SOME INEQUALITIES FOR THE MAXIMUM MODULUS OF RATIONAL FUNCTIONS

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ABSTRACT. For a polynomial \( p(z) \) of degree \( n \), it follows from the Maximum Modulus Theorem that \( \max_{|z|=R \geq 1} |p(z)| \leq R^n \max_{|z|=1} |p(z)| \). It was shown by Ankeny and Rivlin in 1955 that if \( p(z) \neq 0 \) for \( |z| < 1 \) then \( \max_{|z|=R \geq 1} |p(z)| \leq \frac{R^n}{1-R} \max_{|z|=1} |p(z)| \). These two results were extended to rational functions by Govil and Mohapatra [4]. In this paper, we give refinements of these results of Govil and Mohapatra.

1. Introduction and Statement of Results

Let \( \mathcal{P}_n \) denote the set of all complex algebraic polynomials \( p \) of degree at most \( n \) and let \( p' \) be the derivative of \( p \). For a function \( f \) defined on the unit circle \( \mathbb{T} = \{ z \mid |z| = 1 \} \) in the complex plane \( \mathbb{C} \), set \( ||f|| = \sup_{z \in \mathbb{T}} |f(z)| \), the Chebyshev norm of \( f \) on \( \mathbb{T} \).

Let \( \mathbb{D}_- \) denote the region strictly inside \( \mathbb{T} \), and \( \mathbb{D}_+ \) the region strictly outside \( \mathbb{T} \). For \( a_v \in \mathbb{C}, v = 1, 2, \ldots, n \), let \( w(z) = \prod_{v=1}^{n} (z-a_v) \), \( B(z) = \prod_{v=1}^{n} (1-\overline{a_v}z)/(z-a_v) \) being the Blashke product, and \( \mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \ldots, a_n) = \{ p(z)/w(z) \mid p \in \mathcal{P}_n \} \). Then \( \mathcal{R}_n \) is the set of rational functions with possible poles at \( a_1, a_2, \ldots, a_n \) and having a finite limit at \( \infty \). Also note that \( B(z) \in \mathcal{R}_n \).

DEFINITIONS.

(i): For polynomial \( p(z) = \sum_{v=0}^{n} \alpha_v z^v \), the conjugate transpose (reciprocal) \( p^* \) of \( p \) is defined by \( p^*(z) = z^n p(1/z) = \overline{\alpha_0} z^n + \overline{\alpha_1} z^{n-1} + \cdots + \overline{\alpha_n} \).

(ii): For rational function \( r(z) = p(z)/w(z) \in \mathcal{R}_n \), the conjugate transpose, \( r^* \), of \( r \) is defined by \( r^*(z) = B(z) \overline{r(1/z)} = B(z) \overline{r(1/z)} \).

(iii): The polynomial \( p \in \mathcal{P}_n \) is self-inversive if \( p^*(z) = \lambda p(z) \) for some \( \lambda \in \mathbb{T} \).

(iv): The rational function \( r \in \mathcal{R}_n \) is self-inversive if \( r^*(z) = \lambda r(z) \) for some \( \lambda \in \mathbb{T} \).

It is easy to verify that if \( r \in \mathcal{R}_n \) and \( r = p/w \), then \( r^* = p^*/w \) and hence \( r^* \in \mathcal{R}_n \). So \( p/w \) is self-inversive if and only if \( p \) is self-inversive.

If \( p \in \mathcal{P}_n \), then it is well known that

\[
\max_{|z|=R \geq 1} |p(z)| \leq R^n \|p\|.
\]

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This inequality is an immediate consequence of the Maximum Modulus Theorem. Further, if \( p \) has all its zeros in \( T \cup \mathbb{D}_+ \), then

\[
\max_{|z|=R \geq 1} |p(z)| \leq \frac{R^n + 1}{2} ||p||. \tag{1.2}
\]

The inequality (1.2) is due to Ankeny and Rivlin [1]. Both inequalities (1.1) and (1.2) are sharp, inequality (1.1) becomes equality for \( p(z) = \lambda z^n \) where \( \lambda \in \mathbb{C} \), and inequality (1.2) becomes equality for \( p(z) = \alpha z^n + \beta \) where \( |\alpha| = |\beta| \).

Govil and Mohapatra [4] gave a result analogous to inequality (1.1), but for rational functions, as follows.

