Q-DEFORMED STRUCTURES AND GENERALIZED THERMODYNAMICS

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On the basis of the recently proposed formalism [A. Lavagno and P.N. Swamy, Phys. Rev. E \textbf{65}, 036101 (2002)], we show that the realization of the thermostatistics of $q$-deformed algebra can be built on the formalism of $q$-calculus. It is found that the entire structure of thermodynamics is preserved if we use an appropriate Jackson derivative instead of the ordinary thermodynamic derivative. Furthermore, in analogy with the quantum $q$-oscillator algebra, we also investigate a possible $q$-deformation of the classical Poisson bracket in order to extend a generalized $q$-deformed dynamics in the classical regime.

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1. Introduction

In the last few years quantum algebra and quantum groups have been the subject of intensive research in several physical fields. Although some properties of these structures still deserve more study in order to provide full clarity, it was quite clear from the beginning that the basic features of these generalized theories are strongly connected with a deep physical meaning and not merely a mathematical exercise. Quantum algebra and quantum groups, which first emerged in connection with the quantum inverse scattering theory and statistical mechanics model \cite{[1]}, where the quantum Yang-Baxter equation plays a crucial role, have found recent important applications in many physical and mathematical problems, such as conformal field theory, integrable systems, non-commutative geometry, knot theory, thermostatistical models and in a wide range of applications envisaged, from cosmic strings and black holes to the fractional quantum Hall effect and high-$T_c$ superconductors \cite{[2, 3, 4]}.

From the seminal work of Biedenharn \cite{[5]} and Macfarlane \cite{[6]} it was clear that the $q$-calculus, originally introduced at the beginning of this century by Jackson \cite{[7]} in the
study of the basic hypergeometric function, it plays a central role in the representation of the quantum groups [8]. In fact it has been shown that it is possible to obtain a “coordinate” realization of the Fock space of the $q$-oscillators by using the deformed Jackson derivative (JD) or the so-called $q$-derivative operator [9, 10, 11]. In particular, in the paper of Celeghini et al. [10], the quantum Weyl-Heisenberg algebra is studied in the frame of the Fock-Bargmann representation and is incorporated into the theory of entire analytic functions allowing a deeper mathematical understanding of squeezed states, relation between coherent states, lattice quantum mechanics and Bloch functions.

Furthermore, in the recent past there has been increasing emphasis in quantum statistics different from that of standard bosons and fermions. Since the pioneering work of Gentile and Green [12, 13], there have been many extensions beyond the standard statistics, among the others we may list the following: parastatistics, fractional statistics, quon statistics, anyon statistics and quantum groups statistics. In the literature there are two principal methods of introducing an intermediate statistical behavior. The first is to deform the quantum algebra of the commutation-anticommutation relations thus deforming the exchange factor between permuted particles. The second method is based on modifying the number of ways of assigning particles to a collection of states and thus the statistical weight of the many-body system. One interesting realization of the first approach is the study of exactly solvable statistical systems which has led to the theory of the $q$-deformed harmonic oscillator [5, 6], based on the construction of $SU_q(2)$ algebra of $q$-deformed commutation or anticommutation relations between creation and annihilation operators. Such an algebra opens the possibility to study intermediate $q$-boson and $q$-fermion statistical behavior [14, 15, 16] and to introduce a generalized thermostatistics based on the formalism of the $q$-calculus [17]. In this context, based on the formulation of generalized statistical mechanics in connection with $q$-deformed calculus, it has been pointed out that the thermal average of an operator is closely related to quantum algebra [18] in a fully developed scheme.

The purpose of this paper is twofold: first, we want to show that the realization of the thermostatistics of $q$-deformed algebra can be built on the formalism of $q$-calculus and that the entire structure of thermodynamics is preserved if we use the (non-commutative) JD instead of the ordinary thermodynamic derivative. Second, starting from the observation that the creation and annihilation operators in the quantum $q$-deformed $SU_q(2)$ algebra correspond classically to non-commuting coordinates in a $q$-phase-space and the commutation relation between the standard quantum operators corresponds classically to the Poisson bracket (PB), we want to introduce a $q$-deformation of the PB ($q$-PB) in order to define a generalized $q$-deformed dynamics in a non-commutative phase-space. The motivation for this second goal lies in the fact that a full understanding of the physical origin of $q$-deformation in classical physics is still lacking because it is not clear if there exists a classical counterpart to the quantum groups. It is thus an outstanding question: how can classical dynamics be formulated such that, upon canonical quantization, one recovers the deformed quantum $q$-oscillator theory. The problem of a possible $q$-deformation of classical mechanics was dealt with in Ref. [19] where a $q$-PB has been obtained starting from a point of view different from the one adopted in this paper. As the quantum $q$-deformation plays a crucial role in the interpretation of several complex
physical systems, we expect that a classical $q$-deformation of the dynamics can be very relevant in several problems [20]. A remarkable example is the Tsallis nonextensive thermostatistics [21], based on a classical deformation of the Boltzmann-Gibbs entropy, where the thermodynamic functions, such as entropy and internal energy, are deformed, but the whole structure of thermodynamics is preserved. Many investigations are devoted to the relevance of such a classical theory in several physical applications [22].

