A Class of Sixth Order Viscous Cahn-Hilliard Equation with Willmore Regularization in $\mathbb{R}^3$

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Received: 12 August 2020; Accepted: 16 October 2020; Published: 26 October 2020

Abstract: The main purpose of this paper is to study the Cauchy problem of sixth order viscous Cahn–Hilliard equation with Willmore regularization. Because of the existence of the nonlinear Willmore regularization and complex structures, it is difficult to obtain the suitable a priori estimates in order to prove the well-posedness results, and the large time behavior of solutions cannot be shown using the usual Fourier splitting method. In order to overcome the above two difficulties, we borrow a fourth-order linear term and a second-order linear term from the related term, rewrite the equation in a new form, and introduce the negative Sobolev norm estimates. Subsequently, we investigate the local well-posedness, global well-posedness, and decay rate of strong solutions for the Cauchy problem of such an equation in $\mathbb{R}^3$, respectively.

Keywords: sixth order Cahn–Hilliard equation; local well-posedness; global well-posedness; decay estimates

1. Introduction

The phase-field method is a powerful tool for studying the dynamics of heterogeneous materials, such as phase separations in binary mixtures, crystal faceting, epitaxial thin film growth, and multi-phase fluid flows, just to name a few. When compared with the sharp-interface approach, the significant advantage of phase-filed method is that there is no need to track the interface explicitly. Hence, this methods are capable of describing the evolution of complex, morphology-changing surfaces, and they can provide a general framework for considering more physical effects. Usually, the second order Allen–Cahn equations and fourth order Cahn–Hilliard equation, which are sub-classes of phase field models, can be obtained by phase-field approaches and are of interest in materials science.

In [1], the authors pointed out that crystalline anisotropy is a critical contributing factor to the equilibria and dynamic macroscopic shapes of heterogeneous materials. If there is no anisotropy, the microscopic shape is rotationally symmetric. Anisotropy breaks this symmetry as certain directions are endowed with higher energy. Particularly, for sufficiently strong anisotropy, the double-well surface density function may become negative. At equilibrium, the system responds by removing these orientations in the shape of the crystal (Wulff shape) driven by minimizing the total surface energy of the system. As a result, the equilibrium interface is no longer a smooth curve, but it presents sharp corners or facets with slope discontinuities [2]. Consider the three-dimensional phase-filed model, if the gradient energy is non-convex, then the anisotropic phase field model becomes ill-posed. In order to overcome loss of smoothness and ill-posedness of the anisotropic Cahn–Hilliard model, a higher order derivative regularization can be added to the surface energy. Recently, some authors [1,3–5] introduced the Willmore regularization to the Ginzburg–Landau free energy, proposed the following modification of the free energy

$$E(u) = \int \left[ \gamma (v) \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) + \frac{\beta}{2} \omega^2 \right] dx,$$

(1)
where \( u \) is the order parameter, \( \omega = F'(u) - \Delta u \) is the nonlinear Willmore regularization, \( \gamma(v) \) is a function describing the anisotropy effects, \( \beta \) is the bending rigidity, \( \nu = \nabla u / |\nabla u| \), and the nonlinear function \( F(s) = \frac{1}{4}(s^2 - 1)^2 \).

There are some numerical results related to the Cahn–Hilliard model associated with the free energy (1) (see e.g., [1,3,6,7] and the reference therein). However, because of the lack of the anisotropy effects, it is difficult to establish the uniform a priori estimates and study the properties of solutions from the point of mathematical analysis. The effect of anisotropy has to be weakened if we want to establish some mathematical results on the solutions. Actually, Makki and Miranville [8,9] considered a modified free energy, namely,

\[
E(u) = \int_{\Omega} \left( \frac{1}{2} \gamma(v)|\nabla u|^2 + F(u) + \frac{\beta}{2} \omega^2 \right) dx,
\]

(2)

studied the existence and uniqueness of the Cahn–Hilliard model associated with (2). Moreover, assuming isotropy, i.e., \( \gamma(v) \equiv 1 \), we obtain the following modified free energy

\[
E(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + F(u) + \frac{\beta}{2} \omega^2 \right) dx,
\]

(3)

and the associated viscous sixth-order Cahn–Hilliard system

\[
\begin{align*}
    u_t &= \nabla \cdot (M \nabla u), \\
    \mu &= f(u) - \Delta u + \omega f'(u) - \Delta \omega + \epsilon u_t, \\
    \omega &= f(u) - \Delta u.
\end{align*}
\]

(4)

Supposing that the mobility \( M \equiv \text{const} \), and \( \epsilon = 0 \), Miranville [10] studied the asymptotic behavior, in terms of finite-dimensional attractors, for the initial-boundary value problem of Equation (4) without viscous term. Duan et al. [11,12] proved the existence and uniqueness of global weak solutions for the initial-boundary value problem of (4) with degenerate mobility.

\textbf{Remark 1.} Because the study on higher order PDEs is an interesting topic, there are many existing attempts to understand the properties of its solutions and solve such equations. For example, Wang et al. [13], Liu et al. [14], and Coclite and di Ruvo [15] considered the global well-posedness and large time behavior of solutions for some typical higher-order evolution equations, their main methods are frequency decomposition method, Green’s function method, and fixed point theorem. Liu, Wang, and Zhao [16] used the iteration method and Fourier splitting method studied the global existence and large time behavior of the initial problem of Cahn–Hilliard equation. Gatti et al. [17], Liu and Wang [18] investigated the hyperbolic relaxation of the fourth order Cahn–Hilliard equation and sixth order Cahn–Hilliard equation that arise in oil-water-surfactant mixtures, respectively. Boyle [19] considered the instabilities of some types of sixth order Cahn–Hilliard equations. Wise et al. [20,21], Qiao et al. [22,23] and Shen et al. [24,25] have been focused on the numerical study of higher order PDEs for a long time, and the main methods are Fourier spectral method, finite element method, and finite difference method. Kannan et al. [26] solved the Burgers equation while using the Hopf Cole transformation in a high order spectral volume setting. We remark that the spectral volume method (or SV method), which is so interesting, has extensively been developed by Kannan et al. [27] for a variety of hyperbolic and elliptic problems. In particular, he was able to obtain stable solutions from different spectral volume partitions.

