THE CHOW OF $S^{[n]}$ AND THE UNIVERSAL SUBSCHEME

ANDREI NEGUT

Abstract. We prove that any element in the Chow ring of the Hilbert scheme $\text{Hilb}_n$ of $n$ points on a smooth surface $S$ is a universal class, i.e. the push-forward of a polynomial in the Chern classes of the universal subscheme on $\text{Hilb}_n \times S^k$ for some $k \in \mathbb{N}$, with coefficients pulled back from the Chow of $S^k$.

1. Introduction

We study the Hilbert scheme of $n$ points $\text{Hilb}_n = S^{[n]}$ on a smooth algebraic surface over $\mathbb{C}$. As $\text{Hilb}_n$ is smooth, we may consider the Chow rings $A^*(\text{Hilb}_n)$, always with coefficients in $\mathbb{Q}$ throughout the present paper. One of the big sources of elements of $A^*(\text{Hilb}_n)$ are universal classes, see Definition 2.2. During a conversation on Hilbert schemes, Alina Marian suggested that all elements of $A^*(\text{Hilb}_n)$ should be universal, and the purpose of the present note is to prove it.

Theorem 1.1. Any element of $A^*(\text{Hilb}_n)$ is a universal class.

When $S$ is projective, this result follows (see [1]) from an explicit formula for the diagonal of $\text{Hilb}_n$ as a Chern class of the so-called Ext virtual bundle, which in turn is an exterior product of universal classes (see [4] for $S = \mathbb{P}^2$, [9] for $S$ with trivial canonical class, and [5] for the general case). Our proof is quite different, and holds for quasi-projective $S$ as well. We start from [2], which states that:

$$A^*(\text{Hilb}_n) \cong \bigoplus_{\Gamma \in A^*(S^t)_{\text{sym}}} C_{k_1, \ldots, k_t}(\Gamma)$$

(1.1)

where $C_{k_1, \ldots, k_t}(\Gamma)$ are certain correspondences (see [22] for an explicit description, as well as an explanation of the superscript “sym”) expressed in terms of the Heisenberg operators $q_k$ of [6] and [11]. The explicit description of these operators in terms of l.c.i. morphisms from [13] allows us to show that they preserve the subrings of $A^*(\text{Hilb}_n)$ consisting of universal classes, thus implying Theorem 1.1. Moreover, this gives an algorithm for computing the universal classes corresponding to the various summands in (1.1), as we will explain on the last two pages.

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2. Hilbert schemes

2.1. Let $S$ be a smooth algebraic surface over $\mathbb{C}$. Let $\text{Hilb}_n = S^{[n]}$ denote the Hilbert scheme which parametrizes length $n$ subschemes of $S$, i.e. exact sequences:

$$0 \to I \to \mathcal{O}_S \to Z \to 0$$

($I$ will be an ideal sheaf) where $\text{length}(Z) = n$. There exists a universal subscheme:

$$Z_n \subset \text{Hilb}_n \times S$$

whose restriction over any $\{ Z \} \times S$ is supported on the locus $\nu, \mu$.

Proposition 2.4. The set:

$$\text{universal class}$$

is a short exact sequence of coherent sheaves on $\text{Hilb}_n$. For any $n, k \in \mathbb{N}$, we let $\pi, \rho : \text{Hilb}_n \times S^k \to \text{Hilb}_n, S^k$ denote the standard projections, and let $Z_n^{(i)} \subset \text{Hilb} \times S^k$ be the pull-back of $Z_n \subset \text{Hilb} \times S$ via the $i$-th projection.

