GROUND STATES OF NONLINEAR FRACTIONAL CHOQUARD EQUATIONS WITH HARDY-LITTLEWOOD-SOBOLEV CRITICAL GROWTH

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Abstract. We are concerned with nonlinear fractional Choquard equations involving critical growth in the sense of the Hardy-Littlewood-Sobolev inequality. Without the Ambrosetti-Rabinowitz condition or monotonicity condition on the nonlinearity, we establish the existence of radially symmetric ground state solutions.

1. Introduction and main results.

1.1. Overview. In this paper, we are concerned with the following Schrödinger-Newton equations in the fractional setting

\[
\begin{aligned}
(-\Delta)^s u + u &= v f(u), \quad x \in \mathbb{R}^N, \\
(-\Delta)^{\alpha/2} v &= F(u), \quad x \in \mathbb{R}^N, \\
u(x) &\to 0, \quad |x| \to \infty, \\
v(x) &\to 0, \quad |x| \to \infty,
\end{aligned}
\]

(1)

where $N > 2s$, $s \in (0, 1)$, $\alpha \in (0, N)$ and $(-\Delta)^t$ denotes the fractional Laplacian of order $t = s, \alpha/2$. Here $F$ is the prime function of $f$, i.e., $F(t) = \int_0^t f(\tau) d\tau$. By using an explicit Riesz representation formulas for the Poisson equation, equation (1) is equivalent to the following fractional Choquard equation of the form [10]:

\[
(-\Delta)^s u + u = [I_\alpha * F(u)] f(u), \quad x \in \mathbb{R}^N,
\]

(2)

where $I_\alpha$ is the Riesz potential defined for $F$ by

\[
(I_\alpha * F)(x) := A_\alpha \int_{\mathbb{R}^N} |x - y|^{-N+\alpha} F(y) dy, \quad x \in \mathbb{R}^N.
\]

Here $A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\alpha/2) \pi^{N/2}}$ and $\Gamma$ is the Gamma function. It is obvious that the convolution term $I_\alpha * F(u)$ is nonlocal.
If $s = 1$, equation (2) can be reduced to the following Choquard equation
\[-\Delta u + u = [I_\alpha F(u)]f(u), x \in \mathbb{R}^N.\] (3)
For $N = 3$, $\alpha = 2$ and $F(u) = |u|^2/2$, it covers the following Choquard-Pekar equation
\[-\Delta u + u = (I_2 |u|^2)u, x \in \mathbb{R}^3,\] (4)
which was introduced by S. I. Pekar [29] to describe the quantum mechanics of a polaron at rest. In 1976, P. Choquard used (4) to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma [19]. Equation (4) is also known as the stationary nonlinear Hartree equation, in the sense that, if $u$ is a solution of (4), the function $\psi(x,t) = e^{it}u(x)$ is a solitary wave of the focusing time-dependent Hartree equation
\[i\psi_t = -\Delta \psi - (I_2 |\psi|^2)\psi, (t,x) \in (\mathbb{R}^+ \times \mathbb{R}^3).\]
For further physical background, we refer to [27].

1.2. Related results. In the following, let us summarize some results on the Choquard equation (3). In studying equation (3) by variational methods, it seems that the first result is due to E. Lieb [19], who proved the existence and uniqueness of ground state solutions to (4). The more general case of (4) has the form of
\[-\Delta u + u = (I_\alpha |u|^p)|u|^{p-2}u, x \in \mathbb{R}^N.\] (5)
By the Pohožaev identity, it is well known that equation (5) has a nontrivial solution in $H^1(\mathbb{R}^N)$ if and only if
\[\frac{N + \alpha}{N} < p < \frac{N + \alpha}{N - 2}\]
(see [24, 25, 13]), where $\frac{N+\alpha}{N}$ is called the lower critical exponent and $2^*_\alpha := \frac{N+\alpha}{N-2}$ is called the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. As $\alpha \to 0$, the scalar field equation
\[-\Delta u + u = f(u)\] (6)
can be considered as a limiting problem of (3). The existence of ground states to (6) has been studied widely in the subcritical case and critical case, and for more details we refer to [2, 35] and the reference therein.

In contrast to problem (6), problem (3) becomes more complicated. For instance, due to the presence of the nonlocal term $I_\alpha F(u)$, the approaches introduced in [2] such as Schwarz symmetrization and the scaling argument seem to fail. Meanwhile, the standard method of moving planes is difficult to be adopted to deal with the symmetry of positive solutions to (3). By the moving planes method in an integral form introduced in [8], L. Ma and L. Zhao [24] first gave the classification of positive solutions to (5) for some range of $p$. Precisely, they showed that all positive solutions are radially symmetric and monotone decreasing about some point with some assumptions on $\alpha$, $p$ and $N$. In [25], V. Moroz and J. van Schaftingen gave an improvement of [24] and established the existence of ground states in an optimal range of $p$. Recently, V. Moroz and J. van Schaftingen [26] considered the more general Choquard equation (3). In the spirit of Berestycki and Lions, thanks to a Pohožaev approach, they obtained the existence of ground states under a sufficient and almost necessary condition on the nonlinearity $f$, that is,
\[\text{(f11)} \text{ there exists } C > 0 \text{ such that for every } s \in \mathbb{R}, |sf(s)| \leq C(||s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}),\]
(f_{12}) \ f \text{ is subcritical: } \lim_{s \to 0} \frac{F(s)}{|s|^{rac{N+\alpha}{2}}} = 0 \text{ and } \lim_{|s| \to \infty} \frac{F(s)}{|s|^{rac{N+\alpha}{2}}} = 0,

(f_{13}) \text{ there exists } s_0 \in \mathbb{R} \setminus \{0\} \text{ such that } F(s_0) \neq 0.

Here we point out that the results mentioned above are only concerned with the subcritical growth. As for the critical growth, the Brezis-Nirenberg type problem

\begin{align*}
-\Delta u &= (I_{\alpha} * |u|^{2^*_{s,\alpha}}) |u|^{2^*_{s,\alpha}-2} u + \lambda u, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{align*}

was considered by F. Gao and M. Yang \cite{15}, where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \) with Lipschitz boundary. On the whole space \( \mathbb{R}^N \), G. Li and C. Tang \cite{17} considered the more general Choquard equation of the form

\[-\Delta u + u = (I_{\alpha} * F(u)) f(u), \quad x \in \mathbb{R}^N.\]

Under the weaker conditions, the existence of ground states was obtained by using the ideas introduced in \cite{22}. In \cite{5}, using the monotonicity trick, D. Cassani and J. Zhang investigated the ground states of the Choquard equation

\[-\varepsilon^2 \Delta u + V(x) u = \varepsilon^{-\alpha} (I_{\alpha} * F(u)) f(u), \quad x \in \mathbb{R}^N,

where \( f \) satisfies:

(f_{21}) \lim_{t \to 0^+} f(t)/t = 0,

(f_{22}) \lim_{t \to \infty} f(t)/t^{\frac{N+\alpha}{2}} = 1,

(f_{23}) \text{ there exist } \mu > 0, \ q \in (2, (N + \alpha)/(N - 2)) \text{ such that } f(t) \geq t^{(2+\alpha)/(N-2)} + \mu t^{q-1}, \ t > 0.

Here (f_{22}) means that \( f \) satisfies the Hardy-Littlewood-Sobolev upper critical growth. Moreover, by using a truncation approach, they also considered the existence and concentration of single peak solutions concentrating around minimum points of the potential \( V \) as \( \varepsilon \to 0 \). For more progress on this subject, we refer to the survey \cite{27} and the reference therein.

1.3. Choquard equations in the fractional setting. Now, we return our attention to the Choquard equation with fractional Laplacian. The fractional Laplacian \( (-\Delta)^s \) is a classical linear integro-differential operator of order \( s \). The main feature is that it is a nonlocal operator. Recently, a great deal of attention has been devoted to the fractional Laplacian and nonlocal operators of elliptic type, both for their interesting theoretical structure and concrete applications. The fractional Laplacian \( (-\Delta)^s \) arises in the description of various phenomena in the applied science, such as the thin obstacle problem \cite{4, 31}, phase transition \cite{1, 32}, Markov processes \cite{14} and fractional quantum mechanics \cite{16} and the references therein for more details.

In \cite{12}, the regularity, existence, nonexistence, symmetry as well as decays properties are discussed on the equation

\[ (-\Delta)^s u + w u = (I_{\alpha} * |u|^p) |u|^{p-2} u, \quad x \in \mathbb{R}^N. \]

Moreover, the existence of ground states was obtained when \( \frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2s} \). Here we denote the upper critical exponent by \( 2^*_{s,\alpha} := \frac{N+\alpha}{N-2s} \). In \cite{30}, by constructing a Pohozaev-Palais-Smale sequence and using the concentration-compactness arguments, Shen et al. established a counterpart of the result in \cite{26} in the fractional setting. That is, they obtained ground state solutions of equation (2) under some sort of Berestycki-Lions type assumptions. Also in the subcritical case, Chen et al. \cite{7} considered the non-autonomous fractional Choquard equation

\[ (-\Delta)^s u + u = (1 + a(x))(I_{\alpha} * |u|^p) |u|^{p-2} u, \quad x \in \mathbb{R}^N. \]
They established the existence of the ground states with $\lim_{|x| \to \infty} a(x) = 0$ and some other suitable assumptions but without the symmetry property on $a(x)$.

In the critical case, the lack of compactness of the embedding makes the critical problems tough. In [28], T. Mukherjee and K. Sreenadh obtained the existence of solutions for the fractional Choquard Brezis-Nirenberg equation

$$
\{ \begin{array}{ll}
(-\Delta)^s u = \lambda u + (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha-2}u, & x \in \Omega \\
u = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{array} \right.
$$

where $\Omega$ is a bounded domain. By variational methods, they established existence, multiplicity, regularity and nonexistence results for solutions of this equation. In [23], the authors investigated the existence and multiplicity of solutions for the following fractional Choquard equation in the whole space

$$
(-\Delta)^{\alpha/2} u + (\lambda V(x) - \beta)u = (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha-2}u.
$$

1.4. Main result. In this paper, motivated by the works above, we study the fractional Choquard equation (2) with critical growth. Our main purpose is to explore the existence of ground state solutions under a weak assumption. In particular, the Ambrosetti-Rabinowitz condition and monotonicity condition are not required.

