Integrable Structure of Conformal Field Theory III. The Yang-Baxter Relation

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Abstract

In this paper we fill some gaps in the arguments of our previous papers [1,2]. In particular, we give a proof that the $L$ operators of Conformal Field Theory indeed satisfy the defining relations of the Yang-Baxter algebra. Among other results we present a derivation of the functional relations satisfied by $T$ and $Q$ operators and a proof of the basic analyticity assumptions for these operators used in [1,2].
1. Introduction

This paper is a sequel to our works [1,2] where we have introduced the families of operators $T(\lambda)$ and $Q(\lambda)$ which act in a highest weight Virasoro module and satisfy the commutativity conditions

$$[T(\lambda), T(\lambda')] = [T(\lambda), Q(\lambda')] = [Q(\lambda), Q(\lambda')] = 0.$$  \hfill (1.1)

These operators are CFT analogs of Baxter’s commuting transfer-matrices of integrable lattice theory [3,4]. In the lattice theory the transfer-matrices are typically constructed as follows. One first finds an $R$-matrix which solves the Yang-Baxter equation

$$R_{VV'}(\lambda) R_{V'V''}(\lambda') R_{VV''}(\lambda') = R_{V'V''}(\lambda') R_{V'V''}(\lambda) R_{VV'}(\lambda).$$  \hfill (1.2)

Here $R_{VV'}$, $R_{V'V''}$, $R_{V''V''}$ act in the tensor product of the identical vector spaces $V$, $V'$ and $V''$. Then one introduces the $L$-operator

$$L_V(\lambda) = R_{VV_1}(\lambda) R_{V_2V}(\lambda) \ldots R_{VV_N}(\lambda),$$  \hfill (1.3)

which is considered as a matrix in $V$ whose elements are operators acting in the tensor product

$$\mathcal{H}_N = \otimes_{i=1}^N V_i,$$  \hfill (1.4)

where $N$ is the size of the lattice. The space (1.4) is interpreted as the space of states of the lattice theory. The operator (1.3) satisfies the defining relations of the Yang-Baxter algebra

$$R_{VV'}(\lambda/\lambda') L_V(\lambda) L_{V'}(\lambda') = L_{V'}(\lambda') L_V(\lambda) R_{VV'}(\lambda/\lambda').$$  \hfill (1.5)

It realizes, thereby, a representation of this algebra in the space of states of the lattice theory. The “transfer-matrix”

$$T_V(\lambda) = \text{Tr}_V[L_V(\lambda)] : \mathcal{H}_N \to \mathcal{H}_N$$  \hfill (1.6)

satisfies the commutativity condition (1.1) as a simple consequence of the defining relations (1.5).

In many cases the integrable lattice theory defined through the transfer-matrix (1.6) can be used as the starting point to construct an integrable quantum field theory (QFT). If the lattice system has a critical point one can define QFT by taking appropriate scaling
limit (which in particular involves the limit $N \to \infty$). Then the space of states of QFT appears as a certain subspace in the limiting space $\mathcal{H}_{\text{QFT}} \subset \mathcal{H}_{N \to \infty}$. Although many integrable QFT can be constructed and studied this way (and this is essentially the way integrable QFT are obtained in Quantum Inverse Scattering Method \cite{5,6}), the alternative idea of constructing representations of Yang-Baxter algebra directly in the space of states $\mathcal{H}_{\text{QFT}}$ of continuous QFT seems to be more attractive. This idea was the motivation of our constructions in \cite{1,2}.

The natural starting point for implementing this idea is the Conformal Field Theory (CFT) because the general structure of its space of states $\mathcal{H}_{\text{CFT}}$ is relatively well understood \cite{7}. The space $\mathcal{H}_{\text{CFT}}$ can be decomposed as

$$\mathcal{H}_{\text{CFT}} = \bigoplus_{\Delta, \overline{\Delta}} \left[ V_{\Delta} \otimes \overline{V}_{\overline{\Delta}} \right], \quad (1.7)$$

where $V_{\Delta}$ and $\overline{V}_{\overline{\Delta}}$ are irreducible highest weight representations of “left” and “right” Virasoro algebras with the highest weights $\Delta$ and $\overline{\Delta}$ respectively. The sum (1.7) may be finite (as in the “minimal models”), infinite or even continuous. In any case the space (1.7) can be embedded into a direct product $\mathcal{H}_{\text{chiral}} \otimes \overline{\mathcal{H}}_{\text{chiral}}$ of left and right “chiral” subspaces,

$$\mathcal{H}_{\text{chiral}} = \bigoplus_{\Delta} V_{\Delta}. \quad (1.8)$$

In \cite{1,2} we introduced the operators $L(\lambda)$ which realize particular representations of the Yang-Baxter algebra (1.5) in the space (1.8). The commuting operators (1.1) was constructed in terms of these operators $L$. However, the proof that these operators actually satisfy the defining relations (1.5) of the Yang-Baxter algebra was not presented. The main purpose of this paper is to fill this gap.

Here we remind some notations used in \cite{1,2}. Let $\varphi(u)$ be a free chiral Bose field, i.e. the operator-valued function

$$\varphi(u) = iQ + iP u + \sum_{n \neq 0} \frac{a_{-n}}{n} e^{iu}, \quad (1.9)$$

where $P, Q$ and $a_n$ ($n = \pm 1, \pm 2, \ldots$) are operators which satisfy the commutation relations of the Heisenberg algebra

$$[Q, P] = \frac{i}{2} \beta^2; \quad [a_n, a_m] = \frac{n}{2} \beta^2 \delta_{n+m,0}. \quad (1.10)$$
with real $\beta$. The variable $u$ is interpreted as a complex coordinate on $2D$ cylinder of a circumference $2\pi$. The field $\varphi(u)$ is a quasi-periodic function of $u$, i.e.

$$\varphi(u + 2\pi) = \varphi(u) + 2\pi i P .$$  \hfill (1.11)

Let $\mathcal{F}_p$ be the Fock space, i.e. the space generated by a free action of the operators $a_n$ with $n < 0$ on the vacuum vector $| p \rangle$ which satisfies

$$a_n | p \rangle = 0 , \quad \text{for} \quad n > 0 ;$$

$$P | p \rangle = p | p \rangle .$$  \hfill (1.12)

The space $\mathcal{F}_p$ supports a highest weight representation of the Virasoro algebra generated by the operators

$$L_n = \int_0^{2\pi} \frac{du}{2\pi} \left[ T(u) + \frac{c}{24} \right] e^{iu}$$  \hfill (1.13)

with the Virasoro central charge

$$c = 13 - 6 \left( \beta^2 + \beta^{-2} \right)$$  \hfill (1.14)

and the highest weight

$$\Delta = \Delta(p) \equiv \left( \frac{p}{\beta} \right)^2 + \frac{c - 1}{24} .$$  \hfill (1.15)

Here $T(u)$ denotes the composite field

$$-\beta^2 T(u) =: \varphi'(u)^2 : + \left( 1 - \beta^2 \right) \varphi''(u) + \frac{\beta^2}{24}$$  \hfill (1.16)

which is a periodic function, $T(u + 2\pi) = T(u)$. The symbol $:\ :$ denotes the standard normal ordering with respect to the Fock vacuum (1.12). It is well known that if the parameters $\beta$ and $p$ take generic values this representation of the Virasoro algebra is irreducible. For particular values of these parameters, when null-vectors appear in $\mathcal{F}_p$, the irreducible representation $\mathcal{V}_{\Delta(p)}$ is obtained from $\mathcal{F}_p$ by factoring out all the invariant subspaces. In what follows we will always assume that all the invariant subspaces (if any) are factored out, and identify the spaces $\mathcal{F}_p$ and $\mathcal{V}_{\Delta(p)}$.

The space

$$\hat{\mathcal{F}}_p = \oplus_{k=-\infty}^{\infty} \mathcal{F}_{p+k\beta^2}$$  \hfill (1.17)

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admits the action of the exponential fields

$$V_\pm(u) = e^{\pm 2\phi(u)} \equiv \exp \left( \pm \sum_{n=1}^\infty \frac{a_{-n}}{n} e^{inu} \right) \exp \left( \pm 2i (Q + Pu) \right) \exp \left( \pm \sum_{n=1}^\infty \frac{a_n}{n} e^{-inu} \right).$$

(1.18)

The following relations are easily verified from (1.9)-(1.11)

$$V_{\sigma_1}(u_1) V_{\sigma_2}(u_2) = q^{2\sigma_1 \sigma_2} V_{\sigma_2}(u_2) V_{\sigma_1}(u_1), \quad u_1 > u_2,$$

$$P V_\pm(u) = V_\pm(u) (P \pm \beta^2),$$

(1.19)

where $\sigma_1, \sigma_2 = \pm 1$. Moreover,

$$V_\pm(u + 2\pi) = q^{-2} e^{\pm 4\pi i P} V_\pm(u).$$

(1.20)

Any CFT possesses infinitely many local Integrals of Motion (IM) $I_{2k-1}$

$$I_{2k-1} = \int_0^{2\pi} \frac{du}{2\pi} T_{2k}(u), \quad k = 1, 2, \ldots,$$

(1.21)

where $T_{2k}(u)$ are certain local fields, polynomials in $T(u)$ and its derivatives. For example

$$T_2(u) = T(u), \quad T_4(u) =: T^2(u) :,$$

$$T_{2k}(u) =: T^k(u) : + \text{terms with the derivatives}.$$

(1.22)

Here $:\ :$ denote appropriately regularized operator products, see [1] for details. There exists infinitely many densities (1.22) (one for each integer $k$ [10,11]) such that all IM(1.21) commute

$$[I_{2k-1}, I_{2l-1}] = 0.$$

(1.23)

Consider the following operator matrix [12,1]

$$L_j(\lambda) = \pi_j \left[ L(\lambda) \right],$$

(1.24)

$$L(\lambda) = e^{i\pi PH} \mathcal{P} \exp \left\{ \lambda \int_0^{2\pi} du \left( V_-(u) q^H E + V_+(u) q^{-H} F \right) \right\},$$

(1.25)

where the exponential fields $V_\pm(u)$ are defined in (1.18) and $E, F$ and $H$ are the generating elements of the quantum universal enveloping algebra $U_q(sl(2))$ [15],

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}.$$

(1.26)

\[\text{Note that the discrete analog of the operator } L_2(\lambda) \text{ has been used in } [13,14] \text{ in the context of the quantum lattice KdV equation.}\]
with
\[ q = e^{i\pi \beta^2}. \] (1.27)

The symbol \( \pi_j \) in (1.24) stands for the \((2j + 1)\) dimensional representation of \( U_q(sl(2)) \), so that (1.24) is in fact \((2j + 1) \times (2j + 1)\) matrix whose elements are the operators acting in the space (1.17). Following the conventional terminology, we will refer to this space as the “quantum space”. The expression (1.24) contains the ordered exponential (the symbol \( P \) denotes the path ordering) which can be defined in terms of the power series in \( \lambda \) as follows,
\[
L_j(\lambda) = \pi_j \left[ e^{i\pi P H} \sum_{k=0}^{\infty} \lambda^k \int_{2\pi \geq u_1 \geq u_2 \geq \ldots \geq u_k \geq 0} K(u_1)K(u_2)\ldots K(u_k) \ du_1 du_2 \ldots du_k \right], \tag{1.28}
\]
where
\[
K(u) = V_-(u) q^{\frac{u}{2}} E + V_+(u) q^{-\frac{u}{2}} F. \tag{1.29}
\]
The integrals in (1.28) make perfect sense if
\[ -\infty < c < -2. \tag{1.30} \]

For \(-2 < c < 1\) the integrals (1.28) diverge and power series expansion of (1.25) should be written down in terms of contour integrals, as explained in [2] (see also Appendix C of this paper).

