From Schritte and Wechsel to Coxeter Groups

Markus Schmidmeier

Abstract: The PLR-moves of neo-Riemannian theory, when considered as reflections on the edges of an equilateral triangle, define the Coxeter group $\widetilde{S_3}$. The elements are in a natural one-to-one correspondence with the triangles in the infinite Tonnetz. The left action of $\widetilde{S_3}$ on the Tonnetz gives rise to interesting chord sequences. We compare the system of transformations in $\widetilde{S_3}$ with the system of Schritte and Wechsel introduced by Hugo Riemann in 1880. Finally, we consider the point reflection group as it captures well the transition from Riemann’s infinite Tonnetz to the finite Tonnetz of neo-Riemannian theory.

Keywords: Tonnetz, neo-Riemannian theory, Coxeter groups.

1. PLR-moves revisited

In neo-Riemannian theory, chord progressions are analyzed in terms of elementary moves in the Tonnetz. For example, the process of going from tonic to dominant (which differs from the tonic chord in two notes) is decomposed as a product of two elementary moves of which each change only one note.

The three elementary moves considered are the PLR-transformations; they map a major or minor triad to the minor or major triad adjacent to one of the edges of the triangle representing the chord in the Tonnetz. See [4] and [5].

![Figure 1. Triads in the vicinity of the C-E-G-chord](image)

Our paper is motivated by the observation that PLR-moves, while they provide a tool to measure distance between triads, are not continuous as operations on the Tonnetz:

Let $s$ be a sequence of PLR-moves. Applying $s$ to a pair of major chords results in a parallel shift of those two chords. However, applying the sequence $s$ to a major chord and an adjacent minor chord makes the two chords drift apart. For example, applying the sequence $s = RL$ to the chord labeled $(\ast)$ in Figure 1 yields the triad labeled $RL$ on the right,
while \( s \) applied to \( P \) gives \( RLP \) on the left, left of the triangle labelled \( LP \).

In this paper we consider the three reflections \( s_1, s_2, s_3 \) on the edges of a fixed equilateral triangle, see Figure 2.

Figure 2. The reflections \( s_1, s_2, s_3 \)

The three reflections satisfy the relations

\[
s_1^2 = s_2^2 = s_3^2 = 1, \quad (s_1s_2)^3 = (s_2s_3)^3 = (s_3s_1)^3 = 1
\]

which are the defining relations of the Coxeter group \( \tilde{S}_3 \) corresponding to the affine irreducible Coxeter system \( \tilde{A}_2 \)

Figure 3. Affine Coxeter system \( \tilde{A}_2 \)

2. The Coxeter group \( \tilde{S}_3 \)

We collect some results about the Coxeter group \( \tilde{S}_3 \), most of the material is adapted from [1] Section 8.3. The group \( \tilde{S}_3 \) can be realized as the group of **affine permutations**.

\( \tilde{S}_3 = \{ f : \mathbb{Z} \to \mathbb{Z} : f \text{ bijective}, f(-1)+f(0)+f(1) = 0, \forall n : f(n+3) = f(n)+3 \} \),

with multiplication the composition of maps. Due to the last condition, it suffices to record the values of \( f \) on the window \( \{-1, 0, 1\} \), so

\( \tilde{S}_3 = \{ [a, b, c] \in \mathbb{Z}^3 : a+b+c = 0, a, b, c \text{ pairwise incongruent modulo } 3 \} \).
Here, $1 = [-1, 0, 1]$, $s_1 = [0, -1, 1]$, $s_2 = [-1, 1, 0]$, $s_3 = [-2, 0, 2]$ and composition of affine permutations yields the multiplication rule

$$[a, b, c] \cdot s_i = \begin{cases} 
[b, a, c], & \text{if } i = 1 \\
[a, c, b], & \text{if } i = 2 \\
[c - 3, b, a + 3], & \text{if } i = 3.
\end{cases}$$

Affine permutations, unlike sequences of reflections, provide a unique name for each element in $\tilde{S}_3$. The following result permits us to write the Coxeter element into the triangle to which the corresponding sequence of reflections maps the identity element. In Figure 4 we omit the brackets and place minus-signs under the numbers to improve readability.

