DEPENDENCE OF HILBERT COEFFICIENTS

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ABSTRACT. Let \( M \) be a finitely generated module of dimension \( d \) and depth \( t \) over a Noetherian local ring \((A, \mathfrak{m})\) and \( I \) an \( \mathfrak{m} \)-primary ideal. In the main result it is shown that the last \( t \) Hilbert coefficients \( e_{d-t+1}(I, M), \ldots, e_d(I, M) \) are bounded below and above in terms of the first \( d-t+1 \) Hilbert coefficients \( e_0(I, M), \ldots, e_{d-t}(I, M) \) and \( d \).

INTRODUCTION

Let \( M \) be a finitely generated module of dimension \( d \) over a Noetherian local ring \((A, \mathfrak{m})\) and \( I \) an \( \mathfrak{m} \)-primary ideal. The Hilbert-Samuel function \( H_{I,M}(n) = \ell(M/I^{n+1}M) \) agrees with the Hilbert-Samuel polynomial \( P_{I,M}(n) \) for \( n \gg 0 \) and we may write

\[
P_{I,M}(n) = e_0(I, M)\left(\frac{n+d}{d}\right) - e_1(I, M)\left(\frac{n+d-1}{d-1}\right) + \cdots + (-1)^d e_d(I, M).
\]

The numbers \( e_0(I, M), e_1(I, M), \ldots, e_d(I, M) \) are called the Hilbert coefficients of \( M \) with respect to \( I \).

The Hilbert-Samuel function and the Hilbert-Samuel polynomial give a lot of information on \( M \). Therefore, it is of interest to study properties of Hilbert coefficients. Assume that \( A \) is a Cohen-Macaulay ring and \( M \) is a Cohen-Macaulay \( A \)-module. Then Northcott [14] and Narita [13] showed that \( e_1(I, A) \geq 0 \) and \( e_2(I, A) \geq 0 \), respectively. Note that already \( e_3(I, A) \) maybe negative. Later, Rhodes

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[15] showed that the above results also hold for good \( I \)-filtrations of submodules of \( M \). Moreover, Kirby and Mehran [10] were able to show that \( e_1(I, M) \leq \binom{e_0(I,M)}{2} \) and \( e_2(I, M) \leq \binom{e_1(I,M)}{2} \). Subsequently these results were improved by several authors. How about the other coefficients? In 1997, Srinivas and Trivedi [19] and Trivedi [20] obtained a surprising result, stating that all \( |e_i(I, A)| \), \( i \geq 1 \), are bounded by a function depending only on \( e_0(I, A) \) and \( d \).

What happens for non-Cohen-Macaulay modules? Inspired by the previously mentioned result of Srinivas and Trivedi and of Trivedi [21], Rossi-Trung-Valla [17] showed that all \( |e_i(I, A)| \), are bounded by functions depending on the so-called extended degree \( \operatorname{Deg}(I, A) \) and \( d \). These results were extended to modules in [12] and [7]. It is also worth to know, that when \( I \) is a parameter ideal in a generalized Cohen-Macaulay ring, there is a uniform bound for all \( |e_i(I, A)| \), \( i \geq 1 \), which does not depend on the choice of \( I \), see [8]. However from all these results one cannot deduce further relations between Hilbert coefficients.

Using a bound on the Castelnuovo-Mumford regularity in terms of Hilbert coefficients given in [20, Theorem 2] one can immediately see that \((-1)^{i-1}e_i(I, A)\) is bounded above by a (complicated and implicit) function depending only on \( e_0(I, A), \ldots, e_{i-1}(I, A) \) and \( i \), for all \( i \geq 1 \). An explicit bound will be given in Theorem 2.1. However, even in the case \( d = 1 \) an easy example shows that \( |e_1(I, A)| \) is in general not bounded in terms of \( e_0(I, A) \). So, it is natural to ask: how many Hilbert coefficients are enough to be taken such that they completely bound the absolute values of all other ones? The main result of this paper is to show that the first \( d - t + 1 \) Hilbert coefficients have this property, where \( t = \operatorname{depth} M \) (see Theorem 2.4 and Corollary 2.5). As a consequence, we can show that there is only a finite number of Hilbert-Samuel functions if \( e_0(I, M), e_1(I, M), \ldots, e_{d-t}(I, M) \) and \( d \) are fixed (see Theorem 2.6).

In fact, we will deal with a more general situation, namely with good \( I \)-filtrations \( \mathcal{M} \). In this case our bounds also involve the so-called reduction number \( r(\mathcal{M}) \). Our approach is somewhat similar to that of [19, 20] and [17], in the sense that we use the Castelnuovo-Mumford regularity \( \operatorname{reg}(G(\mathcal{M})) \) of the associated module \( G(\mathcal{M}) \) of \( \mathcal{M} \) to bound the Hilbert coefficients (see Proposition 2.3). Then one has to bound \( \operatorname{reg}(G(\mathcal{M})) \) in terms of the first \( d - t + 1 \) Hilbert coefficients. In order to do that, in Section 1, using [20, Theorem 2] we first give a bound for \( \operatorname{reg}(G(\mathcal{M})) \) in terms of all Hilbert coefficients (see Theorem 1.8). Then, combining some idea developed in the proof of [17, Theorem 3.3], and refined in [11, Theorem 4.4] and [7, Theorem 1.5], with bounding the length of certain Artinian modules (see Lemma 1.11), we show in the same section that already the first \( d - t + 1 \) Hilbert coefficients are enough to bound \( \operatorname{reg}(G(\mathcal{M})) \) (see Theorem 1.12). The relations among the Hilbert coefficients are given in the last section (Theorem 2.1 and Theorem 2.4). Finally, we would like to remark, that bounds established in this paper are huge functions. Therefore instead of seeking better bounds we are looking for more compact formulas. In any case the main meaning of the bounds is not their values, but the fact that they exist at all, hence that the last \( t \) Hilbert coefficients are bounded by the first \( d - t + 1 \) ones.
1. Castelnuovo-Mumford Regularity and Hilbert Coefficients

Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian standard graded ring over a local Artinian ring $(R_0, m_0)$ such that $R_0/m_0$ is an infinite field. Let $E$ be a finitely generated graded $R$-module of dimension $d$. For $0 \leq i \leq d$, put

$$a_i(E) = \sup \{ n \mid H^i_{R_+}(E)_n \neq 0 \},$$

where $R_+ = \oplus_{n > 0} R_n$. The Castelnuovo-Mumford regularity of $E$ is defined by

$$\text{reg}(E) = \max \{ a_i(E) + i \mid 0 \leq i \leq d \},$$

and the Castelnuovo-Mumford regularity of $E$ at and above level 1 is defined by

$$\text{reg}^1(E) = \max \{ a_i(E) + i \mid 1 \leq i \leq d \}.$$

We denote the Hilbert function $\ell_{R_0}(E_t)$ and the Hilbert polynomial of $E$ by $h_E(t)$ and $p_E(t)$, respectively. Writing $p_E(t)$ in the form:

$$p_E(t) = \sum_{i=0}^{d-1} (-1)^i e_i(E) \binom{t + d - 1 - i}{d - 1 - i},$$

we call the numbers $e_i(M)$ the Hilbert coefficients of $E$.

