Del Pezzo surfaces with \(\frac{1}{3}(1,1)\) points

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Abstract

We classify non-smooth del Pezzo surfaces with \(\frac{1}{3}(1,1)\) points in 29 qG-deformation families grouped into six unprojection cascades (this overlaps with work of Fujita and Yasutake [14]), we tabulate their biregular invariants, we give good model constructions for surfaces in all families as degeneracy loci in rep quotient varieties, and we prove that precisely 26 families admit qG-degenerations to toric surfaces. This work is part of a program to study mirror symmetry for orbifold del Pezzo surfaces [2].

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1 Introduction

In this paper we:

(I) Classify non-smooth del Pezzo surfaces with \( \frac{1}{3} (1, 1) \) points in precisely 29 qG-deformation families. We further structure the classification into six unprojection cascades, determine their biregular invariants and their directed MMP together with a distinguished configuration of negative curves on the minimal resolution. This overlaps with work of Fujita and Yasutake [1].

(II) Construct good models for surfaces in all families as degeneracy loci in rep quotient varieties. In all but two cases, the rep quotient variety is a simplicial toric variety.

(III) Prove that precisely 26 of the 29 families admit a qG-degeneration to a toric surface.

The classification is summarised in table 1 and table 2 which also plot invariants and provide good model constructions of surfaces in all families.

This work is part of a program to understand mirror symmetry for orbifold del Pezzo surfaces [2] [16] [26] [31] [27] [10] and it is aimed specifically at giving evidence for the conjectures made in [2].

The rest of the introduction is organised as follows: in \( \S 1.1 \) we give precise statements of our main results; in \( \S 1.2 \) we say a few words about the context of [2]; in \( \S 1.3 \) we outline the structure of the paper.

1.1 Our results

1.1.1 The classification and its cascade structure

Definition 1. A \( \frac{1}{n} (a, b) \) point is a surface cyclic quotient singularity \( \mathbb{C}^2 / \mu_n \) where \( \mu_n \) acts linearly on \( \mathbb{C}^2 \) with weights \( a, b \in \left( \frac{1}{n} \mathbb{Z} \right)/\mathbb{Z} \). We always assume no stabilisers in codimension 0, 1, that is, \( \gcd(a, n) = \gcd(b, n) = 1 \).

A del Pezzo surface is a surface \( X \) with cyclic quotient singularities and \( -K_X \) ample.

The Fano index of \( X \) is the largest positive integer \( f > 0 \) such that \( -K_X = fA \) in the Class group \( \text{Cl}_X \).

Remark 2. In this paper we view a del Pezzo surface \( X \) with quotient singularities as a variety. Such a surface is in a natural way a smooth orbifold (or DM stack), but we mostly ignore this structure. Thus for us \( \text{Cl}_X \) is the Class group of Weil divisors on \( X \) modulo linear equivalence. In particular, although \( K_X \) is a Cartier divisor on the orbifold, we think of it as a \( \mathbb{Q} \)-Cartier divisor on the underlying variety (the coarse moduli space of the orbifold) and then to say that it is ample is to say that a positive integer multiple is Cartier and ample.

See [2] for a discussion of qG-deformations of del Pezzo surfaces with cyclic quotient singularities. In particular, it is explained there that the singularity \( \frac{1}{3} (1, 1) \) is qG-rigid and the degree \( d = K^2 \) is locally constant in qG-families.

We classify qG-deformation families of del Pezzo surfaces with \( k \geq 1 \frac{1}{3} (1, 1) \) points. It follows for example from the proof of [2] Lemma 6] that del Pezzo surfaces \( X \) with fixed number \( k \frac{1}{3} (1, 1) \) points and \( d = K^2_X \) form an algebraic stack \( \mathfrak{M}_{k,d} \) of dimension

\[
\dim \mathfrak{M}_{k,d} = -\chi(X, \Theta_X) = -h^0(X, \Theta_X) + h^1(X, \Theta_X)
\]

where \( \Theta_X \) is the sheaf of derivations of \( X \). Note that this dimension can be negative when \( X \) has a continuous group of automorphisms. It is always the case that \( H^2(X, \Theta_X) = 0 \) but in some cases both \( H^0(X, \Theta_X) \) and \( H^1(X, \Theta_X) \) are non-zero.

The following two theorems, together with theorem 12 below, are our main results:

Theorem 3. There are precisely 3 qG-deformation families of del Pezzo surfaces with \( k \geq 1 \frac{1}{3} (1, 1) \) points and Fano index \( f \geq 2 \). They are:

1. \( S_{1, 25/3} = \mathbb{P}(1, 1, 3) \) with \( k = 1, K^2 = \frac{25}{3} \) and \( f = 5 \);
2. \( B_{1, 16/3} \): the family of weighted hypersurfaces \( X_4 \subset \mathbb{P}(1, 1, 1, 3) \) with \( k = 1, K^2 = \frac{16}{3} \) and \( f = 2 \);
3. \( B_{2, 8/3} \): the family of weighted hypersurfaces \( X_6 \subset \mathbb{P}(1, 1, 3, 3) \) with \( k = 2, K^2 = \frac{8}{3} \) and \( f = 2 \).
Theorem 4. There are precisely 26 qG-deformation families of del Pezzo surfaces with $k \geq 1 \frac{1}{3}(1,1)$ points and Fano index $f = 1$. The numerical invariants of these surfaces are shown in table $2$ in § 3. In that table $X_{k,d}$ denotes the unique family with $k \frac{1}{3}(1,1)$ points, $K^2 = d$ and $f = 1$. The table also gives a good model construction of a surface $X$ in all families.

Detailed information on how to read the table is given in § 2. In that section, we also introduce several invariants and explain how to compute some that are not shown in the table. For example, denoting by $X^0 = X^{\text{non-sing}} = X \setminus \text{Sing} X$ the nonsingular locus of $X$, proposition $18$ b) states that $\pi_1(X^0) = (0)$ for all families except $X_6,1$ and $X_6,2$, for which $\pi_1(X^0) = \mathbb{Z}/3\mathbb{Z}$.

Next we discuss the finer structure of the classification.

Definition 5. A negative curve on $X$ is a compact curve $C \subset X$ with negative self-intersection number $C^2 < 0$. We say that a compact curve $C \subset X$ is a $(−m)$-curve if $C^2 = −m$. Note that in general $m \in \mathbb{Q}$. Let $P_1, \ldots, P_k \in X$ be the singular points and denote by $X^0 = X^{\text{non-sing}} = X \setminus \{P_1, \ldots, P_k\}$ the nonsingular locus of $X$. A $(−1)$-curve $C \subset X^0$ is called a floating $(−1)$-curve.

Theorem 4 and theorem 5 are a straightforward logical consequence of the minimal model program and the following, which is proved in § 3.

Theorem 6. Let $X$ be a del Pezzo surface with $k \geq 1 \frac{1}{3}(1,1)$ points. If $X$ has no floating $(−1)$-curves, then $X$ is one of the following surfaces, all constructed in table $1$ and $2$ and in the statement and proof of theorem $35$:

1. $k = 1$ and either $X$ is a surface of the family of weighted hypersurfaces $B_{1,16/3} = X_4 \subset \mathbb{P}(1,1,1,3)$, or $X = S_{1,25/3} = \mathbb{P}(1,1,3)$;
2. $k = 2$ and either $X = X_{2,17/3}$, or $X$ is a surface of the family of weighted hypersurfaces $B_{2,8/3} = X_6 \subset \mathbb{P}(1,1,3,3)$;
3. $k = 3$ and $X = X_{3,5}$;
4. $k = 4$ and $X = X_{4,7/3}$;
5. $k = 5$ and $X = X_{5,5/3}$;
6. $k = 6$ and $X = X_{6,2}$.

Remark 7. With the exception of families $B_{1,16/3}$, $B_{2,8/3}$, all the surfaces in theorem 6 are qG-rigid: in other words, they are the only isomorphism class of surfaces in that family.

Theorem 35 of § 6 is a more precise version of theorem 6 just stated. In particular, the statement of theorem 35 in § 6 has images showing the directed MMP for $X$ that provide a birational construction of $X$, and pictures of a distinguished configuration of negative curves in the minimal resolution $f: Y \to X$.

In all cases, we could have pushed our analysis to the point where we could have made a list of all negative curves on $Y$ and $X$, and computed generators of the nef cones $\text{Nef} Y$, $\text{Nef} X$. We did not pursue this as we don’t have a compelling reason to do so.

Surfaces with a given $k$ are all obtained by a cascade—the terminology is due to [20]—of blow-ups of smooth points starting with one of the surfaces in theorem 6.

Corollary 8. (1) A surface of the family $X_{1,d}$ is the blow-up of $25/3 - d \leq 8$ nonsingular points on $\mathbb{P}(1,1,3)$. If $d < 16/3$, then it is also the blow-up of a surface of the family $B_{1,16/3}$ in $1 \leq 16/3 - d \leq 5$ nonsingular points;

2. A surface of the family $X_{2,d}$ is the blow-up of $17/3 - d \leq 5$ nonsingular points on $X_{2,17/3}$. If $d < 8/3$, then it is also the blow-up of a surface of the family $B_{2,8/3}$ in $1 \leq 8/3 - d \leq 2$ nonsingular points;
3. A surface of the family $X_{3,d}$ is the blow-up of $5 - d \leq 4$ nonsingular points on $X_{3,5}$;
4. A surface of the family $X_{4,d}$ is the blow-up of $7/3 - d \leq 2$ nonsingular points on $X_{4,7/3}$;
5. A surface of the family $X_{5,2/3}$ is the blow-up of a nonsingular point on $X_{5,5/3}$;
(6) $X_{6,1}$ is the blow-up of a nonsingular point on $X_{6,2}$. 

Remark 9. In the cases $k=1$ and $k=2$, corollary \[ \text{[8]} \] is not an immediate consequence of theorem \[ \text{[6]} \]. Indeed, given a surface $X$, it is clear that a sequence of contractions of floating $(-1)$-curves leads to one of the surfaces listed in theorem \[ \text{[6]} \]. We need to show, in addition, that:

1. If $X \to B_{1,16/3}$ is the blow-up of a nonsingular point, there is an alternative sequence of 4 blow-downs of floating $(-1)$-curves starting from $X$ and ending in $\mathbb{P}(1,1,3)$;
2. If $X \to B_{2,8/3}$ is the blow-up of a nonsingular point, then there is an alternative sequence of 4 blow-downs of floating $(-1)$-curves starting from $X$ and ending in the surface $X_{2,17/3}$.

These facts are easy to verify from the explicit birational constructions given in theorem \[ \text{[3]}. \]

1.1.2 Good model constructions

We summarise all of the key features of the constructions provided by table \[ \text{[2]} \].

Definition 10. • A rep quotient variety is a geometric quotient $F = A/G$ where $G$ is a complex Lie group and $A$ a representation of $G$.

• Let $L_1, \ldots, L_c$ be line bundles on $F$ constructed from characters $\rho_i: G \to \mathbb{C}^\times$ ($i = 1, \ldots, c$). A subscheme $X \subset F$ of pure codimension $c$ is a complete intersection of type $(L_1, \ldots, L_c)$ on $F$ if $X = Z(\sigma)$ is the zero-scheme of a section

\[ \sigma \in H^0(F; L_1 \oplus \cdots \oplus L_c) \]

• Let $E_1, E_2$ be vector bundles on $F$ constructed from two representations of $G$. A subscheme $X \subset F$ is a degeneracy locus on $F$ if $X$ is the locus where a vector bundle homomorphism $s: E_1 \to E_2$ drops rank, provided that this locus has the expected codimension.

Examples of rep toric varieties are toric varieties, where $G$ is a torus, but also the weighted Grassmannians of \[ \text{[11]} \].

For all of the 26 families $X_{k,d}$ of theorem \[ \text{[4]} \], we list in table \[ \text{[2]} \] a good model construction of a surface of the family as a degeneracy locus in a rep quotient variety.

For all but one pair $(k,d)$, table \[ \text{[2]} \] shows a rep quotient variety $F$ and line bundles $L_1, \ldots, L_c$ with the following properties:

(a) the line bundles $L_i$ are nef on $F$,

(b) $-K_F - \Lambda$ is ample on $F$, where $\Lambda = \sum_{i=1}^c L_i$,

such that a general complete intersection $X$ of type $(L_1, \ldots, L_c)$ on $F$ is a surface of the family $X_{k,d}$.

Since, by the adjunction formula, $-K_X = -(K_F + \Lambda)|_X$ is ample, the constructions make it manifest that $X$ is a Fano variety. In all cases it is easy to verify that a general complete intersection of type $(L_1, \ldots, L_c)$ on $F$ has $k \frac{1}{2}(1,1)$ singularities, and compute $k$ and the anticanonical degree $d = K_X^2$.

In 24 out of 25 cases $F$ is a toric variety. In 23 of the 24 cases, $F$ is a well-formed simplicial toric variety, and the surface itself is quasi-smooth and well-formed. (These notions are recalled in § 3.4.)

§ 3 summarises conventions and facts about toric varieties and gives model computations demonstrating all of these statements.

Remark 11. Our model construction are not unique. In many cases, several similar constructions exist. It would be nice to understand all model constructions systematically.

We say a few words about the three exceptions:

1. By construction the line bundle $-K_F - \Lambda$ is $G$-linearised, and this uniquely specifies the GIT problem of which $F$ is the solution. Table \[ \text{[2]} \] also gives a complete description of the cone $\text{Nef} F$ of stability conditions.

2. In fact, one can verify, a general surface of the family. We do not claim, however, that every surface of the family has such a description. This may be true, and it is definitely true for rigid surfaces, but we did not check it in general.
Family $X_{1,7/3}$  In §2.2.2 we describe a weighted Grassmannian $F = \text{wGr}(2, 5)$ and a complete intersection $X$ in it which is a surface in this family. $F$ is a well-formed orbifold, and $X$ is quasi-smooth and well-formed.

The surface $X_{5,5/3}$  This family is qG-rigid and it consists of a single surface. This surface has a simple birational construction: blow up the vertices of a pentagon of 5 lines on a smooth del Pezzo surface of degree 5, and blow down the strict transforms of the 5 lines. In §3.3 we construct a model of this surface as codimension 3 degeneracy locus of an antisymmetric vector bundle homomorphism $s : E \otimes L \to E^\vee$ where $E$ is a rank 5 split vector bundle on a simplicial toric 5-fold.

Family $X_{5,2/3}$  In §3.4 we describe a non-simplicial toric variety $F$ and a complete intersection $X$ in it which is a surface in this family. We verify that $X$ misses the non-orbifold locus of $F$. Outside of this locus $F$ is a well-formed orbifold and $X$ is quasi-smooth and well-formed. We did not succeed in finding a good model construction for a surface in this family in a simplicial toric variety. Such a construction may well exist but it is very difficult computationally to look for it, particularly since this family does not admit a toric degeneration. There are, in fact, two difficulties: the software does not exist, and the computations are very large.

1.1.3 Toric qG-degenerations

In §7 we prove the following:

Theorem 12. With the exception of $X_{4,1/3}$, $X_{5,2/3}$ and $X_{6,1}$ (all of which have $h^0(X, -K_X) = 0$), all other families admit a qG-degeneration to a toric surface.

Table 4 and figure 11 list 26 lattice polygons $P$ such that their face-fans $\Sigma(P)$ give toric surfaces $X_P$ that are qG-degenerations of the families in theorem 12.

1.1.4 Comments on the literature and on our proofs

The cascade for the surfaces with $k = 1$ was discovered by Reid and Suzuki in [29].

Del Pezzo surfaces with quotient singularities, also known as log del Pezzo surfaces, are studied in [6, 25, 5, 24, 4, 23, 22].

In two remarkable papers, De-Qi Zhang [32, 33] classifies log del Pezzo surfaces of Picard rank 1 closely related to our surfaces and outlines a general strategy to classify all rank 1 log del Pezzo surfaces. DongSeon Hwang has recently announced a complete classification of rank 1 log del Pezzo surfaces.

While we were working on this project, paper [14] appeared, containing a classification of log del Pezzo surfaces with Gorenstein index 3. While we do not classify all log del Pezzo surfaces of Gorenstein index 3, in some other respect we classify more surfaces than [14]. Indeed, the discussion in §1.2 shows that a del Pezzo surfaces with singularities as example 14(a) and (b) qG-deforms to one of our surfaces. Our classification is done by similar methods: we determine all possibilities for the directed MMP of the surface by a detailed combinatorial study of the configuration of negative curves on the minimal resolution. In our view, our proof is simpler than that in [14]. The cascade structure, our good model constructions, and the statement about toric degenerations, are new.

It took us a significant effort to find the good model constructions. For many of the families, it is comparatively easy to find a construction of a general surface as a complete intersection in a toric variety, but very hard to find one that satisfies properties (a) and (b) of §1.1.2. Initially, we discovered some constructions by hand using birational geometry; then, we found more with the help of a computer search; finally, we learned a systematic way [10]. To determine which of the families admit a toric qG-degeneration we made a computer search for the relevant Fano lattice polygons. A unified picture comprising both these facts—the toric complete intersection model and the toric degeneration—would be desirable. The paper [10] contains the beginning of such a theory.

