Correlation Functions of Classical and Quantum Artin System defined on Lobachevsky Plane and Scrambling Time

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We consider the quantisation of the Artin dynamical system defined on the fundamental region of the Lobachevsky plane. This fundamental region of the modular group has finite volume and infinite extension in the vertical axis that correspond to a cusp. In classical regime the geodesic flow in this fundamental region represents one of the most chaotic dynamical systems, has mixing of all orders, Lebesgue spectrum and non-zero Kolmogorov entropy. The classical correlation functions decay exponentially with an exponent proportional to the entropy. Here we calculated quantum mechanical two- and four-point correlation functions in the approximation when the oscillation modes in the horizontal direction are neglected. The effective influence of these modes is taken into account by using the exact expression for the reflection amplitude. By performing a numerical integration we observed that a two-point correlation function decays exponentially and that the four-point functions decay with a lower pace. With the numerical data available to us we were able to observe a very short time exponential decay of the out-of-time correlation function to almost zero value and then an essential increase with the subsequent large fluctuations. This confirms the existence of the scrambling time in this maximally chaotic system. With more numerical data it seems possible to estimate scrambling time more accurately.

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I. INTRODUCTION

The hyperbolic systems have exponential instability of their trajectories and as such represent the most natural chaotic dynamical systems. Of special interest are dynamical systems which are defined on closed surfaces of the Lobachevsky plane of constant negative curvature. An example of such system has been introduced in a brilliant article published in 1924 by the mathematician Emil Artin. The dynamical system is defined on the fundamental region of the Lobachevsky plane which is obtained by the identification of points congruent with respect to the modular group $SL(2,\mathbb{Z})$, a discrete subgroup of the Lobachevsky plane isometries. The fundamental region $F$ in this case is a hyperbolic triangle on Fig.1. The geodesic trajectories are bounded to propagate on the fundamental hyperbolic triangle. The geodesic flow in this fundamental region represents one of the most chaotic dynamical systems with exponential instability of its trajectories, has mixing of all orders, Lebesgue spectrum and non-zero Kolmogorov entropy.

There is a great interest to consider quantisation of the hyperbolic dynamical systems and investigate their quantum-mechanical properties. Our main interest in this article...
is a study of the behaviour of the correlation functions of the Artin hyperbolic dynamical system in its classical and quantum regimes.

In classical regime the correlation functions are defined as an integral over a pair of functions/observables $A$ and $B$ in which the first one is stationary and the second one evolves with the geodesic flow $g_t$:

$$D_t(A, B) = \int_M A(g) B(gg_t) d\mu(g). \quad (1)$$

The earlier investigation of the classical correlation functions of the geodesic flows was performed in \cite{16-20} by using different approaches including Fourier series for the $SL(2, R)$ group, zeta function for the geodesic flows, relating the poles of the Fourier transform of the correlation functions to the spectrum of an associated Ruelle operator, the methods of unitary representation theory, spectral properties of the corresponding Laplacian and other approaches. In recent articles \cite{38, 39} the authors demonstrated exponential decay of the correlation functions with time on the classical phase space. The result was derived by using the differential geometry, group-theoretical methods of Gelfand and Fomin, the time evolution equations and the properties of automorphic functions on $\mathcal{F}$. The exponential decay rate was expressed in terms of the entropy $h(\mathcal{F})$ of the system

$$|D_t(A, B)| \leq M e^{-K |t|}, \quad (2)$$

where $M$ and $K$ are constants depending on the smoothness of the functions. In classical regime the exponential divergence of the geodesic trajectories resulted into the universal exponential decay of its classical correlation functions \cite{38, 39}.

In order to investigate the behaviour of the correlation functions in quantum-mechanical regime it is necessary to know the spectrum of the system and the corresponding wave functions. In the case of the modular group the energy spectrum has continuous part, which is originating from the asymptotically free motion inside an infinitely long "y-channel" extended in the vertical direction of the fundamental region, as well as infinitely many discrete energy states corresponding to a bounded motion at the "bottom" of the fundamental triangle. The spectral problem has deep number-theoretical origin and was partially solved in a series of pioneering articles \cite{24-27}. It was solved partially because the discrete spectrum and the corresponding wave functions are not known analytically. The general properties of the discrete spectrum have been derived using Selberg trace formula \cite{26-31}. Numerical calculation of the discrete energy levels were performed for many energy states \cite{42-44}.

In the next three sections we shall review the geometry of Lobachevsky hyperbolic plane and of the fundamental region which corresponds to the modular group $SL(2, Z)$, the geodesic flow on that region and the quantisation of the system. The derivation of the Maass wave function \cite{24} for the continuous spectrum will be reviewed in details. We shall use the Poincaré representation for Maass non-holomorphic automorphic wave functions. We introduce a natural physical variable $\tilde{y}$ for the distance in the vertical direction on the fundamental triangle as $\int dy/y = \ln y = \tilde{y}$ and the corresponding momentum $p_y$ in order to represent the Maass wave functions \cite{50} in the form which is appealing to the physical intuition

$$\psi_p(x, \tilde{y}) = e^{-ip\tilde{y}} + \frac{\theta(\frac{1}{2} + ip)}{\theta(\frac{1}{2} - ip)} e^{ip\tilde{y}} + \frac{4}{\theta(\frac{1}{2} - ip)} \sum_{l=1}^{\infty} \tau_{lp}(l) K_{lp} (2\pi l \tilde{y}) \cos(2\pi lx).$$

