Type-Preserving Flow Analysis and Interprocedural Unboxing

Extended Version

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Abstract

Interprocedural flow analysis can be used to eliminate otherwise unnecessary heap allocated objects (unboxing), and in previous work we have shown how to do so while maintaining correctness with respect to the garbage collector. In this paper, we extend the notion of flow analysis to incorporate types, enabling analysis and optimization of typed programs. We apply this typed analysis to specify a type preserving interprocedural unboxing optimization, and prove that the optimization preserves both type and GC safety along with program semantics. We also show that the unboxing optimization can be applied independently to separately compiled program modules, and prove via a contextual equivalence result that unboxing a module in isolation preserves program semantics.

1. Introduction

Many languages and compilers use a uniform object representation in which every source level object is represented at least initially by a heap allocated object. Such a representation allows polymorphic functions to be compiled once and enables the implementation of features that use runtime type information. In this representation machine integers and floating-point numbers are placed in a single-field object, a box, and this operation is called boxing. Operations such as addition require first projecting the number from the box (unboxing), followed by the actual addition, followed by the creation of a new box for the result (boxing). Boxing and unboxing operations add considerable overhead, and thus it is highly desirable to remove them when possible – e.g. when polymorphism or features requiring runtime type information are not being used.

We refer to the general class of optimizations that attempt to remove unnecessary box and unbox operations as unboxing optimizations. We refer to unboxing optimizations that attempt to eliminate boxing and unboxing across function boundaries as interprocedural unboxing. We also include in this latter category optimizations (such as the one given in this paper) which attempt to unbox objects written to and read from other objects in the heap.

Interprocedural unboxing presents additional challenges in a typed setting, since type information must be updated to reflect any unboxing. A box might flow to an argument in an application, and the parameter of the called function might flow to an unbox operation. If the optimization decides to remove the box and unbox operations then it must also remove the box type on the parameter. In other words, typed unboxing requires not just rewriting uses and definitions in the traditional sense, but also rewriting intermediate points in the program through which the unboxed values flow. At a high level then, the optimization can be viewed as selecting a set of box operations, unbox operations, and box types to remove.

Such a selection has a global consistency requirement—a box type should only be removed if all boxes that flow to it are removed, a box operation should only be removed if all unbox operations it flows to are removed, and so on. Thus choosing a set of boxed objects to eliminate and rewriting the program to reflect this choice in a consistent manner requires knowing what things flow to what points in the program, a question that flow analyses are designed to answer. In this paper we use the results of flow analysis to formulate correctness conditions for unboxing and then prove that those conditions ensure correct optimization.

In previous work [8] we considered a simpler problem, that of rewriting garbage-collector (GC) metadata rather than full types. An accurate GC requires specifying for each field of each object and each slot of each stack activation frame whether it contains a pointer into the GC heap or not (contains a machine integer, floating-point number, etc.). As with types, when interprocedurally unboxing such metadata must be rewritten in a globally consistent manner. Our previous paper showed how to do this rewriting correctly using the results of a flow analysis, in a whole program setting. In this paper we extend these ideas to develop a methodology for dealing with interprocedural optimization of statically typed languages (including universal polymorphism) in a type preserving fashion. We also show that this methodology does not depend on whole program compilation, and extends easily to support modular compilation.

In the following sections, we begin by defining a core language with a polymorphic type system that has box and unbox operations. As in our previous paper we formalize a notion of GC safety for our language—that the GC metadata is currently correct—and show that well-typed programs are GC safe throughout execution. Next we specify a set of abstract conditions that a reasonably flow analysis must satisfy, with the property that any flow analysis that satisfies these conditions can be used in our framework to optimize programs. The main section of the paper defines an unboxing optimization parameterized over a choice of objects to unbox, and gives a set of correctness conditions under which such a choice is guaranteed to preserve typing and preserve semantics. We show that this set of correctness conditions is satisfiable by constructing a simple unboxing algorithm which satisfies these conditions. Finally we extend the system slightly by defining a notion of unboxing for modules and show that it is correct in the sense that a module is contextually equivalent to its unboxing.

While our paper is specifically about the concrete optimization of unboxing, the ideas used here generalize naturally to other op-
timizations that change the representation of objects in a non-local fashion. For example, dead-field elimination and dead-parameter elimination impose similar requirements for rewriting types and GC metadata in a globally consistent fashion. Flow analyses can be used to specify and implement these (and others), and we believe (based on practical experience in our compiler) that the framework presented here extends naturally to such optimizations. As far as we know, this and our previous paper are the first to use a flow analysis to rewrite types and GC metadata in a globally consistent fashion, and to use a flow analysis to formulate correctness conditions for this rewriting process and prove these conditions sound.

2. A type and GC safe core language

Consider the following untyped program (using informal notation), where box denotes a boxing operation that wraps its argument in a heap-allocated structure, and unbox denotes its elimination form that projects out the boxed item from the box:

\[
\text{let } f = \lambda x. (\text{box }x) \text{ in unbox(unbox(f(box 3)))}
\]

The only definition reaching the variable \(x\) is the boxed machine integer 3. Information from an interprocedural analysis can be used to rewrite this program to eliminate the boxing as follows:

\[
\text{let } f = \lambda x.x \text{ in } f \ 3
\]

This second version is much better in that it does less allocation, and executes fewer instructions. In this optimized version of the program however, an important property has changed that is not reflected in this untyped syntax. Specifically, the GC status of values reaching \(x\) has changed: whereas in the original program all values reaching \(x\) are represented as heap allocated pointers, in the second program all values reaching \(x\) are represented as machine integers. From the standpoint of a garbage collector, a program however, an important property has changed that is not

2.1 Type safety

How does the problem of unboxing change in a typed setting? Consider again the first example from this section using a still informal but now typed notation:

\[
\text{let } f : \text{box(int)} \rightarrow \text{box(box(int))} = \lambda x : \text{box(int)}. (\text{box }x) \\
\text{in unbox(unbox(f(box 3)))}
\]

As before, it is apparent that the only definition reaching the variable \(x\) is the boxed machine integer 3, and as before we can consider rewriting this program to eliminate (interprocedurally) the boxing. However, simply rewriting the terms of the program is inadequate from the standpoint of type preserving compilation, since the result is not well-typed:

\[
\text{let } f : \text{box(int)} \rightarrow \text{box(box(int))} = \lambda x : \text{box(int)}. x \\
\text{in } f \ 3
\]

The types of both the actual argument and the return value of \(f\) have changed, and are no longer consistent with the type annotation for \(f\) and \(x\). In order to correctly unbox this program then, it is necessary to rewrite not just the terms, but also the types:

\[
\text{let } f : \text{int} \rightarrow \text{int} = \lambda x : \text{int}. x \text{ in } f \ 3
\]

This requirement is a more substantial change than might at first be apparent. In the original (untyped) setting, it was sufficient to have information only about the direct definitions (boxes) and uses (unboxes) of objects. Rewriting the types in this fashion requires information not just about the uses and definitions, but also about intermediate program points (and other objects) through which the boxed objects flow.

In addition to incurring these additional rewriting requirements, the typed setting must still account for GC safety. Consider again the polymorphic variant of the previous untyped example (namely the first application of \(f\) for clarity).

\[
\text{let } f : \forall \alpha. \alpha \rightarrow \text{box(\alpha)} = \lambda \alpha. \lambda x : \text{box(\alpha)}. (\text{box }x) \\
\text{g : box(\forall \alpha. \alpha) \rightarrow box(\alpha))} = \text{f(\forall \alpha. \alpha \rightarrow box(\alpha))(f)} \\
\text{in unbox(unbox(unbox(\forall \alpha. \alpha \rightarrow box(\alpha)))(box 3)))}
\]

Here we have \(f\) applied to itself at a universal type to produce a boxed version of itself \((g)\), which is then unboxed and applied to a boxed integer, at the boxed integer type. Attempting to unbox this example (rewriting types as necessary) immediately illuminates the problem.

\[
\text{let } f : \forall \alpha. \alpha \rightarrow \alpha = \lambda \alpha. \lambda x : \text{box(\alpha)}. x \\
\text{g : \forall \alpha. \alpha \rightarrow \alpha = \text{f}(\forall \alpha. \alpha \rightarrow \alpha)(f)} \\
\text{in g(box(\alpha))}(3)
\]

The function \(f\) is instantiated directly at a universal function type, and via its alias \((g)\) at a machine integer type. As with the untyped example in the previous section, the compiler cannot assign a concrete GC status to the variable \(x\). For correctness then, the compiler must not (fully) unbox this example, and must leave at least the boxing operation on the integer parameter to \(f\).
In order to give a precise account of typed flow analysis and in-
terprocedural unboxing, we begin by defining a type safe core lan-
guage incorporating the essential features of GC safety. The mo-
tivation for the (small) idiosyncrasies of this language lies in the
requirements of the underlying model of garbage collection. We as-
sume that pointers cannot be intrinsically distinguished from non-
pointers, and hence the compiler is required to statically annotate
the program with garbage collection meta-data such that at any
garbage collection point the garbage collector can reconstruct ex-
actly which live variables are roots. Typically, this takes the form of
annotations on variables and temporaries indicating which contain
heap-pointers (the roots) and which do not (the non-roots), along
with information at every allocation site indicating which fields of
the allocated object contain traceable data. This approach is com-
mon in modern systems, and it is this approach that we target in
this paper.

Figure 1 defines the syntax of our core language. The essence of
the language is a standard polymorphic lambda calculus extended
with a fix point operator, implemented via an explicit environment
semantics. For the purposes of the semantics, we also include a
form of degenerate type information we call *traceabilities*. Trace-
abilities describe the GC status of variables: the traceability b (for
bits) indicates something that should be ignored by the garbage
collector, while the traceability r (for reference) indicates a GC-
managed pointer. The traceability b is inhabited by an unspecified
set of constants c and the traceability r is inhabited by functions
(anticipating their implementation by heap-allocated closures) and
by boxed objects. Anticipating the needs of the flow analysis, we
label each type, term, value, and variable binding site with an in-
teger label. We do not assume that labels or variables are unique
within a program.

Types σ consist of type variables, the base type of constants,
b, function types, ∀α.τ₁ → τ₂, and boxed types box(τ). In order to
provide a concrete implementation strategy for the garbage col-
lector, we insist that every type correspond to a traceability so that
we can extract the necessary garbage collection meta-data. Types
are mapped to traceabilities using the function tr(σ), defined in
Figure 2. Polymorphic functions are restricted by well-formedness
rules to only be instantiated with types with the traceability r, and
consequently tr(α) = r. We define substitution of types in the stan-
dard way and define tr[α/τ][σ] = tr[σ/α].

Expressions e consist of labeled terms m¹ and labeled values
v¹. The terms m consist of variables, functions, applica-
tions, box introductions, box eliminations, and frames. Functions
fix f[α][x:τ₁]:τ₂.e are polymorphic recursive and variable binding
sites are decorated with types. We represent heap allocation
in the language via the box e term, which corresponds to alloca-
ting a heap cell containing the value for e. The type r is used by
the dynamic semantics to provide the meta-data with which the
heap-cell will be tagged, allowing the garbage collector to trace the
cell. However, only the top-level traceability of the type (given by
the tr() function in Figure 2) is actually required by the dynamic
semantics, and so the language can be erased into an untyped
language in the obvious way. Objects can be projected out of an allo-
ated object by the unbox operation. Frames ρ(e) are discussed fur-
ther below.

Values consist of either constants, closures, or heap-allocated
boxes. We distinguish between the introduction form (box e) and the
value form ((v¹:τ¹)) for allocated objects. The introduction form
coresponds to the allocation instruction, whereas the value form
corresponds to the allocated heap value. This distinction is key for
the formulation of GC safety and the dynamic semantics. For the
purposes of the dynamic semantics we also distinguish between functions (fix f[α][x:τ₁]:τ₂.e) and the heap allocated closures
that represent them at runtime ((ρ.fix f[α][x:τ₁]:τ₂.e)).

For notational convenience, we will sometimes use the notation
v₀ to indicate that a value v is a non-heap-allocated value (i.e. a
constant c), and v₁ to indicate that a value v is a heap-allocated
value (i.e. either a function value or a boxed value). If t is a
traceability meta-variable, then we use v₁ to indicate that v is a
value of the same traceability as t.

In examples, we use a derived let expression, taking it to be
syntactic sugar for application in the usual manner. Environments
ρ map variables to values. The term ρ(e) executes e in the environ-
ment ρ rather than the outer environment – all of the free variables
of e are provided by ρ. The nested set of these environments at any
point can be thought of as the activation stack frames of the execut-
ing program. The traceability of the typing annotations on variables
in the environments play the role of stack-frame GC meta-data, in-
dicating which slots of the frame are roots (traceability r). The
environments burried in closures ((ρ.fix f[α][x:τ₁]:τ₂.e)) similarly
provide the traceabilities of values reachable from the closure via
the type annotations on the variables in the environment, and hence
provide the GC meta-data for tracing through closures. While we
do not make the process of garbage collection explicit, it should
be clear how to extract the appropriate set of GC roots from the
environment and any active frames.

This core language contains the appropriate information to for-
malize a notion of GC safety consisting of two complementing
pieces. First we define a dynamic semantics in which reductions
that might lead to undefined garbage-collector behavior are explic-
itly undefined. Programs that take steps in this semantics do not
introduce ill-formed heap objects. Secondly, we define a notion of
a traceable program: one in which all heap values have valid
GC meta-data. Reduction steps in the semantics can then be shown
to maintain the traceability property in addition to the usual well-
typedness property. The GC correctness criteria for a compiler op-
timization then becomes simply the usual one: that the optimization
map well-typed programs to semantically equivalent well-typed

\[
\begin{array}{ll}
\text{Labeled Terms} & m ::= x \mid \text{fix } f[\alpha](x:\tau_1):\tau_2.e \mid e_1 [\tau] e_2 \mid \text{box } e \mid \text{unbox } e \mid \rho(e) \\
\text{Values} & v ::= c \mid (\rho.\text{fix } f[\alpha](x:\tau_1):\tau_2.e) \mid (v^1:\tau^1) \\
\text{Environments} & \rho ::= x_1:\tau_1 = v_{1^1}, \ldots, x_n:\tau_n = v_{n^1}
\end{array}
\]
2.3 Operational semantics

We choose to use an explicit environment semantics rather than a standard substitution semantics since this makes the GC meta-data (implicit in the types) for stack frames and closures explicit in the semantics. Thus a machine state \((\rho, e)\) supplies an environment \(\rho\) for \(e\) that provides the values of the free variables of \(e\) during execution. Environments contain typing annotations on each of the variables mapped by the environment, which provide the traceabilities of the variables.

Reduction in this language is for the most part fairly standard. We deviate somewhat in that we explicitly model the allocation of heap objects as a reduction step—hence there is an explicit reduction mapping a function term \(\text{fix} f[a](x:\tau_1):(\tau_2.e)\) to an allocated closure \((\rho, \text{fix} f[a](x:\tau_1):(\tau_2.e))\), and similarly for boxed objects and values. More notably, beta-reduction is restricted to only permit construction of a stack frame when the type for the parameter variable has an appropriate traceability for the actual argument value. This captures the requirement that stack frames have correct meta-data for the garbage collector. In actual practice, incorrect meta-data for stack frames leads to undefined behavior (since incorrect meta-data may cause arbitrary memory corruption by the garbage collector)—similarly here in the meta-theory we leave the behavior of such programs undefined. In a similar fashion, we only define the reduction of the allocation operation to an allocated value \((\text{box}_x, v_i \rightarrow (v_i:\tau))\) when the operation meta-data is appropriate for the value (i.e. \(tr(\tau) = t\)).

It is important to note that this semantics does not model a dynamically checked language, in which there is an explicit check of the meta-data associated with these reductions. The point is simply that the semantics only specifies how programs behave when these conditions are met—in all other cases the behavior of the program is undefined.

2.4 Traceability

The operational semantics ensures that no reduction step introduces mis-tagged values. In order to make use of this, we define a judgment for checking that a program does not have a mis-tagged value in the first place. Implicitly this judgment defines what a well-formed heap and activation stack looks like; however, since our heap and stack are implicit in our machine states, it takes the form of a judgement on terms, values, environments, and machine states.

The value judgement \(\vdash_v e : t\) asserts that a value \(v\) is well-formed, and has traceability \(t\). In this simple language, this corresponds to having the types on the variables in the environment of each function value have traceabilities which are consistent with the values to which they are bound, and the type on each boxed value be consistent with the traceability of the object nested in the box. An environment is consistent, \(\vdash \rho\ \tau\), when the annotation on each variable agrees with the traceability of the value it is bound to. The term judgement \(\vdash \ e\ \tau\) and machine state judgement \(\vdash M\ \tau\) simply check that all values and environments (and hence stack frames) contained in the term or machine state are well-formed.

\[
\begin{align*}
tr(\sigma') &= tr(\sigma) \\
tr(\alpha) &= r \\
tr(B) &= b \\
tr(\forall \alpha. \tau_1 \rightarrow \tau_2) &= r \\
tr(\text{box}(\tau)) &= r \\
\end{align*}
\]

\[
\begin{align*}
(x:\tau = v^j \in \rho) &\rightarrow (\rho, v^j) \\
(r, (\text{fix } f[a](x:\tau_1):(\tau_2.e))^i) &\rightarrow (\rho, (\rho, f[a](x:\tau_1):(\tau_2.e))^i) \\
tr(\tau) &= t \\
(r, (\text{box}_x, v_i)^i) &\rightarrow (\rho, (v_i:\tau)^i) \\
(r, e_1) &\rightarrow (\rho, e_1) \\
(r, (e_1[\tau] e_2)^i) &\rightarrow (\rho, (e_1[\tau] e_2)^i) \\
(r, (e_1[\tau] e_2)^i) &\rightarrow (\rho, (e_1[\tau] e_2)^i) \\
(r, (\rho[\tau][\alpha])^i) &\rightarrow (\rho, (\rho[\tau][\alpha])^i) \\

\end{align*}
\]

Figure 2. Operational Semantics

The key result for traceability is that it is preserved under reduction. That is, if a traceable term takes a well-defined reduction step, then the resulting term will be traceable.

**Lemma 1 (Preservation of traceability)**

If \(\vdash M\ \tau\) and \(M \rightarrow M'\) then \(\vdash M'\ \tau\).

