SPLITTING THEOREMS IN PRESENCE OF AN
IRROTATIONAL VECTOR FIELD

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Abstract. New splitting theorems in a semi-Riemannian manifold which admits an irrotational vector field (not necessarily a gradient) with some suitable properties are obtained. According to the extras hypothesis assumed on the vector field, we can get twisted, warped or direct decompositions. Some applications to Lorentzian manifold are shown and also $S^1 \times L$ type decomposition is treated.

1. Introduction

Warped products are a generalization of direct products, giving sophisticated examples of semi-Riemannian manifolds from simpler ones. They are manageable for computations and sufficiently rich to have a great geometrical and physical interest. The standard spacetime models of the universe and the simplest models of neighborhoods of star and black holes are warped products, therefore, it is of interest to know when a Lorentzian manifold can be decomposed as a warped product. In this paper, we give decomposition theorems without assuming simply connectedness nor the existence of a gradient, obtaining warped and twisted decomposition. Given two semi-Riemannian manifolds $(B, g_B)$, $(L, g_L)$ and a function $f \in C^\infty(B)$, the warped product $M = B \times f L$ is the product manifold furnished with the metric $g = g_B + f^2 g_L$ [10]. When $f$ is a $C^\infty$ function on $B \times L$, it is called a twisted product.

The classical theorems that ensures the metric decomposition of a manifold as a direct product are the De Rham and De Rham-Wu decomposition theorem [3], [28]. They were generalized by Ponge and Reckziegel in [19], where more general decompositions, such as twisted and warped products, were obtained. In all these paper the manifolds are simply connected.

More recent advances, in which a non necessarily simply connected manifold is decomposed as a product, assume the existence of a function without critical points [7], [13], [14], [19], [20]. It is a great simplification because it ensures that the integral curves of the gradient meet the level hypersurfaces of the function for only one value of its parameter. This allows us to construct explicitly a diffeomorphism between the manifold and $R \times L$, where $L$ is a level hypersurface. Some additional properties of the gradient permit to get different types of metric decompositions. The fact that the function has no critical points exclude $S^1 \times L$ decompositions, which are not frequent in the literature.

There are other results in which it is assumed the existence of a vector field which is not necessarily a gradient. In the paper [9] it is obtained a metric decomposition of a manifold as a direct product $R \times L$ assuming some conditions on a timelike...
vector field and its orthogonal distribution. On the other hand, a diffeomorphic
decomposition can be given in a chronological manifold furnished with a special
vector field \[11\].

Sometimes, the decomposition theorems are stated as singularity versus splitting
theorems: if a manifold is not a global product, it must be incomplete \[7\], \[8\], \[9\],
\[22\].

The decomposition process of a manifold has two stages: diffeomorphic and
metric. One of the standard hypothesis to obtain a diffeomorphic decomposition is
simply connectedness, which obviously is not a necessary condition.

Once we have a diffeomorphic decomposition \(B \times L\), we can obtain the metric
decomposition assuming some geometrical properties on the canonical foliation of
the product \(B \times L\) \[19\].

An usual technique used in the literature to split diffeomorphically a mani-
fold is to construct a diffeomorphism using the flow of a suitable vector field. Although
the construction of the diffeomorphism is the same in all cases, each theorem is proved
in a different way depending on the hypotesis assumed on the vector field. In section
two it is shown that the flow of an unitary vector field with an additional property,
present in most splitting theorem in the literature, induces a local diffeomorphism
which is onto. This gives us a common basis to obtain decomposition theorems.
The difficult part is to check the injectivity, which is equivalent to ensure that each
integral curve of the vector field intersects the leaves of the orthogonal distribution
only in one point.

In section three a general decomposition lemma is presented, which is the basic
tool to obtain the splitting theorems.

In section four some decomposition theorems for irrotational vector field with
compact leaves are given, and in section five they are applied to Lorentzian mani-
folds. In section six the \(S^1 \times L\) type decomposition is treated.

All manifolds considered in this paper are assumed to be connected. We follow
the sign convention for curvature of \(R_{XY} = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [\nabla_X, \nabla_Y]Z\)
and we write \(\text{Ric}(X)\) for the quadratic form associated with the Ricci curvature
tensor. Given \(f : M \rightarrow \mathbb{R}\) a \(C^\infty\) function, we call \(\text{grad} f\) the gradient of
\(f\), \(H^f\) its Hessian and \(\Delta f = \text{div} \text{grad} f\) its laplacian.

2. Preliminaries on the flow of an unitary vector field

Let \((M, g)\) be a semi-Riemannian manifold and \(U\) a vector field on \(M\) with
never null norm. The vector field \(U\) has integrable orthogonal distribution if and only if
it is orthogonally irrotational, i.e., \(g(\nabla_X U, Y) = g(X, \nabla_Y U)\) for all \(X, Y \in U^\perp\). In
this situation, we call \(L_p\) the leaf through \(p\), \(E\) its unitary, \(\lambda\) the function such that
\(U = \lambda E\), \(e = g(E, E)\) and \(\Phi\) the flow of \(E\). Usually, the metric that we put on the leaf
is the induced metric. The vector field \(U\) is irrotational if \(g(\nabla_X U, Y) = g(X, \nabla_Y U)\)
for all vector fields \(X, Y\) on \(M\). We say that \(U\) is pregeodesic if its unitary is
geosdesic, or equivalently, if \(\nabla_U U\) is proportional to \(U\).

It is useful to know when the flow of an unitary and orthogonally irrotational
vector field takes leaves into leaves, because it facilitates the construction of a
diffeomorphism using the flow restricted to an orthogonal leaf.

**Lemma 2.1.** Let \(M\) be a semi-Riemannian manifold and \(E\) an unitary, ortho-
ognally irrotational and complete vector field. Then \(\Phi_t\) satisfies \(\Phi_t(L_p) \subset L_{\Phi_t(p)}\)
for all \(t \in \mathbb{R}\) and \(p \in M\) if and only if \(\nabla_E E = 0\).
Proof. Suppose that \( \Phi_t \) takes leaves into leaves. Then, if \( v \in E^\perp \) it follows that 
\[
g(E_{\Phi_t(p)}(\Phi_t)_*v) = 0 \text{ for all } t \in \mathbb{R}, \text{ i.e., } (\Phi_t)^*(g)(E_p, v) = 0 \text{ for all } t \in \mathbb{R}.
\]
Then 
\[
(L_E g)_p(E_p, v) = 0.
\]

But given a vector field \( A \in E^\perp \),
\[
L_E g(E, A) = g(\nabla_E E, A),
\]
then \( g(\nabla_E E, A) = 0 \) for all \( A \in E^\perp \) and being \( E \) unitary, \( \nabla_E E = 0 \).

Now suppose \( \nabla_E E = 0 \). This implies \( E \) is irrotational and so the metrically equivalent one-form \( w \) is closed. Then
\[
L E w = d \circ i_E w + i_E \circ dw = 0,
\]
so \( \Phi_t^* w = w \). Therefore \( \Phi_t \) takes leaves into leaves for all \( t \in \mathbb{R} \). \( \square \)

Remark 2.2. We suppose that \( E \) is a complete vector field for convenience. If we do not assume it, we should say that the flow takes any connected open set of a leaf into a leaf. Note also that being \( E \) unitary and orthogonally irrotational, it is geodesic if and only if it is irrotational.

