COMPACTLY SUPPORTED HAMILTONIAN LOOPS WITH A NON-ZERO CALABI INVARIANT

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Abstract. We give examples of compactly supported Hamiltonian loops with a non-zero Calabi invariant on certain open symplectic manifolds.

1. Introduction

Let $(M, \omega)$ be an open symplectic $2n$-dimensional manifold. Denote by $\text{Ham}(M)$ the group of Hamiltonian diffeomorphisms generated by compactly supported Hamiltonian functions. Denote by $\widetilde{\text{Ham}}(M)$ the universal cover of $\text{Ham}(M)$. We write elements of $\widetilde{\text{Ham}}(M)$ as $\{f_t\}_{t \in [0,1]}$, where $\{f_t\}_{t \in [0,1]}$ is a smooth path of Hamiltonian diffeomorphisms with $f_0 = \text{Id}$, and $\{f_t\}_{t \in [0,1]}$ stands for the homotopy class of $\{f_t\}_{t \in [0,1]}$ with fixed end points. In what follows we use the notation $\text{Vol}(M) := \int_M \omega^n$.

The Calabi homomorphism $\text{Cal}: \widetilde{\text{Ham}}(M) \to \mathbb{R}$ is given by

$$\text{Cal}(\{f_t\}_{t \in [0,1]}) = \int_0^1 \int_M F_t \omega^n dt,$$

where $\{F_t\}_{t \in [0,1]}$ is the compactly supported Hamiltonian function whose flow is $\{f_t\}_{t \in [0,1]}$.

When $\omega$ is exact, the Calabi homomorphism vanishes on $\pi_1(\text{Ham}(M))$ and hence descends to $\text{Ham}(M)$ (see [4, section 10.3]). In this case

$$\text{Cal}(\{f_t\}_{t \in [0,1]}) = \text{Cal}(\{g_t\}_{t \in [0,1]}),$$

whenever $f_1 = g_1$.

Our goal is to give examples of open symplectic manifolds with a non exact symplectic form, for which $\text{Cal}$ does not descend to $\text{Ham}(M)$. Equivalently, we...
wish to find a Hamiltonian loop generated by a compactly supported Hamiltonian function \( \{ H_t \}_{t \in [0,1]} \), such that
\[
\int_0^1 \int_M H_t \omega^n dt \neq 0.
\]
Existence of such an example is indicated by McDuff in [3, Remark 3.10]. In addition, our interest to the problem was stimulated by [7].

An immediate corollary is that we get examples of open symplectic manifolds such that \( \pi_1(\text{Ham}(M)) \neq 0 \).

Let us present one geometric consequence of the non-vanishing of Cal on \( \Pi := \pi_1(\text{Ham}(M)) \subset \tilde{\text{Ham}}(M) \).

For an element \( \gamma \in \Pi \) put
\[
\ell(\gamma) := \inf_{\{f_t\}_{t \in [0,1]}} \int_0^1 (\max F_t - \min F_t) \, dt,
\]
where the infimum is taken over all loops \( \{f_t\}_{t \in [0,1]} \) representing \( \gamma \) and \( F \) stands for the compactly supported Hamiltonian generating \( \{f_t\}_{t \in [0,1]} \). Recall from [5, Chapter 7] that the set
\[
\{ \ell(\gamma) : \gamma \in \Pi \}
\]
forms the Hofer length spectrum of \( \text{Ham}(M) \). One readily checks that
\[
\ell(\gamma) \geq |\text{Cal}(\gamma)| \quad \forall \gamma \in \Pi.
\]
Therefore, when \( \text{Cal} \) does not descend, the Hofer length spectrum is non-trivial.

In section 2 we state the main theorem which enables us to construct examples of Hamiltonian loops with a non-zero Calabi invariant. Its proof is given in section 3.

