A POSITIVE RECURRENT REFLECTING BROWNIAN 
MOTION WITH DIVERGENT FLUID PATH

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Semimartingale reflecting Brownian motions (SRBMs) are diffusion processes with state space the $d$-dimensional nonnegative orthant, in the interior of which the processes evolve according to a Brownian motion, and that reflect against the boundary in a specified manner. The data for such a process are a drift vector $\theta$, a nonsingular $d \times d$ covariance matrix $\Sigma$, and a $d \times d$ reflection matrix $R$. A standard problem is to determine under what conditions the process is positive recurrent. Necessary and sufficient conditions for positive recurrence are easy to formulate for $d = 2$, but not for $d > 2$.

Associated with the pair $(\theta, R)$ are fluid paths, which are solutions of deterministic equations corresponding to the random equations of the SRBM. A standard result of Dupuis and Williams [6] states that when every fluid path associated with the SRBM is attracted to the origin, the SRBM is positive recurrent. Employing this result, El Kharroubi et al. [7, 8] gave sufficient conditions on $(\theta, \Sigma, R)$ for positive recurrence for $d = 3$; Bramson et al. [2] showed that these conditions are, in fact, necessary.

Relatively little is known about the recurrence behavior of SRBMs for $d > 3$. This pertains, in particular, to necessary conditions for positive recurrence. Here, we provide a family of examples, in $d = 6$, with $\theta = (-1, -1, \ldots, -1)^T$, $\Sigma = I$ and appropriate $R$, that are positive recurrent, but for which a linear fluid path diverges to infinity. These examples show in particular that, for $d \geq 6$, the converse of the Dupuis-Williams result does not hold.

1. Introduction. This paper is concerned with the class of $d$-dimensional diffusion processes known as semimartingale reflecting Brownian motions (SRBMs). Such processes arise as approximations for open $d$-station queueing networks (see, e.g., Harrison and Nguyen [10] and Williams [17, 18]). The state space for a process $Z = \{Z(t), t \geq 0\}$ in this class is $S = \mathbb{R}_+^d$, the nonnegative orthant. The data of the process consists of a drift vector $\theta$, a nonsingular covariance matrix $\Sigma$, and a $d \times d$ reflection matrix $R$ that specifies the boundary behavior. In the interior of the orthant, $Z(\cdot)$ behaves as an
ordinary Brownian motion with parameters $\theta$ and $\Sigma$ and, roughly speaking, $Z(\cdot)$ is pushed in direction $R^k$ whenever the boundary $\{z \in S : z_k = 0\}$ is hit, for $k = 1, \ldots, d$, where $R^k$ is the $k$th column of $R$. The process is Feller [16] and so is strong Markov.

A precise description for $Z(\cdot)$ is given by

\begin{equation}
Z(t) = Z(0) + B(t) + \theta t + RY(t), \quad t \geq 0,
\end{equation}

where $B(\cdot)$ is an unconstrained Brownian motion with covariance vector $\Sigma$ and no drift, with $B(0) = 0$, and $Y(\cdot)$ is a $d$-dimensional process with components $Y_1(\cdot), \ldots, Y_d(\cdot)$ such that

\begin{enumerate}
\item[(1.2)] $Y(\cdot)$ is continuous and nondecreasing, with $Y(0) = 0$,
\item[(1.3)] $Y_k(\cdot)$ only increases at times $t$ at which $Z_k(t) = 0$, \quad $k = 1, \ldots, d$,
\item[(1.4)] $Z(t) \in S$ for all $t \geq 0$.
\end{enumerate}

(Display (1.3) means that $Y_k(t_2) > Y_k(t_1)$, for $t_2 > t_1$, implies $Z_k(t) = 0$ at some $t \in [t_1, t_2]$.) For a SRBM with data $(\theta, \Sigma, R)$ to exist, it is necessary and sufficient that $R$ be completely-$S$. Completely-$S$ means that each principal submatrix $R'$ is an $S$-matrix, that is, for some $w \geq 0$, $R'w > 0$ holds. The complete definition and basic properties of $Z(\cdot)$ are reviewed in Appendix A of Bramson et al. [2].

A SRBM is said to be positive recurrent if the expected time to hit an arbitrary open neighborhood of the origin is finite for every starting state. A necessary and sufficient condition for positive recurrence, for $d = 2$, is that

\begin{equation}
R \text{ is nonsingular with } R^{-1} \theta < 0
\end{equation}

and that $R$ is a $P$-matrix (El Kharroubi et al. [7]). (That is, each principal submatrix of $R$ has a positive determinant.) Necessary and sufficient conditions, for $d = 3$, are known, but are more complicated. El Kharroubi et al. [8] gave sufficient conditions; Bramson et al. [2] showed these conditions are necessary. Another proof of the sufficiency of these conditions was recently given in Dai and Harrison [4]. In the special case where $R$ is an $M$-matrix, (1.5) is necessary and sufficient for positive recurrence in all $d$ (Harrison and Williams [11]); (1.5) is always necessary for positive recurrence (El Kharroubi [7]).

Associated with the parameters $\theta$ and $R$ are fluid paths, which are solutions of deterministic equations corresponding to (1.1)–(1.4). More precisely,
a fluid path is a pair of continuous functions \( y, z : [0, \infty) \to \mathbb{R}^d \) that satisfy

\[
\begin{align*}
\text{(1.6)} & \quad z(t) = z(0) + \theta t + Ry(t) \text{ for all } t \geq 0, \\
\text{(1.7)} & \quad y(\cdot) \text{ is continuous and nondecreasing, with } y(0) = 0, \\
\text{(1.8)} & \quad y_k(\cdot) \text{ only increases at times } t \text{ at which } z_k(t) = 0, \quad k = 1, \ldots, d, \\
\text{(1.9)} & \quad z(t) \in S \text{ for all } t \geq 0.
\end{align*}
\]

A fluid path \((y, z)\) is attracted to the origin if \(z(t) \to 0\) as \(t \to \infty\); it is divergent if \(|z(t)| \to \infty\) as \(t \to \infty\) (where \(|u| \overset{\text{def}}{=} \sum_i |u_i|\), for \(u = (u_i) \in \mathbb{R}^d\)).

The following result gives a sufficient condition for positive recurrence of an SRBM in terms of the associated fluid paths.

**Theorem 1.1 (Dupuis-Williams).** Let \(Z(\cdot)\) be a \(d\)-dimensional SRBM with data \((\theta, \Sigma, R)\). If every fluid path associated with \((\theta, R)\) is attracted to the origin, then \(Z(\cdot)\) is positive recurrent.

Theorem 1.1 provides an important ingredient for demonstrating the sufficiency of the conditions in [8] for positive recurrence of an SRBM, for \(d = 3\), that were alluded to above. An open question is whether a converse of Theorem 1.1 holds for \(d > 3\), that is, whether \(Z(\cdot)\) positive recurrent implies that every fluid path is attracted to the origin.

A fluid path \((y, z)\) is linear if \(y(t) = ut\) and \(z(t) = vt\) for given vectors \(u, v \geq 0\). \((u \geq 0\) means \(u_i \geq 0\) for \(i = 1, \ldots, d\).) When \(y(\cdot)\) and \(z(\cdot)\) are linear, the fluid path properties (1.6)-(1.9) can be expressed as solutions of the linear complementarity problem

\[
\begin{align*}
\text{(1.10)} & \quad u, v \geq 0, \quad v = \theta + Ru, \quad u \cdot v = 0,
\end{align*}
\]

where \(u \cdot v \overset{\text{def}}{=} \sum_i u_i v_i\). A solution \((u, v)\) of (1.10) is stable if \(v = 0\) and divergent otherwise. It is nondegenerate if \(u\) and \(v\) together have exactly \(d\) positive components, and it is degenerate otherwise. It is easy to see that, for a converse to Theorem 1.1 to hold, all linear fluid paths associated with a positive recurrent SRBM must be stable.

In this article, we provide a family of examples, in \(d = 6\), for which the SRBM is positive recurrent, yet possesses a divergent linear fluid path. We set

\[
\theta = (-1, -1, \ldots, -1)^T, \quad \Sigma = I
\]

(\(\text{where } "^T" \text{ denotes the transpose}\), and denote by \(R\) the \(6 \times 6\) matrix with

\[
\text{(1.12)} & \quad R = J_1 + J_2,
\]
where $J_1$ satisfies $(J_1)_{i,j} = 1$, for $i, j = 1, \ldots, 6$, and

$$J_2 = \begin{bmatrix}
0 & \delta_2 & \delta_2 & \delta_2 & -\delta_4 \\
0 & 0 & -\delta_3 & -\delta_3 & -\delta_3 \\
0 & -\delta_3 & 0 & -\delta_3 & -\delta_3 \\
0 & -\delta_3 & -\delta_3 & 0 & -\delta_3 \\
\delta_1 & -\delta_3 & -\delta_3 & -\delta_3 & 0
\end{bmatrix}. $$

(1.13)

Here, we assume that $\delta_i > 0$, $i = 1, \ldots, 4$, with

$$\delta_2 + \delta_3 \leq \frac{1}{6}\delta_4$$

(1.14) and

$$\delta_1 \leq \delta_3 \leq .1, \quad \delta_4 < 1.$$  

One can, for example, choose

$$\delta_1 = \delta_2 = \delta_3 = .05, \quad \delta_4 = .6.$$  

(1.16)

The matrix $R$ has been chosen so that $R_{i,i} = 1$ for $i = 1, \ldots, 6$. The roles of the coordinates $i = 2, \ldots, 5$ with respect to $R$ are indistinguishable, and the role of $i = 6$ differs from those of $i = 2, \ldots, 5$ only in its interaction with the coordinate $i = 1$ through $R_{1,6}$ and $R_{6,1}$. Since all entries of $R$ are positive, it is immediate that $R$ is completely-$S$. The role of the relations in (1.14)–(1.15) will be explained in the next subsection.

The main result in this article is the following theorem.

**Theorem 1.2.** Let $Z(\cdot)$ denote the SRBM with $\theta = (-1, -1, \ldots, -1)^T$, $\Sigma = I$ and $R$ satisfying (1.12)–(1.15). Then $Z(\cdot)$ is positive recurrent, but possesses a divergent linear fluid path.

One can check that $(u, v)$, with $u = e_1$ and $v = \delta_1 e_6$, defines a divergent linear fluid path ($e_i$ denotes the $i^{th}$ unit vector). Since $u$ and $v$ together have a total of two positive components, the fluid path is degenerate. (Related divergent fluid paths are easy to construct: for example, $(y, z)$ with $y(t) = e_1 t$ and $z(t) = \sum_{k=2}^{5} e_k + \delta_1 e_6 t$.) In order to demonstrate Theorem 1.2, it suffices to show $Z(\cdot)$ is positive recurrent.

Similar examples exist that satisfy the analog of Theorem 1.2, but with $d > 6$. One can construct such examples by inserting additional coordinates $Z_i(\cdot)$ that are independent of $Z_1(\cdot), \ldots, Z_6(\cdot)$, with $\theta_i = -1$ and $R_{i,i} = 1$.

In the remainder of the section, we summarize how the matrix $R$ affects the evolution of $Z(\cdot)$ and leads to its positive recurrence. We also outline the rest of the paper.
Sketch of positive recurrence. The reflection matrix $R$ that we have chosen has the following properties, which we will use in the next three paragraphs. For $\theta$ given by (1.11), all of the coordinates $Z_k(\cdot), k = 1, \ldots, 6,$ have drift $-1$, which is compensated for by $R$, which pushes a coordinate away from 0 whenever any of the coordinates is being reflected there. (Although the motion induced by $R$ is not absolutely continuous, we will also refer to it as “drift” here.) Because of the choice of $\theta$, for $k, k' = 2, \ldots, 5$,

$$Z_{k'}(\cdot) - Z_k(\cdot) \text{ has no drift}$$

except when one of the coordinates is being reflected; when the coordinate $k$ is reflected, the difference has negative drift because of the term $\delta_3$ in $J_2$. Also, for $k = 2, \ldots, 5$, $Z_6(\cdot) - Z_k(\cdot) \text{ has no drift} \text{ except when } Z_k(\cdot) \text{ is reflected, in which case the difference has negative drift, or when } Z_6(\cdot) \text{ or } Z_1(\cdot) \text{ is reflected, in which case it has positive drift, the last case occurring because of the term } \delta_1 \text{. On the other hand, when the first coordinate is being reflected, for } k = 2, \ldots, 5,$

$$Z_1(\cdot) - Z_k(\cdot) \text{ has no drift}$$

and, when one of the other four coordinates $k = 2, \ldots, 5$ is being reflected, the difference has positive drift because of the term $\delta_2$ in $J_2$. But, when $Z_6(\cdot)$ is reflected, the difference acquires a negative drift because of the term $\delta_4$ in $J_2$ and (1.14).

The process $Z(\cdot)$ is positive recurrent, although its deterministic analog $z(\cdot)$ possesses a divergent linear fluid path in the direction $e_6$ when $u = e_1$. This difference in behavior occurs due to the following interaction between the different coordinates of $Z(\cdot)$. When $Z_1(\cdot)$ is close to 0 (for instance, when $Z_k(\cdot), k = 2, \ldots, 5$, are larger), it may remain small for an extended period of time, with the other coordinates perhaps increasing. Nonetheless, as we will see, after a finite expected time, one of the coordinates $k, k = 2, \ldots, 5,$ will hit 0. Because of the reflections against 0 by this coordinate and perhaps by the other three coordinates, the coordinate $k = 1$ will acquire, on the average, a positive drift and therefore increase linearly. When this occurs, each of the coordinates $k = 2, \ldots, 5$ will drift towards 0 and afterwards remain close to 0.

