Space of isospectral periodic tridiagonal matrices

Anton Ayzenberg

Abstract. A periodic tridiagonal matrix is a tridiagonal matrix with additional two entries at the corners. We study the space $X_{n,\lambda}$ of Hermitian periodic tridiagonal $n \times n$-matrices with a fixed simple spectrum $\lambda$. Using the discretized Schrödinger operator we describe all spectra $\lambda$ for which $X_{n,\lambda}$ is a topological manifold. The space $X_{n,\lambda}$ carries a natural effective action of a compact $(n - 1)$-torus. We describe the topology of its orbit space and, in particular, show that whenever the isospectral space is a manifold, its orbit space is homeomorphic to $S^4 \times T^{n-3}$. There is a classical dynamical system: the flow of the periodic Toda lattice, acting on $X_{n,\lambda}$. Except for the degenerate locus $X^0_{n,\lambda}$, the Toda lattice exhibits Liouville–Arnold behavior, so that the space $X_{n,\lambda}\setminus X^0_{n,\lambda}$ is fibered into tori. The degenerate locus of the Toda system is described in terms of combinatorial geometry: its structure is encoded in the special cell subdivision of a torus, which is obtained from the regular tiling of the euclidean space by permutohedra. We apply methods of commutative algebra and toric topology to describe the cohomology and equivariant cohomology modules of $X_{n,\lambda}$.

1. Introduction

Let $\Gamma = (V, E)$ be a simple graph on a set $V = [n] = \{1, \ldots, n\}$. Let $M_{\Gamma}$ be the vector space of Hermitian $n \times n$-matrices $A = (a_{ij})$, such that $a_{ij} = 0$ for $(i, j) \notin E$. We consider the space $M_{\Gamma,\lambda} \subset M_{\Gamma}$ of all such matrices, which have a given simple spectrum $\lambda = (\lambda_1 < \lambda_2 < \cdots < \lambda_n)$. Note that each space $M_{\Gamma,\lambda}$ carries the conjugation action of a compact torus $T^n$. The action is noneffective: scalar matrices commute with every matrix, hence the diagonal subgroup of $T^n$ acts trivially.

Several examples are well studied. The complete graph $\Gamma = K_n$ corresponds to the space of all isospectral matrices, which is diffeomorphic to the variety $Fl_n$ of complete flags in $\mathbb{C}^n$. The path graph $\Gamma = \mathbb{I}_n$ with $n + 1$ vertices produces the space $M_{\mathbb{I}_n,\lambda}$ of isospectral tridiagonal matrices, which is known to be a smooth $2n$-manifold; its smooth type is independent of $\lambda$. The real version of $M_{\mathbb{I}_n,\lambda}$ is called the Tomei manifold: it was

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introduced and studied in [32]. The $T^n$-action on $M_{n,\lambda}$ is locally standard and its orbit space is diffeomorphic to a simple polytope, the permutohedron [32, 15]. Note that $M_{n,\lambda}$ is not a toric variety, although it is closely related to the permutohedral variety [8].

More generally, the spaces $M_{\Gamma_h,\lambda}$ corresponding to indifferent graphs $\Gamma_h$ are the spaces of staircase matrices. It is more convenient to encode this type of spaces by Hessenberg functions. The Hessenberg function is a function $h: [n] \to [n]$ such that $h(i) \geq i$ and $h(i+1) \geq h(i)$. The space $M_{\Gamma_h}$ is the space of Hermitian matrices $A$ such that $a_{ij} = 0$ for $j > h(i)$. Every space $M_{\Gamma_h,\lambda}$ is a smooth manifold independent of a simple spectrum $\lambda$. Its odd degree cohomology modules vanish, therefore $M_{\Gamma_h,\lambda}$ is equivariantly formal in the sense of Goresky–MacPherson (see Definition 9.5). The equivariant cohomology ring of $M_{\Gamma_h,\lambda}$ can be described using GKM-theory [18, 21]. See [5] for details on the the spaces $M_{\Gamma_h,\lambda}$ and their relation to regular semi-simple Hessenberg varieties.

For the star graph $\Gamma = St_n$ (see Fig. 1), the space $M_{St_n,\lambda}$ is also a smooth manifold, and its diffeomorphism type does not depend on $\lambda$. The effective action of $T = T^{n+1}/\Delta(T^1)$ on $M_{St_n,\lambda}$ is locally standard, therefore the orbit space $Q_{St_n,\lambda} = M_{St_n,\lambda}/T$ is a manifold with corners. Unlike the case of tridiagonal matrices, the orbit space $Q_{St_n,\lambda}$ for $n \geq 3$ is not a simple polytope. The topology of $Q_{St_n,\lambda}$ itself is quite complicated, and it is difficult to state any general result about the manifold $M_{St_n,\lambda}$ itself. However, the topology can be described in details for $n = 4$, which was done in [6].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graphs.png}
\caption{Particular graphs, encoding important isospectral matrix spaces: the path graph $I_n$, indifferent graphs $\Gamma_h$, the star graph $St_n$, and the cycle graph $Cy_n$}
\end{figure}

In this paper we consider the case $\Gamma = Cy_n$, the cyclic graph on $n$ vertices. The Hermitian matrices corresponding to $Cy_n$ have the form

$$L = L(a, b) = \begin{pmatrix}
a_1 & b_1 & 0 & \cdots & b_n \\
b_1 & a_2 & b_2 & \ddots & \\
0 & b_2 & a_3 & \ddots & 0 \\
\vdots & 0 & \ddots & \ddots & b_{n-1} \\
b_n & \cdots & 0 & \bar{b}_{n-1} & a_n
\end{pmatrix},$$

where $a_i \in \mathbb{R}$, $b_i \in \mathbb{C}$. Such matrices are called periodic tridiagonal matrices or periodic Jacobi matrices. We will simply call them periodic. It is assumed throughout the paper that $n \geq 3$. 


The space \( X_{n,\lambda} = M_{C\gamma_n,\lambda} \) of all periodic matrices with a simple spectrum \( \lambda \) has dimension \( 2n \), and carries an effective action of \( T = T^{n-1} \). Hence the torus action has complexity one. The difference between half the real dimension of a manifold and the dimension of a torus is called the complexity of the action: this terminology naturally comes from both algebraic geometry and symplectic geometry.

We prove that under certain conditions on a simple spectrum, the space \( X_{n,\lambda} \) is not a smooth manifold, not even a homology manifold, see Theorem 3.9. This gives a negative answer to our question, posed in [6]. This also settles certain inaccuracy appearing in the work of van Moerbeke [23], who studied the real analogue of \( X_{n,\lambda} \).

For any simple spectrum \( \lambda \), we describe the topology of the orbit space \( X_{n,\lambda}/T \), see Corollary 3.6. If \( X_{n,\lambda} \) is a topological manifold, we prove that the orbit space \( X_{n,\lambda}/T \) is homeomorphic to the product \( S^4 \times T^{n-3} \). When \( n = 3 \), the space \( X_{3,\lambda} \) is the space of all Hermitian matrices with the given spectrum \( \lambda \). This space is diffeomorphic to the full complex flag variety \( \text{Fl}_3 \). Hence, for \( n = 3 \), we recover the result of Buchstaber–Terzic [11, 12, 13], which states that \( \text{Fl}_3/T^2 \cong S^4 \). Note that the action is not free, however the orbit space is still a topological manifold. This fact is consistent with the general theory developed in [4].

The main ingredient of our arguments is the product of off-diagonal elements

\[
B = \prod_{i=1}^{n} b_i \in \mathbb{C}
\]

of the periodic matrix \( L(\mathbf{a}, \mathbf{b}) \). We show that with the matrix spectrum fixed, the number \( B \) takes values inside a compact convex subset \( \mathbb{B} \subset \mathbb{C} \), lying between two confocal parabolas, see Theorem 3.4. This statement may be considered a folklore: its real version was proved in [23, 20], and the complex version is not more complicated. In Section 4 we briefly review the necessary facts about discrete Schrödinger operator, needed for this result.

The value \( B \) is preserved by the torus action, hence there is a map \( \tilde{p} : X_{n,\lambda}/T \to \mathbb{C} \) from the orbit space, evaluating the number \( B \). The set \( \tilde{p}^{-1}(\mathbb{C}\setminus\{0\}) \) consists of free orbits. However the torus action has nontrivial \( T \)-equivariant skeleton, which is a proper subset of \( \tilde{p}^{-1}(0) \). To describe the structure of the equivariant skeleton, we use combinatorial geometry.

It is well known that euclidean space can be tiled by parallel copies of a regular permutohedron. Taking quotients by lattices in a euclidean space, we may produce many interesting permutohedral cell subdivisions of a torus. We show that a certain lattice produces a regular cell subdivision \( \mathcal{PT}^{n-1} \) of an \( (n-1) \)-dimensional torus, which we called the wonderful subdivision. It has several interesting properties. First, it models the equivariant skeleton of the torus action on \( X_{n,\lambda} \). Second, this wonderful subdivision minimizes the number of facets among all possible regular cell subdivisions of a torus. Such subdivisions and their dual simplicial cell subdivisions for general PL-manifolds are known in combinatorial topology under the name of crystallizations [17]. We briefly recall the required combinatorial geometry in Section 5.

Next we describe the topology of the whole space \( X_{n,\lambda} \). Let \( X_{n,\lambda}^0 = \tilde{p}^{-1}(0) \) denote the subset of matrices with \( B = 0 \). The space \( X_{n,\lambda} \) is smooth in vicinity of \( X_{n,\lambda}^0 \); this
actually follows from the properties of non-periodic Toda lattice, see Proposition 3.1. Using the result of [4] concerning the topological classification of complexity one torus actions, we describe the topology of a small neighborhood $X^\leq \varepsilon_{n,\lambda}$ of $X^0_{n,\lambda}$. It happens that, up to homeomorphism, the $T^{n-1}$-action on $X^\leq \varepsilon_{n,\lambda}$ can be extended to a locally standard $T^n$-action on this space. The necessary notions related to complexity one torus actions are given in Section 6.

In a series of works [1, 2, 3, 7] we developed a toolbox to compute cohomology and equivariant cohomology of manifolds with locally standard torus whose orbit spaces have acyclic proper faces. This toolbox is applied to the subspace $X^\leq \varepsilon_{n,\lambda}$. The $T^n$-orbit space of $X^\leq \varepsilon_{n,\lambda}$ is a manifold with corners, whose face structure is the wonderful cell subdivision of a torus, hence all its proper faces are acyclic, so we are in position to apply the general technique. The algebro-topological invariants of $X^\leq \varepsilon_{n,\lambda}$ are computed in terms of combinatorial invariants of the wonderful cell subdivision $\mathcal{P}T^{n-1}$. We recall the theory of $h$, $h'$, and $h''$-numbers of simplicial posets and compute these invariants for the dual simplicial poset of the wonderful subdivision in Section 7.

In Section 8 we describe the additive structure of $T^{n-1}$-equivariant cohomology modules of the neighborhood $X^\leq \varepsilon_{n,\lambda}$. The ordinary Betti numbers of $X_{n,\lambda}$ are calculated in Section 9. There we also prove that $X_{n,\lambda}$ is not equivariantly formal for $n \geq 4$ by comparing equivariant and ordinary Betti numbers of $X_{n,\lambda}$.

2. Torus action and Toda flow

The element $t = (t_1, \ldots, t_n) \in T^n$ acts on a cyclic matrix by the formula
\begin{equation}
(2.1) \quad tL(a; b_1, \ldots, b_n) = L(a; t_1t_2^{-1} \cdot b_1, \ldots, t_{n-1}t_n^{-1} \cdot b_{n-1}, t_1t_2^{-1} \cdot b_n).
\end{equation}

It is easy to see that the torus action preserves the quantity $B = \prod_i^b b_i$. The action is non-effective: the scalar matrices act trivially. Hence we consider the effective action of the quotient torus $T^{n-1} = T^n/\Delta(S^1)$ on $X_{n,\lambda}$.

