A note on the nonzero spectra of irreducible matrices

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Communicated by F. Zhang

(Received 3 April 2011; final version received 16 August 2011)

In this note we extend the necessary and sufficient conditions of Boyle–Handleman [M. Boyle and D. Handelman, The spectra of nonnegative matrices via symbolic dynamics, Ann. Math. 133 (1991), pp. 249–316] and Kim–Ormes–Roush [K.H. Kim, N.S. Ormes, and F.W. Roush, The spectra of nonnegative integer matrices via formal power series, J. Am. Math. Soc. 13 (2000), pp. 773–806] for a nonzero eigenvalue multiset of primitive matrices over $\mathbb{R}^+$ and $\mathbb{Z}^+$, respectively, to irreducible matrices.

Keywords: nonnegative inverse eigenvalue problem; primitive matrices; Boyle–Handleman conditions; Kim–Ormes–Roush conditions

AMS Subject Classifications: 15A18; 15A29; 15A42; 15B36; 15B48

1. Introduction

Denote by $\mathbb{R}^{n \times n} \supset \mathbb{R}^+_{n \times n}$ the algebra of real-valued $n \times n$ matrices and the cone of $n \times n$ nonnegative matrices, respectively. For $A \in \mathbb{R}^{n \times n}$ denote by $\Lambda(A) = \{\lambda_1(A), \ldots, \lambda_n(A)\}$ the eigenvalue multiset of $A$, i.e. $\det(zI - A) = \prod_{i=1}^n (z - \lambda_i(A))$.

An outstanding problem in matrix theory, called NIEP, is to characterize a multiset $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ which is an eigenvalue multiset of some $A \in \mathbb{R}^{n \times n}$. Denote by $\rho(\Lambda) := \max\{[\lambda], \lambda \in \Lambda\}$, and by $\Lambda(r)$ all elements in $\Lambda$ satisfying $|\lambda| = r \geq 0$. For $\lambda \in \Lambda$ denote by $m(\lambda) \in \mathbb{N}$ the multiplicity of $\lambda$ in $\Lambda$. The obvious necessary conditions for $\Lambda = \Lambda(A)$ for some $A \in \mathbb{R}^{n \times n}^+$ are the trace conditions:

$$s_k(\Lambda) := \sum_{i=1}^n \lambda_i^k \geq 0 \quad \text{for} \quad k = 1, \ldots,$$

(1.1)

since $s_k(\Lambda(A)) = \text{tr} A^k$. The following theorem is deduced straightforward from [2, Theorem 2] (see Section 2).

**Theorem 1.1** Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a multiset of complex numbers. Assume that the inequalities in (1.1) hold except for a finite number values of $k$. Then

1. $\tilde{\Lambda} = \Lambda$.
2. $\rho(\Lambda) \in \Lambda$.

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THEOREM 1.4

Let \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \) be an eigenvalue multiset of some \( A \in \mathbb{R}^{n \times n}_+ \). Then in addition to the inequalities (1.1) the following inequalities hold:

\[
s_{km}(\Lambda) \geq \frac{1}{n^{k-1}} (s_m(\Lambda))^k \quad \text{for } m, k - 1 = 1, \ldots. \tag{1.2}
\]

In particular,

\[
\text{if } s_m(\Lambda) > 0 \quad \text{then } s_{km}(\Lambda) > 0 \quad \text{for } k = 2, \ldots. \tag{1.3}
\]

The inequalities (1.1) and (1.2) imply that \( \Lambda \) is an eigenvalue multiset of some \( A \in \mathbb{R}^{n \times n}_+ \) in the following cases: \( n = 3; \ n = 4 \) and \( \Lambda \) is a multiset of real numbers. For \( n = 4 \) and nonreal \( \Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \) the conditions (1.1) and (1.2) are not sufficient [8]. The necessary and sufficient conditions are given in [9]. The inequality \( n s_4(\Lambda) \geq (s_2(\Lambda))^2 \) in (1.2) can be improved to \( (n - 1)s_4(\Lambda) \geq (s_2(\Lambda))^2 \) if \( s_1(\Lambda) = 0 \) and \( n \) is odd [7].

Definition 1.3

A multiset \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C} \) is called a Frobenius multiset if the following conditions hold:

1. \( \bar{\Lambda} = \Lambda \).
2. \( \rho(\Lambda) \in \Lambda \).
3. \( m(\lambda) = 1 \) for each \( \lambda \in \Lambda(\rho(\Lambda)) \).
4. Assume that \( \#\Lambda(\rho(\Lambda)) = p \). Then \( \zeta \Lambda = \Lambda \) for \( \zeta = e^{\frac{2\pi i}{p}} \).

The Frobenius theorem for irreducible \( A \in \mathbb{R}^{n \times n}_+ \), i.e. \( (I + A)^{n-1} \) is a positive matrix, claims that \( \rho(\Lambda( A)) > 0 \) and \( \Lambda( A) \) is a Frobenius set. In particular, an irreducible \( A \in \mathbb{R}^{n \times n}_+ \) is primitive, i.e. \( A^{(n-1)^2 + 1} \) is a positive matrix, if and only if \( \Lambda(\rho( A)) = \{\rho( A)\} \) (see [3, Section 5] and [4, Section 8.5.9]).

