Atomic collapse in graphene and cyclic RG flow

A.Gorsky\textsuperscript{a,b} and F.Popov\textsuperscript{a,b}

\textsuperscript{a} Institute of Theoretical and Experimental Physics, Moscow 117218, Russia
\textsuperscript{b} Moscow Institute of Physics and Technology, Dolgoprudny 141700, Russia

gorsky@itep.ru
popov@itep.ru

Abstract

In this Letter we consider the problem of screening of external charge in graphene from the cyclic RG flow viewpoint. The analogy with conformal Calogero model is used to suggest the interpretation of the tower of resonant states as tower of Efimov states.
1 Introduction

It is usually assumed that the RG flow connects fixed points, starting at a UV repelling point and terminating at a IR attracting point. However it turned out that this open RG trajectory does not exhaust all possibilities and the clear-cut quantum mechanical example of the nontrivial RG limit cycle has been found in [1] confirming the earlier expectations. This finding triggered the search for another examples of this phenomena which was quite successful. The explicit examples have been identified both in the systems with finite number degrees of freedom [2, 3, 4, 5] and in the field theory framework [6, 7, 8]. It was also realized later that the Efimov states predicted long time ago for the three-body system in the context of nuclear physics with the Efimov scaling of the energy levels are just the manifestation of this quite general phenomena. The review on RG interpretation of the Efimov phenomena can be found in [9].

The very phenomenon of the cyclic RG flow has been interpreted in [7] as a kind of generalization of the BKT-like phase transitions in two dimensions. One can start with a usual pattern of an RG flow connecting UV and IR fixed points and then consider a motion in a parameter space which results in a merging of the fixed points. It was argued that when the parameter goes into the complex plane the cyclic behavior of the RG flow gets manifested and a gap in the spectrum arises. This happens similar to the BKT phase transition when the deconfinement of vortices occurs at the critical temperature and the conformal symmetry is restored at lower temperatures. The appearance of the RG cycles can be also interpreted in terms of the peculiar anomaly in the classical conformal group [10]. This anomaly has the origin in a kind of ”falling to the center” UV phenomena which could have quite different reincarnations. Emphasize one more generic feature of this phenomena — the cyclic RG usually occurs in the system with at least two couplings. One of them undergoes the RG cyclic flow while the second determines the period of the cycle.

The RG cycles have been found in the non-relativistic Calogero-like model with $\frac{1}{r}$ potential which enjoys the conformal symmetry [2, 3, 4]. In Calogero model with attraction some small distance boundary conditions are imposed on the wave function. It is usually assumed that the wave function with $E = 0$ at large $r$ does not depend on the cutoff at small $r$. This condition yields the equation for the parameter of a cutoff. It has multiple solutions
which can be interpreted in terms of the tower of shallow bound states with
the Efimov scaling in the regularized Calogero model with attraction. The
scaling factor is determined by the Calogero coupling constant.

In this Letter we shall consider the similar problem in 2+1 dimensions
which physically corresponds to the external charge in the graphene plane.
The problem has a classical conformal symmetry and is the relativistic ana-
logue of the conformal non-relativistic Calogero-like system. Due to confor-
mal symmetry we could expect the RG cycles and Efimov-like states in this
problem upon imposing the short distance cutoff. The issue of the charge in
the graphene plane has been discussed theoretically [12, 13, 14] and experi-
mentally [15, 16]. It was argued that indeed there is the tower of ”quasi-
Rydberg” states with the exponential scaling [11]. The situation can be
interpreted as an atomic collapse phenomena similar to the instability of
$Z > 137$ superheavy atoms in QED [17]. The possible role of the conformal
symmetry in the atomic collapse problem in graphene has been mentioned in [18].

We shall perform the regularization analysis in 2+1 case similar to Calogero
model and identify the Efimov-like states from this perspective. The period
of the RG cycle will be identified with the charge value and the 2+1 rela-
tivistic analog of the anomaly in the commutator has been considered.

The Letter is organized as follows. First, we will briefly review the RG
picture for non-relativistic Calogero model. In Section 3 we will perform
the RG analysis imposing the small distance cutoff. In Section 4 we shall
consider the relativistic version of the anomaly in the commutator. Some
comments on the results obtained can be found in the Conclusion.

