Chiral and nonchiral edge states in quantum Hall systems with charge density modulation
Paweł Szumniak, Jelena Klinovaja, and Daniel Loss
Phys. Rev. B 93, 245308 — Published 27 June 2016
DOI: 10.1103/PhysRevB.93.245308
Chiral and Non-Chiral Edge States in Quantum Hall Systems with Charge Density Modulation

Paweł Szumniak,1,2 Jelena Klinovaja,1 and Daniel Loss1

1Department of Physics, University of Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland
2AGH University of Science and Technology, Faculty of Physics and Applied Computer Science, al. Mickiewicza 30, 30-059 Kraków, Poland

(Dated: June 8, 2016)

We consider a system of weakly coupled wires with quantum Hall effect (QHE) and in the presence of a spatially periodic modulation of the chemical potential along the wire, equivalent to a charge density wave (CDW). We investigate the competition between the two effects which both open a gap. We show that by changing the ratio between the amplitudes of the CDW modulation and the tunneling between wires, one can switch between non-topological CDW-dominated phase to topological QHE-dominated phase. Both phases host edge states of chiral and non-chiral nature robust to on-site disorder. However, only in the topological phase, the edge states are immune to disorder in the phase shifts of the CDWs. We provide analytical solutions for filling factor $\nu = 1$ and study numerically effects of disorder as well as present numerical results for higher filling factors.

PACS numbers: 71.10.Fd; 73.43.-f; 71.10.Pm

I. INTRODUCTION

Over the last decades topological states of matter attracted a lot of attention both theoretically and experimentally. The striking stability of the quantum Hall effect (QHE)1,2 can be linked to topology3. The time-reversal invariant cousins of the QHE are the two-dimensional (2D) topological insulators (TIs) for which many candidate materials were found or synthesized in recent years4. However, despite great progress, the conductance quantization in TI materials is still not as perfect as in the QHE. Thus, the experimental focus has shifted in recent years to a more direct study of the edge state physics in TIs, for instance via Fraunhofer patterns5,6 and SQUID probes7,8. However, the edge states, especially in clean samples, could also be of non-topological origin, for example, due to Tamm-Shockley states9–11.

Recently, edge state behavior was observed in 2D GaSb heterostructures, but in a regime that is believed to be non-topological, and thus challenging the standard interpretation of this system as a TI12. The origin of this unexpected observation is still unclear but it raises the intriguing question whether edge states could not occur in both phases, in the topological as well as in the trivial one, but with different signatures such as e.g. being helical (chiral) in one phase vs. non-helical (non-chiral) in the other. In other words, the system could host topological edge states for one set of parameters while there exist non-topological edge states for another one. It is thus of fundamental interest to see if realistic models can be constructed which demonstrate that, in principle, these two scenarios do not need to exclude each other.

In the present work, we propose a system related to the QHE regime where exactly such a mixed behavior of edge state physics can emerge. The system we consider is given by a 2D array of tunnel coupled wires in the presence of a magnetic field and charge density waves (CDWs) inside the wires. This provides two different mechanisms (QHE and CDW) for inducing gaps and edge states which can compete with each other. Such CDWs may be induced intrinsically by electron-electron interactions13–15, extrinsically by periodically arranged gates inducing spatial modulations of the chemical potential, or by an internal superlattice structure16,17, see Fig. 1. We show that by tuning the ratio between the amplitude of the CDW modulation and the tunneling amplitude between the wires the system undergoes a phase transition between a non-topological (CDW dominated) phase, which supports predominantly non-chiral edge states, and a topological (QHE dominated) phase, which supports predominantly chiral edge states. However, in both phases, one can find both chiral and non-chiral regimes. These results are supported by both numerical and analytical calculations. We confirm numerically that, as expected, the topological chiral states are less susceptible to disorder.

