WALL CROSSING FOR MODULI OF STABLE LOG VARIETIES

KENNETH ASCHER, DORI BEJLERI, GIOVANNI INCHIOSTRO, AND ZSOLT PATAKFALVI

Abstract. We prove, under suitable conditions, that there exist wall-crossing and reduction morphisms for moduli spaces of stable log varieties as one varies the coefficients of the divisor.

1. Introduction

Compactifying moduli spaces is a central problem of algebraic geometry. It has long been apparent that moduli spaces often admit different compactifications depending on some choice of parameters, and so it is natural to ask how these compactifications and their universal families are related as one varies the parameters. The goal of the present article is to answer this question for compact moduli spaces of higher dimensional stable log varieties or stable pairs for short.

A stable pair is a pair \((X, \sum a_i D_i)\) consisting of a variety \(X\) and a \(\mathbb{Q}\)-divisor \(\sum a_i D_i\) satisfying certain singularity and stability conditions which we will recall below. The standard example is a smooth normal crossings pair with \(0 < a_i \leq 1\) and \(K_X + \sum a_i D_i\) ample. Compact moduli spaces of stable pairs with fixed coefficient or weight vector \(a = (a_1, \ldots, a_n)\) and fixed numerical invariants have been constructed using the tools of the minimal model program ([Kol18a] and Section 2). These moduli spaces are quite large and unwieldy in general, and so in practice one studies the closure of a family of interest inside the larger moduli space. Theorem 1.1 below summarizes our main results in a simplified but typical situation. We will state our general results in Section 1.1.

Theorem 1.1. Let \((X, D_1, \ldots, D_n) \to B\) be a family of smooth normal crossings pairs over a smooth connected base \(B\) and let \(P\) be a finite rational polytope of weight vectors \(a = (a_1, \ldots, a_n)\) such that \(a_i < 1\) and \((X, \sum a_i D_i) \to B\) is a family of stable pairs for each \(a \in P\). Let \(N_a\) denote the normalized closure of the image of \(B\) in the moduli space of \(a\)-weighted stable pairs with universal family of stable pairs \((X_a, \sum a_i D_i) \to N_a\). Then there exists a finite rational polyhedral wall-and-chamber decomposition of \(P\) such that the following hold.

(a) For \(a, a'\) contained in the same chamber, there are canonical isomorphisms

\[
\begin{array}{c}
X_a \\ \vee
\end{array} \cong \begin{array}{c}
X_{a'} \\ \vee
\end{array} \\
\begin{array}{c}
N_a \\ \vee
\end{array} \cong \begin{array}{c}
N_{a'} \\ \vee
\end{array}
\]

(b) For \(a, b \in P\) contained in different chambers and satisfying \(b_i \leq a_i\) for all \(i\), there are canonical birational wall-crossing morphisms

\[
\rho_{b,a} : N_a \to N_b
\]

such that for any third weight vector \(c\) with \(c_i \leq b_i\), we have \(\rho_{c,b} \circ \rho_{b,a} = \rho_{c,a}\). Moreover, the map \(\rho_{b,a}\) is induced by a birational map \(h : X_a \dasharrow X_b^\rho_{b,a}\) such that, for a generic \(u \in N_a\), the fiberwise map \(h_u : (X_a)_u \dasharrow (X_b)_{\rho_{b,a}(u)}\) is the log canonical model of \(((X_a)_u, \sum b_i(D_i)_u)\).
Remark 1.2. We note that, to obtain the strongest results, taking the normalization of the closure in the above theorem is necessary; see Section 8.1 for a discussion and example.

Before stating our more general results, let us recap the history and context behind Theorem 1.1. In dimension one, we have the classical moduli space $\mathcal{M}_{g,n}$ of smooth projective $n$-pointed curves $(C, p_1, \ldots, p_n)$ of genus $g$ and the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{g,n}$ parametrizing $n$-pointed stable curves of genus $g$. Inspired by ideas from the minimal model program, Hassett in [Has03] introduced a new family of modular compactifications of $\mathcal{M}_{g,n}$ depending on a rational weight vector $\mathbf{a} = (a_1, \ldots, a_n)$ with $0 < a_i \leq 1$ which parametrizes $\mathbf{a}$-weighted stable curves.

An $\mathbf{a}$-weighted pointed stable curve is a tuple $(C, p_1, \ldots, p_n)$ such that:

- $C$ has genus $g$ and at worst nodal singularities;
- the points $p_i$ lie in the smooth locus of $C$ and for any subset $p_{i_1}, \ldots, p_{i_r}$ of points which coincide, we have $\sum_i a_{i} \leq 1$;
- the divisor $K_C + \sum a_i p_i$ is ample.

When $a_i = 1$ for all $i$, the second condition is the requirement that the $p_i$ are distinct and the third condition is the Deligne-Mumford-Knudsen stability condition, and so we recover $\overline{\mathcal{M}}_{g,n}$.

Weighted stable curves form a proper moduli space $\overline{\mathcal{M}}_{g,\mathbf{a}}$ for $0 < a_i \leq 1$ satisfying the condition that $2g - 2 + \sum_{i=1}^{n} a_i > 0$. These conditions define a finite rational polytope of admissible weight vectors $P$ as in Theorem 1.1, where the family $(X, D_1, \ldots, D_n) \to B$ is the universal family of smooth $n$-pointed curves of $\mathcal{M}_{g,n}$. In particular, Hassett [Has03] proved Theorem 1.1 in this setting. In fact, in this case $h$ is a birational morphism produced as an explicit sequence of contractions of rational tails on which the degree of $K_C + \sum b_i p_i$ is non-positive, that is, $\mathbf{b}$-unstable rational tails.

The natural generalization of a pointed stable curve to higher dimensions, introduced by Kollár and Shepherd-Barron [KSB88] and Alexeev [Ale94], is a stable pair $(X, \sum a_i D_i)$ such that:

1. $(X, \sum a_i D_i)$ has semi-log canonical singularities (slc, see Definition 2.2); and
2. $K_X + \sum a_i D_i$ is an ample $\mathbb{Q}$-Cartier divisor.

Explicit stable pair compactifications of moduli of higher dimensional varieties have been studied extensively in recent years, e.g. weighted hyperplane arrangements [HKT06, Ale15], principally polarized abelian varieties [Ale02], plane curves [Has04], and elliptic surfaces [AB20, Inc20], etc.

Thanks to the combined efforts of many authors (see e.g. [Kol18b, 30] for a historical summary), there exists a proper moduli space $\mathcal{K}_{\mathbf{a}, \mathbf{v}}$ of $\mathbf{a}$-weighted stable pairs with volume of $\text{vol}(K_X + \sum a_i D_i) = v$ in all dimensions. For convenience we often suppress the volume $v$ or consider instead $\mathcal{K}_{\mathbf{a}} := \bigsqcup_v \mathcal{K}_{\mathbf{a}, v}$. This is because unlike the arithmetic genus in dimension one, the volume will vary as a function of the weight vector $\mathbf{a}$ and thus changes under wall-crossing morphisms.

The basic idea then behind Theorem 1.1 is to consider the universal $\mathbf{a}$-weighted stable family $(\mathcal{X}_{\mathbf{a}}, \sum a_i D_i)$ and run the minimal model program with scaling. This produces the log canonical model of $(\mathcal{X}_{\mathbf{a}}, \sum b_i D_i)/\mathbb{N}_{\mathbf{a}}$ and the birational map $h$. We then need to check that this is indeed a stable family of $\mathbf{b}$-weighted pairs which then induces the wall-crossing morphism $\rho_{\mathbf{b}, \mathbf{a}}$. The finite wall-and-chamber decomposition is ultimately a consequence of [BCHM10, Corollary 1.1.5].

One complication of the higher dimensional case is that $h$ is in general not a morphism due to the existence of flips. A more serious challenge is that, contrary to the one dimensional case, $\mathcal{K}_{\mathbf{a}, v}$ is in general very singular with many irreducible components parametrizing non-smoothable reducible varieties [Vak06, PP83]. Moreover, the minimal model program and even finite generation of the log canonical ring can fail for such non-smoothable varieties [Kol11]. In order to overcome this, we need to work with the closure of irreducible loci parametrizing normal crossings, or more generally...
klt pairs. Indeed one of the key insights of this paper is that wall-crossing for moduli of stable pairs is controlled by the minimal model program with scaling on the total spaces of 1-parameter smoothings of the slc pairs on the boundary. Finally, in order to apply the strategy described above, we need to work over some smooth base (e.g. a compactification of $B$ in Theorem 1.1) and then descend to the seminormalization or normalization of the corresponding moduli space.

1.1. Statements of the main results. We are now ready to state our main results in full generality. Fix some weight vector $a = (a_1, \ldots, a_n)$ of rational numbers $a_i \in (0, 1] \cap \mathbb{Q}$. Let $f : (X, \sum a_iD_i) \to B$ be a locally stable family (Definition 2.13).

**Definition 1.3.** We say that a weight vector $b = (b_1, \ldots, b_n)$ is admissible if $(X, \sum b_iD_i) \to B$ is locally stable and $K_X + \sum b_iD_i$ is $f$-big. We say that a polytope $P \subset ((0, 1] \cap \mathbb{Q})^n$ is admissible if every vector $b \in P$ is admissible.

**Notation 1.4.** For $b \leq a$ admissible weight vectors, we define $v(t) = ta + (1 - t)b$ for $t \in [0, 1]$.

**Notation 1.5.** For any weight vector $v = (v_1, \ldots, v_n)$ we denote by $vD$ the divisor $\sum v_iD_i$.

Let $\mathcal{K}^o \subset \mathcal{K}_a$ be a quasicompact locally closed substack of the space of $a$-stable pairs and and suppose that $\mathcal{K}^o$ parametrizes klt pairs. Let $f^\circ : (\mathcal{X}^o, aD^o) \to \mathcal{K}^o$ denote the universal family of klt $a$-stable pairs over $\mathcal{K}^o$. Fix an admissible weight vector $b \leq a$ for $f^\circ$. For each $t \in [0, 1]$ we have a set theoretic map

$$\phi_t : \mathcal{K}^o(k) \to \mathcal{K}_{v(t)}(k)$$

which takes a point $x : \text{Spec} k \to \mathcal{K}^o$ classifying the klt $a$-stable pair $(X, aD)$ to the point $x_{v(t)} : \text{Spec}(k) \to \mathcal{K}_{v(t)}(k)$ classifying the log canonical model of $(X, v(t)D)$.

**Definition 1.6.** For each $t \in [0, 1]$ we let $\mathcal{M}_t$ denote the seminormalization of the closure of the image of $\phi_t$ and we let $\mathcal{N}_t$ denote the normalization of $\mathcal{M}_t$. We let $\mathcal{M}_a$ (resp. $\mathcal{N}_a$) denote $\mathcal{M}_0$ (resp. $\mathcal{N}_1$) and similarly for $N$.

**Remark 1.7.** Note that $\mathcal{M}_t$ and $\mathcal{N}_t$ are proper Deligne-Mumford stacks with families of $v(t)$-stable pairs pulled back from the universal family of $\mathcal{K}_{v(t)}$. Moreover, since seminormalization is functorial, the family over $\mathcal{M}_t$, which we denote $(\mathcal{X}_t, v(t)D_t) \to \mathcal{M}_t$, is universal for $v(t)$-stable families $g : (Z, v(t)\Delta) \to T$ over seminormal base schemes $T$ such that for each $t \in T$, $g_t$ is the limit of a family of log canonical models of the pairs parametrized by $f^\circ$.

**Theorem 1.8** (Theorem 4.4, Corollary 4.12 and Theorem 5.1). There exist finitely many rational numbers $t_i \in [0, 1] \cap \mathbb{Q}$ with $0 < t_1 < \ldots < t_m < 1$ such that the following hold.

(1) For each $t_i < s < s' < t_{i+1}$, $\mathcal{M}_s \cong \mathcal{M}_{s'}$ and the universal families $(\mathcal{X}_s, v(s)D_s)$ and $(\mathcal{X}_{s'}, v(s')D_{s'})$ have isomorphic underlying marked families so that

$$(\mathcal{X}_{s'}, v(s')D_{s'}) \cong (\mathcal{X}_s, v(s)D_s).$$

Moreover, these isomorphisms fit in a commutative diagram below.

$$\begin{array}{ccc}
\mathcal{X}_s & \cong & \mathcal{X}_{s'} \\
\downarrow & & \downarrow \\
\mathcal{M}_s & \cong & \mathcal{M}_{s'}
\end{array}$$
(2) For each consecutive pair \( t_i < t_{i+1} \), and any \( t_i < s < t_{i+1} \) there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}_{t_i} & \longrightarrow & \mathcal{X}_s \\
\downarrow & & \downarrow \\
\mathcal{M}_{t_i} & \longrightarrow & \mathcal{M}_s \\
\alpha_{t_i} & & \beta_{t_{i+1}} \\
\mathcal{M}_{t_i} & \longrightarrow & \mathcal{M}_{t_{i+1}} \\
& & \beta_{t_{i+1}} \\
& & \mathcal{M}_{t_{i+1}}
\end{array}
\]

where the morphism \( \mathcal{X}_s \to \mathcal{M}_s \) in the middle is independent of \( s \) by part (1).

(3) There is a dense open substack \( \mathcal{U} \subset \mathcal{M}_s \) parametrizing klt pairs such that for each \( u \in \mathcal{U} \) classifying the klt \( \mathbf{v}(s) \)-stable pair \((\mathcal{X}_u, \mathbf{v}(s)\mathcal{D}_u), \alpha_{t_i}(u) \) classifies the log canonical model of \((\mathcal{X}_u, \mathbf{v}(t_i)\mathcal{D}_u) \) and \( \beta_{t_{i+1}}(u) \) classifies the log canonical model of \((\mathcal{X}_u, \mathbf{v}(t_i+1)\mathcal{D}_u) \).

In particular, Theorem 1.8 shows that there are finitely many walls \( t_i \) and finitely many moduli spaces parametrizing log canonical models of the fibers of \( f^\circ \) as we reduce weights from \( \mathbf{a} \) to \( \mathbf{b} \) along the line \( \mathbf{v}(t) \). Moreover, around each wall, the moduli spaces are related via the morphisms \( \alpha_{t_i} \) and \( \beta_{t_i} \) which we call flip-like morphisms as they are induced by flips in the mmp with scaling as one reduces weights from \( t_i + \epsilon \) to \( t_i - \epsilon \). This is a higher dimensional phenomenon not witnessed in the case of curves.

In order to obtain reduction morphisms as in [Has03] and in Theorem 1.1, we need to invert \( \beta_{t_i} \).

In general, this is only possible up to normalization (see Section 8.1 for an example).

**Theorem 1.9** (Theorem 6.1 and Theorem 7.6). The morphism \( \beta_{t_i} : \mathcal{M}_{t_i-\epsilon} \to \mathcal{M}_{t_i} \) is quasi-finite, proper, birational and representable. In particular, the induced morphism on normalizations \( \beta_{t_i}^n : \mathcal{N}_{t_i-\epsilon} \to \mathcal{N}_{t_i} \) is an isomorphism.

Theorem 1.9 allows us to define reduction morphisms \( \rho_{b,a} : \mathcal{N}_a \to \mathcal{N}_b \) by composing the induced maps \( \alpha_{t_i}^n \) on normalizations with the inverses of \( \beta_{t_i}^n \) for all \((\mathbf{a} \to \mathbf{b})\)-walls (see Definition 7.3). Under the assumption that the generic fiber of \( f^\circ \) is \( \mathbf{v}(t) \)-stable for all \( t \in [0,1] \), which is the case for example in dimension 1 as well as in the setting of Theorem 1.1, we have the following.

**Theorem 1.10** (Theorem 8.1 and Corollary 7.10). Let \( P \) be an admissible polytope of weight vectors such that that the generic fiber of the universal family \((\mathcal{X}_a, a\mathcal{D}) \to \mathcal{M}_a \) is \( \mathbf{v} \)-stable for all \( \mathbf{v} \in P \). Then for all \( \mathbf{b} \leq \mathbf{a} \) in \( P \), the reduction morphisms \( \rho_{b,a} : \mathcal{N}_a \to \mathcal{N}_b \) are birational and independent of the choice of path from \( \mathbf{a} \) to \( \mathbf{b} \). In particular,

\[ \rho_{c,b} \circ \rho_{b,a} = \rho_{c,a}. \]

In Section 8, we give several examples illustrating that Theorem 1.10 is subtle without the extra assumption on the generic fiber of the universal family.

### 1.2. Relations to other work.

The behavior of stable pairs moduli under changing the coefficients has been studied in a few previous cases. In [Ale15], Alexeev constructed compact moduli spaces of weighted stable hyperplane arrangements. These are moduli spaces parametrizing pairs \((X, \sum a_i H_i)\), where \( X \) is a degeneration of \( \mathbb{P}^n \) and the \( H_i \) are the limits of hyperplanes. Among other things, Alexeev shows that there are wall-crossing morphisms as one varies the weights on the \( H_i \) as in Theorem 1.1. This provides alternate compactifications of the spaces of Hacking-Keel-Tevelev [HKT06]. Similarly, in [AB20] compact moduli spaces of weighted stable elliptic surfaces are constructed (see also [Inc20]). These moduli spaces parametrize pairs of an elliptic surface with the divisor consisting of a section and some weighted (possibly singular) fibers. It is proven that these moduli spaces also satisfy the above wall-crossing morphisms as the weight vector varies.
A similar phenomenon has also been recently studied from the viewpoint of K-moduli [ADL19]. Wall-crossing morphisms play an important role in the study of explicit moduli compactifications, their birational geometry, and for the sake of computations on compact moduli spaces (see e.g. [AB19, AB18], the related Hassett-Keel program [FS10], variation of GIT [DH98, Tha96], and the Hassett-Keel-Looijenga program [LO18, LO19, LO21, ADL20]).

Conventions. We work over an algebraically closed field $k$ of characteristic 0. All schemes are finite type over $k$, unless otherwise stated. In particular, we will omit the adjective “geometrically” when we deal with reduced schemes. A point will be a closed point, unless otherwise stated. Given a morphism $f : \mathcal{X} \to \mathcal{Y}$ between two separated Deligne-Mumford stacks, the closure of the image of $f$ will be defined as follows. If $X$ (resp. $Y$) is the coarse space of $\mathcal{X}$ (resp. $\mathcal{Y}$) and $g$ is the morphism $X \to Y$ induced by $f$, then the closure of the image of $f$ will be $\overline{g(X)} \times_Y Y$. Unless otherwise specified, when we talk about a pair $(X, D)$ we assume that $D > 0$ and that $D$ has rational coefficients. For $a = (a_1, \ldots, a_n) \in \mathbb{Q}^n_{>0}$ and divisors $D_1, \ldots, D_n$, we will adopt the notation

$$aD := \sum a_i D_i.$$ 

We will adopt the same conventions as in [Kol18a, Conventions 4.16.7]. Namely, if $D$ is a Weil divisor such that each irreducible component of $\text{Supp}(D)$ intersects the smooth locus of $X$, we will make no distinction between $D$ and its associated divisorial subsheaf. A divisor $D$ on a normal variety $X$ is big if $X$ admits $s : \tilde{X} \to X$ a small $\mathbb{Q}$–factorial modification (see Theorem 3.1), and $s^{-1}_*(D)$ is big.

Acknowledgements. We thank Dan Abramovich, Kristin DeVleming, Brendan Hassett, Stefan Kebekus, János Kollár, Sándor Kovács, Yuchen Liu, Martin Olsson, Roberto Svaldi, Jakub Witaszek, and Chenyang Xu for helpful discussions. Parts of this paper were completed while authors were in residence at MSRI in Spring 2019 (NSF No. DMS-1440140). K. A. and D. B. were supported in part by NSF Postdoctoral Fellowships. K. A. partially supported by NSF grant DMS-2140781 (formerly DMS-2001408). G. I. was partially supported by funds from NSF grant DMS-1759514. Zs. P. was partially supported by the following grants: grant #200021/169639 from the Swiss National Science Foundation, ERC Starting grant #804334.

2. THE MODULI SPACE OF STABLE LOG VARIETIES

In this section we recall the definitions and basic setup of the moduli of stable log varieties (or stable pairs). We refer the reader to [Kol18a, Kol13] for more details on this formalism, and to [KM98, Section 2.3] for the singularities of the MMP. We begin by recalling the particular kind of singularities appearing on stable log varieties (see [Kol13, Chapter 5]).