**Proof:** If \(\vdash (\rho, e)\ \tau\) then \(\vdash \rho\ \tau\) and \(\vdash e\ \tau\). If \(\vdash (\rho, e)\ \rightarrow (\rho, e')\) then the result follows if we can show \(\vdash e'\ \tau\). The proof of that is by induction on the derivation of \(\vdash (\rho, e)\ \rightarrow (\rho, e')\). Consider the cases for the last rule used to derive it (the cases are in the same order as in the figure):

- In this case, \(e = x^k\) for some \(x\) and \(k\), and \(e' = v^j\) where \(x:\tau = v^j \in \rho\) for some \(\tau, v, j\). Since \(\vdash \rho\ \tau\) then \(\vdash_v v:\tau\) \(\rightarrow\), so by the traceability rules \(\vdash_v v'\ \tau\) as required.
In this case, \( e = (e_1 \tau) \ e_2 \) for some \( e_1, \tau, e_2, \) and \( i, i' = (e_1 \tau) e_2 \) for some \( e_1, \) and \( (\rho, e) \longrightarrow (\rho, e') \) is a subderivation. The first hypothesis is that \( \vdash (e_1 \tau) e_2 \) tr. There is only one rule to derive this judgement and that rule requires that \( \vdash e_1 \tau e_2 \) tr, which in turn can only be derived by one rule that requires both \( \vdash e_1 \tau \) tr and \( \vdash e_2 \) tr. Thus, by the induction hypothesis, \( \vdash e_1 \tau e_2 \) tr. Then, by the rules for traceability, \( \vdash e_1 \tau e_2 \) tr, and by the traceability rules again, \( \vdash (e_1 \tau) e_2 \) tr, as we are required to prove.

• In this case, \( e = (e_1 \tau) e_2 \) for some \( e_1, \tau, e_2, \) and \( i, i' = (e_1 \tau) e_2 \) for some \( e_1, \) and \( (\rho, e) \longrightarrow (\rho, e') \) is a subderivation. The first hypothesis is that \( \vdash (e_1 \tau) e_2 \) tr. There is only one rule to derive this judgement and that rule requires that \( \vdash e_1 \tau e_2 \) tr, which in turn can only be derived by one rule that requires both \( \vdash e_1 \tau \) tr and \( \vdash e_2 \) tr. Thus, by the induction hypothesis, \( \vdash e_1 \tau e_2 \) tr. Then, by the rules for traceability, \( \vdash e_1 \tau e_2 \) tr, and by the traceability rules again, \( \vdash (e_1 \tau) e_2 \) tr, as we are required to prove.

• In this case, \( e = (v' \tau) e_2 \) for some \( v, i, \tau, e_2, \) and \( j, i' = (v' \tau) e_2 \) for some \( e_2, \) and \( (\rho, e_2) \longrightarrow (\rho, e') \) is a subderivation. The first hypothesis is that \( \vdash (v' \tau) e_2 \) tr. There is only one rule to derive this judgement and that rule requires that \( \vdash v' \tau e_2 \) tr, which in turn can only be derived by one rule that requires both \( \vdash v' \tau \) tr and \( \vdash e_2 \) tr. Thus, by the induction hypothesis, \( \vdash e_1 \tau e_2 \) tr. Then, by the rules for traceability, \( \vdash v' \tau e_2 \) tr, and by the traceability rules again, \( \vdash (v' \tau) e_2 \) tr, as we are required to prove.

• In this case, \( e = (v' \tau) e_2 \) for some \( v, i, \tau, e_2, \) and \( j, i' = (v' \tau) e_2 \) for some \( e_2, \) and \( (\rho, e_2) \longrightarrow (\rho, e') \) is a subderivation. The first hypothesis is that \( \vdash (v' \tau) e_2 \) tr. There is only one rule to derive this judgement and that rule requires that \( \vdash v' \tau e_2 \) tr, which in turn can only be derived by one rule that requires both \( \vdash v' \tau \) tr and \( \vdash e_2 \) tr. Thus, by the induction hypothesis, \( \vdash e_1 \tau e_2 \) tr. Then, by the rules for traceability, \( \vdash v' \tau e_2 \) tr, and by the traceability rules again, \( \vdash (v' \tau) e_2 \) tr, as we are required to prove.

For some \( \rho', f, \alpha, x, \tau_1, \tau_2, e'', j, \tau, v_i, k, l \). The first hypothesis is that \( \vdash (v' \tau) e_2 \) tr. There is only one rule to derive that judgement and that rule requires that \( \vdash v' \tau k \) tr, which in turn can only be derived by one rule that requires both \( \vdash v' \tau \) tr and \( \vdash k \) tr. Both of these latter derivations can only be derived by one rule and those rules require that \( \vdash v' \tau (1) \) and \( \vdash v' \tau (2) \) (a simple inspection reveals the traceabilities to be \( \tau \) and \( i \)). Judgement 1 can only be derived by one rule and that rule requires that \( \vdash \rho' \) tr (3) and \( \vdash e'' \) tr (4). By (3), (1), \( \vdash (\tau') \tau_i = t \) (6) we can derive \( \vdash \rho' \) tr (5). By (5) and (4) we can derive \( \vdash \rho'(e'') \) tr, and then \( \vdash e'' \) tr, as required.
\[ \Delta ::= \alpha_1, \ldots, \alpha_n \]
\[ \Gamma ::= x_1 : \tau_1, \ldots, x_n : \tau_n \]
\[ \Delta \vdash \tau \textit{wf} \]
\[ f(t)(\tau) \subseteq \Delta \]
\[ \Delta \vdash \tau \textit{wf} \]
\[ \vdash \tau_1 = \tau_2 \]
\[ \vdash \alpha' = \alpha'' \]
\[ \vdash B' = B'' \]
\[ \vdash \tau_{11} = \tau_{21} \]
\[ \vdash \tau_{12} = \tau_{22} \]
\[ \vdash (\forall \alpha. \tau_{11} \rightarrow \tau_{12}) \equiv (\forall \alpha. \tau_{21} \rightarrow \tau_{22}) \]
\[ \vdash \tau_1 = \tau_2 \]
\[ \vdash \text{box}(\tau_1) = \text{box}(\tau_2) \]
\[ \vdash \rho : \Gamma \]
\[ \emptyset \vdash \tau_1 \textit{wf} \quad \cdots \quad \emptyset \vdash \tau_n \textit{wf} \]
\[ \emptyset \vdash v_{i1} : \tau'_1 \quad \cdots \quad \emptyset \vdash v_{in} : \tau'_n \]
\[ \vdash \tau_1 = \tau'_1 \quad \cdots \quad \vdash \tau_n = \tau'_n \]
\[ \vdash x_1 : \tau_1, \ldots, x_n : \tau_n = v_{1i} : x_1 : \tau_1, \ldots, x_n : \tau_n \]
\[ \vdash M : \tau \]
\[ \vdash \rho : \Gamma \quad \emptyset : \Gamma \vdash e : \tau \]
\[ \vdash (\rho, e) : \tau \]
\[ \frac{\Delta ; \Gamma ; e : \tau}{\Delta ; \Gamma ; x : \tau} \]
\[ \frac{\Delta ; \rho : \Gamma}{\Delta ; \rho ; \Gamma'} \]
\[ \frac{\Delta ; \Gamma ; \rho(e) : \tau}{\Delta ; \Gamma ; \rho(e) : \tau} \]
\[ \frac{\emptyset ; \Gamma ' \vdash (\forall \alpha. \tau_{11} \rightarrow \tau_{12}) \textit{wf} \quad \alpha; \Gamma ', f : (\forall \alpha. \tau_{21} \rightarrow \tau_{22}) \quad \emptyset ; \tau \vdash e : \tau_2}{\Delta ; \Gamma ' \vdash (\rho, \text{fix} f[\alpha](x : \tau_{11} : \tau_{12} : e)) : (\forall \alpha. \tau_{11} \rightarrow \tau_{22}) \textit{wf}} \]
\[ \frac{\emptyset ; \tau \vdash u : \tau' \quad \Delta ; \Gamma ' \vdash \tau' : \tau \quad \tau = \tau'}{\Delta ; \Gamma ' \vdash (v' : \tau') : \text{box}(\tau)^i} \]

Figure 4. Type rules, other constructs

Figure 5. Type rules, expressions

There is no corresponding progress property for our notion of traceability, since in the absence of further guarantees, programs can go wrong. However, typable programs are both traceable and do not go wrong, as we will see in the next section, and so preserving typability ensures GC correctness.

2.5 Typing

The typing rules appear in Figures 4 and 5. They are for the most part standard except for three modifications. First, as types are labelled, we must sometimes ignore the labels in typing. Judgement \[ \vdash \tau_1 = \tau_2 \] states that types \( \tau_1 \) and \( \tau_2 \) are syntactically equivalent except that the labels on their sub-terms might differ. This is important in (for example) the rule for application, where we require only that the parameter type \( \tau_1 \) and the actual argument type \( \tau_2 \) satisfy \( \vdash \tau_1 = \tau_2 \) rather than \( \tau_1 = \tau_2 \); similarly in the rules for environments. Second, in the rules for boxes we require that the traceability of the box equal the traceability of the type of the thing being boxed. This is essential for showing that well-typedness implies traceability. Finally, the instantiation rule for polymorphic functions enforces the property that the type argument have traceability \( \tau \).

One particularly important aspect of our language is that we assume a type erasure semantics. For this interpretation to be correct, we must show that we can compute the correct GC metadata when erasing types. The operational semantics have the application of a polymorphic function step to a frame where the annotation on the function’s parameter is a substituted type. We need that the GC metadata for this substituted type equal the GC metadata for the unsubstituted parameter type of the function. The requirement \( tr(\tau) = \tau \) in the typing rule for application is crucial to that equality, and the following lemma proves it.
Lemma 2
If \( tr(\tau) = \tau \) then \( tr'(\tau') = tr'(\tau'[\alpha]) \).

Proof: The proof is by inspection of the definitions.

We can prove type safety for this language in the standard way, via progress and preservation lemmas. First we need several lemmas: that type equality is an equivalence relation, that equal types have the same traceabilities, that a well-typed value has the same traceability as its type, that type equality respects type substitution, that value typing is independent of the typing context, and a type substitution lemma.

Lemma 3
Type equality is an equivalence relation, that is, \( \vdash \tau = \tau \), \( \vdash \tau_1 = \tau_2 \) implies \( \vdash \tau_2 = \tau_1 \), and \( \vdash \tau_1 = \tau_2 \) and \( \vdash \tau_2 = \tau_3 \) implies \( \vdash \tau_1 = \tau_3 \).

Proof: The proof is by a simple induction on the structure of \( \tau \) for reflexivity or the structure of the derivation(s) for symmetry and transitivity and inspection of the rules.

Lemma 4
\( \vdash \tau_1 = \tau_2 \) then \( \vdash tr(\tau_1) = tr(\tau_2) \).

Proof: The proof is by inspection of the last rule used.

Lemma 5
\( \vdash \tau_1 = \tau_2 \) then \( \vdash \tau_1[\alpha] = \tau_2[\alpha] \).

Proof: The proof is by an easy induction on the derivation of \( \vdash \tau_1 = \tau_2 \).

Lemma 6
\( \Delta; \Gamma \vdash v_i : \tau \) then \( \vdash tr(\tau) = t. \)

Proof: The proof is by inspection of the three rules for value typing.

Lemma 7
\( \Delta; \Gamma \vdash v^i : \tau \) then \( \Delta'; \Gamma' \vdash v^i : \tau \) for any \( \Delta' \) and \( \Gamma' \).

Proof: The proof is by an easy induction on the typing derivation and inspection of the three rules for value typing.

Lemma 8
\( \Delta, \alpha, \Delta'; \Gamma \vdash e : \tau, \Delta \vdash \tau' \text{ wf}, \text{ and } tr(\tau') = \tau \) then \( \Delta, \Delta'; \Gamma[\tau'/\alpha] \vdash e[\tau'/\alpha] : \tau[\tau'/\alpha] \).

Proof: The proof is a straightforward induction over the derivation of \( \Delta, \alpha, \Delta'; \Gamma \vdash e : \tau \). It uses Lemma 2 in the case of the rule for application.

With all these lemmas we can prove Type Preservation and Progress.

Lemma 9 (Type Preservation)
\( \vdash M_1 \quad \text{and} \quad M_1 \Rightarrow M_2 \) then \( \vdash M_2 \) and \( \vdash \tau_1 = \tau_2 \) for some \( \tau_2 \).

Proof: Assume that \( \vdash \rho(e_1) \vdash \tau_1 \) and \( \rho(e_1) \vdash \rho(e_2) \). We will show by induction on the derivation of the latter that \( \vdash \rho(e_2) \vdash \tau_2 \) and \( \vdash \tau_1 = \tau_2 \) for some \( \tau_2 \). By the typing rules, \( \rho(\Gamma) \) and \( \rho(\emptyset) \vdash \rho(e_1) \vdash \tau_1 \) for some \( \Gamma \). By the typing rules, we just need to show that \( \emptyset \vdash \rho(e_2) \vdash \tau_2 \) and \( \vdash \tau_1 = \tau_2 \) for some \( \tau_2 \). Consider the cases, in the same rule as the figure, for the last rule used to derive the reduction:

- (Variable) In this case, \( e_1 = x \), \( e_2 = v_i \), and \( x \tau' = v_i^{\rho} \in \rho \) for some \( x, \iota, v, j, \) and \( \tau' \). The typing judgement can only be derived with one rule and it requires that \( x \tau' \in \Gamma \). The typing judgement (for \( \rho \)) can only be derived in one way and it requires that \( \tau = \tau', \emptyset ; \emptyset \vdash v_i : \tau', v_i \), and \( \vdash \tau = \tau' \). Thus the desired \( \tau_2 \) is \( \tau' \). We just need to show that \( \emptyset ; \Gamma \vdash \tau : \tau_2 \), which follows by Lemma 7.

- (Fix expression) In this case, \( e_1 = (x) \tau_1 \tau_2 \) and \( e_2 = \rho(\text{fix } f[X]:\tau_1 \tau_2) \). The typing judgement can only be derived with one rule and it requires that \( \emptyset \vdash \tau \text{ wf, } \alpha; \Gamma, j : \tau_1 \vdash e_1 : \tau_1 \) and \( \vdash \tau = \tau_1 \). Thus by the typing rules, \( \emptyset ; \Gamma \vdash e_2 \vdash \tau_2 \). By Lemma 3, \( \vdash \tau_1 = \tau_2 \), so the result follows by setting \( \tau_2 = \tau_2 \).

- (Box expression) In this case, \( e_1 = \emptyset \box v_i \) and \( e_2 = (v_i \tau) \). For some \( v, i, \) and \( j \). The typing judgement can only be derived with one rule and it requires that \( \emptyset ; \Gamma \vdash v_i ; \vdash \tau = \tau_1 \), and \( \vdash \tau = \tau_2 \) by the typing rules, \( \emptyset ; \Gamma \vdash e_2 \vdash \tau_1 \). By Lemma 3, \( \vdash \tau_1 = \tau_2 \), so the result follows by setting \( \tau_2 = \tau_2 \).

- (Application function) In this case, \( e_1 = (x) \tau_1 \tau_2 \) and \( e_2 = (x) \tau_1 \tau_2 \). For some \( e_3, e_4, \) and \( e_5 \). The typing judgement can only be derived with one rule and it requires that \( \emptyset ; \Gamma \vdash e_3 \vdash (\forall \alpha. \tau_1 \rightarrow \tau_2) \). By the typing rules, \( \emptyset ; \Gamma \vdash e_4 \vdash \tau_3 \); \( \emptyset ; \Gamma \vdash \tau \text{ wf, } \tau \vdash \tau_3 \); \( \vdash \tau_3 \). By the typing rules, \( \emptyset ; \Gamma \vdash e_2 \vdash \tau_3 \). By Lemma 3, \( \vdash \tau_3 = \tau_3 \), so the result follows by setting \( \tau_3 = \tau_3 \).

- (Application argument) In this case, \( e_1 = \emptyset \box v_i \) and \( e_2 = (v_i) \tau_1 \tau_2 \). For some \( e_3, e_4, \) and \( e_5 \). The typing judgement can only be derived with one rule and it requires that \( \emptyset ; \Gamma \vdash e_3 \vdash (\forall \alpha. \tau_1 \rightarrow \tau_2) \). By the typing rules, \( \emptyset ; \Gamma \vdash e_4 \vdash \tau_3 \); \( \emptyset ; \Gamma \vdash \tau \text{ wf, } \tau \vdash \tau_3 \); \( \vdash \tau_3 \). By the typing rules, \( \emptyset ; \Gamma \vdash e_3 \vdash \tau_3 \). By Lemma 3, \( \vdash \tau_3 = \tau_3 \), so the result follows by setting \( \tau_3 = \tau_3 \).

- (Application beta) In this case:

\[
\begin{align*}
e_1 &= (v_i \tau x) e_2 \\
v_1 &= (\rho(\text{fix } f[X]:\tau_1 \tau_2)) \\
v_2 &= (v_i \tau x) e_2 \\
v_3 &= (v_i \tau x) e_2 \\
v_4 &= (v_i \tau x) e_2 \\
\tau &= (\forall \alpha. \tau_1 \rightarrow \tau_2)
\end{align*}
\]

for some \( \rho, \alpha, \tau_1, \tau_2, e, i, v_2, j, k \). Unpacking the typing judgement, which can only be derived in one way, \( \emptyset ; \Gamma \vdash v_i : \tau(1), \emptyset ; \Gamma \vdash \tau \text{ wf (12)}, \alpha; \Gamma, j : \tau x ; \vdash e_1 : \tau_1 (3), \tau_1 = \tau_2[\tau/[\alpha] (4)), \emptyset ; \Gamma \vdash v_i : \tau_1 (5), \tau_1[\tau/[\alpha] = \tau_1[\tau/[\alpha] (6)), \emptyset ; \Gamma \vdash \tau \text{ wf, and } tr(\tau) = tr(\emptyset) \). By (1) and Lemma 7, \( \emptyset ; \emptyset \vdash v_i : \tau (7) \). By Lemma 3, \( \vdash \tau = \tau (8) \). By (5) and Lemma 7, \( \emptyset ; \emptyset \vdash v_i : \tau_1 (9) \). By (2), (7), (8), and (9), and (6), the typing rules give \( \rho(\Gamma, f; \tau x ; \vdash e_1[\tau/[\alpha] = \tau_2[\tau/[\alpha] (10)) \). By (3) and Lemma 8, \( \emptyset ; \Gamma, j : \tau x ; \vdash e_1[\tau/[\alpha] = \tau_2[\tau/[\alpha] (11) \). By (2) and (12), by inspection of the typing rules, \( \Gamma[\tau/[\alpha] = \Gamma' \) and \( \tau[\tau/[\alpha] = \tau \). Thus, \( \emptyset ; \Gamma, j : \tau x ; \vdash e_1[\tau/[\alpha] = \tau_2[\tau/[\alpha] (13) \). By (10) and (13), the typing rules give \( \emptyset ; \Gamma \vdash \ldots \)
\(\rho''(e'[\tau/\alpha])^k : \tau''[\tau/\alpha]\) (14). By (4), the result follows by setting \(\tau_2 = \tau''[\tau/\alpha]\).

