Canonical Quantization of the Self-Dual Model coupled to Fermions

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Abstract

This paper is dedicated to formulate the interaction picture dynamics of the self-dual field minimally coupled to fermions. To make this possible, we start by quantizing the free self-dual model by means of the Dirac bracket quantization procedure. We obtain, as result, that the free self-dual model is a relativistically invariant quantum field theory whose excitations are identical to the physical (gauge invariant) excitations of the free Maxwell-Chern-Simons theory.

The model describing the interaction of the self-dual field minimally coupled to fermions is also quantized through the Dirac-bracket quantization procedure. One of the self-dual field components is found not to commute, at equal times, with the fermionic fields. Hence, the formulation of the interaction picture dynamics is only possible after the elimination of the just mentioned component. This procedure brings, in turns, two new interactions terms, which are local in space and time while non-renormalizable by power counting. Relativistic invariance is tested in connection with the elastic fermion-fermion scattering amplitude. We prove that all the non-covariant pieces in the interaction Hamiltonian are equivalent to the covariant minimal interaction of the self-dual field with the fermions. The high energy behavior of the self-dual field propagator corroborates that the coupled theory is non-renormalizable. Certainly, the self-dual field minimally coupled to fermions bears no resemblance with the renormalizable model defined by the Maxwell-Chern-Simons field minimally coupled to fermions.

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I. INTRODUCTION

The self-dual (SD) model, put forward by Townsend, Pileh and Van Nieuwenhuizen \cite{1}, has been the object of several investigations. On a semiclassical level, it has been shown to be equivalent to the Maxwell-Chern-Simons (MCS) theory \cite{2,3}. That this equivalence holds on the level of the Green functions was proved in Ref. \cite{4}. Lately, the second-class constraints of the SD model were successfully converted into first-class by means of the embedding procedure proposed by Batalin, Fradkin and Tyutin \cite{5}. It was then found that the SD model and the MCS theory are just different gauge-fixed versions of a parent theory \cite{6,7,8}.

The statements above refer to the free SD and MCS models. The central purpose of this paper is to present the canonical quantization of the SD theory minimally coupled to fermions, while unraveling possible links with the MCS theory minimally coupled to fermions.

Explicit calculations of probability amplitudes for an interacting field theory in 2+1 dimensions are, on general grounds, possible only in perturbation theory. Our main goal will then be to formulate the quantum dynamics of the minimally coupled SD-fermion system in the interaction picture. But to make this operational one must \textit{a priori} solve for the quantum dynamics of the free SD and fermion fields. Hence, we start in section 2 by quantizing the uncoupled SD theory through the Dirac bracket quantization procedure (DBQP) \cite{9,10,11}. The model is found to contain only one independent polarization vector, which is explicitly determined and turns out to be identical to the polarization vector of the MCS in the Landau gauge \cite{12}. By combining this result with the field commutators at different times, obtained afterwards, we are led to conclude that the space of states is positive definite. Ordering ambiguities arise in the construction of the (symmetric) energy-momentum tensor and we are forced to adopt an ordering prescription. The components of the energy-momentum tensor are shown to fulfill the Dirac-Schwinger algebra, which secures the existence of a set of charges obeying the Poincaré algebra \cite{14,15}. The summary of this section is that the SD model is a relativistically invariant quantum field theory describing massive quanta whose spin can be ±1 depending upon the sign of the mass parameter. Thus, the particle content of the SD model is identical to that of the MCS theory in the Coulomb gauge \cite{13,16,17}. This establishes the equivalence of these two models within the operator approach. We emphasize that the developments in this section do not duplicate those in Refs. \cite{2,4,6–8}, which were carried out either within the semiclassical approximation \cite{2} or within the functional approach \cite{6–8}.

The quantization of the free fermion field through the DBQP can be found in the literature \cite{11,18} and leads to standard results quoted in textbooks on quantum field theory. We then turn in section 3 into applying the DBQP for quantizing the SD model minimally coupled to fermions. Aside from technicalities which will be discussed in detail, the main point arising along this process of quantization is that the equal-time commutators (ETC’s) involving the fermions and one of the SD field components do not vanish, thus obstructing the standard way for arriving at the formulation of the quantum dynamics in the interaction picture. The situation resembles that encountered when quantizing the MCS theory minimally coupled to fermions in the Coulomb gauge. There, the ETC’s involving the fermionic fields and the longitudinal component of the momenta canonically conjugate to the gauge
potentials do not vanish, also. The way out from the trouble consists in getting rid off these longitudinal components by means of the Gauss law constraint. This operation brings, in turns, two new interaction terms to the Hamiltonian \[13,19\]. In the present case, the elimination of the corresponding SD field variable also brings two new interaction terms, being both local in space-time and non-renormalizable by power counting. One of these terms is the time component of a Pauli interaction while the other is the time component of a Thirring interaction. After having determined all the vertices of the coupled theory we end this section by presenting the expressions for the free fermionic and SD-field propagators.

The relativistic invariance of the interacting model is tested in section 4. As in Ref. \[13\], we use, for this purpose, the lowest order contribution to the elastic fermion-fermion scattering amplitude. We demonstrate that the combined action of the non-covariant pieces that make up the interaction Hamiltonian can, after all, be replaced by the minimal covariant field-current interaction. The high energy behavior of the self-dual field propagator corroborates that the SD field when minimally coupled to fermions defines a non-renormalizable theory.

Some final remarks and the conclusions are contained in section 5.

