THE SPECTRAL DATA FOR HAMILTONIAN STATIONARY LAGRANGIAN TORI IN \( \mathbb{R}^4 \).

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Abstract. This article determines the spectral data, in the integrable systems sense, for all weakly conformally immersed Hamiltonian stationary Lagrangian in \( \mathbb{R}^4 \). This enables us to describe their moduli space and the locus of branch points of such an immersion. This is also an informative example in integrable systems geometry, since the group of ambient isometries acts non-trivially on the spectral data and the relevant energy functional (the area) need not be constant under deformations by higher flows.

1. Introduction.

A smooth immersion of a surface \( f : M \to \mathbb{R}^4 \) is Hamiltonian stationary Lagrangian \(^1\) (HSL) if \( f(M) \) is a Lagrangian submanifold whose area is stationary for all variations by (compactly supported) Hamiltonian vector fields. This is a natural variational problem for Lagrangian submanifolds and occurs in the study of volume minimisers in families of Lagrangian submanifolds \(^{18, 19}\). The Euler-Lagrange equations were derived by Oh \(^{16}\) and can be phrased in terms of the the mean curvature 1-form \( \sigma_H = f^*(H \mid \omega) \), where \( H \) denotes the mean curvature vector and \( \omega \) is the standard Kähler form on \( \mathbb{R}^4 \simeq \mathbb{C}^2 \). For any Lagrangian submanifold of \( \mathbb{R}^{2n} \) it can be shown that \( d\sigma_H = 0 \), and \( f \) is Hamiltonian stationary when \( d^*\sigma_H = 0 \), i.e., \( \sigma_H \) is harmonic. Consequently, for a compact HSL surface \( \sigma_H \) represents a cohomology class. It turns out that, after scaling, this is the Maslov class \( \mu \in H^1(M, \mathbb{Z}) \), which is an important Hamiltonian isotopy invariant of \( f(M) \). Oh conjectured in \(^{16}\) that the Clifford torus in \( S^3 \subset \mathbb{R}^4 \) minimises area in its Hamiltonian isotopy class and this proved to be a challenging question, which remains unanswered (although see \(^{10, 1, 3}\) for a partial solution).

Rather surprisingly, all weakly conformal HSL immersions of a torus \( \mathbb{C}/\Gamma \) into \( \mathbb{R}^4 \) can be explicitly described in terms of the Fourier components of the immersion: this was discovered by Hélein & Romon \(^{8}\), who showed that the equations can be reduced to a system of linear equations. They arrived at this through an investigation of the HSL equations as the Maurer-Cartan equations for a loop of flat connections, where the connexion form takes values in the Lie algebra of the group \( G \) of symplectic isometries of \( \mathbb{R}^4 \). This is an “integrable

\(^{1}\)The original terminology for these was H-minimal or Hamiltonian minimal.
systems’ approach, and although this proved to be somewhat superfluous to writing down the Fourier component solution, Hélein & Romon showed that the problem does admit the structure normally associated with an integrable system. The loop algebra valued Maurer-Cartan form is quadratic in the loop parameter,
\begin{equation}
\alpha_\zeta = \zeta^{-2}\alpha_{-2} + \zeta^{-1}\alpha_{-1} + \alpha_0 + \zeta\alpha_1 + \zeta^2\alpha_2,
\end{equation}
and satisfies a Lax equation, \( d\alpha_\zeta = [\xi_\zeta, \alpha_\zeta] \) where \( \xi_\zeta \) is a Laurent polynomial in \( \zeta \) (that is to say, HSL tori possess a “polynomial Killing field”). However, the Lax equations have a significant difference from those which are most often seen in the integrable systems study of, say, harmonic maps or minimal surfaces. The group \( G \) is not semisimple (or even reductive): it is the semi-direct product \( U(2) \ltimes \mathbb{C}^2 \) and its complexification is most naturally realised inside a parabolic subgroup of \( GL_5(\mathbb{C}) \). Together with the fundamental linearity of this particular set of equations, this raises the question of what the solution to these Lax equations look like, and particularly whether there is any effective “spectral data” which reproduces the Fourier component solutions. Here we are referring to the expectation of a correspondence between a HSL torus and algebro-geometric data of the type which one typically sees in the solution of Lax equations.

Our aim here is to show that such a correspondence does exist and not only does it reproduce entirely the solutions from the Fourier component method, it provides extra information which that method does not yield so easily. In particular, given that the original formulation allows for branch points it would be nice to know to what extent branched immersions make up the vector space of solutions found in [8]. Our approach leads to the conclusion that unbranched immersions are generic, and generically amongst branch immersions the branch points are isolated (see remark 5.5).

We first show that every weakly conformal HSL immersion \( f : \mathbb{C}/\Gamma \rightarrow \mathbb{R}^4 \) is determined by a triple of data \( (X, \lambda, L) \) consisting of a complete algebraic curve \( X \), a rational function \( \lambda \) on \( X \) and a line bundle \( L \) (or possibly a non-invertible rank 1 sheaf) over \( X \). To achieve this we cannot use just the naive characteristic polynomial spectral curve of one polynomial Killing field \( \xi_\zeta \), since this does not yield enough information. Instead we use a commutative subalgebra of polynomial Killing fields. The linear nature of the problem forces \( X \) to be a rational (and in this case, reducible) curve: its singularities correspond precisely to the non-trivial Fourier modes of \( f \). More surprisingly, the parabolic nature of the group \( G \) means the spectral data is not invariant under symplectic isometries. This is a consequence of the fact not all commutative subalgebras of polynomial Killing fields are isomorphic, so that the choice of subalgebra becomes part of the spectral data.

When this is all put together the full picture has a simple geometry. Fix a conformal type \( \Gamma \) for the torus and fix a Maslov class \( \beta_0 \in \Gamma^* \cong H^1(\mathbb{C}/\Gamma, \mathbb{Z}) \). This fixes the spectral curve \( (X, \lambda) \). Let \( \mathcal{S}(\Gamma, \beta_0) \) be the set of triples \( (X, \lambda, L) \) for weakly conformal HSL immersions \( f : \mathbb{C}/\Gamma \rightarrow \mathbb{R}^4 \) of Maslov class \( \beta_0 \) satisfying the base point condition \( f(0) = 0 \) (and a similar condition on the Lagrangian angle function). This condition is preserved by dilations, which always preserve
the HSL condition, and a subgroup $G_0$ of symplectic isometries isomorphic to $SU(2)$. Let $N$ denote the maximum number of non-trivial Fourier modes such a map can possess. We prove:

**Theorem 1.1.** $S(\Gamma, \beta_0) \cong \mathbb{C}P^{N-1}$ and corresponds to the space of based weakly conformal HSL immersions $f$ modulo dilations and the action of a maximal torus $S^1 \subset G_0$. Varying the base point of $f$ induces an action of the real Lie group $\mathbb{C}/\Gamma$ on $S(\Gamma, \beta_0)$ through a map $p \mapsto (X, \lambda, \mathcal{L}_p)$. The map $f$ has a branch point at $p \in \mathbb{C}/\Gamma$ precisely when $\mathcal{L}_p$ lies on the intersection of two hyperplanes $\Theta_\infty$ and $\Theta_0$ in $S(\Gamma, \beta_0)$.

The quotient of $S(\Gamma, \beta_0)$ by the action of $G_0$ is a $S^2$-bundle over the moduli space $\mathcal{M}(\Gamma, \beta_0)$ of these HSL tori. The natural map $S(\Gamma, \beta_0) \to \mathcal{M}(\Gamma, \beta_0)$ can be realised as the fibration $\mathbb{C}P^{N-1} \to \mathbb{H}P^{N/2-1}$, and the fibres are the $G_0$-orbits of spectral data. We provide an explicit expression for the hyperplanes $\Theta_\infty$, $\Theta_0$ in the homogeneous coordinates on $S(\Gamma, \beta_0)$. Each can be interpreted as a translate of the $\theta$-divisor for the $\theta$-function of a rational curve: this is explained in the appendix. It follows that the parameterisation by spectral data turns the question of locating branch points into a problem in projective geometry.

As one expects, the Jacobi variety of $X$, or more precisely a real subgroup $J_R$ of it, acts on the spectral data to produce new HSL tori. The Maslov class alone determines $(X, \lambda)$ completely, so that $S(\Gamma, \beta_0)$ is a union of $J_R$-orbits of different dimensions. The sheaf $\mathcal{L}$ is only a line bundle on the largest orbit. The base point translation action of $\mathbb{C}/\Gamma$ factors through a homomorphism of $\mathbb{C}/\Gamma$ into $J_R$. In integrable systems language the action of $J_R$ generates the “higher flows”. We show in §6 that these do not all correspond to Hamiltonian variations. This too is a surprising departure from the study of minimal tori, where all higher flows preserve the area. It reflects the fact that the area functional is a non-constant function on the moduli space $S(\Gamma, \beta_0)$.

We are hopeful this study will provide a useful testing ground for deeper investigations in the theory of spectral data in surface theory. In particular, the biggest challenge at this time is to understand how the appropriate energy functional depends on the spectral genus. For HSL tori in $\mathbb{R}^4$ the spectral data is elementary enough to see (cf. remarks 6.2 and 6.3 below) how the area functional depends upon the conformal class and the spectral data. We hope this relationship which may provide insight into how this works more generally in integrable surface theory.

### 2. Lagrangian Surfaces in $\mathbb{R}^4$.

Let $\mathbb{R}^4$ be equipped with its Euclidean metric, its standard complex structure $J$, and Kähler form $\omega$. We will represent its group of symplectic isometries in the form

$$G = \{(g, u) \in SO(4) \times \mathbb{R}^4 : gJg^{-1} = J\} \cong U(2) \times \mathbb{C}^2.$$

Now suppose we have a conformally immersed orientable surface $f : M \to \mathbb{R}^4$. If it is Lagrangian (i.e., $f^*\omega = 0$) then its Gauss map $\gamma : M \to \text{Lag}(\mathbb{R}^4)$ takes values in the Grassmannian $\text{Lag}(\mathbb{R}^4)$ of oriented Lagrangian 2-planes in $\mathbb{R}^4$. Let $H : M \to TM^\perp$ be the mean curvature field for $f$, then one knows that the
mean curvature form $\sigma_H = f^*(H|\omega)$ is closed and $f$ is Hamiltonian stationary precisely when $\sigma_H$ is also co-closed. Now, since $\text{Lag}(\mathbb{R}^4) \simeq U(2)/SO(2)$ we can post-compose $\gamma$ with the well-defined map $\det: U(2)/SO(2) \to S^1$ induced by the determinant on $U(2)$. Let $s : M \to S^1$ be defined by $s = \det \circ \gamma$, then by definition the Maslov form $\mu \in \Omega^1_M$ of $f$ is
\[
\mu = \frac{1}{\pi i} s^{-1} ds.
\]
By a theorem of Morvan [14] this is related to the mean curvature form by
\[
\mu = \frac{2}{\pi} \sigma_H
\]
and therefore
\[
s^{-1} ds = 2i\sigma_H.
\]
Thus $f$ is Hamiltonian stationary if and only if $s$ is a harmonic map.

Now we restrict our attention to the case where $M$ is a torus, represented in the form $M = \mathbb{C}/\Gamma$ where $\Gamma$ is a lattice. We can write $s = \exp(i\beta)$ for a function $\beta : \mathbb{C} \to \mathbb{R}$, which is called the Lagrangian angle, and $f$ is Hamiltonian stationary when $\beta$ is a harmonic function. Note that the action of the centre of $U(2)$ on $f$ is by $f \mapsto e^{i\theta} f$ (for some $\theta \in [0, 2\pi]$), under which the Lagrangian angle changes by $\beta \mapsto \beta + 2\theta$. Therefore we may (and will) assume that $\beta(0) = 0$. Following [8], and using the inner product $\langle z, w \rangle = \text{Re}(z\bar{w})$ on $\mathbb{C}$, we can write $\beta$ as
\[
\beta(z) = 2\pi \langle \beta_0, z \rangle
\]
for a constant $\beta_0 \in \Gamma^* \subset \mathbb{C}$, i.e., $\beta(z) \in 2\pi \mathbb{Z}$ for every $z \in \Gamma$.

**Remark 2.1.** The Maslov class of $f$ is the cohomology class of $[\mu] \in H^1(M, \mathbb{Z})$. Under the natural identification $H^1(M, \mathbb{Z}) \simeq \Gamma^* \subset \mathbb{C}$ we can think of $\beta_0$ as the Maslov class. One knows that HSL surfaces are constrained Willmore surfaces [4] (i.e., the Willmore energy $W(f) = \int_M |H|^2$ is critical for variations through conformal immersions). Because of the relation $H = -\frac{1}{2} J \nabla \beta$ the Willmore energy for a conformally immersed HSL torus $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ is “quantized” by the Maslov class $\beta_0$:
\[
W(f) = \pi^2 |\beta_0|^2 A(\mathbb{C}/\Gamma),
\]
where $A(\mathbb{C}/\Gamma)$ is the area of the flat torus $\mathbb{C}/\Gamma$ with metric $|dz|^2$.

