New bi-Hamiltonian systems on the plane

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Abstract
We discuss several new bi-Hamiltonian integrable systems on the plane with integrals of motion of third, fourth and sixth order in momenta. The corresponding variables of separation, separated relations, compatible Poisson brackets and recursion operators are also presented in the framework of the Jacobi method.

1 Introduction
The Jacobi method of separation of variables is a very important tool in analytical mechanics. Following the classical work [7], the stationary Hamilton-Jacobi equation

\[ H = E \]

is said to be separable in a set of canonical coordinates \( u = (u_1, \ldots, u_m) \) and \( p_u = (p_{u_1}, \ldots, p_{u_m}) \)

\[ \{ u_i, p_{u_j} \} = \delta_{ij}, \quad \{ u_i, u_j \} = \{ p_{u_i}, p_{u_j} \} = 0, \quad i, j = 1, \ldots, m, \]

if there is an additively separated complete integral

\[ W(u_1, \ldots, u_m; \alpha_1, \ldots, \alpha_m) = \sum_{i=1}^{m} W_i(u_i; \alpha_1, \ldots, \alpha_m), \quad \alpha_1, \ldots, \alpha_m \in \mathbb{R}, \]

depending non-trivially on a set of separation constants \( \alpha_1, \ldots, \alpha_m \), and \( W_i \) are found in quadratures as solutions of ordinary differential equations.

In the framework of the Jacobi method, after finding new variables of separation \( u, p_u \) on phase space \( M \), we have to look for new problems to which they can be successfully applied, see lecture 26 in [7]. Indeed, substituting canonical coordinates into separated relations

\[ \Phi_i(u, p_u, H_1, \ldots, H_m) = 0, \quad i = 1, \ldots, m, \quad \text{with } \det \left[ \frac{\partial \Phi_i}{\partial H_j} \right] \neq 0 \quad (1.1) \]

and solving the resulting equations with respect to \( H_1, \ldots, H_m \), we obtain a new integrable system with independent integrals of motion \( H_1, \ldots, H_m \) in the involution.

In [19, 20, 21] we proposed to add one more step to this well-known construction of integrable systems proposed by Jacobi. Namely, we can:

1. take Hamilton-Jacobi equation \( H = E \) separable in variables \( u, p_u \);
2. make auto Bäcklund transformation (BT) of variables \( (u, p_u) \to (\tilde{u}, \tilde{p}_u) \), which conserves not only the Hamiltonian character of the equations of motion, but also the form of Hamilton-Jacobi equation;
3. substitute new canonical variables \( \tilde{u}, \tilde{p}_u \) into the suitable separated relations and obtain new integrable systems.
The main aim of this paper is to extend this scheme by adding one more step associated with a suitable change of time, which is similar to Weierstrass change of time for the Jacobi system on an ellipsoid [22].

Below we take well-known Hamilton-Jacobi equations separable parabolic coordinates on plane $Q = \mathbb{R}^2$

$$u_1 = q_2 - \sqrt{q_1^2 + q_2^2}, \quad u_2 = q_2 + \sqrt{q_1^2 + q_2^2} \quad (1.2)$$

and consider auto Bäcklund transformations $B$ preserving the form of Hamilton and Hamilton-Jacobi equations. After that, using canonical variables on the phase space $T^*\mathbb{R}^2$, we obtain new integrable systems with the Hamilton function

$$H = \sum_{i,j=1}^{2} g_{ij}(u_1, u_2) p_u p_{u_j} + V(u_1, u_2) \quad (1.3)$$

and second integrals of motion, which are polynomials of third, fourth and sixth order in momenta. The corresponding vector fields are bi-Hamiltonian and, moreover, one of these systems is superintegrable.

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The paper is organized as follows. In the remaining part of the first Section we show how our technique allows us to obtain well-known auto BTs which are related with translations on the level manifold. In Section 2 we apply the same technique to construct auto BT that represents a symmetry of the level manifold. Of course, we can not apply such auto BT to discretization of Hamiltonian system because when one iterates such BT one will obtain again original parabolic coordinates, but we can use such auto BT to construct new integrable systems. It is natural to use different types of auto BTs for the different purposes.

1.1 Abel equations and Bäcklund transformations

According [1 2 5 9] Abel differential equation

$$\omega(p_1(t)) + \omega(p_2(t)) + \cdots + \omega(p_n(t)) = 0 \quad (1.5)$$

on a hyperelliptic curve $C$ of genus $g$

$$C: \quad f(x, y) = y^2 - f(x) = 0, \quad f(x) = a_{2g+2}x^{2g+2} + a_{2g+1}x^{2g+1} + \cdots + a_0$$

$$D \approx D', \quad D + D' = D'' \quad \text{and} \quad [\ell]D = D'',$$

where $\approx$, $+$ and $[\ell]$ denotes equivalence, addition and scalar multiplication of divisors $D$ and $D'$, respectively.

Usually authors consider only addition of full degree reduced divisor $D'$ and weight one reduced divisor $D''$ in (1.4). This partial group operation and the corresponding one-point auto Bäcklund transformations have been studied from the different points of view in many publications, in particular see [4 10 14] and references therein. In fact, this construction of auto BTs was specially developed for simplest one-parametric discretization of original continuous systems.

Below we consider auto BTs associated with equivalence relation between two semi-reduced divisors $D$ and $D'$ which belong to a common equivalence class in Jacobian using classical Abel results [1], see also Abel’s papers in Crelle’s Journal, v. 3 and v.4. Such auto BTs represent hidden symmetries of the level manifold which yield new canonical variables on original phase space and, therefore, we can use these new variables to to construction of new integrable systems, i.e. to construction of hetero Bäcklund transformations.

The paper is organized as follows. In the remaining part of the first Section we show how our technique allows us to obtain well-known auto BTs which are related with translations on the level manifold. In Section 2 we apply the same technique to construct auto BT that represents a symmetry of the level manifold. Of course, we can not apply such auto BT to discretization of Hamiltonian system because when one iterates such BT one will obtain again original parabolic coordinates, but we can use such auto BT to construct new integrable systems. It is natural to use different types of auto BTs for the different purposes.
includes regular differential $\omega$ at $n$ points $p_k(t)$ on $C$ moves with a rational parameter $t$. According to Abel’s idea, solutions of (1.5) are points $p_k = (x_k, y_k)$ of $C$ intersecting with the second plane curve defined by equation 

$$h(x, y) = -y + P(x) = 0.$$ 

Here $P(x)$ is the polynomial of $g + 1$ order 

$$P(x) = b_{g+1}x^{g+1} + b_gx^g + \cdots + b_0$$ 

with variable coefficients $b_{g+1}, \ldots, b_0$ depending on $t$. Eliminating $y$ from equations $f(x, y) = 0$ and $h(x, y) = 0$, Abel determines abscissas $x_1, \ldots, x_n$ of the points of intersection as roots of the Abel polynomial 

$$\psi(x) = f(x) - P^2(x).$$ 

If we suppose that $\psi(x)$ has no multiple roots, then dividing roots of the Abel polynomial $\psi$ in two parts we can relate one subset of the roots to the other subset 

