Complex Structures Defined on Dynamically Triangulated Surfaces

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Abstract

A method to define the complex structure and separate the conformal mode is proposed for a surface constructed by two-dimensional dynamical triangulation. Applications are made for surfaces coupled to matter fields such as $n$ scalar fields ($n = 0, 1$ and $4$) and $m$ Ising spins ($m = 1$ and $3$). We observe a well-defined complex structure for cases when the matter central charges are less than and equal to one, while it becomes unstable beyond $c = 1$. This can be regarded as the transition expected in analytic theories.

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1 Introduction

There has been, over the last few years, remarkable progress in the quantum theory of two-dimensional (2-d) gravity coupled to matter fields, such as scalar fields or Ising spins, connected to string theoretic models. Two distinct analytic approaches for quantizing 2-d gravity have been established, these being the discretized and continuous theories. The discretized approach, implemented by the matrix model technique, exhibits behavior found in the continuous approach, given by Liouville field theory, in a continuum limit. There thus seems to exist strong evidence for the equivalence of the two theories. However, we feel that the mutual relationship is not yet fully understood.

Recently numerical methods, such as that of dynamical triangulation and Regge calculus, have drawn much attention as alternative approaches to studying non-perturbative effects, being also capable of handling those cases where analytical theories cannot yet produce meaningful results. In the dynamical triangulation method calculations of the partition function are performed by replacing the path integral over the metric to a sum over possible triangulations. Dual graphs of surfaces created by this method have direct correspondence to Feynman diagrams of the $\phi^3$ matrix model. While the relation between dynamical triangulation and matrix model is rather transparent, its relation to Liouville field theory has not been as evident. However, intuitively, we expect that the manifold treated in the continuous theory can be approximated by the triangulated lattice when the number of triangles are large enough.

On the other hand we still do not understand the nature of the transition at $c = 1$, a barrier which is evident in analytical theories. In spite of serious numerical efforts, all attempts have failed so far to account for the transition, at least by attempting to measure the string susceptibility of simulated surfaces. Of course the string susceptibility, $\gamma_{\text{string}}$, is defined from the asymptotic behavior of the partition function as $Z[N] \approx N^{\gamma_{\text{string}}} e^{\lambda N}$, and it is obviously not an easy task to extract $\log N$ contributions from dominant $\lambda N$ backgrounds unless we have good estimates of the finite size correction.

It follows that it should be much easier to observe this transition in local scale invariant quantities, rather than in the original metric which is comparatively complicated, reflecting the fractal nature of the surface. In this letter we propose a method by which we can define the complex structure of a surface generated by the dynamical triangulation method and separate the conformal mode. Thus this enables us to observe the transition by examining the possibility of defining a smooth background metric.

2 Determination of the complex structure

A fluctuating metric $g_{\mu\nu}$ defined on a two dimensional continuous surface can be decomposed into its complex structure (moduli $\tau$) and a conformal mode $\phi(z)$:

$$g_{\mu\nu} = \hat{g}_{\mu\nu}(\tau, z) e^{\phi(z)}. \quad (1)$$

In the lattice formulation of quantum gravity one expects that surfaces generated by the dynamical triangulation method will tend to continuous surfaces in the limit of large
numbers of triangles. However, such surfaces are known to be fractal\[6, 7\], and it is not at all trivial that the lattice regularization simulates the continuous theory for a small lattice constant. It follows that one of the indispensable problems in the discretized method we are employing is to make sure that we indeed possess a universal continuum limit, i.e. a unique complex structure and conformal mode.

Our method stems from attempts to derive dual amplitudes from planar Feynman diagrams in the large order limit\[8\], by making use of the electrical circuit analogy\[9\]. The basic assumption of these methods is that the network corresponding to a fine planar Feynman diagram can be regarded as a uniform homogeneous conducting sheet with a constant resistivity. Once this is accepted it is straightforward to derive dual amplitudes by evaluating the heat generated by the sheet. Unfortunately, there exists no realistic field theory in which this assumption is literally satisfied so that dual amplitudes are given as the result. For non-critical strings, however, we know matrix models with \(c \leq 1\) reproduce results of the continuum theory (i.e. the Liouville theory). Therefore, we expect the basic assumption is satisfied for these models. In other words random surfaces corresponding to 2-d quantum gravity coupled to \(c \leq 1\) matter fields should behave as uniform homogeneous media with constant resistivities if we regard them as electrical networks.

In order to check the idea numerically it is necessary to measure how the network conducts current, and compare it with the corresponding continuous medium. In the following we first give an algorithm to measure the resistivity from current and voltage distributions. As we shall see, the invariance of resistivity under local scale transformations plays an essential role in the algorithm, and because of this property we can extract information which is independent of the conformal mode such as moduli. We then apply the method to the dynamically triangulated surfaces. Since local scale invariance is not fully realized on discretized surfaces, the resistivity thus obtained may have certain fluctuations. In fact, the value varies from one sample surface to another, and with the position of electrodes for the measurements. Such ambiguities will vanish, however, in the continuum limit, if it exists at all. We shall see that this is indeed the case for matrix models with \(c \leq 1\).

