Three-parameter model of a sand pile

Alexander I. Olemskoï
Department of Physical Electronics, Sumy State University
2, Rimskii-Korsakov St., 244007 Sumy, UKRAINE
E–mail: Alexander@olem.sumy.ua

The theory of a flux steady-state (avalanche) formation is presented for the simplest model of a real sand pile within the framework of Lorenz approach. The stationary values of sand velocity and sand pile slope are derived as functions of controlling parameter (externally driven sandpile slope). The additive noises of above values are taken into account to build the phase diagram, where the noise intensities determine a domain of the avalanche appearance. This domain shows to be crucial to the noise intensity of vertical component of sand velocity.

I. INTRODUCTION

In recent years considerable study has been given to the theory of self–organized criticality (SOC) [1] that explains spontaneous (avalanche-type) dynamics, unlike the typical phase transitions that occur only when a controlling parameter is tuned to a critical value. A main feature of the systems displaying SOC is that they are distributed over avalanche sizes, so that SOC models are mostly studied by making use of scaling-type arguments supplemented with extensive computer simulations (see [2]). On the contrary, in this paper we put forward the theory of a single avalanche formation.

The SOC behavior appears in a variety of systems such as a real sand pile (ensemble of grains of sand moving along increasingly tilted surface), intermittency in biological evolution [3], earthquakes and propagation of forest–fires, depinning transitions in random medium and so on [4]. Among the above models the simplest and most widely studied, analytically [5], [6] and numerically [1], [7], [8], are the sandpile models. The field theory [9], based on nonlinear diffusion equation, has failed to account for the basic feature of self–organized systems – their avalanche dynamics. The reason is that there is no feedback between open subsystem and thermostat within the framework of the one–parameter approach employed in [9]. Recently the two–parameter theories were set forth in [10,11], where the thermostat degree of freedom was either controlling parameter [10], or conjugate field [11]. In the mean field approximation, the approach of [11] allows to obtain critical exponents governing scaling behavior of self–organized system. Our approach is to take into consideration the complete set of degrees of freedom. Owing to this, not only the complimentary results of [10,11] are reproduced, but it is possible to obtain the self–consistent analytical description of the single avalanche formation.

This paper is organized as follows. In Sec. 2 the self-consistent theory of the formation of a flux steady-state is presented. It enables us to treat the problem on the basis of the unified analytical approach. Sec. 3 deals with accounting additive noises of the sand velocity components and sandpile slope. It is shown that the noise intensities increase the possibility of emergence of the SOC regime. Sec.4 contains the obtained results.

II. NOISELESS CASE

Within framework of the simplest model of a sandpile, a dependence \( y = y(t, x) \) defines its surface at given instant of time \( t \). Locally the flow of sand can be described in terms of three quantities: the horizontal and vertical components of the sand velocity, \( \dot{x} = \partial x / \partial t \), \( \dot{y} = \partial y / \partial t \), and the surface slope \( y' = \partial y / \partial x \). The key point of our approach is that the above degrees of freedom are assumed to be of dissipative type, so that, when they are not coupled, their relaxation to the steady state is governed by the Debye-type equations:

\[
\frac{d\dot{x}}{dt} = -\frac{\dot{x}}{\tau_x} \tag{1a}
\]

\[
\frac{d\dot{y}}{dt} = -\frac{\dot{y}}{\tau_y(0)} \tag{1b}
\]
where \( \tau_x, \tau_y \) and \( \tau_S \) are the relaxation times of the velocity components and the slope, respectively. Eqs. (1a) – (1c) imply the sand is at rest in the stationary state, \( \dot{x} = \dot{y} = 0 \) and the equilibrium slope \( y' = y'_0 \neq 0 \) plays the role of a controlling parameter.

