Decays of $B$, $B_s$ and $B_c$ to $D$-wave heavy-light mesons

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Abstract

We study the weak decays of $\bar{B}(s)$ and $B_c$ into $D$-wave heavy-light mesons, including $J^P = 2^-$ ($D_{(s)2}$, $D_{(s)2}'$, $B_{(s)2}$, $B_{(s)2}'$) and $3^-$ ($D_{(s)3}$, $B_{(s)3}$) states. The weak decay hadronic matrix elements are achieved based on the instantaneous Bethe-Salpeter method. The branching ratios for $\bar{B}$ decays are $\mathcal{B}[\bar{B} \to D_2 e \bar{\nu}_e] = 1.1^{+0.3}_{-0.3} \times 10^{-3}$, $\mathcal{B}[\bar{B} \to D_2' e \bar{\nu}_e] = 4.1^{+0.8}_{-0.8} \times 10^{-4}$, and $\mathcal{B}[\bar{B} \to D_3 e \bar{\nu}_e] = 1.0^{+0.2}_{-0.2} \times 10^{-3}$, respectively. For semi-electronic decays of $\bar{B}_s$ to $D_2$, $D_2'$, and $D_3$, the corresponding branching ratios are $1.7^{+0.5}_{-0.5} \times 10^{-3}$, $5.2^{+1.6}_{-1.6} \times 10^{-4}$, and $1.5^{+0.4}_{-0.4} \times 10^{-3}$, respectively. The branching ratios of semi-electronic decays of $B_c$ to $D$-wave $D$ mesons are in the order of $10^{-5}$. We also achieved the forward-backward asymmetry, angular spectra, and lepton momentum spectra. In particular the distribution of decay widths for $2^-$ states $D_2$ and $D_2'$ varying along with mixing angle are presented.

1. Introduction

The $D$-wave $D_{(s)}$ mesons have attracted lots of attention since numerous excited charmed states are discovered by BaBar [1], and LHCb [2–5]. In 2010 BaBar observed four signals $D(2550)^0$, $D^{*}(2600)^0$, $D(2750)^0$, and $D^{*}(2760)^0$ for the first time [1], where the last two are expected to lie in the mass region of four $D$-wave charm mesons [6]. Later LHCb reported two natural parity resonances $D_3^*(2650)^0$ and $D_3^*(2760)^0$ in the $D^{*+}\pi^-$ mass spectrum and measured their angular distribution [2]. The same final states also show the presence of two unnatural parity states, $D_J(2580)^0$ and $D_J(2740)^0$. Here the natural parity denotes states with $J^P = 0^+$, $1^-$, $2^+$, $3^-$, ..., while the unnatural parity indicates series with $J^P = 1^-$, $2^-$, $3^-$, ... .

Then in May 2015, LHCb confirmed that the $D_J^*(2760)^0$ resonance has spin 1 [4]. The mass and width are measured as $m[D_J^*(2760)^0] = 2781 \pm 22$ MeV and $\Gamma[D_J^*(2760)^0] = 177 \pm 38$ MeV, where we have combined the statistical and systematic uncertainties in quadrature for simplicity. Later LHCb determined $D_J^*(2760)^-$ to have spin-parity $3^-$ and it is interpreted as $D_3^*(2760)^-$, namely the $3^D_3\bar{c}\bar{d}$ state. The mass and width are measured as $m[D_3^*(2760)] = 2798 \pm 10$ MeV and $\Gamma[D_3^*(2760)] = 105 \pm 30$ MeV [5].

For the $D$-wave charm-strange meson, BaBar first observed the $D_{sJ}^*(2860)$ [7, 8]. And then LHCb’s results support that $D_{sJ}^*(2860)$ is an admixture of the spin-1 and spin-3 [9, 10]. The measured mass and width for $D_{s3}$ are $2861 \pm 7$ and $53 \pm 10$ MeV, respectively. The two $D$-wave charm-strange mesons with $J = 2$ are $2^-$ states $D_{24}$ and $D_{2s}$ are still undiscovered in experiment.

Identification of these new excited charmed mesons can be found in Refs. [11–21]. We will follow Godfrey’s assignments on $D$-wave $D_{(s)J}^{(*)}$ mesons in Ref. [21], where $D_{s3}^*(2860)$ is identified as $1^3D_3\bar{c}\bar{s}$; $D_3^*(2798)^0$ is identified as $1^3D_3(c\bar{q})$ state; $D_{J}^*(2760)^0$ is interpreted as $1^3D_3(c\bar{q})$; and the $D_J(2750)^0$ reported by BaBar and $D_J(2740)^0$ reported by LHCb are identified as the same state with $1^D_2(c\bar{q})$, where $q$ denotes a light quark $u$ or $d$.

These $D$-wave excited states still need more experimental data to be discovered or confirmed. The identification and spin-parity assignments in above literature are just tentative. As the LHC

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accumulates more and more data, the study of these $D$-wave charm and charm-strange mesons in the weak decay of $B_s$ and $B_c$ meson becomes necessary and important. The properties of $D_{(s)j}^{(*)}$ in $B_s$ and $B_c$ decays would be helpful in identification of these excited $D_{(s)}$ mesons. The semi-leptonic decays of $B_s$ into $D$-wave charmed mesons have been studied by QCD sum rules [22–24] and constituent quark models in the framework of heavy quark effective theory (HQET) [25, 26]. Most of previous work is based on the HQET. The systematic studies on weak decays of $\bar{B}_s$ and constituent quark models in the framework of heavy quark effective theory (HQET) [25, 26].

In this work we will concentrate on the semi-leptonic and non-leptonic decays of $\bar{B}$ ($\bar{B}_s$, $B_c$) into $D$-wave $D$ ($D_s$) meson, including $2^-$ ($D_{(s)2}$, $D'_{(s)2}$) and $3^-$ ($D_{(s)3}^*$) states. For completeness, the weak decays of $B_c$ to $D$-wave bottomed mesons are also studied. We use the Instantaneous Bethe-Salpeter equation (IBS) [32] to get the hadronic transition form factors. BS equation [33] is the relativistic two-body bound states formula. Based on our previous studies [34–37], the relativistic corrections for transitions of higher excited states are larger and more important than that for the ground states, so the relativistic method is more reliable for the processes involved the high excited states. In the instantaneous approximation of the interaction kernel, we can achieve the Salpeter equation. The Salpeter method has been widely used to deal with heavy mesons’ decay constants calculation [38, 39], annihilation rate [40, 41], and hadronic transition [34–37].

This paper is organized as follows: first we present the general formalism of semi-leptonic and non-leptonic decay for $\bar{B}_s$ meson, including decay width, forward-backward asymmetry, and lepton spectra. In Section 3 we compute the form factors in hadronic transition by Salpeter method. In Section 4 we give the numerical results and discussions. Finally we give a short summary of this work.

2. Formalism of semi-leptonic and non-leptonic decays

In this section, firstly we will derive the formalism of transition amplitudes for $\bar{B}_s$ to $D$-wave heavy-light mesons. Then the formalisms of interested observables are presented. We will take the $B\to D_j^{(*)}$ transition as an example to show the calculation details, while results for transition of $B_s$ and $B_c$ will be given directly.

2.1. Semi-leptonic decay amplitude

The Feynman diagram responsible for $\bar{B}$ semi-leptonic decay is showed in Fig. 1, where we use $P$ and $P_F$ to denote the momenta of $\bar{B}$ and $D_j^{(*)}$ respectively. The transition amplitude $\mathcal{A}$ for the process $\bar{B}\to D_j^{(*)}\ell\bar{\nu}$ can be written directly as

\[
\mathcal{A} = \frac{G_F}{\sqrt{2}} V_{cb} l^\mu \langle D_j^{(*)} | J_\mu | \bar{B} \rangle.
\]  

In above equation, $G_F$ denotes the Fermi weak coupling constant; $V_{cb}$ is the Cabibbo-Kobayashi-Maskawa matrix element; the lepton matrix element $l^\mu$ reads

\[
l^\mu = \bar{u}(p_\ell) \Gamma^\mu v(p_\nu),
\]

where $\ell$ ($\bar{\nu}_\ell$) denotes the charged lepton (anti-neutrino), and $p_\ell$ ($p_\nu$) denotes the corresponding momentum, and the definition $\Gamma^\mu = \gamma^\mu (1 - \gamma^5)$ is used; $\langle D_j^{(*)} | J_\mu | \bar{B} \rangle$ is the hadronic transition element, where $J_\mu = c\Gamma_\mu b$ is the weak current.
We use $\mathcal{M}^\mu$ to denote hadronic transition element $\langle D_j^{(*)}|J^\mu|\bar{B}\rangle$, which can be described with form factors. The general form of the hadronic matrix element depends on the total angular momentum $J$ of the final meson. For $D_2$ ($D_2'$) and $D_3^*$ the form factors are defined as

$$
\mathcal{M}^\mu = \begin{cases} 
  e_{\alpha\beta}P^\alpha(s_1P^\beta P^\mu + s_2P^\beta P_F^\mu + s_3g^{\beta\mu} + is_4\epsilon^{\mu\nu\rho\sigma}P_P) & \text{if } J = 2, \\
  e_{\alpha\gamma}P^\alpha P^\beta(h_1P^\gamma P^\mu + h_2P^\gamma P_F^\mu + h_3g^{\gamma\mu} + ih_4\epsilon^{\mu\nu\rho\sigma}P_P) & \text{if } J = 3.
\end{cases}
$$

In above equation, we used the definition $\epsilon_{\mu\nu\rho\sigma}P_P = \epsilon_{\mu\nu\alpha\beta}P^\alpha P_F^\beta$ where $\epsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor; $g^{\mu\nu}$ is the Minkowski metric tensor; $e_{\alpha\beta}$ and $e_{\alpha\gamma}$ are the polarization tensor for $J = 2$ and 3 mesons, respectively, which are completely symmetric; $s_i$ and $h_i$ ($i = 1, 2, 3, 4$) are the form factors for $J = 2$ and 3, respectively. To state it more clearly, we will use $s_i$, $t_i$, and $h_i$ to denote the form factors for transitions $\bar{B}$ to $D_2$, $D_2'$, and $D_3^*$, respectively. $S_i$, $T_i$, and $H_i$ are used to represent the form factors of $B_c^-$ to $D_2$, $D_2'$, and $D_3^*$, respectively. The definition forms are the same with that for transition $\bar{B}\to D_j^{(*)}$, just $s_i$ is replaced by $S_i$, $t_i$ by $T_i$, and $h_i$ by $H_i$. For $\bar{B}_s$ decays, the corresponding form factor behaviors are very similar to $\bar{B}$ decays. The detailed calculations of these form factors will be given in next section.

