Generalised matrix multivariate Pearson type II-distribution

José A. Díaz-García *
Universidad Autónoma Agraria Antonio Narro

Ramón Gutiérrez-Sánchez †
University of Granada

Abstract Matrix multivariate Pearson type II-Riesz distribution is defined and some of its properties are studied. In particular, the associated matrix multivariate beta distribution type I is derived. Also the singular values and eigenvalues distributions are obtained.

Keywords Matrix multivariate; Pearson Type II distribution; Riesz distribution; Kotz-Riesz distribution; real, complex, quaternion and octonion random matrices; real normed division algebras.

1 Introduction

When a new statistic theory is proposed, the statistician known well about the rigourously mathematical foundations of their discipline, however in order to reach a wider interdisciplinary public, some of the classical statistical techniques have been usually published without explaining the supporting abstract mathematical tools which governs the approach. For example, in the context of the distribution theory of random matrices, in the last 20 years, a number of more abstract and mathematical approaches have emerged for studying and generalising the usual matrix variate distributions. In particular, this needing have appeared recently in the

*Corresponding author: José A. Díaz-García, Universidad Autónoma Agraria Antonio Narro, Calzada Antonio Narro 1923, Col. Buena Vista, 25315 Saltillo, Coahuila, México, E-mail: jadiaz@uaaan.mx
†Department of Statistics and O.R, University of Granada, Granada 18071, Spain, E-mail:ramongs@ugr.es
generalisation, by using abstract algebra, of some results of real random matrices to another supporting fields, such as complex, quaternion and octonion, see Ratnarajah et al. [26], Ratnarajah et al. [27], Edelman and Rao [12], Forrester [15], among many others authors. Studying distribution theory by another algebras, beyond real, have led several generalisations of substantial understanding in the theoretical context, and we expect that it is more extensively applied when a an improvement of its unified potential can be explored in other contexts. Two main tendencies have been considered in literature, Jordan algebras and real normed division algebras. Some works dealing the first approach are due to Faraut and Korányi [14], Massam [24], Casalis, and Letac [3], Hassairi and Lajmi [19], Hassairi et al. [20, 21], Kołodziejek [23], and the references therein, meanwhile, the second technique has been studied by Gross and Richards [16], Díaz-García and Gutiérrez-Jáimez [8], Díaz-García [4], Díaz-García [5, 6], among many others.

In the same manner, different generalisations of the multivariate statistical analysis have been proposed recently. This generalised technique studies the effect of changing the usual matrix multivariate normal support by a general matrix multivariate family of distributions, such as the elliptical contoured distributions (or simply, matrix multivariate elliptical distributions), see Fang and Zhang [13] and Gupta and Varga [18]. This family of distributions involves a number of known matrix multivariate distributions such as normal, Kotz type, Bessel, Pearson type II and VII, contaminated normal and power exponential, among many others. Two important properties of these distributions must be emphasise: i). Matrix multivariate elliptical distributions provide more flexibility in the statistical modeling by including distributions with heavier or lighter tails and/or greater or lower degree of kurtosis than matrix multivariate normal distribution; and, ii). Most of the statistical tests based on matrix multivariate normal distribution are invariant under the complete family of matrix multivariate elliptical distributions.

Recently, a slight combination of these two theoretical generalisations have appeared in literature; namely, Jordan algebras has been led to the matrix multivariate Riesz distribution and its associated beta distribution. Díaz-García [6] proved that the above mentioned distributions can be derived from a particular matrix multivariate elliptical distribution, termed matrix multivariate Kotz-Riesz distribution. Similarly, matrix multivariate Riesz distribution is also of interest from the mathematical point of view; in fact most of their basic properties under *the structure theory of normal j-algebras* and *the theory of Vinberg algebras* in place of Jordan
algebras have been studied by Ishi [22] and Boutouria and Hassiri [2], respectively.

In this scenario, we can now propose a generalisation of the matrix multivariate beta, T and Pearson type II distributions based on a matrix multivariate Kotz-Riesz distribution. As usual in the normal case, extensions of beta, T and Pearson type II distributions involves two alternatives, the matricvariate and the matrix multivariate versions, see Díaz-García [4], Díaz-García [3, 6], Díaz-García and Gutiérrez-Jáimez [7, 8, 9] and Díaz-García and Gutiérrez-Sánchez [10].

This article derives the matrix multivariate beta and Pearson type II distributions obtained from a matrix multivariate Kotz-Riesz distribution and some of their basic properties are studied. Section 2 gives some basic concepts and the notation of abstract algebra, Jacobians and distribution theory. The nonsingular central matrix multivariate Pearson type II-Riesz distribution and the corresponding generalised matrix multivariate beta type I distribution are studied in Section 3. Finally, the joint densities of the singular values are derived in Section 4.

2 Preliminary results

A detailed discussion of real normed division algebras can be found in Baez [1] and Neukirch et al. [25]. For your convenience, we shall introduce some notation, although in general, we adhere to standard notation forms.

