Ranks and presentations of some normally ordered inverse semigroups

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Accepted: 12 October 2021 / Published online: 18 March 2022
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Abstract
In this paper we compute the rank and exhibit a presentation for the monoids of all \( P \)-stable and \( P \)-order preserving partial permutations on a finite set \( \Omega \), with \( P \) an ordered uniform partition of \( \Omega \). These (inverse) semigroups constitute a natural class of generators of the pseudovariety of inverse semigroups \( NO \) of all normally ordered (finite) inverse semigroups.

Keywords Transformations · Normally ordered inverse semigroups · Ranks · Presentations

Mathematics Subject Classification 20M20 · 20M05 · 20M07

1 Introduction and preliminaries

Let \( \Omega \) be a set. We denote by \( PT(\Omega) \) the monoid (under composition) of all partial transformations on \( \Omega \), by \( T(\Omega) \) the submonoid of \( PT(\Omega) \) of all full transformations on \( \Omega \), by \( I(\Omega) \) the symmetric inverse semigroup on \( \Omega \), i.e., the inverse submonoid of \( PT(\Omega) \) of all partial
permutations on \( \Omega \), and by \( S(\Omega) \) the symmetric group on \( \Omega \), i.e., the subgroup of \( PT(\Omega) \) of all permutations on \( \Omega \). If \( \Omega \) is a finite set with \( n \) elements (\( n \in \mathbb{N} \)), say \( \Omega = \Omega_n = \{1, \ldots, n\} \), we denote \( PT(\Omega), T(\Omega), I(\Omega) \) and \( S(\Omega) \) simply by \( PT_n, T_n, I_n \) and \( S_n \), respectively.

Now, consider a linear order \( \preceq \) on \( \Omega_n \), e.g., the usual order. We say that a transformation \( \alpha \in PT_n \) is order preserving if \( x \preceq y \) implies \( x\alpha \preceq y\alpha \), for all \( x, y \in \text{Dom}(\alpha) \). Denote by \( PO_n \) the submonoid of \( PT_n \) of all order preserving partial transformations, by \( O_n \) the submonoid of \( T_n \) of all order preserving full transformations of \( \Omega_n \) and by \( POI_n \) the inverse submonoid of \( I_n \) of all order preserving partial permutations of \( \Omega_n \).

A pseudovariety of [inverse] semigroups is a class of finite [inverse] semigroups closed under homomorphic images of [inverse] subsemigroups and finitary direct products.

In the “Szeged International Semigroup Colloquium” (1987) J.-E. Pin asked for an effective description of the pseudovariety (i.e., an algorithm to decide whether or not a finite semigroup belongs to the pseudovariety) of semigroups \( O \) generated by the semigroups \( O_n \), with \( n \in \mathbb{N} \). Although, as far as we know, this question is still open, some progress has been made. First, Higgins [25] proved that \( O \) is self-dual and does not contain all \( R \)-trivial semigroups (and so \( O \) is properly contained in \( A \), the pseudovariety of all finite aperiodic semigroups), although every finite band belongs to \( O \). Next, Vernitskii and Volkov [32] generalized Higgins’s result by showing that every finite semigroup whose idempotents form an ideal is in \( O \) and, in [12], Fernandes proved that the pseudovariety of semigroups \( POI \) generated by the semigroups \( POI_n \), with \( n \in \mathbb{N} \), is a (proper) subpseudovariety of \( O \). On the other hand, Almeida and Volkov [4] showed that the interval \([O, A]\) of the lattice of all pseudovarieties of semigroups has the cardinality of the continuum. Also, Repnitski˘ı and Volkov [30] showed that \( O \) is not finitely based. Another contribution to the resolution of Pin’s problem was given by Fernandes [16] who showed that \( O \) contains all semidirect products of a chain (considered as a semilattice) by a semigroup of injective order preserving partial transformations on a finite chain. This result was later generalized by Fernandes and Volkov [19] for semidirect products of a chain by any semigroup from \( O \). Still related to Pin’s problem, see Auinger’s paper [6].

The inverse counterpart of Pin’s problem can be formulated by asking for an effective description of the pseudovariety of inverse semigroups \( PCS \) generated by \( \{POI_n \mid n \in \mathbb{N} \} \). In [9] Cowan and Reilly proved that \( PCS \) is properly contained in \( A \cap Inv(Inv \) being the class of all inverse semigroups) and also that the interval \([PCS, A \cap Inv]\) of the lattice of all pseudovarieties of inverse semigroups has the cardinality of the continuum. From Cowan and Reilly’s results it can be deduced that a finite inverse semigroup with \( n \) elements belongs to \( PCS \) if and only if it can be embedded into the semigroup \( POI_n \). This is in fact an effective description of \( PCS \). On the other hand, in [13] Fernandes introduced the class \( NO \) of all normally ordered inverse semigroups. This notion is deeply related with the Munn representation of an inverse semigroup \( M \), an idempotent-separating homomorphism that may be defined by

\[
\phi : M \to T(E)
\]

\[
s \iff \phi_s : E s s^{-1} \to E s^{-1}s,
\]

with \( E \) the semilattice of all idempotents of \( M \). A finite inverse semigroup \( M \) is said to be normally ordered if there exists a linear order \( \preceq \) in the semilattice \( E \) of the idempotents of \( M \) preserved by all partial permutations \( \phi_s \) (i.e., for \( e, f \in E s s^{-1}, e \preceq f \) implies \( e \phi_s \preceq f \phi_s \)), with \( s \in M \). It was proved in [13] that \( NO \) is a pseudovariety of inverse semigroups and also that the class of all fundamental normally ordered inverse semigroups consists of all aperiodic normally ordered inverse semigroups. Moreover, Fernandes showed that \( PCS = NO \cap A \),
giving in this way a Cowan and Reilly alternative (effective) description of \( PCS \). In fact, this also led Fernandes [13] to the following refinement of Cowan and Reilly’s description of \( PCS \): a finite inverse semigroup with \( n \) idempotents belongs to \( PCS \) if and only if it can be embedded into \( \mathcal{POI}_n \). Another refinement (in fact, the best possible) was also given by Fernandes [18], by considering only join irreducible idempotents. Notice that, in [13] it was also proved that \( NO = PCS \vee G \) (the join of \( PCS \) and \( G \), the pseudovariety of all groups).

Now, let \( \Omega \) be a finite set.