Now, let \( U \) be an orthogonally irrotational vector field with never null norm in a semi-Riemann manifold. If \( E \) is complete and geodesic, since it is orthogonally irrotational too, we can apply lemma 2.1. Take \( \Phi \) the flow of \( E \) and \( L \) an orthogonal leaf. We construct
\[
\Phi : \mathbb{R} \times L \to M \quad (t, p) \mapsto \Phi_t(p).
\]

Since \( (\Phi_t)_*x(E_p) = E_{\Phi_t(p)} \) and \( \Phi_t \) takes leaves into leaves, \( \Phi \) is a local diffeomorphism which preserves the foliations and identifies \( E \) with \( \frac{\partial}{\partial t} \).

Lemma 2.3. Let \( M \) be a semi-Riemannian manifold and \( E \) a unitary, irrotational and complete vector field. Then the local diffeomorphism \( \Phi : \mathbb{R} \times L \to M \) is onto.

Proof. We show that \( M = \bigcup_{t \in \mathbb{R}} \Phi_t(L) \). It is sufficient to verify that \( \bigcup_{t \in \mathbb{R}} \Phi_t(L) \) is an open and closed set. It is an open set because we know that \( \Phi \) is a local diffeomorphism. If we take \( x \notin \bigcup_{t \in \mathbb{R}} \Phi_t(L) \), then \( x \in \bigcup_{t \in \mathbb{R}} \Phi_t(L) \subset (\bigcup_{t \in \mathbb{R}} \Phi_t(L))^{\circ} \), but \( \bigcup_{t \in \mathbb{R}} \Phi_t(L) \) is also an open set, so \( \bigcup_{t \in \mathbb{R}} \Phi_t(L) \) is a closed set. \( \square \)

If we can ensure that \( \Phi : \mathbb{R} \times L \to M \) is also injective we would have a diffeomorphic decomposition of \( M \). In most of the splitting theorems which we can find in the literature, the vector field (or its unitary) verifies the hypothesis of the lemma 2.3. The injectivity of \( \Phi \) is equivalent to that the integral curves of \( E \) meet the orthogonal leaves for only one value of its parameter.

3. Global decomposition lemma

It is well known that two orthogonally and complementary foliation give rise to a local diffeomorphic decomposition of the manifold. Depending on certain geometrical properties of the foliations, we can get also a metric decomposition. Consider \( g \) a metric on \( M_1 \times M_2 \) such that the canonical foliations are orthogonal. Take \( (p_0, q_0) \in M_1 \times M_2 \), \( F_{p_0} : M_2 \to M_1 \times M_2 \) given by \( F_{p_0}(q) = (p_0, q) \) and \( F^{q_0} : M_1 \to M_1 \times M_2 \) given by \( F^{q_0}(p) = (p, q_0) \). Now, we construct the metrics \( g_1 = (F^{q_0})^*(g) \) and \( g_2 = (F_{p_0})^*(g) \). Then, in [10] it is proven that
(1) If both canonical foliations are geodesic, the metric is the direct product $g_1 + g_2$.
(2) If the first canonical foliation is geodesic and the second one spherical, the metric is the warped product $g_1 + f^2 g_2$, where $f(p_0) = 1$.
(3) If the first canonical foliation is geodesic and the second one umbilic, the metric is the twisted product $g_1 + f^2 g_2$, where $f(p_0, q) = 1$.
(4) If the first canonical foliation is geodesic and the second one spherical, then

$\left(\text{grad} \, \text{div} \, \epsilon g \right) (X, Y) = 0$.

But since $E$ is an irrotational and conformal vector field, its unitary verifies the case two is orthogonally irrotational and geodesic, and therefore it is irrotational, orthogonal conformal and $\text{grad} \, \text{div} \, E$ is proportional to $E$. The following result codifies the properties of the foliations in terms of the normalized vector field.

**Lemma 3.1.** Let $M$ be a semi-Riemannian manifold and $U$ an orthogonally irrotational vector field with never null norm.

1. The foliations $U$ and $U^\perp$ are totally geodesic if and only if $E$ is parallel.
2. The foliation $U$ is totally geodesic and $U^\perp$ spherical if and only if $E$ is irrotational, orthogonally conformal and $\text{grad} \, \text{div} \, E$ is proportional to $E$.
3. The foliation $U$ is totally geodesic and $U^\perp$ umbilic if and only if $E$ is irrotational and orthogonally conformal.

**Proof.** The case 1 is trivial and the if part of case 2 can be found in [21]. We prove first the third case. Assume that $U$ is totally geodesic and $U^\perp$ is umbilic. Then $E$ is orthogonally irrotational and geodesic, and therefore it is irrotational. If we call $II$ the second fundamental form of $U^\perp$, since it is umbilic, there is $b \in C^\infty(M)$ such that $II(X, Y) = g(X, Y) \cdot b E$, for all $X, Y \in E^\perp$. On the other hand, $II(X, Y) = \epsilon g(\nabla_X Y, E)E = -\epsilon g(Y, \nabla_X E)E$, so $\nabla_X E = -\epsilon b X + \alpha(X) E$. But since $E$ is unitary $\alpha(X) = 0$. Therefore $\nabla_X E = -\epsilon b X$ for all $X \in E^\perp$ and then $E$ is orthogonally conformal. The converse is easy. If in addition $U^\perp$ is spherical then $b$ is constant through the leaves and so $\text{grad} \, \text{div} \, E$ is proportional to $E$. This prove the only if part of case 2.

If $U$ is an irrotational and conformal vector field, its unitary verifies the case two of the lemma, and if $U$ is an irrotational, orthogonally conformal and pregeodesic vector field, its unitary verifies the case three. In any case, $\lambda$ is constant through the orthogonal leaves.

It is easy to verify that $U$ is irrotational and conformal if and only if $\nabla U = a \cdot \text{id}$ for some $a \in C^\infty(M)$. In this situation, $a = E(\lambda)$.

On the other hand, $U$ is irrotational, orthogonally conformal and pregeodesic if and only if $\nabla_X U = a X + b g(X, E) E$, where $a, b \in C^\infty(M)$.

The following result is the key to prove the splitting theorems given in this paper.

