GENERAL-RELATIVISTIC MODEL OF MAGNETICALLY DRIVEN JET

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1 Introduction

Powerful bipolar outflows appear to be common features of the wide class of astrophysical objects including protostars, young main-sequence stars, galactic X-ray sources, Active Galactic Nuclei (AGN), and quasars [1]. In the majority of these cases there exists more or less convincing observational evidence about the presence of large-scale magnetic fields in these objects. Ordered magnetic fields should have the crucial role in giving rise to such outflows or jets. Namely, they may be responsible for the collimation of the jets and/or acceleration of matter up to relativistic velocities in them [2].

High compactness and huge energy output, which characterizes some (maybe, the most interesting) concrete kinds of objects featuring bipolar (or, unipolar) outflows (such as AGNs and quasars), may be explained if one assumes that they are produced in result of accretion onto the rotating supermassive black hole. It seems evident that a consideration of the innermost part of such outflows, in the close neighbourhood of the black hole must be performed in the framework of general-relativistic magnetohydrodynamics (MHD). Such treatment enables us to take into account properly the influence of strong gravitational and electromagnetic fields onto the structure of the jet. Since 3 + 1 formulation of black hole electrodynamics [3–5] is the most convenient mathematical apparatus for such purposes, in the present study we shall establish our consideration on it.

It must be noted from the very beginning that the method, which is developed in this paper is the generalization of the advanced methodology used in the number of recent references for non-relativistic [6–9], and for special-relativistic [10–11] magnetically driven jets and winds. By means of the method we derive the set of equations for general-relativistic jets from the equations of the ideal 3 + 1 MHD. Under certain simplifying assumptions we obtain the simple, representative solution of these equations. The features of the solution are discussed and compared with the one, which has been found in [8].
2 Governing equations

In the forthcoming analysis we shall use the following notations: (a) greek indices will range over $t, r, \theta, \phi$ and represent space-time coordinates, components, etc.; (b) Latin indices will range over $r, \theta, \phi$ and represent coordinates in three-dimensional ”absolute” space. We will approve the spacelike signature convention ($- + + +$) and units in which the gravitational constant and the speed of light are equal to one.

The rotation of a central object (i.e. a rapidly rotating Kerr black hole) introduces off-diagonal terms $g_{t\phi}$ in the metric so that the space-time generated by the rotating body is represented by the metric:

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2,$$

(1)

with the metric coefficients independent of $t$ and $\phi$.

In 3+1 notations (1) may be rewritten as [3–5]:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ik}(dx^i + \beta^i dt)(dx^k + \beta^k dt),$$

(2)

where $\alpha$ is the so-called lapse function defined as:

$$\alpha^2 \equiv \frac{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}{g_{\phi\phi}},$$

(3)

$\gamma_{ik}$ is the three-dimensional “absolute” space metric tensor (with diagonal nonzero components $g_{ii} \equiv \gamma_{ii}$) and $\vec{\beta}$ is the spatial (three-dimensional) vector with components

$$\beta^i \equiv \left(0, 0, \frac{g_{t\phi}}{g_{\phi\phi}}\right); \quad \beta_i = \gamma_{ik}\beta^k.$$

(4)

Note that the Kerr metric is the subclass of the general metric (1). In particular,

$$g_{tt} = -\left(1 - \frac{2Mr}{\Sigma}\right),$$

(5)

$$g_{t\phi} = -\frac{2aMr\sin^2\theta}{\Sigma},$$

(6)

$$g_{\phi\phi} = \frac{A\sin^2\theta}{\Sigma},$$

(7)

$$g_{rr} = \frac{\Sigma}{\Delta},$$

(8)
\[ g_{\theta\theta} = \Sigma; \] (9)

where \( a \equiv J/M \) is the specific angular momentum of the black hole per its unit mass, and

\[ \Sigma \equiv r^2 + a^2 \cos^2 \theta, \] (10)
\[ \Delta \equiv r^2 - 2Mr + a^2, \] (11)
\[ A \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta. \] (12)

