THE INDEX THEOREM FOR QUASI-TORI

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Abstract

The Index theorem for holomorphic line bundles on complex tori asserts that some cohomology groups of a line bundle vanish according to the signature of the associated hermitian form. In this article, this theorem is generalized to quasi-tori, i.e. connected complex abelian Lie groups which are not necessarily compact. In view of the Remmert–Morimoto decomposition of quasi-tori as well as the Künneth formula, it suffices to consider only Cousin-quasi-tori, i.e. quasi-tori which have no non-constant holomorphic functions. The Index theorem is generalized to holomorphic line bundles, both linearizable and non-linearizable, on Cousin-quasi-tori using $L^2$-methods coupled with the Kazama–Dolbeault isomorphism and Bochner–Kodaira formulas.
Zusammenfassung

Ein Quasi-Torus ist eine zusammenhängende komplexe abelsche Lie-Gruppe \( X = \mathbb{C}^n / \Gamma \), wobei \( \Gamma \) eine diskrete Untergruppe von \( \mathbb{C}^n \) ist. \( X \) heißt Cousin-Quasi-Torus, wenn alle holomorphen Funktionen auf \( X \) konstant sind. Ist \( X \) kompakt, so ist \( X \) ein komplexer Torus.

Nach einem Satz von Remmert und Morimoto (vgl. [Mo2] oder [CC1, Prop. 1.1]) gibt es für jeden Quasi-Torus \( X \) eine Zerlegung \( X \cong \mathbb{C}^a \times (\mathbb{C}^*)^b \times X' \), wobei \( X' \) ein Cousin-Quasi-Torus ist. Das Ziel des vorliegenden Artikels ist, das Verschwinden von Kohomologiegruppen von Geradenbündeln auf \( X \) zu untersuchen.

Die Künnethformel (vgl. [Kau]) besagt, dass sich die Kohomologiegruppen von \( \mathbb{C}^a \times (\mathbb{C}^*)^b \) und des Cousin-Quasi-Torus \( X' \) zerlegen lassen. Man wird dadurch auf den Fall geführt, dass \( X \) ein Cousin-Quasi-Torus ist, da \( \mathbb{C}^a \times (\mathbb{C}^*)^b \) Steinsch ist und somit alle höheren Kohomologiegruppen (mit Grad \( \geq 1 \)) von kohärenten Garben verschwinden. Es wird also im vorliegenden Artikel angenommen, dass \( X \) ein Cousin-Quasi-Torus ist.

Sei \( F \) der maximale komplexe Unterraum von \( \mathbb{R} \Gamma \) und \( m := \dim_{\mathbb{C}} F \). Wie im kompakten Fall kann jedem holomorphen Geradenbündel \( L \) eine hermitesche Form \( \mathcal{H} \) auf \( \mathbb{C}^n \) zugeordnet werden, deren Imaginärteil \( \text{Im} \mathcal{H} \) mit der ersten Chernklasse \( c_1(L) \) von \( L \) assoziiert ist und ganzzahlige Werte in \( \Gamma \times \Gamma \) annimmt. Im Unterschied zum kompakten Fall ist \( \mathcal{H} \) nicht eindeutig. Lediglich die Einschränkung von \( \text{Im} \mathcal{H} \) auf \( \mathbb{R} \Gamma \times \mathbb{R} \Gamma \), und somit \( \mathcal{H}|_{F \times F} \), ist eindeutig bestimmt. Dies macht zumindest plausibel, dass nur \( \mathcal{H}|_{F \times F} \) für die Eigenschaften von \( L \) verantwortlich ist. Die vorliegende Dissertation widmet sich dem Beweis des folgenden Satzes:

**Index-Satz für Cousin-Quasi-Tori.** Sei \( X = \mathbb{C}^n / \Gamma \) ein Cousin-Quasi-Torus, \( F \) der maximale komplexe Unterraum von \( \mathbb{R} \Gamma \), \( L \) ein holomorphes Geradenbündel auf \( X \) und \( \mathcal{H} \) eine mit \( L \) assoziierte hermitesche Form auf \( \mathbb{C}^n \times \mathbb{C}^n \). Sei \( m := \dim_{\mathbb{C}} F \). Die Einschränkung \( \mathcal{H}|_{F \times F} \) habe \( s^-_F \) negative und \( s^+_F \) positive Eigenwerte. Dann gilt

\[
H^q(X, L) = 0 \quad \text{für} \quad q < s^-_F \quad \text{oder} \quad q > m - s^+_F.
\]

Dieser Satz wird zurückgeführt auf den Index-Satz für komplexe Tori, wie er von Mumford [Mum], Kempf [Kem], Umemura [U], Matsushima [Ma] und Murakami [Mur] für kompakte \( X \) bewiesen wurde. Da \( X \) stark \((m+1)\)-vollständig ist (vgl. [Kaz1]), enthält der Satz auch einen Spezialfall des Resultats von Andreotti und Grauert, das besagt, dass \( H^q(X, \mathcal{F}) = 0 \) ist für alle \( q \geq m + 1 \) und für jede kohärente analytische Garbe \( \mathcal{F} \) auf \( X \) (vgl. [AGr]).

Das Verschwinden von \( H^q(X, L) \) kann unter Verwendung der Dolbeault-Isomorphismen auf gewisse \( \partial \)-Gleichungen für \( L \)-wertige \((0,q)\)-Formen zurückgeführt werden. Diese können mit \( L^2 \)-Methoden gelöst werden. Man zeigt zunächst die Existenz einer formalen Lösung einer \( \partial \)-Gleichung in einem Hilbertraum, indem man die benötigte \( L^2 \)-Abschätzung nachweist, und beweist dann die Glattheit der Lösung. Letzteres
ZUSAMMENFASSUNG

kann mit Hilfe der Regularitätstheorie von $\bar{\partial}$-Operatoren erledigt werden, also ist der
entscheidende Schritt der Nachweis der benötigten $L^2$-Abschätzungen. Diese kann
man durch Anwendung der Bochner–Kodaira-Ungleichungen bekommen.

Jeder Cousin Quasi-Torus $X$ hat eine Faserbündelstruktur über einem komplexen
Torus $T$ mit steinschen Fasern (siehe §5.1 und [eq 2.3]). Mit Hilfe der Lerayschen
Spektralsequenz folgt

$$H^q(X, L) \cong H^q(T, p_*\mathcal{O}_X(L))$$

für alle $q \geq 0$ ,

wobei $p: X \to T$ die Projektion aus [eq 2.3] ist. Die Idee ist jetzt zu zeigen, dass der
Dolbeault Komplex der Garben $(\mathcal{A}^0(L) \otimes \mathcal{O}_T, \bar{\partial})$, eine azyklische Auflösung
von $p_*\mathcal{O}_X(L)$ auf $T$ ist und das Verschwinden der Kohomologie durch Lösen der
$\bar{\partial}$-Gleichungen zu zeigen. Kazama [Kaz2] und Kazama–Umeno [KU2] geben ei-
ne leicht veränderte Formulierung, sie betrachteten die Auflösung von $\mathcal{O}_X(L)$ durch
einen Unterkomplex $(\mathcal{H}^0(L), \bar{\partial})$ von $(\mathcal{A}^0(L), \bar{\partial})$ (siehe [eq 2.3] für die Definition von
$\mathcal{H}^0(L)$). Der Teilkomplex ist ebenfalls eine azyklische Auflösung von $\mathcal{O}_X(L)$ auf
$X$ und liefert damit den Kazama–Dolbeault Isomorphismus (vgl. [Kaz2], siehe auch
Theorem 2.3.1). Letzterer Ansatz wird hier aufgegriffen. Das Ziel der Darstellung
ist dann die Lösung der $\bar{\partial}$-Gleichung $\bar{\partial}\xi = \psi$ für ein gegebenes $\psi \in \Gamma(X, \mathcal{H}^0(L))$
mit $\bar{\partial}\psi = 0$.

Jedes Geradenbündel $L$ auf $X$ kann durch ein System von Automorphiefaktoren
definiert werden, die in eine zur Appell–Humbert-Normalform analoge Normalform
übergeführt werden können, die gegeben ist durch (vgl. [CC1 §2.2] und [V §2])

$$g(\gamma) e^{\pi i(\gamma z) + \bar{\partial} f(\gamma)}$$

wobei $g$ ein Halbcharakter auf $\Gamma$ und $\{f(\gamma)\}_{\gamma \in \Gamma}$ ein additiver Kozykel ist (vgl.
[CC1 §2.2] und [V §2]), siehe auch (eq 2.8)). Wenn $\{f(\gamma)\}_{\gamma \in \Gamma}$ ein Korand ist, so
wird $L$ als linearisierbar bezeichnet; andernfalls als nicht linearisierbar. Indem
man den Trick verwendet, den Murakami in [Mur] für den kompakten Fall benutzt
hat (siehe §3.3), nämlich die Metrik $g$ so abzuändern, dass der von linearen Teil
(dem zahmen Teil) von $L$ in den Basisrichtungen kommende Krümmungsterm von
unten beschränkt ist, wenn $q$ im gegebenen Bereich liegt, kann man die benötigten
$L^2$-Abschätzungen erhalten, wenn $L$ linearisierbar ist (siehe §4). Dies beweist den
Index-Satz für linearisierbare $L$ (siehe Theorem 4.1.1)

Beim Nachweis der benötigten $L^2$-Abschätzungen für nicht linearisierbare $L$ auf
$X$ gibt eine zusätzliche technische Schwierigkeit, die von dem vom nichtlinearen
Teil (dem wilden Teil) von $L$ kommenden Krümmungsterm herrührt. Für diesen
wird Takayama’s schwaches $\partial\bar{\partial}$-Lemma ([Taka2], Lemma 3.14); siehe auch §5.1
angewandt, um den Term auf relativ kompakten Teilmengen von $X$ zu beschränken.
Dadurch erhält man die benötigten $L^2$-Abschätzungen nicht auf $X$, sondern lediglich
auf der ausschöpfenden Familie $\{K_c\}_{c \in \mathbb{R} > 0}$ von pseudokonvexen relativ kompakten
Teilmengen. Man erhält dann eine Folge $\{\xi_\nu\}_{\nu \geq 1}$ von lokalen Lösungen, so dass
$\bar{\partial}\xi_\nu = \psi|_{K_c}$ ist für ein gegebenes $\psi \in \Gamma(X, \mathcal{H}^0(L)) \cap \ker \bar{\partial}$ und für alle ganzen
Zahlen $\nu \geq 1$. Indem man ein Argument im Beweis von Theorem B für Steinsche
Räume in [GR] Ch. IV, §5 nachvollzieht, speziell indem man eine Approximation
vom Runge-Typ verwendet, kann man die lokalen Lösungen $\xi_\nu$ so korrigieren, dass
sie auf jedem $K_c$ konvergieren, was dann eine globale Lösung für alle $q$ im gegebenen
Bereich liefert (siehe §5.3). Der Beweis des Index-Satzes ist damit vollständig.
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CHAPTER 1

Introduction and the main theorem

A quasi-torus is a complex abelian Lie group $X = \mathbb{C}^n/\Gamma$, where $\Gamma$ is a discrete subgroup of $\mathbb{C}^n$. $X$ is said to be a Cousin-quasi-torus if all holomorphic functions on $X$ are constant functions.\(^1\) $X$ is the familiar complex torus when it is compact, i.e. when $\text{rk} \, \Gamma = 2n$.

The study of quasi-tori dates back to the early 20th century when Cousin studied the triply periodic functions of two complex variables (\cite{Cou}). There he showed the existence of 2-dimensional quasi-tori without non-constant holomorphic functions. He also gave, among other things, a complete description of holomorphic line bundles on quasi-tori of dimension 2 and their sections using a method of asymptotic counting of zeros of the sections. In the 60’s, Kopfermann (\cite{Kop}) studied systematically toroidal groups of arbitrary dimensions with a view to generalize the theory of abelian functions on complex tori. He also gave an example of a non-compact toroidal group with no non-constant meromorphic functions. Morimoto (\cite{Mo1} and \cite{Mo2}) studied Cousin-quasi-torus as the maximal toroidal subgroup of a complex (not necessarily abelian) Lie group, aiming to classify non-compact complex Lie groups. He classified all 3-dimensional abelian complex Lie groups. In the early 70’s, Andreotti and Gherardelli gave seminars on quasi-abelian varieties, i.e. Cousin-quasi-tori which possess structures of quasi-projective algebraic varieties (\cite{AGh}). They showed that, among other things, a Cousin-quasi-torus is a quasi-abelian variety if and only if the Generalized Riemann Relations are satisfied on it. Later on, among other contributors, Kazama (\cite{Kaz1} and \cite{Kaz2}), Pothering (\cite{P}), Vogt (\cite{V}), Huckleberry and Margulis (\cite{HM}), Abe (\cite{Ab1} and \cite{Ab2}), Capocasa and Catanese (\cite{CC1} and \cite{CC2}), and Takayama (\cite{Taka2}) made some direct contributions to the theory of quasi-tori and Cousin-quasi-tori. A brief exposition of the historical development of the Generalized Riemann Relations can be found in \cite{CC1}, p. 29, and the Introduction of [AK] describes a brief chronology of the study of toroidal groups in general.

The current research stems from the study of Capocasa and Catanese (ref. \cite{CC1} and \cite{CC2}). In \cite{CC1}, they gave an affirmative answer to a long standing problem of whether the existence of a non-degenerate meromorphic function on a quasi-torus is equivalent to the Generalized Riemann Relations. In \cite{CC2}, they moved on to prove the Lefschetz type theorems on quasi-tori in the best form, based on a statement of Abe with an erroneous proof in \cite{Ab3} Thm. 6.4 (see \cite{CC2} Corollary 1.2)).\(^2\) Abe’s statement is then substituted by a result proven by Takayama (\cite{Taka1} Thm. 1.3 and

\(^1\)A Cousin-quasi-torus is also called a toroidal group or $(H,C)$-group in literature, where the latter means that all holomorphic functions are constant (ref. \cite{AK} Def. 1.1.1)).

\(^2\)Théorème 6.4 in \cite{Ab3} asserts that, on a non-compact toroidal group $X$, there exists a constant $c > 0$ such that, for any holomorphic line bundle $L$ with an associated hermitian form $\mathcal{H}$ on $\mathcal{C}^n$ such that $\mathcal{H} |_{F \times F} > c I_m$ (where $I_m$ is the $m \times m$-identity matrix and $F$ is the maximal complex subspace of $\mathbb{R} \Gamma$; see \cite{2}, $H^0(X, L)$ is non-trivial, and in fact infinite-dimensional.
1. INTRODUCTION AND THE MAIN THEOREM

These results clarify some basic properties of meromorphic functions and global sections of holomorphic line bundles on quasi-tori. This article goes a step further into the investigation of the higher cohomology groups of holomorphic line bundles on quasi-tori. The aim is to generalize the Index theorem on tori to quasi-tori.

1.1. The main theorem

Denote the \( \mathbb{C} \)-span and \( \mathbb{R} \)-span of \( \Gamma \) by \( \mathbb{C} \Gamma \) and \( \mathbb{R} \Gamma \) respectively. Let \( \pi : \mathbb{C}^n \to X \) be the natural projection. Then \( K := \pi(\mathbb{C} \Gamma) = \mathbb{R} \Gamma / \Gamma \) is the maximal compact subgroup of \( X \), and \( F := \mathbb{R} \Gamma \cap \sqrt{-1} \mathbb{R} \Gamma \) is the maximal complex subspace in \( \mathbb{R} \Gamma \).

By a theorem of Remmert and Morimoto (ref. [Mo2], see also [CC1, Prop. 1.1]), if \( X \) is a quasi-torus, there is a decomposition \( X \cong \mathbb{C}^a \times (\mathbb{C}^*)^b \times X' \), where \( X' \) is Cousin. The aim of this article is to investigate the vanishing of cohomology groups on \( X \) stated. In this case, \( \pi \) takes integral values on \( \Gamma \) and corresponds to the first Chern class \( c_1(L) \) of \( L \) (ref. [CC1]). \( \pi \) is uniquely determined only on \( \mathbb{R} \Gamma \times \mathbb{R} \Gamma \), so \( \mathcal{H} \) is uniquely determined only on \( F \times F \).

The following theorem is a generalization of the Index theorem on complex tori (ref. [Mum, p. 150], [Mur] or [BL, §3.4]) to Cousin-quasi-tori, which is the main result of this article.

**Theorem 1.1.1.** Let \( X = \mathbb{C}^n / \Gamma \) be a Cousin-quasi-torus, \( F \) the maximal complex subspace of \( \mathbb{R} \Gamma \), \( L \) a holomorphic line bundle on \( X \), and \( \mathcal{H} \) a hermitian form on \( \mathbb{C}^n \times \mathbb{C}^n \) associated to \( L \). Let \( m := \dim_{\mathbb{C}} F \). Suppose \( \mathcal{H}|_{F \times F} \) has respectively \( s_F^- \) negative and \( s_F^+ \) positive eigenvalues. Then one has

\[
H^q(X, L) = 0 \quad \text{for } q < s_F^- \quad \text{or} \quad q > m - s_F^+.
\]

Let \( \mathcal{O}^p_X \) be the sheaf of germs of holomorphic \( p \)-forms on \( X \), and set \( \mathcal{O}^p_X(L) := \mathcal{O}^p_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(L) \). Since the cotangent bundle of \( X \) is trivial, one has \( \mathcal{O}^p_X(L) \cong \bigoplus^{(n)} \mathcal{O}_X(L) \), and thus \( H^q(X, \mathcal{O}^p_X(L)) \cong \bigoplus^{(n)} H^q(X, L) \). Therefore, one has the following

---

3Theorem 1.3 and 6.1 in [Taka1] together asserts that, for any positive line bundle \( L \) on a non-compact toroidal group \( X \), there exists an explicitly given integer \( \mu_0 > 0 \) such that \( H^0(X, L^{\mu_0}) \) is non-trivial for all \( \mu \geq \mu_0 \). Corollary 1.2 in [CC2] holds true by applying Takayama’s result and Proposition 1.1 in [CC2]. Takayama also gives a different proof of a weaker form of Lefschetz type theorems in [Taka2].

4The Index theorem on complex tori was first proven by Mumford [Mum] and Kempf [Kem] in the algebraic case, and later by Umemura [U], Matsushima [Ma] and Murakami [Mur] in the analytic case.
The required $\overline{\partial}$-equations are solved by exhibiting $A^0(L)$ such that $\overline{\partial}\psi = 0$ and an exhaustive sequence $\{K_{s}\}_{s \in N_{>0}}$ of pseudoconvex relatively compact open subsets of $X$, a sequence $\{\xi_{s}\}_{s \in N_{>0}}$ of weak solutions of $\overline{\partial}\xi_{s} = \psi|_{K_{s}}$ is obtained. Using a Runge-type approximation (see [GR, Ch. IV, §1, Thm. 7], the solutions $\xi_{s}$'s can be adjusted so that they converge to a weak global solution of $\overline{\partial}\xi = \psi$. A strong solution in $\Gamma(X, A^{0,0}(L))$ then exists by the regularity theory for $\overline{\partial}$ or elliptic operators (ref. [Hör3 Thm. 4.2.5 and Cor. 4.2.6] or [Hör2 Thm. 4.1.5 and Cor. 4.1.2]) and the Kazama–Dolbeault isomorphism (ref. [KU2], see also Theorem 2.3.1).
CHAPTER 2

Preliminaries

2.1. A \((\mathbb{C}^*)^{n-m}\)-principal bundle structure on \(X\)

Let \(X = \mathbb{C}^n/\Gamma\) be a Cousin-quasi-torus. Then one has \(\mathbb{C}\Gamma = \mathbb{C}^n\) and \(\text{rk} \Gamma = n+m\) with \(m > 0\). Define \(K := \pi(\mathbb{R}\Gamma) = \mathbb{R}\Gamma/\Gamma\) and \(F := \mathbb{R}\Gamma \cap \sqrt{-1}\mathbb{R}\Gamma\) as before. Fix a basis of \(\mathbb{C}^n\) such that the period matrix of \(X\) is given by

(eq 2.1)

\[
\begin{bmatrix}
I_{n-m} & A_1 + \sqrt{-1}B_1 \\
I_m & A_2 + \sqrt{-1}B_2
\end{bmatrix},
\]

where an empty entry means a zero entry, \(I_r\) denotes the identity matrix of rank \(r\), \(A_i\) and \(B_i\) denotes real matrices such that \(A_1\) and \(B_1\) are of size \((n-m) \times m\), and \(A_2\) and \(B_2\) are square matrices of size \(m \times m\). By re-ordering the basis of \(\mathbb{C}^n\) and respectively the basis of \(\Gamma\), \(B_2\) can be assumed to be invertible (since \(\text{rk} \Gamma = n+m\)).

Take a change of coordinates given by the matrix

(eq 2.2)

\[
\begin{bmatrix}
I_{n-m} & -B_1B_2^{-1} \\
B_2^{-1}
\end{bmatrix},
\]

the period matrix under the new coordinates is then given by

where

\[
\beta_1 = -B_1B_2^{-1}, \quad \alpha_1 = A_1 - B_1B_2^{-1}A_2,
\]

\[
\beta_2 = B_2^{-1}, \quad \alpha_2 = B_2^{-1}A_2 + \sqrt{-1}I_m,
\]

which are all real matrices except for \(\alpha_2\). Let the new coordinates of \(\mathbb{C}^n\) be denoted by \((u,v) := (u^1, \ldots, u^{n-m}, v^1, \ldots, v^n)\), or simply by \(z := (z^1, \ldots, z^n)\). This new coordinate system is called an apt coordinate system (with respect to \(\Gamma\)) (see [CC1, Def. 2.3]; also called a toroidal coordinate system, see [AK, §1.1.12]), which is characterized by the properties

1. \(F = \{(u,v) \in \mathbb{C}^n : u = 0\};\)
2. each coordinate of the imaginary part \(\text{Im} \, u\) of \(u\) is a global function on \(X\) and \(K = \{(u,v) \mod \Gamma \in X : \text{Im} \, u = 0\};\)
3. the standard basic vectors \(e_1, \ldots, e_{n-m}\) in \(\mathbb{C}^n\) can be completed to a basis of \(\Gamma\).