At this time we can not imagine a geometric explanation for the fact that some del Pezzo surfaces do, and some do not, admit a qG-degeneration to a toric surface.
1.2 Context

We put our results in the context of a general program to understand mirror symmetry for orbifold del Pezzo surfaces \([2, 10, 20, 31, 27, 10]\) and answer the question: Why classify del Pezzo surfaces with \(\frac{1}{3}(1, 1)\) points?

qG-deformations of surface singularities is a technical notion that ensures that the canonical class is well-behaved in families: in particular, \(K^2\) is locally constant in a qG-family of proper surfaces. (The Gorenstein index, crucially, is not locally constant in a qG-family.) Thus, it is natural to restrict our attention to qG-deformations.

**Definition 13.**
- A cyclic quotient surface singularity is class \(T\) if it has a qG-smoothing, cf. [19, Definition 3.7];
- A cyclic quotient surface singularity is class \(R\) if it is qG-rigid, cf. [2, 3];
- A del Pezzo surface \(X\) with cyclic quotient singularities is *locally qG-rigid* if \(X\) has singularities of class \(R\) only.

It is well-known that a surface cyclic quotient singularity \((x \in X) \cong \frac{1}{n}(1, q)\) has a unique qG-deformation component and that the general surface of the miniversal family has a unique singularity of class \(R\), the \(R\)-content of \((x \in X)\), cf. [2] and [3, Definition 2.4] and the discussion following it.

**Example 14.**
(a) A singularity is of class \(T\) if and only if it is of the form \(\frac{1}{r}(1, rma - 1)\).
(b) The singularity \(\frac{1}{3}(1, 1)\) is qG-rigid. A singularity has \(R\)-content \(\frac{1}{\beta}(1, 1)\) if and only if it is of the form \(\frac{1}{3(3m+1)}(1, 2(3m + 1) - 1)\).

It is known [2, Lemma 6] that, if \(X\) is a del Pezzo surface with cyclic quotient singularities \(x_i \in X\), the natural transformation of qG-deformation functors:

\[
\text{Def}^{qG} X \rightarrow \prod \text{Def}^{qG}(X, x_i)
\]

is smooth: a choice for each \(i\) of a local qG-deformation of the singularity \((X, x_i)\) can always be globalised to a qG-deformation of \(X\). In other words \(X\) can be qG-deformed to a surface that has only the residues of the \((X, x_i)\) as singularities.

Our point of view here is that, when we classify del Pezzo surfaces, and study mirror symmetry for them, it is natural to classify first the locally qG-rigid ones, for these are the generic surfaces that we are most likely to meet, and study their qG-degenerations as a second step. (Note that the algebraic stack of qG-families of orbifold del Pezzo surfaces \(X\) with fixed \(K^2_X\) is almost always unbounded; for example it is unbounded when \(X = \mathbb{P}^2\), see for instance [20].) This study is motivated by the fact that \(\frac{1}{3}(1, 1)\) is the simplest class \(R\) singularity.

In [2] mirror symmetry for a locally qG-rigid del Pezzo surface is stated in terms of a qG-degeneration to a toric surface: thus it is crucial for us to determine which families admit such a degeneration. The mirror symmetry conjecture B of [2] computes the quantum orbifold cohomology of a locally qG-rigid surface \(X\) from data attached to the toric qG-degeneration. In order to compute the quantum orbifold cohomology of a surface \(X\) by the known technology of abelian/nonabelian correspondence and quantum Lefschetz [8, 9, 30], and thus give evidence for conjecture B of [2], we need a model of \(X\) as a complete intersection in a rep quotient variety. In this context, we need conditions (a) and (b) of \(\S 1.1.2\) to control the asymptotics of certain \(I\)-functions, and this motivates our constructions here. Conditions (a) and (b) are of course also natural from a purely classification-theoretic perspective. Paper [20] computes (part of) the quantum orbifold cohomology of our surfaces.

Part of our motivation in undertaking this classification was to ask seriously how general mirror symmetry is. Out of the 29 families that comprise our classification, 26 admit a toric qG-degeneration and [2] provides a mirror construction for them. (A mirror for these surfaces can also, in principle, be constructed by means of the Gross–Siebert program, see [27].) What about the remaining three families \(X_{4,1/3}, X_{5,2/3}\) and \(X_{6,1}\)? It turns out that we can construct surfaces in these families as complete intersections in toric varieties, thus these families also have mirrors, by the Hori–Vafa construction.
1.3 Structure of the paper

The paper is structured as follows. Section 2 contains the tables that summarise the classification, with instructions on how to read them. We also introduce several invariants and explain how to compute some that are not shown in the tables. In particular, denoting by $X^0 = X^{\text{nonsing}} = X \setminus \text{Sing} X$ the nonsingular locus of $X$, proposition 18(b) of § 2.3 states that $\pi_1(X^0) = (0)$ for all families except $X_{6,1}$ and $X_{6,2}$, for which $\pi_1(X^0) = \mathbb{Z}/3\mathbb{Z}$.

Section 3 summarises some facts from toric geometry needed to validate the tables: in particular, we give the notion of quasi-smooth and well-formed complete intersections in a well-formed simplicial toric variety, and sample computations verifying that the constructions of table 2 really construct what they say they do. Sections 3.3 and 3.4 are extended essay on models of surfaces $X_{5,5/3}$ and $X_{5,2/3}$, and § 3.5 collects special birational constructions in some cases.

Section 4 studies some of the basic invariants introduced in § 2.1 and uses elementary lattice theory and covering space theory to obtain almost optimal bounds for them that we use later on to cut down the number of cases that we need to consider in the proof of theorems 6 and 35. This material is not strictly necessary for the proof of theorems 6 and 35 but it does simplify it. Part of our reason to include it here is that the use of lattice theory and elementary covering space theory is very effective and we think it may have applications in other problems of classification of orbifold del Pezzo surfaces.

Section 5 summarises all that we need from the Minimal Model Program in the proof of theorems 6 and 35 and introduces the directed MMP that we use to organise the combinatorics of the proof.

Section 6 is the heart of the paper. We prove theorems 6 and 35. In particular, the statement of theorem 35 has images showing the directed MMP for all the surfaces that provide birational constructions of them, and pictures of a distinguished configuration of negative curves in the minimal resolution.

In the final § 7 we prove theorem 12. We list and picture 26 lattice polygons and we show that the corresponding toric surfaces are qG-degenerations of the families of theorem 12.

We refer to the short summary at the beginning of each section for more detailed information on the structure and content of that section.

1.4 Acknowledgments

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2 Tables

Tables 1 and 2 summarise the classification, provide model constructions for a surface in each of the families, and display some of their numerical invariants. In § 2.1 we introduce several invariants and state some of the relations that hold between them. We explain how to compute some of the invariants that are not displayed in the table from those that are. In § 2.2 we explain how to read the tables. In § 2.3 we discuss the Fano index and $\pi_1(X^0)$ where $X$ is one of our surfaces and $X^0 = X^{\text{nonsing}} = X \setminus \text{Sing} X$ is the nonsingular locus. Proposition 18 states that: (a) the Fano index of the families of theorem 6 really is as claimed and (b) $\pi_1(X^0) = \mathbb{Z}/3\mathbb{Z}$ for families $X_{6,2}$ and $X_{6,1}$, and it is trivial for all other families.
2.1 Invariants

Here $X$ is a surface of one of the 29 families of Del Pezzo surfaces with $k \geq 1 \frac{1}{3}(1,1)$ points, and $X^0 = X^{\text{nonsing}} = X \setminus \text{Sing} X$ is the nonsingular locus of $X$. We are interested in the following invariants.

(i) $k$, the number of singular points of $X$;
(ii) $K^2 = K^2_X$, the anticanonical degree of $X$. It is obvious that $K_X^2 > 0$ and $K_X^2 \equiv \frac{k}{3}$ (mod $\mathbb{Z}$);
(iii) $h^0(X, -K_X)$, an integer $\geq 0$ and, more generally, $h^0(X, -nK_X)$ for all integers $n \geq 0$;
(iv) $r = \rho(Y) = \rho(X) + k$, the Picard rank of the minimal resolution $f : Y \to X$;
(v) $n = e(X^0) = \varphi_2(X) - k/3 = 2 + \rho(X) - k$, where $e$ is the (homological) topological Euler number and $\varphi_2(X) = \varphi_2(\hat{T}_X)$ is the orbifold second Chern class of $X$;
(vi) $\sigma$, the defect of $X$, defined as follows: let $L = H^2(Y; \mathbb{Z})$, viewed as a unimodular lattice by means of the intersection form, let $N = \{-3\}^{\perp} \subset L$ be the sublattice spanned by the $-3$-curves, and let $\overline{N} = \{v \in L \mid \exists d \in \mathbb{Z} \text{ with } dv \in N\}$ be the saturation of $N$ in $L$, then, for some integer $\sigma > 0$, $\overline{N}/N \cong \mathbb{F}_3^\sigma$. Indeed, note that $\overline{N}/N \subset N^*/N$ where $N^* = \text{Hom}(N, \mathbb{Z})$ and $N \subset N^*$ the natural inclusion given by the quadratic form. Indeed, note that $N^*/N$ is 3-torsion and isomorphic to $(\mathbb{Z}/3\mathbb{Z})^k$, since $\overline{N}/N$ is also 3-torsion. Equivalently, $\sigma = k - \text{rk Im}[N \otimes \mathbb{F}_3 \to L \otimes \mathbb{F}_3]$. We prove in lemma 28 below that $\mathbb{F}_3^\sigma \cong H_1(X^0; \mathbb{Z})$;
(vii) The number of moduli, that is, the dimension dim $\mathfrak{M}$ of the moduli stack. This number is $-\chi(X, \Theta_X) = h^1(X, \Theta_X) - h^0(X, \Theta_X)$ (it is well-known that $H^2(X, \Theta_X)(\hat{T}_X) = (0)$, see for example [2]). By the Riemann–Roch theorem it is a topological invariant constant on qG-families;
(viii) The Fano index $f$ of $X$, defined as follows: $f$ is the largest integer such that $-K_X = fA$ in Cl $X$, for some divisor class $A \in \text{Cl} X$;
(ix) The fundamental group $\pi_1(X^0)$.

Remark 15. The Riemann-Roch [28] § 3] and Noether formula state:

$$h^0(X, -K_X) = 1 + K_X^2 - \frac{k}{3} \text{ and } K_X^2 = 12 - n - \frac{5k}{3}$$

so one can compute $h^0(X, -K_X)$, $n$ and $r$ from $k$ and $K_X^2$ (vanishing implies that $h^0(X, -nK_X) = \chi(X, -nK_X)$ for $n \geq 0$).

It is easy from these data to compute the Poincaré series $P_X(t) = \sum_{n \geq 0} t^n h^0(X, -nK_X)$. We state the result even though it is not logically needed anywhere in the paper:

$$P_X(t) = \frac{1 + (K_X^2 - 1 - \frac{k}{3})t + (K_X^2 + \frac{2k}{3})t^2 + (K_X^2 - 1 - \frac{k}{3})t^3 + t^4}{(1-t)^2(1-t^3)}$$

Remark 16. • If $X$ admits a toric qG-degeneration, then $n = e(X^0) \geq 0$. Indeed, in this case $n$ is the number of $T$-cones of the Fano polygon corresponding to the toric degenerate surface, see [3].

• If $X$ admits a toric qG-degeneration, then $h^0(X, -K_X) \geq 1$. Indeed, by the Riemann–Roch formula, $h^0(X, -K_X)$ is constant on a qG family and if $X_0$ is toric then $H^0(X_0, -K_{X_0}) \neq (0)$ since it contains at least the toric boundary divisor of $X_0$.

• It follows [18, Chapter 10] from the generic semi-positivity of $\hat{T}_X$ [17, 1.8 Corollary] that $\varphi_2(X) \geq 0$.

Remark 17. Table 1 and table 2 show the invariants $k, d, h^0(X, -K_X), r$ and dim $\mathfrak{M}$. As we just explained, the invariants $\rho(X), n, \varphi_2(X)$, and all the $h^0(X, -nK_X)$ are easily computed from these.

The tables do not show the defect $\sigma$ and $\pi_1(X^0)$. Proposition [18] computes $\pi_1(X^0)$ and this, together with lemma 28, determines $\sigma$: $X_{6,2}$ and $X_{6,1}$ have $\pi_1(X^0) = \mathbb{Z}/3\mathbb{Z}$ (hence $\sigma = 1$), and all other families have $\pi_1(X^0) = (0)$ (hence $\sigma = 0$).
2.2 Tables

Tables 1 and 2 summarise the classification and provide constructions for a general surface in each family. We explain how to read the tables. We focus on table 2 since table 1 is a straightforward illustration of theorem 3.

The symbol $X_{k,d}$ in the first column of table 2 signifies the family of surfaces $X$ with $k$ singular points, degree $K_X^2 = d$ and $f = 1$: the next three columns display the invariants $h^0(X,-K_X)$, the rank $r = \text{rk} H^2(Y;\mathbb{Z}) = \rho(Y) = k + \rho(X)$ where $Y$ is the minimal resolution of $X$, and the dimension of the family.

The next column, in all but one case, shows a rep quotient variety $F$ and line bundles $L_1, \ldots, L_n$ on $F$ such that a general complete intersection of type $(L_1, \ldots, L_n)$ on $F$ is a surface of the family. The last column computes the cone $\text{Nef}_F$: this information is necessary to verify that conditions (a) and (b) of §1.2 hold: the $L_i \in \text{Nef}_F$ and $-K_F - \Lambda \in \text{Amp}_F$. We explain in more detail how to read the information in these last two columns.

In all cases except $X_{1,7/3}$, $X_{5,5/3}$ and $X_{5,2/3}$, $F$ is a well-formed simplicial toric variety and $X$ is a quasi-smooth and well-formed complete intersection of nef line bundles $L_i$. All these notions are recalled in §2.2.5. Families $X_{6,2}$ and $X_{6,1}$ are slightly anomalous: the simplicial toric ambient variety $F$ is not in a direct way a rep quotient variety by a torus, but by a product of a torus and a finite group. We discuss these two families in greater detail in §2.2.6 below.

In all cases, because $-K_F - \Lambda$ is Fano, the canonically linearised line bundle $-K_F - \Lambda$ is a stability condition that unambiguously specifies $F$ as a GIT quotient: see §3.1 for more details on this.

2.2.1 The typical entry

All cases except $X_{1,7/3}$, $X_{5,5/3}$, $X_{5,2/3}$, $X_{6,2}$ and $X_{6,1}$ are typical. In a typical case, the table gives the weight matrix of an action of $\mathbb{C}^\times l$ on $\mathbb{C}^m$ such that $F = \mathbb{C}^m/(\mathbb{C}^\times l)$ and, to the right of this and separated by a vertical line, a sequence of column vectors representing the line bundles $L_i$.

A typical entry For example, the entry for $X_{4,4/3}$ shows that an example of a surface $X$ with $k = 4$ singularities and $K^2 = \frac{4}{3}$ can be constructed as a complete intersection of two general sections of the line bundles $L_1 = (2,4)$ and $L_2 = (4,2)$ in the Fano simplicial toric variety $F$ given by weight matrix:

$$\begin{array}{cccccc}
  x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\
  1 & 2 & 2 & 1 & 1 & 0 \\
  0 & 1 & 1 & 2 & 2 & 1
\end{array}$$

and $\text{Nef} F = \langle (2,1), (1,2) \rangle$ (the notation is explained fully in §3.1 below). Here $\Lambda = L_1 + L_2 \sim (6,6)$, $-\langle K_F + \Lambda \rangle \sim (1,1)$ and $-K_F \sim (7,7)$ are all ample.

2.2.2 Model for $X_{1,7/3}$ in $F = \text{wGr}(2,5)$

We refer the reader to [11] for the definition of weighted Grassmannians, as well as notation for complete intersections in them. Consider, as in [11], $F = \text{wGr}(2,5)$ with weights $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3})$: then $F \subset \mathbb{P}(1^3, 2^6, 3)$ and it is easy to see by the methods of [11] that the vanishing locus $X = Z(s)$ of a general section $s \in \Gamma(F, \mathcal{O}_F(2))$ is a surface of this family.

2.2.3 Model for $X_{5,5/3}$

This family is qG-rigid and it consists of a single surface. This surface has a simple birational construction: blow up the vertices of a pentagon of 5 lines on a nonsingular del Pezzo surface of degree 5, and blow down the strict transforms of the 5 lines. In §3.3 we construct a model of this surface as codimension 3 degeneracy locus of an antisymmetric vector bundle homomorphism $s: E \otimes L \to E'$ where $E$ is a rank 5 split vector bundle on a simplicial toric 5-fold.