$$(x - x_1) \cdots (x - x_m) = \frac{f(x) - P(x)^2}{\phi(x)(x - x_{m+1}) \cdots (x - x_n)},$$ 

where $\phi(x)$ is a polynomial with independent on $t$ roots which play the role of constants of integration, see discussion in textbook [2]. According to Abel [1], polynomial $P(x)$ is usually constructed using interpolation because ordinates of the points of intersection are 

$$y_i = P(x_i), \quad i = 1, \ldots, n.$$ 

Following to Euler, mathematicians also investigated algebraic integrals of the Abel equations (1.5), which are equivalent to the correspondence (1.7-1.8) [2]. For instance, at $g = 1$ there are three possible Abel equations 

$$\frac{dx_1}{y_1} + \frac{dx_2}{y_2} = 0, \quad \frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dx_3}{y_3} = 0, \quad \frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dx_3}{y_3} + \frac{dx_4}{y_4} = 0$$ 

on the symmetric products of two, three and four copies of a compactified elliptic curve $C$, respectively. These equations have well-studied algebraic integrals: 

$$\left(\frac{y_1 - y_2}{x_1 - x_2}\right)^2 - a_4(x_1 + x_2)^2 - a_3(x_1 + x_2) = const - \text{the famous Euler integral},$$ 

$$\frac{y_i - y_k}{x_i - x_k} - \sqrt{a_4x_i} = \frac{y_j - y_k}{x_j - x_k} - \sqrt{a_4x_j}, - \text{the Abel integrals}$$ 

and so-called complete integral 

$$\begin{vmatrix}
  x_1^2 & x_1 & 1 & y_1 \\
  x_2^2 & x_2 & 1 & y_2 \\
  x_3^2 & x_3 & 1 & y_3 \\
  x_4^2 & x_4 & 1 & y_4
\end{vmatrix} = 0, \quad (1.9)$$ 

respectively. These algebraic integrals relate two, three and four points of intersection of the elliptic curve $C$ with parabola [2].

Let us show how well known Abel relations (1.7-1.8) yield an auto Bäcklund transformation of Hamilton-Jacobi equations. For instance, we take Hénon-Heiles system with Hamiltonians 

$$H_1 = \frac{p_1^2 + p_2^2}{4} - 4aq_2(q_1^2 + 2q_2^2), \quad H_2 = \frac{p_1(p_2q_2 - q_2p_1)}{2} - aq_1^2(q_1^2 + 4q_2^2) \quad (1.10)$$ 

separable in parabolic coordinates on the plane [1,2]. To describe evolution of $u_{1,2}$ with respect to $H_{1,2}$ we use the canonical Poisson bracket 

$$\{u_i, p_{u_j}\} = \delta_{ij}, \quad \{u_1, u_2\} = \{p_{u_1}, p_{u_2}\} = 0. \quad (1.11)$$

3
and expressions for $H_{1,2}$

$$H_1 = \frac{p_{u_1}^2 u_1 - p_{u_2}^2 u_2}{u_1 - u_2} - a(u_1 + u_2)(u_1^2 + u_2^2), \quad H_2 = \frac{u_1 u_2 (p_{u_1}^2 - p_{u_2}^2)}{u_2 - u_1} + a u_1 u_2 (u_1^2 + u_1 u_2 + u_2^2) \quad (1.12)$$

to obtain

$$\frac{du_1}{dt_1} = \{u_1, H_1\} = \frac{2p_{u_1} u_1}{u_1 - u_2}, \quad \frac{dv_2}{dt_1} = \{u_2, H_1\} = \frac{2p_{u_2} u_2}{u_2 - u_1} \quad (1.13)$$

and

$$\frac{du_1}{dt_2} = \{u_1, H_2\} = \frac{2u_1 u_2 p_{u_1}}{u_2 - u_1}, \quad \frac{dv_2}{dt_2} = \{u_2, H_2\} = \frac{2u_1 u_2 p_{u_2}}{u_1 - u_2}. \quad (1.14)$$

Here $p_{u_{1,2}}$ are the standard momenta associated with parabolic coordinates $u_{1,2}$:

$$p_{u_1} = \frac{p_2}{2} - \frac{p_1 (q_2 + \sqrt{q_1^2 + q_2^2})}{2q_1}, \quad p_{u_2} = \frac{p_2}{2} - \frac{p_1 (q_2 - \sqrt{q_1^2 + q_2^2})}{2q_1}.$$

Using Hamilton-Jacobi equations $H_{1,2} = \alpha_{1,2}$ we can prove that these variables satisfy to the following separated relations

$$(u_i p_{u_i})^2 = u_i (a u_i^3 + \alpha_1 u_i + \alpha_2), \quad i = 1, 2. \quad (1.15)$$

Expressions (1.13, 1.14) yield standard Abel quadratures

$$\frac{du_1}{\sqrt{f(u_1)}} + \frac{dv_2}{\sqrt{f(u_2)}} = 2dt_2, \quad \frac{u_1 du_1}{\sqrt{f(u_1)}} + \frac{u_2 dv_2}{\sqrt{f(u_2)}} = 2dt_1. \quad (1.16)$$

on the hyperelliptic curve $C$ of genus two defined by equation

$$f(x, y) = y^2 - f(x) = 0, \quad f(x) = x(ax^4 + \alpha_1 x + \alpha_2). \quad (1.17)$$

Suppose that transformation of variables

$$\mathcal{B} : (u_1, u_2, p_{u_1}, p_{u_2}) \rightarrow (\tilde{u}_1, \tilde{u}_2, \tilde{p}_{u_1}, \tilde{p}_{u_2}) \quad (1.18)$$

preserves Hamilton equations (1.13, 1.14) and the form of Hamiltonians (1.12). It means that new variables satisfy to the same equations

$$\frac{d\tilde{u}_1}{\sqrt{f(\tilde{u}_1)}} + \frac{d\tilde{u}_2}{\sqrt{f(\tilde{u}_2)}} = 2dt_2, \quad \frac{\tilde{u}_1 d\tilde{u}_1}{\sqrt{f(\tilde{u}_1)}} + \frac{\tilde{u}_2 d\tilde{u}_2}{\sqrt{f(\tilde{u}_2)}} = 2dt_1. \quad (1.19)$$

Subtracting (1.19) from (1.16) one gets two Abel differential equations (1.5)

$$\omega_1(x_1, y_1) + \omega_1(x_2, y_2) + \omega_1(x_3, y_3) + \omega_1(x_4, y_4) = 0, \quad (1.20)$$

$$\omega_2(x_1, y_1) + \omega_2(x_2, y_2) + \omega_2(x_3, y_3) + \omega_2(x_4, y_4) = 0,$$

where

$$x_{1,2} = u_{1,2}, \quad y_{1,2} = u_{1,2} p_{u_{1,2}}, \quad x_{3,4} = \tilde{u}_{1,2}, \quad y_{3,4} = -\tilde{u}_{1,2} \tilde{p}_{u_{1,2}}$$

and $\omega_{1,2}$ form a base of holomorphic differentials on hyperelliptic curve $C$ of genus $g = 2$

$$\omega_1(x, y) = \frac{dx}{y}, \quad \omega_2(x, y) = \frac{xdx}{y}.$$

These Abel equations are closely related to the so-called Picard group which is isomorphic to Jacobian of $C^g$.