Definition 2.1. A scheme $X$ is deminormal if it is $S_2$, and the singularities in codimension one are at worse nodal singularities.

Let $\nu : X^n \to X$ be the normalization of a deminormal scheme. The conductor ideal

$$\text{Ann}(\nu_* \mathcal{O}_{X^n}/\mathcal{O}_X) \subset \mathcal{O}_X$$

define reduced pure codimension 1 closed subschemes $D \subset X$ and $\tilde{D} \subset X^n$ collectively referred to as the double locus.
Definition 2.2. Let \((X, \Delta)\) be a pair consisting of a deminormal variety \(X\) and an effective Weil \(\mathbb{Q}\)-divisor \(\Delta\) whose support does not contain any irreducible component of the double locus. We say \((X, \Delta)\) has semi-log canonical singularities (abbreviated slc) if
- \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier, and
- \((X^n, \nu^* \Delta + \bar{D})\) is log canonical.

Definition 2.3. A stable log variety or stable pair is a pair \((X, \Delta)\) such that \((X, \Delta)\) has semi-log canonical singularities and \(K_X + \Delta\) is ample.

Definition 2.4. Given an slc pair \((X, D)\) with \(K_X + D\) big and semiample, consider the scheme
\[
Y := \text{Proj}(\bigoplus_{n \geq 0} H^0(O_X(n(K_X + D)))) = \text{Proj} R(X, K_X + D).
\]
There is a morphism \(f : X \to Y\), and we refer to the pair \((Y, f_* D)\) as the stable model of \((X, D)\). When \((X, D)\) has klt singularities, this is the log canonical model of the pair.

The following is essential for defining families of stable pairs.

Definition 2.5 ([Kol18a, Definition 4.1 and Theorem 4.3]). A family of pairs \(f : (X, D) \to S\) over a reduced base scheme \(S\) is the data of a morphism \(f : X \to S\) and an effective Weil \(\mathbb{Z}\)-divisor \(D\) of \(X\). This data has to satisfy the following conditions:
- \(f : X \to S\) is flat with reduced fibers of pure dimension \(n\);
- The fibers of \(\text{Supp}(D) \to S\) are pure dimensional of dimension \(n - 1\) or empty;
- \(f\) is smooth at the generic points of \(X_s \cap \text{Supp}(D)\), and
- For every \(s \in S\) we have that \(D\) is Cartier in \(X\), locally around a generic point of \(\text{Supp}(D) \cap X_s\).

Remark 2.6. The last point above is automatic when \(S\) is normal, given the first three (see [Kol18a, Theorem 4.5]). Moreover, by our assumptions we need not distinguish between a Weil divisor and its associated divisorial subscheme (see [Kol18a, Section 4.3]).

In our case, since there is a relatively big open set \(U \subseteq X\) such that \(O_U(-D|_U) \subseteq O_U\) is a relative line bundle, after each base-change \(S' \to S\) the pullback is still a line bundle on the pullback \(U_{S'}\). This gives a pullback operation on \(U\), and we can extend divisorially to get the pulled back family of divisors on \(X_{S'}\). This gives a way to pull back a family of \(\mathbb{Z}\)-divisors, and in the case where we instead have a \(\mathbb{Q}\)-divisor, we can choose an \(m\) divisible enough so that \(mD\) is a \(\mathbb{Z}\)-divisor, pull it back as before, and divide the resulting divisor by \(m\). This is known as the pull-back with the common denominator definition [Kol18a, 4.4.2] and is independent of \(m\) by [Kol18a, Theorem 4.5 (2)].

Notation 2.7. Given a morphism \(g : S' \to S\) and a projective family of pairs \(f : (X, D) \to S\), we will denote with \((X_{S'}, D_{S'}) \to S'\) the pull-back, defined as above, of \(f\) along \(g\).

Finally recall that if \(f : (X, D) \to S\) and \(S' \to S\) are as above, then
\[
\text{Supp}(D_{S'}) = \text{Supp}(h^{-1}(\text{Supp}(D))),
\]
where \(h : X_{S'} \to X\) is the projection (see [Kol18a, Chapter 4]).

Since in our case it is useful to keep track of the various components of \(D\), we recall the following.
Definition 2.8 ([Kol18a, Definition 4.60]). A family of varieties marked with $m$ divisors or an $m$-marked family over a reduced scheme $S$ is the data of $f : (X; D_1, ..., D_m) \rightarrow S$ satisfying the following condition: for every $i$, the pair $(X, D_i) \rightarrow S$ is a family of pairs, and $X \rightarrow S$ is flat with connected and $S_2$-fibers.

Fix $(a_i)_{i=1}^m \in (\mathbb{Q} \cap (0, 1))^m$, and consider an $m$-marked family $f : (X; D_1, ..., D_m) \rightarrow S$ such that for every $s \in S$, the pair $(X_s, a_1(D_1)_s + \ldots + a_m(D_m)_s)$ is stable. The functor of such families is not well-behave. Therefore, on needs the following notion of stable families:

Definition 2.9 ([Kol18a, Definition-Theorem 4.45 and 4.70.3]). A family of varieties marked with divisors $f : (X; D_1, ..., D_m) \rightarrow B$ over a reduced scheme $B$ is stable with coefficients in $a = (a_1, ..., a_m)$ if $K_X/B + \sum a_i D_i$ is $\mathbb{Q}$-Cartier and the fibers $(X_b, \sum a_i(D_i)_b)$ are stable pairs. We will often write that $f : (X, \sum a_i D_i) \rightarrow B$ is a stable family, or that $f$ is stable.

Given $f : (X, \sum a_i D_i) \rightarrow B$ a stable family, let $\delta(a)$ be the smallest common denominator of the $a_i$. Then given a polynomial $p(x)$, we can consider the stable families $f : (X, \sum a_i D_i) \rightarrow B$ such that the Hilbert function $\chi(X_b, \omega_{X_b}^{[\delta(a)]}(\sum m\delta(a)a_i D_i))$ agrees with $p(m)$. Moreover, we can also fix an integer $m$ and require that $L := \mathcal{O}_X(m(K_X + \sum a_i D_i))$ is Cartier, and that $R^i f_* L = 0$ for $i > 0$. This produces a moduli space, which represents a pseudofunctor over seminormal bases:

Theorem 2.10 ([Kol18a, Theorem 4.79]). Fix a function $p : \mathbb{N} \rightarrow \mathbb{N}$, an integer $m$, and a vector of positive rational numbers $a$. Then there is a moduli space $\mathcal{MSP}^{sn}_m(a, p)$ which, for $B$ seminormal, represents the moduli problem of stable families $f : (X, \sum a_i D_i) \rightarrow B$ such that:

1. $\mathcal{O}_X(m(K_X + \sum a_i D_i))$ Cartier;
2. $R^i f_* (\mathcal{O}_X(m(K_X + \sum a_i D_i))) = 0$ for $i > 0$, and
3. $\chi(X_b, \omega_{X_b}^{[\delta(a)]}(\sum r\delta(a)a_i D_i)) = p(r)$ for every $r$.

Remark 2.11. We remark that condition (2) is required in Theorem 2.10 because in [Kol18a, Subsection 4.7] Kollár works with strong polarizations (see [Kol18a, 3.75]) rather than polarizations.

Since seminormalization is functorial (see e.g. [Cor07, Lemma 8.8.6]), we can assume without loss of generality that $\mathcal{MSP}^{sn}_m(a, p)$ is seminormal. If $m_1 | m_2$, then there is a morphism $\mathcal{MSP}^{sn}_{m_2}(a, p) \rightarrow \mathcal{MSP}^{sn}_{m_1}(a, p)$. Moreover, observe that specifying $p$ fixes the volume of the stable pairs parametrized by $\mathcal{MSP}^{sn}_m(a, p)$. By [HMX18] the family of pairs parametrized by $\mathcal{MSP}^{sn}_m(a, p)$ is bounded, in particular by [Kol18a, Lemma 3.63] there is an uniform $m_0$ such that, for every $r$ and for every pair $(X, \sum a_i D_i)$ parametrized by $\mathcal{MSP}^{sn}_{m_0}(a, p)$, (1) and (2) in Theorem 2.10 hold with $m = m_0$.

If we consider the morphism $\mathcal{MSP}^{sn}_{m_0}(a, p) \rightarrow \mathcal{MSP}^{sn}_{m_0 r}(a, p)$ as a morphism of fibered categories, the restriction to the fiber over $\text{Spec}(k)$ is an equivalence of groupoids. Therefore if we show that the source is proper, since the source and the target are seminormal, such a morphism would be an isomorphism (this is a slight generalization of Zariski’s main theorem, see [Inc20, Proposition 8.2]). The properness of $\mathcal{MSP}^{sn}_{m_0}(a, p)$, for $m_0$ divisible enough then follows from [Kol18a, Theorem 4.79 and Complement 4.80].

Notation 2.12. Given a polynomial $p$ and a weight vector $a \in (\mathbb{Q} \cap (0, 1))^n$, we denote by $\mathcal{K}_{a, p}$ the moduli space $\mathcal{MSP}^{sn}_{m_0}(a, p)$ we described above. We denote by $\mathcal{K}_a := \bigcup_p \mathcal{M}_{a, p}$.

Finally, we will need the notion of a locally stable family.
Definition 2.13. [Kol18a, Definition-Theorem 4.45] Let $S$ be a reduced scheme and $f : (X, \Delta) \to S$ a projective family of pairs. Then $f : (X, \Delta) \to S$ is locally stable or slc if the following equivalent conditions hold.

1. $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier and the fibers $(X_s, \Delta_s)$ are slc for all points $s \in S$.
2. $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier and $(X_s, \Delta_s)$ is slc for all closed points $s \in S$.
3. $f_T : (X_T, \Delta_T) \to T$ is locally stable whenever $T$ is the spectrum of a DVR and $q : T \to S$ is a morphism.

Remark 2.14. Note that the definition of a family of stable pairs over a reduced base is étale local. Therefore, the space $\mathcal{K}_a$ represents the functor of stable families with coefficients $a$ for reduced Deligne-Mumford stacks.

Remark 2.15. Kollár has introduced a condition on the reflexive powers of relative pluri-canonical sheaves (see [Kol18a, Chapter 9] and also [AH11, BI21]) and the K-flatness condition on the family of divisors [Kol19] which give a well behaved functor of stable families over arbitrary bases representable by a Deligne-Mumford stack locally of finite type whose seminormalization is the space $\mathcal{K}_a$ introduced above. The reason we avoid this and work with seminormalizations in this paper is twofold. First, checking these conditions over non-reduced bases is subtle and its not clear that K-flatness in particular is preserved by the constructions in this paper (see especially the proof of Theorem 5.1). Second, the reduction morphisms we produce are ultimaely only well defined on the normalization of the moduli space (see Section 8.1).

3. Preliminaries from the MMP

In this section, we collect some preliminary results from the minimal model program that we need for the proofs of the main theorems.

3.1. dlt Modifications and log canonical models. One of the main obstacles in “reducing weights” on the divisor in a stable pair, is that the pair is not necessarily $\mathbb{Q}$-factorial. Indeed while for a pair $(X, D)$ the divisor $K_X + D$ is required to be $\mathbb{Q}$-Cartier, there is no reason for $D$ itself to be $\mathbb{Q}$-Cartier. A somewhat standard approach that allows one to perturb coefficients on a divisor is using dlt modifications.

Theorem 3.1 (Small dlt modification). [Kol13, Corollary 1.37] Let $(X, D)$ be a dlt pair with $D$ a boundary. There is a proper birational morphism $g : \bar{X} \to X$ such that

1. $\bar{X}$ is $\mathbb{Q}$-factorial,
2. the morphism $g$ is small,
3. $\left(\bar{X}, g_*^{-1}D\right)$ is dlt, and
4. $\text{discrep}\left(\bar{X}, g_*^{-1}D\right) = \text{discrep}(X, D)$.

Theorem 3.1 allows us to make the following crucial definition.

Definition 3.2. Let $f : (X, D) \to B$ be a projective morphism such that $(X, D)$ is a dlt pair and let $\Delta$ be any Weil divisor on $X$. We say that $K_X + \Delta$ is $f$-big if $K_{\bar{X}} + g_*^{-1}\Delta$ is $(f \circ g)$-big for $g : \bar{X} \to X$ any small $\mathbb{Q}$-factorial modification.

We will need the following standard lemma and its corollary.
Lemma 3.3. Let \( f : (X, D_X) \to (Y, D_Y) \) be a birational rational map of klt pairs which is an isomorphism in codimension one on both \( X \) and \( Y \), and assume that \( f_*(D_X) = D_Y \). Assume further that the log canonical models of \((X, D_X)\) and \((Y, D_Y)\) exist. Then \( f \) induces an isomorphism of log canonical models.

Proof. Let \( L_X := \mathcal{O}_X(m(K_X + D_X)) \) and \( L_Y := \mathcal{O}_Y(m(K_Y + D_Y)) \) with \( m \) so that they are both line bundles. Then if \( U \) is the open subset where \( X \) and \( Y \) are isomorphic,
\[
H^0(X, L_X^{\otimes m}) = H^0(U, (L_X^{\otimes m})_U) = H^0(U, (L_Y^{\otimes m})_U) = H^0(Y, L_Y^{\otimes m})
\]
since the complement of \( U \) has codimension at least 2 in both \( X \) and \( Y \). Then the log canonical models of \( X \) and \( Y \) are Proj of the same graded algebra. \( \square \)

Corollary 3.4. Let \((X, D)\) be a klt pair, and let \( p : X' \to X \) and \( q : X'' \to X \) be two small dlt modifications. Then:

1. The pairs \((X', p^{-1}_*(D))\) and \((X'', q^{-1}_*(D))\) are klt, and
2. The pairs in (1) have the same log canonical model if it exists.

Proof. (1) follows from [KM98, Lemma 2.30]. We now show (2). Since \( X' \to X \) and \( X'' \to X \) are isomorphisms in codimension one, so is \( X' \to X'' \). The result then follows from Lemma 3.3. \( \square \)

Notation 3.5. Consider a klt pair \((X, \sum a_iD_i)\) and let \( 0 < b_i \leq a_i \). Let \( \widetilde{X} \to X \) be a small dlt modification as above, and let \( \widetilde{D}_i \subseteq \widetilde{X} \) be the proper transform of \( D_i \). We will refer to the log canonical model of \( \left(\widetilde{X}, \sum b_i\widetilde{D}_i\right) \) as “the log canonical model of \((X, \sum b_iD_i)\)”. This is independent of the choice of \( \widetilde{X} \) by Corollary 3.4.

We also need the following version of the base point free theorem for degenerations of klt pairs.

Lemma 3.6. Let \( R \) be a DVR essentially of finite type over \( k \) with closed point \( p \). Let \((X, D)\) be a klt pair with a flat proper morphism \( f : X \to \text{Spec}(R) \). If \( L \) is a nef line bundle such that \( L - K_X - D \) is \( f \)-nef and big, then for \( m \) divisible enough, \( L^{\otimes m} \) is base point free and the morphism induced by \( |L^{\otimes m}| \) on \( X \) restricts to the morphism induced by \( |L^{\otimes m}_{|X_p}| \) on \( X_p \).

In particular, if \((X, D) \to \text{Spec}(R)\) is a stable family (see Section 2) such that \((X, D)\) is klt and \( K_X + D \) is \( f \)-nef and big, then \( m(K_X + D)|_{X_p} = m(K_{X_p} + D_p) \) is semi-ample for \( m \) divisible enough.

Proof. We know from the base point free theorem [KMM87, Theorem 6-1-13], for \( m \) divisible enough, \( L^{\otimes m} \) is globally generated and thus its restriction to a fiber is as well. To conclude, note that \( R^1f_*L = 0 \) from relative Kawamata-Viehweg vanishing and thus by cohomology and base change, \( H^0(X_p, L^{\otimes m}_{|X_p}) = H^0(X, L^{\otimes m})|_{X_p} \). \( \square \)

3.2. MMP with scaling. In this subsection, we recall the version of the MMP with scaling we will use throughout the paper. We refer the reader to [HK10] and [BCHM10] for more details.

Let \((X, D)\) be a \( \mathbb{Q} \)-factorial pair with \( D \) a big divisor. Assume that \( K_X + D \) is big, and let \( H \) be an effective divisor such that the pair \((X, D + H)\) is a klt stable pair. Then to obtain the stable model of \((X, D)\) one can first run an MMP for \((X, D)\) with scaling by \( H \) to obtain a minimal model \((X^{\text{min}}, D^{\text{min}})\) of \((X, D)\) [BCHM10, Corollary 1.4.2]. After that one can apply the base point free theorem to the klt pair \((X^{\text{min}}, D^{\text{min}})\) to get the stable model.
In our setting, we only assume that $K_X + D$ is big but not necessarily that $D$ is big. In this case, we may pick a big effective divisor
\[ D' \sim_{\mathbb{Q}} K_X + D \]
such that $(X, D + D' + H)$ is klt. Then the log canonical model of $(X, D + D')$ is the same as that of $(X, D)$ so we can run MMP with scaling by $H$ on $(X, D + D')$ where now the boundary divisor $D + D'$ is big, and then apply the base point free theorem to compute the log canonical model. In particular, we may apply this method to a small dlt modification to compute the log canonical models of $(X, D + tH)$ for $t \in [0, 1]$ where $H$ is effective and $(X, D + H)$ is a klt stable pair.

**Proposition 3.7.** Let $(X, D) \to B$ be a klt and $\mathbb{Q}$-factorial pair over $B$ with both $D$ or $K_X + D$ big over $B$. Let $H$ be an effective divisor so that the pair $(X, D + H)$ is klt, and let $t_0 \in (0, 1)$. Assume that $K_X + D + tH$ is nef over $B$ for every $t_0 \leq t \leq 1$. Let $(Y, D_Y + t_0H_Y)$ be the log canonical model of $(X, D + t_0H)$ over $B$. Let $(Z, (t_0 - \epsilon)H_Z)$ denote the log canonical model of $(Y, D + (t_0 - \epsilon)H)$ over $B$ for $\epsilon$ small enough (see Notation 3.5).

Then, for all $\epsilon > 0$ small enough, we have that:

(I) There is a birational morphism $Z \to Y$ whose inverse $Y \dasharrow Z$ does not contract divisors, and

(II) $(Z, (t_0 - \epsilon)H_Z)$ is the log canonical model of $(X, D + (t_0 - \epsilon)H)$ over $B$.

The following will be used several times throughout the proof of Proposition 3.7.

**Lemma 3.8.** Let $X$ be a normal variety with two $\mathbb{Q}$-Cartier divisors $D_1$ and $D_2$, and let $\lambda \in (0, 1)$. Assume that $D_1$ is nef and for every curve $C$ that satisfies $(D_2).C < 0$, we have $(\lambda D_1 + (1 - \lambda)D_2).C > 0$. Then $D_\lambda := (\lambda D_1 + (1 - \lambda)D_2)$ is also nef, and $D_\lambda.C = 0$ if and only if $(D_1).C = (D_2).C = 0$.

**Proof.** We need to show that $D_\lambda.C \geq 0$ for any irreducible projective curve $C$. Since $D_1$ is nef, we have $(\lambda D_1).C \geq 0$ since $\lambda \in (0, 1)$. Furthermore, by assumption $D_\lambda.C > 0$ if $(D_2).C < 0$. Therefore, for the first result we only need to check in the case $(D_2).C \geq 0$, but in this case both summands are non-negative and so the conclusion holds. Finally, since $(\lambda D_1).C \geq 0$, the only way for $D_\lambda.C = 0$ is for $(D_2).C \leq 0$. If $(D_2).C = 0$ we are done since this would imply $(\lambda D_1).C = 0$. Otherwise, $(D_2).C < 0$ and by assumption $D_\lambda.C > 0$ which gives a contradiction. \hfill \Box

**Proof of Proposition 3.7.** The base $B$ will not play any significant role throughout the proof, so we will prove the result only in the case where $B = \text{Spec}(k)$. The same proof goes through with any $B$.