In this paper, supposing that the nonlinear function \( f(u) = \int_0^u F(s) ds = u^3 - u \) and the viscous parameter \( \epsilon = 1 \), we consider the Cauchy problem of the sixth-order viscous Cahn–Hilliard equation with Willmore regularization in \( \mathbb{R}^3 \).
First of all, let the constant $\kappa$ be a positive constant to be determined in (33). We rewrite Equation (4) in the following equivalent form:

$$u_t - \Delta u_t - \Delta^3 u + \kappa \Delta^2 u - 6\Delta u + \Delta^2 [u^3 - (1 + \kappa)u] + 3\Delta(u^2 \Delta u) - 3\Delta[u(u^2 - 2)(u^2 + 1)] = 0, \quad x \in \mathbb{R}^3,$$

(5)

together with the initial condition

$$u(x, 0) = u_0(x).$$

(6)

**Remark 2.** In this paper, $\nabla^l$ with an integer $l \geq 0$ stands for the usual spatial derivatives of order $l$. If $l < 0$ or $l$ is not a positive integer, $\nabla^l$ stands for $\Lambda^l$, defined by

$$(-\Delta)^{\frac{l}{2}} f(x) = \Lambda^{2l} f(x) = \int_{\mathbb{R}^3} |x|^{2l} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

We also use the notation $A \lesssim B$ to mean that $A \leq CB$ for a universal constant $C > 0$ that only depends on the parameters that come from the problem.

The first purpose of this paper is to prove the local well-posedness of system (5) and (6). We have the following theorem:

**Theorem 1 (Local well-posedness).** Suppose that $u_0 \in H^3(\mathbb{R}^3)$. Subsequently, there exists a small time $T > 0$ and a unique strong solution $u(x, t)$ to system (5) satisfying

$$u \in L^\infty([0, T]; H^3).$$

(7)

After obtain Theorem 1 on the local well-posedness, we give the following result on the global well-posedness of strong solutions:

**Theorem 2 (Global well-posedness).** Let $N \geq 2$, assume that $u_0 \in H^{N+1}(\mathbb{R}^3)$. Afterwards, there exists a unique global solution $u(x, t)$ for system (5) and (6), such that, for all $t \geq 0$,

$$\|u\|_{H^N}^2 + \|\nabla u\|_{H^N}^2 + \int_0^t \left(\|\nabla u\|_{H^N}^2 + \|\Delta u\|_{H^N}^2 + \|\nabla \Delta u\|_{H^N}^2\right) dt \lesssim \|u_0\|_{H^N}^2,$$

(8)

The main difficulty to study the well-posedness of Cauchy problem of Equation (4) is how to deal with the nonlinear terms $\Delta f(u) + \Delta [\omega f'(u)]$. Similar to [28], note that the principle part of Equation (4) is a sixth-order linear term, and there are second order terms in $\Delta f(u) + \Delta [\omega f'(u)]$. Because of Sobolev’s embedding $L^{\frac{6}{5}}(\mathbb{R}^3) \subset H^3(\mathbb{R}^3)$, we cannot control $\|\nabla f(u)\|_{L^6}$ through $\|\nabla^{k+3} u\|_{L^2}$. To overcome this difficulty, we borrow a fourth-order linear term and a second-order linear term from $\Delta f(u) + \Delta [\omega f'(u)]$, and rewrite (4) as (5). We consider the Cauchy problem of (5) in order obtain the global well-posedness result.

We also establish the following theorem on the decay rate of solutions for system (5) and (6).

**Theorem 3.** If $u_0 \in H^{-s}(\mathbb{R}^3)$ for some $s \in [0, \frac{1}{2}]$, then for all $t \geq 0$,

$$\|\Lambda^{-s} u\|_{L^2} + \|\Lambda^{-s} \nabla u\|_{L^2} \leq C_0,$$

(9)

and for $l = 0, 1, \cdots, N - 1$,

$$\|\nabla^l u\|_{H^{N-l}} + \|\nabla^{l+1} u\|_{H^{N-l}} \lesssim (1 + t)^{-\frac{l+\frac{s}{2}}{2}}.$$

(10)
Note that the Hardy–Littlewood–Sobolev theorem implies that for $p \in [\frac{3}{2}, 2]$, $L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)$ with $s = 3(\frac{1}{p} - \frac{1}{2}) \in \left[0, \frac{1}{2}\right)$. Subsequently, on the basis of Theorem 2, we obtain the optimal decay estimates for system (5).

**Corollary 1.** Under the assumptions of Theorem 2, if we replace the $\dot{H}^{-s}(\mathbb{R}^3)$ assumption by $u_0 \in L^p(\mathbb{R}^3)$, $\frac{3}{2} \leq p \leq 2$, then for $l = 0, 1, \cdots, N - 1$,

$$
\|\nabla^l u(t)\|_{\dot{H}^{N-l}} + \|\nabla^{l+1} u\|_{\dot{H}^{N-l}} \lesssim (1 + t)^{-\left[\frac{3}{2}(\frac{1}{p}-\frac{1}{2})+\frac{l}{2}\right]}.
$$

(11)

One of the main tools to consider the temporal decay rate of dissipative equations is the standard Fourier splitting method. In this paper, because there exists the lower-order linear term on the right side of problem (5), it is difficulty to use the Fourier splitting method to study the decay rate of solutions. Hence, we establish suitable negative Sobolev norm estimates in the negative Sobolev space $\dot{H}^{-s}$ $(0 \leq s \leq \frac{1}{2})$, and obtain the optimal decay rate of problem (5) in $\mathbb{R}^3$.