According [2] there are six points of intersection:

1. one point can be taken at infinity;
2. four points \((x_1, y_1), \ldots, (x_4, y_4)\) are solutions of Abel equations \((1.20)\), which form support of weigh two divisors \(D\) and \(D'\), respectively;

3. one remaining point with coordinates \((\lambda, \mu)\) is independent on \(t\) and belong to support of weight one divisor \(D'\).

Abscissa \(\lambda\) is an arbitrary constant of integration of Abel differential equations \((1.20)\).

Using the corresponding Abel polynomial
\[
\psi(x) = f(x) - P(x)^2 = a(x - \lambda)(x - x_1)(x - x_2)(x - x_3)(x - x_4)
\]
we can determine new variables \(x_{3,4} = \tilde{u}_{1,2}\) as functions on initial variables \(x_{1,2}\) and \(y_{1,2}\)
\[
(x - x_3)(x - x_4) = \frac{f(x) - P(x)^2}{a(x - x_1)(x - x_2)(x - \lambda)}.
\]

In our case the number of degrees of freedom \(m = 2\) is equal to the genus \(g = 2\) and, therefore, we have
\[
P(x) = y_1(x - x_2)(x - \lambda) + y_2(x - x_1)(x - \lambda) + \mu(x - x_1)(x - x_2) + \mu(x - x_1)(x - x_2)
\]
due to the Lagrange interpolation formulæ.

Using \((1.21)\) together with definition of ordinates \(y_{3,4} = P(x_{3,4})\) \((1.18)\) we can explicitly obtain the desired transformation \(B\) \((1.18)\) which can be easily rewritten as canonical transformation of original coordinates on \(T^*\mathbb{R}^2\)
\[
\dot{q}_1 = q_1 - \frac{8aq_1^2(2\lambda q_2 + q_1^2 + 2q_2^2) + p_1(\lambda q_1 - 2\mu q_2 + 2p_2 q_1)}{4a(\lambda^2 - 2\lambda q_2 - q_1^2)} + \frac{q_1(\lambda q_1 p_2^2 + 2q_2 p_2 (p_1 p_1 q_1 - \mu) + p_1 (\lambda - 2q_2)(p_1 p_1 q_1 - 2\mu))}{2a(\lambda^2 - 2\lambda q_2 - q_1^2)}
\]
\[
\dot{q}_2 = q_2 + \frac{8a(q_1^2 + 6q_2^2 + 10\lambda q_1^2 + 16q_2^2 + 2q_1^2 + 2p_2^2)}{8a(\lambda^2 - 2\lambda q_2 - q_1^2)} - \frac{2p_2^2(\mu\mu q_1 - p_2 q_2) + \mu (2p_1 q_1 - \mu (p_1^2 + p_2^2)))}{4a(\lambda^2 - 2\lambda q_2 - q_1^2)}
\]
and
\[
\dot{p}_1 = -p_1 q_1 + \frac{(q_1^2 - q_2^2)(2\mu - \lambda p_2 - p_1 q_1)}{\lambda^2 - 2\lambda q_2 - q_1^2}, \quad \dot{p}_2 = -p_2 + \frac{2(q_2 - q_2)(2\mu - \lambda p_2 - p_1 q_1)}{\lambda^2 - 2\lambda q_2 - q_1^2}.
\]

Auto Bäcklund transformation \(B\) \((1.18)\) is the hidden symmetry of the level set of integrals of motion for any \(\lambda\). It allows us to apply this canonical transformation \(B\) \((1.18)\) to construction of new integrable systems in the framework of the Jacobi method. Indeed, if \(\lambda = 0\) in \((1.21)\), variables \(\tilde{u}_{1,2}\) are the roots of polynomial
\[
(x - \tilde{u}_1)(x - \tilde{u}_2) = x^2 + \left(u_1 + u_2 - \frac{(p_{u_1} - p_{u_2})^2}{a(u_1 - u_2)^2}\right)x + u_1 u_2 + u_1 u_2 + u_2^2 - \frac{p_{u_1}^2 - p_{u_2}^2}{a(u_1 - u_2)}.
\]
whereas momenta are equal to
\[
\dot{p}_{1,2} = \tilde{u}_{1,2}^{-1} P(\tilde{u}_{1,2}), \quad P(x) = \frac{x(x - u_2)}{u_1 - u_2} p_{u_1} + \frac{x(x - u_1)}{u_2 - u_1} p_{u_2}.
\]
It is easy to prove that original Poisson bracket \((1.11)\) in these variables reads as
\[
\{\tilde{u}_i, \tilde{p}_{u_i}\} = \delta_{ij}, \quad \{\tilde{u}_1, \tilde{u}_2\} = \{\tilde{p}_{u_1}, \tilde{p}_{u_2}\} = 0.
\]
Substituting these new canonical variables on \(T^*\mathbb{R}^2\) into a pair of separation relations
\[
\left(\tilde{p}_{u_1}^2 - a\tilde{u}_1^2\right) = \tilde{H}_1 \pm \tilde{H}_2, \quad i = 1, 2
\]
and solving the resulting equations with respect to \(\tilde{H}_{1,2}\), one gets the well-known Hamilton function
\[
\tilde{H}_1 = \left(\tilde{p}_{u_1}^2 - a\tilde{u}_1^2\right) + \left(\tilde{p}_{u_2}^2 - a\tilde{u}_2^2\right) = \frac{\tilde{p}_{u_1}^4}{4} + \frac{\tilde{p}_{u_2}^4}{2} - 2aq_2(3q_1^2 + 8q_2^2)
\]
(1.23)
for another integrable case of the Hénon-Heiles system. Second integral of motion \( \tilde{H}_2 \) is a polynomial of fourth order in momenta \( p_{1,2} \), see details in [21].

In this case the number of degrees of freedom \( m = 2 \) is equal to the genus \( g = 2 \) of hyperelliptic curve \( C \) and relations \([17,18]\) determine group operations on the Jacobian of \( C \). Since the Jacobian of a hyperelliptic curve is the group of degree zero divisors modulo principal divisors, the group operation is formal addition modulo the equivalence relation. Sequently we can say that auto Bäcklund transformation \([1.18]\) is a shift on the Jacobi variety of \( C \) which can be described using the Mumford coordinates of divisors \([10,11]\).

