RESEARCH ARTICLE

The vanishing cohomology of non-isolated hypersurface singularities

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Abstract
We employ the perverse vanishing cycles to show that each reduced cohomology group of the Milnor fiber, except the top two, can be computed from the restriction of the vanishing cycle complex to only singular strata with a certain lower bound in dimension. Guided by geometric results, we alternately use the nearby and vanishing cycle functors to derive information about the Milnor fiber cohomology via iterated slicing by generic hyperplanes. These lead to the description of the reduced cohomology groups, except the top two, in terms of the vanishing cohomology of the nearby section. We use it to compute explicitly the lowest (possibly nontrivial) vanishing cohomology group of the Milnor fiber.

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1 INTRODUCTION

In his search for exotic spheres, Milnor [27] initiated the study of the topology of complex hypersurface singularity germs. For a germ of an analytic map $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ having a singularity at the origin, Milnor introduced what is now called the Milnor fibration $f^{-1}(D^*_\delta) \cap B_\varepsilon \to D^*_\delta$ (in a small enough ball $B_\varepsilon \subset \mathbb{C}^{n+1}$ and over a small enough punctured disc $D^*_\delta \subset \mathbb{C}$), and the Milnor fiber $F := f^{-1}(t) \cap B_\varepsilon$ of $f$ at 0. Around the same time, Grothendieck [8] proved Milnor’s conjecture that the eigenvalues of the monodromy acting on $H^* (F; \mathbb{Z})$ are roots of unity, and Deligne [9] defined the nearby and vanishing cycle functors, $\psi_f$ and $\varphi_f$, globalizing Milnor’s construction. A few years later, Lê [16] extended the geometric setting of the Milnor tube fibration to the case.
of functions defined on complex analytic germs, using the existence of Thom–Whitney stratifications on the zero set $f^{-1}(0)$, which was proved by Hironaka [12].

Since their introduction more than half a century ago, the Milnor fiber and vanishing cycles have found a wide range of applications, in fields like algebraic geometry, algebraic and geometric topology, symplectic geometry, singularity theory, enumerative geometry, computational topology and algebraic statistics. However, despite the enormous interest and vast applications, the very basic question of describing the topology of the Milnor fiber (for example, Betti numbers) for arbitrary hypersurface singularity germs remains largely open.

The Milnor fiber of an isolated hypersurface singularity germ was completely described by Milnor in [27]. The study of hypersurfaces with 1-dimensional singularities was initiated by Yomdin [13], Siersma [39, 40, 42, 43], and continued and refined by Vannier [51], Pellikaan [30, 31], Schrauwen [35], de Jong [2], Zaharia [52], Tibăr [45, 48], and so forth. Milnor fibers for higher dimensional singularities were studied by Lê [15], Zaharia [53], Shubladze [37, 38], Massey [19–23], Nemethi [28, 29], Tibăr [49, 50], Dimca–Saito [4], Libgober [17], Maxim [24], Fernandez de Bobadilla [5, 6], and so forth.

In [15], Lê developed a recurrent method for computing the Betti numbers of the Milnor fiber, based on slicing the singularity by generic hyperplanes, and in [51], Vannier described a general handlebody model for the Milnor fiber of function germs with a 1-dimensional singular locus. These results led Massey to a certain estimation of the Betti numbers of the Milnor fiber, based on the polar multiplicities encoded into what he called the ‘Lê numbers’ (for example, see [21], and also [22] for further developments).

In [4], Dimca and Saito investigated local consequences of the perversity of vanishing cycles, and computed the Milnor fiber cohomology from the restriction of the vanishing cycle complex to the real link of the singularity. In particular, they show that the reduced cohomology groups $\tilde{H}_i(F; \mathbb{Q})$ of the Milnor fiber are completely determined for $i < n - 1$ (and for $i = n - 1$ only partially) by the restriction of the vanishing cycle complex to the complement of the singularity. More precise computations are made in the case when $f^{-1}(0)$ is a divisor with normal crossings in a punctured neighborhood of the singular point, and results are also given on the size of the Jordan blocks of the monodromy in this particular case. If $f$ is a homogeneous polynomial, a more refined dependence of the vanishing cohomology on the singular strata was obtained by Maxim in [24, Proposition 5.1], and also Libgober [17, Theorem 3.1] in the case when $f$ defines a central hyperplane arrangement.

In this paper, we study the Milnor fiber cohomology for hypersurface singularity germs with a singular locus of arbitrary positive dimension. Before discussing our main results, let us introduce some notation and assumptions. Unless otherwise specified, all cohomology groups in this paper are with $\mathbb{Z}$-coefficients.

For $n \geq 1$, we consider a nonconstant holomorphic function germ $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ defined on a pure $(n + 1)$-dimensional complex singularity germ $(X, 0)$ contained in some ambient $(\mathbb{C}^N, 0)$. We assume moreover that

$$r_{\mathbb{H}}(X, \mathbb{Z}) = n + 1,$$

with $r_{\mathbb{H}}(X, \mathbb{Z})$ denoting the \textit{rectified homological depth} of $X$ with respect to the ring $\mathbb{Z}$, as defined by Grothendieck [7], see also [11, 36]. For example, a local complete intersection $X$ of pure complex dimension $n + 1$ satisfies this property. Our assumption on $r_{\mathbb{H}}(X, \mathbb{Z})$ is equivalent to the following two conditions: (i) The shifted constant sheaf $\mathbb{Z}_X[n + 1]$ is a $\mathbb{Z}$-\textit{perversesheaf} on $X$, and (ii) the costalks of $\mathbb{Z}_X[n + 1]$ in the lowest possible degree are free abelian; see Corollary 2.9 for
a precise formulation of this equivalence. If one is only interested in the \( \mathbb{Q} \)-cohomology of the Milnor fiber (for example, Betti numbers), it suffices to assume that \( r_{\text{Hd}}(X, \mathbb{Q}) = n + 1 \), which is equivalent to the fact that \( \mathbb{Q}_X[n + 1] \) is a \( \mathbb{Q} \)-perversesheafon\(X\).

In the formulation of our results, the notation \(X\) will tacitly mean a small enough representative of the germ at 0 of the space \(X\), that is, the intersection of \(X\) with an arbitrarily small ball at the origin. The same convention applies to all the subgerms of \((X, 0)\). We denote by \(\Sigma\) the germ of the stratified singular locus of \(f\) (with respect to some fixed Whitney stratification of \(X\)), and we assume that the complex dimension \(s\) at the origin of \(\Sigma\) satisfies \(0 < s < n\). We denote by \(F\) the Milnor fiber of \(f\) at the origin, and let \(P := \varphi_f^*\mathbb{Z}_X[n]\) be the complex of perverse vanishing cycles. It is then well known that the reduced Milnor fiber cohomology (that is, the vanishing cohomology) of \(F\) is computed by \(P\), and the only possibly nontrivial vanishing cohomology groups \(\tilde{H}^k(F)\) are concentrated in degrees \(n - s \leq k \leq n\). Furthermore, the assumption \(r_{\text{Hd}}(X, \mathbb{Z}) = n + 1\) yields that the \(\mathbb{Z}\)-perverse sheaf \(P\) also satisfies the property of freeness of costalks in lowest possible degree. This property is used in our paper to formulate cohomological substitutes for various homotopy-theoretic statements for the Milnor fiber (which do not necessarily hold if \(X\) is singular).

In the above notations, our first result (Theorem 3.1) uses only the perversity of the vanishing cycles \(P\) to show that, if \(s \geq 2\) and \(k < n - 1\), the group \(\tilde{H}^k(F)\) can be computed from the restriction of the vanishing cycle complex to the singular strata of dimension \( \geq n - k - 1 \) (see also Corollary 3.2 and Proposition 4.3). This is a strengthening of the above-mentioned result of Dimca–Saito (but from a different perspective, in the sense that our result involves the intersection of the whole singular locus of \(f\) with a sufficiently small open ball, while Dimca–Saito formulate their result in terms of the real link; see Remark 3.3(b) for more details), and it reduces the calculation of the vanishing cohomology to a hypercohomology spectral sequence. This computation, while tedious in general, can be made more explicit in the case of the least nontrivial vanishing cohomology group \(\tilde{H}^{n-s}(F)\). In fact, if \(s \geq 1\), Theorem 3.4 gives a monomorphism

\[
\tilde{H}^{n-s}(F) \hookrightarrow \bigoplus_i \tilde{H}^{n-s}(F^{n,i})A_i,
\]

where the summation is over the collection \(\{\Sigma_{s,i}\}_i\) of \(s\)-dimensional singular strata of the germ of \(\Sigma\) at the origin, \(F^{n,i}_{s,i}\) is the transversal Milnor fiber to the \(s\)-dimensional stratum \(\Sigma_{s,i}\), and \(A_i\) denotes the action of \(\pi_1(\Sigma_{s,i})\) on \(\tilde{H}^{n-s}(F^{n,i}_{s,i})\), with invariant subspaces appearing on the right-hand side of \((1.1)\). This represents an extension to arbitrary singularities of Siersma’s results [42] for 1-dimensional singularities in terms of the \textit{vertical monodromy}, as well as of the more recent development in [45], and answers Siersma’s conjecture made in his overview [43]. It also provides an upper bound for the Betti number \(b_{n-s}(F)\). Note that the assumption \(r_{\text{Hd}}(X, \mathbb{Z}) = n + 1\) yields in addition that \(\tilde{H}^{n-s}(F)\) is free, since the right-hand side of \((1.1)\) becomes free in this case free.

If \(s \geq 2\), Theorem 3.4(b) (see also (3.26) and Remark 3.5) expresses the ‘defect’ of \((1.1)\) from being an isomorphism in terms of costalks of the perversive vanishing cycle complex \(P\) at points in the \((s - 1)\)-dimensional strata. In particular, this yields a lower bound for \(b_{n-s}(F)\), which to our knowledge is completely new. We also observe that, if \(s \geq 2\) and \(\Sigma\) does not contain any \((s - 1)\)-dimensional strata, the monomorphism \((1.1)\) becomes an isomorphism.

If \(s \geq 1\), monomorphisms similar to \((1.1)\) are obtained in Theorem 3.1 for all vanishing cohomology groups \(\tilde{H}^k(F), n - s \leq k \leq n - 1\), though not as explicit as in the case \(k = n - s\). The monomorphism \((1.1)\) leads to divisibility results for the characteristic polynomials of Milnor monodromies (Corollary 3.7), as well as to upper bounds for the dimension of the monodromy.
eigenspaces and maximal sizes of Jordan blocks (Corollary 3.7). Furthermore, similar statements can be formulated for the eigenspaces of vanishing cohomology in terms of the generalized eigensheaves of the vanishing cycles (Theorem 3.8 and Corollary 3.9).

We next take a more geometric viewpoint for computing the Milnor fiber cohomology via slicing. In Section 4 we present a sheaf-theoretic interpretation of results of Lê [15] and Tibăr [49] on the computation of the number of vanishing cycles via iterated slicing by generic hyperplanes. In Proposition 4.3, we then give (for $s \geq 2$) an interpretation of the reduced Milnor fiber cohomology in terms of the vanishing cohomology of the nearby section of $f$ (introduced in [49]). Upper bounds for the Betti numbers of the Milnor fiber are derived via slicing in terms of polar multiplicities (Corollary 4.7).

This slicing technique enables us to develop in Section 5 a geometric method for the computation of the lowest (possibly nontrivial) vanishing cohomology group $H^{n-s}(F)$, for $n > s \geq 2$, by using its isomorphism with the vanishing cohomology of the nearby section. It turns out that, by repeated slicing, the computation reduces to the case of a 2-dimensional singular locus. We then use ideas developed in [44] and [45] for computing the homology groups in case of 1-dimensional singularities via admissible deformations of $f$ to obtain in Corollary 5.3 and Theorem 5.4 a description of $H^{n-s}(F)$ in terms of the invariant submodule of a monodromy representation (5.6) on the transversal Milnor fiber of the $s$-dimensional strata, which is in fact a genuine vertical monodromy representation. The similarity with the terms and results of Theorem 3.4 is striking. But surprisingly, this monodromy representation is totally different from (1.1) used in Theorem 3.4, and in general cannot be deduced from it by a Lefschetz slicing argument.

The results obtained from the perverse sheaf side and from the geometric side feed and complete each other in the pursuit of computability. We detail a bunch of examples in Section 6, showing how the general theory works and emphasizing the efficacy of Theorem 5.4 in computing $H^{n-s}(F)$.

2 | PRELIMINARIES

In this section, we review several concepts which play an important role in the remainder of the paper. In Section 2.1, we recall some background on constructible complexes and perverse sheaves. In Section 2.2, we recall the notion of rectified homological depth and indicate its relation to perverse sheaves. In Section 2.3, we introduce the nearby and vanishing cycle functors associated to a holomorphic map, and describe their relation to the cohomology of the Milnor fiber.

2.1 | Perverse sheaves

In this section, we recall the definition of perverse sheaves (for example, see [3, 25] for a quick introduction).

Let $A$ be a noetherian commutative ring of finite global dimension. Let $X$ be a complex analytic variety, and denote by $D^b(X)$ the derived category of bounded complexes of sheaves of $A$-modules.

Recall that a sheaf $\mathcal{F}$ of $A$-modules is said to be constructible if there is a Whitney stratification $\mathcal{W}$ of $X$ so that the restriction $\mathcal{F}|_S$ of $\mathcal{F}$ to every stratum $S \in \mathcal{W}$ is an $A$-local system with finitely generated stalks. A bounded complex $\mathcal{F}' \in D^b(X)$ is said to be constructible if all its cohomology sheaves $H^i(\mathcal{F}')$ are constructible. Denote by $D^b_c(X)$ the full triangulated subcategory of $D^b(X)$
consisting of constructible complexes (that is, complexes which are constructible with respect to some Whitney stratification).

All dimensions below are taken to be complex dimensions.

**Definition 2.1.**

(a) The *perverse t-structure* on $D^b_c(X)$ consists of the subcategories $^pD^{\leq 0}(X)$ and $^pD^{\geq 0}(X)$ of $D^b_c(X)$ defined as:

\[ ^pD^{\leq 0}(X) = \{ F^\cdot \in D^b_c(X) | \dim \text{supp}^{-j}(F^\cdot) \leq j, \forall j \in \mathbb{Z} \}, \]

\[ ^pD^{\geq 0}(X) = \{ F^\cdot \in D^b_c(X) | \dim \text{cosupp}^{+j}(F^\cdot) \leq j, \forall j \in \mathbb{Z} \}, \]

where, for $k_x : \{x\} \hookrightarrow X$ denoting the point inclusion, we define the *j-th support* and, resp., the *j-th cosupport* of $F^\cdot \in D^b_c(X)$ by:

\[ \text{supp}^{j}(F^\cdot) = \{ x \in X | H^j(k_x^*F^\cdot) \neq 0 \}, \]

\[ \text{cosupp}^{j}(F^\cdot) = \{ x \in X | H^j(k_x^!F^\cdot) \neq 0 \}. \]

Here, $k_x^*F^\cdot$ and $k_x^!F^\cdot$ are called the *stalk* and, resp., *costalk* of $F^\cdot$ at $x$.

(b) A complex $F^\cdot \in D^b_c(X)$ is called a *perverse sheaf* on $X$ if $F^\cdot \in ^pD^{\leq 0}(X) \cap ^pD^{\geq 0}(X)$.

(c) We say that $F^\cdot \in D^b_c(X)$ is *strongly perverse* if $F^\cdot \in ^pD^{\leq 0}(X)$ and $D_X F^\cdot \in ^pD^{\leq 0}(X)$, with $D_X$ denoting the dualizing functor.

**Example 2.2.** Assume $X$ is of pure complex dimension with $c : X \to \text{pt}$ the constant map to a point space, and let $A^X = c^*A$ be the constant $A$-sheaf on $X$. Then:

(a) $A^X[\dim X] \in ^pD^{\leq 0}(X)$.

(b) If $X$ is a local complete intersection then $A^X[\dim X]$ is a perverse sheaf on $X$ (for example, see [3, Theorem 5.1.20]).

It is important to note that the categories $^pD^{\leq 0}(X)$ and $^pD^{\geq 0}(X)$ can also be described in terms of a fixed Whitney stratification of $X$. Indeed, the perverse $t$-structure can be characterized as follows (for example, see [25, Theorem 8.3.1]):

**Theorem 2.3.** If $F^\cdot \in D^b_c(X)$ is constructible with respect to a Whitney stratification $\mathcal{W}$ of $X$, then:

(i) *stalk vanishing*:

\[ F^\cdot \in ^pD^{\leq 0}(X) \iff \forall S \in \mathcal{W}, \forall x \in S : H^j(k_x^*F^\cdot) = 0 \text{ for all } j > -\dim S. \]

(ii) *costalk vanishing*:

\[ F^\cdot \in ^pD^{\geq 0}(X) \iff \forall S \in \mathcal{W}, \forall x \in S : H^j(k_x^!F^\cdot) = 0 \text{ for all } j < \dim S. \]
Remark 2.4.

(a) If $S \in \mathcal{W}$ is a stratum of $X$ with inclusion map $k_S : S \hookrightarrow X$, the above costalk vanishing condition for $S$ is equivalent to:

$$H^j(k_S^! F^-) = 0, \quad \text{for all } j < -\dim S. \quad (2.1)$$

(b) If $A$ is a field, the notions of perverse sheaf and strongly perverse sheaf coincide. Indeed, in this case, the universal coefficient theorem yields that $F^- \in \mathcal{P}D^{\geq 0}(X)$ if and only if $D_X F^- \in \mathcal{P}D^{\leq 0}(X)$.

We conclude this section with the following.

**Proposition 2.5.** Assume that the ring $A$ is a principal ideal domain (for example, $A = \mathbb{Z}$). If $F^- \in D^b(X)$ is constructible with respect to a Whitney stratification $\mathcal{W}$ of $X$, then $D_X F^- \in \mathcal{P}D^{\leq 0}(X)$ if and only if the following two conditions are satisfied:

(i) $F^- \in \mathcal{P}D^{\geq 0}(X)$;

(ii) for any stratum $S \in \mathcal{W}$ and any $x \in S$, the costalk cohomology $H^{\dim S}(k_x^! F^-)$ is free.

In particular, $F^-$ is strongly perverse (with respect to $\mathcal{W}$) if and only if $F^-$ is perverse and property (ii) above holds (that is, costalks of $F^-$ in the lowest possible degree are free on each stratum).

