Abstract

Consider the class of \( k \)-independent bond or site percolations with parameter \( p \) on a tree \( T \). We derive tight bounds on \( p \) for both almost sure percolation and almost sure nonpercolation. The bounds are continuous functions of \( k \) and the branching number of \( T \). This extends previous results by Lyons for the independent case \((k = 0)\) and by Balister & Bollobás for 1-independent bond percolations. Central to our argumentation are moment method bounds à la Lyons supplemented by explicit percolation models à la Balister & Bollobás. An indispensable tool is the minimality and explicit construction of Shearer’s measure on the \( k \)-fuzz of \( \mathbb{Z} \).

Keywords: \( k \)-independent, \( k \)-dependent, tree percolation, critical value, percolation kernel, second moment method, Shearer’s measure.

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1 Introduction

If we regard percolation on a tree \( T \), then a natural question is which properties of the percolation and the tree determine the percolation behaviour. One is especially interested in bounds which are not particular to a specific model, but are valid for whole classes of models. The class of models we investigate are \( k \)-independent (also called \( k \)-dependent in the literature) site (bond) percolations with parameter \( p \), i.e. the probability that a single vertex (edge) is open is \( p \) and subsets of vertices (edges) are independent if their distance is greater than \( k \). We look for bounds on the parameter \( p \) which guarantee either a.s. percolation or a.s. nonpercolation.

Lyons [7] first treated this question in the case of independent percolation. He defined the branching number \( br(T) \) as a measure of the size of \( T \). Then he showed that it is the characteristic determining the critical probability for independent percolation (see theorem 1), that is the parameter threshold at which nonpercolation switches to percolation.

A recent work by Balister & Bollobás [2] deals with the class of \( 1 \)-independent bond percolations (see theorem 2). There are two continuous functions of the branching number which give tight bounds for a.s. percolation and a.s. nonpercolation of each model in this class.

In section 3 we present our results: tight bounds for a.s. percolation and a.s. nonpercolation for every \( k \). The bounds are again continuous functions of \( br(T) \), parametrized by \( k \). They are the same for bond and site percolation.
A core ingredient is a probability measure introduced by Shearer [12], which has certain nice minimizing properties, reviewed in section 4.1. We construct it explicitly on the $k$-fuzz of $\mathbb{Z}$ in section 4.2 and show that it is a $(k+1)$-factor. Shearer’s measure minorizes the probability of having an open path of $k$-independent Bernoulli rvs. This property is already exploited implicitly in the work of Balister & Bollobás. We make this argument explicit by using moment method and capacity arguments motivated by Lyons’ proof [7, 8], supplemented with explicit percolation models inspired by Balister & Bollobás’ work [2].

2 Setup and previous results

Let $G := (V,E)$ be a graph. For every subset $H$ of vertices and/or edges of $G$ denote by $V(H)$ the vertices induced by $H$ and by $G(H)$ the subgraph of $G$ induced by $H$. We have the geodesic graph distance $d$ on both vertices and edges, extended naturally to sets of them. Define the equivalence relation $v \leftrightarrow w$ describing connectedness on $G$. We denote by $N(v)$ the neighbours of a vertex $v$. The $k$-fuzz (or $k$th power) of $G$ is the graph $(V,E')$, where $E'$ consists of all distinct pairs of vertices with distance less than or equal to $k$ in $G$.

We primarily work on a locally finite tree $T := (V,E)$. We consider it to be infinite, unless explicitly stated otherwise. Between two nodes $v$ and $w$ we have the unique geodesic path $P(v,w)$. For the following definitions root $T$ at the root $o$ and visualize the tree spreading out downwards from the root. Define the level $l(v) := d(o,v)$ of a node $v$ and let $L(T,n) := \{v : l(v) = n\}$ be the $n$th level of $T$. Downpaths and -rays are finite and infinite geodesics, which start at some vertex $v$ and go downwards, thereby avoiding all ancestors of $v$, respectively. Denote the boundary of $T$ by $\partial T$, which is the set of all ends of $T$, identified with the set of all downrays of $T$ starting at $o$. For all nodes $v \in V \setminus \{o\}$ there is a unique parent denoted by $p(v)$. The confluent $v \wedge w$ is the last common ancestor of two distinct nodes $v$ and $w$. Define a minimal vertex cutset $\Pi$ to be a finite set of vertices containing no ancestors of itself and delineating a connected component containing $o$. Denote by $\Pi(o)$ the set of all minimal vertex cutsets of $o$. Finally let $T^v$ be the induced subtree of $T$ rooted at $v$.

Furthermore we abbreviate $\{1, \ldots, n\}$ by $[n]$. As a convention we interpret $[0] := \emptyset$, $0^0 := 1$, empty products as 1 and empty sums as 0.

Recall that a bond and site percolation on a graph $G := (V,E)$ is an rv taking values in $\{0,1\}^E$ and $\{0,1\}^V$ respectively. A percolation percolates iff it induces an infinite percolation cluster (connected component) in $G$ with nonzero probability.

We investigate percolations on a tree $T := (V,E)$ and look for properties of the percolation and the tree influencing the percolation behaviour. For $k \in \mathbb{N}_0$ we consider the class of $k$-independent site percolations with parameter $p$ on $T$, denoted by $\mathcal{C}_k^p(V)$. A site percolation $Z := \{Z_v\}_{v \in V}$ has parameter $p$ iff the probability that a single site is open equals $p$. For $W \subseteq V$ let $Z_W := \{Z_v\}_{v \in W}$.
The site percolation $Z$ is $k$-independent iff
\[ \forall U, W \subset V : \quad d(U, W) > k \Rightarrow Z_U \text{ is independent of } Z_W, \tag{1} \]
that is, events on subsets at distance greater than $k$ are independent. Independence is synonymous with 0-independence. The present paper investigates bounds on the parameter $p$ guaranteeing either a.s. percolation or a.s. nonpercolation for the whole class. We define the critical values
\[ p^k_{\text{max}}(V) := \inf \{ p \in [0, 1] : \forall \mathcal{P} \in C^k_p(V) : \mathcal{P} \text{ percolates} \} \tag{2a} \]
\[ p^k_{\text{min}}(V) := \inf \{ p \in [0, 1] : \exists \mathcal{P} \in C^k_p(V) : \mathcal{P} \text{ percolates} \}. \tag{2b} \]

Analogously, we define the class $C^k_p(E)$ of $k$-independent bond percolations with parameter $p$ on $T$ and critical values $p^k_{\text{max}}(E)$ and $p^k_{\text{min}}(E)$.

A $\lambda$-flow on $T$ is a function $f : V \mapsto \mathbb{R}_+$ such that
\[ \forall v \in V : \quad 0 \leq f(v) = \sum_{w : \mathcal{P}(w) = v} f(w) \leq \lambda - l(v). \tag{3} \]

Lyons [7] introduced the branching number $br(T)$ as a measure of the size of a tree $T$:
\[ br(T) := \sup \{ \lambda \geq 1 : \text{exists nonzero } \lambda \text{-flow on } T \} \]
\[ = \sup \{ \lambda \geq 1 : \inf_{\mathcal{P} \in \mathcal{P}(o)} \sum_{v \in \mathcal{P}} \lambda^{-l(v)} > 0 \}. \tag{4} \]
The duality in (4) is due to the max-flow min-cut theorem on infinite graphs [5]. The branching number $br(T)$ is independent of the choice of $o$ and equals the exponential of the Hausdorff dimension of the boundary $\partial T$ of $T$ [9, section 1.8]. Throughout this paper we assume $br(T)$ to be finite.

The first known result is due to Lyons [7], where he characterized the critical value of independent percolation ($k = 0$) in terms of $br(T)$:

**Theorem 1** ([7, theorem 6.2]).
\[ p^0_{\text{max}}(V) = p^0_{\text{min}}(V) = p^0_{\text{max}}(E) = p^0_{\text{max}}(E) = \frac{1}{br(T)}. \tag{5} \]

In the independent case the critical values coincide, since for fixed $p$ there is only one percolation. Lyons’ proof is based on moment methods and capacity estimates of percolation kernels. In general, a percolation is quasi-independent [8, section 2.4] iff, using the notation from figure 3 on page 19 with $u := v \lor w$ the confluent of $v$ and $w$, we have an $M > 0$ such that for all $v$ and $w$,
\[ \mathbb{P}(o \leftrightarrow v, o \leftrightarrow w | o \leftrightarrow u) \leq M \mathbb{P}(o \leftrightarrow v | o \leftrightarrow u) \mathbb{P}(o \leftrightarrow w | o \leftrightarrow u). \tag{6} \]
Equivalently, this majorizes the percolation kernel (36c) by
\[ \kappa(v, w) := \frac{\mathbb{P}(o \leftrightarrow v, o \leftrightarrow w)}{\mathbb{P}(o \leftrightarrow v) \mathbb{P}(o \leftrightarrow w)} \leq \frac{M}{\mathbb{P}(o \leftrightarrow u)}. \tag{7} \]
This way Lyons [8, section 2.4] used the weighted second moment method to get bounds for the probability of the percolation reaching subsets of $\partial T$ in terms of
their capacity, extending the independent case in [7].

In a recent work, Balister & Bollobás [2] deal with the class of 1-independent bond percolations:

**Theorem 2** ([2]).

\[ p_{\min}^1(E) = \frac{1}{br(T)^2} \]  \hspace{1cm} (8a)

\[ p_{\max}^1(E) = \begin{cases} 1 - \frac{br(T)-1}{br(T)} & \text{if } br(T) \leq 2 \\ \frac{2}{k} & \text{if } br(T) \geq 2. \end{cases} \]  \hspace{1cm} (8b)

Their proof strategy for \( p_{\min}^1(E) \) is based on the first moment method and a simple explicit model. We generalize it rather straightforwardly to higher \( k \) in section 5.5. Their proof for \( p_{\max}^1(E) \) on the other hand combines a so-called canonical model (discussed in section 5.4.2) with several short and elementary inductive proofs (see [13]). For every \( p \geq \frac{3}{4} \) this canonical model minimizes the probability to percolate. They implicitly retrace the weighted second moment method, percolation kernel capacity estimates based on \( \lambda \)-flows and the minimizing property (22b), (30b) of Shearer’s measure [12] on \( \mathbb{Z} \). Alas this inductive approach exploits a few particularities of the case \( k = 1 \), which we have not been able to abstract from.

### 3 Main results

Our principal result in the setting of section 2 (see also figure 1 on page 6) is:

**Theorem 3.** \( \forall k \in \mathbb{N}^0: \)

\[ p_{\min}^k(V) = p_{\min}^k(E) = \frac{1}{br(T)^{k+1}} \] \hspace{1cm} (9a)

\[ p_{\max}^k(V) = p_{\max}^k(E) = \begin{cases} 1 - \frac{br(T)-1}{br(T)} \frac{k}{k+1} & \text{if } br(T) \leq \frac{k+1}{k} \\ 1 - \frac{k}{k+1} \frac{1}{k+1} & \text{if } br(T) \geq \frac{k+1}{k}, \end{cases} \] \hspace{1cm} (9b)

where we interpret \( \frac{1}{0} := \infty \) in the case \( k = 0 \).