**Lemma 3.2.** Let $M$ be a semi-Riemannian manifold and $E$ an unitary, irrotational and complete vector field. Take $p \in M$ and suppose that the integral curves of $E$ with initial value on $L_p$ intersect $L_p$ at only one point. Then $M$ is isometric to $\mathbb{R} \times L_p$ or $S^1 \times L_p$ with metric $g = \epsilon dt^2 + g_0$ (a semi-Riemannian parametric product) where $g_0 = g \big|_{L_p}$, and $E$ is identified with $\frac{\partial}{\partial t}$. 
Proof. Take \( \Phi : \mathbb{R} \times M \to M \) the flow of \( E \). We know that \( \Phi : \mathbb{R} \times L \to M \) is a local diffeomorphism and onto because of lemma 2.2. If the integral curves \( \Phi_t(q) \), \( q \in L_p \), meet \( L_p \) for only one value of its parameter, then \( \Phi \) is injective. If one of them meet \( L_p \) again, then all the curves \( \Phi_t(q) \), \( q \in L_p \), meet \( L_p \) again since \( \Phi \) takes leaves into leaves. We know that the integral curves intersect \( L_p \) at only one point, so the curves \( \Phi_t(q) \), \( q \in L_p \), must be periodic. It is easy to verify that they have the same period, let us say \( t_0 \), i.e., \( \Phi(t_0, q) = q \) for all \( q \in L_p \). Then, we can define a diffeomorphism

\[
\Psi : S^1 \times L \to M \quad (e^{it}, p) \rightarrow \Phi\left(\frac{t_0 \cdot t}{2\pi}, p\right).
\]

Now, we pull-back the metric \( g \) using \( \Psi \) or \( \Phi \) and obtain a metric on \( \mathbb{R} \times L_p \) or \( S^1 \times L_p \). Using 19, it is easy to see that this metric is \( \varepsilon dt^2 + f^2 g_0 \), where \( g_0 = g \mid_{L_p} \). \( \square \)

Remark 3.3. In the conditions of lemma 3.2, since the flow of \( E \) takes leaves into leaves, in order to ensure that all the integral curves with initial condition on \( L \) do not return to \( L \), is sufficient to check this for only one of them, and therefore we obtain a \( \mathbb{R} \times L \) type decomposition.

If we wish to obtain a \( S^1 \times L \) type decomposition, we have to verify that all the integral curves with initial condition on \( L \) return to \( L \) but intersect it at only one point. But in this case, the existence of an integral curve verifying the above property, does not guarantee it for the others integral curves with initial condition on \( L \).

Corollary 3.4. Let \( M \) be a semi-Riemannian manifold and \( E \) an unitary, irrotational and complete vector field. Take \( p \in M \) such that the integral curves of \( E \) with initial value on \( L_p \) intersect \( L_p \) at only one point.

- If \( E \) is orthogonally conformal then \( M \) is isometric to one of the twisted product \( \mathbb{R} \times_f L_p \) or \( S^1 \times_f L_p \), \( g = \varepsilon dt^2 + f^2 g_0 \), where \( g_0 = g \mid_{L_p} \) and \( f(t, x) = \exp(\int_0^t \frac{\text{div} E(\Phi_s(x))}{n-1} ds) \).
- If \( E \) is orthogonally conformal and \( \text{grad} \text{div} E \) is proportional to \( E \), then \( M \) is isometric to one of the warped product \( \mathbb{R} \times_f L_p \) or \( S^1 \times_f L_p \), \( g = \varepsilon dt^2 + f^2 g_0 \), where \( g_0 = g \mid_{L_p} \) and \( f(t) = \exp(\int_0^t \frac{\text{div} E(\Phi_s(x))}{n-1} ds) \).

Proof. Using the lemma 4.2 \( \mathbb{R} \) is diffeomorphic to \( \mathbb{R} \times L_p \) or \( S^1 \times L_p \). If \( E \) is orthogonally conformal, the orthogonal leaves are umbilic (see lemma 5.1), and therefore we obtain a twisted product \( \mathbb{R} \times_f L_p \) or \( S^1 \times_f L_p \) with metric \( \varepsilon dt^2 + f^2 g_0 \) where \( g_0 = g \mid_{L_p} \) and \( f(0, q) \equiv 1 \). If we take \( v \in U_p \), then \( \frac{\text{div} E}{n-1} v = \nabla_v E \). But using the connexion formulae of a twisted product 19 we get \( \nabla_v E = g(E, \text{grad} \log f) v \), and so \( \frac{\text{div} E}{n-1} = E(\log f) \). Thus \( f(t, x) = \exp(\int_0^t \frac{\text{div} E(\Phi_s(x))}{n-1} ds) \).

If moreover \( \text{grad} \text{div} E \) is proportional to \( E \), then \( \text{div} E(\Phi_s(x)) = \text{div} E(\Phi_p(s)) \) for all \( x \in L_p \) and all \( s \in \mathbb{R} \). Therefore \( f(t) = \exp(\int_0^t \frac{\text{div} E(\Phi_s(x))}{n-1} ds) \). \( \square \)

Remark 3.5. Observe that the conclusion of lemma 3.2 and corollary 3.4 are true locally 15. If \((a, b) \subset \mathbb{R} \), a warped product \( ((a, b) \times_f L, -dt^2 + f^2 g_L) \) is called a Generalized Robertson-Walker spacetime 21.

Example 3.6. If we suppose \( M \) causal instead of the condition about the integral curves with initial value on \( L_p \) the conclusion of corollary 3.4 is not true, compare with 10.
We take $\tilde{M} = \mathbb{R}^2$ with the Minkowski metric and the isometry $\Phi(t, x) = (t, x+1)$. Let $\Gamma$ be the subgroup of isometries generated by $\Phi$ and $M = \tilde{M}/\Gamma$. We consider $X = \sqrt{\frac{3}{2}} \frac{\partial}{\partial t} + \sqrt{\frac{1}{2}} \frac{\partial}{\partial x}$. Since $\Phi$ preserves the vector field $X$, we can define the vector field $U_{\Pi(p)} = \Pi_{e_p}(X_p)$. Both $U$ and $X$ are parallel and complete. The manifold $M$ is causal, but it does not split. This example can be trivially extended to any dimension.

Simply connectedness implies that an irrotational vector field is a gradient. If it has never null norm, it is immediate that the integral curves meet the orthogonal leaves for only one value of its parameter. So, with some additional hypothesis, we can use lemma 3.2 to get a $(\mathbb{R} \times L, e^2 dt^2 + g) \text{ type metric decomposition}$. We can assume directly that the vector field is a gradient and state the following: let $M$ be a semi-Riemannian manifold and $f : M \to \mathbb{R}$ a function which gradient has never null norm and $\frac{\nabla f}{|\nabla f|}$ is complete. If

- $H^t = 0$, then $M$ is isometric to a direct product $\mathbb{R} \times L$.
- $H^t = a \cdot g$, then $M$ is isometric to a warped product $\mathbb{R} \times L$.
- $H^t = a \cdot g + bE^* \otimes E^*$, where $a, b \in C^\infty(M)$, then $M$ is isometric to a twisted product $\mathbb{R} \times L$.

But there are other ways to ensure that the integral curves with initial values on a leaf does not return to the same leaf, as it is shown in the following results.

**Corollary 3.7.** Let $M$ be a semi-Riemannian manifold and $U$ an irrotational and conformal vector field, with never null norm and complete unitary. Suppose that $\lambda$ is not constant. Then

1. If $\text{div}U \geq 0$ (or $\text{div}U \leq 0$) then $M$ is isometric to a warped product $\mathbb{R} \times L$.
2. If $\text{Ric}(U) \leq 0$ then $M$ is isometric to a warped product $\mathbb{R} \times L$.

**Proof.** Since $\nabla U = a \cdot \text{id}$, and $a = E(\lambda)$ it follows that $\text{div}U = n \cdot E(\lambda)$ and $\text{Ric}(U) = -(n-1)U(E(\lambda))$. Since $\lambda$ is not constant, there is a point $p \in M$ such that $\text{div}U_p \neq 0$. Take $L$ the leaf through $p$ and $\gamma(t)$ an integral curve of $E$ with $\gamma(0) \in L$.