Indeed, we can introduce the Jacobi polynomials \([11]\):

\[
U(x) = \prod_{k=1}^{m} (x - x_k), \quad V(x) = \sum_{i=1}^{m} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \quad W(x) = \frac{f(x) - V^2(x)}{U(x)}
\]

and \( 2 \times 2 \) Lax matrix for the Hénon-Heiles system

\[
L(x) = \begin{pmatrix} V(x) & U(x) \\ W(x) & -V(x) \end{pmatrix}.
\]

After similar transformation

\[
L(x) \rightarrow \tilde{L}(x) = ML(x)M^{-1}, \quad M = \begin{pmatrix} U(x) & 0 \\ V(x) - P(x) & U(x) \end{pmatrix}
\]

one gets Lax matrix

\[
\tilde{L}(x) = \begin{pmatrix} P(x) & U(x) \\ \psi(x)U^{-1} & -P(x) \end{pmatrix}
\]

defined by Abel polynomials \( \psi(x), P(x) \) and \( U(x) \). At \( \lambda = 0 \) this Lax matrix was obtained in \([19,21]\).

## 2 Integrable system with integral of motion of sixth order in momenta

Let us take an integrable system with Hamiltonians

\[
H_1 = \frac{p_1^2 + p_2^2}{4} + \frac{a}{q_1^2} - \frac{2bq_2}{q_1^2} + \frac{c(q_1^2 + 4q_2^2)}{q_1^2}, \quad a, b, c \in \mathbb{R},
\]

\[
H_2 = -\frac{p_1(p_1q_2 - p_2q_1)}{2} - \frac{2aq_2}{q_1^2} + \frac{b(q_1^2 + 4q_2^2)}{q_1^2} - \frac{4cq_2(q_1^2 + 2q_2^2)}{q_1^2}.
\]

According to \([18]\) there is integrable deformation of this Hamilton function at \( b = 0 \)

\[
\tilde{H}_1 = H + \Delta H = \frac{p_1^2 + p_2^2}{4} + \frac{a}{q_1^2} + \frac{c(q_1^2 + 4q_2^2)}{q_1^2} + d(q_1^2 + q_2^2) + \frac{e}{q_2^2}.
\]

The corresponding second integral of motion is the polynomial of fourth order in momenta.

Our initial aim was to find variables of separation for this system in the framework of Abel theory. Instead of this, we find new integrable deformation of the Hamiltonian \([2.1]\) with second polynomial integrals of sixth order in momenta. Below we describe construction of this new integrable system in details.
2.1 Separation of variables and Abel differential equation

In parabolic coordinates Hamiltonians $H_{1,2}$ look like

$$
H_1 = \frac{u_1 p_{u_1}^2}{u_1 - u_2} + \frac{u_2 p_{u_2}^2}{u_2 - u_1} - \frac{a}{u_1 u_2} - \frac{b(u_1 + u_2)}{u_1^2 u_2^2} - \frac{c(u_1^2 + u_1 u_2 + u_2^2)}{u_1^2 u_2^2},
$$

$$
H_2 = \frac{u_1 u_2 p_{a_{u_1}}^2}{u_2 - u_1} + \frac{u_1 u_2 p_{a_{u_2}}^2}{u_2 - u_1} + \frac{a(u_1 + u_2)}{u_1 u_2} + \frac{b(u_1^2 + u_1 u_2 + u_2^2)}{u_1^2 u_2^2} + \frac{c(u_1 + u_2)(u_1^2 + u_2^2)}{u_1^2 u_2^2}.
$$

The corresponding Hamilton-Jacobi equations $H_{1,2} = \alpha_{1,2}$ are equivalent to separation relations

$$
\Phi(u, p, H_1, H_2) = (u_i^2 p_{u_i})^2 - (H_1 u_i^4 + H_2 u_i^3 - au_i^2 - bu_i - c) = 0,
$$

which determine the elliptic curve

$$
C : y^2 = f(x), \quad f(x) = \alpha_1 u^4 + \alpha_2 u^3 - au^2 - bu - c,
$$

i.e. variables

$$
x_{1,2} = u_{1,2}, \quad y_{1,2} = u_{1,2}^2 p_{u_{1,2}}
$$

may be identified with abscissas and ordinates of two points on $C$. In this case the number degrees of freedom $m = 2$ does no equal to the genus $g = 1$ of the underlying elliptic curve and, therefore, standard construction of BTs from [4, 10] cannot be applied because it works only when $m = g$.

Equations of motion for $x_{1,2}$ are equal to

$$
\frac{dx_1}{dt} = \{u_1, H_1\} = \frac{\partial H_1}{\partial p_{u_1}} = \frac{2y_1}{x_1(x_1 - x_2)}, \quad \frac{dx_2}{dt} = \{u_2, H_1\} = \frac{\partial H_1}{\partial p_{u_2}} = \frac{2y_2}{x_2(x_2 - x_1)}.
$$

It allows us to obtain Abel quadratures

$$
\frac{x_1 dx_1}{y_1} + \frac{x_2 dx_2}{y_2} = 0, \quad \text{and} \quad \frac{x_1^2 dx_1}{y_1} + \frac{x_2^2 dx_2}{y_2} = 2dt,
$$

which we have to solve with respect to functions $x_{1,2}(t)$ on time $t$. In order to get these functions we can change time $t \rightarrow s$ in (2.3) following to Weierstrass idea [22] and introduce new equations

$$
\frac{du_1}{ds} = \{u_1, H_1\}_W = \frac{2x_1 y_1}{x_1 - x_2}, \quad \frac{du_2}{ds} = \{u_2, H_1\}_W = \frac{2x_2 y_2}{x_2 - x_1}.
$$

in order to reduce Abel quadratures to the following form

$$
\frac{dx_1}{x_1 y_1} + \frac{dx_2}{x_2 y_2} = 0, \quad \frac{dx_1}{y_1} + \frac{dx_2}{y_2} = 2ds_1.
$$

After the Weierstrass change of time second quadrature incorporates standard holomorphic differential on the elliptic curve $C$ that allows us to relate this equation with Jacobian [9]. Equations of motion (2.3) are Hamiltonian equations with respect to the new Poisson bracket

$$
\{u_1, p_{u_1}\}_W = u_1^2 \delta_{ij}, \quad \{u_1, u_2\}_W = \{p_{u_1}, p_{u_2}\}_W = 0,
$$

which is compatible with the original canonical bracket $\{\cdot, \cdot\}$.

