A COMBINATORIAL APPROACH TO THE SET-THEORETIC SOLUTIONS OF THE YANG-BAXTER EQUATION

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Abstract. A bijective map \( r : X^2 \to X^2 \), where \( X = \{x_1, \ldots, x_n\} \) is a finite set, is called a set-theoretic solution of the Yang-Baxter equation (YBE) if the braid relation \( r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23} \) holds in \( X^3 \). A non-degenerate involutive solution \((X, r)\) satisfying \( r(xx) = xx \), for all \( x \in X \), is called square-free solution. There exist close relations between the square-free set-theoretic solutions of YBE, the semigroups of I-type, the semigroups of skew polynomial type, and the Bieberbach groups, as it was first shown in a joint paper with Michel Van den Bergh.

In this paper we continue the study of square-free solutions \((X, r)\) and the associated Yang-Baxter algebraic structures — the semigroup \( S(X, r) \), the group \( G(X, r) \) and the \( k \)-algebra \( A(k, X, r) \) over a field \( k \), generated by \( X \) and with quadratic defining relations naturally arising and uniquely determined by \( r \). We study the properties of the associated Yang-Baxter structures and prove a conjecture of the present author that the three notions: a square-free solution of (set-theoretic) YBE, a semigroup of I type, and a semigroup of skew-polynomial type, are equivalent. This implies that the Yang-Baxter algebra \( A(k, X, r) \) is Poincaré-Birkhoff-Witt type algebra, with respect to some appropriate ordering of \( X \). We conjecture that every square-free solution of YBE is retractable, in the sense of Etingof-Schedler.

1. Introduction

The Yang-Baxter equation appeared in 1967 [33] in Statistical Mechanics and turned out to be one of the basic equations in mathematical physics, and more precisely for introducing the theory of quantum groups. At present the study of quantum groups, and, in particular, the solutions of the Yang-Baxter equation attracts the attention of a broad circle of scientists and mathematicians.

Let \( V \) be a vector space over a field \( k \). We recall that a linear automorphism \( R \) of \( V \otimes V \) is a solution of the Yang-Baxter equation, if the equality

\[
(R \otimes \text{id}_V)(\text{id}_V \otimes R)(R \otimes \text{id}_V) = (\text{id}_V \otimes R)(R \otimes \text{id}_V)(\text{id}_V \otimes R)
\]

holds in the automorphism group of \( V \otimes V \otimes V \). \( R \) is a solution of the quantum Yang-Baxter equation (QYBE) if

\[
R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}
\]

where \( R^{ij} \) means \( R \) acting on the \( i \)-th and \( j \)-th component.
Finding all solutions of the Yang-Baxter equation is a difficult task far from being resolved. Nevertheless many solutions of these equations have been found during the last 20 years and the related algebraic structures (Hopf algebras) have been studied (for example see [19]). Most of these solutions were "deformations" of the identity solution. In 1990 V. Drinfeld [5] posed the problem of studying a class of solutions that are obtained in a different way - the so called set-theoretic solutions.

Definition 1.1. Let $X$ be a nonempty set. Let $r : X \times X \rightarrow X \times X$ be a bijection of the Cartesian product $X \times X$ onto itself. The map $r$ is called a set-theoretic solution of the Yang-Baxter equation, if

$$(r \times \text{id}_X)(\text{id}_X \times r)(r \times \text{id}_X) = (\text{id}_X \times r)(r \times \text{id}_X)(\text{id}_X \times r).$$

Each set-theoretic solution $r$ of the Yang-Baxter equation induces an operator $R$ on $V \otimes V$ for the vector space $V$ spanned by $X$, which is, clearly, a solution of 1.1. Various works dealing with set-theoretic solutions appeared during the last decade, cf. [32], [17], [14], [6], [7], [30], [21], [24], [27].

The purpose of this paper is first to present some recent conjectures on the set-theoretic solutions of the Yang-Baxter equation, and to give an account of the research in this area, and, second to continue the study of the general algebraic and homological properties of the algebraic structures related to the so called square-free solutions. Our approach is combinatorial. To each solution $(X, r)$ we associate a semigroup $S = S(X, r)$, a group $G = G(X, r)$ (the group was also studied in [6]), and a quadratic algebra over a field $k$, $A(k, X, r) \simeq kS$, each of them with a set of $n$ generators $X$ and with quadratic defining relations $\mathfrak{R}(X, r)$ naturally arising and uniquely determined by $r$. We study the "behaviour" of these relations, and use the obtained information for establishing structural and homological properties of the associated algebraic objects. This approach is natural, for usual linear solutions one has similar ideas for instance Manin’s work [23]. In the case of set-theoretic solutions to YBE it was initiated in the joint paper with Michel Van den Bergh [14], and applied to the study of the close relations between different mathematical objects such as set-theoretic solutions of the Yang-Baxter equation, semigroups of I-type (which appeared in the study of Sklyanin algebras) and the semigroups $S_0$ associated with the class of skew-polynomial rings with binomial relations, introduced and studied in [8] and [9]. The semigroups $S_0$ called semigroups of skew-polynomial type are standard finitely presented, more precisely, they are defined in terms of a finite number of generators and quadratic square-free relations, which form a Groebner basis (or equivalently, the algebra $A = kS$ is a PBW algebra) cf. 2.19. It is proven in [14] that each skew-polynomial semigroup $S_0$ defines a nondegenerate set-theoretic solution $r = r(S_0)$ of the Yang-Baxter equation. In connection with this result the present author made the conjecture that under the restriction that $X$ is finite and “square-free” i.e. $r(x, x) = (x, x)$ for each $x \in X$, all nondegenerate involutive solutions can be obtained in this way, cf. 2.18.

In this work we will not be in a position to develop specific physical applications but already we can say that several of the structures we introduce are highly relevant for physics. For example, the groups $G(X, r)$ act on each other to form a matched pair of groups and are hence a natural source of quantum groups of bicrossproduct type. More details are to appear in our sequel [15]. Bicrossproduct quantum groups themselves are increasing importance in noncommutative geometry as for example
the Connes-Kreimer quantum groups associated to renormalisation, the $\kappa$-Poincaré quantum groups related to deformed spacetime, and the original 'Planck-scale' quantum group; see [22] for this background.

2. Basic notions and results

In this section we first recall some basic notions, definitions, and results, from [6], and [14]. They are related to both quantum group theory and noncommutative algebra, so we recall them for convenience of readers with various mathematical background. Next we formulate the main results of the paper and a conjecture about set-theoretic solutions of YBE.

We fix a finite nonempty set $X$ with $n$ elements. We shall often identify the sets $X \times X$ and $X^2$, the set of all monomials of length two in the free semigroup $\langle X \rangle$.

Definition 2.1. [6] Let $r: X \times X \to X \times X$ be a bijective map, we shall refer to it as $(X, r)$. The components of $r$ are the maps $L: X \times X \to X$ and $R: X \times X \to X$ defined by the equality

$$ r(x, y) = (L_x(y), R_y(x)). $$

(i) $(X, r)$ is left nondegenerate if for each $x$ the map $L_x(y)$ is a bijective function of $y$; $(X, r)$ is right nondegenerate if for each $y$ the map $R_y(x)$ is a bijective function of $x$; $(X, r)$ is nondegenerate if it is left and right nondegenerate.

(ii) $(X, r)$ is involutive if

$$ r^2 = \text{id}_{X \times X} $$

(iii) $(X, r)$ is a braided set if $r$ satisfies the braid relation:

$$ r_1 r_2 r_1 = r_2 r_1 r_2, $$

where $r_1 = r \times \text{id}_X$ and $r_2 = \text{id}_X \times r$.

(iv) $(X, r)$ is symmetric if it is braided and involutive.

(v) If $(X, r)$ is a braided, involutive and nondegenerate set we shall call it simply a solution.

Clearly, every braided set presents a set-theoretic solution of the Yang-Baxter equation. A general study of nondegenerate symmetric sets was given in [6].

In [14] was found a special class of solutions, here we call them square-free solutions (cf. 2.2), which are defined via the semigroups with relations of skew-polynomial type. These semigroups were introduced and studied first in [8]. The study continued in [9], [10], [14], [18], cf. also [16].

Definition 2.2. A map $r: X^2 \to X^2$ is square-free if it acts trivially on $\text{diag}(X^2)$, i.e. $r(xx) = xx$, for all $x \in X$.

Example 2.3. Let $X$ be a nonempty set and let $r(xy) = yx$. Then $(X, r)$ is a square-free solution, which is called the trivial solution.

Example 2.4. (Permutational solution, Lyubashenko, [5]). Let $X$ be a non-empty set, let $f, g$ be maps $X \to X$ and let $r(xy) = g(y)f(x)$. Then a) $(X, r)$ is nondegenerate if and only if $f$ and $g$ are bijective; b) $(X, r)$ is braided if and only if $fg = gf$; c) $(X, r)$ is involutive if and only if $f = g^{-1}$.

Remark 2.5. Note that for any permutation $f$ of $X$, the map $r$ defined as $r(xy) = f(y)f^{-1}(x)$, is a solution, but in general $r$ is not square-free. In fact, a permutational involutive solution $r$ is square-free if and only if $f = \text{id}_X$, i.e. $r = \text{id}_{X^2}$. 

Nevertheless, we prove in 3.7 that each square-free solution behaves "locally" as a permutational solution.

Clearly, when the order \( |X| = 2 \), the only square-free solution \((X, r)\) is the trivial one. The lowest order of \(X\) which allows a nontrivial, square-free solution is 3, as shows the following.

**Example 2.6.** Let \(X = \{x_1, x_2, x_3\}\). Up to re-numerating of the set \(X\) there exists a unique non-trivial square-free solution \((X, r)\) namely:

\[
\begin{align*}
\sigma(x_1x_2) &= x_2x_1, \quad \sigma(x_2x_3) = x_3x_1; \\
\sigma(x_3x_1) &= x_3x_2, \\
\sigma(x_1x_2) &= x_2x_3, \\
\sigma(x_1x_3) &= x_1x_2.
\end{align*}
\]

Up to isomorphism of solutions, there exist 5 square-free solutions \((X, r)\) with \( |X| = 4 \). The one with the greatest number nontrivial relations is given in the following example.

**Example 2.7.** Let \(X = \{x_1, x_2, x_3, x_4\}\) and let \(r\) be defined as:

\[
\begin{align*}
\sigma(x_1x_2) &= x_4x_2, \quad \sigma(x_2x_3) = x_3x_2, \\
\sigma(x_3x_1) &= x_3x_4, \\
\sigma(x_1x_3) &= x_2x_3, \\
\sigma(x_2x_1) &= x_2x_4, \\
\sigma(x_1x_4) &= x_2x_1, \\
\sigma(x_4x_1) &= x_4x_3, \\
\sigma(x_3x_2) &= x_3x_4, \\
\sigma(x_2x_4) &= x_2x_3, \\
\sigma(x_4x_3) &= x_4x_1, \\
\sigma(x_1x_4) &= x_1x_2.
\end{align*}
\]

Then \((X, r)\) is a square-free solution. Consider the permutation \(\sigma = (12)(34)\). For \(x, y\) which belong to different orbits of \(\sigma\) one has \(\sigma(xy) = \sigma(y)\sigma^{-1}(x)\), and when \(x\) and \(y\) belong to the same orbit, then \(\sigma(xy) = \sigma^2(y)\sigma^{-2}(x) = yx\).

**Definition 2.8.** The braid group \(B_n\) is the group generated by \(n\) generators \(b_1, \ldots, b_n\) and defining relations

\[
\begin{align*}
(2.3) & \quad b_ib_j = b_jb_i, |i - j| > 1; \\
(2.4) & \quad b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}.
\end{align*}
\]

Recall that the symmetric group \(S_n\) is isomorphic to the quotient of \(B_n\) by the relations \(b_i^2 = 1\).

The following remark is obvious, see for example [6].

**Remark 2.9.** Let \(m \geq 3\) be an integer. (i) The assignment \(b_i \to r^{ii+1}, 1 \leq i \leq m-1\), extends to an action of \(B_m\) on \(X^m\) if and only if \((X, r)\) is a braided set. (ii) The assignment \(b_i \to r^{ii+1}, 1 \leq i \leq m-1\), extends to an action of \(S_m\) on \(X^m\) if and only if \((X, r)\) is a symmetric set. (Here, as usual, \(r^{ii+1} = id_{X^{m-1-i}} \times r \times id_{X^{m-1-i}}\)).

The next well-known fact (see [6]) gives the relation between the braided sets (i.e. the set-theoretic solutions of the Yang-Baxter equation) and the set-theoretic solutions of the quantum Yang-Baxter equation.

**Fact 2.10.** Let \(r : X^2 \to X^2\) be a bijection, \(\sigma : X^2 \to X^2\) be the flip \(\sigma(xy) = yx\), for all \(x, y \in X\). Let \(R = \sigma \circ r\). (i.e. \(R\) is the so called R-matrix corresponding to \(r\)). Then \(r\) satisfies the set-theoretic Yang-Baxter equation if and only if \(R\) satisfies the quantum Yang-Baxter equation:

\[
(2.5) R^{12}R^{23}R^{23} = R^{23}R^{13}R^{12}.
\]
Furthermore, \( r \) is involutive if and only if \( R \) satisfies 2.5 and the unitarity condition
(2.6)
\[
R^{21} R = 1.
\]

In the spirit of a recent trend called a combinatorial approach in algebra, to each bijective map \( r : X^2 \to X^2 \) we associate canonically finitely presented algebraic objects (see precise definition in 2.12) generated by \( X \) and with quadratic defining relations \( \mathcal{R} \) naturally determined as
(2.7)
\[
\mathcal{R} = \mathcal{R}(r) = \{(u = r(u)) \mid u \in X^2, u \neq r(u) \text{ as words in } X^2\}
\]

We study the close relations between the combinatorial properties of the defining relations, e.g. of the map \( r \), and the structural properties of the associated algebraic objects.

**Notation 2.11.** For a non-empty set \( X \), as usual, we denote by \( \langle X \rangle \) the free semigroup generated by \( X \), and by \( k\langle X \rangle \) the free associative \( k \)-algebra generated by \( X \), where \( k \) is an arbitrary field. For a set \( F \subseteq k\langle X \rangle \), \( (F) \) denotes the two sided ideal of \( k\langle X \rangle \), generated by \( F \).

**Definition 2.12.** Assume that \( r : X^2 \to X^2 \) is an involutive, bijective map.
(i) The semigroup
\[
S = S(X, r) = \langle X ; \mathcal{R}(r) \rangle,
\]
with a set of generators \( X \) and a set of defining relations \( \mathcal{R}(r) \), is called the semigroup associated with \( (X, r) \).
(ii) The group \( G = G(X, r) \) associated with \( (X, r) \) is defined as
\[
G = G(X, r) = gr\langle X ; \mathcal{R}(r) \rangle.
\]
(iii) For arbitrary fixed field \( k \), the \( k \)-algebra associated with \( (X, r) \) is defined as
(2.8)
\[
A = A(k, X, r) = k\langle X \rangle / (\mathcal{R}(r)).
\]

Clearly \( A \) is a quadratic algebra, generated by \( X \) and with defining relations \( \mathcal{R}(r) \). Furthermore, \( A \) is isomorphic to the semigroup algebra \( kS(X, r) \).

Manin, [23], introduced the notion of a Yang-Baxter algebra. He calls a Yang-Baxter algebra a quadratic algebra \( A \) with defining relation determined via arbitrary fixed Yang-Baxter operator. In this spirit we give the following definition.