For simplicity, we assume that $f(t) = g(t) + |t|^{2^*_\alpha-2}t$. Then equation (2) is equivalent to the following equation

$$
(-\Delta)^s u + u = [I_\alpha * (G(u) + \frac{1}{2s,\alpha}|u|^{2^*_\alpha})](g(u) + |u|^{2^*_\alpha-2}u), \quad x \in \mathbb{R}^N, \quad (7)
$$

where $G(t) = \int_0^t g(t)dt$. Throughout the paper, we assume that:

- $(g_1)$ $g \in C^1(\mathbb{R}_+, \mathbb{R})$ and $\lim_{t \to 0^+} g(t)/t = 0$,
- $(g_2)$ $\lim_{t \to \infty} g(t)/t^{\frac{\alpha + 2s}{\alpha}} = 0$,
- $(g_3)$ $\lim_{t \to \infty} \frac{G(t)}{t^{\frac{\alpha + 2s}{\alpha}}} = +\infty$, when $N > 4s$,
  $\lim_{t \to \infty} \frac{G(t)}{t^{\frac{\alpha + 2s}{\alpha}}} = +\infty$, when $N = 4s$,
  $\lim_{t \to \infty} \frac{G(t)}{t^{\frac{\alpha + 2s}{\alpha}}} = +\infty$, when $N < 4s$.

We assume $g(t) \equiv 0$ for $t \leq 0$ since we are concerned with the positive solutions.

The main result is as follows.

**Theorem 1.1.** Assume that $\alpha \in ((N - 4s)_+, N)$, $g$ satisfies $(g_1) - (g_3)$, then (7) admits a ground state.

1.5. Main difficulties and ideas. Because of the presence of the nonlocal operator $(-\Delta)^s$ and nonlocal term $I_\alpha * F(u)$, it is more difficult to study the existence of solutions to (7). Firstly, without the classical Ambrosetti-Rabinowitz condition, it is not easy to get the boundedness of the (PS) sequence. To overcome this difficulty, we apply an idea of approximation introduced in [22], precisely, the bounded (PS) sequence of the critical problem is constructed by the ground states of the subcritical problems. Secondly, due to the lack of compactness, the (PS) condition does not hold in general. We overcome this difficulty when the value of energy functional is lower than some positive constant $c^*$ by use of the achieved function of $S_{s,\alpha}$ (Section 4). Thirdly, because of the appearance of the term $I_\alpha * F(u)$, for the bounded (PS) sequence $\{u_n\}$, even $u_n \rightharpoonup u_0$ weakly, it does not hold in general that $u_0$ is a critical point of the energy functional, which brings us more difficulties to get the compactness. We present a splitting lemma to overcome this difficulty. With the help of the relationship between the minimum energy value of subcritical
problem and critical one, we obtain the existence of ground states to the critical problem.

The paper is organized as follows. In Section 2, we give some preliminaries about the fractional Sobolev space and the Hardy-Littlewood-Sobolev inequality. Moreover, the embedding Lemma and Compactness Lemma of Strauss are presented. In Section 3, we introduce the auxiliary problems and the existence result of ground states of these auxiliary problems. In Section 4, we prove a splitting Lemma and obtain an upper estimate of the minimum energy. In Section 5, we complete the proof of Theorem 1.1.

2. Preliminaries. In order to establish the variational setting for (7), we give some useful facts of the fractional Sobolev space and some preliminaries.

The fractional Laplacian \((-\Delta)^{s}\) with \(s \in (0, 1)\) of a function \(u : \mathbb{R}^N \to \mathbb{R}\) is defined by

\[
\mathcal{F}((-\Delta)^{s}u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,
\]

where \(\mathcal{F}\) is the Fourier transform. For \(s \in (0, 1)\), the fractional order Sobolev space \(H^{s}(\mathbb{R}^N)\) is defined by

\[
H^{s}(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 d\xi < \infty \},
\]

endowed with the norm \(\|u\|_{H^{s}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} (|\xi|^{2s} |\hat{u}|^2 + |\hat{u}|^2) d\xi \right)^{\frac{1}{2}}\), where \(\hat{u} \doteq \mathcal{F}(u)\).

By Plancherel’s theorem, we have \(\|u\|_{L^2(\mathbb{R}^N)} = \|\hat{u}\|_{L^2(\mathbb{R}^N)}\) and

\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx = \int_{\mathbb{R}^N} (|\xi|^{s} |\hat{u}|)^2 d\xi.
\]

It follows that \(\|u\|_{H^{s}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u(x)|^2 + |u|^2) dx \right)^{\frac{1}{2}}\). If \(u\) is smooth enough, \((-\Delta)^{s}u\) can be computed by the following singular integral

\[
(-\Delta)^{s} u(x) = c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy.
\]

Here \(c_{N,s}\) is the normalization constant and P.V. is the principal value. So, one can get an alternative definition of the fractional Sobolev space \(H^{s}(\mathbb{R}^N)\) as follows,

\[
H^{s}(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \}.
\]

with the norm

\[
\|u\|_{H^{s}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.
\]

Since the norm \(\|u\|_{H^{s}(\mathbb{R}^N)}\) and \(\|u\|_{H^{s}(\mathbb{R}^N)}\) are equivalent \([11]\), we use \(\|u\|_{H^{s}(\mathbb{R}^N)} := \|u\|\) in the following. \(H^{s}_r(\mathbb{R}^N)\) denotes the space of radial functions in \(H^{s}(\mathbb{R}^N)\), i.e. \(H^{s}_r(\mathbb{R}^N) = \{ u \in H^{s}(\mathbb{R}^N) : u(x) = u(|x|) \}\). The space \(D^s(\mathbb{R}^N)\) denotes the completion of \(C_0^\infty(\mathbb{R}^N)\) with respect to the Gagliardo norm

\[
\|u\|_{D^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 d\xi \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2) dx \right)^{\frac{1}{2}}.
\]

Now, we present the embedding Lemma and Compactness Lemma of Strauss.
In this case, there is equality in (8) if and only if

\[ \text{Lemma 2.1.} \ [20, 9] \text{ For any } s \in (0, 1), H^s(\mathbb{R}^N) \text{ is continuously embedded into } L^q(\mathbb{R}^N) \text{ for } q \in [2, 2_s^*) \text{ and compactly embedded into } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for } q \in [1, 2_s^*).} \]

\[ H^s(\mathbb{R}^N) \text{ is compactly embedded into } L^q(\mathbb{R}^N) \text{ for } q \in (2, 2_s^*) \text{ and } D^{s, 2}(\mathbb{R}^N) \text{ is continuously embedded into } L^{2^*_s}(\mathbb{R}^N), \text{ i.e., there exists } S_s > 0 \text{ such that} \]

\[ S_s \left( \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \right)^{2/2_s} \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx. \]

\[ \text{Lemma 2.2.} \ [6] \text{ Let } X \text{ be a Banach space such that } X \text{ is embedded respectively continuously and compactly into } L^p(\mathbb{R}^N) \text{ for } p \in [p_1, p_2], \text{ where } p_1, p_2 \in (0, \infty). \]

Assume that \{v_n\} \subset X, v : \mathbb{R}^N \to \mathbb{R} \text{ is a measurable function and } P \in C(\mathbb{R}, \mathbb{R}) \text{ such that}

\[ \lim_{|s| \to \infty} \frac{P(s)}{|s|^{p_2}} = 0, \lim_{|s| \to 0} \frac{P(s)}{|s|^{p_1}} = 0, \sup_n \|v_n\| < \infty, \lim_{n \to \infty} P(v_n(x)) = v(x) \text{ a.e. in } \mathbb{R}^N. \]

Then, up to a subsequence, we have

\[ P(v_n) \to v \text{ in } L^1(\mathbb{R}^N). \]

To prove Theorem 1.1, the following Hardy-Littlewood-Sobolev inequality is crucial.

\[ \text{Lemma 2.3.} \ [18, \text{Theorem 4.3}] \text{ Let } s, r > 1 \text{ and } 0 < \alpha < N \text{ with } 1/s + 1/r = 1 + \alpha/N, \ f \in L^s(\mathbb{R}^N) \text{ and } g \in L^r(\mathbb{R}^N), \text{ then there exists a positive constant } C(s, N, \alpha) \text{ (independent of } f, g) \text{ such that} \]

\[ \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x) |x - y|^{\alpha - N} g(y) \, dx \, dy \right| \leq C(s, N, \alpha) \|f\|_s \|g\|_r. \quad (8) \]

In particular, if \( s = r = 2N/(N + \alpha) \), the sharp constant is

\[ C(N, \alpha) := \pi^{\frac{N - \alpha}{2}} \frac{\Gamma(\alpha/2)}{\Gamma((N + \alpha)/2)} \left[ \frac{\Gamma(N/2)}{\Gamma(N)} \right]^{-\alpha/N}. \]

In this case, there is equality in (8) if and only if \( f \equiv Cg \)

or

\[ g(x) = A(\gamma^2 + |x - a|^2)^{-\frac{N + \alpha}{2}} \]

for some \( A \in \mathbb{C}, 0 \neq \gamma \in \mathbb{R} \) and \( a \in \mathbb{R}^N \).

\[ \text{Remark 1.} \text{ From the Hardy-Littlewood-Sobolev inequality above, set } F(u) = |u|^p, u \in H^s(\mathbb{R}^N), \text{ then } \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(u(x)) |x - y|^{\alpha - N} F(u(y)) \text{ is well defined if } F(u) \in L^t(\mathbb{R}^N) \text{ with } t = 2N/(N + \alpha). \text{ Thus by Lemma 2.1, there must hold} \]

\[ 2 \leq tp \leq 2^*_s, \]

that is,

\[ \frac{N + \alpha}{N} \leq p \leq \frac{N + \alpha}{N - 2s}. \]

\[ \text{Remark 2.} \ [33, \text{Theorem 4.3}] \text{ By the Hardy-Littlewood-Sobolev inequality above, for any } v \in L^s(\mathbb{R}^N) \text{ with } s \in (1, N/\alpha), I_\alpha * v \in L^{Ns/(N-\alpha s)}(\mathbb{R}^N). \text{ Moreover, } I_\alpha \in L(L^s(\mathbb{R}^N), L^{Ns/(N-\alpha s)}(\mathbb{R}^N)) \text{ and} \]

\[ \|I_\alpha * v\|_{Ns/(N-\alpha s)} \leq C(s, N, \alpha) \|v\|_s. \]
The energy functional of equation (7) is defined by

\[ I_{2s,\alpha} (u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 + u^2 - \frac{1}{2} \int_{\mathbb{R}^N} [I_\alpha \ast (G(u) + \frac{1}{2s,\alpha} |u|^{2s,\alpha})] (G(u) + \frac{1}{2s,\alpha} |u|^{2s,\alpha}) \]

Obviously, \( I_{2s,\alpha} \) is well defined and the derivative is given by

\[ \langle I'_{2s,\alpha} (u), \phi \rangle = \frac{1}{2} \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u \cdot (-\Delta)^{\frac{\alpha}{2}} \phi + \int_{\mathbb{R}^N} u \phi - \frac{1}{2} \int_{\mathbb{R}^N} [I_\alpha \ast (G(u) + \frac{1}{2s,\alpha} |u|^{2s,\alpha})] (G(u) + \frac{1}{2s,\alpha} |u|^{2s,\alpha}) \phi , \]

for any \( \phi \in H^s(\mathbb{R}^N) \). As is well known, the solutions of equation (7) correspond to the critical points of the functional \( I_{2s,\alpha} \).