The sixth coordinate increases linearly in time when the first coordinate undergoes repeated reflection. However, when the first coordinate is instead increasing, the sixth coordinate will drift back to 0 on account of the terms $(J_2)_{6,j} = -\delta_3, j = 2, \ldots, 5$. Moreover, on account of (1.14), the term $(J_2)_{1,6} = -\delta_4$ is sufficiently smaller than $-\delta_2$ so that, when the sixth
coordinate starts reflecting at 0, the negative drift induced in the first coordinate more than compensates for the positive drift induced in the first coordinate by the reflection of the other four coordinates. As a consequence, the first coordinate acquires a negative net drift. After this occurs, the coordinates \( k = 2, \ldots, 6 \) will all remain close to 0 until the first coordinate hits 0, in which case the behavior outlined above can repeat. This behavior prevents any of the coordinates from typically moving too far from 0, and will ensure that the system is positive recurrent.

The proof of Theorem 1.2 is organized as follows. In Section 2, we give a number of bounds on \( Y(\cdot) \) and \( Z(\cdot) \) that are derived by applying elementary Brownian motion estimates to (1.1). These bounds are employed in the rest of the paper. In Section 3, we demonstrate a version of Foster’s criterion that will be used here. We also recall and then employ the main result in Ratzkin and Treibergs [15], which states that for a Brownian pursuit problem, the presence of four “predators” is enough for them to capture the “prey” in finite expected time. In our context, \( Z_k(\cdot), k = 2, \ldots, 5 \), will play the role of the predators and \( Z_1(\cdot) \) will play the role of the prey. This behavior will justify the claim in the above discussion that one of the coordinates with \( k = 2, \ldots, 5 \) will hit 0 after a finite expected time.

In Section 4, we state the main steps in the proof of Theorem 1.2 in the form of a series of five propositions, and show how the theorem follows from them. Depending on whether or not \( Y_1(\cdot) \) is initially growing quickly, Proposition 4.1 states that, during this time, either the coordinates \( Z_2(\cdot), \ldots, Z_6(\cdot) \) decrease by an appropriate factor or \( Z_6(\cdot) \) increases linearly. In the first case, it follows from Proposition 4.2 that \( Z_1(\cdot) \) will also remain small and so, as desired, the norm of the SRBM decreases by a factor over the time interval. In the second case, the argument proceeds along the lines sketched above in the comparison of \( Z(\cdot) \) with the divergent fluid path, and employs Propositions 4.3, 4.4 and 4.5.

In Section 5, we demonstrate Propositions 4.1 and 4.2 and, in Section 6, we demonstrate Propositions 4.3, 4.4 and 4.5. The reasoning employs the interaction of the different components \( Z_k(\cdot), k = 1, \ldots, 6 \), and draws from the different bounds in Sections 2 and 3.

2. Basic estimates. In this section, we give a number of elementary bounds that will be used in the remainder of the article. In Lemma 2.1, we give bounds on standard one dimensional Brownian motion \( B(\cdot) \). (All of the bounds in the lemma hold in greater generality; see, e.g., [12], page 59, and [13].) These bounds will then be applied in the rest of the section to obtain bounds on the quantities \( Y(\cdot) \) and \( Z(\cdot) \) in (1.1), the equation describing the
evolution of SRBM. Here and elsewhere in the paper, the notation $C_1, C_2, \ldots$ will be employed for positive constants whose precise value is not of interest to us, with the same symbol often being reused.

**Lemma 2.1.** Let $B(\cdot)$ denote a standard Brownian motion. Then, for each $t \geq 0$,

\[ E \left[ \max_{0 \leq s \leq s' \leq t} (B(s') - B(s))^2 \right] \leq 8t. \tag{2.1} \]

For given $\epsilon > 0$, there exist $C_1, \epsilon' > 0$ such that, for each $t \geq 0$,

\[ P \left( \max_{0 \leq s \leq t} |B(s)| \geq \epsilon t \right) \leq C_1 e^{-\epsilon' t}. \tag{2.2} \]

For given $\epsilon > 0$, there exist $C_1, \epsilon' > 0$ such that, for each $u \geq 0$,

\[ P \left( \inf_{i \geq 0} (\epsilon t + u - |B(t)|) \leq 0 \right) \leq C_1 e^{-\epsilon' u}, \tag{2.3} \]

and, for each $u > 0$ and $t \geq 0$,

\[ P \left( \min_{0 \leq s \leq s' \leq t} (\epsilon(s' - s) + u - |B(s') - B(s)|) \leq 0 \right) \leq C_1 (t + 1) e^{-\epsilon' u}. \tag{2.4} \]

**Proof.** Since $(B(s') - B(s))^2 \leq 2(B(s')^2 + B(s)^2)$, it follows from the Reflection Principle that the left side of (2.1) is at most

\[ 4E \left[ \max_{0 \leq s \leq t} B(s)^2 \right] \leq 8E [B(t)^2] = 8t. \]

The bound (2.2) follows by applying the Reflection Principle to both $B(\cdot)$ and $-B(\cdot)$.

Again applying the Reflection Principle to $B(\cdot)$ and $-B(\cdot)$, it follows that, for given $\epsilon > 0$,

\[ P \left( \frac{1}{2}(\epsilon t' + u) - \max_{0 \leq s \leq t'} |B(s)| \leq 0 \right) \leq 4P \left( \frac{1}{2}(\epsilon t' + u) - |B(t')| \leq 0 \right) \]
\[ \leq C_2 \exp\left( -(\epsilon t' + u)^2 / 8t' \right) \leq C_2 e^{-\frac{1}{8} \epsilon u}, \]

where $C_2$ does not depend on $t'$ or $u$. Setting $t' = 2^i$, $i = 0, 1, 2, \ldots$, one obtains bounds whose exceptional probabilities sum to at most $C_1 e^{-\epsilon' u}$, for $\epsilon' = \frac{1}{8} \epsilon$ and appropriate $C_1$. The bound in (2.3) follows quickly from this.
It follows from (2.3) that, for each $i = 0, 1, 2, \ldots$,

$$P\left( \inf_{s' \geq i} (\epsilon (s' - i) + \frac{1}{2} u - |B(s') - B(i)|) \leq 0 \right) \leq C_1 e^{-\frac{1}{2} \epsilon' u}.$$  

Using the Reflection Principle, it is easy to check that, for appropriate $C_3$, $\epsilon'' > 0$ and all $u \geq 0$,

$$P\left( \max_{0 \leq s \leq 1} |B(i + s) - B(i)| \geq \frac{1}{2} u - \epsilon \right) \leq C_3 e^{-\epsilon'' u}.$$  

Together with (2.5), this implies

$$P\left( \inf_{s \in [i, i+1), s' \geq s} (\epsilon (s' - s) + u - |B(s') - B(s)|) \leq 0 \right) \leq C_1 e^{-\epsilon' u}$$

for new choices of $C_1$ and $\epsilon'$. Summing over $i < t$ gives the bounds in (2.4). \hfill \Box

The next lemma provides elementary upper and lower bounds on $Y_k(\cdot)$.

**Lemma 2.2.** For each $t \geq 0$ and $\ell = 2, \ldots, 5$, 

(2.6) \[ \sum_{k=1}^{6} Y_k(t) \geq t - Z_\ell(0) - B_\ell(t) \]

and, for each $\ell = 1, \ldots, 6$, 

(2.7) \[ \sum_{k=1}^{6} Y_k(t) \geq \frac{1}{2} (t - Z_\ell(0) - B_\ell(t)). \]

For each $t \geq 0$ and $k = 1, \ldots, 6$, 

(2.8) \[ Y_k(t) \leq t + \max_{0 \leq s \leq t} (-B_k(s)) \]

and, for a given $\epsilon \geq 0$, there exist $C_1$ and $\epsilon' > 0$ so that 

(2.9) \[ P(Y_k(t) \geq (1 + \epsilon) t) \leq C_1 e^{-\epsilon' t}. \]

**Proof.** Since $\delta_3 \geq 0$, it follows from (1.1) that, for $\ell = 2, \ldots, 5$, 

(2.10) \[ Z_\ell(t) \leq Z_\ell(0) + B_\ell(t) - t + \sum_{k=1}^{6} Y_k(t), \]
from which (2.6) immediately follows. Since each of the entries of $J_2$ in (1.13) is less than 1, the analog of (2.10) holds for $\ell = 1, \ldots, 6$, but with the term $2 \sum_{k=1}^{6} Y_k(t)$. This implies (2.7).

Let $\tau$ denote the time in $[0, t]$ at which $Y_k(t)$ is first attained, for given $k$. It follows from (1.1) that

$$Y_k(t) \leq \tau - B_k(\tau) \leq t + \max_{0 \leq s \leq t} (-B_k(s)),$$

which implies (2.8). The bound (2.9) follows from (2.8) and (2.2). \qed

We next obtain a number of upper bounds on $Z_k(\cdot)$. The following lemma is elementary.

**Lemma 2.3.** Let $B(\cdot)$ denote a standard Brownian motion. For each $k$, $t$ and $x$,

$$P \left( \max_{0 \leq s \leq t} Z_k(s) - Z_k(0) \geq 7t + x \right) \leq 16 P(B(t) \geq x).$$

Consequently, for all $t$, and appropriate $C_1$ and $\epsilon' > 0$,

$$P \left( \max_{0 \leq s \leq t} Z_k(s) - Z_k(0) \geq 8t \right) \leq C_1 e^{-\epsilon' t}.$$

**Proof.** It follows from (1.1) that, since all entries for $J_2$ in (1.13) are at most $\frac{1}{6}$,\n
$$\max_{0 \leq s \leq t} Z_k(s) - Z_k(0) \leq \max_{0 \leq s \leq t} B_k(s) + \frac{7}{6} \sum_{\ell=1}^{6} Y_\ell(t).$$

By (2.8) of Lemma 2.2, this is at most

$$7t + \frac{7}{6} \sum_{\ell=1}^{6} \max_{0 \leq s \leq t} (-B_\ell(s)) + \max_{0 \leq s \leq t} B_k(s).$$

The inequality in (2.11) follows from this and the Reflection Principle. The inequality in (2.12) is an immediate consequence of (2.11). \qed

The following lemma requires a bit more work. Here, we employ the notation $N_k(t), k = 1, \ldots, 6$, with $N_6(t) = Y_1(t)$ and $N_k(t) = 0$ for $k \neq 6; x_+$ denotes the positive part of $x \in \mathbb{R}$. 
Lemma 2.4. For each $k$, $k = 2, \ldots, 6$, $t \geq 0$ and $x$,
\begin{equation}
(2.13) \quad P\left( \max_{0 \leq s \leq t} Z_k(s) - Z_k(0) - \delta_1 N_k(t) \geq x \right) \leq 24 P(B(t) \geq \frac{1}{4}x),
\end{equation}
where $B(\cdot)$ is standard Brownian motion. Consequently, for given $\epsilon > 0$, there exist $C_1, \epsilon' > 0$ such that, for each $t \geq 0$,
\begin{equation}
(2.14) \quad P\left( \max_{0 \leq s \leq t} Z_k(s) - Z_k(0) - \delta_1 N_k(t) \geq \epsilon t \right) \leq C_1 e^{-\epsilon' t}.
\end{equation}
Also, for $k = 2, \ldots, 5$,
\begin{equation}
(2.15) \quad E \left[ \left( \max_{0 \leq s \leq t} Z_k(t) - Z_k(0) \right)^2 \right] \leq 24 \cdot 16t.
\end{equation}

Proof. Let $\tau_k$ denote the last time $r, r \leq s$, at which $Z_k(r) = 0$; if the set is empty, let $\tau_k = 0$. Let $\tau$ denote the last time $r, r \leq s$, at which $Z_k(r) = 0$ for any $\ell = 2, \ldots, 6$; denote this coordinate by $L$. If the set is empty, set $\tau = 0$. We also abbreviate by setting $B_k(r_1, r_2) = B_k(r_2) - B_k(r_1)$ and $N_k(r_1, r_2) = N_k(r_2) - N_k(r_1)$.

We claim that for given $k, k = 2, \ldots, 6$,
\begin{equation}
(2.16) \quad Z_k(\tau) - Z_k(0) \leq B_k(\tau_k, \tau) - B_L(\tau_k, \tau) + \delta_1 N_k(\tau).
\end{equation}
To see this, note that subtraction of the equations for the $k^{th}$ and $L^{th}$ coordinates of (1.1) implies
\begin{align*}
Z_k(\tau) - Z_k(\tau_k) &= Z_L(\tau) - Z_L(\tau_k) + B_k(\tau_k, \tau) - B_L(\tau_k, \tau) \\
&\quad + \delta_1 N_k(\tau_k, \tau) - \delta_1 N_L(\tau_k, \tau) - \delta_1 (Y_L(\tau) - Y_L(\tau_k)) \\
&\leq B_k(\tau_k, \tau) - B_L(\tau_k, \tau) + \delta_1 N_k(\tau).
\end{align*}
When $\tau_k > 0$, $Z_k(\tau_k) = 0$ holds, and so (2.16) follows.

Let $\tau' = \tau \vee \tau_1$. Note that, when $\tau \neq \tau' > 0$,
\begin{equation*}
Z_1(\tau') - Z_1(\tau) = -Z_1(\tau) \leq 0.
\end{equation*}
Also, since $Z_0(r) > 0$ for $r \in (\tau, \tau']$, it follows from the definition of $J_2$ that
\begin{equation*}
(R(Y(\tau') - Y(\tau)))_k \leq (R(Y(\tau') - Y(\tau)))_1.
\end{equation*}
Subtraction of the $k^{th}$ and $1^{st}$ coordinates of (1.1), together with these two inequalities, implies that
\begin{equation}
(2.17) \quad Z_k(\tau') - Z_k(\tau) \leq B_k(\tau, \tau') - B_1(\tau, \tau') + \delta_1 N_k(\tau, \tau').
\end{equation}
It is easy to see that

\[ Z_k(s) - Z_k(\tau') \leq B_k(\tau', s). \]  

Combining (2.16), (2.17) and (2.18) implies

\[ Z_k(s) - Z_k(0) \leq B_k(\tau_k, s) - B_L(\tau_k, \tau) - B_1(\tau, \tau') + \delta_1 N_k(t). \]

One therefore obtains, for all \( x \), that

\[ P \left( \max_{0 \leq s \leq t} Z_k(s) - Z_k(0) - \delta_1 N_k(t) \geq x \right) \leq 6P \left( \max_{0 \leq s \leq s' \leq t} (B(s') - B(s)) \geq \frac{1}{2}x \right). \]

It follows from the Reflection Principle that the right side of (2.20) is at most \( 24P(B(t) \geq \frac{1}{4}x) \), which implies (2.13).