The remaining parts of the present papers are organized, as follows. We begin by giving some preliminary results. Subsequently, in Section 3, we prove the local well-posedness of solutions. Section 4 is devoted to study the global well-posedness of solutions. Finally, the decay estimates are postponed in Section 5.

**2. Preliminaries**

First of all, we introduce the Gagliardo–Nirenberg inequality, which was proved in [29].

**Lemma 1 ([29]).** Let $0 \leq m, \alpha \leq l$, $\nabla^\alpha f \in L^p(\mathbb{R}^3)$, $\nabla^m f \in L^q(\mathbb{R}^3)$ and $\nabla^l f \in L^r(\mathbb{R}^3)$, then we have

$$
\|\nabla^\alpha f\|_{L^p(\mathbb{R}^3)} \lesssim \|\nabla^m f\|_{L^q(\mathbb{R}^3)}^{1-\theta} \|\nabla^l f\|_{L^r(\mathbb{R}^3)}^\theta
$$

(12)

where $\theta \in [0, 1]$ and $a$ satisfies

$$
\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta.
$$

(13)

Here, when $p = \infty$, we require that $0 < \theta < 1$.

The following Kato–Ponce inequality is of great importance in the proofs.

**Lemma 2 ([30]).** Let $1 < p < \infty$, $s > 0$. Assume that $fg \in L^p(\mathbb{R}^3)$, $f \in L^{p_1}(\mathbb{R}^3) \cap W^{s_1, q_1}(\mathbb{R}^3)$ and $g \in L^{p_2}(\mathbb{R}^3) \cap W^{s_2, q_2}(\mathbb{R}^3)$. Then, there exists a positive constant $C$ such that

$$
\|\nabla^s(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|\nabla^s g\|_{L^{p_2}} + \|\nabla^s f\|_{L^{p_1}} \|g\|_{L^{p_2}}
$$

(14)

where $p_1, p_2 > 0$ and $p_2, q_2 \in (1, \infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

The following special Sobolev interpolation lemma will be used in the proof of Theorem 3.

**Lemma 3 ([31, 32]).** Assume that $s, k \geq 0$ and $l \geq 0$. Let $f \in \dot{H}^{-s}(\mathbb{R}^3) \cap H^{l+k}(\mathbb{R}^3)$, then

$$
\|\nabla^l f\|_{L^2} \leq \|\nabla^{l+k} f\|_{L^2}^{1-\theta} \|f\|_{\dot{H}^{-s}}^\theta \quad \text{with} \quad \theta = \frac{k}{l+k+s}.
$$

(15)
3. Proof of Theorem 1

In this section, we are going to prove the local well-posedness of strong solution to problem (5). For this purpose, we first consider the following linearized iteration scheme

\[
\begin{align*}
    u_i^{m} - \Delta u_i^{m} &= - \Delta^2 u^{m} + \kappa \Delta^2 u^{m} - 6 \Delta u^{m} + \Delta^2 [u^{m-1}(u^{m-1} + \sqrt{1+\kappa})(u^{m-1} - \sqrt{1+\kappa})] \\
    + 3 \Delta [u^{m-1}]^2 \Delta u^{m} - 3 \Delta [u^{m-1}]^2 (u^{m-1})^2 + 1) = 0, \\
    u^{m}(x,0) &= u^{0}(x),
\end{align*}
\]

(16)

with \(m \geq 1\) and \(u^{0}(x,t) = 0\). We introduce

\[X^{3}_{1} = \{u(x,t) : \|u\|_{X^{3}} < \infty\}\]

as the suitable space for solutions, where

\[
\|u\|_{X^{3}} = \sup_{0 \leq t \leq T} \left( \|u(\cdot,t)\|_{H^{2}} + \|\nabla u(\cdot,t)\|_{H^{2}} \right).
\]

It is easy to see that \(X^{3}_{1}\) is a non-empty Banach space. To obtain the local solution, we first claim that the sequence \(\{u^{m}\}\) is bounded in \(X^{3}_{1}\), i.e., that \(\|u^{m}\|_{X^{3}} \leq R\) for some constant \(R > 0\). We adopt the inductive method in order to obtain the above result. Clearly, by taking \(R^{2} = 2(\|u^{0}\|_{H^{2}} + \|\nabla u^{0}\|_{H^{2}})\), we have \(\|u^{1}(x,t)\|_{X^{3}} \leq R\). In the following, assuming that \(\|u^{l}\|_{X^{3}} \leq R\) for all \(j \leq m-1\), we need to prove that it also holds for \(j = m\).

**Lemma 4.** Assume that \(T\) is sufficiently small, then there exists a positive constant \(R\) such that \(\|u^{m}\|_{X^{3}} \leq R\).

**Proof.** Multiplying (16) by \(u^{m}\), integrating by parts over \(\mathbb{R}^{3}\), we derive that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt}(\|u^{m}\|_{L^{2}}^{2} + \|\nabla u^{m}\|_{L^{2}}^{2}) &= \int_{\mathbb{R}^{3}} \nabla \left[u^{m}(u^{m-1} + \sqrt{1+\kappa})(u^{m-1} - \sqrt{1+\kappa})\right] \cdot \nabla u^{m} dx - 3 \int_{\mathbb{R}^{3}} (u^{m-1})^2 |\Delta u^{m}|^{2} dx \\
&\quad + 3 \int_{\mathbb{R}^{3}} ((u^{m-1})^2 - 2)(u^{m-1})^2 + 1) \Delta u^{m} dx \\
&= I_{1} + I_{2} + I_{3}.
\end{align*}
\]