Definition 2.2. A universal class is any $\mathbb{Q}$-linear combination of:

$$\pi_* \left[ \prod_{i=1}^{k} a_i \prod_{j=1}^{a_i} \text{ch}_{d_{ij}}(O_{Z_n^{(i)}}) \cdot \rho^* (\Gamma) \right] \in A^*(\text{Hilb}_n)$$

$$\forall k, d_1, ..., d_k \in \mathbb{N} \text{ and all } \Gamma \in A^*(S^k).$$

Since the Chern classes of $O_{Z_n}$ are supported on $Z_n$, the product of Chern characters in (2.2) is supported on the locus $Z_n^{(1)} \cap ... \cap Z_n^{(k)}$, which is proper over $\text{Hilb}_n$.

Remark 2.3. Note that we could equivalently have put any polynomial in the Chern characters of $O_{Z_n^{(i)}}$ in formula (2.2), or in other words, we could have had products of several copies of each $O_{Z_n^{(i)}}$ appearing. This is a consequence of the identity:

$$\pi_* \left[ \prod_{i=1}^{k} a_i \prod_{j=1}^{a_i} \text{ch}_{d_{ij}}(O_{Z_n^{(i)}}) \cdot \rho^* (\Gamma) \right] = \nu_* \left[ \prod_{i=1}^{k} a_i \prod_{j=1}^{a_i} \text{ch}_{d_{ij}}(O_{Z_n^{(a_1+...+a_{i-1}+j)}}) \cdot \mu^* (\Delta, (\Gamma)) \right]$$

where $\nu, \mu : \text{Hilb}_n \times S^{a_1+...+a_k} \to \text{Hilb}_n, S^{a_1+...+a_k}$ are the standard projections, and $\Delta : S^k \to S^{a_1+...+a_k}$ is the diagonal $\{ x_{a_1+...+a_{i-1}+1} = ... = x_{a_1+...+a_i}, \forall 1 \leq i \leq k \}$.

Proposition 2.4. The set:

$$A^*_{\text{univ}}(\text{Hilb}_n) \subseteq A^*(\text{Hilb}_n)$$

of all universal classes is a subring.

Proof. The claim is a consequence of the following identity:

$$\pi_* \left[ \text{ch}_{d_i}(O_{Z_n^{(i)}}) \cdot \text{ch}_{d_k}(O_{Z_n^{(k)}}) \cdot \rho^* (\Gamma) \right] \cdot \pi'_* \left[ \text{ch}_{d'_i}(O_{Z_n^{(i)}}) \cdot \text{ch}_{d'_k}(O_{Z_n^{(k)}}) \cdot \rho'^* (\Gamma) \right] =$$

$$= \nu_* \left[ \text{ch}_{d_i}(O_{Z_n^{(i)}}) \cdot \text{ch}_{d_k}(O_{Z_n^{(k)}}) \cdot \mu^* (\Gamma) \right] \cdot \text{ch}_{d'_i}(O_{Z_n^{(i)}}) \cdot \text{ch}_{d'_k}(O_{Z_n^{(k)}}) \cdot \mu'^* (\Gamma) \right]$$

The notion above is more general than either the small or big tautological classes considered in [10]; the reason is the lack in general of a Künneth decomposition: $A^*(S^k) \nmid A^*(S)^{\otimes k}$.
which is due to base change in the following diagram (even though the maps below are not proper, we will only apply push-forwards to classes with proper support).

\[
\begin{array}{ccc}
\text{Hilb}_n \times S^k & \xrightarrow{\mu} & \text{Hilb}_n \times S^k' \\
\downarrow & & \downarrow \\
\text{Hilb}_n & \xrightarrow{\pi} & \text{Hilb}_n
\end{array}
\]

\[
\begin{array}{ccc}
\text{Hilb}_n \times S^k \times S^k' & \xrightarrow{\delta} & S^k' \\
\downarrow & & \downarrow \\
\text{Hilb}_n \times S^k & \xrightarrow{\pi'} & S^k
\end{array}
\]