This theorem is a corollary of the more general theorem 4 upon setting \( s = k \) and verifying that (un)rooting a percolation does not change its percolation behaviour (see section 5.2).

First we narrow down the definition of the percolation classes we work on. Let \( \mathcal{C}_{p,o}^{k,s}(V) \) be the class of rooted site percolations with parameter \( p \) on \( T \) which are \( k \)-independent along downrays from \( o \) and \( s \)-independent elsewhere, that is among vertices not on the same downray. We define the rooted critical values as

\[ p_{\max}^{k,s}(V) := \inf \{ p \in [0,1] : \forall o \in V : \forall \mathcal{P} \in \mathcal{C}_{p,o}^{k,s}(V) : \mathcal{P} \text{ percolates} \} \] \hspace{1cm} (10a)

\[ p_{\min}^{k,s}(V) := \inf \{ p \in [0,1] : \exists o \in V : \exists \mathcal{P} \in \mathcal{C}_{p,o}^{k,s}(V) : \mathcal{P} \text{ percolates} \}. \] \hspace{1cm} (10b)
Analogously we define the class $c^{k,s}_{p,o}(E)$ of $k,s$-independent, rooted bond percolations with parameter $p$ on $T$ and the critical values $p_{\min}^{k,s}(E)$ and $p_{\max}^{k,s}(E)$. Define the function

$$g_k : [1, \infty) \rightarrow [0, 1] \quad y \rightarrow 1 - \frac{y - 1}{y^{k+1}} \quad (11)$$

and the value

$$p_{sh}^{(k)} := 1 - \frac{k^k}{(k + 1)(k + 1)} \quad (12)$$

We reveal their motivation in proposition 7 and 17 respectively. Our main result determines the critical values (10):

**Theorem 4.** \(\forall k, s \in \mathbb{N}_0:\)

$$p_{\min}^{k,s}(V) = p_{\min}^{k,s}(E) = \frac{1}{br(T)^{k+1}} \quad (13a)$$

$$p_{\max}^{k,s}(V) = p_{\max}^{k,s}(E) = \begin{cases} 1 - \frac{br(T) - 1}{br(T)^{k+1}} &= g_k(br(T)) & \text{if } br(T) \leq \frac{k+1}{k} \\ 1 - \frac{k^k}{(k + 1)(k + 1)} &= p_{sh}^{(k)} & \text{if } br(T) \geq \frac{k+1}{k} \end{cases} \quad (13b)$$

where we interpret \(\frac{1}{0} := \infty\) in the case \(k = 0\).

We give the proof in section 5 and a plot of the results (13) in figure 1.

![Figure 1: (Colour online) The curves of $p_{\max}^{k,s}(V)$ and $p_{\min}^{k,s}(V)$ for $k \in \{0, 1, 2, 3\}$ and branching numbers in $[1, 2.5]$ delimit increasingly shaded regions. The dashed red lines mark the points $\left(\frac{k+1}{k}, p_{sh}^{(k)}\right)$ for $k \geq 1$, where the behaviour of $p_{\max}^{k,s}(V)$ changes.](image-url)

The critical values (13) are independent of the root $o$, the elsewhere-dependence range $s$ and whether we regard bond or site percolation. A change of the root
from $o$ to $a'$ turns a $k$, $s$-independent percolation at worst into a $(k \lor s), (k \lor s)$-independent percolation. Upon closer inspection one sees that this concerns only elements contained in the ball of radius $d(o,a') + (k \lor s)$ around $a'$. They are finitely many and one can ignore them as percolation is a tail-event (see the adaption of Kolmogorov’s zero-one law in lemma 10), hence the percolation essentially remains $k$, $s$-independent. The independence of the parameter $s$ is a consequence of the use of the moment methods, which only take into account the structure of a rooted percolation along downrays. There is a bijection from $E$ to $V \setminus \{o\}$ mapping an edge to its endpoint further away from the root $o$. This implies that $C_{p,o}^{k,s}(E) \subseteq C_{k,o+1}(V)$. Furthermore we have $c_{p,o}^{k,s}(V) = c_{p,o}^{k,s+1}(V) = c_{p,o}^{k,s,1}(V)$ for $k \geq 1$ and $c_{p,o}^{k,0}(V) = c_{p,o}^{k,0}(E)$ as $(V_{p,o}^{k,0}(V) = \emptyset)$. This allows the interpretation of our explicit site percolation models (models 22, 24 and 27) as $k,0$-independent bond percolation models. Hence we focus exclusively on site percolations for the remainder of this paper.

We can generalize the single parameter $s$ to a family of finite and unbounded dependency parameters $\vec{s} := \{s_v\} \in V$. Then the upper bound on $p_{\min}^{k,\vec{s}}(V)$ in proposition 17 does not hold anymore. See the counterexample in model 18 and proposition 19. The lower bound on $p_{\min}^{k,\vec{s}}(V)$ in propositions 26 and hence the value of $p_{\min}^{k,\vec{s}}(V)$ stay valid under these less restrictive conditions and even for $s = \infty$, though.

We determine the critical values by a two-pronged approach. General bounds follow from a direct application of moment arguments [9, sections 5.2/5.3] and capacity estimates of percolation kernels [8, section 1.9] and second moment method our $k$, $s$-independent percolations are quasi-independent (6). Analysis of a number of explicit percolation models (models 22, 24 and 27) renders the bounds tight. All explicit models are in the class $C_{p,o}^{k,0}(V)$ and invariant under automorphisms of the rooted tree.

Shearer’s measure [12] on the $k$-fuzz of $\mathbb{Z}$ (section 4.2) minimizes the conditional probability of the event “open for $m$ more steps | open for $n$ steps” along a path of $k$-independent Bernoulli random variables (see (22b)). Our novel contribution is an explicit construction of Shearer’s measure on the $k$-fuzz of $\mathbb{Z}$ (model 8) as a $(k + 1)$-factor for $p \geq p_{sh}^{Z(k)}$ via a zero-one switch ((27) and figure 2), by reinterpreting calculations from Liggett et al. [6, corollary 2.2]. From the detailed knowledge about Shearer’s measure on the $k$-fuzz of $\mathbb{Z}$ we derive uniform bounds on the percolation kernel over the whole class $C_{p,o}^{k,0}(V)$, leading to $p_{\min}^{k,\vec{s}}(V)$.

A back-of-the-envelope derivation of the critical values (10) goes as follows: The simplest infinite rooted tree is a single ray isomorph to $\mathbb{N}$. Let $Z := (Z_n)_{n \in \mathbb{N}}$ be a collection $k$-independent Bernoulli($p$) rvs on $\mathbb{N}$. We have

$$\xi^n \leq \mathbb{P}(Z_{[n]} = \vec{1}) \leq \eta^n,$$

where the left inequality holds for $p \geq p_{sh}^{Z(k)}$ with the relation $p = 1 - \xi^k(1 - \xi)$, thanks to Shearer (see section 4), and the right one always with the relation $\eta^{k+1} = p$, thanks to $k$-independence. Root $T$ and suppose that (14) carries over
to $k, s$-independent percolation with parameter $p$. Hence we have a comparison with two independent models with parameters $\xi$ and $\eta$, that is
\[
P_\xi(\text{percolates}) \leq P_p(\text{percolates}) \quad \text{and} \quad P_p(\text{percolates}) \leq P_\eta(\text{percolates}) ,
\]
where the left inequality holds for $p \geq p_{sh}^{\xi(k)}$. Plugging in $\frac{1}{br(T)}$, the critical value for independent percolation (5), for $\xi$ and $\eta$ we get the $g_k$ part of $p_{\max}^{k,s}(V)$ for $br(T) \leq \frac{k+1}{k}$ and $p_{\min}^{k,s}(V)$ for all $br(T)$.

This comparison with two independent models in the last paragraph is solely in terms of the probability to percolate. We have no direct relation between the clusters (like a coupling between the percolations) and in particular no stochastic domination (see section 5.7).

Already in the independent case [9, section 5] the percolation behaviour at $p = \frac{1}{br(T)}$ depends on additional properties of the tree. This stays the same for $p_{\min}^{k,s}(V)$ and the $g_k$ part of $p_{\max}^{k,s}(V)$. It is not so for $p = p_{sh}^{\xi(k)}$ and $br(T) > \frac{k+1}{k}$: here proposition 17 asserts that all $C_{\xi(k),p_{sh}}^{k,s}(V)$ percolate.

Recall that the diameter of a percolation cluster is the length of the longest geodesic path contained in it. We call a percolation diameter bounded if its percolation cluster diameters are a.s. bounded, i.e.
\[
\exists D \in \mathbb{N} : \quad P(\sup \{\text{diam}(C) : C \text{ open cluster in } \mathcal{P}\} \leq D) = 1 .
\]

The $p_{sh}^{\xi(k)}$-line admits another interpretation in terms of cluster diameters:

**Theorem 5.** For each $\varepsilon > 0$ there exist $p \in [p_{sh}^{\xi(k)} - \varepsilon, p_{sh}^{\xi(k)}]$ and $\mathcal{P} \in C_{\xi(k),p_{sh}}^{k,s}(V)$ such that $\mathcal{P}$ is diameter bounded. If $p \geq p_{sh}^{\xi(k)}$, then all percolations in $C_{\xi(k),0}^{k,s}(V)$ are not diameter bounded.

### 4 Shearer’s measure

Throughout this section we assume $q := 1 - p$.