Let us first consider a two dimensional conducting medium with conductivity tensor given by \(\sigma^{\mu\nu}\). The Joule heat \(Q\) generated by a potential distribution \(V\) on the surface,

\[
Q = \int d^2 x \sigma^{\mu\nu} \partial_\mu V \partial_\nu V,
\]

leads to an equation for \(V\) by requiring, \(\delta Q = 0;\)

\[
\partial_\mu \sigma^{\mu\nu} \partial_\nu V = 0.
\]

When we define the current density by

\[
j^\mu = \sigma^{\mu\nu} \partial_\nu V,
\]

eq.(3) is just the equation of continuity. By identifying \(\partial_\nu V\) to be the electric field \(E_\nu\), eq.(4) becomes, of course, Ohm’s law, \(j^\mu = \sigma^{\mu\nu} E_\nu\), or as it is often expressed \(E_\mu = \rho_{\mu\nu} j^\nu\).
with the resistivity tensor, $\rho_{\mu\nu}$, defined by

$$\sigma^{\mu\lambda} \rho_{\lambda\nu} = \delta^\mu_\nu. \quad (5)$$

For a uniform homogeneous medium whose surface is specified by the metric tensor $g_{\mu\nu}$ the resistivity tensor is written as

$$\rho_{\mu\nu} = r \frac{1}{\sqrt{g}} g_{\mu\nu}, \quad (6)$$

with $r$ being the resistivity constant. An important property of the resistivity tensor in two dimensions is the invariance under local scale transformations, $g_{\mu\nu} \mapsto g_{\mu\nu} e^{-\sigma}$. The significance of this invariance property is seen by considering the resistance $R$ of a small rectangular section of the conducting sheet with length $a$ and width $b$:

$$R = \frac{a}{b} r, \quad (7)$$

which is invariant under the scale change of $a \mapsto a \xi$ and $b \mapsto b \xi$. This property enables us to determine the complex structure of two dimensional curved surfaces through measurements of the resistivity.

In the case of spherical topology we can regard the surface as an infinite flat sheet. Then the potential at a point with the (complex) coordinate, $z(= x + iy)$, with a source of current $I$ placed at $z_3$, and a sink of the current at $z_4$ is written as

$$V_{34}(z) = -\frac{Ir}{2\pi} \ln \left| \frac{z - z_3}{z - z_4} \right| + \text{Const.}, \quad (8)$$

where the subscript of $V$ indicate points of the source and the sink. In order to avoid the ambiguity arising from Const. in the above equation we measure the potential drop between points 1 and 2 with coordinates $z_1$ and $z_2$ (Fig.1),

$$V_{34}(z_1) - V_{34}(z_2) = -\frac{Ir}{2\pi} \ln \left| [z_1, z_2; z_3, z_4] \right|, \quad (9)$$

where the ratio of four points written as

$$[z_1, z_2; z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

is known as the anharmonic ratio\[^{10}\].

The anharmonic ratio has several interesting properties. For example, its invariance under the projective transformation,

$$z \mapsto \frac{az + b}{cz + d}, \quad (10)$$

with $ad - bc = 1$ allows us to fix three coordinates among the \{$z_i$\} to any desired values without changing the potential drop eq.\(^{(9)}\) by appropriately choosing three complex
parameters among \(a, b, c,\) and \(d\). For example, we can fix three coordinates as \(z_2 = 1,\) \(z_3 = 0,\) and \(z_4 = \infty\). Then the potential drop eq.(11) for \(I = 1A\) is written as
\[
V_{34}^{12} = -\frac{r}{2\pi} \ln |z_1|,
\] (11)
where \(V_{34}^{12}\) stands for the potential drop between points 1 and 2 in the presence of a source and a sink of current \(I = 1A\) at points 3 and 4, respectively.

In addition, by permuting the four points the anharmonic ratios are known to take values such as
\[
[z_1, z_2; z_3, z_4] = \lambda, \\
[z_1, z_3; z_2, z_4] = 1 - \lambda, \\
[z_1, z_2; z_4, z_3] = \frac{1}{\lambda}, \\
[z_1, z_4; z_2, z_3] = 1 - \frac{1}{\lambda}.
\] (12)
For other possible combinations they are given by ratios of these complex numbers.

