Dynamics of threshold solutions for energy critical NLW with inverse square potential

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Received: 5 March 2022 / Accepted: 18 May 2022 / Published online: 23 June 2022
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Abstract
We consider the focusing energy critical NLW with inverse square potential in dimensions \(d = 3, 4, 5\). Solutions on the energy surface of the ground state are characterized. We prove that solutions with kinetic energy less than that of the ground state must scatter to zero or belong to the stable/unstable manifold of the ground state. In the latter case they converge to the ground state exponentially in the energy space as \(t \to +\infty\) or \(t \to -\infty\). When the kinetic energy is greater than that of the ground state, we show that all solutions with finite mass blow up in finite time in both time directions in \(d = 3, 4\). In \(d = 5\), a finite mass solution can either have finite lifespan or lie on the stable/unstable manifolds of the ground state. The proof relies on the detailed spectral analysis of the linearized operator, local invariant manifold theory, and a global Virial analysis.

Keywords Inverse square potential · Ground state solution · Energy critical · NLW

1 Introduction
In this paper, we consider the Cauchy problem of the focusing energy critical nonlinear wave equation with an inverse square potential,

\[
\begin{align*}
\partial_t^2 u + \mathcal{L}_a u &= |u|^4 u, \\
|u|_{t=0} &= u_0, \quad \partial_t u |_{t=0} &= u_1,
\end{align*}
\] (NLW_\alpha)

where \((u_0, u_1) \in \dot{H}^1_\alpha(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\) and we focus on dimensions \(d = 3, 4, 5\). The operator \(\mathcal{L}_a\) is defined as \(\mathcal{L}_a = -\Delta + \frac{a}{|x|^2}\) for some real number \(a > -\frac{(d-2)^2}{2}\). From sharp Hardy’s...
inequality, $\mathcal{L}_a$ is positive and the norm defined by

$$\|f\|_{H^1_a}^2 = \langle \mathcal{L}_a f, f \rangle = \|f\|_{H^1}^2 + \int_{\mathbb{R}^d} \frac{a}{|x|^2} |f(x)|^2 \, dx$$

gives rise to an equivalent norm of Sobolev space $\dot{H}^1(\mathbb{R}^d)$.

In physics, the operator $\mathcal{L}_a$ oftentimes arises in scaling limits of some complicated models: the combustion theory, the Dirac equation with Coulomb potential, and the linearized perturbations of space-time metrics [4, 18, 20, 27, 28]. Besides its physical interest, the operator is mathematically interesting as the potential contains an intrinsic singularity at the origin, thus can neither be treated perturbatively, nor be included into the Kato class.

Throughout this paper, we focus only on the strong solutions, which are spacetime functions obeying the Duhamel’s formula:

$$u(t) = \cos(t \sqrt{\mathcal{L}_a}) u_0 + \sin(t \sqrt{\mathcal{L}_a}) \frac{u_1}{\sqrt{\mathcal{L}_a}} + \int_0^t \sin((t-s) \sqrt{\mathcal{L}_a}) \left( |u|^{\frac{4}{d-2}} u \right)(s) \, ds,$$

and also belonging to certain Strichartz spaces. Such solutions can be constructed via Strichartz methodology when $a$ is some distance away from the endpoint decided by the Sharp Hardy’s inequality: $a > -\left( \frac{d-2}{2} \right)^2 + \left( \frac{d-2}{d+2} \right)^2$ [19]. On the lifespan of the solution, we have the conservation of energy

$$E[\tilde{u}] = \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u|^2 + \frac{1}{2} \frac{a|u|^2}{|x|^2} + \frac{1}{2} |\partial_t u|^2 - \frac{d-2}{2d} |u|^{\frac{2d}{d-2}} \, dx,$$

where $\tilde{u} = (u, \partial_t u)$. The problem is energy critical in the sense that the energy is invariant under the scaling of the equation:

$$(u, \partial_t u) \mapsto (\lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x), \lambda^\frac{d}{2} \partial_t u(\lambda t, \lambda x)).$$

As has been seen in the free case ($a = 0$), for large solutions there is a blowup/scattering dichotomy, separated by the ground state, which is the constrained energy minimizer in our setting. In earlier works [1, 13, 23, 24], it was shown that when $a \in (-\left( \frac{d-2}{2} \right)^2, 0]$, the ground state is a static solution of NLW$_a$:

$$\mathcal{L}_a W - W_{\frac{d+2}{d-2}} = 0,$$

and is given explicitly by

$$W(x) = [d(d-2)\beta^2]^{\frac{d-2}{2}} \left( \frac{|x|^{\beta-1}}{1 + |x|^{2\beta}} \right)^{\frac{d-2}{2}}, \quad \beta = \sqrt{1 + \left( \frac{2}{d-2} \right)^2 a}.$$  \hspace{1cm} (1.1)

When $a > 0$, function defined above remains to be a static solution but fails to be a ground state in the nonradial case. Considering the main purpose of this paper, we restrict our attention to $a < 0$.

The blowup/scattering dichotomy result for NLW$_a$ was proved by Miao et al. [19] by following a standard concentration compactness argument with the adaptation to fit the case with broken symmetries. See for example the case of NLW by Kenig and Merle [11] and the case of NLS$_a$ by Killip et al. [13]. For the convenience of the reference, we cite the following result on NLW$_a$. 

\[ Springer \]
Theorem 1.1 [19] Let $d \in \{3, 4, 5\}$, $0 > a > -\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{d+2}\right)^2$ and $(u_0, u_1) \in \dot{H}^1_\alpha(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Assume that $E(u_0, u_1) < E(W, 0)$. Let $u(t, x)$ be the maximal lifespan solution of NLW$_\alpha$ on $I \times \mathbb{R}^d$.

(i) If $\|u_0\|_{\dot{H}^1_\alpha} < \|W\|_{\dot{H}^1_\alpha}$, then $I = \mathbb{R}$ and $\|u(t)\|_{L^2(t, \infty)} < \infty$. Consequently, the solution scatters in both time directions.

(ii) If $\|u_0\|_{\dot{H}^1_\alpha} > \|W\|_{\dot{H}^1_\alpha}$, then $u(t, x)$ blows up at finite time in both time directions.

In this paper, we focus on the study of the dynamic of solutions on the energy surface of the ground state. This problem was initiated by Duyckaerts–Merle for the focusing energy critical nonlinear Schrödinger and wave equation in their seminal works [7, 8] in dimensions $d = 3, 4, 5$. Their results were later extended to higher dimensions in [14, 15]. See also [3, 22] for the recent progress on the dynamics of threshold solutions of NLS.

In our previous work [25] joint with Zeng, despite the singular external potential, we are able to establish a similar result for NLS$_\alpha$ without symmetry assumption. The goal of this paper is to continue the study of dynamics of solutions on the energy surface of ground state in the setting of energy critical NLW$_\alpha$. Our main results: the existence of smooth stable/unstable manifold of $W$ and the characterization of solutions on the energy surface of $E(W, 0)$, are stated in the following two theorems.

Theorem 1.2 Let $d \in \{3, 4, 5\}$ and $a \in (-\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{d+2}\right)^2, 0)$. There exist solutions $W^+$ and $W^-$ to NLW$_\alpha$ such that

$$E(W^\pm, \partial_t W^\pm) = E(W, 0),$$

$$\|W^\pm(t) - W\|_{\dot{H}^1} \leq C e^{-ct}, \quad \|W^-\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}, \quad \|W^+\|_{\dot{H}^1_a} > \|W\|_{\dot{H}^1_a},$$

for some $C, c > 0$. They are also unique in this class up to time translation. In addition,

$$W^\pm$$

are spherical symmetric solutions,

$$\int_{-\infty}^0 \int_{\mathbb{R}^d} |W^-(t, x)|^{\frac{2(d+1)}{d-2}} \, dx \, dt < \infty, \quad W^\pm - W \in L^2(\mathbb{R}^d).$$

Remark 1.3 Theorem 1.2 indeed constructs the 1D stable manifold of $W$ in $\dot{H}^1(\mathbb{R}^d)$, which is a smooth curve tangent to the linear stable direction at $W$. Its two branches, divided by $W^-$, are exactly the trajectories of $W^+(t)$ and $W^-(t)$. By the time reversal symmetry of NLW$_\alpha$, the 1D unstable manifold of $W$ is given by $W^\pm(-t)$.

Next we characterize solutions on the energy surface. As the identification of solutions hold by modulo symmetries of the equation, it is necessary to clarify the following concept.

Definition 1.4 We say $u(t, x) = v(t, x)$ up to symmetries if there is a time $T$ and $\lambda > 0$ such that $u$ equals one of the four options when alternating the signs:

$$\pm \lambda^{\frac{2-d}{4}} v(T \pm \frac{t}{\lambda}, \frac{x}{\lambda}).$$

We have the following

Theorem 1.5 Let $d \in \{3, 4, 5\}$ and $a \in (-\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{d+2}\right)^2, 0)$. Suppose $(u, \partial_t u) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ is a solution of NLW$_\alpha$ satisfying $E(u, \partial_t u) = E(W, 0)$. We have

a) If $\|u_0\|_{\dot{H}^1_\alpha} = \|W\|_{\dot{H}^1_\alpha}$, then $u(t, x) = W(x)$ up to symmetries.$^2$

1 Although the Theorem in [19] is stated for $d = 3, 4$, their argument can be easily extended to dimension $d = 5$, we thank the authors for clarification.

2 We remark that statement a) follows directly from the variational characterization of the ground state, see Lemma 2.2 below.
b) If $\|u_0\|_{H^1_{\alpha}} < \|W\|_{H^1_{\alpha}}$, then $u$ must be a global solution. In this case, $u$ must conform into the following two cases: Either $\|u\|_{L^{\frac{2(d+1)}{d-2}}(\mathbb{R} \times \mathbb{R}^d)} < \infty$ and $u$ scatters in both time directions, or $u(t, x) = W^-(t, x)$ up to symmetries.

c) If $\|u_0\|_{H^1_{\alpha}} > \|W\|_{H^1_{\alpha}}$ and $u_0 \in L^2(\mathbb{R}^d)$, then in $d = 3, 4$, the lifespan of $u$ is finite; in $d = 5$, $u$ must conform into the following two cases: either the lifespan of $u$ is finite or $u = W^+(t, x)$ up to symmetries.

Let us make some comments on the method we apply to prove these two results. First of all, our method in proving Theorem 1.2, i.e., the construction of stable/unstable manifold (or $W^{\pm}$) is completely different from that in Duyckaerts–Merle’s work [8]. Instead of constructing adhockly a series of Schwartz functions expansion converging to $W$, we apply the Lyapunov–Perron framework, which is based on the exponential trichotomy decomposition for the linearized operator around the ground state $W$. In our previous work [25] with Zeng, we had successfully applied this approach to deal with the energy critical NLS after a comprehensive study on the spectral properties for the linearized operator. In the case of NLW$_{\alpha}$, we apply the Lyapunov–Perron method in the framework of second order differential equation (see also [5, 9, 21]). The proof relies on a thorough asymptotic estimate on the eigenfunction (Lemma 3.4), a different linear estimate (Lemma 5.1) and some previously developed spectral properties (Propositions 3.1, 3.7). It is worthwhile to point out that unlike the NLS$_{\alpha}$ case where we have to restrict $\alpha$ to a much smaller range, in the case of NLW$_{\alpha}$, the restriction for $\alpha$ is the same as that appeared in the local result.

The proof of Theorem 1.5 follows a similar line as that in [8]. Roughly speaking, in a small neighborhood of the ground state manifold, we use the modulation analysis, which says that the behavior of the solution is determined by a number of parameters. Away from the neighborhood, we use the Virial estimates to show that the non-scattering global solution must enter into the small neighborhood of the ground state manifold eventually. The major difference we make here is to unify the choice of scaling parameters so that the proof is more streamlined when piecing these two parts together. This idea has been used in the case of NLS$_{\alpha}$ in our previous work [25] joint with Zeng.

The rest of this paper is organized as follows: in Sect. 2, we record some preliminaries including the variational properties of the ground state $W$, the equivalence of Sobolev norms and the Strichartz estimates for NLW$_{\alpha}$, etc. In Sect. 3, we establish the spectral properties for the linearized operator around the ground state $W$, and based on that, we develop the modulation analysis in Sect. 4. In all the rest sections, we present the details in specific dimension $d = 3$ for the sake of simplicity. We remind that the analysis works for all dimension $d \geq 3$ in Sects. 3 and 4. In Sect. 5, we construct the local stable manifold, and obtain the exponentially convergent solutions $W^{\pm}$. In Sect. 6, we use the monotonicity formula arising from the Virial estimates together with the modulation analysis to control the solution when it is away from the manifold. In Sect. 7, we characterize solutions on the energy surface with less kinetic energy than that of the ground state and show that all non-scattering solutions must coincide with $W^-$ up to symmetries. In Sect. 8, we consider solutions on the energy surface with greater kinetic energy and show that with the additional $L^2(\mathbb{R}^3)$ assumption, all solutions must have finite lifespan. The corresponding result in 4D and 5D is mentioned in Theorem 1.5 part c). In the Appendix, we provide an alternative way to obtain the first eigenfunction $\mathcal{V}$ of $L_+$ (see Remark 3.3).
2 Preliminaries

**Notation.** We explain some notations that will be used throughout the paper.

Occasionally, we use \( \beta = \frac{2}{d-2} \sqrt{a + \frac{(d-2)^2}{2}} \) as a number determined by \( a \). In particular, \( a > -(\frac{d-2}{2})^2 + \frac{(d-2)^2}{2} \) is equivalent to \( \beta > \frac{2}{d+2} \). We use \( \dot{H}^1(\mathbb{R}^d) \) or \( \dot{H}^1_{\sigma}(\mathbb{R}^d) \) to denote the homogeneous Sobolev space as explained in the introduction. \( \dot{H}^1_{rad}(\mathbb{R}^d) \) refers to the spherically symmetric subset in \( \dot{H}^1(\mathbb{R}^d) \). For any two real valued functions \( f \) and \( g \), we use

\[
\langle f, g \rangle = \int_{\mathbb{R}^d} f(x) \cdot g(x) dx
\]

to denote the usual \( L^2 \) inner product and use

\[
\langle f, g \rangle_{\dot{H}^1_{\sigma}} = \langle \mathcal{L}_a f, g \rangle = \int_{\mathbb{R}^d} \mathcal{L}_a f(x) \cdot g(x) dx
\]

to denote the \( \dot{H}^1_{\sigma}(\mathbb{R}^d) \) inner product. We write \( f \perp_{\dot{H}^1_{\sigma}} g \) iff \( \langle f, g \rangle_{\dot{H}^1_{\sigma}} = 0 \).

For any \( \lambda > 0 \) and \( f(x) \), we use

\[
f_{[\lambda]}(x) = \lambda^{-\frac{d-2}{2}} f \left( \frac{x}{\lambda} \right)
\]

to denote the \( \dot{H}^1(\mathbb{R}^d) \) invariant scaling transformation.

We use \( O(F) \) to denote an expression that can be bounded from above by \( CF \) for some constant \( C \).

In a metric space, we use \( B_r(u) \) to denote a ball with center \( u \) and radius \( r \).

**Properties of the ground state** \( W \). Let \( W \) be defined in (1.1). Then it is straightforward to verify the following

**Lemma 2.1** If \( a > -(d-2)^2 + \frac{(d-2)^2}{2} \), then we have

\[
W^{\frac{d+2}{d-2}}, \quad W^{\frac{d}{d-2}} W_1 \in L^2(\mathbb{R}^d).
\]

Here, \( W_1 = x \cdot \nabla W + \frac{d-2}{2} W \) is the generator of scaling transformation (see (3.4) for more explanations).

Next we record the following result obtained earlier in [13] which justifies \( W \) being the ground state.

**Lemma 2.2** [13] Let \( a \in (-\frac{(d-2)^2}{2}, 0) \) and \( f \in \dot{H}^1(\mathbb{R}^d) \). Then

\[
\| f \|_{2^*} \leq \frac{\| W \|_{2^*}}{\| W \|_{\dot{H}^1}} \| f \|_{\dot{H}^1},
\]

where \( 2^* = \frac{2d}{d-2} \) is the Sobolev conjugate. The equality holds if and only if \( f(x) = \alpha W(\lambda x) \) for some \( \alpha \in \mathbb{R} \) and \( \lambda > 0 \). Moreover, if \( \| f \|_{\dot{H}^1} \leq \| W \|_{\dot{H}^1} \), then

\[
\left( \frac{1}{2} - \frac{1}{2^*} \right) \| f \|_{\dot{H}^1}^2 \leq E(f, 0) \leq \frac{1}{2} \| f \|_{\dot{H}^1}^2.
\]

**Equivalence of Sobolev spaces.** The following result on the equivalence of Sobolev spaces is used frequently in our analysis.

