Finite Time Extinction for Stochastic Sign Fast Diffusion and Self-Organized Criticality

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Abstract: We prove finite time extinction for stochastic sign fast diffusion equations driven by linear multiplicative space-time noise, corresponding to the Bak–Tang–Wiesenfeld model for self-organized criticality. This solves a problem posed and left open in several works: (Barbu, Methods Appl Sci 36:1726–1733, 2013; Röckner and Wang, J Lond Math Soc (2) 87:545–560, 2013; Barbu et al. J Math Anal Appl 389:147–164, 2012; Barbu and Röckner, Comm Math Phys 311:539–555, 2012; Barbu et al., Comm Math Phys 285:901–923, 2009, C R Math Acad Sci Paris 347(1–2):81–84, 2009). The highly singular-degenerate nature of the drift in interplay with the stochastic perturbation causes the need for new methods in the analysis of mass diffusion, and several new estimates and techniques are introduced.

1. Introduction

Self-organized criticality (SOC) is a model of complex behavior that has attracted much attention in physics (cf. [BTW88, Zha89, BI92, Jen98, Tur99, CCGS90, DG94, GC98] among many others). We recall from [BI92]: The term “criticality” refers to the power-law behavior of the spatial and temporal distributions, characteristic of critical phenomena. “Self-organized” refers to the fact that these systems naturally evolve into a critical state without any tuning of the external parameters, i.e. the critical state is an attractor of the dynamics. It is this robust tendency to evolve into a critical state that distinguishes SOC from more classical models of criticality as for example observed in phase-transitions.

Based on a cellular automaton algorithm, in [BI92] a continuum limit related to the original sand pile model introduced by Bak–Tang–Wiesenfeld (BTW) in [BTW88] was derived, leading to a highly singular-degenerate PDE of the type

\[
\begin{align*}
\partial_t Z_t &\in \Delta H(Z_t - z_c), & \text{on } [0, T] \times \mathcal{O} \\
0 &\in H(Z_t - z_c), & \text{on } \partial \mathcal{O},
\end{align*}
\]
where $H$ is the Heaviside function, $z_c$ is the critical state and $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded, smooth domain. Rewriting (1.1) as an equation for $X_t = Z_t - z_c$ leads to

$$
\begin{align*}
\partial_t X_t &\in \Delta H(X_t), \quad \text{on } [0, T] \times \mathcal{O} \\
0 &\in H(X_t), \quad \text{on } \partial\mathcal{O}.
\end{align*}
$$

(1.2)

The effect of robust evolution/relaxation in finite time into a subcritical state can now be recast as finite time extinction of $(X_t)^+$, i.e. $X_t \leq 0$ after some finite time $\tau_0$. If we restrict to the relaxation of purely supercritical states (i.e. $Z_0 \geq z^c$ resp. $X_0 \geq 0$) then the relaxation into the critical state corresponds to the extinction of $X_t$ in finite time, i.e. $X_t \equiv 0$ after some finite time $\tau_0$.

As it has been pointed out in [DG94, GDG98, DG92] it is more realistic to include stochastic perturbations in (1.1) modeling the energy randomly added to the system, accounting for the removed microscopic degrees of freedom in the continuum limit and reflecting model uncertainty. As pointed out above, the robustness of self-organization in SOC is crucial. Based on this, the question arises whether this robustness with respect to perturbations is actually satisfied by (1.1), again leading to the study of stochastically perturbed versions of (1.1). Generally speaking, the resulting equations are stochastic partial differential equations (SPDE) of the following type

$$
\begin{align*}
\frac{dX_t}{dt} &\in \Delta H(X_t) + B(X_t)dW_t, \quad \text{on } [0, T] \times \mathcal{O} \\
0 &\in H(X_t), \quad \text{on } \partial\mathcal{O},
\end{align*}
$$

where $B$ are suitable diffusion coefficients. Particular attention (cf. e.g. [RW13, BDPR12, BR12, BDPR09b] among others) has been paid to the case of linear multiplicative space-time noise, i.e. to

$$
\begin{align*}
\frac{dX_t}{dt} &\in \Delta H(X_t) + \sum_{k=1}^N f_k X_t d\beta^k_t, \quad \text{on } [0, T] \times \mathcal{O} \\
0 &\in H(X_t), \quad \text{on } \partial\mathcal{O},
\end{align*}
$$

(1.3)

where $X_0 \geq 0$, $N \in \mathbb{N}$, $f = (f_k)_{k=1,...,N} \in C^2(\bar{\mathcal{O}}; \mathbb{R}^N)$ and $\beta = (\beta^k)_{k=1,...,N}$ is a standard Brownian motion in $\mathbb{R}^N$. Again, the key property of robust relaxation of supercritical states ($X_0 \geq 0$) into subcritical ones can be (re-)stated as the problem of finite time extinction: Let

$$
\tau_0 = \inf\{t \geq 0 | X_t(\xi) = 0 \text{ for a.e. } \xi \in \mathcal{O}\}.
$$

Finite time extinction can then be stated as $\mathbb{P}[\tau_0 < \infty] = 1$ for all nonnegative initial values $X_0 = x \geq 0$.

Despite its fundamental nature, the question of finite time extinction for the stochastic BTW model with linear multiplicative space-time noise (1.3) has remained an open problem for several years. The mathematical difficulty of an analysis of the diffusion of mass and finite time extinction for (1.3) stems from the highly singular-degenerate nature of the drift $\Delta H$ and its interplay with the stochastic perturbation. For example, the problem of finite time extinction for (1.3) has been posed and left as an open problem in the works [Bar13, RW13, BDPR12, BR12, BDPR09b, BDPR09a]. The main purpose