Suppose that transformation of variables

$$
B : (u_1, u_2, p_{u_1}, p_{u_2}) \leftrightarrow (\tilde{u}_1, \tilde{u}_2, \tilde{p}_{u_1}, \tilde{p}_{u_2})
$$

preserves form of Hamilton equations (2.3) and Hamilton-Jacobi equations, i.e.

$$
H_1(u, p_u) = \alpha_1 = H_1(\tilde{u}, \tilde{p}_{u_1}), \quad H_2(u, p_u) = \alpha_2 = H_2(\tilde{u}, \tilde{p}_{u_1}),
$$
where functions $H_{1,2}$ are given by \((2.2)\). It means that variables $x_{1,2, y_{1,2}}$ and
\[
x_{3,4} = \tilde{u}_{1,2}, \quad y_{3,4} = -\tilde{u}_{1,2}^2 \tilde{p}_{u_{1,2}},
\]
satisfy to the Abel differential equation
\[
\frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dx_3}{y_3} + \frac{dx_4}{y_4} = 0. \tag{2.9}
\]
In this case, points on the curve with coordinates $(x_{1,2}, y_{1,2})$ and $(x_{3,4}, y_{3,4})$ form weight two unred-
duced divisors $D$ and $D'$ on $C$. An equivalence relations between these divisors we identify with
desired auto BT. Indeed, the corresponding Abel polynomial is equal to
\[
\psi(x) = f(x) - P(x)^2 = A(x - x_1)(x - x_2)(x - x_3)(x - x_4), \quad A = \alpha_1 - b_2^2,
\]
where $b_2$ is the coefficient of polynomial $P(x) = b_2 x^2 + b_1 x + b_0$. It allows us to explicitly determine
the desired mapping $B$ \((2.8)\) using polynomial
\[
(x - x_1)(x - x_4) = \frac{f(x) - P(x)^2}{A(x - x_1)(x - x_2)}, \tag{2.10}
\]
where $f(x)$ is given by \((2.3)\), and polynomial of the second order
\[
P(x) = x \left( \frac{(x - x_2)y_1}{x_1(x - x_2)} + \frac{(x - x_1)y_2}{x_2(x_2 - x_1)} \right) \tag{2.11}
\]
which is completely defined by its values at $x_k$
\[
y_k = P(x_k), \quad k = 1, \ldots, 4,
\]
and by well known algebraic integral \((1.9)\).

**Proposition 1** If variables $\tilde{u}_{1,2}$ are solutions of equation
\[
\left( H_1 - \left( \frac{u_1 p_{u_1} - u_2 p_{u_2}}{u_1 - u_2} \right)^2 \right) x^2 - \frac{b u_1 u_2 + c (u_1 + u_2)}{u_1^2 u_2^2} x - \frac{c}{u_1 u_2} = 0
\]
and
\[
\tilde{p}_{u_{1,2}} = -\tilde{u}_{1,2}^2 P \big|_{x=\tilde{u}_{1,2}}, \quad P(x) = x \left( \frac{(x - u_2)u_1 p_{u_2}}{u_1 - u_2} + \frac{(x - u_1)u_2 p_{u_2}}{u_2 - u_1} \right),
\]
then mapping $B$ \((2.8)\) preserves the form of Hamilton equations \((2.3)\), the form of Hamiltonians
$H_{1,2}$ \((2.1, 2.2)\) and the form of the Poisson bracket \{.,.\} \((2.7)\), i.e.
\[
\{\tilde{u}_1, \tilde{p}_{u_1}\} = \tilde{u}_1^2 \delta_{ij}, \quad \{\tilde{u}_1, \tilde{u}_2\} = \{\tilde{p}_{u_1}, \tilde{p}_{u_2}\} = 0.
\]
The proof is a straightforward calculation.

Of course, we can not use this mapping $B$ \((2.8)\) for discretization of Hamiltonian flows \((2.5)\),
but this auto BT is the hidden symmetry which describes fundamental properties of the given
Hamiltonian system similar to the Noether symmetries.

**2.2 Construction of the new integrable system on $T^* \mathbb{R}$**

The following Poisson map
\[
\rho : \quad (u_1, u_2, p_{u_1}, p_{u_2}) \to (u_1, u_2, u_1^2 p_{u_1}, u_2^2 p_{u_2})
\]
reduces canonical Poisson bracket \{.,.\} to bracket \{.,.\} \((2.7)\), which allows us to rewrite equations of
motion \((2.5)\) in Hamiltonian form.
Using the composition of Poisson mappings \( \rho \) and \( B \) (2.8) we determine variables \( \hat{u}_{1,2} \), which are solutions of equation

\[
\left( \rho(H_1) - \left( \frac{u_1^2 p_{u_1} - u_2^2 p_{u_2}}{u_1 - u_2} \right)^2 \right) x^2 - \frac{b u_1 u_2 + c(u_1 + u_2)}{u_1^2 u_2^2} x - \frac{c}{u_1 u_2} = 0 \tag{2.12}
\]

where

\[
\rho(H_1) = \frac{u_1^5 p_{u_1}}{u_1 - u_2} + \frac{u_2^5 p_{u_2}}{u_2 - u_1} - \frac{a}{u_1 u_2} - \frac{b(u_1 + u_2)}{u_1^2 u_2^2} - \frac{c(u_1^2 + u_1 u_2 + u_2^2)}{u_1^2 u_2^2}.
\]

The corresponding momenta are equal to

\[
\hat{p}_{u_1,2} = -\hat{u}_{1,2}^4 \rho(P) \bigg|_{x=\hat{u}_{1,2}}, \quad \rho(P) = x \left( \frac{(x - u_2) u_1^3 p_{u_1}}{u_1 - u_2} + \frac{(x - u_1) u_2^3 p_{u_2}}{u_2 - u_1} \right). \tag{2.13}
\]

Straightforward calculation allows us to prove the following statement.

**Proposition 2**  
*Canonical Poisson bracket* 

\[
\{u_i, p_{u_j}\} = \delta_{ij}, \quad \{u_1, u_2\} = \{p_{u_1}, p_{u_2}\} = 0
\]

has the same form

\[
\{\hat{u}_i, \hat{p}_{u_j}\} = \delta_{ij}, \quad \{\hat{u}_1, \hat{u}_2\} = \{\hat{p}_{u_1}, \hat{p}_{u_2}\} = 0
\]

in variables \( \hat{u}_{1,2} \) and \( \hat{p}_{u_{1,2}} \) (2.15, 2.16).

Thus, we obtain new canonical variables on phase space \( T^* \mathbb{R}^2 \), which can be useful to construction of new integrable systems in the frameworks of the Jacobi method.

For instance, let us substitute these variables into the separated relations similar to (1.22)

\[
2\lambda_i = \left( \hat{u}_i^4 \hat{p}_{u_i}^2 + \frac{a}{u_i^2} + \frac{b}{u_i^3} + \frac{c}{u_i^4} \right) = \hat{H}_1 \pm \sqrt{\hat{H}_2}, \quad i = 1, 2, \tag{2.14}
\]

and solve these relations with respect to \( \hat{H}_{1,2} \).

**Proposition 3**  
*Functions on phase space \( T^* \mathbb{R}^2 \)*

\[
\hat{H}_1 = \lambda_1 + \lambda_2, \quad \hat{H}_2 = (\lambda_1 - \lambda_2)^2,
\]

are in involution with respect to the following compatible Poisson brackets

\[
\{\hat{u}_i, \hat{p}_{u_j}\} = \delta_{i,j}, \quad \{\hat{u}_1, \hat{u}_2\} = \{\hat{p}_{u_1}, \hat{p}_{u_2}\} = 0, \tag{2.15}
\]

and

\[
\{\hat{u}_i, \hat{p}_{u_j}\}' = \lambda_i^{-1} \delta_{i,j}, \quad \{\hat{u}_1, \hat{u}_2\}' = \{\hat{p}_{u_1}, \hat{p}_{u_2}\}' = 0. \tag{2.16}
\]

The proof is a straightforward calculation.