Apart from the torus action, there is a classical dynamical system acting on the space of periodic matrices: the periodic Toda lattice. We now briefly recall the definition and properties of this dynamical system.

**Construction 2.1.** For a matrix $L = L(a, b)$ consider the skew-Hermitian matrix
\[
P = P(L) = \begin{pmatrix}
0 & b_1 & \cdots & -\bar{b}_n \\
-\bar{b}_1 & 0 & \cdots & b_{n-1} \\
& \ddots & \ddots & \ddots \\
& b_n & \cdots & -\bar{b}_{n-1} & 0
\end{pmatrix}
\]

The Toda flow (the flow of the periodic Toda lattice) is the flow
\begin{equation}
(2.2) \quad \dot{L} = [L, P] = LP - PL.
\end{equation}
The solution $L(t)$ to (2.2) remains similar to the initial matrix $L(0)$ at all times $t \in \mathbb{R}$, so the Toda flow preserves the spectrum. Therefore the flow acts on the isospectral space $X_{n,\lambda}$.

**Remark 2.2.** The Toda flow commutes with the torus action. Indeed, the action of $T$ on $L$ is given by $DLD^{-1}$, for diagonal Hermitian matrix $D$. We have $P(DLD^{-1}) = DPP(L)D^{-1}$ and therefore $[DLD^{-1}, PP(L)] = D[L, P(L)]D^{-1}$.

The periodic Toda system is well studied for real symmetric matrices. We need a more general Hermitian version of periodic Toda system in order to incorporate torus actions. However, the complex case is not more complicated than the real one. The equations of the flow have the coordinate form:

\[
\begin{align*}
\dot{a}_i &= 2(|b_{i-1}|^2 - 2|b_i|^2), \quad i = 1, \ldots, n; \\
\dot{b}_i &= b_i(a_i - a_{i+1}), \quad i = 1, \ldots, n,
\end{align*}
\]

where $a_i, b_i$ are assumed cyclically ordered. Since $b_i \in \mathbb{C}$, each expression in the second line represents two real equations. We see that the arguments of $b_i \in \mathbb{C}$ remain constant along the flow. The equations on $a_i, |b_i|$ have the form

\[
\begin{align*}
\dot{a}_i &= 2(|b_{i-1}|^2 - 2|b_i|^2), \quad i = 1, \ldots, n; \\
\frac{d}{dt}|b_i| &= |b_i|(a_i - a_{i+1}), \quad i = 1, \ldots, n,
\end{align*}
\]

which coincide with the real form of the periodic Toda flow.

**Construction 2.3.** It is a simple exercise that the quantity $B = \prod_{i=1}^{n} b_i$ is preserved along the flow. In what follows we consider the exceptional subspace

\[
X_{n,\lambda}^0 = \{ L \in X_{n,\lambda} \mid B = 0 \}
\]

This subspace can be represented as the union $X_{n,\lambda}^0 = \bigcup_{i=1}^{n} Y_i$, where $Y_i \subset X_{n,\lambda}$ is the subset of matrices having $b_i = 0$ for a particular index $i \in [n]$. The set $Y_n$ is just the set of isospectral tridiagonal Hermitian matrices, which is known to be a smooth manifold whose smooth type is independent of a simple spectrum $\lambda$ [32]. Moreover, it is known that $Y_n$ is a quasitoric manifold over a permutohedron [8,15] (the reader is advised to consult [10] concerning the terminology of quasitoric manifolds). Each of $Y_i$ for $i \neq n$ is diffeomorphic to $Y_n$. This follows from the fact that the matrix with $b_i = 0$ can be transformed to tridiagonal Hermitian matrix by a cyclic permutation of rows and columns.

Therefore $X_{n,\lambda}^0$ is the union of $n$ submanifolds of dimension $2n - 2$, however, these submanifolds intersect nontrivially. In the intersection of $Y_i$ and $Y_j$ there lies the submanifold of matrices with $b_i = b_j = 0$, which is a torus invariant codimension 2 submanifold of both $Y_i$ and $Y_j$. The combinatorial structure of these intersections will be described in detail in Section 5.

The Toda flow degenerates to a Toda flow of non-periodic Toda lattice on the exceptional set $X_{n,\lambda}^0$. Each submanifold $Y_i$ is preserved by the flow. The Toda flow on $Y_i$ is a gradient flow (see e.g. [33] or [14]), which means that asymptotically each trajectory on $Y_i$ tends to an equilibrium point. The equilibrium points are the diagonal matrices

\[ L_\sigma = \text{diag}(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}), \quad \sigma \in S_n \]
A direct check shows that the subspace $X_{n,\lambda}$ is a smooth manifold in a neighborhood of each equilibrium point $L_\sigma$. The asymptotical properties of the flow on the exceptional set imply that $X_{n,\lambda}$ is a smooth manifold in a neighborhood of $X_{n,\lambda}^0$. It will be shown in Section 3 that $X_{n,\lambda}$ is not always smooth in points with large values of $B$.

**Remark 2.4.** For generic spectrum $\lambda$ the whole space $X_{n,\lambda}$ is a smooth manifold. This easily follows from Sard’s theorem applied to the map sending the periodic tridiagonal matrix $L$ to the tuple $(\text{tr} L, \text{tr} L^2, \ldots, \text{tr} L^n)$.

### 3. The orbit space of the torus action

The action of $T = T^{n-1}$ on $X_{n,\lambda}$ has $n!$ fixed points $L_\sigma, \sigma \in S_n$ which coincide with the equilibria of the Toda flow.

**Proposition 3.1.** The orbit space $Q_{n,\lambda} = X_{n,\lambda}/T$ is a topological manifold in a neighborhood of $X_{n,\lambda}^0/T$. The space $Q_{n,\lambda}$ is a topological manifold for generic $\lambda$.

**Proof.** Note that $\dim X_{n,\lambda} = 2n$ and $\dim T = n - 1$. Consider any fixed point $L_\sigma$. The tangent representation of the action at a point $L_\sigma$ has the weight decomposition

$$T_{L_\sigma} X_{n,\lambda} = V(\alpha_1,\sigma) \oplus \cdots \oplus V(\alpha_n,\sigma), \quad \alpha_i,\sigma \in \text{Hom}(T^n, S^1)$$

where $V(\alpha)$ is the 1-dimensional complex representation

$$tz = \alpha(t) \cdot z.$$ 

In terms of the noneffective action of $n$-dimensional torus $T^n$ we have

$$\alpha_i,\sigma = \epsilon_i - \epsilon_{i+1}, \quad \text{for any } \sigma \in S_n$$

where $\{\epsilon_1 = \epsilon_{n+1}, \epsilon_2, \ldots, \epsilon_n\}$ is the standard basis of $\text{Hom}(T^n, S^1) \cong \mathbb{Z}^n$, as follows from the explicit expression (2.1) for the action.

The following fact was proved in [4]. Suppose a torus $T$ of dimension $n - 1$ acts effectively on a smooth manifold $X$ of dimension $2n$, and assume that each connected component of each equivariant skeleton $X_j$ contains a fixed point. Assume, moreover, that the action has finitely many fixed points, and, at each fixed point, any $n - 1$ of $n$ weights $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}^{n-1}$ of the tangent representation are linearly independent. Then $X/T$ is a closed topological $(n + 1)$-manifold. Applying this result to $X_{n,\lambda}$ in a neighborhood of $X_{n,\lambda}^0$, we get the first part of the proposition.

The second part follows easily from Remark 2.4, since the action of $T$ outside $X_{n,\lambda}^0$ is free. Therefore, whenever $X_{n,\lambda}$ is a smooth manifold, the orbit space $X_{n,\lambda}/T$ is smooth outside $X_{n,\lambda}^0/T$, thus it is a topological manifold.

To describe the topology of $Q_{n,\lambda}$ and $X_{n,\lambda}$, we formulate an result of an independent interest. Let $p: X_{n,\lambda} \to \mathbb{C}$ be the map which associates the number $B = \prod_{i=1}^n b_i$ to a periodic tridiagonal matrix $L(a,b)$. Since the $T$-action preserves $B$, there is an induced continuous map $\tilde{p}: Q_{n,\lambda} \to \mathbb{C}$.

The aim of the following constructions is to describe the image of $\tilde{p}$ and all its preimages. The description is given in Theorem 3.4 below.
**Construction 3.2.** Let a simple spectrum \((\lambda_1 < \ldots < \lambda_n)\) be given. Consider the characteristic polynomial \(F(x) = \prod_{i=1}^{n} (x - \lambda_i)\). Since the polynomial has \(n\) real roots, we have the sequence of real numbers
\[
\tilde{x}_1 < \tilde{x}_2 < \cdots < \tilde{x}_{n-2} < \tilde{x}_{n-1},
\]
where \(x_{n-1}, x_{n-3}, x_{n-5}, \ldots\) are the local minima, and \(x_{n-2}, x_{n-4}, \ldots\) are the local maxima of \(F\). Let
\[
M = \min_{i \text{ is even}} F(x_{n-i}), \quad m = \min_{i \text{ is odd}} -F(x_{n-i}).
\]
We obviously have \(m, M > 0\).

**Figure 2.** The values \(M\) and \(-m\) on the plot of a characteristic polynomial

**Remark 3.3.** The interval \([-m, M]\) represents the set of all \(s \in \mathbb{R}\) such that the polynomial \(F(x) - s\) has \(n\) real roots, see Fig.2.

Let \(n_+\) be the number of local maxima at which \(M\) is achieved and, similarly, \(n_-\) be the number of local minima at which \(-m\) is achieved. For generic \(\lambda\) there holds \(n_+ = n_- = 1\). Fig.2 shows the case \(n_+ = 1, n_- = 2\).

**Theorem 3.4.** The image of \(p: X_{n,\lambda} \to \mathbb{C}\) is the set
\[
\mathbb{B} = \left\{ z \in \mathbb{C} \mid |z| \leq \frac{1}{2} \min \left( \frac{m}{1 + \cos \text{Arg } z}, \frac{M}{1 - \cos \text{Arg } z} \right) \right\}
\]
The preimages of the map \(\tilde{p}: Q_{n,\lambda} \to \mathbb{B}\) are as follows. If \(z \in \mathbb{B}^o\), then \(\tilde{p}^{-1}(z)\) is homeomorphic to a compact torus \(\mathcal{T}^{n-1}\). If \(z \in \partial \mathbb{B}\) and minimum in \(3.2\) is achieved at \(\frac{M}{1 - \cos \text{Arg } z}\), then \(\tilde{p}^{-1}(z)\) is a torus of dimension \(n - 1 - n_+\). If \(z \in \partial \mathbb{B}\) and minimum in \(3.2\) is achieved at \(\frac{m}{1 + \cos \text{Arg } z}\), then \(\tilde{p}^{-1}(z)\) is a torus of dimension \(n - 1 - n_-\). If \(z \in \partial \mathbb{B}\) and \(\frac{m}{1 + \cos \text{Arg } z} = \frac{M}{1 - \cos \text{Arg } z}\), then \(\tilde{p}^{-1}(z)\) is a torus of dimension \(n - 1 - n_+ - n_-\).

The convex set \(\mathbb{B}\) is shown on Fig.3; it is bounded by arcs of two confocal parabolas. The set \(\mathbb{B}\) is a 2-dimensional manifold with corners: we denote by \(F_+\) and \(F_-\) its left and
right sides respectively, and \( F_+ \cap F_- = \{ z_{\text{top}}, z_{\text{bot}} \} \). Note that the minimum is achieved at \( M \) whenever \( z \) lies on the left side of the figure, which explains the notation.