The inequalities in (2.14) and (2.15) follow directly from (2.13).

We will employ (2.13) to show (2.21) of the following lemma. On account of the thin tail of \( \max_{0 \leq s \leq t} Z_k(s) \), restricting its expectation to a set \( F \) decreases the expectation proportionally to \( P(F) \), except for a logarithmic factor; a similar statement holds for the second moment. The lemma will be important for our calculations later in the article.

**Lemma 2.5.** For an appropriate constant \( C_1 \), all \( t \geq 0 \) and all measurable sets \( F \) with \( P(F) > 0 \),

\[ (2.21) \quad E \left[ \max_{0 \leq s \leq t} Z_k(s)^2; F \right] \leq C_1 P(F)(t \log(e/P(F)) + M) \]

for \( k = 2, \ldots, 5 \), when \( Z_k(0) \leq \sqrt{M} \), and

\[ (2.22) \quad E \left[ \max_{0 \leq s \leq t} Z_k(s); F \right] \leq C_1 P(F)(\sqrt{t} \log(e/P(F)) + t + M) \]

for \( k = 1, 2, \ldots, 6 \), when \( Z_k(0) \leq M \).

**Proof.** On account of (2.13), we can construct a standard normal random variable \( W \) on the probability space so that, for \( k = 2, \ldots, 5 \),

\[ (2.23) \quad E \left[ \max_{0 \leq s \leq t} ((Z_k(s) - Z_k(0))_+)^2; F \right] \leq C_2 t E[W^2; F], \]
where $C_2 = 24 \cdot 16$. (The inequality follows by integrating by parts and employing $E[(4B(t))^2] = 16t$.) Choosing $a$ so that $P(W^2 \geq a) = P(F)$, the right side of (2.23) is at most

$$C_2 t \left( aP(F) + \int_a^\infty P(W^2 \geq x) \, dx \right).$$

(2.24)

The random variable $W^2$ has an exponentially tight tail in the sense that, for appropriate $C_3, C_4 > 0$ and all $y, x$ with $0 \leq y \leq x$, (2.25)

$$P(W^2 \geq x) \leq C_3 e^{-C_4(x-y)} P(W^2 \geq y).$$

Setting $y = 0$ and $x = a$, this implies $a \leq \frac{1}{C_4} \log(C_3/P(F)).$ Application of (2.25) with $y = a$ therefore implies (2.24) is at most

$$C_2 t (aP(F)+C_3/C_4 P(W^2 \geq a)) \leq (C_2/C_4)t P(F)(\log(1/P(F)) + C_3 + \log C_3).$$

So, for appropriate $C_5$,

$$E \left[ \max_{0 \leq s \leq t} \left( (Z_k(s) - Z_k(0))^2 \right)^{+}; F \right] \leq C_5 t P(F) \log(e/P(F)).$$

(2.26)

For $Z_k(0) \leq \sqrt{M}$, (2.21) follows from this by considering the complementary events $\{ \max_{s \leq t} Z_k(s) > 2\sqrt{M} \}$ and $\{ \max_{s \leq t} Z_k(s) \leq 2\sqrt{M} \}$, and noting that, on the former,

$$\max_{0 \leq s \leq t} Z_k(s)^2 \leq 4 \max_{0 \leq s \leq t} (Z_k(s) - Z_k(0))^2$$

and, on the latter, $\max_{s \leq t} Z_k(s)^2 \leq 4M$.

In order to show (2.22), we note that, for $k = 1, \ldots, 6$, it follows from (1.1) and (2.8) that

$$Z_k(s) - Z_k(0) \leq B_k(s) - s + \frac{7}{6} \sum_{\ell=1}^6 Y_\ell(s)$$

(2.27)

$$\leq 6s + B_k(s) + \frac{7}{6} \sum_{\ell=1}^6 \max_{0 \leq r \leq s} (-B_\ell(r)).$$

This, together with the Reflection Principle, implies that

$$P \left( \max_{0 \leq s \leq t} Z_k(s) - Z_k(0) - 6t \geq x \right) \leq 7P \left( \max_{0 \leq s \leq t} B(s) \geq \frac{1}{8}x \right) \leq 14P(B(t) \geq \frac{1}{8}x),$$

(2.28)
where $B(\cdot)$ is standard Brownian motion.

Reasoning as in the first part of the proof, we can construct a standard normal random variable $W$ so that

$$E \left[ \max_{0 \leq s \leq t} Z_k(s) - Z_k(0) - 6t; \ F \right] \leq C_2 \sqrt{t} E[W; \ F],$$

where $C_2 = 14 \cdot 8$. Since $W$ has an exponentially tight tail, we can reason as through (2.26) to show that

$$E \left[ \max_{0 \leq s \leq t} Z_k(s) - Z_k(0) - 6t; \ F \right] \leq C_5 \sqrt{t} P(F) \log(e/P(F))$$

for appropriate $C_5$. This implies (2.22) for $Z_k(0) \leq M$ and appropriate $C_1$. \hfill \Box

We now apply Lemma 2.4 to obtain sharper bounds on $Y_k(\cdot)$, with $k = 2, \ldots, 6$, than those in Lemma 2.2, provided bounds on $Y_1(\cdot)$ are given.

**LEMMA 2.6.** For given $\epsilon > 0$, there exist $C_1$ and $\epsilon' > 0$ such that, for all $t \geq 0$ and $k = 2, \ldots, 6$,

$$P \left( Y_k(t) + (1 - \delta_3) \sum_{\ell=2, \ell \neq k}^6 Y_\ell(t) \geq (1 + \epsilon)t + \delta_1 N_k(t) \right) \leq C_1 e^{-\epsilon' t}. $$

There exist $C_1$ and $\epsilon' > 0$ such that, for all $t \geq 0$ and $k = 2, \ldots, 6$,

$$P \left( Y_k(t) \leq \frac{1}{5} t - \frac{1}{5} (Z_k(0) + 2Y_1(t)) \right) \leq C_1 e^{-\epsilon' t}. $$

**PROOF.** It follows from (1.1) that

$$Z_k(t) - Z_k(0) \geq B_k(t) - t + Y_k(t) + (1 - \delta_3) \sum_{\ell=2, \ell \neq k}^6 Y_\ell(t)$$

for $k = 2, \ldots, 6$. Together with (2.14) of Lemma 2.4, (2.33) implies (2.31).

Summing the arguments inside $P(\cdot)$ in (2.31), over $\ell = 2, \ldots, 6$, gives

$$P \left( 1 - \frac{4}{5} \delta_3 \sum_{\ell=2}^6 Y_\ell(t) \geq (1 + \epsilon)t + \delta_1 Y_1(t) \right) \leq 5C_1 e^{-\epsilon' t}. $$

Since $\delta_3 \leq 1/10$,

$$\frac{1 - \delta_3}{1 - \frac{4}{5} \delta_3} \leq 1 - \frac{1}{5} \delta_3 - \frac{1}{10} \delta_3^2,$$

$$\frac{1}{1 - \delta_3} \leq 1 + \delta_3 + \delta_3^2,$$

$$\frac{1}{1 - \delta_3} \leq 1 + \delta_3 + \delta_3^2.$$
which implies that, for small enough $\epsilon$,

$$(2.35) \quad P \left( (1 - \delta_3) \sum_{\ell=2}^{6} Y_{\ell}(t) - t \geq -\frac{1}{3}(1 + \frac{1}{2}\delta_3)\delta_3 t + \delta_1 Y_1(t) \right) \leq 5C_1 e^{-\epsilon t}.$$ 

By (1.1), one has, for $k = 2, \ldots, 6$,

$$Z_k(t) - Z_k(0) \leq B_k(t) - t + \delta_3 Y_k(t) + (1 - \delta_3) \sum_{\ell=2}^{6} Y_{\ell}(t) + (1 + \delta_1) Y_1(t).$$

Off of the exceptional set in (2.35), this is at most

$$-\frac{1}{10}(1 + \frac{1}{2}\delta_3)\delta_3 t + B_k(t) + \delta_3 Y_k(t) + 2Y_1(t).$$

Solving for $Y_k(t)$, together with the obvious exponential bound on $B_k(t)$, produces (2.32) for a new choice of $C_1$ and $\epsilon'$. \hfill $\square$

In Lemma 2.4, we gave upper bounds on $Z_k(\cdot)$ for $k = 2, \ldots, 6$. Here, we employ (2.31) and (2.32) of Lemma 2.6 to obtain an upper bound on $Z_1(\cdot)$. The bound implies in particular that, for large $t$, $Y_1(t) > 0$ and hence $Z_1(s) = 0$ at some $s \leq t$.

**Lemma 2.7.** For given $\epsilon > 0$, there exist $C_1$ and $\epsilon' > 0$ such that, for each $t \geq 0$,

$$(2.36) \quad P \left( Z_1(t) - Z_1(0) - \frac{3}{\delta_3}(Y_1(t) + Z_6(0)) \geq -\frac{1}{30}\delta_4 t \right) \leq C_1 e^{-\epsilon' t}.$$ 

**Proof.** On account of (1.1),

$$(2.37) \quad Z_1(t) - Z_1(0) \leq B_1(t) - t + Y_1(t) + (1 + \delta_2) \sum_{k=2}^{6} Y_k(t) - (\delta_2 + \delta_4) Y_6(t).$$

One bounds $(1 + \delta_2) \sum_{k=2}^{6} Y_k(t)$ by employing (2.31) after summing over $\ell = 2, \ldots, 6$, and one bounds $(\delta_2 + \delta_4) Y_6(t)$ by employing (2.32). It then follows with a little algebra that the right side of (2.37) is at most

$$(2.38) \quad \left( \frac{\delta_3}{10} \delta_2 + \delta_3 - \frac{1}{5} \delta_4 \right) t + \frac{3}{\delta_3} (Y_1(t) + Z_6(0))$$

$$\leq -\frac{1}{30} \delta_4 t + \frac{3}{\delta_3} (Y_1(t) + Z_6(0))$$

off of a set of probability $C_1 e^{-\epsilon' t}$, for appropriate $C_1$ and $\epsilon' > 0$. For the bound on the left side of (2.38), one employs the bounds on $\delta_i$ in (1.14).
and (1.15), together with (2.31), (2.32) and an analog of (2.34). For the inequality in (2.38), one uses $\delta_2 + \delta_3 \leq \frac{1}{6} \delta_4$. It follows from (2.37) and (2.38) that, off of the exceptional set,

$$Z_1(t) - Z_1(0) \leq - \frac{1}{30} \delta_1 t + \frac{2}{3} \delta_3 (Y_1(t) + Z_0(0)),$$

which implies (2.36).

3. A Brownian pursuit model and Foster’s criterion. In this section, we first discuss a Brownian pursuit model, which was mentioned briefly at the end of Section 1. Using a result of Ratzkin and Treibergs [15], it is employed to show that the expected time for at least one of the coordinates $Z_k(\cdot)$, $k = 2, \ldots, 5$, to hit 0 is finite. In Proposition 3.2, we apply this result to obtain a lower bound on $\sum_{k=2}^5 Y_k(\cdot)$ that will be used later in the paper. We then show an appropriate version of Foster’s criterion. Foster’s criterion is a tool for showing the positive recurrence of a Markov process. Since the stopping times we will employ are random, we need a variant of the standard version.

A Brownian pursuit model. The pursuit model consists of $n$ standard 1-dimensional Brownian motions, $X_k(\cdot)$, $k = 2, \ldots, n + 1$, that “pursue” another Brownian motion $X_1(\cdot)$. The $n$ Brownian motions are referred to as predators and the other Brownian motion as the prey. The prey will be said to be captured at time $t$ if $t$ is the first time at which $X_1(t) = X_k(t)$ for some $k = 2, \ldots, n + 1$. All Brownian motions are assumed to move independently.

One wishes to know whether the expected time for capture is finite or infinite. When there are initially predators on each side of the prey, one can show that the expected capture time is finite. When all of the predators are on one side of the prey, the expected capture time is infinite for $n \leq 3$ and finite for $n \geq 4$. This and a number of related problems were considered in Bramson and Griffeath [3] in the context of simple symmetric random walk. There, the behavior for $n \leq 3$ was demonstrated and simulations were given that suggested the behavior for $n \geq 4$. Li and Shao [14] showed finite expected capture time for Brownian motion for $n \geq 5$ and Ratzkin and Treibergs [15] more recently showed this for $n = 4$.

Ratzkin and Treibergs [15] showed finite expected capture time by bounding the tail of the capture time $T$. Their result can be formulated as follows:

**Theorem 3.1.** For any initial state where all four of the predators are within distance 1 and to the right of the prey,

$$P(T > t) \leq C_1 t^{-(1+n)}$$

(3.1)
for appropriate $C_1$ and all $t \geq 0$, where $\eta = .000073$. Consequently,

\begin{equation}
E[T] < \infty.
\end{equation}

The analogous result for $n = 5$ is less delicate, which [15] showed with $\eta = .0634$. The reasoning in both [14] and [15] relies on rephrasing the pursuit model in terms of an eigenvalue problem for the departure time of an $n$-dimensional Brownian motion from an appropriate generalized cone. This type of problem was also studied in DeBlassie [5]. (See [14] for additional references.)