3. The auxiliary equations. In order to prove our main result, similarly to [22], we introduce the following equation

\[ (-\Delta)^s u + u = [I_\alpha \ast (G(u) + \frac{1}{q} |u|^q) + |u|^{q-2} u], x \in \mathbb{R}^N \quad (9) \]

with \( q \in (2, 2s,\alpha) \) and \( g \) satisfying \((g_1) - (g_3)\). The energy functional of equation (9) is denoted by \( I_q \), whose derivative is

\[ \langle I'_q (u), \phi \rangle = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \phi + \int_{\mathbb{R}^N} u \phi - \int_{\mathbb{R}^N} [I_\alpha \ast (G(u) + \frac{1}{q} |u|^q) + |u|^{q-2} u] \phi , \]

for any \( \phi \in H^s(\mathbb{R}^N) \).

**Lemma 3.1** ([30], Proposition 2). Let \( u \) and \( v \) be the nontrivial weak solutions of (7) and (9) respectively, then \( u \) and \( v \) satisfy the following Pohožáev identities

\[
\begin{align*}
\frac{N - 2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} u^2 &= \frac{N + \alpha}{2} \int_{\mathbb{R}^N} [I_\alpha \ast (G(u) + \frac{1}{2s,\alpha} |u|^{2s,\alpha})] (G(u) + \frac{1}{2s,\alpha} |u|^{2s,\alpha}) \\
\frac{N - 2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 + \frac{N}{2} \int_{\mathbb{R}^N} v^2 &= \frac{N + \alpha}{2} \int_{\mathbb{R}^N} [I_\alpha \ast (G(v) + \frac{1}{q} |v|^q)] (G(v) + \frac{1}{q} |v|^q)
\end{align*}
\]

Set

\[ M_{2s,\alpha} = \{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : J_{2s,\alpha} (u) = 0 \}, \quad m_{2s,\alpha} = \inf_{u \in M_{2s,\alpha}} I_{2s,\alpha} (u), \]

and

\[ M_q = \{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : J_q (u) = 0 \}, \quad m_q = \inf_{u \in M_q} I_q (u), \]
where
\[ J_{2^*_s,\alpha}(u) = \frac{N - 2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} u^2 - \frac{N + \alpha}{2} \int_{\mathbb{R}^N} [I_\alpha \ast (G(u) + \frac{1}{2s_\alpha^*}|u|^{2s_\alpha^*})] (G(u) + \frac{1}{2s_\alpha^*}|u|^{2s_\alpha^*}). \]

and
\[ J_q(u) = \frac{N - 2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} u^2 - \frac{N + \alpha}{2} \int_{\mathbb{R}^N} [I_\alpha \ast (G(u) + \frac{1}{q}|u|^{q})] (G(u) + \frac{1}{q}|u|^{q}). \]

Then, \( M_{2^*_s,\alpha} \) and \( M_q \) are good constraints for the solutions of (7) and (9) respectively. That is, if \( u, v \in H^s(\mathbb{R}^N) \) are solutions of (7) and (9) respectively, by the Pohožaev identities, we have \( u \in M_{2^*_s,\alpha} \) and \( v \in M_q \). For the auxiliary equation (9) with \( f(u) = g(u) + |u|^{q-2}u \), we have the result as follows.

**Theorem 3.2** ([30]). Assume \( f \) satisfies,

1. \( |f(t)| \leq C(|t|^2 + |t|^{2^*_s,\alpha}) \), for any \( t \in \mathbb{R} \), where \( C > 0 \) is a constant;
2. \( \lim_{t \to 0} \frac{F(t)}{|t|^2} = 0 \) and \( \lim_{t \to \infty} \frac{F(t)}{|t|^{2^*_s,\alpha}} = 0; \)
3. there exists \( t_0 \in \mathbb{R} \) such that \( F(t_0) \neq 0 \).

Then there exist positive radial symmetric ground states of (9).

**Remark 3.** By Theorem 3.2, it is not difficult to verify that under the conditions of Theorem 1.1, there exist a positive radially symmetric ground state of (9) which is denoted by \( u_q(2 < q < 2^*_s,\alpha) \) in the following sections.

4. **Splitting Lemma and the upper estimate of the minimum energy.**

4.1. **Splitting Lemma.** In the following, we present a splitting lemma which is useful in the proof of our main result.

**Lemma 4.1.** Let \( \{u_n\} \subset H^s(\mathbb{R}^N) \) such that \( u_n \rightharpoonup u \) weakly in \( H^s(\mathbb{R}^N) \) and \( u_n \to u \) a.e. in \( \mathbb{R}^N \), then
\[
\int_{\mathbb{R}^N} |u_n|^{q_n} - |u_n - u|^{q_n} - |u|^{q_n} \frac{2N}{N + \alpha} dx \to 0, \quad as \quad q_n \to 2^*_s,\alpha. \]

**Proof.** In the following, let \( C \) be a positive constant which may change from line to line. For any fixed \( \delta > 0 \), set \( \Omega_n(\delta) := \{ x \in \mathbb{R}^N : |u_n(x) - u(x)| \leq \delta \} \). Thus,
\[
\int_{\mathbb{R}^N} |u_n|^{q_n} - |u_n - u|^{q_n} - |u|^{q_n} \frac{2N}{N + \alpha} dx \leq \int_{\mathbb{R}^N \setminus \Omega_n(\delta)} |u_n|^{q_n} - |u_n - u|^{q_n} - |u|^{q_n} \frac{2N}{N + \alpha} dx + C \int_{\Omega_n(\delta)} |u_n|^{q_n} - |u|^{q_n} \frac{2N}{N + \alpha} + C \int_{\Omega_n(\delta)} |u_n - u|^{2N/(N + \alpha)} := K_1 + CK_2 + CK_3. \]

First, we give the estimate of \( K_3 \).
\[
K_3 = \int_{\Omega_n(\delta)} |u_n - u|^{q_n} \frac{2N}{N + \alpha} \leq \delta \frac{2^{Nq_n} - 2N - 2n}{N + \alpha} \int_{\mathbb{R}^N} |u_n - u|^2 \leq C \delta \frac{2^{Nq_n} - 2N - 2n}{N + \alpha}. \]
By the mean value theorem and Hölder inequality, we have

\[ K_2 \leq \int_{\Omega(n)} \left( \left| u \right| + |u_n| \right)^{\frac{q_n-1}{2}} \left| u_n - u \right| \frac{2^\alpha}{2^{\frac{2\alpha}{\alpha + 1}}} \]

\[ \leq \left( \int_{\Omega_n} \left( \left| u \right| + |u_n| \right)^{(q_n-1)\frac{2N}{N+\alpha} \frac{N+\alpha}{2\alpha+1}} \left( \int_{\Omega_n} \left| u_n - u \right| \frac{2N}{N+\alpha} \frac{N+\alpha}{2\alpha+1} \right)^{\frac{N-2\alpha}{N+\alpha}} \right) \left( \int_{\Omega_n} \left| u_n - u \right| \frac{2N}{N+\alpha} \frac{N+\alpha}{2\alpha+1} \right)^{\frac{N-2\alpha}{N+\alpha}} \]

\[ = \left( \int_{\Omega_n} \left( \left| u \right| + |u_n| \right)^{(q_n-1)\frac{2N}{N+\alpha} \frac{N+\alpha}{2\alpha+1}} \right) \left( \int_{\Omega_n} \left| u_n - u \right| \frac{2N}{N+\alpha} \frac{N+\alpha}{2\alpha+1} \right)^{\frac{N-2\alpha}{N+\alpha}} \]

Set \( \tilde{q}_n = (q_n - 1) \frac{2N}{2\alpha+1} \). It follows from \( q_n \to 2^*_s \) that \( \tilde{q}_n \to \frac{2N}{2s+1} \). By the Young inequality, one has

\[ \left( \left| u \right| + |u_n| \right)^{\tilde{q}_n} = \left( \left| u \right| + |u_n| \right)^{\frac{2\tilde{q}_n}{2_s-\tilde{q}_n}} \left( \left| u \right| + |u_n| \right)^{\frac{2\tilde{q}_n}{2_s-\tilde{q}_n}} \leq \frac{2^*_s - \tilde{q}_n}{2_s - \frac{2N}{2\alpha+1}} \left( \left| u \right| + |u_n| \right) + \tilde{q}_n - 2 \left( \left| u \right| + |u_n| \right) \]

\[ = \left( \left| u \right| + |u_n| \right)^{2^*_s} + o_n(1), \]

where \( o_n(1) \to 0 \) as \( n \to \infty \). Then

\[ K_2 \leq C \delta \frac{4\alpha}{N+\alpha} \left( \int_{\mathbb{R}^N} \left| u_n - u \right|^2 \right)^{\frac{N-2\alpha}{N+\alpha}}. \]

Since \( \{u_n\} \) is bounded in \( H^\varepsilon(\mathbb{R}^N) \), for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( K_2 + K_3 < \frac{\epsilon}{2^s} \). Now, we give the estimate of \( K_1 \).

