BETHE ANSATZ AND QUANTUM GROUPS

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ABSTRACT

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The formulation and resolution of integrable lattice statistical models in a quantum group covariant way is the subject of this review. The Bethe Ansatz turns to be remarkably useful to implement quantum group symmetries and to provide quantum group representations even when \( q \) is a root of unity. We start by solving the six-vertex model with fixed boundary conditions (FBC) that guarantee exact \( SU(2)_q \) invariance on the lattice. The algebra of the Yang-Baxter (YB) and \( SU(2)_q \) generators turns to close and the transfer matrix is \( SU(2)_q \) invariant for FBC. In addition, the infinite spectral parameter limit of the YB generators yields **cleanly** the \( SU(2)_q \) generators. The Bethe Ansatz states constructed for FBC are shown to be **highest weights** of \( SU(2)_q \). The light-cone evolution operator for FBC is introduced and shown to follow from the row-to-row FBC transfer matrix with alternating inhomogeneities. This operator is shown to describe the SOS model after an appropriate gauge choice. Using this FBC light-cone approach, the scaling limit of both six-vertex and SOS models easily follows. Finally, the higher level Bethe Ansatz equations (describing the physical excitations) are explicitly derived for FBC. We then solve the RSOS(\( p \)) models on the light–cone lattice with fixed boundary conditions by disentangling the type II representations of \( SU(2)_q \), at \( q = e^{i\pi/p} \), from the full SOS spectrum obtained through Algebraic Bethe Ansatz. The rule which realizes the quantum group reduction to the RSOS states is that there must not be **singular** roots in the solutions of the Bethe Ansatz equations describing the states with quantum spin \( J < (p - 1)/2 \). By studying how this rule is active on the particle states, we are able to give a microscopic derivation of the lattice \( S \)–matrix of the massive kinks. The correspondence between the light–cone Six–Vertex model and the Sine–Gordon field theory implies that the continuum limit of the RSOS(\( p + 1 \)) model is to be identified with the \( p \)–restricted Sine–Gordon field theory.
1. Introduction

As is by now well known integrability is a consequence of the Yang-Baxter equation (YBE) in two-dimensional lattice models and two-dimensional quantum field theory (QFT), (for recent reviews see for example ref.[1]). More precisely, a statistical model is integrable when the local weights are solutions of the YBE. Analogously, for two dimensional integrable QFT the two-body S-matrix fulfils the YBE.

Quantum groups are closely related to Yang-Baxter algebras[1,2]. However, quantum group invariance holds for an integrable lattice model only for specific choices of the boundary conditions. As we showed in refs.[3-4] (see also refs.[5,6,7]), choosing fixed boundary conditions (FBC), the transfer matrix commutes with the quantum group generators.

The purpose of this paper is to review the work in collaboration with Claudio Destri in refs.[3-4] on integrable lattice models and their scaling limit using a fully quantum group covariant Bethe Ansatz (BA) framework. Let us recall that the YB algebra of monodromy operators acting on the space of physical states, is the main tool to construct the transfer matrix eigenvectors by the (algebraic) Bethe Ansatz. To do that we choose FBC (Dirichlet type) and use the Sklyanin-Cherednik[8,9] construction of the Yang-Baxter algebra. In this framework, besides the R-matrix defining the local statistical weights, there are two matrices $K^\pm(\theta)$ that define the boundary conditions. $K^\pm(\theta)$ must fulfill a set of equations [eqs.(2.12) and (2.15)] in order to respect integrability. (FBC is one special case out of a continuous family of boundary conditions compatible with integrability).

The appropriate monodromy operators $U_{ab}(\lambda, \tilde{\omega})$ for a N-sites line take now the form depicted in fig.1 where arbitrary inhomogeneities $\omega_i , (1 \leq i \leq N)$, are allowed at each site. We compute the large $\theta$ limit of the monodromy operators $U_{ab}(\lambda, \tilde{\omega})$ and find that they are just the quantum group generators. We do that explicitly for the six-vertex model where the quantum group is $SU(2)_q$. We find in this way an explicit representation of the $SU(2)_q$ generators acting on the space of states.

In addition, the $\theta \to \infty$ limit of the Yang-Baxter algebra for the $U_{ab}(\lambda, \tilde{\omega})$ shows that the transfer matrix $t(\lambda, \tilde{\omega})$ commutes with the $SU(2)_q$ generators and that the algebra of the $U_{ab}(\lambda, \tilde{\omega}) (1 \leq a, b \leq 2)$ with the quantum group generators ( $J_\pm$ and $q^{J_z}$ ) closes [see eqs.(2.33)-(2.37)]. Moreover, $t(\lambda, \tilde{\omega})$ in the $\theta \to \infty$ limit yields the $q$-Casimir operator $C_q$ through

$$t(\infty, \tilde{\omega}) = q + q^{-1} + (q - q^{-1})^2 C_q \quad (1.1)$$

It must be recalled that for periodic boundary conditions (PBC) the $\theta \to \infty$ limit of the Yang-Baxter algebra also gives the quantum group generators but that the algebra with the PBC monodromy operators does not close [1].
Then, we investigate the BA construction in this quantum group covariant framework. It is natural to define for FBC a creation operator of pseudoparticles $\hat{B}(\theta)$ (proportional to $U_{12}(\lambda, \tilde{\omega})$) which is odd in $\theta$ [eq.(3.3)]. The exact eigenvectors of the transfer matrix $t(\lambda, \tilde{\omega})$ are then given by

$$\Psi(\vec{v}) = \hat{B}(v_1)\hat{B}(v_2)\ldots\hat{B}(v_r)\Omega$$

(1.2)

where $v_1, v_2, \ldots, v_r$, fulfil the BA equations (3.4) (BAE) and $\Omega$ is the ferromagnetic ground state (3.2). We show that only BAE roots with strictly positive real part must be consider (Re$v_j > 0$). In particular, $v_j = 0$ and purely imaginary roots must be discarded. We map the BAE (3.4) for FBC onto BAE for PBC in $2N$ sites [see eq.(5.2)]. We find the usual PBC BAE plus an extra source and two important constraints:

a) the total number of roots is even and they are symmetrically distributed with respect to the origin,

b) a root at the origin as well as purely imaginary roots are excluded.

Therefore, the antiferroelectric ground state (and the excitations on the top of it) contain always a hole at the origin. This hole combined with the extra source accounts for the surface energy (see for example ref.[12]).

Starting from the general BA state (1.2) we prove that they are highest weights for $SU(2)_q$. That is,

$$J_+ \Psi(\vec{v}) = 0$$

(1.3)

provided the BAE (3.4) hold for $v_1, v_2, \ldots, v_r$. Therefore, the eigenvalues of $t(\theta, \tilde{\omega})$ are degenerate with respect to the quantum group and the eigenvectors:

$$J_- \Psi(\vec{v}), (J_-)^2\Psi(\vec{v}), \ldots, (J_-)^2J_-\Psi(\vec{v}), \ldots$$

(1.4)

are linearly independent from $\Psi(\vec{v})$.

SOS and vertex models are related by the vertex-face correspondence. (The degrees of freedom lie on faces for SOS models whereas they lie on links for the vertex model). The correspondence between them amounts to an application of the $q$-analog of the Wigner-Eckart theorem. As we explain in sec. 4, the SOS space of states is identical to the set of maximal weight six vertex states in a quantum group invariant framework like ours. It then follows from eq.(1.3) that the Solid–On–Solid (SOS) states are just the f.b.c. BA states given by eq.(1.2). The six vertex Hilbert space includes the whole $SU(2)_q$ multiplets and follow then by repeatedly applying the lowering operator $J_-$ to the highest weight BA states (1.2) as shown in eq.(1.4). In other words, we have found the BA solution of the SOS model since we derived the transfer matrix eigenstates. In ref.[10] an alternative but equivalent solution of the SOS model is derived using a BA in face language. In addition PSOS (periodic SOS models where the face states $l$ and $l + p$ are identified) are solved in ref.[10].
The light-cone approach is a direct way to give a field theoretical interpretation to a lattice model and furthermore obtain its scaling limit as a massive QFT. The light-cone approach with periodic boundary conditions has been investigated in refs. [19-20]. Here we consider this approach in the case of fixed boundary conditions leading to a quantum group invariant framework.

In this context we show in sec. 4 that the diagonal-to-diagonal transfer matrix $U(\Theta)$ [fig.6] can be obtained from the row-to-row transfer matrix $t(\lambda, \tilde{\omega})$ choosing the inhomogeneities appropriately. This is analogous to the relation found in ref. [19,20] for PBC. From the transfer matrix $U(\Theta)$ we define the lattice hamiltonian through

$$H = \frac{i}{a} \log U(\Theta) \quad (1.5)$$

where $a$ is the lattice spacing. In the $a \to 0, \Theta \to \infty$ limit this operator defines the continuous QFT hamiltonian provided the renormalized mass scale [see eq.(4.17)] is kept fixed. The evolution operator $U(\Theta)$ can be considered both in the vertex or in the face language. In face language, it has a simple expression provided we make an appropriate gauge transformation. That is, if one transforms each local $R$-matrix to a matrix $\tilde{R}$ such that the $SU(2)_q$ symmetry holds locally. In this way, we show that the associated light-cone evolution operator $\tilde{U}(\Theta)$ just describes the SOS ABF model [13] [eq.(4.27)].

As it is known, the physical states above the AF vacuum are described by the higher level BAE [14]. We obtain the higher level BAE for FBC in sec. 5 [eq.(5.14)]. They are the starting point to study the excitations in SOS and RSOS models. To conclude we show that the BAE for FBC admit solutions at infinity only when $\gamma/\pi$ is a rational number. This is precisely the case when RSOS models can be defined and when the representations of $SU(2)_q$ algebras cease to be isomorphic to usual $SU(2)$. Recall that when $\gamma/\pi$ is rational, type I representations are reducible but indecomposable, type II are irreducible as in $SU(2)$ (see for example ref. [5,6]).

It is known that the Six–Vertex (6V) model, in the so–called light–cone formulation and with periodic boundary conditions (p.b.c.), yields the Sine–Gordon massive field theory in an appropriate scaling limit [19]. Hence the light–cone 6V model can be regarded as an exactly integrable lattice (minkowskian) regularization of the SG model.

Recently, the hidden invariance of the SG model under the quantum group $SU(2)_q$ was exhibited [21]. Our quantum group invariant light-cone formulation, provides a lattice formulation where such hidden invariance appears starting from first principles. That is, not on a bootstrap framework but deriving the field theory as a rigorous scaling limit of the six-vertex model.

We present in section 6 and 7 the derivation of the factorized $S$–matrices on the lattice, i.e. still in the presence of the UV cutoff. This derivation is based on
the “renormalization” of the BA Equations, which consists in removing the infinitely many roots describing the ground state. What is left is once again a f.b.c. BA structure involving the lattice rapidities of the physical excitations (the particles of the model) and the roots of the higher–level BAE obtained in paper I. The explicit form of two–body $S$–matrix for the 6V model and the SOS model can be extracted in a precise way (c.f. eq. (6.11)) from this higher–level BA structure. In the massive scaling limit these lattice scattering amplitudes become the relativistic $S$–matrices of the SG model (or Massive Thirring model) and of the continuum SOS model. Let us remark that the SOS $S$–matrix, although closely related to the 6V and SG $S$–matrices from the analytical point of view, is conceptually different. It describes the scattering of kinks interpolating between renormalized local vacua labelled by integers. This kink $S$–matrix is most conveniently expressed in the so–called face language (see eq.(7.3)).

In section 8 we investigate the f.b.c. BAE (eq.(2.4)) when the quantum group deformation parameter $q$ is a root of unity, say $q^p = \pm 1$, with $p$ some integer larger than 2 (the case $p = 2$ being trivial). In this case it is known that RSOS($p$) model can be introduced by restricting to the finite set $(1, 2, \ldots , p - 1)$ the allowed values of the local height variables of the SOS model [13]. At the same time when $q^p = \pm 1$, the representations of $SU(2)_q$ algebras cease to be isomorphic to usual SU(2). Recall that when $\gamma/\pi$ is rational, type I representations are reducible but indecomposable, type II are irreducible as in SU(2) (see for example ref.[5,6]). The restriction leading to the RSOS model from the SOS model is equivalent to the projection of the full SOS Hilbert space (which is formed by the highest weight states of $SU(2)_q$) to the subspace spanned by the type II representations [5,6]. That is, those representations which remain irreducible when $q$ becomes a root of unity. Our results on this matters, in the BA context, can be summarized as follows:

a) Only when $q$ is a root of unity, the f.b.c. BAE admit singular roots (that is vanishing $z$–roots or diverging $v$–roots, in the notations of sec.1).

b) When $q$ tend to a root of unity, say $q^p = \pm 1$, the BAE solutions can be divided into regular and singular solutions, having, respectively, no singular roots or some singular roots. Regular solutions correspond to irreducible type II representations. Singular solutions with $r$ singular roots correspond to the reducible and generally indecomposable type I representations obtained by mixing two standard $SU(2)_q$ irreps of spin $J$ and $J + r$ (we recall that $J = N/2 - M$, where $N$ is the spatial size of the lattice and $M$ the number of BA roots). Then necessarily $r < p$.

c) The $r$ singular roots $z_1, z_2, \ldots , z_r$ vanish with fixed ratios

$$z_j = \omega^{j-1}z_1, \quad \omega = e^{2\pi i/r}, \quad 1 \leq j \leq r$$

In terms of the more traditional hyperbolic parametrization, with $v$–roots related by $v_j = -\frac{1}{2}\log z_j$ to the $z$–roots, the singular roots form an asymptotic
string–like configuration. They have a common diverging real part and are separated by $\pi/r$ in the imaginary direction.

So we see that the RSOS eigenstates are easily singled out from the full set of BA eigenstates of the 6V or SOS transfer matrix. One must retain all and only those BAE solutions with $M > (N - p + 1)/2$ (which correspond to states with $J < (p - 1)/2$) and with all $M$–roots different from zero. This provides therefore an exact, explicit and quite simple solution for the RSOS model on the lattice (with suitable boundary conditions, as we shall later see). In particular the ground state of the 6V, SOS and RSOS models is the same f.b.c. BA state (in the thermodynamic limit $N \to \infty$ at fixed lattice spacing). It is the unique $SU(2)_q$ singlet with all real positive roots and no holes. The local height configuration which, loosely speaking, dominates this ground state can be depicted as a sequence of bare kinks jumping back and forth between neighboring bare vacua (see fig. 10). In the massive scaling limit proper of the light–cone approach [19], this ground state becomes the physical vacuum of the SG model as well as of and all the Restricted SG field theories.

In the discussion ending sec. 8, we argue that the kink $S$–matrix for the excitations of the RSOS models follows indeed by restriction from that of the SOS model. In the scaling limit it is to be identified with the relativistic $S$–matrix of the Restricted SG models. All these field–theoretical $S$–matrices are naturally related to the Boltzmann weights of the respective lattice models. This is because the higher–level BAE are identical in form to the “bare” BAE, apart from the renormalization of the anisotropy parameter $\gamma$ (related to $q$ by $q = e^{i\gamma}$)

$$\gamma \to \hat{\gamma} = \frac{\pi \gamma}{\pi - \gamma} \quad (1.7)$$

and the replacement of the rapidity cutoff $\pm \Theta$ with the suitably scaled rapidities $\theta_j$ of the physical excitations

$$\pm \Theta \to \frac{\gamma \theta_j}{\pi - \gamma} \quad (1.8)$$

In particular, since $\gamma = \pi/p$ for the RSOS($p$) model, eq.(1.7) yields $p \to p - 1$. This shows that the renormalized local vacua $\hat{\ell}$ run from 1 to $p - 2$ when the bare local heights $\ell_n$ run from 1 to $p - 1$. The higher–level BA structure of the light–cone 6V model (or lattice regularized SG model) thus provides a microscopic derivation of the bootstrap construction of ref. [21] and explains why the $S$–matrix of the $p$–restricted SG field theory has the same functional form of the Boltzmann weights of the lattice RSOS($p + 1$) model. Moreover, the well–known correspondence between the critical RSOS($p$) models and the Minimal CFT Models $M_p$ imply the natural identification of the massive $p$–restricted SG model with a completely massive relevant perturbation of $M_p$. This is generally recognized as the perturbation induced by the primary operator $\phi_{1,3}$ with negative coupling.
Two detailed examples of the BA realization of the quantum group reduction to the RSOS models are presented in section 9. We considered the simplest cases $p = 3$ and $p = 4$. The RSOS(3) model is a trivial statistical system with only one state, since all the local heights are fixed once we choose, for example, the boundary condition $\ell_0 = 1$. In our quantum group covariant f.b.c. construction this corresponds to the existence of one and only one type II state when $\gamma = \pi/3$. We then obtain the following purely mathematical result: for any $N \geq 2$ and real $w = \exp(-2\Theta)$ the set of BAE

$$
\left( \frac{z_j w - e^{\pi i/3}}{z_j w e^{\pi i/3} - 1} \frac{z_j - w e^{\pi i/3}}{z_j e^{\pi i/3} - w} \right)^N = \prod_{k=1, k \neq j}^{[N/2]} \frac{z_j - z_k e^{2\pi i/3}}{z_j e^{2\pi i/3} - z_k} \frac{z_j z_k - e^{2\pi i/3}}{z_j z_k e^{2\pi i/3} - 1}, \quad 1 \leq j \leq N
$$

(1.9)

admit one and only one solution with non-zero roots within the unit disk $|z| < 1$. In addition, these roots are all real and positive.

