Isothermic surfaces in $\mathbb{E}^3$ as soliton surfaces*

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October 27, 2018

Abstract

We show that the theory of isothermic surfaces in $\mathbb{E}^3$ – one of the oldest branches of differential geometry – can be reformulated within the modern theory of completely integrable (soliton) systems. This enables one to study the geometry of isothermic surfaces in $\mathbb{E}^3$ by means of powerful spectral methods available in the soliton theory. Also the associated non-linear system is interesting in itself since it displays some unconventional soliton features and, physically, could be applied in the theory of infinitesimal deformations of membranes.

*The work supported in part by the grants 566/2/91 GR 10 (KBN 2 0168 91 01) and PB 1274/P3/92/02 (KBN 2 2303 91 02).
1 Introduction

There is no doubt today that some fundamental ideas and many concrete results of the modern theory of completely integrable (soliton) systems can be traced back over a century to the classical differential geometry. For instance, one of the most important ingredients of the soliton theory is the theory of Bäcklund transformations. The first example of the Bäcklund transformation (for the celebrated sine-Gordon eq.) originated in the works of the great differential geometers of XIX century (G.Darboux and L.Bianchi) to be finally formulated by A.V.Bäcklund in 1880.

In general, it turns out that a careful study of some other works by G.Darboux and, notably, by L.Bianchi leads to a conclusion that some their results are of a genuine soliton nature. Exactly in this way (by studying the paper [1] by L.Bianchi) in 1991 we realized that, presumably, the theory of the so called isothermic surfaces – one of the oldest branches of differential geometry [2] which recently is a subject of some modern studies [3, 4, 5, 6] – can be reformulated within the modern approach of soliton surfaces [7]. This conjecture has been further confirmed by different tests.

Based on results of this paper (presented in the short report [8]) Pinkall’s Berlin Group have written a number of preprints including [9, 10].

2 Isothermic surfaces and Bonnet surfaces

Consider an arbitrary immersed surface $S$ in $\mathbb{E}^3$ without umbilic points (in umbilic points both principal curvatures coincide). The question is: What conditions one should impose on $S$ to guarantee the existence of infinitesimal isometric deformations preserving principal curvatures (or, equivalently, the mean curvature) invariant?

The answer – well known to the geometers of XIX century – is that the curvature coordinates, say $u$ and $v$, are conformal (after a proper reparameterization), i.e. the fundamental forms read

$$I = e^{2\vartheta}(du^2 + dv^2) \quad \text{(metric)},$$

$$II = e^{2\vartheta}(k_2 du^2 + k_1 dv^2) \quad \text{(2-nd fundamental form)},$$

(1)

where $k_1 = k_1(u, v)$ and $k_2 = k_2(u, v)$ are principal curvatures and $\vartheta = \vartheta(u, v)$. 

2
The triplet \((\vartheta, k_1, k_2)\) has to satisfy the following system of non-linear partial differential eqs. (Gauss-Mainardi-Codazzi eqs.):

\[
\begin{align*}
\vartheta_{uu} + \vartheta_{vv} + k_1 k_2 e^{2\vartheta} &= 0, \\
(k_1 - k_2)\vartheta_u &= 0, \\
(k_2 - k_1)\vartheta_v &= 0,
\end{align*}
\]  

(2a), (2b), (2c)

where comma denotes differentiation.

And vice versa, any solution \((\vartheta, k_1, k_2)\) defines uniquely some isothermic surface in \(E^3\) modulo a rigid motion in \(E^3\). In other words, the geometry of isothermic surfaces in \(E^3\) without umbilic points is completely encoded in the system (2). The theory of isothermic surfaces at umbilics is much more difficult.

Surfaces in \(E^3\) admitting global and non-trivial (rigid motions are excluded) isometries preserving principal curvatures (or, equivalently, the mean curvature \(H := \frac{1}{2}(k_1 + k_2)\)) are called Bonnet surfaces \([3, 5, 6]\) after O. Bonnet who was the first to study such surfaces in 1867.

Bonnet surfaces constitute a proper subset of isothermic surfaces. Indeed, the totality of isothermic surfaces can be parameterized by 4 functions of a single variable each (the order of the system (2) is 4!). For a more detailed proof see \([4]\). On the other hand S.S. Chern \([3]\) has shown that the class of Bonnet surfaces consists of two subclasses: i) surfaces of \(H = \text{const}\) (to select a surface of \(H = \text{const}\) one needs 2 functions of a single variable) and ii) some family of surfaces parameterized by 6 parameters.