If $\text{div}U \geq 0$ (or $\text{div}U \leq 0$), then $\lambda(\gamma(t))$ is increasing (or decreasing), and since $\lambda$ is constant through the leaves, $\gamma$ can not return to $L$.

If $\text{Ric}(U) < 0$ then $\frac{d}{dt} (\lambda(\gamma(t))) \geq 0$, and so $\frac{\text{div} U}{n} + \lambda(p) \leq \lambda(\gamma(t))$ for all $t \in \mathbb{R}$, and therefore $\gamma$ can not return to $L$, since if this happened then $\lambda(\gamma(t))$ would be periodic.

Therefore, in both cases the integral curves of $E$ with initial condition on $L$ intersect $L$ at only one value of its parameter. Since $\frac{\text{div} E}{n} = (n-1)\frac{E(\lambda)}{\lambda}$, it follows from corollary 3.7 that, if we fix a point $p \in M$, then $M$ is isometric to the warped product $\mathbb{R} \times_f L$ where $f(t) = \frac{\lambda(\Phi(t))}{\lambda(p)}$.

**Example 3.8.** Take $(\mathbb{R} \times e^t N, -dt^2 + e^{2t} g)$ where $(N, g)$ is a Riemannian manifold. We know that this warped product is not timelike complete [21]. Then $U = e^t \frac{\partial}{\partial t}$ is an irrotational and conformal vector field with complete unitary and $\text{div} U > 0$. If $\Gamma$ is an isometry group which preserves $U$ and the canonical foliations and the action is properly discontinuous then $(\mathbb{R} \times e^t N)/\Gamma$ is a warped product manifold of $\mathbb{R} \times e^t$ $L$ type.
4. Irrotational vector fields with compact leaves

Completeness is a mild hypothesis but essential in most of splitting theorems. We can give trivial counterexamples to these theorems if we do not assume completeness. Some results change it for the global hyperbolicity hypothesis \([1]\). We can give the following theorems for irrotational vector fields with compact leaves without assuming completeness.

**Theorem 4.1.** Let \(M\) be a non compact semi-Riemannian manifold and \(E\) an unitary and irrotational vector field. Assume \(L_p\) is compact for all \(p \in M\). Then \(M\) is isometric to \((a, b) \times L, -\infty \leq a < b \leq \infty\), where \(E\) is identified with \(\frac{\partial}{\partial t}\). \(L\) is a compact semi-Riemannian manifold and \(g\) is a parametrized semi-Riemannian metric \(\varepsilon dt^2 + g_t\).

**Proof.** Let \(\Phi : A \subset \mathbb{R} \times M \rightarrow M\) be the flow of \(E\). We know that \(\Phi\) take any connected and open set of a leaf into a leaf, see remark 2.2. Given \(L\) a leaf we will show that the maximal definition interval of \(\Phi_p(t)\) is the same for all \(p \in L\). We know that for each \(p \in L\) there exists an open set \(W_p \subset L\) and \(\eta_p = (-\eta_p, \eta_p) \times W_p \subset A\). Since \(L\) is compact, there is \(\eta\) with \((-\eta, \eta) \times L \subset A\).

Let \((a, b)\) be the maximal interval such that \((a, b) \times L \subset A\). We claim that it is the maximal definition interval of each integral curve with initial value on \(L\). In fact, suppose that \(\Phi_t(p_0)\) is defined in \((a, b + \delta)\) for some \(p_0 \in L\). Then, there is a \(\eta\) such that \((-\eta, \eta) \times L_{\Phi_t(p_0)} \subset A\). Since \(\Phi_t\) is injective and a local diffeomorphism and \(L_{\Phi_t(p_0)}\) is compact, it is a diffeomorphism. Therefore, given \(p \in L\) there is \(q \in L_{\Phi_t(p_0)}\) with \(\Phi_t(q) = \Phi_t(p)\), and so \(\Phi_t(p)\) can be defined in \((a, b + \eta)\). Since \(p \in L\) is an arbitrary point, we obtain a contradiction. Then the maximal definition interval of the integral curves with initial values on \(L\) is \((a, b)\). Now, we can define the local diffeomorphism

\[
\Phi : (a, b) \times L \rightarrow M
\]

\[
(t, p) \rightarrow \Phi_t(p).
\]

We can show that \(\Phi\) is onto as in lemma 2.3. Now we see that it is injective. It is sufficient to verify that the integral curves with initial values on \(L\) does not meet \(L\) again. If there is \((s, p) \in (a, b) \times L\) such that \(\Phi_s(p) \in L\), then \(M = \cup_{t \in [0, a]} \Phi_t(L)\), and \(M\) would be compact. Therefore, \(\Phi\) is a diffeomorphism, and using [15], we can show that \(\Phi'(g) = \varepsilon dt^2 + g_t\), where \(g_0 = g|_L\).

**Remark 4.2.** If \(E\) is a complete vector field and a leaf is compact, then all leaves are compact, since given two leaves \(L_p, L_q\) there is a parameter \(t \in \mathbb{R}\) such that \(\Phi_t : L_p \rightarrow L_q\) is a diffeomorphism. In this situation \((a, b) = \mathbb{R}\).

**Remark 4.3.** In the same way as in corollary 3.4 if we assume \(E\) unitary, irrotational and orthogonally conformal in the above theorem, then \(M\) is isometric to a twisted product \((a, b) \times L\), and if moreover \(\text{grad} \text{div} E\) is proportional to \(E\) we obtain a warped product \((a, b) \times L\).

Observe that a chronological Lorentzian manifold is also non compact \([2]\).

**Theorem 4.4.** Let \(M\) be a semi-Riemannian manifold and \(U\) an irrotational and conformal vector field with never null norm such that \(\text{Ric}(U) \geq 0\). Assume that \(L_p\) is compact for all \(p \in M\). Then \(M\) is isometric to a warped product \((a, b) \times L\) where \((a, b) \neq \mathbb{R}\) and \(E\) is identified with \(\frac{\partial}{\partial t}\).
Proof. We can prove in the same way that in the theorem 4.1 that given an orthogonal leaf $L$, all the integral curves with initial value on $L$ have the same maximal definition interval, say $(a, b)$. We also can show that $\Phi : (a, b) \times L \to M$ is a local diffeomorphism which it is into. Now, since $\text{Ric}(U) = -(n - 1)U(E(\lambda))$ it follows that $(a, b) \not\in \mathbb{R}$. If $\Phi$ were not injective, since $\Phi$ takes leaves into leaves, we would obtain that $(a, b) = \mathbb{R}$. So, $\Phi$ is a diffeomorphism and we can show in the same way as in corollary 3.4 that $M$ is isometric to the warped product $\mathbb{R} \times_{f} L$, $g = \varepsilon dt^2 + f^2 g_0$, where $g_0 = g |_L$ and $f(t) = \frac{\lambda(\Phi_p(t))}{\lambda(p)}$ where $p$ is a fixed point in $L$. □

Note that the Closed Friedmann Cosmological Model $(0, \pi) \times_{f} S^3$ verifies the hypothesis of the above theorem with the irrotational and conformal vector field $U = f \frac{\partial}{\partial t}$.

