DIRECTED IMMERSIONS OF CLOSED MANIFOLDS

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Abstract. Given any finite subset $X$ of the sphere $S^n$, $n \geq 2$, which includes no pairs of antipodal points, we explicitly construct smoothly immersed closed orientable hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$ whose Gauss map misses $X$. In particular, this answers a question of M. Gromov.

1. Introduction

To every $(C^1)$ immersion $f: M^n \to \mathbb{R}^{n+1}$ of a closed oriented $n$-manifold $M$, there corresponds a unit normal vector field or Gauss map $G_f: M \to S^n$, which generates a set $G_f(M) \subset S^n$ known as the spherical image of $f$. Conversely, one may ask, c.f. [8, p. 3]: for which sets $A \subset S^n$ is there an immersion $f: M \to \mathbb{R}^{n+1}$ such that $G_f(M) \subset A$? Such a mapping would be called an $A$-directed immersion of $M$ [1, 7, 13, 14]. It is well-known that when $A \neq S^n$, $f$ must have double points (Note 4.1), and $M$ must be parallelizable, e.g., $M$ can only be the torus $T^2$ when $n = 2$ (Note 4.2). Furthermore, the only known necessary condition on $A$ is the elementary observation that $A \cup -A = S^2$, while there is also a sufficient condition due to Gromov [7, Thm. (D'), p. 186]:

Condition 1.1. $A \subset S^n$ is open, and there is a point $p \in S^n$ such that the intersection of $A$ with each great circle passing through $p$ includes a (closed) semicircle.

A great circle is the intersection of $S^n$ with a 2-dimensional subspace of $\mathbb{R}^{n+1}$. Note that, when $n \geq 2$, examples of sets $A \subset S^n$ satisfying the above condition include those which are the complement of a finite set of points without antipodal pairs. Thus the spherical image of a closed hypersurface can be remarkably flexible. Like most $h$-principle or convex integration type arguments, however, the proof does not yield specific examples. It is therefore natural to ask, for instance:

Question 1.2 ([7], p. 186). “Is there a ‘simple’ immersion $T^2 \to \mathbb{R}^3$ whose spherical image misses the four vertices of a regular tetrahedron in $S^2$?”

Here we give an affirmative answer to this question (Section 2), and more generally present a short constructive proof of the sufficiency of a slightly stronger version of Condition 1.1 for the existence of $A$-directed immersions of parallelizable manifolds.
$M^{n-1} \times S^1$, where $M^{n-1}$ is closed and orientable. Any such manifold admits an immersion $f: M^{n-1} \to \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ (Note 4.3). We then extend $f$ to $M^{n-1} \times S^1$ by using the figure eight curve
\begin{equation}
E_\delta(t) := (\cos(t), \delta \sin(2t))
\end{equation}
to put a copy of $S^1 \simeq \mathbb{R}/2\pi$ in each normal plane of $f$, as described below. Note that the midpoint of $GE_\delta(S^1)$ is assumed to be at $(1, 0)$; see Figure 1 which shows $E_{1/2}$ and its spherical image. Further, the unit normal bundle of $f$ may be naturally identified with the pencil of great circles of $S^n$ passing through $(0, \ldots, 0, 1)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}

**Theorem 1.3.** Let $A \subset S^n$ satisfy Condition 1.1 with respect to $p = (0, \ldots, 0, 1)$. Further, if $n \geq 3$, suppose that the semicircle in Condition 1.1 contains $p$, or that no great circle through $p$ is contained in $A$. Let $f: M^{n-1} \to \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ be a smooth ($C^\infty$) immersion of a closed orientable $(n-1)$-manifold, and, for every $q \in M$, $C_q \subset S^n$ be the unit normal space of $f$ at $q$. Then there is a smooth orthogonal frame $\{N_i: M \to S^n\}$, $i = 1, 2$, for the normal bundle of $f$ such that the semicircle in $C_q$ centered at $N_1(q)$ lies in $A$. For any such frame, and sufficiently small $\varepsilon, \delta > 0,$
\begin{equation}
F(q, t) := f(q) + \varepsilon \sum_{i=1}^2 E^i_\delta(t)N_i(q)
\end{equation}
yields a smooth $A$-directed immersion $M \times S^1 \to \mathbb{R}^{n+1}$, where $E^i_\delta$ are the components of the figure eight curve $E_\delta$ given by (1).

It is not known if Condition 1.1 is necessary for the existence of $A$-directed closed hypersurfaces, and the question posed in the first paragraph is open, even for $n = 2$. See [3, 4, 6] for some other recent results on Gauss maps of closed submanifolds, [2, 9, 11, 16] for still more studies of spherical images, and [15] for historical background.

