SYMPLECTIC TORIC ORBIFOLDS

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Abstract. A symplectic toric orbifold is a compact connected orbifold $M$, a symplectic form $\omega$ on $M$, and an effective Hamiltonian action of a torus $T$ on $M$, where the dimension of $T$ is half the dimension of $M$.

We prove that there is a one-to-one correspondence between symplectic toric orbifolds and convex rational simple polytopes with positive integers attached to each facet.

1. Introduction

The main purpose of this paper is to demonstrate the following theorem.

Theorem. Symplectic toric orbifolds are classified by convex rational simple polytopes with a positive integer attached to each facet.

At the same time, however, we would like to build a foundation for further work on symplectic orbifolds. For this reason, we will try to state lemmas in their most natural generality, rather than restricting to our special case.

The above theorem generalizes a theorem of Delzant [D] to the case of orbifolds. He proved that symplectic toric manifolds are classified by the image of their moment maps, that is, by a certain class of rational polytopes. It is easy to see that additional information is necessary in our case:

Example 1.1. Given positive integers $r$ and $s$, let $l$ be their greatest common divisor. Let $K = \mathbb{Z}/(l\mathbb{Z}) \times S^1$ act on $\mathbb{C}^2$ by $(\xi, \lambda) \cdot (x, y) = (\xi^r x, \lambda^s y)$, where $n = r/l$ and $m = s/l$. Let $(M, \omega)$ be the symplectic reduction of $\mathbb{C}^2$ with its standard symplectic form at a positive number. Then $T = (S^1)^2/K$ has an effective Hamiltonian action on $(M, \omega)$. The image of the moment map is always a line interval, but these spaces are not isomorphic. Although $M$ is (topologically) a two sphere, it has two orbifold singularity, which look locally like $\mathbb{C}/(\mathbb{Z}/r\mathbb{Z})$ and $\mathbb{C}/(\mathbb{Z}/s\mathbb{Z})$.

We begin by defining a few terms.

A symplectic toric orbifold is a compact connected orbifold $M$, a symplectic form $\omega$ on $M$, and an effective Hamiltonian action of a torus $T$ on $M$, where the dimension of $T$ is half the dimension of $M$. This is not the
definition used in algebraic geometry. We will see that every symplectic toric orbifold can be given the structure of a projective toric orbifold. Two symplectic toric orbifolds are isomorphic if they are equivariantly symplectomorphic.

Let $\mathfrak{t}$ be a vector space with a lattice $\ell$; let $\mathfrak{t}^*$ denote the dual vector space. A convex polytope $\Delta \subset \mathfrak{t}^*$ is rational if the hyperplanes supporting its facets are defined by the elements of the lattice $\ell$, that is,

$$\Delta = \bigcap_{i=1}^{N} \{ \alpha \in \mathfrak{t}^* \mid \langle \alpha, y_i \rangle \geq \eta_i \}$$

for some $y_i \in \ell$ and $\eta_i \in \mathbb{R}$. Recall that a facet is a face of codimension one. An $n$ dimensional polytope is simple if exactly $n$ facets meet at every vertex. For this paper, we shall adopt the convenient but non standard abbreviation that a weighted polytope is a convex rational simple polytope plus a positive integer attached to each facet. Two weighted polytopes are isomorphic if they differ by the composition of a translation with an element of $\text{SL}(\ell) \equiv \text{SL}(n, \mathbb{Z})$ such that the corresponding facets have the same integers attached to them.

Finally, let $x$ be a point in an orbifold $M$, and let $(\tilde{U}, \Gamma, \phi)$ be a local uniformizing system for neighborhood $U$ of $x$ (see [S]), then the (orbifold) structure group of $x$ is the isotropy group of $\tilde{x} \in \tilde{U}$, where $\phi(\tilde{x}) = x$. This group is well defined.

We are now ready to give a precise statement of our main theorem:

**Theorem 1.2.** For a symplectic toric orbifold $(M, \omega, T)$ the image of the moment map is a rational simple polytope. A positive integer $n$ is attached to every open facet of this polytope as follows: for any $x$ in the preimage of the facet the structure group of $x$ is $\mathbb{Z}/n\mathbb{Z}$.

Two symplectic toric orbifolds are isomorphic if and only if their associated weighted polytopes are isomorphic.

Every weighted polytope occurs as the image of a symplectic toric orbifold.

**Remark 1.3.** We will see in section [7] that all symplectic toric orbifolds have Kähler structures.

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## 2. Group actions on orbifolds

In this section, we recall facts about the action of groups on orbifolds, following Haefliger and Salem [HS]. Although some theorems about group
actions on manifolds hold, there are a few differences. We begin with a few basic definitions.

An orbifold \( M \) is a topological space \(|M|\), plus an atlas of uniformizing charts \((\tilde{U}, \Gamma, \varphi)\), where \( \tilde{U} \) is open subset of \( \mathbb{R}^n \), \( \Gamma \) is a finite group which acts linearly on \( \tilde{U} \) and fixes a set of codimension at least two, and \( \varphi: \tilde{U} \to |M| \) induces a homeomorphism from \( \tilde{U}/\Gamma \) to \( U \subset |M| \). Just as for manifolds, these charts must cover \( |M| \); they are also subject to certain compatibility conditions; and there is a notion of when two atlases of charts are equivalent. For more details, see Satake \[S\]. Given \( x \), we may choose a uniformizing chart \((\tilde{U}, \Gamma, \varphi)\) such that \( \varphi^{-1}(x) \) is a single point which is fixed by \( \Gamma \).

Given \( x \in M \), denote the uniformizing tangent space at \( x \) by \( T_{x}M \); it is \( T_{\varphi^{-1}(x)}\tilde{U} \): the tangent space to \( \varphi^{-1}(x) \) in \( \tilde{U} \). Then \( T_{x}M \), the fiber of the tangent bundle of \( M \) at \( x \) is \( T_{x}M/\Gamma \). A vector field \( \xi \) on \( M \) is a \( \Gamma \) invariant vector field \( \tilde{\xi} \) on each uniformizing chart \((\tilde{U}, \Gamma, \phi)\); of course, these must agree on overlaps. Similar definition apply to differential forms, etc.

Let \( M \) and \( N \) be orbifolds with atlases \( \tilde{U} \) and \( \tilde{V} \). A map of orbifolds \( f: M \to N \) is a map \( f: \tilde{U} \to \tilde{V} \), and an equivariant map \( \tilde{f}_\Gamma: (\tilde{U}, \Gamma) \to (\tilde{V}, \gamma) \) for each \((\tilde{U}, \Gamma) \in \tilde{U}\), where \((\tilde{V}, \gamma) = f(\tilde{U}, \Gamma) \). These \( \tilde{f}_\Gamma \) are subject to a compatibility condition which insures, for instance, that \( f \) induces a continuous map of the underlying spaces. Additionally, there is a notion of when two such maps are equivalent. Again, see \[S\] for details.

We are now ready to recall the definition of the group action on an orbifold.

**Definition 2.1.** Let \( G \) be a Lie group. A smooth action \( a \) of \( G \) on an orbifold \( M \) is a smooth orbifold map \( a: G \times M \to M \) satisfying the usual group laws, that is, for all \( g_1, g_2 \in G \) and \( x \in M \)

\[
a(g_1, a(g_2, x)) = a(g_1g_2, x) \quad \text{and} \quad a(1_G, x) = x,
\]

where \( 1_G \) is the identity element of \( G \).

