Cultural transmission and optimization dynamics

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Abstract

We study the one-dimensional version of Axelrod’s model of cultural transmission from the point of view of optimization dynamics. We show the existence of a Lyapunov potential for the dynamics. The global minimum of the potential, or optimum state, is the monocultural uniform state, which is reached for an initial diversity of the population below a critical value. Above this value, the dynamics settles in a multicultural or polarized state. These multicultural attractors are not local minima of the potential, so that any small perturbation initiates the search for the optimum state. Cultural drift is modeled by such perturbations acting at a finite rate. If the noise rate is small, the system reaches the optimum monocultural state. However, if the noise rate is above a critical value, that depends on the system size, noise sustains a polarized dynamical state.

1 Introduction

Models of social dynamics are instrumental in studying mechanisms that lead from unorganized individual actions to collective social phenomena (Schelling 1971, Schelling 1978). In many cases the collective effects dominate in such a way that a reductionist view in terms of individual psychology might not be appropriate, with many detailed individual characteristics being possibly irrelevant for the collective macrobehavior of the system. One of the paradigms in these collective phenomena is the study of emergence of a social consensus or uniform state versus the emergence of a polarized state with different coexisting social options. Examples are found among models of segregation (Schelling 1971), opinion formation, dissemination of culture (Axelrod 1997a, Axelrod 1997b), general models of social influence (Latane, Nowak, and Liu 1994) and social dynamics (Epstein and Axtell 1996).

An important aspect in the issue of consensus versus polarization is the spatial distribution of individuals that determines the network of social interactions. Models that incorporate this ingredient through local social interactions often lead to a polarized state (Latane, Nowak, and Liu 1994, Axelrod 1997a, Axelrod 1997b): Uniformity is not reached in spite of local mechanisms of convergence. Given the analogy with cooperative phenomena in the Physical Sciences, “order parameters” have been introduced to give quantitative measures of the order emerging in the system (Lewenstein, Nowak, and Latane 1992, Latane, Nowak, and Liu 1994). These are global averaged variables useful to describe changes of macrobehavior in the system.

In most studies of these phenomena the individuals are characterized by one attribute with a two-fold option (black or white, majority or minority viewpoint, pros and cons of some issue, etc.). An exception to this dichotomous world is the Axelrod Culture Model (ACM) for culture transmission (Axelrod 1997a, Axelrod 1997b). In this model culture is defined as the set of attributes subject to social influence. Each individual is characterized by a set of $F$ cultural features, each of which can take $q$ values that represent the possible traits of each feature. In addition to treating culture as multidimensional, a novelty of the model is that its dynamics takes into account the interaction between the different cultural features. The basic premise of the model is that the more similar an actor is to a neighbor, the more likely the actor will adopt one of the neighbor’s traits. This similarity criterion for social influence is an example of social comparison theory in which individuals are mostly influenced by similar others. Processes of social influence and the origins of social networks have been reviewed by Lazer (2001). In the present paper we will restrict ourselves to the simplest situation studied by Axelrod. Namely, individuals are geographically distributed in the sites of a regular grid and they interact with their immediate neighbors according to the similarity criterion.

The ACM illustrates how local convergence can generate global polarization. In a typical dynamical evolution the system freezes in a multicultural state with coexisting spatial domains or clusters of different culture. The number of these domains is taken as a measure of cultural
diversity. It is interesting to notice that it is precisely the similarity criterion that leads to polarization and stops the evolution towards a uniform monocultural state. Indeed, if similarity is not taken into account to weight the probability of social interaction, the system always reaches the consensus or uniform state (Kennedy 1998). Axelrod himself explored how the number of domains in the final state changes with the scope of cultural possibilities given by \( F \) and \( q \), with the range of the interactions and with the size of the system. The robustness of the predictions of the model has been checked by aligning it with the Sugarscape model of Epstein and Axtell (1996). In addition, the ACM has been extended in a number of ways, which include its use as an algorithm for optimizing cognition (Kennedy 1998) and a study with a gradual increase of the range of interaction (Greig 2002), which suggests that the increase in communications promotes the emergence of a global but hybrid culture rather than imposing initially dominant cultural features. The effect of mass media in the cultural evolution has also been incorporated in the ACM (Shibanai, Yasuno, and Ishiguro 2001), with the surprising result that mass media promotes cultural diversity. However, this result, as some of the others mentioned above, was obtained for a fixed set of values of the number of features \( F \) and traits \( q \). It is likely that for a different value of \( q \) the effect of mass media could be the contrary. A systematic analysis of the dependence of \( q \) on the original ACM was carried out from the point of view of Statistical Physics by Castellano, Marsili, and Vespignani (2000) through extensive numerical simulations. Defining an order parameter as the relative size of the largest homogeneous cultural domain, these authors unveil an order-disorder transition: There exists a threshold value \( q_c \) such that for \( q < q_c \) the system orders in a monocultural uniform state, while for \( q > q_c \) the system freezes in a polarized or multicultural state. This result partially modifies the original conclusions of Axelrod, in the sense that consensus or polarization is determined by a parameter \( q \) which measures the degree of initial disorder in the system.