For our purposes: Let $\mathbb{F}$ be a field. An algebra $\mathfrak{A}$ over $\mathbb{F}$ is a pair $(\mathfrak{A}; m)$, where $\mathfrak{A}$ is a finite-dimensional vector space over $\mathbb{F}$ and multiplication $m : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is an $\mathbb{F}$-bilinear map; that is, for all $\lambda \in \mathbb{F}$, $x, y, z \in \mathfrak{A}$,

$$m(x, \lambda y + z) = \lambda m(x; y) + m(x; z)$$
$$m(\lambda x + y; z) = \lambda m(x; z) + m(y; z).$$

Two algebras $(\mathfrak{A}; m)$ and $(\mathfrak{C}; n)$ over $\mathbb{F}$ are said to be isomorphic if there is an invertible map $\phi : \mathfrak{A} \to \mathfrak{C}$ such that for all $x, y \in \mathfrak{A}$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write $m(x; y) = xy$ for all $x, y \in \mathfrak{A}$.\footnote{The term matricvariate distribution was first introduced Dickey [11], but the expression matrix-variate distribution or matrix variate distribution or matrix multivariate distribution was later used to describe any distribution of a random matrix, see Gupta and Nagar [17] and Gupta and Varga [18], and the references therein. When the density function of a random matrix is written including the trace operator then the matrix multivariate designation shall be used.}
Let $\mathcal{A}$ be an algebra over $F$. Then $\mathcal{A}$ is said to be

1. **alternative** if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathcal{A}$,

2. **associative** if $x(yz) = (xy)z$ for all $x, y, z \in \mathcal{A}$,

3. **commutative** if $xy = yx$ for all $x, y \in \mathcal{A}$, and

4. **unital** if there is a $1 \in \mathcal{A}$ such that $x1 = x = 1x$ for all $x \in \mathcal{A}$.

If $\mathcal{A}$ is unital, then the identity $1$ is uniquely determined.

An algebra $\mathcal{A}$ over $F$ is said to be a **division algebra** if $\mathcal{A}$ is nonzero and $xy = 0_\mathcal{A} \Rightarrow x = 0_\mathcal{A}$ or $y = 0_\mathcal{A}$ for all $x, y \in \mathcal{A}$.

The term “division algebra”, comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let $\mathcal{A}$ be an algebra over $F$. Then $\mathcal{A}$ is a division algebra if, and only if, $\mathcal{A}$ is nonzero and for all $a, b \in \mathcal{A}$, with $b \neq 0_\mathcal{A}$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in \mathcal{A}$.

In the sequel we assume $F = \mathbb{R}$ and consider classes of division algebras over $\mathbb{R}$ or “real division algebras” for short.

We introduce the algebras of **real numbers** $\mathbb{R}$, **complex numbers** $\mathbb{C}$, **quaternions** $\mathbb{H}$ and **octonions** $\mathbb{O}$. Then, if $\mathcal{A}$ is an alternative real division algebra, then $\mathcal{A}$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

Let $\mathcal{A}$ be a real division algebra with identity $1$. Then $\mathcal{A}$ is said to be **normed** if there is an inner product $(\cdot, \cdot)$ on $\mathcal{A}$ such that

$$(xy, xy) = (x, x)(y, y) \quad \text{for all } x, y \in \mathcal{A}.$$ 

If $\mathcal{A}$ is a **real normed division algebra**, then $\mathcal{A}$ is isomorphic $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

There are exactly four normed division algebras: real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and octonions ($\mathbb{O}$), see Baez [1]. We take into account that should be taken into account, $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only normed division algebras; furthermore, they are the only alternative division algebras.

Let $\mathcal{A}$ be a division algebra over the real numbers. Then $\mathcal{A}$ has dimension either $1, 2, 4$ or $8$. Finally, observe that

- $\mathbb{R}$ is a real commutative associative normed division algebras,
- $\mathbb{C}$ is a commutative associative normed division algebras,
- $\mathbb{H}$ is an associative normed division algebras,
- $\mathbb{O}$ is an alternative normed division algebras.
Let \( L_{\beta}^{n,m} \) be the set of all \( n \times m \) matrices of rank \( m \leq n \) over \( \mathfrak{A} \) with \( m \) distinct positive singular values, where \( \mathfrak{A} \) denotes a real finite-dimensional normed division algebra. Let \( \mathfrak{A}^{n \times m} \) be the set of all \( n \times m \) matrices over \( \mathfrak{A} \). The dimension of \( \mathfrak{A}^{n \times m} \) over \( \mathbb{R} \) is \( \beta mn \). Let \( \mathbf{A} \in \mathfrak{A}^{n \times m} \), then \( \mathbf{A}^* = \overline{\mathbf{A}}^T \) denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

| Real | Complex | Quaternion | Octonion | Generic notation |
|------|---------|------------|----------|------------------|
| Semi-orthogonal | Semi-unitary | Semi-symplectic | Semi-exceptional type | \( \mathfrak{V}^{\beta}_{m,n} \) |
| Orthogonal | Unitary | Symplectic | Exceptional type | \( \mathfrak{U}^{\beta}(m) \) |
| Symmetric | Hermitian | Quaternion | Octonion | \( \mathfrak{S}^{\beta}_m \) |

We denote by \( \mathfrak{S}^{\beta}_m \) the real vector space of all \( \mathfrak{S} \in \mathfrak{A}^{m \times m} \) such that \( \mathfrak{S} = \mathfrak{S}^* \). In addition, let \( \mathfrak{P}^{\beta}_m \) be the cone of positive definite matrices \( \mathfrak{S} \in \mathfrak{A}^{m \times m} \). Thus, \( \mathfrak{P}^{\beta}_m \) consist of all matrices \( \mathfrak{S} = \mathbf{X}^* \mathbf{X} \), with \( \mathbf{X} \in \mathfrak{L}^{\beta}_{n,m} \); then \( \mathfrak{P}^{\beta}_m \) is an open subset of \( \mathfrak{S}^{\beta}_m \).