II. QUANTIZATION OF THE FREE SD MODEL

The dynamics of the SD theory is described by the Lagrangian density \[1,2\]

\[
L^{SD} = -\frac{1}{2m} \epsilon_{\mu\nu\rho} (\partial_\mu f_\nu) f_\rho + \frac{1}{2} f_\mu f_\mu ,
\]

(2.1)

where \( m \) is a parameter with dimensions of mass. We use natural units \( (c = \hbar = 1) \) and our metric is \( g_{00} = -g_{11} = -g_{22} = 1 \). The fully antisymmetric tensor is normalized such that \( \epsilon^{012} = 1 \) and we define \( \epsilon^{ij} \equiv \epsilon^{0ij} \). Repeated Greek indices sum from 0 to 2 while repeated Latin indices sum from 1 to 2.

By computing the momenta \( (\pi_\mu) \) canonically conjugate to the field variables \( (f^\mu) \) one finds that the theory possesses the primary constraints

\[
P_0 = \pi_0 \approx 0 ,
\]

(2.2a)

\[
P_i = \pi_i + \frac{1}{2m} \epsilon_{ij} f_j \approx 0 , i = 1, 2 ,
\]

(2.2b)

where the sign of weak equality \( (\approx) \) is used in the sense of Dirac \[3\]. Furthermore, the canonical Hamiltonian \( (H^{SD}) \) is found to read

\[
H^{SD} = \int d^3x \left( -\frac{1}{2} f_\mu f_\mu + \frac{1}{m} \epsilon_{ij} f^0 \partial^j f^i \right).
\]

(2.3)

Hence, the total Hamiltonian \( (H^{SD}_T) \) is given by \[3\]

\[
H^{SD}_T = H^{SD} + \int d^3x (u^0 P_0 + u^i P_i),
\]

where the \( u \)'s are Lagrange multipliers.

The persistence in time of \( P_0 \approx 0 \),

\[
\dot{P}_0 = [P_0, H^{SD}_T]_P \approx 0 ,
\]

where \([,]_P\) denotes the Poisson bracket, leads to the existence of the secondary constraint
\[ S = \frac{1}{m} \left( f^0 - \frac{1}{m} \epsilon_{ij} \partial^i f^j \right) \approx 0 . \] (2.4)

On the other hand, \([P_i(\vec{x}), P_j(\vec{y})]_P \neq 0\) implies that by demanding persistence in time of \(P_i \approx 0, i = 1, 2\), one determines the Lagrange multipliers \(u^i, i = 1, 2\). Similarly, \([P_0(\vec{x}), S(\vec{y})]_P \neq 0\), together with the persistence in time of \(S(\vec{x}) \approx 0\), enables one to determine \(u^0\).

From the above analysis follows that all constraints are second-class and, therefore, Dirac brackets with respect to them can be introduced in the usual manner \([9]\). The phase-space variables are, afterwards, promoted to operators obeying an equal-time commutator algebra which is to be abstracted from the corresponding Dirac bracket algebra, the constraints and gauge conditions thereby translating into strong operator relations. This is the DBQP, which presently yields \([20,21]\)

\[
\begin{align*}
[f^0(\vec{x}), f^j(\vec{y})] &= i \partial^0 \delta(\vec{x} - \vec{y}) , \\
[f^k(\vec{x}), f^j(\vec{y})] &= -im \epsilon^{kj} \delta(\vec{x} - \vec{y}) , \\
[f^0(\vec{x}), \pi_k(\vec{y})] &= -i \frac{\epsilon}{2m} \epsilon^{kj} \partial^j \delta(\vec{x} - \vec{y}) , \\
[f^j(\vec{x}), \pi_k(\vec{y})] &= i \frac{\epsilon}{2} g^j_k \delta(\vec{x} - \vec{y}) , \\
[\pi_j(\vec{x}), \pi_k(\vec{y})] &= -i \frac{\epsilon}{4m} \epsilon^{jk} \delta(\vec{x} - \vec{y}) ,
\end{align*}
\] (2.5a-e)

whereas all other ETC’s vanish. As for the quantum mechanical Hamiltonian \((H^{SD})\) it can be read off directly from (2.3), in view of the absence of ordering ambiguities in the classical-quantum transition. This is true although \(f^0\) and \(f^j\) do not commute at equal times (see (2.5a)). Alternative expressions for the operator \(H^{SD}\) can be obtained after recalling that, within the algebra (2.5), the constraints act as strong identities. Thus,

\[ H^{SD} = \int d^2x \left(-\frac{1}{2} f^\mu f_\mu + \frac{1}{m} \epsilon_{ij} f^0 \partial^j f^i \right) = \int d^2x \left(-\frac{1}{2} f^\mu f_\mu + f^0 f_0 \right) . \] (2.6)

What we need next is to solve the equations of motion

\[
\partial_0 f^0(\vec{x}) = i [H^{SD}, f^0(\vec{x})] = -\partial_j f^j(\vec{x}) \implies \partial_\mu f^\mu = 0 ,
\] (2.7)

\[
\partial_0 f^j(\vec{x}) = i [H^{SD}, f^j(\vec{x})] = \partial^j f_0(\vec{x}) + m \epsilon^{jk} f_k(\vec{x}) ,
\] (2.8)

obeyed by the field operators \(f^\mu\). Notice that this last equation and (2.4) can be unified into the single covariant expression

\[ \epsilon^{\mu \sigma \alpha} \partial_\sigma f_\alpha - m f^\mu = 0 , \] (2.9)

which is formally identical to the Lagrange equation of motion deriving from (2.1). Of course, (2.7) follows from (2.9). The solving of (2.9) is greatly facilitated if one observes that its solutions also fulfill the Klein-Gordon equation

\[ (\Box + m^2) f^\mu = 0 , \] (2.10)
the converse not being necessarily true. One can then write

\[ f^\mu(x) = \int d^2z \Delta (x-z) \partial^\mu f^\mu(z) + \int d^2z \partial^\mu \Delta (x-z) f^\mu(z) , \] (2.11)

where the 2+1 dimensional Pauli-Jordan delta function (\(\Delta\)) of mass \(z\) which is easily seen to satisfy the first order differential equation the converse not being necessarily true. One can then write (2.5). In this way one obtains commutators at different times can be computed by using, as input, the equal time algebra favor of the phase-space variables by using Eqs.(2.7) and (2.8). Once this has been done, the commutators at different times can be computed by using, as input, the equal time algebra (2.3). In this way one obtains

\[ [f^\mu(x) , f^\nu(y)] = i \left( m^2 g^{\mu\nu} + \partial^\mu \partial^\nu - m e^{\mu\nu\rho} \partial^\rho \right) \Delta (x-y) , \] (2.12)

which is easily seen to satisfy the first order differential equation

\[ \epsilon_{\alpha\beta\mu} \partial^\beta [f^\mu(x) , f^\nu(y)] - m [f_\alpha(x) , f^\nu(y)] = 0 . \] (2.13)