We first prove (I). Consider $\tilde{Y} \to Y$ a small dlt modification of $Y$. After running an MMP for $(\tilde{Y}, \tilde{D})$ over $Y$, we can assume that $K_{\tilde{Y}} + \tilde{D}$ is nef over $Y$. Then there is a morphism $\pi : \tilde{Y} \to Y$ which is the log canonical model of $(\tilde{Y}, \tilde{D} + t_0\tilde{H})$, so $\pi^*(K_Y + D_Y + t_0H) = K_{\tilde{Y}} + \tilde{D} + t_0\tilde{H}$. In particular, the latter is nef since it is the pull-back of an ample line bundle, and a curve $F$ gets contracted by $\pi$ if and only if $(K_{\tilde{Y}} + \tilde{D} + t_0\tilde{H}).F = 0$. Since $(\tilde{Y}, \tilde{D})$ is a minimal model over $Y$, for every curve $C \subseteq \tilde{Y}$ we have

\[(K_{\tilde{Y}} + \tilde{D}).C < 0 \implies (K_{\tilde{Y}} + \tilde{D} + t_0\tilde{H}).C > 0.\]

Since $\tilde{D}$ is big, there are finitely many $(K_{\tilde{Y}} + \tilde{D})$-negative extremal rays $\{C_i\}$ ([IJK10, Exercise 5.8]). Take any curve $C$ with $(K_{\tilde{Y}} + \tilde{D}).C < 0$. We may write $C = C_+ + \sum_i \lambda_i C_i$, where
\[
\left(K_{\tilde{Y}} + \tilde{D}\right) . C_+ \geq 0 \text{ and } \lambda_i \geq 0. \text{ Then for } \varepsilon > 0 \text{ we have}
\]
\[
\left(K_{\tilde{Y}} + \tilde{D} + (t_0 - \varepsilon)\tilde{H}\right) . C = \varepsilon \left(K_{\tilde{Y}} + \tilde{D} + t_0\tilde{H}\right) . C + (1 - 2\varepsilon) \left(K_{\tilde{Y}} + \tilde{D} + t_0\tilde{H}\right) . C_+
\]
\[
> 0 \text{ by (1)} \quad \geq 0, \text{ as } K_{\tilde{Y}} + \tilde{D} + t_0\tilde{H} \text{ is nef}
\]
\[
+ \varepsilon \left(K_{\tilde{Y}} + \tilde{D}\right) . C_+ + \sum_i \lambda_i \left((1 - 2\varepsilon) \left(K_{\tilde{Y}} + \tilde{D} + t_0\tilde{H}\right) . C_i + \varepsilon \left(K_{\tilde{Y}} + \tilde{D}\right) . C_i\right).
\]
In particular, as for every \( \varepsilon > 0 \) small enough the expression in the big parentheses above is positive for all \( i \), we obtain that for all \( \varepsilon > 0 \) small enough:
\[
\left(K_{\tilde{Y}} + \tilde{D}\right) . C < 0 \implies \left(K_{\tilde{Y}} + \tilde{D} + (t_0 - \varepsilon)\tilde{H}\right) . C > 0.
\]

From Lemma 3.8, for such an \( \varepsilon \) the divisor \( K_{\tilde{Y}} + \tilde{D} + (t_0 - \varepsilon)\tilde{H} \) is nef. Then to take the log canonical model we use the base point free theorem, and by Lemma 3.8 if a curve gets contracted through \( \tilde{Y} \to Z \) then it also gets contracted through \( \tilde{Y} \to Y \). In particular:
- The rational map \( Z \to Y \) is a morphism;
- The contraction \( \tilde{Y} \to Y \) factors through \( \tilde{Y} \to Z \), and
- Since \( \tilde{Y} \to Y \) is small, also \( Z \to Y \) is small.

Thus we have shown (I). We now prove (II).

Let \((X^+, D^+)\) be a minimal model of \((X, D)\) over \( Y \). Proceeding as above, we can assume that \( \varepsilon \) is small enough so that
\[
(K_{X^+} + D^+).C < 0 \implies (K_{X^+} + D^+ + (t_0 - \varepsilon)H^+).C > 0.
\]
In particular, by Lemma 3.8, the divisor \( K_{X^+} + D^+ + (t_0 - \varepsilon)H^+ \) is nef and if \( C \) is a curve that gets contracted taking the log canonical model of \((X^+, D^+ + (t_0 - \varepsilon)H^+)\), then it satisfies \((K_{X^+} + D^+ + t_0H^+).C = 0\). Therefore, if we denote with \( \pi : X^+ \to X^c \) the log canonical model of \((X^+, D^+ + (t_0 - \varepsilon)H^+)\), the morphism \( X^+ \to Y \) factors through \( \pi \). In particular, the rational map \( X^c \dashrightarrow Y \) is a morphism.

We now show that \( X^c \to Y \) is small by showing that it does not contract any divisor. Or in other words, we show that if \( E \) is a divisor in \( X^+ \) contracted by \( X^+ \to Y \), then it is contracted by \( \pi \). We begin by resolving the indeterminacy locus of \( X \dashrightarrow X^+ \) with \( W \) as below

\[
\begin{tikzcd}
W & X^+ \arrow[l, swap, \rightarrow] \arrow[r, \rightarrow] &
\end{tikzcd}
\]

The space \( W \) is obtained by taking the normalization of the closure in \( X \times X^+ \) of the graph of the rational map \( X \dashrightarrow X^+ \). We will denote with \( D_W \) the divisor (a priori not effective) such that \( a^* (K_X + D) = K_W + D_W \). Moreover, we will denote with \( H_W := a^* H \), and \( F \) will be such that \( b^* (K_{X^+} + D^+) = K_W + D_W - F \). Observe that \( F \) is an effective divisor since \( X \dashrightarrow X^+ \) is a minimal model (see [KM98, Definition 3.50]). Moreover, \( F \) is supported on the exceptional locus of \( b \). Finally, observe that

\[
b^* (K_{X^+} + D^+ + t_0H^+) = a^* (K_X + D + t_0H) = K_W + D_W + t_0H_W,
\]
where the first equality can be deduced as follows: $Y$ is the log canonical model of the $(X, D + t_0H)$, where $K_X + D + t_0H$ is nef, hence $K_X + D + t_0H$ is the pullback of $K_Y + D_Y + t_0H_Y$, and then pushing this forward to $X^+$ yields that $K_{X^+} + D^+ + t_0H^+$ is also the pullback of $K_Y + D_Y + t_0H_Y$.

We summarize the previous paragraph below:

- $a^*(K_X + D) = K_W + D_W$ (by definition of $D_W$);
- $a^*(H) = H_W$ (by definition of $H_W$);
- $b^*(K_{X^+} + D^+) = K_W + D_W - F$ with $F \geq 0$ (by [KM98, Definition 3.50]), and
- $b^*(K_{X^+} + D^+ + t_0H^+) = K_W + D_W + t_0H_W$.

We now choose $C' \subseteq E$ a generic curve which satisfies the following conditions:

- $C'$ intersects the locus where $X^+ \to X$ is an isomorphism;
- $C'$ is contracted by $X^+ \to Y$, and
- $C'$ is not contained in $b(\text{Supp}(F))$.

Since $C'$ intersects the locus where $X^+ \to X$ and $b$ is an isomorphism, there is a curve $C$ such that $b_*C = C'$, and $C$ is not contained in $F$. In particular, $F.C \geq 0$ (observe that $F$ is $\mathbb{Q}$-Cartier since it is the difference of two $\mathbb{Q}$-Cartier divisors).

Now, $C'$ gets contracted through $X^+ \to Y$, so

$$0 = (K_{X^+} + D^+ + t_0H^+).C' = (K_W + D_W + t_0H_W).C.$$

Similarly, since $K_{X^+} + D^+$ is nef over $Y$, we have

$$0 \leq (K_{X^+} + D^+).C' = (K_W + D_W - F).C.$$

But then, from the fact that $b^*(K_{X^+} + D^+) = K_W + D_W - F$ with $F \geq 0$, and since $F.C \geq 0$, we have that

$$0 \leq (K_W + D_W - F).C \leq (K_W + D_W).C = (K_X + D).a_*C.$$

Similarly, since $a_*C$ gets contracted taking the stable model of $(X, D + t_0H)$, we have also that $(K_X + D + t_0H).a_*C = 0$. But $K_X + D + H$ is nef, so

$$0 \leq (1-t_0)(K_X + D).a_*C \leq (1-t_0)(K_X + D).a_*C + t_0(K_X + D + H).a_*C = (K_X + D + t_0H).a_*C = 0.$$

Therefore all the inequalities above are equalities, so $0 = (K_W + D_W - F).C = (K_{X^+} + D^+).C'$. We showed that if $E$ is an exceptional divisor for $X^+ \to Y$ and $C'$ is a generic curve in $E$ so that $(K_{X^+} + D^+ + t_0H^+).C' = 0$, we also have that $(K_{X^+} + D^+).C' = 0$. In particular, $(K_{X^+} + D^+ + (t_0 - \epsilon)H^+).C' = 0$ and $C'$ gets contracted by $\pi$. By (I), $X^c$ and $Z$ agree in codimension one, so they are isomorphic by Lemma 3.3.

To show (II) we need to show that if $c : X^+ \to Z$ is the induced morphism, then

$$0 \leq a^*(K_X + D + (t_0 - \epsilon)H) - b^*c^*(K_Y + D_Y + (t_0 - \epsilon)H_Y).$$

As $c^*(K_Y + D_Y + (t_0 - \epsilon)H_Y) = K_{X^+} + D^+ + (t_0 - \epsilon)H^+$, by (2) this is equivalent to showing that $0 \leq \epsilon(b^*H^+ - a^*H)$. However, using (2) and the fact that $a^*(K_X + D) - b^*(K_{X^+} + D^+) = F$ we obtain that

$$\epsilon(a^*H - b^*H^+) = -\frac{\epsilon}{t_0}F \leq 0$$

This concludes point (II).
4. Wall-crossing loci in the moduli space

The goal of this section is to define the natural moduli spaces $M_t$, depending on a parameter $t \in [0, 1]$, which admit a wall-crossing structure. The basic idea is as follows. Let $f : (X, aD) \to B$ an $a$-stable family of interest parametrized by some smooth and irreducible base $B$ and denote by $\mathbf{v}(t) := ta + (1 - t)b$ for $t \in [0, 1]$. Suppose furthermore that $K_{X/B} + bD$ is $f$-big. Then taking the relative log canonical model of $(X, \mathbf{v}(t))$ over $B$ gives us an a priori rational map $B \dashrightarrow \mathcal{K}_{\mathbf{v}(t)}$. We will see in Theorem 4.4 below that, under some mild assumptions, this extends to a morphism $\Phi_t : B \to \mathcal{K}_{\mathbf{v}(t)}$ which on some open set is induced by sending $b \in U \subset B$ to the point classifying the log canonical model of $(X_b, \mathbf{v}(t))$.

Then $M_t$, defined as the seminormalization of the scheme theoretic image of $\Phi_t$, carries a universal family of $\mathbf{v}(t)$-stable pairs which are limits of the log canonical models parametrized by $U$. We will see in Corollary 4.12 that, as $t$ varies, there are only finitely many different moduli spaces $M_t$ and finitely many universal families, up to rescaling the boundary.

We begin with the following two standard lemmas.

**Lemma 4.1.** Let $f : X \to B$ be a flat and proper morphism. Given a closed subset $C \subseteq X$ such that $\text{codim}_X(C) \geq r$, there exists a dense open subset $U \subseteq B$ such that for $b \in U$, we have $\text{codim}_{X_b}(C_b) \geq r$.

**Proof.** This follows from upper semicontinuity of fiber dimension [Sta18, Tag 0D4Q]. Up to replacing $B$ with its irreducible components, we can assume that $B$ is irreducible, so it has a unique generic point. Let $n$ be the dimension of the fibers of $f$. Since for the generic fiber of $C \to B$, we have that $\text{dim}(C) - \text{dim}(B) = n - r$, there is an open subset $U \subseteq B$ such that for $b \in U$, $\text{dim}(C_b) \leq n - r$. □

**Lemma 4.2.** Let $f : X \to B$ and $g : Y \to B$ be two proper and surjective morphisms between normal varieties. Let $\phi : X \dashrightarrow Y$ be a rational map over $B$ and let $\Delta_X$ be a $\mathbb{Q}$-divisor with support $D \subset X$ such that $f|_D$ is surjective. Denote $\Delta_Y := \phi_* (\Delta_X)$. Then there is a dense open subset $U \subseteq B$ where:

1. $f|_{X_U}$, $f|_{D_U}$, $g|_{Y_U}$ and $g|_{\Delta_U}$ are flat;
2. for $u \in U$, the fibers $X_u$ and $Y_u$ are normal;
3. for $u \in U$, $\phi$ is defined on an open subset in $V_u \subseteq X_u$ whose complement has codimension at least two and in particular $\phi$ induces a rational map $\phi_u : X_u \dashrightarrow Y_u$, and
4. we have $(\Delta_Y)|_{Y_u} = (\phi_u)_*(\Delta_X)|_{X_u}$ for every $u \in U$.

**Proof.** (1) is the content of [ACG11, Proposition 3.9]. For (2), let $\eta$ be the generic point of $B$. Then $X_\eta$ and $Y_\eta$ are normal. By [Gro66, Théorème 12.2.6], the locus where the fibers are normal is open. Since such a locus contains the generic point, it is non-empty. As for (3), observe that from the valuative criterion of properness, $\phi$ is defined on points of codimension one. Then Lemma 4.1 with $C$ the indeterminacy locus proves (3). We now show (4). Let $\pi : X' \to X$ be a birational morphism so that the composition $X' \to X \dashrightarrow Y$ extends to a morphism $\psi : X' \to Y$. Then $\psi_*(\pi^{-1}_*(\Delta_X)) = \Delta_Y$ so up to replacing $X$ with $X'$ and $\Delta_X$ with $\pi^{-1}_*(\Delta_X)$, we can assume that $\phi$ is a morphism.

Since $X$ is proper and $\phi$ is a morphism, $\phi(D)$ is closed, and we can write $\phi(D) = D' \cup C$ with $\text{Codim}_Y(C) \geq 2$ and $\text{Supp}(\Delta_Y) = D'$. From Lemma 4.1, up to shrinking $U$, we can assume that $\text{Codim}_{X_u}(C_u) \geq 2$ and that $D'$ does not contain $Y_u$ for every $u \in U$. The result then follows since $\phi(D_b) = \phi(D)_b$. □
Notation 4.3. For coefficient $n$-vectors $\mathbf{a}, \mathbf{b}$, we write $\mathbf{b} \leq \mathbf{a}$ if $b_i \leq a_i$ for all $i = 1, \ldots, n$. For $t \in [0, 1]$, we will denote $\mathbf{v}(t) := t\mathbf{a} + (1-t)\mathbf{b}$.

We are now ready to present the main theorem of this section.

Theorem 4.4. Let $f : (X, aD) \to B$ be a stable family over a smooth irreducible quasi-projective scheme $B$. Suppose that the generic fiber is klt and that $K_X + v(t)D$ is $f$-big for each $t \in [0, 1]$.

1. There exists a unique morphism $\Phi_t : B \to \mathcal{X}_{v(t)}$ and a nonempty open subset $U \subset B$ such that $\Phi_t(u)$ is the point classifying the log canonical model of $(X_u, v(t)D_u)$ for all $u \in U$;

2. There are finitely many $t_i \in \mathbb{Q}$, with $0 = t_0 < t_1 < \ldots < t_m = 1$, which satisfy the following condition. If we denote by $(Z_{t_i}, v(t_i)\Delta_{t_i})$ the family of stable pairs classified by $\Phi_{t_i}$, then for every $t_i < s_1 \leq s_2 < t_{i+1}$ the underlying $m$-marked families $(Z_{s_1}; \Delta_{s_1, 1}, \ldots, \Delta_{s_1, m})$ and $(Z_{s_2}; \Delta_{s_2, 1}, \ldots, \Delta_{s_2, m})$ are equal so that $(Z_{s_2}, v(s_2)\Delta_{s_2}) = (Z_{s_1}, v(s_1)\Delta_{s_1})$.

3. For every $t \in [0, 1]$, the stable family $f_t : (Z_t, v(t)\Delta_t) \to B$ is the relative log canonical model of $(X, v(t)D)$ over $B$.

Remark 4.5. Observe that, in the particular case where the divisor $K_X + v(t)D$ restricted to the generic fiber is ample for every $t \in [0, 1]$, we automatically have a non-empty open subset $U$ and a morphism $U \to \mathcal{X}_{v(t)}$. In this special case, the content of the theorem is that we can extend this morphism to $B$. This is the case, for example, in dimension one [Has03].

The proof proceeds as follows. We first show the existence of the rational numbers $t_i$, the so-called walls. We will begin by defining $f_t : (Z_t, v(t)\Delta_t) \to B$ as the log canonical model of $(X, v(t)D)$ over $B$. Since $B$ is smooth, [Kol18a, Theorem 4.83] guarantees that $f_t$ is stable, whereas [BCHM10] provides us with the finitely many $t_i$. Finally, Lemmas 4.1 and 4.2 furnish the open set $U \subset B$ and the explicit description of the morphism $U \to \mathcal{X}_{v(t)}$.

Proof of Theorem 4.4. We begin by observing that, since the generic fiber of $f$ is klt, the pair $(X, aD)$ is klt from [Kol18a, Corollary 4.85]. If $X$ was also $\mathbb{Q}$-factorial, we would consider the canonical model $(Z_t, v(t)\Delta_t)$ of the pair $(X, v(t)D)$ over $B$. The morphism $(Z_t, v(t)\Delta_t) \to B$ would be stable ([Kol18a, Corollary 4.86]), and would induce the morphisms $\Phi_t$. However, since $X$ may not be $\mathbb{Q}$-factorial, we need to replace $X$ with a small $\mathbb{Q}$-factorial modification in the argument above. In particular, consider a small $\mathbb{Q}$-factorial modification $\pi : \tilde{X} \to X$, let $aD$ be the proper transform of $aD$, and denote by $\tilde{f} : \tilde{X} \to B$ the composition $f \circ \pi$. Since $\pi$ is small, observe that

- $\pi^*(K_X + aD) = K_{\tilde{X}} + a\tilde{D}$, so $K_{\tilde{X}} + a\tilde{D}$ is $\tilde{f}$-big and $\tilde{f}$-nef over $B$ since it is the pull-back of an $f$-ample divisor;
- $\pi_*aD = aD$ and the discrepancies of $(\tilde{X}, a\tilde{D})$ are the same as those of $(X, aD)$, and
- $\pi_*^{-1}aD = a\tilde{D}$.

In particular, the pair $(\tilde{X}, a\tilde{D})$ is a weak canonical model of $(X, aD)$, and from [Kol18a, Corollary 4.86] the morphism $(\tilde{X}, a\tilde{D}) \to B$ is locally stable. Now $\tilde{X}$ is $\mathbb{Q}$-factorial, so for every $t \in [0, 1]$ the morphism $(\tilde{X}, v(t)\tilde{D}) \to B$ is also locally stable. Then we can run MMP with scaling by $(a - b)\tilde{D}$ as described in Subsection 3.2 to take the canonical model $(Z_t, v(t)\Delta_t)$ of the pair $(X, v(t)D)$ over $B$ for all $t \in [0, 1]$. By [Kol18a, Corollary 4.86] the map $(Z_t, v(t)\Delta_t) \to B$ is stable.

Now the key input is [BCHM10, Corollary 1.1.5]. Indeed by loc. cit. there are rational numbers $t_i$ with $0 = t_0 < t_1 < \ldots < t_m = 1$ such that, for every $t_i < s_1 \leq s_2 < t_{i+1}$, the pair $(Z_{s_1}, v(s_1)\Delta_{s_1})$...
is obtained from \((Z_{s_2}, v(s_2)\Delta_{s_2})\) by perturbing the coefficients, i.e. the underlying marked varieties are the same so that

\[(Z_{s_2}, v(s_2)\Delta_{s_2}) = (Z_{s_1}, v(s_2)\Delta_{s_1}).\]

We are left with proving that there exists an open subset \(U \subset B\) such that the morphisms \(\Phi_t\) on \(U\) can be described by sending a pair \((X_p, aD_p)\) to the log canonical model of \((X_u, v(t)D_u)\). It suffices to prove that there is a dense open subset \(U \subset B\) satisfying the following:

(a) For every \(u \in U\), the map \(\tilde{X}_u \rightarrow X_u\) is small, and

(b) For every \(u \in U\), the canonical model of \((\tilde{X}_u, v(t)\bar{D}_u)\) is \((Z_t)_u, v(t)(\Delta_t)_u\).