An ordered partition of \( \Omega \) is a partition of \( \Omega \) endowed with a linear order. By convention, whenever we take an ordered partition \( P = \{X_i\}_{i=1,...,k} \) of \( \Omega \), we will assume that \( P \) is the chain \( \{X_1 < X_2 < \cdots < X_k\} \). We say that \( P = \{X_i\}_{i=1,...,k} \) is a uniform partition of \( \Omega \) if \( |X_i| = |X_j| \) for all \( i, j \in \{1, \ldots, k\} \).

Let \( P = \{X_i\}_{i=1,...,k} \) be an ordered partition of \( \Omega \) and, for each \( x \in \Omega \), denote by \( i_x \) the integer \( i \in \{1, \ldots, k\} \) such that \( x \in X_i \). Let \( \alpha \) be a partial transformation on \( \Omega \). We say that \( \alpha \) is \( P \)-stable if \( X_{i_x} \subseteq \text{Dom}(\alpha) \) and \( X_{i_x} \alpha = X_{i_{\alpha x}} \), for all \( x \in \text{Dom}(\alpha) \); and \( P \)-order preserving if \( i_x \leq i_y \) implies \( i_{\alpha x} \leq i_{\alpha y} \), for all \( x, y \in \text{Dom}(\alpha) \), where \( \leq \) denotes the usual order on \( \{1, \ldots, k\} \).

Denote by \( \mathcal{POI}_{\Omega, P} \) the set of all \( P \)-stable and \( P \)-order preserving partial permutations on \( \Omega \).

Notice that the identity mapping belongs to \( \mathcal{POI}_{\Omega, P} \) and it is easy to check that \( \mathcal{POI}_{\Omega, P} \) is an inverse submonoid of \( I(\Omega) \). Observe also that if \( P \) is the trivial partition of \( \Omega_n \), i.e., \( P = \{\{i\}\}_{i=1,...,n} \), then \( \mathcal{POI}_{\Omega_n, P} \) coincides with \( \mathcal{POI}_n \) and, on the other hand, if \( P = \{\Omega_n\} \) (the universal partition of \( \Omega_n \)) then \( \mathcal{POI}_{\Omega_n, P} \) is exactly the symmetric group \( S_n \), for \( n \in \mathbb{N} \).

These monoids, considered for the first time by Fernandes in [13], were inspired by the work of Almeida and Higgins [3], although they are quite different from the ones considered by these two last authors. The main relevance of the monoids \( \mathcal{POI}_{\Omega, P} \) lies in the fact that they constitute a family of generators of the pseudovariety \( NO \) of normally ordered inverse semigroups. More precisely, Fernandes proved in [13, Theorem 4.4] that \( NO \) is the class of all inverse subsemigroups (up to an isomorphism) of semigroups of the form \( \mathcal{POI}_{\Omega, P} \). In fact, by the proof of [13, Theorem 4.4], it is clear that it suffices to consider semigroups of the form \( \mathcal{POI}_{\Omega, P} \), with \( P \) a uniform partition of \( \Omega \), i.e., we also have the following result:

**Theorem 1.1** The class \( NO \) is the pseudovariety of inverse semigroups generated by all semigroups of the form \( \mathcal{POI}_{\Omega, P} \), where \( \Omega \) is a finite set and \( P \) is an ordered uniform partition of \( \Omega \).

Let \( n \in \mathbb{N} \). An ordered partition of \( n \) is a nonempty sequence of positive integers whose elements sum to \( n \).

Let \( \pi = (n_1, \ldots, n_k) \) be an ordered partition of \( n \). Define the ordered partition \( P_\pi \) of \( \Omega_n \) as the partition into intervals \( P_\pi = \{I_i\}_{i=1,...,k} \) of \( \Omega_n \) (endowed with the usual order), where

\[
I_1 = \{1, \ldots, n_1\} \quad \text{and} \quad I_i = \{n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i\}, \quad \text{for } 2 \leq i \leq k.
\]

Notice that \( \pi = (|I_1|, \ldots, |I_k|) \).

Next, we establish that \( \Omega_n \) and its partitions into intervals allow us to construct, up to an isomorphism, all monoids of type \( \mathcal{POI}_{\Omega, P} \), with \( \Omega \) a set with \( n \) elements and \( P \) an ordered partition of \( \Omega \).

**Theorem 1.2** Let \( \Omega \) be a set with \( n \) elements and let \( P = \{X_i\}_{i=1,...,k} \) be an ordered partition of \( \Omega \). If \( \pi \) is the ordered partition \((|X_1|, \ldots, |X_k|)\) of \( n \), then the monoids \( \mathcal{POI}_{\Omega, P} \) and \( \mathcal{POI}_{\Omega_n, P_\pi} \) are isomorphic.
Now, let $k, m \in \mathbb{N}$ be such that $n = km$. Let $\pi = (m, \ldots, m) \in \Omega_{n}^{k}$. Denote the uniform partition into intervals $P_{\pi}$ of $\Omega_{n}$ by $P_{k \times m}$ (i.e., we have $P_{k \times m} = \{ I_{i} \}_{i=1,\ldots,k}$, with $I_{i} = \{(i-1)m+1, \ldots, im\}$, for $i \in \{1, \ldots, k\}$) and denote the monoid $\mathcal{POI}_{n \times P_{k \times m}}$ by $\mathcal{POI}_{k \times m}$. Therefore, combining Theorems 1.1 and 1.2, we immediately obtain the following result:

**Corollary 1.3** The pseudovariety of inverse semigroups $NO$ is generated by the class $\{ \mathcal{POI}_{k \times m} \mid k, m \in \mathbb{N} \}$.

This fact gave us the main motivation for the work presented in this paper, which is about the monoids $\mathcal{POI}_{k \times m}$, with $k, m \in \mathbb{N}$. We notice that, for $k, m \in \mathbb{N}$, the monoid $\mathcal{POI}_{k \times m}$ is a partial wreath product of $S_{m}$ and $\mathcal{POI}_{k}$ in the sense of Brookes’ paper [7].

The rest of this paper is organized as follows. In Section 2 we calculate their sizes and ranks and in Section 3 we construct presentations for them.

For general background on Semigroup Theory and standard notation, we refer the reader to Howie’s book [27]. For general background on pseudovarieties and finite semigroups, we refer the reader to Almeida’s book [2]. All semigroups considered in this paper are finite.

## 2 Size and rank of $\mathcal{POI}_{k \times m}$

Let $M$ be a monoid. Recall that the quasi-order $\leq_{\mathcal{J}}$ is defined on $M$ as follows: for all $u, v \in M$, $u \leq_{\mathcal{J}} v$ if and only if $MuM \subseteq MvM$. As usual, the $\mathcal{J}$-class of an element $u \in M$ is denoted by $J_{u}$ and a partial order relation $\leq_{\mathcal{J}}$ is defined on the set $M/\mathcal{J}$ by $J_{u} \leq_{\mathcal{J}} J_{v}$ if and only if $u \leq_{\mathcal{J}} v$. Given $u, v \in M$, we write $u <_{\mathcal{J}} v$ or $J_{u} <_{\mathcal{J}} J_{v}$ if and only if $u <_{\mathcal{J}} v$ and $(u, v) \notin \mathcal{J}$.

