On the continuous-time limit of the Barabási–Albert random graph

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Abstract

We prove that, via an appropriate scaling, the degree of a fixed vertex in the Barabási–Albert model appeared at a large enough time converges in distribution to a Yule process. Using this relation we explain why the limit degree distribution of a vertex chosen uniformly at random (as the number of vertices goes to infinity), coincides with the limit distribution of the number of species in a genus selected uniformly at random in a Yule model (as time goes to infinity). We do not assume that the number of vertices increases exponentially over time (linear rates). On the contrary, we retain their natural growth with a constant rate superimposing to the overall graph structure a suitable set of processes that we call the planted model and introducing an ad-hoc sampling procedure.

Keywords: Barabási–Albert model; Preferential attachment random graphs; Planted model; Discrete- and continuous-time models; Yule model.

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1 Introduction

One of the most popular models for network growth is the preferential attachment model proposed by Barabási and Albert [2] to describe the web graph growth. In this model a newly created vertex is connected to one of those already present in the network with a probability proportional to their degrees. An important feature characterizing the preferential attachment growth mechanism is the presence of a power-law distribution for the asymptotic degree of a vertex selected uniformly at random. This property is observed on World Wide Web data [2, 14, 20]. Furthermore, power-law distributions occur frequently in real-world phenomena and many of them are strictly related to the preferential attachment paradigm [7, 9, 10, 19, 26]. This fact determines an increasing interest for the Barabási–Albert model (BA model) and for random graphs growing with the preferential attachment rule in general.

There are other well-known models related to preferential attachment random graphs, such as, the Yule model for macroevolution [32] and the Simon model describing the appearance of new words in the writing of texts [30]. There is indeed an extensive literature on preferential attachment models (see e.g. [12, 21, 23]. For more recent references see chapter 8 in [31] and the papers cited therein). Their study is based mainly on combinatorial arguments and the analysis of the expectation of
specific functions of the degree or in-degree, together with concentration inequalities [5, 8, 11]. Other methods involve continuum and discrete approaches to study large but finite growing graphs [22] and an embedding of the random graph processes into a continuous setting involving a sequence of pure birth continuous Markov chains (see [1, 3, 29]). The technique of embedding a discrete sequence of random variables in continuous time processes is known for almost fifty years. When it is used on random graph processes, asymptotic results about properties of the vertices are obtained through an efficient use of branching process methods. Despite its generality, the application of this technique is not straightforward when the considered graph corresponds to the Barabási–Albert model whose growth allows the simultaneous birth of \( m \geq 1 \) new links. To deal with this problem it would be necessary either to develop suitable “ad hoc” coupling techniques or merge vertices. For this last procedure see [3] for preferential attachment networks.

Recently, in [27] another methodology has been developed. It has been used to show how the Simon model is related (in a sense of weak convergence) to a set of Yule models.

In this paper, we prefer to follow the alternative approach developed in [27], which could also be useful in other cases in which the embedding method does not apply. For example, embedding techniques are problematic for more general preferential attachment models which are non-Markovian, i.e., in which the emergence of future connections to an existing vertex does not depend solely on the present state of its degree (it could also depend on a function of the degree and other external factors, or the connections could be affected by time delays of the random intervals at which the degree of a vertex change, see [3] and the references therein). Other cases in which the approach of [27] and this paper might be applied are models with more general preferential attachment functions, such as models involving individual fitness [6] and/or aging [16], but also models in which hybrid rules are considered such as the uniform/preferential attachment [28].

The innovative idea in the present work is to couple the degree growth process of a vertex in the BA model to a set of Markov processes, and to introduce the planted tree, an auxiliary branching structure superimposed to the random graph. Then, by means of this, we establish rigorously that the degree of a vertex chosen uniformly at random converges in distribution to the size of a uniformly chosen genus in a \( m \)-Yule model. Note that we do not describe the dynamics of the degree of a fixed vertex and the dynamics of the growth of the number of vertices at the same time. Instead, we create a separate mechanism to describe the dynamics of the growth of the number of vertices that does not need the Markov property of the degree growth processes. The paper is organized as follows. The BA, the Yule and the \( m \)-Yule models are described in Section 2. In Section 3 an alternative method for the sampling procedure of a random vertex in a general random graph model is proposed together with the notion of planted model. This method is a key tool to prove our main results. More specifically this procedure is used to prove the relation between a random vertex chosen uniformly at random in the BA model and an individual chosen uniformly at random from one of the \( m \)-Yule models, also chosen uniformly at random from the set of all \( m \)-Yule models in the planted model. In Section 4 the main results are presented and in Section 5 they are proved. Summarizing them briefly, Theorem 1.1 shows that when infinitely many vertices have already appeared in the BA model, the degree distribution of a vertex appearing subsequently coincides with the distribution of the number of individuals in a Yule process starting with \( m \) initial individuals. Theorem 1.2 proves the convergence to the same limit distribution of the degree of a vertex chosen uniformly at random in the BA model (when the number of vertices diverges) and

\[\text{...}\]
the size of a genus chosen uniformly at random in an \(m\)-Yule model when time goes to infinity. Moreover, we also prove that in the BA model the proportion of vertices with a given degree \(k\) converges in probability (as the number of vertices diverges) to the probability that the degree of a vertex chosen uniformly at random is equal to \(k\). Finally, the exact form of the limit distribution of Theorem 4.1 is given in Proposition 4.1.

2 The two models of interest

2.1 The Barabási-Albert (BA) model

In [2] the preferential attachment paradigm was proposed for the first time to model the growth of the World Wide Web. To do so the authors introduced a random graph model in which the vertices were added to the graph one at a time and joined to a fixed number of existing vertices, selected with probability proportional to their degree. In such a model the vertices represented the web pages and the edges their links. In [2] the model is described as follows:

Starting with a small number \((m_0)\) of vertices, at every time step add a new vertex with \(m\) (\(\leq m_0\)) edges that link the new vertex to \(m\) different vertices already present in the system. To incorporate preferential attachment, assume that the probability that a new vertex will be connected to a vertex \(i\) depends on the connectivity \(k_i\) of that vertex, so it would be equal to \(k_i/\sum_j k_j\). Thus, after \(t\) steps the model leads to a random network with \(t + m_0\) vertices and \(mt\) edges.

The model was then defined in rigorous mathematical terms by Bollobás et al. [5]. Here we recall their definition for the growth of the random graph process \((G_t^m)_{t \geq 1}\).

Definition 2.1. For each \(m \geq 1\) and for every \(n \in \mathbb{Z}^+ \cup \{0\}\), the process \((G_t^m)_{t \geq 1}\) is such that,

1. at time \(t = n(m + 1) + 1\) a new vertex \(v_{n+1}\) is added;
2. for \(i = 2, \ldots, m + 1\), at each time \(t = n(m + 1) + i\) an edge from \(v_{n+1}\) to \(v\) is added with \(v\) chosen according to

\[
P(v_{n+1} \rightarrow v) = \begin{cases} 
\frac{d(v, t - 1)}{2(mn + i - 1) - 1}, & v \neq v_{n+1}, \\
\frac{d(v, t - 1) + 1}{2(mn + i - 1) - 1}, & v = v_{n+1},
\end{cases}
\]  

(2.1)

where \(d(v, t)\) denotes the degree of the vertex \(v\) in \(G_t^m\). We explicitly underline that here \((G_t^m)_{t \geq 1}\) starts at time \(t = 1\) with a single vertex, \(v_1\), without loops. However, since at time \(t = 2\) the only existing vertex is \(v_1\), then a loop is produced.

2.2 The Yule model

In the previous section we presented a model of random graph growth in discrete time. Here we introduce a continuous-time model. To avoid misunderstandings, we denote here by \(T \in \mathbb{R}^+ \cup \{0\}\) the continuous-time variable, while \(t \in \mathbb{Z}^+\) indicates the discrete time.
The model of interest here is the classical model introduced in the twenties by Yule [32]. It was proposed to describe the macroevolution of a population characterized by different genera and species belonging to them.

In order to describe the Yule model we first recall the well-known definition of a Yule process, that is a linear birth process in continuous-time. A Yule model will then be defined in terms of a collection of independent Yule processes of different birth intensities.

**Definition 2.2.** A Yule process \( \{N(T)\}_{T \geq 0} \) is a counting process in continuous-time, with the initial condition that \( N(0) = g, \ g \geq 1 \), almost surely. It is such that the infinitesimal transition probabilities

\[
P(N(T + h) = k + \ell \mid N(T) = k) = \begin{cases} 
  k \lambda h + o(h), & \ell = 1, \\
  o(h), & \ell > 1, \\
  1 - k \lambda h + o(h), & \ell = 0,
\end{cases}
\]  

(2.2)

where \( \lambda > 0 \) is the birth intensity and \( h > 0 \).