**Remark 3.5.** It will be convenient to distinguish between the torus, which acts on spaces and the geometrical tori arising in Theorem 3.4. Hence toric groups are denoted by the symbol \( T \), and tori appearing in geometrical considerations are denoted by the symbol \( \mathcal{T} \).

**Corollary 3.6.** With parameters \( n_+ \) and \( n_- \) as above, the orbit space \( Q_{n,\lambda} \) is homeomorphic to \( \Sigma(\mathcal{T}^{n-} \ast \mathcal{T}^{n+}) \times \mathcal{T}^{n-1-n_- - n_+} \).

**Proof of the corollary.** The space \( Q_{n,\lambda} \) is foliated over the contractible space \( \mathbb{B} \) by tori. Hence

\[
Q_{n,\lambda} \cong \mathbb{B} \times \mathcal{T}^{n-1} / \sim,
\]

where certain \( n_+ \)-dimensional subtorus \( \mathcal{T}_+ \) is collapsed over \( F_+ \) and another \( n_- \)-dimensional subtorus \( \mathcal{T}_- \) is collapsed over \( F_- \) (the nature of these tori and their independence is clarified in Section 4). We have \( \mathcal{T} = \mathcal{T}_+ \times \mathcal{T}_- \times \mathcal{T}^{n-1-n_- - n_+} \). The subgroup \( \mathcal{T}^{n-1-n_- - n_+} \) separates as a direct factor of \( Q_{n,\lambda} \). The remaining factor is the suspension space, with the suspension points being the preimages of the points \( z_{\text{top}} \) and \( z_{\text{bot}} \). This suspension is taken over the space \( \tilde{\mathbb{p}}^{-1}(\mathbb{B} \cap \mathbb{R})/K \) which is homeomorphic to the join of \( \mathcal{T}_+ \) and \( \mathcal{T}_- \). \( \square \)

**Corollary 3.7.** For generic spectrum \( \lambda \) there is a homeomorphism \( Q_{n,\lambda} \cong S^4 \times \mathcal{T}^{n-3} \).

**Proof.** In generic case we have \( n_+ = n_- = 1 \). Therefore \( \Sigma(\mathcal{T}^1 \ast \mathcal{T}^1) \cong \Sigma S^3 \cong S^4 \). \( \square \)

**Corollary 3.8 (Theorem of Buchstaber–Terzic [11, 13]).** Consider the effective action of \( T = T^3/\Delta(T^1) \) on the manifold \( \text{Fl}_3 \) of complete complex flags in \( \mathbb{C}^3 \). The orbit space \( \text{Fl}_3 / T \) is homeomorphic to \( S^4 \).
Proof. Note that $X_{3,\lambda}$ is just the set of all Hermitian matrices with the given spectrum. This manifold is diffeomorphic to the flag manifold $\text{Fl}_3$. The statement is the particular case of Corollary 3.7 with $n = 3$.

**Theorem 3.9.** If $\lambda$ is a simple spectrum such that either $n_+ > 1$ or $n_- > 1$, then $X_{n,\lambda}$ is not a homology manifold. In particular, this space is not a smooth manifold.

**Proof.** Assume $n_+ > 1$. The space $Q_{n,\lambda} \cong \Sigma(T^{n-} \times T^{n+}) \times T^{n-1-n_--n_+}$ is not a homology manifold unless $n_+ = n_- = 1$. Its singular points lie over the face $F_+ \subset \partial \mathbb{B}$. Let $q \in Q_{n,\lambda}$ be a singular point such that $\bar{p}(q) \in F_+$ and let $U_q \subset Q_{n,\lambda}$ be a neighborhood of $q$. We have $H_i(U_q, U_q \setminus \{q\}; \mathbb{Z}) \neq 0$ for some $i < n + 1$. The torus action is free over $\partial \mathbb{B}$. Hence, for any point $x \in X_{n,\lambda}$ lying in the orbit $q$, its neighborhood $U_x \ni x$ is homeomorphic to $U_q \times \mathbb{R}^{n-1}$. Therefore $H_i(U_x, U_x \setminus \{x\}; \mathbb{Z}) \neq 0$ for some $i < 2n$, so far $X_{n,\lambda}$ is not a homology manifold.

**Remark 3.10.** Van Moerbeke [23] proves the real analogue of Theorem 3.4. In the real case, there is a family of tori, parametrized by real numbers from the interval $[-M/4, 0] \subset \mathbb{B} \cap \mathbb{R}$. The dimension of all tori is $n - 1$, except for the torus over the endpoint $-M/4$: its dimension decreases by $n_+$. Van Moerbeke calls the union of such family “an open $n$-dimensional torus”. This naming seems misleading, since this union is not even a manifold for $n_+ > 1$, which is proved similarly to Theorem 3.9.

**Remark 3.11.** The most degenerate case appears in the situation when the values at all local minima of the characteristic polynomial $F(x) = \prod_{i=1}^{n} (x - \lambda_i)$ coincide, and the values at all local maxima coincide. For example, this holds for the Chebyshev polynomials $T_n(x)$ (which are defined on the interval $[-1, 1]$ by $T_n(x) = \cos(n \arccos x)$). It can be seen, that a polynomial gives the maximal possible degeneration if and only if it coincides with Chebyshev polynomial up to affine transformation of the image and the domain:

$$F(x) = \gamma T_n(\alpha x + \beta) + \delta,$$

with the natural requirement that $F(x)$ has $n$ distinct real roots.

4. Schrödinger equation and the spectral curve

In this section we prove the first part of Theorem 3.4. It will be assumed that $B = \prod_{i=1}^{n} b_i \neq 0$, i.e. $L \notin X_{n,\lambda}^0$. The action of $T$ is free on such matrices. We may identify $Q_{n,\lambda} = X_{n,\lambda}/T$ with the set of isospectral Hermitian matrices of the form

$$L(w) = \begin{pmatrix}
  a_1 & b_1 & 0 & \cdots & w^{-1}b_n \\
  b_1 & a_2 & b_2 & \ddots & \\
  0 & b_2 & a_3 & \ddots & \\
  \vdots & \ddots & \ddots & \ddots & b_{n-1} \\
  wb_n & \cdots & b_{n-1} & a_n 
\end{pmatrix}$$

where $b_1, \ldots, b_n$ are positive real numbers, and $w \in \mathbb{C}$, $|w| = 1$. Indeed, the arguments of any $n - 1$ off-diagonal terms of a periodic tridiagonal matrix $L(a, b)$ can be rotated to zero
by the torus action \( (2.1) \). We continue denoting \( \prod_{i=1}^{n} b_i \) by the letter \( B \), although in the new notation \( B \) is a positive real number.

**Proposition 4.1.** For a matrix \( L(w) \) with a simple spectrum \( \lambda_1 < \cdots < \lambda_n \) there holds

\[
B \leq \frac{1}{2} \min \left( \frac{M}{1 - \cos \text{Arg } w}, \frac{m}{1 + \cos \text{Arg } w} \right)
\]

where \( M \) and \( m \) are defined by \( (3.1) \).

**Proof.** Matrices of the form \( L(w) \) can be studied using algebro-geometric method in mathematical physics (we refer to \([20]\) for a brief exposition of this subject in relation to periodic Toda flow). Let \( l \) be the space of infinite to both sides sequences \( \{\psi_k\} \):

\[
\psi_k \in \mathbb{C}, \quad k \in \mathbb{Z}
\]

Consider the **periodic discrete Schrödinger** operator given by

\[
H : l \rightarrow l,
\]

\[
H(\psi)_k = b_{k-1}\psi_{k-1} + a_k\psi_k + b_k\psi_{k+1}
\]

where we assume \( a_{k+n} = a_k \) and \( b_{k+n} = b_k \). The eigenfunction \( \psi \in l \) of the Schrödinger operator with eigenvalue \( x \) satisfies the equation

\[
H(\psi) = x\psi
\]

Since \( b_i \neq 0 \), every eigenfunction is determined by its initial values \( (\psi_0, \psi_1) \in \mathbb{C}^2 \). We can define the **monodromy operator** along the period:

\[
M(x) : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad M(x) : (\psi_0, \psi_1) \mapsto (\psi_n, \psi_{n+1}).
\]

Note that the matrix \( L(w) \) has eigenvalue \( x \) if and only if there exists a solution \( \psi \) to \( (4.2) \) such that

\[
\psi_{k+n} = w\psi_k
\]

Such functions are called **Bloch solutions**. We see that whenever there exists a nonzero Bloch solution with parameter \( w \), the number \( w \) is the eigenvalue of the monodromy operator \( M(x) \), so we get a relation

\[
\det(w - M(x)) = 0.
\]

This equation defines a so called **spectral curve** of the periodic Schrödinger equation in the space of parameters \( (w, x) \in \mathbb{C}^2 \). One can show that \( \det M(x) = 1 \) (hint: the operator \( M_i : (\psi_{i-1}, \psi_i) \mapsto (\psi_i, \psi_{i+1}) \) has determinant \( \frac{b_i}{b_i} \), therefore \( M = M_nM_{n-1}\cdots M_1 = \frac{b_{n-1}}{b_n} \frac{b_{n-2}}{b_{n-1}} \cdots \frac{b_0}{b_1} = 1 \)). Hence, the equation \( (4.4) \) of the spectral curve can be rewritten in the form

\[
w^2 - \text{tr } M(x)w + 1 = 0.
\]

It can be shown that \( \text{tr } M(x) = \frac{1}{B}P(x) \), where \( B = \prod_{i=1}^{n} b_i \) as before, and \( P(x) \) is a monic polynomial in \( x \) (Hint: decompose \( M(x) \) as the product of operators \( M_i \) along the period, \( i = 1, \ldots, n \), and count the terms of highest degree of \( x \)). Dividing \( (4.5) \) by \( w \) and denoting \( t = \text{Re } w = \frac{1}{2}(w + w^{-1}) \), we get \( 2t = \frac{1}{B}P(x) \).
The polynomial $P(x) - 2Bt$ is monic and has the given sequence $\lambda_1, \ldots, \lambda_n$ as its roots, therefore
\[
P(x) - 2Bt = \prod_{i=1}^{n} (x - \lambda_i) = F(x),
\]
(4.6)
\[
P(x) = F(x) + 2Bt
\]
Consider the set
\[
\mathcal{A} = \{ s \in \mathbb{R} \mid P(x) = 2Bs \text{ has } n \text{ real roots} \}.
\]
Recalling the definition of $m$ and $M$ and remark 3.3 as well as relation (4.6), we see that $\mathcal{A}$ is the closed interval $[-\frac{m}{2B} + t, \frac{M}{2B} + t]$.

Note that the polynomial $P(x) - 2Bs$ is the characteristic polynomial of the matrix $L(w_s)$, where $|w_s| = 1$, $\Re w_s = s$. Therefore, the equation $P(x) = 2Bs$ necessarily has $n$ real roots for any $s \in [-1, 1]$. Therefore,
\[
[-1; 1] \subseteq \left[-\frac{m}{2B} + t, \frac{M}{2B} + t \right],
\]
from which we deduce $B \leq \frac{1}{2} \min(\frac{M}{1-t}, \frac{m}{1+t})$. Remembering $t = \Re w = \cos \arg w$, we get the required inequality.

**Proposition 4.2.** For $z \in \mathbb{B}$, $z \neq 0$, the preimage $\tilde{p}^{-1}(z)$ is homeomorphic to a torus. The dimension of a torus is $n - 1$ if $z$ lies in the interior of $\mathbb{B}$, $n - 1 - n_+$ if $z$ lies in the relative interior of $F_+$, $n - 1 - n_-$ if $z$ lies in the relative interior of $F_-$, and $n - 1 - n_+ - n_-$ if $z$ is either $z_{\text{top}}$ or $z_{\text{bot}}$.