In the present study we consider jets as being axisymmetric \((\partial_\phi = 0)\) and stationary \((\partial_t = 0)\). We, also, neglect dissipative effects. The basic equations are those of the general-relativistic MHD written in 3 + 1 formalism:

\[ \nabla [\Gamma_{\alpha \beta} (\alpha \vec{v} - \vec{\beta})] = 0, \] (13)
\[ \nabla \times (\alpha \vec{B}) = 4\pi \alpha \vec{J} - \mathcal{L}_{\beta} \vec{E}, \] (14)
\[ \vec{E} + \vec{v} \times \vec{B} = 0, \] (15)
\[ \nabla \times (\alpha \vec{E}) = \mathcal{L}_{\beta} \vec{B}, \] (16)
\[ \nabla \vec{B} = 0, \] (17)

\[ \frac{d\varepsilon}{d\tau} = -\frac{1}{\alpha^2} \nabla (\alpha^2 \vec{S}) - \sigma_{ik} T^{ik}; \] (18)
\[ \frac{d\vec{S}}{d\tau} = \varepsilon \vec{g} + \mathbf{H} \cdot \vec{S} - \frac{1}{\alpha} \nabla (\alpha \mathbf{T}). \] (19)

In these equations we use the following notations:

\[ \Gamma = [1 - (\vec{v} \cdot \vec{v})]^{-1/2}, \] (20)
\[ \frac{d}{d\tau} \equiv \frac{1}{\alpha} [\partial_t - (\vec{\beta} \cdot \nabla)], \] (21)
\[ \vec{g} \equiv -\frac{1}{\alpha} \nabla (\alpha), \] (22)
\[ H_{ik} \equiv \frac{1}{\alpha} \beta_{k;i}, \] (23)
\[ \sigma_{ik} \equiv -\frac{1}{2} (H_{ki} + H_{ik}); \] (24)

and all vector and tensor quantities are defined in the three-dimensional “absolute” space with the metric \( \gamma_{ik} \).
Note also that in these equations \( n \) is the proper baryon number density, while \( \vec{E}, \vec{B}, \) and \( \vec{J} \) are the vectors of the electric field, the magnetic field and the current density, respectively. The “Lie derivative” of a vector \( \vec{A} \) along the vector \( \vec{\beta} \) is defined in the following way:

\[
\mathcal{L}_\beta \vec{A} \equiv (\vec{A}, \nabla) \vec{\beta} - (\vec{\beta}, \nabla) \vec{A},
\]

and \( \vec{\nu} \) is 3-velocity, related to the spatial components of matter 4-velocity \( u^\alpha \equiv (u^t, u^i) \) via the expressions:

\[
v^i = \frac{1}{\alpha} \left( \frac{U^i}{U^t} + \beta^i \right).
\]

In (18) \( \varepsilon \) is the total energy density, defined as [5]:

\[
\varepsilon \equiv \varepsilon_p + \varepsilon_f,
\]

\[
\varepsilon_p \equiv (mn + Pv^2)\Gamma^2,
\]

\[
\varepsilon_f \equiv \frac{1}{8\pi}(E^2 + B^2);
\]

\( \vec{S} \) is the total momentum density, defined as:

\[
\vec{S} \equiv \vec{S}_p + \vec{S}_f,
\]

\[
\vec{S}_p \equiv (mn + P)\Gamma^2 \vec{\nu},
\]

\[
\vec{S}_f \equiv \frac{1}{4\pi}(\vec{E} \times \vec{B}),
\]

and \( T_{ik} \) is defined as

\[
T_{ik} \equiv T_{ik}^p + T_{ik}^f,
\]

\[
T_{ik}^p \equiv (mn + P)\Gamma^2 v_i v_k + Pg_{ik},
\]

\[
T_{ik}^f \equiv \frac{1}{4\pi} \left[ -(E_i E_k + B_i B_k) + \frac{1}{2}(E^2 + B^2)g_{ik} \right].
\]