Since the motion of sand grain along different directions is not independent, Eq. (1a) should be changed by adding the term \( f = \dot{y}/\gamma \) due to liquid friction force along the \( y \)-axis (\( \gamma \) is the kinetic coefficient). Then, we have

\[
\tau_x \ddot{x} = -\dot{x} + a^{-1} \dot{y}
\]

where \( a \equiv \gamma/\tau_x \). Note that, owing to the diffusion equation \( \dot{y} = Dy'' \) (\( D \) is the diffusion coefficient), the friction force appears to be driven by the curvature of the sandpile surface:

\[
f = (D/\gamma)y''.
\]

On the other hand, when \( \ddot{x} = 0 \) (stationary state), solution of Eq. (2) defines the tangent line \( y = ax + \text{const} \), so that the friction force \( f = \tau_x^{-1} \dot{x} \) is proportional to the horizontal component of the sand velocity. Taking into consideration the relation (3) and using the chain rule \( dy'/dt = \dot{y}' + y'^{\prime} \dot{x} \), from Eq. (1c) one obtains the equation of motion for the slope:

\[
\tau_S \ddot{y}' = (y'_0 - y') - (\tau_S/D) \dot{y} \dot{x}.
\]

Following the same line, the equation for the vertical component of the velocity can be deduced

\[
\tau_y \ddot{y} = -\dot{y} + \frac{\tau_y}{\tau_x} \dot{y}' \dot{x}, \quad \frac{1}{\tau_y} \equiv \frac{1}{\tau_y(0)} \left( 1 + \frac{y'_0 - y'_0(0)}{a \tau_x} \right).
\]

Note the higher order terms are disregarded in Eq. (3) and the renormalized relaxation time \( \tau_y \) depending on the stationary slope \( y'_0 \) is introduced.

Eqs. (2), (4), (5) constitute the basis for self-consistent description of the sand flow on the surface with the slope \( y' \) driven by the control parameter \( y'_0 \). The distinguishing feature of these equations is that nonlinear terms that tend to impede the growth of the slope. The positive feedback between \( \dot{x} \) and \( y' \) in Eq. (4) plays an important part in the problem. As we shall see later, it is precisely the reason behind the self-organization that brings about the avalanche generation.

After suitable rescaling, Eqs. (2), (4), (5) can be rewritten in the form of the well-known Lorenz system:

\[
\begin{align*}
\dot{u} &= -u + v, \\
\epsilon \dot{v} &= -v + uS, \\
\delta \dot{S} &= (S_0 - S) - uv,
\end{align*}
\]

where \( u \equiv (\tau_y/\tau_x)^{1/2}(\tau_S/D)^{1/2} \dot{x} \), \( v \equiv (\tau_y/\tau_x)^{1/2}(\tau_S/D)^{1/2} \dot{y}/a \), and \( S \equiv (\tau_y/\tau_x)^{1/2}y'/a \) are the dimensionless velocity components and the slope, respectively; \( \epsilon \equiv \tau_y/\tau_x, \delta \equiv \tau_S/\tau_x \) and the dot now stands for the derivatives with respect to the dimensionless time \( t/\tau_x \). In general, the system (6a) – (6c) cannot be solved analytically, but in the simplest case, when \( \epsilon \ll 1 \) and \( \delta \ll 1 \), the vertical velocity \( v \) and the slope \( S \) can be eliminated by making use of the adiabatic approximation that implies neglecting of the left hand sides of Eqs. (6a), (6c). As a result, the dependencies of \( S \) and \( v \) on the horizontal velocity \( u \) are given by

\[
S = \frac{S_0}{1 + u^2}, \quad v = \frac{S_0u}{1 + u^2}.
\]

Note that, under \( u \) is in the physically meaningful range between 0 and 1, the slope is a monotonically decreasing function of \( u \), whereas the velocity \( v \) increases with \( u \) (at \( u > 1 \) we have \( dv/du < 0 \) and the flow of the sand becomes turbulent).