\[ K_1 = \int_{\mathbb{R}^N} \int_{\Omega_n} \left| u_n \right|^{q_n} - |u_n - u|^{q_n} - \left| u \right|^{q_n} \frac{2N}{N+\alpha} \]

\[ + \int_{B_R(0) \setminus \Omega_n} \left| u_n \right|^{q_n} - |u_n - u|^{q_n} - \left| u \right|^{q_n} \frac{2N}{N+\alpha} := K_{11} + K_{12}. \]

On one hand, by the mean value theorem,

\[ K_{11} \leq \int_{\mathbb{R}^N \setminus B_R(0)} \left| u_n \right|^{q_n} - |u_n - u|^{q_n} - \left| u \right|^{q_n} \frac{2N}{N+\alpha} \]

\[ \leq C \int_{\mathbb{R}^N \setminus B_R(0)} \left| u_n \right|^{q_n} - |u_n - u|^{q_n} \frac{2N}{N+\alpha} + C \int_{\mathbb{R}^N \setminus B_R(0)} \left| u \right|^{q_n} \frac{2N}{N+\alpha} \]

\[ \leq C \int_{\mathbb{R}^N \setminus B_R(0)} \left( \left| u_n \right|^{q_n-1} + |u|^{q_n-1} \right) |u|^{\frac{2N}{N+\alpha}} + |u|^{q_n} \frac{2N}{N+\alpha}. \]

Noting that \( \tilde{q}_n \to 2^*_s \), by the Hölder inequality and Young inequality,

\[ K_{11} \leq C \left( \int_{\mathbb{R}^N \setminus B_R(0)} \left| u_n \right|^{(q_n-1)\frac{2N}{2s+1}} + |u|^{(q_n-1)\frac{2N}{2\alpha+1}} \right)^{\frac{2\alpha}{2s+1}} \left( \int_{\mathbb{R}^N \setminus B_R(0)} |u|^{2^*_s} \right)^{\frac{N-2\alpha}{N+\alpha}} \]

\[ + C \int_{\mathbb{R}^N \setminus B_R(0)} \left| u \right|^{2^*_s} + o_n(1) \]

\[ \leq C \left( \int_{\mathbb{R}^N \setminus B_R(0)} \left| u \right|^{2^*_s} \right)^{\frac{N-2\alpha}{N+\alpha}} + C \int_{\mathbb{R}^N \setminus B_R(0)} \left| u \right|^{2^*_s} + o_n(1). \]

Hence, for \( R \) large enough, there holds \( K_{11} \leq \frac{\epsilon}{4} \).
On the other hand, since $u_n \to u$ a.e. in $\mathbb{R}^N$, by the Severini-Egoroff Theorem, $u_n \to u$ in measure in $B_R(0)$, which implies that
\[
\lim_{n \to \infty} |B_R(0) \setminus \Omega_n(\delta)| = 0.
\]
Thus, for $n$ large, $K_{12} \leq \frac{\delta}{4}$ and then $K_1 \leq \frac{\delta}{4}$. The proof is completed. \qed

4.2. The estimate of the minimum energy. In the following, we still write the nonlinearity as $f(t)$. Let $u_t = u(t)$, then we have the following Lemma.

**Lemma 4.2.** Assume that $N > 2s$ and $(g_1) - (g_3)$ hold. Then

(i) $M_{2^*_{2^*}} \neq \emptyset$,

(ii) for any $u \in H^s(\mathbb{R}^N)$ satisfying
\[
\int_{\mathbb{R}^N} [I_\alpha * F(u)] F(u) > 0,
\]
there exists a unique $t_u > 0$ such that $u_{t_u} \in M_{2^*_{2^*}}$. Moreover, $I_{2^*_{2^*}}(u_{t_u}) = \max_{t>0} I_{2^*_{2^*}}(u_t)$.

**Proof.** Set $u \in H^s(\mathbb{R}^N)$ satisfying $u \geq 0$ and $u \neq 0$, then we have $\int_{\mathbb{R}^N} [I_\alpha * F(u)] F(u) > 0$. Let
\[
\Psi(t) = I_{2^*_{2^*}}(u_t) = \frac{t^{N-2s}}{2} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 + \frac{t^N}{2} \int_{\mathbb{R}^N} u^2 - \frac{t^{N+\alpha}}{2} \int_{\mathbb{R}^N} [I_\alpha * F(u)] F(u).
\]
Since $N + \alpha > N > N - 2s$, one has $\Psi'(t) > 0$ for $t$ small and $\Psi'(t) t < 0$ for $t$ large. Thus, there exists $t_u > 0$ such that $\Psi'(t_u) t_u = 0$, which implies $J_{2^*_{2^*}}(u_{t_u}) = 0$ and thus $u_{t_u} \in M_{2^*_{2^*}}$. So (i) holds.

On the other side, for any $u \in H^s(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^N} [I_\alpha * F(u)] F(u) > 0$, set $\tau = t^N$ and
\[
\Phi(\tau) := \Psi(\tau^s) = \frac{1}{2} \tau^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 + \frac{\tau^N}{2} \int_{\mathbb{R}^N} u^2 - \frac{\tau^{N+\alpha}}{2} \int_{\mathbb{R}^N} [I_\alpha * F(u)] F(u).
\]
Then $\Phi(\tau)$ is a convex function for $\tau > 0$ since $\Phi''(\tau) < 0$. With the similar argument as above, there exists a unique maximum point $\tau_u$. Denote $t_u = \tau_u^s$,
\[
I_{2^*_{2^*}}(u_{t_u}) = \Psi(\tau_u^s) = \max_{t>0} \Phi(t) = \max_{t>0} I_{2^*_{2^*}}(u_t).
\]
The proof is finished. \qed

**Lemma 4.3.** Assume $N > 2s$ and $(g_1) - (g_3)$ hold. Then $0 < \lim \inf_{q \to 2^*_{2^*}} m_q \leq \lim \sup_{q \to 2^*_{2^*}} m_q \leq m_{2^*_{2^*}}$.

**Proof.** First, we prove that lim sup $q \to 2^*_{2^*}$ $m_q \leq m_{2^*_{2^*}}$. Since
\[
m_{2^*_{2^*}} = \inf_{u \in M_{2^*_{2^*}}} I_{2^*_{2^*}}(u),
\]
then for $\epsilon \in (0, \frac{1}{2})$, there exists $u \in M_{2^*_{2^*}}$ such that $I_{2^*_{2^*}}(u) \leq m_{2^*_{2^*}} + \epsilon$ and
\[
\frac{N-2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} u^2 = \frac{N + \alpha}{2} \int_{\mathbb{R}^N} [I_\alpha * F(u)] F(u) > 0.
\]
Hence there exists $T_u > 0$ large enough such that
\[
I_{2^*_{2^*}}(u_{T_u}) = \frac{T_u^{N-2s}}{2} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 + \frac{T_u^N}{2} \int_{\mathbb{R}^N} u^2 - \frac{T_u^{N+\alpha}}{2} \int_{\mathbb{R}^N} [I_\alpha * F(u)] F(u) < 0.
\]
Next, we show that \( \lim \inf \) by the Lebesgue dominated theorem, \( \frac{t^N}{q^2} \int_{\mathbb{R}^N} [I_\alpha * |u|^q] |u|^q \) is continuous in \((t,q) \in [0,T_u] \times (2,2^*_s,\alpha)\). Hence, for \( \varepsilon > 0 \) small, there exists \( \delta > 0 \) such that for any \( t \in [0,T_u] \) and \( q \in (2^*_s,\alpha) - \delta, 2^*_s,\alpha)\), \(|J_1| < \frac{\varepsilon}{5}\). With the similar argument, \(|J_2| < \frac{\varepsilon}{5}\) and \(|J_3| < \frac{\varepsilon}{5}\). Thus, for \( q \in (2^*_s,\alpha - \delta, 2^*_s,\alpha)\) and \( T_u \) large enough,

\[
I_q(u_{T_u}) = I_{2^*_s,\alpha}(u_{T_u}) - J_1 + J_2 + J_3 < I_{2^*_s,\alpha}(u_{T_u}) + \varepsilon < \frac{1}{2}
\]

On the other hand, \( I_q(u_t) > 0 \) for \( t > 0 \) small. Thus, there exists \( t_0 \in (0,T_u) \) such that \( u_{t_0} \in M_q \) and then \( m_q \leq I_q(u_{t_0}) \). By Lemma 4.2, we have \( I_{2^*_s,\alpha}(u_{t_0}) \leq I_{2^*_s,\alpha}(u) \) since \( u \in M_{2^*_s,\alpha} \). Hence, for any \( q \in (2^*_s,\alpha - \delta, 2^*_s,\alpha)\), there hold

\[
m_q \leq I_q(u_{t_0}) \leq I_{2^*_s,\alpha}(u_{t_0}) + \varepsilon \leq I_{2^*_s,\alpha}(u) + \varepsilon \leq m_{2^*_s,\alpha} + 2\varepsilon.
\]

Next, we show that \( \lim \inf_{q \to 2^*_s,\alpha} m_q > 0 \). It follows from Theorem 3.2 that for any \( q_n \in (2,2^*_s,\alpha) \) with \( q_n \to 2^*_s,\alpha \), there exists a positive and radially symmetric sequence \( \{u_n\} \subset H^s_+(\mathbb{R}^N) \) such that

\[
I_{q_n}'(u_n) = 0 \quad \text{and} \quad I_{q_n}(u_n) = m_{q_n}.
\]

Then by the Pohozaev identity, for \( n \) large enough,

\[
m_{2^*_s,\alpha} + 1 \geq m_{q_n} = I_{q_n}(u_n) = I_{q_n}(u_n) - \frac{1}{N + \alpha} J_{q_n}(u_n)
\]

\[
= \frac{\alpha + 2s}{2(N + \alpha)} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |u_n|^2,
\]

which implies that \( \{u_n\} \) is bounded in \( H^s_+(\mathbb{R}^N) \). From \( J_{q_n}(u_n) = 0 \), one has

\[
\frac{N - 2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 + \frac{N}{2} \int_{\mathbb{R}^N} u_n^2
\]

\[
= \frac{N + \alpha}{2} \int_{\mathbb{R}^N} [I_\alpha \left( G(u_n) + \frac{1}{q_n} |u_n|^{q_n} \right)] \left( G(u_n) + \frac{1}{q_n} |u_n|^{q_n} \right).
\]

