Low density expansion for Lyapunov exponents

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Abstract

In some quasi-one-dimensional weakly disordered media, impurities are large and rare rather than small and dense. For an Anderson model with a low density of strong impurities, a perturbation theory in the impurity density is developed for the Lyapunov exponent and the density of states. The Lyapunov exponent grows linearly with the density. Anomalies of the Kappus-Wegner type appear for all rational quasi-momenta even in lowest order perturbation theory.

1 Introduction

A perturbative formula for the Lyapunov exponent of a one-dimensional random medium for weakly coupled disorder was first given by Thouless [Tho] and then proven rigorously by Pastur and Figotin [PF]. Anomalies in the perturbation theory at the band center were discovered by Kappus and Wegner [KW] and further discussed by various other authors [DG, BK, CK]. The Lyapunov exponent is then identified with the inverse localization length of the system. This short note concerns the behavior of the Lyapunov exponent for a low density of impurities, each of which may, however, be large. The presented method is as [JSS, Sch, SSS] a further application of diagonalizing the transfer matrices without perturbation (here the low density of impurities) and then rigorously controlling the error terms by means of oscillatory sums of rotating modified Prüfer phases. Some of the oscillatory sums remain large if the rotation phases (here the quasi-momenta) are rational. This leads to supplementary contributions of the Kappus-Wegner type.

The calculations are carried through explicitly for the one-dimensional Anderson model, but the method transposes also to more complicated models with a periodic background as well as low-density disorder with correlations similar to the random polymer model [JSS]. Extension to a quasi-one-dimensional situation as in [Sch] should be possible, but is even more cumbersome on a calculatory level. It is also straightforward to calculate and control higher order terms in the density.

As one motivation for this study (apart from a mathematical one), let us indicate that a low density of strong impurities seems to describe materials like carbon nanotubes more
adequately than a small coupling limit of the Anderson model. Indeed, these materials have perfect cristaline structure over distances of microns which leads to a ballisistic transport over such a distance [FPWD]. The existing few defects are, on the other hand, quite large. Coherent transport within a one-particle framework should then be studied by a model similar to the one considered here. However, it is possible that the impurties rather play the role of quantum dots so that Coulomb blockade is the determining effect for the transport properties [MBCYL] rather than the coherent transport studied here.

2 Model and preliminaries

The standard one-dimensional Anderson Hamiltonian is given by

$$(H_\omega \psi)_n = -\psi_{n+1} - \psi_{n-1} + v_n \psi_n, \quad \psi \in \ell^2(\mathbb{Z}).$$

Here $\omega = (v_n)_{n \in \mathbb{Z}}$ is a sequence of independent and identically distributed real random variables. The model is determined by their probability distribution $p$ depending on a given density $\rho \in [0, 1]$: $p = (1 - \rho) \delta_0 + \rho \tilde{p}$, (1)

where $\tilde{p}$ is a fixed compactly supported probability measure on $\mathbb{R}$. This measure may simply be a Dirac peak if there is only one type of impurity, but different from $\delta_0$. Set $\Sigma = \text{supp}(p)$ and $\tilde{\Sigma} = \text{supp}(\tilde{p})$. The expectation w.r.t. the $p$’s will be denoted by $E$, that w.r.t. the $\tilde{p}$’s by $\tilde{E}$, while $E_\nu$ and $\tilde{E}_\nu$ is the expectation w.r.t. $p$ and $\tilde{p}$ respectively over one random variable $\nu \in \Sigma$ only.

