LIST ARBORICITY OF FINITARY MATROIDS: A GENERALIZATION OF SEYMOUR’S RESULT

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Abstract. Seymour proved that the chromatic numbers and the list chromatic numbers of loop-free finite matroids are the same. In this paper we prove the same statement for infinite, loop-free finitary matroids.

1. Introduction

Matroids are important objects of finite combinatorics, that can represent the basic properties of independency and rank. The theorems about matroids can be applied in a wide range of fields of mathematics, such as linear algebra or graph theory. One of the most interesting fact about matroids is Seymour’s list coloring theorem [1], that states, that the list chromatic number of a finite matroid is just equal’s to the chromatic number. In this paper, we generalize this result for a class of infinite matroids, called finitary matroid. In section 2 we define the most important properties of these matroids. Then on section 2 we prove the generalization of Seymour’s theorem, when the chromatic number is finite, by using some compactness result. Finally, in section 3 we show the same statement when the chromatic number is an infinite cardinal. This proof uses heavier set theory and logic, inparticular, elementary submodels.

2. Definition and basic properties of finitary matroids

We follow the terminology of [2].

Definition 2.1. Let $S$ be any set. A rank function on $S$ is a function $r : [S]^{<\omega} \to \omega$, with the following properties:
1. $r(\emptyset) = 0$,
2. $\forall A, B \in [S]^{<\omega} \text{ if } A \subseteq B, \text{ then } r(A) \leq r(B) \ (\text{monotony})$,
3. $\forall A \in [S]^{<\omega}, r(A) \leq |A| \ (\text{subcardinality})$,$
4. $\forall A, B \in [S]^{<\omega}, r(A) + r(B) \geq r(A \cap B) + r(A \cup B) \ (\text{submodularity})$.

A finitary matroid is a pair $\mathcal{M} = (S, r)$, where $r$ is a rank function on $S$. A subset $X \subseteq S$ is called independent if $\forall A \in [X]^{<\omega}$, we have $r(A) = |A|$. The set of independent sets in $\mathcal{M}$ is denoted by $\mathcal{I}(\mathcal{M})$.

By this definition it is clear that the subsets of an independent set are also independent.

If $\mathcal{M} = (S, r)$ is a finitary matroid and $A \in [S]^{<\omega}$, then $\mathcal{M}_A = (A, r|_{[A]})$ is a finite matroid. Using this observation, there are some claims, that can be proven

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easily using the fact that these hold for finite matroids. On the other hand, if \( r \) is any function on the finite subsets of \( S \), such that all \( M_A \)'s are finite matroids, then \( M \) is a finitary matroid. This way, we can define several matroids. For example let \( V \) be any vector space and \( S \subseteq V \). The linear matroid of \( S \) is a matroid, where each \( A \in [S]^{<\omega} \) is independent if and only if their elements are linearly independent. Then \( r(A) \) is the dimension of the subspace, generated by the elements of \( A \). The other example is the graphical matroid, where \( V \) is a vertex set and \( S \subseteq [V]^2 \) is an edge set. In this matroid \( A \in [S]^{<\omega} \) is independent, if it does not contain a circuit and \( r(A) \) is the number of vertices covered by \( A \) minus the number of components of these vertices by the edge set \( A \). However, there are many operations of finite matroids, that cannot be used for finitary matroids, such as dualisation.

Let us remark that there are ”natural” infinite matroids which are not finitary: for example the bond matroid of an infinite graph cannot be obtained as a finitary matroid, as it may have sets that are not independent, but all of their finite subsets are independent.

**Lemma 2.2.** If \( M = (S, r) \) is a finitary matroid and \( A \in [S]^{<\omega} \), then \( A \in \mathcal{I}(M) \) if and only if \( r(A) = |A| \). Moreover, \( |B| = r(A) \) for each maximal independent subset \( B \subseteq A \).

**Proof.** Use the same result, (Theorem 1.3.2. in [2]) for the finite matroid \( M_A \). □

**Definition 2.3.** Let \( M = (S, r) \) be a finitary matroid. A subset \( C \subseteq S \) is called a circuit, if \( C \notin \mathcal{I}(M) \), and \( C \) is minimal not independent. The set of circuits of \( M \) is denoted by \( \mathcal{C}(M) \).

If \( X \subseteq S \) is not independent, then there is some \( A \in [X]^{<\omega} \) that is not independent. Then taking the elements one by one, we have that there is a minimal not independent set \( C \subseteq A \subseteq X \), and thus \( C \in \mathcal{C}(M) \). Hence, all not independent subsets of a finitary matroid contain a circuit and all circuits are finite.

It is also clear that \( \emptyset \notin \mathcal{C}(M) \) and if \( C_1 \neq C_2 \in \mathcal{C}(M) \), then \( C_1 \nsubset C_2 \). The following lemma contains two statements, weak and strong circuit axioms, that hold for infinite finitary matroids as well.

**Lemma 2.4.** a) If \( M = (S, r) \) is a finitary matroid, \( C_1 \neq C_2 \in \mathcal{C}(M) \) and \( e \in C_1 \cap C_2 \), then there is a \( C \in \mathcal{C}(M) \), such that \( C \subseteq (C_1 \cup C_2) - e \).

b) If we also have \( e_1 \in C_1 - C_2 \), \( C \) can be chosen such that \( e_1 \in C \).

**Proof.** For a) use Lemma 1.1.3 in [2] for the finite matroid \( M_{C_1 \cup C_2} \). For b) use Proposition 1.4.11 in [2] for the finite matroid \( M_{C_1 \cup C_2} \). □

A circuit consisting of one element is called a loop. A finitary matroid is called loop-free if it does not contain any loop. Equivalently, this means that for all \( x \in S \), we have \( r(\{x\}) = 1 \).

**Definition 2.5.** Let \( M = (S, r) \) be a finitary matroid. A set \( B \in \mathcal{I}(M) \) is a base, if it is a maximal independent set. The set of bases of \( M \) is denoted by \( \mathcal{B}(M) \).

The existence of bases for finite matroids are clear, as we can add elements to an independent sets one by one, until it gets maximal, however since independence depend just on the finite subsets, by Teichmuller-Tukey lemma, we can show that bases exist in all finitary matroids and all independent sets are contained in a base.
Lemma 2.6. If $M = (S, r)$ is a finitary matroid, $B \in \mathcal{B}(M)$ and $x \in S - B$, then there is a unique $C \in \mathcal{C}(M)$, such that $C \subseteq B + x$.

Proof. Since $B$ is maximal independent, $B + x$ is not independent, so it must contain a circuit $C$. Since $B$ is independent, $x \in C$.

For unicity suppose $C_1, C_2 \subseteq B + x$ are circuits. Since no circuit can be the subset of the independent set $B$, we must have $x \in C_1 \cap C_2$. But then by lemma 2.4 (a), there is a $C' \in \mathcal{C}(M)$, such that $C' \subseteq (C_1 \cup C_2) - x \subseteq B$, that is a contradiction. □

Definition 2.7. Let $M = (S, r)$ be a finitary matroid, $B \in \mathcal{B}(M)$ and $x \in S - B$. The main circle of $x$ on $B$, denoted by $C(B, x)$ is the unique $C \in \mathcal{C}(M)$, such that $C \subseteq B + x$.

The next notion we need to introduce are closed sets in finitary matroids. But before, we show some important lemmas. It is clear by submodularity and subcardinality, that if $A \in [S]^{<\omega}$ and $x \in S$, then $r(A + x)$ is either $r(A)$ or $r(A) + 1$.

Lemma 2.8. If $M = (S, r)$ is a finitary matroid, $A, B \in [S]^{<\omega}$, with $A \subseteq B$, and $x \in S$ is such that $r(A + x) = r(A)$. Then $r(B + x) = r(B)$.

Proof. Using the submodularity for the sets $A + x$ and $B$, we have $r(A + x) + r(B) \geq r(A) + r(B + x)$. By our assumption, then $r(A) + r(B) \geq r(A) + r(B + x)$, so $r(B) \geq r(B + x)$. By the monotony, we have $r(B) \leq r(B + x)$, so $r(B)$ and $r(B + x)$ are equal. □

Lemma 2.9. If $M = (S, r)$ is a finitary matroid, $A \in [S]^{<\omega}$ and $x_1, \ldots, x_n \in S$ such that $r(A + x_i) = r(A)$ for all $1 \leq i \leq n$, then $r(A \cup \{x_1, \ldots, x_n\}) = r(A)$.

