The Cosmological Constant as a Function of Extrinsic Curvature and Spatial Curvature

Jin-Zhang Tang*, Qiang Xu†

1Department of Physics, and State Key Laboratory of Nuclear Physics and Technology, Peking University, Beijing 100871, China
(Dated: August 3, 2010)

Abstract

In this paper we suppose that the cosmological constant will change when the universe expands. For a general consideration, the cosmological constant is assumed to be a function of scale factor and Hubble constant. According to the ADM formulation, to the FRW metric, the extrinsic curvature $I$ equals $-6H^2$ and spatial curvature $R$ equals $6k/a^2$. Therefore we suppose cosmological constant is a function of extrinsic curvature and spatial curvature. We investigate the cosmological evolution of FRW universe in these models. At last we investigate two particular models which could fit the observation data about dark energy well. Actually a changeless cosmological constant is not essential. If when the universe expands, the cosmological constant changes slowly and gradually flows to a constant, the observation data about dark energy could also be fitted well by this kind of theory.

PACS numbers: 98.80.Cq

* Electronic address: JinzhangTang@pku.edu.cn
† Electronic address: xuqiang@pku.edu.cn
I. INTRODUCTION

Einstein’s general relativity (GR) has been considered as a fundamental theory of gravity. Even from an effective field theory point of view, it should describe all large scale gravitational physics, in particular the evolution of our universe. However, the discovery of dark matter and dark energy from various observation posed great challenge to the theory. To address the dark energy issues, the most simplest method is adding a cosmological constant which is called \( \Lambda \)CDM model. The cosmological constant problem has been investigated by many papers such as [2–4] and so on. Many other theories also have been proposed for the cosmic acceleration, for instance, Quintessence[5–7], K-essence[8, 9], Phantom[10, 11], etc. And numerous versions of modification or extension of GR have been proposed in the last and this century. The \( f(R) \) theory [12–15] is one of the modifications to GR. In this theory, the Lagrangian density \( f(R) \) is an arbitrary function of \( R(4) \). The theory can explain the cosmic acceleration without introducing a cosmological constant, even though the theory would be very much restricted by the solar system test.

In this paper, we suppose the cosmological constant will change when the universe expands. For the scalar factor and Hubble constant are main variable to characterize a universe, the cosmological constant is supposed to be a function of them. In the ADM formulation [1], the metric is

\[
\text{d} s^2 = -N^2 \text{d} t^2 + g_{ij}(\text{d} x^i - N^i \text{d} t)(\text{d} x^j - N^j \text{d} t). \tag{1}
\]

the building block to construct the action is the gauge invariant quantity: the extrinsic curvature tensor

\[
K_{ij} = \frac{1}{2N} \left( g_{ij} - \nabla_i N_j - \nabla_j N_i \right), \tag{2}
\]

where a dot denotes a derivative with respect \( t \), and the 3-dimensional spatial curvature \( R \) comes from \( g_{ij} \). In terms of them, the action of GR with a cosmological constant is

\[
S_{\text{EH}} = \frac{1}{16\pi G} \int \text{d} t \text{d}^3 x \sqrt{g} N \left[ K_{ij}K^{ij} - K^2 + R - 2\Lambda \right], \tag{3}
\]

where \( K = g^{ij}K_{ij} \). And we define extrinsic curvature \( I \) as

\[
I \equiv K_{ij}K^{ij} - K^2. \tag{4}
\]

The homogeneous and isotropic universe is described by the FRW metric,

\[
ds^2 = -\text{d} t^2 + a(t)^2 \left\{ \frac{\text{d} r^2}{1 - kr^2} + r^2 (\text{d} \theta^2 + \sin^2 \theta \text{d} \phi^2) \right\}, \tag{5}
\]

where \( k = 0, \pm 1 \). To this FRW metric, After some trivial calculations, it is easy to get the extrinsic curvature \( I = -6H^2 \) and spatial curvature \( R = 6k/a^2 \). Obviously \( I \) is proportional to the square of Hubble constant and \( R \) is proportional to the inverse square on scalar factor \( a \). Therefore we suppose the cosmological constant \( \Lambda = \Lambda(I, R) \) as a function of extrinsic curvature \( I \) and spatial curvature \( R \).