In Sect.2 we will show that the operator matrices (1.24) satisfy the relations (1.5),
\[
R_{jj'}(\lambda \mu^{-1}) \ (L_j(\lambda) \otimes 1) \ (1 \otimes L_{j'}(\mu)) = (1 \otimes L_{j'}(\mu)) \ (L_j(\lambda) \otimes 1) \ R_{jj'}(\lambda \mu^{-1}), \tag{1.31}
\]
where the matrix \( R_{jj'}(\lambda) \) is the \( R \)-matrix associated with the representations \( \pi_j, \pi_{j'} \) of \( U_q(sl(2)) \); in particular
\[
R_{\frac{1}{2} \frac{1}{2}}(\lambda) = \begin{pmatrix}
q^{-1} \lambda - q \lambda^{-1} & \lambda - \lambda^{-1} & q^{-1} - q \\
\lambda - \lambda^{-1} & q^{-1} - q & \lambda - \lambda^{-1} \\
q^{-1} - q & \lambda - \lambda^{-1} & q^{-1} \lambda - q \lambda^{-1}
\end{pmatrix}. \tag{1.32}
\]

In fact, we will construct more general \( L \)-operators which satisfy the Yang-Baxter relation (1.3) with the universal \( R \)-matrix for the quantum Kac-Moody algebra \( U_q(\hat{sl}(2)) \). The equation (1.31) will follow then as a particular case.
2. The Yang-Baxter relation

The quantum Kac-Moody algebra $A = U_q(\hat{sl}(2))$ is generated by elements $h_0, h_1, x_0, x_1, y_0, y_1$, subject to the commutation relations

$$[h_i, h_j] = 0, \quad [h_i, x_j] = -a_{ij} x_j, \quad [h_i, y_j] = a_{ij} y_j, \quad (2.1)$$

$$[y_i, x_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad (2.2)$$

and the Serre relations

$$x_i^3 x_j - [3]_q x_i^2 x_j x_i + [3]_q x_i x_j x_i^2 - x_j x_i^3 = 0, \quad (2.3)$$

$$y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0.$$

Here the indices $i, j$ take two values $i, j = 0, 1$; $a_{ij}$ is the Cartan matrix of the algebra $U_q(\hat{sl}(2))$, $a_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, and $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. The sum

$$k = h_0 + h_1 \quad (2.4)$$

is a central element in the algebra $A$. Usually the algebra $A$ is supplemented by the grade operator $d$

$$[d, h_0] = [d, h_1] = [d, x_0] = [d, y_0] = 0, \quad [d, x_1] = x_1, \quad [d, y_1] = -y_1. \quad (2.5)$$

The algebra $A = U_q(\hat{sl}(2))$ is a Hopf algebra with the co-multiplication

$$\delta : \quad A \longrightarrow A \otimes A$$

defined as

$$\delta(x_i) = x_i \otimes 1 + q^{-h_i} \otimes x_i, \quad \delta(y_i) = y_i \otimes q^{h_i} + 1 \otimes y_i, \quad \delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \delta(d) = d \otimes 1 + 1 \otimes d. \quad (2.6)$$

where $i = 0, 1$. As usual we introduce

$$\delta' = \sigma \circ \delta, \quad \sigma \circ (a \otimes b) = b \otimes a \quad (\forall a, b \in A). \quad (2.7)$$
Define also two Borel subalgebras $B_- \subset A$ and $B_+ \subset A$ generated by $d, h_{0,1}, x_0, x_1$ and $d, h_{0,1}, y_0, y_1$ respectively. There exists a unique element \[ R \in B_+ \otimes B_- , \] satisfying the following relations

\[ \delta'(a) R = R \delta(a) \quad (\forall a \in A) , \]
\[ (\delta \otimes 1) R = R^{13} R^{23} , \]
\[ (1 \otimes \delta) R = R^{13} R^{12} , \]

where $R^{12}, R^{13}, R^{23} \in A \otimes A \otimes A$ and $R^{12} = R \otimes 1, R^{23} = 1 \otimes R, R^{13} = (\sigma \otimes 1) R^{23}$. The element $R$ is called the universal $R$-matrix. It satisfies the Yang-Baxter equation

\[ R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12} , \]

which is a simple corollary of the definitions (2.8). The universal $R$-matrix is understood as a formal series in generators in $B_+ \otimes B_-$. Its dependence on the Cartan elements can be isolated as a simple factor. It will be convenient to introduce the “reduced” universal $R$-matrix

\[ \overline{R} = q^{-(h_0 \otimes h_0)/2 + k \otimes d + d \otimes k} R = (\text{series in } y_0, y_1, x_0, x_1) , \]

where $y_i \in B_+ \otimes 1, x_i \in 1 \otimes B_- \ (i = 0, 1)$. There exists an “explicit” expression for the universal $R$-matrix \[\text{(13,19)}\] which, in general case, provides an algorithmic procedure for the computation of this series order by order. Using these results or directly from the definitions (2.8) and (2.9) one can calculate the first few terms in (2.11)

\[ \overline{R} = 1 + (q - q^{-1})(y_0 \otimes x_0 + y_1 \otimes x_1) + \frac{q - q^{-1}}{[2]_q} \left\{ (q^2 - 1)(y_0^2 \otimes x_0^2 + y_1^2 \otimes x_1^2) + y_0 y_1 \otimes (x_1 x_0 - q^{-2} x_0 x_1) + y_1 y_0 \otimes (x_0 x_1 - q^{-2} x_1 x_0) \right\} + \ldots . \]

The higher terms soon become very complicated and their general form is unknown. This complexity should not be surprising, since the universal $R$-matrix contains infinitely many nontrivial solutions of the Yang-Baxter equation associated with $U_q(\widehat{sl}(2))$. A few more terms of the expansion (2.12) are given in the Appendix A.
We are now ready to prove the Yang-Baxter equation \((1.31)\) and its generalizations. Consider the following operator

\[
\mathcal{L} = e^{i\pi P h} \mathcal{P} \exp \left( \int_0^{2\pi} \mathcal{K}(u) du \right),
\]

(2.13)

where

\[
\mathcal{K}(u) = V_-(u) y_0 + V_+(u) y_1.
\]

(2.14)

Here \(h = h_0 = -h_1, y_0, y_1\) are the generators of the Borel subalgebra \(\mathcal{B}_+\) and the \(\mathcal{P}\)-exponent is defined as the series of the ordered integrals of \(\mathcal{K}(u)\), similarly to \((1.28)\). Notice that we assumed here that the central charge \(k\) is zero; considering this case is sufficient for our goals. The operator \((2.13)\) is an element of the algebra \(\mathcal{B}_+\) whose coefficients are operators acting in the quantum space \((1.17)\). It is more general than the one in \((1.25)\) and reduces to the latter for a particular representation of \(\mathcal{B}_+\) (see below). Consider now two operators \((2.13)\)

\[
\mathcal{L} \otimes 1 \in \mathcal{B}_+ \otimes 1, \quad 1 \otimes \mathcal{L} \in 1 \otimes \mathcal{B}_+
\]

(2.15)

belonging to the different factors of the direct product \(\mathcal{B}_+ \otimes \mathcal{B}_+\). Using \((1.19)\) for the product of these operators one obtains

\[
(\mathcal{L} \otimes 1) (1 \otimes \mathcal{L}) = e^{i\pi P \delta(h)} \mathcal{P} \exp \left( \int_0^{2\pi} \mathcal{K}_1(u) du \right) \mathcal{P} \exp \left( \int_0^{2\pi} \mathcal{K}_2(u) du \right),
\]

(2.16)

where

\[
\delta(h) = h \otimes 1 + 1 \otimes h,
\]

(2.17)

and

\[
\mathcal{K}_1(u) = V_-(u) (y_0 \otimes q^h) + V_+(u) (y_1 \otimes q^{-h}),
\]

\[
\mathcal{K}_2(u) = V_-(u) (1 \otimes y_0) + V_+(u) (1 \otimes y_1).
\]

(2.18)

Taking into account \((1.19)\) and \((2.1)\) it is easy to see that

\[
[K_1(u_1), K_2(u_2)] = 0, \quad u_1 < u_2,
\]

(2.19)

therefore the product of the \(\mathcal{P}\)-exponents in \((2.16)\) can be rewritten as

\[
(\mathcal{L} \otimes 1) (1 \otimes \mathcal{L}) = e^{i\pi P \delta(h)} \mathcal{P} \exp \left( \int_0^{2\pi} (K_1(u) + K_2(u)) du \right)
\]

\[
= e^{i\pi P \delta(h)} \mathcal{P} \exp \left( \int_0^{2\pi} (V_-(u) \delta(y_0) + V_+(u) \delta(y_1)) du \right)
\]

\[
= \delta(\mathcal{L}),
\]

(2.20)
where the co-multiplication $\delta$ is defined in (2.6). Similarly

$$\left(1 \otimes L\right)\left(L \otimes 1\right) = \delta(L),$$

(2.21)

with $\delta'$ defined in (2.7). Combining (2.20) and (2.21) with the first equations in (2.9) one obtains the following Yang-Baxter equation

$$R\left(L \otimes 1\right)\left(1 \otimes L\right) = \left(1 \otimes L\right)\left(L \otimes 1\right)R.$$  

(2.22)

Obviously, this equation is more general than (1.31). To obtain the latter from (2.22) we only need to choose appropriate representations in each factor of the direct product $A \otimes A$ involved in (2.22). Consider the so-called evaluation homomorphism $U_q(\hat{sl}(2)) \rightarrow U_q(sl(2))$ of the form

$$x_0 \rightarrow \lambda^{-1}Fq^{-H/2}, \quad y_0 \rightarrow \lambda q^{H/2}E, \quad h_0 \rightarrow H,$$

$$x_1 \rightarrow \lambda^{-1}E q^{H/2}, \quad y_1 \rightarrow \lambda q^{-H/2}F, \quad h_1 \rightarrow -H,$$

(2.23)

where $\lambda$ is a spectral parameter, and $E, F, H$ are the generators of the algebra $U_q(sl(2))$, defined already in (1.26). One could easily check that with the map (2.23) all the defining relations (2.1), (2.2) and (2.3) of the algebra $A = U_q(\hat{sl}(2))$ become simple corollaries of (1.26). For any representation $\pi$ of $U_q(sl(2))$ the formulae (2.23) define a representation of the algebra $A$ with zero central charge $k$, which will be denoted as $\pi(\lambda)$. In particular, the matrix representations of $A$ corresponding to the $(2j + 1)$-dimensional representations $\pi_j$ of $U_q(sl(2))$ will be denoted $\pi_j(\lambda)$. Let us now evaluate the Yang-Baxter equation (2.22) in the representations $\pi_j(\lambda)$ and $\pi_{j'}(\mu)$ for the first and second factor of the direct product respectively. For the $L$-operators one has

$$\pi_j(\lambda)\left[L\right] = L_j(\lambda), \quad \pi_{j'}(\mu)\left[L\right] = L_{j'}(\mu),$$

(2.24)

with $L_j$ given by (1.24), while for the $R$-matrix one obtains

$$\left(\pi_j(\lambda) \otimes \pi_{j'}(\mu)\right)\left[R\right] = \rho_{jj'}(\lambda/\mu) R_{jj'}(\lambda/\mu),$$

(2.25)

where $\rho_{jj'}$ is a scalar factor and the $R_{jj'}$ is the same as in (1.31) [20]. This completes the proof of (1.31).

