ON GERBE DUALITY
AND RELATIVE GROMOV-WITTEN THEORY

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ABSTRACT. We formulate and study an extension of gerbe duality to relative Gromov-Witten theory.

1. Introduction

In this short note, we propose a conjecture about relative Gromov-Witten theory on a general noneffective Deligne-Mumford stack.

Throughout this note, we work over \( \mathbb{C} \). Let \( G \) be a finite group, \( B \) a smooth (proper) Deligne-Mumford stack, and
\[
\pi : \mathcal{Y} \to B
\]
a \( G \)-gerbe. In [10], the dual of \( \pi : \mathcal{Y} \to B \) is a pair \((\hat{\mathcal{Y}}, c)\) where \( \hat{\mathcal{Y}} \) is a disconnected stack with an étale map
\[
\hat{\pi} : \hat{\mathcal{Y}} = \coprod_i \hat{\mathcal{Y}}_i \to B
\]
and \( c \) is a \( \mathbb{C}^* \)-valued 2-cocycle on \( \hat{\mathcal{Y}} \).

We briefly recall the construction of \( \hat{\mathcal{Y}} \) and \( c \), and refer the readers to [14] for the detail. We focus on describing \( \mathcal{Y} \) locally. A chart of \( B \) looks like a quotient of \( \mathbb{C}^n \) by a finite group \( Q \) acting linearly on \( \mathbb{C}^n \). And a \( G \)-gerbe over \([\mathbb{C}^n/Q]\) can be described as follows. There is a group extension
\[
1 \to G \to H \to Q \to 1.
\]
The group \( H \) acts on \( \mathbb{C}^n \) via its homomorphism to \( Q \). Locally a chart of \( \mathcal{Y} \) over \( B \) looks like \([\mathbb{C}^n/H] \to [\mathbb{C}^n/Q]\).

To construct the dual \( \hat{\mathcal{Y}} \), we consider the space \( \hat{G} \), the (finite) set of isomorphism classes of irreducible \( G \)-representations. As \( G \) is a normal subgroup of \( H \), \( H \) acts on \( G \) by conjugation, which naturally gives an \( H \) action on \( \hat{G} \). Furthermore, as \( G \) acts its by conjugation, the \( G \) action on \( \hat{G} \) is trivial. Therefore, the quotient group \( Q \) acts on \( \hat{G} \). The dual space \( \hat{\mathcal{Y}} \) locally looks like \([\hat{G} \times \mathbb{C}^n]/Q\). We notice that the dual \( \hat{\mathcal{Y}} \) has a canonical map to the quotient \( \hat{G}/Q \), and therefore is a disjoint union of stacks over \( \hat{G}/Q \).

The construction of \( c \) is from the Clifford theory of induced representations [7]. Given an irreducible \( G \)-representation \( \rho \) on \( V_\rho \), we want to introduce an \( H \) representation. Let \([\rho]\) be the corresponding point in \( \hat{G} \). Recall that the finite group \( Q \) acts on \( \hat{G} \). We denote \( Q_{[\rho]} \) to
be the stabilizer group of $Q$ action on $[\hat{G}]$ at the point $[\rho]$. For any $q \in Q_{[\rho]}$, define $q(\rho)$, a representation of $G$ on $V^\rho$, by
\[ q(\rho)(g) = \rho(q^{-1}(g)). \]
As $q$ fixes $[\rho]$ in $\hat{G}$, $q(\rho)$, as a $G$-representation, is equivalent to $\rho$. Therefore, there is an intertwining operator $T^\rho_q$ on $V^\rho$ such that
\[ T^\rho_q \rho = q(\rho) T^\rho_q. \]
In general, the operators $\{T^\rho_q\}_{q \in Q_{[\rho]}}$ fail to satisfy
\[ T^\rho_q \circ T^\rho_{q'} = T^\rho_{qq'}. \]
By Schur’s lemma, we can check that there is a number $c^{[\rho]}(q, q') \in \mathbb{C}^*$, such that
\[ T^\rho_q \circ T^\rho_{q'} = c^{[\rho]}(q, q') T^\rho_{qq'}. \]
In \cite{14}, we explained that this function $c^{[\rho]}(q, q')$ glues to a globally defined $\mathbb{C}^*$-gerbe over the dual $\hat{\mathcal{Y}}$.