For the calculational convenience, we define \( F(I, R) \equiv I + R - 2\Lambda(I, R) \) and the action of this theory is

\[
S_F = \frac{1}{16\pi G} \int \text{d} t \text{d}^3 x \sqrt{g} NF(I, R) \tag{6}
\]

This theory could be regard as an extension of \( f(R^{(4)}) \) gravity model. In the next section, we exhibit the basic equations by varying the action with respect to the function \( N, N_k, g_{ij} \). Then we
investigate the cosmological evolution of FRW universe in these models. At last, we investigate two particular models which could fit the observation data about dark energy well. Actually a changeless cosmological constant is not essential. If the cosmological constant changes slowly and gradually flows to a constant, the observation data about dark energy could also be fitted well by this kind of theory.

II. BASIC EQUATIONS

Let us start from the action

$$ S_F = \frac{1}{16\pi G} \int dt d^3 x \sqrt{g} N F(I, R). $$

(7)

Actually the action could be more general if $F$ is an arbitrary function of “$g_{ij}, K, K_{ij},\nabla_i K_{jk}, \cdots, R, R_{ij}, R_{ijkl}, \nabla_i R_{jk}, \cdots$”. In this paper we would like to focus on the simplest case $F = F(I, R)$. Now we consider the case that there are matters couple to the gravity. The action should be

$$ S = S_F + S_m = \int dt d^3 x \sqrt{g} N \left[ \frac{1}{16\pi G} F(I, R) + L_m \right]. $$

(8)

By varying the action with respect to the function $N, N_k, g_{ij}$, we get the Hamiltonian constraint, super-momentum constraint and dynamical equations,

$$ J_N = \frac{1}{16\pi G} \left[ F - 2F_i I \right], $$

(9)

$$ J^k = \frac{1}{8\pi G} \nabla^k \left[ F_i \pi^i \right], $$

(10)

$$ -\frac{1}{2} p^{ij} = \frac{1}{16\pi G} \left\{ -\frac{1}{\sqrt{g} N} \partial_t \left[ \sqrt{g} F_i \pi^i \right] + \frac{1}{N} \nabla_k \left[ F_{,i} \left( \pi^i N^k - \pi^i N^j - \pi^j N^i \right) \right] - 2F_{,i} \left[ K_{m} K^{m,j} - \lambda K K^{ij} \right] - F_{,R} R^{ij} + \left( \nabla^i \nabla_j - g^{ij} \nabla^2 \right) \left( N F_{,R} \right) + \frac{1}{2} F g^{ij} \right\}. $$

(11)

$p^{ij}$ in above equations is defined as $\pi^{ij} = K^{ij} - \lambda K g^{ij}$. And $J_N, J^k, p^{ij}$ are defined as

$$ J_N \equiv -\frac{\delta(N L_m)}{\delta N}; \quad J^k \equiv -\frac{N \delta L_m}{\delta N_k}; \quad p^{ij} \equiv \frac{2}{\sqrt{g}} \frac{\delta \left( \sqrt{g} L_m \right)}{\delta g_{ij}}. $$

(12)

If the matter could be taken to be the perfect liquid, then

$$ \rho_m = J_N, J^k = 0, p^{ij} = p_m g^{ij}. $$

(13)

The quantities $(J_N, J^i, p^{ij})$ should satisfy the conservation laws,

$$ \partial_t g_{jk} p^{jk} + \frac{2}{\sqrt{g}} \partial_t (\sqrt{g} J_N) + \frac{2N_k}{\sqrt{g} N} \partial_t \left( \sqrt{g} J^k \right) = 0, $$

(14)

$$ \nabla_k p^{jk} - \frac{1}{\sqrt{g} N} \partial_t \left( \sqrt{g} J^i \right) - \frac{N^i}{N} \nabla_k J^k - \frac{J^k}{N} \left( \nabla_k N^i - \nabla^i N_k \right) = 0. $$