Note that the roots of the determinants of \(||g_{\alpha\beta}||\) and \(||\gamma_{ik}||\) are equal to:

\[
\sqrt{g} \equiv (det(g_{\alpha\beta}))^{1/2} = \Sigma \sin \theta,
\]

\[
\sqrt{\gamma} \equiv (det(\gamma_{ik}))^{1/2} = \Sigma \sin \theta / \alpha.
\]
3 Main consideration

Let \( \bar{\theta}(r) \) denote the jet outer boundary and

\[
\epsilon \equiv \left( \frac{\theta}{\bar{\theta}} \right)^2,
\]

be the dimensionless angular variable. From now on, we shall consider “narrow jet” i.e., \( \bar{\theta}(r) \) is assumed to be as small that \( \sin \theta \approx \theta \) and \( \cos \theta \approx 1 \). Surely, the same is true for the angular coordinate \( \theta < \bar{\theta}(r) \) inside the jet.

The assumption noticeably simplifies the angular dependence of various quantities appearing in the theory. For example, from (37) we get:

\[
\sqrt{\gamma} \approx \frac{(r^2 + a^2)^{3/2}}{\Delta^{1/2}} \theta,
\]

and owing to the same assumption:

\[
\alpha \approx \left( \frac{\Delta}{r^2 + a^2} \right)^{1/2},
\]

\[
\beta^\theta \approx -\frac{2aMr}{(r^2 + a^2)^2}.
\]

Let us introduce, now, the magnetic flux function \( \psi(r, \theta) \) [8], defined as to have:

\[
B^r = \frac{1}{\sqrt{\gamma}} \partial_\theta \psi(\epsilon),
\]

\[
B^\theta = -\frac{1}{\sqrt{\gamma}} \partial_r \psi(\epsilon),
\]

then it is easy to check out that the ”no monopoles” condition (17) is satisfied automatically, and also

\[
B^\theta = \left[ -\frac{\partial \epsilon}{\partial r} / \frac{\partial \epsilon}{\partial \theta} \right] B^r = \sqrt{\epsilon} \frac{\partial \bar{\theta}(r)}{\partial r} B^r.
\]

Inserting (44) into the (17) and resolving it for \( B^r \) we get:

\[
\bar{B}_r \equiv \sqrt{g_{rr}} B^r = \frac{\Phi}{(r^2 + a^2) \bar{\theta}(r)},
\]

where \( \bar{B}_r \) is the “physical” radial component of the magnetic field vector, while the \( \Phi \) is some constant of integration.
Equations (15) and (16) may be combined to give the so called “nduction equation”:
\[
rot[(\alpha \vec{v} - \vec{\beta}) \times \vec{B}] = 0.
\] (46)

Poloidal components of this equation lead to the simple relation between poloidal components of \(\vec{v}\) and \(\vec{B}\):
\[
v^r B^\theta = v^\theta B^r,\]
(47)
which, together with (44) leads to
\[
v^\theta = \sqrt{\epsilon} \frac{\partial \bar{\theta}(r)}{\partial r} v^r.
\] (48)

The toroidal component of (46), after taking into account of the (17) may be written as
\[
\partial_r [\alpha \sqrt{\gamma} v^r B^\phi] + \partial_\theta [\alpha \sqrt{\gamma} v^\theta B^\phi] = \sqrt{\gamma} B^r \frac{\partial \Omega}{\partial r} + \sqrt{\gamma} B^\theta \frac{\partial \Omega}{\partial \theta}.
\] (49)