**Proof.** Let $S \in \mathcal{W}$ and $x \in S$, with inclusion $k_x : \{x\} \hookrightarrow X$. Properties of the dualizing functor and the universal coefficient theorem yield:

$$H^j(k_x^* D_X F^-) \cong H^j(D_X k_x^! F^-) \cong \text{Hom}(H^{-j}(k_x^! F^-), A) \oplus \text{Ext}(H^{-j+1}(k_x^! F^-), A).$$

The desired equivalence can now be checked easily. \qed

### 2.2 Rectified homological depth

In this section we recall the notion of rectified homological depth, and indicate its relation to (strongly) perverse sheaves.

Let $X$ be a complex analytic space of complex pure dimension. Following [36, Definition 6.0.4], we make the following.

**Definition 2.6.** The rectified homological depth $rHD(X, A)$ of $X$ with respect to the commutative base ring $A$ is $\geq d$ (for some $d \in \mathbb{Z}$) if

$$D_X(A_X[d]) \in \mathcal{P}D^{\leq 0}(X). \quad (2.2)$$

As pointed out in [36], the above definition agrees with the notion of rectified homological depth introduced by Hamm and Lê [11] in more geometric terms.

**Example 2.7.**

(a) One always has $rHD(X, A) \leq \dim(X)$, and $rHD(X, A) = \dim(X)$ if $X$ is smooth and nonempty.
(b) If $X$ is a pure-dimensional local complete intersection, then $\text{rHd}(X, A) = \text{dim} X$ (cf. [36, Example 6.0.11]).

In view of Example 2.2(a) and Definition 2.1(c), we have the following equivalence (see also [36, (6.14)]):

**Proposition 2.8.** For any nonempty pure-dimensional complex analytic space $X$, we have:

$$rHd(X, A) = \text{dim} X \iff A_{\chi}[\text{dim} X] \text{ is strongly perverse.}$$

As a consequence, Remark 2.4(b) and Proposition 2.5 yield the following.

**Corollary 2.9.** Let $X$ be a nonempty pure-dimensional complex analytic space with a Whitney stratification $W$.

(a) If $A$ is a field, then:

$$rHd(X, A) = \text{dim} X \iff A_{\chi}[\text{dim} X] \text{ is perverse.}$$

(b) If $A$ is a principal ideal domain, then $rHd(X, A) = \text{dim} X$ if and only if the following two conditions are satisfied:

(i) $A_{\chi}[\text{dim} X]$ is perverse.

(ii) For any stratum $S \in W$ and any $x \in S$ with $k_{\chi} : \{x\} \hookrightarrow X$, the costalk cohomology $\text{H}^{\text{dim} S}(k^!_{\chi}A_{\chi}[\text{dim} X])$ is free.

## 2.3 Nearby and vanishing cycle functors

Let $n \geq 1$, and consider a nonconstant holomorphic function germ $f : (X, 0) \to (\mathbb{C}, 0)$ defined on a pure $(n + 1)$-dimensional complex singularity germ $(X, 0)$ contained in some ambient $(\mathbb{C}^N, 0)$. Assume as in [49] that

$$rHd(X, \mathbb{Z}) = n + 1,$$

(2.3)

with $rHd(X, \mathbb{Z})$ denoting the *rectified homological depth* of $X$ with respect to the ring $\mathbb{Z}$ (cf. Definition 2.6). As seen in Example 2.7(b), a local complete intersection $X$ of pure complex dimension $n + 1$ satisfies this property. By Proposition 2.8, our assumption on $rHd(X, \mathbb{Z})$ is equivalent to the condition that the shifted constant sheaf $\mathbb{Z}_{\chi}[n + 1]$ is a *strongly perverse sheaf* on $X$, which by Corollary 2.9 means that: (i) the shifted constant sheaf $\mathbb{Z}_{\chi}[n + 1]$ is a perverse sheaf on $X$ in the usual sense, and (ii) on each stratum, the costalks of $\mathbb{Z}_{\chi}[n + 1]$ in the lowest possible degree are free abelian. (Moreover, the weaker condition $rHd(X, \mathbb{Q}) = n + 1$ is equivalent to the fact that $\mathbb{Q}_{\chi}[n + 1]$ is a $\mathbb{Q}$- perverse sheaf on $X$, which is the hypothesis considered in [4].)

Since $Y := f^{-1}(0)$ is a principal divisor on $X$, it follows from [36, Example 6.0.11] that $rHd(Y, \mathbb{Z}) = n$, and hence $\mathbb{Z}_{\chi}[n]$ is a strongly perverse sheaf on $Y$ in the above sense.

We denote by $\psi_f, \varphi_f$ the nearby and vanishing cycle functors associated to $f$ (for example, see [3, 25] for a quick introduction). Recall that if $u : Y = f^{-1}(0) \hookrightarrow X$ is the inclusion map, there is
a distinguished triangle of functors

\[ u^* \to \psi_f \to \varphi_f \to 1 \]  

(2.4)

Moreover, the nearby and vanishing cycle complexes \( \psi_f \mathbb{Z}_X \) and \( \varphi_f \mathbb{Z}_X \) encode the Milnor fiber cohomology at points along \( Y \), in the sense that

\[ H^k(F_x) \cong H^k(\psi_f \mathbb{Z}_X)_x \quad \text{and} \quad \tilde{H}^k(F_x) \cong H^k(\varphi_f \mathbb{Z}_X)_x, \]  

(2.5)

with \( F_x \) denoting the Milnor fiber of \( f \) at \( x \in Y \), and where \( H^*(\mathbb{Z}_X) \) computes the stalk cohomology at \( x \). In particular, the vanishing cycle complex \( \varphi_f \mathbb{Z}_X \) is supported on the stratified singular locus \( \text{Sing}_{\mathbb{W}}(f) \) of \( f \) with respect to a fixed Whitney stratification \( \mathbb{W} \) of \( X \), which is contained in \( Y \) as a set germ at the origin.

If we work with \( \mathbb{C} \)-coefficients, let us denote by \( h \) the monodromy of \( \varphi_f \mathbb{C}_X \), with its Jordan decomposition \( h = h_u h_s \), where \( h_s \) is semi-simple (and locally of finite order) and \( h_u \) is unipotent. For any \( \lambda \in \mathbb{C} \), we set

\[ \psi_f,\lambda \mathbb{C}_X := \ker(h_s - \lambda) \subset \psi_f \mathbb{C}_X \]

and similarly for \( \varphi_f,\lambda \mathbb{C}_X \), in the category of shifted perverse sheaves. It follows from the definition of vanishing cycles that \( \psi_f,\lambda \mathbb{C}_X = \varphi_f,\lambda \mathbb{C}_X \) for \( \lambda \neq 1 \). Moreover, we have decompositions

\[ \psi_f \mathbb{C}_X = \bigoplus_{\lambda} \psi_f,\lambda \mathbb{C}_X, \quad \varphi_f \mathbb{C}_X = \bigoplus_{\lambda} \varphi_f,\lambda \mathbb{C}_X, \]

and for \( x \in Y \) we have isomorphisms

\[ H^k(F_x; \mathbb{C})_\lambda \cong H^k(\psi_f,\lambda \mathbb{C}_X)_x \quad \text{and} \quad \tilde{H}^k(F_x; \mathbb{C})_\lambda \cong H^k(\varphi_f,\lambda \mathbb{C}_X)_x, \]  

(2.6)

with \( H^k(F_x; \mathbb{C})_\lambda \) the corresponding \( \lambda \)-eigenspace of the action of the semi-simple part of the Milnor monodromy on \( H^k(F_x; \mathbb{C}) \).

### 3 MILNOR FIBER COHOMOLOGY VIA PERVERSE VANISHING CYCLES

In this section, we investigate the cohomology of the Milnor fiber as a consequence of the fact that, up to a shift, the nearby and vanishing cycle functors preserve perverse sheaves. In Section 3.1 we show that each reduced cohomology group of the Milnor fiber, except the top two, can be computed from the restriction of the vanishing cycle complex to only singular strata with a certain lower bound in dimension. The same method applies to the study of the monodromy eigenspaces.
of Milnor fiber cohomology; this is discussed in Section 3.3. In Section 3.2, we discuss divisibility results for the characteristic polynomials of Milnor monodromy, and upper bounds for the maximal size of Jordan blocks.

### 3.1 Betti bounds via perverse vanishing cycles

In this section, we derive information on the Milnor fiber cohomology as a consequence of the perversity of vanishing cycles (compare also with [4]).

Under the notations and assumptions of Section 2.3, we consider the stratified singular locus Σ := SingW(f), which is defined as the union of the singular loci Sing f|Wi of the restrictions of f to the strata Wi of fixed Whitney stratification W. Let us remark that Σ is a closed set and a subset of Y. We may and will assume in the following that it is a union of strata in W. Let s := dim0 Σ.

Let Bε be a Milnor ball for f at the origin, that is, the intersection of a small enough ball at the origin of the ambient space C^N with a suitable representative of the germ (X, 0). Let

\[ F := B_ε \cap f^{-1}(γ), \quad 0 < γ \ll ε, \]

be the Milnor fiber of f at the origin.

Assume 0 < s < n and let

\[ P := L \varphi_f^{-1}(\mathbb{Z}_X[n+1]), \]

where \( L \varphi_f^{-1} \) denotes as in Section 2.3 the perverse vanishing cycle functor associated to f. Since \( \mathbb{Z}_X[n+1] \) was assumed to be a perverse sheaf on X, we get that P is perverse on Y = f^{-1}(0). Moreover, since P is supported on Σ, we obtain that

\[ P_0 := P|_Σ \]

is a perverse sheaf on Σ (for example, see [25, Corollary 8.2.10]). For any integer k we have:

\[ \tilde{H}^k(F) \cong H^{k-n}(P_0) \cong H^{k-n}(P_0)_0. \]  

(3.1)

The support condition for the perverse sheaf P_0 on Σ then yields (for example, as in [25, Proposition 10.6.2]) that the only possibly nontrivial integral reduced cohomology \( \tilde{H}^k(F) \) of F is concentrated in degrees \( n - s \leq k \leq n \).

The computation of \( \tilde{H}^k(F) \) is equivalent via (3.1) to the computation of the hypercohomology group \( H^{k-n}(B_ε ∩ Σ; P_0) \), where B_ε is as above a Milnor ball for f at the origin. For convenience of notation, we will follow the convention mentioned in the Introduction, namely we will assume throughout the paper that we work in a sufficiently small open ball B_ε around the origin, and replace B_ε ∩ Σ with simply Σ, B_ε ∩ Y with Y, B_ε with X, and so forth. In particular, we shall write

\[ \tilde{H}^k(F) \cong H^{k-n}(Σ; P_0), \]

(3.2)

with \( H^* \) denoting the hypercohomology functor.

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1 In fact, in Section 3.1 we use only the perversity in the usual sense of \( \mathbb{Z}_X[n+1] \), except for part (d) of Theorem 3.4 where the strong perversity assumption is needed.
Denote by $S$ a Whitney stratification of $Y$ so that $P$ is $S$-constructible. Upon refining $\mathcal{W}$, we can assume that any stratum in $S$ is also a stratum in $\mathcal{W}$. Since we work in a neighborhood of the origin, we can further assume (after shrinking $X$, and restricting to a normal slice through the origin as in Example 6.1 below) that the origin is the only 0-dimensional stratum of $S$ and $\Sigma$ is the union of strata of complex dimension $\leq s$. Let us make the following notations:

- $\Sigma_{\ell}$ is the union of $\ell$-dimensional strata of $S$ (so $\Sigma_0 = \{0\}$);
- $U_{\ell}$ is the union of strata of $S$ of complex dimension $\geq \ell$.

Then each $U_{\ell}$ is an open subset of $\Sigma$, and

$$\Sigma_s = U_s \subseteq U_{s-1} \subseteq \cdots \subseteq U_0 = \Sigma$$

(3.3)

with $\Sigma_\ell = U_\ell \setminus U_{\ell+1}$. Set

$$P_{\ell} := P_0|_{U_\ell}$$

and note that $P_{\ell}$ is a perverse sheaf of $U_\ell$.

We are now ready to prove the following result, which can be seen as an enhancement of results from [4]. However, note that while [4] works with $\mathbb{Q}$-coefficients and the restriction of the vanishing cycle complex to the real link (see Remark 3.3(b) below), our result holds over the integers and is formulated in terms of the germs of strata of the stratified singular locus of $f$.

**Theorem 3.1.** Let $f : (X,0) \to (\mathbb{C},0)$ be a nonconstant holomorphic function defined on a pure $(n+1)$-dimensional complex singularity germ satisfying the property that $\mathbb{Z}_X[n+1]$ is a $\mathbb{Z}$-perverse sheaf on $X$. Assume that the stratified singular locus $\Sigma$ of $f$ is of complex dimension $s > 0$. If $F$ denotes the Milnor fiber of $f$ at the origin, then using the above notations for the stratification of $\Sigma$, the following hold:

(a) For any $j = 0, \ldots, s-1$ there is a monomorphism

$$\tilde{H}^{n-s+j}(F) \hookrightarrow H^{-s+j}(U_{s-j}; P_{s-j}).$$

(b) If $s \geq 2$, then for any $j = 0, \ldots, s-2$ there is an isomorphism

$$\tilde{H}^{n-s+j}(F) \cong H^{-s+j}(U_{s-j-1}; P_{s-j-1}).$$

(3.5)

**Proof.** As already mentioned above, we assume that the singularity germ $X$ is represented by its intersection with a sufficiently small open ball $B_\varepsilon$ at the origin of $\mathbb{C}^N$.

(a) Fix an integer $j = 0, \ldots, s-1$. In the notations preceding the theorem, for any $0 \leq \ell \leq s - j - 1$ consider the inclusions

$$\Sigma_\ell \hookrightarrow U_\ell \leftrightarrow U_{\ell+1}$$

and the attaching distinguished triangle

$$v_{\ell}^*u_{\ell}^*P_{\ell} \to P_{\ell} \to Ru^{*}_{\ell}u_{\ell}^*P_{\ell}^{[1]}$$

(3.6)
with \( v_\\ell^1 = v_\\ell^* \) and \( u_\\ell^* P_\\ell \cong P_\\ell+1 \). The hypercohomology long exact sequence associated to (3.6) contains the terms

\[
\cdots \to H^{-s+j}(\Sigma_\\ell^*; v_\\ell^1 P_\\ell) \to H^{-s+j}(U_\\ell^*; P_\\ell) \to H^{-s+j}(U_\\ell+1^*; P_\\ell+1) \to \cdots
\] (3.7)

The group \( H^{-s+j}(\Sigma_\\ell^*; v_\\ell^1 P_\\ell) \) is computed by a hypercohomology spectral sequence whose \( E_2 \)-term is given by

\[
E_{p,q}^2 = H^p(\Sigma_\\ell^*; H^q(v_\\ell^1 P_\\ell)).
\] (3.8)

Since \( v_\\ell^1 P_\\ell \) is a constructible complex on \( \Sigma_\\ell^* \), its cohomology sheaves are local systems on every connected component of \( \Sigma_\\ell^* \). Hence, by reasons of dimension, \( E_{p,q}^{2} = 0 \) if \( p < 0 \) or \( p > 2\ell \). On the other hand, the costalk condition (2.1) for the perverse sheaf \( P_\\ell \) on \( U_\\ell^* \) (with the induced stratification) yields that \( H^q(v_\\ell^1 P_\\ell) \cong 0 \) for all \( q < -\ell \). Therefore, \( E_{p,q}^{2} = 0 \) if \( q < -\ell \). Altogether, since \( -s + j < -\ell \), we get that \( E_{p,q}^{2} = 0 \) for any pair of integers \((p, q)\) with \( p + q = -s + j \). The spectral sequence (3.8) then implies that

\[
H^{-s+j}(\Sigma_\\ell^*; v_\\ell^1 P_\\ell) \cong 0.
\] (3.9)

By combining (3.7) and (3.9), we get a monomorphism

\[
H^{-s+j}(U_\\ell^*; P_\\ell) \hookrightarrow H^{-s+j}(U_{\ell+1}^*; P_{\ell+1})
\] (3.10)

for any \( 0 \leq \ell \leq s - j - 1 \). Together with (3.2) and noting that \( \Sigma = U_0^* \), the above discussion yields a composition of monomorphisms

\[
\tilde{H}^{n-s+j}(F) \cong H^{-s+j}(U_0^*; P_0) \hookrightarrow H^{-s+j}(U_1^*; P_1) \hookrightarrow \cdots \hookrightarrow H^{-s+j}(U_{s-j}^*; P_{s-j}),
\]

thus completing the proof of (3.4).

(b) Let us now assume that \( s \geq 2 \) and fix an integer \( j = 1, \ldots, s - 1 \).

In view of (3.9), the long exact sequence (3.7) contains the terms:

\[
\cdots \to H^{-s+j-1}(\Sigma_\\ell^*; v_\\ell^1 P_\\ell) \to H^{-s+j-1}(U_\\ell^*; P_\\ell) \to H^{-s+j-1}(U_\\ell+1^*; P_\\ell+1) \to 0 \to \cdots
\] (3.11)

For any \( 0 \leq \ell \leq s - j - 1 \), the same arguments used for studying the spectral sequence (3.8) yield that \( H^{-s+j-1}(\Sigma_\\ell^*; v_\\ell^1 P_\\ell) \cong 0 \) since \( -s + j - 1 < -\ell \). In particular, (3.11) yields isomorphisms

\[
H^{-s+j-1}(U_\\ell^*; P_\\ell) \cong H^{-s+j-1}(U_{\ell+1}^*; P_{\ell+1})
\]

for all \( 0 \leq \ell \leq s - j - 1 \). Together with (3.2), this then yields isomorphisms

\[
\tilde{H}^{n-s+j-1}(F) \cong H^{-s+j-1}(U_0^*; P_0) \cong H^{-s+j-1}(U_1^*; P_1) \cong \cdots \cong H^{-s+j-1}(U_{s-j}^*; P_{s-j}).
\] (3.12)

The isomorphism (3.5) is then obtained by reindexing (that is, replacing \( j \) by \( j + 1 \) in (3.12)).
An immediate consequence of Theorem 3.1(b) is the following.

**Corollary 3.2.** If \( s \geq 2 \), then for any \( j = 0, \ldots, s - 2 \), the group \( \tilde{H}^{n-s+j}(F; \mathbb{Z}) \) depends only on the singular strata of dimension \( \geq s - j - 1 \) of \( \text{Sing}_Y(f) \).