**Proof.** In short, this follows from the fact that the periodic Toda lattice is an integrable dynamical system and its energy levels are the compact submanifolds. Liouville–Arnold theorem then implies that these preimages $\tilde{p}^{-1}(z)$ are tori. To specify the dimensions we give more details on the theory, related to periodic tridiagonal matrices.

As before, consider $P(x) = B \text{ tr } M(x)$, the monic polynomial in $x$ with coefficients depending on $a_i, b_i \in \mathbb{R}$. As follows from the considerations above, the eigenvalues of matrices $L(1)$ and $L(-1)$ are the roots of the polynomials $P(x) - 2B$ and $P(x) + 2B$ respectively. Let
\[
x_1 < x_2 \leq x_3 < \cdots < x_{2n-2} \leq x_{2n-1} < x_2
\]
be the union of all these roots, so that $x_2, x_3, x_{2n-2}, x_{2n-1}, x_{2n-5}, x_{2n-6}, \ldots$ are the roots of $P(x) - 2B$ and $x_{2n-1}, x_{2n-2}, x_{2n-5}, x_{2n-6}, \ldots$ are the roots of $P(x) + 2B$. The intervals
\[
I_1 = [x_2, x_3], \quad I_2 = [x_4, x_5], \quad \ldots, \quad I_{n-1} = [x_{2n-2}, x_{2n-1}]
\]
are called the forbidden zones. We will call $I_{n-1}, I_{n-3}, \ldots$ lower forbidden zones, and $I_{n-2}, I_{n-4}, \ldots$ upper forbidden zones as motivated by Fig.4.

Consider the Riemannian surface $\Theta_g$ of the multivalued function
\[
g(x) = \sqrt{\prod_{i=1}^{2n} (x - x_i)}
\]
Over each forbidden interval $I_k$, $k = 1, \ldots, n - 1$, there lies a circle $S_k$ on $\Theta_g$. If an interval $I_k$ degenerates to a point (i.e. $x_{2k} = x_{2k+1}$), the circle $S_k$ also collapses to a point. Van Moerbeke [23] proved

**Proposition 4.3.** Real periodic tridiagonal symmetric matrices $L_{a,b}$ with the given spectrum $\lambda$, given $B = \prod_i b_i$, and $b_i > 0$, are in one-to-one correspondence with $(n-1)$-tuples $(\mu_1, \ldots, \mu_{n-1})$, where $\mu_k \in S_k$.

Therefore, for real $z$, the preimage $\tilde{p}^{-1}(z)$ is diffeomorphic to a torus $T = \prod_{i}^{n-1} S_i$. The dimension of this torus equals $n - 1$ in general, however, when some forbidden intervals are collapsed, the dimension reduces by the number of collapsed intervals. The upper forbidden intervals collapse if and only if the value $2B$ reaches $M$. The number of collapses among upper intervals equals $n_+$. Similarly, the lower intervals collapse if $2B$ reaches $m$, and the number of collapses among lower intervals is $n_-$. 

Now let $L(\omega)$ be an arbitrary matrix with $\omega \in \mathbb{C}$, $|\omega| = 1$ and the given spectrum $\lambda$. Let $\tilde{\lambda}$ be the set of roots of the polynomial $\prod (x - \lambda_i) + 2B \Re \omega$. It was mentioned in the proof of Proposition 4.1 that the matrix $L(\omega)$ has spectrum $\lambda$ if and only if $L(1)$ has spectrum $\tilde{\lambda}$. Thus Proposition 4.3 implies the required statement for all matrices. $\square$

5. Permutohedral tilings

In this section we study the degenerate locus of the periodic Toda lattice. Recall that $X_{n,\lambda}^0 = p^{-1}(0) \subset X_{n,\lambda}$ is the set of all isospectral matrices with $B = \prod_i b_i = 0$, and $Q_{n,\lambda}^0 = X_{n,\lambda}^0 / T = \tilde{p}^{-1}(0)$.

We recall some standard facts from combinatorial geometry. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis of $\mathbb{Z}^n \cong \text{Hom}(T^n, S^1)$. We assume that $\mathbb{Z}^n \subset \mathbb{R}^n$ and there is a fixed inner product on $\mathbb{R}^n$ such that $\epsilon_1, \ldots, \epsilon_n$ are orthonormal.
Consider the sublattice \( N \subset \mathbb{Z}^n \) of rank \( n - 1 \) given by
\[
N = \text{Hom}(T^n/\Delta(T^1), S^1) = \left\{ \sum_{i=1}^{n} a_i \epsilon_i \,|\, a_i \in \mathbb{Z}, \sum a_i = 0, \text{ and } a_i - a_j \equiv 0 \mod n \right\}
\]
and let \( N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \) be its real span. Consider the vectors \( \alpha_1, \ldots, \alpha_n \):
\[
\alpha_i = (n - 1) \epsilon_i - \sum_{j \neq i} \epsilon_j, \quad i = 1, \ldots, n - 1.
\]
We see that
\[
\sum_{i=1}^{n} \alpha_i = 0,
\]
and any \( n - 1 \) of \( \alpha_1, \ldots, \alpha_n \) generate the lattice \( N \). One can think about \( \alpha_i \)'s as the outward unit normal vectors to the facets of a regular simplex in \( \mathbb{R}^{n-1} \).

For any subset \( S \subset [n] = \{1, \ldots, n\} \) such that \( S \neq \emptyset, [n] \), consider the vector
\[
\alpha_S = \sum_{i \in S} \alpha_i.
\]
Let \( P_{n-1} \) be the Voronoi cell decomposition of \( N_\mathbb{R} \cong \mathbb{R}^{n-1} \) generated by the lattice \( N \). In other words, for any \( \alpha \in N \) we consider the Voronoi cell
\[
P_\alpha = \{ x \in N_\mathbb{R} \mid \text{dist}(x, \alpha) \leq \text{dist}(x, \beta) \text{ for any } \beta \in N, \beta \neq \alpha \},
\]
where dist is the distance determined by the inner product on \( N_\mathbb{R} \subset \mathbb{R}^n \). Each \( P_\alpha \) is a convex \( (n - 1) \)-dimensional polytope and all these polytopes are the parallel copies of each other, \( P_\alpha = P_0 + \alpha \).

**Construction 5.1.** It can be shown that \( P_0 \) is the \( (n - 1) \)-dimensional permutohedron \( \text{Pe}^{n-1} \) determined by the inequalities
\[
\text{Pe}^{n-1} = P_0 = \left\{ x \in N_\mathbb{R} \mid \langle \alpha_S, x \rangle \leq \frac{1}{2} \langle \alpha_S, \alpha_S \rangle, S \in 2^{[n]}, S \neq \emptyset, [n] \right\}
\]
We recall the basic facts about the combinatorics of a permutohedron. The polytope \( \text{Pe}^{n-1} \) is simple, which means that every codimension \( k \) face is contained in exactly \( k \) facets. For a proper subset \( S \subset [n] \) let \( F_S \) denote the facet of \( \text{Pe}^{n-1} \) determined by the support hyperplane \( \langle \alpha_S, x \rangle = \frac{1}{2} \langle \alpha_S, \alpha_S \rangle \). Note that \( \text{Pe}^{n-1} \) is centrally symmetric: the facets \( F_S \) and \( F_{\bar{S}} \) are opposite to each other whenever \( \bar{S} = [n] \setminus S \).

Facets \( F_{S_1}, \ldots, F_{S_k} \) have nonempty intersection in \( \text{Pe}^{n-1} \) if and only if the subsets \( \{S_1, \ldots, S_k\} \) form a chain in the Boolean lattice \( 2^{[n]} \). If \( \sigma = (S_1 \subset \cdots \subset S_k) \) is such a chain, we denote by \( F_\sigma \) the face \( F_{S_1} \cap \cdots \cap F_{S_k} \) of the permutohedron. Each face of \( \text{Pe}^{n-1} \) is known to be a product of permutohedra of smaller dimensions.

We denote by \( F_S(P_\alpha) \) (resp. \( F_\sigma(P_\alpha) \)) the corresponding facets (resp. faces) of the the Voronoi cell \( P_\alpha \) to distinguish different copies of a permutohedron in the Voronoi diagram. It can be seen that
\[
F_S(P_\alpha) = F_S(P_{\alpha + \alpha_S}).
\]
A facet of each cell is adjoint to an opposite facet of a neighboring cell.
We formulate a general construction to precede a particular case needed in the proof of Theorem 3.4.

**Construction 5.2.** Let $\widehat{\mathcal{N}} \subset \mathcal{N}$ be a sublattice of finite index, i.e. $q = |\mathcal{N}/\widehat{\mathcal{N}}| < \infty$. Consider the quotient $\mathcal{N}/\widehat{\mathcal{N}}$. Since $\widehat{\mathcal{N}}$ is a cocompact lattice, this quotient is a torus $\mathcal{T}^{n-1}$. The action of $\widehat{\mathcal{N}}$ by parallel shifts preserves the Voronoi diagram, therefore we have a cell subdivision of the torus $\mathcal{T}^{n-1} \cong \mathcal{N}/\widehat{\mathcal{N}}$. There are $q$ maximal cells in this subdivision, each is a parallel copy of a permutohedron.

**Example 5.3.** A natural example is $\widehat{\mathcal{N}} = \mathcal{N}$. In this case the torus is given by identifying the opposite facets of a single permutohedron. The cell structure on a torus given by this identification is known: the corresponding partially ordered set was introduced and studied by Panina [27] under the name of *cyclopermutohedron*. This poset has a natural combinatorial description.

For the considerations of this paper we need another sublattice.

**Construction 5.4.** Let $\mathcal{N}' \subset \mathcal{N}$ be the sublattice generated by the vectors

$$
\beta_k = \alpha_k - \alpha_{k+1}, \quad k = 1, \ldots, n-1.
$$

Note that $\beta_{n-1} = \alpha_{n-1} - \alpha_n = 2\alpha_{n-1} + \alpha_1 + \cdots + \alpha_{n-2}$ according to (5.1). It can be shown that $\mathcal{N}/\mathcal{N}'$ is the cyclic group of order $n$. Indeed, in the quotient group $\mathcal{N}/\mathcal{N}'$ we have the identities

$$
[\alpha_1] = \cdots = [\alpha_n], \quad n[\alpha_1] = [\alpha_1] + \cdots + [\alpha_{n-2}] + 2[\alpha_{n-1}] = 0. \quad (5.4)
$$

**Definition 5.5.** Let $\mathcal{PT}^{n-1}$ be the cell decomposition of a torus $\mathcal{T}^{n-1}$ obtained as a quotient of Voronoi diagram of the space $\mathcal{N}/\mathcal{N}'$ by the sublattice $\mathcal{N}'$. We call $\mathcal{PT}^{n-1}$ the *wonderful cell decomposition* of a torus.

The wonderful decomposition $\mathcal{PT}^{n-1}$ has $n$ maximal cells. The cells $P_\alpha$ and $P_{\alpha+\beta}$ are identified in $\mathcal{PT}^{n-1}$ whenever $\beta \in \mathcal{N}'$. We denote the resulting cell of $\mathcal{PT}^{n-1}$ by $P_\alpha$. Relations (5.4) imply

$$
[\alpha_S] = |S|[\alpha_1], \quad [n\alpha_1] = [0].
$$

**Lemma 5.6.** Let $1 \leq k < m \leq n$. In the cell complex $\mathcal{PT}^{n-1}$ we have

$$
F_S(P_k[\alpha_1]) = F_S(P_m[\alpha_1]),
$$

where $S$ is any subset of $[n]$ of cardinality $m - k$.