2.5. We will prove Theorem 1.1 by deducing it from another well-known description of \(A^*(\text{Hilb}_n)\): the de Cataldo–Migliorini decomposition \([2]\). To review this construction, we must recall the Heisenberg algebra action introduced independently by Grojnowski \([6]\) and Nakajima \([11]\) on the Chow groups of Hilbert schemes. For any \(n, k \in \mathbb{N}\), consider the closed subscheme:

\[
\text{Hilb}_{n-k,n} = \{ (I \supset I') \text{ s.t. } I/I' \text{ is supported at a single } x \in S \} \subset \text{Hilb}_{n-k} \times \text{Hilb}_n
\]

endowed with projection maps:

\[
\begin{array}{ccc}
\text{Hilb}_{n-k,n} & \xrightarrow{p_-} & \text{Hilb}_{n-k} \\
\downarrow & & \downarrow \\
S & \xrightarrow{p} & \text{Hilb}_n \\
\downarrow & & \downarrow \\
\text{Hilb}_n & \xrightarrow{p_+} & \text{Hilb}_n
\end{array}
\]

that remember \(I, x, I'\), respectively. One may use \(\text{Hilb}_{n-k,n}\) as a correspondence:

\[
\bigoplus_{n=0}^\infty A^*(\text{Hilb}_{n-k}) \xrightarrow{q_k} \bigoplus_{n=0}^\infty A^*(\text{Hilb}_n \times S)
\]

given by:

\[
q_k = (p_+ \times p_S)_* \circ p_-^*
\]

\footnote{The transposed correspondences give rise to operators \(q_{-k}\), which we will not study.}

The main result of \([6]\) and \([11]\) is that the operators \(q_k\) obey the commutation relations in the Heisenberg algebra. More generally, we may consider:

\[
\bigoplus_{n=0}^\infty A^*(\text{Hilb}_n) \xrightarrow{q_{k_1} \cdots q_{k_t}} \bigoplus_{n=0}^\infty A^*(\text{Hilb}_n \times S^t)
\]

where the convention is that the operator \(q_{k_i}\) acts in the \(i\)-th factor of \(S^t = S \times \ldots \times S\). Then associated to any \(\Gamma \in A^*(S^t)\), one obtains an endomorphism of \(\bigoplus_n A^*(\text{Hilb}_n)\):

\[
q_{k_1} \cdots q_{k_t}(\Gamma) = \pi_*(\rho^*(\Gamma) \cdot q_{k_1} \cdots q_{k_t})
\]
where \( \pi, \rho : \Hilb_n \times S^t \to \Hilb_n, S^t \) denote the standard projections (the non-properness of \( \pi \) is not a problem for defining (2.6), because the support of \( q_{k_1} \ldots q_{k_t} \) is proper over \( \Hilb_n \)). One of the main results of [2] is the following decomposition:

\[
A^*(\Hilb_n) = \bigoplus_{\Gamma \in A^*(S^t)}^{k_1 \geq \ldots \geq k_t} q_{k_1} \ldots q_{k_t}(\Gamma) \cdot A^*(\Hilb_0)
\]

where the superscript \( \text{sym} \) refers to the part of \( A^*(S^t) \) which is symmetric with respect to those transpositions \((ij) \in S_t \) for which \( k_i = k_j \). Since \( \Hilb_0 = \text{pt} \), we have \( A^*(\Hilb_0) \cong \mathbb{Q} \), and so Theorem 1.1 follows from (2.7) and the following:

**Proposition 2.6.** The endomorphisms (2.6) preserve the subrings \( A^*_{\text{univ}}(\Hilb_n) \).

2.7. The remainder of our paper will be devoted to proving Proposition 2.6. The problem with proving it directly is that the correspondences (2.3) are rather singular. The exception to this is the case \( k = 1 \), namely:

\[
\Hilb_{n-1,n} \xymatrix{ & \Hilb_n \ar[dl]_{\rho^+} \ar[dr]^{\rho^-} & \\
S \ar[rr]^p & & I \ar[dl]_{x} \ar[dr]^{x} & (I \supset I') \\
& \Hilb_n \ar[dl]_{\rho^+} \ar[dr]^{\rho^-} & \\
& \Hilb_{n-1,n} \ar[dl]_{\rho} 
}
\]