- **(Box argument)** In this case, \(e_1 = (\text{box}_x e')^k\), \(e_2 = (\text{box}_x e')^k\), and \((\rho, e) \rightarrow (\rho, e')\) for some \(\tau, e, i, \text{ and } e'\). The typing judgement can only be derived with one rule and it requires that \(\emptyset \vdash \tau \omega f, \emptyset; \Gamma \vdash e : \tau'_1, \vdash \tau = \tau'\) and \(\tau_1 = \text{box}(\tau)_1\) for some \(\tau'\). By the induction hypothesis, \(\emptyset; \Gamma \vdash e' : \tau''\) and \(\vdash \tau'' = \tau''\) for some \(\tau''\). By the typing rules, \(\emptyset; \Gamma \vdash e_2 : \text{box}(\tau)_1\). By Lemma 3, \(\vdash \tau = \tau''\). By the typing rules, \(\emptyset; \Gamma \vdash e_2 : \text{box}(\tau)_1\).

- **(Unbox argument)** In this case, \(e_1 = (\text{unbox}_x e')^k\), \(e_2 = (\text{unbox}_x e')^k\), and \((\rho, e) \rightarrow (\rho, e')\) for some \(e, i, e'\). The typing judgement can only be derived with one rule and it requires that \(\emptyset; \Gamma \vdash e : \text{box}(\tau)_1\) for some \(j\). By the induction hypothesis, \(\emptyset; \Gamma \vdash e' : \tau'\) and \(\vdash \text{box}(\tau)_1 = \tau'\) for some \(\tau'\). The latter can only be derived with one rule and it requires that \(\tau'' = \text{box}(\tau'')^k\) and \(\vdash \tau_1 = \tau''\) for some \(\tau''\) and \(k\). By the typing rules, \(\emptyset; \Gamma \vdash e_2 : \tau''\), so the result follows by setting \(\tau_2 = \tau''\).

- **(Frame step)** In this case, \(e_1 = \rho'(e')^k\), \(e_2 = \rho'(e')^k\), and \((\rho', e) \rightarrow (\rho', e')\) for some \(\rho', e, i, e'\). The typing judgement can only be derived with one rule and it requires that \(\emptyset; \Gamma \vdash e : \tau_1\) for some \(\Gamma\). By the induction hypothesis, \(\emptyset; \Gamma \vdash e' : \tau'\) and \(\vdash \tau_1 = \tau'\). By the typing rules, \(\emptyset; \Gamma \vdash e_2 : \tau''\), so the result follows by setting \(\tau_2 = \tau''\).

- **(Frame return)** In this case, \(e_1 = \rho'(v')^k\) and \(e_2 = v'\) for some \(\rho', v', i\), and \(j\). The typing judgement can only be derived with one rule and it requires that \(\emptyset; \Gamma \vdash e : \tau_1\). By Lemma 7, \(\emptyset; \Gamma \vdash e' : \tau_1\). By Lemma 3, \(\vdash \tau_1 = \tau_1\), so the result follows by setting \(\tau_2 = \tau_1\).

Lemma 10 (Progress)

If \(\vdash M : \tau\) then either \(M\) has the form \((\rho, v')\) or \(M \rightarrow M'\) for some \(M'\).

Proof: The result follows from: If \(\vdash \rho : \Gamma\) and \(\emptyset; \Gamma \vdash e : \tau\) then either \(e\) has the form \(v'\) or \((\rho, e) \rightarrow (\rho, e')\) for some \(e'\). We will prove this by induction on the typing derivation for \(e\). Consider the last rule, in the same order as the figure, used to derive the judgement:

- **(Variable)** In this case \(e = x'\) and \(x : \tau \in \Gamma\). There is only one rule to derive \(\vdash \rho : \Gamma\) and it requires that \(x : \tau = v\) \(\in \rho\) and other conditions for some \(v\) and \(j\). Then by the variable rule, \((\rho, e) \rightarrow v\), as required.

- **(Fix expression)** In this case \(e = (\text{fix } f[\alpha](x : \tau_1) : \tau_2, e')^k\). Clearly by the fix expression rule:

\[(\rho, e) \rightarrow (\rho, (\text{fix } f[\alpha](x : \tau_1) : \tau_2, e')^k)\]

- **(Application)** In this case, \(e = (e_1[\tau_1/\alpha] e_2)^k\). The typing rule requires that \(\emptyset; \Gamma \vdash e_1 : (\forall x \tau_1 \rightarrow \tau_1)\) (1), \(\emptyset; \Gamma \vdash e_2 : \tau_2\) (2), and \(\vdash \tau_1[\tau_1/\alpha] = \tau_2\) (3) for some \(\tau_1, j, \text{ and } \tau_2\). By the

induction hypothesis, either \(e_1\) is a value or reduces, and \(e_2\) is a value or reduces. There are three subcases:

- **Case 1, \(e_1\) reduces:** In this case there is \(e'_1\) such that \((\rho, e_1) \rightarrow (\rho, e'_1)\). Then by the application function rule, \((\rho, e) \rightarrow (\rho, (e'_1[\tau'_1] e'_2)^k)\), as required.

- **Case 2, \(e_1\) is a value and \(e_2\) reduces:** In this case there is \(e'_2\) such that \((\rho, e_2) \rightarrow (\rho, e'_2)\). Then by the application function rule, \((\rho, e) \rightarrow (\rho, (e'_1[\tau'_1] e'_2)^k)\), as required.

- **Case 3, \(e_1 = v_1\) and \(e_2 = v_2\) for some \(v_1, v_2, \text{ and } l\):** There is only one typing rule to derive (1) and it requires that \(v_1\) have the form \(\langle \rho', \text{fix } f[\alpha](x : \tau_1) : \tau_2, e'\rangle\) for some \(\rho', f, x, \text{ and } e'\). Let \(t\) be the traceability of \(v_2\). By Lemma 6 and (2), \(\text{tr}(\tau_2) = t\). By Lemma 4 and (3), \(\text{tr}(\tau'_1[\tau'_1/\alpha]) = t\).

Then by the application beta rule:

\[(\rho, e) \rightarrow (\rho, (\rho', f: \tau = v_1, x : \tau_1[\tau'_1/\alpha] = v_2))(e'[\tau'/\alpha])^k)\]

where \(\tau' = (\forall x \tau_1 \rightarrow \tau_3)^k\), as required.

- **(Box expression)** In this case, \(e = (\text{box}_x e')^k\) for some \(\tau', e', \text{ and } i\). The typing rule requires that \(\tau = \text{box}(\tau')^k\), \(\emptyset; \Gamma \vdash e' : \tau''(1)\), and \(\vdash \tau'' = \tau''(2)\) for some \(\tau''\). By the induction hypothesis, \(\rho'\) is a value or reduces:

\[\text{If } e' = v'_1 \text{ then by Lemma 6 and (1), } \text{tr}(\tau'') = t. \text{ By (2) and Lemma 4, } \text{tr}(\tau') = t. \text{ So by the box reduction rule, } \text{for } (\rho, e) \rightarrow (\rho, (\rho', f: \tau = v'_1, x : \tau_1[\tau'/\alpha] = v_2))(e'[\tau'/\alpha])^k)\], as required.

- **(Unbox)** In this case, \(e = (\text{unbox}_x e')^k\) for some \(e'\) and \(i\). The typing rule requires that \(\emptyset; \Gamma \vdash e' : \text{box}(\tau)_1\) (1). By the unbox beta rule, \((\rho, e) \rightarrow (\rho, v'_1)^k\), as required.

- **(Frame)** In this case, \(e = (\rho')(e')^k\) for some \(\rho', e', \text{ and } i\). The typing rule requires that \(\rho' : \Gamma\) and \(\emptyset; \Gamma \vdash e' : \tau\) for some \(\Gamma\). By the induction hypothesis, \(\rho'\) is a value or reduces:

\[\text{If } e' = v'_1 \text{ then by the frame return rule, } (\rho, e) \rightarrow (\rho, v'_1)^k\], as required.

- **(Constant)** In this case \(e = c\) for some \(c\) and \(i\) and is clearly a value.

- **(Fix value)** In this case \(e = (\rho'. \text{fix } f[\alpha](x : \tau_1) : \tau_2, e')^k\) for some \(\rho', f, x, \tau_1, \tau_2, e'\) and \(i\) is clearly a value.

- **(Box value)** In this case \(e = (v': \tau')^k\) for some \(\tau', v, i\), and \(j\) is clearly a value.

We can also prove that typability implies traceability and thus typable programs are GC safe and remain so throughout execution.

Lemma 11

- **If \(\vdash M : \tau\) then \(\vdash M \text{ tr}\).**
- **If \(\vdash \rho : \Gamma\) then \(\vdash \rho\).**
- **If \(\Delta; \Gamma \vdash e : \tau\) then \(\vdash e\).**
If $\Delta; \Gamma \vdash v^i : \tau$ then $\vdash v : \text{tr} (\tau)$.

**Proof:** The results are proven simultaneously by induction on the structure of the typing derivation. The cases for the last rule used, in the same order as the figure, are:

- **(Variable)** In this case clearly $\vdash e \ tr$.
- **(Fix expression)** In this case $e = (\text{fix } f[a](x;\tau_1);\tau_2,e')^i$ for some $x, \alpha, \tau_1, \tau_2, e'$, and $i$. Then by the typing rule, $\Delta; \alpha; \Gamma; f; \tau; x; \tau_1 \vdash e' : \tau_2$ is a subderivation. By the induction hypothesis, $\vdash e' \ tr$. So by the rules for traceability, $\vdash e \ tr$, as required.
- **(Application)** In this case $e = (\text{app } \alpha; \tau_1; \tau_2,e')^i$ for some $\tau_1$ and $\tau_2$. By the induction hypothesis, $\vdash e_1 \ tr$ and $\vdash e_2 \ tr$. By the rules for traceability $\vdash e \ tr$, as required.
- **(Box expression)** In this case $e = (\text{box } \tau; e')^i$ for some $\tau'$, and $i$. By the typing rule, $\Delta; \Gamma \vdash e' : \tau''$ for some $\tau''$. By the induction hypothesis, $\vdash e' \ tr$. By the rules for traceability, $\vdash e \ tr$, as required.
- **(Unbox)** In this case $e = (\text{unbox } e')^i$ for some $\rho$, $\rho'$, and $i$. By the typing rule, $\rho : \Gamma'$ and $\rho ; \Gamma \vdash e' : \tau$ for some $\Gamma'$. By the induction hypothesis, $\vdash \rho \ tr$ and $\vdash e' \ tr$. By the rules for traceability, $\vdash e \ tr$, as required.
- **(Frame)** In this case, $e = \rho(e')^i$ for some $\rho$, $\rho'$, and $i$. By the typing rule, $\rho : \Gamma$ and $\Gamma \vdash \rho : e' : \tau'$ for some $\tau'$. By the induction hypothesis, $\vdash \rho \ tr$ and $\vdash e' \ tr$. By the rules for traceability, $\vdash e \ tr$, as required.
- **(Constant)** In this case $e = c$. By the typing rule, $\tau = \rho$ and $\Gamma \vdash c : \rho$. By the rules for traceability, $\vdash c : \text{tr}$, proving the fourth result. By the rules for traceability again, $\vdash e \ tr$, proving the third result.
- **(Fix value)** In this case $e = (\rho, \text{fix } f[a](x;\tau_1);\tau_2,e')^i$ for some $\rho, f, \alpha, x, \tau_1, \tau_2, e'$, and $i$. By the typing rule, $\rho : \Gamma'$ and $\alpha; \Gamma; f; \tau; x; \tau_1 \vdash e' : \tau_2$ for some $\Gamma'$. Also by the typing rules, $\tau$ is a function type, so $\text{tr} (\tau) = \tau$. By the induction hypothesis, $\vdash \rho \ tr$ and $\vdash e' \ tr$. By the rules for traceability, $\vdash (\rho, \text{fix } f[a](x;\tau_1);\tau_2,e')^i$, proving the fourth result. By the rules for traceability again, $\vdash e \ tr$, proving the third result.
- **(Box value)** In this case $e = (\text{box } \tau; e')^i$. By the typing rule, $\Delta; \Gamma \vdash v^i : \tau''$ and $\tau' = \tau''$ for some $\tau''$. Also by the typing rule, $\tau$ is a box type, so $\text{tr} (\tau) = \tau$. By the induction hypothesis, $\vdash v : \text{tr} (\tau'')$. By Lemma 4, $\vdash v : \text{tr} (\tau')$. By the rules for traceability, $\vdash (\text{box } \tau; e')^i$, proving the third result. By the rules for traceability again, $\vdash e \ tr$, as required.
- **(Environment)** In this case $\rho = x_1;\tau_3 = v_1^{s_1}, \ldots, x_n;\tau_n = v_n^{s_n}$. By the typing rule, $\emptyset; \emptyset \vdash v^i : \tau_j$ and $\tau_j = \tau_i$ for $1 \leq j \leq n$ and some $\tau_j$s. By the induction hypothesis, $\vdash v_j : \text{tr} (\tau_j')$ for $1 \leq j \leq n$. By Lemma 4, $\text{tr} (\tau_j) = \tau_j'$ for $1 \leq j \leq n$. Thus $\vdash v_j : \text{tr} (\tau_j)$ for $1 \leq j \leq n$. By the traceability rules, $\vdash \rho \ tr$, as required.
- **(Machine state)** In this case $M = (\rho, e)$. By the typing rule, $\vdash \rho : \Gamma$ and $\emptyset; \emptyset \vdash e : \tau$ for some $\Gamma$. By the induction hypothesis, $\vdash \rho \ tr$ and $\vdash e \ tr$. By the rules for traceability, $\vdash M \ tr$, as required.

### 3. Flow analysis

Our original motivation for this work was to apply interprocedural analysis to the problem of eliminating unnecessary boxing in programs. There is a vast body of literature on interprocedural analysis and optimization, and it is generally fairly straightforward to use these approaches to obtain information about what terms flow to what use sites. This paper is not intended to provide any contribution to the algorithmic side of this body of work, which we will broadly refer to as flow analysis. Our contribution in this paper lies in showing how to extend flow analysis to the type level, and showing that any generic flow analysis so extended can be used to implement an unboxing optimization that preserves type safety.

In order to do this, we must provide some framework for describing what information a flow analysis must provide. For the purposes of our unboxing optimization, we are interested in finding (interprocedurally) for every (unbox $v^i$) operation the set of (box $\tau$, $e$) terms that could possibly reach $v$. Under appropriate conditions, we can then eliminate both the box introductions and the box elimination, thereby improving the program. The core language defined in Section 2 provides labels serving as proxies for the terms, types, and variables on which they occur—so the question above can therefore be re-stated as finding the set of labels $k$ that reach the position labeled with $j$.

More generally, following previous work we begin by defining an abstract notion of analysis. We say that an analysis is a pair $(C, \rho)$. Binding environments $\rho$ simply serve to map variables to the label of their binding sites. The mappings are as usual, global for the program. Consequently, a given environment may not apply to alpha-variants of a term. We do not require that labels be unique within a program—as usual however, analyses will be more precise if this is the case. Variables are also not required to be unique (since reduction may duplicate terms and hence binding sites). However, duplicate variable bindings in a program must be labeled consistently according to $\rho$ or else no analysis of the program can be acceptable according to our definition. This can always be avoided by alpha-varying or relabeling appropriately.

A cache $C$ is a mapping from labels to sets of shapes. Shapes are given by the grammar:

\[
\begin{align*}
\text{Shapes: } s & ::= e^i \mid (\forall i,j \rightarrow k)_{s}^{j} \mid (\text{box } k)_{s}^{j} \mid B^i \mid (\forall i,j \rightarrow k)_{s}^{j} \mid (\text{box } i)_{s}^{j}
\end{align*}
\]

There are two classes of shapes—term shapes and type shapes. The idea behind term shapes is that each shape provides a proxy for a set of terms that might flow to a given location, describing both the shape of the values that might flow there and the labels of the sub-components of those values. For example, for an analysis $(C, \rho)$, $e^i \in C(k)$ indicates that (according to the analysis) the constant $c$, labeled with $i$, might flow to a location labeled with $k$. Similarly, if $(\forall i,j \rightarrow k)_{s}^{j} \in C(l)$, then the analysis specifies that among the values flowing to locations labeled with $l$ might be functions labeled with $k$, whose type parameter is labeled with $i$, whose bodies are labeled with $j$, whose types are labeled with $j$, and whose bodies are labeled with $j$. If $(\text{box } k)_{s}^{j} \in C(l)$ then among the values that might flow to $l$ (according to the analysis) are boxed values labeled with $i$, with meta-data $t$ and whose bodies are labeled by some $j$ such that $C(j) \subseteq C(k)$.

Where term shapes provide a proxy for the set of values that might flow to a given location, type shapes provide a proxy for the types of the locations that values might flow through to get a given location. For example, for an analysis $(C, \rho)$, $B^i \in C(k)$ indicates that (according to the analysis) objects that reach location $k$ might flow through a variable or term of type $B$, labeled with $i$. The function type and box type shapes similarly correspond to the flow of values through locations labeled with function or box types.

It is important to note that the shapes in the cache may not correspond exactly to the terms in the program, since reduction may change program terms (e.g. by instantiating variables with values).
However, reduction does not change the outer shape and labeling of values—it is this reduction invariant information that is captured by shapes.

Clearly, not every choice of analysis pairs is meaningful for program optimization. While in general it is reasonable (indeed, unavoidable) for an analysis to overestimate the set of terms associated with a label, it is unacceptable for an analysis to underestimate the set of terms that flow to a label—most optimizations will produce incorrect results, since they are designed around the idea that the analysis is telling them everything that could possibly flow to them. In order to capture the notion of when an analysis pair gives a suitable approximation of the flow of values in a program we follow the general spirit of Nielson et al. [7], and define a notion of an acceptable analysis. That is, we give a declarative specification that gives sufficient conditions for specifying when a given analysis does not underestimate the set of terms flowing to a label, without committing to a particular analysis. We arrange the subsequent meta-theory such that our results apply to any analysis that is acceptable. In this way, we completely decouple our optimization from the particulars of how the analysis is computed.