The choice of an apt coordinate system fixes a decomposition \(\mathbb{C}^n = E \oplus F\), where \(E\) is the complex vector subspace of \(\mathbb{C}^n\) spanned by \(e_1, \ldots, e_{n-m}\) with \(u\) as the coordinate vector. Set \(\Gamma' := \Gamma \cap E = \mathbb{Z} \langle e_1, \ldots, e_{n-m} \rangle = \mathbb{Z}^{n-m}\). Let \(\tilde{\rho} : \mathbb{C}^n \to F\) be the projection \((u,v) \mapsto v\). It can be seen from (eq 2.2) that \(\tilde{\rho}(\Gamma')\) is a lattice in \(F\), i.e. a discrete subgroup of \(F\) of rank \(2m\). Let \(T^m := F/\tilde{\rho}(\Gamma')\), which is a complex torus of dimension \(m\). Then \(\tilde{\rho}\) induces a holomorphic epimorphism \(\rho : X \to T^m\) with kernel \(E/\Gamma' \cong (\mathbb{C}^*)^{n-m}\). Therefore, \(X\) has a \((\mathbb{C}^*)^{n-m}\)-principal bundle structure given by
the exact sequence of groups

\[
\begin{array}{cccccc}
0 & \to & (\mathbb{C}^\ast)^{n-m} & \overset{\iota}{\to} & X & \overset{p}{\to} & T^m & \to & 0
\end{array}
\]

(eq 2.3)

(ref. [St] \[7.4\] and [Hir] Thm. 3.4.3]). In local coordinates, \( \iota \) is given by \( u \mod \Gamma \mapsto (u,0) \mod \Gamma \) and \( p \) by \( (u,v) \mod \Gamma \mapsto v \mod \bar{p}(\Gamma) \). In view of the fibre bundle structure, the tangential directions with respect to the \( u \)-coordinates are called the fibre directions, while those of the \( v \)-coordinates are called the base directions. These terminologies are used throughout this article to simplify description.

Since the cotangent bundle of \( X \) is trivial, the decomposition \( \mathbb{C}^n = E \oplus F \) induces a decomposition of the holomorphic cotangent bundle \( T^{\ast,0} := T^{\ast,0}_X \) of \( X \) with respect to the fibre and base directions, i.e.

\[
T^{\ast,1,0} = T^{\ast,1,0}_u \oplus T^{\ast,1,0}_v,
\]

where \( T^{\ast,1,0}_u \) and \( T^{\ast,1,0}_v \) are holomorphic subbundles generated at every point of \( X \) respectively by \( du^i \) for \( i = 1, \ldots, n - m \) and \( dv^j \) for \( j = 1, \ldots, m \). For later use, define as usual \( T^{v,p,q}_v := \bigwedge^p T^{1,0}_v \wedge \bigwedge^q T^{1,0}_v \) for any integers \( p, q \geq 0 \), where \( T^{0,0}_v = \bigwedge^0 T^{1,0}_v = \mathcal{O} \) denotes the trivial line bundle on \( X \). Define \( T^{u,p,q}_v \) similarly with \( T^{u}_u \) in place of \( T^{u}_v \).

2.2. An exhaustive family of pseudoconvex subsets

Every Cousin-quasi-torus is pseudoconvex and strongly \((m + 1)\)-complete (cf. [Kaz1] and [Take]; convention of the numbering here following [D1] pp. 512]). Indeed, define \( \varphi(z) := \varphi(\text{Im } u) := ||\text{Im } u||^2 \) (\( ||\cdot|| \) is the Euclidean 2-norm here). Then \( \varphi \) is an exhaustion function on \( X \) whose Levi form is given by

\[
\sqrt{-1} \partial \bar{\partial} \varphi = \frac{1}{2} \sum_{i=1}^{n-m} du^i \wedge \bar{du}^i,
\]

which is semi-positive definite with exactly \( n - m \) positive eigenvalues everywhere on \( X \). Therefore, \( X \) is pseudoconvex and strongly \((m + 1)\)-complete.

For any \( c > 0 \), set \( K_c := \{ z \in X : \varphi(z) < c \} \). Then \( \{ K_c \}_{c>0} \) forms an exhaustive family of open relatively compact subsets of \( X \). Set also \( K_\infty := X \), and \( K_0 := K \), the maximal compact subgroup of \( X \). For every \( c > 0 \), \( K_c \) is of course itself pseudoconvex.

2.3. Kazama sheaves and Kazama–Dolbeault isomorphism

Let \( \mathcal{A} := \mathcal{A}_X \) be the sheaf of germs of smooth functions on \( X \). Fix a choice of an apt coordinate system. Let \( V \) be any holomorphic vector bundle on \( X \). Define on \( X \) the Kazama sheaves as in [KU2] to be

\[
\mathcal{H} := \left\{ f \in \mathcal{A} : \frac{\partial f}{\partial u^i} \equiv 0 \text{ for } 1 \leq i \leq n - m \right\}
\]

and

\[
\mathcal{H}^{0,q} := \mathcal{H} \otimes_{\mathcal{A}} \mathcal{O}_{T^m}, \quad \mathcal{H}^{0,q}(V) := \mathcal{H}^{0,q} \otimes_{\mathcal{O}_X} \mathcal{O}_X(V)
\]

for \( 1 \leq q \leq m \), where \( p \) is the projection given in (eq 2.3) and \( \mathcal{H}^{0,q}(V) \) is the sheaf of germs of \((0,q)\)-forms on the base torus \( T^m \). In words, Kazama sheaf \( \mathcal{H} \) consists of germs of sections of \( \mathcal{A} \) which are holomorphic in the fibre directions, and \( \mathcal{H}^{0,q} \) consists of \( \mathcal{H} \)-valued \((0,q)\)-forms in the base directions. Note that the definitions of the sheaves depend on the choice of the decomposition (eq 2.4). Set also \( \mathcal{H}^{0,0}(V) := \mathcal{H}(V) \). For notational convenience, the space of sections \( \Gamma(U, \mathcal{H}^{0,q}(V)) \) over any subset \( U \) of \( X \),
is also denoted by $\mathcal{H}^{0,q}(U; V)$, and similarly for spaces of sections of other sheaves. The following Kazama–Dolbeault isomorphism is proven in [KU1] and [KU2] (see also [Kaz2]).

**Theorem 2.3.1.** The complex

(eq 2.5) \[ 0 \longrightarrow \mathcal{O}_X(V) \longrightarrow \mathcal{H}^{0,0}(V) \xrightarrow{\partial} \mathcal{H}^{0,1}(V) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \mathcal{H}^{0,m}(V) \longrightarrow 0 \]

is an acyclic resolution of $\mathcal{O}_X(V)$ over $X$, i.e. $H^p(X, \mathcal{H}^{0,q}(V)) = 0$ for any $p \geq 1$ and $0 \leq q \leq m$. Consequently, the natural injection of complexes

\[ (\mathcal{H}^{0,\bullet}(X; V), \partial) \longrightarrow (\mathcal{A}^{0,\bullet}(X; V), \partial) \]

induces the isomorphisms

\[ H^q_\partial(\mathcal{H}^{0,\bullet}(X; V)) \cong H^q_\partial(\mathcal{A}^{0,\bullet}(X; V)) \cong H^q(X, V) \]

for all $q \geq 0$.

In view of the Kazama–Dolbeault isomorphism, to show the vanishing of $H^q(X, V)$ it suffices to show that for any $\partial$-closed $\psi \in \mathcal{H}^{0,q}(X; V)$ there exists $\xi \in \mathcal{A}^{0,q-1}(X; V)$ such that

(eq 2.6) \[ \partial \xi = \psi \].

In fact, (eq 2.6) means that the class $\psi \mod \partial \mathcal{A}^{0,q-1}(X; V)$ is the zero class in $H^q_\partial(\mathcal{A}^{0,\bullet}(X; V))$; so, by the isomorphism, the class $\psi \mod \partial \mathcal{H}^{0,q-1}(X; V)$ is also the zero class in $H^q_\partial(\mathcal{H}^{0,\bullet}(X; V))$. Therefore, $\xi$ in (eq 2.6) can be chosen in $\mathcal{H}^{0,q-1}(X; V)$.

### 2.4. Holomorphic line bundles on $X$

Every holomorphic line bundle $L$ on $X$ can be defined by a system of factors of automorphy, which can be taken into a normal form analogous to the Appell–Humbert normal form, given by (ref. [CC1] Remark 1.11 and §2.2 and [V] §2)

(eq 2.7) \[ \varrho(\gamma)e^{\pi \sqrt{-1} (\gamma, \gamma')} + \varrho'(\gamma, \gamma') + f_\gamma(z) = \forall \gamma \in \Gamma, \]

where

- $\mathcal{H}$ is a hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$, whose imaginary part $\text{Im} \mathcal{H}$ takes integral values on $\Gamma \times \Gamma$ and corresponds to the first Chern class $c_1(L)$ of $L$;
- $\varrho$ is a semi-character for $\text{Im} \mathcal{H}$ on $\Gamma$, i.e.

\[ \varrho(\gamma + \gamma') = \varrho(\gamma)\varrho(\gamma')e^{\pi \sqrt{-1} \text{Im} \mathcal{H}(\gamma, \gamma')} \]

for all $\gamma, \gamma' \in \Gamma$, and $|\varrho(\gamma)| = 1$ for all $\gamma \in \Gamma$; and

- $\{f_\gamma\}_{\gamma \in \Gamma}$ is an additive 1-cocycle with values in $\mathcal{O}_\mathbb{C}^n(\mathbb{C}^n)$, i.e. $f_\gamma \in \mathcal{O}_\mathbb{C}^n(\mathbb{C}^n)$ for all $\gamma \in \Gamma$ and

\[ f_{\gamma + \gamma'}(z) = f_{\gamma'}(z + \gamma) + f_\gamma(z) \]

for all $\gamma, \gamma' \in \Gamma$.

According to [V, Prop. 8], under a fixed apt coordinate system, $f_\gamma(z)$ can be taken to be independent of the variable $v$ for every $\gamma \in \Gamma$. Denote by $\gamma_u$ the image of $\gamma \in \Gamma$ under the projection $\mathbb{C}^n \ni (u, v) \mapsto u \in E$ (see page 4 for the definition of $E$). Also according to [V, Prop. 8] (cf. also [CC1] §1.2), for any $u \in E$, one has

(eq 2.8) \[ \begin{cases} f_{\gamma'}(u) = 0 \\ f_\gamma(u + \gamma'_u) = f_\gamma(u) \end{cases} \]

for all $\gamma' \in \Gamma'$ and $\gamma \in \Gamma$, \]
where \( \Gamma := \Gamma \cap E = \mathbb{Z} \langle e_1, \ldots, e_{n-m} \rangle \) as in §2.1.

It is apparent that \( L \) can be decomposed into \( L_t \otimes L_w \), where \( L_t \) is defined by the linear part
\[
g(\gamma)e^{\pi \mathcal{B}(z,\gamma) + \frac{i}{2} \mathcal{H}(\gamma,\gamma)}
\]
of the factor of automorphy in (eq 2.7), while \( L_w \) is defined by the non-linear part
\[
e^{f(z)}.
\]

Call \( L_t \) and \( L_w \) the **tame part** and **wild part** of \( L \) respectively.

**Definition 2.4.1.** \( L \) is said to be **linearizable** if \( L_w \) is trivial, i.e. there exists a holomorphic function \( g \) on \( \mathbb{C}^n \) such that \( g(z + \gamma) - g(z) = f(\gamma) \) for all \( \gamma \in \Gamma \) and \( z \in \mathbb{C}^n \). \( L \) is said to be **non-linearizable** otherwise.

\( \text{Im} \mathcal{H} \) is uniquely determined only on \( \mathbb{R} \Gamma \times \mathbb{R} \Gamma \) (see [CC1] Remark 1.11 and also [AGH]). Then one has the following proposition.

**Proposition 2.4.2.** Let \( \mathcal{H} \) be a hermitian form associated to \( L \). Suppose in a chosen apt coordinate system the matrix associated to \( \mathcal{H}|_{E \times E} \) is given by \( H_E \). Then, \( \text{Re} H_E \) can be chosen arbitrarily by multiplying the cocycle defining \( L \) by a suitable coboundary.

**Proof.** Fix \( \mathcal{H} \) and an apt coordinate system. Let \( \mathcal{B}(u, u) \) be any symmetric \( \mathbb{C} \)-bilinear form with real coefficients on \( E \times E \) and denote the corresponding \((n-m) \times (n-m)\)-matrix under the chosen apt coordinates by \( B \). Note that \( \gamma_u \) is a real vector by the choice of coordinates (see (eq 2.2)). Then multiplying \( e^{\frac{i}{2} \mathcal{B}(u+\gamma_u,u+\gamma_u) - \frac{i}{2} \mathcal{B}(u,u)} \) (which is a component of a 1-coboundary) to (eq 2.7) gives rise to a system of factors of automorphy defining a line bundle isomorphic to \( L \). The new system of factors of automorphy is of the same form as in (eq 2.7) with \( \mathcal{H} \) replaced by \( \mathcal{H}' \), where \( \mathcal{H}' \) is a hermitian form such that \( \mathcal{H}'(z,\gamma) = \mathcal{H}(z,\gamma) + \mathcal{B}(u,\gamma_u) \) (note that such hermitian \( \mathcal{H}' \) exists since all \( \gamma_u \)'s as well as \( B \) are real). Then \( \text{Re} H'_E = \text{Re} H_E + B \), while the other entries of the matrix of \( \text{Im} \mathcal{H}' \) are the same as the respective entries of \( \text{Im} \mathcal{H} \). Therefore, since \( B \) is arbitrary, \( \text{Re} H_E \) can be chosen arbitrarily. \( \square \)

This shows that one cannot, in general, replace \( s^+_E \) and \( s^-_E \) in Theorem 1.1.1 by \( s^+_t \) and \( s^-_t \), the numbers of positive and negative eigenvalues of \( \mathcal{H} \) (instead of \( \mathcal{H}|_{E \times E} \)) respectively. In fact, if \( L \) is the trivial line bundle, \( \mathcal{H} \) can be chosen such that \( \text{Re} H_E \) is negative definite and the other entries of the matrix associated to \( \text{Im} \mathcal{H} \) are zero. Such \( \mathcal{H} \) has at least 1 negative eigenvalue. However, \( \dim \mathcal{H}^q(X, \mathcal{O}_X) \) cannot be 0 since there exist constant functions on \( X \) (which is true even for any complex manifold). In fact, Kazama has shown in [Kaz2] Thm. 4.3] that \( \mathcal{H}^q(X, \mathcal{O}_X) \) are non-trivial for all \( 1 \leq q \leq m \) for any Cousin-quasi-torus \( X \).

### 2.5. A hermitian metric on \( L \)

Given a holomorphic line bundle \( L \), hermitian metrics \( \eta_t \) on the tame part \( L_t \) and \( \eta_w \) on the wild part \( L_w \) of \( L \) are defined below. The product \( \eta := \eta_t \eta_w \) then defines a hermitian metric on \( L \).

Define a hermitian metric on \( L_t \) by \( \eta_t(z) := e^{-\pi \mathcal{H}(z,z)} \) as in the compact case. Then the corresponding curvature form on \( X \), called the **tame part of the curvature form** of \( L \), is given by
\[
\Theta^t := -\sqrt{-1} \bar{\partial} \partial \log \eta_t = \pi \sqrt{-1} \bar{\partial} \partial \mathcal{H}(z,z).
\]
Next is to define a hermitian metric $\eta_w$ on $L_w$. An apt coordinate system is fixed in what follows. First notice the following

**Proposition 2.5.1.** There exists a smooth function $h$ on $\mathbb{C}^n$ which is holomorphic along the fibre directions under the chosen apt coordinate system and satisfies

\[(eq\ 2.9)\quad h(z + \gamma) - h(z) = f_\gamma(u) \quad \text{for all } \gamma \in \Gamma.\]

**Proof.** This follows from the fact that $H^1(X, \mathscr{H}) = 0$ (ref. [KU2]). A direct proof is given as follows.

Let $\Gamma''$ be the subgroup of $\Gamma$ generated by the last 2$m$ column vectors of the period matrix $[eq\ 2.2]$ of $\Gamma$. Then $\Gamma = \Gamma' \oplus \Gamma''$ ($\Gamma'$ defined as in $[2.1]$). Write $\gamma_v := \tilde{p}(\gamma)$ for all $\gamma \in \Gamma$ ($\tilde{p}$ defined as in $[2.1]$). Note that $\gamma'_v = 0$ for all $\gamma' \in \Gamma'$. Recall that $\tilde{p}(\Gamma) = \tilde{p}(\Gamma'')$ is the lattice defining $T_m$ in $[eq\ 2.3]$, therefore discrete in $F$. Take a suitable smooth function $\rho$ with compact support on $F$ with variable $v$ such that $\sum_{\gamma'' \in \Gamma''} \rho(v + \gamma'_v) \equiv 1$. Note that the sum is a sum of finitely many non-zero terms at each $v \in F$ due to the discreteness of $\Gamma''$. Define

\[h(z) := - \sum_{\gamma'' \in \Gamma''} \rho(v + \gamma'_v) f_{\gamma''}(u).\]

Then $h$ is holomorphic along the fibre directions. To see that it satisfies $[eq\ 2.9]$, note that, for any $\gamma_0 = \gamma'_0 + \gamma''_0 \in \Gamma$ where $\gamma'_0 \in \Gamma'$ and $\gamma''_0 \in \Gamma''$,

\[h(z + \gamma_0) = - \sum_{\gamma'' \in \Gamma''} \rho(v + \gamma'_{0} + (\gamma_0)_{v}) f_{\gamma''}(u + (\gamma_0)_{u}) = - \sum_{\gamma'' \in \Gamma''} \rho(v + \gamma'_{0} + (\gamma_0)_{v}) f_{\gamma''}(u + (\gamma''_0)_{u})
\]

\[= - \sum_{\gamma'' \in \Gamma''} \rho(v + \gamma'_{0} + (\gamma_0)_{v}) \left( f_{\gamma''+\gamma''_0}(u) - f_{\gamma''_0}(u) \right) = h(z) + f_{\gamma''_0}(u),\]

using the fact that $f_{\gamma''}(u + \gamma'_{0}) = f_{\gamma''}(u)$ for all $\gamma' \in \Gamma'$ (see $[eq\ 2.8]$) and $\Gamma'' = \Gamma'' + \gamma''_0$. Applying $[eq\ 2.8]$ again, one obtains

\[f_{\gamma_0}(u) = f_{\gamma''_0}(u + (\gamma'_0)_{u}) + f_{\gamma''_0}(u) = f_{\gamma''_0}(u).\]

This $h$ therefore satisfies $[eq\ 2.9].$ \qed

It follows from $[eq\ 2.9]$ that $\partial \eta_w$ and $\partial \eta_w$ define smooth functions on $X$ (note that $f_\gamma(u)$ are holomorphic). Therefore, $\overline{\partial} h$ is a (smooth) 1-form on $X$, so is $\partial h$. Take any $\delta \in \mathscr{H}(X)$, and let $h_\delta := h - \delta$ for notational convenience. Define a hermitian metric on the wild part $L_w$ of $L$ by $\eta_w(z) := e^{-2\text{Re} h_\delta(z)}$. The corresponding curvature form, called the wild part of the curvature form of $L$, is given by

\[\Theta_{20} := - \sqrt{-1} \overline{\partial} \log \eta_w = 2\sqrt{-1} \overline{\partial} \text{Re} h_\delta
\]

\[= \sqrt{-1} d (\overline{\partial} h_\delta - \partial h_\delta).\]

Note that, since $\overline{\partial} h$ is a smooth $(0,1)$-form on $X$, $\sqrt{-1} d (\overline{\partial} h_\delta - \partial h_\delta)$ is a $d$-exact smooth real $(1,1)$-form on $X$.

The function $\delta$ is an auxiliary function which will be chosen suitably according to the Weak $\overline{\partial}$-Lemma of Takayama [Taka2, Lemma 3.14] (see also Lemma 5.1.1) in order to obtain the required $L^2$ estimates. Details are given in 5.1.
Denote the pointwise 2-norm on $A$ and $T$ the decomposition (eq 2.10)
\[
\eta(z) := \eta_h(z) \eta_w(z) = e^{-\pi b(z,\bar{z}) - 2 \text{Re} b_h(z)}.
\]

The curvature form of $L$ with respect to $\eta$ is then given by
\[
\Theta_\Xi + \Theta_{\Xi_0},
\]
which represents the class $2\pi c_1(L)$ in $H^2(X, \mathbb{R})$ (while $\Theta_\Xi$ represents $2\pi c_1(L)$ in $2\pi H^2(X, \mathbb{Z})$).

2.6. An $L^2$-norm, the $L^2$-spaces $L^2_{\xi,c}(q', q'')$ and differential operators

Let $g$ be a hermitian metric on $X$. Fix an apt coordinate system. For the purpose of this article, $g$ is chosen to be a translational invariant metric such that the decomposition $T^{1,0}_c = T_c^{1,0} \oplus T_c^{0,1}$ is orthogonal. Denote by $\omega := -\text{Im} g$ the associated (1,1)-form as usual.

Fix any holomorphic line bundle $L$. Consider any $0 < c \leq \infty$ and $0 \leq q \leq n$. Denote the pointwise 2-norm on $\mathcal{A}^0(K_c; L)$ induced from the hermitian metrics $g$ and $\eta$ by $|\cdot|_{g,\eta}$. Let also $\tilde{\chi} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a smooth function and set $\chi := \tilde{\chi} \circ \varphi$. For the purpose of this article, $\tilde{\chi}$ is always assumed to be a non-negative convex increasing function. In this case, $\chi$ is plurisubharmonic. Set $|\zeta|_{g,\eta}^2 := |\zeta|^2_{g,\eta} e^{-\chi}$. Let $\mu$ be the measure induced from the volume form $\frac{e^{-\chi}}{m^n}$.

Define \[
\|\zeta\|_{K_c,\chi} := \sqrt{\int_{K_c} |\zeta|_{g,\eta,\chi}^2 d\mu}
\]
for any $\zeta \in \mathcal{A}^{0,q}(K_c; L)$.

