1 Introduction

This is a continuation of papers [16] and [18]. We have studied the effect of each elementary birational map in the minimal model program to the derived categories in the case of toric pairs. In this paper we consider problems of more global nature. We add some cases to the list of affirmative answers to the derived McKay correspondence conjecture and its generalization, the “$K$ implies $D$ conjecture”.

First we remark that the results in [16] and [18] are easily extended to the relative situation (Theorem 1). Then we remark that the derived McKay correspondence holds for any abelian quotient singularity in the following sense; the derived category defined by a finite abelian group contained in a general linear group is semi-orthogonally decomposed into derived categories of a relative minimal model and subvarieties of the quotient space (Theorem 3). As an application, we prove that such derived McKay correspondence also holds for arbitrary quotient singularities in dimension 2 (Theorem 4).

Next we prove that any proper birational morphism between toric pairs which is a $K$-equivalence is decomposed into a sequence of flops (Theorem 5). This is a generalization, in the toric case, of a theorem in [17], which states that any birational morphism between minimal models is decomposed into a sequence of flops. We note that the latter case is easier because the log canonical divisor stays at the bottom and cannot be decreased when the models are minimal, while it is more difficult to preserve the level of the log canonical divisor in the former case.

Now we state the theorems in details. We refer the terminology to the next section.
The following is the reformulation of the results of [16] and [18] to the relative case:

**Theorem 1.** Let \((X, B)\) and \((Y, C)\) be \(\mathbb{Q}\)-factorial toric pairs whose coefficients belong to the standard set \(\{1 - 1/m \mid m \in \mathbb{N}\}\), let \(\tilde{X}\) and \(\tilde{Y}\) be smooth Deligne-Mumford stacks associated to the pairs \((X, B)\) and \((Y, C)\), respectively, and let \(f : X \to Y\) be a toric rational map. Assume one of the following conditions:

(a) \(f\) is the identity morphism, and \(B \geq C\).
(b) \(f\) is a flip, \(C = f_* B\), and \(K_X + B \geq K_Y + C\).
(c) \(f\) is a divisorial contraction, \(C = f_* B\), and \(K_X + B \geq K_Y + C\).
(d) \(f\) is a divisorial contraction, \(C = f_* B\), and \(K_X + B \leq K_Y + C\).
(e) \(f\) is a Mori fiber space and \(C\) is determined by \((X, B)\) and \(f\) as explained in [16].

Then the following hold:

(a), (b), (c): There are toric closed subvarieties \(Z_i (1 \leq i \leq l)\) of \(Y\) such that \(Z_i \neq Y\) for some \(l \geq 0\), and fully faithful functors \(\Phi : D^b(\text{coh}(\tilde{X})) \to D^b(\text{coh}(\tilde{Y}))\) and \(\Psi_i : D^b(\text{coh}(\tilde{Z}_i)) \to D^b(\text{coh}(\tilde{X}))\) for the smooth Deligne-Mumford stacks \(\tilde{Z}_i\) associated to the \(Z_i\) such that there is a semi-orthogonal decomposition of triangulated categories

\[
D^b(\text{coh}(\tilde{X})) = (\Psi_1(D^b(\text{coh}(\tilde{Z}_1))), \ldots, \Psi_l(D^b(\text{coh}(\tilde{Z}_l))), \Phi(D^b(\text{coh}(\tilde{Y}))))
\]

(d): There are \(Z_i \subset Y (1 \leq i \leq l)\) as in (c), and fully faithful functors \(\Phi : D^b(\text{coh}(\tilde{X})) \to D^b(\text{coh}(\tilde{Y}))\) and \(\Psi_i : D^b(\text{coh}(\tilde{Z}_i)) \to D^b(\text{coh}(\tilde{Y}))\) such that there is a semi-orthogonal decomposition

\[
D^b(\text{coh}(\tilde{Y})) = (\Psi_1(D^b(\text{coh}(\tilde{Z}_1))), \ldots, \Psi_l(D^b(\text{coh}(\tilde{Z}_l))), \Phi(D^b(\text{coh}(\tilde{X}))))
\]

(e): There are fully faithful functors \(\Phi_j : D^b(\text{coh}(\tilde{Y})) \to D^b(\text{coh}(\tilde{X}))\) \((1 \leq j \leq m)\) for some \(m \geq 2\) and a semi-orthogonal decomposition

\[
D^b(\text{coh}(\tilde{X})) = (\Phi_1(D^b(\text{coh}(\tilde{Y}))), \ldots, \Phi_m(D^b(\text{coh}(\tilde{Y}))))
\]

Moreover if \(K_X + B = K_Y + C\) in the cases (b) or (c), then \(\Phi\) is an equivalence.