In order to define the Lyapunov exponent, one rewrites the Schrödinger equation $H_\omega \psi = E \psi$ using transfer matrices

$$
\begin{pmatrix}
\psi_{n+1} \\
\psi_n
\end{pmatrix}
= T_E^n
\begin{pmatrix}
\psi_n \\
\psi_{n-1}
\end{pmatrix},
\quad
T_E^n
= \begin{pmatrix}
v_n - E & -1 \\
1 & 0
\end{pmatrix}.
$$

We also write $T_v^n$ for $T_E^n$ if $v_n = v$. Then the Lyapunov exponent at energy $E \in \mathbb{R}$ associated to products of random matrices chosen independently according to $p$ from the family $(T_v^n)_{v \in \Sigma}$ of $\text{SL}(2, \mathbb{R})$-matrices is given by

$$
\gamma(\rho, E) = \lim_{N \to \infty} \frac{1}{N} E \log \left( \left\| \prod_{n=1}^N T_v^n \right\| \right).
$$

The aim is to study the asymptotics of $\gamma(\rho, E)$ in small $\rho$ for $|E| < 2$.

In order to state our results, let us introduce, for $E = -2 \cos(k)$ with $k \in (0, \pi)$, the basis change $M \in \text{SL}(2, \mathbb{R})$ and the rotation matrix $R_k$ by the quasi-momentum $k$:

$$
M = \frac{1}{\sqrt{\sin(k)}} \begin{pmatrix}
\sin(k) & 0 \\
-\cos(k) & 1
\end{pmatrix},
\quad
R_k = \begin{pmatrix}
\cos(k) & -\sin(k) \\
\sin(k) & \cos(k)
\end{pmatrix}.
$$

It is then a matter of computation to verify
\[ MT_v^E M^{-1} = R_k(1 + P_v^E), \quad P_v^E = -\frac{v}{\sin(k)} \begin{pmatrix} 0 & 0 \\ \frac{a}{1} & 1 \end{pmatrix}. \]

Next we introduce another auxillary family of random matrices. Set \( \Sigma = [-\frac{\pi}{2}, \frac{\pi}{2}] \times \hat{\Sigma} \), and, for \((\psi, v) \in \hat{\Sigma}:

\[ \hat{T}_{\psi,v}^E = R_{\psi} MT_v^E M^{-1}. \]

The following probability measures on \( \hat{\Sigma} \) will play a role in the sequel: \( \hat{\nu}_\infty = \frac{d\hat{\nu}}{\pi} \otimes \hat{\nu} \) and
\[
\hat{\nu}_q = \left( \frac{1}{q} \sum_{p=1}^\infty \delta_{p \left( \frac{q}{2} - \frac{q+1}{2q} \right)} \right) \otimes \hat{\nu} \quad \text{for} \quad q \in \mathbb{N}.
\]

The Lyapunov exponents associated to these families of random matrices are denoted by \( \hat{\gamma}_\infty(E) \) and \( \hat{\gamma}_q(E) \) respectively. It is elementary to check that the subgroups generated by matrices corresponding to the supports of \( \hat{\nu}_\infty \) and \( \hat{\nu}_q \) are non-compact, which implies \( [BL] \) that the corresponding Lyapunov exponents are strictly positive.

The matrices \( T_v^E \) and \( \hat{T}_{\psi,v}^E \) induce actions \( \mathcal{S}_{E,v} \) and \( \hat{\mathcal{S}}_{E,\psi,v} \) on \( \mathbb{R} \) via

\[
\nu\mathcal{S}_{E,v}(\theta) = \frac{MT_v^E M^{-1} e_\theta}{\|MT_v^E M^{-1} e_\theta\|}, \quad \hat{\nu}\hat{\mathcal{S}}_{E,\psi,v}(\theta) = \frac{\hat{T}_{\psi,v}^E e_\theta}{\|\hat{T}_{\psi,v}^E e_\theta\|};
\]

where the freedom of phase is fixed by \( \mathcal{S}_{E,0}(\theta) = \theta + k \) and \( \hat{\mathcal{S}}_{E,0}(\theta) = \theta + k + \psi \) as well as the continuity in \( v \). Invariant measures \( \nu^E \), \( \hat{\nu}_\infty^E \) and \( \hat{\nu}_q^E \) for these actions and the probability measures \( p, \hat{p}_\infty \) and \( \hat{p}_q \) are defined by

