THE ELGAMAL CRYPTOSYSTEM OVER CIRCULANT MATRICES

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ABSTRACT. In this paper we study extensively the discrete logarithm problem in the group of non-singular circulant matrices. The emphasis of this study was to find the exact parameters for the group of circulant matrices for a secure implementation. We tabulate these parameters. We also compare the discrete logarithm problem in the group of circulant matrices with the discrete logarithm problem in finite fields and with the discrete logarithm problem in the group of rational points of an elliptic curve.

1. INTRODUCTION

Two of the most popular groups used in the discrete logarithm problem are the group of units of a finite field and the group of rational points of an elliptic curve over a finite field. The obvious question arises, are there any other groups? I write this paper to show, that there are matrix groups – the group of non-singular circulant matrices, which is much better than the finite fields in every aspect and even better than the elliptic curves when one considers the size of the field for a secure implementation. The size of the field for a secure implementation is a huge issue in public key cryptography. One of the reasons, elliptic curves are preferred over a finite field discrete logarithm problem, is the size of the field for a secure implementation. In our current state of knowledge, it is believed that the discrete logarithm problem over \( \mathbb{F}_{2^{1028}} \) offers the same security that of most elliptic curves over \( \mathbb{F}_{2^{160}} \). As our processors get faster and with the advent of distributed computing these sizes will grow bigger with time. In the case of an elliptic curve the rate of growth is much smaller than that of finite fields. We will see, for circulant matrices the size of the field for a secure implementation can get even smaller. The comparison of speed, between circulants and elliptic curves, in an actual implementation is yet to be done. But, since the circulants use smaller field, it is likely that the circulants are faster.

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It is known [6,10] that the group of circulant matrices offers the same security of a finite field of about same size, with half the computational cost. The other interesting fact about circulant matrices is the size of the field for a secure implementation. The arithmetic of the circulant matrices is implemented over a finite field, very similar to the case of elliptic curves, where the arithmetic is also implemented over a finite field. In the case of circulants, the size of the field can be smaller than the one used for elliptic curves. This is extensively studied in Section 5, and the results are tabulated in Table 2. To sum it up, the advantage of circulants is that it uses smaller field and is faster.

In this paper, we denote the group of non-singular circulant matrices of size \( d \) by \( C(d, q) \) and the group of special circulant matrices, i.e., circulant matrices with determinant 1, by \( SC(d, q) \) respectively.

**Definition 1** (Circulant matrix \( C(d, q) \)). A \( d \times d \) matrix over a field \( F \) is called a circulant matrix, if every row except the first row, is a right circular shift of the row above that. So a circulant matrix is defined by its first row. One can define a circulant matrix similarly using columns.

A matrix is a two dimensional object, but a circulant matrix behaves like a one dimensional object – given by the first row or the first column. We will denote a circulant matrix \( C \) of size \( d \), with the first row \( c_0, c_1, \ldots, c_{d-1} \), by \( C = \text{circ} (c_0, c_1, c_2, \ldots, c_{d-1}) \). An example of a circulant 5 \( \times \) 5 matrix is:

\[
\begin{pmatrix}
c_0 & c_1 & c_2 & c_3 & c_4 \\
c_4 & c_0 & c_1 & c_2 & c_3 \\
c_3 & c_4 & c_0 & c_1 & c_2 \\
c_2 & c_3 & c_4 & c_0 & c_1 \\
c_1 & c_2 & c_3 & c_4 & c_0 \\
\end{pmatrix}
\]

One can define a representer polynomial corresponding to the circulant matrix \( C \) as \( \phi_C = c_0 + c_1 x + c_2 x^2 + \ldots + c_{d-1} x^{d-1} \). The circulants form a commutative ring under matrix multiplication and matrix addition and is isomorphic to (the isomorphism being circulant matrix to the representer polynomial) \( \mathcal{R} = \frac{F[x]}{x^d - 1} \). For more on circulant matrices, see [2].