Thus, the field configurations entering the commutator (2.12) are, as required, solutions of Eq.(2.9). Since the Pauli-Jordan delta function admits a decomposition into positive and negative frequency parts [22], we learn from (2.12) that the same is true for the field operators \(f^\mu\), namely,

\[ f^\mu(x) = f^{\mu(+)}(x) + f^{\mu(-)}(x) , \] (2.14)

with

\[ f^{\mu(\pm)}(x) = \frac{1}{2\pi} \int \frac{d^2k}{\sqrt{2\omega_k}} \exp \left[ \pm i (\omega_k x^0 - \vec{k} \cdot \vec{x}) \right] f^{\mu(\pm)}(\vec{k}) \] (2.15)

\(\omega_k \equiv +\sqrt{\vec{k}^2 + m^2}\), and

\[ [f^{\mu(+)}(\vec{k}) , f^{\nu(+)}(\vec{k} \tau)] = [f^{\mu(-)}(\vec{k}) , f^{\nu(-)}(\vec{k} \tau)] = 0 \] (2.16a)
\[ [f^{\mu(-)}(\vec{k}) , f^{\nu(+)}(\vec{k} \tau)] = - \left( m^2 g^{\mu\nu} - k^\mu k^\nu + im e^{\mu\nu\rho} k^\rho \right) \delta(\vec{k} - \vec{k} \tau) . \] (2.16b)

To go further on we must recognize that not all variables spanning the phase are independent variables. Indeed, the system under analysis possesses three coordinates \((f^\mu, \mu = 0, 1, 2)\), three momenta \((\pi_\mu, \mu = 0, 1, 2)\) and four constraints \((P_0, P_1, P_2, S)\). Thus, we are left with only one independent degree of freedom which implies that there is only one polarization vector in the theory, to be designated by \(\epsilon^\mu(\vec{k})\). We can then write

\[ f^{\mu(+)}(\vec{k}) = \epsilon^\mu(\vec{k}) a^{(+)}(\vec{k}) , \] (2.17a)
\[ f^{\mu(-)}(\vec{k}) = \epsilon^*\mu(\vec{k}) a^{(-)}(\vec{k}) , \] (2.17b)

where \(a^{(\pm)}(\vec{k})\) are operators whose commutator algebra will be determined later on. By going back with (2.17) into (2.15) and with this into (2.3) one finds that \(\epsilon^\mu\) verifies the homogeneous equation
\[ \Sigma^\alpha(\vec{k}) \varepsilon_\alpha(\vec{k}) = 0 , \quad (2.18) \]

with
\[ \Sigma^\alpha(\vec{k}) \equiv i \, \varepsilon^\alpha_{\mu} \, k_\mu - m \, g^\alpha . \quad (2.19) \]

The vanishing of the determinant of the matrix \( \Sigma^\alpha \) is a necessary and sufficient condition for Eq.\((2.18)\) to have solution different from the trivial one. The computation of this determinant is straightforward and yields
\[ \det ||\Sigma|| = m \,(k^2 - m^2) . \quad (2.20) \]

Therefore, the theory only propagates particles with mass equal to \(|m|\). To solve \((2.18)\) we adopt the strategy presented in Ref. [13], i.e., we start by relating \( \varepsilon^\mu(\vec{k}) \) with \( \varepsilon^\mu(0) \) through the corresponding Lorentz transformation, namely,
\[ \varepsilon^0(\vec{k}) = \frac{1}{|m|} \vec{k} \cdot \vec{\varepsilon}(0) , \quad (2.21a) \]
\[ \varepsilon^j(\vec{k}) = \varepsilon^j(0) + \frac{\vec{\varepsilon}(0) \cdot \vec{k}}{(\omega_k + |m|) |m|} k^j . \quad (2.21b) \]

This form of the solution is particularly appealing because it shows, explicitly, that \( \varepsilon^\mu(\vec{k}) \) goes continuously to the corresponding value in the rest frame of reference. Now, from \((2.18)\) one finds that
\[ \varepsilon^0(0) = 0 , \quad (2.22a) \]
\[ \varepsilon^j(0) = -i \, \frac{|m|}{m} \varepsilon^j \varepsilon_l(0) , \quad (2.22b) \]

which completes the determination of the polarization vector. One can check that
\[ \varepsilon^\mu(\vec{k}) \varepsilon_\mu(\vec{k}) = -\varepsilon(0) \cdot \varepsilon(0) = 0 , \quad (2.23a) \]
\[ \varepsilon^\mu(\vec{k}) \varepsilon^* \mu(\vec{k}) = -\varepsilon(0) \cdot \varepsilon^*(0) = -2 |\varepsilon^1(0)|^2 , \quad (2.23b) \]

where \(|\varepsilon^1(0)|\) is to be fixed by normalization. This is just the polarization vector of the MCS theory in the Landau gauge [13].