(15)
III. COSMOLOGICAL MODELS

The homogeneous and isotropic universe is described by the FRW metric,

$$ds^2 = -c^2 dt^2 + a(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right\} ,$$

(16)

where $k = 0, \pm 1$. The extrinsic curvature of the spatial slices is

$$K_{ij} = \frac{1}{c a(t)} g_{ij} ; \quad K = \frac{3}{c} \frac{\dot{a}(t)}{a(t)} .$$

(17)

Then $I = -6H^2$ where $H \equiv \dot{a}/a$ which is defined as Hubble constant. The spatial slices are of constant-curvature

$$R_{ij} = \frac{2k}{a^2(t)} g_{ij} ; \quad R = \frac{6k}{a^2(t)} .$$

(18)

It can be shown that the momentum constraint (10) is satisfied identically when $J^k = 0$ for the perfect fluid. From the Hamiltonian constraint (9) and the dynamical equation (11), and “c” is rescaled to unity, we get two cosmological equations as

$$F + 12F_j \left( \frac{\dot{a}}{a} \right)^2 = 16\pi G \rho_m ,$$

(19)

$$2 \left[ F_j \frac{\ddot{a}}{a} + \dot{F}_j \frac{\dot{a}}{a} \right] + 4F_j \left( \frac{\dot{a}}{a} \right)^2 - F_R \frac{2k}{a^2} + \frac{1}{2} F = -8\pi G p_m ,$$

(20)

with the energy conservation equation

$$\dot{\rho}_m + 3 \frac{\dot{a}}{a} (\rho_m + p_m) = 0 .$$

(21)

The FRW equations with the matter and the dark energy could be rewritten as

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} (\rho_m + \rho_{DE}) ,$$

(22)

$$2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = -8\pi G (p_m + p_{DE}) .$$

(23)

Here we have regarded the nonlinear terms in the equations (19), (20) as the dark energy effectively. Comparing (22) to (19), (23) to (20), it is easy to get

$$8\pi G \rho_{DE} = 3 \left( 1 - 2F_j \right) \left( \frac{\dot{a}}{a} \right)^2 + \frac{3k}{a^2} - \frac{1}{2} F ,$$

(24)

$$8\pi G p_{DE} = 2 \left( F_j - 1 \right) \frac{\ddot{a}}{a} + 2\dot{F}_j \frac{\dot{a}}{a} + (4F_j - 1) \left( \frac{\dot{a}}{a} \right)^2 - \left( F_R + \frac{1}{2} \right) \frac{2k}{a^2} + \frac{1}{2} F .$$

(25)

So the equation of state $\omega_{DE}$ is given by

$$\omega_{DE} = -1 + \frac{2 \left( F_j - 1 \right) \dot{H} + 2\dot{F}_j H - (F_R - 1) \left( 2k/a^2 \right)}{3 \left( 1 - 2F_j \right) \dot{H}^2 + (3k/a^2) - (1/2) F} .$$

(26)
From the data of WMAP + BAO + SN, the constraints on $\omega_{DE}$ are $\omega_{DE0} = -0.999^{+0.057}_{-0.056}$ [16]. We now define a deceleration parameter

$$q = -\frac{\dot{a}a}{a^2}. \quad (27)$$

From equations (22) and (23), we get

$$q_0 = -\frac{\rho_{DE0} - (1/2)\rho_{m0}}{\rho_{m0} + \rho_{DE0} - 3k/(8\pi G a_0^2)} = -\Omega_{DE0} + \frac{\Omega_{m0}}{2} \approx -0.592, \quad (28)$$

here we have used “$\Omega_{DE0} = 0.728, \Omega_{m0} = 0.272$” [16] from the data of WMAP + BAO + $H_0$. The critical density of the universe if defined as $\rho_c \equiv 3H^2/8\pi G$. From (22) it is easy to get

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = \rho_{m0} + \rho_{DE0} - \frac{3k}{8\pi G a_0^2}. \quad (29)$$

From above, the curvature density is defined as $\rho_k \equiv -3k/(8\pi G a^2)$. Now we could define

$$\Omega_k = \frac{\rho_k}{\rho_c} = -\frac{k}{a^2H^2}. \quad (30)$$

The data of WMAP + BAO + SN give the constraints on curvature density, $\Omega_{k,0} = -0.0057^{+0.0067}_{-0.0068}$ [16]. Obviously, It is probable that $k > 0$ and the universe is close. If the universe is accelerating as power-law that $a(t) \propto t^\beta$ today, it is easy to get $q_0 = y(1-y)/y^2 = -0.592$ and $y = 2.451$.