Let \( \mathfrak{D}^{\beta}_m \) consisting of all \( \mathfrak{D} \in \mathfrak{A}^{m \times m} \), \( \mathfrak{D} = \text{diag}(d_1, \ldots, d_m) \). Let \( \mathfrak{T}^{\beta}_{U}(m) \) be the subgroup of all upper triangular matrices \( \mathfrak{T} \in \mathfrak{A}^{m \times m} \) such that \( t_{ij} = 0 \) for \( 1 < i < j \leq m \). Let \( \mathbf{Z} \in \mathfrak{L}^{\beta}_{n,m} \), define the norm of \( \mathbf{Z} \) as \( ||\mathbf{Z}|| = \sqrt{\text{tr}\mathbf{Z}^*\mathbf{Z}} \).

For any matrix \( \mathbf{X} \in \mathfrak{A}^{n \times m} \), \( d\mathbf{X} \) denotes the matrix of differentials \( (dx_{ij}) \). Finally, we define the measure or volume element \( (d\mathbf{X}) \) when \( \mathbf{X} \in \mathfrak{A}^{n \times m}, \mathfrak{S}^{\beta}_m, \mathfrak{D}^{\beta}_m \) or \( \mathfrak{V}^{\beta}_{m,n} \), see Díaz-García and Gutiérrez-Jáimez [7] and Díaz-García and Gutiérrez-Jáimez [9].

If \( \mathbf{X} \in \mathfrak{A}^{n \times m} \) then \( (d\mathbf{X}) \) (the Lebesgue measure in \( \mathfrak{A}^{n \times m} \)) denotes the exterior product of the \( \beta mn \) functionally independent variables

\[
(d\mathbf{X}) = \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}. \]

If \( \mathbf{S} \in \mathfrak{S}^{\beta}_m \) (or \( \mathbf{S} \in \mathfrak{T}^{\beta}_{U}(m) \)) with \( t_{ii} > 0, \ i = 1, \ldots, m \) then \( (d\mathbf{S}) \) (the Lebesgue measure in \( \mathfrak{S}^{\beta}_m \) or in \( \mathfrak{T}^{\beta}_{U}(m) \)) denotes the exterior product of the exterior product of the \( m(m-1)\beta/2 + m \) functionally independent variables,

\[
(d\mathbf{S}) = \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i>j}^{m} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}. \]
Observe, that for the Lebesgue measure \((dS)\) defined thus, it is required that \(S \in \mathcal{P}_m^\beta\), that is, \(S\) must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If \(A \in \mathcal{D}_m^\beta\) then \((dA)\) (the Legesgue measure in \(\mathcal{D}_m^\beta\)) denotes the exterior product of the \(\beta m\) functionally independent variables

\[
(dA) = \bigwedge_{i=1}^n \bigwedge_{k=1}^\beta dA_i^{(k)}.
\]

If \(H_1 \in \mathcal{V}_{m,n}^\beta\) then

\[
(H_1^* dH_1) = \bigwedge_{i=1}^m \bigwedge_{j=i+1}^n h_i^* d h_i,
\]

where \(H = (H_1^* H_2^*)^* = (h_1, \ldots, h_m | h_{m+1}, \ldots, h_n)^* \in \Omega^\beta(n)\). It can be proved that this differential form does not depend on the choice of the \(H_2\) matrix. When \(n = 1\); \(\mathcal{V}_{m,1}^\beta\) defines the unit sphere in \(\mathcal{A}^m\). This is, of course, an \((m-1)\beta\)-dimensional surface in \(\mathcal{A}^m\). When \(n = m\) and denoting \(H_1\) by \(H\), \((H dH^*)\) is termed the \(Haar\) measure on \(\Omega^\beta(m)\).

The surface area or volume of the Stiefel manifold \(\mathcal{V}_{m,n}^\beta\) is

\[
\text{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{H_1 \in \mathcal{V}_{m,n}^\beta} (H_1 dH_1^*) = \frac{2^m \pi^m n \beta/2}{\Gamma_m^\beta [n/2]}, \tag{1}
\]

where \(\Gamma_m^\beta[a]\) denotes the multivariate \(Gamma\) function for the space \(\mathcal{G}_m^\beta\) and is defined as

\[
\Gamma_m^\beta[a] = \int_{A \in \mathcal{G}_m^\beta} \text{etr}\{-A\} |A|^{a-(m-1)\beta/2-1} (dA)
= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a - (i - 1)\beta/2],
\]