2.1. **Twistor lift and frames.** Let $E$ denote the pullback $f^{-1} T\mathbb{R}^4$ of the tangent bundle of $\mathbb{R}^4$. Then $E = TM \oplus TM^\perp$ and since $f$ is Lagrangian $TM^\perp = JTM$. Let $J_M$ denote the intrinsic complex structure carried by $M$. Then since $f$ is conformal it induces another complex structure $S = J_M \oplus J J_M J$ on $E$, with the property that $JS = -SJ$. We may think of $S$ as a twistor lift [6] of $f$, i.e., $S : M \to Z$ where $Z$ is the twistor bundle of complex structures on $T\mathbb{R}^4$. Inside $Z$ lies the $S^1$-subbundle of all complex structures which anti-commute with $J$, which is where $S$ takes values. In fact this $S^1$-bundle is the image of a 4-symmetric space $G/G_0$, where $G_0$ is the fixed point subgroup of an order 4 outer automorphism $\tau$ of $G$. To see this, first let $\varepsilon_1, \ldots, \varepsilon_4$ be the standard oriented orthonormal basis of $\mathbb{R}^4$ for which $\varepsilon_2 = J\varepsilon_1$, $\varepsilon_4 = J\varepsilon_3$, and define $L \in SO(4)$ to be the complex structure on $\mathbb{R}^4$ characterised
\[
L\varepsilon_1 = \varepsilon_3, \quad L\varepsilon_2 = -\varepsilon_4.
\]
We observe that $LJ = -JL$ and that every complex structure anti-commuting with $J$ is of the form $gLg^{-1}$ for some $g \in U(2) \subset SO(4)$. Using $L$ we define an order 4 outer automorphism

$$\tau : G \to G; \tau (g, u) = (-Lg, -Lu).$$

The fixed point subgroup is $G_0 = \{(g, 0) \in G : gL = Lg\} \simeq SU(2)$ and the 4-symmetric space $G/G_0$ is an $S^1$-bundle over $\mathbb{R}^4$. On the other hand the twistor bundle has description

$$Z = \{(gJg^{-1}, u) \subset SO(4) \times \mathbb{R}^4 : g \in SO(4)\},$$

which is isomorphic to the homogeneous space $(SO(4) \times \mathbb{R}^4)/U(2)$. Now we have the embedding

$$G/G_0 \to Z; (g, u)G_0 \mapsto (gLg^{-1}, u),$$

whose image is the bundle of complex structures anti-commuting with $J$.

Let us now give a description of $S$ in terms of natural frames, and show that $S$ is essentially the Lagrangian angle function. Any conformal Lagrangian torus possesses a natural frame on the universal cover $\mathbb{C}$, called the fundamental frame, $\tilde{f} : \mathbb{C} \to G$, given by $\tilde{f} = (F, f)$, where $F : \mathbb{C} \to SO(4)$ is chosen so that $F\varepsilon_j = f_j$, where

$$f_1 = e^{-\beta}f_x, \quad f_2 = e^{-\beta}Jf_x, \quad f_3 = e^{-\beta}f_y, \quad f_4 = e^{-\beta}Jf_y,$$

for $|df|^2 = e^{2\beta}|dz|^2$, where $z = x + iy$. Setting $\epsilon = (\varepsilon_1 - i\varepsilon_3)/2$ we see

$$df = e^\beta F(\epsilon dz + \bar{\epsilon}d\bar{z}).$$

It follows that $s = \det(F)^2$.

From the definition of $S$ in terms of the complex structures $J_M$ and $J$ it follows that $Sf_1 = f_3$ and $Sf_2 = -f_4$. Therefore $S = (FLF^{-1}, f)$. But, as observed in [8], we can also take one of two spinor frames for $f$:

$$U_\pm = (\pm \exp(J\beta/2), f).$$

Notice that $\tilde{f} = U_\pm K$ where $K = (\pm \det(F)^{-1/2}F, 0)$. Since $K$ takes values in $G_0$ we have

$$FLF^{-1} = e^{\beta/2}Le^{-J\beta/2} = e^\beta L,$$

and therefore $S$ is essentially the Lagrangian angle function.

**Remark 2.2.** This gives us another perspective on the Hamiltonian stationary condition, namely, a map $f : M \to \mathbb{R}^4$ is conformal Lagrangian if and only if it is $S$-holomorphic for some $S : M \to G/G_0$ (i.e., $Sf_x = if_x$). In that case we necessarily we have $S = e^{\beta}L$ for some function $\beta$. Then $\tilde{f}$ is Hamiltonian stationary if further $\Delta \beta = 0$.

Throughout the remainder of this article we choose to work with the spinor lift $U_+$ since it has a particularly nice Maurer-Cartan form: we will define

$$\alpha = U_+^{-1}dU_+ = \left(\frac{1}{2}Jd\beta, e^{-J\beta/2}df\right).$$

\(^2\)Beware here that $\det(F)$ is the determinant taken in $U(2)$ not in its representation as a subgroup of $O(4)$.
We also assume, without any loss of generality, that \( f(0) = 0 \) and therefore there is a unique spinor lift \( U_+ \) determined by \( \alpha \) satisfying the initial condition \( U_+(0) = (I, 0) \). The subgroup of symplectic isometries preserving the two conditions \( f(0) = 0 \) and \( \beta(0) = 0 \) is \( G_0 \).

2.2. Extended Maurer-Cartan form. The automorphism \( \tau \) induces an order 4 automorphism (which we shall also call \( \tau \)) on \( \mathfrak{g}^C \), the complexification of the Lie algebra \( \mathfrak{g} \) of \( G \). We will represent \( \mathfrak{g}^C \), as vector space, by

\[
\mathfrak{g}^C = \{(X, x) \in \mathfrak{so}_4(C) \times C^4 : [X, J] = 0\}.
\]

The automorphism \( \tau \) takes the form

\[
\tau(X, x) = (-LXL, -Lx),
\]

Let \( \mathfrak{g}_j \subset \mathfrak{g}^C \) be the \( i^j \)-eigenspace for \( \tau \), then one computes

\[
\mathfrak{g}_{-1} = C\epsilon \oplus CJ\epsilon, \quad \mathfrak{g}_0 = \{(X, 0) \in \mathfrak{g}^C) : [X, L] = 0\},
\]

\[
\mathfrak{g}_1 = \bar{\mathfrak{g}}_{-1}, \quad \mathfrak{g}_2 = \{(rJ, 0) : r \in C\}.
\]

Notice that \( \mathfrak{g}_{-1} \) is the \( i \)-eigenspace for \( L \in \text{End}(C^4) \). We notice that the components of \( \alpha \) in this decomposition are

\[
\alpha_{-1} = (0, e^{-J\beta/2} \frac{\partial f}{\partial z} dz), \quad \alpha_1 = (0, e^{-J\beta/2} \frac{\partial f}{\partial \bar{z}} \bar{dz}), \quad \alpha_0 = (0, 0), \quad \alpha_2 = \left( \frac{1}{2} Jd\beta, 0 \right).
\]

We define the extended Maurer Cartan form to be the loop of 1-forms

\[
\alpha_\zeta = \zeta_2^{-1} \alpha_2 + \zeta_1^{-1} \alpha_1 + \alpha_0 + \zeta \alpha_1 + \zeta^2 \alpha_2.
\]

It is the principal observation of Hélein & Romon [8] that the Maurer-Cartan equations for \( \alpha_\zeta \) are satisfied if and only if \( f \) is Hamiltonian stationary. As usual, we think of \( \alpha_\zeta \) as a 1-form with values in a loop algebra. It possesses two symmetries, namely, a real symmetry and \( \tau \)-equivariance:

\[
\tau(\alpha_\zeta) = \alpha_{i\zeta},
\]

where \( (X, x) = (\bar{X}, \bar{x}) \) is simply complex conjugation. Therefore we may think of \( \alpha_\zeta \) as taking values in the twisted loop algebra \( \Lambda^r \mathfrak{g} \) of \( \tau \)-equivariant real analytic maps \( \xi_\zeta : S^1 \to \mathfrak{g}^C \) possessing the real symmetry. This is a real subalgebra of the complex algebra \( \Lambda^r \mathfrak{g}^C \) of \( \tau \)-equivariant real analytic maps \( \xi_\zeta : S^1 \to \mathfrak{g}^C \).

3. Polynomial Killing fields.

Hélein & Romon [8] have shown that every Hamiltonian stationary Lagrangian torus in \( \mathbb{R}^4 \) has an adapted polynomial Killing field, i.e., a map \( \xi_\zeta : \mathbb{C}/\Gamma \to \Lambda^r \mathfrak{g} \) satisfying

\[
(a) \quad d\xi_\zeta = [\xi_\zeta, \alpha_\zeta],
(b) \quad \xi_\zeta = \zeta^{-4d-2} \alpha_{-2} + \zeta^{-4d-1} \alpha_{-1} + \ldots .
\]

However, there are infinitely many linearly independent adapted polynomial Killing fields. Following the principle in [13] we would like to say that, by dropping condition (b) and allowing \( \xi_\zeta \) to take values in \( \Lambda^r \mathfrak{g}^C \), polynomial Killing fields come in complex algebras, and that the solution of the Lax equation (and
the geometry of the original map) should be able to be reconstructed from spectral data determined by this algebra. However, matrix multiplication does not preserve the loop algebra $\Lambda^\rho \mathfrak{g}^\mathbb{C}$. The reason is that $\mathfrak{g}$ itself is not closed under matrix multiplication. We rectify this by working in the larger matrix algebra

$$\mathfrak{p} = \{ ( \begin{array}{cc} A & \rho \\ 0 & b \end{array} ) \in \mathfrak{gl}_5(\mathbb{C}) : A \in \mathfrak{gl}_4(\mathbb{C}), [A, J] = 0, a \in \mathbb{C}^4, b \in \mathbb{C} \}.$$  

This contains $\mathfrak{g}^\mathbb{C}$ as the subalgebra for which $A \in \mathfrak{so}_4(\mathbb{C})$ and $b = 0$. Rather than use the block matrix notation, it will be more convenient to use a hybrid notation,

$$\begin{pmatrix} B & 0 \\ 0 & b \end{pmatrix} = (A, a) + bI,$$

where $I$ stands for the identity matrix in $\mathfrak{gl}_5$ and we have defined $A = B - bI_4$. In this notation the Lie bracket is given by

$$[(X, x) + yI, (A, a) + bI] = ([X, A], Xa - Ax).$$

Notice that $\tau$ extends to $\mathfrak{p}$ as the Lie algebra automorphism for which

$$\tau : (X, x) + yI \mapsto (\lambda Xx, T) = yI.$$  

To properly understand the algebra of polynomial Killing fields and their spectral data, we will work in a different realisation of the twisted loop algebra, in which the twisting is partially removed. First we embed $\Lambda^\rho \mathfrak{g}^\mathbb{C}$ in $\Lambda^\mu \mathfrak{p}$, where $\mu = \tau^2$. The involution $\mu$ is inner and therefore $\Lambda^\rho \mathfrak{p}$ is an algebra with unit under matrix multiplication. To remove the twisting in $\Lambda^\rho \mathfrak{p}$ write $\mu = \text{Ad}Q$ and let $\kappa : S^1 \to K$ be a homomorphism for which $\kappa_1 = I$ and $\kappa_\omega = Q^{-1}$. Since $\mu : \mathfrak{g}^\mathbb{C} \to \mathfrak{g}^\mathbb{C} : (A, a) \mapsto (A, -a)$, we have

$$\text{Ad}\kappa_\xi : (X_\xi, x_\xi) = (X_\xi, \zeta x_\xi).$$

The right hand side is an untwisted loop when considered as a function of $\lambda = \zeta^2$. This extends naturally to $\Lambda^\mu \mathfrak{p}$ to give a matrix algebra automorphism from $\Lambda^\mu \mathfrak{p}$ to the algebra $\Lambda \mathfrak{p}$ of untwisted loops in $\mathfrak{p}$. From now on we shall assume this has been applied to all the objects under study: the notation will imply that the untwisting has been applied by writing all loops as a function of $\lambda = \zeta^2$.

In $\Lambda \mathfrak{p}$ the extended Maurer-Cartan form has the shape

$$\alpha_\lambda = \left( \frac{\pi}{2} (\lambda^{-1} \bar{\beta}_0 dz + \lambda \beta_0 d\bar{z}) J, e^{-J/2}(f_\lambda dz + \lambda f_\bar{z} d\bar{z}) \right).$$

Since $\alpha_\xi$ is both $\tau$-equivariant and satisfies the reality condition (6), this form $\alpha_\lambda$ has the induced symmetries $\bar{\rho}^*(\alpha_\lambda) = \alpha_\lambda$, $\tau^*(\alpha_\lambda) = \alpha_\lambda$, where $\bar{\rho}^*$, $\tau^*$ are the commuting, respectively $\mathbb{R}$-linear and $\mathbb{C}$-linear, algebra involutions of $\Lambda \mathfrak{p}$ defined by

$$\bar{\rho}^*(\xi_\lambda) = \text{Ad}R_\lambda \cdot \xi_{\lambda^{-1}}, \quad \tau^*(\xi_\lambda) = \text{Ad} T \cdot \tau(\xi_{-\lambda}),$$

where

$$R_\lambda = \begin{pmatrix} I_4 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad T = \begin{pmatrix} iI_4 & 0 \\ 0 & 1 \end{pmatrix}.$$  

For $\xi_\lambda = (X_\lambda, x_\lambda) + y_\lambda I$ these look like

$$\bar{\rho}^*(\xi_\lambda) = (\bar{X}_\lambda^{-1}, \lambda \bar{x}_\lambda^{-1}) + \bar{y}_{\lambda^{-1}} I, \quad \tau^*(\xi_\lambda) = (-LX_{-\lambda} L, -iLx_{-\lambda}) + y_{-\lambda} I.$$
Both $\tilde{\rho}^*$ and $\tau^*$ are loop algebra automorphisms. As a result of this symmetry of $\alpha$, if $\xi_\lambda$ is a polynomial Killing field then so are both $\tilde{\rho}^*(\xi_\lambda)$ and $\tau^*(\xi_\lambda)$.