**Lemma 2.3** (Equivalence of Sobolev spaces, [12]) Let \( d \geq 3 \), \( a > -(d-2)^2 \), \( s \in (0, 2) \) and

\[
\sigma = \frac{d-2}{2} - \left[ \frac{(d-2)^2}{2} + a \right]^{\frac{1}{2}}.
\]
• If $p \in (1, \infty)$ satisfies $\frac{s+\sigma}{d} \leq \frac{1}{p} < \min\{1, \frac{d-\sigma}{d}\}$, then
  $$|||\nabla|^s f||_{L^p} \lesssim ||L^\sigma_d f||_{L^p}.$$  
• If $p \in (1, \infty)$ satisfies $\max\{\frac{s}{d}, \frac{\sigma}{d}\} < \frac{1}{p} \leq \min\{1, \frac{d-\sigma}{d}\}$, then
  $$||L^\sigma_d f||_{L^p} \lesssim |||\nabla|^s f||_{L^p}.$$  

Strichartz estimate of $e^{-it\sqrt{-a}}$. We record the following Strichartz estimates for the linear wave equation with inverse-square potential obtained in [2, 29]. For simplicity, we only state the estimates for the operator $e^{\pm it\sqrt{-a}}$ in dimensions $d = 3, 4, 5$.

**Proposition 2.4** (Strichartz [2, 29]) Let $q, r \geq 2$ satisfy the wave admissibility condition
  $$\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4},$$
and let $\gamma$ be defined through the scaling relation
  $$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma.$$
Let $(\tilde{q}, \tilde{r}, \tilde{\gamma})$ be defined in the same way. Assume that $q, \tilde{q} > 2$. Then with the additional restriction on $\gamma$ stated below in (2.1), we have
  $$\|e^{\pm i\sqrt{-a}} f\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|F\|_{\dot{H}^\gamma(\mathbb{R}^d)},$$
  $$\left\|\int_0^t e^{i(t-s)\sqrt{-a}} F(s) \, ds\right\|_{L^q_t L^r_x(I \times \mathbb{R}^d)} \lesssim \|||\nabla|^\gamma \tilde{F}\|_{L^q_t L^r_x(I \times \mathbb{R}^d)}.$$  
Here $I$ is any time interval containing $t_0$.

The conditions on $\gamma$ are
  $$\begin{cases} d = 3, & \gamma \in \left(-\min\{1, \sqrt{a + \frac{9}{4}} - \frac{1}{2}, \sqrt{a + \frac{1}{4}} + 1\}, \min\{\frac{d+1}{2}, v_1 + \frac{1}{2}, v_0 + 1 - \frac{1}{q}\}\right), \\ d \geq 4, & \gamma \in \left(-\min\{\frac{d^2-2d-3}{2(d-1)}, v_1 - \frac{d+3}{2(d-1)}, v_0 + 1\}, \min\{\frac{d+1}{2}, v_1 + \frac{1}{2}, v_0 + 1 - \frac{1}{q}\}\right), \end{cases}$$  
(2.1)
where $v_0 = \sqrt{\left(\frac{d-2}{2}\right)^2 + a}$, $v_1 = \sqrt{\left(\frac{d}{2}\right)^2 + a}$.

We can easily check that in three dimensions, $(q, r) = (8, 8)$ or $(5, 10)$, $\gamma = 1$ and $(\tilde{q}, \tilde{r}, \tilde{\gamma}) = (1, 2), \tilde{\gamma} = 0$ fit into the above conditions. Strichartz estimates with these pairs will be used in later sections.

**Linear Profile decomposition in $\dot{H}^1_a(\mathbb{R}^d)$.** We need the following result on the linear profile decomposition in $\dot{H}^1_a(\mathbb{R}^d)$ proved earlier in [25], Lemma 9.2. This result can be extended to higher dimensions and adapted to the NLW$_a$ flow, see [19].

**Lemma 2.5** Let $\{f_n\}$ be a bounded sequence in $\dot{H}^1_a(\mathbb{R}^d)$. After passing to a subsequence, there exist $J^* \in \{0, 1, 2, \ldots \} \cup \{\infty\}, \{\phi^j\}_{j=1}^{J^*} \subset \dot{H}^1_a(\mathbb{R}^d)$, $\{(\lambda^j_n, x^j_n)\}_{j=1}^{J^*} \subset \mathbb{R}^+ \times \mathbb{R}^d$ such that for every $0 \leq J \leq J^*$, we have the decomposition
  $$f_n = \sum_{j=1}^{J^*} \phi^j_n + r^j_n, \quad \phi^j_n = (\lambda^j_n)^{-\frac{d-2}{2}} \phi^j \left(\frac{x - x^j_n}{\lambda^j_n}\right) := g^j_n \phi^j, \quad r^j_n \in \dot{H}^1_a(\mathbb{R}^d)$$  
$\square$ Springer
satisfying
\[
\lim_{J \to J^*} \limsup_{n \to \infty} \| r_n^f \|_{2^*} = 0; \]
\[
\lim_{n \to \infty} \left( \| f_n \|_{H^1_a}^2 - \sum_{j=1}^{J} \| \phi_n^j \|_{H^1_a}^2 - \| r_n^f \|_{H^1_a}^2 \right) = 0, \quad \forall J; \]
\[
\lim_{n \to \infty} \left( \| f_n \|_{2^*}^2 - \sum_{j=1}^{J} \| \phi_n^j \|_{2^*}^2 - \| r_n^f \|_{2^*}^2 \right) = 0, \quad \forall J. \tag{2.2}
\]

Moreover, for all \( j \neq k \), we have the asymptotic orthogonality property
\[
\lim_{n \to \infty} \left( \left| \left| \left| \frac{\lambda_n^j}{\lambda_n^k} \right| + \frac{\lambda_n^j}{\lambda_n^k} + \frac{|x_n^j - x_n^k|^2}{\lambda_n^j \lambda_n^k} \right| \right|_{X_0} \right) = 0.
\]

Finally we may also assume for each \( j \), either \( |x_n^j|/\lambda_n^j \to \infty \) or \( x_n^j \equiv 0 \), therefore
\[
\| \phi_n^j \|_{H^1_a} \to \| \phi^j \|_{X_0} = \begin{cases} \| \phi^j \|_{H^1_a} \text{ as } x_n^j \to \infty \quad \frac{|x_n^j|}{\lambda_n^j} \to \infty, \\ \| \phi^j \|_{H^1_a} \text{ as } x_n^j \equiv 0. \end{cases} \tag{2.3}
\]

3 Spectral analysis

To analyze the dynamic structure of \( \text{NLW}_a \) near the ground state \( W \), we focus on the equation for the difference \( v(t) = u(t) - W \) and write it into the following form:
\[
\partial_t^2 v + L_+ v = R(v), \tag{3.1}
\]
where
\[
L_+ = L_a - \frac{d+2}{d-2} W \frac{d}{2} v, \quad R(v) = |v + W|^{\frac{4}{d-2}} (v + W) - W^{\frac{4}{d-2}} - W^{\frac{d+2}{d-2}} v.
\]

In the vector form, (3.1) can be written into an equation of \( \vec{v} = (v, v_t) \):
\[
\partial_t \vec{v} = J L \vec{v} + \vec{R}(v),
\]
with \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, L = \begin{pmatrix} L_+ & 0 \\ 0 & 1 \end{pmatrix} \) and \( \vec{R}(v) = \begin{pmatrix} 0 \\ R(v) \end{pmatrix} \).

The kernel and negative directions of \( L \) comes from that of \( L_+ \), for which we know
\[
L_+ W = (L_a - d+2 \frac{d}{d-2} W \frac{d}{2} ) W = -d \frac{d}{d-2} W^{\frac{d+2}{d-2}} < 0, \tag{3.2}
\]
and
\[
L_+ W_1 = (L_a - d+2 \frac{d}{d-2} W \frac{d}{2} ) W_1 = 0. \tag{3.3}
\]

Here \( W_1 \) is the generator of the scaling transformation:
\[
W_1 = -\frac{d}{d\lambda} W(\lambda)|_{\lambda=1} = x \cdot \nabla W + \frac{d+2}{2} W. \tag{3.4}
\]

In our previous work [25], we proved that these are the only two non-positive directions for \( L_+ \). More precisely, we define the following quadratic form \( Q(v) = \langle L_+ v, v \rangle \), then \( Q(v) \) is
non-positive on the two dimensional space spanned by $W$ and $W_1$, see (3.2) and (3.3). $Q(v)$ is elliptic on the remaining codim-2 subspaces as shown in the following proposition.

**Proposition 3.1** [25] Let $X_+ \subset \hat{H}^1_d(\mathbb{R}^d)$ be the codim-2 subspace defined by

$$X_+ = \{ v \in \hat{H}^1_d(\mathbb{R}^d) \mid \langle W, v \rangle_{\hat{H}^1_d} = \langle W_1, v \rangle_{\hat{H}^1_d} = 0 \}.$$ 

Then we have the ellipticity for the quadratic form $Q(v) = \langle L_+ v, v \rangle$ on $X_+$: there exist $c, C > 0$ such that

$$c \| v \|^2_{\hat{H}^1_d} \leq Q(v) \leq C \| v \|^2_{\hat{H}^1_d}, \quad \forall v \in X_+.$$ 

Thus, $L_+$ has one negative direction and

$$\ker L_+ = \text{span}\{W_1\}.$$ 

Based on this, we can cite Theorem 2.1 in [17] to obtain the following exponential trichotomy for the operator $JL$. 

**Proposition 3.2** The flow $e^{tJL}$ is a well-defined operator and there exist closed subspaces $E^u$, $E^s$, $E^c$ such that

\begin{enumerate}
\item $\dim E^u = \dim E^s = 1$,
\item $e^{tJL}(E^{s,u,c}) = E^{s,u,c}$,
\item $\langle Lu, u \rangle = 0$, $\forall u \in E^{u,s}$, and
\item $E^c = \{ u \in (\hat{H}^1)^2; \langle Lu, v \rangle = 0, \forall v \in E^u \oplus E^s \}$,
\end{enumerate}

and

$$|e^{tJL}|_{E^c} \leq C(1 + |t|), \quad \forall t \in \mathbb{R}.$$ 

**Remark 3.3** Applying Proposition 3.2 in $\hat{H}^1_{rad}(\mathbb{R}^d)$ and using the particular form of $L$, we have the following.

There exists $e_0 > 0$ and $\mathcal{Y} \in \hat{H}^1_{rad}(\mathbb{R}^d) \neq 0$ such that

$$L_+ \mathcal{Y} = -e_0^2 \mathcal{Y}$$

and $E^u = \text{Span}\{(\mathcal{Y}, e_0 \mathcal{Y})\}$, $E^s = \text{Span}\{(\mathcal{Y}, -e_0 \mathcal{Y})\}$.

As a consequence of (3.5), we have

$$\|\mathcal{Y}\|^2_2 = \frac{1}{-e_0^2} \langle L_+ \mathcal{Y}, \mathcal{Y} \rangle \lesssim \|\mathcal{Y}\|^2_{\hat{H}^1_d} + \int W^\frac{4}{d-2} \mathcal{Y}^2 dx \lesssim \|\mathcal{Y}\|^2_{\hat{H}^1_d}.$$ 

In the remaining part of the paper, by renormalizing $\mathcal{Y}$, we assume

$$\|\mathcal{Y}\|_2 = 1.$$ 

Knowing only $\mathcal{Y} \in H^1_{rad}(\mathbb{R}^d)$ is not sufficient for future analysis. To get better regularity on $\mathcal{Y}$, we use the method of invariant manifold to extract the exact asymptotics of $\mathcal{Y}$ near $r = 0$ and $r = \infty$. This part of the argument was enlightened by our earlier joint work with Zeng [26].

In the radial variable, Eq. (3.5) can be written as

$$-\mathcal{Y}_{rr} - \frac{d-1}{r} \mathcal{Y}_r + e_0^2 \mathcal{Y} + \frac{a}{r^2} \mathcal{Y} - \frac{d+2}{d-2} W^\frac{4}{d-2}(r) \mathcal{Y} = 0.$$ 

(3.6)

For the asymptotics around $r = \infty$, we have the following lemma.
Lemma 3.4 Let \( Y(r) \neq 0 \) be a radial \( H^1(\mathbb{R}^d) \) solution of (3.6). Then there exists \( c \neq 0 \) such that
\[
\lim_{{r \to \infty}} r^{\frac{d-1}{2}} e^{{e_0}r} Y(r) = c, \quad \lim_{{r \to \infty}} r^{\frac{d-1}{2}} e^{{e_0}r} Y_r(r) = -c. \tag{3.7}
\]

**Proof** Denoting
\[
f\left( \frac{1}{r}, Y \right) = \frac{a}{r^2} Y - \frac{d+2}{2d-2} W(r) \frac{4}{r^2} Y,
\]
we rewrite (3.6) into
\[
-\frac{Y_{rr}}{r} - \frac{d-1}{r} Y_r + c_0 Y + f\left( \frac{1}{r}, Y \right) = 0. \tag{3.8}
\]
Clearly from the exact form of \( W \), we know that
\[
f(\tau, q) \in C^2(\mathbb{R}^2), \quad f\left( \frac{1}{r}, Y \right) = O(\frac{1}{r^2}) Y. \tag{3.9}
\]
As a necessary condition in the invariance manifold approach, we first show the decay of \( Y \) and \( Y_r \):
\[
\lim_{{r \to \infty}} Y(r) = \lim_{{r \to \infty}} Y_r(r) = 0.
\]
Indeed, the decay of \( Y \) follows directly from the radial Sobolev embedding
\[
|r^{\frac{d-1}{2}} Y(r)| \leq C\|Y\|_{{H^1}}.
\]
To see the decay of \( Y_r \), we multiply \( Y_r r^{d-1} \) on both sides of (3.8) and obtain for any \( 0 < r_1 < r_2 < \infty \) that
\[
-\frac{Y_r^2 r^{d-1}|r_2|}{r_1} = \frac{d-1}{2} \int_{r_1}^{r_2} r^{d-2} Y_r^2 dr - \frac{c_0}{2} Y^2 r^{d-1}|r_2| + \frac{c_0}{2} (d-1) \int_{r_1}^{r_2} r^{d-2} Y^2 dr
\]
\[
- \int_{r_1}^{r_2} f\left( \frac{1}{r}, Y \right) r^{d-1} Y_r dr.
\]
From Cauchy–Schwartz and (3.9), we obtain
\[
\left| Y_r^2 r^{d-1}|r_2| \right| \leq C\|Y\|_{{H^1}}^2,
\]
which shows immediately that \( Y_r^2 r^{d-1} \) is Cauchy and
\[
\lim_{{r \to \infty}} Y_r r^{\frac{d-1}{2}} = 0.
\]
Next, we reduce (3.8) into a first order ODE then use
\[
v_1(r) = e_0 Y(r) + Y_r(r), \quad v_2(r) = e_0 Y(r) - Y_r(r), \quad \text{and} \quad \tau = \frac{1}{r}
\]
(3.10) to diagonalize the resulting ODE into
\[
\begin{align*}
\frac{dv_1}{d\tau} &= e_0 v_1 - \frac{d-1}{2} \tau (v_1 - v_2) + f(\tau, \frac{v_1 + v_2}{2e_0}) \\
\frac{dv_2}{d\tau} &= -e_0 v_2 + \frac{d-1}{2} \tau (v_1 - v_2) - f(\tau, \frac{v_1 + v_2}{2e_0}) \\
\frac{d\tau}{d\tau} &= -\tau^2.
\end{align*}
\tag{3.11}
\]
Clearly, \((0, 0, 0)\) is an unstable equilibrium state with the unstable, stable and center directions given by \( v_1, v_2 \) and \( \tau \) respectively. From (3.9), there exists a \( C^2 \) center stable manifold \( W^\infty_{cs} \) given as the graph of a function \( \phi^\infty(\tau, v_2) \):
\[
v_1 = \phi^\infty(\tau, v_2), \quad \phi^\infty(0, 0) = 0, \quad \nabla \phi^\infty(0, 0) = 0.
\tag{3.12}
\]
In addition, \( W^\infty_{cs} \cap \{ \tau \geq 0 \} \) is unique from its positive invariance under the ODE flow. Since \( W^\infty_{cs} \) contains all orbits of (3.11) converging to 0 as \( r \to +\infty \), we know that the trivial solution \((0, 0, \tau = \frac{1}{3}) \in W^\infty_{cs} \) as well as \((v_1, v_2, \tau = \frac{1}{3}) \) for \( v_1, v_2 \) defined in (3.10). The first one gives \( \phi^\infty(\tau, 0) = 0 \), which further implies for \(|v_2|, |\tau| \ll 1\),

\[
v_1 = \phi^\infty(\tau, v_2) - \phi^\infty(\tau, 0) = D_{v_2} \phi^\infty(\tau, \tilde{v}_2)v_2 = O(|v_2| + |\tau|)v_2. \tag{3.13}
\]