We will employ both (3.1) and (3.2) for Proposition 3.2. (The first inequality is not needed, but applying it makes one of the steps more explicit.) We note that, by (1.1), for $k = 2, \ldots, 5$ and all $t \geq 0$,

\begin{equation}
Z_k(t) - Z_1(t) \leq (Z_k(0) - Z_1(0)) + (B_k(t) - B_1(t)) - \delta_2 \sum_{k=2}^{5} Y_k(t) + (\delta_4 - \delta_3) Y_6(t).
\end{equation}

When $Y_6(t) = 0$, this implies

\begin{equation}
Z_k(t) - Z_1(t) \leq (Z_k(0) - Z_1(0)) + (B_k(t) - B_1(t)).
\end{equation}

Set

\begin{align}
T_1(x) &= \min\{t : Z_1(t) - Z_1(0) \geq x\}, \\
T_6 &= \min\{t : Z_6(t) = 0\}.
\end{align}

By employing Theorem 3.1 and (3.4), it is easy to show the following proposition.

**Proposition 3.1.** Suppose that for a given $x \geq 0$, $\max_{k=2,\ldots,5} Z_k(0) \leq x$. Then, for $\eta = .000073$ and an appropriate constant $C_1$ not depending on $x$,

\begin{equation}
P(T_1(x) \land T_6 \geq x^2 t) \leq C_1 t^{-(1+\eta)}
\end{equation}

for all $t \geq 0$. Consequently, for appropriate $C_2$ not depending on $x$,

\begin{equation}
E[T_1(x) \land T_6] < C_2 x^2.
\end{equation}

**Proof.** By scaling space and time by $2x$ and $4x^2$, respectively, it follows from (3.1) of Theorem 3.1 that

\begin{equation}
P\left(B_1(s) - \min_{2 \leq k \leq 5} B_k(s) < 2x \text{ for all } s \leq x^2 t\right) \leq C_1 t^{-(1+\eta)}
\end{equation}
for a new choice of $C_1$. On account of (3.4) and the bounds on $Z_k(0)$, $k = 2, \ldots, 5$, this implies that

$$P(Z_1(s) - Z_1(0) < x \text{ for all } s \leq x^2t; T_6 \geq x^2t)$$

$$\leq P \left( Z_1(s) - Z_1(0) - \min_{2 \leq k \leq 5} Z_k(s) < x \text{ for all } s \leq x^2t; T_6 \geq x^2t \right)$$

$$\leq C_1 t^{-(1+\eta)}.$$  

The inequality in (3.7) follows immediately.  

**Application of Proposition 3.1.** We define the stopping times

$$T_2(x) = \min \left\{ t : \sum_{k=2}^{5} Y_k(t) = \frac{1}{2} (t + x^2) \right\} \wedge T_6 \wedge 5x^{5/\eta},$$

where $\eta$ is as in Theorem 3.1. In Sections 4-6, we will require upper bounds on $E[T_2(x)]$ in order to ensure the linear growth of $Z_1(\cdot)$ mentioned at the end of Section 1. Here, we employ Proposition 3.1 to obtain the following bounds.

**Proposition 3.2.** Suppose that $\max_{k=2, \ldots, 5} Z_k(0) \leq x$, with $x \geq 2$. Then, for appropriate $C_1$ not depending on $x$,

$$E[T_2(x)] \leq C_1 x^2.$$  

In Sections 4-6, we will also require upper bounds on $P(A)$, where

$$A = \{ \omega : T_2(x) = 5x^{5/\eta} \}.$$  

These bounds are obtained in Proposition 3.3, which we state shortly.

In order to demonstrate Propositions 3.2 and 3.3, we need to rule out certain behavior of $Z(\cdot)$ except on sets of small probability. For this, we introduce the following notation. Let $S_1(x)$ denote the last time $t$ before $T_1(x)$ at which $Z_1(t) - Z_1(0) = \frac{1}{2} x$, for given $x$. Set

$$\tau = \min \left\{ t : \min_{2 \leq k \leq 5} Z_k(t) = 0 \text{ for } t \geq S_1(x) \right\}.$$  

(If $\tau$ does not occur, set $\tau = \infty$.) Neither $S_1(x)$ nor $\tau$ is a stopping time. We also set

$$t_e = x^{5/\eta}, \quad t_f = 5x^{5/\eta} \quad \text{and} \quad T_1(x) = 4(T_1(x) + x^2).$$
Using this notation, we define:

\[(3.14)\] \(A_1 = \{\omega : T_1(x) \wedge T_6 > t_e\}\),

\[(3.15)\] \(A_2 = \{\omega : T_1(x) \leq T_6 \wedge \tau \wedge t_e\}\),

\[(3.16)\] \(A_3 = \{\omega : \tau < T_1(x) \leq T_e, T_6 > T'_1(x)\}\),

\[(3.17)\] \(A_4 = \left\{\omega : \sum_{k=2}^{5} Y_k(T'_1(x)) < \frac{1}{2}(T'_1(x) + x^2)\right\}\),

\[(3.18)\] \(A_5 = \{\omega : T_6 > T'_1(x)\}\).

One can check that

\[(3.19)\] \(A_5 \subseteq A_1 \cup A_2 \cup A_3\).

Also, note that

\[(3.20)\] \(T'_2(x) \leq T'_1(x) \wedge T_6 \quad \text{on } A'_5\).

Using this notation, it is not difficult to show the following lemma.

**Lemma 3.1.** For \(A\) as in (3.11),

\[(3.21)\] \(A \subseteq A' \overset{\text{def}}{=} A_1 \cup A_2 \cup (A_3 \cap A_4)\).

**Proof.** Set

\[T'_2(x) = \min \left\{ t : \sum_{k=2}^{5} Y_k(t) = \frac{1}{2}(t + x^2) \right\} \wedge T_6 \]

and \(A_6 = \{\omega : T'_2(x) \geq t_f\}\). It suffices to show that \(A_6 \subseteq A'\).

Since \(T'_1(x) < t_f\) on \(A_3\),

\[A_3 \cap A_6 \subseteq A_3 \cap A_4.\]

Consequently, by (3.19) and the definition of \(A'\),

\[(3.22)\] \(A_5 \cap A_6 \subseteq A_1 \cup A_2 \cup (A_3 \cap A_6) \subseteq A'\).

On the other hand,

\[A'_5 \cap A_6 \subseteq A_1 \subseteq A'.\]

Together with (3.22), this implies \(A_6 \subseteq A'\), as desired. \(\square\)
The bounds on $P(A')$ in Proposition 3.3 will be applied in the proof of Proposition 3.2 and the bounds on $P(A)$ will be applied in the proof of Proposition 4.4. Proposition 3.1, Lemma 2.1 and (1.1) are the main tools in the proof of Proposition 3.3.

**Proposition 3.3.** Suppose that $\max_{k=2, \ldots, 5} Z_k(0) \leq x$, with $x \geq 1$. Then, for an appropriate $C_1$ not depending on $x$,

$$P(A) \leq P(A') \leq C_1 x^{-\frac{\delta}{\eta} - 2}.$$  

**Proof.** In addition to $A_1, A_2, A_3$ and $A_4$, we employ the set

$$A_7 = \{ \omega : Z_1(s) = 0 \text{ for some } s \in [\tau, T_1'(x)] \}.$$  

We proceed to obtain upper bounds on each of $P(A_1), P(A_2), P(A_3 \cap A_7)$ and $P(A_3 \cap A_4 \cap A_7^c)$. We first note that, by applying (3.7) of Proposition 3.1, with $t = x^{\frac{\delta}{\eta} - 2}$,

$$P(A_1) \leq C_2 x^{-\frac{\delta}{\eta} - 2}$$  

for appropriate $C_2$.

In order to bound $P(A_2)$, we need to show that, over $[S_1(x), t_e], T_1(x)$ typically will not occur before $T_6 \wedge \tau$ occurs; on this set, $Z_1(\cdot)$ will drift toward 0 and away from $x$. First note, by (1.1), that when $T_1(x) \leq T_6 \wedge \tau$,

$$Z_1(T_1(x)) - Z_1(S_1(x)) = (B_1(T_1(x)) - B_1(S_1(x))) - (T_1(x) - S_1(x)).$$

It then follows from the definitions of $T_1(x)$ and $S_1(x)$ that

$$B_1(T_1(x)) - B_1(S_1(x)) = T_1(x) - S_1(x) + \frac{1}{2} x.$$  

But, by (2.4) of Lemma 2.1, the probability of (3.26) occurring when $T_1(x) \leq t_e$ is at most

$$C_2(t_e + 1)e^{-\epsilon' x} \leq C_3 e^{-\frac{1}{2}\epsilon' x}$$

for appropriate $C_2, C_3$ and $\epsilon' > 0$. Consequently,

$$P(A_2) \leq C_3 e^{-\frac{1}{2}\epsilon' x}.$$  

We next show that $P(A_3 \cap A_7)$ is small. This event will typically not occur because the coordinates $k = 2, \ldots, 5$ that are reflecting at 0 after $\tau$ will impart a positive drift to $Z_1(\cdot)$. Restricting our attention to the event
A3, let K be the index at which $Z_K(\tau) = 0$. Also, let $\tau'$ be any random time with

$$\tau \leq \tau' \leq T'_1(x) \land \min\{s > \tau : Z_1(s) = 0\}. \tag{3.28}$$

Since $\tau' \leq T'_1(x) < T_6$, it follows from (1.1) that

$$\sum_{k=2}^{5} (Y_k(\tau') - Y_k(\tau)) \geq (\tau' - \tau) - (B_K(\tau') - B_K(\tau)). \tag{3.29}$$

Applying (1.1) for the first coordinate and then substituting in (3.29), one obtains

$$Z_1(\tau') - Z_1(\tau) \geq \tilde{B}(\tau') - \tilde{B}(\tau) + \delta_2(\tau' - \tau), \tag{3.30}$$

where $\tilde{B}(t) \overset{\text{def}}{=} B_1(t) - (1 + \delta_2)B_K(t)$. 

Again applying (2.4) of Lemma 2.1, one has

$$P\left(\tilde{B}(\tau') - \tilde{B}(\tau) + \delta_2(\tau' - \tau) \leq -\frac{1}{2}x\right) \leq C_2(T'_1(x) + 1)e^{-\epsilon'x} \leq C_3 e^{-\frac{1}{2}\epsilon'x}, \tag{3.31}$$

with $T_1(x) \leq t_e$ and the definitions of $T'_1(x)$ and $t_e$ being used in the latter inequality. Applying this to (3.30), one obtains that, since $Z_1(\tau) \geq \frac{1}{2}x$,

$$P(Z_1(\tau') \leq \frac{1}{2}x; A_3) \leq C_3 e^{-\frac{1}{2}\epsilon'x} \tag{3.32}$$

for $\tau'$ as in (3.28). This implies that

$$P(A_3 \cap A_7) \leq C_3 e^{-\frac{1}{2}\epsilon'x}. \tag{3.33}$$

We now show that

$$P(A_3 \cap A_4 \cap A_7^c) \leq C_2 e^{-\frac{1}{4}\epsilon'x} \tag{3.34}$$

for an appropriate choice of $C_2$. On the set $A_3 \cap A_7^c$, it follows from (1.1) that

$$\sum_{k=2}^{5} Y_k(T'_1(x)) \geq (T'_1(x) - \tau) - (B_K(T'_1(x)) - B_K(\tau)), \tag{3.34}$$

where $K$ is the index at which $Z_K(\tau) = 0$. Since $\tau < T_1(x)$, it follows from the definition of $T'_1(x)$ that the right side of (3.34) is at least

$$\frac{1}{2}(T'_1(x) + x^2) + \left[\frac{1}{2}x^2 + \frac{1}{3}(T'_1(x) - \tau) - (B_K(T'_1(x)) - B_K(\tau))\right].$$
Again applying (2.4), this is greater than $\frac{1}{2} (T'_1(x) + x^2)$ off of a set of probability $C_2 e^{-\epsilon' x^2}$, for appropriate $C_2$ and $\epsilon' > 0$. Consequently,

$$P(A_3 \cap A_4 \cap A'_7)$$

(3.35)

$$= P \left( \sum_{k=2}^{5} Y_k(T'_1(x)) < \frac{1}{2} (T'_1(x) + x^2); A_3 \cap A'_7 \right)$$

$$\leq C_2 e^{-\epsilon' x^2}.$$  

One has

$$A \subseteq A' = A_1 \cup A_2 \cup (A_3 \cap A_4) \subseteq A_1 \cup A_2 \cup (A_3 \cap A_7) \cup (A_3 \cap A_4 \cap A'_7).$$

Combining (3.25), (3.27), (3.32) and (3.35) therefore implies (3.23) for an appropriate choice of $C_2$.

Using Proposition 3.3, the demonstration of Proposition 3.2 is quick.

**Proof of Proposition 3.2.** It follows from (3.19) that

$$A'_4 \cup A'_5 \supseteq (A_1 \cup A_2 \cup (A_3 \cap A_4))^c = (A')^c.$$  

Because of (3.20),

(3.36)  

$$T_2(x) \leq T'_1(x) \wedge T_6$$  

on $A'_4$. On the other hand, (3.36) holds trivially on $A'_5$. Along with (3.13), this implies that

(3.37)  

$$T_2(x) \leq 4(T_1(x) + x^2) \wedge T_6 \leq 4(T_1(x) \wedge T_6) + 4x^2$$

on $(A')^c$, and so, by (3.8) of Proposition 3.1,

(3.38)  

$$E[T_2(x); (A')^c] \leq C_3 x^2$$

for appropriate $C_3$.

The bound $T_2(x) \leq 5x^{5/\eta}$ always holds and so, by Proposition 3.3,

(3.39)  

$$E[T_2(x); A'] \leq C_4 / x^2$$

for appropriate $C_4$. Inequality (3.10) follows immediately from (3.38) and (3.39).  

□
**Foster’s criterion.** Foster’s criterion is a standard tool for showing positive recurrence of a Markov process when the process has a “uniformly negative drift” off of a bounded set in the state space (see, e.g., Bramson [1] or Foss and Konstantopoulos [9]). Versions of Foster’s criterion typically employ deterministic stopping times whose length depends only on the initial state. Here, we require a version of Foster’s criterion with random times, which is given below.