Moreover, using the corresponding bivectors \( P \) and \( P' \) it is easy to prove that vector field

\[
X = P d(\lambda_1 + \lambda_2) = P' d \left( \frac{\lambda_1^2 + \lambda_2^2}{2} \right)
\]

is bi-Hamiltonian vector field. This trivial in \( \hat{u}_{1,2} \) and \( \hat{p}_{u_{1,2}} \) variables Hamiltonian \( \hat{H}_1 = \lambda_1 + \lambda_2 \) has more complicated form in original parabolic coordinates and momenta:

\[
\hat{H}_1 = \frac{(bu_1 u_2 + c(3u_1 + u_2)) u_1^4 p_{u_1}^2}{c(u_1 - u_2)} + \frac{(bu_1 u_2 + c(u_1 + 3u_2)) u_2^4 p_{u_2}^2}{c(u_2 - u_1)} - \frac{(bu_1 + c)(au_1^2 + bu_1 + c)}{cu_1^2} - \frac{(bu_2 + c)(au_2^2 + bu_2 + c)}{cu_2^2} - \frac{4ac + b^2}{cu_1 u_2} - \frac{5bb(u_1 + u_2)}{u_1^2 u_2^2} - \frac{4c(u_1^2 + u_1 u_2 + u_2^2)}{u_1^2 u_2^2}.
\]
Second integral is polynomial of sixth order in momenta

\[ \hat{H}_2 = \left( cu_1^2 u_2^4 (u_1^2 p_{u_1} - u_2^2 p_{u_2})^2 + \frac{(u_1 - u_2)^2}{4} \left( 3u_1^2 + 2u_1 u_2 + 3u_2^2 \right)^2 + 2w_1 u_2 \left( 2au_1 u_2 + b(u_1 + u_2) \right) c - b^2 u_2^3 u_2^2 \right) \]

\[ \times \left( \frac{u_1^4 p_{u_1}^2 - u_2^2 p_{u_2}^2}{c(u_1 - u_2)^2} - \frac{a(u_1 + u_2)}{cu_1^2 u_2^2 (u_1 - u_2)} - \frac{b(u_1^2 + u_1 u_2 + u_2^2)}{cu_1^2 u_2^2 (u_1 - u_2)} - \frac{(u_1 + u_2)(u_1^2 + u_2^2)}{u_1^2 u_2^2 (u_1 - u_2)^2} \right)^2 \]

If we put \( a = b = 0 \) and then \( c = 0 \), the second integral of motion \( cH_2 = K^2 \) becomes a complete square. So, we have the geodesic flow on the plane with integrals of motion

\[ \hat{H}_1 = T = \frac{u_1^2 (3u_1^2 + u_2^2)}{u_1 - u_2} + \frac{u_2^2 (u_1 + 3u_2^2)}{u_2 - u_1}, \quad K = \frac{u_1^2 u_2^2 (u_1^2 p_{u_1} - u_2^2 p_{u_2})(u_1^2 p_{u_1}^2 - u_2^2 p_{u_2}^2)}{(u_1 - u_2)^2}. \]

In Cartesian coordinates these integrals of motion read as

\[ T = \frac{(q_1^2 + 6q_2^2)q_1^2 q_2^2}{2} + (5q_1^2 + 12q_2^2)q_1 q_2 p_1 p_2 + \frac{(q_1^4 + 8q_1^2 q_2^2 + 12q_2^4)p_2^2}{2} \]

and

\[ K = \frac{q_1^2 q_2 p_1^3}{4} + \frac{q_2^2 (q_1^2 + 6q_2^2)q_1^2 p_2}{4} + q_1^5 q_2 (q_1^2 + 3q_2^2)p_1 p_2^2 + q_1^4 (q_1^2 + 2q_2^2)q_2^3 p_3. \]

It is easy to prove that Hamiltonian \( T \) has no polynomial integrals of motion of first or second order in momenta.

Such nonstandard Hamiltonians may appear in the study of a wide range of fields such as nonholonomic dynamics, control theory, seismology, biology, in the study of a self gravitating stellar gas cloud, optoelectronics, fluid mechanics etc.

### 3 Other new integrable systems on the plane

If we substitute parabolic coordinates \( u_{1,2} \) and conjugated momenta \( p_{u_{1,2}} \) into a family of separated relations

\[
\begin{align*}
A & : \quad \left( u^2 p_u \right)^2 - (H_1 u^4 + H_2 u^3 + au^2 + bu + c) = 0, \\
B & : \quad \left( u^2 p_u \right)^2 - (au^4 + H_1 u^3 + H_2 u^2 + bu + c) = 0, \\
C & : \quad \left( u^2 p_u \right)^2 - (au^4 + bu^3 + H_1 u^2 + H_2 u + c) = 0, \\
D & : \quad \left( u^2 p_u \right)^2 - (au^4 + bu^3 + ca^2 + H_1 u + H_2) = 0
\end{align*}
\]

and solve the resulting pairs of equations with respect to \( H_{1,2} \), we obtain dual Stäckel systems for which every trajectory of one system is a reparametrized trajectory of the other system. The corresponding integrable diagonal metrics

\[ g_{km} = \begin{pmatrix} u_1^k u_1^m & 0 \\ 0 & u_2^k u_2^m \end{pmatrix}, \quad k = 0, 1; \quad m = 1, \ldots, 4, \]

are geodesically equivalent metrics.

For each of these systems we can construct an analogue of the Bäcklund transformation and Poisson map \( \rho \), associated with the Weierstrass change of time, that allows us to get different canonical variables and different integrable systems on \( T^* \mathbb{R}^2 \). The first case in (3.1) was considered in the previous Section, whereas the third case leads to an integrable system with quadratic integrals of motion. Thus, below we consider only the second and fourth separation relations in (3.1).
3.1 Case B.

In this case
\[ f(x) = ax^4 + H_1 x^3 + H_2 x^2 + bx + c, \quad P(x) = x \left( \frac{(x - x_2)y_1}{x_1(x_1 - x_2)} + \frac{(x - x_1)y_2}{x_2(x_2 - x_1)} \right) \]

and variables \( x_{1,4} = \bar{u}_{1,2} \) are the roots of polynomial
\[
\psi(x) = \left( a - \frac{(u_1p_{u_1} - u_2p_{u_2})^2}{(u_1 - u_2)^2} \right)x^2 + \frac{bu_1u_2 + c(u_1 + u_2)}{u_1^2u_2}x + \frac{c}{u_1u_2},
\]

These coordinates commute with respect to the Poisson brackets
\[
\{u_i, p_{u_j}\}_W = u_i \delta_{ij}, \quad \{u_1, u_2\}_W = \{p_{u_1}, p_{u_2}\}_W = 0. 
\]