Proof. Induction on $n$. For $n = 1$, it is clear. Suppose this is true for some $n$ and show for $n + 1$. Since $A \subseteq A \cup \{x_1, \ldots, x_n\}$ and $r(A + x_{n+1}) = r(A)$, by lemma 2.8 we have $r(A \cup \{x_1, \ldots, x_n\} + x_{n+1}) = r(A \cup \{x_1, \ldots, x_n\}) + x_{n+1} = r(A \cup \{x_1, \ldots, x_n\}) = r(A)$ by the induction hypothesis, so the induction step works. □

For a finite matroid a subset $Z \subseteq S$ is called closed if $r(Z + x) > r(Z)$ for all $x \in S - Z$. For finitary matroids, the rank function if defined just on finite subsets, so we need to refine this definition.

Definition 2.10. Let $M = (S, r)$ be a finitary matroid. A subset $Z \subseteq S$ is closed if for all $Z_0 \in [Z]^{<\omega}$ and $x \in S - Z$, we have $r(Z_0 + x) > r(Z_0)$.

By lemma 2.8 this is clearly equivalent to the original definition of closedness for finite matroids. The whole set $S$ is closed for all matroids and $\emptyset$ is closed if and only $M$ is loop-free.

Lemma 2.11. If $M = (S, r)$ is a finitary matroid, $I$ is an index set and for all $i \in I$, $Z_i \subseteq S$ is a closed subset, then $\bigcap_{i \in I} Z_i$ is closed.

Proof. Let $Z_0 \in [\bigcap_{i \in I} Z_i]^{<\omega}$ and $x \in S - (\bigcap_{i \in I} Z_i)$. Then there is some $i \in I$, such that $x \notin Z_i$. Since $Z_i$ is closed and $Z_0 \in [Z_i]^{<\omega}$, we have $r(Z_0 + x) > r(Z_0)$. Since $Z_0$ and $x$ was arbitrary, we have that $\bigcap_{i \in I} Z_i$ is closed. □

Definition 2.12. Let $M = (S, r)$ be a finitary matroid and $X \subseteq S$, then the closure of $X$, denoted by $\sigma(X)$, is defined by

$$\sigma(X) = \bigcap\{Z \subseteq S : X \subseteq Z, Z \text{ is closed}\}.$$
By lemma 2.14, \( \sigma(X) \) is closed and \( X \subseteq \sigma(X) \), as it was contained in all members of intersection.

**Lemma 2.13.** Let \( \mathcal{M} = (S, r) \) be a finitary matroid, then

a) For all \( X \subseteq S \) if \( X \subseteq Z \subseteq S \) and \( Z \) is closed, then \( \sigma(X) \subseteq Z \),

b) If \( X \subseteq Y \subseteq S \), then \( \sigma(X) \subseteq \sigma(Y) \),

c) If \( X \subseteq S \), the \( \sigma(\sigma(X)) = \sigma(X) \).

**Proof.** a) By the definition of \( \sigma(X) \), \( Z \) was an item in the intersection, so \( \sigma(X) \subseteq Z \).

b) Since \( X \subseteq Y \subseteq \sigma(Y) \) and \( \sigma(Y) \) is closed, by a), we have \( \sigma(X) \subseteq \sigma(Y) \).

c) On one hand \( \sigma(X) \subseteq \sigma(\sigma(X)) \) is by definition, on the other hand, since \( \sigma(X) \subseteq \sigma(X) \) and \( \sigma(X) \) is closed, by a) we must have \( \sigma(\sigma(X)) \subseteq \sigma(X) \), so they are equal. \( \square \)

Now we give a characterization of the elements of \( \sigma(X) \)

**Lemma 2.14.** If \( \mathcal{M} = (S, r) \) is a finitary matroid, \( X \subseteq S \), then

\[ \sigma(X) = \{ x \in S \mid \exists X_0 \in [X]^{< \omega}, r(X_0 + x) = r(X_0) \}. \]

**Proof.** Let

\[ F = \{ x \in S \mid \exists X_0 \in [X]^{< \omega}, r(X_0 + x) = r(X_0) \}. \]

First we will show that \( F \subseteq \sigma(X) \). Suppose \( x \in F \) and let \( X_0 \in [X]^{< \omega} \), such that \( r(X_0 + x) = r(X_0) \). Since \( X_0 \subseteq X \subseteq \sigma(X) \), and \( \sigma(X) \) is closed, so \( x \notin \sigma(X) \) would mean \( r(X_0 + x) > r(X_0) \), in contradiction with our assumption. Thus, \( x \in \sigma(X) \), so \( F \subseteq \sigma(X) \).

Clearly \( X \subseteq F \), as for \( x \in X \), we can write the singleton \( X_0 = \{ x \} \) into the definition of \( F \). Now, we need to show that \( F \) is closed. Let \( Y_0 \in [F]^{< \omega} \), and list all its elements \( Y_0 = \{ y_1, \ldots, y_n \} \) and let \( x \in S - F \). For all \( 1 \leq i \leq n \), since \( y_i \in F \), there is an \( x_i \in [X]^{< \omega} \), such that \( r(X_i + y_i) = r(X_i) \). Since for all \( i \), we have \( X_i \subseteq \bigcup_{j=1}^{n} X_j \), by lemma 2.8, we have \( r(\bigcup_{j=1}^{n} X_j + x_i) = r(\bigcup_{j=1}^{n} X_j) \). Then by lemma 2.9, we have \( r(\bigcup_{j=1}^{n} X_j) = r(\bigcup_{j=1}^{n} X_j) \). Since \( x \notin F \) and \( \bigcup_{j=1}^{n} X_j \in [X]^{< \omega} \), by the definition of \( F \), we must have \( r(\bigcup_{j=1}^{n} X_j + x) \geq r(\bigcup_{j=1}^{n} X_j + x) > r(\bigcup_{j=1}^{n} X_j) \). Then \( r(\bigcup_{j=1}^{n} X_j \cup Y_0 + x) \geq r(\bigcup_{j=1}^{n} X_j + x) > r(\bigcup_{j=1}^{n} X_j) \), so by reverting lemma 2.8, we get that \( r(Y_0 + x) > r(Y_0) \). Since \( Y_0 \) and \( x \) were arbitrary, \( F \) is closed. Then by lemma 2.13, \( \sigma(X) \subseteq F \), and by the first part, \( \sigma(X) \) and \( F \) must be equal. \( \square \)

**Lemma 2.15.** If \( \mathcal{M} = (S, r) \) is a finitary matroid, \( B \subseteq S \), then \( B \in \mathcal{B}(\mathcal{M}) \) if and only if \( B \in \mathcal{I}(\mathcal{M}) \) and \( \sigma(B) = S \).

**Proof.** If \( B \in \mathcal{B}(\mathcal{M}) \), then it is clearly independent. Suppose \( \sigma(B) \neq S \) and let \( x \in S - \sigma(B) \). Then for any \( B_0 \in [B]^{< \omega} \), since \( B_0 \subseteq B \subseteq \sigma(B) \), we have \( r(B_0 + x) > r(B_0) \), so \( r(B_0 + x) = r(B_0) + 1 = |B_0| + 1 = |B_0 + x| \). Since all finite subsets of \( B + x \) are either in this form or subset of \( B \), by definition, we have \( B + x \in \mathcal{I}(\mathcal{M}) \), in contradiction with the maximality of \( B \).

Now suppose \( B \in \mathcal{I}(\mathcal{M}) \) and \( \sigma(B) = S \). We need to show that \( B \) is maximal independent. Let \( x \in S - B \). Then since \( x \in \sigma(B) \), by lemma 2.14, there is a subset \( B_0 \in [B]^{< \omega} \), such that \( r(B_0 + x) = r(B_0) = |B_0| < |B_0 + x| \). Since \( B_0 + x \in [B + x]^{< \omega} \), \( B + x \) is not independent, so \( B \) is maximal. \( \square \)
In the next step, we define the contraction of a finitary matroid $M = (S, r)$, by a subset $Z \subseteq S$. For finite matroids, this is a matroid on the set $S - Z$, defined as for all $A \subseteq S - Z$, $r'(A) = r(A \cup Z) - r(Z)$. However, for finitary matroids, we cannot define the contraction by an infinite set that way, so we need a refined definition.