By substituting Eq.\((2.17)\) into Eq.\((2.16b)\) one arrives to
\[ \varepsilon^\mu(\vec{k}) \varepsilon^\nu(\vec{k}) [a^{-}(\vec{k}) , a^{(+)}(\vec{k})] = -\left( m^2 g^{\mu \nu} - k^\mu k^\nu + im \, \varepsilon^{\mu \rho} \, k_\rho \right) \delta(\vec{k} - \vec{k'}) . \quad (2.24) \]

On the other hand, if we fix \(|\varepsilon^1(0)| = m \) the polarization vector can be seen to fulfill the relationship
\[ \varepsilon^* \mu(\vec{k}) \varepsilon^{\nu}(\vec{k}) = -\left( m^2 g^{\mu \nu} - k^\mu k^\nu + im \, \varepsilon^{\mu \rho} \, k_\rho \right) . \quad (2.25) \]

By combining Eqs.\((2.24)\) and \((2.25)\) one obtains the commutator algebra obeyed by the operators \( a^{(\pm)}(\vec{k}) \), i.e.,
Thus, \( a^{(-)}(\vec{k}) \) and \( (a^{(+)}(\vec{k})) \) are, respectively, destruction and creation operators and the space of states is a Fock space with positive definite metric.

What remains to be done is to investigate whether the DBQP has, for the case of the SD model, led to a relativistically invariant quantum theory and to determine the spin of the corresponding particle excitations. The first of these questions appears to be trivial because the Lagrangian density (2.1) is a Lorentz scalar. However, one is to observe that the space-time and the space-space components of the classical symmetric (Belinfante) energy-momentum tensor derived in Ref. [2] (\( \Theta_{\mu\nu} \)) will be taken as the Poincaré densities of the quantum theory the composite Hermitean operators

\[
\Theta^{\mu\nu} = f^\mu \cdot f^\nu - \frac{1}{2} g^{\mu\nu} f^\alpha f_\alpha .
\]  

(2.27)

After some algebra, one corroborates that the Dirac-Schwinger algebra \([14,15]\),

\[
[\Theta^{00}(x^0, \vec{x}), \Theta^{00}(x^0, \vec{y})] = -i \left( \Theta^{0k}(x^0, \vec{x}) + \Theta^{0k}(x^0, \vec{y}) \right) \partial_k^x \delta(\vec{x} - \vec{y}),
\]

(2.28a)

\[
[\Theta^{00}(x^0, \vec{x}), \Theta^{0k}(x^0, \vec{y})] = -i \left( \Theta^{kj}(x^0, \vec{x}) - g^{kj} \Theta^{00}(x^0, \vec{y}) \right) \partial_j^x \delta(\vec{x} - \vec{y}),
\]

(2.28b)

\[
[\Theta^{0k}(x^0, \vec{x}), \Theta^{0j}(x^0, \vec{y})] = -i \left( \Theta^{0k}(x^0, \vec{x}) \partial_j^x + \Theta^{0j}(x^0, \vec{y}) \partial_k^x \right) \delta(\vec{x} - \vec{y}),
\]

(2.28c)

holds. Then, the generators of space-time translations \((P^\mu)\), Lorentz boosts \((J^{0i})\), and spatial rotations \((J)\),

\[
P^0 \equiv \int d^2 x \Theta^{00}(x^0, \vec{x}) = H^{SD},
\]

(2.29a)

\[
P^i \equiv \int d^2 x \Theta^{0i}(x^0, \vec{x}),
\]

(2.29b)

\[
J^{0i} \equiv -x^0 P^i + \int d^2 x \left[ x^j \Theta^{00}(x^0, \vec{x}) \right],
\]

(2.29c)

\[
J \equiv \epsilon_{ij} \int d^2 x x^i \Theta^{0j}(x^0, \vec{x}),
\]

(2.29d)

fulfill the Poincaré algebra. To write the Poincaré generators in terms of the operators \( a^{(\pm)}(\vec{k}) \) involves a cumbersome calculation whose details will not be given here. We only mention that the functional form of the polarization vector \( \varepsilon^{\mu}(\vec{k}) \) (see Eqs. (2.21) and (2.22)) plays a central role for arriving at the following expressions

\[
P^0 = H^{SD} = \int d^2 k \omega_{\vec{k}} a^{(+)}(\vec{k}) a^{(-)}(\vec{k}) ,
\]

(2.30a)

\[
P^j = \int d^2 k k^j a^{(+)}(\vec{k}) a^{(-)}(\vec{k}) ,
\]

(2.30b)

\[
J = \frac{m}{|m|} \int d^2 k a^{(+)}(\vec{k}) a^{(-)}(\vec{k}) + i \epsilon_{ji} \int d^2 k a^{(+)}(\vec{k}) k^j \frac{\partial}{\partial k_i} a^{(-)}(\vec{k}) .
\]

(2.30c)
By acting with $P^\mu$ on the one particle state $a^{(+)}(\vec{k})|0>\rangle$ one learns that $a^{(+)}(\vec{k})$ creates particles with three-momentum $k^\mu$ ($k^0 = \omega_\vec{k}$, $k^j$). The action of the operator $J$ on a single particle state can also be readily derived. In particular, for the rest frame of reference one finds that

$$J \{a^+(\vec{k} = 0)|0>\} = \frac{m}{|m|} \{a^+(\vec{k} = 0)|0>\}, \tag{2.31}$$

which says that the spin of the SD quanta is $\pm 1$ depending upon the sign of the mass factor.

The quantization of the free SD model is complete. We have demonstrated that the theory only contains particles of mass $|m|$ and spin $m/|m|$. This is just the particle content of the MCS theory in the Coulomb (physical) gauge [23]. On the other hand, the polarization vector in the SD theory is that of the MCS model in the Landau gauge. To summarize, the SD model describes the gauge invariant sector of the MCS model. Therefore, these theories are quantum mechanically equivalent.