Our acceptable-analysis relation is given in Figures 6 and 7—the judgement $C; \rho \vdash (\rho, e)$ determines that an analysis pair $(C, \rho)$ is acceptable for a machine state $(\rho, e)$, and similarly for the environment and expression forms of the judgement. We use the notation $\text{lbl}(e)$ to denote the outermost label of $e$: that is, $i$ where $e$ is of the form $m^i$ or $v^i$. The acceptability judgement generally indicates for each syntactic form what the flow of values is. For example, in the application rule, the judgment insists that for every function value that flows to the applicand position, the set of shapes associated with the parameter of that function is a super-set of the set of shapes associated with the argument of the application; and that the set of shapes associated with the result of the function is a sub-set of the set of shapes associated with the application itself.

The judgement $C; \rho \vdash \tau$ determines that an analysis pair $(C, \rho)$ is acceptable for a labeled type $\tau$. In particular, if a function flows to a function type $\forall \tau_1. \tau_2 \rightarrow \tau$, then the set of values that flow to the function's parameter can flow to the argument type $\tau_1$, and the set of values that can flow from the result of the function can flow to the result type $\tau_2$. And similarly for box types.

Given this definition, we can show that the acceptability relation is preserved under reduction. First we show that the cache is only refined by reduction.

**Lemma 12 (Cache refinement under reduction)**

If $C; \rho \vdash \rho', C; \rho \vdash e_1$, and $(\rho, e_1) \longrightarrow (\rho, e_2)$ then $C(\text{lbl}(e_1)) \supseteq C(\text{lbl}(e_2))$.

**Proof:** The proof is by induction on the derivation of $(\rho, e_1) \longrightarrow (\rho, e_2)$. Consider the cases for the last rule used to it (the cases are in the same order as in the figure):

- **(Variable instantiation.)** In this case, $e_1 = x^k, e_2 = v^l$, and $x : \tau = v : \tau \in \rho$. The assumption $C; \rho \vdash \rho$ requires that $C(\tau) \subseteq C(\text{lbl}(\tau))$ and $\text{lbl}(x) = \text{lbl}(\tau)$. The assumption $C; \rho \vdash e_1$ requires that $C(\text{lbl}(x)) \subseteq C(k)$. Thus $C(\tau) \subseteq C(k)$. Clearly, $\text{lbl}(e_1) = k$ and $\text{lbl}(e_2) = l$ and the result follows.

- **(Fix introduction.)** In this case, clearly $\text{lbl}(e_1) = \text{lbl}(e_2)$ and the result immediately follows.

- **(Box introduction.)** In this case, clearly $\text{lbl}(e_1) = \text{lbl}(e_2)$ and the result immediately follows.

- **(Application left.)** In this case, clearly $\text{lbl}(e_1) = \text{lbl}(e_2)$ and the result immediately follows.

- **(Application right.)** In this case, clearly $\text{lbl}(e_1) = \text{lbl}(e_2)$ and the result immediately follows.

\[
\text{funC}(i,j,k,l) \quad \text{boxC}(i,j)
\]

\[
\text{funC}(i,j,k,l) = \\
\land \forall (\forall j', k' \rightarrow l')^i C(i) : C(j) = C(j') \land C(k) \subseteq C(k') \land C(l') \subseteq C(l)
\]

\[
\land \forall (\forall j', k' \rightarrow l')^i C(i) : C(j) = C(j') \land C(k) \subseteq C(k') \land C(l') \subseteq C(l)
\]

\[\text{boxC}(i,j) = \]

\[\forall \text{boxC}(j_1)^i C(i) : C(j') \subseteq C(j)
\]

\[\forall \text{boxC}(j_1)^i C(i) : C(j') \subseteq C(j)
\]

| $C; \rho \vdash \rho'$ |
|---|
| $C; \rho \vdash e_1$ |
| $C; \rho \vdash \tau$ |
| $C; \rho \vdash e_2$ |
| $\text{funC}(\text{lbl}(e_1), \text{lbl}(\tau), \text{lbl}(e_2), i)$ |
| $C; \rho \vdash (e_1[\tau] e_2)^i$ |
| $C; \rho \vdash \text{boxC}(\text{lbl}(e_1), i)$ |
| $C; \rho \vdash (\text{unboxC})^i$ |
| $C; \rho \vdash \rho'$ |
| $C; \rho \vdash e_1$ |
| $C; \rho \vdash e_2$ |
| $C(\text{lbl}(e)) \subseteq C(i)$ |
| $C; \rho \vdash \rho'$ |
| $C; \rho \vdash e_1$ |
| $C; \rho \vdash e_2$ |
| $C(\text{lbl}(e)) \subseteq C(i)$ |
| $C; \rho \vdash e_1$ |
| $C; \rho \vdash e_2$ |
| $C(\text{lbl}(e)) \subseteq C(i)$ |
| $C; \rho \vdash e_1$ |
| $C; \rho \vdash e_2$ |
| $C(\text{lbl}(e)) \subseteq C(i)$ |
| $C; \rho \vdash e_1$ |
| $C; \rho \vdash e_2$ |
| $C(\text{lbl}(e)) \subseteq C(i)$ |

**Figure 6.** Acceptable Analysis, Expressions
Lemma 13

If \( C; \rho \vdash \tau \) and \( \text{C(lbl(\tau))} = \text{C(\rho(\alpha))} \) then:

- \( \text{If } C; \rho \vdash \tau' \text{ then } C; \rho \vdash \tau'[\alpha/\beta] \).
- \( \text{If } C; \rho \vdash \Gamma \text{ then } C; \rho \vdash \Gamma'[\alpha/\beta] \).
- \( \text{If } C; \rho \vdash e \text{ then } C; \rho \vdash e[\alpha/\beta] \).
- \( \text{If } C; \rho \vdash \rho \text{ then } C; \rho \vdash \rho[\alpha/\beta] \).

Proof: The proof is by induction on the derivation of the \( C; \rho \vdash \tau \) and \( C; \rho \vdash e \). Consider the cases for the rules used to derive it (in the same order as in the figures):

- The cases for expressions are straight forward.
- (Type variable) In this case \( \tau' = \beta' \). If \( \beta \neq \alpha \) then \( \tau'[\alpha/\beta] = \tau' \) and the result is immediate. Otherwise, by the rules for acceptability, \( \text{C(\rho(\alpha'))} = \text{C(i)} \). If \( \tau = \sigma' \) then \( \tau'[\alpha/\beta] = \sigma' \).
- Consider the cases for \( \sigma' \):
  - Subcase 1, \( \sigma = \alpha' \): Then by the rules for acceptability, \( \text{C(\rho(\alpha'))} = \text{C(j)} \). Since \( C(j) = C(\rho(\alpha)) = C(i) \), \( \text{C(\rho(\alpha'))} = \text{C(i)} \), and thus \( C; \rho \vdash \alpha' \), as required.
  - Subcase 2, \( \sigma = \exists \alpha'. \tau_1 \rightarrow \tau_2 \): Since \( C; \rho \vdash \tau \), the rules require:
    - \( C; \rho \vdash \tau_1 \) (1)
    - \( C; \rho \vdash \tau_2 \) (2)
    - \( \forall \rho(\alpha'). \text{C(lbl(\tau_1))} \rightarrow \text{C(lbl(\tau_2))} ) \in C(i) \) (3)
    - \( \text{funC(j, \rho(\alpha'), \text{C(lbl(\tau_1)), C(lbl(\tau_2))})} \) (4)

By (3) and \( C(j) = C(i) \):

\[ \forall \rho(\alpha'). \text{C(lbl(\tau_1))} \rightarrow \text{C(lbl(\tau_2))} \] (5)

By (4) and \( C(j) = C(i) \):

\[ \text{funC(i, \rho(\alpha'), \text{C(lbl(\tau_1)), C(lbl(\tau_2))})} \] (6)

By (1), (2), (5), and (6), by the rules for acceptability, \( C; \rho \vdash \sigma' \).

- Subcase 3, \( \sigma = \text{box}(\tau'') \): Since \( C; \rho \vdash \tau \), the rules require:
    - \( C; \rho \vdash \tau'' \) (1)
    - \( \text{box.C(lbl(\tau''))} \in C(j) \) (2)
    - \( \text{box.C(j, lbl(\tau''))} \) (3)

By (2) and \( C(j) = C(i) \), \( \text{box.lbl(lbl(\tau''))} \) (4). By (3) and \( C(j) = C(i) \), \( \text{box.lbl(lbl(\tau''))} \) (5). By (1), (4), and (5), by the rules for acceptability, \( C; \rho \vdash \sigma' \), as required.

- (Base type) In this case \( \tau'[\alpha/\beta] = \tau \) and the result is immediate.

- (Function type) In this case \( \tau' = (\forall \alpha'. \tau_1 \rightarrow \tau_2) \). The rules for acceptability require:
    - \( C; \rho \vdash \tau_1 \) (1)
    - \( C; \rho \vdash \tau_2 \) (2)
    - \( \forall \rho(\alpha'). \text{C(lbl(\tau_1))} \rightarrow \text{C(lbl(\tau_2))} ) \in C(i) \) (3)
    - \( \text{funC(i, \rho(\alpha'), \text{C(lbl(\tau_1)), C(lbl(\tau_2))})} \) (4)

By (1), (2), and the induction hypothesis:

\[ C; \rho \vdash \tau_1[\alpha/\beta] \] (5)
\[ C; \rho \vdash \tau_2[\alpha/\beta] \] (6)

Since \( \text{C(lbl(\tau_1[\alpha/\beta]))} = \text{C(lbl(\tau_1))} \) and \( \text{C(lbl(\tau_1[\alpha/\beta]))} = \text{C(lbl(\tau_1))} \):

\[ \forall \rho(\alpha'). \text{C(lbl(\tau_1[\alpha/\beta]))} \rightarrow \text{C(lbl(\tau_2[\alpha/\beta]))} ) \in C(i) \] (7)

\[ \text{funC(i, \rho(\alpha'), \text{C(lbl(\tau_1[\alpha/\beta])), C(lbl(\tau_2[\alpha/\beta]))})} \] (8)

Since \( \text{C(lbl(\tau'[\alpha/\beta]))} = \text{i} \), by (5), (6), (7), and (8), \( C; \rho \vdash \tau'[\alpha/\beta] \), as required.

Next we show a type substitution lemma for acceptability.
• (Box type) In this case, \( \tau' = \text{box}(\tau'')^{i} \). The rules for acceptability require:

\[
C; \rho \vdash \tau'' \quad (1)
\]
\[
(\text{box } \text{lbl}(\tau'')^{i})^{k} \in C(i) \quad (2)
\]
\[
\text{boxC}(i, \text{lbl}(\tau'')) \quad (3)
\]

By (1) and the induction hypothesis:

\[
C; \rho \vdash \tau'[\tau/\alpha] \quad (4)
\]

Since \( \text{lbl}(\tau'[\tau/\alpha]) = \text{lbl}(\tau'') \):

\[
(\text{box } \text{lbl}(\tau'[\tau/\alpha])^{i})^{k} \in C(i) \quad (5)
\]
\[
\text{boxC}(i, \text{lbl}(\tau'[\tau/\alpha])) \quad (6)
\]

Since \( \text{lbl}(\tau'[\tau/\alpha]) = i \), by (4), (5), and (6), \( C; \rho \vdash \tau'[\tau/\alpha] \), as required.

• The cases for type and value environments are straightforward.

With these lemmas we can prove that reduction preserves acceptability of the flow analysis.

**Lemma 14 (Preservation of acceptability under reduction)**

If \( C; \rho \vdash M \) and \( M \rightarrow M' \) then \( C; \rho \vdash M' \).

**Proof:** If \( C; \rho \vdash (\rho, e) \) then \( C; \rho \vdash \rho \) and \( C; \rho \vdash e \). If \( (\rho, e) \rightarrow (\rho', e') \), then the result follows if we show that \( C; \rho \vdash (\rho', e') \). The proof of the latter is by induction on the derivation of \( (\rho, e) \rightarrow (\rho', e') \).

Consider the cases for the last rule used to derive it (the cases are in the same order as in the figure):

• In this case \( e = x^{h} \), \( e' = v^{i} \), and \( x^{\tau} = v^{j} \in \rho \). The assumption \( C; \rho \vdash \rho \) requires that \( C; \rho \vdash v^{i} \), what is required.

• In this case \( e = (\text{fix } f[a])(x^{\tau}; \tau, e''^{h})^{i} \) for some \( f, \alpha, x, \tau, \tau, e''^{j} \), and \( j, e' = (\rho, f[x^{\tau}]; \tau, e''^{j})^{i} \). Let \( i = \text{lbl}(\tau) \). The first hypothesis can only be derived by one rule and it requires that \( \rho(f) = j \) \( (5) \), \( \rho(x) = i \) \( (6) \), \( \rho(e) \rightarrow v^{k} \) \( (3) \), and \( (\forall \alpha, \rho(x) \rightarrow \text{lbl}(e'')^{j})^{i} \in \text{C}(j) \) \( (9) \). Instantiating Fact 4 with Fact 9, \( C(\text{lbl}(\tau)) = C(\text{q}(\alpha))^{(10)} \), \( C(k) \subseteq C(i) \) \( (11) \), and \( C(\text{lbl}(e'')) \subseteq C(l) \) \( (12) \). Judgement 7 requires that \( C; \rho \vdash \tau_{1}^{(13)} \). By (13), (2), (10), by Lemma 13, \( C; \rho \vdash \tau_{1}^{(14)} \). Since \( C; \rho \vdash \rho, (5), \tau_{1}^{(15)} \), \( C(j) \subseteq C(j) \), (1), \( \text{lbl}(\tau_{1}/\alpha) = \text{lbl}(\tau) = i \) and \( (6), (15), (11), \) and \( (3) \), we can derive \( C; \rho \vdash (\rho', \tau_{1})^{(16)} \). By (13), (2), and (10), by Lemma 13, \( C; \rho \vdash \text{lbl}(\tau)/\alpha = \text{lbl}(e'')^{(16)} \), and \( (16) \), \( \text{lbl}(\text{lbl}(\tau)/\alpha) = \text{lbl}(e'')^{(12)} \), we can derive \( C; \rho \vdash e' \), as required.

• In this case \( e = (\text{box } v^{i})^{j} \) for some \( t, \tau, e, v, \) and \( e' = (\text{box } v^{i})^{j} \) for some \( e_{1} \) and \( (\rho, e) \rightarrow (\rho, e) \) is a subderivation. The first hypothesis can only be derived by one rule and it requires that \( C; \rho \vdash \text{unbox}(\tau_{1})^{(7)} \), \( C; \rho \vdash e_{1}^{(1)} \), \( (\text{box } \text{v}^{i})^{j} \) \( \subseteq \text{C}(i) \) \( (2) \), some \( j, \) and \( (\text{C}(\text{lbl}(e_{1}))) \subseteq \text{C}(j) \) \( (3) \). By the induction hypothesis and Judgement 1, \( C; \rho \vdash e_{1}^{(4)} \). By Lemma 12, \( C(\text{lbl}(e_{1})) \subseteq C(\text{C}(\text{lbl}(e_{1}))) \) \( (5) \). Combining Facts 3 and 5 gives \( C(\text{C}(\text{lbl}(e_{1}))) \subseteq C(j) \) \( (6) \). Then by Facts 7, 4, 2, and 6, and using the rules for acceptable analysis, \( C; \rho \vdash \text{unbox } v^{i} \) \( (12) \), we are required to prove.

• In this case \( e = \langle e_{1}[\tau] e_{2}^{j} \rangle^{i} \) for some \( e_{1}, \tau, e_{2}, \) and \( e' = \langle e_{1}[\tau] e_{2}^{j} \rangle^{i} \), and \( (\rho, e_{1}) \rightarrow (\rho, e_{1}) \) is a subderivation. The first hypothesis can only be derived by one rule and it requires that \( C; \rho \vdash e_{1}^{(1)} \), \( C; \rho \vdash \tau_{1}^{(7)} \), and \( C; \rho \vdash e_{2}^{(2)} \) \( (3) \), and \( \text{funcC}(\text{lbl}(e_{1})), \text{lbl}(\tau), \text{lbl}(e_{2}), i )^{(4)} \). By the induction hypothesis and Judgement 1, \( C; \rho \vdash e_{1}^{(4)} \). By Lemma 12, \( C(\text{lbl}(e_{1})) \subseteq C(\text{C}(\text{lbl}(e_{1}))) \) \( (5) \). Combining Facts 3 and 5 gives \( C \Rightarrow C(i) \) \( (6) \). Using (1), (4), and (6) we derive \( C; \rho \vdash e^{(5)} \), as required.
In this case \( e = \rho'(v^i)^j \) and \( e' = v^i \). The assumption \( C; \rho \vdash e \) unpackages, requiring that \( C; \rho \vdash v^i \), as required.

**Lemma 15 (Many-step reduction preserves acceptability)**  
If \( C; \rho \vdash M \) and \( M \rightarrow^* M' \) then \( C; \rho \vdash M' \).

**Proof:** The proof is by induction on the length of the reduction sequence and Lemma 14.

We can also show an important connection between typing and acceptable flow analysis—namely that the cache of an expression's type is contained in the cache of that expression.

**Lemma 16**  
If \( \Delta; \Gamma \vdash e : \tau, C; \rho \vdash \Gamma, \) and \( C; \rho \vdash e \) then \( C(\text{lbl}(\tau)) \subseteq C(\text{lbl}(e)) \) and \( C; \rho \vdash \tau \).

