SLOPES OF EUCLIDEAN LATTICES, TENSOR PRODUCT AND GROUP ACTIONS

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ABSTRACT. We study the behaviour of the minimal slope of Euclidean lattices under tensor product. A general conjecture predicts that
\[ \mu_{\text{min}}(L \otimes M) = \mu_{\text{min}}(L) \mu_{\text{min}}(M) \]
for all Euclidean lattices \( L \) and \( M \). We prove that this is the case under the additional assumptions that \( L \) and \( M \) are acted on multiplicity-free by their automorphism group, such that one of them has at most 2 irreducible components.

INTRODUCTION

The notion of slope and the related concept of (semi)stability were initially introduced to study vector bundles over smooth projective curves [Mum63, NS65]. Since then, similar concepts have appeared in a wide range of mathematical contexts, often by analogy with the original geometric setting. In these various theories, one can define a canonical filtration of any object by semistable ones, a fact first recognized by G. Harder and M. Narasimhan in the case of vector bundles on curves [HN75]. A canonical polygon is associated to this filtration, from which the terminology of "slopes" arise. The similarities between the many occurrences of these slope filtrations strongly appealed for a general theoretical framework, which was developed more recently by Y. André [And09] and H. Chen [Che10].

This formalism applies in particular to Euclidean lattices, as observed by U. Stuhler in [Stu76, Stu77], who laid the foundation of the theory in the more general context of \( \mathcal{O} \)-lattices, \( \mathcal{O} \) being the ring of integers of a number field, having in mind applications to the study of arithmetic groups. This program was pushed further by D. Grayson [Gra84, Gra86] who showed that these ideas may be used as an alternative to Borel-Serre compactification to prove the finite presentation of arithmetic groups and the finite generation of their (co)homology. An Arakelov version of this theory, in terms of Hermitian vector bundles, was built by J.-B. Bost in the 1990s, and has been much studied since then. Sticking to the case of ordinary Euclidean lattices, the relevant notions are defined as follows:

- the slope \( \mu(L) \) of a nonzero Euclidean lattice \( L \) is the quantity \( (\det L)^{1/\dim L} \);
- its minimal slope \( \mu_{\text{min}}(L) \) is the minimum of the slopes of all its non zero sublattices;
- a lattice is semi-stable if its slope and minimal slope coincide, or equivalently, if its canonical filtration is trivial.

Note that we adopt here a multiplicative version of the slope, following Stuhler, whereas more recent works, in compliance with the geometric origin of the theory, rather define the slope as
\[ -\log \frac{\det L}{\dim L} = -\log \mu(L), \]
and accordingly consider maximal instead of minimal slope.

Quite naturally, one would like to understand the behaviour of slopes and semistability under standard algebraic operations (direct sum, exact sequences, duality, tensor product). In this respect, the tensor product is rather enigmatic. It is known that the tensor product of semistable vector bundles on a smooth algebraic curve in characteristic zero is semistable [NS65]. Conservation of semistability under tensor product is also known to hold in several other contexts where a slope filtration is available, but it can apparently not be explained.
using general formal arguments: in all cases an *ad hoc* proof is needed, often difficult. In [And09], Y. André coined the term "tensor-multiplicative" to qualify slope filtrations having this property. Surprisingly, the question as to whether tensor-multiplicativity holds for Euclidean lattices is still largely open. Recall that the tensor product of two inner product spaces $E$ and $F$ can be turned into an inner product space by setting

\[(1) \quad x \otimes y : x' \otimes y' = (x \cdot x')(y \cdot y')\]

for all $x, x'$ in $E$ and $y, y'$ in $F$ and extending (1) to $E \otimes F$ by bilinearity. If $L \subseteq E$ and $M \subseteq F$ are two Euclidean lattices, their tensor product $L \otimes M$ thus inherits a structure of Euclidean lattice in $E \otimes \mathbb{R} F$ equipped with the above inner product. In a seminar held in Oberwolfach in July 1997, J.-B. Bost conjectured that the slope filtration of Euclidean lattices is tensor-multiplicative, which reduces to the following statement:

**Conjecture 1.** The minimal slope of the tensor product of two Euclidean lattices $L$ and $M$ is equal to the product of their respective minimal slopes

\[(2) \quad \mu_{\text{min}}(L \otimes M) = \mu_{\text{min}}(L)\mu_{\text{min}}(M).\]

An equivalent formulation of this conjecture, in terms of semistability, is as follows:

**Conjecture 1bis.** The tensor product of two semistable lattices is semistable.

The inequality $\mu_{\text{min}}(L \otimes M) \leq \mu_{\text{min}}(L)\mu_{\text{min}}(M)$ is clear, so the question is whether this inequality can be strict. The equivalence between Conjecture 1 and its variation 1bis is well-known to the experts, although it is somewhat difficult to find a reference for this in the literature (a short argument is presented in Section 2 Proposition 2.3). Note that the initial conjecture was formulated in the wider context of Hermitian vector bundles and is consequently a priori stronger than Conjecture 1.

The hope that the known proofs of the tensor-multiplicativity for vector bundles over curves could inspire a proof of Conjecture 1 is probably overoptimistic, as it was very precisely analyzed by Y. André in [And11]. From another viewpoint, this conjecture is reminiscent of the problem of the first minimum of a tensor product: denoting by $\lambda(L)$ the shortest length of a nonzero vector in a lattice $L$, it is clear that

\[(3) \quad \lambda(L \otimes M) \leq \lambda(L)\lambda(M)\]

for all lattices $L$ and $M$ but it is known, by a non constructive argument due to Steinberg, that in large dimensions, there exist lattices for which inequality (3) is strict (see [MH73, The9]). On the opposite direction, Kitaoka showed that (3) is an equality as long as $L$ or $M$ has dimension $\leq 42$ (see [Kit77], [Kit93, chapter 7]), which makes the construction of an explicit example of lattices $L$ and $M$ for which (3) is not an equality quite difficult, if ever possible. This analogy would apparently advocate against Conjecture 1: one could expect (2) to be true in small dimensions, but false in general for high dimensional lattices. However, our understanding of the minimal slope of a tensor product is at the moment quite modest: the validity of (2) has been established only in very small dimensions by Bost and Chen [BCh], namely when $\dim L \dim M \leq 9$, through a very difficult proof; on the other hand, almost no general result is known for higher dimensional lattices except some cases where (2) holds for trivial reasons (e.g. when $L$ and $M$ are both unimodular).