By \((g_1)\) and \((g_2)\), there exists \( C > 0 \) such that

\[
|G(t)| \leq C(|t|^2 + |t|^{2^*_s,\alpha}), \quad t \in \mathbb{R}
\]

combining with the Hardy-Littlewood-Sobolev inequality, we have

\[
\int_{\mathbb{R}^N} [I_\alpha \left( G(u_n) + \frac{1}{q_n} |u_n|^{q_n} \right)] \left( G(u_n) + \frac{1}{q_n} |u_n|^{q_n} \right)
\]

\[
\leq C \left( \int_{\mathbb{R}^N} |G(u_n) + \frac{1}{q_n} |u_n|^{q_n}|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{N}}
\]

\[
\leq C \left( \int_{\mathbb{R}^N} (|u_n|^2 + |u_n|^{2^*_s,\alpha} + |u_n|^{q_n})^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{N}}
\]
On the other hand, set Hardy-Littlewood-Sobolev inequality, we have This proof is similar to [15]. On one side, for any Lemma 4.4. For the reader's convenience, we give out the proof for completeness.

\[
C > \int \left| (-\Delta)^{\frac{N}{2}} u_n \right|^2 \leq C \left( \| u_n \|^4 + \| u_n \|^{\frac{2(N+\alpha)}{N-2\alpha}} + \| u_n \|^{2q_n} \right),
\]

from which we obtain

\[
\| u_n \|^2 \leq C \left( \| u_n \|^4 + \| u_n \|^{\frac{2(N+\alpha)}{N-2\alpha}} + \| u_n \|^{2q_n} \right).
\]

This inequality implies that there exists \( \delta > 0 \) such that \( \| u_n \| > \delta \) for any \( u_n \in M_{q_n} \). Hence, from (11) we have

\[
m_{q_n} \geq \frac{\alpha}{2(N+\alpha)} \| u_n \|^2 \geq C\delta^2 > 0.
\]

So \( \liminf_{q_n \to 2_s^*} m_{q_n} > 0 \). The proof is finished.

Lemma 4.3 implies \( m_{2_s^*} > 0 \). In order to get the upper estimate of \( m_{2_s^*} \), we first consider the achieved function for \( S_{s, \alpha} \) in [28] which is defined by

\[
S_{s, \alpha} = \inf_{u \neq 0, u \in D^s(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2}{\left( \int_{\mathbb{R}^N} [I_\alpha * |u|^{2_s^*}] |u|^{2_s^*} \right)^{\frac{N}{2^*_s}}},
\]

For the reader's convenience, we give out the proof for completeness.

**Lemma 4.4.** \( S_{s, \alpha} \) is achieved if and only if

\[
u = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2s}{2}},
\]

where \( C > 0 \) is a fixed constant and \( a \in \mathbb{R}^N, b > 0 \) are parameters. Moreover,

\[
S_{s, \alpha} = \frac{S_s}{(C(N, \alpha) A_\alpha)^{\frac{N}{2^*_s}}},
\]

**Proof.** This proof is similar to [15]. On one side, for any \( u \in H^s(\mathbb{R}^N) \), by the Hardy-Littlewood-Sobolev inequality, we have

\[
\int_{\mathbb{R}^N} [I_\alpha * |u|^{2_s^*}] |u|^{2_s^*} \leq A_\alpha C(N, \alpha) \left( \int_{\mathbb{R}^N} \left| u \right|^\frac{\frac{N}{2^*_s} + 2s}{N} \right)^\frac{N}{\frac{N}{2^*_s} + 2s} = A_\alpha C(N, \alpha) \| u \|^{2\frac{N}{2^*_s}}.
\]

Hence,

\[
S_{s, \alpha} \geq \left( A_\alpha C(N, \alpha) \frac{1}{S_{2^*_s}} \right)^{-\frac{N-2s}{2}} = \frac{S_s}{(A_\alpha C(N, \alpha))^{\frac{N-2s}{2^*_s}}}.
\]

On the other hand, set \( u = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2s}{2}} \), from Lemma 2.3, one has

\[
\int_{\mathbb{R}^N} (I_\alpha * |u|^{2_s^*}) |u|^{2_s^*} = A_\alpha C(N, \alpha) \| u \|^{2_s^*} \| u \|^{\frac{N}{N+s}} = A_\alpha C(N, \alpha) \left( \int_{\mathbb{R}^N} |u|^{2_s^*} \right)^\frac{N+s}{N}.
\]

It is well known that \( S_s \) also can be achieved by \( u \) defined above. Thus, we have

\[
S_{s, \alpha} \leq \frac{\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2}{\left( A_\alpha C(N, \alpha) \left( \int_{\mathbb{R}^N} |u|^{2_s^*} \right)^\frac{1}{2^*_s} \right)^{\frac{N-2s}{2}}}. 
\]
The proof is completed.

Let \( u(x) \) be the achieved function for \( S_s \) and \( S_{s,\alpha} \). Defining \( u_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} u(\frac{x}{\varepsilon}) \) and it is well known that \( S_s \) can be achieved by \( u_\varepsilon(x) \). Now, we claim that \( u_\varepsilon(x) \) is also the achieved function for \( S_{s,\alpha} \). From the argument above, it is sufficient to show that

\[
\int_{\mathbb{R}^N} (I_\alpha * |u_\varepsilon|^{2s,\alpha}) |u_\varepsilon|^{2s,\alpha} = A_\alpha C(N,\alpha) \|u_\varepsilon\|_{2s,\alpha}^{2}.
\] (12)

By the definition of \( u_\varepsilon(x) \), we have

\[
\int_{\mathbb{R}^N} (I_\alpha * |u_\varepsilon|^{2s,\alpha}) |u_\varepsilon|^{2s,\alpha} = \int_{\mathbb{R}^N} (I_\alpha * |\varepsilon^{-\frac{N-2s}{2}} u(\frac{x}{\varepsilon})|^{2s,\alpha}) |\varepsilon^{-\frac{N-2s}{2}} u(\frac{x}{\varepsilon})|^{2s,\alpha}
\]

\[
= \varepsilon^{-(N+\alpha)} A_\alpha \int_{\mathbb{R}^{2N}} \frac{|u(\frac{x}{\varepsilon})^{2s,\alpha}| |u(\frac{y}{\varepsilon})^{2s,\alpha}|}{|x-y|^{N-\alpha}} \, dx \, dy = A_\alpha \int_{\mathbb{R}^{2N}} \frac{|u(x)^{2s,\alpha}| |u(y)^{2s,\alpha}|}{|x-y|^{N-\alpha}} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^N} (I_\alpha * |u|^{2s,\alpha}) |u|^{2s,\alpha} = A_\alpha C(N,\alpha) \left( \int_{\mathbb{R}^N} |u|^{2s} \right)^{\frac{N+\alpha}{N}}
\]

where the last equality follows from the fact that the equality of Hardy-Littlewood-Sobolev inequality holds for \( u \). On the other hand, for the right side of (12), we have

\[
A_\alpha C(N,\alpha) \|u_\varepsilon(x)\|_{2s,\alpha}^{2} = A_\alpha C(N,\alpha) \left( \int_{\mathbb{R}^N} |u_\varepsilon|^{\frac{N+\alpha}{N}} \frac{N^{\frac{\alpha}{N-\alpha}}}{2} \, dx \right)^{\frac{N+\alpha}{N}}
\]

\[
= A_\alpha C(N,\alpha) \left( \int_{\mathbb{R}^N} \varepsilon^{-\frac{N-2s}{2}} u(\frac{x}{\varepsilon})^{\frac{N^{\frac{\alpha}{N-\alpha}}}{2}} \, dx \right)^{\frac{N+\alpha}{N}}
\]

\[
= A_\alpha C(N,\alpha) \left( \int_{\mathbb{R}^N} \varepsilon^{-N} |u(x)|^{\frac{2N}{N-\alpha}} \, dx \right)^{\frac{N+\alpha}{N}} = A_\alpha C(N,\alpha) \left( \int_{\mathbb{R}^N} |u|^{2s} \, dx \right)^{\frac{N+\alpha}{N}}.
\]

Thus, the claim is true.

Without loss of generality, in the following, we set \( u_\varepsilon(x) = C \varepsilon^{-\frac{N-2s}{2}} \left( \frac{1}{1+\frac{1}{\varepsilon^2}} \right)^{\frac{N-2s}{2}} \) for some \( C > 0 \) such that \( \|u_\varepsilon\|_{2s} = 1 \). Now, we give the upper estimate of \( m_{2s,\alpha} \).

**Lemma 4.5.** Assume \( g \) satisfies \((g_1) - (g_3)\), \( \alpha \in ((N-4s)_+, N) \), then

\[
m_{2s,\alpha} < \frac{\alpha + 2s}{2(N+\alpha)} \left( \frac{N-2s}{N+\alpha} \right)^{\frac{N+\alpha}{N-\alpha}} S_{s,\alpha}.
\]

**Proof.** Let \( \varphi \in C_0^\infty(\mathbb{R}^N, [0, 1]) \) be a cut-off function satisfying \( \varphi = 1 \) for \( x \in B_1 \) and \( \varphi = 0 \) for \( x \in \mathbb{R}^N \setminus B_2 \). Here \( B_r = \{ x \in \mathbb{R}^N | |x| < r \} \). Define the test function by \( v_\varepsilon(x) = \varphi u_\varepsilon(x) \), where \( u_\varepsilon(x) \in H^s(\mathbb{R}^N) \) is given above. From [34], we have

\[
\int_{\mathbb{R}^N} \frac{|(-\Delta)^{\frac{\alpha}{2}} v_\varepsilon(x)| \, dx \leq S_s + O(\varepsilon^{N-2s}),
\]

and

\[
\|v_\varepsilon\|_{L^2}^2 = \begin{cases} 
O(\varepsilon^{2s}), & N > 4s, \\
O(\varepsilon^{2s} \ln \frac{1}{\varepsilon}), & N = 4s, \\
O(\varepsilon^{N-2s}), & N < 4s.
\end{cases}
\]

It follows from Lemma 4.2, there exists a unique maximum point \( t_\varepsilon > 0 \) such that \( (v_\varepsilon)_{t_\varepsilon} \in M_{2s,\alpha} \). Now, we claim that there exist constants \( t_1 > 0, t_2 > 0 \) such that \( t_\varepsilon \in (t_1, t_2) \) for \( \varepsilon \) small enough. Otherwise, we suppose \( t_\varepsilon \to 0 \) or \( t_\varepsilon \to +\infty \) as
$\varepsilon \to 0$. If $t_\varepsilon^{\varepsilon \to 0} > 0$, then $m_{2s,\alpha} \leq I_{2s,\alpha}((v_\varepsilon)_{t_\varepsilon}) \to 0$, which is a contradiction with $m_{2s,\alpha} > 0$. If $t_\varepsilon^{\varepsilon \to 0} = +\infty$, we have $m_{2s,\alpha} \leq I_{2s,\alpha}((v_\varepsilon)_{t_\varepsilon}) \to -\infty$, which is also a contradiction. Thus, the claim is true.