2.2.4 Model for $X_{5,2/3}$

Section 3.4 is an extended essay on this case.
2.2.5 Model for $X_{6,2}$

This is the toric surface whose fan is the spanning fan of the lattice polygon in Figure 1.

Figure 1: The polygon of $X_{6,2}$

where the origin of the lattice is labelled with a o. It is clear from the picture that the primitive generators $\rho_1, \ldots, \rho_6$ of the rays of the fan generate a sublattice of index 3. This fact easily implies that $\pi_1(X^0) = \mathbb{Z}/3\mathbb{Z}$.

2.2.6 Model for $X_{6,1}$

Every surface in this family is the quotient of a nonsingular cubic surface by a $\mathfrak{g}_{13}$-action as shown by table 2. This fact immediately implies that $\pi_1(X^0) = \mathbb{Z}/3\mathbb{Z}$ for a surface in this family.

2.3 The Fano index and $\pi_1(X^0)$

Proposition 18. (a) $\mathbb{P}(1,1,3)$ has Fano index $f = 5$. The surface $B_{1,16/3}$ and every surface of the family $B_{2,8/3}$ have Fano index $f = 2$. All other surfaces with $k \geq 1 \frac{1}{3}(1,1)$ points have Fano index $f = 1$.

(b) If $X = X_{6,2}$ or a surface of the family $X_{6,1}$, then $\pi_1(X^0) = \mathbb{Z}/3\mathbb{Z}$. If $X$ is any other surface with $k \geq 1 \frac{1}{3}(1,1)$ points, then $\pi_1(X^0) = (0)$.

Remark 19. Our proof of this fact, which we sketch below, is by ad hoc computations. For a quasi-smooth and well-formed complete intersection $X$ of ample (or nef, or irrelevant) line bundles in a well-formed simplicial toric variety $F$, it is natural to imagine that there would be some general Lefschetz type results relating $\text{Cl} F$ to $\text{Cl} X$ and $\pi_1(F^0)$ to $\pi_1(X^0)$. We could not find these results in the literature.

Proof. We give a sketch of the proof, leaving some of the details to the reader. We start by proving (a). Let $f$ be as stated and $A$ the divisor class such that $-K_X = fA$, then we need to show that $A$ is primitive in $\text{Cl} X$. It is clearly enough to show that the class of $A$ in $H^2(X^0, \mathbb{Z})$ is primitive. In all cases, we check this by producing a compact curve $C \subset X^0$ with $A \cdot C = 1$ or a pair of compact curves $C_1, C_2 \subset X^0$ of degrees $d_i = A \cdot C_i \in \mathbb{N}$ such that $\text{hcf}(d_1, d_2) = 1$. If $X$ contains a floating $(-1)$-curve $C \subset X^0$ then $-K_X \cdot C = 1$ and then clearly $f = 1$: thus from now on we work with the 8 families of surfaces of theorem 35. We refer to the constructions and pictures in the statement of theorem 35. The analysis is very simple and we go through each case in turn:

Case $k = 1$ For $X = B_{1,16/3}$ $X$ the picture shows that $X$ has two rulings of rational curves $C$ with $C^2 = 0$ and $A \cdot C = 1$ in both cases.

For $X = \mathbb{P}(1,1,3)$ if $C$ is a general curve of self-intersection 3 then $A \cdot C = 1$.

Case $k = 2$ For $X = B_{2,8/3}$ $X$ the picture again shows that $X$ has two rulings of rational curves $C$ with $C^2 = 0$ and $A \cdot C = 1$ in both cases.

For $X = X_{2,17/3}$ the picture shows that $X^0$ contains a curve $C_1$ with $C_1^2 = 1$ and a ruling $C_2$ with $C_2^2 = 0$, so $A \cdot C_1 = 3$ and $A \cdot C_2 = 2$. 
Case $k = 3$ The surface $X = X_{3,5}$ has a morphism to $\mathbb{P}(1, 1, 3)$ and if $C_1 \subset X^0$ is the proper transform of a general curve of self-intersection 3 then $A \cdot C_1 = 5$. In addition $X$ has a ruling $C_2$ with $C_2^2 = 0$ and $A \cdot C_2 = 2$.

Case $k = 4$ $X = X_{4,7/3}$ has morphisms to $\mathbb{P}^1$ and $\mathbb{P}(1, 1, 3)$.

Case $k = 5$ $X = X_{5,5/3}$ has five morphisms to $\mathbb{P}^1$ and five morphisms to $\mathbb{P}(1, 1, 3)$.

Case $k = 6$ The toric surface $X = X_{6,2}$ has morphisms to $\mathbb{P}^1$ and also to a toric Gorenstein cubic surface $Y$ with $3 \times A_2$ points, hence there are curves $C_1, C_2 \subset X^0$ with $A \cdot C_1 = 2$ and $A \cdot C_2 = 3$.

This concludes the proof of (a). For the proof of (b) we need to equip ourselves with a topological model of $X$. It is clear that if $X \to X_1$ is the blow-down of a floating $(-1)$-curve, then $\pi_1(X^0) = \pi_1(X_1^0)$, so again we only need to consider the 8 families of theorem 35.

To get a topological model of $X^0$ for a surface $X$ of one of these 8 families, we use the toric $qG$-degenerations of §7, a surface $X$ $qG$-degenerates to a toric surface $X_0$ with fan the face-fan of the corresponding polygon in table 4 pictured in figure 11. Denote by $\hat{X}_0$ the maximal crepant partial resolution of $X_0$. From a direct inspection of the polygons we see that, with the exception of $X_{5,5/3}$, $\hat{X}_0$ has $k \cdot (1, 1)$ points $P_i$ ($i = 1, \ldots, k$) and is elsewhere nonsingular. We treat the case of $X_{5,5/3}$ separately below. In all other cases $X^0$ is diffeomorphic to the toric surface $\hat{X}_0^0$ and we determine $\pi_1(X^0) = \pi_1(\hat{X}_0^0)$ by well-known toric methods.

The surface $X_{5,5/3}$ degenerates to the toric surface $X_0$ corresponding to polygon 7: in this case $\hat{X}_0 = X_0$ has two $T$ singularities $Q_1, Q_2$ of type $1/3(1, 1)$. In this case $X^0$ is obtained as a topological space by attaching to $\hat{X}_0^0$ the Milnor fibres of the singularities $Q_1, Q_2$ along their boundaries. The result follows from (i) the fact that $\pi_1(\hat{X}_0^0) = (0)$ and (ii) a calculation of $\pi_1(X^0)$ from $\pi_1(\hat{X}_0^0)$. Here (i) holds because we can see by direct inspection that the rays of the fan of $\hat{X}_0^0$ generate $N$ as a group. As for (ii), it is stated in [20, Proposition 13] that if $F$ is the Milnor fibre of a $qG$-smoothing of a primitive class $T$ singularity $1/(1, ar-1)$ with hcf$(r, a) = 1$, then $\pi_1(F) = \mathbb{Z}/r\mathbb{Z}$ and $\pi_1(\partial F) = \mathbb{Z}/r^2\mathbb{Z}$ and the natural homomorphism $\pi_1(\partial F) \to \pi_1(F)$ is surjective (we only need the $r = 2$ case of this). By two applications of the Seifert–van Kampen theorem, it follows in this case that $\pi_1(X^0) = (0)$.

\[\text{Table 1: del Pezzo surfaces with } 1/3(1, 1) \text{ and } f > 1\]

| Name   | $h^0(X, -K_X)$ | $r$ | No. moduli | Model Construction | $f$ |
|--------|----------------|-----|------------|--------------------|-----|
| $S_{1,25/3}$ | 9              | 2   | -8         | $\mathbb{P}(1, 1, 3)$ | 5   |
| $B_{1,16/3}$ | 6              | 5   | -2         | $X_4 \subset \mathbb{P}(1, 1, 1, 3)$ | 2   |
| $B_{2,8/3}$   | 3              | 8   | 2          | $X_6 \subset \mathbb{P}(1, 1, 3, 3)$ | 2   |

\[\text{Table 2: del Pezzo surfaces with } 1/3(1, 1) \text{ and } f = 1\]

| Name   | $h^0(X, -K_X)$ | $r$ | No. moduli | Weights and Line bundles | Nef $F$ |
|--------|----------------|-----|------------|-------------------------|--------|
| $X_{1,22/3}$ | 8              | 3   | -6         | 1 1 2 0                  | 1 2    |
|         |                |     |            | 0 1 3 1                  | 1 3    |

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| Name   | $h^0(X, -K_X)$ | $r$ | No. moduli | Weights and Line bundles | Nef $F$ |
|--------|----------------|-----|------------|--------------------------|--------|
| $X_{1,19/3}$ | 7 | 4 | -4 | 1 3 3 0 0 | 3 0 0 |
|          |                |    |            | 0 2 1 1 0 | 2 1 0 |
|          |                |    |            | 1 2 0 0 1 | 2 0 1 |
| $X_{1,16/3}$ | 6 | 5 | -2 | 1 1 0 0 0 | 1 0 1 |
|          |                |    |            | 0 0 1 1 3 | 0 1 3 |
| $X_{1,13/3}$ | 5 | 6 | 0  | 1 1 3 1 0 | 1 0 1 |
|          |                |    |            | 0 0 0 1 1 | 0 1 3 |
| $X_{1,10/3}$ | 4 | 7 | 2  | 1 1 2 1 0 0 | 2 2 1 |
|          |                |    |            | 0 0 1 2 1 1 | 2 2 1 |
| $X_{1,7/3}$  | 3 | 8 | 4 | $F = \text{wGr}(2, 5)$ and $\mathcal{O}_F(2)^{\oplus 4}$ | 1 |
|          |                |    |            | $w = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2})$ | 1 |
| $X_{1,4/3}$  | 2 | 9 | 6 | 1 1 2 2 3 | 4 4 1 |
| $X_{1,1/3}$  | 1 | 10 | 8 | 1 2 3 5 | 10 1 |
| $X_{2,17/3}$ | 6 | 5 | -4 | 1 1 2 3 0 | 4 2 1 |
|          |                |    |            | -1 0 1 3 1 | 2 1 1 |
| $X_{2,14/3}$ | 5 | 6 | -2 | 1 1 0 0 | 1 0 1 |
|          |                |    |            | 0 1 1 3 1 | 1 1 1 |
| $X_{2,11/3}$ | 4 | 7 | 0 | 1 0 0 1 0 1 | 2 3 1 1 |
|          |                |    |            | 0 1 0 0 1 1 | 2 1 3 1 |
|          |                |    |            | 0 0 1 1 1 4 | 4 4 4 4 |
| $X_{2,8/3}$  | 3 | 8 | 2 | 1 1 1 1 0 | 3 1 1 |
|          |                |    |            | 0 0 1 3 1 | 3 1 3 |
| $X_{2,5/3}$  | 2 | 9 | 4 | 1 1 2 1 0 4 | 2 2 1 |
|          |                |    |            | 0 1 3 3 1 6 | 3 3 0 0 |
| $X_{2,2/3}$  | 1 | 10 | 6 | 1 2 2 3 3 | 4 6 1 |
| $X_{3,15}$  | 5 | 6 | -4 | 1 0 0 1 0 | 1 1 1 |
|          |                |    |            | 0 1 -1 1 0 | 1 0 0 |
|          |                |    |            | 0 0 1 1 3 | 1 1 1 |
| $X_{3,4}$  | 4 | 7 | -2 | 1 -1 1 0 0 0 | 0 1 0 1 0 1 2 |
|          |                |    |            | -1 1 0 0 0 1 | 0 0 1 0 1 2 1 |
|          |                |    |            | 2 1 0 1 0 0 | 1 2 1 2 2 2 2 |
|          |                |    |            | 1 2 0 0 1 0 | 1 1 2 2 2 4 4 |
| $X_{3,3}$  | 3 | 8 | 0 | 1 1 1 0 0 2 | 1 0 1 |
|          |                |    |            | 0 0 1 1 3 3 | 1 1 1 1 |
| $X_{3,2}$  | 2 | 9 | 2 | 1 3 2 0 -1 | 4 2 0 |
|          |                |    |            | 0 0 1 1 1 2 | 2 1 1 |

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| Name   | $h^0(X, -K_X)$ | $r$ | No. moduli | Weights and Line bundles | Nef $F$ |
|--------|----------------|-----|-------------|--------------------------|---------|
| $X_{3,1}$ | 1             | 10  | 4           | 1 0 0 2 1 1 | 4 2 1 1 |
|         |                |     |             | 0 1 0 1 2 1 | 4 1 2 1 |
|         |                |     |             | 0 0 1 1 1 2 | 4 1 1 2 |
| $X_{4,7/3}$ | 2             | 9   | 0           | 1 0 0 -1 -1 0 | 0 -1 0 1 2 1 6 |
|         |                |     |             | 0 3 3 2 1 | 6 1 2 |
| $X_{4,4/3}$ | 1             | 10  | 2           | 1 0 0 0 2 1 1 1 | 2 3 2 1 1 1 |
|         |                |     |             | 0 1 0 0 1 2 1 1 | 2 3 1 2 1 1 |
|         |                |     |             | 0 0 1 0 1 1 2 1 | 3 2 1 1 2 1 |
|         |                |     |             | 0 0 0 1 1 1 1 2 | 3 2 1 1 1 2 |
| $X_{4,1/3}$ | 0             | 11  | 4           | 2 2 3 3 3 3 | 6 6 | 1 |
| $X_{5,5/3}$ | 1             | 10  | 0           | $F$ and $D(s)$ where | $s: E \otimes L \to E^\vee$ as in § 3.3 |
| $X_{5,2/3}$ | 0             | 11  | 2           | $F$ and $L^{\otimes 2}$ as in § 3.4 |
| $X_{6,2}$  | 1             | 10  | -2          | $F/\mu_3$ (see § 2.2.3) where $F$ has weights | 1 0 0 1 0 1 0 0 1 1 0 1 1 0 0 1 0 0 1 1 1 1 0 1 1 0 1 1 0 0 1 0 0 1 0 1 |
| $X_{6,1}$  | 0             | 11  | 0           | $\mathbb{P}^3/\mu_3$ and $O(3)$ where | $\mu_3$ acts with weights $\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ |

3 Constructions

This section is organised as follows: In § 3.1 we recall basic facts about toric varieties and well-formed complete intersections in them; in § 3.2 we give an extended example of the computations needed to see that the constructions given in table 2 really do what they say they do. § 3.3 and § 3.4 give good model constructions of the surface $X_{5,5/3}$ and of a surface in the family $X_{5,2/3}$. The final § 3.5 contains some Italian-style birational constructions in certain special cases.

3.1 Simplicial toric varieties and well-formed complete intersections in them

As usual $T \cong \mathbb{C}^d$ is a $d$-dimensional torus, $M = \text{Hom}(T, \mathbb{C}^\times)$ is the group of characters, and $N = \text{Hom}(M, \mathbb{Z})$.

**From a fan to a GIT quotient** We recall how to interpret a toric variety as a GIT quotient. To a rational fan $\Sigma$ in $N_\mathbb{R}$ one associates a toric variety $X_\Sigma$. The toric variety is proper if and only if $\Sigma$ is a complete fan, that is, the support $|\Sigma|$ of the fan is all of $N_\mathbb{R}$; the variety is called simplicial if and only if all cones of the fan are simplicial cones. All toric varieties in this section are proper (and, in fact, projective, see below) and simplicial.
A simplicial toric variety $X$ is in a natural way a smooth Deligne-Mumford stack, called the canonical stack of a simplicial toric variety in [13]. Indeed, $X$ is the union of affine open subsets $X_\sigma = \text{Spec} k[M \cap \sigma^\vee]$, where $\sigma \subset N$ is a simplex of maximal dimension. Denote by $\rho_1, \ldots, \rho_n \in N$ the primitive generators of the rays of $\sigma$ (a ray is a 1-dimensional cone), and let $N_\sigma = \sum_{i=1}^n \mathbb{Z} \rho_i$, then the dual lattice $M_\sigma = \text{Hom}(N_\sigma, \mathbb{Z})$ can be constructed as the over-lattice

$$M_\sigma = \{ m \in M \otimes \mathbb{Q} \mid \langle m, n \rangle \in \mathbb{Z} \forall n \in N_\sigma \}$$

The group scheme $\mu_l$ with character group the finite group $N/N_\sigma$ acts on $\mathbb{C}[M_\sigma \cap \sigma^\vee] = \mathbb{C}[N^\vee]$ with ring of invariants $\mathbb{C}[M \cap \sigma^\vee]$ and this shows that $X_\sigma$ is the quotient of the smooth variety $\text{Spec} k[M_\sigma \cap \sigma^\vee]$ by $\mu_l$. These charts give $X_\Sigma$ the structure of a smooth Deligne–Mumford stack with the following two properties:

1. the stabilizer of the generic point is trivial, that is, $X_\Sigma$ is an orbifold;
2. the stabilizers of all codimension 1 points are also trivial.