Technically, by “\(=\)”, we mean “are equivalent as maps of orbifolds.” The action \( a \) induces a continuous action \(|a|\) of \( G \) on the underlying topological space \(|M|\),

\[
|a|: G \times |M| \to |M|.
\]

In particular, this definition states that for every \( g_0 \in G \) and \( x_0 \in M \) there are neighborhoods \( W \) of \( g_0 \) in \( G \), \( U \) of \( x_0 \) in \( M \) and \( U' \) of \( a(g_0, x_0) \) in \( M \), charts \((\tilde{U}, \Gamma, \varphi)\) and \((\tilde{U}', \Gamma', \varphi')\) and a smooth map \( \tilde{a}: W \times \tilde{U} \to \tilde{U}' \) such that \( \varphi'(\tilde{a}(g, \tilde{x})) = |a|(g, \varphi(x)) \) for all \((g, \tilde{x}) \in W \times \tilde{U} \). Note that \( \tilde{a} \) is not unique, it is defined up to composition with elements of the orbifold structure groups \( \Gamma \) of \( x_0 \) and \( \Gamma' \) of \( g_0 \cdot x_0 \).

If \( g_0 = 1_G \), the identity of \( G \), then we may assume \( \tilde{U} \subset \tilde{U}' \), and we can choose \( \tilde{a} \) such that \( \tilde{a}(1_G, x) = x \). Then \( \tilde{a} \) induces a local action of \( G \) on \( \tilde{U} \).

If in addition \( x_0 \) is fixed by the action of \( G \) and \( G \) is compact, then the local
action $G$ on $\tilde{U}$ generates an action of $\hat{G}$ on $\tilde{V} \subset \tilde{U}$, where $\hat{G}$ is a cover of the identity component of $G$. Note that the actions of $\hat{G}$ and $\Gamma$ on $\tilde{U}$ commute.

More generally one can show that for a fixed point $x$ with structure group $\Gamma$ there exists a uniformizing chart $(\tilde{U}, \Gamma, \phi)$ for a neighborhood $U$ of $x$, an exact sequence of groups

$$1 \rightarrow \Gamma \rightarrow \hat{G} \xrightarrow{\pi} G \rightarrow 1,$$

and an action of $\hat{G}$ on $\tilde{U}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\hat{G} \times \tilde{U} & \longrightarrow & \tilde{U} \\
\downarrow & & \downarrow \\
G \times U & \longrightarrow & U
\end{array}$$

The extension $\hat{G}$ of $G$ depends on $x$ and, in particular, is not globally defined.

For any $\xi \in \mathfrak{g}$, there is an associated vector field $\xi_M$ on $M$. On each $\tilde{U}$, it is defined via the local action of $G$ of $\tilde{U}$. The map $\pi : \hat{G} \rightarrow G$ induces an isomorphism of Lie algebras $\pi : \hat{\mathfrak{g}} \rightarrow \mathfrak{g}$. Locally, for any $\xi \in \hat{\mathfrak{g}}$, $\hat{\xi}_U = \pi(\hat{\xi})M|_{\tilde{U}}$.

**Remark 2.2.** If $\hat{\mathfrak{t}}$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$, then $t = \pi(\hat{\mathfrak{t}})$ is also a Cartan subalgebra. Define a lattice $\ell$ by $\ell = \{\xi \in \hat{\mathfrak{t}} \mid \exp(\hat{\xi}) = 1\}$, and define $\ell \subset t$ analogously. Then $\pi : \hat{\ell} \rightarrow \ell$ is not an isomorphism; $\pi(\hat{\xi})$ is in $\ell$ exactly if $\exp(\hat{\xi})$ is in $\Gamma$. Let $\hat{\alpha}_i \in \hat{\ell}^*$ for $i \in I$ be the weights for the action of $\hat{G}$ on $\tilde{U}$. Although $\alpha_i = \pi(\hat{\alpha}_i) \in t$ may not lie in the weight lattice $\ell^*$, $|\Gamma|\alpha_i$ does lie in $\ell^*$, where $|\Gamma|$ is the order of $\Gamma$. We will call $\alpha_i$ the *orbi-weights* for the action of $G$ on $U$.

If $G$ is a compact Lie group acting on an orbifold $M$, we can define local slices to orbits in the differential category. Since the orbit $G \cdot x$ is a *submanifold* of $M$, $\tilde{T}_x(G \cdot x)$ is a $\Gamma$ invariant subspace of $\tilde{T}_x M$. We define the *slice* at $x$ for the action of $G$ on $M$ to be the orbifold $W/\Gamma$, where $W = \tilde{T}_x M/\tilde{T}_x(G \cdot x)$. We may also identify $W$ with the orthogonal complement to $\tilde{T}_x(G \cdot x)$ in $\tilde{T}_x M$ with respect to some invariant metric. It is not hard then to see that the slice theorem takes the following form.

**Proposition 2.3. (Slice theorem)** Suppose a compact Lie group $G$ acts on an orbifold $M$ and $G \cdot x$ is an orbit of $G$. Then a $G$ invariant neighborhood of the orbit is equivariantly diffeomorphic to a neighborhood of the zero section in the associated orbi-bundle $G \times_G W/\Gamma$, where $G_x$ is the isotropy group of $x$ with respect to the action of $G$, $\Gamma$ is the orbifold structure group of $x$ and $W = \tilde{T}_x M/\tilde{T}_x(G \cdot x)$.

**Proof** This is completely analogous to the slice theorem for actions on manifolds, and follows immediately from the fact that metrics can be averaged over compact Lie groups. \qed

**Remark 2.4.** As in the smooth case, the compactness of the group $G$ is not necessary for the existence of slices. It is enough to require that the induced action on the underlying topological space is proper.
For connected $G$, it follows from the existence of slices that the fixed point set is a suborbifold. Therefore, the decomposition of $M$ into infinitesimal orbit types is a stratification into suborbifolds. On the other hand, $M^G$ need not be a suborbifold in general.

Example 2.5. Let $\Gamma = \mathbb{Z}/(2\mathbb{Z})$ act on $\mathbb{C}^2$ by sending $(x, y)$ to $(-x, -y)$ Let $G = \mathbb{Z}/(2\mathbb{Z})$ act on $\mathbb{C}^2/\Gamma$ by sending $[x, y]$ to $[x, -y]$. Then $M^G = \{[x, 0]\} \cup \{[0, y]\} = \mathbb{C}/\Gamma \cup \mathbb{C}/\Gamma$.

Consequently the decomposition of an orbifold according to the orbit type is not a stratification. Fortunately the following lemma still holds.

Lemma 2.6. If $G$ is a compact Lie group acting on a connected orbifold $M$ then there exists an open dense subset of $M$ consisting of points with the same orbit type.

Proof. We first decompose the orbifold into the open dense set of smooth points $M_{\text{smooth}}$ and the set of singular points. Since we assume that all the singularities have codimension 2 or greater, $M_{\text{smooth}}$ is connected. A smooth group action preserves this decomposition. Since $G$ is compact, the action of $G$ on $M_{\text{smooth}}$ has a principal orbit type (see for example Theorem 4.27 in [K]). The set of points of this orbit type is open and dense in $M_{\text{smooth}}$, hence open and dense in $M$.

Corollary 2.7. If a torus $T$ acts effectively on a connected orbifold $M$ then the action of $T$ is free on a dense open subset of $M$.

3. Symplectic local normal forms

In this section, we write down normal forms for the neighborhoods of orbits of compact Lie groups $G$ acting symplectically on $(M, \omega)$; that is, we classify such neighborhoods up to equivariant symplectomorphisms. We also point out consequences of this form, both generally and in the special case of symplectic toric orbifolds.

