_1,x_1),\ldots,(y_n,x_n)} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( h[y_i - f^*[x_i]] - E_{(y,x)}[h[y - f^*[x]]] \right) \right\} \geq \frac{c_h s_{y|x}}{\sqrt{n}t}
\]

\[
E_{(y_1,x_1),\ldots,(y_n,x_n)} \left\{ \frac{\sum_{i=1}^{n} h[y_i - f^*[x_i]] - E_{(y,x)}[h[y - f^*[x]]]}{\sqrt{n}} \right\}^2 / n^2
\]

\[
= \frac{E_{(y_1,x_1)} \left[ h[y_1 - f^*[x_1]] - E_{(y,x)}[h[y - f^*[x]]] \right]^2 / n}{(c_h s_{y|x}/\sqrt{n})^2}
\]

\[
= \frac{E_{(y_1,x_1)} \left[ h[y_1 - f^*[x_1]] - E_{(y,x)}[h[y - f^*[x]]] \right]^2}{(c_h)^2 (s_{y|x})^2} \cdot t^2
\]

\[
\leq \frac{E_{(y,x)} \left[ h[y - f^*[x]] \right]^2}{(c_h)^2 (s_{y|x})^2} \cdot t^2
\]

\[
eq \frac{E_{(y,x)} \left[ h[y - f^*[x]] - h[0] \right]^2}{(c_h)^2 (s_{y|x})^2} \cdot t^2
\]
We can then conclude by plugging this result into Corollary 3.

**Step 2:** We now show that with probability at least $1 - 2t$, it holds that

$$E(y, x) \left[ b \left[ y - f^* [x] \right] \right] \leq E(y, x) \left[ b \left[ y - f^* [x] \right] \right] + 48(b_M)^{l+1} c_h \sqrt{l + 1} \frac{s_x}{\sqrt{n}} + 237 c_h \frac{w_F + s_y |x|}{\sqrt{nt}}.$$  

This step takes care of the Rademacher complexity.

The basis for the proof is a bound for the Rademacher complexity of neural networks from Golowich et al. (2020). Indeed, we use 1. Golowich et al. (2020, Theorem 1), 2. the linearity of integrals, 3. Jensen’s inequality, 4. the linearity of integrals, 5. the i.i.d. assumption on the data, and 6. Definition 1 of $s_x$ to find

$$c_F \leq E(y_1, x_1), ..., (y_n, x_n) \left[ \frac{3(b_M)^{l+1} \sqrt{l + 1}}{\sqrt{n}} \frac{1}{n} \sum_{i=1}^{n} |x_i|_2 \right]$$

$$= \frac{3(b_M)^{l+1} \sqrt{l + 1}}{\sqrt{n}} E(y_1, x_1), ..., (y_n, x_n) \left[ \frac{1}{n} \sum_{i=1}^{n} |x_i|_2 \right]$$

$$\leq \frac{3(b_M)^{l+1} \sqrt{l + 1}}{\sqrt{n}} \left[ \frac{1}{n} \sum_{i=1}^{n} E(y_1, x_1), ..., (y_n, x_n) \left[ |x_i|_2 \right] \right]$$

$$= \frac{3(b_M)^{l+1} \sqrt{l + 1}}{\sqrt{n}} \left[ \frac{1}{n} \sum_{i=1}^{n} E(y, x) \left[ |x|_2 \right] \right]$$

$$= \frac{3(b_M)^{l+1} \sqrt{l + 1} s_x}{\sqrt{n}}.$$  

We can then conclude by plugging this inequality into the result of Step 1.

**Step 3:** We now show that with probability at least $1 - 2t$, it holds that

$$E(y, x) \left[ b \left[ y - \hat{f} [x] \right] \right] \leq E(y, x) \left[ b \left[ y - f^* [x] \right] \right] + 48(b_M)^{l+1} c_h \sqrt{l + 1} \frac{s_x}{\sqrt{n}} + 948 c_h \frac{(b_M)^{l+1} l s_x + s_y |x|}{\sqrt{nt}}.$$  

This step takes care of the size of the envelope. (We do not attempt to optimize constants anywhere in our proofs.)