**Definition 2.13.** Assume \( (X, r) \) is a solution. Then \( S(X, r), G(X, r) \) and \( A(k, X, r) \) are called respectively: the Yang-Baxter semigroup, the Yang-Baxter group, and the Yang-Baxter \( k \)-algebra, associated to \( (X, r) \). We shall also use the abbreviation "YB" for "Yang-Baxter".

In the case when \( (X, r) \) is a solution, \( G(X, r) \) is also called the the structure group of \( (X, r) \), see [6].

**Example 2.14.** Let \( (X, r) \) be the trivial solution, i.e. \( r(xy) = yx \) for all \( x, y \in X \), then clearly, \( S(X, r) = [x_1, \ldots, x_n] \), is the free abelian semigroup generated by \( X \), \( G(X, r) = Z^X \), is the free abelian group generated by \( X \), and \( A(k, X, r) = k[x_1, \ldots, x_n] \) is the commutative polynomial ring over \( k \).

**Definition 2.15.** Let \( S = \langle X ; \mathcal{R} \rangle \) be a semigroup with a set of generators \( X \) and a set of quadratic binomial defining relations:
\[
\mathcal{R} = \{xy = y'x' \mid x, y, x', y' \in X\},
\]
We assume that each monomial $u \in X^2$, occurs in at most one relation in $\mathbb{R}$. Define the map $r = r(S): X^2 \to X^2$ as follows:

(i) $r(xy) = xy$, if $xy$ is a monomial of length 2 which does not occur in any relation in $\mathbb{R}$; and

(ii) if $(xy = y'x') \in \mathbb{R}$, then we set $r(xy) = y'x'$ and $r(y'x') = xy$.

We call $r(S)$ the map associated with the semigroup $S$.

Note that if $r$ is the map defined by the set of relations of a YB-semigroup $S = \langle X; \mathbb{R} \rangle$, then the set $(X; r)$ is always symmetric, since clearly, $r^2 = id_{X^2}$.

We give now an example of a Yang-Baxter semigroup $S$ with 11 generators. In fact, $S$ belongs to the class of semigroups of skew-polynomial type, 2.19, and the map $r(S)$ is a square-free solution.

**Example 2.16.** Let $S = \langle X; \mathbb{R} \rangle$, where the set of generators is $X = \{1, 2, \cdots, 8, a, b, c\}$ and the defining relations are:

- $1a = a2, 2a = a1, 2b = b3, 3b = b2, 3a = a4, 4a = a3, 4c = c1, 1c = c4,$
- $5a = a6, 6a = a5, 6b = b7, 7b = b6, 7a = a8, 8a = a7, 8c = c5, 5c = c8,$
- $1b = b5, 5b = b1, 2c = c6, 6c = c2, 3c = c7, 7c = c3, 4b = b8, 8b = b4,$
- $ab = ca, ac = ba, bc = cb, ij = ji, 1 \leq i, j \leq 8.$

**Remark 2.17.** Let $S_0$ be a semigroup of skew-polynomial type (see 2.19). Let $r = r(S_0)$ be the map defined by the relations of $S_0$. Then $(X, r)$ is a square-free solution (cf [14], Th. 1.2, also Theorem 2.26). Furthermore, $S_0$ is a cancellative semigroup, and has a group of quotients $gr(S_0)$, which is a central localization of $S_0$, see [18]. It is clear, that the groups $gr(S_0)$ and the associated group $G(X, r)$ are isomorphic. Moreover, the set $X$ is embedded in $G(X, r)$.

The semigroups of skew-polynomial type were discovered while the author was searching for a new class of Artin-Schelter regular rings. It turned out that the skew-polynomial rings with binomial relations introduced and studied in [8], [9], [10] provide a class of Artin-Schelter regular rings of arbitrary global dimension. Furthermore, with each ring $A_0$ of this type we associate (uniquely) a semigroup $S_0$ which defines (via its relations) a non-degenerate set-theoretic solution $r(S_0)$ of the Yang-Baxter equation, cf. [14]. It is easy to generalize this result by showing that each skew-polynomial ring with binomial relations defines a solution of the classical Yang-Baxter equation, see Theorem 9.7. The semigroup $S_0$ is called a semigroup of skew-polynomial type. The results in [14] and further study of the combinatorial properties of the solutions inspired the following Conjecture, which we reported first in a talk at the International Conference in Ring Theory, Miskolc 1996, see also [11], [12].

**Conjecture 1 2.18.** [13] Let $(X, r)$ be a square-free (non-degenerate, involutive) solution of the Yang-Baxter equation. Then the set $X$ can be ordered so, that the associated semigroup $S = S(X, r)$ is of skew-polynomial type.

**Definition 2.19.** We say that the semigroup $S_0$ is a *semigroup of skew-polynomial type*, (or shortly, a *skew-polynomial semigroup*) if it has a standard finite presentation as $S_0 = \langle X; \mathbb{R}_0 \rangle$, where the set of generators $X$ is ordered: $x_1 < x_2 < \cdots < x_n$, and the set

$$\mathbb{R}_0 = \{x_jx_i = x_{i'}x_{j'} \mid 1 \leq i < j \leq n, 1 \leq i' < j' \leq n\},$$
contains precisely $n(n - 1)/2$ quadratic square-free binomial defining relations, each of them satisfying the following conditions:

i) each monomial $xy \in X^2$, with $x \neq y$, occurs in exactly one relation in $\mathcal{R}_0$; a monomial of the type $xx$ does not occur in any relation in $\mathcal{R}_0$;

ii) if $(x_jx_i = x_{i'}x_{j'}) \in \mathcal{R}_0$, with $1 \leq i < j \leq n$, then $i' < j'$, and $j > i'$.

[ further studies show that this also implies $i < j'$ see [9]]

iii) the monomials $x_kx_jx_i$ with $k > j > i, 1 \leq i, j, k, \leq n$ do not give rise to new relations in $S_0$, or equivalently, cf. [4], $\mathcal{R}_0$ is a Groebner basis with respect to the degree-lexicographic ordering of the free semigroup $\langle X \rangle$.

**Remark 2.20.** Suppose $S_0$ is a semigroup of skew-polynomial type. It follows from the Diamond Lemma [4] that, each element $w$ of $S$ can be presented uniquely as an ordered monomial

$$w = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

where $\alpha_i \geq 0, 1 \leq i \leq n$. This presentation is called the normal form of $w$ and denoted as $Nor(w)$. It follows from the Diamond lemma, that two monomials $w_1, w_2$ in the free semigroup $\langle X \rangle$ are equal in $S$ if and only if their normal forms coincide, $Nor(w_1) = Nor(w_2)$. Thus $S_0$ can be identified as a set with the set of ordered monomials

$$N_0 = \{x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n} | \alpha_i \geq 0, 1 \leq i \leq n\}.$$  

Furthermore, for an arbitrary field $k$, the set $N_0$ is a $k$- basis of the quadratic algebra

$$A_0 = k\langle X \rangle/(\mathcal{R}_0) \simeq kS_0.$$ 

Clearly, $A_0$ is a Poincaré-Birkhoff-Witt - algebra in the sense of Priddy [25] with $N_0$ as a PBW-basis.

**Remark 2.21.** In [18] the skew-polynomial semigroups $S_0$ are called binomial semigroups.

We now recall the definition of the semigroups of $I$-type, see [14], which are closely related to both- the semigroups of skew-polynomial type and the set-theoretic solutions of Yang-Baxter equation. The rings of $I$-type were introduced and studied by J.Tate, and M. Van den Bergh in their work on the homological properties of Sklyanin Algebras, [31].

**Notation 2.22.** Till the end of the paper we shall denote by

$$U = [u_1, \ldots, u_n],$$

the free commutative multiplicative semigroup generated by $u_1, \ldots, u_n$. 

**Definition 2.23.** [14], A semigroup $S$ generated by $\{x_1, \cdots, x_n\}$ is said to be of (left) $I$-type if there exists a bijection $v : U \rightarrow S$ called (a left $I$-structure), such that $v(1) = 1$, and such that for each $a \in U$ there is an equality of sets $\{v(u_1a), v(u_2a), \cdots, v(u_na)\} = \{x_1v(a), x_2v(a), \cdots, x_nv(a)\}$. Analogously one defines a right $I$-structure $v_1 : U \rightarrow S$.

**Remark 2.24.** It can be extracted from [14], see also 4.1, that if $(X, r)$ is a square-free solution, and $S = S(X, r)$ the associated YB semigroup, then

a) There exists a unique left $I$-structure $v : U \rightarrow S$, such that $v(u_i) = x_i$, for $1 \leq i \leq n$.

b) There exists a unique right $I$-structure $v_1 : U \rightarrow S$, such that $v_1(u_i) = x_i$, for $1 \leq i \leq n$. 


In section 4, Proposition 4.14, we show that a semigroup of $I$-type is a distributive lattice with respect to the order induced from “one-sided” divisibility, defined below.

**Definition 2.25.** For every pair $a, b \in S$ we set:

(i) $a | b$, if and only if there exists a monomial $c \in S$, such that $b = ca$. We call this relation **divisibility with respect to the left multiplication**.

(ii) $a | r b$, if and only if there exists a monomial $c \in S$, such that $b = ac$. This relation is called **divisibility with respect to the right multiplication**.

The following theorem proved in section 6 verifies Conjecture 2.18.

**Main Theorem 2.26.** Assume that $X$ is a finite set of order $n \geq 1$, and $r : X \times X \rightarrow X \times X$ is a square-free involutive bijection. Let $S = S(X, r)$ be the semigroup associated with $(X, r)$, and let $A = A(k, X, r)$ be the quadratic $k$-algebra associated with $(X, r)$, where $k$ is an arbitrary field. Then the following conditions are equivalent.

1. $(X, r)$ is non-degenerate solution of the set-theoretic Yang-Baxter equation.
2. $S = S(X, r)$ is a semigroup of $I$-type.
3. There exists an ordering on $X$, $X = \{x_1 < x_2 < \cdots < x_n\}$, such that $S = S(X, r)$ is a semigroup of skew-polynomial type.
4. There exists an ordering on $X$, $X = \{x_1 < x_2 < \cdots < x_n\}$ such that for every field $k$ the quadratic $k$-algebra $A = A(k, X, r)$ is a Poincaré-Birkhoff-Witt algebra, with a $k$-basis - the set of ordered monomials $N_0$.

Moreover, each of these conditions implies that the solution $(X, r)$ is decomposable, i.e. $X$ a disjoint union of two nonempty $r$-invariant subsets.

**Corollary 2.27.** Let $(X, r)$ be a square-free solution, with associated semigroup $S = S(X, r)$. Then $(S, |)$ is a distributive lattice. Furthermore the left $I$-structure $v : U \rightarrow S$ is an isomorphism of lattices.

Condition 2.26.2 implies cf. [14], various nice algebraic an homological properties of the algebra $A = A(k, X, r)$, like being a Noetherian domain, Koszul, Cohen-Macaulay, Artin-Schelter regular, etc. In particular the semigroup $S$ is cancellative. Hence it is naturally embedded in its group of quotients $gr(S) = G(X, r)$. We recall these results in Theorem 6.1.

My student, M.S. Garcia Roman has shown that for an explicitly given solution $(X, r)$, condition 2.26.3 is equivalent to a standard problem from Linear Programming.

In [15] is presented a matched pairs approach to the set-theoretic solutions of the Yang-Baxter equation. One of the main results in [15], given here as Theorem 5.6 covers all known constructions of solutions $(X, r)$, restricted to the case of square-free solutions, with $X$ a finite set.

In section 8 we study the generalized twisted unions of solutions, and multipermutation solutions.

Section 9 gives an application of the Main Theorem to a particular class of solutions of the classical Yang-Baxter equation.

We close this section with the following conjecture

**Strong Conjecture 2.28.** I. Every square-free solution $(X, r)$, where $X$ is a finite set of order $n \geq 2$, is retractible. Furthermore $(X, r)$ is a multipermutation solution of level $m < n$. 


II. Every multipermutation square-free solution of level \( m \) is a generalized twisted union of multipermutation solutions of levels \( \leq m - 1 \)

3. The cyclic condition and combinatorics in \( S(X, r) \)

In this section we introduce a combinatorial technique for non-degenerate square-free solutions \((X, r)\), which associates cycles in \( Sym(X) \) to each pair of elements \( y, x \) in \( X \). We call the corresponding property of \( r \) cyclic condition. The cyclic condition is the base for all combinatorial techniques in this paper. We use it here to deduce more precise pictures of the left and right actions of the group \( G(X, r) \) on \( X \), and to to show that each involutive square-free solution acts “locally” as a permutational solution. We obtain some important relations of higher degrees

\[
\text{invariant integer } M = M(X, r) \text{ with every solution } (X, r) \text{ called the cyclic degree of } (X, r).
\]

**Definition 3.1.** Let \( r : X \times X \rightarrow X \times X \) be a bijection.

1. We say that \((X, r)\) satisfies the weak cyclic condition, if for every pair \( y, x \in X \), there exist two disjoint cycles \( \mathcal{L}_y^x = (x_1, \ldots, x_m) \) and \( \mathcal{R}_x^y = (y_k, \ldots, y_1) \) in the symmetric group \( Sym(X) \), such that \( x = x_1, y = y_1 \), and for all \( 1 \leq i \leq m, \ 1 \leq j \leq k \), there are equalities:

\[
(3.1) \quad r(y_j x_i) = \mathcal{L}_y^x(x_i) \mathcal{R}_x^y(y_j) = x_{i+1} y_{j-1},
\]

where \( x_{m+1} := x_1 \), and \( y_0 := y_k \).

In particular, \( r(yx) = \mathcal{L}_y^x(x) \mathcal{R}_x^y(y) = x_2 y_k \).

2. \((X, r)\) satisfies the cyclic condition, if for every pair \( y, x \in X \), there exist two disjoint cycles \( \mathcal{L}_y^x = (x_1, \ldots, x_m) \) and \( \mathcal{L}_y^x = (y_1, \ldots, y_k) \) in \( Sym(X) \), such that \( x = x_1, y = y_1 \), and for all \( 1 \leq i \leq m, \ 1 \leq j \leq k \), there are equalities:

\[
(3.2) \quad r(x_i y_j) = y_{j+1} x_{i-1} \quad \text{and} \quad r(y_j x_i) = x_{i+1} y_{j-1},
\]

where \( x_0 = x_m, x_{m+1} := x_1 \), and \( y_0 := y_k, y_{k+1} = y_1 \).

In particular, for every pair \( (y, x) \in X \times X \), the disjoint cycles \( \mathcal{L}_y^x \) and \( \mathcal{L}_x^y \) satisfy:

\[
(3.3) \quad r(y, x) = \mathcal{L}_y^x(x) (\mathcal{L}_x^y)^{-1}(y), \quad \text{and} \quad r(x, y) = \mathcal{L}_x^y(y) (\mathcal{L}_y^x)^{-1}(x).
\]

We call \( \mathcal{L}_y^x \) and \( \mathcal{L}_x^y \), the pair of cycles associated to \((y, x)\).