Recall also that the rank of a (finite) monoid $M$ is the minimum size of a generating set of $M$.

Many manuscripts have been dedicated to the computation of the ranks of certain classes of (transformation) semigroups or monoids. For instance, see [5, 20, 22–24].

Let $P$ be an ordered partition of $\Omega$. Let $\alpha, \beta \in \mathcal{POI}_{\Omega, P}$. Since $\mathcal{POI}_{\Omega, P}$ is an inverse submonoid of $\mathcal{I}(\Omega)$, we immediately have that $\alpha \not\mathcal{R} \beta$ if and only if $\text{Dom}(\alpha) = \text{Dom}(\beta)$ and that $\alpha \not\mathcal{L} \beta$ if and only if $\text{Im}(\alpha) = \text{Im}(\beta)$. If $P$ is uniform, it is easy to check also that $\alpha \not\mathcal{J} \beta$ if and only if $| \text{Im}(\alpha) | = | \text{Im}(\beta) |$ (see [14, Proposition 5.2.2]). In fact, more specifically, we have that $J_{\alpha} \leq_{\mathcal{J}} J_{\beta}$ if and only if $| \text{Im}(\alpha) | \leq | \text{Im}(\beta) |$.

Notice that $\mathcal{POI}_{n \times 1}$ is isomorphic to $\mathcal{POI}_{n}$, whose size is $\binom{2n}{n}$ and rank is $n$ (see [12, Proposition 2.2] and [15, Proposition 2.8]), and $\mathcal{POI}_{1 \times n}$ is isomorphic to $S_{n}^{0}$ (i.e., to $S_{n}$ with a zero adjoined). Observe that the size of $S_{n}$ is well known to be $n!$ and its rank is well known to be $2$, for $n \geq 3$, and $1$, for $n \in \{1, 2\}$.

From now on let $k, m \in \mathbb{N}$ be such that $k, m \geq 2$ and let $n = km$.

Now, we turn our attention to the $\mathcal{J}$-classes of $\mathcal{POI}_{k \times m}$. Let $\alpha \in \mathcal{POI}_{k \times m}$. Then $| \text{Im}(\alpha) | = im$, for some $0 \leq i \leq k$. Hence

$$
\mathcal{POI}_{k \times m}/\mathcal{J} = \{ J_{0} <_{\mathcal{J}} J_{1} <_{\mathcal{J}} \cdots <_{\mathcal{J}} J_{k} \},
$$

where $J_{i} = \{ \alpha \in \mathcal{POI}_{k \times m} \mid | \text{Im}(\alpha) | = im \}$, for $0 \leq i \leq k$.

Let $t \in \{1, \ldots, k\}$. We write

$$
\alpha^{\top} = \left( \begin{array}{c|c|c|c} I_{i_{1}} & I_{i_{2}} & \cdots & I_{i_{t}} \\ \hline I_{j_{1}} & I_{j_{2}} & \cdots & I_{j_{t}} \end{array} \right)
$$
for all transformations \( \alpha \in \mathcal{POI}_{k \times m} \) such that \( \text{Dom}(\alpha) = I_{i_1} \cup I_{i_2} \cup \cdots \cup I_{i_t} \), \( \text{Im}(\alpha) = I_{j_1} \cup I_{j_2} \cup \cdots \cup I_{j_t} \), and \( I_{i_r} \alpha = I_{j_r} \), for \( 1 \leq r \leq t \), assuming that \( 1 \leq i_1 < i_2 < \cdots < i_t \leq k \) and \( 1 \leq j_1 < j_2 < \cdots < j_t \leq k \).

Observe that \( \mathcal{POI}_k \) is a homomorphic image of \( \mathcal{POI}_{k \times m} \) via the map \( \nu : \mathcal{POI}_{k \times m} \to \mathcal{POI}_k \) defined by \( 0 \nu = 0 \) and
\[
\alpha \nu = \begin{pmatrix} i_1 & i_2 & \cdots & i_t \\ j_1 & j_2 & \cdots & j_t \end{pmatrix}
\]
for all \( \alpha \in \mathcal{POI}_{k \times m} \) such that \( \alpha^\vee = \begin{pmatrix} I_{i_1} & I_{i_2} & \cdots & I_{i_t} \\ I_{j_1} & I_{j_2} & \cdots & I_{j_t} \end{pmatrix} \), with \( 1 \leq t \leq k, 1 \leq i_1 < i_2 < \cdots < i_t \leq k \) and \( 1 \leq j_1 < j_2 < \cdots < j_t \leq k \).

Clearly, given \( 1 \leq t \leq k, 1 \leq i_1 < i_2 < \cdots < i_t \leq k \) and \( 1 \leq j_1 < j_2 < \cdots < j_t \leq k \), the set of all transformations \( \alpha \in \mathcal{POI}_{k \times m} \) such that \( \alpha^\vee = \begin{pmatrix} I_{i_1} & I_{i_2} & \cdots & I_{i_t} \\ I_{j_1} & I_{j_2} & \cdots & I_{j_t} \end{pmatrix} \) forms an \( \mathcal{H} \)-class of \( \mathcal{POI}_{k \times m} \) contained in \( J_t \). In particular, it is easy to check that the \( \mathcal{H} \)-class of a transformation \( \alpha \in \mathcal{POI}_{k \times m} \) such that
\[
\alpha^\vee = \begin{pmatrix} I_{i_1} & I_{i_2} & \cdots & I_{i_t} \\ I_{j_1} & I_{j_2} & \cdots & I_{j_t} \end{pmatrix}
\]
constitutes a group isomorphic to \( S'_t \) and so it has \((m!)^t\) elements.

On the other hand, since there are \( \binom{k}{t} \) distinct possibilities for domains (and images) of the transformations of \( J_t \), we deduce that \( |J_t| = \binom{k}{t}^2(m!)^t \).

Thus, we have

**Proposition 2.1** For \( k, m \geq 1 \), the monoid \( \mathcal{POI}_{k \times m} \) has \( \sum_{t=0}^{k} \binom{k}{t}^2 (m!)^t \) elements.