For the fixed $t_\varepsilon$ above, $I_{2s,\alpha}((v_\varepsilon)_{t_\varepsilon}) = \frac{t_\varepsilon^{N-2s}}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}v_\varepsilon|^2 + \frac{t_\varepsilon^N}{2} \int_{\mathbb{R}^N} v_\varepsilon^2$

$$- \frac{t_\varepsilon^{N+\alpha}}{2} \int_{\mathbb{R}^N} \left[ I_\alpha \ast \left( G(v_\varepsilon) + \frac{1}{2s,\alpha} |v_\varepsilon|^{2s,\alpha} \right) \right] \left( G(v_\varepsilon) + \frac{1}{2s,\alpha} |v_\varepsilon|^{2s,\alpha} \right).$$

Next, we give out the estimate of $I_{2s,\alpha}((v_\varepsilon)_{t_\varepsilon})$.

**Step 1.** The estimate of $\int_{\mathbb{R}^N} \left( I_\alpha \ast |v_\varepsilon|^{2s,\alpha} \right) |v_\varepsilon|^{2s,\alpha}$. From the definition of $v_\varepsilon$, we have

$$\int_{\mathbb{R}^N} \left( I_\alpha \ast |v_\varepsilon|^{2s,\alpha} \right) |v_\varepsilon|^{2s,\alpha} \geq A_\alpha \int_{B_1} \int_{B_1} \frac{|u_\varepsilon(x)|^{2s,\alpha} |u_\varepsilon(y)|^{2s,\alpha}}{|x-y|^{N-\alpha}} dxdy$$

$$= A_\alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2s,\alpha} |u_\varepsilon(y)|^{2s,\alpha}}{|x-y|^{N-\alpha}} dxdy - 2A_\alpha \int_{\mathbb{R}^N \setminus B_1} \int_{B_1} \frac{|u_\varepsilon(x)|^{2s,\alpha} |u_\varepsilon(y)|^{2s,\alpha}}{|x-y|^{N-\alpha}} dxdy$$

$$- A_\alpha \int_{\mathbb{R}^N \setminus B_1} \int_{\mathbb{R}^N \setminus B_1} \frac{|u_\varepsilon(x)|^{2s,\alpha} |u_\varepsilon(y)|^{2s,\alpha}}{|x-y|^{N-\alpha}} dxdy.$$ 

Since $S_{s,\alpha}$ can be achieved by $u_\varepsilon$, one has

$$A_\alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2s,\alpha} |u_\varepsilon(y)|^{2s,\alpha}}{|x-y|^{N-\alpha}} dxdy = A_\alpha C(N,\alpha) \left( \int_{\mathbb{R}^N} |u_\varepsilon|^{2s} dx \right)^{\frac{N+\alpha}{N}} = A_\alpha C(N,\alpha).$$

By delicate calculations [15], we have

$$\int_{\mathbb{R}^N \setminus B_1} \int_{B_1} \frac{|u_\varepsilon(x)|^{2s,\alpha} |u_\varepsilon(y)|^{2s,\alpha}}{|x-y|^{N-\alpha}} dxdy \leq O(\varepsilon^{\frac{N+\alpha}{2}})$$

and

$$\int_{\mathbb{R}^N \setminus B_1} \int_{\mathbb{R}^N \setminus B_1} \frac{|u_\varepsilon(x)|^{2s,\alpha} |u_\varepsilon(y)|^{2s,\alpha}}{|x-y|^{N-\alpha}} dxdy \leq O(\varepsilon^{N+\alpha}).$$

Hence, we obtain

$$\int_{\mathbb{R}^N} \left( I_\alpha \ast |v_\varepsilon|^{2s,\alpha} \right) |v_\varepsilon|^{2s,\alpha} \geq A_\alpha C(N,\alpha) - O(\varepsilon^{\frac{N+\alpha}{2}}).$$

By the analysis above, we get

$$I_{2s,\alpha}((v_\varepsilon)_{t_\varepsilon}) = \frac{t_\varepsilon^{N-2s}}{2} (S_\alpha + O(\varepsilon^{N-2s})) - \frac{t_\varepsilon^{N+\alpha}}{2} \left( \frac{1}{2s,\alpha} \right)^2 A_\alpha C(N,\alpha)$$

$$+ O(\varepsilon^{\frac{N+\alpha}{2}}) + \frac{t_\varepsilon^N}{2} \int_{\mathbb{R}^N} |v_\varepsilon|^2 - \frac{t_\varepsilon^{N+\alpha}}{2} \cdot \frac{1}{2s,\alpha} \int_{\mathbb{R}^N} \left[ I_\alpha \ast |v_\varepsilon|^{2s,\alpha} \right] G(v_\varepsilon)$$

$$- \frac{t_\varepsilon^{N+\alpha}}{2} \cdot \frac{1}{2s,\alpha} \int_{\mathbb{R}^N} \left( I_\alpha \ast G(v_\varepsilon) \right) |v_\varepsilon|^{2s,\alpha} - \frac{t_\varepsilon^{N+\alpha}}{2} \int_{\mathbb{R}^N} \left[ I_\alpha \ast G(v_\varepsilon) \right] G(v_\varepsilon)$$

$$:= \frac{t_\varepsilon^{N-2s}}{2} S_\alpha - \frac{t_\varepsilon^{N+\alpha}}{2} \left( \frac{1}{2s,\alpha} \right)^2 A_\alpha C(N,\alpha) + O(\varepsilon^{N-2s}) + O(\varepsilon^{\frac{N+\alpha}{2}})$$

$$+ \frac{t_\varepsilon^N}{2} \int_{\mathbb{R}^N} |v_\varepsilon|^2 - K_1 - K_2 - K_3,$$
where

\[
K_1 = \frac{t_\epsilon^{N+\alpha}}{2} \cdot \frac{1}{2s,\alpha} \int_{\mathbb{R}^N} |I_{\alpha} * |v_\epsilon|^2s,\alpha|G(v_\epsilon),
\]

\[
K_2 = \frac{t_\epsilon^{N+\alpha}}{2} \cdot \frac{1}{2s,\alpha} \int_{\mathbb{R}^N} |I_{\alpha} * G(v_\epsilon)||v_\epsilon|^{2s,\alpha},
\]

\[
K_3 = \frac{t_\epsilon^{N+\alpha}}{2} \int_{\mathbb{R}^N} |I_{\alpha} * G(v_\epsilon)|G(v_\epsilon).
\]

Let \( h(t) = \frac{t^{N-2s}}{2} - s - \frac{t^{N+\alpha}}{2} (\frac{1}{2s,\alpha})^2 A_\alpha C(N, \alpha), \) then

\[
\max_{t \geq 0} h(t) = \frac{\alpha + 2s}{2(N + \alpha)} (2s,\alpha)^{N-2s} S_{N,\alpha}^N.
\]

Define \( \eta(\epsilon) = O(\epsilon^{N-2s}) + O(\epsilon^{\frac{N+\alpha}{2}}) + \frac{t^{N}}{2} \int_{\mathbb{R}^N} |v_\epsilon|^2. \) For \( \epsilon \) small, since \( \alpha \in (N - 4s)_+, \) we have

\[
\eta(\epsilon) = \begin{cases} 
O(\epsilon^{2s}), & N > 4s, \\
O(\epsilon^{2s} \ln \frac{1}{\epsilon} + 1), & N = 4s, \\
O(\epsilon^{N-2s}), & N < 4s.
\end{cases}
\]

**Step 2.** The estimates of \( K_1, K_2 \) and \( K_3. \) It follows from \( v_\epsilon(x) = \varphi \cdot u_\epsilon(x) \) that for \( |x| < \epsilon(\epsilon > 0 \text{ small}), \) we obtain

\[
v_\epsilon(x) = u_\epsilon(x) \leq C \epsilon^{-\frac{N-2s}{2}} \left( \frac{\epsilon^2}{\epsilon^2 + |x|^2} \right) \leq C \epsilon^{-\frac{N-2s}{2}}.
\]

By \((g_3),\) for any \( R > 0, \) there exists \( C_R > 0 \) such that for \( t \in [C_R, +\infty), \)

\[
G(t) \geq \begin{cases} 
R t^{\frac{N+\alpha-4s}{2}}, & N > 4s, \\
R t^{\frac{N+\alpha}{2}} \ln t^{-\frac{1}{2}}, & N = 4s, \\
R t^{\frac{N+\alpha-N}{2}}, & N < 4s.
\end{cases}
\]

Thus, for \( \epsilon \) small and \( N > 4s, \)

\[
K_1 \geq C \int_{B_\epsilon} \int_{B_\epsilon} \frac{|u_\epsilon(x)|^{2s,\alpha} G(u_\epsilon(y))}{|x-y|^{N-\alpha}}
\]

\[
\geq CR \int_{B_\epsilon} \int_{B_\epsilon} \left( \frac{\epsilon^{-\frac{N-2s}{2}}}{|x-y|^{N-\alpha}} \cdot \left( \epsilon^{-\frac{N-2s}{2}} \right)^{\frac{N+\alpha-4s}{2}} \right)
\]

\[
= CR \epsilon^{-\frac{N+\alpha-4s}{2}} \int_{B_\epsilon} \int_{B_\epsilon} \frac{1}{|x-y|^{N-\alpha}}
\]

\[
= CR \epsilon^{-\frac{N+\alpha-4s}{2}} \int_{B_\epsilon} \int_{B_\epsilon} \frac{\epsilon^{2N}}{\epsilon^{N-\alpha}|x-y|^{N-\alpha}}
\]

\[
= CR \epsilon^{2s} \int_{B_\epsilon} \int_{B_\epsilon} \frac{1}{|x-y|^{N-\alpha}},
\]

together with the Hardy-Littlewood-Sobolev inequality, we obtain

\[
K_1 \geq CR \epsilon^{2s}.
\]