In 1903 P. Calapso \([11]\) on performing a series of remarkable transformations was able to replace the system (2) by the following single equation of 4-th order

\[
\left(\frac{w_{,uu}}{w}\right)_{,uu} + \left(\frac{w_{,uv}}{w}\right)_{,vv} + (w^2)_{,uv} = 0.
\]  

(3)

Finally, we mention an interesting physical application of isothermic (in particular: Bonnet) surfaces \([12]\). Imagine an elastic membrane moving isometrically in \(E^3\). We require that the difference of pressures between both sides of the membrane is a constant of motion at each point (though it may vary from point to point). One can show that such a motion is admitted for
Bonnet surfaces only. Obviously, infinitesimally short evolution corresponds to isothermic surfaces.

3 The starting point of the research

In the paper [1] (page 98) one can find the following formulae:

\[
\begin{align*}
\lambda_{uv} &= -\vartheta_{uv} \mu - k_2 e^\vartheta \omega + m \sigma e^\vartheta + me^{-\vartheta} \varphi, \\
\mu_{uv} &= \vartheta_{uv} \lambda, \quad \varphi_{uv} = e^\vartheta \lambda, \quad \omega_{uv} = k_2 e^\vartheta \lambda, \quad \sigma_{uv} = e^{-\vartheta} \lambda, \\
\mu_{vv} &= -\vartheta_{vu} \lambda - k_1 e^\vartheta \omega + m \sigma e^\vartheta - me^{-\vartheta} \varphi, \\
\lambda_{vv} &= \vartheta_{vu} \mu, \quad \omega_{vv} = k_1 e^\vartheta \mu, \quad \sigma_{vv} = -e^{-\vartheta} \mu, \quad \varphi_{vv} = e^\vartheta \mu,
\end{align*}
\]

(4)

where \( \vartheta, k_1 \) and \( k_2 \) is some but fixed solution to the system (2), \( m \) is a non-zero parameter, and \( \lambda, \mu, \omega, \varphi \) and \( \sigma \) are 5 unknowns.

The important point is that the integrability conditions for the system (4) are identical with the system (2) and, moreover, the system (4) contains a free parameter, namely: \( m \).

In the soliton theory such a coexistence of the linear system (linear problem) containing a parameter (spectral parameter) with the corresponding non-linear system strongly suggests (by no means proves !) that the non-linear system could be integrable in the sense of the soliton theory.

The other important fact is that the quadratic form

\[
\lambda^2 + \mu^2 + \omega^2 - 2m\varphi\sigma
\]

(5)
does not depend on \( u \) and \( v \).

One could name the linear problem (4) for the system (2) – a Darboux-Bianchi linear problem since it was introduced by G.Darboux and later on became a subject of intensive studies by L.Bianchi and his Pisa school.

Solving the system (4) one is able to obtain a new solution \( (\vartheta', k_1', k_2') \) of the system (2):

\[
\begin{align*}
e^{\vartheta'} &= \pm \frac{\varphi}{\sigma} e^{-\vartheta}, \\
k_1' e^{\vartheta'} &= \pm \left( k_1 e^\vartheta + \frac{\omega}{\varphi} e^\vartheta - \frac{\omega}{\sigma} e^{-\vartheta} \right),
\end{align*}
\]

(6a) (6b)
\[ k'_2 e^{\vartheta'} = \mp \left( k_2 e^{\vartheta} + \frac{\omega}{\varphi} e^{\vartheta} + \frac{\omega}{\sigma} e^{-\vartheta} \right), \quad (6c) \]

where the upper sign (lower sign) corresponds to \( m > 0 \) (\( m < 0 \)).

One can call (6) which is a kind of Bäcklund transformation a “classical Darboux-Bianchi transformation”. In fact in [13] L.Bianchi proved the abelian property of the transformation (6) which is a characteristic feature of Bäcklund transformations.

The linear problem (4) and the classical Darboux-Bäcklund transformation were the starting point of our research of the subject which we undertook in 1991.

4 Painlevé analysis of the nonlinear systems associated with isothermic surfaces

First of all we applied a relatively simple test of the integrability – the so called “Painlevé test” [14, 15] to the systems (2) and (3).

The Painlevé test is performed in its classical form [14] extended to partial differential equations in [16], i.e. by assuming a solution in the form of a Laurent series about an arbitrary singularity manifold \( \Phi(u, v) = 0 \) and checking compatibility of the resulting recurrence formulae. Detailed discussion of the meaning, validity and techniques of this test may be found in [15]. The test is carried out for system (2) and for the Calapso equation (3). Both the GMC system and the Calapso equation pass the test. For the system (2) we also find the Bäcklund transformation.