II. THEORETICAL MODEL

A. Lattice tight-binding model

We consider a 2D coupled wire construction18–27 in the presence of a perpendicular uniform magnetic field, see Fig. 1. We assume that the propagation is anisotropic in the $xy$ plane, mainly for analytical convenience (In the Appendix B we show numerically that our results can be extended to isotropic systems). The tunneling amplitude along the wires, aligned in the $x$ direction, is thus larger than the one between the wires in the $y$ direction. This allows us to treat the wires as independent one-dimensional channels only weakly coupled to their neighboring wires. In addition, we include a CDW modulation along the wire. The system is then described by
the following tight-binding Hamiltonian,

$$H = \sum_{n,m} \left( -t_{c_{n+m}}^{\dagger} c_{n,m} - t_{y} e^{i(n+m)\phi} c_{n+m+1}^{\dagger} c_{n,m} - U_{0} \cos(2k_{y}n + \varphi) + \mu/2 \right) + H.c. \right) \right),$$

where $c_{n,m}$ is the annihilation operator acting on the electron at a site $(n,m)$ of the lattice with the lattice constant $a_{x}$ ($a_{y}$) in the $x$ ($y$) direction. For simplicity we consider spinless electrons in this work. We choose the hopping amplitude along the $x$ direction $t > 0$ to be much larger than the hopping along the $y$ direction $t_{y} > 0$. The uniform magnetic field applied in the $y$ direction, $B = B_{z}e_{y}$, and the corresponding vector potential $A = B_{x}e_{y}$ is chosen along the $x$ axis, yielding the orbital Peierls phase $\phi = eB_{z}a_{x}/\hbar c$. The chemical potential $\mu$ is modulated with the CDW amplitude $2\Delta_{g} > 0$ and the period $\lambda_{w} = \pi/k_{w}$. The angle $\varphi$ is the phase of the CDW at the left edge of the wire ($n = 0$).

With this choice of the vector potential $A$, the system is translation invariant in the $y$ direction, thus, we can introduce the momentum $k_{y}$ via Fourier transformation $c_{n,m} = \frac{1}{\sqrt{N_{y}}} \sum_{k_{y}} c_{n,k_{y}} e^{-i n k_{y}a_{y}}$, where $N_{y}$ is the number of lattice sites in the $y$ direction. The Hamiltonian becomes diagonal in $k_{y}$ space,

$$H = \sum_{n,k_{y}} \left( -t_{c_{n+1,k_{y}}}^{\dagger} c_{n,k_{y}} + H.c. \right) - t_{y} e^{i n k_{y}a_{y}} c_{n,k_{y}} + \mu/2 \right) + H.c. \right) \right)$$

As a result, the eigenfunctions of $H$ factorize as $e^{ik_{y}y} \psi_{k_{y}}(x)$, with $x = n a_{x}$, $y = n a_{y}$. From now on, we focus on $\psi_{k_{y}}(x)$ and treat $k_{y}$ as a parameter.

The uniform magnetic field $B_{z}$, defined via the chemical potential as $k_{F}a_{x} = \arccos(-\mu/2\ell)$, and the Fermi velocity $v_{F}$ is given by $\hbar v_{F} = 2ta_{x} \sin(k_{F}a_{y})$. The bulk energy spectrum is given by

$$E_{\pm}^{2} = (h_{v_{F}k})^{2} + U_{0}^{2} + t_{y}^{2} + 2U_{0} \cos(\varphi - k_{y}a_{y}) \right),$$

and depends on both $k$ and $k_{y}$ momenta. Here, $E_{+}$ ($E_{-}$) corresponds to the part of the spectrum above (below) $\mu$. The size of the bulk gap $2\Delta_{g} = \min(k_{F}a_{y} - \Delta_{g})$ for given $k_{y}$ becomes

$$\Delta_{g}(k_{y}) \equiv D = \sqrt{U_{0}^{2} + t_{y}^{2} + 2U_{0} \cos(\varphi - k_{y}a_{y})} \right).$$

We note that the system is gapless if $t_{y} = U_{0}$ and $\varphi = k_{y}a_{y} + \pi$ but fully gapped otherwise, see Fig. 2. This closing and reopening of the gap hints to a topological phase transition. For a strip of width $W$, one can thus expect the presence of edge states at the boundaries with energies lying inside the bulk gap. In order to explore the possibility of such edge states we consider a semi-infinite strip $(x \geq 0)$ and exploit the method developed in Ref. [32]. Furthermore we assume that $W$ is much larger than the localization length $\xi$ of the edge state.