We will study the discrete logarithm problem in \( SC(d, q) \), the special circulant matrix. It is fairly straightforward to see that one can develop a Diffie-Hellman key exchange protocol or the ElGamal cryptosystem from this discrete logarithm problem. The ElGamal cryptosystem over \( SL(d, q) \), the special linear group of size \( d \) over \( \mathbb{F}_q \) is described below. Since the special circulant matrix is contained in the special linear group, this description of the ElGamal cryptosystem works for \( SC(d, q) \) as well.

All fields considered in this paper are finite and of characteristic 2.
2. The ElGamal over SL\((d, q)\)

**Private Key:** \(m, m \in \mathbb{N}\).

**Public Key:** \(A\) and \(A^m\). Where \(A \in \text{SL}(d, q)\).

**Encryption.**

\(a:\) To send a message (plaintext) \(v \in \mathbb{F}_q^d\), Bob computes \(A^r\) and \(A^{mr}\) for an arbitrary \(r \in \mathbb{N}\).

\(b:\) The ciphertext is \((A^r, A^{mr}v^T)\). Where \(v^T\) is the transpose of \(v\).

**Decryption.**

\(a:\) Alice knows \(m\), when she receives the ciphertext \((A^r, A^{mr}v^T)\), she computes \(A^{mr}\) from \(A^r\), then \(A^{-mr}\) and then computes \(v\) from \(A^{mr}v^T\).

We show that the security of the ElGamal cryptosystem over \(\text{SL}(d, q)\), is equivalent to the Diffie-Hellman problem in \(\text{SL}(d, q)\). Since \(\text{SC}(d, q)\) is contained in \(\text{SL}(d, q)\), this proves that the security of ElGamal cryptosystem is equivalent to the Diffie-Hellman problem in \(\text{SC}(d, q)\).

Assume that Eve can solve the Diffie-Hellman problem, then from the public information, she knows \(A^m\). From a ciphertext \((A^r, A^{rm}v^T)\) she gets \(A^r\). Since she can solve the Diffie-Hellman problem, she computes \(A^{rm}\) and can decrypt the ciphertext. The converse follows from the following theorem, which is an adaptation of [4, Proposition 2.10]

**Theorem 1.** Suppose Eve has access to an oracle that can decrypt arbitrary ciphertext of the above cryptosystem for any private key, then she can solve the Diffie-Hellman problem in \(\text{SL}(d, q)\).

**Proof.** Let \(g = A^a\) and \(h = A^b\). Eve takes an arbitrary element \(v\) in the vector space of dimension \(d\) on which \(\text{SL}(d, q)\) acts. We use the same basis used for the representation of \(\text{SL}(d, q)\). Then \(v = (v_1, v_2, \ldots, v_d)\) where \(v_i \in \mathbb{F}_q^{\times}\). Let \(\hat{v}_i = (0, \ldots, v_i, \ldots, 0)\) and \(c = \hat{v}_i^T\). She pretends that \(A\) and \(A^a\) is a public key. Sends that information to the oracle. Then asks the oracle to decrypt \((h, c)\). Oracle sends back to Eve, \(h^{-a}c\). Eve knowing \(v\), computes the \(i^{th}\) column of \(A^{-ab}\) from \(h^{-a}c\). In \(d\) tries \(A^{ab}\) is found. This solves the Diffie-Hellman problem. \(\square\)

3. Security of the Proposed ElGamal Cryptosystem

This paper is primarily focused on the discrete logarithm problem in the automorphism group of a vector space over a finite field. There are two kinds of attack on the discrete logarithm problem.

(i) The “so called” generic attacks, like the Pollard’s rho algorithm. These attacks use a black box group algorithm. The time complexity
of these algorithms is about the same as the square-root of the size of the group.

(ii) The other one is an *index calculus* attack. These attacks do not work in any group.

Black box group algorithms work in any group, hence they will work in $\text{SC}(d, q)$ as well. The most efficient way to use black box attack on the discrete logarithm problem, is to use the Pohlig-Hellman algorithm \cite{4} Section 2.9 first. This reduces the discrete logarithm problem to the prime divisors of the order of the element (the base for the discrete logarithm) and then use the Chinese remainder theorem to construct a solution for the original discrete logarithm problem. One can use the Pollard’s rho algorithm to solve the discrete logarithm problem in the prime divisors. So the whole process can be summarized as follows: the security of the discrete logarithm against generic attacks, is the security of the discrete logarithm in the largest prime divisor of the order. We cannot prevent these attacks. These generic attacks are of exponential time complexity and are not of much concern.