The main approach we shall follow to introduce the classical correspondence of quantum $q$-oscillator is based on the following idea. The (undeformed) quantum commutation relations are invariant under the action of $SU(2)$ and, as a consequence, the $q$-deformed commutation relations are invariant under the action of $SU_q(2)$. Analogously, since the (undeformed) PB is invariant under the action of the symplectic group $Sp(1)$, we have to require that $q$-PB must satisfy invariance over the action of the $q$-deformed symplectic group $Sp_q(1)$.

The paper is organized as follows. In Sec. 2 we review the fundamental relations in the $q$-Heisenberg algebra of creation and annihilation operators and we derive the statistical distribution for $q$-boson gas. In Sec. 3 we introduce a self-consistent prescription for the use of the JD in the thermodynamical relations. In Sec. 4, we define the non-commutative $q$-differential calculus in the $q$-deformed phase-space invariant over the action of the $GL_q(2)$ and in Sec. 5 we introduce the $q$-symplectic group and the $q$-PB. Finally, we present our conclusions in Sec. 6.

2. Realization of the $q$-boson algebra and thermal averages

We shall briefly review the theory of $q$-deformed bosons defined by the $q$-Heisenberg algebra of creation and annihilation operators of bosons introduced by Biedenharn and McFarlane [5, 6], derivable through a map from $SU_q(2)$. The $q$-boson algebra is determined by the following commutation relations for annihilation and creation operators $a$, $a^\dagger$ and the number operator $N$, thus (for simplicity we omit the particle index)

\begin{align}
[a, a] &= [a^\dagger, a^\dagger] = 0 , \\
[\{a, a\}^\dagger] &= a - q^2 a^\dagger a = 1 , \\
[N, a^\dagger] &= a^\dagger , \\
[N, a] &= -a .
\end{align}

The $q$-Fock space spanned by the orthonormalized eigenstates $|n\rangle$ is constructed according to

\begin{align}
|n\rangle &= (a^\dagger)^n|0\rangle , \\
[a|0\rangle &= 0 ,
\end{align}

where the $q$-basic factorial is defined as

\begin{align}
[n]! &= [n][n-1] \cdots [1] ,
\end{align}

and the $q$-basic number $[x]$ is defined in terms of the $q$-deformation parameter

\begin{align}
[x] &= \frac{q^{2x} - 1}{q^2 - 1} .
\end{align}
In the limit $q \to 1$, the $q$-basic number $[x]$ reduces to the ordinary number $x$ and all the above relations reduce to the standard boson relations.

The actions of $a$, $a^\dagger$ on the Fock state $|n\rangle$ are given by
\begin{align}
  a^\dagger |n\rangle &= [n + 1]^{1/2} |n + 1\rangle, \\  a |n\rangle &= [n]^{1/2} |n - 1\rangle, \\  N |n\rangle &= n |n\rangle.
\end{align}

From the above relations, it follows that $a^\dagger a = [N], aa^\dagger = [N + 1]$.

We observe that the Fock space of the $q$-bosons has the same structure as the standard bosons but with the replacement $n! \to [n]!$. Moreover the number operator is not $a^\dagger a$ but can be expressed as the nonlinear functional relation $N = f(a^\dagger a)$ which can be explicitly obtained formally in the closed form
\begin{equation}
  N = \frac{1}{2 \log q} \log \left( 1 + (q^2 - 1)a^\dagger a \right).
\end{equation}

The transformation from Fock observables to the configuration space (Bargmann holomorphic representation) may be accomplished by choosing [9, 10, 11]
\begin{equation}
  a^\dagger \to x, \quad a \to D_x,
\end{equation}
where $D$ is the JD [7] defined by
\begin{equation}
  D_x f(x) = \frac{f(q^2 x) - f(x)}{(q^2 - 1)x},
\end{equation}
which reduces to the ordinary derivative when $q$ goes to unity and therefore, the JD occurs naturally in $q$-deformed structures. In an analogous manner, we will see below that JD plays a crucial role in the $q$-deformed classical mechanics also.