The RSOS(4) model can be exactly mapped into an anisotropic Ising model as showed in eqs. (9.1–5). In our case the horizontal and vertical Ising couplings turn out to be $\Theta$–dependent complex numbers. For even $N$, the Ising spins are fixed on both space boundaries. For odd $N$ the spins are fixed on the left and free to vary on the right. We analyzed the BAE for the RSOS(4) model in some detail. In the thermodynamic limit the ground state, as already stated, is common to the 6V and SOS models. The elementary excitations correspond to the presence of holes in the sea of real roots charactering the ground state. Each holes describes a physical particle or kink and may be accompanied by complex roots. In sec. 9 we argue that in the RSOS(4) case a state with $\nu$ holes necessarily contains $[\nu/2]$ two–strings those position is entirely fixed by the holes. Notice that here the number $\nu$ of holes can be odd even for even $N$. This is not the case for the 6V or SOS models, where $\nu$ is always even for $N$ even. What happens is that when $\gamma \to (\pi/4)^-$ the largest real $\nu$–root diverges in the $J = 1$ states of the 6V and SOS models. Therefore, these states get mixed with $J = 2$ states into type I representations and do not belong to the RSOS(4) Hilbert space. The RSOS(4) states with $J = 1$ are obtained by choosing the largest quantum integer $I_{N/2-1} = N/2$ (see the Appendix). There is no root associated to $N/2 + 1$. It follows that these states, from the 6V and SOS viewpoint, have a cutoff dependent term in the energy equal to $\pi/a$, where $a$ is the lattice spacing (loosely speaking, one could say that “there is a hole at infinity”). They are removed from the physical SG spectrum in the continuum limit. We are thus led to propose as RSOS(4) hamiltonian, for even $N$

$$
H_{RSOS(4)} = H_{SOS}(\gamma = \pi/4) - a^{-1}\pi J
$$

(1.10)

where $H_{SOS}$ is given by eqs. (4.14), (4.10), (4.21-27) and $J = 0$ or 1. In this way, the particle content of the light–cone RSOS(4) and the corresponding $S$-matrix
turn out to coincide with the results of the bootstrap–like approach of ref. [21]. Eq. (1.10) defines, in the scaling limit, the hamiltonian of the \((p = 3)\)–restricted SG model. Notice, in this respect, that the higher–level BAE (5.17) and (6.10) completely determine the physical states in terms of renormalized parameters.

To summarize, the picture we get from the BA solution of the f.b.c. 6V, SOS and RSOS\((p)\) lattice models is as follows. Performing the scaling limit within the light–cone approach, these lattice models yields respectively: the SG model (or Massive Thirring Model), a truncated SG and the \((p − 1)\)restricted SG models. For the SG model we have essentially nothing to add to the existing literature, a part from the explicit unveiling, at the regularized microscopic level, of its \(SU(2)_q\) invariance and for a better derivation of the \(S\)–matrix. The truncated SG follows by keeping only the highest weight states with respect to \(SU(2)_q\), that is the kernel of the raising operator \(J_+\). Finally we showed that the RSOS\((p)\) lattice models with trigonometric weights yield in the scaling limit proper of the light–cone approach the \((p − 1)\)restricted SG field theories formulated at the bootstrap level in ref. [21].

2. Boundary conditions in lattice integrable models

Let us consider an integrable vertex model with R-matrix \(R_{ab}^{cd}(\theta)\) (see fig.1). Each element \(R_{cd}^{ab}(\theta)\) defines the statistical weight of this configuration. We assume \(R(\theta)\) to fulfil the Yang-Baxter equations. The indices \(a, b, c, d\), are assumed to run from 1 to \(q\) with \(q \geq 2\).

\[
\begin{bmatrix}
1 & R(\theta - \theta')
\end{bmatrix}
\begin{bmatrix}
R(\theta) & \otimes & 1
\end{bmatrix}
\begin{bmatrix}
1 & \otimes & R(\theta')
\end{bmatrix}
= 
\begin{bmatrix}
R(\theta') & \otimes & 1
\end{bmatrix}
\begin{bmatrix}
1 & \otimes & R(\theta)
\end{bmatrix}
\begin{bmatrix}
R(\theta - \theta') & \otimes & 1
\end{bmatrix}
\]

(2.1)

We shall assume T and P invariance for \(R(\theta)\)

\[
R_{cd}^{ab}(\theta) = R_{ba}^{cd}(\theta) = R_{dc}^{ba}(\theta)
\]

(2.2)

In addition, we assume \(R(\theta)\) to be regular, that is

\[
R(0) = c \ 1 \quad \text{or} \quad R_{cd}^{ab}(0) = c \ \delta_c^a \delta_d^b
\]

(2.3)

where \(c\) is a numerical constant. Eqs. (2.1) and (2.3) imply the unitarity-related equation

\[
R(\theta) \ R(-\theta) = \rho(\theta) \ 1
\]

(2.4)

where \(\rho(\theta)\) is an even c-number function. Furthermore, we assume crossed-unitarity. That is,

\[
\hat{R}(\theta) \ \hat{R}(-\theta - 2 \eta) = \hat{\rho}(\theta) \ 1
\]

(2.5)

where \(\eta\) is a constant and \(\hat{R}_{cd}^{ab}(\theta) \equiv R_{bd}^{ac}(\theta)\).
Let us consider now a $N \times N'$ square lattice with periodic boundary conditions. Then the row-to-row transfer matrix is given by

$$\tau(\theta, \tilde{\omega}) = \sum_a T_{aa}(\theta, \tilde{\omega})$$  \hspace{1cm} (2.6)

where the operators $T_{ab}(\theta, \tilde{\omega})$ defined by (see fig.2)

$$T_{ab}(\theta, \tilde{\omega}) = \sum_{a_1, \ldots, a_{N-1}} t_{a_1b}(\theta - \omega_1) \otimes t_{a_2a_1}(\theta - \omega_2) \otimes \ldots \otimes t_{aa_{N-1}}(\theta - \omega_N)$$  \hspace{1cm} (2.7)

act on the vertical space $V = \bigotimes_{1 \leq i \leq N} V_i , V_i \equiv C^q$, and the simplest choice for the local vertices is $[t_{ab}(\theta)]_{cd} \equiv R_{cd}^{ba}(\theta)$. In eq.(2.7) $\omega_1, \omega_2, \ldots, \omega_N$ stand for arbitrary inhomogeneity parameters. The $T_{ab}(\theta, \tilde{\omega})$ fulfil the YB algebra

$$R(\lambda - \mu) \left[ T(\lambda, \tilde{\omega}) \otimes T(\mu, \tilde{\omega}) \right] = \left[ T(\mu, \tilde{\omega}) \otimes T(\lambda, \tilde{\omega}) \right] R(\lambda - \mu)$$  \hspace{1cm} (2.8)

as follows from eq.(2.1). Thus, the $\tau(\theta, \tilde{\omega})$ are a commuting family

$$\left[ \tau(\theta, \tilde{\omega}), \tau(\theta', \tilde{\omega}) \right] = 0$$  \hspace{1cm} (2.9)

Let us now consider the generalization to other boundary conditions compatible with integrability[8]. Define (see fig.3)

$$U_{ab}(\theta, \tilde{\omega}) = \sum_{cd} T_{ad}(\theta, \tilde{\omega}) K_{dc}^{-}(\theta) T_{cb}^{-1}(-\theta, \tilde{\omega})$$  \hspace{1cm} (2.10)

where $T^{-1}(-\theta, \tilde{\omega})$ is the inverse in both the horizontal and vertical spaces:

$$\sum_b T_{ab}(\theta, \tilde{\omega}) T_{bc}^{-1}(-\theta, \tilde{\omega}) = \delta^a_c \mathcal{I}$$  \hspace{1cm} (2.11)

and $\mathcal{I}$ is the identity on the vertical space $V$. Summation over indices of the vertical space $V$ are omitted both in eqs.(2.10) and (2.11) (c.f. fig. 3). $K^{-}(\theta)$ in eq.(2.10) is a $q \times q$ matrix solely acting on the horizontal space. It must fulfil[8]

$$R(\lambda - \mu) \left[ K^{-}(\lambda) \otimes 1 \right] R(\lambda + \mu) \left[ K^{-}(\mu) \otimes 1 \right] = \left[ K^{-}(\mu) \otimes 1 \right] R(\lambda + \mu) \left[ K^{-}(\lambda) \otimes 1 \right] R(\lambda - \mu)$$  \hspace{1cm} (2.12)

(Notice, that our R-matrix differs from ref.[8] in a permutation matrix $R \rightarrow PR$, $P_{cd}^{ab}(\theta) = \delta^a_c \delta^b_d$).
As is well known, $R_{cd}^{ab}(\theta - \theta')$ has the interpretation of scattering amplitude for a two-body collision where $a(d)$ and $b(c)$ label the initial (final) states of two particles with rapidities $\theta$ and $\theta'$ respectively. In this S-matrix context, $K_{ab}^{-}(\theta)$ is the scattering amplitude for one particle with a rigid wall on the left, $a$ and $b$ labelling the initial and final states and $\theta$ being its final rapidity (see fig.4).

Thanks to eqs.(2.8) and (2.12), $U(\theta, \tilde{\omega})$ fulfils the Yang-Baxter algebra

$$R(\lambda - \mu) [U(\lambda, \tilde{\omega}) \otimes I] R(\lambda + \mu) [U(\mu, \tilde{\omega}) \otimes I] = [U(\mu, \tilde{\omega}) \otimes I] R(\lambda + \mu) [U(\lambda, \tilde{\omega}) \otimes I] R(\lambda - \mu)$$

The transfer matrix is given now by

$$t(\lambda, \tilde{\omega}) = \sum_{ab} K_{ab}^{+}(\lambda + \eta) U_{ab}(\lambda, \tilde{\omega})$$

Here $K_{ab}^{+}(\lambda)$ describes the scattering with a rigid wall on the right (cft. fig.5). It is a solution of the equation :

$$R(\lambda - \mu) \left[ 1 \otimes K^{+}(\lambda) \right] R(\lambda + \mu) \left[ 1 \otimes K^{+}(\mu) \right] = \left[ 1 \otimes K^{+}(\mu) \right] R(\lambda + \mu) \left[ 1 \otimes K^{+}(\lambda) \right] R(\lambda - \mu)$$

It follows from eqs.(2.13)-(2.15) that $t(\lambda, \tilde{\omega})$ is a commuting family

$$[t(\theta, \tilde{\omega}), t(\theta', \tilde{\omega})] = 0$$

As we can see from eqs. (2.10) and (2.14) the boundary conditions associated to $t(\theta, \tilde{\omega})$ follow from the form of $K^{+}(\lambda)$ and $K^{-}(\lambda)$ sitting on the right and left borders, respectively.

From now on we shall consider the case of the six-vertex model, where $q = 2$ and the R–matrix reads, in terms of a generic spectral parameter $\theta$,

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b = b(\theta, \gamma) = \frac{\sinh \theta}{\sinh(i\gamma - \theta)}$$

$$c = c(\theta, \gamma) = \frac{\sinh i\gamma}{\sinh(i\gamma - \theta)}$$

where the anisotropy parameter $\gamma$ (we may assume $0 \leq \gamma \leq \pi$) is related to the quantum group deformation $q$ by $q = \exp(i\gamma)$. This R–matrix (2.17) is unitary for
real $\theta$. Crossed unitarity (2.5) holds for $\eta = i\gamma$. It is also convenient to work with slightly modified local vertices

$$[t_{ab}(\theta)]_{cd} \equiv R^{bd}_{ca}(\theta - i\gamma/2)$$

(2.18)

in order to construct the row-to-row monodromy matrix $T_{ab}(\theta, \bar{\omega})$. For this R-matrix the diagonal solutions $K^\pm(\theta)$ of eqs.(2.12) and (2.15) turn to be[8,9]

$$K^\pm(\theta) = K(\theta, \xi_\pm)$$

where

$$K(\theta, \xi) = \frac{1}{\sinh \xi} \begin{pmatrix} \sinh(\xi + \theta) & 0 \\ 0 & \sinh(\xi - \theta) \end{pmatrix}$$

(2.19)

Here $\xi_+$ and $\xi_-$ are arbitrary numbers that parametrize this boundary conditions compatible with integrability. (For the general solution of eqs.(2.12) and (2.15) in the six-vertex model see ref.[11].)

We are interested on boundary conditions yielding a quantum group covariant framework. $[SU(2)_q$ for the six-vertex model]. For periodic boundary conditions the quantum group transformation properties of $T_{ab}(\theta, \bar{\omega})$ and $t(\theta, \bar{\omega})$ are not simple [15]. A $SU(2)_q$ invariant XXZ hamiltonian requires fixed boundary conditions and special end-point terms[6].

The XXZ hamiltonian follows from $dt(\theta, \bar{\omega} = 0)/d\theta$ evaluated at $\theta = 0$. One finds from eqs.(2.14),(2.17), and (2.19) [8]

$$H_{XXZ} = -\frac{1}{4trK^+(0)} \frac{d}{d\theta} \left[ t(\theta, \bar{\omega} = 0) - trK^+(\theta) \right]$$

$$= -\frac{1}{2} \sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cos \gamma \sigma_n^z \sigma_{n+1}^z) + \frac{1}{2} \sinh(i\gamma)(\sigma_1^+ \coth \xi_- + \sigma_N^- \coth \xi_+).$$

(2.20)

As one can check, the quantum group invariant case corresponds to $\xi_\pm = \pm \infty$. We choose therefore $\xi_\pm = \pm \infty$ in order to built a $SU(2)_q$ covariant framework for the six-vertex model and its scaling limit. This choice corresponds physically to boundary conditions of Dirichlet type. In that case

$$K^\pm(\theta) = \exp(\mp \theta \sigma_z) = \begin{pmatrix} e^{\mp \theta} & 0 \\ 0 & e^{\pm \theta} \end{pmatrix}$$

(2.21)

Now, in order to find the $SU(2)_q$ content of the YB algebra (2.14) and its associate Bethe Ansatz construction, we start by computing the $\theta \to \infty$ limit of the $U_{ab}(\theta, \bar{\omega})$ operators.
The \( \theta \to \pm \infty \) limit of the row-to-row monodromy matrix

\[
T(\theta, \tilde{\omega}) = \begin{pmatrix}
A(\theta) & B(\theta) \\
C(\theta) & D(\theta)
\end{pmatrix}
\] (2.22)

yields \( SU(2)_q \) generators \([1,15]\). We have

\[
\begin{align*}
A(\theta) & \xrightarrow{\theta \to \pm \infty} e^{\mp iN\gamma/2} \exp(\pm i\gamma J_z) [1 + O(e^{ \mp 2\theta})] , \\
D(\theta) & \xrightarrow{\theta \to \pm \infty} e^{\mp iN\gamma/2} \exp(\mp i\gamma J_z) [1 + O(e^{ \mp 2\theta})] , \\
B(\theta) & \xrightarrow{\theta \to \pm \infty} \pm 2e^{\mp iN\gamma/2\mp \theta} \sinh(i\gamma) J_-(\mp \tilde{\omega}, \mp \gamma) [1 + O(e^{ \mp 2\theta})] , \\
C(\theta) & \xrightarrow{\theta \to \pm \infty} \pm 2e^{\mp iN\gamma/2\mp \theta} \sinh(i\gamma) J_+(\pm \tilde{\omega}, \pm \gamma) [1 + O(e^{ \mp 2\theta})] ,
\end{align*}
\] (2.23)

where

\[
J_\pm(\tilde{\omega}, \gamma) = \sum_{k=1}^{N} \prod_{j=1}^{k-1} \exp[-\frac{i\gamma}{2}(\sigma_j)_z] e^{\pm \omega_k(\sigma_\pm)_k} \prod_{l=k+1}^{N} \exp[\frac{i\gamma}{2}(\sigma_l)_z]
\] (2.24)

and

\[
J_z = \frac{1}{2} \sum_{a=1}^{N} (\sigma_a)_z
\] (2.25)

Notice that \( J_+ \equiv J_+(\tilde{\omega}, \gamma) \), \( J_- \equiv J_-(\tilde{\omega}, \gamma) \) and \( J_z \) are \( SU(2)_q \) generators with \( q = e^{i\gamma} \), for they obey the commutation rules

\[
\begin{align*}
[J_+, J_-] &= \frac{\sin(2\gamma J_z)}{\sin \gamma} = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}} , \\
[J_z, J_\pm] &= \pm J_\pm
\end{align*}
\] (2.26)

For the boundary conditions \( \xi_\pm = \pm \infty \), one sets

\[
U(\theta, \tilde{\omega}) = \begin{pmatrix}
A(\theta) & B(\theta) \\
C(\theta) & D(\theta)
\end{pmatrix}
\] (2.27)
From eqs.(2.10) and (2.21) it follows that

\[ \begin{align*}
A(\theta) &= e^{i\gamma/2-\theta} B(\theta) C(-\theta) - e^{-i\gamma/2+\theta} A(\theta) D(-\theta) \\
B(\theta) &= e^{-i\gamma/2+\theta} A(\theta) B(-\theta) - e^{i\gamma/2-\theta} B(\theta) A(-\theta) \\
C(\theta) &= e^{i\gamma/2-\theta} D(\theta) C(-\theta) - e^{-i\gamma/2+\theta} C(\theta) D(-\theta) \\
D(\theta) &= e^{-i\gamma/2+\theta} C(\theta) B(-\theta) - e^{i\gamma/2-\theta} D(\theta) A(-\theta)
\end{align*} \]

(2.28)

Let us compute the \( \theta \to \infty \) limit of these operators. We find from eqs.(2.23) - (2.28)

\[ \begin{align*}
A(\theta) \overset{\theta \to \infty}{=} & -e^{-i\gamma/2+\theta} \exp(2i\gamma J_z) \left[ 1 + O(e^{-2\theta}) \right], \\
B(\theta) \overset{\theta \to \infty}{=} & -e^{-i\gamma/2} \sinh(i\gamma) J_+ \left[ 1 + O(e^{-2\theta}) \right], \\
C(\theta) \overset{\theta \to \infty}{=} & -e^{-i\gamma/2} \sinh(i\gamma) J_+ e^{i\gamma J_z} \left[ 1 + O(e^{-2\theta}) \right], \\
D(\theta) \overset{\theta \to \infty}{=} & 2e^{-i\gamma/2-\theta} \sin [(J + J_z)\gamma] \sin [(J - J_z + 1)\gamma] \\
& - \frac{1}{2} e^{i\gamma/2} e^{-2i\gamma J_z} + O(e^{-3\theta}),
\end{align*} \]

(2.29)

The operator \( J \) is defined through the \( SU(2)_q \) Casimir invariant \( C_q \)

\[ C_q = \frac{1}{2} (J_+ J_- + J_- J_+) + \cos \gamma \frac{\sin^2(\gamma J_z)}{\sin^2(\gamma)} = \frac{\sin \gamma (J + 1) \sin \gamma J}{\sin^2 \gamma} \]

(2.30)

As one can see from eqs.(2.29) the asymptotic form of \( U_{ab}(\lambda, \tilde{\omega}) \) is related with the \( SU(2)_q \) generators in a very clean way.