Next, let \( \psi : \mathcal{POI}_k \to \mathcal{POI}_n \) be the mapping defined by
\[
\text{Dom}(\theta \psi) = \cup \{ I_i \mid i \in \text{Dom}(\theta) \} \quad \text{and} \quad \text{Im}(\theta \psi) = \cup \{ I_i \mid i \in \text{Im}(\theta) \},
\]
for all \( \theta \in \mathcal{POI}_k \). Notice that if
\[
\theta = \begin{pmatrix} i_1 & i_2 & \cdots & i_t \\ j_1 & j_2 & \cdots & j_t \end{pmatrix} \in \mathcal{POI}_k
\]
with \( 1 \leq t \leq k, 1 \leq i_1 < i_2 < \cdots < i_t \leq k \) and \( 1 \leq j_1 < j_2 < \cdots < j_t \leq k \), then
\[
\theta \psi^\vee = \begin{pmatrix} I_{i_1} & I_{i_2} & \cdots & I_{i_t} \\ I_{j_1} & I_{j_2} & \cdots & I_{j_t} \end{pmatrix}.
\]

Moreover, it is a routine matter to show that \( \text{Im}(\psi) = \mathcal{POI}_n \cap \mathcal{POI}_{k \times m} \) and \( \psi \) is an injective homomorphism of monoids.

Let
\[
x_0 = \begin{pmatrix} 2 \cdots k - 1 & k \\ 1 \cdots k - 2 & k - 1 \end{pmatrix}, \quad x_i = \begin{pmatrix} 1 \cdots k - i - 1 & k - i & k - i + 2 \cdots k \\ 1 \cdots k - i - 1 & k - i + 1 & k - i + 2 \cdots k \end{pmatrix},
\]
\[
1 \leq i \leq k - 1,
\]
and take \( \tilde{x}_i = x_i \psi \), for \( 0 \leq i \leq k - 1 \). Observe that \( \tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{k-1} \) are order preserving and \( P \)-order preserving transformations such that
\[
\tilde{x}_0^\vee = \begin{pmatrix} I_2 & \cdots & I_{k-1} & I_k \\ I_1 & \cdots & I_{k-2} & I_{k-1} \end{pmatrix}, \quad \tilde{x}_i^\vee = \begin{pmatrix} I_1 & \cdots & I_{k-i-1} & I_{k-i} \cdots I_k \\ I_1 & \cdots & I_{k-i-1} & I_{k-i+1} \cdots I_k \end{pmatrix}, \quad 1 \leq i \leq k - 1.
\]
Since $\mathcal{POI}_k$ is generated by $\{x_0, x_1, \ldots, x_{k-1}\}$ (see [15]) and $\psi$ is a homomorphism, $\tilde{X} = \{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{k-1}\}$ is a generating set for $\mathrm{Im}(\psi) = \mathcal{POI}_n \cap \mathcal{POI}_{k \times m}$.

Next, recall that it is well known that $S_m$ is generated by the permutations $a = (1 \ 2)$ and $b = (1 \ 2 \ \cdots \ m)$. Take $c = ab = (1 \ 3 \ 4 \ \cdots \ m)$. Thus, since $a = cb^{m-1}$, it is clear that $S_m$ is also generated by the permutations $b$ and $c$. Let
\[
a_i = (1, \ldots, 1, a, 1, \ldots, 1), \quad b_i = (1, \ldots, 1, b, 1, \ldots, 1) \quad \text{and} \quad c_i = (1, \ldots, 1, c, 1, \ldots, 1),
\]
where $a, b$ and $c$ are in the position $i$, for $1 \leq i \leq k$ and $1$ denotes the identity of $S_m$. Clearly,
\[
\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\} \quad \text{and} \quad \{b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_k\}
\]
are generating sets of the direct product $S^k_m$.

Let $d_i = b_ic_{i+1}$, for $1 \leq i \leq k-1$, and $d_k = b_kc_1$. For $1 \leq i \leq k$, we have $b_i^{m} = c_i^{m-1} = 1$, whence $b_i^{(m-1)^2} = b_i$ and $c_i^m = c_i$. Moreover, since $b_ic_j = c_jb_i$, for $1 \leq i, j \leq k$ and $i \neq j$, it is easy to check that $c_1 = d_k^m, c_{i+1} = d_i^m$, for $1 \leq i \leq k-1$, and $b_i = d_i^{(m-1)^2}$, for $1 \leq i \leq k$. Therefore
\[
\{d_1, d_2, \ldots, d_k\}
\]
is also a generating set of $S^k_m$. Observe that as $m, k \geq 2$, the rank of $S^k_m$ is $k$ (for instance, see [33]).

Let $G_{k \times m}$ be the group of units of $\mathcal{POI}_{k \times m}$, i.e., $G_{k \times m} = \{\alpha \in \mathcal{POI}_{k \times m} \mid |\mathrm{Im}(\alpha)| = n\} = S_n \cap \mathcal{POI}_{k \times m}$. We have a natural isomorphism
\[
S^k_m \rightarrow G_{k \times m}
\]
defined by $xz = (x-(i-1)m)z_i+(i-1)m$, for $x \in I_i$, $1 \leq i \leq k$, and $z = (z_1, z_2, \ldots, z_k) \in S^k_m$.

Let $\tilde{A} = [\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k], \tilde{B} = [\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_k], \tilde{C} = [\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_k]$ and $\tilde{D} = [\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_k]$. By the above observations, $\tilde{A} \cup \tilde{B}, \tilde{B} \cup \tilde{C}$ and $\tilde{D}$ are three generating sets of $G_{k \times m}$. Moreover, the number of elements of $\tilde{D}$ is $k$, which is precisely the rank of $G_{k \times m}$.

**Proposition 2.2** For $k, m \geq 2$, the monoid $\mathcal{POI}_{k \times m}$ is generated by $\tilde{X} \cup G_{k \times m}$.

**Proof** Let $\alpha \in \mathcal{POI}_{k \times m}$ be a nonempty transformation (notice that, clearly, $\langle \tilde{X} \rangle$ contains the empty transformation) and suppose that $\alpha \Psi = \left( \begin{array}{cccc}
i_1 & i_2 & \cdots & i_t \\
j_1 & j_2 & \cdots & j_t \end{array} \right) \in \mathcal{POI}_k$, for some $1 \leq t \leq k$, $1 \leq i_1 < i_2 < \cdots < i_t \leq k$ and $1 \leq j_1 < j_2 < \cdots < j_t \leq k$, and let $\tilde{\alpha} = \alpha \Psi \Psi$. Then $\tilde{\alpha} \in \langle \tilde{X} \rangle$.