![Figure 4. Affine permutations](image)

**Proposition 2.1.** The elements in $\tilde{S}_3$ are in one-to-one correspondence with the triangles in the Tonnetz.

The result is well-known, in fact, the Tonnetz picture is commonly used to visualize the tesselation of the affine plane given by the Coxeter system $\tilde{A}_2$, see for example [1, Figure 1.2]. We give the proof to obtain relevant details of this tesselation.

Using the correspondence in Proposition 2.1 we will identify the elements in $\tilde{S}_3$ with the triangles in the Tonnetz. Thus, the group $\tilde{S}_3$ acts on the Tonnetz via left multiplication and via right multiplication, and both actions are simply transitive.

**Proof.** The map given by sending a sequence of reflections to the triangle $\Delta$ obtained by applying the reflections to the triangle marked $(\ast)$ is an onto map: Unless $\Delta$ is the triangle $(\ast)$ itself, there exists at least one axis $s_i$ between the two triangles. Reflecting $\Delta$ on $s_i$ gives a triangle which is closer to $(\ast)$, hence the process of replacing $\Delta$ by $s_i(\Delta)$ terminates after finitely many steps.
Equivalent sequences modulo the relations give the same triangle, hence we obtain a map from $\tilde{S}_3$ to the set of triangles. This map is injective since each triple $[a, b, c]$ records the coordinates of the triangle as described in the following lemma.

**Lemma 2.2.** For an affine permutation $[a, b, c]$ and for $i = 1, \ldots, 3$, define

$$c_i([a, b, c]) = \begin{cases} a + 1 & \text{if } a \equiv i \mod 3 \\ b & \text{if } b \equiv i \mod 3 \\ c - 1 & \text{if } c \equiv i \mod 3 \end{cases}$$

Then the coordinate of the center of the triangle corresponding to the Coxeter element $[a, b, c]$ in the Tonnetz with respect to the axis $s_i$ is $c_i$. (Here, the axis $s_1$ points to the left, $s_2$ to the top right and $s_3$ to the bottom right.)

For example, the three triangles in Figure 4 above and below the intersection of the $s_1$- and the $s_3$-axis all have $a = -2$. Hence the $s_1$-coordinate of their center is $c_1 = -1$.

**Proof.** The numbers $c_i$ are defined since exactly one of the entries $a, b, c$ is congruent to $i$ modulo 3. The formula in the lemma can be verified using induction on the length of a sequence $s_{i_1}s_{i_2}\cdots s_{i_n}$ defining the Coxeter element and the above multiplication formula.

We have seen that for a given triangle $\Delta$, there is a unique Coxeter element which maps $(\ast)$ to $\Delta$. The minimum number of reflections can be computed by counting inversions, see [1, Proposition 8.3.1] or by measuring the distance from $(\ast)$:

**Corollary 2.3.** Suppose the triangle $\Delta$ corresponds to the affine permutation $[a, b, c]$. The minimum number $d$ of reflections needed to map $(\ast)$ to $\Delta$ is the sum of the positive coordinates $c_i$, or $d = \frac{1}{2} \sum_{i=1}^3 |c_i([a, b, c])|$. 

As a product of reflections, each Coxeter element gives rise to an operation on the Tonnetz which may be a translation, a rotation, a reflection or a glide reflection. In Figure 5, we put a hook inside each triangle so that the type of operation given by an element of $\tilde{S}_3$ can be read off from the position of the hooks in two corresponding triangles.