By Lemma 4.1 applied to the exceptional locus of \(\tilde{X} \rightarrow X\), there exists a nonempty open subset \(U\) such that that (a) holds, i.e. for every \(u \in U\), the map \(\tilde{X}_u \rightarrow X_u\) is small. To check (b) we need to check the following conditions (see [KM98, Definition 3.50]):

(I) The birational map \(\tilde{X} \rightarrow Z_t\) is defined over the generic points of \(\tilde{X}_u\), giving a birational map \(\phi_u : \tilde{X}_u \rightarrow (Z_t)_u\).

(II) The \(\phi_u\) does not extract any divisors.

(III) We have \((\phi_u)_* (v(t)\bar{D}_u) = v(t)(\Delta_t)_u\).

(IV) For every divisor \(E\) over \(\tilde{X}_u\), we have

\[a(E, \tilde{X}_u, v(t)\bar{D}_u) \leq a(E, (Z_t)_u, v(t)(\Delta_t)_u)\].

Shrinking \(U\), we can assume that the results of Lemma 4.2 hold. In particular, (I) and (III) follow from Lemma 4.2 points (3) and (4). To show that (II) holds after possibly shrinking \(U\), we apply Lemma 4.1 to the indeterminacy locus of \(Z_t \rightarrow \tilde{X}\) to obtain a \(U \subset B\) where the birational map \((Z_t)_u \rightarrow \tilde{X}_u\) is an isomorphism in codimension one for all \(u \in U\).

Finally, we show (IV). Up to shrinking \(U\), we may assume that all the fibers of \(\tilde{X} \rightarrow U\) and \(Z_t \rightarrow U\) are normal. It suffices to prove that if \(H\) is a smooth hypersurface in \(U\), then the inequality (IV) holds for \(\tilde{X}_u\) and \((Z_t)_u\) replaced by \(\tilde{X}_H\) and \((Z_t)_H\) respectively. Indeed because \(U\) is smooth, each \(u \in U\) is locally the transverse intersection of \(\dim B\) smooth hypersurfaces and so by induction on the dimension of \(B\), (IV) follows from the same claim for \(\tilde{X}_H\) and \((Z_t)_H\).

Consider \(\alpha : Y \rightarrow \tilde{X}\) a log resolution, and assume that there is a morphism \(\beta : Y \rightarrow Z_t\). Up to shrinking \(U\), we can further assume that the results of Lemma 4.2 hold for every \(t \in [0, 1]\) since there are finitely many \(Z_t\) by [BCHM10, Corollary 1.1.5]. Up to further shrinking \(U\), we can assume that for every \(b \in U\) the morphisms \(\alpha_u\) and \(\beta_u\) are birational.

The pair \((Z_t, v(t)\Delta_t)\) is the log canonical model for \((\tilde{X}, v(t)\bar{D})\) over \(U\), thus for every exceptional divisor \(E\) over \(\tilde{X}\) we have \(a(E, \tilde{X}, v(t)\bar{D}) \leq a(E, Z_t, v(t)\Delta_t)\). Now, let us denote by

\[D_Y := a_u^{-1}(v(t)\bar{D}) - \sum_{E \text{ a-exceptional}} a(E; \tilde{X}, v(t)\bar{D})E\]

and
\[
\Delta_Y := \beta^{-1}_*(v(t)\Delta_t) - \sum_{E \beta-\text{exceptional}} a(E; Z_t, v(t)\Delta_t) E
\]

so that \(K_Y + D_Y + Y_H = \alpha^*(K_{\tilde{X}} + v(t)\tilde{D} + \tilde{X}_H)\) and \(K_Y + \Delta_Y + Y_H = \beta^*(K_{Z_t} + v(t)\Delta_t + (Z_t)_H)\). In particular, since \(Z_t \to \tilde{X}\) does not contract any component of \(v(t)\Delta_t\),

\[D_Y - \Delta_Y\] effective.

Now, since \(\tilde{X}_H\) is Cartier and normal, \((K_{\tilde{X}} + v(t)\tilde{D} + \tilde{X}_H)|_{\tilde{X}_H} = K_{\tilde{X}_H} + v(t)\tilde{D}_H\). Similarly, \((K_Y + D_Y + Y_H)|_{Y_H} = K_{Y_H} + (D_Y)|_{Y_H}\) and \((K_{Z_t} + v(t)\Delta_t + (Z_t)_H)|_{(Z_t)_H} = K_{(Z_t)_H} + (v(t)\Delta_t)|_{(Z_t)_H}\).

In particular, if we denote by \(\alpha_H\) (resp. \(\beta_H\)) the restriction of \(\alpha\) (resp. \(\beta\)) to \(Y_H\), we have

\[\alpha_H^*(K_{\tilde{X}_H} + v(t)\tilde{D}_H) = K_{Y_H} + (D_Y)|_{Y_H}\] and \(\beta_H^*(K_{(Z_t)_H} + (v(t)\Delta_t)|_{(Z_t)_H}) = K_{Y_H} + (\Delta_Y)|_{Y_H}\).

Moreover, by definition of the different (see [Kol13, Definition 2.34]),

\[(\alpha_H)_*(v(t)D_Y)|_{Y_H} = v(t)\tilde{D}_H \quad \text{and} \quad (\beta_H)_*(\Delta_Y)|_{Y_H} = (v(t)\Delta_t)|_{(Z_t)_H}.

Thus by [KM98, Lemma 2.30], it suffices to prove that the discrepancies of any divisor over \((Y_H, (D_Y)_H)\) are less than those over \((Y_H, (\Delta_Y)_H)\). But

\[K_{Y_H} + (D_Y)|_{Y_H} - (K_{Y_H} + (\Delta_Y)|_{Y_H}) = (\alpha^*(K_{\tilde{X}} + v(t)\tilde{D}) + Y_H)|_{Y_H} - (\beta^*(K_{Z_t} + v(t)\Delta_t) + Y_H)|_{Y_H}\]

which equals \((D_Y - \Delta_Y)|_{Y_H}\). This is effective since \(D_Y \geq \Delta_Y\) and so the discrepancies of \((Y_H, (D_Y)_H)\) are less than those of \((Y_H, (\Delta_Y)_H)\) by [KM98, Lemma 2.27].

**Remark 4.6.** As phrased, the set of rational numbers \(\{t_i\}_{i=0}^m\) of Theorem 4.4 is not unique as we can always subdivide the interval \([0, 1]\) further by adding extra \(t_j\) and relabeling. However, there is a minimal choice for this set given by the intersection of all the possible sets of \(t_i\). These are the \(t_i\) where the log canonical models \((Z_{t_i}, v(t)\Delta_{t_i})\) actually change.

**Remark 4.7.** We record two consequences of Theorem 4.4:

- For every \(i\) and for every rational \(s \in (t_i, t_{i+1})\), the divisor \(v(s)\Delta_s\) is \(\mathbb{Q}\)-Cartier on \(Z_s\), and
- The pair \((Z_{t_i}, v(t_i)\Delta_{t_i})\) is the log canonical model of \((Z_{s_1}, v(t_i)\Delta_{s_1})\), and \((Z_{t_{i+1}}, v(t_{i+1})\Delta_{t_{i+1}})\) is the log canonical model of \((Z_{s_1}, v(t_{i+1})\Delta_{s_1})\) for \(t_i < s_i < t_{i+1}\).

The first consequence holds since, for every \(t_i < s_1 < t_{i+1}\), the two divisors \(K_{Z_{s_2}} + v(s_2)\Delta_{s_2}\) and \(K_{Z_{s_2}} + v(s_1)\Delta_{s_2}\) are \(\mathbb{Q}\)-Cartier, so their difference is also \(\mathbb{Q}\)-Cartier. The second consequence follows from the definition of the log canonical model. In particular, to check [KM98, Definition 3.50 (4)], one can use that the discrepancies of a pair \((X, \sum a_i D_i)\) are continuous functions of the coefficients \(a_i\).

We are ready to define the moduli spaces \(M_t\) which form the natural setting for wall-crossing.

**Definition 4.8.** Let \(f : (X, aD) \to B\) be an \(a\)-stable family satisfying the conditions of Theorem 4.4 and suppose that \(B\) is proper. Let \(\Phi_t\) be as in the conclusion of the theorem. Define \(M_t\) to be the seminormalization of the image of \(\Phi_t : B \to \mathcal{X}_{v(t)}\) for \(t \in [0, 1]\). We will denote by \((\mathcal{X}_t, v(t)\mathcal{D}_t)\) the universal family of \(v(t)\)-stable pairs over \(M_t\). We will denote by \(M_a\) and \((\mathcal{X}_a, a\mathcal{D}_a)\) (resp. \(M_b\) and \((\mathcal{X}_b, b\mathcal{D}_b)\)) the case when \(t = 1\) (resp. \(t = 0\)).

**Remark 4.9.** Note that \(M_t\) is proper as both \(B\) and \(\Phi_t\) are proper and the seminormalization preserves properness.
Remark 4.10. The reader should keep in mind the following situations which are the most common in practice, noting that the setup of Theorem 4.4 allows us the flexibility to consider more general settings.

- Given a stable family of snc pairs of interest \((X^0, aD^0) \to U\) over a smooth but non-proper base (e.g. \((\mathbb{P}^n, \text{smooth hypersurface})\)) we have an induced map \(U \to \mathcal{K}_a\). This may be compactified to a map \(B \to \mathcal{K}_a\) from a smooth proper base \(B\) using [LMB00, Théorème 16.6], Chow’s Lemma and resolution of singularities. Pulling back the universal family to \(B\) gives us a family \((X, aD)\) of stable pairs for which we can apply the proposition. In this case, \(\mathcal{M}_t\) can be thought of as the seminormalization of the \(v(t)\)-stable pair compactification of the original family of interest.

- Let \(\mathcal{K}_0 \subset \mathcal{K}_a\) be some irreducible component of the moduli space \(\mathcal{K}_a\) which generically parametrizes klt pairs. Then as above, up to taking a finite cover by a scheme and resolving singularities, we obtain an \(a\)-stable family \(f : (X, aD) \to B\) over a smooth and proper base with a morphism \(B \to \mathcal{K}_a\) dominating the component \(\mathcal{K}_0\). In this case, \(\mathcal{M}_1\) is simply the seminormalization of \(\mathcal{K}_0\). If we assume further that a generic pair lying over \(\mathcal{K}_0\) is \(v(t)\)-stable for all \(t \in [0, 1]\), then \(\mathcal{M}_t\) are birational models of \(\mathcal{M}_1\) which carry \(v(t)\)-stable families.

- Let \(\mathcal{K}^0 \subset \mathcal{K}_a\) be a reduced and irreducible locally closed substack which parametrizes klt pairs. After shrinking \(\mathcal{K}^0\), we can assume without loss of generality that it is smooth. After taking a finite cover of the closure of \(\mathcal{K}^0\) and resolving singularities, we obtain an \(a\)-stable family \(f : (X, aD) \to B\) such that \(B\) dominates \(\mathcal{K}^0\) under the morphism \(\Phi_1 : B \to \mathcal{K}_a\). Then \(\mathcal{M}_1\) is the seminormalization of the closure of \(\mathcal{K}^0\) and under the assumptions of Theorem 4.4, the \(a\)-stable klt pairs parametrized by \(\mathcal{K}^0\) are also klt and \(v(t)\)-stable for all \(t \in [0, 1]\) and thus \(\mathcal{K}^0\) admits a monomorphism to \(\mathcal{K}_{v(t)}\) which extend to the morphisms \(\Phi_t : B \to \mathcal{K}_{v(t)}\) given by the theorem. Thus \(\mathcal{M}_t\) are birational models of \(\mathcal{M}_1\) carrying \(v(t)\)-stable families as before. This case is a hybrid of the above two.

Definition 4.11. Given \(b \leq a\) and \(B\) as above, we will denote by \((a \to b)\)-walls, pronounced “\(a\)-to-\(b\) walls”, the minimal choice of numbers \(0 = t_0 < \ldots < t_i < \ldots < t_m = 1\) as in Theorem 4.4.

We have the following immediate corollary of Theorem 4.4.

Corollary 4.12. For each \(t_i < s < s' < t_i+1\), \(\mathcal{M}_s \cong \mathcal{M}_{s'}\) and the universal families \((\mathcal{X}_s, v(s)D_s)\) and \((\mathcal{X}_{s'}, v(s')D_{s'})\) have isomorphic underlying marked families so that \((\mathcal{X}_s, v(s)D_s) \cong (\mathcal{X}_{s'}, v(s')D_{s'})\). Moreover, these isomorphisms fit in a commutative diagram below where each side is cartesian.

\[
\begin{array}{ccc}
Z_s & \xrightarrow{\mathcal{M}_s} & Z_{s'} \\
\downarrow & & \downarrow \\
B & \xrightarrow{\mathcal{M}} & B \\
\downarrow & \swarrow & \downarrow \\
\mathcal{X}_s & \xrightarrow{\mathcal{X}} & \mathcal{X}_{s'} \\
\downarrow & \swarrow & \downarrow \\
\mathcal{M}_s & \xrightarrow{\mathcal{M}} & \mathcal{M}_{s'}
\end{array}
\]

17
Proof. We claim that the morphism \((\mathcal{X}_s, \mathbf{v}(s')D_s) \to M_s\) is locally stable. Since \((\mathcal{X}_s, \mathbf{v}(s')D_s) \to M_s\) is a well defined family of pairs, to prove the claim we can use Definition 2.13. In particular it suffices to check that for every DVR \(R\) and for every morphism \(T = \text{Spec}(R) \to M_s\), the family \((\mathcal{X}_s, \mathbf{v}(s')D_s)_T \to T\) is locally stable. As in the proof of \([\text{Kol18a, Definition-Theorem 4.45}]\), we can replace \(T\) with a DVR \(T' \to T\) possibly ramified over \(T\). In particular, by the valuative criterion of properness, we can assume that the morphism \(T \to M_s\) factors through \(\Phi_s : B \to M_s\) as follows.

\[
\begin{array}{c}
B \\
\Phi_s \\
\downarrow \\
M_s \\
\downarrow T
\end{array}
\]

Thus we can replace \(M_s\) and its universal family with \(B\) and the family lying over \(B\). The claim then follows from Theorem 4.4 (2).

Observe now that \((\mathcal{X}_s, \mathbf{v}(s')D_s) \to M_s\) is in fact stable, i.e. \(K_{\mathcal{X}_s/M_s} + \mathbf{v}(s')D_s\) is relatively ample over \(M_s\). Indeed, by Theorem 4.4 (2) it is relatively ample when pulled back to \(B\), and \(B \to M_s\) is a proper surjection. Therefore, the family \((\mathcal{X}_s, \mathbf{v}(s')D_s) \to M_s\) induces a morphism \(M_s \to M_s\). The argument is symmetric in \(s\) and \(s'\) so we also have a morphism in the other direction.

Finally, the fact that these morphisms are inverses and are induced by isomorphisms

\[
(\mathcal{X}_s', \mathbf{v}(s')D_{s'}) \cong (\mathcal{X}_s, \mathbf{v}(s')D_s)
\]

can be checked pointwise over the moduli space and fiberwise on the universal family and thus follows from Theorem 4.4 (2). Commutativity is clear by construction.

Given the corollary, we will introduce the following notation.

**Notation 4.13.** For consecutive walls \(t_i < t_{i+1}\), we will denote by \((\mathcal{X}_{(t_i,t_{i+1})}, D_{(t_i,t_{i+1})}) \to M_{(t_i,t_{i+1})}\) the moduli space and universal family of varieties marked with divisor for any \(s \in (t_i, t_{i+1})\).

5. **Flip-like morphisms**

In this section we will prove the existence of *flip-like morphisms* that relate the moduli spaces \(M_t\) defined in the previous section as \(t\)-varies across the \((a \to b)\)-walls. With notation as in 4.3, suppose we are in the situation of Theorem 4.4. Recall that the spaces \(M_t\) as in Definition 4.8 admit morphisms \(B \to M_t \to K_{\mathcal{X}_t}\). If \(0 = t_0 < \ldots < t_i < \ldots < t_m = 1\) are the \((a \to b)\)-walls and \(t_i < s_i < s_i' < t_{i+1}\), then the flip-like morphisms assemble into the diagram below.

\[
\begin{array}{ccccccccc}
B & \xrightarrow{\Phi_t} & M_{t_0} & \xrightarrow{\cong} & M_{s_0} & \xrightarrow{\cong} & M_{s_0'} & \xrightarrow{\cong} & M_{t_1} & \xrightarrow{\cong} & M_{s_1} & \xrightarrow{\cong} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_a & \xrightarrow{\cong} & M_{t_0} & \xrightarrow{\cong} & M_{s_0} & \xrightarrow{\cong} & M_{s_0'} & \xrightarrow{\cong} & M_{t_1} & \xrightarrow{\cong} & M_{s_1} & \xrightarrow{\cong} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_{t_0} & \xrightarrow{\Phi_t} & \mathcal{X}_{s_0} & \xrightarrow{\Phi_t} & \mathcal{X}_{s_0'} & \xrightarrow{\Phi_t} & \mathcal{X}_{t_1} & \xrightarrow{\Phi_t} & \mathcal{X}_{s_1} & \xrightarrow{\Phi_t} & \ldots 
\end{array}
\]

By Theorem 4.4, we obtain a diagram without the horizontal arrows where the composition \(B \to \mathcal{X}_t\) is the morphism \(\Phi_t\) and \(M_t\) is the seminormalization of the image of \(\Phi_t\) and by Corollary 4.12, we have the horizontal isomorphisms \(M_{s_0} \cong M_{s_0'}\). We can summarize the situation as follows.
Note here that \( \mathcal{M}_{(t_0, t_1)} \) admit morphisms to \( \mathcal{K}_s \) for each \( t_0 < s < t_1 \) but the target and these morphisms are actually varying even though the source moduli space is independent of \( s \).

For each \( t, B \) carries a \( \mathcal{V}(t) \)-stable family \((Z_t, \mathcal{V}(t)\Delta_t)\) which is pulled back from the universal family \((\mathcal{X}_t, \mathcal{V}(t)D_t) \to \mathcal{M}_t \). We know from Theorem 4.4 that the marked pair \((Z_s, \Delta_s)\) is independent of \( s \) for \( t_i < s < t_{i+1} \) with only the coefficients changing. Moreover, \((Z_{t_i}, \mathcal{V}(t_i)\Delta_{t_i})\) and \((Z_{t_{i+1}}, \mathcal{V}(t_{i+1})\Delta_{t_{i+1}})\) respectively are the log canonical models of \((Z_s, \mathcal{V}(t_i)\Delta_s)\) and \((Z_s, \mathcal{V}(t_{i+1})\Delta_s)\) (Remark 4.7). We showed in Corollary 4.12 that the first fact descends to a statement on the universal family. Putting this together, we have the diagram below where the squares coming out of the paper are cartesian.

Note that the \textit{a priori} rational maps \( Z_s \to Z_{t_i} \) given by taking the log canonical model are actually morphisms. Indeed since \((Z_s, \mathcal{V}(t_i)\Delta_s)\) and \((Z_{s}, \mathcal{V}(t_{i+1})\Delta_s)\) are good minimal models so they admit morphisms to their log canonical models. The idea now is to descend these morphisms the universal families and use them to induce the flip-like morphisms \( \mathcal{M}_{t_i} \leftarrow \mathcal{M}_{(t_i, t_{i+1})} \to \mathcal{M}_{t_{i+1}} \).

**Theorem 5.1** (Flip-like morphisms). In the setting of Theorem 4.4, consider \( t_i < t_{i+1} \) consecutive \((a \to b)\)-walls. There is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}_i & \xleftarrow{\alpha_{t_i}} & \mathcal{X}(t_i, t_{i+1}) & \xrightarrow{\beta_{t_{i+1}}} & \mathcal{X}_{t_{i+1}} \\
\mathcal{M}_{t_i} & \xleftarrow{\alpha_{t_i}} & \mathcal{M}(t_i, t_{i+1}) & \xrightarrow{\beta_{t_{i+1}}} & \mathcal{M}_{t_{i+1}}
\end{array}
\]

which commutes with diagram 5. Moreover, we have

(i) \( \alpha_{t_i} \) is induced by taking a pair \((X, \mathcal{V}(t_i)\Delta)\) to \( \text{Proj}R(K_X + \mathcal{V}(t_i)\Delta) \) with the pushforward divisor.