There are different ways to bound $\text{reg}(E)$. In this section we are interested in bounding this invariant in terms of the Hilbert coefficients. Let $\Delta(E)$ denote the maximal generating degree of $E$. Easy examples show that one cannot bound $\text{reg}(E)$ in terms of $\Delta(E), e_0(E), \ldots, e_{d-1}(E)$. However these invariants bound $\text{reg}^1(E)$, as shown in [3, Theorem 17.2.7] and [20, Theorem 2]. Below we recall the bound by Trivedi which does not depend on the number of generators of $E$ as the one in [3]. Let

$$\Delta'(E) = \max \{ \Delta(E), 0 \}.$$

We inductively define a sequence of integers as follows: $m_1 = e_0(E) + \Delta'(E)$, and for all $i \geq 2$,

$$m_i = m_{i-1} + \sum_{k=0}^{i-1} (-1)^k e_k(E) \binom{m_{i-1} + i - 2 - k}{i - 1 - k}.$$  \hspace{1cm} (1)

Then

**Lemma 1.1.** ([20, Theorem 2]) Assume that $d \geq 1$. Then $\text{reg}^1(E) \leq m_d - 1$.

The above result was originally formulated in [20] for $G_1(M)$, which corresponds to the case $E$ being generated by elements of degree zero. But this assumption is not essential. The proof was eventually given in [19, Lemma 4]. For a more algebraic proof one can use [11, Theorem 2.7].

From the above bound we can derive an explicit bound for $\text{reg}^1(E)$ in terms of $e_i(E)$ and $\Delta(E)$. However this bound is weaker.

**Lemma 1.2.** Let $E$ be a finitely generated graded $R$-module of dimension $d \geq 1$. Put

$$\xi_{d-1}(E) = \max \{ e_0(E), |e_1(E)|, \ldots, |e_{d-1}(E)| \}.$$

Then we have

$$\text{reg}^1(E) \leq (\xi_{d-1}(E) + \Delta'(E) + 1)^d - 2.$$  

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Proof. For short, we put \( e_i := e_i(E), \xi := \xi_{d-1}(E) \) and \( \Delta' := \Delta'(E) \). By Lemma 1.1 it suffices to show that \( m_d \leq (\xi + \Delta' + 1)^{d!} - 1 \). This is a purely arithmetic issue, which is trivial for \( d = 1 \). By the induction hypothesis we may assume
\[
m_{d-1} \leq (\xi + \Delta' + 1)^{(d-1)!} - 1 =: \alpha.
\]
Note that
\[
\sum_{i=0}^{d-1} (-1)^i e_i \left( \frac{\alpha + d - 2 - i}{d - 1 - i} \right) \leq \xi \sum_{i=0}^{d-1} \left( \frac{\alpha + d - 2 - i}{d - 1 - i} \right) = \xi \left( \frac{\alpha + d}{d - 1} \right).
\]
Hence, by the recurrence formula (1) applied to \( i = d \), we get
\[
m_d \leq \alpha + \xi \left( \frac{\alpha + d}{d - 1} \right).
\]
If \( d = 2 \), then \( \alpha = \xi + \Delta' \), and
\[
m_2 \leq \xi + \Delta' + (\xi + \Delta' + 1) = (\xi + \Delta' + 1)(\xi + 1) - 1 \leq (\xi + \Delta' + 1)^2 - 1.
\]
Assume \( d \geq 3 \). Observing that \( \left( \frac{\alpha + d - 1}{d - 1} \right) \leq (\alpha + 1)^{d-1} \) for all \( \alpha \geq 1 \) and \( \alpha \geq (\xi + 1)^2 > \xi + 1 \), we obtain
\[
m_d \leq \alpha + \xi (\alpha + 1)^{d-1} \leq (1 + \xi)(\alpha + 1)^{d-1} - 1 \leq (\alpha + 1)^d - 1 = (\xi + \Delta' + 1)^{d!} - 1.
\]
\( \square \)

We need some more notations and definitions. Let \((A, \mathfrak{m})\) be a Noetherian local ring with an infinite residue field \( K := A/\mathfrak{m} \) and \( M \) a finitely generated \( A \)-module. Given a proper ideal \( I \), a chain of submodules
\[
\mathcal{M} : M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots
\]
is called an \( I \)-filtration of \( M \) if \( IM_i \subseteq M_{i+1} \) for all \( i \), and a good \( I \)-filtration if \( IM_i = M_{i+1} \) for all sufficiently large \( i \). A module \( M \) with a good \( I \)-filtration is called an \( I \)-well filtered module (see [2, III 2.1]). If \( N \) is a submodule of an \( I \)-well filtered module \( M \), then the sequence \( \{M_n + N/N\} \) is a good \( I \)-filtration of \( M/N \) and will be denoted by \( \mathcal{M}/N \).

In this paper we always assume that \( I \) is an \( \mathfrak{m} \)-primary ideal and \( \mathcal{M} \) is a good \( I \)-filtration. The \textit{associated graded module} to the filtration \( \mathcal{M} \) is defined by
\[
G(\mathcal{M}) = \bigoplus_{n \geq 0} M_n/M_{n+1}.
\]
We also say that \( G(\mathcal{M}) \) is the associated module of the filtered module \( M \). This is a finitely generated graded module over the standard graded ring \( G := G_I(A) := \oplus_{n \geq 0} I^n/I^{n+1} \) (see [2, Proposition III 3.3]). In particular, when \( \mathcal{M} \) is the \( I \)-adic filtration \( \{I^n\} \), \( G(\mathcal{M}) \) is just the usual associated graded module \( G_I(M) \).

We call \( H_{\mathcal{M}}(n) = \ell(M/M_{n+1}) \) the Hilbert-Samuel function of \( M \) w.r.t \( \mathcal{M} \). This function agrees with a polynomial - called the Hilbert-Samuel polynomial and denoted by \( P_{\mathcal{M}}(n) \) - for \( n \gg 0 \). If we write
\[
P_{\mathcal{M}}(t) = \sum_{i=0}^{d} (-1)^i e_i(\mathcal{M}) \left( \frac{t + d - i}{d - i} \right),
\]
then the integers \( e_i(\mathcal{M}) \) are called the \textit{Hilbert coefficients} of \( \mathcal{M} \) (see [16, Section 1]). When \( \mathcal{M} = \{I^n\} \), \( H_{\mathcal{M}}(n) \) and \( P_{\mathcal{M}}(n) \) are usually denoted by \( H_{I,M}(n) \) and \( P_{I,M}(n) \), respectively, and \( e_i(\mathcal{M}) = e_i(I, M) \). Note that \( e_i(\mathcal{M}) = e_i(G(\mathcal{M})) \) for \( 0 \leq i \leq d - 1 \).
Now we want to derive a bound for \( \text{reg}(G(\mathcal{M})) \) in terms of Hilbert coefficients. Using Lemma 1.2 we can already bound \( \text{reg}^1(G(\mathcal{M})) \) in terms of \( e_0(\mathcal{M}), \ldots, e_{d-1}(\mathcal{M}) \). If \( \text{depth}(M) > 0 \), by [7, Lemma 1.8], \( \text{reg}(G(\mathcal{M})) = \text{reg}^1(G(\mathcal{M})) \), and so it is bounded in terms of \( e_i(\mathcal{M}), i < d \). The following example shows that this is not true if \( \text{depth}(M) = 0 \).