Thus, we impose vanishing boundary conditions at the end of the strip $\psi_{k_{y}}(0) = 0$, which further imposes the constraint $R(0) = -L(0)$. The energy spectrum of the edge state is then found to be

$$\epsilon(\varphi, k_{y}) = U_{0} \cos(\varphi) + t_{y} \cos(k_{y}a_{y}),$$

under the condition that $U_{0} \sin(\varphi) + t_{y} \sin(k_{y}a_{y}) < 0$. The corresponding wavefunction of the left edge state at energy $\epsilon(\varphi, k_{y})$ is given by $\psi_{k_{y}}(x) \sim \sin(k_{F}x)e^{-x/\xi}$ with the localization length

$$\xi = -h\hbar v_{F}/|U_{0} \sin(\varphi) + t_{y} \sin(k_{y}a_{y})|. \right)$$

These edge states propagate along the boundaries in $y$ direction. They can be considered as 1D extension of fractional fermions of the Jackiw-Rebbi type.
There are two important phases the system can be tuned into: The non-topological phase, dominated by the CDW modulation, and the topological QHE phase at filling factor $\nu = 1$ (higher filling factors are discussed in the Appendix D), dominated by the magnetic field. We study now the transition between these two phases both analytically and by diagonalizing numerically the tight-binding Hamiltonians, see Eqs. (2) and (3). In the calculations we fix the parameters as follows: $t_y/t = 0.1$ and $k_F = k_w = \pi/4a_x$. The topological transition is induced by changing the amplitude of the CDW $U_0$ with respect to the tunneling amplitude $t_y$ between wires. We are interested in the bulk band represented by the edge of the gap $\Delta_y(k_y)$ and in the edge state wave function probability $|\psi(n, k_y)|^2$ and its dispersion $\epsilon(k_y a_y)$.

In the topological phase, $t_y > U_0$, the edge state spectrum merges with the bulk gap at two points $k_y$ (one from the electron band and one from the hole band) determined by the condition $\epsilon(\varphi, k_y) = \pm \Delta_y(k_y)$, leading to $\text{sin}(k_y a_y) = -(U_0/t_y) \text{sin} \varphi$. In other words, for any given value of $\varphi$, the edge state exists only for the range of momenta $k_y \in (k_-, k_+)$, see Fig. 2(a)-(d). Here, we can further distinguish between two regimes. If $\varphi \in (-\pi, 0]$ corresponding to the chiral (piecewise chiral) regime, the sign of the Fermi velocity is independent of (depends on) $\mu$, as illustrated in Fig. 2(a) by the dispersion of the right (chiral) and left (piecewise chiral) edge state. In the piecewise chiral regime, there is a range of $\mu$, for which the edge states are non-chiral, i.e., there are two counterpropagating edge modes at a given boundary in contrast to the single edge mode in the chiral regime, where the velocities are opposite at opposite boundaries. In the topological phase, a non-chiral behaviour is observed for $\mu$ inside the bulk gap. One can also notice the asymmetry in the localization length between the right and left edge states. For example, if $\varphi = -\pi/2$, see Fig. 2(a) [\varphi = \pi/2, see Fig. 2(d)], the left (right) edge state is more strongly localized than the opposite one which is consistent with Eq. (7) and the 2D finite size calculations [see Fig. 3 (b)] even in the presence of disorder [see Figs. 4 (b') and (b'')]. The larger the gap for given $k_y$ the more localized the edge state is.