A symplectic orbifold is an orbifold $M$ with a closed nondegenerate 2-form $\omega$. A group $G$ acts symplectically on $M$ if $G$ preserves $\omega$. A moment map $\phi : M \to \mathfrak{g}^*$ is a map such that for each $\tilde{U}$, the map $\tilde{\phi}$ is a moment map for the local group action of $G$ on $\tilde{U}$. If there is a moment map for $G$, we say that $M$ is a Hamiltonian $G$ space.

If $G$ is a compact Lie group which acts symplectically on $(M, \omega)$, we can define the symplectic slice at a point $x$. The 2-form $\omega$ induces a nondegenerate antisymmetric bilinear form $\omega$ on $T_x M$. Let $\tilde{T}(G \cdot x)^\omega$ be the symplectic perpendicular to the tangent space of $G \cdot x$ with respect to $\omega$. The quotient

$$V = \tilde{T}(G \cdot x)^\omega / (\tilde{T}(G \cdot x) \cap \tilde{T}(G \cdot x)^\omega)$$

is naturally a symplectic vector space. The symplectic slice at $x$ is the symplectic orbifold $V/\Gamma$, where $\Gamma$ is the structure group of $x$. 

Notice that $\tilde{T}(G \cdot x)$ and $\tilde{T}(G \cdot x)^{\omega}$ are both $\tilde{G}_x$ invariant, where $\pi : \tilde{G}_x \to G_x$ is an extension of $G_x$ by $\Gamma$. Therefore there is a symplectic linear action of $G_x$ on $V/\Gamma$, that is, a symplectic linear action of $\tilde{G}_x$ on $V$.

**Remark 3.1.** Let $\pi : \mathfrak{g}_x \to \mathfrak{g}_x$ be the induced map of Lie algebras. Let $\tilde{\phi}_V : V \to \hat{\mathfrak{g}}_x^*$ be the moment map for the action of $\tilde{G}_x$ on $V$. Then $\phi_{V/\Gamma} : V/\Gamma \to \mathfrak{g}_x^*$, the moment map for the action of $G_x$ on $V/\Gamma$, is given by the following diagram:

\[
\begin{array}{ccc}
V & \longrightarrow & V/\Gamma \\
\phi_V & \downarrow & \phi_{V/\Gamma} \\
\hat{\mathfrak{g}}_x^* & \leftarrow & \mathfrak{g}_x^*
\end{array}
\]

Since $\pi^*$ is a vector space isomorphism such that $\pi^*(\ell^*) \subset \hat{\ell}^*$, $\tilde{\phi}_V(V)$ and $\phi_{V/\Gamma}(V/\Gamma)$ are isomorphic as subsets of vector spaces. In particular, if $G$ is Abelian, then $\phi_{V/\Gamma}(V/\Gamma)$ is the rational convex polyhedral cone spanned by the orbi-weights for the action of $G_x$ on $V/\Gamma$. However, since $\pi^*$ need not be an isomorphism of lattices, $\tilde{\phi}_V(V) \cap \hat{\ell}^*$ and $\phi_{V/\Gamma}(V/\Gamma) \cap \ell^*$ may not be isomorphic as semigroups.

As in the case of manifolds, the differential slice at $x$ is isomorphic, as a $G_x$ space, to the product

$$(\mathfrak{g}/\mathfrak{g}_x)^* \times V/\Gamma.$$ 

Thus, by the previous section, a neighborhood of the $G$ orbit of $x$ in $(M, \omega)$ is equivariantly diffeomorphic to a neighborhood of the zero section in the associated orbi-bundle

$$Y = G \times_{G_x} ((\mathfrak{g}/\mathfrak{g}_x)^* \times V/\Gamma).$$

**Lemma 3.2.** Let $G \cdot x$ be an isotropic orbit in $(M, \omega)$. For every $G_x$ equivariant projection $A : \mathfrak{g} \to \mathfrak{g}_M$, there is a symplectic form on $Y$ such that

1. a neighborhood of the zero section in $Y$ is equivariantly symplectomorphic to a neighborhood of $G \cdot x$ in $M$, and
2. the moment map map $\Phi_Y : Y \to \mathfrak{g}^*$ is given by

$$\Phi_Y([g, \eta, [v]]) = Ad^*(g)(\eta + A^*\phi_{V/\Gamma}([v])),$$

where $(\mathfrak{g}/\mathfrak{g}_x)^*$ is identified with the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}^*$, $A^* : \mathfrak{g}_x^* \to \mathfrak{g}^*$ is dual to $A$, and $\phi_{V/\Gamma} : V/\Gamma \to \mathfrak{g}_x^*$ is the moment map for the action of $G_x$ on $V/\Gamma$, as in remark 3.1.

In particular, if $G$ is Abelian then $G \cdot x$ is always isotropic and $Ad^*(g)$ is trivial; the moment map takes the form

$$\Phi_Y([g, \eta, [v]]) = \eta + A^*\phi_{V/\Gamma}([v]).$$

**Proof** The construction is standard in the smooth case (cf [GS]); we simply adapt it for orbifolds. The group $G_x$ acts on $G$ by $g_x \cdot g = gg_x^{-1}$; this lifts to a symplectic action on $T^*G$. The corresponding diagonal action of $G_x$ on
$T^*G \times V/\Gamma$ is Hamiltonian. An equivariant projection $A : g \to g_\ast$ defines a left $G$-invariant connection 1-form on the principal $G_x$ bundle $G \to G/G_x$, and thereby identifies $Y$ with the reduced space $(T^*G \times V/\Gamma)_0$, thus giving $Y$ a symplectic structure. The $G$ moment map on $T^*G \times V/\Gamma$ descends to a moment map for $Y$, giving the formula in (2). The proof that the neighborhoods are equivariantly symplectomorphic reduces to a form of the equivariant relative Darboux theorem; it is identical to the proof in the smooth case.

**Remark 3.3.** The model embedding $i : G \cdot x \hookrightarrow Y$ is **unique** in the following sense. If $i' : G \cdot x \hookrightarrow (N, \sigma)$ is any equivariant isotropic embedding into a symplectic $G$ orbifold $(N, \sigma)$ such that the symplectic slice at $i'(x)$ is the same as the symplectic slice $V/\Gamma$ of the model $Y$ then there exist a neighborhood of $i(x)$ in $Y$, a neighborhood $U'$ of $i'(G \cdot x)$ in $N$ and a symplectic equivariant diffeomorphism $\psi : U \to U'$ such that $i' = \psi \circ i$. The proof of the existence of the map $\psi$ is again, essentially, a form of the equivariant relative Darboux theorem.

The following two lemmas are consequences of lemma 3.2 above.

**Lemma 3.4.** If $G$ is a Hamiltonian torus action on a symplectic orbifold, then the image under the moment map of a neighborhood of an orbit is the neighborhood of a point in a rational polyhedral cone.

**Lemma 3.5.** If $G$ is connected, then $M^G$, the set of points which are fixed by $G$, is a symplectic suborbifold.

We are now ready to specialize to the case of symplectic toric orbifolds.

**Lemma 3.6.** Let $(M, \omega, G)$ be a symplectic toric orbifold with moment map $\Phi_M : M \to g^\ast$. Then for any $x \in M$, the stabilizer of $x$ is connected. Moreover, there is a $G$ invariant neighborhood $U$ of $G \cdot x$ on which

1. $\Phi_M$ induces a homeomorphism form $U/G$ to $\Phi_M(U)$.
2. the image $\Phi_M(U)$ is the neighborhood of a point in a simple rational polyhedral cone.
3. a neighborhood of $G \cdot x$ is classified by $\Phi_M(U)$, plus a positive integer attached to each facet.