For example, going from $(\ast)$ in Figure 5 to (1) is a reflection, to (2) a rotation, to (3) a glide reflection, and to (6) a translation.
3. THE FUNDAMENTAL HEXAGON

It turns out that the Coxeter group $\tilde{S}_3$ has a normal subgroup $\tilde{T}$ of index 6 given by translations. The factor group $\tilde{S}_3/\tilde{T}$ is isomorphic to the symmetric group $S_3$. We call the fundamental domain with respect to the shift by a translation in $\tilde{T}$ the fundamental hexagon. In the next section, we will discuss the role of this fundamental hexagon in music.

It would be desirable to have a “comma subgroup” in $\tilde{S}_3$ to provide a link to the group generated by the PLR-moves on the finite Tonnetz, but the author was not able to detect a suitable normal subgroup in $\tilde{S}_3$. However, there is a related group, the point reflection group $P$, which does have such a “comma subgroup”, as we will see in Section 6.

Note that left multiplication by $s_2s_3s_2 = s_3s_2s_3$ is the reflection on the Tonnetz on the line one unit above the $s_1$-axis. Hence $t_1 = (s_2s_3s_2)s_1$ is the upwards translation by 2 units. Similarly, $t_2 = (s_3s_1s_3)s_2$ and $t_3 = (s_1s_2s_1)s_3$ are translations by 2 units towards the lower right and the lower left, respectively, as indicated in Figure 5.

**Proposition 3.1.** The group of translations $\tilde{T} = \langle t_1, t_2, t_3 \rangle = \langle t_1, t_2 \rangle$ is a free abelian group of rank 2. Moreover, $\tilde{T}$ is a normal subgroup of $\tilde{S}_3$ of index 6, the factor group $\tilde{S}_3/\tilde{T}$ is isomorphic to the symmetric group $S_3$.

**Proof.** From Euclidean geometry it is clear that $t_1t_2 = t_3^{-1} = t_2t_1$, it follows that $\tilde{T} = \langle t_1, t_2 \rangle$ is an abelian group; moreover, the translations

![Figure 5. Left multiplication by affine permutations](image-url)
$t_1, t_2$ span a 2-dimensional lattice in the plane. The group $\widetilde{T}$ is a normal subgroup of $\widetilde{S}_3$ since $s_1 t_1 s_1^{-1} = t_1^{-1}$, $s_2 t_1 s_2^{-1} = t_3^{-1}$, $s_3 t_1 s_3^{-1} = t_2^{-1}$, and the index is 6 since the plane of the Tonnetz can be tiled with hexagons which are in one-to-one correspondence with the elements in $\widetilde{T}$.

![Figure 6. Translations of hexagons](image)

For the last claim, consider $S_3 = \langle s_2, s_3 \rangle$ as a subgroup of $\widetilde{S}_3$. Since $S_3 \cap \widetilde{T} = \{e\}$, the composition

$$S_3 \rightarrow \widetilde{S}_3 \rightarrow \widetilde{S}_3/\widetilde{T}$$

is a one-to-one map, hence a group isomorphism. □ □

**Corollary 3.2.** As a group, $\widetilde{S}_3 = \widetilde{T} \cdot S_3$. □

We call the region in the Tonnetz corresponding to the subgroup $S_3 = \langle s_2, s_3 \rangle$ the **fundamental hexagon** (see Proposition 2.1). The left cosets $tS_3$ for $t \in \widetilde{T}$ form the tiling pictured in Figure 6. Right multiplication by an element in $S_3$ yields a permutation of the triangles within each hexagon, while left multiplication by an element of $\widetilde{T}$ is a parallel shift which preserves the hexagonal pattern.

We conclude this section by briefly listing the four types of elements in $\widetilde{S}_3$ in terms of their action on the Tonnetz given by left multiplication.

- The elements of order 2 are reflections on a line parallel to one of the axes. In particular, the reflection on the $n$-th line parallel to $s_i$ is given by $t_i^n s_i$ ($n \in \mathbb{Z}, i = 1, 2, 3$).
- Products of two reflections on lines which are not parallel are rotations by $\pm 120^\circ$ about a vertex in the Tonnetz. Those are the elements of order 3 in $\widetilde{S}_3$. 