The biggest threat to any cryptosystem using the discrete logarithm problem is a subexponential attack like the index calculus attack \cite{8}. It is often argued \cite{5,9} that there is no index calculus algorithm for most elliptic curve cryptosystems that has subexponential time complexity. This fact is often used to promote elliptic curve cryptosystem over a finite field cryptosystem \cite{5}. So, the best we can hope from the discrete logarithm problem in $\text{SC}(d, q)$ is, there is no index calculus attack or the index calculus attack becomes exponential.

The expected asymptotic complexity of the index calculus algorithm in $\mathbb{F}_{q^k}$ is $\exp\left( (c + o(1)) (\log q^k)^{\frac{1}{2}} (\log \log q^k)^{\frac{1}{2}} \right)$, where $c$ is a constant, see \cite{8} and \cite{5} Section 4. If the degree of the extension, $k$, is greater than $\log^2 q$ then the asymptotic time complexity of the index calculus algorithm becomes exponential. In our case this means, if $d > \log^2 q$, the asymptotic complexity of the index calculus algorithm on circulant matrices of size $d$ becomes exponential.

If we choose $d \geq \log^2 q$, then the discrete logarithm problem in $\text{SC}(d, q)$ becomes as secure as the ElGamal over an elliptic curve, because the index calculus algorithm is exponential; otherwise we can not guarantee. But on the other hand, in the proposed cryptosystem, encryption and decryption works in $\mathbb{F}_q$ and breaking the cryptosystem depends on solving a discrete logarithm problem in $\mathbb{F}_{q^{d-1}}$. Since, implementing the index calculus attack becomes harder as the field gets bigger. It is clear that if we take $d \ll \log^2 q$, then the cryptosystem is much more secure than the ElGamal cryptosystem over $\mathbb{F}_q$. 
4. Is the ElGamal Cryptosystem over $SC(d, q)$ Really Useful?

For a circulant matrix over a field of even characteristic, squaring is fast. It is shown [6, Theorem 2.2] that, if $A = \text{circ}(a_0, a_1, \ldots, a_{d-1})$, then $A^2 = \text{circ}(a^2_{\pi(0)}, a^2_{\pi(1)}, \ldots, a^2_{\pi(d-1)})$. Where $\pi$ is a permutation of $\{0, 1, 2, \ldots, d-1\}$. Now the $a_i$s belong to the underlying field $\mathbb{F}_q$ of characteristic 2. In this field, squaring is just a cyclic shift using a normal basis [7, Chapter 4] representation of the field elements.

It was shown by Mahalanobis [6], that if five conditions are satisfied, then the security of the discrete logarithm problem for circulant matrices of size $d$ over $\mathbb{F}_q$ is the same as the discrete logarithm problem in $\mathbb{F}_{q^{d-1}}$.

The five conditions are:

a. The circulant matrix should have determinant 1.
b. The matrix $A$ should have row-sum 1.
c. The integer $d$ is prime.
d. The polynomial $\chi_A x - 1$ is irreducible.
e. $q$ is primitive mod $d$.

In short, the argument for these five conditions are the following:

Let $A = \text{circ}(a_0, a_1, \ldots, a_{d-1})$ and let $\chi_A$ be the characteristic polynomial of $A$. It is easy to see that the row-sum, $a_0 + a_1 + \cdots + a_{d-1}$, sum of all elements in a row, is constant for a circulant matrix. This row-sum, $\alpha$ is an eigenvalue of $A$ and belongs to $\mathbb{F}_q$. Clearly, $\alpha^m$ is an eigenvalue of $A^m$. This $\alpha$ and $\alpha^m$ can reduce a part of the discrete logarithm problem in $A$, to a discrete logarithm problem in the field $\mathbb{F}_q$. If the row-sum is 1, then there is no such issue. This is the reason behind the condition, the row-sum is 1.