This process describes the growth of a population in which, during any short time interval of length \( h \) each member has probability \( \lambda h + o(h) \), independently one another, to create a new individual. Note that the probability of simultaneous births is \( o(h) \). Yule [32] proposed to use this process to model the growth of the number of species belonging to the same genus in an evolving population. Furthermore, a second Yule process, characterized by a different birth intensity \( \beta \) and independent of the former, was used for the growth of the number of genera. The stochastic process determined by the combination of these two types of Yule processes is now known as a Yule model and can be defined as follows.

**Definition 2.3.** A Yule model describes the growth of a population according to the following rules:

1. genera (each comprising a single species) appear as a Yule process \( \{N_\beta(T)\}_{T \geq 0} \) of parameter \( \beta \) with one genus at time \( T = 0 \) almost surely;
2. each time a new genus appears, a copy of a Yule process of parameter \( \lambda \) with a single initial individual starts. Those copies are independent one another and from the process of appearance of genera. Each copy models the evolution of species belonging to the same genus.

In this paper we also consider an \( m \)-Yule model (denoted by \( \{Y_{m,\lambda,\beta}(T)\}_{T \geq 0} \)), that is a process similar to a classical Yule model but in which the birth processes describing the evolution of the species belonging to each genus start from \( m \in \mathbb{Z}^+ \) initial species almost surely. To underline the initial condition we will add a superscript \( m \) to the Yule process counting the number of species for each genus \( \{N_{m,\lambda}(T)\}_{T \geq 0} \). We explicitly remark that the the letter \( m \), used for the initial value of the \( m \)-Yule model was already used to indicate the number of edges from a vertex in the BA model. This choice is not a coincidence, in the next paragraphs we will in fact show that the initial value \( N_{m,\lambda}(0) \), is determined by the parameter \( m \) of the BA model. Finally, we would like to point out that the 1-Yule model coincides with the original Yule model in [32].

### 3 Sampling a random vertex: a result

To prove the weak convergence of the Simon model to the Yule model, in [27] we introduced a specific procedure of sampling vertices from the random graph related
to the Simon model. Moreover, we showed that this procedure is equivalent to a uniform sampling. Here we propose an alternative method for the uniform sampling procedure of a random vertex from a graph. For a greater generality we consider a sequence \( \{M_n\}_{n \geq 1} \) of random variables taking values in \( \mathbb{Z}^+ \) a.s. and a graph in which \( M_n \) edges start from the vertex appeared at time \( n, n \geq 1 \). Then we apply this procedure to sample a random vertex in the special case of the BA model, i.e. in the case \( M_n = m \) almost surely.

In order to describe the sampling procedure, we introduce first a model that we call the planted model for the random graph \( (G_t)_{t \geq 1} \). The idea underlying the planted model is to superimpose a tree structure on the graph which is independent from the degree growth processes.

We start by noting that, at each time of the form \( T_i = \sum_{r=1}^{i}(M_r + 1) \), the graph \( G_{T_i} \) has exactly \( i \) vertices, \( i \in \mathbb{Z}^+ \). Let \( i > 1 \). To obtain the tree structure at the following times \( T_{n+1}, n \geq i \), we attribute to \( v_{n+1} \) the role of child of a vertex chosen uniformly at random from the set of the existing vertices \( \{v_1, v_2, \ldots, v_n\} \). This procedure determines chains of successive offsprings of each of the planted vertices \( \{v_1, v_2, \ldots, v_i\} \). We call a vertex \( v \) that appeared after \( v_j, j = 1, \ldots, i, \) a descendant of \( v_j \) if both \( v \) and \( v_j \) belong to the same ancestral line. We order the descendant of \( v_j \) by renaming \( v \) as \( v_{j, \ell} \), if \( v \) is the \( \ell \)-th descendant of \( v_j \). We call \( v_{j, 1} \) as \( v_j \), that is, \( v_j \) is in turn, its first descendant. In this way we construct \( 1 < i \leq n \) births in discrete time, \( \{b(v_j, T_n)\}_{n \geq i}, j = 1, \ldots, i \). Here \( b(v_j, T_n), j = 1, \ldots, i, n \geq i \), is the total number of descendants of \( v_j \) at time \( T_n \). Table 1 shows an example of the construction of the planted model. Note that we have:

- \( b(v_j, T_i) = 1, j = 1, \ldots, i; \)
- \( \mathbb{P}[b(v_j, T_{n+1}) = k + 1 \mid b(v_j, T_n) = k] = k/n, k \geq 1, n \geq i, j = 1, \ldots, i. \)

The second equality holds because at time \( T_{n+1}, n \geq i \), each existing vertex in the set \( \{v_1, v_2, \ldots, v_n\} \) may give birth to a new one with probability \( 1/n \).

Note that, the planted model is defined for \( n \geq i \) and \( i > 1 \). Thus for example, given an \( 1 < i \leq n \), there is no process \( \{b(v_j, T_n)\}_{n \geq i} \) with \( j > i \), because \( j \) has to be an element of \( \{1, \ldots, i\} \). The dynamic of the planted model then proceeds for \( n \geq i \). Furthermore, note that the \( i \) discrete-time birth processes are exchangeable.

### 3.1 Sampling from the planted model

Consider the following procedure. Given a realization of \( \{b(v_j, T_n)\}_{n \geq i}, j = 1, \ldots, i, \)

1. choose one of the \( i \) discrete-time birth processes with probability proportional to the number of its vertices;
2. choose a vertex uniformly at random among those belonging to the realization of the selected birth process.

Our focus will be on the selected vertex \( v_{j, \ell}, j = 1, \ldots, i, \ell = 1, \ldots, b(v_j, T_n) \) and on the selected birth process. Let \( W \) denote the index of the birth process chosen. It is a random variable with values in \( \{1, \ldots, i\} \).

**Theorem 3.1.** It holds,

1. \( \mathbb{P}(W = j) = \frac{1}{i} \),
2. \( \mathbb{P}(\{v_{j, \ell} \text{ is selected}\}) = \frac{1}{i}. \)

**Remark 3.1.** The suggested algorithm is a way to select a vertex uniformly at random from \( G_{T_n}, n \geq i \). The procedure considers a given realization of the \( i \) birth processes \( \{b(v_j, T_n)\}_{n \geq i}, j = 1, \ldots, i \). Averaging on all possible realizations of the \( i \)
birth processes we actually select a vertex uniformly at random: we first choose one of the $i$ birth processes belonging to the planted model with uniform probability, then we select a vertex among those belonging to the chosen birth process with uniform probability. Note that this procedure is independent of the dynamics of the degree growth processes in the BA random graph model.

**Remark 3.2.** In the proof of Theorem 3.1 we will make use of the following result for a Pólya urn model. A single urn initially contains $i - 1$ white balls and one black ball. At each time step a ball is drawn from the urn and then replaced together with one ball of the same color. Let $X$ be the number of black balls in $n - i$ drawings. The distribution of $X$ is well known (see e.g. [17], Section 4.2) and its expectation is $E[X] = (n - i)/i$, so that $E[Y_j] = n/i$. Notice that the relationship between $X$ and the number of individuals in the $j$-th birth process of the planted model is $Y_j = X + 1$.

**Proof.** For the sake of simplicity let us rename the random variable $b(v_j, \mathcal{T}_n)$ by $Y_j$ and note that it takes values in $\{1, 2, \ldots, n - i + 1\}$. Furthermore, let $Z_j$ denote the index of a vertex from that birth process, which is a random variable with values in $\{1, \ldots, n - i + 1\}$.

To prove the first part of the theorem, let us consider the probability of choosing one of the $i$ birth processes,

$$P(W = j) = \sum_{k=1}^{n-i+1} P(W = j \mid Y_j = k)P(Y_j = k)$$

$$= \sum_{k=1}^{n-i+1} \frac{k}{n} \frac{E(Y_j)}{n} = \frac{1}{i}.$$ 

Hence, the measure induced by choosing a birth process in the mechanism of sampling a vertex from the planted model, is uniform on the $i$ discrete-time birth processes.
To prove the second part, the probability of selecting the vertex $v_{j,\ell}$ is given by

$$
P(W = j, Z_j = \ell) = \sum_{k=1}^{n-i+1} P(Z_j = \ell \mid W = j, Y_j = k) P(W = j \mid Y_j = k) P(Y_j = k)
$$

(3.2)

where the last equality is obtained by observing that $P(Z_j = \ell \mid W = j, Y_j = k) = 1/k$ and $P(W = j \mid Y_j = k) = k/n$. □

Remark 3.3. Note that Theorem 3.1 can be alternatively proven by exploiting the exchangeability of the $i$ discrete-time birth processes.