The two fundamental obstacles to overcome in order to prove or disprove equality (2) are the computation of the minimal slope of a lattice, which is difficult in general, and the complexity of the sublattices of a tensor product. These problems can nevertheless be notably simplified in the event that a sufficiently big automorphism group is available. For instance, if the automorphism group of either $L$ or $M$ acts absolutely irreducibly on the underlying space, then (2) is true (see [Bos96, Proposition A.3] or [GR13, Proposition 5.1]). This is based on the crucial observation that the canonical filtration of a lattice is fixed by its automorphisms (see Theorem 1.4 (5) below). An intermediate situation is that of a multiplicity-free
action. Indeed, this assumption guarantees that the lattices involved possess only finitely many sublattices fixed by the corresponding automorphism groups. In this situation, the computation of the minimal slope becomes in principle tractable, whatever the dimension, since it amounts to the inspection of finitely many invariant sublattices. Moreover, if both lattices $L$ and $M$ are acted on multiplicity-free by their automorphism groups, then it is easily seen (see Proposition 2.2) that the minimal slope of $L \otimes M$ is achieved on a subspace $U$ which is "split", i.e. of the shape

$$ U = \bigoplus_{i=1}^{r} E_i \otimes F_i $$

where the $E_i$s and $F_i$s are subspaces of $\mathbb{R}L$ and $\mathbb{R}M$ respectively. This is of course a very specific situation, but nevertheless, even in that case, the verification of (2) is nontrivial, except if $r = 1$. Thus, it seems natural to study the following special case of Conjecture 1:

**Conjecture 2.** Let $L$ and $M$ be lattices acted on multiplicity-free by their automorphism groups. Then $\mu_{\text{min}}(L \otimes M) = \mu_{\text{min}}(L)\mu_{\text{min}}(M)$.

Our main result is a proof of the first nontrivial case of Conjecture 2 that is when $r = 2$ in (4). In other words, we will establish the following result:

**Theorem.** Let $L$ and $M$ be lattices acted on multiplicity-free by their automorphism groups, such that one of them has at most two irreducible components. Then $\mu_{\text{min}}(L \otimes M) = \mu_{\text{min}}(L)\mu_{\text{min}}(M)$.

This result, and more generally Conjecture 2 (if true), partly offsets the impression, based on the analogy with the first minimum, that Conjecture 1 could be false.

Here is an outline of the paper: in Section 1 we recall, essentially without proofs, the basics about slopes of lattices. General considerations about tensor products, group actions and slope filtration are developed in Section 2. The final section is devoted to the proof of our main result.

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**NOTATION**

In the sequel, we define the determinant of a $n$-dimensional lattice $L$ as the volume of the fundamental parallelootope associated to any of its $\mathbb{Z}$-basis $B = (e_1, \ldots, e_n)$. It is thus the square root of the determinant of the Gram matrix $\text{Gram} B = (e_i \cdot e_j)_{1 \leq i,j \leq n}$:

$$ \det L = \sqrt{\det \text{Gram} B}. $$

Note that this convention differs from the one adopted e.g. in [CS99], which take the determinant of a Gram matrix as the definition of $\det L$.

To avoid confusions between ordinary and orthogonal direct sums, we will use the symbol $\perp$ for the latter and $\oplus$ for the former.
1. Preliminaries on slopes

General references for this introductory section are \[\text{Stu76, Gra84, Cas04, And09, And11}\].

A Euclidean lattice is a free \(\mathbb{Z}\)-module equipped with a positive definite quadratic form (inner product). In the sequel, in accordance with the convention in \[\text{Gra84}\], the word sublattice stands for "primitive" (or "pure") sublattice (recall that a sub \(\mathbb{Z}\)-module \(M\) of a \(\mathbb{Z}\)-module \(L\) is primitive if the quotient \(L/M\) is torsion free, or equivalently if \(M\) is a direct summand of \(L\)). We will have to consider in places non primitive submodules of a given lattice \(L\), in particular submodules of finite index, and will consequently refrain from using the term "sublattice" in such situations.

If \(M\) is a sublattice of \(L\), then the quotient \(L/M\) is also a Euclidean lattice, the inner product being inherited from the identification of \(L/M \otimes \mathbb{R}\) with \((\mathbb{R}M)^\perp\). By a morphism of lattices, we mean a morphism \(f\) of the underlying \(\mathbb{Z}\)-modules, with operator norm \(\leq 1\).

The notion of exact sequence is defined accordingly. For instance, whenever \(M\) is a submodule of \(L\), we have an exact sequence
\[
0 \rightarrow M \rightarrow L \rightarrow L/M \rightarrow 0.
\]

**Definition 1.1.**

1. The slope of a lattice \(L\) is defined as
\[
\mu(L) = (\det L)^{1/\dim L}.
\]
2. The minimal and maximal slopes of a lattice are defined as
\[
\mu_{\text{min}}(L) = \min_{0 \neq M \subseteq L} \mu(M)
\]
and
\[
\mu_{\text{max}}(L) = \max_{N \subseteq L} \mu(L/N).
\]

We collect in the following lemma some elementary properties of slopes, for further reference.

**Lemma 1.2.** Let \(L, M,\) and \(N\) be Euclidean lattices of dimension \(\ell = \dim(L), m = \dim(M),\) and \(n = \dim(N)\).