**Remark 3.3.**

(a) Assuming \( s \geq 2 \) and fixing \( j = 0, \ldots, s - 2 \), if there are no strata of dimension \( s - j - 1 \), then \( U_{s-j} = U_{s-j-1} \), so in this case (3.5) is a finer result than (3.4). In general, the right-hand side of either (3.4) or (3.5) can be computed via the hypercohomology spectral sequence, though explicit computations can be tedious.

(b) The results of Theorem 3.1 and Corollary 3.2 show that \( \tilde{H}^j(F) \) for \( j < n - 1 \) (resp., \( j = n - 1 \)) is completely (resp., partially) determined by the restriction of the vanishing cycle complex \( \varphi_f \mathbb{Z}_X \) to the complement of the singular point at the origin. A similar statement, though not as explicit as Corollary 3.2, can be derived from [4, Theorem 0.1], where one considers the restriction of the vanishing cycle complex to the real link of the singularity. Specifically, if \( K \) denotes the real link of \( 0 \in Y = f^{-1}(0) \), that is, the intersection of \( Y \) with a sufficiently small sphere around 0 in a smooth ambient space \( \mathbb{C}^N \), then under our assumptions and notations one gets isomorphisms

\[
\tilde{H}^k(F; \mathbb{Z}) \cong \mathbb{H}^k(K; \varphi_f \mathbb{Z}_X | K) \tag{3.13}
\]

for all \( k < n - 1 \), and a monomorphism

\[
\tilde{H}^{n-1}(F; \mathbb{Z}) \hookrightarrow \mathbb{H}^{n-1}(K; \varphi_f \mathbb{Z}_X | K). \tag{3.14}
\]

For the benefit of the reader, we include here the elementary proof of (3.13) and (3.14). Recall that \( X \) is represented by its intersection with a Milnor ball \( B_\epsilon \) at the origin. Denote by \( i : \{0\} \hookrightarrow Y \) and \( j : Y \setminus \{0\} \hookrightarrow Y \) the inclusion maps. Let \( P \) be a \( \mathbb{Z} \)-perverse sheaf on \( Y \). The costalk conditions for \( P \) yields the vanishing:

(i) \( H^k(i! P) = 0 \) if \( k < 0 \). Moreover, it is well known that

(ii) \( H^k(i^* R j_* j^* P) \cong \mathbb{H}^k(K; P|_K) \), where \( K \) is the real link of \( 0 \in Y \).

By applying the pullback \( i^* \) to the attaching triangle

\[
i_! i^! P \rightarrow P \rightarrow R j_* j^* P \xrightarrow{[1]} \]

and using the fact that \( i^* i_! \simeq id \), one gets the distinguished triangle

\[
i^! P \rightarrow i^* P \rightarrow i^* R j_* j^* P \xrightarrow{[1]} . \tag{3.15}
\]

In view of (i) and (ii), the long exact sequence of hypercohomology groups associated to (3.15) yields isomorphisms

\[
H^k(i^* P) \cong \mathbb{H}^k(K; P|_K), \quad \forall \ k < -1, \tag{3.16}
\]
and a monomorphism

$$H^{-1}(i^* P) \hookrightarrow H^{-1}(K; P|_K). \quad (3.17)$$

To obtain (3.13) and (3.14), one simply applies (3.16) and (3.17) to the perverse vanishing cycles $P := p \varphi_f(Z_X[n + 1])$. It should be noted that Dimca and Saito worked in [4] with $\mathbb{Q}$-coefficients, but as seen above their result extends easily to the integers.

Finally, let us compare the above arguments with the statement and proof of our Theorem 3.1. It is well known that complex analytic sets are locally conelike (see [1]), hence $U_1 = \Sigma \setminus \{0\}$ is stratwise topologically equivalent with the product of the link of 0 in $\Sigma$ (that is, $K \cap \Sigma$) with an open interval $(0, \varepsilon)$. Therefore, in the special case $s - j = 2$, the isomorphism (3.5) reproves (3.13) with $k = n - 2$. In all other cases, our isomorphism (3.5) is strictly finer than (3.13).

Moreover, the monomorphism (3.4) specializes, after setting $s - j = 1$, to (3.14), while none of the other cases addressed by (3.4) (except at $j = 0$) has a counterpart in [4]. This justifies our assertion that Theorem 3.1 provides a new perspective and an enhancement of some of the results from [4].

(c) If $f$ is a homogeneous polynomial, a more refined dependence of the vanishing cohomology on the singular strata was obtained in [24, Proposition 5.1] (see also [17, Theorem 3.1] for the case when $f$ defines a central hyperplane arrangement).

In what follows, we specialize Theorem 3.1 to the case $j = 0$, to derive more explicit information about $\widetilde{H}^{n-s}(F)$, that is, the lowest (possibly nontrivial) cohomology group of $F$ (compare also with [4, Section 3.5] for a related discussion). Just like in Theorem 3.1, most arguments used in the proof of Theorem 3.4 below use only the perversity in the usual sense of $P$. But the statement about the freeness of $\widetilde{H}^{n-s}(F)$ (part (d) of Theorem 3.4) requires the strong perversity of $P$, deduced via Proposition 2.8 from the assumption $r_{Hd}(X, Z) = n + 1$. We first introduce some notations.

Recall that

$$U_s = \Sigma_s = \bigsqcup_i \Sigma_{s, i},$$

where $\Sigma_{s, i}$ are the $s$-dimensional (connected) strata of $\Sigma$. Denote by $F^n_{s, i}$ the Milnor fiber of $f$ at a point $x_{s, i} \in \Sigma_{s, i}$. Let $N$ be a normal slice to the stratum $\Sigma_{s, i}$ at the point $x_{s, i}$ (that is, a smooth analytic subvariety of $\mathbb{C}^N$, transversal to $\Sigma_{s, i}$ at $x_{s, i}$). By the base change formula for vanishing cycles (for example, see [36, Lemma 4.3.4]), we get

$$\widetilde{H}^k(F^n_{s, i}) \cong H^k(\varphi_f Z_X)_{x_{s, i}} \cong H^k(\varphi_f |_N Z_X \cap N)_{x_{s, i}}, \quad (3.18)$$

that is, $\widetilde{H}^k(F^n_{s, i})$ can be identified with the $k$-th reduced cohomology of the Milnor fiber of the restriction $f|_N$ of $f$ to the normal slice $N$ to the stratum $\Sigma_{s, i}$. For this reason, we will simply refer to $F^n_{s, i}$ as the transversal Milnor fiber of $f$ along $\Sigma_{s, i}$. Furthermore, by transversality, the function $f|_N$ has an isolated (stratified) singularity at $x_{s, i}$, and hence the point $x_{s, i}$ is an isolated point in the support of the perverse sheaf $\varphi_f |_N Z_X \cap N [n - s]$. Therefore, the stalk cohomology of $\varphi_f |_N Z_X \cap N [n - s]$ at $x_{s, i}$ is concentrated in degree 0. This then implies that $\widetilde{H}^k(F^n_{s, i})$ is trivial except possibly in degree $k = n - s$. Denote by $\mu^n_{s, i}$ the rank of $\widetilde{H}^{n-s}(F^n_{s, i})$. 
For any \( i \), the fundamental group \( \pi_1(\Sigma_{s,i}) \) of the stratum \( \Sigma_{s,i} \) acts on \( \tilde{H}^{n-s}(F_{s,i}^\circ) \) via a homomorphism

\[
A_i : \pi_1(\Sigma_{s,i}) \longrightarrow \text{Aut} \left( \tilde{H}^{n-s}(F_{s,i}^\circ) \right),
\]

which determines a local system \( \mathcal{L}_{s,i} \) on \( \Sigma_{s,i} \), with stalk \( \tilde{H}^{n-s}(F_{s,i}^\circ) \). We refer to \( A_i \) as the local system monodromy along the stratum \( \Sigma_{s,i} \). If \( \pi_1(\Sigma_{s,i}) \cong \mathbb{Z} \) (for example, if \( s = 1 \)), the homomorphism \( A_i \) can be regarded as an automorphism \( A_i : \tilde{H}^{n-s}(F_{s,i}^\circ) \to \tilde{H}^{n-s}(F_{s,i}^\circ) \); in this case, it will be referred to as the vertical monodromy along \( \Sigma_{s,i} \), by analogy with the case \( s = 1 \) considered in [42].

Let us also denote by \( \{\Sigma_{s-1,j}\}_j \) the collection of connected singular strata of dimension \( s-1 \), and for each \( j \) we fix a point \( x_j \in \Sigma_{s-1,j} \).

With the above notations and under the hypotheses of Theorem 3.1, we can now prove the following.

**Theorem 3.4.**

(a) There is a monomorphism

\[
\tilde{b}_{n-s}(F) \hookrightarrow \bigoplus_i \tilde{H}^{n-s}(F_{s,i}^\circ)A_i,
\]

where \( F_{s,i}^\circ \) is the transversal Milnor fiber to the \( s \)-dimensional stratum \( \Sigma_{s,i} \), and \( A_i \) is the local system monodromy along \( \Sigma_{s,i} \).

In particular, taking ranks yields the inequalities:

\[
\tilde{b}_{n-s}(F) \leq \sum_i \text{rank } \tilde{H}^{n-s}(F_{s,i}^\circ)A_i \leq \sum_i \mu_{s,i}^\circ.
\]

(b) If, moreover, \( s \geq 2 \), then

\[
\tilde{b}_{n-s}(F) \geq \sum_i \text{rank } \tilde{H}^{n-s}(F_{s,i}^\circ)A_i - \sum_j \text{rank } H^{s-1}(i_{x_j}^! p)^\circ(\pi_1(\Sigma_{s-1,j})),
\]

where \( x_j \) is some point in the \( (s-1) \)-dimensional stratum \( \Sigma_{s-1,j} \).

(c) If \( s \geq 2 \) and the germ of the singular locus \( \Sigma \) at the origin has no strata of dimension \( s-1 \), then (3.20) is an isomorphism and the first inequality in (3.21) becomes an equality.

(d) If \( r\text{Hd}(X,\mathbb{Z}) = n+1 \), then \( \tilde{H}^{n-s}(F) \) is free.

**Proof.**

(a) We continue to use here the notations from Theorem 3.1. First note that (3.4) yields a monomorphism

\[
\tilde{H}^{n-s}(F) \hookrightarrow \mathbb{H}^{-s}(\Sigma; P_s),
\]

\[\text{M. Saito communicated to us that parts (a)–(c) of Theorem 3.4 can also be deduced from arguments implicit in [4].}\]

\[\text{Since we are interested only in the } A_i \text{-invariant part of } \tilde{H}^{n-s}(F_{s,i}^\circ), \text{it is clear that this is equal to the intersection of the invariant submodules over some set of generators of } \pi_1(\Sigma_{s,i}). \text{See also (5.6) and the conjecture after it.}\]
The hypercohomology spectral sequence together with the support condition for perverse sheaves then yields that (for example, see [3, Proposition 5.2.20])

\[ H^{-s}(\Sigma_s; P_3) \cong H^0(\Sigma_s; H^{-s}(P_3)) \cong \bigoplus_i H^0(\Sigma_{s,i}; H^{-s}(P_3)|_{\Sigma_{s,i}}). \]

By constructibility, \( H^{-s}(P_3)|_{\Sigma_{s,i}} \) is a local system on \( \Sigma_{s,i} \) with stalk \( \tilde{H}^{n-s}(F^\circ_{s,i}) \), which in our previous notations is exactly \( \mathcal{L}_{s,i} \). Finally, it is well known (for example, see [25, Exercise 4.2.16]) that

\[ H^0(\Sigma_{s,i}; \mathcal{L}_{s,i}) \cong \tilde{H}^{n-s}(F^\circ_{s,i})^A_i, \]

with the right-hand side denoting the fixed part of \( \tilde{H}^{n-s}(F^\circ_{s,i}) \) under the \( A_i \)-action. Altogether, this proves (3.20), while (3.21) follows by computing ranks in (3.20).

(b) Let us next assume that \( s \geq 2 \). By setting \( j = 0 \) in (3.5) we get an isomorphism

\[ \tilde{H}^{n-s}(F) \cong H^{-s}(U_{s-1}; P_{s-1}). \]

Consider the inclusions

\[ \Sigma_{s-1} \xrightarrow{\alpha} U_{s-1} \xleftarrow{\beta} U_s = \Sigma_s \]

(that is, in the notations of Theorem 3.1, \( \alpha = v_{s-1} \) and \( \beta = u_{s-1} \)) and the corresponding attaching triangle

\[ \alpha_! \alpha^! P_{s-1} \to P_{s-1} \to R\beta_! \beta^* P_{s-1} \xrightarrow{[1]} \]

(3.24)

with \( \alpha_! = \alpha_\ast \) and \( \beta^* P_{s-1} \cong P_s \). In view of (3.23), the hypercohomology long exact sequence associated to (3.24) contains the terms

\[ \cdots \to H^{-s}(\Sigma_{s-1}; \alpha^! P_{s-1}) \to \tilde{H}^{n-s}(F) \to H^{-s}(U_s; P_s) \to H^{-s+1}(\Sigma_{s-1}; \alpha^! P_{s-1}) \to \cdots. \]

As in the proof of Theorem 3.1, a spectral sequence computation yields that

\[ H^{-s}(\Sigma_{s-1}; \alpha^! P_{s-1}) \cong 0, \]

and, as in the proof of (3.20) above, we have:

\[ H^{-s}(U_s; P_s) \cong \bigoplus_i \tilde{H}^{n-s}(F^\circ_{s,i})^A_i. \]

Therefore, (3.25) yields the exact sequence

\[ 0 \to \tilde{H}^{n-s}(F) \to \tilde{H}^{n-s}(F^\circ_{s,i})^A_i \to H^{-s+1}(\Sigma_{s-1}; \alpha^! P_{s-1}) \to \cdots. \]
Since \( \alpha^! P_{s-1} \) is a constructible complex on \( \Sigma_{s-1} \), its cohomology sheaves are local systems on every connected component \( \Sigma_{s-1,j} \) of \( \Sigma_{s-1} \). Moreover, the hypercohomology spectral sequence yields that

\[
\mathcal{H}^{-s+1}(\Sigma_{s-1}; \alpha^! P_{s-1}) \cong H^0(\Sigma_{s-1}; \mathcal{H}^{-s+1}(\alpha^! P_{s-1})) \\
\cong \bigoplus_j H^0(\Sigma_{s-1,j}; \mathcal{H}^{-s+1}(\alpha^! P_{s-1})|_{\Sigma_{s-1,j}}).
\]

(3.27)

If \( x \in \Sigma_{s-1} \), with inclusions

\[
\{x\} \xhookrightarrow{k_x} \Sigma_{s-1} \xhookrightarrow{\alpha} U_{s-1}
\]

and \( i'_x := \alpha \circ k_x : \{x\} \xhookrightarrow{U_{s-1}} \), then we have as in [36, Remark 6.0.2(1)] that

\[
k^*_x \alpha^! \cong k^!_x \alpha^!|2(s-1) \cong (i'_x)^!|2(s-1).
\]

In particular, the stalk of \( \mathcal{H}^{-s+1}(\alpha^! P_{s-1}) \) at a point \( x \in \Sigma_{s-1} \) is computed by:

\[
\mathcal{H}^{-s+1}(\alpha^! P_{s-1})_x = H^{-s+1}(k^*_x \alpha^! P_{s-1}) \cong H^{s-1}((i'_x)^! P_{s-1}).
\]

(3.28)

Furthermore, if \( i_x : \{x\} \xhookrightarrow{Y} \) is the inclusion map, then it follows (for example, using the proof of [25, Corollary 8.2.10]) that

\[
(i'_x)^! P_{s-1} \cong i^!_x P.
\]

(3.29)

Therefore, after choosing a point \( x_j \) in each \((s-1)\)-dimensional stratum \( \Sigma_{s-1,j} \), we get from (3.27), (3.28), and (3.29) that

\[
\mathcal{H}^{-s+1}(\Sigma_{s-1}; \alpha^! P_{s-1}) \cong \bigoplus_j H^{s-1}(i^!_{x_j} P|_{\Sigma_{s-1,j}}).
\]

(3.30)

The inequality (3.22) follows now by taking ranks in (3.26) and using (3.30).

The assertion in (c) follows immediately from (3.26).

To prove (d), we deduce from (3.20) that it suffices to show that \( \tilde{H}^{n-s}(F_{s,i}^\text{fr}) \) is free, for each \( i \). In the notations of (3.18), we have:

\[
\tilde{H}^{n-s}(F_{s,i}^\text{fr}) \cong H^{n-s}(\varphi_{f|_{N}} Z_{X\cap N})_{x_{s,i}} \cong H^0(\varphi_{f|_{N}} Z_{X\cap N}[n-s])_{x_{s,i}}.
\]

Since the point \( x_{s,i} \) is an isolated point in the support of the perverse sheaf \( \varphi_{f|_{N}} Z_{X\cap N}[n-s] \), the stalk and costalk cohomology of \( \varphi_{f|_{N}} Z_{X\cap N}[n-s] \) at \( x_{s,i} \) are isomorphic, and they are concentrated in degree 0. It is therefore enough to show that \( \varphi_{f|_{N}} Z_{X\cap N}[n-s] \) is a strongly perverse sheaf.