The authors of \cite{10} propose a gerbe duality principle which suggests that there is an equivalence between the geometry of $\mathcal{Y}$ and that of the pair $(\hat{\mathcal{Y}}, c)$. Several aspects of such an equivalence have been proven in \cite{14}. In \cite{14} Conjecture 1.8, the following conjecture is also explicitly formulated:

**Conjecture 1.1.** As generating functions, the genus $g$ Gromov-Witten theory of $\mathcal{Y}$ is equal to the genus $g$ Gromov-Witten theory\footnote{It is also called $c$-twisted Gromov-Witten theory of $\hat{\mathcal{Y}}$.} of $(\hat{\mathcal{Y}}, c)$,
\[ GW_g(\mathcal{Y}) = GW_g(\hat{\mathcal{Y}}, c). \]

Conjecture 1.1 has been proven in increasing generalities, see \cite{3}, \cite{4}, \cite{5}, and \cite{15}. In particular, we have obtained the following theorem.

**Theorem 1.2.** (\cite{15} Theorem 1.1) When $\mathcal{Y}$ is a banded $G$-gerbe over $\mathcal{B}$, Conjecture 1.1 holds true.

A toric Deligne-Mumford stack $\mathcal{Y}$ is a banded gerbe over an effective DM stack $\mathcal{B}$, see e.g. \cite{9}. As a corollary to Theorem 1.2, we can compute the Gromov-Witten theory of $\mathcal{Y}$ in terms of the (twisted) Gromov-Witten theory of the dual toric DM stack $\hat{\mathcal{Y}}$.

From the perspective of Gromov-Witten theory, Gromov-Witten theory relative to a divisor is important. Therefore it is natural to consider an extension of Conjecture 1.1 to the relative setting. Let
\[ D \subset \mathcal{B} \]
be a smooth (irreducible) divisor. The inverse images
\[ \mathcal{D} := \pi^{-1}(D) \subset \mathcal{Y}, \quad \hat{\mathcal{D}} := \hat{\pi}^{-1}(D) \subset \hat{\mathcal{Y}} \]
are smooth divisors. The $c$-twisted Gromov-Witten theory of the relative pair $(\hat{\mathcal{Y}}, \hat{\mathcal{D}})$ can be defined using the construction of \cite{2}, \cite{11}, \cite{12}, and \cite{13}. A natural extension of Conjecture 1.1 is the following.
**Conjecture 1.3.** As generating functions, the genus $g$ Gromov-Witten theory of $(\mathcal{Y}, \mathcal{D})$ is equal to the genus $g$ Gromov-Witten theory of $((\hat{\mathcal{Y}}, \hat{\mathcal{D}}), c)$. Symbolically, 

$$GW_g(\mathcal{Y}, \mathcal{D}) = GW_g((\hat{\mathcal{Y}}, \hat{\mathcal{D}}), c).$$

The purpose of this note is to present some evidence to Conjecture 1.3.

2. Evidence to Conjecture 1.3

By taking the divisor $D$ to be empty, we see that Conjecture 1.3 implies Conjecture 1.1.

Next, we explain how to derive Conjecture 1.3 from the full strength of Conjecture 1.1.

Let $r \geq 1$. We consider the stack of $r$-th roots of $Y$ along $\mathcal{D}$, denoted by $Y_{D,r}$.

Consider also the stack of $r$-th roots of $B$ along $\mathcal{D}$, denoted by $B_{D,r}$.