**Proof** Let $H$ be the stabilizer of $x$; choose a projection $A : g \to h$. By lemma 3.2, there is a neighborhood of $G \cdot x$ which is equivariantly symplectomorphic to a neighborhood of the zero section of the model space $Y = G \times_H ((g/h)^* \times V/\Gamma)$, where $\Gamma$ is the orbifold structure group at $x$, and $V/\Gamma$ is the symplectic slice at $x$. Therefore, it suffices to prove the above claims for the model $Y$.

Since $G$ is abelian, $H$ does not act on $(g/h)^*$. By assumption the action of $G$ on $M$ is effective and therefore, by corollary 2.7, generically free. Therefore the action of $H$ on $V/\Gamma$ is generically free as well.

Since the action of $\Gamma$ on $V$ is generically free, it follow that the action of $\tilde{H}$, the extension $H$ by $\Gamma$, on $V$ is generically free. Hence the symplectic
representation \( \hat{H} \to Sp(V, \omega_V) \) is faithful. That is to say we may think of \( \hat{H} \) as a compact subgroup of \( Sp(2h, \mathbb{R}) \), where \( 2h = \dim V \), hence as a subgroup of the unitary group \( U(h) \), the maximal compact subgroup of the symplectic group.

Also by assumption \( \dim G = \frac{1}{2} \dim Y \). Consequently \( \dim H = \frac{1}{2} \dim V = h \). The group \( \hat{H} \), the connected component of 1 in \( \hat{H} \) is a cover of the component of 1 in \( H \), hence is a torus of dimension \( h \). Therefore, \( \hat{H} \) is a maximal torus of \( U(h) \). Consequently we may identify \( V \subset \mathbb{C}^h \) and \( \hat{H} \) with the standard torus \( T^h \) of diagonal unitary matrices. Since \( \hat{H} \) commutes with the action of \( \Gamma \) and since the centralizer of the maximal torus in \( U(h) \) is the torus itself, \( \Gamma \) must be a subgroup of \( \hat{H} \). In particular \( \Gamma \) is abelian.

We next argue that \( H \) has to be connected. Since \( H \) is abelian, all the elements of \( H \) commute with the elements of the identity component of \( H \). By continuity they lift to the elements of \( U(h) \) that commute with the elements of \( \hat{H} \), hence by the same argument as before, have to lie in \( \hat{H} \). Therefore \( \hat{H} \) is connected and consequently \( H \) itself is connected.

The moment maps \( \hat{\phi}_V : V \to \mathfrak{h} \) and \( \hat{\phi}_{V/\Gamma} : V/\Gamma \to \mathfrak{h} \) are orbit maps. The moment map map \( \Phi_Y : Y \to \mathfrak{g}^* \) is given by \( \Phi_Y([g, \eta, [v]]) = \eta + A^* \hat{\phi}_{V/\Gamma}([v]) \). Therefore \( \Phi_Y \) is also an orbit map. Hence the original moment map \( \Phi_M \) is an orbit map.

Because the pair \((\hat{H}, V)\) can be identified with \((T^h, \mathbb{C}^h)\), the image \( \hat{\phi}_V(V) \subset \hat{\mathfrak{h}}^* \) is the positive orthant. Let \( x_i = \pi(e_i) \in \ell \), where \( e_i \) is a standard basis vector in \( \mathfrak{h} = \mathbb{R}^h \) and \( \pi : \mathfrak{h} \to \mathfrak{h} \) is the obvious projection (it is actually an isomorphism). Since \((\pi^*)^{-1}(\hat{\phi}_V(V)) = \hat{\phi}_{V/\Gamma}(V/\Gamma) \), \( \hat{\phi}_{V/\Gamma}(V/\Gamma) \) is the image of the positive orthant under \((\pi^*)^{-1} \). An easy computation shows that \( \Phi_Y(Y) = \bigcap_{i=1}^h \{ \xi \in \mathfrak{g}^* \mid \langle \xi, x_i \rangle \geq 0 \} \). Therefore, \( \Phi_Y(Y) \) is a simple rational convex polyhedral cone.

For \( y = [g, \eta, [v]] \in Y \), notice that \( \Phi_Y(y) \) lies in the interior of the \( i \)th facet exactly if \( v_i = 0 \), but the other components do not. Here we think of \( v \in V \) as being the \( h \) tuple \( v = (v_1, \ldots, v_h) \in \mathbb{C}^h \). In this case, it is easy to check that the structure group of \( v \) is \( \mathbb{Z}/(m_i \mathbb{Z}) \), where \( m_i \) is the length of \( x_i \). Let \( m_i \) be the number assigned to the \( i \)th face.

Finally, we must show that \( Y \) is determined, up to equivariant symplectomorphism, by \( \hat{\phi}_Y(Y) \) plus the positive integers attached to facets. But \( \mathfrak{h} \) is determined by the cone \( \hat{\phi}_Y(Y) \), since \( H \) is connected, it is also determined by \( \hat{\phi}_Y(Y) \). Moreover, the projection \( \pi : \mathbb{R}^h \to \mathfrak{h} \) could be read off from the data, because \( \pi(e_i) = m_i y_i \), where \( m_i \) is the integer attached to the \( i \)th facet and \( y_i \) is the unique primitive outward normal to the \( i \)th facet. This allows us to recover the structure group \( \Gamma \) and thereby the symplectic slice.

Since symplectically and equivariantly a neighborhood of an orbit \( G \cdot x \) in \((M, \omega, G)\) is uniquely determined by the representation of the isotropy group of \( x \) on the symplectic slice at \( x \) (cf. remark 3.3), we are done.
4. Morse Theory

In this section, we extend Morse theory to orbifolds. Since orbifolds are stratified spaces, this is simply a special case of Morse theory on stratified spaces [GM]. Nevertheless, it is an important special case which does not seem to be readily available in literature. We need Morse theory for the following result.

**Lemma 4.1.** $M$ be a connected compact $n$ dimensional orbifold, and $f : M \to \mathbb{R}$ be a Bott-Morse function with no critical suborbifold of index $1$ or $n - 1$. Then $M(a,b) = f^{-1}(a,b)$ is connected for all $a, b \in \mathbb{R}$.

We will use this result in the next section to prove that the fibers of a torus moment map are connected, and that the image of a compact symplectic orbifold under a torus moment map is a convex polytope.

There are two main theorems in Morse theory; both relate the Morse polynomial to the Poincaré polynomial. The first states that $\mathcal{M}_i \geq \mathcal{P}_i$, where $\mathcal{M}_i$ is the number of critical points of index $i$, and $\mathcal{P}_i$ is the $i^{th}$ Betti number. The second, stronger theorem states that $\mathcal{M}(x) - \mathcal{P}(x) = (1 + x)Q(x)$, where $Q$ is a polynomial with nonnegative coefficients, $\mathcal{M}(x) = \sum \mathcal{M}_i x^i$ is the Morse polynomial, and $\mathcal{P}(x) = \sum \mathcal{P}_i x^i$ is the Poincaré polynomial.

Although the first theorem holds for orbifolds, the second statement no longer holds; see the following counterexample. Luckily, most applications of Morse theory to symplectic geometry rely only on the first theorem.

**Example 4.2.** Let $M$ be torus, stood on end (see Figure 1 below). Let $f : M \to \mathbb{R}$ be the height function. Let $\Gamma = \mathbb{Z}/(2\mathbb{Z})$ act on $M$ by rotating it 180 degrees. Then $H_0(M/\Gamma) = H_2(M/\Gamma) = \mathbb{R}$, but $H_1(M/\Gamma) = 0$. On the other hand, $\mathcal{M}(x) = 1 + 2x + x^2$.