Next we see how to interpret $X_\Sigma$ as a global GIT quotient. Denote by $\rho_1, \ldots, \rho_m \in N$ the primitive generators of the rays of the fan. Here and below we assume for simplicity that the $\rho_i$ generate $N$ as a group. Then we have an exact sequence:

$$(0) \rightarrow L \rightarrow \mathbb{Z}^m \xrightarrow{L} N \rightarrow (0)$$

and a dual exact sequence$^3$

$$(0) \rightarrow M \rightarrow \mathbb{Z}^* \xrightarrow{D} \mathbb{L}^* \rightarrow (0)$$

Note that the homomorphism $\rho$ is not enough to reconstruct the fan $\Sigma$. Below we identify a simplex $\sigma = \langle \rho_{i_1}, \ldots, \rho_{i_k} \rangle$ of $\Sigma$ with the list of indices $[i_1, \ldots, i_k]$.

**Definition 20.** It is well-known [12] that $\mathbb{L}^* = \text{Cl} X_\Sigma$ is the divisor class group of $X_\Sigma$. We call $D$ the divisor homomorphism of the toric variety.

The homomorphism $D : \mathbb{Z}^* \rightarrow \mathbb{L}^*$ is dual to a group homomorphism $\mathbb{G} \rightarrow \mathbb{C}^* \mathbb{m}$ where $\mathbb{G}$ is the torus with character group $\mathbb{L}^*$; $\mathbb{G}$ acts on $\mathbb{C}^m$ via this homomorphism and as we next explain in greater detail $X_\Sigma$ is a GIT quotient $\mathbb{C}^m//\mathbb{G}$. Identify $\mathbb{L}^*$ with the group of $\mathbb{G}$-linearised line bundles on $\mathbb{C}^n$. An element $\omega \in \mathbb{L}^*$ is also called a stability condition. The choice of a stability condition $\omega \in \mathbb{L}^*$ determines a GIT quotient $\mathbb{C}^m//\omega \mathbb{G}$ and we need to state what choices of $\omega$ produce $X_\Sigma$. Denoting by $x_i$ the standard basis of $\mathbb{Z}^* \mathbb{m}$, and writing $D_i = D(x_i) \in \mathbb{L}^*$, it is well-known that the nef cone of $X_\Sigma$ is:

$$\text{Nef } X_\Sigma = \bigcap_{\sigma \in \Sigma} \langle D_i \mid i \notin \sigma \rangle$$

In this paper we always assume that $X_\Sigma$ is projective; in other words, $\text{Nef } X_\Sigma$ has nonempty interior $\text{Amp } X_\Sigma$ (the ample cone). Then, for all $\omega \in \text{Amp } X_\Sigma$, we have that $X_\Sigma = \mathbb{C}^n//\omega \mathbb{G}$.

In fact, we can be more explicit than this. Thinking of $x_i$ as a coordinate function on $\mathbb{C}^m$, define the irrelevant ideal as:

$$\text{Irr}_{\Sigma} = \left( \prod_{i \notin \sigma} x_i \mid \sigma \in \Sigma \right) \subset \mathbb{C}[x_1, \ldots, x_m]$$

and let $Z_\Sigma = V(\text{Irr}_{\Sigma}) \subset \mathbb{C}^m$ the variety of $I$. Then for all $\omega \in \text{Amp } X_\Sigma$, $Z_\Sigma \subset \mathbb{C}^m$ is the set of $\omega$-unstable points, $\mathbb{C}^m \setminus Z_\Sigma$ is the set of stable points (there are no strictly semi-stable points in this situation), and

$$X_\Sigma = \mathbb{C}^m//\omega \mathbb{G} = (\mathbb{C}^m \setminus Z_\Sigma)/\mathbb{G}$$

is a “bona fide” quotient.

The action of $\mathbb{G}$ on $\mathbb{C}^m \setminus Z_\Sigma$ has finite stabilizers; this naturally endows the quotient $(\mathbb{C}^m \setminus Z_\Sigma)/\mathbb{G}$ with a structure of a smooth Deligne-Mumford stack which we denote by $[(\mathbb{C}^m \setminus Z_\Sigma)/\mathbb{G}]$; this stack structure is the same as the canonical stack structure on $X_\Sigma$. Below we describe in detail an atlas of charts.

---

$^3$If the map $\rho$ is not surjective but has finite cokernel, then one must work instead with the Gale dual sequence, cf. [7].
From a GIT quotient to a fan Conversely, consider a rank $r$ lattice $\mathbb{L}^r \cong \mathbb{Z}^r$ and denote by $G$ the torus with character group $\mathbb{L}^r$. Consider now $\mathbb{Z}^m$, denote by $x_i$ the standard basis elements, let $D: \mathbb{Z}^m \to \mathbb{L}^r$ be a group homomorphism such that the $D_i = D(x_i)$ span a strictly convex cone $C \subset \mathbb{L}^r_\mathbb{R}$. $D$ dualises to a group homomorphism $G \to \mathbb{C}^\times m$ and hence $G$ acts on $\mathbb{C}^m$.

**Definition-Remark 21.** It is easy to see \[ \text{that:} \]

1. Choose a basis of $\mathbb{L}^r \cong \mathbb{Z}^r$ and identify $D$ with a $r \times m$ matrix, which we call the weight matrix. $G$ acts faithfully if and only if the rows of $D$ span a saturated sublattice of $\mathbb{Z}^r$, if and only if the hcf of all the $r \times r$ minors of $D$ is 1. A matrix satisfying this condition is called standard.

2. $G$ acts faithfully on the divisor $D_i = (x_i = 0) \subset \mathbb{C}^m$ if and only if the matrix $D_i = (D_1, \ldots, \hat{D}_i, \ldots, D_m)$ obtained from $D$ by removing the $i$-th column, is standard.

**Definition 22.** The homomorphism $D: \mathbb{Z}^m \to \mathbb{L}^r$ is well-formed if both the weight matrix $D$ and the $D_i$ for all $i = 1, \ldots, m$ are standard.

**Remark 23.** We can take GIT quotients for any $D$: however, if $D$ is the divisor homomorphism of some toric variety $X_\Sigma$, then $D$ is well-formed. The aim of the considerations that follow is precisely to state that the converse is also true.

Given a stability condition $\omega \in \mathbb{L}^r_\mathbb{R}$, we can form the GIT quotient

$$X_\omega := \mathbb{C}^m/\omega G$$

There is a wall-and-chamber decomposition of $C \subset \mathbb{L}^r_\mathbb{R}$, called the secondary fan, and if stability conditions $\omega_1, \omega_2$ lie in the same chamber then the GIT quotients $X_{\omega_1}, X_{\omega_2}$ coincide. In more detail, the walls of the decomposition are the cones of the form $(D_{i_1}, \ldots, D_{i_k}) \subset \mathbb{L}^r_\mathbb{R}$ that have codimension one. The chambers are the connected components of the complement of all the walls; these are $r$-dimensional open cones in $C$. By construction, a chamber is the intersection of the interiors of the simplicial $r$-dimensional cones $(D_{i_1}, \ldots, D_{i_k}) \subset \mathbb{L}^r_\mathbb{R}$ that contain it. Choose now a chamber, and pick a stability condition $\omega$ in it. Given such an $\omega$, the irrelevant ideal $I_\omega \subset \mathbb{C}[x_1, \ldots, x_m]$ is

$$I_\omega = (x_{i_1} \cdots x_{i_r} \mid \omega \in (D_{i_1}, \ldots, D_{i_k}))$$

the unstable locus is $Z_\omega = V(I_\omega)$; and the GIT quotient is the bona fide quotient

$$X_\omega = (\mathbb{C}^m \setminus Z_\omega)/G.$$  

Note that $I_\omega, Z_\omega$ and the quotient $X_\omega$ depend only on the chamber that $\omega$ sits in and not on $\omega$ itself. Given such an $\omega$ we can also form a simplicial fan $\Sigma$ where

$$\sigma \in \Sigma \quad \text{if and only if} \quad \omega \in (D_i \mid i \notin \sigma)$$

and $\Sigma$ also depends only on the chamber that $\omega$ sits in. The following Proposition collects a few well-known facts that can easily be synthesized from the literature \[12\] \[7\] \[13\] \[11\]:

**Proposition 24.** Let $D: \mathbb{Z}^m \to \mathbb{L}^r$ be a homomorphism as above, choose a chamber in the secondary fan as described above, and let $\omega$ be a stability condition in it. Let $I_\omega, Z_\omega, X_\omega, \Sigma$, be as above:

1. If (as we are assuming) $C$ is a strictly convex cone, then $\Sigma$ is a complete fan;

2. If $D$ is well-formed, then $X_\omega = [(\mathbb{C}^m \setminus Z_\omega)/G]$ is isomorphic to $X_\Sigma$ as a Deligne–Mumford stack, $D$ is the divisor homomorphism of the toric variety $X_\Sigma$, and the chosen chamber is $\text{Amp} X_\Sigma$;

3. If $D$ is not well-formed, there is a natural non-representable morphism of Deligne–Mumford stacks $X_\omega = [(\mathbb{C}^m \setminus Z_\omega)/G] \to X_\Sigma$ and the two stacks have the same coarse moduli space.\[4\]

4. In general, if $D$ is not well-formed, \[11\] describes an algorithm to construct a well-formed $D'$ such that $D'$ is the divisor homomorphism of $X_\Sigma$.

---

\[4\] In this case the chamber canonically determines a stacky fan in the sense of \[7\], such that $X_\omega$ is isomorphic to the toric Deligne–Mumford stack associated to the stacky fan. We don’t need this construction in this paper.
Charts on GIT quotients

We explain how to set up an explicit atlas of charts on \( X_\omega = [(C^m \setminus Z_\omega)/G] \), which we use repeatedly in the calculations needed to validate the entries of table \([2]\). Fix a well-formed \( D: \mathbb{Z}^* \to \mathbb{L}^* \), choose a basis of \( \mathbb{L}^* \), identify \( D \) with an integral \( r \times m \) matrix. We have that \( C^m \setminus Z_\omega \) is a union of \( G \)-invariant open subsets:

\[
C^m \setminus Z_\omega = \bigcup_{\{(i_1, \ldots, i_r) \in (D_1, \ldots, D_r)\}} U_{i_1, \ldots, i_r} \quad \text{where} \quad U_{i_1, \ldots, i_r} = \{x_{i_1} \neq 0, \ldots, x_{i_r} \neq 0\} \subset C^m
\]

Let now \( V_{i_1, \ldots, i_r} = \{x_{i_1} = \cdots = x_{i_r} = 1\} \subset C^m \), then \( [U_{i_1, \ldots, i_r}/G] = [V_{i_1, \ldots, i_r}/G] \) where \( G \) is the finite subgroup of \( \mathbb{G} \) that fixes \( V_{i_1, \ldots, i_r} \). Concretely, \( G \) is the finite group with character group \( A \), the cokernel of the homomorphism:

\[
D_{i_1, \ldots, i_r} = (D_{i_1}, \ldots, D_{i_r}): \mathbb{Z}^r \to \mathbb{L}^*
\]

Complete intersections in toric varieties

Consider a well-formed \( D: \mathbb{Z}^* \to \mathbb{L}^* \) as above. Fix a chamber of the secondary fan, a stability condition in it, and let \( x \in X \) be a divisor of \( F \), the homomorphism:

3.2 Sample computations

In the column labelled “Weights and Line bundles,” all lines of table \([2]\) except those corresponding to families \( X_{1,7/3}, X_{5,5/3}, X_{5,2/3}, X_{6,2} \) and \( X_{6,1} \), list a well-formed weight matrix

\[
D: \mathbb{Z}^* \to \mathbb{L}^* = \mathbb{Z}^r
\]

for constructing a simplicial toric variety \( F \) and, to the right of it and separated by a vertical line, a list of column vectors \( L_i \in \mathbb{Z}^r \), representing line bundles on \( F \) such that \( X \) is a complete intersection of general members of the \( \{L_i\} \). The last column is a list of column vectors in \( \mathbb{Z}^r \), the generators of \( \text{Nef} F \), which is the chamber of the secondary fan that contains the stability conditions that give \( F \) as GIT quotient. In all cases it is immediate to verify that the \( L_i \in \text{Nef} F \) and that \( -K_F - \Lambda \in \text{Amp} F \) where \( \Lambda = \sum L_i \). In particular it follows from this that \( X \) is a Fano variety.

Example: family \( X_{1,10/3} \)

As stated in corollary \([8]\) a surface \( X \) in this family is either: (i) The blow-up of \( \mathbb{P}(1,1,3) \) at \( d = 5 \) general points; or (equivalently) (ii) The blow-up of \( B_{1,16/3} \) at \( d = 2 \) general points.

According to the table, a surface in this family can be constructed as a codimension 2 complete intersection of type \( L_1 = (2,2) \), \( L_2 = (2,2) \) in the (manifestly well-formed) simplicial toric variety \( F \) with weight matrix:

\[
\begin{array}{cccccc}
  x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\
  1 & 1 & 2 & 1 & 0 & 0 \\
  0 & 0 & 1 & 2 & 1 & 1
\end{array}
\]

More precisely, line bundles on the canonical DM stack of \( X_\Sigma \).
and Nef $F = (L + 2M, 2L + M)$, where $L = (1, 0)$ and $M = (0, 1)$ are the standard basis vectors of $\mathbb{L}^*$. Note that both $L_1$, $L_2$ are ample, and $-(K_F + L_1 + L_2) \sim L + M$ is ample.

First we examine all the charts of $F$ and verify that $X$ is a quasi-smooth well-formed complete intersection with $1 \times \frac{1}{3}(1, 1)$ singularities. Finally we calculate $K_X^2 = 10/3$.

The chamber is $(x_2, x_3)$, so the irrelevant ideal for the given stability condition is $\text{Irr} = (x_0, x_1, x_2)(x_3, x_4, x_5)$, and the charts are the $U_{ij}$ with $i \leq 2, 3 \leq j$.

Let us first look at the chart $U_{03} = \{x_0 \neq 0, x_3 \neq 0\}$. Considering $V_{03} = \{x_0 = x_3 = 1\} \subset \mathbb{C}^6$ it is immediate that:

$$U_{03} = \frac{1}{2}(0, 1, 1, 1)_{x_1, x_2, x_4, x_5}$$

the quotient of $V_{03} \cong \mathbb{C}^4$ with coordinates $x_1, x_2, x_4, x_5$ by the action of $\mu_2$ with weights $(0, 1, 1, 1)$. We see that the $x_1$-axis $C$ is a curve toric stratum of the 4-fold $F$ with stabilizer $\mu_2$ at the generic point. We claim that $C \cap X = \emptyset$. Indeed $C = \{x_2 = x_4 = x_5 = 0\}$ is the toric variety with weight matrix:

$$\begin{pmatrix}
 x_0 & x_1 & x_3 \\
 1 & 1 & 1 \\
 0 & 0 & 2
\end{pmatrix}$$

Note, however, that this matrix is not well-formed. Applying the algorithm in [1], we see that $C$, together with the line bundles $L_{1|C}, L_{2|C}$, is the toric variety with well-formed weight matrix

$$\begin{pmatrix}
 x_0 & x_1 & x_3 \\
 1 & 1 & 0 \\
 0 & 0 & 1
\end{pmatrix}$$

and line bundles $L_{1|C} = L_{2|C} = (1, 1)$, which is manifestly the same as $\mathbb{P}^1$ with $L_{1|C} = L_{2|C} = O(1)$. It is clear that the two restriction maps $H^0(F, L_i) = \langle x_0, x_3, x_1, x_3 \rangle \rightarrow H^0(C, L_{i|C})$ are surjective and thus two general members of $L_1$ and $L_2$ do not intersect anywhere on $C$.

The chart $U_{13}$ is very similar; and the charts $U_{04}, U_{05}, U_{14}, U_{15}$ are smooth and it is immediate that none of the strata passing through those charts are contained in the base locus of $|L_i|$; thus, we only need to look at $U_{23}$.

Considering $V_{23} = \{x_2 = x_3 = 1\} \subset \mathbb{C}^6$ it is easy to see that:

$$U_{23} = \frac{1}{3}(1, 1, 1, 1)_{x_0, x_1, x_4, x_5}$$

the quotient of $V_{23} \cong \mathbb{C}^4$ with coordinates $x_0, x_1, x_4, x_5$ by the action of $\mu_3$ with weights $(1, 1, 1, 1)$. Denote by $f_i \in H^0(F, L_i)$ general members: the monomials $x_0 x_3, x_1 x_3, x_2 x_4, x_3 x_4$ all appear in $f_i$ with nonzero coefficient, thus the surface $X$ must contain the origin of this chart, it is quasi-smooth there, and it has a singularity $1/3(1, 1)$ there. This completes the verifications that $X$ is well-formed and has $1 \times 1/3(1, 1)$ singularities.