Then $\|\cdot\|_{K_c,\chi}$ defines an $L^2$-norm with weight $e^{-\chi}$ (or simply $\chi$) on $\mathcal{A}^{0,q}(K_c; L)$, the space of sections in $\mathcal{A}^{0,q}(K_c; L)$ with compact support. To simplify notation, $d\mu$ in the integral is made implicit in what follows. The inner product corresponding to $\|\cdot\|_{K_c,\chi}$ is denoted by $\langle \cdot, \cdot \rangle_{K_c,\chi}$. The norm is written as $\|\cdot\|_{K_c; g,\eta,\chi}$ to emphasize its dependence on $g$ and $\eta$ when necessary.

Denote by $L^2_{\xi,c}(q', q'') := L^2_{\xi,c}^{0,q}(K_c; L)$ the Hilbert space of $(\mu)$-measurable $L$-valued $(0, q)$-forms $\zeta$ on $K_c$ such that $\|\zeta\|_{K_c,\chi} < \infty$. It is well known that $\mathcal{A}^{0,q}(K_c; L) \subset L^2_{\xi,c}(q', q'')$ is a dense subspace under the norm $\|\cdot\|_{K_c,\chi}$.

For any $0 \leq p', q' \leq n - m$ and $0 \leq p'', q'' \leq m$, define
\[
\mathcal{A}^{(p', q'), (q'', q'')} := \mathcal{A}^{(p', q')}(T^{p', q'}_u \wedge T^{p'', q''}_v),
\]
i.e. a sheaf of germs of smooth sections of $T^{p', q'}_u \wedge T^{p'', q''}_v$ (defined in eq 2.4). For other values of $p', p'', q', q''$, set $\mathcal{A}^{(p', q'), (q', q'')} := 0$. Note that, for $0 \leq p, q \leq n$, there is a decomposition (eq 2.11)
\[
\mathcal{A}^{p, q} = \bigoplus_{p' + p'' = p, q' + q'' = q} \mathcal{A}^{(p', q'), (q', q'')}.
\]
This decomposition depends on the choice of the decomposition (eq 2.4). Since the fibre and base directions are orthogonal to each other with respect to $g$, the decomposition is also orthogonal with respect to $g$. As only those sheaves with $p' + p'' = 0$ are considered in what follows, set $\mathcal{A}^{0, q' q''} := \mathcal{A}^{(0,0), (q', q'')}$. 
for notational convenience. Notice that \( \mathcal{H}^{0,q''}(L) \) is a subsheaf of \( \mathcal{A}^{0,(0,q'')} (L) \) for 
\( 0 \leq q'' \leq m \). For any \( c > 0 \), denote also the space of sections in \( \mathcal{A}^{0,(q',q'')} (K_c; L) \) with compact support by \( \mathcal{A}^{0,(q',q'')} (K_c; L) \). Define

\[
L^0_{c,\mathcal{X}}(q',q'') := L^0_{\mathcal{X}}(q',q'')(K_c; L) := \mathcal{A}^{0,(q',q'')} (K_c; L),
\]

i.e. the closure of \( \mathcal{A}^{0,(q',q'')} (K_c; L) \) in \( \left( L^0_{c,\mathcal{X}} q',q'' \right. \), \( \| \cdot \|_{K_c,\mathcal{X}} \). Note that the decomposition

(eq 2.12)

\[
L^0_{c,\mathcal{X}} q, q = \bigoplus_{q'\!+\!q''=q} L^0_{c,\mathcal{X}}(q',q''),
\]

induced from (eq 2.11) is also an orthogonal decomposition.

The operator \( \partial \) is decomposed into \( \overline{\partial}_{[u]} + \overline{\partial}_{[v]} \) according to the decomposition

(eq 2.4)

where \( \overline{\partial}_{[u]} \) and \( \overline{\partial}_{[v]} \) are operators such that

\[
\overline{\partial}_{[u]} : \mathcal{A}^{0,(q',q'')} (K_c; L) \rightarrow \mathcal{A}^{0,(q'+1,q'')} (K_c; L) \quad \text{and} \quad \overline{\partial}_{[v]} : \mathcal{A}^{0,(q',q'')} (K_c; L) \rightarrow \mathcal{A}^{0,(q',q''+1)} (K_c; L).
\]

Denote the formal adjoints of \( \overline{\partial}_{[u]} \) and \( \overline{\partial}_{[v]} \) above respectively by

\[
\overline{\partial}_{[u]}^* : \mathcal{A}^{0,(q'+1,q'')} (K_c; L) \rightarrow \mathcal{A}^{0,(q',q'')} (K_c; L) \quad \text{and} \quad \overline{\partial}_{[v]}^* : \mathcal{A}^{0,(q',q''+1)} (K_c; L) \rightarrow \mathcal{A}^{0,(q',q'')} (K_c; L)
\]

(see, for example, [D1 Ch. VI, 1.5] for the definition).

Some basic facts about differential operators on Hilbert spaces are recalled here. Extend the action of these operators to \( L^0_{c,\mathcal{X}}(q',q'') \) in the sense of distributions (or currents). Then, they define closed (i.e. having closed graph) and densely defined linear operators on \( L^0_{c,\mathcal{X}}(q',q'') \) (see, for example, [Hör2 Ch. 1] and [D2 Prop. 4.9]) with domain given by

(eq 2.13)

\[
\text{Dom}_{K_c,\mathcal{X}}(q',q'') T \quad \text{(or Dom } T) := \left\{ \zeta \in L^0_{c,\mathcal{X}}(q',q'') : \| T\zeta \|_{K_c,\mathcal{X}} < \infty \right\},
\]

where \( T \) denotes any of the above operators. Note that \( T \) is densely defined since \( \mathcal{A}^{0,(q',q'')} (K_c; L) \subset \text{Dom}_{K_c,\mathcal{X}}(q',q'') T \). An operator will be written as \( (T,\text{Dom } T) \) when the domain is emphasized.

Given \( \overline{\partial}_{[u]} : L^0_{c,\mathcal{X}}(q',q'') \rightarrow L^0_{c,\mathcal{X}}(q'+1,q'') \) and \( \overline{\partial}_{[v]} : L^0_{c,\mathcal{X}}(q',q'') \rightarrow L^0_{c,\mathcal{X}}(q',q''+1) \) with domains given as in (eq 2.13), their Hilbert space adjoints (also called Von Neumann’s adjoints, see for example [D1 Ch. VIII, §1] for a discussion on them) are denoted respectively by

\[
\overline{\partial}_{[u]}^* : L^0_{c,\mathcal{X}}(q'+1,q'') \rightarrow L^0_{c,\mathcal{X}}(q',q'') \quad \text{and} \quad \overline{\partial}_{[v]}^* : L^0_{c,\mathcal{X}}(q',q''+1) \rightarrow L^0_{c,\mathcal{X}}(q',q''),
\]

which are closed and densely defined operators on \( L^0_{c,\mathcal{X}}(q'+1,q'') \) and \( L^0_{c,\mathcal{X}}(q',q''+1) \) respectively. Denote also their domains of definition respectively by \( \text{Dom}_{K_c,\mathcal{X}}(q'+1,q'') \overline{\partial}_{[u]}^* \) and \( \text{Dom}_{K_c,\mathcal{X}}(q',q''+1) \overline{\partial}_{[v]}^* \).

In general, one has \( \text{Dom}_{K_c,\mathcal{X}}(q'+1,q'') \overline{\partial}_{[u]}^* \subset \text{Dom}_{K_c,\mathcal{X}}(q'+1,q'') \overline{\partial}_{[u]} \) and \( \overline{\partial}_{[u]}^* \zeta = \overline{\partial}_{[u]} \zeta \) for all \( \zeta \in \text{Dom}_{K_c,\mathcal{X}}(q'+1,q'') \overline{\partial}_{[u]}^* \) (see, for example, [D1 Ch. VIII, §3]). The same holds true for \( \overline{\partial}_{[v]}^* \) and \( \overline{\partial}_{[v]} \).
CHAPTER 3

$L^2$ estimates

3.1. Existence of a solution of $\overline{\partial} \xi = \psi$

The aim of this section is to show that, for $0 \leq q \leq m$, given $\psi \in H^{0,q}(K_c; L) \cap L^2_{c,\text{ext}}$ such that $\overline{\partial} \psi = 0$ on $K_c$, there exists a weak solution $\xi \in L^2_{c,\text{ext}}(0,q-1)$ of the $\overline{\partial}$-equation $\overline{\partial} \xi = \psi$ provided that an $L^2$ estimate is satisfied. When $c = \infty$, there exists a strong solution which lies in $H^{0,q-1}(X; L)$.

First recall the following classical theorems for $L^2$ estimates (see, for example, Hörmander, Lemmas 4.1.1 and 4.1.2) or [DI], Ch. VIII, Thm. 1.2). Let $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)$, $(\mathcal{H}_2, \langle \cdot, \cdot \rangle_2)$ and $(\mathcal{H}_3, \langle \cdot, \cdot \rangle_3)$ be some Hilbert spaces, and let $(S, \text{Dom } S)$ and $(T, \text{Dom } T)$ be two closed (i.e. closed graph) and densely defined linear operators with domains $\text{Dom } S \subset \mathcal{H}_2$ and $\text{Dom } T \subset \mathcal{H}_1$ respectively such that

$\mathcal{H}_1 \xrightarrow{T} \mathcal{H}_2 \xrightarrow{S} \mathcal{H}_3$

and $S \circ T = 0$, i.e. $T(\text{Dom } T) \subset \text{ker } S := \{ \xi \in \text{Dom } S : S\xi = 0 \}$. Let $S^\ast$ and $T^\ast$ denote the Hilbert space adjoints of $S$ and $T$ respectively, which are also closed, densely defined and satisfies $T^\ast \circ S^\ast = 0$ (see, for example, [DI], Ch. VIII, Thm. 1.1).

**Theorem 3.1.1** (see Hörmander, Lemmas 4.1.1 and 4.1.2]). If there exists a constant $C > 0$ such that

(eq 3.1) \[ \| S\xi \|_3^2 + \| T^\ast \xi \|_1^2 \geq C \| \xi \|_2^2 \quad \text{for all } \xi \in \text{Dom } S \cap \text{Dom } T^\ast, \]

then

1. for every $\psi \in \text{ker } S$, there exists $\xi \in \text{im } T^\ast \cap \text{Dom } T$ such that $T\xi = \psi$ and $\|\xi\|_1^2 \leq \frac{1}{C} \|\psi\|_2^2$. In other words, $\text{ker } S = \text{im } T$ (and thus $\text{im } T$ is closed as $\text{ker } S$ is so);

2. for every $\Psi \in (\text{ker } T)^\perp = \text{im } T^\ast$, there exists $\Xi \in \text{im } T \cap \text{Dom } T^\ast$ such that $T^\ast \Xi = \Psi$ and $\|\Xi\|_2^2 \leq \frac{1}{C} \|\Psi\|_1^2$. In other words, $\text{im } T^\ast = \text{im } T^\ast$.

**Remark 3.1.2.** By exchanging the roles of $S$ and $T^\ast$, one also gets $\text{ker } T^\ast = \text{im } S^\ast$ and $\text{im } S = \text{im } S$ if the $L^2$ estimate (eq 3.1) is satisfied.

When $X$ is compact, consider the complex

$L^2_{0,q-1}(X; L) \xrightarrow{\overline{\partial}} L^2_{0,q}(X; L) \xrightarrow{\partial} L^2_{0,q+1}(X; L).$

Murakami [Mur] shows that the $L^2$ estimates (eq 3.1) hold for $q < s^-$ or $q > n - s^+$ by choosing the hermitian metric $g$ suitably. The $L^2$ estimate on $L^2_{0,q}(X; L)$ implies that the harmonic $L$-valued $(0,q)$-forms must vanish. Elements in $H^q(X, L)$ are represented by harmonic forms when $X$ is compact, so this proves the vanishing of $H^q(X, L)$ in the compact case.

In the current situation, although elements in $H^q(X, L)$ are not represented by harmonic forms in general, the $L^2$ estimate (eq 3.1) is still useful in solving $\overline{\partial}$-equations which leads to the vanishing of $H^q(X, L)$ for suitable $q$'s according to Theorem 3.1.1 ([II]).
Due to the existence of non-linearizable line bundles, it turns out it is necessary to solve the $\overline{\partial}$-equation on $K_c$ for any $0 < c < \infty$ (see §5.1). Therefore, the aim now is to solve the $\overline{\partial}$-equation $\overline{\partial}K = \psi|_{K_c}$ for a given $\psi \in \mathcal{H}^{0,q}(X; L)$ with $\overline{\partial}\psi = 0$. In view of the fibre bundle structure (eq 2.3), instead of considering the complex $L^2_{c,\psi}[\mathbb{R}] \xrightarrow{\overline{\partial}} L^2_{c,\psi} \xrightarrow{\overline{\partial}} L^2_{c,\psi}$, it is natural (see the discussion in §1.2) to consider the subcomplex

\begin{equation}
L^2_{c,\psi} \xrightarrow{\overline{\partial}} L^2_{c,\psi} \xrightarrow{S_q} L^2_{c,\psi} \xrightarrow{T_q} L^2_{c,\psi},
\end{equation}

where $T_q$ and $S_q$ act as $\overline{\partial}$ on $L^2_{c,\psi}$ and $L^2_{c,\psi}$ respectively, and $T_q^*$ and $S_q^*$ are their Hilbert space adjoints. The Hilbert spaces in the complex are defined as

\begin{align*}
\mathcal{A}^{0,q}_{2,\psi}(K; L) &:= \mathcal{A}^{0,1,q-1}(K; L) \oplus \mathcal{A}^{0,0,q}(K; L), \\
\mathcal{A}^{0,q+1}_{2,\psi}(K; L) &:= \mathcal{A}^{0,1,q}(K; L) \oplus \mathcal{A}^{0,0,q+1}(K; L), \\
L^2_{c,\psi} &:= \mathcal{A}^{0,q}_{2,\psi}(K; L) = L^2_{c,\psi}, \\
L^2_{c,\psi+1} &:= \mathcal{A}^{0,q}_{2,\psi}(K; L) = L^2_{c,\psi} + L^2_{c,\psi}.
\end{align*}

Recall from (eq 2.11) and (eq 2.12) that all the direct sums on the right hand sides above are orthogonal decompositions. Denote the norms on $L^2_{c,\psi}$, $L^2_{c,\psi}$ and $L^2_{c,\psi}$ respectively by $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$, and their inner products by $(\cdot, \cdot)$, with the corresponding subscripts.

Write the Hilbert space adjoint of $\overline{\partial}$: $L^2_{c,\psi} \rightarrow L^2_{c,\psi}$ as $\overline{\partial}$. Let $\text{pr}: L^2_{c,\psi} \rightarrow L^2_{c,\psi}$ be the orthogonal projection. For later use, $(T_q^*, \text{Dom } T_q^*)$ is described more explicitly.

**Proposition 3.1.3.** With the notation described above, one has

\begin{align*}
\text{Dom } T_q^* &= \text{Dom}_{K_{c,\psi}} \overline{\partial}^* \cap L^2_{c,\psi}, \\
&= \text{Dom}^{(1,q-1)}_{K_{c,\psi}} \overline{\partial}^* \oplus \text{Dom}^{(0,q)}_{K_{c,\psi}} \overline{\partial}^*.
\end{align*}

Moreover, for any $\zeta = \zeta' + \zeta'' \in \text{Dom } T_q^*$ where $\zeta' \in \text{Dom}^{(1,q-1)}_{K_{c,\psi}} \overline{\partial}^*$ and $\zeta'' \in \text{Dom}^{(0,q)}_{K_{c,\psi}} \overline{\partial}^*$, one has $T_q^* \zeta = \text{pr} \overline{\partial}^* \zeta = \overline{\partial}^*_u \zeta' + \overline{\partial}^*_v \zeta''$.

**Proof.** Define operators $(W_1, \text{Dom } W_1)$ and $(W_2, \text{Dom } W_2)$ from $L^2_{c,\psi}$ into $L^2_{c,\psi}$ such that

\begin{align*}
\text{Dom } W_1 &= \text{Dom}_{K_{c,\psi}} \overline{\partial}^* \cap L^2_{c,\psi}, \\
\text{Dom } W_2 &= \text{ Dom}^{(1,q-1)}_{K_{c,\psi}} \overline{\partial}^* \oplus \text{Dom}^{(0,q)}_{K_{c,\psi}} \overline{\partial}^*.
\end{align*}

and

\begin{align*}
W_1 \zeta &= \text{pr} \overline{\partial}^* \zeta \quad \text{for } \zeta \in \text{ Dom } W_1, \\
W_2 \zeta &= \overline{\partial}^*_u \zeta' + \overline{\partial}^*_v \zeta'' \quad \text{for } \zeta = \zeta' + \zeta'' \in \text{ Dom } W_2.
\end{align*}

\[\text{The symbol } T_q^*(\text{resp. } S_q^*) \text{ is used instead of } \overline{\partial}^* \text{ so that the domains and codomains of the two operators can be distinguished. More precisely, if } i: L^2_{c,\psi} \rightarrow L^2_{c,\psi} \text{ and } \text{pr}: L^2_{c,\psi} \rightarrow L^2_{c,\psi} \text{ are respectively the inclusion and projection, then } T_q = \text{pr} \circ \overline{\partial} \circ i. \text{ Therefore, } T_q^* \text{ and } \overline{\partial}^* \text{ are different operators.}\]
These are closed and densely defined linear operators on \( L^2_{c \times \chi < 2} \). Since \( \| T_{q-1} \zeta \|_2^2 = \| \overline{\partial} \zeta \|_2^2 = \| \overline{\partial}_{[v]} \zeta \|_2^2 + \| \overline{\partial}_{[u]} \zeta \|_2^2 \) for all \( \zeta \in L^2_{c \times \chi < 2} \), it follows that

\[
\text{Dom } T_{q-1} = \text{Dom } \overline{\partial} \cap \overline{L}^2_{c \times \chi < 2} = \text{Dom}^{(0,q-1)}_{K_{c \times \chi}} \overline{\partial}_{[v]} \cap \text{Dom}^{(0,q-1)}_{K_{c \times \chi}} \overline{\partial}_{[u]}.
\]

First is to show that \( (T^*_q, \text{Dom } T^*_q) = (W_1, \text{Dom } W_1) \). Note that, for any \( f \in L^2_{c \times \chi < 2} \) and any \( \zeta \in \text{Dom } W_1 \), one has

\[
\langle f, W_1 \zeta \rangle_1 = \langle f, \text{pr } \overline{\partial} \zeta \rangle_1 = \langle f, \overline{\partial} \zeta \rangle_{K_{c \times \chi}}.
\]

For any \( \tilde{\zeta} \in L^2_{c \times \chi} = L^2_{c \times \chi < 2} \oplus \left( L^2_{c \times \chi < 2} \right)^{\perp} \), write \( \tilde{\zeta} = \zeta + \zeta' \) where \( \zeta \in \text{Dom } L^2_{c \times \chi < 2} \) and \( \zeta' \in \left( L^2_{c \times \chi < 2} \right)^{\perp} = \bigoplus_{q'=2} L^2_{c \times \chi < 2} \). Note that \( \overline{\partial} \zeta = \bigoplus_{q'=1} q^{-q-1} L^2_{c \times \chi} \), thus \( \langle f, \overline{\partial} \zeta \rangle_{K_{c \times \chi}} = 0 \) for any \( f \in L^2_{c \times \chi < 2} \). Therefore, for any \( f \in L^2_{c \times \chi < 2} \), one has

\[
f \in \text{Dom } W_1^* \quad \iff \quad \exists C > 0 : \forall \zeta \in \text{Dom } W_1, \quad |\langle f, W_1 \zeta \rangle_1| = |\langle f, \overline{\partial} \zeta \rangle_{K_{c \times \chi}}| \leq C \| \zeta \|_2
\]

(see [DI] Ch. VIII, §1 for the definition of the Hilbert space adjoints), and thus \( \text{Dom } W_1^* = \text{Dom } T_{q-1} \). It follows that \( \langle f, W_1 \zeta \rangle_1 = \langle f, \overline{\partial} \zeta \rangle_{K_{c \times \chi}} = \langle \overline{\partial} f, \zeta \rangle_2 = \langle T_{q-1} f, \zeta \rangle_2 \) for any \( f \in \text{Dom } T_{q-1} \) and \( \zeta \in \text{Dom } W_1 \). As a result, \( (T_{q-1}, \text{Dom } T_{q-1}) = (W_1^*, \text{Dom } W_1^*) \), and hence \( (T_{q-1}^*, \text{Dom } T_{q-1}^*) = (W_1, \text{Dom } W_1) \) (see [DI] Ch. VIII, Thm. 1.1).

The proof of \( (T_{q-1}^*, \text{Dom } T_{q-1}^*) = (W_2, \text{Dom } W_2) \) is similar. Notice that \( \| \zeta \|_2^2 = \| \zeta' \|_2^2 + \| \zeta'' \|_2^2 \) and thus \( \| \zeta' \|_2 + \| \zeta'' \|_2 \leq \sqrt{2} \| \zeta \|_2 \) for all \( \zeta = \zeta' + \zeta'' \in L^2_{c \times \chi < 2} \). Then, for any \( f \in L^2_{c \times \chi < 2} \), one has

\[
f \in \text{Dom } W_2^* \quad \iff \quad \exists C > 0 : \forall \zeta = \zeta' + \zeta'' \in \text{Dom } W_2, \quad |\langle f, W_2 \zeta \rangle_1| = |\langle f, \overline{\partial} \zeta \rangle_{K_{c \times \chi}}| \leq C \| \zeta \|_2
\]

and thus \( \text{Dom } W_2^* = \text{Dom } T_{q-1} \). Note that \( \langle f, W_2 \zeta \rangle_1 = \langle \overline{\partial}_{[v]} f, \zeta' \rangle_2 + \langle \overline{\partial}_{[u]} f, \zeta'' \rangle_2 = \langle \overline{\partial}_{[v]} f + \overline{\partial}_{[u]} f, \zeta' + \zeta'' \rangle_2 = \langle T_{q-1} f, \zeta \rangle_2 \) for any \( f \in \text{Dom } T_{q-1} \) and \( \zeta \in \text{Dom } W_2 \), since
Suppose now given $0 < c \leq \infty$ and $\psi \in \mathcal{H}^{0,q}(K_c; L)$ \& $L^2_{c,\chi} \subset L^2_{0,\chi} < 2\varsigma$ such that $S\chi \psi = \partial \psi = 0$. Theorem 3.1.1 (II) asserts that, if the $L^2$ estimate (eq 3.1) is satisfied, then there exists $\xi \in \mathrm{im} \mathcal{T}_{\psi^{-1}} \subset L^2_{c,\chi}$ such that 

\[(eq 3.3) \quad \mathcal{T}_{\psi^{-1}}(\xi) = \bar{\partial} \psi = \psi \text{ in } L^2_{c,\chi}.\]

One can have a further reduction. When $c = \infty$, since $(X, g)$ is complete in the sense of Riemannian geometry, $\mathcal{H}^{0,q}(X; L)$ is dense in $\mathrm{Dom}_X \mathcal{T}_{\psi^{-1}} \cap \mathrm{Dom}_X S\nu$ under the above graph norm (see, for example, [D1] Ch. VIII, Thm. 3.2). Therefore, it suffices to establish the required $L^2$ estimates (eq 3.1) for $\zeta \in \mathcal{H}^{0,q}(X; L)$.

