Abstract

We prove that every tetrahedron $T$ has a simple, closed quasigeodesic that passes through three vertices of $T$. Equivalently, every $T$ has a face whose “exterior angles” are at most $\pi$.

1 Introduction

A quasigeodesic $Q$ is a curve on the surface of a polyhedron that is convex to both sides in the sense that each point $p \in Q$ has surface angle $\leq \pi$ to each side. In a vertex-free region, a quasigeodesic is a geodesic with exactly $\pi$ to each side, but unlike geodesics, a quasigeodesic can pass through a vertex. Of particular interest are the simple (non-self-intersecting) closed quasigeodesics, which we henceforth abbreviate to “quasigeodesic” without qualifiers. The main result of this note is that every tetrahedron $T$ has a quasigeodesic $Q$ passing through three vertices. So $Q$ is the boundary $\partial F$ of a face $F$ of $T$. This is of interest because Pogorelov [Pog49] proved that every convex polyhedron has at least three quasigeodesics, a generalization of the 3-geodesics theorem of Lysternick and Schnirleman\footnote{See also [DO07] p. 374]. So our main theorem (Theorem 1) identifies at least one of the three guaranteed quasigeodesics on tetrahedra.

Theorem 1 may also be stated without reference to the notion of a quasigeodesic: Every tetrahedron $T$ has a face $F$ such that the angles to the other side of the vertices of $F$ is at most $\pi$. By the “other side” is meant: the
two incident face angles not in \( F \), in some sense the “exterior angles” of \( F \). This is a fundamental relation among the 12 face angles of any tetrahedron. It could have been known since antiquity, but perhaps was of no interest without the notion of a quasigeodesic.

2 Notation

The detailed argument concerning the 12 angles requires precise notation.

- Vertices of tetrahedron \( T \): \( a, b, c, d \).
- Face \( A \) is opposite \( a \), and so does not include \( a \); etc.
- So: \( A = \text{bdc}, B = \text{cda}, C = \text{adb}, D = \text{abc} \).
• Face angles are specified by vertex and face. So the three face angles incident to vertex \( a \) are: \( aB, aC, aD \); etc. So \( A \), which is opposite \( a \), is not incident to \( a \); etc. See Fig. 1.

• Vertex curvature: \( \omega(a) = 2\pi - (aB + aC + aD) \).

To simplify the calculations, angles will be represented in inequalities in units of \( \pi \): 1 \( \equiv \) \( \pi \), 2 \( \equiv \) \( 2\pi \), etc. Thus under this convention, each of the 12 face angles of a tetrahedron lies in \((0, 1)\).

3 Lemmas

We establish two preliminary lemmas that will be used in the proof.

**Lemma 1.** Let \( \alpha_1, \alpha_2, \alpha_3 \) be the face angles incident to vertex \( v \) of a tetrahedron \( T \). Then the angles satisfy the triangle inequality: \( \alpha_1 < \alpha_2 + \alpha_3 \), and similarly \( \alpha_2 < \alpha_1 + \alpha_3 \), and \( \alpha_3 < \alpha_1 + \alpha_2 \). The inequalities are strict unless \( T \) is flat.

**Proof.** Surround \( v \) with a sphere \( S \) centered on \( v \). Then the planes containing the faces incident to \( v \) cut \( S \) in great-circle geodesics, forming a spherical triangle on \( S \). The length of each geodesic arc is the measure of the corresponding face angle \( \alpha_i \) incident to \( v \). The triangle inequality holds on \( S \), so \( \alpha_1 < \alpha_2 + \alpha_3 \), strictly less than because the triangle cannot degenerate to a geodesic segment (unless the tetrahedron is flat).

In the flat case, \( \alpha_1 = \alpha_2 + \alpha_3 \).

This lemma holds at any degree-3 vertex of a convex polyhedron.

We say that “face \( F \) fails at vertex \( v \)” if the two angles incident to \( v \) not in \( F \) exceed \( \pi \). So, for face \( A \) to fail on vertex \( b \), then among the three face angles \( bA, bC, bD \) incident to \( b \), the two angles not in \( A \) satisfy \( bC + bD > 1 \). This means that \( \partial A \) is not a quasigeodesic, because to one side—the other side from \( bA \)—the angle exceeds \( \pi \).

