Baryon Hyperfine Mass Splittings in Large N QCD

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Abstract

The hyperfine mass splittings of baryons in large $N$ QCD are proved to be proportional to $J^2$. Hyperfine mass splittings are first allowed at order $1/N$ in the $1/N$ expansion.
Consistency requirements of the large $N$ limit of QCD have recently led to a new quantitative understanding of the interactions of baryons with pions \[1\][2][3]. The starting point of this work is the realization that many physical quantities diverge with $N$ for arbitrary baryon-pion couplings. Correct large $N$ behavior requires that the leading order in $N$ contributions to these quantities cancel exactly. The condition of exact cancellation implies relations between baryon-pion couplings at leading order in $N$. These large $N$ consistency conditions are extremely predictive; all baryon-pion couplings are determined by a single coupling constant in large $N$ \[1\][2]. The couplings obtained in large $N$ QCD are identical to the couplings of the large $N$ Skyrme and non-relativistic quark models \[1\][4][5]. Furthermore, the large $N$ baryon-pion couplings respect light quark spin-flavor symmetry relations. The $1/N$ correction to these symmetry relations vanishes \[3\], so that violation of the symmetry relations first occurs at order $1/N^2$. Constraints on other parameters of the baryon-pion chiral Lagrangian can also be obtained in large $N$ \[3\]. These parameters also satisfy light quark spin-flavor symmetry relations at leading order in the $1/N$ expansion. The emergence of an effective light quark spin-flavor symmetry for baryon couplings in large $N$ explains the phenomenological success of these symmetry relations for baryon couplings.

The purpose of this paper is to study the hyperfine mass splittings of baryons in large $N$ QCD. In large $N$, the spectrum of baryon states for $N_f = 2$ light flavors consists of a degenerate tower of isospin and angular momentum multiplets $(I, J)$ with $I = J$. For $N$ odd, the tower of baryon states is $(1/2, 1/2), (3/2, 3/2), (5/2, 5/2), ..., (N/2, N/2)$. A consistent large $N$ expansion for baryons requires that the baryon multiplets are degenerate up to mass splittings of order $1/N$ \[1\][3]. In this work, it is shown that hyperfine mass splittings amongst multiplets in a degenerate tower of baryon states must satisfy large $N$ consistency conditions. These consistency conditions determine all the hyperfine mass splittings in terms of the mass splitting of the two lowest spin states. This unique solution of the consistency conditions yields hyperfine mass splittings which are identical to those produced by the operator $J^2$. Thus, one concludes that to first non-vanishing order in the $1/N$ expansion, the hyperfine mass splittings are generated by $J^2$. Further, such a mass splitting is first allowed at order $1/N$. These large $N$ results are reminiscent of the Skyrme model \[7\] where hyperfine mass splittings are given by $J^2/2I$, and the moment of inertia $I$ of the baryon states is $O(N)$.
The above results can be generalized to baryons which contain a single heavy quark. In the limit $m_Q \to \infty$, the low-energy strong interactions of heavy quark baryons are independent of the heavy quark mass, flavor and spin [8]. Thus, low-energy pion interactions of heavy quark baryons can be analyzed in terms of the flavor and angular momentum quantum numbers of the light degrees of freedom of the baryons in the heavy quark symmetry limit. For large $N$, $N$ odd, and for $N_f = 2$ light flavors, the light degrees of freedom of baryons containing a single heavy quark consists of the degenerate tower of $(I, J)$ states