5. Application to Lorentzian Manifolds

We can use the above results to get decomposition theorems on Lorentzian manifolds.

Theorem 5.1. Let $M$ be a Lorentzian manifold with positive sectional curvature on timelike planes and $U$ a timelike, irrotational and conformal vector field with complete unitary. Then $M$ is isometric to a warped product $\mathbb{R} \times L$ where $L$ is a Riemann manifold and $E$ is identified with $\frac{\partial}{\partial t}$.

Proof. Take $L$ a leaf through and $p \in L$. Given $v \in T_p L$, a direct computation gives us $K_{\{v, u_p\}} = \frac{f(t)\lambda(\lambda)}{\lambda(p)}$. Therefore, the sectional curvature of a plane which $U_p$ only depends on $p$. Let $\gamma$ be the integral curve of $E$ with $\gamma(0) = p$ and take $y : \mathbb{R} \to \mathbb{R}$ given by $y(t) = \frac{\lambda(\gamma(t))}{\lambda(\gamma(t))}$. Then $K_{\{U_\gamma(t)\}} = \frac{y''(t)}{y(t)}$, and if we define $f(t) = \log \frac{y(t)}{y(t)}$, we obtain that $0 < K_{\{U_\gamma(t)\}} = f'' + f^2$. Now, it is easy to show that $f : [0, \infty) \to \mathbb{R}$ has a finite number of zeros. If there exists $t_0 > 0$ such that $\gamma(t_0) \in L$, then, since the flow of $E$ takes leaves into leaves, $\gamma(nt_0) \in L$ for all $n \in \mathbb{N}$. But $\lambda$ is constant through the leaves, thus $\lambda(\gamma(nt_0)) = \lambda(p)$ and therefore $f(nt_0) = 0$ for all $n \in \mathbb{N}$, which it is a contradiction. Then, $\gamma$ does not return to $L$, and using corollary 3.2 we can conclude that $M$ is isometric to the warped product $\mathbb{R} \times \frac{\lambda(\gamma)}{\lambda(p)} L$. □

Theorem 5.2. Let $M$ be a Lorentzian manifold with positive sectional curvature on all timelike planes and $U$ a timelike, irrotational and conformal vector field with complete unitary. Then $M$ is isometric to a warped product $\mathbb{R} \times L$ where $L$ is a compact Riemann manifold and $E$ is identified with $\frac{\partial}{\partial t}$.

Theorem 5.3. Let $M$ be a complete and non compact Lorentzian manifold and $U$ a timelike, irrotational and conformal vector field. Suppose that $\text{Ric}(v) \geq 0$ for all $v \perp U$ and $|U|$ is not constant and bounded from above. Then $M$ is isometric to a warped product $\mathbb{R} \times L$ where $L$ is a compact Riemann manifold and $E$ is identified with $\frac{\partial}{\partial t}$.

Proof. Observe that $E$ is complete, since it is geodesic. We show that $X = \lambda^0 E$ is a complete vector field. Let us suppose it is not true. Take $\gamma : \mathbb{R} \to M$ an integral curve of $E$ and $\beta : (c, d) \to M$ an integral curve of $X$ with the same initial condition. Then $\beta(t) = \gamma(h(t))$ where $h : (c, d) \to \mathbb{R}$ is a diffeomorphism. But
satisfies \( \text{Ric} \), connected, we know that it is isometric to a warped product on \( P \).

If we take the universal covering \( \tilde{X} \) and \( \tilde{f} \) \( \in \text{E} \), let \( \lambda \) be a unitary and orthogonally irrotational vector field such that the leaves of \( \tilde{X} \) are finally irrotational and timelike. Since \( \tilde{X} \) is complete and \( \tilde{M} \) is simply connected, we know that it is isometric to a warped product \( \mathbb{R} \times \tilde{L}_e \), where \( \tilde{L}_e \) is an orthogonal leaf of \( \tilde{U} \). Since \( \tilde{M} \) is complete, \( \tilde{L}_e \) is complete too, and so \( \tilde{L}_e \) is complete [21].

If we take \( e \in \tilde{M} \) such that \( P(e) = q \) then \( P(L_e) = L_q \) and since \( P \) is a local isometry \( L_q \) is complete. Then, \( L_q \) is a complete Riemann manifold which satisfies \( \text{Ric}_{L_q}(v) > c > 0 \) for all unitary vector \( v \in TL_q \) (for the induced metric on \( L_q \)) and so, using Myers theorem [19], it is compact. Since \( \Phi : \mathbb{R} \times L_q \rightarrow M \) is a local diffeomorphism and it is onto, all the leaves are compact. Then, using theorem 5.1, \( M \) is isometric to a warped product \( \mathbb{R} \times L \).

\[ \mathbf{Theorem 5.4.} \text{Let } M \text{ be a non compact Lorentzian manifold and } E \text{ a timelike, unitary and orthogonally irrotational vector field such that the leaves of } E^\perp \text{ are compact and simply connected. If } \text{div}(E) \geq -\frac{(\text{div}(E))^2}{n-1} \text{ and } \text{Ric}(E) \geq 0 \text{ then } M \text{ splits isometrically as a twisted product } (a, b) \times_f L, \text{ where } L \text{ is an orthogonal leaf and } f(t, p) = \frac{\text{div}(E)(t)}{n-1} + 1, \text{ and the above inequalities are equalities.} \]

\[ \mathbf{Proof.} \text{We take } p \in M \text{ and } \{e_2, ..., e_n\} \text{ an orthonormal basis of } E_p^\perp \text{ and we consider } \text{div}(E)(t, p) = E_p \rightarrow E_p^\perp \text{ the endomorphism given by } E_p(X) = \nabla_X E. \text{ Since } E \text{ is orthogonally irrotational and timelike, } A_p \text{ is diagonalizable. Thus } \frac{1}{n-1} \text{tr}(A_p)^2 \leq || A_p ||^2, \text{ where } \text{tr}(A_p) \text{ denote the trace of } A_p \text{ and } || A_p ||^2 = \sum_{i=2}^n g(A_p(e_i), A_p(e_i)). \text{ The equality holds if and only if } A_p(X) = \frac{\text{tr}(A_p)}{n-1} X. \text{ Let } \{E_1, ..., E_n\} \text{ be a frame near } p, \text{ with } E_1(p) = E_p \text{ and } E_i(p) = e_i. \text{ A straightforward computation shows that } \]

\[ \text{Ric}(E)_p = \text{div}\nabla E - E(\text{div}E)_p - || A_p ||^2. \]

Using that \( \text{Ric}(E) \geq 0 \) and \( \text{div}(E) \geq -\frac{(\text{div}(E))^2}{n-1} \) we obtain that

\[ || A_p ||^2 - \frac{(\text{div}(E))^2}{n-1} \leq \text{div}\nabla E. \]