Note that

\[
I_{1} = \int_{\mathbb{R}^{3}} \nabla \left[u^{m}(u^{m-1} + \sqrt{1+\kappa})(u^{m-1} - \sqrt{1+\kappa})\right] \cdot \nabla u^{m} dx
\]

\[
\leq \|\nabla u^{m}\|_{L^{2}} \left( \|\nabla u^{m}\|_{L^{2}} \|u^{m-1}\|_{L^{\infty}} \|u^{m-1} - \sqrt{1+\kappa}\|_{L^{\infty}} \\
+ \|u^{m}\|_{L^{5}} \|\nabla u^{m-1} + \sqrt{1+\kappa}\|_{L^{5}} \|u^{m-1} - \sqrt{1+\kappa}\|_{L^{5}} \\
+ \|u^{m}\|_{L^{5}} \|\nabla (u^{m-1} - \sqrt{1+\kappa})\|_{L^{5}} \|u^{m-1} + \sqrt{1+\kappa}\|_{L^{5}} \right)
\]

\[
\leq C \|\nabla u^{m}\|_{L^{2}} \left( \|\nabla u^{m}\|_{L^{2}} \|u^{m-1}\|_{L^{2}}^{2} + \|\nabla u^{m}\|_{L^{2}} \|\Delta u^{m-1}\|_{L^{2}} \|\nabla u^{m-1}\|_{L^{2}} \right)
\]

\[
\leq \frac{1}{4} \|\nabla u^{m}\|_{L^{2}}^{2} + C \|u^{m-1}\|_{L^{2}}^{4} \|\nabla u^{m}\|_{L^{2}}^{2}
\]

\[
\leq \frac{1}{4} \|\nabla u^{m}\|_{L^{2}}^{2} + CR^{4} \|\nabla u^{m}\|_{L^{2}}^{2},
\]
and

\[ I_2 + I_3 = 3 \int_{\mathbb{R}^3} \nabla ((u^{m-1})^2 u^m) \cdot \nabla u^m dx + 3 \int_{\mathbb{R}^3} |u^m((u^{m-1})^2 - 2((u^{m-1})^2 + 1))u^m| dx \]

\[ \leq 3 \| \nabla u^m \|_{L^2} (\| \nabla u^m \|_{L^2} \| u^{m-1} \|_{L^2}^2 + 2 \| u^{m-1} \|_{L^2} \| \nabla u^m \|_{L^2}) \]

\[ + 3 \| \Delta u^m \|_{L^2} \| \nabla u^m \|_{L^2} \| u^{m-1} + \sqrt{2} \| \nabla u^m \|_{L^2} (\| u^{m-1} \|_{L^2}^2 + 1) \| \nabla u^m \|_{L^2} \]

\[ \leq 3C \| \nabla u^m \|_{L^2} \| \nabla u^m \|_{L^2} \| u^{m-1} \|_{L^2}^2 + 3C \| \nabla u^m \|_{L^2} (\| \nabla u^m \|_{L^2} \| u^{m-1} \|_{L^2}) \]

\[ \| u^{m-1} \|_{L^2} \| \nabla u^m \|_{L^2} \| \nabla u^m \|_{L^2} \| u^{m-1} \|_{L^2} \| \nabla u^m \|_{L^2} \]

\[ \leq \frac{1}{4} \| \nabla u^m \|_{L^2}^2 + C (R^4 + R^8) \| \nabla u^m \|_{L^2}^2. \]

While taking \( \Delta \) to (16), multiplying it by \( \Delta u^m \) and integrating over \( \mathbb{R}^3 \), we deduce that

\[ \frac{1}{2} \frac{d}{dt} (\| \Delta u^m \|_{L^2}^2 + \| \nabla \Delta u^m \|_{L^2}^2) + \| \Delta \Delta u^m \|_{L^2}^2 + \kappa \| \Delta u^m \|_{L^2}^2 \]

\[ = 3 \int_{\mathbb{R}^3} \nabla ((u^{m-1})^2 u^m) \cdot \nabla \Delta u^m dx \]

\[ + 3 \int_{\mathbb{R}^3} \nabla (u^{m-1})^2 u^m dx - 3 \int_{\mathbb{R}^3} \nabla [u^m((u^{m-1})^2 - 2((u^{m-1})^2 + 1)] \cdot \nabla \Delta u^m dx \]

\[ =: I_4 + I_5 + I_6. \]

By (25), we have

\[ I_4 = \int_{\mathbb{R}^3} \nabla \Delta [u^m(u^{m-1} + \sqrt{1 + \kappa})(u^{m-1} - \sqrt{1 + \kappa})] \cdot \nabla \Delta u^m dx \]

\[ \leq \| \Delta \Delta u^m \|_{L^2} (\| \nabla u^m \|_{L^2} \| u^{m-1} + \sqrt{1 + \kappa} \|_{L^2} \| u^{m-1} - \sqrt{1 + \kappa} \|_{L^2}) \]

\[ + \| u^m \|_{L^\infty} \| \nabla \Delta (u^{m-1} + \sqrt{1 + \kappa}) \|_{L^\infty} \| u^{m-1} - \sqrt{1 + \kappa} \|_{L^\infty} \]

\[ + \| u^m \|_{L^\infty} \| \nabla \Delta (u^{m-1} - \sqrt{1 + \kappa}) \|_{L^\infty} \| u^{m-1} \|_{L^\infty} \]

\[ \leq C \| \Delta \Delta u^m \|_{L^2} (\| \nabla u^m \|_{L^2} \| u^{m-1} \|_{L^2}) \| \nabla u^{m-1} \|_{L^2} \| \Delta u^{m-1} \|_{L^2} \]

\[ + \| u^m \|_{L^\infty} \| \nabla u^{m-1} \|_{L^2} \| \Delta u^{m-1} \|_{L^2} \]

\[ \| \nabla u^{m-1} \|_{L^2} \| \Delta u^{m-1} \|_{L^2} \]

\[ \leq \frac{1}{4} \| \Delta \Delta u^m \|_{L^2}^2 + C \| \nabla u^{m-1} \|_{L^2}^2 \| \Delta u^{m-1} \|_{L^2} \]

\[ + C (\| \nabla u^{m-1} \|_{L^2} \| \Delta u^{m-1} \|_{L^2}) \| \nabla u^{m-1} \|_{L^2} \| \Delta u^{m-1} \|_{L^2} \]

\[ \leq \frac{1}{4} \| \Delta \Delta u^m \|_{L^2}^2 + CR^4 (\| \nabla u^m \|_{L^2}^2 + \| u^m \|_{L^2}^2 + \| \Delta u^m \|_{L^2}^2). \]