For this, we first note that the assumption \( \text{Hd}(X,\mathbb{Z}) = n+1 \) implies via Proposition 2.8 that \( Z_{X}[n+1] \) is strongly perverse. Then the fact that \( N \) is transversal to the stratification of \( X \) can be used to show that \( Z_{X\cap N}[n+1-s] \) is strongly perverse on \( X \cap N \). Indeed, by transversality, \( X \cap N \) gets an induced stratification with strata of the form \( S \cap N \) for \( S \) a stratum in \( X \). For a stratum \( S \)
of $X$ and $x \in S \cap N$, we denote by $k_x' : \{x\} \hookrightarrow X \cap N$ and $k_x : \{x\} \hookrightarrow X$ the point inclusions. If we let $i_N : X \cap N \hookrightarrow X$ be the inclusion map, then $i_N \circ k_x' = k_x$, and $i_N^* = i_N^*[-2s]$ (see, for example, [36, (6.44)]). The fact that $\mathbb{Z}_{X \cap N}[n+1-s]$ is perverse on $X \cap N$ follows immediately from the perversity of $\mathbb{Z}_X[n+1]$ (see, for example, [36, Lemma 6.0.4]). Moreover,

$$H^{\dim S}(k_x' \mathbb{Z}_X[n+1]) \cong H^{\dim S}((k_x')^* i_N^* \mathbb{Z}_X[n+1]) \cong H^{\dim S}((k_x')^* i_N^* \mathbb{Z}_X[n+1-2s])$$

$$\cong H^{\dim S-s}((k_x')^* \mathbb{Z}_{X \cap N}[n+1-2s]),$$

which translates to the fact that the perverse sheaf $\mathbb{Z}_{X \cap N}[n+1-s]$ on $X \cap N$ is strongly perverse. Finally, the stability of strong perversity under the perverse vanishing cycle functor $\varphi_f |_{N}[-1]$ implies that $\varphi_f |_{N} \mathbb{Z}_{X \cap N}[n-s]$ is a strongly perverse sheaf, as desired.\footnote{Similar arguments regarding freeness are used by the authors in the proof of [26, Theorem 1.2], in the context of vanishing cohomology of complex projective hypersurfaces.}

Let us point out that the freeness is also proved in Theorem 5.4(a).

\begin{remark}
The exact sequence (3.26) shows that the ‘correction’ of (3.20) from being an isomorphism depends only on the $(s-1)$-dimensional strata of the singular locus $\Sigma$ and of the costalks of the perverse vanishing cycle complex $\mathcal{P}$ at points in these strata. A similar remark also follows from Theorem 5.4 after reducing to $s = 2$ by iterated slicing. As we will see in some of the examples in Section 6, the monomorphism (3.20) is not in general an isomorphism if $(s-1)$-dimensional strata are present in $\Sigma$.
\end{remark}

\begin{remark}
If we work with $\mathbb{Q}$-coefficients, the statements and proofs of Theorems 3.1 and 3.4 hold in the category of mixed Hodge modules, provided that $\mathcal{Q}_X$ exists in the derived category of mixed Hodge modules (for example, if $X$ is a complete intersection in a complex manifold, cf. [34, Proposition 2.19]). In particular, under this assumption, (3.4) is a monomorphism in the category of mixed Hodge structures and (3.5) is an isomorphism of mixed Hodge structures.
\end{remark}

### 3.2 Characteristic polynomials of Milnor monodromy. Jordan blocks

In this section, cohomology groups are taken with $\mathbb{C}$-coefficients. Let $h$ and $h_i$ denote the Milnor monodromy on the cohomology of $F$, and on the transversal Milnor fiber $F_{s,i}^h$ to some $s$-dimensional stratum, respectively. Let $\text{char}_{h|\bar{H}^{n-s}(F)}$ denote the characteristic polynomial of the monodromy $h$ acting on $\bar{H}^{n-s}(F)$. Let $b_\lambda(V, \mu)$ denote the dimension of the eigenspace corresponding to the eigenvalue $\lambda$ of the linear operator $\mu$ acting on the vector space $V$, and let $J_\lambda(V, \mu)$ denote the maximum of the sizes of the Jordan blocks.

Since the monomorphism (3.20) is compatible with the Milnor monodromy actions, with these notations we get the following.

\begin{corollary}
The characteristic polynomial $\text{char}_{h|\bar{H}^{n-s}(F)}$ divides $\prod_i \text{char}_{h_i|\bar{H}^{n-s}(F_{s,i}^h)}$. In particular, $\text{char}_{h|\bar{H}^{n-s}(F)}$ divides the product $\prod_i \text{char}_{h_i|\bar{H}^{n-s}(F_{s,i}^h)}$.
\end{corollary}

\begin{remark}
\end{remark}
Moreover, we have the inequalities:

(i) \( b_\lambda(\widetilde{H}^{n-s}(F), h) \leq \sum_i b_\lambda(\widetilde{H}^{n-s}(F_{s,j}^{\lambda}), h_i) \).

(ii) \( J_\lambda(\widetilde{H}^{n-s}(F), h) \leq \sum_i J_\lambda(\widetilde{H}^{n-s}(F_{s,j}^{\lambda}), h_i) \).

Corollary 3.7 extends the results [49, Corollaries 3.2 and 3.3]; compare also with [4, Section 3.3].

### 3.3 Eigenspaces of monodromy

The proofs of Theorems 3.1 and 3.4 apply to any perverse sheaf supported on the stratified singular locus of \( f \), for example, to the generalized eigensheaves \( \varphi_{f,\lambda} \mathbb{C}_X \) \((\lambda \in \mathbb{C})\) of vanishing cycles. With the notations of Section 2.3, consider the \( \mathbb{C} \)-perverse sheaf

\[
\mathcal{P}_\lambda := p \varphi_{f,\lambda}(\mathbb{C}_X[n+1])
\]
on \( Y \), and let \( \Sigma^\lambda \) be the support of \( \mathcal{P}_\lambda \) with \( s_\lambda = \dim \Sigma^\lambda \). Then

\[
\mathcal{P}_\lambda|_{\Sigma^\lambda} =: \mathcal{P}_0^\lambda
\]
is a perverse sheaf on \( \Sigma^\lambda \) and we have the isomorphisms:

\[
\widetilde{H}^k(F; \mathbb{C})_\lambda \cong H^{k-n}(\mathcal{P}_0^\lambda)|_0 \cong H^{k-n}(\mathcal{P}_0^\lambda)|_0 \cong H^{k-n}(\Sigma^\lambda; \mathcal{P}_0^\lambda).
\]  

(3.31)

The support condition for perverse sheaves then yields immediately that

\[
\widetilde{H}^{n-s_\lambda-j}(F; \mathbb{C})_\lambda = 0
\]  

(3.32)

for all \( j > 0 \), so the only interesting \( \lambda \)-eigenspaces of Milnor monodromy are \( \widetilde{H}^k(F; \mathbb{C})_\lambda \) with \( k = n-s_\lambda, \ldots, n \). Let us next note that \( \Sigma^\lambda \subseteq \Sigma \) is a closed union of strata in the Whitney stratification of \( Y \), and it is exhausted by opens as in (3.3), where we use the superscript \( \lambda \) when considering strata which are contained in \( \Sigma^\lambda \). Let us set

\[
\mathcal{P}_0^\lambda := \mathcal{P}_0^\lambda|_{\Sigma^\lambda} \subseteq \mathcal{P}_0^\lambda
\]

and note that \( \mathcal{P}_0^\lambda \) is a perverse sheaf of \( U_{\epsilon'}^\lambda \) (that is, on the union of strata in \( \Sigma^\lambda \) of complex dimension \( \geq \epsilon' \)). With these notations, the proof of Theorem 3.1 adapted to the perverse sheaf \( \mathcal{P}_0^\lambda \) on \( \Sigma^\lambda \) yields the following (for \( j = 0 \), compare also with [4, Section 3.5]).

**Theorem 3.8.**

(a) For any \( j = 0, \ldots, s_\lambda - 1 \) there is a monomorphism

\[
\widetilde{H}^{n-s_\lambda+j}(F; \mathbb{C})_\lambda \hookrightarrow H^{-s_\lambda+j}(U_{s_j}^\lambda; \mathcal{P}_0^\lambda_{s_j}).
\]  

(3.33)
(b) If \( s_\lambda \geq 2 \), then for any \( j = 0, \ldots, s_\lambda - 2 \) there is an isomorphism

\[
\widetilde{H}^{n-s_\lambda + j}(F; \mathbb{C}_\lambda) \cong H^{n-s_\lambda + j}(U^\lambda_{s_\lambda - j - 1}; \mathbb{P}^1_{s_\lambda - j - 1}).
\]  

(3.34)

By specializing (3.33) to \( j = 0 \), and denoting by \( A^\lambda_i \) the local system monodromy representation along the \( i \)-th component \( \Sigma^\lambda_{s_\lambda, i} \) of the top dimensional stratum of \( \Sigma^\lambda \), with transversal Milnor fiber \( F^\lambda_{s_\lambda, i} \), we get as in Theorem 3.4 the following (see also [4, Section 3.3]).

Corollary 3.9.

\[
\widetilde{H}^{n-s_\lambda}(F; \mathbb{C}_\lambda) \hookrightarrow \bigoplus_i \left( \widetilde{H}^{n-s_\lambda}(F^\lambda_{s_\lambda, i}; \mathbb{C}_\lambda) \right)^{A^\lambda_i}.
\]  

(3.35)

Moreover, (3.35) becomes an isomorphism if \( \Sigma^\lambda \) contains no strata of dimension \( s_\lambda - 1 \).

Taking dimensions, Corollary 3.9 yields (with self-explanatory notations) the following Betti number estimate refining Corollary 3.7(i):

\[
b^\lambda_{n-s_\lambda}(F) \leq \sum_i \dim \widetilde{H}^{n-s_\lambda}(F^\lambda_{s_\lambda, i}; \mathbb{C})^{A^\lambda_i} \leq \sum_i b^\lambda_{n-s_\lambda}(F^\lambda_{s_\lambda, i}).
\]  

(3.36)

4 | MILNOR FIBER COHOMOLOGY VIA ITERATED SLICING

In this section, we use nearby and vanishing cycle functors to derive information about the Milnor fiber cohomology via slicing. We also make use here of the vanishing neighborhood of the nearby section of [49], which gives the geometric counterpart of the slicing via perverse sheaves.

4.1 | Setup and notations

As before, let \( n \geq 1 \) and consider a nonconstant holomorphic function germ \( f : (X, 0) \to (\mathbb{C}, 0) \) defined on a pure \((n+1)\)-dimensional complex singularity germ \((X, 0)\) contained in some ambient \((\mathbb{C}^N, 0)\). We work under the notations and assumptions of Section 2.3, in particular we assume that \( r_{\text{Hd}}(X, Z) = n + 1 \) or, equivalently, that \( Z_X[n + 1] \) is a strongly perverse sheaf on \( X \) in the sense of Definition 2.1(c); see Proposition 2.8 and Corollary 2.9(b).

Let \( l : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0) \) be a general linear function germ, that is, transversal to all the strata of \( \mathcal{W} \) of \( X \) except at 0. The restriction of \( l \) to \( X \), to \( f^{-1}(0) \), or to any other subset shall be clear from the context and will also be denoted by \( l \). Set

\[
Y := f^{-1}(0), \quad H := l^{-1}(0), \quad f' := f|_H, \quad Y' := f'^{-1}(0) = Y \cap H.
\]

In what follows, we will consider the composition of inclusion maps

\[
\{0\} \hookrightarrow Y' \hookrightarrow Y \hookrightarrow X, \quad \text{with} \quad i := u' \circ i' : \{0\} \to Y.
\]
As in Section 3.1, we consider a Milnor ball $B_\varepsilon \subset X$ for $f$ at 0. The generic choice of $l$ implies that the stratified singular locus $\text{Sing}_W(l, f)$ of the map germ $(l, f) : (X, 0) \to (\mathbb{C}^2, 0)$ is the union $\Gamma(l, f) \cup \Sigma$ of the stratified singular locus $\Sigma$ of $f$ with the so-called polar curve $\Gamma(l, f)$, see, for example, [16, 49, 50]. It follows that $\Gamma(l, f)$ intersects the fiber $(l, f)^{-1}(0, 0)$ only at the origin. In turn, this implies that the map germ $(l, f)$ is open at the origin of the target, and that there exists a fibration outside the discriminant $\Delta := (l, f)(\Gamma(l, f) \cup \Sigma)$, in the following sense: There exist disks (denoted by $D_\eta$) of small enough radii $0 < \eta' \ll \eta'' \ll \varepsilon$ such that the map $(l, f) : B \to \mathbb{C}^2$, with $B := B_\varepsilon \cap l^{-1}(D_{\eta''}) \cap f^{-1}(D_{\eta'})$, restricts to a locally trivial fibration:

$$(l, f)_* : B \setminus (l, f)^{-1}(\Delta) \to \mathbb{C}^2 \setminus \Delta.$$ (4.1)

It also follows that the fibration (4.1) is also independent of the choices of the constants, cf. [16, 49, 50]. Since the complement of the discriminant $\Delta$ is path connected, it follows that the fiber of the map (4.1) is unique. Let us denote it by $F'$. In particular, $F'$ is the Milnor fiber of $f'$ and also

$$F' \simeq B_\varepsilon \cap f^{-1}(\gamma) \cap l^{-1}(\eta)$$

for $0 < \gamma \ll \eta \ll \varepsilon \ll 1$.

We refer to Remark 4.5 for map germs involving several general functions, and for a different approach by [18] where the functions are non-generic.

### 4.2 Cohomological version of Lê’s attaching result

Here we recall the cohomological version of a well-known result of Lê [15] concerning the CW structure of the Milnor fiber. It has already been discussed in this form in [33] or [20].

By applying the distinguished triangle (2.4) for the nearby and vanishing cycle functors associated to $l : (Y, 0) \to (\mathbb{C}, 0)$ to the complex $\psi_f \mathbb{Z}_{\cdot X}$ on $Y$, we get the distinguished triangle

$$u'^* \psi_f \mathbb{Z}_{\cdot X} \to \psi_l \psi_f \mathbb{Z}_{\cdot X} \to \varphi_l \psi_f \mathbb{Z}_{\cdot X} \to [1].$$ (4.2)

Applying the functor $i'^*$ to (4.2) yields the distinguished triangle

$$i^* \psi_f \mathbb{Z}_{\cdot X} \to i'^* \psi_l \psi_f \mathbb{Z}_{\cdot X} \to i'^* \varphi_l \psi_f \mathbb{Z}_{\cdot X} \to [1].$$ (4.3)

Applying the cohomology functor $H^*(-)$ to (4.3) and using (2.5) yields the following long exact sequence of groups:

$$\cdots \to H^k(F) \to H^k(\psi_l \psi_f \mathbb{Z}_{\cdot X})_0 \to H^k(\varphi_l \psi_f \mathbb{Z}_{\cdot X})_0 \to H^{k+1}(F) \to \cdots.$$ (4.4)

We include the following well-known result and its proof for future reference:

**Lemma 4.1.** For any integer $k$,

$$H^k(\psi_l \psi_f \mathbb{Z}_{\cdot X})_0 \cong H^k(F') \cong H^k(\psi_f \psi_l \mathbb{Z}_{\cdot X})_0.$$ (4.5)
Proof. With $\gamma, \eta, \varepsilon$ as in Section 4.1, the generic choice of the linear function $l$ implies that $l^{-1}(\eta)$ is smooth in the ambient space $C^N$, and transversal to a fixed Whitney stratification $\mathcal{W}$ of the set germ $(X \setminus \{0\}, 0)$. Letting

$$\hat{f} := f|_{X \cap l^{-1}(\eta)},$$

there a base change isomorphism (for example, see [36, Lemma 4.3.4]):

$$(\psi_f \mathbb{Z}_X)_{|f^{-1}(0) \cap l^{-1}(\eta)} \cong \psi_{\hat{f}} \mathbb{Z}_{X \cap l^{-1}(\eta)}. \quad (4.6)$$

Then using (2.5) we get:

$$\begin{align*}
H^k(\psi_{\hat{f}} \mathbb{Z}_{X})_{0} & \cong H^k(B_\varepsilon \cap l^{-1}(\eta) \cap f^{-1}(0); \psi_{\hat{f}} \mathbb{Z}_X) \\
& \cong H^k(B_\varepsilon \cap l^{-1}(\eta) \cap f^{-1}(0); \psi_f \mathbb{Z}_{X \cap l^{-1}(\eta)}) \cong H^k(B_\varepsilon \cap l^{-1}(\eta) \cap f^{-1}(\gamma); \mathbb{Z}) \cong H^k(F').
\end{align*}$$

To show the isomorphism $H^k(\psi_f \mathbb{Z}_X)_{0} \cong H^k(F')$ we repeat the above procedure. What makes it possible is the genericity of $l$ as discussed in Section 4.1. This genericity implies, roughly speaking, that commuting the functors $\psi_f$ and $\psi_l$ yields the isomorphism (4.5). □

By combining (4.4) and (4.5), we get that

$$H^k(\varphi_l \psi_f \mathbb{Z}_X)_{0} \cong H^{k+1}(F,F'). \quad (4.7)$$

Since $l$ is generic, the origin $\{0\}$ is an isolated point in the support of the strongly perverse sheaf $p_l \psi_f(\mathbb{Z}_X[n+1])$ on $Y'$, and hence the cohomology of $i^* p_l \psi_f(\mathbb{Z}_X[n+1])$ is concentrated in degree zero. Moreover, since

$$\begin{align*}
H^k(\varphi_l \psi_f \mathbb{Z}_X)_{0} & \cong H^k(i^* \varphi_l \psi_f \mathbb{Z}_X) \cong H^{k-n+1}(i^* p_l \psi_f(\mathbb{Z}_X[n+1])) \\
& \cong H^{k-n+1}((i^! p_l \psi_{\hat{f}} \mathbb{Z}_X)[n+1]),
\end{align*}$$

we get that $H^k(\varphi_l \psi_f \mathbb{Z}_X)_{0} = 0$ for all $k \neq n - 1$, and

$$H^{n-1}(\varphi_p \psi_f \mathbb{Z}_X)_{0} \cong H^n(F,F') \cong \mathbb{Z}^{\mathbb{T}_f}$$

is free.

The long exact sequence (4.4) together with (4.5) and (4.7) then yield the isomorphisms

$$H^k(F) \cong H^k(F'), \quad \text{for } k < n - 1, \quad (4.8)$$

and an exact sequence

$$0 \to H^{n-1}(F) \to H^{n-1}(F') \to \mathbb{Z}^{\mathbb{T}_f} \to H^n(F) \to 0, \quad (4.9)$$
where
\[ \tau_f = \text{int}_0(\Gamma(l, f), f^{-1}(0)) \]
is a polar intersection multiplicity at 0 (see Section 4.1 for the definition of the polar locus \( \Gamma(l, f) \)). Note that by (4.9) one has that \( b_n(F) \leq \tau_f \), where \( b_n(F) \) is the \( n \)-th Betti number of the Milnor fiber of \( f \) at the origin. We will give a sharper bound in (4.33) below.

### 4.3 Milnor fiber cohomology and the vanishing neighborhood of the nearby section

We give here a sheaf-theoretic version of the cohomological results obtained by Tibăr in [49, Theorem 2.2], which will play a fundamental role in this paper, and which we could not locate in this form in the literature.