Suppose $c < \infty$. Note that $\mathcal{H}^{0,q}(K_c; L) \subset \mathrm{Dom}_{S\nu}$. Since $\partial K_c$ is smooth and $\chi$ is smooth on a neighborhood of $K_c$, using [Hör1] Prop. 2.1.1] together with an argument of partition of unity, it yields the following

**Proposition 3.1.4.** $\mathcal{H}^{0,q}(K_c; L) \cap \mathrm{Dom} \mathcal{T}_{\psi^{-1}}$ is dense in $\mathrm{Dom} \mathcal{T}_{\psi^{-1}} \cap \mathrm{Dom} S\nu$ under the graph norm $\sqrt{\| \mathcal{T}_{\psi^{-1}}(\xi) \|^2 + \| S\nu \xi \|^2 + \| \zeta \|^2}$.

**Proof.** Note that the statement follows from [Hör1 Prop. 2.1.1] when $\mathcal{T}_{X}^{0,q}$ and $L$ are both trivial by using a partition of unity. The aim now is to handle the case when $L$ is non-trivial.

Take a locally finite open cover $\{U_\alpha\}_{\alpha \in A}$ of $X$ such that every $U_\alpha$ is a coordinate chart of $X$ and $L$ is trivialized on each $U_\alpha$ with transition functions $\sigma_{\alpha \beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in A$. Then, for any $\zeta \in L^2_{0,q}(X; L)$ with $\zeta_\alpha$ representing $\zeta$ over $U_\alpha$ under the trivialization, one has $\zeta_\alpha = \sigma_{\alpha \beta} \zeta_\beta$ on $U_\alpha \cap U_\beta$.

Fix any $\zeta \in \mathrm{Dom} \mathcal{T}_{\psi^{-1}} \cap \mathrm{Dom} S\nu$. It suffices to show that $\zeta$ can be approximated by a sequence $\{\zeta^{(n)}\}_{n \in \mathbb{N}} \subset \mathcal{H}^{0,q}(K_c; L) \cap \mathrm{Dom} \mathcal{T}_{\psi^{-1}}$ under the given graph norm.

Extend $\zeta$ by zero to a section on $X$. Using a partition of unity which decomposes $\zeta$ into a sum of finitely many compactly supported sections, one can assume that $\zeta$ is compactly supported in a coordinate chart $U := U_0 \in \{U_\alpha\}_{\alpha \in A}$. Then the hermitian metric $\eta$ on $L$ can be viewed as a function $\tilde{\eta} := \eta_0$ on $U = U_0$ (under the given trivialization), and any $L$-valued form $f \in L^2_{0,q}(W; U)$ can be viewed as a $\mathcal{O}_X$-valued form $\tilde{f} := f_0 \in L^2_{0,q}(\tilde{U}; U)$. Let $W := U \cap K_c$. Note that one has

\[
\| \tilde{f} \|_{W, g_0, \eta_0} = \| f \|_{W, g_0, \eta_0} \quad \| \tilde{\eta} f \|_{W, g_0, \eta_0} = \| \partial \tilde{\eta} f \|_{W, g_0, \eta_0} \quad \text{and} \quad \| \tilde{\eta} f \|_{W, \bar{g}, \eta_0} = \| \partial \tilde{\eta} f \|_{W, g_0, \eta_0}
\]

for all $f \in L^2_{0,q}(W; L)$. Then $\zeta$ is in $\mathrm{Dom} \mathcal{T}_{\psi^{-1}} \cap \mathrm{Dom} S\nu$ implies $\zeta \in \mathrm{Dom} \mathcal{T}_{\psi^{-1}} \cap \mathrm{Dom} \mathcal{T}_{\psi^{-1}}$ and $\mathcal{T}_{\psi^{-1}} \mathcal{T}_{\psi^{-1}}$ implies $\zeta \in \mathcal{T}_{\psi^{-1}} \mathcal{T}_{\psi^{-1}}$.

Since $g$ and $\chi$ are fixed in what follows, subscripts of them are omitted from the notations below.

By [Hör1 Prop. 2.1.1] (or applying [Hör1 Prop. 1.2.4] directly), there exists a sequence $\{\tilde{\zeta}^{(n)}\}_{n \in \mathbb{N}} \subset \mathcal{H}^{0,q}(W) \cap \mathrm{Dom} \tilde{\mathcal{T}}$ such that

\[
\| \tilde{\partial} (\tilde{\zeta}^{(n)} - \bar{\tilde{\zeta}}) \|^2_{W, \bar{\tilde{\partial}}} + \| \tilde{\partial} (\tilde{\zeta}^{(n)} - \bar{\tilde{\zeta}}) \|^2_{W, \bar{\tilde{\partial}}} + \| \tilde{\zeta}^{(n)} - \bar{\tilde{\zeta}} \|^2_{L, \bar{\tilde{\partial}}} \to 0
\]

as $n \to \infty$ and $\sup \tilde{\zeta}^{(n)} \subset \mathcal{H}^{0,q}(W)$. As $\zeta^{(n)}$’s are obtained from convolutions between smoothing kernels and $\tilde{\zeta}$ which do not change the type of forms, it follows that $\zeta^{(n)} \in \mathcal{H}^{0,q}(W)$ and $\zeta^{(n)} \to \zeta$ on $U_\alpha \cap U \neq \emptyset$ are compactly supported in $U$ (hence $\zeta^{(n)} \in \mathcal{H}^{0,q}(K_c; L)$ and...
satisfy \( \tilde{\zeta}^{(\nu)} = \tilde{\zeta}^{(\nu)} \). Therefore, one obtains a sequence \( \{\zeta^{(\nu)}\}_{\nu \in \mathbb{N}} \subset \text{Dom}_{K,\eta} \tilde{\partial}^* \cap \mathcal{A}_{<2>}(\overline{K}; L) = \text{Dom} T_{q-1}^* \cap \mathcal{A}_{<2>}(\overline{K}; L) \) (see Proposition 3.1.3) such that

\[
\|T_{q-1}^* (\zeta^{(\nu)} - \zeta)\|_1^2 + \|S_q (\zeta^{(\nu)} - \zeta)\|_3^2 + \|\zeta^{(\nu)} - \zeta\|_2^2
\leq \left\| \tilde{\partial} (\zeta^{(\nu)} - \zeta) \right\|_{W,q}^2 + \left\| \tilde{\partial} (\zeta^{(\nu)} - \zeta) \right\|_{W,q}^2 + \|\zeta^{(\nu)} - \zeta\|_{W,q}^2
= \left\| \tilde{\partial} (\tilde{\zeta}^{(\nu)} - \tilde{\zeta}) \right\|_{W,\tilde{\eta}}^2 + \left\| \tilde{\partial} (\tilde{\zeta}^{(\nu)} - \tilde{\zeta}) \right\|_{W,\tilde{\eta}}^2 + \|\tilde{\zeta}^{(\nu)} - \tilde{\zeta}\|_{W,\tilde{\eta}}^2
\rightarrow 0
\]

as \( \nu \rightarrow \infty \) as required. \( \square \)

As a result, it suffices to establish the required \( L^2 \) estimates for \( \zeta \in \mathcal{A}_{<2>}(\overline{K}; L) \cap \text{Dom} T_{q-1}^* \).

The above discussion is summarized in the following

**Proposition 3.1.5.** Suppose \( 0 < c \leq \infty \). If there exists a constant \( C > 0 \) such that

\[
\langle 3.4 \rangle \quad \|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 \geq C \|\zeta\|_2^2
\]

for all \( \zeta \in \left\{ \mathcal{A}_{<2>}(\overline{K}; L) \cap \text{Dom} T_{q-1}^* \quad \text{when} \ c < \infty \right\}
\left\{ \mathcal{A}_{0<2>}(X; L) \quad \text{when} \ c = \infty \right\}

then, for every \( \psi \in \mathcal{H}^0,q(K; L) \cap L^2_{\chi}(K; L) \) such that \( \tilde{\partial} \psi = 0 \), there exists \( \xi \in L^2_{\chi}(0,q-1)(K; L) \) such that \( \tilde{\partial} \xi = \psi \) in \( L^2_{\chi}(K; L) \).

**Remark 3.1.6.** Let \( L^2_{0,q-1}(K; L; \text{loc}) \) denote the space of locally \( L^2 \) \( (0,q-1) \)-forms on \( K_c \), which contains \( L^2_{\chi}(0,q-1)(K_c; L) \) as a subspace. It follows from the classical regularity theory for \( \tilde{\partial} \)-operator or elliptic operators (ref. [Hör3] Thm. 4.2.5 and Cor. 4.2.6 or [Hör2] Thm. 4.1.5 and Cor. 4.1.2) that the existence of \( \xi \in L^2_{0,q-1}(K; L; \text{loc}) \) satisfying the equation \( \langle 3.3 \rangle \) in \( L^2_{\chi}(K; L; \text{loc}) \) implies that there exists \( \xi \in \mathcal{A}_{0,q-1}(K; L) \) (but not necessarily in \( \mathcal{A}_{0,q-1}(K; L) \)) satisfying the same equation in \( \mathcal{A}_{0,q}(K; L) \). In case \( c = \infty \), Theorem 2.3.1 implies that there even exists a solution \( \xi \in \mathcal{H}^0,q-1(X; L) \) such that \( \tilde{\partial} \xi = \psi \) on \( X \).

**Remark 3.1.7.** Write \( \mathcal{H}^0,q_{c,\chi}(K; L) := \mathcal{H}^0,q(K; L) \cap L^2_{c,\chi}(q) \). Following the idea discussed in [12], it would be more natural to consider the \( L^2 \) estimate on \( \mathcal{S}_{c,\chi}^0 := \mathcal{H}^0,q_{c,\chi}(K; L) \) rather than \( L^2_{c,\chi<2>} \), where the closure is taken in \( L^2_{c,\chi<2>} \). However, the author faces the difficulty in obtaining the required estimate from the Bochner–Kodaira inequalities when \( \mathcal{S}_{c,\chi}^0 \) instead of \( L^2_{c,\chi<2>} \) is considered. Write \( \tilde{\partial}^*_{\mathcal{H}} \) as the Hilbert space adjoint of \( \tilde{\partial} = \tilde{\partial}_{[c]} : \mathcal{S}_{c,\chi}^0 \rightarrow \mathcal{S}_{c,\chi}^{0,q+1} \) It can be shown that \( \tilde{\partial}^*_{\mathcal{H}} = \text{pr}_c \circ \tilde{\partial}^*_{\mathcal{H}} \) on \( \text{Dom}_{K,\chi} \tilde{\partial}^*_{\mathcal{H}} \), where \( \text{pr}_c : L^2_{c,\chi}(q) \rightarrow \mathcal{S}_{c,\chi}^0 \) is the orthogonal projection. Set \( \tilde{\partial}_c := \tilde{\partial} - \tilde{\partial} \) and \( \tilde{\partial}_c \) are orthogonal to each other for all \( \zeta \in \text{Dom}_{K,\chi} \tilde{\partial}^*_{\mathcal{H}} \) and

\[
\|\tilde{\partial}_{\mathcal{H}} \zeta\|_{K_c,\chi}^2 = \|\tilde{\partial}_{\mathcal{H}} \zeta\|_{K_c,\chi}^2 + \|\tilde{\partial}_c \zeta\|_{K_c,\chi}^2.
\]
From the Bochner–Kodaira inequalities, one obtains
\[ \left\| \partial_{\zeta} \zeta \right\|^2_{K\otimes} + \left\| \partial_{\zeta} \zeta \right\|^2_{K\otimes} \geq \int_{K\otimes} \text{Curv}(\zeta, \zeta) \]
for all \( \zeta \in \mathcal{H}_L^0(K; L) \cap \text{Dom}_K^0(\mathcal{F}) \) for some constant \( C > 0 \). Therefore, in order to obtain the desired estimate \( \left\| \partial_{\zeta} \zeta \right\|^2_{K\otimes} \) for some constant \( C'' > 0 \), one has to show that \( \left\| \partial_{\zeta} \zeta \right\|^2_{K\otimes} \leq C'' \left\| \zeta \right\|^2_{K\otimes} \) for some constant \( C'' > 0 \) such that \( C > C'' \). However, the constant \( C'' \) depends on \( g \) in general and one may not be able to make \( C'' \) smaller than \( C \) by altering \( g \). That's why the \( L^2 \) estimate on \( \mathcal{H}_L^0(\mathcal{F}) \) instead of \( \mathcal{F}_{L,0}^0(\mathcal{F}) \) is considered in this article.

### 3.2. Bochner–Kodaira formulas

Let
\[ \nabla : \mathcal{A}(T^{**} \otimes L) \to \mathcal{A}(T^* \otimes T^{**} \otimes L) , \]
where \( T^{*} := T^{1,0} \oplus T^{0,1} \), be the connection on \( T^{**} \otimes L \) induced from the Chern connections on the holomorphic hermitian vector bundles \( (T^{1,0}, g) \) and \( (L, \eta e^{-\chi}) \). Therefore, \( \nabla \) is compatible with the pointwise norm \([ \cdot ]_{g,\eta e^{-\chi}} \).

Under a chosen apt coordinate system, set \( \partial_k := \frac{\partial}{\partial z^k} \) and \( \partial_{\zeta} := \frac{\partial}{\partial \zeta^k} \) for \( 1 \leq k \leq n \). These define global vector fields on \( X \). Set \( \nabla_k := \nabla_{\partial_k} \) and \( \nabla_{\zeta} := \nabla_{\partial_{\zeta}} \) for \( 1 \leq k \leq n \). Set also \( \nabla_{v^j} := \nabla_{v^j} \) and \( \nabla_{V^j} := \nabla_{V^j} \) (and define \( \partial_{v^j} \) and \( \partial_{V^j} \) similarly) for \( 1 \leq j \leq m \) for notational convenience. Since the hermitian metric tensor \( g \) is translation invariant on \( X \), the Christoffel symbols given from \( g \) vanish and thus one has locally
\[ \nabla_k = \partial_k + \partial_k \log \left( \eta e^{-\chi} \right), \]
\[ \nabla_{\zeta} = \partial_{\zeta} \]
for \( 1 \leq k \leq n \). For later use, note that the commutator of \( \nabla_k \) and \( \nabla_{\zeta} \) is given by
\[ [\nabla_k, \nabla_{\zeta}] = -\partial_k \partial_{\zeta} \log \left( \eta e^{-\chi} \right), \]
and the curvature form of \( L \) endowed with the metric \( \eta e^{-\chi} \) is given by
\[ \Theta := -\sqrt{-1} \partial \bar{\partial} \log \left( \eta e^{-\chi} \right) = \sqrt{-1} \sum_{k,l=1}^n \Theta_{k\bar{l}} \ d\bar{z}^k \wedge dz^l \].

Write the curvature tensor associated to \( \Theta \) as
\[ \mathcal{R} := \sum_{k,l=1}^n \Theta_{k\bar{l}} \ d\bar{z}^k \otimes dz^l \].

Since the base and fibre directions are orthogonal to each other with respect to \( g \), the identification between \( \mathcal{A}^{\rho \sigma} \) and \( \mathcal{A}_{\rho \sigma} := \mathcal{A}(T^{\rho \sigma}) \) induced from \( g \) respects the decomposition (eq 2.1) \( (\mathcal{A}_{\rho \sigma}, \mathcal{A}_{p \sigma}, \mathcal{A}_{\rho q}) \) here means the complex conjugate of \( \mathcal{A}_{\rho \sigma} \). For later use, set \( \mathcal{A}^{(p', q', q'')} := \mathcal{A}(T^{p' \otimes q' \otimes q''}) \) and \( \mathcal{A}^{(p', q', q''), 0} := \mathcal{A}(T^{p' \otimes q' \otimes q''}, 0) \) for \( 0 \leq p', q' \leq n - m \) and \( 0 \leq p'', q'' \leq m \). For any \( \zeta \in \mathcal{A}^{\rho \sigma}, \mathcal{A}^{\rho q} \), let \( \zeta' \) denote the image of \( \zeta \) in \( \mathcal{A}_0^p \otimes \mathcal{A}_{q,0} \) via the isomorphism induced from \( g \). Then, for example, if \( \zeta \in \mathcal{A}^{0, (q', q'') \}, \) one has \( \zeta' \in \mathcal{A}^{(q', q'')} \).
As a bilinear form on $\mathcal{A}_{1,0} \otimes \mathcal{A}_{1,0}$, $\mathcal{R}$ can be decomposed according to the decomposition (eq 2.4) into the sum of
\[
\mathcal{R}_{u} := \mathcal{R}|_{\mathcal{A}_{1,0} \otimes \mathcal{A}_{1,0} \otimes \mathcal{A}_{1,0}} ; \quad \mathcal{R}_{v} := \mathcal{R}|_{\mathcal{A}_{1,0} \otimes \mathcal{A}_{1,0} \otimes \mathcal{A}_{1,0}} ;
\]
\[
\mathcal{R}_{uv} := \mathcal{R}|_{\mathcal{A}_{1,0} \otimes \mathcal{A}_{1,0} \otimes \mathcal{A}_{1,0}} ; \quad \mathcal{R}_{vu} := \mathcal{R}|_{\mathcal{A}_{1,0} \otimes \mathcal{A}_{1,0} \otimes \mathcal{A}_{1,0}} .
\]
Since $\mathcal{R}$ is a hermitian form, it follows that $\mathcal{R}_{u} = \overline{\mathcal{R}_{u}}$, $\mathcal{R}_{v} = \overline{\mathcal{R}_{v}}$ and $\mathcal{R}_{uv} = \overline{\mathcal{R}_{uv}}$.

Let $\text{Tr}_{L} : \mathcal{A}^{0,q} \otimes \mathcal{A}^{0,q} \rightarrow \mathcal{A}^{0,0}$ be the trace operator which is defined in such a way that $\zeta \otimes \xi \mapsto \xi^\vee \llcorner \zeta$, where $\zeta \in \mathcal{A}^{0,q}$, $\xi \in \mathcal{A}^{q,0}$ and $\xi^\vee \llcorner \zeta$ denotes the complete contraction between $\zeta$ and $\xi^\vee$. Denote by $\text{Tr}_{\phi,0}$ the similar contraction for $L$-valued forms.

Fix any $0 < c < \infty$. Denote the Hilbert space adjoint of $\overline{\partial} : L^{0,q,1}_{c} \rightarrow L^{0,q}_{c}$ by $\overline{\partial}^* : L^{0,q}_{c} \rightarrow L^{0,q-1}_{c}$. Identify $\mathcal{A}^{1,1}$ and $\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}$ via the isomorphism $dz^k \wedge dz^\ell \mapsto dz^k \otimes dz^\ell$ for any $1 \leq k, \ell \leq n$. Let $\mathcal{R}^\vee (\zeta \otimes \zeta)$ (resp. $(\overline{\partial} \mathcal{F})^\vee (\zeta \otimes \zeta)$) denotes the natural contraction between $\mathcal{R}^\vee$ (resp. $(\overline{\partial} \mathcal{F})^\vee$) and $\zeta \otimes \zeta$. Let $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$ be the decomposition of $\nabla$ into $(1,0)$- and $(0,1)$-types. The $\nabla$-Bochner–Kodaira formula (cf. [Siu] (2.1.4) and (1.3.3)) is then given by
\[
\left\| \nabla \mathcal{F} \right\|^2_{K,c} + \left\| \overline{\partial} \mathcal{F} \right\|^2_{K,c} = \int_{\partial K,c} e^{-c} \mathcal{F} \mathcal{R}_{\phi,0} (\overline{\partial} \mathcal{F})^\vee (\zeta \otimes \zeta) \]
\[
+ \left\| \nabla^{(0,1)} \right\|_{K,c}^{2} + \int_{K,c} e^{-c} \mathcal{R}^\vee (\zeta \otimes \zeta)
\]
for all $\zeta \in \mathcal{A}^{0,q}(\overline{K}; L) \cap \text{Dom}_{K,c} \overline{\partial}^*$.