We note that, when \(f\) is a divisorial contraction, the direction of the inclusion of the derived categories is unrelated to the direction of the morphism, but is the same as the direction of the inequality of the canonical divisors.
If $X$ and $Y$ are smooth and $f$ is a blowing up of a smooth center in the case (b), or if $X$ and $Y$ are smooth and $f$ is a standard flip in the case (c), then it is a theorem by Bondal-Orlov ([1]). The derived equivalence in the case of flops (case (c) with $K_X + B = K_Y + C$) was proved for general (non-toric) case under the additional assumptions as follows: if dim $X = 3$ and $X$ is smooth by Bridgeland ([5]), if dim $X = 3$ and $X$ has only Gorenstein terminal singularities by Chen ([8]) and Van den Bergh ([20]), if dim $X = 3$ and $X$ has only terminal singularities by [14], and if $X$ is a holomorphic simplectic manifold by Kaledin ([11]). There are also interesting results concerning the last case by Cautis ([7]) by using the categorification of linear group actions, and by Donovan-Segal ([9]), Ballard-Favero-Katzarkov ([1]) and Halpern-Leistner ([10]) by using the theory of variations of GIT quotients.

As a corollary we obtain

**Corollary 2.** Let $(X, B)$ be a $Q$-factorial toric pair whose coefficients belong to the standard set $\{1 - 1/m \mid m \in \mathbb{N}\}$. Assume that there is a projective toric morphism $f : X \to Z$ to another $Q$-factorial toric variety. Then there exist toric closed subvarieties $Z_i$ ($1 \leq i \leq m$) of $Z$ for some $m \geq 1$ such that there are fully faithful functors $\Phi_i : D^b(\text{coh}(\tilde{Z}_i)) \to D^b(\text{coh}(X))$ with a semi-orthogonal decomposition

$$D^b(\text{coh}(X)) \cong \langle \Phi_1(D^b(\text{coh}(\tilde{Z}_1))), \ldots, \Phi_m(D^b(\text{coh}(\tilde{Z}_m))) \rangle$$

where $\tilde{X}$ and the $\tilde{Z}_i$ are smooth Deligne-Mumford stacks associated to $(X, B)$ and the $Z_i$.

In other words, one can say that the derived category $D^b(\text{coh}\tilde{X})$ is generated by a relative exceptional collection. The proofs of the above statements are the same as in [15], [16] and [18] except Theorem 1(a).

The following is the derived McKay correspondence for finite abelian groups:

**Theorem 3.** Let $G \subset GL(n, \mathbb{C})$ be a finite abelian subgroup acting naturally on an affine space $A = \mathbb{C}^n$, and let $X = A/G$ be the quotient space. Let $f : Y \to X$ be a $Q$-factorial terminal relative minimal model of $X$. Then there exist toric closed subvarieties $Z_i$ ($1 \leq i \leq m$) of $X$ with $Z_i \neq X$ for some $m \geq 0$, with possible repetitions, such that there is a semi-orthogonal decomposition

$$D^b(\text{coh}([A/G])) \cong \langle D^b(\text{coh}(\tilde{Z}_1)), \ldots, D^b(\text{coh}(\tilde{Z}_m)), D^b(\text{coh}(\tilde{Y})) \rangle$$
where $\tilde{Y}$ and the $\tilde{Z}_i$ are smooth Deligne-Mumford stacks associated to $Y$ and the $Z_i$, respectively. Moreover if $G \subset SL(n, \mathbb{C})$, then $m = 0$.

In the case of dimension 2, we do not need the assumption that the group is abelian:

**Theorem 4.** Let $G \subset GL(2, \mathbb{C})$ be a finite subgroup acting naturally on an affine space $A = \mathbb{C}^2$, let $X = A/G$, and let $f : Y \to X$ be the minimal resolution. Then there exist smooth closed subvarieties $Z_i$ $(1 \leq i \leq m)$ of $X$ with $Z_i \neq X$ for some $m \geq 0$, with possible repetitions, such that there is a semi-orthogonal decomposition

$$D^b(\text{coh}(A/G)) \cong \langle D^b(\text{coh}(Z_1)), \ldots, D^b(\text{coh}(Z_m)), D^b(\text{coh}(Y)) \rangle.$$ Moreover if there are no quasi-reflections in $G$, the elements whose invariant subspaces are of codimension 1, then $\dim Z_i = 0$ for all $i$. Thus the semi-orthogonal complement of $D^b(\text{coh}(Y))$ in $D^b(\text{coh}(A/G))$ is generated by an exceptional collection in this case.