$$(0, 0), (1, 1), (2, 2), \ldots, ((N - 1)/2, (N - 1)/2),$$

where $J$ is the angular momentum of the light degrees of freedom. The hyperfine mass splittings of these states are generated by the operator $J^2$ in large $N$. The mass splittings are first allowed at order $1/N$ in the $1/N$ expansion. In order to obtain the hyperfine mass splittings of the heavy baryon states, heavy baryon states must be constructed from the spin of the heavy quark $S_Q = \frac{1}{2}$ and the tower of states $(I, J)$ for the light degrees of freedom. The $(0, 0)$ state in the tower corresponds to the spin-$\frac{1}{2}$ $\Lambda_Q$ baryon, with the spin of the $\Lambda_Q$ determined by the spin of the heavy quark. All other $(I, J)$ states in the tower correspond to a degenerate doublet of heavy baryon multiplets with isospin $I$ and total spin equal to $I \pm \frac{1}{2}$, since $J = I$ for the given tower of states. Thus, for example, the $(1, 1)$ state corresponds to the spin-$\frac{1}{2}$ $\Sigma_Q$ and the spin-$\frac{3}{2}$ $\Sigma_Q^*$. In the $m_Q \to \infty$ limit, the hyperfine mass splittings of the heavy quark baryons are the same as the hyperfine mass splittings of the light degrees of freedom†. At order $1/N$ and order $1/m_Q$ in the heavy quark mass expansion, the doublet of heavy baryon multiplets for each $(I, J)$ state in the tower is no longer degenerate. For instance, a $(\Sigma_Q^* - \Sigma_Q)$ mass splitting is generated at order $1/(N m_Q)$. The factor of $1/N$ for this splitting is required since only mass splittings which are suppressed by $1/N$ are consistent with the large $N$ limit [1]. Thus, the hyperfine mass splittings for heavy quark baryons contain contributions of order $1/N$ which preserve heavy quark spin symmetry and contributions of order $1/(N m_Q)$ which violate heavy quark spin symmetry. The purely $1/N$ splittings are related by the large $N$ consistency conditions described in this work. The $1/(N m_Q)$ splittings are related by large $N$ consistency conditions derived in Ref. [4]. Ref. [4] proves that these splittings are proportional to the operator $J \cdot S_Q$. These results agree with recent calculations performed in the Skyrme model [10][11].

† Note that the heavy quark baryon hyperfine mass splittings are proportional to $J^2 = I^2$, where $J = I$ is the angular momentum of the light degrees of freedom, not the total spin of the heavy quark baryon.
quark arise in the Skyrme picture as heavy quark meson-soliton bound states \[12\], where the solitons are the ordinary baryons containing no heavy quark of the Skyrme model. This work shows that these successes of the Skyrme model actually are consequences of large $N$ QCD.

The results of this paper are obtained by studying the renormalization of baryon masses due to pion loop corrections. The pion loop correction can be calculated in chiral perturbation theory. The correction to a baryon hyperfine splitting is of the form

\[
\Delta M \to \Delta M + \alpha \frac{m^2_{\pi}}{16\pi^2 f_{\pi}^2} \ln \left( \frac{m_{\pi}^2}{\mu^2} \right) + \beta(\mu),
\]

where $m_{\pi}$ is the pion mass, $f_{\pi}$ is the pion decay constant, and $\mu$ is a renormalization group subtraction point. The $\mu$-dependence of the chiral logarithmic correction is exactly compensated for by the $\mu$-dependence of the counterterm $\beta(\mu)$ so that the right-hand side of Eq. (1) is independent of $\mu$. The coefficient $\alpha$, which is proportional to baryon mass splittings, is calculable in chiral perturbation theory. Consistency of the large $N$ expansion requires that this coefficient be the same order or higher order in the $1/N$ expansion as the leading contribution $\Delta M$.