**Example 1.3.** Let \( A = K[[x, y]]/(x^2, xy^s), s \geq 1 \). Then \( G_m(A) \cong k[x, y]/(x^2, xy^s) \). Since \( (x^2, xy^s) \) is a so-called stable ideal, \( \text{reg}(G_m(A)) = s \) can be arbitrarily large, while \( e_0(A) = 1 \).

Our first goal is to show that also using \( e_d(\mathcal{M}) \) we can bound \( \text{reg}(G(\mathcal{M})) \). For that we need some more preparations. We denote \( M/H^0_m(M) \) by \( \mathcal{M} \) and the filtration \( M/H^0_m(M) \) of \( \mathcal{M} \) by \( \mathcal{M} \) and let \( h^0(M) = \ell(H^0_m(M)) \). Then

**Lemma 1.4.** ([16, Proposition 2.3]) For all \( n \) we have
\[
h^0(M) = P_{\mathcal{M}}(n) - P_{\mathcal{M}}(n) = (-1)^d[e_d(\mathcal{M}) - e_d(\mathcal{M})].
\]

Applying the Grothendieck-Serre formula to \( G(\mathcal{M}) \) and the arguments in the proof of [11, Lemma 3.4], we get

**Lemma 1.5.** \( P_{\mathcal{M}}(n) = H_{\mathcal{M}}(n) \) for all \( n \geq \text{reg}(G(\mathcal{M})) \).

**Lemma 1.6.** \( h^0(M) \leq P_{\mathcal{M}}(n) \) for all \( n \geq \text{reg}(G(\mathcal{M})) \).

**Proof.** By Lemma 1.5, \( P_{\mathcal{M}}(n) = H_{\mathcal{M}}(n) \) for all \( n \geq \text{reg}(G(\mathcal{M})) \). Hence, by Lemma 1.4, \( h^0(M) = P_{\mathcal{M}}(n) - P_{\mathcal{M}}(n) = P_{\mathcal{M}}(n) - H_{\mathcal{M}}(n) \leq P_{\mathcal{M}}(n) \) for all \( n \geq \text{reg}(G(\mathcal{M})) \). \( \square \)

We call
\[
r(\mathcal{M}) = \min\{r \geq 0 \mid M_{n+1} = IM_n \text{ for all } n \geq r\}
\]
the reduction number of \( \mathcal{M} \) (w.r.t. \( I \)). Then we have

**Lemma 1.7.** ([7, Lemma 1.9]) \( \text{reg}(G(\mathcal{M})) \leq \max\{\text{reg}(G(\mathcal{M})), r(\mathcal{M})\} + h^0(M) \).

In the sequel we will often use the following notation:
\[
\xi_s(\mathcal{M}) = \max\{e_0(\mathcal{M}), |e_1(\mathcal{M})|, \ldots, |e_s(\mathcal{M})|\}
\]
where \( 0 \leq s \leq d \). Now we can state and prove the first bound on \( \text{reg}(G(\mathcal{M})) \) in terms of Hilbert coefficients.

**Theorem 1.8.** Let \( \mathcal{M} \) be a good I-filtration of \( M \) of dimension \( d \geq 1 \). Then
\[
\text{reg}(G(\mathcal{M})) < (\xi_d(\mathcal{M}) + r(\mathcal{M}) + 1)^{d+1} - 2.
\]

**Proof.** Let \( r = r(\mathcal{M}), e_i = e_i(\mathcal{M}) \) and \( \xi := \xi_d(\mathcal{M}) \). By [7, Lemma 1.8] we have \( \text{reg}(G(\mathcal{M})) = \text{reg}^1(G(\mathcal{M})) \). By Lemma 1.7,
\[
\text{reg}(G(\mathcal{M})) \leq \max\{\text{reg}^1(G(\mathcal{M})), r\} + h^0(M).
\]

Set \( \alpha := (\xi + r + 1)^d - 2 \geq r \). By Lemma 1.4, \( e_i(G(\mathcal{M})) = e_i(\mathcal{M}) = e_i \) for all \( i \leq d - 1 \). As mentioned above, \( G(\mathcal{M}) \) is generated by elements of degrees at most
\(r(\mathbb{M}) \geq 0\). Therefore, by Lemma 1.2, \(\text{reg}^1(\mathbb{G}(\mathbb{M})) \leq \alpha\). Using (2) and Lemma 1.6 we then get

\[
\text{reg}(\mathbb{G}(\mathbb{M})) \leq \alpha + P_{\mathbb{M}}(\alpha) \\
\leq \alpha + \xi \sum_{i=0}^{d} \left( \frac{\alpha + d - i}{d - i} \right) \\
= \alpha + \xi \left( \frac{a + d + 1}{d} \right) \\
= (\xi + r + 1)^d - 2 + \xi \left( \frac{(\xi + r + 1)^{d-1} + d}{d} \right) \\
\leq (\xi + r + 1)^d - 2 + \xi (\xi + r + 1)^{dd!} \\
< (\xi + r + 1)^{dd!+1} - 2.
\]

\(\square\)

The above bound is a huge number when \(d \gg 1\). In the case of \(I\)-adic filtrations of an one-dimensional module there is a sharp bound given in a recent paper [6].

Our next goal is to show that in order to bound \(\text{reg}(\mathbb{G}(\mathbb{M}))\) one can use \(\xi_{d-t}(\mathbb{M})\), where \(t = \text{depth} \, M\). For this we need some more auxiliary results.

An element \(x \in I\) is called \(\mathbb{M}\)-superficial element for \(I\) if there exists a non-negative integer \(c\) such that \((M_{n+1} :_M x) \cap M_c = M_n\) for every \(n \geq c\) and we say that a sequence of elements \(x_1, ..., x_r\) is an \(\mathbb{M}\)-superficial sequence for \(I\) if, for \(i = 1, 2, ..., r, x_i\) is an \(\mathbb{M}/(x_1, ..., x_{i-1})M\)-superficial sequence for \(I\) (see [16, Section 1.2]). Note that \(x \in I \setminus I^2\) is an \(\mathbb{M}\)-superficial element for \(I\) if and only if its initial form \(x^* \in G\) is a filter-regular element on \(G(\mathbb{M})\), i.e. \([0 :_{G(\mathbb{M})} x^*]_n = 0\) for all \(n \gg 0\).

