Absence of backscattering at integrable impurities in
one-dimensional quantum many-body systems

Alexander Punnoose
Department of Physics, Indian Institute of Science, Bangalore, 560 012, India

Hans-Peter Eckle
LMPM, Département de Physique, Université François Rabelais, F-37200 Tours, France

Rudolf A. Römer
Institut für Theoretische Physik C, RWTH Aachen, Templergraben 55, D-52056 Aachen, Germany

(Version: December 18, 1995; printed September 22, 2018)

Abstract

We study interacting one dimensional (1D) quantum lattice gases with integrable impurities. These model Hamiltonians can be derived using the quantum inverse scattering method for inhomogeneous models and are by construction integrable. Absence of backscattering at the impurities is shown to be the characteristic feature of these disordered systems. The value of the effective carrier charge and the Sutherland-Shastry relation are derived for the half-filled XXX model and are shown to be independent of the impurity concentration and strength. For the half-filled XXZ model we show that there is no enhancement of the persistent currents for repulsive interactions. For attractive interactions we identify a crossover regime beyond which enhance-
ment of the currents is observed.
Much work has been done recently towards a deeper understanding of the effect of impurities in Fermi- and Luttinger liquids. Conformal field theories with boundaries have been used to study the multichannel Kondo problem [1]. Bosonization techniques and renormalization group methods have given new insights into the problem of potential scatterers in Luttinger liquids [2-4]. Also, the observed discrepancy between the experimental value of the persistent current in moderately disordered ensembles of quasi one-dimensional (1D) rings [5] and the theoretical predictions based on calculations in a disordered but non-interacting electron gas [6], have generated much interest in the interplay between interactions and impurities [7-11].

Besides the spin impurity model of Andrei and Johannesson [12], no other exactly solvable interacting quantum many-body problems with impurities have been studied. This is in retrospect somewhat surprising as the existence of models with site-dependent defects has been mentioned throughout the literature on the method of quantum inverse scattering (QISM) [13-17]. Recently, Bares [18] and, independently, Schmitteckert, Schwab and Eckern (SSE) [19] have used these remarks and explicitly investigated a $tJ$ model and an XXZ model with integrable defects, respectively.

In the XXZ model, SSE find the following surprising properties: (i) The energy spectrum is independent of the spatial distribution of impurities; (ii) there is no localisation of the ground state wave function. In contrast measurements of the transport properties in mesoscopic systems show a very sensitive dependence on the impurity configuration [21]. Further, it is well understood that in the thermodynamic limit in 1D a generic impurity drives the system from a metallic to an insulating phase even in the presence of weak interactions [2,3,22].

In the present work, we shall first introduce the (antiferromagnetic) XXX Heisenberg model as a simple example of a system with integrable impurities. We will then show that a complete physical understanding of the anomalous properties of this class of models can be got by studying the properties of a single particle scattering off a single impurity. The system is then threaded with an external flux $\Phi$. For the half-filled band, we establish the charge
of the effective carriers to be $-q/2$. The Sutherland-Shastry relation \[ D\chi = 1/2\pi \]
still holds in the presence of impurities. For the XXZ model with integrable impurities at half filling the charge stiffness $D$ shows enhancement of the persistent currents for attractive interactions in agreement with studies of XXZ models with potential impurities \cite{9}.

We construct our model from the $R$ and $L_n$ matrices of the Heisenberg XXX model \cite{14}, i.e. $R(\lambda) = (\lambda P + i)/(\lambda + i)$ and $L_n(\lambda) = \lambda + i\sum^3_{a=1} \sigma^a \otimes \sigma^a_n$. $P$ is a $4 \times 4$ permutation matrix and $\{\sigma^a\}$ are the Pauli matrices. These operators satisfy the local Yang-Baxter (YB) equation, i.e. $R(\lambda - \lambda') [L_n(\lambda) \otimes L_n(\lambda')] = [L_n(\lambda') \otimes L_n(\lambda)] R(\lambda - \lambda')$. The main idea that allows for the construction of models with integrable impurities is that the YB equation continues to be satisfied under an arbitrary \emph{local} shift in the spectral parameter $L_n(\lambda) \rightarrow L_n(\lambda + \nu_n)$.