4 Main Results

Let us first describe a heuristic approach explaining the relation between the discrete time process for the degree growth of a fixed vertex and a Yule process. In the BA model, $m$ directed edges sequentially connect each new vertex to the others with probabilities proportional to the degrees of the existing vertices. Thus, at the time at which there are $n$ vertices, that is at time $t = n(m + 1)$, we have $mn$ directed edges, and by the preferential attachment rule,

$$
P[d(v, (n+1)(m+1)) = k + 1 \mid d(v, n(m+1)) = k] \approx \frac{km}{2mn} = \frac{k}{2n},
$$

(4.1)

where $d(v, t)$ is the degree of $v$ in the BA model. The approximation in (4.1) consists in connecting all the $m$ edges simultaneously instead of sequentially. That is, we consider $m$ chances of increasing the degree of $v$ from $k$ to $k + 1$. Then, considering the time interval between the instants at which the degree of $v$ changes from $k$ to $k + 1$, and neglecting the increasing of the number of vertices, by (4.1) the distribution of this random time interval is Geometric with parameter $k/(2n)$. In this approximations, when $n \to \infty$ we obtain a convergence to an exponential random variable of parameter $k\lambda$, with $\lambda = 1/2$. Furthermore, neglecting the possibility of loops, the initial degree of $v_i$, $i \geq n$, turns out to be equal to $m$. These two observations suggest to approximate the distribution of the degree of a vertex in the BA model by the distribution of the number of individuals in a Yule process with parameter $\lambda = 1/2$ and initial condition $N_{\lambda}(0) = m$.

In Theorem 4.1 we prove this convergence substituting the above approximations with rigorous convergence results. To do that, we start by showing that the transition probabilities of the degree process of a vertex in the BA model are bounded (Lemma 5.1). Then, by using these bounds we construct two Markovian processes coupled with the degree process (Lemma 5.2 and Corollary 5.1). Finally, we use this coupling to show that the finite-dimensional distribution of the degree of a vertex in the BA model, converges to the finite-dimensional distribution of the number of individuals in a Yule process with initial population size equal to $m$ (Lemma 5.3).

However, the interest is to study the asymptotic degree of a vertex chosen uniformly at random. Therefore, we start by relating the deterministic appearance of new vertices in the BA model with a continuous-time Yule process; for this aim, in Theorem 4.2 we make use of the planted model and the sample procedure from the planted model described in Section 3. The key point is that, by Theorem 3.1 the choice of a vertex with uniform distribution in the BA model is equivalent to
choosing a birth process from the planted model with uniform distribution and then choosing a vertex among those belonging to the selected birth process, again with uniform distribution. In Lemma 5.4, we prove that the number of individuals in each birth process in the planted model, that is \( \{b(v_j, n(m + 1))\}_{n \geq 1}, \ 1 \leq j \leq i, \) converges in distribution as \( i \to \infty, \) to the size of a Yule process with parameter \( \beta = 1 \) and with one initial progenitor.

Finally, in Theorem 4.2, by using Theorem 4.1 and Lemma 5.4, we show that the BA model is related to a sequence of suitably scaled \( m \)-Yule models. Exploiting this relation we prove that the asymptotic degree distribution of a vertex chosen uniformly at random in the BA model coincides with the asymptotic distribution of the size of a genus chosen uniformly at random in the \( m \)-Yule model.

**Theorem 4.1.** Let \( z(i, w) : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{N}, \ i > 1, \ w \in \mathbb{R}^+ \) be a function such that \( c(w) \equiv \lim_{i \to \infty} z(i, w)/i \) exists finite, and \( c(w) : \mathbb{R}^+ \to \mathbb{R}^+ \) is an increasing function in \( w \). Let \( b \geq 1 \) and \( w_1 < w_2 < \cdots < w_b \) be positive real numbers. Then, we have that
\[
\lim_{i \to \infty} \mathbb{P} \left[ d(v_i, (i + z(i, w_1))(m + 1)) = k_1, \ldots, d(v_i, (i + z(i, w_b))(m + 1)) = k_b \right]
= \mathbb{P}[N_{i/2}^m(\ln(1 + c(w_1))) = k_1, \ldots, N_{i/2}^m(\ln(1 + c(w_b))) = k_b]
= \prod_{i=1}^{b} \left( \frac{k_i - 1}{k_i - k_{i-1}} \right) e^{-\frac{k_{i-1}}{2} \ln\left(1 + \frac{1 + c(w_i)}{1 + c(w_{i-1})}\right) / (1 - e^{-\frac{1}{2} \ln\left(1 + \frac{1 + c(w_i)}{1 + c(w_{i-1})}\right)})}^{k_i - k_{i-1}}.
\]
Here \( w_0 = 0, k_0 = m, \) and \( m \leq k_1 \leq \cdots \leq k_b \in \mathbb{N}. \)

**Remark 4.1.** Notice that the required scaling of time \( i \mapsto i + z(i, w_1) \) behaves asymptotically as the linear function \( i \mapsto i + ic(w_1). \) Moreover, the logarithm of its slope, \( 1 + c(w_1), \) is the time at which the Yule process is evaluated. Regarding the existence of the function \( z(i, w_1), \) possible choices can be \( z(i, w_1) = \lfloor iw_1 \rfloor \) or \( z(i, w_1) = \lfloor (i - 1)w_1 \rfloor \).

**Remark 4.2.** Theorem 4.1 states that the joint distribution of the degrees of \( v_i \) at times \( z(i, w_1)(m + 1), \ldots, z(i, w_b)(m + 1) \) after its first appearance in \( (G_m^i)^{\ell=1} \), converges, as \( i \to \infty, \) to the joint distribution of the number of individuals of a Yule process with \( m \) initial individuals and parameter \( \lambda = 1/2, \) evaluated at times \( \ln(1 + c(w_1)), \ldots, \ln(1 + c(w_b)). \)

**Theorem 4.2.** Consider an \( m \)-Yule model \( \{Y_{i/2,1}^m(T)\}_{T \geq 0}, \) and let \( N_T^m \) be the size of a genus chosen uniformly at random at time \( T \) in \( \{Y_{i/2,1}^m(T)\}_{T \geq 0}. \) Consider the random graph process \( (G_m^i)^{\ell=1} \) defining the BA model with \( N_{k,t} \) vertices with degree \( k. \) Let \( d(V_i) \) indicate the degree of a vertex chosen uniformly at random at time \( t \) in \( (G_m^i)^{\ell=1}. \) Then, for \( t = n(m + 1) \) we have
\[
p_k := \lim_{n \to \infty} \mathbb{P}(d(V_i) = k) = \lim_{T \to \infty} \mathbb{P}(N_T^m = k), \quad k \geq m, \tag{4.3}
\]
and for any \( C > m\sqrt{3}, \)
\[
\mathbb{P}\left( \max_k \bigg| \frac{N_{k,t}}{n} - \mathbb{P}(d(V_i) = k) \bigg| \geq C\sqrt{\frac{(m + 1)\ln(n(m + 1))}{n}} \right) = o(1). \tag{4.4}
\]
Furthermore as \( n \to \infty, \frac{N_{k,t}}{n} \to p_k \) in probability.
Remark 4.3. To prove the first part of Theorem 4.2 we will use the result of Theorem 4.1. The idea is to take $n := n(i, w)$, a function of $i$ and a positive real number $w$, such that, $i/n(i, w) \to 1/(1 + c(w))$ as $i \to \infty$, with $c(w)$ as in Theorem 4.1. Thus, $\lim_{w \to \infty} \lim_{i \to \infty} i/n(i, w) = 0$.

Using the previous theorem and directly exploiting the properties of the $m$-Yule model we are able to recover the well known result for the asymptotic degree distribution of the BA random graph.

Proposition 4.1. Consider an $m$-Yule model $\{Y^m_{T/2,1}(T)\}_{T \geq 0}$ and the size $N^m_T$ of a genus chosen uniformly at random at time $T$ from it as in Theorem 4.2. Then,

$$p_k = m(m + 1)B(k, 3),$$

where $k \geq m$ and $B(a, b)$ is the Beta function.

Remark 4.4. Notice that the distribution (4.5) clearly coincides with the degree distribution of the BA model [5].

5 Proofs

5.1 Auxiliary Lemmas and the Proof of Theorem 4.1

In order to prove Theorem 4.1 within the construction of the BA model, we identify two different processes in discrete time, one for the appearing of in-links of each specific vertex and the other related to the creation of new vertices. Then, we prove that these two processes converge to the two birth processes which are at the basis of the definition of an $m$-Yule model. Before starting the construction of the process for the appearance of in-links of a fixed vertex we introduce the following definition.

Definition 5.1. We say that a vertex $v_i$ appears “complete” when it has appeared in the BA random graph process together with all the directed edges originated from it.

Note that the degree of a complete vertex is at least $m$, and at time $t = n(m + 1)$, the BA model has for the first time exactly $n$ complete vertices.

Next we determine how the degree of a fixed vertex changes during the time until a new complete vertex appears.