1. If \(M\) is \(\mathbb{Z}\)-submodule of \(L\) of finite index, then \(\ell = m\) and \([L : M] = \left(\frac{\mu(M)}{\mu(L)}\right)^\ell\).
2. \(\mu(L^*) = \mu(L)^{-1}\).
3. \(\mu(L \otimes M) = \mu(L)\mu(M)\).
4. For any exact sequence
\[
0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0
\]
one has
\[
\mu(L)^\ell = \mu(M)^m \mu(N)^n.
\]
Consequently,
\[
\min(\mu(M), \mu(N)) \leq \mu(L) \leq \max(\mu(M), \mu(N))
\]
and both inequalities are strict, unless \(\mu(M) = \mu(N) = \mu(L)\).
5. In particular
\[
\mu(L/M)^{\ell-m} = \mu(L)^\ell \mu(M)^{-m}
\]
\[
\mu(M \perp N)^{m+n} = \mu(M)^m \mu(N)^n.
\]

A slightly less immediate property of slopes is the following inequality, which we call the "parallelogram constraint", following Grayson:
Lemma 1.3. [Stu76, Proposition 2][Gra84, Theorem 12] If $L_1$ and $L_2$ are sublattices of a lattice $L$, one has:

$$\mu(L_1 / L_1 \cap L_2) \geq \mu(L_1 + L_2 / L_2).$$

The invariant $\mu_{\text{min}}$ (or $\mu_{\text{max}}$) induces a canonical filtration, starting with the so-called destabilizing sublattice. We collect below without proofs some properties of this filtration, which we name Grayson-Stuhler filtration in reference to its discoverers (the proofs essentially rely on repeated applications of Lemma 1.3).

Theorem 1.4. [Stu76, Satz 1][Gra84] Let $L$ be a non zero Euclidean lattice.

1. The set of sublattices $M \subset L$ such that $\mu(M) = \mu_{\text{min}}(L)$, admits a maximum, with respect to inclusion, called the destabilizing sublattice of $L$.
2. The set of sublattices $N \subset L$ such that $\mu(L / N) = \mu_{\text{max}}(L)$ admits a minimum, with respect to inclusion, called the co-destabilizing sublattice of $L$.
3. To any lattice $L$ is associated a canonical filtration

$$\{0\} = L_{(0)} \subset L_{(1)} \subset L_{(2)} \subset \cdots \subset L_{(m)} = L$$

defined recursively by the conditions that

a) $L_{(1)}$ is the destabilizing sublattice of $L$.
b) $\{0\} = L_{(1)} / L_{(1)} \subset L_{(2)} / L_{(1)} \subset \cdots \subset L_{(m)} / L_{(1)} = L / L_{(1)}$ is the canonical filtration of $L / L_{(1)}$.
4. With the previous notation, $L_{(1)}$ and $L_{(m-1)}$ are respectively the destabilizing and co-destabilizing sublattices of $L$.
5. The canonical filtration is invariant under any automorphism of $L$.
6. One has $\mu(L_{(i)}/L_{(i-1)}) < \mu(L_{(i+1)}/L_{(i)})$ for any $1 \leq i \leq m - 1$ and this set of inequalities characterizes the canonical filtration.

Note that, since a sublattice $M$ of $L$ is entirely determined by the subspace $F = \mathbb{R}M$ it generates in $E = \mathbb{R}L$, the canonical filtration can be viewed as a filtration by vector subspaces of the underlying vector space $E$, namely

$$\{0\} \subset F_{(1)} \subset F_{(2)} \subset \cdots \subset F_{(m)} = E$$

where $F_{(i)} = \mathbb{R}L_{(i)}$ and, conversely, $L_{(i)} = L \cap F_{(i)}$. One can consequently speak of the destabilizing (resp. co-destabilizing) subspaces of $E$ with respect to $L$.

Although we won’t develop further this point of view, one should also point out the geometric interpretation of this canonical filtration, in terms of the associated canonical polygon, which justifies the terminology of slope (resp. minimal and maximal slope) see [Gra84].

The notion of semi-stability can be formulated using the previous proposition:

Definition 1.5. A lattice $L$ is semistable if it satisfies one of the following equivalent conditions:

1. $\mu(M) \geq \mu(L)$ for every sublattice $M$ of $L$.
2. $L$ coincides with its destabilizing sublattice $L_{(1)}$.
3. The co-destabilizing sublattice $L_{(m-1)}$ of $L$ is reduced to $\{0\}$.

We now review the properties of $\mu$ and $\mu_{\text{min}}$ with respect to quotients and duality. Suppose $L$ is a lattice with underlying space $E = \mathbb{R}L$. For any sublattice $M$, with underlying vector space $F = \mathbb{R}M$, the quotient $L/M$ inherits a Euclidean structure via the identification of $E / F$ with $F^\perp$. One can as well identify $L / M$ with $\pi_{F^\perp}(L)$, the orthogonal projection of $L$ on $F^\perp$.

The dual $L^* = \text{Hom}(L, \mathbb{Z}) \subset (\mathbb{R}L)^*$ of a lattice $L$ inherits a Euclidean structure from the identification between $\mathbb{R}L$ and $(\mathbb{R}L)^*$ via the inner product.

If $M$ is any sublattice of $L$, then to the exact sequence

$$0 \rightarrow M \rightarrow L \rightarrow L / M \rightarrow 0$$
corresponds an exact sequence

\[ 0 \rightarrow (L/M)^* \rightarrow L^* \rightarrow M^* \rightarrow 0. \]

Consequently, setting \( M^2 := \text{Im} ((L/M)^* \rightarrow L^*) \), we have

\[ \mu(L^* / M^2) = \mu(M^*) = \mu(M)^{-1}. \] (5)

Note that, under the identification between \( L/M \) and \( \pi_F(L) \), where \( F = \mathbb{R}M \), the lattice \( M^* \) identifies with \( \pi_F(L^*) \) and \( M^2 \) with \( L^* \cap F^\perp \). As a consequence, we get the following proposition:

**Proposition 1.6.** The map

\[ M \mapsto M^2 := \text{Im} ((L/M)^* \rightarrow L^*) \]

induces a bijection between the sets of sublattices of \( L \) and \( L^* \) respectively, which exchanges the destabilizing and co-destabilizing sublattices of \( L \) and \( L^* \).

In particular,

\[ \mu_{\text{max}}(L^*) = (\mu_{\text{min}}(L))^{-1}. \]

More generally, if

\[ \{0\} = L_{(0)} \subset L_{(1)} \subset L_{(2)} \subset \cdots \subset L_{(m)} = L \]

is the canonical filtration of \( L \) then

\[ \{0\} = L^2_{(m)} \subset L^2_{(m-1)} \subset L^2_{(m-2)} \subset \cdots \subset L^2_{(1)} \subset L^2_{(0)} = L^* \]

is that of \( L^* \).