We conclude this section with following observation concerning the structure of the $L$-operator (2.13). As one could expect the equation (2.22) is, in fact, a specialization
of the Yang-Baxter equation (2.10) for the universal \( R \)-matrix. To demonstrate this it would be sufficient to find an appropriate realization of the algebra \( \mathcal{A} \) in the third factor of the product \( \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \) involved in (2.10), such that (2.10) reduces to (2.22). A little inspection shows that each side of (2.10) is an element of \( \mathcal{B}_+ \otimes \mathcal{A} \otimes \mathcal{B}_- \) rather than an element of \( \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \). Therefore we do not need a realization of the full algebra \( \mathcal{A} \) in third factor; realization of the Borel subalgebra \( \mathcal{B}_- \) is sufficient. Let us identify the generators \( x_0, x_1 \in \mathcal{B}_- \) of this Borel subalgebra with the integrals of the exponential fields

\[
x_0 = \frac{1}{q-q^{-1}} \int_0^{2\pi} V_-(u) \, du, \quad x_1 = \frac{1}{q-q^{-1}} \int_0^{2\pi} V_+(u) \, du.
\]  \( \tag{2.26} \)

One can check that these generators satisfy \([11]\) the Serre relations (2.3). To do this one should express the fourth order products of \( x \)'s in (2.3) in terms of the ordered integrals of products of the exponential fields \( V_\pm(u) \). The calculations are simple but rather technical. We present them in the Appendix A.

Substituting the expressions (2.26) for the generators \( x_0, x_1 \) into the “reduced” universal \( R \)-matrix \( \overline{R} \) (2.11), (2.12) one obtains a vector in \( \mathcal{B}_+ \) whose coordinates are operators acting in the quantum space (1.17). It is natural to expect that it coincides with the \( \mathcal{P} \)-ordered exponent from (2.13).

**Conjecture.** The specialization of the “reduced” universal \( R \)-matrix \( \overline{R} \in \mathcal{B}_+ \otimes \mathcal{B}_- \) (2.11) for the case when \( x_0, x_1 \in \mathcal{B}_- \) are realized as in (2.26) coincide with the \( \mathcal{P} \)-exponent

\[
\overline{L} = \mathcal{P} \exp \left( \int_0^{2\pi} \mathcal{K}(u) \, du \right),
\]  \( \tag{2.27} \)

where \( \mathcal{K}(u) \) is defined in (2.14).

One can check this conjecture in a few lowest orders in the series expansion for the universal \( R \)-matrix. Substitute (2.26) into (2.12). It is not difficult to see that every polynomial of \( x \)'s appearing in (2.12) as a coefficient to the monomial in \( y \)'s can be written as a single ordered integral of the vertex operators (rather than their linear combination). For example, the second order terms read

\[
x_0^2 = \frac{[2]_q}{q(q-q^{-1})^2} J(-,-), \quad (x_0x_1 - q^{-2}x_1x_0) = \frac{[2]_q}{(q-q^{-1})} J(+,-),
\]

\[
x_1^2 = \frac{[2]_q}{q(q-q^{-1})^2} J(+,+), \quad (x_1x_0 - q^{-2}x_0x_1) = \frac{[2]_q}{(q-q^{-1})} J(-,+),
\]  \( \tag{2.28} \)

\( ^2 \) This statement requires no restrictions on the value of the central element \( k \).
where

$$J(\sigma_1, \sigma_2, \ldots, \sigma_n) = \int_{2\pi \geq u_1, \ldots, u_n \geq 0} V_{\sigma_1}(u_1) V_{\sigma_2}(u_2) \cdots V_{\sigma_n}(u_n) \, du_1 \ldots du_n , \quad (2.29)$$

with $\sigma_i = \pm 1$. Using (2.26) and (2.29) one can rewrite the RHS of (2.12) as

$$\mathcal{R} = 1 + y_0 J(-) + y_1 J(+) + y_0^2 J(-,-) + y_0 y_1 J(-,+) + y_1 y_0 J(+,-) + \ldots , \quad (2.30)$$

which coincides with the first three terms of the expansion of $P$-exponent (2.27). We have verified this conjecture to within the terms of the fourth order in the generators $x_0$ and $x_1$ (see Appendix A). Notice that starting from the fourth order one has to take into account the Serre relations (2.3). The above conjecture suggests that the operators (2.27) can be reexpressed through algebraic combinations of the two elementary integrals (2.26) instead of the ordered integrals (2.29).\(^4\) Conversely, this statement combined with the uniqueness\(^1\) of universal $R$-matrix satisfying (2.8) and (2.9) implies the above conjecture.

Finally let us stress that our proof of the Yang-Baxter equations (1.31) and (2.22) is independent of this conjecture.

### 3. Commuting T- and Q-operators

It is well known and simple consequence of the Yang-Baxter relation (1.31) that appropriately defined traces of the operator matrices $L_j(\lambda)$ give rise to the operators $T_j(\lambda)$ which commute for different values of the parameter $\lambda$, i.e.

$$[T_j(\lambda), T_{j'}(\lambda)] = 0 . \quad (3.1)$$

In fact, there is an certain ambiguity in the construction of these operators. Below we show that this ambiguity is eliminated if we impose additional requirement that the operators $T_j(\lambda)$ also commute with the local IM (1.21),

$$[T_j(\lambda), I_{2k-1}] = 0 . \quad (3.2)$$

\(^3\) Note that this definition differs by the factor $q^n$ from that given in Eq.(2.31) of Ref. \cite{RefB}.\(^4\) Perhaps this statement is less trivial than it might appear. In fact, one can always write any product of $x_0$ and $x_1$ from (2.26) as a linear combination of the integrals (2.29), but not vice versa, since the elementary integrals (2.26) are algebraically dependent due to the Serre relations.
It is easy to check that the Yang-Baxter equation (1.31) is not affected if one multiplies the $L$-operator (1.25) by an exponent of the Cartan element $H$

$$L(\lambda) \rightarrow L^{(f)}(\lambda) = e^{i f H} L(\lambda), \quad (3.3)$$

where $f$ is an arbitrary constant. Therefore the operators

$$T_j^{(f)}(\lambda) = \text{Tr}_{\pi_j} \left[ e^{i f H} L_j(\lambda) \right] \quad (3.4)$$

satisfy the commutativity relations (3.1) for any value of $f$. Moreover, this commutativity is not violated even if the quantity $f$ is a function of $P$ rather than a constant (despite the fact that in this case the operators (3.3) do not necessarily satisfy the ordinary Yang-Baxter equation (1.31)). This is obvious if one uses the standard realization of the spin-$j$ representations $\pi_j$ of the algebra (1.26)

$$\pi_j [E] |k\rangle = [k]_q [2j-k+1]_q |k-1\rangle, \quad \pi_j [F] |k\rangle = |k+1\rangle, \quad \pi_j [H] |k\rangle = (2j-2k) |k\rangle, \quad (3.5)$$

where $[k]_q = (q^k - q^{-k})/(q - q^{-1})$ and the vectors $|k\rangle$ ($k = 0, 1, \ldots, 2j$) form a basis in the $(2j+1)$-dimensional space. Then, using (1.19) it is easy to show that all the diagonal entries of the $(2j+1) \times (2j+1)$ matrices $L_j(\lambda)$ commute with the operator $P$. As an immediate consequence the quantity $f = f(P)$ in (3.4) can be treated as a constant and therefore the commutativity (3.1) remains valid. It follows also that the operators (3.4) invariantly act in each Fock module $F_p$.

The commutativity (3.2) requires a special choice of the function $f = f(P)$. We show in the Appendix C that the operators (3.4) commute with $I_1 = L_0 - c/24$ if

$$f = \pi (P + N) . \quad (3.6)$$

Here $N$ is an arbitrary integer which obviously has no other effect on (3.4) than the overall sign of this operator; in what follows we set $N = 0$ and define

$$T_j(\lambda) = \text{Tr}_{\pi_j} \left[ e^{i \pi P H} L_j(\lambda) \right]. \quad (3.7)$$

In fact, with this choice of $f$ the operators (3.7) commute with all the local IM (1.21). This is demonstrated in Appendix C. The operators (3.7) act invariantly in each Fock module $F_p$ and satisfy both (3.1) and (3.2).
The above operators $T_j(\lambda)$ are CFT analogs of the commuting transfer-matrices of the Baxter’s lattice theory. Besides these commuting transfer-matrices the “technology” of the solvable lattice models involves also another important object - the Baxter’s $Q$-matrix [3]. It turns out that another specialization of the general $L$ operator (2.13) leads to the CFT analog of the $Q$-matrix [2].

Consider the so-called $q$-oscillator algebra generated by the elements $H, E_+, E_-$ subject to the relations

$$q E_+ E_- - q^{-1} E_- E_+ = \frac{1}{q - q^{-1}}, \quad [H, E_\pm] = \pm 2 E_\pm . \quad (3.8)$$

One can easily show that the following two maps of the Borel subalgebra $B_-$ of $U_q(\widehat{sl}(2))$ into the q-oscillator algebra (3.8)

$$h = h_0 = -h_1 \rightarrow \pm H, \quad y_0 \rightarrow \lambda E_+, \quad y_1 \rightarrow \lambda E_- \quad (3.9)$$

(here one has to choose all the upper or all the lower signs) are homomorphisms. Under these homomorphisms the operator (2.13) becomes an element of the algebra (3.8)

$$L \rightarrow L_{\pm}(\lambda) = e^{\pm i\pi P H} \mathcal{P} \exp \left( \lambda \int_0^{2\pi} du \left( V_- (u) q^{\frac{\pi}{2}} E_\pm + V_+ (u) q^{\frac{\pi}{2}} E_\mp \right) \right) . \quad (3.10)$$

Let $\rho_\pm$ be any representations of (3.8) such that the trace

$$Z_\pm (p) = \text{Tr}_{\rho_\pm} [ e^{\pm 2\pi i p H} ] \quad (3.11)$$

exists and does not vanish for complex $p$ belonging to the lower half plane, $\Im p < 0$. Then define two operators

$$A_{\pm}(\lambda) = Z_{\pm}^{-1}(p) \text{Tr}_{\rho_\pm} [ e^{\pm \pi i P H} L_{\pm}(\lambda) ] . \quad (3.12)$$

Since we are interested in the action of these operators in $F_p$ the operator $P$ in (3.12) can be substituted by its eigenvalue $p$. The definition (3.12) applies to the case $\Im p < 0$. However the operators $A_{\pm}(\lambda)$ can be defined for all complex $p$ (except for some set of singular points on the real axis) by an analytic continuation in $p$. As was shown in [2], the trace in (3.12) is completely determined by the commutation relations (3.8) and the cyclic property of the trace, so the specific choice of the representations $\rho_{\pm}$ is not significant as long as the above property is maintained.
The $Q$-operators (the CFT analogs of the Baxter’s $Q$-matrix) are defined as

$$Q_{\pm}(\lambda) = \lambda^{\pm 2\beta^2} A_{\pm}(\lambda).$$

(3.13)

Similarly to the $T$-operators they act invariantly in each Fock module $F_p$

$$Q_{\pm}(\lambda) : F_p \rightarrow F_p,$$

(3.14)

and commute with the local IM (1.21). The operators $Q_{\pm}(\lambda)$ with different values of $\lambda$ commute among themselves and with all the operators $T_j(\lambda)$

$$[Q_{\pm}(\lambda), Q_{\pm}(\lambda')] = [Q_{\pm}(\lambda), T_j(\lambda')] = 0.$$

(3.15)

This follows from the appropriate specializations of the Yang-Baxter equation (2.22).