On the other hand, define $\gamma \in T_n$ by $x \gamma = (x + (i_r - j_r)m)\alpha$, for $x \in I_{j_r}, 1 \leq r \leq t$, and $x \gamma = x$, for $x \in \Omega_n \setminus \mathrm{Im}(\alpha)$. Then $\gamma \in G_{k \times m}$ and it is a routine matter to check that $\alpha = \tilde{\alpha} \gamma$, which proves the result. \qed

From this we deduce immediately

**Corollary 2.3** For $k, m \geq 2$, $\tilde{A} \cup \tilde{B} \cup \tilde{X}, \tilde{B} \cup \tilde{C} \cup \tilde{X}$ and $\tilde{D} \cup \tilde{X}$ are generating sets of $\mathcal{POI}_{k \times m}$.

Notice that, in particular, $\tilde{D} \cup \tilde{X}$ is a generating set of $\mathcal{POI}_{k \times m}$ with $2k$ elements.

**Proposition 2.4** For $k, m \geq 2$, the rank of $\mathcal{POI}_{k \times m}$ is $2k$.\[\square\]
Proof Let \( L \) be a generating set of \( \mathcal{POI}_{k \times m} \). Since \( |\bar{D} \cup \bar{X}| = 2k \), it suffices to show that \( |L| \geq 2k \). As \( \mathcal{POI}_{k \times m} \) is a finite monoid, \( L \) contains a generating set for its group of units \( \mathcal{G}_{k \times m} \). Recall that \( \mathcal{G}_{k \times m} (\cong S_{m}^{k}) \) has rank \( k \). Therefore \( |L \cap \mathcal{G}_{k \times m}| \geq k \). Since \( \mathcal{POI}_{k} \) is a homomorphic image of \( \mathcal{POI}_{k \times m} \) via the map \( \nu \), it follows that \( L \nu \) is a generating set for \( \mathcal{POI}_{k} \). Since the rank of \( \mathcal{POI}_{k} \) is also \( k \), we have that \( L \nu \) contains at least \( k \) elements of rank \( m(k - 1) \). Thus, \( L \) contains at least \( k \) elements of rank \( m(k - 1) \). We conclude that \( |L| \geq 2k \), as required.

\[ \square \]

3 Presentations for \( \mathcal{POI}_{k \times m} \)

We begin this section by recalling some notions and facts on presentations. Let \( A \) be a set and denote by \( A^* \) the free monoid generated by \( A \). Usually, the set \( A \) is called the alphabet and the elements of \( A \) and \( A^* \) are called letters and words, respectively. A monoid presentation is an ordered pair \( \langle A \mid R \rangle \), where \( A \) is an alphabet and \( R \) is a subset of \( A^* \times A^* \). An element \( (u, v) \) of \( A^* \times A^* \) is called a relation and it is usually represented by \( u = v \). To avoid confusion, given \( u, v \in A^* \), we will write \( u \equiv v \), instead of \( u = v \), whenever we want to state precisely that \( u \) and \( v \) are identical words of \( A^* \). A monoid \( M \) is said to be defined by a presentation \( \langle A \mid R \rangle \) if \( M \) is isomorphic to \( A^*/\rho_{R} \), where \( \rho_{R} \) denotes the smallest congruence on \( A^* \) containing \( R \). For more details see [28, 31].

A direct method to find a presentation for a monoid is described by the following well-known result (e.g., see [31, Proposition 1.2.3]).

Proposition 3.1 Let \( M \) be a monoid generated by a set \( A \) (also considered as an alphabet) and let \( R \subseteq A^* \times A^* \). Then \( \langle A \mid R \rangle \) is a presentation for \( M \) if and only if the following two conditions are satisfied:

1. the generating set \( A \) of \( M \) satisfies all the relations from \( R \);
2. if \( u, v \in A^* \) are any two words such that the generating set \( A \) of \( M \) satisfies the relation \( u = v \), then \( u \rho_{R} v \).

Given a presentation for a monoid, a method to find a new presentation consists in applying Tietze transformations. For a monoid presentation \( \langle A \mid R \rangle \), the four elementary Tietze transformations are as follows:

– adding a new relation \( u = v \) to \( \langle A \mid R \rangle \), provided that \( u \rho_{R} v \);
– deleting a relation \( u = v \) from \( \langle A \mid R \rangle \), provided that \( u \rho_{R}\{u=v\} v \);
– adding a new generating symbol \( b \) and a new relation \( b = w \), where \( w \in A^* \);
– if \( \langle A \mid R \rangle \) possesses a relation of the form \( b = w \), where \( b \in A \), and \( w \in (A\backslash\{b\})^* \), then deleting \( b \) from the list of generating symbols, deleting the relation \( b = w \), and replacing all remaining appearances of \( b \) by \( w \).

The next result is also well known (e.g., see [31, Proposition 3.2.5]):

Proposition 3.2 Two finite presentations define the same monoid if and only if one can be obtained from the other by applying a finite number of elementary Tietze transformations.

Another tool that we will use is given by the following proposition (e.g., see [26]):

Proposition 3.3 Let \( M \) and \( N \) be two monoids defined by the monoid presentations \( \langle A \mid R \rangle \) and \( \langle B \mid S \rangle \), respectively. Then the monoid presentation \( \langle A, B \mid R, S, ab = ba, a \in A, b \in B \rangle \) defines the direct product \( M \times T \).
In 1962, Aženšat [1] and Popova [29] exhibited presentations for the monoids \( O_n \) and \( PO_n \), respectively, and from the sixties until our days several authors obtained presentations for many classes of monoids. See the survey [17] and references therein. See also, for example, [8, 10, 11, 15, 21, 26].

Our strategy for obtaining a presentation for \( POI_{k \times m} \) will use well-known presentations of \( S_m \) and \( POI_k \).

First, we consider the following (monoid) presentation of \( S_m \), with \( m + 1 \) relations in terms of the generators \( a \) and \( b \) defined in the previous section:

\[ \langle a, b \mid a^2 = b^m = (ba)^{m-1} = (ab^{m-1}ab)^3 = (ab^{m-j}ab^j)^2 = 1, \ 2 \leq j \leq m - 2 \rangle \]

(for instance, see [17]). From this presentation, applying Tietze transformations, we can easily deduce the following presentation for \( S_k \) of the following 2

By Proposition 3.3, the monoid \( S_m \) is formed by the following 2

\[ \langle b, c \mid (cb^{m-1})^2 = b^m = (bcb^{m-1})^{m-1} = (cb^{m-2}c)^3 = (cb^{m-j-1}cb^{j-1})^2 = 1, \ 2 \leq j \leq m - 2 \rangle \]

(recall that \( c = ab \) and \( a = cb^{m-1} \)). Notice that \( c^{m-1} = 1 \).