**Proof:** The proof is by induction on the derivation of \( \Delta; \Gamma \vdash e : \tau \). Consider the cases for the last rule used (in same order as figure):

- **(Variable)** In this case \( e = x^i \). By the rules for acceptable analysis, \( g(x) = \text{lbl}(\tau) \) (1), \( C; \rho \vdash (2) \), and \( C(g(x)) \subseteq C(i) \) (3). By (1) and (3), \( C(\text{lbl}(\tau)) \subseteq C(\text{lbl}(e)) \) (4), and The result is (4) and (2).
- **(Fix expression)** In this case, \( e = (\text{fix } f[\alpha][x : \tau])^j \) and \( \tau = (\forall \alpha. \tau_1 \rightarrow \tau_2)^j \). The first part is immediate since \( \text{lbl}(\tau) = \text{lbl}(e) \). The second part is required by \( C; \rho \vdash e \).
- **(Application)** In this case:

  \[
  e \vDash (e_1 \tau_1 e_2)^j \\
  \Delta; \Gamma \vdash e_1 : (\forall \alpha. \tau_1 \rightarrow \tau_3)^j \quad (1) \\
  \tau = \tau_3[\tau / \alpha]
  \]

  By the rules for acceptability, \( C; \rho \vdash e_1 (2) \), \( C; \rho \vdash \tau' (3) \), \( C; \rho \vdash e_2 \), and \( \text{funC}(\text{lbl}(e_1), \text{lbl}(\tau'), \text{lbl}(e_2), i) \) (4). By (1), (2), and the induction hypothesis, \( C(j) \subseteq C(\text{lbl}(e_1)) \) (5) and \( C; \rho \vdash (\forall \alpha. \tau_1 \rightarrow \tau_3)^j \) (6). By (5) and the rules for acceptability, \( C; \rho \vdash \tau_2 (7) \) and \( (\forall \alpha(x). \text{lbl}(\tau_1) \rightarrow \text{lbl}(\tau_3)) \subseteq C(j) \) (8). By (5), instantiating (4) with (8), \( C(\text{lbl}(\tau')) = C(\text{g}(\alpha)) \) (9) and \( C(\text{lbl}(\tau_3)) \subseteq C(i) \), so since \( \text{lbl}(\tau) = \text{lbl}(\tau_3) \), \( C(\text{lbl}(\tau)) \subseteq C(i) \) (10). By (3), (7), and (9), \( C; \rho \vdash \tau_3[\tau / \alpha] \) (11). The result is (10) and (11).
- **(Box expression)** In this case \( e = (\text{box } e')^j \) and \( \tau = \text{box } (\tau')^j \). The first part holds as \( \text{lbl}(e) = \text{lbl}(\tau) \). The second part is required by \( C; \rho \vdash e \).
- **(Fix value)** In this case, \( e = (\rho. \text{fix } f[\alpha][x : \tau])^j \), \( \tau = (\forall \alpha. \tau_1 \rightarrow \tau_2)^j \). The first part holds as \( \text{lbl}(e) = \text{lbl}(\tau) \). The second part is required by \( C; \rho \vdash e \).
- **(Box value)** In this case \( e = (v^i. \tau)^j \) and \( \tau = \text{box}(\tau')^j \). The first part holds as \( \text{lbl}(e) = \text{lbl}(\tau) \). The second part is required by \( C; \rho \vdash e \).

**4. Unboxing**

The goal of the unboxing optimization is to use the information provided by a flow analysis to replace a boxed object with the contents of the box. Doing so may change the traceability, since the object in the box may not be a GC-managed reference. Moreover, the object in the box may itself be a candidate for unboxing; consequently, determining the traceability of boxed objects depends on exactly which objects are unboxed. Function parameters may be instantiated with objects from multiple different definition sites, some of which may be unboxed and some of which may not.

Consider again the first example from Section 2, written out with explicit type information and labels:

\[
\begin{aligned}
\text{let } & f[[x : B^0]:B^2.x^7] = z_1 = (\text{box } 2014)^{11} \\
& z_2 = f[[z_1^{12}]]^{13} \\
& \text{in } z_2^{15}
\end{aligned}
\]

It is fairly easy to see that this program is unboxable. The binding site for \( x \) is only reached by the term labeled with 11 (the outer box introduction), and hence there should be no problems with changing its type annotation. Each box elimination is reached only by a single box introduction, and hence the box/unbox pairs in this program should be eliminable, yielding an optimized program:

\[
\begin{aligned}
\text{let } & f[[x : B^0]:B^2.x^7] = z_1 = 3^{10} \\
& z_2 = f[[z_1^{12}]]^{13} \\
& \text{in } z_2^{15}
\end{aligned}
\]

Notice that in order to rewrite the program, we have had to change the types of both \( f \) and \( x \), since we have eliminated the box introductions on the argument and in the body of \( f \). The change in type of \( x \) has changed its traceability from \( \text{box } B \) to \( B \). If we choose (perhaps because of limitations on the precision of the analysis, or perhaps because of other constraints) to only eliminate the outer box/unbox pair, then we must similarly adjust types on the remaining box introduction (labeled with 8).

Notice that in order to rewrite the program, we have had to change the types of both \( f \) and \( x \), since we have eliminated the box introductions on the argument and in the body of \( f \). The change in type of \( x \) has changed its traceability from \( \text{box } B \) to \( B \). If we choose (perhaps because of limitations on the precision of the analysis, or perhaps because of other constraints) to only eliminate the outer box/unbox pair, then we must similarly adjust types on the remaining box introduction (labeled with 8).

\[
\begin{aligned}
\text{let } & f[[x : B^0]:B^2.x^7] = z_1 = 3^{10} \\
& z_2 = f[[z_1^{12}]]^{13} \\
& \text{in } z_2^{15}
\end{aligned}
\]

Clearly then, to optimize these programs in a type preserving fashion, we must rewrite types along with the terms in the program. In the rest of this section, we first develop a framework for specifying an unboxing assignment regardless of any correctness concerns, and then separately define a judgement specifying when such an assignment is a reasonable one.
We specify a particular choice of unboxing via an unboxing set \( \Upsilon \).

4.1 The unboxing optimization

We specify a particular choice of unboxing via an unboxing set \( \Upsilon \) which contains the set of labels of terms and types to be unboxed. A choice of a particular \( \Upsilon \) then induces an unboxing function as defined in Figure 8. The unboxing function is defined in a straightforward compositional manner. Box introductions are dropped when their labels are in the unboxing set, box type constructors are dropped when their labels are in the unboxing set, box eliminations are dropped when the labels of their arguments are in the unboxing set, and all other terms and types are left unchanged.

An important observation about the unboxing optimization as we have defined it is unlike many previous interprocedural approaches (Section 7), it only improves programs and never introduces new instructions or allocation. This is easy to see, since the unboxing function only removes boxes (which allocate and have an instruction cost), and unboxes (which have an instruction cost) and never introduces any new operations at all.

4.2 Acceptable unboxings

While any choice of \( \Upsilon \) defines an unboxing, not every unboxing set is reasonable in the sense that it defines a type and semantics preserving optimization. Just as we defined a notion of acceptable analysis in Section 3, we will define a judgement that captures sufficient conditions for ensuring correctness of an unboxing, without specifying a particular method of choosing such an unboxing. By using analyses of different precisions or choosing different optimization strategies, we may end up with quite different choices of unboxings; however, so long as they satisfy our notion of acceptability we can be sure that they will preserve correctness.

Informally, a choice of an unboxing set is reasonable if it meets two criteria. Firstly, it must make uniform choices in the sense that if a box introduction is eliminated, then all of the types and elimination forms to which it flows must also be unboxed, and vice versa. Secondly, we must ensure that types remain consistent with their uses in polymorphic instantiations, since we do not allow polymorphism over base types.

We use the notation \( i \triangleleft j \) to indicate when an unboxing agrees at two labels \( i \) and \( j \).

\[ i \triangleleft j \quad \text{iff} \quad \text{either } i, j \in \Upsilon \text{ or } i, j \notin \Upsilon \]

The first requirement is then specified via the cache consistency judgement, which enforces that for any label \( i \), the unboxing set must agree on \( i \) and the labels of any shapes in the cache of \( i \).

\[ \forall i, s : s \in C(i) \implies \exists \Upsilon \quad \text{lb}(i) \]

The second requirement is specified via the consistent unboxing judgement of Figure 9. The type rules determine the traceability of the unboxed type: that is, the judgement \( \Upsilon \vdash \tau : t \) indicates that unboxing \( \tau \) with \( \Upsilon \) will result in a type of traceability \( t \). The key use of the type judgement is in the term level polymorphic instantiation rule, which requires that the traceability of the unboxed type be \( \tau \).

4.3 Type Preservation

Our goal is to show that the unboxing function induced by any acceptable unboxing is in some sense correct as an optimization. The first part of this is to show that unboxing preserves typing. One key property is that types have non-empty caches.

Lemma 17 (Type Inhabitance)

If \( \tau \) is not a type variable and \( C; \Upsilon \vdash \tau \) then \( C(\text{lb}(\tau)) \neq \emptyset \).

Proof: The proof is by inspection of the rules for acceptability.

We also need several technical properties: labels agree when their caches intersect, unboxing preserves type well formedness, type traceability, and type equality, and unboxing commutes with type substitution.

Lemma 18 (Agreement)

If \( C \vdash \Upsilon \) and \( C(i) \cap C(j) \neq \emptyset \) then \( i \triangleleft j \).

Proof: The proof is by inspection of the rules for cache consistency.

Lemma 19

If \( \Delta \vdash \tau : \omega \) then \( \Delta \vdash \tau : \omega \).

Proof: The proof is a straightforward induction on the structure of \( \tau \).

Lemma 20

If \( C; \Upsilon \vdash \tau_2 \) and \( C(\text{lb}(\tau_2)) = C(\text{lb}(\alpha)) \) then:

- If \( C; \Upsilon \vdash \tau_1 \) then \( \tau_1[\tau_2/\alpha]_{\Upsilon} = \tau_1[\tau_2/\alpha]_{\Upsilon} \).
- If \( C; \Upsilon \vdash e \) then \( e[\tau_2/\alpha]_{\Upsilon} = e[\tau_2/\alpha]_{\Upsilon} \).

Proof:
The proof is a straight forward induction on the structure of e.

Lemma 21
If $\vdash \tau : t$ then $\text{tr}(\tau[\tau]) = t$.

Proof: The proof is a straight forward induction on the derivation of $\vdash \tau : t$.

Lemma 22
If $\vdash \tau_1 = \tau_2$, $C \vdash \tau_1$, and either $C(\text{lbl}(\tau_1)) \subseteq C(\text{lbl}(\tau_2))$ or $C(\text{lbl}(\tau_2)) \subseteq C(\text{lbl}(\tau_1))$ then $\vdash \tau_1_{\tau} = \tau_2_{\tau}$.

Proof: The proof is by induction on the derivation of $\vdash \tau_1 = \tau_2$.

Consider the last rule used (in the same order as the figure):

- (Type variable) In this case $\tau_1 = \alpha^i$ and $\tau_2 = \alpha^k$. By definition, $\tau_1_{\tau} = \tau_2_{\tau}$, and the result is immediate.

- (Base) In this case $\tau_1 = \emptyset$ and $\tau_2 = \emptyset$. By definition, $\tau_1_{\tau} = \tau_1$ and $\tau_2_{\tau} = \tau_2$, and the result is immediate.

- (Function) In this case:
  \[
  \begin{align*}
  \tau_1 &= (\forall \alpha. \tau_1 \rightarrow \tau_2)^i_1 \\
  \tau_2 &= (\forall \alpha. \tau_1 \rightarrow \tau_2)^i_2 \\
  \vdash \tau_1 &= \tau_2_1 \\
  \vdash \tau_2 &= \tau_2_2
  \end{align*}
  \]

WLOG, assume $C(i_1) \subseteq C(i_2)$ (3). By the rules for acceptability:

\[
\begin{align*}
C; \emptyset &\vdash \tau_1_1 \quad \text{(4)} \\
C; \emptyset &\vdash \tau_1_2 \quad \text{(5)} \\
(\forall \alpha. \text{lbl}(\tau_1_1) &\rightarrow \text{lbl}(\tau_1_2))_i^i &\in C(i_1) \quad \text{(6)} \\
C; \emptyset &\vdash \tau_2_1 \quad \text{(7)} \\
C; \emptyset &\vdash \tau_2_2 \quad \text{(8)} \\
f\alpha C(i_2, \emptyset, \text{lbl}(\tau_1_2), \text{lbl}(\tau_1_2)) &\in \text{C}(i_2) \quad \text{(9)}
\end{align*}
\]

By (6), (3), and (9), $C(\text{lbl}(\tau_1_2)) \subseteq C(\text{lbl}(\tau_1_2))$ (10) and $C(\text{lbl}(\tau_1_2)) \subseteq C(\text{lbl}(\tau_2_2))$ (11). By (1), (4), (7), (10), and the induction hypothesis, $\vdash \tau_1_{\tau_1} = \tau_2_{\tau_2}$ (12). By (2), (5), (8), (11), and the induction hypothesis, $\vdash \tau_2_{\tau_2} = \tau_2_{\tau_2}$ (13).
By (12), (13), and the typing rules:

\[ \vdash (\forall \alpha. r_{11} \rightarrow r_{12} \rhd) \alpha = (\forall \alpha. r_{21} \rightarrow r_{22} \rhd) \alpha^2 \]

By definition:

\[ \vdash [(\forall \alpha. r_{11} \rightarrow r_{12}) \alpha] = [(\forall \alpha. r_{21} \rightarrow r_{22}) \alpha] \]

as required.

- (Box) In this case \( r_{11} = \text{box}(r_{11}^{11}), r_{21} = \text{box}(r_{21}^{12}) \), and \( \vdash r_{11} = r_{21} (1) \). WLOG, assume \( C(i_1) \subseteq C(i_2) \) (2). By the rules for acceptability, \( C; \rho \vdash r_{12} (3) \), \( \text{box} \{ C(i_2) \} \subseteq C(i_1) \) (4), \( C; \rho \vdash r_{12} (5) \), and \( \text{box} \{ C(i_2), \text{lbl}(r_{12}) \} \subseteq C(rell(i_1)) \) (6). By (4), (2), (7), and the induction hypothesis, \( \vdash r_{12} = r_{22} \). By Lemmas 17 and 18, \( i_1 \preceq i_2 \). There are two cases:

  - Case 1, \( i_1 \in \Upsilon \): In this case, \( r_{12} = r_{22} \), and the result is (8).
  - Case 2, \( i_2 \notin \Upsilon \): In this case, \( r_{12} = \text{box}(r_{12}^{11}) \), \( r_{22} = \text{box}(r_{22}^{12}) \), and the result follows from (8) and the typing rules.

Now we can prove that unboxing preserves typing.

**Theorem 1 (Consistent unboxings preserve typing)**

**If** \( C \vdash \Upsilon : \tau \) **then**:

- \( \text{If } \Delta; \Gamma \vdash e : \tau, C; \rho \vdash \Gamma, C; \rho \vdash e \), and \( \Upsilon \vdash e \) **then** \( \Delta; \Gamma \vdash \text{val}_e : \tau \) **then** \( \Delta; \Gamma \vdash \text{val}_e : \tau \).
- \( \text{If } \rho \vdash \Gamma, C; \rho \vdash M, \) and \( \Upsilon \vdash \Gamma \) **then** \( \rho \vdash M : \tau \).

**Proof:** The proof is by induction on the structure of the typing judgement. Consider the cases, in the same order as the figure, for the last rule used in the derivation:

- (Variable) In this case \( e = x \) and \( e : \tau \in \Gamma \). Then \( e : \tau \in \Gamma \) and clearly \( e : \tau \in \Gamma \), so the result follows by the typing rules.

- (Fix expression) In this case \( e = (\text{fix } f [\alpha](x; r_{12}; e) : \tau_2) \).

  The typing rule requires that both \( \tau = (\forall \alpha. r_{12} \rightarrow r_{12} \rhd) \) and \( \Delta; \Gamma, f: \tau, x: \tau \vdash e : \tau_2 \). The assumption \( C; \rho \vdash e \) requires \( g(f) = x : \text{lbl}(\tau_2), C; \rho \vdash \tau, \) and \( C; \rho \vdash \tau_2 \). From \( C; \rho \vdash \tau \) and the rules for acceptability, \( C; \rho \vdash \tau \). From these facts, \( C; \rho \vdash \Gamma, f: \tau, x: \tau \), the assumption \( \Upsilon \vdash e \) requires \( \tau \vdash e \). By the induction hypothesis, \( \Delta; \Gamma, f: \tau, x: \tau \vdash e : \tau_2 \).

  Since:

  \[ \vdash (\forall \alpha. r_{12} \rightarrow r_{12} \rhd) \alpha = \vdash (\forall \alpha. r_{22} \rightarrow r_{22} \rhd) \alpha \]

by the typing rules:

\[ \Delta; \Gamma \vdash \text{fix } f [\alpha](x; r_{12} \rhd) : \vdash (\forall \alpha. r_{12} \rightarrow r_{12} \rhd) \]

The result follows since:

\[ e \vdash (\forall \alpha. r_{12} \rhd) \alpha = \vdash \text{fix } f [\alpha](x; r_{12}) : \vdash (\forall \alpha. r_{22} \rhd) \alpha \]

- (Application) In this case, \( e = (e_1 [\tau_2] e_2) \). The typing rule, \( C; \rho \vdash e \), and \( \Upsilon \vdash e \).

By (12), (13), (10), (2), (7), (11), and the induction hypothesis:

\[ \Delta; \Gamma \vdash e_1 : \vdash (\forall \alpha. r_{12} \rightarrow r_{12} \rhd) \alpha \]

\[ \Delta; \Gamma \vdash e_2 : \tau_2 \]

\[ \Delta \vdash \tau \vdash \text{wf} \]

\[ \vdash (\forall \alpha. r_{12} \rightarrow r_{12} \rhd) \alpha = \tau_2 \]

\[ C; \rho \vdash e_1 \]

\[ C; \rho \vdash \tau \]

\[ C; \rho \vdash e_2 \]

\[ \Upsilon \vdash \tau \]

\[ \Upsilon \vdash \tau : \tau \]

Now we can prove that unboxing preserves typing.

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(Unbox) In this case, \( e = \text{unbox}_{e'} \) for some \( e' \) and \( i \). The typing rule requires that \( \Delta; \Gamma \vdash e' : \boxC(\tau') \) for some \( j \). The assumption \( C; \Gamma \vdash e' \) requires \( C; \Gamma \vdash e' \). The assumption \( \Upsilon \vdash e \) requires \( \Upsilon \vdash e \). By the induction hypothesis, \( \Delta; \Gamma \vdash e' : \boxC(\tau') \). By Lemma 16, \( C(j) \subseteq \text{C}(\text{lbl}(e')) \). By Lemmas 17 and 18, \( j \leq \text{lbl}(e) \). There are two subcases:

- If \( j \in \Upsilon \) then \( e|\Upsilon = e'|\Upsilon \) and \( \text{box}(\tau')|\Upsilon = \tau'|\Upsilon \) and the result is immediate.

- If \( j \not\in \Upsilon \) then \( e|\Upsilon = (\text{unbox } e'|\Upsilon) \) and \( \text{box}(\tau')|\Upsilon = \text{box}(\tau')|\Upsilon \). The result then follows by the typing rules.

(Frame) In this case, \( e = \rho'(e')' \) for some \( \rho', e', i \). The typing rule requires that \( \rho' : \Gamma' \) and \( \Delta; \Gamma' \vdash e' : \tau' \) for some \( \tau' \). The assumption \( C; \rho' \vdash e' \) requires that \( C; \rho' \vdash \rho' \) and \( C; \rho' \vdash e' \). The former requires that \( C; \rho' \vdash \Gamma' \). The assumption \( \Upsilon \vdash e' \) requires \( \Upsilon \vdash e' \). By the induction hypothesis, \( \uparrow \rho' : \Gamma' \vdash e': \tau' \). So by the typing rules, \( \Delta; \Gamma \vdash \rho' : (e'|\Upsilon) : \tau'|\Upsilon \). The result follows since \( e|\Upsilon = e'|\Upsilon \).