### III. THE SD FIELD MINIMALLY COUPLED TO FERMIONS

We now bring the fermions into the game [24]. The use of Dirac’s method [9–12] enables one to conclude that the dynamics of the SD model minimally coupled to fermions is described by the canonical Hamiltonian

$$H_1 = \int d^2x \left[ \frac{i}{2} (\partial_\mu \bar{\psi}^k \gamma^k \psi) - \frac{i}{2} \bar{\psi}^k (\partial_\mu \psi) + M \bar{\psi} \psi - \frac{g}{m} \bar{\psi} \gamma^k \psi k^k - \frac{1}{2} f_\mu f^\mu + f^0 \left( \frac{1}{m} \epsilon_{ij} f^j + \frac{g}{m} \bar{\psi} \gamma_0 \psi \right) \right], \tag{3.1}$$

the primary constraints

$$P_0 = \pi_0 \approx 0, \tag{3.2a}$$

$$P_i = \pi_i + \frac{1}{2m} \epsilon_{ij} f^j \approx 0, \ i = 1, 2, \tag{3.2b}$$

$$\theta_a = \pi_\psi - \frac{i}{2} \gamma^0 \bar{\psi} \approx 0, \ a = 1, 2, \tag{3.2c}$$

$$\bar{\theta}_a = \pi_{\bar{\psi}} - \frac{i}{2} \bar{\psi} \gamma^0 \approx 0, \ a = 1, 2, \tag{3.2d}$$

and the secondary constraint

$$S^F = \frac{1}{m} \left( f^0 - \frac{1}{m} \epsilon_{ij} f^j + \frac{g}{m} \bar{\psi} \gamma_0 \psi \right) \approx 0, \tag{3.3}$$

where $\pi_\psi$ and $\pi_{\bar{\psi}}$ are the momenta canonically conjugate to $\psi_a$ and $\bar{\psi}_a$, respectively. Furthermore, all constraints are second-class, as can be easily checked.

The next step consists in introducing Dirac brackets. The direct computations of Dirac brackets with respect to all constraints is difficult. It is easier to compute first partial Dirac brackets with respect to the fermionic constraints, $\theta_a$ and $\bar{\theta}_a$, and then use this result as input for computing the full Dirac brackets [11]. However, this procedure is valid if and only
if \( \{ \theta_a, \tilde{\theta}_a \} \cap \{ P_0, P_f, S^F \} = \{ \phi \} \) which is not the case here, since \( [\theta_a(\bar{x}), S^F(\bar{y})]_P \neq 0 \), \( [\bar{\theta}_a(\bar{x}), S^F(\bar{y})]_P \neq 0 \). Nevertheless, we recall that any combination of constraints is also a constraint and we can, therefore, replace \( S^F \) by \( \tilde{S}^F \), which is to be constructed such that \( [\theta_a(\bar{x}), \tilde{S}^F(\bar{y})]_P \approx 0, [\bar{\theta}_a(\bar{x}), \tilde{S}^F(\bar{y})]_P \approx 0 \), while keeping \( [P_0(\bar{x}), \tilde{S}^F(\bar{y})]_P \neq 0 \). It has been shown that \( \tilde{S}^F(\bar{x}) = S^F(\bar{x}) + \bar{\alpha}_a(\bar{x}) \theta_a(\bar{x}) + \bar{\theta}_a(\bar{x}) \alpha_a(\bar{x}) \approx 0 \),

with \( \alpha_a = -i \frac{2}{m^2} \psi_a \) and \( \bar{\alpha}_a = -i \frac{2}{m^2} \bar{\psi}_a \) verifies all the above requirements.

For the partial Dirac brackets with respect to the fermionic constraints (\( \Delta \)-brackets) one obtains

\[
[\psi_a(\bar{x}), \psi_b(\bar{y})]_\Delta = 0 ,
\]

\[
[\psi_a(\bar{x}), \bar{\psi}_b(\bar{y})]_\Delta = -i \gamma_{ab}^0 \delta(\bar{x} - \bar{y}) ,
\]

\[
[\bar{\psi}_a(\bar{x}), \bar{\psi}_b(\bar{y})]_\Delta = 0 ,
\]

while all \( \Delta \)-brackets involving bosonic variables equal the corresponding Poisson brackets. For any pair of functionals, \( \Lambda \) and \( \Omega \), of the phase space variables, the full Dirac bracket (D-bracket) is now to be computed as follows

\[
[\Lambda, \Omega]_D = [\Lambda, \Omega]_\Delta - \sum_{q=1}^4 \sum_{p=1}^4 \int d^2u \int d^2v [\Lambda, \xi_p(u)]_\Delta R^{pq}(u, v) [\xi_q(v), \Omega]_\Delta ,
\]

where \( \xi_1 \equiv P_0, \xi_j \equiv P_j, j = 1, 2, \xi_4 \equiv S^F, \| R \| = \| Q^{-1} \| \) and \( Q_{pq}(u, v) \equiv [\xi_p(u), \xi_q(v)]_\Delta \). Notice that the fermionic constraints hold as strong identities within the \( \Delta \)-algebra (3.4) and, consequently, we can use \( S^F \) instead of \( \tilde{S}^F \) when computing the D-brackets through Eq.(3.5).