**Remark 3.2.** Clearly, the (strong) cyclic condition implies that \( r \) is involutive. We will show that every involutive square-free solution \((X, r)\) satisfies the cyclic condition and use this to study the left (and the right) action of \( G(X, r) \) on \( X \). Note that if the cyclic condition holds, and we set

\[
\sigma = \sigma_{y,x} = \sigma_{x,y} = (x_1, \ldots, x_m)(y_1, \ldots, y_k) \in Sym(X),
\]

the map \( r \) is expressible "locally" as a permutational solution

\[
r(y_j x_i) = \sigma(x_i) \sigma^{-1}(y_j) \quad \text{and} \quad r(x_i y_j) = \sigma(y_j) \sigma^{-1}(x_i).
\]

If we do not assume involutiveness for \( r \), then, in general, only the weak cyclic condition is satisfied. We give an example, see 3.3, of a non-involutive solution in which the cyclic condition does not hold.
Example 3.3. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, and suppose the map $r : X^2 \to X^2$ is defined as:

\[
\begin{align*}
  x_1x_2 &\leftrightarrow x_2, x_1; x_3x_4 \leftrightarrow x_4x_3; \\
  x_3x_5 &\leftrightarrow x_5x_3; x_3x_6 \leftrightarrow x_6x_3; \\
  x_4x_5 &\leftrightarrow x_5x_4; x_4x_6 \leftrightarrow x_6x_4; xx \leftrightarrow xx, \text{ for all } x \in X
\end{align*}
\]

Then $(X, r)$ is a non-involutive solution, with $r^4 = id_X$. Furthermore

\[
L_{x_1} = (x_3x_4)(x_5x_6); R_{x_1} = (x_3x_6)(x_4x_5), \text{ and } R_{x_1} \neq (L_{x_1})^{-1}.
\]

Recall first a well known fact from [6].

Fact 3.4. [6] Let $(X, r)$ be nondegenerate, $G = G(X, r)$. Then $(X, r)$ is a braided set if and only if the following three conditions are satisfied:

1. The assignment $x \mapsto L_x$ induces a left action of $G$ on $X$;
2. The assignment $x \mapsto R_x$ induces a right action of $G$ on $X$;
3. The following equality holds for any $x, y, z \in X$:

\[
L_{yz}R_x = R_yL_{xz} = R_yL_{yz}R_x,
\]

Notation 3.5. We shall denote by $O_G(x)$ the orbit of $x, x \in X$, under the left action of $G$ on $X$.

Lemma 3.6. With notation being as in 3.1,

1. $(X, r)$ satisfies the weak cyclic condition if and only if for all $i, j, 1 \leq i \leq m, 1 \leq j \leq k$, there are equalities

\[
L_{y_j}^x = L_y^x = (x_1, \cdots, x_m), \text{ and } R_{x_i}^y = R_y^x = (y_1, \cdots, y_k).
\]

2. $(X, r)$ satisfies the cyclic condition if and only if for all $i, j, 1 \leq i \leq m, 1 \leq j \leq k$ there are equalities

\[
L_{y_j}^x = (R_{x_i}^y)^{-1} = (x_1, \cdots, x_m),
\]

and

\[
L_{x_i}^y = (R_{y_j}^x)^{-1} = (y_1, \cdots, y_k).
\]

The following theorem gives an account of various conditions on the bijective maps $r : X^2 \to X^2$ and the corresponding semigroup $S(X, r)$. For some of them we assume neither that $r$ is necessarily a solution of the Yang-Baxter equation, nor we assume that $r$ is involutive.

Theorem 3.7. Let $r : X^2 \to X^2$ be a bijective map, denoted by $(X, r)$. Let $S = S(X, r)$ be the semigroup associated to $(X, r)$. Let $L_x$ and $R_x$ be the left and right components of $r$, introduced in 2.1. Consider the following conditions:

1. a) $(X, r)$ is left nondegenerate; b) $(X, r)$ is right nondegenerate.
2. a) (Right Ore condition) For every pair $a, b \in X$ there exists a unique pair $x, y \in X$, such that $ax = by$; b) (Left Ore condition) For every pair $a, b \in X$ there exists a unique pair $z, t \in X$, such that $za = tb$.
3. $(X, r)$ is square-free and nondegenerate.
(4) $L_x$ is a bijection and $L_x(y) \neq x$, for each $y \neq x$; $R_y$ is a bijection and $R_y(x) \neq y$, for each $y \neq x$.

Then the following is true:

A. The conditions 1 a), and 2 a) are equivalent; the conditions 1 b), and 2 b) are equivalent;

B. The conditions 3 and 4 are equivalent.

C. If $(X, r)$ is a non-degenerate square-free solution of the Yang-Baxter equation, (not necessarily involutive) then the weak cyclic condition 3.11 holds.

D. If $(X, r)$ is a non-degenerate involutive square-free solution of the Yang-Baxter equation, then the cyclic condition 3.1.2 holds.

Proof. A. (1 a) $\Rightarrow$ (2 a)) Let $a, b \in X$. By our assumption the function $L_a$ is a bijection of $X$ onto itself, so there exists a unique $y$ such that $L_a(y) = b$, hence the equality $r(a y) = L_a(y) R_y(a)$ gives $r(a y) = b z$, for some $z \in X$. But $r$ is a bijective map on $X^2$ onto itself, so $z$ is also determined uniquely. The implication (1 b) $\Rightarrow$ (2 b)) is analogous.

The implications (2 a) $\Rightarrow$ (1 a)) and (2 b) $\Rightarrow$ (1 b)) are obvious.

B. 3 $\Rightarrow$ 4. Let $x, y \in X, x \neq y$. By assumption $r(x x) = x x$, so $L_x(x) = x \neq L_x(y)$. 4 $\Rightarrow$ 3. Let $x \in X$, clearly there is an equality of sets

$$\{L_x(y) \mid y \in X, y \neq x\} = X \setminus \{x\}$$

so $L_x(x) = x$. Similarly $R_x(x) = x$, thus $r(x x) = x x$.

For the following lemmas we assume the hypothesis of the theorem.

**Lemma 3.8.** If $(X, r)$ is nondegenerate and square-free, then $r(x y) \neq x y$ if and only if $x \neq y$.

Proof. The statement of the lemma follows immediately from B. and from the equation $r(x y) = L_x(y) R_y(x)$.

Lemma 3.9. If $(X, r)$ is a non-degenerate and square-free solution of the Yang-Baxter equation (not necessarily involutive), then the following conditions hold in $S$:

\[(3.8) \quad [yx = x'y', x \neq y] \Rightarrow [yx' = x''y', y' x = x'y''],\]

for some $x'', y'' \in X$.

Furthermore, there are equalities:

\[(3.9) \quad yxx = x'x'y'', \text{ and } yyx = x''y'y'.\]

Proof. Let $x \neq y$ and let $yx = x'y'$, or equivalently, $r(y x) = x'y'$. It follows from 3.8 that $yx \neq x'y'$, as monomials in the free semigroup $(X)$ Assume that

\[(3.10) \quad r(y x') = x''y''.\]
Now consider the "Yang-Baxter diagram"

$\begin{align*}
    & yyx \\ & \xrightarrow{r \times \text{id}_X} \\
\end{align*}$

\begin{align*}
    & yx'y' \\
    & \xrightarrow{\text{id}_X \times r} \\
    & yx'y' \\
    & \xrightarrow{r \times \text{id}_X} \\
    & x''y''y' \\
\end{align*}

It follows then that $r(y''y') = y'y'$, which, since $r$ is square-free, is possible only if $y'' = y'$. We have shown that

\begin{equation}
    (yx = x'y') \implies (yx' = x''y').
\end{equation}

$(y' = y$ is possible). Note that $x'' \neq y, y'$.

Similarly, we prove that

\begin{equation}
    (yx = x'y') \implies (y'x = x'y'').
\end{equation}

for some appropriate $y'' \in X$

The equality $yx = x''y'y'$ in $S$ also follows from the diagram 3.11. \qedbold

The validity of conditions C and D can be deduced from the following lemma. Note that in the hypothesis of the lemma we do not assume that $(X, r)$ is a solution.

**Lemma 3.10.** i) $(X, r)$ satisfies the weak cyclic condition 3.1.1 if and only if $r$ is non-degenerate and satisfies condition (3.8).

ii) Suppose $(X, r)$ satisfies the weak cyclic condition. Then $r$ is involutive if and only if for every pair $y, x \in X$ there is a solution.

**Proof.** Clearly, the weak cyclic condition 3.1.1 implies (3.8) and $r$ is non-degenerate. Assume now that $r$ is non-degenerate and condition (3.8) holds.

Suppose $y, x \in X$, $y \neq x$, and $r(yx) = x'y'$ ($x' = x$, or $y' = y$ are possible.) We denote $x_1 = x$, $x_2 = x'$, and apply 3.8 successively to obtain a sequence of pairwise distinct elements $x_1, \ldots, x_m \in X$, such that

\begin{equation}
    r(yx_i) = x_{i+1}y', \quad 1 \leq i \leq m-1, \text{ and } r(yx_m) = x_1y'.
\end{equation}

Similarly, (after an appropriate re-numeration) we obtain $y_1 = y, y_2, \ldots, y_k = y' \in X$, such that

\begin{equation}
    r(y_jx_1) = x_2y_{j-1}, \quad 2 \leq j \leq k, \text{ and } r(y_1x_1) = x_2y_m.
\end{equation}

We claim that

\begin{equation}
    r(y_jx_i) = x_{i+1}y_{j-1}, \quad 1 \leq i \leq m, 1 \leq j \leq k,
\end{equation}

where $x_{m+1} := x_1$, $y_0 := y_m$. We prove 3.16 by induction on $j$.

Step 1. $j = 1$. Clearly 3.14, with $y_k = y'$, give the base for the induction. Assume 3.16 is satisfied for all $j, 1 \leq i \leq j - 1$. We shall prove 3.16 for $j = j_0$, $1 \leq i \leq m - 1$, using induction on $i$. The base of the induction:

\begin{equation}
    r(y_{j_0}x_1) = x_2y_{j_0-1}.
\end{equation}

follows from 3.15. Assume now 3.16 is true for all $i < i_0$. In particular,

\begin{equation}
    r(y_{j_0}x_{i_0-1}) = x_{i_0}y_{j_0-1}.
\end{equation}
Then by (3.8) one has:

\[(3.19) \quad r(y_{j_0}x_{i_0}) = ty_{j_0}^{-1} \text{, for some } t \in X.\]

we apply (3.8) again and obtain

\[(3.20) \quad r(y_{j_0+1}x_{i_0}) = tz,\]

for some \(z \in X\). It follows from the inductive assumption that:

\[(3.21) \quad r(y_{j_0}x_{i_0}) = x_{i_0+1}y_{j_0-1},\]

which together with (3.20) gives \(t = x_{i_0+1}\) thus \(r(y_{j_0}x_{i_0}) = x_{i_0+1}y_{j_0-1}\). We have proved that (3.16) holds for all \(i, 1 \leq i \leq m\), and \(j = j_0\), which verifies (3.16). This proves i).

We set \(L_y = (x_1, \cdots, x_m) \in Sym(X)\), and \((R^y_x)^{-1} = (y_1, \cdots, y_k) \in Sym(X)\). Consider the permutation

\[
\sigma_{y,x} = (x_1, \cdots, x_m)(y_1, \cdots, y_k).
\]

Clearly,

\[(3.22) \quad r(y_jx_i) = \sigma_{y,x}(x_i)\sigma_{y,x}^{-1}(y_j).
\]

Assume now that \(r\) is involutive, and apply \(r\) to (3.16) to obtain \(r(x_{i+1}y_{j-1}) = y_jx_i\). This implies for \(1 \leq i \leq m\) and \(1 \leq j \leq k\):

\[(3.23) \quad L_y = L_y x_i = (y_1, \cdots, y_k) = (R^y_x)^{-1} = (L_y x_i)^{-1},\]

\[(3.24) \quad L_y = L_y y_j = (x_1, \cdots, x_m) = (R^y_x)^{-1} = (L_y y_j)^{-1}.
\]

Conversely, 3.23 and 3.24 imply that \(\sigma_{y,x} = \sigma_{x,y}\) therefore \(r\) is involutive. This proves the lemma, and completes the proof of the theorem. \(\square\)

\textbf{Remark 3.11.} Let \((X, r)\) be an arbitrary square-free non-degenerate solution (not necessarily involutive). Consider the left and the right actions of \(G\) on \(X\), see 3.4, extending the assignment \(y \rightarrow L_y\), \(x \rightarrow R_x\), where \(L_y, R_x \in Sym(X)\) are the permutations defined via \(r(yx) = L_y(x)R_x(y)\). Since each permutation has a presentation as a product of disjoint cycles in \(Sym(X)\), (unique up to commutation of multiples) we obtain that the cycle \(L^y_x = (x_1, \cdots, x_m)\), \((x_1 = x)\) occurs as a multiple in such a presentation of \(L_y\) and the cycle \(R^y_x = (y_1, \cdots, y_k)\) is a multiple of the corresponding presentation for \(R_x\). The surprising part is that each pair \(y_j, x_i\) with \(1 \leq j \leq k\) and \(1 \leq i \leq m\), produces the same pair of cycles: \(L^y_{y_j} = L^y_x = (x_1, \cdots, x_m)\), and \(R^y_x = R^y_{y_j} = (y_1, \cdots, y_k)\). Therefore although in general \(L_{y_j} \neq L_y\), each permutation \(L_{y_j}\), \(1 \leq j \leq k\) contains the same cycle \((x_1, \cdots, x_m)\) in its presentation as products of disjoint cycles in \(Sym(X)\). Analogously, the cycle \((y_1, \cdots, y_k)\) participates in the presentation of each \(R_{y_j}\), \(1 \leq j \leq m\), as a product of disjoint cycles. We do not know how the cycles \(L^y_x\) and \(R^y_x\), are related to each other, in the general (non-involutive) case of square-free non-degenerate solutions, besides the obvious property, that each of them contains \(x\), see example 3.3. In the case of involutive solutions \((X, r)\) there is a "symmetry" \(L^x_y = (R^x_y)^{-1} = (x_1, \cdots, x_m)\) for each pair \(y \neq x, y, x \in X\).

\textbf{Notation 3.12.} To avoid complicated expressions, sometimes we shall use also the notation \(\overline{y} = L_x(y)\) and \(\overline{x} = R_x(y)\).
The following corollary is a "translation" of the cyclic condition in the new notation. It can be extracted from a more general result in [15].

**Corollary 3.13.** Let \( r : X \times X \to X \times X \) be a non-degenerate involutive bijection. Consider the following conditions:

1. \( x^x = x \) for every \( x \in X \).
2. \( r(x, x) = (x, x) \) for every \( x \in X \).
3. \((X, r)\) satisfies the cyclic condition.
4. For every \( x, y \in X \) there are equalities:
   \[
   (xy)^x = yx; \quad (yx)^y = y(x^y) = x.
   \]

Then the following is true:

a) Conditions 1 and 2 are equivalent.

b) Conditions 3 and 4 are equivalent.

**Convention 3.14.** In the rest of the paper we shall consider only involutive non-degenerate square-free solutions \((X, r)\) of the Yang-Baxter equation; they will be briefly called square-free solutions.

Let \( x \in X \). Clearly, for \( t \in X \) the cycle \( L^x_t \) is of length one if and only if \( xt = tx' \).

**Notation 3.15.** We denote by \( G_L = G_L(X, r) \) the image of \( G(X, r) \) under the group homomorphism \( L : G \to \text{Sym}(X) \), which extends the assignment \( x \mapsto L^x_x \).

\( G_R = G_R(X, r) \) denotes the image of \( G(X, r) \) under the group homomorphism \( R : G \to \text{Sym}(X) \), which extends the assignment \( x \mapsto R^x_x \).

**Lemma 3.16.** Let \((X, r)\) be a square-free solution, \( L_x \), and \( R_x \) be the left and right components of \( r \), which are extended to a left, respectively right action of \( G(X, r) \) on \( X \). Then

1. The permutation \( L_x \) is presented as a product of disjoint cycles in \( \text{Sym}(X) \) via the equality:
   \[
   L_x = L_{x_1}^{t_1} L_{x_2}^{t_2} \cdots L_{x_s}^{t_s},
   \]
   where \( t_1, \ldots, t_s \) are representatives of all disjoint orbits of \( L_x \) in \( X \).