In particular, \( L \) is semistable if and only if \( L^* \) is.

We end this section with a last useful property of \( \mu_{\text{min}} \) with respect to quotients, communicated to the first author by Gaël Rémond:

**Lemma 1.7.** If \( M \subset L \) then

\[ \mu_{\text{min}}(M) \geq \mu_{\text{min}}(L) \geq \min(\mu_{\text{min}}(M), \mu_{\text{min}}(L/M)). \]

In particular, if \( \mu_{\text{min}}(M) \leq \mu_{\text{min}}(L/M) \) then \( \mu_{\text{min}}(L) = \mu_{\text{min}}(M) \).

**Proof.** Let \( L' \) be the destabilizing sublattice of \( L \); one has the short exact sequence

\[ 0 \rightarrow L' \cap M \rightarrow L' \rightarrow L'/L' \cap M \rightarrow 0 \] (6)

from which we derive the inequality

\[ \mu_{\text{min}}(L) = \mu(L') \geq \min (\mu(L' \cap M), \mu(L'/L' \cap M)). \] (7)

On the other hand, from the parallelogram inequality ([Stu76, Proposition 2], [Gra84, Theorem 12]), one has

\[ \mu(L'/L' \cap M) \geq \mu(L' + M/M) \geq \mu_{\text{min}}(L/M). \] (8)

Finally,

- if \( \mu(L' \cap M) \leq \mu(L'/L' \cap M) \), then \( \mu_{\text{min}}(L) = \mu(L') \geq \mu(L' \cap M) \geq \mu_{\text{min}}(M) \),
- if \( \mu(L' \cap M) \geq \mu(L'/L' \cap M) \), then \( \mu_{\text{min}}(L) = \mu(L') \geq \mu(L'/L' \cap M) \geq \mu(L' + M/M) \geq \mu_{\text{min}}(L/M) \).

In all cases, one has

\[ \mu_{\text{min}}(M) \geq \mu_{\text{min}}(L) \geq \min(\mu_{\text{min}}(M), \mu_{\text{min}}(L/M)). \]

\[ \square \]

2. Grayson-Stuhler filtration in the presence of a group action

This section, still preparatory, gives some structural properties of the destabilizing sublattice of lattices endowed with group actions and their tensor products.
2.1. Review on the tensor product of group representations.

Definition 2.1. Let $K$ be a field and $G$ a group.

1. A $K[G]$-module $E$ is absolutely irreducible if the $\overline{K}[G]$-module $E_{\overline{K}} := E \otimes_K \overline{K}$ is irreducible. Here $\overline{K}$ denotes an algebraic closure of $K$.
2. A $K[G]$-module $E$ is multiplicity-free if it splits as a direct sum $E = \bigoplus_{i=1}^r E_i$ of pairwise non isomorphic absolutely irreducible $K[G]$-submodules $E_i$.

The following proposition is certainly classical. We nevertheless include it, together with its proof, for sake of completeness.

Proposition 2.2. Let $G$, $H$ be finite groups, and $K$ a field of characteristic either zero or prime to both $|G|$ and $|H|$. Let $E$ be a finite dimensional $K[G]$-module, $F$ a finite dimensional $K[H]$-module.

If $E$ is a multiplicity-free $K[G]$-module, i.e. $E = \bigoplus_{i=1}^r E_i$ where the $E_i$s are pairwise non isomorphic absolutely irreducible $K[G]$-submodules, then for any $K[G \times H]$-submodule $U$ of $E \otimes F$, there exist $K[H]$-submodules $F_1, \ldots, F_r$ of $F$ (possibly zero) such that

$$U = \bigoplus_{i=1}^r E_i \otimes F_i.$$ 

Note that the $E_i$s and $F_j$s play asymmetrical roles in the above statement: in particular, the $F_j$s are not irreducible in general, even if $F$ is multiplicity free as a $K[H]$-module, and may intersect each other non trivially.

Proof of the proposition. Let us first observe that if $E'$ is an absolutely irreducible $K[G]$-module and $F'$ an irreducible $K[H]$-module, then $E' \otimes F'$ is $G \times H$-irreducible over $K$. Indeed, under these assumptions, one has $\text{End}_{K[G]}(E') = K \text{Id}_{E'}$ (see e.g. [CR81, Theorem 3.43]) and $\text{End}_{K[H]}(F') = D$ is a division algebra over $K$ (Schur’s lemma), so that $\text{End}_{K[G \times H]}(E' \otimes F') = K \text{Id}_{E'} \otimes D \simeq D$, whence we deduce that $E' \otimes F'$ is $G \times H$-irreducible over $K$, since the algebra $K[G \times H]$ is semisimple.

Let now $F = \bigoplus F_j$ be the decomposition of $F$ into $H$-isotypic components (note that the $F_j$s are not $H$-irreducible in general). From the previous observation, we deduce easily that the subspaces $E_i \otimes F_j$ are the $G \times H$-isotypic components of $E \otimes F$. The proposition will be finally proved if we can show that any $G \times H$-stable subspace $U$ of $E_i \otimes F_j$ is equal to $E_i \otimes F_j$ for a suitable $H$-stable subspace $F_j$ of $F_j$. The semi simplicity of $K[G \times H]$ insures that such a subspace $U$ admits a $G \times H$-invariant complement $V$, and the projector $p$ corresponding to the direct sum decomposition $U \oplus V$ is an element of $\text{End}_{K[G \times H]}(E_i \otimes F_j) \simeq K \text{Id}_{E_i} \otimes \text{End}_{K[H]}(F_j) = K \otimes \text{End}_{K[H]}(F_j)$. In other words, $p = \lambda \otimes f$, with $\lambda \in K$ and $f \in \text{End}_{K[H]}(F_j)$, so that $U = p(E_i \otimes F_j) = E_i \otimes f(F_j)$ is of the required form $E_i \otimes F_j$, setting $F_j = f(F_j)$. \[ \square \]

2.2. Application to the canonical filtration of a tensor product. We will apply the above general considerations to derive some properties of the destabilizing sublattice of a tensor product. From now on, $L$ and $M$ will be Euclidean lattices with automorphism groups $G$ and $H$ respectively. When dealing with Conjecture 1, we make no further assumption on $G$ and $H$ (they might even be trivial), while in the case of Conjecture 2, we assume that $G$ and $H$ act multiplicity-free on the underlying real vector spaces $E = \mathbb{R}L$ and $F = \mathbb{R}M$ respectively. Note also that, in this Euclidean context, and since $G$ and $H$ consist of orthogonal automorphisms, the isotypic components of $E$ and $F$ are mutually orthogonal.