Using an additional Poisson map
\[
\rho_B : (u_1, u_2, p_{u_1}, p_{u_2}) \to (u_1, u_2, u_1p_{u_1}, u_2p_{u_2}) ,
\]
we can define canonical variables \( \hat{u}_{1,2} \) on \( T^*\mathbb{R}^2 \), which are the roots of polynomial
\[
\rho_B \left( \frac{\psi(x)}{(x - u_1)(x - x_2)} \right) = \left( a - \frac{(u_1^2p_{u_1} - u_2^2p_{u_2})^2}{(u_1 - u_2)^2} \right)x^2 + \frac{bu_1u_2 + c(u_1 + u_2)}{u_1^2u_2}x + \frac{c}{u_1u_2},
\]

and the conjugated momenta
\[
\hat{p}_{u_{1,2}} = -\hat{u}_{1,2}^3 \rho_B(P)|_{x=\hat{u}_{1,2}}, \quad \rho_B(P) = x \left( \frac{(x - u_2)u_1^2p_{u_1}}{u_1 - u_2} + \frac{(x - u_1)u_2^2p_{u_2}}{u_2 - u_1} \right),
\]

so that
\[
\{\hat{u}_i, \hat{p}_{u_i}\} = \delta_{ij}, \quad \{\hat{u}_1, \hat{u}_2\} = \{\hat{p}_{u_1}, \hat{p}_{u_2}\} = 0. 
\]

Substituting these canonical variables into the separated relations
\[
2\lambda_i = \left( \hat{u}_i^3p_{u_i} - a\hat{u}_i - \frac{b}{\hat{u}_i} - \frac{c}{\hat{u}_i^2} \right) = \hat{H}_1 \pm \sqrt{\hat{H}_2}, \quad i = 1, 2,
\]

one gets Hamilton function
\[
\hat{H}_1 = \frac{u_1^3(bu_1u_2 + c(3u_1 + u_2))}{c(u_1 - u_2)}p_{u_1}^2 + \frac{u_2^3(bu_1u_2 + c(u_1 + 3u_2))}{c(u_2 - u_1)}p_{u_2}^2 - a \left( \frac{bu_1u_2}{c} + 3(u_1 + u_2) \right)
+ \frac{b^2(u_1 + u_2)}{cu_1u_2} + \frac{b(u_1 + 2u_2)(2u_1 + u_2)}{u_1^3u_2^2} + \frac{c(u_1 + u_2)(u_1^2 + 3u_1u_2 + u_2^2)}{u_1^2u_2^2}
\]

and the second integral of motion, which is polynomial of sixth order in momenta
\[
\hat{H}_2 = \left( \frac{u_1^3u_2^3(u_1p_{u_1} - u_2p_{u_2})^2}{c(u_1 - u_2)^2} - \frac{a(u_1^3u_2^3)}{c} + \frac{b^2u_1^2u_2^2}{4c^2} + \frac{b(u_1 + u_2)u_1u_2}{2c} + \frac{(u_1 + u_2)^2}{4} \right)
\]
\times \frac{4}{(u_1 - u_2)^2} \left( a(u_1 - u_2) + b \left( \frac{1}{u_1} - \frac{1}{u_2} \right) + c \left( \frac{1}{u_1} - \frac{1}{u_2} \right) - u_1^3p_{u_1}^3 + u_2^3p_{u_2}^3 \right)^2.
\]

Vector field associated with Hamiltonian \( \hat{H}_1 \) is a bi-Hamiltonian vector field with respect to the compatible Poisson brackets \( (2.15, 2.16) \).

If we put \( a = b = 0 \) and then \( c = 0 \), we obtain a geodesic flow on the plane with an integral of motion of third order in momenta
\[
\hat{H}_1 = T = u_1^3(3u_1 + u_2)p_{u_1}^2 + u_2^3(u_1 + 3u_2)p_{u_2}^2,
\]
\[
K = \frac{\frac{3/2}{u_1^2}u_2^3(u_1p_{u_1} - u_2p_{u_2})(u_1^3p_{u_1}^3 - u_2^3p_{u_2}^3)}{(u_1 - u_2)^2}.
\]

(3.2)
In original Cartesian coordinates these integrals of motion have the form

\[ \dot{H}_1 = T = \frac{3q_1^2 q_2^2}{2} p_1^2 + (q_1^2 + 6q_2^2)q_1 p_1 p_2 + \frac{q_2 (5q_1^2 + 12q_2^2)}{2} p_2^2 \]

and

\[ K = q_1^5 p_1^3 + 6q_1^2 q_2^2 p_2^2 + q_1^4 (q_1^2 + 12q_2^2) p_1 p_2^2 + 2(q_1^2 + 4q_2^2)q_1^3 q_2 p_2^3. \]

Construction and classification of all the integrable geodesic flows on Riemannian manifolds is a classical problem in Riemannian geometry [3, 8, 12]. We propose to use auto BTs to solution of this problem. Similar to [8] second integral of motion \( K \) is factorized on two polynomials in momenta but in contrast with [8] factors itself do not commute with \( T \). In three dimensional case similar examples of factorized integrals of motion are discussed in [18].

### 3.2 Case D.

In this case

\[ f(x) = ax^4 + bx^3 + cx^2 + H_1 x + H_2, \quad P(x) = x \left( \frac{(x-x_2)y_1}{x_1(x_1-x_2)} + \frac{(x-x_1)y_2}{x_2(x_2-x_1)} \right) \]

and variables \( x_{3,4} = u_{1,2} \) are the roots of polynomial

\[ \psi(x) = \left( a - \frac{(u_1 p_u - u_2 p_u)^2}{(u_1 - u_2)^2} \right) x^2 + \left( a(u_1 + u_2) + b - \frac{u_1^2 p_u - u_2^2 p_u}{u_1 - u_2} \right) x + \left( a(u_2^2 + u_1 u_2) + b(u_1 + u_2) + c - \frac{u_1^3 p_u - u_2^3 p_u}{u_1 - u_2} \right). \]

These coordinates commute with respect to the Poisson brackets

\[ \{u_i, p_{u_j}\}_W = u_i^{-1} \delta_{ij}, \quad \{u_1, u_2\}_W = \{p_{u_1}, p_{u_2}\}_W = 0 \]

associated with the Weierstrass change of time.