From now on we extend the definition of a polynomial Killing field to include any solution of (a) with values in $\Lambda$. It follows that $x_\lambda$ for which $\pi_\lambda R, x_\lambda$ are polynomial Killing fields of the form $(X_\lambda, x_\lambda) + y_\lambda I$ the polynomial Killing field equations become

\begin{align}
  dX_\lambda &= 0 \\
  dy_\lambda &= 0 \\
  dx_\lambda + \frac{\pi}{2} (\lambda^{-1} \beta_0 dz + \lambda \beta_0 d\bar{z}) J x_\lambda &= X_\lambda e^{-J\beta/2} (f_z dz + \lambda f_{\bar{z}} d\bar{z}).
\end{align}

Therefore $X_\lambda$ and $y_\lambda$ depend on $\lambda$ alone. Since these equations are linear over $\mathbb{C}[\lambda, \lambda^{-1}]$ we may assume that $X_\lambda$ is a polynomial of degree $N$, $X_\lambda = \sum_{j=0}^N X_j \lambda^j$. Expanding $x_\lambda$ similarly we obtain the equations

\begin{align}
  \partial x_j / \partial z + \frac{\pi}{2} \beta_0 J x_{j+1} &= X_j e^{-J\beta/2} f_z, \\
  \partial x_j / \partial \bar{z} + \frac{\pi}{2} \beta_0 J x_{j-1} &= X_{j-1} e^{-J\beta/2} f_{\bar{z}}.
\end{align}

It follows that $x_j = 0$ for $j \leq 0$ and $j > N$. It is also clear that if $X_\lambda$ is identically zero then so is $x_\lambda$, i.e., there are no non-trivial polynomial Killing fields of the form $(0, x_\lambda)$. Now consider polynomial Killing fields for which $X_\lambda = q_\lambda R$, where $q_\lambda \in \mathbb{C}[\lambda]$ and $R$ is a constant projection matrix (i.e., $R^2 = R$), which we will assume to be orthogonal projection onto some subspace $V$ of $\mathbb{C}^4$, since $[R, J] = 0$ this subspace $V$ is $J$-invariant.$^3$

**Lemma 3.1.** For each orthogonal projection matrix $R$ commuting with $J$ there exists a unique monic polynomial $p(\lambda)$, of minimal degree $N$ with $p(0) \neq 0$, such that there is a polynomial Killing field of the form $(p(\lambda) R, x_\lambda)$. Further, for any polynomial Killing field of the form $(q(\lambda) R, y_\lambda)$, with $q(\lambda) \in \mathbb{C}[\lambda, \lambda^{-1}]$, there exists $r(\lambda) \in \mathbb{C}[\lambda, \lambda^{-1}]$ for which $q(\lambda) = r(\lambda) p(\lambda)$ and $y_\lambda = r(\lambda) x_\lambda$. Finally, $x_\lambda$ takes values in the image of $R$.

**Proof.** Suppose $(p(\lambda) R, x_\lambda), (p'(\lambda) R, x'_\lambda)$ are both non-trivial polynomial Killing fields for which $p(\lambda), p'(\lambda)$ are monic of the same degree, minimal for polynomial Killing fields of this type. Then $((p(\lambda) - p'(\lambda)) R, x_\lambda - x'_\lambda)$ is again a polynomial Killing field and so $p(\lambda) = p'(\lambda)$, otherwise the degree of their difference is less than the minimal degree. It follows that $x_\lambda = x'_\lambda$.

Now suppose $(q(\lambda) R, y_\lambda)$ is a polynomial Killing field, then there exists $k \in \mathbb{N}$ for which $\lambda^k q(\lambda)$ is a polynomial. Clearly we can always find a polynomial $s(\lambda)$ which $\lambda^k q(\lambda) - s(\lambda) p(\lambda)$ has degree less than $p(\lambda)$. But the polynomial Killing field equations are linear over $\mathbb{C}[\lambda, \lambda^{-1}]$, therefore must have

$$\lambda^k q(\lambda) - s(\lambda) p(\lambda) = 0, \quad \lambda^k y_\lambda - s(\lambda) x_\lambda = 0.$$

$^3$We know such polynomial Killing fields must exist since in [8] it is shown that there are polynomial Killing fields of the form $(p(\lambda) J, y_\lambda)$ and multiplication on the left by $-RJ$ produces solutions of (11) by linearity.
Finally, \( x_\lambda = R x_\lambda + R_\perp x_\lambda \), where \( R_\perp \) is the complementary orthogonal projection onto \( U_\perp \). Since \([R_\perp, J] = 0\), \( R_\perp x_\lambda \) satisfies (11) with \( X_\lambda = 0 \), hence it is identically zero.

For convenience, we will refer to the polynomial Killing field with this unique, minimal degree, monic polynomial multiplier of \( R \) as the minimal polynomial Killing field for the projector \( R \). Note that if \( \dim(V) = 1 \) then \( x_\lambda = s_\lambda v \) for some non-zero \( v \in V \) and some function \( s_\lambda(z) \) which is polynomial in \( \lambda \).

Everything we need to know about the spectral curve is encoded in the minimal polynomial Killing field of the form \((p(\lambda)I_4, x_\lambda)\). From now on we will use \( p(\lambda), x_\lambda \) exclusively for this polynomial Killing field, and denote the degree of \( p(\lambda) \) by \( N \). To understand \( p(\lambda) \) we note first that \( x_\lambda \) is completely determined by \( p(\lambda) \), given \( u \), using the recursion implicit in the equations (12):

\[
x_{j+1} = -\left(\frac{2}{\pi \beta_0} J\right)^{j+1} \sum_{k=0}^{j} \left(-\frac{\pi \beta_0}{2} J\right)^k p_k \frac{\partial^{j-k} u}{\partial z^{j-k}}, \quad 0 \leq j \leq N,
\]

where for simplicity we set \( u := \alpha_1(\partial/\partial z) = e^{-J\beta/2}f_z \) (following the notations in \([8, \S3.2]\)). Recall also that the Maurer-Cartan equation for \( \alpha_\xi \) imply in particular

\[
\bar{u}_z = \frac{\pi}{2} \bar{\beta}_0 J \bar{u},
\]

This recursion has an important consequence for the roots of \( p(\lambda) \). Indeed, considering (14) for \( j = N \), we see that, since \( x_{N+1} = 0 \),

\[
\sum_{k=0}^{N} \left(-\frac{\pi \beta_0}{2} J\right)^k p_k \frac{\partial^{N-k} u}{\partial z^{N-k}} = 0.
\]

Now use the Fourier expansions of \( f \) and \( u \),

\[
f = \sum_{\gamma \in \Gamma^*} f_\gamma e_\gamma, \quad u = \sum_{\gamma \in \frac{1}{2} \Gamma^*} u_\gamma e_\gamma,
\]

where \( e_\gamma(z) = \exp(2\pi i \langle \gamma, z \rangle) \) and we recall from \([8]\) that \( u \) is \textit{a priori} only \( 2\Gamma \)-periodic. Let \( \Delta_f \) denote the set of frequencies \( \gamma \in \frac{1}{2} \Gamma^* \) such that \( u_\gamma \neq 0 \). It is a subset of

\[
\Gamma^*_{\beta_0} = \{ \gamma \in \Gamma^* : |\gamma| = \frac{\beta_0}{2}, \gamma \neq \pm \frac{\beta_0}{2} \}
\]

which is invariant under \( \gamma \mapsto -\gamma \) (see \([8]\)). Define also \( u^+ := \frac{1}{2}(1 - Ji)u \), the projection on \( i \)-eigenspace of \( J \) (and similarly define \( u^- \) for the \((-i)\)-eigenspace). Then equation (16) projects down to the \( i \)-eigenspace as

\[
\forall \gamma, \quad 0 = (i\pi \gamma)^N \sum_{k=0}^{N} \left(-\frac{\beta_0}{2\gamma} \right)^k p_k u^+_\gamma.
\]
and similarly for the \((-i)\)-eigenspace. One easily shows that \(u^+_\gamma\) and \(u^-_{-\gamma}\) are both non-zero for \(\gamma \in \Delta_f\) so that
\[
\frac{\beta_0}{2\bar{\gamma}} = 0 \text{ whenever } \gamma \in \Delta_f.
\]
Since \(|\gamma| = |\beta_0/2|\) these roots of \(p(\lambda)\) lie on the unit circle. On top of this, the \(\bar{\rho}^*\) and \(\tau^*\) symmetries of the polynomial Killing field equations, together with the uniqueness result in lemma 3.1, tell us that
\[
\begin{align*}
(p(\lambda^{-1})I_4, \lambda x(\lambda^{-1})) &= p(0)\lambda^{-N}(p(\lambda)I_4, x(\lambda)), \\
(p(-\lambda)I_4, -iLx(-\lambda)) &= (-1)^N(p(\lambda)I_4, x(\lambda)),
\end{align*}
\]
We note that \(p(0) = 1/p(0)\) and \(N\) must be even for \(p\) to have minimal degree. Taken altogether this information allows us to completely determine \(p(\lambda)\).

**Lemma 3.2.** The minimal polynomial Killing field \((p(\lambda)I_4, x_{\lambda})\) has
\[
p(\lambda) = \prod_{\gamma \in \Delta_f} (\lambda - 2\gamma/\beta_0).
\]
In particular, \(p(\lambda)\) is even, its roots are simple and all lie on the unit circle.

We will label these roots \(s_j, j = 1, \ldots, N\) and note that
\[
s_j = 2\gamma_j/\beta_0 = \bar{\beta}_0/2\bar{\gamma}_j,
\]
where \(\{\gamma_j : j = 1, \ldots, N\} = \Delta_f\). It will be convenient for us later to label these so that, for \(s_{j+N/2} = -s_j\) (i.e., \(\gamma_{j+N/2} = -\gamma_j\)).

**Proof.** Assume that \(p(\lambda)\) is given by (19) and define \(x_{\lambda}\) according to (14). This necessarily satisfies (12): we must show that it also satisfies (13) (equivalently, that it has the real symmetry in (18)). But that follows from (15) by a straightforward computation.

**Remark 3.3.** A particular consequence of (14) and (18) is that \(x_N = -\frac{2}{\pi \beta_0} J\bar{u} = \bar{p}_0x_1\). Under the assumption that \(f\) is conformal this means \(x_1, x_N\) never vanish. Later we will consider the possibility that \(f\) is only weakly conformal: the branch point of \(f\) will occur precisely when the degree of \(x_{\lambda}\) drops.

Finally, consider the effect of the group of symplectic isometries on the minimal polynomial Killing field. Since we wish to retain the conditions \(f(0) = 0\) and \(\beta(0) = 0\) we are only interested in the action of \(G_0 \subset G\).

**Lemma 3.4.** If \(f : \mathbb{C}/\Gamma \to \mathbb{R}^4\) has minimal polynomial Killing field \((p(\lambda)I_4, x_{\lambda})\) then, for any \(g \in G_0\), \(gf\) has minimal polynomial Killing field \((p(\lambda)I_4, gx_{\lambda})\).

The proof follows at once from the construction above.

4. The spectral data.

As with other surface geometries which arise from integrable systems, the spectral data for a HSL torus consists of a complete algebraic curve, a rational function on that curve and a line bundle over that curve. Two features which distinguish this geometry are that: a) the spectral curve is rational (indeed
reducible), b) the spectral data is not invariant under the action of ambient symmetries (the symplectic isometries of $\mathbb{R}^4$). Both of these features oblige us to take extra care when formulating the correspondence between spectral data and HSL tori.

4.1. The spectral curve. As a general principle (see e.g. [13]) the spectral data should be a geometric realisation of the algebra $\mathcal{K}$ of all polynomial Killing fields. As a straightforward consequence of lemma 3.1 we obtain a complete description of $\mathcal{K}$, by thinking of it as a module over the algebra $\mathcal{B} = \mathbb{C}[\lambda^{-1}, \lambda]$ in the obvious way.

**Lemma 4.1.** Let $(p(\lambda))_{i\lambda}$ be the minimal polynomial Killing field of this type for $f$ and let $R \in \text{End}(\mathbb{C}^4)$ be an orthogonal projection commuting with $J$. Then the minimal polynomial Killing field for $R$ is $\left( \frac{2}{r}, \frac{1}{r} R x_\lambda \right)$ where $r(\lambda)$ is the monoic polynomial whose zeroes are exactly the common zeroes of $p(\lambda)$ and $Rx_\lambda$. Thus $\mathcal{K}$ is generated over $\mathcal{B}$ by the set of all these minimal polynomial Killing fields together with the constant polynomial Killing field $J$.

In particular, since $p(\lambda)$ and $x_\lambda$ do not have common zeroes, $(p(\lambda)R, Rx_\lambda)$ is the minimal polynomial Killing field for $R$ unless $x_\lambda$ lies in $\ker(R)$ at some zeroes of $p(\lambda)$.

Since $\mathcal{K}$ is non-commutative it is necessary to construct the spectral curve by taking a maximal abelian subalgebra $\mathcal{A}$ of $\mathcal{K}$, as the completion by smooth points of the affine curve $\text{Spec}(\mathcal{A})$. Given our symmetries $\mathcal{A}$ should correspond to a choice of real maximal torus $t$ in the commutator $\mathfrak{g}^J \subset \text{End}_R(\mathbb{R}^4)$ of $J$ for which $\text{Ad}L \cdot t = t$. Let us first consider the maximal torus $t_0 \subset \mathfrak{g}^J$ generated by

\[
\begin{pmatrix}
I_2 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix} J_2 & 0 \\
0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\
0 & I_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\
0 & J_2 \end{pmatrix},
\]

where $I_2$ is the identity, and $J_2$ the standard complex structure, on $\mathbb{R}^2$. This maximal torus is characterised by its decomposition of $\mathbb{C}^4$ into (Hermitian orthogonal) invariant lines given by

\[
V_1 = \mathbb{C}.(\varepsilon_1 - i\varepsilon_2), V_2 = \mathbb{C}.(\varepsilon_3 - i\varepsilon_4), V_3 = V_1 = LV_2, V_4 = V_2 = LV_1.
\]

Notice that $t_0$ is completely determined by $V_1$, which is a complex line in the $i$-eigenspace $V \subset \mathbb{C}^4$ of $J$. Every other maximal torus of $\mathfrak{g}^J$ for which $\text{Ad}L \cdot t = t$ is given by $\text{Ad}g \cdot t_0$ for some $g \in G_0$.

For a fixed $f$ we can define

\[\mathcal{A}_f = \{(q(\lambda)R, y_\lambda) + u(\lambda)I \in \mathcal{K} : R \in \mathfrak{t}_0^C\}.\]

By lemma 4.1 $\mathcal{A}_f$ is abelian. Further, it is maximal abelian. For if $\xi \in \mathcal{K}$ commutes with every element of $\mathcal{A}_f$ then $\xi = (q(\lambda)R, y_\lambda)$ where $R \in \mathfrak{t}_0^C$ since $\mathfrak{t}_0^C$ is maximal.