The next proposition, of interest in itself, shows in particular that it is enough to consider Conjecture 1 and 2 for semi-stable lattices (in other words: Conjecture 1 and 1bis are equivalent, as already mentioned in the introduction).

Proposition 2.3. Let $(L, M)$ be a minimal counter-example to either Conjecture 1 or Conjecture 2 (by "minimal", we mean: "such that $\dim L + \dim M$ is minimal"). Denote by $E$ and $F$ respectively
the Euclidean spaces spanned by $L$ and $M$ respectively, and by $U$ the destabilizing subspace of $E \otimes F$ with respect to $L \otimes M$. Then:

1. $L$ and $M$ are semi-stable.
2. If $U$ splits as $U = \bigoplus_{i=1}^{r} E_i \otimes F_i$ where the $E_i$s are pairwise orthogonal subspaces of $E$, and the $F_i$s are subspaces of $F$, then

$$\sum_{i=1}^{r} E_i = E, \quad \sum_{i=1}^{r} F_i = F \text{ and } \bigcap_{i=1}^{r} F_i = \{0\}.$$ 

Remark. The situation in Proposition 2.3(2) is of course very specific: there is a priori no reason that the destabilizing subspace of $E \otimes F$ with respect to $L \otimes M$ admits a splitting of the form $U = \bigoplus_{i=1}^{r} E_i \otimes F_i$ in general.

Proof. (1) The reduction to the semi-stable case is classical and was explained to the first author by Gaël Rémond. We reproduce the argument here, by lack of an appropriate reference. Assume, by way of contradiction, that the pair $(L, M)$ violate either Conjecture 1 or 2, and that $L$ is not semi-stable. Let $L' \subseteq L$ be the destabilizing sublattice of $L$. Thanks to Lemma 1.7, we have:

$$\mu_{\min}(L' \otimes M) \geq \mu_{\min}(L \otimes M) \geq \min\left(\mu_{\min}(L' \otimes M), \mu_{\min}(L/L' \otimes M)\right).$$

We deduce from the minimality assumption on the pair $(L, M)$ that both $(L', M)$ and $(L/L', M)$ satisfy Conjecture 1 or 2 respectively (note that for the latter, we use the fact that the multiplicity free assumption is preserved for $G$-submodules and quotients). Consequently,

$$\mu_{\min}(L' \otimes M) = \mu_{\min}(L')\mu_{\min}(M) = \mu_{\min}(L)\mu_{\min}(M)$$

and

$$\mu_{\min}(L/L' \otimes M) = \mu_{\min}(L/L')\mu_{\min}(M).$$

We also know from Theorem 1.4(6) that $\mu_{\min}(L/L') > \mu_{\min}(L)$ if $L'$ is the destabilizing sublattice of $L$; together with (9), this implies that

$$\mu_{\min}(L \otimes M) = \mu_{\min}(L)\mu_{\min}(M),$$

contradicting our initial assumption.

(2) Let $\hat{E} = \sum_{i=1}^{r} E_i$, $\hat{F} = \sum_{i=1}^{r} F_i$, $\hat{L} = L \cap \hat{E}$, and $\hat{M} = M \cap \hat{F}$. The sublattice $\hat{L} \otimes \hat{M}$ is primitive, i.e. $\hat{L} \otimes \hat{M} = (L \otimes M) \cap \hat{E} \otimes \hat{F}$, and since $U \subset \hat{E} \otimes \hat{F}$, one has

$$(L \otimes M) \cap U = (\hat{L} \otimes \hat{M}) \cap U.$$ 

By the very definition of $U$ we get

$$\mu_{\min}(L \otimes M) = \mu(L \otimes M \cap U) = \mu(\hat{L} \otimes \hat{M} \cap U).$$

Assuming that $(L, M)$ violates Conjecture 1, we have

$$\mu_{\min}(L)\mu_{\min}(M) > \mu_{\min}(L \otimes M)$$

so that (10), together with the trivial inequality

$$\mu(\hat{L} \otimes \hat{M} \cap U) \geq \mu_{\min}(\hat{L} \otimes \hat{M})$$

implies that

$$\mu_{\min}(L)\mu_{\min}(M) > \mu_{\min}(\hat{L} \otimes \hat{M}).$$

Yet, if we assume that either $\hat{E} \neq E$ or $\hat{F} \neq F$, then by the minimality assumption for the pair $(L, M)$, we infer that $(\hat{L}, \hat{M})$ satisfies Conjecture 1 so that

$$\mu_{\min}(\hat{L} \otimes \hat{M}) = \mu_{\min}(\hat{L})\mu_{\min}(\hat{M}) \geq \mu_{\min}(L)\mu_{\min}(M)$$

contradicting (11).
It remains to prove that $\bigcap_{i=1}^r F_i = \{0\}$, or alternatively that $\sum_{i=1}^r F_i^\perp = F$ (note that the corresponding statement for the $E_i$s is automatically satisfied from the assumption that these subspaces are mutually orthogonal). Having proven that $E = \bigoplus_{i=1}^r E_i$, and using the pairwise orthogonality of the $E_i$s, a simple calculation yields

$$U^\perp = \left( \bigoplus_{i=1}^r E_i \otimes F_i \right)^\perp = \bigoplus_{i=1}^r E_i \otimes F_i^\perp.$$  