**Definition 2.16.** Let $M = (S, r)$ be a finitary matroid and $Z \subseteq S$. The *contraction of $M$ by $Z$* is the pair $M' = (S \setminus Z, r')$, where for $A \in [S - Z]^{<\omega}$ we let

$$r'(A) = \min\{r(A \cup Z) - r(Z) : Z_0 \in [Z]^{<\omega}\}.$$ 

Since all values of this set are nonnegative integers, $r'$ is well-defined (we still need to show that this construction makes a matroid).

For $A \in [S - Z]^{<\omega}$, we say that a $Z_0 \in [Z]^{<\omega}$ fits $A$ if $r'(A) = r(A \cup Z_0) - r(Z_0)$.

**Lemma 2.17.** Let $M = (S, r)$ be a finitary matroid, $Z \subseteq S$, and $r'$ is the function defined by the contraction.

a) If $A \in [S - Z]^{<\omega}$ and $Z_0, Z_1 \in [Z]^{<\omega}$ with $Z_0 \subseteq Z_1$ and $Z_0$ fits $A$, then $Z_1$ also fits $A$.

b) If $A_1, \ldots, A_n \in [S - Z]^{<\omega}$, then there is a $Z_0 \in [Z]^{<\omega}$, that fits all of them.

**Proof.** a) In one hand, by the definition of $r'$, we clearly have $r(A \cup Z_1) - r(Z_1) \geq r'(A)$. On the other hand, we can write the submodular inequality for $A \cup Z_0$ and $Z_1$ to get that $r(A \cup Z_0) + r(Z_1) \geq r(A \cup Z_1) + r(Z_0)$, that can be transformed as $r(A \cup Z_1) - r(Z_1) \leq r(A \cup Z_0) - r(Z_0) = r'(A)$, so the two sides are equal, $Z_1$ fits $A$.

b) For each $1 \leq i \leq n$ choose $Z_i \in [Z]^{<\omega}$, that fits $A_i$. Then by a) $\bigcup_{i=1}^n Z_i$ fits all.

**Lemma 2.18.** Let $M = (S, r)$ be a finitary matroid, $Z \subseteq S$, and $r'$ is the function defined by the contraction, then $r'(A) \leq r(A)$ for all $A \in [S - Z]^{<\omega}$, and $M' = (S - Z, r')$ is a finitary matroid.

**Proof.** By writing $Z_0 = \emptyset$ into the definition of $r'$, we get that for $A \in [S - Z]^{<\omega}$ $r'(A) \leq r(A \cup \emptyset) - r(\emptyset) = r(A) - 0 = r(A)$. To see that $M'$ is a matroid, we need to show properties 1-4 from Definition 2.1 for $r'$.

1. Clear by $0 \leq r'(\emptyset) \leq r(\emptyset) = 0$

2. Let $A, B \in [S - Z]^{<\omega}$ with $A \subseteq B$ and choose $Z_0 \in [Z]^{<\omega}$ that fits both. Then since $A\cup Z_0 \subseteq B\cup Z_0$, we have $r(A\cup Z_0) \leq r(B\cup Z_0)$, so $r'(A) = r(A\cup Z_0) - r(Z_0) \leq r(B\cup Z_0) - r(Z_0) = r'(B)$.

3. Comes from $r'(A) \leq r(A) \leq |A|$

4. Let $A, B \in [S - Z]^{<\omega}$ and choose $Z_0 \in [Z]^{<\omega}$, that fits all of $A, B, A \cap B, A \cup B$. Then since $(A \cup Z_0) \cap (B \cup Z_0) = (A \cap B) \cup Z_0$, and $(A \cup Z_0) \cup (B \cup Z_0) = (A \cup B) \cup Z_0$, we have $r'(A) + r'(B) = r(A \cup Z_0) + r(B \cup Z_0) - 2r(Z_0) \geq r((A \cap B) \cup Z_0) + r((A \cup B) \cup Z_0) - 2r(Z_0) = r'(A \cap B) + r'(A \cup B)$.

**Lemma 2.19.** Let $M = (S, r)$ be a finitary matroid, $Z \subseteq S$, and $M' = (S - Z, r')$ be the contraction matroid. Then for $X \subseteq S - Z$, we have $X \in I(M')$ if and only if for all $Y \in I(M) \cap P(Z)$, $X \cup Y \in I(M)$.

**Proof.** First suppose $X \in I(M')$ and let $Y \in I(M)$ with $Y \subseteq Z$. We need to show that $X \cup Y$ is independent. Let $X_0 \cup Y_0 \in [X \cup Y]^{<\omega}$, where $X_0 \subseteq X, Y_0 \subseteq Y$. 

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Then since $Y$ is independent, we have $r(Y_0) = |Y_0|$ and since $X$ is independent in $M'$, we have $r(X_0 \cup Y_0) - |Y_0| = r(X_0 \cup Y_0) - r(Y_0) \geq r'(X_0) = |X_0|$, so $r(X_0 \cup Y_0) \geq |X_0| + |Y_0|$, so they are equal. Since all finite subsets of $X \cup Y$ is in this form, $X \cup Y$ is independent.

Now suppose that $X$ has this property and let $X_0 \in [X]^{<\omega}$ be arbitrary. Choose a $Z_0 \in [Z]^{<\omega}$ that fits $X_0$, and let $Y \subseteq Z_0$ be maximal independent subset of $Z_0$, so by lemma 2.22 we have $|Y| = r(Z_0)$. Since $Y$ is independent, by our assumption $X \cup Y$ is also independent, so $r(X_0 \cup Y) = |X_0 \cup Y| = |X_0| + |Y| = |X_0| + r(Z_0)$, but then $r'(X_0) = r(X_0 \cup Z_0) - r(Z_0) \geq r(X_0 \cup Y) - r(Z_0) = |X_0| + r(Z_0) - r(Z_0) = |X_0|$. Since $X_0$ was arbitrary, we have $X \in \mathcal{I}(M')$. □

Lemma 2.20. Let $M = (S, r)$ be a finitary matroid, $Z \subseteq S$. Then the contraction matroid $M' = (S - Z, r')$ is loop-free if and only if $Z \subseteq S$ is closed (M does not need to be loop-free).

Proof. First suppose $Z \subseteq S$ is closed. Then for all $x \in S - Z$ and $Z_0 \in Z^{<\omega}$, we have $r(Z_0 + x) = r(Z_0) + 1$, so $r(\{x\} \cup Z_0) - r(Z_0) = 1$. Then clearly, by the definition, $r'(\{x\}) = 1$, so $M'$ is loop-free.

Now suppose that $M'$ is loop-free. Let $Z_0 \in [Z]^{<\omega}$ and $x \in S - Z$. Then $r(Z_0 + x) - r(Z_0) = r(\{x\} \cup Z_0) - r(Z_0) \geq r'(\{x\}) = 1$, so $r(Z_0 + x) > r(Z_0)$. Since $Z_0$ and $x$ were arbitrary, we have that $Z$ is closed.

Now we define the proper colorings of finitary matroids, the chromatic number the list chromatic numbers.

Definition 2.21. Let $M = (S, r)$ be a finitary matroid $K$ be any color set. A proper coloring of $M$ by $K$ is a function $\Phi : S \rightarrow K$, such that for all $i \in K$, we have $\Phi^{-1}(i) \in \mathcal{I}(M)$.

It is clear that $\Phi$ is a proper coloring if and only if there is no $C \in C(M)$, such that $C \subseteq \Phi^{-1}(i)$ for some $i \in K$.

First we need to see what finitary matroids have any proper colorings

Lemma 2.22. A finitary matroid $M = (S, r)$ has a proper coloring to some color set, if and only if it is loop-free.

Proof. If $M$ is loop-free, then let $K = S$, and for all $x \in S$ put $\Phi(x) = x$. Then clearly for all $x$, $\Phi^{-1}(x) = \{x\} \in \mathcal{I}(M)$. If $M$ is not loop-free, there is an $x \in S$, that $\{x\}$ is not independent. But then for any $K$ and $\Phi : S \rightarrow K$, we have $\Phi^{-1}(\Phi(x)) \supseteq \{x\} \not\in \mathcal{I}(M)$, so it has no proper colorings. □

For loop-free finitary matroids, we can define the chromatic number the following way.