2. EXAMPLES FOR HAMILTONIAN LOOPS WITH A NON-ZERO CALABI INVARIANT

**Definition 2.1.** Let \((X^{2n},\omega)\) be a closed symplectic manifold, and let \(\{f_t\}_{t \in [0,1]}\) be a Hamiltonian \(S^1\)-action on \(X\). Let \(z_0 \in X\) be a fixed point of the action. We say that \(z_0\) is a Maslov-zero fixed point if the loop
\[
\{d_{z_0} f_t\}_{t \in [0,1]} \subset Sp(T_{z_0} X, \omega)
\]
has Maslov index 0.

**Theorem 2.2.** Let \((X^{2n},\omega)\) be a closed symplectic manifold, and let \(\{f_t\}_{t \in [0,1]}\) be a Hamiltonian \(S^1\)-action generated by a Hamiltonian function \(F\). Let \(z_0\) be a Maslov-zero fixed point of \(\{f_t\}_{t \in [0,1]}\) which satisfies
\[
\int_X F \omega^n \neq \text{Vol}(X) \cdot F(z_0).
\]

Then there exists an open neighborhood \(B\) that contains \(z_0\) such that there exists a compactly supported Hamiltonian loop \(\{h_t\}_{t \in [0,1]}\) in the open manifold \(M = X \setminus B\), which satisfies
\[
\text{Cal}(\{h_t\}_{t \in [0,1]}) \neq 0
\]
and which is also homotopic to \(\{f_t\}_{t \in [0,1]}\) in \(\text{Ham}(X)\).
Example 2.3. Let
\[ X = S^2(r_1) \times S^2(r_2) \subset \mathbb{R}^3 \times \mathbb{R}^3 \]
with \( r_1 \neq r_2 \), where \( r_1, r_2 \) stand for the radii. This is a closed symplectic manifold with the symplectic form \( \omega := \text{area}_{S^2(r_1)} \oplus \text{area}_{S^2(r_2)} \). Define the Hamiltonian function
\[ F((\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)) := 2\pi r_1 \alpha_3 + 2\pi r_2 \beta_3. \]
The Hamiltonian flow of \( F \) is an \( S^1 \)-action. The point
\[ s = ((0, 0, r_1), (0, 0, -r_2)) \]
is a fixed point of the flow of \( F \).

Claim: The fixed point \( s \) is a Maslov-zero fixed point, and
\[ \int_X F\omega^n \neq \text{Vol}(X) \cdot F(s). \]

Proof: Near \( s \) we can find an \( S^1 \)-invariant neighborhood that is \( S^1 \)-equivariantly symplectomorphic to an open neighborhood of 0 in \( \mathbb{C}^2 \) with the Hamiltonian \( S^1 \)-action generated by\(^1\)
\[ F' = \pi \cdot (|z_1|^2 + |z_2|^2), \]
where \((z_1, z_2) \in \mathbb{C}^2\). We see that the Hamiltonian flow in \( \mathbb{C}^2 \) is the loop of symplectic matrices \( e^{-2\pi i t} \oplus e^{2\pi i t} \in Sp(4) \), which has Maslov index 0. We get that \( s \) is a Maslov-zero fixed point.

We can calculate
\[ \int_{S^2(r_1) \times S^2(r_2)} \omega^2 \cdot F(s) = \text{Vol}(S^2(r_1) \times S^2(r_2)) \cdot (2\pi r_1^2 - 2\pi r_2^2) \neq 0, \]
while
\[ \int_{S^2(r_1) \times S^2(r_2)} F\omega^2 = 0. \]

This proves the claim and we see that all the requirements of Theorem 2.2 are satisfied.

Remark 2.4. We can extend Example 2.3 to higher dimensions.

Look at
\[ (S^2)^m = S^2(r_1) \times S^2(r_1) \times \ldots \times S^2(r_1) \times S^2(r_2), \]
with the Hamiltonian
\[ F = 2\pi (r_1 h_1 + r_1 h_2 + \ldots + r_1 h_{m-1} + (m-1) \cdot r_2 h_m), \]
where \( h_i \) is the height function of the \( i \)-th copy of \( S^2 \) and \( r_1 \neq r_2 \). Define \( s \) to be the fixed point
\[ s = ((0, 0, r_1), \ldots, (0, 0, r_1), (0, 0, -r_2)). \]
Now \( s \) is a Maslov-zero fixed point, \( F(s) \neq 0 \) and \( \int F\omega^m = 0 \). We get again that all the requirements of Theorem 2.2 are satisfied.

