In this paper the problem of constructing spacetime from string theory is addressed in the context of D–brane physics. It is suggested that the knowledge of discrete configurations of D–branes is sufficient to reconstruct the motivic building blocks of certain Calabi-Yau varieties. The collections of D–branes involved have algebraic base points, leading to the notion of $K$–arithmetic D–crystals for algebraic number fields $K$. This idea can be tested for D0–branes in the framework of toroidal compactifications via the conjectures of Birch and Swinnerton-Dyer. For the special class of D0–crystals of Heegner type these conjectures can be interpreted as formulae that relate the canonical Néron-Tate height of the base points of the D–crystals to special values of the motivic L–function at the central point. In simple cases the knowledge of the D–crystals of Heegner type suffices to uniquely determine the geometry.
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1 Introduction

It has become apparent over the past few years that motivic $L-$functions encode interesting physical information. In the context of Calabi-Yau manifolds and their natural generalizations there is a distinguished $L-$function, associated to the $\Omega-$motive of a variety $X$, that has been shown to be modular in examples that cover all physically relevant dimensions. In these instances the $L-$function leads to modular forms which can be expressed in terms of modular forms of the corresponding conformal field theory on the worldsheet, leading to relations of the type

$$f_\Omega(X, q) = \prod_i \Theta^{k_i}_{\ell_i, m_i}(q^{\alpha_i}) \otimes \chi_K.$$  \hspace{1cm} (1)

Here the Mellin transform of the cusp form $f_\Omega(X, q) \in S_w(\Gamma_0(N), \epsilon_N)$ is the $L-$function of the $\Omega-$motive $M_\Omega$ of $X$, the functions $\Theta^{k_i}_{\ell_i, m_i}(q)$, with $q = e^{2\pi i \tau}$, are Hecke indefinite modular forms, and $\chi_K$ is the quadratic character associated to a number field determined by the conformal field theory. These results show that the arithmetic information of an exactly solvable variety carries the essential information contained in the worldsheet model. Several examples of this type have been constructed in different dimensions in [1, 2, 3] for diagonal varieties, and ref. [4] for families of manifolds. Viewing this relation as a map from the worldsheet conformal field theory $\text{CFT}_\Sigma$ to the Calabi-Yau variety $X$ gives meaning to the notion of an emergent spacetime in string theory as a direct construction $\text{CFT}_\Sigma \longrightarrow X$ via automorphic motives.

It is natural to ask whether motivic $L-$functions contain information about the string vacuum other than the worldsheet modular forms, in particular whether there are physical objects in spacetime that are described by the $L-$function. The purpose of this paper is to address this question by showing that the $L-$function contains information about particular types of D$-$branes on the compactification variety, and that these D$-$branes in turn contain enough information to identify the basic building blocks of the compact manifold. Roughly speaking, the $L-$function measures the detailed structure of D$-$branes via the canonical height of the D$-$branes. This leads to a notion of D$-$brane multiplicities, and in the process leads to a direct spacetime interpretation of the $L-$function.
The concrete definition of the ‘arithmetic’ D–branes adopted here is motivated by the results of [5], where it was shown that certain algebraic number fields have both a geometric and a conformal field theory interpretation. This suggests that a natural probe for the structure of spacetime is provided by D–branes that are defined over complex algebraic number fields $K$, i.e. finite extensions of the rational numbers $\mathbb{Q}$. In this framework D–branes that wrap around cycles $C$ of the manifold are viewed as objects that are defined over the extensions $K/\mathbb{Q}$. This construction should not necessarily be viewed as a replacement of the archimedean fields common in physics, but instead should be interpreted as a tool to probe the domain of D–branes in various ways, depending on the nature of the field $K$. The structure of $C$–wrapped D–brane defined over $K$ will be denoted by $\mathcal{D}_K(C)$ and called $K$–rational D–crystals. More generally D–crystals are associated to all arithmetic cycles, denoted by $\mathcal{D}_K(C)$, where $C$ is the set of all cycles.

The simplest framework in which the idea of relating D–brane probes to L-functions can be made precise and tested is given by D0–branes in toroidal compactifications. In this case the moduli space $\mathcal{M}_{D0}(X)$ of D0–branes on a variety $X$ is the manifold $X$ itself, $\mathcal{M}_{D0}(X) = X$. Put differently, in the case of D0–branes there are no non-trivial cycles that can be wrapped and in the arithmetic setting the base points of the D0–branes can be located at all $K$–rational points of the variety, leading to $\mathcal{D}_K(X) = X(K)$. In the general the connection between the $K$–arithmetic D–crystals and the $L$–function is based on one of the central themes in arithmetic geometry, the conjecture of Birch and Swinnerton-Dyer and its generalizations, the Bloch-Kato conjecture. For D0–crystals on elliptic curves the key measure with which they can be characterized in a numerically precise way is given by the Néron-Tate canonical height. The conjecture of Birch and Swinnerton-Dyer (BSD conjecture [6, 7]), relates this height, in combination with other factors, to the motivic $L$–function evaluated at certain critical points. Very roughly, the conjectured relation

$$L^{(r)}(E, 1) = b_E \Omega_E R_E$$

links the Taylor coefficient $L^{(r)}(E, s)$ of the elliptic curve $E$ at a special point $s = 1$ to the regulator $R_E$ constructed from the height function and the period $\Omega_E$. Here $b_E$ is a rational factor which will be made explicit later in this paper. The $L$–function in turn can be used
to reconstruct (motives of) the compactification manifold, and determine string worldsheet modular forms. In this way it is possible to determine the structure of spacetime via D0-brane physics.

For arbitrary complex number fields $K$ it is difficult to determine the structure of the base points of the associated D-crystals because no constructive procedure is known at present. This, in combination with the results of [5], motivates the consideration of a subclass of $K$-algebraic D-crystals whose base points take values in number fields that are more general than the rational numbers, but special enough to be under control. This class of fields is given by the sequence of imaginary quadratic fields $K_D := \mathbb{Q}(\sqrt{-D})$ of discriminant $-D$.

For elliptic curves over $K_D$ a BSD type theorem has been proven by Gross-Zagier [8]. Their results can be interpreted as providing an explicit construction of $K_D$-rational D0-branes via the notion of Heegner points $P_D$ associated to certain imaginary quadratic extensions $K_D$. For elliptic curves of rank one over $K_D$ the Néron-Tate height $\hat{h}(P_K)$ of these objects leads to the derivative of the $L$-function of the curve $E/K$ defined over $K$

$$L'(E/K, 1) = c_{E,K} \hat{h}(P_K),$$

where $c_{E,K}$ is a number that depends on the geometry of $E$ and on $K$. The relation between Heegner type D-branes and $L$-functions established in this way is completely general, independent of any knowledge of a worldsheet interpretation of the motivic $L$-function. It therefore allows a motivic analysis via D-branes for families of varieties, independently of the conformal field theoretic structure. Using results from Faltings then makes it possible to construct the curve $E$ up to isogeny from the inverse Mellin transform of the $L$-function $L(E, s)$.

The outline of this paper is as follows. Sections 2 and 3 contain an outline of the conjectures of Birch and Swinnerton-Dyer as well as a discussion of arithmetic D-branes. Section 4 explains the notion of Heegner type D0-branes and Section 5 describes how they lead to motivic $L$-functions that allow to reconstruct the compactification geometry. Section 6 describes how Heegner type D-crystals determine the compactification variety and Section 7 contains two examples, one to illustrate the constructions in the context of an exactly solvable elliptic
curve, the second to illustrate that the ideas here are more general than the framework of exactly solvable compactifications.

\section{Rational D–crystals and $L$–functions}

The idea that some special sub-class of D–branes should encode sufficient information to reconstruct the (motivic) structure of spacetime leads to the question whether some particular class of D–branes can be related to motivic $L$–functions derived from a variety. In this section this problem is addressed in the simplest possible context, provided D0–branes on a toroidal manifold. This provides a framework that is quite challenging mathematically, with many open conjectures, but simple enough to contain some proven results. It is also a good starting point because it avoids the complications introduced in higher dimensions by the necessity to consider motives. In the elliptic case the situation is simpler because the question raised translates into the problem of whether there exists a relation between the Hasse-Weil $L$–function $L(E, s)$ of an elliptic curve and particular types of points on the variety which define the base points of the D–branes. When considering possible types of D0–brane base points as candidates it is natural to ask whether base points given by rational points, or some subset thereof, provide sufficiently sensitive probes of the spacetime manifold. This question motivates a closer investigation of the relation between rational points on elliptic curves and $L$–functions.