Definition 2.23. A loop-free finitary matroid $M = (S, r)$ is $\kappa$-colorable for a given (finite or infinite) cardinal $\kappa$ if there is a proper coloring $\Phi : S \rightarrow \kappa$ of $M$. The chromatic number of $M$, denoted by $\text{Chr}(M)$, is the smallest cardinal $\kappa$, such that $M$ is $\kappa$-colorable.

By the similar proof as lemma 2.22 we can see, that for all loop-free matroids $M = (S, r)$, $\text{Chr}(M)$ exists and $\text{Chr}(M) \leq |S|$.

Definition 2.24. For a loop-free, finitary matroid $M = (S, r)$ and a cardinal $\kappa$ a $\kappa$-listing is a function $L$ from $S$, such that for all $x \in S$, we have $|L(x)| \geq \kappa$. 

An \(L\)-coloring of \(\mathcal{M}\) is a function \(\Phi : S \to \bigcup_{x \in S} L(x)\), such that \(\Phi\) is a proper coloring of \(\mathcal{M}\) and for all \(x \in S\), we have \(\Phi(x) \in L(x)\). \(\mathcal{M}\) is \(\kappa\) list-colorable, if it is \(L\)-colorable for all \(\kappa\) listing \(L\). The list chromatic number denoted by \(\text{List}(\mathcal{M})\) is the smallest cardinal \(\kappa\), such that \(\mathcal{M}\) is \(\kappa\) list-colorable.

If \(\mathcal{M}\) is \(\kappa\) list-colorable, then it is also \(\kappa\)-colorable, as we can write \(L(x) = \kappa\) for all \(x \in S\), and then the \(L\)-colorings are exactly the \(\kappa\) colorings. By this, if \(\text{List}(\mathcal{M})\) exists, then \(\text{Chr}(\mathcal{M}) \leq \text{List}(\mathcal{M})\).

**Lemma 2.25.** For all loop-free finitary matroids \(\mathcal{M} = (S, r)\) and \(\text{List}(\mathcal{M})\) exists and \(\text{List}(\mathcal{M}) \leq |S|\).

**Proof.** Let \(\prec\) be a well-ordering of \(S\) in order type \(|S|\) and let \(L\) be any \(|S|\)-listing. Then by transfinite recursion, for all \(x \in S\), since \(|L(x)| > |\{\Phi(y) : y < x\}|\), we can choose \(\Phi(x) \in L(x)\), such that \(\Phi(x) \neq \Phi(y)\), for \(y < x\). Then all values of \(\Phi\) are different, so it is clearly a proper \(L\)-coloring.

Then we have that for all loop-free finitary matroids \(\text{Chr}(\mathcal{M}) \leq \text{List}(\mathcal{M}) \leq |S|\).

Seymour’s famous list-coloring theorem [1] Theorem 2] states that for any finite matroid \(\text{Chr}(\mathcal{M}) = \text{List}(\mathcal{M})\) holds. In this paper, we generalize this result for finitary matroid. In the next section \(3\) we consider the easy case when \(\text{Chr}(\mathcal{M})\) is finite, and in Section \(4\) we investigate the case when \(\text{Chr}(\mathcal{M})\) is an infinite cardinal.

### 3. The finite chromatic number case

**Theorem 3.1.** Let \(\mathcal{M} = (S, r)\) be a loop-free finitary matroid and \(k \in \omega\). Then the following statements are equivalent:

1. \(\text{Chr}(\mathcal{M}) \leq k\)
2. \(\text{List}(\mathcal{M}) \leq k\)

**Proof.** (2) \(\Rightarrow\) (1) is clear.

(1) \(\Rightarrow\) (2)

Let \(\Phi : S \to k\) be a \(k\)-coloring of \(\mathcal{M}\). Then clearly for all \(A \in [S]^{<\omega}\), \(\Phi|_A\) is a \(k\)-coloring of \(\mathcal{M}_A\), so \(\text{Chr}(\mathcal{M}_A) \leq k\). Since \(\mathcal{M}_A\) is a finite matroid, by Seymour’s list coloring theorem, we have that \(\text{List}(\mathcal{M}_A) \leq k\) for all \(A \in [S]^{<\omega}\).

Let \(L\) be an arbitrary \(k\)-listing, that we want to construct an \(L\)-coloring of \(\mathcal{M}\). We may suppose that \(|L(x)| = k\) for all \(x \in S\), as if not, we can choose an arbitrary \(L'(x) \subseteq L(x)\) for all \(x \in S\) with \(|L'(x)| = k\), and then the constructed \(L'\)-coloring is also an \(L\)-coloring.

For any \(A \in [S]^{<\omega}\), let \(T_A = \{X \in [S]^{<\omega} : A \subseteq X\}\), and let us denote \(T_x = T_{\{x\}}\). Clearly for any \(A\), we have \(A \in T_A\) and for \(A_1, \ldots, A_n \in [S]^{<\omega}\), we have \(T_{A_1} \cap \ldots \cap T_{A_n} = T_{A_1 \cup \ldots \cup A_n}\). Since \(\mathcal{T} = \{T_A : A \in [S]^{<\omega}\} \subseteq \mathcal{P}([S]^{<\omega})\) is a set that is closed under finite intersection and \(\emptyset \notin \mathcal{T}\), there is an ultrafilter \(\mathcal{U} \subseteq \mathcal{P}([S]^{<\omega})\), such that \(\mathcal{T} \subseteq \mathcal{U}\).

Let \(x \in S\) be arbitrary. Then for each \(A \in T_x\), since \(x \in A\), the coloring \(\Phi_A\) is defined on \(x\) and \(\Phi_A(x) \in L(x)\). For each \(i \in L(x)\), let \(T_{x,i} = \{A \in T_x : \Phi_A(x) = i\}\). Then the finite union \(\bigcup_{i \in L(x)} T_{x,i} = T_x \subseteq \mathcal{U}\), so there is a unique \(i \in L(x)\), with \(T_{x,i} \in \mathcal{U}\). Define the coloring \(\Phi_\mathcal{U} : S \to \bigcup_{x \in S} L(x)\), be such that for all \(x \in S\), \(\Phi_\mathcal{U}(x) \in L(x)\) is the unique element with \(T_{x,\Phi_\mathcal{U}(x)} \in \mathcal{U}\).

We need to show that \(\Phi_\mathcal{U}\) is a proper coloring. Suppose for contradiction, that there is a \(C \in \mathcal{C}(\mathcal{M})\) and some \(i\), such that \(C \subseteq \Phi_\mathcal{U}^{-1}(i)\). Then for all \(x \in C\), we have \(i \in L(x)\) and \(T_{x,i} \in \mathcal{U}\). Then the finite intersection \(\bigcap_{x \in C} T_{x,i} \in \mathcal{U}\), so...
\[ \bigcap_{x \in C} T_{x,i} \neq \emptyset. \] Let \( A \subseteq \bigcap_{x \in C} T_{x,i} \). Then for all \( x \in C \), \( A \subseteq T_{x,i} \), so \( x \in A \) and \( \Phi_A(x) = i \). But then \( C \subseteq \Phi_A^{-1}(i) \), that is a contradiction, since \( \Phi_A \) is a proper coloring of \( M_A \). Thus, \( \Phi \) is a \( L \)-coloring. Hence, \( M \) is \( k \) list-colorable, so \( \text{List}(k) \leq k \).

4. The infinite chromatic number case

In this final section, we will show that \( \text{Chr}(M) = \text{List}(M) \), when it is an infinite cardinal. For this, first we define a notion.

**Definition 4.1.** If \( M = (S, r) \) is a loop-free finitary matroid, and we say that \( (B, \leq) \) is a well-ordered base of \( M \), if \( B \) is a base of \( M \) and \( \leq \) is a well-order on \( B \).