**Remark 3.2.1.** Note that the measure for the boundary integral is induced from $\left( \frac{d\mathcal{F}}{d\mathcal{F}} \right)^\vee = \frac{\omega_n}{n!} d\mathcal{F}$ on $\partial K$. In order to compare notations in [Siu] (2.1.4) and those in (eq 3.7), write $[x]_{\text{Siu}}$ to mean the symbol $x$ used in [Siu]. Then
\[
\left[ \nabla \right]_{\text{Siu}} = \nabla^{(1,0)} , \quad \left[ \nabla \right]_{\text{Siu}} = \nabla^{(0,1)} , \quad \left[ \rho \right]_{\text{Siu}} = \frac{\varphi - c}{d\mathcal{F}}, \quad \left[ R_{\overline{\partial}^*} \right]_{\text{Siu}} = 0 ,
\]
and
\[
\left[ -\Omega_{\overline{\partial}^*} \right]_{\text{Siu}} = \text{components of} \; \mathcal{R} = \Theta_{\phi,0}.
\]
Note that $[R_{\overline{\partial}^*}]_{\text{Siu}} = 0$ as the Chern connection on $(\mathcal{T}^{1,0}, g)$ is flat. Also be aware of the typos of the signs preceding the curvature integrals involving $\left[ \Omega_{\overline{\partial}^*} \right]_{\text{Siu}}$ and $[R_{\overline{\partial}^*}]_{\text{Siu}}$ in [Siu] (2.1.4)]. The correct signs can be found in [Siu] (1.3.3)]. To see that the boundary term in (eq 3.7) coincides with the one in [Siu] (2.1.4)], note that at every $z \in \partial K$, $d\mathcal{F} = \overline{\partial}^* (\varphi - c)$
\[
\overline{\partial}^* (\varphi - c) (z) = \frac{\partial \overline{\partial} \varphi (z)}{d\mathcal{F}} - \frac{\partial |d\mathcal{F}|^2 (z)}{|d\mathcal{F}|^2 (z)} \overline{\partial}^* (\varphi - c) (z).
\]
After taking $\vee$ and contracting with $\zeta \otimes \zeta$ where $\zeta \in \mathcal{A}^{0,q}(\overline{K}; L) \cap \text{Dom}_{K,c} \overline{\partial}^*$, the last two terms on the right hand side vanish because, for $\zeta \in \mathcal{A}^{0,q}(\overline{K}; L)$, $(|\partial \mathcal{F}|^2) \llcorner \zeta = 0$ on $\partial K$ if and only if $\zeta \in \text{Dom}_{K,c} \overline{\partial}^*$ (ref. [Hörk pg. 101] or [Siu] (2.1.1)]. The boundary terms therefore coincide.

When the subcomplex (eq 3.2) is considered, the $\nabla$-Bochner–Kodaira formula (eq 3.7) is restricted to $\zeta \in \mathcal{A}^{0,q}_{\mathcal{L}^2 \phi}(\overline{K}; L) \cap \text{Dom}_{K,c} \overline{\partial}^* = \mathcal{A}^{0,q}_{\mathcal{L}^2 \phi}(\overline{K}; L) \cap \text{Dom}_{K,c} \mathcal{T}^{*,-1}$ (see Proposition 3.1.3). The $(0,1)$-connection splits into $\nabla^{(0,1)} = \nabla^{(0,1)}_u + \nabla^{(0,1)}_v$
according to the decomposition \((eq \ref{2.4})\). Write \(\nabla_\pi := \nabla_u^{(0,1)}\) and \(\nabla_\pi := \nabla_v^{(0,1)}\) for notational convenience. Let also \(pr_F : \mathcal{A}^{0,q} \otimes \mathcal{A}^{0,s} \rightarrow \mathcal{A}^{0,(0,q)} \otimes \mathcal{A}^{0,(0,s)}\) be the canonical projection (where \(\mathcal{A}^{0,s}\) (resp. \(\mathcal{A}^{0,(0,s)}\)) is the complex conjugate of \(\mathcal{A}^{0,s}\) (resp. \(\mathcal{A}^{0,(0,s)}\))). Set

\[
(eq \ref{3.8}) \quad \Bd(\zeta, \zeta) := \int_{\partial K_e} e^{-\chi} \frac{\nu}{\nu} \text{Tr}_{g,\eta} \left( \partial \varphi \right) \left( \zeta \otimes \zeta \right)
\]

for notational convenience. Then \((eq \ref{3.7})\) gives the following

**Lemma 3.2.2.** For any \(\zeta = \zeta' + \zeta'' \in \mathcal{A}^{0,q}_{<2>}(K_c; L) \cap \text{Dom}\ T_{q-1}^*, \) where \(\zeta' \in \mathcal{A}^{0,(1,q-1)}(K_c; L) \cap \text{Dom} \mathcal{J}_{[\zeta]}\) and \(\zeta'' \in \mathcal{A}^{0,(0,q)}(K_c; L),\) one has

\[
||S_{q} \zeta||^2_{3} + ||T_{q-1}^* \zeta||^2_{1} = \Bd(\zeta, \zeta) + ||\mathcal{J}_{[\zeta]} \zeta'\|_{3}^2 + ||\mathcal{J}_{[\zeta]} \zeta''\|_{3}^2
\]

\[
+ ||\nabla_\zeta' \zeta||^2_{K_e,\chi} + ||\nabla_\zeta'' \zeta||^2_{K_e,\chi}
\]

\[
+ \int_{K_e} e^{-\chi} \text{Tr}_{g,\eta} pr_F \left( \mathcal{R}^\chi (\zeta \otimes \zeta) \right).
\]

**Proof.** On \(\text{Dom} K_{e,\chi} \mathcal{J}_{[\zeta]},\) one has \(\mathcal{J}^* = \partial_{(\zeta)} + \partial_{(\zeta)}\). Then, for all \(\zeta = \zeta' + \zeta'' \in \text{Dom} T_{q-1}^* = \text{Dom} K_{e,\chi} \mathcal{J} \cap L^2_{c,\chi <2>}\) (see Proposition 3.1.3), one has

\[
\mathcal{J}_{[\zeta]} = \partial_{(\zeta)} + \partial_{(\zeta)}\zeta''\zeta' + \partial_{(\zeta)}\zeta' = T_{q-1}^* \zeta + \partial_{(\zeta)}\zeta',
\]

as \(T_{q-1}^* \zeta = \overline{\mathcal{J}}_{(\zeta)} \zeta' + \overline{\mathcal{J}}_{(\zeta)} \zeta''\) (see Proposition 3.1.3) and \(\partial_{(\zeta)}\zeta'' = 0\). Note also that \(\nabla(1)^{\zeta} = \nabla_\zeta' \zeta' + \nabla_\zeta' \zeta'' + \nabla_\zeta'' \zeta' + \nabla_\zeta'' \zeta''\), and \(\overline{\zeta} = S_{q} \zeta\). Since the decomposition \((eq \ref{2.4})\) is orthogonal with respect to \(g\), it follows that

\[
||\nabla(1)^{\zeta} \zeta||^2_{K_e,\chi} = ||\partial_{(\zeta)}\zeta''\|_{3}^2
\]

\[
+ ||\nabla_\zeta' \zeta''||^2_{K_e,\chi} + ||\nabla_\zeta'' \zeta'\|_{3}^2 + ||\nabla_\zeta'' \zeta''\|_{K_e,\chi}^2.
\]

Note that \(||\nabla_\zeta'' \zeta''\|_{K_e,\chi}^2 = ||\mathcal{J}_{[\zeta]} \zeta''\|_{3}^2\).

Following the argument in \([\text{Hör1}]\) pg. 101] with \(\mathcal{J}_{[\zeta]}\) in place of \(\mathcal{J}\), it follows that, for any \(\zeta \in \mathcal{A}^{0,q}((q',q''))(K_c; L), \zeta \in \mathcal{A}^{0,q}((q',q''))(\mathcal{J}_{[\zeta]}; L)\) if and only if \((\partial_{(\zeta)} \varphi) + \zeta = 0\) on \(\partial K_e\).

Since \(\partial_{(\zeta)} \varphi = 0\), it follows that \(\mathcal{A}^{0,q}((q',q''))(K_c; L) \subset \text{Dom} \mathcal{J}_{[\zeta]}\). In particular, \(\zeta' \in \text{Dom}^{(1,q-1)} \mathcal{J}_{[\zeta]}\) for all \(\zeta' \in \mathcal{A}^{0,(1,q-1)}(K_c; L)\). Then, since the decomposition \((eq \ref{2.4})\) is orthogonal with respect to \(g\), by taking the analogy between the decompositions \(\mathcal{A}^{r} = \bigoplus_{p+q=r} \mathcal{A}^{p,q}\) and \(\mathcal{A}^{p,q} = \bigoplus_{p+q' = p'} \mathcal{A}((p',q''))\) and putting \(\mathcal{J}_{[\zeta]}\) in place of \(\mathcal{J}_{[\zeta]}\), one can follow the derivation of \((eq \ref{3.7})\) as in \([\text{Si]}\) §1 and §2 to obtain

\[
||\mathcal{J}_{[\zeta]} \zeta'||^2_{K_e,\chi} + ||\partial_{(\zeta)} \zeta'||^2_{K_e,\chi} = \int_{\partial K_e} e^{-\chi} \text{Tr}_{g,\eta} \left( \partial_{(\zeta)} \mathcal{J}_{[\zeta]} \varphi \right) \left( \zeta' \otimes \zeta' \right)
\]

\[
+ ||\nabla_\zeta' \zeta'||^2_{K_e,\chi} + \int_{K_e} e^{-\chi} \text{Tr}_{g,\eta} \mathcal{R}^\chi (\zeta' \otimes \zeta')
\]

for any \(\zeta' \in \mathcal{A}^{0,(1,q-1)}(K_c; L)\). The boundary term vanishes as \(\partial_{(\zeta)} \mathcal{J}_{[\zeta]} \varphi = 0\). Therefore, combining the above results with \((eq \ref{3.7})\) yields

\[
||S_{q} \zeta||^2_{3} + ||T_{q-1}^* \zeta||^2_{1} = \Bd(\zeta, \zeta) + ||\mathcal{J}_{[\zeta]} \zeta''\|_{3}^2 + ||\mathcal{J}_{[\zeta]} \zeta'\|_{3}^2 + ||\nabla_\zeta'' \zeta'\|_{K_e,\chi}^2 + ||\nabla_\zeta'' \zeta''\|_{K_e,\chi}^2
\]

\[
+ \int_{K_e} e^{-\chi} \text{Tr}_{g,\eta} \mathcal{R}^\chi (\zeta' \otimes \zeta') - \int_{K_e} e^{-\chi} \text{Tr}_{g,\eta} \mathcal{R}^\chi (\zeta' \otimes \zeta').
\]
For every fixed \( z \in K_c \), \( \text{Tr}_{g,\eta} \mathcal{R}^\nu(\zeta \otimes \zeta) \) is a hermitian form in \( \zeta \). Again, since the decomposition [eq 2.4] is orthogonal with respect to \( g \), it follows that
\[
\text{Tr}_{g,\eta} \mathcal{R}^\nu(\zeta' \otimes \zeta') = \text{Tr}_{g,\eta} \mathcal{R}^\nu_{\nu=0}(\zeta' \otimes \zeta') + \text{Tr}_{g,\eta} \mathcal{R}^\nu_{\nu=2}(\zeta' \otimes \zeta'),
\]
\[
\text{Tr}_{g,\eta} \mathcal{R}^\nu(\zeta'' \otimes \zeta'') = \text{Tr}_{g,\eta} \mathcal{R}^\nu_{\nu=2}(\zeta'' \otimes \zeta'').
\]
On the other hand, one has
\[
\text{pr}_F \left( \mathcal{R}^\nu(\zeta' \otimes \zeta') \right) = \mathcal{R}^\nu_{\nu=0}(\zeta' \otimes \zeta'), \quad \text{pr}_F \left( \mathcal{R}^\nu(\zeta'' \otimes \zeta'') \right) = \mathcal{R}^\nu_{\nu=2}(\zeta'' \otimes \zeta'').
\]
Therefore, it follows that
\[
\text{Tr}_{g,\eta} \mathcal{R}^\nu(\zeta \otimes \zeta) = \text{Tr}_{g,\eta} \mathcal{R}^\nu_{\nu=0}(\zeta' \otimes \zeta') + \text{Tr}_{g,\eta} \mathcal{R}^\nu_{\nu=2}(\zeta' \otimes \zeta')
\]
and hence the lemma. \( \square \)

Let \( g_F := \text{pr}_F g \), and let \( (g_F)_{j,j'}^l \)'s for \( 1 \leq j, j' \leq m \) be the entries of the inverse of the matrix of \( g_F \) under the chosen coordinates. Denote by \((.,.)_{g,\eta,\chi}\) the pointwise inner product induced from \( |.|_{g,\eta,\chi} \). Write \( \nabla^{(1,0)} = \nabla_u^{(1,0)} + \nabla_v^{(1,0)} \) as the splitting of \( \nabla^{(1,0)} \) according to the decomposition [eq 2.4], and set \( \nabla_u := \nabla_u^{(1,0)} \) and \( \nabla_v := \nabla_v^{(1,0)} \) for convenience. The following integration by parts argument is put into a lemma for clarity.

**Lemma 3.2.3.** For all \( \zeta'' \in \mathcal{A}^{0,0}(\nabla^{(1,0)}|_{K_c}; L) \), one has
\[
\|\nabla_v \zeta''\|^2_{K_c,\chi} = \|\nabla_v \zeta''\|^2_{K_c,\chi} - \int_{K_c} \left( \text{Tr}_g \mathcal{R}_{\nu,\chi} \right) |\zeta''|^2_{g,\eta,\chi}.
\]

**Proof.** Recall that \( d\mu := \omega^{\nu,\eta}_{|}\frac{\partial}{\partial \nu} \) is the volume element on \( K_c \), while that on \( \partial K_c \) is given by \( d\sigma := \left( \frac{\partial}{\partial \nu} \right)^\nu_{\eta=\nu} d\mu \bigg|_{\partial K_c} \). Einstein summation convention is applied in what follows. Fix a \( \zeta'' \in \mathcal{A}^{0,0}(\nabla^{(1,0)}|_{K_c}; L) \). Let
\[
Y := (\nabla_v \nu^{\nu''} \zeta''; g,\eta,\chi)_{g,\eta,\chi}(g_F)_{j,j'}^l \frac{\partial}{\partial \nu} \quad \text{and} \quad W := (\nabla_v \nu^{\nu''} \zeta''; g,\eta,\chi)(g_F)_{j,j'}^l \frac{\partial}{\partial \nu} \]
be two vector fields in \( \mathcal{A}^{0,0}(\nabla^{(1,0)}|_{K_c}) \) and \( \mathcal{A}^{0,0}(\nabla^{(1,0)}|_{K_c}) \) respectively. Then, using the fact that \( \nabla_v = \text{pr}_F g \), it follows that
\[
|\nabla_v \zeta''|^2_{g,\eta,\chi} d\mu = \left( \frac{\partial}{\partial \nu} \left( (g_F)_{j,j'}^l \nabla_v \nu^{\nu''} \zeta''; g,\eta,\chi \right) \right)_{g,\eta,\chi} d\mu - \left( (g_F)_{j,j'}^l \nabla_v \nu^{\nu''} \zeta''; g,\eta,\chi \right)_{g,\eta,\chi} d\mu
\]
\[
= d(Y \cdot d\mu) - (g_F)_{j,j'}^l \nabla_v \nu^{\nu''} \zeta''; g,\eta,\chi)_{g,\eta,\chi} d\mu - \left( (g_F)_{j,j'}^l \nabla_v \nu^{\nu''} \zeta''; g,\eta,\chi \right)_{g,\eta,\chi} d\mu
\]
\[
= d(Y \cdot d\mu) - \left( \frac{\partial}{\partial \nu} \left( (g_F)_{j,j'}^l \nabla_v \nu^{\nu''} \zeta''; g,\eta,\chi \right) \right)_{g,\eta,\chi} d\mu + |\nabla_v \zeta''|^2_{g,\eta,\chi} d\mu
\]
\[
= d(Y \cdot d\mu) - (W \cdot d\mu) + |\nabla_v \zeta''|^2_{g,\eta,\chi} d\mu - \left( \text{Tr}_g \mathcal{R}_{\nu,\chi} \right) |\zeta''|^2_{g,\eta,\chi} d\mu.
\]
Since $\partial K_c = \{ \varphi = c \}$ and $(d\varphi)|_{\partial K_c} = 0$, it follows that for any vector field $V$ such that \( (V \cdot (\frac{d\varphi}{|d\varphi|})_g) = 0 \), one has \( (V \cdot d\mu)|_{\partial K_c} = 0 \). The component of $Y - W$ in the direction of $\frac{d\varphi}{|d\varphi|}_g$ is $(Y - W, (\frac{d\varphi}{|d\varphi|})_g)$. Therefore, by integrating over $K_c$ and applying Stokes’ theorem, it yields
\[
\|\nabla_v \zeta''\|^2_{K_{c,\chi}} = \int_{\partial K_c} \left( Y - W, (\frac{d\varphi}{|d\varphi|})_g \right) d\sigma + \|\nabla_v \zeta''\|^2_{K_{c,\chi}} - \int_{K_c} (Tr_g R_{\varphi,\sigma}) |\zeta''|^2_{g,\eta,\chi} d\mu.
\]
But $(d\varphi)^\vee \in \mathcal{A}(1,0,0) \oplus \mathcal{A}(0,1,0) (X)$, so $(Y - W, (\frac{d\varphi}{|d\varphi|})_g) = 0$ and hence the lemma.

Combining the result above with (eq 3.9) yields (eq 3.10)
\[
\|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 = Bd(\zeta, \zeta) + \|\mathcal{T}_{(\omega)}\zeta''\|_3^2 + \|\mathcal{T}_{(\eta)}\zeta''\|_3^2 + \|\nabla_v \zeta''\|^2_{K_{c,\chi}} - \int_{K_c} (Tr_g R_{\varphi,\sigma}) |\zeta''|^2_{g,\eta,\chi} + \int_{K_c} e^{-\chi} Tr_{g,\eta} pr_F (R^\vee (\zeta \otimes \overline{\zeta})).
\]
This formula is analogous to the usual $\nabla$-Bochner–Kodaira formula (see [Sim 2.2.1]). However, it contains term involving $\nabla_v$, but not $\nabla_u$, and the boundary term is the same as the one in the $\nabla$-Bochner–Kodaira formula.

Consider the boundary term $Bd(\zeta, \zeta)$ in (eq 3.8). Since $\sqrt{-1} \partial \overline{\partial} \varphi$ is non-negative on $\partial K_c$, i.e. $K_c$ is pseudoconvex, by choosing coordinates at any point in $\partial K_c$ such that $\sqrt{-1} \partial \overline{\partial} \varphi$ and $g$ are simultaneously diagonalized, one sees that $Bd(\zeta, \zeta)$ is non-negative for all $\zeta \in \mathcal{A}_{<2} (\overline{K_c}; L)$. Noting that all other norm-square terms are also non-negative, one then obtains
\[
\|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 \geq \int_{K_c} e^{-\chi} Tr_{g,\eta} pr_F (R^\vee (\zeta \otimes \overline{\zeta}))
\]
and
\[
\|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 \geq - \int_{K_c} (Tr_g R_{\varphi,\sigma}) |\zeta''|^2_{g,\eta,\chi} + \int_{K_c} e^{-\chi} Tr_{g,\eta} pr_F (R^\vee (\zeta \otimes \overline{\zeta}))
\]
for all $\zeta \in \mathcal{A}_{<2} (\overline{K_c}; L) \cap \text{Dom}_{K_{c,\chi}} T_{q-1}^*$. These are the Bochner–Kodaira inequalities for $T_{q-1}^*$ and $S_q$ which are used to obtain the required $L^2$ estimates.

### 3.3. Murakami’s trick

From (eq 3.6) and log $(\eta e^{-\chi}) = \log \eta + \log \eta_w - \chi$ (see §2.5 for notation), it follows that
\[
\Theta = \Theta_T + \Theta_{2\theta} + \sqrt{-1} \partial \overline{\partial} \chi = \pi \sqrt{-1} \partial \overline{\partial} \mathcal{H} + 2 \sqrt{-1} \partial \overline{\partial} \text{Re} \mathfrak{h} + \sqrt{-1} \partial \overline{\partial} \mathcal{H}
\]
where $\Theta_T$ and $\Theta_{2\theta}$ are respectively the tame and wild curvature forms of $L$ defined in §2.5 and $\mathcal{H}$ is a hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$ associated to $L$. Therefore, by abusing
Let $H$ to mean the associated hermitian form in $\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}(X)$, the curvature integral $\int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F \left( H^\vee (\zeta \otimes \zeta) \right)$ in (eq 3.11) and (eq 3.12) can be split into the sum of

$$\mathcal{X}(\zeta, \zeta) := \pi \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F \left( H^\vee (\zeta \otimes \zeta) \right),$$

(eq 3.14)

$$\mathcal{M}(\zeta, \zeta) := \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F \left( (2 \partial \overline{\partial} \text{Re} \, h_8)^\vee (\zeta \otimes \zeta) \right),$$

$$\mathcal{W}(\zeta, \zeta) := \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F \left( (\partial \overline{\partial} \chi)^\vee (\zeta \otimes \zeta) \right),$$

(recall that $\mathcal{A}^{1,1}$ is identified with $\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}$ via $dz^k \wedge d\overline{z}^\ell \mapsto dz^k \otimes d\overline{z}^\ell$ for all $1 \leq k, \ell \leq n$).

One of the essential ingredients for obtaining the required $L^2$ estimates for $q < s_F^-$ or $q > m - s_F^+$ is Murakami’s trick used in [Mur]. The trick is applied to the part of the curvature integral $\mathcal{X}(\zeta, \zeta)$ involving $H_F := H|_{F \times F}$. For any $\zeta = \zeta' + \zeta'' \in \mathcal{A}^{0,1}(\overline{K}_c; L)$ where $\zeta' \in \mathcal{A}^{0,1,q-1}(\overline{K}_c; L)$ and $\zeta'' \in \mathcal{A}^{0,0,q}(\overline{K}_c; L)$, that part is given by

$$\mathcal{X}_F(\zeta, \zeta) := \pi \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F \left( H^\vee_F (\zeta \otimes \zeta) \right)$$

$$= \pi \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F H^\vee_F (\zeta'' \otimes \zeta'') = \mathcal{X}_F (\zeta'', \zeta'').$$

**Definition 3.3.1.** An $H$-apt coordinate system is an apt coordinate system such that the matrix of $H_F$ under such coordinate system is given by

(eq 3.15)

$$H_F = D := \text{diag} \left( \frac{1}{s_F^+}, \ldots, \frac{1}{s_F^+}, \frac{1}{m-s_F^-}, \ldots, \frac{1}{m-s_F^-} \right).$$

Under a chosen apt coordinate system, an $H$-apt coordinate system can be obtained by a linear change of coordinates only in the variable $v$ (which preserves the decomposition in (eq 2.4)).