We consider the simple case that

$$-2\Lambda(I, R) = g(I) + f(R). \quad (31)$$

From (24) and (25), it is easy to get

$$8\pi G \rho_{DE} = g, I - \frac{1}{2}g(I) - \frac{1}{2}f(R), \quad (32)$$

$$8\pi G \rho_{DE} = 2(g, I + 2g, II I) H - g, I - f, R \frac{2k}{a^2} + \frac{1}{2}g(I) + \frac{1}{2}f(R). \quad (33)$$

The equation of state is

$$\omega_{DE} = -1 + \frac{2(g, I + 2g, II I) H - f, R(2k/a^2)}{g, I - (1/2)g(I) - (1/2)f(R)}. \quad (34)$$

Now we investigate a concrete model $-2\Lambda(I, R) = \alpha e^{\beta R}$, here $R = 6k/a^2$ and we assume $k > 0$. The state equation of this model is

$$\omega_{DE} = -1 + \frac{2}{3} \beta R. \quad (35)$$

It is easy to get $-0.083 < \beta R_0 < 0.087$ from $\omega_{DE0} = -0.999^{+0.057}_{-0.056}$ [16]. The coefficient $\alpha$ can be fixed from the equation $8\pi G \rho_{DE0} = (-1/2)\alpha e^{\beta R_0}$. From (21) and (22), the evolution equation of $a$ is

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{k,0} \left(\frac{a_0}{a}\right)^4 + \Omega_{DE0} e^{\beta R_0(a_0/a)^2 - 1} + \Omega_{m,0} \left(\frac{a_0}{a}\right)^3 + (1 - \Omega_{DE0} - \Omega_{m,0} - \Omega_{k,0}) \left(\frac{a_0}{a}\right)^4,$$
where \( \Omega_{DE0} = 0.73, \Omega_m0 = 0.27, \Omega_r0 = 0.014 \) [16]. The details about the universe evolution are shown in Fig. 1 and 2. Obviously in Fig. 1, the evolutions of deceleration parameter \( q \) for \( \beta R_0 = -0.01 \) or 0.01 are nearly identical to the case \( \beta R_0 = 0 \) which corresponds to the \( \Lambda CDM \) model. Fig. 2 shows that the state function \( \omega_{DE} \) flow to -1 quickly for the cases \( \beta R_0 = -0.01 \) or \( \beta R_0 = 0.01 \) when \( \omega_{DE0} \) identically equal to -1 for \( \beta R_0 = 0 \).

Another specific model is \(-2\Lambda(I, R) = \gamma e^\eta I\). The corresponding state equation is

\[
\omega_{DE} = -1 + \frac{2\eta I}{3} \frac{1 + 2\eta I}{1 - 2\eta I H^2}. \tag{36}
\]

When \( \omega_{DE0} = -1 \), \( \eta \) has two solutions \( \eta = 0 \) and \( \eta I_0 = -1/2 \). When \( \eta = 0 \), it just is a \( \Lambda CDM \) model. The attention will be put on the second case \( \eta I_0 = -1/2 \). The coefficient \( \gamma \) could be fixed...
from

\[ 8\pi G \rho_{DE0} = \gamma \eta e^{\eta t_0} I_0 - \frac{1}{2} \gamma e^{\eta t_0}, \]  

(37)

and \( \gamma = -3e^{1/2} \Omega_{DE0} H_0^2 \). The evolution equation of \( a \) is

\[
\left( \frac{H}{H_0} \right)^2 = \Omega_{k0} \left( \frac{a_0}{a} \right)^2 + \frac{1}{2} \Omega_{DE0} \left[ 1 + \left( \frac{H}{H_0} \right)^2 \right] e^{\frac{\gamma}{2} \left[ 1 - \left( H/H_0 \right)^2 \right]} + \Omega_{m0} \left( \frac{a_0}{a} \right)^3 + (1 - \Omega_{DE0} - \Omega_{m0} - \Omega_{k0}) \left( \frac{a_0}{a} \right)^4.
\]

Similar to the discussion above, Fig. 3 is the evolution of deceleration parameter \( q \) which is similar to \( \Lambda \) CDM model. Fig. 4 shows that the state function \( \omega_{DE} \) flow to \(-1\) quickly.