In this paper we assume that \(\Omega = \Omega(r)\). It allows us to neglect the last term on the right in (49). Remained equation may be solved separately as inside \([at \theta < \bar{\theta}(r)]\) as outside the jet \([when \theta > \bar{\theta}(r)]\). In such a way we get:
\[
B^\phi(r) = \frac{\Phi(\Omega - \Omega_0)}{(r^2 + a^2) v^r \theta^2(r)}, \quad \theta \leq \bar{\theta},\]
(50a)
\[
B^\phi(r) = \frac{\Phi(\Omega - \Omega_0)}{(r^2 + a^2) v^r \theta^2(r)}, \quad \theta > \bar{\theta}.
\] (50b)

Let us consider, now, the momentum conservation equation (18). For stationary case it may be rewritten as:
\[
-\frac{1}{\alpha}(\vec{\beta} \cdot \nabla) \vec{S} = \epsilon \vec{g} + \vec{H} \cdot \vec{S} - \frac{1}{\alpha} \nabla (\alpha \tilde{T}).
\] (51)

For the \(\phi\)-component of this equation taking into account that \(g_{\phi} = 0\) and
\[
\frac{1}{\alpha} \beta^k S_{\phi;k} = H_{\phi k} S^k,
\] (52)
we get simple equation:
\[
(\alpha \sqrt{\gamma} T^k_{\phi}),_k = 0.
\] (53)
For the poloidal components of the same equation, taking into account that

\[ \beta^k S_{i;k} = H_{\phi i} S^\phi \quad i = r, \theta, \]  

we derive the following equation

\[ \varepsilon g_i + (H_{\phi i} + H_{i\phi}) S^\phi - \frac{1}{\alpha \sqrt{\gamma}} [\alpha \sqrt{\gamma} T^k_i],k + \frac{1}{2} T^{mn} g_{mn,i} = 0. \]  

(55)

Note that for the narrow jet:

\[ g_r = -\frac{M(r^2 - a^2)}{(r^2 + a^2) \Delta}, \]  

(56a)

\[ g_\theta = \frac{2Mr^2a^2 \theta}{(r^2 + a^2)^2}. \]  

(56b)

It is also easy to prove that

\[ (H_{\phi i} + H_{i\phi}) S^\phi = \frac{1}{\alpha} S_{\phi i} \beta^\phi, r. \]  

(57)

Energy conservation equation in stationary the case may be written simply as

\[ \nabla (\alpha^2 \tilde{S}) = -\alpha^2 \sigma_{ik} T^{ik}. \]  

(58)

If we consider that

\[ \sigma_{ik} T^{ik} = -\frac{1}{\alpha} T_i^j \beta^\phi, i \approx -\frac{1}{\alpha} T_i^r \beta^\phi, r, \]  

then we can rewrite (58) in the following form:

\[ \nabla (\alpha^2 \tilde{S}) = \alpha T_r^r \beta^\phi. \]  

(60)

It must be noted that our equations contain as “special-relativistic” effects (Lorentz factors and all that) as, also, purely gravitational effects related to the curving of absolute space (\(\alpha \neq 1\)) and the “frame dragging” (\(\beta \neq 0\)). In the present paper, in purpose to simplify the consideration, we shall assume that in the innermost region of the jet, where gravitational effects are perceptible, matter moves with non-relativistic velocity (i.e., \(v < 1\) and \(\Gamma \approx 1\)). Thus we shall deal with the general-relativistic but slow (“non-relativistic” in the sense of special relativity) jet. Hereafter, we shall
need to integrate some of our equations over \( r = \text{const} \) surfaces. According to the general theory element of such surface is:

\[
d^2\vec{x} = \sqrt{g_{\phi\phi} g_{\theta\theta}} d\phi d\theta = \alpha \sqrt{\gamma} d\phi d\theta \approx (r^2 + a^2) \theta d\phi d\theta. \tag{61}
\]

First of all, let us integrate continuity equation:

\[
\partial_r [\alpha \sqrt{\gamma} \Gamma mnv^r] + \partial_\theta [\alpha \sqrt{\gamma} \Gamma mn v^\theta] = 0. \tag{62}
\]