**Proof.** Choose any subset $S'$ such that $|S'| = k$ and $S'$ is disjoint from $S$. According to (5.3) we have

$$
F_S(P_k[\alpha_1]) = F_S(P_{\alpha_{S'}}) = F_S(P_{\alpha_{S'} + \alpha_1}) = F_S(P_{[\alpha_{S'+1}]} = F_S(P_m[\alpha_1]).
$$

which proves the statement.
In the following, we denote the maximal cells \( P_k(\alpha_1) \) by \( \mathcal{PT}_k \). Now we return to the space of tridiagonal matrices. Recall that \( Y_k \) denotes the space of all isospectral matrices \( L(\alpha, \beta) \) with \( b_k = 0 \), for \( k = 1, \ldots, n \). Let \( Q_k \) denote the orbit space \( Y_k/T \). We have \( Q_{n,\lambda}^0 = \bigcup_{i=1}^n Q_k \). For convenience introduce the cyclic notation: \( Q_k = Q_{k+n} \), for any \( k \in \mathbb{Z} \).

**Theorem 5.7.** The space \( Q_{n,\lambda}^0 \) can be identified with \( \mathcal{PT}^{n-1} \) so that the subspaces \( Q_k \) are identified with \( \mathcal{PT}_k \).

**Proof.** The orbit space \( Q_0 = Q_n \) is identified with the space of all tridiagonal symmetric real matrices

\[
L = \begin{pmatrix}
  a_1 & b_1 & 0 & \cdots & 0 \\
  b_1 & a_2 & b_2 & 0 \\
  0 & b_2 & a_3 & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & b_{n-1} \\
  0 & 0 & \cdots & b_{n-1} & a_n 
\end{pmatrix}
\]

with \( b_i \geq 0 \) and the given simple spectrum \( \lambda \). It is known (see [32]) that \( Q_0 \) is diffeomorphic to a permutohedron \( \text{Pe}^{n-1} \) as a manifold with corners. The facet \( F_S(\text{Pe}^{n-1}) \) corresponds to the subset of \( Q_0 \), which consists of matrices \( L \) such that \( b_{|S|} = 0 \) and the eigenvalues \( \{\lambda_i \mid i \in S\} \) are distributed in the first \((|S| \times |S|)\)-block.

Similar considerations are valid for other spaces \( Q_k \): this can be shown by cyclic permutation of rows and columns of \( L \). Indeed, the set \( Q_k \) can be identified with \( \text{Pe}^{n-1} \) in such way that the facet \( F_S(\text{Pe}^{n-1}) \) consists of all matrices with the property

\[ b_k = 0, \quad b_{k+|S|} = 0, \]

and the block between \( k \)-th and \((k + |S|)\) rows and columns has eigenvalues \( \{\lambda_i \mid i \in S\} \).

It can be seen that the faces \( F_S(Q_k) \) and \( F_S(Q_m) \) represent the same set of matrices for \( 1 \leq k < m \leq n \) and \( |S| = m - k \). Therefore, \( F_S(Q_k) = F_S(Q_m) \) in \( Q_{n,\lambda}^0 \). These gluing rules for the cells in \( Q_{n,\lambda}^0 \) coincide with the gluing rules for \( \mathcal{PT}_k \) in \( \mathcal{PT}^{n-1} \) according to Lemma 5.6.

**Example 5.8.** Right part of Fig. 5 shows the space \( Q_{3,\lambda}^0 = \mathcal{PT}^2 \). This example was described in details by van Moerbeke [23]. The 1-skeleton of \( \mathcal{PT}^2 \) is shown on the left. As an abstract graph, it is isomorphic to the complete bipartite graph \( K_{3,3} \). This graph is a GKM-graph of the complete flag variety \( \text{Fl}_3 \), see details in [4].

**Remark 5.9.** Let us briefly sketch the phase portrait of the Toda flow on the degenerate set of orbits \( Q_{n,\lambda}^0 \). Let \( v \in N_{\mathbb{R}}^* \) be a generic linear function on \( N_{\mathbb{R}} \). Take any face of any permutohedron of the Voronoi diagram in \( N_{\mathbb{R}} \). On each such polytope consider a flow, which moves all points in the interior of \( P \) to the vertex maximizing the linear function \( v \). The flow looks the same on all Voronoi cells, thus we have an induced flow on the torus \( \mathcal{PT}^{n-1} = N_{\mathbb{R}}/N' \).

This picture describes the Toda flow on \( Q_{n,\lambda}^0 \approx \mathcal{PT}^{n-1} \). Indeed, Toda flow degenerates to the flow of a non-periodic Toda lattice on each permutohedron \( Q_i \), and its Morse-like
behavior is well-known (see [16]). For any block tridiagonal matrix, the Toda flow “sorts” the diagonal elements within each block [32].

The phase portrait for \( n = 3 \) is shown on the left part of Fig.6. The oddity of the phase portrait near equilibria points is explained by the fact that the orbit space \( Q_{n,\lambda}^0 \) is not smooth at these points.

Note that for \( B \neq 0 \), the Toda flow exhibits Liouville–Arnold behavior. The equilibria points disappear, however the flow still follows some direction \( v \) on a torus, see Fig.6 right part.

Remark 5.10. Each \( k \)-dimensional cell of the cell subdivision \( \mathcal{P}T^{n-1} \) lies in exactly \( n-k \) different maximal cells. This means there exists a dual simplicial cell subdivision \( K^{n-1}_{\mathcal{P}T} \). In Section 7 we recall the definition of a simplicial poset which is a useful combinatorial notion to study simplicial cell subdivisions.

Note that the simplicial poset \( K^{n-1}_{\mathcal{P}T} \) minimizes the number of vertices among all simplicial cell subdivisions of the torus \( T^{n-1} \). Indeed, any \((n-1)\)-dimensional simplex of such subdivision has \( n \) distinct vertices, therefore a simplicial cell subdivision of \( T^{n-1} \) should have at least \( n \) vertices. This number is achieved at \( K^{n-1}_{\mathcal{P}T} \).
Remark 5.11. Note that any closed connected $(n - 1)$-manifold admits a simplicial cell subdivision with exactly $n$ vertices. This result was proved in [17], where such subdivisions (or, their equivalent combinatorial representations) were called the crystallizations. Previous remark shows that $K_{\mathcal{P}T}^n$ provides an explicit crystallization for the torus $T^{n-1}$.

Propositions 4.1, 4.2 and Theorem 5.7 conclude the proof of Theorem 3.4.

6. Topology near degeneration locus

In this section we study the topology of a small neighborhood of $X^0_{n,\lambda}$. The space $X_{n,\lambda}$ is a smooth manifold in vicinity of $X^0_{n,\lambda}$; see Construction 2.3.

Remark 6.1. Note that the $T$-action is free outside $X^0_{n,\lambda}$ and admits a section given by the formula (4.1). However, the free part of the action is larger than $X^0_{n,\lambda}$; the action is also free over the interiors of facets of $Q^0_{n,\lambda} \cong \mathcal{P}T^{n-1}$. The whole free action $X^0_{n,\lambda} \rightarrow X^0_{n,\lambda}/T$ does not admit a section, as explained below.

Recall that $p: X_{n,\lambda} \rightarrow \mathbb{C}$ maps a matrix $L(a, b)$ to the product $B = \prod_i b_i$. For a small $\varepsilon$ consider the preimage of points close to zero:

$$X_{n,\lambda}^{\leq \varepsilon} = p^{-1}(\{z \in \mathbb{C} \mid |z| \leq \varepsilon\}).$$

According to Proposition 4.2, $X_{n,\lambda}^{\leq \varepsilon}$ is a manifold with boundary, the boundary $\partial X_{n,\lambda}^{\leq \varepsilon}$ being the subset

$$(6.1) \quad X_{n,\lambda}^{\leq \varepsilon} = p^{-1}(\{|z| = \varepsilon\}) \cong S^1_{\varepsilon} \times T^{n-1} \times T^{n-1} \cong T^{2n-1},$$

where $S^1_{\varepsilon} = \{z \mid |z| = \varepsilon\}$, $T^{n-1}$ is the Liouville–Arnold torus, and $T^{n-1}$ is the acting torus.

It will be useful to incorporate the circle $S^1_{\varepsilon}$ into the action to obtain a $T^n$-action on $X_{n,\lambda}^{\leq \varepsilon}$.

Construction 6.2. Consider a topological manifold with boundary $W = T^{n-1} \times [0, 1]$. Its boundary consists of two connected components

$$\partial W = \partial_0 W \sqcup \partial_1 W, \quad \partial_0 W = T^{n-1} \times \{0\}, \quad \partial_1 W = T^{n-1} \times \{1\}.$$ 

On the left component $\partial_0 W$, we introduce the wonderful cell structure $\mathcal{P}T^{n-1}$, constructed in Section 5. This procedure subdivides $\partial_0 W$ into $n$ permutohedra $\mathcal{P}T_1, \ldots, \mathcal{P}T_n$ of dimension $n - 1$ so that every cell of dimension $k$ lies in $n - k$ top-dimensional cells. This makes $W$ a manifold with corners (understood in a broad topological sense). We leave the right boundary component $\partial_1 W$ unchanged: no face structure is imposed on $\partial_1 W$.

Let $T^n = \{t = (t_1, \ldots, t_n) \mid |t_i| = 1\}$ be a compact $n$-torus and $T_I, I \subseteq [n]$ be its coordinate subtorus,

$$T_I = \{t \in T^n \mid t_j = 1, j \notin I\}.$$ 

Consider the space

$$Y = W \times T^n / \sim$$

where $(r, t)$ and $(r', t')$ are identified whenever $r = r'$ lies in the intersection of facets $\{\mathcal{P}T_i \mid i \in I\}$ and $t^{-1}t' \in T_I$ for some subset $I \subseteq [n]$. This construction can be considered
as particular case of either moment-angle manifold construction for simplicial posets (see [22, 10]) or the construction of locally standard actions (see [34]). The space \( Y \) is a particular case of the collar models introduced in [3].

The space \( Y \) is a manifold with boundary \( \partial Y = \partial_1 W \times T^n \cong \mathcal{T}^{n-1} \times T^n \). It carries the action of \( T^n \) which is free on the boundary and its orbit space is \( W \).

Consider the induced action of the subtorus \( T^{n-1} = \{ t_1 t_2 \cdots t_n = 1 \} \subset T^n \) on the space \( Y \). It can be checked (see details in [4]) that the orbit space \( Y/T^{n-1} \) is homeomorphic to \( \mathcal{T}^{n-1} \times D^2 \).

**Theorem 6.3.** The space \( X_{n, q}^{< \epsilon} \) is \( T^{n-1} \)-equivariantly homeomorphic to the collar model \( Y \).

**Proof.** In [4] we developed a topological theory of complexity one torus actions. The main concepts are recalled here. By definition, an effective action of \( T \cong T^{n-1} \) on \( X = X^{2n} \) is called a *strictly appropriate action in general position*, if the following conditions hold.

1. The action has finitely many fixed points.
2. Stabilizers of all points are connected.
3. Each connected component of each equivariant skeleton \( X_1 \) contains a fixed point.
4. For every fixed point \( x \), the weights \( \alpha_1, \ldots, \alpha_n \in \text{Hom}(T^{n-1}, S^1) \cong \mathbb{Z}^{n-1} \) of the tangent representation are in general position, which means that every \( n - 1 \) of them are linearly independent.