Given a well-ordered base \( (B, \leq) \) we define the function \( M_B : S \to B \) in the following way:

\[
M_B(x) = \begin{cases} 
  x & \text{if } x \in B, \\
  \max_{\leq}(C(B, x) \cap B) & \text{if } x \notin B,
\end{cases}
\]

where \( C(B, x) \) denotes the main circle of \( x \) in \( B \), see Definition 2.7.

**Theorem 4.2.** Let \( M = (S, r) \) be a loop-free finitary matroid, \( \kappa \) be an infinite cardinal. Then the following statements are equivalent:

1. \( \text{Chr}(M) \leq \kappa \)
2. \( \text{List}(M) \leq \kappa \)
3. There is a well-ordered base \( (B, \leq) \) of \( M \), such that for all \( b \in B \),

\[
|\{x \in S \mid M_B(x) = b\}| \leq \kappa.
\]

**Proof.** (2) \( \Rightarrow \) (1) is clear.

Before proving (3) \( \Rightarrow \) (2) we need some preparation.

**Lemma 4.3.** If \( M = (S, r) \) is a finitary matroid \( Z \subseteq S \) is closed, and \( C \in \mathcal{C}(M) \) such that there is an \( x \in C \) with \( C - x \subseteq Z \), then \( C \subseteq Z \).

**Proof.** Suppose for contradiction that \( x \notin Z \). Then \( C - x \in [Z]^{\omega} \) and \( r((C - x) + x) = r(C) = |C| - 1 = r(C - x) \), in contradiction with \( Z \) is closed. \( \square \)

**Lemma 4.4.** If \( M = (S, r) \) is a loop-free finitary matroid \( (B, \leq) \) is a well-ordered base and \( C \in \mathcal{C}(M) \). Then there are \( x, y \in C \), with \( x \neq y \) and \( M_B(x) = M_B(y) \).

**Proof.** Let \( B^* = (C \cap B) \cup (\bigcup_{x \in C \setminus B} C(B, x) \cap B) \subseteq B \). This is a finite subset of \( B \). Let \( b = \max_{\leq}(B^*) \). List the elements of \( C \) by \( C = \{x_1, \ldots, x_n\} \). Clearly \( M_B(x) = b \) holds for at least one element of \( C \), as either \( b \in C \) or \( b \in C(B, x) \cap B \) for some \( x \in C \), and \( b \) is even the maximal element of the whole \( B^* \). We will show that \( M_B(x) = b \) holds for at least two elements of \( C \). Suppose for contradiction that this is not true. By symmetry, we may suppose, that \( M_B(x_n) = b \), and for all \( 1 \leq j \leq n - 1 \), \( M_B(x_j) \neq b \). Then for \( 1 \leq j \leq n - 1 \), if \( x_j \in B \), then \( x_j \in B^* - b \). If \( x_j \notin B \), then \( C(B, x_j) \cap B = C(B, x_j) - x_j \subseteq B^* - b \), as if \( b \) would be in \( C(B, x_j) \), it would be maximal. Thus, by lemma 4.3 we have that \( x_1, \ldots, x_{n-1} \in \sigma(B^* - b) \). Applying lemma 4.3 now for \( C \) we also get \( x_n \in \sigma(B^* - b) \). If \( x_n \in B \), then \( x_n = M_B(x_n) = b \) would mean that \( b \in \sigma(B^* - b) \). If \( x_n \notin B \), then we have \( C(B, x_n) - x_n - b \subseteq B^* - b \) and \( x_n \in \sigma(B^* - b) \), so \( C(B, x_n) - b \subseteq \sigma(B^* - b) \), using lemma 4.3 again, we also get that \( b \in \sigma(B^* - b) \). But since \( B^* \) is independent,
for all \( B_0 \in [B^* - b]^{<\omega} \), we have \( r(B_0 + b) = |B_0 + b| = r(B_0) + 1 > r(B_0) \) in contradiction with lemma 2.14. Thus, \( M_b(x) = b \) holds for at least two \( x \in C \). \( \square \)

(3) \( \Rightarrow \) (2)

Suppose, that (3) holds, and let \((B, \leq)\) be a well-ordered basis of \( \mathcal{M}\) such that for all \( b \in B \), \( |S_b| = |\{x \in S|MB(x) = b\}| \leq \kappa \). We need to show that \( \mathcal{M}\) is \( \kappa \) list-colorable. Let \( L \) be an arbitrary \( \kappa \)-listing. First, we will show that we can choose a \( \Phi_b \) function on \( S_b \), such that for \( x \in S_b \), we have \( \Phi_b(x) \in L(x) \) and \( \Phi_b \) is one-to-one. Let \( \prec_b \) be a well-ordering of \( S_b \) in order type \( \leq \kappa \) and let us define \( \Phi_b \) by transfinite recursion. For \( x \in S_b \), since \( |L(x)| \geq \kappa \) and \( |\{\Phi_b(y), y \prec_b x\}| < \kappa \), we can choose \( \Phi_b(x) \in L(x) \), such that \( \Phi_b(x) \neq \Phi_b(y) \) for \( y \prec_b x \). The \( \Phi_b \) defined this way is clearly one-to-one. Let \( \Phi = \bigcup_{b \in B} \Phi_b \). Then clearly for all \( x \in S \), we have \( \Phi(x) = \Phi_{MB(x)}(x) \in L(x) \). We need to show that this is a proper coloring. Suppose for contradiction, that there is some \( C \in \mathcal{C}(\mathcal{M}) \) and an \( i \in \bigcup_{x \in S} L(x) \), such that \( C \subseteq \Phi^{-1}(i) \). Then by lemma 1.4 there are \( x, y \in C \), \( x \neq y \) and \( MB(x) = MB(y) = b \in B \). Then \( x, y \in S_b \), so \( \Phi(x) = \Phi_b(x) \neq \Phi_b(y) = \Phi(y) \), since \( \Phi_b \) is one-to-one. This is in contradiction with \( \Phi(x) = i = \Phi(y) \), so \( \Phi \) is a proper coloring.

Before proving (1) \( \Rightarrow \) (3) we need to prove some lemmas.

**Lemma 4.5.** Let \( \mathcal{M} = (S, r) \) be a finitary matroid, \( A \in [S]^{<\omega} \), with \( |A| = n \), and \( x_1, ..., x_{n+1} \in S - A \) be such that \( r(A + x_i) = r(A) \) for all \( 1 \leq i \leq n+1 \). Then the set \( \{x_1, ..., x_{n+1}\} \) is not independent.

**Proof.** By lemma 2.9 we have that \( r(A \cup \{x_1, ..., x_{n+1}\}) = r(A) \leq |A| = n \), so by monotony \( r(\{x_1, ..., x_{n+1}\}) \leq r(A \cup \{x_1, ..., x_{n+1}\}) < n+1 = |\{x_1, ..., x_{n+1}\}| \), that it is not independent. \( \square \)

**Lemma 4.6.** Let \( \mathcal{M} = (S, r) \) be a loop-free finitary matroid, \( \kappa \) be an infinite cardinal, with \( \text{Chr}(\mathcal{M}) \leq \kappa \). Then for all \( A \in [S]^{<\omega} \), we have

\[
|\{x \in S \setminus A \mid r(A + x) = r(A)\}| \leq \kappa.
\]

**Proof.** Let \( \Phi : S \rightarrow \kappa \) be a \( \kappa \)-coloring of \( \mathcal{M}\) and for all \( \alpha < \kappa \) let \( A_\alpha = \{x \in S - A | r(A + x) = r(A), \Phi(x) = \alpha\} \). Let \( n = |A| \). First we will show that \( |A_\alpha| \leq n \) for all \( \alpha \). Suppose for contradiction, that for some \( \alpha \), we have \( |A_\alpha| \geq n+1 \), and let \( x_1, ..., x_{n+1} \in A_\alpha \). Then by lemma 4.5 we have that \( \{x_1, ..., x_{n+1}\} \notin \mathcal{I}(\mathcal{M}) \) and \( \{x_1, ..., x_{n+1}\} \subseteq A_\alpha \subseteq \Phi^{-1}(\alpha) \) in contradiction with \( \Phi \) is a proper coloring. Then we have \( |\{x \in S - A | r(A + x) = r(A)\}| = |\bigcup_{\alpha < \kappa} A_\alpha| \leq \kappa \cdot n = \kappa \). \( \square \)