This together with the decay estimate for \( W \) gives

\[
\frac{v_1}{r} = O\left(\frac{1}{r^2} + \frac{v_2}{r}\right)v_2, \quad f(\tau, \frac{v_1 + v_2}{2e_0}) = O\left(\frac{1}{r^2}\right)v_2. \tag{3.14}
\]

Inserting (3.13) into the \( v_2 \) equation in (3.11), we have

\[
\frac{d}{dr} v_2 = -e_0 v_2 + O\left(\frac{1}{r}\right) v_2 \quad \text{for} \quad |v_2|, |\tau| \ll 1,
\]

which clearly yields

\[
\frac{1}{2} \frac{d}{dr} (e^{d_0 r} v_2^2(r)) = e^{d_0 r} v_2^2(-\frac{e_0}{2} + O(\frac{1}{r})) < 0.
\]

For \( r \) large enough, \( e^{d_0 r} v_2^2 \) is monotone decreasing, and

\[
0 < |v_2(r)| \leq C e^{-\frac{e_0}{r}}. \tag{3.15}
\]

Using this, we see the first bound in (3.14) can be refined to

\[
\frac{v_1}{r} = O\left(\frac{1}{r^2}\right)v_2. \tag{3.16}
\]

Finally we upgrade the estimate for \( v_2 \) by plugging (3.16) and (3.14) into the \( v_2 \) equation, we obtain

\[
\frac{d}{dr} v_2 = \left(-e_0 - \frac{d}{2} - \tau + O\left(\frac{1}{r}\right)\right) v_2.
\]

The solution is given explicitly by

\[
v_2(r) = v_2(r_0) e^{-\int_{r_0}^r e_0 + \frac{d}{2} + ds} e^{\int_{r_0}^r O\left(\frac{1}{r}\right)ds} = v_2(r_0) e^{e_0 r_0} \frac{d}{2} e^{\int_{r_0}^r O\left(\frac{1}{r}\right)ds} e^{-e_0 r r^{-\frac{d}{2}}}.
\]

From the convergence of \( \int_{r_0}^\infty \frac{1}{s^2} ds \), we have

\[
v_2(r) = c(r) r^{-\frac{d}{2}} e^{-e_0 r},
\]

where \( c(r) \to 2e \) as \( r \to \infty \) for some constant \( c \neq 0 \). Thus, \( v_1(r) = O(r^{-\frac{d}{2}} e^{-e_0 r}) \) by (3.16) (3.17) then follows quickly from the asymptotics of \( v_1 \) and \( v_2 \).

Let us turn to the asymptotics of \( \mathcal{Y} \) near \( r = 0^+ \). We have the following lemma.

**Lemma 3.5** Let \( \mathcal{Y}(r) \neq 0 \) be a radial \( H^1(\mathbb{R}^d) \) function solving (3.6). Then there exists \( c \neq 0 \) such that

\[
\lim_{r \to 0^+} r^{\frac{(d-2)(1-\beta)}{2}} \mathcal{Y}(r) = c, \quad \lim_{r \to 0^+} r^{\frac{(d-2)(1-\beta)}{2}} + 1 \mathcal{Y}_r(r) = \frac{(d-2)(\beta-1)}{c}. \tag{3.17}
\]

**Proof** Denote

\[
u_1(r) = r^{\frac{d-2}{2}} \mathcal{Y}(r), \quad u_2(r) = r^{\frac{d}{2}} \mathcal{Y}_r(r).
\]

We first show the decay of \( u_1, u_2 \) as \( r \to 0^+ \) as this will justify the trajectory lying on the invariant manifold we constructed below.
Indeed, the decay of \( u_1 \) follows directly from the fundamental theorem of calculus and Cauchy-Schwartz, we have
\[
\left| u_1^n(r) \right|^2_{r_1} = \int_{r_1}^{r_2} 2u_1(u_1)dr = (d-2) \int_{r_1}^{r_2} r^{d-3} \gamma^2(r)dr + 2 \int_{r_1}^{r_2} r^{d-2} \gamma Y_2 dr \\
\lesssim \| \nabla \gamma \| L^2(r_1 \leq |x| \leq r_2) + \| \nabla \gamma \| L^2(r_1 \leq |x| \leq r_2).
\]
Thus, \( u_1(r) \) is Cauchy and \( u_1(r) \to 0 \) as \( r \to 0^+ \).

For \( u_2 \), we use the equation for \( Y \) to compute
\[
(u_2^2)_r = (2-d)r^{d-1} \gamma_2^2 + 2e_0^2r^d \gamma Y_r + 2ar^{d-2} \gamma Y_r - \frac{2(d+2)}{d-2} W^{\frac{d}{d-2}} r^d \gamma Y_r.
\]

Following the similar argument and using the fact that \( W^{\frac{d}{d-2}} r^d \sim r^{d-2+2\beta} \), we obtain the convergence of \( u_2 \).

Now let \( s = -\ln r \), we write (3.6) into the following ODE system
\[
\begin{align*}
\frac{d}{ds} u_1 &= -\frac{d-2}{2} u_1 - u_2, \\
\frac{d}{ds} u_2 &= -au_1 + \frac{d-2}{2} u_2 - r^2(e_0^2 - \frac{d+2}{d-2} W^{\frac{d}{d-2}})u_1, \\
r_s &= -r.
\end{align*}
\]

Denote
\[
\gamma = \sqrt{(d-2)^2 + 4a} = (d-2)\beta,
\]
we diagonalize the ODE system by defining:
\[
u_1 = v_1 + v_2, \quad u_2 = (-\frac{d-2}{2} - \frac{1}{2} \gamma)v_1 + (-\frac{d-2}{2} + \frac{1}{2} \gamma)v_2.
\]
Considering \( W(r) \sim r^{\frac{(d-2)(\beta-1)}{2}} \beta = \sqrt{1 + \frac{4}{(d-2)^2} a} \) near 0, we have
\[
r^2 W^{\frac{d}{d-2}} \sim (r^{2\beta}), \quad \text{as} \quad r \to 0^+,
\]
which fails to be \( C^1 \) near \( r = 0^+ \) when \( \beta \) is small. This non-essential non-smoothness can be removed by a change of variable. Defining
\[
\tau = r^\beta,
\]
we write the final diagonal system in \( v_1(s), \ v_2(s), \ \tau(s) \) into
\[
\begin{align*}
\frac{d}{ds} v_1 &= \frac{1}{2} \gamma v_1 + \frac{1}{2} \gamma^2 \left( e_0^2 - \frac{d+2}{d-2} W(\tau^\frac{1}{2})^\frac{d}{d-2} \right) (v_1 + v_2), \\
\frac{d}{ds} v_2 &= -\frac{1}{2} \gamma v_2 - \frac{1}{2} \gamma^2 \left( e_0^2 - \frac{d+2}{d-2} W(\tau^\frac{1}{2})^\frac{d}{d-2} \right) (v_1 + v_2), \\
\tau_s &= -\beta \tau.
\end{align*}
\]
Here the right sides are \( C^2 \) in \( v_1, v_2, \tau \). As the equation is linear in \( v_1, v_2 \), from the standard invariant manifold theory, there exists a unique \( C^2 \) stable manifold \( W^s_\tau \) near \( (\tau, v_1, v_2) = (0, 0, 0) \) represented by the graph of a linear function
\[
v_1 = \omega(\tau)v_2, \quad \text{for} |\tau| \ll 1,
\]
where \( \omega(\tau) \in C^1 \) and \( \omega(0) = 0 \), thus
\[
v_1 = \omega(\tau)v_2 - \omega(0)v_2 = O(\tau)v_2. \tag{3.18}
\]
Plugging this into the $v_2$ equation and using the fact that $O \left( r^{\frac{2}{\beta}} W(\tau^\frac{1}{\beta})^{\frac{4}{d+2}} \right) = O(\tau^2)$, we have
\[ \frac{d}{ds} v_2 = -\frac{1}{2} \gamma v_2 + O(\tau^2) v_2 = v_2(-\frac{1}{2} \gamma + O(e^{-2\beta s})), \]
which, by the method of integrating factors, gives
\[ v_2(s) = v_2(s_0)e^{\int_{s_0}^{s} (-\frac{1}{2} \gamma + O(e^{-2\beta s}))\,ds} := C(s, s_0)e^{-\frac{1}{2} \gamma s}, \]
for sufficiently large $s, s_0$. Due to the non-trivialness of $\mathcal{Y}$, we know $v_2(s_0) \neq 0$, hence
\[ C(s, s_0) \to c \neq 0. \]
Recalling the relation $r = e^{-s}$ and using (3.18), we immediately have
\[ r^{-\frac{1}{2} \gamma} v_2(r) \to c, r^{-\frac{1}{2} \gamma} v_1(r) \to 0, \text{ as } r \to 0^+. \]
This further gives the asymptotics of $u_1, u_2$
\[ r^{-\frac{1}{2} \gamma} u_1(r) \to c, r^{-\frac{1}{2} \gamma} u_2(r) \to (-\frac{d+2}{2} + \frac{\gamma}{2})c, \]
from which (3.17) follows.

This completes the proof of this lemma. \qed

Combining the above asymptotics of $\mathcal{Y}$ with the restriction for $\beta$: $\beta > \frac{2}{d+2}$, we immediately have

**Corollary 3.6** *For the eigenfunction $\mathcal{Y}$, we also have*
\[ \mathcal{Y} \in H^1, \mathcal{Y} \in L^{2(d+2)}(\mathbb{R}^d). \]

This additional regularity will be used later in constructing the local stable/unstable manifolds.

We end this section with the ellipticity of the quadratic form in a slightly different codim-2 subspaces.

**Proposition 3.7** *Let $\mathcal{Y}$ be the eigenfunction appearing above. Let $G_+$ be the codim-2 subspace defined by*
\[ G_+ = \{ v \in \dot{H}^1(\mathbb{R}^d) \mid \langle \mathcal{Y}, v \rangle_{\dot{H}^1} = \langle W_1, v \rangle_{\dot{H}^1} = 0 \}. \]
*Then there exist $c, C > 0$ such that the quadratic form $Q(v) = \langle L_+ v, v \rangle$ satisfies*
\[ c\|v\|_{\dot{H}^1}^2 \leq Q(v) \leq C\|v\|_{\dot{H}^1}^2, \quad \forall v \in G_+. \]

**Proof** The upper bound is a direct result of Hölder inequality and is already contained in Proposition 3.1. It remains to show the lower bound.

To this end, we decompose $v$ and $\mathcal{Y}$ in the orthogonal framework $\{ W, W_1, X_+ \}$ as follows:
\[ v = \alpha W + h, \quad \mathcal{Y} = \theta W + \eta W_1 + \tilde{h} \quad (3.19) \]
for some $h, \tilde{h} \in X_+$. Clearly from Proposition 3.1 we know
\[ \|h\|_{\dot{H}^1}^2 \sim Q(h), \quad \|\tilde{h}\|_{\dot{H}^1}^2 \sim Q(\tilde{h}). \]
As from the orthogonality,
\[ \|v\|_{\dot{H}^1}^2 = \alpha^2 \|W\|_{\dot{H}^1}^2 + \|h\|_{\dot{H}^1}^2, \]
\[ \text{ Springer} \]
the matter of the problem is then reduced to controlling $\alpha^2$ and $Q(h)$.

From direct calculation using (3.19), the orthogonality, $W_1 \in \ker L_+, L_+ Y = -e_0^2 Y$ and that $\|Y\|_2 = 1$, we have

$$Q(v) = \langle L_+ v, v \rangle = \alpha^2 Q(W) + Q(h), \quad -e_0^2 = Q(Y) = \theta^2 Q(W) + Q(\tilde{h}),$$

or,

$$-\alpha^2 Q(W) = Q(h) - Q(v), \quad -\theta^2 Q(W) = e_0^2 + Q(\tilde{h}).$$

(3.20)

Multiplying these two yields:

$$\alpha^2 \theta^2 Q(W)^2 = (Q(h) - Q(v))(e_0^2 + Q(\tilde{h})).$$

(3.21)

To continue, we use the fact that $v \perp Y$ in $L^2$ and $Y$ is the eigen-function to compute

$$0 = \langle v, L_+ Y \rangle = \alpha \theta Q(W) + \langle \tilde{h}, L_+ \tilde{h} \rangle,$$

which from Cauchy–Schwartz, gives

$$\alpha^2 \theta^2 Q(W)^2 = \left\langle \tilde{h}, L_+ \tilde{h} \right\rangle^2 \leq Q(h) Q(\tilde{h}).$$

Plugging this upper bound into (3.21), we obtain the following estimate for $Q(h)$

$$Q(h) \leq \frac{e_0^2 + Q(\tilde{h})}{e_0^2} Q(v).$$

(3.22)

Inserting this into the first equation in (3.20) gives

$$-\alpha^2 Q(W) \leq \frac{Q(\tilde{h})}{e_0^2} Q(v).$$

Note that $Q(\tilde{h}) \sim \|\tilde{h}\|_{H_1^d}^2 \leq \|Y\|_{H_1^d}^2$, we prove the Proposition. \qed

## 4 Modulation analysis

In this section, we perform the modulation analysis for solutions in a small neighborhood of 1D manifold of the ground state $\{\pm W_{[\mu]}, \mu > 0\}$. We first show that as part of the variational characterization of the ground state, on the energy surface of the ground state $W$, the distance to this manifold can be measured by

$$d(u(t), \partial_t u(t)) = \left| \| u(t) \|_{H_1^d}^2 - \| W \|_{H_1^d}^2 \right| + \| \partial_t u(t) \|_2^2.$$

The proof is similar to the case of energy critical NLS$_\alpha$ [25], we give the proof for the sake of completeness. In the free case, the variational characterization of $W$ was a classical result in [1, 16, 24].

**Proposition 4.1** Assume $(f, g) \in \dot{H}_1^d(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and $E(f, g) = E(W, 0)$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $d(f, g) = \left| \| f \|_{H_1^d}^2 - \| W \|_{H_1^d}^2 \right| + \| g \|_2^2 < \delta$, we have

$$\inf_{\mu, \pm} \| f_{[\mu]} \pm W \|_{H_1^d} < \varepsilon.$$
Proof We argue by contradiction. Suppose the statement fails, there must exist $\varepsilon_0 > 0$ and a sequence of functions $\{(f_n, g_n)\} \subset \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ such that

$$E(f_n, g_n) = E(W, 0), \quad \left\| f_n \right\|_{\dot{H}^1}^2 - \left\| W \right\|_{\dot{H}^1}^2 + \left\| g_n \right\|_2^2 \to 0,$$

but

$$\inf_{\mu \in \pm} \left( f_n \right) \pm W \gg 0.$$  \hfill (4.2)

From (4.1), we get the following convergence

$$\left\| f_n \right\|_{\dot{H}^1} \to \left\| W \right\|_{\dot{H}^1}, \quad \left\| f_n \right\|_{2^*} \to \left\| W \right\|_{2^*}, \quad \left\| g_n \right\|_2 \to 0.$$  \hfill (4.3)

We will derive a contradiction by showing $f_n$ converges to the manifold. This will be achieved by using the approach of linear profile decomposition.

