SOM E CURVATURE ESTIMATES
FOR R IEMANNIAN MANIFOLDS
EQUIPPED WITH FOLIATIONS OF RANK 2

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ABSTRACT. Some curvature estimates are derived from geometrical data concerning quasi-conformality properties of some commuting linearly independent vector fields on a compact Riemannian manifold.

1. Introduction. Calculating the rank, i.e. the maximal number of pointwise linearly independent commuting vector fields, of a manifold $M$ is a well known problem in differential topology. If rank $M = k$, then $M$ admits a locally free action of $\mathbb{R}^k$. Such actions (and foliations generated by them) are classified up to the either topological or differentiable type in several cases ([AC], [AS], [CRW], [RR], etc.). Also, for some pairs $(k, n)$, $n$-dimensional closed manifolds of rank $k$ are classified. For example, if $k = n - 1$, then $M$ is a $T^n$-bundle over a torus $T^{n-m}$ for some $m$ ([CR], [R], [RRW], etc.).

On the other hand, in several geometric situations (like infinitesimal isometries [Be] or infinitesimal conformal transformations on closed Riemannian manifolds of positive sectional curvature, see [W] and [Y], for instance) vector fields have to vanish at some points or have to be somewhere collinear whenever commute. The result presented here is of this sort: Assuming the existence of two commuting linearly independent vector fields $X$ and $Y$ on a Riemannian manifold $(M, g)$ we estimate the curvature of $M$ in terms of some geometric data (like the divergence, the norm of the Ahlfors operator) obtained from the $\mathbb{R}^2$-action generated by the flows of $X$ and $Y$. More precisely, our result reads as follows.

Theorem. For any $\epsilon > 0$ there exists a constant $K(\epsilon) > 0$ such that any closed $n$-dimensional $(n \geq 2)$ Riemannian manifold of sectional curvature $K_M > K(\epsilon)$ admits no $(C^1, \epsilon)$-quasi-conformal $\mathbb{R}^2$-actions.

Roughly speaking, a vector field $X$ is $C^1$-quasi-conformal if the Ahlfors operator $S$ on $(M, g)$ evaluated at $X$ has the $C^1$-norm small enough. An action $\phi$ of $\mathbb{R}^2$ is $(C^1, \epsilon)$-quasi-conformal if all the vector fields $Z_u$ ($u \in \mathbb{R}^2$) given by $Z_u(p) = (t \mapsto \phi(tu, p))(0)$ ($p \in M$) are $(C^1, \epsilon)$-quasi-conformal. For the precise definition of infinitesimal $(C^1, \epsilon)$-quasi-conformal transformations, see Section 3.

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The proof of the Theorem is given in Sections 2 - 5 below. In general, it follows the line of that in \([Y]\). Several identities in there which hold for infinitesimal conformal transformations are replaced by inequalities produced from the geometric data related to the vector fields under consideration. At some points, the argument differs essentially. For instance, instead of studying the eigenvalues and eigenvectors of some linear operator \(A\) we estimate the inner products of \(A(v)\) with some particular vector \(y\). Section 6 contains an application of the estimates derived before: We obtain an inequality which involves the Gaussian curvature of the orbits of a locally free \((C^1, \epsilon)\)-quasi-conformal action of \(\mathbb{R}^2\) and their second fundamental form.

Remark. Our Theorem together with the mentioned above results on the rank of closed manifolds justify to some extend the following Yang's remark \([Y]\): One might expect that any two commuting vector fields on a closed manifold of positive sectional curvature must be linearly dependent at some points, in other words, if \(M\) is closed and has positive sectional curvature, then \(\text{rank} \, M \leq 1\). However, the problem of calculating (or, estimating) the rank of all manifolds which admit positively curved Riemannian metrics is probably pretty difficult. First, there is a lot of examples of positively curved closed manifolds of different topological types (\([AW]\), \([E1]\), \([E2]\), \([KS]\), etc.) and still new examples arrive (see \([Ba]\) and \([T]\) for instance). Second, even if \(M\) is simply connected, the curvature of \(M\) is \(\delta\)-pinched with \(\delta\) close enough to 1 (for example, \(\delta = 0.681\)) and, therefore, \(M\) is diffeomorphic to the standard sphere \(S^n\) \([S]\), the rank of \(M\) is unknown in odd dimensions. Of course, \(\text{rank} \, S^n = 0\) when \(n\) is even. Moreover, \(\text{rank} \, S^3 = 1\) \([L]\) and there are no locally free actions of \(S^1 \times \mathbb{R}\) on any standard sphere \([ASc]\).