2. The permutations \( L_x \) and \( R_x \) satisfy the equality:
   \[
   R_x = (L_x)^{-1}.
   \]
   Furthermore, the two permutation groups determined by the left and right action of \( G(X, r) \) on \( X \) coincide:
   \[
   G_R = G_L.
   \]

3. The assignment \( x \mapsto L_x, x \in X \), determines the solution \( r \) uniquely, via the formula:
   \[
   r(x, y) = L_x(y)(L_y)^{-1}(x).
   \]

To each solution we associate an invariant integer number \( M = M(X, r) \) defined as follows.

**Definition 3.17.**

1. For every \( x \in X \) we denote by \( M_x \) the order of the permutation \( L_x \) in \( \text{Sym}(X) \), i.e. (in the notation of 3.16) the least common multiple of the lengths of the cycles \( L_{x_i}^{t_i}, 1 \leq i \leq s \).
(2) By \( M = M(X,r) \) we denote the least common multiple of all \( M_x \), where \( x \in X \), and call \( M \) the cyclic degree of the solution \( (X,r) \).

**Lemma 3.18.** Suppose \( ax = y'a' \), for some \( x, y, a, a' \in X \) Then \( M_x = M_y \).

**Proof.** It will be enough to show that the length \( k \) of each cycle \( L^t_x \) occurring in \( L_x \) divides \( M_y \). \( \square \)

**Proposition 3.19.** Assume \( x, y \in X \), and \( O_G(x) = O_G(y) \). Then \( M_y = M_x \).

**Corollary 3.20.** Suppose \( M_x \neq M_y \), for some \( x, y \in X \). Then \( G \) acts non-transitively on \( X \), and \( X \) is decomposable into a disjoint union of two \( r \)-invariant subsets.

The following proposition follows easily from the cyclic condition, and 3.17.

**Proposition 3.21.** Let \( (X,r) \) be a square-free solution of cyclic degree \( M \). Let \( p, \) and \( q \) be arbitrary natural numbers. Suppose \( y, x \in X, y \neq x, \) and let \( k, m \) be the natural numbers defined in 3.1. Let \( M_x, \) denote the order of \( L_x \). Then the following equalities hold.

\[
\begin{align*}
(3.28) & \quad y^m x = xy^m. \\
(3.29) & \quad y^p x^q = (x')q(y')p, \text{ where } x' = (L_y)^p(x), \text{ and } y' = (L_x)^{-q}(y). \\
(3.30) & \quad x^M y = y(x_m)^M. \\
(3.31) & \quad x^M y^M = y^M x^M. 
\end{align*}
\]

The next corollary follows immediately from 3.31.

**Corollary 3.22.** Let \( (X, r) \) be a square-free solution, then the center of the Yang-Baxter algebra \( A(k, X, r) \) contains all symmetric functions in \( x_x, x_x^2, \ldots, x_x^n \).

**Corollary 3.23.** Let \( (X, r) \) be a square-free solution. Then the group \( A = gr[x_1^M, \ldots, x_n^M] \) is a free abelian subgroup of \( G(X,r) \) of index \( M^n \).

### 4. The lattice structure of \( S(X,r) \)

In this section we show that for a semigroup \( S \) of left \( I \)-type, the relation \( | \) of left divisibility, defined in 2.25, and the left \( I \)-structure \( v : U \rightarrow S \), see 2.23, are compatible, and prove that \( (S, |) \) is a distributive lattice. Analogous results are true for semigroups with right \( I \)-structure \( v_1 : U \rightarrow S \). As a corollary we obtain that the Yang-Baxter semigroup \( S = S(X, r) \) has a structure of distributive lattice, induced by its left \( I \)-structure \( v \). We keep the notation from the previous sections, In particular,

\[
(4.1) \quad U = \langle u_1, \ldots, u_n \rangle
\]

is the free commutative multiplicative semigroup generated by \( u_1, \ldots, u_n \), and \( \langle X \rangle \) denotes the free semigroup generated by \( X \). The definition of an \( I \)-structure is given in 2.23.

The following result can be extracted from [14], Theorem 1.3.

**Theorem 4.1.** Let \( (X, r) \) be a square-free solution, and \( S = S(X, r) \) be the associated Yang-Baxter semigroup. Then

A. There exists a unique left \( I \)-structure \( v : U \rightarrow S \), which is inductively defined by the following conditions:
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(1) \( v_1(1) = 1, \) \( v(u_i) = x_i, \) for \( 1 \leq i \leq n. \)
(2) For every \( b \in \mathcal{U} \) and every \( i, 1 \leq i \leq n, \) there exists an \( x_{b,i} \in X, \) such that \( v(u_i b) = x_{b,i} v(b). \) Moreover, there is an equality of sets

\[
\{x_{b,i} \mid 1 \leq i \leq n\} = \{x_1, \cdots, x_n\}. 
\]

(3) For every \( b \in \mathcal{U}, \) and \( 1 \leq i, j \leq n, \) there is a relation in \( S: \)

\[
x_{u_i b, i} x_{b,j} = x_{u_i b, j} x_{b,i}. 
\]

B. There exists a unique right I-structure \( v_1 : \mathcal{U} \to S, \) which satisfies the following conditions:

(1) \( v_1(1) = 1, \) \( v_1(u_i) = x_i, \) for \( 1 \leq i \leq n. \)
(2) For every \( b \in \mathcal{U} \) and every \( i, 1 \leq i \leq n, \) there exists an \( x_{i,b} \in X, \) such that \( v(b u_i) = v(b) x_{i,b}. \) Furthermore, there is an equality of sets

\[
\{x_{i,b} \mid 1 \leq i \leq n\} = \{x_1, \cdots, x_n\}. 
\]

(3) For every \( b \in \mathcal{U}, \) and \( 1 \leq i, j \leq n, \) there is a relation in \( S: \)

\[
x_{i,b u, j} = x_{i, b u, j} x_{b,i}. 
\]

Remark 4.2. Suppose \( \mathcal{S}, \) is a semigroup of (left) \( I\)-type generated by \( x_1, \cdots, x_n, \) with a left \( I\)-structure \( v : \mathcal{U} \to \mathcal{S}. \) Then in general, \( v \) satisfies a modified version on condition A where condition 1 is modified to

\[
v(u_j) = x_{i_j}, 1 \leq j \leq n, \]

where \( i_1, \cdots, i_n \) is a permutation of \( 1, \cdots, n, \) and conditions 2, and 3 are unchanged. Moreover 4.6 determines the bijection \( v \) uniquely. Analogous statement is true for right \( I\)-structures. Without loss of generality we can consider only the special \( I\)-structures \( v \) and \( v_1 \) defined in theorem 4.1.

Notation 4.3. Throughout this section \( \mathcal{S} \) will denote a semigroup of \( I\)-type generated by \( x_1, \cdots, x_n \) with a left \( I\)-structure \( v \) and a right \( I\)-structure \( v_1. \) We assume that \( v \) and \( v_1 \) satisfy conditions 4.1 A, and B, respectively.

Remark 4.4. Note that given \( a \in \mathcal{U}, \) in finitely many steps one can find effectively the monomials \( v(a) \) and \( v_1(a). \) In particular, it is easy to see that for any \( i, 1 \leq i \leq n, \) and any positive integer \( k \) there are equalities \( v(u_i^k) = v_1(u_i^k) = x_i^k. \) In general, for a monomial \( u \in \mathcal{U} \) there might be inequality \( v(u) \neq v_1(u) \) (as elements of \( \mathcal{S}, \) see 4.11.

We study first the properties of the relations "\( |_l \)"- divisibility with respect to left multiplication or shortly- left divisibility and "\( |_r \)"- right divisibility on \( \mathcal{S}, \) defined as

\[
a |_l b, \text{if there exists a } c \in \mathcal{S}, \text{ such that } b = ca. 
\]

\[
a |_r b, \text{if there exists a } d \in \mathcal{S}, \text{ such that } b = ad. 
\]

The following lemma shows that the left \( I\)-structure \( v \) is compatible with the left divisibility.

Lemma 4.5. \( |_l \) is a partial order on \( \mathcal{S}, \) compatible with the left multiplication. Furthermore, this order is compatible with the left \( I\)-structure \( v. \) More precisely, the following two conditions hold:

a) If \( a |_l b \in \mathcal{U} \) (i.e. \( b = ca \) is an equality in \( \mathcal{U} \)) then \( v(a) |_l v(b); \)

b) Conversely, let \( a, b, c \in \mathcal{S} \) satisfy \( b = ca. \) Let \( a_0, b_0 \) be the unique elements of \( \mathcal{U} \) which satisfy \( v(a_0) = a \) and \( v(b_0) = b. \) Then \( b_0 = c_0a_0, \) for some \( c_0 \in \mathcal{U}. \)
Proof. First we show that $|_1$ is an ordering on $S$ as a set. Clearly, $a |_1 a$ for every $a \in S$. It is known that each semigroup $S$ of $I$-type is with cancellation low, see [14]. It follows then that $a |_1 b$ and $b |_1 a$ imply $a = b$. The transitiveness follows at once from the definition of $|_1$.

Next we prove a). Assume $b = ca$, for $a, b, c \in U$. We use induction on the length $| c |$ of $c$ to find a monomial $c' \in S$, such that $v(b) = c' v(a)$. If $c = u_i$, then by the definition of $v$ we have $v(b) = v(u_i a) = x_{a,i} v(a)$. Assume that the statement of the proposition is true for all $c$ of length $\leq m$. Let $b = ca$, where $| c | = m + 1$. Then $c = u_i d$, where $1 \leq i \leq n$, and $| d | = m$. We have $v(b) = v(u_i d a) = x_{a,i} v(da)$. By the inductive assumption $v(da) = d v(a)$, so $v(b) = x_{a,i} d v(a)$, which proves a). Assume now that $a, b \in S$, and $b = ca$, for a $c \in S$. By definition, $v$ is a bijection, so there are unique $a_0$ and $b_0$ in $U$, such that $v(a_0) = a$, and $v(b_0) = b$. We have to find a $c_0 \in U$, such that $b_0 = c_0 a_0$. We show this again by induction on the length $| c |$ of $c$. If $| c | = 1$, then $c = x_i \in X$. It follows from 4.1 that there is an equality of sets

$$\{v(u_1 a_0), \ldots, v(u_n a_0)\} = \{x_{1} v(a_0), \ldots, x_{n} v(a_0)\}. \quad (4.9)$$

Clearly, then there exists a $j$, such that $v(u_j a_0) = x_{j} v(a_0) = x_{j} a = b$. This gives $b_0 = u_j a_0$. Assume b) is true for all $c \in S$ with length $| c | \leq k$. Let $b = ca$, where $| c | = k + 1$. Then $c = x d$, for some $x \in X$ and $| d | = k$. It follows from the inductive assumption that there is a $d_0 \in U$, such that

$$v(d_0 a_0) = d v(a) \quad (4.10)$$

In addition an equality of sets similar to 4.9 shows that there exists an $u_j$, such that $v(u_j d_0 a_0) = x v(d_0 a_0)$. The last equality together with 4.10 gives $v(u_j d_0 a_0) = x v(d_0 a_0) = x d a = c a$, so $c a = u_j d_0 a_0$ satisfies the desired equality $b_0 = c_0 a_0$. □

An analogous statement is true for the right $I$-structure $v_1$.

**Lemma 4.6.** Let $a, b \in S$. a) There exist a uniquely determined least common multiple of $a$ and $b$, with respect to $|_1$, that is a monomial $w$ of minimal length, such that $w = w_1 a = w_2 b$, for some $w_1, w_2 \in S$. b) There exist a uniquely determined least common multiple, of $a$ and $b$, with respect to $|_r$, that is a monomial $w'$ of minimal length, such that $w' = w'_1 a = w'_2 b$, for some $w'_1, w'_2 \in S$.

**Proof.** The map $v$ is bijective, so $a = v(a_0)$, and $b = v(b_0)$, for some uniquely determined $a_0$ and $b_0$ in $U$. Let $w_0$ be the least common multiple $a_0 \cup b_0$ of $a_0$ and $b_0$ in $U$. It follows from 4.5 that $v(w_0) = \xi v(a_0) = \eta v(b_0)$. Thus $w = v(w_0)$ satisfies

$$w = \xi u = \eta b. \quad (4.11)$$

is a common multiple of $a$ and $b$ (with respect to $|_1$). That $w$ is of minimal possible length among the monomials satisfying 4.11 follows from 4.5. This proves a). An analogous argument proves b). □

**Notation 4.7.** By $a \cup b$ we denote the least common multiple of $a$ and $b$ with respect to $|_1$. $a \vee b$ denotes the the least common multiple of $a$ and $b$ with respect to $|_r$.

**Lemma 4.8.** Let $v, v_1$ be the left and the right $I$-structures on $S$, defined in 4.3. Then a) $v$ is a lattice isomorphism for $(U, |_1)$ and $(S, |_1)$; b) $v_1$ is a lattice isomorphism for $(U, |_r)$ and $(S, |_r)$.
Definition 4.9. Let $u \in S$. We say that $h \in X$ is a head of $u$ (as an element of $S$), if $u$ can be presented as $u = hu'$, for some $u' \in X$. The element $t \in X$ is called a tail of $u$ (in $S$) if $u = u't$ is an equality in $S$, for some $u' \in S$.

Note that a monomial may have more than one heads (respectively tails).

Example 4.10. The relation $(xy = y'x') \in \mathcal{R}$ implies that the heads of $xy$ are $x$ and $y'$, and its tails are $y$ and $x'$. Furthermore, $xy = x \lor y' = y \lor x'$.

Example 4.11. Consider the YB semigroup $S = \langle X; \mathcal{R} \rangle$, where $X = \{x_1, x_2, x_3, x_4\}$ and the set of relations is

$$x_4x_1 = x_2x_3, x_4x_2 = x_1x_3, x_3x_1 = x_2x_4, x_3x_2 = x_1x_4, x_1x_2 = x_2x_1, x_3x_4 = x_4x_3.$$ Then

$$v(u_2u_4) = x_1x_4 = x_3x_2 = v_1(u_1u_4), v_1(u_2u_4) = x_4x_1 = x_2x_3 = v(u_1u_3).$$

Clearly, $v(u_2^2u_4) \neq v_1(u_2^2u_4)$ as elements of $S$. In fact, $v(u_2^2u_4) = v_1(u_2^2u_4)$. For $w = x_1^2x_4$ there are equalities in $S$

$$w = x_1^2 \lor x_4 = x_1^2 \lor x_3.$$ 

Remark 4.12. In general, for $w \in \mathcal{U}$ there might be an inequality $v(w) \neq v_1(w)$, and it is not true that $a \lor b = a \lor b$, cf. 4.11. Still for the special monomial

$$W_0 = u_1u_2 \cdots u_n$$

one has

$$v(W_0) = v_1(W_0) = x_1 \lor x_2 \lor \cdots \lor x_n = x_1 \lor x_2 \lor \cdots \lor x_n.$$ 

Lemma 4.13. Let $w_0 \in \mathcal{U}$. Suppose $w_0 = u_{i_1}^{\alpha_{i_1}}u_{i_2}^{\alpha_{i_2}} \cdots u_{i_k}^{\alpha_{i_k}}$, where $1 \leq i_1 < i_2 < \cdots i_k \leq n$, and all $\alpha_j$ are positive integers. Then

1. $v(w_0) = x_{i_1}^{\alpha_{i_1}} \lor x_{i_2}^{\alpha_{i_2}} \lor \cdots \lor x_{i_k}^{\alpha_{i_k}}$,
2. $v_1(w_0) = x_{i_1}^{\alpha_{i_1}} \lor x_{i_2}^{\alpha_{i_2}} \lor \cdots \lor x_{i_k}^{\alpha_{i_k}}$.