Lemma 5.1. Let $(G_t)_{t \geq 1}$ be the random graph process defining the BA model and let $d(v_i, t)$ denote the degree of an existing vertex $v_i$ at time $t$, $i \leq n$. Given that $d(v_i, n(m + 1)) = k$, $k \geq m$, then there exist four constants $b_2 > b_1 > 0$ and $c_1, c_2 > 0$ such that

$$\frac{k}{2(n + 1)} + \frac{c_2}{n^2} < P[d(v_i, (n + 1)(m + 1)) = k + 1 | d(v_i, n(m + 1)) = k] < \frac{k}{2n} + \frac{c_1}{n^2},$$

and

$$\frac{b_2}{n^2} < P[d(v_i, (n + 1)(m + 1)) = k + \ell | d(v_i, n(m + 1)) = k] < \frac{b_1}{n^2}, \quad 1 < \ell \leq m.$$
Furthermore,
\[
\mathbb{P}[d(v_{n+1}, (n+1)(m+1)) = m] = \prod_{\ell=2}^{m+1} \left(1 - \frac{1}{2(mn + \ell - 1) - 1} \right) = 1 - O(1/n).
\] (5.3)

**Proof.** Our aim is to determine the change of degree of a fixed vertex during the time interval \((n(m+1), (n+1)(m+1))\), i.e., during the time interval necessary to switch from \(n\) to \((n+1)\) complete vertices. Let us fix \(t = n(m+1)\) and follow the graph growth during the considered interval. At time \(n(m+1) + 1\) a new vertex \(v_{n+1}\) (without edges) appears. Then from time \(n(m+1)+2\) to \((n+1)(m+1)\) a directed edge from \(v_{n+1}\) to an existing vertex \(v_i\), \(i \leq n+1\), is added. The vertex \(v_i\) is chosen with probability given by (2.1). Let \(Y^n_{v_i}\) be the total number of incoming edges to \(v_i\), \(i \leq n\), added to \(v_i\) during the time interval \((n(m+1), (n+1)(m+1))\). Note that 
\[
\mathbb{P}[d(v_i, n(m+1)) = k] = \mathbb{P}[Y^n_{v_i} = k] = \mathbb{P}[d(v_i, n(m+1) + 1) = k],
\] 
\(\ell = 0, \ldots, m\). To estimate the latter conditional probabilities we distinguish the cases \(Y^n_{v_i} = 0, Y^n_{v_i} = 1, \) and \(Y^n_{v_i} \geq 2\).

In the first case, considering the probabilities (2.1) we have
\[
\mathbb{P}[Y^n_{v_i} = 0 \mid d(v_i, n(m+1)) = k] = \prod_{\ell=2}^{m+1} \left(1 - \frac{k}{2(mn + \ell - 1) - 1} \right). \tag{5.4}
\]
Since
\[
\prod_{\ell=2}^{m+1} \left(1 - \frac{k}{2(mn + \ell - 1) - 1} \right) \leq \left(1 - \frac{k}{2(mn+m) - 1} \right)^m,
\]
we get the upper bound for (5.4),
\[
\left(1 - \frac{k}{2(mn + m) - 1} \right) = 1 - \frac{mk}{2m(n+1) - 1} + O \left(\frac{1}{n^2}\right). \tag{5.5}
\]

Furthermore, since
\[
\prod_{\ell=2}^{m+1} \left(1 - \frac{k}{2(mn + \ell - 1) - 1} \right) \geq \left(1 - \frac{k}{2(mn+m+1) - 1} \right)^m,
\]
we get the lower bound
\[
\left(1 - \frac{k}{2(mn + n + 1) - 1} \right) = 1 - \frac{mk}{2mn + 1} + O \left(\frac{1}{n^2}\right). \tag{5.6}
\]

As far as the third case is concerned we observe that if \(m = 1\) then \(\mathbb{P}(Y^n_{v_i} \geq 2 \mid d(v_i, n(m+1)) = k) = 0\). Thus we calculate such probability for the case \(m > 1\) only. Furthermore, since we do not need a closed form of \(\mathbb{P}(Y^n_{v_i} \geq 2 \mid d(v_i, n(m+1)) = k)\), we limit ourselves to estimate its order of magnitude. For each \(y = 2, \ldots, m\), the event \(\{Y^n_{v_i} = y\}\) means that \(v_i\) gets \(y\) new incoming edges joining \(v_i\) during the times \(t = n(m+1) + \ell, \ell = 2, \ldots, m+1\). Given the value of the degree of \(v_i\) at time \(t - 1\), by (2.1), a new edge is attached to \(v_i\) at time \(t = n(m+1) + \ell\) with probability
\[
p^{n, \ell}_{v_i} := \frac{d(v_i, n(m+1) + \ell - 1)}{2(mn + \ell - 1) - 1}.
\]

Let \(\Omega\) be the space of all sequences of \(m\) dichotomous independent experiments, performed at times \(t = n(m+1) + \ell, \ell = 2, \ldots, m+1\), with exactly \(y\) successes. Assume that \(p^{n, \ell}_{v_i}, \ell = 2, \ldots, m+1\), are the probabilities of success. Note that the cardinality of \(\Omega\) is equal to that of the set of all \(y\)-combinations from a given set of \(m\) distinct elements, i.e. \(|\Omega| = \binom{m}{y}\). Take the set \(\{2, \ldots, m+1\}\) and consider its \(y\)-combinations, say \(C_y = \{e_1, \ldots, e_{\binom{y}{y}}\}\) (e.g. ordered by their smallest element).
For each $e \in C_y$, let $e(j)$ denote the element in the $j$-th position of $e$, $j = 1, \ldots, y$. We have,

$$
P(Y^n_{v_i} = y | d(v_i, n(m + 1)) = k) = \prod_{e \in C_y} p_{v_i}^{n, e(j)} \prod_{\ell \in \{2, \ldots, m+1\}, \ell \notin e(1), \ldots, e(y)} (1 - p_{v_i}^{n, \ell})
$$

(5.7)

$$
= \left(\begin{array}{c} m \\ y \end{array}\right) O \left(\frac{1}{n^y}\right) \left(1 - O\left(\frac{1}{n}\right)\right)^{m-y}
$$

$$
= \left(\begin{array}{c} m \\ y \end{array}\right) O \left(\frac{1}{n^y}\right) \sum_{\ell=0}^{m-y} \binom{m-y}{\ell} (-1)^\ell O\left(\frac{1}{n^\ell}\right)
$$

$$
= O\left(\frac{1}{n^y}\right), \quad 2 \leq y \leq m.
$$

Hence

$$
P(Y^n_{v_i} \geq 2 | d(v_i, n(m+1)) = k) = O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n^3}\right) + \cdots + O\left(\frac{1}{n^{m}}\right)
$$

(5.8)

$$
= O\left(\frac{1}{n^2}\right).
$$

Finally, by (5.6) and (5.8) we obtain that $P(Y^n_{v_i} = 1)$ is at most

$$
1 - \left[1 - \frac{mk}{2mn+1} + O\left(\frac{1}{n^2}\right)\right] - O\left(\frac{1}{n^2}\right) < k \frac{2}{2n} + O\left(\frac{1}{n^2}\right)
$$

(5.9)

and by (5.5) and (5.8), $P(Y^n_{v_i} = 1)$ is at least

$$
1 - \left[1 - \frac{mk}{2m(n+1) - 1} + O\left(\frac{1}{n^2}\right)\right] - O\left(\frac{1}{n^2}\right) > k \frac{2}{2(n+1)} + O\left(\frac{1}{n^2}\right).
$$

(5.10)

Therefore, there exist $b_1 > b_2 > 0$ such that (5.8) gives (5.2), and $c_2, c_1 > 0$ such that (5.9) and (5.10) give (5.1).