The operators $T_j(\lambda)$ and $A_{\pm}(\lambda)$ enjoy remarkable analyticity properties as the functions of the variable $\lambda^2$. Namely, all the matrix elements and eigenvalues of these operators are entire functions of this variable [1,2]. The proof is carried out in the Appendix B. It is based on the result of [21] on the analyticity of certain Coulomb partition functions which was obtained through the Jack polynomial technique. Currently there is a complete proof of the above analyticity for $T_j(\lambda)$ for all values of $\beta^2$ in the domain $0 < \beta^2 < 1/2$ (which corresponds to (1.30) ) and “almost complete” proof of this analyticity for $A_{\pm}(\lambda)$ which extends to all rational values of $\beta^2$ and to almost all irrational values of $\beta^2$ (i.e. to all irrationals values except for some set of the Lebesgue measure zero, see Appendix B for the details) in the above interval. It is natural to assume that this analyticity remains valid for those exceptional irrationals as well.

4. The functional relations

It is well know from the lattice theory that analyticity of the commuting transfer matrices become extremely powerful condition when combined with the functional relations which the transfer matrices satisfy, and, in principle, allows one to determine all the eigenvalues. Therefore, the functional relations (FR) for the operators $Q_{\pm}(\lambda)$ and $T_j(\lambda)$ are of primary interest. We start our consideration with the “fundamental” FR (fundamental in the sense that it implies all the other functional relations involving the operators $T_j(\lambda)$ or $Q_{\pm}(\lambda)$).
\textbf{i.) Fundamental Relation.} The “transfer-matrices” $T_j(\lambda)$ can be expressed in terms of $Q_{\pm}(\lambda)$ as

\begin{equation}
2i \sin(2\pi P) T_j(\lambda) = Q_+(q^{j + \frac{1}{2}}\lambda)Q_-(q^{-j - \frac{1}{2}}\lambda) - Q_+(q^{-j + \frac{1}{2}}\lambda)Q_-(q^{j + \frac{1}{2}}\lambda),
\end{equation}

where $j$ takes (half-) integer values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \infty$.

Before going into the proof of (4.1) let us mention its simple but important corollary

\textbf{ii.) $T$-$Q$ Relation.} The operators $Q_{\pm}(\lambda)$ satisfy the Baxter’s $T$-$Q$ equation

\begin{equation}
T(\lambda)Q_{\pm}(\lambda) = Q_{\pm}(q\lambda) + Q_{\pm}(q^{-1}\lambda),
\end{equation}

where $T(\lambda) \equiv T_{1/2}(\lambda)$. This equation can be thought of as the finite-difference analog of a second order differential equation so we expect it to have two linearly independent solutions. As $T(\lambda)$ is a single-valued function of $\lambda^2$, i.e. it is a periodic function of $\log \lambda^2$, the operators $Q_{\pm}(\lambda)$ are interpreted as two “Bloch-wave” solutions to the equation (4.2). The operators $Q_{\pm}(\lambda)$ satisfy the “quantum Wronskian” condition

\begin{equation}
Q_+(q^{\frac{1}{2}}\lambda)Q_-(q^{-\frac{1}{2}}\lambda) - Q_+(q^{-\frac{1}{2}}\lambda)Q_-(q^{\frac{1}{2}}\lambda) = 2i \sin(2\pi P),
\end{equation}

which is just a particular case of (4.1) with $j = 0$.

To prove these relations consider more general $T$-operators which correspond to the infinite dimensional highest weight representations of $U_q(sl(2))$. These new $T$-operators are defined by the same formula as (3.7)

\begin{equation}
T_j^+(\lambda) = \text{Tr}_{\pi_j^+} [e^{i\pi P H} L_j^+(\lambda)], \quad L_j^+(\lambda) = \pi_j^+ [L(\lambda)]
\end{equation}

except that the trace is now taken over the infinite dimensional representation $\pi_j^+$ of (1.26). The corresponding representation matrices $\pi_j^+[E], \pi_j^+[F]$ and $\pi_j^+[H]$ for the generators of (1.26) are defined by the equations

\begin{equation}
\pi_j^+[E] |k\rangle = [k]_q [2j - k + 1]_q |k - 1\rangle, \quad \pi_j^+[F] |k\rangle = |k + 1\rangle, \quad \pi_j^+[H] |k\rangle = (2j - 2k) |k\rangle,
\end{equation}

which are similar to (3.5), but the basis $|k\rangle$, is now infinite, $k = 0, 1, \ldots, \infty$. The highest weight $2j$ of the representation $\pi_j^+$,

\begin{equation}
\pi_j^+(H) |0\rangle = 2j |0\rangle,
\end{equation}
can take arbitrary complex values. Since we are interested in the action of the operators \( T_j(\lambda) \) in \( \mathcal{F}_p \) the operator \( P \) in (4.4) can be substituted by its eigenvalue \( p \). Similarly to (3.12) the definition (4.4) makes sense only if \( \Im p < 0 \), but it can be extended to all complex \( p \) (except for some set of singular points on the real axis) by the analytic continuation in \( p \). The operators (4.4) thus defined satisfy the commutativity conditions

\[
[T_j(\lambda), T_{j'}(\mu)] = [T_j^+(\lambda), T_{j'}^+(\mu)] = 0,
\]

which follow from the appropriate specializations of the Yang-Baxter equation (2.22).

If \( j \) takes a non-negative integer or half-integer value the matrices \( \pi_j^+ [E] \), \( \pi_j^+ [F] \) and \( \pi_j^+ [H] \) in (4.5) have a block-triangular form with two diagonal blocks, one (finite) being equivalent to the \( (2j+1) \) dimensional representation \( \pi_j \) and the other (infinite) coinciding with the highest weight representation \( \pi_{-j}^+ \). Hence the following simple relation holds,

\[
T_j^+(\lambda) = T_j(\lambda) + T_{-j-1}^+(\lambda), \quad j = 0, 1/2, 1, 3/2, \ldots .
\]  

(4.6)

In some ways the operators \( T_j^+(\lambda) \) are simpler than \( T_j(\lambda) \). Making a similarity transformation

\[
E \rightarrow \lambda E, \quad F \rightarrow \lambda^{-1} F,
\]

which does not affect the trace in (4.4) and observing that

\[
\lambda^2 \pi_j^+ [E] |k\rangle = \frac{[k]_q}{q - q^{-1}} \left( \lambda_+^2 q^{-k} - \lambda_-^2 q^k \right) |k - 1\rangle, \quad k = 0, 1, \ldots, \infty,
\]

where

\[
\lambda_+ = \lambda q^{j+\frac{1}{2}}, \quad \lambda_- = \lambda q^{-j-\frac{1}{2}},
\]  

(4.7)

one can conclude that the operator \( T_j^+(\lambda) \) can be written as

\[
T_j^+(\lambda) = T_j^+(0) \Phi(\lambda_+, \lambda_-),
\]  

(4.8)

where

\[
T_j^+(0) = \frac{e^{2\pi i (2j+1)P}}{2i \sin(2\pi P)}
\]  

(4.9)

and \( \Phi(\lambda_+, \lambda_-) \) is a series in \( \lambda_+^2 \) and \( \lambda_-^2 \) with the coefficients which do not depend on \( j \) and the leading coefficient being equal to 1. Remarkably, the expression (4.8) further simplifies since the quantity \( \Phi(\lambda_+, \lambda_-) \) factorizes into a product of two operators (3.12)

\[
2i \sin(2\pi P) \ T_j^+(\lambda) = e^{2\pi i (2j+1)P} \ A_+ (\lambda q^{j+\frac{1}{2}}) \ A_- (\lambda q^{-j-\frac{1}{2}}).
\]  

(4.10)
This factorization can be proved algebraically by using decomposition properties of the tensor product of two representations of the q-oscillator algebra (the latter are also representations of the Borel subalgebra of \( U_q(\hat{sl}(2)) \) with respect to the co-multiplication from \( U_q(\hat{sl}(2)) \). The detail of the calculations are presented in the Appendix D. The functional relation (4.1) trivially follows from (4.6) and (4.10).

The relation (4.3) shows that the operators \( Q_+^{\lambda} \) and \( Q_-^{\lambda} \) are functionally dependent. Using this dependence one can write (4.1) as

\[
T_j(\lambda) = Q(q^{j+\frac{1}{2}}\lambda) Q(q^{-j-\frac{1}{2}}\lambda) \sum_{k=-j}^{j} \frac{1}{Q(q^{k+\frac{1}{2}}\lambda) Q(q^{k-\frac{1}{2}}\lambda)},
\]

where \( Q(\lambda) \) is any one of \( Q_+^{\lambda} \) and \( Q_-^{\lambda} \).

The last group of FR we want to discuss here is the relations involving solely the transfer matrices \( T_j(\lambda) \) and usually referred to as the “fusion relations”[22]. Note that these are again simple corollaries of the “fundamental relation” (4.1).

iii.) Fusion relations. The transfer matrices \( T_j(\lambda) \) satisfy

\[
T_j(q^{\frac{j}{2}}\lambda) T_j(q^{-\frac{j}{2}}\lambda) = 1 + T_{j+\frac{1}{2}}(\lambda) T_{j-\frac{1}{2}}(\lambda),
\]

where \( T_0(\lambda) \equiv 1 \). These can also be equivalently rewritten as

\[
T(\lambda) T_j(q^{j+\frac{1}{2}}\lambda) = T_{j-\frac{1}{2}}(q^{j+1}\lambda) + T_{j+\frac{1}{2}}(q^j\lambda),
\]

or as

\[
T(\lambda) T_j(q^{-j-\frac{1}{2}}\lambda) = T_{j-\frac{1}{2}}(q^{-j-1}\lambda) + T_{j+\frac{1}{2}}(q^{-j}\lambda).
\]

Considerable reductions of the FR occur when \( q \) is a root of unity. Let

\[
q^N = \pm 1 \quad \text{and} \quad q^n \neq \pm 1 \quad \text{for any integer} \quad 0 < n < N,
\]

where \( N \geq 2 \) is some integer. When using (4.11) it is easy to obtain that

\[
e^{2\pi i N P} T_j(\lambda) + T_{\frac{N}{2} - 1 - j}(\lambda q^{N}) = \frac{\sin(2\pi N P)}{\sin(2\pi P)} Q_+(\lambda q^{j+\frac{1}{2}}) Q_-^{\lambda}(\lambda q^{-j-\frac{1}{2}})
\]

In fact, all the above FR can also be called the fusion relation since they all follow from (4.10) which describes the “fusion” of the q-oscillator algebra representations.
for \( j = 0, \frac{1}{2}, \ldots , \frac{N}{2} - 1 \). Similarly,

\[
T_{\frac{N}{2} - j} (\lambda) = \frac{\sin(2\pi NP)}{\sin(2\pi P)} Q_+ (\lambda q^{\frac{N}{2}}) Q_- (\lambda q^{\frac{N}{2}}) .
\] (4.17)

Moreover in this case there is an extra relation involving only \( T \)-operators

\[
T_{\frac{N}{2}} (\lambda) = 2 \cos(2\pi NP) + T_{\frac{N}{2} - 1} (\lambda) ,
\] (4.18)

as it readily follows from (4.1). As is shown in [23] this allows to bring the FR (4.12) to the form identical to the functional TBA equations (the \( Y \)-system) of the \( D_N \) type.