Using an additional Poisson map

\[ \rho_D : \ (u_1, u_2, p_{u_1}, p_{u_2}) \rightarrow (u_1, u_2, u_1^{-1} p_{u_1}, u_2^{-1} p_{u_2}) \],

we can define canonical variables \( u_{1,2} \) on \( T^* \mathbb{R}^3 \), which are the roots of polynomial

\[ \rho_D \left( \frac{\psi(x)}{(x-u_1)(x-u_2)} \right) = \left( a - \frac{(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^2} \right) x^2 + \left( a(u_1 + u_2) + b - \frac{p_{u_1}^2 - p_{u_2}^2}{u_1 - u_2} \right) x + \left( a(u_1^2 + u_1 u_2 + u_2^2) + b(u_1 + u_2) + c - \frac{u_1^3 p_{u_1} - u_2^3 p_{u_2}}{u_1 - u_2} \right), \]

and the conjugated momenta

\[ \dot{p}_{u_1,2} = -u_i^{-1} \rho_D(P)_{x=u_{1,2}}, \quad \rho_D(P) = x \left( \frac{(x-u_2)p_{u_1}}{u_1 - u_2} + \frac{(x-u_1)p_{u_2}}{u_2 - u_1} \right), \]

so that

\[ \{\dot{u}_i, \dot{p}_{u_j}\} = \delta_{ij}, \quad \{\dot{u}_1, \dot{u}_2\} = \{\dot{p}_{u_1}, \dot{p}_{u_2}\} = 0. \]

Substituting these variables into the separation relations

\[ 2\lambda_i = \left( \dot{u}_i p_{u_i}^2 - a u_i^3 - b u_i^2 - c u_i \right) = \dot{H}_1 \pm \sqrt{\dot{H}_2}, \quad i = 1, 2, \]
one gets Hamilton function

\[
\hat{H}_1 = \frac{u_1(2u_1 + u_2)p_{u_1}^2}{u_1 - u_2} + \frac{u_2(u_1 + 2u_2)p_{u_2}^2}{u_2 - u_1} - a(u_1 + u_2)(2u_1^2 + u_1u_2 + 2u_2^2) \\
- b(2u_1^2 + 3u_1u_2 + 2u_2^2) - 2c(u_1 + u_2)
\]

and the second integral of motion, which is polynomial of fourth order in momenta

\[
\hat{H}_2 = \frac{u_1^2u_2^2}{u_1 - u_2} \left( \frac{(p_{u_1} - p_{u_2})^3}{(u_1 - u_2)^3} \right) \left( (3u_1u_2)p_{u_1} + (u_1 + 3u_2)p_{u_2} \right) - \frac{4c(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^2} \\
- 2a \left( \frac{3(u_1^2 + u_2^2)(p_{u_1}^2 + p_{u_2}^2) - 4(u_1^2 + u_1u_2 + u_2^2)p_{u_1}p_{u_2}}{(u_1 - u_2)^2} \right) \\
+ a^2(3u_1^2 + 2u_1u_2 + 3u_2^2) + a(2b(u_1 + u_2) + 4c) - b^2.
\]

Vector field associated with Hamiltonian \(\hat{H}_1\) is a bi-Hamiltonian vector field with respect to the compatible Poisson brackets (2.17) and (2.18).

If we put \(a = b = 0\) and then \(c = 0\), we obtain a geodesic flow with an integral of motion, which is polynomial of fourth order in momenta

\[
\hat{H}_1 = T = \frac{u_1(2u_1 + u_2)p_{u_1}^2}{u_1 - u_2} + \frac{u_2(u_1 + 2u_2)p_{u_2}^2}{u_2 - u_1}, \\
K = \frac{u_1^2u_2^2}{u_1 - u_2} \left( \frac{(p_{u_1} - p_{u_2})^3}{(u_1 - u_2)^3} \right) \left( (3u_1u_2)p_{u_1} + (u_1 + 3u_2)p_{u_2} \right).
\]

This integrable system is superintegrable system with one more integral of motion, which is a polynomial of third order in momenta

\[
J = \frac{u_1u_2(p_{u_1} - p_{u_2})^2(u_1^2p_{u_1} - u_2^2p_{u_2})}{(u_1 - u_2)^3},
\]

so that

\[
\{T, K\} = 0, \quad \{T, J\} = 0, \quad \{J, K\} \neq 0.
\]

In original Cartesian coordinates these integrals of motion are

\[
T = q_2 \left( \frac{p_2^2}{2} + \frac{p_1^2}{2} \right) + \frac{q_1p_1p_2}{2}, \quad K = \frac{(q_1^2 - q_2^2)p_1^2}{4} + \frac{q_1q_2p_1^2p_2}{2}, \quad J = (q_1p_1 + 2q_2p_2)p_2^2.
\]

Using additional canonical transformation we can reduce polynomial \(T\) to the following form

\[
T = m_1(q_1, q_2)p_1^2 + m_2(q_1, q_2)p_2^2
\]

and obtain new example of superintegrable system with the position dependent masses, see [13] and references therein.

4 Conclusion

We prove that auto Bäcklund transformation of Hamilton-Jacobi equation which represents symmetry of the level manifolds can be also useful in classical mechanics as the auto Bäcklund transformation associated with addition law in Jacobian.

In particular, we prove that geodesic flow with diagonal in parabolic coordinates metric

\[
\hat{g}^{(km)} = \begin{pmatrix}
\frac{(ku_1 + u_2)u_{1}^m}{u_1 - u_2} & 0 \\
0 & \frac{(ku_2 + u_1)u_{2}^m}{u_2 - u_1}
\end{pmatrix}
\]

and references therein.
is integrable for \( k = 3; m = 3, 4 \) and for \( k = 2; m = 1 \). The corresponding integrals of motion

\[
K = (u_1^2 p_{u_1} - u_2^2 p_{u_2}) \cdot \frac{u_1^m u_2^m (u_1^m p_{u_1}^2 - u_2^m p_{u_2}^2)}{(u_1 - u_2)^2}, \quad m = 3, 4
\]

and

\[
K = (u_1^2 p_{u_1} - u_2^2 p_{u_2}) \cdot \frac{u_1^m u_2^m (p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^3}, \quad m = 1
\]

are polynomials of the third order in momenta with a common factor. It allows us to suppose, that there are similar integrable systems for other values of \( k \) and \( m \) in (4.4), see discussion in [3, 8, 12]. Indeed, substituting

\[
T = \sum_{i,j=1}^2 \hat{g}^{(km)}_{ij}(u)p_{u_i}p_{u_j}
\]

and

\[
K = (u_1^2 p_{u_1} - u_2^2 p_{u_2}) \cdot (f(u_1, u_2)p_{u_1}^2 + g(u_1, u_2)p_{u_1}p_{u_2} + h(u_1, u_2)p_{u_2}^2)
\]

into the equation \( \{T, K\} = 0 \) and solving the resulting system of PDE’s with respect to functions \( f, g \) and \( h \) we can prove the following proposition.

**Proposition 4** Metric \( \hat{g}^{(km)} \) yields integrable geodesic flow on the plane at \( m = 1, k = 2 \) and

\[
\begin{align*}
m &= 3, & k &= \pm 1, 3, \frac{1}{2}, \quad \text{and} \quad m = 4, & k &= \pm 1, \pm 3, -\frac{3}{5}, -\frac{1}{7}, \frac{1}{5}, \frac{1}{2}.
\end{align*}
\]

In the similar manner we can study common properties of the obtained variables of separation, compatible Poisson brackets and recursion operators. It can be useful for investigation of other integrable systems with integrals of motion of third, fourth and sixth order in momenta. In particular we hope to use obtained experience for the study of Toda lattice associated with \( G_2 \) root system.

Starting with other well-known separable Hamilton-Jacobi equations on the plane, sphere, ellipsoid, Lorentzian space, de Sitter and anti-de Sitter spaces we can also obtain new integrable systems with polynomial integrals of motion. We plan to describe some of these systems in the forthcoming publications.

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