The derived McKay correspondence was already proved for finite subgroups of $SL(2, \mathbb{C})$, $SL(3, \mathbb{C})$ (Bridgeland-King-Reid [6]), and $Sp(2n, \mathbb{C})$ (Bezrukavnikov-Kaledin [2]).

Finally, we prove the following decomposition theorem for $K$-equivalent toric birational map:

**Theorem 5.** Let $f : X \dashrightarrow Y$ be a toric birational map between projective $\mathbb{Q}$-factorial toric varieties which is an isomorphism in codimension 1, and let $B$ be a toric $\mathbb{R}$-divisor on $X$ whose coefficients belong to the interval $(0,1)$, and let $f_*B = C$. Assume that $K_X + B = K_Y + C$. Then the birational map $f : (X, B) \dashrightarrow (Y, C)$ is decomposed into a sequence of flops.

As a corollary of Theorems 4 and 5, we obtain an affirmative answer to “$K$ implies $D$ conjecture” in the toric case:

**Corollary 6.** Assume additionally that the coefficients of $B$ belong to a set $\{1 - 1/m \mid m \in \mathbb{Z}_{>0}\}$. Let $\tilde{X}$ and $\tilde{Y}$ be the smooth Deligne-Mumford stacks associated to the pairs $(X, B)$ and $(Y, C)$, respectively. Then there is an equivalence of triangulated categories $D^b(\text{coh}(\tilde{X})) \cong D^b(\text{coh}(\tilde{Y}))$. 

4
2 Preliminaries

We fix the terminology in this section. A pair \((X, B)\) consisting of a normal algebraic variety and an \(\mathbb{R}\)-divisor is said to be KLT (resp. terminal) if there is a projective birational morphism \(p : Z \to X\) from a smooth variety with an \(\mathbb{R}\)-divisor \(D\) whose support is a normal crossing divisor such that an equality \(p^*(K_X + B) = K_Z + D\) holds and all the coefficients of \(D\) are smaller than 1 (resp. all the coefficients of \(D - p_*^{-1}B\) are smaller than 0), where the canonical divisors \(K_X\) and \(K_Z\) are defined by using the same rational differential forms. It is called \(\mathbb{Q}\)-factorial if any prime divisor on \(X\) is a \(\mathbb{Q}\)-Cartier divisor.

A birational map \(h : X \dashrightarrow Y\) between varieties is said to be proper if there exists a third variety \(Z\) with proper birational morphisms \(f : Z \to X\) and \(g : Z \to Y\) such that \(g = h \circ f\).

Let \(f : (X, B) \dashrightarrow (Y, C)\) be a proper birational map between KLT pairs. We say that there is an equality of log canonical divisors \(K_X + B = K_Y + C\) (resp. an inequality \(K_X + B \geq K_Y + C\)) if \(p^*(K_X + B) = q^*(K_Y + C)\) (resp. \(p^*(K_X + B) \geq q^*(K_Y + C)\)) as an inequality (resp. an equality) of \(\mathbb{R}\)-divisors for some \(p, q\) as above. We note that such equality or inequality can be defined only when the birational map \(f\) is fixed, and the definition is independent of the choice of \(p, q\). Such an equality (resp. an inequality) is called a \(K\)-equivalence (resp. \(K\)-inequality). \(f\) is also said to crepant if it is a \(K\)-equivalence.

A proper birational map \(f : X \dashrightarrow Y\) is said to be isomorphic in codimension 1 (resp. surjective in codimension 1) if it induces a bijection (resp. surjection) between the sets of prime divisors.

A flop (resp. flip) is a proper birational map \(f : (X, B) \dashrightarrow (Y, C)\) between \(\mathbb{Q}\)-factorial KLT pairs satisfying the following conditions: \(f_*B = C\), \(K_X + B = K_Y + C\) (resp. \(K_X + B > K_Y + C\)), and there exist projective birational morphisms \(s : X \to W\) and \(t : Y \to W\) to a normal variety which are isomorphisms in codimension 1 and the relative Picard numbers \(\rho(X/W)\) and \(\rho(Y/W)\) are equal to 1.

A divisorial contraction is a projective birational morphism \(f : (X, B) \to (Y, C)\) between \(\mathbb{Q}\)-factorial KLT pairs satisfying the following conditions: \(f_*B = C\), and the exceptional locus of \(f\) consists of a single prime divisor. We have always \(\rho(X/Y) = 1\), and we have either \(K_X + B = K_Y + C\), \(K_X + B > K_Y + C\), or \(K_X + B < K_Y + C\). The inverse birational map \(f^{-1} : (Y, C) \dashrightarrow (X, B)\) of a divisorial contraction is called a divisorial extraction.