An explicit formula for the coefficient $\alpha$ can be obtained by studying the chiral logarithmic correction to an individual baryon mass $M_i$. The chiral logarithmic correction to a baryon mass $M_i$ is equal to all one-loop diagrams with a single mass insertion $M_j$ (summed over intermediate states $j$) minus wavefunction renormalization $Z_i$ times the tree-level mass $M_i$, as depicted in fig. 1. Because the emission of a pion can only change the isospin or spin of the initial baryon by one unit, the allowed states $j$ are the $(i-1)$, $i$, and $(i+1)$ states of the baryon tower. Thus, the chiral logarithmic correction of a baryon mass $M_i$ depends on the masses $M_{i-1}$, $M_i$, and $M_{i+1}$. The diagrams displayed in fig. 1 can be reduced to a simpler set of graphs by noting that a one-loop diagram with a mass insertion can be rewritten in terms of the diagram without a mass insertion times the mass of the intermediate baryon, see fig. 2. This graphical identity is easily verified using the Feynman rules for baryon chiral perturbation theory in which the baryon is treated as a heavy static fermion \[13\][14][15]. The chiral correction to a baryon mass difference can be obtained by taking the difference of the chiral logarithmic corrections to each mass. It is sufficient to consider hyperfine mass splittings of neighboring multiplets in the baryon tower. The chiral correction to the mass difference ($M_{i+1} - M_i$) is given by the diagrams shown in fig. 3, where $(i-1)$, $i$, $(i+1)$, and $(i+2)$ are sequential multiplets in the baryon tower, with $(i-1)$ being the smallest dimensional spin state.
in the sequence. The chiral correction to \((M_{i+1} - M_i)\) depends on the mass differences \((M_i - M_{i-1})\), \((M_{i+1} - M_i)\), and \((M_{i+2} - M_{i+1})\). The kinematic factors of the one-loop graphs produce the chiral logarithm in Eq. (1) and are all identical. Thus, the coefficient \(\alpha\) is proportional to a sum of mass differences times Clebsch-Gordan factors. An explicit expression for this linear combination will be derived below. Note that fig. 3 assumes that the baryon mass difference being renormalized is not the mass difference of the two lowest spin states, so that the state \((i - 1)\) exists. The chiral logarithmic correction to the first mass splitting is given by the truncated version of fig. 3 shown in fig. 4. In QCD with \(N = 3\), where the baryon tower consists of only two multiplets, the chiral logarithmic correction to the single baryon hyperfine mass splitting \((M_2 - M_1)\) reduces to the term proportional to \((M_2 - M_1)\) in fig. 4 since there is no third multiplet in the tower. This is the renormalization equation found for the \(\Delta - N\) mass difference in Ref. [14].

The large \(N\) power counting for the one-loop graph given in fig. 3 and fig. 4 is determined by the large \(N\) behavior of baryon-pion couplings. A general baryon-pion coupling is of the form

\[
B_2 G^{ai} B_1 \frac{\partial^a \pi^i}{f_\pi},
\]

where \(a = 1, 2, 3\) labels the angular momentum channel of the \(p\)-wave pion, \(i = 1, 2, 3\) labels the isospin of the pion, and \(G^{ai}\) is an operator with unit spin and isospin. Refs. [1] and [2] prove that the pion couplings amongst baryon multiplets in the degenerate tower can be parametrized by a single coupling constant \(g\),

\[
\langle I_2 I_{2z}, J_2 J_{2z} | G^{ai} | I_1 I_{1z}, J_1 J_{1z} \rangle = N g \sqrt{\frac{2J_1 + 1}{2J_2 + 1}} \left( \begin{array}{c} I_1 \\ I_{1z} \end{array} \right) \left( \begin{array}{c} I_2 \\ I_{2z} \end{array} \right) \left( \begin{array}{c} J_1 \\ J_{1z} \end{array} \right) \left( \begin{array}{c} J_2 \\ J_{2z} \end{array} \right),
\]

where \(I_1 = J_1\) and \(I_2 = J_2\) for the assumed tower of states and \(g\) is \(O(1)\). An explicit factor of \(N\) has been factored out of the matrix elements to keep all \(N\) dependence manifest. Note that since \(f_\pi \sim \sqrt{N}\), each baryon-pion coupling Eq. (2) produces a net factor of \(\sqrt{N}\). Thus the chiral logarithmic correction to a baryon mass difference grows with one additional power of \(N\) than a baryon mass difference. Consistency of large \(N\) power counting therefore requires that the linear combinations of mass differences given in fig. 3 and fig. 4 vanish identically at leading order in \(N\). These constraints are enough to determine all the baryon hyperfine mass splittings in terms of one splitting.