Now assume that $\frac{\chi_A}{x - 1} = f_1^e_1 f_2^e_2 \cdots f_n^e_n$, where each $f_i$ is an irreducible polynomial and $e_i$s are positive integers. Then it follows, the discrete logarithm problem in $A$, can be reduced to discrete logarithm problems in $\frac{\mathbb{F}_q[x]}{f_i}$, for each $i$. Then one can solve the individual discrete logarithms in extensions of $\mathbb{F}_q$, put those solutions together using the Chinese remainder theorem and solve the discrete logarithm problem in $A$. The degree of these extensions, the size of which provides us with the better security, is maximized when $\frac{\chi_A}{x - 1}$ is irreducible. This is the reason for $\frac{\chi_A}{x - 1}$ is irreducible.

The ring of circulant matrices is isomorphic to $\frac{\mathbb{F}_q[x]}{x^{d-1}}$, moreover $\frac{\mathbb{F}_q[x]}{x^d - 1}$ is isomorphic to $\frac{\mathbb{F}_q[x]}{x - 1} \times \frac{\mathbb{F}_q[x]}{\Phi(x)}$, where $\Phi(x) = \frac{x^d - 1}{x - 1}$ is the $d$th cyclotomic

\footnote{Condition e. ensures that $e_i = 1$ for all $i$.}
polynomial. If $d$ is prime and $q$ is primitive modulo $d$, then the cyclotomic polynomial $\Phi(x)$ is irreducible. In this case, the discrete logarithm problem in circulant matrices reduce to the discrete logarithm problem in $\mathbb{F}_{q^{d-1}}$.

4.1. **What are the advantages of using circulant matrices?** The advantages of using circulant matrices are:

- Multiplying circulant matrices of size $d$ over $\mathbb{F}_q$ is twice as fast compared to multiplication in the field of size $\mathbb{F}_{q^d}$.
- Computing the inverse of a circulant matrix is easy.

Since any circulant matrix $A$ can be represented as a polynomial of the form 
$f(x) = c_0 + c_1 x + \ldots + c_{d-1} x^{d-1}$. This polynomial is invertible, implies that, $\gcd(f(x), x^d - 1) = 1$. Then one can use the extended Euclid’s algorithm to find the inverse. In our cryptosystem, we need to find that inverse, and it is easily computable.

We now compare the following three cryptosystems for security and speed. We do not compare the key sizes and the size of the ciphertext, as these can be decided easily.

1. The ElGamal cryptosystem using the circulant matrices of size $d$ over $\mathbb{F}_q$.
2. The ElGamal cryptosystem using the group of an elliptic curve.
3. The ElGamal cryptosystem over $\mathbb{F}_{q^d}$.

4.2. **ElGamal over $\mathbb{F}_{q^d}$ vs. the circulants of size $d$ over $\mathbb{F}_q$.** Clearly the circulants are the winner in this case. The circulants provide almost the same security as the ElGamal over the finite field $\mathbb{F}_{q^d}$, but multiplication in the circulants is twice as fast compared to the multiplication in the finite field $\mathbb{F}_{q^d}$. See Silverman [10][11] for more details.

To understand the difference, we need to understand the standard field multiplication. A field $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$, an extension of degree $d$, is a commutative algebra of dimension $d$ over $\mathbb{F}_q$. Let $\alpha_0, \alpha_1, \ldots, \alpha_{d-1}$ be a basis of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$. Let $A := (a_0\alpha_0 + a_1\alpha_1 + \ldots + a_{d-1}\alpha_{d-1}), B := (b_0\alpha_0 + b_1\alpha_1 + \ldots + b_{d-1}\alpha_{d-1})$ and

$$C := A \cdot B = (c_0\alpha_0 + c_1\alpha_1 + \ldots + c_{d-1}\alpha_{d-1})$$

be elements of $\mathbb{F}_{q^d}$.