A Mori fiber space is a projective morphism \(f : (X, B) \to Y\) from a
A *Q*-factorial KLT pair to a normal variety satisfying the following conditions:

\[ \rho(X/Y) = 1, \quad -(K_X + B) \text{ is ample}, \quad \text{and} \quad \dim X > \dim Y. \]

For a *Q*-factorial KLT pair \((X, B)\), there exists a sequence of crepant divisorial contractions

\[ (Y, C) = (X_m, B_m) \to \cdots \to (X_1, B_1) \to (X_0, B_0) = (X, B) \]

such that \((Y, C)\) is terminal and *Q*-factorial [3]. The composition \(f : (Y, C) \to (X, B)\) is called a *Q*-factorial crepant terminalization of the pair \((X, B)\). We note that even if \(B\) has standard coefficients, \(C\) may not be so.

Let \(f : (Y, C) \to (X, B)\) be a projective birational morphism from a *Q*-factorial terminal pair to a KLT pair. It is called a *Q*-factorial terminal relative minimal model if \(C = f^{-1}_* B\) and \(K_Y + C\) is relatively nef over \(X\). We note that \(f\) is not necessarily crepant. The existence of such a model is also proved in [3].

A rational number is said to be standard if it is of the form \(1 - 1/m\) for a positive integer \(m\). A KLT pair \((X, B)\) is said to be of quotient type, if there is a quasi-finite surjective morphism \(\pi' : X' \to X\) from a smooth variety \(X'\), which is not necessarily irreducible, such that \(K_{X'} = (\pi')^*(K_X + B)\). In this case \(B\) has only standard coefficients, and \(X\) is automatically *Q*-factorial. If the pair \((X, B)\) is toric, \(X\) is *Q*-factorial, and if \(B\) has standard coefficients, then it is of quotient type.

There is a smooth Deligne-Mumford stack \(\tilde{X}\) associated to the pair \((X, B)\) of quotient type in such a way that a sheaf \(\mathcal{F}\) on \(\tilde{X}\) is nothing but a sheaf \(\mathcal{F}'\) on \(X'\) with an isomorphism \(p_i^* \mathcal{F}' \cong p_i^* \mathcal{F}\) satisfying the cocycle condition, where \(p_i : X' \times_X X' \to X'\) (\(i = 1, 2\)) are projections. In this case, there is a finite birational morphism \(\pi : \tilde{X} \to X\) such that \(\pi^*(K_X + B) = K_{\tilde{X}}\).

For an algebraic variety or a Deligne-Mumford stack \(X\), let \(D^b(\text{coh}(X))\) denotes the bounded derived category of coherent sheaves on \(X\). There are variants of this notation; if a finite group \(G\) acts on \(X\), then \(D^b(\text{coh}^G(X))\) denotes the bounded derived category of \(G\)-equivariant coherent sheaves on \(X\). Thus \(D^b(\text{coh}^G(X)) \cong D^b(\text{coh}([X/G]))\).

A triangulated category \(\mathcal{T}\) is said to have a semi-orthogonal decomposition \(\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_m \rangle\) by triangulated full subcategories \(\mathcal{T}_i\) if the following conditions are satisfied: (a) \(\text{Hom}(a, b) = 0\) for \(a \in \mathcal{T}_i\) and \(b \in \mathcal{T}_j\) if \(i > j\), and (b) for any object \(a \in \mathcal{T}\) there is a sequence of objects \(a_1 \in \mathcal{T}_1\) and \(b_i \in \mathcal{T}_i\) for \(1 \leq i \leq m\) such that \(a = a_1\) and there are distinguished triangles \(a_{i+1} \to a_i \to b_i\), where \(a_{m+1} = 0\). In particular, if the \(\mathcal{T}_i\) are equivalent to \(D^b(\text{Spec } k)\) for all \(i\), then \(\mathcal{T}\) is said to have a full exceptional collection.
A toric pair \((X, B)\) consists of a toric variety and an \(\mathbb{R}\)-divisor whose irreducible components are invariant under the torus action. \(X\) is \(\mathbb{Q}\)-factorial if and only if the corresponding fan is simplicial. In this case it has only abelian quotient singularities. Conversely, the quotient of an affine space by any finite abelian group is a \(\mathbb{Q}\)-factorial toric variety. Any \(\mathbb{Q}\)-factorial crepant terminalization and \(\mathbb{Q}\)-factorial terminal relative minimal model of a toric pair is toric.

For a pair of quotient type, we can define a derived category whose homological dimension is finite. Therefore we would like to ask the following:

**Question 7.** Let \(f : (X, B) \dasharrow (Y, C)\) be a flop or a flip. If \((X, B)\) is of quotient type, then is \((Y, C)\) too?