- (Constant) In this case \( e = c \) for some \( c \) and \( i \). The typing rule requires that \( \tau = \tau' \). Clearly, \( e|\Upsilon = e'|\Upsilon = b' \), and the result follows by the typing rules.

- (Fix Value) In this case \( e = (\rho; \text{fix } \rho(x) : (x:\tau_1) : \tau_2) \). The typing rule requires that \( \tau = (\forall x. \tau_1) \rightarrow \tau_2 \), \( \rho : \Gamma \), and \( \Delta; \alpha; \Gamma' ; \forall x. \tau_1 \vdash \tau_2 : \tau \). The assumption \( C; \Gamma \vdash e \) requires \( C; \Gamma \vdash \rho \) and \( C; \Gamma \vdash e \). From these facts, \( C; \rho \vdash \Gamma' \). The assumption \( \Upsilon \vdash e \) requires \( \Upsilon \vdash e \). By the induction hypothesis, \( \uparrow \rho : \Gamma' \vdash (\forall x. \tau_1) \rightarrow \tau_2 \). The result follows since \( e|\Upsilon = (\rho|\Upsilon) \text{fix } f[\alpha](x|\Upsilon : \tau_1) \vdash \tau_2|\Upsilon \).

- (Box Value) In this case, \( e = (\nu_1 : \tau''_1) \) for some \( v, i, j \). The typing rule requires that \( \tau = \boxC(\tau'') \), \( \Delta \vdash \tau'' \) \( uf \), \( \Delta; \Gamma \vdash v_1 : \tau' \), and \( i \vdash \tau'' = \tau' \). The assumption \( C; \rho \vdash e \) requires that \( C \vdash \tau \) and \( C; \rho \vdash v \). The assumption \( \Upsilon \vdash e \) requires \( \Upsilon \vdash v \). For the induction hypothesis, \( \Delta; \Gamma \vdash v_1 : \tau' \). There are two subcases:

  - If \( i \in \Upsilon \) then \( e|\Upsilon = \nu_1|\Upsilon \) and \( \tau'|\Upsilon = \tau'|\Upsilon \) and the result is immediate.

  - If \( i \not\in \Upsilon \) then \( e|\Upsilon = \nu_1|\Upsilon \) and \( \tau'|\Upsilon = \text{box}(\tau'')|\Upsilon \). The result then follows by the typing rules if \( \Delta \vdash \nu_1: \tau'' \) \( uf \), which holds by Lemma 19, and \( \tau'' \vdash \tau'' \) \( \vdash \tau'' \) \( \vdash \tau'' \). The result follows by the typing rules if \( \Delta \vdash \nu_1: \tau'' \) \( uf \), which holds by Lemma 22 if its other three premises hold. Since \( C \vdash \tau \), by the rules for acceptability, \( \text{box}(\tau''|\Upsilon) \) (1) and \( \Upsilon \vdash \tau'' \), showing the first premise. Since \( \Delta; \Gamma \vdash v_1 : \tau' \), by Lemma 16, \( \text{C}(\text{lbl}(\tau'')) \subseteq \text{C}(\tau') \) (2) and \( \Upsilon \vdash \tau'' \), showing the second premise. By the induction hypothesis, \( C; \rho \vdash e \). By the rules for acceptability, \( \Box C(\text{fix}(\tau'')) \) \( \Upsilon \subseteq \text{C}(\tau'') \) (3). By (1), \( C(j) \subseteq \text{C}(\text{lbl}(\tau'')) \), so by (2), \( \text{C}(\text{lbl}(\tau'')) \subseteq \text{C}(\text{lbl}(\tau'')) \), showing the third premise, as required.

- (Environment) In this case \( \rho = \tau_1 : v_1, \ldots, \tau_n : v_n \) and \( \Gamma = \tau_1 : v_1, \ldots, \tau_n : v_n \). The typing rule requires that \( \Upsilon \vdash v_j : \tau_j \) for \( 1 \leq j \leq n \) and some \( \tau_j \). The assumption \( C; \rho \vdash \rho \) requires \( C; \Upsilon \vdash \tau_j \) and \( C; \rho \vdash v_j \) for \( 1 \leq j \leq n \). Clearly, \( C; \Upsilon \vdash \Upsilon \) for \( \Gamma \) empty. The assumption \( \Upsilon \vdash \rho \) requires \( \Upsilon \vdash v_j \) for \( 1 \leq j \leq n \). By the induction hypothesis, \( \Upsilon \vdash v_j : \tau_j \) for \( 1 \leq j \leq n \). By the rules for acceptability, \( C(\tau_j) \subseteq \text{C}(\text{lbl}(\tau_j)) \) for \( 1 \leq j \leq n \). By Lemma 16, \( C(\tau_j) \subseteq C(\tau_j) \) and \( \Upsilon \vdash \tau_j \) for \( 1 \leq j \leq n \). Thus \( C(\tau_j) \subseteq C(\tau_j) \) for \( 1 \leq j \leq n \). By Lemma 22, \( \Upsilon \vdash \tau_j \) for \( 1 \leq j \leq n \). Then by the typing rules \( \Upsilon \vdash \tau_1 : v_1, \ldots, \tau_n : v_n \). The result follows since \( \text{C}(\tau_j) \subseteq \text{C}(\tau_j) \) and \( \Upsilon \vdash \tau_j \), showing the first premise.

The result follows since \( |\rho| \vdash e|\Upsilon = e|\Upsilon \).

A consequence of type preservation is that unboxed well-typed programs are traceable.

**Theorem 2**

\[ \forall \rho \vdash M : \tau, C \vdash \Upsilon, \text{ and } C; \rho \vdash M \text{ then } \Upsilon \vdash |M| \vdash \Upsilon \text{ tr}. \]

**Proof:** The proof follows from Theorem 1 and Lemma 11.

### 4.4 Coherence

The other part of proving correctness is to show that unboxing preserves semantics in some appropriate sense. That requires two key lemmas—that a step of the program can be matched by zero or more steps of the unboxed program and that consistency is preserved under reduction.

To show the first lemma, we need three technical lemmas—that a value’s cache is nonempty, that reduction preserves the unboxing or not of the outermost label, and a multistep compositionality property.

**Lemma 23 (Inhabitation)**

\[ \text{If } C; \rho \vdash v \text{ then } \exists s \in C(k) \text{ such that } \text{lbl}(s) = k. \]

**Proof:** By inspection of the acceptable analysis and acceptable instantiation rules.

**Lemma 24 (Unboxing set preservation)**

\[ \text{If } C; \rho \vdash e, C \vdash \Upsilon, \text{ and } (\rho; e) \rightarrow (\rho; e) \text{ then } \text{lbl}(e) \subseteq \text{lbl}(e'). \]

**Proof:** All of the cases for which \( \text{lbl}(e_1) = \text{lbl}(e_2) \) follow immediately. For the remaining cases:

- If \( (\rho; e) \rightarrow (\rho; e') \) then by the assumptions we have that \( g(x) = \text{lbl}(\tau) \) (1), \( C(j) \subseteq \text{C}(\text{lbl}(\tau)) \) (2) and \( C(g(x)) \subseteq \text{C}(k) \) (3), so by transitivity we have \( C(j) \subseteq \text{C}(k) \) (4). By Inhabitability (Lemma 23) we have an \( s \in C(k) \) (5) such that \( \text{lbl}(s) = j \) (6), and so by Agreement (Lemma 18) we have \( k \leq j \). Since \( \text{lbl}(e) = k \) and \( \text{lbl}(e') = j \), the result follows.
• If \((\rho, (\text{unbox} (e', t'))^k) \rightarrow (\rho, v')\) then we must show that 
k \leq i. By Inhabitance we have \(s \in C(i)\) with \(\text{lbl}(s) = i\), so by Agreement, it suffices to show that \(C(i) \subseteq C(k)\). By the box rule for an acceptable analysis, there is a \(s = (\text{box}, l)^j \in C(j)\) such that \(C(i) \subseteq C(l)\). Since \(s \in C(j)\), by the rule for unbox, \(C(l) \subseteq C(k)\), so \(C(i) \subseteq C(k)\) and we’re done.

• If \((\rho, \rho'(e')^j) \rightarrow (\rho, v')\) then we must show that \(j \geq i\). By Inhabitance, there is an \(u \in C(i)\), and by the acceptable analysis rule for frames we have that \(C(i) \subseteq C(j)\), so by Agreement we have that \(j \geq i\).

Lemma 25 (Many step compositionality)
If \((\rho, e_1) \rightarrow* (\rho, e_3)\) then:

- \(\rho, (e_1 e)^j) \rightarrow* (\rho, (e_2 e)^j)\)
- \(\rho, (v e')^j) \rightarrow* (\rho, (v e')^j)\)
- \(\rho, (\text{box}, e)^j) \rightarrow* (\rho, (\text{box}, e)^j)\)
- \(\rho, (\text{unbox}, e_1)^j) \rightarrow* (\rho, (\text{unbox}, e_2)^j)\)
- \(\rho, \rho'(e_1')^j) \rightarrow* (\rho, \rho'(e_2')^j)\)

Proof: The proof is by an easy induction on the length of the reduction sequences.

Theorem 3 (Single step reduction coherence)
If \(\vdash M : t, C; \emptyset \vdash M, C \vdash \top, \top \vdash M, C \vdash \top, \top \vdash M'\) and \(M \rightarrow M'\) then \(\vdash M \rightarrow* M'\).

Proof: The proof is by induction on the derivation of \(M \rightarrow M'\), consider the cases for the last rule used to derive it:

- If \((\rho, v) \rightarrow (\rho, v')\) where \(x : \tau \rightarrow v' \in \rho\) then by definition \(\vdash M_T = (\rho_T, x, k)\), \(M'_T = (\rho_T, x, v')\), and \(x : \tau \rightarrow v' \in \rho_T\). Thus \(\vdash M_T \rightarrow* M'_T\) by the same rule.

- If: \((\rho, (\text{fix} f(x) : (x : \tau_1) : t_2, e_1)^j) \rightarrow (\rho, (\text{fix} f(x) : (x : \tau_2) : t_2, e_1)^j)\)
then the unboxes of the \(e\) and \(e'\) are of the same form, and the same reduction step applies.

- If \((\rho, (\text{box}, e'')^j) \rightarrow (\rho, (e' : \tau')^j)\) where \(tr(\tau') = t\) then:
  - If \(j \notin \top\) then:
    By the definition of unboxing, \(\vdash (\text{box}, \tau' : x, \tau') \rightarrow (\rho, (e' : \tau')^j)\) where \(\rho''_\tau = \rho''_{\tau'}\).
    By hypothesis, \(\vdash \top : \Gamma, \vdash v' : \tau' \rightarrow \tau'' \in \rho'\) for some \(\tau''\). By hypothesis, \(\vdash \tau' \rightarrow \tau''\). By Theorem 1, \(\vdash \Gamma, \vdash v' : \tau' \rightarrow \tau''\). The proof of that theorem also showed that \(\vdash (\tau''_\tau) = (\tau''_{\tau'}), \top \vdash (\text{box}, \tau' : x, \tau') \rightarrow (\rho, (e' : \tau')^j)\).
    By Theorem 6, \(\vdash (\text{box}, \tau' : x, \tau') \rightarrow (\rho, (e' : \tau')^j)\).
    By definition of reduction \((\vdash (\rho, (\text{box}, \tau' : x, \tau'))^j) \rightarrow (\vdash (\rho, (\text{box}, \tau' : x, \tau'))^j)\).

By definition of unboxing, \(\vdash (\text{unbox} (e', t'))^k \rightarrow (\rho, v')\) where \(\rho''_\tau = \rho''_{\tau'}\).

- If \(j \in \top\) then:
  By definition of unboxing \(\vdash (\text{unbox} (e', t'))^k \rightarrow (\rho, v')\) where \(\rho''_\tau = \rho''_{\tau'}\).
  By definition of reduction \(\vdash (\rho, (\text{unbox} (e', t'))^k) \rightarrow* (\rho, (\text{unbox} (e', t'))^k)\).

- If \((\rho, (e_1 : \tau')^j \rightarrow (\rho, (e_2 : \tau')^j)) \rightarrow (\rho, (e_1 : \tau')^j)\) then:
  By definition of unboxing \(\vdash (\text{unbox} (e', t'))^k \rightarrow (\rho, v')\) where \(\rho''_\tau = \rho''_{\tau'}\).
  By definition of reduction \(\vdash (\rho, (\text{unbox} (e', t'))^k) \rightarrow* (\rho, (\text{unbox} (e', t'))^k)\).

By definition of C; \(\vdash M, C \vdash \top, \top \vdash M\) we have that \(C; \emptyset \vdash \top, \top \vdash e_1, \top \vdash e_1\). Hence we have that \(C; \emptyset \vdash (\rho, (e_1 : \tau')^j)\). By the typing rules we also have that \(\vdash \rho : \Gamma, \emptyset, \Gamma, \vdash e_1 : \tau_1\) for some \(\Gamma, \rho, \tau_1\). So \(\vdash (\rho, (e_1 : \tau')^j) \vdash \tau_1\). Hence by induction we have that \((\vdash (\rho, (e_1 : \tau')^j) \rightarrow* (\rho, (e_2 : \tau')^j)\).

By Lemma 25
\((\vdash (\rho, (e_1 : \tau')^j) \rightarrow* (\rho, (e_2 : \tau')^j)\)

By definition of unboxing
\((\vdash (\rho, (e_1 : \tau')^j) \rightarrow* (\rho, (e_2 : \tau')^j)\)
If $j \notin \tau$ then:

By definition of unboxing:

\[
\begin{align*}
(\rho^-)(\mathbf{e}'_{\Upsilon})_{\Upsilon} & = (\mathbf{e}_{\Upsilon})_{\Upsilon}^\rho \quad \text{(by definition of unboxing)}
\end{align*}
\]

Hence by the induction hypothesis and Lemma 25 we have that:

\[
\begin{align*}
(\rho^-)(\mathbf{e}'_{\Upsilon})_{\Upsilon} & \implies (\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon}
\end{align*}
\]

If $(\rho, (\text{unbox } \mathbf{e}')) \implies (\rho, (\text{unbox } \mathbf{e}'))$ then let $i = \text{lb}l(e)$ and $i' = \text{lb}l(e')$. By Lemma 24 we have that $i \leq i'$. By the rules for acceptability, $\Upsilon; \; \rho \vdash \rho$ and $\Upsilon; \; e \vdash e$, so $\Upsilon; \; (\rho, (e)).$

By the typing rules, $\Upsilon; \; \rho \vdash \rho$, $\Upsilon; \; \epsilon \vdash \epsilon$, so $\Upsilon; \; (\rho, (e)).$

By the inducction hypothesis, $\Upsilon; \; \rho$ and $\Upsilon; \; \epsilon \vdash \rho$ and $\Upsilon; \; \epsilon \vdash \epsilon$, so $\Upsilon; \; (\rho, (e)).$ So by the induction hypothesis, $(\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon} \implies (\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon}$.

If $i, i' \notin \Upsilon$ then:

By definition of unboxing

\[
\begin{align*}
(\text{unbox } \mathbf{e})_{\Upsilon} & = \mathbf{e}_{\Upsilon}
\end{align*}
\]

By definition of unboxing

\[
\begin{align*}
(\text{unbox } \mathbf{e'})_{\Upsilon} & = \mathbf{e'}_{\Upsilon}
\end{align*}
\]

By induction

\[
\begin{align*}
(\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon} & \implies (\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon}
\end{align*}
\]

If $(\rho, (\text{unbox } \mathbf{e})) \implies (\rho, (\text{unbox } \mathbf{e}))$ then:

By definition of unboxing

\[
\begin{align*}
(\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon} & = (\mathbf{e}_{\Upsilon})_{\Upsilon}^\rho \quad \text{(by definition of unboxing)}
\end{align*}
\]

By induction

\[
\begin{align*}
(\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon} & \implies (\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon}
\end{align*}
\]

By the typing rules, $\Upsilon; \; \rho \vdash \rho$, $\Upsilon; \; \epsilon \vdash \epsilon$, so $\Upsilon; \; (\rho, (e)).$ By the inducction hypothesis, $\Upsilon; \; \rho$ and $\Upsilon; \; \epsilon \vdash \rho$ and $\Upsilon; \; \epsilon \vdash \epsilon$, so $\Upsilon; \; (\rho, (e)).$ So by the induction hypothesis, $(\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon} \implies (\rho^-)(\mathbf{e}_{\Upsilon})_{\Upsilon}$.

To show preservation of consistency we need a type substitution lemma.

**Lemma 26**

If $\Upsilon \vdash \tau : \mathbf{r}, \tau$ is not a type variable, and $\Upsilon; \; \rho \vdash \tau[\alpha]: \mathbf{r}$ then:

- $\Upsilon; \; \rho \vdash \tau[\alpha]: \mathbf{r}$ and $\Upsilon; \; \epsilon \vdash \epsilon[\alpha]: \mathbf{r}$.
- $\Upsilon; \; \rho \vdash \epsilon[\alpha]: \mathbf{r}$.
- $\Upsilon; \; \rho \vdash \epsilon[\alpha]: \mathbf{r}$.

**Proof:**

The proof is by simultaneous induction on the derivation of $\Upsilon \vdash \tau : \mathbf{r}$, $\Upsilon \vdash \epsilon$, and $\Upsilon \vdash \rho$. The cases for expressions and environments are straightforward. Consider the cases for types:

- Case 1. $\tau = \alpha^i$. If $\tau = \alpha^j$ then $\tau[\alpha]: \mathbf{r} = \alpha^j$. Since $\Upsilon; \; \rho \vdash \alpha^i, \Upsilon; \; \epsilon \vdash \alpha^i$, so $\Upsilon; \; (\rho, \epsilon)$. By Lemmas 17 and 18, $i \leq j$. Then by inspection of the rules, $\Upsilon \vdash \alpha^i : \mathbf{r}$ as $\Upsilon \vdash \alpha^j : \mathbf{r}$.
- Case 2. $\tau = \beta^i$ and $\alpha \neq \beta$. Then $\tau[\alpha]: \mathbf{r} = \tau[\alpha]$ and the result is immediate.
- Case 3. $\tau = \mathbf{r}^i$. Then $\Upsilon \vdash \tau[\alpha]: \mathbf{r}$ is not possible.
- Case 4. $\tau = \langle \alpha \bracket{\tau_1 \tau_2}^i \rangle$. Then $\Upsilon \vdash \tau[\alpha]: \mathbf{r}$, as required.
- Case 5. $\tau = \langle \alpha \mathbf{r}^i \rangle$. $\Upsilon \vdash \tau[\alpha]: \mathbf{r}$. Assumption $\Upsilon; \; \rho \vdash \tau[\alpha]$ requires $\Upsilon; \; \epsilon \vdash \tau[\alpha]$.