Thus, by Lemma 4.6 for each for loop-free finitary matroid \( \mathcal{M}\) with \( \text{Chr}(\mathcal{M}) \leq \kappa \) we can fix a bookkeeping function \( h \), i.e. a function \( h : [S]^{<\omega} \times \kappa \rightarrow S \) such that for all \( A \in [S]^{<\omega} \)

\[
\{x \in S : r(A + x) = r(A)\} \subset \{h(A, \alpha) : \alpha < \kappa\}.
\]

The last thing, we need for this proof is some model theoretic approach, using elementary submodels. For this, let \( H(\theta) = \{x : |TC(x)| < \theta\} \) for some infinite cardinal \( \theta \). Here \( TC \) denotes the transitive closure, so \( TC(x) = \bigcup_{n \in \omega} U_n(x), \) where \( U_0(x) = x \) and \( U_{n+1}(x) = U_n(x) \cup U_n(x) \), for any set \( x \). If \( \theta \) is a regular cardinal, then \( H(\theta) \) is a set and all ZFC axioms except Power Set Axiom holds in the model \( H(\theta) \). However, if \( x \in H(\theta) \) and \( 2^{|x|} < \theta \), then we also have \( P(X) \in H(\theta) \). Since all sets belong to some \( H(\theta) \), in practice we may say that \( \theta \) is sufficiently large. That means, that all sets defined in the proof are in \( H(\theta) \). An \( M \) is an elementary
submodel of $H(\theta)$ if for set-theoretic formulae $\phi$ and $x_1, \ldots, x_n \in M$ (where $n$ is the number of free variables of $\phi$) we have $M \vDash \phi(x_1, \ldots, x_n) \equiv H(\theta) \vDash \phi(x_1, \ldots, x_n)$. By Löwenheim-Skolem theorem, for all $R \subseteq H(\theta)$, there is an elementary submodel $M$, such that $R \subseteq M$ and $|M| = \max(|R|, \omega)$.

(1) $\Rightarrow$ (3)

Let $\kappa$ be fixed. We define the statement $Q(\lambda)$ for each infinite cardinal $\lambda$ in the following way:

(Q(\lambda)) If $\mathcal{M} = (S, r)$ is a loop-free matroid, with $\text{Chr}(\mathcal{M}) \leq \kappa$ and $|S| = \lambda$, then there is a well-ordered base $(B, \leq)$ of $\mathcal{M}$, such that for all $b \in B$, we have $|\{x \in S \mid M_B(x) = b\}| \leq \kappa$.

The statement (1) $\Rightarrow$ (3) would mean, that $Q(\lambda)$ holds for all cardinals $\lambda$. We prove it by induction on $\lambda$.

If $\lambda \leq \kappa$, then $Q(\lambda)$ clearly holds, as any well-ordered base $(B, \leq)$ would fit. Now suppose that $\lambda > \kappa$ and $Q(\mu)$ holds for all $\mu < \lambda$. We need to prove that $Q(\lambda)$ also holds.

Let $\mathcal{M} = (S, r)$ be a finitary matroid with $|S| = \lambda$ and $\text{Chr}(\mathcal{M}) \leq \kappa$. Fix a proper coloring $\Phi : S \rightarrow \kappa$ and a bookkeeping function $h : |S|^{<\omega} \times \kappa \rightarrow S$. Let $(S_\alpha)_{\alpha < cf(\lambda)}$ be such that $S_\alpha \subseteq S$, $|S_\alpha| < \lambda$ for all $\alpha < cf(\lambda)$ and $\bigcup_{\alpha < cf(\lambda)} S_\alpha = S$.

Let $\theta$ be a sufficiently large regular cardinal. We construct an increasing sequence of elementary submodels $M_\alpha \subseteq H(\theta)$ for $\alpha < cf(\lambda)$. Let $M_0 \subseteq H(\theta)$ be an elementary submodel, such that $\kappa \cup \{\mathcal{M}, \Phi, h\} \subseteq M_0$ and $|M_0| = \kappa$. For $\alpha < cf(\lambda)$, let $M_{\alpha+1} \subseteq H(\theta)$ be an elementary submodel with $M_\alpha \cup S_\alpha \subseteq M_{\alpha+1}$, and $|M_{\alpha+1}| = |M_\alpha \cup S_\alpha|$.

If $\alpha < cf(\lambda)$ is a limit ordinal, then let $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$. We also have that for all $\alpha < cf(\lambda)$, $|M_\alpha| < \lambda$. For $M_0$, we have $|M_0| = \kappa < \lambda$, for successor ordinals, since $|M_\alpha| < \lambda$ and $|S_\alpha| < \lambda$, we have by definition, that $|M_{\alpha+1}| < \lambda$. For limit ordinals $M_\alpha$ is a $< cf(\lambda)$ union of $< \lambda$ sets, so $|M_\alpha| < \lambda$ also holds. For all $\alpha < cf(\lambda)$, let $Z_\alpha = M_\alpha \cap S$.

**Lemma 4.7.** For all $\alpha < cf(\lambda)$, the set $Z_\alpha \subseteq S$ is closed.

*Proof.* Suppose for contradiction, that $Z_\alpha$ is not closed and let $A \in [Z_\alpha]^{<\omega}$, and $x \in S - Z_\alpha$ be such that $r(A+x) = r(A)$. Then since $A \subseteq Z_\alpha \subseteq M_\alpha$ is a finite subset of an elementary submodel, we have $A \subseteq M_\alpha$. Since $h$ is a bookkeeping function, there is some $\gamma < \kappa$, such that $h(A, \gamma) = x$. Clearly, we also have $h, \gamma \in M_0 \subseteq M_\alpha$, so $x = h(A, \gamma) \in M_\alpha$. Thus, $x \in Z_\alpha$, that is a contradiction. \qed

Now since for all $\alpha < cf(\lambda)$, we have $S_\alpha \subseteq Z_{\alpha+1}$ and $S = \bigcup_{\alpha < cf(\lambda)} S_\alpha$, for all $x \in S$ there is some $\alpha < cf(\lambda)$, such that $x \in Z_\alpha$. Let us define the rank function $\rho : S \rightarrow cf(\lambda)$, such that $\rho(x) = \min\{\alpha : x \in Z_\alpha\}$. Moreover, $\rho(x)$ must be a successor ordinal, as for limit ordinals $Z_\alpha$ is just the union of former ones.

We construct an increasing chain $(B_\alpha, \leq_\alpha)_{\alpha < cf(\lambda)}$ of well-ordered independent sets in $\mathcal{M}$, such that

(i) $B_\alpha \in B(\mathcal{M}_Z_\alpha)$,

(ii) $(B_\alpha, \leq_\alpha)$ is an initial segment of $(B_\beta, \leq_\beta)$ for $\alpha < \beta < cf(\lambda)$,

(iii) $|\{x \in Z_\alpha \mid M_{B_\alpha}(x) = b\}| \leq \kappa$ for each $\alpha < cf(\lambda)$ and $b \in B_\alpha$.

We construct those $B_\alpha$s by transfinite recursion. First, since $Q(\kappa)$ clearly holds and $|Z_0| \leq |M_0| = \kappa$, we can construct $(B_0, \leq_0)$. 
Now suppose that $\alpha$ is a limit ordinal and for $\beta < \alpha$, $(B_\beta, \leq_\beta)$ is already constructed. Then clearly $Z_\alpha = M_\alpha \cap S = (\bigcup_{\beta < \alpha} M_\beta) \cap S = \bigcup_{\beta < \alpha} (M_\beta \cap S) = \bigcup_{\beta < \alpha} Z_\beta$. Let $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$ and $\leq_\alpha = \bigcup_{\beta < \alpha} \leq_\beta$.

We need to show that $B_\alpha \in B(M_{Z_\alpha})$. First we show that $B_\alpha$ is independent. Let $B' \in [B_\alpha]^{<\omega}$ and let $B' = \{x_1, ..., x_n\}$. Since for all $1 \leq i \leq n$, we have $x_i \in B_\alpha = \bigcup_{\beta < \alpha} B_\beta$, there is some $\beta_i < \alpha$, such that $x_i \in B_{\beta_i}$. Let $\beta = \max_{1 \leq i \leq n} (\beta_i)$. Then for all $i$, $x_i \in B_{\beta_i} \subseteq B_\beta$, so $B' \subseteq B_\beta$. Since $B_\beta$ is independent, we have $r(B') = |B'| = n$. Since it was an arbitrary finite subset, we have $B_\alpha \in I(M_{Z_\alpha})$.