The transfer matrix $T$ with local shifts $\{\nu_1, \cdots, \nu_N\}$ is given as $T(\lambda, \{\nu_n\}) = \prod^N_{n=1} L_n(\lambda + \nu_n)$. The Hamiltonian $H$ is then given as usual by the logarithmic derivative of the transfer matrix $T(\lambda, \{\nu_n\})$ evaluated at the special point $\lambda_0 = i/2$. Note that this construction is different from the one used by Andrei and Johannesson \cite{12} and does indeed yield inequivalent, albeit similar, Bethe Ansatz (BA) equations.

As is customary, we now Jordan-Wigner transform the spin model into a lattice gas of spinless fermions and impose periodic boundary conditions. The resulting Hamiltonian on a ring of size $N$ with $M$ spinless fermions and a single impurity at site $m$ with strength $\nu$ is given as $H = H_{XXX} + H_{m,\text{imp}}$, where $H_{XXX} = -\sum^N_{j=1} (c_j^\dagger c_j + h.c.) + 2 \sum^N_{j=1} \rho_j \rho_{j+1} + 2M$ is the usual Hamiltonian of the clean XXX model in units of the hopping energy. The operators $c_j^\dagger$ and $c_j$ create and annihilate fermions on site $j$ and $\rho_j = c_j^\dagger c_j$ is the density operator. The impurity interaction at site $m$ is $H_{m,\text{imp}} = H_{m,t} + H_{m,\rho} + H_{m,j}$ with

\begin{align}
H_{m,t} &= -(u + iw)[c_m^\dagger c_{m-1} + c_{m+1}^\dagger c_m + c_{m-1}^\dagger c_{m+1}] \\
&\quad + h.c., \quad (1a)
\end{align}

\begin{align}
H_{m,\rho} &= -2u[\rho_m - \rho_m(\rho_{m-1} + \rho_{m+1}) + \rho_{m-1}\rho_{m+1}], \quad (1b)
\end{align}

\begin{align}
H_{m,j} &= 2u(c_{m+1}^\dagger c_{m-1} + c_{m-1}^\dagger c_{m+1})\rho_m \\
&\quad + 2iw(c_m^\dagger c_{m-1} - c_{m-1}^\dagger c_m)\rho_{m+1}
\end{align}
\[ + 2iw(c_{m+1}^\dagger c_m - c_m^\dagger c_{m+1})\rho_{m-1}, \] (1c)

where the coupling constants as a function of \( \nu \) are \( u = -\nu^2/(1 + \nu^2) \) and \( w = -\nu/(1 + \nu^2) \).

In Eq. (1), we have three types of terms: (i) \( H_{m,t} \): Nearest and next nearest neighbour hopping matrix elements are modified involving the sites \( m \) and \( m \pm 1 \). (ii) \( H_{m,\rho} \): The onsite potential at site \( m \) and both nearest and next nearest neighbour density-density interaction terms are induced. (iii) \( H_{m,j} \): This term corresponds to a current density interaction. Thus we see that a local translation of the spectral parameter \( \lambda \rightarrow \lambda + \nu_m \) effectively involves the site \( m \) and its two neighbors \( m \pm 1 \) and in addition involves next nearest neighbour terms involving the sites \( m - 1 \) and \( m + 1 \). We note that the impurity terms involve qualitatively the same processes as in the \( tJ \) model with integrable impurity [18].

The Hamiltonian in the presence of more than one impurity can now easily be constructed: In case the impurities are well separated (\( |m - m'| > 1 \)) the Hamiltonian reduces to a sum over the isolated impurities i.e. \( \sum_{\alpha=1}^{F} H_{m(\alpha),\text{imp}} \) for \( F \) such well-separated impurities at positions \( m(\alpha), \alpha = 1, \ldots, F \). In case two impurities are on neighboring sites, say \( m \) and \( m + 1 \), the Hamiltonian will acquire similar terms involving the sites \( m - 1 \) to \( m + 2 \). As we will show later, this does not change any of the physics of these systems.