THEOREM A. If

\[
r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathcal{R}_n
\]

is a rational function with \( |a_v| > 1 \) for \( 1 \leq v \leq n \), then for \( |z| \geq 1 \),

\[
|r(z)| \leq ||r|| |B(z)| \tag{1.3}
\]

This result is best possible and equality holds for \( r(z) = \lambda \prod_{v=1}^{n} \frac{1 - a_v z}{z - a_v} = \lambda B(z) \) where \( \lambda \in \mathbb{C} \).

In the same paper, Govil and Mohapatra [4] also proved a result given below, that is analogous to inequality (1.2) for rational functions.

THEOREM B. Let

\[
r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathcal{R}_n
\]

with \( |a_v| > 1 \) for \( 1 \leq v \leq n \). If all the zeros of \( r \) lie in \( T \cup \mathbb{D}_+ \), then for \( |z| \geq 1 \)

\[
|r(z)| \leq ||r|| \left| B(z) \right| + 1 \tag{1.4}
\]

This result is best possible and equality holds for the rational function \( r(z) = \alpha B(z) + \beta \) where \( |\alpha| = |\beta| \).

In this paper we prove the following refinements of the above two theorems. Here \( p(z) = \sum_{v=0}^{n} \alpha_v z^v \) is a polynomial of degree \( n \).

**Theorem 1.1.** If

\[
r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathcal{R}_n
\]

is a rational function with \( |a_v| > 1 \), \( 1 \leq v \leq n \), then for \( |z| \geq 1 \),

\[
|r(z)| \leq ||r|| \left| B(z) \right| \left\{ 1 - \frac{(||r|| - |r^*(0)|)(|z| - 1)}{|r^*(0)| + |z||r||} \right\}. \tag{1.5}
\]

The result is best possible and equality holds for \( r(z) = \lambda B(z) \) where \( \lambda \in \mathbb{C} \).

It is clear that Theorem 1 sharpens Theorem A. Also, we can use Theorem 1 to derive a sharpening form of Bernstein’s Inequality for polynomials. For this, let
\[ p(z) = \sum_{v=0}^{n} \alpha_v z^v \] be a polynomial of degree \( n \). Then \[ r(z) = \frac{p(z)}{\prod_{v=1}^{n}(z - a_v)} \in \mathcal{R}_n \] and hence by Theorem 1, for \( |z| \geq 1 \),

\[
(1.6) \quad \frac{|r(z)|}{B(z)} = \left| \frac{p(z)}{\prod_{v=1}^{n}(1 - \alpha_v z)} \right| \leq ||r|| \left\{ 1 - \frac{(||r|| - |r^*(0)|)(|z| - 1)}{|r^*(0)|} \right\}.
\]

If \( z^* \) is such that \( |z| = 1 \) is such that

\[
(1.7) \quad ||r|| = |r(z^*)| = \frac{|p(z^*)|}{\prod_{v=1}^{n}(z^* - a_v)}
\]

then we get from (1.6)

\[
(1.8) \quad \frac{p(z)}{\prod_{v=1}^{n}(1 - \alpha_v z)} \leq \left| \sum_{v=1}^{n} \frac{1 - \alpha_v z}{z^* - a_v} \right| 1 - \left( \frac{|p(z^*)| - |r^*(0)|}{\prod_{v=1}^{n}|z^* - a_v|}(|z| - 1) \right) \left( \frac{|r^*(0)|}{\prod_{v=1}^{n}|z^* - a_v| + |z||p(z^*)|} \right).
\]

Since \( p(z) = \sum_{v=0}^{n} \alpha_v z^v \) and \( r^*(z) = \frac{p^*(z)}{\prod_{v=1}^{n}(z - a_v)} \), we get \( |r^*(0)| = \frac{|\alpha_n|}{\prod_{v=1}^{n}|a_v|} \) and therefore from (1.8) we have for \( |z| > 1 \),

\[
|p(z)| \leq |p(z^*)||z^*| \left\{ 1 - \left( \frac{|p(z^*)| - |\alpha_n|}{\prod_{v=1}^{n}|z^* - a_v|}(|z| - 1) \right) \right\}.
\]