The objective of multiplication is to find $c_k$ for $k = 0, 1, \ldots, (d - 1)$. Now notice that, if

$$\alpha_i \alpha_j = \sum_{k=0}^{d-1} t^k_{ij} \alpha_k,$$

we can define a $d \times d$ matrix $T_k$ as $(t^k_{ij})_{ij}$. It follows that $c_k = AT_kB^t$. The number of nonzero entries in the matrix $T_k$, which is constant over
$k$, is called the complexity of the field multiplication [7, Chapter 5]. The following theorem is well known [7, Theorem 5.1]:

**Theorem 2.** For any normal basis $N$ of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$, the complexity of multiplication is at least $2d - 1$.

Note that in an implementation of a field exponentiation, one must use a normal basis to use the square and multiply algorithm.

In our case, circulants of size $d$ over a finite field $\mathbb{F}_q$, the situation is much different. We need a normal basis implementation for $\mathbb{F}_q$. However, to implement multiplication of two circulants, i.e., multiplication in $\mathbb{R} = \mathbb{F}_q[x]/(x^d - 1)$ we can use the basis $\{1, x, x^2, \ldots, x^{d-1}\}$.

In a very similar way as before, if $A := a_0 + a_1 x + \ldots + a_{d-1} x^{d-1}$ and $B := b_0 + b_1 x + \ldots + b_{d-1} x^{d-1}$ then $C := A \cdot B = c_0 + c_1 x + \ldots + c_{d-1} x^{d-1}$. Our job is to compute $c_k$ for $k = 0, 1, \ldots, d - 1$. It follows that

$$c_k = \sum_{i=0}^{d-1} a_i b_j \text{ where } i + j = k \mod d \text{ and } 0 \leq i, j \leq d - 1$$

It is now clear that the complexity of the multiplication is $d$. Compare this to the best case situation for the optimal normal basis [7, Chapter 5], in which case it is $2d - 1$. So multiplying circulants take about half the time that of finite fields.

It is clear that the keysizes will be the same for both these cryptosystems.

### 4.3. The elliptic curve ElGamal vs. the circulants of size $d$.

In this case there is no clear winner. On one hand, take the case of embedding degree. For most elliptic curves the embedding degree is very large. The embedding degree, that we refer to as the security advantage, for a circulant is tied up with the size of the matrix. For a matrix of size $d$, it is $d - 1$. So with circulants, it is hard to get very large embedding degree, without blowing up the size of the matrix. On the other hand, a very large embedding degree is not always necessary.

On the other hand, in elliptic curves, the order of the group is about the same as the size of the field. For 80-bit security, we must take the field to be around $2^{160}$, to defend against any square-root algorithms. In the case of circulants, the order of a circulant matrix can be large. This enables us to use smaller field for the same security. In circulants, one can use the extended Euclid’s algorithm to compute the inverse.

So, as we said before, we are not in a position to declare a clear winner in this case. However, if the size of the field is important in the implementation, and a moderate embedding degree suffices for security, then circulants
are a little ahead in the game. We explain this by some examples in the next section.

It is clear that the keysize for circulant matrices will be larger than that of the elliptic curve cryptosystem, both satisfying the following:

1: Security of 80 bits or more from generic algorithms.
2: Security from index-calculus comparable to the field $\mathbb{F}_{2^{1000}}$, i.e., index calculus security of 1000 bits.

5. An Algorithm

Recall that $C(d, q)$ is isomorphic to $\frac{\mathbb{F}_q[x]}{x - 1} \times \frac{\mathbb{F}_q[x]}{\Phi(x)}$. We now describe an algorithm to find a circulant matrix satisfying the above five conditions.

**Algorithm 1** (Construct a circulant matrix satisfying five conditions).

**Input** $q, d$.

- **construct** $\mathbb{F}_q$.
- $\tau(x) \leftarrow$ A primitive polynomial of degree $d - 1$ over $\mathbb{F}_q$.
- **order** $\leftarrow$ Order of the determinant of the companion matrix of $\tau(x)$.
- **Use Chinese remainder theorem to find** $\psi(x)$ **such that** $\psi(x) = 1 \mod (x - 1)$ **and** $\psi(x) = \tau(x) \mod \Phi(x)$.
- $\psi(x) \leftarrow \psi(x) \mod (x^d - 1)$.
- $A \leftarrow$ The circulant matrix with the first row $\psi(x)$.
- $A \leftarrow A_{\text{order}}$.