To understand the properties of integrable impurities, we first study the scattering of a single particle by a single impurity at site \( m \). The terms in the Hamiltonian involving only a single particle then are \( H_s = -\sum_i (c_i^\dagger c_i + \text{h.c.}) + H_t - 2u\rho_m + 2 \). Let \( \Psi_{\text{in}}(n) = e^{ikn} + R(k)e^{-ikn} \) for \( n < m \) be the incoming and \( \Psi_{\text{out}}(n) = T(k)e^{ikn} \) for \( n > m \) be the outgoing wave functions and \( R(k) \) and \( T(k) \) the reflection and transmission amplitudes, respectively. Then with \( \epsilon(k) = -2(\cos k + 1) \) the usual nearest-neighbor one-particle dispersion, we find that (i) \( R(k) = 0 \) for all momenta \( k \). Thus there is no backscattering of a particle due to the impurity. This is consistent with the understanding that the number of scattering channels in a BA-solvable Hamiltonian is conserved. In contrast, a generic impurity would give rise to a reflected and a transmitted wave thus doubling the number of scattering channels.

We remark that the parametrisation of \( u \) and \( w \) given in Eq. (1) is unique, i.e. it is the
only such parametrisation that will lead to \( R(k) = 0 \) for all \( k \) and \( \nu \). (ii) Unitarity of the scattering matrix requires \(|T|^2 + |R|^2 = 1\) and we consequently have \(|T(k)|^2 = 1\). Writing
\[
T(k) = e^{i\delta(k,\nu)},
\]
we then find the phase-shift of the particle wave function due to the presence of the impurity to be \( \delta(k,\nu) = -k + 2 \arctan(\tan(k/2) - 2\nu) \).

To study the problem with more than one particle, we use the BA equations in the presence of the impurities which are given for the XXX model by QISM as [14]
\[
Nk_j = 2\pi I_j - \sum_{k=1}^{M} \theta(k_j, k_k) - \sum_{\alpha=1}^{F} \delta(k_j, \nu_{m(\alpha)}),
\]
where \(\theta(k) = 2 \arctan[(\tan(k/2) - \tan(k'/2))/2]\) is the two-body phase shift. Note that although the total momentum \(P\) is no longer a good quantum number in the presence of impurities, the individual BA pseudo-momenta \(k\) still represent the eigenvalues of the BA equations and are thus good quantum numbers.

We now make the important observation that the only change in the BA equations w.r.t. the clean case is the additional phase shift \(\delta(k,\nu)\) which is identical to the forward scattering phase shift acquired by a single particle scattering off the impurity. This is true although the BA equations include many-body interactions and the interactions \(H_{m(\alpha),\rho}\) and \(H_{m(\alpha),j}\) induced by the impurities. Thus it is also irrelevant, if the impurities are well separated or not. For all configurations of the impurity the net effect will be the same phase shift \(\delta(k,\nu_{m})\) for each impurity of strength \(\nu_{m}\). Further, Eq. (2) is insensitive to the spatial distribution of the \(F\) impurities. This is a consequence of the absence of backscattering. Quantum interference of multiply backscattered waves would in general lead to a non trivial dependence of the energy spectrum on the impurity configuration. Note that, although the energy spectrum is insensitive to the impurity configuration the wave function is no longer given by the simple coordinate BA form \(\Psi(x_1,\ldots,x_M) = \sum_P A(P) \exp(i \sum_{j=1}^{M} x_j k_{Pj})\) with \(P\) a permutation of the \(M\) momenta \(k_j\). Presumably, one might calculate the wave function from the reference state of the QISM, which then by construction includes a site dependence.

We believe that these results are generic for all such integrable impurities constructed by QISM. This is implied by the very existence of the BA equations. Besides the XXX
model, the $tJ$ model of Bares [18] and the XXZ ring of SSE [20], this should therefore also apply to, e.g., the Hubbard model with integrable impurities constructed from two coupled inhomogeneous six-vertex models [15].

To study the transport properties of these integrable impurity models, we thread the system with an external flux $\Phi$, which accelerates all the particles around the ring. The variation of the energy spectrum as a function of flux is made nontrivial for an interacting quantum system only in the absence of Galilean invariance. In the present problem this invariance is destroyed both due to the presence of the lattice and the existence of the impurities.