Substituting second Eq. (5) into Eq. (6a) yields the Landau–Khalatnikov equation:
with the kinetic energy given by
\[ E = \frac{1}{2} u^2 - \frac{1}{2} S_0 \ln (1 + u^2). \]

For \( S_0 < 1 \), the \( u \)-dependence of \( E \) is monotonically increasing and the only stationary value of \( u \) equals zero, \( u_0 = 0 \), so that there is no avalanches in this case. If the slope \( S_0 \) exceeds the critical value, \( S_c = 1 \), the kinetic energy assumes the minimum with non–zero steady state velocity components \( u_e = v_e = (S_0 - 1)^{1/2} \) and the slope \( S_e = 1 \).

The above scenario represents supercritical regime of the avalanche formation and corresponds to the second–order phase transition. The latter can be easily seen from the expansion of the kinetic energy (9) in power series of \( u^2 \approx 1 \). So the critical exponents are identical to those obtained within the framework of the mean field theory [11].

The drawback of the outlined approach is that it fails to account for the subcritical regime of the self–organization that is the reason for the appearance of avalanches and analogous to the first–order phase transition, rather than the second–order one. So one has to modify the above theory by taking the assumption that the effective relaxation time \( \tau_x(x) \) increases with velocity \( u \) and its further smooth increase is determined by Eq.(12). If the parameter \( u \) has the maximum, the lower one corresponds to the stable state \( u_e \). The corresponding value of the stationary slope
\[ S^m = \frac{1 + u^2}{2} + \sqrt{(1 + u^2)^2 - (1 - u^2) S_0} \]
smoothly increases from the value
\[ S_{\text{min}} = 1 + u_0 \sqrt{m/(1 - u^2_0)} \]
at the parameter \( S_0 = S_{c_0} \) with
\[ S_{c_0} = (1 - u^2_0) S_{\text{min}} \]
to the marginal value \( S_c = 1 + m \) at \( S_0 = S_c \). The \( S_0 \)-dependencies of \( u_e \), \( u_m \), and \( S_e \) are presented in Fig.1. As is shown in Fig.1a, under the adiabatic condition \( \tau_x < \tau_x \) is met and the parameter \( S_0 \) slowly increases being below \( S_c (S_0 \leq S_c) \), no avalanches can form. At the point \( S_0 = S_c \) the velocity \( u_e \) jumps upward to the value \( \sqrt{2u_0} \) and its further smooth increase is determined by Eq.(12). If the parameter \( S_0 \) then goes downward the velocity \( u_e \) continuously decreases up to the point, where \( S_0 = S_{c_0} \) and \( u_e = u_0 \). At this point the velocity jump–like goes down to zero. Referring to Fig.1b, the stationary slope \( S_e \) shows a linear increase from 0 to \( S_e \) with the parameter \( S_0 \) being in the same interval and, after the jump down to the value \( (1 - u^2_0)^{-1} \) at \( S_0 = S_c \), \( S_e \) smoothly decays to 1 at \( S_0 \gg S_c \). Under the parameter \( S_0 \) then decreases from above \( S_c \) down to \( S_{c_0} \) the slope grows. When the point (14) is reached, the avalanche stops, so that the slope undergoes the jump from the value (13) up to the one defined by Eq.(13). For \( S_0 < S_{c_0} \) again the parameter \( S_e \) does not differ from \( S_0 \). Note that this subcritical regime is realized provided the parameter \( m \), that enters the dispersion law (10), is greater than...
Clearly, according to the picture described, the avalanche generation is characterized by the well pronounced hysteresis, when the grains of sand initially being at rest begin to move downhill only if the slope of the surface exceeded its limiting value $S_c = 1 + m$, whereas the slope $S_o$ needed to stop the avalanche is less than $S_c$ (see Eqs.(4), (5)). This is the case in the limit $\tau_S/\tau_x \to 0$ and the hysteresis loop shrinks with the growth of the adiabaticity parameter $\delta \equiv \tau_S/\tau_x$. In addition to the smallness of $\delta$, the adiabatic approximation implies the ratio $\tau_y/\tau_x \equiv \epsilon$ is also small. In contrast to the former, the latter does not seem to be realistic for the system under consideration, where, in general, $\tau_y \approx \tau_x$. So it is of interest to study to what extent the finite value of $\epsilon$ could change the results.