\[
\int_0^\pi d\nu^E(\theta) f(\theta) = \int_0^\pi d\nu^E(\theta) \mathbf{E}_v f(\mathcal{S}_{E,v}(\theta) \mod \pi), \quad f \in C(\mathbb{R}/\pi \mathbb{Z}),
\]

and similar formulas for \( \hat{\nu}_\infty^E \) and \( \hat{\nu}_q^E \). Due to a theorem of Furstenberg \([BL]\), the invariant measures exist and are unique whenever the associated Lyapunov exponent is positive. Let us note that one can easily verify that the invariant measure \( \hat{\nu}_\infty^E \) is simply given by the Lebesgue measure \( \frac{d\hat{\nu}}{\pi} \). Furthermore \( \hat{\nu}_\infty^E \) and \( \hat{\nu}_q^E \) do not depend on \( \rho \) (but \( \nu^E \) does).

Next let us write out a more explicit formula for the new Lyapunov exponent \( \hat{\gamma}_\infty(E) \). First of all, according to Furstenberg’s formula \([BL], [JS]\),

\[
\hat{\gamma}_\infty(E) = \mathbf{E}_{\psi,v} \int_0^\pi d\hat{\nu}_\infty^E(\theta) \log(\|\hat{T}_{\psi,v}^E e_\theta\|), \quad \text{where} \quad e_\theta = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.
\]

As \( \hat{\nu}_\infty^E \) is the Lebesgue measure, rotations are orthogonal and the integrand is \( \pi \)-periodic, one gets

\[
\hat{\gamma}_\infty(E) = \frac{1}{2} \mathbf{E}_v \int_0^{2\pi} \frac{d\theta}{2\pi} \log \left( e_\theta (MT_v^E M^{-1})^* (MT_v^E M^{-1}) e_\theta \right).
\]

Now \( (MT_v^E M^{-1})^* (MT_v^E M^{-1}) = |1 + P_v^E|^2 \) is a positive matrix with eigenvalues \( \lambda_v \geq 1 \) and \( 1/\lambda_v \) where \( \lambda_v = 1 + \frac{a}{2} + \sqrt{a + \frac{a^2}{4}} \) with \( a = \frac{\sqrt{d}}{\sin(k)} \). As it can be diagonalized by an orthogonal transformation leaving the Lebesgue measure invariant, we deduce that
\[ \hat{\gamma}_\infty(E) = \frac{1}{2} \hat{E}_v \int_0^{2\pi} \frac{d\theta}{2\pi} \log \left( \lambda_v \cos^2(\theta) + \frac{1}{\lambda_v} \sin^2(\theta) \right) = \frac{1}{2} \int d\hat{p}(v) \log \left( \frac{1 + \lambda_v^2}{2\lambda_v} \right). \]

This formula shows immediately that \( \hat{\gamma}_\infty(E) > 0 \) unless \( \hat{p} = \delta_0 \) (in which case \( \lambda_v = \lambda_0 = 1 \)).

## 3 Result on the Lyapunov exponent

**Theorem** Let \( E = -2 \cos(k) \) with either \( k/\pi \) rational or \( k \) satisfying the weak diophantine condition

\[ |1 - e^{2\pi i m/k}| \geq c e^{-|m|}, \quad \forall \; m \in \mathbb{Z}, \tag{5} \]

for some \( c > 0 \) and \( \xi' > 0 \). Then

\[ \gamma(\rho, E) = \begin{cases} \rho \hat{\gamma}_\infty(E) + \mathcal{O}(\rho^2) & k \text{ satisfies } (5), \\ \rho \hat{\gamma}_q(E) + \mathcal{O}(\rho^2) & k = \pi \frac{p}{q}, \end{cases} \]

where \( p \) and \( q \) are relatively prime. Furthermore, for \( \xi \) depending only on \( \hat{p} \),

\[ |\hat{\gamma}_q(E) - \hat{\gamma}_\infty(E)| \leq c e^{-|q|}. \]