We now compute the degree of $X$. The Chow ring of $F$ is generated by $L = (1, 0)$ and $M = (0, 1)$ with the relations $L^2(2L + M) = 0, (L + 2M)M^2 = 0$ (corresponding to the components $(x_0, x_1, x_2), (x_3, x_4, x_5)$ of Irr), and, for example, $L^2M^2 = 1/3$ obtained by looking at the chart $U_{23}$. From this information, we get that $L^3M = -(1/2)L^2M^2 = -1/6$ and $L^4 = -(1/2)L^3M = 1/12$ and similarly $M^4 = 1/2, M^3L = -1/6$ and then it is easy to compute:

$$K_X^2 = L_1L_2(-K_F - L_1 - L_2)^2 = (2L + 2M)^2(L + M)^2 = 4(L + M)^4 = 4\left(\frac{1}{12} - \frac{4}{6} + \frac{6}{3} - \frac{4}{6} + \frac{1}{12}\right) = \frac{10}{3}$$

3.3 The surface $X_{5,5/3}$

We construct this surface as the degeneracy locus of an antisymmetric homomorphism $s: E \otimes L \rightarrow E^\vee$ where $E$ is a rank 5 (split) homogeneous vector bundle, and $L$ a line bundle, on a simplicial toric Fano variety $F$. 

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Specifically, $F$ is the Fano toric variety with Cox coordinates and weight matrix:

\[
\begin{array}{cccccccc}
y_1 & y_2 & y_3 & y_4 & y_5 & x_1 & x_2 & x_3 & x_4 & x_5 \\
1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 \\
\end{array}
\]

The Nef cone of $F$ is the simplicial cone generated by the last 5 vectors of the weight matrix. One can draw the secondary fan; $F$ is covered by 32 charts; it has isolated orbifold singularities $5 \times \frac{1}{3}(1,1,1,1,1), 10 \times \frac{1}{3}(1,1,2,2,2), 10 \times \frac{1}{3}(1,1,1,3,3), 5 \times \frac{1}{3}(1,1,1,1,4), \frac{1}{6}(1,1,1,1,1)$.

Consider the following line bundles on $X$:

\[ L_1 = \mathcal{O}(2), L_2 = \mathcal{O}(2), L_3 = \mathcal{O}(2), L_4 = \mathcal{O}(3), L_5 = \mathcal{O}(3), \]

and

\[ L = \mathcal{O}(6) \]

**Claim** Writing $E = \bigoplus_{i=1}^{5} L_i$, $X \subset F$ is the degeneracy locus of a general antisymmetric homomorphism $s: E \otimes L \rightarrow E^\vee$.

We can take $s$ to be defined by the $5 \times 5$ antisymmetric matrix

\[
A = \begin{pmatrix}
0 & y_1^2 y_2 y_3 y_4 & x_1 & x_2 & y_1 y_2^2 y_4 y_5 \\
y_1^2 y_2 y_3 y_4 & 0 & x_3 & x_4 & y_1 y_2 y_3^2 y_5 \\
x_1 & x_2 & 0 & y_1 y_2 y_3 y_4 & x_5 \\
y_1 y_2 y_3 y_4 & x_3 & y_1 y_2 y_3^2 y_5 & 0 & x_6 \\
x_1 y_2 y_3 y_4 y_5 & x_4 & y_1 y_2 y_3^2 y_5 & y_2 y_3^2 y_4 y_5 & 0
\end{pmatrix}
\]

and the equations of $X$ are the five $4 \times 4$ Pfaffians of the matrix $A$:

\[
\begin{align*}
-x_3 x_5 + x_4 y_1 y_2 y_3 y_4^2 + y_1^2 y_2^2 y_3 y_4^3 y_5^3 &= 0 \\
-x_5 x_1 + x_2 y_1 y_2 y_3 y_4 y_5 + y_1^2 y_2 y_3 y_4^2 y_5^2 &= 0 \\
-x_2 x_4 + x_3 y_1 y_2 y_3 y_4 y_5 + y_1^2 y_2^2 y_3^2 y_4 y_5^2 &= 0 \\
-x_4 x_1 + x_5 y_1 y_2 y_3 y_4^2 + y_1^2 y_2 y_3^2 y_4 y_5^3 &= 0 \\
-x_1 x_3 + x_2 y_1 y_2 y_3 y_4 y_5 + y_1^2 y_2^2 y_3 y_4^2 y_5^2 &= 0
\end{align*}
\]

One can check that $X$ given by these equations avoids all the singularities of $F$ except 5 of the points with $y_3$-stabilizer, that $X$ is quasismooth and has $5 \times \frac{1}{3}(1,1)$ singularities at those points, and that $X$ is nonsingular everywhere else.

All relevant information about $X$, including the Poincaré series and $K_X^2$, can be obtained from the resolution of $\mathcal{O}_X$:

\[
(0) \rightarrow L^{\nu_1} \rightarrow E \otimes L^{A} \xrightarrow{E^\vee \nu_1} \mathcal{O}_F \rightarrow \mathcal{O}_X \rightarrow (0)
\]

In particular, the shape of the resolution shows that

\[
-K_X = \mathcal{O}_X \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
\]
We denote by $Z$ the chart corresponding to the cone spanned by the vectors $z$ and $V$.

In what follows we denote by $S_i$. In terms of these, the monomial basis of $H^0(F,L)$ is

$$x_0 = y_{01}y_{02}y_{03}y_{04}, x_1 = y_{01}y_{12}y_{13}y_{14}, x_2 = y_{02}y_{12}y_{23}y_{24}, x_3 = y_{03}y_{13}y_{23}y_{34}, x_4 = y_{04}y_{14}y_{24}y_{34}$$ (1)

### 3.4.1 Simplicial charts

We denote by $U_0, ..., U_4$ the simplicial charts of $F$ where for example $U_0 = (y_{12} = y_{13} = ... = y_{34}) = 1$ is the chart corresponding to the cone spanned by the vectors $\rho_{01}, \rho_{02}, \rho_{03}, \rho_{04}$. These vectors generate a sublattice of index 3 in $\mathbb{Z}^4$ and in fact

$$U_0 = \frac{1}{3}(1,1,1,1)_{y_{01}+y_{02}+y_{03}+y_{04}}$$

it is clear that $X$ has 5 isolated $\frac{1}{3}(1,1)$ singularities, one in each of these charts.

### 3.4.2 Octahedral charts

We denote by $V_0, ..., V_4$ the octahedral charts of $F$ where, for example, $V_0 = (y_{01} = y_{02} = y_{03} = y_{04} = 1)$ is the chart corresponding to the cone $\sigma$ over the octahedron $[\rho_{12}, ..., \rho_{34}]$. From the exact sequence

$$(0) \to \mathbb{Z}^2 \to \mathbb{Z}^6 \xrightarrow{\rho} \mathbb{Z}^4$$

where $\rho = (\rho_{12}, ..., \rho_{34})$ and $N_0 = \text{Im}(\rho) = \{x_1, ..., x_4 \mid x_1 + ... + x_4 \equiv 0 \pmod{2}\}$, we see that $V_0 = \mathbb{C}^6_{y_{12}, ..., y_{34}}/\mathbb{C}^2 \times \mathbb{R}_2$ where $\mathbb{C}^2$ acts with weights

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}$$

and $\mathbb{R}_2$ acts with weights 0, 0, 0, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2} \in (\frac{1}{2} \mathbb{Z})/\mathbb{Z}$. It follows that:

$$V_0 = \text{Spec} \mathbb{C}[y_{12}, ..., y_{34}]^{\mathbb{C}^2 \times \mathbb{R}_2} = \text{Spec} \mathbb{C}[z_1, ..., z_8]^{\mathbb{R}_2}$$

where $z_1 = y_{12}y_{13}y_{14}$, $z_2 = y_{12}y_{14}y_{24}$, $z_3 = y_{12}y_{23}y_{24}$, $z_4 = y_{12}y_{13}y_{23}$, $z_5 = y_{13}y_{14}y_{34}$, $z_6 = y_{14}y_{24}y_{34}$, $z_7 = y_{23}y_{24}y_{34}$, $z_8 = y_{13}y_{23}y_{34}$ are generators corresponding to the vertices of a cube in $M_0 = \text{Hom}(N_0, \mathbb{Z})$ dual to the octahedron. Finally

$$V_0 = V(z_1z_6 - z_2z_5, z_1z_3 - z_2z_4, z_1z_8 - z_4z_5, z_3z_6 - z_2z_7, z_3z_8 - z_4z_7, z_6z_8 - z_5z_7) \subset \frac{1}{2}(0, 1, 0, 1, 0, 1, 0,$$
where the 6 relations correspond to the six faces of the cube. It follows that $V_0 \simeq A/\mu_2$ where $A \subset \mathbb{C}^6$ is the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding. The $\mu_2$-fixed locus $(z_2 = z_4 = z_5 = z_7 = 0)$ intersects $A$ in the 4 curves $z_1 z_6 = z_1 z_3 = z_1 z_8 = z_2 z_6 = z_2 z_8 = z_6 z_8 = 0$, i.e. the $z_1$-, $z_3$-, $z_6$-, $z_8$-axes. It follows that $V_0$ has index 2 at the origin, where 4 curves meet where $F$ has generically $\frac{1}{2}(1,1,0,0)$ singularities. $L$ is a Cartier divisor at the origin, thus $X$ also avoids the origin (this was clear from the start) and $X$ avoids the 1-dimensional singular stratum in this chart, which consists of the 4 curves just mentioned.

### 3.4.3 The embedding $F \subset \mathbb{P}(2^5,3^5)$

Let $x_0, \ldots, x_4$ and $z_0, \ldots, z_4$ be homogeneous coordinates on $\mathbb{P}(2^5,3^5)$. Then with $x_i$ as in equation 1 and

\[
\begin{align*}
  z_0 &= y_{12}y_{13}y_{14}y_{23}y_{24}y_{34} \\
  z_1 &= y_{02}y_{03}y_{04}y_{12}y_{24}y_{34} \\
  z_2 &= y_{01}y_{03}y_{04}y_{13}y_{14}y_{24} \\
  z_3 &= y_{01}y_{02}y_{04}y_{12}y_{14}y_{24} \\
  z_4 &= y_{01}y_{02}y_{03}y_{12}y_{13}y_{23}
\end{align*}
\]

$F$ embeds into $\mathbb{P}(2^5,3^5)$ with 14 equations:

1. $x_0z_0 = x_1z_1$ ($i \in \{1,2,3,4\}$), and
2. $z_0z_1 = x_2x_3x_4$, $z_0z_2 = x_1x_3x_4$, etc (10 equations)\[^{[6]}\]

### 3.5 Italian-style constructions

Here we list a few Italian-style constructions of some of the surfaces.

**$k$-gons** Take a $k$-gon of $(-1)$-curves on a nonsingular del Pezzo surface $S$ of degree $2 \le k = K_S^2 \le 6$. Blowing up the vertices of the $k$-gon and then blowing down the strict transforms of the $k$ $(-1)$-curves—which have now become $(-3)$-curves—one obtains a surface with $k \times 1/3(1,1)$ singularities and degree $d = k/3$. Hence this construction gives $X_{2,2/3}$, $X_{3,1}$, $X_{4,4/3}$, $X_{5,5/3}$, $X_{6,2}$.

**The family $X_{5,2/3}$** A surface in this family is the blow up of the $10 = \binom{5}{2}$ points of intersection of pairs of 5 general lines in $\mathbb{P}^2$ followed by contraction of the proper preimages of the 5 lines. A floating $(-1)$-curve on this surface can be seen as follows: choose 5 of the 10 points of intersection of the 5 general lines in such a way that no three of them are collinear. The proper transform of the unique conic through these 5 points is a floating $(-1)$-curve.

**The family $X_{6,1}$** A surface in this family is the blow up of the 9 points of intersections of pairs of lines (one in each ruling) of a grid of $6 = 3 + 3$ lines, three in each ruling, on $\mathbb{P}^1 \times \mathbb{P}^1$ followed by contraction of the proper preimages of the 6 lines.

### 4 Invariants

The main result of this section is proposition \[^{[29]}\] where we derive an almost exact table of invariants of del Pezzo surfaces with $\frac{1}{2}(1,1)$ points from elementary lattice theory and elementary covering space theory. These methods are surprisingly effective in producing an almost exact table of invariants and we hope that they can be useful in other problems of classification of orbifold del Pezzo surfaces. We use the result in §6 to cut down on the cases we need to consider in the proof of theorems \[^{[6]}\] and \[^{[35]}\]. We start with a study of the defect invariant introduced in §2.1.

**Lemma 26.** Using the notation introduced in §2.1, $k - r/2 \le \sigma \le k/2$.\[^{[6]}\]

\[^{[6]}\]One can see that there are 35 syzygies between these equations, as is typical of codimension 5 Gorenstein ideals with 14 generators.
Proof. \( \mathcal{N}/N \subset N^*/N \) is totally isotropic where \( N^*/N \) is endowed with the discriminant quadratic form, hence \( \sigma = \dim_{\mathbb{F}_2} \mathcal{N}/N \leq \frac{1}{2} \dim_{\mathbb{F}_2} N^*/N = \frac{5}{2} \). Also, \( \text{Im}[N \otimes F_3 \to L \otimes F_3]\) is totally isotropic, hence it has dimension \( \leq r/2 \), thus the kernel has dimension \( \geq k - r/2 \).

\[ \square \]

Remark 27. In fact one can do better, but we won’t need to do so here. For example, if \( k = 2 \), then the discriminant bilinear form \( A(x, y) = x^2 + y^2 \) has no isotropic vector, hence \( \sigma = 0 \) in this case.

Lemma 28. \( H_1(X^0; \mathbb{Z}) \cong \mathbb{F}_3^2 \).

Proof. Denote by \( E = \cup_{i=1}^k E_i \subset Y \) the exceptional divisor of the minimal resolution morphism \( Y \to X \), and note that of course \( X^0 = Y \setminus E \). Because \( Y \setminus E \) is smooth, the Poincaré homomorphism \( H_1(Y \setminus E; \mathbb{Z}) \to H_{1-1}(Y; \mathbb{Z}) \) is an isomorphism. The long exact sequence for compactly supported cohomology fits into a commutative diagram:

\[
\begin{array}{cccccc}
H^2(Y; \mathbb{Z}) & \longrightarrow & H^2(E; \mathbb{Z}) & \longrightarrow & H^3_c(Y \setminus E; \mathbb{Z}) & \longrightarrow & H^3(Y; \mathbb{Z}) = (0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \longrightarrow & N^* & \longrightarrow & F_3^2 & \longrightarrow & (0)
\end{array}
\]

\[ \square \]

The following is the main result of this section.

Proposition 29. \( k \leq 6 \) and moreover:

1. If \( k = 1 \) then \( K^2 \equiv 1/3 \pmod{\mathbb{Z}} \) and \( 1/3 \leq K^2 \leq 25/3 \);
2. If \( k = 2 \) then \( K^2 \equiv 2/3 \pmod{\mathbb{Z}} \) and \( 2/3 \leq K^2 \leq 20/3 \);
3. If \( k = 3 \) then \( K^2 \equiv 0 \pmod{\mathbb{Z}} \) and \( 1 \leq K^2 \leq 5 \);
4. If \( k = 4 \) then \( K^2 \equiv 1/3 \pmod{\mathbb{Z}} \) and \( 1/3 \leq K^2 \leq 10/3 \);
5. If \( k = 5 \) then \( K^2 \equiv 2/3 \pmod{\mathbb{Z}} \) and \( 2/3 \leq K^2 \leq 8/3 \);
6. If \( k = 6 \) then \( K^2 \equiv 0 \pmod{\mathbb{Z}} \) and \( 1 \leq K^2 \leq 2 \).

Remark 30. It follows from the proof of theorems 6 and 35 that the possibilities \( k = 2, K^2 = 20/3; k = 4, K^2 = 10/3; k = 5, K^2 = 8/3 \) do not actually occur.