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$\text{Figure 6. Translations of hexagons}$
• Products of two reflections on parallel lines are translations, they form the normal subgroup $\tilde{T}$ considered above.
• The remaining elements are odd, they act as glide reflections on the Tonnetz; all have infinite order.

4. THE FUNDAMENTAL HEXAGON IN MUSIC

The hexagon encapsulates fundamental concepts in music theory, and leads to some, perhaps weird, sequences of chords. Each hexagon in the Tonnetz consists of six triangles which represent major and minor chords which have one note in common, see Figure 7.

Consider the E-hexagon from Figure 7. The three reflections $s_2, s_3, s_2s_3s_2$ in $S_3$ describe the PLR-moves locally.
Reflection on the $s_3$-axis is the leading tone exchange:

\[ L : \quad C-E-G \leftrightarrow E-G-B, \quad A-C\#-E \leftrightarrow C\#-E-G\# \]

Reflection on the $s_2$-axis yields the relative major or minor:

\[ R : \quad C-E-G \leftrightarrow A-C-E, \quad E-G\#-B \leftrightarrow C\#-E-G\# \]

and reflection on the $s_2s_3s_2$-axis the parallel major or minor:

\[ P : \quad A-C\#-E \leftrightarrow A-C-E, \quad E-G\#-B \leftrightarrow E-G-B. \]

The elements $s_2s_3$ and $s_3s_2$ in $S_3$ have order 3. Iterated multiplication by $s_3s_2$ gives rise to a 3-cycle of major chords within the E-hexagon:

\[ C-E-G \rightarrow E-G\#-B \rightarrow A-C\#-E \rightarrow C-E-G \]

and a 3-cycle of minor chords:

\[ C\#-E-G\# \rightarrow A-C-E \rightarrow E-G-B \rightarrow C\#-E-G\#. \]

The three steps in the cycle: The move to the upper left, then the horizontal move to the right and the move to the lower left, mark three stripes in the Tonnetz.
• The horizontal stripe given by a chord contains all possibly higher subdominant and dominant chords.
• The stripe in NE-SW direction pictures the (infinite) hexatonic system to which the chord belongs, see [2, Part III].
• The stripe in NW-SE direction represents the (infinite) octatonic system for the given chord, see [3].

We notice that while the rotations in $S_3$ mark the directions of the three stripes, the translations in $\bar{T}$ can be used to move between parallel systems.

We would like to point out that the three opening chords of Ludwig van Beethoven’s *Moonlight Sonata* take place within the $E$-hexagon. The $C_3$-minor chord leads to the $C_3$-minor sept chord $C_3^\#E_2^\#G_1^\#$ (which contains the relative $E$-major chord), then to the (subdominant) $A$-major chord. The neighboring $A$-hexagon captures the transition from the $A$-major chord to the following subdominant $D$-major chord...

There are more, perhaps even weirder, chord sequences outside of the central hexagon. For example, the rotation by $s_3s_2$ permutes the major chords which have one edge in common with the $E$-hexagon:

$$G-B-D \rightarrow C_3^\#E_2^\#G_1^\# \rightarrow F-A-C \rightarrow G-B-D$$

and similarly the minor chords:

$$C-E_2^\#G \rightarrow G_1^\#B-D_2^\# \rightarrow F_2^\#A-C_3^\# \rightarrow C-E_2^\#G.$$  

Another type of chord sequence is obtained from translations of the hexagons in the Tonnetz. Consider the tiling pictured in Figure 8.

![Figure 8. The tiling by hexagons](image)

The translations which define the tiling satisfy the identity $t_3t_2t_1 = e$. Applying successively $t_1,t_2t_1,t_3t_2t_1$ to $C-E-G$ yields the sequence of major chords

$$C-E-G \rightarrow C_2^\#E_2^\#G_1^\# \rightarrow B-D_2^\#F_2^\# \rightarrow C-E-G$$
But the alteration of elements of order 6, so the rotation by 60°
the succession of

\begin{align*}
\text{A-C-E} & \rightarrow A^\#_2-C^\#_2-E^\#_2 \\
& \rightarrow G^\#_2-B-D^\#_2 \\
& \rightarrow \text{A-C-E}
\end{align*}

(also in the E-, E^\#-, and D^\#-hexagons).

