Distribution of reducible polynomials with a given coefficient set

Shane Chern

Abstract. For a given set of integers $S$, let $R_n^*(S)$ denote the set of reducible polynomials $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ over $\mathbb{Z}[X]$ with $a_i \in S$ and $a_0a_n \neq 0$. In this note, we shall give an explicit bound of $|R_n^*(S)|$. We also present an application of this bound to reducible bivariate polynomials over $\mathbb{Z}[X,Y]$.

Keywords. Reducible polynomial, bivariate polynomial, counting function, Euler’s identity.

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1. Introduction

Here and throughout this note, we say a polynomial is reducible if it is reducible over $\mathbb{Z}[X]$ or $\mathbb{Z}[X,Y]$. Furthermore, the notation $\mathbb{P}(F \text{ reducible})$ denotes the probability of $F$ being reducible under a given coefficient set. In a recent paper [2], L. Bary-Soroker and G. Kozma proved the following

Theorem A. Let $F = F(X,Y) = \sum_{i,j \leq n} \varepsilon_{i,j}X^iY^j$ be a bivariate polynomial of degree $n$ with random coefficients $\varepsilon_{i,j} \in \{\pm 1\}$. Then

$$\lim_{n \to \infty} \mathbb{P}(F \text{ reducible}) = 0.$$  

This result originates from similar distribution problems of reducible univariate polynomials, which were studied for a long period. Let the height of a polynomial $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ with coefficients $a_i \in \mathbb{Z}$ be defined as $H(f) = \max\{|a_i| : i = 0, 1, \ldots, n\}$. For a fixed integer $n \geq 2$ and a real parameter $h \geq 1$, let $\mathcal{R}_n(h)$ denote the set of reducible polynomials $f(X)$ over $\mathbb{Z}$ with degree $n \geq 2$ and height $H(f) \leq h$, and $\mathcal{R}_n^*(h)$ the subset of $\mathcal{R}_n(h)$ with $f(0) \neq 0$. The bound of $|\mathcal{R}_n(h)|$ given by G. Kuba [7] reads

$$h^n \leq |\mathcal{R}_n(h)| \leq C_nh^n \quad \text{for all } n \geq 3 \text{ and } g \geq 1,$$

(1.1)

where $C_n > 0$ is a constant depending only on $n$. In fact, the left hand side comes directly from the reducibility of polynomials with $f(0) = 0$. On the other hand, the upper bound has been studied by many authors; see, e.g., [3, 5, 8, 9]. Furthermore, if we restrict that the coefficients of polynomials should be chosen from a given set $S$, it is also natural to ask for the bound of number of such reducible polynomials with degree $n$, or at least the probability $p_{n,S}$ of such random polynomials as $n \to \infty$; see [6] for the case $S = \{0, 1\}$ and [10] for the case $S = \{\pm 1\}$.

However, considering the notorious difficulty of proving

$$\lim_{n \to \infty} p_{n,S} = 0$$
for some $S$, as Bary-Soroker and Kozma mentioned, they wanted to seek for a modest generalization, that is, adding one degree of freedom, or more precisely, adding one more variable — just like that given in the above theorem.

2. Revisit of Bary-Soroker and Kozma’s proof and our main result

Before presenting our main result, let us go back to Bary-Soroker and Kozma’s proof of Theorem A. In my personal opinion, the most crucial part of their proof is the following proposition listed as Eq. (3) of their paper.

**Proposition A.** Let

$$
\Omega(n, h) = \left\{ f = \sum_{i=0}^{n} a_i X^i : a_i \text{ odd and } H(f) \leq 2h - 1 \right\}.
$$

Then there exists an absolute constant $C > 0$ such that for any $n > 1$ and $h > 2$ the probability that a random uniform polynomial $f \in \Omega(n, h)$ is reducible satisfies

$$
\mathbb{P}_{\Omega(n, h)}(f \text{ reducible}) \leq C \cdot \frac{n(\log h)^2}{h} \left(1 + \frac{1}{2h}\right)^n.
$$

In view of their proof of this proposition, whose idea is due to I. Rivin [9], I note that we can even step further. Again, let $S = \{s_1, s_2, \ldots, s_k\}$ be a given set of integers, and $S^* = S \setminus \{0\}$. We denote by $R^*_n(S)$ the set of reducible polynomials $f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$ with $a_i \in S$ and $a_0 a_n \neq 0$. At last, let $d(n) = \sum_{d|n} 1$ be the divisor function whose summation runs over all positive divisors of $n$. Our result is