We start by evaluating the distinguished triangle (2.4) for the nearby and vanishing cycle functors associated to \( l : (Y, 0) \to (\mathbb{C}, 0) \) on the complex \( \varphi_f \mathbb{Z}_X \) on \( Y \), to get the distinguished triangle on \( Y' \)

\[
u'_* \varphi_f \mathbb{Z}_X \to \psi_l \varphi_f \mathbb{Z}_X \to \varphi_l \varphi_f \mathbb{Z}_X \to \mathbb{Z}^1.
\]

(4.10)

Applying \( i'^* \), and then taking the long exact sequence obtained by applying the cohomology functor \( H^*(\cdot) \), one then gets by using (2.5) the long exact sequence

\[
\cdots \to \tilde{H}^k(F) \to H^k(\psi_l \varphi_f \mathbb{Z}_X)_0 \to H^k(\varphi_l \varphi_f \mathbb{Z}_X)_0 \to \tilde{H}^{k+1}(F) \to \cdots.
\]

(4.11)

Since \( i'^* \psi_l^p \varphi_f(\mathbb{Z}_X[n+1]) \) is a perverse sheaf on \( Y' \), the support condition for perverse sheaves yields that

\[
H^k(\psi_l \varphi_f \mathbb{Z}_X)_0 = H^{k-n+1}(i'^* \psi_l^p \varphi_f(\mathbb{Z}_X[n+1])) = 0 \quad \text{if} \quad k > n - 1.
\]

Furthermore, since \( l \) is generic, we obtain as before that \( H^k(\varphi_l \varphi_f \mathbb{Z}_X)_0 = 0 \) for all \( k \neq n - 1 \). Altogether, we get from (4.11) isomorphisms

\[
\tilde{H}^k(F) \cong H^k(\psi_l \varphi_f \mathbb{Z}_X)_0 \quad \text{if} \quad k < n - 1,
\]

(4.12)

and an exact sequence

\[
0 \to \tilde{H}^{n-1}(F) \to H^{n-1}(\psi_l \varphi_f \mathbb{Z}_X)_0 \to H^{n-1}(\varphi_l \varphi_f \mathbb{Z}_X)_0 \to \tilde{H}^n(F) \to 0.
\]

(4.13)

Let us next give a geometric interpretation of (4.12) and (4.13). Since \( \psi_l \) is a functor of triangulated categories, by applying it to the distinguished triangle on \( Y \)

\[
\mathbb{Z}_Y \to \psi_f \mathbb{Z}_X \to \varphi_f \mathbb{Z}_X \to \mathbb{Z}_{Y}^1,
\]

(4.14)
we get a new distinguished triangle
\begin{equation}
\psi_i \mathbb{Z}_Y \to \psi_i \psi_j \mathbb{Z}_X \to \psi_i \varphi_j \mathbb{Z}_X \to [1]
\end{equation}
of constructible complexes on $Y'$. Applying the functor $H^*(-)_0$ of taking the stalk cohomology at the origin, one gets the long exact sequence
\begin{equation}
\cdots \to H^k(\psi_i \mathbb{Z}_Y)_0 \to H^k(\psi_i \psi_j \mathbb{Z}_X)_0 \to H^k(\psi_i \varphi_j \mathbb{Z}_X)_0 \to \cdots.
\end{equation}
We have that
\begin{equation}
H^k(\psi_i \mathbb{Z}_Y)_0 \cong H^k(B_\varepsilon \cap Y \cap l^{-1}(\eta)) \cong H^k(B_\varepsilon \cap f^{-1}(D_\gamma) \cap l^{-1}(\eta)),
\end{equation}
where $D_\gamma \subset \mathbb{C}$ is a closed disk of radius $\gamma$ centered at the origin. Following the notations from [49] and from Section 4.1, we set
\[ F'_D := B_\varepsilon \cap f^{-1}(D_\gamma) \cap l^{-1}(\eta) \]
and note that $F \cap F'_D = F'$. In the last isomorphism of (4.17) we used the fact that $F'_D$ deformation retracts to the complex link $\text{Clk}(Y, 0) = B_\varepsilon \cap Y \cap l^{-1}(\eta)$ of the hypersurface $Y = f^{-1}(0)$ at the origin. Plugging (4.5) and (4.17) in (4.16), we obtain for any $k \leq n - 1$ an isomorphism
\begin{equation}
H^k(\psi_i \varphi_j \mathbb{Z}_X)_0 \cong H^{k+1}(F'_D, F').
\end{equation}
We can therefore restate (4.12) as an isomorphism
\begin{equation}
\widetilde{H}^k(F) \cong H^{k+1}(F'_D, F') \quad \text{if } k < n - 1.
\end{equation}
Furthermore, since by excision we get $H^*(F \cup F'_D, F) \cong H^*(F'_D, F')$, the long exact sequence (4.13) can be identified with the long exact sequence of (reduced) cohomology of the pair $(F \cup F'_D, F)$. So in particular we have the identification
\[ H^{n-1}(\psi_i \varphi_j \mathbb{Z}_X)_0 \cong \widetilde{H}^n(F \cup F'_D), \]
and (4.13) becomes the exact sequence from [49, Theorem 2.2], namely,
\begin{equation}
0 \to \widetilde{H}^{n-1}(F) \to H^n(F'_D, F') \to \widetilde{H}^n(F \cup F'_D) \to \widetilde{H}^n(F) \to 0.
\end{equation}
For future reference, let us record here the fact that the above discussion also shows that the reduced cohomology $\widetilde{H}^*(F \cup F'_D)$ is concentrated in degree $n$ (where it is free).

Let $T$ be a small tubular neighborhood of the complex link $\text{Clk}(\Sigma, 0) = B_\varepsilon \cap \Sigma \cap l^{-1}(\eta)$ of $\Sigma$ in the slice $l^{-1}(\eta)$. By retraction we get (as in [49]) the isomorphism:
\begin{equation}
H^*(F'_D, F') \cong H^*(T, T \cap F'),
\end{equation}
which provides us with a replacement of the pair \((F_D', F')\) appearing in (4.19) and (4.20) by the pair \((T, T \cap F')\).

**Definition 4.2.** We call the pair \((T, T \cap F')\) the **vanishing neighborhood of the nearby section**, and we call \(H^*(T, T \cap F')\) the **vanishing cohomology of the nearby section of** \(f\).

The Milnor monodromy \(h\) of the cohomology \(H^*(F)\) is induced by a geometric monodromy which acts on \(F\) by moving along a small circle \(y(t) = \exp(2\pi t)\gamma\), and on the slice \(F' = B_\varepsilon \cap f^{-1}(y) \cap l^{-1}(\eta)\) by moving along the circle \(\{y(t) = \exp(2\pi t)\gamma\} \cap \{x = \eta\} \subset \mathbb{C}^2\). It acts as the identity on \(F_D'\) and on the neighborhood \(T\).

There is another geometric monodromy, defined by moving the slice \(l = \eta\) around the circle \(\exp(2\pi t)\eta\) for \(t \in [0, 1]\). We call it the **sectional l-monodromy**. This acts on \(F'\), on \(F_D'\), and on \(T \cap F'\). It acts as the identity on \(F\). Its induced action on the \(\mathbb{Z}\)-cohomology groups of all these spaces will be denoted by \(L\), which in the previous notations can also be identified with the action on \(H^*(\psi_0 \varphi \mathbb{Z}_\chi)\) induced from the monodromy action on \(\psi_0\).

In particular the sectional monodromy acts on the exact sequence (4.20). Looking only to the left-hand side of it, by the identification (4.21), we get the following monomorphism, cf. [49, Corollary 2.4]:

\[
\tilde{H}^{n-1}(F) \hookrightarrow \ker \left[ L - \text{id} | H^n(T, T \cap F') \right].
\] (4.22)

### 4.4 | Iterated slicing, iterated nearby, and vanishing cycles

We focus now on the computation of the vanishing cohomology of the nearby section which, as can be seen from (4.19), yields the cohomology of the Milnor fiber \(F\), except in the two top dimensions.

In what follows, we calculate the Milnor fiber cohomology using repeated slicing by general hyperplanes \(H_k := \{l_k = 0\}\) for \(1 \leq k \leq s - 2\), with each \(l_k\) general linear functions with \(l_k(0) = 0\), as in Section 4.2 and Section 4.3.

Let \(X^{(k)} := X \cap H_1 \cap \cdots \cap H_k\), with \(X^{(0)} = X\), and consider the function germ

\[
f^{(k)} : (X^{(k)}, 0) \to (\mathbb{C}, 0)
\]

with Milnor fiber \(F^{(k)}\) at the origin. In particular, \(F^{(1)} = F'\) and we set \(F^{(0)} := F\). Let \(Y^{(k)} := (f^{(k)})^{-1}(0) = Y \cap H_1 \cap \cdots \cap H_k\). By the genericity of the hyperplane slices, \(\Sigma^{(k)} := \Sigma \cap H_1 \cap \cdots \cap H_k\) is the singular locus of \(f^{(k)}\) and its Whitney stratification is induced by that of \(\Sigma\).

The Milnor fiber \(F^{(k)}\) is also identified with \(B_\varepsilon \cap (l_k, f^{(k-1)})^{-1}(\eta, \gamma)\), the tube \(F^{(k)}_D\) is defined similarly to \(F'_D\) in Section 4.3, and \(T^{(k)}\) is a tubular neighborhood of the complex link \(\text{Clk}(\Sigma^{(k)}, 0)\) of the singular locus \(\Sigma^{(k)}\), where \(T^{(0)} := T\). As in Section 4.3, the reduced integral cohomology \(\tilde{H}^*(F^{(k-1)}_D \cup F^{(k)}_D)\) is concentrated in degree \(n - k + 1\).

The case of a 1-dimensional singular locus having been considered before, we focus now on the higher dimensional case \(n > s \geq 2\). Combining the slice isomorphism (4.8) on the one hand, with the isomorphism (4.19) and the exact sequences (4.20) on the other hand, we get the following:
Proposition 4.3. Let $n > s \geq 2$. For each fixed $2 \leq k \leq s$, there are isomorphisms

$$
\tilde{H}^{n-k}(F) \cong \tilde{H}^{n-k}(F^{(j)}) \cong H^{n-k+1}(T^{(j)}, T^{(j)} \cap F^{(j+1)})
$$

for any $j = 1, \ldots, k-2$. Moreover, for $1 \leq k \leq s-1$ we have the following exact sequence:

$$
0 \to \tilde{H}^{n-k}(F^{(k-1)}) \to H^{n-k+1}(T^{(k-1)}, T^{(k-1)} \cap F^{(k)}) \to \tilde{H}^{n-k+1}(F^{(k-1)} \cup F^{(k)} \cap D) \to \tilde{H}^{n-k+1}(F^{(k-1)}) \to 0,
$$

where $\tilde{H}^{n-k}(F) \cong \tilde{H}^{n-k}(F^{(k-1)})$.

The groups $H^q(T^{(j)}, T^{(j)} \cap F^{(j+1)})$ appearing in Proposition 4.3 can be described by an iteration of nearby and vanishing cycle functors, as follows.

Proposition 4.4. For any integers $q \geq 1$ and $j \geq 1$,

$$
H^q(T^{(j)}, T^{(j)} \cap F^{(j+1)}) \cong H^q(F^{(j+1)} \cap F^{(j+1)}),
$$

with

$$
F^{(j+1)}_D := B_\gamma \cap (f^{(j)})^{-1}(0) \cap l_{j+1}^{-1}(\eta_{j+1}) = B_\gamma \cap (f^{(j)})^{-1}(0) \cap l_{j+1}^{-1}(\eta_{j+1}) = \text{Clk}(Y^{(j)}), 0),
$$

with $\gamma, \eta_{j+1}$ sufficiently small. To prove (4.24) it is therefore sufficient to show that

$$
H^q(F^{(j+1)}_D, F^{(j+1)}) \cong H^q(F^{(j+1)}_D, F^{(j+1)}),
$$

Applying the functor $\psi_{l_1} \psi_{l_2} \ldots \psi_{l_k}$ to the distinguished triangle (4.14) on $Y$, we get a new distinguished triangle on $Y^{(j+1)}$:

$$
\psi_{l_1} \psi_{l_2} \ldots \psi_{l_k} \psi_{l_{j+1}} \psi_{l_1} Y \to \psi_{l_1} \psi_{l_2} \ldots \psi_{l_k} \psi_{l_{j+1}} X \to \psi_{l_1} \psi_{l_2} \ldots \psi_{l_k} \psi_{l_{j+1}} X \to \ldots
$$

By applying the cohomological functor $H^*(-)_0$ to (4.28), we get a long exact sequence

$$
\ldots \to H^q(\psi_{l_{j+1}} \psi_{l_1} Y)_0 \to H^q(\psi_{l_{j+1}} \psi_{l_1} \psi_{l_2} X)_0 \to H^q(\psi_{l_{j+1}} \psi_{l_1} \psi_{l_2} \ldots \psi_{l_k} X)_0 \to \ldots
$$
As already shown in (4.23),

$$H^q(\psi_{l_j+1} \ldots \psi_{l_1} \psi_f \mathbb{Z}_X)_0 = H^q(F^{(j+1)}),$$

(4.30)

and the same formula yields that

$$H^q(\psi_{l_j+1} \ldots \psi_{l_1} \mathbb{Z}_Y)_0 \cong H^q\left( B_\varepsilon \cap Y \cap l_1^{-1}(\eta_1) \cap \ldots \cap l_{j+1}^{-1}(\eta_{j+1}) \right)$$

(4.31)

for small enough $\eta_1, \ldots, \eta_{j+1}$. By the genericity of the linear functions $l_1, \ldots, l_{j+1}$, there is a homotopy equivalence

$$B_\varepsilon \cap Y \cap l_1^{-1}(\eta_1) \cap \ldots \cap l_{j+1}^{-1}(\eta_{j+1}) \cong B_\varepsilon \cap Y \cap l_1^{-1}(0) \cap \ldots \cap l_{j+1}^{-1}(0) = \text{Clk}(Y^{(j)}, 0),$$

defined by moving $(\eta_1, \ldots, \eta_j, \eta_{j+1})$ along a straight path to $(0, \ldots, 0, \eta_{j+1})$. Combining this fact with (4.26) and (4.31), we therefore have an isomorphism

$$H^q(\psi_{l_j+1} \ldots \psi_{l_1} \mathbb{Z}_Y)_0 \cong H^q(F_{D}^{(j+1)}).$$

(4.32)

Plugging (4.30) and (4.32) into the long exact sequence (4.29) yields (4.27), thus completing the proof. □

Remark 4.5. Note that, by the genericity of the linear forms $l_i$, the map $(f, l_1, \ldots, l_j)$, for $j \geq 1$, defines a stratified isolated singularity at the origin (for example, it is an isolated complete intersection singularity (ICIS) in case of $X = \mathbb{C}^{n+1}$), and therefore it is an open map at the origin of the target and defines a stratified fibration outside its discriminant (which is actually included in a hypersurface). This is a particular case of a map ‘sans éclatement’ in the terminology of Sabbah [32]. Consequently, $H^q(\psi_{l_j} \ldots \psi_{l_1} \mathbb{Z}_X)_0$ in formula (4.23) is independent of the order of the nearby cycle functors in the sequence. A non-generic situation occurs in [18, Section 3], where the iterated nearby cycles do not commute anymore, where the map may not be open† anymore, and where the stratified fibration has a very special meaning depending on the order of the functions in the map. A formula similar to (4.23) also holds in this general situation after fixing an ordering of the defining functions, see [18, Formula (8)].

4.5 | Betti bounds and polar multiplicities

In this section we indicate how the slicing technique yields Betti bounds for the Milnor fiber.

We begin with describing an upper bound of the top Betti number of the Milnor fiber. While this result is known (for example, see [49, Corollary 2.3]), for completeness we include here a proof in the spirit of Section 4.3.

**Corollary 4.6** [49, Corollary 2.3], [21, Theorem 3.3].

$$b_n(F) \leq \lambda^0 + b_n(\text{Clk}(X, 0)),$$

(4.33)

where $\lambda^0 := \tau_f - \tau_l = \text{int}_0(\Gamma, f^{-1}(0)) - \text{int}_0(\Gamma, l^{-1}(0))$.

† We may refer to [14] for a study of images of map germs in relation with singular fibrations.
Proof. By applying the functor $\varphi_l$ to the distinguished triangle (4.14) on $Y$, we get a new distinguished triangle

$$
\varphi_l\mathbb{Z}_Y \to \varphi_l\psi_l\mathbb{Z}_X \to \varphi_l\varphi_j\mathbb{Z}_X \to [1]
$$
on $Y'$, which, upon applying the functor $H^*(-)_0$ of taking the stalk cohomology at the origin and the fact mentioned earlier that $H^k(\varphi_l\varphi_j\mathbb{Z}_X)_0 = 0$ for all $k \neq n - 1$, yields the short exact sequence

$$
0 \to H^{n-1}(\varphi_l\mathbb{Z}_Y)_0 \to H^{n-1}(\varphi_l\psi_l\mathbb{Z}_X)_0 \to H^{n-1}(\varphi_l\varphi_j\mathbb{Z}_X)_0 \to 0.
$$

(4.34)

Using the notation from (4.9), we get from (4.20) and (4.34) together with (2.5) that

$$
\tilde{b}_n(F) \leq \tau_f - \tilde{b}_{n-1}(\text{Clk}(Y,0)).
$$

(4.35)

On the other hand, since $\mathbb{Z}_X[n + 1]$ is strongly perverse on $X$, the complex link $\text{Clk}(X,0)$ of $X$ at the origin has its reduced cohomology concentrated in degree $n$ (for example, see [25, Corollary 10.6.3]), where it is free (by the freeness of lowest degree costalks). Similarly, since $\mathbb{Z}_Y[n]$ is strongly perverse on $Y$ (cf. §2.3), the reduced cohomology of the complex link $\text{Clk}(Y,0)$ of $Y$ at the origin is concentrated in degree $n - 1$, where it is free (again, by the freeness of costalks). The long exact sequence for the reduced cohomology of the pair $(\text{Clk}(X,0), \text{Clk}(Y,0))$ then yields that

$$
H^n(\text{Clk}(X,0), \text{Clk}(Y,0)) \cong \mathbb{Z}^{\tau_l}
$$
is free, with

$$
\tau_l = \tilde{b}_n(\text{Clk}(X,0)) + \tilde{b}_{n-1}(\text{Clk}(Y,0)).
$$

It is known by work of Lê (see, for example, [49, Facts 1.1(b,c)] and the references therein) that

$$
\tau_l = \text{int}_0(\Gamma, l^{-1}(0))
$$
is a corresponding polar intersection number for $l$. Our claim follows now from (4.35). □

The following two relations are consequences of iterated slicing and repeated application of (4.8) and (4.9):

$$
b_{n-k}(F) = b_{n-k}(F^{(k-1)}) \quad \text{and} \quad b_{n-k}(F) \leq b_{n-k}(F^{(k)}).
$$

(4.36)

These yield the inequality:

$$
b_{n-k}(F^{(k-1)}) \leq b_{n-k}(F^{(k)}), \quad \text{for} \quad k \geq 1,
$$

(4.37)

which can be used to get the following generalization of Corollary 4.6 (compare also with [49, Corollary 2.3] and [22]):
Corollary 4.7. Let $s \geq 1$. For any $k = 0, \ldots, s - 1$, we have the bounds:

$$b_{n-k}(F) \leq b_{n-k}(F^{(k)}) \leq \lambda^k + b_{n-k}(\text{Clk}^k(X, 0)), \quad (4.38)$$

where

$$\lambda^k := \text{int}_0(\Gamma(l_k, f^{(k)}), (f^{(k)})^{-1}(0)) - \text{int}_0(\Gamma(l_k, f^{(k)}), (l_k)^{-1}(0)), \quad (4.39)$$

Remark 4.8. The numbers $\lambda^*$ which occur in (4.39) are the analogues of the sectional Milnor numbers $\mu^*$ defined by Teissier [46, 47], in the sense that $\mu^*$ verify the same equality (4.39) in case $X = \mathbb{C}^{n+1}$ and $f$ has an isolated singularity. In Massey’s terminology, $\lambda^*$ are called ‘Lê numbers’, see, for example, [20].