2. A preliminary lemma. Let \(X\) be an arbitrary vector field on a Riemannian manifold \((M, g), p_0 \in M\) and \(X(p_0) \neq 0\).

Lemma. For any vector \(v \in T_{p_0}M, v \neq 0\), orthogonal to \(X(p_0)\) there exists a vector field \(V\) in a neighbourhood \(U\) of \(p_0\) such that

\[
(1) \quad V(p_0) = v, \quad \langle V, X \rangle = 0, \quad [V, X] = \theta \cdot X \quad \text{and} \quad \nabla_v V = a \cdot X(p_0)
\]

for some \(\theta \in C^\infty(U)\) and \(a \in \mathbb{R}\).

Proof. In a neighbourhood \(U\) of \(p_0\), there exists a chart \(\phi = (x, y, z_1, \ldots, z_{n-2})\) \((n = \dim M)\) such that \(X = \frac{\partial}{\partial x}, v = \frac{\partial}{\partial y}(p_0)\) and \(\nabla_v \frac{\partial}{\partial y} = 0\). Find a function \(f \in C^\infty(U)\) such that \(V = \frac{\partial}{\partial y} + f \cdot \frac{\partial}{\partial x}\) is orthogonal to \(X\). \(V\) satisfies (1) with

\[
(2) \quad \theta = -\frac{\partial f}{\partial x} \quad \text{and} \quad a = \frac{\partial f}{\partial y}(p_0). \quad \Box
\]

3. Infinitesimal \((C^1, \epsilon)\)-quasi-conformal transformations. Given \(\epsilon \geq 0\), a vector field \(X\) on a Riemannian manifold \((M, g)\) will be called an infinitesimal \((C^1, \epsilon)\)-quasi-conformal transformation (an infinitesimal \((C^1, \epsilon)\)-QCT, for short) if

\[
(3) \quad \|SX(p)\| \leq \epsilon \|X(p)\| \quad \text{and} \quad \|\nabla SX(p)\| \leq \epsilon^2 \|X(p)\|
\]

at any point \(p\) of \(M\).
Here $S$ is the Ahlfors operator on $M$. Recall that $SX$ is a traceless symmetric 2-form on $M$ defined by
\begin{equation}
SX = L_X g - f_X \cdot g, \tag{4}
\end{equation}
where $L_X$ denotes the Lie derivation with respect to $X$, $f_X = \frac{2}{n} \text{div} X$ and $n = \dim M$. Equivalently, $SX$ can be defined by
\begin{equation}
SX(V, W) = \langle \nabla V X, W \rangle + \langle V, \nabla W X \rangle - f_X \cdot \langle V, W \rangle, \tag{5}
\end{equation}
where $V$ and $W$ are arbitrary vector fields and $\nabla$ is the Levi-Civita connection on $(M, g)$. Recall also that $\|SX\|$ describes the rate of quasi-conformality of the flow maps $(\phi_t)$ of $X$, namely, any map $\phi_t$ is $k_t$-quasi-conformal with $k_t \leq \exp(|t| \cdot \|SX\|)$, $\|\cdot\|$ being the supremum norm here $[P]$.

**Remark.** Observe that our notion of quasi-conformality differs from the standard one. Usually, a vector field $X$ is said to be $k$-quasi-conformal just when $\|SX\| \leq k$. Also, observe that if the $C^1$-norm of $\nabla X$ is small,
\begin{equation}
\|\nabla X\| \leq \delta \cdot \|X\| \quad \text{and} \quad \|\nabla^2 X\| \leq \delta^2 \cdot \|X\|, \tag{6}
\end{equation}
then $X$ is $(C^1, \epsilon)$-quasi-conformal with $\epsilon = 4\delta$. Finally, note that if $X$ is an infinitesimal $(C^1, \epsilon)$-QCT then, for any $c \neq 0$, $cX$ is an infinitesimal $(C^1, \epsilon)$-QCT as well, and if $X$ is an infinitesimal $(C^1, \epsilon)$-QCT on $(M, g)$ and $g' = c^2 g$ for some $c \in \mathbb{R}_+$, then $X$ is an infinitesimal $(C^1, \epsilon')$-QCT on $(M, g')$ with $\epsilon' = c^{-1}\epsilon$.