As in the free field case, the quantization consists in promoting all phase-space variables to operators obeying an equal-time commutation algebra abstracted from the corresponding D-bracket algebra. After a lengthy calculation one finds that the nonvanishing ETC’s and anticommutators read as follows

\[
[f^0(\bar{x}), f^j(\bar{y})] = i \partial^j_x \delta(\bar{x} - \bar{y}) ,
\]

\[
[f^k(\bar{x}), f^j(\bar{y})] = -im \epsilon^{kj} \delta(\bar{x} - \bar{y}) ,
\]

\[
[f^0(\bar{x}), \pi_k(\bar{y})] = -\frac{i}{2m} \epsilon_{kj} \partial^j_x \delta(\bar{x} - \bar{y}) ,
\]

\[
[f^j(\bar{x}), \pi_k(\bar{y})] = \frac{i}{2} g^{jk} \delta(\bar{x} - \bar{y}) ,
\]

\[
[\pi_j(\bar{x}), \pi_k(\bar{y})] = -\frac{i}{4m} \epsilon_{jk} \delta(\bar{x} - \bar{y}) ,
\]

\[
[f^0(\bar{x}), \psi_a(\bar{y})] = \frac{g}{m} \psi_a(\bar{x}) \delta(\bar{x} - \bar{y}) ,
\]

\[
[f^0(\bar{x}), \bar{\psi}_a(\bar{y})] = -\frac{g}{m} \bar{\psi}_a(\bar{x}) \delta(\bar{x} - \bar{y}) ,
\]

\[
[\psi_a(\bar{x}), \bar{\psi}_b(\bar{y})] = \gamma^0_{ab} \delta(\bar{x} - \bar{y}) .
\]

The Hamiltonian operator \( (H) \) is the quantum counterpart of \( H_1 \) (see Eq.(3.1)), i.e.,
\[
H = \int d^2x \left[ \frac{i}{2} (\partial_k \bar{\psi}) \cdot \gamma^k \psi - \frac{i}{2} \bar{\psi} \cdot \gamma^k (\partial_k \psi) + M \bar{\psi} \cdot \psi - \frac{g}{m} \bar{\psi} \cdot \gamma_k \psi f_k^D \\
+ \frac{1}{2} f^0 f^0 + \frac{1}{2} f^i f^i \right].
\] (3.7)

Clearly, Eq. (3.7) follows from Eq. (3.1) after appropriate ordering of fermionic factors \((\psi_a \cdot \bar{\psi}_b \equiv 1/2(\psi_a \bar{\psi}_b - \bar{\psi}_b \psi_a))\). The expression for \(H\) was, furthermore, simplified by using the constraint (3.3). We are entitled to do so because, within the algebra (3.6), all constraints hold as strong operator relations [25].

The main observation, concerning the equal-time commutation algebra (3.6), is that the bosonic field variable \(f^0\) does not commute with the fermion fields \(\psi\) and \(\bar{\psi}\). Hence, the implementation of the quantum dynamics in the interaction picture will only be possible after the elimination of \(f^0\). This can be done by using the constraint relation (3.3) which ultimately leads to

\[
H^D = H^D_0 + H^D_1,
\] (3.8)

where

\[
H^D_0 = \int d^2x \left[ \frac{1}{2m^2} \epsilon^{ij} \epsilon^{kl} (\partial_i f^D_j) (\partial_k f^D_l) + \frac{1}{2} f^D_i f^D_i \right] \\
+ \int d^2x \left[ \frac{i}{2} (\partial_k \bar{\psi}^D) \cdot \gamma^k \psi^D - \frac{i}{2} \bar{\psi}^D \cdot \gamma^k (\partial_k \psi^D) + M \bar{\psi}^D \cdot \psi^D \right]
\] (3.9)

and

\[
H^D_1 = \int d^2x \left[ - \frac{g}{m} \bar{\psi}^D \cdot \gamma^k \psi^D f^D_k - \frac{g}{m^2} \epsilon^{ij} \partial_i f^D_j (\bar{\psi}^D \cdot \gamma^0 \psi^D) \\
+ \frac{g^2}{2m^2} (\bar{\psi}^D \cdot \gamma^0 \psi^D) (\bar{\psi}^D \cdot \gamma^0 \psi^D) \right].
\] (3.10)

Here, the superscript \(D\) denotes field operators belonging to the interaction picture; the Heisenberg field operators, we were so far dealing with, bear no picture superscript.

Since different pictures are connected by unitary transformations, the equal-time commutation rules obeyed by the interaction picture field operators can be read off directly from Eq. (3.6). Then, the equation of motion obeyed by the operator \(\psi^D (\partial_0 \psi^D = i[H^D_0, \psi^D])\) is just the free Dirac equation and the corresponding momentum space fermion propagator \((S(p))\) is well known to be

\[
S(p) = i \frac{M + \gamma \cdot p}{p^2 - M^2 + i\epsilon}.
\] (3.11)

The equations of motion satisfied by the operators \(f^D_i, i = 1, 2\) are exactly those studied in detail in section 2 of this paper. Hence, from Eqs. (2.15), (2.17), (2.21), (2.22) and (2.26) one finds that the momentum space Feynman propagator \((D_{ij}(k))\) is given by

\[
D_{ij}(k) = \frac{i}{k^2 - m^2 + i\epsilon} \left( -m^2 g_{ij} + k_i k_j - i m \epsilon_{ij} k_0 \right) = D_{ij}(-k).
\] (3.12)
To complete the derivation of the Feynman rules, we turn into analyzing the vertices of the SD model minimally coupled to fermions. We observe that $H^D$ in Eq.(3.10) is made up by three different kind of monomial terms. The first monomial is the spatial part of the standard field-current interaction, whereas the second and third monomials arised as by products in the process of elimination of $f_0$. Unlike the cases of MCS minimally coupled to fermions [13] and Quantum Electrodynamics, these extra terms are strictly local in space-time. Also, they are non-renormalizable by power counting. The second term in Eq.(3.10) is the time component of a magnetic coupling, while the third one is the time component of a four-fermions (Thirring) interaction.

The Feynman rules derived in this section are non-manifestly covariant. Then, we must elucidate whether or not this set of rules leads to a relativistically invariant $S$-matrix. As in section 2, one may first build the symmetric energy-momentum tensor for the interacting theory and then use the equal-time commutation rules (3.6) to check the fulfillment of the Dirac-Schwinger algebra. However, the validity of such procedure would now be dubious, since the components of the energy-momentum tensor necessarily involve (ill defined) products of Heisenberg field operators evaluated at the same space-time point. The next section is dedicated to test the relativistic invariance of the coupled theory in connection with the specific process of elastic fermion-fermion scattering. Since we are dealing with a non-renormalizable theory, our computations will be restricted to the tree approximation.