We say that a multiset \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \), where \( \lambda_i \neq 0 \) for \( i = 1, \ldots, n \), is a nonzero eigenvalue multiset of a nonnegative matrix if there exists an integer \( N \geq n \) and \( A \in \mathbb{R}^{N \times N}_{\geq 0} \), such that \( \Lambda \) is obtained from \( \Lambda( A) \) by removing all zero eigenvalues. The following remarkable theorem was proved by Boyle and Handelman [1]. Namely, a multiset \( \Lambda \subset \mathbb{C} \setminus \{0\} \) is a nonzero spectrum of a nonnegative primitive matrix if and only if \( \Lambda(\rho( A)) = \{\rho( A)\} \), and the inequalities (1.1) and (1.3) hold. See the recent proof of Laffey [6] of a simplified version of this result. The aim of this note is to extend the theorem of Boyle–Handelman to a nonzero eigenvalue multiset of nonnegative irreducible matrices.

Theorem 1.4

Let \( \Lambda \) be a multiset of nonzero complex numbers. Then \( \Lambda \) is a nonzero eigenvalue multiset of a nonnegative irreducible matrix if and only if \( \Lambda \) is a Frobenius set, and (1.1) and (1.3) hold.
Similarly, we extend the results of Kim et al. [5] to a nonzero eigenvalue multiset of nonnegative irreducible matrices with integer entries.

2. Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1 For \( \Lambda = \{0, \ldots, 0\} \) the theorem is trivial. Assume that \( \rho(\Lambda) > 0 \). Consider the function

\[
f_\Lambda(z) = \sum_{\ell=1}^{n} \frac{1}{1 - \ell \alpha} = \sum_{k=0}^{\infty} s_k(\Lambda)z^k.
\]

Assume that \( s_k(\Lambda) \geq 0 \) for \( k > N \). Then by subtracting a polynomial \( P(z) \) of degree \( N \) at most, we deduce that \( f_\Lambda(z) = f'_\Lambda(z) - P(z) \) has real nonnegative MacLaurin coefficients. So \( f_0(z) = f_\Lambda(z) \). Hence \( \Lambda = \Lambda \). The radius of convergence of this series is \( R(f_\Lambda) = \frac{1}{\rho(\Lambda)} \). The principal part of \( f \) is \( f_1 := \sum_{i,j} \frac{1}{i-j} \). So \( \pi(f_\Lambda(z)) = \{(\lambda_1, m(\lambda_1), 1), \ldots, (\lambda_q, m(\lambda_q), 1)\} \), where \( \lambda_1, \ldots, \lambda_q \) are all pairwise distinct elements of \( \Lambda(\rho(\Lambda)) \) (see [2, Definition 1]). Then parts 2–4 follow from [2, Theorem 2].

Proof of Theorem 1.4 First, assume that \( \Lambda \) is a nonzero eigenvalue multiset of a nonnegative irreducible matrix. The Frobenius theorem yields that \( \Lambda \) has to be a Frobenius set, and (1.1) and (1.3) hold. Now assume that \( \Lambda \) is a Frobenius set, and (1.1) and (1.3) hold. In view of the Boyle–Handelman theorem, it is enough to consider the case

\[
\Lambda(\rho(\Lambda)) = \{\rho(\Lambda), \zeta \rho(\Lambda), \ldots, \zeta^{p-1} \rho(\Lambda)\}, \quad \text{for } \zeta = e^{\frac{2\pi i}{p}} \text{ and } 1 < p \in \mathbb{N}. \quad (2.1)
\]

First, observe that \( s_k(\Lambda) = 0 \) if \( p \not| k \). Let \( \phi: \mathbb{C} \to \mathbb{C} \) be the map \( z \mapsto z^p \). Since \( \zeta \Lambda = \Lambda \), it follows that for \( z \in \Lambda \) with multiplicity \( m(z) \), the multiplicity of \( z^p \) in \( \phi(\Lambda) \) is \( pm(z) \). Hence \( \phi(\Lambda) \) is a union of \( p \) copies of a Frobenius set \( \Lambda_1 \), where \( \rho(\Lambda_1) = \rho(\Lambda)^p \) and \( \Lambda(\rho(\Lambda_1)) = \{\rho(\Lambda_1)\} \). Moreover, \( \Lambda_p = \Lambda_1 \). Hence \( \Lambda_1 \) satisfies the assumptions of the Boyle–Handelman theorem. Thus there exists a primitive matrix \( B \in \mathbb{R}_+^{M \times M} \) whose nonzero eigenvalue multiset is \( \Lambda_1 \). Let \( A = [A_{ij}]_{i,j=1}^{p} \) be the following nonnegative matrix of order \( pM \):

\[
A = \begin{bmatrix}
0_{m \times m} & I_m & 0_{m \times m} & 0_{m \times m} & \cdots & 0_{m \times m} \\
0_{m \times m} & 0_{m \times m} & I_m & 0_{m \times m} & \cdots & 0_{m \times m} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0_{m \times m} & 0_{m \times m} & 0_{m \times m} & 0_{m \times m} & \cdots & I_m \\
B & 0_{m \times m} & 0_{m \times m} & 0_{m \times m} & \cdots & 0_{m \times m} \\
\end{bmatrix}.
\]

Then \( A \) is irreducible and the nonzero part of eigenvalue multiset \( \Lambda(A) \) is \( \Lambda \).