Proof of proposition 29. By \[ 17 \] 1.8 Corollary the orbi-tangent bundle is generically semi-positive, and then by \[ 18 \] Chapter 10 \( e_2 = n + k/3 \geq 0 \). It follows from this that \( K^2 = 12 - n - \frac{2k}{3} \leq 12 - \frac{4k}{3} \). By using \( K^2 \equiv k/3 \pmod{\mathbb{Z}} \) and:

\[ 0 < K^2 \leq 12 - \frac{4k}{3} \text{ and } h^0(X, -K) = 1 + K^2 - k/3 \geq 0 \]

we immediately conclude that \( k \leq 7 \). Indeed, if \( k = 8 \) the first set of inequalities forces \( K^2 = 2/3 \) but this would imply \( h^0(X, -K) = 1 + K^2 - 8/3 = 1 + 2/3 - 8/3 = -1 \leq 0 \), a contradiction. Similarly, when \( k = 7 \) the first inequality gives \( 1/3 \leq K^2 \leq 7/3 \), but \( K^2 = 1/3 \) does not occur because it would imply \( h^0(X, -K) = 1 + 1/3 - 7/3 = -1 \), again a contradiction. With a bit more work we can exclude a few more cases: if \( k = 1 \), then \( K^2 = 31/3 \) implies \( r = 0 \), \( K^2 = 28/3 \) implies \( r = 1 \) and both are impossible because \( r = k + p(X) > k \). Similarly, \( k = 2 \) and \( K^2 = 26/3 \) implies \( r = 2 \), impossible; \( k = 3 \) and \( K^2 = 8 \) implies \( r = 3 \), also impossible. Thus \( k \leq 7 \) and we are left with the following possibilities:

1. If \( k = 1 \) then \( K^2 \equiv 1/3 \pmod{\mathbb{Z}} \) and \( 1/3 \leq K^2 \leq 25/3 \);
2. If \( k = 2 \) then \( K^2 \equiv 2/3 \pmod{\mathbb{Z}} \) and \( 2/3 \leq K^2 \leq 23/3 \);
3. If \( k = 3 \) then \( K^2 \equiv 0 \pmod{\mathbb{Z}} \) and \( 1 \leq K^2 \leq 7 \);
(4) If \( k = 4 \) then \( K^2 \equiv 1/3 \pmod{\mathbb{Z}} \) and \( 1/3 \leq K^2 \leq 19/3 \);
(5) If \( k = 5 \) then \( K^2 \equiv 2/3 \pmod{\mathbb{Z}} \) and \( 2/3 \leq K^2 \leq 14/3 \);
(6) If \( k = 6 \) then \( K^2 \equiv 0 \pmod{\mathbb{Z}} \) and \( 1 \leq K^2 \leq 4 \);
(7) If \( k = 7 \) then \( K^2 \equiv 1/3 \pmod{\mathbb{Z}} \) and \( 4/3 \leq K^2 \leq 7/3 \).

We are still quite some way from proving what we need. We exclude the remaining possibilities by studying the invariant \( \sigma \). The key observation is that, by lemma 26, we have that \( \sigma \geq k - r/2 \) so, for example, if \( k = 2 \) and \( K^2 = 23/3 \), we must have \( r = 3 \) and then \( \sigma > 0 \). It is easy to see that this case does not occur: by lemma 28 \( H_1(X^0, \mathbb{Z}) \cong \mathbb{F}_3 \), so by covering space theory there is a 3-to-1 covering \( Y \to X \), étale above \( X^0 \), from a surface \( Y \), necessarily a del Pezzo surface, with \( 1/3(1,1) \) points and degree \( K_Y^2 = 3 \times 23/3 = 23 \) and we already know that such a surface does not exist.

As another example, \( k = 7, K^2 = 4/3 \) implies \( \sigma \geq 2 \) so there is a 9-to-1 cover \( Y \to X \) from a del Pezzo surface \( Y \) with \( 1/3(1,1) \) points and \( K_Y^2 = 12 \) and we know that such a surface does not exist.

In table 3 we summarise the cases where we can definitely conclude \( \sigma > 0 \). All but two are excluded at once by the same method (the other two cases actually occur) and the result follows.

| \( k \) | \( K^2 \) | \( r \) | \( \sigma \) | Occurs |
|---|---|---|---|---|
| 2 | 23/3 | 3 | > 0 | No |
| 3 | 6 | 5 | > 0 | No |
| 3 | 7 | 4 | > 0 | No |
| 4 | 13/3 | 7 | > 0 | No |
| 4 | 16/3 | 6 | > 0 | No |
| 4 | 19/3 | 5 | > 1 | No |
| 5 | 8/3 | 9 | > 0 | No |
| 5 | 11/3 | 8 | > 0 | No |
| 5 | 14/3 | 7 | > 1 | No |
| 6 | 1 | 11 | > 0 | Yes |
| 6 | 2 | 10 | > 0 | Yes |
| 6 | 3 | 9 | > 1 | No |
| 7 | 4/3 | 11 | > 1 | No |
| 7 | 7/3 | 10 | > 1 | No |

Table 3: Necessarily defective possibilities

All other possibilities are excluded by the same method, except \( k = 5, K^2 = 8/3, \sigma \geq 1 \): this possibility is not excluded at this point, and it is not excluded by the statement of proposition 29. Table 3 states that it does not occur, but this fact will only follow from the proof of theorems 6 and 35 in \( \S 6 \).

5 MMP

In our proof of theorem 6 in \( \S 6 \) we systematically use the following elementary result, which we state without proof. Analogous statements for surfaces with canonical singularities can be found in [21, 15].

**Theorem 31.** Let \( X \) be a projective surface having \( k \times 1 \times 1 = (1,1), n_2 \times A_2, \) and \( n_1 \times A_1 \) singularities.

Assume that \( k + 2n_2 + n_1 \leq 6 \).

Let \( f : X \to X_1 \) be an extremal contraction. Then exactly one of the following holds:

(E) \( f : (X,E) \to (X_1,P) \) is a divisorial contraction. Denote by \( Y \to X \) and \( Y_1 \to X_1 \) the minimal resolutions, and \( E' \subset Y \) the proper transform of the exceptional curve. Then \( E' \subset Y \) is a \((-1)\)-curve meeting transversely at most one exceptional curve of \( Y \to X \) above each singularity, and one of the following holds:

(E.1) \( E \) is contained in the nonsingular locus. Then \( E \) is a \((-1)\)-curve and we call it a floating \((-1)\)-curve;

(E.2) \((A1\) contraction\) \( E \) contains one \( A_1 \)-singularity, \( P \in X_1 \) is a nonsingular point;
(E.3) (A2 contraction) \( E \) contains one \( A_2 \)-singularity, \( P \in X_1 \) is a nonsingular point;

(E.4) \( E \) contains one \( \frac{1}{3}(1,1) \)-singularity, \( P \in X_1 \) is a \( A_1 \)-point;

(E.5) \( E \) contains one \( \frac{1}{3}(1,1) \)-singularity and one \( A_1 \) singularity, \( P \in X_1 \) is a nonsingular point;

(E.6) \( E \) contains two \( \frac{1}{3}(1,1) \)-singularities, \( P \in X_1 \) is an \( A_2 \)-point.

(C) \( X_1 = \mathbb{P}^1 \), that is, \( f \) is generically a conic bundle. Denote by \( F \subset X \) a special fibre of \( f \), and by \( Y \to X \) and \( Y_1 \to X_1 \) the minimal resolutions and \( F' \subset Y \) the proper transform of \( F \). Then \( F' \) is a \((-1)\)-curve and one of the following holds:

(C.1) \( F \) contains two \( A_1 \)-singularities, and \( F' \) meets each of the \((-2)\)-curves transversely;

(C.2) \( F \) contains one \( \frac{1}{3}(1,1) \)-singularity and one \( A_2 \) singularity, and \( F' \) meets the \((-3)\)-curve and one of the \((-2)\)-curves transversely.

(D) \( X_1 = \{ \text{pt} \} \) is a point, that is, \( X \) is a del Pezzo surface of Picard rank one, and \( X \) is one of the following surfaces:

(D.1) \( \mathbb{P}^2 \);

(D.2) \( \mathbb{P}(1,1,2) \) (this surface has exactly one \( A_1 \) singular point);

(D.3) \( \mathbb{P}(1,2,3) \) (this surface has exactly one \( A_1 \) and one \( A_2 \) singularities);

(D.4) \( \mathbb{P}^2/\mu_3 \) where \( \mu_3 \) acts with weights \( 1, \omega, \omega^2 \). This surface has exactly \( 3 \times A_2 \) singularities\(^7\)

(D.5) \( \mathbb{P}(1,1,3) \).

\( \square \)

Remark 32. Consider the class of projective surfaces \( X \) be having \( k \times \frac{1}{3}(1,1), n_2 \times A_2 \), and \( n_1 \times A_1 \) singularities and \( k + 2n_2 + n_1 \leq 6 \). It follows from the previous statement that a MMP starting from a surface in the class only involves surfaces in the class.

The directed minimal model program In the proof of theorems \( \[6 \] \) and \( \[35 \] \) in the following section, we run the MMP starting with a del Pezzo surface with \( \frac{1}{3}(1,1) \) points.

In all cases, we perform extremal contractions in the order that they are listed in theorem \( \[31 \] \) above: that is, we first contract all the floating \((-1)\)-curves, then we contract a ray of type (E.2) if available, or else one of type (E.3), etc.

We call this the directed MMP.

Lemma 33. Let \( X \) be a del Pezzo surface with \( k \geq 1 \frac{1}{3}(1,1) \) singular points. Assume that \( X \) contains no floating \((-1)\)-curves. Denote by

\[
X = X_0 \xrightarrow{\varphi_0} \cdots \xrightarrow{} X_{i-1} \xrightarrow{\varphi_{i-1}} X_i \xrightarrow{} \cdots
\]

the contractions and surfaces occurring in a MMP for \( X \) (not necessarily directed).

(1) All surfaces \( X_i \) are del Pezzo surfaces.

(2) Denote by \( f_i : Y_i \to X_i \) the minimal resolution of \( X_i \) and let \( C \subset Y_i \) be a (reduced and irreducible) curve with negative self-intersection \( C^2 = -m \). Then:

(2.1) if \( C \) is \( f_i \)-exceptional, then \( m = 2 \) or \( 3 \),

(2.2) if \( C \) is not \( f_i \)-exceptional, then \( m = 1 \) and \( C \) intersects at least one \( f_i \)-exceptional curve. In particular, none of the surfaces \( X_i \) contain a floating \((-1)\)-curve.

\(^7\) \( X \) is the toric surface obtained by blowing up 3 vertices on the hexagon of lines of a degree 6 nonsingular del Pezzo surface.
Proof of Lemma 33. We prove the statement by induction on $i$. We first show that $X_i$ is a del Pezzo surface. Suppose $X_{i-1}$ is del Pezzo and let $E \subset X_{i-1}$ be the effective divisor such that $K_{X_{i-1}} = \varphi_{i-1}^*K_{X_i} + aE$, $a > 0$. Let $\Gamma \subset X_i$ be a curve. Denoting by $\Gamma' \subset X_{i-1}$ the proper transform, we have that:

$$K_{X_i} \cdot \Gamma = K_{X_i} \cdot \varphi_{i-1}^* \Gamma' = \varphi_{i-1}^*(K_{X_i}) \cdot \Gamma' = (K_{X_{i-1}} - aE) \cdot \Gamma' < 0.$$  

As $K_{X_i}^2 > K_{X_{i-1}}^2$, by the Nakai-Moishezon criterion we conclude that $-K_{X_i}$ is ample.

Assuming that (2.1) holds for $X_{i-1}$, then it also holds for $X_i$, by the structure of divisorial contractions listed in theorem 31.

Let now $C$ be a $(-m)$-curve on $Y_i$ that is not contracted by $f_i$, then since $-K_{Y_i} + f_i^*K_{X_i} \geq 0$ we have that:

$$-K_{Y_i} \cdot C = f_i^*(-K_{X_i}) \cdot C + (-K_{Y_i} + f_i^*K_{X_i}) \cdot C \geq f_i^*(-K_{X_i}) \cdot C = -K_{X_i} \cdot f_i \cdot C > 0.$$  

Then $K_C = (K_{Y_i} + C)|_C < 0$, therefore $C$ is a rational curve and $K_{Y_i} \cdot C = m - 2 < 0$ implies $m = 1$, that is, $C \subset Y_i$ is a $(-1)$-curve, and the image $C_i = f_i(C) \subset X_i$ is a floating $(-1)$-curve. Now $C_i$ does not contain the image of the $\varphi_{i-1}$-exceptional curve: otherwise, the proper transform $C' \subset Y_{i-1}$ would be a curve of negative self-intersection $C'^2 < -1$ not contracted by $f_{i-1}: Y_{i-1} \rightarrow X_{i-1}$, contradicting (2.1) for $X_{i-1}$. Thus, $C_i$ is the image of a floating $(-1)$-curve in $X_{i-1}$ and then in fact, by descending induction on $i$, $C_i$ is the image of a floating $(-1)$-curve on $X_i$ a contradiction to our main assumption that there are no such things. This shows (2.2). 

Remark 34. In section 6 we use the following type of argument very frequently. Suppose that 

$$X = X_0 \xrightarrow{\varphi_0} \ldots \rightarrow X_i \xrightarrow{\varphi_i} X_{i+1} \xrightarrow{\varphi_{i+1}} X_{i+2} \ldots$$  

is the sequence of contractions and surfaces occurring in a directed MMP for $X$. If $\varphi_i$ is of type (E.3), then $\varphi_{i+1}$ is not of type (E.3). Indeed denote by $f_i: Y_i \rightarrow X_i$ the minimal resolution. If $\varphi_{i+1}$ is of type (E.3), the proper transform $C \subset Y_{i+1}$ of the curve contracted by $\varphi_{i+1}$ is a $(-1)$-curve, and its proper transform on $Y_i$ is a $(-1)$-curve that shows that a contraction of type (E.3) or (E.4) was available on $X_i$ in the first place, and this is a contradiction.

6 Trees

This section is the heart of the paper. We prove theorem 35 from which theorem 6 of the introduction immediately follows. The proof uses proposition 29.

Theorem 35. Let $X$ be a del Pezzo surface with $k \geq 1 \frac{1}{4}(1,1)$ singular points. If $X$ has no floating $(-1)$-curves, then it is one of the following surfaces. The images show the sequence of contractions and surfaces of the directed MMP for $X$ providing a birational construction of it, followed by—and separated by a double horizontal rule—a picture of the minimal resolutions of the surfaces of the MMP showing a configuration of curves on them. We hope that these are self-explanatory:

- In the images showing the sequence of contractions we record the singularities on each intermediate surface. For example, "2 $\times$ 1/3 + $A_2$" signifies a surface with two $\frac{1}{3}(1,1)$ singularities and one $A_2$ singularity.

- In the pictures showing the minimal resolutions the contracted curves are in bold and their images are denoted by a bold point.
(1) $k = 1$ and either $X = B_{1, \frac{16}{3}}$ (the first case pictured), or $X = \mathbb{P}(1, 1, 3)$ (the second case pictured):

$$1 \times 1/3 \rightarrow \mathbb{P}(1,1,3) \rightarrow (E.2) \rightarrow A_1 \rightarrow (E.4) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

(2) $k = 2$ and either $X = B_{2, \frac{8}{3}}$ (the first case pictured), or $X = \mathbb{P}(1, 1, 3)$ (the second case pictured):

$$2 \times 1/3 \rightarrow \mathbb{P}(1,1,3) \rightarrow (E.2) \rightarrow 1 \times 1/3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$$
(3) $k = 3$ and $X = X_{3,5}$:

\[
\begin{array}{c}
3 \times 1/3 \quad \stackrel{(E.4)}{\longrightarrow} \quad 2 \times 1/3 + A_1 \quad \stackrel{(E.5)}{\longrightarrow} \quad \mathbb{P}(1,1,3)
\end{array}
\]

(4) $k = 4$ and $X = X_{4,7/3}$:

\[
\begin{array}{c}
4 \times 1/3 \quad \stackrel{(E.4)}{\longrightarrow} \quad 3 \times 1/3 + A_1 \quad \stackrel{(E.2)}{\longrightarrow} \quad 3 \times 1/3 \quad \stackrel{(E.4)}{\longrightarrow} \quad 2 \times 1/3 + A_1 \quad \stackrel{(E.5)}{\longrightarrow} \quad \mathbb{P}(1,1,3)
\end{array}
\]
(5) $k = 5$ and $X = X_{5, 5/3}$:

\[
\begin{align*}
5 \times 1/3 & \xrightarrow{(E.4)} 4 \times 1/3 + A_t & \xrightarrow{(E.4)} 3 \times 1/3 + 2 \times A_t & \xrightarrow{(E.5)} P(1,1,3) \\
& \xrightarrow{(E.5)} 2 \times 1/3 + A_t & \xrightarrow{(E.5)} & \\
\end{align*}
\]

Proof. In all cases we run the directed MMP for $X$. In other words, at each step we choose rays exactly in the order that they are listed in theorem 3[1]. The figures in the statement show the sequence of contractions as they occur in the directed minimal model program.

We begin the proof by drawing a tree representing the directed MMPs that can potentially occur (figure 2 is an example). For each branch, corresponding to a sequence of contractions, we construct a configuration of curves on the minimal resolution $Y$. In many cases, the configuration of curves shows that at some stage in the MMP there was the option of performing a contraction higher up in the list of theorem 3[1] that is, the MMP represented by that branch is not directed and hence it does not actually occur. At the end we are left with the directed MMPs that actually take place.

Here we only treat in detail the cases $k = 4$ and $k = 6$; the other cases are very similar and can be done.
by the same methods. Figures [7] to [10] at the end of the proof list the remaining trees for all \( k \). We leave it up to the interested reader to finish the proof.

**The \( k = 4 \) case** We first argue that the sequence of extremal contractions of the directed MMP must be one of those shown on figure [2].