In the non-topological phase, $t_y < U_0$, the edge states exist only for particular values of the CDW phase shift $\varphi$, see Figs. 2(b'') and (d''). Generally, there are three possible scenarios, see Fig. 4. If $t_y < -U_0 \text{sin} \varphi$, the edge state exists inside the bulk gap without touching the bulk spectrum, see Figs. 2(a'') and (d''). These edge states are non-chiral and disorder e.g. due to random

III. TOPOLOGICAL TRANSITION BETWEEN CDW AND QHE PHASE

FIG. 2. The energy spectrum $E(k_y a_y)$ near the band gap around the Fermi level defined at $\mu = -\sqrt{2}t$: (a)-(d) in the topological phase ($t_y > U_0 = 0.05t$), (a')-(d') at the phase transition point ($t_y = U_0 = 0.1t$), (a'')-(d'') in the non-topological phase ($t_y < U_0 = 0.15t$) for $\nu = 1$. The panel columns (a), (b), (c), and (d) correspond to the phase shift of the CDW modulation $\varphi = -\pi/2, -0.1\pi, 0, \pi/2$, resp. The energy of the edge state and the bulk spectral edge is found both numerically (solid line) and analytically (dashed line), while the color map represents $|\Psi(n, k_y)|^2$ for the edge state wave function probability. The position at site $n$ (energy) is marked on the left (right) axis. We note that edge states can be found both in the topological and non-topological phase. By changing $\mu$, the edge states can be tuned between being chiral and non-chiral independent in both phases, see panels (a) and (b'').
impurities can result in backscattering inside the same channel, reducing the conductance. If $t_y < U_0 \sin \varphi$, the system is in the trivial phase without edge states. In the regime $U_0 > t_y > -U_0 \sin \varphi$ ($U_0 > t_y > U_0 \sin \varphi$) there are again two wavevectors $k_\pm$ at which edge states merge with the bulk electron (hole) spectrum. As a result, there is a range of chemical potentials (corresponding to the Fermi wavevectors between $k_-$ and $k_+$) for which edge states are chiral even in the non-topological phase, see Fig. 2(b”). However, these values are not in the bulk gap, so the edge states coexist with the bulk modes. The previous analysis was relying on the fact that $k_y$ is a good quantum number in the absence of disorder. Similarly to Weyl semimetals$^{38–43}$, one can expect to detect$^{44}$ such chiral edge states in a gapless bulk by searching for an enhanced response at the boundaries. Similarly, weak disorder cannot eliminate these merging points at $k_\pm$ (e.g. by combining them) as they are protected by continuity of our analytical solutions.

![FIG. 3. The wave functions $|\Psi(n,m,E \approx 0)|^2$ of edge states for a 2D finite size lattice inside the bulk gap (near $E = 0$) in the topological phase, $t_y > U_0 = 0.05t$ for (a) $\varphi = 0$, (b) $\varphi = 3/2\pi$, and (c) in the non-topological phase, $t_y < U_0 = 0.15t$ and $\varphi = 3/2\pi$. The lower panels (a’)-(c’) and (a”)-(c”) correspond to the case of onsite disorder ($\sigma_V = 0.05t$) and disorder in the CDW phase ($\sigma_\varphi = \pi/8$). In the topological phase, the presence of edge states is not affected by weak disorder (with effective amplitude not exceeding the size of the gap), while edge states in the non-topological phase are affected by weak disorder leading to Anderson localization (around some random edge site). However, even in the latter case edge states can survive some small amount of disorder of both types when $\sigma_V \lesssim 0.1t$ and $\sigma_\varphi \lesssim \pi/16$.]

![FIG. 4. Phase diagram for filling factor $\nu = 1$. If $t_y/U_0 > 1$ ($t_y/U_0 < 1$), the system is in the topological (non-topological) phase, indicated as QHE (CDW) phase. The phase transition between phases occurs at $t_y/U_0 = 1$, where the gap closes. The non-topological phase is subdivided into following subphases: (I) (orange area) edge states are totally inside the bulk gap, (II) (blue area) no edge states, and edge states spread between two wavevectors $k_-$ and $k_+$ belonging either both to the electron (III) (light green area) or both to the hole (III’) (dark green areas) band. At the phase boundary between the subphases (dashed lines), we have $k_- = k_+$. Generally, depending on the phase $\varphi$ the edge states can be either chiral for all values of $\mu$ or be piecewise chiral, such that by shifting $\mu$, opposite chiralities can be observed.]