A substantial collection of Tonnetz models can be found in [8]. An interesting musical case for the application of \( S_3 \) are the “pitch retention loops” in which the chords in the hexagon occur in cyclical order, see in particular [8] Figure 6.3]. The Coxeter group \( S_3 \) does not contain any elements of order 6, so the rotation by 60° may be difficult to explain. But the alteration of \( s_2 \) and \( s_3 \) is still a more effective description than the succession of \( L, P \) and \( R \).

In the above, we have identified the infinite triadic Tonnetz with the Coxeter group \( \widetilde{S}_3 \) and studied the left action of the group \( \widetilde{S}_3 \) on itself. Considering the right action, note that alternating right multiplication by \( s_3 \) and \( s_2 \) generates the triads in each of the hexagons with base triad in \( \widetilde{T} \), they are pictured in Figure [8] The remaining hexagons in the Tonnetz have their base triad in either \( s_3s_2\widetilde{T} \) or \( s_2s_3\widetilde{T} \), there the triads in the cycle are generated by alternating right multiplication by \( s_1 \) and \( s_3 \), or by \( s_2 \) and \( s_1 \), respectively.

5. Schritte and Wechsel, revisited

In [8], Hugo Riemann presents two kinds of operations on the Tonnetz, Schritte and Wechsel. Under a Schritt, each note in a major triad moves up or down a certain number of scale degrees, while the notes in a minor triad move in the opposite direction. For example, using neo-Riemannian terminology, the Quintschritt is given by the RL-move, the Terzschritt by the PL-move. Under a Wechsel, major and minor triads correspond to each other, for example the Seitenwechsel \( w \) yields the parallel triad given by the P-move.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{schritte-wechsel.png}
\caption{Quintschritt RL, Terzschritt PL, and Seitenwechsel P}
\end{figure}
For a detailed description of Riemann’s system of Schritte and Wechsel we refer to [7] which also discusses the composition of Schritte and Wechsel. The Schritte group $T$ is generated by the Quintschritt $RL$ and the Terzschritt $PL$, it is isomorphic to the additive group $\mathbb{Z} \times \mathbb{Z}$. Note that under a Schritt, major and minor triads move in opposite directions, see Figure 9.

Each Wechsel can be obtained as a composition of a Schritt with the Seitenwechsel $w$, hence the product of a Wechsel with itself is the identity operation on the Tonnetz. More generally, if $t, t’ \in T$ are Schritte, then then the product $(t’w) \cdot (tw) = t’t^{-1}$ of two Wechsel is the composition of a Schritt $t’$ with the opposite of the Schritt $t$ — hence the product of two Wechsel is always a Schritt. The map $\varphi : T \to T, t \mapsto t^{-1}$ given by conjugation by $w$ is an automorphism of order two (since $T$ is an abelian group), and the group $R$ of Schritte and Wechsel as described in [7, Appendix III] is the semi-direct product $\mathbb{Z}_2 \rtimes \varphi T$.

We compare the Schritt-Wechsel group $R$ and the Coxeter group $\tilde{S}_3$.

As for the Coxeter group $\tilde{S}_3$, the elements in $R$ are in one-to-one correspondence with the triangles in the infinite Tonnetz. Using this identification, the Schritt-Wechsel group $R$ acts on itself via left multiplication and via right multiplication; each action is simply transitive.

There is a normal subgroup $N$ in $\tilde{S}_3$ of index 2, it is given by all sequences of reflections of even length. Under the identification of $\tilde{S}_3$ with the triangles in the Tonnetz, the subgroup $N$ corresponds to the triangles of shape $\Delta$, which are the triangles in even distance from $(\ast)$ (see Corollary 2.3). As a subset of $\tilde{S}_3$, the group $N$ consists of the rotations and the translations.