For \( I_5 \) and \( I_6 \), we have

\[ I_5 + I_6 = 3 \int_{\mathbb{R}^3} \nabla ((u^{m-1})^2 \Delta u^m) \cdot \nabla \Delta u^m dx - 3 \int_{\mathbb{R}^3} \nabla [u^m((u^{m-1})^2 - 2((u^{m-1})^2 + 1)] \cdot \nabla \Delta u^m dx \]

\[ \leq 3 \| \Delta \Delta u^m \|_{L^2} (\| \nabla u^m \|_{L^2} \| u^{m-1} \|_{L^2} + 2 \| u^{m-1} \|_{L^\infty} \| \nabla u^{m-1} \|_{L^\infty} \| \Delta u^m \|_{L^\infty}) \]

\[ + 3 \| \nabla \Delta u^m \|_{L^2} (\| \nabla u^m \|_{L^\infty} \| u^{m-1} + \sqrt{2} \| \nabla u^m \|_{L^\infty}) \| u^{m-1} - \sqrt{2} \| \nabla u^m \|_{L^\infty} \| \Delta u^m \|_{L^\infty}) \]

\[ + 2 \| u^m \|_{L^\infty} \| \nabla \Delta u^m \|_{L^\infty} \| u^{m-1} \|_{L^\infty} \| (u^{m-1})^2 + 1 \| \| \nabla u^m \|_{L^\infty} \| \nabla u^{m-1} \|_{L^\infty} \| \Delta u^{m-1} \|_{L^\infty} \]

\[ \leq \frac{1}{4} \| \Delta \Delta u^m \|_{L^2}^2 + C \| \nabla \Delta u^m \|_{L^2} \| u^{m-1} \|_{L^2} \| \Delta u^{m-1} \|_{L^2} \]

\[ + C (\| \nabla u^m \|_{L^2} \| \Delta u^m \|_{L^2}) \| \nabla u^{m-1} \|_{L^2} \| \Delta u^{m-1} \|_{L^2} \]

\[ + \| \nabla u^{m-1} \|_{L^2} \| \Delta u^{m-1} \|_{L^2} \| \Delta u^{m-1} \|_{L^2} \]

\[ \leq \frac{1}{4} \| \Delta \Delta u^m \|_{L^2}^2 + C (R^4 + R^8) (\| \nabla u^m \|_{L^2}^2 + \| \nabla u^{m-1} \|_{L^2}^2 + \| \Delta u^m \|_{L^2}^2). \]
Combining (17)–(22) together, we obtain
\[
\frac{d}{dt} \left( \|u^m\|_{L^2}^2 + \|\nabla u^m\|_{L^2}^2 + \|\Delta u^m\|_{L^2}^2 + \|\nabla \Delta u^m\|_{L^2}^2 \right) \\
+ \|\Delta^2 u^m\|_{L^2}^2 + \|\nabla \Delta u^m\|_{L^2}^2 + \|\Delta u^m\|_{L^2}^2 + \|\nabla u^m\|_{L^2}^2 \\
\leq C(R^4 + R^8) \left( \|u^m\|_{L^2}^2 + \|\nabla u^m\|_{L^2}^2 + \|\Delta u^m\|_{L^2}^2 + \|\nabla \Delta u^m\|_{L^2}^2 \right),
\]
which gives
\[
\sup_{0 \leq t \leq T} \left( \|u^m\|_{L^2}^2 + \|\nabla u^m\|_{L^2}^2 + \|\Delta u^m\|_{L^2}^2 + \|\nabla \Delta u^m\|_{L^2}^2 \right) \\
\leq \|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2 + \|\nabla \Delta u_0\|_{L^2} \\
+ C(R^4 + R^8) T \sup_{0 \leq t \leq T} \left( \|u^m\|_{L^2}^2 + \|\nabla u^m\|_{L^2}^2 + \|\Delta u^m\|_{L^2}^2 + \|\nabla \Delta u^m\|_{L^2}^2 \right).
\]
For sufficiently small \( T \), we obtain
\[
\sup_{0 \leq t \leq T} \left( \|u^m\|_{L^2}^2 + \|\nabla u^m\|_{L^2}^2 + \|\Delta u^m\|_{L^2}^2 + \|\nabla \Delta u^m\|_{L^2}^2 \right) \\
\leq 2(\|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2 + \|\nabla \Delta u_0\|_{L^2}) \leq R^2,
\]
this complete the proof. \( \square \)

Next, we show that \( \{u^m\} \) is a Cauchy sequence.

**Lemma 5.** Assume that \( T \) is sufficiently small, then \( \{u^m\} \) is a Cauchy sequence in \( X^3_T \).

**Proof.** Denote
\[
u^m = u^m - u^{m-1},
\]
we have
\[
\nu^m - \Delta \nu^m - \Delta^2 \nu^m + \kappa \Delta^2 \nu^m - 6 \Delta \nu^m + \Delta^2 \left( (u^m)^2 - 1 - \kappa \right) - u^{m-1}((u^{m-2})^2 - 1 - \kappa) \\
+ 3 \Delta \left[ (u^{m-1})^2 \Delta u^m - (u^{m-2})^2 \Delta u^{m-1} \right] \\
- 3 \Delta \left[ (u^{m-1})^2 - 2 \right] ((u^{m-1})^2 + 1) - \mu^m - 2 ((u^{m-2})^2 - 2)((u^{m-2})^2 + 1) = 0.
\]

Multiplying (26) by \( \nu^m \) and \( \Delta^2 \nu^m \), integrating over \( \mathbb{R}^3 \), respectively, and using the Gronwall’s inequality, taking \( T \) small enough, we arrive
\[
\|\nu^m\|_{X^3_T} \leq C R^6 T \|\nu^{m-1}\|_{X^3},
\]
for some \( \theta \in \mathbb{R}^+ \). Subsequently, \( \{u^m\} \) is convergent in \( X^3_T \) and the proof is complete. \( \square \)

Now, we give the proof of Theorem 1.