In what follows, write $d\overline{v}^{J_q} := d\overline{v}^{j_1} \wedge \ldots \wedge d\overline{v}^{j_q}$ for every $q$-multiindex $J_q = (j_1, \ldots, j_q)$. Moreover, let $\zeta''_q$ be the component of $\zeta'' \in \mathcal{A}^{0,0,q}(L)$ corresponding to $d\overline{v}^{J_q}$, and $\zeta'_{i,j,\ldots}$ the component of $\zeta' \in \mathcal{A}^{0,1,q-1}(L)$ corresponding to $d\overline{w}^i \wedge d\overline{v}^{J_q-1}$.

**Lemma 3.3.2** (Murakami’s trick for $q > m - s_F^+$). For any $q > m - s_F^+$ and given any constant $M > 0$, one can choose the translational invariant hermitian metric $g$ suitably such that $\mathcal{X}_F (\zeta'', \zeta'') \geq \pi M ||\zeta''||^2_2$ for every $\zeta'' \in \mathcal{A}^{0,0,q}(\overline{K}_c; L)$.

**Proof.** Fix an $H$-apt coordinate system. Choose $g$ such that it is diagonal in the chosen $H$-apt coordinates and its matrix is given by

(eq 3.16)

$$\text{diag} \left( \frac{1}{g_1^F}, \ldots, \frac{1}{g_1^F}, \frac{1}{g_2^F}, \ldots, \frac{1}{g_2^F}, \frac{1}{g_3^F}, \ldots, \frac{1}{g_3^F}, \frac{1}{g_4^F},\ldots, \frac{1}{g_4^F} \right),$$

where $g_1^F, g_2^F$ and $g_3^F$ are positive numbers. Given $M > 0, g_1^F, g_2^F$ and $g_3^F$ are chosen as

$$g_1^F := s_F^+ + M, \quad g_2^F := 1 \quad \text{and} \quad g_3^F := 1.$$
Under the chosen \(\mathcal{H}\)-apt coordinates, since \(\mathcal{H}_F\) and \(g\) are both diagonal, the monomial forms \(\zeta^n_{J_q} d\nu_{J_q} \in \mathcal{A}^{0,(0,q)}(\mathcal{K}_c; L)\) with different multiindices \(J_q\) are orthogonal to one another with respect to \(\mathcal{T}_F\) and \(\langle \cdot, \cdot \rangle_2\). Therefore, it suffices to show that

\[
\mathcal{T}_F \left( \zeta^n_{J_q} d\nu_{J_q}, \zeta^n_{J_q} d\nu_{J_q} \right) \geq \pi M \left\| \zeta^n_{J_q} d\nu_{J_q} \right\|^2_2
\]

for all monomial forms \(\zeta^n_{J_q} d\nu_{J_q} \in \mathcal{A}^{0,(0,q)}(\mathcal{K}_c; L)\).

In fact, for each multiindex \(J_q = (j_1, \ldots, j_q)\), one has

\[
\mathcal{T}_F \left( \zeta^n_{J_q} d\nu_{J_q}, \zeta^n_{J_q} d\nu_{J_q} \right) = \pi \int_{K_c} \left( \sum_{\nu=1}^{q} (g_F)_{J_q}^{\nu} (H_F)_{J_q}^{\nu} \right) \left\| \zeta^n_{J_q} d\nu_{J_q} \right\|^2_{g,\pi,\chi}
\]

\[
= \pi \left( \sum_{\nu=1}^{q} (g_F)_{J_q}^{\nu} (H_F)_{J_q}^{\nu} \right) \left\| \zeta^n_{J_q} d\nu_{J_q} \right\|^2_2,
\]

where \((g_F)_{J_q}^{\nu}\)'s are the diagonal components of \((g_F)^{-1} := (pr_F) \cdot (H_F)_{J_q}^{\nu} \cdots (H_F)_{J_q}^{\nu}\)'s are the diagonal entries of \(H_F\) in \(\text{eq } 3.15\), which are either 1, \(-1\) or 0. Define

\[
R^+(J_q) := \# \{ j_{\nu} \in J_q : 1 \leq j_{\nu} \leq s_F^+ \}
\]

\[
R^-(J_q) := \# \{ j_{\nu} \in J_q : s_F^++1 \leq j_{\nu} \leq s_F^++s_F^- \}.
\]

Then, the sum in the parenthesis becomes

\[
\text{eq } 3.17
\]

\[
\sum_{\nu=1}^{q} (g_F)_{J_q}^{\nu} (H_F)_{J_q}^{\nu} = g_F^+ R^+(J_q) - g_F^E R^-(J_q)
\]

\[
= (s_F^-+M)R^+(J_q) - R^-(J_q).
\]

Since \(q > m - s_F^+\), it follows that \(R^+(J_q) \geq 1\) for any multiindex \(J_q\). Note also that \(R^-(J_q) \leq s_F^-\) for any \(J_q\). Therefore, by the choice of \(g_F^+\) and \(g_F^-\), one obtains \(g_F^+ R^+(J_q) - g_F^- R^-(J_q) \geq M\) and thus \(\pi\) follows. This completes the proof. \(\square\)

In order to apply Lemma 3.3.2 to \(\text{eq } 3.11\), note that for any \(\zeta'' \in \mathcal{A}^{0,(0,q)}(\mathcal{K}_c; L)\), one has

\[
\|\zeta''\|^2_2 = \int_{K_c} e^{-x} \text{Tr}_{g,\pi} \zeta'' \otimes \zeta'' .
\]

Decompose \(\mathcal{H} \in \mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}(X)\) into \(\mathcal{H}_{E} + \mathcal{H}_{\pi} + \mathcal{H}_{\nu} + \mathcal{H}_{F}\) according to the decomposition \(\text{eq } 2.4\) as is done to \(\mathcal{R}\) (write \(\mathcal{H}_{E}\) for \(\mathcal{H}_{\nu}\) and \(\mathcal{H}_{F}\) for \(\mathcal{H}_{\nu}\), to respect previous notations). Now note that, for any \(\zeta = \zeta' + \zeta'' \in \mathcal{A}^{0,(0,q)}(\mathcal{K}_c; L)\) such that \(\zeta' \in \mathcal{A}^{0,(1,q-1)}(\mathcal{K}_c; L)\) and \(\zeta'' \in \mathcal{A}^{0,(0,q)}(\mathcal{K}_c; L)\), one has

\[
\mathcal{T}(\zeta, \zeta) = \pi \int_{K_c} e^{-x} \text{Tr}_{g,\pi} (\mathcal{H}_E(\zeta' \otimes \zeta') + \mathcal{H}_\nu(\zeta' \otimes \zeta'))
\]

\[
+ \mathcal{H}_\nu(\zeta'' \otimes \zeta'') + \mathcal{H}_F(\zeta'' \otimes \zeta'') .
\]

If \(q > m - s_F^+\), Lemma 3.3.2 then implies that, given \(M > 0\), \(g\) can be chosen such that

\[
\mathcal{T}(\zeta, \zeta) \geq \pi \int_{K_c} e^{-x} \left[ \text{Tr}_{g,\pi} (\mathcal{H}_E(\zeta' \otimes \zeta') + \mathcal{H}_\nu(\zeta' \otimes \zeta')) + \mathcal{H}_\nu(\zeta'' \otimes \zeta'') \right]
\]

\[
+ M \text{Tr}_{g,\pi} \zeta'' \otimes \zeta'' .
\]
Define $\mathcal{H}(M)$ to be an element in $\mathcal{A}^{0,0} \oplus (\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}) (X)$ such that
\[
\text{pr}_F \left( \left( \mathcal{H}(M) \right)^{\vee} (\zeta \otimes \overline{\zeta}) \right) = \mathcal{H}^{\vee}_F (\zeta' \otimes \overline{\zeta'}) + \mathcal{H}^{\vee}_{\pi}(\zeta' \otimes \overline{\zeta''}) + \mathcal{H}^{\vee}_{\pi} (\zeta'' \otimes \overline{\zeta''})
\]
for any $\zeta = \zeta' + \zeta'' \in \mathcal{A}^{0,q}_{\leq 2} (\mathcal{K}^c; L)$ (note that $\zeta' = 0$ when $q = 0$). Then one has
\[
\mathcal{F}(\zeta, \zeta) \geq \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left( \left( \mathcal{H}(M) \right)^{\vee} (\zeta \otimes \overline{\zeta}) \right) + \mathcal{M}(\zeta, \zeta) + \text{rot}(\zeta, \zeta)
\]
when $q > m - s_F^+$. Therefore, the consequence of Lemma 3.3.2 applied to (eq 3.11) can be stated as follows.

**Corollary 3.3.3.** Suppose $q > m - s_F^+$. Then, given any constant $M > 0$, the translational invariant hermitian metric $g$ can be chosen suitably such that \[eq 3.11\] yields
\[
\|S_q\zeta\|^2 + \|T_{q-1}^*\zeta\|_1^2 \geq \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left( \left( \mathcal{H}(M) \right)^{\vee} (\zeta \otimes \overline{\zeta}) \right) + \mathcal{M}(\zeta, \zeta) + \text{rot}(\zeta, \zeta)
\]
for all $\zeta \in \mathcal{A}^{0,q}_{\leq 2} (\mathcal{K}^c; L) \cap \text{Dom} T_{q-1}^*$.

Now consider the integral involving $\text{Tr}_g \mathcal{R} \mathcal{H}$ in \[eq 3.12\]. Note that
\[
\text{pr}_F \Theta = \pi \sqrt{-1} \partial \bar{\partial} \mathcal{H} + 2 \sqrt{-1} \partial \bar{\partial} \left( \text{Re} \, h_\delta \right).
\]
Here no term involving $\chi$ appears since $\sqrt{-1} \partial \bar{\partial} \chi = 0$. Again, by abusing $\mathcal{H}_F$ to mean the associated hermitian form in $\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1} (X)$, the curvature integral $-\int_{K_c} (\text{Tr}_g \mathcal{R}_0) |\zeta''^2_{g,\eta,\chi}|$ in \[eq 3.12\] can be split into the sum of
\[
\mathcal{F}'_F (\zeta'', \zeta'') := -\pi \int_{K_c} \left( \text{Tr}_g \mathcal{H}_F \right) |\zeta''^2_{g,\eta,\chi}|
\]
and
\[
\mathcal{M}'_F (\zeta'', \zeta'') := \int_{K_c} (2 \text{Tr}_g \partial \bar{\partial} \mathcal{H} \text{Re} \, h_\delta) |\zeta''^2_{g,\eta,\chi}|.
\]
Similar argument as in the proof of Lemma 3.3.2 yields

**Lemma 3.3.4** (Murakami’s trick for $q < s_F^+$). Suppose that $q < s_F^+$. Then for any given constant $M > 0$, one can choose the translational invariant hermitian metric $g$ suitably such that
\[
\mathcal{F}'_F (\zeta'', \zeta'') + \mathcal{F}_F (\zeta'', \zeta'') \geq \pi M \|\zeta''\|^2_2
\]
for all $\zeta'' \in \mathcal{A}^{0,(0,q)} (\mathcal{K}^c; L)$.

**Proof.** Fix an $\mathcal{H}$-apt coordinate system. Choose $g$ as in \[eq 3.10\]. Given $M > 0$, $g^+_F$, $g^+_E$ and $g^+_0$ are chosen as
\[
g^+_F := 1, \quad g^+_E := s_F^+ + M \quad \text{and} \quad g^+_0 := 1.
\]
Using the $\mathcal{H}$-apt coordinates, one sees that $\text{Tr}_g \mathcal{H}_F = g^+_F s^+_F - g^+_E s^-_F$ and therefore
\[
\mathcal{F}'_F (\zeta'', \zeta'') = \pi \left( g^+_F s^+_F - g^+_E s^-_F \right) \|\zeta''\|^2_2.
\]
Again, since $\mathcal{H}_F$ and $g$ are both diagonal under the chosen $\mathcal{H}$-apt coordinates, the monomial forms $\zeta^q_{\frac{q}{2}} d^{1/2} \in \mathcal{A}^{0,(0,q)} (\mathcal{K}^c; L)$ with different multiindices $J_q$ are
orthogonal to one another with respect to $\Sigma_F$ and $\langle \cdot, \cdot \rangle_2$. Therefore, it suffices to show that
\[
(\ast\ast) \quad \pi \left( g^E_s \mathbf{F}_s - g^E_{s'} \mathbf{F}_{s'} \right) \left\| \zeta''_q \, dv^{J_q} \right\|_2^2 + \Sigma_F \left( \zeta''_q \, dv^{J_q} , \zeta''_q \, dv^{J_q} \right) \geq \pi M \left\| \zeta''_q \, dv^{J_q} \right\|_2^2
\]
for all monomial forms $\zeta''_q \, dv^{J_q} \in H^0(0,0L) (K_c; L)$.

Taking into account (eq. 3.17) and the expression of $\Sigma_F$ in the proof of Lemma 3.3.2 it follows that
\[
\pi \left( g^E_s \mathbf{F}_s - g^E_{s'} \mathbf{F}_{s'} \right) \left\| \zeta''_q \, dv^{J_q} \right\|_2^2 + \Sigma_F \left( \zeta''_q \, dv^{J_q} , \zeta''_q \, dv^{J_q} \right) = \pi \left( g^E_s \left( \mathbf{F}_s - R^{-}(J_q) \right) - g^E_{s'} \left( \mathbf{F}_{s'} - R^{+}(J_q) \right) \right) \left\| \zeta''_q \, dv^{J_q} \right\|_2^2.
\]
Since $q < s_{\mathbf{F}}$, it follows that $s_{\mathbf{F}} - R^{-}(J_q) \geq 1$ for any multiindex $J_q$. Note also that $s_{\mathbf{F}} - R^{+}(J_q) \leq s_{\mathbf{F}}$ for any $J_q$. Therefore, by the choice of $g^E_s$ and $g^E_{s'}$, one obtains $g^E_s \left( \mathbf{F}_s - R^{-}(J_q) \right) - g^E_{s'} \left( \mathbf{F}_{s'} - R^{+}(J_q) \right) \geq M$ and thus (\ast\ast) follows. This completes the proof. $\square$

Considering the definition of $\tilde{H}(M)$ in (eq. 3.18), Lemma 3.3.4 then implies that, if $q < s_{\mathbf{F}}$, then, given $M > 0$, $g$ can be chosen such that
\[
\Sigma_F(\zeta'', \zeta'') + \Sigma(\zeta', \zeta) \geq \pi \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F \left( \left( \tilde{H}(M) \right)^\vee (\zeta \otimes \zeta) \right)
\]
for all $\zeta = \zeta' + \zeta'' \in H^0(0,0L) (K_c; L)$. Combining this with (eq. 3.12) yields

**COROLLARY 3.3.5.** Suppose $q < s_{\mathbf{F}}$. Then, given any constant $M > 0$, the translational invariant hermitian metric $g$ can be chosen suitably such that (eq. 3.13) yields
\[
\left\| S_q \zeta \right\|_2 + \left\| T_{q-1}^{*} \zeta \right\|_1 \geq \pi \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F \left( \left( \tilde{H}(M) \right)^\vee (\zeta \otimes \zeta) \right) + 2M(\zeta', \zeta) + 2\mathbb{M}(\zeta', \zeta) + 2\mathbb{M}(\zeta, \zeta)
\]
for all $\zeta = \zeta' + \zeta'' \in H^0(0,0L) (K_c; L) \cap \text{Dom} T_{q-1}^{*}$, where $\zeta'' \in H^0(0,0L) (K_c; L)$ and $\zeta' \in H^0(1,0L) (K_c; L) \cap \text{Dom} T_{q-1}^{*}$.

The remaining part of this section is devoted to getting a suitable estimate of the integral
\[
\pi \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F \left( \left( \tilde{H}(M) \right)^\vee (\zeta \otimes \zeta) \right)
\]
by varying $K_c$ in $\tilde{H}(M)$ (see (eq. 3.18)) according to Proposition 2.4.2.

**LEMMA 3.3.6.** Given a constant $M > 0$ and a fixed translational invariant hermitian metric $g$ on $X$ such that the decomposition (eq. 2.4) is orthogonal, one can choose $K_c$ sufficiently positive according to Proposition 2.4.2 such that
\[
\pi \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{pr}_F \left( \left( \tilde{H}(M) \right)^\vee (\zeta \otimes \zeta) \right) \geq \frac{\pi}{4} M \left\| \zeta \right\|_2^2
\]
for all $\zeta \in H^0(0,0L) (K_c; L)$.
Proof. For \( q = 0 \), it follows from (eq 3.18) that
\[
\pi \int_{K_c} e^{-x} \text{Tr}_{g,\eta} \text{Pr}_F \left( \left( \tilde{\mathcal{H}}(M) \right) \left( \zeta \otimes \bar{\zeta} \right) \right) = \pi M \| \zeta \|_2^2 \geq \frac{4}{\pi} M \| \zeta \|_2^2 ,
\]
so this case is done.

Assume \( q \neq 0 \). Since \( \mathcal{H}^\nu_{\alpha\sigma} \) is a bounded linear operator on \( L_2^{0,(1,0)} \cap \mathcal{N} \) (where \( L_2^{0,(1,0)} \) here means the complex conjugate of \( \mathcal{H}^{0,(1,1)} \)), it follows that there is a bounded linear operator \( N: L_2^{0,(q,0)} \to L_2^{0,(q-1,0)} \) such that
\[
\int_{K_c} e^{-x} \text{Tr}_{g,\eta} \mathcal{H}^\nu_{\alpha\sigma}(\zeta' \otimes \bar{\zeta''}) = \langle \zeta', N\zeta'' \rangle_2
\]
for all \( \zeta' \in L_2^{0,(q-1,0)} \) and \( \zeta'' \in L_2^{0,(0,q)} \). In fact, after a linear change of coordinates such that \( g \) becomes the Euclidean metric while keeping the decomposition (eq 2.4) orthogonal, one has
\[
\text{Tr}_{g,\eta} \mathcal{H}^\nu_{\alpha\sigma}(\zeta' \otimes \bar{\zeta''}) = \eta \sum_{J_{q-1}} \sum_{i=1}^{m} \sum_{j=1}^{n-m} \zeta'_i (\mathcal{H}^{\nu}_{\alpha\sigma})_i^j \zeta''_j ,
\]
where \( \sum_{J_{q-1}} \) denotes summation over all ordered multiindices \( J_{q-1} \) such that \( 1 < j_1 < \cdots < j_{q-1} \leq m \), and \( (\mathcal{H}^{\nu}_{\alpha\sigma})_i^j \)'s are the components of \( \mathcal{H}^{\nu}_{\alpha\sigma} = \overline{\mathcal{H}^{\nu}_{\alpha\sigma}} \). Therefore, under such coordinates,
\[
(N\zeta'')_{J_{q-1}} = \sum_{j=1}^{m} (\mathcal{H}^{\nu}_{\alpha\sigma})_j^i \zeta''_j .
\]
Moreover,
\[
|N\zeta''|^2_{g,\eta} = \eta \sum_{J_{q-1}} \sum_{i=1}^{m} \sum_{j=1}^{n-m} |(\mathcal{H}^{\nu}_{\alpha\sigma})_i^j \zeta''_j|^2 \leq \eta \sum_{J_{q-1}} \sum_{i=1}^{m} \left( \sum_{j=1}^{n-m} |(\mathcal{H}^{\nu}_{\alpha\sigma})_i^j|^2 \right) \left( \sum_{j=1}^{n-m} |\zeta''_j|^2 \right) \text{ by Cauchy–Schwarz ineq.,}
\]
\[
= |\mathcal{H}^{\nu}_{\alpha\sigma}|_{g,\eta}^2 \cdot q |\zeta''|^2_{g,\eta} = |\mathcal{H}^{\nu}_{\alpha\sigma}|_{g,\eta}^2 \cdot q |\zeta''|^2_{g,\eta} \text{ as } \mathcal{H}^{\nu}_{\alpha\sigma} = \overline{\mathcal{H}^{\nu}_{\alpha\sigma}} .
\]
Since both \( \mathcal{H}^{\nu}_{\alpha\sigma} \) and \( g \) are translational invariant forms, \( |\mathcal{H}^{\nu}_{\alpha\sigma}|_{g,\eta}^2 \) is a constant. Set \( \nu := \sqrt{q} |\mathcal{H}^{\nu}_{\alpha\sigma}|_{g,\eta} \). Then, one has
\[
|N\zeta''|^2_{g,\eta} \leq \nu \| \zeta'' \|_2
\]
for all \( \zeta'' \in L_2^{0,(0,q)} \). Note that \( \nu \) depends only on \( q, \mathcal{H}^{\nu}_{\alpha\sigma} \) and \( g \). It is independent of \( \mathcal{H}_E \) in particular.

Since the decomposition (eq 2.4) is orthogonal with respect to \( g, g \) can be decomposed into \( g_E + g_F \) such that \( g_E \) is a hermitian metric on \( T^{1,0}_u \) and \( g_F \) is that on \( T^{1,0}_v \). Choose a real number \( \lambda > 0 \) such that
\[
(\#, \#) \quad \lambda \geq \max \left\{ \frac{M}{2}, \frac{2\nu^2}{M}, 4\nu \right\} .
\]
Since \( \nu \) is independent of \( \mathcal{H}_E \), by varying the real part of the matrix of \( \mathcal{H}_E \) under the chosen apt coordinates according to Proposition 2.4.2, \( \mathcal{H}_E \) can be chosen such that
\[
\mathcal{H}_E \geq \lambda g_E ,
\]
and therefore,
\[ \int_{K_c} e^{-\chi} \text{Tr}_{g;\eta} \mathcal{K}^\vee_{E}(\zeta' \otimes \overline{\zeta'}) \geq \lambda \| \zeta' \|_2^2 \]
for all \( \zeta' \in \mathfrak{g}^{0,(1,q-1)}(K_c; L) \).