Applying Lemma 2.5 to $\{f_n\}$, we obtain

$$f_n = \sum_{j=1}^J \phi_n^j + r_n^j,$$

for each $J \in \{1, \ldots, J^*\}$ with the stated properties.

On the one hand, the $\dot{H}^1$ decoupling in Lemma 2.5 together with (4.3) implies

$$\left\| W \right\|_{\dot{H}^1}^2 = \lim_{n \to \infty} \left( \sum_{j=1}^J \left\| \phi_n^j \right\|_{\dot{H}^1}^2 + \left\| r_n^j \right\|_{\dot{H}^1}^2 \right) = \sum_{j=1}^J \left\| \phi_j \right\|_{X^j}^2 + \lim_{n \to \infty} \left\| r_n^j \right\|_{\dot{H}^1}^2.$$

Here $\left\| \cdot \right\|_{X^j} = \left\| \cdot \right\|_{\dot{H}^1}$ if $x_n^j \equiv 0$ and $\left\| \cdot \right\|_{X^j} = \left\| \cdot \right\|_{\dot{H}^1}$ if $\left\| x_n^j \right\|_{X^j} \to \infty$. Taking $J \to J^*$, we obtain

$$\sum_{j=1}^{J^*} \left\| \phi_j \right\|_{X^j}^2 + \lim_{J \to J^*} \lim_{n \to \infty} \left\| r_n^j \right\|_{\dot{H}^1}^2 = \left\| W \right\|_{\dot{H}^1}^2.$$  \hfill (4.5)

On the other hand, using the $L^{2^*}$-norm decoupling, (4.3) and Lemma 2.2, we have

$$\left\| W \right\|_{2^*}^2 = \lim_{n \to \infty} \left\| f_n \right\|_{2^*}^2 = \sum_{j=1}^{J^*} \left\| \phi_j \right\|_{2^*}^2 \leq \sum_{j=1}^{J^*} \left\| \phi_j \right\|_{\dot{H}^1}^2 \left\| W \right\|_{2^*}^2,$$

which implies

$$\left\| W \right\|_{\dot{H}^1}^2 \leq \sum_{j=1}^{J^*} \left\| \phi_j \right\|_{\dot{H}^1}^2.$$  \hfill (4.6)

This together with (4.5) yields

$$\left( \sum_{j=1}^{J^*} \left\| \phi_j \right\|_{X^j}^2 + \lim_{J \to J^*} \lim_{n \to \infty} \left\| r_n^j \right\|_{\dot{H}^1}^2 \right)^{2^*/2} \leq \sum_{j=1}^{J^*} \left\| \phi_j \right\|_{\dot{H}^1}^2.$$
Hence, \( \|\phi^1\|_{\dot{H}^1}\leq \|W\|_{\dot{H}^1}\), \( \|\phi^1\|_{2^*} = \|W\|_{2^*}\). Applying Lemma 2.2, we know there exists \( \mu_0 \) such that \( \phi^1 = W_{(\mu_0)} \) or \( \phi^1 = -W_{(\mu_0)} \), contradicting (4.2) in view of (4.6) and (4.4). The proposition is proved. \( \Box \)

The next result gives the decomposition in the local orthogonal frame when the distance function \( d(f, g) \) is small enough. The coordinates in this frame, which are called "modulation parameters", are crucial to understand the behavior of solutions.

**Lemma 4.2** There exist \( \delta_0, \epsilon_0 > 0 \) such that for any \((f, g) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) satisfying \( E(f, g) = E(W, 0) \) and \( d(f, g) < \delta_0 \), there exists a unique \( \mu > 0 \) such that

\[
f_{[\mu]} \perp W_1 \text{ in } \dot{H}^1(\mathbb{R}^d): \|f_{[\mu]} - W\|_{\dot{H}^1} < \epsilon_0 \text{ or } \|f_{[\mu]} + W\|_{\dot{H}^1} < \epsilon_0
\]

and we have the orthogonal decomposition in the minus sign case

\[
f_{[\mu]} = W + \alpha W + \tilde{u} = W + v, \quad \tilde{u} \perp \dot{H}^1_1 \{W, W_1\}.
\]

In the positive sign case, the same equation holds true if we change \( f \) to \( -f \). Moreover, the modulation parameters obey

\[
|\alpha| \sim \|v\|_{\dot{H}^1_1} \sim \|g\|_2 + \|\tilde{u}\|_{\dot{H}^1_1} \sim d(f, g).
\]

**Proof** The orthogonal decomposition (4.8) follows directly from (4.7), thus we only need to prove (4.7) and (4.9).

We first prove (4.7) when \( f \) is in the small neighborhood of \( W \). We will show that there exist small constants \( \epsilon_0, \eta_0 \) such that when \( f \in B_{\epsilon_0}(W) \), there exists a unique \( \mu_1 \in B_{\eta_0}(1) \) such that

\[
f_{[\mu_1]} \perp \dot{H}^1_1 W_1, \text{ or } \langle f_{[\mu_1]}, W_1 \rangle_{\dot{H}^1_1} = 0.
\]

Indeed, this will be a result of the implicit function theorem as we show below.

Define the functional \( J: \mathbb{R}^+ \times \dot{H}^1(\mathbb{R}^d) \rightarrow \mathbb{R} \)

\[
J(\mu, f) = \langle f_{[\mu]}, W_1 \rangle_{\dot{H}^1_1},
\]

we can verify easily that \( J \) is linear in \( f, C^1 \) in \( \mu \) and

\[
J(1, W) = 0, \quad \frac{\partial J}{\partial \mu}(1, W) = -\|W_1\|^2_{\dot{H}^1_1} \neq 0.
\]

An application of the implicit function theorem gives rise to the above mentioned statement. Moreover, the mapping \( \gamma: \dot{H}^1(\mathbb{R}^d) \supset B_{\epsilon_0}(W) \rightarrow B_{\eta_0}(1) \subset \mathbb{R}^+ \) defined by \( \gamma(f) = \mu \) is \( C^1 \).

Next, let \((f, g)\) be the functions satisfying the conditions in this lemma. Then from Proposition 4.1, for the constant \( \epsilon_0 \), there exists \( \delta_0 > 0 \) such that when \( d(f, g) < \delta_0 \), we have

\[
\|f_{[\mu_2]} - W\|_{\dot{H}^1} < \frac{1}{2} \epsilon_0 \text{ or } \|f_{[\mu_2]} + W\|_{\dot{H}^1_1} < \frac{1}{2} \epsilon_0 \text{ for some } \mu_2 > 0.
\]

In the minus sign case, applying the statement from the first step to \( f_{[\mu_2]} \), we immediately obtain for \( \mu = \mu_1 \mu_2 \) that

\[
f_{[\mu]} \perp \dot{H}^1_1 W_1, \text{ and } \|f_{[\mu]} - W\|_{\dot{H}^1_1} \leq \|f_{[\mu_2]} - W\|_{\dot{H}^1} + \|W_{[\mu_1^{-1}] - W\|_{\dot{H}^1_1}}
\]

\[
\leq \frac{\epsilon_0}{2} + C \eta_0 \leq \epsilon_0
\]
if \( \eta_0 \) is chosen so that \( C \eta_0 < \frac{1}{2} \varepsilon_0 \). The plus sign case can be dealt with similarly if we change \( f \) to \(-f\).

Finally, we remark that the uniqueness of \( \mu \) can be proved in the same way as in our previous work (Lemma 4.2 in [25]), we skip that part. Claims in (4.7) are proved.

It remains to verify (4.9). Expanding the energy functional around \( W, \) we have for \( q = \min\{3, \frac{2d}{d-2}\} \) that

\[
0 = E(W + v, g) - E(W, 0) = Q(v) + \frac{1}{2} \| g \|^2_2 + O(\| v \|^q_{\dot{H}^1_d})
= \alpha^2 Q(W) + Q(\tilde{u}) + \frac{1}{2} \| g \|^2_2 + O(\| \alpha \|^q + \| \tilde{u} \|^q_{\dot{H}^1_d}).
\]

Then the ellipticity of \( L_+ \) on \( \{ W, W_1 \} \) from Proposition 3.1 and \( Q(W) < 0 \) implies

\[
\| \tilde{u} \|^2_{\dot{H}^1_d} + \| g \|^2_2 \sim \alpha^2 + O(\| \alpha \|^q + \| \tilde{u} \|^q_{\dot{H}^1_d}).
\]

Picking \( \delta_0 \) small enough, we then have

\[
\alpha^2 \sim \| \tilde{u} \|^2_{\dot{H}^1_d} + \| g \|^2_2.
\]

This together with \( \| v \|^2_{\dot{H}^1_d} = \alpha^2 \| W \|^2_{\dot{H}^1_d} + \| \tilde{u} \|^2_{\dot{H}^1_d} \) yields

\[
\alpha \sim \| v \|_{\dot{H}^1_d}.
\]

From (4.8), we get

\[
\| f \|^2_{\dot{H}^1_d} - \| W \|^2_{\dot{H}^1_d} = (2\alpha + \alpha^2) \| W \|^2_{\dot{H}^1_d} + \| \tilde{u} \|^2_{\dot{H}^1_d} \sim \alpha.
\]

The definition of \( d(f, g) \) and above equivalent relations imply (4.9). This completes the proof of the lemma.

In the sequel, we simply write

\[
d(u(t)) = d(u(t), \partial_t u(t))
\]

and we assume \( u(t) \) is a solution of NLW\(_a\) on a connected time interval \( I \) satisfying

\[
E(u(t), \partial_t u(t)) = E(W, 0), \ d(u(t)) < \delta_0, \ \forall t \in I.
\]

From Lemma 4.2, and changing \( u(t) \) to \(-u(t)\) if necessary, there exists \( \mu(t) \) for each \( t \in I \) such that we can decompose

\[
u(t)_{[\mu(t)]} = W + \alpha(t) W + \tilde{u}(t) := W + v(t), \ \tilde{u}(t) \perp_{\dot{H}^1_d} \{ W, W_1 \}.
\]

We are ready to obtain the temporal derivative estimates on the modulation parameters \( \alpha(t) \) and \( \mu(t) \).

**Lemma 4.3** The modulation parameters in the decomposition (4.13) obey

\[
|\alpha(t)| \sim \| v(t) \|_{\dot{H}^1_d} \sim \| \tilde{u}(t) \|_{\dot{H}^1_d} + \| \partial_t u(t) \|_2 \sim d(u(t)),
\]

and

\[
\left| \frac{\alpha'(t)}{\mu(t)} \right| + \left| \frac{\mu'(t)}{\mu(t)^2} \right| \lesssim d(u(t)).
\]

All the implicit constants are time independent.
Proof Estimate (4.14) follows directly from Lemma 4.2, and it remains to prove (4.15). Denote \( \rho(t, y) = u_{(\mu(t)}(t) = \mu(t)^{-\frac{d-2}{2}} u(t, y/\mu(t)) \), and let \( y = \frac{x}{\mu(t)} \). Then
\[
\frac{\partial_t u(t, x)}{\mu} = \frac{d - 2}{2} \mu^{-\frac{d-2}{2}} \rho(t, y) + \mu^{-\frac{d-2}{2}} (\partial_t \rho(t, y) + \frac{\mu'}{\mu} y \cdot \nabla_y \rho(t, y)).
\]
Plugging the decomposition (4.13), we get
\[
\mu^{-\frac{d}{2}} \partial_t u(t, y) = \frac{d - 2}{2} \mu^{-\frac{d-2}{2}} \rho(t, y) + \frac{1}{\mu} (\partial_t \rho(t, y) + \frac{\mu'}{\mu} y \cdot \nabla_y \rho(t, y))
\]
\[
= \frac{\mu'}{\mu^2} W_1 + \frac{\alpha'}{\mu} W + \left( \frac{\alpha \mu'}{\mu^2} W_1 + \frac{\mu'}{\mu^2} \frac{d - 2}{2} + y \cdot \nabla_y \mu \right).
\]
Taking inner product with \( W, W_1 \) gives
\[
\left< \mu^{-\frac{d}{2}} \partial_t u(t, y), W \right>_H = \frac{\alpha'}{\mu} \| W \|_{H^1}^2 + \left< y \cdot \nabla_y \mu, W \right>_H,
\]
\[
\left< \mu^{-\frac{d}{2}} \partial_t u(t, y), W_1 \right>_H = \left( 1 + \frac{\alpha}{\mu^2} \right) \| W_1 \|_{H^1}^2 + \left< y \cdot \nabla_y \mu, W_1 \right>_H.
\]
These two equations together with (4.14), and the Hölder inequality yield
\[
\left| \frac{\alpha'}{\mu} \right| \lesssim \| \partial_t u(t) \|_2 + \| y \cdot \nabla_y \mu, L_t W \|_2 \lesssim \| \partial_t u(t) \|_2 + \| y \cdot \nabla_y \mu, \|_{H^\frac{d-2}{2}} \]
\[
\lesssim \| \partial_t u(t) \|_2 + \| \mu \|_{H^1} \lesssim \textbf{d}(u(t)),
\]
and
\[
\left| \frac{\mu'}{\mu^2} \right| \lesssim \left| \frac{\alpha \mu'}{\mu^2} \right| + \| \partial_t u(t) \|_2 + \| \mu \|_{H^1} \lesssim \textbf{d}(u(t)) \left| \frac{\mu'}{\mu^2} \right| + \textbf{d}(u(t)).
\]
By taking \( \delta_0 \) small enough, we obtain \( \left| \frac{\mu'}{\mu^2} \right| \lesssim \textbf{d}(u(t)). \) Equality (4.15) is finally proved. \( \square \)

5 Construction of local stable manifold

The goal of this section is to construct the local stable/unstable manifold, which is ultimately reduced to constructing the exponential decaying solution for the difference equation
\[
\partial_t^2 v + L_+ v = R(v)
\]
(5.1)
as mentioned in Sect. 3. Recalling also that with the notation \( \tilde{v}(t) = (v(t), \partial_t v(t)) \) and other notation in Sect. 3, we can write (5.1) into the following the equivalent form
\[
\partial_t \tilde{v} - JL \tilde{v} = \tilde{R} (v).
\]
In the rest of the paper, we will present the proof in the specific dimension \( d = 3 \), the dimensions \( d = 4, 5 \) can be dealt with by the same argument after some minor notational change.

To solve (5.1), we first use the following orthogonal decomposition:
\[
\begin{cases}
v(t) = y(t) \mathcal{V} + v^c(t), \ y(t) = (v(t), \mathcal{V}), \ v^c(t) \in \mathcal{V}^\perp \\
R(v) = R^s (v) \mathcal{V} + R^c (v), \ R^s (v) = \langle R(v), \mathcal{V} \rangle, \ R^c (v) \perp \mathcal{V}
\end{cases}
\]
(5.2)
to reduce (5.1) further to the decoupled system
\[
\begin{aligned}
&\{ y''(t) - e_0^2 y(t) = R^a(v) \\
&\partial_t^2 v^c + L_+ v^c = R^c(v).
\end{aligned}
\tag{5.3}
\]

Next we derive some linear estimates for the second equation in (5.3). To simplify the notation, for the vector \(\tilde{v}(t)\), we denote
\[
\|\tilde{v}(t)\| = \|v(t)\|_{\dot{H}_a^1} + \|\partial_t v(t)\|_2, \quad \|\tilde{v}(t)\|_2 = \|v(t)\|_2 + \|\partial_t v(t)\|_2,
\]
and we use \(S(I)\) to denote the following Strichartz space
\[
S(I) = L^8_{t,x}(I \times \mathbb{R}^3) \cap L^5_{t} L^{10}_{x}(I \times \mathbb{R}^3).
\]

We also use \(\mathcal{Y}^\perp\) to denote the \(L^2\) orthogonal compliment in \(\dot{H}_a^1(\mathbb{R}^3)\), namely, \(\mathcal{Y}^\perp = \{u \in \dot{H}_a^1(\mathbb{R}^3), \langle u, \mathcal{Y} \rangle = 0\}\).

We have the following linear estimate.