The ground state energy $E_0(\Phi)$ for a given flux $\Phi$ is periodic in $\Phi$ with period $2\pi$. However, the adiabatic variation of the energy $E(\Phi)$ of the state that begins as the ground state at $\Phi = 0$ may have a periodicity greater than $2\pi$. The clean ($\nu = 0$) XXX model has been studied previously by Sutherland and Shastry [24]. They find a periodicity of $4\pi$ implying carriers with charge $-q/2$ in terms of the charge $q$ of the fundamental carriers. The second quantity of interest is the stiffness $D = (N/2)\partial^2 E_0(\Phi)/\partial \Phi^2|_{\Phi=0}$. Disorder will in general destroy the phase sensitivity of the ground state and therefore lead to a non trivial renormalization of the stiffness.

Let us consider the half-filled case $N = 2M$. In order to study the periodicity of the ground state, we follow the particle with the maximum momentum $k_M$ characterised by the ground state quantum number $I_M = (M - 1)/2$ as a function of $\Phi$. At the special point $k_M = \pi$, the BA Eq. (2) for $k_M$ with $\theta(k_i, \pi) = \pi$ for $i < M$ and $\delta(\pi, \nu) = 0$ for all finite $\nu$ gives the flux to be $2\pi$. Conversely, we notice that as long as $|\Phi| < 2\pi$, all $k'$s stay within $|k| < \pi$ until at $\Phi = 2\pi$, the maximum momentum takes the value $k_M = \pi$. Hence, it follows that for $2\pi < \Phi < 4\pi$ the $k'$s are the same as in the case $|\Phi| < 2\pi$ with the states relabelled as $k_{M-1} > k_{M-2} > \cdots > k_1 > k_M$ modulo $2\pi$. We next substitute the values $k_M = \pi$ and $\Phi = 2\pi$ into the remaining $M - 1$ equations. These reduce to

$$Nk_j = 2\pi I_j' - \sum_{l \neq M} \theta(k_j, k_l) - \sum_{\alpha=1}^{F} \delta(k_j, \nu_{m(\alpha)}).$$  \hspace{1cm} (3)
We immediately notice that these equations with the quantum numbers \( I_j' = I_j + \frac{1}{2} \) are just the BA equations for the ground state of \( M - 1 \) particles in the absence of flux. The energy of the ground state of the \( M \) particle system \( E(\Phi) \) at flux \( \Phi = 2\pi \) is thus related to the ground state of \( M - 1 \) particles at \( \Phi = 0 \), i.e. \( E_M(2\pi) = E_{0,M-1}(0) - 2(1 + \cos(k_M)) = E_{0,M-1}(0) \). Therefore

\[
\Delta E_M(\Phi) = E_M(2\pi) - E_M(0) = 4\pi^2 D/N = E_{0,M-1}(0) - E_{0,M}(0) = 2\pi \chi^{-1}/N, \tag{4}
\]

with \( \chi \) the susceptibility. This remarkable property of the BA Eq. (2) relates the stiffness and the susceptibility through \( D\chi = 1/2\pi \) even in the presence of impurities. This is the Sutherland-Shastry relation derived previously in the clean XXZ [23], Hubbard [24] and SC model [25].

We next turn to explicitly calculating the stiffness \( D \) of a system with integrable impurities. For convenience, we will use the XXZ spinless fermion model with impurities where we can easily vary the interaction strength \( \Delta = \cos(2\eta) \). The clean system is given by the Hamiltonian, \( H_{XXZ} = -\sum_{j=1}^{N}(c_{j+1}^\dagger c_j + h.c.) - 2\Delta \sum_{j=1}^{N} \rho_j \rho_{j+1} \). The stiffness is calculated by studying the leading order finite size corrections to the thermodynamic value of the ground state energy in the presence of a flux with the help of the Wiener-Hopf technique [26]. For the XXZ model, this was already done by SSE for impurities chosen symmetric w.r.t. the origin and of equal strength \( \nu \). They get for the stiffness