\subsection{Rational D–crystals on tori}

The problem of understanding the rational points on abelian varieties in general, and of elliptic curves in particular, is a quite difficult one that has a long history. In the context of elliptic curves the first structural result is the theorem of Mordell [9], which says that the group of rational points $E(\mathbb{Q})$ is finitely generated as

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \times E(\mathbb{Q})_{\text{tor}},$$

(4)
where $r$ is called the rank of the group (or elliptic curve) and $E(\mathbb{Q})_{\text{tor}}$ denotes the points of finite order. The torsion points can be determined algorithmically and the possible groups are known by a theorem of Mazur [10], which proves that a previously established list of groups is complete [11]. No such algorithm is known for the free part of the Mordell-Weil group. It is in particular not known whether the rank is bounded.

A connection between the structure of the Mordell-Weil group and the $L$–function has not been proven in full generality at this point, but partial results are known, confirming a rather detailed picture of such a relation. The precise form of this link involves several other quantities that characterize the arithmetic structure of the curve and is formulated in two deep conjectures by Birch and Swinnerton-Dyer that basically relate the Taylor series structure of the $L$–series at the critical point to numerical information associated to the Mordell-Weil group, mediated by further arithmetic information [6, 7].

Geometric $L$–functions can be viewed as tools that characterize the fine structure of a manifold. In the case of an elliptic curve $E$ of discriminant $\Delta$ the $L$–function can be defined via the Euler product

$$L(E, s) = \prod_{p \mid \Delta} (1 - a_p p^{-s})^{-1} \prod_{p \nmid \Delta} (1 - a_p p^{-s} + p^{1-2s})^{-1},$$  

(5)

where the coefficients $a_p$ are given by $a_p = p + 1 - N_p$, where $N_p = \#(E(\mathbb{F}_p)$ is the number of points of $E$ over the finite field $\mathbb{F}_p$. A bound determined by Hasse for the coefficients shows that this function converges in the half-plane $\text{Re}(s) > 3/2$. In order to formulate the conjectures of Birch and Swinnerton-Dyer it is necessary to evaluate the $L$–function at $s = 1$. The resulting problem of analytic continuation was solved by Wiles et al. [12, 13] in the context of the proof of the Shimura-Taniyama-Weil conjecture. According to the resulting elliptic modularity theorem there exists a modular form $f \in S_2(\Gamma_0(N))$ such that the $L$–function is essentially given by the Mellin transform, more precisely

$$\int_0^\infty f(iy)y^{s-1}dy = (2\pi)^{-s}\Gamma(s)L(f, s).$$  

(6)

It was shown by Carayol that $f$ is a newform and an eigenfunction of the Fricke involution

$$f(-1/N\tau) = \lambda N\tau^2 f(\tau),$$  

(7)
where $\lambda = \pm 1$. This transformation behavior motivates the introduction of a renormalized completed $L$–function as

$$L^*(f, s) := N^{s/2} \frac{\Gamma(s)}{(2\pi)^s} L(f, s),$$

which satisfies the functional equation

$$L^*(f, 2 - s) = \epsilon L^*(f, s),$$

where $\epsilon = -\lambda \in \{\pm 1\}$. This functional equation shows that the point $s = 1$ is of particular significance since $L^*(f, 1) = \epsilon L^*(f, 1)$. Thus if $\epsilon = -1$ the $L$–function necessarily vanishes.

### 2.2 The Birch–Swinnerton-Dyer conjectures

The idea described above of probing the spacetime geometry by D–branes and relating their structure to the worldsheet physics via their associated $L$–functions translates into the question whether the base points of the D–crystals can be related to the motivic $L$–function. The rational positioning of the D–crystal leads to the Mordell-Weil group, hence the $\mathbb{Q}$–rational base points lead to the conjectures of Birch and Swinnerton-Dyer.

The difficulties that arise in the problem of understanding the Mordell-Weil group $E(\mathbb{Q})$ led Birch and Swinnerton-Dyer in the 1960s to perform extensive computer computations that suggested relations between the $L$–function and certain other characteristics that encode the arithmetic structure of elliptic curves. The first part of these conjectures [6, 7] relates the rank $\text{rk } E(\mathbb{Q})$ of the Mordell-Weil group $E(\mathbb{Q})$ to the vanishing order $\text{ord}_{s=1} L(E, s)$ of the $L$–series $L(E, s)$ at the central critical point $s = 1$. This is one of the Millenium Prize Problem established by the Clay Mathematics Institute [14].

**BSD rank conjecture.**

$$\text{BSD}_{\mathbb{Q}I} : \text{ord}_{s=1} L(E, s) = \text{rk } E(\mathbb{Q}).$$

An intuitive observation concerning the structure of the $L$–function motivates this conjecture. If one were to evaluate the $L$–function (5) at $s = 1$, the second factor, denoted by $\tilde{L}(E, s)$,
takes the form

\[ \bar{L}(E, 1) = \prod_{p \mid N} \frac{p}{p - a_p + 1}. \]  

(11)

With \( a_p = p + 1 - N_p(E) \), where \( N_p(E) := \#E/F_p \) is the cardinality of the elliptic curve over the finite field \( F_p \), this leads to

\[ \bar{L}(E, 1) = \prod_{p \mid N} \frac{p}{N_p(E)}. \]  

(12)

The expectation now is that if the Mordell-Weil group \( E(\mathbb{Q}) \) is large then there are many points that can be obtained via reduction by \( p \) from the rational points. If many of the \( N_p(E) \) are very large it is reasonable to expect \( L(E, 1) \) given by (12) to vanish.

For elliptic curves with vanishing order \( \leq 1 \) the rank conjecture is known to hold. The arithmetic structure of elliptic curves however is much richer than just its rank and it is natural to ask whether the coefficients in the Taylor expansion of the \( L \)–function admit a geometric interpretation as well. The vision of Birch and Swinnerton-Dyer, guided by their experimental results, led to the conclusion that the \( L \)–function does in fact contain highly nontrivial information about the geometry. More precisely, the rank conjecture suggests to consider the Taylor expansion of the \( L \)–function around the critical point. If the rank of the Mordell-Weil group is abbreviated by \( r = \text{rk} E(\mathbb{Q}) \) this expansion takes the form

\[ L(E, s) = d_r (s - 1)^r + \cdots \]  

(13)

for some value \( d_r \). The question then becomes whether one find a formula for \( d_r \) in terms of the geometry of \( E \). This leads to the second part of the BSD conjectures, which can be formulated as follows.

**BSD Taylor conjecture.**

\[ \text{BSD}_{\mathbb{Q} \Pi} : \lim_{s \to 1} \frac{L(E, s)}{(s - 1)^r} = c_T \cdot \Omega \frac{|\Sha|}{|E(\mathbb{Q})_{\text{tor}}|^2} R, \]  

(14)

where \( c_T = \prod_p c_p \) is the product of the Tamagawa numbers, which are determined by the behavior of the curve at the bad primes as \( c_p = [E(\mathbb{Q}_p) : E_0(\mathbb{Q}_p)] \), with \( E_0(\mathbb{Q}_p) \) the set of
points that reduce to smooth points over $\mathbb{F}_p$. If $E$ has good reduction at $p$ then $c_p = 1$, hence $c_T$ is determined by a finite number of primes

$$c^T = \prod_{p \mid \Delta(E)} c_p,$$

(15)

where $\Delta(E)$ is the discriminant of $E$. The period $\Omega$ of an elliptic curve given via its generalized Weierstrass equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

(16)
is defined via the Néron differential

$$\omega = \frac{dx}{2y + a_1 x + a_3}$$

(17)
as

$$\Omega = \int_{E(\mathbb{R})} \omega.$$ 

(18)

Very little is known about the Tate-Shafarevich group $\Sha$. It is conjectured to be finite, but can be arbitrarily large. It can be shown [15] that for some fixed $c > 0$ and infinitely many $N \in \mathbb{N}$ there exists an elliptic curve such that $\Sha(E) \gg N^{c/\log \log N}$. This group is a torsion group which measures the failure of the Hasse principle, i.e. it detects whether a variety admits local points but no global points. It can be defined in terms of cohomology groups associated to the absolute Galois group $\text{Gal}(\overline{K}/K)$ of any number field $K$. With the short-hand notation $H^1(K, E) = H^1(\text{Gal}(\overline{K}/K), E)$ the Tate-Shafarevich group is given by

$$\Sha(E/\mathbb{Q}) = \ker \left( H^1(\mathbb{Q}, E) \to \prod_v H^1(\mathbb{Q}_v, E) \right),$$

(19)

where $\mathbb{Q}_v$ is the completion of $\mathbb{Q}$ at the place $v$. It turns out that this group is the most difficult part to compute and much effort has gone into attempts to develop algorithms that allow to determine it. For elliptic curves of rank $\leq 1$ this group is known to be finite, but nothing is known for higher rank curves. While mathematically difficult, the Shafarevich-Tate group is in many examples either trivial or small, hence it is not a particularly sensitive characteristic of elliptic curves and will not play an essential role in this paper.