**4. First estimates.** Assume now that $X$ and $Y$ are pointwise linearly independent commuting vector fields on a compact Riemannian manifold $M$ such that $u_1 X + u_2 Y$ is an infinitesimal $(C^1, \epsilon)$-QCT for any $u_1$ and $u_2 \in \mathbb{R}$.

For any $p \in M$ and $t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ put
\begin{equation}
Z(p, t) = \cos t \cdot X(p) + \sin t \cdot Y(p). \tag{7}
\end{equation}

The function $h : M \times S^1 \to \mathbb{R}$ given by $h(p, t) = \|Z(t)(p)\|^2$ is continuous, therefore it attains its minimum at a point $(p_0, t_0)$ of $M \times S^1$. Since all the fields $Z(\cdot, t)$ are $(C^1, \epsilon)$-quasi-conformal, we may assume without losing generality that that $t_0 = 0$.

Let $x = X(p_0)$, $y = Y(p_0)$ and $v \in T_{p_0} M$ be an unit vector. The standard analysis of partial derivatives of $h$ at $p_0$ shows that
\begin{equation}
v\langle X, X \rangle = 0, \quad \langle x, y \rangle = 0, \tag{8}
\end{equation}
\begin{equation}
V^2 \langle X, X \rangle(p_0) \geq 0, \tag{9}
\end{equation}
\begin{equation}
\|y\| \geq \|x\| \tag{10}
\end{equation}
and
\begin{equation}
d = \left| \frac{V^2 \langle X, X \rangle(p_0)}{v \langle X, Y \rangle} \frac{v \langle X, Y \rangle}{\|y\|^2 - \|x\|^2} \right| \geq 0, \tag{11}
\end{equation}
where $V$ is the Levi-Civita connection.
where $V$ is an arbitrary vector field on a neighbourhood of $p_0$, $V(p_0) = v$.

Now, take any unit vector $v \perp x$ and extend it to a vector field $V$ satisfying the conditions of Lemma in Section 2. Then, at the point $p_0$ as above,

(12) $SX(x, x) = -f_X(p_0) \cdot \|x\|^2$, therefore $|f_X(p_0)| \leq \epsilon \|x\|$, 

(13) $SX(x, v) = -x\langle X, V \rangle - \langle X, [X, V] \rangle = \theta(p_0) \|x\|^2$, therefore $|\theta(p_0)| \leq \epsilon$

and

(14) $SX(v, v) = -2\langle x, \nabla_v V \rangle - f_X(p_0) = -2a\|x\|^2 - f_X(p_0)$, therefore $a\|x\| \leq \epsilon$.

Moreover, $\langle \nabla_x X, x \rangle = 0$ and $\langle \nabla_x X, v \rangle = -\langle x, \nabla_x V \rangle = -\langle x, \nabla_x X \rangle - \langle x, [X, V](p_0) \rangle = \theta(p_0) \cdot \|x\|^2$. From (13) it follows that

(15) $\|\nabla_x X\| \leq \epsilon \|x\|^2$.

Also, since $SX(v, v) = x\langle V, V \rangle - f_X(p_0)$, we get from (3) and (12) that

(16) $|X\langle V, V \rangle(p_0)| \leq 2\epsilon \|x\|$.

Passing to the covariant derivative $\nabla SX$ we observe that

(17) $(\nabla_x SX)(v, v) = X^2 \langle V, V \rangle(p_0) - Xf_X(p_0) - f_X(p_0)X\langle V, V \rangle(p_0)$

$- 2SX(\nabla_x V, v)$,

(18) $(\nabla_v SX)(x, v) = v\theta \cdot \|x\|^2 - SX(\nabla_x V, v) - \theta(p_0)^2 \|x\|^2 - a \cdot SX(x, x)$,

and

(19) $(\nabla_x SX)(x, x) = X^2 \langle X, X \rangle(p_0) - Xf_X(p_0) \cdot \|x\|^2 - 2SX(\nabla_x X, x)$.