#### 4.1 Definition and general properties

The graph $G := (V,E)$ is a dependency graph of a random field $Z := \{Z_v\}_{v \in V}$ iff
\[
\forall A,B \subset V : \quad d(A,B) > 1 \Rightarrow Z_A \text{ is independent of } Z_B ,
\]
that is non-adjacent subsets index independent subfields. The random field $Z$ may have several different dependency graphs [11, section 4.1], in particular one can always add edges. A question which arose naturally in the context of the probabilistic method [4] is: If we take $Z$ to be a Bernoulli random field with parameter $p$ and dependency graph $G$, what are the parameters $p$ for which we can guarantee that $P(Z_V = \vec{1}) > 0$?
Shearer [12] answered this question for finite $G$. He defined Shearer’s (signed) measure $\mu_{G,p}$ on set $\{0,1\}^V$ by setting the marginals (18a) and constructing the other events by the inclusion-exclusion principle (18b):

$$\forall B \subseteq V: \mu_{G,p}(Y_B = \vec{0}) := \begin{cases} q^{|B|} & B \text{ independent} \\ 0 & B \text{ not independent} \end{cases}$$

(18a)

$$\forall B \subseteq V: \mu_{G,p}(Y_B = \vec{0}, Y_{V \setminus B} = \vec{1}) := \sum_{B \subseteq T \subseteq V \atop T \text{ independent}} (-1)^{|T|-|B|} q^{|T|} q^{|B|}.$$ (18b)

Recall that an independent set of vertices (in the graph-theoretic sense) contains no adjacent vertices. It is the second part of (18a), assigning zero probability to every realization with adjacent 0s, that renders Shearer’s measure special among all measures with parameter $p$, that assigns probability to every realization. Define the critical function

$$\Xi_G(p) := \mu_{G,p}(Y_V = \vec{1}) = \sum_{T \subseteq V \atop T \text{ independent}} (-q)^{|T|}.$$ (19)

It satisfies a fundamental identity: $\forall v \in V, v \not\in W \subseteq V, p \in [0,1]:$

$$\Xi_{G(W \cup \{v\})}(p) = \Xi_{G(W)}(p) - q \Xi_{G(W \setminus \{v\})}(p),$$ (20)

derived by discriminating between independent sets containing $v$ and those not. Shearer’s measure is a priori signed and only becomes a probability measure starting at a critical value [11, theorem 4.1 and proposition 2.18]

$$p_{sh}^G := \max\{p : \Xi_G(p) \leq 0\} = \min\{p : \mu_{G,p} \text{ is a probability measure}\}.$$ (21)

We emphasize that $\Xi_G(p_{sh}^G) = 0$. For $p \geq p_{sh}^G$, the critical function $\Xi_G(p)$ is the strictly monotone increasing probability that our realization contains only 1s [11, proposition 2.18]. The key property of Shearer’s probability measure is:

**Lemma 6** ([12]). Let $Z$ be a random Bernoulli field with parameter $p \geq p_{sh}^G$ and dependency graph $G$. Let $Y$ be $\mu_{G,p}$-distributed. Then $\forall W \subseteq V$:

$$\mathbb{P}(Z_W = \vec{1}) \geq \mu_{G,p}(Y_W = \vec{1}) = \Xi_{G(W)}(p) \geq 0$$ (22a)

and $\forall W \subseteq \vec{W} \subseteq V$: if $\Xi_{G(W)}(p) > 0$, then

$$\mathbb{P}(Z_{\vec{W}} = \vec{1} | Z_W = \vec{1}) \geq \mu_{G,p}(Y_{\vec{W}} = \vec{1} | Y_W = \vec{1}) = \frac{\Xi_{G(\vec{W})}(p)}{\Xi_{G(W)}(p)} \geq 0.$$ (22b)

**Proof.** It suffices to prove (22b) inductively for one-vertex extensions with $\vec{W} = W \cup \{v\}$. We prove (22) jointly by induction over the cardinality of $W$. The induction base for $W = \{w\}$ is:

$$\mathbb{P}(Z_w = 1) = p = \mu_{G,p}(Y_w = 1) = \Xi_{\{w\}, \emptyset}(p).$$

Induction step $W \to \vec{W}$: suppose that $\mu_{G,p}(Y_W = \vec{1}) = 0$. Hence also $\mu_{G,p}(Y_{\vec{W}} = \vec{1}) = 0$ and (22a) holds trivially. If $\mu_{G,p}(Y_W = \vec{1}) > 0$, then $\mathbb{P}(Z_{\vec{W}} = \vec{1}) > 0$.
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by the induction hypothesis. Let $W \cap \mathcal{N}(v) =: \{w_1, \ldots, w_m\}$ and $W_i := W \setminus \{w_1, \ldots, w_m\}$. If $m = 0$, then we revert to the equality in the induction base. If $m \geq 1$ then

$$
\begin{align*}
\mathbb{P}(Z_v = 1|Z_W = \bar{1}) &= \frac{\mathbb{P}(Z_v = 1, Z_W = \bar{1})}{\mathbb{P}(Z_W = \bar{1})} \\
&\geq \frac{\mathbb{P}(Z_W = \bar{1}) - q \mathbb{P}(Z_W \setminus \mathcal{N}(v) = \bar{1})}{\mathbb{P}(Z_W = \bar{1})} \\
&= 1 - \frac{q}{\prod_{i=1}^m \mathbb{P}(Z_{w_i} = \bar{1}|Z_{W_i} = \bar{1})} \\
&\geq 1 - \frac{q}{\prod_{i=1}^m \mu_{G,p}(Y_{w_i} = \bar{1}|Y_{W_i} = \bar{1})} \\
&= \mu_{G,p}(Y_v = 1, Y_W = \bar{1}) \mu_{G,p}(Y_W = \bar{1}) \\
&= \mu_{G,p}(Y_v = 1|Y_W = \bar{1}) \mu_{G,p}(Y_W = \bar{1}) \\
&= \mu_{G,p}(Y_v = 1|Y_W = \bar{1}).
\end{align*}
$$

as $Z$ has dependency graph $G$ (17)

$$
\begin{align*}
\mathbb{P}(Z_W = \bar{1}) &= \mathbb{P}(Z_v = 1|Z_W = \bar{1}) \mathbb{P}(Z_W = \bar{1}) \\
&\geq \mu_{G,p}(Y_v = 1|Y_W = \bar{1}) \mu_{G,p}(Y_W = \bar{1}) \\
&= \mu_{G,p}(Y_v = 1|Y_W = \bar{1}) \mu_{G,p}(Y_W = \bar{1}).
\end{align*}
$$

This proves (22b). To obtain (22a) it suffices to see that

$$
\mathbb{P}(Z_W = \bar{1}) = \mathbb{P}(Z_v = 1|Z_W = \bar{1}) \mathbb{P}(Z_W = \bar{1}) \\
\geq \mu_{G,p}(Y_v = 1|Y_W = \bar{1}) \mu_{G,p}(Y_W = \bar{1}) = \mu_{G,p}(Y_W = \bar{1}).
$$

Finally we see that for $p \geq p_{G}^{sh}$ the probability measure $\mu_{G,p}$

has dependency graph $G$, (23a)

has marginal parameter $p$, i.e. $\forall v \in V : \mu_{G,p}(Y_v = 1) = p$, (23b)

and forbids neighbouring $0$s, i.e. $\forall (v, w) \in E : \mu_{G,p}(Y_v = Y_w = 0) = 0$. (23c)

Every probability measure $\nu$ on $\{0, 1\}^V$ fulfilling (23) can be constructed by (18) and thus coincides with $\mu_{G,p}$. Hence (23) characterizes $\mu_{G,p}$.

If $G$ is an infinite graph we define

$$
p_{sh}^G := \sup \{p_{sh}^H : H \text{ finite subgraph of } G\}. 
$$

(24)

This is well-defined as $p_{sh}^W$ is a monotone increasing function over the lattice of finite subgraphs (strictly monotone increasing for connected subgraphs) [11, proposition 2.15]. For $p \geq p_{sh}^G$ the family $\{\mu_{G,W,p} : W \subseteq V, W \text{ finite}\}$ forms a consistent family à la Kolmogorov [3, (36.1) & (36.2)]. Hence Kolmogorov’s existence theorem [3, theorem 36.2] establishes the existence of an extension of this family, which we call $\mu_{G,p}$. The uniqueness of this extension is given by the $\pi$-$\lambda$ theorem [3, theorem 3.3]. Furthermore $\mu_{G,p}$ fulfills (23) on the infinite graph $G$. Conversely let $\nu$ be a probability measure having the properties (23). Then
all its finite marginals have them, too and they coincide with Shearer’s measure. Hence by the uniqueness of the Kolmogorov extension ν coincides with μ_{G,p} and (23) characterizes μ_{G,p}. It follows that the minimal p for (23) to have a solution is p_{sh}^G.

The reader can find more about Shearer’s measure in the seminal work by Scott & Sokal [11], especially the rich connection with hard core lattice gases in statistical mechanics and the Lovász Local Lemma of the probabilistic method in graph theory [4].

4.2 On the \( k \)-fuzz of \( \mathbb{Z} \)

In this section we deal with Shearer’s measure on \( \mathbb{Z}(k) \), the \( k \)-fuzz of \( \mathbb{Z} \). It is the graph with vertices \( \mathbb{Z} \) and edges for every pair of integers at distance less than or equal to \( k \). Recall that an \( X \)-valued process indexed by \( \mathbb{Z} \) is called a \((k+1)\)-factor iff there exists a measurable function \( f : [0,1]^{(k+1)} \to X \) such that for every \( n \in \mathbb{Z} : X_n = f(U_n, \ldots, U_{n+k}) \), where \( \{U_n\}_{n \in \mathbb{Z}} \) is a i.i.d. sequence of Uniform([0,1]) rvs. It follows that every \((k+1)\)-factor is \( k \)-independent, stationary and has \( \mathbb{Z}(k) \) as dependency graph.

We derive the critical value \( p_{\mathbb{Z}(k)}^{sh} \) in proposition 7 (thus validating (12)), construct \( \mu_{\mathbb{Z}(k),p} \) explicitly in model 8 as a \((k+1)\)-factor and derive asymptotic properties in proposition 9. For \( k \in \mathbb{N}_0 \) define the function

\[
 h_k : [0,1] \to \mathbb{R} \quad z \mapsto z^k(1 - z).
\]  

(25)

It attains its maximum at \( \frac{k}{k+1} \) with value \( \frac{k^k}{(k+1)^{k+1}} \). If \( p \in [p_{\mathbb{Z}(k)}^{sh}, 1] \), then the equation

\[
 h_k(\xi) = q
\]  

has a unique solution \( \xi := \xi(p,k) \) lying in the interval \([k/(k+1), 1]\). Denote by \( [N]_{(k)} \) the \( k \)-fuzz of a line of \( N \) points and by \( N_{(k)} \) the \( k \)-fuzz of \( N \). It is easy to see that \( p_{\mathbb{Z}(k)}^{sh} = p_{\mathbb{Z}(k)}^{N(1)} = p_{\mathbb{N}(k),p} \) and \( \mu_{\mathbb{Z}(k),p} \) is just the projection of \( \mu_{\mathbb{Z}(k),p} \). Hence all the properties of and estimates for \( \mu_{\mathbb{Z}(k),p} \) stated in the following also hold for \( \mu_{\mathbb{N}(k),p} \).