To determine (5.3), let $X^n_{v_{n+1}}$ be number of incoming edges from $v_{n+1}$ to itself during the time interval $(n(m+1), (n+1)(m+1))$, that is the number of loops. Note that during this period $X^n_{v_{n+1}}$ can be at most equal to $m$, since at time $n(m+1)+1$ no edge is added. Thus, by (2.1), the order of magnitude of the probability of no loops for $v_{n+1}$ during such time interval is given by

$$
P(X^n_{v_{n+1}} = 0) = \prod_{i=2}^{m+1} \left(1 - \frac{1}{2(mn + i - 1)}\right) = [1 - O(1/n)]^m = 1 - O(1/n).
$$

(5.11)

Note that if the number of loops for $v_{n+1}$ is zero, this is equivalent to say that when $v_{n+1}$ appears complete, its degree is equal to $m$. Thus, by (5.11) we can write:

$$
P[d(v_{n+1}, (n+1)(m+1)) = m] = \prod_{\ell=2}^{m+1} \left(1 - \frac{1}{2(mn + \ell - 1)}\right) = 1 - O(1/n).
$$

(5.12)
Now we consider for each \( i \geq 1 \) the degree process \( \{d(v_i, n(m+1))\}_{n \geq 1} \), indexed by \( n \), where \( d(v_i, n(m+1)) \) satisfies (5.1), (5.2) and (5.3). Let \( E = \{m, m+1, \ldots \} \) be the state space of \( d(v_i, n(m+1)) \) and let \( \mathcal{M}(E) \) be the class of probability measures on the space \( E \) endowed with the \( \sigma \)-algebra \( \mathcal{F} = \mathcal{P}(E) \), the power set of \( E \). The degree process \( \{d(v_i, n(m+1))\}_{n \geq 1} \) is defined on the product space \( (E^\infty, \mathcal{F}^\infty) = (\times_{m=n} E_n, \otimes_{m=n} \mathcal{F}_n) \), where \( (E_n, \mathcal{F}_n), n \geq i \) are copies of \( (E, \mathcal{F}) \). The process from time \( i \) to time \( i+h \), \( \{d(v_i, n(m+1))\}_{n \geq 1}^{i+h} \) takes values in the product space \( (E^h, \mathcal{F}^h) \) and \( (E^\infty, \mathcal{F}^\infty) \) will be denoted by \( x^h = (x_i, x_{i+1}, \ldots, x_{i+h}) \) and \( x^\infty = (x_i, x_{i+1}, \ldots) \), respectively. We say that \( x^h \leq y^h \) if and only if \( x_{i+j} \leq y_{i+j} \) for all \( 0 \leq j \leq h \).

To prove Theorem 4.1, we proceed through three steps. First we define two Markovian processes, on the same probability space as \( \{d(v_i, n(m+1))\}_{n \geq 1} \), determined by known Markovian kernels. Then we show that these two processes bound from above and below the degree process of the BA model. Finally we prove that these two processes, each one evaluated at a convenient time, converge in distribution as \( n \to \infty \) to a unique process evaluated at a unique time.

As far as the first step is concerned we define \( p_{n+1} \) be a positive function on \( E^n \times E_{n+1}^\infty \), measurable with respect to \( \mathcal{F}^n \otimes \mathcal{F}_{n+1}^\infty \), and given by

\[
p_{n+1}(x^n, x_{n+1}) = p_{n+1}(x^n, x_{n+1}, \ell) = \begin{cases} 
\frac{x_{\ell} + c_{\ell}}{n}, & \text{if } \ell = 1, \\
\frac{b_{\ell}}{n}, & \text{if } \ell = 2, \\
1 - \frac{x_{\ell}}{2^{n+1}} - \frac{b_{\ell} + c_{\ell}}{n}, & \text{if } \ell = 0, \\
0, & \text{otherwise.}
\end{cases} \tag{5.13}
\]

Then we define the following Markov transition kernel \( K_{n+1}^p \) from \( E^n \times E_{n+1}^\infty \) into \([0,1]\\):

\[
K_{n+1}^p(x^n, B) = \sum_{x_{n+1} \in B} p_{n+1}(x^n, x_{n+1}), \quad x^n \in E^n, B \in \mathcal{F}_{n+1}^\infty. \tag{5.14}
\]

The mapping \( B \to K_{n+1}^p(x^n, B) \) is a measure \( P_{n+1} \in \mathcal{M}(E_{n+1}^\infty) \) for every \( x^n \in E^n \). In an analogous way we define the function

\[
r_{n+1}(z^n, z_{n+1}) = r_{n+1}(z^n, z_{n+1}, \ell) = \begin{cases} 
\frac{z_{\ell}}{2^n}, & \text{if } \ell = 1, \\
\frac{b_{\ell}}{2^n}, & \text{if } \ell = m, \\
1 - \frac{z_{\ell}}{2^n} - \frac{b_{\ell} + c_{\ell}}{n}, & \text{if } \ell = 0, \\
0, & \text{otherwise,}
\end{cases} \tag{5.15}
\]

that we associate to the Markov transition kernel \( K_{n+1}^r \), where \( B \to K_{n+1}^r(z^n, B) \), is a measure \( R_{n+1} \in \mathcal{M}(E_{n+1}^\infty) \) for every \( z^n \in E^n \). Furthermore, it is easy to see from (5.13) and (5.15) that there exists a function \( q_{n+1}(y^n, y_{n+1}) = q_{n+1}(y^n, y_{n+1}, \ell) \), such that:

\[
p_{n+1}(x^n, x_{n+1} + 1) < q_{n+1}(y^n, y_n + 1) < r_{n+1}(z^n, z_{n+1}),
\]
\[
p_{n+1}(x^n, x_{n+1} + 2) < \sum_{\ell=2}^m q_{n+1}(y^n, y_{n+1} + \ell) < r_{n+1}(z^n, z_{n+1} + m),
\]
\[
q_{n+1}(y^n, y_n) = 1 - \sum_{\ell=1}^m q_{n+1}(y^n, y_{n+1} + \ell),
\]

whenever \( x_n \leq y_n \leq z_n, n \geq i \). We associate this function to another Markov transition kernel \( K_{n+1}^q \) in the same way as \( K_{n+1}^p \) and \( K_{n+1}^r \), where \( B \to K_{n+1}^q(y^n, B) \) is a measure \( Q_{n+1} \in \mathcal{M}(E_{n+1}^\infty) \) for every \( y^n \in E^n \).
In order to prove that there exist two processes bounding respectively from above and below the degree process of the BA model we have first to prove a similar result for random variables. This is attained through the following

**Lemma 5.2.** Let \( X_i, Y_i, \) and \( Z_i, \) \( i \geq 1, \) be random variables on \( E_i \) with distributions \( P_i, Q_i \) and \( R_i, \) respectively, and satisfying \( \mathbb{P}(X_i = Y_i = Z_i) = 1. \) Then there exist random variables \( X_{n+1}, Y_{n+1}, \) and \( Z_{n+1}, n \geq 1, \) taking values in \( E_{n+1}, \) such that the conditional distributions of \( X_{n+1} \) given \( X_n = x_n, \) \( Y_{n+1} \) given \( Y_n = y_n, \) and \( Z_{n+1} \) given \( Z_n = z_n, \) are exactly \( p_{n+1}(x^n, \cdot), q_{n+1}(y^n, \cdot), \) and \( r_{n+1}(z^n, \cdot), \) respectively. Moreover,

\[
\mathbb{P}(X_n \leq Y_n \leq Z_n, n = i, i + 1, \ldots) = 1. \tag{5.17}
\]

**Proof.** We seek to prove a stochastic ordering for \( K_{n+1}^p(x^n, \cdot), K_{n+1}^q(y^n, \cdot) \) and \( K_{n+1}^r(z^n, \cdot). \) For this aim, take the set \( B \in \mathcal{F}_{n+1} \) such that \( B := \{b, b+1, \ldots\}, \) where \( b \) is any integer \( b \geq m. \) Then,

\[
K_{n+1}^p(x^n, B) = \sum_{j \geq b} p_{n+1}(x^n, j) = \begin{cases} 1 & \text{if } b \leq x_n, \\ \frac{x_n}{n+1} + \frac{b}{n+1} & \text{if } b = x_n + 1, \\ b & \text{if } b = x_n + 2, \\ 0 & \text{if } b \geq x_n + 3. \end{cases} \tag{5.18}
\]

\[
K_{n+1}^q(y^n, B) = \sum_{j \geq b} q_{n+1}(y^n, j) = \begin{cases} 1 & \text{if } b \leq y_n, \\ \sum_{\ell=0}^{m-1} q_{n+1}(y^n, y_n + \ell) & \text{if } b = y_n + \ell, \\ q_{n+1}(y^n, y_n + m) & \text{if } b \geq y_n + m, \\ 0 & \text{if } b \geq x_n + m + 1. \end{cases} \tag{5.19}
\]

and

\[
K_{n+1}^r(z^n, B) = \sum_{j \geq b} r_{n+1}(z^n, j) = \begin{cases} 1 & \text{if } b \leq z_n, \\ \frac{z_n}{n} + \frac{b}{n} & \text{if } b = z_n + 1, \\ b & \text{if } b = z_n + 2, \\ 0 & \text{if } b \geq z_n + m + 1. \end{cases} \tag{5.20}
\]

Comparing these three last kernels (5.18), (5.19), and (5.20), and by (5.13), (5.15) and (5.16), if \( (x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \leq (z_1, \ldots, z_n), \) then

\[
K_{n+1}^p(x^n, B) \leq K_{n+1}^q(y^n, B) \leq K_{n+1}^r(z^n, B).
\]

Equivalently, \( K_{n+1}^p(x^n, \cdot) \) is stochastically smaller than \( K_{n+1}^q(y^n, \cdot), \) and this last one is in turn stochastically smaller than \( K_{n+1}^r(z^n, \cdot), \) whenever \( x^n \leq y^n \leq z^n. \) To show that (5.17) holds we recall that \( X_i = Y_i = Z_i \) almost surely, then we apply Theorem 2 in [18].