It follows from (eq. 3.18) that, for any \( \zeta = \zeta' + \zeta'' \in \mathfrak{g}^{0,q}_{2,(1)}(K_c; L) \),
\[ \int_{K_c} e^{-\chi} \text{Tr}_{g;\eta} \text{pr}_F \left( \mathcal{H}(M) \right)^\vee (\zeta \otimes \overline{\zeta}) \]
\[ \geq \lambda \| \zeta' \|_2^2 + 2 \text{ Re} \langle \zeta', N\zeta'' \rangle_2 + M \| \zeta'' \|_2^2 \]
\[ = \lambda \left\| \zeta' + \frac{1}{\lambda} N\zeta'' \right\|^2_2 - \frac{1}{\lambda} \| N\zeta'' \|_2^2 + M \| \zeta'' \|_2^2 \]
\[ \geq \lambda \left\| \zeta' + \frac{1}{\lambda} N\zeta'' \right\|^2_2 - \frac{\nu^2}{\lambda} \| \zeta'' \|_2^2 + M \| \zeta'' \|_2^2 \]
by completing square,
\[ \geq \frac{M}{2} \left( \left\| \zeta' + \frac{1}{\lambda} N\zeta'' \right\|^2_2 + \| \zeta'' \|_2^2 \right) \]
by \( \ast \), thus \( \frac{\nu^2}{\lambda} \leq \frac{M}{2} \),
\[ = \frac{M}{2} \left\| \zeta + \frac{1}{\lambda} N\zeta'' \right\|^2_2 \]
by \( \ast \), thus \( \| \zeta'' \|_2 \leq \| \zeta \|_2 \),
as \( L^2_{c,\chi} \perp L^2_{c,\chi} \).

Furthermore, since
\[ \left\| \zeta + \frac{1}{\lambda} N\zeta'' \right\|^2_2 \geq \| \zeta \|_2^2 - \frac{\nu^2}{\lambda} \| \zeta'' \|_2^2 \]
\[ \geq \| \zeta \|_2^2 - \frac{\nu^2}{\lambda} \| \zeta'' \|_2^2 \]
by \( \ast \),
\[ \geq \left( 1 - \frac{\nu^2}{\lambda} \right) \| \zeta \|_2 \]
as \( \| \zeta'' \|_2 \leq \| \zeta \|_2 \),
\[ \geq \frac{3}{4} \| \zeta \|_2 \geq 0 \]
by \( \ast \),
one has
\[ \frac{M}{2} \left\| \zeta + \frac{1}{\lambda} N\zeta'' \right\|^2_2 \geq \frac{M}{2} \cdot \left( \frac{3}{4} \right)^2 \| \zeta \|_2^2 \geq \frac{M}{4} \| \zeta \|_2^2 . \]

This completes the proof. \( \square \)
CHAPTER 4

The linearizable case

4.1. Proof of Theorem 1.1.1 for linearizable $L$

The proof of Theorem 1.1.1 for linearizable $L$ is given here so that one can see clearly how the proof works without having to handle additional technicality required for the case of non-linearizable line bundles.

**Theorem 4.1.1.** Suppose $L$ is linearizable and $q < s_F^-$ or $q > m - s_F^+$. Then, for any $\psi \in \mathcal{H}^{0,q}(X;L)$ such that $\overline{\partial} \psi = 0$, there exists $\xi \in \mathcal{H}^{0,q-1}(X;L)$ such that $\overline{\partial} \xi = \psi$ on $X$. (In case $q = 0 < s_F^-$, this means $\psi = 0$.) In other words, by virtue of Theorem 2.3.1, $H^q(X,L) = 0$ for any $q$ in the given range.

**Proof.** Fix any $\psi \in \mathcal{H}^{0,q}(X;L) \cap \ker \overline{\partial}$.

An $L^2$-norm $\|\cdot\|_{X,\chi}$ is chosen as follows. Since $L$ is linearizable, one can take $h = 0$ (see (2.5) for the definition of $h$). Then, choose $\delta = 0$ and thus $h_\delta = h - \delta = 0$. Choose the translational invariant hermitian metric $g$ of the form as described in the proof of Lemma 3.3.2 for $q > m - s_F^+$ or Lemma 3.3.4 for $q < s_F^-$, with $M = 1$. For the hermitian form $\mathcal{H}$ associated to $L$, choose $\mathcal{H}_E := \mathcal{H}|_{E \times E}$ as described in the proof of Lemma 3.3.6. A hermitian metric $\eta$ on $L$ is then defined as in (2.5).

Choose a convex increasing smooth function $\tilde{\chi}$ (thus $\chi := \tilde{\chi} \circ \varphi$ is plurisubharmonic, i.e. $\sqrt{-1} \partial \overline{\partial} \chi \geq 0$) such that $\|\psi\|_{X,\chi} < \infty$. An $L^2$-norm $\|\cdot\|_{X,\chi}$ is then fixed and $\psi \in L^2_{\chi} (X;L)$.

Note that every $\zeta \in A^0_{0<2,q}(X;L)$ is contained in $A^0_{0<2,q}(K_\zeta;L)$ for some sufficiently large but finite $c > 0$. Consequently, the conclusion of Corollary 3.3.3 when $q > m - s_F^+$ or Corollary 3.3.5 when $q < s_F^-$, as well as that of Lemma 3.3.6 holds for all $\zeta = \zeta' + \zeta'' \in A^0_{0<2,q}(X;L)$, where $\zeta' \in A^0_{0,(q-1)}(X;L)$ and $\zeta'' \in A^0_{0,(q)}(X;L)$.

Since $h_\delta = 0$, $\mathcal{W}(\zeta, \zeta)$ (see (eq 3.14)) and $\mathcal{W}_F(\zeta'', \zeta'')$ (see (eq 3.19)) both vanish for all $\zeta = \zeta' + \zeta'' \in A^0_{0<2,q}(X;L)$.

Since $\chi$ is plurisubharmonic on $X$ and $\partial |_{\chi} \chi = 0 = \partial |_{\zeta} \chi$, one can choose at every point $z \in X$ the coordinates such that both $g$ and $\sqrt{-1} \partial |_{\zeta} \chi$ are simultaneously diagonalized while keeping the decomposition $\{eq 2.4\}$ orthogonal, and see that

$$\text{Tr}_{g,\eta} \text{pr}_F \left( (\partial \overline{\partial} \chi)^\prime (\zeta \otimes \overline{\zeta}) \right) = \text{Tr}_{g,\eta} \text{pr}_F \left( (\partial |_{\chi} \overline{\partial} |_{\chi})^\prime (\zeta' \otimes \overline{\zeta'}) \right) \geq 0.$$ 

Therefore, $\mathfrak{m} (\zeta, \zeta) \geq 0$ (see (eq 3.14)).

As a result, combining Lemma 3.3.6 as well as the above facts about $\mathcal{W}$, $\mathcal{W}_F$ and $\mathfrak{m}$ with Corollary 3.3.3 or Corollary 3.3.5 one obtains

$$\|S_\chi \zeta\|^2_3 + \|T_{q-1}^\prime \zeta\|^2_1 \geq \frac{\pi}{4} \|\zeta\|^2_2$$

for all $\zeta \in A^0_{0<2,q}(X;L)$. This is the required $L^2$ estimate. Proposition 3.1.5 and Remark 3.1.6 then assert that there exists $\xi \in \mathcal{H}^{0,q-1}(X;L)$ such that $\overline{\partial} \xi = \psi$ on $X$. \hfill $\square$
CHAPTER 5
The non-linearizable case

For a non-linearizable line bundle \( L \) (see (eq 3.14)) and \( \mathcal{W}_F \) (see (eq 3.19)) are not identically zero. In order to get the estimates for these terms, Takayama’s Weak \( \partial \bar{\partial} \)-Lemma (ref. [Taka2, Lemma 3.14]) is invoked. One is then forced to restrict attention to each of the \( K_c \)'s and obtain the required \( L^2 \) estimates there. What then remains is to show that the existence of a solution of the \( \bar{\partial} \)-equation \( \bar{\partial} \xi = \psi \) on every \( K_c \) implies the existence of a global solution. The argument for this latter part is essentially the same as the one in [GR] Ch. IV, §1, Thm. 7.

An apt coordinate system is fixed throughout this section.

5.1. Bounds on the wild curvature terms

Takayama proves in [Taka2] the following Weak \( \partial \bar{\partial} \)-Lemma.

**Weak \( \partial \bar{\partial} \)-Lemma 5.1.1** (cf. [Taka2, Lemma 3.14]). Let \( \omega \) be a positive real \((1,1)\)-form on \( X \), and let \( \theta \) be a smooth real \(1\)-form on \( X \) such that \( \theta = \beta + \bar{\beta} \) for some smooth \((0,1)\)-form \( \beta \), and \( d\theta \) is of type \((1,1)\). Then for every positive number \( \varepsilon \) and every relatively compact open subset \( W \) of \( X \), there exists a smooth function \( \delta \) on \( X \) such that

\[
-\varepsilon \omega < d\theta - 2\sqrt{-1}\partial \bar{\partial} \text{Re} \delta < \varepsilon \omega \quad \text{on} \ W.
\]

Moreover, if \( \beta \in \mathcal{H}^{0,1}(X) \), then \( \delta \) can be chosen such that \( \delta \in \mathcal{H}(X) \).

In the current situation, the role of \( \beta \) in Lemma 5.1.1 is taken by \( \sqrt{-1}\partial h \) (therefore \( d\theta = 2\sqrt{-1}\partial \bar{\partial} \text{Re} h \)), and that of \( W \) by \( K_c \).

**Remark 5.1.2.** In Takayama’s formulation, the assertion of the Weak \( \partial \bar{\partial} \)-Lemma is that there exists a smooth real valued function \( f_{\varepsilon W} := 2(\text{Im} f_0 + \text{Im} \Psi_{M_0}) \) on \( X \) such that \( -\varepsilon \omega < d\theta - \sqrt{-1}\partial \bar{\partial} f_{\varepsilon W} < \varepsilon \omega \) on \( W \), in which \( f_0 \) is a smooth function on \( X \) such that \( \beta = \phi + \bar{\phi} f_0 \) for some real analytic \((0,1)\)-form \( \phi \) in \( \mathcal{H}^{0,1}(X) \), and \( \Psi_{M_0} \) is some real analytic function in \( \mathcal{H}(X) \). Therefore, the smooth function \( \delta \) here is given by \( \delta := -\sqrt{-1}(f_0 + \Psi_{M_0}) \) in Takayama’s notation. If \( \beta \in \mathcal{H}^{0,1}(X) \), then one has \( f_0 \in \mathcal{H}(X) \) as \( \partial_{[\omega]} f_0 = 0 \), so \( \delta \in \mathcal{H}(X) \) also.

**Remark 5.1.3.** As a side remark, following the construction of \( \delta \) in [Taka2, Lemma 3.14], \( \partial h_\delta = \partial h - \partial \delta \) is real analytic on \( X \), so \( h_\delta \) is real analytic on \( \mathbb{C}^n \). It follows that the hermitian metric \( \eta \) on \( L \) is real analytic.

Suitable estimates for the wild curvature terms \( \mathcal{W} \) and \( \mathcal{W}_F \) are obtained by choosing a proper \( \delta \in \mathcal{H}(X) \) according to the Weak \( \partial \bar{\partial} \)-Lemma.

**Lemma 5.1.4.** Suppose a hermitian metric \( g \) on \( X \) and a choice of \( \mathcal{H}_E \) are fixed. Then, on every \( K_c \) where \( 0 < c < \infty \), given any real number \( \varepsilon_w > 0 \) and for any
\( q \geq 0 \), one can choose \( \delta_c \in \mathcal{H}(X) \) which yields a hermitian metric \( \eta_c \) on \( L \) such that, for any given weight \( \chi \),

\[
|\mathcal{W}(\zeta, \zeta)| \leq \varepsilon_w \|\zeta\|^2_{K_c,\eta_c,\chi}
\]

(eq 5.1)

\[
|\mathcal{W}^\prime_\ast(\zeta'', \zeta'')| \leq \varepsilon_w m \|\zeta''\|^2_{K_c,\eta_c,\chi} \leq \varepsilon_w m \|\zeta\|^2_{K_c,\eta_c,\chi}
\]

(eq 5.2)

for all \( \zeta = \zeta' + \zeta'' \in \mathcal{A}^0,q (\overline{K}; L) \) where \( \zeta' \in \mathcal{A}^0,1,q-1 (\overline{K}; L) \) and \( \zeta'' \in \mathcal{A}^0,0,q (\overline{K}; L) \).

PROOF. First the estimate for \( \mathcal{W}^\prime_\ast \) is considered. Recall that \( \omega \) is the (1, 1)-form associated to \( g \). The weak \( \partial\bar{\partial} \)-Lemma asserts that, for any \( \varepsilon_w > 0 \), there exists \( \delta_c \in \mathcal{H}(X) \) such that

\[
-2\varepsilon_w \omega < 2\sqrt{-1}\partial\bar{\partial} \text{Re} h_{\delta_c} < 2\varepsilon_w \omega \quad \text{on} \quad K_c .
\]

Such \( \delta_c \) yields a hermitian metric \( \eta_c \) on \( L \) given the fixed choice of \( \mathcal{K}_E \). Then, it follows from (eq 3.19) that, for any weight \( \chi \),

\[
-\varepsilon_w \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta_c} \text{pr}_F (g^\prime (\zeta \otimes \overline{\zeta})) \leq \mathcal{W}^\prime_\ast (\zeta, \zeta) \leq \varepsilon_w \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta_c} \text{pr}_F (g^\prime (\zeta \otimes \overline{\zeta}))
\]

for any \( \zeta = \zeta' + \zeta'' \in \mathcal{A}^0,q (\overline{K}; L) \) (\( \varepsilon_w \) instead of \( 2\varepsilon_w \) in the bounds because of the factor \( \frac{1}{2} \) in \( \omega = - \text{Im} g = \frac{-1}{2} \sum_{k,\ell} g_{k,\ell} dz^k \wedge d\overline{z}^\ell \)). Note that

\[
\int_{K_c} e^{-\chi} \text{Tr}_{g,\eta_c} \text{pr}_F (g^\prime (\zeta \otimes \overline{\zeta})) = |\zeta'|^2_{K_c,\eta_c,\chi} + q \|\zeta''\|^2_{K_c,\eta_c,\chi} \leq q \|\zeta\|^2_{K_c,\eta_c,\chi}
\]

when \( q \geq 1 \). When \( q = 0 \), the integral on the left hand side is zero, so the above inequality is still valid. As a result, one obtains

\[
-\varepsilon_w \omega \|\zeta\|^2_{K_c,\eta_c,\chi} \leq \mathcal{W}^\prime_\ast (\zeta, \zeta) \leq \varepsilon_w \omega \|\zeta\|^2_{K_c,\eta_c,\chi}
\]

and hence (eq 5.1).

For the estimate for \( \mathcal{W}^\prime_\ast \), note that (eq 5.3) implies

\[
-2\varepsilon_w \omega \text{pr}_F \omega < 2\sqrt{-1}\partial\bar{\partial} \text{Re h}_{\delta_c} < 2\varepsilon_w \omega \text{pr}_F \omega \quad \text{on} \quad K_c .
\]

Then, one has \( -\varepsilon_w m < 2 \text{Tr}_g \partial\bar{\partial} \omega \text{Re h}_{\delta_c} < \varepsilon_w m \) with the same \( \varepsilon_w \) and \( \delta_c \) as above. Therefore, it follows from (eq 3.19) that

\[
-\varepsilon_w m \|\zeta''\|^2_{K_c,\eta_c,\chi} \leq \mathcal{W}^\prime_\ast (\zeta'', \zeta'') \leq \varepsilon_w m \|\zeta''\|^2_{K_c,\eta_c,\chi}
\]

for any \( \zeta'' \in \mathcal{A}^0,0,q (\overline{K}; L) \), and hence (eq 5.2). \( \square \)

### 5.2. Existence of weak solutions on \( K_c \)

With the bounds given in (eq 5.1) for the wild curvature terms, it is easy to follow the proof of Theorem 4.1.1 and get the following

**Proposition 5.2.1.** Suppose \( L \) is a holomorphic line bundle on \( X \) (which can possibly be non-linearizable), and suppose \( q < s_F^- \) or \( q > m-s_F^+ \). Then, there exists a suitable hermitian metric \( g \) on \( X \) such that the following holds: for any \( 0 < c < \infty \), a hermitian metric \( \eta_c \) on \( L \) can be chosen such that, given any plurisubharmonic weight \( \chi \), the \( L^2 \) estimate

\[
\|S_q \zeta\|^2_{K_c,\eta_c,\chi} + \|T^\ast_{q-1} \zeta\|^2_{K_c,\eta_c,\chi} \geq \frac{\pi}{4} \|\zeta\|^2_{K_c,\eta_c,\chi}
\]

for all \( \zeta \in \mathcal{A}^0,q (\overline{K}; L) \cap \text{Dom}_{K_c,\eta_c,\chi} T^\ast_{q-1} \) is satisfied.
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**Proof.** Choose the translational invariant hermitian metric $g$ as described in the proof of Lemma 3.3.2 for $q > m - s_F^+$ or Lemma 3.3.4 for $q < s_F^-$, with $M = 2$. For the hermitian form $\mathcal{H}$ associated to $L$, choose $\mathcal{H}_E$ as described in the proof of Lemma 3.3.6. These choices are independent of $c$.

Consider $K_c$ for some fixed $0 < c < \infty$. Take any $\varepsilon_w > 0$ such that

\[
\varepsilon_w(q + m) \leq \frac{\pi}{4}
\]

and choose $\delta_c \in \mathcal{H}(X)$ according to Lemma 5.1.4 such that, for any given weight $\chi$, the inequalities (eq 5.1) and (eq 5.2) hold under the induced $L^2$-norm $\|\cdot\|_{K_c,\eta_c,X}$.

By the choices of the metrics, the conclusion of Corollary 3.3.3 when $q > m - s_F^+$ or Corollary 3.3.4 when $q < s_F^-$, as well as that of Lemma 3.3.6 holds for all $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{<2>(\overline{K}_c; L) \cap \text{Dom}_{K_c,\eta_c,X} T_{q-1}}$, where $\zeta' \in \mathcal{A}_{0,1,q-1}(\overline{K}_c; L) \cap \text{Dom}_{K_c,\eta_c,X}^{(1,q-1)} \text{Dom}_{K_c,\eta_c,X}^{(1,q-1)}$ and $\zeta'' \in \mathcal{A}_{0,0,q}(\overline{K}_c; L)$.

Since $\chi$ is plurisubharmonic, $\text{wt}(\zeta, \zeta) \geq 0$ for all $\zeta \in \mathcal{A}_{0,q}(\overline{K}_c; L)$ as in the proof of Theorem 4.1.1.

As a result, from Corollary 3.3.3 or 3.3.4 as well as Lemma 3.3.6, one obtains

\[
\|S_q \zeta\|^2_{K_c,\eta_c,X} + \|T_{q-1}^* \zeta\|^2_{K_c,\eta_c,X} \leq \begin{cases} \frac{\alpha}{2} \|\zeta\|^2_{K_c,\eta_c,X} + 2\mathcal{W}(\zeta, \zeta) & \text{for } q > m - s_F^+ \\ \frac{\alpha}{2} \|\zeta\|^2_{K_c,\eta_c,X} + 2\mathcal{W}(\zeta', \zeta') + 2\mathcal{W}(\zeta', \zeta') & \text{for } q < s_F^+ \\ \frac{\alpha}{2} \|\zeta\|^2_{K_c,\eta_c,X} - \varepsilon_w(m + q) \|\zeta\|_{K_c,\eta_c,X} & \text{by (eq 5.1)} \text{ and (eq 5.2),} \\ \frac{\alpha}{2} \|\zeta\|^2_{K_c,\eta_c,X} & \text{by (i).} 
\end{cases}
\]

This gives the required $L^2$ estimate. \hfill \square

Since, for any $\psi \in \mathcal{H}^{0,q}(X; L)$, one has $\psi|_{K_c} \in L^2_{0,0,q}(K_c; L)$ (unweighted) for any $0 < c < \infty$, it follows the following corollary of Propositions 3.1.5 and 5.2.1.

**Corollary 5.2.2.** Consider the exhaustive sequence $\{K_c\}_{c \in \mathbb{N}_{>0}}$ of relatively compact open subsets of $X$. Suppose $q < s_F^-$ or $q > m - s_F^+$. Then one can choose a suitable hermitian metric $g$ on $X$ and a sequence of hermitian metrics $\{\eta_c\}_{c \in \mathbb{N}_{>0}}$ on $L$ as in Proposition 5.2.1 such that, for any $\psi \in \mathcal{H}^{0,q}(X; L) \cap \ker \nabla$, there exists a sequence of solutions $\{\xi'_c\}_{c \in \mathbb{N}_{>0}}$ such that $\xi'_c \in L^2_{0,0,q-1}(K_c; L)$ (unweighted) and $\nabla \xi'_c = \psi|_{K_c}$ in $L^2_{0,0,q}(K_c; L)$.

**Remark 5.2.3.** Since $\chi$ has to be smooth on a neighborhood of $\overline{K}_c$ (as required by [Hör1, Prop. 2.1.1] so that $\mathcal{A}_{<2>(\overline{K}_c; L) \cap \text{Dom} T_{q-1}}$ is dense in $\text{Dom} T_{q-1} \cap \text{Dom} S_q$ under the suitable graph norm), if $\psi \in \mathcal{H}^{0,q}(K_c; L)$, there may not exist such $\chi$ such that $\|\psi\|_{K_c,\chi} < \infty$. To avoid technical difficulty, the author does not attempt to solve the $\nabla$-equation for any $\psi \in \mathcal{H}^{0,q}(K_c; L)$ such that $\nabla^2 \psi = 0$ by means of $L^2$ estimates directly.

### 5.3. A Runge-type approximation

This section is devoted to proving a Runge-type approximation which is required to construct a global solution to the equation $\nabla^2 \xi = \psi$ from the solutions on $K_c$’s given in Corollary 5.2.2.
In what follows, $q$ is assumed to be $0 < q < s_F^-$ or $q > m - s_F^+$, and the hermitian metric $g$ as well as the family of hermitian metrics $\{\eta_c\}_{c>0}$ as asserted by Proposition 5.2.1 is fixed. Then, according to the choices of the $\eta_c$’s in the proof of Proposition 5.2.1 for any $c’, c > 0$, one has
\[
\eta_c = \eta_c e^{2\Re(\delta_{c’} - \delta_c)} =: \eta_c e^{\delta_{c’}}.
\]
Note that $e^{\delta_{c’}} > 0$ on $X$. It is understood that the hermitian metric $\eta_c$ on $L$ is chosen when the $L^2$-norm on $K_c$ is considered, so write $L^2_{\eta_c}(K_c; L)$ as $L^2_{\chi_c}(K_c; L)$, $\langle \cdot, \cdot \rangle_{K_c, \eta_c}$ as $\langle \cdot, \cdot \rangle_{K_c, \chi}$ and so on to simplify notation. When the weight $\chi$ is absent from the notation, e.g. $L^2_{\eta_c}(K_c; L)$ or $\langle \cdot, \cdot \rangle_{K_c}$, it is understood that the corresponding object is unweighted, i.e. $\chi = 0$.