Owing to the condition $\delta \ll 1$, Eq.(6c) is still algebraic and $S$ can be expressed in terms of $u$ and $v$. As a result, we derive the system of two nonlinear differential equations that can be studied by the phase portrait method [12]. The phase portraits for various values of $\epsilon$ are displayed in Fig.2, where the point O represents the stationary state and the point S is related to the maximum of the kinetic energy. As is seen from the figure, independently of phase portraits for various values of $\epsilon$, there is the universal section that attains all phase trajectories and its structure is appeared to be almost insensitive to changes in $\epsilon$. Analysis of time dependencies $v(t)$ and $u(t)$ reveals that the velocity components slow down appreciably on this section in comparison to the rest parts of trajectories that are almost rectilinear (it is not difficult to see that this effect is caused by the smallness of $\delta$). Since the most of time the system is in vicinity of this universal section, we arrive at the conclusion that finite values of $\epsilon$ do not affect qualitatively the above results obtained in the adiabatic approximation.

### III. NOISE INFLUENCE

To take into account additive noises of the velocity components $u$, $v$ and the slope $S$ it needs to add to right-hand parts of Eqs.(17) - (20) the stochastic terms $I_u^{1/2}\xi$, $I_v^{1/2}\xi$, $I_S^{1/2}\xi$, respectively (here the noise intensities $I_u,v,S$ are measured in units $(\tau_x/\tau_u)(D/\tau_S)$, $a^2(\tau_x/\tau_u)(D/\tau_S)$, $a^2(\tau_x/\tau_u)$, correspondingly, and $\xi(t)$ is $\delta$-correlated stochastic function) [13]. Then, within the adiabatic approximation Eq.(5) acquires the stochastic addition

$$\left\{I_u^{1/2} + I_v^{1/2}g_v(u) + I_S^{1/2}g_S(u)\right\}\xi(t),$$

where we introduce the multiplicative functions $g_v(u) = ug_u(u) = u/(1 + u^2)$. As a result the extreme points of the distribution $P(u) \propto \exp\{-U(u)\}$ of the stochastic variable $u$ is given by the effective energy [14]

$$U(u) = \ln I(u) + \int \frac{\partial E/\partial u}{I(u)} du$$

(17)

where the bare energy $E$ is determined by Eq.(8) and the expression for the effective noise intensity

$$I(u) = I_v g_v^2(u) + I_S g_S^2(u)$$

(18)

follows from the known property of additivity of squares of variances of independent Gaussian random quantities [13]. Combining expressions (8), (17), (18), we can find the explicit form of $U(u)$, which is too cumbersome to be reproduced here. Much simpler is the equation

$$x^3 - S_0x^2 - 2I_Sx + 4(I_S - I_u) = 0, \quad x = 1 + u^2,$$

(19)

which defines the locations of the maxima of distribution function $P(u)$. According to Eq.(18), they are insensitive to changes in the intensity of noise $I_u$ of the velocity component $u$, but are determined by the stationary value $S_0$ of the sandpile slope and the intensities $I_v, I_S$ of the noises of vertical velocity component $v$ and slope $S$, which acquire the multiplicative character in Eq.(18). Hence, it can put for simplicity $I_u = 0$ and Eqs.(8), (17), (18) give the follow expression for the effective energy:

$$U(u) = \frac{1}{2} \left[\frac{u^4}{2} + (2 - S_0 - i)u^2 + (1 - i)(1 - S_0 - i)\ln(i + u^2)\right] + I_S \ln[g_S^2(u) + ig_v^2(u)], \quad i = I_v/I_S.$$
According to Eq. (19), the effective energy \( E \) has a minimum at \( v = 0 \) if the stationary slope \( S_0 \) does not exceed the critical level

\[
S_c = 1 + 2I_S - 4I_v,
\]