It will become clear further below that one of the most relevant quantities in the present discussion is the regulator $R_E$ of the elliptic curve. This quantity is defined as the determinant
of a height pairing defined in terms of the Néron-Tate height of the generators $P_i$ defining a basis of the Mordell-Weil group of $E$. More precisely, the Néron-Tate canonical height of a rational point is defined in terms of the logarithmic height $h(P)$ of a point $P = (x, y) \in E$. Writing $x = r/s$ where $r, s \in \mathbb{Z}$ have no common factor, the latter is defined as

$$h(P) = \log \max\{|r|, |s|\},$$

leading to the definition of the Néron-Tate height as

$$\hat{h}(P) = \lim_{n \to \infty} \frac{h(2^n P)}{4^n}.$$  \hspace{1cm} (21)

This function satisfies the homogeneity condition $\hat{h}(nP) = n^2 \hat{h}(P)$ and it vanishes if and only if the point $P$ is a torsion point. The Néron-Tate height in turn can be used to define the height pairing

$$\langle P, Q \rangle = \frac{1}{2} \left( \hat{h}(P + Q) - \hat{h}(P) - \hat{h}(Q) \right)$$

which defines a nondegenerate real quadratic form on the free part of the Mordell-Weil group $E(\mathbb{Q})/E(\mathbb{Q})_{\text{tor}}$ because of the triviality of the Néron-Tate height on the torsion part. The regulator $R_E$, finally, is defined as

$$R_E = \det(\langle P_i, P_j \rangle)_{i,j=1,\ldots,\text{rk}(E)}.$$  \hspace{1cm} (23)

An intuitive physical interpretation of the logarithmic height is in terms of the complexity of D0–brane in the sense of measuring the amount of energy, or time, needed to communicate the structure of the algebraic points at which the D0–branes are located. The conjecture of Birch and Swinnerton-Dyer thus relates the complexity of D0–brane crystals to the critical values of the motivic $L$–function.

The BSD conjectures have been proven for certain types of elliptic curves, which will be made more explicit further below. They provide the first realization of the sought-after link between a certain class of base points of D0–branes on the simplest possible compactification. With this framework in place one can ask whether rational D0–branes lead to interesting information about the varieties on which they live.
2.3 String modular examples of rank 0 and torsion D–crystals

With the conjectures of Birch and Swinnerton-Dyer in place it is possible to check what the relations BSD$_Q$I and BSD$_Q$II imply in the context of a special class of string compactifications given by exactly solvable elliptic curves of Gepner type. For these models the underlying conformal field theory is a tensor product of $\mathcal{N} = 2$ superconformal minimal models whose partition function is described by characters that are constructed from Kac-Petersson string functions of the underlying $A_1^{(1)}$ affine Lie algebra. These string functions in turn are determined by Hecke indefinite modular forms and it was shown in ref. [16] that the Mellin transform of twisted products of these Hecke indefinite modular forms are identical to the $L$–function of the corresponding elliptic curves via equations of type (1). These results show that the arithmetic structure of the compactification manifold encodes the physics on the worldsheet without the need to probe the geometry over continuous fields, such as $\mathbb{R}$ or $\mathbb{C}$. The link between the worldsheet physics and the spacetime geometry is provided by the $L$–function and its associated modular form. This provides a first physical interpretation of the $L$–function.

The detailed structure of the modular forms $f(E, q)$ such that $L(f, s) = L(E, s)$ for these three diagonal elliptic curves involve Hecke indefinite modular forms $\Theta_{\ell, m}^k$, where $k$ indicates the level of the conformal field theory and $\ell, m$ are quantum numbers within the CFT. The results are summarized in Table 1.

| Curve $E_d$ | Worldsheet representation | Space |
|-------------|---------------------------|-------|
| $E_3 \subset \mathbb{P}_2$ | $f(E_3, q) = \Theta_{1,1}^1(q^3)\Theta_{1,1}^1(q^9)$ | $S_2(\Gamma_0(27))$ |
| $E_4 \subset \mathbb{P}_{(1,1,2)}$ | $f(E_4, q) = \Theta_{1,1}^2(q^4)^2 \otimes \chi_2$ | $S_2(\Gamma_0(64))$ |
| $E_6 \subset \mathbb{P}_{(1,2,3)}$ | $f(E_6, q) = \Theta_{1,1}^1(q^6)^2 \otimes \chi_3$ | $S_2(\Gamma_0(144))$ |

Table 1. Modular $L$–functions of diagonal elliptic curves [16].

The exactly solvable models listed in Table 1 have rank 0, hence the base points of the $\mathbb{Q}$-rational D–crystals are of torsion type. Since the height function is trivial on the torsion factor of the Mordell-Weil group the regulator is trivial and is set to $R = 1$. The conjecture
of Birch and Swinnerton-Dyer therefore specializes for these models to

\[ L(E, 1) = c_T \cdot \Omega \frac{|\Sha|}{|E(\mathbb{Q})_{\text{tor}}|^2}. \]  

(24)

As mentioned above, the Tamagawa number \( c_T \) receives contributions only from the bad primes \( p | N \), where \( N \) is the conductor of the curve. For the examples here these conductors are given by \( N = 27, 64, 144 \), respectively, hence there are at most two bad primes. The Brieskorn-Pham curves listed in Table 1 admit complex multiplication over some imaginary quadratic field \( K \) for which \( L(E/K, 1) \neq 0 \). For CM curves Rubin has shown that the Shafarevich-Tate group \( \Sha_K(E) \) is finite and that the BSD conjecture holds [17]. It is therefore possible to compute the order of the Shafarevich-Tate group by using the BSD II relation (24). For the exactly solvable elliptic curves of Brieskorn-Pham type the numerical results collected in Table 2 show that the formula of Birch and Swinnerton-Dyer leads to trivial Shafarevich-Tate group \( |\Sha| = 1 \) for all three curves.

| Elliptic curve | \( L(E, 1) \) | \( \Omega \) | \( |E(\mathbb{Q})_{\text{tor}}|^2 \) | \( \prod_p c_p \cdot r \) | \( |\Sha| \) |
|----------------|----------------|-------|----------------|----------------|-------|
| \( E_{27} \)   | 0.588879...    | 1.766638... | 3            | 3              | 0     | 1     |
| \( E_{64} \)   | 0.927037...    | 3.708149... | 2            | 1              | 0     | 1     |
| \( E_{144} \)  | 1.214325...    | 2.428650... | 2            | 1 \cdot 2      | 0     | 1     |

Table 2. Birch-Swinnerton-Dyer data for the elliptic curves \( E_N \) analyzed in [16].

It becomes clear from the results collected in Table 2 that the Shafarevich-Tate group, while the most difficult to compute, is not a very sensitive characteristic, a result that is confirmed by the data base developed by Cremona. Essentially then, the BSD conjectures show that in the case of exactly solvable curves the \( L \)-functions of the modular forms on the worldsheet evaluated at the central value contain global information about the compactification geometry, such as the periods, and the torsion points. More precisely, for the examples summarized in Tables 1 and 2 the data shows in particular that the quotient \( L(E, 1)/\Omega_E \) of the \( L \)-function evaluated at the critical point by the period determines the size of the group of the base points of the \( \mathbb{Q} \)-rational D-brane crystal of torsion type on the curve. Since the period itself is determined by the modular form it follows that the order of the group of torsion type D0-branes is determined
completely by the modular form derived from the worldsheet conformal field theory. Thus the
worldsheet modular form determines the size of the rational D-crystal $\mathcal{D}_\mathbb{Q}(E)$.