Above and hereafter, we write \( I \supset I' \) if \( I \supset I' \) and \( I/I' \cong \mathbb{C}_x \). It is well-known that \( \Hilb_{n-1,n} \) is smooth of dimension \( 2n \). Consider the line bundle:

\[
\mathcal{L} \mid_{(I \supset I')} = I/I'
\]

If \( \mathcal{E} = [W \to V] \) is a complex of locally free sheaves on a scheme \( X \), then we define:

\[
P_X(\mathcal{E}) \leftrightarrow \mathbb{P}_X(\mathcal{V})
\]

to be the zero locus of the map:

\[
\rho^*(\mathcal{W}) \to \rho^*(\mathcal{V}) \to \mathcal{O}(1)
\]

where \( \rho : \mathbb{P}_X(\mathcal{V}) \to X \) is the standard projection. In all cases considered in the present paper, the closed subscheme (2.4) is a local complete intersection, cut out by the cosection (2.11). The following result is closely related to Lemma 1.1 of [3]:

**Proposition 2.8.** Let \( \mathcal{L}_n \) be the universal ideal sheaf on \( \Hilb_n \times S \), i.e. the kernel (2.1) of the map \( \mathcal{O}_{\Hilb_n} \to \mathcal{O}_{\mathbb{Z}_n} \). Then we have an isomorphism:

\[
\Hilb_{n-1,n} \xymatrix{ & \mathbb{P}_{\Hilb_n \times S}(-\mathcal{T}_n' \otimes \omega_S) \ar[dl]_{\rho} \\
& \Hilb_n \times S \ar[dl]_{\rho \times p_S} \ar[dr]^{p_+ \times p_S} & \\
& \Hilb_n \ar[dl]_{\rho} 
}
\]

The line bundle \( \mathcal{L} \) on \( \Hilb_{n-1,n} \) is isomorphic to \( \mathcal{O}(-1) \) on \( \mathbb{P}_{\Hilb_n \times S}(-\mathcal{T}_n' \otimes \omega_S) \).
We refer the reader to Section 4 of [12] for details on why (2.12) is a special case of (2.10). In a few words, there is a short exact sequence with $W, V$ locally free:

\[(2.13) 0 \to W \to V \to I_n \to 0\]

Then the notation $-I_n^\vee$ in (2.12) stands for the complex $[V^\vee \to W^\vee]$. Finally, $\omega_S$ denotes both the canonical line bundle on $S$ and its pull-back to $\text{Hilb}_n \times S$.

2.9. Let us consider the following more complicated cousin of the scheme $\text{Hilb}_{n-1,n}$:

\[(2.14) \text{Hilb}_{n-1,n,n+1} = \left\{ (I, I', I'') \text{ such that } I \supset x I' \supset x I'' \text{ for some } x \in S \right\} \]

where $I \in \text{Hilb}_{n-1}, I' \in \text{Hilb}_n$ and $I'' \in \text{Hilb}_{n+1}$. We have shown in [12] that $\text{Hilb}_{n-1,n,n+1}$ is smooth of dimension $2n+1$. Consider the line bundles:

\[(2.15) L, L' \downarrow \downarrow \text{Hilb}_{n-1,n,n+1} \]

\[L|_{(I \supset x I' \supset x I'')} = I'/I'', \quad L'|_{(I \supset x I' \supset x I'')} = I/I' \]