**Proposition 7.**

\[
 p_{\mathbb{N}(k),p}^{[N]_{(k)}} \xrightarrow{N \to \infty} 1 - \frac{k^k}{(k+1)^{(k+1)}} = p_{\mathbb{Z}(k)}^{sh} = p_{\mathbb{N}(k),p}^{sh}.
\]  

(27)

An explicit construction of Shearer’s measure on \( \mathbb{Z}(k) \) is given by:

**Model 8.** Let \( p \geq p_{\mathbb{Z}(k)}^{sh} \) and \( X := \{X_n\}_{n \in \mathbb{Z}} \) be i.i.d. Bernoulli rvs with parameter \( \xi \) as in (26). Define \( Z := \{Z_n\}_{n \in \mathbb{Z}} \) by

\[
 \forall n \in \mathbb{Z} : \quad Z_n := 1 - (1 - X_n) \prod_{i=1}^{k} X_{n-i},
\]  

(28)

then \( Z \) is \( \mu_{\mathbb{Z}(k),p} \)-distributed.
If $k = 0$, then the empty product in (28) disappears and $Z = X$, that is $\mu_{Z(0), p}$ is a Bernoulli product measure with parameter $p$. Accordingly $p_{sh}^{Z(0)} = 0$.

A result of Aaronson, Gilat, Keane & de Valk [1, result 4(i) on page 140] on the question of the representability of certain stationary 1-independent $\{0, 1\}$-valued processes on $Z$ as 2-factors implies that $\mu_{[n], p}$ is not representable as a 2-factor for $p < \frac{3}{4}$. This statement is easily extended to assert non-representability of $\mu_{[n], k, p}$ as a $(k + 1)$-factor for every $k, n \in \mathbb{N}$ and $p < p_{sh}^{Z(k)}$. It follows from the fact that for $p < p_{sh}^{Z(k)}$ the sequence $(\beta_n)_{n \in \mathbb{N}}$ in the proof of proposition 7 does not remain positive.

On the other hand, if one fixes $N$ and $p \in [p_{sh}^{Z(k)}, p_{sh}^{Z(k)}]$, one can get something close to a factor representation. Let $(X_n)_{n=1}^{N}$ be a collection of independent rvs, with $X_n$ Bernoulli($\beta_n$)-distributed. Then the same rule as in (28), truncated for the first $k$ indices, yields a $\mu_{[N], k, p}$-distributed $(Z_n)_{n=1}^{N}$.

Figure 2: A partial view of Shearer’s measure on the 2-fuzz of $Z$. The lower row shows a realization of $X$, the upper row the resulting one of $Z$. We point out a $0$ in $Z$, the realizations on its underlying nodes in $X$ (solid downward arrows) and the effect of the zero-one switch (dashed upward arrows), resulting in $1$s on its neighbours up to distance 2.

**Proof of proposition 7.** The inequality $p_{sh}^{Z(k)} \geq 1 - \frac{k^k}{(k + 1)^{k+1}}$ follows from [6, theorem 2.1]. We repeat it for completeness. Define $b_n := \Xi_{[n], k}(p)$, then $b_n = 1 - nq$ for $n \in [k + 1]$ and $b_n = b_{n-1} - q b_{n-k}$ for $n > k$, both times using the fundamental identity (20). We show by induction that $\beta_n := \frac{b_n}{b_{n-1}}$ is a strictly monotone falling sequence:

$n \in [k + 1]: \beta_n = \frac{1 - nq}{1 - (n-1)q} > \frac{k}{1 - (n+1)q} = \beta_{n+1}$ as $n^2 > (n-1)(n+1)$.

$n \to n + 1: \beta_{n-k} > \beta_n \Leftrightarrow \frac{b_{n-k}}{b_n} > \frac{b_{n-k-1}}{b_{n-1}}$ by the induction hypothesis, hence

$$\beta_{n+1} = 1 - \frac{b_{n-k}}{b_n} < 1 - \frac{b_{n-k-1}}{b_{n-1}} = \beta_n.$$

The sequence $(\beta_n)_{n \in \mathbb{N}}$ is positive and well-defined iff $p \geq p_{sh}^{Z(k)}$. Upon taking the limit $\beta = \lim_{n \to \infty} \beta_n$ we arrive at the identity $\beta = 1 - q \beta^{-k}$. Rewrite it to $q = \beta^k (1 - \beta) = h_k(\beta)$, which has solutions only for $q \leq \frac{k^k}{(k + 1)^{k+1}}$. Hence

$$1 - \frac{k^k}{(k + 1)^{k+1}} \leq p_{sh}^{Z(k)}.$$

The second inequality $p_{sh}^{Z(k)} \leq 1 - \frac{k^k}{(k + 1)^{k+1}}$ follows from model 8.
Proof of model 8. By construction $\mathbb{P}(Z_n = 0) = \xi^k(1 - \xi) = h_k(\xi) = q$ and

$$\mathbb{P}(Z_n = 0) = \mathbb{P}(X_{n-1} = \ldots = X_{n-k} = 1, X_n = 0)
= \mathbb{P}(Z_{n-k} = \ldots = Z_{n-1} = 1, Z_n = 0, Z_{n+1} = \ldots = Z_{n+k} = 1).$$

This zero-one switch (see figure 2) guarantees that vertices with distance less than or equal to $k$ can never index a 0 in the same realization. Therefore $Z$ has no realizations containing neighbouring 0s with respect to $Z_{(k)}$ as well as the right dependency graph and marginals. Using the characterization (23) we see that $Z$ is $\mu_{Z_{(k)}}, p$-distributed.

\[ \forall Z \in \mathbb{Z} \quad \text{no realizations containing neighbouring 0s with respect to } Z_{(k)}. \]

For every $k \in \mathbb{N}_0$ fixed define the strictly monotone decreasing function

$$f_k : \{0, \ldots, k\} \to \mathbb{R} \quad g \mapsto \begin{cases} \frac{(q+1)\xi - q}{q\xi - q(1 - \xi)} & \text{if } k \geq 1, \\ \xi & \text{if } k = 0. \end{cases} \quad \text{(29)}$$

**Proposition 9.** We have for every $k \in \mathbb{N}_0$ and $p \geq p_{sh}^{Z_{(k)}}$ the minoration

$$\forall \text{ finite } B \subseteq \mathbb{Z} \setminus \{0\} : \quad \mu_{Z_{(k)}, p}(Y_0 = 1|Y_B = \vec{1}) \geq f_k(g_B), \quad \text{(30a)}$$

where $g_B := 0 \vee (k + 1 - d_B)$ and $d_B := \min \{|n| : n \in B\}$. In particular we have for each $n \in \mathbb{N}$:

$$\mu_{Z_{(k)}, p}(Y_n = 1|Y_{[n-1]} = \vec{1}) \geq \xi \quad \text{and} \quad \Xi_{[n], (k)}(p) \geq \xi^n. \quad \text{(30b)}$$

And likewise the majoration

$$\forall \varepsilon > 0 : \exists C > 0, \exists N \in \mathbb{N} : \forall n \geq N : \quad \Xi_{[n], (k)}(p) \leq C[(1 + \varepsilon)\xi]^n. \quad \text{(30c)}$$

**Remark.** The minimality of Shearer’s measure (22) implies that these lower bounds also hold for every $k$-independent Bernoulli random field on $\mathbb{Z}$ and $\mathbb{N}$ with marginal parameter $p \geq p_{sh}^{Z_{(k)}}$ respectively.

**Proof.** Fix $p \geq p_{sh}^{Z_{(k)}}$. In the proof of proposition 7 we see that $(\beta_n)_{n \in \mathbb{N}}$ is a strictly monotone falling sequence with $\beta_n \xrightarrow{n \to \infty} \beta \geq \frac{k}{k+1}$. As $\beta$ fulfills $q = h_k(\beta)$ we have $\beta = \xi$. Hence $\beta_n \geq \xi$, yielding (30b). The monotonicity of $(\beta_n)_{n \in \mathbb{N}}$ implies that

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \quad \beta_n \leq (1 + \varepsilon)\beta = (1 + \varepsilon)\xi.$$

Hence for $n \geq N$ we have

$$\Xi_{[n], (k)}(p) = \prod_{i=1}^{n} \beta_i \leq \prod_{i=N+1}^{n} \beta_i \leq (1 + \varepsilon)^{n-N}\xi^{n-N} = \frac{1}{[(1 + \varepsilon)\xi]^N}[(1 + \varepsilon)\xi]^n.$$ 

This proves (30c) upon setting $C(\varepsilon) := [(1 + \varepsilon)\xi]^{-N}$.

For (30a) we differentiate according to the shape of $B$. If $d_B > k$, then $k$-independence implies that $\mu_{Z_{(k)}, p}(Y_0 = 1|Y_B = \vec{1}) = p \geq \xi = f_k(0).$
If $d_B \leq k$ let $B_\pm := B \cap \mathbb{Z}_\pm$ and $d_{B_\pm} := \inf \{|n| : n \in B_\pm\}$. Thus $d_B = d_{B_+} \land d_{B_-}$. In the first case $d_{B_-} > k$ and $d_{B_+} \leq k$. Let $\{b_1, \ldots, b_m\} := B_+ \cap [k]$ with $b_1 < \ldots < b_m$. Hence

$$1 - \mu_{Z(k), p}(Y_0 = 1|Y_B = \vec{1}) = 1 - \mu_{Z(k), p}(Y_0 = 1|Y_{B_+} = \vec{1})$$

by $k$-independence

$$= \frac{\prod_{i=1}^m \mu_{Z(k), p}(Y_{b_i} = 1|Y_{B_+ \setminus \{b_1, \ldots, b_{i-1}\}} = \vec{1})}{q}$$

fundamental identity (20)

$$\leq \frac{q}{\xi^m}$$

by induction over $|B_+|$

$$\leq (1 - \xi)\xi^{k-m}$$

as $q = (1 - \xi)\xi^k$

$$\leq 1 - \xi$$

as $m \leq k$.

This also holds in the symmetric case with $d_{B_-} \leq k$ and $d_{B_+} > k$.