(ii) \( \beta_{t_{i+1}} \) is induced by taking a pair \((X, \mathcal{V}(t_{i+1})\Delta)\) to \( \text{Proj}R(K_X + \mathcal{V}(t_{i+1})\Delta) \) with the pushforward divisor.

In particular, over the dense open subset \( \mathcal{U} \subset \mathcal{M}(t_i, t_{i+1}) \) parametrizing klt pairs, \( \alpha_{t_i} \) and \( \beta_{t_{i+1}} \) can be be described by taking fiberwise log canonical models.
Proof. We will denote \((X_{(t_i, t_{i+1})}, D_{(t_i, t_{i+1})}) \rightarrow M_{(t_i, t_{i+1})}\) by \((X, D) \rightarrow M\) for convenience.

We need to construct a well defined family of pairs over \(M\) but with coefficients \(v(t_i)\), that pulls back to \((Z_{t_i}, v(t_i)\Delta_{t_i})\) over \(B\) and similarly for coefficients \(v(t_{i+1})\). In particular, we have to show that the log canonical model map

\[(Z_s, v(t)\Delta_s) \rightarrow (Z_t, v(t)\Delta_t)\]

is a morphism which is pulled back from a morphism of families of \(M\) for any \(s \in (t_i, t_{i+1})\) and \(t = t_i, t_{i+1}\). Note that (7) in fact a morphism for \(t = t_i, t_{i+1}\) by the basepoint free theorem applied to \(K_{Z_t} + v(t)\Delta_t\). By construction \((Z_s, \Delta_s)\) is the pullback of the universal pair \((X, D) \rightarrow M\).

Our task is to descend the stable family \((Z_t, v(t)\Delta_t) \rightarrow B\) to a stable family \((\mathcal{Y}, v(t)D_{\mathcal{Y}}) \rightarrow M\) along with a log canonical linear series \(X \rightarrow \mathcal{Y}\) whose construction is compatible with basechange.

By 4.12, we know that for every \(s \in (t_i, t_{i+1})\) the family \((X, v(s)D) \rightarrow M\) is stable. Therefore, since both having lc singularities and being nef are closed conditions on the coefficients of the divisor, the morphism \(\pi : (X, v(t)D) \rightarrow M\) is locally stable, and \(K_{X/M} + v(t)D\) is \(\pi\)-nef. We define, for \(d\) divisible enough,

\[\mathcal{Y} := \text{Proj}_M \left( \bigoplus_{m \in \mathbb{N}} \left( H^0(X, md(K_{X/M} + v(t)D)) \right) \right)\]

That is, \(\mathcal{Y}\) is the relative log canonical model of \((X, v(t)\Delta)\).

We claim that the construction of \(X \rightarrow \mathcal{Y}\) commutes with base change. By cohomology and base change it suffices to prove that for \(d\) and \(m\) divisible enough, and for every \(p \in M\), we have

\[H^1(X_p, md(K_{X_p} + v(t)D_p)) = 0.\]

Recall that from the definition of \(M\), every pair appearing as a fiber of \(\pi\) can be obtained as the degeneration of a klt pair over a DVR and moreover that \(K_{X_p} + v(t)D_p\) is big and nef. Now the desired vanishing follows from relative Kawamata-Viehweg vanishing as in [Inc20, Theorem 8.1].

In particular, for every \(p \in M\),

\[\mathcal{Y}_p = \text{Proj} \left( \bigoplus_{m \in \mathbb{N}} \left( H^0(X_p, nd(K_{X_p} + v(t_i)D_p)) \right) \right)\]

is the log canonical model of \((X_p, v(t)D_p)\). We conclude moreover that the rational map \(h : X \rightarrow \mathcal{Y}\) is in fact a morphism as it basechanges to the morphism (7) via the surjective map \(B \rightarrow M\), and that \(h\) induces the fiberwise log canonical model for each \(p \in M\).

We now need to produce a family of divisors \(D_{\mathcal{Y}}\) (see Section 2). On \(X\) we have \(n\) well-defined families of divisors \((X, D^{(i)}) \rightarrow M\) and a universal pair \((X, \sum(sa_i + (1 - s)b_i)D^{(i)}) \rightarrow M\). We wish to define \(D^{(i)}_{\mathcal{Y}}\) as the pushforward to get \(D^{(i)}_{\mathcal{Y}} := h_* (\mathcal{D}^{(i)})\) to \(\mathcal{Y}\). We need to check that \((\mathcal{Y}, D^{(i)}_{\mathcal{Y}})\) is a well defined family of divisors for each \(i\) [Kol18a, Definition 4.1] (see also Section 2), that is,

(a) \(\mathcal{Y} \rightarrow M\) is flat with seminormal fibers;
(b) for every \(i\), \(\text{Supp}(D^{(i)}_{\mathcal{Y}}) \rightarrow M\) is equidimensional of dimension \(n - 1\);
(c) for every \(p \in M\), the fiber \(\mathcal{Y}_p\) is smooth at the generic point of \(\text{Supp}(D^{(i)}_{\mathcal{Y}})\), and
(d) the assumptions of [Kol18a, Theorem 4.5 (2)] (which we recall below) apply.

Each of the statements (a), . . . , (d) can be checked étale locally so we can pull back to an étale cover \(U \rightarrow M\) by a scheme. We denote by \(f : X \rightarrow Y\) the pullback of \(h\) and let \(D, D^{(i)}\) and \(D^{(i)}_{\mathcal{Y}}\) be the divisorial pullbacks of \(D, D^{(i)}\) and \(D^{(i)}_{\mathcal{Y}}\) respectively as defined in Section 2. Note moreover
that \( f_*D^{(i)} = D_u^{(i)} \) as taking divisorial part and scheme theoretic image both commute with étale base change. The situation is summarized in the following diagram.

\[
\begin{array}{ccc}
\text{Supp} \left( D^{(i)} \right) & \xrightarrow{f} & X \\
\downarrow & & \downarrow \phi \\
\text{Supp} \left( D_Y^{(i)} \right) & \xrightarrow{g} & Y \\
\downarrow & & \downarrow \psi \\
U & \xrightarrow{h} & M
\end{array}
\]

Recall now that for every \( u \in U \), we have:

- The morphism \( f_u : X_u \to Y_u \) is the stable model of \( \left( X_u, \sum_j (t_i a_j + (1 - t_i) b_j) D_u^{(j)} \right) \), and
- \( \text{Supp} \left( D^{(i)} \right)_u = \text{Supp} \left( D_u^{(i)} \right) \) since \( (X, D^{(i)}) \to U \) is a family of pairs.

In particular, for each \( u \in U \), the fiber \( Y_u \) comes with \( n \) divisors (namely \( (f_u)_*(D_u^{(i)}) \)).

**Claim:** For each \( u \in U \), we have

\[
\text{Supp} \left( f_*D^{(i)} \right)_u \overset{(**)}{=} \text{Supp} \left( (f_u)_*D_u^{(i)} \right).
\]

First we prove the inclusion \( \subseteq \) of the claim. Consider \( p \in \text{Supp} \left( f_*D^{(i)} \right)_u \), then there is an irreducible component \( \Delta \subseteq \text{Supp} \left( D^{(i)} \right) \) such that \( \Delta \to f(\Delta) \) is generically finite, and \( p \in f(\Delta) \). Since \( \Delta \) is a component of \( D^{(i)} \), then \( \Delta_u \subseteq \text{Supp} \left( D^{(i)} \right)_u \) where \( \text{Supp} \left( D^{(i)} \right)_u = \text{Supp} \left( D_u^{(i)} \right) \) is of pure codimension one by the second bullet point above. Hence, \( \text{codim}_{X_u}(\Delta_u) \geq 1 \). On the other hand, by upper semicontinuity of fiber dimension, \( \text{codim}_{X_u}(\Delta_u) \leq 1 \) so \( \Delta_u \) is a union of irreducible components of \( \text{Supp}(D_u^{(i)}) \). Since \( f_u : X_u \to Y_u \) is the stable model map, \( \text{codim}_{Y_u}(f_u(\Delta_u)) > 0 \). On the other hand, \( \Delta \to f(\Delta) \) is generically finite so \( f_{\mu}(\Delta_u) \) has codimension 1 at the generic point \( \mu = g(\eta_\Delta) \). Therefore, \( \text{codim}(f_u(\Delta_u)) = 1 \) by upper semicontinuity of fiber dimension. Thus \( f_u(\Delta_u) \) is a union of components of \( (f_u)_*D_u^{(i)} \), giving us the required inclusion.

Next, we prove the inclusion \( \supseteq \). Consider a point \( p \in \text{Supp} \left( (f_u)_*D_u^{(i)} \right) \), and let \( \eta \in \text{Supp} \left( D_u^{(i)} \right) \) be a generic point of an irreducible component \( \Delta_u \subseteq \text{Supp} \left( D_u^{(i)} \right) \) such that \( f_u(\Delta_u) \) is a divisor with \( p \in f_u(\Delta_u) \). Since \( (X, D^{(i)}) \to U \) is a family of pairs, \( \text{Supp} \left( D_u^{(i)} \right) \subseteq (\text{Supp}D^{(i)})_u \). In particular, there is an irreducible component \( \Delta \subseteq \text{Supp} \left( D^{(i)} \right) \) containing the generic point of \( \Delta_u \), but since \( \Delta \) is closed, \( \Delta_u \subseteq \Delta \). Therefore it suffices to show that \( p \in \text{Supp}(f_*(\Delta)) \). We already know that \( p \in f(\Delta) \), thus we need to show that the push forward \( f_*(\Delta) \neq 0 \), or in other words that the map \( \Delta \to f(\Delta) \) is generically finite. By assumption, \( f_*(\Delta) : \Delta_u \to f_*(\Delta_u) \) is generically finite, so by upper semicontinuity of fiber dimension it is finite on an open subset.

Now we are ready to check conditions (a), . . . , (d).

For (a), it suffices to check the conditions after pullback along all morphisms \( \xi : \text{Spec}(R) \to M \) from the spectrum of a DVR by the valuative criterion for flatness (see also e.g. [Kol18a, Lemma 10.48]). Now the construction of \( \mathcal{Y} \to M \) via the relative Proj commutes with base change so
the pullback is the log canonical model of a locally stable family over a DVR which is flat with
deminormal fibers by the construction of stable limits.

Next, (b) and (c) are properties of the fibers over points so can be checked for each \( u \in U \). Thus
they follow from the claim that

\[
\text{Supp}(f_nD^{(i)})_u = \text{Supp}((f_n)_*D^{(i)}_u)
\]

and the fact that \( \left( Y_u, \sum_j (t_ia_j + (1-t_i)b_j)(f_n)_*D^{(i)}_u \right) \) is a stable pair.

We now show (d), using [Kol18a, Theorem 4.5 (2)]. With the notations of the previous paragraphs,
we need to show the following. Consider \( \nu : U^n \rightarrow U \) the normalization, and let \( X_n \)
(resp. \( D^{(i)}_n, Y_n \) and \( f_n \)) be the pullback of \( X \) (resp. \( D^{(i)}, Y \) and \( f \)). Then \( (Y_n, (f_n)_*(D^{(i)}_n)) \) is a
well defined family of pairs from [Kol18a, Theorem 4.5 (1)]. We need to show that for every two points
\( u, v \in U \) with \( \nu(u) = \nu(v) \), we have \( (Y_u, ((f_n)_*(D^{(i)}_n))_u) = (Y_v, ((f_n)_*(D^{(i)}_n))_v) \). But from
the claim above, we know that \( \text{Supp}((f_n)_*(D^{(i)}_n))_u \) is the support of the push forward of \( (D^{(i)}_n)_u \),
via the map that takes the stable model of \( (X_u, \sum_j (t_ia_j + (1-t_i)b_j)(D^{(i)}_n)_u) \). In particular, it is
uniquely determined by \( (X_u, \sum_j (t_ia_j + (1-t_i)b_j)(D^{(i)}_n)_u) \). But since \( \nu(s) = \nu(t) \) and the family
\( (X_n, \sum_j (t_ia_j + (1-t_i)b_j)D^{(i)}_n) \) is pulled back via \( \nu \), we have

\[
(X_u, \sum_j (t_ia_j + (1-t_i)b_j)(D^{(i)}_n)_u) = (X_v, \sum_j (t_ia_j + (1-t_i)b_j)(D^{(i)}_n)_v).
\]

Putting this together, we conclude that \( M \) carries a canonical \( v(t) \)-stable family \( (Y, h_*v(t)D) \)
which induces the required morphism \( M \rightarrow M_t \).

Finally, as we showed above, the formation of the log canonical morphism

\[
f : X \rightarrow \text{Proj}_B \text{R}(X/B, K_{X/B} + v(t)D)
\]
of the log canonical ring commutes with basechange for all \( v(s) \)-stable families parametrized by \( M \)
and similarly the formation of the Weil divisor \( \text{Weil}(f_*D) \) also commutes. Therefore the resulting
morphism \( M \rightarrow M_t \) can be described pointwise as taking a point \( p \) corresponding to the \( v(s) \)-stable
pair \( (X_p, v(s)D_p) \) to the point of \( M_t \) classifying the \( v(t) \)-stable pair \( (\text{Proj}_B(K_{X_p} + v(t)D_p), f_*v(t)D) \).
In particular, over the locus where \( X \) is normal, the morphism is induced by taking the fiberwise
log canonical model.

\[\square\]

**Remark 5.2.** If \( (X, D) \rightarrow B \) is a locally stable family of pairs with \( B \) smooth, then the log
canonical model over \( B \) is a stable family by [Kol18a, Corollary 4.86]. The main difficulty in
the above Theorem then is descending the conditions on a stable family along the non-smooth
morphism \( B \rightarrow M \).

The following Corollary will be useful in the proof of Theorem 7.6.

**Corollary 5.3.** Following the notation of Theorem 5.1, the morphisms \( \beta_{t_{i+1}} \) and \( \alpha_t \) are surjective.

**Proof.** We prove the desired statement for \( \alpha_t \), the case of \( \beta_{t_{i+1}} \) is analogous.

From Theorem 4.4 and Definition 4.8, we have a surjective morphism \( p : B \rightarrow M_{(t_i, t_{i+1})} \) with \( B \)
a smooth projective variety, induced by the family \( (Z_s, v(s)\Delta_s) \rightarrow B \) for any \( s \in (t_i, t_{i+1}) \). Then
to show that \( \alpha_t \) is surjective, it suffices to show that \( \alpha_t \circ p \) is surjective.
The composition $B \to \mathcal{M}(t_i,t_{i+1}) \xrightarrow{\alpha_{t_i}} \mathcal{M}_{t_i}$ is induced by taking the log canonical model of the pair $(Z_s,v(t_i)\Delta_s)$ over $B$, which from Proposition 3.7 agrees with $(Z_{t_i},v(t_i)\Delta_{t_i})$. Now the desired statement follows from the definition of $\mathcal{M}_{t_i}$. \hfill \Box

Finally, we end the section with a discussion of the name “flip-like morphisms”.

**Notation 5.4.** When working around a single $(a \to b)$-wall $t_i$, we will denote $\mathcal{M}(t_i-1,t_i)$ (resp. $\mathcal{M}(t_i,t_i+1)$) by $\mathcal{M}_{t_i-\varepsilon}$ (resp. $\mathcal{M}_{t_i+\varepsilon}$).

Theorem 5.1 guarantees the existence of a diagram

$$
\begin{array}{ccc}
X_{t_i-\varepsilon} & \rightarrow & X_{t_i+\varepsilon} \\
\downarrow & & \downarrow \\
X_{t_i} & & \\
\end{array}
$$

of universal families.

These universal families lie over different moduli spaces. However, we can pull back the above diagram to the fiber product $\mathcal{F} := \mathcal{M}_{t_i-\varepsilon} \times_{\mathcal{M}_{t_i}} \mathcal{M}_{t_i+\varepsilon}$ to obtain a diagram

$$
\begin{array}{ccc}
Z_{t_i-\varepsilon} & \rightarrow & Z_{t_i+\varepsilon} \\
\downarrow & & \downarrow \\
Z_{t_i} & & \downarrow \mathcal{F} \\
\end{array}
$$

which one can think of as a sort of *universal generalized log flip* (see [AB20, Proposition 8.4] and the preceding discussion). Indeed this diagram pulling back along the natural morphism $B \to \mathcal{F}$ yields the generalized log flip

$$
\begin{array}{ccc}
Z_{t_i-\varepsilon} & \rightarrow & Z_{t_i+\varepsilon} \\
\downarrow & & \downarrow \\
Z_{t_i} & & \downarrow \\
\downarrow & & \downarrow \\
B & & \\
\end{array}
$$

Here we say generalized to emphasize the fact that the log canonical contraction $Z_{t_i+\varepsilon} \to Z_{t_i}$ can be the contraction of a higher dimensional extremal face and thus can contract both divisorial and higher codimension exceptional loci.

Theorem 5.1 can be summarized then by saying that this universal generalized log flip induces the flip-like diagram

$$
\begin{array}{ccc}
\mathcal{M}_{t_i-\varepsilon} & \rightarrow & \mathcal{M}_{t_i+\varepsilon} \\
\beta_{t_i} & & \alpha_{t_i} \\
\downarrow & & \downarrow \\
\mathcal{M}_{t_i} & & \\
\end{array}
$$

In the following sections we will see that $\beta_{t_i}$ is in fact an isomorphism after passing to the normalizations of the moduli spaces.
6. Quasi-finiteness of the flip-like morphism below a wall

The goal of this section is to prove that for any \((a \to b)\)-wall \(t_i\), the flip-like morphism \(\beta_{t_i} : M_{t_i-\epsilon} \to M_{t_i}\) of Theorem 5.1 is quasi-finite.

**Theorem 6.1.** The morphism \(\beta_{t_i}\) does not contract any curves. In particular, \(\beta_{t_i}\) is quasi-finite.

6.1. Preliminaries and setup. To prove Theorem 6.1, we consider the following situation. Let \(C\) be an affine curve mapping to \(M_{t_i-\epsilon}\) that is contracted by \(\beta_{t_i}\). More precisely, suppose that there exists a point \(p \in M_{t_i}(k)\) with residual gerbe \(G_p\) such that the composition \(C \to M_{t_i}\) factors through the closed substack \(G \subset M_{t_i}\). Up to taking a finite étale cover we may suppose that the composition in fact factors through the point \(\text{Spec}(k) \to G\) so we obtain a diagram as follows.

![Diagram](image)

We can then pull back the universal families of \(M_{t_i-\epsilon}\) and \(M_{t_i}\) to \(C\), to get the following diagram:

\[
(Y, v(t_i - \epsilon)D) \quad \xrightarrow{\gamma} \quad (X_{C}, v(t_i)D_C) \quad \xrightarrow{f} \quad (X, v(t_i)D_X) \quad \xrightarrow{g} \quad C
\]

First, note that \(X_C := X \times C\) and \(D_C := D \times C\) as the family \(g\) is a product since the morphism \(C \to M_{t_i}\) factors through the point \(p : \text{Spec}(k) \to M_{t_i}\). Second, note that by the construction of the morphism \(\beta_{t_i} : M_{t_i-\epsilon} \to M_{t_i}\), \(g\) is the relative log canonical model of the pair \((Y, v(t_i)D)\) over \(C\); see Lemma 6.4 for the fiberwise properties of \(\gamma\). Theorem 6.1 now follows from the following claim.

**Claim 6.2.** There are finitely many isomorphism classes of slc pairs in the fibers of \(f\).

**Notation 6.3.** For a pair \((X, D)\) and \(B\) a variety, we will denote by \((X_B, D_B)\) the pair \((X \times B, D \times B)\).

We begin by recalling the fiberwise properties of the map \(\gamma\) induced by \(\beta_{t_i} : M_{t_i-\epsilon} \to M_{t_i}\).

**Lemma 6.4.** Let \((Y, v(t_i - \epsilon)D)\) be an slc pair corresponding to \(p \in M_{t_i-\epsilon}(k)\), and let the pair \((X, v(t_i)D_X)\) be the image \(\beta_{t_i}(p)\). Then there is a morphism \(h : Y \to X\) with the following properties:

1. A curve \(C\) gets contracted by \(h\) if and only if \((K_Y + v(t_i)D).C = 0\);
2. \(h\) has connected fibers;
3. \(\text{Exc}(h) \subseteq \text{Supp}(v(t_i)D)\), in particular \(h\) does not contract any component of \(Y\), and
4. \(h^*(K_X + v(t_i)D_X) = K_Y + v(t_i)D\).