Let us consider the process \( \{d(v_i, n(m+1))\}_{n \geq 1}, \) and its probability space \( (\Omega, \mathcal{A}, \mathbb{P}). \) On the same probability space let us define two Markov processes \( \{d^1(v_i, n(m+1))\}_{n \geq 1} \) and \( \{d^2(v_i, n(m+1))\}_{n \geq 1}, i > 1, \) with their initial states such that \( \mathbb{P}(d^1(v_i, i(m+1)) = d(v_i, i(m+1)) = d^2(v_i, i(m+1))) = 1, \) and transition probabilities given by (5.15) and (5.13), respectively.
Corollary 5.1. There exist versions \( \{ \tilde{d}^2(v_i, n(m + 1)) \}_{n \geq i}, \{ \tilde{d}^2(v_i, n(m + 1)) \}_{n \geq i}, \{ \tilde{d}(v_i, n(m + 1)) \}_{n \geq i}, \{ \tilde{d}(v_i, n(m + 1)) \}_{n \geq i}, \) of the processes \( \{ d^1(v_i, n(m + 1)) \}_{n \geq i}, \{ d^2(v_i, n(m + 1)) \}_{n \geq i}, \) and \( \{ d(v_i, n(m + 1)) \}_{n \geq i}, \) respectively, such that
\[
P[\tilde{d}^2(v_i, n(m + 1)) \leq \tilde{d}(v_i, n(m + 1)) \leq d^2(v_i, n(m + 1)), n = i, i + 1, \ldots] = 1.
\] (5.21)

Proof. It immediately follows by applying Lemma 5.2 to the processes \( \{ d^1(v_i, n(m + 1)) \}_{n \geq i}, \{ d^2(v_i, n(m + 1)) \}_{n \geq i}, \) and \( \{ d(v_i, n(m + 1)) \}_{n \geq i}. \)

Lemma 5.3. Let \( \{ \tilde{d}(v_i, n(m + 1)) \}_{n \geq i}, i > 1, \) be the process of Corollary 5.1, \( w \in \mathbb{R}^+ \) and let \( z(i, w) : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{N} \) be a function such that \( c(w) := \lim_{i \to \infty} z(i, w)/i \) exists finite, where \( c(w) : \mathbb{R}^+ \to \mathbb{R}^+ \) is an increasing function in \( w. \) Let \( b \geq 1 \) and \( w_1 < w_2 < \cdots < w_b \) be positive real numbers. Then, the random vector
\[
(\tilde{d}(v_i, i + z(i, w_1))(m + 1)), \ldots, \tilde{d}(v_i(i + z(i, w_b))(m + 1))
\]
converges in distribution to \( (N_{1/2}^{m_1}(1 + c(w_1))), \ldots, N_{1/2}^{m_b}(1 + c(w_b))) \) as \( i \to \infty. \) Here \( N_{1/2}^{m_1}(T), T \geq 0, \) is the number of individuals at time \( T \) in a Yule process with parameter \( 1/2 \) and \( m \) initial individuals.

Proof. In order to prove the convergence we make use of the processes \( \{ d^1(v_i, n(m + 1)) \}_{n \geq i}, \{ d^2(v_i, n(m + 1)) \}_{n \geq i}, \) and of their behaviour as \( i \) goes to infinity. We focus now only on the process \( \{ d^1(v_i, n(m + 1)) \}_{n \geq i}, \) as the case of \( \{ d^2(v_i, n(m + 1)) \}_{n \geq i} \) can be treated in an analogous way.

Let \( i > 1 \) be fixed, and let \( T_0^i \) be 0. For any \( x \geq 1 \) we introduce the times
\[
T_i^x = \sum_{n=i}^{i+x-1} 1/n.
\]
This way we obtain a partition of \( (0, T_i^x],
\]
\[
(0, T_i^1] \cup (T_i^1, T_i^2] \cup \cdots \cup (T_i^{x-1}, T_i^x].
\] (5.22)
The intervals of this partition have lengths \( h_n = 1/n, n = i, i + 1, \ldots, i + x - 1. \) See Figure 5.1.

We introduce the point process \( \{ \mathfrak{X}^{1, i}(T) \}_{T \geq 0}, \) jumping at times \( T_i^x, x \geq 1, \) and determined by the following principles:

1. At time \( T = 0 \) the process starts with an initial random number of individuals with distribution only asymptotically degenerate on \( m, \) i.e.
\[
P(\mathfrak{X}^{1, i}(0) \neq m) = 1 - \prod_{\ell=2}^{m+1} \left( 1 - \frac{1}{2(m\ell + \ell - 1)} \right) = O(1/i).
\] (5.23)

2. The transition probabilities of this point process coincide with \( \mathfrak{X}^{1, i}(T_{i}^x \rightarrow T_{i}^{x-1} = h_n = 1/n,\)
\[
P(\mathfrak{X}^{1, i}(T_{i}^x) = k + \ell | \mathfrak{X}^{1, i}(T_{i}^{x-1}) = k] = \begin{cases} 
\frac{k}{2} h_n + o(h_n), & \text{if } \ell = 1, \\
o(h_n), & \text{if } \ell = m, \\
1 - \frac{k}{2} h_n + o(h_n), & \text{if } \ell = 0, \\
0, & \text{otherwise.}
\end{cases}
\] (5.24)

Observe that the sample paths of \( \{ \mathfrak{X}^{1, i}(T) \}_{T \geq 0}, \) and those of \( \{ \tilde{d}(v_i, n(m + 1)) \}_{n \geq i}, \) are non-decreasing right continuous and integer-valued step functions. However
the lengths of the steps in \( \{ \tilde{d}^i(v, n(m + 1)) \}_{n \geq 1} \) always equal unity, while those of \( \{ \mathcal{X}^{1,i}(T) \}_{T \geq 0} \) admit the rational values \( h_n \).

Using the well-known relation \( \sum_{n=1}^{M} 1/n = \ln(M) + \gamma + O(1/M) \), where \( \gamma \) is the Euler–Mascheroni constant, we have that

\[
T_i^x = \ln \left( 1 + \frac{x}{(i-1)} \right) + O(1/i),
\]

so, if \( z(i, w\ell) \geq 1 \) for any \( \ell = 1, \ldots, b \),

\[
T_i^{z(i, w\ell)} = \ln \left( 1 + \frac{z(i, w\ell)}{(i-1)} \right) + O(1/i) \to \ln(1 + c(w\ell)),
\]
as \( i \to \infty \).

Analogously, we introduce the times \( T_i^y = \sum_{n=1}^{i} 1/(n + 1) \), \( y \geq 1 \), and \( T_i^0 = 0 \). We divide \( (0, T_i^y) \) into \( y \) disjoint subintervals of lengths \( h_n^y = 1/(n + 1) \),

\[
(0, T_i^y] = (0, T_i^1] \cup (T_i^1, T_i^2] \cup \ldots \cup (T_i^{y-1}, T_i^y].
\]

We introduce the point process \( \{ \mathcal{X}^{2,i}(T) \}_{T \geq 0} \), jumping at times \( T_i^y \), \( y \geq 1 \), and determined by the following properties:

1. This process starts with an initial random number of individuals such that

\[
\mathbb{P}(\mathcal{X}^{2,i}(0) \neq m) = 1 - \prod_{\ell=2}^{m+1} \left( 1 - \frac{1}{2(mi + \ell - 1)} \right) = O(1/i). \tag{5.25}
\]

2. Its transition probabilities coincide with \( (5.13) \). Hence, for each \( y \geq 1 \), \( n = (i + y - 1) \) and \( T_i^y - T_i^{y-1} = h_n^y = 1/(n + 1) \), we can write

\[
\mathbb{P}[\mathcal{X}^{2,i}(T_i^y) = k + \ell | \mathcal{X}^{2,i}(T_i^{y-1}) = k] = \begin{cases} \frac{\ell}{2} h_n^y + o(h_n^y), & \ell = 1, \\
o(h_n^y), & \ell = 2, \\
1 - \frac{\ell}{2} h_n^y + o(h_n^y), & \ell = 0, \\
0, & \text{otherwise.} \end{cases} \tag{5.26}
\]
Then we get that
\[ T^y_i = \ln \left( 1 + \frac{\ell}{\ell} \right) + O(1/i). \]
Therefore, if \( z(i, w) \geq 1 \) for any \( \ell = 1, \ldots, b \),
\[ T^{z(i, w)}_i = \ln \left( 1 + \frac{z(i, w)}{\ell} \right) + O(1/i) \to \ln(1 + c(w)), \]
as \( i \to \infty \).