2 RG cycles in nonrelativistic quantum me-
chanics

In this Section we consider the example of the RG limit cycle in the non-
relativistic system with the attractive $\frac{1}{r^2}$ Calogero potential regularized at
short distances in some way. Two most popular regularizations involve the
square-well potential [3, 4] or the delta-shell potential [2]. In both cases the
depth of the potential is logarithmic function of the short distance cutoff.
Consider the particle in the attractive potential

\[ V(r) = -(1/4 + \mu^2)r^{-2}, \quad r > R \]  
\[ V_{\text{short}} = \text{const}, \quad r < R \]

It turns out that the potential regularized at short distances has infinitely many shallow S-wave bound states. They accumulates at the threshold energy \( E = 0 \). Near the threshold the ratio of the energies of the successive states approaches \( e^{\frac{2\mu}{\pi}} \) and the bound state energies behave as

\[ E_n \rightarrow ce^{-\frac{2\mu}{\pi}(n-n_0)} \]  

The model before regularization respects the conformal symmetry which is broken by the boundary conditions imposed by the short-distance cutoff.

The RG flow is formulated in terms of the tuning parameter \( \lambda \) which enters the short distance potential

\[ V(r, \lambda) = -\frac{\lambda}{R^2}, \quad r < R \]

for the "spherical shell" potential. We assume that \( \lambda \) is R dependent function. The dependence on the RG scale is fixed by some condition introduced by hands. The most convenient RG condition is the requirement that the regularized potential reproduces the \( E = 0 \) wave function at \( r > R \). It can be shown that the shallow bound states are reproduced as well in the leading approximation. Note that we could require the RG condition not only for zero energy state but for any shallow state. The delta-shell regularization is defined by the following cutoff potential

\[ V_{\text{delta}} = -\frac{\lambda}{R}\delta(r - R_-), \quad r \leq R \]

where \( R_- \) is close to \( R \) but lies inside the interval \( 0 < r < R \).

In the spherical square wall the RG condition implies the equation

\[ \sqrt{\lambda} \cot(\sqrt{\lambda}) = \frac{1}{2} + \mu \cot(\mu \ln(\frac{R}{R_0})) \]

which has infinitely many solutions. In the delta-well regularization the RG equation reads as

\[ \lambda(R) = \frac{1}{2} - \mu \cot(\mu \ln(\frac{R}{R_0})) \]
In both cases the period of the RG cycle is defined by the Calogero coupling constant and equals $\frac{2\pi}{\mu}$.

Therefore we see that in the conformal non-relativistic quantum mechanics one can define the cyclic RG flow for the parameter of the cutoff potential. At each cycle one bound state appears or disappears from the spectrum.

3 RG cycle in graphene

In this Section we discuss the similar problem in 2+1 dimensions which physically corresponds to the external charge in the graphene plane. Consider an electron in graphene which interacts with an external charge. The two-dimensional Hamiltonian reads as,

$$H_D = v_F \sigma_i p^i - V(r), \quad i = 1, 2.$$  \hspace{1cm} (8)

where the external charge creates a Coulomb potential

$$V(r) = -\frac{\alpha}{r}, \quad r \geq R.$$  \hspace{1cm} (9)

As we shall see, the solution in presence of the potential (9) oscillates at the origin and needs to be regularized by some cutoff $R$. Hence close enough to the origin $r \leq R$ the potential (9) gets replaced by some constant potential $V_{\text{reg}}(r, \lambda(R))$. The renormalization condition for the $\lambda$ parameter is that the zero-energy wave function is independent on the short-distance regularization. This condition is chosen similarly to that of the renormalization in the Calogero system. Hence our primary task is to find the zero-energy solution to the Dirac equation,

$$H_D \psi_0 = 0.$$  \hspace{1cm} (10)

Since the Hamiltonian commutes with the $J_3$ operator,

$$J_3 = i \frac{\partial}{\partial \varphi} + \sigma_3, \quad [H_D, J_3] = 0,$$  \hspace{1cm} (11)

we can look for the solutions of (10) in the form:

$$\psi_0 = \begin{pmatrix} \chi_0(r) \\ \xi_0(r)e^{i\varphi} \end{pmatrix}, \quad J_3 \psi_0 = \psi_0.$$  \hspace{1cm} (12)
In polar coordinates the equation (10) reads as:
\[
\begin{cases}
-\ii \hbar v_F \left( \partial_r + \frac{1}{r} \right) \xi_0 = -V(r) \chi_0, \\
-\ii \hbar v_F \partial_r \chi_0 = -V(r) \xi_0.
\end{cases}
\] (13)

which is equivalent to
\[
\begin{cases}
\xi_0(r) = i \hbar v_F (V(r))^{-1} \partial_r \chi_0, \\
\partial^2_r \chi_0 + \left( \frac{1}{r} - \frac{V'(r)}{V(r)} \right) \partial_r \chi_0 + \frac{V^2(r)}{r^2 v_F^2} \chi_0 = 0.
\end{cases}
\] (14)

For the potential \( V = -\frac{a}{r} \) we get the following equation on \( \chi_0(r) \):
\[
\partial^2_r \chi_0 + \frac{2}{r} \partial_r \chi_0 + \beta^2 \frac{1}{r^2} \chi_0 = 0, \quad \beta = \frac{\alpha}{\hbar v_F}. \] (15)