Using the second of the above equations we get
\[
V_{24}^{13} = -\frac{r}{2\pi} \ln |1 - z_1|.
\] (13)
All the other possible permutations of \(\{z_i\}\) only give the potential drops obtained by linear combinations of eq.(11) and (13). Therefore, using this set we can determine \(z_1\) within the uncertainty of its complex conjugate, if we know the resistivity constant \(r\) from other means. There may be several methods to determine \(r\). We describe here one of the methods\(^*\) which we have employed in our numerical simulations. If we introduce an additional point for the measurement (Fig.2), we get three more relations in addition to eq.(11) and (13) which together are sufficient to specify five unknowns, namely two points in the complex plane and \(r\). Denoting the coordinate added as \(z_5\) the three extra equations are given by
\[
V_{34}^{52} = -\frac{r}{2\pi} \ln |z_5|,
\] (14)
\[
V_{24}^{53} = -\frac{r}{2\pi} \ln |1 - z_5|,
\] (15)
\[
V_{14}^{52} = -\frac{r}{2\pi} \ln |[z_5, z_2; z_1, z_4]|
= -\frac{r}{2\pi} \ln \frac{|z_5 - z_1|}{|1 - z_1|}.
\] (16)

Now we apply this algorithm to a random surface generated by the dynamical triangulation method. The dual graph of a surface consisting of \(N\) triangles is regarded as a trivalent network, where we fix the resistance of the link connecting two neighbouring vertices to be 1\(\Omega\). What we want to do is to examine whether such a network behaves like a continuous medium in the limit of \(N \to \infty\) or not. For the measurement we pick

\(^*\)We will refer to this method as the five point method in later discussion.
five vertices in the dual graph and perform the five point method as explained above. We note that this sometimes produces two solutions for $r$ due to the uncertainty explained below eq.(13). This, however, is not a great problem, since the probability of such cases is about 10 to 20 per cent, so we can simply throw them away without adversely affecting the statistics.

The practical method we employ for the determination of potential drops is as follows; we pick two vertices at $P_{in}$ and $P_{out}$ for the source and the sink of current with unit intensity ($1\text{A}$). By writing the potential of the vertex at $P$ as $V(P)$ current conservation reads

$$\frac{V(P) - V(P_a)}{1} + \frac{V(P) - V(P_b)}{1} + \frac{V(P) - V(P_c)}{1} = 0,$$

expressing the zero net flow to the three neighboring vertices $P_a$, $P_b$ and $P_c$, which gives

$$V(P) = \frac{1}{3}\{V(P_a) + V(P_b) + V(P_c)\},$$

except at two vertex points where the current enters or exits,

$$V(P_{in}) = \frac{1}{3}\{V(P_a) + V(P_b) + V(P_c) + 1\},$$

and

$$V(P_{out}) = \frac{1}{3}\{V(P_a) + V(P_b) + V(P_c) - 1\},$$

respectively. It is straightforward to obtain potentials at all the vertex points on the surface numerically by the Jacobi iteration method \footnote{In practice we employ the successive over-relaxation(SOR) method for rapid convergence.}

3 Numerical results and discussions

Before presenting our numerical results let us sketch our simulation. Random surfaces are generated by the dynamical triangulation method with fixed topology ($S^2$) and a fixed number of triangles (the micro-canonical simulation) prohibiting tadpole and self-energy diagrams. For matter fields coupled to gravity we put $n$ scalar fields with $n = 0, 1$ and 4 and/or $m$ Ising spins $m = 1$ and 3 on the triangle. For thermalization of Ising spins we employ the Wolf algorithm\cite{11} to avoid severe critical slowing down.

The distribution of the resistivity constant is determined in the following manner; we first pick-up 40 to 50 independent configurations from the ensemble \footnote{Here we consider dual graphs of triangulated surfaces.}. For each configuration we pick five vertices randomly to perform the five point method to determine the resistivity. We have repeated this procedure 50 times for each configuration. Thus a graph of the distribution of $r$ consists in total of about 2,000 to 2,500 data values.

Let us first take a look at the measurement for pure gravity(Fig.3). The distributions of $r$ for three different lattice sizes show distinct peaks at about 2.6, and the peaks get...
narrower as the size grows. The value $r \approx 2.6$ should be compared to $\sqrt{3}$ of the flat network (i.e. 6 triangles around each apex). An increase in $r$ is understood to be a reflection of the fractal nature of the surface. As the number of triangles gets larger, finite size effects due to the discreteness of the surface diminish and eventually the peak grows infinitely. This is what we have expected as the continuum limit of a network of resistors. We find similar tendency in the one Ising ($c = \frac{1}{2}$) and the one scalar ($c = 1$) (Fig. 4 (a)) cases.