**Theorem 2.1.** Let $M$ be a positive integer such that

$$
s_i \not\equiv s_j \pmod{M} \text{ for all } i \neq j \quad (i, j = 1, 2, \ldots, k).
$$

Then

$$
|R^*_n(S)| \leq 4(n-1)M^{n-2} \left(\sum_{a \in S^*} d(a)\right)^2.
$$

(2.1)

**Remark 2.1.** One readily notes that a possible value of $M$ is $\max S - \min S + 1$. However, for some $S$, we could even find smaller $M$. For example, in the case of Bary-Soroker and Kozma’s Proposition A, that is, $S$ being the set of odd integers in the interval $[-2h + 1, 2h - 1]$, they chose $M = 2h + 1$.

**Proof.** We only need to slightly modify Bary-Soroker and Kozma’s proof of Proposition A. Let $\Omega_n(S)$ be the set of polynomials with $a_i \in S$ and $a_0 a_n \neq 0$. We also fix $s, t > 0$ with $s + t = n$ and $b_0, c_0, b_s, c_t \in \mathbb{Z}$ with $a_0 = b_0 c_0$ and $a_n = b_s c_t$ where $a_0, a_n \in S^*$. Now we need to count the set $V = V(s, t, b_0, c_0, c_t)$ containing all polynomials $f \in \Omega_n(S)$ such that $f = pq$ with $\deg p = s$, $\deg q = t$, $p(0) = b_0$, $q(0) = c_0$, and leading coefficients of $p$ and $q$ being $b_s$ and $c_t$, respectively. This implies

$$
|R^*_n(S)| \leq \sum_{a_0, a_n} \sum_{b_0, c_0} \sum_{a_s, a_n} |V(s, t, b_0, b_s, a_0/b_0, a_n/b_s)|.
$$

Next we bound $|V(s, t, b_0, b_s, c_t)|$. The method is essentially the same as that of Bary-Soroker and Kozma. We consider the map $\phi: \Omega_n(S) \to \mathbb{Z}/M\mathbb{Z}[X]$ with

$$
\phi(f) \equiv f \pmod{M}
$$
for \( f \in \Omega_n(S) \). Since \( s_i \neq s_j \mod M \) for all \( i \neq j \) \((i, j = 1, 2, \ldots, k)\), it follows that \( \phi \) is injective. For any \( \bar{p} \) (resp. \( \bar{q} \)) in \( \mathbb{Z}/M\mathbb{Z}[X] \) with \( \deg \bar{p} = s \) (resp. \( \deg \bar{q} = t \)), \( \bar{p}(0) \equiv b_0 \mod M \) (resp. \( \bar{q}(0) \equiv c_0 \mod M \)), and leading coefficient \( \bar{b}_s \equiv b_s \mod M \) (resp. \( \bar{c}_t \equiv c_t \mod M \)), we claim that the pair \( (\bar{p}, \bar{q}) \) will identify at most one \( f \in V(s, t, b_0, b_s, c_0, c_t) \) through the relation

\[
\phi(\bar{pq}) = \phi(f),
\]

since \( \phi \) is injective. On the other hand, for any \( f \in V(s, t, b_0, b_s, c_0, c_t) \) with \( f = pq \), we can always find a pair \( (\bar{p}, \bar{q}) = (\phi(p), \phi(q)) \) such that

\[
\phi(\bar{pq}) = \phi(f).
\]

We therefore conclude that

\[
|V(s, t, b_0, b_s, c_0, c_t)| \leq \sum_{(\bar{p}, \bar{q})} 1 = M^{s-1}M^{t-1} = M^{n-2}.
\]

To complete our proof, we have

\[
|P_n^c(S)| \leq \sum_{a_0, a_n \mid b_0, b_s} \sum_{a_{s+t=n}} \sum_{s+t=n} |V(s, t, b_0, b_s, a_0, a_n/b_s)|
\]

\[
\leq (n-1)M^{n-2} \sum_{a_0, a_n \mid b_0, b_s} \sum_{a_{s+t=n}} 1
\]

\[
= (n-1)M^{n-2} \left(2 \sum_{a \in S^*} d(a)\right)^2.
\]

\[\square\]

It is also noteworthy to mention Kuba’s bound (1.1). In fact, he counted the set

\[
P_n^c(h) = \{(p, q) \in (\mathbb{Z}[X]/\mathbb{Z})^2 : \deg p + \deg q = n \text{ and } H(p)H(q) \leq e^n(h)\}.
\]

Comparing with our proof, in which we restrict the coefficients of \( p \) and \( q \) to \( \mathbb{Z}/M\mathbb{Z} \), we conclude that Kuba’s bound works better for \( n = o(\log h) \).