We state the proposition for SRBM defined on the induced $\mathbb{Z}$-path space, consisting of continuous paths on $\mathbb{R}_6^+$ with the natural filtration, in order to facilitate the definition of the sequence of stopping times employed in its proof. The SRBM can always be projected onto this space. The proof of the proposition employs an elementary martingale argument that extends to more general Feller processes.

Here and later on in the article, we employ the norm

$$
\|z\| = z_1 + \sum_{k=2}^{5} z_k^2 + z_6 \quad \text{for } z = (z_1, \ldots, z_6), z_k \geq 0.
$$

We set, for $\delta > 0$,

$$
\tau_A(\delta) = \inf\{t \geq \delta : Z(t) \in A\};
$$

$E_z[\cdot]$ denotes the expectation for the process with $Z(0) = z$, and $\mathcal{F}(t), t \geq 0$, denotes the filtration of $\sigma$-algebras associated with the SRBM.

**Proposition 3.4.** Suppose that, for some $\delta, \epsilon, \kappa > 0$ and a family of stopping times $\sigma(z), z \in \mathbb{R}_6^+$, with $\sigma(z) \geq \delta, E_z[\sigma(z)]$ is measurable in $z$ and the SRBM $Z(\cdot)$ satisfies

$$
E_z[\|Z(\sigma(z))\|] \leq (\|z\| \vee \kappa) - \epsilon E_z[\sigma(z)]
$$

for all $z$. Then

$$
E_z[\tau_A(\delta)] \leq \frac{1}{\epsilon}(\|z\| \vee \kappa) \quad \text{for all } z,
$$

where $A = \{z : \|z\| \leq \kappa\}$. Hence, $Z(\cdot)$ is positive recurrent.

**Proof.** The argument is a slight modification of that for the generalized Foster’s criterion given on page 94 of [1]. Set $\sigma_0 = 0$, and let $\sigma_1 < \sigma_2 < \ldots$ denote the stopping times defined inductively, with $\sigma_n - \sigma_{n-1}$, conditioned on $Z(\sigma_{n-1}) = z$, having the same law as $\sigma(z)$ given $Z(0) = z$. By (3.41) and the strong Markov property, for all $z$,

$$
E_z[\|Z(\sigma_n)\| \mid \mathcal{F}(\sigma_{n-1})] \leq (\|Z(\sigma_{n-1})\| \vee \kappa) - \epsilon E_{Z(\sigma_{n-1})}[\sigma(Z(\sigma_{n-1}))]
$$

(3.43)
for almost all $\omega$.

Set $M(0) = \|z\| \vee \kappa$ and

$$M(n) = \|Z(\sigma_n)\| + \epsilon \sigma_n \quad \text{for} \ n \geq 1. \tag{3.44}$$

Also, set $G(n) = F(\sigma_n)$. On account of (3.43),

$$E_z[M(n) \mid G(n - 1)] \leq M(n - 1) \quad \text{for} \ n \leq \rho, \tag{3.45}$$

where $\rho$ is the first time $n > 0$ at which $M(n) \in A$. So, $M(n \wedge \rho)$ is a nonnegative supermartingale on $G(n)$.

It follows from the Optional Sampling Theorem that

$$E_z[M(\rho)] \leq \|z\| \vee \kappa. \tag{3.46}$$

Note that $\tau_A(\delta) \leq \sigma_\rho$. Therefore, by (3.44) and (3.46),

$$\epsilon E_z[\tau_A(\delta)] \leq E_z[M(\rho)] \leq \|z\| \vee \kappa, \tag{3.47}$$

which implies (3.42) as desired.

4. Main steps of the proof of Theorem 1.2. Here, we present the main steps of the proof of Theorem 1.2, postponing their proofs until Sections 5 and 6. Our goal is to show that (3.41) of Proposition 3.4 is satisfied for each SRBM satisfying the conditions of Theorem 1.2. It then follows from the proposition that the SRBM is positive recurrent.

We employ the notation $D_1, D_2, \ldots$ and $\epsilon_1, \epsilon_2, \ldots$, as well as the previous notation $C_1, C_2, \ldots$, to denote positive constants. As earlier, $C_i$ denote terms whose precise value is not of interest to us, with the same symbol sometimes being reused. The terms $D_i$ and $\epsilon_i$ will sometimes take general values in the statements of the propositions, in which case specific values will be employed at the end of the section to demonstrate (3.41). We state the values of $D_i$ and $\epsilon_i$ we will apply, in most cases, when they are first introduced.

Proposition 4.1 is the first result. It states in essence that, after an appropriate time, either the norm of the initial state of the process decreases by a large factor or the sixth coordinate is bounded away from 0. In the first case, (3.41) will be demonstrated by using Proposition 4.2. In the second case, this will be done by using Propositions 4.3-4.5. In the statement of Proposition 4.1, one can choose $D_1 = 24 \cdot 16 \cdot 4 + 4$ and $D_2 = 24 \cdot 16 \cdot 40 \delta_1 \delta_3$. At the end of the section, we will set $\epsilon_1 = \epsilon_2^2$; the term $\epsilon_2 \in (0, \delta_1 \delta_2 \delta_3 / 1200]$, with the exact value being specified then.
Proposition 4.1. Suppose that \( Z(0) = z \) with \( z_1 \leq M \), \( z_k^2 \leq M \), for \( k = 2, \ldots, 5 \), and \( z_6 \leq M \). (a) For given \( \epsilon_1 > 0 \), there exist \( C_1, D_1 \geq 1 \) and \( \epsilon' > 0 \) such that, for all \( M \),

\[
\begin{align*}
(4.1) & \quad P(Z_k(M) > D_1M) \leq C_1 e^{-\epsilon'M} \quad \text{for } k = 1, 6, \\
(4.2) & \quad P(Z_k(M) > \epsilon_1M) \leq C_1 e^{-\epsilon'M} \quad \text{for } k = 2, \ldots, 5, \\
(4.3) & \quad E[Z_k(M)^2] \leq D_1M \quad \text{for } k = 2, \ldots, 5.
\end{align*}
\]

(b) For appropriate \( D_2 > 0 \) and each \( \epsilon_2 \in (0, \frac{1}{30}\delta_1\delta_3] \), there exist sets \( F_1 \in \mathcal{F}(M) \), \( F_2 \in \mathcal{F}(M) \) and \( \epsilon' > 0 \) such that, for large enough \( M \),

\[
\begin{align*}
(4.4) & \quad P((F_1 \cup F_2)^c) \leq e^{-\epsilon'M} \quad \text{for } k = 1, 6, \\
(4.5) & \quad Z_0(M) \leq \epsilon_2M \quad \text{on } F_1, \\
(4.6) & \quad E[Z_k(M)^2; F_1] \leq \epsilon_2D_2M \quad \text{for } k = 2, \ldots, 5, \\
(4.7) & \quad Z_0(M) \geq \epsilon_2M \quad \text{on } F_2.
\end{align*}
\]

Depending on whether \( F_1 \) or \( F_2 \) holds, we proceed in different ways. Under \( F_1 \), we consider the evolution of the SRBM for an additional time \( D_3M \). For this, we employ Proposition 4.2, which is given below.

We introduce the following terminology for Proposition 4.2. Set

\[
(4.8) \quad \epsilon_3 = 6\epsilon_2/\delta_3 \quad \text{and} \quad D_3 = 250D_1/\delta_3\delta_4,
\]

for given \( \epsilon_2 > 0 \) and \( D_1 \geq 1 \). Let \( U_1 \) be the first time \( t \) on the interval \([0, (D_3 - \epsilon_3)M]\) at which \( Z_1(t) = \frac{12\epsilon_2}{\delta_3}M \), with \( U_1 = (D_3 - \epsilon_3)M \) if this does not occur. Set \( U_2 = U_1 + \epsilon_3M \leq D_3M \). The proposition states that \( Z_1(U_2) \), \( Z_6(U_2) \) and \( Z_k(U_2) \), \( k = 2, \ldots, 5 \), are all small in an appropriate sense. The argument requires \( Z_1(t) > 0 \) for \( t \leq U_2 \), which enables all other coordinates to drift toward 0.

Proposition 4.2. Suppose \( Z(0) = z \) satisfies

\[
\begin{align*}
(4.9) & \quad (12\epsilon_2/\delta_3)M \leq z_1 \leq D_1M, \\
(4.10) & \quad z_k \leq \epsilon_2M, \quad \text{for } k = 2, \ldots, 6,
\end{align*}
\]

for given \( D_1 \geq 1 \) and \( \epsilon_2 \in (0, 1] \). Then, for \( U_2 \) as given above and large enough \( M \),

\[
\begin{align*}
(4.11) & \quad E[Z_k(U_2)] \leq (70\epsilon_2/\delta_3)M \quad \text{for } k = 1, 6, \\
(4.12) & \quad E[Z_k(U_2)^2] \leq (24 \cdot 97\epsilon_2/\delta_3)M \quad \text{for } k = 2, \ldots, 5.
\end{align*}
\]
When $F_2$ occurs, we follow the sketch given near the end of Section 1. In this case, we restart the SRBM at time $M$ and apply Proposition 4.3. In the proposition, we employ the stopping times $T_3(\cdot)$ and $T_4(\cdot)$. We define

\begin{equation}
T_3(M) = \min \left\{ t : \sum_{k=2}^{5} Y_k(t) = \frac{1}{6}(\delta_1 t + \epsilon_2 M) \right\}
\end{equation}

for given $\epsilon_2 \in (0, 1]$. We then set $T_4(M) = T_3(M)$ off of a set $G_M$ that will be specified in the proof of the proposition, with $T_4(M) \leq T_3(M) \land T_6$ holding on $G_M$, where $T_6$ is the stopping time that was defined in (3.6). ($T_3(M) < T_6$ will hold off of $G_M$.) The set $G_M$ will be negligible in the sense of (4.23) and (4.24).

In addition to the bounds on $G_M$ in (4.23) and (4.24), Proposition 4.3 gives upper and lower bounds on $T_3(M)$ and $Z_k(T_3(M))$, for $k = 2, \ldots, 5$. We will set the constant $\epsilon_4$ in the proposition equal to $\frac{1}{18}$ at the end of the section.

**Proposition 4.3.** Suppose $Z(0) = z$ satisfies

\begin{align}
&z_k \leq D_1 M \quad \text{for } k = 1, 6, \\
&z_k \leq \epsilon_2^2 M \quad \text{for } k = 2, \ldots, 5, \\
&z_6 \geq \epsilon_2 M,
\end{align}

for given $M$, $D_1$ and $\epsilon_2 \in (0, \frac{1}{20}]$. Then, on $G_M^c$ and $T_3(M) < \infty$,

\begin{align}
&T_3(M) \geq \frac{1}{30} \epsilon_2 M, \\
&T_3(M) \leq T_6, \\
&Z_k(T_3(M)) \leq \frac{31 D_1}{\epsilon_2} T_3(M) \quad \text{for } k = 1, 6, \\
&Z_k(T_3(M)) \leq 31 \epsilon_2 T_3(M) \quad \text{for } k = 2, \ldots, 5, \\
&Z_1(T_3(M)) \geq \frac{1}{12} \delta_1 \delta_2 T_3(M), \\
&Z_6(T_3(M)) \geq \frac{1}{2} \delta_1 T_3(M).
\end{align}

For given $\epsilon_4 > 0$ and large enough $M$,

\begin{align}
&E[Z_k(T_4(M)) ; G_M] \leq \epsilon_4 \quad \text{for } k = 1, 6, \\
&E[Z_k(T_4(M))^2 ; G_M] \leq \epsilon_4 \quad \text{for } k = 2, \ldots, 5.
\end{align}

We define stopping times $T'_3(M)$ as follows. For given $M > 0$ and $z = (z_1, \ldots, z_6)$, set

\begin{equation}
T'_3(M) = T_3(M) \land T_6 \land 5 N_M(z)^{5/2 \eta},
\end{equation}
where

\[(4.26) \quad N_M(z) = \left( \max_{k=2,\ldots,5} z_k^2 \right) \vee M \]

and \( \eta = .000073 \) as in Section 3. Assuming random initial conditions that satisfy the analog of \((4.3)\) in Proposition 4.1, we give, in Proposition 4.4, bounds on \(E[T'_3(M)]\). Moreover, the truncation event

\[(4.27) \quad A_M = \{ \omega : T'_3(M) = 5N_M(z)^{5/2\eta} \} \]

is small in the sense of \((4.30)\) and, under further initial conditions, is small as in \((4.31)\). Propositions 3.2 and 3.3 are the key ingredients in the proof.

**Proposition 4.4.** Suppose that \(Z(0)\) satisfies

\[(4.28) \quad E[Z_k(0)^2] \leq D_1 M \quad \text{for } k = 2, \ldots, 5, \]

and given \(M \geq 4\) and \(D_1\). Then, for appropriate \(C_2\) not depending on \(M\),

\[(4.29) \quad E[T'_3(M)] \leq C_2 M \]

and

\[(4.30) \quad E[Z_k(T'_3(M))^2; A_M] \leq C_2/\sqrt{M}. \]

If, in addition, \(Z_k(0) \leq D_1 M\) for \(k = 1, 6\), then

\[(4.31) \quad E[Z_k(T'_3(M)); A_M] \leq C_3/M \]

for appropriate \(C_3\) not depending on \(M\).

On the set \(G'_M \cap A'_M\), we continue to follow the evolution of \(Z(\cdot)\) after the elapsed time \(M + T_3(M)\). (Note that, on \(G'_M \cap A'_M\), \(T_3(M) = T'_3(M)\).) We wish to show that, provided \(Z_k(\cdot), k = 1, 6\), are initially “large” but \(Z_k(\cdot), k = 2, \ldots, 5\), are initially “small”, then all coordinates will typically be small at an appropriate random time. This is done in Proposition 4.5. The bounds \((4.39)\) and \((4.40)\) will allow us to demonstrate \((3.41)\) under the event \(F_2\) in Proposition 4.1.