Indeed, the first two terms describe the incoming and outgoing plane waves. The plane wave $e^{-ip\tilde{y}}$ incoming from infinity of the $y$ axis on Fig. [\ref{fig:1}] (the vertex $D$) elastically scatters on the boundary $ACB$ of the fundamental triangle $\mathcal{F}$ on Fig. [\ref{fig:1}]. The reflection amplitude is a pure phase and is given by the expression in front of the outgoing plane wave $e^{ip\tilde{y}}$

$$\frac{\theta(\frac{1}{2} + ip)}{\theta(\frac{1}{2} - ip)} = \exp [i \phi(p)]. \quad (4)$$
The rest of the wave function describes the standing waves \( \cos(2\pi lx) \) in the \( x \) direction between boundaries \( x = \pm 1/2 \) with the amplitudes \( K_{l\mu}(2\pi lx) \), which are exponentially decreasing with index \( l \). Fig[2] The continuous energy spectrum is given by the formula

\[
E = p^2 + \frac{1}{4} .
\]

The wave functions of the discrete spectrum have the form [24–27, 42–44]

\[
\psi_n(z) = \sum_{l=1}^{\infty} c_l(n) K_{lu\mu}(2\pi le\bar{y}) \begin{cases} 
\cos(2\pi lx) \\
\sin(2\pi lx) 
\end{cases} .
\]

where the spectrum \( E_n = \frac{1}{4} + u_n^2 \) and the coefficients \( c_l(n) \) are not known analytically, but have been computed numerically for many values of \( n \).

Having explicit expressions of the wave functions one can analyse a quantum-mechanical behaviour of the correlation functions [36], which we investigate in the sixth section. Here we calculated the quantum-mechanical correlation functions in the approximation when the oscillation modes in the horizontal direction are neglected. The effective influence of these modes is taken into account by using the exact expression for the reflection amplitude. By performing a numerical integration we observed that a two-point correlation function decays exponentially, Fig[3] and that a four-point function demonstrates tendency to decay with lower pace, Fig[4]. With the numerical data available to us we were able to observe a very short time exponential decay of the out-of-time correlation function [36] to almost zero value and then an essential increase with the subsequent large fluctuations on Fig[4,5]. This confirms the existence of the scrambling time in this maximally chaotic system. With more numerical data it seems possible to estimate scrambling time \( t^* \) more accurately.

II. LOBACHEVSKY PLANE AND ITS ISOMETRY GROUP

Let us start with Poincare model of the Lobachevsky plane, i.e. the upper half of the complex plane: \( H = \{ z \in \mathbb{C}, \Im z > 0 \} \) supplied with the metric (we set \( z = x + iy \))

\[
dl^2 = \frac{dx^2 + dy^2}{y^2}.
\]

with the Ricci scalar \( R = -2 \). Isometries of this space are given by \( SL(2, \mathbb{R}) \) transformations. The \( SL(2, \mathbb{R}) \) matrix \( (a,b,c,d) \) are real and \( ad - bc = 1 \)

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

acts on a point \( z \) by linear fractional substitutions

\[
z \rightarrow \frac{az + b}{cz + d} .
\]

Note also that \( g \) and \( -g \) give the same transformation, hence the effective group is \( SL(2, \mathbb{R})/\mathbb{Z}_2 \). We’ll be interested in the space of orbits of a discrete subgroup \( \Gamma \subset SL(2, \mathbb{R}) \) in \( H \). Our main example will be the modular group \( \Gamma = SL(2, \mathbb{Z}) \). A nice choice of the fundamental region \( \mathcal{F} \) of \( SL(2, \mathbb{Z}) \) is displayed in Fig[1]. The fundamental region \( \mathcal{F} \) of the modular group consisting of those points between the lines \( x = -\frac{1}{2} \) and \( x = +\frac{1}{2} \) which lie outside the unit circle in Fig[1]. The modular triangle \( \mathcal{F} \) has two equal angles \( \alpha = \beta = \frac{\pi}{3} \) and with the third one equal to zero, \( \gamma = 0 \), thus \( \alpha + \beta + \gamma = 2\pi/3 < \pi \). The area of the fundamental region is finite and equals to \( \frac{\pi}{3} \) and gets a topology of sphere by “gluing” the opposite edges of the triangle. The invariant area element on the Lobachevsky plane is proportional to the square root of the determinant of the metric (7):

\[
d\mu(z) = \frac{dxdy}{y^2} .
\]
FIG. 1. The non-compact fundamental region $\mathcal{F}$ of a finite area is represented by the hyperbolic triangle $ABD$. The vertex $D$ is at infinity of the $y$ axis and corresponds to a cusp. The edges of the triangle are the arc $AB$, the rays $AD$ and $BD$. The points on the edges $AD$ and $BD$ and the points of the arcs $AC$ with $CB$ should be identified by the transformations $w = z + 1$ and $w = -1/z$ in order to form a closed non-compact surface $\overline{\mathcal{F}}$ by “gluing” the opposite edges of the modular triangle together. The hyperbolic triangle $OAB$ can be considered equally well as the fundamental region. The modular transformations of the fundamental region $\mathcal{F}$ create a regular tessellation of the whole Lobachevsky plane by congruent hyperbolic triangles. $K$ is the geodesic trajectory passing through the point $(x, y)$ of $\mathcal{F}$ in the $\vec{v}$ direction.

thus

$$\text{Area}(\mathcal{F}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\frac{1}{2}}^{\infty} dy \frac{\sqrt{1-x^2}}{y^2} = \frac{\pi}{3}.$$  