2. Example

If $A = S^2 \setminus X$ for a finite set $X$ without antipodal pairs, we may always find a point $p \in S^2$ with respect to which $A$ satisfies the hypothesis of Theorem 1.3 (e.g., let $p \not\in X$ be in the complement of all great circles which pass through at least two points of $X$ other than $-p$). After a rigid motion (which may be arbitrarily small)
we may assume that $p = (0, 0, 1)$ or $(0, 0, -1)$, and let $f(\theta) := (\cos(\theta), \sin(\theta), 0)$ be the standard immersion of $S^1 \simeq \mathbb{R}/2\pi$ in $\mathbb{R}^3$. Then the desired framing for the normal bundle of $f$ may always take the form

$$N_1(\theta) := f'(\theta) \times N_2(\theta), \quad N_2(\theta) := \left( \frac{\cos(\theta), \sin(\theta), z(\theta)}{\sqrt{1 + z^2(\theta)}} \right),$$

where $z: \mathbb{R}/2\pi \to \mathbb{R}$ is a smooth function with $z(\theta) = -z(\theta + \pi)$ and such that $X$ is contained entirely in one of the components of $S^2 - N_2(S^1)$. For instance, when $X$ is the vertices of a regular tetrahedron, we may set $z(\theta) := \cos(3\theta)$ in (3). Then, for $\varepsilon, \delta \leq 1/8$, the mapping $F(\theta, t)$ given by (2) yields an immersion $T^2 \simeq \mathbb{R}/2\pi \times \mathbb{R}/2\pi \to \mathbb{R}^3$ which answers Question 1.2. The resulting surface, for $\varepsilon = \delta = 1/8$, is depicted in Figure 2 together with its spherical image (note that

![Figure 2.](image)

here $p = (0, 0, -1))$. To find $z(\theta)$ in general, we may order the points in $X' \cup -X'$, where $X' := X \setminus \{-p\}$, according to their “longitude” $\theta$, and connect them by geodesic segments to obtain a simple closed symmetric curve $\gamma(\theta)$. A perturbation of $\gamma$ then yields a smooth symmetric curve $\tilde{\gamma}$ such that $X$ is contained in one of the components of $S^2 - \tilde{\gamma}(S^1)$. The third coordinate of $\tilde{\gamma}$ gives our desired height function $z$.

3. Proof of Theorem 1.3

3.1. First we construct the frame $\{N_i\}$. For every $q \in M$, $C_q$ is a great circle passing through $p$. So it contains a semicircle in $A$ by assumption (Condition 1.1). Let $m_q \subset C_q$ be the set of midpoints of all such semicircles. We need to find a smooth map $N_1: M \to S^n$ such that $N_1(q) \in m_q$ for all $q \in M$. To this end note that $m_q$ is open and connected. Further, if $m_q$ contains any pairs of antipodal points, then $m_q = C_q$; otherwise, $m_q$ lies in the interior a semicircle of $C_q$. Consequently,

$$\text{Cone}(m_q) := \{ \lambda x \mid x \in m_q, \text{ and } \lambda \geq 0 \},$$

is a convex set in $\mathbb{R}^{n+1}$. In particular, for any finite set of points $x_i \in \text{Cone}(m_q)$ and numbers $\lambda_i \geq 0$, $\sum \lambda_i x_i \in \text{Cone}(m_q)$. Now let $B$ be the set of all points $q \in M$ such that $m_q \neq C_q$. Then $B$ is closed (and therefore compact) since $M \setminus B$ is open; indeed the set of great circles contained in $A$ is open, since $A$ is open. Further note
that for any point \( q \in M \), normal vector \( x \in m_q \), and continuous local extension \( v \) of \( x \) to a normal vector field of \( M \), we have \( v(q') \in m_{q'} \) for all \( q' \) in an open neighborhood \( U \) of \( q \) (because the set of semicircles contained in \( A \) is open). Let \( \{ v_i : U_i \to S^n \} \), \( i = 1, \ldots, k \), be a finite collection of such local vector fields so that \( \bigcup U_i \) covers \( B \) and \( v_i \) are smooth. Also let \( \{ \phi_i : M \to \mathbb{R} \} \) be a smooth partition of unity subordinate to \( \{ U_i \} \), and, for \( q \in \bigcup_i U_i \), set

\[
N_1(q) := \frac{\sum_{i=1}^k \phi_i(q) v_i(q)}{\| \sum_{i=1}^k \phi_i(q) v_i(q) \|}.
\]

