Superconformal Invariance
and
The Geography of Four-Manifolds

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The correlation functions of supersymmetric gauge theories on a four-manifold $X$ can sometimes be expressed in terms of topological invariants of $X$. We show how the existence of superconformal fixed points in the gauge theory can provide nontrivial information about four-manifold topology. In particular, in the example of gauge group $SU(2)$ with one doublet hypermultiplet, we derive a theorem relating classical topological invariants such as the Euler character and signature to sum rules for Seiberg-Witten invariants. A short account of this paper can be found in [1].

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1. Introduction

Expeditions in the current Age of Exploration of supersymmetric quantum field theory have recovered a number of impressive trophies. Chief amongst these has been the delivery of boatloads of new conformal and superconformal fixed points in four (and even higher) dimensions. At the same time, explorations of a rather different nature into the application of supersymmetric quantum field theory to the differential topology of four-manifolds have brought to light another collection of trophies: the Seiberg-Witten invariants and the exact formulae for the Donaldson-Witten path integral in \( d = 4, \mathcal{N} = 2 \) topologically twisted gauge theories. In view of this, one might hope that the new superconformal fixed points will be a source of further insights into the topology of four-manifolds. Conversely, advances in 4-manifold theory might prove to be very useful in understanding aspects of superconformal fixed points.

The present paper takes a small step in the program of combining superconformal invariance with four-dimensional topological field theory. We use the behavior of the Donaldson-Witten function at superconformal points to prove an interesting and nontrivial property of the topology of a compact, oriented, four-manifold \( X \). Loosely stated, our main result is the following. Let \( \chi, \sigma \) be the Euler character and signature of \( X \). Then, either \( 7\chi + 11\sigma \geq -12 \), or the Seiberg-Witten invariants of \( X \) satisfy a collection of sum rules. The message is summarized in Fig. 2 of section 6.2 below. A more precise statement of the result can be found in the theorems in sections 6.1 and 6.3 below. The sum rules that we have found give very strong constraints on the structure of the basic classes and the values of the Seiberg-Witten invariants, and they show that the line \( 7\chi + 11\sigma = -12 \) plays an important role in four-manifold topology.

The organization of the paper is as follows.

The evaluation of the Donaldson-Witten function for \( SU(2) \) gauge theories with \( N_f = 1, 2, 3, 4 \) fundamental hypermultiplets was carried out in [2], but the overall normalization of the path integral, which is a function of the hypermultiplet masses and the quantum scale \( \Lambda_{N_f} \), was left undetermined. We fill in this gap in section two using the standard tools of dimensional analysis, holomorphy, RG flow, and anomalies. In the present paper we will be concentrating on the superconformal fixed point of the theory with \( N_f = 1 \) studied in [3]. In section three we summarize a few expansions in the scaling variable \( z \) near this fixed point. These expansions are needed in the subsequent technical development. In section four we combine the previous results to examine in detail the analytic structure of
the Donaldson-Witten function $Z_{DW}(z)$ as a function of the scaling variable $z$. Our key physical observation is that the function $Z_{DW}(z)$ must be regular at $z = 0$. Given the structure of $Z_{DW}(z)$ as a function of topological invariants, this is a nontrivial fact. In section five we review for the benefit of the reader (and the authors) some basic facts in the problem of the “geography of four-manifolds.” Loosely put, this is the problem of mapping out which regions in the $\chi, \sigma$ plane are inhabited by four-manifolds and which are not. As we shall see, there are definitely regions of terra incognita. In section six we state our main result on geography following from the analytic structure of $Z_{DW}(z)$. As stated above we show that either $7\chi + 11\sigma \geq -12$, or the Seiberg-Witten invariants of $X$ satisfy a collection of sum rules. The precise statement of the sum rules is, unfortunately, rather intricate. However, we introduce a notion of a four-manifold of superconformal simple type. Roughly speaking, these are manifolds which satisfy our sum rules in the most natural way. As another byproduct of our results, we show that manifolds with only one basic class have to satisfy $7\chi + 11\sigma \geq -12$. In section seven, we give what we hope is compelling evidence for our results: we study in detail complex surfaces, as well as many important constructions of four-manifolds, and we show that all four-manifolds of $b_2^+ > 1$ of simple type (of which we are aware) are of superconformal simple type. In section eight, we discuss upper and lower bounds on the number of basic classes for manifolds of superconformal simple type. In particular, we find a lower bound which generalizes the Noether inequality. Finally, in section nine we state our conclusions and some possible applications of our work.

2. Remarks on the $\alpha$ and $\beta$ functions

In this somewhat technical section we discuss the overall normalization of the Donaldson-Witten partition function. For the sake of brevity we will assume (in this section only) some familiarity with the results and notation of section 11 of [2], which discusses the $u$-plane integral for gauge group $SU(2)$ or $SO(3)$ in theories with matter hypermultiplets.

When the matter hypermultiplets are in the fundamental representation, the usual twisting procedure is consistent only if the second Stiefel-Whitney class of the gauge bundle equals that of the four-manifold, i.e. if $w_2(E) = w_2(X)$. One can in fact consider more general twisting procedures (by twisting for instance the baryon number of the hypermultiplet [4]), and this would allow us to study more general situations. In this paper we will study only the usual twist, although the generalization might be interesting. The $u$-plane
integral for the theories with matter hypermultiplets on a simply-connected four-manifold has the form [2]:

\[ Z_u(p, S; m_i, \Lambda) = \int_C \frac{dud\bar{u}}{y^{1/2}} \mu(\tau)e^{2pu+S^2T(u)}\Psi, \]  

(2.1)

where the measure \( \mu(\tau) \) is given by

\[ \mu(\tau) = -\frac{\sqrt{2}}{2} \alpha \beta^\sigma \frac{d\bar{\tau}}{d\tau} \left( \frac{da}{du} \right)^{1-\frac{1}{2}x} \Delta^{\sigma/8}. \]  

(2.2)

In this equation, \( \Delta \) is the discriminant of the corresponding Seiberg-Witten curve, and is a polynomial in \( u, m_i, i = 1, \ldots, N_f, \) and \( \Lambda_{N_f} \) for the asymptotically free theories with \( N_f < 4 \). The definitions of the other terms in (2.1) can be found in [2] and will not be needed here.

The measure (2.2) involves functions \( \alpha, \beta \), which depend in principle on \( m_i, \Lambda_{N_f} \). These functions can be obtained in the \( N_f = 0 \) theory by comparing the results obtained via the \( u \)-plane integral to the mathematical results in Donaldson theory, as in [2]. Our aim here is to be more precise about the functions \( \alpha, \beta \). In particular we will be interested in their mass dependence. In order to do this, it is important to recall the physical origin of the \( \alpha, \beta \) functions in more detail [6][2].

When the twisted \( N = 2 \) theories are considered in a gravitational background, there are extra couplings involving the curvature tensor that can be generated in the effective action on the \( u \)-plane. The requirement of topological invariance tells us that the only possible allowed terms are in fact the signature and the Euler character densities. These densities have dimension 4, and will couple to holomorphic “functions” in \( u, m_i, \Lambda_{N_f} \), therefore the effective action will contain the extra couplings

\[ (\log A(u, m_i, \Lambda_{N_f}))\chi + (\log B(u, m_i, \Lambda_{N_f}))\sigma. \]  

(2.3)

Notice that \( A, B \) are dimensionless. The \( u \)-dependence of these functions was determined in [2], where it was found that

\[ A(u, m_i, \Lambda_{N_f}) = \alpha(m_i, \Lambda_{N_f}) \left( \frac{du}{da} \right)^{1/2}, \]

\[ B(u, m_i, \Lambda_{N_f}) = \beta(m_i, \Lambda_{N_f}) \Delta^{1/8}. \]  

(2.4)

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1 In the theory with \( N_f = 4 \) doublet hypermultiplets, as well as in the mass-deformed \( N = 4 \) theory, the \( \alpha, \beta \) functions depend in principle on the masses \( m_i \) and the microscopic coupling constant \( \tau_0 \). The mass-deformed \( N = 4 \) theory has been studied in detail from the point of view of the \( u \)-plane integral in [5], where a proposal for the \( \alpha, \beta \) functions has been made for this theory. In this paper, we will only consider the asymptotically free theories, \( N_f \leq 3 \).
This explains the presence of these terms in the measure of the $u$-plane integral, in (2.2).

To find the structure of the functions $\alpha, \beta$, we have to take into account some physical criteria that are similar to those used in the analysis of superpotentials in $\mathcal{N} = 1$ supersymmetric gauge theories (see [7] for a review) and in the derivation of the Seiberg-Witten curves in $\mathcal{N} = 2$ supersymmetric gauge theories [8][9].

1. Holomorphy: $\alpha, \beta$ are local holomorphic functions of $m_i$, $i = 1, \ldots, N_f$ and $\Lambda_{N_f}$.

2. Dimensional analysis: $Z_u$ is dimensionless.

3. RG-flow: in the double scaling limit where one of the hypermultiplet masses goes to infinity in such a way that $m_{N_f} \Lambda_{N_f}^{4-N_f} \equiv \Lambda_{N_f-1}^{4-(N_f-1)}$ is fixed, we expect the general behavior,

$$Z_u^{N_f} \rightarrow \left( \frac{m_{N_f}}{\Lambda_{N_f}} \right)^{\theta_{N_f}} Z_u^{N_f-1}, \quad (2.5)$$

where $\theta_{N_f}$ is some exponent that can depend on the number of flavors and on the topological invariants $\chi, \sigma$.

The constraint of dimensional analysis on $\alpha, \beta$ is easily solved. We have canonical dimensions for all the quantities involved in the Donaldson-Witten generating function:

$$[u] = 2, \quad [\tau] = 0, \quad [\alpha] = 1,$$
$$[\Psi] = 0, \quad [\Delta] = 12. \quad (2.6)$$

The dimension of $\Delta$ follows from the structure of the Seiberg-Witten curve, where $[x] = 2, [y] = 3$. It follows from (2.4) that

$$[\alpha] = -1/2, \quad [\beta] = -3/2. \quad (2.7)$$

Notice that, for $Z_u$ to be dimensionless, we have to include an overall factor $1/(c_{N_f} \Lambda_{N_f})$ in the $u$-plane measure, where $c_{N_f}$ is a constant.

Imposing the other constraints is more involved and depends on $N_f$. We will give two arguments that $\alpha, \beta$ are independent of the masses and given by $\alpha = \alpha_0/\Lambda_{N_f}^{1/2}$, and $\beta = \beta_0/\Lambda_{N_f}^{3/2}$, where $\alpha_0, \beta_0$ are constants. The second argument only applies to the case $N_f = 1$, which is the case of primary interest here.

First of all, one can argue on general grounds that the functions $\alpha, \beta$ must be independent of the masses. If they had such a dependence, since they are locally holomorphic functions of the masses, they would have singularities at some special values $m_i^*$, and for any value of $u$. Therefore, the functions $A(u, m_i, \Lambda_{N_f})$ and $B(u, m_i, \Lambda_{N_f})$ appearing in
the $u$-plane integral would then have extra singularities along complex codimension one varieties of the form $(m_i^*, u)$, for any $u \in \mathbb{C}$. But there is no physical reason to expect such behavior on manifolds of $b_2^+(X) = 1$. The only points in the $(m_i, u)$ space where we can have zeroes or poles for these functions are at the singularities in the $u$-plane, which have the form $(m_i, u_m(m_i))$ (where the $u_m(m_i)$ are the zeroes of the discriminant $\Delta(u, m_i, \Lambda_{N_f})$).

One could think that there are in fact special values of the masses, namely the critical values giving superconformal fixed points, where some singular behavior can show up. But, again, the superconformal points are a special class of singularities in the $u$-plane, and they only occur for special values of $u$. Moreover, if we consider the contributions of the SW singularities (as we will do in this paper), the $\alpha$ and $\beta$ functions will be overall factors for the contribution of all the singularities. If they had zeroes or poles at the values of the $m_i$ associated to the superconformal points, for example, we would find a singular behavior even in the contributions of the singularities not involved in a nonlocal collision. This is clearly unreasonable physically. 

The argument we have just given is subject to a possible subtlety for the theories $N_f > 1$ related to the appearence of noncompact Higgs branches. Thus, while we regard the argument as reasonable, we also present a second argument which applies in the case $N_f = 1$ and leads to the same result.

The second argument proceeds by using another physical input which is crucial in the analysis of the functions $A, B$ explained in [6]. These functions have to satisfy the general constraints listed above, but they also have to reproduce the anomaly associated to the fields that have been integrated out. In fact, we can understand the terms in (2.4) as anomaly functionals in the effective theory that reproduce the anomaly in the $R$-current due to a background gravitational field. This physical input was enough to find the $u$-dependence of $A, B$ in [6]. As we will see, a slight modification of this analysis also fixes the dependence of $\alpha, \beta$ on $m, \Lambda_{N_f=1}$.

We want to analyze the terms $A, B$ using the general constraints above as well as the anomaly condition. Since the hypermultiplet mass explicitly breaks the $U(1)_R$ symmetry, in order to have the classical $R$-symmetry we use Seiberg’s trick and give $R$-charge two to the mass. Therefore, under a $U(1)_R$ rotation, we have that:

$$u \rightarrow e^{4i\phi_R}u, \quad m \rightarrow e^{2i\phi_R}m. \quad (2.8)$$

Notice that the same arguments rule out any mass dependence in the overall factor that we introduced in the measure for dimensional reasons.
The scale $\Lambda_{N_f=1}$ doesn’t rotate under $U(1)_R$. Let’s now consider the anomaly analysis of $A$. By dimensional analysis, we know that $1/\alpha^2$ is a homogeneous function of degree one in $\Lambda_{N_f=1}$, $m$, that we will denote by $P(m, \Lambda_{N_f=1})$. It will be enough to analyze the behavior at the semiclassical region, where $(du/da)^{1/2} \sim u^{1/4}$. The term

$$\log\left(\frac{u^{1/4}}{P(m, \Lambda_{N_f=1})}\right) \chi$$

has to reproduce the part of the anomaly proportional to $\chi$, and corresponding to the massive $W$ bosons that have been integrated out. This means that the function of $u$, $\Lambda_{N_f=1}$ and $m$ in the argument of the logarithm must have $R$-charge $1$ [2]. As the mass has $R$-charge two now, the function $P(m, \Lambda_{N_f=1})$ has to be independent of $m$. This means that $\alpha$ is independent of the mass, and is given by $\alpha = \alpha_0/\Lambda_{N_f=1}^{1/2}$, where $\alpha_0$ is a constant.

A similar analysis can be done for $B$, where the anomaly in the semiclassical region corresponds now to the massive components of the hypermultiplet and depends on $\sigma$. Again, to reproduce the anomaly $\beta$ must be independent of $m$, and we obtain $\beta = \beta_0/\Lambda_{N_f=1}^{3/2}$, $\beta_0$ a constant.

Using the expressions for $\alpha$, $\beta$ that we have derived, as well as the RG relation $\Lambda_{N_f}^{4-N_f} m_{N_f} = \Lambda_{N_f-1}^{5-N_f}$, we find that in fact the $u$-plane integral changes as in (2.3), with

$$\theta_{N_f} = \frac{3 + \sigma}{5 - N_f},$$

where we have used that $\chi + \sigma = 4$ for simply-connected four-manifolds with $b_2^+ = 1$.

Finally, we remark that the $\alpha, \beta$ functions can in principle be derived from a weak-coupling one-loop calculation at large values of $u$. It should also be possible to derive them from field theory limits of string gravitational corrections such as the quantum correction $F_1 \text{tr} R \wedge R$ in $\mathcal{N} = 2$, $d = 4$ compactifications of string theory. The string theory approach to deriving the $\alpha, \beta$ functions raises many interesting further issues which are beyond the scope of this paper.

3. Some properties of the superconformal points

3.1. Superconformal divisors in the $SU(2)$ theories with matter

At some special divisors in the moduli space of $\mathcal{N} = 2$ supersymmetric gauge theories, the low-energy theory is a non-trivial superconformal field theory. The first example of
such a point was found in \[10\] in pure $SU(3)$ Yang-Mills, and subsequently many other superconformal divisors were found in the $SU(2)$ theories with matter \[3\], as well as in higher rank theories with matter \[11\]. These superconformal divisors are characterized by the fact that at them, two or more mutually non-local BPS states become massless simultaneously.

In the $SU(2)$ theories with $N_f$ massive hypermultiplets, the superconformal points appear as follows: for generic values of the masses, there are $2 + N_f$ points in the $u$-plane with massless BPS states. For special values of the masses, these singularities can collide. There are two possibilities for these collisions. The first possibility is that the $k$ states that come together are mutually local, and the low-energy theory will be $\mathcal{N} = 2$ QED with $k$ hypermultiplets. This happens, for instance, in the massless case, and more generally along the lines where $m_i = \pm m_j$. The second possibility is to tune the values of the hypermultiplet masses, in such a way that the singularities that collide are mutually non-local. This will give nontrivial superconformal points. When two singularities collide to give a superconformal point, one of them will correspond to $k$ mutually local massless states, while the other one will be associated to a massless state which is mutually non-local with respect to the $k$ states in the other singularity. For this reason, these points are called $(k, 1)$ points \[3\]. The $SU(2)$ theories with $N_f < 4$ hypermultiplets will have superconformal points of type $(k, 1)$ for $k = 1, \ldots, N_f$. The $N_f = 4$ theory has no additional collisions, and it will have $(k, 1)$ points with $k = 1, 2, 3$. Some of these superconformal points are the following:

a) In the $N_f = 1$ theory, there are $(1, 1)$ superconformal points when the mass of the hypermultiplet satisfies $m_1^3 = (3\Lambda_1/4)^3$, and these points are located at $u_1^3 = (3\Lambda_1^2/4)^3$.

b) In the $N_f = 2$ theory, there are $(2, 1)$ points that occur when $m_1 = \pm m_2$ and $m_1^2 = \Lambda_2^2/4$.

c) Finally, in the $N_f = 3$ theory, a $(3, 1)$ point appears when $m_1 = m_2 = m_3 = \Lambda_3/8$.