While random surfaces coupled to matter with central charge less than or equal to one seem to approach to a uniform homogeneous medium, simulations with matter central charge larger than one shows a different behavior. In Fig.4 (b) the distributions of $r$ in the $c = 4$ case is compared for the sizes $N = 2,000$ and 8,000. Here no sharpening of peaks as increasing sizes is seen unlike the previous cases, which suggests that the surface does not approach a smooth continuum limit due to instability, which is expected from the complex string susceptibility in the Liouville theory. Existence of the transition at about $c = 1$ is also supported in the simulations with 3 Ising spins (Fig. 5). This system behaves as a model with $c = \frac{3}{2}$ matter at the critical point, and $c = 0$ off the critical point. We observe the peaking phenomena only in the latter case as the size increases from 2,000 to 8,000.

In conclusion we have observed the so called $c = 1$ barrier for two dimensional quantum gravity coupled to matter as the transition from a well-defined complex structure to an ill-defined one. Although our simulation with 8,000 triangles has been enough to observe the indication of such a transition, larger size simulations should exhibit the phenomena much more clearly.

Next, we discuss how to extract information on the conformal mode. We have seen, so far, that the resistivity of the dual graph of a dynamically triangulated surface takes a well-defined value, if the matter central charge is less than or equal to one. Once we have found the value of the resistivity, we can assign a complex coordinate to each vertex of the dual graph from eq.(11) and (13). Although these two equations alone cannot specify in which half-plane (upper or lower) it lies, this ambiguity can be easily resolved by the five point method.

When we plot the obtained coordinates of all the vertices on a complex plane the point density around a point $z$ should be proportional to $\sqrt{g(z)}$, because each vertex is supposed to carry the same space-time volume. In the conformal gauge which we have been employing, we have $\sqrt{g(z)} = \exp \phi(z)$, where $\phi(z)$ is the conformal mode. The value of the conformal mode is then given by taking the logarithm of the point density. This argument may be over-simplified in the sense that we did not pay attention to the quantum fluctuation of $\phi(z)$ and the renormalization of the composite operator $\exp \phi(z)$. What we can say safely is that the point density at $z$ after taking an ensemble average should be equal to the following expression in Liouville theory,

$$< e^{\alpha \phi(0)} e^{\alpha \phi(1)} e^{\alpha \phi(\infty)} e^{\alpha \phi(z)} >$$

with $\alpha$ being the renormalization factor for the conformal mode.

Lastly, we briefly discuss a way to define complex structures on dynamically triangulated surfaces with higher genus. For simplicity here we consider the case of a torus, but
the generalization to higher genus would be straightforward. In the continuous formulation the period $\tau$ is obtained by the following procedure. First we introduce harmonic 1-forms $j_\mu dx^\mu$ which by definition satisfy the divergence and rotation free conditions
\[
\partial_\mu j^\mu = 0,
\]
and
\[
\partial_\mu j_\nu - \partial_\nu j_\mu = 0,
\]
where $j^\mu = \sqrt{g} g^{\mu\nu} j_\nu$. Since there are two linearly independent solutions, we can impose two conditions such as
\[
\oint_a j_\mu dx^\mu = 1,
\]
and
\[
\oint_a \tilde{j}_\mu dx^\mu = 0,
\]
where $\tilde{j}_\mu$ is the dual of $j_\mu$ given by
\[
\tilde{j}_\mu = \epsilon_{\mu\nu} \sqrt{g} g^{\nu\lambda} j_\lambda,
\]
and $a$ is a cycle on the torus. Then the period $\tau$ is defined by
\[
\tau = \oint_b j_\mu dx^\mu + i \oint_b \tilde{j}_\mu dx^\mu,
\]
where $b$ is a cycle which intersects with $a$ just once.

This procedure can be easily translated to the case of a triangulated surface by identifying $j_\mu$ with the current on the dual graph. With this identification (18) and (19) respectively mean current conservation and zero potential drop along contractible loops. It is easy to show that two linearly independent solutions exist as in the continuous case. Then the rest is almost trivial, once one recognizes that $\oint_a j_\mu dx^\mu$ and $\oint_a \tilde{j}_\mu dx^\mu$ respectively mean the potential drop around $a$ and the total current going across $a$.

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Figure 1: A measurement of the potential drop between $z_1$ and $z_2$. An electric current $I$ enters at $z_3$ and exits at $z_4$. $V$ represents a voltmeter.

Figure 2: A configuration of five electrodes. $z_5$ is fixed on the upper half plane for the measurement of resistivity $r$. The uncertainty in resistivity comes from the ambiguity of choosing $z_1$ or $z_1^*$. 
Figure 3: A comparison of resistivity distributions for the case of pure gravity with 4000, 8000 and 16000 triangles.

Figure 4: A comparison of resistivity distributions for (a) 1 scalar and (b) 4 scalars with 2000 and 8000 triangles.
Figure 5: A comparison of resistivity distributions for 3 Ising spins (a) off-critical and (b) near-critical with 2000 and 8000 triangles.
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