However, quite surprisingly, it is not true that $\mathcal{A}_{gf} \simeq \mathcal{A}_f$ for every $g \in G_0$ (recall that $G_0$ is the group of symplectic isometries preserving the base point conditions $f(0) = 0$ and $\beta(0) = 0$). This isomorphism is only true generically in $G_0$, as we will show below. The generic algebra is isomorphic to an algebra $\mathcal{A}$ which we will now describe. For each $k = 1, \ldots, 4$ let $R_k \in \text{End}_\mathbb{C}(\mathbb{C}^4)$ denote
the unitary projection matrix corresponding to orthogonal projection onto \( V_k \). These span \( \ell^C_0 \) and satisfy \( R_j R_k = \delta_{jk} R_k \) (where \( \delta_{jk} \) is the Kronecker delta) and

\[
R_3 = -LR_2 L, \quad R_4 = -LR_1 L, \quad R_3 = R_1, \quad R_4 = R_2.
\]

(22) Now define \( \xi_k = (p(\lambda) R_k, R_k x_\lambda) \). Then

\[
\xi_j \xi_k = \delta_{jk} p(\lambda) \xi_k.
\]

Finally, define

\[
A = \mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4, \lambda I, \lambda^{-1} I]
\]

This is clearly an abelian subalgebra of \( K \). Notice that any \( \eta \in A \) can be uniquely written in the form

\[
\eta = \sum_{j=1}^{4} q_j \xi_j + q_5 I, \quad q_j \in \mathbb{C}[\lambda, \lambda^{-1}].
\]

It is easy to see from this that \( \eta = 0 \) if and only if each \( q_j = 0 \) and therefore \( A \) has no non-trivial relations other than those in (23). We deduce the following.

**Lemma 4.2.** \( A \cong \mathbb{C}[Z_1, Z_2, Z_3, Z_4, Z_5, Z^{-1}_5]/I \) where \( I \) is the ideal generated by \( Z_j^2 - p(Z_5) Z_k, Z_j Z_k \) for \( j,k = 1, \ldots, 4 \) and \( j \neq k \).

The next lemma describes the situation under which \( A \) is maximal abelian.

**Lemma 4.3.** \( A \subseteq A_f \), and \( A = A_f \) if and only if \( R_k(x(s_j, 0)) \neq 0 \) for \( k = 1, \ldots, 4, j = 1, \ldots, N \).

**Proof.** By definition \( A \neq A_f \) if and only if there exists a minimal polynomial Killing field for \( R \in \ell^C_0 \), which we write as \( (qR, y) \), not in \( A \). Since it is minimal there exists \( r \in \mathbb{C}[\lambda] \) such that \( (rqR, ry) = (pR, Rx) \). This is equivalent to saying that \( p \) and \( Rz \) have at least one common zero, i.e., there exists at least one \( s_j \) for which \( R_k(x(s_j, z)) = 0 \), for some \( k = 1, \ldots, 4 \) and for all \( z \in \mathbb{C} \). Further, the common zeroes are independent of \( z \), for at the zeroes of \( p(\lambda) \) (11) reduces to

\[
d(\exp(1/2 \beta_\lambda J)x_\lambda) = 0, \quad \beta_\lambda = \pi(\lambda^{-1} \bar{\beta}_0 z + \lambda \bar{\beta}_0 \bar{z}),
\]

hence

\[
x(s_j, z) = \exp(-1/2 \beta(s_j, z)J)x(s_j, 0), \quad j = 1, \ldots, N.
\]

(24) It follows that

\[
R_k(x(s_j, z)) = e^{\pm i\beta(s_j, z)/2} R_k(x(s_j, 0)),
\]

and therefore \( A \neq A_f \) if and only if \( R_k(x(s_j, 0)) = 0 \) for some \( j, k \). \( \square \)

Now we can show that generically \( A = A_f \). To be precise, consider the set

\[
\mathcal{U}_f = \{ g \in G_0 : R_k(gx(s_j, 0)) \neq 0 \ \forall \ k = 1, \ldots, 4, \ j = 1, \ldots, N, \ z \in \mathbb{C} \}.
\]

**Lemma 4.4.** The non-empty subset \( \mathcal{U}_f \subset G_0 \) is proper, open and \( A_{gf} \cong A \) if and only if \( g \in \mathcal{U}_f \).
Proof. Recall that $G_0 \subset U(2) \subset SO(4)$ is the commutator of $J$ and $L$. Therefore $G_0$ preserves both $V = V_1 \oplus V_2$ and $\bar{V}$, the $\pm i$-eigenspaces of $J$. Further, $G_0$ acts transitively on $\mathbb{P}V$ and the subset of $g \in G_0$ for which $gV_1 \cap V_1 = \{0\}$ is an open subset of $G_0$ (hence, the same statements are true for the action of $G_0$ on $\bar{V}$ and the orbit of $V_1$). Finally, whenever $|\lambda| = 1$ the real symmetry in (18) implies that the line $\ell = \mathbb{C}x(\lambda, 0)$ has $\ell = \ell$ and therefore has non-trivial components in both $V$ and $\bar{V}$. It follows that for each root $s_j$ of $p(\lambda)$ there is a proper non-empty open subset $U_{k_j} \subset G_0$ for which each $R_k(gx(s_j, 0))$ is non-zero: $U_f$ is the intersection of the finitely many open subsets $U_{k_j}$. It follows from lemmas 4.2 and 4.3 that $A_{gf} \simeq A$ if and only if $g \in U_f$. \hfill $\square$

Given $f$ we define its affine spectral curve to be the affine scheme $X_0 = \text{Spec}(A)$. The natural inclusion of algebras $\mathcal{B} \to A$ is dual to a finite morphism $\lambda : X_0 \to \mathbb{C}^\times$. It follows from (23) that $A$ is a free rank 5 module over $\mathbb{C}[\lambda, \lambda^{-1}]$, hence this morphism has degree 5. We define the spectral curve $X$ to be the completion of $X_0$ by smooth points. By lemma 4.2 $X_0$ is birational to the curve in $\mathbb{C}^5 \setminus \{Z_5 = 0\}$ determined by the equations

$$Z_j(Z_j - p(Z_5)) = 0, \ Z_jZ_k = 0, \ j, k = 1, \ldots, 4, \ k \neq j.$$  

Its normalisation $\varphi : \tilde{X}_0 \to X_0$ is dual to the algebra monomorphism

$$\varphi^* : A \to \tilde{A} : \sum_{j=1}^4 q_j \xi_j + q_5 I \to (q_1 p + q_5, \ldots, q_4 p + q_5),$$

where $\tilde{A}$ denotes $\mathcal{B}^5$ with the direct product structure. We obtain the following structure for the spectral curve $X$.

Proposition 4.5. $X$ is a reducible rational curve with five irreducible components $C_1, \ldots, C_5$ each of which is a smooth rational curve. Any two intersect along the 0-dimensional subscheme $\mathfrak{S} \subset X_0$ given by $\mathfrak{S} = \text{Spec}(A/J)$ where $J$ is the ideal in $A$ generated by $p, \xi_1, \ldots, \xi_4$. As a divisor, $\mathfrak{S}$ is just the union of the $N$ singular points corresponding to the zeroes of $p(\lambda)$. In particular, $X$ has arithmetic genus $g = 4(N - 1)$.

For simplicity, we will abuse notation by also using $\mathfrak{S}$ to denote the set $\lambda(\mathfrak{S}) = \{s_1, \ldots, s_N\}$.

It will be convenient for us to identify the irreducible components as follows. The component $C_5$ has affine part given by $Z_j = 0$ for $j = 1, \ldots, 4$, while the affine part of $C_j$, for $j \leq 4$, is the curve with equations

$$Z_j = p(Z_5), \ Z_k = 0, \ k \neq j, \ k \leq 4.$$  

The holomorphic function $\lambda = Z_5$ on $X_0$ extends to a rational function $\lambda : X \to \tilde{\mathbb{C}}$, where $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. It has 5 points over $\lambda = 0$, which we will denote by $P_1, \ldots, P_5$, with $P_j$ lying on $C_j$. The corresponding points over $\lambda = \infty$ will be denoted $Q_1, \ldots, Q_5$.

Since $o_\lambda$ possesses the symmetries $\rho^*$ and $\tau^*$ these act as involutions on $A$. Therefore they induce involutions on $X$, which we will call $\rho$ and $\tau$, for which $h \mapsto \rho h = \rho^* h$ and $h \mapsto \tau h = \tau^* h$ for any $h \in \mathbb{C}[X_0]$. It is straightforward to establish the following characterisation of these involutions.
Proposition 4.6. The involutions \( \rho \) and \( \tau \) on \( X \) are, respectively, anti-holomorphic and holomorphic, and act as follows.

\[
\begin{align*}
(Z_1 \circ \rho, Z_2 \circ \rho, Z_3 \circ \rho, Z_4 \circ \rho, Z_5 \circ \rho) &= \left( \frac{Z_4}{Z_5^N p(0)}, \frac{Z_4}{Z_5^N p(0)}, \frac{Z_1}{Z_5^N p(0)}, \frac{Z_2}{Z_5^N p(0)}, \frac{1}{Z_5} \right), \\
\tau(Z_1, Z_2, Z_3, Z_4, Z_5) &= (Z_4, Z_3, Z_2, Z_1, -Z_5).
\end{align*}
\]

Finally, let us explain the difference between \( \text{Spec}(A_f) \) and \( \text{Spec}(A) \) when \( A \neq A_f \). For each \( k = 1, \ldots, 4 \) define

\[
\mathcal{S}_k = \{ s_j : R_k(x(s_j, 0)) \neq 0 \}.
\]

Note that every \( s_j \) belongs to at least one \( \mathcal{S}_k \) since \( p(\lambda) \) and \( x(\lambda, 0) \) have no common zeroes. Now define

\[
p_k(\lambda) = \prod_{s_j \in \mathcal{S}_k} (\lambda - s_j).
\]

As a consequence of the symmetries (18) and (22) we see that \( \mathcal{S}_1 = \mathcal{S}_3 \), \( \mathcal{S}_2 = \mathcal{S}_4 \) and \( \mathcal{S}_5 = -\mathcal{S}_1 \). Therefore \( p_3 = p_1 \), \( p_4 = p_2 \) and \( p_2(\lambda) = \pm p_1(-\lambda) \). It follows that \( A_f = \mathbb{C}[\eta_1, \ldots, \eta_4, \lambda I, \lambda^{-1} I] \) where

\[
\eta_k = \frac{p_k}{\lambda} \xi_k = \left( \frac{p_k}{\lambda} R_k, \lambda, \frac{p_k}{\lambda} x(\lambda) \right)
\]

and therefore \( \text{Spec}(A_f) \) is biregular to the affine curve in \( \mathbb{C}^5 \setminus \{ Z_5 = 0 \} \) with equations

\[
Z_j(Z_j - p_j(Z_5)) = 0, \quad Z_jZ_k = 0, \quad j, k = 1, \ldots, 4, \quad k \neq j.
\]

We conclude from this the following structure.

Lemma 4.7. Let \( X_f \) denote the completion by smooth points of \( \text{Spec}(A_f) \). It has five irreducible components \( C'_j, \; j = 1, \ldots, 5 \), each with \( C'_j \cong \hat{\mathbb{C}} \). All five components intersect along \( \mathcal{S}_1 \cap \mathcal{S}_2 \), with further intersection relations

\[
C'_1 \cap C'_3 = \mathcal{S}_1, \quad C'_2 \cap C'_4 = \mathcal{S}_2, \quad C'_k \cap C'_5 = \mathcal{S}_k, \quad k = 1, \ldots, 4.
\]

In particular, \( X_f \) has arithmetic genus \( 4(N_1 - 1) \), where \( N_1 = \# \mathcal{S}_1 \).

The picture one should have in mind is that the natural inclusion \( A \hookrightarrow A_f \) is dual to a finite morphism \( \pi_f : X_f \to X \) which realises \( X_f \) as the desingularisation of \( X \) obtained by pulling apart the components \( C_1, \ldots, C_5 \) so that only the above intersection relations remain and the symmetries \( \rho \) and \( \tau \) persist. Notice that, because of the symmetries \( \rho \) and \( \tau \), the structure of \( X_f \) is completely determined by knowing how \( C'_1 \) and \( C'_5 \) intersect.

Remark 4.8. In case this seems an overly elaborate way of obtaining the spectral curve, consider the alternatives. Taking the characteristic polynomial of \( \xi_\lambda = (p(\lambda)I_4, x(\lambda)) \) we obtain an unreduced planar curve with equation \( \mu(\mu - p(\lambda))^2 = 0 \). A slightly more sophisticated approach is to consider the curve of eigenlines of \( \xi_{\hat{\lambda}} \), but its eigenlines are generated by \( \varepsilon_1, \ldots, \varepsilon_4 \), which are constant, and the vector

\[
(x_1(\lambda, z), \ldots, x_4(\lambda, z), -p(\lambda)), \quad x = \sum_{j=1}^{4} x_j \varepsilon_j.
\]
This leads to a disconnected union of Riemann spheres. Neither of these approaches permits the vector \( x(\lambda, z) \) to be encoded in the spectral data using a line bundle, or sheaf, over the spectral curve. The next section show how our spectral data achieves this.

4.2. Line bundles over the spectral curve. In actuality \( \mathcal{A} \) is a family of algebras. By evaluating every polynomial Killing field at a point \( z \) we construct an algebra \( \mathcal{A}(z) \): these are all isomorphic under the map

\[
\mathcal{A}(0) \rightarrow \mathcal{A}(z); \quad \xi_\lambda(0) \mapsto \text{Ad} U_\lambda(z)^{-1} \xi_\lambda(0),
\]

where \( U_\lambda \) is the extended frame, i.e., \( U_\lambda^{-1} d U_\lambda = \alpha_\lambda \) and \( U_\lambda(0) = I \). The vector space \( \mathcal{M} = \mathcal{B} \otimes \mathbb{C}^5 \) is an \( \mathcal{A}(z) \)-module for each \( z \). We will will show that it determines a line bundle over each irreducible component and then explain how these fit together over \( X \). Before we do this, we need to review the moduli space \( \operatorname{Pic}(X) \) of line bundles over \( X \).