We note that a divisorial contraction of a smooth 3-fold may produce a singularity of non-quotient type even if it is crepant. We note also that even if \(X\) is smooth and \(f\) is a flop, \(Y\) is not necessarily smooth, but we have still the derived equivalence in some examples when we consider the associated Deligne-Mumford stacks (cf. [13]).

The following lemma is standard:

**Lemma 8.** Let \(f : (X, B) \dasharrow (Y, C)\) be a proper birational map between \(\mathbb{Q}\)-factorial terminal pairs, and assume that \(K_X + B \geq K_Y + C\). Then \(f\) is surjective in codimension 1, and \(f_*B \geq C\). Moreover, if \(K_X + B = K_Y + C\), then \(f\) is an isomorphism in codimension 1.

**Proof.** Suppose that there is a prime divisor \(C_1\) on \(Y\) which is not the strict transform of a prime divisor on \(X\). Let \(c_1 \geq 0\) be the coefficient of \(C_1\) in \(C\). Let \(p : Z \to X\) and \(q : Z \to Y\) be projective birational morphisms from a smooth variety, and let \(D_1\) be the strict transform of \(C_1\). Since we have an equality \(p^*(K_X + B) \geq q^*(K_Y + C)\), the coefficient of \(D_1\) in \(p^*(K_X + B) - K_Z\) should be non-negative, a contradiction to the fact that \((X, B)\) is terminal. Therefore \(f\) is surjective in codimension 1. If we apply the first assertion to \(f\) and \(f^{-1}\), then we obtain the second assertion. \(\square\)

### 3 Decomposition theorem for toric pairs

We prove Theorem 5 in this section.

First we prove a condition for the decomposition theorem that is also valid for the non-toric case. Let \(f : (X, B) \dasharrow (Y, C)\) be a birational map between
projective $\mathbb{Q}$-factorial KLT pairs which is an isomorphic in codimension 1 and such that $K_X + B = K_Y + C$. Let $p : Z \to X$ and $q : Z \to Y$ be projective birational morphisms such that $f \circ p = q$. Let $\{l_\lambda\}$ be a set of curves on $X$ whose classes generate the cone of curves $\overline{NE}(X)$ as a closed convex cone. Let $L_Y$ be an ample and effective divisor on $Y$ such that $L_Y - (K_Y + C)$ is also ample, and let $L_X = f^{-1}_* L_Y$ be its strict transform.

Lemma 9. Assume that, if $(L_X, l_\lambda) < 0$ for some $\lambda$, then $((K_X + B), l_\lambda) = 0$. Then $f$ is decomposed into a sequence of flops.

Proof. We shall find suitable extremal rays in order to find a path from $(X, B)$ to $(Y, C)$.

Assume first that $L_X$ is nef. We claim that $L_X - (K_X + B)$ is also nef. Otherwise, there exists a curve $l_\lambda$ such that $((L_X - (K_X + B)), l_\lambda) < 0$. By the assumption, it follows that $((K_X + B), l_\lambda) = 0$. Therefore $(L_X, l_\lambda) < 0$, a contradiction.

By the base point free theorem ([12]), $L_X$ is semiample. Since $f$ is an isomorphism in codimension 1, the associated morphism coincides with $f$. Since both $X$ and $Y$ are $\mathbb{Q}$-factorial, we conclude that $f$ is an isomorphism.

Next assume that $L_X$ is not nef. We take a small positive number $\epsilon$ such that the pair $(X, B + \epsilon L_X)$ is KLT. We would like to run a minimal model program for this pair in order to move from the pair $(X, B)$ to the pair $(Y, C)$. Here we have to note that there may be extremal rays of $(X, B + \epsilon L_X)$ which do not come from the negativity of $L_X$.

Since $L_X$ is not nef, there exists a curve $l_\lambda$ such that $(L_X, l_\lambda) < 0$. By the assumption, $((K_X + B), l_\lambda) = 0$ for such curve. Therefore $K_X + B + \epsilon L_X$ is not nef.

The part of the closed cone of curves $\overline{NE}(X)$ where $L_X$ is negative is contained in the part where $K_X + B + \epsilon L_X$ is negative. Thus there exists an extremal ray $R$ of $\overline{NE}(X)$ on which $L_X$ is negative. Since $K_X + B$ is numerically trivial on $R$, the flip of $R$ for the pair $(X, B + \epsilon L_X)$ is a flop for $(X, B)$. Since $B + \epsilon L_X$ is big, this process terminates by [3], and $f$ is decomposed into the flops. \hfill \Box

It is important note that $\{l_\lambda\}_{\lambda \in \Lambda}$ is not necessarily the set of all curves.