**Proof.** By Theorem 5.1 and the construction of \(\beta_{t_i}\), \(X\) is the Proj of the log canonical ring of \((X, v(t_i)D)\), the a priori rational map \(h : X \to Y\) is a morphism, \(D_X = h_* D\), and the formation of the Proj and \(h_* D\) as a Weil divisor both commute with base change. If \(Y\) is klt, then \(X\) is klt and (1), (2) and (4) follow from basic properties of the log canonical model of log terminal model.

In general, every point of \(M_{t_i-\epsilon}\) is smoothable to a klt pair. Therefore, consider a one parameter family \((Y, v(t_i - \epsilon)D) \to \text{Spec}(R)\) in \(M_{t_i-\epsilon}\) with closed fiber isomorphic to \((Y, v(t_i - \epsilon)D)\) and generic fiber klt, and consider the relative log canonical model of \((Y, v(t_i)D)\) over \(\text{Spec}(R)\), namely
Then the pair \((X, v(t_i)D_X)\) is the closed fiber of \((\mathcal{X}, v(t_i)D_{\mathcal{X}}) \to \text{Spec}(R)\) and the total spaces \(\mathcal{X}\) and \(\mathcal{Y}\) are normal so from the construction of the log canonical model, we have that:

- The morphism \(\gamma : \mathcal{Y} \to \mathcal{X}\) has connected fibers, and
- A curve \(C \subseteq Y\) gets contracted by \(\gamma\) if and only if \(C.(K_Y + v(t_i)D) = C.(K_Y + v(t_i)D) = 0\).

In particular we have shown (1). Moreover, since a fiber of \(h : Y \to X\) is also a fiber of \(\gamma : \mathcal{Y} \to \mathcal{X}\), we have also shown (2).

To prove (3), we only need to check that a curve \(C\) which is not contained in \(\text{Supp}(v(t_i)D)\), satisfies \((K_Y + v(t_i)D).C > 0\). Note that \((K_Y + v(t_i - \epsilon)D).C > 0\) since the pair \((Y, v(t_i - \epsilon)D)\) is stable. Moreover, \((v(t_i)D).C \geq (v(t_i - \epsilon)D).C\) since \(C\) is not contained in \(\text{Supp}(v(t_i)D)\). Therefore \((K_Y + v(t_i)).C \geq (K_Y + v(t_i - \epsilon)).C > 0\).

Finally to show (4), let us denote the closed point by \(c \in \text{Spec}(R)\). By [Kol13, Lemma 1.28], the pair \((\mathcal{X}, v(t_i)D_{\mathcal{X}} + X_C)\) is the stable model of \((\mathcal{Y}, v(t_i)D + Y_C)\). In particular, there is a morphism \(\gamma : \mathcal{Y} \to \mathcal{X}\) which restricts to \(h\), such that \(\gamma^*(K_Y + v(t_i)D_\mathcal{X} + X_C) = K_Y + v(t_i)D + Y_C\). But \((K_Y + v(t_i)D_\mathcal{X} + X_C)|_{X_C} = K_X + v(t_i)D_X\) and \((K_Y + v(t_i)D + Y_C)|_{Y_C} = K_Y + v(t_i)D\), so (4) follows from the commutative diagram below, and functoriality of pull back:

$$
\begin{array}{ccc}
Y & \longrightarrow & \mathcal{Y} \\
\downarrow h & & \downarrow \gamma \\
X & \longrightarrow & \mathcal{X}.
\end{array}
$$

\[\square\]

### 6.2. Reduction to the log canonical case

We now show that we can assume that the fibers of \(f\) and \(g\) are normal. This will be achieved with the following two lemmas.

**Lemma 6.5.** In the setting of Section 6.1, consider the normalization

\[\nu : (\mathcal{Y}^n, v(t_i - \epsilon)D^n + \Delta) \to (\mathcal{Y}, v(t_i - \epsilon)D)\]

with \(\Delta\) the conductor. Then, up to shrinking \(C\), we can assume that:

1. \(f \circ \nu : (\mathcal{Y}^n, v(t_i - \epsilon)D^n + \Delta) \to C\) is stable with lc (a priori, not connected) fibers, and
2. For every \(p \in C\), the restriction \(\nu_p : \mathcal{Y}^n_p \to \mathcal{Y}_p\) is the normalization.

Similarly, if we denote by \(\nu_X : (X^n, D_X^n + \Delta_X) \to (X, D)\) the normalization of \(X\) with conductor \(\Delta_X\), then the morphism \((X^n, v(t_i)D^n_C + (\Delta_X)_C) \to C\) also satisfies (1) and (2).

**Proof.** First notice that the normalization of the generic fiber is the generic fiber of the normalization. In particular, if we denote with \(\eta\) the generic point of \(C\), then \(\mathcal{Y}^n_\eta \to \mathcal{Y}_\eta\) is the normalization, the conductor is \(\Delta_\eta\), whereas the preimage of \(v(t_i - \epsilon)D_\eta\) is \(v(t_i - \epsilon)D^n_\eta\). Then by [Kol13, Theorem 5.13], the family \((\mathcal{Y}^n_\eta, v(t_i - \epsilon)D^n_\eta + \Delta_\eta) \to \text{Spec}(k(\eta))\) is stable.

Since \(C\) is normal, in order for the morphism \(f \circ \nu : (\mathcal{Y}^n, v(t_i - \epsilon)D^n + \Delta) \to C\) to be a well defined family of stable pairs, it suffices to show that it satisfies [Kol18a, 4.1.2-4.1.4] (see [Kol18a, Theorem 4.2]). But the conditions in [Kol18a, 4.1.2-4.1.4] are open, so up to shrinking \(C\) we can assume that \(f \circ \nu\) is a well defined family of pairs. Similarly, since the generic fiber is normal, we can further assume that all the fibers are normal (see [Gro66, Théorème 12.2.1]). By [Kol18a, Corollary 4.49], up to shrinking \(C\), we can assume that \(f \circ \nu\) is locally stable. Moreover, \(f \circ \nu\) is stable since \(\nu^*(K_Y + v(t_i - \epsilon)D) = K_Y + v(t_i - \epsilon)D^n + \Delta\), and the pull-back of an ample line bundle through a finite morphism is ample.
Now, if we apply Lemma 4.1 with the locus where $\nu$ is not an isomorphism, we can assume that for every $p \in B$, the locus where $\nu_p : Y^n_p \to \mathcal{Y}_p$ is not an isomorphism is of codimension at least one in $\mathcal{Y}_p$. But $\nu_p$ is finite with the source being normal, so $\nu_p$ is the normalization.

Finally, we can repeat the same argument with $X$ instead of $\mathcal{Y}$.

\[\square\]

**Lemma 6.6.** In the setting of Section 6.1, denote by $\mu : X^n_C \to X_C$ the normalization of $X_C$. Then:

1. There is a morphism $\gamma^n : Y^n \to X^n_C$ such that $\gamma \circ \nu = \mu \circ \gamma^n$:

\[
\begin{array}{ccc}
Y^n & \xrightarrow{\gamma^n} & X^n_C \\
\downarrow{\nu} & & \downarrow{\mu} \\
Y & \xrightarrow{\gamma} & X_C
\end{array}
\]

2. The pair $(X^n_C, \nu(t_i)D^n_C + (\Delta_X)_C)$ is the log canonical model of $(Y^n, \nu(t_i)D^n + \Delta)$ over $C$;

3. $\text{Exc}(\gamma^n) \subseteq \text{Supp}(\nu(t_i)D^n + \Delta)$, and

4. If there are finitely many isomorphism classes of pairs in the fibers of $\nu \circ f$, then there are finitely many isomorphism classes of pairs in the fibers of $f$.

**Remark 6.7.** Lemma 6.6 (2) and (3), guarantee that Lemma 6.4 holds for $\gamma^n$ as well.

**Proof of Lemma 6.6.** We first note that, if we prove (1) and (2), then (3) follows as in the proof of Lemma 6.4. Let $n$ be the dimension of $X$. To prove (1), we need to produce a morphism $H^n$ as below:

\[
(Y^n, \nu(t_i - \epsilon)D^n + \Delta) \xrightarrow{\gamma^n} (X^n_C, \nu(t_i)D^n_C + (\Delta_X)_C)
\]

We construct $\gamma^n$ using the universal property of the normalization (see [Sta18, Tag 0BB4]). We need to check that the composition $Y^n \to Y \to X_C$ is such that every irreducible component of $Y^n$ dominates an irreducible component of $X_C$. If $Z \subseteq Y^n$ is an irreducible component, it suffices to show that $(\gamma \circ \nu)(Z)$ is an irreducible component of $X_C$. Notice that the irreducible components of $X^n_C$ are its irreducible closed subschemes of dimension $n + 1$. Now, $(\gamma \circ \nu)(Z)$ is irreducible and closed since it is the image of an irreducible proper scheme through a proper morphism: it suffices to check that $(\gamma \circ \nu)(Z)$ has dimension $n + 1$. But $\nu$ is the normalization, so $\nu(Z)$ is an irreducible component of $Y$. Then $\gamma(\nu(Z))$ has dimension $n + 1$ since $\gamma$ does not contract any irreducible component (see Lemma 6.4).

Now we prove (2). First observe that the divisor $K_{Y^n} + \nu(t_i)D^n + \Delta$ is nef over $C$. Indeed, $K_Y + \nu(t_i)D$ is nef over $C$, and $K_{Y^n} + \nu(t_i)D^n + \Delta = \nu^*(K_Y + \nu(t_i)D)$. Thus it suffices to check that a curve $E$ contained in a fiber of $Y^n \to C$ gets contracted by $\gamma^n$ if and only if $(K_{Y^n} + \nu(t_i)D^n + \Delta).E = 0$. But the normalizations $\nu : Y^n \to Y$ and $\mu : X^n_C \to X_C$ do not contract curves, therefore:

\[\gamma^n \text{ contracts } E \iff \mu \circ \gamma^n \text{ contracts } E \iff \gamma \circ \nu \text{ contracts } E \iff \gamma \text{ contracts } \nu(E).\]

From Lemma 6.4, $H$ contracts $\nu(E)$ if and only if

\[0 = (K_Y + \nu(t_i)D).\nu_*(E) = (\nu^*(K_Y + \nu(t_i)D)).E,\]
if and only if $E$ gets contracted by taking the stable model of $(Y^n, v(t)D^n + \Delta)$.

Finally, we prove (4). By [Kol13, Theorem 5.13], it suffices to prove that if we have a normal stable pair $(X, D + \Delta)$ (a priori not connected) such that the divisor $\Delta$ has coefficient 1, then there are finitely many involutions of $(\Delta^n, \text{Diff}_{\Delta^n}(D))$ along which we can glue to obtain an slc stable pair. The latter is a stable pair by adjunction and so has finitely many automorphisms by [KP17, Proposition 5.5].

Therefore the proof of Theorem 6.1 reduces to the following situation. Consider a stable family of lc pairs (a priori, not connected) $(Y, (v(t - \epsilon)D) \to C$ such that

- $C$ is connected,
- when we increase some of the weights from $v(t - \epsilon)$ to $v(t)$, we still have a locally stable family, but with nef log canonical divisor over $C$, and
- when we take the stable model over $C$, we obtain a product $(X_C, v(t)D_C)$ for $(X, v(t)D)$ and $v(t)$-weighted stable pair.

It suffices to prove that all the fibers of $Y \to C$ are isomorphic. We adopt the following notation.

$$ (Y, v(t - \epsilon)D) \xrightarrow{\gamma} (X_C, v(t)D_C) \xleftarrow{g} C \xrightarrow{f} (Y, v(t)D) $$

**Remark 6.8.** Observe that since $\gamma$ is the log canonical model, it has connected fibers. In particular, up to replacing $C$ with the Stein factorization of $g$, we can assume that the fibers of $f$ are connected.

**6.3. Proof of Theorem 6.1.** In this subsection we prove Theorem 6.1, using the reduction obtained in Subsection 6.2. We begin with two preparatory lemmas.

**Lemma 6.9.** Let $\{E^{(i)}\}$ be the exceptional divisors of $\gamma$. Then, up to shrinking $C$ and taking an étale cover, we can assume that each $E^{(i)} \to C$ is flat with geometrically integral fibers. Moreover, for each $p \in C$, the set $\{E^{(i)}_p\}$ contains the exceptional divisors of $\gamma_p$.

**Proof.** By [ACG11, Proposition 3.9], we can assume that $E^{(i)} \to C$ are flat. Let $\eta$ be the generic point of $C$, and consider $\{E^{(i)}_\eta\}_i$. Up to replacing $C$ with an étale covering (i.e. up to replacing $k(\eta)$ with a finite extension) we can assume that each $E^{(i)}_\eta$ is geometrically integral. Then by [Gro66, Théorème 12.2.1], up to shrinking $C$, we can assume that $E^{(i)} \to C$ are flat with geometrically integral fibers.

For the claim about the exceptional $\gamma_p$ consider $E \subseteq Y$ the exceptional locus for $\gamma$. We can write $E = \text{Weil}(E) \cup S$ where $\text{Weil}(E) = \cup E^{(i)}$ is the divisorial part, whereas $S$ has codimension greater $\geq 2$. Now we apply Lemma 4.1 to $\text{Weil}(E)$ and $S$. \hfill $\Box$

Before stating the next lemma, we recall a definition.

**Definition 6.10.** (see [Kol18a, Definition 1.96]) Let $(X, D = \sum_{i \in J} a_i D_i)$ be a simple normal crossing (snc) pair. A stratum of $(X, D)$ is any irreducible component of an intersection $\cap_{i \in J} D_i$ for some $J \subset I$.

**Lemma 6.11.** Let $(X, D)$ be an lc pair, with $D$ a priori not effective, such that $(X, \text{Supp}(D))$ is snc. Let $E \subseteq W$ be a divisor on a birational model $W \to X$, with $a(E; X, D) < 0$. Let $R := \mathcal{O}_{W, E}$ be the local ring at the generic point of $E$ in $W$, and let $\xi$ be the closed point of $\text{Spec}(R)$. Then
there is a sequence of blow-ups \( X_m \xrightarrow{p_m} X_{m+1} \xrightarrow{p_{m-1}} \ldots \xrightarrow{p_2} X_1 := X \) so that, if we denote with \( D_i := p_i^*(D_{i-1}) \) and with \( q_i \) the image of \( \xi \) through the morphism \( \text{Spec}(R) \to X_i \), then:

1. \( (X_i, \text{Supp}(D_i)) \) is snc;
2. \( X_i \to X_{i-1} \) is the blow-up of a stratum of \( X \), and
3. \( q_m \) has codimension one in \( X_m \).

**Proof.** Let \( v \) be the valuation associated to \( R \). Now, [KM98, Lemma 2.45] gives us a recipe for producing a sequence of blow-ups \( X_m \to X_{m-1} \to \ldots \to X_0 \) so that \( E \) is a divisor in \( X_m \). In particular, each morphism \( X_i \to X_{i-1} \) is the blow-up of the closure of \( q_{i-1} \).

Therefore, since if we blow-up a stratum in a log-smooth pair, we still get a log-smooth pair, it suffices to proceed by induction showing that:

- The closure of \( q_i \) is a stratum in \( X_i \), and
- For every divisor \( F \) over \( X \), we have \( a(F; X_i, D_i) = a(F; X_{i+1}, D_{i+1}) \).

The first claim follows from [Kol13, 2.10.1] and the next line, the second from [KM98, Lemma 2.30]. □

**Remark 6.12.** Consider a smooth variety \( C \), a log-smooth pair \( (X, D) \), and let \( \{S_J \subseteq D\}_{J \subseteq I} \) be the strata of \( (X, D) \). For a closed subvariety \( Z \subseteq Y \), let us denote with \( B_Z(Y) \) the blow-up of \( Y \) along \( Z \). Then the pair \( (X_C, D_C) \) is snc, its strata are \( S_J \times C \), and we have an isomorphism \( BS(X) \times C \cong BS_C(X \times C) \).

**Proof.** The first two claims follow easily from the definition. The last claim follows since a blow-up commutes with flat base change and \( C \to \text{Spec}(k) \) is flat. □

The following proposition allows us to bound the exceptional divisors of each \( \gamma_p \):

**Proposition 6.13.** In the notation of Subsection 6.1, there is a log-resolution \( (Z, B) \to (X, D) \) such that all the exceptional divisors of \( \gamma : Y \to X_C \) have divisorial center on \( Z_C \). In particular, they are also exceptional divisors for \( Z_C \to X_C \).

**Proof.** Consider any log resolution \( \pi : X' \to X \) of the pair \( (X, \text{Supp}(D)) \) and let \( D' \) be a divisor on \( X' \) such that \( \pi^*(K_X + v(t)D) = K_{X'} + D' \). Then for any divisor \( F \) over \( X_C \), we have

\[
a(F; X_C, v(t)D_C) = a(F; X'_C, v(t)D'_C).
\]

Now, the following is the main point of the proof:

\[
\gamma^*(K_{X_C} + v(t)D_C) = K_Y + v(t)D = K_Y + \gamma_a^{-1}(v(t)D_C) + \sum_{E \text{ exceptional}} -a(E; X_C, v(t)D_C)E.
\]

Indeed, the first equality follows from Lemma 6.6 (3), whereas the last equality follows from the definition of discrepancies. But by Lemma 6.6 (4), we have that for every exceptional \( E \), the containment \( \text{Supp}(E) \subseteq \text{Supp}(v(t)D) \) holds. Since the entries of \( v(t) \) are all positive, for every \( \gamma \)-exceptional divisor \( E \) as above we have

\[
a(E; X_C, v(t)D_C) < 0.
\]

Then we have also \( a(E; X'_C, (D_{X'})_C) < 0 \). Now the desired \( Z \) can be constructed from Lemma 6.11 and Remark 6.12. □

We also need the following lemma.
Lemma 6.14. Let $f : X \rightarrow Y$ a birational rational map between normal proper varieties. For every divisor $D \subseteq X$, let $R_D := O_{X,D}$ the DVR at the generic point of $D$ in $X$, with $\xi_D$ the closed point of $\text{Spec}(R_D)$, and with $f_D : \text{Spec}(R) \rightarrow Y$ the morphism induced by the valuative criteria of properness. Assume that for every divisor $D$, $f_D(\xi_D)$ has codimension one in $Y$. Then the open subset $U \subseteq X$ such that $f|_U : U \rightarrow Y$ is an isomorphism has complement of codimension at least 2.

Proof. From the valuative criteria of properness, $f$ is defined on all codimension one points of $X$. We need to check that for every divisor $D$ on $X$ we have $\xi_D \in U$. Consider $S := O_{Y,fD(\xi_D)}$ the local ring of $Y$ at $f_D(\xi_D)$. By assumption, this is a DVR, and $f$ induces a morphism of local rings $\phi : S \rightarrow R_D$. Our goal is to show that $\phi$ is injective and surjective. We will drop the subscript $D$ on $R_D$.

Since $X$ and $Y$ are birational, $\phi$ is an isomorphism on fraction fields. Then the composition $S \rightarrow R \rightarrow \text{Frac}(R) = \text{Frac}(S)$ is injective since it is the inclusion of a DVR into its fraction field, so also $\phi$ is injective.

For surjectivity, it suffices to check that the uniformizer $x$ of $R$ belongs to $S$. If we denote with $v$ the valuation on $\text{Frac}(R)$ induced by $S$, we need to show that $v(x) \geq 0$. But if $v(x) < 0$ then $v\left(\frac{1}{x}\right) > 0$ so $\frac{1}{x} \in S$ and since $S \subseteq R$, we would have $\frac{1}{x} \in R$ which is a contradiction. □

Proof of Theorem 6.1. First, up to shrinking $C$ and taking an étale cover, we can assume that Lemma 6.9 holds. Then it suffices to prove that, up to shrinking $C$ we have

(1) For $p,q \in C$, the fibers $\mathcal{V}_p$ and $\mathcal{V}_q$ are isomorphic in codimension one, and

(2) the isomorphism sends $v(t - \epsilon)\mathcal{D}_p$ to $v(t - \epsilon)\mathcal{D}_q$.