III. GEODESIC FLOW IN HAMILTONIAN GAUGE

Consider geodesic flow on $\mathcal{F}$, which is conveniently described by the least action principle $\delta S = 0$, where (cf. with (7))

$$S = \int L dt = \int \sqrt{\dot{x}^2 + \dot{y}^2} \frac{dy}{y} dt.$$  

(10)

By varying the action, we immediately get the equations of motion

$$\frac{d}{dt} \frac{\dot{x}}{y\sqrt{\dot{x}^2 + \dot{y}^2}} = 0,$$

$$\frac{d}{dt} \frac{\dot{y}}{y\sqrt{\dot{x}^2 + \dot{y}^2}} + \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y^2} = 0.$$  

(11)

Notice the invariance of the action and of the equations under time reparametrizations $t \rightarrow t(\tau)$. Presence of a local (“gauge”) symmetry indicates that we have a constrained dynamical system. One particularly convenient choice of gauge fixing specifying the time parameter $t$ proportional to the proper time, is archived by imposing the condition

$$\frac{\dot{x}^2 + \dot{y}^2}{y^2} = 2H,$$  

(12)
where $H$ is a constant. In this gauge the equations (11) will take the form

\[
\frac{d}{dt} \left( \frac{\dot{x}}{y^2} \right) = 0 \\
\frac{d}{dt} \left( \frac{\dot{y}}{y^2} + \frac{2H}{y} \right) = 0.
\]

(13)

Defining the canonical momenta $p_x, p_y$, conjugate to the coordinates $x, y$ as

\[
p_x = \frac{\dot{x}}{y^2}, \quad p_y = \frac{\dot{y}}{y^2},
\]

(14)

we shall get the geodesic equations (13) in the Hamiltonian form:

\[
\dot{p}_x = 0, \quad \dot{p}_y = -\frac{2H}{y}.
\]

(15)

Indeed, after defining the Hamiltonian as

\[
H = \frac{1}{2} y^2 (p_x^2 + p_y^2)
\]

(16)

the corresponding equations will take the form:

\[
\dot{x} = \frac{\partial H}{\partial p_x} = y^2 p_x, \quad \dot{y} = \frac{\partial H}{\partial p_y} = y^2 p_y \\
\dot{p}_x = -\frac{\partial H}{\partial x} = 0, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -y(p_x^2 + p_y^2) = -\frac{2H}{y}
\]

(17)

and coincide with (14) and (15). The advantage of the gauge (12) is that the Hamiltonian (16) coincides with the constraint.

FIG. 2. The incoming and outgoing plane waves. The plane wave $e^{-ip\tilde{y}}$ incoming from infinity of the $y$ axis on Fig. 1 (the vertex $D$) elastically scatters on the boundary $ACB$ of the fundamental triangle $F$ on Fig. 1. The reflection amplitude is a pure phase and is given by the expression in front of the outgoing plane wave $e^{ip\tilde{y}}$. The rest of the wave function describes the standing waves in the $x$ direction between boundaries $x = \pm1/2$ with the amplitudes, which are exponentially decreasing.
IV. QUANTIZATION

Now it is fairly standard to quantize this Hamiltonian system. We simply replace in (16)
\[ p_x = -i \frac{\partial}{\partial x}, \quad p_y = -i \frac{\partial}{\partial y} \]
and consider the (time independent) Schrödinger equation
\[ H \psi = E \psi. \]
The resulting equation explicitly reads:
\[ -y^2 (\partial_x^2 + \partial_y^2) \psi = E \psi. \] (18)

On the lhs one easily recognises the Laplace operator \[ \nabla^2 \] (with an extra minus sign) in Poincare metric (7). It is easy to see that the Hamiltonian is positive semi-definite Hermitian operator. Indeed, for any quadratically integrable function \( \psi(x, y) \)
\[
- \int \psi^*(x, y) y^2 (\partial_x^2 + \partial_y^2) \psi(x, y) \frac{dx dy}{y^2} = 
\int (|\partial_x \psi(x, y)|^2 + |\partial_y \psi(x, y)|^2) dx dy \geq 0.
\] (19)

It is convenient to introduce parametrization of the energy \( E = s(1 - s) \) and to rewrite this equation as
\[ -y^2 (\partial_x^2 + \partial_y^2) \psi(x, y) = s(1 - s) \psi(x, y). \] (20)

As far as \( E \) is real and semi-positive and parametrisation is symmetric with respect to \( s \leftrightarrow 1 - s \) it follows that the parameter \( s \) should be chosen within the range
\[ s \in [1/2, 1] \text{ or } s = 1/2 + iu, \quad u \in [0, \infty]. \] (21)

One should impose the "periodic" boundary condition on the wave function with respect to the modular group
\[ \psi \left( \frac{az + b}{cz + d} \right) = \psi(z), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}) \] (22)
in order to have the wave function which is properly defined on the fundamental region \( \tilde{F} \) shown in Fig. 1. Taking into account that the transformation \( T : z \to z + 1 \) belongs to \( SL(2, \mathbb{Z}) \), one has to impose the periodicity condition \( \psi(z) = \psi(z + 1) \). Thus we have a Fourier expansion
\[ \psi(x, y) = \sum_{n=-\infty}^{\infty} f_n(y) \exp(2\pi inx). \] (23)

Inserting this into Eq. (20), for the Fourier component \( f_n(y) \) we get
\[ \frac{d^2 f_n(y)}{dy^2} + (s(1 - s) - 4\pi^2 n^2) f_n(y) = 0. \] (24)