\(^1\)More explicitly, if we look at \( S^2 \times S^2 \) with cylindrical coordinates \((\theta_1, \alpha_3, \theta_2, \beta_3)\) and on \( \mathbb{C}^2 \) with polar coordinates \((\sigma_1, \rho_1, \sigma_2, \rho_2)\), then we can write the symplectomorphism explicitly as
\[ (\theta_1, \alpha_3, \theta_2, \beta_3) \mapsto (\theta_1, \sqrt{2r_1^2 - 2r_1 \alpha_3}, -\theta_2, \sqrt{2r_2 \beta_3 + 2r_2^2}). \]
Example 2.5. Let \((X', \omega)\) be a closed symplectic manifold with an \(S^1\)-action which has a Maslov-zero fixed point \(s\), and is generated by a non-constant Hamiltonian function \(F\). Under these conditions we can always construct an open symplectic manifold which admits a compactly supported Hamiltonian loop with a non-zero Calabi invariant.

Normalize \(F\) so that \(F(s) = 0\). Take another fixed point \(s'\) with \(F(s') \neq 0\) (there are always at least two fixed points with different \(F\)-values because \(F\) achieves maximum and minimum on \(X'\)). If \(\int_{X'} F \omega^n \neq \text{Vol}(X') \cdot F(s)\) then the requirements of Theorem 2.2 are satisfied right away. Assume that \(\int_{X'} F \omega^n = 0\). We first deal with the case where \(F(s') > 0\). Choose a neighborhood \(U\) of \(s'\) so that \(F|_U > 0\). Perform an equivariant symplectic blow-up at \(s'\) (see [2, section 6.1]) with a small enough weight so that the symplectic ball \(B\) that we cut out in the blow-up will be inside \(U\). Call the resulting symplectic manifold \((X, \tilde{\omega})\), and the resulting Hamiltonian function \(\tilde{F}\). The mean value of \(\tilde{F}\) will be less than zero because

\[
0 = \int_{X'} F \omega^n = \int_X \tilde{F} \omega^n + \int_B F \omega^n.
\]

Hence,

\[
\int_X \tilde{F} \omega^n = -\int_B F \omega^n.
\]

Note that in a neighborhood of \(s\), we have \(\tilde{F} = F\). We get that \(s\) is a Maslov-zero fixed point of \(\tilde{F}\) and \(\tilde{F}(s) = 0\). Hence, the requirements of Theorem 2.2 are satisfied.

If \(F(s') < 0\) then we can define \(U\) to be a neighborhood of \(s'\) so that \(F|_U < 0\) and then continue as before.

Note that we can always perform an equivariant symplectic blow-up if the weight is small enough. From the equivariant Darboux theorem, we get that there is a small neighborhood of \(s'\) which is equivariantly symplectomorphic to a neighborhood of zero in \(\mathbb{C}^n\) with a linear symplectic \(S^1\)-action inside \(U(n)\). Hence, we can choose a ball inside the neighborhood with radius \(\lambda\) and get that the action naturally extends to the blow-up with weight \(\lambda\).

Example 2.6. Here we give examples for open monotone symplectic manifolds which admit a Hamiltonian loop with a non-zero Calabi invariant.