Let us now consider the transfer matrix \( t(\lambda, \tilde{\omega}) \) [eq.(2.14)]. We find when \( \xi_\pm = \pm \infty \)

\[ t(\theta, \tilde{\omega}) = e^{-i\gamma/2-\theta} A(\theta) + e^{i\gamma/2+\theta} D(\theta) \]

(2.31)

Now, when \( \theta \to \infty \), inserting eq.(2.29) in (2.31) yields

\[ t(\theta, \tilde{\omega}) \overset{\theta \to \infty}{=} 2 \cos[\gamma(2J + 1)] + O(e^{-2\theta}) \]

(2.32)

That is, the asymptotics of the transfer matrix expresses up to numerical constants in terms of the \( q \)-Casimir \( C_q \) [eq.(2.30)].
The operators $U_{ab}(\theta, \tilde{\omega})$ fulfil the Yang-Baxter algebra (2.13) with R-matrix (2.17). Letting $\theta' \to \infty$ in eq. (2.13) we can compute the relevant commutation of $U_{ab}(\theta, \tilde{\omega})$ with the $SU(2)_q$ generators. We find

$$[A(\theta), J_-] = -B(\theta) e^{i\gamma J_z + \theta} \quad [D(\theta), J_-] = B(\theta) e^{i\gamma (J_z+1) + \theta}$$

It is well known that $B(\theta)$ and $C(\theta)$ act as lowering and raising operators for $J_z$, while $A(\theta)$ and $D(\theta)$ commute with it[1]. We find here the same properties for the elements of $U(\theta, \tilde{\omega})$,

$$[B(\theta), J_z] = B(\theta), \quad [C(\theta), J_z] = -C(\theta)$$

$$[A(\theta), J_z] = [D(\theta), J_z] = [t(\theta, \tilde{\omega}), J_z] = 0$$

In addition, we find

$$[e^{-i\gamma J_z} B(\theta), J_-] = 0 \quad B(\theta) J_+ = e^{-i\gamma} J_+ B(\theta) + [e^{i\theta - i\gamma} D(\theta) - e^{-\theta} A(\theta)] e^{i\gamma J_z}$$

Using now eq. (2.31) yields

$$[t(\theta, \tilde{\omega}), J_-] = 0$$

One can analogously prove that

$$[t(\theta, \tilde{\omega}), J_+] = 0$$

Therefore, the transfer matrix $t(\theta, \tilde{\omega})$ is $SU(2)_q$ invariant. As a corollary, we see that $H_{XXZ}$ (see eq. (2.20)) is $SU(2)_q$-invariant, as shown in ref.[6] by direct calculation.

We investigate in the next section the Bethe Ansatz construction of eigenvectors of $t(\theta, \tilde{\omega})$. The first consequence of the the $SU(2)_q$ invariance of $t(\theta, \tilde{\omega})$ is the degeneracy of its eigenvalues with respect to the quantum group.
3. The Bethe Ansatz and the $SU(2)_q$ group

In section II we developed the Yang-Baxter framework with boundary conditions adapted to the $SU(2)_q$ invariance. We call this a quantum group covariant framework. Let us now investigate in such scheme the eigenvectors of the transfer matrix $t(\theta, \tilde{\omega})$ in the algebraic Bethe Ansatz[8]. These eigenvectors can be written as

$$
\Psi(\vec{v}) = \hat{B}(v_1)\hat{B}(v_2)\ldots\hat{B}(v_r)\Omega
$$

where $\Omega$ is the ferromagnetic ground state and $\vec{v} \equiv (v_1, v_2, \ldots, v_r)$.

$$
\Omega = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes \ldots \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
$$

and

$$
\hat{B}(\theta) \equiv \frac{\sinh 2\theta}{\sinh(2\theta - i\gamma)} e^{i\gamma/2} B(\theta) = e^{-\theta} B(\theta) A(-\theta) - e^{\theta} B(-\theta) A(\theta)
$$

Notice that use has been made of the Yang-Baxter algebra (2.8) to obtain the last form in eq.(3.3) from the expression (2.28) for $B(\theta)$. The numbers $v_1, v_2, \ldots, v_r, (0 \leq r \leq N/2)$ must be all distinct roots of the set of algebraic equations

$$
\prod_{k=1}^{N} \frac{\sinh[v_m - \omega_k + i\gamma/2] \sinh[v_m + \omega_k + i\gamma/2]}{\sinh[v_m - \omega_k - i\gamma/2] \sinh[v_m + \omega_k - i\gamma/2]} = \\
\prod_{k=1, k\neq m}^{r} \frac{\sinh[v_m - v_k + i\gamma] \sinh[v_m + v_k + i\gamma]}{\sinh[v_m - v_k - i\gamma] \sinh[v_m + v_k - i\gamma]} \quad (3.4)
$$

These Bethe Ansatz Equations (BAE) guarantee the vanishing of the so-called unwanted terms. That is, vectors in $t(\theta, \tilde{\omega})\Psi(\vec{v})$ which are not proportional to $\Psi(\vec{v})$. They arise, together with the wanted terms, from the repeated commutation of $t(\theta, \tilde{\omega})$ with $\hat{B}(v_1)\hat{B}(v_2)\ldots\hat{B}(v_r)$ by means of the Yang-Baxter algebra (2.13). Let us point out that the order of
the factors $\hat{B}(v_j)$ in eq.(3.1) is irrelevant since the Yang-Baxter algebra (2.13) implies

\[
\left[ \hat{B}(\theta), \hat{B}(\theta') \right] = 0
\]  

(3.5)

From eq.(3.3) it is evident that $\hat{B}(\theta)$ is an odd function of $\hat{B}(-\theta) = -\hat{B}(\theta)$  

(3.6)

Consequently, a direct check shows that the BAE (3.4) are invariant under negation of any single unknown $v_j$. This implies that it is enough to consider BAE roots with strictly positive real parts. In particular, purely imaginary pairs ($\pm i\eta$) are ruled out. Moreover, one verifies by explicit calculation that $\hat{B}(i\pi/2)\Omega = 0$. So that the selfconjugate root $v = i9/2$ is also ruled out.

The eigenvalue of $t(\theta, \tilde{\omega})$ on $\Psi(\vec{v})$ is given by

\[
\Lambda(\theta; \vec{v}) = \Lambda_+(\theta; \vec{v}) + \Lambda_-(\theta; \vec{v})
\]  

(3.7)

\[
\Lambda(\theta; \vec{v}) = \frac{\sinh(2\theta \pm i\gamma)}{\sinh(2\theta + i\gamma)} \prod_{k=1}^{N} \frac{\sinh[\theta - \omega_k + i\gamma/2] \sinh[\theta + \omega_k + i\gamma/2]}{\sinh[\theta - \omega_k - i\gamma/2] \sinh[\theta + \omega_k - i\gamma/2]}
\]

\[
\prod_{m=1}^{r} \frac{\sinh[\theta - v_m \mp i\gamma]}{\sinh[\theta - v_m]} \frac{\sinh[\theta + v_m \mp i\gamma]}{\sinh[\theta + v_m]}
\]  

(3.8)

Taking logarithms, eq.(3.8) become

\[
\sum_{k=1}^{N} \left[ \phi(v_m - \omega_k, \gamma/2) + \phi(v_m + \omega_k, \gamma/2) \right]
\]

\[
= 2\pi I_m + \sum_{k=1, k\neq m}^{r} \left[ \phi(v_m - v_k, \gamma) + \phi(v_m + v_k, \gamma) \right]
\]

(3.9)

\[
1 \leq m \leq r
\]

where the $I_m$ are positive integers[16] and

\[
\phi(\lambda, x) \equiv i \log \frac{\sinh(ix + \lambda)}{\sinh(ix - \lambda)} \quad \text{with} \quad \phi(0, x) = 0.
\]

(3.10)

Before solving these equations in the $N \to \infty$ limit, let us study the $\theta \to \infty$ behavior
of $\Lambda(\theta; \vec{v})$ in order to match with the asymptotics (2.29). We find from eqs.(3.7)-(3.8)

$$\Lambda(\theta; \vec{v}) \overset{\theta \to \infty}{=} 2 \cos [\gamma (N + 1 - 2r)] \left[ 1 + O(e^{-2\theta}) \right] \quad (3.11)$$

Here all roots $v_1, v_2, \ldots, v_r$ are assumed to be finite. Actually this must be the case unless $\gamma/\pi$ is rational (see sec.5). Comparison of eqs.(2.32) and (3.11) shows that there is only one positive (or zero) solution

$$J = \frac{N}{2} - r \quad (3.12)$$

except when $\gamma/\pi$ is rational where other possibilities may happen. In addition, we have from eqs.(2.34) and (3.2) that

$$J_z \Psi(\vec{v}) = \left( \frac{N}{2} - r \right) \Psi(\vec{v}) \quad (3.13)$$

Eqs.(3.12) and (3.13) show that $J = J_z$ for Bethe-Ansatz eigenvectors. That is, they are maximal weight vectors for the quantum group for non-rational values of $\gamma/\pi$. Thus,

$$J_+ \Psi(\vec{v}) = 0 \quad (3.14)$$

In addition, the kernel of $J_+$ is stable under variations of $\gamma$. Therefore eq.(3.14) holds even when $\gamma/\pi$ is rational (see below). Each eigenvalue $\Lambda(\theta; \vec{v})$ has (at least) a degeneracy $(2J+1)$ for $t(\theta, \vec{\omega})$, since the vectors

$$J_- \Psi(\vec{v}), \ (J_-)^2 \Psi(\vec{v}), \ldots, \ (J_-)^{2J} \Psi(\vec{v}),$$

are all eigenvectors of $t(\theta, \vec{\omega})$ with the same eigenvalue thanks to eqs.(2.36)-(2.37).

Of course, when $\gamma/\pi$ is rational $J$ may not be equal to $J_z$ for some BA states $\Psi(\vec{v})$ as follows from eqs.(2.32),(3.11) and (3.13).

Let us give an algebraic proof of the highest weight condition (3.14) for the BA states. We shall apply $J_+$ to $\Psi(\vec{v})$ given by eq.(3.1) permuting $J_+$ through the
\( \mathcal{B}(v_j), (1 \leq j \leq r) \) with the help of eq.(2.35). Finally we shall use that

\[ J_+ \Omega = 0 \quad (3.15) \]

Actually it is more convenient to use the operator \( \hat{D}(\theta) \)

\[ \hat{D}(\theta) = \frac{1}{\sinh(2\theta - i\gamma)} \left[ \mathcal{D}(\theta) \sinh 2\theta - \mathcal{A}(\theta) \sinh(\gamma) \right] \quad (3.16) \]

eq  (3.15) can now be written as

\[ J_+ \mathcal{B}(\theta) = e^{\gamma} \mathcal{B}(\theta) J_+ + [e^{-\theta} \mathcal{A}(\theta) - e^{\theta} \hat{D}(\theta)] \quad (3.17) \]

We find from eq.(3.1) after using \( r \) times eq.(3.17):

\[ J_+ \Psi(\vec{v}) = \sum_{j=1}^{r} \exp[i\gamma(\frac{N}{2} - r + 2j)] \mathcal{B}(v_1) \mathcal{B}(v_2)\ldots \mathcal{B}(v_{j-1}) \]

\[ \left[ e^{-v_j} \mathcal{A}(v_j) - e^{v_j} \hat{D}(v_j) \right] \mathcal{B}(v_{j+1}) \ldots \mathcal{B}(v_r) \Omega \quad (3.18) \]

where we also used eqs.(3.13) and (3.15). Now we can push the operators \( \mathcal{A}(v_j) \) and \( \hat{D}(v_j) \) in eq.(3.18) to the right using the Yang-Baxter algebra (2.13). That is,

\[ \mathcal{A}(\theta) \mathcal{B}(\theta') = \frac{\sinh(\theta - \theta' - i\gamma) \sinh(\theta + \theta' - i\gamma)}{\sinh(\theta - \theta') \sinh(\theta + \theta')} \mathcal{B}(\theta') \mathcal{A}(\theta) + \]

\[ \frac{\sinh(2\theta - i\gamma) \sinh(i\gamma)}{\sinh(2\theta)} \mathcal{B}(\theta) \left[ \frac{\mathcal{A}(\theta')}{\sinh(\theta - \theta')} - \frac{\hat{D}(\theta')}{\sinh(\theta + \theta')} \right] \]

\[ \hat{D}(\theta) \mathcal{B}(\theta') = \frac{\sinh(\theta - \theta' + i\gamma) \sinh(\theta + \theta' + i\gamma)}{\sinh(\theta - \theta') \sinh(\theta + \theta')} \mathcal{B}(\theta') \hat{D}(\theta) + \]

\[ \frac{\sinh(2\theta + i\gamma) \sinh(i\gamma)}{\sinh(2\theta)} \mathcal{B}(\theta) \left[ \frac{\mathcal{A}(\theta')}{\sinh(\theta + \theta')} - \frac{\hat{D}(\theta')}{\sinh(\theta - \theta')} \right] \quad (3.19) \]

We find from eqs.(3.18)-(3.19) an expression of the form

\[ J_+ \Psi(\vec{v}) = \sum_{j=1}^{r} c_j(\vec{v}) \mathcal{B}(v_1) \mathcal{B}(v_2)\ldots \mathcal{B}(v_{j-1}) \mathcal{B}(v_{j+1}) \ldots \mathcal{B}(v_r) \Omega \quad (3.20) \]

where the \( c_j(\vec{v}) \) are c-number coefficients. The easier is to first compute \( c_1(\vec{v}) \) and then deduce the rest by symmetry of the arguments (cfr. eq.(3.5)). \( c_1(\vec{v}) \) follows by
permuting in the first term of eq.(3.18) the bracket \[ e^{-v_j \mathcal{A}(v_j)} - e^{v_j \hat{\mathcal{D}}(v_j)} \] through \( \hat{\mathcal{B}}(v_2) \ldots \hat{\mathcal{B}}(v_r) \) using eqs.(3.19) and keeping just the first terms in eqs.(3.19). That is, those proportional to \( \hat{\mathcal{B}}(v_j) \mathcal{A}(v_1) \) and to \( \hat{\mathcal{B}}(v_j) \hat{\mathcal{D}}(v_1) \), \( 2 \leq j \leq r \). Collecting all factors, it is easy to see that \( c_1(\vec{v}) \) and the \( c_j(\vec{v}) \) \( 1 \leq j \leq r \) are proportional to the BAE (3.4) and hence identically zero. We have therefore proved that all BA vectors are highest weights [3,17,18].

4. The quantum group covariant light-cone approach.

The light-cone approach is a direct way to give a field theoretical interpretation to a lattice model and furthermore obtain its scaling limit as a massive QFT. We start from a diagonal lattice that we interpret as a discretized two-dimensional Minkowski spacetime. The sites in the light-cone lattice are considered as world events. Each site has then microscopic amplitudes associated to it which describe the different processes that may take place and must verify general properties like unitarity.