For the case \( n \neq 0 \) the solution which exponentially decays at large \( y \) reads
\[ f_n(y) = \sqrt{y} K_{s-\frac{1}{2}}(2\pi n |y|) \] (25)
and for \( n = 0 \) one simply gets
\[ f_0(y) = c_0 y^s + c_0' y^{1-s}. \] (26)
Thus the solution can be represented in the form

$$\psi(x, y) = c_0 y^s + c_0'y^{1-s} + \sqrt{y} \sum_{n=-\infty}^{\infty} c_n K_{s-\frac{1}{2}}(2\pi n|y|) \exp(2\pi inx),$$

(27)

where the coefficients $c_0, c_0', c_n$ should be defined such that the wave function will fulfill the boundary conditions (22). Thus one should impose also the invariance with respect to the second generator of the modular group $SL(2, \mathbb{Z})$, that is, with respect to the transformation $S: z \rightarrow -1/z$:

$$\psi(z) = \psi(-1/z).$$

(28)

This functional equation defines the coefficients $c_0, c_0', c_n$. We have found that it is much easier to resolve it using the full group of $SL(2, \mathbb{Z})$ transformations acting on a particular solution (26). The wave function generated in this way will be invariant with respect to the $SL(2, \mathbb{Z})$ transformations. We shall review this approach in the next section.

V. CONTINUOUS SPECTRUM AND THE REFLECTION AMPLITUDE

As we just mentioned above in order to get $SL(2, \mathbb{Z})$ invariant solutions, one should define the coefficients $c_0, c_0'$ and $c_n$ in (27). Another option is to start from a particular solution and perform summation over all nonequivalent shifts of the argument by the elements of $SL(2, \mathbb{Z})$, that is, using the Poincaré series representation. We'll demonstrate this strategy using the simplest solution (26), (27) with $c_0 = 1, c_0' = 0$:

$$\psi(z) = y^s = (3z)^s.$$

Let us denote by $\Gamma_\infty$ the subgroup of $\Gamma = SL(2, \mathbb{Z})$, generating shifts $z \rightarrow z + n, n \in \mathbb{Z}$. Explicitly the elements of $\Gamma_\infty$ are given by $2 \times 2$ matrices:

$$g_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

(29)

Since $y^s$ is already invariant with respect to $\Gamma_\infty$, we should perform summation over the conjugacy classes $\Gamma_\infty \setminus \Gamma$. Let us define these conjugacy classes. If two $SL(2, \mathbb{Z})$ matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

belong to the same class, then by definition for some $n \in \mathbb{Z}$

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that $c' = c$, $d' = d$, $a' - a = nc$ and $b' - b = nd$. Since $ad - bc = 1$ it follows that $a$ and $c$ do not have a common divisor. In fact, the opposite is also true. Given a pair of mutually prime integers $(c, d)$ it is always possible to find a pair of integers $(a, b)$ such that $ad - bc = 1$. For any other pair $(a', b')$ satisfying the same condition $a'd' - b'c' = 1$, the relations $a' - a = nc$ and $b' - b = nd$ are satisfied for some integer $n$. Thus we established a bijection between the set of mutually prime pairs $(c, d)$ with $(c, d) \neq (0, 0)$ and the set of conjugacy classes $\Gamma_\infty \setminus \Gamma$. The fact that the integers $(c, d)$ are mutually prime integers means that their greatest common divisor $(gcd)$ is equal to one: $gcd(c, d) = 1$. As a result, it is defined by the classical Poincaré series representation and for the sum of our interest
Eisenstein series representation of the wave function

\[ \psi_s(z) = \frac{1}{2} \sum_{\gamma \in \Gamma \setminus \Gamma} (3(\gamma z))^s = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2} \sum_{\gcd(c,d)=1} \frac{y^s}{((cx + d)^2 + c^2y^2)^s}, \tag{30} \]

where, as explained above, the sum on r.h.s. is taken over all mutually prime pairs \((c,d)\). The series (30) is convergent when \(\Re s > 1\). We have used also the simple relation

\[ \Im \gamma z \equiv \Im \frac{az + b}{cz + d} = \frac{y}{(cx + d)^2 + c^2y^2}. \]

To simplify further the sum let us multiply both sides of the eq. (30) by \(\frac{1}{n^{2s}}\) so that we shall get

\[ \frac{1}{n^{2s}} \equiv \zeta(2s) \]

It is easy now to get convinced that the set of all pairs \((nc,nd)\) with \(n\) a positive integer and \((c,d)\) - mutually prime, coincides with the set of all pairs of integers \((m,k)\) which are not simultaneously zero. Indeed, given a pair \((m,k)\) we can factor out the greatest common divisor \(n\) and represent it as \((nc,nd)\) with mutually prime \((c,d)\). Thus we arrive at the Eisenstein series representation of the wave function

\[ \zeta(2s) \psi_s(z) = \frac{1}{2} \sum_{(m,k) \in \mathbb{Z}^2} \sum_{\gcd(m,k) \neq (0,0)} \frac{y^s}{((mx + k)^2 + m^2y^2)^s}. \tag{32} \]

Since the r.h.s. of this equation is periodic in \(x\) with period 1, we can expand it in Fourier series. Our next goal is to find the coefficients of this expansion

\[ c_1(y) = \frac{1}{2} \sum_{(m,k) \in \mathbb{Z}^2} \sum_{\gcd(m,k) \neq (0,0)} \int_{0}^{1} \frac{y^s e^{-2\pi i lx} dx}{((mx + k)^2 + m^2y^2)^s}. \tag{33} \]