If \( N = 4s, \) for \( \epsilon \) small, by the similar argument above,

\[
K_1 \geq CR \int_{B_\epsilon} \int_{B_\epsilon} \left( \epsilon^{-\frac{N-2s}{2}} \right)^{2s,\alpha} \cdot \left( \epsilon^{-\frac{N-2s}{2}} \right)^{\frac{N+\alpha-4s}{2}} \cdot \ln(\epsilon^{-\frac{N-2s}{2}})^{\frac{1}{2}}
\]

\[
= CR \epsilon^{2s} \int_{B_\epsilon} \int_{B_\epsilon} \frac{1}{|x-y|^{N-\alpha}}.
\]
Thus, and
\[ R \text{By the arbitrary of } \varepsilon \text{, which implies that for } \varepsilon > 0, \]
\[ \int_{B_1} \frac{1}{|x-y|^{N+\alpha-2s}} \epsilon^{-1} \int_{B_1} B_1 \ln \epsilon^{-1} \int_{B_1} \epsilon^{2N} |x-y|^{N-\alpha} = CR^2 \ln \epsilon^{-1}. \]

Similarly, if \( N < 4s \),
\[ K_1 \geq CR^2 N - 2s, \]
\[ K_2 \geq \begin{cases} \text{RC}^2 s, & N > 4s, \\ \text{RC}^2 s \ln \epsilon^{-1}, & N = 4s, \\ \text{RC}^2 N - 2s, & N < 4s, \end{cases} \]
and
\[ K_3 \geq \begin{cases} R^2 C \epsilon^{4s}, & N > 4s, \\ R^2 C \epsilon^{4s} (\ln \epsilon^{-1})^2, & N = 4s, \\ R^2 C \epsilon^{2N-4s}, & N < 4s. \end{cases} \]

Thus,
\[ K_1 + K_2 + K_3 \geq \begin{cases} \text{RC}^2 s, & N > 4s, \\ \text{RC}^2 s \ln \epsilon^{-1}, & N = 4s, \\ \text{RC}^2 N - 2s, & N < 4s. \end{cases} \]

By the arbitrary of \( R \), we have
\[ \lim_{\epsilon \to 0^+} \frac{K_1 + K_2 + K_3}{\eta(\varepsilon)} = +\infty, \]
which implies that for \( \varepsilon \) small,
\[ m_{2s,\alpha} \leq I_{2s,\alpha}((v_\varepsilon)_n) < \frac{\alpha + 2s}{2(1 + \alpha)} \frac{N_{\alpha}}{S_{\alpha}} \frac{N_{\alpha}}{S_{\alpha}}. \]

The proof is finished. \( \square \)

5. Proof of Theorem 1.1.

Proof. By Theorem 3.2, for any \( q_n \in (2, 2^*_\alpha) \), equation (9) admits a positive radially symmetric ground state \( u_{q_n} \). In the following, we write it as \( u_n \). It follows from Lemma 4.3 that \( \{u_n\} \) is bounded in \( H^s_2(\mathbb{R}^N) \). So there exists a non-negative radially symmetric function \( u \in H^s_2(\mathbb{R}^N) \) such that \( \{u_n\} \) (up to a subsequence) convergent weakly to \( u \) in \( H^s_2(\mathbb{R}^N) \). By Lemma 2.1, for any \( r \in (2, 2^*_\alpha) \), \( u_n \to u \) strongly in \( L^r(\mathbb{R}^N) \) and \( u_{q_n} \to u \) in \( \mathbb{R}^N \).

In the following, we prove that \( u \) is a ground state solution to problem (7) by three steps.

Step 1: We prove that \( u \) is a critical point of \( I_{2s,\alpha} \).

Because of the presence of the term \( I_\alpha * F(u) \), for the bounded (PS) sequence \( \{u_n\} \), even \( u_n \to u_0 \) weakly, it does not hold in general that \( u_0 \) is a critical point of the energy functional. In order to prove that \( u \) is a critical point of \( I_{2s,\alpha} \), it is crucial to prove that for any \( \varphi \in C_0^\infty(\mathbb{R}^N) \), there holds as \( n \to \infty \),
\[ \int_{\mathbb{R}^N} [I_\alpha * (G(u_n) + \frac{1}{q_n}|u_n|^q)] (g(u_n) + |u_n|^q u_n) \varphi \]
\[ \to \int_{\mathbb{R}^N} [I_\alpha * (G(u) + \frac{1}{2s} |u|^{2s})] (g(u) + |u|^{2s} u) \varphi. \] (13)
Firstly, we prove that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, as $n \to \infty$,

$$\int_{\mathbb{R}^N} [I_\alpha * G(u_n)]g(u_n)\varphi \to \int_{\mathbb{R}^N} [I_\alpha * G(u)]g(u)\varphi. \quad (14)$$

Since

$$\int_{\mathbb{R}^N} [I_\alpha * G(u_n)]g(u_n)\varphi - \int_{\mathbb{R}^N} [I_\alpha * G(u)]g(u)\varphi = \int_{\mathbb{R}^N} [I_\alpha * (G(u_n) - G(u))]g(u_n)\varphi + \int_{\mathbb{R}^N} [I_\alpha * G(u)](g(u_n) - g(u))\varphi.$$

On one side, by the Hardy-Littlewood-Sobolev inequality,

$$\left| \int_{\mathbb{R}^N} [I_\alpha * (G(u_n) - G(u))]g(u_n)\varphi \right| \leq C \left( \int_{\mathbb{R}^N} |G(u_n) - G(u)|^{\frac{2N}{2N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \left( \int_{\mathbb{R}^N} |g(u_n)|^{\frac{2N}{2N+\alpha}} |\varphi|^{\frac{2N}{2N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \to 0.$$

From $(g_1)-(g_2)$, for any $\varepsilon > 0$ and $r \in (1, 2^*_s, 1)$, there exist $C(\varepsilon) > 0$, such that

$$|g(t)| \leq \varepsilon |(t| |t|^{2^*_s,1}) + C(\varepsilon)|t|^r, \quad t \in \mathbb{R}.$$

Since $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*_s)$, $\int_{\mathbb{R}^N} |G(u_n) - G(u)|^{\frac{2N}{2N+\alpha}} \to 0$. Together with the integrability of $\int_{\mathbb{R}^N} |g(u_n)|^{\frac{2N}{2N+\alpha}}$, we obtain that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} [I_\alpha * (G(u_n) - G(u))]g(u_n)\varphi \to 0$ as $n \to \infty$.

On the other hand, by the Hardy-Littlewood-Sobolev inequality and $u_n \to u$ weakly in $H^s(\mathbb{R}^N)$, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, there holds as $n \to \infty$,

$$\int_{\mathbb{R}^N} [I_\alpha * G(u)](g(u_n) - g(u))\varphi \leq \left( \int_{\mathbb{R}^N} |G(u)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \left( \int_{\mathbb{R}^N} |g(u_n) - g(u)|^{\frac{2N}{2N+\alpha}} |\varphi|^{\frac{2N}{2N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \to 0.$$

So $(14)$ is true.

Next, we prove that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, as $n \to \infty$,

$$\int_{\mathbb{R}^N} [I_\alpha * |u_n|^{q_n}]|u_n|^{q_n-2}u_n\varphi \to \int_{\mathbb{R}^N} [I_\alpha * |u|^{2^*_s,\alpha}]|u|^{\frac{N+\alpha}{N-1}}u\varphi. \quad (15)$$

By the Hardy-Littlewood-Sobolev inequality and $u_n \to u$ weakly in $H^s(\mathbb{R}^N)$, the similar argument as above, we have

$$\int_{\mathbb{R}^N} [I_\alpha * |u_n|^{q_n}]|u_n|^{q_n-2}u_n\varphi = \int_{\mathbb{R}^N} [I_\alpha * |u_n|^{q_n}]|u|^{2^*_s,\alpha-2}u\varphi + o_n(1).$$

Moreover,

$$\left| \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^{q_n} - |u|^{q_n} - |u_n - u|^{q_n})]|u|^{2^*_s,\alpha-2}u\varphi \right| \leq C \left( \int_{\mathbb{R}^N} ||u_n|^{q_n} - |u|^{q_n} - |u_n - u|^{q_n}| \right)^{\frac{N+\alpha}{2N}} \left( \int_{\mathbb{R}^N} |u|^{(2^*_s,\alpha-1)(\frac{N}{N-1})} |\varphi|^{\frac{2N}{N-1}} \right)^{\frac{N+\alpha}{2N}}.$$
Thus (15) holds. by the H"older inequality and Lemma 2.1, 

\[
\left( \int_{\mathbb{R}^N} |u|^{(2^*_s)_s-1}\frac{2^N}{N+s}|\varphi|^2 \frac{2^N}{N+s} \right)^{\frac{N+s}{2^*_s}} 
\leq \left( \int_{\mathbb{R}^N} |u|^{(2^*_s)_s-1}\frac{2^N}{N+s}|\varphi|^\frac{2^N}{N+s} \right)^{\frac{N+s}{2^*_s}} \left( \int_{\mathbb{R}^N} |\varphi|^\frac{2^N}{N+s} \right)^{\frac{N+s}{2^*_s} - \frac{2^*_s}{2^*_s}} 
\leq \left( \int_{\mathbb{R}^N} |u|^{2^*_s} \left( \int_{\mathbb{R}^N} |\varphi|^{2^*_s} \right) \right)^{\frac{N+s}{2^*_s} - \frac{2^*_s}{2^*_s}} 
\leq C\|\varphi\|.
\]

Hence, for any \( \varphi \in C_0^\infty(\mathbb{R}^N) \), as \( n \to \infty \), 

\[
\int_{\mathbb{R}^N} \left[ I_\alpha * (|u_n|^{q_n} - |u|^{q_n} - |u_n - u|^{q_n}) \right] |u|^{2^*_s-2} u \varphi 
\to 0.
\]

As \( q_n \xrightarrow{n \to \infty} 2^*_s \), combining Young inequality, 

\[
|u_n|^{q_n} \leq \frac{2^*_s \alpha - q_n}{2^*_s \alpha - 2} |u_n|^2 + \frac{q_n - 2}{2^*_s \alpha - 2} |u_n|^{2^*_s},
\]

which implied that \( \{|u_n|^{q_n}\} \) is bounded in \( L^{\frac{2^*_s}{2^*_s-2}}(\mathbb{R}^N) \) and up to a subsequence, 