The thermal average of an operator is written in the standard form
\begin{equation}
  \langle O \rangle = \frac{\operatorname{Tr} (O e^{-\beta H})}{\mathcal{Z}},
\end{equation}
where $\mathcal{Z}$ is the grand canonical partition function defined as
\begin{equation}
  \mathcal{Z} = \operatorname{Tr} (e^{-\beta H}),
\end{equation}
and $\beta = 1/T$. Henceforward we shall set Boltzmann constant to unity.

By using the definition in Eq.(5) of the $q$-basic number, the mean value of the occupation number $n_i$ can be calculated starting from the relation
\begin{equation}
  [n_i] = \frac{1}{\mathcal{Z}} \operatorname{Tr} \left( e^{-\beta H} a_i^\dagger a_i \right).
\end{equation}
As outlined before, the consistency of this approach is warranted by the fact that the thermal averages of an operator is related with quantum algebra in a fully consistent scheme [18].
In the grand canonical ensemble, the Hamiltonian of the non-interacting boson gas is expected to have the following form
\[ H = \sum_i (\epsilon_i - \mu) N_i \]
where the index \( i \) is the state label, \( \mu \) is the chemical potential and \( \epsilon_i \) is the kinetic energy in the state \( i \) with the number operator \( N_i \). Therefore from Eq.(14), after simple manipulations the explicit expression for the mean occupation number can be determined as
\[ n_i = \frac{1}{2} \log q \log \left( \frac{z^{-1} e^{\beta \epsilon_i} - 1}{z^{-1} e^{\beta \epsilon_i} - q^2} \right), \]  
where \( z = e^{\beta \mu} \) is the fugacity. It is easy to see that the above equation reduces to the standard Bose-Einstein distribution when \( q \to 1 \). The total number of particles is given by \( N = \sum_i n_i \).

3. \( q \)-calculus in the deformed thermodynamic relations

From the definition of the partition function, and the Hamiltonian, it follows that the logarithm of the partition function has the same structure as that of the standard boson
\[ \log Z = -\sum_i \log(1 - ze^{-\beta \epsilon_i}) . \]
This is due to the fact that we have chosen the Hamiltonian to be a linear function of the number operator but it is not linear in \( a^\dagger a \) as seen from Eq.(9). For this reason, the standard thermodynamic relations in the usual form are ruled out. It is verified, for instance, that
\[ N \neq z \frac{\partial}{\partial z} \log Z . \]
As the coordinate space representation of the \( q \)-boson algebra is realized by the introduction of the JD (see Eq.(10)), we stress that the key point of the \( q \)-deformed thermodynamics is in the observation that the ordinary thermodynamic derivative with respect to \( z \), must be replaced by the JD [17]
\[ \frac{\partial}{\partial z} \Rightarrow \mathcal{D}_z . \]
Consequently, the number of particles in the \( q \)-deformed theory can be derived from the relation
\[ N = z \mathcal{D}_z \log Z \equiv \sum_i n_i , \]
where \( n_i \) is the mean occupation number expressed in Eq.(15).

The usual Leibniz chain rule is ruled out for the JD and therefore derivatives encountered in thermodynamics must be modified according to the following prescription. First we observe that the JD applies only with respect to the variable in the exponential form such as \( z = e^{\beta \mu} \) or \( y_i = e^{-\beta \epsilon_i} \). Therefore for the \( q \)-deformed case, any thermodynamic derivative of functions which depend on \( z \) or \( y_i \) must be converted to derivatives in one of these variables by using the ordinary chain rule and then applying the JD with respect
to the exponential variable. For example, the internal energy in the $q$-deformed case can be written as

$$U = \sum_i \partial y_i / \partial \beta \, D_y \log(1 - z y_i) \bigg|_z .$$

(20)

In this case we obtain the correct form of the internal energy

$$U = \sum_i \epsilon_i n_i .$$

(21)