For such actions we proved that the orbit space \( Q = X/T^{n-1} \) is a topological manifold of dimension \( n + 1 \). The orbits of dimensions less than \( n - 1 \) form a subset \( Z \subset Q \) which is called a *sponge*. A sponge is an \( (n-2) \)-dimensional subset of \( Q \) locally modeled by a \( (n-2) \)-skeleton of \( \mathbb{R}^n_{\geq 0} \). The free part of action gives the principal \( T^{n-1} \)-bundle

\[
X^{\text{free}} \rightarrow Q \setminus Z.
\]

This bundle is classified by the cohomology class \( e \in H^2(Q \setminus Z; H_1(T^{n-1})) \), which is called the *Euler class* of the action. Proposition 3.7 of [4] asserts that equivariant topological type of \( X \) is uniquely determined by the triple \( (Q, Z, e) \) (which essentially means that the information on stabilizers of the action can be recovered from the class \( e \)).

The inclusion \( i_x : U_x \rightarrow Q \) induces a homomorphism

\[
i_x^* : H^2(Q, Q \setminus Z; H_1(T^{n-1})) \rightarrow H^2(U_x, U_x \setminus Z; H_1(T^{n-1})).
\]

The class \( e_x = i_x^*(e) \in H^2(U_x, U_x \setminus Z; H_1(T^{n-1})) \) is called the local Euler class at \( x \). It was noted in [4] that local Euler classes are always nonzero. In particular, the global Euler class is always non-zero for suitable actions of complexity one.

These constructions work similarly if \( X \) is a manifold with boundary, and the torus action is free on the boundary. In this case, \( Q = X/T^{n-1} \) is a manifold with boundary \( \partial X/T^{n-1} \). The sponge of the action lies in the interior of \( Q \). Under certain conditions the local Euler classes at fixed points determine the space \( X \) uniquely.

Assume \( Q \) has the form \( Q_M = M \times D^2 \), where \( M \) is a closed \( (n-1) \)-manifold with a fixed simple cell decomposition. Assume that the sponge \( Z_M \) is the \( (n-2) \)-skeleton of this
cell structure, and we have

\[ Z_M = M^{(n-2)} = M^{(n-2)} \times \{0\} \subset M \times D^2 = Q_M. \]

**Proposition 6.4 ([4 Prop.5.7]).** Let X be a manifold with boundary, which carries a strictly appropriate torus action in general position such that the orbit space and the sponge of the action are given by \((Q_M, Z_M)\). Assume that the free action of T on the boundary is a trivial principal bundle. Then the local Euler classes at fixed points uniquely determine the \(T^{n-1}\)-equivariant homeomorphism type of X.

Apply this proposition to spaces \(X_{n,\lambda}\) and \(Y\). The orbit space is \(T^{n-1} \times D^2\) in both cases. The sponge of the action is the \((n-2)\)-skeleton of the wonderful cell subdivision \(\mathcal{PT}^{n-1}\), defined earlier. The free action on the boundary is a trivial principal bundle. This is true for \(X_{n,\lambda}\) since there is a section of the action, see remark 6.1. This is true for \(Y\) since \(Y = P \times T^n/\approx\), and the \(T^n\)-action over \(\hat{\partial}_1 P\) is a trivial principal bundle.

Finally, consider any fixed point \(x = L_\sigma\) of \(X_{n,\lambda}\). The tangent representation at \(x\) is isomorphic to the standard action of \(T^{n-1}\) on \(C^n\) (the infinitesimal action just rotates off-diagonal entries, so that the angles of rotation sum to zero). However the action of \(T^{n-1}\) in the neighborhood of the corresponding fixed point of \(Y\) is exactly the same by the definition of \(Y\). Therefore the local Euler classes of \(X_{n,\lambda}\) and \(Y\) coincide at each fixed point.

Proposition 6.4 then implies the existence of \(T^{n-1}\)-homeomorphism \(X_{n,\lambda} \cong Y\).

**7. Enumerative combinatorics of the wonderful subdivision**

In this section we study the enumerative invariants of the permutoheral cell complex \(\mathcal{PT}^{n-1}\) or, equivalently, its dual simplicial poset \(K^{n-1}_{\mathcal{PT}}\). These invariants will be used further to describe the homological structure of \(X_{n,\lambda}\). At first, we recall several standard definitions from commutative algebra and combinatorics.

**Definition 7.1.** A finite partially ordered set \(S\) is called **simplicial** if it has the minimal element \(\hat{0} \in S\) and, for any \(I \in S\), the order interval \(\{J \in S \mid J \leq I\}\) is isomorphic to the poset of faces of a \(k\)-dimensional simplex, for some number \(k \geq 0\).

The elements of \(S\) are called **simplices**. The number \(k\) from the definition is called the dimension of a simplex \(I\). A simplex of dimension \(0\) is called a **vertex**. The geometrical realization of \(S\) is the simplicial cell complex, obtained by gluing geometrical simplices according to the order relation in \(S\), see [9] for details. In the following we only consider **pure** simplicial posets, which means that all maximal elements of \(S\) have the same dimension. A simplicial poset is called a **homology sphere** (resp. a **homology manifold**) if its geometrical realization is a homology sphere (resp. a homology manifold).

**Construction 7.2.** Let \(f_j\) denote the number of \(j\)-dimensional simplices of \(S\) for \(j = -1, 0, \ldots, n-1\), in particular, \(f_{-1} = 1\) (the empty simplex \(\hat{0}\) has dimension \(-1\)).
$h$-numbers of $S$ are defined from the relation:

$$\sum_{j=0}^{n} h_j t^{n-j} = \sum_{j=0}^{n} f_{j-1}(t-1)^{n-j},$$

where $t$ is a formal variable. Let $\tilde{\beta}_j(S) = \dim \tilde{H}_j(S)$ be the reduced Betti number of the geometric realization of $S$. $h'$- and $h''$-numbers of $S$ are defined as follows

$$h'_j = h_j + \left(\sum_{i=0}^{j} \frac{(-1)^{i} \beta_{i-1}(S)}{i+1}\right) \quad \text{for } 0 \leq j \leq n;$$

$$h''_j = h'_j - \left(\sum_{i=0}^{j} \frac{(-1)^{i} \tilde{\beta}_{i-1}(S)}{i+1}\right) \quad \text{for } 0 \leq j \leq n - 1, \text{ and } h''_n = h'_n.$$ Sums over empty sets are assumed zero.

Let $[m] = \{1, \ldots, m\}$ be the vertex set of $S$, $m = f_0$. Let $R$ be a field or the ring $\mathbb{Z}$, and let $R[m] = R[v_1, \ldots, v_m]$, $\deg v_i = 2$, denote the graded polynomial algebra with $m$ generators, corresponding to the vertices of $S$. Slightly abusing the terminology, we call the elements of degree 2 linear, when working with such polynomial rings. For a graded $R$-module $V^* = \bigoplus_{j=0}^{\infty} V_j$ we denote by Hilb$(V^*; t)$ its Hilbert–Poincare function $\sum_{j=0}^{\infty} t^j \text{rk}_R V_j \in \mathbb{Z}[t]$.

**Definition 7.3** (see [31]). The face ring of a simplicial poset $S$ is the commutative associative graded algebra $R[S]$ over a ring $R$ generated by formal variables $v_I$, one for each simplex $I \in S$, with relations

$$v_{I_1} \cdot v_{I_2} = v_{I_1 \cap I_2} \cdot \sum_{J \in I_1 \vee I_2} v_J, \quad v_0 = 1.$$ 

Here $I_1 \vee I_2$ denotes the set of least upper bounds of $I_1, I_2 \in S$, and $I_1 \cap I_2 \in S$ is the intersection of simplices (it is well-defined and unique when $I_1 \vee I_2 \neq \emptyset$). We take the doubled grading on the ring, in which $v_I$ has degree 2($\dim I + 1$). The natural graded ring homomorphism $R[m] = R[v_1, \ldots, v_m] \to R[S]$ defines the structure of the $R[m]$-module on $R[S]$.

If $R$ is an infinite field, and $\dim S = n - 1$, then a generic set of linear elements $\theta_1, \ldots, \theta_n \in R[S]_2$ is a linear system of parameters (we remark that linear systems of parameters can be constructed using characteristic functions on $S$, see e.g. [10], Lm.3.5.8]). Let $\Theta$ denote the parametric ideal of $R[S]$ generated by $\theta_1, \ldots, \theta_n$.

**Proposition 7.4** (Reisner, Stanley [28, 30]). For a pure simplicial poset $S$ of dimension $n - 1$ there holds

$$\text{Hilb}(R[S]; t) = \frac{h_0 + h_1 t^2 + \cdots + h_n t^n}{(1 - t^2)^n}.$$ 

For a homology sphere $S$ there holds $\text{Hilb}(R[S]/\Theta; t) = \sum_i h_i t^{2i}$. 
Proposition 7.5 (Schenzel, Novik-Swartz [29, 25, 26]).

(1) For a homology manifold $S$ there holds

$$\text{Hilb}(R[S]/\Theta; t) = \sum_i h_i t^{2i}.$$ 

(2) Let $S$ be a connected $R$-orientable homology manifold of dimension $n - 1$. The $2j$-th graded component of the module $R[S]/\Theta$ contains a vector subspace $(I_{NS})_{2j} \cong \binom{n}{j} \tilde{H}^{j-1}(S; R)$, which is a trivial $R[m]$-submodule (i.e. $R[m] \cdot (I_{NS})_{2j} = 0$). Let $I_{NS} = \bigoplus_{j=0}^{n-1} (I_{NS})_{2j}$ be the sum of all these submodules except the top-degree component. Then the quotient module $R[S]/\Theta/I_{NS}$ is a Poincare duality algebra, and there holds

$$\text{Hilb}(R[S]/\Theta/I_{NS}; t) = \sum_i h''_i t^{2i}.$$ 

We now compute the combinatorial characteristics of the simplicial poset $K_{\mathcal{PT}}^{n-1}$ dual to $\mathcal{PT}^{n-1}$. Combinatorially, the simplicial cell complex $K_{\mathcal{PT}}^{n-1}$ can be defined as a poset, whose elements are the faces of the wonderful cell decomposition $\mathcal{PT}^{n-1}$ and the order is given by the reversed inclusion. It can be seen that $K_{\mathcal{PT}}^{n-1}$ is a simplicial poset. Recall that $\binom{n}{k}$ denotes the Stirling number of the second kind, that is the number of unordered partitions of the set $\{n\}$ into $k$ nonempty subsets.

Proposition 7.6. For the simplicial poset $K_{\mathcal{PT}}^{n-1}$ there holds

$$f_{k-1} = n(k-1)! \binom{n}{k}$$ for $k = 1, 2, \ldots, n; \quad f_{-1} = 1;$$

(7.4) $$h_l = (-1)^l \binom{n}{n-l} + \sum_{k=1}^{l} (-1)^{l-k} \binom{n-k}{n-l} n(k-1)! \binom{n}{k}$$ for $l = 0, 1, \ldots, n$$

(7.5) $$h'_l = (-1)^l \binom{n}{n-l} + \sum_{k=1}^{l} (-1)^{l-k} \binom{n-k}{n-l} n(k-1)! \binom{n}{k} + \binom{n}{l} \sum_{k=2}^{l-1} (-1)^{l-k-1} \binom{n-1}{k-1}$$ for $l = 0, 1, \ldots, n$$

(7.6) $$h''_l = (-1)^l \binom{n}{n-l} + \sum_{k=1}^{l} (-1)^{l-k} \binom{n-k}{n-l} n(k-1)! \binom{n}{k} + \binom{n}{l} \sum_{k=2}^{l-1} (-1)^{l-k-1} \binom{n-1}{k-1}$$ for $l = 0, 1, \ldots, n-1$, and $h''_n = 1$. 