Let \( \tilde{X} \) be the normalisation of \( X \): it is the disjoint union of the smooth rational curves \( C_1, \ldots, C_5 \). A line bundle over \( X \) can be thought of as a divisor equivalence class \([D]\) for a divisor \( D \) of smooth points on \( X \). Recall (from e.g., [17]) that the equivalence is characterised by the property of being trivial if and only if \( D \) is the divisor of a rational function on \( X \). Thus \( \operatorname{Pic}(X) \) is isomorphic to the group of these divisor equivalence classes. Since \( X \) is reducible such a class cannot be assigned a single integer for its degree, but rather

\[
\deg : \operatorname{Pic}(X) \rightarrow \mathbb{Z}^5; \quad \deg([D]) = (\deg([D \cap C_1], \ldots, \deg([D \cap C_5])).
\]

In particular, the Jacobi variety \( \operatorname{Jac}(X) \) is the subgroup of line bundles whose degree on each component is zero. The structure of \( \operatorname{Jac}(X) \) is given by the next lemma, whose proof is straightforward.

Lemma 4.9. Let \( \operatorname{Div}_0(X) \) denote the set of all divisors with \( \deg(D) = (0, \ldots, 0) \). For each \( D \in \operatorname{Div}_0(X) \) there is a rational function \( f \) on \( \tilde{X} \), unique up to scale, with divisor \( D \). The map \( \operatorname{Jac}(X) \rightarrow (\mathbb{C}^\times)^{4(N-1)} \) which assigns to \([D]\) the coordinates

\[
t_{kj} = \frac{f_k(s_j)}{f_k(s_N)} \frac{f_j(s)}{f_j(s_N)} \bigg|, \quad 1 \leq k \leq 4, \quad 1 \leq j \leq N - 1,
\]

in which \( f_k = f|_{C_k} \), is an isomorphism of linear algebraic groups.

Lemma 4.10. The \( \mathcal{A}(z) \)-module \( \mathcal{M} \) determines a rank 1 sheaf \( \mathcal{L}_f(z) \) over \( X \). Either:

(a) \( \mathcal{A}(z) = \mathcal{A}_f(z) \) and \( \mathcal{L}_f(z) \) is a line bundle of degree \((N, N, N, N, 0)\), or,

(b) \( \mathcal{A}(z) \neq \mathcal{A}_f(z) \) and \( \mathcal{L}_f(z) \) is the direct image of a line bundle over \( X_f \) of degree \((N_1, N_1, N_1, N_1, 0)\).

In either case \( \mathcal{L}_f \) possesses the symmetries

\[
\rho^* \mathcal{L}_f \simeq \mathcal{L}_f(P_3 - Q_5), \quad \tau^* \mathcal{L}_f \simeq \mathcal{L}_f.
\]

For the purpose of the proof, and for subsequent use, it will be convenient to work with the Hermitian orthonormal basis \( v_1, \ldots, v_4 \) for \( \mathbb{C}^4 \), thought of as
the first four dimensions in $\mathbb{C}^5$, with $v_j \in V_j$ defined by
\begin{equation}
(30) \quad v_1 = \frac{1}{\sqrt{2}}(\varepsilon_1 - i\varepsilon_2), \quad v_2 = \frac{1}{\sqrt{2}}(\varepsilon_3 - i\varepsilon_4), \quad v_3 = v_1, \quad v_4 = \bar{v}_2.
\end{equation}
We note that $v_2 = L\bar{v}_1, v_4 = Lv_1$. In such a basis we write $x_\lambda(z) = \sum_{j=1}^4 \chi_j(\lambda, z)v_j$, so that $R_kx_\lambda = \chi_k(\lambda)v_k$. We also define $v_5 = (0, 0, 0, 0, 1)$.

**Proof.** First assume $\mathcal{A} = \mathcal{A}_f$ (for simplicity we drop the explicit dependence of the notation on $z$). Let $I_j \subset \mathcal{A}$ denote the prime ideal corresponding to the component $C_j \cap X_0$. In terms of generators we have
\begin{align*}
I_j &= \langle \xi_j - pI, \xi_k : k \neq j \rangle, \quad j \leq 4 \\
I_5 &= \langle \xi_1, \ldots, \xi_4 \rangle
\end{align*}
Now define $\mathcal{A}_j = \mathcal{A}/I_j$ (this is the coordinate ring for $C_j \cap X_0$) and $\mathcal{M}_j = \mathcal{M}/I_j\mathcal{M}$. It is easy to see that $\mathcal{A}_j \simeq \mathcal{B}$ for each $j = 1, \ldots, 5$ and the points over $\lambda = 0, \infty$ on $C_j$ correspond to the two gradings carried by $\mathcal{A}_j$, namely, the degree in $\lambda^{-1}$ and $\lambda$, respectively. For $j = 1, \ldots, 4$ we have
\[I_j\mathcal{M} = \mathcal{B}(pv_k, \chi_kv_k, \chi_jv_j - pv_5 : k \neq j).\]
But $p(\lambda)$ and $\chi_k(\lambda)$ have no common zeroes so the ideal in $\mathcal{B}$ generated by $p$ and $\chi_k$ is $\mathcal{B}$ itself. Therefore
\[I_j\mathcal{M} = \mathcal{B}(v_k, \chi_jv_j - pv_5 : k \neq j).\]
It follows that
\[\mathcal{M}_j \simeq \mathcal{B}(v_j, v_5)/\langle \chi_jv_j - pv_5 \rangle.
\]
This is clearly a rank one module over $\mathcal{A}_j$ and torsion free since $p$ and $\chi_j$ have no common zeroes. This determines a line bundle $L_j$ over $C_j$ with sections $\sigma_j, \sigma_5$, corresponding to $v_j, v_5$, satisfying $\chi_j\sigma_j - p\sigma_5 = 0$. Now
\[I_5\mathcal{M} = \mathcal{B}(v_1, \ldots, v_4)\]
and therefore $\mathcal{M}_5 \simeq \mathcal{B}(v_5)$, which is clearly a rank one torsion free module over $\mathcal{A}_5$. It determines a trivial line bundle $L_5$ with nowhere vanishing global section $\sigma_5$.

Therefore as an $\mathcal{A}$-module $\mathcal{M}$ determines a sheaf $\mathcal{L}_f$ over $X$ for which $\mathcal{L}_f(C_j) = L_j$. It has globally holomorphic section $\sigma_5$ which vanishes exactly at the zeroes of $\chi_j(\lambda, z)$: this gives a degree $N$ divisor $D_j(z)$ on $C_j$, so $\deg(L_j) = N$. Since $L_5$ is trivial it has degree 0. Hence $\sigma_5$ has divisor
\begin{equation}
(31) \quad D(z) = D_1(z) + \ldots + D_4(z).
\end{equation}
Since this divisor includes no singular points $\mathcal{L}(z)$ is invertible (i.e., a line bundle) with $\mathcal{L}(z) \simeq \mathcal{O}_X(D(z))$.

Now consider the case $\mathcal{A} \neq \mathcal{A}_f$. A simple adaptation of the arguments above shows that the $\mathcal{A}_f$-module $\mathcal{M}$ determines a line bundle over $X_f$ of degree $(N_1, N_1, N_1, N_1, 0)$. Therefore its direct image $\mathcal{L}_f$ corresponds to $\mathcal{M}$ as an $\mathcal{A}$-module.
Finally, from the symmetries (18) and (30) we obtain
\begin{equation}
\chi_3(\lambda) = \lambda^{N+1}p(0)\chi_1(\lambda^{-1}), \\
\chi_4(\lambda) = \lambda^{N+1}p(0)\chi_2(\lambda^{-1}), \\
\chi_4(\lambda) = -i\chi_3(-\lambda), \\
\chi_3(\lambda) = i\chi_2(-\lambda).
\end{equation}

The first expression gives the equation of divisors
\[ D_3 - NQ_3 = (N + 1)(P_3 - Q_3) + \rho^*(D_1 - NQ_1). \]
It follows that \( D_3 + Q_3 - P_3 = \rho^*D_1 \). We obtain a similar equation relating \( D_4 \) and \( \rho^*D_2 \), and deduce
\[ \rho^*D = D + \sum_{j=1}^{4}(Q_j - P_j) \sim D + P_5 - Q_5, \]
using the fact that \( \sum_{j=1}^{5}(P_j - Q_j) \) is the divisor of \( \lambda \). Similarly \( \tau^*D = D \). \( \square \)

Remark 4.11. In this lemma we are implicitly assuming that the degree of \( \chi_j(\lambda, z) \) is exactly \( N \) for each \( z \in \mathbb{C} \). This means that \( D_j(z) \) actually lies on \( C_j \setminus \{Q_j\} \). But it is possible for the degree of \( \chi_j \) to drop (e.g., at branch points of \( f \); see below). We shall keep \( \deg(D_j(z)) = N \) by allowing it to include the points at \( \infty \).

It is an inevitable consequence of lemma 4.4 that the group \( G_0 \) acts non-trivially on the spectral data. Indeed, the \( G_0 \)-orbit of \( f \) takes \( \mathcal{L}_f \) outside the Jacobi variety \( \text{Jac}(X) \) into a compactification which includes non-invertible sheaves, since there is at least one point in this orbit at which \( \chi_k \) has a zero at some \( s_j \). We will explain precisely what happens when we describe the moduli space of HSL tori in the next section. Meanwhile let us observe that the previous proof provides a convenient characterisation of the sheaf \( \mathcal{L}_f \), since it shows that \( \mathcal{L}_f \) is entirely determined by the divisor of zeroes of \( \chi_1 \), even when these are allowed to include singularities or points at infinity.

Corollary 4.12. Up to isomorphism \( \mathcal{L}_f \) is determined by the positive divisor \( E = D_1 - P_1 \) of degree \( N - 1 \) on \( C_1 \), where \( D_1 \) is the divisor of zeroes of \( \chi_1 \). The map \( f \) has a branch point at \( z = 0 \) precisely when \( E \) has the form \( E = P_1 + Q_1 + E' \) for some positive divisor \( E' \) of degree \( N - 3 \).

Proof. From (12) and (13) we know \( x_\lambda \), and hence \( \chi_1 \), has a zero at \( \lambda = 0 \). From remark 3.3 we know \( f \) has a branch point at \( z = 0 \) precisely when the degree of \( x_\lambda \) drops. Using the symmetries (32) we see that this occurs precisely when \( \chi_1 \) has an additional zero at \( P_1 \) and degree less then \( N - 1 \), i.e., a zero at \( Q_1 \). \( \square \)

We finish this section by describing how the sheaf \( \mathcal{L}_f(z) \) moves with base point translation. Since we started with a Lax equation we expect this motion to be linear, and that is indeed the case.

Proposition 4.13. The map \( \ell : \mathbb{C}/\Gamma \to \text{Jac}(X) \) given by \( \ell = \mathcal{L}(z) \otimes \mathcal{L}(0)^{-1} \) has image in the real analytic subgroup \( J_R \subset \text{Jac}(X) \) consisting of all line bundles possessing the symmetries
\begin{equation}
\rho^*\ell \simeq \ell, \quad \tau^*\ell \simeq \ell.
\end{equation}
Further, $\ell$ is a homomorphism of real groups characterised by the equation
\begin{equation}
\frac{\partial \ell}{\partial z}|_{z=0} = \frac{i\pi \beta_0}{2} \frac{\partial A}{\partial \lambda}|_{\lambda=0}
\end{equation}
where $A : \mathbb{C} \to \text{Jac}(X)$ is defined by
\begin{equation}
A(\lambda) = \mathcal{O}(\sum_{j=1}^{4} (P_j(\lambda) - P_j)),
\end{equation}
with $P_j(\lambda)$ the point on $C_j$ given by $Z_j = \lambda$ for $j = 1, 2$ but $Z_j = -\lambda$ for $j = 3, 4$.

**Proof.** The isomorphisms in (33) are a direct consequence of (29). To prove (34) we use the coordinates (28) on $\text{Jac}(X)$, in which $L(z)$ has coordinates,
\begin{equation}
t_{kj}(z) = \frac{\chi_k(s_j, z) \chi_k(s_N, 0)}{\chi_k(s_j, 0) \chi_k(s_N, z)}, \quad 1 \leq k \leq 4, \quad 1 \leq j \leq N - 1.
\end{equation}
Here we are using the fact that $D(z) - D(0)$ is the divisor for the rational function $\chi$ on $\tilde{X}$ whose restriction to $\tilde{C}_k$ is $\chi_k$ for $1 \leq k \leq 4$. Now using (24) we compute
\begin{equation}
\frac{\partial t_{kj}}{\partial z}|_{z=0} = \epsilon_k \frac{-i\pi \beta_0}{2} (s_j^{-1} - s_N^{-1}),
\end{equation}
where $\epsilon_k$ equals $+1$ for $k = 1, 2$ and $-1$ for $k = 3, 4$. On the other hand we compute, in analogy with the computation in the appendix, the coordinates of $A(\lambda)$ to be
\begin{equation}
a_{kj}(\lambda) = \frac{\epsilon_k \lambda - s_j s_N}{\epsilon_k \lambda - s_N s_j}, \quad 1 \leq k \leq 4, \quad 1 \leq j \leq N - 1.
\end{equation}
Therefore
\begin{equation}
\frac{\partial a_{kj}}{\partial \lambda}|_{\lambda=0} = \epsilon_k (s_N^{-1} - s_j^{-1}).
\end{equation}
Equation (34) follows. \[\square\]

The real subgroup $J_R$ acts on the set of spectral data by tensor product $L \mapsto L \otimes L$, for $L \in J_R$. This action gives what are usually called the “higher flows” in integrable systems language.

**Lemma 4.14.** The real subgroup $J_R \subset \text{Jac}(X)$ is isomorphic, as a real group, to $\mathbb{C}^\times \setminus \mathbb{C}^\times$. In the coordinates (28) on $\text{Jac}(X)$ it corresponds to the subgroup
\begin{equation}
\{ t_{kj} \in \mathbb{C}^\times : t_{1j} = \frac{t_j}{t_N}, \quad t_{2j} = \frac{\bar{t}_{j+N}}{t_{N/2}}, \quad t_{3j} = \frac{\bar{t}_j}{t_N}, \quad t_{4j} = \frac{t_j + N}{t_{N/2}}, \quad \exists t_j \in \mathbb{C}^\times, \quad j \in \mathbb{Z}_N \}.
\end{equation}

This is straightforward to prove: the action of $\text{Jac}(X)$ equates to the action of $(\mathbb{C}^\times)^{4(N-1)}$ as the group of diagonal matrices in $(\text{SL}_N)^4$ on the $4N$ vector $\chi_k(s_j)$. The subgroup $J_R$ is that which preserves the symmetries (32).
5. The moduli spaces of spectral data and of HSL tori.