(21)
whose value increases at increasing intensity of noise of the sandpile slope, but decreases with one of the velocity. In this case, sand grains are at rest. In the simple case \( I_v = 0 \), the avalanche creation corresponds to solutions

\[
u_\pm^2 = \frac{1}{2} \left[ S_0 - 3 + \sqrt{(3 - S_0)^2 + 4(2S_0 - 3 + 2I_S)} \right],
\]

(22)
which are obtained from Eq. (19) after eliminating the root \( \nu^2 = 0 \). The magnitude of this solution has its minimum

\[
u_0^2 = \frac{1}{2} \left[ (S_0 - 3) - \sqrt{(S_0 + 7)(S_0 - 1)} \right]
\]

(23)
on the line defined by expression (21) with \( I_v = 0 \). At \( S_0 < 4/3 \), the roots \( \pm u_\nu \) are complex, at \( S_0 = 4/3 \) they become zero, at \( S_0 > 4/3 \) they are real, and \( u_+ = -u_- \). In this way, the tricritical point

\[
S_0 = 4/3, \quad I_S = 1/6
\]

(24)
corresponds to the appearance of roots \( u_\pm \neq 0 \) of Eq. (19) (avalanche creation). If condition (21) is satisfied, the root \( u = 0 \) corresponds to the minimum of the effective energy (20) at \( S_0 < 4/3 \), whereas at \( S_0 > 4/3 \) this root corresponds to the maximum, and the roots \( u_\pm \) to symmetrical minima.

Now we find another condition of stability of roots \( u_\pm \). Setting the discriminant of Eq. (19) equal to zero, we get the equations

\[
I_S = 0, \quad I_S^2 - I_S \left[ \frac{27}{2} \left( 1 - \frac{S_0}{3} \right) - \frac{S_0^2}{8} \right] + \frac{S_0^3}{2} = 0,
\]

(25)
the second of which gives

\[
2I_S = \left[ \frac{27}{2} \left( 1 - \frac{S_0}{3} \right) - \frac{S_0^2}{8} \right] \pm \left\{ \left[ \frac{27}{2} \left( 1 - \frac{S_0}{3} \right) - \frac{S_0^2}{8} \right]^2 - 2S_0^3 \right\}^{1/2}.
\]

(26)
This equation defines a bell-shaped curve \( S_0(I_S) \), which intersects with the horizontal axis at the points \( I_S = 0 \) and \( I_S = 27/2 \), and has a maximum at

\[
S_0 = 2, \quad I_S = 2.
\]

(27)
It is easy to see that for \( I_v = 0 \) this line touches the curve (21) at point (24).

Let us now consider the more general case of two multiplicative noises \( I_v, I_S \neq 0 \). Introducing the parameter \( a = 1 - i, i = I_v/I_S \) and the renormalized variables \( \tilde{I} = I_S/a^2, \tilde{S}_0 = S_0/a, \tilde{u}^2 = (1 + a^2)/a - 1 \), at \( i < 1 \) we may reproduce all above expressions with the generalized energy \( \tilde{U}/\tilde{I} \) in Eq. (24). Then the action of the noise of the vertical component of the velocity \( v \) is reduced to the renormalization of the extremum value of the horizontal one by the quantity \( (a^{-1} - 1)^{1/2} \), so that the region of divergence \( \tilde{u} \approx 0 \) becomes inaccessible.

The condition of extremum of the generalized energy (20) splits into two equations, one of which is simply \( u = 0 \), and the other is given by Eq. (19). As pointed out above, analysis of the latter indicates that the line of existence of the zero solution is defined by an expression (21). The tricritical point has the coordinates

\[
S_0 = \frac{4}{3}(1 - I_v), \quad I_S = \frac{1}{6}(1 + 8I_v).
\]

(28)
The phase diagram for the fixed intensities \( I_v \) is shown in Fig. 3. Here the curves 1, 2 define the thresholds of absolute loss of stability for the fluxless and flux steady-states, respectively. Above line 1 the system occurs in a stable flux state, below curve 2 it is in fluxless one, and between these lines the two-phase domain is realized. For \( I_v < 1/4 \) situation is generally the same as in the simple case \( I_v = 0 \) (see Fig. 3a). At \( I_v > 1/4 \) the SOC regime is possible for small intensities \( I_S \) of the slope noise (Fig. 3b). According to (28) the tricritical point occurs on the \( I_S \) axis at \( I_v = 1 \), and for the noise intensity \( I_v \) larger than the critical value \( I_v = 2 \) the stable fluxless state disappears (see Fig. 3c).
IV. CONCLUSION