![Figure 3: The deceleration parameter \( q = -\ddot{a}/a^2 \) as a function of \( a \).](image1)

![Figure 4: The parameter \( \omega_{DE} \) as a function of \( a \).](image2)

The two models discussed above could be regarded as the modification to the \( \Lambda \) CDM model. To address the dark energy issues, a changeless cosmological constant is not essential. If the
cosmological constant changes slowly and gradually flows to a constant, the observation data about dark energy could also be fitted well by this kind of theory.

Acknowledgments

The work was partially supported by NSFC Grant No. 10775002, 10975005 and RFDP. We would like to thank Professor Bin Chen very much for some useful suggestions and his works on the modification of this paper.

[1] R.L. Arnowitt, S. Deser and C.W. Misner, The dynamics of general relativity, “Gravitation: an introduction to current research”, Louis Witten ed., (Wilew 1962), chapter 7, pp 227-265, [arXiv:gr-qc/0405109].
[2] S. Weinberg, “The cosmological constant problem”, Rev. Mod. Phys. 61, 1(1989).
[3] Sean M. Carroll, “The Cosmological Constant”, LivingRev. Rel. 4:1,2001, [arXiv:astro-ph/0004075v2].
[4] P. J. E. Peebles, Bharat Ratra, “The Cosmological Constant and Dark Energy”, Rev. Mod. Phys. 75: 559-606 (2003), [arXiv:astro-ph/0207347v2].
[5] Ivaylo Zlatev, Limin Wang, Paul J. Steinhardt, “Quintessence, Cosmic Coincidence, and the Cosmological Constant”, Phys. Rev. Lett. 82:896-899 (1999), [arXiv:astro-ph/9807002v2].
[6] A. Yu. Kamenshchik, U. Moschella, V. Pasquier, “An alternative to quintessence”, Phys. Lett. B 511:265-268 (2001), [arXiv:gr-qc/0103004v2].
[7] Sean M. Carroll, “Quintessence and the Rest of the World”, Phys. Rev. Lett. 81: 3067-3070 (1998), [arXiv:astro-ph/9806099v2].
[8] C. Armendariz-Picon, V. Mukhanov, Paul J. Steinhardt, “Essentials of k-essence”, Phys. Rev. D 63:103510 (2001), [arXiv:astro-ph/0006373v1].
[9] Takeshi Chiba, “Tracking K-essence”, Phys. Rev. D 66 (2002), [arXiv:astro-ph/0206298v2].
[10] R.R. Caldwell, Phys.Lett.B 545:23-29 (2002), [arXiv:astro-ph/9908168v2].
[11] Robert R. Caldwell, Marc Kamionkowski, Nevin N. Weinberg, “Phantom Energy and Cosmic Doomsday”, Phys. Rev. Lett. 91(2003) 071301, [arXiv:astro-ph/0302506v1].
[12] S. Nojiri and S. D. Odintsov, eConf C0602061, 06 (2006)[Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007)][arXiv:hep-th/0601213].
[13] T. P. Sotiriou and V. Faraoni, arXiv:0805.1726 [gr-qc].
[14] A. de Felice and S. Tsujikawa, arXiv:1002.4928 [hep-th].
[15] Sean M. Carroll, Vikram Duvvuri, Mark Trodden, Michael S. Turner, “Is Cosmic Speed-Up Due to New Gravitational Physics?”, Phys. Rev. D 70, 043528 (2004) [arXiv:astro-ph/0306438].
[16] E. Komatsu et al., “Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation”, [arXiv:astro-ph/1001.4538v2].