Cultural evolution might be thought of as an optimizing process. For instance, it can be argued that in social comparison theory individuals seek to optimize their social integration (Lazar 2001). The ACM itself, and extensions thereof, have been shown to be able to optimize complex functions (Kennedy 1998), suggesting that social interaction is an optimization process. Likewise, the dissemination of technological innovations can be also seen as the search for an optimum, but in which the system can lock-in a suboptimal state (Leydesdorff 2001), as in the well known example of the QWERT-keyboard. From this perspective one might think that the consensus or monocultural uniform state is an optimum state, while the polarized multicultural state represent suboptimal states in which the system gets trapped. To make such ideas on optimization dynamics quantitative one needs to have a Lyapunov potential (Guckenheimer and Holmes 1983) for the system dynamics. The Lyapunov potential is a functional of the configuration of the system such that it can only decrease or remain equal during the dynamical evolution of the system. Although having a Lyapunov potential does not determine the dynamics of the system (Montagut, Hernández-García, and San Miguel 1996, San Miguel and Toral 2000), it can be stated quite generally that the system will evolve minimizing the potential until it is trapped in an attractor of the dynamics.

In this paper we address the question of the ACM as a dynamical optimization process by constructing a Lyapunov potential for the system. The potential is only shown to exist for a one-dimensional system: individuals are distributed at regular intervals along a line. This simplest geographical set-up allows us to discuss and make clear most of the concepts and mechanisms also occurring in higher dimensional systems. Considering one-dimensional systems for clarity of concepts is within the tradition of studies of models of social dynamics (Schelling 1971). In addition, this one dimensional configuration was also considered by Axelrod to exemplify dialect dynamics. We show that the global minimum of the Lyapunov potential for the one-dimensional ACM model is the uniform monocultural state. In addition, we show that the potential has no other local minima. The other attractors of the dynamics, corresponding to multicultural states, are shown to have nearby configurations of the same or lower value of the potential. This implies that when the system is trapped in one of these states for \( q > q_c \), any small perturbation will take the system away from the multicultural attractor and the optimization process will continue. Such perturbation can be seen as the effect of cultural drift and this result answers the question posed by Axelrod (Axelrod 1997a, Axelrod 1997b): “Perhaps the most interesting extension and, at the same time, the most difficult to analyze is cultural drift”. In this sense, cultural drift, against the naive expectation of promoting differentiation, is an efficient mechanism to take the system to the optimum uniform monocultural state. The result is reminiscent of the effect of randomness (named temperature) in the studies of social impact theory (Latane, Nowak, and Liu 1994) which was also shown to increase the self-organization tendencies in the system.

The paper is organized as follows. Section 2 reviews the formal definitions involved in the ACM model. Section 3 describes the main features of the ACM in a one dimensional world, including the classification of dynamical attractors and the order-disorder transition observed for a threshold value of \( q \). In section 4 we introduce the Lyapunov potential and we use it to characterize the order-disorder transition and the stability of the multicultural or polarized attractors of the dynamics. Any perturbation acting on these states is shown to take the system to the uniform monocultural state. Section 5 is devoted to a discussion of cultural drift. Concluding remarks and an outlook of this work is given in section 6.
2 Axelrod model

The model we study is defined (Axelrod 1997a, Axelrod 1997b) by considering $N$ individuals or agents as the sites of a network. The state of agent $i$ is a vector of $F$ components (cultural features) $(\sigma_{i1}, \sigma_{i2}, \cdots, \sigma_{if})$. Each $\sigma_{if}$ is one of the $q$ integer values (cultural traits) $1, \ldots, q$, initially assigned independently and with equal probability $1/q$. The time-discrete dynamics is defined as iterating the following steps:

1. Select at random a pair of sites of the network connected by a bond $(i,j)$.
2. Calculate the overlap (number of shared features) $l(i,j) = \sum_{f=1}^{F} \delta_{\sigma_{if},\sigma_{jf}}$.
3. If $0 < l(i,j) < F$, the bond is said to be active and sites $i$ and $j$ interact with probability $l(i,j)/F$. In case of interaction, choose $g$ randomly such that $\sigma_{ig} \neq \sigma_{jg}$ and set $\sigma_{ig} = \sigma_{jg}$.