More generally, for elliptic curves with complex multiplication it was shown first by Coates
and Wiles [18] that $L(E, 1) \neq 0$ implies that the rank of the elliptic curve is zero. Work by
Gross-Zagier [8], Kolyvagin [19, 20, 21], Bump-Friedberg-Hoffstein [22, 23] and Murty-Murty
[24] generalizes the Coates-Wiles result to all modular curves and furthermore shows that if
$L(E, 1) = 0$ and $L'(E, 1) \neq 0$ then the rank of $E$ is equal to 1. Combining this with the
elliptic modularity theorem of Wiles et. al. leads to the conclusion that for any curve $E$ with
$L(E, 1) \neq 0$ the Mordell-Weil group $E(\mathbb{Q})$ is finite and that for all elliptic curves with
$L(E, 1) = 0$ and $L'(E, 1) \neq 0$ the rank of the Mordell-Weil group is one. Thus the conjecture
$BSD_{\mathbb{Q}}$ is proven for curves whose rank is at most one.

The analysis of exactly solvable compactification shows that while $\mathbb{Q}$-rational D-branes do
provide the sought after link between the worldsheet physics and spacetime D-branes, this
type of probe is not particularly sensitive. As noted earlier, the possible torsion subgroups
of the Mordell-Weil group have been classified in work by Ogg and Mazur, with the result
that the number of different groups is very limited, in particular the order of these groups is
bounded by $|E(\mathbb{Q})_{tor}| \leq 12$. As a result the torsion part of the Mordell-Weil group $E(\mathbb{Q})_{tor}$ is
an invariant that is enormously degenerate on the space of all elliptic curve, hence pure torsion
D-crystals are not particularly discriminating probes. This motivates the consideration of
more general D-crystals with base points that are not restricted to have $\mathbb{Q}$-rational base point
coordinates, but instead live in more general algebraic number fields $K$ that are extensions of
$\mathbb{Q}$.

3 $K$-rational D-crystals

The results described above concerning torsion type D-crystals and their relation to the
worldsheet physics raise the question whether extensions to more general algebraic $K$-rational
D-branes might be useful as more sensitive probes of the geometry of the compactification
manifold than $\mathbb{Q}$-rational $D$-crystals. The fact that the diagonal curves considered in the previous section have complex multiplication suggests that a natural class of fields to consider are imaginary quadratic fields. The purpose of this section is to describe the more general notion of $K$-rational $D$-crystals, which in the next Section are shown to provide a physical spacetime interpretation of the $L$-function. Later in this paper it will become clear that in the case of elliptic curves (and their derived toroidal compactifications) these $K$-rational $D$-crystals contain sufficient information to reconstruct the compactification manifold itself. In the case of exactly solvable varieties they also are linked to the theory on the worldsheet via certain modular forms of the conformal field theory. Basic background for elliptic curves can be found in the books by Silverman [25, 26].

The shift from $\mathbb{Q}$-rational to $K$-rational $D$-crystals necessitates the generalization of Mordell-Weil group to the group of $K$-rational base points $E(K)$. This in turn implies that a generalization of the BSD conjectures is necessary. Such a formulation has been provided by Tate [27]. The focus of the first BSD conjecture now is the relation between the rank of $E(K)$ and the $L$-function of the curve $E/K$ considered over $K$. The generalized rank conjecture takes the form

$$\text{BSD}_K: \; \text{ord}_{s=1} L(E/K, s) = \text{rk}_K E(K),$$

and the generalized Taylor conjecture becomes

$$\text{BSD}_K: \; \frac{1}{r!} L^{(r)}(E/K, 1) = c \cdot \Omega_{E/K} \frac{\text{I}(E/K)}{|E(K)_{\text{tor}}|^2} R_{E/K},$$

where $c = \left( \prod_p c_p \right)$ is again the Tamagawa number.

The formulation of the general BSD conjectures for extensions $K/\mathbb{Q}$ of the rational numbers shows that it is possible in principle to address the question whether $K$-rational $D$-crystals for such number fields can be used to determine the underlying modular forms on the worldsheet. While it is difficult to make general statements about the structure of the Mordell-Weil group $E(K)$, it will become clear further below that it sufficient for the framework developed in this paper to consider extensions of $\mathbb{Q}$-rational $D$-crystals that are defined by imaginary quadratic fields $K_D = \mathbb{Q}(\sqrt{-D})$ of discriminant $-D$. For such fields a systematic construction of generators of $E(K_D)$ exists, based on the work of Birch and Gross-Zagier. This will lead
to an infinite number of $D$–crystals defined over $K_D$ for running $D$, as described in the next section.

4 D–crystals of Heegner type

The discussion of the $\mathbb{Q}$–rational $D$–crystals above shows that while these objects do contain information about the $L$–functions they are not sensitive enough to provide probes that are sensitive enough to contain all the information about the compactification geometry, i.e. they do not allow to identity isogeny classes of elliptic curves. It was also noted that general $K$-rational Mordell-Weil group $E(K)$ are very difficult to understand and to construct, hence are problematic as practical tools. This motivates the search for types of $D$–crystals whose base points are more general than rational points, but are special enough to be under better control and which are rich enough to identify isogeny classes of elliptic curves. Such fields are in fact provided by the simplest type of generalization of rational numbers given imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ of discriminant $D$. This leads to the question whether there exist non-torsion rational points when the curve is considered over such fields. The problem of constructing $K$–rational points is nontrivial and has first been considered by Heegner in his solution of Gauss’ class number problem [28]. The question precisely when his construction leads to points of infinite order has been discussed more systematically first by Birch whose construction will be described in this Section.

The framework formulated in [29, 30] is indirect, starting with points on the modular curve $X_0(N) = \mathcal{H}/\Gamma_0(N)$, followed by a projection via the modular parametrization $X_0(N) \rightarrow E_N$ to an elliptic curve of conductor $N$ associated to a modular form $f \in S_2(\Gamma_0(N))$. A second issue that arises is that the field of definition of the points obtained on the elliptic curve via the modular parametrization map are defined not over the imaginary quadratic field $K_D$ but over its ring class field $H_D$. This necessitates a trace map that projects the coordinates from $H_D$ to $K_D$. 

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4.1 Modular parametrization of elliptic curves and CM points

It was shown by Wiles et.al. [12, 13] that for every elliptic curve $E_N$ with conductor $N$ there exists a modular curve $X_0(N) = \mathcal{H}/\Gamma_0(N) \cup \{\text{cusps}\}$ and a modular parametrization map

$$\Phi_f : X_0(N) \rightarrow E_N(\mathbb{C}) = \mathbb{C}/\Lambda_f$$

that is determined by a modular cusp form $f(q) \in S_2(\Gamma_0(N))$ of weight two and level $N$ with respect to Hecke’s congruence subgroup defined by

$$\Gamma_0(N) = \\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N}\}.$$  \hspace{1cm} (28)

The modular curves $X_0(N)$ are in general of higher genus

$$g(X_0(N)) = \dim S_2(\Gamma_0(N))$$

and the elliptic curve can be obtained via Shimura’s quotient construction by choosing an element $f(q) \in S_2(\Gamma_0(N))$ of weight two and level $N$ with

$$q = e^{2\pi i \tau}$$

Expanding $f(q)$ as a Fourier series $f(q) = \sum_n a_n q^n$ allows to write the series expansion of the map $\Phi_f$ as

$$\Phi_f(\tau) := -\int_{\tau}^{i\infty} \omega_f = \sum_n \frac{a_n}{n} e^{2\pi i n z},$$

where the differential form is defined $\omega_f := 2\pi i f(z)dz$. The lattice $\Lambda_f$ of the image is given by

$$\Lambda_f := \left\{ \int_{\gamma p}^{\gamma p} \omega_f \mid \gamma \in \Gamma_0(N) \right\}.$$  \hspace{1cm} (30)