For any finite $c’ > c > 0$ and for any $\Psi \in L^2_{\eta_c}(K_c; L)$, if $\Psi$ is extended by zero to a section in $L^2_{\eta_{c’}}(K_{c’}; L)$, then it follows that
\[
\langle \zeta, \Psi \rangle_{K_c} = \langle \zeta, \Psi e^{\delta_{c’}} \rangle_{K_{c’}},
\]
for any $\zeta \in L^2_{\eta_{c’}}(K_{c’}; L)$.

Define $(\ker_{K_c} T_{q - 1})_{K_c}$ to be the image of $\ker_{K_c} T_{q - 1}$ under the restriction map $L^2_{\eta_c}(K_c; L) \to L^2_{\eta_{c’}}(K_{c’}; L)$. Note that $T_{q - 1}$ commutes with the restriction map (as $c > 0$), so one has
\[
(\ker_{K_c} T_{q - 1})_{K_c} \subset \ker_{K_{c’}} T_{q - 1}.
\]

The following proof of the required Runge-type approximation is an analogue of the one for strongly pseudoconvex manifolds given in [Hör3 Lemma 4.3.1].

**Proposition 5.3.1.** Suppose $0 < q < s_F^-$ or $q > m - s_F^+$, and $g$ and $\eta_c$’s are chosen according to Proposition 5.2.1. Then, for any finite $c’ > c > 0$, the closure of $(\ker_{K_c} T_{q - 1})_{K_c}$ in $L^2_{\eta_c}(K_c; L)$ is $\ker_{K_c} T_{q - 1}$. In other words, $(\ker_{K_c'} T_{q - 1})_{K_c}$ is dense in $\ker_{K_c} T_{q - 1}$.

**Proof.** By virtue of the Hahn-Banach theorem, it suffices to show that for every $\Psi \in L^2_{\eta_c}(K_c; L)$, if the induced bounded linear functional
\[
L^2_{\eta_{c’}}(K_{c’}; L) \ni \zeta \mapsto \langle \zeta, \Psi \rangle_{K_c}
\]
vanishes on $(\ker_{K_c} T_{q - 1})_{K_c}$, then it also vanishes on $\ker_{K_c} T_{q - 1}$.

Suppose that $\Psi \in L^2_{\eta_{c’}}(K_{c’}; L)$ satisfies the above assumption. Extend $\Psi$ by zero to $K_c$ as a section in $L^2_{\eta_{c}}(K_c; L)$. Now it suffices to show that there exists $\Xi \in L^2_{\eta_{c’}}(K_{c’}; L)$ such that $\Xi \equiv 0$ on $K_c \setminus K_c$ and
\[
\langle \zeta, \Psi e^{\delta_{c’}} \rangle_{K_{c’}} = \langle T_{q - 1} \zeta, \Xi \rangle_{K_c}
\]
for any $\zeta \in \text{Dom}_{K_c} T_{q - 1}$, which then implies that
\[
\langle \zeta, \Psi \rangle_{K_c} = \langle T_{q - 1} \zeta, \Xi e^{-\delta_{c’}} \rangle_{K_c}
\]
for any $\zeta \in \text{Dom}_{K_c} T_{q - 1}$ due to (eq 5.4). The equality (1) holds true for $\zeta \in \mathcal{A}^0_{\eta_{c}}(K_c; L)$ in particular, and $\mathcal{A}^0_{\eta_{c}}(K_c; L)$ is dense in $\text{Dom}_{K_c} T_{q - 1}$ under the graph norm $\sqrt{\|\zeta\|_{K_c}^2 + \|T_{q - 1} \zeta\|_{K_c}^2}$ by [Hör1 Prop. 2.1.1], so (1) also holds.

\[\text{If there exists } \zeta \in \ker_{K_c} T_{q - 1} \text{ which does not lie in the closure of } (\ker_{K_c} T_{q - 1})_{K_c} \text{ in } L^2_{\eta_{c’}}(K_{c’}; L), \text{ then the Hahn-Banach theorem asserts that there is a bounded linear functional } \Lambda \text{ such that } (\ker_{K_c} T_{q - 1})_{K_c} \subset \ker \Lambda \text{ and } \Lambda \zeta = 1.\]

true for \( \zeta \in \text{Dom}_{K_c} T_{q-1} \). It follows that
\[
\langle \zeta, \Psi \rangle_{K_c} = \langle T_{q-1} \zeta, \Xi e^{-\delta c_\nu} \rangle_{K_c} = 0
\]
for all \( \zeta \in \ker_{K_c} T_{q-1} \subset \text{Dom}_{K_c} T_{q-1} \) as required. It remains to show the existence of such \( \Xi \).

Take a sequence of smooth convex increasing functions \( \tilde{\chi}_\nu : \mathbb{R} \to \mathbb{R} \) such that \( \tilde{\chi}_\nu(x) = 0 \) for all \( x \leq c \), and \( \tilde{\chi}_\nu(x) \nearrow +\infty \) as \( \nu \to \infty \) for every \( x > c \). Note that \( \tilde{\chi}_\nu \geq 0 \) for any \( \nu \geq 0 \) by such choice. Set \( \chi_\nu := \tilde{\chi}_\nu \circ \varphi \) as before. A sequence of weighted norms \( \|\cdot\|_{c,\nu} := \|\cdot\|_{K_c,\chi_\nu} \) on \( K_c \) is then defined. Let the corresponding inner products, Hilbert spaces and \( \text{Dom} \) also be distinguished by using the subscripts \( c,\nu \), and the corresponding adjoint of \( T_{q-1} \) by \( T_{q-1}^{\ast} \).

For any \( q \) in the given range, the \( L^2 \) estimate in Proposition 5.2.1 holds under each of the above weighted norms with \( T_{q-1}^{\ast} \) replaced by \( T_{q-1}^{\ast} \). Since \( \langle \zeta, \Psi e^{\delta c_\nu \varphi} \rangle_{c,\nu} = \langle \zeta, \Psi e^{\delta c_\nu \varphi} \rangle_{c,\nu} \equiv \langle \zeta, \Psi e^{\delta c_\nu \varphi} \rangle_{c,\nu} \)
for all \( \zeta \in \ker_{K_c} T_{q-1} = \ker_{c,\nu} T_{q-1} \) by the assumption on \( \Psi \), it follows that
\[
\Psi e^{\delta c_\nu \varphi} \in \left( \ker_{c,\nu} T_{q-1} \right) ^\perp = \text{im}_{c,\nu} T_{q-1}^{\ast}.
\]

Given the \( L^2 \) estimate, Theorem 3.1.1 (2) then asserts that there exists \( \tilde{\Xi}_\nu \in \text{Dom}_{c,\nu} T_{q-1}^{\ast} \) such that \( T_{q-1}^{\ast} \tilde{\Xi}_\nu = \Psi e^{\delta c_\nu \varphi} \). Therefore, one has
\[
\langle \zeta, \Psi e^{\delta c_\nu \varphi} \rangle_{c,\nu} = \langle \zeta, T_{q-1} \tilde{\Xi}_\nu \rangle_{c,\nu} = \langle T_{q-1} \zeta, \Xi e^{-\delta c_\nu} \rangle_{K_c}
\]
for all \( \nu \in \mathbb{N} \) and for all \( \zeta \in \text{Dom}_{c,\nu} T_{q-1} = \text{Dom}_{K_c} T_{q-1} \). By defining \( \Xi_\nu := \tilde{\Xi}_\nu e^{-\delta c_\nu} \),
one obtains
\[
(*) \quad \langle \zeta, \Psi e^{\delta c_\nu \varphi} \rangle_{K_c} = \langle T_{q-1} \zeta, \Xi_\nu \rangle_{K_c}.
\]

Moreover, notice that the constant in the \( L^2 \) estimate is independent of \( \nu \) (which is chosen to be \( \frac{\pi}{4} \) in Proposition 5.2.1). The estimate on the solution \( \Xi_\nu \) from Theorem 3.1.1 (2) then implies that
\[
(**) \quad \frac{\pi}{4} \int_{K_c} |\Xi_\nu|^2_{g,\eta_{c,\nu}} \varphi e^{\chi_\nu} \leq \int_{K_c} |\Psi e^{\delta c_\nu \varphi}|^2_{g,\eta_{c,\nu}} \varphi e^{\chi_\nu} = \int_{K_c} |\Psi^2|_{g,\eta_{c,\nu}} e^{\delta c_\nu \varphi},
\]
where the last equality is due to the fact that \( \Psi \) vanishes on \( K_c \setminus \overline{K_c} \). Since \( \tilde{\chi}_\nu(\varphi) \) is independent of \( \nu \) when \( \varphi \leq c \), the integral on the right hand side is independent of \( \nu \), so the left hand side is a bounded sequence in \( \nu \). This in turn implies that there exists a subsequence of \( \{ \Xi_\nu \}_{\nu \in \mathbb{N}} \) which converges to some \( \Xi \in L^2_{0,\varphi} (K_c; \mathcal{L}) \) (unweighted) in the weak topology. From (**) since \( \tilde{\chi}_\nu(\varphi) \nearrow +\infty \) for \( \varphi > c \), it follows that \( \Xi \equiv 0 \) when \( \varphi > c \), i.e. on \( K_c \setminus \overline{K_c} \). Moreover, from (**) it follows that (i) holds for all \( \zeta \in \text{Dom}_{K_c} T_{q-1} \). This is what is desired. \( \square \)

5.4. Proof of Theorem 1.1.1 for general \( L \)

First notice that, if \( q = 0 < s \), then the \( L^2 \) estimate in Proposition 5.2.1 holds when the metrics are chosen suitably, and thus for any \( \psi \in \mathcal{H} (X; L) \cap \ker \partial \) one has
\[
0 = \|\partial \psi\|^2_{K_c} \geq \frac{\pi}{4} \|\psi\|^2_{K_c}
\]
(note that $T_{-1}^*\xi = 0$ for all $\xi \in \mathcal{A}^i(K_c; L)$). This means that $\psi|_{K_c} = 0$ for any $c > 0$, and thus $\psi = 0$ on $X$. Therefore, one has the following

**Theorem 5.4.1.** If $s_F^- > 0$, one has $H^0(X, L) = 0$.

Assume $0 < q < s_F^-$ or $q > m - s_F^+$ in what follows. The metrics $g$ and $\eta_r$'s from Corollary 5.2.2 are fixed for this section. Again, write $L^2_{0,0,q}(K_{\nu}; L)$ as $L^2_{0,0,q}(K_{\nu}; L)$ and so on, and notations like $L^2_{0,0,q}(K_{\nu}; L)$ or $\|\cdot\|_{K_{\nu}}$ are understood as unweighted objects, i.e. $\chi = 0$.

For every integer $\nu \geq 1$, as $\delta_{\nu+1} - \delta_{\nu}$ is smooth on $X$ and $K_{\nu+1}$ is compact, there exists a constant $M_{\nu+1}' \geq 1$ such that

(eq 5.5) $\|\zeta\|_{K_{\nu}} \leq M_{\nu+1}' \|\zeta\|_{K_{\nu+1}}$

for all $\zeta \in L^2_{0,0,q}(K_{\nu+1}; L)$. Define also $M_1 := 1$ and $M_\nu := \prod_{k=2}^\nu M_k'$ for $\nu \geq 2$.

Proposition 5.3.1 is used to complete the proof of Theorem 1.1.1. The following argument is adopted from [GR, Ch. IV, §1, Thm. 7].

**Theorem 5.4.2.** Suppose $0 < q < s_F^-$ or $q > m - s_F^+$. Then one has $H^q(X, L) = 0$ for any $q$ in the given range.

**Proof.** Given any $\psi \in \mathcal{H}^{0,q}(X; L) \cap \ker \partial$, Corollary 5.2.2 provides a sequence of local solutions $\{\xi_{\nu}\}_{\nu \geq 1}$ such that $\xi_{\nu} \in L^2_{0,0,q-1}(K_{\nu}; L)$ and $\partial \xi_{\nu} = \psi|_{K_{\nu}}$ for all integers $\nu \geq 1$. First a sequence of local solutions $\{\xi_{\nu}\}_{\nu \geq 1}$ such that $\xi_{\nu} \in L^2_{0,0,q-1}(K_{\nu}; L)$, $\partial \xi_{\nu} = \psi|_{K_{\nu}}$ and

\[(*) \quad \|\xi_{\nu+1} - \xi_{\nu}\|_{K_{\nu}} < \frac{1}{M_\nu 2^{\nu}}\]

for all $\nu \geq 1$ is defined inductively as follows. Set $\xi_1 := \xi'_{1}$. Suppose $\xi_1, \ldots, \xi_{\nu}$ are defined for some $\nu \geq 1$. Let $\gamma_{\nu}' := \xi_{\nu+1}|_{K_{\nu}} - \xi_{\nu}$. Notice that $\gamma_{\nu}' \in \ker K_{\nu}, \partial g_{-1} \subset L^2_{0,0,q-1}(K_{\nu}; L)$. Proposition 5.3.1 then implies that there exists $\gamma_{\nu} \in \ker K_{\nu+1}, \partial g_{-1} \subset L^2_{0,0,q-1}(K_{\nu+1}; L)$ such that

\[\|\gamma_{\nu}' - \gamma_{\nu}\|_{K_{\nu}} < \frac{1}{M_\nu 2^{\nu}}.\]

Set $\xi_{\nu+1} := \xi_{\nu+1}' - \gamma_{\nu}$. Then one has $\partial \xi_{\nu+1} = \partial \xi_{\nu+1}' = \psi|_{K_{\nu+1}}$ and the inequality $(\ast)$ is satisfied. The required sequence $\{\xi_{\nu}\}_{\nu \geq 1}$ is therefore defined.

Notice that, for every $\nu \geq 1$, the sequence $\{\xi_{\mu}|_{K_{\nu}}\}_{\mu \geq \nu}$ converges in $L^2_{0,0,q-1}(K_{\nu}; L)$. Indeed, for any $\mu \geq \nu \geq 1$ and for any integer $k > 0$,

\[\|\xi_{\mu+k} - \xi_{\mu}\|_{K_{\nu}} \leq \sum_{r=0}^{k-1} \|\xi_{\mu+r+1} - \xi_{\mu+r}\|_{K_{\nu}} \leq \sum_{r=0}^{k-1} \frac{M_{\mu+r}}{M_{\mu}} \|\xi_{\mu+r+1} - \xi_{\mu+r}\|_{K_{\mu+r}} \quad \text{by (eq 5.5)},\]

\[\leq \frac{1}{M_{\mu}} \sum_{r=0}^{k-1} \frac{1}{2r+1} \quad \text{by (\ast)},\]

\[\leq \frac{1}{M_{\nu} 2^{\nu-1}},\]
which tends to 0 as $\mu \to \infty$, so $\{\xi_\mu|_{K_\nu}\}_{\mu \geq \nu}$ is a Cauchy sequence in $L^2_{0,(0,q-1)}(K_\nu; L)$. Let $\xi^{(\nu)}$ be the limit of $\{\xi_\mu|_{K_\nu}\}_{\mu \geq \nu}$ in $L^2_{0,(0,q-1)}(K_\nu; L)$. Since $\overline{\partial}\xi_\mu|_{K_\nu} = \psi|_{K_\nu}$ for all $\mu \geq \nu$, and $\overline{\partial}$ is a closed operator, one has $\overline{\partial}\xi^{(\nu)} = \psi|_{K_\nu}$ for all $\nu \geq 1$. Now notice that restriction from $K_{\nu+1}$ to $K_\nu$ is continuous by (eq 5.5), so

$\xi^{(\nu+1)}|_{K_\nu} - \xi^{(\nu)} = \lim_{\mu \to \infty} (\xi_\mu|_{K_\nu} - \xi_\mu|_{K_\nu}) = 0$ in $L^2_{0,(0,q-1)}(K_\nu; L)$. On every $K_\nu$, different choices of $\delta_\nu \in \mathcal{H}(X)$ yield equivalent norms. Therefore, by fixing one $\delta \in \mathcal{H}(X)$, one can consider $L^2_{0,q-1}(X; L; \text{loc})$, the space of locally $L^2$ $L$-valued $(0,q-1)$-forms on $X$, and there exists $\xi' \in L^2_{0,q-1}(X; L; \text{loc})$ such that

$\xi'|_{K_\nu} = \xi^{(\nu)}$ for all $\nu \geq 1$, and

$\overline{\partial}\xi' = \psi$ in $L^2_{0,q-1}(X; L; \text{loc})$.

Remark 3.1.6 then assures that there exists $\xi \in \mathcal{H}_{0,q-1}(X; L)$ such that $\overline{\partial}\xi = \psi$ on $X$.

Since $\psi \in \mathcal{H}_{0,q}(X; L) \cap \ker \overline{\partial}$ is arbitrary, this shows that $H^q(X, L) = 0$. This completes the proof. \hfill \Box
List of Symbols

\[ \mathcal{A} \] .......................... p. 5
\[ \mathcal{A}_{p,q} \] ........................... p. 16
\[ \mathcal{A}(p',q',q''),(q',q'') \] ....... p. 16
\[ \mathcal{A}(p',q',0) \] ........................ p. 16
\[ \mathcal{A}(p',q'',0) \] ........................ p. 9
\[ \mathcal{A}_0(q',q'') \] ........................ p. 10
\[ \mathcal{A}_0(q',q'')(K_c;L) \] .......... p. 10
\[ \mathcal{A}_0,q \] ............................. p. 12
\[ \mathcal{A}_{c,\chi}^{0,q+1}(K_c;L) \] ....... p. 12
\[ \delta \] ....................................... p. 8
\[ \partial_{\mathcal{F}} \] ............................. p. 16
\[ \partial_{\mathcal{F}_{\mathcal{V}}} \] ........................ p. 16
\[ \overline{\partial}_{[\mathcal{V}]} \] ........................ p. 10
\[ \overline{\partial}_{[\mathcal{V}]} \] ........................ p. 10
\[ \overline{\partial}_{[\mathcal{V}]} \] ........................ p. 10
\[ d_{\mathcal{V}} \] ................................ p. 21
\[ \eta \] ...................................... p. 7
\[ \eta_{\mathcal{T}} \] ................................ p. 8
\[ \eta \] ...................................... p. 9
\[ E \oplus F \] .............................. p. 4
\[ F \] ....................................... p. 2
\[ f_{\gamma} \] ................................ p. 6
\[ g \] ...................................... p. 9
\[ h \] ...................................... p. 8
\[ h_\delta \] ................................ p. 8
\[ \mathcal{H} \] ................................ p. 6
\[ \mathcal{H}_F \] ................................ p. 21
\[ \mathcal{H}_E \oplus \mathcal{H}_\infty \oplus \mathcal{H}_\infty \] ........................ p. 22
\[ \mathcal{H}_F \] ................................ p. 22
\[ \mathcal{H}_E \] ................................ p. 22
\[ \mathcal{H}_{\mathcal{F},0} \] ........................ p. 5
\[ \mathcal{H}^{0,q} \] ............................. p. 5
\[ \mathcal{H}^{0,q}(U;V) \] ....................... p. 6
\[ K \] ...................................... p. 2
\[ K_c \] ..................................... p. 5

\[ K_\infty \] ................................. p. 5
\[ L \] ....................................... p. 6
\[ L_t \] ..................................... p. 7
\[ L_{c,\chi}^{0,q}(K_c;L) \] ................... p. 9
\[ L_{c,\chi}^{0,q}(K_c;L) \] ................... p. 10
\[ L_{0,q;\chi}(K_c;L) \] .................... p. 12
\[ \nabla \] .................................... p. 16
\[ \nabla_\mathcal{F} \] ............................. p. 16
\[ \nabla_\mathcal{V} \] ............................. p. 16
\[ \nabla_{\mathcal{F};\mathcal{V}} \] ........................ p. 16
\[ \nabla = \nabla_{(1,0)} + \nabla_{(0,1)} \] ........ p. 17
\[ \nabla_{(1,0)} = \nabla_{\mathcal{V}} \] ............. p. 18
\[ \nabla_{(0,1)} = \nabla_{\mathcal{F}} \] ............. p. 19
\[ \varphi \] ................................ p. 5
\[ \mathcal{P}_F \] ................................ p. 18
\[ \mathcal{R} \] ................................ p. 16
\[ \mathcal{R}_m + \mathcal{R}_n + \mathcal{R}_m + \mathcal{R}_n \] .... p. 17
\[ s_F \] ................................ p. 2
\[ T_m \] ................................ p. 4
\[ \mathcal{T}_{\mathcal{V}}^{1,0} \] ........................ p. 5
\[ \mathfrak{X}(\zeta,\zeta) \] ........................ p. 21
\[ \mathfrak{W}(\zeta,\zeta) \] ........................ p. 21
\[ \mathfrak{W}_F(\zeta,\zeta) \] ........................ p. 21
\[ \mathfrak{T}_{\mathcal{F}}(\zeta,\zeta) \] ................. p. 21
\[ \mathfrak{T}_{\mathcal{F}}(\zeta,\zeta) \] ................. p. 21
\[ T_{\mathfrak{G}} \] ................................ p. 17
\[ T_{\mathfrak{G}} \] ................................ p. 17
\[ T_{\mathfrak{G}} \] ................................ p. 17
\[ \Theta \] ................................ p. 16
\[ \varphi \] ................................ p. 16
\[ \varphi \] ................................ p. 16
|\(g,\eta)\|_1 \] ........................ p. 9
\[ |\(g,\eta)\| \] ........................ p. 12
\[ \|\cdot\|_{K_c,\chi} \] ........................ p. 9
\[ \|\cdot\|_{(\cdot,\cdot)}^{1,\infty} \] ............. p. 12
\[ \|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_3 \] .......... p. 12
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