The final case is $d_{B_+} \leq k$ and $d_{B_-} \leq k$. Assume without loss of generality that $d_B = d_{B_-} \leq d_{B_+}$ and let $\{a_0, \ldots, a_1\} := B_- \cap \{-k, \ldots, -1\}$ with $a_0 < \ldots < a_1$. Applying the fundamental identity (20) and induction over $|B|$ we get

$$1 - \mu_{Z(k), p}(Y_0 = 1|Y_B = \vec{1}) = \prod_{j=1}^m \mu_{Z(k), p}(Y_{a_j} = 1|Y_{B_- \setminus \{a_1, \ldots, a_{j-1}\}} = \vec{1}, Y_{B_+} = \vec{1})$$

$$\leq \prod_{j=1}^m \frac{q}{f_k(k + a_j)} \prod_{i=1}^k f_k(0)$$

$$\leq \prod_{j=1}^k f_k(k - j) \prod_{i=1}^k f_k(0)$$

$$\leq \frac{(1 - \xi)\xi^k}{[k + 1 - d_B] \xi - (k - d_B)}$$

It follows that

$$\mu_{Z(k), p}(Y_0 = 1|Y_B = \vec{1}) \geq 1 - \frac{1 - \xi}{(k + 1 - d_B) \xi - (k - d_B)} = \frac{(k + 2 - d_B) \xi - (k + 1 - d_B)}{(k + 1 - d_B) \xi - (k - d_B)} = f_k(k + 1 - d_B).$$

\[ \Box \]

5 Proofs

5.1 Proof outline of theorems 4 and 5

Proof of theorem 4. We start with some obvious relations between the rooted percolation classes and their critical values, based on the restrictions imposed
by \( k \) and \( s \). For all \( k, k', s, s' \in \mathbb{N}_0 \):

\[
\text{if } k \leq k' \text{ and } s \leq s' \text{ then } \begin{cases} 
C_{p, o}^{k,s}(V) \subseteq C_{p, o}^{k',s'}(V) \\
p_{\text{max}}^{k,s}(V) \leq p_{\text{max}}^{k',s'}(V) \\
p_{\text{min}}^{k,s}(V) \leq p_{\text{min}}^{k',s'}(V)
\end{cases}
\]

(31)

The first part is the proof for \( p_{\text{max}}^{k,s}(V) \) in sections 5.3 and 5.4. To get an upper bound on \( p_{\text{max}}^{k,s}(V) \) we need to show that every \( k, s \)-independent percolation percolates for \( p \) close enough to 1. Our approach uses a classical second moment argument, recalled in lemma 14. We relate it to \( b_r(T) \) in proposition 15, with the core ingredient being a sufficient condition for percolation in terms of an exponential bound on the percolation kernel, defined in 36c. For \( k, s \)-independent percolation proposition 16 reduces this to the problem of bounding the conditional probabilities of extending open geodesic downrays by the right exponential term. Finally proposition 17 uses the minimality of Shearer’s measure from lemma 6 and detailed estimates about its structure on \( \mathbb{Z}_{(k)} \) in proposition 9 to uniformly guarantee the right exponential term and arrive at (39):

\[
\forall k, s \in \mathbb{N}_0 : \quad p_{\text{max}}^{k,s}(V) \leq \begin{cases} 
g_k(b_r(T)) & \text{if } b_r(T) \leq \frac{k+1}{k} \\
p_{sh} & \text{if } b_r(T) \geq \frac{k+1}{k}
\end{cases}
\]

(44)

For the lower bound on \( p_{\text{max}}^{k,s}(V) \) it suffices to exhibit \( k, 0 \)-independent percolation models that do not percolate. We describe two such models, the canonical model 22 and the cutup model 24, both constructed from Shearer’s measure. More precisely, in section 5.4.1 we describe a general procedure, called tree-fission, to create a \( k, 0 \)-independent percolation with identical distributions along all downrays from a given \( k \)-independent Bernoulli random field over \( \mathbb{N} \). When applied to Shearer’s measure on \( \mathbb{N}_{(k)} \) and a derivative of \( n_{(k)} \) it yields the canonical model 22 and the cutup model 24 respectively. We then use the first moment method, recalled in lemma 13, to establish their nonpercolation, leading to the following results from (44) and (45):

\[
p_{\text{max}}^{k,0}(V) \geq g_k(b_r(T)) \quad \text{if } b_r(T) \in \left[ 1, \frac{k+1}{k} \right] \quad \text{and} \quad p_{\text{max}}^{k,0}(V) \geq p_{sh}^{Z_{(k)}}.
\]

(45)

Conclude by applying the inequality from (31).

The second part is the proof for \( p_{\text{min}}^{k,s}(V) \) in section 5.5. Here the argumentation is the reverse of the one for \( p_{\text{max}}^{k,s}(V) \). To get a lower bound on \( p_{\text{min}}^{k,s}(V) \) we need to show that every \( k, s \)-independent percolation does not percolate for \( p \) close enough to 0. We achieve this by a first moment argument in proposition 26, using solely \( k \)-independence along downrays. It culminates in (46):

\[
\forall k, s \in \mathbb{N}_0 : \quad p_{\text{min}}^{k,s}(V) \geq \frac{1}{b_r(T)}.
\]

(46)

For the upper bound on \( p_{\text{min}}^{k,s}(V) \) we differentiate between \( k = 0 \) and \( k \geq 1 \). In the case \( k = 0 \) we already have a matching upper bound in the upper bound for \( p_{\text{max}}^{k,s}(V) \) in (39). For \( k \geq 1 \) we describe a percolating \( k, 0 \)-independent percolation model, called the minimal model 27. It is constructed by the tree-fission procedure from section 5.4.1. In proposition 28 we show that it percolates
by bounding its percolation kernel with the help of proposition 16 and applying the second moment method adaption from proposition 15, leading to (47):

$$\forall k \geq 1 : \quad p^k_{min}(V) \leq \frac{1}{b^r(T)^{k+1}}.$$ 

Conclude by applying the inequality from (31), using the upper bound for $p^0_{min}(V)$ in the case of $k = 0$. 

**Proof of theorem 5.** By (27) for every $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that $p_{sh}^{(N)} > p_{sh}^{(N)} - \varepsilon$. Then proposition 25 asserts that the cut percolation $\mathcal{P}_{\text{cut}}(k,N)$ (model 24) is diameter bounded with $D = 4N - 4$.

On the other hand, let $p \geq p_{sh}^{(N)}$ and $Z := \{Z_v\}_{v \in V}$ be in $C^{k,a}_{p,\varepsilon}(V)$. We have

$$\forall n \in \mathbb{N}, v \in L(T,n) : \quad P(Z_{P(o,v)} = \overline{I}) \geq \mu_{\{v\} \to (v)}(Y_n = \overline{I}) \geq \xi^n > 0,$$

where $Y$ is $\mu_{\{v\} \to p}$-distributed, we use the minimality of Shearer’s measure (22a) and the minoration from (30b), with $\xi > 0$ from 26. This implies that $Z$ is not diameter bounded. 

**5.2 General tools for percolation on trees**

In this section we list some general tools for percolations on trees which allow us to shorten the following proofs. The following extension of Kolmogorov’s zero-one law [3, theorem 36.2] is well known. In particular it encompasses k-independent rvs on a graph $G$, as they have the $k$-fuzz of $G$ as their dependency graph.

**Lemma 10.** Let $G = (V,E)$ be a locally finite, infinite graph. Let $X := \{X_v\}_{v \in V}$ be a random field with dependency graph $G$. Then the tail $\sigma$-algebra of $X$ is trivial.

**Proof.** Let $(V_n)_{n \in \mathbb{N}}$ be an exhausting, strictly monotone growing sequence of finite subsets of $V$. For $W \subseteq V$ let $\mathcal{A}_W := \sigma(X_W)$ and define the tail $\sigma$-algebra $\mathcal{A}_{\infty} := \bigcap_{n=1}^{\infty} \mathcal{A}_{V_n}$. For an event $B \in \mathcal{A}_{\infty}$ set $Z_n := E[1_B A_{V_n}] = 1_B$. Then we have a.a.s. constant martingale with $\lim_{n \to \infty} Z_n = 1_B$:

$$E[Z_{n+1} | A_{V_n}] = E[E[1_B | A_{V_{n+1}}] | A_{V_n}] = E[E[1_B | A_{V_n}] | A_{V_{n+1}}] = E[1_B | A_{V_n}] = Z_n.$$

Hence $P(B)^2 = E[1_B P(B)] = E[1_B^2] = P(B)$ and $P(B) \in \{0,1\}$. 

Next we introduce some notation for rooted percolation on $T$.

**Notation 11.** In the context of rooted percolation and for $v \in V$ we write

$$O_v^\Pi := \{v \leftrightarrow \Pi \cap V(T^v)\} \quad \Pi \in \Pi(o) \quad (32a)$$

$$O_v := \{v \leftrightarrow \infty\} = \{v \leftrightarrow \partial T^v\}, \quad (32b)$$

where those events mean “there is an open downray from $w$ to the cutset $\Pi$” and “there is an open downray starting at $v$”.

The following lemma allows us to concentrate exclusively on rooted percolation (see [13] for a proof):

by bounding its percolation kernel with the help of proposition 16 and applying the second moment method adaption from proposition 15, leading to (47):
Lemma 12. Let $\mathcal{P} \in \mathcal{C}_p^k(V)$, for finite $k$. Then

\begin{align}
(\exists v \in V : P(O_v) > 0) &\iff P(\mathcal{P} \text{ percolates on } T) = 1, \quad (33a) \\
(\forall v \in V : P(O_v) = 0) &\iff P(\mathcal{P} \text{ percolates on } T) = 0. \quad (33b)
\end{align}

In the case $k = s = 0$ we can change the $\exists$ to $\forall$ in (33a), which is needed in the proof of proposition 28. Finally the obvious relationship between rooted percolation reaching a cutset $\Pi \in \Pi(o)$ or the boundary $\partial T$ from $o$ is:

$$\forall w \in V : O_w = \bigcap_{\Pi \in \Pi(o)} O^\Pi_w. \quad (34)$$

This holds already for the intersection over an exhaustive sequence of cutsets $\{\Pi_m\}_{m \in \mathbb{N}}$, i.e. $\forall v \in V : \exists m_v \in \mathbb{N} : \exists w \in \Pi_{m_v} : v$ is an ancestor of $w$. A central tool is the following two moment methods:

**Lemma 13** (First moment method [9, section 5.2]). We have

$$P(O_o) = P(o \leftrightarrow \infty) \leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} P(o \leftrightarrow v). \quad (35)$$

**Lemma 14** (Weighted second moment method [9, section 5.3]).

$$P(O_o) = P(o \leftrightarrow \infty) \geq \inf_{\Pi \in \Pi(o)} \sup_{\mu \in \mathcal{M}_1(\Pi)} \frac{1}{E(\mu)}, \quad (36a)$$

where $\mathcal{M}_1(\Pi)$ is the set of probability measures on the vertex cutset $\Pi$ and the energy $E(\mu)$ of $\mu \in \mathcal{M}_1(\Pi)$ is determined by

$$E(\mu) = \sum_{v,w \in \Pi} \mu(v)\mu(w)\kappa(v,w). \quad (36b)$$

and $\kappa$ is the symmetric percolation kernel

$$\kappa : V^2 \to \mathbb{R}^+ \quad (v,w) \mapsto \kappa(v,w) := \frac{P(o \leftrightarrow v, o \leftrightarrow w)}{P(o \leftrightarrow v)P(o \leftrightarrow w)}. \quad (36c)$$

### 5.3 Upper bound on $p^{k,s}_{\max}(V)$

The task is to establish an upper bound on $p^{k,s}_{\max}(V)$. In other words, we want to guarantee percolation for high enough $p$. The first step in section 5.3.1 is to use the second moment method to translate this problem into the search for a suitable exponential bound on the percolation kernel. Then we use $k,s$-independence to bound the percolation kernel in terms of a conditional probability along a single downray. Hence we can guarantee percolation as soon as we can bound this conditional probability from below in sufficient exponential terms. The percolation along a single downray is just a Bernoulli random field with parameter $p$ and dependency graph $N_{(k)}$. In the second step in section 5.3.2 we apply the generic minimality of Shearer’s measure and a lower bound on $\mu_{N_{(k)},p}$ to get such an exponential lower bound of parameter $\xi$. Finally we relate $\xi$ and $br(T)$ and derive the upper bound.
### 5.3.1 Percolation kernel estimates

In proposition 15 we state a sufficient condition on the percolation kernel in order to percolate. This condition relates the second moment method to the branching number. In proposition 16 we bound the percolation kernel for \( k, s \)-independent percolation in terms of conditional probabilities along a single downray, hence providing a simpler means to derive the sufficient condition in subsequent steps.