Consider the \(S^1\)-action defined on \(\mathbb{CP}^n\) by

\[
f_t([x_0 : x_1 : \ldots : x_n]) = [x_0 : e^{2\pi i a_1 t} x_1 : \ldots : e^{2\pi i a_n t} x_n]
\]

for some \((a_1, \ldots, a_n) \in \mathbb{Z}^n\) where \(\sum_{i=1}^n a_i = 0\) and \((a_1, \ldots, a_n) \neq 0\). Denote the moment map by \(F\). The point \(s = [1 : 0 : \ldots : 0]\) is a Maslov-zero fixed point of \(F\). Perform a monotone \(T^n\)-equivariant symplectic blow-up at another fixed point where the value of the moment map is different. Call the resulting manifold \(X\). We get that

\[
\int_X F \omega^n \neq \text{Vol}(X) \cdot F(s).
\]

Hence, all the requirements of Theorem 2.2 are satisfied so we can construct an open monotone symplectic manifold \(M := X \setminus B\) which admits a Hamiltonian loop with a non-zero Calabi invariant. Note that \(M\) remains monotone.
In Remark 2.7 below we shall use the monotonicity in order to show that the non-contractible Hamiltonian loop that we have constructed in $\text{Ham}(M)$ is also non-contractible in $\text{Ham}(X)$.

**Remark 2.7.** In the previous examples we considered an open manifold $M$ which is a subset of a bigger closed manifold $X$. One could ask if the non-contractible loops that we have constructed in $\text{Ham}(M)$ remain non-contractible in $\text{Ham}(X)$. The constructed loop in $\text{Ham}(M)$ is always homotopic to the original $S^1$-action in $\text{Ham}(X)$ so we get that it remains to check whether the original $S^1$-action is non-contractible in $\text{Ham}(X)$.

The answer to this question is, in general, no, as can be seen in the following example: let $X = S^2(r_1) \times S^2(r_2)$ with $r_1 \neq r_2$ and consider the Hamiltonian function

$$F((\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)) := 4\pi r_1 \alpha_3 + 4\pi r_2 \beta_3.$$  

The Hamiltonian flow of $F$ is the 2-turn rotation around the $\alpha_3$-axis times the 2-turn rotation around the $\beta_3$-axis. It is a known fact that

$$\pi_1(\text{Ham}(S^2)) = \mathbb{Z}_2,$$

and that the generator is the 1-turn rotation (see [5, section 7.2]). We get that the 2-turn rotation is a contractible loop in $\text{Ham}(S^2)$ and hence the flow of $F$ is contractible in $\text{Ham}(X)$.

The point $s = ((0, 0), (0, 0), (0, -r_2))$ is a Maslov-zero fixed point and

$$\int_X F \omega^n \neq \text{Vol}(X) \cdot F(s).$$

The requirements of Theorem 2.2 are satisfied so we can construct the open manifold $M$ and the Hamiltonian loop $\{h_t\}_{t \in [0, 1]}$ such that $\{h_t\}_{t \in [0, 1]}$ is non-contractible in $\text{Ham}(M)$. The loops $\{h_t\}_{t \in [0, 1]}$ and $\{f_t\}_{t \in [0, 1]}$ are in the same homotopy class in $\pi_1(\text{Ham}(X))$ so we get that $\{h_t\}_{t \in [0, 1]}$ is contractible in $\text{Ham}(X)$ and non-contractible in $\text{Ham}(M)$.

We will now show that the situation is different for monotone manifolds. For a closed monotone symplectic manifold $X$ the Hamiltonian loops that we are considering will always be non-contractible in $\text{Ham}(X)$.

**Proposition 2.8.** Let $X$ be a closed monotone symplectic manifold which admits a Hamiltonian $S^1$-action $\{f_t\}_{t \in [0, 1]}$ with a Maslov-zero fixed point $s$ such that

$$\int_X F \omega^n \neq \text{Vol}(X) \cdot F(s),$$

where $F$ is the moment map. Then the loop $\{f_t\}_{t \in [0, 1]}$ is non-contractible in $\text{Ham}(X)$.