and \(\text{Re}(a) > (m-1)\beta/2\). This can be obtained as a particular case of the \(generalised\) \(\gamma\) \(function of weight \(\kappa\)) for the space \(\mathcal{G}_m^\beta\) with \(\kappa = (k_1, k_2, \ldots, k_m) \in \mathbb{R}^m\), taking \(\kappa = (0, 0, \ldots, 0) \in \mathbb{R}^m\) and which for \(\text{Re}(a) \geq (m-1)\beta/2 - k_m\) is defined by, see Gross and Richards \[16\] and Faraut and Korányi \[14\],

\[
\Gamma_m^\beta[a, \kappa] = \int_{A \in \mathcal{G}_m^\beta} \text{etr}\{-A\} |A|^{a-(m-1)\beta/2-1} g_\kappa(A) (dA) \tag{2}
= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a + k_i - (i - 1)\beta/2]
= [a]_\kappa^{\beta} \Gamma_m^\beta[a], \tag{3}
\]
where $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$, $|\cdot|$ denotes the determinant, and for $A \in \mathfrak{S}_m^\beta$

$$q_\kappa(A) = |A_m|^{k_m} \prod_{i=1}^{m-1} |A_i|^{k_i - k_{i+1}}$$

(4)

with $A_p = (a_{rs}), r, s = 1, 2, \ldots, p, p = 1, 2, \ldots, m$ is termed the highest weight vector, see Gross and Richards [16], Faraut and Korányi [14] and Hassairi and Lajmi [19]; And, $[a]^\beta_\kappa$ denotes the generalised Pochhammer symbol of weight $\kappa$, defined as

$$[a]^\beta_\kappa = \prod_{i=1}^{m} (a - (i - 1)\beta/2)_{k_i}$$

$$= \frac{\pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a + k_i - (i - 1)\beta/2] \Gamma^\beta_m[a]}{\Gamma^\beta_m[a]}$$

where $\text{Re}(a) > (m - 1)\beta/2 - k_m$ and

$$(a)_i = a(a + 1) \cdots (a + i - 1),$$

is the standard Pochhammer symbol.

Additional, note that, if $\kappa = (p, \ldots, p)$, then $q_\kappa(A) = |A|^p$. In particular if $p = 0$, then $q_\kappa(A) = 1$. If $\tau = (t_1, t_2, \ldots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, then $q_{\kappa+\tau}(A) = q_\kappa(A)q_\tau(A)$, and in particular if $\tau = (p, p, \ldots, p)$, then $q_{\kappa+\tau}(A) \equiv q_{p+k}(A) = |A|^p q_\kappa(A)$. Finally, for $B \in \mathfrak{S}_U^\beta(m)$ in such a manner that $C = B^*B \in \mathfrak{S}_m^\beta$, $q_\kappa(B^*AB) = q_\kappa(C)q_\kappa(A)$, and $q_\kappa(B^{*-1}AB^{-1}) = (q_\kappa(C))^{-1}q_\kappa(A) = q_{-\kappa}(C)q_\kappa(A)$, see Hassairi et al. [21].

Finally, the following Jacobians involving the $\beta$ parameter, reflects the generalised power of the algebraic technique; the can be seen as extensions of the full derived and unconnected results in the real, complex or quaternion cases, see Faraut and Korányi [14] and Díaz-García and Gutiérrez-Jáimez [7]. These results are the base for several matrix and matric variate generalised analysis.

**Proposition 2.1.** Let $X$ and $Y \in \mathfrak{S}_{n,m}^\beta$ be matrices of functionally independent variables, and let $Y = AXB + C$, where $A \in \mathfrak{S}_{n,n}^\beta$, $B \in \mathfrak{S}_{m,m}^\beta$ and $C \in \mathfrak{S}_{n,m}^\beta$ are constant matrices. Then

$$(dY) = |A^*A|^{m_\beta/2}|B^*B|^{mn_\beta/2}(dX).$$

(5)
Proposition 2.2 (Singular Value Decomposition, SVD). Let $X \in L^{\beta}_{n,m}$ be matrix of functionally independent variables, such that $X = W_1 D V^*$ with $W_1 \in V^\beta_{m,n}$, $V \in U^\beta(m)$ and $D = \text{diag}(d_1, \ldots, d_m) \in \mathcal{D}^1_m$, $d_1 > \cdots > d_m > 0$. Then

\[
(dX) = 2^{-m} \pi^\beta \prod_{i=1}^{m} d_i^{\beta(n-m+1)-1} \prod_{i<j}^{m} (d_i^2 - d_j^2)^{\beta/2} (dD)(V^*dV)(W_1^*dW_1),
\]

where

\[
\varrho = \begin{cases} 
0, & \beta = 1; \\
-m, & \beta = 2; \\
-2m, & \beta = 4; \\
-4m, & \beta = 8.
\end{cases}
\]

Proposition 2.3. Let $X \in L^{\beta}_{n,m}$ be matrix of functionally independent variables, and write $X = V_1 T$, where $V_1 \in V^\beta_{m,n}$ and $T \in \mathcal{U}^\beta_U(n)$ with positive diagonal elements. Define $S = X^*X \in P^{\beta}_m$. Then

\[
(dX) = 2^{-m}|S|^{\beta(n-m+1)/2-1}(dS)(V_1^*dV_1).
\]

Finally, to define the matrix multivariate Pearson type II-Riesz distribution we need to recall the following two definitions of Kotz-Riesz and Riesz distributions.