First let us handle the term with \(m = 0\):

\[ \frac{1}{2} \sum_{k \in \mathbb{Z}} \int_{0}^{1} \frac{y^s e^{-2\pi i lx} dx}{k^{2s}} = \delta_{l,0} \zeta(2s)y^s. \tag{34} \]

For the sum over non-zero \(m\)'s let's notice that we may drop the factor \(1/2\) and sum over \(m \geq 1\). Indeed, the sum over negative \(m\)'s can be reverted to a sum over positive ones through redefinition \(k \rightarrow -k\). For fixed positive \(m\) it is instructive to represent \(k\) as \(k = nm + s\) thus splitting the initial sum over \(k \in \mathbb{Z}\) into double sum over \(r = 0, 1, \ldots, m - 1\) and \(n \in \mathbb{Z}\). In this way after few simple manipulations we get

\[ \sum_{m=1}^{\infty} \sum_{r=0}^{m-1} \sum_{n \in \mathbb{Z}} \int_{0}^{1} \frac{y^s e^{-2\pi i lx} dx}{((mx + n + r)^2 + m^2y^2)^s} = \]

\[ = \sum_{m=1}^{\infty} \sum_{r=0}^{m-1} \int_{-\infty}^{\infty} \frac{y^s e^{-2\pi i lx} dx}{((mx + r)^2 + m^2y^2)^s} = \]

\[ = \sum_{m=1}^{\infty} \sum_{r=0}^{m-1} m^{-2s} y^{1-s} e^{\frac{2\pi i r}{m}} \int_{-\infty}^{\infty} \frac{\cos(2\pi |l| yx) dx}{(x^2 + 1)^s}. \tag{35} \]
The last integral is expressed in terms of modified Bessel's $K$ function:

$$
\int_{-\infty}^{\infty} \cos(2\pi |l|yx)\, dx = \frac{2\pi^s |l|y^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |l|y), \quad \text{if } l \neq 0}{\sqrt{\pi}(s-\frac{1}{2})}, \quad \text{if } l = 0.
$$

A useful alternative representation of modified Bessel's $K$ function which makes its properties more transparent is given by

$$
K_{iu}(y) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-y\cosh t} e^{iut} \, dt.
$$

This expression allows analytical continuation of the wave function from the region $\Re s > 1$ in (30) into the whole complex plane $s$ because the Bessel's $K_s(y)$ functions are well defined for any $s$. Besides, an easy examination shows that the finite sum

$$
\sum_{r=0}^{m-1} e^{2\pi ilr/m} = \begin{cases} m & \text{if } m \text{ divides } l \\ 0 & \text{otherwise} \end{cases}
$$

To summarise, for the Fourier coefficients (33) we shall get

$$
c_l(y) = \frac{2\pi^s}{\Gamma(s)} \tau_{s-\frac{1}{2}}(l) \sqrt{y} K_{s-\frac{1}{2}}(2\pi |l|y) \quad \text{if } l \neq 0,
$$

where

$$
\tau_{\nu}(n) = \sum_{a \cdot b = n} \left(\frac{a}{b}\right)^\nu,
$$

while for $l = 0$:

$$
c_0(y) = \frac{\sqrt{\pi} \Gamma(s-\frac{1}{2}) \zeta(2s-1)}{\Gamma(s)} y^{1-s}.
$$

Thus we recovered the second solution $y^{1-s}$ in (27) and calculated the coefficient $c_0$ in front of it. Thus the invariant solution (32) takes the following form:

$$
\zeta(2s) \psi_s(x, y) = \zeta(2s) y^s + \frac{\sqrt{\pi} \Gamma(s-\frac{1}{2}) \zeta(2s-1)}{\Gamma(s)} y^{1-s} + \sqrt{y} \frac{4\pi^s}{\Gamma(s)} \sum_{l=1}^{\infty} \tau_{s-\frac{1}{2}}(l) K_{s-\frac{1}{2}}(2\pi ly) \cos(2\pi lx).
$$

Using Riemann's reflection relation

$$
\zeta(s) = \frac{\pi^{s-\frac{1}{2}} \Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s)
$$

and introducing the notation

$$
\theta(s) = \pi^{-s} \zeta(2s) \Gamma(s)
$$

we arrive at the elegant final expression for the energy eigenfunctions obtained by Maass:

$$
\theta(s) \psi_s(z) = \theta(s) y^s + \theta(1-s) y^{1-s} + 4\sqrt{y} \sum_{l=1}^{\infty} \tau_{s-\frac{1}{2}}(l) K_{s-\frac{1}{2}}(2\pi ly) \cos(2\pi lx),
$$
This wave function is well defined in the complex $s$ plane and has a simple pole at $s = 1$. The physical continuous spectrum was defined in \([21]\), where $s = \frac{1}{2} + iu$, $u \in [0, \infty)$ so that

$$E = s(1 - s) = \frac{1}{4} + u^2.$$ \hfill (46)