The critical exponents and scaling dimensions of the operators in these superconformal theories can be obtained by looking at the structure of the Seiberg-Witten curves near the colliding singularities \[3\] or by using the expansions of the Seiberg-Witten periods in the critical theory \[12\] \[13\], and are in fact completely determined by the value of $k$.
3.2. Expansions around the superconformal point

In this paper, we will be interested in the simplest superconformal point arising in the $SU(2)$ theories with matter: the superconformal point of type $(1,1)$ in the $N_f = 1$ theory. For simplicity, we will only consider the $(1,1)$ point in the $m$-plane, occurring at $m_\ast = 3\Lambda_1/4$ (the other $(1,1)$ points are obtained by $\mathbb{Z}_3$ symmetry on the complex $m$-plane). We want to study the behavior of the theory as $m$ approaches the critical value $m_\ast$. As we will see, the quantities appearing in the topological correlation functions are natural quantities associated to the elliptic curve describing the low-energy theory, and when $b_2^+ > 1$ and the manifold is of simple type, they involve the evaluation of these quantities at the singularities in the $u$-plane. When the theory is nearly critical and $m = m_\ast + z$, the $u$-coordinates of the two singularities that collide at the superconformal point (that will be denoted by $u_{\pm}$) can be expanded in terms of the parameter $z$. Therefore, all the quantities evaluated at $u_{\pm}$ will have an expansion in $z$ as well. We will present here explicitly the first few terms of these expansions, since we will need them in the next section.

We have to evaluate some quantities associated to the Seiberg-Witten curve at the $u$-plane singularities. To do this, we follow the procedure in section 11 of [2] and write the SW curves

$$y^2 = x^3 + a_2 x^2 + a_4 x + a_6$$

in the form

$$y^2 = x^3 - \frac{c_4}{48} x - \frac{c_6}{864},$$

where

$$c_4 = 16(a_2^2 - 3a_4),$$
$$c_6 = -64a_2^3 + 288a_2a_4 - 864a_6.$$ (3.3)

The value of the period at the singularity $u = u_\ast$ is given by

$$\left(\frac{da}{du}\right)_\ast = \frac{c_4(u_\ast)}{2c_6(u_\ast)}.$$ (3.4)

Another quantity that enters in the Donaldson-Witten function for manifolds of simple type is the following. Consider the effective coupling $\tau$ in the duality frame appropriate to the singularity $u = u_\ast$ (in particular, $\tau \to i\infty$ as $u \to u_\ast$). We define $\kappa$ as $\kappa = (du/dq)$, where $q = e^{2\pi i\tau}$. The value of this quantity at the singularity can be expressed in terms of (3.3) and the discriminant of the curve as

$$\kappa_\ast = \frac{c_4^3(u_\ast)}{\Delta'(u_\ast)}.$$ (3.5)
We will now consider the behavior of these quantities (evaluated at the $u$-plane singularities) for the $N_f = 1$ theory with a massive hypermultiplet, when the value of the mass is near the critical value $m_*= 3\Lambda_1/4$. The Seiberg-Witten curve in this case is [9]:

$$y^2 = x^2(x - u) + \frac{1}{4}m\Lambda_1^3x - \frac{1}{64}\Lambda_1^6,$$

(3.6)

with discriminant

$$\Delta_1(u, m, \Lambda_1) = \frac{\Lambda_1^6}{16}\left[-u^3 + m^2u^2 + \frac{9}{8}\Lambda_1^3mu - \Lambda_1^3m^3 - \frac{27}{256}\Lambda_1^6\right].$$

(3.7)

One finds

$$c_4(u, m, \Lambda_1) = 16u^2 - 12\Lambda_1^3m,$$

$$c_6(u, m, \Lambda_1) = 64u^3 - 72\Lambda_1^3mu + \frac{27}{2}\Lambda_1^6.$$

(3.8)

For the critical value of the mass $m = m_* = 3\Lambda_1/4$, two singularities on the $u$-plane will collide at $u_* = 3\Lambda_1^2/4$. If we write

$$m = m_* + z, \quad u = u_* + \Lambda_1z + \delta u,$$

(3.9)

and we introduce the shifted variable $x = u/3 + \tilde{x}$, the Seiberg-Witten curve (3.4) becomes, at leading order,

$$y^2 = \tilde{x}^3 - \frac{\Lambda_1^3}{4}z\tilde{x} - \frac{\Lambda_1^4}{16}\delta u,$$

(3.10)

which is a deformation of the cuspidal cubic $y^2 = \tilde{x}^3$. The variables $z$ and $\delta u$ correspond to operators in the conformal field theory at the $(1,1)$ singularity. The scaling dimensions of these operators can be deduced from (3.10) after taking into account that $a \sim (\delta u/y)d\tilde{x}$ has dimension 1, and one finds that $z$ has dimension 4/5, while $\delta u$ has dimension 6/5.

If $u_\pm = u_* + \Lambda_1z + \delta u_\pm$ denotes the position of the two colliding singularities, the deformation parameter $\delta u_\pm$ will depend on $z$ as well, and of course $\delta u_\pm \to 0$ when $z \to 0$. The dependence of $\delta u_\pm$ on $z$ can be obtained as a power series expansion by looking at the zeros of the discriminant (3.7), $\Delta_1(u, m, \Lambda_1) = 0$ in terms of the variables (3.9). The most appropriate normalization for $\Lambda_1$, for our purposes, is $4\sqrt{3}\Lambda_1^{3/2} = 1$ (with this normalization, the leading term of $(du/da)^2$ is $z^{1/2}$.) One then finds,

$$\delta u_\pm = \pm \left(\frac{16}{243}\right)^{1/3}z^{3/2} + \frac{4}{9}z^2 + \mathcal{O}(z^{5/2}).$$

(3.11)
Notice that
\[
\delta u_-(z^{1/2}) = \delta u_+(z^{1/2}).
\] (3.12)

This is due to the fact that, for \( z < 0 \), one has \( \pm z^{1/2} = \pm i|z|^{1/2} \). But in this case the roots \( u_\pm \) of the equation \( \Delta_1(m, u, \Lambda_1) = 0 \) must be related by complex conjugation, therefore we must have (3.12). Also notice that the leading term of (3.11) follows from the structure of the curve near the (1, 1) singularity given in (3.10). Using the expansion (3.11) and the explicit expressions (3.8)(3.4)(3.5), we have the following expansions of \((du/da)^2\) and \(\kappa\) at the singularities \(u_\pm\) for \( m = m_* + z \):

\[
\begin{align*}
\left(\frac{du}{da}\right)_\pm &= \pm z^{1/2} \left\{ 1 \pm \left(\frac{4}{3}\right)^{4/3} z^{1/2} + \mathcal{O}(z) \right\} \\
\kappa_\pm &= \mp 2^{31/3} \cdot 3^{4/3} z^{3/2} \left\{ 1 \pm \left(\frac{2048}{3}\right)^{1/3} z^{1/2} + \mathcal{O}(z) \right\}.
\end{align*}
\] (3.13)

Notice that, due to (3.12), one has the following property:

\[
\left(\frac{du}{da}\right)_-(z^{1/2}) = \left(\frac{du}{da}\right)_+(z^{1/2}),
\] (3.14)

and a similar equation for \(\kappa_\pm\). This will be important when we consider the analytic properties of the correlation functions as a function of \( z \). Another important remark is that the leading powers of \( z \) in the expansion of \( \delta u, (du/da)^2 \) and \( \kappa \) are determined by the anomalous scaling weights of the operators near the (1, 1) superconformal point.

### 4. Correlation functions near the superconformal point

In this section, we will study the behavior of the Donaldson-Witten function for the \( SU(2) \) theory with one massive hypermultiplet, and for manifolds with \( b_2^+ > 1 \) and of simple type. The general answer for the generating function in the theories with matter was obtained in [2], and we briefly review it in 4.1. Using this expression, we will study the behavior of the generating function near the superconformal point. In 4.2 we extract from the generating function a Laurent expansion in \( z \), that we denote by \( F(z) \), and we analyze some of its properties. In 4.3 we argue on physical grounds that this Laurent expansion is in fact a Taylor expansion (equivalently, that the generating function is regular at the superconformal point). This will be the main result of our paper. In the rest of the sections, we will develop the consequences of this fact for the geography of four-manifolds and the structure of the SW invariants.
4.1. The Donaldson-Witten function with massive hypermultiplets

The Donaldson-Witten function of $N = 2$, $SO(3)$ theories with matter hypermultiplets can be written explicitly in terms of the SW invariants, for manifolds with $b_{1}^{+} > 1$ and of simple type, but not necessarily simply connected. The expression for generic values of the masses is:

$$Z(m_i, \Lambda_{N_f}; p, S) = 2^{1+\frac{3\sigma+\chi}{2}}(-i)^{\chi_h} \left(\frac{\pi^2 \beta^8}{2^8}\right)^{\sigma/8} \left(-\frac{\pi \alpha^4}{2}\right)^{\chi/4} \sum_{j=1,...,2+N_f} \kappa \chi_h \left(\frac{da}{du}\right)_j^{-(\chi_h+\sigma)} 
\cdot \sum_{\lambda} SW(\lambda) \exp \left[2pu_j + S^2T_j - i\left(\frac{du}{da}\right)_j(S, \lambda)\right] e^{2\pi i(\lambda^2_0 + \lambda \cdot \lambda_0)}. \tag{4.1}$$

The notation is the following:

1. The sum on $j$ is a sum over the $2 + N_f$ singularities at finite values on the $u$-plane. The subindex $j$ in the different quantities means that they are evaluated at the $j$-th singularity. The values of $da/du$ and $\kappa$ at the singularities of the $u$-plane are given in (3.4) and (3.5). The contact term at the singularity can be written as [2]

$$T_j = -\frac{1}{24}\left(\frac{du}{da}\right)_j^2 - 8u_j. \tag{4.2}$$

2. $\chi_h$ denotes

$$\chi_h = \frac{\chi + \sigma}{4} = \frac{1 - b_1 + b_{1}^{+}}{2}. \tag{4.3}$$

Note in particular that we have not assumed $b_1 = 0$ and hence $\chi_h$ can be negative. The notation for this particular combination of $\chi$, $\sigma$ comes from the fact that, if the four-manifold we are considering is a complex surface, then $\chi_h$ is the holomorphic Euler characteristic.

3. The sum over $\lambda$ is a sum over basic classes. As explained in [2], the theories with matter can be considered on any smooth, compact, oriented four-manifold $X$ if the non-abelian magnetic flux of the $SO(3)$ gauge bundle $E$ satisfies $w_{2}(E) = w_{2}(X)$, where $w_{2}(E)$, $w_{2}(X)$ are the second Stiefel-Whitney classes of the bundle $E$ and of the manifold $X$, respectively. In the expression (4.1), we have to choose an integer lifting of $w_{2}(X)$ (this lifting always exists, by a theorem of Hirzebruch and Hopf [14]), which is denoted by $c_0 = 2\lambda_0$. Notice that $\lambda \in \Gamma + \frac{1}{2}w_{2}(X)$, where $\Gamma = H^2(X; \mathbb{Z})$. Therefore, $x = 2\lambda$ is an integral cohomology class which is congruent to $w_{2}(X)$ mod 2, in other words, $x$ is the first Chern class of the determinant line bundle associated.
to a Spin$^c$-structure. In particular, $x$ is characteristic. The exponential involving $\lambda_0$ in (4.1) can be written then as

$$(-1)^{c_0 \frac{x + c_0^2}{2}},$$  \hspace{1cm} (4.4)

and since $x$ is characteristic one can easily prove that the exponent is in fact an integer. A change of the lifting $c_0 \rightarrow \tilde{c}_0$ multiplies the above generating function by the factor

$$(-1)^{\left(\frac{\tilde{c}_0 - c_0}{2}\right)^2}.$$  \hspace{1cm} (4.5)

Remarks:

1. Equation (4.1) generalizes Witten’s expression [15] for pure Yang-Mills and was obtained in section 11 of [2]. The extension of the $N_f = 0$ Donaldson-Witten function to non-simply connected four-manifolds was begun in [2] [16], and completed in [17]. The extension to $N_f > 0$ is trivially obtained following the arguments in [17], where the $u$-plane integral and the SW contributions were determined in the nonsimply connected case for $N_f = 0$. Somewhat later the same result for $N_f = 0$ was stated, less precisely, in [18].

2. Notice that, if a manifold is of simple type and has non-trivial SW invariants, the dimension of the moduli space of solutions to the SW monopole equations has to be zero, and this implies that the index of the Dirac operator coupled to the Spin$^c$-structure associated to $\lambda$ is:

$$\frac{1}{2} \lambda^2 - \frac{\sigma}{8} = \chi_h$$  \hspace{1cm} (4.6)

where we have used $(2\lambda)^2 = 2\chi + 3\sigma$. Therefore, if (4.1) is not identically zero, $\chi_h$ must be an integer [13].

3. For nongeneric masses (for instance, in the massless case) and $N_f > 1$, there are singularities where two or more mutually local states become massless simultaneously. The analysis of the $u$-plane integral in this situation has been done in [19]. Notice that at these singularities, the monopole equations involve more than one spinor field, giving rise to some generalized Seiberg-Witten invariants. These invariants have not been studied in detail, but we should expect a noncompact moduli space of solutions, in correspondence with the noncompact Higgs branch in the physical theory.

4. The UV theory with $N_f = 1$ hypermultiplet describes the nonabelian monopole theory. This theory has been formulated from the point of view of topological quantum field theory in [20] [21] [22] [23], and from a mathematical point of view in [24] [25] [26].
The mass terms for the hypermultiplet have been interpreted in [27] [16] [28] [29] as an equivariant extension of the Thom form with respect to a $U(1)$ action on the moduli space of nonabelian monopoles. For a detailed review of the different aspects of the nonabelian monopole theory, see [30].

4.2. The function $F(z)$

We will now concentrate on the generating function (4.1) for the theory with one massive hypermultiplet. For generic values of the mass, there are three different singularities on the $u$-plane. At each singularity, one of the periods of the Seiberg-Witten curve goes to infinity (corresponding to the degeneration of one of the cycles of the elliptic fiber over the $u$-plane), while the period associated to the appropriate duality frame near the singularity remains finite and different from zero. Likewise, $\kappa$ is finite and well-defined. Therefore, the expression in (4.1) is well-defined. A different situation arises when we approach a superconformal point, $m \to m_*$. In this case, at the colliding singularities, both periods go to infinity and (3.4) diverges, as we have seen in the previous section. Notice that the contribution to (4.1) of the singularity which is not involved in the collision is still perfectly regular as we approach the critical value of the mass, since nothing special happens there. As we showed in section 2, the $\alpha, \beta$ functions only depend on the scale $\Lambda_1$, so they don’t affect the behavior of the generating functional as a function of the mass, and we will drop them as well as the other constant factors in (4.1). Therefore, to study the behavior at $z = 0$ of the generating function, where $z$ is defined in (3.9), we can focus on the contribution of the two colliding singularities $u_\pm$. This contribution is given by:

$$
\sum \pm \kappa_{\pm}^{\chi_h} \left[(\frac{du}{da})^2\right]_{\pm}^{\chi_h + \sigma} e^{2pu_\pm + S^2T_\pm} \sum \lambda SW(\lambda) e^{-i\left(\frac{4\pi}{\Lambda_1}\right)_{\pm}(S,\lambda)} e^{2\pi i(\lambda_0^2 + \lambda \cdot \lambda_0)}. \tag{4.7}
$$

We can now study the properties of this function as a power series in $z$, focusing on the issue of regularity at $z = 0$.

First, we express $\chi, \sigma$ in terms of more convenient linear combinations:

$$
\chi_h = \frac{\chi + \sigma}{4}, \quad c_1^2 = 2\chi + 3\sigma. \tag{4.8}
$$

The quantity $\chi_h$ was introduced in (1.3). When the four-manifold is a complex surface (or, more generally, an almost complex manifold), the combination $c_1^2$ is the square of the first
Chern class of the holomorphic tangent bundle, but we will still use the above notation for this particular combination of \( \chi \) and \( \sigma \) for an arbitrary compact, oriented four-manifold. If the manifold \( X \) is of simple type, then \( x^2 = c_1^2 \) for any basic class. Notice that we can express \( \chi \) and \( \sigma \) in terms of \( \chi_h \) and \( c_1^2 \) as follows:

\[
\chi = 12\chi_h - c_1^2, \quad \sigma = c_1^2 - 8\chi_h. \tag{4.9}
\]

The reason that we introduce these combinations is to facilitate comparison with the results on the geography of four-manifolds.