Let $\rho : B_{D,r} \to B$ be the natural map. Consider the Cartesian diagram:

$$
\begin{array}{ccc}
\rho^*Y & \longrightarrow & Y \\
\downarrow & & \downarrow \pi \\
B_{D,r} & \rho & B.
\end{array}
$$

The pull-back $\rho^*Y \to B_{D,r}$ is a $G$-gerbe. By functoriality property of root constructions, there is a natural map $Y_{D,r} \to \rho^*Y$, which is an isomorphism. Therefore we have

(2.1) 

$$GW_g(Y_{D,r}) = GW_g(\rho^*Y).$$

Since $\rho^*Y \to B_{D,r}$ is a $G$-gerbe, applying Conjecture 1.1, we have

(2.2) 

$$GW_g(\rho^*Y) = GW_g(\hat{\rho}^*\hat{Y}, c'),$$

where $(\hat{\rho}^*\hat{Y}, c')$ is the dual pair of the gerbe $\rho^*Y$. By construction of the dual pair, we have $\hat{\rho}^*\hat{Y} = \rho^*\hat{Y}$ and $c' = \rho^*c$. Here $\rho^*\hat{Y}$ fits in the Cartesian diagram:

$$
\begin{array}{ccc}
\rho^*\hat{Y} & \longrightarrow & \hat{Y} \\
\downarrow & & \downarrow \# \\
B_{D,r} & \rho & B.
\end{array}
$$

By functoriality property of root constructions, we have $\rho^*\hat{Y} \simeq \hat{Y}_{D,r}$. Therefore

(2.3) 

$$GW_g(\rho^*\hat{Y}, c') = GW_g(\hat{Y}_{D,r}, \rho^*c).$$
Combining (2.1)–(2.3), we have

\[(2.4) \quad GW_g(Y_{D,r}) = GW_g(\hat{Y}_{\hat{D},r}, \rho^*c).\]

The left hand side of (2.4), \(GW_g(Y_{D,r})\), is a polynomial in \(r\) for \(r\) large. Furthermore, taking the \(r^0\)-coefficient, we obtain the relative Gromov-Witten invariants:

\[(2.5) \quad \text{Coeff}_{r^0}GW_g(Y_{D,r}) = GW_g(Y, D).\]

This follows from the arguments of [16], suitably extended to the setting of Deligne-Mumford stacks, see [17], [6].

The right hand side of (2.4), \(GW_g(\hat{Y}_{\hat{D},r}, \rho^*c)\), is also a polynomial in \(r\) for \(r\) large. Taking the \(r^0\)-coefficient, we obtain the relative Gromov-Witten invariants:

\[(2.6) \quad \text{Coeff}_{r^0}GW_g(\hat{Y}_{\hat{D},r}, \rho^*c) = GW_g((\hat{Y}, \hat{D}), c).\]

This again follows from a suitable extension of [16] as in [17], [6]. Note that the \(\rho^*c\)-twist plays no role in applying the arguments of [16], as they are essentially done at the level of virtual cycles (see [8]), while \(\rho^*c\)-twist takes place at insertions.

Combining (2.5) and (2.6), we arrive at Conjecture 1.3.

**Remark 2.1.** In genus 0, the argument about polynomiality in \(r\) can be replaced by (a suitable extension of) the arguments of [1].

The following result follows directly from Theorem 1.2 and the above discussion.

**Theorem 2.1.** When \(\mathcal{Y}\) is a banded \(G\)-gerbe over \(\mathcal{B}\), as generating functions, the genus \(g\) Gromov-Witten theory of \((\mathcal{Y}, D)\) is equal to the genus \(g\) Gromov-Witten theory of \((\hat{\mathcal{Y}}, \hat{D}), c)\).

Symbolically,

\[GW_g(\mathcal{Y}, D) = GW_g((\hat{\mathcal{Y}}, \hat{D}), c).\]

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**References**

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