The light-cone approach with periodic boundary conditions has been investigated in refs.[19-20]. Here we consider this approach in the case of fixed boundary conditions leading to a quantum group invariant framework. For the six-vertex model, this can be done as follows. Consider the \( SU(2)_q \) invariant transfer matrix \( t(\theta, \tilde{\omega}) \) constructed in sec.2, and identify the inhomogeneity parameters \( \omega_1, \omega_2, \ldots, \omega_N \) with the cutoff rapidities of the diagonal lattice. Namely set

\[
\omega_j = (-1)^j \Theta
\]

(4.1)

Then define

\[
U(\Theta) = t(\Theta + i\gamma/2, \omega_j = (-1)^j \Theta) \frac{\sinh(2\Theta + i\gamma)}{\sinh(2\Theta + 2i\gamma)}
\]

(4.2)

We shall now show that \( U(\Theta) \) has the natural interpretation of the unit time evolution operator on the diagonal lattice. To this end we introduce some convenient compact notation. We denote by 0 the auxiliary horizontal space and label by \( j \), \( 1 \leq j \leq N \), the vertical spaces. Then

\[
L_j(\theta) = R_{0j}(\theta - \omega_j) P_{0j}
\]

(4.3)

is a local vertex acting only on the \( j \)th vertical space. Here \( R_{jk}(\theta) \) stand for the six-vertex R-matrix (2.17) acting on the spaces \( j \) and \( k \), \( 0 \leq j, k \leq N \) and \( P_{jk} \) is the usual exchange operator. In practice, the \( ab \) matrix element of \( L_j(\theta) \) in the 0th space act on the \( j \)th space as the operators \( t_{ab}(\theta - \omega_j) \) defined in sec.2. In terms of
the $L_j(\theta)$ we can write the transfer matrix as

$$t(\theta + i \gamma / 2, \omega) = Tr_0 \{ K_0^+(\theta + i \gamma) L_N(\theta) .... L_1(\theta) K_0^-(\theta) L_1(-\theta)^{-1} .... L_N(-\theta)^{-1} \} \quad (4.4)$$

Since the R-matrix is regular, $R(0) = 1$, we have

$$L_j(\theta) \big|_{\theta = \omega_j} = P_{0j} \quad (4.5)$$

Hence, as soon as $\omega_j = (-1)^j \Theta$, one obtains

$$L_{2k}(\Theta) = P_{0, 2k}, \quad L_{2k+1}(\Theta) = S_{0, 2k+1} \quad (4.6)$$

where $S_{jk} = R_{jk}(2\Theta) P_{jk}$. Inserting eq.(4.6) into eq.(4.4) yields for odd $N$

$$t(\theta + i \gamma / 2, \omega_j = (-1)^j \Theta)$$

$$= Tr_0 \{ K_0^+(\theta + i \gamma) S_{0, N} P_{0, N-1} S_{0, N-2} P_{0, N-3} .... S_{0, 3} P_{0, 2} S_{0, 1} K_0^-$$

$$P_{0, 1} S_{0, 2} P_{0, 3} .... S_{0, N-1} P_{0, N} \}$$

$$= Tr_0 \{ K_0^+(\theta + i \gamma) S_{0, N} S_{N-2, N-1} .... S_{1, 2} P_{0, N-1} P_{0, N-3} .... P_{0, 2} K_0^-(\Theta)$$

$$P_{0, 1} P_{0, 3} .... P_{0, N} S_{2, 3} .... S_{N-1, N} \}$$

$$= Tr_0 \{ K_0^+(\theta + i \gamma) S_{0, N} P_{0, N} \} S_{12} .... S_{N-2, N-1} P_{0, N} P_{0, N-1} .... P_{0, 2}$$

$$P_{0, 1} P_{0, 3} .... P_{0, N} K_1^-(\Theta) S_{23} .... S_{N-1, N}$$

$$= Tr_0 \{ K_0^+(\theta + i \gamma) R_{0, N}(2\Theta) \} R_{12}(2\Theta) .... R_{N-2, N-1}(2\Theta) K_1^-(\Theta) R_{23}(2\Theta) .... R_{N-1, N}(2\Theta)$$

Actually, the computation up to now is completely general and would hold for any R-matrix and K-matrices. Let us exploit now the explicit form of $R(\theta)$ and $K^\pm(\theta)$ for the six-vertex model [eqs.(2.17)-(2.19)]. Specialising to the $SU(2)_q$ invariant case, we obtain

$$K_1^-(\Theta) = g_1(\Theta)^{-1}, \quad Tr_0 \{ K_0^+(\theta + i \gamma) R_{0, N}(2\Theta) \} = g_N(\Theta) \frac{\sinh(2\Theta + i \gamma)}{\sinh(2\Theta + 2i \gamma)} \quad (4.8)$$

where

$$g(\theta) = \exp[-\theta \sigma_z] = \begin{pmatrix} e^{-\theta} & 0 \\ 0 & e^{+\theta} \end{pmatrix} \quad (4.9)$$

Thus

$$U(\Theta) = R_{12} R_{34} .... R_{N-1-\epsilon, N-\epsilon} g_N g_1^{-1} R_{23} R_{45} .... R_{N-2+\epsilon, N-1+\epsilon} \quad (4.10)$$

where $\epsilon = [1 - (-1)^N]/2$, $g_j$ is the matrix $\exp[\Theta \sigma_z^+]$ acting nontrivially only on the two-dimensional vector space attached to the $j^{th}$ link, and $R_{ij}$ is the 6V R–matrix $R(2\Theta)$ acting nontrivially only on the tensor product of the $j^{th}$ and $k^{th}$ vector spaces.
$U(\Theta)$ is clearly the unit time evolution operator on a diagonal lattice with "reflecting" type boundary conditions, $g(\theta) [g(\theta)^{-1}]$ acting for collisions on the right [left] wall. Graphically $U(\Theta)$ can be depicted as in fig.6. By taking powers of $U(\Theta)$ one then obtains the evolution on the diagonal lattice for any discrete time (Fig.7). By construction, this time evolution is $SU(2)_q$ invariant.

The eigenvectors of $U(\Theta)$ can now be written as in sec.3, namely

$$\Psi(\vec{v}) = \hat{B}(v_1)\hat{B}(v_2)...\hat{B}(v_r)\Omega$$ \hspace{1cm} (4.11)

where $\omega_j = (-1)^j \Theta$ and the Bethe Ansatz equations (3.4) take the form

$$\begin{vmatrix} \sinh[v_m - \Theta + i\gamma/2] \sinh[v_m + \Theta + i\gamma/2] \\ \sinh[v_m - \Theta - i\gamma/2] \sinh[v_m - i\gamma/2] \\ \sinh[v_m - v_k + i\gamma] \sinh[v_m + v_k + i\gamma] \\ \sinh[v_m - v_k - i\gamma] \sinh[v_m + v_k - i\gamma] \end{vmatrix} = \prod_{k=1, k\neq m}^{r} \frac{\sinh[v_m - v_k + i\gamma] \sinh[v_m + v_k + i\gamma]}{\sinh[v_m - v_k - i\gamma] \sinh[v_m + v_k - i\gamma]} \hspace{1cm} (4.12)$$

The associated eigenvalue of $U(\Theta)$ is given by

$$U(\Theta, v_1, v_2, \ldots, v_M) = \prod_{k=1}^{r} \frac{\sinh[\Theta - v_m - i\gamma/2] \sinh[\Theta + v_m - i\gamma/2]}{\sinh[\Theta - v_m + i\gamma/2] \sinh[\Theta + v_m + i\gamma/2]} \hspace{1cm} (4.13)$$

Let us now discuss the problem of unitarity for the time evolution defined by $U(\Theta)$. The R–matrix (2.17) is unitary for real $\theta$. Hence the evolution operator $U$ is unitary too, for real $\Theta$, up to the boundary effects due to the term $g_N g_1^{-1}$ in eq.(4.10). As is evident from figs.(6-7) [eqs.(4.10)] there exist unitarity violating boundary effects (factors $g(\theta)_N$ and $g(\theta)_1^{-1}$). Then, the reflection of the bare particles on the boundaries is affected by "leaking". However, on average this leaking compensates since we have opposite factors $g(\theta)_N$ and $g(\theta)_1^{-1}$ on right and left sides respectively. Thus one can expect $U(\Theta)$ to still have unimodular eigenvalues. Indeed, this is the case from eq.(4.13) provided, as it is usually the case, the BA roots are either real, self-conjugate ($\text{Im } v = \pi/2$) or organised in complex conjugated pairs. As seen below, the exact diagonalization of $U(\Theta)$, shows that all its eigenvalues are unimodular, so that there exists a similarity transformation mapping $U(\Theta)$ to an explicitly unitary operator. On the other hand, it is natural to expect that in the thermodynamic limit ($N \to \infty$) different boundary conditions become equivalent (the model has a finite mass gap). This suggest that the above mentioned similarity transformation reduces to the identity as $N \to \infty$. We conclude therefore that the light–cone 6V model with fixed b.c. described by the evolution operator $U$ of eq.(4.10) is another good,
integrability–preserving regularization of the SG model. Its advantage over the more conventional setup with periodic b.c. is that $U(\Theta)$ is explicitly $SU(2)_q$ invariant even for finite $N$.

According to the general light-cone construction the physical lattice hamiltonian can be defined in terms of $U(\Theta)$ as

$$H = \frac{i}{a} \log U(\Theta) \quad (4.14)$$

where $a$ is the lattice spacing. By a judicious choice of the logarithmic branches the energy eigenvalues can be written from eqs.(4.13)-(4.14)

$$E = a^{-1} \sum_{j=1} e_0(v_j) \quad (4.15)$$

where

$$e_0(v) = -\pi + 2 \arctan \left( \frac{\cosh 2\Theta \cos \gamma - \cosh 2v}{\sinh 2\Theta \sin \gamma} \right) \quad (4.16)$$

Notice that $e_0(v)$ is smooth and negative definite for real $v$. The specific choice of the logarithmic branch in passing from eq.(4.13) to eqs.(4.15), (4.16) is dictated by the requirement that $e_0(v)$ should correspond to the negative energy branch of the spectrum of a single spin wave over the ferromagnetic state $\Omega$.

The physical ground state and the particle–like excitations are obtained by filling the interacting sea of negative energy states. This sea is described by a set of real $v$–roots of the BAE with no “holes” (in the Appendix we give a detailed exposition on the treatment of the BAE, also to clarify some rather subtle matters). Excitations correspond to solutions of the BAE which necessarily contain holes and possibly complex $v$–roots. The crucial point is that in the limit $N \to \infty$ only the number of real roots of the sea grows like $N$, while the number of holes and complex roots stays finite to guarantee a finite energy above the ground state energy. Hence these solutions of the BAE can be described by densities $\rho(v)$ of real roots plus a finite number of parameter associated to the positions $x_1, x_2, \ldots x_\nu$ of the $\nu$ holes and to the location of the complex $v$–roots.

The continuous theory is defined by the double limit $a \to 0, \Theta \to \infty$ taken in such a way that a finite mass gap $m$ emerges. The explicit result in the periodic case which will be shown to hold also here is [1,19,20]

$$m = \lim_{a \to 0} \frac{4}{\Theta} e^{-\frac{4}{a} \Theta} \quad (4.17)$$

[The double limit is taken such that the l.h.s. is finite and non-zero]. The invariance of $U(\Theta)$ under the quantum group $SU(2)_q$ allows to perform a transformation of the
original vertex model where states are assigned to the links and interactions as 'bare' scattering amplitudes, to the vertices, into a diagonal lattice face model, where states are defined on the plaquettes. The plaquettes are the sites of the dual lattice and the interaction involves the four plaquettes around each vertex, or equivalently, the four sites around each face of the dual lattice.

On the faces of the diagonal lattice one introduces positive integer–valued local height variables \( \ell_n(t), t \in \mathbb{Z} \), according to (see fig. 2)

\[
\begin{align*}
\ell_0(t) &= 1; \quad \ell_1(t) = 2 \\
\ell_{n+1}(t) &= \ell_n(t) \pm 1, \quad n = 1, 2, \ldots N - 1
\end{align*}
\]

The configurations \( \{\ell_1(t), \ldots \ell_N(t)\} \) at any fixed discrete time \( t \) are in one–to–one correspondence with the \( SU(2)_q \) multiplets of the 6V model of dimension \( \ell_n(t) = 2J + 1 \) (notice that \( \ell_N(t) \) can be chosen to be time–independent thanks to \( SU(2)_q \) invariance). Now the matrix elements of \( U \) between these highest weight states define the unit–time evolution in the face language.

This vertex-IRF correspondence is well known: in practice, it amounts to an application of the q-analog of the Wigner-Eckart theorem. Indeed, by \( SU(2)_q \) invariance, the matrix elements of \( U(\Theta) \) in the subspace with definite total q-spin \( J = j \), for \( \gamma/\pi \) not a rational, can be written

\[
< jm; \tau | U(\Theta) | j'm'; \tau' > = \delta_{jj'} \delta_{mm'} \quad < \tau || U(\Theta) || \tau' >_j
\]

(4.19)

where \( m = -j, -j + 1, \ldots, +j \) labels the \( J_z \) eigenvalues and \( \tau \) the degeneracy of the \( J = j \) subspace which has dimension

\[
\left( \begin{array}{c} N \\ N/2 - j \end{array} \right) - \left( \begin{array}{c} N \\ N/2 - j - 1 \end{array} \right) \equiv \Gamma_j^{(N)}
\]

(4.20)

\( < \tau || U(\Theta) || \tau' >_j \) thus denotes the reduced matrix elements, and a specific choice of basis is required to give explicit expressions for them. The most natural basis, which is the basis useful for the vertex-IRF correspondence, is that obtained by the successive composition of the q-spin 1/2 basic constituents assigned to the links. Consider a 'time zero' line cutting the diagonal lattice as in fig. 8. By intersecting \( N \) links, it identifies the Hilbert space of the vertex model as \( (\otimes [1/2])^N \), where \([j]\) denotes an irreducible representation of weight \( j \). On the other hand, the line passes also through a well defined set of plaquettes, including the half-plaquette at the extreme left and right. Then, a set of configurations of a local height variable \( l \), assigned to each plaquette, can be constructed as follows. Assign \( l = l_0 = 1 \) to the half plaquette on the left, the 0th plaquette. Then pass to the next one on the right, the first plaquette, following
the time zero line. This cuts a link carrying a spin 1/2 representation. So that, at this stage the Hilbert space is just the representation \([1/2]\) itself, i.e. \(J = 1/2\). We then set \(l_1 = 2J + 1 = 2\). Following the line, we now cut another spin 1/2 link and arrive at the second plaquette. This Hilbert space is now the direct sum \([J = 0] \oplus [J = 1]\) and the local height \(l\) can take two values, \(l_2 = 2j + 1 = 1\) or 3. By repeating this procedure \(n\) times (with \(n \leq N\)), we land in the \(n\)th plaquette after having cut \(n\) spin 1/2 links: the Hilbert space is \((\otimes [1/2])^n\) and contains irreps with \(J\) running from 0 or 1/2 up to \(n/2\). Hence the local variables \(l_n\) take the values

\[
 l_n = 2J + 1 = \begin{cases} 1, 3, 5, \ldots, n + 1, & n \text{ even} \\ 2, 4, 6, \ldots, n + 1, & n \text{ odd} \end{cases}
\]

By construction, \(|l_n - l_{n-1}| = 1\), since the \(l_n\) follow the composition laws of \(SU(2)_q\) representations

\[
[J] \otimes [1/2] = [J + 1/2] \oplus [J - 1/2]
\]

which for \(q = \exp(i\gamma)\) not a root of unity, are just those of the usual \(SU(2)\).

When \(n = N\) we arrive at the half-plaquette on the right. Here the values of \(l_n\) are just the dimensions of the irreducible representations in which the full Hilbert space \((\otimes [1/2])^N\) can be decomposed. All together, a specific choice of \(l_2, l_3, \ldots, l_N\) (\(l_0\) and \(l_1\) are fixed to \(l_0 = 1\), \(l_1 = 2\) by construction) defines one and only one of the \(\Gamma_j(N)\) irreps with \(J = j\). , if \(l_N = 2j + 1\). In other words, we have constructed a map from the \(N + 1\) plaquettes at 'time zero' and the Bratteli diagrams giving the composition rules for the tensor product of \(N\) spin 1/2 representations (see fig.9). We are now in position to identify the degeneracy label \(\tau\) with a specific path in the Bratelli diagram, i.e. \(\tau = (l_0, l_1, \ldots, l_N)\), with \(l_0 = 1\) and \(l_N = 2j + 1\).

It remains to evaluate the reduced matrix elements

\[
< \tau || U(\Theta) || \tau' >_j = < l_0, l_1, \ldots, l_N | U(\Theta) | l'_0, l'_1, \ldots, l_N >
\]

(4.21)

where the height variables \(l'_0, l'_1, \ldots, l_N\) are associated to the plaquettes crossed by the 'time one' line (see fig.8). Notice that \(l_0 = l'_0 = 1\) by definition, while \(l_N = l'_N = 2j + 1\) by \(SU(2)_q\) invariance. However, in spite of the factorised form of \(U(\Theta)\) [cft. eqs.(4.10)], these matrix elements cannot be written in a factorised form, with each factor depending locally on the variables \(l_0, l_1, \ldots, l_N\) and \(l'_0, l'_1, \ldots, l_N\). This is so because the full unit time evolution operator \(U(\Theta)\) is \(SU(2)_q\) invariant, but each single \(R_{k,k+1}(2\Theta)\) which enters is not. Only the peculiar boundary terms \(g_1\) and \(g_N\) enforce \(SU(2)_q\) invariance. Nevertheless, it is easy to pass to a different vertex representation where the \(SU(2)_q\) symmetry holds in a local sense. We define
a similarity transformation on \( U(\Theta) \)

\[
\tilde{U}(\Theta) = GU(\Theta)G^{-1}
\]  

(4.22)

where

\[
G = g_1^{1/2} g_2^{-1/2} g_3^{1/2} \cdots g_N^{\epsilon-1/2}
\]  

(4.23)

\( \epsilon = \frac{1 - (-1)^N}{2} \) and \( g_j \) is \( \exp[-\Theta \sigma_z] \) acting on the \( j \)-th spin 1/2 space. Now comparing eqs.(4.10) and (4.22)-(4.23) it is easy to verify that \( \tilde{U}(\Theta) \) can be written, e. g. for even \( N \)

\[
\tilde{U}(\Theta) = \tilde{R}_{12} \tilde{R}_{34} \cdots \tilde{R}_{N-1,N} \tilde{R}_{23} \tilde{R}_{45} \cdots \tilde{R}_{N-2,N-1}
\]  

(4.24)

where

\[
\tilde{R}_{k,k+1} = g_k^{-1/2} g_{k+1}^{1/2} R_{k,k+1}(2\Theta) g_k^{1/2} g_{k+1}^{-1/2}
\]  

(4.25)

One can explicitly verify that \( \tilde{R}_{k,k+1} \) is \( SU(2)_q \) invariant; that is

\[
\{ \tilde{R}_{k,k+1}, (\sigma_z)_m + (\sigma_z)_{m+1} \} = 0 \\
\{ \tilde{R}_{k,k+1}, (q^{-\sigma_z/2})_m (\sigma_\pm)_{m+1} + (\sigma_\pm)_m (q^{\sigma_z/2}) \} = 0
\]  

(4.26)

The reduced matrix elements of \( \tilde{U}(\Theta) \) can now be expressed in a factorised form ,

\[
< l_0, l_1, \ldots, l_N | U(\Theta) | l'_0, l'_1, \ldots, l_N > = W'_1 W'_3 W'_5 \cdots W'_{N-1+\epsilon} W_2 W_4 \cdots W_N-\epsilon,
\]  

where

\[
W_m = W(l_{m-1}, l_{m+1} | l_m, l'_m; \Theta) \quad W'_m = W(l'_{m-1}, l'_{m+1} | l_m, l'_m; \Theta)
\]  