In any finite network the dynamics settles into an absorbing state, characterized by the absence of active bonds. Obviously all the completely homogeneous configurations are absorbing. Homogeneous means here that all the sites have the same value of the cultural trait for each cultural feature. Inhomogeneous states consisting of two or more homogeneous domains separated by inactive bonds with zero overlap are absorbing as well. A domain is here defined by a set of sites connected by bonds.

3 A one-dimensional world

We consider the case of a one-dimensional lattice formed by $N$ agents with first neighbors interaction with open boundary conditions. Each agent $i$ can only interact with his right $i+1$ and left $i-1$ neighbors. We define a cultural domain as a contiguous set of agents with the same cultural traits for all the features. Then the system settles in an absorbing state consisting of cultural domains separated by bonds with no overlap. These constitute the barriers through which no interaction occurs. Thus the absorbing states can be classified according to the number of barriers or equivalently by the number of cultural domains. A polarized or multicultural configuration corresponds to an absorbing state containing several cultural domains while a uniform or monocultural state is formed by a single culture spanning along all the sites of the lattice.

Extensive numerical simulation show that in not too small systems only monocultural or extremely polarized configurations are reached. This behavior is captured by an order parameter defined here as the relative size of the largest homogeneous cultural domain $S_{\text{max}}/N$. Clearly, if this quantity is unity, one culture spans the whole system, corresponding to a monocultural state. On the other hand, if none of the cultural domains reaches a size that is visible on the scale of the system size, $S_{\text{max}} \ll N$, the configuration is extremely polarized. Then an agent shares his cultural attributes with only a small neighborhood.

Figure 1 shows, for $F = 10$, the values of the average order parameter $\langle S_{\text{max}} \rangle/N$ in the final absorbing state as a function of the number of available traits $q$. For $q < 8$ we always find a monocultural absorbing state. Increasing $q$ beyond 8, $\langle S_{\text{max}} \rangle/N$ drops towards zero, the more rapidly the larger the system, indicating the existence of a transition for $q \approx 8$. This change of behavior between monocultural and polarized states is emphasized when looking at the outcomes of the realizations themselves (without averaging) in Fig. 2. We observe that this transition is not accompanied by a regime of bistability close to $q_c$. This means that it does not exist a finite range of $q$-values for which a similar number of realizations finish either in the monocultural or in the multicultural state. The absence of bistability suggests that the transition can be classified as continuous, while a similar type of transition observed in two-dimensional lattices is accompanied by a bistable regime (Castellano, Marsili, and Vespignani 2000, Klemm, Eguíluz, Toral, and San Miguel 2002b), indicating that in general the transition is discontinuous or first order.

It is important here to note that the control parameter $q$ that governs this transition or change of behavior is not a parameter that can be tuned in a given system. Rather it enters in the definition of the system, and therefore the transition corresponds to a change of behavior in a class of systems that we explore by changing $q$. On the other hand, the dynamic rules do not change with $q$ and the crucial way through which $q$ enters in the dynamic evolution is in the initial condition. We have chosen random initial conditions with a uniform probability distribution for the value taken by each feature. With this choice, $q$
gives a measure of initial disorder. The transition that we discuss refers to an average behavior, the average being taken over an ensemble of such initial conditions, and the value of \(q_c\) reflects this choice of initial conditions. For initial conditions with large initial disorder \(q > q_c\) the system freezes in a multicultural configuration, while for a small initial disorder \(q < q_c\) the system reaches the monocultural state. If a different choice of random initial conditions is made, for example taking a Poisson distribution (Castellano, Marsili, and Vespignani 2000), the value of \(q_c\) changes. Of course, if the ensemble of initial conditions were restricted, for instance, to homogeneous states, no transition would be observed.

4 Lyapunov potential

In the one-dimensional version of the model the dynamics can be described in terms of a Lyapunov potential. A function of the state of the system \(L(\{\sigma\})\) is a Lyapunov potential if its value does not increase during the dynamical evolution. If \(L(t)\) represents the Lyapunov potential of the system at time \(t\), then \(L(t) \geq L(t+1)\). We state that the negative total overlap

\[
    L = - \sum_{i=1}^{N} l(i, i+1)
\]

is a Lyapunov potential of the one-dimensional Axelrod model.

**Proof:** We have to show that the negative total overlap cannot be increases by an interaction\(^1\). At a given time step the bond \((i, i + 1)\) is selected so that agent \(i\) acquires one of the traits of agent \(i + 1\). Then the overlap across that bond increases by one unit. For the overlap across the other bond \((i-1, i)\) of site \(i\) there are three possibilities (see Fig. 3):

1. It is the same as before, if the acquired trait is shared by agent \(i - 1\) or the discarded trait was shared by agent \(i - 1\). Then \(L(t+1) = L(t) - 1\) (Fig. 3a),

2. it increases by one unit, if the acquired trait is also shared by the agent \(i - 1\). Then \(L(t+1) = L(t) - 2\) (Fig. 3b), or

3. it decreases by one unit, if the change occurred with respect to one of the shared traits with \(i - 1\). Then \(L(t+1) = L(t)\) (Fig. 3c).