5.1. Reconstruction from the spectral data. We begin by showing that the spectral data reconstructs the HSL torus $f$, up to dilations and base point preserving isometries which preserve the maximal torus $t_0$. These isometries comprise

$$T_0 = \{ g \in G_0 : \text{Ad} g \cdot t_0 = t_0 \} \simeq S^1.$$  

Fix a Maslov class $\beta_0 \in \Gamma^*$ for which $\Gamma^*_\beta_0$ is non-empty, and choose a non-empty subset $\Delta \subseteq \Gamma^*_\beta_0$ which is closed under $\gamma \mapsto -\gamma$. Now define

$$p_{\Delta}(\lambda) = \prod_{\gamma \in \Delta} (\lambda - 2\gamma/\beta_0)$$  

and let $\Theta_\Delta = \{ 2\gamma/\beta_0 : \gamma \in \Delta \}$ (whose elements we label $s_1, \ldots, s_N$ as above). Let $X_\Delta$ denote the completion by smooth points of the curve given by using $p_{\Delta}$ in equations (25), and let $\lambda$ be the rational function on it given by $Z_5$. Let $E_\Delta$ denote the set of all positive divisors $E$ of degree $N - 1$ on the smooth component $C_1$ of $X_\Delta$ and with the property that if $s_j$ lies on $E$ then $-s_j$ does not. Let $\text{Pic}(X)$ be the set of rank 1 coherent sheaves over $X$. We can define a map

$$\mathcal{L} : \mathcal{E}_\Delta \to \overline{\text{Pic}(X_\Delta)}, \quad E \mapsto O(E' + \rho^*E' + \tau^*E' + (\rho\tau)^*E' + \sum_{k=1}^4 P_k).$$

where

$$E' = E - \sum_{s_j \in \text{Supp}(E)} s_j$$

and whenever $E' \neq E$ we mean $\mathcal{L}(E)$ is the direct image of that line bundle over the partial desingularisation of $X_\Delta$ obtained in the manner of lemma 4.7.

**Lemma 5.1.** The map (38) is injective.

**Proof.** Suppose $\mathcal{L}(E) = \mathcal{L}(F)$ for $E, F \in \mathcal{E}_\Delta$. Then there is a rational function $h$ on $X$ for which $h.\mathcal{L}(E)_P = \mathcal{L}(F)_P$ at every stalk. So both must be invertible on $X$ if either is, in which case $h$ has divisor $E - F$. Therefore $h$ is constant on the component $C_5$ and hence constant globally since it has degree $N$ on the other components. When $\mathcal{L}(E)$ is non-invertible, it and $\mathcal{L}(F)$ must both come by direct image of a line bundle over the same curve, and these are therefore equivalent line bundles upstairs. Again $h$ is constant on $C_5$ and on the other components has degree equal to the number of intersection points, hence $h$ is constant globally. \qed

**Proposition 5.2.** To each triple $(X_\Delta, \lambda, \mathcal{L}(E))$, with $E \in \mathcal{E}_\Delta$, there corresponds a based map $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$, determined up to dilations and the action of $T_0$, which is a weakly conformal HSL immersion with Maslov class $\beta_0$ for which the triple is its spectral data.

**Proof.** Given $E$ there is a unique, up to complex scaling, function $\chi_1(\lambda, 0)$ whose divisor is $E + P_1 - N.\infty$. Using the relations (32) this gives $x(\lambda, 0) = \sum \chi_j \epsilon_j$. Now define $x(s_j, z)$ using (24). Then $x(\lambda, z)$ is uniquely determined by all
x(s_j, z) by linear algebra since p(λ) has N zeroes and \( x(λ, z) \) has at most N non-trivial coefficients \( x_1, \ldots, x_N \) as a polynomial in \( λ \). Finally, set

\[
(39) \quad f(z) = \frac{1}{p(1)}(\exp(\frac{1}{2} \beta J)x(1, z) - x(1, 0)).
\]

This is a based map into \( \mathbb{R}^4 \) and it is easy to check that effect of the scaling is to dilate and rotate \( f \), with rotations from \( T_0 \).

It remains to show that this map is a HSL immersion. Since by definition we have \( \beta_z = 0 \) it suffices to show that \( Sf_z = if_z \), where \( S = e^{J\beta/2}L e^{-J\beta/2} \) (cf. remark 2.2). Differentiating (39) gives

\[
\frac{d}{dz}(\exp(\frac{1}{2} \beta J)x(1, z)) = p(1)df,
\]

which is (11) at \( λ = 1 \). Since \( x(λ, z) \) also satisfies (24) equation (11) must hold (with \( X_λ = p_Δ(λ)I \)) at the \( N + 1 \) values \( λ = 1, s_1, \ldots, s_N \), and therefore holds for all \( λ \) since both sides are polynomials in \( λ \) with at most \( N + 1 \) non-trivial terms. Consequently (12) holds. Since \( p_Δ(λ) \) is an even polynomial, its coefficient \( p_j \neq 0 \) only if \( j \) is even. When \( p_j \) is non-zero we have

\[
(40) \quad e^{-J\beta/2}f_z = \frac{1}{p_j} (\partial x_j/\partial z + \frac{\pi}{2\beta_0} J x_{j+1}).
\]

Now we use the symmetry \( L x(-\lambda, z) = ix(\lambda, z) \) to deduce that for \( j \) even \( L x_j = ix_j \) and \( L x_{j+1} = -ix_{j+1} \). So when \( L \) is applied to both sides of (40) we obtain

\[
L e^{-J\beta/2}f_z = ie^{-J\beta/2}f_z,
\]

using \( LJ = -JL \). The real symmetry in (32) ensures that \( f \) takes values in \( \mathbb{R}^4 \) and the \( Γ \)-periodicity is ensured by the fact that the Fourier frequencies of \( f \) come from \( Γ^* \).

Finally, the construction above is clearly the reconstruction from spectral data since it works via the polynomial Killing field \( (p_Δ(λ)I, x_λ) \).

\[ \square \]

5.2. The moduli space of spectral data. From now on we fix a torus \( \mathbb{C}/Γ \) and a Maslov class \( β_0 ∈ Γ^* \). We fix \( N = #Γ^*_β_0 \) and set

\[
p(λ) = \prod_{j=1}^{N}(λ - s_j),
\]

so that \( X \) is now the most singular curve possible to obtain a HSL torus with Maslov class \( β_0 \). From this point of view, \( (X, λ) \) is completely determined by \( (Γ, β_0) \). As before, we identify the singular set \( S \) with \( \{s_j : j = 1, \ldots, N\} \).

For every non-empty \( Δ ⊂ Γ^*_β_0 \) of the type used in the previous section \( X_Δ \) is a partial desingularisation of \( X \), obtained by pulling apart the intersections at \( S \setminus \mathcal{S}_Δ \). By proposition 5.2 the moduli space of spectral data for HSL tori with Maslov class \( β_0 \) is \( \mathcal{S}(Γ, β_0) = \cup_Δ \mathcal{E}_Δ \). Also by that proposition, we can think of \( \mathcal{S}(Γ, β_0) \) as the moduli space of based, weakly conformal HSL immersions \( f : \mathbb{C}/Γ → \mathbb{R}^4 \), with Maslov class \( β_0 \), up to dilations and isometries from \( T_0 \). We will show that this is naturally isomorphic to \( \mathbb{C} \mathbb{P}^{N-1} \) and provide
coordinates in which the action of $J_R \subset \text{Jac}(X)$ corresponds to the natural action on Fourier components.

First, let $\text{Div}_{N-1}(C_1)$ denote the set of all positive divisors of degree $N - 1$ on $C_1$. For each $\Delta$ the map
\begin{equation}
(41) \quad \mathcal{E}_\Delta \to \text{Div}_{N-1}(C_1); \quad E \mapsto E + \sum_{s_j \notin \Theta_\Delta} s_j,
\end{equation}
is clearly injective. In fact it gives a bijection from $\mathcal{S}(\Gamma, \beta_0)$ to $\text{Div}_{N-1}(C_1)$, since $E \in \mathcal{E}_\Delta$ is not permitted to contain both $s_j$ and $-s_j$ for any $s_j$. On the other hand, $\text{Div}_{N-1}(C_1)$ is clearly isomorphic to the projective space
\[\mathbb{P}F_{N-1} = \mathbb{P}\{h(\lambda) \in \mathbb{C}[\lambda] : \deg(h) \leq N - 1\}\]
since every polynomial is determined up to scale by its divisor of zeroes. Now we require a result (which must surely be classical) whose proof is given in appendix A.

**Proposition 5.3.** The map
\begin{equation}
(42) \quad \iota : \mathbb{P}F_{N-1} \to \mathbb{C}P^{N-1}; \quad [h] \mapsto [h(s_1), \ldots, h(s_N)],
\end{equation}
is an isomorphism whose inverse is given by $[t] \mapsto [\varphi(\lambda, t)]$ where
\begin{equation}
(43) \quad \varphi(\lambda, t) = \prod_{j=1}^{N}(\lambda - s_j) \det \begin{pmatrix}
1 & s_1 & \ldots & s_1^{N-2} & t_1/(\lambda - s_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & s_N & \ldots & s_N^{N-2} & t_N/(\lambda - s_N)
\end{pmatrix},
\end{equation}
whenever $[t_1, \ldots, t_N] = [t]$. The polynomial $\varphi(\lambda, t)$ has degree less than $N - 1$ if and only if $[t]$ lies on the hyperplane
\begin{equation}
(44) \quad \Theta_\infty = \{[t] \in \mathbb{C}P^{N-1} : \sum_{j=1}^{N} (\prod_{m \neq j} \frac{1}{s_m - s_j})t_j = 0\},
\end{equation}
and has a zero at $\lambda = 0$ precisely when $[t]$ lies on the hyperplane
\begin{equation}
(45) \quad \Theta_0 = \{[t] \in \mathbb{C}P^{N-1} : \sum_{j=1}^{N} (\prod_{m \neq j} \frac{1}{s_m - s_j})\frac{t_j}{s_j} = 0\}.
\end{equation}

Consequently we have fixed an isomorphism $\mathcal{S}(\Gamma, \beta_0) \simeq \mathbb{C}P^{N-1}$, which we think of as assigning (homogeneous) coordinates to each based HSL immersion $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ with Maslov class $\beta_0$, and the proposition shows how to invert this. Before we describe the inversion, note that as a corollary to lemma 4.14 the action of the real subgroup $J_R \subset \text{Jac}(X)$ on $\mathcal{S}(\Gamma, \beta_0)$ corresponds in these coordinates to the standard action of $(\mathbb{C}^\times)^N/\mathbb{C}^\times$ on $\mathbb{C}P^{N-1}$ by diagonal action on homogeneous coordinates precomposed with the isomorphism
\begin{equation}
(46) \quad (\mathbb{C}^\times)^{N-1} \to (\mathbb{C}^\times)^N/\mathbb{C}^\times; \quad (a_1, \ldots, a_{N-1}) \mapsto [a_1, \ldots, a_{N-1}, 1].
\end{equation}
In particular, it follows from (36) that the action of base point translation on $\mathcal{S}(\Gamma, \beta_0)$ corresponds to the action of $\mathbb{C}/\Gamma$ via the real homomorphism
\begin{equation}
(47) \quad \ell : \mathbb{C}/\Gamma \to (\mathbb{C}^\times)^N/\mathbb{C}^\times; \quad z + \Gamma \mapsto [e^{-\gamma_1}(z), \ldots, e^{-\gamma_N}(z)],
\end{equation}
where we recall that $e_\gamma(z) = \exp(2\pi i \langle \gamma, z \rangle)$. 
Now we describe the explicit inversion of the isomorphism from $S(\Gamma, \beta_0)$ to $\mathbb{C}P^{N-1}$ which follows from proposition 5.3. First, by combining (39) with (30) and (32) we have the formula

\begin{equation}
(48) \quad f(z) = 2\text{Re}\left[\frac{1}{p(1)}(\chi_1(1, z)e_{\beta_0/2} - \chi_1(1, 0))v_1 \right] \\
+ 2\text{Im}\left[\frac{1}{p(1)}(\chi_1(-1, z)e_{-\beta_0/2} - \chi_1(-1, 0))Lv_1 \right],
\end{equation}

Thus, as a consequence of corollary 4.12 and proposition 5.3, we achieve the coordinate inversion and characterise the locus of branch points of $f$.

**Proposition 5.4.** The isomorphism $S(\Gamma, \beta_0) \rightarrow \mathbb{C}P^{N-1}$ given above is inverted by assigning to $[t] = [t_j] \in \mathbb{C}P^{N-1}$ the map $f$ given by (48), in which

$$
\chi_1(\lambda, z) = \lambda \varphi(\lambda, t(z)), \quad t_j(z) = t_j e_{-\gamma_j}(z).
$$

Further, $f$ has a branch point at $f(z)$ precisely when $[t(z)]$ lies on the intersection of hyperplanes $\Theta_0 \cap \Theta_\infty$.

Later we will require the explicit expression for $f$, which follows from

\begin{equation}
(49) \quad \chi_1(1, z) = \sum_{j=1}^{N} (-1)^{j-1} \Delta_j t_j e_{-\gamma_j},
\end{equation}

\begin{equation}
(50) \quad \chi_1(-1, z) = \sum_{j=1}^{N} (-1)^{j-1} \tilde{\Delta}_j t_j e_{-\gamma_j},
\end{equation}

where

$$
\Delta_j = \prod_{0 \leq k < \ell \leq N \atop k, k \neq j} (s_{\ell} - s_k), \quad \tilde{\Delta}_j = \prod_{1 \leq k < \ell \leq N+1 \atop k, k \neq j} (s_{\ell} - s_k)
$$

with $s_0 = 1$ and $s_{N+1} = -1$.