\[
D(\eta, \nu) = \frac{\pi}{8} \frac{\sin(2\eta)/(\eta(\pi - 2\eta))}{1 - f + f \cosh[\pi \nu/(\pi - 2\eta)]}, \tag{5}
\]

where \( f = F/N \) is the impurity density. By the same arguments as above, \( D\chi = 1/2\pi \) also in the XXZ model and we have the susceptibility in the presence of impurities. Using hydrodynamical arguments, we further have \( D\chi^{-1} = v^2 \) [27], where \( v \) is the Fermi velocity. Thus we see that the impurities simply renormalize the values of the stiffness, inverse susceptibility and the Fermi velocity of the clean system by the factor \( (1 - f + f \cosh[\pi \nu/(\pi - 2\eta)])^{-1} \). This then further implies that the critical exponents remain unchanged and that the correlation functions are modified by the renormalization of \( \nu \) only. We remark here that these
calculations can also be done by choosing a more general, say Gaussian, distribution for the impurity strengths. However, the resulting expressions are lengthy and will not be included in this letter.

A finite stiffness also implies that the ground state is extended. The stiffness in the $\nu \neq 0$ case as seen from Eq. (5) has a quadratic dependence on the strength of the impurity and only in the limit of very large disorder does one get an exponential dependence. This is contrary to the understanding that due to coherent backscattering in 1D a generic impurity would lead to localisation for arbitrarily weak disorder. However this result is consistent with the observation that these are ”special” impurities with zero reflection.

The stiffness provides an operational definition for the persistent currents as $J = D\Phi$. In Fig. 1, we plot $D$ of the half-filled XXZ model with impurity density $f = 0.001$ in the attractive and the repulsive interaction regimes as a function of impurity strength $\nu$. In agreement with previous studies of the XXZ model in the presence of scalar potential scatterers [9], we find that there is no enhancement in the persistent currents for repulsive interaction ($\Delta < 0$ i.e. $\pi/4 < \eta \leq \pi/2$). For attractive interactions ($\Delta > 0$ i.e. $0 \leq \eta < \pi/4$), we observe such a crossover into a regime where the currents for non-zero attraction $\eta$ are larger than the currents of the non-interacting case at the same strength of the impurities [7]. Thus, although there is no backscattering in the present case of integrable impurities, the situation regarding enhancement of persistent currents is qualitatively the same as for potential scatterers.

Away from half-filling, the situation is more complicated. Here we only remark that the degeneracy of levels observed in the clean XXZ for $1/r$ fillings with $r$ rational [28] is lifted. Details will be published elsewhere.

In conclusion, we have shown that an interacting quantum many-body system with impurities, that can be constructed by the QISM, is characterised by an absence of reflection at the impurities. As a direct consequence of the absence of back scattering, the ground state of the interacting system remains extended even in the presence of these special impurities. These systems are therefore unique in that there is no localisation in the presence of disorder.
in 1D. The value of the effective carrier charge $-q/2$ and the Sutherland-Shastry relation $D\chi = 1/2\pi$ are derived for the half-filled XXX and XXZ models and are shown to be independent of the impurity strength. For the half-filled XXZ model, we show enhancement of the persistent currents only for attractive interactions.

A.P. would like to thank Pinaki Majumdar and Ramesh Pai for fruitful discussions. H.-P.E. and R.A.R. gratefully acknowledge financial support through the European Union’s "Human Capital and Mobility" program and the Alexander von Humboldt Foundation, respectively.
REFERENCES

* Permanent address: Institut für Physik, TU Chemnitz-Zwickau, D-09107 Chemnitz, Germany.

[1] I. Affleck, in *Springer Series in Solid-State Sciences* 118, ed. by A. Okiji and N. Kawakami, (Springer Verlag 1994).

[2] T. Giamarchi and H. Schulz, Europhys. Lett. 3, 1287 (1987).

[3] T. Giamarchi and H. Schulz, Phys. Rev. B 37, 325 (1988).

[4] C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. 46, 15233 (1992).