Consider also the proper maps which forget either $I''$ or $I'$:

\[(2.16) \text{Hilb}_{n-1,n,n+1} \xrightarrow{\pi_-} \text{Hilb}_{n-1,n} \xleftarrow{\pi_+} \text{Hilb}_{n,n+1} \]

\[\pi_- \quad \pi_+ \quad \pi_- \quad \pi_+ \quad \pi_- \quad \pi_+ \quad \pi_- \quad \pi_+ \]

\[(I \supset x I' \supset x I'') \]

\[\pi_- \quad \pi_+ \quad \pi_- \quad \pi_+ \quad \pi_- \quad \pi_+ \quad \pi_- \quad \pi_+ \]

\[\text{Hilb}_{n,n+1} \]

Let $\Gamma : \text{Hilb}_{n,n+1} \hookrightarrow \text{Hilb}_{n,n+1} \times S$ be the graph of the map $p_S$, and let $L$ be the line bundle on $\text{Hilb}_{n,n+1}$. We showed in [13] that there is an injective map:

\[L^{-1} \hookrightarrow \ker [\Gamma^*(V^\vee) \to \Gamma^*(W^\vee)] \]

with $W, V$ as in (2.13). We will use the notation $-\Gamma^*(I^\vee) + L^{-1}$ for the complex:

\[\Gamma^*(V^\vee)/L^{-1} \to \Gamma^*(W^\vee) \]

of coherent sheaves on $\text{Hilb}_{n,n+1}$. The following was proved in [13]:

**Proposition 2.10.** Let $\mathcal{I}_n$ be the universal ideal sheaf on $\text{Hilb}_{n,n+1} \times S$. Then:

\[(2.17) \text{Hilb}_{n-1,n,n+1} \xrightarrow{\sim} \mathbb{P}\text{Hilb}_{n,n+1}(-\Gamma^*(I_n^\vee) \otimes \omega_S + L^{-1} \otimes \omega_S) \]

The line bundle $L'$ on $\text{Hilb}_{n-1,n,n+1}$ is isomorphic to $O(-1)$ on the projectivization.

In both (2.12) and (2.17), we considered projectivization $\mathbb{P}(\ast)$ where $\ast$ is written as a $K$–theory class instead of a complex of sheaves. The reason for this is that we are only interested in $\ast$ inasmuch as it helps us compute push-forwards. In fact, the definition of Chern/Segre classes implies that we have, for all $k \geq 0$:

\[(2.18) (p_+ \times p_S)_\ast(c_1(L))^k = (-1)^k c_{k+2}(I_n \otimes \omega_S^{-1}) \]
\begin{equation}
\pi_*(c_1(L')^k) = (-1)^k c_{k+1} \left( \Gamma^*(-I_n) \otimes \omega_S^{-1} - L \otimes \omega_S^{-1} \right)
\end{equation}

2.11. Our reason for defining the smooth schemes $\text{Hilb}_{n-1,n,n+1}$ is that it allows us to produce a resolution of the singular scheme $\text{Hilb}_{n,n'}$, for any $n < n'$, in the following sense. Consider the following diagram of spaces and maps (113):

\[ \begin{array}{ccc}
\text{Hilb}_{n,n+1} & \xrightarrow{\iota_*} & \text{Hilb}_{n,n+1,n+2} \\
\pi_+ & & \pi_+ \\
\text{Hilb}_{n} & \xrightarrow{\pi_+} & \text{Hilb}_{n,n+1,n+2} \\
\pi_- & & \pi_- \\
\text{Hilb}_{n+1} & \xrightarrow{\pi_-} & \text{Hilb}_{n,n+1,n+2} \\
\end{array} \]

for all $n < n'$. Then we have the following formula (110):

\[ q_k = (p_+ \times p_S)^* \circ (\pi_+ \circ \pi_+)^{k-1} \circ p_* \]

Indeed, the right-hand side of (2.20) is a $2n + k + 1$ dimensional cycle $C$ supported on the $2n + k + 1$ dimensional locus $\text{Hilb}_{n,n+k}$. It is well-known that the latter locus has a single irreducible component of top dimension, namely the closure of the locus $U$ of pairs $(I \supset I')$ where $I/I'$ is isomorphic to a length $k$ subscheme of a curve supported at a single point. But in this case, there exists a unique full flag of ideals $I = I_0 \supset I_1 \supset \cdots \supset I_k = I'$, which implies that $C|_U \cong U$, hence (2.20) follows.