![Figure 1](image-url)
The basic definitions for Morse theory on orbifolds are identical to their smooth counterparts. Let \( M \) be a compact orbifold. Let \( f : M \to \mathbb{R} \) be a smooth function. We say a critical point \( x \) of \( f \) is non-degenerate if the Hessian \( H(f)_x \) of \( f \) is non-degenerate. In this case, the index of \( f \) at \( x \) is the dimension of the negative eigenspace of the Hessian \( H(f)_x \). We say \( f \) is Morse if every critical point \( x \) is non-degenerate.

**Notation** Denote \( f^{-1}(-\infty, a) \) by \( M^-_a \) for all \( a \in \mathbb{R} \).

**Lemma 4.3.** Choose \( a < b \in \mathbb{R} \) such that \([a, b]\) contains no critical values. Then \( M^-_a \) is diffeomorphic to \( M^-_b \), and \( f^{-1}(a) \) is diffeomorphic to \( f^{-1}(b) \).

**Proof** The usual proof still applies, i.e., given a Riemannian metric on \( M \), one simply flows along the (renormalized) gradient of \( f \).

For critical points, the situation is only slightly more complicated.

**Lemma 4.4. (Morse Lemma)** Let \( p \) be a non-degenerate critical point for \( f : M \to \mathbb{R} \) on an \( n \) dimensional orbifold \( M \). There exists a neighborhood \( U \) of \( p \) in \( M \) and an uniformizing chart \((\tilde{U}, \Gamma, \phi)\) for \( U \) such that \( U \subset \mathbb{R}^n \), \( \Gamma \) acts linearly, \( \phi^{-1}(p) \) is a single point \( \tilde{p} \), and the Hessian of \( f \circ \phi \) at \( \tilde{p} \) equals \( H(f \circ \phi)_{\tilde{p}} = f \circ \phi \).

Note that the action of \( \Gamma \) on \( \tilde{U} \) preserves the positive and negative eigenspaces of the Hessian of \( f \circ \phi \).

**Proof** This is simply an equivariant version of the Morse lemma for manifolds.

Given a critical point \( p \) of a function \( f \), we can choose \( \epsilon > 0 \) such that \( a = f(p) \) is the only critical value in \([f(p) - \epsilon, f(p) + \epsilon]\). Suppose further that \( p \) is the only critical point of \( f \) in the level set \( f^{-1}(a) \) and that the index of \( p \) is \( \lambda \). Then the manifold \( M^-_{a+\epsilon} \) has the homotopy type of the space obtained by attaching the “cell” \( D^\lambda/\Gamma \) to \( M^-_{a-\epsilon} \) by a map from \( S^{\lambda-1}/\Gamma \) to \( f^{-1}(a - \epsilon) \). Here, \( D^\lambda \) is is the standard closed disk in the negative eigenspace of \( H(f)_p \) with the action of \( \Gamma \) given as above; \( S^{\lambda-1} \) is its boundary. As before, it suffices to check that the usual proof is equivariant. Therefore, \( H_*(M^-_{a+\epsilon}, M^-_{a-\epsilon}) = H_*(D^\lambda/\Gamma, S^{\lambda-1}/\Gamma) \). When we consider coefficients in \( \mathbb{Z} \), this can be complicated. Over \( \mathbb{R} \), however, the following lemma is immediate:

**Lemma 4.5.** Let \( f \) be a Morse function on an orbifold \( M \). Suppose that \( p \) is the only critical point of \( f \) in \( f^{-1}(a - \epsilon, a + \epsilon) \) and that it has index \( \lambda \). Then \( H_i(M^-_{a+\epsilon}, M^-_{a-\epsilon}; \mathbb{R}) = 0 \) for \( i \neq \lambda \); whereas \( H_\lambda(M^-_{a+\epsilon}, M^-_{a-\epsilon}; \mathbb{R}) = \mathbb{R} \) if \( \Gamma \) preserves the orientation of \( D^\lambda \), and is trivial otherwise.

**Corollary 4.6.** The number of critical points with index \( i \) is greater than or equal to the dimension of \( H_i(M) \).
Remark 4.7. We will see in the next section that every moment map corresponding to a circle action with isolated fixed points is a Morse function with even indices. Moreover, it is clear that the orbifold structure group $\Gamma$ preserves the symplectic form, and hence the orientation, on $D^\lambda$. Therefore, these moment maps are perfect Morse functions, i.e., the $i^{\text{th}}$ coefficient of the Morse polynomial equals the $i^{\text{th}}$ coefficient of the Poincaré polynomial: $\mathcal{M}_i = \dim(H_i(M))$.

We must also consider moment maps which correspond to circle actions with non-isolated fixed points, that is, consider Bott-Morse theory.

Definition 4.8. A smooth function $f : M \to \mathbb{R}$ is Bott-Morse if the set of critical points is the disjoint union of suborbifolds, and if for every point $x$ of such a suborbifold $F \subset M$, the null space of the Hessian $H(f)_x$ is precisely the tangent space to $F$.

If $f : M \to \mathbb{R}$ is Bott-Morse and $F$ is a critical orbifold of $f$, the normal orbi-bundle of $F$ splits as a direct sum of vector orbi-bundles $E^-$ and $E^+$ corresponding to the negative and positive spectrum of the Hessian of $f$ along $F$.

Given any metric on $M$, let $D = D_F$ denote a disc bundle of $E^-$ and $S = S_F$ denote the corresponding sphere bundle. The index of $f$ at $F$ is the dimension of $D_F$. In this case, we have the following result:

Lemma 4.9. For small $\epsilon$, $H_*(M^-_{f(F)+\epsilon}, M^-_{f(F)-\epsilon}) = H_*(D_F, S_F)$. Moreover, the boundary map from $H_0(M^-_{f(F)+\epsilon}, M^-_{f(F)-\epsilon})$ to $H_{-1}(M^-_{f(F)-\epsilon})$ in the long exact sequence of relative homology is the composition of the boundary map from $H_0(D_F, S_F)$ to $H_{-1}(S_F)$ and the map on homology induced by the “inclusion” map from $S_F$ to $M^-_{f(F)-\epsilon}$.

Proof Again, the manifold proof (see for example [9]) can be adapted to the case of orbifolds.

We now prove lemma 4.11. We do it in a sequence of lemmas.

Lemma 4.10. Let $F$ be an orbifold, $\pi : E \to F$ be a $\lambda$ dimensional real vector orbi-bundle and $D(E)$ and $S(E)$ the corresponding disk and M sphere orbi-bundles with respect to some metric. If $\lambda > 1$, then $H_1(D(E), S(E)) = 0$.

Proof By the long exact sequence in relative homology it suffices to show that the maps $H_0(S(E)) \to H_0(D(E))$ and $H_1(S(E)) \to H_1(D(E))$ induced by inclusion are surjective. But this follows from two facts: the fibers of $\pi : E \to F$ are path connected and any path in the base $F$ can be lifted to a path in the sphere bundle $S(E)$.

Lemma 4.11. Let $M$ be a connected compact orbifold, and $f : M \to \mathbb{R}$ be a Bott-Morse function with no critical suborbifold of index 1. Then

\footnote{Of course one has to be careful when talking about direct sums of vector orbi-bundles.}
1. \( M_a^- = \{ m \in M : f(m) < a \} \) is connected for all \( a \in \mathbb{R} \), and
2. if \( M_a^- \neq \emptyset \), then \( H_1(M_a^-) \to H_1(M) \) is a surjection.

**Proof** Let \( F \subset M \) be a critical suborbifold of \( f \) index \( \lambda \). Let \( D_F \) and \( S_F \) be the disk and sphere bundles of the negative orb-bundle of \( f \) along \( F \). Let \( a = f(F) \), and let \( \epsilon > 0 \) be small. We assume, for simplicity, that no other critical suborbifold maps to \( a \).