In the argument that follows we denote by

\[
X = X_0 \xrightarrow{\varphi_0} \ldots \xrightarrow{\varphi_{i-1}} X_i \xrightarrow{\varphi_i} \ldots
\]

the sequence of contractions and surfaces occurring in a directed MMP for \( X \). Also we denote by \( f_i : Y_i \to X_i \) the minimal resolutions. By theorem 31, \( \varphi_0 \) is either of type (E.6) or (E.4) and we claim that (E.6) does not occur.

Suppose for a contradiction that \( \varphi_0 \) is an (E.6) contraction. By theorem 31, \( \varphi_1 \) is of type (E.3), (E.4) or (E.6): indeed, \( \varphi_1 \) can not be a conic bundle because \( X_1 \) has an odd number of singular points and from the classification of fibres every special fibre has two singularities on it, and it is clear from the classification that \( X_1 \) is not a del Pezzo surface with \( \rho = 1 \).

By remark 34, (E.3) can not follow (E.6). If \( \varphi_1 \) is of type (E.4), then this contraction would have been already available on \( X_0 \), a contradiction. Finally, if \( \varphi_1 \) is of type (E.6), \( X_2 \) has \( 2 \times A_2 \) singularities and then, by theorem 31, \( \varphi_2 \) is of type (E.3): just as before, none of the \( \rho = 1 \) del Pezzo surfaces have \( 2 \times A_2 \) singularities, and from the classification of fibres \( \varphi_2 \) can not be a conic bundle. But again (E.3) can not follow (E.6).

All of this shows that \( \varphi_0 \) is of type (E.4), therefore \( X_1 \) has \( 3 \times \frac{1}{4}(1,1) + A_1 \) singularities. Thus \( \varphi_1 \) can be of type (E.2), (E.4), (E.5) or (E.6), and we claim that the last two do not occur.

Suppose for a contradiction that \( \varphi_1 \) is of type (E.5). \( X_2 \) is a del Pezzo surface with \( 2 \times \frac{1}{4}(1,1) \) singularities. By the case \( k = 2 \) of the theorem, which we assume to have already proved, \( \varphi_2 \) is of type (E.4) and this contraction was available on \( X_1 \), a contradiction.

If \( \varphi_1 \) is of type (E.6) the surface \( X_2 \) has \( A_1 + A_2 + \frac{1}{4}(1,1) \) singularities. The contraction \( \varphi_2 \) can not be of type (E.2), (E.4) or (E.5) because otherwise the same contraction would have been available on \( X_1 \). It can not be of type (E.3) either because by remark 34 (E.3) can not follow (E.6). By theorem 31 these were the only possibilities thus this case does not occur.

If \( \varphi_1 \) is of type (E.2) then \( X_2 \) is a del Pezzo surface with \( k = 3 \) and the tree continues as in the \( k = 3 \) case, which we assume already known.

If \( \varphi_1 \) is of type (E.4) then \( X_2 \) has \( 2 \times A_1 + 2 \times \frac{1}{4}(1,1) \) singularities.

The next contraction \( \varphi_2 \) is not of type (E.2) because it would have been available earlier.

If \( \varphi_2 \) were of type (E.6) then \( X_3 \) would have \( A_2 + 2A_1 \) singularities. The next contraction \( \varphi_3 \) is not of type (E.2) because \( X_3 \) has an odd number of singularities. The next contraction \( \varphi_3 \) is not of type (E.3) because \( X_3 \) has an odd number of singularities. The next contraction \( \varphi_3 \) is not of type (E.4) because \( X_3 \) has an odd number of singularities. The next contraction \( \varphi_3 \) is not of type (E.5) because \( X_3 \) has an odd number of singularities.

Thus, \( \varphi_2 \) is not of type (E.2) or (E.6) and it can be only of type (E.4) or (E.5), which can be shown to lead respectively to the two remaining possibilities in figure [2].
We now explore the branches of this tree one at a time and show that only one actually occurs.

**Case 1** \((E.4) + (E.2) + (E.4) + (E.5)\)

If this sequence of contractions occurs, then \(Y\) must contain the configuration of curves depicted in Figure 3 below. The figure shows the effect of the contractions of the MMP on the minimal resolutions: the contracted curves are in bold, as are the points onto which they map.

Looking more closely at how \(Y\) is built from \(Y_3 = F_3\) by a sequence of blow-ups, we argue that \(Y\) must have more negative curves, shown in Figure 4 below.

We use the following result:

**Remark 36.** Let \(X\) be a del Pezzo surface and \(C \subset X\) an irreducible rational curve with positive self intersection. Then \(C\) moves in a free linear system. Indeed, the map \(H^0(X, O_X(C)) \to H^0(C, O_C(C))\) is surjective because \(-K_X\) is ample. Since \(C^2 > 0\) we have that \(O_C(C)\) is base point free and the conclusion immediately follows from the vanishing of \(H^1(X, O_X)\).
Indeed, the point $P_4 \in Y_4$ that is the image of the exceptional curve in $Y_3$ does not lie on the $(−3)$-curve and then by remark 36 we can choose a configuration of curves as shown in the Figure displaying $P_4$ as the intersection of a fibre and a curve of self-intersection $+3$. At the next step we need to blow up a point $P_3 \in Y_3$ on the $(−2)$-curve and not contained in any other negative curve. By remark 36 again, we can “move” the curve with self-intersection $1$ until it contains $P_3$ as in the figure. At the next step again we need to blow up a nonsingular point $P_2 \in Y_2$ not lying on any negative curve, and we use remark 36 to “move” the two curves with self-intersection $0$ until they both contain $P_2$.

We are left with the minimal resolution of the surface $X_{4,7/3}$.

**Case 2** $(E.4) + (E.4) + (E.4) + (E.4)$

We contract one $(−1)$-curve intersecting each $(−3)$-curve on $Y$ and end up with a surface fibering over $\mathbb{P}^1$, denoted by $Y_4$, corresponding to an extremal contraction $X_4 \to \mathbb{P}^1$ having two singular fibers of type (C.1). As $Y_4$ is a nonsingular surface, we next run the classical Minimal Model Program for $Y_4$ relative to the existing fibration $Y_4 \to \mathbb{P}^1$, which ends in a Segre surface $\mathbb{F}_k$, and we claim that we can assume that $k = 0$:

Indeed, since all negative curves left on $Y_4$ have self-intersection $\geq −2$, the same is true about $\mathbb{F}_k$. We are left with $k \in \{0, 1, 2\}$. If $k = 2$, this leads to a $(−2)$-curve on $Y$ not contracted by the morphism to $X$, contradicting lemma 33. If on the other hand $k = 1$ then by choosing the last contraction differently we would have landed on $\mathbb{F}_0$. 

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The horizontal ruling of $F_0$ transforms to a free linear system of $(0)$-curves intersecting two of the opposing $(-2)$-curves in the special fibers on $Y_4$, depicted as a dashed line in the figure. As both of these $(-2)$-curves contain a point that is to be blown up in the process of building $Y$, we denote by the dotted line the $(0)$-curve passing through the point $P_4 \in Y_4$ that is the image of the exceptional curve in $Y_3$ corresponding to the last $(E.4)$-type contraction (as well as its strict transform on $Y$). Exactly before this contraction is performed, in $Y_3$ (and, ultimately, also in $Y$), the dotted line is a $(−1)$-curve intersecting only the $(−1)$-curve that is to be contracted and the $(−2)$-curve on the other special fiber. This means that an $(E.2)$-type contraction was available on $X_3$, which contradicts the fact that a directed MMP was used in obtaining $X_4$. Therefore this case does not occur.

**Case 3**  $(E.4) + (E.4) + (E.5) + (E.5)$

In this case the minimal resolutions must contain the following configurations of curves:

Since the resulting surface $X_4$ is smooth and rationally connected and, by lemma [33] it contains no negative curves, it is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In the case of the projective plane, looking more closely at how $Y$ is built from $Y_4 = \mathbb{P}^2$ by a sequence of blow-ups, we see that $Y$ must have the negative curves shown in the following figure:

Note that after having done the first two steps of the MMP, $Y_2$ contains two $(-1)$-curves showing that two $(E.2)$-type contractions were available on $X_2$, which proves that the MMP is not directed and that this case does not occur.

If $Y_4 = \mathbb{P}^1 \times \mathbb{P}^1$ we reach a contradiction without having to study the minimal resolution of $X$. Indeed, the following figure shows the contraction $Y_3 \to Y_4$:
From the picture it is clear that an (E.4)-type contraction was available on X_3, a contradiction. This case also does not occur, and the only surface with four $\frac{1}{3}(1,1)$ points is the one described in Case 1.

**The k = 6 case** We argue that the sequence of extremal contractions of the directed MMP is one of the two shown in figure 5.

We use the same notations as in the k = 4 case, i.e we denote by

$$X = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_{i-1}} X_i \xrightarrow{\varphi_i} \ldots$$

the sequence of contractions and surfaces occurring in a directed MMP for X and by $f_i$: $Y_i \rightarrow X_i$ the minimal resolutions. Proposition 29 which we use repeatedly in the course of the proof, implies that $\rho(X) \in \{1 \ldots 5\}$. By theorem 31 $\varphi_0$ is either of type (E.4) or (E.6) and we claim that (E.4) does not occur.

Suppose for a contradiction that $\varphi_0$ is an (E.4) contraction, then $X_1$ has $5 \times \frac{1}{3}(1,1) + A_1$ singularities. Theorem 31 implies that $\varphi_1$ is either of type (E.2), (E.4), (E.5) or (E.6): indeed, $\varphi_1$ cannot be a conic bundle because $X_1$ contains singularities of type $\frac{1}{3}(1,1)$ but none of type $A_2$, and it is clear from the classification that $X_1$ is not a del Pezzo surface with $\rho = 1$.

If $\varphi_1$ is of type (E.2), then $X_2$ is a del Pezzo surface with $k = 5$ singularities. As we assume the case $k = 5$ is known, this implies that $X_2 = X_{5,5/3}$, a contradiction since $\rho(X_2) \leq 3$ while $\rho(X_{5,5/3}) = 5$.

If $\varphi_1$ is a type (E.4) contraction, the surface $X_2$ has $4 \times \frac{1}{3}(1,1) + 2 \times A_1$ singularities and its Picard number is at most three. From this surface, only contractions of type (E.2), (E.4) and (E.6) are possible: once again $\varphi_2$ cannot be of fiber type because of the presence of $\frac{1}{3}(1,1)$ singularities and the absence of those of type $A_2$, and by theorem 31 it is not a del Pezzo surface of $\rho = 1$.

We show that none of these possibilities occur. Indeed, if $\varphi_2$ is of type (E.2), the del Pezzo surface $X_3$ has $\rho \leq 2$ and $4 \times \frac{1}{3}(1,1) + A_1$ singularities. The following contraction $\varphi_3$ cannot be a fibration because the number of singularities is even, and $X_3$ cannot be of Picard rank one by the classification in theorem 31. Since a second (E.2) contraction should have already been performed as $\varphi_1$, $\varphi_3$ can only be of type (E.4), (E.5) or (E.6). These lead to surfaces of $\rho = 1$ and singularities of type $3 \times \frac{1}{3}(1,1) + 2 \times A_1$, $3 \times \frac{1}{3}(1,1)$ and $2 \times \frac{1}{3}(1,1) + A_1 + A_2$ respectively, none of which appear in the list in theorem 31.

The same reasoning holds if $\varphi_2$ is of type (E.4). In this case, the del Pezzo surface $X_3$ has $\rho \leq 2$ and $3 \times \frac{1}{3}(1,1) + 3 \times A_1$ singularities. By the classification in theorem 31, $X_3$ is clearly not a conic bundle over $\mathbb{P}^1$, nor does it have $\rho = 1$. The next contraction $\varphi_3$ can be either of type (E.2), (E.4), (E.5) or (E.6) and leads to a surface $X_4$ of Picard rank one. The first three cases result in at least two points of type $\frac{1}{3}(1,1)$ on $X_4$, while the contraction of type (E.6) means that $X_4$ has $\frac{1}{3}(1,1) + A_2 + 3 \times A_1$ singularities. None of these correspond to one of the $\rho = 1$ surfaces in theorem 31, since the only surface of this type containing a $\frac{1}{3}(1,1)$ point is $\mathbb{P}(1,1,3)$.

Finally, if $\varphi_2$ is of type (E.6), then $X_3$ has $2 \times \frac{1}{3} + 2 \times A_1 + A_2$ singularities. Theorem 31 implies that the only possible contractions on $X_3$ are of type (E.3) and (E.6). Indeed, the odd number of singularities and the presence of two $\frac{1}{3}(1,1)$ points allow us to respectively eliminate the possibilities of $\varphi_3$ being of fiber type and $X_3$ having $\rho = 1$. A contraction of type (E.2) would have been available earlier in the directed MMP since by performing $\varphi_2$ no new singularities of type $A_1$ were created. The same is true for contractions of type (E.4) and (E.5) which, if available, should have been done prior to the one of type (E.6). As before, both contractions would lead to a non-existent del Pezzo surface of $\rho = 1$: if $\varphi_3$ is of type (E.3), then $X_4$ has $2 \times \frac{1}{3} + 2 \times A_1$ singularities and if $\varphi_3$ is of type (E.6) we obtain $2 \times A_1 + 2 \times A_2$ singularities, a contradiction to theorem 31.
If \( \varphi_1 \) is of type (E.5), the del Pezzo surface \( X_2 \) has exactly four \( \frac{1}{3} (1, 1) \) points. From our discussion in the case \( k = 4 \) we obtain that \( X_2 = X_{4,7/3} \) and since \( \rho(X_2) \leq 2 \) and \( \rho(X_{4,7/3}) = 5 \), this is a contradiction.

If \( \varphi_1 \) is of type (E.6), \( X_2 \) has \( 3 \times \frac{1}{3} + A_1 + A_2 \) singularities. This surface is not of Picard rank one, and by theorem [31] it is not a conic fibration, as it has an odd number of singularities and too many \( \frac{1}{3} (1, 1) \) points. Since by remark [34] a contraction of type (E.3) cannot follow one of type (E.6) and considering the order of the contractions in the directed MMP, the only possibility left is that \( \varphi_2 \) is of type (E.6). \( X_3 \) now has singularities of type \( \frac{1}{3} + A_1 + 2 \times A_2 \) and no further divisorial contractions are available, again because we cannot follow with one of type (E.3) and all other possibilities would have been performed earlier. The singularities on \( X_3 \) do not however correspond to any of the surfaces of \( \rho = 1 \) in theorem [31], nor can they be paired on singular fibers of a conic fibration, for instance because there is only one point of type \( A_1 \). We have thus exhausted all the possibilities for \( \varphi_1 \).

All of this shows that \( \varphi_0 \) can not be of type (E.4), thus it is a contraction of type (E.6) and all the divisorial contractions that follow must be of the same type. The surface \( X_1 \) can not be of \( \rho = 1 \) or of fibering type since it has too many \( \frac{1}{3} (1, 1) \) points and an odd number of singularities, thus \( \varphi_1 \) is also an (E.6) contraction. As depicted in figure 5 by theorem 31 we have two possibilities: either \( X_2 \) is a conic bundle with two special fibres of type (C.2), or one last divisorial contraction is available, leading to the \( \rho = 1 \) surface (D.4).

### Figure 5: \( k = 6 \) tree of possibilities

| \( \rho \leq 5 \) | \( \rho \leq 4 \) | \( \rho \leq 3 \) | \( \rho \leq 2 \) |
|------------------|------------------|------------------|------------------|
| \( 6 \times \frac{1}{3} \) | \( 4 \times \frac{1}{3} + A_2 \) | \( 2 \times \frac{1}{3} + 2 \times A_2 \) | \( 3 \times A_2 \) |

Both instances occur and, as we will see, they lead to the same surface \( X \). This makes sense since at the very beginning of the MMP there are a total of six contractions of the same type available and at each step we choose one at the expense of two others. Depending on their configuration we stop after either two or three contractions, thus obtaining two end products of the directed MMP for the same \( X \).

**Case 1** (E.6) + (E.6) + (E.6)

The curve configuration for this case is:

Looking more closely at how \( Y \) is built from \( Y_3 \)—the surface of Picard rank one having three singular points of type \( A_2 \) described in theorem 31—by a sequence of blow-ups, we see that \( Y \) must have the negative curves shown in the following figure:
Case 2 \((E.6) + (E.6)\)

This sequence ends with a conic fibration having two singular fibers. On the minimal resolutions, the contractions are the following:

Figure 6: Configuration of curves for \(k = 6\), Case 2

We again proceed to run the nonsingular minimal model program for the surface \(Y_2\) relative to the current fibration over \(\mathbb{P}^1\). As before, we eventually reach a surface \(\mathbb{F}_k\), where \(k \in \{0, 1, 2\}\), and we choose our sequence of contractions such that \(k\) is maximal. Figure 6 shows that all \((-3)\)-curves on \(Y\) either already existed on the special fibers of \(Y_2\) or they come from blowing up points inside these fibers. Thus if \(k = 2\), the \((-2)\)-section remains as such even on \(Y\), which is impossible according to lemma 33. By maximality, \(k = 1\), and then the minimal resolutions must have the negative curves shown in the following figure:

Cases 1 and 2 are two different directed MMPs starting from the del Pezzo surface \(X_{6,2}\).