**IV. LOWEST ORDER ELASTIC FERMION-FERMION SCATTERING AMPLITUDE**

From the inspection of Eq.(3.10) follows that the contributions of order $g^2$ to the lowest order elastic fermion-fermion scattering amplitude ($R^{(2)}$) can be grouped into four different kind of terms,

$$R^{(2)} = \sum_{\alpha=1}^{4} R^{(2)}_\alpha ,$$

where

$$R^{(2)}_1 = -\frac{g^2}{2} (\gamma^k)_{ab} (\gamma^l)_{cd} \int d^3 x \int d^3 y < \Phi_f | \frac{1}{m} : \bar{\psi}_a^D(x) \psi_b^D(x) f_k^D(x) : $$

$$\times \frac{1}{m} : \bar{\psi}_c^D(y) \psi_d^D(y) f_l^D(y) : \{ \Phi_i > ,$$

$$R^{(2)}_2 = -\frac{ig^2}{2} (\gamma^0)_{ab} (\gamma^0)_{cd} \int d^3 x \int d^3 y \delta(x-y)$$

$$\times < \Phi_f | \frac{1}{m^2} : \bar{\psi}_a^D(x) \psi_b^D(x) \bar{\psi}_c^D(y) \psi_d^D(y) : \{ \Phi_i > ,$$

$$R^{(2)}_3 = -g^2 (\gamma^k)_{ab} (\gamma^0)_{cd} \int d^3 x \int d^3 y < \Phi_f | \frac{1}{m} : \bar{\psi}_a^D(x) \psi_b^D(x) f_k^D(x) : $$

$$\times : \frac{1}{m^2} \epsilon^{li} \left( \partial^l f_i^D(y) \right) \bar{\psi}_c^D(y) \psi_d^D(y) : \{ \Phi_i > ,$$

$$R^{(2)}_4 = -\frac{g^2}{2} (\gamma^0)_{ab} (\gamma^0)_{cd}$$
corresponding creation and annihilation operators goes as usual. wave expansion of the free fermionic operators where

\[ T \]

Here, \( T \) is the chronological ordering operator, whereas \( |\Phi_i> \) and \( |\Phi_f> \) denote the initial and final state of the reaction, respectively. For the case under analysis, both \( |\Phi_i> \) and \( |\Phi_f> \) are two-electron states. Fermion states obeying the free Dirac equation in 2+1-dimensions were explicitly constructed in Ref. \[26\], where the notation \( |\bar{\Phi}_f\rangle \) was employed to designate the two-component spinor describing a free electron of two-momentum \( \bar{p} \), energy \( p^0 = (+p^2 + m^2)^{1/2} \) and spin \( s = M/|M| \) in the initial (final) state. The plane wave expansion of the free fermionic operators \( \bar{\psi} \) and \( \bar{\psi} \) in terms of these spinors and of the corresponding creation and annihilation operators goes as usual.

In terms of the initial \((p_1, p_2)\) and final momenta \((p'_1, p'_2)\), the partial amplitudes in (4.2) are found to read

\[
R_{1}^{(2)} = \frac{1}{2\pi} \delta^{(3)}(p'_1 + p'_2 - p_1 - p_2) \\
\times \left\{ [\bar{\psi}^{(+)}(\bar{p}'_1)(ig_{\gamma^j})\psi^{(-)}(\bar{p}_1)] [\bar{\psi}^{(+)}(\bar{p}'_2)(ig_{\gamma^j})\psi^{(-)}(\bar{p}_2)] \frac{1}{m^2} D_{ji}(k) - p'_1 \leftrightarrow p'_2 \right\},
\]

\[
R_{2}^{(2)} = \frac{1}{2\pi} \delta^{(3)}(p'_1 + p'_2 - p_1 - p_2) \\
\times \left\{ \frac{i}{m^2} [\bar{\psi}^{(+)}(\bar{p}'_1)(ig_{\gamma^0})\psi^{(-)}(\bar{p}_1)] [\bar{\psi}^{(+)}(\bar{p}'_2)(ig_{\gamma^0})\psi^{(-)}(\bar{p}_2)] - p'_1 \leftrightarrow p'_2 \right\},
\]

\[
R_{3}^{(2)} = \frac{1}{2\pi} \delta^{(3)}(p'_1 + p'_2 - p_1 - p_2) \\
\times \left\{ [\bar{\psi}^{(+)}(\bar{p}'_1)(ig_{\gamma^j})\psi^{(-)}(\bar{p}_1)] [\bar{\psi}^{(+)}(\bar{p}'_2)(ig_{\gamma^0})\psi^{(-)}(\bar{p}_2)] \frac{1}{m^2} \Gamma_j(k) \\
+ [\bar{\psi}^{(+)}(\bar{p}'_1)(ig_{\gamma^0})\psi^{(-)}(\bar{p}_1)] [\bar{\psi}^{(+)}(\bar{p}'_2)(ig_{\gamma^j})\psi^{(-)}(\bar{p}_2)] \frac{1}{m^2} \Gamma_j(-k) - p'_1 \leftrightarrow p'_2 \right\},
\]

\[
R_{4}^{(2)} = \frac{1}{2\pi} \delta^{(3)}(p'_1 + p'_2 - p_1 - p_2) \\
\times \left\{ [\bar{\psi}^{(+)}(\bar{p}'_1)(ig_{\gamma^0})\psi^{(-)}(\bar{p}_1)] [\bar{\psi}^{(+)}(\bar{p}'_2)(ig_{\gamma^0})\psi^{(-)}(\bar{p}_2)] \frac{1}{m^2} \Lambda(k) - p'_1 \leftrightarrow p'_2 \right\},
\]