**Lemma 1.9.** Let \(x\) be an \(\mathbb{M}\)-superficial element for \(I\). Then

\[
\text{reg}(G(\mathbb{M}/xM)) \leq \text{reg}(G(\mathbb{M})).
\]

**Proof.** We have the following exact sequence:

\[
0 \longrightarrow \bigoplus_{n \geq 0} xM \cap M_n \longrightarrow G(\mathbb{M})/x^*G(\mathbb{M}) \longrightarrow G(\mathbb{M}/xM) \longrightarrow 0.
\]

By [7, Lemma 1.3(ii)] (see also [22, Lemma 4.4]), \(xM \cap M_n = xM_{n-1}\) for \(n \gg 0\). Hence

\[
\text{reg}(G(\mathbb{M}/xM)) \leq \text{reg}(G(\mathbb{M})/x^*G(\mathbb{M})) \leq \text{reg}(G(\mathbb{M})).
\]

\(\square\)

**Lemma 1.10.** Let \(x_1, x_2, ..., x_d\) be an \(\mathbb{M}\)-superficial sequence for \(I\). Set \(M_i = M/(x_1, ..., x_i)M\) and \(\mathbb{M}_i = \mathbb{M}/(x_1, ..., x_i)\mathbb{M}\), where \(M_0 = M\) and \(\mathbb{M}_0 = \mathbb{M}\). Then, for all \(0 \leq i \leq d - 1\), we have

\[
h^0(M_i) \leq (i + 1)\xi_d(\mathbb{M})(\text{reg}(G(\mathbb{M})) + 2)^d \leq d\xi_d(\mathbb{M})(\text{reg}(G(\mathbb{M})) + 2)^d.
\]

**Proof.** Set \(a := \text{reg}(G(\mathbb{M}))\) and \(\xi := \xi_d(\mathbb{M})\). We proceed by induction on \(i\). Note by Lemma 1.9 that \(\text{reg}(G(\mathbb{M}_i)) \leq \text{reg}(G(\mathbb{M})) \leq \text{reg}(G(\mathbb{M})) = a\).

For \(i = 0\), by Lemma 1.6, we have

\[
h^0(M_0) = h^0(M) \leq P_{\mathbb{M}}(a) \leq \xi \sum_{j=0}^{d} \left( \frac{d + a - j}{a - j} \right) = \xi \left( \frac{a + d + 1}{d} \right) \leq \xi (a + 2)^d.
\]
For $0 < i \leq d - 1$, by [16, Proposition 1.2], we have $e_j(M_i) = e_j(M_{i-1})$ for all $0 \leq j \leq d - i - 1$ and
\[|e_{d-i}(M_i)| = |e_{d-i}(M_{i-1}) + (-1)^{d-i} \ell(0 : _{M_{i-1}} x_i)| \leq |e_{d-i}(M_{i-1})| + h^0(M_{i-1}) \leq \xi_{d-i+1}(M_{i-1}) + h^0(M_{i-1}) \leq \xi + h^0(M_{i-1}).\]

Hence, by Lemma 1.6 and the induction hypothesis we get
\[h^0(M_i) \leq P_{M_i}(a) \leq \xi \sum_{j=0}^{d-i-1} \binom{d-i+a-j}{d-i-j} + |e_{d-i}(M_i)| \leq \xi \binom{a+d+i+1}{d-i} - \xi + |e_{d-i}(M_i)| \leq \xi(a+2)^{d-i} - \xi + \xi + h^0(M_{i-1}) \leq \xi(a+2)^{d-i} + \xi \xi(a+2)^d \leq (i+1)\xi(a+2)^d.\]

\[\square\]

**Lemma 1.11.** Set $B = \ell(M/(x_1, x_2, \ldots, x_d)M)$, where $x_1, x_2, \ldots, x_d$ is an $\mathcal{M}$-superficial sequence for $I$. Then
\[B < (d+1)\xi_d(\mathcal{M})(\text{reg}(G(\mathcal{M})) + 2)^d.\]

**Proof.** Keep the notation in the proof of the previous lemma. Since $\dim(M_{d-1}) = 1$, $M_{d-1}$ is a generalized Cohen-Macaulay module. By [5, Lemma 1.5],
\[B - e_0(x_d; M_{d-1}) = \ell(M_{d-1}/x_dM_{d-1}) - e_0(x_d; M_{d-1}) \leq h^0(M_{d-1}).\]

Since $e_0(x_d; M_{d-1}) = e_0(x_1, \ldots, x_d, M) = e_0(\mathcal{M}) = e_0$, we get
\[B \leq e_0 + h^0(M_{d-1}) \leq \xi + h^0(M_{d-1}).\]

By Lemma 1.10, $h^0(M_{d-1}) \leq d\xi(a+2)^d$. From this estimate we immediately get $B < (d+1)\xi(a+2)^d$. \[\square\]

Finally we can state and prove the second bound on $\text{reg}(G(\mathcal{M}))$, which only uses the first $d - t + 1$ Hilbert coefficients.

**Theorem 1.12.** Let $\mathcal{M}$ be a good $I$-filtration of $M$ with $\dim(M) = d \geq 1$ and $\text{depth}(M) = t$. Then
\[\text{reg}(G(\mathcal{M})) \leq (\xi_{d-t}(\mathcal{M}) + r(\mathcal{M}) + 1)^{2(d-t+1)d} - 2.\]

**Proof.** For short we write $e_i = e_i(\mathcal{M})$, $\xi_i := \xi_i(\mathcal{M})$ and $r := r(\mathcal{M})$. We do induction on $t$. The case $t = 0$ follows from Theorem 1.8.

Assume that $t \geq 1$. In the case $t = d$, i.e. $M$ is a Cohen-Macaulay module, the statement follows from the following bounds given in [7, Theorem 1.5]:
\[\text{reg}(G(\mathcal{M})) \leq \begin{cases} e_0 + r - 1 & \text{if } d = 1, \\
 e_0 + r + 1)^{3(d-1)!-1} - d & \text{if } d > 1.\end{cases}\]

Let $t < d$, and so $d \geq 2$. The first part of the following arguments uses the idea of the proof of [17, Theorem 3.3] (see also [11, Theorem 4.4] and [7, Theorem 1.5]). Let $x = x_1, \ldots, x_d$ be an $\mathcal{M}$-superficial sequence for $I$. Let $N = M/xM$ and $\mathcal{N} = \mathcal{M}/xM$. Then $\dim N = d - 1$ and $\text{depth} N = t - 1$. By [16, Proposition 1.2], $e_i(\mathcal{N}) = e_i$ for
all $i \leq d - 1$. Hence $\xi_{d-t}(N) = \xi_{d-t}$. It is clear that $r(N) \leq r$. Let $m$ be an integer such that

$$m \geq \max\{r, \reg(G(N))\}.$$  

From the exact sequence (3) it follows that

$$\reg^1(G(M)/x^*G(M)) = \reg^1(G(N)) \leq m.$$  

Hence, by [11, Theorem 2.7],

$$\reg^1(G(M)) \leq m + p_{G(M)}(m).$$  

By [7, Lemma 1.6] and [7, Lemma 1.7(i)],

$$p_{G(M)}(m) \leq H_{I,N}(m) \leq \binom{m + d - 1}{d - 1} \ell(N/(x_2, \ldots, x_d)N) = B\binom{m + d - 1}{d - 1}.$$  

Since $\reg(G(M)) = \reg^1(G(M))$ (see [7, Lemma 1.8]),

$$\reg(G(M)) \leq m + B\binom{m + d - 1}{d - 1} \leq m + B(m + 1)^{d-1} < (B + 1)(m + 1)^{d-1}. \quad (5)$$  

Let $M_t := M/(x_1, \ldots, x_t)M$. Then $\dim(M_t) = d - t$. Again, by [16, Proposition 1.2], $e_t(M_t) = e_t$ for all $i \leq d - t$, which yields $\xi_{d-t}(M_t) = \xi_{d-t}$. Let $a_t := \reg(G(M_t))$. Applying Theorem 1.8 to $M_t$, we have

$$a_t \leq \omega^{(d-t)(d-t)!+1} - 2,$$

where $\omega = \xi_{d-t} + r + 1$. Note that $\omega^{d-1} \geq 2^{d-1} \geq d$. Since $t \geq 1$, applying Lemma 1.11 to $M_t$ we get

$$B = \ell(M_t/(x_{t+1}, \ldots, x_d)M_t) \leq (d - t + 1)\xi_{d-t}(a_t + 2)^{d-t} < d\omega^{(d-t)! + d-t} \leq \omega^{(d-t)! + 2d-t}. \quad (6)$$  

We distinguish two cases.