After summing the polarization of all the final states, including the charged lepton, anti-neutrino and the final $D_j^{(*)}$, we obtain

$$
|A|^2 = \frac{G_{F}^2}{2}|V_{cb}|^2L^{\mu\nu}H^{\mu\nu},
$$

where the lepton tensor $L^{\mu\nu}$ has the following form

$$
L^{\mu\nu} = 8(p_\ell\gamma^\mu p_\nu^\nu + p_\ell^\mu p_\nu^\nu - p_\ell^\nu p_\nu^\mu)g^{\mu\nu} - i\epsilon^{\mu\nu\rho\sigma}P_P.
$$

$H^{\mu\nu}$ is the hadronic tensor describing the propagator-meson interaction vertex, which depends on $P$, $P_F$ and the corresponding form factors. It can be written as

$$
H^{\mu\nu} = \sum_{s=-J}^{J} \mathcal{M}_s^{(s)}\mathcal{M}_s^{(s)*} = N_1P_\mu P_\nu + N_2(P_\mu P_{F\nu} + P_\nu P_{F\mu}) + N_4P_{F\mu}P_{F\nu} + N_5g^{\mu\nu} + iN_6\epsilon^{\mu\nu\rho\sigma}P_P,
$$

where the summation is over the polarization of final $D_j^{(*)}$ meson; $N_i$ is related to the form factors $s_i$ for $D_2$, $t_i$ for $D_2'$ or $h_i$ for $D_3^*$. The detailed expressions for $N_i$ can be found in appendix A.

### 2.2. Non-leptonic decay amplitude

The Feynman diagram for the non-leptonic decay of $\bar{B}$ to $D_j^{(*)}$ and a light meson $X$ is showed in Fig. 2. As a preliminary study for non-leptonic decays of $\bar{B}$ to $D$-wave $D$ mesons, we will work in the framework of naive factorization approximation [42–45], which has been widely used in heavy mesons’ weak decays [46–50]. The factorization assumption is expected to hold for process that
involves a heavy meson and a light meson, provided the light meson is energetic [51–53]. Also we only consider the processes when the light meson $X$ is $\pi$, $\rho$, $K$, or $K^*$. In the naive factorization approximation, the decay amplitude can be factorized as the product of two parts, the hadronic transition matrix element and an annihilation matrix element. Then we can write the non-leptonic approximation, the decay amplitude can be factorized as the product of two parts, the hadronic decay amplitude as

$$A[\bar{B} \to D^{(*)}_j X] \simeq \frac{G_F}{\sqrt{2}} V_{cb} V_{uq} a_1(\mu) \langle D^{(*)}_j | J_\mu | \bar{B} \rangle \langle X | (\bar{q}u)_{V-A} | 0 \rangle,$$  

(7)

where we have used the definition $(\bar{q}u)_{V-A} = \bar{q} \Gamma^\mu u$ and $q$ denotes a $d$ or $s$ quark field; $V_{uq}$ denotes the corresponding CKM matrix element; $a_1 = a_1 + \frac{1}{N_c} c_2$, where $N_c = 3$ is the number of colors. For $b$ decays, we take $\mu = m_b$, and $a_1 = 1.14$, $a_2 = -0.2$ [48] are used in this work. The annihilation matrix element can be expressed by decay constant as

$$\langle X | (\bar{q}u)_{V-A} | 0 \rangle = \begin{cases} i p^\mu_X f_P & X \text{ is a pseudoscalar meson (} \pi, K), \\ e^\mu M_X f_V & X \text{ is a vector meson (} \rho, K^*). \end{cases}$$  

(8)

$M_X$, $P_X$ are the mass and momentum of $X$ meson, respectively; the meson polarization vector $e^\mu$ satisfies $e^\mu P_X^\mu = 0$ and the completeness relation is given by $\sum_s e^\mu(s) e^\mu(s) = P^\mu_M P_X^\mu - g^\mu\nu$, where $s$ denotes the polarization state; $f_P$ and $f_V$ are the corresponding decay constants.

Then the $|A|^2$ can be expressed by hadronic tensor $H_{\mu\nu}$, which is just the same with that in the corresponding semi-leptonic decays, and light meson tensor $X^{\mu\nu}$ as

$$|A|^2 = \frac{G_F^2}{2} |V_{cb}|^2 |V_{uq}|^2 a_1^2 H_{\mu\nu} X^{\mu\nu},$$  

(9)

where $X^{\mu\nu}$ has the following expression

$$X^{\mu\nu} = \langle X | (\bar{q} \Gamma^\mu u) | 0 \rangle \langle X | \bar{q} \Gamma^\nu u | 0 \rangle^* = \begin{cases} P^\mu_X P^\nu_X f_P^2 & X \text{ is a pseudoscalar meson}, \\ (P^\mu_X P^\nu_X - M_X^2 g^{\mu\nu}) f_V^2 & X \text{ is a vector meson}. \end{cases}$$  

(10)

2.3. Several observables

One of the interested quantity in $\bar{B}$ semi-leptonic decay is the angular distribution of the decay width $\Gamma$, which can be described as

$$\frac{d\Gamma}{d\cos \theta} = \int \frac{1}{(2\pi)^3} \frac{|p^\mu_\ell| |p^\nu_{\bar{v}}| |A|^2}{16 M^3} dm^2_{\ell\nu},$$  

(11)

where $M$ is the initial $\bar{B}$ mass; $m^2_{\ell\nu} = (p_\ell + p_{\bar{v}})^2$ is the invariant mass square of $\ell$ and $\bar{v}$; $p^\mu_\ell$ and $p^\nu_{\bar{v}}$ are the three momenta of $\ell$ and $D^{(*)}_j$ in the $\ell\bar{v}$ rest frame, respectively; $\theta$ is the angle between $p^\mu_\ell$ and

![Feynman Diagram](image_url)

**Fig. 2:** The Feynman diagram of the nonleptonic decay of $\bar{B}(s)$ meson into $D$-wave charmed meson. $X$ denotes a light meson.
\( p_F^\ast; \vec{p}_F^\ast = \frac{1}{2m_{F^\ast}} \lambda^2(m^2_{F^\ast}, M^2_F, M^2_{F^\ast}) \) and \( p_F = \frac{1}{2m_v} \lambda^2(m^2_v, M^2_v, M^2_F) \), where we have used the Källén function \( \lambda(a, b, c) = (a^2 + b^2 + c^2 - 2ab - 2bc - 2ac) \), \( M_F \) and \( m_v \) are the lepton mass and anti-neutrino mass, respectively. Another quantity we are interested is the forward-backward asymmetry \( A_{FB} \), which is defined as

\[
A_{FB} = \frac{\Gamma_{\cos \theta > 0} - \Gamma_{\cos \theta < 0}}{\Gamma_{\cos \theta > 0} + \Gamma_{\cos \theta < 0}}.
\]  

(12)

The decay width varying along with charged lepton 3-momentum \( |\vec{p}_\ell| \) is given by

\[
\frac{d\Gamma}{d|\vec{p}_\ell|} = \int \frac{1}{(2\pi)^3} \frac{|\vec{p}_\ell|}{16M_F^2E_\ell} |\mathcal{A}|^2 dm^2_\ell,
\]  

(13)

where \( E_\ell \) denotes the charged lepton energy in the rest frame of initial state meson.

The non-leptonic decay width of the \( B \) meson is given by

\[
\Gamma = \frac{|\vec{p}|}{8\pi M_F^2} |\mathcal{A}|^2,
\]  

(14)

where \( \vec{p} \) represents the 3-momentum of the final \( D_j^{(s)} \) in \( B \) rest frame, which is expressed as \( |\vec{p}| = \frac{1}{2M} \lambda^2(M^2, M^2_F, M^2_{F^\ast}) \).

3. Hadronic transition matrix element

The hadronic transition matrix element \( \langle D_j^{(s)}|J^\mu|B \rangle \) plays an key role in the calculations of \( B \) semi-leptonic and non-leptonic decays. In this section we will give details to calculate the hadronic transition matrix element by Bethe-Salpeter method in the framework of constituent quark model.