2.12. We henceforth let $\pi: \text{Hilb}_n \times S^k \to \text{Hilb}_n \times S^l$ be the map which forgets the factors labeled by $t+1, \ldots, k$ of $S^k$, for all fixed $t \geq 0$ and arbitrary $k \geq t$. Define:

\begin{align*}
(2.21) & \quad A^*_{univ}(\text{Hilb}_n \times S^t) \subset A^*(\text{Hilb}_n \times S^t) \\
(2.22) & \quad A^*_{univ}(\text{Hilb}_{n,n+1} \times S^t) \subset A^*(\text{Hilb}_{n,n+1} \times S^t) \\
(2.23) & \quad A^*_{univ}(\text{Hilb}_{n,n+1,n+2} \times S^t) \subset A^*(\text{Hilb}_{n,n+1,n+2} \times S^t)
\end{align*}

to be the subsets of $\mathbb{Q}$-linear combinations of, respectively:

\begin{align*}
(2.24) & \quad \pi_* \left[ \chi_{d_1}(O_{Z(n)}^{(1)}) \chi_{d_2}(O_{Z(n)}^{(b)}) \cdot \rho^*(\Gamma) \right] \\
(2.25) & \quad \pi_* \left[ \chi_{d_1}(O_{Z(n)}^{(1)}) \chi_{d_2}(O_{Z(n)}^{(b)}) \cdot \rho^*(\Gamma) \cdot c_1(L)^a \right] \\
(2.26) & \quad \pi_* \left[ \chi_{d_1}(O_{Z(n)}^{(1)}) \chi_{d_2}(O_{Z(n)}^{(b)}) \cdot \rho^*(\Gamma) \cdot c_1(L)^a c_1(L)^b \right]
\end{align*}

for all $a, b, d_1, \ldots, d_k \in \mathbb{N}$ and $\Gamma \in A^*(S^k)$, where $\rho: \text{Hilb}_n \times S^k \to S^k$ is the usual projection. By analogy with Proposition 2.21, the subsets (2.22), (2.23) are subrings. Then Proposition 2.16 is a consequence of (2.20) and the following:

**Proposition 2.13.** For any $t \geq 0$, the maps $(p_- \times \text{Id}_{S^t})^*$, $(\pi_- \times \text{Id}_{S^t})^*$, $(\pi_+ \times \text{Id}_{S^t})^*$, $(p_+ \times p_S \times \text{Id}_{S^t})^*$ preserve the universal subrings, as defined above.

Indeed, formula (2.20) and Proposition 2.13 imply that if $x \in A^*_{univ}(\text{Hilb}_n)$, then $y = q_{k_1} \ldots q_{k_t}(x) \in A^*_{univ}(\text{Hilb}_{n+k_1+\cdots+k_t} \times S^t)$, in the sense of (2.21). If we multiply $y$ by the pull-back of any $\Gamma \in A^*(S^t)$ and then push it forward to $\text{Hilb}_{n+k_1+\cdots+k_t}$,
it will remain in the subring of universal classes, and this establishes Proposition 2.6.