**Case 1:** $t = 1$. Then depth $N = 0$. By Theorem 1.8 we can take $m = \omega^{(d-1)(d-1)!+1} - 2$. By (5) and (6) we get

$$\reg(G(M)) \leq \omega^{(d-1)! + 2d-1} (\omega^{(d-1)(d-1)!+1} - 1)^d \leq \omega^{(d-1)! + (d-1)! + 3d-1} + 2 - 2 \leq \omega^{2d!} - 2.$$  

**Case 2:** $t > 1$. Then $d \geq 3$. By the induction hypothesis we can take $m = \omega^{2(d-t+1)(d-1)!} - 2$. Again, by (5) and (6) we obtain

$$\reg(G(M)) \leq \omega^{(d-t)! + 2d-t} (\omega^{2(d-t+1)(d-1)!} - 1)^{d-1} \leq \omega^{2d-t! + 2d-2 + 2d(d-t+1)} - 2.$$  

We have

$$2(d - t + 1)d! = 2(d - 1)(d - t + 1)(d - 1)! + 2(d - t + 1)(d - 1)!.$$
Since \( d > t \geq 2 \), the following hold

\[
2(d-t+1)(d-1)! \geq 2(d-1)(d-t+1)(d-t)! \geq 2(d-1)(d-t)! + 2(d-t)^2(d-t)! \geq 2(d-1) + (d-t)^2(d-t)!. 
\]

Hence \( \text{reg}(G(\mathcal{M})) \leq \omega^{2(d-t+1)d} - 2 \), as required. \( \square \)

**Remark 1.13.** Keep the notation of Lemma 1.10 and Lemma 1.11. Set

\[
B(M) = \ell(M/(x_1, \ldots, x_d)M) \quad \text{and} \quad \kappa(M) = \max\{h^0(M_j) \mid 0 \leq i \leq d-1\}.
\]

In the first version of this paper (see http://viasm.edu.vn/2012/05/preprints-2012, Preprint ViAsM12.25) we proved that

(i) \( \text{reg}(G(\mathcal{M})) \leq B(M) + \kappa(M) + r(\mathcal{M}) - 1 \) if \( d = 1 \),

(ii) \( \text{reg}(G(\mathcal{M})) \leq [B(M) + \kappa(M) + r(\mathcal{M}) + 1]^{3(d-1)i-1} - d \) if \( d \geq 2 \).

Note that \( B(M) = B(M_t) \) and \( \kappa(M) = \kappa(M_t) \), where \( t = \text{depth} M \). Using this result, Lemma 1.10, Lemma 1.11 and Theorem 1.8 we can get another bound for \( \text{reg}(G(\mathcal{M})) \) in terms of \( \xi_{d-t} \), which is smaller than the one of Theorem 1.12 if \( d-t \) is very small (compared with \( d \)). However, when \( d-t \) is big, the bound presented in Theorem 1.12 is better.

### 2. Relations between Hilbert coefficients

In this section we always assume that \( M \) is an \( A \)-module of positive dimension \( d \) and \( \mathcal{M} \) is a good \( I \)-filtration of \( M \), where \( I \) is an \( \mathfrak{m} \)-primary ideal. First we give an upper bound for \( (-1)^{i-1}e_i(\mathcal{M}) \) in terms of the preceding Hilbert coefficients. The first statement of the following theorem is implicitly contained in [16].

**Theorem 2.1.** (i) \( e_1(\mathcal{M}) \leq \binom{e_0(\mathcal{M})}{2} \).

(ii) Let \( \xi_{i-1} := \xi_{i-1}(\mathcal{M}) \). For \( i \geq 2 \) we have

\[
(-1)^{i-1}e_i(\mathcal{M}) \leq \xi_{i-1}\left(\binom{\xi_{i-1} + r + 1}{i} \right) < (\xi_{i-1} + r + 1)^{i+1}.
\]

**Proof.** We do induction on \( d \). Let \( d = 1 \). Then the inequality \( e_1(\mathcal{M}) \leq \binom{e_0(\mathcal{M})}{2} \) follows from [16, Proposition 2.8 and Lemma 2.3].

Assume that \( d \geq 2 \). First we prove the statement for \( i \leq d-1 \). Let \( \overline{\mathcal{M}} = \mathcal{M}/H^0_{\mathfrak{m}}(M) \). Since \( e_j(\mathcal{M}) = e_j(\overline{\mathcal{M}}) \) for all \( j \leq d-1 \), we may assume that \( \text{depth} M > 0 \). Let \( x \) be an \( \mathcal{M} \)-superficial element for \( I \). Then \( \dim(M/xM) = d-1 \) and by [16, Proposition 1.2], \( e_j(\mathcal{M}) = e_j(\mathcal{M}/xM) \) for all \( j \leq d-1 \). Hence, the inequalities follow from the induction hypothesis applied to \( \mathcal{M}/xM \).

Finally let \( i = d \). Since \( G(\overline{\mathcal{M}}) \) is generated by elements of degrees at most \( r(\mathcal{M}) \geq 0 \), by [7, Lemma 1.8] and Lemma 1.2 we have

\[
\text{reg}(G(\overline{\mathcal{M}})) = \text{reg}^1(G(\overline{\mathcal{M}})) \leq (\xi_{d-1} + r + 1)^d - 2 =: \alpha.
\]
By Lemma 1.7 and Lemma 1.6 we then get
\[
\text{reg}(G(M)) \leq \max\{\text{reg}^1(G(M)), r\} + h^0(M)
\]
\[
\leq \max\{\text{reg}^1(G(M)), r\} + P_M(\alpha)
\]
\[
\leq \alpha + \sum_{i=0}^{d-1} e_i(M)\left(\frac{a+d-i}{d-i}\right) + (-1)^d e_d(M)
\]
\[
\leq \xi_{d-1}[\alpha - 1 + \sum_{i=0}^{d} \left(\frac{a+d-i}{d-i}\right)] + (-1)^d e_d(M)
\]
\[
= \xi_{d-1}[\alpha - 1 + \left(\frac{a+d+1}{d}\right)] + (-1)^d e_d(M)
\]
\[
< \xi_{d-1}\left[\left(\frac{a+d+1}{d}\right) + \left(\frac{a+d+1}{d}\right)\right] + (-1)^d e_d(M)
\]
\[
= \xi_{d-1}\left(\frac{(a+d+2)}{d}\right) + (-1)^d e_d(M).
\]