The result can be interpreted as follows: if the density of the impurities is small, then the incoming (Prüfer) phase at the impurity is uniformly distributed for a sufficiently irrational rotation angle (i.e. quasi-momentum) because the sole invariant measure of an irrational rotation is the Lebesgue measure. For a rational rotation, the mixing is to lowest order in \( \rho \) perfect over the orbits of the rational rotation, which leads to the definition of the family \( (\hat{T}_E^{p,\sigma})_{(p,\sigma) \in \hat{\Sigma}_q} \) and its distribution \( \hat{\nu}_q \). As indicated above, the proof that this is the correct image is another simple application of modified Prüfer phases and an oscillatory sum argument.

Let us note that \( \hat{\gamma}_q(E) \neq \hat{\gamma}_\infty(E) \); more detailed formulas for the difference are given below. As a result, one can expect a numerical curve of the energy dependence of the Lyapunov exponent at a given fixed low density to have spikes at energies corresponding to rational quasimomenta with small denominators. Moreover, the invariant measures \( \nu^E \) and \( \hat{\nu}_q^E \) are not close to the Lebesgue measure, but has higher harmonics as is typical at Kappus-Wegner anomalies. Furthermore, let us add that at the band center \( E = 0 \) the identity \( \gamma(\rho, 0) = \rho \hat{\gamma}_0(0) \) holds with no higher order correction terms and where \( \hat{\gamma}_0(0) \) is the center of band Lyapunov exponent of the usual Anderson model with distribution \( \hat{p} \).

Finally, let us compare the above result with that obtained for a weak-coupling limit of the Anderson model \([PF, JSS]\): the Lyapunov exponent grows quadratically in the coupling constant of the disordered potential, while it grows linearly in the density. The reason is easily understood if one thinks of the change of the coupling constant also rather as a change of the probability distribution on the space of matrices. At zero coupling, the measure is supported on one critical point (or more generally, on a commuting subset), and the weight in its neighborhood grows as the coupling constant grows. In \([4]\) the weight may grow far from the critical point, and this leads to the faster increase of the Lyapunov exponent.
4 Proof

For fixed energy \( E \), configuration \((v_n)_{n \in \mathbb{N}}\) and \((\psi_n)_{n \in \mathbb{N}}\), as well as an initial condition \( \theta_0 \), let us define iteratively the sequences

\[
\theta_n = S_{E,v_n}(\theta_{n-1}), \quad \hat{\theta}_n = \hat{S}_{E,\psi_n,v_n}(\hat{\theta}_{n-1}) = S_{E,v_n}(\hat{\theta}_{n-1}) + \psi_n. \tag{6}
\]

When considered modulo \( \pi \), these are also called the modified Prüfer phases. They can be efficiently used in order to calculate the Lyapunov exponent as well as the density of states. For the Lyapunov exponent, let us first note that one can make a basis change in (2) at the price of boundary terms vanishing at the limit, and furthermore, that according to [BL, A.III.3.4] it is possible to introduce an arbitrary initial vector, so that

\[
\gamma(\rho, E) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \left( \left\| \prod_{n=1}^{N} M T_n^E M^{-1} e_\theta \right\| \right). \tag{7}
\]

Now using the modified Prüfer phases with initial condition \( \theta_0 = \theta \), this can be developed into a telescopic sum:

\[
\gamma(\rho, E) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \sum_{n=1}^{N} \log \left( \left\| MT_n^E M^{-1} e_{\theta_{n-1}} \right\| \right) = \rho \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \sum_{n=1}^{N} \tilde{E}_v \log \left( \left\| MT_v^E M^{-1} e_{\theta_{n-1}} \right\| \right),
\]