The map from a lattice realization $E_\Lambda = \mathbb{C}/\Lambda$ of an elliptic curve to its algebraic form $E^w$ is obtained via the Weierstrass function

$$w_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

as

$$\mathbb{C}/\Lambda \rightarrow E^w$$

$$z \mapsto (w_\Lambda(z), w_\Lambda'(z)).$$  \hspace{1cm} (32)
The functions \((x, y) = (w_\Lambda(z), w'_\Lambda(z))\) satisfy the algebraic relation

\[ y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda) \]  

(33)

where the coefficients \(g_2, g_3\) are given in terms of the defining lattice \(\Lambda\) as

\[
g_2 = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^4} \\
g_3 = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^5}. \tag{34}\]

The modular form \(f(q) \in S_2(\Gamma_0(N))\) associated to the curve \(E\) has a geometric interpretation as a power series whose Fourier coefficients \(a_p\) for primes \(p\) essentially measure the number of points \(E(\mathbb{F}_p)\) of the elliptic curve over the finite fields \(\mathbb{F}_p\). This leads to the Hasse-Weil L-series

\[
L(E, s) = \sum_{n} \frac{a_n}{n^s} \tag{35}
\]

which in turn leads to \(f(E, q)\) via inverse Mellin transform. For any specific curve \(E\) such a modular parametrization can be found by explicit computation and the general machinery of Wiles et. al. is not necessary.

### 4.2 The construction of D-crystals of Heegner-Birch type

In this section a class of \(K\)-arithmetic D-crystals is described whose existence can be proven. These D-crystals are defined such that their base points are given by the special class of Heegner points associated to imaginary quadratic fields \(K_D = \mathbb{Q}(\sqrt{-D})\) with discriminant \(-D\). For such fields nontriviality criteria for \(E(K_D)\) are known. Furthermore, Gross and Zagier have proven a precise form of the conjecture of Birch-Stephens [30] concerning a relation between the height of Heegner-Birch points and the derivative of the \(L\)-function at the central point in the rank 1 case. This result shows that in this case the derivative of \(L(E/K_D, s)\) at \(s = 1\) controls the existence of nontrivial \(K\)-rational points and thereby provides the sought-after relation between D-crystals and \(L\)-functions.
The first step in this construction is to consider special points in the upper half plane. A point \( z \in \mathcal{H} \) is called a complex multiplication point (CM point) if it is the root of a quadratic equation

\[
Az^2 + Bz + C = 0,
\]

with \( A, B, C \in \mathbb{Z} \) and \(-D = B^2 - 4AC < 0\). Such points are special because the \( j \)–function evaluates to algebraic instead of transcendental values on CM points.

The next step of the construction of Heegner points given by Birch [29] selects points \( z \in \mathcal{H} \) of the upper half plane such that \( z \) and \( Nz \) satisfy quadratic equations with the same discriminant \( D(z) = D(Nz) \).

The motivation for this condition arises from the combination of two facts. The first is that in general the \( j \)–function applied to arbitrary \( z \in \mathcal{H} \) and \( Nz \) defines values \((x, y) = (j(z), j(Nz))\) that are not independent but satisfy an equation \( F_N(x, y) = 0 \), where the structure of the defining polynomial depends on the level \( N \). \( F_N \) defines an algebraic curve \( Z_0(N) = \{F_N(x, y) = 0\} \) which is singular in general and whose resolution determines the modular curve \( X_0(N) \). The second fact is that the nature of the \( j \)–function values \( j(z) \) depends on the number theoretic character of \( z \). As noted above, if \( z \) is a CM point in some imaginary quadratic field \( K_D \) of conductor \(-D\) then \( j(z) \) is algebraic, more precisely it takes values in the ring class field \( R_D \) of \( K_D \). The key now is that \( j(Nz) \) takes values in a different field, hence the point \((j(z), j(Nz))\) is an element of \( X_0(N)(\mathbb{C}) \) but is not defined over the ring class field. The constraint \((37)\) ensures that \((j(z), j(Nz)) \in X_0(N)(R_D)\). This will be of importance below.

The existence of points that satisfy the discriminant constraint \((37)\) depends on the structure of \( D \) relative to the level \( N \). Solutions exist when the discriminant can be written as a square mod \( 4N \)

\[
r^2 \equiv -D(\text{mod } 4N).
\]

This congruence condition can be formulated in a more conceptual way by noting that for such imaginary quadratic extensions all prime divisors \( p \) of the conductor \( N \) are split or ramified, i.e.

\[
\chi_D(p) = \left(\frac{-D}{p}\right) = 1, \quad \text{for primes } p | N,
\]

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where $\chi_D$ denotes the Legendre symbol. This constraint is called the Heegner constraint, and the discriminants which satisfy this condition are called Heegner discriminants.

Given a solution $r$ one can define points in the upper half-plane $\mathcal{H}$ as

$$z_{D,r} = \frac{-B + \sqrt{-D}}{2A} \in \mathcal{H},$$  \hspace{1cm} (40)

where $N|A$ and $B \equiv r (\text{mod } 2N)$. If the constraint (39) is not satisfied then there are no such points, and if it is then the set $\{z_{D,r}\}$ of such points is invariant under $\Gamma_0(N)$, leading to $h(D)$ orbits $[z_{D,r}] \in X_0(N)$ in the modular curve $X_0(N)$. Here $h(D)$ denotes the class number of the field $K_D$. In order to lighten the notation the orbits $[z_{D,r}]$ will again be denoted by $z_{D,r}$.

The Heegner points on the curve $X_0(N)$ can now be mapped to an elliptic curve $E_N$ of conductor $N$ via the modular parametrization $\Phi_f$ map associated to a modular form $f \in S_2(\Gamma_0(N))$. This leads to points on the torus $E_N = \mathbb{C}/\Lambda_f$ defined by the period lattice $\Lambda_f$ of the modular form $f$. The image points $\Phi_f(z_{D,r})$ do not live in the original field. It follows from the theory of complex multiplication that instead of taking values in the imaginary quadratic field $K_D$ they are defined over ring class fields $R_D$ associated to $K_D$. If $K_D$ satisfies the splitting condition of Birch (37) this extension is the Hilbert class field $H_D$, i.e. the maximal abelian unramified extension of $K_D$, and they are permuted amongst themselves by the Galois group $\text{Gal}(H_D/K_D)$, whose order is the same as the class number $h(D)$. Taking the trace of an image point over this Galois group

$$\text{tr} \Phi(z_{D,r}) = \sum_{\sigma \in \text{Gal}(H_D/K_D)} \sigma(\Phi(z_{D,r}))$$  \hspace{1cm} (41)

therefore leads to points on the elliptic curve that are defined over $K_D$. The trace is divisible by $u_D = \frac{1}{2}|\mathcal{O}_{K_D}^\times|$, half the number of units of the field $K_D$. This motivates the definition of a Heegner point in $E(K_D)$ as

$$P_{D,r} := \frac{1}{u_D} \text{tr} \Phi(z_{D,r}) \in E(K_D).$$  \hspace{1cm} (42)

For any given discriminant $D$ there may or may not exist nontrivial points that are defined over the field $K_D$ and the question arises whether non-trivial $D$–crystals can be obtained for a
given elliptic curve. If so then the complexity of these arithmetic $D-$crystals can be measured in terms of the Néron-Tate height $\hat{h}(P_D)$ which defines a real valued function on $E(K_D)$. This existence problem will be addressed next.