Thus an interaction the value of \(L\) will be less than before or the same as before. Taking into account that all other terms in \(L\) do not vary, we find that in any interaction, \(L\) never increases. **End of proof.**

It is also convenient to relate the Lyapunov potential to the number of bonds \(n_k\) with overlap \(k\):

\[
    L = - \sum_{k=0}^{F} n_k k
\]

with \(\sum_{k=0}^{F} n_k = N\). The absorbing configurations correspond to the case \(n_k = 0\), for \(0 < k < F\). For these configurations we obtain

\[
    L_{\text{absorbing}} = -n_F F = -(N - n_0) F
\]

where \(n_0\) is the number of bonds with zero overlap, that is the number of barriers. Therefore, the absorbing states can be ordered according to the number of barriers. In the monocultural homogeneous states all the bonds have overlap \(F\) and thus \(n_k = 0, \forall k \neq F\) and \(n_F = N\). They

\[^1\]If there is no interaction, the state of the system does not change and then the Lyapunov potential remains unchanged.

\[^2\]In the example in Fig. 3, \(F = 3\) and \(\sigma_{i+1}(t+1) = \sigma_{(i+1)2}(t) = 8\)

Figure 2: Scatter plot of the order parameter in one-dimensional lattices as a function of \(q\) for system size \(N = 1000\) and \(F = 10\) features. For each value of \(q\) the outcome of 100 independent runs is plotted.

Figure 3: Three possible outcomes of an interaction between agents \(i\) and \(i + 1\) for a system with \(F = 3\) features and \(q = 10\). Shared features are indicated by grey background. The trait of feature \(\sigma_{i3} = 3\) is switched to \(\sigma_{i3} = \sigma_{(i+1)3} = 8\). The new acquired trait by agent \(i\) increases the overlap with its \(i + 1\) neighbor but (a) has no effect on \(i - 1\) \([L(t+1) = L(t) - 1]\); (b) increases the overlap with \(i - 1\) \([L(t+1) = L(t) - 2]\); (c) decreases the overlap with \(i - 1\) \([L(t+1) = L(t)]\).
correspond to the absolute minima of the Lyapunov potential or optimum states with \( L_0 = -NF \). There is a multiplicity of these minima corresponding to the \( \eta_0 = q^F \) equivalent different cultures \( (\sigma_i \neq \sigma_j \forall i, j \text{ and } f) \), characterized by the combination of cultural traits. It is important to notice that they are equivalent. Which of the optimum states is selected for \( q < q_c \) depends on the initial conditions and the stochastic realization of the dynamics.

Inhomogeneous multicultural states consisting of two or more homogeneous domains separated by barriers are absorbing as well. We can order the multicultural absorbing configurations according to their Lyapunov potential. The first absorbing configurations (different from the monocultural states) in potential correspond to the coexistence of two different cultural domains separated by one barrier. There are \( \eta_1 = [q(q - 1)]^FN \) configurations with a value of the Lyapunov potential \( L_1 = -(N - 1)F \). In this case \( n_0 = 1, n_F = N - 1 \) and all other \( n_k = 0 \). The next level corresponds to three cultural domains (and two bonds of zero overlap), with a potential \( L_2 = -(N - 2)F \) \( (n_0 = 2, n_F = N - 2 \text{ and all other } n_k = 0 \) and \( \eta_2 = [q(q - 1)]^2F(N(N - 1)/2 \) equivalent configurations. In general an absorbing state with \( K + 1 \) cultural domains and \( K \) barriers will have

\[
L_K = -(N - K)F \quad (4)
\]

and

\[
\eta_K = [q(q - 1)^K]^F \left( \frac{N}{K} \right) \quad (5)
\]

equivalent configurations.

The properties of the Lyapunov potential can give valuable insight into the dynamics of the system. For instance, the minima of the Lyapunov potential are absorbing states of the dynamics. However the opposite is not true. As the dynamics never increases the potential, once a minimum is reached the dynamics stops there because any neighboring configuration has a larger potential. However, an absorbing configuration can have neighboring configurations with lower or equal potential (see Fig. 2 described below). The reason why the dynamics stops in such absorbing states is not included in the potential but in the dynamical rules.