Indeed, if this is the case, we can apply Lemma 3.3 to conclude that for every $p,q \in C$ we have $(\mathcal{V}_p,v(t - \epsilon)\mathcal{D}_p) \cong (\mathcal{V}_q,v(t - \epsilon)\mathcal{D}_q)$.

To prove (1), we apply Lemma 6.14. We show that if $v$ is a valuation of the fraction field of $X$, the center of $v$ in $\mathcal{V}_p$ is a divisor if and only if it is a divisor in $\mathcal{V}_q$. If the center of $v$ in $X$ is a divisor $G$, its center in $\mathcal{V}_p$ (resp. $\mathcal{V}_q$) is $(\gamma_p)_*^{-1}(G)$ (resp. $(\gamma_q)_*^{-1}(G))$. We focus now on the exceptional divisors of $\gamma_p$ and $\gamma_q$. We need to show that for every valuation $v$ of the fraction field of $X$, the following holds.

Claim: (**) The center of $v$ in $\mathcal{V}_p$ is an exceptional divisor of $\gamma_p$ if and only if the same holds for $\mathcal{V}_q$ and $\gamma_q$.

We fix some notation. Consider $(Z,B) \rightarrow (X,v(t - \epsilon)D)$ as in Proposition 6.13, let $\{E^{(i)}\}_i$ be the exceptional divisors of $\gamma$, and let $F_i$ be those of $Z \rightarrow X$. Then the exceptional divisors of $Z_C \rightarrow X_C$ are $(F_i)_C$. Moreover, given a normal variety $W$ with a divisor $G \subseteq W$, we will denote by $v_G$ its valuation.

From Proposition 6.13, the induced rational morphism $Z \times C \rightarrow \mathcal{V}$ is a birational contraction, namely its inverse does not contract any divisor. More explicitly, there is an open subset $U \subseteq \mathcal{V}$ where the inverse $\iota : U \rightarrow Z \times C$ is an open embedding, and $\text{Codim}(\mathcal{V} \setminus U) \geq 2$.
In particular, for every $i$ we have $E^{(i)} \cap U \neq \emptyset$. Since the diagram above commutes, for every $i$ there is an exceptional divisor $F_i \times C$ of $Z \times C \to X \times C$ such that

$$\iota(E^{(i)} \cap U) = (F_i \times C) \cap \iota(U).$$

In other terms, $\iota$ maps exceptional divisors to exceptional divisors. From Lemma 4.1 we can assume that, for every $p \in C$, we have $\text{Codim}(\mathcal{Y}_p \setminus \mathcal{U}_p) \geq 2$. In particular, for every $p \in C$ we have $E_p^{(i)} \cap \mathcal{U}_p \neq \emptyset$, and therefore

$$\iota(E^{(i)}_p \cap \mathcal{U}_p) = \iota((E^{(i)} \cap U) \cap \mathcal{U}_p) = \iota(E^{(i)} \cap U) \cap \iota(\mathcal{U}_p) = ((F_i \times C) \cap \iota(U)) \cap \iota(\mathcal{U}_p) = F_i \cap \iota(\mathcal{U}_p).$$

In particular, $v_{E_p^{(i)}} = v_{F_i}$, and since all the exceptional divisors $\mathcal{Y}_p \to X$ are of the form $E_p^{(i)}$ (see Lemma 6.9) and the right hand side does not depend on $p$, we proved the claim (**).

Now we prove (2). Let us denote by $\{G_p^{(i)}\}$ (resp. $\{G_q^{(i)}\}$) the divisors in $\text{Supp}(v(t - \epsilon)D_p)$ (resp. $\text{Supp}(v(t - \epsilon)D_q)$) that are not contracted by $\gamma_p$ (resp. $\gamma_q$). Then we can write

$$v(t - \epsilon)D_p = \sum d_iG_p^{(i)} + \sum e_iE_p^{(i)}$$

and

$$v(t - \epsilon)D_q = \sum d_iG_q^{(i)} + \sum e_iE_q^{(i)}.$$

From (**), since the valuation $v_{E_p^{(i)}}$ coincides with $v_{E_q^{(i)}}$, the birational map $\mathcal{Y}_p \to \mathcal{Y}_q$ sends $E_p^{(i)}$ to $E_q^{(i)}$. Since it is a birational map over $X$, it sends $G_p^{(i)}$ to $G_q^{(i)}$, proving (2).

7. Reduction morphisms up to normalization

The goal of this section is to construct reduction morphisms $\rho_{a,b}$ for weight vectors $b \leq a$ generalizing Hassett’s reduction morphisms [Has03, Theorem 4.1] to higher dimensions. To accomplish this we need to normalize the moduli space (see Section 8.1 for an example showing this is necessary).

**Definition 7.1.** In the setting of Definition 4.8, we let $\mathcal{N}_t$ denote for $t \in [0, 1]$ denote the normalization of $\mathcal{M}_t$. We denote by $\mathcal{N}_a$ (resp. $\mathcal{N}_b$) the normalization of $\mathcal{M}_a$ (resp. $\mathcal{M}_b$).

**Theorem 7.2.** Let $b \leq a$ be weight vectors and $0 < t_1 < \ldots < t_m = 1$ the $(b \to a)$-walls. Then for any $t_i$, the flip-like morphism $\beta_{t_i} : \mathcal{M}_{t_i - \epsilon} \to \mathcal{M}_{t_i}$ induces an isomorphism $\beta_{t_i}^n : \mathcal{N}_{t_i - \epsilon} \to \mathcal{N}_{t_i}$.

The proof of Theorem 7.2 proceeds as follows:

1. $\beta_{t_i}$ is quasi-finite by Theorem 6.1 in the previous section, and
2. $\beta_{t_i}$ is proper, representable, and an isomorphism on a dense open subset (see Theorem 7.6 below).

Then Theorem 7.2 follows then from Zariski’s main theorem.
Definition 7.3. Composing \((\beta_{t_i}^n)^{-1}: N_{t_i} \to N_{t_i-\epsilon}\) with \(\alpha_{t_{i-1}}^{n}: N_{t_{i-1}+\epsilon} \to N_{t_{i-1}}\) for all \(i\) gives the desired reduction morphisms:

\[
\rho_{b,a}: N_a \to N_b
\]
\[
\rho_{b,a} := \alpha_{t_0}^n \circ (\beta_{t_1}^n)^{-1} \circ \alpha_{t_1}^n \circ \ldots \circ \alpha_{t_{m-1}}^n \circ \beta_{t_m}^{-1}.
\]

Remark 7.4. Note that both \(\alpha_{t_i}\) and \(\beta_{t_i}\) are dominant by Corollary 5.3 so they induce morphisms \(\alpha_{t_i}^n\) and \(\beta_{t_i}^n\) between normalizations.

Remark 7.5. For any weight vector \(v(t) = ta + (1-t)b\), the reduction morphisms are compatible by definition: \(\rho_{b,v(t)} \circ \rho_{v(t),a} = \rho_{b,a}\). In general, for weight vectors \(c \leq b \leq a\) that are not co-planar we may have

\[
\rho_{c,b} \circ \rho_{b,a} \neq \rho_{c,a}.
\]

This is because the construction of the moduli spaces and morphisms a priori depends on the MMP with scaling we used to get from weights \(a\) to \(b\). In section 8, we will give some examples showing this can occur and state conditions under which the reduction morphisms are compatible for all weights (Theorem 8.1).

Theorem 7.6. The morphism \(\beta_{t_i}: M_{t_i-\epsilon} \to M_{t_i}\) is representable, proper and birational.

Proof. Recall that by Theorem 4.4, the morphism \(\beta: M_{t_i-\epsilon} \to M_{t_i}\) can be described as follows on the dense open subset parametrizing klt pairs. Given a point \(p \in M_{t_i-\epsilon}(k)\) corresponding to a stable pair \((Y, v(t_i - \epsilon)D)\), \(\beta_{t_i}(p)\) classifies the log canonical model of the pair \((Y, v(t_i)D)\) which we will denote by \((X, v(t_i)D_X)\).

The morphism \(\beta_{t_i}\) is proper since the source and target are proper, and it is surjective by Corollary 5.3. It is generically injective since we can recover \((Y, v(t_i - \epsilon)D)\) from the pair \((X, v(t_i)D_X)\) in the klt case. Indeed, by Proposition 3.7 the pair \((Y, v(t_i - \epsilon)D)\) is the log canonical model of \((X, v(t_i - \epsilon)D_X)\) when \((X, v(t_i)D_X)\).

To show representability, consider a stable pair \((Y, v(t_i - \epsilon)D)\) which corresponds to a point \(p \in M_{t_i-\epsilon}(k)\), and suppose \(\tau\) is an automorphism of the pair. Let \((X, v(t_i)D_X)\) be the pair corresponding to \(\beta_{t_i}(p)\). Then \(\tau\) induces an automorphism \(\tau_X\) of \((X, v(t_i)D_X)\) by functoriality of the construction of \(\beta_{t_i}\). We need to prove that \(\tau_X = Id \implies \tau = Id\). This is proved as in [Inc20, Observation 8.4], using Lemma 6.4.

We are left with showing that \(\beta_{t_i}\) is birational. To do this, we produce a dense open substack \(U \subseteq M_{t_i}\) such that

\[
\beta_{t_i}: \beta_{t_i}^{-1}(U) \to U
\]

is an isomorphism.

First, observe that if we denote with roman letters the coarse moduli spaces, the induced morphism \(\beta_{t_i}^c: M_{t_i-\epsilon} \to M_{t_i}\) is proper as the source is proper and the target is separated. Moreover, \(\beta_{t_i}^c\) is quasi-finite, surjective and generically injective, since these are properties that we can check on algebraically closed points, they hold for \(\beta_{t_i}: M_{t_i-\epsilon} \to M_{t_i}\), and we have a bijection \(|M_t(Spec(k))| \cong |M_{t_i}(Spec(k))|\) for \(k\) algebraically closed. Since we are in characteristic 0, \(\beta_{t_i}^c\) is also generically unramified. Therefore by Zariski’s main theorem, the morphism \(\beta_{t_i}^c\) induces an isomorphism of normalizations.

As \(M_{t_i}, M_{t_i-\epsilon}, M_{t_i}\) and \(M_{t_i-\epsilon}\) are are reduced and irreducible, there exist normal dense open subsets of all these spaces. Moreover, these can be chosen to fit in the following diagram
where the left and right squares are cartesian. Since normalization is an isomorphism on the locus that is already normal, then up to shrinking the open subsets, we can summarize the situation as follows:

- $U \subset M_{t_i}$ and $V = (\beta_{t_i}^c)^{-1}(U) \subset M_{t_i-\epsilon}$ are open and dense subsets such that $(\beta_{t_i})_{|V} : V \to U$ is an isomorphism,
- $V$ (resp. $U$) is a normal dense open substack of $M_{t_i}$ (resp. $M_{t_i-\epsilon}$), with coarse space $U$ (resp. $V$), and
- $U$ and $V$ are contained in the locus parametrizing klt pairs.

Then the restriction $\beta_{t_i} \, |_V : V \to U$ is a representable morphism between normal Deligne-Mumford stacks which is an isomorphism on coarse moduli spaces. We wish to show that this is an isomorphism on the level of stacks. By construction, it induces an isomorphism of coarse spaces and hence a bijection on geometric points. Moreover, since we are in characteristic 0, then up to shrinking $U$ and $V$, we may assume that $\beta_{t_i} \, |_V$ is étale. Thus by [AK16, Lemma 3.1], it suffices to show that the morphism is stabilizer preserving.

As we already know the morphism is representable, we are left with showing surjectivity of automorphism groups. In the notation above, for $p \in V(k)$ classifying a klt stable pair $(Y, \mathbf{v}(t_i-\epsilon)D)$, we need to show that any automorphism of the pair $(X, \mathbf{v}(t_i)D_X)$ comes from an automorphism of $(Y, \mathbf{v}(t_i-\epsilon)D)$. Recall that by construction of $\beta_{t_i}$, $(X, \mathbf{v}(t_i)D_X)$ is the log canonical model of $(Y, \mathbf{v}(t_i)D)$. On the other hand, by Proposition 3.7, $(Y, \mathbf{v}(t_i - \epsilon)D)$ is the log canonical model of $(X, \mathbf{v}(t_i - \epsilon)D_X)$ and there is a small birational morphism $Y \to X$.

In particular, there is an open subscheme $W \subseteq Y$ such that:

1. $W \to X$ is an isomorphism with its image (which we denote with $W_X$);
2. $W$ is $\mathbb{Q}$-factorial, and
3. The complement of $W$ and $W_X$ have codimension at least 2 in $Y$ and $X$ respectively.

Therefore $Y = \text{Proj}(\bigoplus_n H^0(W_X, nd(K_{W_X} + \mathbf{v}(t_i - \epsilon)(D_X)|_{W_X})))$ for $d$ divisible enough. Recall that we $\mathbf{v}(t_i)D$ is a shorthand for

$$\sum (t_ia_i + (1 - t_i)b_i)D^{(i)},$$

so any automorphism of $(X, \mathbf{v}(t_i)D)$ fixes the components $D^{(i)}$. In particular, it induces an automorphism of $X$ which sends $D^{(i)}$ to itself. Thus it induces an automorphism of

$$Y = \text{Proj}(\bigoplus_n H^0(W_X, nd(K_{W_X} + \mathbf{v}(t_i - \epsilon)(D_X)|_{W_X}))),$$

and since it preserves the $D^{(i)}$ it induces an automorphism of the pair $(Y, \mathbf{v}(t_i - \epsilon)D)$, completing the proof. \qed

**Proof of Theorem 7.2.** This follows from Theorem 6.1, Theorem 7.6 and Zariski’s main theorem for representable morphisms of algebraic stacks [LMB00, Theorem 16.5]. \qed
Next we show that under some natural assumptions, the flip-like morphism $\alpha_{t_i} : M_{t_i+\epsilon} \to M_{t_i}$ is also birational.

**Proposition 7.7.** Let $t_i$ be an $(a \to b)$-wall and suppose that there exists a dense open substack $U \subset M_{t_i+\epsilon}$. Denote by $(X_{t_i}, v(t_i+\epsilon)D_U) \to U$ the universal $v(t_i+\epsilon)-$ stable family over $U$. Suppose that the family $(X_{t_i}, v(t_i)D_U) \to U$ is also a stable. Then up to shrinking $U$, we have that

- $\alpha_{t_i}(U)$ is a dense open substack of $M_{t_i}$,
- $(\alpha_{t_i})_U : U \to \alpha_{t_i}(U)$ is an isomorphism, and
- the pullback of $(X_{t_i}, v(t_i)D_U) \to U$ along $(\alpha_{t_i})^{-1}$ is the universal $v(t_i)-$ stable family over $\alpha_{t_i}(U)$.

In particular, $\alpha_{t_i} : M_{t_i+\epsilon} \to M_{t_i}$ is birational.

**Remark 7.8.** Note that in contrast to $\beta_{t_i}$, the flip-like morphism $\alpha_{t_i}$ is not finite in general. This happens already in the case of weighted stable curves [Has03] where $\alpha_{t_i}$ can contract high dimensional loci parametrizing rational tails with $n \geq 4$ special points which are contracted to a point by the log canonical model for coefficients $v(t_i)$.

**Remark 7.9.** The hypothesis of Proposition 7.7 is often satisfied in practice. For example (see also Remark 4.10), one often begins with a family of pairs over an open base $U$ which are stable for all $t \in [0, 1]$ and asks how the stable pairs compactification changes as $t$ varies. In this case, the image of $U$ inside $M_t$ is constructible by Chevalley’s Theorem and dense by construction. Therefore the image of $U$ contains a dense open substack of $M_t$. By assumption, the pairs over this substack are stable for all $t \in [0, 1]$ so the hypothesis of the proposition is satisfied.

**Proof.** We will denote $\alpha_{t_i}$ by $\alpha$ for convenience.

Let $f : U \to M_{t_i}$ be the coarse space map of the restriction $\alpha_U : U \to M_{t_i}$. Then $f$ is dominant since $U$ is dense in $M_{t_i+\epsilon}$ and $\alpha$ is surjective by Corollary 5.3. Therefore, the image $f(U)$ is dense and constructible so by e.g. [Har77, Chapter 2, Exercises 3.18-3.19] there exists a dense open subset $V \subset f(U) \subset M_{t_i}$.

Then, defining $\mathcal{V} := M_{t_i} \times_{M_{t_i}} V$ and up to replacing $U$ with $\alpha^{-1}(\mathcal{V})$, we can assume that:

1. $\mathcal{U}$ is an open and dense substack of $M_{t_i+\epsilon}$,
2. $V$ is an open and dense substack of $M_{t_i}$, and
3. $\alpha(\mathcal{U}) = V$ and $\alpha^{-1}(\mathcal{V}) = \mathcal{U}$.

Now we proceed as in the proof of Theorem 7.6. We know that the induced map on $k$-points $U(k) \to V(k)$ is injective since by assumption, for any $v(t_i+\epsilon)$-stable pair $(X, v(t_i+\epsilon)D)$ parametrized by $U$ the pair $(X, v(t_i)D)$ is also stable. Moreover, it is surjective since $\alpha(U) = V$. Thus $\alpha_U : U \to V$ is a morphism between normal separated stacks of finite type in characteristic 0 which is a bijection on geometric points so up to shrinking $U$ and $V$ further, we may assume that it is étale. Then by [AK16, Lemma 3.1], to show that $\alpha_U$ is an isomorphism, it suffices to show it is stabilizer preserving. But this again follows by assumption since over $U$, the pair corresponding to a point $\alpha_{t_i}(p)$ is simply $(X, v(t_i)D)$ where $p$ corresponds to $(X, v(t_i+\epsilon)D)$.

**Corollary 7.10.** Let $b \leq a$, and let $f : (X, D_a) \to M_a$ be the universal pair. Suppose that the restriction of $f$ to the generic point is $b-$ stable. Then the reduction morphism $\rho_{b,a} : N_a \to N_b$ is birational. 

33
Proof. Since the normalization of a reduced stack is birational, it suffices to check the desired statement for the rational map $\mathcal{M}_a \dashrightarrow \mathcal{M}_b$. But according to Theorem 5.1, this factors as a finite sequence of flip-like morphisms $\mathcal{M}_a \to \mathcal{M}_b$. Such morphisms are birational by Theorem 7.6 and Proposition 7.7, so their composition is birational. $\square$

We conclude the section with an application of representability of $\beta_t$ to the isotrivial families studied in Section 6.

Corollary 7.11. In the situation of Section 6.1, suppose that $C$ is a smooth and irreducible curve. Then the family $f : (\mathcal{Y}, \nu(t_i - e)\mathcal{D}) \to C$ is isomorphic to a product.

Proof. Let $\varphi : C \to \mathcal{M}_{t_i - e}$ be the moduli map induced by the stable family $f$. By assumption, the composition $\beta_{t_i} \circ \varphi : C \to \mathcal{M}_{t_i}$ with $\beta_{t_i}$ is constant, that is, it factors through a closed point $x : \text{Spec} \mathcal{O} \to \mathcal{M}_{t_i}$. Since $C$ is a smooth connected curve, $\varphi$ factors through a connected component of the reduced preimage $(\beta_{t_i}^{-1}(x))_{\text{red}}$. By Theorems 6.1 and 7.6, $\beta_{t_i}^{-1}(x)$ is a finite and representable over $\text{Spec} \mathcal{O}$. Therefore $(\beta_{t_i}^{-1}(x))_{\text{red}}$ is a finite union of points and so $\varphi$ is the constant map. $\square$

8. Examples, counterexamples, and natural questions

In this section, we discuss several natural generalizations of our main results one might hope for and give examples showing some of these are not possible.

8.1. Normalizing the moduli space. It is natural to ask if Theorem 7.2 holds without taking the normalization of the moduli space $\mathcal{M}_t$. However, the following example shows that the morphism $\beta_{t_i}$ is not injective in general and thus not an isomorphism. In particular, reduction morphisms $\rho_{b,a}$ can only be well defined on the normalization of the moduli space.