4.3 The existence of $K_D-$type $D-$crystals

The conjecture of BSD$_{K_{II}}$ shows that in order to obtain nontrivial $D-$crystals whose base points take values in $K_D$ it is necessary to have curves $E/K_D$ such that the derivative of their $L-$functions does not vanish at the critical point. The $L-$function of $E/K_D$ can be computed in terms of curves over the rational field by considering the twisted curve $E^D$ of the the curve $E$

$$L(E/K_D, s) = L(E, s) L(E^D, s).$$

(43)

If the curve $E$ is described by a cubic polynomial $p(x)$ as

$$E : y^2 = x^3 + ax^2 + bx + c,$$

(44)

its twist is defined as $E^D : Dy^2 = p(x)$, and can be brought into the standard cubic form by

$$E^D : y^2 = x^3 + aDx^2 + bD^2x + cD^3.$$

(45)

If the Hasse-Weil $L-$function of $E$ is expanded as

$$L(E, s) = \sum_n a_n n^{-s}$$

(46)

and $\chi_D(p)$ is again the character associated to $K_D$ via the Legendre symbol, the $L-$function of the twisted curve $E^D$ is given by twisting the coefficients $a_n$ by this character

$$L(E^D, s) = \sum_n a_n \chi_D(n)n^{-s} =: L(E, s) \otimes \chi_D.$$

(47)

The conductor of the twisted curve $E^D$ for discriminants $D$ that are coprime to the conductor $N(E)$ of are given by

$$N(E^D) = N(E)D^2,$$

(48)
and the sign of the functional equation for the twisted curve is finally given in terms of the sign \( \epsilon = \epsilon(E) \) of the original curve and the character \( \chi_D \) by

\[
\epsilon(E^D) = \epsilon(E)\chi_D(-N(E)).
\]

These relations make it possible to understand the conjecture of Birch and Swinnerton-Dyer for \( E/K_D \) in terms of the behavior of the curve \( E \) and its twists \( E^D \) defined over the rational number field.

It follows from (43) that the non-vanishing of the \( L \)-series at the critical point can be guaranteed either by having (twisted) curves with \( L'(E,1) \neq 0 \neq L(E^D,1) \) or via \( L(E,1) \neq 0 \neq L'(E^D,1) \). Information about the existence of such twisted curves can be obtained by considering the behavior of twists of modular forms of weight 2. This follows from the elliptic modularity theorem of Wiles and Taylor, as well as Breuil-Conrad-Diamond-Taylor, which says that for each elliptic curve \( L(E,s) = L(f,s) \) where \( f \in S_2(\Gamma_0(N)) \). Furthermore, the \( L \)-function of \( E^D \) is given by \( L(E^D,s) = L(f \otimes \chi,s) \). Results by Bump-Friedberg-Hoffstein [22, 23], Murty-Murty [24] and Waldspurger [31] then imply the existence of infinitely many algebraic \( D \)-crystals. The details depend on the sign of the functional equation as follows.

Let \( f \in S_{2w}(\Gamma_0(N)) \) be a cusp new-form with trivial character and the completed \( L \)-function normalized as in eq. (8) which satisfies the functional equation (9), where \( \epsilon = \pm 1 \). Heegner points exist for both signs of the functional equation.

First, it was shown in [23, 24] that if the sign \( \epsilon \) of the functional equation is positive then there exist infinitely many imaginary fields \( \mathbb{Q}(\sqrt{-D}) \) of discriminant \( -D \) prime to \( N \) such that the Heegner constraint is satisfied, i.e. all prime divisors of \( N \) split in \( K_D \), and the twisted \( L \)-function \( L(f \otimes \chi_D,s) \) has a first order zero at \( s = w \). If for \( w = 1 \) the modular form \( f \in S_2(\Gamma_0(N)) \) is such that \( L(E,1) = L(f,1) \neq 0 \) the existence of these twists therefore implies the nonvanishing of the height of the Heegner point associated to \( K_D \).

If the sign \( \epsilon \) is negative then \( \text{ord}_{s=1} L(E,s) \geq 1 \) and a result of Waldspurger [31] shows that there exist infinitely many quadratic characters \( \chi_D \) such that the \( L \)-function of the twisted curve \( E^D \) does not vanish, \( L(E^D,1) \neq 0 \). It follows from the work of Kolyvagin that the twisted curves \( E^D \) have rank zero. Combining this result with the theorem of Gross and
Zagier again guarantees the nonvanishing of the height of the Heegner points associated to $K_D$. It has been conjectured by Goldfeld [32] that for newforms of weight two the twisted $L$–functions $L(f \otimes \chi_D, 1)$ do not vanish for $\frac{1}{2}$ of the square-free integers $D$.

The results just described therefore show that for general elliptic curves there are an infinite number of discriminants $-D$ such that the Heegner points of $K_D$ define base points of algebraic $D$–crystals

$$D_H(E) = \bigcup_{K_D} \mathbb{Z}P_D.$$  \hfill (50)

Such $D$–crystals can then be used as probes the compactification geometry as well as the worldsheet theory, the latter mediated by the $L$–function.

### 4.4 General Heegner points

Less well understood, but also of interest for the physics of D0–branes, are constructions of algebraic points on elliptic curves that go beyond Birch’s systematization [29, 30] of Heegner points. The aim of these generalization is to relax the Heegner constraint on the imaginary quadratic extensions $K_D$ over which the elliptic curve is defined since it is of interest to try to construct rational points over such $K_D$ for which the prime divisors of the level $N$ does not necessarily split in $K$. In the present paper this will not be described in a systematic way, but will be exemplified for a particular curve which allows not only to relate Heegner type $D$–crystals to the geometry of the compactification manifold, but also to the structure on the string worldsheet via the modular form associated to the motivic $L$–function.

### 5 Heights of $K_D$–type $D$–crystals and $L$–functions

The rank part of the conjecture of Birch and Swinnerton-Dyer BSD$_\mathbb{Q}$I implies that for curves with positive rank the $L$–function at the central critical point vanishes, $L(E, 1) = 0$. It is therefore natural to ask what the role is of the derivative at this point.
5.1 The formula of Birch, Gross-Zagier and Zhang

In the 1960s Birch conjectured, based on numerical experiments, that the height of Heegner points associated to imaginary quadratic fields $K_D = \mathbb{Q}(\sqrt{-D})$ of discriminant $-D$ should be related to the special value of the derivative of the Hasse-Weil $L$–function of $E/K_D$ evaluated at the central critical point [33]

$$\hat{h}(P_D) = c_{E,D}L'(E/K_D, 1), \quad (51)$$

where $K_D$ satisfies the Heegner condition and $c_{E,D}$ is a constant that depends on the elliptic curve $E$ and the field $K_D$, but was not completely specified by the experimental data.

This conjecture was made precise by Gross and Zagier who proved the formula conjectured by Birch (51), in the process providing an explicit form of the coefficient $c_{E,D}$. More precisely, for imaginary quadratic fields $K_D$ with odd $D$ that satisfy the Heegner condition it was shown in [8] that

$$L'(E/K_D, 1) = \frac{1}{c^2 \cdot u_D^2 \cdot \sqrt{D}} \hat{h}(P_D), \quad (52)$$

where $\hat{h}(P_D)$ is the Néron-Tate height of the Heegner point $P_D \in E(K_D)$. The Manin constant $c$ relates the Néron differential $\omega$ on $E$ to the differential $\omega_f = 2\pi i f(z) dz$ on the modular curve $X_0(N)$ via the modular representation $\Phi_N^*(\omega) = c\omega_f$. It is expected that $c = 1$, which has been shown for square-free $N$ in ref. [34]. For the Néron differential $\omega$ the norm $\|\omega\|^2$ is defined as

$$\|\omega\|^2 = \int_{E(\mathbb{C})} \omega \wedge \overline{\omega}. \quad (53)$$

Finally, $u_D$ denotes one half of the number of units in $K_D = \mathbb{Q}(\sqrt{-D})$. One finds that $u_D = 1$ except for the Gauss field $\mathbb{Q}(\sqrt{-1})$, for which $u_1 = 2$, and the Eisenstein field $\mathbb{Q}(\sqrt{-3})$, for which $u_3 = 3$. The Gross-Zagier theorem was extended later for general discriminants by Zhang [35].

The key consequence of the Gross-Zagier formula is that the point $P_D$ has infinite order if and only if the $L'(E/K_D, 1) \neq 0$.

Comparing the theorem of Gross and Zagier with the conjecture of Birch and Swinnerton-Dyer...
leads to the expectation that the subgroup of $E(K)$ generated by the Heegner point $P_D$ is related to the order of the Shafarevich-Tate group as
\[
\left( e \prod_p c_p \right)^2 |\Sha_K(E)| = |E(K_D) : \mathbb{Z}P_D|^2.
\] (54)

The details of the Gross-Zagier relation depend on the type of the elliptic curve under consideration. In the present paper the focus will be on the two most important situations which together make up the vast majority of all the cases encountered for elliptic curves. These types are characterized by the rank of the Mordell-Weil group. It is not known whether the rank of this group is bounded, but the existing databases all indicate that the ranks 0 and 1 combined provide by far the majority of all known curves. The following discussion therefore aims at this most important class of curves.