**Output** $A$.

Using Magma [1] and Algorithm [1], we were able to compute several circulant matrices over many different fields of characteristic 2. We produce part of that data in Table [1]. The row with $q$ is the size of the field extension and the row with $d$ is the size of the circulant matrix over that field extension.

To construct the table, we considered all possible field extensions of size $q$, where $q$ varies from $2^{40}$ to $2^{100}$. For each such extension, we took all the primes, $d$, from 11 to 50. We then checked and tabulated the ones for which $q$ is primitive modulo $d$. For every extension $q$ and for all primes $d$, satisfying the primitivity condition, Algorithm [1] was used and the output matrix was checked for all the five conditions and moreover the order of the matrix $A$ was found to be at least $q^{d-3}$. So, if $q$ is primitive modulo $d$, our algorithm produces the desired matrix $A$, satisfying all five conditions. The computation was fast on a standard workstation.
| $q$ | $2^{41}$ | $2^{43}$ | $2^{47}$ | $2^{49}$ | $2^{53}$ | $2^{55}$ |
|-----|----------|----------|----------|----------|----------|----------|
| $d$  | 11, 13, 19, 29, 37 | 11, 13, 19, 29, 37 | 11, 13, 19, 37 | 11, 13, 19, 29, 37 | 13, 19, 29, 37 | 13, 19, 29, 37 |
| $q$ | $2^{53}$ | $2^{61}$ | $2^{65}$ | $2^{67}$ | $2^{71}$ | $2^{73}$ |
| $d$  | 11, 13, 19, 29, 37 | 11, 13, 19, 29, 37 | 13, 19, 29, 37 | 11, 13, 19, 29, 37 | 11, 13, 19, 29, 37 | 11, 13, 19, 29, 37 |
| $q$ | $2^{77}$ | $2^{79}$ | $2^{83}$ | $2^{85}$ | $2^{89}$ | $2^{95}$ |
| $d$  | 11, 13, 19, 37 | 11, 13, 19, 37 | 11, 13, 19, 37 | 11, 13, 19, 37 | 13, 19, 29, 37 | 37 |

**Table 1.** Fields from size $2^{40}$ to $2^{100}$ and matrices from size 11 to 50 that satisfy those five conditions.

So now it is clear, that there are a lot of choices for parameters for the ElGamal cryptosystem over circulant matrices. We describe our findings with some arbitrary examples. For more data see Table 2.

In the case, $q = 2^{89}, d = 13$, we found the largest prime factor of the order of $A$ to be

$$7993364465170792998716337691033251350895453313.$$  

The base two logarithm of this prime is 152.5. So even if we use the Pohlig-Hellman algorithm to reduce the discrete logarithm in $A$, to the discrete logarithm problem in the prime factors of the order of $A$, we still have the security very close to the 80-bit security from generic attacks. The security against the index calculus is the same as in $\mathbb{F}_{2^{1068}}$.

In case of $q = 2^{39}, d = 29$, the largest prime factor of $A$ was

$$3194753987813988499397428643895659569.$$  

The logarithm base 2 of which is about 120. So from generic attack, the security is about $2^{60}$ or sixty bit security. From index calculus the security is the same as the security of a field of size $\mathbb{F}_{2^{1092}}$.

In the case of $q = 2^{45}, d = 29$, the largest prime factor of the order of $A$ is $15169173997557864184867895400813639018421$ with more than 60 bit security. The security against the index calculus is equivalent to $\mathbb{F}_{2^{1260}}$.

In the case of $q = 2^{97}, d = 11$, the largest prime divisor of $A$ is

$$50996843392805314313033252108853668830963472293743769141 - 06957559915561,$$  

the logarithm base 2 is 231. Security from generic attacks is 115 bits and from index calculus is equivalent to the field $\mathbb{F}_{2^{970}}$, i.e., 970 bits security.
In the case of \( q = 2^{43}, \ d = 29 \), the largest prime factor of the order is
\[
15971330269144846039246876225999124906449282490944114 -
1855981389550399714935349
\]
the logarithm of that is 253. So this has about 125 bit security from the
generic attacks and 1204 bit security from index calculus attack.