5 | THE GEOMETRIC COMPUTATION OF $\widetilde{H}^{n-s}(F)$

In this section we compute the lowest (possibly nontrivial) group $\widetilde{H}^{n-s}(F)$ by using the isomorphism with the vanishing cohomology of the nearby section. Throughout this section we continue to use the previous assumptions, namely that $(X, 0)$ is a pure $(n+1)$-dimensional space germ with $\text{rHd}(X, \mathbb{Z}) = n+1$, $n \geq 3$ and $\dim \text{Sing} f = s \geq 2$.

From Proposition 4.3 for $k = s$ and $j = s - 2$, we have:

$$\widetilde{H}^{n-s}(F) \cong \widetilde{H}^{n-s}(F^{(s-2)}) \cong H^{n-s+1}(T^{(s-2)}, T^{(s-2)} \cap F^{(s-1)}). \quad (5.1)$$

Consequently,

$$\widetilde{H}^{n-s}(F) \cong \widetilde{H}^{m-2}(F_g),$$

where $g := f|_H$, $H := H_1 \cap \cdots \cap H_{s-2}$, $m := n - s + 2$, with $\dim \text{Sing} \cap H(g) = 2$, and $F_g$ denotes the Milnor fiber of the function germ $g$.

By the above isomorphism, it is sufficient to compute $\widetilde{H}^{m-2}(F_g)$ in order to obtain $\widetilde{H}^{n-s}(F)$, and therefore the remainder of this section will be devoted to this computation. For the sake of simplicity we will use the notation $f$ instead of $g$, assume that $\dim_0 \text{Sing} f = 2$, and thus compute $\widetilde{H}^{n-2}(F)$.

5.1 | Computing the vanishing cohomology of the nearby section

Let $n \geq 3$ and $s = \dim \text{Sing} f = 2$. As before, we denote by $\Sigma_2$ the union of the strata of $\text{Sing} f$ of dimension 2, and by $\Sigma_1$ the union of the 1-dimensional strata of $\text{Sing} f$. We may have singular 1-dimensional strata which are inside or outside $\Sigma_2$.

The cohomology of the Milnor fiber $\widetilde{H}^*(F)$ is known to be concentrated only in degrees $n$, $n - 1$ and $n - 2$. From (4.19) and (4.20) we know that it is computed by the vanishing cohomology of

\[\text{All these computations also apply for an admissible deformation of } f.\]
the nearby section in dimension $n - 2$, namely, we have the isomorphism:

$$
\widetilde{H}^{n-2}(F) \cong H^{n-1}(T, T \cap F(1)),
$$

and the inclusion:

$$
\widetilde{H}^{n-1}(F) \hookrightarrow H^n(T, T \cap F(1)),
$$

where we recall that $T$ is a small tubular neighborhood of $B_z \cap \text{Sing} f \cap l^{-1}(\eta)$ in the slice $l^{-1}(\eta)$, and $F(1) := B_z \cap l^{-1}(\eta) \cap f^{-1}(\gamma)$.

We therefore focus on the computation of the vanishing cohomology $H^*(T, T \cap F(1))$ of the nearby section.

The singular slice

$$
S := \Sigma_2 \cap l^{-1}(\eta)
$$

is a union of curves, so let $S = S_1 \cup ... \cup S_p$ be its decomposition into irreducible components. The finite set of points $\Sigma_1 \cap l^{-1}(\eta)$ is the union $R \cup \bigcup_{i=1}^p Q_i$, where $R$ is the set of all isolated singularities outside $S$ of the slice $B_z \cap \text{Sing} f \cap l^{-1}(\eta)$, and where $Q_i := S_i \cap \Sigma_1$ will be called the set of special points of $S_i$. Note that if the curve components $S_i$ and $S_j$ intersect, for $i \neq j$, then the intersection points belong to $\Sigma_1$, and thus $S_i \cap S_j \subset Q_i$ for any $i \neq j$.

Let $S^* := S \setminus \bigcup_{i=1}^p Q_i$, and $S^*_i := S_i \setminus Q_i$ for each $i$.

Let $B_r$ be a Milnor ball at the point $r \in R$ in the slice $l^{-1}(\eta)$, and let $T_S$ denote a small tubular neighborhood of $S$ in the same slice. Then $F_r := B_r \cap F(1)$ is the Milnor fiber of an isolated hypersurface singularity at $r$, and therefore its reduced cohomology is concentrated in the top dimension. With these notations one has:

**Lemma 5.1.**

$$
\widetilde{H}^{n-2}(F) \cong H^{n-1}(T, T \cap F(1)) \cong H^{n-1}(T_S, T_S \cap F(1))
$$

$$
\widetilde{H}^{n-1}(F) \hookrightarrow H^n(T, T \cap F(1)) \cong H^n(T_S, T_S \cap F(1)) \oplus \bigoplus_{r \in R} H^{n-1}(F_r). \tag{5.2}
$$

**Proof.** We have the direct sum decomposition:

$$
H^*(T, T \cap F(1)) \cong \bigoplus_{r \in R} H^*(B_r, B_r \cap F(1)) \oplus H^*(T_S, T_S \cap F(1)).
$$

Since $H^*(B_r, B_r \cap F(1)) \cong H^{n-1}(F_r)$ is concentrated in dimension $*=n$, the first term is a direct summand of $H^n(T, T \cap F(1))$ and the assertion follows. \qed

**5.2 Cohomology of $(T_S, T_S \cap F(1))$**

To study the cohomology of the pair $(T_S, T_S \cap F(1))$, we inspire ourselves from the technique displayed in [45] and [44].
The small tubular neighborhood $T_S$ is the union $T^n_S \cup B_Q$ of the tubular neighborhood $T^n_S$ of $S^*$, together with $B_Q := \bigcup_{q \in Q} B_q$, where $B_q$ denotes a small enough Milnor ball at $q \in Q$ in the slice $l^{-1}(\eta)$.

Each component $S_i$ has a generic transversal Milnor fiber $F^\#_i$, on the cohomology of which $\pi_1(S^*_i)$ acts. Let $\mu^\#_i$ denote its Milnor number.

The pair

$$(T_S, T_S \cap F^{(1)}) = (T^n_S \cup B_Q, (T^n_S \cap F^{(1)}) \cup (B_Q \cap F^{(1)}))$$

comes into a relative Mayer–Vietoris long exact sequence

$$\cdots \to H^*(T_S, T_S \cap F^{(1)}) \to H^*(T^n_S, T^n_S \cap F^{(1)}) \oplus H^*(B_Q, B_Q \cap F^{(1)}) \to$$
$$\to H^*(T^n_S \cap B_Q, (T^n_S \cap F^{(1)}) \cap (B_Q \cap F^{(1)})) \to \cdots$$

(5.3)

that we analyze in the following.

### 5.2.1 The intersection term $(T^n_S \cap B_Q, (T^n_S \cap F^{(1)}) \cap (B_Q \cap F^{(1)}))$

This is a disjoint union of pairs localized at points $q \in Q$, namely, one pair for each local irreducible branch of the germ $(S, q)$.

Let $K_q$ be the set of indices for the irreducible branches at $q \in Q$. We use the same set of indices $K_q$ for the corresponding small loops around $q$ in $S^*$.

When we are counting the local irreducible branches at some point $q \in Q_i$ on a specified component $S_i$ then the set of indices for the local branches of the curve germs $(S_i, q)$ will be denoted by $K_{qi}$.

With these notations we get the following decomposition:

$$H^*(T^n_S \cap B_Q, (T^n_S \cap F^{(1)}) \cap (B_Q \cap F^{(1)})) \cong \bigoplus_{q \in Q} \bigoplus_{k \in K_q} H^*(Z_k, C_k).$$

(5.4)

One such local pair $(Z_k, C_k)$ is the bundle over the component of the link of the corresponding irreducible branch of the curve germ $(S, q)$, having as fiber the local transversal Milnor data $(E^\#_k, F^\#_k)$. These data depend only on the component $S_i$ containing the local branch indexed by $k$, and therefore we have $(E^\#_k, F^\#_k) = (E^\#_i, F^\#_i)$, and in particular the Milnor numbers equality $\mu^\#_k = \mu^\#_i$.

The relative cohomology groups in the direct sum decomposition (5.4) are concentrated in dimensions $n - 1$ and $n$, and depend on the local system monodromy $\nu_k$ along the link $C_k$, by the following Wang sequence:

$$0 \to H^{n-1}(Z_k, C_k) \to H^{n-1}(E^\#_k, F^\#_k) \xrightarrow{\nu_k - \text{id}} H^{n-1}(E^\#_k, F^\#_k) \to H^n(Z_k, C_k) \to 0,$$

(5.5)

from which one gets the isomorphisms:

$$H^{n-1}(Z_k, C_k) \cong \ker (\nu_k - \text{id} | H^{n-2}(F^\#_k)), \quad H^n(Z_k, C_k) \cong \text{coker} (\nu_k - \text{id} | H^{n-2}(F^\#_k))$$

for any $k \in K_q, q \in Q$. 

5.2.2  |  Cohomology of \((B_Q, B_Q \cap F^{(1)})\)

Since \(B_Q\) is a disjoint union, one has the direct sum decomposition

\[
H^*(B_Q, B_Q \cap F^{(1)}) \cong \bigoplus_{q \in Q} H^*(B_q, F_q),
\]

into the local Milnor data of the hypersurface germs \((f^{-1}(0) \cap l^{-1}(\eta), q)\) which have 1-dimensional singular locus. Therefore, the relative cohomology \(H^k(B_q, F_q) \cong \tilde{H}^{k-1}(F_q)\) is non-zero only for \(k = n - 1\) and \(k = n\).

5.2.3  |  Cohomology of \((T^*_S, T^*_S \cap F^{(1)})\)

Each irreducible 1-dimensional component \(\Sigma^0_i\) of \(\text{Sing}_l f\big|_{l=0}\) is a curve germ, and hence its link \(\partial B_{\varepsilon} \cap \Sigma^0_i\) is a circle. Our \(S \sqcup R\) is an admissible deformation of \(\text{Sing}_l f\big|_{l=0}\) in the sense\(^\dagger\) of [45, §3.1], that is, an isotopy at the boundary \(\partial B_{\varepsilon}\). Therefore, the boundary \(\partial B_{\varepsilon} \cap S_i\) of the irreducible curve \(S_i\) is diffeomorphic to the link of the corresponding \(\Sigma^0_i\), hence it is diffeomorphic to a circle.

We call this circle the outside loop of \(S^*_i\), for any \(i\), and denote by \(U\) the set of outside loops of \(S^*\). Note that over any loop \(u_i \in U\) we have a local system monodromy \(\nu_{u_i} : Z^{\mu_i^0} \to Z^{\mu_i^0}\), and this monodromy is invariant in the admissible deformation.

Starting from the outside loop \(u_i\), we retract the Riemann surface with boundary \(S^*_i\) to a bouquet configuration \(\Gamma_i\) of loops connected by simple paths which have a single base point \(z_i \in S^*_i\). These loops will be indexed by the set \(W_i\).

\[
\tau_i = \sum_{q \in Q_i} \# K_q
\]

loops, where the set \(K_q\) introduced at Section 5.2.1 is indexing the local branches of \(S_i\) at \(q \in Q_i\). We thus have \(\# W_i = 2g_i + \tau_i\).

The pair \((T^*_i, T^*_i \cap F^{(1)})\) decomposes into the connected pairs \((T^*_i, T^*_i \cap F^{(1)})\) along the curve components \(S^*_i\). Let us denote the projection of the tubular neighborhood by \(\pi_S : T^*_i \to S^*\). Each pair \((T^*_i, T^*_i \cap F^{(1)})\) is then homotopy equivalent to the pair \((\pi_S^{-1}(\Gamma_i), \pi_S^{-1}(\Gamma_i)) \cap F^{(1)})\). For each loop in the bouquet of loops \(\Gamma_i\), we have a pair similar to \((Z_k, C_k)\) defined above. The difference is that the pairs \((Z_k, C_k)\) are disjoint whereas in \(S^*_i\) the loops meet at a single point \(z_i\). We therefore take as reference the transversal fiber \(F^{\mu_i}_i := F^{(1)} \cap \pi_S^{-1}(z_i)\) over the point \(z_i\).

For any \(w \in W_i\), we denote by \(\nu_w\) the vertical monodromy along the loop in \(S^*_i\) indexed by \(w\). Let

\[
\hat{A}_i : \pi_1(S^*_i, z_i) \to \text{Aut}(H^{n-2}(F^{\mu_i}_i, Z))
\]

be the vertical monodromy representation, and let \(H^{n-2}(F^{\mu_i}_i)\hat{A}_i\) denote the submodule of invariants. Let us point out that, in general, this representation cannot be related by a Lefschetz slicing argument\(^\ddagger\) to the monodromy representation defined at \((3.19)\), because the Lefschetz argument needs the rectified homotopical depth condition for \(\text{Sing}_l f\) (see [11], [50, §9 and §10]). However, if

\(^\dagger\) Moreover, \(f_{l=0}\) is an admissible deformation of \(f_{l=0}\) in \(B_{\varepsilon}\).

\(^\ddagger\) That is, the surjectivity of the map \(\pi_1(S^*_i, z_i) \to \pi_1(\Sigma_{\mu, j}, z_i)\) induced by inclusion.
the singular set has maximal rectified homological depth, then by [50, Theorem 9.3.1], the equality

\[ H^{n-2}(F^n_i)^{\hat{A}_i} = H^{n-2}(F^n_i)^{\bar{A}_i} \]

holds for any \( i \in I \). In view of Theorem 3.4(c) and of the forthcoming Corollary 5.3, we conjecture that this later equality holds in general.

**Lemma 5.2.**

(a) We have the isomorphisms:

\[
H^{n-1}(T^*_i, T^*_i \cap F^{(1)}) \cong \bigcap_{w \in W_i} \ker (v_w - \text{id} | H^{n-2}(F^n_i)^{\hat{A}_i}),
\]

\[
H^n(T^*_i, T^*_i \cap F^{(1)}) \cong \text{coker} \left( \bigoplus_{w \in W_i} (v_w - \text{id} | H^{n-2}(F^n_i)^{\hat{A}_i}) \right).
\]

(b) The Mayer–Vietoris exact sequence (5.3) is trivial except of the 6-term sequence:

\[
0 \to H^{n-1}(T_S, T_S \cap F^{(1)}) \to \bigoplus_{i \in I} H^{n-2}(F^n_i)^{\hat{A}_i} \oplus \bigoplus_{q \in Q} H^{n-2}(F_q) \to^j \bigoplus_{q \in Q} \bigoplus_{k \in K_q} \ker (v_k - \text{id})
\]

\[
\to H^n(T_S, T_S \cap F^{(1)}) \to \bigoplus_{i \in I} \text{coker} \left( \bigoplus_{w \in W_i} (v_w - \text{id} | H^{n-2}(F^n_i)^{\hat{A}_i}) \right) \oplus \bigoplus_{q \in Q} H^{n-1}(F_q)
\]

\[
\to \bigoplus_{q \in Q} \bigoplus_{k \in K_q} \text{coker} (v_k - \text{id} | H^{n-2}(F^n_k)^{\bar{A}_k}) \to 0.
\]

(c) The following Euler characteristic formulae hold:

\[
\chi(T^*_i, T^*_i \cap F^{(1)}) = (-1)^n (2g_i + \tau_i - 1) \mu^n_i, \tag{5.8}
\]

where \( g_i \) is the genus of the normalization of \( S_i \) and \( \tau_i \) is the total number of the local branches of \( \Sigma_i \) at the points \( Q_i \), and:

\[
\chi(T, T \cap F^{(1)}) = - \sum_{q \in Q} (\chi(F_q) - 1) + (-1)^n \sum_{i \in I} (2g_i + \tau_i - 1) \mu^n_i + (-1)^n \sum_{r \in R} \mu_r. \tag{5.9}
\]

**Proof.** From (4.19), (4.20), and (4.21) we know that \( H^j(T_S, T_S \cap F^{(1)}) = 0 \) for \( j \leq n - 2 \) and \( j > n \). By (5.4) and (5.5), the cohomology

\[
H^*(T^*_S \cap B_Q, (T^*_S \cap F^{(1)}) \cap (B_Q \cap F^{(1)})) \cong \bigoplus_{q \in Q} \bigoplus_{k \in K_q} H^*(\mathbb{Z}_k, C_k)
\]
is concentrated in dimensions $n-1$ and $n$, and its Euler characteristic is zero since it is a direct sum of relative circle bundles. And by Section 5.2.2, the cohomology $H^*(B_Q, B_Q \cap F(1))$ is also concentrated in dimensions $n-1$ and $n$.