where

\[
\frac{1}{m^2} D_{ji}(k) = \frac{i}{k^2 - m^2 + i\epsilon} \left( -g_{ji} + \frac{k_j k_i}{m^2} - \frac{i}{m} \epsilon_{ji} k_0 \right),
\]

\[
\frac{1}{m^2} \Gamma_j(k) = \frac{i}{k^2 - m^2 + i\epsilon} \left( \frac{i\epsilon_{ji} k^j}{m} + \frac{k_j k_0}{m^2} \right),
\]

\[
\frac{1}{m^2} \Lambda(k) = \frac{i}{k^2 - \theta^2 + i\epsilon} \left( \frac{-k^i k_j}{m^2} \right),
\]

and

\[
k = p'_1 - p_1 = -(p'_2 - p_2)
\]
is the momentum transfer (our convention for the Fourier integral representation is \( f(x) = 1/(2\pi)^3 \int d^3k f(k) \exp(-ik \cdot x) \)). By substituting (4.3) into (4.1) and after taking into account (4.4) one arrives to

\[
R^{(2)} = \left( -\frac{g^2}{2\pi} \right) \delta^{(3)}(p_1' + p_2' - p_1 - p_2) \times \left\{ [\bar{v}^+(\vec{p}_1) \gamma^\mu v^-(\vec{p}_1)] [\bar{v}^+(\vec{p}_2) \gamma^\nu v^-(\vec{p}_2)] \frac{1}{m^2} D_{\mu\nu}(k) - p_1' \leftrightarrow p_2' \right\},
\]

where

\[
\frac{1}{m^2} D_{\mu\nu}(k) = -\frac{i}{k^2 - m^2 + i\epsilon} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} + i m \epsilon_{\mu\nu\sigma} \frac{k^\sigma}{m} \right).
\]

The amplitude in Eq. (4.6) is a Lorentz scalar. The theory has then passed the test on relativistic invariance. Also, one is to observe that \( D_{\mu\nu}(k)/m^2 \) in Eq. (4.6) is contracted into conserved currents and, hence, terms proportional to \( k^\mu \) can be added at will. Thus, we can replace \( D_{\mu\nu}(k)/m^2 \), given at Eq. (4.7), by

\[
\frac{1}{m^2} D^F_{\mu\nu}(k) = -\frac{i}{k^2 - m^2 + i\epsilon} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} + i m \epsilon_{\mu\nu\sigma} \frac{k^\sigma}{m} \right),
\]

which, up to contact terms, is the free field propagator arising from (2.12). In the tree approximation one is, therefore, allowed to replace all the non-covariant terms in \( H_D \) (see Eq. (3.10)) by the minimal covariant interaction \(-\frac{g}{m} \bar{\psi} D \cdot \gamma^\mu \psi D^\mu \).

We close this section by noticing that the high energy behavior of the propagator in (4.8) is radically different from that of the MCS theory in the Landau gauge [13,17],

\[
D^L_{\mu\nu}(k) = -\frac{i}{k^2 - m^2 + i\epsilon} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + i m \epsilon_{\mu\nu\sigma} \frac{k^\sigma}{k^2} \right).
\]

In fact, the SD model coupled to fermions is a non-renormalizable theory. This corroborates the arguments in the previous section, based on power counting.

**V. FINAL REMARKS**

The equivalence between the free SD and MCS models was first observed, at the level of the field equations, in Ref. [2]. As known [17], the Lagrangian density describing the dynamics of the free MCS theory,

\[
\mathcal{L}^{MCS} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m}{4} \epsilon^{\mu\alpha\sigma} F_{\mu\nu} A_\alpha,
\]

where \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \), leads to the equation of motion

\[
\partial_{\alpha} F^{\alpha\beta} + \frac{m}{2} \epsilon^{\alpha\mu\alpha} F_{\mu\alpha} = 0.
\]

It is now easy to see that the mapping
\[ F^\beta \equiv \frac{1}{2} \epsilon^{\beta \mu \alpha} F_{\mu \alpha} \rightarrow f^\beta , \]  

(5.3)

carries Eq.(5.2) into Eq.(2.9). In section 2 we proved that this equivalence holds rigorously at the quantum level. Since the SD field \( f^\beta \) is identified with the vector \( F^\beta \), dual of the tensor \( F^{\mu \nu} \), only the gauge invariant excitations of the MCS theory must show up in the SD model. As demonstrated in this work, this is the case.

The proof of equivalence of the quantized versions of the free SD and MCS models, given in this paper, was entirely carried out within the operator approach. One the main advantages of our presentation is that relativistic invariance is kept operational in all stages. This is to be compared with the outcomes in Refs. [4,6–8]. Frequently, the embedding of a second-class system in a larger phase-space, with the aim of making it first-class [5], implies in losing relativistic invariance [27].

The formulation of the interaction picture dynamics for the SD model minimally coupled to fermions was possible only after the elimination of the degree of freedom \( f^0 \), whose ETC’s with the fermionic fields do not vanish. As consequence, the Hamiltonian formulation did not retain any relic of relativistic covariance. Nonetheless, we proved that the non-covariant pieces in \( H^D_I \) are equivalent to the minimal covariant field-current interaction. The high energy behavior of the \( f \)-propagator signalizes that the coupled theory is non-renormalizable.

Thus, the SD model minimally coupled to fermions bears no resemblance with the renormalizable model defined by the MCS field minimally coupled to fermions. To give further support to this conclusion we recall that it has recently been shown [28] that the SD model minimally coupled to fermions is equivalent to the MCS model with non-minimal magnetic coupling to fermions. Also, for the fermionic sectors of the two theories to agree one is to add a Thirring like interaction in one of the models. These are, of course, non-renormalizable field theories.
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