3.1. Formalism of hadronic transition matrix element with Bethe-Salpeter method

According to the Mandelstam formalism [54], the hadronic transition amplitude \( \mathcal{M}^\mu \) can be written by Beter-Salpeter (BS) wave function as

\[
\mathcal{M}^\mu = -i \int \frac{d^4q d^4q'}{(2\pi)^4} \text{Tr}[\bar{\Psi}_D(q', P_F) \Gamma^\mu \Psi_B(q, P)(m_2 + p_2') \delta^4(p_2 - p_2')],
\]  

(15)

where \( \Psi_B(q, P) \) and \( \Psi_D(q', P_F) \) are the BS wave functions of the \( B \) meson and the final \( D_j^{(s)} \), respectively; \( \bar{\Psi} \) is defined as \( \gamma^0 \bar{\Psi} \gamma^0 \); \( q \) and \( q' \) are respectively the inner relative momenta of \( B \) and \( D_j^{(s)} \) system, which are related to the quark (anti-quark) momentum \( p_i^{(s)} \) by \( p_i = \alpha_i P + (-1)^{i+1} q \) and \( p_i' = \alpha_i' P_F + (-1)^{i+1} q' \) (\( i = 1, 2 \)). And here we defined the symbols \( \alpha_i = \frac{m_i}{m_1 + m_2} \) and \( \alpha_i' = \frac{m_i'}{m_1' + m_2'} \), where \( m_i \) and \( m_i' \) are masses of the constituent quarks in the initial and final bound states, respectively (see Fig. 1). Here in \( B \) decays we have \( m_1 = m_b, m_1' = m_c, m_2 = m_d' = m_d \). As there is a delta function in above equation, the relative momenta \( q \) and \( q' \) are related by \( q' = q - (\alpha_2 P - \alpha_2' P_F) \).

In the instantaneous approximation [32], the inner interaction kernel between quark and anti-quark in bound state is independent of the time component \( q_F = (q \cdot P) \) of \( q \). By performing the contour integral on \( q_F \) and then we can express the hadronic transition amplitude as [37]

\[
\mathcal{M}^\mu = \int \frac{d^3q}{(2\pi)^3} \text{Tr} \left[ \frac{\not{P}}{M} \bar{\psi}_D(q^\perp) \Gamma^\mu \psi_B(q^\perp) \right],
\]  

(16)

where we have used the definitions \( q^\perp \equiv q - \frac{\not{P} \not{q}}{M^2} P \) and \( q'^\perp \equiv q' - \frac{\not{P} \not{q}}{M^2} P \). Here \( \psi \) denotes the 3-dimensional positive Salpeter wave function (see appendix B), \( \psi_B \) and \( \psi_D \) denote the positive Salpeter wave functions for \( B \) and \( D_j^{(s)} \), respectively, and \( \bar{\psi}_D \) is defined as \( \gamma^0 \psi_D \gamma^0 \).
where we have the following constraint conditions,

\[ A_1 = \frac{M}{2} \left[ \omega_1 + \omega_2 k_1 + k_2 \right], \quad A_3 = -\frac{M(\omega_1 - \omega_2)}{m_1 \omega_2 + m_2 \omega_1}, \]
\[ A_2 = \frac{M}{2} \left[ k_1 + \frac{m_1 + m_2}{\omega_1 + \omega_2} k_2 \right], \quad A_4 = -\frac{M(m_1 + m_2)}{m_1 \omega_2 + m_2 \omega_1} A_1. \]  

The definition \( \omega_i = \sqrt{m_i^2 - q_{i\perp}^2} (i = 1, 2) \) is used. The derivation of Eq. (17) and (18) can be found in appendix B. So there are only two undetermined wave function \( k_1 \) and \( k_2 \) here, which are the functions of \( q_{i\perp} \). The positive Salpeter wave function for \( 3^- \) \((^3D_3)\) state with unequal mass of quark and anti-quark has the following forms [56]

\[
\psi_D(^3D_3) = e_{\mu\nu\alpha} q_{\perp}^\mu q_{\perp}^\nu \left[ q_{\perp}^\alpha (n_1 + n_2 \frac{\not{P}_F}{M_F} + n_3 \frac{\not{q}_i}{M_F} + n_4 \frac{\not{P}_F \not{q}_{i\perp}}{M_F}) + \gamma^\mu (n_5 M_F + n_6 \not{P}_F) \right. \\
\left. + n_7 (\gamma^\mu \not{P}_F \not{q}_{i\perp} + \not{P}_F q_{i\perp}^\mu) \right]. 
\]  

In above equation \( n_i \) \( (i = 1, 2, \ldots, 8) \) can be expressed with 4 wave functions \( u_i \) \( (i = 3, 4, 5, 6) \) as below,

\[
n_1 = \frac{(\omega_1' + \omega_2')(q_{i\perp}^2 u_3 + M_F^2 u_5) + (m_1' + m_2')(q_{i\perp}^2 u_4 - M_F^2 u_6)}{2 M_F (m_1' \omega_2' + m_2' \omega_1')},
\]
\[
n_2 = \frac{(m_1' - m_2')(q_{i\perp}^2 u_3 + M_F^2 u_5) + (\omega_1' - \omega_2')(q_{i\perp}^2 u_4 - M_F^2 u_6)}{2 M_F (m_1' \omega_2' + m_2' \omega_1')},
\]
\[
n_3 = \frac{1}{2} \left[ u_3 + \frac{m_2' + m_1'}{\omega_1' + \omega_2'} u_4 - \frac{2 M_F^2}{m_1' \omega_2' + m_2' \omega_1'} u_6 \right],
\]
\[
n_4 = \frac{1}{2} \left[ u_4 + \frac{\omega_1' + \omega_2'}{m_1' + m_2'} u_4 - \frac{2 M_F^2}{m_1' \omega_2' + m_2' \omega_1'} u_5 \right],
\]
\[
n_5 = \frac{1}{2} \left[ u_5 - \frac{\omega_1' + \omega_2'}{m_1' + m_2'} u_6 \right], \quad n_6 = \frac{1}{2} \left[ u_6 - \frac{m_1' + m_2'}{\omega_1' + \omega_2'} u_5 \right],
\]
\[
n_7 = \frac{M_F (\omega_1' - \omega_2')}{(m_1' \omega_2' + m_2' \omega_1')} n_5, \quad n_8 = \frac{M_F (\omega_1' + \omega_2')}{(m_1' \omega_2' + m_2' \omega_1')} n_6.
\]

In above Salpeter positive wave functions \( \psi_B \) and \( \psi_D \), the undetermined wave functions \( k_1, k_2 \) for \( 0^- \) and \( u_i \) \( (i = 3, 4, 5, 6) \) for \( 3^- \) can be achieved by solving the full Salpeter equations numerically (see appendix B). The positive Salpeter wave functions for \( ^1D_2 \) [41], and \( ^3D_2 \) [56] states can be seen in appendix C. \( e^{\mu\nu\alpha} \) is the symmetric polarization tensor for spin-3 and satisfies the following relations [57]

\[
e^{\mu\nu\alpha} g_{\mu\nu} = 0, \quad e^{\mu\nu\alpha} P_{F\mu} = 0,
\]
\[
\sum_s e^{(s)\mu\nu\alpha} e^{(s)\mu\nu\alpha} = \frac{1}{6} \Omega^1_{abc,\mu\nu\alpha} - \frac{1}{15} \Omega^2_{abc,\mu\nu\alpha},
\]
where

\[
\Omega_{1}^{abc\mu\nu} = g_{1}^{ab} g_{1}^{bc} + g_{1}^{ab} g_{1}^{bc} + g_{1}^{ab} g_{1}^{bc} + g_{1}^{ab} g_{1}^{bc},
\]

\[
\Omega_{2}^{abc\mu\nu} = g_{1}^{ab} g_{1}^{bc} + g_{1}^{ab} g_{1}^{bc} + g_{1}^{ab} g_{1}^{bc} + g_{1}^{ab} g_{1}^{bc} + g_{1}^{ab} g_{1}^{bc} + g_{1}^{ab} g_{1}^{bc}
\]

and we have used the definition \( g_{1}^{\mu\nu} = -g^{\mu\nu} + \frac{P_{F}^{\mu} P_{F}^{\nu}}{M_{F}^{2}} \).

Inserting the initial \( B \) wave function \( \psi_{B}(1S_{0}) \) (Eq. 17) and final \( D_{1}^{*} \) wave function \( \psi_{D} (3D_{3}) \) (Eq. 19) into the hadronic transition amplitude Eq. (16), after calculating the trace and performing the integral in Eq. (16) we achieve the form factors \( h_{i} \) for \( B \to D_{1}^{*} \) transition defined in Eq. (3). When performing the integral over \( q \) in the rest frame of the initial meson, the following formulas are used.

\[
\int \frac{d^{3}q}{(2\pi)^{3}} q_{\mu}^{\prime} = C_{1} P_{F,\perp}^{\mu},
\]

\[
\int \frac{d^{3}q}{(2\pi)^{3}} q_{\mu}^{\prime} q_{\mu}^{\prime} = C_{21} P_{F,\perp}^{\mu} + C_{22} g_{T}^{\mu\nu} P_{F,\perp}^{\nu},
\]

\[
\int \frac{d^{3}q}{(2\pi)^{3}} q_{\mu}^{\prime} q_{\mu}^{\prime} q_{\mu}^{\prime} q_{\mu}^{\prime} = C_{31} P_{F,\perp}^{\mu} P_{F,\perp}^{\nu} + C_{32} g_{T}^{\mu\nu} P_{F,\perp}^{\nu} + C_{33} P_{F,\perp}^{\mu} + C_{33} P_{F,\perp}^{\nu} + g_{T}^{\mu\nu} P_{F,\perp}^{\nu},
\]

where \( g_{T}^{\mu\nu} \) are defined as \( (g^{\mu\nu} - \frac{P_{F}^{\mu} P_{F}^{\nu}}{M_{F}^{2}}) \) and \( P_{F,\perp}^{\mu} = (P_{F}^{\mu} - \frac{P_{F} P_{F}^{\mu}}{M_{F}^{2}} P_{F}) \). From above equations we can easily obtain the following expressions of \( C_{i} \),

\[
\begin{align*}
C_{1} &= |q| \cos \eta, \\
C_{22} &= \frac{1}{2} |q|^{2}(\cos^{2} \eta - 1), \\
C_{32} &= \frac{1}{2} |q|^{3}(\cos^{3} \eta - \cos \eta), \\
C_{42} &= \frac{1}{8} |q|^{4}(5 \cos^{4} \eta - 6 \cos^{2} \eta + 1),
\end{align*}
\]

\[
\begin{align*}
C_{21} &= \frac{1}{2} |q|^{2}(3 \cos^{2} \eta - 1), \\
C_{31} &= \frac{1}{2} |q|^{3}(5 \cos^{3} \eta - 3 \cos \eta), \\
C_{41} &= \frac{1}{8} |q|^{4}(35 \cos^{4} \eta - 30 \cos^{2} \eta + 3), \\
C_{43} &= \frac{1}{8} |q|^{4}(\cos^{4} \eta - 2 \cos^{2} \eta + 1),
\end{align*}
\]

where \( \eta \) is the angle between \( q \) and \( P_{F} \).