The key idea here is to apply a Lipschitz property of neural networks derived in Taheri et al. (2020). We use 1. Definition 1 of $w_F$, 2. the specification of the set $\mathcal{F}$ on Page 4 and the
assumption that on \( M \) on Page 4, 3. (Taheri et al., 2020, Proposition 2) and the definition of the set \( M \) on Page 4, 4. the fact that \((u - v)^2 \leq 2u^2 + 2v^2\), 5. again the definition of \( M \), 6. a consolidation and the linearity of integrals, and 7. Definition 1 of \( s_x \) to find

\[
(w_{F})^2 = E_{(y,x)} \left[ \sup_{f \in F} |f[x] - f^*|_2^2 \right]
\]

\[
\leq E_{(y,x)} \left[ \sup_{\Theta, \Gamma \in M} |f_{\Theta}[x] - f_{\Gamma}[x]|_2^2 \right]
\]

\[
\leq E_{(y,x)} \left[ \sup_{\Theta, \Gamma \in M} \left\{ 4(b_M)^2 l \|x\|_2^2 \sum_{j=0}^{l} \|\Theta^j - \Gamma^j\|_F^2 \right\} \right]
\]

\[
\leq E_{(y,x)} \left[ 4(b_M)^2 l \|x\|_2^2 \left( 2 \sum_{j=0}^{l} \|\Theta^j\|_F^2 + 2 \sum_{j=0}^{l} \|\Gamma^j\|_F^2 \right) \right]
\]

\[
= 16(b_M)^2 l^2 E_{(y,x)} \left[ \|x\|_2^2 \right]
\]

\[
= 16(b_M)^2 l^2 (s_x)^2 ,
\]

and, hence, \( w_F \leq 4(b_M)^{l+1} l s_x \). We can then conclude by putting this result back into the result of Step 2.

**Step 4:** The first inequality in Theorem 4 finally follows from consolidating the result of Step 3 and using the fact that \( t \in (0, 1) \).

The second inequality follows from the first one and this derivation:

\[
E_{(y,x)} \left[ h[y - f^*[x]] \right]
\]

\[
\leq E_{(y,x)} \left[ |h[y - f^*[x]] - h[0]| \right]
\]

\[
\leq E_{(y,x)} \left[ c_h |y - f^*[x] - 0| \right]
\]

\[
= c_h E_{(y,x)} \left[ |y - f^*[x]| \right]
\]

\[
\leq c_h \sqrt{E_{(y,x)} \left[ |y - f^*[x]|^2 \right]}
\]

\[
= c_h s_{y|x} ,
\]

where we use similar techniques as in the other parts of the proof.

\[\blacksquare\]

### 5. Discussion

Our statistical guarantees show that replacing the standard least-squares loss with a Lipschitz-continuous loss renders weight decay an effective method for regression for a broad spectrum of data. This spectrum includes benign data (such as sub-Gaussian or bounded data) but also corrupted data (having outliers that are caused by an adversary or by other means).
More generally, our results provide theoretical support for the use of robust loss functions in deep learning.

We have formulated our bounds for weight decay, because it is arguably the most popular type of regularization in view of its ability to avoid overfitting and accelerate computations (Krizhevsky et al., 2012). But one can easily transfer our derivations to other types of regularization—as long as there are appropriate bounds for the Rademacher complexities.

Some robust loss functions, such as Huber and Cauchy loss, involve an additional parameter: see Figure 1. Ideas for how to calibrate this parameter in practice can be found in Chichignoud and Lederer (2014) and Loh (2018).

It is straightforward to generalize our results from empirical-risk minimizers to approximate empirical-risk minimizers. Such generalizations take into account that minimizers can rarely be computed exactly. But our theories do not apply to local minima: this is a limitation that our paper has in common with most statistical literature on deep learning.

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