In order to state Proposition 4.5, we define

\[(4.32) \quad T_5(M_1) = \min\{t : Z_1(t) = \epsilon_5 M_1\} \wedge D_4 M_1, \]

\[(4.33) \quad T'_5(M_1) = T_5(M_1) + \frac{1}{2}\epsilon_5 M_1, \]
for given $M_1 > 0$, $D_4$ and $\epsilon_5 > 0$. Note that
\begin{equation}
(4.34) \quad T'_5(M_1) \leq (D_4 + \frac{1}{2}\epsilon_5)M_1
\end{equation}
always holds. We employ the constants $\epsilon_5$, $\epsilon_6$, $\epsilon_7$, $\epsilon_8$ and $D_4$, $D_5$ in the proposition. A specific value of $\epsilon_5 \in (0, \frac{1}{12}\delta_1\delta_2]$ will be assigned at the end of the section; there, we will also employ $\epsilon_6 = 31\epsilon_2$, $\epsilon_7 = \frac{1}{12}\delta_1\delta_2$, $\epsilon_8 = \frac{1}{20}\delta_3\epsilon_5$ and $D_5 = 31D_1/\epsilon_2$; $D_4$ is specified in the proposition.

**Proposition 4.5.** Let $T_5(\cdot)$ and $T'_5(\cdot)$ be as in (4.32) and (4.33) for given $\epsilon_5 > 0$. Suppose $Z(0) = z$ satisfies
\begin{equation}
(4.35) \quad z_k \leq \epsilon_6 M_1 \quad \text{for } k = 2, \ldots, 5,
\end{equation}
\begin{equation}
(4.36) \quad \epsilon_7 M_1 \leq z_k \leq D_5 M_1 \quad \text{for } k = 1, 6,
\end{equation}
for given $M_1 > 0$, $\epsilon_6 > 0$, $\epsilon_7 \geq 6\epsilon_5 \lor 3\epsilon_6$ and $D_5 > 0$. Then, for given $\epsilon_8 > 0$ and $D_4 = 10D_5\delta_4/\delta_2\delta_3$,
\begin{equation}
(4.37) \quad P(T_5(M_1) = D_4 M_1) \leq C_1 e^{-\epsilon' M_1},
\end{equation}
\begin{equation}
(4.38) \quad P(Z_k(T_5(M_1)) \geq \epsilon_8 M_1) \leq C_1 e^{-\epsilon' M_1} \quad \text{for } k = 2, \ldots, 6,
\end{equation}
for appropriate $C_1$ and $\epsilon' > 0$ not depending on $M_1$. Moreover,
\begin{equation}
(4.39) \quad E[Z_k(T'_5(M_1))] \leq 6\epsilon_5 M_1 + C_4 \quad \text{for } k = 1, 6,
\end{equation}
\begin{equation}
(4.40) \quad E[Z_k(T'_5(M_1))^2] \leq 24 \cdot 8\epsilon_5 M_1 + C_4 \quad \text{for } k = 2, \ldots, 5,
\end{equation}
for appropriate $C_4$ not depending on $M_1$.

**Demonstration of Theorem 1.2.** It suffices to consider the SRBM $Z(\cdot)$ on the induced $Z$-path space. We will show that, for $z \in \mathbb{R}_+^d$ and an appropriate stopping time $\sigma(z)$, the assumption (3.41) of Proposition 3.4 is satisfied. The proposition will then imply $Z(\cdot)$ is positive recurrent. We abbreviate by setting $\sigma(z) = \sigma$ and dropping the subscript $z$ from $E_z[\cdot]$.

We will express $\sigma$ in terms of a related stopping time $\sigma'$, which we construct piecemeal by using the sets appearing in the previous propositions. Assume that $\|z\| = M$. Then $z_1 \leq M$, $z_k^2 \leq M$, for $k = 2, \ldots, 5$, and $z_6 \leq M$, and so the assumptions of Proposition 4.1 are satisfied. It follows from the proposition that (4.1)–(4.3) hold for given $M$ and (4.4)–(4.7) hold for large enough $M$. Let $H_1$ denote the union of the set where $(F_1 \cup F_2)^c$ occurs and where either the event in (4.1) or the event in (4.2) occurs. On $H_1$, we set $\sigma' = M$. It follows from Lemma 2.5, (4.1), (4.2) and (4.4) that, for large enough $M$,
\begin{equation}
(4.41) \quad E[\|Z(\sigma')\|; H_1] \leq 1.
\end{equation}
Suppose next that the event $F_1 \cap H_1^c$ holds. Then either (a) $Z_1(M) < (12 \epsilon_2 / \delta_3)M$ or (b) $Z_1(M) \geq (12 \epsilon_2 / \delta_3)M$; denote the former of these events by $H_2$ and the latter by $H_3$. Under $H_2$, we set $\sigma' = M$. Then, on account of (4.5) and (4.6) of Proposition 4.1, with $D_2 = 24 \cdot 16 \cdot 40 / \delta_1 \delta_3$,

$$
E[\|Z(\sigma')\|; H_2] \leq \left( \frac{12 \epsilon_2}{\delta_3} + 4 \epsilon_2 D_2 + \epsilon_2 \right) M 
\leq (97 \cdot 16 \cdot 40 \epsilon_2 / \delta_1 \delta_3)M.
$$

When $H_3$ occurs, we set $\sigma' = M + U_2$, where $U_2$ is defined below (4.8). (Here and later on, stopping times such as $U_2$ refer to the restarted process. )

The process restarted at time $M$ satisfies conditions (4.9) and (4.10) of Proposition 4.2. It follows from (4.11) and (4.12) of the proposition that

$$
E[\|Z(\sigma')\|; H_3] \leq (140 + 96 \cdot 97) \frac{\epsilon_2}{\delta_3} M \leq 972 \frac{\epsilon_2}{\delta_3} M.
$$

The bounds (4.41)–(4.43) consider the behavior of $Z(\sigma)$ off of $F_2 \cap H_1^c$. We now consider the behavior on $F_2 \cap H_1^c$, for which there are two cases. Denote by $H_4$ the subset of $F_2 \cap H_1^c$ corresponding to the union of the events $G_M$ and $A_M$ for the restarted process, which appear in the proof of Proposition 4.3 and in (4.27). Let

$$
\sigma' \overset{\text{def}}{=} M + (T_4(M) \wedge 5N_M(Z(M))^{5/\eta}) \leq M + T_3(M),
$$

that is, $\sigma'$ is the earlier of the times at which either the event $G_M$ or $A_M$ occurs. The restarted process satisfies both (4.14)–(4.16) of Proposition 4.3 and (4.28) of Proposition 4.4. It therefore follows from (4.23)–(4.24), with $\epsilon_4 = \frac{1}{10}$, and (4.30)–(4.31) that

$$
E[\|Z(\sigma')\|; H_4] \leq 1
$$

for large enough $M$.

We also consider the behavior of $Z(\sigma')$ on $H_5 \overset{\text{def}}{=} F_2 \cap H_1^c \cap H_4^c$. On account of (4.19)–(4.22) of Proposition 4.3, the conditions (4.35)–(4.36) of Proposition 4.5 are satisfied for the process restarted at time $M + T_3(M) = M + T_3(M)$, for $M_1 = T_3(M)$ and $D_5$, $\epsilon_6$ and $\epsilon_7$ as specified before Proposition 4.5. Also, $\epsilon_7 \geq 6 \epsilon_5 \vee 3 \epsilon_6$ holds for $\epsilon_2 \leq \delta_1 \delta_3 / 1200$ and $\epsilon_5$ as specified before the proposition. Inequalities (4.39) and (4.40) therefore hold for $T_3'(M_1)$ chosen as in (4.33). Setting $\sigma' = M + T_3(M) + T_3'(T_3(M))$, it follows from these inequalities that

$$
E[\|Z(\sigma')\|; H_5] \leq 97 \cdot 8 \epsilon_5 E[T_3(M); H_5] + C_4 \leq 97 \cdot 8 \epsilon_5 E[T_3'(M)] + C_4
$$
for appropriate $C_4$. On account of (4.3) of Proposition 4.1, one can apply Proposition 4.4 to $Z(\cdot)$ restarted at time $M$, which gives the upper bound in (4.29) on $E[T'_3(M)]$. Applying this to (4.45), one obtains

$$E[\|Z(\sigma')\|; H_5] \leq 98 \cdot 8\epsilon_5 C_2 M$$

for large enough $M$ and appropriate $C_2$.

Adding the bounds in (4.41)–(4.46) for $E[Z(\sigma'); H_i], i = 1, \ldots, 5$, one obtains

$$E[\|Z(\sigma')\|] \leq C_5(\epsilon_2 + \epsilon_5) M$$

for large enough $M$, with $C_5$ depending on $\delta_1$ and $\delta_3$. So far, we have not specified the values of $\epsilon_2$ and $\epsilon_5$; we now set

$$\epsilon_2 = \epsilon_5 = (1/4C_5) \wedge (\delta_1\delta_2\delta_3/1200).$$

It follows that

$$E[\|Z(\sigma')\|] \leq \frac{1}{2} M$$

for $\|z\| = M$ and $M \geq M_0$, for appropriate $M_0 \geq 1$.

We define $\sigma$ in terms of $\sigma'$, by setting $\sigma = \sigma'$ when $\|z\| = M$ for $M \geq M_0$, and $\sigma = M \vee 1$ for $M \leq M_0$. When $\|z\| = M$ and $M \geq M_0$, this implies

$$E[\|Z(\sigma')\|] \leq \frac{1}{2} M.$$

On the other hand, by applying (2.22) of Lemma 2.5 to (4.1), it follows for all $M$ that

$$E[Z_k(M \vee 1)] \leq C_1(M \vee 1) \quad \text{for } k = 1, 6$$

and appropriate $C_1 \geq D_1 \vee 1$. Together with (4.3), this implies

$$E[\|Z(M \vee 1)\|] \leq 6C_1(M \vee 1)$$

for all $M$. Setting $\kappa = 12C_1(M_0 \vee 1)$, it follows from (4.48) and (4.49) that

$$E[\|Z(\sigma)\|] \leq (\|z\| \vee \kappa) - \frac{1}{2}(M \vee 1)$$

for $\|z\| = M$ and all $M$.

We also wish to show that, for $\|z\| = M$,

$$E[\sigma] \leq C_3(M \vee 1)$$

for some $C_3$. This is a quick consequence of the definition of $\sigma$ on $H_1, \ldots, H_5$ for $\|z\| \geq M_0$. On $H_1 \cup H_2, \sigma = M$; on $H_3, \sigma \leq D_3 M$; on $H_4, \sigma \leq M + T'_3(M)$;
and on $H_5$, $\sigma = M + T_3'(M) + T_5'(T_3'(M))$. It therefore follows from (4.29) of Proposition 4.4 and (4.34) that
\begin{equation}
E[\sigma] \leq M + D_3 M + E[T_3'(M)] + E[T_5'(T_3'(M))]
\leq (1 + D_3 + C_2 + C_2(D_4 + \frac{1}{2} \epsilon_5))M \leq C_3 M
\end{equation}
for $\|z\| \geq M_0$ and appropriate $C_2$ and $C_3$. Together with $\sigma = M \vee 1$ for $\|z\| < M_0$, this implies (4.51).

Combining (4.50) and (4.51), one obtains
\begin{equation}
E[\|Z(\sigma)\|] \leq (\|z\| \vee \kappa) - (1/2C_3)E[\sigma].
\end{equation}
This implies (3.41) of Proposition 3.4, with $\epsilon = 1/2C_3$. Since $Z(\cdot)$ is Feller and $\sigma$ is defined in terms of hitting times of closed sets, one can check that $E_z[\sigma(z)] = E[\sigma]$ is measurable in $z$. By applying the proposition, (3.42) follows and hence $Z(\cdot)$ is positive recurrent. This demonstrates Theorem 1.2.

5. Demonstration of Propositions 4.1 and 4.2. Proposition 4.1 constitutes the first step of the proof of Theorem 1.2. It provides elementary upper bounds (4.1)–(4.3) on $Z_k(M)$, $k = 1, \ldots, 6$, and on $E[Z_k(M)^2]$, $k = 2, \ldots, 5$, that are valid over all $M$. It states that, off of the exceptional set in (4.4), either $Z_k(M)$ will be small for all $k = 2, \ldots, 6$ or $Z_6(M)$ will be large, in the sense of (4.5)–(4.7). This dichotomy depends on the rate of growth of $Y_1(\cdot)$ as given by the set $F_3$ in (5.1) of the proof, although the actual correspondence is a bit more complicated. The proof of Proposition 4.1 relies on the application of lemmas from Section 2 to the equation (1.1) of the SRBM.

Proof of Proposition 4.1. Both inequalities in (4.1) follow directly from (2.12) of Lemma 2.3, with $D_1 \geq 9$. Inequality (4.2) follows from (2.14) of Lemma 2.4, with a new choice of $C_1$. For (4.3), one can restrict the expectation to the set \{${Z_k(M) > 2\sqrt{M}}$\} and its complement. One then applies (2.15) to the first part and a trivial bound to the second part to obtain (4.3), with $D_1 \geq 24 \cdot 16 \cdot 4 + 4$.

For the inequalities (4.4)–(4.7), we first set
\begin{equation}
F_3 = \{\omega : Y_1(M) - Y_1(\tau_k) > \epsilon_9 M \text{ for some } k = 2, \ldots, 5\}.
\end{equation}
Here, $\epsilon_9 \overset{\text{def}}{=} 2\epsilon_2/\delta_1$ and $\tau_k$ is the last time before $M$ at which $Z_k(t) = 0$ for any $t$; if the set is empty, let $\tau_k = 0$. On $F_3$, we denote by $K$ one of the indices $k$ satisfying (5.1).
We consider the behavior on $F_3$ and $F_3^c$ separately, first considering the behavior on $F_3$. One has, by applying (1.1) to the $K$th and 6th coordinates,

\begin{equation}
Z_0(M) - Z_K(M) = (Z_0(\tau_K) - Z_K(\tau_K)) + (B_0(M) - B_0(\tau_K)) - (B_K(M) - B_K(\tau_K)) + \delta_1(Y_1(M) - Y_1(\tau_K)) + \delta_3(Y_0(M) - Y_0(\tau_K)).
\end{equation}

(5.2)

On $F_3$, it follows from (2.3) of Lemma 2.1 that, except on a set $F_1 \in \mathcal{F}(M)$ of exponentially small probability in $M$,

\begin{equation}
Z_0(M) \geq \delta_1 \epsilon_9 M - \epsilon M \geq \frac{1}{2} \delta_1 \epsilon_9 M = \epsilon_2 M
\end{equation}

(5.3)

for $\epsilon = \frac{1}{2} \delta_1 \epsilon_9$ and large enough $M$. This gives the inequality in (4.7) on the set $F_3 \cap F_4^c$.