**Proposition 15.** Let \( \mathcal{P} \in C_{k,o}^{k,s}(V) \) and \( \alpha < br(\mathcal{T}) \), \( C \in \mathbb{R}_+ \) such that \( \forall v, w \in V: \)

\[
\kappa(v, w) \leq C \alpha^{l(v, w)},
\]

(37)

then \( \mathcal{P} \) percolates.

**Remark.** The "?" in \( C_{k,o}^{k,s}(V) \) means that we place no restriction yet on the marginals of \( \mathcal{P} \). The confluent of \( v \) and \( w \) is \( v \approx w \). See also figure 3.

**Proof.** Take \( \beta \in \alpha, br(\mathcal{T}) \) and let \( g \) be a \( \beta \)-flow. Define \( \mu(v) := g(v) \) and \( \mu(w) = g(w) \), hence \( \mu|_{\Pi} \in M_1(\Pi) \) for each vertex cutset \( \Pi \in \Pi(o) \). We have

\[
\mathcal{E}(\mu|_{\Pi}) = \sum_{v, w \in \Pi} \mu|_{\Pi}(v) \mu|_{\Pi}(w) \kappa(v, w) \\
\leq \sum_{v, w \in \Pi} \mu(v) \mu(w) C \alpha^{l(v, w)} \\
= C \sum_{n=0}^{\infty} \sum_{v, w \in \Pi} \sum_{u \in L(T, n)} \frac{g(v)g(w)}{g(o)^2} \\
= C \frac{g(o)}{\alpha} \sum_{n=0}^{\infty} \sum_{u \in L(T, n)} g(u)^2 \\
\leq C \frac{g(o)}{\alpha} \sum_{n=0}^{\infty} \left( \frac{\alpha}{\beta} \right)^n \sum_{u \in L(T, n)} g(u) \\
= C \frac{g(o)}{\beta} \frac{\alpha}{\beta} \sum_{n=0}^{\infty} \left( \frac{\alpha}{\beta} \right)^n \\
= C \frac{\alpha}{\beta} \frac{g(o)}{\beta} \\
\beta - \alpha \alpha < \beta,
\]

which is a finite bound independent of \( \Pi \). Apply the weighted second moment method (see lemma 14) to see that \( \mathcal{P}(o \leftrightarrow \infty) > 0 \) and conclude.

**Proposition 16.** We use the notation from figure 3. Then \( \forall k, s \in \mathbb{N}_0, \mathcal{P} \in C_{p,o}^{k,s}(V), v, w \in V: \)

\[
\kappa(v, w) \leq \frac{1}{\mathcal{P}(o \leftrightarrow t | t \leftrightarrow w)}.
\]

(38)

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Figure 3: Decomposition of the percolation kernel $\kappa(v, w)$ for $k, s$-independent, rooted site percolation. The node $t \in P(u, w)$ has distance $(k \lor s) + 1$ from $u$ if the path $P(u, w)$ is longer than this (left side), otherwise $t = w$ (right side).

**Proof.** We use the notation from figure 3. In the case $d(u, w) > (k \lor s) + 1$ we have

$$
\kappa(v, w) = \frac{P(o \leftrightarrow v, o \leftrightarrow w)}{P(o \leftrightarrow v)P(o \leftrightarrow w)} = \frac{P(o \leftrightarrow v, o \leftrightarrow w)}{P(o \leftrightarrow v)P(t \leftrightarrow w)P(o \leftrightarrow t | t \leftrightarrow w)} = \frac{1}{P(o \leftrightarrow t | t \leftrightarrow w)} \quad \text{using } (k \lor s)\text{-independence}
$$

In the case $d(u, w) \leq (k \lor s) + 1$ we have $t = w$ and

$$
\kappa(v, w) = \frac{P(o \leftrightarrow v, o \leftrightarrow w)}{P(o \leftrightarrow v)P(o \leftrightarrow w)} = \frac{P(u \leftrightarrow t | o \leftrightarrow v)}{P(o \leftrightarrow t | t \leftrightarrow w)} \leq \frac{1}{P(o \leftrightarrow t | t \leftrightarrow w)} .
$$

---

### 5.3.2 Uniform bound by Shearer’s measure

The following proposition combines our knowledge of $\mu_{Z(k), p}$ and its properties with the simplified condition on the percolation kernel from proposition 16 to ensure uniform percolation.

**Proposition 17.**

$$
\forall k, s \in \mathbb{N}_0 : \quad p_{\text{max}}^{\kappa, s}(V) \leq \begin{cases} 
    g_k(\text{br}(T)) & \text{if } \text{br}(T) \leq \frac{k+1}{k} \\
    z(k) & \text{if } \text{br}(T) \geq \frac{k+1}{k} .
\end{cases}
$$

(39)

Furthermore for $\text{br}(T) > \frac{k+1}{k}$ every percolation in $C_{\mu_{Z(k)}^{\kappa, s}}(V)$ percolates. In the case $k = 0$ we interpret $\frac{1}{0} := \infty$. 

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Proof. Let \( p \geq p_{zh}^{Z(k)} \). Use the notation from figure 3. Let \( \xi \) be the unique solution of the equation \( 1 - p = \xi (1 - \xi)^k \) from (26). In a first step we use (38), the minimaliy of Shearer's measure (22a), the explicit minoration of Shearer's measure on \( \mathbb{N}(k) \) (30b) and the fact that \( l(t) \leq l(u) + (k \vee s) + 1 \) to majorize the percolation kernel as follows:

\[
\kappa(v, w) \leq \frac{1}{\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)} \leq \frac{1}{\mu_{Z_{zh}^{Z(k)}}, p(o \leftrightarrow t | t \leftrightarrow w)} \leq \frac{1}{\xi(t)} \leq \xi^{-(k \vee s) - 1} \xi^{-l(u)}.
\]

In the second step we want to apply the sufficient exponential bound condition on the percolation kernel from proposition 15, hence we have to relate \( \xi \) with \( \text{br} (T) \). The function \( g_k (11) \) satisfies \( g_k (\frac{1}{\xi}) = p \), has a global minimum in \( \frac{k + 1}{k} \) with value \( p_{zh}^{Z(k)} \) and induces a strictly monotone decreasing bijection between \( [1, \frac{k + 1}{k}] \) and \( [p_{zh}^{Z(k)}, 1] \).

Case \( \text{br} (T) \leq \frac{k + 1}{k} \) and \( g_k (\text{br} (T)) < p = \xi^k (1 - \xi) \): Apply proposition 15 with \( C := \xi^{-(k \vee s) - 1} \) and \( \alpha := \frac{1}{\xi} < \text{br} (T) \) to show that we percolate. This proves the \( g_k \) part of (39).

Case \( \text{br} (T) > \frac{k + 1}{k} \) and \( p_{zh}^{Z(k)} \leq p \): Apply proposition 15 with \( C := \xi^{-(k \vee s) - 1} \) and \( \alpha := \frac{1}{\xi} < \frac{k + 1}{k} < \text{br} (T) \) to show that we percolate. This proves the \( p_{zh}^{Z(k)} \) part of (39) and the percolation statement at \( p_{zh}^{Z(k)} \).

We show that we need uniformly bounded elsewhere-dependences to guarantee percolation for high \( p \). The counterexample consists of multiplexing a distribution indexed by \( \mathbb{N}_0 \) over the corresponding level of \( T \).

**Model 18.** For \( p \geq p_{zh}^{Z(k)} \) let \( Z := \{ Z_n \}_{n \in \mathbb{N}_0} \) be a collection of \( k \)-independent Bernoulli(p) rvs. Define a site percolation \( Z := (Z_v)_{v \in V} \) on the rooted tree \( T \) by

\[
Z_v := Z_{l(v)}.
\]

**Proposition 19.** For every \( s \in \mathbb{N} \) we have \( Z \notin \mathcal{C}_{p, \sigma}^{k, s}(V) \) and \( Z \) percolates iff

\[
p = 1.
\]

**Proof.** All the sites on a chosen level of \( T \) realize a.s. in the same state. Therefore the elsewhere-dependence \( s_v \) of \( v \) is in the range \( 2l(v) \leq s_v \leq 2l(v) + k \) and unbounded in \( v \). Using (40) and \( k \)-independence we get

\[
\forall n \in \mathbb{N} : \mathbb{P}(o \leftrightarrow L(T, n)) = \mathbb{P}(Z_0 = \ldots = Z_n = 1) \leq p^{n/(k + 1)}.
\]

This exponential upper bound implies that \( \mathbb{P}(O_o) = 0 \) if \( p < 1 \).

### 5.4 Lower bound on \( p_{\max}^{k, s}(V) \)

To derive a lower bound on \( p_{\max}^{k, s}(V) \) we exhibit appropriate nonpercolating percolation models. The proof of proposition 17 suggests to look for percolations being \( \mu_{Z_{zh}^{Z(k)}}, p \)-distributed along downrays. To be as general as possible we also want \( s = 0 \). Section 5.4.1 presents a procedure to construct a \( k, 0 \)-independent percolation model with given distribution along downrays. We then apply this construction to probability distributions derived from \( \mu_{Z_{zh}^{Z(k)}}, p \) and \( \mu_{l[N(k)], p} \). Applying the first moment method and relating the relevant parameters to \( \text{br} (T) \) yields the lower bounds.
5.4.1 Tree fission

In this section we show how to create a $k, 0$-independent percolation model from a $k$-independent Bernoulli random field $Z$ indexed by $\mathbb{N}_0$. Additionally the resulting model has the same distribution along all downrays, namely the one of $Z$, and is invariant under automorphisms of the rooted tree. The generic construction is presented in proposition 20 and specialized to our setting in corollary 21.