The physical \( 2^{-} D \)-wave states \( D_{2} \) and \( D_{2}^{'} \) are the mixing states of \( 3D_{2} \) and \( 1D_{2} \) states, whose wave functions are what we solve directly from the full Salpeter equations. Here we will follow Ref. [58] and Ref. [59], where the mixing form for \( D \)-wave states is defined with the mixing angle \( \alpha \) as

\[
\begin{align*}
|D_{2}\rangle &= + \cos \alpha |1D_{2}\rangle + \sin \alpha |3D_{2}\rangle, \\
|D_{2}^{'}\rangle &= - \sin \alpha |1D_{2}\rangle + \cos \alpha |3D_{2}\rangle.
\end{align*}
\]

In the heavy quark limit \( (m_{Q} \to \infty) \), the \( D \) mesons are described in the \( |J, j_{l}\rangle \) basis, where \( m_{Q} \) denotes the heavy quark mass and \( j_{l} \) denotes the total angular momentum of the light quark. The relations between \( |J, j_{l}\rangle \) and \( |J, S\rangle \) for \( L = 2 \) are showed by

\[
\begin{bmatrix}
|2, 5/2\rangle \\
|2, 3/2\rangle
\end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix}
\sqrt{2} + 1 & \sqrt{2} \\
-\sqrt{2} & \sqrt{2} + 1
\end{bmatrix} \begin{bmatrix}
|1D_{2}\rangle \\
|3D_{2}\rangle
\end{bmatrix}.
\]

Then the mixing angle for \( L = 2 \) can be expressed as \( \alpha = \arctan \sqrt{2/3} = 39.23^\circ \). So in this definition \( D_{2} \) corresponds to the \( |J, j_{l}\rangle = |2^{-}, 5/2\rangle \) state and \( D_{2}^{'} \) corresponds to the \( |2^{-}, 3/2\rangle \) state. In this
work the same mixing angle will also be used for $2^-$ states $D_{s2}^{(i)}$ and $B_{(s)}^{(i)}$. Here the mixing angle is the ideal case predicted by the HEQT in the limit of $m_Q \to \infty$. The dependence for decay widths varying over the mixing angle can be seen in equations (29) and (30).

The wave functions of $^{1}D_2$ and $^{3}D_2$ states can be achieved by solving the corresponding Salpeter equations directly. Then the amplitude for physical $2^-$ states can be considered as the mixing of the transition amplitudes for $^{1}D_2$ and $^{3}D_2$ states, namely

$$\mathcal{M}^\mu(D_2) = + \cos \alpha \mathcal{M}^\mu(^{1}D_2) + \sin \alpha \mathcal{M}^\mu(^{3}D_2),$$
$$\mathcal{M}^\mu(D_2') = - \sin \alpha \mathcal{M}^\mu(^{1}D_2) + \cos \alpha \mathcal{M}^\mu(^{3}D_2).$$  \hspace{1cm} (27)

By using Eq. (27), replacing the final state’s wave function $\psi_{D}(^{3}D_2)$ by $\psi_{D}(^{1}D_2)$ and $\psi_{D}(^{3}D_2)$, and then repeating the above procedures for $^{3}D_2$ state, we can get the form factors $s_i$ for $D_2$ and $t_i$ for $D_2'$ defined in Eq. (3).

3.2. Form factors

To solve the Salpeter equations, in this work we choose the Cornell potential as the inner interaction kernel as before [55], which is a linear scalar potential plus a vector interaction potential as below

$$V(q) = (2\pi)^3V_s(q) + \gamma^0 \otimes \gamma_0(2\pi)^3V_v(q),$$
$$V_s(q) = -(\frac{\lambda}{\alpha} + V_0)\delta^3(q) + \frac{\lambda}{\pi^2(q^2 + \alpha^2)^2},$$
$$V_v(q) = -\frac{2\alpha_s(q)}{3\pi^2(q^2 + \alpha^2)}, \alpha_s(q) = \frac{12\pi}{27 \ln(a + \frac{q^2}{\Lambda_{QCD}^2})}.$$  \hspace{1cm} (28)

In above equations, the symbol $\otimes$ denotes that the Salpeter wave function is sandwiched between the two $\gamma^0$ matrices. The model parameters we used are the same with before [35], which read

$$a = e = 2.7183,$$
$$m_u = 0.305 \text{ GeV}, \quad m_d = 0.311 \text{ GeV}, \quad m_s = 0.500 \text{ GeV},$$
$$m_c = 1.62 \text{ GeV}, \quad m_b = 4.96 \text{ GeV}, \quad \Lambda_{QCD} = 0.270 \text{ GeV}.$$

The free parameter $V_0$ is fixed by fitting the mass eigenvalue to experimental value.

With the numerical Salpeter wave function we can obtain the form factors.

![Graphs of Form Factors](image-url)
Form Factors for $B \to D_3^*(3^-)$.

Fig. 3: Form factors for transitions $B \to D_3^*(2^-, 3^-)$ and $B_c^- \to \bar{D}_2^*(2^-, 3^-)$. $t^2 = (P - P_F)^2$ denotes the square of momentum transfer. To make the dimension consistent, $s_3$, $t_3$ and $h_3$ are divided by $M_{B_3}^2$, $S_3$, $T_3$ and $H_3$ are divided by $M_{B_c}^2$.

Here we plot the $B \to D_j^*(s)$ form factors $s_i$, $t_i$, and $h_i$ ($i = 1, 2, 3, 4$) changing with the square of momentum transfer $t^2 = (P - P_F)^2$ in Fig. 3(a) $\sim$ Fig. 3(c), respectively, where $s_3$, $t_3$, and $h_3$ are divided by $M_{B_3}^2$ in order to keep the dimension consistent. Fig. 3(d) $\sim$ Fig. 3(f) are the distribution of form factors $S_i$, $T_i$, and $H_i$ for $B_c^- \to \bar{D}_2^*(s)$ transitions. Also we divided $S_3$, $T_3$, and $H_3$ by $M_{B_c}^2$ to keep the dimension consistent. From Fig. 3, we can see that in all the range concerned the form factors are quite smooth along with $t^2$. And for transitions $\bar{B} \to D_j^*(s)$, the form factors change slowly and almost linearly when $t^2$ varies from 0 to $(M - M_F)^2$. For transitions $B_c^- \to \bar{D}_2^*(s)$, the form factors change dramatically over $t^2$, especially in the range with large momentum transfer.

4. Numerical Results and Discussions

Firstly we specify the meson mass, lifetime, CKM matrix elements and decay constants used in this work. For the mass of $B$, $B_s$, and $B_c$ mesons we take the values from PDG [60]. We follow the mass predictions and $J^P$ assignments of Ref. [21] for $D$-wave charm and charm-strange mesons. For $D$-wave bottom mesons $B_2$, $B'_2$, and $B'_3$ we use the average values of Ref. [61] and Ref. [59]. Predictions of Ref. [59] and Ref. [62] are averaged to achieve the mass of $D$-wave bottom-strange
mesons $B_{s2}$, $B'_{s2}$, and $B'_{s3}$. These mass values we used can been seen below

\[
\begin{align*}
M_B &= 5.280 \, \text{GeV}, \quad M_{B_s} = 5.367 \, \text{GeV}, \quad M_{B_c} = 6.276 \, \text{GeV}, \\
M_{D_2} &= 2.750 \, \text{GeV}, \quad M_{D'_2} = 2.780 \, \text{GeV}, \quad M_{D'_3} = 2.800 \, \text{GeV}, \\
M_{D_{s2}} &= 2.846 \, \text{GeV}, \quad M_{D'_{s2}} = 2.872 \, \text{GeV}, \quad M_{D'_{s3}} = 2.860 \, \text{GeV}, \\
M_{B_2} &= 6.060 \, \text{GeV}, \quad M_{B'_2} = 6.100 \, \text{GeV}, \quad M_{B'_3} = 6.050 \, \text{GeV}, \\
M_{B_{s2}} &= 6.150 \, \text{GeV}, \quad M_{B'_{s2}} = 6.210 \, \text{GeV}, \quad M_{B'_{s3}} = 6.190 \, \text{GeV}.
\end{align*}
\]

The lifetime of initial mesons we used are as below [60]

\[
\tau_B = 1.519 \times 10^{-12} \, \text{s}, \quad \tau_{B_s} = 1.512 \times 10^{-12} \, \text{s}, \quad \tau_{B_c} = 0.452 \times 10^{-12} \, \text{s}.
\]

The involved CKM matrix element values are [60]

\[
|V_{ud}| = 0.974, \quad |V_{us}| = 0.225, \quad |V_{ub}| = 0.0042, \quad |V_{cd}| = 0.23, \quad |V_{cs}| = 1.006, \quad |V_{cb}| = 0.041.
\]

In the calculation of non-leptonic decays, the decay constants we used are [48, 60]

\[
f_\pi = 130.4 \, \text{MeV}, \quad f_K = 156.2 \, \text{MeV}, \quad f_\rho = 210 \, \text{MeV}, \quad f_{K^*} = 217 \, \text{MeV}.
\]

For the theoretical uncertainties, here we will just discuss the dependence of the final results on our model parameters $\lambda$, $\Lambda_{\text{QCD}}$ in the Cornell potential, and the constituent quark mass $m_b$, $m_c$, $m_s$, $m_d$ and $m_u$. The theoretical errors, induced by these model parameters, are determined by varying every parameter by $\pm 5\%$, and then scanning the parameters space to find the maximum deviation. Generally, this theoretical uncertainties can amount to $10\% \sim 30\%$ for the semi-leptonic decays. The theoretical uncertainties show the robustness of the numerical algorithm.