Equations (2) are cast into a polynomial form by substitution

\[ k_1 = K \exp(-\vartheta), \quad k_2 = M \exp(-\vartheta) \quad (7) \]

The Laurent expansion of \( \vartheta \) is supplemented by a logarithmic term due to potential character of this variable [15]. Variables \( K \) and \( M \) begin their series with \( \Phi^{-1} \). The resonances (sometimes called ‘indices’ [17]) arise at terms of number \( r = 0 \) and \( r = 2 \) in the Laurent expansions of \( \vartheta, K, M \) (the logarithmic term is not numbered). The resonance at \( r = 2 \) is double. All of them are compatible.

Truncation of these Laurent series on terms of order \( \Phi^0 \) yields a Bäcklund transformation. The transformation between \( \vartheta, K, M \) and \( \vartheta_0, K_1, M_1 \) reads
\[ \vartheta = \pm \ln(\Phi) + \vartheta_0, \quad K = iS/\Phi + K_1, \quad M = \mp iS/\Phi + M_1, \quad (8a) \]
\[
\Phi_{uv} \vartheta_{0;u} + \Phi_{,u} \vartheta_{0;v} \pm \Phi_{;uv} = 0, \quad (8b) 
\]
\[
\pm (i\Phi_{,u} M_1 + S \vartheta_{0;u}) + S_{,u} = 0, \quad (8c) 
\]
\[
i\Phi_{,v} K_1 \mp S \vartheta_{0;v} - S_{,v} = 0, \quad \text{where} \quad (8d) 
\]
\[
S = (\Phi_{,u}^2 + \Phi_{,v}^2)^{1/2} \quad (8e)
\]

The first three of these equations (8a) are truncated expansions of \( \vartheta, K, M \) (the indices in \( \vartheta_0, K_1, M_1 \) correspond to the numbering of terms in the Laurent series). Condition (8b) may easily be recognized as vanishing of the next term \( \vartheta_1 \) in the Laurent series of the potential \( \vartheta \) while the last two conditions, (8c) and (8d) are recurrence relations for \( M_1 \) and \( K_1 \), respectively, when \( \vartheta_1 \) vanishes.

Compatibility conditions for this overdetermined system are indeed equations (2). Namely, (2b) and (2c) (substituted according to (7)) ensure compatibility \( S_{uv} = S_{vu} \) of (8c) and (8d). We were also able to obtain equation (2a) as a fairly complicated combination of compatibility conditions for (8) (useful hints for construction of that combination are provided by the fact that second order coefficients in the Laurent expansions of \( \vartheta, K, M \) should vanish).

The Backlund transformation (8) is different from the transformation (4), (6); when applied to a solution which is real for real \( (u, v) \), it takes at least one of the dependent variables out of the real axis.

The Calapso equation is given a polynomial form, by multiplication of both hand sides of (3) by \( w^3 \). The Laurent series for \( w \) begins with a term of order \( \Phi^{-1} \). Compatibility conditions arise at \( r = 2, r = 3 \) and \( r = 4 \) (one condition per resonance). The check for compatibility is a bit cumbersome but straightforward. The conditions are satisfied at all the resonances.

5 Further developments

It is not difficult to notice that the invariant quadratic form (5) is of the signature \((+++-+-)\). This enables one to rewrite the Darboux-Bianchi linear problem as an \( \text{so}(4,1) \)-linear problem (here we use the standard terminology...
of the soliton theory). Namely, assuming \( m > 0 \) we perform the following transformation

\[
\tilde{\varphi} = \sqrt{\frac{m}{2}} (\sigma - \varphi) , \quad \text{and} \quad \tilde{\sigma} = \sqrt{\frac{m}{2}} (\sigma + \varphi) ,
\]

and then the Darboux-Bianchi linear problem (4) can be rewritten as follows

\[
\begin{align*}
\psi, u &= \left( -\vartheta \, f_{12} - k_2 e^\vartheta f_{13} + \zeta \sinh \vartheta \, f_{14} + \zeta \cosh \vartheta \, f_{15} \right) \psi , \\
\psi, v &= \left( \vartheta \, f_{12} - k_1 e^\vartheta f_{23} + \zeta \cosh \vartheta \, f_{24} + \zeta \sinh \vartheta \, f_{25} \right) \psi ,
\end{align*}
\]