3. An extension of Kim–Ormes–Roush theorem

In this section we give necessary and sufficient conditions on a multiset \( \Lambda \) of nonzero complex number to be a nonzero eigenvalue multiset of a nonnegative irreducible matrix with integer entries. Recall the Möbius function \( \mu: \mathbb{N} \to \{-1, 0, 1\} \). First, \( \mu(1) = 1 \). Assume that \( n > 1 \). If \( n \) is not square free, i.e. \( n \) is divisible by \( l^2 \) for some positive integer \( l > 1 \), then \( \mu(n) = 0 \). If \( n > 1 \) is square free, let \( \omega(n) \) be the number of
distinct primes that divide \( n \). Then \( \mu(n) = (-1)^{\omega(n)} \). The following theorem is a generalization of the Kim–Ormes–Roush theorem [5].

**Theorem 3.1** Let \( \Lambda \) be a multiset of nonzero complex numbers. Then \( \Lambda \) is a nonzero eigenvalue multiset of a nonnegative irreducible matrix with integer entries if and only if the following conditions hold:

1. \( \Lambda \) is a Frobenius set.
2. The coefficients of the polynomial \( \prod_{\lambda \in \Lambda} (z - \lambda) \) are integers.
3. \( t_k(\Lambda) := \sum_{d|k} \mu(d) s_d(\Lambda) \geq 0 \) for all \( k \in \mathbb{N} \).

The case \( \Lambda(\rho(\Lambda)) = \{\rho(\Lambda)\} \) is the Kim–Ormes–Roush theorem.

**Proof** Assume that \( \Lambda \) is a nonzero spectrum of a nonnegative irreducible matrix with integer entries, i.e. \( A \in \mathbb{Z}^{N \times N} \). Then part 1 follows from the Frobenius theorem. Since \( \det(zI - A) \) has integer coefficients, we deduce part 2. It is known that \( t_k(\Lambda) = t_k(\Lambda(A)) \) is the number of minimal loops of length \( k \) in the directed multigraph induced by \( A \) (see [1]). Hence part 3 holds.

Suppose that \( \Lambda \) satisfies (1)–(3). In view of the Kim–Ormes–Roush theorem, it is enough to assume the case (2.1). We now use the notations and the arguments of the proof of Theorem 1.4. First, \( s_k(\Lambda) = 0 \) if \( p \nmid k \). Second, \( \prod_{\lambda \in \Lambda}(z - \lambda) = \prod_{\kappa \in \Lambda_1}(z^p - \kappa) \). Hence, \( \prod_{\kappa \in \Lambda_1}(z - \kappa) \) has integer coefficients. A straightforward calculation shows that \( t_k(\Lambda) = p t_k(\Lambda_1) \). Hence, \( t_k(\Lambda_1) \geq 0 \). Kim–Ormes–Roush theorem yields the existence of \( B \in \mathbb{Z}^{M \times M} \) such that \( \Lambda_1 \) is the nonzero eigenvalue multiset of \( B \). Hence, \( \Lambda \) is the nonzero eigenvalue set of \( A \in \mathbb{Z}^{pM \times pM} \) given by (2.2).

**References**

[1] M. Boyle and D. Handelman, *The spectra of nonnegative matrices via symbolic dynamics*, Ann. Math. 133 (1991), pp. 249–316.

[2] S. Friedland, *On inverse problem for nonnegative and eventually nonnegative matrices*, Israel J. Math. 29 (1978), pp. 43–60.

[3] F.R. Gantmacher, *The Theory of Matrices*, Vols. I and II, Chelsea Publ. Co., New York, 1959.

[4] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1988.

[5] K.H. Kim, N.S. Ormes, and F.W. Roush, *The spectra of nonnegative integer matrices via formal power series*, J. Am. Math. Soc. 13 (2000), pp. 773–806.

[6] T.J. Laffey, *A constructive version of the Boyle–Handelman theorem on the spectra of nonnegative matrices*, arXiv:1005.0929.

[7] T. Laffey and E. Meehan, *A refinement of an inequality of Johnson, Loewy and London on nonnegative matrices and some applications*, Electron. J. Linear Algebra 3 (1998), pp. 119–128.

[8] R. Loewy and D. London, *A note on an inverse problem for nonnegative matrices*, Linear Multilinear Alg. 6 (1978), pp. 83–90.

[9] J. Torre-Mayo, M.R. Abril-Raymundo, E. Alarcia-Estvez, C. Marijun, and M. Pisonero, *The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. EBL digraphs*, Linear Algebra Appl. 426 (2007), pp. 729–773.