By the induction hypothesis, $\Upsilon \vdash \tau[\alpha]$. By the consistency rules, $\Upsilon \vdash \langle \alpha \mathbf{r}^i \rangle[\alpha]: \mathbf{r}$. By the induction hypothesis, $\Upsilon \vdash \langle \alpha \mathbf{r}^i \rangle[\alpha]: \mathbf{r}$.
- Case 6. $\tau = \langle \alpha \mathbf{r}^i \rangle[\alpha]$. $\Upsilon \vdash \tau[\alpha]: \mathbf{r}$, as required.
...the same leaf reductions. Two of them take expression forms to value forms, but otherwise leave the term unchanged. One of them takes unbox of box of a value to that value. One of them takes a frame of a value to that value. Thus if we measure a term by adding its size, number of lambda expressions, and number of box expressions, then this metric strictly decreases for these three leaf reductions. Therefore, in any infinite reduction sequence, there must be an infinite number of steps whose leaf reduction is a variable reduction or an application beta reduction. Then observe in the proof of Theorem 3 that the unboxing of a variable redex or of an application redex will always take a step, and that Lemma 25 preserves this. Thus the unboxing will also take an infinite number of steps.

Theorem 4 shows that if two terms are related by reduction, then their images under the unboxing function are also related by the many step reduction relation given that the unboxing pair is acceptable; and that if a term diverges under reduction, then its image under the unboxing function also diverges. In other words, for an acceptable analysis and an acceptable unboxing, the induced unboxing function preserves the semantics of the original program up to elimination of boxes. Since the semantics of the core language only defines reduction steps that preserve GC safety, this theorem implies that the image of a GC safe program under unboxing is also GC safe.

5. Construction of an acceptable unboxing

The previous section gives a declarative specification for when an unboxing set \( T \) is correct but does not specify how such a set might be chosen. In this section we give a simple algorithm for constructing an acceptable unboxing given an arbitrary acceptable flow analysis.

The idea behind the algorithm is that given a program and an acceptable flow analysis for it, we use the results of the flow analysis to construct the connected components of the interprocedural flow graph of the program. All of the elements of a connected component will then either be unboxed together, or not unboxed at all. Any such choice of unboxing (as we will show) satisfies the cache coherence property. The only remaining requirement is that the choice of unboxing set be consistent, which is easily satisfied by ensuring that any connected component which includes a type expression has traceability r. In the rest of the section, we make this informal algorithm concrete and show that the choice of unboxing that it produces is in fact acceptable.

For the purposes of this section we ignore environments and the intermediate forms \( \rho(e) \), \( \langle \rho, \text{fix } f[\alpha](x:\tau_1):\tau_2,e \rangle \) and \( \langle \mu, t \rangle \). These constructs are present in the language solely as mechanisms to discuss the dynamic semantics—in this sense they can be thought of as intermediate terms, rather than source terms. It is straightforward to incorporate these into the algorithm if desired.
Given a flow analysis \((C, g)\) and program \(e\) such that \(C; p \vdash e\), we define the induced undirected flow graph \(FG\) as an undirected graph with a node for every label in \(C\), and edges as follows:

- For every label \(i\) and every shape \(s \in C(i)\), we add an edge between \(i\) and \(\text{lbl}(s)\).

The edges simply connect up each program point with all of its shapes.

Given a flow graph \(FG\), we can find the connected components in the usual way. Let \(CC\) be a mapping which maps labels to the connected component in which they occur. Note that by definition each label occurs in exactly one connected component. It is easy to show that any connected component is cache consistent.

**Lemma 28 (Cache consistency of a connected component)**

Given any acceptable analysis \((C, g)\) with induced flow graph \(FG\), and any connected component \(S\) of \(FG\), \(S\) is cache consistent: that is, \(C \vdash S\).

**Proof:** To show that \(C \vdash S\) we must show that \(\forall i, s : s \in C(i) \implies i \not\in \text{lbl}(s)\). But note that by the construction of the induced flow graph \(FG\), whenever \(s \in C(i)\) there is an edge between \(i\) and \(\text{lbl}(s)\), and consequently by definition of a connected component, \(i\) and \(\text{lbl}(s)\) must be in the same connected component. Since every label occurs in exactly one connected component, either both \(i\) and \(\text{lbl}(s)\) are in \(S\) or both are not in \(S\). By definition then, \(i \not\in \text{lbl}(s)\).

It is also easy to show that the union of any two disjoint cache consistent sets is also cache consistent.

**Lemma 29 (Cache consistency (unary) closure)**

Given any acceptable analysis \((C, g)\) and disjoint label sets \(S_1\) and \(S_2\), then if \(C \vdash S_1\) and \(C \vdash S_2\) then \(C \vdash S_1 \cup S_2\).

**Proof:** To show that \(C \vdash S_1 \cup S_2\) we must show that \(\forall i, s : s \in C(i) \implies i \not\in \text{lbl}(s)\). Consider an arbitrary label \(i\). If \(i\) is not in \(S_1 \cup S_2\), then we have that \(i\) is not in \(S_1\) and not in \(S_2\), and hence by assumption, \(\text{lbl}(s)\) is not in \(S_1\) and not in \(S_2\), and hence we have agreement. If \(i\) is in \(S_1 \cup S_2\), then it must be in either \(S_1\) or \(S_2\). WLOG assume that \(i \in S_1\). By assumption, \(i \not\in \text{lbl}(s)\), and \(\text{lbl}(s)\) is in \(S_1\), and hence \(\text{lbl}(s)\) is in \(S_1 \cup S_2\) and we have agreement.

Consequently, we can show that any set consisting of a union of connected components of the induced flow graph is cache consistent.

**Lemma 30 (Cache consistency closure)**

Given any acceptable analysis \((C, g)\) with induced flow graph \(FG\), and any set \(SS\) of connected components of \(FG\), \(\bigcup SS\) is cache consistent.

**Proof:** By Lemma 28, each connected component is cache consistent. By definition, any two connected components are disjoint, and so by Lemma 29 the union of any two connected components are cache consistent, and are disjoint from any other connected component. The cache consistency of \(\bigcup SS\) follows directly by induction.

### 5.1 The algorithm

Given the set of connected components for the induced flow graph, the algorithm begins with an initial unboxing set \(\Upsilon\) consisting of the union of all of the connected components. By Lemma 30, we have that \(C \vdash \Upsilon\). The algorithm then proceeds by considering in turn each application sub-term \(e_1[\tau]e_2\) as follows:

- For each sub-term of \(e\) of the form \(e_1[\tau]e_2\):
  - if \(\text{lbl}(\tau) \not\in \Upsilon\), and if \(\Upsilon \vdash \tau : b\), then:
    - \(\Upsilon \leftarrow \Upsilon - \text{CC}(\text{lbl}(\tau))\).

That is, for any application for which the current unboxing results in the type argument being unboxed to a non-reference type, we remove the connected component for the type from the unboxing set. Note that after removing a connected component from \(\Upsilon\), the new unboxing set \(\Upsilon\) is still cache consistent since it is still a union of connected components (just a union of one less connected component).

With the help of some technical lemmas, it is straightforward to show that the final unboxing set \(\Upsilon\) computed by the algorithm is an acceptable unboxing for the program.

To begin with, we observe that if a type’s label is not in the unboxing set \(\Upsilon\), then it is consistent and its traceability is unchanged by the unboxing.

**Lemma 31 (Type consistency)**

For any unboxing set \(\Upsilon\) and type \(\tau\), if \(\text{lbl}(\tau) \not\in \Upsilon\) then \(\Upsilon \vdash \tau : \text{tr}(\tau)\).

**Proof:** By inspection.

- (Variable) \(\text{tr}(\alpha^i) = r\), and \(\Upsilon \vdash \alpha^i : r\).
- (Base type) \(\text{tr}(b^i) = b\), and \(\Upsilon \vdash b^i : b\).
- (Fun type) \(\text{tr}(\forall \alpha.\tau_1 \to \tau_2^i) = r\), and \(\Upsilon \vdash \forall \alpha.\tau_1 \to \tau_2^i : r\).
- (Box type) \(\text{tr}(\text{box}(\tau^i)) = r\), and by assumption \(i \not\in \Upsilon\), so we have that \(\Upsilon \vdash \text{box}(\tau^i) : r\).

It is also the case that the consistent type judgement defines a total function on types, and hence for any type we either have that it is consistent at traceability \(r\) or that it is consistent at traceability \(b\).

**Lemma 32 (Type consistency is a total function)**

For any unboxing set \(\Upsilon\) and type \(\tau\), either \(\Upsilon \vdash \tau : b\), or \(\Upsilon \vdash \tau : r\).

**Proof:** By induction on types. All of the cases follow immediately except when \(\tau = \text{box}(\tau^i)\) and \(i \in \Upsilon\). In that case, by induction we have that both \(\Upsilon \vdash \tau^i : b\), or \(\Upsilon \vdash \tau^i : r\), and so by construction either \(\Upsilon \vdash \tau : b\), or \(\Upsilon \vdash \tau : r\).

**Theorem 5**

If \(\Delta; \Upsilon \vdash e : \tau; C; g \vdash \Gamma; C; g \vdash e\) and if \(\Upsilon\) is the unboxing set computed by the algorithm in this section, then \(\Upsilon\) is an acceptable unboxing for \(e\). That is, \(C \vdash \Upsilon\) and \(\Upsilon \vdash e\).

**Proof:** The conclusion that \(C \vdash \Upsilon\) follows almost immediately from Lemma 30. The initial choice of \(\Upsilon\) is a union of connected components, and hence is cache consistent. At every step of the algorithm, we may remove a single connected component from \(\Upsilon\). The result is still a union of connected components (since connected components are disjoint), and hence the result of removing a connected component is still cache consistent by Lemma 30.

The conclusion that \(\Upsilon \vdash e\) follows by induction on the structure of the typing derivation.

- (Variable) In this case, \(e = x^i\), consistency is immediate.
- (Fix) In this case \(e = (\text{fix} f[\alpha](x;\tau_1);\tau_2,e^i))\). To get consistency, we must show that \(\Upsilon \vdash e^i\). The last rule applied in the typing judgement must have been the fix rule, and by its premises we have that \(\Delta \vdash \forall \alpha.\tau_1 \to \tau_2^i : \text{wf}(1)\), and that
\( \Delta, \alpha; \Gamma, f : \forall \alpha. \tau_1 \rightarrow \tau_2 \delta, x : \tau_1 \vdash e' : \tau_2 \) (2). The last rule applied in the acceptable analysis judgement must also have been the fix rule, and by its premises we have that \( C; \varrho \vdash e' \) (3). To apply the induction hypothesis, we need (1), (3), and that \( C; \varrho \vdash \Gamma : f : \forall \alpha. \tau_1 \rightarrow \tau_2 \delta, x : \tau_1 \) (4). To show (4), it is sufficient to show that:

- \( \varrho(f) = i \) which is a premise of the acceptable analysis derivation
- \( C; \varrho \vdash \forall \alpha. \tau_1 \rightarrow \tau_2 \delta \) which is a premise of the acceptable analysis derivation
- \( \varrho(x) = \text{lb}(\tau_1) \) which is a premise of the acceptable analysis derivation
- \( C; \varrho \vdash \tau_1 \) which is a sub-premise of the derivation of \( C; \varrho \vdash \forall \alpha. \tau_1 \rightarrow \tau_2 \delta \).

So by (1), (3), and (4), we have by induction that \( \Upsilon \vdash e' \).

- **(Application)** In this case \( e = (e_1 \tau|e_2) \). To prove consistency, we need that \( \Upsilon \vdash e_1 \) (1), \( \Upsilon \vdash e_2 \) (2), and \( \Upsilon \vdash \tau : \tau \) (3). Inverting the typing derivation and the acceptable analysis derivation immediately gives us the premises we need to apply the induction hypothesis to get (1) and (2). To prove (3), note that a premise of the typing derivation gives us that \( tr(\tau) = \tau \) (4). If \( \text{lb}(\tau) \notin \Upsilon \), then by Lemma 31 we have that \( \Upsilon \vdash \tau : \tau \) (5) and so by (4) we’re done. If \( \text{lb}(\tau) \in \Upsilon \), then by the definition of the algorithm, we must have that \( \Upsilon \vdash \tau : \beta \) does not hold (since otherwise the algorithm would have removed the connected component containing \( \text{lb}(\tau) \) from \( \Upsilon \)), and so by Lemma 32 we must have that \( \Upsilon \vdash \tau : \tau \) and we’re done.

- **(Box)** All of the premises need to apply the induction hypothesis are available immediately by inverting the typing derivation and the acceptable analysis derivation.

- **(Unbox)** All of the premises need to apply the induction hypothesis are available immediately by inverting the typing derivation and the acceptable analysis derivation.

- **(Constant)** Follows immediately.

Thus we have shown by construction that the specification defined in Section 4 is a useful one in the sense that it is satisfiable.

### 6. Open Terms

The paper so far has considered whole-program optimization and proved that unboxing in that setting is correct. We would like to be able to optimize program fragments where we have part of the program but know nothing about the rest of the program. Such a setting adds one more correctness criteria—since we are not optimizing the rest of the program, anything that flows across the boundary to or from the rest of the program must remain as boxed as it originally was. We can ensure this requirement by simply requiring that nothing on the boundary is in the unboxing set. This section formalizes these ideas and proves them correct.

For our purposes, a program fragment is a module, which is a triple \( \Gamma : e : \tau \). \( \Gamma \) specifies the imports of the module, \( e \) specifies the body of the module, which exports only one thing—the value that \( e \) evaluates to, and \( \tau \) specifies the type of the export. We wish to optimize modules without making any assumptions about the code that the module is linked to. In particular that means we cannot unbox any of the imports nor unbox anything exported. This requirement can be achieved by not unboxing any subterm of any type in \( \Gamma \) nor in \( \tau \).

We can extend the definitions of well typedness, acceptability of flow analysis, unboxing, and consistency of unboxing to modules,
Lemma 33
If \( M_1 \sim M_2 \) and \( M_1 \mapsto M_3 \) then there exists \( M_4 \) such that \( M_3 \sim M_4 \) and \( M_2 \mapsto M_4 \).

Proof: The proof is by a straightforward induction on the derivation of \( M_1 \mapsto M_3 \).

Next we prove three lemmas about unboxing preservation. In the first two we show that something’s unboxing is that something because either the not unboxed judgement (the first lemma) or the labels in the context are not in the unboxing set (the second lemma). In the third we show the unboxing of an expression is the same if the unboxing set is the same on the labels in the expression. To state and prove these and subsequent lemmas we need a function to return all the labels in an expressions, type, or environment. It is defined in Figure 13.

Lemma 34

- If \( \Gamma \vdash \tau \) not unboxed then \( |\tau|_\Gamma = \tau \).
- If \( \Gamma \vdash \Gamma \) not unboxed then \( |\Gamma|_\Gamma = \Gamma \).

Proof:

- The proof is by induction on the structure of \( \tau \). Consider the cases:
  - Case 1, \( \tau = \alpha_i \): Then by definition \( |\tau|_\Gamma = \tau \), as required.
  - Case 2, \( \tau = \text{B}^i \): Then by definition \( |\tau|_\Gamma = \tau \), as required.
  - Case 3, \( \tau = (\forall \alpha \tau_1 \rightarrow \tau_2) \): Then \( \Gamma \vdash \tau \) not unboxed requires \( \tau_1 \) not unboxed and \( \tau_2 \) not unboxed. By the induction hypothesis, \( |\tau_1|_\Gamma = \tau_1 \) and \( |\tau_2|_\Gamma = \tau_2 \). By definition, \( |\tau|_\Gamma = \tau \), as required.
  - Case 4, \( \tau \) = box\((\tau')\): Then \( \Gamma \vdash \tau \) not unboxed requires \( i \notin \Gamma \) and \( \Gamma \vdash \tau \) not unboxed. By the induction hypothesis, \( |\tau'|_\Gamma = \tau' \). By definition, \( |\tau|_\Gamma = \tau \), as required.

- If \( \Gamma = x_1 : \tau_1 , \ldots , x_n : \tau_n \) then: \( \Gamma \vdash \Gamma \) not unboxed requires \( \tau_j \) not unboxed for \( 1 \leq j \leq n \). So by the first item, \( |\tau_j|_\Gamma = \tau_j \) for \( 1 \leq j \leq n \). Then by definition, \( |\Gamma|_\Gamma = \Gamma \), as required.

Lemma 35

- If \( |\text{lbs}(\rho) \cap \Gamma| = \emptyset \) then \( |\rho|_\Gamma = \rho \).
- If \( |\text{lbs}(E) \cap \Gamma| = \emptyset \) then \( |E|_\Gamma = E \).

Proof: The proof is a straightforward induction on the structure of \( \rho \) and \( E \).

Lemma 36

If \( \Upsilon_1 \cap \text{lbs}(e) = \Upsilon_2 \cap \text{lbs}(e) \) then \( |e|_{\Upsilon_1} = |e|_{\Upsilon_2} \).

Proof: The proof is a straightforward induction on the structure of \( e \).

Next we state and prove our main technical lemma. This lemma states that we can rewrite the context and flow analysis to have certain desirable properties, namely that the flow analysis covers the context and the module, that the context is not unboxed, that the module is unboxed as before, and the unboxing set and flow analysis remain consistent and consistent with the module and context.

Lemma 37

If:

\[
\emptyset \vdash \Gamma \text{ unif} \\
\emptyset, \Gamma \vdash e : \tau \\
C; g \vdash \Gamma \\
C; g \vdash e \\
C; g \vdash \tau \\
C \vdash \Upsilon \\
\Upsilon \vdash \Gamma \text{ not unboxed} \\
\Upsilon \vdash e \\
\Upsilon \vdash \tau \text{ not unboxed} \\
\vdash \rho : \Upsilon \\
\Gamma \vdash E : \text{B}^i(\tau)
\]

then there exists \( \rho', E', C', \rho' \), and \( \Upsilon' \) such that:

\[
\rho \sim \rho' \\
E \sim E' \\
\vdash \rho' : \Upsilon \\
\Gamma \vdash E' : \text{B}^i(\tau) \\
C' ; \rho' \vdash \rho', E'(e) \\
C' \vdash \Upsilon' \\
\Upsilon' \vdash \rho', E'(e) \\
\text{lbs}(\rho') \cap \Upsilon' = \emptyset \\
\text{lbs}(E') \cap \Upsilon' = \emptyset \\
\Upsilon \cap \text{lbs}(e) = \Upsilon' \cap \text{lbs}(e)
\]

Proof: Let \( V \) be the set of variables that occur in \( e \). Let \( A \) be the set of type variables that occur in \( \Gamma, e, \) or \( \Upsilon \). Both these sets are finite.