In terms of \( \chi_h \), \( c_1^2 \), the expansion of the factors that are independent of \( \lambda \) in (4.7) reads as follows:

\[
\kappa^\chi_h \left[ \left( \frac{du}{da} \right)^2 \right] \chi_h^{\chi+h} e^{2pu+2^2T+} = c\chi_h e^{u_+(2p+S^2/3)z} \frac{z^2-\chi_h}{4} \left\{ 1 + \left( \frac{4}{3} \right)^{4/3} \left( \frac{c_1^2 + 5\chi_h}{2} \right) - \frac{S^2}{24} \right\} z^{1/2} + O(z) \right), \tag{4.10}
\]

\[
kappa^\chi_h \left[ \left( \frac{du}{da} \right)^2 \right] \chi_h^{\chi-h} e^{2pu-2^2T-} = c\chi_h e^{u_-(2p+S^2/3)z} \frac{z^2-\chi_h}{4} \left\{ 1 - \left( \frac{4}{3} \right)^{4/3} \left( \frac{c_1^2 + 5\chi_h}{2} \right) - \frac{S^2}{24} \right\} z^{1/2} + O(z) \right),
\]

where \( c = -(2^{31/3} \cdot 3^{4/3}) \) is an overall constant in the expansion of \( \kappa_{\pm} \). The power of the leading term in \( z \) in the expansions (4.10) is

\[
\frac{c_1^2 - \chi_h}{4} = \frac{7\chi + 11\sigma}{16}. \tag{4.11}
\]

Notice that the change of sign in \( z^{1/2} \) introduces a relative phase between the contributions at \( u_{\pm} \). It is clear that, to study the regularity of (4.7), we don’t have to worry about the overall \( c\chi_h \exp[u_+(2p+S^2/3)] \). We will define the function \( F(z) \) as

\[
F(z) = \sum_{\pm} \left( c^{-1} \kappa_{\pm} \right)^{\chi_h} \left[ \left( \frac{du}{da} \right)^2 \right] \chi_h^{\chi+h} e^{2p(u_{\pm} - u_+) + S^2(T_{\pm} - u_+/3)} \cdot \sum_{\lambda} SW(\lambda) e^{-i \left( \frac{du}{da} \right)^p(S,\lambda)} e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)}. \tag{4.12}
\]

The generating function of the \( N_f = 1 \) theory will be regular at the superconformal point if and only if \( F(z) \) is regular at \( z = 0 \). In the rest of this section, we will focus on the properties of \( F(z) \).
First, we will find the structure of $F(z)$ as a power series. In principle, as $F(z)$ contains the quantity $(du/da)_\pm$, the expansion is in powers of $z^{1/4}$. However, as we now show, $F(z)$ in fact has an expansion in integral powers of $z$. Consider the $\lambda$-dependent piece of (4.12), denoted by:

$$SW^\pm(z^{1/4}) = \sum_\lambda SW(\lambda)e^{-i\left(\frac{\lambda_0}{4}\right)_{\pm}(S,\lambda)}e^{2\pi i(\lambda_0^2+\lambda\cdot\lambda_0)}.$$

(4.13)

An important property of these functions is that, if $\chi_h+\sigma$ is even (odd), they only contain even (odd) powers of $(du/da)_\pm$. This is due to the fact that, if $\lambda$ is a basic class, then $-\lambda$ is also a basic class, and

$$SW(-\lambda) = (-1)^{\chi_h}SW(\lambda).$$

(4.14)

On the other hand, changing $\lambda$ to $-\lambda$ in the phase in (4.13) introduces a global factor

$$e^{-4\pi i\lambda_0\cdot\lambda} = (-1)^\sigma,$$

(4.15)

as one can easily check using Wu’s formula. If one decomposes $F(z)$ as the sum of the contributions at $u_\pm$, $F(z) = F_+(z) + F_-(z)$, it follows from the above analysis that $F_\pm(z)$ only contain even powers of $(du/da)_\pm$, respectively. Therefore, as $\chi_h$ is an integer, the functions $F_\pm$ have a series expansion in powers of $z^{1/2}$. Actually, more is true. As we remarked in section 3.2, we have $F_-(z^{1/2}) = F_+(-z^{1/2})$, therefore $F(z)$ has in fact an expansion in integral powers of $z$. We have proved the following

**Proposition 4.2.1.** The function $F(z)$ defined in (4.12) has a Laurent series expansion in integral powers of $z$ around $z = 0$, i.e., there is no monodromy around $z = 0$.

We finish this section by rewriting the expansion of $F(z)$ in a way that will be useful in section 6.1. Inserting the expansions (4.10) in $F(z)$, we obtain the expression:

$$F(z) = z^{\frac{c_1^2-\chi_h}{4}} \left\{ SW^+(z^{1/4}) + e^{\frac{\pi i}{2}(c_1^2-\chi_h)}SW^-(z^{1/4}) + z^{1/2}\left(\frac{4}{3}\right)^{4/3}\left(c_1^2 + \frac{5\chi_h}{2}\right) - \frac{S^2}{24}\right\} \left[SW^+(z^{1/4}) - e^{\frac{\pi i}{2}(c_1^2-\chi_h)}SW^-(z^{1/4})\right] + O(z).$$

(4.16)

Notice that the functions $SW^\pm(z^{1/4})$ are themselves series expansions in $z^{1/4}$. Nevertheless, we will see, however, that this way of organizing the terms turns out to be very useful. It is clear that the possible poles of this expression are due to the power of $z$ in front of the expression, as all the terms inside the curly brackets are regular at $z = 0$. Also notice that the terms that we haven’t written explicitly involve linear combination of $SW^\pm$ with coefficients that depend on $p$, $S^2$, $\chi_h$ and $c_1^2$, and to write them we need the explicit expansions of $(du/da)$ and $\kappa$ to arbitrary order.
4.3. The main result: $F$ is regular

We now state the main result in this paper. We are studying the theory with one massive quark hypermultiplet on a compact, oriented manifold $X$ of simple type, with $b^+_2 > 1$. There are two sources of divergence in the correlation functions of a topological quantum field theory: one is due to the noncompactness of spacetime, and the other is the noncompactness of a moduli space of vacua. In our case, neither source of divergence is present: by assumption, the spacetime manifold is compact. On the other hand, when $b^+_2 > 1$, as it was proved in [2], the only contributions to the correlation functions come from a finite set of points in the $u$-plane, and the moduli space of vacua associated to the low-energy theory -the moduli space of the SW monopole equations- is also compact. We then have the following

(Physical) Theorem 4.3.1. The function $F(z)$ is analytic at $z = 0$, i.e., it has a Taylor series expansion around the origin.

One of the key physical ingredients to guarantee the regularity of $F(z)$ is the absence of Higgs branches in the moduli space, in the case of $N_f = 1$. These branches are noncompact and could be a source of divergences in the path integral. For the theories with $N_f \geq 2$, there are superconformal points of type $(k,1)$, with $k > 1$. In this case, the low-energy colliding singularity with $k$ mutually local hypermultiplets will have a noncompact Higgs branch, therefore we can not guarantee the regularity of the generating function at the superconformal point. This noncompactness of the moduli space of vacua is the source of the noncompactness of the moduli space of solutions to the monopole equations with $k > 1$ monopole fields. See, for examples, eq. (6.4) of [16] or equations (3.26) and (3.27) of [19]. When there is more than one term on the right-hand-side of the $F^+ = \sum M\Gamma M$ equation then the moduli space is noncompact.

One could worry that, although for $N_f = 1$ the moduli space of vacua is compact for any value of the mass arbitrarily close to its critical value, a Higgs branch might appear in the moduli space precisely when $m = m_*$. Indeed, the phenomenon of a “jump” in the Higgs branch for special values of parameters is a well-known phenomenon in supersymmetric gauge theory and string theory. Indeed, when mutually local singularities collide the Higgs branch does jump leading to the noncompact moduli spaces discussed above in theories with $N_f \geq 2$. However, it was shown in [3] that there are no jumps in the Higgs branch at the superconformal points where non-mutually local singularities collide.
This distinction was called the “Higgs branch criterion” in [3] and was a key technical tool used to find the nontrivial superconformal theories in the $SU(2)$ theories with matter. According to the Higgs branch criterion, the Higgs branch at the $(k, 1)$ points is the Higgs branch of the theory with $k$ hypermultiplets: there is no extra source of noncompactness when the masses take their critical values. This shows in particular that, in the $N_f = 1$ theory that we are studying, the moduli space of vacua will remain compact for $m = m_*$. 

From a strictly mathematical point of view, we should consider Theorem 4.3.1 as a conjecture, but we should also emphasize that the above result is rigorous at the physical level. As we will see, this theorem has far-reaching implications for the topology of four-manifolds: looking at (4.16), it is clear that the analyticity of $F(z)$ will give a strong set of constraints on the structure of $SW^\pm$ if the overall factor has a negative exponent. As this exponent depends on the numerical invariants of the manifold under consideration, we conclude that, for certain values of these invariants, the basic classes of the manifold and their SW invariants will have to satisfy certain nontrivial relations.

This result is a generalization of the constraint $x^2 = 2\chi + 3\sigma$ on basic classes for manifolds of simple type, and shows that the superconformal points of the $\mathcal{N} = 2$ gauge theories give predictions on four-manifold topology with a very different flavor from previous results on the “geography problem” (described below). The new constraints reveal a relation between two a priori unrelated quantities: the classical topological invariant $c_1^2 - \chi_h$, and the diffeomorphism invariants obtained from gauge theory.

In section 6 we develop the consequences of this physical theorem. In order to appreciate these we first review, in the next section, the geography problem, and in particular what is known about possible values of $c_1^2$ and $\chi_h$. Since our theorem is not mathematically rigorous we check it in a large number of examples in section 7.

5. A brief review of the geography of four-manifolds

The geography problem is one of the most active areas of research in four-manifold topology. Here we will give a brief review of the results on geography, focusing on the most relevant aspects for our work. An updated review with a list of references can be found in [31].
5.1. The geography of four-manifolds

The numerical invariants of a manifold are topological invariants given by integers. The simplest examples are the Euler characteristic $\chi$ and the signature $\sigma$. The geography problem for four manifolds can be formulated as follows: which pairs of integers can be realized as $(\chi, \sigma)$ of a smooth four-manifold $X$, and what is the influence of these values on the geometry of the four-manifold? To investigate the geography problem, we can also use the quantities $c_1^2$ and $\chi_h$ introduced in (4.8), and this is what we will do to facilitate comparison with mathematical results. Notice that $\chi_h$ (defined in (4.3) above) is not necessarily an integer for a generic four-manifold. The condition that $\chi_h$ be an integer is called the Noether condition. In particular, any almost-complex manifold satisfies it.

To analyze the geography problem, one should take into account that there are simple topological constructions that can change the value of the numerical invariants. The most important of these constructions is the blow-up process. The smooth blow-up of a four-manifold $X$ is simply the connected sum $\hat{X} = X \# \mathbb{CP}^2$. This construction can be considered in the complex and the symplectic categories (i.e., one can perform the blow-up in such a way that the resulting manifolds are complex or symplectic, respectively). Notice that, under a blowup, the numerical invariants of the four-manifold change as follows:

\[
c_1^2(\hat{X}) = c_1^2(X) - 1, \quad \chi_h(\hat{X}) = \chi_h(X).
\]

Therefore, we can generate arbitrarily low values of $c_1^2$ just by considering a succession of blow-ups. This is why the study of geography of complex and symplectic four-manifolds focuses on minimal four-manifolds. A complex surface is minimal if it doesn’t contain a holomorphically embedded sphere of square $(-1)$. A minimal symplectic four-manifold can be defined in a similar way by changing “holomorphically embedded curve” to “symplectically embedded sphere” in the above definition. Notice that a spin complex or symplectic manifold is always minimal: by the Wu formula $(w_2(X), \alpha) = \alpha^2 \mod 2$, so if there is a class with $\alpha^2 = -1$ then $w_2(X) \neq 0$.

We have stated the geography problem for general four-manifolds, but in fact the problem has been studied by restricting to more specific subsets, where it can be analyzed in more detail, and then enlarging progressively the set of manifolds under consideration. The most complete results have been obtained for complex surfaces. Symplectic manifolds have been also considered in some detail, but are not classified. We will consider the case of complex surfaces separately, and then we will briefly review what is known in the symplectic case, as well as in more general situations.
5.2. Geography of complex surfaces

We will review now some useful facts about the geography of minimal complex surfaces.

Complex surfaces can be classified according to the Kodaira dimension $\kappa(S)$. This quantity measures the number of holomorphic $(2n,0)$ forms for large $n$ and can take the values $-\infty, 0, 1$ and $2$ \cite{34} \cite{33}. In fact, the Kodaira dimension already gives important restrictions on the possible values of the numerical invariants. We have the following possibilities:

![Diagram of the geography of four-manifolds](image)

**Fig. 1:** The geography of four-manifolds. The various lines are explained in the text. No known irreducible manifolds lie below the $\chi_h$ axis. The non-existence of such manifolds is the “3/2 conjecture.” No known spin manifolds lie below the “11/8 line.” The non-existence of such manifolds is the “11/8 conjecture”.

\footnote{Since we have no wish to write a textbook on 4-manifolds we use several terms here without definition. The reader should consult, e.g., \cite{31} \cite{32} \cite{33} or \cite{34} Chapter V, section 5, pages 146-147 and Chapter VI.}
• The minimal surfaces with $\kappa(S) = -\infty$ are $\mathbb{C}P^2$, geometrically ruled (i.e. a sphere bundle over a Riemann surface) or of Kodaira class VII. All these surfaces have $b_2^+ = 0$ (for class VII) or 1. For a geometrically ruled surface, one has $c_1^2 = 8(1 - g)$ and $\chi_h = 1 - g$. We won’t consider surfaces with $\kappa(S) = -\infty$ in the following, since they have $b_2^+ \leq 1$.

• The minimal surfaces with $\kappa(S) = 0$ have $c_1^2 = 0$ and $\chi_h \geq 0$. In fact, there are only five types of surfaces in this class: Enriques surfaces, bielliptic surfaces, Kodaira surfaces (primary and secondary), abelian surfaces (tori), and $K3$ surfaces. Enriques, bielliptic and secondary Kodaira surfaces have $b_2^+ \leq 1$. Abelian varieties and primary Kodaira surfaces have $\chi_h = 0$, and $K3$ surfaces have $\chi_h = 2$. Again, we will be only interested in the surfaces with $b_2^+ > 1$, which in this case are tori, $K3$ and primary Kodaira surfaces. All of them are in fact elliptic fibrations, which we consider now.

• An elliptic fibration is a complex surface $S$ together with a holomorphic fibration $\pi : S \rightarrow \Sigma_g$ over a Riemann surface of genus $g$, where the generic fibres are elliptic curves. All the minimal surfaces with Kodaira dimension $\kappa(S) = 1$ are elliptic fibrations, but the converse is not true, as the examples above show. Any minimal elliptic surface has $c_1^2 = 0$ and $\chi_h \geq 0$. Actually, all the nonnegative values of $\chi_h$ are realized. Therefore, the minimal elliptic fibrations fill the line $c_1^2 = 0$ in the $(\chi_h, c_1^2)$ plane (see fig. 1).

• The surfaces with Kodaira dimension $\kappa(S) = 2$ are called surfaces of general type. For minimal surfaces of general type, one has $c_1^2 > 0$, $\chi_h > 0$. The geography problem for minimal surfaces of general type has been investigated extensively, and the following important bounds have been obtained:

$$2\chi_h - 6 \leq c_1^2 \leq 9\chi_h. \quad (5.2)$$

The lower bound is given by the Noether line $c_1^2 = 2\chi_h - 6$, and the corresponding inequality is called the Noether inequality, while the upper bound corresponds to the Bogomolov-Miyaoka-Yau (BMY) line. It has also been proved that all the positive integers $(c_1^2, \chi_h)$ in the region $2\chi_h - 6 \leq c_1^2 \leq 8\chi_h$ are, in fact, actually realized by minimal surfaces of general type (see [31] for a detailed discussion). The line $c_1^2 = 8\chi_h$ is called the 0-signature line. It is not yet known if all the points in the “arctic region” $8\chi_h \leq c_1^2 \leq 9\chi_h$ are inhabited by simply-connected minimal surfaces of general type.

This is all we will need about complex surfaces. We will consider now the geography problem for more general manifolds.
5.3. Geography of symplectic manifolds and beyond

Symplectic manifolds have played an important role in recent developments on four-manifold topology. For some time it was thought that, although the class of symplectic manifolds was strictly larger than the class of Kähler manifolds, the non-Kähler symplectic manifolds were rather special, comprising perhaps only a few examples in the simply connected case. The new constructions of symplectic manifolds by Gompf [35] shattered this view, and now many examples of non-Kähler symplectic manifolds are known. Therefore, one can wonder about the geography problem for minimal, symplectic manifolds. Interestingly enough, some of the results have been obtained using the results by Taubes about the SW invariants of symplectic manifolds [36][37]. This is a simple consequence of Taubes’ results that, for minimal, simply-connected, symplectic four-manifolds one has $c_1^2 \geq 0$. Moreover, it can be shown that there are manifolds in this category that violate the Noether inequality. Indeed, all the values $(c_1^2, \chi_h)$ satisfying $0 \leq c_1^2 \leq 2\chi_h - 6$ are realized by minimal, simply connected, symplectic manifolds (for some examples of these manifolds, see [35]).

A natural extension of symplectic manifolds is that of irreducible manifolds. A smooth four-manifold is called irreducible if, for every smooth connected sum decomposition $X = X_1 \# X_2$, either $X_1$ or $X_2$ is homeomorphic to $S^4$. In other words, an irreducible four-manifold is not the connected sum of non-trivial four-manifolds. It also follows from Taubes’ theorems on SW invariants for symplectic manifolds that a simply connected, minimal, symplectic four-manifold with $b_2^+ > 1$ is irreducible. It was thought for some time that symplectic manifolds are the building blocks of irreducible, simply connected manifolds, but once again this view was shattered by the discovery of many examples of irreducible, simply-connected manifolds that do not admit a symplectic structure [38][39]. Again, SW invariants played a very important role in these constructions.

Many questions regarding the geography problem for irreducible four-manifolds have not been solved yet. It has been conjectured, for example, that the inequality $c_1^2 \geq 0$ holds for irreducible manifolds (this conjecture is called the $3/2$ conjecture). In addition, irreducible manifolds do not necessarily satisfy the Noether condition. When this condition does not hold, all the SW invariants are zero and very little is known.