4.1. Lepton spectra and $A_{FB}$

The distribution of $\bar{B}$ and $B_c^-$ decay width $\Gamma$ varying along with $\cos \theta$ for $e$ and $\tau$ modes can be seen in Fig. 4, from which we can see that, for $\bar{B}$ decays, the distribution of semi-electronic decay widths are much more symmetric than that for the semi-taunic mode. These asymmetries over $\cos \theta$ can also be reflected by the forward-backward asymmetries $A_{FB}$, which are showed in Tab. I. We can see that $A_{FB}$ is sensitive to lepton mass $m_\ell$ and is the monotonic function of $m_\ell$. Considering the absolute values of $A_{FB}$, we find that for $\bar{B} \to D^{(*)}_J$ and $B_c^- \to D^{(*)}_J$, the $\mu$ decay mode has the smallest $|A_{FB}|$.

(a) Angular spectrum for $\bar{B} \to D^{(*)}_J e\bar{\nu}$.

(b) Angular spectrum for $\bar{B} \to D^{(*)}_J \tau\bar{\nu}$.
Fig. 4: The spectra of relative width vs $\cos \theta$ for semi-leptonic decays $\bar{B} \to D_j^*(\ell \nu)$ and $B_c^- \to \bar{D}_j^*(\ell \nu)$. $\theta$ is the angle between charged lepton $\ell$ and final charmed meson in the rest frame of $\ell \bar{\nu}$ pair.

Table I: $A_{FB}$ for semi-leptonic decays of $\bar{B}, \bar{B}_s$ and $B_c$ to $D$-wave heavy-light mesons.

| Channels          | $A_{FB}$ | Channels          | $A_{FB}$ | Channels          | $A_{FB}$ |
|-------------------|----------|-------------------|----------|-------------------|----------|
| $\bar{B} \to \bar{D}_2e\bar{\nu}$ | -0.08    | $\bar{B} \to \bar{D}_2' e\bar{\nu}$ | -0.08    | $\bar{B} \to \bar{D}_3 e\bar{\nu}$ | -0.10    |
| $\bar{B} \to \bar{D}_2\mu\bar{\nu}$ | -0.05    | $\bar{B} \to \bar{D}_2' \mu\bar{\nu}$ | -0.05    | $\bar{B} \to \bar{D}_3' \mu\bar{\nu}$ | -0.07    |
| $\bar{B} \to \bar{D}_2\tau\bar{\nu}$ | 0.20     | $\bar{B} \to \bar{D}_2' \tau\bar{\nu}$ | 0.21     | $\bar{B} \to \bar{D}_3' \tau\bar{\nu}$ | 0.12     |
| $B_s \to D_2e\bar{\nu}$ | -0.10    | $B_s \to D_2' e\bar{\nu}$ | -0.09    | $B_s \to D_3 e\bar{\nu}$ | -0.10    |
| $\bar{B}_s \to \bar{D}_2\mu\bar{\nu}$ | -0.07    | $\bar{B}_s \to \bar{D}_2' \mu\bar{\nu}$ | -0.06    | $\bar{B}_s \to \bar{D}_3' \mu\bar{\nu}$ | -0.08    |
| $B_s \to D_2\tau\bar{\nu}$ | 0.17     | $B_s \to D_2' \tau\bar{\nu}$ | 0.20     | $B_s \to D_3' \tau\bar{\nu}$ | 0.11     |
| $B_c^- \to \bar{D}_2e\bar{\nu}$ | -0.28    | $B_c^- \to \bar{D}_2' e\bar{\nu}$ | -0.43    | $B_c^- \to \bar{D}_3 e\bar{\nu}$ | -0.24    |
| $B_c^- \to \bar{D}_2\mu\bar{\nu}$ | -0.28    | $B_c^- \to \bar{D}_2' \mu\bar{\nu}$ | -0.42    | $B_c^- \to \bar{D}_3' \mu\bar{\nu}$ | -0.23    |
| $B_c^- \to \bar{D}_2\tau\bar{\nu}$ | -0.03    | $B_c^- \to \bar{D}_2' \tau\bar{\nu}$ | -0.19    | $B_c^- \to \bar{D}_3' \tau\bar{\nu}$ | -0.01    |
| $B_c^+ \to B_2e^+\nu$ | 0.04     | $B_c^+ \to B_2' e^+\nu$ | -0.07    | $B_c^+ \to B_3 e^+\nu$ | 0.03     |
| $B_c^+ \to B_2\mu^+\nu$ | 0.23     | $B_c^+ \to B_2' \mu^+\nu$ | 0.18     | $B_c^+ \to B_3' \mu^+\nu$ | 0.24     |
| $B_c^+ \to B_2e^+\nu$ | 0.03     | $B_c^+ \to B_2' e^+\nu$ | -0.03    | $B_c^+ \to B_3 e^+\nu$ | 0.01     |

The spectra of decay widths for $\bar{B}$ and $B_c^-$ varying along with $|p_\ell|$, the absolute value of the three-momentum for charged leptons, are showed in Fig. 5. This distribution is almost the same for $\bar{B}$ decays into $D_2, D'_2$ or $D'_3$. For $B_c^- \to \bar{D}_j^*(\ell \nu)$, the momentum spectrum of $\bar{D}_2'$ is sharper than that of $\bar{D}_2$ and $\bar{D}_3'$. 

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4.2. Branching ratios of semi-leptonic decays

The semi-electronic decay widths we got are $\Gamma(\bar{B} \to D_2^2 e \bar{\nu}_e) = 4.9 \times 10^{-16}$ GeV, $\Gamma(\bar{B} \to D_2^s e \bar{\nu}_e) = 1.8 \times 10^{-16}$ GeV, and $\Gamma(\bar{B} \to D_2^* e \bar{\nu}_e) = 4.5 \times 10^{-16}$ GeV. The branching ratios of $\bar{B}$ to $D$-wave charmed mesons are listed in Tab. II. We have listed others’ results for comparison if available. Our results are about 5 times greater than that in Ref. [23]. It’s noticeable that our results for decays into $D_2$ and $D_2^*$ are in the same order, while in the results of QCD sum rules [23] $\mathcal{B}(\bar{B} \to D_2)$ is about 25 times larger than $\mathcal{B}(\bar{B} \to D_2^*)$. The branching ratios for semi-leptonic decays of $B_s$ into $D_{s2}$, $D_{s2}'$, $D_{s2}^*$, and $D_{s3}^*$ are listed in Tab. III. Our results for $B_s$ to $D$-wave charm-strange mesons are also much larger than the results of QCD sum rules in Ref. [24].

The branching ratios for $B_c$ to $D$-wave $\bar{D}_J^{(*)}$ are listed in Tab. IV. The branching ratios for semi-leptonic decays of $B_c^-$ to $\bar{D}_2$ and $\bar{D}_2^*$ are in the order of $10^{-5}$, and for $B_c^+ \to D_2^*$ the results are in the order of $10^{-6}$. These results are about 100 times smaller than that for $\bar{B}_s(\xi)$ decays owing to the different CKM matrix elements.

For completeness of this research, we also give the corresponding results for $B_c$ to the $D$-wave $B_J^{(*)}$ and $B_{sJ}^{(*)}$ in Tab. IV, although their branching ratios are quite small due to the tiny phase space. For $D$-wave bottom mesons, the semi-taunic mode is not available and for $D$-wave bottom-bottom-strange mesons, both the $\mu$ and $\tau$ modes are unavailable since the constraints of phase space. The branching ratios for $B_c^+ \to B_J^{(*)}$ are less than $10^{-8}$ and that for $B_c^+ \to B_{sJ}^{(*)}$ are less than $10^{-9}$. Based on our results, the possibilities for the $D$-wave bottomed mesons to be detected in $B_c$ decays are quite small by current experiments.
following equations for semi-taunic decay over semi-electronic decay for phase space. By simple integral over the phase space, we can find that, the phase space ratio of our fit results give that the parameters are as

\[ \Gamma(\bar{B} \to D_2 e\bar{\nu}) = \Gamma_1 [1 + \lambda_1 \cos(2\alpha + \Theta_1)], \]  

\[ \Gamma(\bar{B} \to D'_2 e\bar{\nu}) = \Gamma_2 [1 - \lambda_2 \cos(2\alpha + \Theta_2)]. \]

Our fit results give that the parameters are as

\[ \Gamma_1 = 3.46 \times 10^{-16}, \quad \lambda_1 = 0.709, \quad \Theta_1 = -23.7^\circ, \]  

\[ \Gamma_2 = 3.06 \times 10^{-16}, \quad \lambda_2 = 0.711, \quad \Theta_2 = -24.1^\circ. \]
Table IV: Semi-leptonic decay branching ratios of $B_c$ to $D$-wave heavy-light mesons with $\tau_{B_c} = 0.452 \times 10^{-12}$ s.