**Remark 5.5.** Proposition 5.4 above allows to make some general conclusions about the existence of branch points, by exploiting the projective geometry. For simplicity consider the case where $\beta_0/2 \in \Gamma^*$, so that $\Gamma^*_{\beta_0} \subset \Gamma^*$. This is the case referred to in [8] as “truly periodic”, in that the immersion does not factor through a covered torus. In that case the real homomorphism $\ell$ in (47) has a complexification obtained in the following way. Fix generators $\tau_1, \tau_2$ for $\Gamma$ and identify $\mathbb{C}/\Gamma$ with $S^1 \times S^1$ via

$$
\mathbb{C}/\Gamma \rightarrow S^1 \times S^1 \subset \mathbb{C}^\times \times \mathbb{C}^\times; \quad u\tau_1 + v\tau_2 + \Gamma \mapsto (e^{2\pi i u}, e^{2\pi i v}), \quad u, v \in \mathbb{R}.
$$

The complexification of $\ell$ is

\begin{equation}
(51) \quad \ell^C : \mathbb{C}^\times \times \mathbb{C}^\times \rightarrow (\mathbb{C}^\times)^N / \mathbb{C}^\times; \quad (a, b) \mapsto [a^1 b^{m_1}, \ldots, a^N b^{m_N}],
\end{equation}

where $l_j = -\langle \gamma_j, \tau_1 \rangle$ and $m_j = -\langle \gamma_j, \tau_2 \rangle$ are integers. This gives a homomorphism of algebraic groups. Through every point $[t] \in \mathbb{C}P^{N-1}$ this puts a $\mathbb{C}^\times \times \mathbb{C}^\times$-orbit $\Theta_t$. This extends to provide an algebraic map

$$
\ell^C : \mathbb{C}^1 \times \mathbb{C}^1 \rightarrow \mathbb{C}P^{N-1},
$$

\begin{equation}
([a_0, a_1], [b_0, b_1]) \mapsto \left[\left(\frac{a_1}{a_0}\right)^{l_1}(b_1)^{m_1}t_1, \ldots, \left(\frac{a_1}{a_0}\right)^{l_N}(b_1)^{m_N}t_N \right]
\end{equation}
whose image is the Zariski closure $\bar{O}_t$ of the orbit. We can think of $\bar{O}_t$ as foliated by $\mathbb{C}/\Gamma$-orbits. This gives a $(\mathbb{R}^+)^2$ family, although some will be degenerate orbits.

For generic $[t]$ the pull-back along $\ell^C_t$ of the hyperplanes $\Theta_\infty, \Theta_0$ determines curves $\Sigma_\infty, \Sigma_0$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Since $\Theta_\infty$ and $\Theta_0$ are clearly transverse so are these curves, therefore for generic $[t]$ the $\mathbb{C}/\Gamma$-orbit of $[t]$ will miss these points, and the corresponding HSL map will be an unbranched immersion.

For any branched immersion lying in $\bar{O}_t$ we can give an upper bound to the number of isolated branch points it can possess, using the intersection number $\Sigma_0 \cdot \Sigma_\infty$. Since $\Gamma_{\beta_0}$ is invariant under $\gamma \mapsto -\gamma$ the set of indices $\{l_j, m_j : j = 1, \ldots, N\}$ in (51) is also invariant under change of sign. The degree of the map $\ell^C$ restricted to the first factor $\mathbb{CP}^1 \times \{\text{pt}\}$ is, generically, $\max_{j,m} |l_j - m|$. Since the $\gamma_j$ lie on a circle this equals $2l$, where $l = \max_{j} \{l_j\}$. Similarly $\ell^C$ has degree $2m$ where $m = \max_{j} \{m_j\}$ when restricted to the second factor. Hence $\Sigma_0$ and $\Sigma_\infty$ are curves of type $(2l, 2m)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and have intersection number $8lm$.

5.3. The moduli space of HSL tori. Here we will give a simple description of the moduli space $\mathcal{M}(\Gamma, \beta_0)$ of all weakly conformal HSL tori $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$, based at $f(0) = 0$ and with Maslov class $\beta_0$, modulo base point preserving symplectic isometries and dilations of $\mathbb{R}^4$. As in the previous section, we still assume $N = \#\Gamma_{\beta_0}^*$ for each $j = 1, \ldots, N/2$. We have seen that $x(\lambda, 0)$ is uniquely determined by its values at the roots of $p(\lambda)$, and by the symmetries (18) it suffices to specify the $N/2$ vectors

$$w_j = \frac{1}{\sqrt{p(0)s_j^{N+1}}} x(s_j, 0) \in \mathbb{R}^4, \ j = 1, \ldots, N/2. \tag{52}$$

It is convenient now to identify $\mathbb{R}^4$ with $\mathbb{H}$ so that $J$ represents left multiplication by $i \in \mathbb{H}$ and $L$ represents left multiplication by $j \in \mathbb{H}$. In that case the action of $G_0$ corresponds to the action by right multiplication of the group $\text{Spin}(3)$ of unit quaternions. It follows from lemma 3.4 that the assignment

$$(w_1, \ldots, w_{N/2}) \mapsto x(\lambda, 0) \mapsto f \tag{53}$$

given by (39) intertwines the action of $\mathbb{H}^*$ (by right multiplication) with the action of $\mathbb{R}^+ \times G_0$ (by dilations and symplectic isometries).

We may allow $w_j$ to take any value in $\mathbb{H}$ provided they are not all zero: the subset on which they are non-zero will correspond to $\Delta f$. So (53) descends to a bijection. In fact if we give the latter the manifold structure it inherits from the Banach space $C^2(S^1 \times S^1, \mathbb{R}^4)$, equipped with norm of uniform convergence on derivatives up to second order, we see this must be a diffeomorphism.

**Proposition 5.6.** Set $n = \frac{N}{2} - 1$. Whenever $N > 0$, the map $\mathbb{H}^n \to \mathcal{M}(\Gamma, \beta_0)$ defined by (53) is a diffeomorphism.

Of course, the two moduli spaces $\mathcal{S}(\Gamma, \beta_0)$ and $\mathcal{M}(\Gamma, \beta_0)$ are not isomorphic, since the first labels $T_{\beta}$-orbits and the second labels $G_0$-orbits. Hence the former will be a $G_0/T_0 \simeq S^2$ bundle over the latter. The next result explains exactly what this is.
Proposition 5.7. The action of $G_0$ on $\mathcal{S}(\Gamma, \beta_0)$ via base point preserving isometries is equivalent, in appropriate homogeneous coordinates, to the action of $SU(2)$ on $\mathbb{C}P^{n-1}$ whose orbits are the fibres of $\mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$.

Proof. If we take the coordinates on $\mathcal{M}(\Gamma, \beta_0)$ from (53) then we must make a change coordinates on the real manifold $\mathcal{S}(\Gamma, \beta_0)$: we identify the spectral data for $f$ with the point

$$\chi(1) = \chi(k_1, 0), \chi(2)(s_1, 0), \ldots, \chi(1)(s_{n+1}, 0), \chi(2)(s_{n+1}, 0) \in \mathbb{C}P^{2n+1}.$$

Here we are using the fact, from (32), that $\chi_2(s_1, 0) = is_j^{N+1}p(0)\chi_1(s_1+N/2, 0)$ for $1 \leq j \leq n + 1$. On the other hand the point in $\mathcal{M}(\Gamma, \beta_0)$ corresponding to $f$ is identified by the quaternionic homogeneous coordinates

$$[w_1, \ldots, w_{n+1}]_{\mathbb{H}} \in \mathbb{H}P^n, \quad w_j = \frac{1}{\sqrt{p(0)s_j^{N+1}}}x(s_j, 0).$$

Here the identification of $\mathbb{R}^4$ with $\mathbb{H}$ is via

$$w_j = \sum_{k=1}^{4} w_{jk}e_k = (w_{j1} + w_{j2}J + w_{j3}L + w_{j4}JL)e_j.$$

Now we observe that

$$w_j = \frac{1}{\sqrt{2}}[(w_{j1} + iw_{j2})v_1 + (w_{j3} + w_{j4})v_2 + (w_{j1} - iw_{j2})v_3 + (w_{j3} - w_{j4})v_4],$$

and therefore

$$\chi_1(s_1, 0) = \sqrt{\frac{s_j^{N+1}p(0)}{2}}(w_{j1} + iw_{j2}), \quad \chi_2(s_1, 0) = \sqrt{\frac{s_j^{N+1}p(0)}{2}}(w_{j3} + iw_{j4}).$$

Therefore the natural fibration $\mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$ obtained by the identification $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ matches the map $\mathcal{S}(\Gamma, \beta_0) \rightarrow \mathcal{M}(\Gamma, \beta_0)$ in the chosen coordinates.

\[\square\]

6. Higher flows and Hamiltonian variations.

The action of the real subgroup $J_R \subset \text{Jac}(X)$ generates the so-called higher flows. These give Lagrangian variations for our immersed Lagrangian tori, and we will show that a codimension 1 subspace of these are actually Hamiltonian variations. First we need to explain what we mean by Lagrangian and Hamiltonian vector fields along a Lagrangian immersion.

For a Lagrangian immersion $f : M \rightarrow N$ into a Kähler manifold $N$ a section $T \in \Gamma(f^{-1}TN)$ is called a Lagrangian vector field along $f$ if $\sigma_T = f^*(T|\omega)$ is a closed 1-form, and $T$ is Hamiltonian when $\sigma_T$ is exact. We denote the vector space of Lagrangian vector fields along $f$ by $\mathcal{X}_{\text{Lag}}(f)$ and use $\mathcal{X}_{\text{Ham}}(f)$ to denote the subspace of Hamiltonian vector fields. As one expects, families of Lagrangian immersions give rise to Lagrangian vector fields along an immersion [7].

Now we consider the Lagrangian variations corresponding to the higher flows. For simplicity set $S = S(\Gamma, \beta_0)$. Let $\mathcal{H} \simeq \mathbb{C}^N \setminus \{0\}$ be the space of HSL tori in $\mathbb{R}^4$ given by (48), (49) and (50) parameterised by $t \in \mathbb{C}^N \setminus \{0\}$. The map $\mathbb{C}^N \rightarrow$
$\mathbb{CP}^{N-1}$ which sends $t$ to $[t]$ is then the parameterisation of the map $\mathcal{H} \to \mathcal{S}$ which assigns to each HSL torus its spectral data. Suppose $f \in \mathcal{H}$ corresponds to $t = (t_1, \ldots, t_N)$. To each $a = (a_j) \in \mathbb{C}^N$ we can assign the Lagrangian vector field $T = (\partial f / \partial t)_t = 0$ along $f$ corresponding to the Lagrangian deformation $f(t)$ determined by the real curve

$$
(55) \quad t \mapsto (t_1 e^{ia_1}, \ldots, t_N e^{ia_N}), \quad t \in \mathbb{R},
$$

in $\mathcal{S}$. This gives a linear map

$$
(56) \quad T_f : \mathbb{C}^N \to \mathcal{X}_{\text{Lagr}}(f); \quad T_f(a) = T.
$$

It induces a linear map $\mathbb{C}^N / \mathbb{C} \to T[t] \mathcal{S}$ which is tangent to the action of $(\mathbb{C}^\times)^N / \mathbb{C}^\times$ which generates the higher flows for $f$ (46). Notice that what we lose in the quotient are all dilations and the rotations from $T_0$.

Now recall that $Z^1(\mathbb{C}/\Gamma)$ carries the Hodge inner product

$$
\langle \sigma_1, \sigma_2 \rangle = \frac{1}{A(\mathbb{C}/\Gamma)} \int_{\mathbb{C}/\Gamma} \sigma_1 \wedge \ast \sigma_2,
$$

from which we obtain the orthogonal decomposition $Z^1(\mathbb{C}/\Gamma) = \mathcal{H}^1(\mathbb{C}/\Gamma) \oplus B^1(\mathbb{C}/\Gamma)$, where the first summand is the space of harmonic 1-forms. Since $d\beta$ and $\ast d\beta$ span $\mathcal{H}^1(\mathbb{C}/\Gamma)$ we know that $\sigma_T \in Z^1(\mathbb{C}/\Gamma)$ is exact, and thus $T$ is Hamiltonian, precisely when

$$
(\sigma_T, d\beta) = 0 = (\sigma_T, \ast d\beta).
$$

Notice that we can write

$$
\sigma_T = JT \cdot df,
$$

using dot product notation for the metric on $\mathbb{R}^4$ and its complex bilinear extension.

**Theorem 6.1.** Let $T = T_f(a) \in \mathcal{X}_{\text{Lagr}}(f)$ be given by (56). Then $T$ is Hamiltonian along $f$ precisely when both real equations

$$
(57) \quad \sum_{j=1}^N \frac{\text{Re}(a_j)|t_j|^2}{|s_j - s_j|^2} = 0, \quad \sum_{j=1}^N \frac{\text{Re}(s_j)}{\text{Im}(s_j)} \frac{\text{Re}(a_j)|t_j|^2}{|s_j - s_j|^2} = 0,
$$

are satisfied.

The first equation in (57) is the statement that $T_f(a)$ is a stationary variation for area (cf. the area formula (62) below). Together, equations (57) can also be derived as the condition that the Liouville form, which when pulled back along $f$ is the 1-form $\sigma_f = J f \cdot df$ on $\mathbb{C}/\Gamma$, has stationary $\Gamma$-periods for the variation $T_f(a)$. These periods are well-known to be Hamiltonian isotopy invariants.