Combining identities (17) - (19) and using inequalities (3) and (12) - (16) we arrive at

(20) $|X^2 \langle V, V \rangle(p_0) - 2v\theta \cdot \|x\|^2 - \|x\|^{-2}X^2 \langle X, X \rangle(p_0)| \leq 12\epsilon^2 \|x\|^2$.

All the inequalities above will be applied in the next Sections to get some curvature estimates.

5. Estimates involving sectional curvature. Let us keep all the notation introduced in Section 4 and assume that the sectional curvature of $M$ is positive. (Otherwise, the statement of the Theorem holds trivially.) Also, denote by $A$ the linear transformation of $T_{p_0} M$ given by

(21) $Aw = \nabla_w X$

and choose a vector $v$ in such a way that

(22) $Av = bv$ for some $b \in \mathbb{R}$.
(Such a vector exists. Since \( A : \{x\}^\perp \rightarrow \{x\}^\perp \), either \( \ker A|\{x\}^\perp \) is nontrivial and \( Av = 0 \) for some \( v \perp x \) or \( A|\{x\}^\perp \) is an isomorphism.)

As before, extend \( v \) to a vector field \( V \) defined in a neighbourhood of \( p_0 \) as in our Lemma. From the definition of the curvature tensor \( R \) on \((M, g)\) we obtain easily the identity

\[
V^2\langle X, X\rangle(p_0) + X^2\langle V, V\rangle(p_0) = -2K_M(x \wedge v)\|x\|^2 + 2\|Av\|^2
\]

\[+ 2v\theta \cdot \|x\|^2 - 2\theta(p_0)\langle Ax, v\rangle,\]

where

\[
K_M(x \wedge v) = \|x\|^{-2}\langle R(v, x)x, v\rangle
\]
is the sectional curvature of \( M \) in the direction of the plane spanned by \( x \) and \( v \).

Since both \( X \) and \( Y \) are infinitesimal \((C^1, \epsilon)\)-QCTs,

\[
|\langle \nabla_x Y, v \rangle + \langle x, \nabla_y Y \rangle| \leq \epsilon\|x\| \cdot \|y\|
\]

and

\[
|\langle \nabla_y X, v \rangle + \langle y, \nabla_y X \rangle| \leq 2\epsilon\|x\| \cdot \|y\|.
\]

Since \( X \) and \( Y \) commute, \( \nabla_x Y = \nabla_y X \) and (25) together with (26) imply that

\[
|v\langle X, Y \rangle - 2b\|y\|^2| \leq 3\epsilon\|x\| \cdot \|y\|.
\]

Consequently,

\[
-(v\langle X, Y \rangle)^2 \leq -4b^2\|y\|^4 + 12\epsilon b\|x\| \cdot \|y\|^3 + 9\epsilon^2\|x\|^2\|y\|^2.
\]

The determinant \( d \) in (11) satisfies

\[
d = (\|y\|^2 - \|x\|^2)(-2K(x \wedge z)\|x\|^2 - X^2\langle V, V\rangle(p_0) + 2\|Av\|^2
\]

\[+ 2\theta(p_0)\langle Ax, v\rangle + 2v\theta \cdot \|x\|^2) - (v\langle X, Y \rangle)^2.
\]

Since the sectional curvature of \( M \) is positive, combining inequalities (9), (13), (15), (20) and (28), and using identity (22) we obtain the inequality

\[
0 \leq d \leq (\|y\|^2 - \|x\|^2)(14\epsilon^2\|x\|^2 + 2b^2\|y\|^2) - b^2\|y\|^4
\]

\[+ 12\epsilon b\|x\| \cdot \|y\|^3 + 9\epsilon^2\|x\|^2\|y\|^2.
\]