(4.27)

and

\[
W(x, y | u, v; \Theta) = \delta_{uv} - \frac{\sinh 2\Theta}{\sinh(2\Theta + i\gamma)} \sqrt{\frac{|u||v|}{|x|}} \delta_{xy}
\]  

(4.28)

with the standard notation \( [x]_q = (q^x - q^{-x})/(q - q^{-1}) = \sin(\gamma x)/\sin \gamma \). This completes the transformation from vertex to faces description. The global time evolution in the faces (or heights) language is obtained by taking matrix products of

\[
< l_0, l_1, \ldots, l_N | U(\Theta) | l'_0, l'_1, \ldots, l_N >
\]

The boundary conditions are \( l_0 = \text{constant} = 1 \) on the left, while \( l_N = \text{constant} = 2j+1 \) on the right, if the reduction is performed onto the \( \Gamma_j^{(N)} \) dimensional space of irreps of \( q \)-spin \( J = j \). With the natural constraint that \( |l - l'| = 1 \), when \( l \) and \( l' \) sit on neighboring faces, the weights given by eqs.(4.27)-(4.28) define the ABF-SOS model in the trigonometric regime[13].
5. Analysis of the Bethe Ansatz equations

We investigate in this section the solution of the BAE (4.12) associated to the quantum group covariant BA in the light-cone approach. It is convenient to relate them with the BAE for periodic boundary conditions (see for example [1]). Define 2r variables \( \lambda_j \) as

\[
\lambda_j = v_j , \quad \lambda_{j+r} = -v_{r-j+1} , \quad 1 \leq j \leq r
\]  

(5.1)

Then eqs.(4.12) can be rewritten as

\[
\left[ \frac{\sinh[\lambda_m - \Theta + i\gamma/2] \sinh[\lambda_m + i\gamma/2]}{\sinh[\lambda_m - \Theta - i\gamma/2] \sinh[\lambda_m - i\gamma/2]} \right]^N \frac{\sinh(2\lambda_m + i\gamma)}{\sinh(2\lambda_m - i\gamma)} = -\prod_{k=1}^{2r} \frac{\sinh[\lambda_m - \lambda_k + i\gamma]}{\sinh[\lambda_m - \lambda_k - i\gamma]}, \quad 1 \leq m \leq 2r.
\]  

(5.2)

These equations are like the BAE for periodic boundary conditions on a 2N sites line and with an additional source factor

\[
\frac{\sinh(2\lambda_m + i\gamma)}{\sinh(2\lambda_m - i\gamma)} = -\frac{\sinh(\lambda_m + i\gamma/2) \sinh(\lambda_m - i\pi^2/2)}{\sinh(\lambda_m - i\gamma/2) \sinh(\lambda_m + i\pi^2/2)}
\]  

(5.3)

More important, we have the following constraints on the roots \( \lambda_m \):

a) the total number of roots is even (2r) and they are symmetrically distributed with respect to the origin according to eq.(5.1).

b) \( \lambda_m = 0 \) and purely imaginary \( \lambda_m \) are excluded as roots.

Let us start by considering the antiferroelectric ground state. It is formed by roots with fixed imaginary part equal to \( \pi/2 \). Therefore, we shift \( \lambda_m \rightarrow \lambda_m + i\pi/2 \) and take \( \lambda_m \) real for the ground state. Now \( \lambda_m = 0 \) is excluded whereas it may be present for periodic boundary conditions (PBC). Therefore, the ground state for eq.(4.12) is a one-hole solution of eq.(5.2) with the hole at \( \lambda_m = 0 \). We also redefine \( \gamma \) into \( \pi - \gamma \) in order to agree with the conventions of refs.[1,19,20]. With this choice, \( \gamma \) is related to the sine-Gordon coupling constant \( \beta \) by \( \beta^2/(8\pi) = 1 - \gamma/\pi \).

Let us now consider the \( N \rightarrow \infty \) limit where a continuous density of BAE real roots can be introduced

\[
\rho(\lambda_m) = \lim_{N \rightarrow \infty} \frac{1}{N(\lambda_{m+1} - \lambda_m)}
\]  

(5.4)

This function must always be even in our fixed boundary conditions case. Taking logarithms in both sides of eq.(5.2) and using eqs.(5.3) and (5.4) the BAE yield linear
integral equations through the usual procedure

\[ N \left[ \Phi'(\lambda + \Theta, \gamma/2) + \Phi'(\lambda - \Theta, \gamma/2) \right] + \Phi'(\lambda, \gamma/2) - \Phi'(\lambda, \frac{\pi - \gamma}{2}) \]

\[ = 2\pi N \rho(\lambda) + N \int_{-\infty}^{+\infty} d\mu \rho(\mu) \Phi'(\lambda - \mu, \gamma) + \]

\[ 2N \sum_{l=1}^{2N_c} \left[ \Phi'(\lambda - z_l, \gamma) + \Phi'(\lambda - \bar{z}_l, \gamma) \right] + 2\pi \delta(\lambda) + 2\pi \sum_{h=1}^{2M_h} \delta(\lambda - \theta_h) \] (5.5)

where we assume \(2N_c\) pairs of complex roots \((z_k, \bar{z}_k)\) and \(2M_h\) holes \(\theta_h\), symmetrically distributed with respect to the origin. The roots density \(\rho(\lambda)\) is connected with \(\sigma(\lambda)\), the derivative of the counting function through

\[ \rho(\lambda) = \sigma(\lambda) - \frac{1}{N} \delta(\lambda) - \frac{1}{N} \sum_{h=1}^{2M_h} \delta(\lambda - \theta_h) \] (5.6)

The Fourier transform solution of eq.(5.5) gives for the ground state

\[ \sigma_0(\lambda) = \int_{-\infty}^{+\infty} dk \frac{1}{2\pi} e^{ik\lambda} \tilde{\sigma}_0(k) \]

\[ \tilde{\sigma}_0(k) = \frac{\cos(k\Theta) + 1/(2N)}{\cosh(k\gamma/2)} + \frac{1}{2N} \frac{\sinh[k(\pi - 3\gamma)/4]}{\cosh(k\gamma/2) \sinh[k(\pi - \gamma)/4]} \] (5.7)

Notice that the first term is twice the PBC ground state density (cf. ref.[1]). The second term is the boundary effect and leads to a \(N^{-1}\) surface correction to the free energy[12].

The densities of real roots for excited states, that is in the presence of (non-zero) holes and complex roots are analogous to the PBC case. We find from eq.(5.5) :

\[ \sigma(\lambda) = \sigma_0(\lambda) + \frac{1}{N} [\sigma_h(\lambda) + \sigma_c(\lambda)] \] (5.8)

Here \(\sigma_h(\lambda)\) and \(\sigma_c(\lambda)\) stand for the holes and complex roots contributions, respec-
The explicit expressions of these densities are:

$$\sigma_h(\lambda) = \sum_{h=1}^{2M_h} p(\lambda - \theta_h), \quad \sigma_c(\lambda) = \sum_{h=1}^{2N_c} Q(\lambda - \sigma_l, \eta_l)$$  \hspace{1cm} (5.9)

where $z_k = \sigma_k + i\eta_k$ with $\eta_k > 0$ and

$$p(\lambda) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\sinh(\pi/2 - \gamma)k}{2\sinh(\pi - \gamma)k/2 \cosh(\gamma k/2)} e^{ik\lambda}$$

$$Q(\lambda, \eta) = -p(\lambda - i\eta) - p(\lambda + i\eta), \quad \eta < \gamma < \pi/2.$$  \hspace{1cm} (5.10)

We find from eq.(5.7)

$$\sigma_0(\lambda) + \sigma_0(\lambda - i\gamma)$$

We find from eq.(5.7)

$$\sigma_0(\lambda) + \sigma_0(\lambda - i\gamma) = \frac{1}{2\pi N} \frac{d}{d\lambda} \Phi_\gamma(2\lambda - i\gamma, \gamma)$$  \hspace{1cm} (5.12)

The r.h.s. comes entirely from the second term in eq.(5.7) . Notice that the l.h.s. of eq.(5.12) exactly vanishes in the PBC case.

We define the parameters $\chi_j$ as in the PBC case[14] :

$\chi = \text{real part of the two-string position,}$

$\chi = z_c - i\gamma/2, \bar{z}_c + i\gamma/2$ for a quartet $(z_c, \bar{z}_c, z_c - i\gamma, \bar{z}_c + i\gamma),$  

$\chi = z_w - i\gamma/2, \bar{z}_w + i\gamma/2$ for a wide pair $(z_w, \bar{z}_w).$
Then, the BAE take the following form in the $N = \infty$ limit

$$\prod_{h=1}^{\nu} \frac{\sinh \alpha(\chi_j - \theta_h + i\gamma/2)}{\sinh \alpha(\chi_j - \theta_h - i\gamma/2)} = -e^{i\Phi(2\chi_j, \gamma)} \prod_{k=1}^{M_p} \frac{\sinh \alpha(\chi_j - \chi_k + i\gamma)}{\sinh \alpha(\chi_j - \chi_k - i\gamma)}, \quad 1 \leq j \leq M_p. \quad (5.13)$$

where $\alpha \equiv 1/(1 - \gamma/\pi)$ and the use of eq.(5.12) yields the phase factor $e^{i\Phi(2\chi_j, \gamma)}$ in the r.h.s. The $\nu = 2M_h$ holes and the $M_p$ complex pairs are all distributed symmetrically with respect to the Re$\lambda = 0$ axis. Hence, restricting to positive real parts, we have

$$\prod_{h=1}^{M_h} \frac{\sinh \alpha(\chi_j - \theta_h + i\gamma/2)}{\sinh \alpha(\chi_j - \theta_h - i\gamma/2)} \prod_{k=1}^{M_p/2} \frac{\sinh \alpha(\chi_j - \chi_k + i\gamma)}{\sinh \alpha(\chi_j - \chi_k - i\gamma)}, \quad 1 \leq j \leq M_p/2. \quad (5.14)$$

To conclude, let us come back to the BAE (4.12). They can be rewritten in a manifestly algebraic form by introducing the variables $z_j \equiv \exp(-2v_j)$, $(1 \leq j \leq r)$.

$$\left( \frac{z_j w - q}{z_j w q - 1} \frac{z_j - w q}{z_j - w} \right)^N = \prod_{k=1}^{M_p} \frac{z_j - z_k q}{z_j q^2 - z_k} \frac{z_j z_k - q^2}{z_j z_k q^2 - 1} \quad (5.15)$$

where $w = \exp(-2\Theta)$ and $q = \exp(i\gamma)$ is the quantum group deformation parameter. According to the general discussion following eq.(3.6), we can restrict the search for solutions to eq.(5.15) strictly within the unit circle $|z_j| < 1$. Therefore, to any unordered set of distinct numbers fulfilling eq.(5.15), $z_1, \ldots, z_r$, with $|z_j| < 1$ , there corresponds one BA eigenstate of the transfer matrix (4.4) and hence of the evolution operator on the light cone-lattice. Special attention should be paid to the possibility that one or more roots $z_j$ lay exactly at the origin. This corresponds to Re$v_j \rightarrow \infty$, and therefore to the reduction of the corresponding $\hat{B}(v_j)$ to a multiple of the lowering operator $J_-$ of $SU(2)_q$ [c.f. eq.(2.29)].

$$\hat{B}(\infty) = (1 - q^2)J_- \quad (5.16)$$

In effect, the point at infinity constitutes the only exception to the requirement that the roots $v_1, \ldots, v_r$ be all distinct. However, this is a very special possibility. Indeed,
suppose that a given root in eq.(5.15), which we can always identify with $z_1$, lays at the origin $z_1 = 0$. We immediately obtain from eq.(5.15):

$$q^{2N} = q^{4(r-1)} \implies q^{4(J_z+1)} = 1$$

(5.17)

that is, $q$ must be a root of unit and $\gamma$ a rational multiple of $\pi$. This correspond to the special cases when certain irreducible representations of $SU(2)_q$ mix into type I reducible but indecomposable representations. This interesting phenomena is discussed in secs.7-9 within the Bethe Ansatz framework in connection with the RSOS models.

Eqs.(5.14) can also be written in algebraic form:

$$\prod_{h=1}^{\nu} \frac{\xi_j w_h - \hat{q} - \xi_j - w_h \hat{q}}{\xi_j w_h \hat{q} - 1} = \prod_{k=1}^{\hat{M}} \frac{\xi_j - \xi_k \hat{q}^2}{\xi_j \hat{q}^2 - \xi_k} \frac{\xi_j \xi_k - q^2}{\xi_j \xi_k \hat{q}^2 - 1}$$

(5.18)

where

$$\xi_j = e^{-2\alpha x_j}, \quad w_h = e^{-2\alpha x_h}, \quad \hat{q} \equiv e^{i\hat{\gamma}} = e^{i\alpha \gamma}, \quad \alpha = \frac{\pi}{\pi - \gamma}$$

(5.19)

These new BAE involve only a finite number of parameters and are formally identical to the “bare” BAE (4.12)-(5.15), with the holes acting as sources in the place of the alternating rapidities $\pm \Theta$. Moreover, from (5.18) one reads out the renormalization of the quantum group deformation parameter

$$q \to \hat{q} \quad i.e. \quad \gamma \to \hat{\gamma} = \frac{\pi \gamma}{\pi - \gamma}$$

(5.20)

As we shall see later on, this renormalization has a nice physical interpretation for the SOS and RSOS models related to the 6V model by the vertex–face correspondence.
6. Higher–level Bethe Ansatz and S–matrix

The holes in the sea of real BAE roots are the particles of the light–cone 6V model. They are \(SU(2)_q\) doublets and can be present in even (odd) number for \(N\) even (odd). Consider first the even \(N\) sector, setting \(N = 2N'\). The energy of a BA state with an even number \(\nu\) of holes located at \(x_1, x_2, \ldots x_\nu\) can be calculated to be, in the \(N' \to \infty\) limit,

\[
E = E_0 + a^{-1} \sum_{h=1}^{\nu} e(x_h) + O(a^{-1}N^{-1}) , \quad e(x) = 2 \arctan \left( \frac{\cosh \pi x/\gamma}{\sinh \pi \Theta/\gamma} \right) \quad (6.1)
\]

where \(E_0\) is the energy of the ground state, that is the state with no holes. \(E_0\) is of order \(N\) and is explicitly given in the appendix along with some detail on the derivation of eq.(6.1). Now suppose \(N\) odd, with \(N = 2N' - 1\). To compare this situation with the previous one, we need to slightly dilate the lattice spacing \(a\) to \(a' = 2N'a/(2N' - 1)\), in order to keep constant the physical size \(L = Na\) of the system. Then eq.(6.1) remains perfectly valid, as \(N' \to \infty\), also for the case of \(N\) odd, with the same ground state energy \(E_0\). The only difference is that now \(\nu\) is odd. Hence, altogether, we obtain that the number of holes can be arbitrary (unlike in the treatment with periodic b.c.) and that the total energy, relative to the ground state and in the \(L \to \infty\) limit, is the uncorrelated sum of the energy of each single hole, independently of the complex pair structure of the corresponding BAE solution. The BA state with one hole has \(J = N/2 - M = (2N' - 1)/2 - (N' - 1) = 1/2\) and is therefore a \(SU(2)_q\) doublet. The \(2^\nu\) polarizations of a state with \(\nu\) holes are obtained by considering all solutions of the higher–level BAE (5.18) with \(0 \leq \hat{M} \leq \nu/2\). We see in this way that the holes can be consistently interpreted as particles.

As a lattice system the light–cone 6V model is not critical. The (dimensionless) mass gap is the minimum value of the positive definite \(e(x)\), the energy of a single hole, that is

\[
e(0) = 2 \arctan \left( \frac{1}{\sinh \pi \Theta/\gamma} \right) \quad (6.2)
\]

This gap vanishes in the limit \(\Theta \to \infty\). Hence the continuum limit \(a \to 0\) can be reached provided at the same time \(\Theta \to \infty\) in such a way that the physical mass

\[
m = a^{-1}e(0) \simeq 4a^{-1}e^{-\pi\Theta/\gamma} \quad (6.3)
\]

stays constant [19]. In the same limit we obtain the relativistic expression

\[
a^{-1}e(x) \to m \cosh \pi x/\gamma \quad (6.4)
\]

so that \(\pi x/\gamma\) is naturally interpreted as the physical rapidity \(\theta\) of the hole. This is consistent also in our fixed b.c. framework, provided the limit \(L \to \infty\) is taken before
the continuum limit. Indeed the total momentum becomes a conserved quantity in the
infinite volume limit and its eigenvalues can be expressed in terms of the BA $v$–roots
exactly as in the periodic b.c. formulation. Then one finds that the momentum of a
single particle takes the required form $m \sinh \theta$ in the continuum limit.