Having characterized the absorbing configurations in terms of the Lyapunov potential now we turn our attention to the description of the transition in terms of the potential. In order to facilitate systematic comparison between systems with different sizes \( N \) and different numbers of features \( F \), we use the normalized Lyapunov potential \( \rho = (L - L_0)/NF = L/NF + 1 \) with values ranging from zero to unity. In absorbing configurations \( \rho \) is simply the density of barriers. For instance, \( \rho = 6/100 \) for a configuration with 6 barriers in a system of size \( N = 100 \).

In Fig. 3 we show the average value of \( \rho \) in the absorbing state reached by the dynamics. The average is take over 100 simulations for each data point. We observe that, as a function of \( q \), the Lyapunov potential increases continuously from the value of a monocultural configuration, incorporating barriers, and thus increasing the potential and \( \rho \) as \( q \) increases. It is apparent in this figure that a change of behavior between the monocultural state with \( \langle L \rangle = -NF, \rho = 0 \) and the multicultural states occurs for \( F \approx q_c \), in agreement with the result in Fig. 1. This change of behavior also manifests itself in the difference between the value of \( \rho \) in the initial random configuration and the final absorbing state. For the average over our set of random initial conditions \( \rho = 1 - 1/q \). The difference with the final value shows a maximum for \( F \approx q_c \) (see inset of Fig. 1). We have checked that the value of the maximum increases linearly with \( F \). The fact that \( F \) and \( q \) are not two independent relevant parameters, but that rather the scaling parameter \( F/q \) is the proper one to describe these phenomena is made more explicit in Fig. 2. In this figure the x-axis has been rescaled as \( q/F \) for the same data as in Fig. 1. We observe a scaling phenomena in the sense that \( q_c \) is seen to be a function only of \( q/F \). These results, together with Fig. 1, suggest the existence of a transition for \( q = q_c \approx F \). To make this statement rigorous we have to find some singular behavior. We find such behavior in the dynamical evolution of the normalized Lyapunov potential \( \rho(t) \) (Fig. 1). In the early steps of the dynamical evolution \( \rho(t) \) remains constant or decays slowly for a large number of iterations. For values of \( q \) below the transition point \( q_c \) the density decays as \( \rho(t) \sim t^{-0.5} \). For \( q \) above the transition point \( q_c \), \( \rho \) saturates at a finite value. For \( q > q_c \), an absorbing state is reached after a time span several orders of magnitude shorter than in the case \( q < q_c \). This clearly identifies the continuous transition from a monocultural to a multicultural state for a critical value of \( q_c = q_c \approx F \).

Is the Lyapunov potential useful in understanding why the system is trapped in a multicultural configuration for
Figure 5: The data from Fig. 4 with $N = 1000$ and $F = 10, 20, 30$ collapse when plotted as a function of the rescaled parameter $q/F$.

$q > q_c$? Due to the multiplicity of configurations with a given a value of the potential one might conjecture that the system is trapped in an absorbing configuration which is close to the initial configuration, making a “short excursion”. Consequently, the difference of the Lyapunov potential to the random initial and the absorbing configuration should be small. However, we observe that it assumes large values and has a peak at exactly the transition point $q_c$ (see inset in Fig. 4). This fact rules out the argument that the excursion is short in potential. In particular it is not true that when the system reaches the optimum state is because the initial condition has a value of the Lyapunov potential close to the optimum state. In fact for $q \leq q_c$, when the system reaches the optimum state, the excursion in value of the potential is much larger than for $q \gg q_c$. What this means is that the dynamics is not gradient (Montagne, Hernández-García, and San Miguel 1996; San Miguel and Toral 2000). It does not follow the trajectory of the steepest descent of the potential. This represents a genuine non-equilibrium dynamics which cannot be completely described just by the optimization of a potential. In addition, in the case considered in this paper there are entropic contributions (the degeneracy of equivalent configurations with the same potential) which are also at play. Even though the dynamics is not fully determined by the potential, the potential is very useful in understanding the stability properties of the absorbing states.

The homogeneous configurations are local and at the same time global minima of $L$. They are the optimum states and deviating from these in any local step in configuration space, i.e. assigning a different trait to one of the features of an agent, always increases $L$. All other absorbing configurations are neither local nor global minima: there are always neighboring configurations with the same or a lower value of $L$. From these configurations, one can always make a local step to a non-absorbing state. From the so reached non-absorbing state the dynamics does not necessarily return to the original absorbing state. The dynamics does not drive spontaneously the system towards an adjacent disordered absorbing configuration when the system is trapped in an absorbing state. These excursions are caused by exogenous perturbations that randomly flip a cultural trait. In terms of the classification of stationary states the absorbing monocultural states are stable. Properly, they are unstable since a perturbation can take them to a state of higher potential and from there to an equivalent monocultural state which is also optimum. The absorbing multicultural states are not meta-stable. There exist perturbations of the smallest size (a change in a single trait of an agent) that take the system to a configuration of the same or lower value of the potential. In the first case we talk about marginally stable states and in the second on unstable states. Therefore the disordered absorbing configurations are meta-stable.