If \( \lambda = 0 \) then \( S_F \) is empty. Otherwise, since \( \lambda \neq 1 \), \( \lambda \) is greater than one and \( H_1(D_F, S_F) \) is trivial by lemma \[1.10\]. In either case the map \( H_1(D_F, S_F) \to H_0(S_F) \), and hence also the map \( H_1(D_F, S_F) \to H_0(M_{a-\epsilon}^-) \), is trivial. Therefore, the following sequence is exact:

\[
0 \to H_0(M_{a-\epsilon}^-) \to H_0(M_{a+\epsilon}^-) \to H_0(D_F, S_F) \to 0
\]

Notice that \( \dim(H_0(M_{a-\epsilon}^-)) \geq \dim(H_0(M_{a-\epsilon}^-)) \). Since \( M \) is connected, this completes part (1).

If \( \lambda = 0 \), then \( \dim(H_0(M_{a+\epsilon}^-)) > \dim(H_0(M_{a-\epsilon}^-)) \). Therefore, since \( M \) is connected, the minimum is the unique critical value of index 0. For any other critical value \( a \), \( H_1(M_{a+\epsilon}, M_{a-\epsilon}) = 0 \), and the map \( H_1(M_{a-\epsilon}) \to H_1(M_{a+\epsilon}) \) is a surjection. \( \square \)

**Proof of lemma \[4.1\]** We may assume that \( a \) and \( b \) are regular and that \( M_{(a,b)} = \{ m \in M : a < f(m) < b \} \) is not empty. By lemma \[1.11\] \( H_1(M_a^-) \oplus H_1(M_b^+) \to H_1(M) \) is a surjection, where \( M_a^+ = f^{-1}(a, \infty) \). Therefore, by Mayer-Vietoris, the following sequence is exact:

\[
0 \to H_0(M_{(a,b)}) \to H_0(M_a^-) \oplus H_0(M_b^+) \to H_0(M) \to 0.
\]

Finally, by lemma \[1.11\], \( M_a^- \) and \( M_b^+ \) are connected. \( \square \)

**Remark 4.12.** Since \( f : M \to \mathbb{R} \) is proper, it follows from lemma \[4.1\] and a simple point set topology argument that for any \( a \in \mathbb{R} \) the fiber \( f^{-1}(a) \) is connected.

### 5. Connectedness and Convexity

Let \( (M, \omega, T) \) be a symplectic toric orbifold with moment map \( \phi : M \to t^* \).

In this section we show that \( \phi(M) \subset t^* \) is a convex rational simple polytope.

We will prove this as a corollary of the Atiyah-Guillemin-Sternberg convexity theorem \[\text{AGS}\] for orbifolds; our proof is similar to Atiyah’s. We also prove that the fibers of the moment map \( \phi \) are connected. In fact, as was observed by Atiyah (op. cit.) in the manifold case, convexity is an easy consequence of connectedness. In turn, the fibers of toral moment maps are connected because the components of these moment maps are Bott-Morse functions with even indices. We now give precise statements of the main results of the section.
Theorem 5.1. Let $M$ be a Hamiltonian $T$ orbifold, $T$ a torus, with a momentum map $\phi : M \to t^*$. The fibers of $\phi$ are connected.

Theorem 5.2. Let $M, T$ and $\phi : M \to t^*$ be as in theorem 5.1 above. Then $\Delta = \phi(M) \subset t^*$ is a rational convex polytope. In particular it is the convex hull of the image of the points in $M$ fixed by $T$,

$$\phi(M) = \text{convex hull } (\phi(M^T)).$$

Corollary 5.3. Let $(M, \omega, T)$ be a symplectic toric orbifold with moment map $\phi : M \to t^*$. Then $\Delta = \phi(M)$ is a rational simple polytope.

We begin the proof of the above statements with the following lemma.

Lemma 5.4. Let $G \times (M, \omega) \to (M, \omega)$ be a Hamiltonian group action of a compact Lie group on a symplectic orbifold with moment map $\phi : M \to g^*$. Then for any $\xi \in g$ the $\xi$th component of the moment map $\phi^\xi := \xi \circ \phi$ is Bott-Morse and the indices of its critical orbifolds are all even.

Proof This is a generalization of Theorem 5.3 of [GS] and of Lemma (2.2) of [A] to the case of orbifolds. The proof is the same, except one has to use the orbifold version of the equivariant Darboux theorem (cf. lemma 3.2 which specializes to the equivariant Darboux theorem when the orbit is a point).

Proof of Theorem 5.1 Because the moment map $\phi$ is continuous and proper, the connectedness of fibers is implied by the connectedness of the preimages of balls. We will prove this stronger statement by induction on the dimension of $T$.

By lemma 5.4 moment maps for circle actions are Bott-Morse functions of even index. Then the preimages of balls are connected by lemma 4.1.

Suppose now that $T$ is a $k$ dimensional torus and let $B$ be a ball in $t^*$. Let as before $\ell \subset t$ denote the lattice of circle subgroups. Then for every $0 \neq \xi \in \ell$ the map $\phi^\xi \equiv \xi \circ \phi$ is a moment map for the action of the circle $S_\xi := \{ \exp t\xi : t \in \mathbb{R} \}$. Let $R_\xi$ denote the set of regular values of $\phi^\xi$. For every $a \in R_\xi$ the reduced space $M_{a,\xi} := (\xi \circ \phi)^{-1}(a)/S_\xi$ is a symplectic orbifold. The $k-1$ dimensional torus $H := T/S_\xi$ acts on $M_{a,\xi}$ and the action is Hamiltonian. By inductive assumption the preimages of balls under the $H$ moment maps $\phi^H : M_{a,\xi} \to h^*$ are connected.

The affine hyperplane $\{ \eta \in t^* : \xi(\eta) = a \}$ is naturally isomorphic to the dual of the Lie algebra of $h$, and we can identify the restriction of $\phi$ to $(\phi^\xi)^{-1}(a)$ with the pull-back of $\phi^H$ by the orbit map $\pi : (\phi^\xi)^{-1}(a) \to M_{a,\xi}$. It follows that $\phi^{-1}(B \cap \{ \eta : \xi(\eta) = a \}) = \pi^{-1}((\phi^H)^{-1}(B))$ is connected.

Now the set

$$U = \bigcup_{\xi \in \ell} \bigcup_{a \in R_\xi} B \cap \{ \eta : \xi(\eta) = a \}$$
is connected and dense in the ball $B$, and its preimage $\phi^{-1}(U)$ is connected. Therefore the closure $\overline{\phi^{-1}(U)}$ in $M$ is connected. Since $\phi$ is proper, $\overline{\phi^{-1}(U)}$ is the preimage of the closure of $U$ in $t^*$, which is $\phi^{-1}(B)$. Hence $\phi^{-1}(B)$ is connected.

**Proof of Theorem 5.2** It is no loss of generality to assume that the action of $T$ is effective, hence by corollary 2.7 free on a dense subset. Consequently the interior of the image $\phi(M)$ is nonempty.

To prove that $\phi(M)$ is convex it suffices to show that for any affine line $L \subset t^*$, the intersection $L \cap \phi(M)$ is connected. In fact it is enough to prove this for just rational lines, i.e. the lines of the form $\mathbb{R}v + a$ where $a \in t^*$ and $v \in \ell^*$, the weight lattice of $T$.