For a reflection in $\tilde{S}_3$, say $s_1$, the conjugation by $s_1$ defines a group automorphism $\psi : N \to N, n \mapsto s_1ns_1^{-1}$. The group $\tilde{S}_3$ is isomorphic to the semi-direct product $\mathbb{Z}_2 \rtimes \psi N$.

Despite these structural similarities, the groups $R$, $\tilde{S}_3$ are not isomorphic.

6. The point reflection group

To shed light on the interplay between Riemannian theory on the infinite Tonnetz and neo-Riemannian theory on the finite Tonnetz, we exhibit a third group (besides $\tilde{S}_3$ and $R$), the point reflection group $P$. It has three main features:

- The group $P$ is generated by three reflections (that is, elements of order 2).
• The group $P$ is naturally isomorphic to the opposite group of $R$.
• There is a normal subgroup $K$ in $P$ with factor the dihedral group $D_{12}$.

Recall that $D_{12}$ is the group of 24 elements, generated by the PLR-moves on the finite Tonnetz, see [6, Chapter 5]. The point reflection group $P$ is the subgroup of the group of Euclidean plane isometries generated by the $180^\circ$ rotations $\pi_1, \pi_2, \pi_3$, where each $\pi_i$ is the point reflection about the midpoint of the edge of (*) on the $s_i$-axis, see Figure 10.

\[ \begin{matrix}
C^*_2 & \cdots & G^*_2 \\
\downarrow & \cdots & \downarrow \\
\pi_2 \pi_1 & \cdots & \pi_3 \pi_1 \\
\pi_2 & \cdots & \pi_3 \\
\pi_2 \pi_3 & \cdots & (\ast) \\
\pi_1 \pi_3 & \cdots & \pi_1 \pi_2 \\
A_b & \cdots & E_b & \cdots & B_b' \\
\end{matrix} \]

**Figure 10. Point reflections on the Tonnetz**

The action on the Tonnetz given by left multiplication by point reflections is pictured in Figure 11. By comparison, Figure 5 in Section 2 shows the action by affine permutations.

\[ \begin{matrix}
& & s_2 & & \\
& \downarrow & \cdots & \downarrow & & \\
& & \ast & & \\
& \downarrow & \cdots & \downarrow & & \\
& & s_1 & & \\
& \downarrow & \cdots & \downarrow & & \\
& & s_3 & & \\
\end{matrix} \]

**Figure 11. Left multiplication by point reflections**

The product of two point reflections is the translation by twice the difference between the centers, so for example, $\pi_3 \pi_2$ is the shift by one
unit to the right in parallel to the $s_1$-axis. Hence the collection of all products of an even number of point reflections forms the group of translations $T$.

Each remaining element in $P$ is a product of a point reflection and a translation, hence a point reflection itself, about a vertex or about the midpoint of a triangle edge in the Tonnetz. Each such element has order 2.

**Proposition 6.1.** The point reflection group $P$ is in a natural way isomorphic to the opposite group of Riemann’s Schritte-Wechsel group $R$.

**Proof.** Identify the elements of $P$ with the triads in the Tonnetz, as indicated in Figure 11. Then right multiplication by $\pi_3\pi_2$ is the Quintschritt $RL$, by $\pi_3\pi_1$ the Terzschritt $PL$, and by $\pi_1$ the Seitenwechsel $P$, see Figure 10. Quintschritt, Terzschritt and Seitenwechsel generate the group $R$, and the elements $\pi_3\pi_2, \pi_3\pi_1$ and $\pi_1$ generate $P$. Hence the left action of $R$ on the Tonnetz coincides with the right action of $P$. Both actions are simply transitive; it follows that the groups $R^\text{op}$ and $P$ are isomorphic. □ □

It seems to be well known that Riemann’s Schritt-Wechsel group $R$ has the Comma-Schritte subgroup as a normal subgroup such that the Schritt-Wechsel group of the finite Tonnetz, which is isomorphic to the dihedral group $D_{12}$, is a factor. The corresponding result for the (isomorphic) point reflection group can be obtained directly.