But \( \text{div}E_p = \sum_{i=2}^n g(\nabla e_i, E, e_i) = \text{tr}(A_p) \). So,

\[ 0 \leq || A_p ||^2 - \frac{1}{n-1} \text{tr}(A_p)^2 \leq \text{div}\nabla E. \]
Since $p$ is arbitrary, $\text{div} \nabla_E E \geq 0$ on $M$. Now, it is known that through the leaves, the one form $g(\nabla_E E, \cdot)$ is closed \[\nabla E \leq 0\]. Let $L$ be the orthogonal leaf through $p$. Since it is simply connected, there is a function $f : L \rightarrow \mathbb{R}$ such that $\text{grad} f = \nabla_E E$. Now, a direct computation shows that

$$\Delta_L e^l = e^l : \text{div} \nabla_E E.$$

Since $L$ is compact and $\Delta_L e^l \geq 0$ on $L$, $f$ is constant. Therefore $\nabla_E E = 0$ and $\|A_p\|^2 = \frac{1}{n-1} \text{tr}(A_p)^2$ but $p$ is arbitrary, thus the above equalities remain valid on $M$. So, $E$ is geodesic and $\nabla_X E = \frac{\text{tr}(A)}{\text{tr}(A)} X$ for all $X \in E^\perp$. Therefore, $E$ is unitary, irrotational and orthogonally conformal. Now, using remark \[\nabla E \leq 0\] $M$ is isometric to a twisted product $(a, b) \times L$ where $f(t, p) = \exp(\int_0^t \frac{\text{div} E(\Phi(t))} {n-1} ds)$. But, since $\nabla_E E = 0$, the inequalities are equalities, so $E(\text{div} E) = -\frac{(\text{div} E)^2}{n-1}$ and therefore $f(t, p) = \frac{\text{div} E(p)} {n-1} t + 1$. \hfill \Box

**Corollary 5.5.** Let $M$ be a non compact Lorentzian manifold and $E$ an unitary and orthogonally irrotational vector field such that $\text{Ric}(E) \geq 0$ and $E(\text{div} E) \geq 0$. Assume that the orthogonal leaves are compact and simply connected. Then $M$ is isometric to a direct product $(a, b) \times L$, where $L$ is a compact and simply connected Riemann manifold.

**Proof.** Since $E(\text{div} E) \geq 0 \geq -\frac{(\text{div} E)^2}{n-1}$ it follows from the above theorem that $M$ is isometric to $(a, b) \times L$, where $f(t, p) = \frac{\text{div} E(p)} {n-1} t + 1$, and the equality holds. It is $0 \leq E(\text{div} E) = -\frac{(\text{div} E)^2}{n-1} \leq 0$. So $\text{div} E = 0$ and $f(t, p) = 1$. \hfill \Box

In a warped product $\mathbb{R} \times_f L$, if $\frac{\partial}{\partial t}$ is complete and $\text{Ric} \left( \frac{\partial}{\partial t} \right) \geq 0$ then $f \equiv cte$. On the other hand, in a twisted product $\mathbb{R} \times_f L$, the same conditions on $\frac{\partial}{\partial t}$ implies that $f$ is independent of the variable $t$. Therefore, in the above theorem or corollary if we assume that $E$ is complete, or $M$ timelike complete, then $(a, b) = \mathbb{R}$ and we get a direct product. This shows that if we want to get more general decomposition theorems with $\text{Ric}(E) \geq 0$, then we must drop the completeness hypothesis.

A leaf is achronal if a timelike and future directed curve meets the leaf at most once. Particularly, the integral curves of $E$ only meet the leaves one time. The achronality of the leaves is more restrictive than the chronology, and it is well known that a chronological manifold is non compact. So, the achronality of the leaves implies $M$ is non compact. Then, the above theorem and corollary are generalizations of theorem 1 in \[\nabla E \leq 0\].

Observe that the following twisted product verifies the hypothesis of the theorem \[\nabla E \leq 0\]. Take $(-1, \infty) \times S^1$ with the metric $g = -dt^2 + f^2 g_{\text{can}}$, where the function is $f(t, e^{is}) = t + 2 + \cos(s)$.

6. **Irrotational vector fields with periodicity**

Let $M$ be a semi-Riemannian manifold and $U$ an irrotational and pregeodesic vector field with never null norm and complete unitary. Then, the one form $w = g(\cdot, U)$ is closed, because $U$ is irrotational. Following \[\nabla E \leq 0\], we take the homomorphism

$$\Psi : H_1(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$[\sigma] \rightarrow \int_\sigma w.$$
Then, $G = \Psi(H_{1}(M, \mathbb{R}))$ is a subgroup of $\mathbb{R}$, so there are three possibilities

1. $G = 0$, and therefore $U$ is a gradient. Then $M$ is isometric to $(\mathbb{R} \times L, g)$, where $g = \varepsilon dt^2 + g_t$.
2. $G \approx \mathbb{Z}$, and $M$ is a fibre bundle over $S^1$, with fibres the leaves of $U^-.$
3. $G$ is dense in $\mathbb{R}$.

Then, if for example, $\Pi_1(M)$ is finite, $U$ is a gradient and $M = \mathbb{R} \times L$.

The $S^1 \times L$ type decomposition is not frequent, and it is more complicated than the $\mathbb{R} \times L$ type, as it were commented in remark 3.3. The following example shows the typical difficulty that presents this type of decompositions.

**Example 6.1.** Take $\mathbb{R} \times f S^3(\frac{1}{2})$, $g = dt^2 + f^2 g_{can}$, where $f(t) = \sqrt{3 + \sin(2t)}$.

Since each factor is Riemannian and complete, this warped product is complete. Take the isometry $\eta: \mathbb{R} \times f S^3(\frac{1}{2}) \to \mathbb{R} \times f S^3(\frac{1}{2})$ given by $\eta(t, p) = (t + \pi, -p)$. We call $\Gamma$ the isometry group generated by $\eta$. Then it is easy to check that $\Gamma$ acts in a properly discontinuous manner. We consider the quotient $\Pi: \mathbb{R} \times f S^3(\frac{1}{2}) \to M = (\mathbb{R} \times f S^3(\frac{1}{2})) / \Gamma$ and take the irrotational and conformal vector field $V = f \frac{\partial}{\partial t}$. Since $\eta$ preserves the vector field $V$, there is a vector field $U$ on $M$ such that $\Pi_*(V) = U$ and it is irrotational and conformal too. The integral curves of $U$ are periodic, but $M$ is not isometric to a product $S^1 \times L$ since the integral curves of $U$ intersect each leaf in two different points. In fact, take $\Pi(0, p) = \Pi(\pi, -p) \in M$ and call $L = \Pi((0) \times S^3(\frac{1}{2})) = \Pi((\pi) \times S^3(\frac{1}{2}))$ the leaf through it. Observe that $\Pi(t, p)$ is the integral curve of $\frac{d}{dt}$ through $\Pi(0, p)$ and it intersects the above leaf at $\Pi(0, p)$ and $\Pi(\pi, p)$.

A foliation is regular if for each $p \in M$ there exists an adapted coordinated system such that each slice belongs to a unique leaf $[17]$.

**Theorem 6.2.** Let $M$ be a chronological Lorentzian manifold and $U$ a timelike, irrotational and conformal vector field with complete unitary. Suppose that the foliation $U^\perp$ is regular and let $L$ be a leaf of $U^\perp$. Then $M$ is isometric to a warped product $\mathbb{R} \times L$ or there is a lorentzian covering map $\Psi : M \to S^1 \times N$, where $N$ is a quotient of $L$ and $S^1 \times N$ is a warped product.