Proof. of Proposition 2.13. The statements about the pull-back maps \((p_+ \times \text{Id}_{S'})^*\) and \((\pi_+ \times \text{Id}_{S'})^*\) preserving the universal rings are obvious given the definitions of (2.22) and (2.23). Concerning the push-forward \((\pi_+ \times \text{Id}_{S'})_*\), we must show that:

\[
(2.27) \quad x \text{ as in (2.26)} \implies (\pi_+ \times \text{Id}_{S'})_*(x) \in A^*_\text{univ}(\text{Hilb}_{n+1,n+2} \times S')
\]

We have the following short exact sequence on \(\text{Hilb} \subset \text{Hilb}_{n,n+2}\) where \(\Delta \subset S \times S\) is the diagonal, and \(p_S : \text{Hilb}_{n+1,n+2} \to S\) is the map which remembers the point \(x\) in (2.14). Then we can replace formula (2.26) by:

\[
(2.28) \quad 0 \to \mathcal{L}' \otimes (p_S \times \text{Id}_{S'})^*(\mathcal{O}_\Delta) \to \mathcal{O}_{Z_{n+1}} \to \mathcal{O}_{Z_n} \to 0
\]

where \(\Delta \subset S \times S\) is the diagonal, and \(p_S : \text{Hilb}_{n+1,n+2} \to S\) is the map which remembers the point \(x\) in (2.14). Then we can replace formula (2.26) by:

\[
x = \pi_* \left[ \prod_{i=1}^{k} \left( \text{ch}_{d_i}(\mathcal{O}_{Z_{n+1}^{(i)}}) - \text{ch}_{d_i}(\mathcal{L}' \otimes \mathcal{O}_{\Delta_i}) \right) \cdot \rho^*(\Gamma') \right] \cdot c_1(\mathcal{L})^a c_1(\mathcal{L}')^b
\]

where \(\Delta_i \in \text{Hilb}_{n,n+2} \times S^k\) is the codimension 2 diagonal pulled-back via \(p_S\) and the \(i\)-th projection \(S^k \to S\). The element \(x\) above is a linear combination of:

\[
\pi_* \left[ \prod_{i \in A} \text{ch}_{d_i}(\mathcal{O}_{Z_{n+1}^{(i)}}) \cdot \Delta_i \cdot \rho^*(\Gamma') \right] \cdot c_1(\mathcal{L})^a c_1(\mathcal{L}')^b
\]

for various \(b' \in \mathbb{N}, \Gamma' \in A^*(S^k)\), and partitions \(\{1, \ldots, k\} = A \sqcup B\). Since the product of diagonals corresponds to a map \(S^{k-\#B} \hookrightarrow S^k\), then \(x\) is a linear combination of:

\[
\tilde{\pi}_* \left[ \prod_{i=1}^{k-\#B} \text{ch}_{d_i}(\mathcal{O}_{Z_{n+1}^{(i)}}) \cdot \rho^*(\tilde{\Gamma}') \right] \cdot c_1(\mathcal{L})^a \cdot (\pi_+ \times \text{Id}_{S'})_*(c_1(\mathcal{L}')^{b'})
\]

where \(\tilde{\pi}, \tilde{\rho}\) are defined just like \(\pi, \rho\), but using \(S^{k-\#B}\) instead of \(S^k\). Since the term \(\tilde{\pi}_*[\ldots]\) above is pulled-back from \(\text{Hilb}_{n+1,n+2} \times S^t\) via \(\pi_* \times \text{Id}_{S'}\), we conclude that \((\pi_+ \times \text{Id}_{S'})_*\) is a linear combination of expressions of the form:

\[
\tilde{\pi}_* \left[ \prod_{i=1}^{k-\#B} \text{ch}_{d_i}(\mathcal{O}_{Z_{n+1}^{(i)}}) \cdot \rho^*(\tilde{\Gamma}') \right] \cdot c_1(\mathcal{L})^a \cdot (\pi_+ \times \text{Id}_{S'})_*(c_1(\mathcal{L}')^{b'})
\]

The first two factors in the formula above are universal by definition, while the third is universal due to (2.13). Thus (2.27) is proved.