Additional simplifications occur when the operators \( T_j \) act in Fock spaces \( F_p \) with special values of \( p \),

\[
p = \frac{\ell + 1}{2N} ,
\] (4.19)

where \( \ell \geq 0 \) is an integer such that \( 2p \neq n\beta^2 + m \) for any integers \( n \) and \( m \). Then the RHS’s of (4.16) and (4.17) vanish and these relations lead to

\[
T_{\frac{N}{2} - j - 1} (\lambda) = (-1)^{\ell} T_j (q^{\frac{N}{2}} \lambda) , \quad \text{for} \quad j = 0, \frac{1}{2}, 1, \ldots , \frac{N}{2} - 1 ; \quad T_{\frac{N}{2} - \frac{1}{2}} (\lambda) = 0 .
\] (4.20)

Further discussion of this case can be found in [1] and [23].

Finally, some remarks concerning the lattice theory are worth making. Although our construction of the \( Q \)-operators in terms of the \( q \)-oscillator representations was given here specifically for the case of continuous theory, it is clear that the lattice \( Q \)-matrices admit similar construction. In particular the \( Q \)-matrix of the six-vertex model can be obtained as a transfer matrix associated with infinite dimensional representations of the \( q \)-oscillator algebra (3.8). In the case of the six-vertex vertex model with nonzero (horizontal) field this construction gives rise to two \( Q \)-matrices, \( Q_{\pm} \). As the structure of the FR (1.1), (4.2), (4.10) is completely determined by the decomposition properties of products of representations of \( U_q(\hat{sl}(2)) \), all these FR are valid in the lattice case, with minor modifications mostly related to the normalization conventions of the lattice transfer matrices.

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6 Using this construction it is possible, in particular, to reproduce a remarkably simple expression for an arbitrary matrix element of the \( Q \)-matrix of the zero field six-vertex model in the “half-filling” sector [3].
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Appendix A.

Here we present the results on the series expansion verification of our conjecture on the structure of the universal $R$-matrix for the quantum Kac-Moody algebra $U_q(\widehat{sl}(2))$.

We will need expressions for products of the basic contour integrals (2.26) in terms of linear combinations of the ordered integrals (2.29). To derive them one only has to use the commutation relation (1.19) for the vertex operators. For example, consider the second order product

$$(q-q^{-1})^2 x_0 x_1 = \int_0^{2\pi} \int_0^{2\pi} :e^{-2\varphi(u_1)} :e^{2\varphi(u_2)} :du_1 du_2$$

$$= \int_{2\pi>u_1>u_2>0} \ldots du_1 du_2 + \int_{2\pi>u_2>u_1>0} \ldots du_1 du_2$$

$$= J(-,+) + q^2 J(+,-),$$

where $J$’s are defined in (2.29).

For the $n$-th order products one has to split the domain of integration in $n!$ pieces corresponding to all possible orderings of the integration variables and then rearrange the products of the vertex operators using the commutation relations (1.19). Below we present the results of these calculations for the products of orders less or equal to four,

$$x_0 = \frac{1}{(q-q^{-1})} J(-),$$

$$x_0^2 = \frac{q^{-1}[2]_q}{(q-q^{-1})^2} J(-,-),$$

$$x_0^3 = \frac{q^{-3}[2]_q[3]_q}{(q-q^{-1})^3} J(-,-,-),$$

$$x_0^4 = \frac{q^{-6}[2]_q[3]_q[4]_q}{(q-q^{-1})^4} J(-,-,-,-),$$

$$x_1 = \frac{1}{(q-q^{-1})} J(+),$$

$$x_1^2 = \frac{q^{-1}[2]_q}{(q-q^{-1})^2} J(+,+),$$

$$x_1^3 = \frac{q^{-3}[2]_q[3]_q}{(q-q^{-1})^3} J(+,+,+),$$

$$x_1^4 = \frac{q^{-6}[2]_q[3]_q[4]_q}{(q-q^{-1})^4} J(+,+,+,+),$$

(A.2)
\[
\begin{align*}
\begin{bmatrix}
x_0x_1 \\
x_1x_0
\end{bmatrix} &= \frac{1}{(q - q^{-1})^2} \begin{bmatrix} 1 & q^2 \\ q^2 & 1 \end{bmatrix} \begin{bmatrix} J(-, +) \\ J(+, -) \end{bmatrix}, \quad (A.3) \\
\begin{bmatrix}
x_0^2x_1 \\
x_0x_1x_0 \\
x_1x_0^2 \\
x_0^2x_1
\end{bmatrix} &= \frac{[2]_q}{q(q - q^{-1})^3} \begin{bmatrix} 1 & q^2 & q^4 \\ q^2 & q^2 & q^2 \\ q^4 & q^2 & 1 \end{bmatrix} \begin{bmatrix} J(-, -, +) \\ J(-, +, -) \\ J(+, -, -) \end{bmatrix}, \quad (A.4) \\
\begin{bmatrix}
x_1^2x_0 \\
x_1x_0x_1 \\
x_0x_1^2 \\
x_0^2x_1
\end{bmatrix} &= \frac{[2]_q}{q(q - q^{-1})^3} \begin{bmatrix} 1 & q^2 & q^4 \\ q^2 & q^2 & q^2 \\ q^4 & q^2 & 1 \end{bmatrix} \begin{bmatrix} J(+, -) \\ J(+, -) \\ J(-, -) \end{bmatrix}, \quad (A.5) \\
\begin{bmatrix}
x_0^3x_1 \\
x_0^2x_1x_0 \\
x_0x_1^2x_0 \\
x_1x_0^3 \\
x_0^3x_1
\end{bmatrix} &= \frac{[2]_q}{(q - q^{-1})^4} \begin{bmatrix} q^{-3}[3]_q & q^{-1}[3]_q & q[3]_q & q^3[3]_q \\ q^{-1}[3]_q & q + 2q^{-1} & 2q + q^{-1} & q[3]_q \\ q[3]_q & 2q + q^{-1} & q + 2q^{-1} & q^{-1}[3]_q \\ q^3[3]_q & q[3]_q & q^{-1}[3]_q & q^{-3}[3]_q \end{bmatrix} \begin{bmatrix} J(-, -, -) \\ J(-, -, -) \\ J(+, -, -) \\ J(+, -, -) \end{bmatrix}, \quad (A.6) \\
\begin{bmatrix}
x_1^3x_0 \\
x_1^2x_0x_1 \\
x_1x_0^2x_1 \\
x_0x_1^3 \\
x_0^3x_1
\end{bmatrix} &= \frac{[2]_q}{(q - q^{-1})^4} \begin{bmatrix} q^{-3}[3]_q & q^{-1}[3]_q & q[3]_q & q^3[3]_q \\ q^{-1}[3]_q & q + 2q^{-1} & 2q + q^{-1} & q[3]_q \\ q[3]_q & 2q + q^{-1} & q + 2q^{-1} & q^{-1}[3]_q \\ q^3[3]_q & q[3]_q & q^{-1}[3]_q & q^{-3}[3]_q \end{bmatrix} \begin{bmatrix} J(+, +, -) \\ J(+, +, -) \\ J(+, +, -) \\ J(+, +, -) \end{bmatrix}, \quad (A.7) \\
\begin{bmatrix}
x_0^2x_1 \\
x_0x_1x_0 \\
x_0x_1^2 \\
x_1x_0^2 \\
x_1x_0x_1 \\
x_2^2x_1
\end{bmatrix} &= \frac{q^2[2]_q}{(q - q^{-1})^4} \begin{bmatrix} q^{-4} & q^{-2} & 1 & 1 & q^2 & q^4 \\ q^{-2} & 2q^{-1} & [2]_q & 1 & 1 & q^2 \\ 1 & 1 & [3]_q - 2 & 1 & 1 & 1 \\ q^2 & 2q^{-1} & [2]_q & 1 & 1 & 1 \\ q^4 & q^2 & 1 & q^{-2} & q^2 & q^{-4} \end{bmatrix} \begin{bmatrix} J_{12} \\ J_{13} \\ J_{14} \\ J_{23} \\ J_{24} \\ J_{34} \end{bmatrix}, \quad (A.8)
\end{align*}
\]

where
\[
J_{12} = J(-, -, +, +), \quad J_{13} = J(-, +, -, +), \quad J_{14} = J(-, +, +, -), \\
J_{23} = J(+, -, -, +), \quad J_{24} = J(+, +, -, +), \quad J_{34} = J(+, +, -). \quad (A.9)
\]

We can now invert most of these relations (except (A.6) and (A.7)) to express J’s in terms of products of x’s. This is not possible for (A.4) and (A.7) because the products of x’s in
the left hand sides are linearly dependent (the rank of the four by four matrix therein is equal to three) as a manifestation of the Serre relations. In fact, using (A.6) and (A.7) one can easily check that the Serre relations (2.3) for the basic contour integrals (2.20) are indeed satisfied. It is, perhaps, not surprising that the $J$-integrals entering (A.6) and (A.7) appear in the expansion of the $L$-operator (2.13) only in certain linear combinations which can be expressed through the products of $x$’s. We will need the following combinations,

$$J(-,+,+,+) + J(+,+,+,-) = \frac{(q - q^{-1})^2}{[2]_q^2} \left\{ \frac{3}{[3]_q} x_0 x_1^3 - 2 x_1 x_0 x_1^2 + x_1^2 x_0 x_1 \right\},$$

$$J(+,-,+,+) - [3]_q J(+,+,+,-) = \frac{(q - q^{-1})^2}{[2]_q^2} \left\{ -2 x_0 x_1^3 + ([3]_q + q^{-2}) x_1 x_0 x_1^2 - q^{-1}[2]_q x_1^2 x_0 x_1 \right\},$$

$$J(+,+,-,+) + [3]_q J(+,+,-,-) = \frac{(q - q^{-1})^2}{[2]_q^2} \left\{ x_0 x_1^3 - q^{-1}[2]_q x_1 x_0 x_1^2 + q^{-2} x_1^2 x_0 x_1 \right\},$$

$$J(-,-,-,+) + J(+,-,-,-) = \frac{(q - q^{-1})^2}{[2]_q^2} \left\{ \frac{3}{[3]_q} x_0^3 x_1 - 2 x_0^2 x_1 x_0 + x_0 x_1 x_0^2 \right\},$$

$$J(-,-,+,-) - [3]_q J(+,-,-,-) = \frac{(q - q^{-1})^2}{[2]_q^2} \left\{ -2 x_0^3 x_1 + ([3]_q + q^{-2}) x_0^2 x_1 x_0 - q^{-1}[2]_q x_0 x_1 x_0^2 \right\},$$

$$J(-,+,-,-) + [3]_q J(+,-,-,-) = \frac{(q - q^{-1})^2}{[2]_q^2} \left\{ x_0^3 x_1 - q^{-1}[2]_q x_0^2 x_1 x_0 + q^{-2} x_0 x_1 x_0^2 \right\}.$$

(A.10)