**Lemma 5.1** (Linear estimate) Estimates for solutions of the homogeneous/inhomogeneous equations are given in a), b), respectively.

a) Let \(w(t, x) \in \mathcal{Y}^\perp\) solve \(\partial_t^2 w + L_+ w = 0\), then \(\tilde{w}(t, x) = e^{tJL} \tilde{w}(0)\) satisfies
\[
\|\tilde{w}(t)\| \lesssim \langle t \rangle \|\tilde{w}(0)\|, \quad \|\tilde{w}(t)\|_2 \lesssim \langle t \rangle \|\tilde{w}(0)\|, \quad \|w\|_{S((0,T))} \lesssim \langle T \rangle^2 \|\tilde{w}(0)\|.
\]

b) Let \(f(t, x) \perp \mathcal{Y}\) and \(v(t, x) \in \mathcal{Y}^\perp\) solve
\[
v_{tt} + L_+ v = f, \quad v(0) = v_t(0) = 0.
\]

Then \(v(t, x)\) and \(\tilde{v}(t, x) = \int_0^t e^{(t-s)JL} \tilde{f}(s) \, ds\) satisfy
\[
\|\tilde{v}(t)\| \lesssim \langle t \rangle \|f\|_{L^1_t L^2_x((0,t])}, \quad \|v\|_{S((0,T))} \lesssim \langle T \rangle^2 \|f\|_{L^1_t L^2_x((0,T])}.
\]

**Proof** We first prove a).

Recalling \(G_+\) from Proposition 3.7, we know \(\mathcal{Y}^\perp = \text{span}\{W_1\} \oplus G_+\) and we can write
\[
w(t) = \gamma(t)W_1 + g(t) \quad \text{and} \quad \gamma(t) = \frac{\langle w(t), W_1 \rangle_{\dot{H}_a^1}}{\|W_1\|^2_{\dot{H}_a^1}}.
\]

Moreover,
\[
\|w(t)\|^2_{\dot{H}_a^1} = \|\gamma(t)\|^2 \|W_1\|^2_{\dot{H}_a^1} + \|g(t)\|^2_{\dot{H}_a^1}, \quad \langle L+g(t), g(t) \rangle \sim \|g(t)\|^2_{\dot{H}_a^1}.
\]

This together with the following energy estimate
\[
\frac{d}{dt}(\|w_t\|^2_2 + \langle L+w, w \rangle) = \frac{d}{dt}(\|w_t\|^2_2 + \langle L+g(t), g(t) \rangle) = 0
\]

yields that
\[
\|w_t(t)\|^2_2 + \|g(t)\|^2_{\dot{H}_a^1} \lesssim \|w_t(0)\|^2_2 + \|g(0)\|^2_{\dot{H}_a^1} \leq \|\tilde{w}(0)\|. \tag{5.4}
\]

To obtain the estimate for \(\|\tilde{w}(t)\|\), it remains to estimate \(\gamma(t)\). Note from Lemma 2.1
\[
|\gamma'(t)| \leq \frac{1}{\|W_1\|^2_{\dot{H}_a^1}} |\langle w_t, W_1 \rangle_{\dot{H}_a^1}| \lesssim \|w_t\|_2 \|L_a W_1\|_2 \lesssim \|w_t\|_2 \|W_{\frac{d-2}{2}} W_1\|_2 \lesssim \|w_t\|_2 \lesssim \|\tilde{w}(0)\|.
\]
The estimate on \(\|\vec{\phi}(t)\|\) then follows.

To get the estimate for \(\|\vec{\phi}(t)\|\), it suffices to estimate \(\|w(t)\|\), which by the triangle inequality, the fundamental theorem of calculus, and (5.4), can be estimated as

\[
\|w(t)\| \leq \|w(t) - w(0)\| + \|w(0)\| \leq \int_0^t \|\partial_s w(s)\| ds + \|w(0)\|
\]

Finally we estimate the Strichartz norm of \(w\) based on Strichartz estimate of NLW. Let \(\eta\) be a small number, we partition

\[
[0, T] = \cup_{j=0}^{N-1} I_j, \quad I_j = [j\eta, (j+1)\eta], \quad N\eta \in (T-\eta, T).
\]

Noting the alternative form of the equation of \(w\):

\[
\partial_t^2 w + \mathcal{L}_a w = 5W^4 w,
\]

we can apply the Strichartz estimate Proposition 2.4 on each \(I_j\) and the estimate on \(\|\vec{\phi}(t)\|\) to obtain

\[
\|w\|_{S(I_j)} \lesssim \|\vec{\phi}(j\eta)\| + \eta^\frac{4}{5} \|w\|_{S(I_j)} \lesssim (j\eta) \|\vec{\phi}(0)\| + \eta^\frac{4}{5} \|w\|_{S(I_j)}.
\]

This yields \(\|w\|_{S(I_j)} \lesssim (\eta j) \|\vec{\phi}(0)\|\). Summing in \(j\), we obtain the last estimate for \(w\).

We turn to the estimate for \(v\). Applying the homogeneous estimate in a), we have the energy estimate

\[
\|v(t)\| \leq \int_0^t \|e^{(t-s)J} f(s)\| ds \lesssim \int_0^t \|f(s)\| ds \lesssim \langle t\rangle \|f\|_{L^1_t L^2_x([0,t])}.
\]

With this estimate, we are able to follow the same strategy as in a) to estimate the Strichartz norm of \(v\).

Indeed, rewriting the equation for \(v\) into \(v_{tt} + \mathcal{L}_a v = 5W^4 v + f\), we partition the interval \([0, T]\) and apply the Strichartz estimate on each \(I_j\) to obtain

\[
\|v\|_{S(I_j)} \lesssim \|\vec{v}(j\eta)\| + \|5W^4 v + f\|_{L^1_t L^2_x(I_j)} \lesssim \|\vec{v}(j\eta)\| + \eta^\frac{4}{5} \|v\|_{S(I_j)} + \|f\|_{L^1_t L^2_x(I_j)},
\]

which yields

\[
\|v\|_{S(I_j)} \lesssim \|\vec{v}(j\eta)\| + \|f\|_{L^1_t L^2_x(I_j)}.
\]

Inserting the estimate for \(\|\vec{v}(j\eta)\|\) and summing in \(j\) gives the estimate in b). \(\square\)

Next we construct the stable manifold of \(W\), which contains trajectories of solutions exponentially convergent to \(W\) as \(t \to \infty\). The construction of unstable manifold can be obtained by reversing the time direction. We have the following.

**Theorem 5.2** For any \(\lambda \in (0, e_0]\), there exist \(\delta = \delta(\lambda)\) such that whenever \(|\gamma_0| \leq \delta\), there exists a unique solution to

\[
\partial_t^2 v + L_+ v = R(v) \tag{5.5}
\]
satisfying
\[
\langle v(0), \mathcal{Y} \rangle = y_0, \text{ and } \| \tilde{v}(t) \| \leq 2|y_0|e^{-\lambda t}. \quad (5.6)
\]

Moreover,
\[
\begin{aligned}
\| \tilde{v}(t) \| + \| \tilde{v}(t) \|_2 + \| v \|_{S([t, \infty))} & \leq C|y_0|e^{-\lambda t}, \quad t \geq 0, \\
\| v(0) - y \|_{\mathcal{H}} & \leq C|y_0|^2.
\end{aligned} \quad (5.7)
\]

Furthermore, For any \( y_0, \tilde{y}_0 \) such that \( y_0 \tilde{y}_0 > 0 \) and \( |y_0|, |\tilde{y}_0| < \delta \), the corresponding solutions \( v(t, x) \) and \( \tilde{v}(t, x) \) obey \( v(t) = \tilde{v}(t + T) \) for some \( T = T(y_0, \tilde{y}_0) \).

**Proof** From the orthogonal decomposition and using the same notation as in (5.2), solving (5.5) is equivalently to solving (5.3). In the exponential decaying class, (5.3) can be further written into the following Duhamel’s formulation
\[
\begin{aligned}
y(t) &= e^{-\lambda t} y_0 + \frac{1}{2e\omega} \int_0^\infty e^{-\lambda(t-s)} R^*(v(s))ds \\
&\quad - \frac{1}{2e\omega} \int_t^{\infty} e^{-\lambda(s-t)} R^*(v(s))ds - \frac{1}{2e\omega} \int_0^t e^{-\lambda(t-s)} R^*(v(s))ds := \tilde{y}(t), \\
\tilde{v}^c(t) &= - \int_0^\infty e^{\lambda L(t-s)} R^c(v(s))ds := \tilde{v}^c(t).
\end{aligned} \quad (5.8)
\]

Define the solution map
\[ \Phi: \; v(t) = y(t)\mathcal{Y} + v^c(t) \rightarrow \tilde{v}(t) = \tilde{y}(t)\mathcal{Y} + \tilde{v}^c(t), \]
we will obtain the unique solution to (5.8) by proving that \( \Phi \) is a contraction on the ball
\[
B_{|y_0|, \lambda} = \{ v = y(t)\mathcal{Y} + v^c(t), \; y(t) \in C([0, \infty)), \; v^c(t) \in C([0, \infty); \mathcal{Y}^\perp) , \}
\]
and \( \| v \|_X \leq 2y_0 \}. \quad (5.9)
\]

Here,
\[ \| v \|_X = \max \{ \sup_{t \geq 0} e^{\lambda t} |y(t)|, \; \sup_{t \geq 0} e^{\lambda t} \| \tilde{v}(t) \|, \; \sup_{t \geq 0} e^{\lambda t} \| v^c \|_{S([t, \infty))} \}. \]

Refined uniqueness and other properties for solutions as stated in this theorem will be proved later.

The proof of contraction relies heavily on the following nonlinear estimates for any \( v, v_1, v_2 \in B_{|y_0|, \lambda} \):
\[
\begin{aligned}
|R^*(v(s))| &\leq C|y_0|^2e^{-2\lambda s}, \; s \geq 0; \quad \| R^c(v) \|_{L^1_tL^2_x([T_0, T_0+1])} \leq C|y_0|^2e^{-2\lambda s} \| v_1 - v_2 \|_X; \\
|R^*(v_1(s)) - R^*(v_2(s))| &\leq C|y_0|e^{-2\lambda s} \| v_1 - v_2 \|_X; \\
\| R^c(v_1) - R^c(v_2) \|_{L^1_tL^2_x([T_0, T_0+1])} &\leq C|y_0|e^{-2\lambda T_0} \| v_1 - v_2 \|_X.
\end{aligned} \quad (5.10)
\]

Indeed, noting \( R(v) = O(|v|^2W^3 + |v|^5) \) and \( R^*(v) = \langle R(v), \mathcal{Y} \rangle \), from the Hölder inequality we have
\[
|R^*(v(s))| \leq C \| R(v) \|_x \mathcal{Y} \|_6 \leq C (\| v \|^2_6 + \| v \|^5_6),
\]
\[
\leq C (\| y(s) \|^2 + \| v^c \|^2_6 + \| v^c \|^5_6) \leq C|y_0|^2e^{-2\lambda s}.
\]

To prove the estimate for \( R^c(v) \), we use the fact that \( \mathcal{Y} \in L^{10}_x(\mathbb{R}^3) \), \( W \in L^{10}_x(\mathbb{R}^3) \) and the Hölder inequality to obtain
\[
\| R^c(v) \|_{L^1_tL^2_x([T_0, T_0+1])} \leq \| R(v) \|_{L^1_tL^2_x([T_0, T_0+1])}
\]
\[
\leq C(\| W \|^3_{10} \| v \|^3_{L^{10}_x([T_0, T_0+1])} + \| v \|^5_{L^{10}_x([T_0, T_0+1])}) \leq C|y_0|^2e^{-2\lambda T_0}.
\]
The estimates of the differences are similar if we apply the following point-wise estimate

\[ |R(v_1) - R(v_2)| \leq C|v_1 - v_2| \sum_{j=1}^{2} (W^3|v_j| + |v_j|^4). \]

With (5.10), we first show that for any given \( v \in B_{|y_0|, \lambda} \), \( \bar{v} = \Phi(v) \in B_{|y_0|, \lambda} \). As it is easy to check that \( \bar{v}^c(t) = (\bar{v}^c(t), \bar{v}^c(t)) \) and that \( \bar{v}^c(t) \perp \mathcal{Y} \) are built into the definition, we are left to only control the X-norm of \( \bar{v} \).

For the contribution from \( \mathcal{Y} \)-coordinate \( \bar{y}(t) \), we use the right hand side of (5.8) to estimate

\[
\begin{align*}
e^{\lambda t} |\bar{y}(t)| &\leq e^{(\lambda - \epsilon_0) t} y_0 + C|y_0|^2 \int_0^\infty e^{-\epsilon_0 s} e^{-2\lambda s} \, ds \\
&\quad + C|y_0|^2 e^{\epsilon_0 t} \int_t^\infty e^{-\epsilon_0 s} e^{-2\lambda s} \, ds + C|y_0|^2 e^{-\epsilon_0 t} \int_0^t e^{\epsilon_0 s} e^{-2\lambda s} \, ds \\
&\leq |y_0| + C|y_0|^2 \leq 2|y_0|.
\end{align*}
\]

For the contribution from \( \bar{z}^c \), we use the second equation in (5.8) and the homogeneous estimate in Lemma 5.1 to write

\[
\begin{align*}
\left\| \bar{z}^c(t) \right\| &\leq \int_t^{[t]+1} \left\| e^{(t-s)JL} \bar{R}^c(v(s)) \right\| \, ds + \sum_{N \geq [t]+1} \int_N^{N+1} \left\| e^{(t-s)JL} \bar{R}^c(v(s)) \right\| \, ds \\
&\leq \int_t^{[t]+1} \| \bar{R}^c(v(s)) \|_2 \, ds + \sum_{N \geq [t]+1} \int_N^{N+1} (N - [t]) \| \bar{R}^c(v(s)) \|_2 \, ds,
\end{align*}
\]

which in view of (5.10), can be continued as

\[
\begin{align*}
\left\| \bar{z}^c(t) \right\| &\leq C|y_0|^2 (e^{-2\lambda t} + \sum_{N \geq [t]+1} (N - [t]) e^{-2\lambda n}) \\
&\leq C|y_0|^2 e^{-2\lambda t} (1 + \frac{1}{\lambda^2}) \leq C|y_0|^2 e^{-2\lambda t}.
\end{align*}
\]

Hence

\[
\left\| \bar{z}^c(t) \right\| e^{\lambda t} \leq C|y_0|^2 e^{-\lambda t} \leq 2|y_0|, \quad (5.11)
\]

as desired.

We turn to estimating the Strichartz norm of \( \bar{v}^c \), for which we partition into

\[
\| \bar{v}^c \|_{S([T, \infty))} \leq \sum_{N \geq 1} \| \bar{v}^c \|_{S([T + N - 1, T + N])}.
\]

For each summand, we use (5.8) to write the corresponding vector into

\[
\begin{align*}
\bar{v}^c(t) &= -\int_t^\infty e^{(t-s)JL} \bar{R}^c(v(s)) \, ds \\
&= -\int_t^{T+N} e^{(t-s)JL} \bar{R}^c(v(s)) \, ds - \int_{T+N}^\infty e^{(t-s)JL} \bar{R}^c(v(s)) \, ds \\
&=: -(\bar{v}_+ + \bar{v}_-).
\end{align*}
\]

As \( \bar{v}_\pm = (v_\pm, \partial_t v_\pm) \), we know \( \bar{v}^c(t) = -(v_+(t) + v_-(t)) \), thus it suffices to estimate \( \| v_\pm \|_{S([T + N - 1, T + N])} \).
The estimate of $v_+$ follows directly from the inhomogeneous estimate in Lemma 5.1 and (5.10):

$$\|v_+\|_{S([T+N-1, T+N])} \leq C \|R^c\|_{L^1_tL^2_x}(T+N) \leq C e^{-2\lambda(T+N)}|y_0|^2.$$ 

To control $v_-$, we further partition the integral

$$\tilde{v}_- = \int_{T+N}^{\infty} e^{(t-s)JL} \tilde{R}^c(v(s))ds = \sum_{M \geq N+1} \int_{T+M}^{T+N} e^{(t-s)JL} \tilde{R}^c(v(s))ds$$

and apply the homogeneous estimate in Lemma 5.1 to obtain

$$\|v_-\|_{S([T+N-1, T+N])} \leq \sum_{M \geq N+1} \int_{T+M}^{T+N} (M - N + 1)^2 \|\tilde{R}^c(v(s))\|_2 ds \leq C|y_0|^2 \sum_{M \geq N+1} (M - N + 1)^2 e^{-2\lambda(T-M)} \leq C \frac{e^{-2\lambda T}}{T^3}.$$ 

Combining the estimates for $v_+$, we obtain

$$\|\tilde{v}^c\|_{S(T, \infty)} \leq \sum_{N \geq 1} \|v_+\|_{S([T+N-1, T+N])} + \|v_-\|_{S([T+N-1, T+N])} \leq C(1 + \frac{1}{\lambda^2})|y_0|^2 e^{-2\lambda T} \sum_{N \geq 1} e^{-2\lambda N} \leq C \frac{1}{\lambda^2} e^{-2\lambda T} \leq 2|y_0|e^{-2\lambda T}$$

as desired. We proved $\tilde{v}(t) \in B_{|y_0|, \lambda}$, thus the solution map $\Phi$ maps the ball $B_{|y_0|, \lambda}$ to itself.