The continuous spectrum wave functions $\psi_s(x, y)$ are delta function normalisable \([24-29]\). The wave function (45) can be conveniently represented also in the form

$$\psi_{\frac{1}{2} + iu}(z) = y^{\frac{1}{2} + iu} + \frac{\theta\left(\frac{1}{2} - iu\right)}{\theta\left(\frac{1}{2} + iu\right)} y^{\frac{1}{2} - iu} +$$

$$+ \frac{4\sqrt{y}}{\theta\left(\frac{1}{2} + iu\right)} \sum_{l=1}^{\infty} \tau_{in}(l) K_{iu}(2\pi ly) \cos(2\pi lx), \hfill (47)$$

where

$$K_{-iu}(y) = K_{iu}(y), \quad \tau_{-iu}(l) = \tau_{iu}(l). \hfill (48)$$

The physical interpretation of the wave function becomes more transparent when we introduce the new variables

$$\tilde{y} = \ln y, \quad p = -u, \quad E = p^2 + \frac{1}{4}. \hfill (49)$$

as well as alternative normalisation of the wave function

$$\psi_p(x, \tilde{y}) = y^{-\frac{1}{2}} \psi_{\frac{1}{2} + iu}(z) =$$

$$= e^{-ip\tilde{y}} + \frac{\theta\left(\frac{1}{2} + ip\right)}{\theta\left(\frac{1}{2} - ip\right)} e^{+ip\tilde{y}} +$$

$$+ \frac{4}{\theta\left(\frac{1}{2} - ip\right)} \sum_{l=1}^{\infty} \tau_{ip}(l) K_{ip}(2\pi le^{\tilde{y}}) \cos(2\pi lx). \hfill (50)$$

Indeed, the first two terms describe the incoming and outgoing plane waves. The plane wave $e^{-ip\tilde{y}}$ incoming from infinity of the $y$ axis on Fig. 1-2 (the vertex $D$) elastically scatters on the boundary $ACB$ of the fundamental region $\mathcal{F}$ on Fig. 1. The reflection amplitude is a pure phase and is given by the expression in front of the outgoing plane wave $e^{+ip\tilde{y}}$

$$\frac{\theta\left(\frac{1}{2} + ip\right)}{\theta\left(\frac{1}{2} - ip\right)} = \exp[i \varphi(p)]. \hfill (51)$$

The rest of the wave function describes the standing waves $\cos(2\pi lx)$ in the $x$ direction between boundaries $x = \pm 1/2$ with the amplitudes $K_{ip}(2\pi ly)$, which are exponentially decreasing with index $l$.

In addition to the continuous spectrum the system (20) has a discrete spectrum \([24-29]\). The number of discrete states is infinite: $E_0 = 0 < E_1 < E_2 < \ldots \to \infty$, the spectrum is extended to infinity - unbounded from above - and lacks any accumulation point except infinity. Let us denote the wave functions of the discrete spectrum by $\psi_n(z)$ so that the expansion into the full set of basis vectors will take the form

$$f(x, \tilde{y}) = \sum_{n \geq 0} a_n \psi_n(x, \tilde{y}) + \frac{1}{2\pi} \int_0^{\infty} a_p \psi_p(x, \tilde{y}) dp \hfill (52)$$

and the Parseval identity will be

$$||f(z)||^2 = \sum_{n \geq 0} |a_n|^2 + \frac{1}{2\pi} \int_0^{\infty} |a_p|^2 dp, \hfill (53)$$
and
\[ \sum_{n \geq 0} \psi_n(x, \tilde{y}) \psi_n^*(x_1, \tilde{y}_1) + \frac{1}{2\pi} \int_{0}^{\infty} \psi_p(x, \tilde{y}) \psi_{-p}(x_1, \tilde{y}_1) dp = \delta^{(2)}(z - z_1). \]

(54)

The wave functions of the discrete spectrum have the form\textsuperscript{24–27,42–44}
\[ \psi_n(z) = \sum_{l=1}^{\infty} c_l(n) \sqrt{y} K_{iu_n}(2\pi l y) \left\{ \cos(2\pi lx) \sin(2\pi lx) \right\}, \]

(55)

where the spectrum \( E_n = \frac{1}{4} + u_n^2 \) and the coefficients \( c_l(n) \) are not known analytically, but were computed numerically for many values of \( n \textsuperscript{42–44} \). Having explicit expressions of the wave functions one can analyse the quantum-mechanical behaviour of the correlation functions, which we shall investigate in the next sections.

VI. CORRELATION FUNCTIONS

A. Two-point correlation function

First let us calculate the two-point correlation function
\[ D_2(\beta, t) = \langle A(t) B(0) e^{-\beta H} \rangle = \sum_n <n| e^{iHt} A(0) e^{-iHt} B(0) e^{-\beta H} | n> = \sum_{n,m} e^{i(E_n - E_m)t - \beta E_n} <n|A(0)|m> <m|B(0)|n>. \]

(56)

The energy eigenvalues \textsuperscript{46} are parametrised by \( n \sim \frac{1}{2} + iu \) and \( m \sim \frac{1}{2} + iv \), thus
\[ D_2(\beta, t) = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} du \, dv \, e^{i(u^2 - v^2)t - \beta (\frac{1}{4} + u^2)} \int_{x} \psi_{\frac{1}{2} - iu}(z) A \psi_{\frac{1}{2} + iv}(z) d\mu(z) \int_{x} \psi_{\frac{1}{2} - iv}(w) B \psi_{\frac{1}{2} - iu}(w) d\mu(w), \]

(57)

where the complex conjugate function is \( \psi^*_{\frac{1}{2} - iu}(z) = \psi^*_{\frac{1}{2} - iv}(z) \). Defining the basic matrix element as
\[ A_{uv} = \int_{x} \psi_{\frac{1}{2} - iu}(z) A \psi_{\frac{1}{2} + iv}(z) d\mu(z) = \int_{-1/2}^{1/2} dx \int_{-\infty}^{\infty} \frac{dy}{y^2} \psi_{\frac{1}{2} - iu}(z) A \psi_{\frac{1}{2} + iv}(z), \]