IV. DISORDER EFFECTS

In realistic systems one cannot avoid disorder. In our 2D finite size lattice model, we study effects of disorder by introducing (i) a random on-site potential $\sum_{n,m} V_{n,m} c_{n,m}^\dagger c_{n,m}$ and (ii) a random phase $\tilde{\varphi}_m$ for the CDW modulation in each wire, i.e., $2U_0 \sum_{n,m} \cos(2k_{y,n} a_x + \varphi + \tilde{\varphi}_m) c_{n,m}^\dagger c_{n,m}$. Here, $V_{n,m}$ ($\tilde{\varphi}_m$) is taken according to a Gaussian distribution with zero mean and standard deviation $\sigma_V$ ($\sigma_\varphi$). By analyzing the edge state wave functions $|\Psi(n,m,\varphi)|^2$ in different phases (see Fig. 3), we see that, for the given variance $\sigma_V \lesssim 0.1t$, the onsite disorder does not destroy the edge states in both topological [also the asymmetry in localization lengths is preserved, see Fig. 3 (b’)] and non-topological, phases, see Figs. 3 (a’)-(c’)$^{30}$. Interestingly, chiral QHE edge states survive any amount of disorder in the phase $\varphi$, while in the non-topological phase, the edge states survive only up to a certain small amount of disorder $\sigma_\varphi$ with stronger disorder leading to Anderson localization around a random location along the edge. Importantly, the magnetic field stabilizes the edge states induced by the CDW also in the non-topological regime by suppressing backscattering caused by disorder. In our numerics on finite size systems such localization effects were always negligible in the parameter regimes considered.

V. SUMMARY

We have studied a system of weakly coupled and CDW modulated wires in a perpendicular magnetic field. The
system supports edge states in both the non-topological (CDW dominated) and topological (QHE dominated) phase. Interestingly, both phases host chiral and non-chiral edge states depending on the chemical potential position. Numerical calculations showed that, in general, the edge states in the non-topological phase are more affected by the disorder than in the topological phase, however, the former can still survive a finite amount of disorder. We propose that our predictions can be tested in semiconducting nanowires with CDW modulations, heterostructures\textsuperscript{16,17}, organic conductors\textsuperscript{45}, but also in optical lattices\textsuperscript{46–48} or photonic crystals\textsuperscript{49}. Finally, it would be interesting to see if our results can be extended to other models in 2D and 3D, which include TI phases with (piecewise) helical edge states.

ACKNOWLEDGMENTS

We acknowledge support from the Swiss NSF, NCCR QSIT, and SCIEX.

Appendix A: Effective linearized Hamiltonian

In the weak coupling regime $t_y, |U_0| \ll t$, when tunneling and CDW modulation can be treated as small perturbations, the continuum version of the Hamiltonian $H_x = \sum_{n,k_y} [-t_{\text{lin}} c_{n+1,k_y} c_{n,k_y} + H.c.] - \sum_{n,k_y} \mu c_{n,k_y}^\dagger c_{n,k_y}$ can be linearized in the vicinity of the Fermi points, $\pm k_F$\textsuperscript{29,32}. The corresponding electron annihilation operator can be expressed in terms of slowly varying left $[L(x)]$ and right movers $[R(x)]$ as

$$\Psi(x) = R(x)e^{ik_Fx} + L(x)e^{-ik_Fx}. \quad (A1)$$

As a result by dropping out all fast oscillating terms, we rewrite the kinetic-energy term $H_x$ in the linearized model as

$$H_{x,\text{lin}} = i\hbar v_F \int dx \left[ L(x) \frac{\partial L(x)}{\partial x} - R(x) \frac{\partial R(x)}{\partial x} \right]. \quad (A2)$$

where $v_F$ is the Fermi velocity. The Hamiltonian corresponding to the charge density modulation $U(x) = -2U_0 \cos(2k_w x + \varphi)$ that resonantly couples the left and the right movers for $k_w = k_F$ and can be written as

$$H_{\text{CDW,lin}} = -U_0 \int dx \left[ e^{i\varphi} R(x) L(x) + H.c. \right]. \quad (A3)$$

Here, we again neglected all fast-oscillating terms. We note that Eq. (3) can be use also in the case if the system is slightly out of the resonance $k_w = k_F + \delta k$ if $\hbar v_F \delta k \ll |U_0|$. Thus, the deviation can be quite substantial if the amplitude $U_0$ is large. However, we also note that if the CDW is generated not by gates but intrinsically by electron-electron interactions then the relation $k_w = k_F$ emerges by itself (similar to a Peierls transition): The gap opened at the Fermi energy lowers the total energy of the system and the energy gain is maximum at $k_w = k_F$.