If \( q \in B \), then \( v_i(q) \in m_q \) which lies in the interior of a semicircle \( S \subset C_q \), and so \( \| \sum_{i=1}^k \phi_i(q) v_i(q) \| \neq 0 \). Indeed, if \( x \) is the midpoint of \( S \), then \( \langle \sum_{i=1}^k \phi_i(q) v_i(q), x \rangle = \sum_{i=1}^k \phi_i(q) \langle v_i(q), x \rangle > 0 \). Thus \( N_1 \) is well defined (and smooth) on an open neighborhood \( V \) of \( B \). Further, \( N_1(q) \in m_q \) for all \( q \in V \), since \( \text{Cone}(m_q) \) is convex. In particular we are done if \( B = M \); otherwise, note that we may write

\[
N_1(q) = \cos(\theta(q)) p + \sin(\theta(q)) G_f(q),
\]

for some function \( \theta : V \to \mathbb{R} \), since \( G_f \) is well defined due to the orientability of \( M \), and thus \( \{ p, G_f(q) \} \) forms an orthonormal basis for the normal plane \( df(T_q M) \perp \). Further, it is easy to see that we may choose \( \theta \) continuously (and therefore smoothly) if \( n = 2 \). This also holds for \( n > 2 \) if each \( C_q \) contains a semicircle passing through \( p \); for then \( \theta \) is uniquely determined within the range \([-\pi/2, \pi/2]\). Indeed, we may choose the vectors \( v_i \) above so that \( \langle v_i(q), p \rangle \geq 0 \) which would in turn yield that \( \langle N_1(q), p \rangle \geq 0 \). Now let \( V' \) be an open neighborhood of \( B \) with closure \( \overline{V'} \subset V \). Using Tietze’s theorem, followed by a perturbation and a gluing, we may extend \( \theta |_{V'} \) smoothly to all of \( M \). Then (4) yields the desired vector field on \( M \), since for any \( q \in M \setminus B \), \( N_1(q) \in C_q = m_q \). Finally, set

\[
N_2(q) := \sin(\theta(q)) p - \cos(\theta(q)) G_f(q).
\]

3.2. It remains to show that \( G_F(M \times S^1) \subset A \), for small \( \varepsilon, \delta > 0 \). For all \( q \in M \), \( C_q \cap A \) contains an arc of length \( \geq \pi + \alpha \) with midpoint \( N_1(q) \) for some uniform constant \( \alpha > 0 \). Indeed, if we let \( g(q) \) be the supremum of lengths of all arcs in \( C_q \cap A \) with midpoint \( N_1(q) \), then \( g : M \to \mathbb{R} \) is lower semicontinuous, i.e., \( \lim_{q \to q_0} g(q) \geq g(q_0) \), since \( A \) is open. Thus, since \( g \geq \pi + \alpha \) and \( M \) is compact, \( g \geq \pi + \alpha \). Now choose \( \delta > 0 \) so small that the length \( \ell \) of the spherical image of \( E_\delta \) is \( \leq \pi + \alpha \) (this is possible since \( \ell \to \pi \) as \( \delta \to 0 \)). Next, for \( (q, t) \in M \times S^1 \), let \( G_F(q, t) \) be the normalized projection of \( G_F(q, t) \) into \( df(T_q M) \perp \), i.e.,

\[
\tilde{G}_F(q, t) := \frac{\sum_{i=1}^2 \langle G_F(q, t), N_i(q) \rangle N_i(q)}{\sqrt{\sum_{i=1}^2 \langle G_F(q, t), N_i(q) \rangle^2}}.
\]

Also, for fixed \( t \in S^1 \), let \( F_t(q) := F(q, t) \). Then, by the tubular neighborhood theorem, \( F_t : M \to \mathbb{R}^{n+1} \) is a smooth immersion for small \( \varepsilon \). Further, as \( \varepsilon \to 0 \), \( F_t \) converges to \( f \) with respect to the \( C^1 \)-topology. Thus, for each \( q \in M \), the normal
plane \( dF_t(T_q M)^\perp \) (which contains \( G_F(q, t) \)) converges to \( df(T_q M)^\perp \). Consequently \( G_F \) is well-defined for small \( \varepsilon \), and converges to \( \tilde{G}_F \) as \( \varepsilon \to 0 \). So it suffices to check that \( \tilde{G}_F(M \times S^1) \subset A \), which follows from our choice of \( \delta \). Indeed for each fixed \( q \in M, G_F(\{q\} \times S^1) \) is the spherical image of the figure eight curve \( \sum_{i=1}^{2} E_i^1(t)N_i(q) \) in \( df(T_q M)^\perp \), which is an arc of \( C_q \) with midpoint \( N_1(q) \) and length \( \leq \pi + \alpha \). □