(58)

for the two-point correlation function we shall get
\[ D_2(\beta, t) = \int_{-\infty}^{+\infty} e^{i(u^2 - v^2)t - \beta (\frac{1}{4} + u^2)} A_{uv} B_{vu} du \, dv. \]

(59)
In terms of the new variables \( \tau \), the basic matrix element \( \langle 58 \rangle \) will take the form

\[
A_{pq} = \int_{-1/2}^{1/2} dx \int_{1/2 \log(1-x^2)}^{\infty} dy \, \psi_p^*(x, y) \left( e^{-\frac{i}{2} y A e^{\frac{1}{2} y}} \right) \psi_q(x, y) dx \ dy = \tag{60}
\]

\[
= \int_{-1/2}^{1/2} dx \int_{1/2 \log(1-x^2)}^{\infty} dy \, \psi_p^*(x, y) \left( e^{-\frac{i}{2} y A e^{\frac{1}{2} y}} \right) \psi_q(x, y) \ .
\]

The matrix element \( \langle 58 \rangle \) plays a fundamental role in the investigation of the correlation functions because all correlations can be expressed through it. One should choose also appropriate observables \( A \) and \( B \). The operator \( y^{-2} \) seems very appropriate for two reasons. First, the convergence of the integrals over the fundamental region \( \mathcal{F} \) will be much stronger (in fact, one can consider more general observables \( y^{-N}, \ N = 1, 2, ... \) without additional difficulties, the other interesting observable is \( A = \cos(2\pi N x), \ N = 1, 2, ... \)). This operator is reminiscent of the exponentiated Liouville field since \( y^{-2} = e^{-2\theta} \). Thus we have to calculate the matrix element \( \langle 60 \rangle \) and for the observable \( A = e^{-2y} \) we shall get

\[
A_{pq} = \int_{-1/2}^{1/2} dx \int_{1/2 \log(1-x^2)}^{\infty} dy \, \psi_p^*(x, y) e^{-2y} \psi_q(x, y) \ . \tag{61}
\]

In all the above formulas one should also include the contribution coming from the discrete spectrum, and we always mean that they are present. But as far as the region \( \Im z \gg 1, \Im w \gg 1 \) is concerned the contribution of the sum over Bessel’s functions in \( \langle 50 \rangle \) and the contribution of the discrete spectrum \( \langle 55 \rangle \) we shall neglect. Indeed, the wave functions of the discrete spectrum \( \langle 55 \rangle \) are entirely expressed in terms of the Bessel’s functions and also will be neglected in our approximation because of the their exponential decay at large \( \Im z \gg 1 \) as one can observe in \( \langle 37 \rangle \). In this approximation of the wave function \( \langle 50 \rangle \) we shall have

\[
A_{pq}^{(0)} = \frac{1}{2\pi} \int_{-1/2}^{1/2} dx \int_{1/2 \log(1-x^2)}^{\infty} dy \, \left( e^{-ip\eta + \frac{\theta(1/2 + ip)}{\theta(1/2 - ip)} e^{ip\eta}} \right)
\]

\[
e^{-2y} \left( e^{ip\eta + \frac{\theta(1/2 - iq)}{\theta(1/2 + iq)} e^{-ip\eta}} \right) =
\]

\[
= \frac{1}{2\pi} \int_{-1/2}^{1/2} dx \int_{1/2 \log(1-x^2)}^{\infty} dy \, \left( e^{-ip\eta + e^{ip\eta - i\varphi(p)}} \right)
\]

\[
e^{-2y} \left( e^{ip\eta + e^{-ip\eta + i\varphi(q)}} \right) =
\]

\[
= \frac{1}{4\pi} \int_{-1/2}^{1/2} dx \left( \frac{(1 - x^2)^{-1 - \frac{p + q}{2\pi}}}{1 - \frac{p - q}{2\pi}} + \frac{(1 - x^2)^{-1 + \frac{p + q}{2\pi}} e^{i\varphi(q)}}{1 - \frac{p - q}{2\pi}} \right)
\]

\[
+ \frac{(1 - x^2)^{-1 - \frac{p + q}{2\pi}} e^{-i\varphi(p)}}{1 + \frac{p + q}{2\pi}} + \frac{(1 - x^2)^{-1 - \frac{p - q}{2\pi}} e^{-i(\varphi(p) - \varphi(q))}}{1 + \frac{p - q}{2\pi}} \right), \tag{62}
\]

where we used also \( \langle 51 \rangle \). Integration over \( x \) can be performed exactly with the result for the basic matrix element

\[
8\pi A_{pq}^{(0)} = 2 F_1 \left( \frac{1}{2}, 1 - \frac{p - q}{2\pi}, \frac{3}{2}, 1 \right) +
\]

\[
+ 2 F_1 \left( \frac{1}{2}, 1 - \frac{p + q}{2\pi}, \frac{3}{2}, 1 \right) e^{i\varphi(q)} +
\]

\[
+ 2 F_1 \left( \frac{1}{2}, 1 + \frac{p + q}{2\pi}, \frac{3}{2}, 1 \right) e^{-i\varphi(p)} +
\]

\[
+ 2 F_1 \left( \frac{1}{2}, 1 + \frac{p - q}{2\pi}, \frac{3}{2}, 1 \right) e^{-i(\varphi(p) - \varphi(q))}. \tag{63}
\]
In this approximation for the two-point correlation function we shall get

$$D_2^{(0)}(\beta, t) = \int_{-\infty}^{+\infty} e^{i(p^2 - q^2)t - \beta(\frac{1}{4} + p^2)} A_{pq}^{(0)} A_{qp}^{(0)} dpdq.$$

This expression is very convenient for the numerical calculation. On Fig. 3 one can see exponential decay of the two-point correlation function with time at different temperatures. It is expected that the two-point correlation function should decay as

$$D_2(\beta, t) \sim e^{-t/t_d},$$

where $t_d \sim \beta$.