From Díaz-García [6].

Definition 2.1. Let $\Sigma \in \Phi^\beta_{m,n}$, $\Theta \in \Phi^\beta_{n,m}$, $\mu \in L^\beta_{n,m}$ and $\kappa = (k_1, k_2, \ldots, k_m) \in \mathbb{R}^m$. And let $Y \in \mathcal{L}^\beta_{n,m}$ and $\mathcal{U}(B) \in \mathcal{U}^\beta_U(n)$, such that $B = \mathcal{U}(B)^* \mathcal{U}(B)$ is the Cholesky decomposition of $B \in \mathcal{G}^\beta_m$. Then it is said that $Y$ has a Kotz-Riesz distribution of type I and its density function is

\[
\frac{\beta^{mn\beta/2+\sum_{i=1}^{m} k_i} \Gamma_m[n\beta/2]}{\pi^{mn\beta/2} \Gamma_m[n\beta/2, \kappa] |\Sigma|^{n\beta/2} |\Theta|^{m\beta/2}}
\times \det \left\{-\beta \text{tr} \left[\Sigma^{-1}(Y - \mu)^* \Theta^{-1}(Y - \mu)\right]\right\}
\times q_\kappa \left[\mathcal{U}(\Sigma)^{-1}(Y - \mu)^* \Theta^{-1}(Y - \mu) \mathcal{U}(\Sigma)^{-1}\right] (dY)
\]

with $\text{Re}(n\beta/2) > (m-1)\beta/2 - k_m$; denoting this fact as

\[
Y \sim \mathcal{K}_{P^\beta_{n,m}}^{\beta, I}(\kappa, \mu, \Theta, \Sigma).
\]

From Hassairi and Lajmi [19] and Díaz-García [4].
Definition 2.2. Let \( \Xi \in \Phi^\beta_m \) and \( \kappa = (k_1, k_2, \ldots, k_m) \in \mathbb{R}^m \). Then it is said that \( V \) has a Riesz distribution of type I if its density function is

\[
\frac{\beta^{am + \sum_{i=1}^m k_i}}{\Gamma_m[a, \kappa]|\Xi|^{a}q_\kappa(\Xi)} \text{etr}\{ -\beta \Xi^{-1}V\}|V|^{a-(m-1)\beta/2-1}q_\kappa(V)(dV) \tag{9}
\]

for \( V \in \mathcal{P}_\beta^m \) and \( \text{Re}(a) \geq (m-1)\beta/2 - k_m \); denoting this fact as \( V \sim \mathcal{R}_m^a, I(a, \kappa, \Xi) \).

3 Matrix multivariate Pearson type II-Riesz distribution

A detailed discussion of Riesz distribution may be found in Hassairi and Lajmi [19] and Díaz-García [4]. In addition the Kotz-Riesz distribution is studied in detail in Díaz-García [6]. For your convenience, we adhere to standard notation stated in Díaz-García [4], Díaz-García [6].

Theorem 3.1. Let \( \left( S_1^{1/2}\right)^2 = S_1 \sim \mathcal{R}_{1}^{\beta,I}(\nu \beta/2, k, 1) \), \( k \in \mathbb{R} \) and \( \text{Re}(\nu \beta/2) > -k \); independent of \( Y \sim \mathcal{K}\mathcal{R}_{n \times m}^{\beta,I}(\tau, 0, I_n, I_m) \), \( \text{Re}(n \beta/2) > (m-1)\beta/2 - t_m \). In addition, define \( R = S^{-1/2}Y \) where \( S = S_1 + ||Y||^2 \). Then \( S \sim \mathcal{R}_{1}^{\beta,I}(\nu + mn, \beta/2 + \sum_{i=1}^m t_i, k, 1) \) independent of \( R \). Furthermore the density of \( R \) is

\[
\frac{\Gamma_m[n \beta/2] \Gamma_1[(\nu + mn) \beta/2 + k + \sum_{i=1}^m t_i]}{\pi^{mn/2} \Gamma_m[n \beta/2, \tau] \Gamma_1[\beta \beta/2 + k]} (1 - ||R||^2)^{\nu \beta/2 + k - 1} q_\tau(R^*R) (dR),
\]

where \( (1 - ||R||^2) > 0 \); which is termed the standardised matrix multivariate Pearson type II-Riesz type distribution and is denoted as \( R \sim \mathcal{P}_{II} \mathcal{R}_{m \times n}^{\beta,I}(\nu, k, \tau, 1, 0, I_n, I_m) \).