**Proof.** Let us begin by writing $\sigma_T = cdz + \bar{c}d\bar{z}, \ c = J T \cdot f_z$. Since we are working over a torus, we can simplify computations significantly by observing that the orthogonal projection of $\sigma_T$ onto $\mathcal{H}^1(\mathbb{C}/\Gamma)$ is just its Fourier zero mode, i.e., the coefficient of $e_0$ in the Fourier decomposition. Hence, if $c_0$ denotes the Fourier zero mode of $c$ then

$$
(58) \quad (\sigma_T, d\beta) = \pi(c_0 dz + \bar{c}_0 d\bar{z}, \beta_0 dz + \bar{\beta}_0 d\bar{z}) = 4\pi \text{Re}(c_0 \beta_0),
$$

$$
(59) \quad (\sigma_T, \ast d\beta) = \pi(c_0 dz + \bar{c}_0 d\bar{z}, -i\beta_0 dz + i\bar{\beta}_0 d\bar{z}) = -4\pi \text{Im}(c_0 \beta_0).
$$
To compute $c_0$ we first simplify the formula (48) by writing it as

$$f = A v_1 + \tilde{A} \tilde{v}_1 + B v_2 + B \tilde{v}_2,$$

using

$$A = \frac{1}{p(1)} (\chi_1(1,z) e^{\beta_0/2} - \chi_1(1,0)), \quad B = \frac{1}{ip(1)} (\chi_1(-1,z) e^{-\beta_0/2} - \chi_1(-1,0)).$$

Since $f(0) = 0$ we may write each of the functions $A, B : \mathbb{C}/\Gamma \to \mathbb{C}$ as

$$A = \sum_{j=1}^{N} A_j (e^{-\gamma_j + \beta_0/2} - 1), \quad B = \sum_{j=1}^{N} B_j (e^{-\gamma_j - \beta_0/2} - 1),$$

and, using equations (49) and (50),

$$A_j = \frac{(-1)^{j-1}}{p(1)} \Delta_j t_j, \quad B_j = \frac{(-1)^{j-1}}{ip(1)} \tilde{\Delta}_j t_j.$$ 

The deformation $f(t)$ is given by

$$A(t) = \sum_{j=1}^{N} e^{\lambda t} A_j (e^{-\gamma_j + \beta_0/2} - 1), \quad B(t) = \sum_{j=1}^{N} e^{\lambda t} B_j (e^{-\gamma_j - \beta_0/2} - 1).$$

Since $v_1, \tilde{v}_1, v_2, \tilde{v}_2$ is a Hermitian orthonormal frame for $\mathbb{C}^4$ we find that

$$J f_t \cdot f_z = i \left[ \frac{\partial A}{\partial t} \frac{\partial \tilde{A}}{\partial \tilde{z}} - \frac{\partial \tilde{A}}{\partial \tilde{t}} \frac{\partial A}{\partial z} - \frac{\partial B}{\partial t} \frac{\partial \tilde{B}}{\partial \tilde{z}} + \frac{\partial \tilde{B}}{\partial \tilde{t}} \frac{\partial B}{\partial z} \right].$$

We compute at $t = 0$

$$\frac{\partial A}{\partial t} = \sum_{j=1}^{N} a_j A_j (e^{\beta_0/2 - \gamma_j} - 1), \quad \frac{\partial \tilde{A}}{\partial t} = \sum_{j=1}^{N} \tilde{a}_j \tilde{A}_j (e^{\gamma_j - \beta_0/2} - 1)$$

and similar expressions for $B$. The Fourier zero mode $c_0$ of $J T \cdot f_z$ is given by the following calculation. To simplify it, we let $V_N$ stand for the Vandermonde determinant $\prod_{1 \leq k < \ell \leq N} (s_{\ell} - s_k)$ (cf. appendix A below).

$$c_0 = \pi \left( \sum_{j=1}^{N} \left( \frac{\beta_0}{2} - \tilde{\gamma}_j \right) (a_j + \tilde{a}_j) |A_j|^2 - \sum_{j=1}^{N} \left( -\frac{\beta_0}{2} - \gamma_j \right) (a_j + \tilde{a}_j) |B_j|^2 \right)$$

$$= 2\pi \sum_{j=1}^{N} \left( \left( \frac{\beta_0}{2} - \tilde{\gamma}_j \right) |A_j|^2 + \left( \frac{\beta_0}{2} + \gamma_j \right) |B_j|^2 \right) \text{Re}(a_j)$$

$$= \frac{\pi \beta_0}{|p(1)|^2} \sum_{j=1}^{N} \left( (1 - s_j) |\Delta_j|^2 + (1 + s_j) |\tilde{\Delta}_j|^2 \right) \text{Re}(a_j) |t_j|^2$$

$$= \pi \beta_0 |V_N|^2 \sum_{j=1}^{N} \left( \frac{1}{1 - s_j} + \frac{1}{1 + s_j} \right) \prod_{k \neq j} \left| s_k - s_j \right|^2 \text{Re}(a_j) |t_j|^2.$$
where we have used the property $p(1) = p(-1)$ to write
\[
|\Delta_j|^2 = \frac{|p(1)|^2|V_N|^2}{|1 - s_j|^2 \prod_{k \neq j} |s_k - s_j|^2} = \frac{|1 + s_j|^2}{|1 - s_j|^2} |\bar{\Delta}_j|^2.
\]
It follows that
\[
(\sigma_T, d\beta) = 4\pi^2 |\beta_0|^2 |V_N|^2 \sum_{j=1}^{N} \frac{\text{Re}(a_j)|t_j|^2}{\prod_{k \neq j} |s_k - s_j|^2}
\]
\[
(\sigma_T, *d\beta) = -4\pi^2 |\beta_0|^2 |V_N|^2 \sum_{j=1}^{N} \frac{\text{Re}(s_j)\text{Re}(a_j)|t_j|^2}{\text{Im}(s_j) \prod_{k \neq j} |s_k - s_j|^2}.
\]
\[\square\]

Remark 6.2. It is interesting to note that, given the isomorphism $J_R \simeq (S^1)^{N-1} \times (\mathbb{R}^+)^{N-1}$, all the higher flows tangent to the compact factor of $J_R$ are Hamiltonian, but not all higher flows are Hamiltonian, in the sense that there are vector fields $T_J(a)$ which are not Hamiltonian and whose projections onto $S$ give non-trivial higher flows. This is a reflection of the fact that the area functional is non-constant on $S$. Using the expressions above it is straightforward to show that the area of $f(\mathbb{C}/\Gamma)$, when $f$ corresponds to $[t] \in \mathbb{CP}^{N-1}$, is given by
\[
A(f) = \int_{\mathbb{C}/\Gamma} |f_z|^2 |dz|^2 = A(\mathbb{C}/\Gamma) \frac{\pi^2 |\beta_0|^2 |V_N|^2}{2} \sum_{j=1}^{N} \frac{|t_j|^2}{\prod_{k \neq j} |s_k - s_j|^2}.
\]
Here, of course, we run into the problem that the dilations act on the area but have been factored out of the space $S$ but we can restrict $A(f)$ to the unit sphere $S^{2N-1} \subset \mathbb{C}^N$: it then descends to $\mathbb{CP}^{N-1}$. It is tantalising to observe that, up to a constant independent of $t$, $A(f)$ equals
\[
\int_{\mathbb{C}/\Gamma} |\theta_{\infty}(z)|^2 |dz|^2,
\]
where $\theta_{\infty}$ is defined in (65).

Remark 6.3. Our final remark concerns the dependence of $A(f)$ on the spectral genus of $f$. Here we interpret the spectral genus of $f$ to be the arithmetic genus $g_f$ of $X_f$. From the earlier discussion $g_f = 4(N_1 - 1)$ where $N_1$ is the complex dimension of the $J_R$-orbit of the spectral data for $f$ in $S(\Gamma, \beta_0^*)$. The closure of these $J_R$-orbits provides a stratification of $S(\Gamma, \beta_0^*)$, and as we have observed this agrees with the stratification of $\mathbb{CP}^{N-1}$ by $X_{\mathbb{CP}^{N-1}}$. In particular, one can decrease the spectral genus by going to the boundary of a $J_R$-orbit. In our parameterisation this corresponds to taking the limit as some $t_j \to 0$.

To make proper sense of the dependence of $A(f)$ on spectral genus we can restrict $A(f)$ to the unit sphere $S^{2N-1} \subset \mathbb{C}^N$: it then descends to $\mathbb{CP}^{N-1}$. Clearly “going to the boundary of spectral genus” decreases $A(f)$. Moreover, it is easy to show, using Lagrange multipliers, that $A(f)$ has critical points only at the homogeneous tori (those which arise as orbits of a homomorphism from $\mathbb{C}/\Gamma$ into $G$). These are all minima of $A(f)$. These are also the HSL tori with lowest
(non-trivial) spectral genus. While both of these observations are elementary for this geometry, they are an echo of what one hopes to find in other, more complicated, integrable surface theory (cf. the work of Kilian and Schmidt [11] on deformations of CMC tori into cylinders, which is a form of “going to the boundary of spectral data” to decrease spectral genus, for example).

**APPENDIX A. THE \( \theta \)-FUNCTION.**

Here we will prove proposition 5.3 and also explain its geometric meaning: it is essentially the Riemann vanishing theorem for a singular curve related to our spectral curve \( X \).

At an algebraic level the proof is straightforward. For any \( h \in F_{N-1} \) write \( h(\lambda) = \sum_{k=0}^{N-1} h_k \lambda^k \). Given \( [t] = [t_1, \ldots, t_N] \in \mathbb{C}P^{N-1} \) we have \( \iota([h]) = [t] \) precisely when we can solve the linear system

\[
\begin{pmatrix}
1 & s_1 & \cdots & s_1^{N-1} \\
\vdots & \vdots & & \vdots \\
1 & s_{N-1} & \cdots & s_{N-1}^{N-1} \\
1 & s_N & \cdots & s_N^{N-1}
\end{pmatrix}
\begin{pmatrix}
h_0 \\
h_{N-2} \\
h_{N-1} \\
t_{N-1} \\
t_N
\end{pmatrix}
= 
\begin{pmatrix}
t_1 \\
\vdots \\
t_{N-1} \\
t_N
\end{pmatrix}
\tag{64}
\]

The matrix has determinant \( V_N = \prod_{m>j}(s_m - s_j) \) (it is a Vandermonde matrix), hence it is invertible whenever \( s_1, \ldots, s_N \) are distinct. Hence \( \iota \) is an isomorphism. Further, \( \iota([\varphi(\lambda, t)]) = [t] \) since

\[
\varphi(s_j, t) = \left( \prod_{m\neq j}(s_m - s_j) \right) \det
\begin{pmatrix}
1 & s_1 & \cdots & s_1^{N-2} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
1 & s_j & \cdots & s_j^{N-2} & t_j \\
\vdots & \vdots & & \vdots & \vdots \\
1 & s_N & \cdots & s_N^{N-2} & 0
\end{pmatrix}
= t_j V_N.
\]

Finally, by applying Cramer’s rule to (64) it follows that \( h_{N-1} = 0 \) precisely when

\[
\theta_\infty(t) = \det
\begin{pmatrix}
1 & s_1 & \cdots & s_1^{N-2} & t_1 \\
\vdots & \vdots & & \vdots & \vdots \\
1 & s_N & \cdots & s_N^{N-2} & t_N
\end{pmatrix} = V_N \sum_{j=1}^{N} \left( \prod_{m \neq j} \frac{1}{s_m - s_j} \right) t_j = 0.
\tag{65}
\]

Similarly, \( h_0 = 0 \) when

\[
\theta_0(t) = \begin{pmatrix}
t_1 & s_1 & \cdots & s_1^{N-1} \\
\vdots & \vdots & & \vdots \\
t_N & s_N & \cdots & s_N^{N-1}
\end{pmatrix} = (\prod_{j=1}^{N} s_j) V_N \sum_{j=1}^{N} \left( \prod_{m \neq j} \frac{1}{s_m - s_j} \right) \frac{t_j}{s_j} = 0.
\tag{66}
\]

This completes the proof.

The geometric meaning is as follows (cf. the discussion of rational nodal curves in [15, 3.251]). Let \( Y \) be the rational singular curve obtained from \( \hat{\mathcal{C}} \) by identifying the points \( s_1, \ldots, s_N \) into one singularity \( s \). This curve has
arithmetic genus $g = N - 1$ and one can show that \( \text{Jac}(Y) \simeq (\mathbb{C}^\times)^{N-1} \). It has an Abel map
\[
A_\infty : Y \setminus \{s\} \to \text{Jac}(Y); \quad Q \mapsto \mathcal{O}_Y(Q - \infty).
\]
We can identify \( Y \setminus \{s\} \) with \( \mathbb{C} \setminus \mathcal{S} \) and \( \text{Jac}(Y) \) with
\[
\{[t_1, \ldots, t_{N-1}, t_N] \in \mathbb{CP}^{N-1} : t_j \neq 0 \forall j\}.
\]
Since \( \mathcal{O}_Y(Q - \infty) \) has unique, up to scale, global section \( \lambda - Q \) the Abel map extends holomorphically to the map
\[
A_\infty : \mathbb{C} \to \mathbb{CP}^{N-1}; \quad A_\infty(Q) = [s_1 - Q, \ldots, s_N - Q].
\]
By analogy with the case of compact Riemann surfaces, we say the \( \theta \)-divisor for \( Y \) (given base point \( \infty \)) is the image of \( (Y \setminus \{s\})^{(N-2)} \) in \( \text{Jac}(Y) \) under the Abel map on divisors (note that \( N - 2 = g - 1 \)). This image is the set of all divisor classes of the form \( [E - (N - 2)\infty, \infty] \), hence corresponds to polynomials of degree strictly less than \( N - 1 \). Therefore this \( \theta \)-divisor is \( \text{Jac}(Y) \cap \Theta_\infty \). Similarly, \( \Theta_0 \) is a translate of this \( \theta \)-divisor. The function \( \varphi \) plays the role of a translate of the \( \theta \)-function pulled back to \( Y \) along the Abel map. In this context proposition 5.3 is the analogue of Riemann’s vanishing theorem.

The link with our spectral curve \( X \) is that there is an isomorphism of real groups \( \text{Jac}(Y) \simeq J_R \) given by identifying \( Y \setminus \{s\} \) with \( C_1 \setminus \mathcal{S} \) and mapping
\[
\mathcal{O}_Y(D) \to \mathcal{O}_X(D + \rho^*D + \tau^*D + (\rho\tau)^*D), \quad D \in \text{Div}_0(Y \setminus \{s\}).
\]

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