Next, we use these presentations of \( S_m \) for getting two presentations of \( S_m^k \).

Consider the alphabets \( A = \{ a_i \mid 1 \leq i \leq k \} \) and \( B = \{ b_i \mid 1 \leq i \leq k \} \) (with \( k \) letters each) and the set \( R \) formed by the following \( 2k^2 + (m - 1)k \) monoid relations:

\[
\begin{align*}
(R_1) & \ a_i^2 = 1, \ 1 \leq i \leq k; \\
(R_2) & \ b_i^m = 1, \ 1 \leq i \leq k; \\
(R_3) & \ (b_i a_i)^{m-1} = 1, \ 1 \leq i \leq k; \\
(R_4) & \ (a_i b_i^{m-1} a_i b_i)^3 = 1, \ 1 \leq i \leq k; \\
(R_5) & \ (a_i b_i^{m-j} a_i b_i^j)^2 = 1, \ 2 \leq j \leq m - 2, \ 1 \leq i \leq k; \\
(R_6) & \ a_i a_j = a_j a_i, b_i b_j = b_j b_i, \ 1 \leq i < j \leq k; \ a_i b_j = b_j a_i, \ 1 \leq i, j \leq k, \ i \neq j.
\end{align*}
\]

Then, by Proposition 3.3, the monoid \( S_m^k \) is defined by the presentation \( \langle A, B \mid R \rangle \).

Now, consider the alphabet \( C = \{ c_i \mid 1 \leq i \leq k \} \) (with \( k \) letters) and the set \( U \) formed by the following \( 2k^2 + (m - 1)k \) monoid relations:

\[
\begin{align*}
(U_1) & \ (c_i b_i^{m-1})^2 = 1, \ 1 \leq i \leq k; \\
(U_2) & \ b_i^m = 1, \ 1 \leq i \leq k; \\
(U_3) & \ (b_i c_i b_i^{m-1})^{m-1} = 1, \ 1 \leq i \leq k; \\
(U_4) & \ (c_i b_i^{m-2} c_i)^3 = 1, \ 1 \leq i \leq k; \\
(U_5) & \ (c_i b_i^{m-j-1} c_i b_i^j)^2 = 1, \ 2 \leq j \leq m - 2, \ 1 \leq i \leq k; \\
(U_6) & \ b_i b_j = b_j b_i, c_i c_j = c_j c_i, \ 1 \leq i < j \leq k; \ b_i c_j = c_j b_i, \ 1 \leq i, j \leq k, \ i \neq j.
\end{align*}
\]

By Proposition 3.3, the monoid \( S_m^k \) is also defined by the presentation \( \langle B, C \mid U \rangle \).

Let us also consider the \( k \)-letters alphabet \( D = \{ d_i \mid 1 \leq i \leq k \} \). Recall that, as elements of \( S_m^k \), we have \( d_i = b_i c_i + 1, \ 1 \leq i \leq k - 1 \), and \( d_k = b_k c_1 \). Moreover, \( c_1 = d_i^m, c_i + 1 = d_i^m, \) for \( 1 \leq i \leq k - 1 \), and \( b_i = d_i^{(m-1)} \), for \( 1 \leq i \leq k \). Also, notice that \( d_i^{m(m-1)} = 1 \), whence \( b_i^{m-1} = d_i^{(m-1)} = d_i^{m-1} \), for \( 1 \leq i \leq k \). By applying Tietze transformations to the previous presentation, it is easy to check that \( S_m^k \) is also defined by the presentation \( \langle D \mid V \rangle \), where \( V \) is formed by the following \( 2k^2 + (m - 2)k \) monoid relations:

\[
\begin{align*}
(V_1) & \ (d_i^m d_i^{m-1})^2 = 1; \ (d_i^m d_i^{m-1})^2 = 1, \ 1 \leq i \leq k - 1; \\
(V_2) & \ d_i^{m(m-1)} = 1, \ 1 \leq i \leq k;
\end{align*}
\]
(V3) \( (a_i^{(m-1)}d_k^m d_i^{m-1})^{m-1} = 1 \); \( (d_i^{(m-1)}d_{i+1}^m d_i^{m-1})^{m-1} = 1, 1 \leq i \leq k - 1 \);

(V4) \( (d_k^n d_i^{(m-1)^2(m-2)} d_k^m)^3 = 1 \); \( (d_i^n d_{i+1}^{(m-1)^2(m-2)} d_i^m)^3 = 1, 1 \leq i \leq k - 1 \);

(V5) \( (d_k^n d_i^{(m-1)^2(m-j-1)} d_k^m d_j^{(m-1)^2(j-1)})^2 = 1, 2 \leq j \leq m - 2 \);

(V6) \( d_i^{(m-1)^2} d_j^{(m-1)^2} d_i^{(m-1)^2} d_j^{(m-1)^2} = 1, 1 \leq i < j \leq k \);

We move on to the monoid \( \mathcal{P} \mathcal{O} \mathcal{I}_k \). Let \( X = \{ x_i \mid 0 \leq i \leq k - 1 \} \) be an alphabet (with \( k \) letters). For \( k \geq 2 \), let \( W \) be the set formed by the following \( \frac{1}{2}(k^2 + 5k - 4) \) monoid relations:

(W1) \( x_i x_0 = x_0 x_{i+1}, 1 \leq i \leq k - 2 \);

(W2) \( x_i x_j = x_j x_i, 2 \leq i + 1 < j \leq k - 1 \);

(W3) \( x_i^2 x_0 = x_i^2 = x_{k-1} x_0 \);

(W4) \( x_i x_{i+1} x_i x_{i+1} = x_i x_{i+1} x_i, 1 \leq i \leq k - 2 \);

(W5) \( x_i x_{i+1} \ldots x_{k-1} x_0 x_1 \ldots x_{i-1} x_i = x_i, 0 \leq i \leq k - 1 \);

(W6) \( x_i x_1 \ldots x_{k-1} x_0 x_1 \ldots x_{i-1} x_i^2 = x_i^2, 1 \leq i \leq k - 1 \).

The presentation \( \langle X \mid W \rangle \) defines the monoid \( \mathcal{P} \mathcal{O} \mathcal{I}_k \) (see [15, 17]).