Substituting \( s = b\|x\|^{-1}\|y\| \), dividing both sides of (30) by \( \|y\|^2 \) and performing some other elementary transformations one can see easily that (30) implies

\[
-2s^2 + 12\epsilon s + 23\epsilon^2 \geq 0.
\]

Therefore,

\[
s \leq (3 + \frac{1}{2}\sqrt{82})\epsilon\|x\|
\]

and

\[
b \leq (3 + \frac{1}{2}\sqrt{82})\epsilon\|x\| \cdot \|y\|^{-1}.
\]

Applying formula (22) once again we can see that

\[
K(x \wedge v) = \|x\|^{-2}(\frac{1}{2}V^2\langle X, X\rangle(p_0) - \frac{1}{2}X^2\langle V, V\rangle(p_0) + \|Av\|^2
\]

\[+ v\theta - \theta(p_0)\langle Ax, v\rangle) \leq \|x\|^{-2}(6\epsilon^2\|x\|^2 + b^2\|y\|^2)
\]

\[\leq \epsilon^2(36.5 + 3\sqrt{82}) \approx 63.66615541 \ldots \epsilon^2.
\]

Therefore, the statement of the Theorem holds with

\[
K(\epsilon) = \epsilon^2(36.5 + 3\sqrt{82}). \quad \square
\]
Corollary [Y]. Any two commuting conformal vector fields on a closed Riemannian manifold of positive sectional curvature must be linearly dependent at least at one point.

6. Estimates involving Gaussian curvature of the leaves. Let \( \mathcal{F} \) be a foliation of a compact positively curved manifold \( M \) defined by a locally free action of \( \mathbb{R}^2 \). Then \( \dim \mathcal{F} = 2 \) and the tangent bundle of \( \mathcal{F} \) is spanned by two commuting linearly independent vector fields \( X \) and \( Y \). These fields as well as all the linear combinations \( u_1X + u_2Y, \ u_i \in \mathbb{R} \), become infinitesimal \((C^1, \epsilon)\)-QCTs for some \( \epsilon \geq 0 \). Therefore, with this \( \epsilon \), \( \mathcal{F} \) is defined by a \((C^1, \epsilon)\)-quasi-conformal action of \( \mathbb{R}^2 \).

Consider fields \( X \) and \( Y \), and find \( p_0 \in M \) as before. Let \( v \) be a unit vector tangent to \( \mathcal{F} \) and orthogonal to \( x = X(p_0) \), say \( v = y/\|y\| \). Then

\[
Av = by + B(x, v),
\]

where \( B \) is the second fundamental form of \( \mathcal{F} \). The argument analogous to that of Section 5 shows that

\[
b \leq \|x\| \cdot \|y\|^{-1} \left( (3 + \frac{1}{2} \sqrt{82}) \epsilon + \|B(p_0)\| \right),
\]

where \( x_1 = x/\|x\| \). Applying formula (22) as in Section 5 we can get the estimate

\[
K_M(x \wedge v) \leq \epsilon^2 (36.5 + 3\sqrt{82}) + \epsilon \sqrt{2} (3 + 0.5\sqrt{82}) \|B(p_0)\| + \frac{3}{2}\|B(p_0)\|^2.
\]

Finally, the classical Gauss equation reads

\[
K_\mathcal{F}(p_0) = K_M(x \wedge v) + \langle B(x_1, x_1), B(v, v) \rangle - \|B(x_1, v)\|^2,
\]

where \( K_\mathcal{F} \) is the Gaussian curvature of the leaves of \( \mathcal{F} \) and \( x_1 = x/\|x\| \). Therefore, we can conclude by the following.

**Proposition.** The Gaussian curvature \( K_\mathcal{F} \) of a foliation \( \mathcal{F} \) defined by a locally free \((C^1, \epsilon)\)-quasi-conformal action of \( \mathbb{R}^2 \) on a closed positively curved Riemannian manifold \( M \) satisfies at some point \( p_0 \) of \( M \) the inequality

\[
K_\mathcal{F}(p_0) \leq \epsilon^2 (36.5 + 3\sqrt{82}) + (6 + \sqrt{82}) \|B(p_0)\| + \|B(p_0)\|^2
\]

with \( B \) being the second fundamental form of \( \mathcal{F} \). \( \square \)

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