Note that \( \mathfrak{X}^{1,i}(T^y_i) \) and \( \mathfrak{X}^{2,i}(T^y_i) \) have the same law and initial condition as \( d^1(v, (i + x)(m + 1)) \) and \( d^2(v, (i + y)(m + 1)) \), \( x, y \geq 0 \), respectively. By \((5.24)\) and \((5.25)\), these processes start with \( m \) initial individuals, as \( i \to \infty \). Moreover, as \( i \to \infty \), \( h_n \) and \( h^*_n \) decrease to zero as infinitesimals of the same order. We emphasize that both \( T^{y(i, w)}_i \) and \( T^{z(i, w)}_i \) converge to the same time \( T(\ell) = \ln(1 + c(w)), \) \( \ell = 1, \ldots, b \), and \((5.24)\) and \((5.26)\) can be read as the infinitesimal transition probabilities of a Yule process (see \((2.2)\)) with intensity \( 1/2 \). Since the transition probabilities and the initial condition determine uniquely the finite-dimensional distributions of a Markov process, we obtain the convergence of the random vectors
\[ \left( \mathfrak{X}^{1,i}(T^{y(i, w)}_i), \ldots, \mathfrak{X}^{1,i}(T^{y(i, w)}_i) \right) \to \left( N_{1/2}^m(T(1)), \ldots, N_{1/2}^m(T(b)) \right), \]
and
\[ \left( \mathfrak{X}^{2,i}(T^{y(i, w)}_i), \ldots, \mathfrak{X}^{2,i}(T^{y(i, w)}_i) \right) \to \left( N_{1/2}^m(T(1)), \ldots, N_{1/2}^m(T(b)) \right), \]
in distribution, as \( i \to \infty \), where \( N_{1/2}^m(T) \) is the number of individuals of a Yule process at time \( T \) with intensity \( 1/2 \) and initial population size equal to \( m \). Consequently, we have that
\[ (d^1(v, (i + z(i, w))(m + 1)), \ell = 1, \ldots, b) \to \left( N_{1/2}^m(T(\ell)), \ell = 1, \ldots, b \right), \quad (5.27) \]
and
\[ (d^2(v, (i + z(i, w))(m + 1)), \ell = 1, \ldots, b) \to \left( N_{1/2}^m(T(\ell)), \ell = 1, \ldots, b \right), \quad (5.28) \]
in distribution, as \( i \to \infty \).

Observe that at time \( n(m + 1), n \geq i \), by \((5.21)\) the random variables \( d^1(v, n(m + 1)) \) and \( d^2(v, n(m + 1)) \) are almost surely ordered, that is
\[ \mathbb{P}(d^2(v, (i + z(i, w))(m + 1)) \leq d^1(v, (i + z(i, w))(m + 1)) \leq d^2(v, (i + z(i, w))(m + 1)), \ell = 1, \ldots, b) = 1. \]

This implies that for any \( k \geq m \),
\[ \mathbb{P}(\widetilde{d}(v, (i + z(i, w))(m + 1)) \leq k_\ell, \ell = 1, \ldots, b) \leq \mathbb{P}(\widetilde{d}(v, (i + z(i, w))(m + 1)) \leq k_\ell, \ell = 1, \ldots, b) \leq \mathbb{P}(\widetilde{d}(v, (i + z(i, w))(m + 1)) \leq k_\ell, \ell = 1, \ldots, b). \]

Thus, from \((5.27), (5.28)\) and \((5.30)\) we obtain the convergence of the random vector
\[ (\widetilde{d}(v, (i + z(i, w))(m + 1)), \ell = 1, \ldots, b) \to \left( N_{1/2}^m(T(\ell)), \ell = 1, \ldots, b \right), \quad (5.31) \]
in distribution, as \( i \to \infty \).
Proof of Theorem 4.1 Using Lemma 5.1 Corollary 5.1 and Lemma 5.3 we obtain the convergence to the $b$-finite-dimensional distributions of a Yule process, for all $b \geq 1$. To obtain the exact formula we make use of the independence of the increments and of the distribution of the number of individuals in a Yule process with $k_\ell$ initial progenitors, $\ell = 0, \ldots, b$. Thus,

$$
\mathbb{P}[N_{1/2}^m(1 + c(w_1))] = k_1, \ldots, N_{1/2}^m(1 + c(w_b))] = k_b
$$

$$
= \prod_{\ell=1}^b \mathbb{P}
\left(N_{1/2}^{k_{\ell-1}} \left( \ln \left( \frac{1 + c(w_\ell)}{1 + c(w_{\ell-1})} \right) \right) = k_\ell \right).
$$

Finally, we use equation (3.5) in ([15], Section XVII.3).

5.2 A Lemma and the Proof of Theorem 4.2

In this section we make use of the planted model described in Section 3. Remember that the BA random graph model corresponds to the case in which all the random variables $M_a$, $a \geq 1$, are concentrated on $m$, so that $\mathbb{E}_a = n(m+1)$ almost surely.

Formally, let $(G'_m)_{m \geq 1}$ be the random graph process defining the BA model as in subsection 2.1. For each $1 \leq j \leq i$ consider the birth processes in discrete time $(b(v_j, n(m+1)))_{n \geq 1}$, with state space given by $\mathbb{N}$ and determined by the transition probabilities

$$
\mathbb{P}[b(v_j, (n+1)(m+1)) = k + \ell | b(v_j, n(m+1)) = k] = \begin{cases} k/n, & \ell = 1, \\ 0, & \text{otherwise}, \end{cases}
$$

and initial condition $b(v_j, i(m+1)) = 1$ almost surely.

Lemma 5.4. Let $z(i, w) : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ be a function such that

$$
c(w) := \lim_{i \to \infty} z(i, w)/i
$$

exists finite, where $c(w) : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function in $w$, and let $w_1 < \cdots < w_b$, $b \in \mathbb{N}$, be positive real numbers. For any $1 \leq j \leq i$ we have

$$
(b(v_j, (i + z(i, w_\ell))(m+1)), \ell = 1, \ldots, b) \to (N_1^i(1 + c(w_\ell))), \ell = 1, \ldots, b)
$$

in distribution as $i \to \infty$, where $N_1^i(T)$ is the number of individuals of a Yule process at time $T$, with one initial individual and parameter $1$.

Proof. For each $1 \leq j \leq i$, we consider a convergence in distribution in the same way as we did in the proof of Lemma 5.3 for $\mathcal{R}_1^{i\mathbb{N}}(T^\mathbb{N})$, but now with $\mathcal{R}_i(0) = 1$, i.e., the process starts with only one individual, and transition probabilities given by (5.32). Thus, as $i \to \infty$, the probabilities (5.32) become the infinitesimal transition probabilities of a Yule process with intensity 1, starting with one individual. Since the process is Markov, the transition probabilities and the initial condition determine uniquely the finite-dimensional distributions.

Proof of Theorem 4.2 We start by proving (1.3). Consider the BA model at time $t = n(m+1)$, $n \geq i$, $i > 1$, and the planted model of section 3. Recall that in the planted model we have $i$ discrete-time birth processes $(b(v_j, n(m+1)))_{n \geq 1}$, $j = 1, \ldots, i$, which are exchangeable. By Theorem 5.1 the event of choosing a vertex uniformly at random in the BA model is equivalent to that of selecting
first uniformly at random one of the \( i \) processes \( \{ b(v_j, n(m + 1)) \}_{n \geq i}, j = 1, \ldots, i \), and then choosing uniformly at random a vertex belonging to it. Thus, the degree of \( V_t^i \) can be studied through the analysis of the degree of a random vertex chosen with uniform probability between the vertices in any of the \( i \) processes \( \{ b(v_j, n(m + 1)) \}_{n \geq i}, j = 1, \ldots, i \). Let \( V_t^j \) be a vertex chosen uniformly at random from the vertices in the \( j \)-th process \( \{ b(v_j, n(m + 1)) \}_{n \geq i} \), and let \( \epsilon(i, n) \) be a function we will use to measure the error. Using the notation of section 3.1, where \( W \) denotes the index of the birth process chosen, \( Y_j \) is a random variable taking values in \( \{ 1, 2, \ldots, n - i + 1 \} \) which denotes the number of vertices in \( b(v_j, n(m + 1)) \), we have

\[
P[d(V_t) = k] = \sum_{j=1}^{i} \mathbb{P}[d(V_{t,t}^j) = k, V_{t,t}^j \neq v_j, W = j] \]

\[
+ \sum_{j=1}^{i} \mathbb{P}[d(V_{t,t}^j) = k, V_{t,t}^j = v_j, W = j]
\]

\[
= \mathbb{P}[d(V_{t,t}^1) = k \mid V_{t,t}^1 \neq v_1, W = 1] \sum_{j=1}^{i} \mathbb{P}[V_{t,t}^j \neq v_j, W = j]
\]

\[
+ \sum_{j=1}^{i} \mathbb{P}[d(V_{t,t}^j) = k \mid V_{t,t}^j = v_j, W = j] \mathbb{P}[V_{t,t}^j = v_j, W = j] \]

\[
= \mathbb{P}[d(V_{t,t}^1) = k \mid V_{t,t}^1 \neq v_1, W = 1] + \epsilon(i, n).
\]

The last two equalities are obtained by considering the following two observations. First, permuting the labels of the \( i \) birth processes \( \{ b(v_j, n(m + 1)) \}_{n \geq i}, j = 1, \ldots, i \), will not change the distribution of the process of the new vertices and their degrees, thus for any \( j = 1, \ldots, i \), we can write \( \mathbb{P}[d(V_{t,t}^j) = k \mid V_{t,t}^j \neq v_j, W = j] = \mathbb{P}[d(V_{t,t}^1) = k \mid V_{t,t}^1 \neq v_1, W = 1] \). Second,

\[
\sum_{j=1}^{i} \mathbb{P}[d(V_{t,t}^j) = k \mid V_{t,t}^j = v_j, W = j] \mathbb{P}[V_{t,t}^j = v_j, W = j] \leq \sum_{j=1}^{i} \mathbb{P}[V_{t,t}^j = v_j, W = j]
\]

\[
= \sum_{j=1}^{i} \sum_{\ell=1}^{n-i+1} \frac{1}{\ell n} \mathbb{P}(Y_j = \ell)
\]

\[
= \frac{i}{n},
\]

that is, \( \epsilon(i, n) = O(i/n) \).