We also show that this is maximal. Let $x \in Z_\alpha - B_\alpha$ be arbitrary. Then for $\beta = \rho(x) < \alpha$, since $x \in Z_\beta$, $B_\beta + x$ is not independent, thus $B_\alpha + x$ neither. Hence, $B_\alpha$ is maximal independent in $Z_\alpha$, so it is a base.

By (ii), $\leq_\alpha$ is a well-ordering of $B_\alpha$, and $(B_\beta, \leq_\beta)$ is an initial segment of $(B_\alpha, \leq_\alpha)$.

For (iii), let $b \in B_\alpha$ be arbitrary, and let $\beta = \min\{\gamma : b \in Z_\gamma\} < \alpha$. Now, for any $x \in Z_\alpha$ with $M_{B_\alpha}(x) = b \in B_\beta$, we have either $x = b$ or $x \notin B_\alpha$ and $C(B_\alpha, x) - x = C(B_\alpha, x) \cap B_\alpha \subseteq B_\beta \subseteq Z_\beta$, as $B_\beta$ is an initial segment. Then by lemma 4.8, $Z_\alpha$ is closed, applying lemma 4.8 we get that $x \in B_\beta$. Here we also have that $M_{B_\beta}(x) = M_{B_\alpha}(x)$. If $x \in B_\beta \subseteq B_\alpha$, this is clear, otherwise $x \notin B_\beta$, so $C(B_\beta, x) \subseteq B_\beta + x \subseteq B_\alpha + x$, that is a circuit, so by lemma 2.20 we have that $C(B_\alpha, x) = C(B_\beta, x)$, thus $M_{B_\alpha}(x) = M_{B_\beta}(x)$. Hence, $|\{x \in Z_\alpha : M_{B_\alpha}(x) = b\}| = |\{x \in Z_\beta : M_{B_\beta}(x) = b\}| \leq \kappa$, so we are done.

Now the successor case: let $\alpha < cf(\lambda)$ and assume that $(B_\alpha, \leq_\alpha)$ is constructed. In the matroid $M_{Z_{\alpha+1}}$, the set $Z_\alpha$ is closed by lemma 4.7. Then, by lemma 2.20 the contracted matroid $M' = (Z_{\alpha+1} - Z_\alpha, r')$ is loop-free. Next we will see that $Chr(M') \leq \kappa$ also holds.

**Lemma 4.8.** For the contracted matroid $M' = (Z_{\alpha+1} - Z_\alpha, r')$, the restriction $\Phi|_{Z_{\alpha+1} - Z_\alpha}$ is a proper coloring of $M'$, and thus $Chr(M') \leq \kappa$.

**Proof.** Suppose for contradiction, that $\Phi$ is not a proper coloring. Then there is an $X \subseteq Z_{\alpha+1} - Z_\alpha$ and a $\gamma < \kappa$, such that $X \subseteq \Phi^{-1}(\gamma)$ and $X \notin I(M')$. Then by lemma 2.13 there is a $Y \subseteq Z_\alpha$, such that $Y \in I(M)$, but $X \cup Y \notin I(M)$. Let $C \in C(M)$, be such that $C \subseteq X \cup Y$. Then since $Y$ is independent, we must have $C \subseteq Y$, and since $C \cap Z_\alpha \subseteq Y$, we have $C \subseteq Z_\alpha$. Moreover, for all $x \in C - Z_\alpha$, we have $x \in X \subseteq \Phi^{-1}(\gamma)$, so $\Phi(x) = \gamma$.

Let $k = \min\{|A| : A \in [Z_\alpha]^{<\omega} \land \exists C' \in C(M_{Z_{\alpha+1}})$
\hspace{1cm} $C' \not\subseteq Z_\alpha \land \forall x \in C' \setminus A \ (\Phi(x) = \gamma)\}.

As we can place $A_0 = C - \Phi^{-1}(\gamma) \subseteq Z_\alpha$ into this definition, so $k$ is well-defined. We also must have $k \geq 1$, as for $k = 0$, we would have $C' \subseteq \Phi^{-1}(\gamma)$ in contradiction with $\Phi$ is a proper coloring of $M$. Let $A \in [Z_\alpha]^{<\omega}$, be a set with $|A| = k$, and $C_1 \in C(M_{Z_{\alpha+1}})$ be a circuit with $A \subseteq C_1$, such that $C_1 \not\subseteq Z_\alpha$ and $C_1 - A \subseteq \Phi^{-1}(\gamma)$. Let $l = |C_1| - k = |C_1 - A|$. Then we have that $H(\theta) \not\models \exists x_1 ... \exists x_l, \Phi(x_1) = \gamma \land ... \land \Phi(x_l) = \gamma \land A \cup \{x_1, ..., x_l\} \in C(M)$.

Since $A \subseteq Z_\alpha \subseteq M_\alpha$ is a finite subset, we have $A \in M_\alpha$. As $\Phi, \gamma, M \in M_0 \subseteq M_\alpha$ and $M_\alpha$ is an elementary submodel of $H(\theta)$, we have $M_\alpha \models \exists x_1 ... \exists x_l, \Phi(x_1) = \gamma \land ... \land \Phi(x_l) = \gamma \land A \cup \{x_1, ..., x_l\} \in C(M)$. 

Then there is a $C_2 \in \mathcal{C}(\mathcal{M})$, with $A \subseteq C_2 \subseteq Z_\alpha$ and for all $x \in C_2 - A$, $\Phi(x) = \gamma$. Let $e \in A$, then clearly $e \in C_1 \cap C_2$. Also choose an $e_1 \in C_1 - Z_\alpha \subseteq C_1 - C_2$. Then we can use lemma 2.19, so there is a $C_3 \in \mathcal{C}(\mathcal{M})$, with $C_3 \subseteq (C_1 \cup C_2) - e$ and $e_1 \in C_3$. Then clearly $C_3 \subseteq C_1 \cup C_2 \subseteq Z_{\alpha+1}$, and $C_3 \not\subseteq Z_\alpha$, as $e_1 \in C_3$. Let $A_1 = A \cap C_3$. Since $A_1 \subseteq A - e$, $|A_1| \leq \kappa - 1$. Moreover, for all $x \in C_3 - A_1$, we either have $x \in C_1 - A$, or $x \in C_2 - A$, thus $\Phi(x) = \gamma$. Then the set $A_1$ is also good with the circuit $C_3$, that is a contradiction with the minimality of $k$. Hence, $\Phi|_{Z_{\alpha+1} - Z_\alpha}$ is a proper coloring of $\mathcal{M}'$.

Now let $\mu = |Z_{\alpha+1} - Z_\alpha| \leq |Z_{\alpha+1}| \leq |M_{\alpha+1}| < \lambda$. By $Q(\mu)$, $\mathcal{M}'$ has a well ordered base $(B_\alpha, \leq_\alpha)$, such that $\{|x \in Z_{\alpha+1} \setminus Z_\alpha : M_{B_\alpha}(x) = b\} \leq \kappa$ for each $b \in B'_\alpha$.