Remembering that \( \Gamma \approx 1 \) and integrating (62) by \( d^2\vec{x} \) we get the following equation:

\[
\dot{\tilde{M}} \equiv 2\pi \int_0^\theta \alpha \sqrt{\gamma} mn v^r d\theta = \pi (r^2 + a^2) \tilde{\theta}^2 \tilde{m} \tilde{v}^r = \text{const}. \tag{63}
\]

Note that deriving (63) we made the following assumption about the angular dependence of \( n(r, \theta) \):

\[
n(r, \theta) = \bar{n}(r) f(\epsilon), \tag{64}
\]

where \( f(\epsilon) > 0 \) is the dimensionless number-density “profile-function” normalized in such a way as to get:

\[
\int_0^1 f(\epsilon) d\epsilon = 1. \tag{65}
\]

Similarly, under same assumptions, we can integrate toroidal component of momentum conservation equation (53). We assume that the jet matter may be treated as the medium with ultrarelativistic temperature

\[
P = nK\tilde{T}, \tag{66}
\]

\[
e = mn + (\tilde{\gamma} - 1)^{-1} P. \tag{67}
\]

where \( \tilde{\gamma} = 5/3 \). In this case we have:

\[
P + \varepsilon = mn \left( 1 + \frac{5K\tilde{T}}{2\mu} \right). \tag{68}
\]

Taking into account (68), after integration of (53) we get:

\[
\frac{\dot{M}_\xi}{2\pi\alpha} \left( 1 + \frac{5K\tilde{T}}{2\mu} \right) (r^2 + a^2) (\beta^\phi + \Omega) \tilde{\theta}^2(r) - \frac{\Phi^2 \alpha (\Omega - \Omega_0)}{16\pi v^r} = L, \tag{69}
\]
where $L$ is some constant of integration and $\xi$ is defined as:

$$\xi \equiv \int_0^1 \epsilon f(\epsilon) d\epsilon. \quad (70)$$

Integration of the energy conservation equation also leads to another algebraic equation of the following form:

$$\frac{\alpha \dot{M}}{2\pi} \left(1 + \frac{5KT}{2\mu}\right) - \frac{\alpha \Phi^2(\Omega - \Omega_0)(\Omega_0 - \omega)}{16\pi v^r} = L(\bar{\omega} - \omega), \quad (71)$$

where $\omega \equiv -\beta \phi$ and $\bar{\omega}$ is the another constant of integration.

If we assume that $5KT/2\mu \ll 1$, then (69) can be rewritten as:

$$\Omega = \frac{\alpha \dot{M} \xi(r^2 + a^2)\bar{\theta}^2 \omega + L}{\frac{\alpha \dot{M}}{2\pi} (r^2 + a^2)\bar{\theta}^2(r)v^r - \frac{\alpha \Phi^2}{16\pi}}. \quad (72)$$

For the physical jet solution, the numerator and denominator of (72) must vanish simultaneously at a distance $r = r_A$, termed the Alfven point of the flow. At this point:

$$v_A^r = \frac{\alpha A^2 \Phi^2}{8\dot{M} \xi (r_A^2 + a^2)\bar{\theta}^2_A}, \quad (73)$$

$$L = \frac{\dot{M} m}{2\pi \alpha_A} (r_A^2 + a^2)\bar{\theta}^2_A (\Omega_0 - \omega_A) = \frac{\alpha A \Phi^2 (\Omega_0 - \omega_A)}{16\pi v_A^r}. \quad (74)$$

If we neglect similarly term containing temperature in (71), then it may be soled together with (72) for the angular velocity. In such a way we get:

$$\Omega = \omega + \frac{\alpha[\alpha + (2\pi L/\dot{M})(\Omega_0 - \bar{\omega})]}{\xi (r^2 + a^2)\bar{\theta}^2(\Omega_0 - \omega)}. \quad (75)$$