Finally, we define three sets of relations that enfold the letters from \( X \) together with the previous alphabets considered. Foremost, let \( R' \) be the set formed by the following \( 2k^2 + 2k \) monoid relations over the alphabet \( A \cup B \cup X \):

(R1') \( a_1 x_0 = x_0, b_1 x_0 = x_0 \);

(R2') \( x_0 a_i = a_{i+1} x_0, x_0 b_i = b_{i+1} x_0, 1 \leq i \leq k - 1 \);

(R3') \( x_i a_{k-i} = x_i, x_i b_{k-i} = x_i, 0 \leq i \leq k - 1 \);

(R4') \( x_i x_{k-i+1} = x_{k-i+1}, x_i b_{k-i+1} = b_{k-i+1}, 2 \leq i \leq k \);

(R5') \( x_i b_{k-i+1} = a_{k-i} x_i, x_i b_{k-i+1} = b_{k-i+1}, 1 \leq i \leq k - 1 \);

(R6') \( x_i a_j = a_j x_i, x_i b_j = b_j x_i, 1 \leq i \leq k - 1, 1 \leq j \leq k, j \neq [k - i, k - i + 1] \).

Secondly, consider the set \( U' \) formed by the following \( 2k^2 + 2k \) monoid relations over the alphabet \( B \cup C \cup X \):

(U1') \( c_i b_i^{-m} x_0 = x_0, b_1 x_0 = x_0 \);

(U2') \( x_0 c_i b_i^{-m} = c_{i+1} b_i^{m+1} x_0, x_0 b_i = b_{i+1} x_0, 1 \leq i \leq k - 1 \);

(U3') \( x_i c_k b_k^{m-i} = x_i, x_i b_k^{-i} = x_i, 0 \leq i \leq k - 1 \);

(U4') \( c_i b_i^{-m} x_{k-i+1} = x_{k-i+1}, b_i x_{k-i+1} = x_{k-i+1}, 2 \leq i \leq k \);

(U5') \( x_i c_k b_k^{m-i+1} = c_{k-i} b_k^{-m} x_i, x_i b_{k-i+1} = b_{k-i+1}, 1 \leq i \leq k - 1 \);

(U6') \( x_i c_j b_j^{m-i} = c_j b_j^{m-i} x_i, x_i b_j = b_j x_i, 1 \leq i \leq k - 1, 1 \leq j \leq k, j \neq [k - i, k - i + 1] \).

Lastly, let \( V' \) be the set formed by the following \( 2k^2 + 2k \) monoid relations over the alphabet \( D \cup X \):

(V1') \( d_k^{m-1} x_0 = x_0, d_i^{m-1} x_0 = x_0 \);

(V2') \( x_0 d_k^{m-1} d_i^{m-1} = d_i^{m-1} x_0, x_0 d_i^{m-1} = d_{i+1}^{m-1} x_0, 1 \leq i \leq k - 2 \);

(V3') \( x_i d_{k-i}^{m-1} x_i = x_i, 0 \leq i \leq k - 2 \);

(V4') \( d_i^{m-1} x_i = x_{k-i}, d_i^{m-1} x_{k-i} = x_{k-i}, 1 \leq i \leq k - 1 \).

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Lemma 3.4 The generating set \( \widehat{A} \cup \widehat{B} \cup \widehat{X} \) of \( \mathcal{POI}_{k \times m} \) satisfies (via \( \varphi \)) all the relations from \( R \cup W \cup R' \).

Observe that it follows from the previous lemma that \( w_1 \varphi = w_2 \varphi \) for all \( w_1, w_2 \in (A \cup B \cup X)^* \) such that \( w_1 \rho_{R\cup W\cup R'} = w_2 \).

Lemma 3.5 Let \( e \in A \cup B \) and \( x \in X \). Then there exists \( f \in A \cup B \cup \{1\} \) such that \( ex \rho_{R'} = xf \).

Proof The result follows immediately from relations \( (R'_1) \) and \( (R'_2) \) for \( x = 0 \), and from relations \( (R'_0), (R'_j) \) and \( (R''_0) \) for \( x \in X \setminus \{0\} \).

Let us denote by \(|w|\) the length of a word \( w \in (A \cup B \cup X)^* \).

Lemma 3.6 For each \( w \in (A \cup B \cup X)^* \) there exist \( u \in X^* \) and \( s \in (A \cup B)^* \) such that \( w \rho_{R'} u s \).

Proof We will prove the lemma by induction on the length of \( w \in (A \cup B \cup X)^* \).

Clearly, the result is trivial for any \( w \in (A \cup B \cup X)^* \) such that \(|w| \leq 1 \).

Let \( t \geq 2 \) and, by the induction hypothesis, assume the result for all \( w \in (A \cup B \cup X)^* \) such that \(|w| < t \).

Let \( w \in (A \cup B \cup X)^* \) be such that \(|w| = t \). Then there exist \( v \in (A \cup B \cup X)^* \) and \( x \in A \cup B \cup X \) such that \( w \equiv vx \) and \(|v| = t - 1 \).

Since \(|v| < t \), by the induction hypothesis, there exist \( u_1 \in X^* \) and \( s_1 \in (A \cup B)^* \) such that \( v \rho_{R'} u_1 s_1 \). Hence \( w \rho_{R'} u_1 s_1 x \).

If \(|s_1| = 0 \) or \( x \in A \cup B \) then the result is proved.

So, suppose that \(|s_1| \geq 1 \) and \( x \in X \). Let \( s' \in (A \cup B)^* \) and \( e \in A \cup B \) be such that \( s_1 \equiv s'e \). By Lemma 3.5, there exists \( f \in A \cup B \cup \{1\} \) such that \( ex \rho_{R'} f \). On the other hand, since \(|s'| < |s_1| \leq |v| < t \), we have \(|s'x| < t \) and so, by the induction hypothesis, there exist \( u_2 \in X^* \) and \( s_2 \in (A \cup B)^* \) such that \( s'x \rho_{R'} u_2 s_2 \). Thus, \( w \rho_{R'} u_1 u_2 s_2 f \), where \( u_1 u_2 \in X^* \) and \( s_2 f \in (A \cup B)^* \), as required.

From now on let us denote the congruence \( \rho_{R\cup W\cup R'} \) simply by \( \rho \).
Lemma 3.7  For all $j \in \{1, 2, \ldots, k\}$ and $u \in X^*$ such that $I_j \not\subseteq \text{Im}(u\varphi)$, we have $u a_j \rho u$ and $u b_j \rho u$.