Proposition 4.14. Let $S$ be a semigroup of I-type, let $v$ and $v_1$ be the left and right structures on $S$ as in 4.3. Then following conditions hold.

1. $(S, \lceil \cdot \rceil)$ is a distributive lattice. More precisely, any monomial $w \in S$ has a unique presentation as $w = x_1^{\alpha_1} \lor x_2^{\alpha_2} \lor \cdots \lor x_n^{\alpha_n}$, where $\alpha_i$ is a uniquely determined nonnegative integer for each $i, 1 \leq i \leq n$. In particular, for each $i$, with $\alpha_i \geq 1$, there is an equality $w = w_1x^{\alpha_i}_i$, where $w_1 \in S$, and $x_i$ does not occur as a tail of $w_1$.

2. The properties of the lattice $(S, \lceil \cdot \rceil)$ are analogous. In particular, every element $w \in S$ has a unique presentation as $w = x_1^{\beta_1} \lor x_2^{\beta_2} \lor \cdots \lor x_n^{\beta_n}$, where all $\beta_i$ are nonnegative integers. Moreover

3. The following are equalities in $S$:

$$W_0 = v_1(u_1u_2 \cdots u_n) = x_1 \lor x_2 \lor \cdots \lor x_n = x_1 \lor x_2 \lor \cdots \lor x_n = v(u_1u_2 \cdots u_n)$$

Proof. It is well known that $\mathcal{U}$ is a distributive lattice with respect to the order of divisibility, $a \mid b$. In particular, every element $a \in \mathcal{U}$ has a unique presentation $a = u_{i_1}^{k_1}u_{i_2}^{k_2} \cdots u_{i_n}^{k_n}$, where $k_1, \ldots, k_n$ are nonnegative integers, and $a = v_1^{k_1} \lor u_{i_2}^{k_2} \lor \cdots \lor u_{i_n}^{k_n}$ ($v \lor w$ denotes the least common multiple of $v, w$ in $\mathcal{U}$). Lemma 4.8 implies
5. Unions of solutions and matched pairs of groups

In this section we briefly recall some definitions and properties of unions of solutions. We also state a recent result from [15], in which matched pairs approach is used to describe extensions of solutions.

**Definition 5.1.** [6] Let \((X, r)\) be a solution. A subset \(Y \subseteq X\) is \(r\)-invariant, if \(r\) restricts to a bijection \(r_Y : Y \times Y \rightarrow Y \times Y\). \((X, r)\) is decomposable if it can be presented as a union of two non-empty disjoint \(r\)-invariant subsets. A solution \((Z, r)\) is a union of the solutions \((X, r_X)\) and \((Y, r_Y)\), if \(X \cap Y = \emptyset\), \(Z = X \cup Y\), as a set, and the bijection \(r\) extends \(r_X\), and \(r_Y\).

Clearly, \((Z, r)\) is a union of two nonempty \(r\)-invariant subsets, if and only if it is decomposable.

**Remark 5.2.** [6] Suppose the solution \((Z, r)\) is a union of \((X, r_X)\) and \((Y, r_Y)\). Then the map \(r\) induces bijections

\[
X \times Y \rightarrow Y \times X, \text{ and } Y \times X \rightarrow X \times Y.
\]

Note that a (disjoint) union \((Z, r)\) of two square-free solutions \((X, r_X)\), and \((Y, r_Y)\) is also a square-free solution. The cyclic condition implies then that for every \(z \in Z\), there is an equality \(R_z = L_z^{-1}\). Therefore the equality \(r(x, y) = (L_{z|Y}(y), L_{y|X}^{-1}(x))\) defines a left action of the groups \(G(X, r_X)\) on the set \(Y\) and a left action of the group \(G(Y, r_Y)\) on the set \(X\). Furthermore for every \(z \in Z\) there is an equality of permutations in \(Sym(Z)\): \(L_z = L_{z|X} L_{z|Y}\) The following lemma is straightforward.

**Lemma 5.3.** Let \((X, r)\) be a solution. Suppose \(X_1, X_2, \ldots, X_k\) are all disjoint orbits of the left action of \(G(X, r)\) on \(X\). Then \(r\) induces solutions \((X_i, r_i)\), \(1 \leq i \leq k\), where each \(r_i\) is the restriction of \(r\) on \(X_i \times X_i\). Furthermore, \(X\) is a disjoint union of \((X, r_i)\), \(1 \leq i \leq k\).

Clearly, \((X, r)\) is decomposable if and only if \(G(X, r)\) acts non-transitively on \(X\).

**Remark 5.4.** W. Rump [27] proved that every square-free solution \((X, r)\) is decomposable.

Therefore to understand the structure of a solution and also for constructing solutions it is essential to study extensions of solutions.

**Definition 5.5.** [6] Suppose \((X, r_X)\) and \((Y, r_Y)\) are (disjoint) solutions. The set of extensions of \(X\) by \(Y\), denoted by \(Ext(X, Y)\), is defined as the set of all decomposable solutions \(Z\) which are unions of \(X\) and \(Y\).

It is shown in [6], that given \((X, r_X)\), and \((Y, r_Y)\), an element \(Z\) of \(Ext(X, Y)\) is uniquely determined by the function \(r_{X,Y} : X \times Y \rightarrow Y \times X\).

The fact that every square-free solution \((Z, r)\) can be presented as a union of two disjoint solutions \((X, r_X)\), and \((Y, r_Y)\), where the bijective map \(r : Z \times Z \rightarrow Z \times Z\) extends the maps \(r_X\), and \(r_Y\), implies that the following theorem covers all known constructions of solutions restricted to the square-free case.
Theorem 5.6. [15] Let \((X, r_X)\) and \((Y, r_Y)\) be disjoint solutions, \(G_X = G(X, r_X)\), \(G_Y = G(Y, r_Y)\) be the groups associated with \((X, r_X)\), and \((Y, r_Y)\), respectively. Suppose that \(Z = X \cup Y\), and the bijective map \(r : Z \times Z \to Z \times Z\) is an extension of the maps \(r_X\) and \(r_Y\). Then \((Z, r)\) is a solution if and only if \((G_X, G_Y)\) is a matched pair of groups, in the sense of Majid [22]. Moreover \((Z, r)\) is square-free if and only if \((X, r_X)\) and \((Y, r_Y)\) are square-free solutions.

6. The equivalence of the notions square-free set-theoretic solution of the Yang-Baxter equation, semigroup of I type, and semigroup of skew-polynomial type

We keep all notation and conventions from the previous sections. As usual \((X, r)\) is a square-free solution, where \(X = \{x_1, \ldots, x_n\}\) is a finite set with \(n\) elements, \(S = S(X, r)\), \(G = G(X, r)\), and \(A = k\langle X; \mathcal{R}(r) \rangle\) are the associated Yang-Baxter semigroup, group and algebra over a field \(k\), defined in 2.12. In this section we prove Theorem 2.26.

For convenience of the reader, we first recall some basic algebraic and homological properties of \(S = S(X, r)\) and \(A = k\langle X, r \rangle\).

Theorem 6.1. [14] Let \(X\) be a finite set of \(n\) elements, \((X, r)\) be a square-free solution. Let \(S = S(X, r)\), \(G = G(X, r)\), and \(A = k\langle X; \mathcal{R}(r) \rangle\) be the associated Yang-Baxter semigroup, group, and algebra over a field \(k\), respectively. Then the following conditions hold.

1. The semigroup \(S\) is of I-type.
2. \(S\) is a semigroup with cancellation, and \(G(X, r)\) is its group of quotients.
3. \(S\) is Noetherian.
4. The algebra \(A\) is a Noetherian domain.
5. The Hilbert series of \(A\) is \(H_A(t) = \frac{1}{1 - t^n}\), the same as the Hilbert series of the commutative polynomial rings in \(n\) variables over \(k\).
6. \(A\) is Koszul.
7. \(A\) satisfies the Auslander condition.
8. \(A\) is Cohen-Macaulay.
9. \(A\) is Artin-Schelter regular ring of global dimension \(n\).
10. The Koszul dual \(A^!\) of \(A\) is a Frobenius algebra.
11. \[16\] \(A\) satisfies a polynomial identity. Moreover, \(S\) satisfies a semigroup identity.
12. \(A\) is catenary.

Sketch of the proof. For the definition of "Cohen-Macaulay" and the "Auslander condition" see [20]. Artin-Schelter regular rings are defined in [3]. Conditions 6.1.1 till 6.1.9 can be extracted from [14] (cf. [14], Theorems 1.3, 1.4)

Condition 6.1.11 follows from a more general result in [16]. It is proved (cf. [16], Theorem 3.1 and Corollary 3.2) that if a semigroup \(S\) has homogeneous defining relations, and the semigroup algebra \(k[S]\) is right Noetherian and has finite Gelfand-Kirillov dimension, then \(k[S]\) satisfies a polynomial identity, and \(S\) satisfies a semigroup identity.

Condition 6.1.12 follows from [28]. The Koszul dual algebra \(A^!\) is defined in [23]. Condition 6.1.10 follows from the fact that a Koszul algebra \(A\) of finite global dimension is Gorenstein if and only if \(A^!\) is Frobenius, cf [29], Proposition 5.10.

The following theorem proofs Conjecture 2.18
Theorem 6.2. Let \((X, r)\) be a square-free solution, where \(X\) is a finite set with \(n\) elements, \(n \geq 2\). Then there exists an ordering of \(X = \{x_1 < x_2 < \cdots < x_n\}\), such that the Yang-Baxter semigroup \(S(X, r)\) is of skew-polynomial type (with respect to this ordering), and the Yang-Baxter algebra \(A(k, X, r)\), over an arbitrary field \(k\) is a PBW algebra with a \(k\)-basis the set of ordered monomials:

\[
N_0 = \{x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0, 1 \leq i \leq n\}.
\]

Under the hypothesis of the theorem we first prove some lemmas.

Lemma 6.3. There exist an ordering on \(X, X = \{x_1 < x_2 < \cdots < x_n\}\), such that for any pair \(x, t \in X\) the following holds.

\[
(tx = x't') \in \mathbb{R}(X, r), \text{ and } (t > x) \implies (x' < t')
\]

**Proof.** We use induction on \(n = |X|\). Assume that the statement of the lemma is true for all solutions \((X, r)\), with \(|X| \leq n - 1\). It follows from a theorem of Rump, [27], that every square-free solution \((X, r)\), where \(X\) is a finite set, is decomposable into a disjoint union \(X = Y \cup Z\) of two nonempty \(r\)-invariant subsets \(Y, Z\). Suppose \(|Y| = k, |Z| = m, k + m = n\). Let \(r_Y\) and \(r_Z\) be the restrictions on \(r\) on \(Y^2\) and \(Z^2\), respectively. It follows from the inductive assumption that there exist orderings \(Y = \{y_1 < \cdots < y_k\}\), and \(Z = \{z_1 < \cdots < z_m\}\), which satisfy condition 6.1. We set: \(y_1 < \cdots < y_k < z_1 < \cdots < z_m\) and verify that this is an ordering on \(X\), which satisfies 6.1. Assume

\[
tx = x't' \in \mathbb{R}(X, r), \text{ and } t > x.
\]

We have to show that \(x' < t'\). Clearly if \(t, x \in Y\), or \(t, x \in Z\), then by the inductive assumption and by the choice of the ordering \(<\), condition 6.1 is satisfied. Assume now \(x \in Y\), and \(t \in Z\). (Note that the case \(t \in Y, x \in Z\) is impossible since we assume \(t > x\)). The sets \(Y\), and \(Z\), are \(r\)-invariant, therefore by 5.2 \(r\) induces a map \(Z \times Y \to Y \times Z\). In particular \(tx = x't' \in \mathbb{R}(X, r)\), and \(t \in Z, x \in Y\), imply that \(x' \in Y, t' \in Z\). Hence, by the choice of \(<\), there is an inequality \(x < t'\), which proves 6.1.

Lemma 6.4. Suppose condition 6.1 holds. Let \(x, t \in X\), and let \(L^x_t = (x_1, \cdots, x_k), L^x_r = (t_1, \cdots, t_m)\) be their associated disjoint cycles, see 3.1. Then \(t_i > x_i\) implies \(t_j > x_i\), for all \(i, j, 1 \leq i \leq k, 1 \leq j \leq m\).

**Proof.** Using induction on \(i\), we first show that

\[
(6.3) \quad t_1 > x_i, 1 \leq i \leq k.
\]

By hypothesis \(t_1 > x_1\), which gives the base for the induction. Assume

\[
(6.4) \quad t_1 > x_s, \text{ for } 1 \leq s \leq i - 1.
\]

We claim \(t_1 > x_i\). Assume the contrary,

\[
(6.5) \quad t_1 < x_i.
\]

Note that \(t_1 = x_i\) is impossible, since the cycles \(L^x_t\) and \(L^x_r\) are disjoint. By the cyclic condition, 3.1 one has:

\[
(6.6) \quad t_1x_{i-1} = x_it_m,
\]

and

\[
(6.7) \quad t_1x_i = \begin{cases} x_{i+1}t_m & \text{if } i < k \\ x_1t_m & \text{if } i = k \end{cases}
\]
In the case when \( i = k \), we obtain immediately a contradiction with 6.1, since
\[
(6.8) \quad t_1 x_k = x_1 t_m, \text{ and } x_1 < t_1 < x_k < t_m.
\]
Assume now \( i < k \). Then 6.1, 6.7 and the assumption 6.5, imply
\[
(6.9) \quad x_{i+1} > t_m.
\]
At the same time, the equality 6.6 and 6.4 give
\[
(6.10) \quad t_m > x_i.
\]
We have obtained:
\[
(6.11) \quad x_{i+1} > t_m > x_i > t_1 > x_1.
\]
Induction on \( j \) and analogous argument show, that for \( 1 \leq j \leq k - i \), the following inequalities hold:
\[
(6.12) \quad x_{i+j} > t_m > x_i > t_1 > x_1.
\]
In particular,
\[
(6.13) \quad x_k > t_m > x_i > t_1 > x_1.
\]
Now the equality \( t_1 x_k = x_1 t_m \) together with 6.13 give a contradiction with 6.1. We have shown that
\[
(6.14) \quad t_1 > x_i, \text{ for all } i, 1 \leq i \leq k.
\]
Induction on \( j \) and analogous argument show that
\[
(6.15) \quad t_j > x_i, \text{ for all } i, 1 \leq i \leq k.
\]
This proves the lemma. \( \square \)

**Lemma 6.5.** Let \((X, r)\) be a square-free solution, with an ordering \(<\) on \(X\) which satisfies 6.1, \( S = S(X, r)\) be the associated Yang-Baxter semigroup. Then the following two conditions are satisfied:

1. (6.16) \( tx = x't' \in R(X, r), \text{ and } (t > x) \implies (x' < t'), \text{ and } (t > x') \).

2. The relations \( R(X, r) \) form a Groebner basis, with respect to the degree-lexicographic ordering in the free semigroup \( \langle X \rangle \), induced by \(<\), or equivalently the monomials \( txu \), where \( t, x, u \in X \) and \( t > x > u \) do not give rise to new relations in \( S(X, r) \).