The magnetic field at the filling factor $\nu = 1$ results in the phase $\phi = 2k_F a_y$ for tunneling matrix elements\textsuperscript{14,20}. Thus, the tunneling Hamiltonian $H_y$ also couples resonantly the left and the right movers,

$$H_{y,\text{lin}} = -t_y \int dx \left[ e^{i\varphi_{a_{y}}} R^\dagger(x)L(x) + H.c. \right]. \quad (A4)$$

To conclude, we note that the total Hamiltonian $H_{y,\text{lin}} + H_{\text{CDW,lin}} + H_{x,\text{lin}}$ can be conveniently represented in terms of Pauli matrices acting on the left-right mover subspace, see Eq. (3) of the main text.

FIG. 5. The energy spectrum $E(k_y a_y)$ near the band gap at the Fermi level defined by $\mu = -\sqrt{2t}$ for the isotropic system ($t_y = t$) (a) in the topological phase ($U_0 = 0.5t$), (b) at the phase transition point $(U_0 = t)$, (c) in the non-topological phase ($U_0 = 1.5t$) found numerically. The phase of the CDW modulation is set to $\varphi = \pi/2$ (a’), (b’), (c’) for $\varphi = -0.1\pi$). The color map represents $|\Psi(n,k_y)|^2$ for the edge state wave function probability. The position at site $n$ (energy) is marked on the left (right) axis. We note that edge states can be found both in the topological and non-topological phase. By changing $\mu$, the edge states can be tuned between being chiral and non-chiral, see the panels (a) and (c’).

Appendix B: Isotropic limit

The continuum model we used in the main text is derived in the anisotropic regime, i.e., under the assumption of weak tunneling $t_y \ll t$. While the isotropic model...
with $t_y = t$ is difficult to study analytically it can be easily addressed numerically by performing exact diagonalization of the tight-binding Hamiltonian given by Eq. (2) in the main text. Also in this case we find the phase transition between the topological and non-topological phases separated by the gap closing at $t_y = U_0$ (see Fig. 1) as it was shown in the main text in the anisotropic model ($t_y \ll t$). Importantly, we again observe both chiral and non-chiral edge states in both phases.

The wavefunctions of in-gap edge states can be also obtained by the exact diagonalization of the isotropic ($t_y = t$) 2D Hamiltonian without introducing the momentum $k_y$, see Fig. 2. In the topological phase, chiral edge states are present at all four sides of the 2D sample, whereas, in the nontopological phase, there are localized edge states only at one side. This difference in edge state behavior could help to distinguish between the two phases experimentally. A further possibility to distinguish the two phases is to study their robustness against disorder. In the presence of disorder, the conductance of non-chiral edges deviates from well-quantized values typical for chiral edge states.

**Appendix C: Dependence on the width $W$ of the strip**

In the numerical results presented in the main text, the number of lattice sites is chosen such as to cover an integer number of Fermi wavelengths inside the strip. As a result, the discretized CDW potential has an inversion symmetry point for $\varphi = -\pi, 0, \pi$. However, one can choose the number of sites such that the length of the wire ($i.e.$, the width $W$ of the strip) corresponds to a half integer number of Fermi wavelengths. In this case the CDW phase shift $\varphi$ at the right end of the strip is different from those in the main text, see Fig. 7. As expected, the left edge modes are not affected by this new choice, but the dispersion of the right edge mode is changed. For example, the discretized CDW potential has now an inversion symmetry point for the phase value $\varphi = 3\pi/2$. Interestingly, in the non-topological phase, $U_0 > t_y$, one can find again non-chiral edge states that do not touch the bulk modes. The 2D wave functions $|\Psi(n, m, \varepsilon \approx 0)|^2$ for both edge states present in the non-topological regime are depicted in Fig. 8.