Note that the degree of the planted vertices behaves differently as they have appeared in the very early history of the graph evolution. Also, in the limit, the number of planted vertices becomes negligible compared to the total size of the graph.

Now take \( n(i, w) = i + z(i, w) \), where \( z(i, w) \) is defined in Lemma 5.3. Lemma 5.4 and Theorem 4.1. As \( i \to \infty \),

- by Lemma 5.4 we have that \( b(v_1, n(m + 1)) \), converges in distribution to the size of a Yule process evaluated at time \( T = \ln(1 + c(w)) \), with intensity 1 and starting with one initial individual;
- by Lemma 5.3 the degree of each vertex belonging to \( \{ b(v_1, n(m + 1)) \} \), given that it is different to \( v_1 \), converges in distribution to the size of a Yule process with intensity 1/2 and \( m \) initial individuals.
The above Yule processes describe an \( m \)-Yule model \( \{Y_{m/2,1}^m(T)\}_{T \geq 0} \) of parameters \( \lambda = 1/2 \) and \( \beta = 1 \). For \( i \to \infty \), the degree of \( V_i^1 \) given that \( V_i^1 \neq v_1 \), converges in distribution to the size of a genus chosen uniformly at random in the \( m \)-Yule model at time \( T = \ln(1 + c(w)) \), given in turn that such a random genus is different to the first genus appeared, \( g_1 \). Thus, if \( N_T^m \) denotes the size of a genus \( G_T \) chosen uniformly at random at time \( T \) in \( \{Y_{m/2,1}^m(T)\} \),

\[
\lim_{i \to \infty} \mathbb{P}(d(V_i^1) = k \mid V_i^1 \neq v_1, W = 1) = \mathbb{P}(N_T^m = k \mid G_{\ln(1 + c(w))} \neq g_1).
\]

By (5.34) and (5.35)

\[
\lim_{i \to \infty} \mathbb{P}(d(V_i^1) = k) = \mathbb{P}(N_T^m = k \mid G_{\ln(1 + c(w))} \neq g_1) + \varepsilon(w),
\]

(5.36)

where \( \varepsilon(w) = O(1/(1 + c(w))) \). Since \( c(w) \) is an increasing function and a Yule process is supercritical, then

\[
\lim_{w \to \infty} \mathbb{P}(N_T^m = k) = \lim_{w \to \infty} \mathbb{P}(N_T^m = k \mid G_{\ln(1 + c(w))} \neq g_1).
\]

(5.37)

Therefore, by (5.36) and (5.37),

\[
\lim_{w \to \infty} \lim_{i \to \infty} \mathbb{P}(d(V_i^1) = k) = \lim_{w \to \infty} \mathbb{P}(N_T^m = k \mid G_{\ln(1 + c(w))} = k).
\]

(5.38)

To prove 4.1, note that

\[
\mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(d(v_i, t) = k) \right) = \frac{\mathbb{E} N_{k,t}}{n} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(d(v_i, t) = k) = \mathbb{P}(V_i^1 = k).
\]

Let \( \mathcal{F}_t \) be the natural filtration generated by the process \( \{N_{k,t}\}_{t \geq 1} \) up to time \( t \), and define \( Z_s = \mathbb{E}(N_{k,t} \mid \mathcal{F}_s) \). Observe that \( Z_s \) is a martingale as \( \mathbb{E}[\mathbb{E}(N_{k,t} \mid \mathcal{F}_s) \mid \mathcal{F}_r] = \mathbb{E}(N_{k,t} \mid \mathcal{F}_r) \), for \( r \leq s \leq t \). Considering that at each time interval \( (s - 1, s] \) a new vertex \( v_s \) appears and \( m \) directed edges from it are attached to existing vertices, then \( v_s \) is attached to at most \( m \) different vertices, say \( v_1, \ldots, v_m \). This does not affect the degree of the other existing vertices \( w \neq v_1, \ldots, v_m \), neither the attachment probabilities related to them. Thus, it follows that \( |Z_s - Z_{s-1}| \leq 2m \).

Since \( Z_0 = N_{k,t} \) and \( Z_0 = \mathbb{E} N_{k,t} \), then by taking \( x = C \sqrt{T \ln T} \), with \( C > m/\sqrt{8} \) and applying the Azuma–Hoeffding inequality (see Lemma 4.1.3 in [13]), we obtain

\[
\mathbb{P} \left( \left| \frac{N_{k,t}}{n} - \frac{\mathbb{E} N_{k,t}}{n} \right| > C \sqrt{\frac{(m + 1) \ln(n(m + 1))}{n}} \right) \leq o \left( \frac{1}{n} \right).
\]

(5.39)

Now observe that \( N_{k,t} = 0 \) when \( k \geq n(m + 1), n \geq 1 \). Therefore,

\[
\mathbb{P} \left( \max_{k \leq n(m+1)} \left| \frac{N_{k,t}}{n} - \frac{\mathbb{E} N_{k,t}}{n} \right| > C \sqrt{\frac{(m + 1) \ln(n(m + 1))}{n}} \right)
\]

\[
= \sum_{k=1}^{n(m+1)-1} \mathbb{P} \left( \max_{k < n(m+1)} \left| \frac{N_{k,t}}{n} - \frac{\mathbb{E} N_{k,t}}{n} \right| > C \sqrt{\frac{(m + 1) \ln(n(m + 1))}{n}} \right)
\]

Thus, by (5.39) we get the desired result. \( \square \)
5.3 Proof of Proposition 4.1

Proof. Let us consider an \( m \)-Yule model \( \{Y_{1/2}(T)\}_{T \geq 0} \). It is known that by conditioning on the number of genera present at time \( T \), the random times at which novel genera appear are distributed as the order statistics of i.i.d. random variables distributed with distribution function given by (see e.g. [24] or [25] and the references therein)

\[
P(\mathcal{T} \leq \tau) = \frac{e^{\tau - 1} - 1}{e^\tau - 1}, \quad 0 \leq \tau \leq T.
\] (5.40)

As above, let \( \mathcal{N}_T^m \) denote the size of a genus chosen uniformly at random at time \( T \). Then, for any \( k \geq m \) and recalling the distribution of a Yule process starting with \( m \) initial individuals,

\[
P(\mathcal{N}_T^m = k) = \int_0^T P(\mathcal{N}_{1/2}(T) = k | \mathcal{N}_{1/2}^m(\tau) = m) P(\mathcal{T} \in d\tau)
\] (5.41)

\[
= \int_0^T \binom{k-1}{m-1} e^{-m} \frac{\tau^m}{m} (1 - e^{-\frac{T-\tau}{T}})^{k-m} \frac{e^\tau}{e^T - 1} d\tau
\]

\[
= \frac{1}{1 - e^{-T}} \int_0^T \binom{k-1}{m-1} e^{-y} e^{-m} \frac{y^m}{m} (1 - e^{-y})^{k-m} dy.
\] (5.42)

By letting \( z = 1 - e^{-y} \), we can write (5.41) as

\[
P(\mathcal{N}_T^m = k) = \frac{2}{1 - e^{-T}} \int_0^{1 - e^{-T}} \binom{k-1}{m-1} z^{k-m} (1 - z)^{m+1} dz.
\] (5.42)

Our interest is in the asymptotic behaviour when \( T \to \infty \). In this case (5.42) reduces to

\[
\lim_{T \to \infty} P(\mathcal{N}_T^m = k) = 2 \binom{k-1}{m-1} B(k-m+1, m+2)
\] (5.43)

\[
= m(m+1)B(k, 3),
\]

where \( B(a, b) \) denotes the Beta function. \( \square \)

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