In the case of \( q = 2^{29}, \ d = 37 \), the largest prime factor is
\[
32801702501410292344998866375296008086511412965881
\]
with logarithm 167, i.e., security of more than 80 bits from generic attacks
and 1044 bits from index calculus.

Using GAP \([3]\), we created Table 2. In this table, all extensions \( q, q \) from
\( 2^{45} \) to \( 2^{90} \) and all primes from 10 to 20 are considered. For those extensions
and primes, it was checked if \( q \) is primitive mod \( d \). If that was so, then
the circulant matrix \( A \) was constructed and both the generic and the index
calculus security was tabulated.

5.1. Complexity of exponentiation of a circulant matrix of size \( d \). Let
us assume, that the circulant matrix of size \( d \) is \( A \) and we are raising it to
power \( m \), i.e., compute \( A^m \). We are using the square and multiply algo-

\[
R \text{EFERENCES}
\]
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| Size of the extension $q$ | Size of the matrix $d$ | Logarithm of the largest prime | Index-calculus security in bits |
|--------------------------|------------------------|-------------------------------|-------------------------------|
| $2^{47}$                 | 11                     | 115                           | 470                           |
|                          | 13                     | 77                            | 564                           |
|                          | 19                     | 207                           | 846                           |
| $2^{49}$                 | 11                     | 157                           | 490                           |
|                          | 13                     | 83                            | 588                           |
|                          | 19                     | 112                           | 882                           |
| $2^{51}$                 | 11                     | 92                            | 510                           |
|                          | 11                     | 129                           | 530                           |
|                          | 13                     | 92                            | 636                           |
|                          | 19                     | 312                           | 954                           |
| $2^{53}$                 | 11                     | 80                            | 660                           |
|                          | 13                     | 239                           | 990                           |
|                          | 19                     | 232                           | 590                           |
| $2^{55}$                 | 11                     | 232                           | 590                           |
|                          | 13                     | 91                            | 708                           |
|                          | 19                     | 262                           | 1062                          |
| $2^{57}$                 | 11                     | 157                           | 610                           |
|                          | 13                     | 120                           | 732                           |
|                          | 19                     | 294                           | 1098                          |
| $2^{59}$                 | 11                     | 123                           | 630                           |
|                          | 13                     | 96                            | 780                           |
|                          | 19                     | 131                           | 1170                          |
| $2^{61}$                 | 11                     | 248                           | 670                           |
|                          | 13                     | 106                           | 804                           |
|                          | 19                     | 274                           | 1206                          |
| $2^{63}$                 | 11                     | 176                           | 690                           |
|                          | 13                     | 242                           | 710                           |
|                          | 19                     | 281                           | 1278                          |
| $2^{65}$                 | 11                     | 111                           | 852                           |
|                          | 13                     | 103                           | 876                           |
|                          | 19                     | 258                           | 1314                          |
| $2^{67}$                 | 11                     | 184                           | 730                           |
|                          | 13                     | 121                           | 924                           |
|                          | 19                     | 359                           | 1386                          |
| $2^{69}$                 | 11                     | 279                           | 790                           |
|                          | 13                     | 140                           | 948                           |
|                          | 19                     | 209                           | 1422                          |
| $2^{71}$                 | 11                     | 143                           | 810                           |
|                          | 13                     | 284                           | 830                           |
|                          | 19                     | 432                           | 996                           |
| $2^{73}$                 | 11                     | 101                           | 1020                          |
|                          | 13                     | 245                           | 1530                          |
|                          | 19                     | 227                           | 890                           |
| $2^{75}$                 | 11                     | 151                           | 870                           |
|                          | 13                     | 152                           | 1068                          |
|                          | 19                     | 323                           | 1602                          |

Table 2. Security for $q$ from $2^{45}$ to $2^{90}$ and $d$ from 10 to 20
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