**Proposition 20.** Let $Z := \{Z_n\}_{n \in \mathbb{N}_0}$ be a Bernoulli random field and $T := (V,E)$ be a tree rooted at $o$. Then there exists a unique probability measure $\nu$, called the $T$-fission of $Z$, under which the Bernoulli field $Z := \{Z_v\}_{v \in V}$ has the following properties:

$$\forall W \subseteq V: \text{ if } \forall v,w \in W: v \nsubseteq V(T^w),$$

then the subfields $\{Z_{V(T^w)}\}_{w \in W}$ are independent. \hspace{1cm} (41a)

$$\forall v \in V: \text{ } Z_{P(o,v)} \text{ has the same law as } \{Z_{l(w)}\}_{w \in P(o,v)}.$$ \hspace{1cm} (41b)

Furthermore $Z$ is invariant under automorphisms of the rooted tree.

**Proof.** For $v \in V$ let $A(v) := P(o,v) \setminus \{v\}$ be the set of all ancestors of $v$. Let $S$ be the family of vertices of finite connected components of $V$ containing $o$. For $R \in S$ define the probability measure $\nu_R$ on $\{0,1\}^R$ by setting

$$\forall s_R \in \{0,1\}^R: \text{ } \nu_R(Y_R = s_R) := \prod_{v \in R} \mathbb{P}(Z_{l(w)} = s_v) \forall w \in A(v): Z_{l(w)} = s_w. \hspace{1cm} (42)$$

We claim that $\{\nu_R\}_{R \in S}$ is a consistent family à la Kolmogorov. Furthermore each $\nu_R$ has properties (41). One can prove these claims by induction over the cardinality of $R$ (omitted). Hence Kolmogorov’s existence theorem [3, theorem 36.2] yields an extension $\nu$ of the above family. The probability measure $\nu$ fulfills (41) because all its marginals $\nu_R$ do so. Uniqueness follows from the fact that the properties (41) imply the construction of the marginal laws $\nu_R$ via (42) and the $\pi - \lambda$ theorem [3, theorem 3.3]. \hfill $\square$

**Corollary 21.** If $Z$ from proposition 20 is $k$-independent and has marginal parameter $p$ then $\nu$, the $T$-fission of $Z$, is the law of a percolation in $C^k_0,(0) (V)$ invariant under automorphisms of the rooted tree.

**Proof.** The definition of $\nu$ implies that it is the law of a rooted site percolation which is invariant under automorphisms of the rooted tree. $k$-independence and the fact that $\nu(Y_o = 1) = p$ follow from (41b), while $s = 0$ follows from the independence over disjoint subtrees in (41a). \hfill $\square$

5.4.2 The canonical model

For $p \geq p_{sh}$ we derive a $k, 0$-independent percolation model from $\mu_{Z_{(k)},p}$. It does not percolate for small $br(T)$ if $p$ is smaller than the $g_{sh}$ part of (39), leading to a lower bound on $p_{sh}(T)$.\hfill $\square$

**Model 22.** Let $k \in \mathbb{N}$, $p \geq p_{sh}$ and $Z := \{Z_n\}_{n \in \mathbb{N}_0}$ be $\mu_{s_{(k)},p}$-distributed (shifting indices by 1). Define the canonical model of $k$-independent site percolation with parameter $p$, abbreviated $\mathcal{P}^{\text{can}(k)}_p$, as the $T$-fission of $Z$. 

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Remark. We named our canonical model after the canonical model of Balister & Bollobás [2]. Their model is a bond percolation model, whose limit case is defined in the following way: for \( p \geq \frac{3}{2} \) let \( \xi \geq \frac{1}{2} \) be the unique solution of \( 1 - p = \xi(1 - \xi) \) (compare with (26)). Define the bond percolation \( Z := \{ Z_e \}_{e \in E} \) by
\[
Z_e := 1 - (1 - X_{p(v)}X_v),
\] where \( e := (p(v), v) \). See also figure 4. Hence it has dependency parameters \( k = s = 1 \). We see that \( Y_e \) is closed iff \( (X_{p(v)}, X_v) = (0, 1) \) and comparing it with (28) we deduce that it is \( \mu_{Z,p} \)-distributed along downrays. Balister & Bollobás do not mention this link explicitly, though. They not only use this model in its role as nonpercolating counterexample for a lower bound on \( p_{\text{max}}(E) \), as we do with our canonical model in proposition 23, but also show that it has the smallest probability to percolate among all percolations in \( C_{p,o}^{1,1}(E) \), their equivalent to our calculations in section 5.3.

Balister & Bollobás’ explicit construction is easily generalizable to bond models with higher \( k \), but only for \( s \geq 2k - 1 \). Furthermore their inductive approach fails us already for \( k \geq 2 \). Thus its main inspiration has been to look for \( k,0 \)-independent percolation models being \( \mu_{Z,v} \)-distributed along all downrays, leading to the tree-fission and our construction in model 22.

![Figure 4: Construction of Balister & Bollobás’ canonical model. See (43).](image-url)

**Proposition 23.** For all \( k \in \mathbb{N} \) : \( P^\text{can}_{P}^{(k)}(V) \in C_{p,o}^{k,0}(V) \). If \( br(T) \leq k+1 \) and \( p \in \left[ P_{sh}^k, g_k(br(T)) \right] \), then \( P^\text{can}_{P}^{(k)} \) does not percolate. This implies that
\[
\forall k \in \mathbb{N}, \ br(T) \in \left[ 1, \frac{k + 1}{k} \right] : \ P_{\text{max}}^{k,0}(V) \geq g_k(br(T)).
\] **Proof.** As \( Z \) from model 22 is \( k \)-independent and has marginal parameter \( p \) corollary 21 asserts that \( P^\text{can}_{P}^{(k)} \in C_{p,o}^{k,0}(V) \).

Remember that \( p < g_k(br(T)) \) is equivalent to \( \xi < \frac{1}{br(T)} \), hence we can choose
\[ \varepsilon > 0 \text{ such that } (1 + \varepsilon) \xi < \frac{1}{\left| r(T) \right|}. \]

The first moment method (lemma 13) yields

\[ \mathbb{P}(o \leftrightarrow \infty) \leq \inf_{\Pi \in \Pi} \sum_{v \in \Pi} \mathbb{P}(o \leftrightarrow v) \leq \inf_{\Pi \in \Pi} \sum_{v \in \Pi} C\left[ (1 + \varepsilon) \xi \right]^{l(v) + 1} \]

by (30c)

\[ = C(1 + \varepsilon) \xi \inf_{\Pi \in \Pi} \sum_{v \in \Pi} \frac{1}{(1 + \varepsilon) \xi} - l(v) \]

\[ = 0 \text{ by definition of } \br(T) \text{ in (4)}. \]

Therefore \( \mathcal{P}_{p}^{\text{cut}(k)} \) does not percolate and (44) follows directly.

5.4.3 The cutup model

For \( N \in \mathbb{N} \) and \( p_{k,0}^{[N]} < p_{sh}^{[N]} \) we derive a \( k,0 \)-independent percolation model from \( \mu_{[N]_{(k)}}^{[N]}_{p_{sh}} \). It never percolates. In the limit \( N \to \infty \) this yields a lower bound of \( p_{k,0}^{[N]} \) for \( p_{k,0}^{[N]}_{\text{max}}(V) \).

**Model 24.** Let \( k, N \in \mathbb{N} \) and \( Z := \{ Z_n \}_{n \in \mathbb{N}} \) be distributed like independent copies of \( \mu_{[N]_{(k)}}^{[N]}_{p_{sh}} \) on \( \{mN, mN + 1, \ldots, (m + 1)N - 1\} \) for all \( m \in \mathbb{N}_{0} \). Define the \( N \)-cutup model of \( k \)-independent site percolation, abbreviated \( \mathcal{P}^{\text{cut}(k,N)} \), as the \( T \)-fission of \( Z \).

**Proposition 25.** For all \( k, N \in \mathbb{N} : \mathcal{P}^{\text{cut}(k,N)} \in \mathcal{C}^{k,0}_{p_{k,0}^{[N]}_{\text{sh}}} (V) \). It has percolation cluster diameters a.s. bounded by \( 4N - 4 \). Hence it does not percolate. This implies that

\[ \forall k \in \mathbb{N} : \ p_{k,0}^{[N]}_{\text{max}}(V) \geq p_{sh}^{[N]_{(k)}}. \quad (45) \]

**Remark.** It is possible to generate models like the cutup model for every \( p < p_{Z}^{[N]}_{sh} \) \[12, \text{proof of theorem 1}\].

**Proof.** As \( Z \) from model 24 is \( k \)-independent and has marginal parameter \( p \) \( p_{k,0}^{[N]} \) corollary 21 asserts that \( \mathcal{P}^{\text{cut}(k,N)} \in \mathcal{C}^{k,0}_{p_{k,0}^{[N]}_{\text{sh}}} (V) \).

To bound cluster diameters note that \( \mu_{[N]_{(k)}}^{[N]}_{p_{sh}} \) blocks going more than \( 2N - 2 \) steps up or down along a downray. Hence cluster diameters are a.s. bounded by \( 4N - 4 \) and \( \mathcal{P}^{\text{cut}(k,N)} \) does not percolate. Thus \( p_{k,0}^{[N]}_{\text{max}}(V) \geq p_{sh}^{[N]_{(k)}} \).

Finally we know from (27) that \( p_{sh}^{[N]_{(k)}} \xrightarrow{N \to \infty} p_{sh} \).

5.5 Determining \( p_{k,s}^{[V]}(V) \)

To determine \( p_{k,s}^{[V]}(V) \) we take the opposite approach from \( p_{k,0}^{[V]}(V) \). For a uniform lower bound we use the first moment method in proposition 26 on percolations with small enough \( p \). An upper bound follows from the so-called minimal model 27, again built by tree-fission from section 5.4.1. We show that it
percolates for sufficiently high $p$ employing the sufficient conditions on the percolation kernel from section 5.3.1, effectively using the second moment method.

**Proposition 26.**

$$\forall k \in \mathbb{N}_0, s \in \mathbb{N}_0 \cup \{\infty\} : p_{\min}^{k,s}(V) \geq \frac{1}{br(T)^{k+1}}. \quad (46)$$

**Proof.** Let $P \in C_{p,0}^{k,s}(V)$ with $p < \frac{1}{br(T)^{k+1}}$. Then the first moment method (lemma 13) results in

$$P(o \leftrightarrow \infty) \leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} \lambda^{(o)}(v) \leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} p^{(o,v)} \leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} \lambda^{(v)} \leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} \lambda^{(v)} \leq 0 \quad \text{as } br(T) < \frac{1}{p - \frac{1}{k+1}}.$$

Hence $P$ does not percolate and (46) follows trivially. \qed

**Model 27.** Let $X := \{X_n\}_{n \in \mathbb{N}_0}$ be an i.i.d. Bernoulli field with parameter $\hat{p} := p^{1/(k+1)}$. Define $Z := \{Z_n\}_{n \in \mathbb{N}_0}$ by $\forall n \in \mathbb{N}_0 : Z_n := \prod_{s=0}^{k} X_{n+s}$. Define the minimal model of $k$-independent site percolation with parameter $p$, abbreviated $P_{p,0}^{\min} \subset C_{p,0}^k(V)$, as the $T$-fission of $Z$.