The contraction of $\Phi$ follows a similar argument after using the last two estimates in (5.10), we skip the details.

Next we prove additional properties for the solution in (5.7). Indeed, by repeating the same contraction argument, we can easily see that $\Phi$ is also a contraction on a smaller ball $B_{|y_0|, \lambda_0}$, thus the unique solution in $B_{|y_0|, \lambda}$ also belongs to $B_{|y_0|, \lambda_0}$. That leaves us only the estimate of $\|\tilde{v}(t)\|_2$ in the first row of (5.7) to verify.

Noting $\|y\|_2 = 1$, we have

$$\|\tilde{v}(t)\|_2 \leq |y(t)| + |y'(t)| + \|\tilde{v}^c(t)\|_2,$$

thus the matter is reduced to estimating $y'(t)$ and $\|\tilde{v}^c(t)\|_2$. The estimate of $y'(t)$ comes from its exact expression obtained by differentiating $y(t)$-equation together with a similar estimate for $y(t)$. The estimate of $\|\tilde{v}^c(t)\|_2$ follows by the same argument as we use when estimating $\|\tilde{v}^c(t)\|_2$. Combing these two pieces together, we verify the first row in (5.7).

The second row in (5.7), which says that the $v^c$ component is of quadratic order around 0 is a clear implication of the estimate in (5.11).

Now let us turn to refining the uniqueness by eliminating the assumption on the exponential decay of the Strichartz norm. This can be done by showing that the exponential decay of $\|\cdot\|$ implies the exponential decay of Strichartz norm, i.e. given solution of (5.5) satisfying

$$\|\tilde{v}(t)\| \leq e^{-\lambda t}|y_0|, \forall t \geq 0,$$
Indeed, to prove (5.12), we take \( \eta \ll 1 \) such that \( C \eta^4 \leq \frac{1}{2} \). Applying Strichartz estimate on the interval \([\tau, \tau + \eta]\) gives that

\[
\|v\|_{S(\tau, \tau + \eta)} \leq C \|\tilde{v}(\tau)\| + C \eta^4 \|v\|_{S(\tau, \tau + \eta)} + C \|v\|_{S(\tau, \tau + \eta)}^5 \\
\leq C|y_0|e^{-\lambda \tau} + \frac{1}{2} \|v\|_{S(\tau, \tau + \eta)} + C \|v\|_{S(\tau, \tau + \eta)}^5.
\]

Choosing \( \delta \) sufficiently small if necessary and using the standard continuity argument we have

\[
\|v\|_{S(\tau, \tau + \eta)} \leq C|y_0|e^{-\lambda \tau}
\]

for a different constant \( C \). The estimate (5.12) then follows from partitioning the time interval \([\tau, \infty]\) into equal \( \eta \)-length intervals and summing up estimates from all subintervals.

Finally, we show the uniqueness up to time translation of the solutions with the same signed coordinates in \( Y \).

Let \( y_0, \tilde{y}_0 \) satisfy \( |y_0| < |\tilde{y}_0| < \delta \) and \( y_0 \tilde{y}_0 > 0 \) and let \( v(t), \tilde{v}(t) \) be solutions with initial \( Y \)-coordinates given by \( y_0, \tilde{y}_0 \). From the exponential decay of \( \tilde{v}(t) \), there must exist \( T > 0 \) such that

\[
\tilde{v}(T) = y_0 = v(0),
\]

then the uniqueness gives immediately that

\[
v(t) = \tilde{v}(t + T).
\]

Lemma (5.2) is proved. \( \square \)

The implication of Theorem 5.2 is twofold. On the one hand, it implies that there are two solutions lying on the stable manifold, denoted by \( W^+ \) and \( W^- \), and

\[
W^\pm = W + v_\pm, \quad v_\pm(0) = y_0^\pm Y + v^c(0) \quad \text{for some } |y_0^\pm| < \delta \quad \text{and} \quad y_0^+ > 0, \quad y_0^- < 0.
\]

Due to (5.7), it is easy to see that

\[
\|W^+(0)\|_{\dot{H}_a^1} > \|W\|_{\dot{H}_a^1} \quad \text{and} \quad \|W^-(0)\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1}.
\]

On the other hand, if there is a solution \( u(t) \) satisfying

\[
v(t) = u(t) - W \quad \text{and} \quad \|v(t)\| \leq C e^{-\lambda t}
\]

for some constant \( C \) and \( \lambda \in (0, e_0) \). Then for sufficiently large time \( T > 0 \) we have

\[
\|v(t)\| \leq \frac{1}{2} e^{-(\lambda + \epsilon) t} \quad \forall \ t \geq T.
\]

From the time translation symmetry and the uniqueness in \( B_{\frac{1}{2}, \lambda - \epsilon} \), we know \( u(t + T) = W^+(t) \) or \( u(t + T) = W^-(t) \). We conclude

**Corollary 5.3** There exist exactly two solutions (up to time translation) \( W^\pm \) of \( N L W_a \) satisfying

\[
\begin{align*}
\|W^\pm - W\| &\leq C e^{-e_0 t}, \quad \forall t \geq 0, \\
\|W^+(0)\|_{\dot{H}_a^1} &> \|W\|_{\dot{H}_a^1}, \quad \|W^-(0)\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1}.
\end{align*}
\]
Moreover, if a solution \( u(t, x) \) of NLW\(_a\) satisfy
\[
\|u(t) - W\|_{H^1_a} + \|\partial_t u(t)\|_2 \leq Ce^{-\lambda t}, \ \forall t \geq 0
\]
for some \( C > 0 \) and \( \lambda \in (0, e_0] \), there must exist unique \( T^\pm \) such that
\[
\begin{cases}
    u(t) = W^+(t + T^+) & \text{if } \|u\|_{H^1_a} > \|W\|_{H^1_a}, \\
    u(t) = W^-(t + T^-) & \text{if } \|u\|_{H^1_a} < \|W\|_{H^1_a}.
\end{cases}
\]

This corollary does not characterize the behavior of \( W^\pm \) for \( t < 0 \), we will discuss this problem in later sections, and then complete the picture of the dynamics of all solutions on the energy surface.

6 Virial estimates

In this section, we establish the Virial estimates by incorporating the modulation estimates in Sect. 4. Some arguments here are motivated by Duyckaerts and Merle [8] and our previously joint work with Zeng [25]. We will apply these estimates in both Sects. 7 and 8 to control the non-scattering global solutions.

Let \( \phi(x) \) be a smooth radial function such that
\[
\phi(x) = \begin{cases} 
1, & |x| \leq 1 \\
0, & |x| > 2 
\end{cases}
\]
Denote \( \phi_R(x) = \phi(\frac{x}{R}) \), we define the truncated Virial as
\[
G_R(t) = \int_{\mathbb{R}^3} [\partial_t u(t)u(t) + \partial_t u(t)x \cdot \nabla u(t)]\phi_R(x)dx.
\]
Let \( u(t) \) be a solution of NLW\(_a\) with \( E(u, \partial_t u) = E(W, 0) \), from direct computation we have
\[
\partial_t G_R(t) = \|\partial_t u(t)\|^2_{H^1_a} + \|u(t)\|^2_{H^1_a} - \|W\|^2_{H^1_a} + A_R(u(t)) + B_R(u(t)),
\]
where
\[
A_R(u(t)) = \int_{\mathbb{R}^3} (1 - \phi_R)[\frac{|\nabla u(t)|^2}{2} + \frac{a|u(t)|^2}{2|x|^2} - \frac{|u(t)|^6}{6}]dx + \frac{1}{2} \int_{\mathbb{R}^3} \Delta \phi_R |u(t)|^2 dx - \int_{\mathbb{R}^3} \nabla \phi_R \cdot \nabla u(t)x \cdot \nabla u(t)dx + \int_{\mathbb{R}^3} x \cdot \nabla \phi_R \left( \frac{|\nabla u(t)|^2}{2} + \frac{a|u(t)|^2}{2|x|^2} - \frac{|u(t)|^6}{6} \right) dx,
\]
\[
B_R(u(t)) = \int_{\mathbb{R}^3} (1 - \phi_R - x \cdot \nabla \phi_R) \frac{\partial_t u(t)|^2}{2} dx.
\]
It is straightforward to see that for any \( \mu > 0 \) that
\[
A_R(u(t)) = A_{\mu R}(u(t)|_{\mu}) \quad \text{and} \quad A_R(W) = 0. \quad (6.1)
\]
Using the definition of $d(u(t))$, we can rewrite $\partial_t G_R(t)$ as

$$
\partial_t G_R(t) = \begin{cases} 
2\|\partial_t u(t)\|_2^2 - d(u(t)) + A_R(u(t)) + B_R(u(t)), & \text{if } \|u(t)\|_{\dot{H}^1}^6 < \|W\|_{\dot{H}^1}^6; \\
\hat{d}(u(t)) + A_R(u(t)) + B_R(u(t)), & \text{if } \|u(t)\|_{\dot{H}^1}^6 > \|W\|_{\dot{H}^1}^6.
\end{cases}
$$

We estimate $G_R(t)$, $A_R(u(t))$, and $B_R(u(t))$ in the following lemma.

**Lemma 6.1** (Virial estimate) Let $u(t)$ be a solution of NLW$^a$ satisfying $E(u, \partial_t u) = E(W, 0)$. Then

$$
|G_R(t)| \lesssim R\hat{d}(u(t)), \quad \|\partial_t u(t)\|_2 \lesssim d(u(t)),
$$

and

$$
|A_R(u(t))| + |B_R(u(t))| \lesssim \int_{|x| > R} |u(t)|^6 + \frac{|u(t)|^2}{|x|^2} + |\nabla u(t)|^2 + |\partial_t u(t)|^2 dx.
$$

If in addition $d(u(t)) < \delta_0$ and the modulation parameter $\mu(t)$ satisfies $\mu(t)R \gtrsim 1$, we have

$$
|A_R(u(t))| + |B_R(u(t))| \lesssim (\mu(t)R)^{-\frac{\beta}{2}}\hat{d}(u(t)) + d(u(t))^2.
$$

Here all the implicit constants are independent of $d(u(t)), R, \|\partial_t u(t)\|_2$ and $\|u(t)\|_{\dot{H}^1}$.

**Proof** We first prove (6.3) by discussing two cases depending on the size of $d(u(t))$. When $d(u(t)) \geq \delta_0$, we use Hardy and Cauchy–Schwarz to obtain

$$
|G_R(t)| \lesssim 2R \int_{|x| \leq 2R} \frac{|u(t)|}{|x|} |\partial_t u(t)| + \frac{|x \cdot \nabla u(t)|}{|x|} |\partial_t u(t)| dx \\
\lesssim R\|\partial_t u(t)\|_2 \|u(t)\|_{\dot{H}^1} \lesssim R(\|\partial_t u(t)\|_2^2 + \|u(t)\|_{\dot{H}^1}^2) \\
\lesssim R(d(u(t)) + \|W\|_{\dot{H}^1}^2),
$$

$$
\|\partial_t u(t)\|_2 \leq \sqrt{\hat{d}(u(t))} \leq \frac{1}{\sqrt{\delta_0}}d(u(t)) \lesssim d(u(t)).
$$

When $d(u(t)) < \delta_0$, we are able to apply the modulation analysis to find $\mu(t)$ and write $u(t)|_{\mu(t)} = W + v(t)$.

From the scaling invariance and applying Lemma 4.3 we have the desired control for $\|\partial_t u(t)\|_2$ as well as

$$
|G_R(t)| \lesssim R\|\partial_t u(t)\|_2 \|u(t)\|_{\dot{H}^1} \lesssim R\|\partial_t u(t)\|_2 \|u(t)|_{\mu(t)}\|_{\dot{H}^1} \lesssim R\hat{d}(u(t))(\|W\|_{\dot{H}^1}^2 + \|v(t)\|_{\dot{H}^1}^2) \lesssim R\hat{d}(u(t)).
$$

Next we turn to estimating $A_R(u(t))$ and $B_R(u(t))$ in (6.4) and (6.5). We only consider (6.5) as (6.4) follows directly from the definition of $\phi$. With the condition $d(u(t)) < \delta_0$, we are able to use (6.6) and both identities in (6.1) to write

$$
|A_R(u(t))| = |A_{\mu(t)}R(u(t)|_{\mu(t)})| = |A_{\mu(t)}R(W + v(t)) - A_{\mu(t)}R(W)| \\
\lesssim \|\nabla W\|_{L^2(|x| \geq \mu(t)R)} \|\nabla v(t)\|_2 + \|\nabla v(t)\|_2^2 \\
+ \|v\|_{6}(\|W\|_{L^6(|x| \geq \mu(t)R)}^5 + \|v(t)\|_{6}^5) + \|W/|x|\|_{L^2(|x| \geq \mu(t)R)} \|\nabla v(t)\|_2
$$

The estimate (6.5) then follows from Lemma 4.3 and the asymptotics of $W(x): W(x) = O(|x|^{-\frac{1}{2}-\frac{1}{2\beta}})$ as $|x| \gtrsim 1$.

\[ \square \]
7 Exponential convergence in the sub-critical case

In this section, we characterize the global non-scattering solutions on the energy surface of $E(W, 0)$ when the kinetic energy is less than that of the ground state $W$. The main result is the following.

**Theorem 7.1** Let $u$ be a solution of $\text{NLW}_a$ satisfying

$$E(u, \partial_t u) = E(W, 0), \|u_0\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1}, \|u\|_{\dot{S}((0, \infty))} = \infty. \quad (7.1)$$

Then there exist $\mu > 0$ and a unique time $T = T(u)$ such that

$$u(t, x) = \pm \mu^{\frac{1}{2}} W(\mu t + T, \mu x). \quad (7.2)$$

In the opposite time direction, $u$ exists globally and obeys $\|u\|_{\dot{S}((\infty, 0])} < \infty$.

According to Corollary 5.3, it suffices to prove that there exist $\mu > 0$ and $c, C > 0$ such that the scaled solution $T_\mu u(t, x) = \mu^{-\frac{1}{2}} u(\frac{t}{\mu}, \frac{x}{\mu})$ satisfies

$$\|\partial_t T_\mu u(t)\|_2 + \|T_\mu u(t) - W\|_{\dot{H}_a^1} \leq Ce^{-ct}, \forall t \geq 0. \quad (7.3)$$

The rest of this section is devoted to proving (7.3). The rough idea is to get the exponential decay of the distance function from the Virial estimates. This implies that eventually the solution will enter into the small neighborhood of the ground state manifold. Then one can prove the convergence of the solution to $\pm W$ by showing the exponential decay of the modulation parameters. The underlying reason that the error in the truncated Virial is under control comes from the compactness of the solution, which in turn is a result for minimality of the ground state energy $E(W, 0)$ (see Theorem 1.1). Therefore it is necessary to review some of the key properties of solutions satisfying (7.1). As this part of the argument bears some similarities with our previous work on $\text{NLS}_a$, we will skip details in some of the discussions, for example, the choice for the scaling parameter $\lambda(t)$ in Lemma 7.3. One refers to see Section 7 in [25] for the detailed proof of Lemma 7.3.

7.1 Properties of solutions satisfying (7.1)

An immediate result for solutions on the energy surface is the coercivity of the kinetic energy, we have the following lemma.