Another conjecture in geography deals with simply connected, spin manifolds. The intersection form of this class of manifolds is equivalent to

$$Q = 2kE_8 \oplus \ell I^{1,1},$$  \hfill (5.3)
where $E_s$ denotes the Cartan matrix of this Lie algebra, and $II^{1,1}$ is the even unimodular rank two lattice. Notice that $k$ can be positive or negative, depending on the sign of the signature $\sigma = 16k$. The 11/8 conjectures states that $l \geq 3|k|$. For manifolds with negative signature, this corresponds to the inequality

$$c_1^2 \geq \frac{8}{3}(2 - \chi_h).$$

This is the line denoted by “11/8” in fig. 1. Evidently, the 3/2 conjecture implies the 11/8 conjecture.

6. New results on the geography of four-manifolds

In this section, we will use our main result in section 4 (namely, that the correlation functions are finite) to extract some interesting information about the geography of four manifolds. We will find a set of constraints that relate the structure of the SW invariants and the basic classes to the value of $c_1^2 - \chi_h$. First we will find a sufficient condition for the correlation functions to be finite and we will introduce the important concept of manifolds of superconformal simple type. Then we will consider more general possibilities for analyticity of $F(z)$ in some detail. Finally, we study manifolds of simple type with only one basic class, and we prove (using our main result) that their numerical invariants must satisfy the inequality $c_1^2 \geq \chi_h - 3$.

6.1. A sufficient condition for regularity: manifolds of superconformal simple type

In this subsection, we give a simple sufficient condition for $F(z)$ to be analytic. To do this, it is convenient to work with Laurent series in integral powers of $z$.

We consider first

$$SW_0(z) = z^{c_1^2 - \chi_h} \left( SW^+(z^{1/4}) + e^{\frac{2\pi i}{3}(c_1^2 - \chi_h)} SW^-(z^{1/4}) \right),$$

$$SW_1(z) = z^{c_1^2 - \chi_h - 2} \left( SW^+(z^{1/4}) - e^{\frac{2\pi i}{3}(c_1^2 - \chi_h)} SW^-(z^{1/4}) \right).$$

Notice that these are precisely the combinations that appear in (4.16). It is easy to prove that these functions have a power series expansion in integral powers of $z$. There are two

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4 Actually, there is a small region below the 11/8 line but above the $\chi_h$ axis. It is easy to check that all the integral points in this region are inconsistent with the spin condition.
cases to consider, depending on the parity of $\chi_h + \sigma$. If $\chi + \sigma$ is even, then $\mathcal{S}\mathcal{W}^\pm$ have a series expansion in powers of $z^{1/2}$, as we saw in section 4. But in this case

$$c_1^2 - \chi_h = 7\chi_h + \sigma$$  \hfill (6.2)

is also even. Therefore, $z^\frac{c_1^2 - \chi_h}{4}\mathcal{S}\mathcal{W}^+(z^{1/4})$ only contains even powers of $z^{1/4}$. Under $z^{1/2} \to -z^{1/2}$, this function changes as

$$z^{-\frac{c_1^2 - \chi_h}{4}}\mathcal{S}\mathcal{W}^+(z^{1/4}) \to e^{\frac{\pi i}{2}(c_1^2 - \chi_h)z^{-\frac{c_1^2 - \chi_h}{4}}}\mathcal{S}\mathcal{W}^-(z^{1/4}).$$  \hfill (6.3)

Therefore, the sum of these two terms only contains integral powers of $z$, and their difference only contains half-integral powers of $z$. As $\mathcal{S}\mathcal{W}_1$ has an extra power of $z^{1/2}$, we have proved our claim. The analysis when $\chi + \sigma$ is odd is completely analogous.

In an analogous way we can define

$$\mathcal{A}^{(0)} \equiv \frac{1}{2}z^{-\frac{(c_1^2 - \chi_h)}{4}}\left(\mathcal{A}^+ + e^{-\frac{\pi i}{2}(c_1^2 - \chi_h)}\mathcal{A}^-\right),$$

$$\mathcal{A}^{(1)} \equiv \frac{1}{2}z^{-\frac{(c_1^2 - \chi_h+2)}{4}}\left(\mathcal{A}^+ - e^{-\frac{\pi i}{2}(c_1^2 - \chi_h)}\mathcal{A}^-\right),$$

$$\mathcal{A}^\pm \equiv (c^{-1}k_{\pm})^{\chi_h}\left[\left(\frac{du_\pm}{da}\right)^2\right]^\chi_h\frac{\chi_h+\sigma}{2}e^{2p(u_\pm - u_*+S^2(T_\pm - u_*/3)).}$$  \hfill (6.4)

In terms of these functions we may write

$$F(z) = \mathcal{A}^{(0)}(z)\mathcal{S}\mathcal{W}_0(z) + \mathcal{A}^{(1)}(z)\mathcal{S}\mathcal{W}_1(z).$$  \hfill (6.5)

The advantage of this representation is that, by an argument analogous to that for $\mathcal{S}\mathcal{W}_0, \mathcal{S}\mathcal{W}_1$ we see that $\mathcal{A}^{(0)}, \mathcal{A}^{(1)}$ have power series in integral nonnegative powers of $z$:

$$\mathcal{A}^{(0)} = \sum_{k=0}^{\infty} A_k^{(0)} z^k,$$

$$\mathcal{A}^{(1)} = \sum_{k=0}^{\infty} A_k^{(1)} z^k,$$  \hfill (6.6)

where the coefficients are polynomials in $p, S^2, \chi_h$ and $c_1^2$. For instance, working at next-to-leading order in the expansions in (3.13), we find

$$A_0^{(0)} = 1, \quad A^{(1)}_0 = \frac{4}{3} \left(\frac{c_1^2 + 5\chi_h}{2}\right) - \frac{S^2}{24}.$$  \hfill (6.7)
corresponding to the first terms written in (4.16). Of course, the next terms become increasingly complicated.

It would be extremely useful to have a precise criterion for the regularity of \( F(z) \) at \( z = 0 \). It is clear from (6.5) that, if \( SW_0(z) \), \( SW_1(z) \) are regular at \( z = 0 \), then \( F(z) \) will be regular as well. We will now state a theorem that gives a simple condition for \( F(z) \) to be regular. In this theorem, the basic classes are regarded as functionals acting on \( H^2(X; \mathbb{Z}) \) through the intersection form: \( \lambda(S) = (\lambda, S) \). More generally, the functionals \( \lambda^n \) act on \( \text{Sym}^n(H^2(X; \mathbb{Z})) \) as follows:

\[
\lambda_n(S_1 \cdots S_n) = (\lambda, S_1) \cdots (\lambda, S_n). \tag{6.8}
\]

**Theorem 6.1.1.**

a.) If \( \chi_h - c_1^2 - 4 < 0 \) (i.e., \( c_1^2 \geq \chi_h - 3 \)) then \( F(z) \) is regular at \( z = 0 \).

b.) If \( \chi_h - c_1^2 - 4 \geq 0 \) and the following relations are satisfied

\[
\sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} \chi^k = 0, \quad k = 0, \ldots, \chi_h - c_1^2 - 4, \tag{6.9}
\]

then the function \( F(z) \) is regular at \( z = 0 \).

Notice that, if \( \chi_h + \sigma \) is even (odd), the expressions of the form (6.9) with \( k \) odd (even) are automatically zero. Therefore, there are actually \((\chi_h - c_1^2 - 4)/2 + 1\) nontrivial conditions if \( \chi_h + \sigma \) is even, and \((\chi_h - c_1^2 - 3)/2\) nontrivial conditions if \( \chi_h + \sigma \) is odd.

**Proof:** If condition (a) holds then the leading power of \( z \) in \( SW_0(z) \) and \( SW_1(z) \) is greater than or equal to \( z^{-3/4} \). Since these series have no monodromy, they must be regular. Therefore we may assume condition (b) holds.

To prove the theorem, we will show that the condition (6.9) is equivalent to the regularity of \( SW_0(z) \) and \( SW_1(z) \) at \( z = 0 \). To do this, we have to consider the different values of \( c_1^2 - \chi_h \). If \( c_1^2 - \chi_h \) is even, then \( \chi_h + \sigma \) is even, and \( SW^+(z^{1/4}) \) has the expansion

\[
SW^+(z^{1/4}) = \sum_{n=0}^{\infty} a_{2n} z^{n/2}. \tag{6.10}
\]

If \( c_1^2 - \chi_h \) is odd, \( \chi_h + \sigma \) is odd as well and one has

\[
SW^+(z^{1/4}) = \sum_{n=0}^{\infty} a_{2n+1} z^{(2n+1)/4}. \tag{6.11}
\]

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The structure of the series \( SW_0(z) \) and \( SW_1(z) \) depends on the value of \( c_1^2 - \chi_h \mod 4 \).

We then have four different cases:

1) Suppose that \( c_1^2 - \chi_h = 4r \). By condition \((b)\), \( r \) is a negative integer. Using the definitions \((6.1)\), one finds

\[
SW_0(z) = 2 \sum_{p=0}^{\infty} a_{4p} z^{p+r}, \quad SW_1(z) = 2 \sum_{p=0}^{\infty} a_{4p+2} z^{p+r+1}.
\] (6.12)

It follows that \( SW_0 \) and \( SW_1 \) are regular at \( z = 0 \) if and only if

\[
a_{2n} = 0 \quad \text{for} \quad 0 \leq 2n \leq \chi_h - c_1^2 - 4 = 4(|r| - 1).
\]

2) If \( c_1^2 - \chi_h = 4r + 2 \), \( r < -1 \) by condition \((b)\). In this case, one finds:

\[
SW_0(z) = 2 \sum_{p=0}^{\infty} a_{4p+2} z^{p+r+1}, \quad SW_1(z) = 2 \sum_{p=0}^{\infty} a_{4p} z^{p+r+1}.
\] (6.13)

Again, we find that \( SW_0 \) and \( SW_1 \) are regular at \( z = 0 \) if and only if \( a_{2n} = 0 \) for \( 2n \leq \chi_h - c_1^2 - 4 = 4|r| - 6 \).

3) If \( c_1^2 - \chi_h = 4r + 1 \), \( r < -1 \) by condition \((b)\). The series \( SW_0(z) \) and \( SW_1(z) \) have the form

\[
SW_0(z) = 2 \sum_{p=0}^{\infty} a_{4p+3} z^{p+r+1}, \quad SW_1(z) = 2 \sum_{p=0}^{\infty} a_{4p+1} z^{p+r+1}.
\] (6.14)

These functions are regular if and only if \( a_{2n+1} = 0 \) for \( 2n + 1 \leq \chi_h - c_1^2 - 4 = 4|r| - 5 \).

4) Finally, if \( c_1^2 - \chi_h = 4r + 3 \), condition \((b)\) imposes again \( r < -1 \). The series \( SW_0(z) \) and \( SW_1(z) \) are now

\[
SW_0(z) = 2 \sum_{p=0}^{\infty} a_{4p+1} z^{p+r+1}, \quad SW_1(z) = 2 \sum_{p=0}^{\infty} a_{4p+3} z^{p+r+2}.
\] (6.15)

The necessary and sufficient condition for regularity of these series at \( z = 0 \) is again that \( a_{2n+1} = 0 \) for \( 2n + 1 \leq \chi_h - c_1^2 - 4 = 4|r| - 7 \).

In all the cases, we find that \( SW_0 \) and \( SW_1 \) are regular at \( z = 0 \) if and only if \( a_k = 0 \) for \( k \leq \chi_h - c_1^2 - 4 \), and \( k \) even (odd) for \( \chi_h + \sigma \) even (odd). In order to relate the coefficients \( a_k \) in the expansions \((6.10)-(6.11)\) to the SW invariants, we have to expand \( (du/da)^2_+ \) in powers of \( z^{1/2} \):

\[
(du/da)^2_+ = z^{1/2} (1 + b_1 z^{1/2} + \ldots),
\] (6.16)
where \( b_1 = (4/3)^{1/3} \). One finds that

\[
a_k = \frac{(-i)^k}{k!} \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda, S)^k + p(a_{k-2}, \ldots),
\]

(6.17)

where \( p(a_{k-2}, \ldots) \) is a linear function of the previous \( a_i, i < k, i \equiv k \mod 2 \), whose coefficients depend on the coefficients in the expansion (6.16). For example,

\[
\begin{align*}
a_0 &= \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda), \\
a_1 &= -i \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda, S), \\
a_2 &= -\frac{1}{2} \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda, S)^2, \\
a_3 &= \frac{i}{3!} \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda, S)^3 + \frac{b_1}{2} a_1, \\
a_4 &= \frac{1}{4!} \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda, S)^4 + b_1 a_2,
\end{align*}
\]

(6.18)

and so on. Clearly, \( a_k = 0 \) for \( k \leq \chi_h - c_1^2 - 4 \) if and only if

\[
\sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda, S)^k = 0, \quad k = 0, \ldots, \chi_h - c_1^2 - 4,
\]

(6.19)

and this must be true for every \( S \in H^2(X; \mathbb{Z}) \). As the intersection form is nondegenerate, we obtain the condition (6.9) stated in the theorem. ♠

Therefore, we see that, if a manifold of simple type satisfies the conditions (6.9), regularity of \( F(z) \) will be automatically guaranteed, without any further knowledge of the expansions of the physical quantities around the superconformal point. We stress that (6.9) is only a sufficient condition for regularity, and we will see in the next section that there are other possibilities for \( F(z) \) to be regular. These possibilities are much more complicated and involve next-to-leading terms in the expansion in \( z \). The simplicity of the condition (6.9) suggests the following

**Definition 6.1.2.** Let \( X \) be a compact, oriented manifold with \( b_2^+ > 1 \) and of simple type. We say that \( X \) is of superconformal simple type if it satisfies any of the conditions stated in the previous theorem, i.e. if \( c_1^2 \geq \chi_h - 3 \) or \( \chi_h - c_1^2 - 4 \geq 0 \) and the relations (6.9) hold.

**Remarks**

1. Notice that any manifold with trivial SW invariants is of superconformal simple type.
2. The concept of a manifold of superconformal simple type has some resemblances with the concept of manifold of simple type. From the results of [13][2], it has become clear
that a manifold is of simple type if, in the expansion of the different quantities around
the monopole singularity, only the leading term is relevant. Similarly, a manifold is of
superconformal simple type if, in the analysis of regularity around the superconformal
point, only the leading term is relevant. This means in particular that the constraints
on the geometry of the manifold that follow from the regularity of \( F(z) \) are dictated
only by the universal behavior of the critical theory (i.e. by the anomalous dimensions
of the operators). Notice that the quantum field theory analysis does not imply that
a manifold with \( b_2^+ > 1 \) is of simple type, and in the same way our result that the
generating function is regular at the superconformal point does not imply that a
manifold is of superconformal simple type.

3. From the mathematical point of view, there are also some similarities between the two
concepts. We will show in section 7 that all complex surfaces are of superconformal
simple type, and that this property is preserved under blowup and other standard
constructions. Indeed, we haven’t found any example of a manifold of \( b_2^+ > 1 \) and of
simple type which is not of superconformal simple type.

4. An alternative characterization of manifolds of superconformal simple type is the
following. For any manifold \( X \) with \( b_2^+ > 1 \) and of simple type, one can define the
following SW series with magnetic flux \( w_2(E) = w_2(X) \):

\[
SW_X w_2(X) = \sum_x (-1)^{c_2 + c_0} x SW(x) e^x, \tag{6.20}
\]

where the notations are as in section 4.1. As in (6.9), the exponential \( e^x \) is understood
here as a multilinear map on \( \text{Sym}^*(H^2(X, \mathbb{Z})) \). Notice that (6.20) is not the usual
SW series considered in the mathematical literature (see, for example, [39]), which
is defined with zero magnetic flux. According to the result of Witten [15], the series
(6.20) is related to the Donaldson series for \( w_2(E) = w_2(X) \) as follows:

\[
D_X w_2(X) = 2^{2+c_1^2-\chi_h} e^{Q/2} SW_X w_2(X), \tag{6.21}
\]

where \( Q \) is the intersection form. Consider now the following holomorphic function of
\( z \), obtained from the SW series after replacing \( x \to zx \):

\[
SW_X w_2(X)(z) = \sum_x (-1)^{c_2 + c_0} x SW(x) e^{zx}. \tag{6.22}
\]

However, from the QFT viewpoint the conjecture that all manifolds of \( b_2^+ > 1 \) are of simple
type is rather natural.
If we expand this holomorphic function around $z = 0$, we find

$$SW_{X}^{w2}(X)(z) = \sum_{n=0}^{\infty} \left( \sum_{x} (-1)^{\frac{c_{2}^{x}+c_{0}^{x}}{2}} SW(x)x^{n} \right) \frac{z^{n}}{n!}.$$  \hspace{1cm} (6.23)

According to our definition, $X$ is of superconformal simple type if $c_{1}^{2} \geq \chi_{h} - 3$ or $\chi_{h} - c_{1}^{2} - 4 \geq 0$ and the first $\chi_{h} - c_{1}^{2} - 4$ coefficients of (6.23) are zero. Therefore,

**Proposition 6.1.3.** $X$ is of superconformal simple type if and only if $SW_{X}^{w2}(X)(z)$ has a zero at $z = 0$ of order $\geq \chi_{h} - c_{1}^{2} - 3$.

Notice that, depending on the parity of $\chi_{h} + \sigma$, we will have even or odd powers of $z$ in (6.23). If a manifold is of superconformal simple type and $\chi_{h} - c_{1}^{2} - 4 \geq 0$, the order of the zero in the series (6.22) is in fact greater or equal than $\chi_{h} - c_{1}^{2} - 2$. Proposition 6.1.3 will be very useful in section seven: given a manifold of $b_{2}^{+} > 1$ and of simple type, to check that it is of superconformal simple type we only have to compute the SW series (6.22) and examine the order of vanishing at $z = 0$. Notice that, in (6.22), $z$ is a formal variable, different from the physical expansion parameter considered before, although it plays a similar role from the point of view of the analyticity. As we will see in section 7, the sign in (6.22) depending on the second Stiefel-Whitney class is crucial. It is interesting that the analysis of the Donaldson series as a series in the holomorphic variable $z$ is one of the key steps in the proof of the structure theorem of Kronheimer and Mrowka [11].