**Proof.** Consider the mixed action-Maslov invariant of the loop $\{f_t\}_{t \in [0, 1]}$ (see [6]). We calculate it via the constant loop $\{f_t(s)\}_{t \in [0, 1]} \subset X$ with the constant spanning disk $\varpi : D^2 \to X$. Normalize $F$ so that its mean value will be zero, and calculate the symplectic action

$$A(\{f_t\}_{t \in [0, 1]}; \varpi) = \int_{D^2} \varpi^* \omega - \int_0^1 F(s) dt = -F(s) \neq 0.$$
The Maslov index of the loop \( \{d_s f_t\}_{t \in [0,1]} \) is zero so we get that the mixed action-Maslov invariant is

\[
I = A(\{f_t\}_{t \in [0,1]}, \overline{s}) \neq 0.
\]

Hence the loop \( \{f_t\}_{t \in [0,1]} \) is non-contractible in \( Ham(X) \). \(\square\)

3. Proof of Theorem 2.2

In order to prove Theorem 2.2 we will need the following lemma.

**Lemma 3.1.** Let \( \{A_t\}_{t \in [0,1]} \subset Sp(2n) \) be a contractible loop of symplectic matrices. Then for any \( \epsilon > 0 \) and any open ball \( B \) around zero in the standard symplectic \( \mathbb{R}^{2n} \), there is a smaller ball \( B' \) around zero and a Hamiltonian loop \( \{g_t\}_{t \in [0,1]} \subset Ham(\mathbb{R}^{2n}) \) such that

1. \( g_t \) is supported in \( B \).
2. \( g_t|_{B'} = A_t|_{B'} \).
3. \( |Cal([g_t\}_{t \in [0,1]})| < \epsilon \).
4. \( \{g_t\}_{t \in [0,1]} \) is a contractible loop in \( Ham(\mathbb{R}^{2n}) \).

**Proof:** Let \( A_{s,t} \) be a smooth homotopy of the loop \( \{A_t\}_{t \in [0,1]} \) to the identity, such that \( A_{0,t} = Id, A_{1,t} = A_t \), and \( A_{s,0} = A_s = Id \).

**Claim:** There is a function \( H: \mathbb{R}^{2n} \times I \times I \to \mathbb{R} \) such that for a fixed \( t \), \( \{H(\cdot, s, t)\}_{s \in [0,1]} \) generates \( \{A_{s,t} \}_{s \in [0,1]} \) as the Hamiltonian flow with respect to the time variable \( s \), and if we put \( H_{s,t} := H(\cdot, s, t) \), then

1. \( H \) is smooth.
2. \( H_{s,t}(0) = 0 \).
3. \( H_{s,0} = H_{s,1} = 0 \).

**Proof:** A path of symplectic matrices \( \{B_t\}_{t \in [0,1]} \) is generated as Hamiltonian flow by a Hamiltonian function of the form \( \langle x, Q_t x \rangle \), where \( Q_t = \frac{1}{2} J \frac{\partial B_t}{\partial t} B_t^{-1} \) and \( J \) is the standard linear complex structure on \( \mathbb{R}^{2n} \). In our situation \( H_{s,t} = \langle x, Q_{s,t} x \rangle \) is a Hamiltonian function that generates the flow \( s \mapsto A_{s,t} \) for \( Q_{s,t} = \frac{1}{2} J \frac{\partial A_{s,t}}{\partial s} A_{s,t}^{-1} \).

We get that \( H \) is smooth,

\[
H_{s,t}(0) = \langle 0, Q_{s,t} 0 \rangle = 0,
\]

and

\[
H_{s,0} = H_{s,1} = 0.
\]

The latter assertion holds because

\[
A_{s,0} = A_{s,1} = Id,
\]

so

\[
\frac{\partial A_{s,0}}{\partial s} = \frac{\partial A_{s,1}}{\partial s} = 0.
\]

This proves the claim.

Choose three balls \( B_1 \subset B_2 \subset B_3 \subset B \) such that for each \( t \in [0,1] \), \( A_t B_1 \subset B_2 \).

Choose a cut-off function \( a: \mathbb{R}^{2n} \to \mathbb{R} \) such that \( a|_{B_2} = 1 \) and \( a|_{\mathbb{R}^{2n}\setminus B_3} = 0 \).