Now we consider the toric case. We take $p : Z \to X$, $q : Z \to Y$ and $L_Y$ to be also toric. Let $\Delta_X$ and $\Delta_Y$ be the simplicial fans corresponding to $X$ and $Y$, respectively, in the same vector space $N_\mathbb{R}$. By assumption, the sets
of primitive vectors on the edges of $\Delta_X$ and $\Delta_Y$ coincide. Since $\overline{NE}(X)$ is generated by toric curves, Theorem 5 is a consequence of the following:

**Lemma 10.** (1) Let $w$ be a wall in $\Delta_X$ and let $l_w$ be the corresponding curve on $X$. Assume that $(L_X, l_w) \leq 0$. Then $w$ is not a wall in $\Delta_Y$.

(2) Let $w$ be a wall in $\Delta_X$ which does not belong to $\Delta_Y$. Then $((K_X + B), l_w) = 0$.

**Proof.** (1) Since $f$ is an isomorphism in codimension 1, the set of primitive vectors $\{v_i\}$ on the edges of $\Delta_X$ and $\Delta_Y$ coincide. Let $g_X$ and $g_Y$ be functions on $\mathbb{R}$ corresponding to the divisors $L_X$ and $L_Y$, respectively. $g_X$ and $g_Y$ have the same values at the $v_i$, and are linear inside each simplex belonging to $\Delta_X$ and $\Delta_Y$, respectively. Since $L_Y$ is ample, $g_Y$ is convex along walls of $\Delta_Y$.

Assuming that $w$ is also a wall in $\Delta_Y$, we shall derive a contradiction. Under this assumption, the values of $g_X$ and $g_Y$ are equal on $w$.

Let $v_1, v_2$ be two vertexes adjacent to $w$ in $\Delta_X$. We take a general point $P$ on $w$ such that the line segments $I_i$ connecting $P$ to $v_i$ for $i = 1, 2$ pass only the interiors of simplexes and walls of $\Delta_Y$. Then the function $g_Y$ is convex along the $I_i$.

Since $g_X$ is linear on the $I_i$ and coincides with $g_Y$ at the end points, we have an inequality $g_X \geq g_Y$ on $I_1 \cup I_2$. On the other hand, $g_X = g_Y$ on $w$. Since $g_Y$ is convex along $w$, we conclude that $g_X$ is also convex along $w$, hence $(L_X, l_w) > 0$, a contradiction.

(2) Let $D_i$ be the the prime divisors on $X$ corresponding the $v_i$. We write $B = \sum d_i D_i$, and let $h$ be the function on the vector space $\mathbb{R}$ which takes values $1 - d_i$ at the $v_i$ and linear inside each simplex of $\Delta_X$. Since $K_X + B = K_Y + C$, $h$ is also linear inside each simplex of $\Delta_Y$. There is a point inside $w$ which is an interior point of a simplex in $\Delta_Y$. Then $h$ is linear across $w$. Therefore $((K_X + B), l_w) = 0$. \qed

**Remark 11.** The relative version is similarly proved. We need to assume that $X$ and $Y$ are projective and toric over a toric variety $S$ and $f$ is a toric birational map over $S$. We only consider curves relative over $S$, i.e., those mapped to single points on $S$.

If dim $X = 3$, then the condition of Lemma 9 is easily verified even in the non-toric case:
Proposition 12. Assume that \( \dim X = 3 \). Then any \( K \)-equivalent birational map between projective \( \mathbb{Q} \)-factorial KLT pairs \( f : (X, B) \to (Y, C) \) which is an isomorphism in codimension 1 is decomposed into a sequence of flops.

Proof. Assume that \( (L_X, l) \leq 0 \) for a curve \( l \). Since \( \dim X = 3 \), there are only finitely many such curves \( l \). Then \( l \) is the image of a fiber of \( q \) by \( p \). Therefore we have \( ((K_X + B), l) = 0 \). \( \square \)

4 Derived McKay correspondence

We prove Theorems 3 and 4 as well as Theorem 1 (a) in this section.

proof of Theorem 3. Let \( B \) be a \( \mathbb{Q} \)-divisor on \( X \) such that \( p^*(K_X + B) = K_A \). Since the action of \( G \) is diagonalizable, the pair \( (X, B) \) is toric. We have \( D^b(\text{coh}([A/G])) \cong D^b(\text{coh}(\tilde{X})) \) for the smooth Deligne-Mumford stack \( \tilde{X} \) associated to the pair \( (X, B) \).

There exists a toric relative minimal model of \( X \). Any other relative minimal model is obtained by a sequence of flops from the toric relative minimal model by [17]. Since the extremal rays are toric, the sequence is automatically toric, hence any relative minimal model is again toric. By Theorem 1, it is sufficient to prove the theorem for a particular relative minimal model.