Supposing that \( \beta^2 = \frac{1}{4} + \nu^2 \) we write the solution as
\[
\chi_0 = \sqrt{r} \left( c_- \left( \frac{r}{r_0} \right)^{-i\nu} + c_+ \left( \frac{r}{r_0} \right)^{i\nu} \right) \propto \sqrt{r} \sin \left( \nu \log \frac{r}{r_0} + \varphi \right). \] (16)

It shares the properties of the ground-state Calogero wave-function and at nonzero \( c_\pm \) generates its own intrinsic length scale. In order to fix the constant we need to introduce a cut-off potential:
\[
V(r) = \begin{cases}
-\frac{a}{r}, & r > R, \\
V_{\text{reg}} = -\hbar v_F \frac{\lambda}{R}, & r \leq R.
\end{cases} \] (17)

The dilatation acts on \( \chi \) as following:
\[
r \partial_r \chi_0 = \left( \frac{1}{2} + \nu \cot \left( \nu \log \frac{r}{r_0} \right) \right) \chi_0.
\] (18)

For the constant potential \( V_{\text{reg}} \) we get from (14):
\[
\partial^2_r \chi_{0\text{reg}} + \frac{1}{r} \partial_r \chi_{0\text{reg}} + \frac{\lambda^2}{R^2} \chi_{0\text{reg}} = 0.
\] (19)

and solution of (19) regular at the origin is
\[
\chi_{0\text{reg}} \propto J_0 \left( \frac{\lambda}{R} r \right). \] (20)
Computing the action of the dilatation on the solution at short distance and equating it to the action of the dilatation (18) we get the equation on the parameter of regularization:

\[ \frac{1}{2} + \nu \cot \left( \nu \log \left( \frac{R}{r_0} \right) \right) = -\lambda \frac{J_1(\lambda)}{J_0(\lambda)}. \quad (21) \]

The equation (21) defines \( \lambda \) as a multi-valued function of \( R \). The period of the RG flow corresponds to jump from one branch of the \( \lambda(R) \) function to another.

Now we shall derive the bound states in the (9) potential and consider the Dirac equation,

\[ H_D \psi_\kappa = -\hbar v_F \kappa \psi_\kappa. \quad (22) \]

The equation on \( \chi \) analogous to (14) reads as:

\[ \partial_r^2 \chi_\kappa + \frac{2\beta - \kappa r}{\beta - \kappa r} \partial_r \chi_\kappa + \left( \frac{\beta}{r} - \kappa \right)^2 \chi_\kappa = 0. \quad (23) \]

Asymptotically at \( r \gg \frac{\beta}{\kappa} \) the solution to (23) regular at infinity is given by the Hankel function,

\[ \chi_\kappa \propto H_0^{(1)}(i\kappa r). \quad (24) \]

At small \( r \ll \frac{\beta}{\kappa} \) the solution is not regular at the origin,

\[ \chi_\kappa \propto \sqrt{r} \sin \left( \nu \log \left( \frac{r}{r_0} \right) \right), \quad (25) \]

and we need for the regulator potential. Solving the Dirac equation (22) in presence of the constant potential \( V_{reg} \) and computing the action of the dilation operator,

\[ r \partial_r \chi_{\kappa \kappa}^{reg} = -\left( \lambda - \kappa R \right) \frac{J_1(\lambda - \kappa R)}{J_0(\lambda - \kappa R)} \chi_{\kappa \kappa}^{reg}, \quad (26) \]

we can equate (26) to the action of the dilatation operator on (25) and get the equation on the spectrum of the bound states,

\[ \frac{1}{2} + \nu \cot \left( \nu \log (\kappa R) \right) = -\left( \lambda - \kappa R \right) \frac{J_1(\lambda - \kappa R)}{J_0(\lambda - \kappa R)}. \quad (27) \]
This condition gives the spectrum of infinitely many shallow bound states, with Efimov scaling
\[ \kappa_n = \kappa_* \exp \left( -\frac{\pi n}{\nu} \right), \quad \kappa \to \infty. \] (28)

### 4 Anomalous commutator algebra

Let us make some comments on the algebraic counterpart of the phenomena considered following [10]. As we have mentioned above the conformal symmetry is the main player since Hamiltonians under consideration are scale invariant before regularization. Actually this group can be thought of as the example of spectrum generating algebra when the Hamiltonian is identified with one of the generators or is expressed in terms of the generators in a simple manner. This situation is familiar from the exactly or quasi-exactly solvable problems in quantum mechanics when the dimension of the group representation selects the size of the algebraic part of the spectrum.