**Proposition 28.** For all $k \in \mathbb{N}$ : $P_{p,0}^{\min}(k) \subset C_{p,0}^k(V)$. If $p > \frac{1}{br(T)^{k+1}}$, then $P_{p,0}^{\min}(k)$ percolates, which entails that

$$\forall k \in \mathbb{N} : p_{\min}^{k,0}(V) \leq \frac{1}{br(T)^{k+1}}. \quad (47)$$

**Proof.** As $Z$ from model 27 is $k$-independent and has marginal parameter $p$ corollary 21 asserts that $P_{p,0}^{\min}(k) \subset C_{p,0}^k(V)$.

Let $Z := \{Z_v\}_{v \in V}$ be $P_{p,0}^{\min}(k)$-distributed and $p > \frac{1}{br(T)^{k+1}}$. Looking at model 27, we see that $P(Z_{[n]} = \overline{1}) = P(X_{[n+k]} = \overline{1}) = \hat{p}^{n+k}$, with $\hat{p} := p^{1/(k+1)}$. Use the notation from figure 3 and apply the bound on the percolation kernel (38) to arrive at:

$$\kappa(v, w) \leq \frac{1}{P(o \leftrightarrow \overline{1} \mid \overline{1} \leftrightarrow w)} \leq \frac{1}{\hat{p}^{l(u)}} \leq \hat{p}^{-k-1} \hat{p}^{-l(u)}$$

Apply proposition 15 with $C := \hat{p}^{-k-1}$ and $\alpha := \frac{1}{\hat{p}^{l(u)}} < br(T)$ to show that we percolate. This proves (47). \qed
5.6 The connection with quasi-independence

In this section we show that in both cases (propositions 17 and 28) where we apply the second moment method via exponential bounds on the percolation kernel our $k,s$-independent percolations are also quasi-independent (6). This gives an a posteriori connection with Lyons’ work and explains why we have been able to exploit percolation kernels so effectively.

**Proposition 29.** Let $p > p_{zh}^Z$. Then $\forall P \in \mathcal{C}_k^{k,s}(V), \forall v, w \in V$:

\[
\kappa(v, w) \leq \frac{\xi^{k-(k \lor s)}}{(k + 1) \xi - k} \cdot \frac{1}{P(o \leftrightarrow u)},
\]

hence $P$ is quasi-independent.

**Remark.** (by Temmel) It is an artefact of our use of (38) that we can not show (48) to hold for $p = p_{zh}^Z$, where $\xi = \frac{k}{k + 1}$, and the rhs of (48) explodes. I believe this artefact to be genuine and conjecture that quasi-independence does not hold for $P_{zh}^Z$. This is related to the relation of Shearer’s measure with hard-core lattice gases and non-physical singularities of the partition function [11, section 8].

**Proof.** Let $p > p_{zh}^Z$. We use the notation from figure 3. Then the minimality of Shearer’s measure (22b), the explicit minoration on $Z_k$ in (30a) and the fact that $l(t) \leq l(u) + (k \lor s) + 1$ imply that

\[
P(o \leftrightarrow t | t \leftrightarrow w) = P(u \leftrightarrow t | o \leftrightarrow u, t \leftrightarrow w)P(o \leftrightarrow u | t \leftrightarrow w)
\]

\[
= P(u \leftrightarrow t | o \leftrightarrow u, t \leftrightarrow w)P(o \leftrightarrow u)
\]

\[
\geq \mu_{zh}^Z P(Y_{t(i+1)}, \ldots, l(t) - 1) = \bar{I} Y_{0, \ldots, l(u)} = \bar{I}, Y_{l(t), \ldots, l(u)} = \bar{I} P(o \leftrightarrow u)
\]

\[
\geq \prod_{i=1}^k f_k(i) \cdot f_k(0)^{k \lor s - k} P(o \leftrightarrow u)
\]

\[
= \left[ (k + 1) \xi - k \right] \xi^{k \lor s - k} P(o \leftrightarrow u).
\]

Together with the bound on $k, s$-independent percolation kernels (38) on $\kappa(v, w)$ this yields (48) and quasi-independence.

**Proposition 30.** The minimal percolation model $P_p^{\min(k)}$ is quasi-independent.

**Proof.** We use the notation from figure 3. The explicit construction in model 27 with $\tilde{p} = p^{1/(k+1)}$ and the fact that $l(t) \leq l(u) + k + 1$ imply that

\[
P(o \leftrightarrow t | t \leftrightarrow w) = \tilde{p}^{l(t)} \geq \tilde{p}^{l(u) + k + 1} = P(o \leftrightarrow u).
\]

Together with the bound on $k, s$-independent percolation kernels (38) we get quasi-independence

\[
\kappa(v, w) \leq \frac{1}{P(o \leftrightarrow t | t \leftrightarrow w)} \leq \frac{1}{P(o \leftrightarrow u)}.
\]
5.7 A comment on stochastic domination

Recall that a percolation $X$ stochastically dominates a percolation $Y$ iff there is a coupling of $X$ and $Y$ such that $P(X \geq Y) = 1$. Here the natural order is the partial component-wise order on $\{0, 1\}^E$. We show that for $k \geq 1$ our bounds do not imply stochastic domination of an independent percolation by all $k$-independent percolations for high enough $p$.

**Proposition 31.** $\forall k \geq 1, p \in [0, 1], b \in [1, \infty] : \exists \hat{p} \in [p, 1]$ and $T$ with $br(T) = b$ and a $k$-independent site percolation $Z$ on $T$ with parameter $\hat{p}$ such that $Z$ stochastically dominates only the trivial Bernoulli product field.

**Remark.** It is possible to extend proposition 31 to all $(\hat{p}, b) \in [0, 1] \times [1, \infty]$, using [12, proof of theorem 1].

**Proof.** Denote the $d$-regular tree by $T_d$. We know that $p_{sh}^T = 1 - \left(\frac{d}{d-1}\right)^{d-1}$ [12, theorem 2]. Choose $d$ such that $p_{sh}^T > p$. By the definition of $p_{sh}^T$ (24) there is a finite subtree $\hat{T}$ of $T$ with $p < \hat{p} := p_{sh}^T < p_{sh}^T$.

Root $\hat{T}$ at some vertex $\hat{o}$. Replace every edge of $\hat{T}$ by a length $(k+1)$ path. Add an extra path of $(k+1)$ edges at $\hat{o}$ with endpoint $\bar{o}$. Extend this finite tree further to some arbitrary infinite tree $\bar{T}$ with branching number $b$ and root it at $\bar{o}$.

For every length $(k+1)$ path in the previous paragraph take its last edge and denote their union by $S$. Place $\mu_{\hat{T}, \hat{p}}$ on $S$ and fill up the other edges with i.i.d. Bernoulli($\hat{p}$) variables independently of $\mu_{\hat{T}, \hat{p}}$ on $S$. The resulting percolation is $k$,0-independent. By (21) $\mu_{\hat{T}, \hat{p}}$ fulfills $\mu_{\hat{T}, \hat{p}}(Y_{V(\hat{T})} = \vec{1}) = 0$ and hence the superpercolation on $S$ dominates only the trivial Bernoulli product field. \qed

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6 Additional material

In the proof of proposition 20 the consistency and properties of the family \( \{ \nu_R \}_{R \in S} \) have not been shown.

**Proof.** Let \( R, T \in S \). Then \( \forall s_{R,T} \in \{0,1\}^{R \cup T} \):

\[
\nu_{R,T}(Y_{R,T} = s_{R,T}) = \nu_{R,T}(Y_{R,T} = s_{R,T})
\times \left( \prod_{v \in R \setminus T} P(Z_{I(v)} = s_v | \forall w \in A(v) : Z_{I(w)} = s_w) \right) \left( \prod_{v \in T \setminus R} \cdots \right).
\]
Hence $\nu_S$ and $\nu_T$ coincide on their common support $\{0, 1\}^{R\cap T}$. This implies consistency of the family $\{\nu_R\}_{R \in S}$.

It remains to show that $\nu_R$ is a probability measure on $\{0, 1\}^R$ with properties (41). We prove this by induction over the cardinality of $R$. The induction base for $R = \{o\}$ is

$$
\nu_{\{o\}}(Y_o = 0) + \nu_{\{o\}}(Y_o = 1) = \mathbb{P}(Z_0 = 0) + \mathbb{P}(Z_0 = 1) = 1.
$$

The induction step reduces $R$ to $T := R \setminus \{v\}$ for some leaf $v$ of $G(R)$. Hence

$$
\sum_{\tilde{s}_R} \nu_R(Y_R = \tilde{s}_R) = \sum_{\tilde{s}_T} \sum_{s \in \{0, 1\}} \nu_R(Y_v = s_v, Y_T = \tilde{s}_T)
$$

$$
= \sum_{\tilde{s}_T} \nu_R(Y_T = \tilde{s}_T) \sum_{s \in \{0, 1\}} \mathbb{P}(Z_{l(v)} = s_v | \forall w \in A(v) : Z_{l(w)} = \tilde{s}_w) = 1.
$$

For independence suppose that $W \subseteq R \in S$ fulfills the condition of (41a). Let $U := \bigcup_{w \in W} A(w)$ and for $w \in W$ let $V_w := V(T_w) \cap R$. Then (42) entails that

$$
\nu_R(\forall w \in W : Z_{V_w} = \tilde{s}_{V_w} | Z_U = \tilde{s}_U) = \mathbb{P}(\forall w \in P : Z_{l(w)} = s_w).
$$

Conditional independence on $Z_U$ implies independence as in (41a).

We turn to the distribution along downpaths. For $v \in V$ we have $P := P(o, v) =: \{o := w_0, \ldots, w_{l(v)} := v\} \in S$. Hence $\forall \tilde{s}_P \in \{0, 1\}^P$:

$$
\nu_P(Z_P = \tilde{s}_P) = \prod_{i=0}^{l(v)} \nu_P(Z_{w_i} = s_{w_i} | Z_{A(w_i)} = \tilde{s}_{A(w_i)})
$$

$$
= \prod_{i=0}^{l(v)} \mathbb{P}(Z_i = s_{w_i} | \forall w \in A(w_i) : Z_{l(w)} = s_w)
$$

$$
= \mathbb{P}(\forall w \in P : Z_{l(w)} = s_w).
$$

Finally the invariance under automorphisms of the rooted tree is a result of the obliviousness of the construction to the ordering of the children. \qed