Since (1.9) holds for all \( |a_v| \geq 1 \), where \( 1 \leq v \leq n \), making \( |a_v| \to \infty \), where \( 1 \leq v \leq n \), we get that for \( |z| \geq 1 \),

\[
|p(z)| \leq \frac{|p(z^*)||z^*|^n}{\alpha_n + |z||p(z^*)|^n} \left\{ 1 - \left( \frac{|p(z^*)| - |\alpha_n|}{\prod_{v=1}^{n}|z^* - a_v|}(|z| - 1) \right) \right\}.
\]

We show in Lemma 5 in the next section that (1.10) implies for \( |z| \geq 1 \)

\[
|p(z)| \leq ||p|| |z|^n \left\{ 1 - \left( \frac{|p| - |\alpha_n|}{\alpha_n + |z||p|}(R - 1) \right) \right\},
\]

which is equivalent to that for \( |z| = R \geq 1 \),

\[
|p(z)| \leq R^n \left\{ 1 - \left( \frac{|p| - |\alpha_n|}{\alpha_n + |z||p|}(R - 1) \right) \right\} ||p||.
\]

This rate of growth result for a polynomial, which is a sharpening of Bernstein Inequality, first appeared as Lemma 3 of [2].

As a refinement of Theorem B, we shall prove

**Theorem 1.2.** Let

\[
r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n}(z - a_v)} \in \mathcal{R}_n
\]

with \( |a_v| > 1 \) for \( 1 \leq v \leq n \). If all the zeros of \( r \) lie in \( \mathbb{T} \cup \mathbb{D}_+ \), then for \( |z| \geq 1 \)

\[
|r(z)| \leq \frac{1}{2} \left( ||r||(B(z)| + 1) - (|B(z)| - 1) \min_{|z|=1} |r(z)| \right).
\]

Clearly Theorems 1.1 and 1.2 without any additional hypotheses, give bounds that are sharper than those obtainable from Theorems A and B respectively.
2. Lemmas

The following is a well known generalization of Schwarz’s Lemma (see, for example, [3]).

Lemma 2.1. If \( f \) is analytic inside and on the circle \(|z| = 1\), then for \(|z| \leq 1\),

\[
|f(z)| \leq \|f\| \frac{||f|| |z| + |f(0)|}{|f(0)||z| + ||f||}.
\]

The next two results are due to Govil and Mohapatra [4].

Lemma 2.2. Let \( r \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). If \( r \) has all its zeros in \( \mathbb{T} \cup \mathbb{D}_+ \), then for all \(|z| \geq 1\), \(|r(z)| \leq |r^*(z)|\).

Lemma 2.3. Let \( r \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). Then for \(|z| \geq 1\),

\[
|r(z)| + |r^*(z)| \leq \|r\|(|B(z)| + 1).
\]

Lemma 2.4. Let \( r \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). If \( r \) has all its zeros in \( \mathbb{T} \cup \mathbb{D}_+ \), then for \(|z| \geq 1\), we have

\[
|r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq |r^*(z)|.
\]

**Proof.** Since the rational function \( r \) has no zeros in \( \mathbb{D}_- \) hence for every \( \alpha \in \mathbb{C} \) with \(|\alpha| < 1\), the rational function \( r(z) - \alpha \min_{|z|=1} |r(z)| \) has no zero in \( \mathbb{D}_- \) and has all its poles, like \( r \), in \( \mathbb{D}_+ \). Applying Lemma 2.2 to \( r(z) - \alpha \min_{|z|=1} |r(z)| \) we get that for \(|z| \geq 1\)

\[
\left| r(z) - \alpha \min_{|z|=1} |r(z)| \right| \leq \left| r^*(z) - B(z) \min_{|z|=1} |r(z)| \right|,
\]

and so for \(|z| \geq 1\),

\[
|r(z)| - |\alpha| \min_{|z|=1} |r(z)| \leq \left| r^*(z) - B(z) \min_{|z|=1} |r(z)| \right|.
\]

With the appropriate choice of \( \arg(\alpha) \) we then have for \(|z| \geq 1\),

\[
|r(z)| - |\alpha| \min_{|z|=1} |r(z)| \leq |r^*(z)| - |\alpha| |B(z)| \min_{|z|=1} |r(z)|.
\]