If $v \in \ell^*$ then $\ker v \subset t$ is the Lie algebra of a subtorus $H = H_v$ of $T$. A moment map $\phi^H$ for the action of $H$ on $M$ is given by the formula $\phi^H = i^* \circ \phi$ where $i^*$ is the dual of the inclusion $i : h \to t$. The fibers $(\phi^H)^{-1}(\alpha)$ are connected by theorem 5.1. On the other hand

$$(\phi^H)^{-1}(a) = \phi^{-1}(i^* \alpha - 1(a)) = \phi^{-1}(\phi(M) \cap (a + \mathbb{R}v))$$

for some $a \in (i^*)^{-1}(\alpha)$. This proves that the image $\phi(M)$ is convex.

Since $\phi(M)$ is compact it is, by Minkowski’s theorem, the convex hull of its extreme points. Recall that a point $a$ in the convex set $A$ is extreme for $A$ if it cannot be written in the form $\alpha = \lambda \beta + (1 - \lambda) \gamma$ for any $\beta, \gamma \in A$ and $\lambda \in (0, 1)$. Lemma 3.2 shows that for any point $x$ in a Hamiltonian $T$ orbifold $M$ the image $\phi(M)$ contains an open ball in the affine plane $\phi(x) + t^*_x$, where $t^*_x$ is the annihilator of the isotropy Lie algebra of $x$ in $t^*$. Therefore the preimage of extreme points of $M$ consists entirely of fixed points.

Since the set of fixed points $M^T$ is closed and $M$ is compact, $H_0(M^T)$ is finite. Since $\phi$ is locally constant on $M^T$, $\phi(M^T)$ is finite. Therefore $\phi(M) = \text{convex hull } (\phi(M^T))$ is a convex polytope.

To prove that $\phi(M)$ is rational we need to show that the hyperplanes supporting its facets are defined by elements of the circle subgroup lattice $\ell \subset t$ (cf. the introduction). Suppose $\xi \in t$ defines a hyperplane supporting a facet $F$ of $\phi(M)$. Then the function $\phi^\xi$ takes a global minimum on the preimage of the facet $\phi^{-1}(F)$. Therefore the points in $\phi^{-1}(F)$ are fixed by the closure $H$ of $\{ \exp t \xi : t \in \mathbb{R} \}$ in $T$. It follows from the local normal form (lemma 3.2) that the images of the connected components of $M^H$ lie in the affine translates of the annihilator of $h$ in $t^*$. Since the affine hull of the facet $F$ has codimension 1, $h$ is one-dimensional, i.e. $H$ is a circle. Therefore $\xi$ is in $\ell$.

**Proof of corollary 5.3** We saw in the proof of theorem 5.2 that facets of the image $\phi(M)$ correspond to one dimensional isotropy subgroups. Therefore in order to prove that $\phi(M)$ is simple it is enough to show that in a neighborhood of the preimage of a vertex the number of circle isotropy
groups that can occur is the same as the dimension of the torus. But this is precisely the content of the second assertion of lemma 3.6.

6. From Local to Global

Let \((M, \omega, T)\) and \((M', \omega', T')\) be symplectic toric orbifolds with isomorphic associated weighted polytopes. In this section, we show that the orbifolds are isomorphic, that is, equivariantly symplectomorphic. The first step is to show that \(M\) and \(M'\) are locally isomorphic. Then, by extending proposition 2.4 in [HS] to the symplectic category, we show that we can “glue” these local isomorphisms together to construct a global isomorphism. Note that we may assume that \(T = T'\), and that the associated weighted polytopes are equal.

Let \(T\) be a torus, and let \((M, \omega)\) be a Hamiltonian \(T\) orbifold of any dimension. Let \(\pi : M \to M/T\) be the orbit map. Since the moment map \(\phi : M \to T^*\) is \(T\) invariant, we descend to a map \(\phi : M/T \to T^\star\). Denote \(\text{im}(\phi) \subset T^*\) by \(\Delta\).

Suppose \(M'\) is another Hamiltonian \(T\) orbifold. We say \(M'\) is an orbifold over \(M/T\) if there is a continuous map \(\pi' : M' \to M/T\) which induces a homeomorphism from \(M'/T\) to \(M/T\). In this case, define \(\phi' : M' \to T^*\) by \(\phi' = \phi \circ \pi'\).

For the purposes of this section two Hamiltonian \(T\) orbifolds \((M, \omega, \pi)\) and \((M', \omega', \pi')\) over \(M/T\) are isomorphic if there exists a \(T\) equivariant symplectomorphism \(f : M \to M'\) such that \(\pi \circ f = \pi'\).

Two Hamiltonian \(T\) orbifolds \((M, \omega, \pi)\) and \((M', \omega', \pi')\) over \(M/T\) are locally isomorphic over \(\Delta\) if every point in \(\Delta\) has an open neighborhood \(U\) and a \(T\) equivariant symplectomorphism \(f : \phi'^{-1}(U) \to \phi^{-1}(U)\) such that \(\pi \circ f = \pi'\). In this case, \(\phi' : M' \to T^*\) is a moment map for the action of \(T\) on \((M', \omega')\).

**Remark 6.1.** If \(\dim T = \frac{1}{2} \dim M\), then \((M, \omega, \pi)\) and \((M', \omega', \pi')\) are isomorphic exactly if they are equivariantly symplectomorphic. Furthermore, \(\Delta = M/T\), so there is no need to distinguish the two. In contrast, if \(\dim T < \frac{1}{2} \dim M\), then \((M, \omega, \pi)\) and \((M', \omega', \pi')\) may be equivariantly symplectomorphic but not isomorphic. Furthermore, \(\Delta \neq M/T\), so it is important to notice that although we consider isomorphisms of neighborhoods of fibers of the moment map, we demand that these isomorphisms fix the orbits of \(M/T\). Since we only need the former case, the reader may wish to assume that \(\dim T = \frac{1}{2} \dim M\).

**Lemma 6.2.** Let \((M, \omega, T)\) and \((M', \omega', T)\) be symplectic toric orbifolds. Let \(\Phi : M \to T^*\) and \(\Phi' : M \to T^*\) be the associated moment maps. Assume that \(\Phi(M) = \Phi'(M')\) and that the integers associated to each facet are the same. Then \(M\) and \(M'\) are locally isomorphic.
Proof The proof is a direct corollary of the local normal form for symplectic toric orbifolds (lemma 3.6), the connectedness of fibers of the moment map (theorem 3.1), and the properness of the moment map.

Lemma 6.3. Let \((M, \omega)\) be a Hamiltonian \(T\) orbifold; let \(\Delta\) denote the image of its moment map. Let \(S\) be the sheaf over \(\Delta\) defined as follows: for each open \(U \subset \Delta\), \(S(U)\) is the set of isomorphisms of \(\phi^{-1}(U)\), that is, the set \(T\)-equivariant symplectomorphisms of \(\phi^{-1}(U)\) which preserve the orbits of \(T\). Then isomorphisms classes of Hamiltonian \(T\) orbifolds over \(M/T\) which are locally isomorphic to \(M\) are classified by \(H^1(\Delta, S)\).

Proof Let \(U = \{U_i\}_{i \in I}\) be a covering of \(\Delta\) such that that there is an isomorphism \(h_i : \phi^{-1}(U_i) \to \phi^{-1}(U_i)\) for each \(i \in I\). Define \(f_{ij} : \phi^{-1}(U_i \cap U_j) \to \phi^{-1}(U_i \cap U_j)\) by \(f_{ij} = h_i^{-1} \circ h_j\). These \(f_{ij}\)'s give a closed element of \(C^1(U, S)\). Moreover, the cohomology class of this element is independent of the choices of the isomorphisms \(h_i\).