**Appendix D: Filling factor $\nu = 2$**

One can also observe phase transitions similar to the ones descibed in the main text for $\nu = 1$ for the filling factor $\nu = 2$, which corresponds to an indirect resonant magnetic field with $\phi = k_F a_x^{14,20}$. In this case, the size of the gap opened by the resonant tunneling in the $y$ direction is smaller ($\Delta \approx 2t_y^2/\mu$) than for $\nu = 1$, thus the phase transitions occurs for correspondingly smaller values of $U_0$. In the topological phase, there are always two chiral edge states present at each edge inside the gap, see Figs. 3 (a)-(b). Interestingly, in the non-topological phase, there can be two or four modes present at the same edge, see Fig. 9(b'). Similarly to $\nu = 1$, we observe chiral and non-chiral edge states in both topological and non-topological regimes.

**Appendix E: Connection to 2D TIs**

Mentioning of the 2D TI just served as a motivation for our work to underline the importance of studying systems that can support both topological and non-topological edge states. While the focus of our work is on QHE and CDW, we shall point out here a connection to 2D TIs and our work on a qualitative level. The spin Hall effect (present in a 2D TI) can be considered as resulting from two uncoupled layers of spin up and spin down electrons, both in the QHE regime. Importantly, the direction of the magnetic field must then be opposite in the two layers (which models the spin-orbit interaction and preserves time reversal symmetry). Given this analogy, our study of the competition between the QHE and the CDW dominated phases can hint on the existence of similar effects also for 2D TIs, where one can address the interplay between helical and non-helical edge states. However, explicit modeling of a 2D TI is a more subtle issue which is beyond the scope of the present work and we leave this study to future work.

In this work we focus on 2D systems. However, our ideas on coexistence of two types of edge states of course could be extended to 1D$^{32,50–52}$ and 3D systems.

![FIG. 6. The wave functions $|\Psi(n, m, E_0)|^2$ (color maps) of edge states for an isotropic ($t_y = t$) 2D finite size lattice inside the bulk gap (near the center of the gap $E_0$) (a) in the topological phase, $U_0 = 0.5t$ and (b) in the non-topological phase $U_0 = 1.5t$ for $\varphi = \pi/2$. The discrete energy spectrum in the gap is plotted in the insets with the corresponding state marked by the red dot. In the non-topological regime localized edge states are present only on one side of the 2D sample.](image-url)
FIG. 7. The energy spectrum in the anisotropic regime ($t_y = 0.1t$) $E(k_y)$ near the band gap for the system (a-d) in the topological (QHE dominated) phase ($t_y > U_0 = 0.05t$), (a'-d') at the phase transition point ($t_y = U_0 = 0.1t$), and (a''-d'') in the non-topological (CDW dominated) phase ($t_y < U_0 = 0.15t$). The panels (a), (b), (c), and (d) refer to the phase of CDW $\phi = \pi, 5\pi/4, 3\pi/2, 7\pi/4$, respectively. The spectrum of localized edged state is found numerically (dashed line) and analytically (red line). The color map represents $|\Psi(n,k_y)|^2$ for the edge state. The position, site $n$, (energy) is marked on the left (right) axis. The found behavior of edge states is, generally, the same as described in the main text, confirming that proposed phases do not depend qualitatively on the system width.

FIG. 8. The wavefunction $|\Psi(n,m,E \approx 0)|^2$ for the zero edge states inside the gap in the non-topological phase $|t_y| < |U_0| = 0.15t$ and $\varphi = 3/2\pi$ where both edge states are present, see Fig. 7.

FIG. 9. The same as in Fig. 2 of the main text ($t_y = 0.1t$ and $\mu = -\sqrt{2}t$) but for the filling factor $\nu = 2$ (resulting in $\phi = k_F a_y$). (a, b) In the topological phase ($U_0 = 0.02t$, $\varphi = 0, -\pi/2$), there are two chiral edge states present at each edge. (a', b') In the non-topological phase ($U_0 = 0.1t$, $\varphi = 0, -\pi/2$), depending on the chemical potential, there could be two or four non-chiral edge states.
[References are not available in the provided image.]