**Proof.** If the integral curves of $E$ with initial value on $L$ do not meet $L$ again, we know that $M$ is isometric to $\mathbb{R} \times \chi(\Phi_{t_0}(q)) L$ with $p \in L$ (corollary 3.3).

Suppose there is an integral curve $\gamma$ that meets $L$ again. Now, we can define $t_0 = \inf\{t > 0 : \gamma(t) \in L\}$. Since $U^\perp$ is a regular foliation, $t_0 > 0$ and it is a minimum. Since $\Phi$ takes leaves into leaves it is easy to verify that $\Phi_{t_0}(q) \in L_q$ for all $q \in M$. Now, since $U$ is conformal, the diffeomorphisms $\Phi_t : L_q \to L_{\Phi_t(q)}$ are conformal with constant factor $\left(\frac{\lambda(\Phi_t(q))}{\lambda(q)}\right)^2$. But $\Phi_{t_0}(q) \in L_q$ and $\lambda$ is constant through the leaves, so $\Phi_{t_0} : L_q \to L_{\Phi_{t_0}(q)}$ is an isometry. Since $\Phi_{t_0}$ preserves the vector field $E$, we have that $\Phi_{t_0} : M \to M$ is an isometry.

Let $\Gamma$ be the subgroup of isometries generates by $\Phi_{t_0}$. We can suppose that $U$ is future pointing. Since $M$ is chronological, $\Gamma$ is isomorphic to $\mathbb{Z}$.

Now we show that given $q \in M$ there is an open set $B$ with $q \in B$, such that for all $z \in B$ the integral curve of $E$ with initial value $z$ leaves $B$ before $t_0$ and it does not return to $B$.

We know that there is an open set $B$ such that $\Phi : (-\varepsilon, \varepsilon) \times \chi(\Phi_{t_0}(q)) W \to B$ is an isometry, where $W \subset L_q$ (see remark 5.5). We can assume that $W$ is the
convex ball $B_q(\delta)$ in $L_q$ with $\delta < \frac{\varepsilon^2}{4}$. Suppose that there is $z \in B$ such that the integral curve $\Phi_t(z)$ returns to $B$. Then, since $E$ is unidentified with $\frac{\partial}{\partial t}$, there are $a, b \in W$ such that $\Phi_t(a) = b$ for some $s \in \mathbb{R}$. Take $\alpha : [0, 1] \to W$ a geodesic in $W$ with $\alpha(0) = b$ and $\alpha(1) = a$. Now we consider the curve $\beta(t) = \Phi(-\frac{\varepsilon}{2}(1 - t), \alpha(t))$, $t \in [0, 1]$. This curve joins $\Phi(-\frac{\varepsilon}{2}, b)$ with $\Phi(0, a) = a$, and

$$g(\beta'(t), \beta'(t)) = -\frac{\varepsilon^2}{4} + g(\alpha'(t), \alpha'(t)) < -\frac{\varepsilon^2}{4} + \delta < 0.$$  

So, using the curve $\Phi_t(a)$ and $\beta(t)$ we can construct a piecewise smooth closed timelike curve, which is a contradiction with the chronological hypothesis.

Now, we take the action of $\Gamma$ on $M$. Let us see that this is a properly discontinuous action. We have to see:

1. Given $p \in M$ there exists an open set $U$, $p \in U$, such that $U \cap \Phi_{nt_0}(U) = \emptyset$ for all $n \in \mathbb{Z}$.

It is sufficient to take an open set $B$ with the above property.

2. Given $p, q \in M$ with $p \neq \Phi_{nt_0}(q)$ for all $n \in \mathbb{Z}$, there is open sets $U, V$ such that $p \in U, q \in V$ and $U \cap \Phi_{nt_0}(V) = \emptyset$ for all $n \in \mathbb{Z}$.

We suppose that this is not true and show that there is $m_n \in \mathbb{N}$ such that $\lim_{n \to \infty} \Phi_{m_nt}(q) = p$.

We take $U_n = \Phi((-\frac{1}{n}, \frac{1}{n}) \times B_p(\frac{1}{2n}))$ and $V_n = \Phi((-\frac{1}{n}, \frac{1}{n}) \times B_q(\frac{1}{2n}))$. Since property 2 is not true, there is $m_n$ with $U_n \cap \Phi_{m_n t_0}(V_n) \neq \emptyset$. Using the fact that $\Phi_{m_n t_0} : L_q \to L_q$ are isometries, it is easy to verify that

$$\Phi_{m_n t}(V_n) = \Phi((-\frac{1}{n}, \frac{1}{n}) \times B_p(\frac{1}{2n})),$$

then it follows that $\lim_{n \to \infty} \Phi_{m_n t}(q) = p$.

We claim that $m_n$ is constant from a $n_1$ forward. If this were not true, we take the open set $B$ with $p \in B$, such that the integral curves of $E$ with initial values in $B$, leave it before $t_0$ and it does not return to it. Since $p \in B$, there is $n_0$ such that if $n \geq n_0$ we have $\Phi_{m_n t_0}(q) \in B$. But, there are $m_r, m_s$ such that $m_r - m_s = k \geq 1$ and $m_r, m_s \geq n_0$, thus $\Phi_{k t_0}(\Phi_{m_n t_0}(q))$ is outside $B$ and this is in contradiction with $\Phi_{k t_0}(\Phi_{m_n t_0}(q)) = \Phi_{m_n t_0}(q) \in B$.

Therefore $\Phi_{k t_0}(q) = p$ for some $k$, and this is a contradiction.

Now we take the quotient $P : M \to M/\Gamma$. We can take a metric on $M/\Gamma$ such that $P$ is a local isometry. Since $\lambda$ is constant through the orthogonal leaves, $\Phi_{t_0}$ preserves the vector field $U$. So there is a timelike, irrotational and conformal vector field $Y$ on $M/\Gamma$ such that $P_{\alpha}(U_x) = Y_{p(e)}$. The integral curves of $Y$ intersect the leaf of $Y^\perp$ given by $p(L) = N$ at only one point, and since the integral curves of $Y$ are diffeomorphic to $S^1$, $M/\Gamma$ is isometric to $S^1 \times \frac{1}{\pi(p)} N$. It is easy to prove that $\Gamma$ acts on $L$ and $L/\Gamma = N$. \hfill $\Box$

Remark 6.3. In the above theorem, we can not expect that the covering map would be a diffeomorphism because $S^1 \times N$ is not chronological. On the other hand, the example 6.10 satisfies the conditions of the above theorem, and we obtain the covering map $p : M \to T^2$.

Given a foliation of arbitrary dimension we can define the holonomy of a leaf of the foliation [3]. In some sense it measures how intertwine the leaves through a small transversal manifold around a fixed point. If the foliation is defined by the integral curves of a vector field, then the holonomy is given by its flow. In this case,
an integral curve without holonomy means that it is diffeomorphic to \(\mathbb{R}\) or if it is periodic, then all other nearby integral curves are periodic. Using this notion, we can prove the following result.