**Example.** Fig. 2 shows a tetrahedron with \( \partial B \) a quasigeodesic, but none of the other face boundaries is a quasigeodesic. Its vertex coordinates are:

\[
(a, b, c, d = (0, 0, 0), (1, 0, 0), (4.91, 3.24, 0), (-3.54, 1.98, 4.58)).
\]

For example, face \( B \) does not fail at vertex \( a \): \( aC + aD = 125^\circ + 33^\circ = 159^\circ < \pi \). Face \( A \) fails at vertex \( b \): \( bC + bD = 48^\circ + 140^\circ = 188^\circ > \pi \).
Figure 2: The (red) boundary of shaded face $B = cda$ is a quasigeodesic, but none of $\partial A, \partial C, \partial D$ are quasigeodesics.
Lemma 2. If a face $A$ fails at a vertex $b$, then $\omega(b) < 1$.

Proof. Since face $A$ fails at $b$, by definition, $bC + bD > 1$. Therefore

$$\omega(b) = 2 - (bA + bC + bD)$$
$$\omega(b) = 2 - (bC + bD) - bA$$
$$\omega(b) < 1 - bA$$
$$\omega(b) < 1$$

This establishes the claim of the lemma. \hfill \Box

4 Case Analysis

We now undertake a case analysis to show that it is not possible for all four faces of tetrahedron $T$ to fail at vertices. The cases, illustrated in Fig. 3, distinguish first the number of distinct vertices among the four face-failures, and second, the pattern of the failures.

The proof analyzes the 12 face angles of $T$, and shows the set of solutions in $(0, 1)^{12}$ is empty (under the convention that each angle is in $(0, 1)$). So we are representing tetrahedra by their 12 face angles. The four faces each have a total of $\pi$ angle, which reduces the dimension of the tetrahedron configuration space from 12 to 8. It is known that in fact the configuration space is 5-dimensional, not 8-dimensional, but the proof to follow works without including the various additional trigonometric relations that tetrahedron angles must satisfy. It suffices to use linear equalities and inequalities among the 12 face angles.

Case 1: 4 vertices. Suppose first that each of the four faces $A, B, C, D$ fail on four distinct vertices. Then Lemma 2 shows that $\omega(v) < 1$ for each vertex $v$. But then $\sum \omega(v) < 4$, contradicting the Gauss-Bonnet theorem.

Case 2a: 2 vertices, $3+1$. Suppose now that the four faces fail on a total of two vertices. This can occur in two distinct ways: three faces fail on one vertex, which we call Case 2a, or two faces fail each on two vertices, Case 2b. Say that $b$ is the vertex at which three faces fail. We then have:
Figure 3: Failures. Case 1: 4 vertices. Case 2: 2 vertices. Case 3: 3 vertices.
It turns out that we do not need to use the fact that \( C \) and \( D \) fail at some vertices, so the implied inequalities are suppressed. Summing the failure inequalities above leads to a contradiction:

\[
\begin{align*}
(aC + bC) + (aD + bD) & > 2 \\
(1 - dC) + (1 - cD) & > 2 \\
2 & > 2 + (dC + cD) \\
0 & > dC + cD
\end{align*}
\]

This is a contradiction because all angles have positive measure.

**Case 2b: 2 vertices, 2 + 2.** This follows the exact same proof, as again \( C \) and \( D \) failures are not needed to reach a contradiction.

**Case 3a: 3 vertices, double outside.** The three vertices at which faces fail bound a face, say \( A \). One vertex of \( A \), say \( b \), is “doubled” in the sense that two faces fail at \( b \). Case 3a is distinguished in that neither face failing on \( b \) is the three-vertex face \( A \). (Swapping \( B \) to fail on \( c \) and \( A \) to fail of \( d \) is symmetrically equivalent to the case illustrated.)

We again do not need all failures, in particular, we only need those for faces \( B \) and \( D \):

\[
\begin{align*}
A \text{ fails at } c \\
B \text{ fails at } d & : \ dA + dC > 1 \\
C \text{ fails at } b \\
D \text{ fails at } b & : \ bA + bC > 1
\end{align*}
\]
Adding these inequalities leads to the same contradiction:

\[(bA + dA) + (bC + dC) > 2\]
\[(1 - cA) + (1 - aC) > 2\]
\[0 > aC + cA\]

Again a contradiction.

**Case 3b: 3 vertices, double inside.** In contrast to Case 3a, in this case, one of the faces that fail on \(b\) is the three-vertex face \(A\). (Swapping \(B\) to fail on \(c\), \(D\) to fail on \(b\), and \(C\) to fail on \(d\), is symmetrically equivalent.) This is the only difficult case, and the only case in which the triangle inequalities guaranteed by Lemma 1 are needed.

The angles of face \(A\) satisfy \(bA + cA + dA = 1\). Assume without loss of generality that \(bA \leq cA \leq dA\). Three faces, \(B, C, D\) fail at the three vertices of face \(A\): \(d, b, c\) respectively.

