On Kropina Change of \(m\)-th Root Finsler Metrics

A. Tayebi, T. Tabatabaeifar and E. Peyghan

September 26, 2014

Abstract

In this paper, we consider Kropina change of \(m\)-th root Finsler metrics. We find necessary and sufficient condition under which the Kropina change of an \(m\)-th root Finsler metric be locally dually flat. Then we prove that the Kropina change of an \(m\)-th root Finsler metric is locally projectively flat if and only if it is locally Minkowskian.

Keywords: Locally dually flat metric, projectively flat metric, \(m\)-th root metric.

1 Introduction

Let \(M\) be an \(n\)-dimensional \(C^\infty\) manifold, \(TM\) its tangent bundle. Let \(F = \sqrt[A]{A}\) be a Finsler metric on \(M\), where \(A\) is given by \(A := a_{i_1...i_m}(x)y^{i_1}y^{i_2}\ldots y^{i_m}\) with \(a_{i_1...i_m}\) symmetric in all its indices. Then \(F\) is called an \(m\)-th root Finsler metric. Suppose that \(A_{ij}\) define a positive definite tensor and \(A_{ij}^{-1}\) denotes its inverse. For an \(m\)-th root metric \(F\), put

\[
A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{xi} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i}y^i.
\]

Then the following hold

\[
g_{ij} = \frac{A^{\frac{m-2}{m}}}{m^2}[mA_{ij} + (2 - m)A_iA_j],
\]

\[
y^iA_i = mA, \quad y^iA_{ij} = (m - 1)A_j, \quad y_i = \frac{1}{m}A^{\frac{m-1}{m}}A_i,
\]

\[
A^{ij}A_{jk} = \delta^i_k, \quad A^{ij}A_i = \frac{1}{m-1}y^i, \quad A_iA_jA^{ij} = \frac{m}{m-1}A.
\]

Let \((M, F)\) be a Finsler manifold. For a 1-form \(\beta(x, y) = b_i(x)y^i\) on \(M\), we have a change of Finsler which is defined by following

\[
F(x, y) \to \bar{F}(x, y) = f(F, \beta),
\]

\(^1\) 2000 Mathematics subject Classification: 53C60, 53C25.
where $f(F, \beta)$ is a positively homogeneous function of $F$. This is called a $\beta$-change of metric. It is easy to see that, if $||\beta||_F := \sup_{F(x,y)=1} |b_i(x)y^i| < 1$, then $\bar{F}$ is again a Finsler metric [7].

In this paper, we consider a special case of $\beta$-change, namely

$$\bar{F}(x, y) = \frac{F^2(x, y)}{\beta(x, y)}$$

which is called the Kropina change. If $F$ reduces to a Riemannian metric $\alpha$, then $\bar{F}$ reduces to a Kropina metric $\beta(x, y)$. Due to this reason, the transformation (5) has been called the Kropina change of Finsler metrics. It is remarkable that, the Kropina metrics are closely related to physical theories. These metrics, was introduced by Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by Kropina [5].

In [2], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory [1]. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [6]. A Finsler metric $F$ on an open subset $U \subset \mathbb{R}^n$ is called dually flat if it satisfies $(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}$. In this paper, we find necessary and sufficient condition under which a Kropina change of an $m$-th root metric be locally dually flat.

**Theorem 1.1.** Let $F = \sqrt[2]{A}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\bar{F} = \frac{F^2}{\beta}$ be Kropina change of $F$ where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x)y^l$ on $U$ such that the following hold

$$\beta_0 \beta_0 - 3 \beta_1 \beta_0 = 2 \beta \beta_0,$$

$$A_{xl} = \frac{1}{3m} [m A \theta_1 + 4 \theta A_1],$$

$$\beta_0 A_0 = -\beta_1 A_0,$$

where $\beta_0 = \beta_{x^k y^k}$, $\beta_x = (b_i)_{x^i y^i}$, $\beta_0 = \beta_{x^i y^i}$ and $\beta_0 = (b_0)_{x^0}$.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if $G_i = Py^i$, where $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$ [4].

In this paper, we prove that the Kropina change of an $m$-th root Finsler metric is locally projectively flat if and only if it is locally Minkowskian.

**Theorem 1.2.** Let $F = \sqrt[2]{A}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\bar{F} = \frac{F^2}{\beta}$ be Kropina change of $F$ where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.
2 Proof of the Theorem 1.1

A Finsler metric $F = F(x, y)$ on a manifold $M$ is said to be locally dually flat if at any point there is a standard coordinate system $(x^i, y^i)$ in $TM$ such that $L = F^2(x, y)$ satisfies

$$L_{x^i y^j} y^k = 2L_{x^i}. \quad (9)$$

In this case, the coordinate $(x^i)$ is called an adapted local coordinate system.