Note that \((a+d_d) < a^d\) for all \(a \geq 4\) and \(d \geq 2\). Since \(\text{reg}(G(M)) \geq r(M) \geq 0\), we therefore get
\[
(-1)^{d-1}e_d(M) \leq \xi_{d-1}\left(\frac{(a+d+2)}{d}\right)
\]
\[
= \xi_{d-1}\left(\xi_{d-1} + r + 1\right)^{d^d}
\]
\[
< \left(\xi_{d-1} + r + 1\right)^{d^d+1}.
\]

\[\Box\]

**Remark 2.2.** Using Lemma 1.1 and induction one can derive a better bound for \((-1)^{i-1}e_i(M), i \leq d - 1\). Since this bound is of almost the same complexity as the one in the above theorem, we do not give it here. The fact, that \((-1)^{i-1}e_i(I, A)\) is bounded above by a function depending on \(e_0(I, A), ..., e_{i-1}(I, A)\), if \(i \leq d - 1\), was mentioned in [1, Remark 3.10], provided that \(A\) is an equicharacteristic local ring. Also no explicit bound was given there.

It is easy to see that in general \(|e_i(M)|\) is not bounded above by \(\xi_{i-1}(M)\) (see Examples 2.7 below). In order to prove the main result of this paper, we also need bounds on Hilbert coefficients in terms of the Castelnuovo-Mumford regularity.

**Remark:** The following result was published in the original but should be removed; see Corrigendum.

**Proposition 2.3.** Let \(x_1, \ldots, x_d \in I\) be an \(M\)-superficial sequence for \(I\) and \(B = \ell(M/(x_1, \ldots, x_d)M)\). Then

(a) For all \(1 \leq i \leq d - 1\), \(|e_i(M)| \leq B(\text{reg}^1(G(M)) + 1)^i\);

(b) \(|e_d(M)| \leq B(d + 1)(\text{reg}(G(M)) + 1)^d\).

**Proof.** (a) The inequalities in (a) immediately follow from [4, Theorem 4.6] by noticing that \(\text{reg}(G(M)) = \text{reg}^1(G(M))\) and that \(G(M)\) is generated in non-negative degrees. In fact, the proof of [4, Theorem 4.6] is based on [4, Theorem 4.5(ii)]. In its turn, [4, Theorem 4.5(ii)] follows from [4, Theorem 4.2] and by local duality. These results were formulated for graded modules over a polynomial ring over a field. However, with a small modification, one can show that [4, Theorem 4.5(ii)] and therefore also [4, Theorem 4.6] remain true for any polynomial ring over an Artinian local ring. There is yet another way: in order to show [4, Theorem 4.5(ii)] for the case of Artinian local ring one can rewrite the proof of [4, Theorem 4.2] in terms of local cohomology modules. This was done in [9, Theorem 4.1.3]. For convenience of the reader we sketch the proof here.
**Claim:** Let $E$ be a graded $R$-module of dimension $d$ and let $s = \text{reg}(E)$. Assume that $y_1, \ldots, y_d \in R_1$ is an $E$-filter-regular sequence of $R$, that is $[0 : E/(y_1, \ldots, y_i-1)E]_n = 0$ for all $n \gg 0$. Put $h^i_E(t) := \ell_{R_0}(H^i_{R_+}(E)_t)$. Our immediate aim is to show that for all $i \geq 1$ and $s' \geq s$ we have

$$h^i_E(t) \leq \binom{s' - 1 - t}{i - 1} h_{E/(y_1, \ldots, y_i-1)E}(s'). \quad (7)$$

Since $h^i_E(t) = 0$ for all $t \geq s$, we may assume that $t \leq s - 1 \leq s' - 1$. We proceed by induction on $i$. For $i = 1$, let $E' := \oplus_{n \geq s'} E_n$. Then $\text{reg}(E') = s'$ and $y_1$ is regular on $E'$. The exact sequence

$$H^0_{R_+}(E'/y_1E')_t \to H^i_{R_+}(E')_t \to H^i_{R_+}(E'/y_1E')_t$$

implies

$$h^1_{E'}(t - 1) - h^1_{E'}(t) \leq h^0_{E'/y_1E'}(t) \leq h^1_{E'/y_1E'}(t).$$

Hence

$$h^1_E(t) \leq h^1_{E'}(t) = \sum_{i=t+1}^{s'} (h^1_E(i - 1) - h^1_E(i)) \leq \sum_{i=t+1}^{s'} h^1_{E'/y_1E'}(i) = h^1_{E'}(s') = h^1_E(s').$$

The case $i \geq 2$ follows from the induction hypothesis and the inequality

$$h^i_E(t - 1) - h^i_E(t) \leq h^{i-1}_{E/(y_i-1)E}(t).$$

So, the proof of the claim (7) is completed. Now, taking $s' = s$ and using [4, Lemma 4.4(i)], we obtain

$$h^i_E(t) \leq \ell(E/(y_1, \ldots, y_d)E) \binom{\text{reg}(E) - 1 - t}{i - 1} \binom{\text{reg}(E) + d - i}{d - i}.$$

This is similar to the inequality in [4, Theorem 4.5(ii)] and it is exactly the inequality applied in the proof of [4, Theorem 4.6] in order to derive (a).

(b) Let $a = \text{reg}(G(M))$ and $e_i = e_i(M)$. By Lemma 1.5, $H_M(a) = P_M(a)$. By [7, Lemma 1.7],

$$H_M(a) = \ell(M/M_a+1) \leq \ell(M/I^{a+1}M) \leq B\left(\frac{a + d}{d}\right).$$

Since $\binom{a+j}{j} \leq (a + 1)^j$ and $\sum_{i=0}^{d} (-1)^i e_i \binom{a+d-i}{d-i} = H_M(a)$, by (a) we get

$$|e_d| \leq H_M(a) + \sum_{i=0}^{d-1} |e_i| \binom{a+d-i}{d-i} \leq B\left(\frac{a+d}{d}\right) + B \sum_{i=0}^{d-1} \binom{a+d-i}{d-i}(a + 1)^i \leq B(a+1)^d + B \sum_{i=0}^{d-1} (a + 1)^{d-1}(a + 1)^i = B(d+1)(a+1)^d.$$
Theorem 2.4. Let $\mathcal{M}$ be a good $I$-filtration of $M$. Assume that $\dim(M) = d \geq 1$ and $\depth(M) = t \geq 1$. Then $|e_d(\mathcal{M})|$, $|e_{d-1}(\mathcal{M})|$, $\ldots$, $|e_{d-t+1}(\mathcal{M})|$ are bounded by a function depending only on $d, e_0(\mathcal{M}), e_1(\mathcal{M}), \ldots, e_{d-t}(\mathcal{M})$ and $r(\mathcal{M})$. Namely, for all $j \geq d - t + 1$ we have

$$|e_j(\mathcal{M})| \leq (\xi_{d-t}(\mathcal{M}) + r(\mathcal{M}) + 1)^{3(d-t)!}.$$