**Proof.** Condition 6.16 follows immediately from Lemma 6.4. Therefore the set of defining relations \( R = R(X, r) \) for the Yang-Baxter semigroup \( S(X, r) \) satisfies the following
\[
(6.17) \quad (x_j x_i = x_i' x_j') \in R \text{ and } (j > i) \implies (i' < j'), \text{ and } (j > i') \).
\]
We have to show that \( R \) is Groebner basis. It follows from the theory of Groebner bases, that each monomial \( u \in \langle X \rangle \) has a unique normal form, denoted by \( \text{Nor}(u) \), with respect to the so called reduced Groebner basis, \( R_0 \) which is uniquely determined by the set \( R \) and, \( R \subseteq R_0 \). As a set \( S \) can be identified with the set of normal monomials
\[
(6.18) \quad \mathcal{N}(S) = \{ \text{Nor}(u) \mid u \in \langle X \rangle \}.
\]
Knowing the normal monomials one can uniquely restore the set of obstructions, i.e. the set of highest monomials in the reduced Groebner basis, \( R_0 \). To verify
the equality $\mathbb{R} = \mathbb{R}_0$, therefore $\mathbb{R}$ is a Groebner basis, it will be enough to show that the "ambiguities" $x_k x_j x_i$, where $n \geq k > j > i \geq 1$, do not give rise to new relations in $S(X, r)$, or equivalently, that each monomial of the shape $x_i x_j x_k$, with $1 \leq i \leq j \leq k \leq n$ is normal, with respect to $\mathbb{R}_0$. This will follow immediately from a stronger statement:

**Lemma 6.6.** Each ordered monomial $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $a_i \geq 0, 1 \leq i \leq n$ is in normal form with respect to the reduced Groebner basis $\mathbb{R}_0$ in $\langle X \rangle$.

**Proof.** Each relations in $\mathbb{R}$ satisfies 6.17, so its highest monomial is $x_j x_i$, with $j > i$, therefore the normal form $\text{Nor}(u)$ of each $u \in \langle X \rangle$ does not contain $x_j x_i, j > i$ as a sub word. This shows that

\begin{equation}
S = \mathcal{N}(S) \subseteq \mathcal{N}_0, \tag{6.19}
\end{equation}

where $\mathcal{N}_0$ is the set of ordered monomials $\mathcal{N}_0 = \{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} | a_i \geq 0, 1 \leq i \leq n\}$.

The existence of the $I$-structure $v$ on $S(X, r)$ (by definition $v : \mathcal{U} \rightarrow S$ is a bijection) implies the equality $\mathcal{N}(S) = \mathcal{N}_0$.

We have proved 6.5.

**Proof of the theorem.** The theorem follows from Lemma 6.5. Note that the Diamond Lemma 2.2 implies that the Yang-Baxter algebra $\mathcal{A} = \mathcal{A}(k, X, r)$ is PBW in the sense of Priddy [25], and the set of ordered monomials $\mathcal{N}_0$ projects to a $k$-basis of $\mathcal{A}$ (as a $k$- vector space).

**Proof of theorem A.** The equivalence of 2.26.1 and 2.26.2 follow from [14], Theorem 1.4. The implications $2.26.1 \Rightarrow 2.26.3$, and $2.26.1 \Rightarrow 2.26.4$ follow from theorem 6.2. Clearly, the theory of Groebner basis implies the equivalence of conditions 2.26.3 and 2.26.4. Theorem 1.2, [14], proves the implication $2.26.3 \Rightarrow 2.26.1$.

7. More about $S(X, r)$ and $G(X, r)$

In this section, as usual $(X, r)$ denotes a square-free solution, where $X$ is a finite set of $n$ elements. We show that $G = G(X, r)$ acts by conjugation on the set $X^M = \{x_1^M, \cdots, x_n^M\}$, where $M = M(X, r)$ is the cyclic degree of $(X, r)$ defined in 3.17. We compare this action with the left action of $G(X, r)$ on the set $X$. Next we prove that $G(X, r)$ contains a free abelian subgroup $A$ of index $M^n$, and prove that the quotient group $\overline{G} = G/A$ can be presented as a product of its Sylow subgroups (cf. 7.10). This implies a presentation of the group $G(X, r)$ as a product of its Sylow subgroups. As a corollary we obtain a result of Etingof-Schedler-Solovyev, [6], that the group $G(X, r)$ is solvable.

**Notation 7.1.** For any positive integer $k$ we set $X^{(k)} = \{x_1^k, \cdots, x_n^k\}$. By $S^k = \langle X^{(k)} \rangle$ we denote the submonoid of $S = S(X, r)$ generated by $X^{(k)}$. If $A, B \subseteq S$, then as usual, $AB$ denotes the set of all elements $u$ of the form $u = ab$, with $a \in A, b \in B$.

**Proposition 7.2.** Let $k$ be a positive integer, $X^{(k)}$ and $S^k$ as in 7.1. Then the following conditions hold.

1. The map $r$ induces a map $r_k : X^{(k)} \times X^{(k)} \rightarrow X^{(k)} \times X^{(k)}$ such that $(X^{(k)}, r_k)$ is a square-free solution.
2. $S^k$ is of $I$-type.
Remark 7.6. It is a routine fact, that the order of each orbit \( O(x^M) \), \( x \in X \) is a
divisor of the order \( M^n \) of \( G \), see for example [2], 6.1.

Corollary 7.3. Suppose \((X, r)\) is a square-free solution. Then

\begin{enumerate}
\item \( S(X, r) \) contains the free abelian semigroup \( [x_1^M, \ldots, x_n^M] = S^M \).
\item \( S(X, r) \) is left and right Noetherian.
\item The group \( A = \text{gr} [x_1^M, \ldots, x_n^M] \) is a free abelian normal subgroup of \( G \) of index \( M^n \).
\item The group \( G = G(X, r) \) acts by conjugation on the set \( X^M \). Moreover the action of \( A \) on \( X^M \) is trivial, thus the quotient group \( \overline{G} = G/A \) acts on \( X^M \) by conjugation. Clearly, \( \overline{G} \) is a finite group of order \( M^n \).
\item The group \( A \) is contained in the kernel \( \ker \mathcal{L} \) of the homomorphism \( \mathcal{L} : G \rightarrow \text{Sym}(X) \). Therefore there exists an epimorphism \( \overline{\mathcal{L}} \circ \nu \rightarrow \overline{G} \), induce
ded by \( \mathcal{L} \), satisfying the equality: \( \mathcal{L} = \overline{\mathcal{L}} \circ \nu \), where \( \nu \) is the natural epimorphism \( \nu : G \rightarrow \overline{G} \).
\item The order of \( \overline{G} \) divides \( M^n \).
\end{enumerate}

Notation 7.4. For every \( y \in X \) we denote by \( O(y^M) \) the orbit of \( y^M \) under
the action if \( G \) on \( X^M \). For \( x, y \in X \) we define an equivalence on \( X \) by setting \( x \equiv y \)
iff \( O(x^M) = O(y^M) \). By \( Y(y) \) we denote the equivalence class of \( y, y \in X \).

The lemma below follows straightforward from the definition of the actions of \( G \)
on the sets \( X \) and \( X^M \), and from Proposition 3.21.

Lemma 7.5. The following conditions hold.

\begin{enumerate}
\item There exists a one-to-one correspondence between the \( G \)-orbits of \( X^M \) and
the \( G \)-orbits of \( X \). More precisely for every \( \xi \in X \), there are equalities
\( O_G(\xi) = X(\xi) = \{x \in X \mid x^M \in O(\xi^M)\} \). Furthermore, the orbits \( O_G(\xi) \)
can be obtained simply by acting with the "semigroup" elements of \( G \): e.g.
y \in O_G(x) if and only if, there exist monomials \( a, b \in S, a = a_1 \cdots a_k \), and
\( b = b_1 \cdots b_k \) \( (a_i, b_i \in X) \) and elements \( y_1, \ldots, y_k \in X \), such that there are
equalities:
\( a_k x = y_k b_k, a_{k-1} y_k = y_{k-1} b_{k-1}, \cdots, a_1 y_1 = y b_1 \).
\item If \( x \in X(a) \) and \( y \in X(b) \), for some \( a, b \in X \) (not necessarily \( a \neq b \)) then
there is an equality \( xy = y' x' \), with \( y' \in X(b), x' \in X(a) \).
\item Each orbit \( O_G(\xi), \xi \in X \) is \( r \)- invariant.
\item \( X \) is \( r \)- decomposable if and only if \( \overline{G} \) does not act transitively on \( X^M \). More
precisely, if \( O_G(\xi_i^M), 1 \leq i \leq k \) are all disjoint orbits of this action then
\( X \) splits into a disjoint union of \( k \) nonempty \( r \)-invariant subsets: \( X_1 = O_G(\xi_1), \cdots, X_k = O_G(\xi_k) \).
\end{enumerate}
A sufficient condition for $r$-decomposability of $X$ follows immediately from 7.6. As a corollary we obtain a result from [6], that every solution $(X, r)$, where $X$ is of prime order $p$ is decomposable.

**Corollary 7.7.** If $M$ is not divisible by some prime divisor $p$ of $n$, then the action of $G$ (and of $G$) on $X^{(M)}$ is not transitive and $X$ is a disjoint union of $k$ $r$-invariant subsets, where $k \geq 2$ is the number of orbits in $X^{(M)}$.

**Corollary 7.8.** [6] If $n = p$ is a prime number, then $X$ is a disjoint union of two nonempty $r$-invariant subsets.

Next we study the relations between the cyclic degree $M = M(X, r)$, the Sylow subgroups of $\overline{G}$ and the cyclic properties of the semigroup $S(X, r)$. Note that

**Notation 7.9.** Let $M = M(X, r)$ be the cyclic degree of the solution $(X, r)$ defined in 3.17. Suppose $M = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $p_1, \cdots, p_k$ are distinct prime numbers, and $\alpha_1 \cdots \alpha_k$ are positive integers. For $i = 1, \cdots, k$, we set

$$q_i = M/p_i^{\alpha_i},$$

$$S^{q_i} = \langle x_1^{q_i}, \cdots, x_n^{q_i} \rangle,$$

the sub-monoid of $S$ generated by $x_1^{q_i}, \cdots, x_n^{q_i}$, $1 \leq i \leq k$. We denote by $\overline{S^{q_i}}$ the natural image of $S^{q_i}$ in the quotient group $\overline{G}$, and by $\mathcal{L}(S^{q_i})$ the image of $S^{q_i}$ under the homomorphism $\mathcal{L} : G \to \mathcal{G}_L \subset S\text{ym}(X)$, defined by the left action of $G$ on $X$.

Clearly, the integers $q_1, \cdots, q_k$ are pairwise coprime, and $\overline{S^{q_i}}$ are submonoids of $\overline{G}$.

The next theorem gives a presentation of $\overline{G}$ as a product of its Sylow subgroups. Surprisingly it also allows to consider each element of $\overline{G}$ as an element of the monoid $\overline{S}$.

**Theorem 7.10.** The following conditions hold.

1. For every $i$, $1 \leq i \leq r$, the submonoid $\overline{S^{q_i}}$ is a subgroup of order $p_i^{\alpha_i}$ in $\overline{G}$. In particular, it is a Sylow $p_i$-subgroup of $\overline{G}$.
2. For every pair $q_i, q_j$, $1 \leq i, j \leq r$, there is an equality $\overline{S^{q_i}S^{q_j}} = \overline{S^{q_j}S^{q_i}}$.
3. The group $\overline{G}$ is a product of its Sylow subgroups: $\overline{G} = \overline{S^{q_1}} \cdots \overline{S^{q_k}}$. In particular, $\overline{G} = \overline{S}$.
4. For each $i, 1 \leq i \leq k$, such that $\mathcal{L}(S^{q_i}) \neq id_X$, $\mathcal{L}(S^{q_i})$ is a $p_i$-Sylow subgroup of $\mathcal{G}_L$.
5. Let $1 \leq i_1, \cdots, i_s \leq k$ be all indices, for which $\mathcal{L}(S^{q_{i_j}}) \neq \{id_X\}, 1 \leq j \leq s$.
   Then the group $\mathcal{G}_L = \mathcal{G}_L(X, r)$ is a product of its Sylow subgroups:

   $$\mathcal{G}_L = \mathcal{L}(S^{q_{i_1}}) \cdots \mathcal{L}(S^{q_{i_s}}).$$

   In particular, $\mathcal{G}_L = \mathcal{L}(S)$.
6. The groups $\mathcal{G}_L$, $\overline{G}$, and $G$ are solvable.

**Proof.** Consider $\overline{S^{q_i}}$, where $1 \leq i \leq k$. Note first that as a finite submonoid of the group $\overline{G}$, $\overline{S^{q_i}}$ is a subgroup of $\overline{G}$. We claim that the order of $\overline{S^{q_i}}$ is exactly $p_i^{\alpha_i}$. The equalities 3.29 imply that every element $w$ of $\overline{S^{q_i}}$ can be presented as

$$w = v((u_1^{q_i})^{\beta_1} \cdots (u_n^{q_i})^{\beta_n}),$$

where $0 \leq \beta_s \leq p_i^{\alpha_i}$ for all $s$, $1 \leq s \leq n$.  

(7.2)
We set \( \beta = (\beta_1, \beta_2, \cdots, \beta_n) \), and \( w = w(\beta) \), for the monomial \( w \) determined by 7.2. It follows from the properties of the I-structure \( v \) on \( S \) and from 3.29 that each inequality \( \beta' \neq \beta'' \) implies an inequality in \( S \):

\[
(7.3) \quad w(\beta') \neq w(\beta'').
\]

This implies that \( S^{\beta'} \) is a group of order \( (p_1^{\alpha_1})^n \) thus a Sylow \( p_i \) subgroup of \( S \), which proves 1.

Next we recall that for every pair of integers \( i, j, 1 \leq i, j \leq k \), and for every pair \( x, y \in X \) there exist \( z, t \in X \), such that the equality

\[
(7.4) \quad x^by^q = z^st^q
\]

holds in \( S \). This implies that \( S^{\beta'} = S^{\beta''} \) for all \( i, j \), which verifies 2. Let \( S' = \langle S^n, \cdots, S^{qk} \rangle \) be the submonoid of \( S \), generated by \( S^n, \cdots, S^{qk} \). It follows from 7.4 that there is an equality

\[
(7.5) \quad S' = S^n \cdots S^{qk}.
\]

Hence

\[
(7.6) \quad S = S^n \cdots S^{qk}.
\]

is a presentation of \( S' \) as a product of subgroups with pairwise co-prime orders: \( p_1^{\alpha_1}, \cdots, p_k^{\alpha_k} \), respectively. It follows then that the order of \( S' \) is exactly \( p_1^{\alpha_1} \cdots p_k^{\alpha_k} = M^n \), thus \( G = S^n \cdots S^{qk} \). This proves 3. The proof of 4, and 5 is routine. \( \square \)

Note that, in general the Sylow subgroups \( S^{\beta'k} \) might not be normal subgroups of \( \overline{G} \), as shows the following example.