**6.2. Examples of more general conditions**

As we have remarked, the conditions (6.9) are not the most general conditions to achieve regularity of $F(z)$. In this section, we will briefly examine the general condition for regularity of $F(z)$ that can be derived from the expression (6.5). We just expand this expression in powers of $z$ in the usual way and write the conditions derived from it. The exact form of the conditions depends on the residue class of $c_{1}^{2} - \chi_{h}$ mod 4. We will only consider in some detail the case of $c_{1}^{2} - \chi_{h} = 4r$. The other cases are similar and can be easily worked out.

If $c_{1}^{2} - \chi_{h} = 4r$, the series $SW_{0}, SW_{1}$ have the structure given in (6.12). Using (6.7), we find that $F(z)$ has the following expansion:

$$F(z) = 2a_{0} \sum_{k=0}^{\infty} A_{k}^{(0)} z^{r+k} + 2 \sum_{k=0}^{\infty} \left( \sum_{p=0}^{k} A_{k-p}^{(0)} a_{4p+4} + A_{k-p}^{(1)} a_{4p+2} \right) z^{r+k+1}. \hspace{1cm} (6.24)$$
Suppose that \( r < 0 \). Then, as \( A_0^{(0)} = 1 \), if \( F(z) \) is regular we necessarily have

\[
a_0 = \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} = 0.
\]

(6.25)

If \( r < -1 \), the next conditions are more involved. It follows from (6.24) that we need

\[
\sum_{k=0}^{\lfloor |r| - 2 \rfloor} A_k^{(0)} a_{4p+4} + A_k^{(1)} a_{4p+2} = 0, \quad k = 0, 1, \ldots, |r| - 2.
\]

(6.26)

If the manifold is of superconformal simple type, the coefficients \( a_{4p} \) and \( a_{4p+2} \) will vanish separately for \( p \leq |k| - 2 \), but in principle we could have cancellations between the coefficients with indices 0 mod 4 or 2 mod 4. It is in fact instructive to write the first condition in (6.26), corresponding to \( k = 0, r < -1 \). We have:

\[
a_4 + A_0^{(1)} a_2 = 0,
\]

(6.27)

where \( a_4, a_2 \) are given explicitly in (6.18). Notice that in analyzing these equations, we have to consider the terms with the same powers of \( S \). Therefore, the equation (6.27) contains in fact the two equations,

\[
\left( \frac{c_1^2 + 5 \chi_h}{2} + 1 \right) \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda, S)^2 = 0,
\]

\[
2 \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda, S)^4 + S^2 \sum_{\lambda} SW(\lambda) e^{2\pi i (\lambda_0^2 + \lambda \cdot \lambda_0)} (\lambda, S)^2 = 0.
\]

(6.28)

The first equation has two possible solutions: either \( c_1^2 + 5 \chi_h + 2 = 0 \), or the second factor (which is one of the sum rules we have found in (6.9)) vanishes. If the latter condition is true, then the second equation tells us that we also have the sum rule of (6.9) for \( k = 4 \). Notice that, in the first equation, if the second factor is different from zero but the first vanishes, we still have to satisfy the second equation, that involves a constraint in the intersection form of the manifold. The general picture follows this pattern: for each of the equations in (6.26), either we have a set of simple sum rules like (6.3), or we have conditions involving \( c_1^2, \chi_h \) and the intersection form of the manifold, and probing the higher order terms in the expansion of the physical quantities. The same situation holds for the other values of \( c_1^2 - \chi_h \) mod 4.

Fortunately, in the concrete manifolds that we have analysed, the more general (and complicated) possibilities for the regularity of \( F(z) \) do not play any role: as we spell out
in the next section, all known 4-manifolds are of superconformal simple type. We should stress, however, that regularity of $F(z)$ is not equivalent to the superconformal simple type condition: as we have seen, more general possibilities are allowed. The analysis of these possibilities depends very much on the manifold under consideration, but in some simple cases it can be done in detail. We will consider such an example in the next section, which also illustrates the general conditions we have been discussing.

**Fig. 2:** The line $c_1^2 = \chi_h - 3$ and the geography of four-manifolds. The manifolds under this line have to satisfy sum rules for the SW invariants.

At this point, we can discuss the results we have obtained so far in the context of the geography of four-manifolds. Our analysis shows that there is a special line in the $(\chi_h, c_1^2)$ plane that separates two well-distinguished regions (see figure 2). For $c_1^2 \geq \chi_h - 3$, the function $F(z)$ is always regular, as we have seen in section 6.1. When $c_1^2 \leq \chi_h - 4$, our result about the regularity of $F(z)$ implies a series of sum rules for the SW invariants. These rules can take a very simple form, as in (6.3), or follow more complicated patterns. It is interesting that the region where our constraints are non trivial is precisely
the most intriguing from the point of view of geography. For instance, our results put severe constraints on the SW invariants and basic classes of possible irreducible manifolds with $c^2_1 < 0$, or possible spin manifolds that violate the 11/8 conjecture.

As far as we know, the significance of this particular combination of numerical invariants, $c^2_1 - \chi_h = (7\chi + 11\sigma)/4$, has not been discussed in the mathematical literature. However, this quantity does show up as an “experimental” bound to construct manifolds with only one basic class. We will explain why this is so in the next section. Also, this invariant appears in the famous expression for the Donaldson series due to Witten [41][15] (see (6.21)), and very much for the same reasons, since it enters through the $\chi$, $\sigma$-dependent factors in the measure of the twisted theory.

6.3. Manifolds with one basic class

The purpose of this section is to extract a very concrete prediction from the regularity of $F(z)$. It follows from (4.14) that, if $x$ is a basic class, then so is $-x$. For this reason, if $B_X$ denotes the set of basic classes of $X$, we say that $X$ has $B$ basic classes if the set $B_X/\{\pm 1\}$ consists of $B$ elements. There are many examples of manifolds with only one basic class, for example the minimal surfaces of general type. There also examples of noncomplex and nonsymplectic manifolds with only one basic class [42][43][44]. The examples constructed with this property satisfy $c^2_1 \geq \chi_h - 3$, but as far as we know there is not a clear relation between this bound and the existence of only one basic class in the manifold. Here we will prove that, as a consequence of the regularity of $F(z)$, one has the following

**Theorem 6.3.1.** Let $X$ be a smooth, compact, oriented four-manifold with $b^+_2 > 1$ and of simple type. If $X$ has one basic class, then

$$c^2_1(X) \geq \chi_h(X) - 3.$$  \hspace{1cm} (6.29)

**Proof:** To prove this theorem, we will show that, if $X$ has one basic class and $c^2_1(X) < \chi_h(X) - 3$, then $F(z)$ cannot be regular, contradicting our physical theorem. We will use in fact the general conditions for regularity derived from (5.3). In this case, the analysis is relatively easy because the quantities involved in the general sum rules are simple. Assume then that $X$ has only one basic class. There are two different cases; the case when $\chi_h + \sigma$ is even, and the case when $\chi_h + \sigma$ is odd. In the first case, $c^2_1 - \chi_h$ can be 0 or 2 mod 4, and in the second case it can be 1 or 3 mod 4. We will denote the only basic class of $X$...
by $K$, therefore $SW(K) \neq 0$ by assumption. We can choose the integral lifting $c_0 = K$. Notice that, if $K = 0$, then both $\chi_h$ and $\sigma$ must be even, due to (4.14) and to the fact that $K^2 = 2\chi + 3\sigma$. As in section 6.1, we have to consider four different cases:

1) Suppose that $c_1^2 - \chi_h = 4r$, and that $r < 0$. In this case, according to (6.25), we must have $a_0 = 0$. But

$$a_0 = 2(-1)^\sigma SW(K)$$

(6.30)

if $K \neq 0$, and $a_0 = SW(0)$ if $K = 0$. Hence, $a_0 \neq 0$ and we get a contradiction.

2) Suppose that $c_1^2 - \chi_h = 4r + 2$ and $r < -1$. The condition analogous to (6.27) is now $A_0^{(1)} a_0 + a_2 = 0$. As we discussed in the previous section, this equality gives in fact two independent equations, and one of them involves the intersection form. Using the explicit expressions (6.18)(6.7) and taking into account that $SW(K) \neq 0$, we find equations analogous to (6.28):

$$c_1^2 + 5\chi_h = 0, \quad S^2 + 3(K,S)^2 = 0,$$

(6.31)

where we have taken into account that $K = 2\lambda$. The second equation in (6.31) is incompatible with unimodularity of the intersection matrix because $b_2^+ > 1$, so $b_2 > 1$ so the intersection form is more than one-dimensional. Therefore, we find again a contradiction.

3) Suppose now that $c_1^2 - \chi_h = 4r + 1$. If $r < -1$, one finds that regularity of $F(z)$ implies that $A_0^{(1)} a_1 + a_3 = 0$. This gives again two equations,

$$(c_1^2 + 5\chi_h + 1)(K,S) = 0, \quad S^2(K,S) + (K,S)^3 = 0,$$

(6.32)

where both equations hold for any $S$. Since $\chi_h + \sigma$ is odd, $K \neq 0$, so, by the nondegeneracy of the intersection form, $(c_1^2 + 5\chi_h + 1) = 0$. Now consider the second equation. We can put $S = K$, and take into account that $K^2 = c_1^2$. We then find that $c_1^2 = 0$ or $-1$. If $c_1^2 = 0$, the first equation has no solution for an integral $\chi_h$. If $c_1^2 = -1$, the first equation gives $\chi_h = 0$, but this contradicts our assumption $c_1^2 - \chi_h = 4r + 1$. Therefore, (6.32) has no solution and we find again a contradiction.

4) The case $c_1^2 - \chi_h = 4r + 3$ is similar to the first one. If $r < -1$, one finds as a necessary condition for analyticity of $F(z)$

$$a_1 = -i(-1)^\sigma (K,S)SW(K) = 0,$$

(6.33)

for any $S$. Since $K \neq 0$ (because $\chi_h + \sigma$ is odd in this case), and the intersection form is nondegenerate, we find again a contradiction. This ends the proof. ♠
This theorem proves that the lower bound for $c_1^2$ that has been found for manifolds with one basic class is in fact sharp. Therefore, the examples on the line $c_1^2 = \chi_h - 3$ saturate the bound (these examples include $E(3)$ and the manifolds $Y(n)$ constructed in [42], which will be considered in some detail in section 7.4).

A corollary of this theorem is that, if a manifold of simple type with $b_2^+ > 1$ has only one basic class, it is necessarily of superconformal simple type. In other words, we have seen that the more general conditions for regularity of $F(z)$ cannot be achieved. In fact, we suspect that this will also be the case for other manifolds, although, as we have seen in this simple case, the analysis of the conditions is rather delicate.

7. All available 4-manifolds of $b_2^+ > 1$ are of superconformal simple type

In section 4, we have argued that, on physical grounds, the generating function of the theory near the superconformal $(1,1)$ point has to be regular on every compact four-manifold with $b_2^+ > 1$ and of simple type. This leads to some nontrivial constraints relating the value of the numerical invariant $c_1^2 - \chi_h$ to the SW invariants and the structure of the basic classes. This is of course a strong statement, and the consequences are rather surprising. In this section, we will put our result to the test by actually checking that it is true for a large family of four-manifolds. We think that the analysis in this section provides compelling evidence for our result.

The strategy is the following: first, we will consider minimal complex surfaces. The analysis of this class of manifolds is fairly systematic due to the Kodaira-Enriques classification. We will show that any minimal compact complex surface with $b_2^+ > 1$ is in fact of superconformal simple type. This will be done in sections 7.1 and 7.2. As we will see, due to the results in section 6, the only nontrivial check is for elliptic fibrations. In section 7.3, we show that the blowup of a manifold of superconformal simple type is also of superconformal simple type. This, together with the results in sections 7.1 and 7.2, proves that actually any complex surface with $b_2^+ > 1$ (minimal or not) is of superconformal simple type. In sections 7.4, 7.5 and 7.6 we analyze three different topological procedures to construct four-manifolds: rational blowdowns, fiber sums along tori, and knot surgery. As we will see, these three procedures preserve the superconformal simple type condition. This result is very interesting, because most of the exotic constructions of four-manifolds (symplectic and non-symplectic) that we are aware of use these constructions, and take as their building blocks complex surfaces. Therefore, without further ado,
we can state that all the manifolds constructed using these operations and starting with manifolds of superconformal simple type will also be of superconformal simple type. The detailed examination of these constructions will show to the skeptical reader the “magic” of the superconformal simple type condition: in all the cases, one finds a remarkable balance between the value of \( c_1^2 - \chi_h \) and the order of the zero of the SW series (6.22). In section 7.7, we consider an “exotic” example: a family of symplectic manifolds under the Noether line. These manifolds are not complex, and are clearly in a region where the sum rules hold (see fig. 2). We analyze in some detail a family of manifolds filling the wedge \( 0 \leq c_1^2 \leq 2\chi_h - 6 \). Although these manifolds are of superconformal simple type due to the general results about fiber sum along tori and rational blowdown, the construction is a good example of the uses of these nice topological constructions. Finally, in section 7.8, motivated by these results, we state the conjecture that any manifold of \( b_2^+ > 1 \) and of simple type is of superconformal simple type.

7.1. Minimal surfaces of general type

We want to consider minimal complex surfaces, and see if they satisfy our physical theorem or not. If we come back for a while to the Kodaira-Enriques classification, summarized in section 5.2, we see that the only minimal models which possibly have \( b_2^+ > 1 \) are elliptic fibrations (including the complex surfaces with \( \kappa(S) = 0 \) and \( b_2^+ > 1 \) listed in section 5.2) and minimal surfaces of general type. Both of them are of simple type.

We first consider minimal surfaces of general type. As we have remarked in section 5.2, a classical result in geography states that \( c_1^2 > 0 \) and \( 2\chi_h - 6 \leq c_1^2 \). In particular, \( c_1^2 \geq \chi_h - 3 \) (see fig. 2), therefore all the minimal complex surfaces are of superconformal simple type, according to our results in section 6. Notice that minimal surfaces of general type only have one basic class up to sign, the canonical bundle \( K \). It is a very happy fact that the Noether line turns out to be above the boundary line for manifolds with only one basic class that we found in section 6.3!

7.2. Elliptic fibrations

In many aspects, elliptic fibrations are the canonical examples for our result, because they have \( c_1^2 = 0 \) and arbitrarily large \( \chi_h \). If our result is true, the structure of the SW series for these surfaces is highly constrained, and we will find that this is indeed the case. We will then explain in some detail how to characterize these manifolds topologically, and we will present the results for the structure of their basic classes and SW invariants. This
material is covered in detail in [45][46][47]. We will focus on relatively minimal elliptic surfaces, \textit{i.e.}, elliptic surfaces with no exceptional spheres of self-intersection \(-1\) in the fibers. All relatively minimal elliptic surfaces are minimal, except for the blowup of \(\mathbb{C}P^2\) at nine points.

Recall that an elliptic fibration is a complex surface \(S\) together with a holomorphic fibration \(\pi : S \to \Sigma_g\). Given an elliptic fibration, we can associate to it a line bundle \(L\) over the Riemann surface \(\Sigma_g\) with the property that \(\deg(L) = \chi_h \geq 0\). In the fibration we will have a simple fiber \(f\) as well as \(r\) multiple fibers \(f_i, i = 1, \ldots, r\) of multiplicity \(m_i\), and with the following relation in (co)homology: \(f = m_if_i\). The canonical bundle of \(S\) can be written in terms of the canonical bundle of the Riemann surface, the line bundle \(L\), and the holomorphic line bundles associated to the multiple fibers. The general expression is

\[
K_S = \pi^*(K_{\Sigma_g} \otimes L) + \mathcal{O}_S\left(\sum_i (m_i - 1)f_i\right),
\]

which has first Chern class

\[
c_1(K_S) = (\chi_h + 2g - 2)f + \sum_i (m_i - 1)f_i,
\]

and this gives \(c_1^2(S) = 0\) (since \(f^2 = 0\)). Therefore, \(\chi = 12\chi_h\). There are two different cases for the study of elliptic fibrations: elliptic fibrations with nonzero Euler number (equivalently, with \(\chi_h > 0\)) and elliptic fibrations with zero Euler number. Elliptic fibrations with \(\chi_h = 0\) and \(b_2^+ > 1\) are automatically of superconformal simple type (these include tori and primary Kodaira surfaces). Therefore, we can focus on elliptic fibrations with \(\chi_h > 0\), \textit{i.e.}, with positive Euler number. In this case, one has \(b_1(S) = 2g\) [34]. Another useful numerical invariant is the geometric genus \(p_g(S)\), which for elliptic fibrations with \(\chi_h > 0\) is given by

\[
p_g(S) = \chi_h + g - 1.
\]

On the other hand, a well-known theorem by Kodaira [45][34] states that \(b_2^+(S) = 2p_g(S) + 1\) if \(b_1(S)\) is even, therefore \(b_2^+(S) > 1\) if and only if \(p_g \geq 1\).