We now consider the behavior of $Z(\cdot)$ on $F_3^c$. Set $t_1 = (1 - 20 \epsilon_9 / \delta_3) M$; since $\epsilon_2 \leq \frac{1}{20} \delta_1 \delta_3$, $t_1 \geq 0$ holds. It follows from (2.14) of Lemma 2.4 that, except on a set $F_5 \in \mathcal{F}(M)$ of exponentially small probability in $M$,

\begin{equation}
Z_k(t_1) \leq Z_k(0) + \epsilon M \leq \epsilon_9 M
\end{equation}

(5.4)

for $k = 2, \ldots, 5$, $\epsilon = \epsilon_9 / 2$ and large enough $M$. Restarting $Z(\cdot)$ at time $t_1$, it follows from (2.32) of Lemma 2.6 and (5.4) that, except on a set $F_6 \in \mathcal{F}(M)$ of exponentially small probability,

\begin{equation}
Y_k(M) - Y_k(t_1) \geq \frac{4 \epsilon_9}{\delta_3} M - \frac{\epsilon_9}{\delta_3} M - \frac{2}{\delta_3} (Y_1(M) - Y_1(t_1)).
\end{equation}

(5.5)

On $F_3^c$, when $\tau_k < t_1$, the last term on the right side of (5.5) is at most $2 \epsilon_9 M / \delta_3$, which implies

\begin{equation}
Y_k(M) - Y_k(t_1) > 0,
\end{equation}

(5.6)

and hence $Z_k(\tau_k') = 0$ for some $\tau_k' \in [t_1, M]$. This contradicts the definition of $\tau_k$, and so $\tau_k \geq t_1$.

Let $\tau_k'$ be the smallest such time. Since $\tau_k'$ is a stopping time, we may restart $Z(\cdot)$ at $\tau_k'$. Applying (2.15) of Lemma 2.4, it follows that

\begin{equation}
E[\{Z_k(M)^2; F_3^c \cap F_5^c \cap F_6^c\}] \leq 24 \cdot 16 \cdot 2 \frac{\epsilon_9}{\delta_3} M = \epsilon_2 D_2 M
\end{equation}

(5.7)

for $k = 2, \ldots, 5$ and $D_2 = 24 \cdot 16 \cdot 40 / \delta_1 \delta_3$.

We now conclude the demonstration of (4.4)–(4.7). Denoting the set on which the inequality in (4.7) holds by $F_2$, one has by (5.3) that $F_2 \supseteq F_3 \cap F_4^c$. Setting $F_1 = F_2^c \cap F_5^c \cap F_6^c$, then (4.5) is automatically satisfied and
(4.6) holds because of (5.7). Since \((F_1 \cup F_2)^c \subseteq F_4 \cup F_5 \cup F_6\), (4.4) follows, for appropriate \(\epsilon' > 0\), from the upper bounds on the probabilities of \(F_4\), \(F_5\) and \(F_6\). It follows from the definition of \(F_2\) that \(F_2 \in \mathcal{F}(M)\); since \(F_i \in \mathcal{F}(M), i = 2, \ldots, 6\), one also has \(F_1 \in \mathcal{F}(M)\). \(\square\)

Proposition 4.2 states that, if \(z_k, k = 2, \ldots, 6\), are all small and \(z_1\) is bounded below, but is not too large, then \(Z_k(U_2), k = 1, \ldots, 6\), are all small in the sense of (4.11) and (4.12). The proof considers the behavior of \(Z(t)\) over \([U_1, U_2]\). The stopping time \(U_1\) was defined so that \(Z_1(U_1)\) is relatively small, but large enough so that, over \([0, U_2]\) with \(U_2 = U_1 + \epsilon_3 M\), \(Z_1(t) > 0\) holds. The interval \([U_1, U_2]\) is both large enough to obtain the desired behavior of \(Z_k(U_2), k = 2, \ldots, 5\), in (4.12) and short enough so (4.11) holds for \(Z_k(U_2), k = 1, 6\). As with Proposition 4.1, the proof applies the lemmas of Section 2 to (1.1).

**Proof of Proposition 4.2.** We first show (4.11) for \(k = 1\). It follows from Lemma 2.7, (4.9) and (4.10) that, on the set where \(Z_1(t) > 0\) for \(t \in [0, \frac{1}{2} D_3 M]\),

\[
Z_1(\frac{1}{2} D_3 M) \leq D_1 M + \frac{3\epsilon_3}{\delta_3} M - \frac{1}{60} \delta_1 D_3 M
\]  

(5.8) except for a set \(F_7\) of exponentially small probability in \(M\). Since the right side of (5.8) is negative for \(D_3\) satisfying (4.8) and \(Z_1(0) \geq \frac{12\epsilon_3}{\delta_3} M\), \(Z_1(t) = \frac{12\epsilon_3}{\delta_3} M\) must occur at some \(t \leq \frac{1}{2} D_3 M\); hence \(Z_1(U_1) = \frac{12\epsilon_3}{\delta_3} M\) on \(F_7^c\). By (2.12) of Lemma 2.3 and (4.8), this in turn implies that \(Z_1(U_2) \leq \frac{60\epsilon_3}{\delta_3} M\) off of an additional set of exponentially small probability. Together with (2.22) of Lemma 2.5, this implies (4.11) for \(k = 1\) and large \(M\).

Restarting \(Z(\cdot)\) at \(U_1\), it follows from (1.1) and (4.8) that, except on a set \(F_8^c\) of exponentially small probability in \(M\),

\[
Z_1(U_1 + s) \geq \frac{12\epsilon_3}{\delta_3} M + B_1(s) - s > 0
\]  

(5.9) for \(s \leq \epsilon_3 M\). Consequently, on \(F_8^c\),

\[
Z_1(t) > 0 \quad \text{for } t \leq U_2.
\]

Since \(Z_0(0) \leq \epsilon_2 M\), one can therefore employ (2.14) of Lemma 2.4, with small enough \(\epsilon > 0\), together with (2.22) of Lemma 2.5, to obtain (4.11) for \(k = 6\).

We still need to show (4.12). For this, one can employ the conditions (4.8), (4.10) and (5.9) and argue similarly to (5.4) through (5.6), in the proof of Proposition 4.1, to conclude that, for \(k = 2, \ldots, 5\),

\[
Z_k(\tau'_k) = 0 \quad \text{for some } \tau'_k \in [U_1, U_2],
\]
off of a set $F_0$ of exponentially small probability in $M$. Letting $\tau'_k$ denote the
first such time, we restart $Z(\cdot)$ at $\tau'_k$. Applying (2.15) and (2.21), it follows that
\[
E[Z_k(U_2)^2] \leq E[Z_k(U_2)^2; F_0^c] + E[Z_k(U_2)^2; F_0] \\
\leq (24 \cdot 16 + 1)\epsilon_3 M \leq (24 \cdot 97\epsilon_2/\delta_3) M
\]
for large enough $M$. This implies (4.12). \hfill \Box

6. Demonstration of Propositions 4.3, 4.4 and 4.5. The proofs of Propositions 4.3, 4.4 and 4.5 rely on the application of the lemmas in Section 2 to the equation (1.1) of the SRBM $Z(\cdot)$. Proposition 4.4 also relies on Propositions 3.2 and 3.3. The reasoning behind the proofs follows in spirit the sketch given near the end of Section 1 and in Section 4.

We first demonstrate Proposition 4.3. The proposition states that, off of the exceptional set $G_M$ defined in the proof, the inequalities (4.17)–(4.22) all hold. In particular, $Z_k(T_3(M))$, $k = 2, \ldots, 5$, will be small and $Z_k(T_3(M))$, $k = 1, 6$, will be bounded below, but not too large. These inequalities, except for (4.21), will follow from their analogs (6.1)–(6.4) that hold over $[\frac{1}{30}\epsilon_2 M, T_3(M)]$ and $[0, T_3(M)]$. The exceptional set $G_M$ will be shown to be small in the sense of (4.23) and (4.24).

The lower bounds on $Z_k(T_3(M))$, $k = 1, 6$, constitute the more delicate part of the argument and depend on the condition $z_0 \geq \epsilon_2 M$ in (4.16).

Arguing as in (6.17)–(6.20), we will show that the growth of $Y_1(\cdot)$ causes $Z_6(\cdot)$ to increase linearly. On the other hand, as shown below (6.6), the growth of $Y_k(\cdot)$, $k = 2, \ldots, 5$, together with $Y_6(T_3(M)) = 0$, causes $Z_1(\cdot)$ to eventually increase linearly. The stopping time $T_3(M)$ has been chosen so that both features are present.

Proof of Proposition 4.3. We first specify the set $G_M$ used in the definition of $T_4(M)$. We abbreviate by setting $M' = \frac{1}{30}\epsilon_2 M$. Writing $G_M = \bigcup_{i=1}^5 G_i$, the sets $G_i$ are defined as follows:

(6.1) \hspace{1cm} G_1 = \left\{ \omega : \sum_{k=2}^5 Y_k(M') \geq 5M' \right\},

(6.2) \hspace{1cm} G_2 = \left\{ \omega : Z_k(s) \geq \frac{31}{\epsilon_2} D_1 s \text{ for some } s \in [M', T_3(M)], k = 1, 6 \right\},

(6.3) \hspace{1cm} G_3 = \left\{ \omega : Z_k(s) \geq 31\epsilon_2 s \text{ for some } s \in [M', T_3(M)], k = 2, \ldots, 5 \right\},

(6.4) \hspace{1cm} G_4 = \left\{ \omega : Z_6(s) \leq \frac{1}{T_3} \delta_3 s \text{ for some } s \in [0, T_3(M)] \right\},

(6.5) \hspace{1cm} G_5 = \left\{ \omega : B_2(s) - B_1(s) \geq \frac{1}{T_3} \delta_2 s \text{ for some } s \in [M', T_3(M)] \right\}.
Inequality (4.17) follows from (6.1) and the definition of \( T_3(M) \). Inequalities (4.19) and (4.20) follow by setting \( s = T_3(M) \) in (6.2) and (6.3); both (4.18) and (4.22) follow from (6.4). The demonstration of (4.21) requires a little work. First note that, on \( G_4^e \), (1.1), (2.6) of Lemma 2.2 and (4.15) imply that

\[(6.6)\]

\[
Z_1(T_3(M)) \geq B_1(T_3(M)) - T_3(M) + \sum_{k=1}^{6} Y_k(T_3(M)) + \delta_2 \sum_{k=2}^{5} Y_k(T_3(M)) \\
\geq \frac{1}{6} \delta_1 \delta_2 T_3(M) + (\frac{1}{6} \delta_2 \epsilon_2 - \epsilon_2^2) M + B_1(T_3(M)) - B_2(T_3(M)) \\
\geq \frac{1}{6} \delta_1 \delta_2 T_3(M) + B_1(T_3(M)) - B_2(T_3(M))
\]

which, on \( G_5^e \), is at least \( \frac{1}{12} \delta_1 \delta_2 T_3(M) \). So,

\[
Z_1(T_3(M)) \geq \frac{1}{12} \delta_1 \delta_2 T_3(M) \quad \text{on } G_4^e \cap G_5^e,
\]

which demonstrates (4.21).

We need to show (4.23) and (4.24). For this, we define \( V_i \), \( i = 2, 3, 4, 5 \), to be the first time at which the event in \( G_i \) occurs, with

\[
V_1 = \inf \left\{ s : \sum_{k=2}^{5} Y_k(s) \geq 5M' \right\}
\]

if \( G_1 \) occurs; off of these sets, define \( V_i = T_3(M) \) for \( i = 1, \ldots, 5 \). We complete our definition of \( T_4(M) \) in (4.13) by setting

\[(6.7)\]

\[
T_4(M) = V_1 \wedge V_2 \wedge V_3 \wedge V_4 \wedge V_5.
\]

Note that \( V_4 \leq T_6 \). It follows from this and (6.7) that \( T_4(M) \leq T_3(M) \wedge T_6 \); moreover, \( T_4(M) \) is a stopping time.

We note that, by (2.9) of Lemma 2.2,

\[(6.8)\]

\[
P(G_1) \leq C_1 e^{-\epsilon'M}
\]

for appropriate \( C_1 \) and \( \epsilon' > 0 \). Using (2.21) and (2.22) of Lemma 2.5, it therefore follows that, for given \( \epsilon_{10} > 0 \) and large enough \( M \),

\[(6.9)\]

\[
E[Z_k(T_4(M)); G_1] \leq \epsilon_{10} \quad \text{for } k = 1, 6,
\]

\[(6.10)\]

\[
E[Z_k(T_4(M))^2; G_1] \leq \epsilon_{10} \quad \text{for } k = 2, \ldots, 5.
\]

We require more detailed estimates for \( G_2, \ldots, G_5 \). For each \( i = 2, \ldots, 5 \) and \( j = 1, 2, \ldots \), we denote by \( G_i(j) \) the event for which \( G_i \) first occurs on
We first consider the behavior on $G_3$. We recall that, by (2.14) of Lemma 2.4, for $k = 2, \ldots, 5$ and given $\epsilon > 0$,

$$(6.11) \quad P(Z_k(s) - Z_k(0) \geq \epsilon j \text{ for some } s \leq j) \leq C_1 e^{-\epsilon'j}$$

for each $j = 1, 2, \ldots$, and appropriate $C_1$ and $\epsilon' > 0$. On account of (4.15), it follows for small enough $\epsilon$ that

$$(6.12) \quad P(Z_k(s) \geq 31\epsilon_2 s \text{ for some } s \in [j - 1, j]) \leq C_1 e^{-\epsilon'j}$$

for $j \geq M'$. It therefore follows from (2.21) and (2.22) that

$$E[Z_k(V_3); G_3(j)] \leq e^{-\frac{1}{2}\epsilon'j} \quad \text{for } k = 1, 6,$$
$$E[Z_k(V_3)^2; G_3(j)] \leq e^{-\frac{3}{4}\epsilon'j} \quad \text{for } k = 2, \ldots, 5,$$

for $j \geq M'$ and large enough $M$. Summing over $j$ gives

$$E[Z_k(V_i); G_i] \leq \epsilon_{10} \quad \text{for } k = 1, 6,$$
$$E[Z_k(V_i)^2; G_i] \leq \epsilon_{10} \quad \text{for } k = 2, \ldots, 5,$$

for $i = 3$, given $\epsilon_{10} > 0$ and large enough $M$.