From part (1) we can moreover assume that $L$ and $M$ are semi-stable, as well as $L^*$ and $M^*$. In particular

$$\mu_{\text{min}}(L^*) \mu_{\text{min}}(M^*) = \mu(L^*) \mu(M^*),$$

As $U$ is the destabilizing subspace of $E \otimes F$ with respect to $L \otimes M$, we infer that $U^\perp$ is the co-destabilizing subspace of $E^* \otimes F^*$ with respect to $L^* \otimes M^*$, so that the destabilizing subspace $V$ of $E^* \otimes F^*$ with respect to $L^* \otimes M^*$ is contained in $U^\perp = \bigoplus_{i=1}^r E_i \otimes F_i^\perp$. In particular, since $U^\perp \subseteq E \otimes F$, as $U \neq \{0\}$, we have

$$\mu(L^* \otimes M^*) > \mu_{\text{min}}(L^* \otimes M^*) = \mu(L^* \otimes M^* \cap V)$$

by definition of $V$. Setting $\hat{F} = \sum_{i=1}^r F_i^\perp$ and $\hat{M}^* = M^* \cap \hat{F}$, and noticing that $V \subset E \otimes \hat{F}$, we can reproduce essentially the same argument as before: one has

$$(L^* \otimes M^*) \cap V = \left( L^* \otimes \hat{M}^* \right) \cap V$$

so that

$$\mu_{\text{min}}(L^* \otimes M^*) = \mu(L^* \otimes M^* \cap V) = \mu(L^* \otimes \hat{M}^* \cap V)$$

and

$$\mu(L^* \otimes M^*) > \mu_{\text{min}}(L^* \otimes M^* \cap V) \geq \mu_{\text{min}}(L^* \otimes \hat{M}^*).$$

If $\hat{F} \neq F$, then $\dim L^* + \dim \hat{M}^* < \dim L + \dim M$ and the minimality assumption on the pair $(L, M)$ again implies that

$$\mu_{\text{min}}(L^* \otimes \hat{M}^*) = \mu_{\text{min}}(L^*) \mu_{\text{min}}(\hat{M}^*) \geq \mu_{\text{min}}(L^*) \mu_{\text{min}}(M^*) = \mu(L^* \otimes \hat{M}^*),$$

contradicting (15). Therefore $\hat{F} = F$, or in other words $\bigcap_{i=1}^r F_i = \{0\}$.

\[\square\]

3. Main result

**Theorem 3.1.** Let $L$ and $M$ be Euclidean lattices, respectively in $E = \mathbb{R} M$ and $F = \mathbb{R} M$. Let $G \leq \text{Aut} L$ and $H \leq \text{Aut} M$ and assume $E$ and $F$ to be respectively $G$ and $H$ multiplicity free. Denote by $r$ and $s$ the number of irreducible components of $E$ and $F$ respectively. Then, if $\min(r,s) \leq 2$, one has

$$\mu_{\text{min}}(L \otimes M) = \mu_{\text{min}}(L) \mu_{\text{min}}(M).$$

The proof of the theorem relies on the reduction allowed by Proposition 2.3. The following classical lemma will also be used several times:

**Lemma 3.2.** Let $L$ be a lattice in a Euclidean space $E$.

1. If $F$ is a subspace of $E$ and $F^\perp$ its orthogonal, then one has the following isomorphisms:

$$L/ \left( L \cap F \perp L \cap F^\perp \right) \cong \pi_F(L)/L \cap F \cong \pi_{F^\perp}(L)/L \cap F^\perp \cong \pi_F(L) \perp \pi_{F^\perp}(L)/L.$$

2. Let $F_1, F_2, F_3$ be subspaces of $E$, such that $F_3 = F_1 \perp F_2$, and let $\pi_1, \pi_2$ and $\pi_3$ denote the orthogonal projections onto $F_1, F_2$ and $F_3$ respectively. For $i = 1, 2, 3$, set $L_i = L \cap F_i$.

   a. There are natural injective morphisms

   $$L_3/(L_1 \perp L_2) \hookrightarrow \pi_1 L/L_1 \text{ and } L_3/(L_1 \perp L_2) \hookrightarrow \pi_2 L/L_2.$$
(b) There are natural surjective morphisms
$$\pi_1 L / L_1 \twoheadrightarrow (\pi_1 L \perp \pi_2 L) / \pi_3 L$$
and
$$\pi_2 L / L_2 \twoheadrightarrow (\pi_1 L \perp \pi_2 L) / \pi_3 L.$$

Proof. (1) The kernel of the surjective morphism $L \twoheadrightarrow \pi_F(L)/L \cap F$ is clearly equal to $L \cap F \perp L \cap F^\perp$ and similarly exchanging $F$ and $F^\perp$, whence the first two isomorphisms. As for the last one, it amounts to show that the injective morphism $\pi_F(L)/L \cap F \hookrightarrow \pi_F(L) \perp \pi_{F^\perp}(L)/L$ is onto, which is clear from the observation that $\pi_F(x) + \pi_{F^\perp}(y) \equiv \pi_F(x - y) \mod L$.

(2) We observe that $\pi_3 = \pi_1 + \pi_2$. The kernel of the natural morphism
$$L_3 \longrightarrow \pi_1 L / L_1$$
$$x \longmapsto \pi_1(x) \mod L_1$$
consists of the elements $x \in L_3 = L \cap (F_1 \perp F_2)$ such that $\pi_1(x) \in L$ which means exactly that $x \in L_1 \perp L_2$ since for any $x \in L_3$, one has $x = \pi_3(x) = \pi_1(x) + \pi_2(x)$, so that $\pi_1(x) \in L$ if and only if $\pi_2(x) \in L$. The same holds for the natural morphism $L_3 \rightarrow \pi_2 L / L_2$, which proves (2a).

As for (2b), one can consider the morphism
$$\pi_1 L \longrightarrow (\pi_1 L \perp \pi_2 L) / \pi_3 L$$
$$\pi_1(x) \longmapsto \pi_1(x) \mod \pi_3 L$$
and observe that an element $\pi_1(x)$ is in the kernel if and only if there exists $y \in L$ such that
$$\pi_1(x) = \pi_3(y) = \pi_1(y) + \pi_2(y)$$
in which case
$$\pi_1(x - y) = \pi_2(y) = 0$$
so that $y$ belongs to $L \cap E_1 = L_1$. Consequently $\pi_1(x) = \pi_1(y) = y$ also belongs to $L_1$ so that the kernel of the morphism (17) is $L_1$. As for the surjectivity, it amounts to notice that for any $x$ and $y$ in $L$
$$\pi_1(x) + \pi_2(y) = \pi_1(x - y) + \pi_3(y)$$
which shows that any element in $\pi_1 L \perp \pi_2 L$ is equivalent to an element of $\pi_1 L$ modulo $\pi_3 L$. The same argument apply to the natural morphism $\pi_2 L \rightarrow (\pi_1 L \perp \pi_2)L / \pi_3 L$ and achieves the proof of (2b).