For a fixed \( t \), define \( \{g_{s,t}\}_{s \in [0,1]} \) to be the Hamiltonian flow generated by the Hamiltonian function \( \{a \cdot H_{s,t}\}_{s \in [0,1]} \) with the time variable \( s \).

For a fixed \( s \), \( \{g_{s,t}\}_{t \in [0,1]} \) is a loop with respect to \( t \). That is true because \( H_{s,0} = H_{s,1} = 0 \) for every \( s \), so \( g_{s,0} = g_{s,1} = Id \).
Define $\{G_{s,t}\}_{t \in [0,1]}$ to be the Hamiltonian function that generates $g_{s,t}$ for a fixed $s$ with the time variable $t$, normalized so that $G_{s,t}(0) = 0$. Denote $g_t := g_{1,t}$. Note that $g_t|_{B_1} = A_t|_{B_1}$ for each $t$, because $a|_{B_2} = 1$ and $A_tB_1 \subset B_2$. Note also that $g_{0,t} = Id$ because it is the time-0 map of the flow with the time variable $s$, so we get that $G_{0,t} = 0$

for each $t \in [0,1]$.

**Claim:** For each $\epsilon > 0$ there is $\delta > 0$ such that if $B_3$ is with radius less than $\delta$, then

$$\left| \int_0^1 \int_{\mathbb{R}^n} G_{1,t} \omega_0^n \ dt \right| < \epsilon,$$

where $\omega_0$ is the standard symplectic form on $\mathbb{R}^{2n}$.

**Proof:** The Hamiltonian functions $G_{s,t}$ generates $g_{s,t}$ with time $t$, and $a \cdot H_{s,t}$ generates $g_{s,t}$ with time $s$. We can use a known formula (see [5, section 6.1]) and get that

$$\frac{\partial G_{s,t}}{\partial s} = a \cdot \frac{\partial H_{s,t}}{\partial t} - \{G_{s,t}, a \cdot H_{s,t}\}.$$ 

Note that $G_{0,t} = 0$, so if we integrate over $s$ we will get

$$G_{1,t} = \int_0^1 a \cdot \frac{\partial H_{s,t}}{\partial t} - \{G_{s,t}, a \cdot H_{s,t}\} ds.$$ 

Note that $G_{1,t} = 0$ outside $B_3$. Multiply with $\omega_0^n$ and integrate over $\mathbb{R}^{2n}$ to get

$$\int_{\mathbb{R}^{2n}} G_{1,t} \omega_0^n = \int_{B_3} a \cdot \int_0^1 \frac{\partial H_{s,t}}{\partial t} ds \omega_0^n - \int_0^1 ds \int_{B_3} \{G_{s,t}, a \cdot H_{s,t}\} \omega_0^n.$$ 

For any two smooth functions $F_1, F_2$, the form $\{F_1, F_2\}_0^n$ is exact, so from this we get that

$$\int_0^1 ds \int_{B_3} \{G_{s,t}, a \cdot H_{s,t}\} \omega_0^n = 0.$$ 

Now we get that

$$\left| \int_{\mathbb{R}^{2n}} G_{1,t} \omega_0^n \right| = \left| \int_{B_3} a \cdot \int_0^1 \frac{\partial H_{s,t}}{\partial t} ds \omega_0^n \right| \leq \int_{B_3} \int_0^1 \left| \frac{\partial H_{s,t}}{\partial t} \right| ds \omega_0^n.$$ 

We know that $\frac{\partial H_{s,t}}{\partial t}$ is a smooth function, so it is bounded on $B \times I \times I$. Hence for every $t, s \in [0,1]$ and for every $\delta$ such that $B_3 \subset B$, we have that $\left| \frac{\partial H_{s,t}}{\partial t} \right| < K$ for some $K > 0$. We get that

$$\int_{B_3} \int_0^1 \left| \frac{\partial H_{s,t}}{\partial t} \right| ds \omega_0^n \leq K \cdot \int_{B_3} \omega_0^n.$$ 

This means that we can choose $\delta$ such that

$$\int_{B_3} \int_0^1 \left| \frac{\partial H_{s,t}}{\partial t} \right| ds \omega_0^n < \epsilon.$$ 

If we integrate over $t$, we will get that

$$\left| \int_0^1 \int_{\mathbb{R}^{2n}} G_{1,t} dt \right| \leq \int_0^1 \int_{B_3} \int_0^1 \left| \frac{\partial H_{s,t}}{\partial t} \right| ds \omega_0^n \ dt < \epsilon.$$
This proves the claim.