3. An application of Theorem 2.1

We first step back to the last step of Bary-Soroker and Kozma’s proof. As they showed in their Section 3, by substituting \( Y = 2 \) in \( F(X, Y) \), they got

\[
F(X, 2) = \sum_{i=0}^n \left( \sum_{j=0}^n \pm 2^j \right) X^i.
\]  

(3.1)

Now they only need to use the straightforward argument that if \( F(X, Y) \) is reducible, then either of the following holds: 1) \( F(X, 2) \) is reducible; 2) \( F(2, Y) \) is reducible; 3) \( F(X, Y) = f(X)g(Y) \) for some polynomials \( f \) and \( g \).

At a glimpse of the inner summation of the right hand of (3.1), the following identity of Euler may immediately come to the reader’s mind:

\[
\prod_{n=0}^{\infty} \left( x^{-3^n} + 1 + x^{3^n} \right) = \sum_{n=-\infty}^{\infty} x^n.
\]

(3.2)
This identity was given in Chapter 16 of Euler’s *Introductio in analysin infinitiorum* which is entitled “De Partitio Numerorum”. The reader may refer to J. Blanton’s translation [4] of Euler’s book. In fact, one may readily prove by induction that

\[
\prod_{n=0}^{N-1} \left( x^{-3^n} + 1 + x^{3^n} \right) = \sum_{n=-(3^N-1)/2}^{(3^N-1)/2} x^n; \tag{3.3}
\]

see [1, Eq. (5.4)], which is also an excellent expository article describing Euler’s pioneering work.

Now this identity of Euler along with Theorem 2.1 immediately give

**Theorem 3.1.** Let \( F = F(X, Y) = \sum_{i,j \leq n} \varepsilon_{i,j} X^i Y^j \) be a bivariate polynomial of degree \( n \) with random coefficients \( \varepsilon_{i,j} \in \{0, \pm 1\} \). Then

\[
\lim_{n \to \infty} \mathbb{P}(F \text{ reducible}) = 0.
\]

**Proof.** We substitute \( Y = 3 \) in \( F(X, Y) \). Then

\[
F(X, 3) = \sum_{i=0}^{n} \left( \sum_{j=0}^{n} \varepsilon_{i,j} 3^j \right) X^i, \tag{3.4}
\]

where \( \varepsilon_{i,j} \in \{0, \pm 1\} \). Thanks to Euler’s identity, we immediately see that the right hand side of (3.4) consists of all integer coefficient polynomials with degree \( \leq n \) and height \( \leq (3^{n+1} - 1)/2 = h^* \). Note also that the number of such polynomials with \( a_0a_n = 0 \) is less than \( 2(2h^* + 1)^n \). This implies that we only need to consider the probability \( \mathbb{P}(f \text{ reducible}) \) where \( f \) is a random integer coefficient polynomial with \( \deg f = n, H(f) \leq h^* \), and \( f(0) \neq 0 \). Now by Theorem 2.1, we have

\[
|R_n^* (h^*)| \leq 4(n-1)(2h^* + 1)^{n-2} \left( 2 \sum_{n=1}^{h^*} d(n) \right)^2,
\]

where we put \( M = 2h^* + 1 \). Hence

\[
\mathbb{P}(F(X, 3) \text{ reducible}) \ll \frac{|R_n^* (h^*)|}{(2h^* + 1)^n+1} \ll \frac{n^3}{3^n} \quad (n \to \infty).
\]

Here we use the approximation

\[
\sum_{n \leq x} d(x) \sim x \log x \quad (x \to \infty).
\]

At last, similar to Bary-Soroker and Kozma’s argument, we notice that if \( F(X, Y) \) is reducible, then either of the following holds: 1) \( F(X, 3) \) is reducible; 2) \( F(3, Y) \) is reducible; 3) \( F(X, Y) = f(X)g(Y) \). We also have

\[
\mathbb{P}(F(X, Y) = f(X)g(Y)) \leq \frac{3^{n+1} \cdot 3^{n+1}}{3(n+1)^2} \ll 3^{-n} \quad (n \to \infty),
\]

since both \( f \) and \( g \) have coefficients in \( \{0, \pm 1\} \). Hence

\[
\mathbb{P}(F(X, Y) \text{ reducible}) \ll \frac{n^3}{3^n} \to 0 \quad (n \to \infty).
\]

This ends our proof. \( \square \)
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DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, USA
E-mail address: shanechern@psu.edu