Explicit multi-time step computing method in the form of non-integer step ratio

Zhiqiang Ma\textsuperscript{1,2}, Yunfeng Lou\textsuperscript{1,2} and Xianlong Jin\textsuperscript{1,2}\* \\
\textsuperscript{1}School of Mechanical Engineering, Shanghai Jiaotong University, Shanghai 200240, China \\
\textsuperscript{2}State Key Laboratory of Mechanical System and Vibration, Shanghai Jiaotong University, Shanghai 200240, China \\
*Corresponding author’s e-mail: jxlong@sjtu.edu.cn

Abstract. Due to the requirement of stability, the time step in explicit dynamics must be small enough. This inevitably leads to a waste of computing time to adopt a unified time step for dynamics models with partial refined mesh. Multi-time step or subcycling method especially the integer step ratio is a classical method to deal with these problems. An explicit non-integer ratio multi-time step computing method is proposed in this paper. The finite element model is divided into different subdomains by node partition and the coupling region is formed by multiple overlapping boundary nodes according to the step ratio between adjacent subdomains. Each subdomain chooses explicit integration time step according to mesh characteristics. Non-integer step ratio is used among adjacent subdomains. The method provides a new implement for explicit finite element analysis.

1. Introduction

Explicit algorithm is suitable for solving wave propagation problems [1]. The central difference method is the most commonly used explicit integration method and most explicit method is conditionally stable [2]. In order to ensure the computational stability, the time step must be small enough, and the minimum time size depends on the minimum mesh size in the structural dynamics model.

However, at present, the commercial explicit finite element dynamics software mostly uses a single time step. For finite element model with complexly structured, the mesh size may differ from a hundred times, so the uniform time step will inevitably waste a lot of computing time and resource.

The multi-time step method is commonly used to deal with structure with local refined mesh [3]. A smaller time step is used in the area of fine mesh and larger step in the area of coarse mesh. Boundary data are transferred between smaller and larger time step by constraining the boundary node variable. The common multi-time step method is typical integer time step ratio [4-5]. Traditional subcycling methods have some problems in performance, such as poor stability and low accuracy [6-7].

In this paper, An explicit non-integer time step ratio computing method is proposed. Non-integer time step ratio means the non-integer ratio between larger time step and smaller time step [8]. This extends the application scope of the traditional integer form step ratio subcycling method.

2. Explicit predictor corrector non-integer step ratio computing method

The multiple overlapping node method is proposed first to explain the model partition method [9]. An
explicit predictive correction Newmark method is used in all subdomains.

2.1. Multiple overlapping nodes for explicit partition

The node partition method is used to divide the finite element model into several subdomains [10]. The node in the finite element model can be divided into internal node, boundary node and external node. The boundary and external nodes are shared by adjacent subdomain which also consist of the coupling region. The multi-layer overlapping node is needed in explicit multi-time step computation. The internal, boundary node and external node are represented by subscript I, B and E. The step ratio \( m \) is defined as the ratio between larger time step and smaller time step. Non-integer ratio means \( m \) can be non-integer expressed as a fraction. The multiple overlapping node with ratio \( m=3/2 \) is shown in figure 1.

![Node overlapping method with ratio m=3/2](image)

2.2. Explicit multi-time step computational process with non-integer ratio

The predictive process for the explicit predictive correction Newmark method can be written as follows [1]

\[
\begin{align*}
\vec{u}_{n+j} &= u_{n+j-1} + \vec{v}_{n+j-1} \Delta t + \frac{1}{2} \beta a_{n+j-1} \Delta t^2 \\
\vec{v}_{n+j} &= \vec{v}_{n+j-1} + (1-\gamma) a_{n+j-1} \Delta t
\end{align*}
\]

(1)

Where the subscript \( n \) means system time step, \( j \) is subcycling time step. \( \beta \) and \( \gamma \) are two parameters for Newmark algorithm. \( \Delta t \) is the time step. \( \vec{u}, \vec{v} \) and \( \vec{a} \) are the displacement, velocity and acceleration vectors.

The partitioned subdomain dynamic equation can be written as follows

\[
\begin{bmatrix}
M_I & K_I & K_{IB} \\
K_I & K_B & K_{BE} \\
K_{IB} & K_{BE} & K_E
\end{bmatrix}
\begin{bmatrix}
\vec{u}_{I,n+j} \\
\vec{u}_{B,n+j} \\
\vec{u}_{E,n+j}
\end{bmatrix}
= \begin{bmatrix}
\vec{f}_{I,n+j} \\
\vec{f}_{B,n+j} \\
\vec{f}_{E,n+j}
\end{bmatrix}
\]

(2)

Where \( M \) is mass matrix which adopted as lumped mass, \( K \) is the stiffness matrix. \( \vec{f} \) is the external force. Subscript I, B and E are represented as internal node, boundary node and external node respectively. The acceleration formulae for inner node and boundary node are computed as

\[
\begin{align*}
M_I \vec{a}_{I,n+j} &= \vec{f}_{I,n+j} - (K_I \vec{u}_{I,n+j} + K_{IB} \vec{u}_{B,n+j}) \\
M_B \vec{a}_{B,n+j} &= \vec{f}_{B,n+j} - (K_B \vec{u}_{B,n+j} + K_{BE} \vec{u}_{E,n+j}) - K_{IB} \vec{u}_{I,n+j}
\end{align*}
\]

(3)

The predictive displacement, velocity of the internal node and boundary node can be computed correctly. From the equation (3), we know that the acceleration for external node can’t be computed correctly. The multiple overlapping method offers a data transfer method that the external node is exactly the boundary node as adjacent subdomain.