Proof. From the combinatorial description of a permutohedron it follows that the number $f_{n-k}(\text{Pe}^{n-1})$ is equal to $k! \binom{n}{k}$. The wonderful subdivision $\mathcal{PT}^{n-1}$ consists of $n$
permutohedra and each \((n - k)\)-dimensional face of \(PT^{n-1}\) lies in exactly \(k\) permutohedral cells, since the subdivision is simple. Therefore,
\[
f_{k-1}(K_{PT}^{n-1}) = f_{n-k}(PT^{n-1}) = \frac{n}{k} f_{n-k}(\text{Pe}^{n-1}) = n(k-1)! \binom{n}{k}
\]
for \(k \geq 1\). The identity \(f_{-1} = 1\) holds automatically.

Since \(K_{PT}^{n-1}\) is a simplicial cell subdivision of the torus \(T^{n-1}\), we have \(\tilde{\beta}_{j}(K_{PT}^{n-1}) = \binom{n-1}{j}\) for \(j \geq 1\). Expressions (7.4), (7.5), and (7.6) follow from the general definitions of \(h_{-}\), \(h'_{-}\), and \(h''\)-numbers.

8. Equivariant cohomology

Let \(X\) be a 2\(n\)-manifold with a locally standard action of \(T^{n}\). The orbit space \(P = X/T^{n}\) is a manifold with faces. It means that every codimension \(k\) face of \(P\) lies in exactly \(k\) different facets of \(P\). Let \(S_{P}\) denote the simplicial poset dual to the poset of faces of \(P\). In [7] we proved

**Proposition 8.1.** Assume that all proper faces of \(P\) are acyclic and the projection map \(X \to P\) admits a section. Then \(H^{*}_{\text{eq}}(X; \mathbb{Z}) \cong \mathbb{Z}[S_{P}] \oplus H^{*}(P; \mathbb{Z})\) as the rings, and as the modules over \(\mathbb{Z}[n] \cong H^{*}(BT^{n})\). The components of degree 0 are identified in the direct sum. The ring \(H^{*}(P; \mathbb{Z})\) is considered a trivial \(\mathbb{Z}[n]\)-module.

Now we apply this statement to the space \(X^{\leq \varepsilon}_{n, \lambda}\) which is \(T^{n-1}\)-equivariantly homeomorphic to \(Y\) (see construction [6.2]).

**Theorem 8.2.** The Hilbert–Poincare series of the \(T^{n-1}\)-equivariant cohomology ring of \(X^{\leq \varepsilon}_{n, \lambda}\) is given by
\[
\text{Hilb}(H^{*}_{T^{n-1}}(X^{\leq \varepsilon}_{n, \lambda}); t) = \frac{\sum_{i=0}^{n} h_{i} t^{2i}}{(1 - t^{2})^{n-1}} + (1 + t)^n - 1 - t.
\]

Here \(h_{i}\), the \(h\)-numbers of the simplicial poset \(K_{PT}^{n-1}\), are given by (7.4).

**Proof.** Recall that \(Y\) is the collar model, that is the locally standard \(T^{n}\)-space over \(T^{n-1} \times [0, 1]\). Proposition 8.1 implies the following isomorphism for the \(T^{n}\)-equivariant cohomology
\[
H^{*}_{T^{n}}(Y) \cong \mathbb{Z}[K_{PT}^{n-1}] \oplus H^{*}(T^{n-1}).
\]

There is an induced action of the \((n - 1)\)-dimensional subtorus
\[
T^{n-1} = \{t_{1} \cdots t_{n} = 1\}
\]
on \(Y\), and Theorem 6.3 states that \(Y\) and \(X^{\leq \varepsilon}_{n, \lambda}\) are \(T^{n-1}\)-equivariantly homeomorphic. To compute the \(T^{n-1}\)-equivariant cohomology of \(Y\), we first note that there is a Serre fibration
\[
Y_{T^{n-1}} \xrightarrow{S^{1}} Y_{T^{n}}, \quad S^{1} = T^{n}/T^{n-1}
\]
where \(Y_{T^{n-1}}\) and \(Y_{T^{n}}\) are the Borel constructions of \(T^{n-1}\) and \(T^{n}\)-actions on \(Y\) respectively. Consider the corresponding Serre spectral sequence:
\[
E_{2}^{p,q} = H^{p}_{T^{n}}(Y) \otimes H^{q}(S^{1}) \Rightarrow H^{p+q}_{T^{n-1}}(Y).
\]
The sequence has only two nonzero rows, hence it collapses at the $E_3$-term. Let $\omega$ denote a generator of $H^1(S^1)$. The second differential $d_2 \colon H^1(S^1) \to H^2_{T^n}(Y)$ of the spectral sequence coincides with the composition

\[ H^1(T^n/T^{n-1}) \cong H^2(B(T^n/T^{n-1})) \to H^2(BT^n) \to H^2_{T^n}(Y), \]

where the middle map is induced by the projection $T^n \to T^n/T^{n-1}$ and the right map is the defining map for the $H^*(BT^n)$-module structure on $H^*_{T^n}(Y)$. It follows that

\[ d_2(\omega) = \eta \in \mathbb{Z}[K^\alpha_{PT}]_2 \subset H^2_{T^n}(Y), \]

where $\eta = \sum_{i=1}^n v_i$, according to the definition (8.1) of the subtorus $T^{n-1}$.

**Lemma 8.3.** $\eta$ is not a zero divisor in the face ring $\mathbb{Z}[K^\alpha_{PT}]$, or, equivalently, $\eta$ is a regular element.

**Proof.** We use the standard argument in the theory of face rings. For any non-empty simplex $I \cong \Delta^k$ in $K^\alpha_{PT}$ consider the epimorphism

\[ \varphi_\sigma : \mathbb{k}[K^\alpha_{PT}] \to \mathbb{k}[I] \]

defined by sending $v_I$ to 0 for all $J \subsetneq I$. Notice that $\mathbb{k}[I]$ is just the polynomial algebra in $\dim I + 1$ generators. The map $\varphi_\sigma$ is a homomorphism of $\mathbb{k}[n]$-algebras, with the $\mathbb{k}[n]$-structure on $\mathbb{k}[I]$ is defined by an epimorphism $\psi_I$ sending the excess variables to zeroes.

Assume that there exists $\beta \in \mathbb{k}[K^\alpha_{PT}]$ such that $\eta \cdot \beta = 0$. Then $\psi_I(\eta) \cdot \varphi_\sigma(\beta) = 0$ in $\mathbb{k}[I]$. Since there are no zero divisors in $\mathbb{k}[I]$, and $\psi_I(\eta) = \sum_{i \in I} v_i \neq 0$, we have $\varphi_\sigma(\beta) = 0$ for any simplex $I$ of $K^\alpha_{PT}$. The homomorphism

\[ \bigoplus_\sigma \varphi_\sigma : \mathbb{k}[K^\alpha_{PT}] \to \bigoplus_\sigma \mathbb{k}[I_\sigma] \]

is known to be injective [10, Thm 3.5.6]. Therefore $\beta = 0$. \hfill $\square$

According to the lemma, for any nonzero element $\beta \in \mathbb{Z}[K^\alpha_{PT}] \subset H^*_{T^n}(Y)$, there holds

\[ d_2(\omega \beta) = (d_2 \omega) \beta \pm \omega d_2(\beta) = \eta \beta \neq 0 \]

In other words, $d_2$ is injective on the submodule $\mathbb{Z}[K^\alpha_{PT}] \otimes H^1(S^1) \subset E^2_{2,1}.

On the other hand, for any element $\alpha \in H^i(T^{n-1}) \subset H^i_{T^n}(Y)$, $i > 0$, we have

\[ d_2(\omega \alpha) = (d_2 \omega) \alpha \pm \omega d_2(\alpha) = \eta \cdot \alpha = 0, \]

since the products of elements from the components $H^+(T^{n-1})$ and $\mathbb{Z}[K^\alpha_{PT}]_+$ of the ring $H^*_{T^n}(Y)$ vanish. Therefore the differential $d_2$ vanishes on $H^+(T^{n-1})$. Finally, we have

\[ E^p_{3,q} \cong \begin{cases} \mathbb{Z}[K^\alpha_{PT}]/(\eta)^p \oplus H^p(T^{n-1}), & \text{for } q = 0; \\ H^p(T^{n-1}), & \text{for } q = 1. \end{cases} \]

The Hilbert–Poincare series of $\mathbb{Z}[K^\alpha_{PT}]$ is given by $(\sum_{i=0}^n h_i t^{2i})/(1 - t^2)^n$ according to Proposition 7.4. We have

\[ \text{Hilb}(\mathbb{Z}[K^\alpha_{PT}]/(\eta); t) = \sum_{i=0}^n h_i t^{2i} \]

\[ (1 - t^2)^n - 1. \]
The statement now follows from the degeneration of the spectral sequence at $E_3$-term. □

**Corollary 8.4.** For the whole isospectral space $X_{n,\lambda}$ there holds

$$\text{Hilb}(H^*_T(X_{n,\lambda}); t) = \sum_{i=0}^{n} h_i t^{2i} + R(t),$$

where $R(t)$ is a polynomial, and $h_i$ are given by (7.4).

**Proof.** The space $X_{n,\lambda}$ is patched from $X^{\leq \varepsilon}_{n,\lambda} \cup X^{\geq \varepsilon}_{n,\lambda}$. Both subsets are preserved by the torus action, hence the equivariant cohomology groups can be computed via Mayer–Vietoris exact sequence. However, the torus action on $X^{\geq \varepsilon}_{n,\lambda}$ and $X^{\leq \varepsilon}_{n,\lambda}$ is free, so the equivariant cohomology groups of these subsets coincide with the ordinary cohomology groups of their orbit spaces. Hence they are concentrated in a finite range of degrees. The statement now follows from Theorem 8.2. □

**Example 8.5.** We check the calculations for the case $n = 3$. The isospectral space $X_{3,\lambda}$ coincides with the manifold $\text{Fl}_3$ of complete complex flags. Its equivariant cohomology are known to satisfy

$$\text{Hilb}(H^*_T(\text{Fl}_3); t) = \frac{1 + 2t^2 + 2t^4 + t^6}{(1 - t^2)^2}.$$  

Using formulas (7.4) compute the h-numbers of $K^2_{pT}$: $(h_0, h_1, h_2, h_3) = (1, 0, 6, -1)$. There holds

$$\frac{1 + 2t^2 + 2t^4 + t^6}{(1 - t^2)^2} = \frac{1 + 6t^4 - t^6}{(1 - t^2)^2} + 2t^2,$$

which confirms Corollary 8.4.

9. **Betti numbers**

In this section we describe the additive structure of the cohomology ring of $X_{n,\lambda}$. As a first step, we compute the homological structure of the subset $X_{n,\lambda}^{\leq \varepsilon}$, containing the essential information on the torus action. It is assumed in this section, that all coefficients are taken in a fixed field. The next proposition follows from the general technique developed in [1].

**Proposition 9.1.** The homology modules of $Y \cong X_{n,\lambda}^{\leq \varepsilon}$ admit the double grading: $H_j(Y) \cong \bigoplus_{p+q=j} H_{p,q}(Y)$. There holds

1. $H_{p,q}(Y) \cong H_p(T^{n-1}) \otimes H_q(T^n)$ for $q < p < n$.
2. $H_{p,q}(Y) = 0$ for $q > p$.
3. The dimension of $H_{p,p}(Y)$ equals

$$h_p + \binom{n}{p} \sum_{k=2}^{p+1} (-1)^{p+k-1} \binom{n-1}{k-1},$$

for $p = 0, 1, \ldots, n - 1$. In particular, for $p \geq 2$ there holds $\dim H_{p,p}(Y) = h_p^n + \binom{n}{p} \binom{n-1}{p}$. 