We recall the following construction due to Hassett (see [Kol18a, Example 2.39 and 1.44]). Consider the cone $S \subseteq \mathbb{P}^5$ over the degree four rational curve in $\mathbb{P}^4$. This is a surface with an $\mathbb{A}^2/\mathbb{A}^1$ singularity on the vertex of the cone, and it can be obtained as a flat degeneration of both $\mathbb{P}^2$ (see [Kol18a, Example 1.44]) and $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, there are two DVRs, which we will denote by $R_1$ and $R_2$, and two projective families $f_i : X_i \to \text{Spec}(R_i)$, so that the special fiber of $f_i$ is isomorphic to $S$, and the generic fiber of $f_1$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ whereas the one of $f_2$ is isomorphic to $\mathbb{P}^2$.

Moreover, there are families of divisors $D_i \subseteq X_i$ that can be described as follows. First, fix a natural number $r$ and let $D_0 \subseteq S$ be the union of $2r$ generic lines through the cone point. Now for $i = 1$, the divisor $D_1$ is the union of $r$ lines of one ruling on the generic fiber with divisorial limit $D_0$. Note that in this case $D_0$ is not the flat limit of the generic fiber $(D_1)_m$ but merely the divisorial component of the flat limit. Similarly, the divisor $D_2$ is $r$ general lines on the generic fiber $\mathbb{P}^2$ with divisorial limit $D_0$. In this case it turns out that $D_0$ is actually the flat limit $(D_2)_{\eta_2}$. The pairs $(X_i, \tilde{D}_i) \to \text{Spec}(R_i)$ are projective locally stable families with special fiber $(S, \tilde{D}_0)$.

In particular, we can pick an $f_1$-ample line bundle $L_i$ satisfying $R^1(f_i)_*(\mathcal{L}_i^{2m}) = 0$ for every $m > 0$. Then any section of $H^0((\mathcal{L}_i)|_S)$ extends to a section of $H^0(\mathcal{L}_i)$. This follows from the following exact sequence, where $t_i$ is the uniformizer of $\text{Spec}(R_i)$:

$$0 \to \mathcal{L}_i \to \tilde{L}_i \to (\mathcal{L}_i)|_S \to 0.$$
In [Kol18a, Example 2.39] it is shown that the divisor $K_X + \frac{1}{r}D_i$ is $\mathbb{Q}$-Cartier and anti-ample, so if we choose $L_i := \mathcal{O}_X(-(K_X + \frac{1}{r}D_i))$ for $n$ divisible enough, then $R^1(f_i)_*(\mathcal{L}_i^{\otimes m}) = 0$ for $m > 0$, and $(\mathcal{L}_1)|_S = (\mathcal{L}_2)|_S$. In particular, we can choose two generic hyperplane sections $\mathcal{H}_i \subseteq \mathcal{X}_i$ of an appropriate multiple of $L_i$ so that $(\mathcal{H}_1)|_S = (\mathcal{H}_2)|_S$ and the divisor $(\mathcal{H}_1)|_S$ avoids the singular locus of $S$ and intersects $D_0$ transversally. In particular the pair $(S, \frac{1}{r}D_0 + \frac{1}{2}(\mathcal{H}_i)|_S)$ is lc, so by inversion of adjunction the morphisms $(\mathcal{X}_i, \frac{1}{r}D_i + \frac{1}{2}\mathcal{H}_i) \to \text{Spec}(R_i)$ are stable.

This produces two stable families $(\mathcal{X}_i, \frac{1}{r}D_i + \frac{1}{2}\mathcal{H}_i) \to \text{Spec}(R_i)$ such that, if $\eta_i$ is the generic point of $\text{Spec}(R_i)$, then:

- the generic fiber is klt;
- the special fibers are the same, but
- $K^2_{(\mathcal{X}_1)_{n_1}} \neq K^2_{(\mathcal{X}_2)_{n_2}}$.

In particular, let $a := (\frac{1}{r}, \frac{1}{2})$. Then for every $\epsilon > 0$ the pairs $(\mathcal{X}_i, (1 - \epsilon)(\frac{1}{r}D_i + \frac{1}{2}\mathcal{H}_i))_{\eta_i}$ have different volumes. Therefore their stable limits along $\text{Spec}(R_i)$ are two different points in $\mathcal{M}_a - \epsilon$, but they have the same $a$-stable limit and thus map to the same point in $\mathcal{M}_a$. Therefore the morphism $\beta_a$ is not injective.

### 8.2. Multiple ways to reduce weights.

In Theorem 5.1, we construct wall-crossing morphisms for the $(a \to b)$-walls along the line segment connecting the two weights vectors. However, the wall-and-chamber structure ultimately is a result of [BCHM10, Theorem E] which gives a decomposition of a polytope of weight vectors rather than just a line segment. Thus, it is natural to ask how the wall-crossing morphisms behave over the whole polytope.

In particular, given weight vectors $a \geq b \geq c$ we have reduction morphisms $\rho_{b,a}$, $\rho_{c,b}$ and $\rho_{c,a}$ defined as a composition of flip-like morphisms and their inverses for the straight line segment $(a \to b)$, $(b \to c)$ and $(a \to c)$ respectively. Do these reduction morphisms commute? That is, do we have $\rho_{c,b} \circ \rho_{b,a} = \rho_{c,a}$ in general (see Remark 7.5)? Recall that the construction of the flip-like morphisms proceeds by running a minimal model program with scaling as we reduce the coefficients along the corresponding line segment. The example below illustrates that these mmp with scaling do not commute in general and therefore the $\rho$ do not necessarily commute.

We refer the reader to [Mir89] for the background about on elliptic fibrations (see also [AB17]). Consider a Weierstrass elliptic fibration $f : X \to \mathbb{P}^1$ with section $S$ and assume that the fundamental line bundle $\mathcal{L}$ on $\mathbb{P}^1$ has degree $3$. Consider five generic fibers $F_1, ..., F_5$ and let $F := \sum F_i$. Then we have that $K_X + (1 - \epsilon)S + \frac{1}{2}F$ is ample, and the pair $(X, S + \frac{1}{2}F)$ is stable. For a suitable $\epsilon$ small enough, the pair $(X, dS + \frac{d}{2}F)$ is also stable and klt, with $d := 1 - \epsilon$.

Recall now that

- $K_X = f^*(\omega_{\mathbb{P}^1} \otimes \mathcal{L}) = f^*(\mathcal{O}_{\mathbb{P}^1}(1))$, therefore
- $-2 = (K_X + S).S = K_X.S + S^2 = 1 + S^2$ so $S^2 = -3$.

We aim at reducing the weights on $S$ and on $F$. We first reduce the weight on $S$ from $d$ to $\epsilon$, for $\epsilon > 0$ small enough. It is easy to check that the pair $(X, (td + (1 - t)\epsilon)S + \frac{d}{2}F)$ is stable for every $t \in [0, 1]$. Now we can reduce the weight on $F$ from $\frac{d}{2}$ to $\frac{\epsilon}{2}$. Again, if $\epsilon$ is small enough, it is easy to check that the pair $(X, \epsilon S + (t\frac{d}{2} + (1 - t)\frac{\epsilon}{2})F)$ is stable for every $t \in [0, 1]$.

On the other hand, we can first reduce the weights on $F$ first, and then on $S$. If we reduce the weights on $F$ from $\frac{d}{2}$ to $\frac{\epsilon}{2} - \epsilon$, then $(K_X + dS + (\frac{d}{2} - \epsilon)F).S < 0$. In particular, the section $S$ must be contracted in the stable model. This gives a contraction morphism $g : X \to Y$, and a pseudoeuclidian pair $(Y, g_*(\frac{d}{2} - \epsilon)F))$. We can now keep reducing the weights from $\frac{d}{2} - \epsilon$ to $\frac{\epsilon}{2}$.
which produces a stable surface \((Z,D)\) with a contraction morphism \(X \to Z\) which factors through \(g\). In particular, \(X\) and \(Z\) are not isomorphic despite being the result of starting with the same \((1 - \epsilon, (1 - \epsilon)/2)\)-stable pair and reducing to coefficients \((\epsilon, \frac{1}{\epsilon})\).

One can produce a positive dimensional family of varieties of the above type by considering a Weierstrass fibration defined over the field \(\mathbb{C}(t)\). This gives a morphism \(\varphi : \text{Spec}(\mathbb{C}(t)) \to \mathcal{K}_a\) whose closure of its image, assuming that \(\varphi\) is non-isotrivial, will be a higher dimensional family of elliptic surfaces with generic fiber as in the example. In this case, the objects parametrized by the interior of the moduli spaces \(\mathcal{M}_t\) in Theorem 5.1 depends on the chosen path from \(a \to c\).

This shows that the moduli spaces \(\mathcal{M}_t\) and the flip-like morphisms depend \emph{a priori} on the choice of path. However, if we assume that we have a family such that the generic fiber has the same stable model for all coefficient vectors, then we can avoid this issue. More generally, suppose that there exists a polytope \(P\) of admissible weight vectors and moduli spaces \(\mathcal{M}_v\) of \(v\)-stable models for each \(v \in P\) such that

- there are dense open substacks \(\mathcal{U}_v \subset \mathcal{M}_v\) with reduction morphisms \(r_{b,a} : \mathcal{U}_a \to \mathcal{U}_b\) for \(b \leq a\), and
- for every \(c \leq b \leq a\) in \(P\), we have \(r_{c,b} \circ r_{b,a} = r_{c,a}\).

Then since the moduli spaces are separated, we must have that \(\rho_{c,b} \circ \rho_{b,a} = \rho_{c,a}\) (see [DH21, Lemma 7.2]). This applies for example in the hypothesis of Theorem 1.1.

More generally, we have proved the following.

**Theorem 8.1.** Let \(\pi : (\mathcal{X}, \mathcal{D}_1, ..., \mathcal{D}_n) \to \mathcal{M}_a\) be the universal family of pairs over \(\mathcal{M}_a\) and assume that for every \(v \leq a\) in an admissible polytope of weight vectors \(P\) (Definition 1.3), the generic point of \(\pi : (\mathcal{X}, \mathcal{vD}) \to \mathcal{M}_a\) is a \(v\)-stable family. Then the moduli spaces \(\mathcal{M}_v\) and morphisms \(\rho_{b,c} \circ \rho_{a,b} = \rho_{a,c}\) are well defined for every pair \(c \leq b\) in \(P\).

### 8.3. Reduction morphisms not birational

We give an example where the reduction morphisms are not birational if we don’t assume that the generic fiber is stable for each \(t \in [0,1]\).

Consider for simplicity a smooth stable surface \(X\) and let \(x \subset X\) a general point. Let \(D \subset X\) be a non-reduced closed subscheme supported on \(x\) corresponding to a tangent direction. Equivalently, \(D\) corresponds to choosing a line in the tangent space \(T_x,X\).

Let \(Z = Bl_p(X)\) be the blowup of \(X\) along the non-reduced subscheme \(D\). We claim that \(D\) is klt. Indeed, étale locally around \(x \in X\), this is the blowup of the ideal \((x,y^2)\) in \(\mathbb{A}^2_{x,y}\) which has a single \(A_1\) singularity.

In fact we can describe \(Z\) as follows. The Consider first the blowup \(Y = Bl_v(X)\). The exceptional divisor is isomorphic to the projectivized tangent space \(E = \mathbb{P}(T_x,X)\) and \(D\) corresponds to a point \(p \in E\). We consider the further blowup \(Z' = Bl_pY\). Now \(Z\) can be obtained by contracting the strict transform \(E'\) of \(E\) in \(Z'\) to a point. This produces the \(A_1\) singularity of \(Z\) mentioned above.

Now let \(\Delta\) be the exceptional divisor of the blowup \(\pi : Z \to X\) and let \(A\) be the strict transform of a generic ample divisor \(B\) on \(X\) passing through \(x\) and suppose that \(B\) has a node at \(x\). Consider the pair \((Z, a\Delta + bA)\). This pair is klt for \(a,b < 1\) and stable for \(a \ll b = 1 - \epsilon\). Indeed \(K_Z = \pi^*K_X + 2\Delta\) and \(\pi^*K_X\) is nef therefore \(K_Z.C \geq 0\) for all curves except \(\Delta\). We have \(A.\Delta = 2\) by construction and moreover we can compute that \(\Delta^2 = -1/2\) and \(K_Z.\Delta = -1\). Therefore \((K_Z + a\Delta + bA).\Delta > 0\) for \(a \ll b\). On the other hand there is a wall where \((a\Delta + bA).\Delta = 0\). At this wall, the log canonical model of \((Z, a\Delta + bA)\) is \((X,bB)\).
Thus the family of such pairs at this wall gets contracted by $\alpha$ onto a single point by in the moduli corresponding to $(X, B)$. However, $Z$ varies in a 1-dimensional moduli depending on the chosen tangent direction $D$. This shows that $\alpha$ need not be birational even under the klt assumption.

8.4. Further questions. We end with a few natural open questions.

**Question 1.** Let $\mathcal{U} \subset \mathcal{M}_a$ be the locus parametrizing klt pairs. Is $\rho_{b,a}(\mathcal{U})$ open inside $\mathcal{M}_b$?

This is true in the case of curves but note that even in that case, the image of $\mathcal{U}$ is not the locus of $\mathcal{M}_b$ parametrizing klt pairs.

**Question 2.** Suppose $\mathcal{N}_a$ is the normalization of an irreducible component of $\mathcal{K}_a$ with generic point parametrizing klt pairs. Then is the image $\mathcal{N}_b$ under reduction morphisms itself the normalization of an irreducible component of $\mathcal{K}_b$?

The question is essentially whether the $b$-stable model $(Z, \sum b_iD'_i)$ of a klt $a$-stable pair $(X, \sum a_iD_i)$ can be deformed in the Kollár moduli functor outside the locus of stable models of such pairs.

Next, in the examples of Sections 8.2 and 8.3, what seems to go wrong is that the stable model contracts marked divisors of our pairs. Thus it is natural to ask if this is the only thing that can go wrong.

**Question 3.** In the setting of Theorem 1.1, do the conclusions hold if we only assume that the stable models of $(X, \sum a_iD_i) \to B$ are isomorphic in codimension one for each $a \in \mathbb{P}$?

Finally, we make use throughout of the klt assumption for the generic fiber of our universal family in order to apply the results of [BCHM10] among other things. It is natural to ask the following.

**Question 4.** Do the wall-crossing results of this paper hold if we only assume that the generic fiber of the universal family over the moduli space is log canonical rather than klt?

References

[AB17] Kenneth Ascher and Dori Bejleri. Log canonical models of elliptic surfaces. *Adv. Math.*, 320:210–243, 2017.

[AB18] Kenneth Ascher and Dori Bejleri. Stable pair compactifications of the moduli space of degree one del pezzo surfaces via elliptic fibrations. *arXiv preprint arXiv:1802.00805*, 2018.

[AB19] Kenneth Ascher and Dori Bejleri. Compact moduli of elliptic K3 surfaces. *arXiv preprint arXiv:1902.10686*, 2019.

[AB20] Kenneth Ascher and Dori Bejleri. Moduli of weighted stable elliptic surfaces and invariance of log plurigenera. *Proceedings of the London Mathematical Society*, To appear, 2020.

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[ADL19] Kenneth Ascher, Kristin DeVleming, and Yuchen Liu. Wall crossing for k-moduli spaces of plane curves, 2019.

[ADL20] Kenneth Ascher, Kristin DeVleming, and Yuchen Liu. K-moduli of curves on a quadric surface and K3 surfaces. *J. Inst. Math. Jussieu, to appear*, 2020. [arXiv:2006.06816].

[AH11] Dan Abramovich and Brendan Hassett. Stable varieties with a twist. In *Classification of algebraic varieties*, EMS Ser. Congr. Rep., pages 1–38. Eur. Math. Soc., Zürich, 2011.

[AK16] Jarod Alper and Andrew Kresch. Equivariant versal deformations of semistable curves. *Michigan Math. J.*, 65(2):227–250, 2016.

[Ale94] Valery Alexeev. Boundedness and $K^2$ for log surfaces. *Internat. J. Math.*, 5(6):779–810, 1994.
[Ale02] Valery Alexeev. Complete moduli in the presence of semiabelian group action. *Ann. of Math. (2)*, 155(3):611–708, 2002.

[Ale15] Valery Alexeev. *Moduli of weighted hyperplane arrangements*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Basel, 2015. Edited by Gilberto Bini, Martí Lahoz, Emanuele Macrì and Paolo Stellari.

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[BI21] Dori Bejleri and Giovanni Inchiostro. Stable pairs with a twist and gluing morphisms for moduli of surfaces. *Selecta Math. (N.S.)*, 27(3):Paper No. 40, 44, 2021.

[Cor07] Alessio Corti, editor. *Flips for 3-folds and 4-folds*, volume 35 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2007.

[DH98] Igor V. Dolgachev and Yi Hu. Variation of geometric invariant theory quotients. *Inst. Hautes Études Sci. Publ. Math.*, (87):5–56, 1998. With an appendix by Nicolas Ressayre.

[DH21] Anand Deopurkar and Changho Han. Stable log surfaces, admissible covers, and canonical curves of genus 4. *Trans. Amer. Math. Soc.*, 374(1):589–641, 2021.

[FS10] Maksym Fedorchuk and David Smyth. Alternate compactifications of moduli spaces of curves. *Handbook of Moduli*, edited by G. Farkas and I. Morrison, 2010.

[Gro66] Alexander Grothendieck. *Éléments de géométrie algébrique* (rédigés avec la collaboration de Jean Dieudonné): IV. étude locale des schémas et des morphismes de schémas, troisième partie. *Inst. Hautes Études Sci. Publ. Math.*, 28, 1966.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Has03] Brendan Hassett. Moduli spaces of weighted pointed stable curves. *Adv. Math.*, 173(2):316–352, 2003.

[HK10] Christopher D. Hacon and Sándor J. Kovács. *Classification of higher dimensional algebraic varieties*, volume 41 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2010.

[HKT06] Paul Hacking, Sean Keel, and Jenia Tevelev. Compactification of the moduli space of hyperplane arrangements. *J. Algebraic Geom.*, 15(4):657–680, 2006.

[HMX18] Christopher D. Hacon, James McKernan, and Chenyang Xu. Boundedness of moduli of varieties of general type. *J. Eur. Math. Soc. (JEMS)*, 20(4):865–901, 2018.

[Kol11] János Kollár. Two examples of surfaces with normal crossing singularities. *Sci. China Math.*, 54(8):1707–1712, 2011.

[Kol13] János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

[Kol18a] János Kollár. Families of varieties of general type. Available at https://web.math.princeton.edu/~kollar/, 2018.

[Kol18b] János Kollár. Mumford’s influence on the moduli theory of algebraic varieties, 2018.

[Kol19] János Kollár. Families of divisors. *arXiv:1910.00937*, 2019.

[KP17] Sándor Kovács and Zsolt Patakfalvi. Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension. *J. Amer. Math. Soc.*, 30(4):959–1021, 2017.

[KSB88] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Invent. Math.*, 91(2):299–338, 1988.
Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

Radu Laza and Kieran G. O’Grady. GIT versus Baily-Borel compactification for quartic K3 surfaces. In *Geometry of moduli*, volume 14 of *Abel Symp.*, pages 217–283. Springer, Cham, 2018.

Radu Laza and Kieran O’Grady. Birational geometry of the moduli space of quartic K3 surfaces. *Compos. Math.*, 155(9):1655–1710, 2019.

Radu Laza and Kieran O’Grady. GIT versus Baily-Borel compactification for K3’s which are double covers of $\mathbb{P}^1 \times \mathbb{P}^1$. *Adv. Math.*, 383:Paper No. 107680, 63, 2021.

Rick Miranda. *The basic theory of elliptic surfaces*. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1989.

Ulf Persson and Henry Pinkham. Some examples of nonsmoothable varieties with normal crossings. *Duke Math. J.*, 50(2):477–486, 1983.

Michael Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996.

Ravi Vakil. Murphy’s law in algebraic geometry: badly-behaved deformation spaces. *Invent. Math.*, 164(3):569–590, 2006.

Department of Mathematics, University of California, Irvine, CA, 92697

Email address: kascher@uci.edu

Harvard University, One Oxford Street, Cambridge, MA 02138 USA

Email address: bejleri@math.harvard.edu

University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195 USA

Email address: giovanni@math.brown.edu

École Polytechnique Fédérale de Lausanne (EPFL), MA C3 635, Station 8, 1015 Lausanne, Switzerland

Email address: zsolt.patakfalvi@epfl.ch