**Proof of Theorem 1.** By Lemma 5, we know that \( \{u^m\} \) is a Cauchy sequence in Banach space \( X^3_T \). Moreover, by using Lemma 4, we see that \( u^m \) is bounded in \( X^3_T \). Therefore, the limit function \( u \) of \( u^m \) is a local solution of problem (5). \( \square \)

4. **Proof of Theorem 2**

In this section, we give the proof of Theorem 2.
Proof of Theorem 2. Using system (5) itself, it yields that

\[
\frac{\partial E(u)}{\partial t} \leq 0.
\]  
(27)

Hence,

\[
\frac{1}{2} \| \nabla u \|^2_{L^2} + \frac{1}{4} \| u^2 - 1 \|^2_{L^2} + \frac{1}{2} \| \omega \|^2_{L^2} \\
\leq \frac{1}{2} \| \nabla u_0 \|^2_{L^2} + \frac{1}{4} \| u_0^2 - 1 \|^2_{L^2} + \frac{1}{2} \| \omega_0 \|^2_{L^2} = E(u_0).
\]  
(28)

Applying \( \nabla^k \) to (5), multiplying the resulting identity by \( \nabla^k u \), then integrating over \( \mathbb{R}^3 \) by parts, we derive that

\[
\frac{1}{2} \int_{\mathbb{R}^3} \nabla^k \Delta^2(u^3 - (1 + \kappa)u) \cdot \nabla^k u \, dx \\
= -\int_{\mathbb{R}^3} \nabla^k \Delta^2(u^3 - (1 + \kappa)u) \cdot \nabla^k u \, dx - 3 \int_{\mathbb{R}^3} \nabla^k \Delta(u^2 \Delta u) \cdot \nabla^k u \, dx \\
+ 3 \int_{\mathbb{R}^3} \nabla^k \Delta(u(u^2 - 2)(u^2 + 1)) \cdot \nabla^k u \, dx.
\]  
(29)

The first term of the right hand side of (29) can be estimated as

\[
- \int_{\mathbb{R}^3} \nabla^k \Delta^2(u^3 - (1 + \kappa)u) \cdot \nabla^k u \, dx \\
\lesssim \| \nabla^{k+3} u \|_{L^2} \| \nabla^{k+1} [u(u - \sqrt{1 + \kappa})(u + \sqrt{1 + \kappa})] \|_{L^2} \\
\lesssim \| \nabla^{k+3} u \|_{L^2} (\| \nabla^{k+1} u \|_{L^6} \| u - \sqrt{1 + \kappa} \|_{L^6} \| u + \sqrt{1 + \kappa} \|_{L^6} \\
+ \| \nabla^{k+1} [u(u - \sqrt{1 + \kappa})(u + \sqrt{1 + \kappa})] \|_{L^6} \| u \|_{L^6} \\
+ \| \nabla^{k+1}[u(u - \sqrt{1 + \kappa})(u + \sqrt{1 + \kappa})] \|_{L^6} \| u \|_{L^6})
\]  
(30)

The second term of the right hand side of (29) satisfies

\[
-3 \int_{\mathbb{R}^3} \nabla^k \Delta(u^2 \Delta u) \cdot \nabla^k u \, dx \lesssim \int_{\mathbb{R}^3} u^2 \Delta u \nabla^{k+1} u \, dx \\
\lesssim \int_{\mathbb{R}^3} u^2 \nabla^2 u \cdot \nabla^{k+3} u \, dx \\
\lesssim \int_{\mathbb{R}^3} \nabla^k (u^2) \cdot \nabla^{k+2} u \, dx + \int_{\mathbb{R}^3} |\nabla^{k+2} (u^2 \nabla u) \cdot \nabla^{k+1} u \, dx| \\
\lesssim \| \nabla^{k+2} u \|_{L^6} \sum_{0 \leq l \leq m \leq k+1} \| \nabla^l u \|_{L^6} \| \nabla^{m-l} \nabla u \|_{L^6} \| \nabla^{k-m} u \|_{L^2} \\
+ \| \nabla^{k+1} u \|_{L^6} (\| \nabla^{k+2} u \|_{L^6} \| u \|_{L^6} \| u \|_{L^2} + \| u \|_{L^2} \| \nabla^{k+3} u \|_{L^2}) \\
\lesssim \| \nabla^{k+2} u \|_{L^6} \sum_{0 \leq l \leq m \leq k+1} \| \nabla^{k+2} u \|_{L^2} \| \nabla u \|_{L^6} \| \nabla^{k+2} u \|_{L^2} + \| u \|_{L^2} \| \nabla^{k+3} u \|_{L^2} \\
+ \| \nabla^{k+2} u \|_{L^2} (\| \nabla^{k+3} u \|_{L^2} \| u \|_{L^2} + \| u \|_{L^2} \| \nabla^{k+3} u \|_{L^2}) \\
\leq \frac{1}{4} \| \nabla^{k+3} u \|_{L^2}^2 + C_0 \| \nabla u \|_{L^2}^4 \| \nabla^{k+2} u \|_{L^2}^2 \\
\leq \frac{1}{4} \| \nabla^{k+3} u \|_{L^2}^2 + 4 C_0 [E(u_0)]^2 \| \nabla^{k+2} u \|_{L^2}^2.
\]  
(31)
Moreover, we estimate the last term of the right hand side of (29) as
\[
3 \int_{\mathbb{R}^3} \nabla^k \Delta [u(u^2 - 2)(u^2 + 1)] \cdot \nabla^k u \, dx
= 3 \int_{\mathbb{R}^3} \nabla^k [u^3 + u](u^2 - 2) \cdot \nabla^k u \, dx
= 3 \int_{\mathbb{R}^3} \nabla^k [u^3(u^2 - 2)] \cdot \nabla^k u \, dx + 3 \int_{\mathbb{R}^3} \nabla^k \Delta [u(u^2 - 2)] \cdot \nabla^k u \, dx
\leq ||\nabla^{k+1} u||_{L^6} ||\nabla^{k+1} [u^3(u^2 - 2)]||_{L^6} + ||\nabla^{k+1} u||_{L^2} ||\nabla^{k+1} [u(u^2 - 2)]||_{L^2} \tag{32}
\]
Combining (29)–(32) together gives
\[
\frac{d}{dt}(||\nabla u||^2_{L^2} + ||\nabla^{k+1} u||^2_{L^2}) + ||\nabla^{k+3} u||^2_{L^2} + 2(2\kappa - 24C_0[E(u_0)]^2) ||\nabla^{k+2} u||^2_{L^2} + 6 ||\nabla^{k+1} u||^2_{L^2} \leq 0. \tag{33}
\]
It follows from (33), we can assume that the parameter \( \kappa = 24C_0[E(u_0)]^2 \). Summing up the estimate (34) from \( k = 0 \) to \( N \), we obtain
\[
\frac{d}{dt} \left( ||u||^2_{H^N} + ||\nabla u||^2_{H^N} + ||\nabla^3 u||^2_{H^N} + ||\nabla^2 u||^2_{H^N} + ||\nabla u||^2_{H^N} \right) \lesssim 0. \tag{34}
\]
which implies that
\[
||u||^2_{H^N} + ||\nabla u||^2_{H^N} + \int_0^t (||\nabla u||^2_{H^N} + ||\nabla u||^2_{H^N}) \, d\tau \lesssim ||u_0||^2_{H^N}, \tag{35}
\]
we complete the proof of Theorem 2. \( \square \)