For the rest of $J$’s one has

$$\begin{bmatrix} J(-,+), & J(+,+) \end{bmatrix} = \frac{(q - q^{-1})}{[2]_q} \begin{bmatrix} -q^{-2} & 1 \\ 1 & -q^{-2} \end{bmatrix} \begin{bmatrix} x_0 x_1 \\ x_1 x_0 \end{bmatrix},$$

(A.11)

$$\begin{bmatrix} J(-,-,+) & J(-,+,-) & J(+,-,-) \end{bmatrix} = \frac{(q - q^{-1})}{[2]_q^2} \begin{bmatrix} -q^{-2} & -q^{-1}[2]_q & 1 \\ -q^{-1}[2]_q & q^{-1}[4]_q & -q^{-1}[2]_q \\ 1 & -q^{-1}[2]_q & q^{-2} \end{bmatrix} \begin{bmatrix} x_0^2 x_1 \\ x_0 x_1 x_0 \\ x_1 x_0^2 \end{bmatrix},$$

(A.12)

This happens again due to the Serre relation but now for the generators $y_0$ and $y_1$. 22
\[
J(+, +, -) = \frac{(q - q^{-1})}{[2]_q} \begin{pmatrix}
q^{-2} & -q^{-1}[2]_q & 1 \\
-q^{-1}[2]_q & q^{-1}[4]_q & -q^{-1}[2]_q \\
1 & -q^{-1}[2]_q & q^{-2}
\end{pmatrix}
\begin{pmatrix}
x^2 x_0 \\
x_1 x_0 x_1 \\
x_0 x_1^2
\end{pmatrix},
\quad (A.13)
\]

\[
J(+, -, +) = q^{-2} \frac{(q - q^{-1})}{[4]_q[2]_q} \left\{ q^2 \frac{2}{[2]_q} x_0^2 x_1^2 - q^2 [2]_q x_0 x_1 x_0 x_1 + (q - q^{-1}) x_0 x_1^2 x_0 +
\right.
\left.
(q - q^{-1}) x_1 x_0^2 x_1 + q^{-2} [2]_q x_0 x_1 x_0 x_1 - q^{-2} \frac{2}{[2]_q} x_1^2 x_0^2 \right\},
\]

\[
J(+, +, -) = q^{-2} \frac{(q - q^{-1})^2}{[4]_q[2]_q} \left\{ q^{-2} \frac{2}{[2]_q} x_0^2 x_1^2 - (2q + q^3 + q^5) x_0 x_1 x_0 x_1 -
\right.
\left.
(q - q^{-1}) [3]_q x_0 x_1 x_0 x_0 - (q - q^{-1}) [3]_q x_1 x_0^2 x_1 -
\right.
\left.
(q^{-5} + q^{-3} + 2q^{-1}) x_1 x_0 x_1 x_0 + q^{-2} [2]_q x_1^2 x_0^2 \right\},
\]

\[
J(+, -, +) = q^{-2} \frac{(q - q^{-1})^2}{[4]_q[2]_q} \left\{ x_0^2 x_1^2 - [3]_q x_0 x_1 x_0 x_1 + x_0 x_1^2 x_0 +
\right.
\left.
[3]_q x_1 x_0^2 x_1 - [3]_q x_0 x_1 x_0 x_0 + x_1^2 x_0^2 \right\},
\]

\[
J(-, +, -) = q^{-2} \frac{(q - q^{-1})^2}{[4]_q[2]_q} \left\{ x_0^2 x_1^2 - [3]_q x_0 x_1 x_0 x_1 + [3]_q x_0 x_1^2 x_0 +
\right.
\left.
x_1 x_0^2 x_1 - [3]_q x_0 x_1 x_0 x_0 + x_1^2 x_0^2 \right\},
\]

\[
J(-, +, +) = q^{-2} \frac{(q - q^{-1})}{[4]_q[2]_q} \left\{ q^{-2} [2]_q x_0^2 x_1^2 - (q^{-5} + q^{-3} + 2q^{-1}) x_0 x_1 x_0 x_1 -
\right.
\left.
(q - q^{-1}) [3]_q x_0 x_1 x_0 x_0 - (q - q^{-1}) [3]_q x_1 x_0^2 x_1 +
\right.
\left.
(2q + q^3 + q^5) x_1 x_0 x_1 x_0 - q^2 [2]_q x_1^2 x_0^2 \right\},
\]

\[
J(-, -, +) = q^{-2} \frac{(q - q^{-1})}{[4]_q[2]_q} \left\{ -q^{-2} \frac{2}{[2]_q} x_0^2 x_1^2 + q^{-2} [2]_q x_0 x_1 x_0 x_1 + (q - q^{-1}) x_0 x_1^2 x_0 +
\right.
\left.
(q - q^{-1}) x_1 x_0^2 x_1 - q^{-2} [2]_q x_0 x_1 x_0 x_0 + q^2 \frac{2}{[2]_q} x_1^2 x_0^2 \right\}.
\quad (A.14)
\]

Expanding the $P$-exponent in (2.27) in a series one obtains

\[
\exp \left( \int_0^{2\pi} K(u) du \right) = 1 + \sum_{n=1}^{\infty} \sum_{\{\sigma_i = \pm 1\}} y_{\sigma_1} y_{\sigma_2} \cdots y_{\sigma_n} J(-\sigma_1, -\sigma_2, \ldots, -\sigma_n),
\quad (A.15)
\]
where
\[ y_+ = y_0, \quad y_- = y_1. \]

Let us restrict our attention to the terms in (A.15) of the order four or lower. One can exclude the products \( y_0 y_1^3 \) and \( y_0^3 y_1 \) using the Serre relations (2.3). Then one can substitute the \( J \)-integrals in (A.15) with the corresponding expressions (A.14)-(A.10). There is no need to rewrite (A.15) again since this substitution is rather mechanical and no cancelation can occur. The resulting expression is to be compared with the corresponding expansion of the universal \( R \)-matrix. The latter can be found using the results of [17-19]. The notation for the generators of the \( U_q(\hat{sl}(2)) \) used in those papers is different from ours. The generators \( e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}, h_\alpha, h_\beta \) in [17-19] are related to \( x_0, x_1, y_0, y_1, h_0, h_1 \) in (2.1)-(2.3) as follows
\[
\begin{align*}
  e_\alpha &= q^{-h_0} y_0, \quad e_{-\alpha} = x_0 q^{h_0}, \quad h_\alpha = h_0, \\
  e_\beta &= q^{-h_1} y_1, \quad e_{-\beta} = x_1 q^{h_1}, \quad h_\beta = h_1.
\end{align*}
\]

The expression for the "reduced" universal \( R \)-matrix (2.11) follows from Eq.(5.1) of Ref.[19]
\[
\overline{R} = \left( \prod_{n \geq 0} \exp_{q^{-2}} \left( (q - q^{-1}) e_{\alpha + n \delta} q^{h_{\alpha + n \delta}} \otimes q^{-h_{\alpha + n \delta}} e_{-\alpha - n \delta} \right) \right) \times \\
\exp \left( \sum_{n > 0} (q - q^{-1}) \frac{n e_{n \delta} q^{h_{n \delta}} \otimes q^{-h_{n \delta}} e_{-n \delta}}{[2n]_q} \right) \times
\left( \prod_{n \geq 0} \exp_{q^{-2}} \left( (q - q^{-1}) e_{\beta + n \delta} q^{h_{\beta + n \delta}} \otimes q^{-h_{\beta + n \delta}} e_{-\beta - n \delta} \right) \right),
\]

where
\[
\exp_p(x) = \sum_{n=0}^{\infty} \frac{p^{(n-1)(2-n)/2} x^n}{[n]_p!}, \quad [n]_p! = [1]_p [2]_p \cdots [n]_p
\]
and \( h_{\gamma+n\delta} = h_\gamma + n (h_\alpha + h_\beta) \) (\( h_\gamma = 0, \ h_\alpha, \ h_\beta \)). The index \( n \) of the multipliers increases from left to right in the first ordered product above and decreases in the second one. The root vectors \( e_{\alpha+n\delta}, e_{-\alpha-n\delta}, \) etc. appearing in (A.17) are defined recursively by Eqs.(3.2)-(3.5) of Ref.[13]. Applying these formulae one obtains first few of them
\[
\begin{align*}
  e_{\alpha+\delta} &= \frac{1}{2} q\left( e_\alpha^2 e_\beta - (1 + q^{-2}) e_\alpha e_\beta e_\alpha + q^{-2} e_\beta e_\alpha^2 \right), \\
  e_{\beta+\delta} &= \frac{1}{2} q\left( e_\beta^2 e_\alpha - (1 + q^{-2}) e_\beta e_\alpha e_\beta + q^{-2} e_\alpha e_\beta^2 \right),
\end{align*}
\]
\[ e_{-\alpha-\delta} = \frac{1}{[2]_q} (e_{-\beta} e^2_{-\alpha} - (1 + q^2) e_{-\alpha} e_{-\beta} e_{-\alpha} + q^2 e^2_{-\alpha} e_{-\beta}), \] (A.19)

\[ e_{-\beta-\delta} = \frac{1}{[2]_q} (e^2_{-\beta} e_{-\alpha} - (1 + q^2) e_{-\beta} e_{-\alpha} e_{-\beta} + q^2 e_{-\alpha} e^2_{-\beta}), \] (A.20)

\[ e_\delta = e_\alpha e_\beta - q^{-2} e_\beta e_\alpha, \quad e_{-\delta} = e_{-\beta} e_{-\alpha} - q^2 e_{-\alpha} e_{-\beta}, \]

\[ e_{2\delta} = \frac{1}{2q^2[2]_q} \left\{ 2q^2 e^2_\alpha e_\beta e_\alpha - q^2 [2]_q e_\alpha e_\beta e_\alpha + (q^2 - q^{-2}) (e_\alpha e_\beta e_\alpha + e_\beta e_\alpha e_\beta) + q^{-2} [2]_q e_\beta e_\alpha e_\beta e_\alpha - 2q^{-2} e^2_\beta e^2_\alpha \right\}, \]

\[ e_{-2\delta} = -\frac{q^2}{2[2]_q} \left\{ 2q^2 e^2_\alpha e^{-2}_\beta - q^2 [2]_q e^{-2}_\alpha e^{-2}_\beta e_{-\alpha} e_{-\beta} + (q^2 - q^{-2}) (e_{-\alpha} e_{-\beta} e_{-\alpha} + e_{-\beta} e_{-\alpha} e_{-\beta}) + q^{-2} [2]_q e_{-\beta} e_{-\alpha} e_{-\alpha} e_{-\beta} - 2q^{-2} e^2_{-\beta} e^2_{-\alpha} \right\}. \] (A.21)

These formulae enable us to calculate the expansion of the universal \( R \)-matrix (A.17) to within the fourth order terms. Substituting (A.18)-(A.21) into (A.17), expanding the exponents and calculating their product one gets precisely the result obtained above from the expansion of the \( L \)-operator (2.13) given by (A.15) and (A.14)-(A.10).

Finally notice the that the negative root vectors (A.19)-(A.21) have particular simple expressions in terms of \( J \)-integrals (2.29), namely they all reduce to just a single \( J \)-integral as one easily obtains from (A.19)-(A.21) and (A.14)-(A.13),

\[ q^{-h_\delta} e_{-\delta} = -q^4 \frac{[2]_q}{q - q^{-1}} J(+, -), \]

\[ q^{-h_{\alpha+\delta}} e_{-\alpha-\delta} = q^6 \frac{[2]_q}{q - q^{-1}} J(+, -, -), \] (A.22)

\[ q^{-h_{\beta+\delta}} e_{-\beta-\delta} = q^6 \frac{[2]_q}{q - q^{-1}} J(+, +, -), \]

\[ q^{-h_{2\delta}} e_{-2\delta} = -q^4 \frac{[2]_q [4]_q}{2(q - q^{-1})} J(+, +, -, -). \]

Appendix B.