We further give the trees of possibilities for the four remaining cases. When constructing the tree for a surface with \(k_0\) singularities, we are allowed to use the final working cases for the surfaces with \(k < k_0\) points of type \(\frac{1}{3}(1, 1)\). This not only shortens the process, but also allows us to further exclude certain sequences.
of contractions. Indeed, suppose that in the $k_0$ tree a branch leads to a del Pezzo surface with singularity content $k \times \frac{1}{3}(1, 1)$, where $k < k_0$. If the sequence in the statement of theorem 35 for $k$ doesn't correlate with the directed MMP obtained thus far in the $k_0$ tree, then the entire branch can be removed.

Figure 7: $k = 1$ Tree

| $\rho \leq 9$ | $\rho \leq 8$ | $\rho \leq 7$ |
|--------------|--------------|--------------|
| $1 \times 1/3$ | $A_1$ | $\mathbb{P}^i$ or $\mathbb{P}_x \mathbb{P}^i$ |

**Figure 8: $k = 2$ Tree**

| $\rho \leq 8$ | $\rho \leq 7$ | $\rho \leq 6$ | $\rho \leq 5$ | $\rho \leq 4$ |
|--------------|--------------|--------------|--------------|--------------|
| $2 \times 1/3$ | $1 \times 1/3 + A_1$ | $1 \times 1/3$ | $A_1$ | $\mathbb{P}^i$ or $\mathbb{P}_x \mathbb{P}^i$ |

**Figure 9: $k = 3$ Tree**

| $\rho \leq 7$ | $\rho \leq 6$ | $\rho \leq 5$ | $\rho \leq 4$ | $\rho \leq 3$ |
|--------------|--------------|--------------|--------------|--------------|
| $3 \times 1/3$ | $2 \times 1/3 + A_1$ | $2 \times 1/3$ | $3 \times 1/3 + A_1$ | $1 \times 1/3$ |

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Finally, using the techniques presented so far in this section will lead to a systematic elimination of the branches so that we are left with exactly the surfaces in the statement.

\[ \rho \leq 6 \]

\[ \rho = 1 \]

7 Toric Degenerations

**Proof of theorem [12]** We first argue that \( X_{4, 1/3}, X_{5, 2/3} \) and \( X_{6, 1} \) do not admit a toric qG-degeneration. For all of these \( h^0(X, -K_X) = 0 \). But we know that \( h^0(X, -K_X) \) is a qG-deformation invariant, and if \( X_0 \) is a toric surface then \( H^0(X, -K_X) \neq 0 \) as \( -K_X \) always contains at least the toric boundary divisor. We show that the remaining 26 families do indeed admit a toric qG-degeneration.

Let \( N = \mathbb{Z}^2 \). A Fano polygon is a convex lattice polygon \( P \subset N_\mathbb{Q} \) such that: (a) the origin lies strictly in the interior of \( P \) and (b) the vertices of \( P \) are primitive lattice vectors. Table 4 lists 26 Fano polygons—each specified by the list of its vertices—\( P \subset N_\mathbb{Q} \) with singularity content \( (n, \{ k \times \frac{1}{3}(1, 1) \}) \), and figure 11 shows pictures of all these polygons in turn and the location of the origin in their interiors. The singularity content of a polygon is defined in [3, Definition 2.4 and 3.1]. It is easy to see that the singularity content of \( P \) is what it says on the table by looking at the picture of the polygon in fig. 11; for all cones of the polygon, all you have to do is eyeball the residue of that cone and persuade yourself that is is either empty or a \( \frac{1}{3}(1, 1) \) cone, and count the primitive \( T \)-cones. Then \( n \) is the total number of \( T \)-cones over the whole polygon, and \( k \) is the number of nonempty residue cones.

For all \( P \) in the table, let \( \Sigma(P) \) be the face-fan of \( P \) and denote by \( X_P \) the toric surface constructed from \( \Sigma(P) \). Then \( -K_{X_P}^2 = 12 - n - \frac{5k}{3} \).

It is explained for instance in [2, Lemma 6] that qG-deformations of del Pezzo surfaces with cyclic quotient singularities are unobstructed. It follows from this that if \( P \) is a polygon in the table, and it has singularity content \( (n, \{ k \times \frac{1}{3}(1, 1) \}) \), then \( X_P \) qG-deforms to a locally qG-rigid del Pezzo surface \( X \) with \( k \frac{1}{3}(1, 1) \) points and \( K_X^2 = K_{X_P}^2 \), that is, one of the 26 remaining families that can, in principle, have a toric qG-degeneration.

To prove the theorem, all that is left to do is to determine which toric surfaces qG-deform to which locally qG-rigid families. For all \( i \in \{ 1, \ldots, 26 \} \), denote by \( P_i \) the \( i \)th polygon of the list. Looking at the table, the only cases where there is any ambiguity are \( P_12, P_13 \)—they both have singularity content \( (6, 2 \times \frac{1}{3}(1, 1)) \)—and \( P_{21}, P_{22} \)—they both have singularity content \( (5, 1 \times \frac{1}{3}(1, 1)) \). We argue that:

(a) \( X_{P_{12}} \) qG-deforms to \( B_{2, 8/3} \) and \( X_{P_{13}} \) to \( X_{2, 8/3} \);

(b) \( X_{P_{21}} \) qG-deforms to \( B_{1, 16/3} \) and \( X_{P_{22}} \) to \( X_{1, 16/3} \).

The best way to show this here is to write down explicitly the qG-deformation inside the ambient variety \( F \) of table 2. This is not hard to do by hand.
For instance $X_{P_{21}} = \mathbb{P}(1, 3, 4)$. Denote by $u, v, w$ be the co-ordinates of weights $1, 3, 4$ on $\mathbb{P}(1, 3, 4)$ and by $x_0, x_1, x_2, y$ the coordinates of weights $1, 1, 1, 3$ on $\mathbb{P}(1, 1, 1, 3)$. We define an embedding $i : \mathbb{P}(1, 3, 4) \hookrightarrow \mathbb{P}(1, 1, 1, 3)$, such that $i^*\mathcal{O}(1) = \mathcal{O}(4)$, as follows:

$$i^*(x_0) = u^4, \ i^*(x_1) = u, \ i^*(x_2) = w, \ i^*(y) = v^4$$

and it is immediate that the image is the degree $4$ weighted hypersurface given by the binomial equation $y x_0 - x_1^4 = 0$. This shows that $X_{P_{21}}$ belongs to the family $B_{1,16/3}$.

The case of $X_{P_{12}}$ is similar and only slightly harder. The vertices

$$u_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \ u_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \ v = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

of $P_{12}$ satisfy the relation $u_0 + u_1 + 2v = 0$ but together they only generate an index $3$ subgroup of $N = \mathbb{Z}^2$. This identifies $X_{P_{12}}$ with the quotient $\mathbb{P}(1,1,2)/\mu_3$ where $\mu_3$ acts on the homogeneous coordinates $u_0, u_1, v$ with weights $\frac{1}{3}, 0, \frac{1}{3} \in \left(\frac{1}{3}\mathbb{Z}\right)/\mathbb{Z}$. Note, indeed, that $X_{P_{12}}$ has $2 \times \frac{1}{3}(1,1)$ points at $(1 : 0 : 0), (0 : 1 : 0)$, and $1 \times \frac{1}{3}(1,-1)$ at $(0 : 0 : 1)$. Let $L$ be the line bundle on $X_{P_{12}}$ of integer weight $2$ and $\mu_3$-weight $\frac{1}{3}$. We define an embedding $i : X_{P_{12}} \hookrightarrow \mathbb{P}(1,1,3,3)$ with homogeneous coordinates $x_0, x_1, y_0, y_1$, such that $i^*\mathcal{O}(1) = L$, as follows:

$$i^*(x_0) = v, \ i^*(x_1) = u_0 u_1, \ i^*(y_0) = u_0^6, \ i^*(y_1) = u_1^6$$

and it is immediate that the image $i(X_{P_{12}}) \subset \mathbb{P}(1,1,3,3)$ is the degree $6$ weighted hypersurface given by the binomial equation $y_0 y_1 - x_1^6 = 0$. This shows that $X_{P_{12}}$ belongs to the family $B_{2,8/3}$.

Consider now $P_{13}$. The vertices

$$u_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \ u_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \ v_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

of $P_{13}$ satisfy the relations:

$$u_0 + u_1 + v_0 = 0 \quad \text{and} \quad v_0 + v_1 = 0$$

but together they only generate an index $3$ subgroup of $N = \mathbb{Z}^2$. This identifies $X_{P_{13}}$ with the quotient $\mathbb{P}(1)/\mu_3$ where $\mu_3$ acts on $\mathbb{P}_1$ as follows. Choose Cox coordinates and weight matrix for $\mathbb{P}_1$ as:

$$\begin{array}{cccc}
u_0 & u_1 & v_0 & v_1 \\
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{array}$$

The action of $\mu_3$ on $\mathbb{P}_1$ is by weights $\frac{1}{3}, 0, \frac{1}{3} \in \left(\frac{1}{3}\mathbb{Z}\right)/\mathbb{Z}$ on the Cox coordinates. Also denote by $M_1$ the line bundle on $\mathbb{P}_1$ with of bidegree $(1,0)$ and trivial $\mu_3$-weight $\frac{1}{3}$ (any $\mu_3$-weight will do here), and by $M_2$ the line bundle of bidegree $(0,1)$ and $\mu_3$-weight $\frac{1}{3}$ (we need this particular $\mu_3$-weight here). We construct an embedding $i : X_{P_{13}} \hookrightarrow F$ where $F$ is the Fano simplicial toric 3-fold with weight matrix and Cox coordinates:

$$\begin{array}{cccc}
s_0 & s_1 & x_0 & y \\
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 3 \\
\end{array}$$

where we also denote by $L_1, L_2$ the standard basis of $\text{Cl} F$, such that $i^*(L_1) = 3 M_1$ and $i^*(L_2) = M_2$, as follows:

$$i^*(s_0) = u_0^3, \ i^*(s_1) = u_1^3, \ i^*(x_0) = u_0 v_1 v_0, \ i^*(y) = v_0^3, \ i^*(x_1) = v_1$$

and it is immediate that the image $i(X_{P_{13}}) \subset F$ is the hypersurface of bidegree $(3,3)$ given by the binomial equation $s_0 s_1 y - x_0^3 = 0$. This shows that $X_{P_{13}}$ belongs to the family $X_{2,8/3}$.

The final case $P_{22}$ is very similar and, in fact, easier. The vertices

$$u_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \ u_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ v_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ v_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$
of \( P^{22} \) generate \( N = \mathbb{Z}^2 \) and satisfy the relations:

\[
    u_0 + 3u_1 - v_1 = 0 \quad \text{and} \quad v_0 + v_1 = 0
\]

This identifies \( X_{P^{22}} \) with the toric surface with Cox coordinates and weight matrix:

\[
\begin{pmatrix}
    u_0 & u_1 & v_0 & v_1 \\
    1 & 3 & 0 & -1 \\
    0 & 0 & 1 & 1
\end{pmatrix}
\]

Denote by \( M_1, M_2 \) the standard basis of \( \text{Cl} X_{P^{22}} \). We construct an embedding \( i : X_{P^{22}} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}(1,1,3) \) such that \( i^* \mathcal{O}(1,0) = 3M_1, i^* \mathcal{O}(0,1) = M_2 \) and, choosing homogeneous coordinates \( s_0, s_1 \) on \( \mathbb{P}^1 \) and \( x_0, x_1, y \) on \( \mathbb{P}(1,1,2) \):

\[
i^*(s_0) = u_0^3, \quad i^*(s_1) = u_1, \quad i^*(x_0) = v_0, \quad i^*(x_1) = u_0v_1, \quad i^*(y) = u_1v_1^3
\]

and it is immediate that the image \( i(X_{P^{22}}) \subset \mathbb{P}^1 \times \mathbb{P}(1,1,3) \) is the hypersurface of bidegree \( (1,3) \) given by the binomial equation \( s_0y - s_1x_1^3 = 0 \). This shows that \( X_{P^{22}} \) belongs to the family \( X_{1,16/3} \).

Remark 37. Our proof of theorem12 is by very efficient ad hoc considerations. The Gross–Siebert program is a systematic approach to constructing qG-deformations of toric surfaces, see [27]. Paper [10] gives a systematic approach to constructing qG-deformations as toric complete intersections, when they exist.

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Table 4: 26 Fano polygons $P \subset N_\mathbb{Q}$ with singularity content $(n, \{k \times F(1,1)\})$

| #  | $V(P)$                  | $n$ | $k$ | Deforms to |
|----|------------------------|-----|-----|------------|
| 1  | $(7,5), (-3,5), (-3,-5)$ | 10  | 1   | $X_{1,1/3}$ |
| 2  | $(3,2), (-3,2), (-3,-2), (3,-2)$ | 8   | 2   | $X_{2,2/3}$ |
| 3  | $(3,1), (3,2), (-1,2), (-2,1), (-2,-3), (-1,-3)$ | 6   | 3   | $X_{3,1}$  |
| 4  | $(3,2), (-1,2), (-2,1), (-2,-3)$ | 9   | 1   | $X_{1,4/3}$ |
| 5  | $(2,1), (1,2), (-1,2), (-2,1), (-2,-1), (-1,-2), (1,-2), (2,-1)$ | 4   | 4   | $X_{4,4/3}$ |
| 6  | $(3,2), (-1,2), (-2,1), (-2,-1), (-1,-2)$ | 7   | 2   | $X_{2,5/3}$ |
| 7  | $(2,1), (1,2), (-1,2), (-2,1), (-2,-1), (-1,-2), (1,-1)$ | 2   | 5   | $X_{5,5/3}$ |
| 8  | $(2,1), (1,2), (-1,2), (-2,1), (-2,-1), (-1,-2)$ | 5   | 3   | $X_{3,2}$  |
| 9  | $(1,1), (-1,2), (-2,1), (-1,-1), (1,-2), (2,-1)$ | 0   | 6   | $X_{6,2}$  |
| 10 | $(1,1), (-1,2), (-1,-2), (1,-2)$ | 8   | 1   | $X_{1,7/3}$ |
| 11 | $(1,1), (-1,2), (-2,1), (-1,-1), (2,-1)$ | 3   | 4   | $X_{4,7/3}$ |
| 12 | $(3,1), (-3,1), (0,-1)$ | 6   | 2   | $B_{2,8/3}$ |
| 13 | $(1,1), (-1,2), (-1,-1), (2,-1)$ | 6   | 2   | $X_{2,8/3}$ |
| 14 | $(1,1), (-1,2), (-2,1), (-1,-1), (1,-1)$ | 4   | 3   | $X_{3,3}$  |
| 15 | $(1,1), (-1,2), (-1,-1), (1,-1)$ | 7   | 1   | $X_{1,10/3}$ |
| 16 | $(1,1), (-1,2), (-1,0), (0,-1), (2,-1)$ | 5   | 2   | $X_{2,11/3}$ |
| 17 | $(1,0), (1,1), (-1,2), (-2,1), (-1,-1), (0,-1)$ | 3   | 3   | $X_{3,4}$  |
| 18 | $(1,0), (0,1), (-1,1), (-1,-3)$ | 6   | 1   | $X_{1,13/3}$ |
| 19 | $(1,1), (-1,2), (-1,1), (0,-1), (2,-1)$ | 4   | 2   | $X_{2,14/3}$ |
| 20 | $(1,1), (-1,2), (-2,1), (-1,-1), (0,-1)$ | 2   | 3   | $X_{3,5}$  |
| 21 | $(1,1), (-1,2), (-1,-2)$ | 5   | 1   | $B_{1,16/3}$ |
| 22 | $(1,1), (-1,2), (-1,-1), (0,-1)$ | 5   | 1   | $X_{1,16/3}$ |
| 23 | $(1,1), (-1,2), (0,-1), (2,-1)$ | 3   | 2   | $X_{2,17/3}$ |
| 24 | $(0,1), (-1,2), (-2,1), (-1,0), (1,-1)$ | 4   | 1   | $X_{1,19/3}$ |
| 25 | $(0,1), (-1,2), (-2,1), (1,-1)$ | 3   | 1   | $X_{1,22/3}$ |
| 26 | $(-1,2), (-2,1), (1,-1)$ | 2   | 1   | $S_{1,25/3}$ |
Figure 11: 26 Fano polygons $P \subset N_{\mathbb{Q}}$ with singularity content $(n, \{k \times \frac{1}{4}(1, 1)\})$. See also Table 4.