Note that \( r \) has no zeros in \( \mathbb{D}_- \) and so is analytic in \(|z| \leq 1\). Hence by the Minimum Modulus Theorem, we have \(|r(z)| > |\alpha| \min_{|z|=1} |r(z)|\) for \(|z| \leq 1\). Therefore for \(|z| \geq 1\) we get

\[
|r^*(z)| = \left| B(z) r(1/\overline{z}) \right| = |B(z)| |r(1/\overline{z})| > |\alpha| |B(z)| \min_{|z|=1} |r(z)|,
\]

which clearly implies that the right-hand side of (2.3) is positive. Making \(|\alpha| \to 1\) in (2.3), we easily get

\[
|r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq |r^*(z)|, \text{ for } |z| \geq 1,
\]

which is (2.2), and thus the proof of Lemma 2.4 is complete. \(\square\)
Lemma 2.5. The function
\[ g(x) = x \left\{ 1 - \frac{(x - |a_n|)(|z| - 1)}{|a_n| + |z|x} \right\}, \]
where \(a_n, z \in \mathbb{C}\) with \(z \neq 0\), is an increasing function for \(x \geq 0\).

Proof. We have
\[ g'(x) = \frac{|z|^2 + 2|a_n||z| + |z|^2|a_n|^2}{(|a_n| + |z|x)^2} \geq 0 \]
for \(x \geq 0\). So \(g\) is an increasing function for \(x \geq 0\), as claimed. \(\square\)

3. Proofs of Theorems

Proof of Theorem 1.1. Since
\[ r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n}(z - a_v)} \in \mathcal{R}_n \]
with \(|a_v| > 1\) for \(1 \leq v \leq n\), the function \(r^*(z) = p^*(z)/\prod_{v=1}^{n}(z - a_v)\) is analytic in \(|z| \leq 1\). Therefore by Lemma 2.1 we get that, for \(|z| \leq 1\),
\[ |r^*(z)| \leq ||r^*|| |z| + |r^*(0)|/|r^*(0)||z| + ||r^*|| \]
and since \(||r^*|| = ||r||\), inequality (3.1) is in fact equivalent to the inequality that, for \(|z| \leq 1\),
\[ |r^*(z)| \leq ||r|| |z| + |r^*(0)|/|r^*(0)||z| + ||r|| \]
Since by definition \(r^*(z) = B(z)r(1/\overline{z})\), we get from (3.2) that for \(|z| \leq 1\),
\[ |r(z)| \leq \frac{||r|| |z| + |r^*(0)|}{|B(z)|} \frac{|r(z)| + |r^*(0)|}{|r^*(0)|} + ||r|| \]
which clearly gives that for \(|z| \geq 1\),
\[ |r(z)| \leq \frac{||r|| + |r^*(0)| |z|}{|B(1/\overline{z})| + ||r|| |z|} \]
It is clear from the definition of \(B(z)\) that \(|B(1/\overline{z})| = 1/|B(z)|\) and this, when combined with (3.3), gives that for \(|z| \geq 1\),
\[ |r(z)| \leq ||r|| |B(z)| \frac{|r(z)| + |r^*(0)| |z|}{|r^*(0)| + ||r|| |z|} \]
\[ = ||r|| |B(z)| \left( 1 - \frac{(||r|| - |r^*(0)|)(|z| - 1)}{|r^*(0)| + ||r|| |z|} \right) \]
which is (1.5) and this completes the proof of the Theorem 1.1. \(\square\)

Proof of Theorem 1.2. Since \(r \in \mathcal{R}_n\) and has all its poles in \(\mathbb{D}_+\) hence, by Lemma 2.3, for \(|z| \geq 1\) we have
\[ |r(z)| + |r^*(z)| \leq ||r|| (|B(z)| + 1). \]
Because $r$ has all its zeros in $\mathbb{T} \cup \mathbb{D}_+$, therefore we can apply Lemma 2.4 to $r$, and this will give that for $|z| \geq 1$,

\[(3.5) \quad |r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq |r^*(z)|.\]

Combining the conclusion of (3.5) with (3.4) we get that for $|z| \geq 1$.

\[2|r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq \|r\|(|B(z)| + 1),\]

which is clearly equivalent to

\[|r(z)| \leq \frac{1}{2} \left( \|r\|(|B(z)| + 1) - (|B(z)| - 1) \min_{|z|=1} |r(z)| \right),\]

and the proof of Theorem 1.2 is thus complete. \qed

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