Explicit formulae for the large \(N\) consistency conditions can be obtained by evaluating the Clebsch-Gordan factors of the one-loop graphs in fig. 3 and fig. 4. The Clebsch-Gordan
factor for the one-loop diagram with initial baryon \((I_1, J_1)\) and intermediate baryon \((I_2, J_2)\) is given by

\[
\sum_{I_{2z}, J_{2z}, i, a} (-1)^i (-1)^a \left( \begin{array}{c c c}
I_1 & 1 \\
I_{1z} & I_{2z}
\end{array} \right) \left( \begin{array}{c c c}
J_1 & 1 \\
J_{1z} & J_{2z}
\end{array} \right) \left( \begin{array}{c c c}
I_2 & 1 \\
I_{2z} & -i
\end{array} \right) \left( \begin{array}{c c c}
J_2 & 1 \\
J_{2z} & -a
\end{array} \right) = \frac{(2J_2 + 1)}{(2J_1 + 1)},
\]

where the factors of \((-1)^i\) and \((-1)^a\) are required since \((\pi^\ast)^\ast = -\pi^\ast\) in the Condon-Shortley phase convention. Thus, the constraint that fig. 3 vanishes at leading order in \(N\) is

\[
(M_i - M_{i-1}) \frac{(2J_{i-1} + 1)}{(2J_i + 1)} - (M_{i+1} - M_i) \left[ \frac{(2J_i + 1)}{(2J_{i+1} + 1)} + \frac{(2J_{i+1} + 1)}{(2J_i + 1)} \right] + (M_{i+2} - M_{i+1}) \frac{(2J_{i+2} + 1)}{(2J_{i+1} + 1)} = 0,
\]

where

\[
J_{i+1} = J_i + 1
\]

for the tower of baryon states. The consistency condition for the hyperfine mass splitting of the two smallest dimensional spin multiplets is given by

\[
-(M_2 - M_1) \left[ \frac{(2J_1 + 1)}{(2J_2 + 1)} + \frac{(2J_2 + 1)}{(2J_1 + 1)} \right] + (M_3 - M_2) \frac{(2J_3 + 1)}{(2J_2 + 1)} = 0.
\]

There is a unique solution to these recursion relations. Given a hyperfine mass splitting \((M_2 - M_1)\), Eq. (7) determines the next consecutive mass splitting \((M_3 - M_2)\). Given \((M_2 - M_1)\) and \((M_3 - M_2)\), Eq. (3) then determines the mass splitting \((M_4 - M_3)\). All remaining mass splittings are determined recursively using Eq. (5).

The unique solution to the recursion equations (5) and (7) produces hyperfine mass splittings with the same ratios as the operator \(J^2\). The proof of this assertion is as follows. Consider the initial recursion relation Eq. (3). This equation fixes the ratio of the first two sequential hyperfine mass splittings of the baryon tower,

\[
\frac{(M_3 - M_2)}{(M_2 - M_1)} = \frac{(2J_1 + 1)(2J_1 + 1) + (2J_2 + 1)(2J_2 + 1)}{(2J_1 + 1)(2J_3 + 1)}.
\]

This result is to be compared with the ratio produced by a mass contribution proportional to \(J^2\),

\[
\frac{(M_3 - M_2)}{(M_2 - M_1)} = \frac{J_3(J_3 + 1) - J_2(J_2 + 1)}{J_2(J_2 + 1) - J_1(J_1 + 1)} = \frac{J_3}{J_2}.
\]
since $J_3 = J_2 + 1$ and $J_2 = J_1 + 1$. These two expressions are not equivalent for general $J_1$. However, for the two special cases of interest, baryon towers starting with $J_1 = \frac{1}{2}$ or $J_1 = 0$, the two expressions are equal. For these towers, the initial ratios are

$$(M_3 - M_2) = \frac{5}{3} (M_2 - M_1)$$

(10)

for the $J_1 = \frac{1}{2}$ tower, and

$$(M_3 - M_2) = 2 (M_2 - M_1)$$

(11)

for the $J_1 = 0$ tower. It remains to be shown that given that the initial ratio equals $J_3/J_2$, all subsequent ratios satisfy

$$\frac{(M_{i+2} - M_{i+1})}{(M_{i+1} - M_i)} = \frac{J_{i+2}}{J_{i+1}}.$$  

(12)