**Definition:** The **comma subgroup** $K$ is the subgroup of the point reflection group $P$ generated by $(\pi_3\pi_1)^3$ and $(\pi_1\pi_2)^4$.

One can see that $K$ is the smallest subgroup of $P$ which contains the “Lesser-Diesis-Schritt” $(\pi_3\pi_1)^3$, the “Greater-Diesis-Schritt” $(\pi_1\pi_2)^4$, the “Syntonic-Comma-Schritt” $(\pi_3\pi_2)^3\pi_1\pi_2$, and the “Pythagorean-Comma-Schritt” $(\pi_3\pi_2)^{12}$.

**Proposition 6.2.** The comma subgroup $K$ is a normal subgroup of $P$ of index 24. The factor group $P/K$ is isomorphic to the dihedral group $D_{12}$.

**Proof.** Using the formula from the proof of Proposition 6.1, we obtain $\pi_i(\pi_3\pi_1)^3\pi_i^{-1} = (\pi_1\pi_3)^3 = ((\pi_3\pi_1)^3)^{-1} \in K$ and similarly, $\pi_i(\pi_1\pi_2)^4\pi_i^{-1} \in K$, so $K$ is normal in $P$. The map $\psi : \mathbb{Z} \times \mathbb{Z} \to P, (a, b) \mapsto (\pi_3\pi_1)^a(\pi_1\pi_2)^b$ is one-to-one and has image $T$. The subgroup generated by $(3, 0)$ and $(0, 4)$ of $\mathbb{Z} \times \mathbb{Z}$ corresponds to $K$ under $\psi$. Hence $K$ has index 12 in $T$ and index 24 in $P$. □
Let \( h = (\pi_1 \pi_2)^{-1}(\pi_3 \pi_1) \) (the semitone) and \( \rho = \pi_1 \). Then \( hK \) has order 12 in \( P/K \), \( \rho K \) has order 2, and \( (\rho K)(hK)(\rho K)^{-1} = (hK)^{-1} \). Thus, \( hK \) and \( \rho K \) generate a subgroup in \( P/K \) isomorphic to \( D_{12} \). □ □

7. Conclusion

In Riemannian theory, the vertices in the infinite Tonnetz are labeled by notes such that the fifth marks the horizontal direction. All operations on the Tonnetz preserve the horizontal direction: The Schritte are translations, and the Wechsel are products of a translation and a flip on the horizontal axis.

Neo-Riemannian theory purifies the Tonnetz by removing the labels attached to the vertices, and by identifying the triangles with chords. This allows to redefine the operations in terms of more basic reflections, which in turn give rise to new moves, in particular to rotations.

The group which defines the operations in Riemannian theory is the semi-direct product \( R = \mathbb{Z}_2 \rtimes \phi \) of the cyclic group of two elements by the group of affine translations in the plane.

By comparison, three reflections corresponding to PLR-moves generate the Coxeter group \( \tilde{S}_3 \) which acts on the infinite Tonnetz. Like \( R \), the group \( \tilde{S}_3 \) contains a subgroup, say \( \tilde{T} \), of translations; actually \( \tilde{T} \) is isomorphic to \( T \). Unlike \( R \), the translation subgroup has index six, so there are many more elements in \( \tilde{S}_3 \): reflections, \( 120^\circ \)-rotations and glide reflections.

What has changed since Hugo Riemann introduced Schritte and Wechsel? We still visualize music in the Tonnetz... We still use algebra to describe the development of harmony... Yet, the building blocks are more fundamental and the operations have more variety. Riemannian theory is very much alive.

Happy 170th Birthday, Hugo Riemann!

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