Conversely, if \(\{f_{ij}\} \in C^1(U, S)\) is closed, we can construct a Hamiltonian \(T\) orbifold over \(M/G\) by taking the disjoint union of the \(\phi^{-1}(U_i)\)'s and gluing \(\phi^{-1}(U_i)\) and \(\phi^{-1}(U_j)\) together using \(f_{ij}\).

Proposition 6.4. Let \(M\) and \(M'\) be Hamiltonian \(T\) orbifolds over \(M/T\) which are locally isomorphic. Then \(M\) and \(M'\) are isomorphic.

Proof Let \(C^\infty\) denote the sheaf of germs of smooth functions on \(\Delta\). Let \(\ell \times \mathbb{R}\) denote the sheaf of locally constant functions with values in \(\ell \times \mathbb{R}\). Since \(C^\infty\) is a fine sheaf, \(H^i(\Delta, C^\infty) = 0\) for all \(i > 0\). Since \(\Delta\) is contractable, \(H^1(\Delta, \ell \times \mathbb{R}) = 0\) for all \(i > 0\).

By lemma 6.3 above, it suffices to show that \(H^1(\Delta, S) = 0\). Therefore, by the above comments, it is sufficient to show that the following sequence of sheaves is exact:

\[
0 \to \ell \times \mathbb{R} \xrightarrow{j} C^\infty \xrightarrow{\Lambda} S \to 0.
\]

First we construct the map \(\Lambda : C^\infty \to S\). For \(U \subset \Delta\), let \(f : U \to \mathbb{R}\) be a smooth function. Then the Hamiltonian vector field \(X\) on \(M\) of the function \(f \circ \phi\) is a \(T\) invariant symplectic vector field. Moreover, \(X\) preserves \(T\) orbits. Therefore \(\exp(X)\) is an isomorphism of \(\phi^{-1}(U)\), i.e., \(\exp(X) \in S(U)\).

To define \(j : \mathbb{R} \times \ell \to C^\infty\), for any \((c, \xi) \in \mathbb{R} \times \ell\) and \(\eta \in \mathfrak{t}^*\), let \(j(c, \xi)\)(\(\eta\)) = \(c + \langle \xi, \eta \rangle\). It is clear that \(j\) is injective, and that \(\text{im}(j) = \ker(\Lambda)\).

The final step is to show that \(\Lambda\) is surjective. Let \(\psi\) be an isomorphism of \(\phi^{-1}(U)\), that is, a \(T\)-equivariant symplectic diffeomorphism which preserves orbits. Then obviously \(\psi\) is a \(T\)-equivariant diffeomorphism of \(\phi^{-1}(U)\) which preserves orbits. Therefore, by Theorem 3.1 in [HS], there exists a smooth \(T\) invariant map \(h : \phi^{-1}(U) \to T\) such that \(\psi(x) = h(x) \cdot x\). For sufficiently small \(U\), there exists a smooth \(T\) invariant map \(\theta : \phi^{-1}(U) \to \mathfrak{t}\) such that \(\exp \circ \theta = h\). Define a vector field \(X_\theta\) on \(M\) by \(X_\theta(x) = \frac{d}{ds}\big|_{s=0} \exp(s\theta(x)) \cdot x\).
A computation using a local normal form at the points where the action of $T$ is free show that $X_{\theta}$ is symplectic. Hence locally $X_{\theta}$ is Hamiltonian. Choose $f$ such that $df = iX_{\theta}\omega$. Since $X_{\theta}t$ is tangent to orbits of $T$, the function $f$ Poisson commutes with all $T$ invariant functions on $M$. Since the arguments in [L] works in the case of orbifolds, it follows that there is a smooth function $\tilde{f} : \Delta \to \mathbb{R}$ such that $\phi^*\tilde{f} = f$. Finally, it is not hard to see that $\Lambda(\tilde{f}) = \psi$.

**Theorem 6.5.** Let $(M,\omega,T)$ and $(M',\omega',T')$ be symplectic toric orbifolds with isomorphic weighted polytopes. Then $(M,\omega,T)$ and $(M',\omega',T')$ are isomorphic.

**Proof** By lemma 6.2, $M$ and $M'$ are locally isomorphic. By proposition 6.4, locally isomorphic implies isomorphic, so we are done.

**Remark 6.6.** Given any weighted polytope $\Delta$, one can construct local models for the symplectic toric orbifold associated to $\Delta$. Since we’ve shown that $H^2(S,\Delta) = 0$, the arguments in [HS] allow one to show that there exists a symplectic toric orbifold which corresponds to the given weighted polytope. However, in section 7, we give a more explicit construction.

7. **Surjectivity**

Finally, for every weighted polytope we construct a corresponding Kähler toric orbifold. Our construction is a slight variation of Delzant’s construction.

Let $t$ be a vector space with a lattice $\ell$. Let $\Delta \subset t^*$ be a rational simplicial polytope with $N$ facets, and a positive integer $m_i$ associated to each facet. Then $\Delta$ can be written uniquely as

$$\Delta = \cap_{i=1}^N \{ \alpha \in t^* \mid \langle \alpha, y_i \rangle \geq \eta_i \},$$

where $y_i \in \ell$ is primitive.

Let $x_i = m_i y_i$. Define a linear projection $\pi : \mathbb{R}^N \to t$ by $\pi(e_i) = x_i$. Let $\mathfrak{k}$ be the kernel of $\pi$; let $j : \mathfrak{k} \to \mathbb{R}^N$ denote the inclusion map. Let $K$ be the kernel of the map from $\mathbb{R}^N/\mathbb{Z}^N$ to $t/\ell$ induced by $\pi$.

Let $\omega$ be the standard symplectic form on $\mathbb{C}^N$. The standard action of $(S^1)^N$ on $\mathbb{C}^N$ has moment map $\phi_{(S^1)^N}(z_1, \ldots, z_N) = |z_1|^2 + \cdots + |z_N|^2$. Since $K$ is a subset of $\mathbb{R}^N/\mathbb{Z}^N$, the identification of $\mathbb{R}^N/\mathbb{Z}^N$ with $(S^1)^N$ induces an action of $K$ on $\mathbb{C}^N$; its moment map is given by $\phi_K = j^* \circ \phi_{S^1}$.

Let $(M,\sigma)$ be the symplectic reduction of $\mathbb{C}^N$ by $K$ at $j^*(m_1 \eta_1, \ldots, m_N \eta_N)$. $T = t/\ell = (S^1)^N/K$ acts symplectically on $(M,\sigma)$. Since the action of $K$ on $\mathbb{C}^N$ preserves the Kähler structure on $\mathbb{C}^N$, $(M,\sigma,T)$ has an induced Kähler structure.

It is easy to check that

$$\text{im}(\phi_T) = \cap_{i=1}^N \{ \alpha \in t^* \mid \langle \alpha, m_i y_i \rangle \geq m_i \eta_i \} = \Delta,$$
where \( \phi_T : M \to t^* \) is the moment map for \( T \).

Consider \([z] \in M\), where \( z = (z_1, \ldots, z_N) \in \mathbb{C}^N\). It is clear that \( \phi_T([z]) \) lies in the interior of the \( i \)th facet exactly if \( \phi_{(S^1)^N}(z) \) lies in the interior of the \( i \)th coordinate hyperplane, that is, exactly if \( z_i = 0 \), but \( z_j \neq 0 \) for \( j \neq i \).

The structure group of such points is just the intersection of \( K \) with the \( i \)th \( S^1 \), that is, \( \mathbb{Z}/(m_i \mathbb{Z}) \).

Therefore, \((M, \sigma, T)\) is a Kähler toric variety, and \((\Delta, \{m_i\})\) is the associated weighted polytope.

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