### 5.2 Curves with $\text{rk } E(\mathbb{Q}) = 0$

The elliptic curves of Brieskorn-Pham type considered above for which the reconstruction from string worldsheet is completely known [16] have D-crystals with base points that are pure torsion. For the case $\text{rk } E(\mathbb{Q}) = 0$ the rank part of the conjecture of Birch and Swinnerton-Dyer implies that the $L$-function does not vanish at the central critical point, $L(E, 1) \neq 0$. The conjecture of BSD$_{\mathbb{Q}II}$ then implies that it essentially determines the period of the curve, as well as the order of the torsion group and the order of the Shafarevich-Tate group $\Sha$ via the specialized form (24).

For the Brieskorn-Pham curves in particular, and rank 0 curves in general, the goal therefore is to find twists $E^D$ that lead to root numbers $\epsilon(f \otimes \chi_D) = -1$, hence to vanishing $L$-functions. The existence of such $D$ is guaranteed by the result described in Section 4. The generator in these cases thus comes from $E^D$ and the Gross-Zagier formula takes the form
\[
\hat{h}(P_D) = \frac{u_D^2 \sqrt{|D|}}{||\omega_E||^2} L(E, 1)L'(E^D, 1)
\] (55) for $\text{rk } E(\mathbb{Q}) = 0$. 

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This result can be read in two ways. For exactly solvable curves one might view the modular form derived from the worldsheet conformal field theory as the primary physical object. Associated to this modular form is the $L$–function and the formula of Gross and Zagier can be interpreted as providing information about spacetime objects, namely the height of $D$–crystals, in terms of the worldsheet theory, mediated by the $L$–function. Inverting this point of view one can consider $D$–crystals as probes of the geometry of spacetime to gain information about the worldsheet theory, again mediated by the $L$–function and its associated modular form.

5.3 Curves with $\text{rk } E(\mathbb{Q}) = 1$

For curves $E$ of $\mathbb{Q}$–rank 1 the conjecture of Birch and Swinnerton-Dyer predicts that the $L$–function vanishes at the central critical point, $L(E, 1) = 0$. Hence the derivative of $L$–function of $E$ over the imaginary quadratic field $K_D$ takes the form $L'(E/K_D, 1) = L'(E, 1)L(E^D, 1)$. The Gross-Zagier formula therefore reduces to

$$h(P_D) = \frac{u^2 \sqrt{|D|}}{||\omega||^2} L'(E, 1)L(E^D, 1). \quad (56)$$

It follows that the generator of $E(K_D)$ comes from the rational Mordell-Weil group $E(\mathbb{Q})$. Recalling that $L(E^D, s) = L(f \otimes \chi_D, s)$ with $L(E, s) = L(f, s)$ shows that the height determines the elliptic curve $E$ itself. This will be made more precise below. It is therefore possible to compare the $K_D$–rational points with the generator $P \in E(\mathbb{Q})$, and to ask whether the integral multiples $c_D$ of

$$P_D = c_D P \quad (57)$$

have a geometric interpretation. This turns out to be the case. The work of Gross-Kohnen-Zagier shows that the values $c_D$ are the coefficients of a modular form $F = \sum_n c_n q^n$ of weight $3/2$ whose image under the Shimura map [36, 37]

$$\text{Sh} : S_{3/2}(N) \rightarrow S_2(\Gamma_0(N)) \quad (58)$$

is the motivic form $f(E, q) \in S_2(\Gamma_0(N))$ if $E$ has conductor $N$. By a result of Waldspurger the coefficients $c_D$ are determined by special values of the motivic $L$–function of the twisted curve $E^D$. 
6 Characterization of the compactification geometry via $D$–crystals

An essential ingredient of the program described in [3] (and references therein) to construct spacetime in string theory via arithmetic methods is that the information contained in automorphic forms is discrete. Using worldsheet modular forms to construct automorphic forms and automorphic motives works as a tool to construct the geometry of spacetime works because the discrete information contained in the associated motivic $L$–function is sufficient to characterize the motive. The key result thus is that in automorphic geometries probes of finite sensitivity completely determine the geometry. Following the same logic in the present context of $D$–branes motivates the question whether $K$–arithmetic $D$–crystals can be viewed as probes that are sufficiently sensitive to determine the geometry uniquely (up to isogeny).

The results described in the context of the Brieskorn-Pham exactly solvable elliptic curves suggest that in general $\mathbb{Q}$-rational points on elliptic curves will not be sensitive enough to determine the geometry. These curves are of rank 0, hence the $D$–crystals are of torsion type, of which there are only 15 different types. This means that there is an enormous degeneracy among all the elliptic curves of rank 0 as far as their $\mathbb{Q}$–rational structure is concerned. The next logical step is to consider Heegner type $D$–crystals because they involve the simplest possible generalization of $\mathbb{Q}$.

It is known that the geometry of an elliptic curve is determined by its Hasse-Weil $L$–function up to isogeny, a result that was proven by Faltings [38]. The relation between the height of $D0$–branes and special values of $L$–functions raises the question whether it is possible to characterize the geometry also in terms of the special class of Heegner type $D$–crystals associated to imaginary quadratic fields $K_D = \mathbb{Q}(\sqrt{-D})$, where $D$ is viewed as a variable parameter. This is in fact possible by interpreting a result by Luo-Ramakrishnan [39] as a statement about $D$–crystals.

**Theorem.** Let $E, E'$ be two elliptic curves over $\mathbb{Q}$ of conductors $N, N'$ such that for a non-zero
scalar $C$ one has

$$\hat{h}(c_{f,K}) = C\hat{h}(c'_{f',K})$$

where $f, f'$ are the corresponding modular forms and $c_{f,K}, c'_{f',K}$ are the Heegner divisor components in $J(K) \otimes \mathbb{Q}$ for imaginary quadratic fields $K$. Then $N = N'$ and $E$ and $E'$ are isogenous over $\mathbb{Q}$.

The proof of this result is based on [40]. This theorem shows that the Heegner type structure is sensitive enough to probe differences between elliptic curves at the level of isogenies. The theorem is precisely the result needed to see that the $D-$crystals of Heegner type are sufficiently sensitive to probe the compactification manifold in the needed detail because the $L-$function is an isogeny invariant. Since the $L-$function is determined for the exactly solvable curves by the modular forms on worldsheet it follows that the manifold is determined by the conformal field theory up to isogeny, as shown in [1, 16].

## 7 Examples

In this penultimate section two examples are discussed that illustrate the behavior for both elliptic curves $E$ that are of rank 0 and 1 respectively. The curves for which the relation between spacetime geometry and worldsheet physics has been analyzed so far are of the former type, while for rank 1 no such relation is known at present. It is therefore of interest to show that the context of $D-$branes does provide another route from string theoretic objects to the reconstruction of the compactification geometries. The two examples are motivated by the choice of an example with known string worldsheet construction [16] and an example for which no such interpretation is known. Many other examples can be analyzed similarly by making use of the databases of Cremona, Stein and others.
7.1 The exactly solve curve $E_{64}$

As an example of a Brieskorn-Pham type elliptic curve consider the curve embedded in the weighted projective plane $\mathbb{P}_{(1,1,2)}$. This is a curve of conductor $N = 64$ and in affine coordinates takes the form

$$E_{64} : \quad y^2 = x^3 + x.$$  

(59)

For this example the precise relation is known between the modular form $f(E_{64}, q) \in S_2(\Gamma_0(64))$ associated to its Hasse-Weil $L-$function $L(E_{64}, s)$ and the modular form on the string worldsheet, as noted in Section 2. The latter is a Hecke indefinite modular form $\Theta_{1,1}^2(\tau) = \eta(\tau)\eta(2\tau)$ and the relation involves a twist via the quadratic character associated to its field of quantum dimensions of the underlying conformal field theory [16]

$$f(E_{64}, q) = \Theta_{1,1}^2(4\tau)^2 \otimes \chi_2.$$  

(60)

The curve $E_{64}$ has $\mathbb{Q}$-rank 0, hence the base points of the $\mathbb{Q}$-rational $D-$crystals are pure torsion. In order to obtain non-torsion type crystals it is useful to consider the curve $E$ over extensions $K$ of the rational numbers.