This prescription is a crucial point of our approach because this allows us to maintain the whole structure of thermodynamics and the validity of the Legendre transformations in a fully consistent manner. For instance, in light of the above discussion, we have the recipe to derive the entropy of the $q$-bosons which leads to

$$S = \left. - \frac{\partial \Omega}{\partial T} \right|_{\mu} = \log Z + \beta \sum_i \partial \kappa_i / \partial \beta \, D^{(q)}(1 - \kappa_i)$$

$$= \log Z + \beta U - \beta \mu N ,$$

(22)

where $\kappa_i = z e^{-\beta \epsilon_i}$, $U$ and $N$ are the modified functions expressed in Eqs.(20) and (19) and $\Omega = -T \log Z$ is the thermodynamic potential. After some manipulations, we obtain the entropy as follows

$$S = \sum_i \left\{ -n_i \log [n_i] + (n_i + 1) \log [n_i + 1] - \log ([n_i + 1] - [n_i]) \right\} .$$

(23)

4. Non-commutative differential calculus

In the above sections we have seen that the non-commutative JD and the $q$-calculus play a crucial role in the definition of the $q$-deformed quantum mechanics and quantum thermodynamics. We would now like to introduce an analogous $q$-deformation in classical theory. Since the creation and annihilation operators in the quantum $q$-deformed $SU_q(2)$ algebra correspond classically to non-commuting coordinates in a $q$-phase-space, in this section we introduce the $q$-deformed plane which is generated by the non-commutative elements $\hat{x}$ and $\hat{p}$ fulfilling the relation [24]

$$\hat{p} \hat{x} = q \hat{x} \hat{p} ,$$

(24)

which is invariant under GL$_q$(2) transformations (see later). Henceforward, for simplicity, we shall limit ourselves to consider the two-dimensional case.

From Eq.(24) the $q$-calculus on the $q$-plane can be obtained formally through the introduction of the $q$-derivatives $\hat{\partial}_x$ and $\hat{\partial}_p$ [25]

$$\hat{\partial}_p \hat{x} = \hat{\partial}_x \hat{p} = 1 ,$$

(25)

$$\hat{\partial}_x \hat{x} = \hat{\partial}_p \hat{p} = 0 .$$

(26)
They fulfill the $q$-Leibniz rule

\begin{align}
\hat{\partial}_p \hat{p} &= 1 + q^2 \hat{p} \hat{\partial}_p + (q^2 - 1) \hat{x} \hat{\partial}_x, \\
\hat{\partial}_p \hat{x} &= q \hat{x} \hat{\partial}_p, \\
\hat{\partial}_x \hat{p} &= q \hat{p} \hat{\partial}_x, \\
\hat{\partial}_x \hat{x} &= 1 + q^2 \hat{x} \hat{\partial}_x, \\
\hat{\partial}_p \hat{\partial}_x &= q^{-1} \hat{\partial}_x \hat{\partial}_p.
\end{align}

(27) \hfill (28) \hfill (29) \hfill (30)

together with the $q$-commutative derivative

\begin{equation}
\hat{\partial}_p \hat{\partial}_x = q^{-1} \hat{\partial}_x \hat{\partial}_p.
\end{equation}

(31)

It is easy to see that in the $q \to 1$ limit one recovers the ordinary commutative calculus. Let us outline the asymmetric mixed derivative relations Eq.(27) and Eq.(30) in $\hat{x}$ and in $\hat{p}$. These properties come directly from the non-commutative structure of the $q$-plane defined in Eq.(24).

We recall now that the most general function on the $q$-plane can be expressed as a polynomial in the $q$-variable $\hat{x}$ and $\hat{p}$

\begin{equation}
f(\hat{x}, \hat{p}) = \sum_{i,j} c_{ij} \hat{x}^i \hat{p}^j.
\end{equation}

(32)

where we have assumed the $\hat{x}$-$\hat{p}$ ordering prescription (it can be always accomplished by means of Eq. (24)). Thus, taking into account Eqs.(27)-(30), we obtain the action of the $q$-derivatives on the monomials

\begin{align}
\hat{\partial}_x (\hat{x}^n \hat{p}^m) &= [n] \hat{x}^{n-1} \hat{p}^m, \\
\hat{\partial}_p (\hat{x}^n \hat{p}^m) &= [m] q^n \hat{x}^n \hat{p}^{m-1},
\end{align}

(33) \hfill (34)

where $[n]$ is the same $q$-basic number introduced in Eq.(5).