| Channels | $B_c^{-}\to D_2e\bar{\nu}$ | $B_c^{-}\to D_2^0e\bar{\nu}$ | $B_c^{-}\to D_3^+e\bar{\nu}$ | $B_c^{-}\to D_3^0e\bar{\nu}$ |
|----------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $B_c^{-}\to D_2\mu\bar{\nu}$ | $2.2_{-0.7}^{+0.7} \times 10^{-5}$ | $1.2_{-0.2}^{+0.2} \times 10^{-5}$ |
| $B_c^{-}\to D_2\mu\bar{\nu}$ | $2.2_{-0.7}^{+0.7} \times 10^{-5}$ | $1.2_{-0.2}^{+0.2} \times 10^{-5}$ |
| $B_c^{-}\to D_3\tau\bar{\nu}$ | $7.7_{-2.0}^{+2.0} \times 10^{-6}$ | $3.1_{-1.1}^{+1.1} \times 10^{-6}$ |
| $B_c^{-}\to B_2 e^+\nu$ | $9.4_{-1.0}^{+1.0} \times 10^{-9}$ | $1.4_{-0.3}^{+0.3} \times 10^{-10}$ |
| $B_c^{-}\to B_2 e^+\nu$ | $1.7_{-0.2}^{+0.2} \times 10^{-9}$ | $2.0_{-0.4}^{+0.4} \times 10^{-11}$ |
| $B_c^{-}\to B_2 e^+\nu$ | $3.3_{-0.2}^{+0.2} \times 10^{-9}$ | $5.6_{-1.7}^{+1.7} \times 10^{-13}$ |

Table V: $\mathcal{R}(D_s^{(*)}) = \frac{\Gamma(B\to D_s^{(*)}\tau\bar{\nu})}{\Gamma(B\to D_s^{(*)}e\bar{\nu})}$, $\mathcal{R}(D_s^{(*)}) = \frac{\Gamma(B_s\to D_s^{(*)}\tau\bar{\nu})}{\Gamma(B_s\to D_s^{(*)}e\bar{\nu})}$, and ratios of semi-taonic branching ratio to semi-electronic branching ratio for $B_s$, $\bar{B}_s$, and $B_c$ to $D$-wave charmed mesons.

| Modes | $D_2$ | $D_2'$ | $D_s$ | $D_s'$ | $D_2$ | $D_2'$ | $D_s$ | $D_s'$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\mathcal{R}$ | 0.0071 | 0.0065 | 0.0052 | 0.0079 | 0.0066 | 0.0064 | 0.35 | 0.29 | 0.25 |

The tiny differences in parameters for $D_2$ and $D_2'$ come from the small difference between $m_{D_2}$ and $m_{D_2'}$. In Fig. 6(c) and Fig. 6(d), we also show the ratios $\Gamma(B\to D_2 e\bar{\nu})/\Gamma(B\to D_2' e\bar{\nu})$ and $H(B_s\to D_2 e\bar{\nu})/\Gamma(B_s\to D_2' e\bar{\nu})$, which are very sensitive to the mixing angle.

4.3. Non-leptonic decay widths and branching ratios

The non-leptonic decay widths are listed in Tab. VI, where we have kept the Wilson coefficient $a_1$ in order to facilitate comparison with other models. The corresponding branching ratios are listed in Tab. VII, where we have specified the values $a_1^b = 1.14$ for $b\to c(u)$ transition and $a_1^c = 1.2$ for $c\to d(s)$ transition [48]. From the non-leptonic decay results we can see that, with the same final $D$ meson, the $\rho$ mode has the largest branching ratio and can reach $10^{-3}$ order in $B_s$ decays, and $10^{-6}$ order in $B_c$ decay. When the light mesons have the same quark constituents, the width for decay into vector meson ($\rho, K^*$) mode is about $2 \sim 3$ times greater than its pseudoscalar meson ($\pi, K$) mode.

5. Summary

In this work we calculated semi-leptonic and non-leptonic decays of $\bar{B}_s$ into $D$-wave charmed mesons ($D(s)2, D(s)2', D(s)3$) and $B_c$ into $D$-wave charmed and bottomed excited mesons. Form factors of hadronic transition are calculated by instantaneous Bethe-Salpeter methods. The semi-electronic branching ratios for $B_s\to D_s^{(*)}$ we got are about $10^{-3}$ order, and for $B_c$ to $D$-wave charmed mesons are about $10^{-5}$ order. The non-leptonic branching ratios for decays to $\rho$ mode can reach $10^{-3}$ order for $\bar{B}_s$ decays. So the $D$-wave $D$ and $D_s$ mesons are hopefully to be detected in $\bar{B}_s$ decays by current experiments. Our results reveal the branching fractions for $B_c$ to $D$-wave bottomed mesons are less than $10^{-8}$, which makes the $D$-wave bottomed mesons almost impossible to be discovered in $B_c$ decays by current experiments.

We also present the angular distribution and charged lepton spectra for $\bar{B}$ and $B_c$ decays. The $2^-$ states $D_2$ and $D_2'$ are the mixing states of $^1D_2-^3D_2$, so we present the dependence of the decay width varying along with the mixing angle. Based on our results, the semi-leptonic and non-leptonic branching ratios for $\bar{B}_s$ decays to the $D$-wave charm and charm-strange mesons have
Fig. 6: Decay widths $\Gamma(\bar{B} \to D_2^0 e\bar{\nu})$ and $\Gamma(B_c^- \to \bar{D}_2^0 (\bar{D}_2^0) e\bar{\nu})$ vary along with the mixing angle. The vertical solid line shows the results when mixing angle $\alpha = 39.23^\circ$, where the decay width ratio is 2.73 for $\bar{B} \to D_2^0 D_2^0 e\bar{\nu}$ and 5.63 for $B_c^- \to \bar{D}_2^0 (\bar{D}_2^0) e\bar{\nu}$.

reached the experimental detection thresholds. These results would be helpful in future detecting and understanding these new $D$-wave excited $D_{(s)}$ mesons.

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Table VI: Non-leptonic decay widths of $\bar{B}$, $\bar{B}_s$ and $B_c$ to $D$-wave heavy-light meson with general Wilson coefficient $a_1$.

| Channels | Width | Channels | Width | Channels | Width |
|----------|-------|----------|-------|----------|-------|
| $\bar{B}\rightarrow D_2\pi^-$ | $2.8^{+0.7}_{-0.6}\times 10^{-16}$ | $\bar{B}\rightarrow D_2^*\pi^-$ | $9.8^{+2.0}_{-1.4}\times 10^{-17}$ | $\bar{B}\rightarrow D_3\pi^-$ | $2.2^{+0.4}_{-0.3}\times 10^{-16}$ |
| $\bar{B}\rightarrow D_2K^-$ | $2.0^{+0.5}_{-0.4}\times 10^{-17}$ | $\bar{B}\rightarrow D_2^*K^-$ | $6.7^{+1.4}_{-1.5}\times 10^{-18}$ | $\bar{B}\rightarrow D_3^*K^-$ | $1.4^{+0.3}_{-0.2}\times 10^{-17}$ |
| $\bar{B}\rightarrow D_2\rho^-$ | $5.5^{+1.4}_{-1.2}\times 10^{-16}$ | $\bar{B}\rightarrow D_2^*\rho^-$ | $2.0^{+0.4}_{-0.3}\times 10^{-17}$ | $\bar{B}\rightarrow D_3^*\rho^-$ | $4.7^{+0.9}_{-0.8}\times 10^{-16}$ |
| $\bar{B}\rightarrow D_2K^{**}$ | $2.9^{+0.7}_{-0.6}\times 10^{-17}$ | $\bar{B}\rightarrow D_2^*K^{**}$ | $1.0^{+0.2}_{-0.2}\times 10^{-17}$ | $\bar{B}\rightarrow D_3^*K^{**}$ | $2.5^{+0.4}_{-0.4}\times 10^{-17}$ |
| $\bar{B}_s\rightarrow D_{s2}\pi^-$ | $4.0^{+1.0}_{-1.1}\times 10^{-16}$ | $\bar{B}_s\rightarrow D_{s2}\pi^-$ | $1.2^{+0.4}_{-0.3}\times 10^{-16}$ | $\bar{B}_s\rightarrow D_{s3}\pi^-$ | $2.9^{+0.8}_{-0.7}\times 10^{-16}$ |
| $\bar{B}_s\rightarrow D_{s2}K^-$ | $2.8^{+0.8}_{-0.7}\times 10^{-17}$ | $\bar{B}_s\rightarrow D_{s2}^*K^-$ | $8.2^{+2.4}_{-2.3}\times 10^{-18}$ | $\bar{B}_s\rightarrow D_{s3}K^-$ | $1.9^{+0.5}_{-0.4}\times 10^{-17}$ |
| $\bar{B}_s\rightarrow D_{s2}\rho^-$ | $8.1^{+2.3}_{-2.2}\times 10^{-16}$ | $\bar{B}_s\rightarrow D_{s2}\rho^-$ | $2.4^{+0.7}_{-0.6}\times 10^{-16}$ | $\bar{B}_s\rightarrow D_{s3}\rho^-$ | $6.4^{+1.7}_{-1.6}\times 10^{-16}$ |
| $\bar{B}_s\rightarrow D_{s2}K^{**}$ | $4.2^{+1.2}_{-1.1}\times 10^{-17}$ | $\bar{B}_s\rightarrow D_{s2}^*K^{**}$ | $1.3^{+0.4}_{-0.3}\times 10^{-17}$ | $\bar{B}_s\rightarrow D_{s3}^*K^{**}$ | $3.4^{+0.9}_{-0.8}\times 10^{-17}$ |
| $B_c^-\rightarrow \bar{D}_2\pi^-$ | $1.1^{+0.4}_{-0.3}\times 10^{-18}$ | $B_c^-\rightarrow \bar{D}_2\pi^-$ | $2.5^{+0.7}_{-0.6}\times 10^{-19}$ | $B_c^-\rightarrow \bar{D}_3\pi^-$ | $1.1^{+0.3}_{-0.2}\times 10^{-18}$ |
| $B_c^-\rightarrow \bar{D}_2K^-$ | $9.0^{+0.3}_{-0.3}\times 10^{-20}$ | $B_c^-\rightarrow \bar{D}_2K^-$ | $2.0^{+0.6}_{-0.5}\times 10^{-20}$ | $B_c^-\rightarrow \bar{D}_3^*K^-$ | $8.4^{+2.2}_{-2.1}\times 10^{-20}$ |
| $B_c^-\rightarrow \bar{D}_2\rho^-$ | $3.5^{+1.9}_{-1.8}\times 10^{-18}$ | $B_c^-\rightarrow \bar{D}_2\rho^-$ | $7.8^{+2.2}_{-2.1}\times 10^{-19}$ | $B_c^-\rightarrow \bar{D}_3\rho^-$ | $3.1^{+1.8}_{-1.7}\times 10^{-18}$ |
| $B_c^-\rightarrow \bar{D}_2K^{**}$ | $2.1^{+0.6}_{-0.5}\times 10^{-19}$ | $B_c^-\rightarrow \bar{D}_2K^{**}$ | $4.7^{+1.3}_{-1.2}\times 10^{-20}$ | $B_c^-\rightarrow \bar{D}_3K^{**}$ | $1.9^{+0.5}_{-0.4}\times 10^{-19}$ |
| $B_c^+\rightarrow B_2\pi^+$ | $1.4^{+0.1}_{-0.0}\times 10^{-19}$ | $B_c^+\rightarrow B_2\pi^+$ | $1.3^{+0.3}_{-0.2}\times 10^{-21}$ | $B_c^+\rightarrow B_3\pi^+$ | $1.9^{+0.4}_{-0.3}\times 10^{-21}$ |