Let $B_{\alpha+1} = B_\alpha \cup B'_\alpha$. First we need to show that this is a base. For this, we prove the properties of lemma 2.19. Since $B_\alpha$ is independent, and $B'_\alpha$ is independent in the contracted matroid, by lemma 2.19, we have $B_{\alpha+1}$ is also independent. Now we need to show that $\sigma(B_{\alpha+1}) = Z_{\alpha+1}$. Since by lemma 4.3, $Z_{\alpha+1}$ is closed, we have $\sigma(B_{\alpha+1}) \subseteq Z_{\alpha+1}$. Let $x \in Z_{\alpha+1}$. If $x \in Z_\alpha$, then using lemma 2.19 we get $x \in \sigma(B_\alpha) \subseteq \sigma(B_{\alpha+1})$. Suppose $x \in Z_{\alpha+1} - Z_\alpha$. If $x \in B_\alpha'$, then we are done, so suppose $x \notin B_\alpha'$. Then $B_\alpha' + x$ is not independent in $\mathcal{M}'$, so by lemma 2.19, there is an independent $Y \subseteq Z_\alpha$, such that $(Y \cup B_\alpha') + x$ is not independent. Let $C \in \mathcal{C}(\mathcal{M})$, with $C \subseteq (Y \cup B_\alpha') + x$. Since $B_\alpha' + x \in \mathcal{I}(\mathcal{M}')$, by lemma 2.19, we have $Y \cup B_\alpha' \in \mathcal{I}(\mathcal{M})$, we must have $x \in C$. In one hand, we have $Y \subseteq Z_\alpha = \sigma(B_\alpha) \subseteq \sigma(B_{\alpha+1})$, on the other hand $B_\alpha' \subseteq B_{\alpha+1} \subseteq \sigma(B_{\alpha+1})$, thus $C - x \subseteq Y \cup B_\alpha' \subseteq \sigma(B_{\alpha+1})$, so by lemma 4.3, we have $x \in \sigma(B_{\alpha+1})$. Since $x \in Z_{\alpha+1}$ was arbitrary, $\sigma(B_{\alpha+1}) = Z_{\alpha+1}$ by lemma 2.19, $B_{\alpha+1}$ is a base.

For the well ordering $\leq_{\alpha+1}$, we simply put $B_\alpha'$ into the top of $B_\alpha$, more formally,

$$\leq_{\alpha+1} = \leq_\alpha \cup (B_\alpha \times B'_\alpha) \cup \leq'_\alpha.$$  

Clearly this is a well ordering and $B_\alpha$ (and hence all $B_\beta$-s for $\beta < \alpha$) is an initial segment. Now we need to show that for all $b \in B_{\alpha+1}$, we have $\{|x \in Z_{\alpha+1} \setminus M_{B_\alpha}(x) = b\} \leq \kappa$. First suppose $b \in B_\alpha$. We will show that for any $x \in Z_{\alpha+1}$ with $M_{B_\alpha}(x) = b$, we have $x \in Z_\alpha$ and $M_{B_\alpha}(x) = b$. If $x \in B_{\alpha+1}$, then $x = b$, so this is clear. Suppose that $x \notin B_{\alpha+1}$. Then since $B_\alpha$ is an initial segment of $B_{\alpha+1}$, we have that $C(B_{\alpha+1}, x) = C(B_\alpha, x) \cap B_{\alpha+1} \subseteq B_\alpha \subseteq Z_\alpha$. By lemma 4.3, $Z_\alpha$ is closed, so by lemma 4.3, we have $x \in Z_\alpha$. Moreover, $C(B_\alpha, x) \subseteq B_\alpha + x \subseteq B_{\alpha+1} + x$ is a circuit, so by lemma 2.6, $C(B_{\alpha+1}, x) = C(B_\alpha, x)$, thus $b = M_{B_{\alpha+1}}(x) = M_{B_\alpha}(x)$. Hence, $\{|x \in Z_{\alpha+1} \setminus M_{B_{\alpha+1}}(x) = b\} = \{|x \in Z_\alpha \setminus M_{B_\alpha}(x) = b\}$ $\leq \kappa$. Now suppose $b \in B'_\alpha$. We will show that for any $x \in Z_{\alpha+1}$ with $M_{B_\alpha}(x) = b$, we have $x \in Z_{\alpha+1} - Z_\alpha$ and $M_{B_\alpha}(x) = b$. Again if $x \in B_{\alpha+1}$, then $x = b$, so this is clear, so suppose that $x \notin B_{\alpha+1}$. If $x$ was in $Z_\alpha$, we would have a circuit $C(B_\alpha, x) \subseteq B_\alpha + x \subseteq B_{\alpha+1} + x$, so by lemma 2.6, we would have $C(B_{\alpha+1}, x) = C(B_\alpha, x)$, thus $b = M_{B_{\alpha+1}}(x) = M_{B_\alpha}(x)$, that is a contradiction. So we must have $x \in Z_{\alpha+1} - Z_\alpha$. Let $C' = C(B_{\alpha+1}, x) - Z_\alpha$. Then $C' \subseteq B_\alpha + x$. We will show that $C'$ is the main circuit of $x$ on $B_\alpha'$. Then by lemma 2.6, we only need to prove, that it is a circuit in the contraction matroid. First $C'$ is not independent in $\mathcal{M}'$, as if it was independent, then by lemma 2.19, $C(B_{\alpha+1}, x) = C'(\cup(C(B_{\alpha+1}, x) \cap Z_\alpha) = C'(\cup(C(B_{\alpha+1}, x) \cap B_\alpha)$ would be independent in $\mathcal{M}$, that is not true. Now we need to show that for all $X \subseteq C'$ with $X \notin C'$, we have $X \in \mathcal{I}(\mathcal{M}')$. If $x \notin X$, then $X \subseteq B_\alpha' \in \mathcal{I}(\mathcal{M}')$. Suppose $x \in X$, 

and suppose for contradiction, that \( X \not\in \mathcal{I}(\mathcal{M}') \) Then by lemma 2.19 there is a \( Y \subseteq Z_\alpha \) independent set, such that \( X \cup Y \) is not independent. Let \( C \in \mathcal{C}(\mathcal{M}) \) be such that \( C \subseteq X \cup Y \). As \( X - x \subseteq B'_\alpha \in \mathcal{I}(\mathcal{M}') \), by lemma 2.19 we have \( (X - x) \cup Y \notin \mathcal{I}(\mathcal{M}) \), so we must have \( x \in C \). Then by lemma 2.15 we have \( Y \subseteq Z_\alpha = \sigma(B_\alpha) \subseteq \sigma(B_\alpha \cup (X - x)) \), thus \( C - x \subseteq Y \cup (X - x) \subseteq \sigma(B_\alpha \cup (X - x)) \), so by lemma 4.3 we have \( x \in \sigma(B_\alpha \cup (X - x)) \). Then by lemma 2.11 there is an \( A \in [B_\alpha \cup (X - x)]^{<\omega} \), such that \( r(A + x) = r(A) \leq |A| < |A + x| \), thus \( A + x \) is not independent. Let \( C'' \in \mathcal{C}(\mathcal{M}) \) be such that \( C'' \subseteq A + x \). But then \( C'' \subseteq A + x \subseteq B_\alpha \cup X \subseteq B_{\alpha + 1} + x \), then by lemma 2.6 we have \( C'' = C(B_{\alpha + 1}, x) \). But \( C'' - Z_\alpha \subseteq X \), witch is a proper subset of \( C' \) and \( C(B_{\alpha + 1}, x) - Z_\alpha = C' \), that is a contradiction. Hence, \( C' \) is a circuit in \( \mathcal{M}' \), so it is the main circuit of \( x \) for \( B'_\alpha \). 

Finally, let \( B = \bigcup_{\alpha < \omega} B_\alpha \), and we define the well ordering \( \leq \) on \( B \) by taking \( \leq = \bigcup_{\alpha < \omega} \leq_\alpha \) Similarly as in the proof for the limit step, we can see that \( |\{ x \in S | M_B(x) = b \}| \leq \kappa \), thus \( Q(\lambda) \) for all \( b \in B \).

Thus, by transfinite induction, we have proved, that \( Q(\lambda) \) holds for all cardinals, so (1) \( \Rightarrow \) (3) holds.

\section*{Theorem 4.9} For any loop-free finitary matroid \( \mathcal{M} = (S, r) \), we have \( \operatorname{Chr}(\mathcal{M}) = \operatorname{List}(\mathcal{M}) \).

\textbf{Proof.} If \( \operatorname{Chr}(\mathcal{M}) = k \in \omega \), by theorem 3.1 we have \( \operatorname{List}(\mathcal{M}) \leq k = \operatorname{Chr}(\mathcal{M}) \). If \( \operatorname{Chr}(\mathcal{M}) \) is an infinite cardinal, then by theorem 4.2 \( \operatorname{List}(\mathcal{M}) \leq k = \operatorname{Chr}(\mathcal{M}) \). As clearly \( \operatorname{Chr}(\mathcal{M}) \leq \operatorname{List}(\mathcal{M}) \), \( \operatorname{Chr}(\mathcal{M}) = \operatorname{List}(\mathcal{M}) \) holds for all loop-free finitary matroids. \( \square \)

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