To build intuition, we first run through the proof for specific \(A\)-face angles:

\[(bA, cA, dA) = (0.1, 0.3, 0.6)\]

\(A\) fails at \(b\)

\(B\) fails at \(d\) : \(dA + dC > 1 : dC > 0.4\)

\(C\) fails at \(b\) : \(bA + bD > 1 : bD > 0.9\)

\(D\) fails at \(c\) : \(cA + cB > 1 : cB > 0.7\)

Note \(0.4 + 0.9 + 0.7 = 2\); this holds for arbitrary \(A\) angles. Now apply the triangle inequality to each of \(dB, cB, dC\):

\[bD < bA + bC : bC > bD - bA : bC > 0.8\]
\[cB < cA + cD : cD > cB - cA : cD > 0.4\]
\[dC < dA + dB : dB > dC - dA : dB > -0.2\]

Note \(0.8 + 0.4 - 0.2 = 1\); this again holds for arbitrary \(A\) angles.

Triangle face \(D\) satisfies: \(bD + cD + aD = 1\).

\[bD > 0.9\]
\[cD > 0.4\]
\[bD + cD > 1.3\]
\[bD + cD + aD > 1.3 > 1\]
which contradicts $bD + cD + aD = 1$.

Without specific angles assigned to $(bA,cA,dA)$, the argument is less transparent. Again assume that $bA \leq cA \leq dA$.

\[
\begin{align*}
A \text{ fails at } b \\
B \text{ fails at } d &: dA + dC > 1 : dC > 1 - dA \\
C \text{ fails at } b &: bA + bD > 1 : bD > 1 - bA \\
D \text{ fails at } c &: cA + cB > 1 : cB > 1 - cA \\
\end{align*}
\]

Note the sum of the above three right-hand sides is $3 - (dA + bA + cA) = 2$.

Now apply the triangle inequality to $dB, cB, dC$:

\[
\begin{align*}
bD &< bA + bC : bC > bD - bA : bC > 1 - 2 \cdot bA \\
cB &< cA + cD : cD > cB - cA : cD > 1 - 2 \cdot cA \\
dC &< dA + dB : dB > dC - dA : dB > 1 - 2 \cdot dA \\
\end{align*}
\]

Note the sum of the above three right-hand sides is $3 - 2(dA + bA + cA) = 1$.

Face $D$’s angles satisfy $bD + cD + aD = 1$. Now we reach a contradiction using the inequalities above.

\[
\begin{align*}
bD &> 1 - bA \\
cD &> 1 - 2 \cdot cA \\
bD + cD &> 2 - (bA + 2 \cdot cA) \\
\end{align*}
\]

We have $(bA + 2 \cdot cA) \leq 1$ because $bA + cA + dA = 1$ and $cA \leq dA$. And of course every angle is positive, so $aD > 0$. So we have

\[
\begin{align*}
bD + cD &> 1 \\
bD + cD + aD &> 1 \\
\end{align*}
\]

which contradicts $bD + cD + aD = 1$.

That the inequalities for each of the above cases cannot be simultaneously satisfied has been verified by Mathematica’s `FindInstance[]` function, which
uses Linear Programming over the rationals\footnote{https://mathematica.stackexchange.com/q/255494/194} to conclude that the set of solutions in $\mathbb{R}^{12}$ is empty.

Replacing the triangle inequalities with equalities when the tetrahedron is flat (e.g., $aB = aC + aD$ instead of $aB < aC + aD$) again leads to the same contradiction.

5 Conclusion

**Theorem 1.** Every tetrahedron has at least one face $F$ whose boundary $\partial F$ is a simple, closed quasigeodesic $Q$, passing through the three vertices of $F$. So $Q$ is a 3-vertex quasigeodesic.

In Open Problem 18.13 [OV21], we conjecture that every convex polyhedron either has a simple closed geodesic, or a simple closed quasigeodesic through just one vertex, i.e., a 1-vertex quasigeodesic. This remains for future work.

**References**

[DO07] Erik D. Demaine and Joseph O’Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, 2007. \url{http://www.gfalop.org}.

[OV21] Joseph O’Rourke and Costin Vâlcu. Reshaping Convex Polyhedra. arXiv 2107.03153: \url{https://arxiv.org/abs/2107.03153}, July 2021.

[Pog49] Aleksei V. Pogorelov. Quasi-geodesic lines on a convex surface. *Mat. Sb., 25(62):275–306*, 1949. English transl., *Amer. Math. Soc. Transl.* 74, 1952.