5. Proof of Theorem 3

Now, we give the proof of Theorem 3 on the large time behavior of problem (5) and (6).

Proof of Theorem 3. Applying \( \Lambda^{-s} \) to (5) and multiplying the resulting identities by \( \Lambda^{-s} u \), then integrating over \( \mathbb{R}^3 \), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}^3} |\Lambda^{-s} u|^2 \, dx + \int_{\mathbb{R}^3} |\Lambda^{-s} \nabla u|^2 \, dx \right) \\
+ \int_{\mathbb{R}^3} |\nabla^3 \Lambda^{-s} u|^2 \, dx + \kappa \int_{\mathbb{R}^3} |\nabla^2 \Lambda^{-s} u|^2 \, dx + 6 \int_{\mathbb{R}^3} |\nabla \Lambda^{-s} u|^2 \, dx \\
= - \int_{\mathbb{R}^3} \Lambda^{-s} [\Delta^2 (u^3 - (1 + \kappa)u)] \cdot \Lambda^{-s} u \, dx - 3 \int_{\mathbb{R}^3} \Lambda^{-s} \Delta (u^2 \Delta u) \cdot \Lambda^{-s} u \, dx \\
+ 3 \int_{\mathbb{R}^3} \Lambda^{-s} \Delta (u^2 \Delta u) - 3 \Delta [u(u^2 - 2)(u^2 + 1)] \cdot \Lambda^{-s} u \, dx. \tag{36}
\]
If \( s \in (0, \frac{1}{2}] \), we estimate the right hand side of (36) as

\[
- \int_{\mathbb{R}^3} \Lambda^{-s}[\Delta^2(u^3 - (1 + \kappa)u)] \cdot \Lambda^{-s} u dx
\]

\[
\leq \|\Lambda^{-s} u\|_{L^2} \left[ \|\Lambda^{-s}[\Delta^2(u^3 - (1 + \kappa)u)]\|_{L^2} + \|\Lambda^{-s} u\|_{L^2} \right]
\]

\[
\leq \|\Lambda^{-s} u\|_{L^2} \left[ \|\Delta u\|_{L^2} \|\Delta u\|_{L^2} + \|\Delta u\|_{L^2} \right]
\]

\[
+ \|\Lambda^2 u\|_{L^2} \left( \|\Delta u\|_{L^2} \|\Delta u\|_{L^2} \right)^2
\]

\[
\leq \|\Lambda^{-s} u\|_{L^2} \|\nabla u\|_{L^2}^2,
\]

and

\[
3 \int_{\mathbb{R}^3} \Lambda^{-s}[\Delta(2u^2)u] \cdot \Lambda^{-s} u dx
\]

\[
\leq \|\Lambda^{-s} u\|_{L^2} \|\Lambda^{-s}[\Delta(u^2 - 2)(u^2 + 1)]\|_{L^2}
\]

\[
\leq \|\Lambda^{-s} u\|_{L^2} \left( \|\Delta u\|_{L^2} + \|\Delta u\|_{L^2} + \|\Delta(u^2 - 2)\|_{L^2} \right)
\]

\[
\leq \|\Lambda^{-s} u\|_{L^2} \left( \|\Delta u\|_{L^2} + \|\Delta u\|_{L^2} + \|\Delta(u^2 - 2)\|_{L^2} \right)
\]

\[
+ \|\Delta u\|_{L^2} \|u + \sqrt{2}L\|_{L^2} + \|\Delta(u + \sqrt{2})\|_{L^2} \|u + \sqrt{2}L\|_{L^2}
\]

\[
+ \|\Delta(u - \sqrt{2})\|_{L^2} \|u - \sqrt{2}L\|_{L^2}
\]

\[
\leq \|\Lambda^{-s} u\|_{L^2} \|\nabla u\|_{L^2}^2.
\]