The recursion relation Eq. (5) gives

$$\frac{(M_{i+2} - M_{i+1})}{(M_{i+1} - M_i)} = \frac{(2J_{i+1} + 1)}{(2J_{i+2} + 1)} \times \left\{ \frac{(2J_{i+1} + 1)}{(2J_{i+1} + 1)} + \frac{(2J_{i+1} + 1)}{(2J_{i+1} + 1)} - \frac{(M_i - M_{i-1})}{(M_{i+1} - M_i)} \frac{(2J_{i-1} + 1)}{(2J_{i+1} + 1)} \right\}. \tag{13}$$

Assuming that the ratio

$$\frac{(M_{i+1} - M_i)}{(M_i - M_{i-1})} = \frac{J_{i+1}}{J_i}, \tag{14}$$

Eq. (13) then implies that Eq. (12) is satisfied. This result is most easily seen by making the substitution $J_i = \frac{n}{2}$. Then the sought result is equivalent to the identity

$$\frac{(n + 3)}{(n + 5)} \left\{ \frac{(n + 3)}{(n + 1)} + \frac{(n + 1)}{(n + 3)} \right\} - \frac{n}{(n + 2)} \frac{(n - 1)}{(n + 1)} = \frac{(n + 4)}{(n + 2)}, \tag{15}$$

which is true for arbitrary $n$.

In summary, this work shows that there is a unique solution of large $N$ consistency conditions following from pion loop renormalization of baryon hyperfine mass splittings. This solution relates baryon hyperfine mass splittings at leading order in a $1/N$ expansion. The solution produces baryon hyperfine mass splittings proportional to $J^2$. Previous work [1][3] has proven that mass splittings are first allowed at order $1/N$. 

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Figure Captions

Fig. 1. The chiral logarithmic correction to a baryon mass $M_i$. All baryon states $j$ which are accessible by single pion exchange occur as intermediate states in the one-loop mass and wavefunction renormalization diagrams. Square vertices denote baryon mass insertions.

Fig. 2. The one-loop mass insertion diagram with external state $i$ and intermediate state $j$ is equal to the mass of the intermediate baryon $M_j$ times the one-loop diagram with intermediate state $j$ and no mass insertion.

Fig. 3. The chiral logarithmic correction to the baryon mass difference ($M_{i+1} - M_i$). The baryon states $(i-1)$, $i$, $(i+1)$, and $(i+2)$ are sequential multiplets in the baryon tower, with $(i-1)$ being the smallest dimensional spin state of the sequence. The chiral correction to $(M_{i+1} - M_i)$ depends on the mass differences $(M_i - M_{i-1})$, $(M_{i+1} - M_i)$, and $(M_{i+2} - M_{i+1})$. Consistency of the large $N$ limit requires that this linear combination of mass differences vanishes at leading order in $N$.

Fig. 4. The chiral logarithmic correction to the $(M_2 - M_1)$ mass difference. The correction involves only two mass differences $(M_2 - M_1)$ and $(M_3 - M_2)$ since the first state is the smallest spin state in the baryon tower.
Figure 1
Figure 2
(M_i - M_{i-1})

\[ M_{i+1} - M_i \]

+ (M_{i+2} - M_{i+1})
\[ - (M_2 - M_1) \begin{bmatrix} 1 \quad 2 \quad 1 \end{bmatrix} + \begin{bmatrix} 2 \quad 1 \quad 2 \end{bmatrix} \]

\[ + (M_3 - M_2) \begin{bmatrix} 2 \quad 3 \quad 2 \end{bmatrix} \]

Figure 4