The corrected displacement and velocity of the internal nodes are calculated as
\[ u_{t,n+1} = \dot{u}_{t,n} + \frac{\beta}{\gamma} \Delta t \dot{v}_{t,n} + \beta \Delta t \Delta v_{t,n} \]

\[ v_{t,n+1} = \dot{v}_{t,n} + \gamma \Delta t \Delta u_{t,n} \]  

(4)

The data transfer flow for explicit non-integer ratio \( m = 3/2 \) computation is shown in figure 2. In a boundary data exchange period, a larger step subdomain is computed twice and the smaller step subdomain is computed three times. The coupling data are transferred through MPI.

![Figure 2. The data transfer flow for explicit non-integer computation](image)

3. Numerical case

The tube model subjected to impact load is used to verify the validity of the proposed method. The model is symmetrical and only half of the tube model is used in the calculation. The parameters of the tube finite element model are as follows: density \( \rho = 7800 \text{ kg/m}^3 \), modulus of elasticity \( E = 208 \times 10^9 \text{ Pa} \), Poisson's ratio \( \nu = 0.25 \).

The radius of the tube is 1m and the thickness of the wall is 8mm. External force \( f_{\text{ext}} \) is loaded on the middle part of the tube which maximum is \( 2 \times 10^4 \text{N} \). The tube is discretized by triangular shell element. The model is divided into two subdomains. Subdomain 2 adopts a refined mesh. The dimensions of the tube and external load \( f_{\text{ext}} \) are shown in figure 3. Point A is the reference point.

![Figure 3. The dimensions of the tube and external load time curve](image)

The finite element model contains node 8695, element 17271. Under the same time step, the node number for subdomain 2 is 2033 and element number is 4869. The critical time step for subdomain 2 is \( \Delta t_{cr} = 1.87 \times 10^{-5} \text{s} \). We use the time step \( \Delta t_2 = 1 \times 10^{-5} \text{s} \) for submain 2. Subdomain 1 adopts a larger time step \( m \Delta t_2 \) in the following calculations. \( m \) is the time step ratio.

The lateral displacement of point A is calculated by different methods and time step ratios. The results are shown in figure 4 and figure 5.

Under the time step ratio \( m = 3.5 \), the result is compared with explicit non-integer subcycling methods [8]. The step ratio \( m = 3.5 \) which means a time step \( 3.5 \times 10^{-5} \text{s} \) is used in subdomain 1. The model is also calculated by the central difference method as reference result. The results are shown in
From the figure 4, we can know that the displacement of the proposed method is more approximate than that of the central difference method. The algorithm is effective and shows high accuracy.

We compare \( m=1.5 \), \( m=3.5 \) and \( m=6.5 \) three different time step ratios in figure 5 with the same proposed method. The result is shown in figure 5.

From the figure 5, it can be seen that with the increase of the time step ratio, the calculation results of the proposed algorithm conform to the displacement calculation pattern.

4. Conclusions
An explicit non-integer ratio dynamics computational method is proposed in this paper. The format of non-integer extends the range of time step ratio which can be selected. The coupled region is formed by multiple overlapping nodes between adjacent partitions. The predictive waveform can propagate completely in the whole coupling area of the multiple overlapping nodes. The truncation process is not involved in the computational process.

The decoupling characteristics of explicit algorithm and the modularization of implementation process make it easy to extend the method to a multiple subdomain. This is also the focus of future work.

Acknowledgements
This work is supported by the National Key Research and Development Program of China (2016YFB0201800) and the National Natural Science Foundation of China (Number:11772192).
References

[1] Hughes T J R 2012. *The finite element method: linear static and dynamic finite element analysis*. Courier Corporation.

[2] Sun X, Ariza M P, Ortiz M, et al 2017. Acceleration of diffusive molecular dynamics simulations through mean field approximation and subcycling time integration. *Journal of Computational Physics*, 350: 470-492.

[3] Belytschko T, Lu Y Y 1993. Explicit multi-time step integration for first and second order finite element semidiscretizations. *Computer Methods in Applied Mechanics and Engineering*, 108(3):353-383.

[4] Karimi S, Nakshatrala K B 2014. On multi-time-step monolithic coupling algorithms for elastodynamics. *Journal of Computational Physics*, 273: 671-705.

[5] Daniel W J T 2003. A partial velocity approach to subcycling structural dynamics. *Computer Methods in Applied Mechanics and Engineering*, 192(3-4): 375-394.

[6] Bunting G, Prakash A, Dyke S, et al 2016. Characterizing errors and evaluating performance of transient simulations using multi-time-step integration. *Journal of Computing in Civil Engineering*, 30(5):04016013.

[7] Klisinski M, Mostrom A 1998. On stability of multi-time step integration procedures. *Journal of Engineering Mechanics*, 124(7):783–793.

[8] Smolinski P 1996. Subcycling integration with non-integer time steps for structural dynamics problems. *Computers & Structures*, 59(2):273-281.

[9] Fernier A, Faucher V, Jamond O 2017. Multi-model Arlequin method for transient structural dynamics with explicit time integration. *International Journal for Numerical Methods in Engineering*, 112(9): 1194-1215.

[10] Karypis G, Kumar V 2018. Metis. A software package for partitioning unstructured graphs, partitioning meshes and computing fill-reducing orderings of sparse matrices, Version, 2018.