Proof of Theorem 3.7. Assume, by a way of contradiction, that there exists a pair of lattices $(L, M)$ satisfying the assumptions of the theorem and violating its conclusion, i.e. such that
$$\mu_{\min}(L \otimes M) < \mu_{\min}(L) \mu_{\min}(M).$$

By Proposition 2.3(1), we can assume that both $L$ and $M$ are semistable. If $\min(r,s) = 1$, that is if either $E$ or $F$ is absolutely irreducible, then the relation $\mu_{\min}(L \otimes M) = \mu_{\min}(L) \mu_{\min}(M)$ is well-known to hold (see introduction), in contradiction with (18). Thus $\min(r,s) = 2$ and we may assume, without loss of generality, that $r = 2$. In other words, $E$ splits as
$$E = E_1 \perp E_2,$$
where $E_1$ and $E_2$ are non isomorphic, and consequently mutually orthogonal, $G$-absolutely irreducible subspaces of $E = \mathbb{R}M$. Let $U$ be the destabilizing subspace of $E \otimes F$ with respect to $L \otimes M$. As $U$ is $G \times H$-invariant, we know by Proposition 2.2 that there exists $H$-stable subspaces $F_1$ and $F_2$, such that
$$U = E_1 \otimes F_1 \perp E_2 \otimes F_2.$$

Furthermore, by Proposition 2.3(2) we can assume that $F = F_1 \oplus F_2$. Since each $F_i$ is, by the multiplicity free assumption, a sum of irreducible, pairwise non isomorphic, hence
mutually orthogonal, subrepresentations, we infer that $F_1$ and $F_2$ are themselves mutually orthogonal, so that

\[(21) \quad F = F_1 \perp F_2.\]

Let $\ell = \dim L$ and $m = \dim M$. We set $L_i = L \cap E_i$ and $M_i = M \cap F_i (i = 1, 2)$, denote by $\ell_i$ and $m_i$ their respective dimensions and by $\pi_i$ and $\pi'_i$ the orthogonal projection onto $E_i$ and $F_i$ respectively. Then $\ell = \ell_1 + \ell_2$, $m = m_1 + m_2$, $L$ contains $L_1 \perp L_2$ and $M$ contains $M_1 \perp M_2$, both with finite indices

\[(22) \quad a = [L : L_1 \perp L_2], \quad b = [M : M_1 \perp M_2].\]

The destabilizing sublattice

\[P := U \cap (L \otimes M)\]

of $L \otimes M$ contains $L_1 \otimes M_1 \perp L_2 \otimes M_2$ with an index $x \geq 1$.

As a consequence of (19), (20) and (21), one has

\[(23) \quad U^\perp = E_1 \otimes F_2 \perp E_2 \otimes F_1\]

and we infer from Proposition 1.6 that the co-destabilizing sublattice of $(L \otimes M)^*$ is

\[P^\sharp = (L \otimes M)^* \cap U^\perp = (\pi_{U^\perp}(L \otimes M))^*.\]

The situation is summarized in the following diagram:

\[
\begin{align*}
\pi_1 L \otimes \pi'_1 M & \perp \pi_2 L \otimes \pi'_2 M \\
\pi_1 L \otimes \pi'_2 M & \perp \pi_2 L \otimes \pi'_1 M \\
\pi_{U^\perp}(L \otimes M) & \perp \pi_{U^\perp}(L \otimes M) \\
\pi_1 L \otimes \pi'_1 M & \perp \pi_2 L \otimes \pi'_2 M \\
L \otimes M & \perp L \otimes M \\
L_1 \otimes M_1 \perp L_2 \otimes M_2 & \perp L_1 \otimes M_2 \perp L_2 \otimes M_1
\end{align*}
\]

On the left-hand side of the above diagram, we observe that for $i = 1, 2$ the map $\pi_i \otimes \pi'_i$ induces a monomorphism

\[
P \quad \xrightarrow{P} \quad \pi_i L \otimes \pi'_i M \quad \xrightarrow{L_1 \otimes M_1 \perp L_2 \otimes M_2}
\]

(this is Lemma 3.2 (2a)), whence the upper bound

\[(25) \quad x \leq \min(a^{m_1}b^{\ell_1}, a^{m_2}b^{\ell_2}).\]

Set

\[
\alpha_i = \left( \frac{\mu(L_i)}{\mu(L)} \right)^{\ell_i} \quad \text{and} \quad \beta_i = \left( \frac{\mu(M_i)}{\mu(M)} \right)^{m_i}, \quad i = 1, 2.
\]