It is easy to see that every locally Minkowskian metric satisfies in the a bove equation, hence is locally dually flat \[11][12].

In this section, we are going to prove the Theorem 1.1. To prove it, we need the following.

**Lemma 2.1.** Suppose that the equation $ΦA^2 + Ψ A + Θ = 0$ holds, where $Φ, Ψ, Θ$ are polynomials in $y$ and $m > 2$. Then $Φ = Ψ = Θ = 0$.

**Proof of Theorem 1.1** Let $\bar{F}$ be a locally dually flat metric. We have

$$\bar{F}^2 = \frac{A^2}{\beta^2},$$

$$(\bar{F}^2)_{x^k} = \frac{1}{\beta^2} \frac{4}{m} A^\frac{2}{m} - 1 A_{x^k} - \frac{2}{\beta^2} A^\frac{2}{m} \beta_{x^k},$$

$$(\bar{F}^2)_{x^k y^l} y^k = \frac{1}{\beta^2} \left[ \frac{4}{m} A^\frac{2}{m} - 1 A_{0l} + \left( \frac{4}{m} \right) (\frac{4}{m} - 1) A^\frac{2}{m} - 2 A_{0l} \right]$$

$$- \frac{2}{\beta^2} \left[ \frac{4}{m} A^\frac{2}{m} - 1 A_{0} \beta_{0} + \frac{4}{m} A^\frac{2}{m} - 1 A_{0} \beta_{i} + A^\frac{2}{m} \beta_{0i} + \frac{6}{\beta^2} [A^\frac{2}{m} \beta_{0i}] \right]$$

Thus, we get

$$\frac{A^\frac{2}{m} - 2}{\beta^4} \left[ \frac{4}{m} \beta^2 \left[ \frac{4}{m} - 1 \right] A_{0} A_{l} + A A_{0l} - 2 AA_{x^i} \right] - \frac{8}{m} A \beta [A_{i} \beta_{0} + A_{0} \beta_{i}]$$

$$+ 2 A^2 [3 \beta_{0} \beta_{i} + 2 \beta_{x^i} - \beta_{0l}] = 0 \quad (10)$$

By Lemma 2.1 we have

$$\frac{4}{m} - 1) A_{i} A_{0} + A A_{0i} = 2 A A_{x^i}, \quad (11)$$

$$\beta_{0i} A_{l} = - A_{0i} \beta_{l}, \quad (12)$$

$$\beta_{0i} \beta_{l} - 3 \beta_{i} \beta_{0} = 2 \beta_{x^i} \beta_{l}, \quad (13)$$

One can rewrite (11) as follows

$$A (2 A_{x^i} - A_{0i}) = \frac{4}{m} - 1 A_{i} A_{0}. \quad (14)$$

Irreducibility of $A$ and $deg(A_i) = m - 1$ imply that there exists a 1-form $θ = θ_{y} y^i$ on $U$ such that

$$A_{0} = θ A. \quad (15)$$
Plugging (15) into (14), yields
\[ A_{0l} = A\theta_l + \theta A_l - A_{xl}. \] (16)
Substituting (15) and (16) into (14) yields (7). The converse is a direct computation. This completes the proof.

3 Proof of the Theorem 1.2

A Finsler metric \( F(x, y) \) on an open domain \( U \subseteq \mathbb{R}^n \) is said to be locally projectively flat if its geodesic coefficients \( G^i \) are in the form
\[ G^i(x, y) = P(x, y)y^i, \]
where \( P : TU = U \times \mathbb{R}^n \to \mathbb{R} \) is positively homogeneous with degree one, \( P(x, \lambda y) = \lambda P(x, y), \lambda > 0 \). We call \( P(x, y) \) the projective factor of \( F \).

In this section, we are going to prove the Theorem 1.2. To prove it, we need the following.

**Proposition 3.1.** Let \( F = A \hat{F} \) be an \( m \)-th root Finsler metric on an open subset \( U \subseteq \mathbb{R}^n \) (\( n \geq 3 \)), where \( A \) is irreducible. Suppose that \( \bar{F} = F^2_{\beta} \) be Kropina change of \( F \) where \( \beta = b_i(x)y^i \). If \( \bar{F} \) is projectively flat metric then it reduces to a Berwald metric.