**Example 7.11.** Let \( S = \langle X; \mathcal{R} \rangle \), where \( X = \{x_i \mid 1 \leq i \leq 6\} \cup \{y_j \mid 1 \leq j \leq 4\} \) and the relations \( \mathcal{R} \) are defined by the permutation

\[
(7.7) \quad \sigma = (x_1x_2x_3x_4x_5x_6)(y_1y_2y_3y_4);
\]

as follows:

\[
(7.8) \quad y_jx_i = \sigma(x_i)\sigma^{-1}(y_j), \text{ and } x_ijy = \sigma_1(y_j)\sigma^{-1}(x_i) \text{ for } 1 \leq i \leq 6; \ 1 \leq j \leq 4;
\]

\[
(7.9) \quad x_ix_k = \sigma^3(x_k)\sigma^{-3}(x_i), \text{ for all } i \neq k \text{ (mod 3)}, \ 1 \leq i, k \leq 6;
\]

\[
(7.10) \quad x_ix_k = x_kx_i, \text{ for all } i = k \text{ (mod 3)}, \ 1 \leq i, k \leq 6
\]

\[
(7.11) \quad y_jy_k = \sigma^2(y_k)\sigma^{-2}(y_j), \text{ for all } j \neq k \text{ (mod 2)}, \ 1 \leq j, k \leq 4.
\]

\[
(7.12) \quad y_jy_k = y_ky_j, \text{ for all } j = k \text{ (mod 2)}, \ 1 \leq j, k \leq 4.
\]

It is easy to verify that the set of relations \( \mathcal{R} \) defines naturally a square-free solution, \( r \), thus \( S \) is an YB semigroup. The set of all lengths of cycles is \( 6, 4, 2 \), thus \( M = 12 = 2^2 \cdot 3 \), and (in the notation 7.9), \( q_1 = 3 \), and \( q_2 = 4 \). Thus, by Theorem 7.10, \( \overline{G} = S^n \overline{S} \). Note that none of the subgroups \( S^n, \overline{S} \) is normal in \( \overline{G} \).

One can use Theorem 7.10 to give a straightforward proof of the \( r \)- decomposability of \( (X, r) \) in all cases when the cycles are not enough "dense" on \( X \). More precisely, the following corollary is true.

**Corollary 7.12.** Suppose that there exists a prime divisor \( p \) of \( n \), and an \( x \in X \), such that \( x \) does not belong to a cycle of length divisible by \( p \). Then the action of \( \overline{G} \) on \( X \) is non-transitive, therefore \( (X, r) \) is decomposable.
8. Multipermutation solutions and generalized twisted unions

We give a description of the generalized twisted unions of solutions $Z = X \cup Y$, showing that the group $G_Y = G(Y, r_Y)$ acts as automorphisms on $X$, and all the elements $\xi$ of an orbit $O(x) = O_G(x)$ have the same action on $Y$ see 8.3. Lemma 8.10 generalizes the cyclic condition. We give a conjecture that every multipermutation solution of level $m$ is a generalized twisted union of multipermutation solutions of level $\leq m - 1$. We keep the notation and conventions from the previous sections. In particular, to avoid complicated expressions sometimes we shall use both notation $xy = L_x(y)$ and $y^x = R_x(y)$.

**Definition 8.1.** [6] Let $(Z, r)$, be a disjoint union of the solutions $(X, r_X)$, and $(Y, r_Y)$.

1. $(Z, r)$ is called a twisted union of $X$ and $Y$ if the maps $r_{XY} : X \times Y \to Y \times X$ and $r_{YX} : Y \times X \to X \times Y$ are defined as

   \[
   r_{XY}(x, y) = (g(y), f^{-1}(x))
   \]

   and

   \[
   r_{YX}(y, x) = (f(x), g^{-1}(y)),
   \]

   where $f \in \text{Sym}(X)$, and $g \in \text{Sym}(Y)$ are fixed.

2. $(Z, r)$ is a generalized twisted union of $X$ and $Y$ if the map $r$ is determined by the formula:

   \[
   r_{XY}(x, y) = (L_x|_Y(y), R_{y|X}(x)),
   \]

   where the permutations $L_x|_Y \in \text{Sym}(Y)$, and $R_{y|X} \in \text{Sym}(X)$ satisfy the following condition:

   (*): For every $y \in Y$ the permutation $L_x|_Y : Y \to Y$ is independent of $y$, and for every $x \in X$, the permutation $R_{y|X} : X \to X$ is independent of $x$.

**Notation 8.2.** When the element $\xi \in Z$ is specified we shall simply write, as usual, $L_x(\xi)$, or $x^\xi$ instead of $L_x|_Y(\xi)$, respectively $L_y(\xi), y^\xi$ instead of $L_y|_X(\xi)$.

**Proposition 8.3.** Let $(Z, r)$ be union of the disjoint solutions $(X, r_X)$, and $(Y, r_Y)$. Then $(Z, r)$ is a generalized twisted union of $X$ and $Y$ if and only if for every pair $x, y, x \in X, y \in Y$ the following equalities hold:

   \[
   L_{x^y|Y} = L_{x^y|Y} = L_{x^y|Y};
   \]

   \[
   L_{y|x|X} = L_{y|x|X} = L_{y|x|X};
   \]

**Proof.** Note first that the equalities 8.4 and 8.5 imply that $(Z, r)$ is a generalized twisted union of $X$ and $Y$. Assume now that $(Z, r)$ is a generalized twisted union of $X$ and $Y$. Let $x \in X$, $y \in Y$. We have to show that for every $z \in Y$ there is an equality

   \[
   L_x(z) = L_x(z).
   \]

By definition 8.1 the map $L_{x^y|Y} : Y \to Y$ is independent of $y \in Y$. Hence for every pair $y, z \in Y$ there is an equality

   \[
   L_{x^y}(z) = L_{x^y}(z)
   \]
By the cyclic condition in \((Z, r)\), see 3.25, one has:

\begin{equation}
\mathcal{L}_{x^*}(z) = \mathcal{L}_x(z).
\end{equation}

Now the equations 8.7 and 8.8 imply

\begin{equation}
\mathcal{L}_{x^y}(z) = \mathcal{L}_x(z)
\end{equation}

for every \(z \in Y\). We have shown that

\begin{equation}
\mathcal{L}_{x^y|Y} = \mathcal{L}_{x|Y}
\end{equation}

for arbitrary \(x \in X\) and \(y \in Y\). We apply this to the pair \(y^y x \in X\) and \(y \in Y\) and obtain

\begin{equation}
\mathcal{L}_{y y^y|Y} = \mathcal{L}_{y x|Y}.
\end{equation}

By 3.25 there is an equality:

\begin{equation}
(y^y x)^y = x,
\end{equation}

which together with 8.11, 8.10 implies \(\mathcal{L}_{y x|Y} = \mathcal{L}_{x^y|Y} = \mathcal{L}_{x^y|Y}\).

This completes the proof of 8.4. Analogous argument proves 8.5.

**Theorem 8.4.** Let \((Z, r)\) be a generalized twisted union of the solutions \((X, r_X)\) and \((Y, r_Y)\), and let \(G_X = G(X, r_X)\), \(G_Y = G(Y, r_Y)\) be the associated Yang-Baxter groups. Suppose \(O_{G_Y}(\xi_1), \ldots, O_{G_Y}(\xi_p)\) are the (distinct) orbits of the action of the group \(G_Y\) on \(X\), and \(O_{G_X}(\eta_1), \ldots, O_{G_X}(\eta_q)\) are the (distinct) orbits of the action of \(G_X\) on \(Y\). Then the following conditions are satisfied.

1. The assignment \(x \mapsto \mathcal{L}_{x|Y}\), for all \(x \in X\) extends to a group homomorphism

\[
L_X : G(X, r_X) \rightarrow \text{Aut}(Y, r_Y)
\]

2. Let \(H_X\) denotes the kernel \(\text{Ker} L_X\). Then each orbit \(O_{G_Y}(\xi_i)\), is contained in the left coset \(\xi_i H_X\), i.e. \(O_{G_Y}(\xi_i) \subseteq \xi_i H_X\). In particular, for every \(x \in O_{G_Y}(\xi_i)\), \(1 \leq i \leq p\) there is an equality

\begin{equation}
\mathcal{L}_{x|Y} = \mathcal{L}_{\xi_i|Y}.
\end{equation}

3. The assignment \(y \mapsto \mathcal{L}_{y|x}\), for all \(y \in Y\) extends to a group homomorphism

\[
L_Y : G(Y, r_Y) \rightarrow \text{Aut}(X, r_X).
\]

4. Let \(H_Y\) denotes the kernel \(\text{Ker} L_Y\). Then \(O_{G_X}(\eta_j) \subseteq \eta_j H_Y\), for \(1 \leq j \leq q\). In particular, for every \(y \in O_{G_X}(\eta_j)\), there is an equality:

\begin{equation}
\mathcal{L}_{y|x} = \mathcal{L}_{\eta_j|x}.
\end{equation}

**Definition 8.5.** [6] Let \((X, r)\) be a square-free solution. Define an equivalence relation on \(X\) as \(x \sim y\) iff \(\mathcal{L}_x = \mathcal{L}_y\).

Clearly, since \(\mathcal{R}_x = \mathcal{L}_x^{-1}\), one has also \(x \sim y\) iff \(\mathcal{R}_x = \mathcal{R}_y\). Let \(X^\sim = X/\sim\). It is known, see [6], that the solution \(r : X \times X \rightarrow X \times X\) induces a bijection \(r^\sim : X^\sim \times X^\sim \rightarrow X^\sim \times X^\sim\), so that \((X^\sim, r^\sim)\) is a solution. It is not difficult to see that this solution is also square-free. The solution \((X^\sim, r^\sim)\) is called the retraction of \((X, r)\) and is denoted by \(\text{Ret}(X, r)\). The solution is retractible if \(\sim\) is a nontrivial
Lemma 8.6. For any \( x, y \in X \) the equivalence \( x \sim y \) implies \( xy = yx \).

Definition 8.7. Inductively, for \( 1 < k \) we define the retractions of higher level as \( \Ret^k(X, r) = \Ret(\Ret^{k-1}(X, r)) \).

We denote by \( x^{(k)} \) the image of \( x \) in \( \Ret^k(X, r) \). The set
\[
[x^{(k)}] := \{ \xi \in X \mid \xi^{(k)} = x^{(k)} \}
\]
is called the \( k \)th retract orbit of \( x \).

Definition 8.8. [6], A solution \((X, r)\) is called multipermutation solution of level \( m \) if \( m \) is the minimal nonnegative integer, such that \( \Ret^m(X, r) \) is finite of order 1.

Lemma 8.9. For any positive integer \( k \), and any \( x \in X \) the \( k \)th retract orbit \([x^{(k)}]\) is \( r \)-invariant. Furthermore, if we denote by \( r_{x,k} \) the corresponding solution induced by \( r \), then \( \langle x^{(k)} \rangle, r_{x,k} \) is a multipermutation solution of level \( k \).

Lemma 8.10. Let \((X, r)\) be a square-free solution. Then the following conditions hold:

(1) For every \( x, y, t \in X \), and \( k \) a positive integer,
\[
y^{(k)} = t^{(k)} \implies (y)x^{(k-1)} = (t)x^{(k-1)}.\]
(8.16)

(2) For every \( x, y, t \in X \)
\[
y^{(2)} = t^{(2)} \implies yx \sim tx, \text{ in particular, } yt \sim t, \text{ and } ty \sim y.\]
(8.17)

Proof. We first prove 1. By hypothesis, \( y^{(k)} = t^{(k)} \), or equivalently
\[
y^{(k-1)} \sim t^{(k-1)}.\]
(8.18)

Let \( x \in X \). Clearly,
\[
yx = \xi y_1, tx = \xi_1 t_1, \text{ for some } \xi, \xi_1, y_1, t_1 \in X.
\]
(8.19)

This implies the following equalities in \( \Ret^{k-1}(X, r) \)
\[
y^{(k-1)}x^{(k-1)} = \xi^{(k-1)}y_1^{(k-1)}, \text{ and } t^{(k-1)}x^{(k-1)} = \xi_1^{(k-1)}t_1^{(k-1)}.
\]
(8.20)

It follows then from 8.18 that
\[
\xi^{(k-1)} = \xi_1^{(k-1)}, \text{ or equivalently, } \xi^{(k-2)} \sim \xi_1^{(k-2)}.
\]
(8.21)

By 8.19, one has \( \xi = yx \), and \( \xi_1 = tx \), which proves 1. Condition 2 follows straightforward from 1, with \( k = 2 \), and the cyclic condition.

Proof. Clear.

Corollary 8.11. Let \((X, r)\) be a multipermutation solution of level \( m \), \( G_X = G(X, r) \) be the associated Yang-Baxter group. Then for every \( y \in X \) one has:
\[
O_{G_X}(y) \subseteq [y^{(m-1)}],
\]
where \( O_{G_X}(y) \) is the \( G_X \) orbit of \( y \) in \( X \), and \([y^{(m-1)}]\) is the \((m-1)\)th retract orbit of \( y \). In particular, \( O_{G_X}(y) \) is a multipermutation solution of level at most \( m - 1 \).

The cyclic condition, \( (xy)(y) = x^y \) is extended to the class \([y^{(2)}]\) by the following lemma.

Lemma 8.12. Let \((X, r)\) be a solution. Then the following conditions hold.
(1) For every \( x \in X \), and \( z \in [y^{(2)}] \) there is an equality
\[
(x^z)(z) = xz.
\]

and
\[
\mathcal{L}_{x|y^{(2)}} = \mathcal{L}_{x|y^{(2)}}.
\]

(2) Suppose that \([x^{(2)}] \neq [y^{(2)}]\), and the set \([y^{(2)}]\) is invariant under the left action of \( G([x^{(2)}], r_{x, 2}) \), respectively, \([x^{(2)}]\) is invariant under the left action of \( G([y^{(2)}], r_{y, 2}) \). Then the disjoint union \( Z = [x^{(2)}] \cup [y^{(2)}] \) is a generalized twisted union of \([x^{(2)}]\) and \([y^{(2)}]\). Moreover, \((Z, r_Z)\) is a multipermutation solution of level 3, where \( r_Z \) is the restriction of \( r \) on \( Z \times Z \).

Proof. Let \( x \in X \), and let \( z \in y^{(2)} \). We will show that 8.22 holds. It follows from 8.17 that
\[
y_x \sim x z.
\]
So, by the definition of \( \sim \), and by the cyclic condition,
\[
(x^z)(z) = (x^z)(z) = xz.
\]

We have shown 8.22. Clearly, 8.22 implies 8.23. Condition 2 follows easily from 1. \( \square \)

Corollary 8.13. Let \((X, r)\) be a multipermutation solution of level 3. Then \((X, r)\) is a generalized twisted union of multipermutation solutions of level \( \leq 2 \).

Example 8.14. Let \( X = \{x, x_1, \xi, t_1, t, \eta, \eta_1, y, y_1\} \) and let \( r \) be determined via
\[
\mathcal{L}_x = \mathcal{L}_{x_1} = (tt_1)(\eta\eta_1)(yy_1);
\]
\[
\mathcal{L}_{\xi} = \mathcal{L}_{t_1} = (\eta\eta_1)(tt_1)(yy_1);
\]
\[
\mathcal{L}_t = \mathcal{L}_{t_1} = \mathcal{L}_\eta = \mathcal{L}_{\eta_1} = \text{id}_X
\]
\[
\mathcal{L}_y = \mathcal{L}_{y_1} = (x\xi)(x_1\xi_1).
\]
Then \( Ret(X, r) = (X^\sim, r^\sim) \), where \( X^\sim = \{x^\sim, \xi^\sim, t^\sim, y^\sim\} \), and \( r^\sim \) is determined by \( \mathcal{L}_y^\sim = (x^\sim \xi^\sim), \mathcal{L}_x^\sim = \mathcal{L}_{\xi^\sim} = \mathcal{L}_{t^\sim} = \text{id}_X^\sim \). Clearly, \( X^{(2)} = \{y^{(2)}, x^{(2)}\} \), and \( Ret^2(X, r) \) is the trivial solution, therefore \( Ret^3(X, r) = 1 \). In this case \((X, r)\) is a multipermutation solution of level 3.

9. Binomial solutions of the classical Yang-Baxter equation

In this section we study a particular class of solutions of the classical Yang-Baxter equation, called binomial solutions. We show that there is a close relation between a class of Artin-Schelter regular rings, which we call skew-polynomial rings with binomial relations and the square-free binomial solutions of the classical Yang-Baxter equation.