**Lemma 7.2** Let the solution $u$ of $\text{NLW}_a$ satisfy $E(u, u_t) = E(W, 0)$. If $\|u(0)\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1}$ or $\|u(0)\|_{\dot{H}_a^1} > \|W\|_{\dot{H}_a^1}$, then for all $t$ in its lifespan we have

$$\|u(t)\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1} \quad \text{or} \quad \|u(t)\|_{\dot{H}_a^1} > \|W\|_{\dot{H}_a^1}. \quad (7.4)$$

In the former case, we also have

$$\|u(t)\|_{\dot{H}_a^1}^2 + \frac{3}{2} \|\partial_t u(t)\|_2^2 \leq \|W\|_{\dot{H}_a^1}^2, \text{ and } \|\partial_t u(t)\|_2^2 \leq \frac{2}{3} d(u(t)). \quad (7.4)$$

Guaranteed by Theorem 1.1, we know that if $u$ does not scatter at the minimal energy $E(W, 0)$ forward in time, it must have compactness modulo scaling symmetry, namely, there exists $\lambda(t) : [0, \infty) \rightarrow \mathbb{R}^+$ such that

$$\{(u(t)_{\lambda(t)}), \lambda(t)^{-\frac{3}{2}} \partial_t u(t, \lambda(t)^{-1} x)\}_{t \in [0, \infty)} \text{ is precompact in } \dot{H}_a^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3). \quad (7.5)$$
Consequently, there exists \( C(\varepsilon) > 0 \) such that
\[
\int_{|x| > \frac{C(\varepsilon)}{\lambda(t)}} |\partial_t u(t)|^2 + |\nabla u(t)|^2 + |u(t)|^6 + \frac{|u(t)|^2}{|x|^2} dx < \varepsilon. \tag{7.6}
\]

Clearly, there is a freedom to choose the scaling parameter \( \lambda(t) \). In particular, by following a similar argument as in Section 7 \[25\], we are able to take \( \lambda(t) \) to satisfy the following properties.

**Lemma 7.3** There exists a continuous function \( \lambda(t) \) satisfying (7.5) and all the properties below:
1) \( \lim_{t \to \infty} \lambda(t)t = \infty \).
2) On the interval of \( t \) when \( d(u(t)) < \delta_0 \), \( \lambda(t) = \mu(t) \), here \( \mu(t) \) is the modulation parameter appearing in Sect. 4.
3) \( \lambda(t) \) is almost differentiable and satisfies
\[
\left| \frac{1}{\lambda(a)} - \frac{1}{\lambda(b)} \right| \leq C \int_a^b d(u(t)) dt \forall [a, b] \subset [0, \infty). \tag{7.7}
\]

We first prove the convergence of \( d(u(t)) \) along a sequence of time by combining the Virial estimate and item 1) in Lemma 7.3.

**Lemma 7.4** Let \( u \) be the solution of NLW \( a \) satisfying (7.1). Then there exists a sequence of time \( t_n \to \infty \) such that \( d(u(t_n)) \to 0 \).

**Proof** Lemma 7.4 will follow eventually from a proper control on the error in \( \partial_t G_R(t) \) and then integrating in \( t \). To this end, for any \( \varepsilon > 0 \) and \( C(\varepsilon) \) in (7.6), we make use of item 1) in Lemma 7.3 to find \( T_0(\varepsilon) \) such that
\[
\lambda(t) > C(\varepsilon) / \varepsilon, \quad \text{thus } \varepsilon T > \frac{C(\varepsilon)}{\lambda(t)} \quad \forall t \in [T_0, T].
\]

This together with (6.4) and (7.6) gives for \( R = \varepsilon T > \frac{C(\varepsilon)}{\lambda(t)} \) that
\[
|A_R(u(t))| + |B_R(u(t))| \lesssim \int_{|x| > \frac{C(\varepsilon)}{\lambda(t)}} |\nabla u(t)|^2 + |u(t)|^2 + \frac{|u(t)|^2}{|x|^2} + |\partial_t u(t)|^2 dx \leq \varepsilon.
\]

With this we can estimate
\[
\partial_t G_R(t) = 2\|\partial_t u(t)\|_2^2 - d(u(t)) + A_R(u(t)) + B_R(u(t))
\leq - \frac{d(u(t))}{5} + |A_R(u(t))| + |B_R(u(t))|
\leq - \frac{d(u(t))}{5} + C\varepsilon.
\]

Noting also from (7.4) and (6.3) that
\[
G_R(t) \lesssim R,
\]
we integrate the estimate for \( \partial_t G_R(t) \) over \([T_0, T]\) then divide by \( T \) to get
\[
\frac{1}{T} \int_{T_0}^T d(u(t)) dt \lesssim \frac{\varepsilon(T - T_0) + R}{T} \lesssim \varepsilon.
\]
This implies
\[
\frac{1}{T} \int_0^T d(u(t))dt = \frac{1}{T} \int_0^{T_0} d(u(t))dt + \frac{1}{T} \int_{T_0}^T d(u(t))dt \leq C\varepsilon
\]
for all \( T \geq \frac{T_0(\varepsilon)}{\varepsilon} \). We have
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T d(u(t))dt = 0,
\]
and the convergence of \( d(u(t)) \) along a sequence of time is then proved. 

Next we prove the integral estimate of \( d(u(t)) \) by another application of the Virial estimate.

**Lemma 7.5** Let \( u \) be the solution of NLW\(_a\) satisfying (7.4) and (7.5), then there exists \( C > 0 \) such that for any \([a, b] \subset [0, \infty)\),
\[
\int_a^b d(u(t))dt \leq C \sup_{t \in [a, b]} \frac{1}{\lambda(t)} [d(u(a)) + d(u(b))].
\]

**Proof** The proof starts with a proper control on the error in the Virial identity. Let \( \eta_0 \) be a small constant and \( C(\eta_0) \) be the constant in (7.6). Take \( \eta_0 \) small enough so that \( \eta_0 < \frac{1}{10} \delta_0 \) and \( CC(\eta_0)^{-\frac{3}{2}} < \frac{1}{10} \). Then with \( R = C(\eta_0) \sup_{t \in [a, b]} \frac{1}{\lambda(t)} \), we can control \( A_R \) and \( B_R \) as
\[
|A_R(t)| + |B_R(t)| \leq \frac{d(u(t))}{10}.
\]
Indeed, when \( d(u(t)) > \delta_0 \), this follows from (6.4) and compactness (7.6); in the case when \( d(u(t)) \leq \delta_0 \), this follows from (6.5) and the fact that \( \lambda(t) = \mu(t) \) from Lemma 7.3.

Continuing from (7.9), we further use (6.2) and (7.4) to estimate \( \partial_t G_R(t) \) as
\[
\partial_t G_R(t) = 2\|\partial_t u(t)\|^2_2 - d(u(t)) + A_R(u(t)) + B_R(u(t))
\leq - \frac{d(u(t))}{5} + |A_R(u(t))| + |B_R(u(t))|
\leq - \frac{d(u(t))}{10}.
\]
Then the conclusion follows from applying the fundamental theorem of calculus and the upper bound in (6.3):
\[
\int_a^b d(u(t))dt \leq \int_a^b -\partial_t G_R(t)dt = G_R(a) - G_R(b) \lesssim R[d(u(a)) + d(u(b))]
\leq C(\eta_0) \sup_{t \in [a, b]} \frac{1}{\lambda(t)} [d(u(a)) + d(u(b))].
\]

Next we remove the \( \lambda \)-dependance in (7.8) by showing the uniform lower bound of \( \lambda(t) \).

**Lemma 7.6** Let \( u(t) \) be the solution of NLW\(_a\) satisfying (7.1), there exists a constant \( c > 0 \) such that
\[
\inf_{t \in [0, \infty)} \lambda(t) \geq c.
\]
**Proof** Let \( t_n \) be the time sequence in Lemma 7.4 along which \( d(u(t_n)) \rightarrow 0 \). Then there exists \( N \in \mathbb{N} \) such that for all \( m \geq N \),

\[
C(d(u(t_N))) + d(u(t_m)) \leq \frac{1}{2}.
\]

For any \( \tau \in [t_N, t_m] \), we apply (7.7) on \([t_N, \tau]\) then (7.8) to obtain

\[
\left| \frac{1}{\lambda(\tau)} - \frac{1}{\lambda(t_N)} \right| \leq C \int_{t_N}^{\tau} d(u(t))dt \leq C \int_{t_N}^{t_m} d(u(t))dt \\
\leq C \sup_{t \in [t_N, t_m]} \frac{1}{\lambda(t)} (d(u(t_N)) + d(u(t_m))) \\
\leq \frac{1}{2} \sup_{t \in [t_N, t_m]} \frac{1}{\lambda(t)}.
\]

Thus,

\[
\frac{1}{\lambda(\tau)} \leq \frac{1}{2} \sup_{t \in [t_N, t_m]} \frac{1}{\lambda(t)} + \frac{1}{\lambda(t_N)}, \forall \tau \in [t_N, t_m].
\]

Taking supremum in \( \tau \) on \([t_N, t_m]\) yields

\[
\sup_{\tau \in [t_N, t_m]} \frac{1}{\lambda(\tau)} \leq \frac{2}{\lambda(t_N)},
\]

which gives

\[
\sup_{\tau \in [t_N, \infty)} \frac{1}{\lambda(\tau)} \leq \frac{2}{\lambda(t_N)}
\]

by taking \( m \rightarrow \infty \). Thus, on \([t_N, \infty)\), \( \lambda(t) \geq \frac{\lambda(t_N)}{2} \); this together with the lower bound of \( \lambda(t) \) on finite interval \([0, t_N]\) proves the lemma. \( \square \)

Finally we are ready to prove Theorem 7.1.

### 7.2 Proof of Theorem 7.1

**Proof** The proof proceeds in four steps.

**Step 1.** We will show that there exist \( c, C, \lambda_\infty > 0 \) such that

\[
\int_t^\infty d(u(s))ds + |\lambda(t) - \lambda_\infty| \leq Ce^{-ct}.
\]  

(7.10)

The integral estimate of \( d(u(t)) \) is straightforward. Let \( t_n \) be the sequence such that \( d(u(t_n)) \rightarrow 0 \). Applying Lemma 7.5 and Lemma 7.6, we have

\[
\int_t^{t_n} d(u(s))ds \leq C[d(u(t)) + d(u(t_n))],
\]

which gives

\[
\int_t^\infty d(u(s))ds \leq C d(u(t)) \forall t \geq 0,
\]

by taking \( t_n \rightarrow \infty \). Then the integral estimate of \( d(u(t)) \) follows directly from Grönwall’s inequality.
We turn to the convergence of $\lambda(t)$. Taking an arbitrary interval $[a, b] \subset [0, \infty)$, from (7.7) and the estimate for $d(u(t))$ in (7.10), we have

$$\left| \frac{1}{\lambda(a)} - \frac{1}{\lambda(b)} \right| \lesssim \int_a^b d(u(t)) dt \leq Ce^{-ca}.$$ 

This implies that $\frac{1}{\lambda(t)}$ is Cauchy as $t \to \infty$, hence there exists $l_\infty \in [0, \infty)$ such that

$$\left| \frac{1}{\lambda(t)} - l_\infty \right| \leq Ce^{-ct}. \quad (7.12)$$

If $l_\infty > 0$, (7.12) yields (7.10) with $\lambda_\infty = \frac{1}{l_\infty}$. It remains to preclude

$$\lim_{t \to \infty} \frac{1}{\lambda(t)} = 0. \quad (7.13)$$

Let $N$ be a positive integer to be defined later and consider a subinterval $[t_n, t_{n+1}] \subset [t_N, \infty)$. Applying (7.7) then Lemma 7.5, we have

$$\left| \frac{1}{\lambda(t_n)} - \frac{1}{\lambda(t)} \right| \leq C \int_{t_n}^t d(u(t)) dt \leq C \int_{t_N}^t d(u(t)) dt \leq C \sup_{t \in [t_N, t_{n+1}]} \frac{1}{\lambda(t)} [d(u(t_n)) + d(u(t_{n+1}))].$$

If (7.13) holds, we can use (7.13) and the convergence $d(u(t_n)) \to 0$ to take the limit as $n \to \infty$, we obtain

$$\frac{1}{\lambda(t_n)} \leq C \sup_{t \in [t_N, \infty)} \frac{1}{\lambda(t)} d(u(t_{n+1})).$$

Choosing $N$ sufficiently large so that $Cd(u(t_N)) \leq \frac{1}{2}$, we get a contradiction by taking supremum in $t_n$ over $[t_N, \infty)$. Thus, (7.13) does not hold, we prove the decay of $\lambda(t)$ in (7.10).

**Step 2.** We show the uniform decay of $d(u(t))$:

$$\lim_{t \to \infty} d(u(t)) = 0. \quad (7.14)$$

We argue by contradiction. If (7.14) does not hold, there must exist a constant $\varepsilon_0 \in (0, \delta_0)$, a subsequence in $\{t_n\}$ (which we still denote as $\{t_n\}$) and $b_n \in (t_n, t_{n+1})$ such that

$$d(u(b_n)) = \varepsilon_0 \text{ and } d(u(t)) < \varepsilon_0 \ \forall t \in (t_n, b_n). \quad (7.15)$$

Moreover, from (4.14), we also have

$$\alpha(b_n) \sim d(u(b_n)) = \varepsilon_0, \ \alpha(t_n) \sim d(u(t_n)) \to 0. \quad (7.16)$$

But this will contradict the following estimate which can be obtained by applying the fundamental theorem of calculus, Lemme 4.3 and (7.10):

$$|\alpha(b_n) - \alpha(t_n)| = \int_{t_n}^{b_n} \alpha'(t) dt \lesssim \int_{t_n}^{b_n} \frac{\alpha'(t)}{\mu(t)} dt \lesssim \int_{t_n}^{b_n} d(u(t)) dt \lesssim e^{-ct_n}. \quad (7.17)$$

(7.14) is then proved.

**Step 3.** Finally we show the exponential convergence of modulation parameters as well as the solution.
In view of (7.14), we can find a $T_0 \gg 1$ such that $d(u(t)) < \delta_0$ for all $t \geq T_0$ on which the modulation parameters are well defined. The convergence of $\mu(t)$ comes from the coincidence between $\lambda(t)$ and $\mu(t)$ in Lemma 7.3 and the estimate of $\dot{\lambda}(t)$ in (7.10). We have

$$|\mu(t) - \lambda_{\infty}| = |\lambda(t) - \lambda_{\infty}| \leq Ce^{-ct} \forall t \geq T_0.$$  

By repeating the estimate in (7.17), we have for any $\tau, t \in [T_0, \infty)$ that

$$|\alpha(t) - \alpha(\tau)| \leq Ce^{-ct} \forall \tau \geq t \geq T_0.$$  

Letting $\tau \to \infty$ and using the fact that $\alpha(\tau) \sim d(u(\tau)) \to 0$, we have

$$|\alpha(t)| \leq Ce^{-ct} \forall t \geq T_0.$$  

Plugging these estimates in Lemma 4.3 yields

$$\|\dot{\alpha}u(t)\|_2 + \|\tilde{u}(t)\|_{\dot{H}_d^1} \sim \|v(t)\|_{\dot{H}_d^1} \sim d(u(t)) \leq Ce^{-ct} \forall t \geq T_0. \tag{7.18}$$  

Consequently,

$$\|u(t)\|_{\dot{H}_d^1} - W \|\mu\|_{\dot{H}_d^1} = \|u(t)\|_{\dot{H}_d^1} - W\|\mu(t)\|_{\dot{H}_d^1} \leq \|u(t)\|_{\dot{H}_d^1} - W\|\mu(t)\|_{\dot{H}_d^1} + \|W - W\|_{\dot{H}_d^1} \leq \|v(t)\|_{\dot{H}_d^1} + C|\mu(t) - \lambda_{\infty}| \leq Ce^{-ct} \forall t \geq T_0.$$  

With a slight modification on the constant $c$, $C$, we have proved (7.3) for $\mu = \lambda_{\infty}$. An application of Corollary 5.3 then yields (7.2). The last step is to show the scattering of $u$ in the opposite direction: $\|u\|_{S((\infty, 0])} < \infty$. After rescaling, this amounts to showing the next step.

**Step 4.** $\|W^-(t)\|_{S((\infty, 0])} < \infty$.

Assume otherwise $\|W^-(t)\|_{S((\infty, 0])} = \infty$, then all the above results also hold for $W^-(t)$ on $t \in (-\infty, 0]$. From Lemma 7.5 and Lemma 7.6, we get

$$\int_a^b d(W^-(t))dt \leq C\left(d(W^-(a)) + d(W^-(b))\right), \quad \forall a, b \in \mathbb{R}.$$  

Since $\lim_{|t|\to\infty} d(W^-(t)) = 0$, taking the limit as $a \to -\infty$, $b \to \infty$, we obtain $\int_{-\infty}^\infty d(W^-(t))dt = 0$, which implies $W^- \equiv W$ up to symmetry. It contradicts the conditions in Corollary 5.3 and with this we complete the proof of Theorem (7.1).