Proof. From definition 2.1 and 2.2 the joint density of \( S_1 \) and \( Y \) is

\[
\propto s_1^{\nu \beta/2 + k - 1} \text{etr}\left\{ -\beta (s_1 + ||Y||^2) \right\} q_\tau(Y^*Y) (dS_1)(dY)
\]

where the constant of proportionality is

\[
c = \frac{\beta^{\nu \beta/2 + k}}{\Gamma_1[\beta \beta/2 + k]} \cdot \frac{\beta^{mn \beta/2 + \sum_{i=1}^m t_i} \Gamma_m[n \beta/2]}{\pi^{mn \beta/2} \Gamma_m[n \beta/2, \tau]}.
\]

Making the change of variable \( S = S_1 - ||Y||^2 \) and \( Y = S_1^{1/2}R \), by (5)

\[
(ds_1) \wedge (dY) = s^{\beta mn/2}(ds) \wedge (dR).
\]
Now, observing that \( S = S_1 - \|Y\|^2 = S_1 (1 - \|R\|^2) \), the joint density of \( S \) and \( R \) is

\[
\propto (1 - \|R\|^2)^{\beta/2 + k - 1} s^{\beta/2 + k - 1} \text{etr} \left\{ -\beta s \right\} q_r (sR^*R) (ds)(dR).
\]

Also, note that

\[
q_r (sR^*R) = q_r \left( (s^{1/2} I_m) R^* R (s^{1/2} I_m) \right) = q_r (sI_m) q_r (R^*R) = s^\sum t_i q_r (R^*R).
\]

From where, the joint density of \( S \) and \( R \) is

\[
= \frac{\beta^{(\nu + mn)/2} \Gamma_{\nu}^2 \Gamma_{\beta}^2 \Gamma_{m}^2}{\Gamma_{\nu + mn/2} \Gamma_{\beta + 1}^2 \Gamma_{m}^2} \text{etr} \left\{ -\beta s \right\} s^{(\nu + mn)/2 + \sum t_i - 1} (ds)
\]

\[
\times \frac{\Gamma_{m}^2}{\pi^{mn/2}} \frac{\Gamma_{m}^2 \Gamma_{\beta}^2 \Gamma_{\nu}^2}{\Gamma_{\beta/2}^2 \Gamma_{\beta/2 + 1}^2 \Gamma_{m}^2} (1 - \|R\|^2)^{\nu/2 + k - 1} q_r (R^*R) (dR),
\]

which shows that \( S \sim R_{1,1} (\nu + mn) \beta/2 + \sum t_i, k, 1 \), and is independent of \( R \), where \( R \) has the density \([10]\).

**Corollary 3.1.** Let \( R \sim \mathcal{P}_{\mathcal{T}} \mathcal{R}_{m \times n}^{\beta,1} (\nu, k, \tau, 1, 0, I_n, I_m) \) and define

\[
C = \rho^{-1/2} U(\Theta)^* R U(\Sigma) + \mu
\]

where \( U(B) \in \mathcal{U}_\nu^\beta (n) \), such that \( B = U(B)^* U(B) \) is the Cholesky decomposition of \( B \in \mathcal{S}_m^\beta \), \( \Theta \in \mathcal{P}_{m}^{\beta} \), \( \Sigma \in \mathcal{P}_{n}^{\beta} \), \( \rho > 0 \) constant and \( \mu \in \mathcal{U}_m^\beta \) is a matrix of constants. Then the density of \( S \) is

\[
\propto (1 - \rho \text{tr} \Sigma^{-1} (C - \mu)^* \Theta^{-1} (C - \mu))^{\nu/2 + k - 1}
\]

\[
\times q_r \left[ U(\Sigma)^{-1} (C - \mu)^* \Theta^{-1} (C - \mu) U(\Sigma)^{-1} \right] (dS)
\]

where \( (1 - \rho \text{tr} \Sigma^{-1} (C - \mu)^* \Theta^{-1} (C - \mu)) > 0 \); with constant of proportionality

\[
\frac{\Gamma_{m}^2}{\pi^{mn/2}} \frac{\Gamma_{m}^2 \Gamma_{\nu}^2 \Gamma_{\beta}^2 \Gamma_{\nu}^2}{\Gamma_{\beta/2}^2 \Gamma_{\beta/2 + 1}^2 \Gamma_{m}^2} (\nu + mn) \beta/2 - k - \sum t_i \rho^{mn \beta/2 - \sum t_i}
\]

which is termed the matrix multivariate Pearson type II-Riesz distribution and is denoted as \( C \sim \mathcal{P}_{\mathcal{T}} \mathcal{R}_{m \times n}^{\beta,1} (\nu, k, \tau, \rho, \Theta, \Sigma) \).

**Proof.** Observe that \( R = \rho^{1/2} U(\Theta)^* U(\Sigma)^{-1} (C - \mu) U(\Sigma)^{-1} \) and

\[
(dR) = \rho^{mn \beta/2} |\Sigma|^{-\beta n/2} |\Theta|^{-\beta m/2} (dC).
\]

The desired result is obtained making this change of variable in \([10]\).
Next we derive the corresponding matrix multivariate beta type I distribution.