(10)

where we put \( \psi := (\lambda, \mu, \omega, \tilde{\varphi}, \tilde{\sigma})^T \) and \( \zeta := \sqrt{2m} \), the latter as a “spectral parameter”, and, finally the matrices \( f_{ij} \) \((1 \leq i < j \leq 5)\) constitute the standard basis of the Lie algebra \( \text{so}(4,1) \):

\[
\begin{align*}
(f_{ij})_{\alpha\beta} &= \delta_{i\alpha} \delta_{j\beta} - \delta_{i\beta} \delta_{j\alpha} \quad \text{(for } i < 5, \ j < 5\), \\
(f_{i5})_{\alpha\beta} &= \delta_{i\alpha} \delta_{5\beta} + \delta_{i\beta} \delta_{5\alpha} ,
\end{align*}
\]

(11)

where \( 1 \leq \alpha, \beta \leq 4 \) and \( \delta_{jk} \) is Kronecker’s delta.

Certainly, the integrability conditions for (10) are still the same, i.e. they are given by the system (2).

There are some obvious disadvantages of the linear problem (10): too many zero-entries in the matrices of the problem and the large dimension of the matrices. In this context a natural question arises: what is a minimal Lie algebra containing both matrices of the linear problem (10) for an arbitrary choice of \( \vartheta, k_1, k_2 \) and \( \zeta \in \mathbb{R} \)? This problem turned out to be non-trivial. It took a few months to find the answer [18].

The obtained minimal Lie algebra is the Lie algebra of the rigid motions in \( \mathbb{E}^3 \) (semidirect sum of the Lie algebra of rotations and translations). It was a discouraging result: usually soliton systems are related to semi-simple Lie algebras.

The only way out is to make use of the well known isomorphism between \( \text{so}(4,1) \) and \( \text{sp}(1,1) \) [19]. For instance, this isomorphism can be given by

\[
\text{so}(4,1) \ni f_{jk} \quad \mapsto \quad \frac{1}{2} e_{jk} := \frac{1}{2} e_j e_k \in \text{sp}(1,1) .
\]

(12)
where complex $4 \times 4$ matrices $e_j$ ($j = 1, \ldots, 5$) are defined as follows

$$e_1 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}.$$  \hfill (13)

and $\sigma_k$ ($k = 1, 2, 3$) are standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (14)

One can check in a straightforward way the following properties

$$e_k e_j = -e_j e_k \quad (k \neq j),$$

$$e_1^2 = e_2^2 = e_3^2 = e_4^2 = -e_5^2 = I,$$

$$ie_1 e_2 e_3 e_4 e_5 = I,$$  \hfill (15)

(I is an identity matrix) which mean that $i e_1, \ldots, i e_5$ generate an algebra isomorphic to the subalgebra of even elements of the Clifford algebra $\mathcal{C}(1, 4)$.

The isomorphism (12) enables one to rewrite the $\mathfrak{so}(4, 1)$-linear problem (10) as the following $\mathfrak{sp}(1, 1)$-linear problem

$$\Psi_{,u} = \frac{1}{2} e_1 \left(-\vartheta_{,v} e_2 - k_2 e_1 e_3 + \zeta \sinh \vartheta \ e_4 + \zeta \cosh \vartheta \ e_5\right) \Psi,$$  \hfill (16a)

$$\Psi_{,v} = \frac{1}{2} e_2 \left(-\vartheta_{,u} e_1 - k_1 e_1 e_3 + \zeta \cosh \vartheta e_4 + \zeta \sinh \vartheta e_5\right) \Psi,$$  \hfill (16b)

where $\Psi = \Psi(u, v; \zeta)$ is a non-degenerate complex $4 \times 4$ matrix.

We conclude this section mentioning that in 1992 we asked our colleagues (Ruud Martini’s group) of the Math. Dep. of the University of Twente (The Netherlands) to apply their original technique (based on symmetries) to give an independent proof of the integrability of the underlying non-linear system (3). Indeed, in 1993 Theo van Bemmelen [20] found the so-called recursion operator for the system (2) providing us with yet another proof of the soliton nature of the system.
6 Isothermic surfaces as soliton surfaces

The integrability of the system (2) which is an implicit description of the isothermic surfaces in $E^3$ means that the class of all (local) isothermic surfaces is yet another example of integrable geometry. This subject can be studied within the approach of soliton surfaces [7].

We recall that the fundamental forms (e.g. those of isothermic surfaces in $E^3$) define a surface in $E^3$ uniquely (modulo rigid motion in $E^3$). As a rule, it is very difficult to recover the explicit expression for the position vector to the surface from the knowledge of its fundamental forms.