4. Notes

4.1. It is well-known that \( G_f(M) = S^n \) for any embedding \( f: M^n \to \mathbb{R}^{n+1} \) of a closed oriented \( n \)-manifold [7, p. 187]. More generally, this also holds for “Alexandrov embeddings”, i.e., immersions \( f: M \to \mathbb{R}^{n+1} \) which may be extended to an immersion \( \tilde{f}: \tilde{M} \to \mathbb{R}^{n+1} \) of a compact \( (n+1) \)-manifold \( \tilde{M} \) with \( \partial \tilde{M} = M \). Indeed if \( \nu \) is any vector field along \( M \) which points “outward” with respect to \( \tilde{M} \), then for \( p \in M \), the normalized projection of \( df(\nu(p)) \) into the line \( df(T_p M)^\perp \) defines a normal vector field \( \nu : M \to \mathbb{S}^n \) which coincides with \( G_f \) (after a reflection of \( G_f \) if necessary). Then, for any \( u \in \mathbb{S}^n \), if \( p \) is a point which maximizes the height function \( \langle \cdot, u \rangle \) on \( M \), we have \( G_f(p) = u \). On the other hand, being only regularly homotopic to an embedding, is not enough to ensure that \( G_f(M) = \mathbb{S}^n \). Indeed the example in Figure 2 is regularly homotopic to an embedded torus of revolution [12].

4.2. If \( G_f(M) \neq \mathbb{S}^n \) for an immersion \( f: M^n \to \mathbb{R}^{n+1} \) of an oriented \( n \)-manifold, then, as is well-known [11], \( M \) must be parallelizable. Here we include a brief geometric argument for this fact. If \( (0, \ldots, 0, 1) \notin G_f(M) \), we may define a continuous map \( F: TM \to \mathbb{R}^n \simeq \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1} \) as follows, c.f. [5, Lemma 2.2]. There is a continuous map \( \mathbb{S}^n \setminus \{0, \ldots, 0, 1\} \xrightarrow{\rho} SO(n+1), \ u \mapsto \rho_u \) such that \( \rho_u(0) = (0, \ldots, 0, -1) \). Let \( \pi: TM \to M \) be the canonical projection, and for \( X \in TM \) set \( F(X) = \rho_{\pi(G_f(X))(df(X))} \). Also let \( F_p := F|_{T_p M} \). Then \( \{F_p^{-1}(e_i)\} \), where \( \{e_i\} \) is a fixed basis of \( \mathbb{R}^n \), gives a framing for \( TM \) as desired. So in particular, when \( M \) is closed and \( n = 2 \), we have \( M = \mathbb{T}^2 \). The last observation also follows from Gauss-Bonnet theorem via degree theory when \( f \) is \( C^2 \); since if \( G_f(M) \neq \mathbb{S}^2 \), then

\[
0 = \deg(G_f) = \frac{1}{4\pi} \int_M \det(df_G) = \frac{1}{4\pi} \int_M K = \frac{1}{2} \chi(M),
\]

where \( K \) is the Gaussian curvature and \( \chi \) is the Euler characteristic.

4.3. To generate some concrete examples of the immersions \( f: M^{n-1} \to \mathbb{R}^n \simeq \mathbb{R}^n \times \{0\} \) in Theorem 1.3, note that if \( f_0: M_0^{n-k-1} \to \mathbb{R}^{n-k} \times \{0\} \) is any immersion such that \( f_0(M_0) \) is disjoint from the subspace \( L := \mathbb{R}^{n-k} \times \{(0, 0)\} \), then spinning \( f_0 \) about \( L \) yields an immersion \( f_1: M_0 \times S^1 \to \mathbb{R}^{n-k+1} \) given by

\[
f_1(q, t) := \begin{bmatrix}
1 & 0 \\
0 & \cos(t) & \sin(t) \\
-\sin(t) & \cos(t) & 0
\end{bmatrix} \begin{bmatrix}
f_0^1(q) \\
\vdots \\
f_0^{n-k}(q)
\end{bmatrix},
\]
where $f_0^i$ are the components of $f_0$. Thus, for instance, one may inductively construct immersions of $S^{n-k-1} \times T^k$ in $\mathbb{R}^n$, for $k = 1, \ldots, n-1$. More generally, if $M^{n-1} \times S^1$ is parallelizable, then so is the open manifold $M^{n-1} \times (0,1)$, which may be immersed in $\mathbb{R}^n$ [10] by the $h$-principle [7], or more specifically, the “holonomic approximation theorem” of Eliashberg-Mishachev [1, 5].

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