**Theorem 3.2.** Let $\mathbf{R} \sim \mathcal{P}_{\mathbb{I}^n} \mathcal{R}^{\beta, I}_{n \times m}(\nu, k, \rho, 0, \mathbf{I}_n, \Sigma)$, and define $\mathbf{B} = \mathbf{R}^* \mathbf{R} \in \mathcal{F}^\beta_m$, with $n \geq m$. Then the density of $\mathbf{B}$ is,

$$\alpha |\mathbf{B}|^{(n-m+1)\beta/2-1} (1 - \rho \text{tr} \Sigma^{-1} \mathbf{B})^{\nu/2 + k - 1} q_r(\mathbf{B})(d\mathbf{B}), \quad (12)$$

where $1 - \rho \text{tr} \Sigma^{-1} \mathbf{B} > 0$; and with constant of proportionality

$$\frac{\Gamma_1^n [(\nu + mn)\beta/2 + k + \sum_{i=1}^m t_i]}{\Gamma_m^n [n\beta/2, \tau]} \frac{\Gamma_1^{\beta}[\nu\beta/2 + k]}{\Gamma_1^{\beta}[(\nu + mn)\beta/2 + k]} |\mathbf{B}|^{n-m+1} (1 - \rho \text{tr} \mathbf{B})^{\nu/2 + k - 1} q_r(\mathbf{B})(d\mathbf{B}), \quad (13)$$

where $1 - \rho \text{tr} \mathbf{B} > 0$. $\mathbf{B}$ is said to have a nonstandardised matrix multivariate beta-Riesz type I distribution.

**Proof.** The desired result follows from (10), by applying (7) and then (11); and observing that $q_r(\mathcal{U}(\Sigma)^{-1} \mathbf{B} \mathcal{U}(\Sigma)^{-1}) = q_r(\Sigma) q_r(\mathbf{B})$. \hfill $\Box$

In particular if $\Sigma = \mathbf{I}_m$ in Theorem 3.2, we obtain:

**Corollary 3.2.** Let $\mathbf{R} \sim \mathcal{P}_{\mathbb{I}^n} \mathcal{R}^{\beta, I}_{n \times m}(\nu, k, \rho, 1, 0, \mathbf{I}_n, \mathbf{I}_m)$, and define $\mathbf{B} = \mathbf{R}^* \mathbf{R} \in \mathcal{F}^\beta_m$, with $n \geq m$. Then the density of $\mathbf{B}$ is,

$$\frac{\Gamma_1^n [(\nu + mn)\beta/2 + k + \sum_{i=1}^m t_i]}{\Gamma_m^n [n\beta/2, \tau]} \frac{\Gamma_1^{\beta}[\nu\beta/2 + k]}{\Gamma_1^{\beta}[(\nu + mn)\beta/2 + k]} |\mathbf{B}|^{n-m+1} (1 - \rho \text{tr} \mathbf{B})^{\nu/2 + k - 1} q_r(\mathbf{B})(d\mathbf{B}), \quad (13)$$

where $1 - \rho \text{tr} \mathbf{B} > 0$. $\mathbf{B}$ is said to have a matrix multivariate beta-Riesz type I distribution.

**Remark 3.1.** Observe that alternatively to classical definitions of generalised matrix variate beta function (for symmetric cones), see Díaz-García [5], Faraut and Korányi [14] and Hassairi et al. [20], defined as

$$\mathcal{B}^\beta_m[a, \kappa; b, \tau] = \int_{b \in \mathbb{I}^m} |\mathbf{B}|^{b-(m-1)\beta/2-1} q_r(\mathbf{B})(\mathbf{I}_m - \mathbf{B})^{a-(m-1)\beta/2-1} q_r(\mathbf{I}_m - \mathbf{B})(d\mathbf{B})$$

$$= \int_{\mathbf{F} \in \mathcal{F}^\beta_m} |\mathbf{F}|^{b-(m-1)\beta/2-1} q_r(\mathbf{F})(\mathbf{I}_m + \mathbf{F})^{-(a+b)\beta/2-1} q_r(\mathbf{I}_m + \mathbf{F})(d\mathbf{F})$$

$$= \frac{\Gamma_m^\beta[a, \kappa]}{\Gamma_m^\beta[a + b, \kappa + \tau]},$$

where $\kappa = (k_1, k_2, \ldots, k_m) \in \mathbb{R}^m$, $\tau = (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m$, $\text{Re}(a) > (m-1)\beta/2 - k_m$ and $\text{Re}(b) > (m-1)\beta/2 - t_m$. From Corollary 3.2 and Díaz-García and Gutiérrez-Sánchez [10, Theorem 3.3.1], we have the following alternative definition:
Definition 3.1. The matrix multivariate beta function is defined and denoted as:

\[ \mathcal{B}^\beta_{m}[a, k; b, \tau] = \int_{1-\text{tr}B>0} |B|^{b-(m-1)\beta/2-1}(1-\text{tr}B)^{a+k-1}q_\tau(B)(dB) \]

\[ = \int_{\mathbb{R}^{\mathbb{Q}_m^\beta}} |F|^{b-(m-1)\beta/2-1}(1+\text{tr}F)^{-(a+mb+k+\sum_{i=1}^{m} t_i)}q_\tau(F)(dF) \]

\[ = \frac{\Gamma_1^\beta[a+k]\Gamma_1^\beta[b, \tau]}{\Gamma_1^\beta[a+mb+k+\sum_{i=1}^{m} t_i]} . \]

Also, observe that, when \( m = 1 \), then \( \tau = t \) and \( \kappa = k \) and

\[ \mathcal{B}^\beta_{1}[a, k; b, t] = \frac{\Gamma_1^\beta[a+k]\Gamma_1^\beta[b+t]}{\Gamma_1^\beta[a+b+k+t]} = \mathcal{B}^\beta_{1}[a, k; b, t] \]

Finally observe that if in results in this section are defined \( k = 0 \) and \( \tau = (0, \ldots, 0) \), the results in Díaz-García and Gutiérrez-Jáimez \[8\] are obtained as particular cases.