A realization of the above $q$-algebra and its $q$-calculus can be accomplished by the replacements [26]

\begin{align}
\hat{x} &\to x, \\
\hat{p} &\to p D_x, \\
\hat{\partial}_x &\to D_x, \\
\hat{\partial}_p &\to D_p D_x,
\end{align}

(35) \hfill (36) \hfill (37) \hfill (38)

where

\begin{equation}
D_x = q^x \partial_x, \quad D_x f(x, p) = f(q x, p),
\end{equation}

(39)

is the dilatation operator along the $x$ direction (reducing to the identity operator for $q \to 1$), whereas

\begin{equation}
\mathcal{D}_x = \frac{q^2 x \partial_x - 1}{(q^2 - 1) x}, \quad \mathcal{D}_p = \frac{q^2 p \partial_p - 1}{(q^2 - 1) p}.
\end{equation}

(40)
are the JD with respect to $x$ and $p$. Their action on an arbitrary function $f(x, p)$ is
\[ D_x f(x, p) = \frac{f(q^2 x, p) - f(x, p)}{(q^2 - 1)x} , \quad D_p f(x, p) = \frac{f(x, q^2 p) - f(x, p)}{(q^2 - 1)p} . \] (41)

Therefore, as a consequence of the non-commutative structure of the $q$-plane, in this realization the \( \hat{x} \) coordinate becomes a \( c \)-number and its derivative is the JD whereas the \( \hat{p} \) coordinate and its derivative are realized in terms of the dilatation operator and JD.

5. \( q \)-Poisson bracket and \( q \)-symplectic group

After the formulation of the \( q \)-differential calculus, we are now able to introduce a \( q \)-PB. As previously stated, since the undeformed PB is invariant under the action of the undeformed symplectic group \( \text{Sp}(1) \), we will assume, as a fundamental point, that the \( q \)-PB must satisfy the invariance property under the action of the \( q \)-deformed symplectic group \( \text{Sp}_q(1) \) with the same value of the deformed parameter \( q \) used in the construction of the quantum plane.

Let us start by recalling the classical definition of a 2-Poisson manifold, which is a two dimensional Euclidean space \( \mathbb{R}^2 \) generated by the position and momentum variables \( x \equiv x^1 \) and \( p \equiv x^2 \) and equipped with a PB. If \( f(x, p) \) and \( g(x, p) \) are smooth functions, the PB is defined as \[ \{ f, g \} = \partial_x f \partial_p g - \partial_p f \partial_x g . \] (42)

Eq. (42) can be expressed in a compact form
\[ \{ f, g \} = \partial_i f J^{ij} \partial_j g , \] (43)
where \( J \) is the unitary symplectic matrix given by
\[ (J)_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \] (44)

Remarkably, Eq.(43) does not change under the action of a symplectic transformation on the phase-space:
\[ X \rightarrow X' = T X , \] (45)
where \( X^i = (x^1, x^2) \) and \( T \in \text{Sp}(1) \) with \( T^j_i \) belonging to the fundamental representation of \( T \). This property follows from the relation
\[ J^{rs} T^i_r T^j_s = J^{ij} . \] (46)

As is well known Eq. (43) can also be expressed as
\[ \{ f, g \} = \{ x^i, x^j \} \partial_i f \partial_j g , \] (47)
so that, if we know the PB between the generators \( x^i \) we can compute the PB between any pair of functions.
Let us now introduce the $q$-symplectic group $\text{Sp}_q(1)$, generated by $\hat{T}_i^j$, the elements of a $2 \times 2$ matrix $\hat{T}$ belonging to the fundamental representation of $\text{Sp}_q(1)$. The $q$-commutativity of the elements of $\hat{T}$ are controlled by the RTT relation:

$$
R_{kl}^{ij} \hat{T}_k^r \hat{T}_s^r = \hat{T}_i^r \hat{T}_j^s R_{rs}^{kl},
$$

(48)

where the matrix $R$ is defined by

$$
(R)_{ij}^{kl} = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & q^{-1} & 0 & 0 \\
0 & \lambda & q^{-1} & 0 \\
0 & 0 & 0 & q
\end{pmatrix},
$$

(49)

with $\lambda = q^{-1}(q^2 - q^{-2})$ (row and column are numbered as 11, 12, 21, 22, respectively). It satisfies the quantum Yang-Baxter equation

$$
R_{ij}^{kl} R_{kr}^{st} R_{lt}^{uv} = R_{jr}^{lt} R_{it}^{kv} R_{kl}^{su},
$$

(50)

a sufficient condition for the consistency of the RTT relation (48). Let us observe that by virtue of the isomorphism $\text{SL}_q(2) = \text{Sp}_q(1) \text{GL}_q(2)$ the matrix $q^{-1} R_q$ coincides with the matrix $R_{q^2}$ of the $\text{GL}_q(2)$.