These quantities satisfy the relations

\[(26) \quad \alpha_1 \alpha_2 = a \quad \text{and} \quad \beta_1 \beta_2 = b.\]
Indeed, using Lemma 1 we have
\[
a = [L : L_1 \perp L_2] = \frac{\det(L_1) \det(L_2)}{\det(L)} = \frac{\mu(L_1)^{\ell_1} \mu(L_2)^{\ell_2}}{\mu(L)^{\ell_1 + \ell_2}} = \alpha_1 \alpha_2
\]
which proves the first identity in (26), the second being identical. Using Lemma 1.2 and the fact that \(L\) and \(M\) are semistable, the assumption that \(\mu(P) < \mu_{\min}(L)\mu_{\min}(M)\) amounts to say that
\[
x = [P : (L_1 \otimes M_1 \perp L_2 \otimes M_2)] = \frac{(\mu(L_1)\mu(M_1))^{\ell_1 m_1} (\mu(L_2)\mu(M_2))^{\ell_2 m_2}}{\mu(P)^{\ell_1 m_1 + \ell_2 m_2} > \frac{(\mu(L_1)\mu(M_1))^{\ell_1 m_1} (\mu(L_2)\mu(M_2))^{\ell_2 m_2}}{(\mu(L)\mu(M))^{\ell_1 m_1 + \ell_2 m_2}} = \alpha_1^{m_1} \beta_1^{\ell_1} \beta_2^{\ell_2}
\]
Together with (26) this yields
\[
x > \alpha_1^{m_1} \beta_1^{\ell_1} \alpha_2^{m_2} \beta_2^{\ell_2} = \alpha_1^{m_1 - m_2} \beta_1^{\ell_1 - \ell_2} \alpha_2^{m_2} \beta_2^{\ell_2}
\]
The combination of (25) and (27) implies that
\[
\alpha_1^{m_1 - m_2} \beta_1^{\ell_1 - \ell_2} \alpha_2^{m_2} \beta_2^{\ell_2} < \min(a^{m_1}b^{\ell_1}, a^{m_2}b^{\ell_2})
or equivalently
\[
\alpha_1^{m_1 - m_2} \beta_1^{\ell_1 - \ell_2} < \min(1, a^{m_1 - m_2}b^{\ell_1 - \ell_2}).
\]
As a consequence of the semi-stability of \(L\) and \(M\), one has \(\alpha_i \geq 1\) and \(\beta_i \geq 1\), \(i = 1, 2\). Together with the relations \(\alpha_1 \alpha_2 = a\) and \(\beta_1 \beta_2 = b\), it implies that \(1 \leq \alpha_1 \leq a\) and \(1 \leq \beta_1 \leq b\). Consequently, inequality (28) cannot hold unless \(m_1 - m_2\) and \(\ell_1 - \ell_2\) have opposite signs, that is
\[
(m_1 - m_2)(\ell_1 - \ell_2) < 0.
\]
We now consider the right-hand side of the diagram, using duality. First, we have an upper bound for
\[
y' = \left[\pi_1 L \otimes \pi_2' M \perp \pi_2 L \otimes \pi_1' M : \pi_{1\perp}(L \otimes M)\right], \text{ similar to (25), namely}
\]
\[
y' \leq \min(a^{m_1}b^{\ell_1}, a^{m_2}b^{\ell_1}),
\]
obtained either by using the natural epimorphisms \(\pi_1 L \otimes \pi_2' M / L_1 \otimes M_2 \to \pi_1 L \otimes \pi_2' M \perp \pi_2 L \otimes \pi_1' M / \pi_{1\perp}(L \otimes M)\) and \(\pi_2 L \otimes \pi_1' M / L_2 \otimes M_1 \to \pi_1 L \otimes \pi_2' M \perp \pi_2 L \otimes \pi_1' M / \pi_{1\perp}(L \otimes M)\) (see Lemma 3.2 (2b)), or by dualizing (25).
Furthermore, one has
\[
y' = \left[\pi_1 L \otimes \pi_2' M \perp \pi_2 L \otimes \pi_1' M\right]^* = \left[\frac{\mu(\pi_1 L \otimes \pi_2' M \perp \pi_2 L \otimes \pi_1' M)^*}{\mu(P^*)}\right]^{\ell_1 m_1 + \ell_2 m_2}
\]
\[
= \left[\frac{(\mu(\pi_1 L)\mu(\pi_2' M))^{\ell_1 m_2} (\mu(\pi_2 L)\mu(\pi_1' M))^{\ell_2 m_1}}{\mu(P^*)^{\ell_1 m_2 + \ell_2 m_1}}\right]^*
\]
\[
= \left[\frac{(\mu(\pi_1 L)\mu(\pi_2' M))^{\ell_1 m_2} (\mu(\pi_2 L)\mu(\pi_1' M))^{\ell_2 m_1}}{\mu(P^*)^{\ell_1 m_2 + \ell_2 m_1}}\right]
\]
Using Lemma 1.2 again, one has

\[ \mu(P_m^{\ell_1} \otimes M_m^{\ell_2}) = \mu(L^* \otimes M^*) \mu \left( \frac{L^* \otimes M^*}{P_m^{\ell_1} \otimes M_m^{\ell_2}} \right) = (\mu(L) \mu(M))^{-\ell_1} \mu \left( \frac{L^* \otimes M^*}{P_m^{\ell_1} \otimes M_m^{\ell_2}} \right)^{-\ell_2} \]

Since \( P_m^{\ell_1} \) is the co-destabilizing sublattice of \( (L \otimes M)^* \), and \( L \) and \( M \) are semi-stable, we have

\[ \mu \left( \frac{L^* \otimes M^*}{P_m^{\ell_1}} \right) > \mu_{\max}(L^*) \mu_{\max}(M^*) = \mu(L)^{-1} \mu(M)^{-1}. \]

Plugging this into the above calculation of \( y' \), we get

\[ y' > \frac{\mu(L) \mu(M)}{(\mu(\pi_1 L) \mu(\pi_2^* M))^{\ell_1 m_2 + \ell_2 m_1}} \]

whence finally, using (26),

(31)

\[ y' > a_m b_{\ell_1}^m + a_m^2 b_{\ell_2} + a_m^1 b_{\ell_1}^m - a_m^2 b_{\ell_2} - a_m^1 b_{\ell_2}^m = a_m b_{\ell_1}^m a_{\ell_2}^m - a_m^2 b_{\ell_2} - a_m^1 b_{\ell_2}^m \]

where we also used the relations \( \mu(\pi_1 L) = a_m b_{\ell_1}^m \) and \( \mu(\pi_2^* M) = b_{\ell_2}^m \mu(M_i) \). The combination of (30) and (31) yields

\[ a_m b_{\ell_1}^m a_{\ell_2}^m - a_m^2 b_{\ell_2} - a_m^1 b_{\ell_2}^m < \min(a_m b_{\ell_2}^m, a_m^2 b_{\ell_2}^m) \]

or equivalently

(32)

\[ a_{\ell_2}^m - a_m^2 b_{\ell_2} - a_m^1 b_{\ell_2}^m < \min(1, a_m b_{\ell_2}) \]

The same argument we used to derive (29) now yields

(33)

\[ (m_1 - m_2)(\ell_2 - \ell_1) < 0 \]

which is incompatible with (29). This concludes the proof of the theorem.

\[ \square \]

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