**Theorem 6.4.** Let \(M\) be a compact and orientable Riemann manifold with odd dimension and let \(U\) be an irrotational and conformal vector field. Suppose that \(K_{\Pi} \geq 0\) for all planes \(\Pi \perp U\), the norm \(|U|\) is not constant and the integral curves are without holonomy. Then \(M\) is isometric to a warped product \(S^1 \times_f L\) where \(L\) is a compact and simply connected Riemannian manifold.

**Proof.** We know that \(E\) is geodesic and therefore complete. If we take the universal covering \(\tilde{M}\) we know that it is isometric to \(\mathbb{R} \times \tilde{L}_c\). Since \(M\) is compact it follows that \(\mathbb{R} \times \tilde{L}_c\) is complete. Then \(\tilde{L}_c\) is complete \[21\], and so is \(L\). Now we show that there exists a compact leaf with positive sectional curvature. Let \(\gamma\) be an integral curve of \(E\). If \(\gamma\) does not meet the leaf \(L_{\gamma(0)}\) again then the integral curves with initial condition on \(L_{\gamma(0)}\) do not meet again \(L_{\gamma(0)}\) and \(M\) would be diffeomorphic to \(\mathbb{R} \times L_{\gamma(0)}\) which is a contradiction with the compactness of \(M\). Then \(\gamma\) meets again \(L_{\gamma(0)}\) and therefore \(f(t) = \lambda(\gamma(t))\) is periodic and non constant. Then there exists \(s\) with \(f'(s) > 0\). Now, let \(L\) be the leaf trough \(\gamma(s)\). If \(\Pi\) is a plane of \(L\) then we obtain \(K_{\Pi}^L = K_{\Pi}^M + \left(\frac{\Phi(\lambda)}{\lambda}\right)^2 \geq \left(\frac{L(\lambda)}{\lambda}\right)^2 > 0\). Since \(L\) is a complete and orientable Riemann manifold with even dimension and \(K_{\Pi}^L > c > 0\) for all planes \(\Pi\), it follows that it is compact and simply connected \[5\]. Take \(p \in L\) and consider \(t_0 = \inf\{t > 0 : \Phi_t(p) \in L\}\). Now we show that \(t_0 > 0\). Suppose \(t_0 = 0\). Then there exists \(t_n \to 0, t_n > 0\), such that \(p_n = \Phi(t_n, p) \in L\). Since \(L\) is compact we can assume that \(p_n\) converges, necessarily to \(p\). We know that \(\Phi : (\varepsilon, \varepsilon) \times W \to \theta\) is a diffeomorphism, where \(W\) is an open set in \(L\). So, we can suppose that \(p_n \in W\), but then \(t_n = 0\) for all \(n\), and this is a contradiction. Now it is easy to verify that \(t_0\) is a minimum and it is the minimum value which \(\Phi_t(q) \in L\) for all \(q \in L\). Since \(U\) is irrotational and conformal, \(\Phi_t\) are conformal diffeomorphism with constant factor \(\left(\frac{\lambda(\Phi_t(p))}{\lambda(p)}\right)^2\). Then, \(\Phi_{t_0} : L \to L\) is an isometry. If \(\omega\) is the volume form of \(M\) then \(i_U \omega\) is a volume form of \(L\). It is easy to verify that \(\Phi_{t_0}\) preserve this orientation. Now, using the theorem of Synge \[5\], we can ensure that there exists \(q \in L\) such that \(\Phi_{t_0}(q) = q\). Since the integral curves have not holonomy, \(\Phi_{t_0}\) is the identity near \(q\), but since it is an isometry, \(\Phi_{t_0} = id\). Since the integral curves with initial condition on \(L\) intersect it at only one point, it follows from \[3, 4\] that \(M\) is isometric to the warped product \(S^1 \times_{\lambda(\Phi_{t_0}(p))} L\) where \(p \in L\). \(\square\)

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**References**

1. L. Andersson, G.J. Galloway, R. Howard: *A strong maximum principle for weak solutions of quasi-linear elliptic equations with applications to Lorentzian geometry*, Comm. Pure Appl. Math., 51, no.6, (1998) 581-624.
2. J.K. Beem, P.E. Ehrlich and K.L. Easley: Global Lorentzian geometry, 2nd. ed. Marcel Dekker (1996).
3. C. Camacho and A. L. Neto: Geometric theory of foliations , *Birkhäuser* (1985).
[4] G. De Rham: *Sur la reductibilité d’un espace de Riemann*, Comment. Math. Helv., 26, (1952) 328–344.

[5] M.P. Do Carmo: Riemannian Geometry, *Birkhäuser* (1992).

[6] A.E. Fischer: *Riemannian Submersion and the Regular Interval Theorem of Morse Theory*, Ann. Global Anal. Geom., 14, (1996) 263-300.

[7] E. Garcia-Rio and D.N. Kupeli: *Some Splitting Theorems for Stably Causal Spacetimes*, General Relat. Grav., 30, no.1, (1998) 35-44.

[8] E. Garcia-Rio and D.N. Kupeli: *Singularity versus splitting theorems for stably causal spacetimes*, Ann. Glob. Anal. Geom., 14, (1996) 301-312.

[9] E. Garcia-Rio and D.N. Kupeli: *A rigid singularity theorem for spacetimes admitting irrotational references frames*, J. Geom. and Phys., 28, (1998) 158-162.

[10] S.G. Harris: *A Characterization of Robertson-Walker Spaces by Null Sectional Curvature*, Gen. Relat. Grav., 17, no.5, (1985) 493-498.

[11] S.G. Harris and R.J. Low: *Causal monotonicity, omnicient foliations and the shape of space*, Class. Quantum Grav., 18, no.1, (2001) 27–43.

[12] G. Hector and U. Hirsch: Introduction to the geometry of foliations. Part B, *Friedr. Vieweg & Sohn* 1987.

[13] N. Innami: *Splitting theorems of riemannian manifolds*, Compositio Math., 47, no.3, (1982) 237-247.

[14] M. Kanai: *On a Differential Equation Characterizing a Riemannian Structure of a Manifold*, Tokyo J. Math., 6, no.1, (1983) 143-151.

[15] S. Montiel: *Unicity of constant mean curvature hypersurfaces in some riemannian manifold*, Indiana Univ. Math. J., 48, (1999) 711-748.

[16] B. O’Neill: Semi-Riemannian geometry with applications to relativity, *Academic Press* (1983).

[17] R.S. Palais: *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc., no.22, (1957).

[18] P. Petersen and G. Walschap: *Observer fields and the strong energy condition*, Class. Quantum Grav., 13, (1996) 1901-1908.

[19] R. Ponge and H. Recziegel: *Twisted products in pseudo-Riemannian Geometry*, Geom. Dedicata, 48, (1993) 15-25.

[20] T. Sakai: *On riemannian manifolds admitting a function whose gradient is of constant norm*, Kodai Math. J., 19, (1996) 39-51.

[21] M. Sanchez: *On the Geometry of Generalized Robertson-Walker Spacetimes: Geodesics*, Gen. Relat. Grav., 30, no.6, (1998) 915-932.

[22] G. Walschap: *Spacelike metric foliations*, J. Geom. and Phys., 32, (1999) 97-101.

[23] H. Wu: *On the De Rham decomposition theorem*, Illinois J. Math., 8, (1964) 291–311.

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