The cohomology $H^*(T_S^*, T_S^* \cap F(1)) = \bigoplus_{i \in I} H^*(T_i^*, T_i^* \cap F(1))$ is also concentrated in degrees $n-1$ and $n$ since the pair $(T_i^*, T_i^* \cap F(1))$ is the relative bundle $(\pi_*^{-1}(\Gamma_i), \pi_*^{-1}(\Gamma_i) \cap F(1))$ over a bouquet of loops indexed by $W_i$ with fiber $(E_i^0, F_i^0)$. More precisely we have a relative Wang sequence for a bundle over a wedge of circles, the nontrivial part of which is the following:

$$H^{n-1}(T_i^*, T_i^* \cap F(1)) \hookrightarrow H^{n-1}(E_i^0, F_i^0) \oplus \bigoplus_{w \in W_i} H^{n-1}(E_i^0, F_i^0) \Rightarrow H^n(T_i^*, T_i^* \cap F(1)). \tag{5.10}$$

The relative cohomology group $H^{n-1}(T_i^*, T_i^* \cap F(1))$ identifies to the kernel of the boundary map $\partial = \bigoplus_{w \in W_i} (\nu_w - \text{id})$, and this kernel is the intersection $\bigcap_{w \in W_i} \ker(\nu_w - \text{id} | H^{n-2}(F_i^0))$ which is by definition the submodule of invariants $H^{n-2}(F_i^0)A_i$ defined by the representation (5.6). This proves (a). Part (b) follows by using all the above facts proved in the Mayer–Vietoris sequence (5.7).

(c) Counting the ranks in the exact sequence (5.10) yields the above claimed formula (5.8). The second formula (5.9) is obtained by taking the Euler characteristic in the Mayer–Vietoris long exact sequence (5.7), and using Lemma 5.2(c). □

Let us give a quick consequence of the above study for the particular case $Q = \emptyset$, yielding a slight extension of Theorem 3.4(c):

**Corollary 5.3.** Let $n \geq 3$ and $\dim_0 \text{Sing}_W f = 2$. If $\Sigma_2 \cap \Sigma_1 = \emptyset$ then:

$$H^{n-2}(F) \cong \bigoplus_{i \in I} H^{n-2}(F_i^0)A_i, \quad H^{n-1}(F) \cong \bigoplus_{i \in I} \ker\left( \bigoplus_{w \in W_i} (\nu_w - \text{id} | H^{n-2}(F_i^0)) \right) \oplus \bigoplus_{r \in R} H^{n-1}(F_r).$$

**Proof.** By the above construction we have

$$(T_S, T_S \cap F(1)) = \bigsqcup_{i \in I} (T_i^*, T_i^* \cap F(1))$$

and our result follows by Lemma 5.2(b), by (5.2) and from the Mayer–Vietoris sequence (5.7) which reduces to two isomorphisms since $Q = \emptyset$. □

### 5.3 | **End of the computation of $H^{n-2}(F)$**

We continue the computation without any restrictions on the set $Q$.

The function germ $f |_{l=\eta}$ at some point $q \in Q$ is a function germ with a 1-dimensional singularity, and we have denoted by $F_q$ its Milnor fiber. The pair $(T_S^* \cap B_Q, T_S^* \cap B_Q \cap F(1))$ is a disjoint
union of pairs localized at the points of $Q$. Let us denote $(Z_q, C_q) := \bigsqcup_{k \in K_q} (Z_k, C_k)$, where the pair $(Z_k, C_k)$ is defined in (5.4) and corresponds to the branch $S_{q,k}$ of the germ of the singular locus $(S, q)$. We claim that we have the natural monomorphism† (induced by the inclusion of pairs):

$$\tilde{H}^{n-2}(F_q) \cong H^{n-1}(B_q, F_q) \hookrightarrow H^{n-1}(Z_q, C_q) \cong \bigoplus_{k \in K_q} \ker(\nu_k - \text{id} | \tilde{H}^{n-2}(F_k^n)) \quad (5.11)$$

where the last isomorphism follows directly from (5.5), and where $\nu_k$ is here the vertical monodromy acting on the branch $S_{q,k}$ of the curve germ $(S, q)$.

To prove our claim, we apply the results and notations of Section 4.3 for $f$ to the function germ $g_q := f |_{l=\eta}$ at $q \in Q$, see in particular Definition 4.2 and the notations following it. Its singular locus $\text{Sing}_{\{w\cap\{|l=\eta|\}}g_q$ coincides with $S$, viewed as set germs at $q$. We shall use the index $q$ to designate similar objects for $g_q$, namely, $T_q$ will be the tubular neighborhood of the curve germ $(S, q)$, and $L_q$ the slice monodromy acting on the slice $F'_q$ of the Milnor fiber $F_q$ of $g_q$.

By (4.22), we have the monomorphism:

$$\tilde{H}^{n-2}(F_q) \hookrightarrow \ker\left[L_q - \text{id} | H^{n-1}(T_q, T_q \cap F'_q)\right]. \quad (5.12)$$

The pair $(T_q, T_q \cap F'_q)$ is a disjoint union, over $k \in K_q$, of the transversal Milnor data $(B_{k,j}, F_k^n)$ at some point of the branch cut out by a generic transversal slice $H$ of the space germ $(X \cap l^{-1}(\eta), q)$ passing close enough to the point $q$. The number of those points where the generic slice intersects the branch $S_{q,k}$ is the local multiplicity $\text{mult}_q(S_{q,k})$, and let us index these points by the set $\Lambda_{q,k}$. With these notations at hand, we have the isomorphism:

$$H^{n-1}(T_q, T_q \cap F'_q) \cong \bigoplus_{k \in K_q} \bigoplus_{j \in \Lambda_{q,k}} H^{n-1}(B_{k,j}, F_k^n), \quad (5.13)$$

where the Milnor data $(B_{k,j}, F_k^n)$ are the same for all $j \in \Lambda_{q,k}$, up to homeomorphisms.

The $l$-monodromy acts on the direct sum of (5.13) for each fixed $k$, and this action corresponds to the cyclic movement of a singular point of $S_{q,k} \cap H$. As explained in [41], [48], [49, Proof of Prop. 3.1], this yields a particular shape of the matrix of the monodromy matrix $L$, which gives the following direct sum splitting:

$$\ker\left[L_q - \text{id} | H^{n-1}(T_q, T_q \cap F'_q)\right] \cong \bigoplus_{k \in K_q} \ker(\nu_k - \text{id} | H^{n-2}(F_k^n)).$$

By comparing it with (5.12), this finishes the proof of our claimed monomorphism (5.11).

Coming back to the global picture, let us point out that in the direct sum of (5.11), each fiber $F_k^n$ identifies with the transversal fiber $F_i^n$ for the $i \in I$ corresponding to $k$ in the decomposition $K_q = \bigsqcup_{i \in I} K_{qi}$. Let us denote by $\iota_{q,k}$ the composition of the injection $i$ from (5.11) with the projection on the direct summand $\ker(\nu_k - \text{id} | H^{n-2}(F_k^n))$.

By the exact sequence (5.3) via Lemma 5.2(a) and the exact sequence (5.7), and from the expression of the second direct summand given in Section 5.2.2, we have $H^{n-2}(F) \cong \ker j$, where $j$ is the

† Remark that the monomorphism (5.11) can be viewed as a particular case of (3.20).
following morphism which occurs in (5.7):

$$\bigoplus_{i \in I} H^{n-2}(F_i)_{\tilde{A}_i} \oplus \bigoplus_{q \in Q} H^{n-2}(F_q) \xrightarrow{j} \bigoplus_{q \in Q} \bigoplus_{k \in K_q} \ker (\nu_k - \text{id} | H^{n-2}(F_k)).$$

(5.14)

Let

$$j_1 : \bigoplus_{i \in I} H^{n-2}(F_i)_{\tilde{A}_i} \rightarrow \bigoplus_{q \in Q} \bigoplus_{k \in K_q} \ker (\nu_k - \text{id} | H^{n-2}(F_k))$$

denote the restriction of \(j\) to the first summand. We will denote by \(j_2\) the restriction of \(j\) to the second summand. By Lemma 5.2(a) and (b), \(j_1\) identifies with the map:

$$\bigoplus_{i \in I} \bigcap_{w \in W_i} \ker (\nu_w - \text{id} | H^{n-2}(F_i)) \xrightarrow{j_1} \bigoplus_{q \in Q} \bigoplus_{k \in K_q} \ker (\nu_k - \text{id} | H^{n-2}(F_k)),$$

(5.15)

which is the diagonal map, in the sense that, for each fixed \(i \in I\) such that \(Q_i \neq \emptyset\), the intersection from the left side injects into each of the members of the direct sum from the right taken over \(q \in Q_i\). Indeed, if \(Q_i \neq \emptyset\), the intersection \(\bigcap_{w \in W_i} \ker (\nu_w - \text{id})\) from the left side of (5.15) injects in each kernel \(\ker (\nu_k - \text{id})\) from the right side, for any \(k \in K_q\) and \(q \in Q_i\). On the other hand, if \(Q_i = \emptyset\), then \(\bigcap_{w \in W_i} \ker (\nu_w - \text{id} | H^{n-2}(F_i))\) is clearly included in \(\ker j\).

So, if \(I_0 := \{i \in I \mid Q_i = \emptyset\}\), let \(j'_1\) denote the restriction of \(j_1\) from (5.15) where in the source we take only the direct sum \(\bigoplus_{i \in I \setminus I_0}\) corresponding to the indices \(I \setminus I_0\).

Let us denote by \(j_1(H^{n-2}(F_i)_{\tilde{A}_i})\) the subgroup in \(\ker (\nu_k - \text{id} | H^{n-2}(F_k))\) for every \(k \in K_{q_i}\) and \(q \in Q_i\). With all these notations and preliminaries we obtain the following:

**Theorem 5.4.** Let \((X, 0)\) be a pure \((n + 1)\)-dimensional space germ with \(\text{rHd}(X, \mathbb{Z}) = n + 1, n \geq 3\) and \(\dim \text{Sing}_{\mathcal{W}_f} = 2\). Then:

(a) \(H^{n-2}(F)\) is free. The morphisms \(j'_1\) and \(j_2\) are injective.

(b) \(H^{n-2}(F) \cong G \oplus \bigoplus_{i \in I_0} H^{n-2}(F_i)_{\tilde{A}_i}\), where \(G \cong \text{Im} j'_1 \cap \text{Im} j_2\) is a submodule of

$$\bigoplus_{i \in I \setminus I_0} \left[ j_1(H^{n-2}(F_i)_{\tilde{A}_i}) \cap \bigcap_{q \in Q_i, k \in K_{q_i}} \tau_{q_k}(H^{n-2}(F_q)) \right].$$

**Proof.**

(a). We have \(H^{n-2}(F) \cong \ker j\), where \(j\) is given in (5.14), and this is a consequence of the assumption \(\text{rHd}(X, \mathbb{Z}) = n + 1\). The freeness follows then from the fact that the source of \(j\) is a free \(\mathbb{Z}\)-module, since \(H^{n-2}(F_i)_{\tilde{A}_i}\) are free, and \(H^{n-2}(F_q)\) are also free due to (5.11), both being consequences of the assumption \(\text{rHd}(X, \mathbb{Z}) = n + 1\) (see also the proof of Theorem 3.4(d)).

We have seen before that \(j'_1\) is injective and it is actually the diagonal map for each fixed \(i \in I\). Let us now consider

$$j_2 : \bigoplus_{q \in Q} H^{n-2}(F_q) \rightarrow \bigoplus_{q \in Q} \bigoplus_{k \in K_q} \ker (\nu_k - \text{id} | H^{n-2}(F_k)).$$
as the restriction of \( j \) to the second summand. After taking the direct sum \( \bigoplus_{q \in Q} \) on both sides of (5.11), we deduce the injectivity of \( j_2 \).

(b). By the same argument as for Corollary 5.3, we get the inclusion \( \ker j \supset \bigoplus_{i \in I_0} H^{n-2}(F_i^n)_{\hat{A}_i} \).

Since \( j'_1 \) and \( j_2 \) are injective, we deduce:

\[
\ker j \cong (\text{Im } j'_1 \cap \text{Im } j_2) \oplus \bigoplus_{i \in I_0} H^{n-2}(F_i^n)_{\hat{A}_i}.
\]

Next, for any fixed \( i \notin I_0 \), let us consider the following restriction, where in the target we take the summands over \( K_{qi} \) only:

\[
j_1 : H^{n-2}(F_i^n)_{\hat{A}_i} \oplus \bigoplus_{q \in Q_i} H^{n-2}(F_q) \longrightarrow \bigoplus_{q \in Q_i} \bigoplus_{k \in K_{qi}} \ker (\nu_k - \text{id} | H^{n-2}(F_k^n)).
\]

Its kernel is then isomorphic \( j_1(H^{n-2}(F_i^n)_{\hat{A}_i}) \cap \bigcap_{q \in Q_i, k \in K_{qi}} \iota_{qk}(H^{n-2}(F_q)), \) due to the fact that \( j_1 \) is the diagonal map.

Nevertheless, when we take the direct sum over \( i \in I \setminus I_0 \), the kernels may not add up since there is interaction between different \( S_i \) due to the fact that whenever \( q \in Q \) belongs to several irreducible components \( S_i \), then the fiber \( F_q \) contributes with \( \iota_{qk}(H^{n-2}(F_q)) \) for all those indices \( k \) such that \( k \in K_{qi} \). Therefore we do not get \( \ker j \) as a direct sum but as a submodule only; our claim is proved.

We immediately get from the above theorem the following:

**Corollary 5.5.** Let \( n \geq 3 \). Let \( I_1 := \{i \in I \setminus I_0 \mid H^{n-2}(F_q) = 0 \text{ for some } q \in Q_i\} \).

(a) \( G \) is isomorphic to a submodule of \( \bigoplus_{q \in Q_i} H^{n-2}(F_q) \). In particular, if \( I_1 = I \), then \( H^{n-2}(F) = 0 \).

(b) \( b_{n-2}(F) \leq \sum_{i \in I \setminus I_1} \min\{b_{n-2}(F_i^n), b_{n-2}(F_q) \mid q \in Q_i\} \).

### 6. EXAMPLES

In this section, we apply our results to specific computations on examples. Let us refer here again to [29, 53], not only for very interesting and sharp results about the Milnor fiber in a class of functions with \( s = 2 \), but also for examples of computations in homology. More computations in homology are done in [45] in case of admissible deformations of 1-dimensional singularities, in which case, as we have already mentioned before, has common grounds with our case \( s = 2 \), whereas the computations in cohomology are different.

**Example 6.1.** If the singularity at the origin of \( f : (X, 0) \to (C, 0) \) is contained in a positive dimensional stratum \( S \), say of complex dimension \( r \leq s < n \), then by the local product structure around the origin we get:

\[
\tilde{H}^{n-r+1}(F) \cong \cdots \cong \tilde{H}^n(F) \cong 0.
\]
So in this case the reduced cohomology of the Milnor fiber $F$ is concentrated in degrees $[n -s, n -r]$. In terms of vanishing cycles, this can be seen from the base change property (for example, see [36, Lemma 4.3.4]) as follows: If $H$ is a normal slice (in the ambient space $\mathbb{C}^N$) through the origin to the stratum $S$ containing the origin, then we have for $f' := f|_H$ and with $G'$ a bounded constructible complex that:

$$\varphi_{f'}(G'|_H) \cong (\varphi_f G')|_H.$$  \hspace{2cm} (6.1)

Let us consider the example $f : \mathbb{C}^4 \to \mathbb{C}$ given by $f = xyz$. The singular locus of $f$ is given by $\Sigma = H_{xy} \cup H_{xz} \cup H_{yz}$, with $H_{xy} := \{x = y = 0\}, H_{xz} := \{x = z = 0\}, H_{yz} := \{y = z = 0\}$. These three complex 2-planes intersect mutually along the line $H_{xyz} := \{x = y = z = 0\} \cong \mathbb{C}$. Then $\Sigma$ has a Whitney stratification with three 2-dimensional strata

$$\Sigma_{2,1} := H_{xy} \setminus H_{xyz}, \Sigma_{2,2} := H_{xz} \setminus H_{xyz}, \Sigma_{2,3} := H_{yz} \setminus H_{xyz},$$

each homotopy equivalent to a circle, and a 1-dimensional stratum $\Sigma_1 = H_{xyz}$. The transversal type of each of the three 2-dimensional strata is $A_1$, and each vertical monodromy $A_i$ is the identity. Formula (3.20) yields in this case that $H^1(F) \cong \mathbb{Z}^3$. It appears that this inclusion is strict, which also shows that the monomorphism in (3.20) is not in general an isomorphism. Indeed, since the singularity of $f$ at the origin is contained in a 1-dimensional stratum, by slicing with a generic $\mathbb{C}^3$ through the origin reduces the calculation of $H^1(F)$ to the case of the 1-dimensional singularity defined by $xyz = 0$ in $\mathbb{C}^3$, for which the origin is a 0-dimensional stratum. This was considered by Siersma in [42], who computed that the Milnor fiber $F$ at the origin is homotopy equivalent to $S^1 \times S^1$, hence $H^1(F) \cong \mathbb{Z}^2$. This calculation can also be performed via Theorem 5.4(a): Indeed, we have $j_1 = \text{id}$, and hence $H^1(F) \cong \ker j \cong \text{Im}(j_2 : \mathbb{Z}^2 \to \mathbb{Z}^3) \cong \mathbb{Z}^2$, since $j_2$ is injective.

**Example 6.2.** Let $f : \mathbb{C}^4 \to \mathbb{C}$ be given by $f = xyzu$. The singular locus of $f$ is 2-dimensional and is given by

$$\Sigma = H_{xy} \cup H_{xz} \cup H_{xu} \cup H_{yz} \cup H_{yu} \cup H_{zu},$$

where $H_{xy} := \{x = y = 0\}$, and so forth, thus each component is a 2-plane in $\mathbb{C}^4$. Any two of the six components of $\Sigma$ intersect either along a complex line, for example, $H_{xy} \cap H_{xz} = H_{xyz} = \{x = y = z = 0\}$ is the $u$-line, or at the origin, for example, $H_{xy} \cap H_{zu} = \{(0, 0, 0, 0)\}$. Moreover, any three (or more) of the six components of $\Sigma$ intersect at the origin, for example, $H_{xy} \cap H_{xz} \cap H_{xu} = H_{xyzu} = \{x = y = z = u = 0\} = \{(0, 0, 0, 0)\}$. A Whitney stratification of $\Sigma$ can be given with six 2-dimensional strata, one for each component of $\Sigma$, which are of the form

$$\Sigma_{2,1} := H_{xy} \setminus (H_{xyz} \cup H_{xyu}) \cong \mathbb{C}^* \times \mathbb{C}^*,$$

and so forth. There are four 1-dimensional strata of the form $H_{xyz} \setminus H_{xyzu} \cong \mathbb{C}^*$, and the origin $H_{xyzu} = \{(0, 0, 0, 0)\}$ is a 0-dimensional stratum.