The above analysis shows that a relativistic particle spectrum appears in the
$a \to 0$ limit above the antiferromagnetic ground state. For $\gamma > \pi/2$ one can show
also that bound states appear, associated to appropriate strings of complex roots, just
as in the periodic b.c setup. In the sequel we shall anyway restrict our analysis to
the repulsive $\gamma < \pi/2$ region, where the only particles are the holes, to be identified
with the solitons of the SG model. Of course at this moment, since the infinite
volume limit is already implicit, the particles are in their asymptotic, free states: the
rapidities $\theta_h = \pi x_h / \gamma$ may assume arbitrary continuous values and the total excitation
energy is the sum of each particle energy and does not depend on the internal state
of the particles. The situation changes if we consider $L$ very large but finite, since in
this case the hole parameters $x_1, \ldots, x_\nu$ are still quantized through the “bare” BAE
(4.12). Indeed, by definition the holes are real distinct numbers satisfying

$$Z_N(x_h; v_1, v_2, \ldots, v_M) = \frac{2\pi \tilde{I}_h}{N} ; \quad h = 1, \ldots, \nu$$  \hspace{1cm} (6.5)

where $Z_N(x; v_1, \ldots, v_M)$ is the “counting function” defined in the Appendix and the
positive integers $\tilde{I}_1, \ldots, \tilde{I}_\nu$ are all distinct from the integers $I_1, \ldots, I_r$ labelling the
$r$ ($r \leq M$) real $v$–roots of the BAE. For $N$ very large, $J = N/2 - M$ finite and
$x < (\gamma/\pi) \ln N$, the counting function can be approximated as

$$Z_N(x; v_1, \ldots, v_M) = Z_\infty(x) + N^{-1} F(x; x_1, \ldots, x_M; \chi_1, \ldots, \chi_M) + O(N^{-2})$$  \hspace{1cm} (6.6)

where $Z_\infty(x)$ is the ground state counting function at the thermodynamic limit

$$Z_\infty(x) = 2 \arctan \left(\frac{\sinh \pi x / \gamma}{\cosh \pi \Theta / \gamma}\right)$$  \hspace{1cm} (6.7)

and

$$F(x; x_1, \ldots, x_M; \chi_1, \ldots, \chi_M) = -i \log \prod_{h=1}^{\nu} S_0\left(\frac{x}{\pi}(x - x_h)\right) S_0\left(\frac{x}{\pi}(x + x_h)\right)$$

$$- i \log \prod_{j=1}^{M} \frac{\sinh \alpha(x - \chi_j + i\gamma/2) \sinh \alpha(x + \chi_j + i\gamma/2)}{\sinh \alpha(x - \chi_j - i\gamma/2) \sinh \alpha(x + \chi_j - i\gamma/2)}$$  \hspace{1cm} (6.8)

In the last expression, the numbers $\chi_j$ are the roots of the higher level BAE (5.18),
(5.19), while $S_0(\theta)$ coincides with the soliton-soliton scattering amplitude of the SG
model

\[ S_0(\theta) = \exp i \int_0^\infty dk \frac{\sinh(\pi/2\gamma - 1)k \sin k\theta/\pi}{\sinh(\pi k/2\gamma) \cosh k/2} \]  

(6.9)

under the standard identification \( \gamma/\pi = 1 - \beta^2/8\pi \).

Combining eqs. (6.5) and (6.6), and taking the continuum limit one obtains the “higher level” expression

\[
\exp (-imL \sinh \theta_h) = \prod_{n=1}^{\nu} S_0(\theta_h - \theta_n) S_0(\theta_h + \theta_n) \prod_{j=1}^{M} \frac{\xi_j w_h - \hat{q}}{\xi_j w_h \hat{q} - 1} \frac{\xi_j \hat{q} - w_h}{\xi_j - w_h \hat{q}}
\]  

(6.10)

where, according to (5.19), \( \xi_j = e^{-2\alpha x_j} \) and \( w_h = e^{-2\hat{\gamma}\theta_h/\pi} \). Together with the higher level BAE (5.18), this last equation provides the exact Bethe ansatz diagonalization of the commuting family formed by the \( \nu \) renormalized one–soliton evolution operators (see fig. 3)

\[
\hat{U}_h = S_{h,h-1}(\theta_h - \theta_{h-1}) \ldots S_{1,1}(\theta_1 - \theta_1) g_h(-\theta_h) S_{h,1}(\theta_h + \theta_1) \ldots S_{h,\nu}(\theta_h + \theta_\nu) \times \]  

\[
g_h(\theta_h) S_{h,\nu}(\theta_h - \theta_\nu) \ldots S_{h,h+1}(\theta_h - \theta_{h+1})
\]  

(6.11)

where \( g_h(\theta) = \exp \theta \sigma^z_\theta \) and \( S(\theta) \) is the complete \( 4 \times 4 \) SG soliton \( S \)-matrix in the repulsive regime \( \gamma < \pi/2 \) (for brevity we set \( \hat{\theta} = \gamma \theta/(\pi - \gamma) \)).

\[
S(\theta) = S_0(\theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \hat{b} & \hat{c} & 0 \\ 0 & \hat{c} & \hat{b} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
\hat{b} = b(\hat{\theta}, \hat{\gamma}) = \frac{\sinh \hat{\theta}}{\sinh(\hat{\gamma} - \hat{\theta})}
\]

\[
\hat{c} = c(\hat{\theta}, \hat{\gamma}) = \frac{\sin \hat{\gamma}}{\sinh(\hat{\gamma} - \hat{\theta})}
\]  

(6.12)

The operators \( \hat{U}_h \) are the values at \( \theta = \theta_h \) of the fully inhomogeneous Sklyanin–type transfer matrix \( T(\theta; \theta_1, \ldots, \theta_\nu) \) constructed with \( S(\theta - \theta_h) \) as local vertices. Let us stress that the higher–level Bethe Ansatz structure just described follows directly, after specification of the BAE solution corresponding to the ground state and without any other assumptions, from the “bare” BA structure of the light–cone 6V model. In particular, this provides a derivation “from first principles” of the SG \( S \)-matrix. We want to remark that \( S(\theta) \) is also the exact \( S \)-matrix for the elementary excitations of the 6V model on the infinite lattice: it is a bona–fide lattice \( S \)-matrix. Of course,
in this case $\theta = \theta_1 - \theta_2$ is the difference of lattice rapidities, which are related to the energy and momentum of the scattering particles through the lattice uniformization

$$
P_j = a^{-1} Z_\infty (\gamma_\theta j / \pi) = 2a^{-1} \arctan \left( \frac{\sinh \theta_j}{\cosh \pi \Theta / \gamma} \right)
$$

$$
E_j = a^{-1} \epsilon (\gamma_\theta j / \pi) = 2a^{-1} \arctan \left( \frac{\cosh \theta_j}{\sinh \pi \Theta / \gamma} \right)
$$

(6.13)

implying the lattice dispersion relation

$$
\cos a E_j / 2 = \tanh (\pi \Theta / \gamma) \cos a P_j / 2
$$

(6.14)

Thus, we see that the continuum limit only changes these energy–momentum relations to the standard relativistic form, $(E_j, P_j) = m (\cosh \theta_j, \sinh \theta_j)$, without affecting the $S$–matrix as a function of $\theta$.

7. SOS and RSOS reductions and kink interpretation

Under the vertex–face correspondence previously described, the light-cone 6V model is mapped into the SOS model. At the level of the Hilbert space, this corresponds to the restriction to the highest weight states of $SU(2)_q$: each $(2J + 1)$–dimensional multiplet of spin $J$ (for generic, non–rational values of $\gamma/\pi$) is regarded as a single state of the SOS model. In particular, in such a state the local height variables $\ell_n$ have well defined boundary values $\ell_0 = 1$ and $\ell_N = 2J + 1$. The ground state of the 6V model, which is a $SU(2)_q$ singlet, is also the ground state of the SOS model: it has $\ell_0 = \ell_N = 1$ and is “dominated”, in the thermodynamic limit $N \to \infty$ by the see-saw configuration depicted in fig. 1. Our boundary conditions allow for only one such “ground state dominating” configuration, with $\ell_n = (3 - (-)^n)/2$, while periodic b.c. on the SOS variables $\ell_n$ would allow any possibility: $\ell_n = (2\ell + 1 \pm (-)^n)/2$, with $\ell$ any positive integer.

Consider now a BA state with one hole. Then $N$ is necessarily odd while $J = 1/2$ and $\ell_N = 2J + 1 = 2$. From the SOS point of view this state is “dominated” by height configurations of the type depicted in fig.4. Upon “renormalization”, we can replace the ground state and one hole configurations with the smoothed ones of fig. 5. The hole corresponds, in configuration space, to a kink of the SOS model which interpolates between two neighboring vacuum states: the vacuum on the left of the kink has $\ell_n = (3 - (-)^n)/2$, corresponding to a constant “renormalized” height $\hat{\ell}_n = 1$, while the vacuum on the right has $\ell_n = (5 + (-)^n)/2$, corresponding to $\hat{\ell}_n = 2$. Of course, many kink configurations like that depicted in fig. 4 are to be combined into a standing wave (a plane wave in the infinite volume limit which turns the one hole BA state into an eigenstate of momentum). Let us observe that, by
expressions like “dominating configuration”, we do not mean that, e.g. the ground state, becomes an eigenstate of $\ell_n$ as $N \to \infty$. We expect local height fluctuations to be present even in the thermodynamic limit or, in other words, that the ground state remains a superpositions of different height configurations. The identification of a dominating configuration is made possible by the integrability of the model, which guarantees the existence of the higher–level BA. In turns, the higher–level BA allows us to consistently interpret the ground state or the one hole state as in figs. 1, 4 and 5, since the presence of fluctuations only renormalizes in a trivial way the scattering of physical excitations relative to the bare, or microscopic, $R-$matrix (2.17), as evident from eq.(6.12). Therefore, we can reinterpret at the renormalized level the vertex-face correspondence: the holes, that is the solitons of the SG model, are $SU(2)_q$ doublets acting as SOS kinks that increase or decrease by 1 the renormalized heights $\hat{\ell}_n$. The higher–level BA makes sure that the total number of internal states of $\nu$ kinks interpolating between $\hat{\ell} = 1$ and $\hat{\ell} = 2J$ is just the number of highest weight states of spin $J$ in the tensor product of $\nu$ doublets

$$d_{\nu}(J) = \begin{pmatrix} \nu \\ \hat{M} \end{pmatrix} - \begin{pmatrix} \nu \\ \hat{M} - 1 \end{pmatrix} ; \quad \hat{M} = [\nu/2] - J$$

(7.1)

This parallels exactly the original BA, which provides the highest weight states of $N$ doublets with total spin $J$.

Comparing eqs. (2.17), (4.28) and (6.12), we can directly write down the $S-$matrix of the SOS kinks:

$$S(\theta)^{is, sj}_{ir, rj} = S_0(\theta) \left\{ \delta_{rs} + \delta_{ij} b(\hat{\theta}, \hat{\gamma}) \left( \frac{\sin \hat{\gamma}r \sin \hat{\gamma}s}{\sin \hat{\gamma}i \sin \hat{\gamma}j} \right)^{1/2} \right\}$$

(7.3)

It defines the two–body scattering as follows. The first ingoing kink interpolates between the local vacuum with $\hat{\ell} = i$ and that with $\hat{\ell} = r$ ($r = i \pm 1$), while the second interpolates between $\hat{\ell} = r$ and $\hat{\ell} = j$ ($j = r \pm 1$). The outgoing kinks are interpolating between $\hat{\ell} = i$ and $\hat{\ell} = s$ ($s = i \pm 1, i.e., s = r, r \pm 2$) and between $\hat{\ell} = s$ and $\hat{\ell} = j$ ($j = s \pm 1$).

When $\gamma = \pi/p$ for $p = 3, 4, \ldots$, the SOS models can be restricted to the RSOS($p$) models, by imposing $\ell_n < p$. In the vertex language of the 6V model this corresponds to the Hilbert space reduction to the subspace formed by the so–called type II representations of $SU(2)_q$ with $q^p = -1$ [5,6]. In the next section we shall describe in detail
how the restriction takes place in our BA framework. Here we simply observe that
the local height restriction, when combined with the finite renormalization $\gamma \to \hat{\gamma}$ of eq.(5.20), provides a strong support for the kink interpretation presented above. Indeed, if $\gamma = \pi/p$, then $\hat{\gamma} = \pi/(p - 1)$ and the renormalized heights $\hat{\ell}_n$ can take the values $1, 2, \ldots, p - 2$. This appears now obvious, since each constant configuration of $\hat{\ell}_n$ corresponds to an $\ell_n-$configuration oscillating two neighboring values. Moreover under the standard identification of the critical RSOS($p$) models with the minimal CFT series $M_p$, we see that each light–cone RSOS($p$) model has kink excitations whose $S-$matrix (7.3) is proportional to the (complex) microscopic Boltzmann weights of the RSOS($p-1$) model, under the replacement of $2\Theta = \Theta - (-\Theta)=$ rapidity difference of light-cone right and left movers, with $(p-1)$ times $\theta = \theta_1 - \theta_2 =$rapidity difference of physical particles. This is exactly the pattern found by bootstrap techniques [21] for the minimal model $M_p$ perturbed by the primary operator $\phi_{1,3}$ (with negative coupling).

8. BA roots when $q$ is a root of unity and Quantum Group reduction.

As previously explained to each $M$ roots solution of the BAE there corresponds a highest weight state of the quantum group $SU(2)_q$ with spin $J = N/2 - M$. It is well known that when $q$ is a root of unity, say $q^p = \pm 1$, then $(J_+)^p = (J_-)^p = 0$ and the representations of $SU(2)_q$ divide into two very different types. Type I representations are reducible and generally indecomposable. They can be described as pairwise mixings of standard irreps (that is the irreducible representations for $q$ not a root of unity) with spin $J$ and $J'$ such that

$$|J - J'| < p , \quad J + J' = p - 1 \mod p$$

(8.1)

Notice that the sum of $q-$dimensions for this pair of reps vanishes.

Type II representations are all the others. They are still fully irreducible and structured just like the usual $SU(2)$ irreps. Since $(J_\pm)^p = 0$, type II representations have necessarily dimension smaller than $p$, that is

$$J < \frac{p - 1}{2}$$

(8.2)

We shall now show how the BAE reflect these peculiar properties of the $SU(2)_q$ representations for $q$ a root of unity. First of all let us stress that when $q$ is not a root of unity (i.e.$\gamma/\pi$ is irrational), then the BAE cannot possess $v-$roots at (real) infinity. Indeed, since $\mathcal{B}(\infty)$ is proportional to $J_-$ (eq.(5.16)), a root at infinity means that the corresponding BA state is obtained by the action of the lowering operator $J_-$ on some other state with higher spin projection $J_z$. But for $q$ not a root of unity,
the BA states have $J = J_z$ and therefore cannot be obtained by applying $J_-$ on any
other state. On the other hand, if we assume that one $z-$root, say $z_1$, of the BAE
(4.12) lay at the origin, i.e. $Re v_1 = +\infty$, then eqs. (4.12) for $j = 1$ imply

$$q^{4(J+1)} = 1$$

That is, $q$ must be a root of unity. It is now crucial to observe that the remaining
equations for the non–zero roots (those labelled by $j = 2, 3, \ldots, M$) are precisely the
BAE for $M - 1$ unknowns. This invariance property holds only for the quantum group
covariant, fixed boundary conditions BAE. It does not hold for the p.b.c. BAE where
$v-$roots at infinity twist the remaining equations for finite roots. This twisting reflect
the fact that the corresponding periodic row–to–row transfer matrix is not quantum
group invariant but gets twisted under $SU(2)_q$ transformations [1]. Thus, when $q$ is
a root of unity, BA states (4.11) with one $v-$root at infinity take the form

$$\Psi(v_1 = \infty, v_2, \ldots, v_M) = (1 - q^2)J_- \Psi(v_2, v_3, \ldots, v_M)$$

(8.4)

Let us recall that the BA state on the l.h.s. is annihilated by $J_+$ for any $q$ (including $q$
a root of unity) [3]. Hence eq.(8.4) represents the mixing of two reps with spin $J$ and
$J' = J + 1$ into a type I representation. Indeed, applying the mixing rule (8.1) into
the necessary condition (8.3) for one root at infinity, yields an identity as required:

$$1 = q^{4(J+1)} = q^{2(J+J'+1)} = (q^n)^2 = 1$$

(8.5)

where $n$ is a suitable ($J-$ dependent) positive integer.

We can generalize this analysis to any number of vanishing BA $z-$roots. Let us
identify the roots going to the origin with $z_1, z_2, \ldots, z_r, 1 \leq r \leq M$. The BAE for
the remaining non–zero roots $z_{r+1}, \ldots, z_M$ take the standard form (4.12) valid for a
BA state formed by $M - r$ $B-$operators, which has therefore spin $J + r$. The BAE
for the vanishing roots take the form

$$q^{4J+2(r+1)} = F_j \equiv \prod_{k=1}^{r} \frac{z_j q^2 - z_k}{z_j - z_k q^2}, \quad 1 \leq j \leq r$$

(8.6)

where the definition of $F_j$ is understood in the limit of vanishing $z_1, \ldots, z_r$. We can
obtain complete agreement with the quantum group mixing rules (8.1) by setting
Indeed the presence of \( r \) vanishing \( z \)-roots imply the mixing of two representations with spin \( J \) and \( J' = J + r \), so that
\[
q^{4J+2(r+1)} = q^{2(J+J'+1)} = (q)^{2n} = 1
\]
with \( n \) an integer depending on \( J \) and \( J' \). Eq. (8.7) ha a very simple solution for the limiting behaviour of the vanishing roots. We find that if
\[
z_j = \omega^{j-1} z_1, \quad 1 \leq j \leq r
\]
with \( \omega^r = 1 \), then identically
\[
F_j = \prod_{k=1, k \neq j}^{r} \frac{\sin[-\gamma + \pi(k-j)/r]}{\sin[\gamma + \pi(k-j)/r]} = 1, \quad 1 \leq j \leq r
\]
therefore, the RSOS reduction is obtained by retaining only those states with spin
\( J < \frac{(p-1)}{2} \), that is with \( M > \frac{(N-p+1)}{2} \), such that all BA roots \( z_1, z_2, \ldots, z_r \) are
non–zero. This constitutes the general and simple prescription to select the RSOS\((p)\)
subspace of eigenstates of the evolution operator when \( q^p = \pm 1 \).