**Remark 7.7** The only missing part is to show all the solutions on the energy surface with less kinetic energy than that of the ground state $W$ must be global. A contradiction argument together with the uniform control on the kinetic energy (7.4) and the compactness of the solution yields this conclusion, see Corollary 2.9 [8] and [11, 19] for details.

**8 Exponential convergence in the supercritical case**

This section is devoted to characterizing the non-scattering solutions on the energy surface of $E(W, 0)$ when the kinetic energy is greater than that of the ground state $W$. The main result is the following.
Theorem 8.1 Let $u$ be a solution of NLW$_a$ satisfying
\[ u_0 \in L^2(\mathbb{R}^3); \]  
(8.1)
\[ E(u, \partial_t u) = E(W, 0), \|u_0\|_{\dot{H}^1_a} > \|W\|_{\dot{H}^1_a}. \]  
(8.2)

Then $u$ must blow up in finite time in both time directions.

Proof We argue by contradiction. Without loss of generality, we assume $u$ is globally defined in the positive time direction. The argument for the reverse direction is a mere repetition.

As $u_0 \in L^2$, we define
\[ y(t) = \int_{\mathbb{R}^3} u(t, x)^2 \, dx. \]

From direct computation, we have
\[ y'(t) = 2 \int_{\mathbb{R}^3} u(t, x) \partial_t u(t, x) \, dx, \quad y''(t) = 8 \|\partial_t u(t)\|_{L^2}^2 + 4 \|u(t)\|_{H^1_a}^2 - 4 \|W\|_{H^1_a}^2. \]

Using coercivity Lemma 7.2 and Hölder inequality, we know
\[ y''(t) > \max(8\|u_t\|_{L^2}^2, 4d(u(t))) > 0, \quad (y')^2 \leq 4\|u(t)\|_{L^2}^2\|u_t(t)\|_{L^2}^2 \leq \frac{1}{2} y(t) y''(t). \]

Thus, $y'(t)$ is monotone increasing.

The proof after this point will proceed in three steps as indicated below.

Step 1. We show there exists $y_\infty \in (0, \infty)$ such that
\[ \lim_{t \to +\infty} y(t) = y_\infty, \quad \lim_{t \to +\infty} y'(t) = 0. \]

(8.4)

As a preliminary step, we first show the negativity of $y'(t)$:
\[ y'(t) < 0 \quad \text{when} \quad t \geq 0. \]

(8.5)

Indeed, if for some $t_0 \geq 0$, $y'(t_0) \geq 0$, then the monotonicity of $y'(t)$ guarantees that
\[ y'(t) \geq y'(t_0 + 1) > 0 \quad \forall t \geq t_0 + 1. \]

On this interval we can rewrite the second inequality in (8.3) into
\[ \frac{y''(t)}{y'(t)} \leq \frac{2y'(t)}{y(t)} \quad \text{or} \quad (\ln y'(t))' \geq 2(\ln y(t))'. \]

A simple ODE argument shows that
\[ \frac{y'(t)}{y'(t_0 + 1)} \geq \left( \frac{y(t)}{y(t_0 + 1)} \right)^2, \quad \forall t \in [t_0 + 1, \infty), \]

and consequently, $y(t)$ blows up at finite time, contradicting to that $u$ is global. Therefore the negativity (8.5) is proved. As a result of (8.3) and (8.5), $y'(t)$ is monotone increasing and $y(t)$ is monotone decreasing. Consequently, there exist $c \leq 0$ and $y_\infty \geq 0$ such that
\[ \lim_{t \to \infty} y'(t) = c, \quad \lim_{t \to \infty} y(t) = y_\infty. \]

Now if $c < 0$, $y(t)$ decreases at least at a linear rate, thus $y(t) \to -\infty$ (as $t \to \infty$) which is a contradiction. This proves that
\[ \lim_{t \to \infty} y'(t) = 0. \]

It remains to verify $y_\infty > 0$. 

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On the one hand, from the second equation in (8.4), and the lower bound for $y''(t)$ in (8.3), we integrate $y''$ over $[t, \infty)$ to obtain
\[
-y'(t) = \int_t^\infty y''(\tau) d\tau \geq \int_t^\infty d(u(\tau)) d\tau.
\] (8.6)

On the other hand, the second inequality in (8.3) together with the control for $\|u_t\|_2$ in (6.3) implies
\[
-y'(t) \leq C \|\partial_t u(t)\|_2 \leq C d(u(t)).
\]

Combining the two estimates we have
\[
\int_t^\infty d(u(\tau)) d\tau \leq C d(u(t)),
\]

which, by the Grönwall inequality, implies
\[
\int_t^\infty d(u(\tau)) d\tau \leq C e^{-ct}.
\] (8.7)

For future use, we notice that (8.7) implies the existence of a sequence $t_n \to \infty$ such that
\[
\lim_{n \to \infty} d(u(t_n)) = 0.
\] (8.8)

Now suppose otherwise $y_\infty = 0$, we integrate $y'$ over $[t, \infty)$ and obtain
\[
y(t) = -\int_t^\infty y'(\tau) d\tau \leq C \int_t^\infty d(u(\tau)) d\tau.
\]

This together with (8.6) and the second equation in (8.4) gives
\[
\int_t^\infty d(u(\tau)) d\tau \leq -y'(t) \leq 2\|\partial_t u(t)\|_2 \sqrt{y(t)} \leq C d(u(t)) \sqrt{\int_t^\infty d(u(\tau)) d\tau}.
\]

Or equivalently,
\[
\sqrt{\int_t^\infty d(u(\tau)) d\tau} \leq C d(u(t)).
\]

Denote $Z(t) = \int_t^\infty d(u(\tau)) d\tau$, we have the following ODE inequality for $Z(t)$.
\[
(\sqrt{Z(t)})' \leq -C, \text{ or } C(T-t) \leq \sqrt{Z(t)} - \sqrt{Z(T)} \leq \sqrt{Z(t)}.
\]

We get a contradiction by taking $T$ sufficiently large. This finally proves $y_\infty > 0$, hence (8.4).

**Step 2.** Convergence of $u(t)$ in $L^2(\mathbb{R}^3)$ and $d(u(t))$.

To see $\{u(t), \ t \geq 0\}$ is Cauchy, we take any $0 \leq t < T$ and use Minkowski’s inequality, (6.3) and (8.7) to obtain
\[
\|u(T) - u(t)\|_2 = \left\| \int_t^T \partial_t u(\tau) d\tau \right\|_2 \leq \int_t^T \|\partial_t u(\tau)\|_2 d\tau \\
\leq \int_t^T C d(u(\tau)) d\tau \leq C e^{-ct}.
\]

Thus there exists $u_\infty \in L^2(\mathbb{R}^3)$ such that $u(t) \to u_\infty$ in $L^2(\mathbb{R}^3)$ as $t \to +\infty$. As $y_\infty \neq 0$, we know $u_\infty \neq 0$. 

\[ \text{Springer} \]
We turn to the convergence of \( d(u(t)) \) for which we argue the same way as the NLS case \([25]\).

Let \( I \) be a time interval such that
\[
d(u(t)) < \delta_0, \quad \forall t \in I.
\]

We first show the uniform lower bound of \( \mu(t) \) on \( I \). Indeed, on the interval \( I \), we can use Lemma 4.3 to write
\[
u(t) = W + v(t).
\]
Computing the \( L^2 \) norm on both sides of above equation gives
\[
\mu(t) \|
u(t)\|^2_2 \geq \|W + v(t)\|^2_{L^2(|x| \leq 1)} \geq \|W\|^2_{L^2(|x| \leq 1)} - C\|v(t)\|_6^6 \geq \|W\|^2_{L^2(|x| \leq 1)} - C\delta_0,
\]
which together with \( \|
u(t)\|^2_2 \lesssim 1 \) shows
\[
\mu(t) \gtrsim 1, \quad \forall t \in I. \tag{8.9}
\]

Next we show the uniform upper bound of \( \mu(t_n) \) for \( t_n \) obtained in (8.8). Noticing from Lemma 4.3, along the sequence \( \{t_n\} \), we also have
\[
|\alpha(t_n)| \sim \|v(t_n)\|_{\dot{H}^1_a} \sim d(u(t_n)) \to 0,
\]
and
\[
u(t_n)_{\mu(t_n)} \to W \text{ in } \dot{H}^1_a(\mathbb{R}^3). \tag{8.10}
\]

If otherwise there exists a subsequence in \( \{t_n\} \) (which we still denote as \( \{t_n\} \)) such that
\[
\lim_{t \to \infty} \mu(t_n) = \infty,
\]
we can use the strong convergence (8.10) to get for any \( \varepsilon > 0 \) that
\[
\int_{|x| \geq \varepsilon} |\nu(t_n)|^6dx = \int_{|x| \geq \mu(t_n)\varepsilon} |\nu(t_n)_{\mu(t_n)}|^6dx \to 0.
\]
Thus for any \( R \ll 1 \) and the indicator function \( \chi_R \) on the ball \( B_0(R) \), we have
\[
\int_{\mathbb{R}^3} u(t_n)^2 \chi_R(x)dx = \int_{|x| \leq \varepsilon} u(t_n)^2 \chi_R(x)dx + \int_{|x| > \varepsilon} u(t_n)^2 \chi_R(x)dx
\]
\[
= \int_{|x| \leq \varepsilon} u^2_\infty dx + O\left(\int u^2_\infty (1 - \chi_R)dx + R^2\left(\int_{|x| > \varepsilon} |\nu(t_n)|^6dx\right)^{\frac{1}{2}}\right) + o_n(1).
\]
Here we have used the convergence \( u(t_n) \to u_\infty \) in \( L^2 \) and the Hölder inequality. Letting \( n \to \infty \) then \( \varepsilon \to 0 \), we obtain
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} u(t_n)^2 \chi_R(x)dx = \int_{\mathbb{R}^3} (u_\infty)^2 \chi_R(x)dx = 0.
\]
Taking \( R \to \infty \) shows that \( u_\infty = 0 \) which contradicts (8.4). Hence, the uniform upper bound for \( \mu(t_n) \) is proved. Combining the lower bound (8.9), we have
\[
\mu(t_n) \sim 1. \tag{8.11}
\]
We are ready to prove the convergence of \( d(u(t)) \):

\[
\lim_{t \to \infty} d(u(t)) = 0. \tag{8.12}
\]

Let \( \{t_n\} \) be the time sequence in (8.8). If otherwise (8.12) is not true, there exist \( \varepsilon_0 \in (0, \delta_0) \), a subsequence in \( \{t_n\} \) (still denote as \( \{t_n\} \)), and \( b_n \in (t_n, t_n + 1) \) such that

\[
d(u(b_n)) = \varepsilon_0, \quad d(u(t)) \in (0, \varepsilon_0], \quad \forall t \in [t_n, b_n]. \tag{8.13}
\]

On the one hand, from the comparison estimate in Lemma 4.3, we know

\[
\alpha(t_n) \to 0, \quad \text{and} \quad \alpha(b_n) \sim \varepsilon_0. \tag{8.14}
\]

On the other hand, applying the derivative estimate for \( \mu(t) \) in Lemma 4.3, we have

\[
\left| \frac{1}{\mu(t_n)} - \frac{1}{\mu(t)} \right| \leq \int_{t_n}^t \left| \frac{\mu'(\tau)}{\mu(\tau)^2} \right| d\tau \lesssim \int_{t_n}^\infty d(u(\tau)) d\tau \to 0. \tag{8.15}
\]

This together with (8.11) yields

\[
\mu(t) \sim 1 \quad \forall t \in [t_n, b_n]. \tag{8.16}
\]

Plugging this into estimate of \( \alpha'(t) \) in Lemma 4.3 further gives

\[
|\alpha(t_n) - \alpha(b_n)| \leq \int_{t_n}^{b_n} |\alpha'(t)| dt \lesssim \int_{t_n}^{b_n} \frac{|\alpha'(t)|}{\mu(t)} dt \lesssim \int_{t_n}^\infty d(u(t)) dt \to 0,
\]

which clearly contradicts (8.14).

Finally we show the last step.

**Step 3.** Blowup of \( u(t) \) at finite time.

In view of (8.12) and Lemma 4.3, we can perform the orthogonal decomposition for all \( t \geq T_0 \) and repeat the same argument as in (8.11) to show \( \mu(t) \sim 1 \). From here we can repeat the same argument as in Sect. 7 to show the exponential convergence of all parameters as well as the solution. The latter together with Corollary 5.3 implies for some \( \mu > 0 \) and \( T \in \mathbb{R} \) that

\[
u(t, x) = \pm \mu^{\frac{1}{2}} W_+(\mu t + T, \mu x).
\]

Since \( u(t) \in L^2(\mathbb{R}^3) \) we know \( W_+ \in L^2(\mathbb{R}^3) \), which together with \( W_+ - W \in L^2(\mathbb{R}^3) \) from Theorem 5.2 gives \( W \in L^2(\mathbb{R}^3) \). We get a contradiction, Theorem 8.1 is then proved.

\[\square\]

**Acknowledgements** We thank the referee for careful reading of our manuscript and helpful comments. We are grateful to Prof. Chongchun Zeng for several discussions. K.Y. was supported by the “Jiangsu Shuang Chuang Doctoral Plan” and NSF of Jiangsu(China): BK20200346. X.Z. was supported by Simons Collaboration Grant.

## 9 Appendix

In this section, we present an alternative way to obtain the eigenfunction \( \mathcal{V} \) of \( L_+ \) as mentioned in Remark 3.3. Recalling from Proposition 3.1, \( L_+ \) has only one dimensional negative direction, the existence of \( \mathcal{V} \) then amounts to showing that the essential spectrum of \( L_+ \) lies in \([0, +\infty)\) (see e.g. [6], Chapter 4). The latter follows from that \( \frac{d+2}{d-2} W^\frac{4}{d-2} \) is a relative compact perturbation of \( L_a \), which is equivalent to showing that \( \frac{d+2}{d-2} W^\frac{4}{d-2} (1 + L_a)^{-1} \) is a compact operator on \( L^2(\mathbb{R}^d) \). For the sake of completeness, we present details for all dimensions \( d \geq 3 \).
Claim 9.1 Let $d \geq 3$ and $0 > a > -(d-2)^2/2 + (d-2)^2/24$ $\iff 1 > \beta > \frac{2}{d+2}$, then $W \frac{1}{d+2}(1 + \mathcal{L}_a)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$, where $W$ is the ground state solution.

Proof Taking a bounded sequence $f_n$ in $L^2(\mathbb{R}^d)$, we denote $g = \frac{r^{\beta-1}}{1+r^{2d}}$ and recall that $W \frac{1}{d+2} = C g^2$. Then, by taking $\varepsilon > 0$ sufficiently small, we have
\[
\|\nabla^\varepsilon (W \frac{1}{d+2}(1 + \mathcal{L}_a)^{-1} f_n)\|_2 \leq \| (|\nabla|^\varepsilon g^2) (1 + \mathcal{L}_a)^{-1} f_n\|_2 + \| g^2 |\nabla|^\varepsilon (1 + \mathcal{L}_a)^{-1} f_n\|_2 \leq \| |\nabla|^\varepsilon g\|_2 \frac{d(d+2)}{d^2-2d-(4+2d)} \| g\|_2 \frac{d(d+2)}{d^2-2d-(4+2d)} \| \nabla|^\varepsilon (1 + \mathcal{L}_a)^{-1} f_n\|_2 \frac{d(d+2)}{d^2-2d-(4+2d)} \lesssim \| \nabla g\|_2^2 \| f_n\|_2 \lesssim 1.
\]

To prove compactness of the operator, it remains to verify the decay estimate:
\[
\| \chi_{> R} W \frac{1}{d+2}(1 + \mathcal{L}_a)^{-1} f_n\|_2 \lesssim \| \chi_{> R} g^2\|_\infty \| (1 + \mathcal{L}_a)^{-1} f_n\|_2 \lesssim R^{-2-2\beta}.
\]
The proof is complete. \qed

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