where in the second step we have evaluated the partial expectation over the last random variable \( v_n \) by using the fact that for \( v_n = 0 \) the contribution vanishes. Next let us note that the function \( e^{i\theta} \mapsto \tilde{E}_v \log(\left\| MT_v^E M^{-1} e_\theta \right\|) \) has an analytic extension to \( \mathbb{C} \setminus \{0\} \), contains only even frequencies so that its Fourier series

\[
\tilde{E}_v \log \left( \left\| MT_v^E M^{-1} e_\theta \right\| \right) = \sum_{m \in \mathbb{Z}} a_m e^{2im\theta},
\]

has coefficients satisfying for any \( \xi > 0 \) a Cauchy estimate of the form

\[
|a_m| \leq c_\xi e^{-\xi|m|}. \tag{8}
\]

Comparing with (4), we deduce

\[
a_0 = \hat{\gamma}_\infty(E).
\]

Introducing now the oscillatory sums

\[
I_m(N) = \mathbb{E} \frac{1}{N} \sum_{n=1}^{N} e^{2m\theta_n}, \quad \hat{I}_m(N) = \hat{\mathbb{E}} \frac{1}{N} \sum_{n=1}^{N} e^{2m\hat{\theta}_n},
\]

the Lyapunov exponent now reads

\[
\gamma(\rho, E) = \rho \sum_{m \in \mathbb{Z}} a_m \lim_{N \to \infty} I_m(N) \tag{9}.
\]
Hence we need to calculate $I_m(N)$ perturbatively in $\rho$. Clearly $I_0(N) = 1$. Furthermore, integrating over the initial condition w.r.t. the invariant measure gives for all $N \in \mathbb{N}$

$$\int d\nu^E(\theta) \ I_m(N) = \int d\nu^E(\theta) \ e^{2im\theta} .$$

Hence calculating $I_m(N)$ perturbatively also gives the harmonics of $\nu^E$ perturbatively (similar statements hold for $\hat{I}_m(N)$, of course). Going back in history once, one gets

$$I_m(N) = \frac{1}{N} \mathbb{E} \sum_{n=1}^{N} \left( (1 - \rho) e^{2imk} e^{2im\theta_{n-1}} + \rho \hat{E}_v(e^{2imS_{E,v}(\theta_{n-1})}) \right) = (1 - \rho) e^{2imk} I_m(N) + O(\rho, N^{-1}) .$$

For $k$ satisfying (5), one deduces

$$|I_m(N)| \leq \frac{1}{|1 - (1 - \rho) e^{2imk}|} \ O(\rho, N^{-1}) \leq c \ e^{\xi|m|} \ O(\rho, N^{-1}) .$$

Replacing this and (8) with $\xi > \xi'$ into (9) concludes the proof in this case because only the term $m = 0$ gives a contribution to order $\rho$.

If now $k = \pi \frac{p}{q}$, the same argument implies

$$I_{nq+r}(N) = O(\rho, N^{-1}) , \quad \forall \ n \in \mathbb{Z} , \ r = 1, \ldots, q - 1 . \quad (10)$$

Setting

$$\hat{E}_v(e^{2imS_{E,v}(\theta)}) = \sum_{l \in \mathbb{Z}} \hat{b}_l^{(m)} e^{2i(m+l)\theta} , \quad \hat{E}_v(e^{2imS_{E,v}(\theta)}) = \sum_{l \in \mathbb{Z}} \hat{b}_l^{(m)} e^{2i(m+l)\theta} ,$$

one deduces for the remaining cases

$$I_{nq}(N) = (1 - \rho) I_{nq}(N) + O(N^{-1}) + \rho \sum_{l \in \mathbb{Z}} b_l^{(mq)} (I_{nq+l}(N) + O(N^{-1})) .$$

Due to (10), this gives the following equations

$$I_{nq}(N) = \sum_{r \in \mathbb{Z}} b_{rq}^{(mq)} I_{(n+r)q}(N) + O((\rho N)^{-1}, \rho) .$$