Since \( X \) is \( \mathbb{Q} \)-factorial and \( (X, B) \) is KLT, we can construct a sequence toric birational morphisms

\[
(X_t, B_t) \to (X_{t_1}, B_{t_1}) \to \cdots \to (X_0, B_0) = (X, B)
\]

which satisfy the following conditions:

(1) Each step \( f_i : (X_i, B_i) \to (X_{i-1}, B_{i-1}) \) is a divisorial contraction such that \( B_i = f_{i-1}^{-1}B_{i-1} \) and \( K_{X_i} + B_i \leq K_{X_{i-1}} + B_{i-1} \) for \( 1 \leq i \leq t \).

(2) The pair \( (X_t, B_t) \) is terminal.

Next we run a toric minimal model program for \( X_t \) over \( X \) to obtain a sequence of birational maps

\[
X_t = Y_0 \to Y_1 \to \cdots \to Y_s = Y
\]

consisting of divisorial contractions and flips such that \( K_Y \) is nef over \( X \). We have \( K_{Y_{j-1}} > K_{Y_j} \) for \( 1 \leq j \leq s \).

We apply Theorem 1 and Corollary 2 to each step of the above sequences to obtain our result. \( \square \)
proof of Theorem 4. Let $H$ be the normal subgroup of $G$ defined by $H = G \cap SL(2, \mathbb{C})$. The residue group is a cyclic group $\mathbb{Z}_r = G/H$ for $r = [G : H]$. By [6], a component of the Hilbert scheme of $H$-invariant subschemes of $A$ gives a crepant resolution $f : M \to A/H$, and there is a derived equivalence $\Phi : D^b(\text{coh}(M)) \to D^b(\text{coh}^H(A)) = D^b(\text{coh}([A/H]))$. More precisely, $M$ is the moduli space of closed subschemes $P \subset A$ which is $H$-invariant and such that length $O_P = \#H$. For a generic point $[P] \in M$, $P$ is a reduced subscheme, and corresponds to a generic orbit of $H$ in $A$. There exists a universal subscheme $W \subset M \times A$ with projections $p : W \to M$ and $q : W \to A$ such that the functor $\Phi = q_\ast p^*$ gives the equivalence.

For $[P] \in M$ and $g \in G$, $gP$ is also $H$-invariant, since $H$ is a normal subgroup. Hence $[gP] \in M$, and $G$ acts on $M$ such that its subgroup $H$ acts trivially. The equivariant derived category $D^b(\text{coh}^{\mathbb{Z}_r}(M))$ is identified with the category where the objects are $G$-equivariant complexes on which $H$ acts trivially, and the morphisms are $G$-equivariant morphisms.

$G$ acts on the product $M \times A$ diagonally and preserves the subscheme $W$. We claim that the functor $\Phi' = q_\ast p^* : D^b(\text{coh}^{\mathbb{Z}_r}(M)) \to D^b(\text{coh}^G(A))$ is an equivalence. Indeed if $a, b$ are $\mathbb{Z}_r$-equivariant objects, then the isomorphism $\text{Hom}_M(a, b) \cong \text{Hom}_A(\Phi(a), \Phi(b))^H$ implies an isomorphism

$$\text{Hom}_M(a, b)^{\mathbb{Z}_r} \cong (\text{Hom}_A(\Phi(a), \Phi(b))^H)^{\mathbb{Z}_r} = \text{Hom}_A(\Phi(a), \Phi(b))^G.$$ 

Let $g : M \to M/\mathbb{Z}_r$ be the map to the quotient space. We define a boundary divisor $B_{M/\mathbb{Z}_r}$ on $M/\mathbb{Z}_r$ by $g^\ast(K_{M/\mathbb{Z}_r} + B_{M/\mathbb{Z}_r}) = K_M$. Since $M$ is smooth, $M/\mathbb{Z}_r$ has only cyclic quotient singularities. Let $h : Y' \to M/\mathbb{Z}_r$ be the minimal resolution, and let $k : Y' \to Y$ be the contraction morphism to the minimal resolution of $A/G$. The former is a toroidal and decomposed into toroidal divisorial contractions. The latter is a composition of divisorial contractions of $(-1)$-curves, which are toroidal too.