Combining (36)–(39) together gives

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^3} |\Lambda^{-s} u|^2 dx + \int_{\mathbb{R}^3} |\Lambda^{-s} \nabla u|^2 dx \right)
\]

\[
+ \int_{\mathbb{R}^3} \nabla^3 \Lambda^{-s} u^2 dx + \kappa \int_{\mathbb{R}^3} \nabla^2 \Lambda^{-s} u^2 dx + \int_{\mathbb{R}^3} 6|\nabla \Lambda^{-s} u|^2 dx \leq \|\Lambda^{-s} u\|_{L^2} \|\nabla u\|_{L^2}.
\]

For convenience, define

\[
\mathcal{E}_s(t) := \|\Lambda^{-s} u(t)\|_{L^2}^2 + \|\Lambda^{-s} \nabla u(t)\|_{L^2}^2.
\]
Subsequently, on the basis of (8) and (40), we have
\[ E_{-s}(t) \leq E_{-s}(0) + C \int_0^t \| \nabla u \|^2_{L^2} \sqrt{E_{-s}(\tau)} d\tau \leq C_0 \left( 1 + \sup_{0 \leq \tau \leq t} \sqrt{E_{-s}(\tau)} d\tau \right), \] (41)
which implies (9) holds for \( s \in (0, \frac{1}{2}] \), which is
\[ \| \Lambda^{-s}u(t) \|^2_{L^2} + \| \Lambda^{-s}\nabla u(t) \|^2_{L^2} \leq C_0. \] (42)
In the following, if \( l = 1, 2, \cdots, N - 1 \), we may use Lemma 3 to have
\[ \| \nabla^{l+1}f \|^2_{L^2} \geq C \| \Lambda^{-s} f \|^2_{L^2} \| \nabla^{l}f \|^2_{L^2}. \]
Subsequently, by this facts and (42), we find that
\[ \| \nabla^{l+2}u \|^2_{L^2} + \| \nabla^{l+1}u \|^2_{L^2} \geq C_0 \| \nabla^{l+1}u \|^2_{L^2} + \| \nabla^{l}u \|^2_{L^2} \right)^{1 + \frac{1}{N_1}}. \] (43)
For \( l = 1, 2, \cdots, N - 1 \),
\[ \| \nabla^{l+2}u \|^2_{H^{N-1}} + \| \nabla^{l+1}u \|^2_{H^{N-1}} \geq C_0 \| \nabla^{l+1}u \|^2_{H^{N-1}} + \| \nabla^{l}u \|^2_{H^{N-1}} \right)^{1 + \frac{1}{N_1}}. \]
Thus, let \( E^N_{-l} = \sum_{k \leq N} (\| \nabla^k u \|^2_{L^2} + \| \nabla^{k+1} u \|^2_{L^2}) \), we deduce from (33) that
\[ \frac{d}{dt} E^N_{-l} + C_0 \left( E^N_{-l} \right)^{1 + \frac{1}{N_1}} \leq 0, \quad \text{for } l = 1, 2, \cdots, N - 1. \] (44)
Solving this inequality directly gives
\[ E^N_{-l}(t) \leq C_0 (1 + t)^{-\frac{l-1}{s}}, \quad \text{for } l = 1, 2, \cdots, N - 1, \] (45)
which means
\[ \| \nabla^l u(t) \|^2_{H^{N-l}} \leq C (1 + t)^{-\frac{l-1}{s}}, \quad \text{for } l = 0, 1, \cdots, N - 1, \]
and, hence, (10) holds for \( s \in (0, \frac{1}{2}] \), this complete the proof. \( \Box \)

6. Conclusions

The Cahn–Hilliard equation is a partial differential equation of mathematical physics that describes the process of phase separation or solution, by which the two components of a binary fluid spontaneously separate and form domains pure in each component. Recently, in order to study crystalline anisotropy, some authors introduced the anisotropy Cahn–Hilliard model, in order to overcome the difficulties that are caused by the anisotropy, some authors introduced the Willmore regularization to the free energy, obtained the sixth order Cahn–Hilliard equation with Willmore regularization. Because of the lack of the anisotropy effects, it is difficulty to establish the uniform a priori estimates and study the properties of solutions for such equation from the point of mathematical analysis. Until now, only a few papers related to this filed. Hence, the mathematical analysis of sixth order Cahn–Hilliard equation with Willmore regularization is an interesting problem. In this paper, we consider the well-posedness and large time behavior of solutions for the Cauchy problem of such equation, prove the local well-posedness, global well-posedness, and decay estimates of solutions. The results of this paper can be seen as a preliminary attempt to study the Cauchy problem of sixth order Cahn–Hilliard equation with Willmore regularization. We will continue to study the properties of solutions for the equation with non-constant (include degenerate) mobility in order to describe the original model in more detail.
Author Contributions: X.Z. proved the local well-posedness and global well-posedness, N.D. proved the decay estimates. All authors have read and agreed to the published version of the manuscript.

Funding: This paper was supported by the Fundamental Research Funds for the Central Universities (grant No. N2005031).

Acknowledgments: We would like to thank Wei Li for her suggestions. Moreover, we appreciate very much the useful suggestions of the reviewers. The suggestions will benefit the improvement of the paper and our future research.

Conflicts of Interest: The authors declare no conflict of interest.

List of Symbols

\( \nabla \) \( \nabla f \) is the gradient of \( f \)

\( \nabla^l \) \( \text{the usual spatial derivatives of order } l \text{ with } l \geq 0 \)

\( A^l \) \( \text{the usual spatial derivatives of order } l \text{ with } l < 0 \)

\( \preceq \) \( A \preceq B \) means \( A \leq CB \) for a universal constant \( C > 0 \)

\( \varrho \) \( \text{the symbol of embedding} \)

\( \mathbb{R}^3 \) \( \text{3-dimensional Euclidean space} \)

\( u \) \( \text{the order parameter} \)

\( \omega \) \( \text{the nonlinear Willmore regularization} \)

\( \beta \) \( \text{the bending rigidity} \)

\( \nu \) \( \text{the unit norm vector} \)

\( \gamma(v) \) \( \text{a function describing the anisotropy effects} \)

\( E(u) \) \( \text{the free energy} \)

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