Definition 9.1. Let \( V \) be a finite dimensional vector space over a field \( k \) with a \( k \)-basis \( X = \{x_1, \cdots, x_n\} \). Suppose the linear automorphism \( R : V \otimes V \rightarrow V \otimes V \) is a solution of the Yang-Baxter equation. We say that \( R \) is a binomial solution of the (classical) Yang-Baxter equation or shortly binomial solution if the following conditions hold.
(1) for every pair $i \neq j, 1 \leq i, j \leq n$,

$$R(x_j \otimes x_i) = c_{ij}x_i \otimes x_j, R(x_i \otimes x_j) = \frac{1}{c_{ij}}x_jx_i,$$

where $c_{ij} \in k, c_{ij} \neq 0$.

(2) $R$ is non-degenerate, that is the associated set-theoretic solution $(X, R(R))$, where $r = r(R) : X \times X \to X \times X$ is defined as

$$r(x_j, x_i) = (x_i, x_j')$$

is non-degenerate.

We call the binomial solution $R$ square-free if $R(x_i \otimes x_i) = x_i \otimes x_i$, or equivalently, $(X, r)$ is square-free.

**Notation 9.2.** By $(k, X, R)$ we shall denote a square-free binomial solution of the classical Yang-Baxter equation.

Each square-free binomial solution $(k, X, R)$ defines a quadratic algebra $A_R = A(k, X, R)$, namely the associated Yang-Baxter algebra, in the sense of Manin [23]. The algebra $A(k, X, R)$ is generated by $X$ and has quadratic defining relations, $R(R)$ determined by $R$ similarly to 2.7:

$$R(R) = \{ (x_jx_i - c_{ij}x_i'x_j') \mid R(x_j \otimes x_i) = c_{ij}x_i' \otimes x_j' \}$$

Sometimes it will be more convenient to work with the free associative algebra $k(X)$, instead of working with the tensor algebra, generated by $V$. Similarly to the identification of $X \times X$ and the set of $X^2$, now we identify the vector spaces $V^\otimes m$ and $\text{Span}_k X^m$, $m \geq 1$. We will show that the square-free binomial solutions of the classical Yang-Baxter equation are closely related with a class of quadratic PBW-algebras, the so called skew-polynomial rings with binomial relations and will prove an analogue of Theorem 2.26. We recall the definition.

**Definition 9.3.** [8]. Let $A_0 = A_0(k, X, R_0) = k < X > / (R_0)_{0}$ be a finitely presented quadratic algebra.

a) We say that $A_0(k, X, R_0)$ is an algebra with binomial relations of skew-polynomial type, if the set of generators $X$ is ordered: $X = \{ x_1 < x_2 < \cdots < x_n \}$, and the set of defining relations

$$R_0 = \{ x_jx_i = c_{ij}x_i'x_j' \mid 1 \leq i < j \leq n, \},$$

contains precisely $n(n-1)/2$ quadratic square-free binomial relations such that the following three conditions hold:

1) each monomial $xy$, with $x \neq y, x, y \in X$ occurs in exactly one relation in $R_0$; a monomial of the type $xx$ does not occur in any relation in $R_0$

2) $c_{ij} \neq 0$, for all $i, j$ with $1 \leq i < j \leq n$.

3) For every pair $i, j$ with $1 \leq i < j \leq n$, there are inequalities: $j > i'$, $i' < j'$.

b) An algebra $A_0 = A_0(k, X, R_0)$ with binomial relations of skew-polynomial type is called a skew-polynomial ring with binomial relations if

4) $R_0$ is a Groebner basis of the ideal $I = (R_0)$ in the free associative algebra $k < X >$, with respect to the degree-lexicographic ordering of the free semigroup $\langle X \rangle$.

**Remark 9.4.** It follows from the Diamond Lemma, cf. [4], that condition 9.3.4) is equivalent to each of the conditions 4') and 4'') below.

4') The set of ordered monomials,

$$N_0 = \{ x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0, 1 \leq i \leq n \}$$
is a \( k \)-basis of \( A_0 \), as a \( k \)-vector space.

4” The monomials \( x_k x_j x_i \), with \( k > j > i \) do not give rise to new relations in \( A_0 \).

Note that given the relations \( R_0 \), condition 4”) is recognizable.

**Definition 9.5.** Let \( A_0 = A_0(k, X, R_0) \) be an algebra with binomial relations of skew-polynomial type.

Let \( V \) be the \( k \)-vector space with a basis \( x_1, \ldots, x_n \). Consider the linear automorphism \( R = R(R_0) \) of \( V \otimes V \) defined as follows:

a) for each pair \( i, j, 1 \leq i < j \leq n \), we set
\[
R(x_j \otimes x_i) = c_{ij} x_{i'} \otimes x_{j'}, 1 \leq i < j \leq n,
\]
\[
R(x_{i'} \otimes x_{j'}) = \frac{1}{c_{ij}} x_j x_i, 1 \leq i < j \leq n,
\]
b) for each \( i, 1 \leq i \leq n \)
\[
R(x_i \otimes x_i) = x_i \otimes x_i.
\]

We say that \( R \) is the automorphism associated with the relations \( R_0 \), and denote it by \( R(R_0) \). We also define the bijection \( r = r(R_0) \) of \( X^2 \) onto itself, as
\[
r(xx) = xx, \text{ for all } x \in X, r(x_j x_i) = (x_i x_j)
\]
and
\[
r(x_{i'} x_{j'}) = x_j x_i, \text{ whenever } x_j x_i = c_{ij} x_{i'} x_{j'} \in R_0.
\]

**Lemma 9.6.** Assume that \( A_0(k, X, R_0) = k\langle X \rangle/\langle R_0 \rangle \) is an algebra with binomial relations of skew-polynomial type, and let \( R = R(R_0) \) be the automorphism of \( V \otimes V \) associated with the relations \( R_0 \). Then \( R \) is a solution of the classical Yang-Baxter equation if and only if \( R_0 \) is Groebner basis.

**Proof.** Assume that \( R = R(R_0) \) is a solution of the Yang-Baxter equation. We will prove that \( R_0 \) is a Groebner basis. It will be enough to show that each monomial \( x_k x_j x_i \), with \( k > j > i \), can be reduced by means of reductions defined via \( R_0 \) to a unique element of the shape \( \alpha_{i'j'k'} x_{i'} x_{j'} x_{k'} \), where \( 1 \leq i' < j' < k' \leq n \), and \( \alpha_{i'j'k'} \) is a uniquely determined coefficient, \( 0 \neq \alpha_{i'j'k'} \in k \). Let \( (X, r(R)) \) be the associated set-theoretic solution, see 9.2. Denote \( r_1 = r \times id_X, r_2 = id_X \times r \). Then the group \( g_r \langle r_1, r_2 \rangle \), which is isomorphic to the symmetric group \( S_3 \), acts on the set \( X^3 \).

Consider the orbit \( O \) of \( w = x_k x_j x_i \) under this action. It is not difficult to see that it has precisely 6 elements. By Lemma 6.5, the relations \( R(r) \) form a Groebner basis, therefore the orbit \( O \) contains exactly one ordered monomial, namely some \( w_0 = x_{i'} x_{j'} x_{k'} \), such that \( 1 \leq i' < j' < k' \leq n \).

Clearly, the orbit \( O \) of \( x_k x_j x_i \) under the action of \( g_r \langle R^{12}, R^{23} \rangle \) on \( kX^3 \) contains the same monomials of \( X^3 \) as \( O \), but, in general, they occur with non-zero coefficients which might be different from 1. In particular, \( O \) contains exactly one element in normal form modulo \( R_0 \), namely \( \alpha x_{i'j'} x_{k'} \), where \( \alpha \in k, \alpha \neq 0 \). It is also clear that each sequence of reductions (in the sense of [4]) reduces the monomial \( x_k x_j x_i \) to some element of the orbit \( O \). It follows then, that the ambiguity \( x_k x_j x_i, k > j > i \) is solvable, therefore \( R_0 \) is Groebner basis.

Conversely, let \( R_0 \) be a Groebner basis. Consider the associated linear automorphism \( R(R_0) \) and the associated bijective map \( r = r(R(R_0)) : X^2 \rightarrow X^2 \). By [14], Theorem 1.4, \( r \) is a solution of the set-theoretic Yang-Baxter equation. Now one can easily deduce that \( R(R_0) \) is a solution of the classical Yang-Baxter equation. \( \square \)
Theorem 9.7. Let $V$ be finite-dimensional vector space over a field $k$, with a $k$-basis $X$. Suppose $R$ is a linear automorphism of $V \otimes V$. Then the following conditions are equivalent:

1. $(k, X, R)$ is a square-free binomial solution of the classical Yang-Baxter equation.
2. There exists an ordering of $X$, $X = \{x_1 < x_2 \cdots < x_n\}$, such that the associated quadratic algebra $A = A(k, X, R) = k\langle X \rangle / (\mathcal{R}(R))$ is a skew-polynomial ring with binomial relations.

Furthermore, each of the above conditions implies that $A$ is a Yang-Baxter algebra which satisfies conditions 4 through 11 of Theorem 6.1. In particular, $A$ is a Noetherian domain and an Artin-Schelter regular ring of global dimension $n$.

Proof. 1) $\implies$ 2). Assume $(k, X, R)$ is a square-free binomial solution of the classical Yang-Baxter equation. Consider the associated set-theoretic solution $(X, r(R))$. It follows from 2.26 that there exists an ordering $X = \{x_1 < \cdots < x_n\}$ such that the relations $\mathcal{R}(r(R))$ are of skew-polynomial type. Then the relations $\mathcal{R}(R)$ of the Yang-Baxter algebra $A$ associated to $(k, X, R)$ are also of skew-polynomial type. Now Lemma 9.6 implies that $\mathcal{R}(R)$ is a Groebner basis, therefore $A(k, X, R)$ is a skew-polynomial ring. The implication 1) $\implies$ 2) follows from Lemma 9.6.

The remaining part of the theorem presents properties of the skew-polynomial rings with binomial relations, $A_0$, which can be extracted from our previous works. The Noetherian properties were proved in [9], a combinatorial proof of the Artin-Schelter regularity of $A_0$ was first given in [10]. Conditions 4, through 11 of Theorem 6.1, have been deduced in [14] from algebraic and homological properties of the semigroups $S$ of $I$-type and the associated semigroup algebras $kS$. □

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References

[1] D. Anick, On the homology of associative algebras, Trans. AMS 296 (1986), 641-659.
[2] M. Artin, Algebra, Prentice Hall, 1991.
[3] M. Artin and W. Schelter, Graded algebras of global dimension 3, Adv. in Math. 66 (1987), 171-216.
[4] G. M. Bergman, The diamond lemma for ring theory, Adv. in Math. 29 (1978), 178-218.
[5] V. G. Drinfeld, On some unsolved problems in quantum group theory, Quantum Groups (P. P. Kulish, ed.), Lecture Notes in Mathematics, vol. 1510, Springer Verlag, 1992, pp. 1-8.
[6] P. Etingof, T. Schedler, A. Soloviev Set-theoretical solutions to the quantum Yang-Baxter equation, Duke Math. J. 100 (1999), pp. 169–209.
[7] P. Etingof, R. Guralnick, A. Soloviev Indecomposable set-theoretical solutions to the quantum Yang-Baxter equation on a set with a prime number of elements, J. Algebra 249 (2001), pp. 709-719.
[8] T. Gateva-Ivanova, Noetherian properties of skew polynomial rings with binomial relations, Trans. Amer. Math. Soc. 343 (1994), 203–219.
[9] T. Gateva-Ivanova, Skew polynomial rings with binomial relations, J. Algebra 185 (1996), pp. 710–753.
[10] T. Gateva-Ivanova, *Regularity of the skew polynomial rings with binomial relations*, Preprint (1996).

[11] Tatiana Gateva-Ivanova, *Conjectures on the set-theoretic solutions of the Yang-Baxter equation*, Abstract, NATO Advanced Study Institute, and Euroconference "Rings, Modules and Representations", Constanta 2000, Romania; http://at.yorku.ca/cgi-bin/amca/cafe-34, http://www.mathematik.uni-stuttgart.de/~ovid/

[12] T. Gateva-Ivanova, *Set-theoretic solutions of the Yang-Baxter equation*, Mathematics and education in Mathematics, Proc. of the Twenty Ninth Spring Conference of the Union of Bulgarian Mathematicians, Lovetch, '2000 (2000), pp. 107-117.

[13] T. Gateva-Ivanova and M. Van den Bergh, *Regularity of skew polynomial rings with binomial relations*, Talk at the International Algebra Conference, Miskolc, Hungary, 1996.

[14] T. Gateva-Ivanova and M. Van den Bergh, *Semigroups of I-type*, J. Algebra 206 (1998), pp. 97–112.

[15] T. Gateva-Ivanova and Sh. Majid, *Matched pairs approach to set theoretic solutions of the Yang-Baxter equation*, Preprint (2004).

[16] T. Gateva-Ivanova, E. Jespers, and J. Okninski *Quadratic algebras of skew polynomial type and the underlying semigroups*, arXiv:math.RA/0210217 v1, pp. 1-25. J. Algebra, 270 (2003) pp. 635-659.

[17] J. Hietarinta, *Permutation-type solution to the Yang-Baxter and other n-simplex equations*, J. Phys. A 30 (1997), pp. 4757–4771.

[18] E. Jespers and J. Okninski, *Binomial Semigroups*, J. Algebra 202 (1998), pp. 250–275.

[19] C. Kassel *Quantum Groups*, Graduate Texts in Mathematics, Springer Verlag, 1995.

[20] T. Levasseur, *Some properties of non-commutative regular rings*, Glasgow Math. J. 34 (1992), pp. 277–300.

[21] J. Lu, M. Yan, Y. Zhu *On the set-theoretical Yang-Baxter equation*, Preprint (1999), pp. 1–25.

[22] Sh. Majid, *foundations of the quantum groups*, Cambridge University Press, 1995, Ch. 6.

[23] Yu. I. Manin, *Quantum groups and non-commutative geometry*, Les publications CRM, Universite de Montreal (1988) pp. 1–87.

[24] Odesskii *On the set-theoretical Yang-Baxter equation*, Preprint (2002), pp. 1–25.

[25] St. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. 152 (1970), pp. 39–60.

[26] N. Yu. Reshetikhin, L. A. Takhtadzhyan, L. D. Faddeev *Quantization of Lie groups and Lie algebras* (in Russian), Algebra i Analiz 1 (1989), pp. 178–206; English translation in Leningrad Math. J. 1 (1990), pp. 193–225.

[27] W. Rump, *A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation*, Preprint 2003, pp. 1–15.

[28] W. Schelter, *Noncommutative affine rings with polynomial identity are catenary*, J. Algebra 51 (1978), pp. 12–18.

[29] P. Smith *Some finite-dimensional algebras related to elliptic curves*, Representation theory of algebras and related topics (Mexico City, 1994) CMS Conf. Proc. Amer. Math. Soc., Providence, RI 19 (1996), pp. 315–348.

[30] A. Soloviev *Set-theoretical solutions to QYBE*, Preprint (1999), pp. 1–19.

[31] J. Tate and M. Van den Bergh, *Homological properties of Sklyanin algebras*, Invent. Math. 124 (1996), pp. 619–647.

[32] A. Weinstein and P. Xu *Classical solutions of the quantum Yang-Baxter equation*, Comm. Math. Phys. 148 (1992), pp. 309–343.

[33] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. 19 (1967), pp. 1312–1315.