A. Expressions for $N_s$s in the Hadronic Tensor $H_{\mu\nu}$

The hadronic tensor $N_i$ ($i = 1, 2, 4, 5, 6$) for $\bar{B}$ to $D_2$ meson are

\[
N_1 = \frac{2M^4p_F^2s_1}{3M_F^2} - \frac{4M^2p_F^2s_1s_3}{3M_F^2} - \frac{1}{2}M^2p_F^2s_2^2 + \frac{s_3^2}{6}.
\]

\[
N_2 = \frac{2E_FM^3p_F^2s_1s_3}{3M_F^2} + \frac{E_FM^3p_F^2s_2^2}{2M_F^2} - \frac{E_FM^2s_2^2}{6M_F^2} + \frac{2M^4p_F^4s_1s_2}{3M_F^2} - \frac{2M^2p_F^2s_2s_3}{3M_F^2}.
\]

\[
N_4 = \frac{4E_FM^3p_F^2s_2s_3}{3M_F^2} + \frac{2M^4p_F^4s_2^2}{2M_F^2} - \frac{M^2p_F^2s_3^2}{2M_F^2} + \frac{M^2s_2^2(M_F^2 + 4p_F^2)}{6M_F^2},
\]

\[
N_5 = -\frac{M^4p_F^4s_2^2}{2M_F^2} - \frac{M^2p_F^2s_3s_4}{2M_F^2},
\]

\[
N_6 = -\frac{M^2p_F^2s_3s_4}{M_F^2}.
\]

Here $p_F$ denotes the three-momentum of final $D$ systems and $E_F = \sqrt{M_F^2 + p_F^2}$. For $\bar{B}$ to $D_2$ the relations between $N_i$ and form factors $t_k$ ($k = 1, 2, 3, 4$) have the same form with that for $D_2$, just $s_k$ are replaced with $t_k$. Both $s_k$ and $t_k$ are functions of $q^2_\perp$. 


Table VII: Branching ratios of non-leptonic decays for $\bar{B}$, $\bar{B}_s$ and $B_c$ to $D$-wave heavy-light mesons. $a^b_l = 1.14$ for $b$ quark decay and $a^c_l = 1.2$ for $c$ quark decay.

| Channels | Br   | Channels | Br   | Channels | Br   |
|----------|------|----------|------|----------|------|
| $\bar{B}\to D_2^0\pi^-$ | $8.5^{+2.2}_{-1.1}$ | $\bar{B}\to D_2^0\pi^-$ | $2.9^{+0.6}_{-0.5}$ | $\bar{B}\to D_3^0\pi^-$ | $6.5^{+1.2}_{-1.1}$ |
| $\bar{B}\to D_2^0K^-$ | $5.9^{+1.5}_{-1.4}$ | $\bar{B}\to D_2^0K^-$ | $2.0^{+0.4}_{-0.3}$ | $\bar{B}\to D_3^0K^-$ | $4.3^{+0.8}_{-1.0}$ |
| $\bar{B}\to D_2^0\rho^-$ | $1.7^{+0.8}_{-0.7}$ | $\bar{B}\to D_2^0\rho^-$ | $5.9^{+1.3}_{-1.2}$ | $\bar{B}\to D_3^0\rho^-$ | $1.4^{+0.3}_{-0.3}$ |
| $\bar{B}\to D_2^0K^{*-}$ | $8.6^{+2.2}_{-2.1}$ | $\bar{B}\to D_2^0K^{*-}$ | $3.1^{+0.7}_{-0.6}$ | $\bar{B}\to D_3^0K^{*-}$ | $7.5^{+1.3}_{-1.2}$ |
| $\bar{B}_s\to D_{s2}\pi^-$ | $1.2^{+0.3}_{-0.3}$ | $\bar{B}_s\to D_{s2}^0\pi^-$ | $3.6^{+1.0}_{-1.1}$ | $\bar{B}_s\to D_{s3}\pi^-$ | $8.5^{+2.3}_{-2.2}$ |
| $\bar{B}_s\to D_{s2}^0K^-$ | $8.3^{+2.3}_{-2.1}$ | $\bar{B}_s\to D_{s2}^0K^-$ | $2.5^{+0.7}_{-0.6}$ | $\bar{B}_s\to D_{s3}^0K^-$ | $5.7^{+1.6}_{-1.5}$ |
| $\bar{B}_s\to D_{s2}\rho^-$ | $2.4^{+0.7}_{-0.6}$ | $\bar{B}_s\to D_{s2}\rho^-$ | $7.3^{+2.1}_{-2.2}$ | $\bar{B}_s\to D_{s3}\rho^-$ | $1.9^{+0.5}_{-0.4}$ |
| $\bar{B}_s\to D_{s2}K^{*-}$ | $1.3^{+0.4}_{-0.3}$ | $\bar{B}_s\to D_{s2}K^{*-}$ | $3.8^{+1.1}_{-1.2}$ | $\bar{B}_s\to D_{s3}K^{*-}$ | $1.0^{+0.3}_{-0.3}$ |
| $B_c^-\to D_2^0\pi^-$ | $1.0^{+0.3}_{-0.3}$ | $B_c^-\to D_2^-\pi^-$ | $2.2^{+0.6}_{-0.5}$ | $B_c^-\to D_3^-\pi^-$ | $9.6^{+2.6}_{-2.5}$ |
| $B_c^-\to D_2^-K^-$ | $8.0^{+2.6}_{-2.5}$ | $B_c^-\to D_2^-K^-$ | $1.7^{+0.8}_{-0.7}$ | $B_c^-\to D_3^-K^-$ | $7.5^{+2.1}_{-2.0}$ |
| $B_c^-\to D_2^-\rho^-$ | $3.1^{+1.0}_{-1.1}$ | $B_c^-\to D_2^-\rho^-$ | $6.9^{+2.0}_{-2.0}$ | $B_c^-\to D_3^-\rho^-$ | $2.8^{+0.7}_{-0.7}$ |
| $B_c^-\to D_2^-K^{*-}$ | $1.9^{+0.6}_{-0.6}$ | $B_c^-\to D_2^-K^{*-}$ | $4.2^{+1.2}_{-1.2}$ | $B_c^-\to D_3^-K^{*-}$ | $1.7^{+0.4}_{-0.4}$ |
| $B_c^+\to B_2^0\pi^+$ | $1.4^{+0.1}_{-0.1}$ | $B_c^+\to B_2^+\pi^+$ | $1.3^{+0.2}_{-0.2}$ | $B_c^+\to B_3^0\pi^+$ | $1.9^{+0.4}_{-0.4}$ |

The hadronic tensor $N_i$ for $\bar{B}$ to $D^*_c$ are expressed with form factors $h_k$ ($k = 1, 2, 3, 4$) as

\begin{align}
N_1 &= \frac{2M^6p^2_Fh^4_1}{5M_F^8} - \frac{4M^4p^2_Fh^2_1h^2_3}{5M_F^6} - \frac{4M^4p^2_Fh^2_1h^2_3}{15M_F^4} + \frac{2M^2p^2_Fh^2_3}{15M_F^2}, \\
N_2 &= \frac{2ETF^6p^4_Fh^2_3}{5M_F^6} + \frac{4ETF^6p^2_Fh^2_1}{15M_F^4} - \frac{2ETF^4p^2_Fh^2_1h^2_3}{15M_F^2} + \frac{2M^2p^2_Fh^2_3}{15M_F^2}, \\
N_4 &= \frac{4ETF^6p^4_Fh^2_3}{15M_F^4} + \frac{2M^6p^2_Fh^2_3}{15M_F^2} - \frac{4M^6p^2_Fh^2_3}{15M_F^4} + \frac{2M^4p^2_Fh^2_3(2M^2_F + 3p^2_F)}{15M_F^4}, \\
N_5 &= -\frac{4M^6p^2_Fh^2_3}{15M_F^4} - \frac{4M^4p^2_Fh^2_3}{15M_F^4}, \\
N_6 &= -\frac{8M^4p^2_Fh^2_3}{15M_F^4}. 
\end{align}

B. Full Salpeter equations and the numerical solutions

B.1. Salpeter equations

Salpeter wave function $\varphi(q_\perp)$ is related to BS wave function $\Psi(q)$ by the following definition

\begin{equation}
\varphi(q_\perp) = i \int \frac{dp_\parallel}{2\pi} \Psi(q), \quad \eta(q_\perp) = \int \frac{d^3k_\perp}{(2\pi)^3} \varphi(k_\perp)V(|q_\perp - k_\perp|),
\end{equation}

where the 3-dimensional integration $\eta(q_\perp)$ can be understood as the BS vertex for bound states, and $V(|q_\perp - k_\perp|)$ denotes the instantaneous interaction kernel.
The projection operators \( \Lambda_i^\pm(q_\perp) \) (\( i = 1 \) for quark and \( 2 \) for anti-quark) are defined as
\[
\Lambda_i^\pm = \frac{1}{2\omega_i} \left[ \hat{p} M \omega_i \pm (-1)^i 1 (m_i + \not{q}_\perp) \right].
\tag{B.2}
\]

Then we define four wave functions \( \varphi^{\pm\pm} \) by \( \varphi \) and \( \Lambda_i^\pm \) as
\[
\varphi^{\pm\pm} = \Lambda_i^\pm(q_\perp) \frac{\hat{p}}{M} \varphi(q_\perp) \frac{\hat{p}}{M} \Lambda_j^\pm(q_\perp),
\tag{B.3}
\]
where \( \varphi^{++} \) and \( \varphi^{--} \) are called the positive and negative Salpeter wave function, respectively. And we can easily check that \( \varphi = \varphi^{++} + \varphi^{--} + \varphi^{+-} + \varphi^{-+} \).