Proof. As usual, we write $e_i := e_i(\mathcal{M})$, $\xi_i = \xi_i(\mathcal{M})$, $r = r(\mathcal{M})$ and $\omega = \xi_{d-t} + r + 1$. The case $t = d$, i.e. $M$ is a Cohen-Macaulay module, follows from the following bound given in [7, Theorem 1.10]:

$$|e_j| \leq (e_0 + r + 1)^{3j-1}.$$

So we can assume that $t < d$ and $d \geq 2$. First we prove our claim in the case $i = d$. Let $x_1, \ldots, x_d$ be an $\mathcal{M}$-superficial sequence for $I$. Keep the notation of the proof of Theorem 1.12. Then by (6) we get

$$B \leq \omega^{(d-t)!+2d-t}.$$  

Note that $\omega \geq 2$. Hence, by Proposition B in the Corrigendum and Theorem 1.12 we have

$$|e_d| \leq B 2^d (\reg(G(\mathcal{M}))) + 1)^d \leq \omega^{(d-t)!+3d-t} \omega^{2d(d-t+1)!} < \omega^{(d-t)!+2d(d-t+1)!}.$$

Since $t < d$ and $d \geq 2$, it holds

$$d(d-t+1)d! \geq 2(d-t+1)d! \geq 2(d-t)d(d-1)! + 2d > (d-t)^2(d-t)! + d + 2d.$$  

Hence

$$|e_d| < \omega^{3d(d-t+1)!}.$$

Now let $d - t + 1 \leq j \leq d - 1$. Since $\depth(M) = t$, by [16, Proposition 1.2], $e_j(\mathcal{M}) = e_j(\mathcal{M}_{d-j})$. Note that $\dim(M_{d-j}) = j$, $\depth(M_{d-j}) = t + j - d \geq 1$ and $r(\mathcal{M}_{d-j}) \leq r(\mathcal{M})$. Therefore $\xi_{d-t}(\mathcal{M}_{d-j}) = \xi_{d-t}(\mathcal{M}) = \xi_{d-t}$. Applying (8) to $\mathcal{M}_{d-j}$, we then get

$$|e_j(\mathcal{M})| = |e_j(\mathcal{M}_{d-j})| < \omega^{3j(d-t+1)!}.$$  

For the $I$-ad filtration $\mathcal{M} = \{I^n M\}_{n \geq 0}$ we have $r(\mathcal{M}) = 0$. Hence, as an immediate consequence of Theorem 2.4, we get the following extension of [19, Theorem 1] to the non-Cohen-Macaulay case. In the Cohen-Macaulay case our bound is much bigger than that of [7, Theorem 1.10] (see also [17, Theorem 4.1] and [19, Theorem 1]).

Corollary 2.5. Assume that $\dim(M) = d \geq 1$ and $\depth(M) = t \geq 1$. Then for all $d - t + 1 \leq j \leq d$ we have

$$|e_j(I, M)| < (\xi_{d-t} + 1)^{3j(d-t+1)!},$$

where

$$\xi_{d-t} = \max\{e_0(I, M), |e_1(I, M)|, \ldots, |e_{d-t}(I, M)|\}.$$  

In other words, if $d - t + 1 \leq j \leq d$, $|e_j(I, M)|$ is bounded in terms of $d, e_0(I, M)$, $e_1(I, M), \ldots, e_{d-t}(I, M)$.
Finally we can state and prove a result about the finiteness of Hilbert-Samuel functions.

**Theorem 2.6.** Let \( d \geq t \geq 0, e_0, \ldots, e_{d-t} \) be positive integers. Then there exists only a finite number of Hilbert-Samuel polynomials \( P_{t,M}(n) \) such that \( e_j(I, M) \leq e_j \) for all \( 0 \leq j \leq d - t \).

**Proof.** By Corollary 2.5, there exists only a finite number of Hilbert-Samuel polynomials \( P_{t,M}(n) \) such that \( e_j(I, M) \leq e_j \) for all \( 0 \leq j \leq d - t \). By Lemma 1.5, \( H_{I,M}(n) = P_{t,M}(n) \) for \( n \geq \text{reg}(G_t(M)) =: a \). By Theorem 1.12, \( a \) is bounded in terms of \( e_0, e_1, \ldots, e_{d-t} \) and \( d \). Since \( H_{I,M}(n) = 0 \) for \( n < 0 \) and \( H_{I,M}(n) \) is an increasing function for \( n \geq 0 \), \( H_{I,M}(n) \leq P_{t,M}(a) \) for all \( n \leq a \). This implies that the number of these functions is bounded in terms of \( e_0, e_1, \ldots, e_{d-t} \) and \( d \). \( \square \)

**Example 2.7.** The following examples show that one cannot reduce the number of “independent” coefficients in Theorem 2.4.

(i) Let \( A = K[[x_1, \ldots, x_{d+1}]]/(x_1^2, x_1 x_2, \ldots, x_1 x_d, x_1 x_{d+1}) \), where \( s \geq 1 \), and \( I = \mathfrak{m} = (\bar{x}_1, \ldots, \bar{x}_{d+1}) \). Then \( \dim A = d \), \( \text{depth } A = 0 \), \( e_0 = 1 \), \( e_1 = \cdots = e_{d-1} = 0 \), while \( e_d = (-1)^d s \).

(ii) Even under certain additional assumption on \( A \) we cannot reduce the number of “independent” coefficients. For example, in [18] there were constructed a complete regular local ring \( R \) and an infinite sequence of prime ideals \( \mathfrak{p}_n \) of \( R \) such that \( \dim(R/\mathfrak{p}_n) = 2 \), \( e_0(R/\mathfrak{p}_n) = 4 \), but \( e_1(R/\mathfrak{p}_n) = 8 - n \).

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**References**

[1] Blancafort, C.: *Hilbert functions of graded algebras over Artinian rings*. J. Pure Appl. Algebra 125, 55 - 78 (1998).

[2] Bourbaki, N.: *Algèbre commutative*. Hermann, Paris (1961 - 1965).

[3] Brodmann, M. P., Sharp, R. Y.: *Local cohomology: an algebraic introduction with geometric applications*. Cambridge Studies in Advanced Mathematics, 60. Cambridge University Press, Cambridge (1998).

[4] Chardin, M., Ha, D. T., Hoa, L. T.: *Castelnuovo-Mumford regularity of Ext modules and homological degree*. Trans. Amer. Math. Soc. 363, 3439 - 3456 (2011).

[5] Cuong, N. T., Schenzel, P., Trung, N. V.: *Über verallgemeinerte Cohen-Macaulay Moduln*. Math. Nachr. 85, 57 - 73 (1978).

[6] Dung, L. X.: *Castelnuovo-Mumford regularity of associated graded modules in dimension one*. Acta Math. Vietnam. 38, 541 - 550 (2013).

[7] Dung L. X., Hoa, L. T.: *Castelnuovo-Mumford regularity of associated graded modules and fiber cones of filtered modules*. Comm. Algebra 40, 404 - 422 (2012).