In this Appendix we show that for \( 0 < \beta^2 < 1/2 \) the operators \( T_j(\lambda) \) (3.7) and \( A_{\pm}(\lambda) \) (3.12) are entire functions of the variable \( \lambda^2 \).
Consider the simplest nontrivial $T$-operator $T(\lambda) = T_{\frac{3}{2}}(\lambda)$ which corresponds to the two-dimensional representation of $U_0(sl(2))$. In this case

$$\pi_{\frac{3}{2}}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi_{\frac{3}{2}}(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \pi_{\frac{3}{2}}(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (B.1)$$

Using these expressions to compute the trace in (3.7) one finds

$$T(\lambda) = 2 \cos(2\pi P) + \sum_{n=1}^{\infty} \lambda^{2n} G_n, \quad (B.2)$$

where

$$G_n = q^n e^{2i\pi P J(\ldots, +, \ldots, +, +)} + q^n e^{-2i\pi P J(\ldots, -, \ldots, -)} \quad (B.3)$$

with $J$’s defined in (2.29). The operators $G_n$ are the “nonlocal integrals of motion” (NIM) [1] which commute among themselves and with all operators $T_j(\lambda)$. They act invariantly in each Fock module $F_p$. In particular, the vacuum state $|p\rangle \in F_p$ is an eigenstate of all operators $G_n$

$$G_n |p\rangle = G_n^{(vac)}(p) |p\rangle, \quad (B.4)$$

where the eigenvalues $G_n^{(vac)}(p)$ are given by the integrals [1]

$$G_n^{(vac)}(p) = \int_0^{2\pi} du_1 \int_0^{u_1} dv_1 \int_0^{v_1} du_2 \int_0^{u_2} dv_2 \cdots \int_0^{v_{n-1}} du_n \int_0^{u_n} dv_n \prod_{j>i} \left( 4 \sin \left( \frac{u_i - u_j}{2} \right) \sin \left( \frac{v_i - v_j}{2} \right) \right)^{2\beta^2} \prod_{j=i}^{n} \left( 2 \sin \left( \frac{u_i - v_j}{2} \right) \right)^{-2\beta^2} \quad (B.5)$$

Let us now examine the convergence properties of the series

$$T^{(vac)}(\lambda) = \cos(2\pi P) + \sum_{n=1}^{\infty} \lambda^{2n} G_n^{(vac)}(p) \quad (B.6)$$

for the vacuum eigenvalue of the operator $T(\lambda)$. A similar problem was studied in [24] for the series

$$Z(\lambda) = 1 + \sum_{n=1}^{\infty} \lambda^{2n} Z_n \quad (B.7)$$
with

\[ Z_n = \frac{1}{(n!)^2} \int_0^{2\pi} du_1 \int_0^{2\pi} du_2 \cdots \int_0^{2\pi} du_n \int_0^{2\pi} dv_1 \int_0^{2\pi} dv_2 \cdots \int_0^{2\pi} dv_n \prod_{j>i}^{n} \left| 4 \sin \left( \frac{u_i - u_j}{2} \right) \sin \left( \frac{v_i - v_j}{2} \right) \right|^{2\beta^2} \prod_{j,i=1}^{n} \left| 2 \sin \left( \frac{u_i - v_j}{2} \right) \right|^{-2\beta^2}, \]  

(B.8)

where 0 < \beta^2 < 1/2. It was shown (using the Jack polynomial technique) that the leading asymptotics of the integrals (B.8) for large \( n \) is given by

\[ \log Z_n = 2 (\beta^2 - 1) n \log n + O(n), \quad n \to \infty \]  

(B.9)

and hence series (B.7) defines an entire function of the variable \( \lambda^2 \). It is easy to see that

\[ |G_n^{(vac)}(p)| < Z_n \]  

(B.10)

and therefore the eigenvalue (B.7) is also an entire function of \( \lambda^2 \). Similar considerations apply to an arbitrary matrix elements of \( T(\lambda) \) between the states in \( F_p \). Thus all matrix elements and eigenvalues of \( T(\lambda) \) are entire functions of \( \lambda^2 \).

Consider now the vacuum eigenvalue \( A^{(vac)}(\lambda) \) of the operator \( A(\lambda) \equiv A_+(\lambda) \) defined in (3.12). It can be written as a series

\[ A^{(vac)}(\lambda) = 1 + \sum_{n=1}^{\infty} \sum_{\sigma_1 + \cdots + \sigma_{2n} = 0} \lambda^{2n} a_n(-\sigma_1, \ldots, -\sigma_{2n}) J^{(vac)}(\sigma_1, \ldots, \sigma_{2n}) \]  

(B.11)

where the sum is taken over all sets of variables \( \sigma_1, \ldots, \sigma_{2n} = \pm 1 \) with zero total sum and \( J^{(vac)}(\sigma_1, \ldots, \sigma_{2n}) \) denote vacuum eigenvalues of the operators (2.29). The numerical coefficients \( a_n \) defined as

\[ a_n(\sigma_1, \ldots, \sigma_{2n}) = q^n Z_+^{-1}(p) \text{Tr}_{\rho_+} (e^{2\pi ipH} \mathcal{E}_{\sigma_1} \cdots \mathcal{E}_{\sigma_{2n}}), \]  

(B.12)

where trace is taken over the representation \( \rho_+ \) of the \( q \)-oscillator algebra (3.8) and \( Z_+(p) \) is given by (3.11). It is easy to see that

\[ \sum_{\sigma_1 + \cdots + \sigma_{2n} = 0} \left| J^{(vac)}(\sigma_1, \ldots, \sigma_{2n}) \right| \leq Z_n. \]  

(B.13)

\[ \text{The higher spin operators } T_j(\lambda) \text{ with } j > 1/2 \text{ can be polynomially expressed through } T_{1/2}(\lambda) \text{ (as it follow from (4.13)) and obviously enjoy the same analyticity properties.} \]
To estimate the coefficients \((B.12)\) it is convenient to use the explicit form of the representation matrices \(\rho_+ (\mathcal{E}_\pm)\) and \(\rho_+ (\mathcal{H})\) given in \((D.6)\). Using these one can show

\[
|a_n(\{\sigma\})| < \frac{2^{2n}}{\prod_{j=1}^{n} (1 - q^{-2j} e^{4 \pi i P})} \tag{B.14}
\]

where we have assumed that

\[
2p \neq n \beta^2 + m \tag{B.15}
\]

for any integer \(m\) and any positive integer \(n\). For rational \(\beta^2\) the relation \((B.14)\) obviously imply

\[
|a_n(\{\sigma\})| < C^n \tag{B.16}
\]

is \(C\) is some constant. Combining \((B.16)\), \((B.13)\) and \((B.9)\) we conclude the series \((B.11)\) in this case converges in a whole complex plane of \(\lambda^2\). In fact, the same inequality \((B.16)\) holds for (almost all) irrational \(\beta^2\). This follows from a remarkable result of \([24,25]\)

\[
\lim_{n \to \infty} \frac{1}{n} \log \left| \prod_{j=1}^{n} (1 - q^{-2j} e^{4 \pi i P}) \right| = \int_{0}^{2\pi} \log(2 \sin x) \, dx = 0 \tag{B.17}
\]

which is valid for all irrational \(\beta^2\) satisfying \((B.13)\) except a set of some exceptional irrationals of the linear Lebesgue measure zero (see \([24,25]\) for the details).

**Appendix C.**

Using \((1.9)\), \((1.13)\), \((1.16)\) one can write the Virasoro generator \(L_0\) as

\[
L_0 = \frac{P^2}{\beta^2} + \frac{c - 1}{24} + \frac{2}{\beta^2} \sum_{n>0} a_{-n} a_n \tag{C.1}
\]

Then it easy to show that

\[
[L_0, \varphi(u)] = -i \partial_u \varphi(u) \tag{C.2}
\]

Therefore the adjoint action of the the operator \(\exp(i \varepsilon L_0)\) on \((3.4)\)

\[
e^{i \varepsilon L_0} T_j^{(f)}(\lambda) \ e^{-i \varepsilon L_0} = \text{Tr}_{\pi_j} \left[ e^{i(\pi P + f) H} \mathcal{P} \exp \left( \int_{\varepsilon}^{2 \pi + \varepsilon} K(u) \, du \right) \right] \tag{C.3}
\]
reduces to a shift of the limits of integration in the $\mathcal{P}$-exponent on the amount of $\varepsilon$, where $\varepsilon$ is assumed to be real. Here $K(u)$ is the same as in (1.29). Retaining in (C.3) linear in $\varepsilon$ terms only one gets

$$[L_0, T_j^{(f)}(\lambda)] = -i \text{Tr}_{\pi_j} \left[ e^{i(\pi P + f)H} \left( K(2\pi) e^{-i\pi PH} L_j(\lambda) - e^{-i\pi PH} L_j(\lambda) K(0) \right) \right]. \quad (C.4)$$

Expanding the $\mathcal{P}$-exponent as in (2.30), using (1.20), the commutations relations (1.19) and (1.26) and the cyclic property of the trace one obtains

$$[L_0, T_j^{(f)}(\lambda)] = \sin(\pi P - f) \sum_{\sigma_0 + \ldots + \sigma_n = 0} a^{(f)}(\sigma_0, \sigma_1, \ldots, \sigma_n) : e^{-2\sigma_0 \varphi(2\pi)} : J(-\sigma_1, \ldots, -\sigma_n), \quad (C.5)$$

where

$$a^{(f)}(\sigma_0, \sigma_1, \ldots, \sigma_n) = -2\sigma_0 e^{i\sigma_0 (\pi P - f)} \text{Tr}_{\pi_j} \left[ e^{i(\pi P + f)H} y_{\sigma_0} y_{\sigma_1} \ldots y_{\sigma_n} \right] \quad (C.6)$$

with

$$y_+ = \lambda q^{H/2} E, \quad y_- = \lambda q^{-H/2} F,$$

and the ordered integrals $J(\sigma_1, \ldots, \sigma_n)$ defined in (2.29). Obviously, RHS of (C.5) vanishes if

$$f = \pi(P + N),$$

where $N$ is arbitrary integer. We set $N = 0$, since (3.4) depends on $N$ only through a trivial sign factor $(-1)^{2jN}$.