We are going to illustrate these properties with the help of Fig. 4. For the sake of concreteness we consider 11 agents interacting through $N = 10$ bonds with $F = 3$ and $q = 10$. For this case we know that $q > q_c \simeq 3$. Thus the system should reach a disordered configuration. The overlap between the state of the agents in the initial condition is indicated in the figure giving $L = -3$ in accordance with the average estimation $\langle L \rangle = -NF/q = 3$. The second row (b) shows the final configuration after the evolution of the system. The multicultural configuration is formed by 7 cultural domains (indicated in the figure) having 6 barriers. Thus the Lyapunov potential of this absorbing configuration is $L = -(10 - 6)3 = -12$. Now an exogenous perturbation switches $\sigma_{62}$ (indicated in bold in the third row) activating the bond with its right neighbor. Note that this perturbation has not changed the Lyapunov potential. Thus it is a neighboring configuration (c) with the same Lyapunov potential: if we now let the system relax, it reaches an equivalent configuration (7 domains) with the same potential (d). Thus an exogenous perturbation has lead the system to another equivalent configuration without a modification of the potential. However, there are other perturbations that can decrease the potential as indicated in the next row. The new perturbation $\sigma_{32}$ activates two bonds and then $L = -14$ (e). Therefore the absorbing state (d) is unstable. As the potential cannot increase the new evolution cannot recover the previous potential level. Instead an absorbing configuration with lower potential is reached (f). This configuration reached is composed of 5 domains and has Lyapunov potential $L = -18$. One may expect that by repetition of these cycles of perturbation-relaxation the number of domains and the Lyapunov potential are reduced further until a homogeneous configuration is reached.

In order to analyze the consequences of the lack of stability of the absorbing multicultural states, we have devised simulations of the model including exogenous perturbations. The absorbing states are subject to single feature perturbations, defined as randomly choosing
For this particular realization $L = -3$ in accordance with the average estimation. The overlap is indicated by the horizontal lines. (b) The dynamics leads to an attractor which is a multicultural configuration ($q > q_c = 3$) formed by 7 cultural domains indicated by the bar on the top. For this configuration $L = -12$. (c) An exogenous perturbation switches on of the traits (indicated in bold). The Lyapunov does not change but it opens the possibility to interact with its right neighbor. (d) After the evolution the new absorbing configuration is an equivalent configuration (it also contains 7 cultural domains) with the same value of the Lyapunov potential. (e) A new perturbation switches one trait that decreases the potential. (f) The system cannot come back to a state with higher potential but reaches a new configuration of 5 cultural domains with $L = -18$.

Figure 7: (a) Initial condition for a system formed by $N = 10$ agents, with $F = 3$ features, and $q = 10$. The average potential for the initial condition is $\langle L \rangle_0 = -NF/q$. For this particular realization $L = -3$ in accordance with the average estimation. The overlap is indicated by the horizontal lines. (b) The dynamics leads to an attractor which is a multicultural configuration ($q > q_c = 3$) formed by 7 cultural domains indicated by the bar on the top. For this configuration $L = -12$. (c) An exogenous perturbation switches on of the traits (indicated in bold). The Lyapunov does not change but it opens the possibility to interact with its right neighbor. (d) After the evolution the new absorbing configuration is an equivalent configuration (it also contains 7 cultural domains) with the same value of the Lyapunov potential. (e) A new perturbation switches one trait that decreases the potential. (f) The system cannot come back to a state with higher potential but reaches a new configuration of 5 cultural domains with $L = -18$. 

Figure 6: Time evolution of the normalized Lyapunov potential for $F = 10$ and $q = 2, 5, 7, 8, 9, 10, 11, 12, 15$ (solid curves, bottom to top) in systems of size $N = 10000$. For $q < q_c = 8$ the normalized Lyapunov potential approaches zero according to a power law. The dashed line has slope $-0.5$.

In other words, whenever an absorbing configuration has been reached, we measure $L$ and $S_{\text{max}}$, perform a perturbation and restart the dynamics from the perturbed configuration. This mimics the effect of a random influence on the system which acts much more seldomly than the dynamics of cultural imitation in the original model. We find that in this case the system is driven to complete order, i.e. $L$ gradually decreases to the minimum value $-NF$ and $S_{\text{max}}$ gradually increases to the maximum value $N$. For a typical simulation run, Fig. 6 displays the evolution. One observes that the normalized Lyapunov potential decays exponentially. Recalling that in absorbing states the normalized Lyapunov potential is simply the fraction of bonds that constitute a barrier, we see that the number of barriers decreases exponentially. The probability for a given barrier to vanish during a perturbation cycle is constant, i.e. it does not depend on the number of barriers present in the system. Barriers dissolve independently.