They determine the invariant measure $\nu^E$ to lowest order in $\rho$. This shows, in particular, that $\nu^E$ is already to lowest order not given by the Lebesgue measure. We will not solve these equations, but rather show that the oscillatory sum $I_{nq}(N)$ satisfy the same equations, and hence, up to higher order corrections, the measure $\hat{\nu}^E_q$ can be used instead of $\nu^E$ in order to calculate the Lyapunov exponent. Indeed, it follows directly from (6) and the definition of $\hat{p}_q$ that

$$\hat{b}_l^{(m)} = \delta_{m \mod q = 0} b_l^{(m)} .$$
In particular, \( \hat{I}_m(N) = 0 \) if \( m \mod q \neq 0 \). Thus we deduce

\[
\hat{I}_{nq}(N) = \sum_{r \in \mathbb{Z}} \hat{I}^{(nq)}_{rq}(N) + \mathcal{O}(N^{-1}) = \sum_{r \in \mathbb{Z}} b_{rq}^{(nq)} \hat{I}_{(n+r)q}(N) + \mathcal{O}(N^{-1}).
\]

Comparing the equations for \( I_{nq}(N) \) and \( \hat{I}_{nq}(N) \) (which have a unique solution because the invariant measures are unique by Furstenberg’s theorem), it follows that

\[
\hat{I}_{nq}(N) = I_{nq}(N) + \mathcal{O}(\rho, (\rho N)^{-1}).
\]

Replacing this into (9), one deduces

\[
\gamma(\rho, E_v) = \rho \sum_{m \in \mathbb{Z}} a_m \int d\tilde{\nu}^E(\theta) e^{2m\rho} + \mathcal{O}(\rho^2) = \rho \int d\tilde{\nu}^E(\theta) \tilde{E}_v \log \| MT_v^E M^{-1} e_{\theta} \| + \mathcal{O}(\rho^2).
\]

Now due to the orthogonality of rotations one may replace \( MT_v^E M^{-1} \) by \( T_{\psi,v}^E \), and then the r.h.s. contains exactly the Furstenberg formula for \( \hat{\gamma}_q(E) \) as claimed. The estimate comparing \( \hat{\gamma}_q(E) \) and \( \hat{\gamma}_{\infty}(E) \) follows directly from the Cauchy estimate (8).

\section{Result on the density of states}

Another ergodic quantity of interest is the integrated density of states, defined by

\[
\mathcal{N}(\rho, E) = \lim_{N \to \infty} \frac{1}{N} E \# \{ \text{negative eigenvalues of the restriction of } H_{\omega} - E \text{ to } l^2(\{1, \ldots, N\}) \}.
\]

Recall, in particular, that \( \mathcal{N}(0, E) = k \) if \( E = -2 \cos(k) \). Defining the mean phase shift of the impurities by

\[
\tilde{\varphi}(\theta) = \tilde{E}_v (S_{E,v}(\theta) - \theta),
\]

the low density expansion of the density of states reads as follows:

\[
\mathcal{N}(\rho, E) = \begin{cases} 
(1 - \rho) k + \rho \int \frac{d\theta}{2\pi} \tilde{\varphi}(\theta) + \mathcal{O}(\rho^2) & k \text{ satisfies (5)}, \\
(1 - \rho) k + \rho \int d\tilde{\nu}_q^E(\theta) \tilde{\varphi}(\theta) + \mathcal{O}(\rho^2) & k = \pi \frac{p}{q}
\end{cases}
\]

with \( \rho \) and \( q \) relatively prime. The proof of this is completely analogous to the above when the rotation number calculation (e.g. [JSS] for a proof) is applied:

\[
\mathcal{N}(\rho, E) = \lim_{N \to \infty} \frac{1}{N} E \sum_{n=1}^{N} (S_{E,v}(\theta_{n-1}) - \theta_{n-1})
\]

\[
= (1 - \rho) k + \rho \lim_{N \to \infty} \frac{1}{N} E \sum_{n=1}^{N} \tilde{\varphi}(\theta_{n-1})
\]
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