The full coupled Salpeter equations then can be expressed as [32]
\[
\varphi^{+-} = \varphi^{-+} = 0,
\tag{B.4}
\]
\[
(M - \omega_1 - \omega_2) \varphi^{++} = +\Lambda_1^+(q_\perp) \eta(q_\perp) \Lambda_2^+(q_\perp),
\tag{B.5}
\]
\[
(M + \omega_1 + \omega_2) \varphi^{--} = -\Lambda_1^-(q_\perp) \eta(q_\perp) \Lambda_2^-(q_\perp).
\tag{B.6}
\]

From above equations, we can see that in the weak binding condition \( M \sim (\omega_1 + \omega_2) \), \( \varphi^{--} \) is much smaller compared with \( \varphi^{++} \), and can be ignored in the calculations. The normalization condition for Salpeter wave function reads
\[
\int \frac{d^3 q_\perp}{(2\pi)^3} \left[ \varphi^{++} \frac{\hat{p}}{M} \varphi^{++} - \varphi^{--} \frac{\hat{p}}{M} \varphi^{--} \right] = 2M.
\tag{B.7}
\]

**B.2. Numerical solutions of \( 0^- \) state**

Now we take the \( 0^- (^1S_0) \) state as an example to show the details of achieving Saperter equations' numerical results. The Salpeter wave function for \( 0^- (^1S_0) \) state has the following general form [55]
\[
\varphi(^1S_0) = M \left[ k_1 \frac{\hat{p}}{M} + k_2 + k_3 \frac{\not{q}_\perp}{M} + k_4 \frac{\hat{p} \not{q}_\perp}{M^2} \right] \gamma^5.
\tag{B.8}
\]

By utilizing the Eq. (B.4), we have the following two constraint conditions
\[
k_3 = -\frac{M(\omega_1 - \omega_2)}{m_1 \omega_2 + m_2 \omega_1} k_2, \quad k_4 = -\frac{M(\omega_1 + \omega_2)}{m_1 \omega_2 + m_2 \omega_1} k_1.
\tag{B.9}
\]
In above wave function, the only undetermined wave functions are \( k_1 \) and \( k_2 \), which are the functions of \( q_\perp^2 \).

By using the definition Eq. (B.3), we can easily get the positive Salpeter wave function of \( ^1S_0 \) state as Eq. (17), and the corresponding constraint conditions Eq. (18). Similarly, the Salpeter negative wave function \( \varphi^{--}(^1S_0) \) is expressed as
\[
\varphi^{--}(^1S_0) = \left[ Z_1 + Z_2 \frac{\hat{p}}{M} + Z_3 \frac{\not{q}_\perp}{M} + Z_4 \frac{\hat{p} \not{q}_\perp}{M^2} \right] \gamma^5.
\tag{B.10}
\]

\( Z_i \) (\( i = 1, 2, 3, 4 \)) has the following forms
\[
Z_1 = \frac{M}{2} \left[ k_2 - \frac{\omega_1 + \omega_2}{m_1 + m_2} k_1 \right], \quad Z_3 = -\frac{M(\omega_1 - \omega_2)}{m_1 \omega_2 + m_2 \omega_1} Z_1,
\tag{B.11}
\]
\[
Z_2 = \frac{M}{2} \left[ k_1 - \frac{m_1 + m_2}{\omega_1 + \omega_2} k_2 \right], \quad Z_4 = +\frac{M(m_1 + m_2)}{m_1 \omega_2 + m_2 \omega_1} Z_1.
\]
And now the normalization condition Eq. (B.7) becomes

\[ \int \frac{d^3 q}{(2\pi)^3} \frac{8M\omega_1\omega_2k_1k_2}{(m_1\omega_2 + m_2\omega_1)} = 1. \] (B.12)

Inserting the Salpeter positive wave function Eq. (17), and negative wave function Eq. (B.10) into Salpeter equations Eq. (B.5) and (B.6) respectively, we can obtain two coupled eigen equations on \( k_1 \) and \( k_2 \) [55] as

\[
\begin{cases}
(M - \omega_1 - \omega_2) [ck_1(q) + k_2(q)] &= \frac{1}{2\omega_1\omega_2} \int d^3 k \left[ \beta_1 k_1(k) + \beta_2 k_2(k) \right], \\
(M + \omega_1 + \omega_2) [k_2(q) - ck_1(q)] &= \frac{1}{2\omega_1\omega_2} \int d^3 k \left[ \beta_1 k_1(k) - \beta_2 k_2(k) \right],
\end{cases}
\] (B.13)

where we have used definition \( c = \frac{\omega_1 + \omega_2}{m_1 + m_2} \) and the shorthand

\[
\beta_1 = k \cdot q(V_s + V_v) \frac{(\nu_1 + \nu_2)(\omega_1 + \omega_2)}{m_1\nu_2 + m_2\nu_1} - (V_s - V_v)(m_1\omega_2 + m_2\omega_1), \\
\beta_2 = k \cdot q(V_s + V_v) \frac{(\nu_1 - \nu_2)(m_1 - m_2)}{m_1\nu_2 + m_2\nu_1} - (V_s - V_v)(m_1m_2 + \omega_1\omega_2 + q^2). \] (B.14)

In above equations, \( V_s \) and \( V_v \) are the scalar and vector parts defined in Cornell potential (see Eq. 28) respectively; we have used the definition \( \nu_i = \sqrt{m_i^2 + k^2} \) \( (i = 1, 2) \).

Then by solving the two coupled eigen equations numerically, we achieve the mass spectrum and corresponding wave functions \( k_1 \) and \( k_2 \). Repeating the similar procedures we can obtain the numerical wave functions for \( ^1D_2 \), \( ^3D_2 \) and \( ^3D_3 \) states. Interested reader can see more details on solving the full Salpeter equations in Refs. [35, 55, 56].

C. Positive Salpeter wave function for \( ^1S_0 \), \( ^1D_2 \) and \( ^3D_2 \)

The positive Salpeter wave function and its constraint conditions for \( ^1D_2 \) state [41] are displayed in C.1 and C.2. And the undetermined wave function are \( f_1 \) and \( f_2 \).

\[
\psi_D(^1D_2) = \epsilon_{\mu\nu} q_1^\mu q_2^\nu \left[ b_1 + b_2 \frac{P_F}{M_F} + b_3 \frac{q_1^\mu M_F}{P_F} + b_4 \frac{P_F q_1^\mu}{M_F^2} \right] \gamma^5, \] (C.1)

\[
b_1 = \frac{1}{2} \left[ f_1 + \frac{\omega_1' + \omega_2'}{m_1' + m_2'} f_2 \right], \quad b_3 = -M_F(\omega_1' - \omega_2') b_1, \quad b_2 = \frac{1}{2} \left[ f_2 + \frac{m_1' + m_2'}{\omega_1' + \omega_2'} f_1 \right], \quad b_4 = -M_F(\omega_1' + \omega_2') b_1. \] (C.2)

The positive Salpeter wave function of \( ^3D_2 \) state [56] and constraint conditions can be written as

\[
\psi_D(^3D_2) = i\epsilon_{\mu\nu\alpha\beta} \frac{P_F}{M_F} q_1^\mu e^{-i\beta} q_2^\nu \gamma^\alpha \left[ i_1 + i_2 \frac{P_F}{M_F} + i_3 \frac{q_1^\mu}{M_F} + i_4 \frac{P_F q_1^\mu}{M_F^2} \right]. \] (C.3)

\[
i_1 = \frac{1}{2} \left[ v_1 - \frac{\omega_1' + \omega_2'}{m_1' + m_2'} v_2 \right], \quad i_3 = + M_F(\omega_1' - \omega_2') i_1, \quad i_2 = \frac{1}{2} \left[ v_2 - \frac{m_1' + m_2'}{\omega_1' + \omega_2'} v_1 \right], \quad i_4 = -M_F(\omega_1' + \omega_2') i_1. \] (C.4)

Here we also only have two undetermined wave function \( v_1 \) and \( v_2 \).

In above equations C.1 ~ C.4 the indeterminate wave functions, such as \( f_1 \) and \( f_2 \) in \( \psi_D(^1D_2) \), \( v_1 \) and \( v_2 \) in \( \psi_D(^3D_2) \), which are functions of \( q_2^\mu \) and can be determined numerically by solving the
coupled Salpeter eigen equations B.5 and B.6.

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