4 Singular value densities

In this section, the joint densities of the singular values of random matrix \( R \sim \mathcal{P}_{\mathbb{Q}}R_{m}^{\beta, I}(\nu, k, \tau, 1, 0, I_n, I_m) \) are derived. In addition, and as a direct consequence, the joint density of the eigenvalues of matrix multivariate beta-Riesz type I distribution is obtained for real normed division algebras.

Theorem 4.1. Let \( \delta_1, \ldots, \delta_m, 1 > \delta_1 > \cdots > \delta_m > 0 \), be the singular values of the random matrix \( R \sim \mathcal{P}_{\mathbb{Q}}R_{m}^{\beta, I}(\nu, k, \tau, 1, 0, I_n, I_m) \). Then its joint density is

\[ \frac{2^{2m}n^{\beta m^2/2+\rho}}{\Gamma_{m}[\beta m/2]|B^\beta_{m}[\nu\beta/2, k; n\beta/2, \tau]|} \prod_{i=1}^{m} \left( \delta_i^2 \right)^{(n-m-1)\beta/2-1/2} \left( 1 - \rho \sum_{i=1}^{m} \delta_i^2 \right)^{\nu\beta/2+k-1} \]

\[ \times \prod_{i<j}^{m} (\delta_i^2 - \delta_j^2)^{\beta} \frac{C_{\tau}^{\beta}(D^2)}{C_1^{\beta}(I_m)} \left( \bigwedge_{i=1}^{m} d\delta_i \right) \]

for \( 1 - \rho \sum_{i=1}^{m} \delta_i^2 > 0 \). Where \( \rho \) is defined in Lemma \[2,2\], \( D = \text{diag}(\delta_1, \ldots, \delta_m) \), and \( C_{\kappa}(\cdot) \) denotes the zonal spherical functions or spherical polynomials, see Gross and Richards \[16\] and Faraut and Korányi \[14\], Chapter XI, Section 3.

Proof. This follows immediately from \[11\]. First using \[9\], then applying \[11\] and observing that, from \[16\], Equation 4.8(2) and Definition 5.3 and Faraut and Korányi \[14\], Chapter XI, Section 3, we have that for \( L \in \mathbb{Q}_m^\beta \),

\[ C_{\tau}^{\beta}(Z) = C_{\tau}^{\beta}(I_m) \int_{H \in \mathbb{Q}_m^\beta} q_{\kappa}(HZH^*)(dH) , \]
Finally, observe that \( \delta_i = \sqrt{\text{eig}_i(\mathbf{R}^*\mathbf{R})} \), where \( \text{eig}_i(\mathbf{A}) \), \( i = 1, \ldots, m \), denotes the \( i \)-th eigenvalue of \( \mathbf{A} \). Let \( \lambda_i = \text{eig}_i(\mathbf{R}^*\mathbf{R}) = \text{eig}_i(\mathbf{B}) \), observing that, for example, \( \delta_i = \sqrt{\lambda_i} \). Then

\[
\bigwedge_{i=1}^{m} d\delta_i = 2^{-m} \prod_{i=1}^{m} \lambda_i^{-1/2} \bigwedge_{i=1}^{m} d\lambda_i,
\]

the corresponding joint densities of \( \lambda_1, \ldots, \lambda_m \), \( 1 > \lambda_1 > \cdots > \lambda_m > 0 \) is obtained from (14) as

\[
\frac{\pi^{\beta m^2/2+\theta}}{\Gamma_m^{\beta m/2} B_m^{\beta}} \prod_{i=1}^{m} \lambda_i^{(n-m+1)\beta/2-1} \left( 1 - \sum_{i=1}^{m} \lambda_i \right)^{v\beta/2+k-1} \times \prod_{i<j}^{m} (\lambda_i - \lambda_j)^\beta C_\gamma^\beta (\mathbf{G}) C_\mu^\beta (\mathbf{I}_m) \left( \bigwedge_{i=1}^{m} d\lambda_i \right)
\]

for \( 1 - \sum_{i=1}^{m} \lambda_i > 0 \), where \( \mathbf{G} = \text{diag}(\lambda_1, \ldots, \lambda_m) \).

5 Conclusions

As visual examples, different Pearson type II-Riesz densities for \( m = 1 \) are showed in figures 1 and 2.

Recall that in octonionic case, from the practical point of view, we must keep in mind the fact from Baez [1], there is still no proof that the octonions are useful for understanding the real world. We can only hope that eventually this question will be settled on one way or another. In addition, as is established in Faraut and Korányi [14] and Sawyer [28] the result obtained in this article are valid for the algebra of Albert, that is when hermitian matrices (\( \mathbf{S} \)) or hermitian product of matrices (\( \mathbf{X}^*\mathbf{X} \)) are \( 3 \times 3 \) octonionic matrices.

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Figure 1: With $\nu = 15$, $n = 18$ and $t = 7$

Figure 2: With $\nu = 3$, $n = 18$ and $k = 0$
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