Concerning the push-forward \((p_+ \times p_S \times \text{Id}_{S'})_*\), we must show that:

\[
(2.29) \quad y \text{ as in (2.26)} \implies (p_+ \times p_S \times \text{Id}_{S'})_*(y) \in A^*_\text{univ}(\text{Hilb}_{n+1} \times S \times S')
\]

By a short exact sequence analogous to (2.28), formula (2.26) may be replaced by:

\[
y = \pi_* \left[ \prod_{i=1}^{k} \left( \text{ch}_{d_i}(\mathcal{O}_{Z_{n+1}^{(i)}}) - \text{ch}_{d_i}(\mathcal{L} \otimes \mathcal{O}_{\Delta_i}) \right) \cdot \rho^*(\Gamma') \right] \cdot c_1(\mathcal{L})^a
\]

where \(\Delta_i \in \text{Hilb}_{n,n+2} \times S^k\) denotes the codimension 2 diagonal pulled-back via \(p_S\) and the \(i\)-th projection \(S^k \to S\). The element \(y\) above is a linear combination of:

\[
\pi_* \left[ \prod_{i \in A} \text{ch}_{d_i}(\mathcal{O}_{Z_{n+1}^{(i)}}) \cdot \Delta_i \cdot \rho^*(\Gamma') \right] \cdot c_1(\mathcal{L})^a
\]
for various $\alpha' \in \mathbb{N}$, $\Gamma' \in A^*(S^k)$, and partitions $\{1, \ldots, k\} = A \sqcup B$. Since the product of diagonals corresponds to a map $S^{k-\# B} \hookrightarrow S^k$, then $x$ is a linear combination of:

$$\hat{\pi}_* \left[ k-\# B \prod_{i=1}^{k-\# B} \text{ch}_d \left( \mathcal{O}_{\mathbb{P}^{n+1}}(i) \right) \cdot \hat{\rho}^*(\Gamma') \right] \cdot c_1(L)^{\alpha'}$$

where $\hat{\pi}_*, \hat{\rho}$ are defined just like $\pi, \rho$, but using $S^{k-\# B}$ instead of $S^k$. Since the term $\hat{\pi}_*[\ldots]$ above is pulled-back from $\text{Hilb}_{n+1} \times S \times S^t$ via $p_+ \times p_S \times \text{Id}_{S^t}$, we conclude that $(p_+ \times p_S \times \text{Id}_{S^t})_*(y)$ is a linear combination of expressions of the form:

$$\hat{\pi}_* \left[ k-\# B \prod_{i=1}^{k-\# B} \text{ch}_d \left( \mathcal{O}_{\mathbb{P}^{n+1}}(i) \right) \cdot \hat{\rho}^*(\Gamma') \right] \cdot (p_+ \times p_S \times \text{Id}_{S^t})_*(c_1(L)^{\alpha'}) =$$

$$(2.30) \quad = \hat{\pi}_* \left[ k-\# B \prod_{i=1}^{k-\# B} \text{ch}_d \left( \mathcal{O}_{\mathbb{P}^{n+1}}(i) \right) \cdot \hat{\rho}^*(\Gamma') \right] \cdot (-1)^{\alpha'} c_{\alpha'+2} \left( \omega_S^{-1} - \mathcal{O}_{\mathbb{P}^{n+1}}(0) \otimes \omega_S^{-1} \right)$$

where $\mathcal{O}_{\mathbb{P}^{n+1}}(0)$ is the universal sheaf on $\text{Hilb}_{n+1} \times S \times S^t$ corresponding to the first factor of $\hat{S}$. The Chern class in the expression above can be written as a product of $\text{ch}_d(\mathcal{O}_{\mathbb{P}^{n+1}}(i))$ with coefficients pulled back from $S^t$, for various $d$, so we conclude that (2.30) is a linear combination of universal classes. This establishes (2.29).

\[ \square \]

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MIT, DEPARTMENT OF MATHEMATICS, 77 MASSACHUSETTS AVE, CAMBRIDGE, MA 02139, USA

SIMION STOILOW INSTITUTE OF MATHEMATICS, BUCHAREST, ROMANIA

E-mail address: andrei.negut@gmail.com