Fortunately, when one deals with the integrable geometry of surfaces (submanifolds), the problem of reconstruction of a surface (submanifold) can be simplified greatly. Namely, it is the existence of the associated $\zeta$-dependent linear problem (e.g. (10) or (16) for isothermic surfaces in $E^3$) for the “wave function” $\Psi$. One can prove [7, 21] that the formula

$$R = \Psi^{-1}\zeta |_{\zeta=\zeta_0}$$

(17)

defines a class of surfaces (submanifolds) immersed into the associated Lie algebra of the linear problem with exactly the same underlying nonlinear system as the one of the initial integrable geometry. In many cases ([1], [2] and references quoted therein) the formula (17) reconstructs our initial integrable geometry.

It is interesting that in the case of the integrable geometry of isothermic surfaces in $E^3$ the formula (17) requires some modification. Indeed, one can show that in this case the formula (17) ($\zeta_0 = 0$) defines class of surfaces immersed in a 6-dim. linear subspace of the Lie algebra $\mathfrak{sp}(1, 1)$ spanned by $e_k e_4$, $e_k e_5$ ($k = 1, 2, 3$).

The main results of the paper reads (for proof see [23]):

Given a solution $(\vartheta, k_1, k_2)$ to the system (2), the corresponding isothermic surface may be recovered as follows.

(a). Insert the solution $(\vartheta, k_1, k_2)$ into matrices of the linear problem (16).

(b). Compute the corresponding wave function $\Psi = \Psi(u, v; \zeta)$.

(c). Compute

$$r = P\Psi^{-1}\zeta |_{\zeta=0}$$

(18)
where $P$ is the constant projector given by $P := \frac{1}{2}(1 - e_{45})$.

(d). Decompose $r$ in the basis $f_k := \frac{1}{4}e_k(e_4 + e_5), \ k = 1, 2, 3$:

$$r = Xf_1 + Yf_2 + Zf_3.$$  \hfill (19)

The map $\mathbb{R}^2 \ni (u,v) \mapsto (X,Y,Z) \in \mathbb{R}^3$ describes explicitly the surface we look for.

Performing in the step (c) the projection $I - P$ instead of $P$ we obtain the so called dual surface, or Christoffel transform of $r$. This surface is isothermic as well and its fundamental forms are parameterized by $\vartheta' = -\vartheta, \ k_1' = e^{2\vartheta}k_1, \ k_2' = -e^{2\vartheta}k_2$ (compare [1]).

In the above algorithm the step (b) is certainly the most difficult. However, if the triplet $(\vartheta, k_1, k_2)$ is the $N$-soliton solution, the formalism of the soliton theory (e.g. [22]) enables one to compute the corresponding $\Psi$ explicitly (see [23, 24]). In this way one arrives at the expression for 1-soliton isothermic surface

$$r_1 = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} + \frac{2}{\cosh v \cosh \gamma - \cos u} \begin{pmatrix} \sin u \\ -\sinh v \cosh \gamma \\ -\sinh \gamma \end{pmatrix}$$ \hfill (20)

where $\gamma$ is a constant parameter. A sample of such surfaces is shown on Fig. 1.

We conclude with the statement that within the approach of soliton surfaces one can reconstruct and generalize all the classical findings by G.Darboux and L.Bianchi [1]. In particular one can derive (by the standard dressing method) the classical Darboux-Bäcklund transformation [8]. The detailed discussion of these results is given in [23].

Acknowledgements

Special thanks are due to our colleagues and friends: Peter Gragert, Reinhard Meinel and Theo van Bemmelen, for their interests, comments and useful hints.
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[18] In the Spring of 1991 we asked this question Peter Gragert (Enschede University, The Netherlands) on his visit to Warsaw. Soon he was able to show that the dimension of such a Lie algebra is 6 but he was unable to find it explicitly. Then, in the December 1991, one of us (A.S.) on his visit to Jena University (Germany) repeated the same question to Reinhard Meinel who succeeded to construct the minimal Lie algebra explicitly and, finally, Adam Doliwa (Warsaw University) identified it as a semidirect sum of so(3) and t³.

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**Figure caption**

Fig. 1. One-soliton isothermic surfaces (20) for three values of the parameter $\gamma$. In the limit of high $\gamma$ the surface becomes a self-intersecting pipe whose cross section has a shape of the Wadati-Konno-Ichikawa loop soliton [25].