[5] L. P. Levy, G. Dolan, J. Dunsmuir and H. Bouchiat, Phys. Rev. Lett. 64, 2074 (1990).
  V. Chandrasekhar, R. A. Webb, M. J. Brady, M. B. Ketchen, W. J. Galager and A. Kleinsasser, Phys. Rev. Lett. 67, 3578 (1991). D. Mailly, C. Chapelier and A. Benoit, Phys. Rev. Lett. 70, 2020 (1993).

[6] B. Altshuler, Y. Gefen and Y. Imry, Phys. Rev. Lett. 66, 88 (1991).

[7] R. A. Römer and A. Punnoose, to be published in Phys. Rev. B (1995).

[8] T. Giamarchi and B. S. Shastry, Phys. Rev. B 51, 10915 (1995).

[9] G. Bouzerar, D. Poilblanc and G. Montambaux, Phys. Rev. B 49, 8258 (1993); M. Ramin, B. Reulet and H. Bouchiat, Phys. Rev. B 51, 5582 (1994).

[10] M. Kamal, Z. H. Musslimani and A. Auerbach, preprint (1995).

[11] H. Mori and M. Hamada, preprint (1995).

[12] N. Andrei and H. Johannesson, Phys. Lett. A 100, 108 (1984); K.-J.-B. Lee and P. Schlottmann, Phys. Rev. B 37, 379 (1988); P. Schlottmann, J. Phys. CM 3, 6617 (1991);
  E. S. Sørensen, S. Eggert and I. Affleck, J. Phys. A 26, 6757 (1993).

[13] R. J. Baxter, *Exactly solved models in statistical mechanics*, (Academic Press, London,
1982).

[14] L. A. Takhtajan, in Vol 242 Exactly Solvable Problems in Condensed Matter and Relativistic Field Theory, edited by B. S. Shastry, V. Singh and S. S. Jha (Springer, New York, 1985).

[15] B. S. Shastry, J. of Stat. Phys. 50, 57 (1988).

[16] E. K. Sklyanin, preprint HU-TFT-91-51 (1991).

[17] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, Quantum Inverse Scattering Method and Correlation Functions, (Cambridge, New York, 1993).

[18] P. A. Bares, preprint (1994).

[19] P. Schmitteckert, P. Schwab and U. Eckern, Europhys. Lett. 30, 543 (1995).

[20] Note that for example in the XXZ model of SSE the impurity phase shift $\delta$ is not only a function of $k$ and $\nu$ but also of the interaction strength $\Delta$.

[21] B. L. Altshuler, P. A. Lee and R. A. Webb, (eds.), Mesoscopic Phenomena in Solids, (North-Holland, New York, 1991).

[22] N. F. Mott and W. D. Twose, Adv. Phys. 10, 107 (1961) R. E. Borland, Proc. R. Soc. London Ser. A 274, 529 (1963).

[23] B. Sutherland and B. S. Shastry, Phys. Rev. Lett. 65, 1833 (1990).

[24] B. S. Shastry and B. Sutherland, Phys. Rev. Lett. 65, 243 (1990).

[25] R. A. Römer and B. Sutherland, Phys. Rev. B 50, 15389 (1994).

[26] F. Woynarovich and H. P. Eckle, J. Phys. A. 20, L97 (1987); C. J. Hamer, G. R. W. Quispel, and M. T. Batchelor, J. Phys. A 20, 5677 (1987).

[27] B. Sutherland, unpublished; R. A. Römer and B. Sutherland, Phys. Lett. A 190, 295 (1994); B. Sutherland, R. A. Römer and B. S. Shastry, Phys. Rev. Lett. 73, 2154 (1994).
[28] R. A. Römer, H.-P. Eckle and B. Sutherland, Phys. Rev. B 52, 1656 (1995).
FIGURES

FIG. 1. Stiffness of the half-filled XXZ model with impurity density $f = 0.001$ for various interaction strengths $\Delta$ as a function of impurity strength $\nu$. Note that increasing repulsion and impurity strength will always reduce $D$ whereas there is a crossover for all values of the attraction as we increase the impurity strength.
Repulsive $\Delta \leq 0$

Attractive $\Delta \geq 0$

Impurity Strength $\nu$

Stiffness $D$