The symplectic condition on the element $\hat{T}_i^j$ is taken into account by means of a matrix $C$ given by

$$
(C)_{ij}^{kl} = \begin{pmatrix}
0 & q^{-1} \\
-q & 0
\end{pmatrix},
$$

(51)

through the relation

$$
C_{rs} \hat{T}_r^i \hat{T}_s^j = C_{ij}.
$$

(52)

In analogy with the classical PB, we introduce the following bilinear operator on the generators $\hat{x}^i$

$$
\left\{ \hat{x}^i, \hat{x}^j \right\}_q = (\hat{\partial}_i \hat{x}^i) J^{rs} (\hat{\partial}_k \hat{x}^j),
$$

(53)

where we have defined the unitary $q$-symplectic matrix $J^{rs}$ in

$$
(J)^{ij} = q (C)^{ij}.
$$

(54)

Eq.(54) is the $q$-analogue of Eq. (44) which is recovered in the $q \rightarrow 1$ limit.

By construction, taking into account Eq. (52), it immediately follows that Eq. (53) is invariant under the action of the $q$-symplectic group $\text{Sp}_q(1)$:

$$
\hat{x}_i \rightarrow \hat{x}_i' = \hat{x}_j \hat{T}_j^i,
$$

(55)

where we assume the commutation between group elements and the plane elements.

Explicitly Eq. (53) becomes

$$
\left\{ \hat{x}_i, \hat{x}_j \right\}_q = \hat{\partial}_x \hat{x}_i \hat{\partial}_p \hat{x}_j - q^2 \hat{\partial}_p \hat{x}_i \hat{\partial}_x \hat{x}_j.
$$

(56)
It is easy to verify the following fundamental relations

\[
\{ \hat{x}, \hat{x} \}_q = \{ \hat{p}, \hat{p} \}_q = 0 , \\
\{ \hat{x}, \hat{p} \}_q = 1 , \\
\{ \hat{p}, \hat{x} \}_q = -q^2 ,
\]

which coincide with the one obtained in Ref. [19].

In particular, from Eqs. (58) and (59) it follows that Eq. (53) is not antisymmetric. A similar behavior appears also in the quantum q-oscillator theory introduced earlier.

By means of Eqs.(35)-(38), a realization of our generalized q-PB can be written as

\[
\{ f, g \}_q = D_x f(x, p D_x) D_p g(q x, p D_x) - q^2 D_p f(q x, p D_x) D_x g(x, p D_x) ,
\]

where \( f \) and \( g \) are identified with \( x \) or \( p \), respectively.

6. Conclusions

In this paper, we have shown that the q-calculus plays a crucial role in the definition of the quantum mechanics of q-oscillators, thermodynamics and in a q-classical theory, defined by means of the introduction of a q-PB. We have shown that the realization of the thermostatistics of q-deformed algebra can be built on the formalism of q-calculus and that the entire structure of thermodynamics is preserved if we use the (non-commutative) JD instead of the ordinary thermodynamic derivative. In analogy with quantum group invariance properties of the quantum q-oscillation theory, the q-PB has been defined by assuming the invariance under the action of \( \text{Sp}_q(1) \) group and its derivatives act on the q-deformed non-commutative plane invariant under \( \text{GL}_q(2) \) transformations. Therefore such a classical q-deformation theory can be seen as the analogue of q-oscillator deformation in the quantum theory. This opens the possibility of introducing a classical counterpart of the quantum q-deformations and we expect that such a classical q-deformed dynamics can be very relevant in several physical applications, in a manner similar to the classical Tsallis’ thermostatistics, based on a deformation of the Boltzmann-Gibbs entropy [21, 22].

Although a complete treatment of this matter lies out the scope of this paper, we would like to state that, upon simple canonical quantization of the q-classical theory which has been under investigation, it is possible to obtain, consistently, the q-deformed Heisenberg uncertainty relations and the q-deformed oscillator algebra, introduced in Sec. 2. Finally, for future investigations we would like to mention some interesting developments in the q-classical harmonic oscillators, equipartition theorem, Euler’s theorem, Liouville theorem and in q-deformed classical thermodynamics.

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