Now, we can analyze the SW invariants and the basic classes for minimal elliptic surfaces, that have been worked out in [46][17]. The basic classes have the form

\[
x_{d,a_i} = (\chi_h + 2g - 2 - 2d)f + \sum_i (m_i - 1 - 2a_i)f_i, \quad 0 \leq d \leq \chi_h + 2g - 2, \quad 0 \leq a_i \leq m_i - 1,
\]

\footnote{For details, see [45], section 1.3.5. Technically \(L = (R^1\pi_*\mathcal{O}_S)^{-1}\).}
and the corresponding SW invariants are

\[
SW(x_{d,a_i}) = (-1)^d \left( \frac{\chi_h + 2g - 2}{d} \right).
\] (7.5)

Notice that, since \( p_g \geq 1 \) and \( g \geq 0 \), we always have \( \chi_h + 2g - 2 \geq 0 \). We want to compute the SW series (6.22). As \( c_1(K_S) \equiv w_2(X) \mod 2 \), we can choose the integral lifting \( c_0 = c_1(K_S) \). As we noticed in (4.5), another choice of the lifting will give an overall \( \pm 1 \) in the series, therefore it won’t change the analyticity property that we want to verify. With this choice of the lifting, the series (6.22) is just the usual one and we find the holomorphic function

\[
SW_X^{w_2(X)}(z) = 2^{\chi_h+2g-2} \left( \frac{\sinh(zf)}{\sinh(zf_i)} \right)^{\chi_h+2g-2} \prod_{i=1}^{r} \frac{\sinh(zf)}{\sinh(zf_i)}. \] (7.6)

This has a zero at \( z = 0 \) of order \( \chi_h + 2g - 2 \), which is greater than \( \chi_h - 3 \), for any \( g \). Note that, since \( f = m_i f_i \), the factors in the product in (7.6) are polynomials \( U_{m_i-1}(\cosh zf_i) \) where \( U_n(x) \) is the Chebyshev polynomial of the second kind.

In conclusion, elliptic fibrations are of superconformal simple type. Notice that the presence of multiple fibers does not affect the order of vanishing of the SW series. This is a consequence of a general result that will be proved in section 7.4: the log transform of a manifold of superconformal simple type is also of superconformal simple type.

There is a special class of elliptic fibrations that will play an important role in the examples of section 7.5. These are the simply connected elliptic fibrations over \( \mathbb{C}P^1 \), without multiple fibers, and with \( \chi_h = n \). They are usually denoted by \( E(n) \). One can see that \( E(2) \) is a K3 surface (it corresponds to \( \chi_h = 2, g = 0 \) in the computation above), and \( E(1) \) (which has \( b_2^+ = 1 \)) is the rational elliptic surface obtained by blowing up \( \mathbb{C}P^2 \) at nine points at the intersection of two different cubics. The mechanism for the regularity of \( F(z) \) is very clear in these examples: as we increase \( \chi_h \) and the degree of divergence of the prefactor in (1.16), new basic classes appear in such a way that the SW series has a zero with the appropriate order to compensate the divergence.

In conclusion, we have shown that all minimal complex surfaces are of superconformal simple type. We consider now the behavior under blowup, another canonical construction in four-manifold topology.
The blowup process is a crucial ingredient in the study of algebraic geometry and four-manifolds. We study it here in the smooth category.

The effect of the blowup on the numerical invariants is to decrease the value of \( c_1^2 \), keeping \( \chi_h \) fixed, as we have seen in (5.1). By performing an arbitrarily large number of blow-ups on a four-manifold, we can decrease \( c_1^2 - \chi_h \) as much as we want, so if our result is true the SW invariants and the structure of the basic classes have to change in such a process. We will prove here the following

**Theorem 7.3.1.** Let \( X \) be a manifold of superconformal simple type. Then, the blown up manifold \( \hat{X} = X \# \mathbb{C}P^2 \) is also of superconformal simple type.

**Proof:** The proof is rather easy using the known behavior of the SW invariants under blowup [12]. First, recall that manifolds of superconformal simple type, according to the definition given in section 6.1, must have \( b_2^+ > 1 \) and must be of simple type. The blowup preserves these two properties, so the statement of the theorem makes sense (the fact that the blowup of a manifold of simple type is of simple type is also a result of Fintushel and Stern [12].) The basic classes of \( \hat{X} \) are given by \( x \pm E \), where \( x \) is a basic class in \( X \) (more precisely, the proper transform of \( x \) in \( \hat{X} \)), and \( E \) is the exceptional divisor, with \( E^2 = -1 \). The SW invariants of these basic classes are given by:

\[
SW(x \pm E) = SW(x).
\]

By assumption, the holomorphic function \( SW_{X}^{w_2(X)}(z) \) (6.22) has a zero of order \( \geq \chi_h(X) - c_1^2(X) - 3 \) at \( z = 0 \). To compute the SW series (6.22) of the new manifold \( \hat{X} \), we have to be careful with \( w_2(\hat{X}) \), because the blowup changes the second Stiefel-Whitney class. We can take the integral lifting \( \hat{c}_0 = c_0 + E \), where \( c_0 \) is an integral lifting of \( w_2(X) \). The computation of the SW series of the blowup manifold is now straightforward, and we find

\[
SW_{\hat{X}}^{w_2(\hat{X})}(z) = -2 \sinh(zE)SW_{X}^{w_2(X)}(z).
\]

Using (5.1), it is immediate to see that the order of the zero of \( SW_{\hat{X}}^{w_2(\hat{X})}(z) \) is \( \geq \chi_h(\hat{X}) - c_1^2(\hat{X}) - 3 \), i.e., \( \hat{X} \) is of superconformal simple type. ♠

This computation shows that the inclusion of the phase factor in (6.22) associated to the second Stiefel-Whitney class is extremely important. The usual SW series changes
under blowup by a \( \cosh E \), and therefore our statement for the order of vanishing at \( z = 0 \) is simply false for it.

As a corollary of this result, together with the analysis in sections 7.1 and 7.2, we see that any complex surface with \( \text{b}_2^+ > 1 \) is of superconformal simple type, since any complex surface can be obtained from a minimal model by a series of blowups.

7.4. Rational blowdowns

The blowup process is perhaps the simplest procedure to construct new manifolds starting from a given one, but we would like to consider other constructions in order to give more evidence for our result, and to illustrate some properties of the manifolds of superconformal simple type. An important construction is the rational blowdown introduced by Fintushel and Stern [42]. This is a surgical procedure that has three important outcomes: first, it generalizes in a nice way the usual blowdown. Second, it gives a very useful description of another classical construction, the log transform. Finally, one can construct using this procedure manifolds which lie on the line \( c_1^2 = \chi_h - 3 \), providing in this way a very nice confirmation of the picture that we have been developing in this paper. In addition, the rational blowdown will be one of the ingredients in the construction of the manifolds in the next section, and our remarks here will also provide a useful background for these constructions.

To construct a rational blowdown, we must first consider a special construction in four-manifold topology called “plumbing” (see [31] for details). Consider two spheres \( S_1 \), \( S_2 \) together with disk bundles over them (these are fiber bundles whose fiber is a disk \( D^2 \)). Restricting the disk bundles to two nonintersecting hemispheres \( H_1 \subset S_1 \), \( H_2 \subset S_2 \), (which are also disks), we get two trivial bundles \( H_1 \times D^2_1 \), \( H_2 \times D^2_2 \). The plumbing of these two configurations consists in gluing the two total spaces along these trivial bundles, but interchanging the factors, i.e. we identify \( H_1 \) with \( D^2_2 \), and \( H_2 \) with \( D^2_1 \) (see fig. 3 ).

Consider now \( p - 1 \) disjoint two-spheres, \( u_1, u_2, \ldots, u_{p-2}, u_{p-1} \) (where \( p > 2 \)) and disk bundles \( D_i \to u_i \) with Euler class \(-2\) for \( D_1, \ldots, D_{p-2} \), and \(-(p + 2)\) for \( D_{p-1} \). If one plumbs these bundles pairwise, following the sequence \( u_1, \ldots, u_{p-1} \), one obtains a four-manifold with boundary which is denoted by \( C_p \). According to [31], Lemma 8.5.2, the boundary \( \partial C_p \) is the Lens space \( L(p^2, 1 - p) \). The embedded spheres in \( C_p \) satisfy \( u_1^2 = \ldots = u_{p-2}^2 = -2 \), \( u_{p-1}^2 = -(p + 2) \), and they can be oriented in such a way that \( (u_j, u_{j+1}) = 1 \), \( 1 \leq j \leq p - 2 \). When \( p = 2 \), the configuration \( C_2 \) is just a single two-sphere \( u_1 \) together with the disk bundle of Euler number \(-4\).
Fig. 3: This figure, adapted from Fig. 4.33 of [31], illustrates the plumbing construction. The two disk bundles over $S_1, S_2$ become the trivial bundles $H_1 \times D^2_1$, $H_2 \times D^2_2$ after restriction to the disks $H_1, H_2$ in the base manifolds. These two trivial bundles are then identified after interchanging the factors.

The Lens space $L(p^2, 1 - p)$ also bounds another four-manifold known as the “rational ball,” denoted by $B_p$. The rational ball $B_p$ can be constructed as follows [12]: consider the connected sum $\#(p - 1)\mathbb{C}P^2$, which has $p - 1$ embedded spheres $v_1, v_2, \ldots, v_{p-2}, v_{p-1}$ with $p \geq 2$ and such that $v_1^2 = \ldots = v_{p-2}^2 = 2$, $v_{p-1}^2 = p + 2$. (Again we must separate cases $p = 2$ and $p > 2$.) These spheres can be easily constructed from the hyperplane divisors in each copy of $\mathbb{C}P^2$, and we have

$$v_{p-1} = 2h_1 - h_2 + \ldots \pm h_{p-1},$$
$$v_{p-2} = h_1 + h_2, \ldots, v_1 = h_{p-2} + h_{p-1},$$

(7.9)

where $h_i$ is the hyperplane divisor of the $i$th copy of $\mathbb{C}P^2$. The two-homology classes $v_1, \ldots, v_{p-1}$ are a basis for $H_2(\#(p - 1)\mathbb{C}P^2, \mathbb{Z})$. Consider now a regular neighbourhood
of this configuration of $p - 1$ spheres. The complement of this regular neighborhood in $\sharp(p - 1)\mathbb{C}P^2$ is a manifold with boundary $L(p^2, 1 - p)$, which is precisely the rational ball $B_p$. One can show that the rational ball has $\pi_1(B_p) = \mathbb{Z}_p$. Also, $H^2(B_p, \mathbb{Q}) = 0$, as all the rational two-cohomology classes of $\sharp(p - 1)\mathbb{C}P^2$ are in the regular neighborhood of the configuration of $p - 1$ spheres.

Let $X$ be a closed four-manifold with an embedded $C_p$ configuration. The rational blowdown of $X$ along $C_p$, that will be denoted by $X_p$, is obtained by removing the interior of $C_p$ and replacing it with $B_p$. Under this operation, the numerical invariants $\chi_h$, $c_1^2$ change as follows:

$$\chi_h(X_p) = \chi_h(X), \quad c_1^2(X_p) = c_1^2(X) + p - 1. \quad (7.10)$$

We see that the rational blowdown of a configuration $C_2$, which is associated with a single sphere of self-intersection $(-4)$, has the same effect on the numerical invariants as a usual blowdown. This blowdown along $(-4)$-spheres is precisely the construction that we will use in section 7.7.

Of course, in order to verify our results for rational blowdowns, we have to relate the basic classes and SW invariants of $X_p$ to those of the original manifold $X$. There are two general results that give a partial answer to this problem. Recall that any basic class is a characteristic element. Fintushel and Stern proved that the characteristic elements of $X_p$, $\bar{x}$, come from characteristic elements of $X$, denoted by $x$. $x$ is called a lift of $\bar{x}$. This means that the basic classes of $X_p$ will be essentially a subset of the set of basic classes of $X$. Moreover, if $\bar{x}$ is in fact a basic class of $X_p$, then one has $SW(\bar{x}) = SW(x)$. Unfortunately, this doesn’t tell us which $\bar{x}$ are in fact the basic classes of $X_p$. To clarify this, we need the details of the embedding $C_p \subset X$. It can be shown that, if $X$ is of simple type, then so is $X_p$. We will consider in this section two examples of rational blowdowns, where we are able to obtain a precise description of the resulting basic classes: the generalized log transform, and the rational blowdown along configurations $C_{n-2}$ in the elliptic fibrations $E(n)$ (for $n \geq 4$).

### 7.4.1 Generalized log transform

We first need a couple of definitions that will be also useful in section 7.5. For more details, see for example [31][32][33]. We say that a smooth four-manifold contains a cusp neighborhood if it contains an embedded submanifold $N$ which is fibered by tori and contains a singular cusp fiber. An example of manifolds with cusp neighborhoods are the
elliptic fibrations, where the cusp neighborhood can be taken as a regular neighborhood of a singular cusp fiber together with a section, and it is often called the nucleus of the elliptic fibration. A *c-embedded torus* is a smoothly embedded torus $T$, representing a non-trivial homology class $[T]$, which is a smooth fiber in a cusp neighborhood. In an elliptic fibration, the generic fiber near a cusp fiber is a c-embedded torus. Notice that a c-embedded torus has self-intersection zero.

Let $X$ be a smooth four-manifold which contains a cusp neighborhood, with generic fiber $f$, and consider the blowup $X^\#(p - 1)\mathbb{C}\mathbb{P}^2$ along an embedded sphere in the singular fiber (see [31], Example 8.5.5. (a), for more details). It can be shown that the blownup manifold contains a $C_p$ configuration [12] [31]. The rational blowdown of $X^\#(p - 1)\mathbb{C}\mathbb{P}^2$ along $C_p$ is, by definition, the *generalized p-log transform* of $X$, and will be denoted by $X(p)$. The effect of this transform is to create a multiple fiber in the fibration, $f_p$, where $f = pf_p$. When $X = E(n)$, this construction is equivalent to the usual $p$-log transform of Kodaira. Notice that, because of (5.1) and (7.10), the generalized $p$-log transform does not change the value of the numerical invariants $c_1^2$ and $\chi_h$. The SW invariants and basic classes of $X(p)$ can be obtained by combining the results on blowup reviewed in the previous section with the results on rational blowdowns. The results are as follows. Let $x$ be a basic class of $X$. Then, the basic classes of $X(p)$ have the form

$$x_r = x + (p - 1 - 2r)f_p, \quad r = 0, \ldots, p - 1,$$  \hspace{1cm} (7.11)

and $SW(x_r) = SW(x)$.

Using this information, one can compute the SW series (6.22) of $X(p)$ in terms of the SW series of $X$. Again, one has to be careful with the value of $w_2(X(p))$. A convenient choice of an integral lifting is $\hat{c}_0 = c_0 + (p - 1)f_p$, where $c_0$ is an integral lifting of $w_2(X)$. To compute the SW series, the only subtlety is the computation of the phase factor involving $w_2(X(p))$. In order to do that, we need a particular case of the generalized adjunction inequality proved in [48], which gives a very useful relation between the basic classes and the smooth topology of four-manifolds. The generalized adjunction inequality can be stated as follows: let $X$ be a smooth four-manifold with $b_2^+ > 1$, and assume that $\Sigma \subset X$ is an embedded, oriented, connected surface with self-intersection $[\Sigma]^2 \geq 0$. For every basic class $x$ of $X$, one has the inequality

$$2g(\Sigma) - 2 \geq [\Sigma]^2 + |(x, \Sigma)|.$$  \hspace{1cm} (7.12)
In particular, if $T$ is an embedded torus of self-intersection zero, one has $([T], x) = 0$ for every basic class in $X$. This applies, in particular, to the generic fiber $f$ in the cusp neighbourhood, hence $(f, x) = 0$ for every basic class. Using this, one obtains:

$$SW_{X(p)}^{w_2}(X) \left( \frac{-1}{\sinh(zf)} \right) = \frac{-1}{\sinh(zf_p)} SW_X^{w_2}(X)(z).$$

We recognize in this equation the factor associated to a multiple fiber in the SW series of an elliptic fibration, given in (7.6). Notice that the order of the zero of $SW_X^{w_2}(X)(z)$ at $z = 0$ is not changed under the $p$-log transform, nor is the value of $c_1^2(X) - \chi_h(X)$. We have then proved the following

**Theorem 7.4.1.** Let $X$ be a manifold of superconformal simple type that contains a cusp neighborhood. Then, the generalized $p$-log transform, $X(p)$, is also of superconformal simple type.

**7.4.2. Rational blowdowns of elliptic fibrations**

Once more we summarize some definitions and technical results from [42]. Let $X$ be a manifold of simple type with a $C_p$ configuration. We say that $C_p$ is tautly embedded if, for every basic class $x$ of $X$, one has $(u_i, x) = 0$ for $i = 1, \ldots, p - 2$, and $|u_{p-1}, x| \leq p$. When the configuration $C_p$ is tautly embedded, the basic classes of the rational blowdown $X_p$ can be obtained in a simple way [42]. Let $\bar{x}$ be a basic class of $X_p$, and let $x$ be a lift in $X$. Then, one must have

$$|(u_{p-1}, x)| = p.$$  (7.14)

Basic classes of $X$ satisfying this are in one-one correspondence with basic classes $\bar{x}$ of $X_p$. Moreover, the self-intersection changes by $\bar{x}^2 = x^2 + (p - 1)$.