Boundary condition $\Omega(r_0) = \Omega_0$ implies that

$$\frac{2\pi L}{\dot{M}}(\Omega_0 - \bar{\omega}) = -\alpha_0 + \frac{\xi (r_0^2 + a^2)\bar{\theta}_0^2(\Omega_0 - \omega_0)^2}{\alpha_0},$$

and (75) may be rewritten in the following form:

$$\Omega = \omega + \frac{\alpha}{(r^2 + a^2)(\Omega_0 - \omega)} \left[\frac{\alpha - \alpha_0}{\xi \bar{\theta}^2} + \frac{(r_0^2 + a^2)(\Omega_0 - \omega_0)^2}{\alpha_0} \left(\frac{\bar{\theta}_0}{\bar{\theta}}\right)^2\right]. \quad (76)$$
Knowing explicit analytical expression for $\Omega(r)$ we can calculate all other physical quantities connected with it. In particular, we can get for toroidal magnetic field:

$$B^\phi = \left(\frac{8\dot{M}\xi}{\Phi}\right) \frac{1}{\alpha (r^2 + a^2) \bar{\theta}^2} \left[ \frac{(r^2 + a^2) \bar{\theta}^2 (\Omega - \omega)}{\alpha} - \left(1 - \frac{\alpha_0^2 \epsilon}{\xi} \right) \frac{(r_0^2 + a^2) \bar{\theta}_0^2 (\Omega_0 - \omega_0)}{\alpha_0} \right] \quad (77)$$

and then, we can find the expression for radial velocity $v^r$ all other jet physical variables.

It must be emphasized that solutions contain unknown ”jet profile” function $\bar{\theta}(r)$. It may be calculated from the radial component of momentum conservation equation, since it leads to the first order differential equation for $\bar{\theta}(r)$.

4 Conclusion

In this letter we demonstrated the general scheme for the construction of the general-relativistic model of the magnetically driven jet. The method is based on the usage of the $3 + 1$ MHD formalism. It is shown that the critical points of the flow and the explicit radial behavior of the physical variables may be derived. All jet characteristics may be expressed through one quantity: the jet “profile function” $\bar{\theta}(r)$. The latter quantity may be modelled in some way (i.e., by adopting the simplest constant open angle jet approximation, or by using of some phenomenological jet profile form). Alternative, and more self-consistent, approach should imply the solution of the complex first order ordinary differential equation for the $\bar{\theta}(r)$ function.

However, the full examination of the problem is beyond the scope of this work. The results of this letter are of the preliminary character. Full consideration of the problem is in preparation and will be published elsewhere.
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REFERENCES

1. R. D. Blandford, in Astrophysical Jets, Eds. D. Burgarella, M. Livio and C. O’Dea, (Cambridge University Press, Cambridge, 1993).
2. M. C. Begelman, R. D. Blandford & M. J. Rees, rev. Mod. Phys. 56, 255, (1984).
3. K. S. Thorne, & D. A. MacDonald, Mon. Not. R. Astr. Soc. 198, 339, (1982).
4. D. A. MacDonald & K. S. Thorne, Mon. Not. R. Astr. Soc. 198, 345, (1982).
5. K. S. Thorne, R. H. Price, & D. A. MacDonald, eds. Black Holes: The Membrane Paradigm (Yale University Press, New Haven 1986).
6. R. V. E. Lovelace, C. Mehanian, C. Mobarry & M. E. Sulkanen, Ap.J.S. 62, 1, (1986).
7. R. V. E. Lovelace, J. C. L. Wang & M. E. Sulkanen, Ap.J. 315, 504, (1987).
8. R. V. E. Lovelace, H. L. Berk & J. Contopoulos, Ap.J. 394, 459, (1991).
9. J. Contopoulos & R. V. E. Lovelace, Ap.J. 429, 139, (1994).
10. J. Contopoulos, Ap.J. 432, 508, (1994).
11. J. Contopoulos, Ap.J. 446, 67, (1995).