We get that \( \{ g_t \}_{t \in [0,1]} \) is supported in \( B_3 \subset B \), \( g_t|_{B_3} = A_t|_{B_3} \) and for any \( \epsilon > 0 \) we can choose the radius of \( B_3 \) so that \( |\text{Cal}(\{g_t\}_{t \in [0,1]}))| < \epsilon \), so we can define \( B' := B_3 \). Note also that the loop \( \{ g_t \}_{t \in [0,1]} \) is homotopic to the identity with the homotopy \( \{ g_{s,t} \}_{s \in [0,1]} \).

This proves the lemma.

Proof of Theorem 2.2. Assume that \( B_2 \) is an equivariant Darboux ball around \( z_0 \) (see [1], section 3.1). We choose the Darboux coordinates on \( B_2 \) such that \( z_0 \) is identified with 0 and the path \( f_t|_{B_2} \) is identified with the path \( A_t \in \text{Symp}(2n) \). From the fact that \( z_0 \) is a Maslov-zero fixed point, we know that the loop \( \{ A_t \}_{t \in [0,1]} \) has Maslov index 0, and hence it is a contractible loop in \( \text{Sp}(2n) \). Normalize \( F \) so that \( F(z_0) = 0 \) and \( \int_X F\omega^n \neq 0 \). Set

\[
\epsilon = \frac{\int_X F\omega^n}{2}.
\]

Use Lemma 3.1 to define the contractible loop \( \{ g_t \}_{t \in [0,1]} \) and the ball \( B_1 \) such that for each \( t \in [0,1] \), \( g_t \) is supported in \( B_2 \), \( g_t|_{B_1} = A_t|_{B_1} \), and

\[ |\text{Cal}(\{g_t\}_{t \in [0,1]}))| < \epsilon. \]

Define

\[ h_t := g_t^{-1} \circ f_t. \]

From the fact that \( \{ g_t \}_{t \in [0,1]} \) is a contractible loop we know that \( \{ h_t \}_{t \in [0,1]} \) is homotopic to \( \{ f_t \}_{t \in [0,1]} \) in \( \text{Ham}(X) \). Note also that

\[ h_t|_{B_1} = Id. \]

Define \( H_t \) as the Hamiltonian function generating \( h_t \):

\[ H_t := (F - G_t) \circ g_t. \]

Note that \( G_t|_{B_1} = F|_{B_1} \). Hence we get that \( H_t|_{B_1} = 0 \). Choose a ball \( \overline{B} \subset B_1 \). The flow \( h_t \) is defined on \( X \setminus \overline{B} \) and \( H_t|_{B_1 \setminus \overline{B}} = 0 \) so we get that \( H_t \) is compactly supported on \( X \setminus \overline{B} \). From the definition of the loop \( \{ g_t \}_{t \in [0,1]} \) we have that

\[
\left| \int_0^1 \int_X H_t \omega^n \, dt - \int_X F\omega^n \right| = \left| \int_0^1 \int_X G_t \omega^n \, dt \right| = |\text{Cal}(\{g_t\}_{t \in [0,1]}))| < \epsilon = \frac{\int_X F\omega^n}{2}.
\]

From this and from the fact that \( \int_X F\omega^n \neq 0 \) we get that \( \int_0^1 \int_X H_t \omega^n \, dt \neq 0 \). However,

\[
\int_0^1 \int_X H_t \omega^n \, dt = \int_0^1 \int_{X \setminus \overline{B}} H_t \omega^n \, dt = \text{Cal}(\{h_t\}_{t \in [0,1]})).
\]

This completes the proof.

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