To see how this works, consider the elliptic fibrations $E(n)$ with fiber $f$, and $n \geq 4$. In $E(n)$ one has a sphere $u_{n-3}$ of square $(-n)$, which is a section of the fibration (i.e., $(u_{n-3}, f) = 1$), together with $n - 4$ spheres $u_i, i = 1, \ldots, n - 4$ with self-intersection $(-2)$ that are disjoint from the fiber, $(u_i, f) = 0$ for $i = 1, \ldots, n - 4$. One can see that this configuration of spheres in $E(n)$ is in fact a $C_{n-2}$ configuration [31] [42]. The basic classes of $E(n)$ are of the form $x_r = (n - 2 - 2r)f$, with $r = 0, \ldots, n - 2$. Therefore, the above configuration $C_{n-2}$ is tautly embedded. Suppose that we perform a rational blowdown of $E(n)$ along this configuration, to obtain a manifold that will be denoted by $Y(n)$. On $E(n)$ we have $|(u_{n-3}, x_r)| = |n - 2 - 2r|$, and the only basic classes that satisfy (7.14)
are \( \pm(n-2)f \). These two basic classes give the only two basic classes of \( Y(n) \), \( \pm\lambda_n \), with \( \lambda_n^2 = n - 3 \). Notice that in this process we have “killed” most of the basic classes of \( E(n) \), to obtain a manifold \( Y(n) \) with only one basic class (in the sense explained in section 6.3). This can be potentially dangerous for Theorem 4.3.1: It is clear from the analysis in section 7.2 that, for the elliptic fibrations \( E(n) \), we need all the basic classes \( x_r \) to have a zero of the appropriate order. Remarkably, using (7.10), we find

\[
c^1_1(Y(n)) = n - 3, \quad \chi_h(Y(n)) = n, \tag{7.15}
\]

therefore \( c^2_1 = \chi_h - 3 \) for this family of manifolds! This is in perfect agreement with our result in section 6.3, and shows that \( Y(n) \) is also of superconformal simple type. The manifolds \( Y(n) \), first constructed by Fintushel and Stern in [42], therefore saturate our inequality for manifolds with only one basic class.

This ends our remarks about rational blowdowns. It would be interesting to prove in full generality that the rational blowdown of a manifold of superconformal simple type is also of superconformal simple type. We have only been able to prove this assertion in two important but special cases: generalized log transforms, and rational blowdowns of \( E(n) \) manifolds. This is due to the fact that, in contrast to the blowup, the change in the basic classes and SW invariants under a rational blowdown depends on the particular embedding of the configuration \( C_p \). Even for tautly embedded configurations, a precise knowledge of the intersection numbers of the embedded spheres \( u_i \) with the basic classes is needed, and is not known in general. In particular, the intersection form of the manifolds does not behave in a simple way under the blowdown, because one has in general nonzero intersections between the embedded spheres and the other two-homology classes in the manifold.

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7 The elliptic surfaces \( E(n) \) contain in general two \( C_{n-2} \) configurations, but a further blowdown of the remaining configuration takes us to the line \( c^2_1 = 2\chi_h - 6 \). In general, when performing blowdowns along tautly embedded configurations, the most dangerous possibility for our inequalities is to perform only one. Further blowdowns increase the value of \( c^2_1 \) and maintain the number of basic classes, see the examples discussed in [42].
7.5. Fiber sum along tori

The fiber sum is one of the most common procedures to obtain new four-manifolds starting with simple building blocks, and in fact all the recent constructions in the geography of four-manifolds are essentially based on this procedure. Another advantage of the fiber sum is that, under certain conditions, the SW invariants of the resulting manifold can be computed from the invariants of the original manifolds. In this section, we will review some of the results concerning this operation and we will prove that (in some appropriate situations) it preserves the superconformal simple type condition.

First, we define the fiber sum. Again, see [31], section 7.1, for further details and more precise statements. Let $X_i$, $i=1,2$ be smooth four-manifolds, and let $F_i \subset X_i$ two-dimensional embedded surfaces with equal genus $g$ and $[F_i]^2 = 0$. To construct the fiber sum of $X_1$ and $X_2$, we consider tubular neighborhoods of $F_i$ in $X_i$, $\nu F_i$, and we glue the two manifolds $X_i - \nu F_i$ along their boundaries using a diffeomorphism $f : F_1 \to F_2$. The resulting manifold, the fiber sum of $X_1$ and $X_2$ along $F_1$, $F_2$, is denoted by $X_1#F_1=X_2$.

Notice that, in general, the diffeomorphism type of the sum depends on the gluing map that we have chosen, but this won’t be the case in the examples we will consider here (essentially because the fiber sums that we will analyze are along c-embedded tori). The numerical invariants of $X_1#F_1=X_2$ can be computed in terms of the numerical invariants of $X_i$ and the genus $g$ of $F_i$. One has [35]:

$$c_1^2(X_1#F_1=X_2) = c_1^2(X_1) + c_1^2(X_2) + 8(g - 1),$$

$$\chi_h(X_1#F_1=X_2) = \chi_h(X_1) + \chi_h(X_2) + (g - 1).$$

(7.16)

In the construction that follows, the fiber sum will be performed along tori, therefore the numerical invariants will simply add under this operation. Another important aspect of the fiber sum is that, in appropriate situations, it can be done on symplectic manifolds in such a way that the symplectic condition is preserved [35].

The behavior of the SW invariants under fiber sum along tori has been analyzed in [49][33]. We will only need theorems 2.1 and 2.2 in [33]. Suppose $X_i$, $i=1,2$ are smooth four-manifolds with $b_2^+ > 1$, and that they contain smoothly embedded tori $T_i$ representing nontrivial homology classes $[T_i]$ with self-intersection 0. Let $SW_X$ be the usual SW series

$$SW_X = \sum_x SW(x)e^x.$$

(7.17)
Then, if $X_1\sharp T_1 = T_2 X_2$ has $b^+_2 > 1$, the SW series of the fiber sum is given by

$$SW_{X_1\sharp T_1 = T_2 X_2} = SW_{X_1\sharp T_1 = F E(1)} \cdot SW_{X_2\sharp T_2 = F E(1)},$$

(7.18)

where $E(1)$ is the rational elliptic surface with fiber $[F]$.

To obtain more concrete results, we clearly need the expression of each of the factors in the right hand side of (7.18). Assume $X$ is as before, and that the torus $T$ is c-embedded. In this case, one has:

$$SW_{X\sharp T = F E(1)} = (e^T - e^{-T}) \cdot SW_X.$$  

(7.19)

If the torus is not c-embedded, we have to use in principle other techniques to compute the invariants. We can now prove the following

**Theorem 7.5.1.** Let $X_i$, $i = 1, 2$ be smooth four-manifolds with $b^+_2 > 1$, and let $T_i$ be c-embedded tori $T_i \subset X_i$. Let $X = X_1\sharp T_1 = T_2 X_2$ be the fiber sum, and suppose that $b^+_2(X) > 1$. If $X_i$ are of superconformal simple type, then $X$ is also of superconformal simple type.

**Proof:** Using (7.16) for the fiber sum along tori, one has

$$c_1^2(X) = c_1^2(X_1) + c_1^2(X_2), \quad \chi_h(X) = \chi_h(X_1) + \chi_h(X_2).$$

(7.20)

Notice that, if $X_i$ and $X$ are simply-connected, then $X$ will have $b^+_2 > 1$. On the other hand, as a consequence of a theorem of Morgan, Mrowka and Szabó [49], any smooth four-manifold with $b^+_2 > 1$ and containing a c-embedded torus is of simple type. Therefore, the statement of our theorem makes sense. According to (7.18) and (7.19), the SW series (7.17) of $X$ is given by

$$SW_X = 4 (\sinh T)^2 SW_{X_1 \cdot SW_{X_2}}.$$  

(7.21)

Here we denote $[T] = [T_1] = [T_2]$ as a class in $X$. If we denote by $\{x_i^{(1)}\}_{i=1,...,n_1}$, $\{x_j^{(2)}\}_{j=1,...,n_2}$ the basic classes of $X_1$, $X_2$, respectively, (7.21) says that the basic classes of $X$ have the form

$$x_{i,j,s} = x_i^{(1)} + x_j^{(2)} + (2 - 2s)T,$$

(7.22)

---

8 This is a kind of gluing formula familiar from axiomatic approaches to topological field theory. The derivation from the path integral remains an interesting challenge.
where \( s = 0, 1, 2 \), with SW invariant

\[
SW(x_{i,j,s}) = (-1)^s \binom{2}{s} SW(x^{(1)}_i)SW(x^{(2)}_j).
\]

(7.23)

Notice that \( T = T_1 = T_2 \) has self-intersection 0, and due to the adjunction formula (7.12), \((T, x^{(1)}_i) = (T, x^{(2)}_j) = 0 \) for any \( i, j \). As \( X \) is of simple type, one has \((x^{(1)}_i + x^{(2)}_j + (2 - 2s)T)^2 = c^2(X) \), for any \( i, j \). We then see that \((x^{(1)}_i, x^{(2)}_j) = 0 \), for any \( i, j \). With this information, we can already compute the SW series (6.22). The simplest choice of integral liftings of \( w_2(X_1), w_2(X_2) \) is to pick any basic class of \( X_1, X_2 \), say \( x^{(1)}_1, x^{(2)}_1 \). We also choose the lifting \( w_2(X) = x^{(1)}_1 + x^{(2)}_1 \). A simple computation shows that

\[
SW_{X}^{w_2(X)}(z) = 4 (\sinh zT)^2 SW_{X_1}^{w_2(X_1)}(z) \cdot SW_{X_2}^{w_2(X_2)}(z).
\]

(7.24)

With this result, the proof of the theorem is easy. By assumption, \( X_i \) are of superconformal simple type. There are three different cases to consider:

1) Assume \( c^2_1(X_i) \geq \chi_h(X_i) - 3 \), for \( i = 1, 2 \). Therefore, there are no constraints on the SW series appearing in the right hand side of (7.24). According to (7.20), one has \( c^2(X) \geq \chi_h(X) - 6 \), i.e. \( \chi_h(X) - c^2(X) = 3 \leq 3 \). If we have strict inequality, then we are done by Proposition 6.1.3, because the order of vanishing of (7.24) is at least 2, due to the factor \((\sinh zT)^2\). If \( \chi_h(X) - c^2(X) = 6 \), we must have \( c^2_1(X_i) = \chi_h(X_i) - 3 \), therefore \( \chi_h(X_i) + \sigma(X_i) \) is odd for \( i = 1, 2 \) and the series \( SW_{X_1}^{w_2(X_1)}(z) \) have at least a zero of order 1 at \( z = 0 \) (the sum in (6.3) with \( k = 0 \) vanishes due the the symmetry properties of the SW series). Therefore, the order of the zero at \( z = 0 \) is at least 4 > \( \chi_h(X) - c^2(X) - 3 \), and we see that the manifold \( X \) is of superconformal simple type.

2) Assume \( c^2_1(X_1) \geq \chi_h(X_1) - 3, \chi_h(X_2) - c^2(X_2) - 4 \geq 0 \). In this case, \( SW_{X_2}^{w_2(X_2)}(z) \) has a zero of order \( \geq \chi_h(X_2) - c^2(X_2) - 2 \), and \( SW_{X_1}^{w_2(X)}(z) \) has a zero of order \( \geq \chi_h(X_2) - c^2(X_2) \). On the other hand, we have by assumption that \( \chi_h(X_2) - c^2(X_2) \geq \chi_h(X) - c^2(X) - 3 \), therefore \( X \) is of superconformal simple type.

3) Finally, there is the case \( \chi_h(X_i) - c^2(X_i) - 4 \geq 0 \). In this case, \( SW_{X_1}^{w_2(X_1)}(z) \) has a zero of order \( \geq \chi_h(X_i) - c^2(X_i) - 2 \), therefore by (7.24), \( SW_{X}^{w_2}(z) \) has a zero of order \( \geq \chi_h(X) - c^2(X) - 2 \). This ends the proof. \( \spadesuit \)
7.6. Knot surgery

The knot surgery construction, introduced by Fintushel and Stern in [39], is a powerful technique to generate exotic manifolds. The starting point is a knot $K$ in $S^3$, with (symmetric) Alexander polynomial

$$\Delta_K(t) = a_0 + \sum_{j=1}^{n} a_j(t^j + t^{-j}). \quad (7.25)$$

If we perform 0-surgery on $K$, we obtain a three-manifold $M_K$ with $b_1(M_K) = 1$, where the generator of $H_1(M_K, \mathbb{Z})$ is the meridian $m$ of the knot. In the four-manifold $M_K \times S^1$ there is a smoothly embedded torus $T_m = S^1 \times m$. Consider now a simply connected manifold $X$ with $b_2^+ > 1$ and containing a $c$-embedded torus $T$. The knot surgery manifold $X_K$ is defined as the fiber sum

$$X_K = X \sharp T = T_m(M_K \times S^1). \quad (7.26)$$

One can prove that $X_K$ is homeomorphic to $X$, and in particular has the same values for the numerical invariants. The torus $T_m$ is not $c$-embedded in $M_K \times S^1$, therefore we cannot use the result (7.21) to compute the SW invariants of $X_K$. However, the SW series (7.17) of $X_K$ is computed in [39] and is given by

$$SW_{X_K} = SW_X \cdot \Delta_K(t), \quad (7.27)$$

where $t = e^{2T}$. A simple computation, following the lines in the previous section, shows that

$$SW_{X_K}^{w_2(X_K)}(z) = SW_X^{w_2(X)}(z) \cdot \Delta_K(e^{2T}). \quad (7.28)$$

Therefore, if $X$ is of superconformal simple type, so is $X_K$.

7.7. An exotic example

Thus far, we have accumulated some evidence in favour of our main result about the regularity of $F(z)$. Moreover, we have seen that all the complex surfaces are in fact of superconformal simple type, and we have seen that important constructions in four-manifold topology (like the blowup, the generalized log transform, the rational blowdown of $E(n)$, the fiber sum along tori, and the knot surgery) preserve this condition. In the last few years, such constructions have been used to find “exotic” manifolds, like noncomplex,
symplectic manifolds or nonsymplectic, irreducible manifolds. Since the building blocks of these constructions are complex surfaces, which are of superconformal simple type, and the constructions preserve the superconformal simple type condition, the manifolds constructed in this way will be of superconformal simple type as well. In this section, we will present an example of such an exotic family of manifolds in order to illustrate the interplay between the above constructions and the superconformal simple type condition.

As we mentioned in section 5.3, the “wedge” \( 0 < c_1^2 < 2\chi - 6 \) in the \((\chi, c_1^2)\) plane cannot be filled with minimal complex surfaces, but it can be filled with minimal (therefore noncomplex) symplectic manifolds. This wedge is relevant to our results because “half” of the manifolds in it are below our line \( c_1^2 = \chi - 3 \), and there are some nontrivial sum rules for the SW invariants that these manifolds have to satisfy. An interesting example of this construction of a family of symplectic manifolds that fills this region is described in Theorem 10.2.12 of [31]. This construction uses elliptic fibrations as the building blocks, and the operations of symplectic fiber sum introduced in [35], as well as the rational blowdown discussed in section 7.4.

The starting point of the construction is the elliptic surface \( E(4) \). This surface contains nine disjoint spheres of square \((-4)\) that are also sections of the fibration. One of these spheres was used in section 7.4 to perform a rational blowdown along a configuration \( C_2 \), but in fact one has nine different configurations of this type. Consider now \( k \) copies of \( E(4)_i \), and perform \( b_i \leq 8 \) rational blowdowns along embedded \((-4)\)-spheres (i.e. \( C_2 \) configurations) in each \( E(4)_i \). The resulting manifolds will be denoted by \( W(b_i) \), \( i = 1, \ldots, k \). They were first considered in the Example 5.2 of [35], and their Donaldson invariants were computed in [12].

Before proceeding with the construction, we will compute the SW series (6.22) of the manifolds \( W(b_i) \). To do this, we have to be more precise about the possible values of \( b_i \). Let’s denote by \( \alpha, \beta \) and \( \gamma \) the corresponding subsets of \( 1, \ldots, k \) with \( b_\alpha \) odd, \( b_\beta \) even and different from zero, and \( b_\gamma \) zero. We relabel the indices in such a way that \( \alpha = 1, \ldots, n_1 \), \( \beta = 1, \ldots, n_2 \), and \( \gamma = 1, \ldots, n_3 \), and of course \( n_1 + n_2 + n_3 = k \). We also write,

\[
b_\alpha = 2m_\alpha + 1, \quad b_\beta = 2n_\beta, \quad m_\alpha \geq 0, \quad n_\beta > 0. \tag{7.29}
\]

Suppose that we perform \( b_i \neq 0 \) rational blowdowns in the \( i \)th copy of \( E(4) \). The configurations \( C_2 \) we are considering are tautly embedded in \( E(4)_i \). According to the results in section 7.4.2, after performing \( b_i \) rational blowdowns, we only have two basic classes \( \pm \tilde{f}_i \)
in $W(b_i)$, satisfying $\bar{f}_i^2 = b_i$ (each rational blowdown increases the square by one unit, and we started with $f_i^2 = 0$). As we have explained, there are three different cases. If $b_\alpha = 2m_\alpha + 1$, and we choose the lifting $c_0 = \bar{f}_\alpha$, we get

$$SW_{W(b_\alpha)}^{w_2}(z) = -2 \sinh(z\bar{f}_\alpha). \quad (7.30)$$

If $b_\beta = 2n_\beta$, one finds (with the same choice of lifting)

$$SW_{W(b_\beta)}^{w_2}(z) = 2 \cosh(z\bar{f}_\beta). \quad (7.31)$$

Finally, if $b_\gamma = 0$, $W(b_\gamma) = E(4, \gamma)$ and the SW series is given by the SW series of $E(4)$:

$$SW_{W(b_\gamma)}^{w_2}(z) = 4(\sinh(zf_\gamma))^2. \quad (7.32)$$

Notice that these manifolds have $c^2_1(W(b_i)) = b_i$, $\chi_h(W(b_i)) = 4$, as a consequence of (7.10), and they are of superconformal simple type, as $c^2_1(W(b_i)) \geq \chi_h(W(b_i)) - 3$.