The canonical divisors satisfy the following inequalities:

$$K_Y \leq K_{Y'} \leq K_{M/\mathbb{Z}_r} \leq K_{M/\mathbb{Z}_r} + B_{M/\mathbb{Z}_r} = K_{A/G}.$$ 

By Theorem 1 we obtain fully faithful embeddings

$$D^b(\text{coh}(\tilde{Y})) \to D^b(\text{coh}(\tilde{Y}')) \to D^b(\text{coh}([M/\mathbb{Z}_r])) = D^b(\text{coh}([A/G]))$$

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whose semi-orthogonal complements are generated by the derived categories of subvarieties of $X = \mathbb{A}/G$. \hfill \square

**proof of Theorem 7** (a). It is sufficient to consider the following situation: $X = Y$ is an affine toric variety corresponding to a simplicial cone $\sigma \subset N_{\mathbb{R}}$ generated by primitive vectors $v_1, \ldots, v_n \in N$. Let $B_i$ be prime divisors on $X$ corresponding to the $v_i$, and let $B = \sum (1 - 1/r_i)B_i$ and $C = \sum (1 - 1/s_i)B_i$, where $r_i > s_i$ and $r_i = s_i$ for $i = 2, \ldots, n$. Let $t_1 = \text{LCM}(r_1, s_1)$ and $t_i = r_i$ for $i = 2, \ldots, n$.

The stacks $\tilde{X}$, $\tilde{Y}$ and $\tilde{W}$ are defined by the sublattices $N_X$, $N_Y$ and $N_W$ of $N$ generated by the $r_iv_i$, $s_iv_i$ and $t_iv_i$, respectively. Let $\tilde{B}_i^X$, $\tilde{B}_i^Y$ and $\tilde{B}_i^W$ be prime divisors on $\tilde{X}$, $\tilde{Y}$ and $\tilde{W}$ over $B_i$, respectively. They correspond primitive vectors $r_iv_i$, $s_iv_i$ and $t_iv_i$, respectively. Let $p : \tilde{W} \to \tilde{X}$ and $q : \tilde{W} \to \tilde{Y}$ be the natural morphisms. Then the fully faithful functor $\Phi : D^b(\text{coh}(\tilde{Y})) \to D^b(\text{coh}(\tilde{X}))$ is constructed as $\Phi = p_*q^*$ in [15] Theorem 4.2 (1). We have

$$\Phi(\mathcal{O}_{\tilde{Y}}(k\tilde{B}_1^Y)) = \mathcal{O}_{\tilde{X}}(\sqcup kr_1/s_1 \downarrow \tilde{B}_1^X)$$

for $k = 1, \ldots, s_1 - 1$.

We consider the semi-orthogonal complement. It is generated by the sheaves $\mathcal{O}_{\tilde{B}_1^X}(l\tilde{B}_1^X)$ for $1 \leq l \leq r_1 - 1$ such that $l \neq \sqcup kr_1/s_1 \downarrow$ for any $k$. We have

$$\text{RHom}_{\tilde{X}}(\mathcal{O}_{\tilde{X}}(\sqcup kr_1/s_1 \downarrow \tilde{B}_1^X), \mathcal{O}_{\tilde{B}_1^X}(l\tilde{B}_1^X)) = 0$$

for such $k, l$, and

$$\text{RHom}_{\tilde{X}}(\mathcal{O}_{\tilde{B}_1^X}(l\tilde{B}_1^X), \mathcal{O}_{\tilde{B}_1^X}(l'\tilde{B}_1^X)) = 0$$

for such $l, l'$ with $l < l'$.

We define a toric boundary $D$ on $B_1$ from the pair $(X, B)$ as follows (cf. [16] §5). Let $\tilde{N} = N/\mathbb{Z}v_1$, and write $v_i + \mathbb{Z}v_1 = d_i\tilde{v}_i \in \tilde{N}$ for primitive vectors $\tilde{v}_i$ ($i = 2, \ldots, n$). Then define $D = \sum_{i=2}^{n}(1 - 1/d_i)D_i$ for $D_i = B_i \cap B_i'$. Let $\tilde{B}_1$ be the associated smooth Deligne-Mumford stack above $(B_1, D)$. Let $\tilde{B}_1'$ be the normalization of the fiber product $\tilde{B}_1 \times_X \tilde{X}$ with natural morphisms $p_1 : \tilde{B}_1' \to \tilde{B}_1$ and $p_2 : \tilde{B}_1' \to \tilde{X}$. Then as in [16], the semi-orthogonal complement is generated by subcategories

$$p_2p_1^*D^b(\text{coh}(\tilde{B}_1)) \otimes \mathcal{O}_{\tilde{X}}(l\tilde{B}_1^X)$$

$1 \leq l \leq r_1 - 1$ such that $l \neq \sqcup kr_1/s_1 \downarrow$ for any $k$. Moreover, there is no morphism between objects belonging to different $k$’s. Therefore we conclude the proof. \hfill \square
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Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Meguro, Tokyo, 153-8914, Japan
kawamata@ms.u-tokyo.ac.jp