**B. Four-point correlation function**

The out-of-time four-point correlation function of interest is

$$D_4(\beta, t) = \langle A(t)B(0)A(t)B(0)e^{-\beta H} \rangle = \sum_{n,m,l,r} e^{i(E_n - E_m + E_l - E_r)t - \beta E_n} <n|A(0)|m><m|B(0)|l><l|A(0)|r><r|B(0)|n>,$$

as well as the behaviour of the commutator $[A(t), B(0)]^2 e^{-\beta H}$. In order to compute the square of the commutator one should consider also the following four-point correlation functions

$$D'_4(\beta, t) = \langle A(t)B(0)B(0)A(t)e^{-\beta H} \rangle = \sum_{n,m,l,r} e^{i(-E_m + E_r)t - \beta E_n} <n|A(0)|m><m|B(0)|l><l|B(0)|r><r|A(0)|n>,$$

$$D''_4(\beta, t) = \langle B(0)A(t)A(t)B(0)e^{-\beta H} \rangle = \sum_{n,m,l,r} e^{i(E_m - E_r)t - \beta E_n} <n|B(0)|m><m|A(0)|l><l|A(0)|r><r|B(0)|n>,$$

$$D'''_4(\beta, t) = \langle B(0)A(t)B(0)A(t)e^{-\beta H} \rangle = \sum_{n,m,l,r} e^{i(-E_n + E_m - E_l + E_r)t - \beta E_n} <n|B(0)|m><m|A(0)|l><l|B(0)|r><r|A(0)|n>.$$
The corresponding energy eigenvalues one shall parametrise as \( n \sim \frac{1}{2} + iu \), \( m \sim \frac{1}{2} + iv \), \( l \sim \frac{1}{2} + il \) and \( r \sim \frac{1}{2} + ir \), thus

\[
D_4(\beta, t) = \int_{-\infty}^{+\infty} e^{i(u^2-v^2+l^2-r^2)t-\beta(\frac{1}{4}+u^2)} A_{uv} B_{vl} A_{lr} B_{ru} dudvdldr = 
\]

\[
= \int_{-\infty}^{+\infty} e^{i(u^2+v^2)t-\beta(\frac{1}{4}+u^2)} (AB)_{ul}(t) (AB)_{lu}(t) du dl ,
\]

where

\[
(AB)_{ul}(t) = \int_F A_{uv} e^{-iv^2t} B_{vl} dv.
\]

In our plane wave approximation (analogue of so called mini-superspace approximation in Liouville theory) the four-point correlation functions will take the form

\[
D_4^{(0)}(\beta, t) = \int_{-\infty}^{+\infty} e^{i(p^2-q^2+l^2-r^2)t-\beta(\frac{1}{4}+p^2)} A_{pq}^{(0)} A_{ql}^{(0)} A_{lr}^{(0)} A_{rp}^{(0)} dpdqdl dr
\]

(72)

On Fig. 4 one can see the behaviour of the four-point correlation function as a function of time.

It was speculated that the most important correlation functions indicating the traces of the classical chaotic dynamics in quantum regime is

\[
C(t, \beta) = - < [A(t), B(0)]^2 e^{-\beta H} >= 
\]

\[
= -D_4(\beta, t) + D_4^{(0)}(\beta, t) + D_4''^{(0)}(\beta, t) - D_4'''^{(0)}(\beta, t)
\]

(73)

and that the parts of the above commutation relation include the correlation functions which have the following behaviour:

\[
D_4(\beta, t) \sim 1 - f_0 e^{\frac{2\pi}{\beta} t} .
\]

(74)

These formulas and the explicit form of the wave functions allow to calculate the above correlation functions at least using numerical integration. In our plane wave approximation we calculated the commutator

\[
C^{(0)}(t, \beta) = - < [A(t), A(0)]^2 e^{-\beta H} >= 
\]

\[
= (-D_4^{(0)} + D_4^{(0)}(\beta, t) + D_4''^{(0)}(\beta, t) - D_4'''^{(0)}(\beta, t)) = 
\]

\[
- \int_{-\infty}^{+\infty} e^{-\beta(\frac{1}{4}+p^2)} \{ \cos(p^2 - q^2 + l^2 - r^2)t - \cos(q^2 - r^2)t \}
\]

\[
A_{pq}^{(0)} A_{ql}^{(0)} A_{lr}^{(0)} A_{rp}^{(0)} dpdqdl dr .
\]

(75)
The results of the numerical integration are presented on the Fig. 5. Thus by performing numerical integration we observed that a two-point correlation function decays exponentially and that a four-point function demonstrates tendency to decay with a lower pace. With the numerical data available to us we were able to observe a very short time exponential decay of the out-of-time correlation function at almost zero value and then an essential increase with the subsequent large fluctuations on Fig. 4. This confirms the existence of the scrambling time in this maximally chaotic system. With more numerical data it seems possible to estimate scrambling time more accurately. It is also true that with a more powerful computer in hand it will be possible to calculate the correlation functions (57), (58) and (70), (71) using exact wave function.

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