[8] Goto, S., Ozeki, K.: *Uniform bounds for Hilbert coefficients of parameters*. In "Commutative algebra and its connections to geometry", pp. 97 – 118. Contemp. Math., 555, Amer. Math. Soc., Providence, RI (2011).

[9] Ha, D. T.: *Castelnuovo-Mumford regularity of some classes of modules*. PhD. Thesis, Vinh University (in Vietnamese), (2010).

[10] Kirby, D., Mehran, H. A.: *A note on the coefficients of the Hilbert-Samuel polynomial for a Cohen-Macaulay module*. J. London Math. Soc. (2) 25, 449 - 457 (1982).
[11] Linh, C. H.: *Upper bound for Castelnuovo-Mumford regularity of associated graded modules*. Comm. Algebra. 33, 1817 - 1831 (2005).

[12] Linh, C. H.: *Castelnuovo-Mumford regularity and degree of nilpotency*. Math. Proc. Cambridge Philos. Soc. 142, 429 - 437 (2007).

[13] Narita, M.: *A note on the coefficients of Hilbert characteristic functions in semi-regular local rings*. Proc. Cambridge Philos. Soc., 59, 269 - 275 (1963).

[14] Northcott, N. G.: *A note on the coefficients of the abstract Hilbert function*. J. London Math. Soc. (1), 35, 209 - 214 (1960).

[15] Rhodes, C. P. L.: *The Hilbert-Samuel polynomial in a filtered module*. J. London Math. Soc. (1) 3, 73 - 85 (1971).

[16] Rossi, M. E., Valla, G.: Hilbert functions of filtered modules. Lecture Notes of the Unione Matematica Italiana, 9. Springer, Heidelberg (2010).

[17] Rossi, M. E., Trung, N. V., Valla, G.: *Castelnuovo-Mumford regularity and extended degree*. Trans. Amer. Math. Soc. 355, 1773 - 1786 (2003).

[18] Srinivas, V., Trivedi, V.: *A finiteness theorem for the Hilbert functions of complete intersection local rings*. Math. Z. 225, 543 - 558 (1997).

[19] Srinivas, V., Trivedi, V.: *On the Hilbert function of a Cohen-Macaulay local ring*. J. Algebraic Geom. 6, 733 - 751 (1997).

[20] Trivedi, V.: *Hilbert functions, Castelnuovo-Mumford regularity and uniform Artin-Rees numbers*. Manuscripta Math. 94, 485 - 499 (1997).

[21] Trivedi, V.: *Finiteness of Hilbert functions for generalized Cohen-Macaulay modules*. Comm. Algebra 29, 805 - 813 (2001).

[22] Trung, N. V.: *Castelnuovo-Mumford regularity of the Rees algebra and the associated graded rings*. Trans. Amer. Math. Soc 350, 2813 - 2832 (1998).
CORRIGENDUM

Unfortunately there was a gap in the proof of Proposition 2.3 and we have to delete it. Keeping the notation in the above original version, then the proof of Proposition 2.3 only gives the following result.

**Proposition A.** Assume that \(y_1, \ldots, y_d \in R_1\) is an \(E\)-filter-regular sequence of \(R\), that is \([0 : E/(y_1, \ldots, y_{n-1})E] y_i]n = 0\) for all \(n \gg 0\). Put \(B^* = \ell_{R_0} (E/(y_1, \ldots, y_d)E)\). Then \(|e_i(E)| \leq B^*(\text{reg}^i(E) + 1)^i\), for all \(1 \leq i \leq d - 1\).

These inequalities could be useful elsewhere. For the local case we can only prove

**Proposition B.** Let \(x_1, \ldots, x_d \in I\) be an \(M\)-superficial sequence for \(I\) and \(B = \ell(M/(x_1, \ldots, x_d)M)\). Then \(|e_i(M)| < B(\text{reg}(G(M)) + 2)^i\) for all \(1 \leq i \leq d\).

**Proof.** We do induction on \(d\). Let \(a = \text{reg}(G(M))\) and \(e_i = e_i(M)\). By Lemma 1.5,

\[
H_M(a) = P_M(a) = \sum_{i=0}^{d} (-1)^i e_i \binom{a + d - i}{d - i}.
\]

By [7, Lemma 1.7],

\[
H_M(a) = \ell(M/M_{a+1}) \leq \ell(M/I_{a+1}^a M) \leq B \binom{a + d}{d}.
\]

Note that \(\binom{a+j}{j} \leq (a + 1)^j\) and \(e_0 = e_0(I, M) \leq B\).

If \(d = 1\), then

\[
|e_1| = |H_M(a) - e_0(a + 1)| \leq \max\{B(a + 1), e_0(a + 1)\} = B(a + 1).
\]

Let \(d \geq 2\). First we prove the statement for \(0 < i \leq d - 1\). Assume that \(\text{depth}(M) = 0\). Then \(\dim(M/x_1 M) = d - 1\) and by [16, Proposition 1.2], \(e_i(M) = e_i(M/x_1 M)\) for all \(i \leq d - 1\). By Lemma 1.9, \(\text{reg}(M/x_1 M) \leq a\). Hence, by the induction hypothesis applied to \(M/x_1 M\) and the sequence \(x_2, \ldots, x_d\), we get

\[
|e_i(M)| < B(\text{reg}(G(M/x_1 M)) + 2)^i \leq B(2a + 2)^i.
\]

We now assume that \(\text{depth}(M) = 0\). Let \(M = M/H_0^m(M)\) and \(\overline{M} = M/H^0_m(M)\). Note that \(e_i(M) = e_i(\overline{M})\) for all \(i \leq d - 1\) and \(\ell(\overline{M}/(x_1, \ldots, x_d)\overline{M}) \leq B\). In the proof of [7, Lemma 1.9], it was shown that there is an exact sequence

\[
0 \rightarrow K \rightarrow G(M) \rightarrow G(\overline{M}) \rightarrow 0,
\]

where \(K\) has a finite length. Hence \(\text{reg}(G(\overline{M})) \leq \text{reg}(G(M)) = a\), and

\[
|e_i(M)| = e_i(\overline{M}) < \ell(\overline{M}/(x_1, \ldots, x_d)\overline{M})(2\text{reg}(G(\overline{M})) + 2)^i \leq B(2a + 2)^i.
\]

Finally, we have

\[
|e_d| \leq H_M(a) + \sum_{i=0}^{d-1} |e_i| \binom{a + d - i}{d - i} < B \binom{a + d}{d} + B \sum_{i=0}^{d-1} 2^i (a + 1)^i \binom{a + d - i}{d - i} \leq B(a + 1)^d + B \sum_{i=0}^{d-1} 2^i (a + 1)^i(a + 1)^{d-i} = B2^d(a + 1)^d.
\]
Using Proposition B instead of Proposition 2.3 in the original proof of Theorem 2.4 we can still derive the same bound, because there we used a very rough estimation $d + 1 \leq \omega^{d+1}$, and now instead of it we only need to use the estimation $2^d \leq \omega^d$. Also note that there were some misprints in establishing the inequality (8) in the original proof of Theorem 2.4, but the inequality is correct. All these remarks were taken into account in the above corrected version.