Heegner type $D-$crystals obtained on $E/K_D$ via imaginary quadratic fields $K_D$ of discriminant $D$ involves the family of twisted curves $E_{64}^D : \quad Dy^2 = x^3 + x$, which can alternatively be written in the standard Weierstrass form as

$$E_{64}^D : \quad y^2 = x^3 + D^2x.$$  

(61)

For discriminants $D$ such that $(N, D) = 1$ the conductor of the twisted curve is given by $N(E_{64}^D) = 64D^2$, and the question then becomes for which conductor the twist curve has rank 1. It should be noted that the standard Heegner construction of Birch does not detect the existing rational points in this case. Direct computation (in part with the help of Stein’s database) leads to the results of Table 3, which contains enough information to establish the relations of Birch and Swinnerton-Dyer in this case.
| $D$ | 3  | 7   | 11  | 19  |
|-----|----|-----|-----|-----|
| $N(E_{64}^D)$ | 576 | 3136| 7744| 23104 |
| $h(K_D)$ | 1   | 1   | 1   | 1   |
| $\Omega_{E_{64}}$ | 2.140901... | 1.401548... | 1.1180... | 0.850707... |
| $R_{E_{64}}$ | 1.777251... | 4.268241... | 8.5062... | 14.457238... |
| $L'(E_{64}^D, 1)$ | 1.902460... | 2.991074... | 4.7552... | 6.149442... |
| $|E_{64}(\mathbb{Q})_{tor}|$ | 2   | 2   | 2   | 2   |
| $c_p$ | 1,2 | 2   | 2   | 2   |

Table 3. Rank 1 twists in the family of curves $E_{64}^D$.

This then shows that the heights of the rational D0–branes on the elliptic curve $E_{64}$ are given in terms of the $L$–function $L(E_{64}, s)$, which in turn leads to the string theoretic modular form on the worldsheet via eq. (60).

### 7.2 The rank 1 curve $E_{37}$

This section exemplifies the strategy of constructing the compactification geometry from the structure of CM D0–branes in a simple case. Consider the elliptic curve defined in its affine form by

$$E_{37} : \ y^2 + y = x^3 - x,$$

which has conductor $N = 37$ and which is birationally equivalent to the Weierstrass form

$$E_{37}^W : \ y^2 = x^3 - 16x + 16.$$  

This curve is not of Brieskorn-Pham type and goes beyond the framework discussed in [16].

The rank of the Mordell-Weil group is 1, in agreement with the prediction of the BSD-I formula since the sign of the functional equation of the modular for associated to the Hasse-Weil L-series

$$S_2(\Gamma_0(37)) \ni f_{37}(q) = q - 2q^2 - 3q^3 - 2q^5 - q^7 - 5q^{11} - 2q^{13} + 2q^{23} + 6q^{29} + \cdots$$
is $\epsilon = -1$ and

$$L(E_{37}, s) = L(f_{37}, s). \tag{65}$$

The Mordell-Weil group has no torsion, hence the group of rational points is given by

$$E(\mathbb{Q}) \cong \mathbb{Z}, \tag{66}$$

with generator $P_\text{gen} = (0, 0)$. This example has been discussed in some detail by Zagier [41] and much of the necessary data for this curve can also be obtained from the tables constructed by Cremona [42]. The remaining BSD ingredients of this curve are given by

$$R_{E_{37}} = 0.051111...$$

$$\Omega_{E_{37}}^R = 2.99345...$$

$$L'(E_{37}, 1) = 0.305999...$$

$$c_{37} = 1, \tag{67}$$

which leads to the analytic rank 1 of the Shafarevich-Tate group. The complex periods of this curve is given by $\Omega_{E_{37}}^2 = 2.451389...i$, leading to the norm of the Néron differential $||\omega||^2 = 7.33813...$

The twists $E_{37}^D$ of $E_{37}$ can be described in a Weierstrass form as the family of curves

$$E_{37}^D : \; y^2 = x^3 - 16D^2x + 16D^3. \tag{68}$$

The Heegner condition $\chi_D(N) = 1$ can easily be computed and for the solutions it is possible to use the Gross-Zagier relation for the rank 1 case (56) to relate the $L$–function values to the height of the resulting Heegner points. Table 4 illustrates this situation for a few Heegner points.

| $D$       | 7   | 11  | 47  | 67  |
|-----------|-----|-----|-----|-----|
| $N(E_{37}^D)$ | 1813 | 4477 | 81733 | 166093 |
| $L(E_{37}^D, 1)$ | 1.853... | 1.475... | 0.715... | 21.562... |
| $\hat{h}(P_D)$ | 0.204... | 0.204... | 0.204... | 7.360... |

Table 4. Results for Heegner points $P_D$ on $E_{37}/K_D$. 
8 Outlook

This paper has shown that the notion of D–crystals as discrete D–brane configuration based on number fields \( K \) is useful as a probe that to detect the physical structure of the compactification manifold. These D–crystals are linked to the motivic \( L \)–function of the manifold, which according to the Langlands reciprocity conjecture is expected to lead to automorphic forms. In several cases where this automorphic form is known it was shown that the underlying motive is in fact modular that the corresponding modular form encodes the physics of worldsheet via modular forms that arise from the underlying conformal field theory.

The results of this paper raise the question whether the strategy described here can be generalized to higher dimensions. While much less is known in this case, and even less has been proven, it is possible to indicate the framework into which higher dimensional D–crystals could be placed. On the motivic side there are concrete examples of K3 surfaces and Calabi-Yau threefolds of weighted hypersurface type for which motivic modular forms have been derived from the motivic \( L \)–functions \([2, 3]\). For diagonal hypersurfaces these motivic \( L \)–functions and their automorphic forms are related to CFT theoretic forms on the worldsheet, hence the discussion of the exactly solvable elliptic curves in this paper should generalize. Furthermore, there exist a framework that generalizes the Mordell-Weil groups to higher dimensions, and which allows to formulate conjectures that generalize the conjectures of Birch–Swinnerton-Dyer and the results of Gross and Zagier.

The higher dimensional conjectures generalizing the BSD framework are formulated in terms of certain types of algebraic cycles, which can be viewed as base schemes with D–branes wrapped around them. The analog of the rank conjecture in this framework is the conjecture of Beilinson and Bloch, which relates again the rank of a particular group, the Chow group \( \text{CH}^r(X)_{\text{hom}} \) of null-homologous algebraic cycles of codimension \( r \), to the vanishing order of the \( L \)–function assigned to the intermediate cohomology of the variety

\[
\text{ord}_{s=r} L(H^{2r-1}(X), s) = \text{rk} \text{CH}^r(X)_{\text{hom}}.
\]

For \( r = 1 \) the Chow group \( \text{CH}^1(X)_{\text{hom}} \) of codimension 1 algebraic cycles describes points on
the elliptic curves, and the Beilinson-Bloch conjecture reduces to the rank conjecture of Birch and Swinnerton-Dyer.

Embedded in the Chow group of cycles one can consider generalizations of the Heegner points, in this case Heegner cycles, or more generally complex multiplication cycles. In the case of CM cycles no proof exists for the analog of the Gross-Zagier theorem in this case, but the conjectural framework can be used for guidance. It would be of great interest to generalize the framework of the present paper to the context of higher dimensional $D$–crystals. Furthermore, it would be of importance to generalize this framework from a cohomological to a motivic setting, in which the $L$–functions considered are that of motives, i.e. the lhs is concerned with the vanishing order of $L(M, s)$, with $M$ the motive.

Acknowledgement.

It is a pleasure to thank Ulf Kühn and Monika Lynker for conversations. This work has been conducted over a number of years, during which I have benefitted from the support and hospitality of several institutions. First, I thank the Mathematics Institute Oberwolfach for support during an extended stay. Some of the results of this paper were presented at the Banff International Research Station and I thank Chuck Doran, Noriko Yui and Don Zagier for the opportunity to present this work. Visits to the Werner Heisenberg Max Planck Institute in Munich and to CERN greatly facilitated the conclusion of this project and I thank the Max Planck Gesellschaft and CERN for their support. Thanks are due in particular to Dieter Lüst in Munich and Wolfgang Lerche at CERN for making these visits possible, and I thank the string theory groups at both institutions for their friendly hospitality. This work was supported in part by an IUSB Faculty Research Grant and a grant by the National Science Foundation under grant No. 0969875.
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