Each of these 2-dimensional strata has the homotopy type of $S^1 \times S^1$ and $A_1$-transversal type. The two generators of the fundamental group $\mathbb{Z}^2$ of a 2-dimensional stratum act trivially on the first cohomology group, $\mathbb{Z}$, of the corresponding transversal Milnor fiber, so the vertical mon-
odromy along each of the 2-dimensional strata is the identity. Formula (3.20) yields the inclusion $H^1(F) \hookrightarrow \mathbb{Z}^6$.

Let us further use Theorem 5.4. A generic slice $l = \eta$ has 1-dimensional singularities. We have that $S$ is a configuration of 6 lines intersecting at 4 points, where 3 lines pass through each of the 4 points. Thus $S^*_i \simeq \mathbb{C}^*$, $i = 1, \ldots, 6$, and each local vertical monodromy around such a puncture is the identity. We compute $j$ and find that the image of $j_1$ has 6 generators. Intersecting it with the image of $j_2$ introduces 3 relations among these generators, more precisely a symmetric linear relation between 3 generators in each of the 4 punctures, and resolving the system yields finally 3 linear relations among the 6 generators. We get $H^1(F) \cong \text{ker } j \cong \mathbb{Z}^3$.

On the other hand, by direct computation, the fiber $F = \{xyzu = 1\}$ is homotopy equivalent to $S^1 \times S^1 \times S^1$, hence $H^1(F) \cong \mathbb{Z}^3$, confirming the above result.

Example 6.3. Let $f : \mathbb{C}^4 \to \mathbb{C}$ be given by $f = x^2z + y^2u$. The hypersurface $f = 0$ has a 2-dimensional singular locus $\Sigma = \{x = y = 0\}$, with Whitney strata $\Sigma_0 = \{(0,0,0,0)\}$, $\Sigma_1 = \{(0,0,0,u) | u \neq 0\} \cup \{(0,0,z,0) | z \neq 0\}$, and $\Sigma_2 = \{(0,0,z,u) | z \neq 0, u \neq 0\}$. The transversal Milnor fiber $F^\text{th}$ to the stratum $\Sigma_2$ is the Milnor fiber of the singularity at $(0,0)$ of the curve $x^2 + y^2 = 0$ in $\mathbb{C}^2$, so $F^\text{th} \simeq S^1$. It follows from Theorem 3.4(a) that if $F$ denotes the Milnor fiber of $f$ at the origin, then

$$H^1(F) \hookrightarrow \mathbb{Z}.$$ 

On the other hand, it can be seen by using the Thom–Sebastiani theorem that $F \cong S^1 \times S^1 \simeq S^3$, so in fact $H^1(F) = 0$.

In the setting of Theorem 5.4, slicing with the hyperplane $u + z = 1$ and taking advantage of the homogeneity which implies that all the fibrations are global, we get $S^*_i \simeq \mathbb{C}^*$, where the vertical monodromies $\nu_i$ around the two special points are $-\text{id}$. This implies that $\text{ker}(\nu_i - \text{id}) = 0$ and therefore Theorem 5.4(b) yields indeed that $H^1(F) = 0$.

Example 6.4. Let $f : \mathbb{C}^4 \to \mathbb{C}$ be given by $f = x^2 + x(y^2 + z^2 + u^2)$. The singular locus of the hypersurface $f = 0$ has a Whitney stratification with a 0-dimensional stratum $\Sigma_0 = \{(0,0,0,0)\}$ and a 2-dimensional stratum

$$\Sigma_2 = V(x, y^2 + z^2 + u^2) \setminus \{(0,0,0,0)\}.$$ 

The transversal Milnor fiber $F^\text{th}$ to the stratum $\Sigma_2$ is the fiber of a $A_1$ singularity. The stratum $\Sigma_2$ is homotopy equivalent to the link of a quotient surface singularity of type $A_1$, hence $\pi_1(\Sigma_2) = \mathbb{Z}/2$. Since there are no 1-dimensional strata, Theorem 3.4(c) shows that the first cohomology group of the Milnor fiber $F$ of $f$ at the origin is computed as:

$$H^1(F) \cong H^1(F^\text{th})\mathbb{Z}/2.$$ 

Note that we can write $f = P(h, g)$, with $h = x : \mathbb{C}^4 \to \mathbb{C}$, $g = y^2 + z^2 + u^2 : \mathbb{C}^4 \to \mathbb{C}$ and $P = h^2 + hg$, so the Milnor fiber $F$ at 0 can be deduced from [28, Theorem A] as having the homotopy type of $S^1 \vee S^3 \vee S^3$, which gives $H^1(F) \cong \mathbb{Z}$. Alternatively, after a change of coordinates, we note

---

† One may refer to [45, §5.4] for a related computation in homology in case of an admissible deformation of a 4 planes central arrangement.
that $F$ is the Milnor fiber of the singularity at the origin of the polynomial $x^2 + (y^2 + z^2 + u^2)^2$, and its homotopy type can be easily deduced via the Thom–Sebastiani theorem as the suspension on two disjoint $S^2$'s.

Let us indicate how this isomorphism follows from Theorem 5.4(b). By this result we get $H^1(F) \cong H^1(F^\eta)\mathbb{Z}$ since the stratum $S$ is the complex link of $\Sigma_2$ hence homotopy equivalent to a circle. We compute the vertical monodromy $\nu$ of $H^1(F^\eta)$ along $S$. We first slice near the origin with $\{l := u = \eta\}$ and consider the restriction $f|_{l=\eta} = f'(x, y, z) = x^2 + x(y^2 + z^2 + \eta^2)$ with singular locus $S = Z(x, y^2 + z^2 + \eta^2)$. We consider a circle $y^2 + z^2 = t > 0$ (in real coordinates) which is a geometric generator of the fundamental group of $S$. The monodromy action $\nu$ on $H^1(F^\eta) \cong \mathbb{Z}$ along this circle is the identity, thus $H^1(F) \cong \ker(\nu - \text{id} | H^1(F^\eta)) = \mathbb{Z}$.

More generally, consider the following example.

**Example 6.5.** Let $f : \mathbb{C}^4 \to \mathbb{C}$ be given by $f = x^p + (y^2 + z^2 + u^2)^q$, with $p, q \geq 2$. This is very similar to Example 6.4 above, in which $p = q = 2$. Indeed, the singular locus of the hypersurface $f = 0$ has a Whitney stratification with a 0-dimensional stratum $\Sigma_0 = \{(0, 0, 0, 0)\}$ and a 2-dimensional stratum $\Sigma_2 = V(x, y^2 + z^2 + u^2) \setminus \{(0, 0, 0, 0)\}$.

Outside of the origin in $\mathbb{C}^4$ the functions $x$ and $g = y^2 + z^2 + u^2$ are part of a coordinate system and define the smooth singular set.

The transversal Milnor fiber may be found as follows: At the two points of the set $\Sigma_2 \cap \{y = z = a\} = \{x = 0, y = z = a, u^2 + 2a^2 = 0\}$, for some $a \neq 0$, we consider two Milnor balls in the ‘vertical’ slice $\{y = z = a\}$. At each of the two points, one has local coordinates $\nu := x$ and $w := g$ in the slice $\{y = z = a\}$. The transversal Milnor fibers at the two points are both described by the equation $x^p + (2a^2 + u^2)^q = \eta$. Therefore, the transversal singularity of $f$ at a point in $\Sigma_2$ is locally expressed by $x^p + g^q = 0$. So the transversal Milnor fiber $F^\eta$ to the stratum $\Sigma_2$ is a bouquet of $(p - 1)(q - 1)$ circles $S^1$, with $H^1(F^\eta) \cong \mathbb{Z}^{(p-1)(q-1)}$.

Once again, $\pi_1(\Sigma_2) = \mathbb{Z}/2$, and since there are no 1-dimensional strata, Theorem 3.4(c) shows that the first cohomology group of the Milnor fiber $F$ of $f$ at the origin is computed as:

$$H^1(F) \cong H^1(F^\eta)\mathbb{Z}/2 \cong \mathbb{Z}^{(p-1)(q-1)}.$$ 

Let us also note that an application of the Thom–Sebastiani theorem shows that the Milnor fiber $F$ of $f$ at the origin is homotopy equivalent to the join of $p$ points (corresponding to $x^p = 1$) with a disjoint union of $q$ spheres $S^2$ (each given by $y^2 + z^2 + u^2 = 1$). In particular, $H^1(F) \cong \mathbb{Z}^{(p-1)(q-1)}$. This implies that $\pi_1(\Sigma_2)$ acts trivially on $H^1(F^\eta)$.

In order to apply Theorem 5.4(b), we compute the vertical monodromy on the stratum $S$ which is by definition the complex link of the 2-dimensional stratum $\Sigma_2$. It may be defined by $y^2 + z^2 = \varepsilon$, thus it is homotopy equivalent to a circle. A geometric generator of its $\pi_1$ may be given by the same equation in real coordinates, considering $\varepsilon > 0$ as a real constant.

While the real coordinates $y = \varepsilon \cos t, z = \varepsilon \sin t$, for $t \in [0, 2\pi]$, describe this circle, each of the two transversal Milnor balls move along with fixed coordinates $x$ and $u$. This yields in particular a geometric monodromy which is the identity on all coordinates.
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REFERENCES

1. D. Burghelea and A. Verona, Local homological properties of analytic sets, Manuscripta Math. 7 (1972), 55–66.
2. T. de Jong, Some classes of line singularities, Math. Z. 198 (1988), no. 4, 493–517.
3. A. Dimca, Sheaves in topology, Universitext, Springer, Berlin, 2004.
4. A. Dimca and M. Saito, Some consequences of perversity of vanishing cycles, Ann. Inst. Fourier (Grenoble) 54 (2004), no. 6, 1769–1792.
5. J. Fernández de Bobadilla, Topological equisingularity of hypersurfaces with 1-dimensional critical set, Adv. Math. 248 (2013), 1199–1253.
6. J. Fernández de Bobadilla and M. Marco-Buzunáriz, Topology of hypersurface singularities with 3-dimensional critical set, Comment. Math. Helv. 88 (2013), no. 2, 253–304.
7. A. Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA2), Masson & North-Holland Ed., Paris, 1968.
8. Groupes de monodromie en géométrie algébrique. I. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. Lecture Notes in Mathematics, Vol. 288. Springer, Berlin-New York, 1972.
9. Groupes de monodromie en géométrie algébrique. II. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II). Dirigé par P. Deligne et N. Katz. Lecture Notes in Mathematics, Vol. 340. Springer, Berlin-New York, 1973.
10. H. Hamm and D. T. Lê, Un théorème de Zariski du type de Lefschetz, Ann. Sci. École Norm. Sup. (4) 6 (1973), 317–355.
11. H.A. Hamm and D. T. Lê, Rectified homotopical depth and Grothendieck conjectures, The Grothendieck Festschrift, Vol. II, 311–351, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990.
12. H. Hironaka, Stratification and flatness, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 199–265. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
13. I.N. Iomdin, Variétés complexes avec singularités de dimension un, Sibirskii Mat. Zhurnal 15 (1974), 1061–1082.
14. C. Joița and M. Tibăr, Images of analytic map germs and singular fibrations, Europ. J. Math 6 (2020), no. 3, 888–904.
15. D. T. Lê, Calcul du Nombre de Cycles Évanouissants d’une Hypersurface Complexe, Ann. Inst. Fourier (Grenoble) 23 (1973), 261–270.
16. D. T. Lê, Some remarks on relative monodromy, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 397–403. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
17. A. Libgober, Eigenvalues for the monodromy of the Milnor fibers of arrangements, in A. Libgober, M. Tibăr (eds.), Trends in singularities, Trends Math., Birkhäuser, Basel, 2002, pp. 141–150.
18. C. McCrory and A. Parusiński, Complex monodromy and the topology of real algebraic sets, Compositio Math. 106 (1997), no. 2, 211–233.
19. D.B. Massey, Milnor fibres of non-isolated hypersurface singularities, Singularity theory (Trieste, 1991), 458–467, World Sci. Publ., River Edge, NJ, 1995.
20. D.B. Massey, Numerical invariants of perverse sheaves, Duke Math. J. 73 (1994), no. 2, 307–369.
21. D. B. Massey and B. David, *Lê cycles and hypersurface singularities*, Lecture Notes in Mathematics, 1615 (Springer, Berlin, 1995). xii+i+131 pp.

22. D. B. Massey and B. David, *Numerical control over complex analytic singularities*, Mem. Amer. Math. Soc. **163** (2003), no. 778, xii+i+268 pp.

23. D. B. Massey, *Milnor fibers and links of local complete intersections*, Internat. J. Math. **25** (2014), no. 11, 1450110.

24. L. Maxim, *Intersection homology and Alexander modules of hypersurface complements*, Comment. Math. Helv. **81** (2006), no. 1, 123–155.

25. L. Maxim, *Intersection homology & perverse sheaves, with applications to singularities*, Graduate Texts in Mathematics, vol. 281, Springer, Cham, 2019, 270 pp.

26. L. Maxim, L. Păunescu, and M. Tibăr, *Vanishing cohomology and Betti bounds for complex projective hypersurfaces*, Ann. Inst. Fourier (Grenoble), to appear.

27. J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math Studies, No. 61, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968, iii+i+122 pp.

28. A. Némethi, *The Milnor fiber and the zeta function of the singularities of type $f = P(h, g)$*, Compositio Math. **79** (1991), no. 1, 63–97.

29. A. Némethi, *Hypersurface singularities with 2-dimensional critical locus*, J. London Math. Soc. (2) **59** (1999), no. 3, 922–938.

30. R. Pellikaan, *Hypersurface singularities and resolutions of Jacobi modules*, Dissertation, Rijksuniversiteit te Utrecht, Utrecht, 1985. Utrecht, 1985. vii+168 pp.

31. R. Pellikaan, *Deformations of hypersurfaces with a one-dimensional singular locus*, J. Pure Appl. Algebra **67** (1990), no. 1, 49–71.

32. C. Sabbah, *Morphismes analytiques stratifiés sans éclatement et cycles évanescents*, C. R. Math. **294** (1982), no. 1, 39–41.

33. C. Sabbah, *Proximité évanescente. I. La structure polaire d’un $\mathcal{D}$-module*, Compositio Math. **62** (1987), no. 3, 283–328.

34. M. Saito, *Mixed Hodge modules*, Publ. RIMS, Kyoto Univ. **26** (1990), 221–333.

35. R. Schrauwen, *Topological series of isolated plane curve singularities*, Enseign. Math. (2) **36** (1990), no. 1-2, 115–141.

36. J. Schürmann, *Topology of singular spaces and constructible sheaves*, Monografie Matematyczne, vol. 63. Birkhäuser, Basel, 2003.

37. M. Shubladze, *On nonisolated hypersurface singularities*, Algebra and geometry. J. Math. Sci. (N.Y.) **160** (2009), no. 6, 833–842.

38. M. Shubladze, *Singularities with critical locus a complete intersection and transversal type $A_1$*, Real and complex singularities, Contemp. Math., vol. 569 (Amer. Math. Soc., Providence, RI, 2012), pp. 193–202.

39. D. Siersma, *Isolated line singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40 (Amer. Math. Soc., Providence, RI, 1983) 485–496.

40. D. Siersma, *Singularities with critical locus a 1-dimensional complete intersection and transversal type $A_1$*, Topology Appl. **27** (1987), 51–73.

41. D. Siersma, *The monodromy of a series of hypersurface singularities*, Comment. Math. Helvetici **65** (1990), 181–197.

42. D. Siersma, *Variation mappings on singularities with a 1-dimensional critical locus*, Topology **30** (1991), no. 3, 445–469.

43. D. Siersma, *The vanishing topology of non isolated singularities*, New developments in singularity theory (Cambridge, 2000), NATO Sci. Ser. II Math. Phys. Chem., vol. 21 (Kluwer Acad. Publ., Dordrecht, 2001) 447–472.

44. D. Siersma and M. Tibăr, *Vanishing homology of projective hypersurfaces with 1-dimensional singularities*, Europ. J. Math. **3** (2017), 565–586.

45. D. Siersma and M. Tibăr, *Milnor fibre homology via deformation*, in W. Decker, G. Pfister, M. Schulze (eds.), *Singularities and computer algebra*, Springer, Cham, 2017, pp. 305–322.

46. B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972), pp. 285–362. Astérisque, Nos. 7 et 8, Soc. Math. France, Paris, 1973.
47. B. Teissier, *Variétés polaires. II*. Multiplicités polaires, sections planes, et conditions de Whitney, in Algebraic geometry (La Rábida, 1981), Lecture Notes in Math., vol. 961 (Springer, Berlin, 1982) 314–491.

48. M. Tibăr, *A supplement to the Iomdin-Lê theorem for singularities with one-dimensional singular locus*, Singularities and differential equations (Warsaw, 1993), Banach Center Publ., vol. 33 (Polish Acad. Sci., Warsaw, 1996) 411–419.

49. M. Tibăr, *The vanishing neighbourhood of non-isolated singularities*, Israel J. Math. 157 (2007), 309–322.

50. M. Tibăr, *Polynomials and vanishing cycles*. Cambridge Tracts in Mathematics, vol. 170. Cambridge University Press, Cambridge, 2007.

51. J-P. Vannier, *Familles à un paramètre de fonctions analytiques à lieu singulier de dimension un*, C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), no. 8, 367–370.

52. A. Zaharia, *On some derivations of negative weights and simple germs with nonisolated singularities*, Bull. London Math. Soc. 25 (1993), no. 5, 467–475.

53. A. Zaharia, *Topological properties of certain singularities with critical locus a 2-dimensional complete intersection*, Topology Appl. 60 (1994), no. 2, 153–171.