From the explicit time evolution of all the barriers in the system (upper panel of Fig. 6) it is also apparent that no new barriers are created. This can be understood geometrically: Let us first consider only one arbitrary feature $f$. An interface within the feature $f$ is a bond $(i, i+1)$ with disagreeing traits $\sigma_{i,f} \neq \sigma_{i+1,f}$. The dynamics does
not create new interfaces. When agent $i$ adopts the trait of agent $i + 1$, the interface merely moves from $(i, i + 1)$ to $(i - 1, i)$. If the latter bond has been an interface already, the two interfaces either merge or annihilate. Considering the whole system, in an absorbing state the interfaces are the same for all features. Trivially, all features have exactly the same number of interfaces. In order to increase the number of barriers by perturbation and subsequent relaxation the number of interfaces would equally have to increase in all features. However, all but one feature (the one in which the perturbation is performed) are guaranteed not to increase the number of interfaces, as shown before. So the number of barriers and the number of cultural domains do not increase. This again proves that only configurations without barriers are stable. Once such a homogeneous configuration has been reached, the perturbations cannot drive the system to a different absorbing configuration. In consequence, all but the completely ordered absorbing configurations are not stable, meaning that minimal perturbations drive the system away from these states.

We mention again that the system is always allowed to relax to an absorbing configuration before a perturbation is performed. However, when perturbations occur simultaneously with the original dynamics, their effects may accumulate. At a sufficiently large rate of perturbations this may result in a disordered system with many cultures. On the other hand, for very low rate of perturbations, the scenario will be close to the alternating perturbation and relaxation studied in this section above, resulting in a homogeneous system. The following section is dedicated to the study of this case of ongoing perturbations at different rates.

5 Cultural drift

In this section we address the role that cultural drift has on the behavior of Axelrod’s model. The previous section has shown that even infinitesimal noise has a non–trivial effect and, therefore, we expect that cultural drift, modelled as random perturbations acting at a constant rate $r$, will have a relevant role in the model. To be more specific, we implement cultural drift by adding a fourth step in the iterated loop of the model defined in Section 2:

4. With probability $r$, perform a single feature perturbation.

This is intended to be a more realistic effect of uncertainty in the agent’s behavior. As this kind of noise acts continuously on the dynamics, the difference with the scenario discussed in previous section is that the system is not necessarily in an absorbing configuration when a perturbation occurs. Therefore, it is not straightforward to generalize the previous results based on the existence of fixed points of the dynamics and their stability properties, since the dynamics is not allowed to relax to them before a perturbation acts. We will see, however, that a simple argument based on the random walk, is able to give us some quantitative predictions.

We first show the results of the numerical simulations of the model modified to take into account the cultural drift. Figure 8 shows the variation of the order parameter $\langle S_{\max} \rangle/N$ with the noise rate $r$ for different system sizes. As expected, disorder appears for sufficiently large noise rate $r$. The exact location of the transition point strongly depends on the system size $N$, but it is only weakly dependent on the number of traits $q$. In fact, the variation of $q$ from $q = 5$ to $q = 50$, which in the absence of noise or perturbations lead to qualitatively different outcomes (remember that $q_c \approx 10$ in that case), causes an almost negligible shift of the transition towards slightly lower values of the noise rate.

Although at a sufficiently low rate $r$ the situation might appear to be close to the case of alternating perturbation and relaxation studied in previous sections, we must stress that there is an essential difference: for noise acting continuously the system can explore continuously nearly
homogeneous regions (with a high value of the order parameter $\langle S_{\text{max}} \rangle / N$) by jumping from one region to another at a time scale that grows with $N$. This reflects the metastability of the different equivalent optimum states (homogeneous cultures). We can give an intuitive explanation of the existence of a transition from ordered to disordered states in the presence of cultural drift: if the noise rate is such that the typical time $1/r$ between perturbations is shorter than the average relaxation time $T$, the effect of the perturbations adds up in the system and disorder appears. The system is in a polarized noisy dynamical state. On the contrary, if the noise rate is small, it becomes efficient in taking the system to explore nearby configurations of lower potential and the optimization dynamics proceeds escaping from absorbing states. The minima of the potential is then reached and a monocultural state emerges. This simple picture tells us that disorder will set in when the average relaxation time $T$ of perturbations of a homogeneous state satisfies $rT = O(1)$.