The inclusion map \( i: \mathcal{T}^{n-1} \times T^n \cong \partial Y \rightarrow Y \) induces the homomorphism in homology, which respects the double grading:
\[
i_*: H_p(\mathcal{T}^{n-1}) \otimes H_q(T^n) \rightarrow H_{p,q}(Y).
\]
This homomorphism is an isomorphism for \( q < p \), injective for \( q = p \), and zero for \( q > p \).

Note that the subspace \( X^{\leq \varepsilon}_{n,\lambda} \) does not depend on the parameters \( n_+, n_- \) discussed in the previous sections. Now we are in position to compute the Betti numbers of \( X_{n,\lambda} \). Homology of \( X_{n,\lambda} \) will certainly depend on parameters \( n_+, n_- \), which encode “the degree of degeneration” of this space.

**Theorem 9.2.** The Hilbert–Poincaré series for homology of \( X_{n,\lambda} \) is given by the formula
\[
\sum_{i=0}^{2n} \beta(X_{n,\lambda}) t^i = H^{\geq \varepsilon}(t) + H^{\leq \varepsilon}(t) - H^\varepsilon(t) + (1 + t) \cdot H^{\ker}(t),
\]
where
\[
H^{\leq \varepsilon}(t) = (1 + t)^{2n-1},
\]
\[
H^{\geq \varepsilon}(t) = (1 + t)^{2n-n_+ - n_- - 2(1 - t + t(1 + t)^{n_+} + t(1 + t)^{n_-})},
\]
\[
H^{\leq \varepsilon}(t) = \sum_{p=0}^{n-1} \left( \begin{array}{c} n \\ p \end{array} \right) \left( \begin{array}{c} p+1 \\ k \end{array} \right) (-1)^{p+k-1} \binom{n-1}{k-1} t^{2p} + \sum_{q=p<n} \left( \begin{array}{c} n-1 \\ p \end{array} \right) \left( \begin{array}{c} n-1 \\ q \end{array} \right) t^{p+q},
\]
\[
H^{\ker}(t) = \sum_{(p,e,q,s,r) \in \mathcal{Y}} \left( \begin{array}{c} n-1 - n_+ - n_- \\ p \\ e \\ q \\ s \\ r \end{array} \right) \binom{1}{e} \binom{n_+}{q} \binom{n_-}{s} \binom{n-1}{r} t^{p+e+q+s+r},
\]
The indexing subset \( \mathcal{Y} \) in the last expression is defined by the conditions
\[
0 \leq p \leq n - 1 - n_+ - n_-; \quad 0 \leq e \leq 1;
\]
\[
0 \leq q \leq n_+; \quad 0 \leq s \leq n_-; \quad 0 \leq r \leq n - 1;
\]
\[
either (e = 0 and q + s > 0) or (e = 1 and q > 0 and s > 0).
\]
The \( h \)-numbers are given by (7.4).

**Proof.** Although the result looks awkward, the idea behind this calculation is straightforward: we analyze the Mayer–Vietoris sequence for the union \( X_{n,\lambda} = X^{\leq \varepsilon}_{n,\lambda} \cup X^{\geq \varepsilon}_{n,\lambda} \). Let \( X^{\leq \varepsilon}_{n,\lambda}(t) \) denote the intersection \( X^{\leq \varepsilon}_{n,\lambda} \cap X^{\geq \varepsilon}_{n,\lambda} \). Then there is a long exact sequence
\[
H_i(X^{\leq \varepsilon}_{n,\lambda}) \rightarrow H_i(X^{\leq \varepsilon}_{n,\lambda}) \oplus H_i(X^{\geq \varepsilon}_{n,\lambda}) \rightarrow H_i(X_{n,\lambda}) \rightarrow H_{i-1}(X^{\leq \varepsilon}_{n,\lambda}) \rightarrow H_{i-1}(X^{\geq \varepsilon}_{n,\lambda}) \rightarrow \dots
\]
Note, that given an exact sequence
\[
A_i \rightarrow B_i \rightarrow C_i \rightarrow A_{i-1} \rightarrow B_{i-1},
\]
the dimension of the vector space in the middle is given by
\[
\dim C_i = \dim B_i - \dim A_i + \dim \ker \alpha_i + \dim \ker \alpha_{i-1}
\]
After introducing a natural notation
\[ H^{-ε}(t) = \sum_{i} \dim H_i(X_{n,λ}^-)t^i, \]
\[ H^{≥ ε}(t) = \sum_{i} \dim H_i(X_{n,λ}^{≥})t^i, \]
\[ H^{≤ ε}(t) = \sum_{i} \dim H_i(X_{n,λ}^{≤})t^i, \]
\[ H^{Ker}(t) = \sum_{i} \dim (\ker \iota_i : H_i(X_{n,λ}^ε) \to H_i(X_{n,λ}^{≤}) \oplus H_i(X_{n,λ}^{≥}))t^i. \]

Formula (9.8) implies (9.1). We need to check formulas (9.2)–(9.5).

(1) According to (6.1), \( X_{n,λ}^{≥} \approx T^{2n-1} \), which implies (9.2).

(2) The space \( X_{n,λ}^{≥} \) supports a free action of \( T^{n-1} \), which admits a section. Therefore,
\[ X_{n,λ}^{≥} = Q_{n,λ}^{≥} \times T^{n-1}, \]
where \( Q_{n,λ}^{≥} = X_{n,λ}^{≥}/T^{n-1} \). The structure of the orbit space was described in detail in Section 3.

(3) Homology of \( X_{n,λ}^{≤} \) are described by Theorem 9.1. Formula (9.4) is its simple consequence.

(4) To derive (9.5), we need to count the dimension of the vector space of all homology cycles of \( X_{n,λ}^{≥} \), annihilated by both maps
\[ \iota_ε^≤ : H_*(X_{n,λ}^{≥}) \to H_*(X_{n,λ}^{≤}), \]
\[ \iota_ε^≥ : H_*(X_{n,λ}^{≥}) \to H_*(X_{n,λ}^{≥}). \]

The torus \( X_{n,λ}^{≥} \) decomposes into the product
\[ X_{n,λ}^{≥} \approx S_ε^1 \times T^{n-1} \times T^{n-1} \approx T^{n-1-n+} \times S_ε^1 \times T^{n+} \times T^{n-} \times T^{n-1}, \]
where

(1) the component \( T^{n-1} \) corresponds to the acting torus;
(2) \( T^{n+} \), \( T^{n-} \) are the components of Liouville–Arnold tori, collapsing over the sides of the biangle \( B \);
(3) \( T^{n-1-n+} \) is the surviving component of the Liouville–Arnold torus.
(4) \( S_ε^1 \) is the circle \( \{|z| = ε\} \), lifted to the total space.

These five components of the torus explain the appearance the five-element indexing subset in the statement. Let \( θ = ω_p \otimes α_e \otimes ω_q^− \otimes ω_s^− \otimes ν_r \) be a homology cycle from
\[ H_*(X_{n,λ}^{≥}) \approx H_*(T^{n-1-n+} \times S_ε^1 \times T^{n+}) \otimes H_*(T^{n-}) \otimes H_*(T^{n-1}), \]
where the degrees of the factors are indicated in their subscripts. A straightforward calculation shows that both maps \( \iota_ε^≤ \) and \( \iota_ε^≥ \) annihilate \( θ \) if and only if the 5-tuple \( (p,e,q,s,r) \) satisfies the conditions (9.6). This proves (9.5).
Example 9.3. The Betti numbers computed for small values of \( n \) are shown in the tables 7 and 8.

Corollary 9.4. The space \( X_{n,\lambda} \) has nonzero Betti numbers in odd degrees for \( n \geq 4 \).

Proof. Theorem 9.2 implies that \( \beta_1(X_{n,\lambda}) = n - 1 - n_+ - n_- \). This number is nonzero unless \( n_+ \) and \( n_- \) are the maximal possible, representing the most degenerate case. For the most degenerate case, and \( n \geq 4 \), \( \beta_2(X_{n,\lambda}) \) is nonzero. To prove this, it is sufficient to estimate the coefficient at \( t^{2n-1} \) in (9.5) by 2 from below. Indeed, the only negative term in (9.1) comes from \( \text{Hilb}^*(t) \), and its coefficient at \( t^{2n-1} \) is one. The estimation is straightforward.

Recall that with any action of a torus \( T \) on a space \( X \) one can associate a fibration \( r: X \times_T ET \to BT \), where \( ET \) is a contractible space, carrying the free action of \( T \) and \( BT \) is the classifying space of a torus. It can be assumed that \( ET = (S^k)^k \) and \( BT = (\mathbb{C}P^k)^k \), where \( k = \text{dim} T \).

Definition 9.5. The space \( X \) is called equivariantly formal in the sense of Goresky–Macpherson [18] if the Serre spectral sequence (9.11)

\[
E_2^{*,*} \cong H^*(BT) \otimes H^*(X) \Rightarrow H^*(X \times_T ET) = H^*_T(X),
\]

i.e. the spectral sequence of the fibration \( r \), degenerates at its second page.

Proposition 9.6. The space \( X_{n,\lambda} \) is not equivariantly formal for \( n \geq 4 \).

Proof. Assume that \( X_{n,\lambda} \) is equivariantly formal. The degeneration of the Serre spectral sequence (9.11) at a second page then implies

\[
\text{Hilb}(H^*_T(X_{n,\lambda}); t) = \text{Hilb}(H^*(X_{n,\lambda}); t) \cdot \text{Hilb}(H^*(BT_n-1); t) = \text{Hilb}(H^*(X_{n,\lambda}); t) \cdot (1 + t^2)^{n-1}.
\]

This identity and Corollary 9.4 imply that \( H^*_T(X_{n,\lambda}) \) has nontrivial components of arbitrarily high odd degree. This contradicts to Corollary 8.4.
Remark 9.7. The fundamental group of $X_{n,\lambda}$ can be explicitly described as well. As in the computations above, we consider the decomposition $X_{n,\lambda} = X_{n,\lambda}^{\leq \varepsilon} \cup X_{n,\lambda}^{\geq \varepsilon}$ and apply van Kampen theorem. For the intersection there holds

$$\pi_1(X_{n,\lambda}^{\leq \varepsilon}) = \pi_1(T^n \times T^{-1} \times T^{-1-n_+ - n_-} \times S^1 \times T^{-1}) = \mathbb{Z}^{n_+} \oplus \mathbb{Z}^{n_-} \oplus \mathbb{Z}^{n_+ - n_-} \oplus \mathbb{Z}^n$$

The summands $\mathbb{Z}^{n_+}$ and $\mathbb{Z}^{n_-}$ vanish in $\pi_1(X_{n,\lambda}^{\geq \varepsilon})$ since the corresponding components of Lioville–Arnold tori are collapsed over the facets of biangle $\mathcal{B}$. The summand $\mathbb{Z}^n = \pi_1(S^1 \times T^{-1})$ vanishes in $\pi_1(X_{n,\lambda}^{\leq \varepsilon})$ according to [35]. The result of this paper asserts that for a locally standard action of $T$, $\dim T = n$, on $M$, $\dim M = n$, having a fixed point, there holds $\pi_1(M) \cong \pi_1(M/T)$. In other words, any loop on the acting torus can be contracted via a fixed point. This result is applied to $X_{n,\lambda}^{\leq \varepsilon} \cong Y$, carrying the extended action of $T^n$.

No other loops appear in $X_{n,\lambda}^{\geq \varepsilon}$ and $X_{n,\lambda}^{\leq \varepsilon}$, therefore

$$\pi_1(X_{n,\lambda}) \cong \mathbb{Z}^{n_+ - n_-}.$$

Hence $X_{n,\lambda}$ is simply connected if and only if the spectrum $\lambda$ satisfies $n_+ + n_- = n - 1$. This corresponds to the most degenerate situation, considered in Remark 3.11.

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Faculty of computer science, Higher School of Economics

E-mail address: ayzenberga@gmail.com