Let us introduce the generators of the $SO(2,1)$ conformal algebra $J_1, J_2, J_3$ : the Calogero Hamiltonian,
\[ J_1 = H = p^2 + V(r), \] (29)

the dilatation
\[ J_2 = D = tH - \frac{1}{2} (pr + rp), \] (30)

and the generator of special conformal transformations,
\[ J_3 = K = t^2 H - \frac{t}{2} (pr + rp) + \frac{1}{2} r^2. \] (31)

They satisfy the relations of the $\mathfrak{so}(2,1)$ algebra:
\[ [J_2, J_1] = -iJ_1, \quad [J_3, J_1] = -2iJ_2, \quad [J_2, J_3] = iJ_3 \] (32)

The singular behavior of the potential at the origin amounts to a kind of anomaly in the $\mathfrak{so}(2,1)$ algebra [10],
\[ A(r) = -i[D, H] + H, \] (33)
which in $d$ space dimensions can be presented in the following form:

$$A(r) = -\frac{d - 2}{2} V(r) + (\nabla_i r^i) V(r). \quad (34)$$

The simple arguments imply the following relation

$$\frac{d}{dt} \langle D \rangle_{\text{ground}} = E_{\text{ground}}, \quad (35)$$

where the matrix element is taken over the ground state.

It turns out that (35) is fulfilled for the singular potentials in Calogero-like model or in models with contact potential, $V(r) = g\delta(r)$. The expression for anomaly does not depend on the regularization chosen. Moreover the detailed analysis demonstrates that the anomaly is proportional to the $\beta$-function of the coupling providing the UV regularization as can be expected.

The similar calculation of the anomaly for the 2+1 case can be performed for arbitrary state.

$$\langle \frac{d D}{dt} \rangle_\psi = \Xi_\psi = -\int d^2 x \psi^*(V(x) + x_i \partial_i V(x)) \psi, \quad (36)$$

which yields with square-well regularization

$$\Xi_\psi = \hbar v_F \frac{\lambda(R)}{r} \int_0^R \frac{r |\psi|^2 dr}{r \int_0^\infty |\psi|^2 dr} \quad (37)$$

Wave function for the very shallow state has the asymptotic behavior

$$\chi = \begin{cases} A r^{\frac{1}{2}} \sin(\nu \log(\kappa r)), & L \gg r > R \\ B J_0(\frac{\nu}{R}), & r < R \end{cases} \quad (38)$$

$$\phi = \begin{cases} A_2^{\nu} r^{\frac{1}{2}} \cos(\nu \log(\kappa r)) + A_1^\nu r^{\frac{1}{2}} \sin(\nu \log(\kappa r)), & L \gg r > R \\ B J_1(\frac{\nu}{R}), & r < R \end{cases} \quad (39)$$

Since function behaves well near the origin ($\sqrt{r} \sin(r) \to 0, r \to 0$ and $J_\alpha(r) \sim r^\alpha$) we get

$$\int_0^\infty |\psi|^2 r dr = A^2 N + O(R), R \to 0 \quad (40)$$
Due to continuity we can use the following condition $\chi(R - 0) = \chi(R + 0), \phi(R - 0) = \phi(R + 0)$ and obtain

$$\Xi(R) = \lambda(R) \frac{\hbar v_F}{RA^2N} \int_0^R |\psi|^2 r dr = \frac{\hbar v_F}{N} \left( \frac{1}{2} + \frac{\nu^2}{2\beta^2} + \frac{1}{8\beta^2} \right) \lambda(R) R^2$$  \hspace{1cm} (41)

Hence, if we choose log-periodic behavior of $\lambda(R)$, we immediately get in the $R \to 0, \Xi(R) \to 0$. On the other hand if we choose continuous branch of $\lambda(R)$ there is no a well-defined limit.

5 Conclusion

In this note we have interpreted the atomic collapse problem considered in [11] in terms of the RG cycle. The consideration was similar to the non-relativistic Calogero case. Introducing the short distance cutoff and imposing the RG condition in terms of zero energy wave function we get the RG equation with limit cycle. It was found that the period of the RG cycle is fixed by the value of the external charge. The spectrum of the states with the Efimov scaling gets rearranged upon the cycle and $E_{n+1}$ starts to play the role of $E_n$.

It would be interesting to push forward the analogy with $Z > 137$ phenomena further. The possible conjecture could sound as follows. When one discusses the Efimov state the general picture involves the pair of particles near the threshold while the third particle provides the three-body bound state. We could imagine that in the atomic collapse problem the electron pair plays the role of two particles near threshold while the external source plays the role of the third body. It would be also interesting to discuss the possibility of the similar cyclic RG flow in the bilyer graphene with possible Mexican hat potential discussed in [19].

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