We have considered up to here the RSOS reduction at the microscopic level, that
is for the BA states (4.11) described by the bare BAE (4.12). It should be clear,
however, that the analysis of the singular BAE solutions when \( q \) is a root of unity
holds equally well for the higher–level BAE (5.18), due to their structural identity with
the bare BAE. The crucial problem is whether a quantum group reduction carried out
the higher level would be equivalent to that described above in the “bare” framework.
This equivalence can actually be established as follows. Suppose that \( \gamma/\pi \) is irrational,
but as close as we like to a specific rational value. Then all solutions of the bare BAE
are regular and can be correctly analyzed, in the limit \( N \to \infty \), using the density
description for the real roots. As described in paper I, this yields the higher–level
BAE (5.18), those roots are in a precise correspondence with the complex roots of the
bare BAE. When \( \gamma \) tends to \( \pi/p \), then \( \hat{\gamma} \) tends to \( \pi/(p-1) \) and singular solutions of
the bare BAE with two or more singular roots are in one–to–one correspondence with
singular solutions of the higher–level BAE. Thus in this case quantum group reduction
and renormalization of the BAE commute. The only potentially troublesome cases
are those of singular solutions with only one real singular root. Indeed the density
description of the \( v \)–roots cannot account, by construction, for real roots at infinity,
and no sign of the type I nature of the corresponding BA state would show up at
the higher level. Notice that spin \( J \) BA states with a single singular root are mixing
with spin \( J + 1 \) states, so that, by the quantum group mixing rule (8.1), necessarily
\( J = \frac{p}{2} - 1 \). On the other hand, a direct application of the constraint (8.2) at
the higher level, that is with the replacement \( p \to \hat{p} = p - 1 \), would overdo the
job, by incorrectly ruling out all BA states with \( J = \frac{p}{2} - 1 = (\hat{p} - 1)/2 \). To be
definite, consider \( N \) even. Then the type II BA states with \( J = \frac{p}{2} - 1 \), which ar
e superpositions of SOS configurations with \( \ell_n \leq p - 1 = 2J + 1 = \ell_N \), have an
effective, higher–level spin \( \hat{J} \equiv (\hat{\ell}_N - 1)/2 = J - 1/2 \). This higher–level spin does
satisfy (8.2). The type I states are those in which \( \ell_n \) somewhere exceeds \( p - 1 \). In
particular, the simplest change on a type II configuration, turning it into type I, is to
flip the oscillating \( \ell_n \) close to the right wall (as shown in fig. 6). \( \ell_N \) and hence the
spin \( J \) are left unchanged, but clearly \( \hat{\ell}_N \) increases by one, causing \( \hat{J} \) to violate the
bound (8.2). In other words, RSOS\((p)\)–acceptable BA states with \( J = \frac{p}{2} - 1 \) have
one kink less than the corresponding SOS states. From the detailed analysis of the
\( p = 4 \) case, to be discussed below, it appears that the removal of one kink corresponds
giving infinite rapidity to the hole representing that kink in the higher–level BAE.

Thank to the possibility of performing the restriction directly at the renormalized
level, the kink \( S \)–matrix of the RSOS\((p)\) model follows by direct restriction on the
SOS \( S \)–matrix given by eq.(7.3). Namely, one must set \( \gamma = \pi/p \), that is \( \hat{\gamma} = \pi/(p-1) \),
and consider all indices as running from 1 to \( p - 2 \).
9. The models RSOS(3) and RSOS(4)

In order to clarify matters about the BA RSOS reduction discussed in the previous section, we present here the more details about the two simplest examples, when \( \gamma = \pi/3 \) and \( \gamma = \pi/4 \). For these two cases the RSOS reductions correspond, respectively, to a trivial one–state model and to the Ising model (at zero external field and non–critical temperature). Let us recall that the light–cone approach yields in the continuum limit massive field theories with the same internal symmetry of the corresponding critical regimes. Therefore the RSOS(4) reduction of the light–cone 6V model coincides with the \( \mathbb{Z}_2 \)–preserving perturbation of the \( c = 1/2 \) minimal model (which is obtained upon quantum group restriction of the critical 6V model).

9.1. The case \( p = 3 \)

From the RSOS viewpoint, the case \( \gamma = \pi/3 \) is particulary simple. Since \( p = 3 \) the local height variables \( \ell_n \) can assume only the values 1 and 2. Then the SOS adjacency rule \( |\ell - \ell'| = 1 \) implies that the global configuration is completely determined once the height of any given site is chosen. In our light–cone formulation the first height on the left is frozen to the value 1 (see eq.(4.18) and fig. 2), so that the restricted Hilbert space contains only one state: an \( SU(2)_q \) singlet when \( N \) is even and the spin up component of a \( J = 1/2 \) doublet when \( N \) is odd.

For \( \gamma = \pi/3 \) the BAE still have many solutions which reproduce the full SOS Hilbert space. Indeed, from the SOS point of view, nothing particular happens when \( \gamma \to \pi/3 \). However, at this precise value of the anisotropy, only one BAE solution corresponds to the unique type II representation: for \( N \) even (odd) it contains \( N/2 \) (\( N/2 - 1/2 \)) non–zero roots in the \( z \)–plane. All other BAE solutions with \( M = [N/2] \) contain at least one vanishing \( z \)-root and correspond to type I SOS states. Moreover, there unique RSOS state is the ground state of the SOS model and is formed by real positive roots labelled by consecutive quantum integers (no holes). We reach therefore the following rather non–trivial conclusion: the complicated system of algebraic equations (4.12) admit, for \( q^3 = \pm 1 \) and \( w \) real, one and only one solution with \( M = [N/2] \) non–zero roots within the unit circle. In addition, these roots are real and positive. For few specific choices of \( N \) we also verified this picture numerically.

9.2. The case \( p = 4 \)

The local height variables can assume now the three values \( \ell = 1, 2, 3 \). However, each configuration can be decomposed into two sub–configurations laying on the two sublattices formed by even and odd faces, respectively. On one of the two, the SOS adjacency rule \( |\ell - \ell'| = 1 \) freezes the local heights to take the constant value \( \ell = 2 \). Then on the other sub–lattice we are left with two possibilities, \( \ell = 1 \) and 3. Moreover,
the interaction round–a–site of the original model reduces in this way to a nearest–neighbor interaction in the vertical and horizontal directions. The framework is that of the Ising model.

To obtain the standard Ising formulation, we can set
\[ \sigma_n(t) = \ell_{2n}(t) - 2 \] (9.1)
where the numbering of the lattice faces can be read from fig. 2. The fixed b.c. on the \( \ell_n \) now correspond to
\[ \sigma_0(t) = -1, \quad \sigma_R(t) = (-1)^{J-1} \quad (N = 2R) \]
\[ \sigma_0(t) = -1, \quad \sigma_R(t) = \pm 1 \quad (N = 2R + 1) \] (9.2)
where \( J = 0, 1 \) for \( N \) even, due to the bound (8.2). For \( N \) odd we must consider only the possibility \( J = 1/2 \), which implies \( \ell_N = 2 \), since the line of half-plaquettes on the extreme right belong to the frozen sublattice. Then \( \ell_{N-1} \) is left free to fluctuate between 1 and 3, leading to free b.c. on \( \sigma \). The matrix elements of the unit time evolution operator \( \tilde{U} \) can be calculated from the explicit form of the SOS weights (4.27)-(4.28). They can be written in the “lagrangian” form
\[ t+1 \langle \sigma'_0, \sigma'_1, \ldots \sigma'_R | \tilde{U} | \sigma_0, \sigma_1, \ldots \sigma_R \rangle_t = e^{iL(t)} \] (9.3)
where
\[ L(t) = \beta_v \sum_{n=1}^{R-1} \sigma_n(t)\sigma_n(t+1) - \beta_h \sum_{n=0}^{R-1} \sigma_n(t)\sigma_{n+1}(t) + \text{const} \] (9.4)
and
\[ \beta_v = \frac{1}{4}(\pi + 2i \ln \tanh 2\Theta), \quad \beta_h = \arctan \tanh 2\Theta \] (9.5)
Alternatively, standard simple manipulations allow to rewrite \( \tilde{U} \) explicitly in terms of Pauli matrices
\[ \tilde{U} = e^{-i\beta_h H_2} e^{-i\beta_h H_1} \]
\[ H_1 = \sum_{n=1}^{R-1} (\sigma_n^x + 1), \quad H_2 = \sum_{n=0}^{R-1} \sigma_n^z \sigma_{n+1}^z \] (9.6)
In either cases, one sees that the complex Boltzmann weights formally belong to the critical line
\[ \sin 2\beta_v \sin 2\beta_h = 1 \] (9.7)
Of course, this follows from the original definition of the light–cone 6V model in terms of complex trigonometric Boltzmann weights, which, under the replacement \( \Theta \to i\Theta \),
would correspond to the critical standard 6V model. Nevertheless, just as for the vertex model, also in this “light–cone” Ising model, a massive field theory can be constructed in the Re $\Theta \to +\infty$ limit.

In the BA diagonalization of $\tilde{U}$, we must restrict ourselves to the BAE solutions with $M = R$ or $M = R - 1$ for even $N$ and $M = R$ for odd $N$. For sufficiently large even $N$, the ground state has $R$ real roots with consecutive quantum integers. Actually, as already stated above, this is a general fact valid for all RSOS($p$) models. Namely, the infinite volume ground state of the SG model, of the SOS model and of all its restrictions RSOS($p$) is the same f.b.c. BA state. It is the unique $SU(2)_q$ singlet with all real positive roots and no holes. It is described in the thermodynamic limit ($N \to \infty$, $a$ fixed) by the density of roots given in eq.(5.7) of paper I.

Excited states with $J = 0$ have an even number of holes and a certain number of complex roots such that no $z$–root lays at the origin. For instance a two–particle state contains two holes (holes are naturally identified with the particles) and a two–string with imaginary parts $\pm [\pi/8 +$ corrections exponentially small in $N]$ laying between the two holes. This state (that is this precise choice of quantum integers) is just the two–particle state of the SG or SOS model, with $\gamma$ fixed to the precise value $\pi/4$. We have explicitly checked, by numerically solving the BAE for various values of $N$, that indeed all $z$–roots stay away from the origin as $\gamma$ crosses $\pi/4$ while the quantum integers are kept fixed to the two–hole configuration. Now consider a state with four holes. Apart from the multiplicity of the rapidity phase space, there are two distinct type of such states: in $v$–space one contains two two–strings, while the other contains one sigle wide pair, that is a complex pair with imaginary part larger than $\gamma$. This situation holds for generic values of $\gamma/\pi$ and simply reflects the fact that the holes are $SU(2)_q$ doublets. The crucial point is that, as $\gamma$ reaches $\pi/4$ (from below), the wide pair moves towards infinity, while its real part gets closer and closer to the (diverging) value of the largest real root and the imaginary part approaches $\pi/3$. This picture is confirmed by a careful numerical study of the BAE and is in perfect agreement with the general picture presented in sec. 5. In addition, there exists numerical evidence that the real parts diverge like $\log(\pi/4 - \gamma)^{-1/6}$ while the imaginary part of the wide pair goes to $\pi/3$ like $(\pi/4 - \gamma)^{1/3}$. When $\gamma > \pi/4$, the $v$–root corresponding to the largest quantum integer has the largest real part, but is no longer real, having an imaginary part equal to $\pi/2$. As $\gamma \to (\pi/4)^+$, this real part again diverges together with real part of the wide pair. Hence the four–hole state with a wide pair is type I when $\gamma = \pi/4$. On the other hand, the four–hole state with two two–string is type II, since all its $v$–roots stay finite.

From the study of the two– and four–hole states, we are led to the following general conjecture: in the BA framework, the RSOS(4) $J = 0$ states are all and only the states with $\nu = 2k$ holes and $k$ two–strings. Then, in the higher–level BAE (5.18), we recognize this state as that corresponding to the unique solution with $k$ real $\chi$–roots. In other words, this BA state is completely determined once the location of the holes (that is the rapidities of the physical particles) is given.

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Consider now the states with $J = 1$. In this sector, the lowest energy type II state contains exactly one hole. This is an unusual situation for BA systems on lattices with even $N$, where holes are always treated in pairs. As long as $\gamma \neq \pi/4$, the same is true in our f.b.c. BA: the lowest energy state with $J = 1$ contains two holes, in agreement with the interpretation of holes as SG solitons with quantum spin 1/2. For $\gamma < \pi/4$ all roots are real. For $\gamma > \pi/4$ the root $v_{N/2-1}$ corresponding to the largest quantum integer $I_{N/2-1} = N/2 + 1$ acquires an imaginary part $\pi/2$ (see the appendix for details). But when $\gamma = \pi/4$ then $\text{Re} v_{N/2-1} = +\infty$, and the $J = 1$ two-hole state is mixed with some $J = 2$ state into a type I representation. It does not belong to the RSOS(4) Hilbert space. This picture can be easily verified numerically.

To prevent the largest root $v_{N/2-1}$ from diverging, it is sufficient to consider a $J = 1$ state with only one hole and $I_{N/2-1} = N/2$. From the point of view of the SG or SOS models this one–hole state has a cutoff–dependent energy which diverges as $a^{-1}$ in the continuum limit. If $x_1$ is the position of the single hole, the energy relative to the ground state reads

$$E - E_0 = a^{-1} e(x_1) + \pi a^{-1}$$

(9.8)

where the renormalized energy function $e(x)$ is given in eq.(6.1). Notice that this result coincides with the limit $x_2 \to \infty$ of a two–hole state (cf. eq.(6.1)). Strictly speaking, however, the density method leading to (6.1) has no justification when “one hole is at infinity”. We trust eq.(9.8) nonetheless because it passes all our numerical checks. In the continuum limit $a \to 0$, $\Theta \to \infty$, $4a^{-1}e^{-\pi\Theta/\gamma} = m$ (fixed), this one–hole state is removed from the physical spectrum, as required from the SG point of view: holes are spin 1/2 solitons while here $J = 1$.

The preceding discussion easily extends to generic states in the $J = 1$ sector containing an odd number of holes and a given set of complex pairs. The infinite–volume energy of these multiparticle states is given by

$$E = E_0 + a^{-1} \sum_{h=1}^\nu e(x_h) + \pi a^{-1}$$

(9.9)

with the same divergent constant $\pi a^{-1}$ appearing irrespective of the physical content of the state. Our conjecture for the $J = 0$ states naturally extends to these $J = 1$ states: if there are $\nu = 2k + 1$ holes, then the $\nu$–roots are all finite, implying that the state is type II, provided there are also $k$ two-strings. Then the higher–level BAE imply that these states are completely defined by the hole rapidities. To retain these type II $J = 1$ BA states in the RSOS(4) model, an extra $J$–dependent subtraction is necessary to get rid of the divergent constant. Namely, for $N$ even and $J = 0$ or 1, we set

$$H_{\text{RSOS}(4)} = H_{\text{SOS}}(\gamma = \pi/4) - \pi a^{-1} J$$

(9.10)

where $H_{\text{SOS}}$ is given by eqs. (4.10), (4.14), (4.22)-(4.28).
Alltogether, we see that the BA picture for the excitations of the RSOS(4) model is fully consistent with the kink interpretation for the holes. Indeed $J = 0$ corresponds to even Dirichlet b.c. for the Ising field, while $J = 1$ corresponds to odd Dirichlet b.c. (c.f. eqs. (9.2)). In the Ising model the kink description is valid below the critical temperature, with the disorder field as natural interpolator for the kinks. In this case the $S$–matrix must be $-1$ (it is $+1$ when the asymptotic particles are the free massive Majorana fermions) and this is exactly what follows from eq.(7.3) upon setting $\hat{\gamma} = \pi/3$ and restricting all indices to run from $1$ to $p - 2 = 2$.

The analysis of the BA spectrum for odd $N$ does not contain real new features. Now $J$ is fixed to $1/2$ and the b.c. on the Ising field are of mixed fixed–free type, as shown in the second of eqs. (9.2). The lowest energy state corresponds to the BA solution formed by real roots and one single hole as close as possible to the the $v$–origin. In other words, the quantum integers are given by $I_j = j + 1$, for $j = 1, 2, \ldots , (N - 1)/2$. By letting this hole to move away along the positive real axis, we reconstruct the energy spectrum of a one–particle state. Of course, to compare this state to the global ground state, which contains no holes, a judicious choice of the odd value of $N$ is required. If in the ground state $N = 2R$, then the new state is indeed an excited state if we choose $N = 2R - 1$, rather than $N = 2R + 1$, because of the antiferromagnetic nature of the interaction.

**APPENDIX**

In this appendix, we present for completeness a (rather non–standard) treatment of the BAE (4.12). As explained in paper I, it is convenient to first rewrite them in the p.b.c. form

\[
\begin{aligned}
\left[ \frac{\sinh(\lambda_j - \Theta + i\gamma/2) \sinh(\lambda_j + \Theta + i\gamma/2)}{\sinh(\lambda_j - \Theta - i\gamma/2) \sinh(\lambda_j + \Theta - i\gamma/2)} \right]^N \sinh(2\lambda_j + i\gamma) \\
= - \prod_{k=-M+1}^{M} \frac{\sinh(\lambda_j - \lambda_k + i\gamma)}{\sinh(\lambda_j - \lambda_k - i\gamma)}
\end{aligned}
\]  
(A.1)

where the numbers $\lambda_{-M+1}, \lambda_{-M+2}, \ldots , \lambda_M$ are related to the $v$–roots by

\[
\lambda_j = v_j = -\lambda_{1-j} , \quad j = 1, 2, \ldots , M
\]  
(A.2)

and $\text{Re} v_j > 0$, thanks to the symmetries of the BAE (4.12). Let us concentrate our attention on those BAE solutions which are mostly real, that is those which contain an arbitrary but fixed number of complex pairs interspersed in a sea of order $N$ of real
roots. The reason for this restriction will become clear in the sequel. The counting function associated to a given solution \( v_1, \ldots, v_M \) is defined to be

\[
Z_N(x; v_1, \ldots, v_M) = \phi_{\gamma/2}(x + \Theta) + \phi_{\gamma/2}(x - \Theta) + N^{-1}\phi_{\gamma}(2x) - N^{-1} \sum_{j=-M+1}^{j} \phi_{\gamma}(x - \lambda_j) \tag{A.3}
\]

where

\[
\phi_{\alpha}(x) \equiv i \log \frac{\sinh(i\alpha + x)}{\sinh(i\alpha - x)} \tag{A.4}
\]

The logarithmic branch in eq.(A.4) is chosen such that \( \phi_{\alpha}(x) \), and as direct consequence \( Z_N(x; v_1, \ldots, v_M) \), are odd.