The derivation of \( C; g \vdash \Gamma, C; g \vdash e, \) and \( C; g \vdash \tau \) will for each type that is not a type variable require a particular type shape with some label on it in the cache of the label of that type, similarly for each box expression and box value require a box shape with some label of its contents in the cache. Let \( L \) be one such label for each such type and such box as well as \( g(V) \cup g(A) \cup \text{lbs}(\Gamma) \cup \text{lbs}(e) \cup \text{lbs}(\tau) \). Note that \( L \) is a finite set.

Let \( g' \) be \( g \) on \( V \) and \( A \) and on every other variable or type variable let it map to a fresh label (distinct from each other and
$$E ::= \emptyset \mid (E[\tau] \ e) \mid (\text{unbox } E)$$

\[
\Gamma \vdash : \tau(\tau)
\]

\[
\Gamma \vdash (E[\tau] \ e)^i : \tau_2[\tau/\alpha]^i
\]

\[
\Gamma \vdash : \text{box}(\tau')^i(\tau)
\]

\[
\Gamma \vdash : \text{unbox } E)^i : \tau'(\tau)
\]

\[
M_1 \equiv M_2 \quad \text{def} \quad \forall c : (M_1 \longrightarrow \ast c' \equiv M_2 \longrightarrow \ast c')) \wedge (M_1 \longrightarrow \cdots \equiv M_2 \longrightarrow 
\]

\[
\Gamma \vdash : \emptyset \mid \Gamma \vdash : \tau \wedge \emptyset \mid \Gamma \vdash : \tau \wedge
\]

\[
\forall \rho, \Gamma \vdash : \emptyset \mid \Gamma \vdash : \emptyset \wedge \emptyset \mid \Gamma \vdash : \emptyset \wedge \emptyset \mid \Gamma \vdash : \emptyset \wedge \emptyset
\]

\[
\Gamma \vdash : \emptyset \mid \Gamma \vdash : \tau \wedge \tau \wedge (\Gamma \vdash : \emptyset \mid \Gamma \vdash : \tau \wedge \tau)
\]

\[
\vdash (\Gamma_1 \rightarrow e_1 : \tau_1) \equiv (\Gamma_2 \rightarrow e_2 : \tau_2)
\]

From $L$. Define:

$$C''(i) = \begin{cases} \{s \mid s \in C(i) \wedge \text{lbs}(s) \subseteq L\} & i \in L \\ \emptyset & \text{else} \end{cases}$$

Claim: $C''; \rho' \vdash \Gamma, C''; \rho' \vdash e$, and $C''; \rho' \vdash \tau$. The proof is by induction on the derivation, consider the last rule used:

- (Variable) In this case $e = x^i$ and $C(\rho(x)) \subseteq C(\rho(x))$. Since $x \in V$, $\rho(x) = \rho(x)$ and $\rho(x) \in L$. Also $i \in L$. Therefore, $C''(\rho(x)) \subseteq C''(i)$, as required. Thus by the same rule, $C''; \rho' \vdash e$, as required.

- (Fix expression) In this case $e = \text{fix } f[\alpha](x : \tau_1) : \tau_2(\rho(x), \text{lbs}(\alpha))$. Since $\rho(x) = \rho(x)$ and $\rho(x) \in L$. Also $\alpha \in L$. Therefore, $C''(\rho(x)) \subseteq C''(i)$, as required. Thus by the same rule, $C''; \rho' \vdash e$, as required.

- (Application) In this case $e = (e_1[\tau] \ e_2)^i$, $C; \rho \vdash e_1, C; \rho \vdash \tau$, $C; \rho \vdash e_2$, and $\text{fun } (\text{lbs}(e_1), \text{lbs}(\tau), \text{lbs}(e_2), \text{lbs}(i))$. By the induction hypothesis, $C''; \rho' \vdash e_1, C''; \rho' \vdash \tau$, and $C''; \rho' \vdash e_2$. Since $\text{lbs}(e_1) \in L$, $\text{lbs}(\tau) \in L$, $\text{lbs}(e_2) \in L$, and $i \in L$, it is easy to see that $\text{fun } (\text{lbs}(e_1), \text{lbs}(\tau), \text{lbs}(e_2), \text{lbs}(i))$ (for $C''$). Thus by the same rule, $C''; \rho' \vdash e$, as required.

- Other cases are similar . . .

Let $A'$ be the set of type variables that appear in $\Gamma$ and $\tau$. We construct $\rho'$ and $E'$ as alpha variants and relabellings of $\rho$ and $E$ as follows. Since $\rho : \Gamma, \rho$ contains $\Gamma$, so we keep that part the same. Type variables that are in $A'$ we keep the same. All other type variables and variables we pick an alpha variant that is fresh (distinct from each other and from $A$ respectively $V$). The outermost label on types on variables we relabel to the binding label for that variable. All other labels we relabel to be fresh. Clearly $\rho \sim_{\kappa} \rho'$ and $E \sim_{\kappa} E'$.

Claim: $\rho' : \Gamma$ and $\Gamma \vdash : \emptyset$ for some $j$. The proof is a straightforward induction on the structure of $\rho'$ and $E'$. 

Figure 12. Contextual Equivalence

Figure 13. The labels in an type, expression, or environment
Now we need to build a $C$ such that $C'; g' \vdash (\rho', E'(e))$. We start from $C''$. First we add into the caches, shapes required directly for the rules for $C'; g' \vdash \rho'$ and $C'; g' \vdash E'$ (such things are already there for $e$). In the case of types we add shapes using the label of the type as the label of the shape. In the case of box expressions and values we use the label of the contents of the box as the label of the contents of the shape. What remains is a bunch of subset and equality constraints between cache entries, so we pick $C'$ to be the smallest largest cache that satisfies these constraints. Clearly such a $C'$ exists and by construction, $C'; g' \vdash (\rho', E'(e))$.

Set $\mathcal{Y} = \mathcal{Y} \cap L$. Clearly, $\mathcal{Y} \cap \text{lbs}(e) = \mathcal{Y} \cap \text{lbs}(e)$ as $\text{lbs}(e) \subseteq \mathcal{L}$. By construction, the labels of $\rho'$ and $E'$ are in the labels of $\mathcal{Y}$ or $\tau$ are not in $\mathcal{L}$. Since $\mathcal{Y} \cap \mathcal{Y} \not= \text{unboxed}$ and $\mathcal{Y} \cap \tau \not= \text{unboxed}$, the labels of $\mathcal{Y}$ and $\tau$ are not in $\mathcal{L}$. Therefore, $\text{lbs}(\mathcal{Y}) \cap \mathcal{Y}' = \emptyset$ and $\text{lbs}(\tau) \cap \mathcal{Y}' = \emptyset$. In fact, if $A'$ is a set of type variables in $\rho'$ and $E'$ then $g'(A') \cap \mathcal{Y}' = \emptyset$ too.

Claim: any flow from the interface to a $\text{boxC}(i, j)$ condition has a box type at the interface (**), and similarly for $\text{funC}(j, j, l, k)$. The proof is by induction on the flow conditions noting that in all cases the two end points have the same type.

Case: $\mathcal{Y} \cap \mathcal{Y}'$. Let $i$ and $j$ be such that $s \in C(i)$. If $s \in C'(i)$ then $s \in C(i)$, $i \in L$, and $\text{lbs}(s) \subseteq \mathcal{L}$, and in particular, $\text{lbs}(s) \subseteq L$. Since $\mathcal{Y} \cap \mathcal{Y}$, $i \not\in \text{lbs}(s)$. Since $\mathcal{Y} = \mathcal{Y} \cap L$, $i \in L$, and $\text{lbs}(s) \subseteq L$, $i \not\in \text{lbs}(s)$, as required. Otherwise, we claim that $i, \text{lbs}(s) \not\subseteq \mathcal{Y}$. Let $L_C = \text{lbs}(\rho') \cup \text{lbs}(E') \cup g'(A')$, $L_i = \text{lbs}(\mathcal{Y}) \cup \text{lbs}(\tau)$, $L_M = L - L_C$, and $L_A = \text{lbs}(\mathcal{X}) \cup \mathcal{L}$. First notice that $C''$ has entries only for labels in $L$ and with shapes whose labels are in $L$. The first part of computing $C''$ added shapes to cache entries for labels in $L_C$ with shapes whose labels are in $L_C$. The second part of computing $C'$ only propagates existing shapes from one cache entry to another, and only from/to cache entries in $L_A$ or in labels in shapes in the cache entries. Thus, the cache entries of $C'$ are only for $L_A$ with shapes with labels in $L_M$. If $\text{lbs}(s) \subseteq L_M$ then by previous argument $\text{lbs}(s) \not\subseteq \mathcal{Y}$, as required. If $\text{lbs}(s) \subseteq L_M$ then we will show that $s \in C(i)$ for some $j \in L_I$. That $s \not\in C(i)$, and since $\mathcal{Y} \cap \mathcal{Y}$ and $j \not\in \mathcal{Y}$, $\text{lbs}(s) \not\subseteq \mathcal{Y}$ so $\text{lbs}(s) \not\subseteq \mathcal{Y}$, as required. If $i, j \not\in \text{lbs}(s)$ then previous argument $i \not\in \mathcal{Y}$, as required. If $i \in L_M$ then we will show that $\mathcal{Y}'(j) \subseteq C''(i)$ for some $j \in L_I$. Then $\mathcal{Y}'(j) = \mathcal{Y}'(i)$, and since that required type shape has labels in $L$, $\emptyset \not= \mathcal{Y}'(j) \subseteq \mathcal{C}(i)$. Then since $\mathcal{Y} \cap \mathcal{Y}$ and $j \not\in \mathcal{Y}$, $i \not\in \mathcal{Y}$, so $\mathcal{Y}'(i) = \mathcal{Y}'(j)$, as required. It remains to show the two conditions we claimed. Since $C''$ was computed using a least fixed point, we prove these claims by induction on when $s$ was added to $C(i)$. Consider the cases:

- Case 1, $s$ was added to $i$ because $s \in C'(j)$, $s \not\in C'(i)$, and $C'(j) \subseteq C(i)$ is required by the rules for variables, box expressions, frames, box values, or environments. In this case, $i$ and $j$ have to come from the same term, that is, either $i, j \in L_C$ or $i, j \in L$. If $\text{lbs}(s) \subseteq L_M$ then $s$ must have been added to $C(j)$ previously in the second phase of constructing $C'$, so by the induction hypothesis, $\text{lbs}(s) \subseteq C''(k)$ for some $k \in L_I$. If $i \in L_M$ then $j \in L$. First notice that the condition on $j$ and $i$ is also required to show that $C''; g' \vdash \mathcal{Y}$, $C''; g' \vdash \mathcal{Y}$, or $C''; g' \vdash \mathcal{Y}$, so $C''(j) \subseteq C'(i)$. If $j \in L_I$ then we have what we need. Otherwise $j \in L_M$, so $s$ was added to $C(j)$ previously in the second phase of constructing $C'(j)$, so by the induction hypothesis, $C''(k) \subseteq C''(j)$ for some $k \in L_I$. Then $C''(k) \subseteq C''(i)$, as required.

- Case 2, $s$ was added to $i$ because $C(g'(\alpha)) = C(i)$, required by the rule for type variables, did not hold and $s \in C(g'(\alpha))$. Similar to Case 1.

- Case 3, $s$ was added to $i$ because $\text{boxC}(j, i)$ is required, either $\text{boxC}(j, i) \not= C'(j)$ or $\text{boxC}(j, i) \not= C'(j)$, and $s$ was already in $C(i')$. In this case, $i$ and $j$ have to come from the same term, that is, either $i, j \in L_C$ or $i, j \in L$. Now that the labels in any shape under consideration come from the same term, that is, either they are all in $L$ or they are all in $L_C$.

- If $\text{lbs}(s) \subseteq L_M$ then:
  - If $s$ was added to $C'(i')$ previously in the second phase of constructing $C'$ then by the induction hypothesis, $s \not\in C''(k)$ for some $k \in L_I$.
  - Otherwise $s \not= C''(i')$ and $i', j' \in L$. Since $s$ was not already in $C(i')$ then the box shape was added to $C''(i')$ previously in the second phase of constructing $C'$, so by the induction hypothesis, the box shape is in $C''(k)$ for some $k \in L_I$. By (**), $k$ labels a box type. By the rules for acceptability, $\text{boxC}(k, k')$ for $C''(k)$ for some $k' \in L_I$. Thus, $C''(i') \subseteq C''(k')$. Thus $s \not= C''(i')$, as required.

- If $i \in L_M$ then $j \in L$ and $\text{boxC}(j, i)$ holds for $C''$.

  - If the box shape was added to $C''(j)$ previously in the second phase of the constructing $C'$ then by the induction hypothesis, $C''(j) \subseteq C''(k)$ for some $k \in L_I$. By (**), $k$ labels a box type. Then by the rules for acceptability, $\text{boxC}(i') \not\in C''(k)$ for some $i' \in L_I$, so $\text{boxC}(i') \not\in C''(j)$. By $\text{boxC}(j, i)$, $C''(i') \subseteq C''(i)$, as required.

- Otherwise, the box shape was in $C''(j)$ and $i', j' \in L$. By $\text{boxC}(j, i)$, $C''(i') \subseteq C''(i)$. Since $s$ was not already in $C(i')$ then it was previously added to $C''(i')$ in the second phase of constructing $C'$, so by the induction hypothesis, $C''(k) \subseteq C''(i')$ for some $k \in L_I$. Then by transitivity $C''(k) \subseteq C''(i)$, as required.

- Case 4, $s$ was added because a $\text{funC}(j, j, j, j, j)$ is required. Similar to Case 3.

Claim: $\mathcal{Y} \vdash (\rho', E'(e))$. The proof is by a straight forward induction on the structure of $\rho'$, $E'(e)$. The only interesting case is application. In that case, we have $(e_1 | \tau) e_2$. By the induction hypothesis we get $\mathcal{Y} \vdash e_1$ and $\mathcal{Y} \vdash e_2$. We just need to show that $\mathcal{Y} \vdash \tau \vdash \tau$. If the application came from $e$ then since the labels of $\tau$ are in $L$, the result follows from $\mathcal{Y} \vdash \tau$ which holds by assumption $(\mathcal{Y} \vdash e)$. Otherwise, the labels of $\tau$ are not in $\mathcal{Y}$ so clearly $\mathcal{Y} \vdash \tau \vdash \tau$. Then $\mathcal{Y} \vdash \tau$ holds by the typing rules.

We these definitions we can prove that unboxing for modules is correct.
Since the operational semantics is deterministic, we just need to show that $\rho(E[e])$ matches $(\rho, E[e])$ in behaviour. There are two cases:

- If $(\rho, E[e]) \mapsto^* (\rho', c')$ then by Lemma 33, $(\rho', E'[e]) \mapsto^* (\rho', c')$. Hence by Theorem 4, $(\rho, E'[e]) \mapsto^* (\rho', c')$. Therefore by Lemma 33 again, $(\rho, E'[e]) \mapsto^* (\rho', c')$, for some $j'$. It is not too hard to see that $j = j'$, as required.

- If $(\rho, E[e]) \mapsto \cdots$ then by Lemma 33, $(\rho', E'[e]) \mapsto \cdots$. By Theorem 4, $(\rho', E'[e]) \mapsto \cdots$. By Lemma 35, $(\rho', E'[e]) \mapsto \cdots$. By Lemma 36, $(\rho, E'[e]) \mapsto \cdots$, as required.

### 7. Related work

This paper provides a modular approach to showing correctness of a realistic compiler optimization that rewrites the structure of program data structures in significant ways. Our approach uses an arbitrary inter-procedural reaching definitions analysis to eliminate unnecessary heap allocation in an intermediate representation in which object representation has been made explicit. Our optimization can be staged freely with other optimizations. Unlike any previous work that we are aware of, we account for correctness with respect to the meta-data requirements of the garbage collector. For presentational purposes, we have restricted our attention to the core concern of GC safety, but additional issues such as value size, dynamic type tests, etc. are straightforward to incorporate.

There has been substantial previous work addressing the problem of unboxing. Peyton Jones [3] introduced an explicit distinction between boxed and unboxed objects to provide a linguistic account of unboxing, and hence to allow a high-level compiler to locally eliminate unboxes of syntactically apparent box introduction operations. Leroy [4] defined a type-driven approach to adding coercions into and out of specialized representations. The type driven translation represented monomorphic objects natively (unboxed, in our terminology), and then introduced wrappers to coerce polymorphic uses into an appropriate form. To a first-order approximation, instead of boxing at definition sites this approach boxes objects at polymorphic use sites. This style of approach has the problem that it is not necessarily beneficial, since allocation is introduced in places where it would not otherwise be present. This is reflected in the slowdowns observed on some benchmarks described in the original paper. This approach also has the potential to introduce space leaks. In a later paper [5] Leroy argued that a simple untyped approach gives better and more predictable results.

Henglein and Jørgensen [2] defined a formal notion of optimality for local unboxes and gave two different choices of coercion placements that satisfy their notion of optimality. Their definition of optimality explicitly does not correspond in any way to reduced allocation or reduced instruction count and does not seem to provide uniform improvement over Leroy’s approach. The MLton compiler [11] largely avoids the issue of a uniform object representation by completely monomorphizing programs before compilation. This approach requires whole-program compilation. More limited monomorphization schemes could be considered in an incremental compilation setting. Monomorphization does not eliminate the need for boxing in the presence of dynamic type tests or reflection. Just in time compilers (e.g. for .NET) may monomorphize dynamically at runtime.

The TIL compiler [1, 10] uses intentional type analysis in a whole-program compiler to allow native data representations without committing to whole-program compilation. As with the Leroy coercion approach, polymorphic uses of objects require conditionals and boxing coercions to be inserted at use sites, and consequently there is the potential to slow down, rather than speed up, the program.

Serrano and Feeley [9] described a flow analysis for performing unboxing substantially similar in spirit to our approach. Their algorithm attempts to find a monomorphic typing for a program in which object representations have not been made explicit, which they then use selectively to choose whether to use a uniform or non-uniform representation for each particular object. Their approach differs in that they define a dedicated analysis rather than using a generic reaching definitions analysis. They assume a conservative garbage collector and hence do not need to account for the requirements of GC safety, and they do not prove a correctness result.

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