**Proof** Let $\alpha = (u\varphi)v \in \mathcal{POI}_k$. Since $I_j \not\subseteq \text{Im}(u\varphi)$, it follows that $j \notin \text{Im}(\alpha)$. Since $(W_j)-(W_k)$ are defining relations for $\mathcal{POI}_k$, we have $u \rho v x_k \cdot j$, for some $v \in X^*$. By using also relation $(R'_3)$, we deduce that $u a_j \rho v x_{k-j} a_j \rho v x_{k-j} \rho u$. The proof for $u b_j \rho u$ is analogous. \hfill \square

Lemma 3.8  For each $w \in (A \cup B \cup X)^*$ there exist $u \in X^*$ and $s \in (A \cup B)^*$ such that $w \rho u s$ and $\ell(s\varphi) = \ell$, for all $\ell \in \Omega_n \setminus \text{Im}(u\varphi)$.

**Proof** Let $u \in X^*$ and $s_0 \in (A \cup B)^*$ be such that $w \rho u s_0$ (by applying Lemma 3.6). Hence, we may take $s \in (A \cup B)^*$ such that $w \rho u s$ and $s$ has minimum length among all $s' \in (A \cup B)^*$ such that $w \rho u s'$.

Let $\ell \in \Omega_n \setminus \text{Im}(u\varphi)$. Then $\ell \in I_j$, for some $1 \leq j \leq k$.

Suppose that $a_j$ or $b_j$ occur in $s$. Let $s_1 \in ((A \cup B) \setminus \{a_j, b_j\})^*$, $y \in \{a_j, b_j\}$ and $s_2 \in (A \cup B)^*$ be such that $s \equiv s_1 y s_2$. Then, clearly, $s_1 y \rho y s_1$ and so $s \rho y s_1 s_2$. On the other hand, as $I_j \not\subseteq \text{Im}(u\varphi)$, by Lemma 3.7, we have $u y \rho u$, whence $w \rho u s_1 s_2$, $s_1 s_2 \in (A \cup B)^*$ and $|s_1 s_2| = |s| - 1$, which is a contradiction.

Therefore, $a_j$ and $b_j$ do not occur in $s$ and so the restriction of $s\varphi$ to $I_j$ is the identity of $I_j$. In particular, $\ell(s\varphi) = \ell$, as required. \hfill \square

Let $u \in X^*$ and $s \in (A \cup B)^*$. Notice that as $s\varphi \in G_{k \times m}$, then $I_j(s\varphi) = I_j$, for all $1 \leq j \leq k$, and so $\text{Dom}((u s)\varphi) = \text{Dom}(u\varphi)$ and $\text{Im}((u s)\varphi) = \text{Im}(u\varphi)$.

We are now in a position to prove our last lemma.

Lemma 3.9  Let $w_1, w_2 \in (A \cup B \cup X)^*$. If $w_1 \varphi = w_2 \varphi$ then $w_1 \rho w_2$.

**Proof** By Lemma 3.8 we can consider $u_1, u_2 \in X^*$ and $s_1, s_2 \in (A \cup B)^*$ such that $w_1 \rho u_1 s_1$ and $w_2 \rho u_2 s_2$, $\ell(s_1 \varphi) = \ell$ for all $\ell \in \Omega_n \setminus \text{Im}(u_1 \varphi)$, and $\ell(s_2 \varphi) = \ell$ for all $\ell \in \Omega_n \setminus \text{Im}(u_2 \varphi)$.

Observe that $\text{Dom}(u_1 \varphi) = \text{Dom}((u_1 s_1) \varphi) = \text{Dom}(u_1) = \text{Dom}(w_1) = \text{Dom}(u_2) = \text{Dom}(u_2 \varphi) = \text{Dom}(u_1 \varphi)$ and $\text{Im}(u_1 \varphi) = \text{Im}((u_1 s_1) \varphi) = \text{Im}(u_1) = \text{Im}(u) = \text{Im}(u_2 \varphi) = \text{Im}(u_1 \varphi)$.

Since $u_1 \varphi, u_2 \varphi \in \mathcal{POI}_{k \times m} \cap \mathcal{POI}_n$, it follows that $u_1 \varphi = u_2 \varphi$. On the other hand, $\mathcal{POI}_{k \times m} \cap \mathcal{POI}_n \cong \mathcal{POI}_k$ and the monoid $\mathcal{POI}_k$ is defined by the presentation $(B \mid W)$. Therefore $u_1 \rho u_2$.

Next, we turn our attention to $s_1 \varphi, s_2 \varphi \in G_{k \times m}$. Let $\ell \in \Omega_n \setminus \text{Im}(u_1 \varphi) = \Omega_n \setminus \text{Im}(u_2 \varphi)$. Then $\ell(s_1 \varphi) = \ell = \ell(s_2 \varphi)$. On the other hand, let $\ell \in \text{Im}(u_1 \varphi) = \text{Im}(u_2 \varphi)$. Take $t \in \Omega_n$ such that $t(u_1 \varphi) = \ell$. Then $\ell(s_1 \varphi) = (t(u_1 \varphi)) (s_1 \varphi) = t ((u_1 s_1) \varphi) = t(u_1) = t(u_2) = t(u_2) = t((u_2 s_2) \varphi) = t((u_2 \varphi)(s_2 \varphi)) = \ell(s_2 \varphi)$. Hence $s_1 \varphi = s_2 \varphi$. Since $G_{k \times m} \cong S_m^k$ and $S_m^k$ is defined by the presentation $(A, B \mid R)$, it follows that $s_1 \rho s_2$.

Thus, $u_1 s_1 \rho u_2 s_2$ and so $w_1 \rho w_2$, as required. \hfill \square

In view of Proposition 3.1, we deduce immediately from Lemmas 3.4 and 3.9

**Theorem 3.10**  For $k, m \geq 2$, the monoid $\mathcal{POI}_{k \times m}$ is defined by the presentation $(A \cup B \cup X \mid R \cup W \cup R')$ on $3k$ generators and $\frac{1}{2}(9k^2 + (2m + 7)k - 4)$ relations.

In view of Proposition 3.2, as corollaries of the previous theorem, the following statements also hold:

**Theorem 3.11**  For $k, m \geq 2$, the monoid $\mathcal{POI}_{k \times m}$ is defined by the presentation $(B \cup C \cup X \mid U \cup W \cup U')$ on $3k$ generators and $\frac{1}{2}(9k^2 + (2m + 7)k - 4)$ relations.
Theorem 3.12 For \( k, m \geq 2 \), the monoid \( \mathcal{POI}_{k \times m} \) is defined by the presentation \( \langle D \cup X \mid V \cup W \cup V' \rangle \) on \( 2k \) generators and \( \frac{1}{2}(9k^2 + (2m + 5)k - 4) \) relations.

Acknowledgements We acknowledge the anonymous referee for the valuable suggestions. We wish to express to him/her our thanks.

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