It is possible to introduce an approximate argument for the calculation of $T$. Imagine a completely ordered state as the initial condition at $t = 0$. A single feature perturbation of this state induces a “damage” of size $x(t = 0) = 1$ in one of the features. In the following time steps the damage may spread until an ordered state is reached again by $x(t) = 0$ or $x(t) = N$. Therefore we can envisage the system as a damage cluster and an undamaged background separated by 2 active bonds (interfaces). These interfaces execute a random walk type of diffusion and the average time needed for them to merge in such a way that an ordered region spans the whole system is well known (Grimmett and Stirzaker 1982) to scale as

$$T \sim N^2,$$

so that the average relaxation time of perturbations diverges quadratically with increasing system size.

This result is confirmed, see Fig. 9, by showing that the data of Fig. 9 collapse into a single curve when plotted as a function of a rescaled noise rate $rN^2$, which incorporates noise rate $r$ and system size $N$. This shows indeed that for increasingly larger system sizes, a vanishingly small noise rate can alter dramatically the behavior, showing that cultural drift, as modelled by continuous random perturbations, has a relevant role in the behavior of Axelrod's model.

Finally, notice that our argumentation based on relaxation times of perturbations so far does not involve the value of $q$ and it may explain the weak $q$–dependence of the system in the presence of noise.

## 6 Conclusions and Outlook

We have shown that the ACM, a model of cultural transmission, in a one-dimensional world can be understood as an optimization process in which the global uniform state is the optimum state corresponding to the global minimum of a Lyapunov potential. When the initial cultural diversity is large enough the system freezes in an attractor of the dynamics. The system can always escape from these attractors by any small perturbation, since there are always nearby configurations with the same or lower value of the potential. Cultural drift gives rise to such perturbations, and therefore it is an instrument to promote cultural globalization giving to the system the necessary input to proceed in the optimization dynamics. However, if cultural drift acts at high enough rate it leads to a noisy polarized dynamical state.

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3We are assuming here that the system has periodic boundary conditions. The role of the boundaries should be negligible for sufficiently large system size $N$. 

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Figure 9: Dependence of the relative size of the largest cultural domain with noise rate $r$ in one-dimensional lattices of size $N = 100$ (circles), $N = 1000$ (squares), $N = 10000$ (diamonds), for $q = 5$ (filled symbols) and $q = 50$ (open symbols). Agents have $F = 10$ features.

Figure 10: Scaling of the relative size of the largest cultural domain in one-dimensional lattices. Symbols as in Fig. 9. $q = 50$. 
As we have previously mentioned, we have only found a Lyapunov potential for the one-dimensional version of the ACM. However, most of our qualitative findings persist in simulations in a two-dimensional regular network (Klemm, Eguíluz, Toral, and San Miguel 2002a). In the two-dimensional world there is a transition between the uniform and multicultural states for a critical value of $q$, but the attractors are not easily classified in terms of the number of barriers, and the uniform state is not known to be the global minimum of a potential. The dynamical stability of the other attractors is also unknown. Still, simulations indicate that perturbations acting on the polarized states take the system to the uniform state. In the presence of cultural drift there is also a transition from uniform states to a polarized multicultural state controlled by the noise rate.

Our discussion has been here restricted to regular networks with interactions between nearest neighbors. However, social networks are known to be in many cases different from regular or random networks. The question of the influence of network topologies reflecting social cleavages was already posed by Axelrod (Axelrod 1997a, Axelrod 1997b). Two types of networks very much studied recently are the small world networks (Watts and Strogatz 1998), representing an intermediate situation between regular and random networks, and the scale free networks (Barabási and Albert 1999), characterized by a power law tail in the probability distribution for the number of bonds connecting a site in the network. Such power law indicates the presence of few sites with a very large number of links. Simulations of the Axelrod model in these networks (Klemm, Eguíluz, Toral, and San Miguel 2002b) indicate that the small world connectivity favors cultural globalization, in the sense that the value of $q_c$ for the transition to a polarized multicultural state is larger than in the regular network. A maximum value of $q_c$ is obtained for the random network, but the scale free connectivity is still more efficient than the random connectivity in promoting global culture, giving a larger value of $q_c$. In fact this value depends on the system size, and in the limit of a very large system size, the system reaches the uniform multicultural state for any value of $q$. The interesting unsolved question so far is to take into account that if the cultural evolution of the individual is molded by the network of social interactions, the network is also constructed by the individuals. Properly the network can not be taken as given and fixed (Lazer 2001). Such co-evolution of individual culture and social network could be modelled similarly to studies of cooperation in which the social network emerges from the results of the dynamics of cooperation (Zimmermann, Eguíluz, and San Miguel 2001).

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