**Proof.** Let \( \bar{F} \) be projectively flat metric. We have
\[
\bar{F}_{x^k} = \frac{2}{m\beta} A^{\hat{F}}_{-1}A_{x^k} - \frac{1}{\beta^2} A^{\hat{F}} \beta_{x^k},
\]
\[
\bar{F}_{x^i}y^jy^k = \frac{1}{\beta} \left[ \frac{2}{m} A^{\hat{F}}_{-1}A_{0l} + \left( \frac{2}{m} \right) \left( \frac{2}{m} - 1 \right) A^{\hat{F}}_{-2}A_0A_l \right]
- \frac{1}{\beta^2} \left[ \frac{2}{m} A^{\hat{F}}_{-1}A_l\beta_0 + \frac{2}{m} A^{\hat{F}}_{-1}A_0\beta_l + A^{\hat{F}} \beta_{0l} + \frac{2}{\beta^2} \left( \beta \beta_{x^l} - \beta \beta_{0l} \right) \right]
\]
Thus, we get
\[
\frac{A^{\hat{F}}_{-2}}{\beta^3} \left[ \frac{2}{m} \beta^2 \left( \frac{2}{m} - 1 \right) A_0A_l + AA_{0l} - AA_{x^l} \right]
- \frac{2}{m} A\beta[A_l\beta_0 + A_0\beta_l] + A^2 \left[ 2\beta_0\beta_l + \beta\beta_{x^l} - \beta\beta_{0l} \right] = 0
\]
By Lemma 2.1, we have
\[ mA(A_{0l} - A_{x^l}) = (m - 2)A_0A_l. \] (17)
Then irreducibility of \( A \) and \( \text{deg}(A_l) = m - 1 < \text{deg}(A) \) implies that \( A_0 \) is divisible by \( A \). This means that, there is a 1-form \( \theta = \theta_l y^l \) on \( U \) such that the following holds
\[ A_0 = 2mA\theta. \]
Then \( G^i = Py^i \), where \( P = \theta \). Then \( F \) is a Berwald metric. \( \square \)
Proof of Theorem 1.2: By Proposition 3.1 if $F$ is projectively flat then it reduces to a Berwald metric. Now, if $n \geq 3$ then by Numata’s theorem every Berwald metric of non-zero scalar flag curvature $K$ must be Riemannian. This is contradicts with our assumption. Then $K = 0$, and in this case $F$ reduces to a locally Minkowskian metric.

References

[1] S.-I. Amari, *Differential-Geometrical Methods in Statistics*, Springer Lecture Notes in Statistics, Springer-Verlag, 1985.

[2] S.-I. Amari and H. Nagaoka, *Methods of Information Geometry*, AMS Translation of Math. Monographs, Oxford University Press, 2000.

[3] V. Balan and N. Brinzei, *Einstein equations for $(h, v)$-Berwald-Moór relativistic models*, Balkan. J. Geom. Appl. 11(2)(2006), 20-26.

[4] B. Li and Z. Shen, *On projectively flat fourth root metrics*, Canad. Math. Bull. 55(2012), 138-145.

[5] M. Matsumoto, *Theory of Finsler spaces with $(\alpha, \beta)$-metric*, Rep. Math. Phys. 31(1992), 43-84.

[6] Z. Shen, *Riemann-Finsler geometry with applications to information geometry*, Chin. Ann. Math. 27(2006), 73-94.

[7] C. Shibata, *On invariant tensors of $\beta$-changes of Finsler metrics*, J. Math. Kyoto Univ. 24(1984), 163-188.

[8] H. Shimada, *On Finsler spaces with metric $L = \sqrt[\alpha_{i_1i_2...i_m}] y_{i_1}y_{i_2}...y_{i_m}$*, Tensor, N.S. 33(1979), 365-372.

[9] A. Tayebi and B. Najafi, *On $m$-th root Finsler metrics*, J. Geom. Phys. 61(2011), 1479-1484.

[10] A. Tayebi and B. Najafi, *On $m$-th root metrics with special curvature properties*, C. R. Acad. Sci. Paris, Ser. I. 349(2011), 691-693.

[11] A. Tayebi, E. Peyghan and H. Sadeghi, *On locally dually flat $(\alpha, \beta)$-metrics with isotropic S-curvature*, Indian J. Pure. Appl. Math, 43(5) (2012), 521-534.

[12] A. Tayebi, E. Peyghan and H. Sadeghi, *On a class of locally dually flat Finsler metrics with isotropic S-curvature*, Iran. J. Sci. Tech, Trans A, Vol 36, No. A3 (2012), 377-382.

[13] A. Tayebi, E. Peyghan and M. Shahbazi Nia, *On generalized $m$-th root Finsler metrics*, Linear. Algebra. Appl. 437(2012), 675-683.
Akbar Tayebi and Tayebeh Tabatabaeifar
Department of Mathematics, Faculty of Science
University of Qom
Qom, Iran
Email: akbar.tayebi@gmail.com
Email: t.tabaei@gmail.com

Esmaeil Peyghan
Department of Mathematics, Faculty of Science
Arak University
Arak 38156-8-8349, Iran
Email: epeyghan@gmail.com