The inequalities (6.15) and (6.16) hold for $i = 2$ and $i = 5$ for the same reasons, except that one applies (2.12) of Lemma 2.3 in place of (2.14) and (4.14) in place of (4.15) for $i = 2$, and one applies (2.3) of Lemma 2.1 for $i = 5$. The inequalities (6.15) and (6.16) also hold for $i = 4$, although this requires more work; we now do this.

We note that, by (2.6) of Lemma 2.2, (4.13) and (4.15),

$$Y_1(s) \geq s - (\epsilon_2^2 M + B_2(s)) - \frac{1}{6}(\delta_1 s + \epsilon_2 M)$$

for $s \leq T_3(M) \land T_6$. Together with (1.1), (4.16) and $\epsilon_2 \leq \frac{1}{20}$, this implies

$$Z_6(s) \geq \epsilon_2 M + B_6(s) - s + (1 + \delta_1)Y_1(s) \geq \frac{3}{4}(\delta_1 s + \epsilon_2 M) + (B_6(s) - (1 + \delta_1)B_2(s)).$$

It follows from (2.2) of Lemma 2.1 that

$$P(Z_6(s) \leq \frac{1}{2}\delta_1 s \text{ for some } s \in [j - 1, j]) \leq C_1 e^{-\epsilon'j}$$

for $j \geq M'$, and appropriate $C_1$ and $\epsilon' > 0$. Also, by (2.3),

$$P(Z_6(s) \leq \frac{1}{2}\delta_1 s \text{ for some } s \leq M') \leq C_1 e^{-\epsilon'M}$$
for appropriate $C_1$ and $\epsilon' > 0$. Proceeding similarly to (6.12)–(6.16), the inequalities (6.15) and (6.16) with $i = 4$ also hold.

On account of (6.9)–(6.10) and (6.15)–(6.16), for $i = 2, \ldots, 5$, it follows for large enough $M$ that

(6.21) $E[Z_k(T_4(M)); G_M] \leq 5\epsilon_{10}$ for $k = 1, 6$,

(6.22) $E[Z_k(T_4(M))^2; G_M] \leq 5\epsilon_{10}$ for $k = 2, \ldots, 5$,

where $\epsilon_{10}$ is as in (6.9)–(6.10). This implies (4.23) and (4.24) for $\epsilon_4 = 5\epsilon_{10}$, and completes the proof of the proposition.

The demonstration of Proposition 4.4 is based on the comparison between $T'_3(M, z)$ and $T_2(\sqrt{N_M(z)})$ in (6.23). This enables one to employ the upper bounds on $E[T_2(x)]$ and $P(A)$ from Propositions 3.2 and 3.3.

**Proof of Proposition 4.4.** Let $T'_3(M, z)$ and $A_M(z)$ denote the analogs of $T'_3(M)$ and $A_M$, with $Z(0) = z$ being specified. Comparing $T'_3(M, z)$ with $T_2(x)$ in (3.9), for $x = \sqrt{N_M(z)}$, it is easy to see that

(6.23) $T'_3(M, z) \leq T_2(\sqrt{N_M(z)})$.

It therefore follows from Proposition 3.2 that, for appropriate $C_1$,

(6.24) $E[T'_3(M, z)] \leq E\left[T_2\left(\sqrt{N_M(z)}\right)\right] \leq C_1 N_M(z)$.

Integrating (6.24) over $z$ and applying (4.28), one obtains

(6.25) $E[T'_3(M)] = E[E[T'_3(M)| Z(0) = z]] \leq C_1 E[N_M(Z(0))] \leq C_1 \left( E\left[\max_{k=2,\ldots,5} Z_k(0)^2\right] + M\right) \leq C_1 (5D_1 + 1) M$.

This implies (4.29) with $C_2 = C_1 (5D_1 + 1)$.

Since the truncated values $T_6 \land 5N_M(z)^{5/\eta}$ and $T_6 \land 5x^{5/\eta}$ in (4.25) and (3.9) are equal, it is easy to check that

(6.26) $A_M(z) \subseteq A$

for given $z$, where $A$ is the event in (3.11) with $x = \sqrt{N_M(z)}$. It therefore follows from Proposition 3.3 that, for appropriate $C_1$,

(6.27) $P(A_M(z)) \leq P(A) \leq C_1 N_M(z)^{-\frac{5\eta}{2\eta - 1}}$. 

Together with (2.21), (4.27) and (6.27) imply that, for given \( z \),

\[
E[Z_k(T_3'(M))]^2; A_M(Z(0)|Z(0) = z] \leq C_2/\sqrt{N_M(z)} \leq C_2/\sqrt{M}
\]

for \( k = 2,\ldots,5 \) and appropriate \( C_2 \). Similarly, by (2.22), (4.27) and (6.27),

\[
E[Z_k(T_3'(M))]; A_M(Z(0)|Z(0) = z] \leq C_3/N_M(z) \leq C_3/M
\]

for \( k = 1,6 \) and appropriate \( C_3 \). Integrating (6.28) and (6.29) over \( z \) produces (4.30) and (4.31).

We now demonstrate Proposition 4.5. We first show (4.37) and (4.38), which are then used to show (4.39) and (4.40). On account of the upper bounds on \( z_1 \) and \( z_6 \) in (4.36), \( Z_1(\cdot) \) will drift toward 0 so that \( Z_1(t) = \epsilon_5 M_1 \) typically occurs before time \( D_4 M_1 \). This will imply (4.37). Since \( Y_1(T_5(M_1)) = 0 \), the coordinates \( k = 2,\ldots,6 \) drift toward 0 over \( [0,T_5(M_1)] \), implying (4.38). The elapsed time between \( T_5(M_1) \) and \( T_5'(M_1) \) is short enough so \( Z_k(T_5'(M_1)), k = 1,6 \), will typically still be small, and so (4.39) will hold. It is also short enough so \( Y_1(T_5'(M_1)) = 0 \) and long enough for \( Z_k(t) = 0, k = 2,\ldots,5 \), to typically occur, from which (4.40) will follow.

**Proof of Proposition 4.5.** We first demonstrate (4.37). To do so, we analyze \( Z(t) \) when \( Y_1(t) = 0 \), for given \( t \geq 0 \). By (1.1),

\[
Z_1(t) = Z_1(0) + B_1(t) - t + (1 + \delta_2) \sum_{k=1}^{6} Y_k(t) - (\delta_2 + \delta_4) Y_6(t).
\]

Since \( Y_1(t) = 0 \), it follows from (2.32) of Lemma 2.6 and (4.36) that

\[
Y_6(t) \geq \frac{1}{\delta} t - \frac{D_3}{\delta_z} M_1
\]

off of a set of exponentially small probability in \( M_1 \). Together, (1.14), (4.36), (6.30) and (6.31) imply that

\[
Z_1(t) \leq \frac{2 \delta_4 D_5}{\delta_3} M_1 + B_1(t) - \left(1 + \frac{\delta_2 + \delta_4}{5}\right) t + (1 + \delta_2) \sum_{k=2}^{6} Y_k(t).
\]

Now, by (2.31) of Lemma 2.6, one has that, for given \( \epsilon > 0 \),

\[
- \left(1 + \frac{\delta_2 + \delta_4}{5}\right) t + (1 + \delta_2) \sum_{k=2}^{6} Y_k(t)
\]

\[
\leq \left[\frac{(1 + \epsilon)(1 + \delta_2)}{1 - \delta_3} - 1 - \frac{\delta_2 + \delta_4}{5}\right] t
\]
off of a set of exponentially small probability in $M_1$. One can check that, because of (1.14) and $\delta_3 \leq \frac{1}{10}$, the right side of (6.33) is less than $-(\frac{1}{5}\delta_2 + \epsilon)t$ for $\epsilon$ chosen small enough. Combining (6.32), (6.33) and applying (2.2) of Lemma 2.1, one therefore obtains

\[(6.34) \quad Z_1(t) < \frac{2\delta_4D_5}{\delta_3}M_1 - \frac{1}{5}\delta_2t\]

off of a set of exponentially small probability in $M_1$, provided $Y_1(t) = 0$. But, since $Z_1(t) \geq 0$, it follows from (6.34) that, off this set, $Y_1(D_1M_1) > 0$ for $D_1 = (10D_5\delta_4/\delta_2\delta_3)M_1$. So, $Z_1(t) = 0$ for some $t \leq D_1M_1$, which implies (4.37).

Set $\tau_k = \min\{t : Z_k(t) = 0\}$ for $k = 2, \ldots, 6$. In order to demonstrate (4.38), we show that

\[(6.35) \quad \tau_k \leq T_5(M_1) \quad \text{for } k = 2, \ldots, 6,\]

off of a set of exponentially small probability in $M_1$. Inequality (4.38) then follows from (4.37) and (2.14) of Lemma 2.4 for a small enough choice of $\epsilon > 0$.

We claim that, off of a set of exponentially small probability in $M_1$,

\[(6.36) \quad \tau_6 < T_5(M_1).\]

To see this, note that, when $Y_6(t) = 0$, it follows from (6.30), (2.6) of Lemma 2.2, (4.35)–(4.36) and $\epsilon_7 \geq 3\epsilon_6$ that

\[(6.37) \quad Z_1(t) \geq \frac{1}{3}\epsilon_7M_1 + \delta_2t + (B_1(t) - (1 + \delta_2)B_2(t)).\]

Since $\epsilon_7 \geq 6\epsilon_5$, on account of (2.3) of Lemma 2.1, this is greater than $\epsilon_5M_1$ off of a set of exponentially small probability in $M_1$. Together with (4.37), this implies the claim.

Also note that, for each $k = 2, \ldots, 5$, it follows from (1.1) and (4.35)–(4.36) that

\[Z_k(t) - Z_6(t) \leq B_k(t) - B_6(t) - \frac{2}{3}\epsilon_7M_1\]

on $Y_k(t) = 0$. Together with (2.2) of Lemma 2.1, this implies

\[(6.38) \quad \tau_6 \geq \tau_k \wedge D_1M_1 \geq \tau_k \wedge T_5(M_1) \quad \text{for } k = 2, \ldots, 5,\]

off of a set of exponentially small probability in $M_1$. Together with (6.36), this implies (6.35).
In order to demonstrate \((4.39)\) for \(k = 1\) and \(k = 6\), we restart \(Z(\cdot)\) at time \(T_5(M_1)\) and then apply \((2.12)\) of Lemma 2.3. Using the definition of \(T_5(M_1)\), for \(k = 1\), and \((4.38)\), with \(\epsilon_8 \overset{\text{def}}{=} \frac{1}{20} \delta_3 \epsilon_5 \leq \epsilon_5\), for \(k = 6\), one obtains
\[
P(Z_k(T'_5(M_1))) \geq 5\epsilon_5 M_1 \leq C_1 e^{-\epsilon' M_1} \text{ for } k = 1, 6,
\]
for appropriate \(C_1\) and \(\epsilon' > 0\). Inequality \((4.39)\) then follows from \((2.22)\) of Lemma 2.5.

In order to demonstrate \((4.40)\), we restart \(Z(\cdot)\) at time \(T_5(M_1)\), denoting the new process by \(\tilde{Z}(\cdot)\) and the corresponding Brownian motion by \(\tilde{B}(\cdot)\). By \((1.1)\),
\[
\tilde{Z}_1(t) \geq \epsilon_5 M_1 - t + \tilde{B}_1(t),
\]
and so \(\tilde{Z}_1(t) > 0\) on \([0, \frac{1}{2} \epsilon_5 M_1]\) off of a set of exponentially small probability in \(M_1\). Since \(\epsilon_8 < \frac{1}{19} \delta_3 \epsilon_5\), it follows, by \((2.32)\) of Lemma 2.6 and \((4.38)\), that
\[
\tilde{Y}_k(\frac{1}{2} \epsilon_5 M_1) \geq (\frac{1}{19} \epsilon_5 - \frac{8}{19} \epsilon_5) M_1 > 0 \quad \text{for } k = 2, \ldots, 5,
\]
off of a set \(F_{10}\) of exponentially small probability in \(M_1\).

We denote by \(\tilde{\tau}_k\) the first time at which \(\tilde{Z}_k(t) = 0\). Restarting the process at \(\tilde{\tau}_k\), it follows from \((2.15)\) of Lemma 2.4 that
\[
E[\tilde{Z}_k(\frac{1}{2} \epsilon_5 M_1)^2; F_{10}] \leq 24 \cdot 8 \epsilon_5 M_1 \quad \text{for } k = 2, \ldots, 5.
\]
On account of the upper bounds on \(P(F_{10})\) and \((2.21)\) of Lemma 2.5, one obtains
\[
(6.41) \quad E[\tilde{Z}_k(\frac{1}{2} \epsilon_5 M_1)^2] \leq 24 \cdot 8 \epsilon_5 M_1 + C_4 \quad \text{for } k = 2, \ldots, 5
\]
for appropriate \(C_4\), which depends on \(\epsilon_5\) but not on \(M_1\). This implies \((4.40)\). \(\square\)

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