Now, we can construct a family of manifolds filling the wedge $0 < c^2_1 < 2\chi_h - 6$. In $E(4)$ there are two c-embedded tori disjoint from the nine $C_2$ configurations, $T^\pm$ (none of these tori is the fiber of the elliptic fibration, and they are in fact disjoint from the fiber). Therefore, $T^\pm$ are also c-embedded tori in the rational blowdowns $W(b_i)$. Using these tori, we can perform the following fiber sum:

$$W(b_1)\sharp_{T_1^+ = T_2^-} W(b_2)\sharp_{T_2^+ = T_3^-} \cdots W(b_{k-1})\sharp_{T_{k-1}^+ = T_k^-} W(b_k), \quad (7.33)$$

i.e. we first sum $W(b_1)$ and $W(b_2)$ along $T_1^+ = T_2^-$. The resulting manifold still has a $T_2^+$ coming from $W(b_2)$, which we use to sum the result of the first sum with $W(b_3)$, and so on. We now sum $l$ copies of $(E(1), F)$ to $W(b_k)$ along parallel copies of $T_k^+$, where $0 \leq l \leq 3$. The resulting manifold will be denoted by $Y_{b_1, \ldots, b_k}$. Let’s first compute the numerical invariants of $Y_{b_1, \ldots, b_k}$. Using the expression (7.16), we obtain:

$$c^2_1(Y_{b_1, \ldots, b_k}) = b, \quad \chi_h(Y_{b_1, \ldots, b_k}) = 4k + l, \quad (7.34)$$

where

$$b = b_1 + b_2 + \ldots + b_k. \quad (7.35)$$

Notice that these values realize all the possible values of $(c^2_1, \chi_h)$ in the wedge: as we can do up to $8k$ rational blowdowns in total, we can obtain any value of $b \leq 2\chi_h - 6 = 8k + 2l - 6$. These manifolds are also symplectic, because both the rational blowdown
along $C_2$ configurations and the fiber sum along tori can be done in such a way that the symplectic condition is preserved \[35\]. A simple computation using \((7.24)(7.19)(7.30)(7.31)\) and \((7.32)\) shows that

$$SW_{w_2(Y_{b_1,\ldots,b_k})}(z) = (-1)^{n_1} 2^{3k+l+n_3-2} \prod_{\alpha=1}^{n_1} \sinh(z\overline{f}_\alpha) \prod_{\beta=1}^{n_2} \cosh(z\overline{f}_\beta) \prod_{\gamma=1}^{n_3} (\sinh(zf_\gamma))^2$$

$$\cdot (\sinh(zT_k))^{k-1} \prod_{i=1}^{k-1} (\sinh(zT_i))^2.$$  \hfill (7.36)

Of course, due to the results in section 7.5, the manifold $Y_{b_1,\ldots,b_k}$ is of superconformal simple type. One can check it directly by looking at \((7.36)\). The order of the zero of this series at $z = 0$ is $2k - 2 + l + 2n_3 + n_1$. On the other hand,

$$\chi_h - c_1^2 - 3 = 4k + l - b - 3.$$ \hfill (7.37)

To see that the manifold $Y_{b_1,\ldots,b_k}$ is of superconformal simple type, it is enough to prove that the order of the zero is greater or equal than \((7.37)\), or equivalently, that $b + 2n_3 + n_1 - 2k + 1 \geq 0$. Using the definition, $b + 2n_3 + n_1 - 2k + 1 = b - n_1 - 2n_2 + 1$, but $b \geq n_1 + 2n_2$. Therefore, $Y_{b_1,\ldots,b_k}$ is indeed of superconformal simple type. This example illustrates very nicely the delicate balance between the value of $\chi_h - c_1^2 - 3$ and the number of zeroes of the SW series.

In the same vein, one can examine many other examples of noncomplex and even nonsymplectic manifolds in \[35\][38][50][51], and it is easy to check that all of them are of superconformal simple type. In most of the cases, the constructions use the operations we have analyzed in the previous sections, and the superconformal simple type condition is automatically satisfied.

### 7.8. A conjecture

As we have seen in section 6, the superconformal simple type condition is the most natural way to guarantee regularity of $F(z)$ at $z = 0$ from the physical point of view. In the last subsections, we have seen that this condition is also very natural from the point of view of four-manifold topology: all available four-manifolds are of superconformal simple type. Based on the evidence we have reviewed above, we would like to state the following

**Conjecture 7.8.1.** Every compact, oriented manifold with $b_2^+ > 1$ of simple type is of superconformal simple type.
8. Bounds on the number of basic classes

8.1. A lower bound and a generalized Noether inequality

In section 6.3, we found a lower bound for $c_2^1 - \chi_h$ for manifolds with only one basic class. This bound was a consequence of the sum rules imposed by our physical theorem 4.3.1. If we assume that the manifold $X$ is of superconformal simple type, we can prove a very interesting result: there is a lower bound on the number of basic classes in terms of $c_2^1 - \chi_h$. This is a simple consequence of the sum rules (6.9). In view of our conjecture 7.8.1, we expect this bound to be true for any manifold with $b_2^+ > 1$ and of simple type.

**Theorem 8.1.1.** Let $X$ be a four-manifold of superconformal simple type. If $X$ has $B$ distinct basic classes (in the sense of section 6.3), and $B > 0$, then

$$B \geq \left\lfloor \frac{\chi_h - c_2^1}{2} \right\rfloor,$$

(8.1)

where $[\cdot]$ is the integral part function.

**Proof:** To prove this theorem, we will analyze, as usual, the two different cases $\chi_h + \sigma \equiv 0, 1 \mod 2$.

a) If $\chi_h + \sigma$ is even, the sum rules (6.9) are nontrivial when $k$ is even, $k = 0, 2, \ldots, \chi_h - c_2^1 - 4$. Let $x_i$, $i = 1, \ldots, B$ be the distinct basic classes in $B_X/\{\pm 1\}$. Notice that we have modded out by the involution $x_i \rightarrow -x_i$, therefore no two basic classes in $B_X/\{\pm 1\}$ can differ by a sign. We introduce the notation:

$$n_i = 2SW(x_i)(-1)^{(c_0^2 + c_0 \cdot x_i)/2},$$

(8.2)

if $x_i \neq 0$, and $n_i = SW(0)(-1)c_0^2/2$ otherwise. By assumption, $n_i \neq 0$ for all $i = 1, \ldots, B$.

To analyze the sum rules, we will use the equivalent form (6.19), which holds for any $S \in H^2(X, \mathbb{Z})$. Suppose that the number of nontrivial equations exceeds the number of basic classes. Then we have

$$\frac{\chi_h - c_2^1 - 4}{2} + 1 \geq B.$$

(8.3)

If we consider the first $B$ of these equations we obtain a linear system of $B$ equations with $B$ unknowns:

$$\begin{pmatrix}
1 & \cdots & 1 \\
(x_1, S)^2 & \cdots & (x_B, S)^2 \\
\vdots & \ddots & \vdots \\
(x_1, S)^{2B-2} & \cdots & (x_B, S)^{2B-2}
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2 \\
\vdots \\
n_B
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.$$

(8.4)
Since the $n_i$ are not zero, the determinant of the matrix has to be zero. Therefore,

$$\prod_{i<j}((x_i, S)^2 - (x_j, S)^2) = \prod_{i<j}(x_i + x_j, S) \prod_{i<j}(x_i - x_j, S) = 0, \quad (8.5)$$

for any $S$. This determinant is a product of (linear) polynomials in the coordinates of $S$. A polynomial ring is a domain, so at least one of the factors in this product must be zero, i.e., there is a pair $i < j$ with $i, j \in \{1, \cdots, B\}$, so that

$$(x_i + x_j, S) = 0 \text{ or } (x_i - x_j, S) = 0 \quad (8.6)$$

for any $S$. Because the intersection form is nondegenerate, this means that $x_i = \pm x_j$ for this pair $i, j$. This contradicts the hypothesis that the basic classes $x_i, x_j$ are in $B_X/\{\pm 1\}$ (as they differ by a sign). Therefore, we must have (taking into account that $\chi_h - c_1^2$ is even):

$$B \geq \frac{\chi_h - c_1^2}{2}. \quad (8.7)$$

b) If $\chi_h + \sigma$ is odd, the sum rules apply when $k$ is odd. Notice that, in this case, $x = 0$ cannot be a basic class. Using the notation $n_i$ as before, and assuming again that the number of equations is at least equal to the number of basic classes, $\chi_h - c_1^2 - 3 \geq 2B$, we find the system of equations:

$$\begin{pmatrix}
(x_1, S) & \cdots & (x_B, S) \\
\vdots & \ddots & \vdots \\
(x_1, S)^{2B-1} & \cdots & (x_B, S)^{2B-1}
\end{pmatrix}
\begin{pmatrix}
n_1 \\
\vdots \\
n_B
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}. \quad (8.8)$$

The discriminant has to be zero again, hence

$$\prod_i (x_i, S) \prod_{i<j}(x_i + x_j, S) \prod_{i<j}(x_i - x_j, S) = 0. \quad (8.9)$$

As $x_i \neq 0$, for any $i = 1, \cdots, B$, we find again a contradiction. This means that

$$B \geq \frac{\chi_h - c_1^2 - 1}{2}, \quad (8.10)$$

and this ends the proof. ♠
The inequality (8.1) is a remarkable fact. It encodes in a simple way the pattern observed in the previous section, where decreasing the quantity $c_1^2 - \chi_h$ led to an increase in the number of basic classes. In fact, theorem 8.1.1 has the following

**Corollary 8.1.2.** (Generalized Noether inequality). If $X$ is of superconformal simple type and has $B > 0$ basic classes, then the following inequality holds

$$c_1^2 \geq \chi_h - 2B - 1.$$  \hspace{1cm} (8.11)

Notice that, if $\chi_h + \sigma$ is even, we cannot have $c_1^2 - \chi_h = -2B - 1$ (as the right hand side is odd), and (8.11) is equivalent to (8.7). The inequality (8.11) is a generalization of Theorem 6.3.1 under the assumption of the superconformal simple type condition (if our conjecture is true, this inequality is valid for any manifold of simple type with $b_2^+ > 1$.) It can also be regarded as a far-reaching generalization of the Noether inequality. We should mention that the bound (8.11) is in fact sharp: as it is easy to check, it is saturated by the simply-connected elliptic fibrations $E(n)$.
8.2. Upper bounds for the number of basic classes

It is natural to ask if there are upper bounds for the number of basic classes. A moment’s thought shows that in general there can be no upper bound depending only on topological invariants. For example, if we consider a manifold with a cusp neighborhood and we perform logarithmic transforms, the numerical invariant \( c_1^2 - \chi_h \) will remain the same, and at the same time we will introduce many new basic classes in the manifold (associated to the multiple fibers.)

On the other hand, if we introduce a metric \( g \) on \( X \) then it was already noticed by Witten in [15] that there is an upper bound for the number of basic classes depending on the curvature. We can combine this with our lower bound to obtain an interesting corollary in the theory of Riemannian functionals.

First, let us recall Witten’s upper bound. If \( 2F \) is the curvature of a Spin\(^c\) structure for a solution to the monopole equations then Witten showed that

\[
\frac{1}{4\pi^2} \int_X \left( |F^+|^2 - |F^-|^2 \right) = \frac{c_1^2(X)}{4} \tag{8.12}
\]

\[
\int_X |F^+|^2 \leq \frac{1}{16} \int_X d^4x \sqrt{g} (R(g))^2.
\]

Here \( R(g) \) is the scalar curvature of the metric \( g \). Now, let us choose a basis of harmonic forms \( \omega_\alpha, \alpha = 1, \ldots, b_2(X) \equiv b \), so that they represent an integral basis of \( H^2(X; \mathbb{Z}) \). Then we have

\[
\left[ \frac{2F}{2\pi} \right] = (2n_\alpha + c_\alpha)[\omega_\alpha] \tag{8.13}
\]

where \( c_0 = [c_\alpha\omega_\alpha] \), and the \( n_\alpha \) are integers. Since harmonic representatives minimize the \( L^2 \) norm equation (8.12) implies:

\[
(n_\alpha + \frac{1}{2}c_\alpha)D_{\alpha\beta}(n_\beta + \frac{1}{2}c_\beta) < \rho^2 \equiv \frac{1}{32\pi^2} \int_X d^4x \sqrt{g} (R(g))^2 - \frac{c_1^2(X)}{4} \tag{8.14}
\]

Here \( D_{\alpha\beta} = \int_X \omega_\alpha \wedge * \omega_\beta \) is a positive definite quadratic form (depending on \( g \)) defined on \( H^2(X; \mathbb{Z}) \otimes \mathbb{R} \). Equation (8.14) says that the basic classes are contained in an ellipsoid of the form

\[
\lambda_1^2y_1^2 + \cdots + \lambda_b^2y_b^2 \leq \rho^2 \tag{8.15}
\]

where the \( \lambda_i^2 \) are the eigenvalues and \( y_i \) are coordinates in the directions of the principal axes of \( D_{\alpha\beta} \). We may order the axes so that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_b > 0 \). (The \( \lambda_i \) depend on the choice of integral basis \( \omega_\alpha \).) Since the \( n_\alpha \) are integers the basic classes can be
surrounded by nonoverlapping $b$-dimensional cubes of volume one in the Euclidean metric $\sum (dy_\alpha)^2$. Therefore, the Euclidean volume of any set completely containing all the cubes surrounding the basic classes gives an upper bound on the number of classes. The vertices of any such cube are vectors of the form $n_\alpha + \frac{1}{2}c_\alpha + e_\alpha$ where $e_\alpha \in \{\pm \frac{1}{2}\}$. Using the Schwarz inequality and (8.14) one can show that all such cubes are completely contained in the ellipsoid of the form (8.15) with $\rho \to \rho + \frac{1}{2}\sqrt{b\lambda_1}$. Thus we obtain the inequality on the number of basic classes:

$$2B \leq \frac{\pi^{b/2}}{\Gamma(\frac{b}{2} + 1)} \frac{(\rho + \frac{1}{2}\sqrt{b\lambda_1})^b}{\sqrt{\det D_{\alpha\beta}}}.$$  \hspace{1cm} (8.16)

(If $c_1^2 = 0$ then $x = 0$ can be a basic class and we should replace the LHS of (8.16) by $2B - 1$.) When combined with our lower bound, Theorem 8.1, we find a topological lower bound on the Riemannian functional in (8.16) which must hold whenever $X$ supports basic classes. The maximal eigenvalue $\lambda_1$ depends on the choice of integral basis of harmonic forms, while $B$ does not. The inequality holds for any choice of basis, so we may in fact take the infimum over the $\text{SL}(b, \mathbb{Z})$ orbit of bases.

9. Conclusions

In this paper, we have shown that the existence of superconformal points in gauge theories gives surprising constraints on the topology of four-manifolds. Superconformal field theory thus provides a new tool to study the relations between the geography problem and gauge invariants. We have analyzed in detail the simplest realization of this scenario, the $(1, 1)$ point of $\mathcal{N} = 2$, $SU(2)$ supersymmetric gauge theory with one massive hypermultiplet. Even in this simple example, the constraints derived from the analyticity of the Donaldson-Witten function give nontrivial information, revealing a new line in the $(\chi_h, c_1^2)$ plane, the line $c_1^2 = \chi_h - 3$, demarcating two different kinds of behaviors for the Seiberg-Witten invariants. In the region below the line the Seiberg-Witten invariants must satisfy nontrivial sum rules. Using these sum rules we have proved that any manifold of simple type and $b_2^+ > 1$ with only one basic class has to satisfy the inequality $c_1^2 \geq \chi_h - 3$. Our sum rules also motivate the definition of manifolds of superconformal simple type, and in section seven we have given what we hope is convincing evidence that all manifolds with $b_2^+ > 1$ and of simple type are in fact of superconformal simple type. We have also shown that the sum rules (6.9) encoding the superconformal simple type condition lead
to a bound for the number of basic classes (8.1). This bound should be regarded as a
generalization of the Noether inequality.

We hope that these results will be also useful in further exploration of the important
open problems such as the “3/2 conjecture” and the “11/8 conjecture.” In particular,
we have shown that, in the relevant regions, manifolds of $b_2^+ > 1$ which would violate
these conjectures must satisfy very strong sum rules on their SW invariants. Indeed, our
lower bound on the number of basic classes, $B$, suggests a new strategy to approach these
problems. If one could establish, for instance, an upper bound on $B$ which holds for
minimal manifolds of $c_1^2 < 0$ and which violates our bound then the “3/2 conjecture”
would follow. Note that the property of minimality must play a crucial role in such a
hypothetical upper bound since log transforms on a blowup of $E(n)$ provide examples of
nonminimal manifolds with $c_1^2 < 0$ and arbitrarily large values for $B$.

An interesting consequence of our results is that, as $z \to 0$, only a finite number of
correlators survive in the $F(z)$ function, and this number depends on the values of $\chi, \sigma$. It
is tempting to conjecture that the correlation functions encoded in $F(0)$ are essentially the
topological correlation functions of the superconformal theory. The fact that only a finite
number of these survive is a strong hint of a selection rule where the anomalous $U(1)_R$
charge depends on the Euler character and signature of the four-manifold, as suggested
in [52] in a closely related context. This opens the possibility of studying topological
correlators of the still mysterious superconformal points of $\mathcal{N} = 2$ gauge theories.

The techniques we have used can probably be considerably generalized. Supercon-
formal points can be associated with many choices of vectormultiplet gauge group and
hypermultiplet matter representation. In addition one could consider other topological
twists. It should, for example, be straightforward to generalize our discussion to certain
multicritical points in higher rank theories with matter. Another obvious generalization
concerns the higher critical points in $SU(2)$ gauge theory with $N_f = 2, 3, 4$. As we have
explained, these new critical points involve noncompact moduli spaces and new kinds of
monopole invariants, so further work needs to be done before interesting information on
4-manifold topology can be extracted from these points. But these are matters for future
investigation.

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