Thus the operators $T_j(\lambda)$ (3.7) commute with the simplest local IM $I_1 = L_0 - c/24$. As it follows from (4.12) and (B.2) the coefficients of the series expansions of $T_j(\lambda)$ in the variable $\lambda^2$ can be algebraically expressed in terms the nonlocal IM (B.3). Therefore the above commutativity is equivalent to

$$[G_n, I_1] = 0, \quad n = 1, 2, \ldots, \infty. \quad (C.7)$$

In fact, the operators $G_n$ commute with all local IM (1.21). To check this one has to transform the ordered integrals in (B.3) to contour integrals. For example, $G_1$ can be written as

$$G_1 = (q^2 - q^{-2})^{-1} \int_0^{2\pi} du_1 \int_0^{2\pi} du_2 \left\{ \left( q e^{-2\pi i P} - q^{-1} e^{2\pi i P} \right) \times 
 V_-(u_1 + i0)V_+(u_2 - i0) + \left( q e^{2\pi i P} - q^{-1} e^{-2\pi i P} \right) V_+(u_1 + i0)V_-(u_2 - i0) \right\}. \quad (C.8)$$
Characteristic property of the local IM is that their commutators with the exponential fields (1.18) reduces to a total derivative [10,11]

\[
[I_{2n-1}, V_{\pm}(u)] = \partial_u \left\{ : O_n^{\pm}(u)V_{\pm}(u) : \right\} \equiv \partial_u X_n^{\pm}(u). \tag{C.9}
\]

Here $O_n^{\pm}(u)$ are some polynomials with respect the field $\partial_u \varphi$ and its derivatives. It follows then that the commutator of (C.8) with $I_{2n-1}$,

\[
C_n = [I_{2n-1}, G_1], \tag{C.10}
\]

is expressed as a double contour integral of a linear combination of products of the form $\partial_{v_1} X_n^{\pm}(v_1) V_{\mp}(v_2)$ and $V_{\pm}(v_1) \partial_{v_2} X_n^{\mp}(v_2)$. It is important to note that the operator product expansion for these products does not contain any terms proportional to negative integer powers of the difference $(v_1 - v_2)$. Therefore the above integrand for (C.10) does not contain any contact terms (i.e. the terms proportional to the delta function $\delta(u_1 - u_2)$ and its derivatives). Thus we can easily perform one integration

\[
C_n = (q^2 - q^{-2})^{-1} (qe^{-2\pi iP} - q^{-1}e^{2\pi iP}) (qe^{2\pi iP} - q^{-1}e^{-2\pi iP}) \times \\
\int_0^{2\pi} du \left[ qe^{2\pi iP} V_{\pm}(u)X^{\pm}(0) - q^{-1}e^{-2\pi iP} X^{\pm}(0)V_{\mp}(u) + qe^{-2\pi iP} V_{\mp}(u)X^{\pm}(0) - \right. \\
\left. q^{-1}e^{2\pi iP} X^{\pm}(0)V_{\mp}(u) \right], \tag{C.11}
\]

where we have used the periodicity property

\[
X_{\pm}(u + 2\pi) = q^{-2} e^{\pm 4\pi iP} X_{\pm}(u). \tag{C.12}
\]

Using now the commutation relations

\[
X_{\pm}(0) V_{\mp}(u) = q^2 V_{\mp}(u) X_{\pm}(0), \quad u > 0, \tag{C.13}
\]

where $\sigma_1, \sigma_2 = \pm 1$, one can see that the RHS of (C.11) is equal to zero. The higher nonlocal IM $G_n$ also admit contour integral representations similar to (C.8) and their commutativity with $I_{2k-1}$ can, in principle, be proved in the same way. However, these representations become more and more complicated for high orders and in general unknown. It would be interesting to obtain a general proof of the above commutativity to all orders.\footnote{B.Feigin and E.Frenkel have pointed out [26] that such proof can be obtained by extending the results of [10,11].}
Appendix D.

In this Appendix we present the derivation of the factorization (4.10). Using the definition (3.12) one can write the product of the operators $A_\pm$ from (4.10) in the form

$$A_+ (\lambda \mu) A_- (\lambda \mu^{-1}) = \left( Z_+ (P) Z_- (P) \right)^{-1} \text{Tr}_{\rho_+ \otimes \rho_-} \left[ e^{i \pi P \mathcal{H}} L_+ (\lambda \mu) \otimes L_- (\lambda \mu^{-1}) \right], \quad (D.1)$$

where

$$\mu = q^{i + \frac{1}{2}}, \quad (D.2)$$

and the trace is taken over the direct product of the two representations $\rho_+ \otimes \rho_-$ of (3.8) (these are defined after (3.10) in the main text) and

$$\mathcal{H} = \mathcal{H} \otimes 1 - 1 \otimes \mathcal{H} \quad (D.3)$$

It is convenient to choose the representation space of $\rho_+ (\rho_-)$ as a highest module generated by a free action of the operator $\rho_+ [\mathcal{E}_-] (\rho_- [\mathcal{E}_+])$ on a vacuum vector defined respectively as

$$\rho_\pm [\mathcal{E}_\pm] |0\rangle_\pm = 0, \quad \rho_\pm [\mathcal{H}] |0\rangle_\pm = 0. \quad (D.4)$$

Defining natural bases in these modules

$$|k\rangle_\pm = \rho_\pm [\mathcal{E}_\pm^k] |0\rangle_\pm, \quad k = 0, 1, 2, \ldots, \infty, \quad (D.5)$$

with the upper signs for $\rho_+$ and the lower signs for $\rho_-$ one can easily calculate the matrix elements

$$\rho_\pm [\mathcal{E}_\pm] |k\rangle_\pm = \frac{1 - q^{2k}}{(q - q^{-1})^2} |k - 1\rangle_\pm, \quad \rho_\pm [\mathcal{E}_\pm] |k\rangle_\pm = |k + 1\rangle_\pm, \quad (D.6)$$

$$\rho_\pm [\mathcal{H}] |k\rangle_\pm = \mp 2k |k\rangle_\pm.$$

Notice that the trace in (3.11) for this choice of $\rho_\pm$ reads

$$Z_+ (P) = Z_- (P) = \frac{e^{2 \pi i P}}{2i \sin(2 \pi P)}. \quad (D.7)$$

Specializing now the formula (2.20) for the product of the two operators $L_\pm$ in (D.1) one obtains

$$L_+ (\lambda \mu) \otimes L_- (\lambda \mu^{-1}) = e^{i \pi P \mathcal{H}} \mathcal{P} \exp \left( \lambda \int_0^{2 \pi} (V_-(u) q^{\mp \mathcal{E}} + V_+(u) q^{-\mp \mathcal{F}}) du \right), \quad (D.8)$$

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where $\mathcal{H}$ is given by (D.3) and
\[
\mathcal{E} = \mu \mathcal{E}_+ \otimes q^{-\frac{H}{2}} + \mu^{-1} q^{-\frac{H}{2}} \otimes \mathcal{E}_-,
\]
\[
\mathcal{F} = \mu \mathcal{E}_- \otimes q^{\frac{H}{2}} + \mu^{-1} q^{\frac{H}{2}} \otimes \mathcal{E}_+.
\]

The last two equations can be written in a compact form
\[
\mathcal{E} = a_+ + b_-, \quad \mathcal{F} = a_- + b_+,
\]
if one introduces the operators
\[
a_\pm = \mu \mathcal{E}_\mp \otimes q^{\pm \frac{H}{2}}, \quad b_\pm = \mu^{-1} q^{\pm \frac{H}{2}} \otimes \mathcal{E}_\pm,
\]
acting in $\rho_+ \otimes \rho_-$. These operators satisfy the commutation relations
\[
a_{\sigma_1} b_{\sigma_2} = q^{2\sigma_1 \sigma_2} b_{\sigma_2} a_{\sigma_1}, \quad [\mathcal{H}, a_\pm] = \mp 2 a_\pm, \quad [\mathcal{H}, b_\pm] = \mp 2 b_\pm,
\]
\[
q a_- a_+ - q^{-1} a_+ a_- = \frac{\mu^2}{q - q^{-1}}, \quad q b_+ b_- - q^{-1} b_- b_+ = \frac{\mu^{-2}}{q - q^{-1}},
\]
where $\sigma_1, \sigma_2 = \pm 1$.

Further, the direct product of the modules $\rho_\pm$ can be decomposed in the following way
\[
\rho_+ \otimes \rho_- = \bigoplus_{m=0}^{\infty} \rho^{(m)},
\]
where each $\rho^{(m)}$, $m = 0, 1, 2, \ldots, \infty$, is again a highest weight module spanned on the vectors
\[
\rho^{(m)} : \quad |\rho^{(m)}_k\rangle = (a_+ + b_+)^k (a_+ - \gamma b_+)^m |0\rangle_+ \otimes |0\rangle_- , \quad k = 0, 1, 2, \ldots, \infty .
\]
The constant $\gamma$ here is constrained by the relation
\[
\gamma \neq -q^{-2n}, \quad n = 0, 1, 2, \ldots, \infty ,
\]
but otherwise arbitrary. To prove that the modules $\rho^{(m)}$ are linearly independent (as subspaces in the vector space $\rho_+ \otimes \rho_-$) it is enough to prove that $\ell + 1$ vectors $|\rho^{(\ell-k)}_k\rangle$, $k = 0, 1, \ldots, \ell$, on each “level” $\ell = 0, 1, \ldots, \infty$ are linearly independent (the vectors on different levels are obviously linearly independent). To see this let us use the commutation relations (D.12) and rewrite
\[
z_k = (a_+ + b_+)^k (a_+ - \gamma b_+)^{\ell-k}, \quad k = 0, 1, \ldots, \ell
text{(D.16)}
\]
as ordered polynomials in the variables $a_+$ and $b_+$

$$
    z_k = \sum_{m=0}^{\ell} C_{km}^{(\ell)} (a_+)^{\ell-m} (b_+)^m.
$$

(D.17)

If $\gamma$ satisfies (D.15) the determinant of the coefficients of these polynomials

$$
    \det \left| C_{km}^{(\ell)} \right|_{0 \leq k, m \leq \ell} = \prod_{n=0}^{\ell-1} (\gamma + q^{-2n})^{\ell-n}
$$

(D.18)

does not vanish. That implies the required linear independence.

From the above definitions it is easy to see that the operators $\overline{\mathcal{H}}$ and $\mathcal{F}$ entering (D.8) act invariantly in each module $\rho^{(m)}$

$$
    \overline{\mathcal{H}}, \mathcal{F} : \rho^{(m)} \rightarrow \rho^{(m)} ,
$$

(D.19)

while for the operator $\mathcal{E}$ acts as

$$
    \mathcal{E} : \rho^{(m)} \rightarrow \rho^{(m)} \oplus \rho^{(m-1)}
$$

(D.20)

with $\rho^{(-1)} \equiv 0$. The matrix element of these operators can be easily found from (D.10), (D.12), (D.14),

$$
    (\rho_+ \otimes \rho_-) [\overline{\mathcal{H}}] |\rho_k^{(m)}\rangle = -2 (m + k) |\rho_k^{(m)}\rangle ,
$$

$$
    (\rho_+ \otimes \rho_-) [\mathcal{F}] |\rho_k^{(m)}\rangle = |\rho_{k+1}^{(m)}\rangle ,
$$

$$
    (\rho_+ \otimes \rho_-) [\mathcal{E}] |\rho_k^{(m)}\rangle = [k]_q [2j - 1 + k]_q |\rho_{k-1}^{(m)}\rangle + c_k^{(m)} |\rho_k^{(m-1)}\rangle ,
$$

(D.21)

where we have used (D.2). The values of $c_k^{(m)}$ can be calculated but not necessary in that following. Thus the matrices (D.21) have the block triangular form with an infinite number of diagonal blocks. It is essential to note that in each diagonal block these matrices coincide with those of the highest weight representation $\pi_j^+$ given by (4.5) (up to an overall shift in the matrix elements of $(\rho_+ \otimes \rho_-) [\overline{\mathcal{H}}]$ in different blocks). Substituting now (D.21) into (D.8) and then into (D.1) and using the definition (4.4) one easily arrives to the factorization (4.10).
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