According to the above consideration, the dissipative dynamic of grains flow in a real sand pile can be represented within the framework of Lorenz model, where the horizontal and vertical velocity components play a role of an order parameter and its conjugate field, respectively, and the sandpile slope is a controlling parameter. In Sec.2, the noiseless case is examined to show that an avalanche creates if the externally driven sandpile slope \( y'_0 \) is larger than the critical magnitude \( \gamma(\tau_x\tau_y)^{-1/2} \). In this sense, the systems with small values of the kinetic coefficient \( \gamma \) and large relaxation times \( \tau_x, \tau_y \) of the velocity components are preferred. However, the sand flow appears here as a phase transition because the spontaneous avalanche creation is impossible in the noiseless case. Taking into account the additive noises of the above degrees of freedom, we show in Sec.3 that the stochasticity influence is non-essential for the horizontal velocity component and is crucial for the vertical one. The SOC appears if the noise intensity of the latter exceeds the value \( 2^{-2}(D\gamma^2/\tau_x\tau_y\tau_S) \), provided the noise is small for the sandpile slope. If the noise intensity of the vertical velocity component is larger than the critical value \( 2(D\gamma^2/\tau_x\tau_y\tau_S) \), the fluxless steady-state disappears at all.

[1] P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. Lett. 59, 381 (1987).
[2] M. Parzuski, S. Maslov, P. Bak, Phys. Rev. E 53, 414 (1996).
[3] P. Bak, K. Sneppen, Phys. Rev. Lett. 71, 4083 (1993).
[4] T. Halpin–Healy, Y.–C. Zhang, Phys. Rep. 254, 215 (1995).
[5] D. Dhar, R. Ramaswamy, Phys. Rev. Let. 63, 1659 (1989); D. Dhar, ibid. 64, 1613 (1991).
[6] L. Pietronero, A. Vespignani, S. Zapperi, Phys. Rev. Lett. 72, 1690 (1994); Phys. Rev. E 51, 1711 (1995).
[7] L. P. Kadanoff, S. R. Nagel, L. Wu, S. Zhu, Phys. Rev. A 39, 6524 (1989).
[8] P. Grassberger, S. S. Manna, J. Phys. (Paris) 51, 1077 (1990).
[9] T. Hwa, M. Kadar, Phys. Rev. A 45, 7002 (1992).
[10] L. Gil, D. Sornette, Phys. Rev. Lett. 76, 3991 (1996).
[11] A. Vespignani, S. Zapperi, Phys. Rev. Lett. 78, 4793 (1997); Phys. Rev. E 57, 6345 (1998).
[12] A.I. Olemskoi, A.V. Khomenko, JETP 83, 1180 (1996).
[13] H. Risken, *The Fokker–Planck equation* (Springer, Berlin, 1989).
[14] A.I. Olemskoi, Physics–Uspekhi, 41, 269 (1998).
FIGURE CAPTIONS

Fig.1. The $S_0$-dependencies of a) the velocities $u_e$, $u_m$, and b) the equilibrium slope $S_e$. The arrows indicate the hysteresis loop.

Fig.2. Phase portraits in the $v-u$ plane at $S_0 = 1.25S_c0$ for a) $\epsilon = 10^{-2}$; b) $\epsilon = 1$; c) $\epsilon = 10^2$.

Fig.3. Phase diagrams for fixed values $I_v$ of the noise intensities of the vertical velocity component: a) $I_v = 0$, b) $I_v = 1$, c) $I_v = 2$. Curves 1 and 2 define the boundary of stability of avalanche and non-avalanche phases; A – avalanche phase, N – non-avalanche.