\(|u_n|^{q_n} \to |u|^{2^*_s}\) weakly in \( L^{\frac{2^*_s}{2^*_s-2}}(\mathbb{R}^N) \) as \( n \to \infty \). Due to \( |u|^{2^*_s-2} u \varphi \in L^{\frac{2^*_s}{2^*_s-2}}(\mathbb{R}^N) \), 

by the Hardy-Littlewood-Sobolev inequality, we have, for any \( \varphi \in C_0^\infty(\mathbb{R}^N) \), 

\[
\psi \in L^{\frac{2^*_s}{2^*_s-2}}(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} \left[ I_\alpha * (|u|^{2^*_s-2} u \varphi) \right] \psi
\]

is a linear and bounded functional. Then 

\[
\int_{\mathbb{R}^N} \left[ I_\alpha * |u_n|^{q_n} \right] |u|^{2^*_s-2} u \varphi
\]

\[
= \int_{\mathbb{R}^N} \left[ I_\alpha * (|u|^{2^*_s-2} u \varphi) \right] |u_n|^{q_n}
\]

\[
= \int_{\mathbb{R}^N} \left[ I_\alpha * (|u|^{2^*_s-2} u \varphi) \right] |u|^{2^*_s} + o_n(1)
\]

\[
= \int_{\mathbb{R}^N} \left[ I_\alpha * |u|^{2^*_s} \right] |u|^{2^*_s-2} u \varphi + o_n(1).
\]

So, we have, as \( n \to \infty \), 

\[
\int_{\mathbb{R}^N} \left[ I_\alpha * |u_n|^{q_n} \right] |u|^{2^*_s-2} u \varphi \to \int_{\mathbb{R}^N} \left[ I_\alpha * |u|^{2^*_s} \right] |u|^{2^*_s-2} u \varphi, \forall \varphi \in C_0^\infty(\mathbb{R}^N).
\]

Thus (15) holds.

Similarly, we can prove that 

\[
\int_{\mathbb{R}^N} \left[ I_\alpha * G(u_n) \right] |u_n|^{q_n-2} u_n \varphi \to \int_{\mathbb{R}^N} \left[ I_\alpha * G(u) \right] |u|^{\frac{n+4}{N-2^*_s}} u \varphi \tag{16}
\]

and 

\[
\int_{\mathbb{R}^N} \left[ I_\alpha * |u_n|^{q_n} g(u_n) \right] \varphi \to \int_{\mathbb{R}^N} \left[ I_\alpha * |u|^{2^*_s} \right] g(u) \varphi. \tag{17}
\]

By (14), (15), (16) and (17), we have (13). Hence, \( u \) is a critical point of \( I_{2^*_s} \).
Step 2: We prove \( u \neq 0 \). Suppose that \( u_n \to 0 \) weakly in \( H^s(\mathbb{R}^N) \). By (10), we have
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 + \int_{\mathbb{R}^N} u_n^2
\]
\[
= \int_{\mathbb{R}^N} I_\alpha \left( G(u_n) + \frac{1}{q_n} |u_n|^{q_n} \right) (g(u_n)u_n + |u_n|^{q_n})
\]
\[
= \int_{\mathbb{R}^N} I_\alpha \left( G(u_n) + \frac{1}{q_n} |u_n|^{q_n} \right) g(u_n)u_n
\]
\[
+ \int_{\mathbb{R}^N} (I_\alpha * G(u_n)) |u_n|^{q_n} + \frac{1}{q_n} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_n}) |u_n|^{q_n}.
\]
Let \( P(t) = |g(t)t|^\frac{2N}{N+\alpha} \). By \((g_1) - (g_2)\),
\[
\lim_{t \to 0} \frac{P(t)}{|t|^2} = 0, \quad \lim_{t \to +\infty} \frac{P(t)}{|t|^2} = 0.
\]
It follows from \( u_n \to 0 \) weakly in \( H^s(\mathbb{R}^N) \) and \( u_n^{a.e. \to 0} \) that \( P(\lim t_n(x)) \to 0 \). Moreover, from the argument in Lemma 4.3, \( \{u_n\} \) is bounded in \( H^s(\mathbb{R}^N) \). Then, by Lemma 2.1 and Lemma 2.2, \( P(u_n(x)) \to 0 \) in \( L^1(\mathbb{R}^N) \). Consequently, \( g(u_n)u_n \to 0 \) strongly in \( L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \). Thus, by the Hardy-Littlewood-Sobolev inequality,
\[
\left| \int_{\mathbb{R}^N} I_\alpha \left( G(u_n) + \frac{1}{q_n} |u_n|^{q_n} \right) g(u_n)u_n \right|
\]
\[
\leq \left( \int_{\mathbb{R}^N} \left| G(u_n) + \frac{1}{q_n} |u_n|^{q_n} \right|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \left( \int_{\mathbb{R}^N} |g(u_n)u_n|^\frac{2N}{N+\alpha} \right)^{\frac{N+\alpha}{2N}} \to 0,
\]
where we used the fact that \( \{G(u_n) + \frac{1}{q_n} |u_n|^{q_n}\} \) is bounded in \( L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \).

Similarly, \( G(u_n) \to 0 \) strongly in \( L^{\frac{N}{N+\alpha}}(\mathbb{R}^N) \), which implies that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * G(u_n)) |u_n|^{q_n} = 0.
\]

Then
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 + \int_{\mathbb{R}^N} u_n^2 = \frac{1}{q_n} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_n}) |u_n|^{q_n} + o_n(1).
\]
Combining with the Young inequality and \( q_n \to 2^*_s - \), we have
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 + \int_{\mathbb{R}^N} u_n^2
\]
\[
\leq \frac{1}{q_n} \int_{\mathbb{R}^N} \left\{ I_\alpha \left[ \frac{2^*_s - q_n}{2^*_s - 2} |u_n|^2 + \frac{q_n - 2}{2^*_s - 2} |u_n|^{2^*_s} \right] \right\}
\]
\[
= \frac{1}{2^*_s} \int_{\mathbb{R}^N} [I_\alpha * |u_n|^{2^*_s}] |u_n|^{2^*_s} + o_n(1).
\]
By the definition of \( S_{s,\alpha} \),
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \leq \frac{1}{2^*_s} \left( \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2}{S_{s,\alpha}} \right)^{\frac{N+\alpha}{N-\alpha}} + o_n(1).
\]
So, either \( \int_{\mathbb{R}^N} |(-\Delta) \hat{z} u_n|^2 \rightarrow 0 \) or

\[
\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |(-\Delta) \hat{z} u_n|^2 \geq \left( 2^*_{s, \alpha} \cdot S_{s, \alpha}^{\frac{N+\alpha}{N - 2\alpha}} \right)^{\frac{N - 2s}{N - 2\alpha}}.
\]

If \( \int_{\mathbb{R}^N} |(-\Delta) \hat{z} u_n|^2 \rightarrow 0 \), there hold \( \int_{\mathbb{R}^N} |I_{s, \alpha} |u_n|^{2^*_{s, \alpha}} |u_n|^{2s_{s, \alpha}} \rightarrow 0 \) and \( \int_{\mathbb{R}^N} |u_n|^2 \rightarrow 0 \). Then we have \( m_{q_n} = I_{q_n}(u_n) \rightarrow 0 \), which is a contradiction with Lemma 4.3. This means \( \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |(-\Delta) \hat{z} u_n|^2 \geq \left( 2^*_{s, \alpha} \cdot S_{s, \alpha}^{\frac{N+\alpha}{N - 2\alpha}} \right)^{\frac{N - 2s}{N - 2\alpha}} \). By Lemma 4.3,

\[
m_{2 s_{s, \alpha}} \geq \limsup_{n \rightarrow \infty} m_{q_n}
\]

\[
= \limsup_{n \rightarrow \infty} [I_{q_n}(u_n) - \frac{1}{N + \alpha} J_{q_n}(u_n)]
\]

\[
= \limsup_{n \rightarrow \infty} \left( \frac{\alpha + 2s}{2(N + \alpha)} \int_{\mathbb{R}^N} |(-\Delta) \hat{z} u_n|^2 + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |u_n|^2 \right)
\]

\[
\geq \frac{\alpha + 2s}{2(N + \alpha)} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |(-\Delta) \hat{z} u_n|^2
\]

\[
\geq \frac{\alpha + 2s}{2(N + \alpha)} \left( 2^*_{s, \alpha} \cdot S_{s, \alpha}^{\frac{N+\alpha}{N - 2\alpha}} \right)^{\frac{N - 2s}{N - 2\alpha}}
\]

\[
= \frac{\alpha + 2s}{2(N + \alpha)} (2^*_{s, \alpha})^{\frac{N - 2s}{N - 2\alpha}}.
\]

which is a contradiction with Lemma 4.5. Hence we have \( u \not\equiv 0 \).

**Step 3:** We claim \( m_{2 s_{s, \alpha}} = I_{2 s_{s, \alpha}}(u) \). Since \( I_{2 s_{s, \alpha}}(u) = 0 \), by the weakly lower semi-continuity of the norm, we have

\[
m_{2 s_{s, \alpha}} \leq I_{2 s_{s, \alpha}}(u)
\]

\[
= I_{2 s_{s, \alpha}}(u) - \frac{1}{N + \alpha} J_{2 s_{s, \alpha}}(u)
\]

\[
= \frac{\alpha + 2s}{2(N + \alpha)} \int_{\mathbb{R}^N} |(-\Delta) \hat{z} u|^2 + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |u|^2
\]

\[
\leq \liminf_{n \rightarrow \infty} \left( \frac{\alpha + 2s}{2(N + \alpha)} \int_{\mathbb{R}^N} |(-\Delta) \hat{z} u_n|^2 + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |u_n|^2 \right)
\]

\[
= \liminf_{n \rightarrow \infty} (I_{q_n}(u_n) - \frac{1}{N + \alpha} J_{q_n}(u_n))
\]

\[
= \liminf_{n \rightarrow \infty} m_{q_n}
\]

\[
\leq \limsup_{n \rightarrow \infty} m_{q_n} \leq m_{2 s_{s, \alpha}}.
\]

which means \( I_{2 s_{s, \alpha}}(u) = m_{2 s_{s, \alpha}} \). The proof is completed. 

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