The BAE (A.1) can now be written in compact form

\[
Z_N(\lambda_j; v_1, \ldots, v_M) = 2\pi N^{-1} I_j, \quad j = -M + 1, -M = 2, \ldots, M \tag{A.5}
\]

where the quantum integers \( I_j \) entirely fix the specific BAE solution, and therefore the BA eigenstate, and by construction satisfy \( I_{1-j} = -I_j \) for \( j = 1, 2, \ldots, M \). Notice also that by definition \( Z_N(0; v_1, \ldots, v_M) = 0 \), but \( \lambda = 0 \) is not a root, due to (A.2). Rather, \( \lambda = 0 \) is always a hole, from the p.b.c. point of view.

To begin, consider the case when \( N \) is even, \( M = N/2 \) (i.e. \( J = 0 \)) and all roots are real. This state (the groud state for even \( N \)) is unambiguously identified by the quantum integers

\[
I_j = j \quad j = 1, 2, \ldots, N/2 \tag{A.6}
\]

and the corresponding counting function is indeed monotonically increasing on the real axis, justifying its name. To our knowledge, the existence itself of this BAE solution has not been proven in a rigorous analytical way. But it is very easy to obtain it numerically for values of \( N \) in the thousands and precisions of order \( 10^{-15} \) on any common workstation.

Next consider removing \( J \) roots from the ground state. \( J \) is the quantum spin of the corresponding new BA state. For \( \gamma \) sufficently small, one now finds that

\[
N/2 + J < Z_N(+\infty; v_1, \ldots, v_M) < N/2 + J + 1 \tag{A.7}
\]

so that, together with the actual roots satisfying eq.(A.5), there must exist positive numbers \( x_1, x_2, \ldots, x_\nu \), with \( \nu \geq 2J \), satisfying

\[
Z_N(x_h; v_1, \ldots, v_M) = 2\pi N^{-1} \bar{I}_h, \quad h = 1, 2, \ldots, \nu \tag{A.8}
\]

where the \( \bar{I}_h \) are positive integers. If \( Z_N(x) \) is monotonically increasing, then necessarily \( \nu = 2J \) and the integers \( \{I_1, \ldots, I_{N/2-J}, \bar{I}_1, \ldots, \bar{I}_{2J}\} \) are all distinct. The
numbers $x_1, x_2, \ldots, x_r$ are naturally called *holes*. For $J$ held fixed as $N$ becomes larger and larger, it is natural to expect that the counting function is indeed monotonically increasing, and numerical calculations confirm this expectation. For larger values of $\gamma$ the situation becomes more involved. Numerical studies show that, first of all, $Z_N(x)$ develops a local maximum beyond the largest root, while still satisfying the bounds (A.7), as $\gamma$ exceeds a certain ($J$-dependent) value. For even larger values of $\gamma$, the asymptotic value of $Z_N(x)$ becomes smaller than $Z^* \equiv 2\pi(1/2 + J/N)$, but its maximum stays larger than $Z^*$, provided there is indeed a root corresponding to $N/2 + J$ (i.e. $I_M = N/2 + J$). Up to now, $v_M$ is obviously located where $Z_N(x)$ reaches $N/2 + J$ from below, and one could say that there exists an extra hole $x_{2J+1}$ further beyond, where $Z_N(x)$ reaches $Z^*$ from above. When $\gamma$ reaches a certain critical value, the local maximum lowers till $Z^*$. At this point the root and the extra hole exchange their places, and for slightly larger values of $\gamma$ the hole is located where $Z_N(x)$ reaches $Z^*$ from below, while the root lays further away, where $Z_N(x)$ reaches $Z^*$ from above. As the numerical calculations show, however, this extra hole with the same quantum integer $I_M = N/2 + J$ of the largest root hole is spurious, since no energy increase is really associated to its presence. When none of $N/2 - J$ root has $N/2 + J$ as quantum integer, then $Z_N(x)$ does never reach $Z^*$ for sufficiently large $\gamma$, and, strictly speaking there are only $2J - 1$ holes. This time, however, we find that the energy increases with respect to the ground state in the same way as if there was "a hole at infinity", that is a hole beyond the largest root. Thus $\nu$ can always be regarded to be even, when $N$ is even.

As an important example consider the definite choice $J = 1$. Then for $\gamma < \pi/6$ we find $Z_N(+\infty) > Z^*$, and there are two holes, with $1 \leq I_1 < I_2 \leq N/2 + 1$. Assume $I_2 \leq N/2 + 1$ and consider the interval $\pi/6 < \gamma < \pi/4$. The counting function has a maximum $Z_{\max}$ (larger than $Z^*$) situated to the right of the largest root $v_{N/1-1}$, as long as $\gamma$ is smaller than the critical value $\gamma^*$ at which the maximum lowers to $Z^*$. For $\gamma > \gamma^*$, the maximum is still larger than $Z^*$ but is located to the left of $v_{N/1-1}$. In any case, for $\pi/6 < \gamma < \pi/4$ the asymptotic value $Z_N(+\infty)$ is smaller than $Z^*$. When $\gamma \to (\pi/4)^-$, then the largest root as well as the maximum $Z_{\max}$ are pushed to infinity, and $Z_N(x)$ is once again monotonic with $Z_N(+\infty) = Z^*$. As $\gamma$ exceeds $\pi/4$, the last root $v_{N/1-1}$ passes, through the point at infinity, from the real line to the line $\text{Im } v = \pi/2$. This pictures generalizes to arbitrary $J$ with the two special values $\gamma = \pi/6$ and $\gamma = \pi/4$ replaced, respectively, by $\gamma = \pi/(4J + 2)$ and $\gamma = \pi/(2J + 2)$. In fig. 7 the salient portion of the numerically calculated counting function is plotted for $J = 1$, $N = 64$ and a specific choice of $I_1$, $I_2$. In this case we approximatively find $\gamma^* \simeq 0.21 \pi$.

Finally, consider a BAE solution containing, in addition to a number of order $N$ of real roots, also a certain configuration of complex roots. In the $v-$space, these complex roots appear either in complex conjugate pairs or with fixed imaginary part equal to $i\pi/2$, so that the counting function is real analytic: $\overline{Z_N(x)} = Z_N(\bar{x})$. Moreover, it is fairly easy to show, by looking at the value of the counting function a real
infinity, that the presence of complex roots implies the existence of holes in the sea
of real roots. For our next purposes, we shall now consider \( \gamma/\pi \) irrational, so that
all \( v \)-roots are finite. Denoting with \( u_q, q = 1, 2, \ldots, M_c \) the values of the complex
roots and with \( M_r = M - M_c \) the number of real roots, we now write the derivative
of the counting function as

\[
2\pi \rho_N(x) \equiv Z'_N(x) = F'_\Theta(x) + N^{-1} \left[ F'_0(x) + F'_c(x) - F'_h(x) \right] + (K * \rho_\delta)(x)
\]  \( (A.9) \)

where

\[
F_\Theta(x) = \phi_{\gamma/2}(x + \Theta) + \phi_{\gamma/2}(x - \Theta)
\]

\[
F_0(x) = \phi_{\gamma}(2x)
\]

\[
F_c(x) = \sum_{q=1}^{M_c} [\phi_{\gamma}(x - u_q) + \phi_{\gamma}(x + u_q)]
\]

\[ (A.10) \]

\[
F_h(x) = \phi_{\gamma}(x) + \sum_{n=1}^{\nu} [\phi_{\gamma}(x - x_n) + \phi_{\gamma}(x + x_n)]
\]

\[
N \rho_\delta(x) = \sum_{j=1}^{M_r} \delta(x - \lambda_j) + \phi_{\gamma}(x) + \sum_{n=1}^{\nu} [\delta(x - x_h) + \delta(x + x_h)]
\]

and \( K* \) is the convolution defined by

\[
(K * f)(x) = \int_{-\infty}^{+\infty} dy \phi'_{\gamma}(x - y) f(y)
\]

\( (A.11) \)

The so-called “density approach” consists in replacing, as \( N \to \infty \), both \( \rho_N \) and \( \rho_\delta \)
with the same smooth function \( \rho \), representing the density of roots and holes on the
real line. This function is therefore the unique solution of the linear integral equation
(cfr. eq.(A.9))

\[
2\pi \rho = F_\Theta + N^{-1} (F_0 + F_c - F_h) + K * \rho
\]  \( (A.12) \)

which can be easily solved by Fourier transformation. Combining this equation with
eq.(A.9), we now obtain, after some simple manipulations

\[
\rho_N = \rho + G * (\rho_N - \rho_\delta)
\]  \( (A.13) \)

where \( G* = (2\pi + K)^{-1} * K* \) stands for the convolution with kernel

\[
G(x) = \int \frac{dk}{2\pi} e^{ikx} \frac{\sinh(\pi/2 - \gamma)k}{\sinh(\pi - \gamma)k/2 \cosh \gamma k/2}
\]

\( (A.14) \)

Finally, a simple application of the residue theorem to the analytic function \( \rho_N(1 - e^{-iNZ_N})^{-1} \) plus an integration by parts lead to the following formal nonlinear integral
equation for the counting function

\[ Z_N = Z + G \ast L_N \]  \hspace{1cm} (A.15)

where

\[ L_N(x) = -iN^{-1} \log \frac{1 - e^{iNZ_N(x+i0)}}{1 - e^{-iNZ_N(x-i0)}} \]  \hspace{1cm} (A.16)

and \( Z \) is the odd primitive of \( 2\pi \rho \), namely

\[ Z = Z_\infty + N^{-1}(2\pi + K) \ast (F_0 + F_c - F_h) \]  \hspace{1cm} (A.17)

with

\[ Z_\infty(x) = ((2\pi + K) \ast F_\Theta)(x) = 2 \arctan \left( \frac{\sinh \pi x/\gamma}{\cosh \pi \Theta/\gamma} \right) \]  \hspace{1cm} (A.18)

Eq.(A.15) is a formal integral equation since the knowledge of the exact position of holes and complex roots is required in \( Z \). It becomes a true integral equation for the ground state counting function. In any case it is an exact expression satisfied by \( Z_N \) where the number of site \( N \) enters only in an explicit, parametric way, except for the positions of the holes and of the complex roots. After having fixed the corresponding quantum integers, these parameters retain an implicit, mild dependence on \( N \), for large \( N \).

We shall now show that eq.(A.15) is very effective for establishing the result (6.6), which is of crucial importance for the calculation of the \( S \)-matrix. It is sufficient to check that the second nonlinear term in the r.h.s. of eq.(A.15) is indeed of higher order in \( N^{-1} \) relative to the first. To this purpose observe that the integration contour of the convolution in eq.(A.15) can be deformed away from the upper and lower edges of the real axis, since for sufficiently large \( N \) no complex roots can appear in the whole strip \( |\text{Im } x| < \gamma/2 \) and \( G(x) \) is analytic there. Indeed \( Z_N \) tends to \( Z_\infty \) as \( N \to \infty \), and one can explicitly check that the imaginary part of \( Z_\infty \) is positive definite for \( 0 < \text{Im } x < \gamma/2 \) and negative definite for \( 0 > \text{Im } x > -\gamma/2 \). This also implies that the contribution to the convolution integral is exponentially small in \( N \) for all values of the integration variable (let's call it \( y \)) where \( \text{Im } Z_\infty \) of order 1. For \( |\text{Re } y| \) of order \( \log N \), we find \( \text{Im } Z_\infty(y) \) of order \( N^{-1} \), so that the nonlinearity \( L_N \), rather than exponentially small, is also of order \( N^{-1} \). But now the exponential damping in \( y \) of the kernel \( G(x - y) \) guarantees that the convolution integral is globally of order \( N^{-2} \) or smaller, provided \( |x| \) is kept smaller than \( (\gamma/\pi) \log N \). Hence we can write

\[ Z_N = Z + O(N^{-2}) \]  \hspace{1cm} (A.19)

Finally, the coefficient of the \( N^{-1} \) term of \( Z \), in eq.(A.17), can be calculated in the
$N \to \infty$ limit, with the techniques described at length in paper I. After some straightforward albeit cumbersome algebra, this yields

$$
\lim_{N \to \infty} \left( (2\pi + K) \ast (F_0 + F_c - F_h) \right)(x) = F(x; x_1, \ldots, x_M; \chi_1, \ldots, q\chi_M) \quad (A.20)
$$

where the higher–level quantity $F$ is defined in eq.(6.8). Together with eqs. (A.19) and (A.17), this proves eq.(6.6) of section 3, as claimed.

Let us now consider the problem of calculating the energy of a given BA state, through eq.(4.15). We rewrite first the “bare” energy function $e_0(x)$ as

$$
e_0(x) = -2\pi + \phi_{\gamma/2}(x + \Theta) - \phi_{\gamma/2}(x - \Theta) \quad (A.21)
$$

Then we calculate

$$
\sum_{j=1}^{\infty} \phi_{\gamma/2}(x) = \frac{1}{2} N \int_{-\infty}^{+\infty} \rho(x) \phi_{\gamma/2}(x) + \sum_{q=1}^{M_c} \phi_{\gamma/2}(u_q) - \sum_{h=1}^{\nu} \phi_{\gamma/2}(x_h) - \frac{1}{2} \phi_{\gamma/2}(0) 
\quad (A.22)
$$

Through the residue theorem and an integration by parts, the last term can be transformed, as done before for the counting function, into an integral of the nonlinear term $L_N$, namely the integral

$$
\frac{1}{2} N \int_{-\infty}^{+\infty} \frac{L_N(x)}{\gamma \cosh \pi x/\gamma} 
$$

By the same argument used above, this last expression is globally of order $N^{-1}$. Finally, inserting the explicit form of the continuum density $\rho(x)$ into eq.(A.22) and recalling eqs. (4.15) and (A.21), for the energy we obtain eq.(6.1) of the main text

$$
E = E_0 + a^{-1} \sum_{h=1}^{\nu} e(x_h) + O(a^{-1}N^{-1}) \quad (A.23)
$$
where

\[
E_0 = -\pi a^{-1}N + a^{-1} \int_0^\infty \frac{dk}{k} \frac{\sinh(\pi - \gamma)k/2 \sin k\Theta}{\sinh \pi k/2 \cosh \gamma k/2} \left[ 2N \cos k\Theta + 1 + \frac{\sinh(\pi - 3\gamma)k/4}{\sinh(\pi - \gamma)k/4} \right] \tag{A.24}
\]

is the energy of the ground state.

We would like to close this appendix with a comment on the limitations of the density approach, where one deals only with the solution \( \rho \) of the linear equation (A.12). Regarding \( \rho(x) \) as the actual density of real roots and holes in the \( N \to \infty \) limit, it is natural to use it to replace summations with integrals. What one learns from the exact treatment presented above as well as from computer calculations, is that the error made in such a replacement depends crucially on the large \( x \) behaviour of the quantity which is to be summed. This error is down by \( N^{-1} \) only when there is exponential damping in \( x \). This means, for instance, that the integral of \( \rho(x) \) does not reproduce in general the exact number of real roots and holes, but rather some \( \gamma \)-dependent quantity close to it. Misunderstanding this for the actual number of real roots and holes would lead to the absurd result that the holes have a \( \gamma \)-dependent value of the \( SU(2)_q \) spin, which is instead necessarily integer or half-integer and, in the particular case of the holes, just 1/2 for any value of \( \gamma \).

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11. Figure Captions

Fig.1. Graphical representation of the $R$-matrix.

Fig.2. The standard row-to-row monodromy matrix. The numbers from 1 to N label the vertical spaces and each vertical line represents a couple of free indices in the corresponding vertical space. Indices on the internal horizontal lines are summed over.

Fig.3. The doubled monodromy matrix used to describe systems with fixed boundary conditions. The correct contraction over the internal indices is dictated by the position of the arcs.

Fig.4. Graphical representation of the compatibility relations (2.12) between the two-body scattering and reflections on the left wall.

Fig.5. Graphical representation of the compatibility relations (2.15) between the two-body scattering and reflections on the right wall.

Fig.6. The unit time evolution operator $U(\Theta)$ for even N (top) and odd N (bottom). As usual, indices over internal lines are summed over.

Fig.7. Reconstruction of the diagonal light-cone lattice through powers of the unit time evolution operator (case N even).

Fig.8. Assignment of the local height variables on the plaquettes cut by the time-zero and time-one lines. As is evident from the figure, these two lines sandwich the unit time evolution operator $U(\Theta)$.

Fig.9. The Bratteli diagram corresponding to the case N=6. Right-moving paths on this diagram arriving at height $j$ define the basis in the space of irreps of weight $j$.

Fig.10. The “ground state dominating” configuration of local height variables.

Fig.11. Graphical representation of the vertex–face correspondence. The strip sandwiched between the two time lines represents the unit time evolution.
Fig. 12. Graphical representation of the one–soliton operator corresponding to the particle with rapidity $\theta_2$ in a system with $\nu = 4$ particles. At each intersection, the appropriate two–body $S$–matrix acts. In the collisions against the wall, $\theta_2$ is flipped and the two–by–two matrix $g_2(\pm \theta_2)$ acts on the internal states of the soliton. The choice of $\theta_2$ as largest rapidity is done purely for graphical convenience.

Fig. 13. One of the local height configurations that dominate the one–hole BA state of spin $J = 1$.

Fig. 14. The “renormalized” version of Fig. 13.

Fig. 15. Flipping the last portion of the configuration from the solid to the dotted line does not change the quantum spin $J = p/2 - 1$, but transform the type II state into type I.

Fig. 16. Plot of the $N/2\pi$ times the counting function $Z_N(x)$ versus $\tanh x$ for $N = 64$, $J = 1$, $\Theta = .15$ and various values of $\gamma$ from $\pi/6$ to $0.999\pi/4$. The quantum integers associates to the two holes are $I_1 = 8$ and $I_1 = 21$. The critical value of $(N/2\pi)Z_N$ is $N/2 + J = 33$, while that of $\gamma$ is, roughly, $\gamma^* = 0.21\pi$. 

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