Edge States from
Defects on the Noncommutative Plane*

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Abstract

We illustrate how boundary states are recovered when going from a noncommutative manifold to a commutative one with a boundary. Our example is the noncommutative plane with a defect, whose commutative limit was found to be a punctured plane - so here the boundary is one point. Defects were introduced by removing states from the standard harmonic oscillator Hilbert space. For Chern-Simons theory, the defect acts as a source, which was found to be associated with a nonlinear deformation of the $\mathfrak{w}_\infty$ algebra. The undeformed $\mathfrak{w}_\infty$ algebra is recovered in the commutative limit, and here we show that its spatial support is in a tiny region near the puncture.

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1 Introduction

Field theory on the noncommutative plane has been well studied.\cite{1} It is a nontrivial deformation of field theory on the commutative plane and has implications for strings, gravity, renormalization and the fractional quantum Hall effect (FQHE). There are good reasons for finding analogous deformations of other commutative manifolds. Of particular interest, especially for topological theories, are manifolds with boundaries. A primary example is Chern-Simons theory, which is empty on the plane. The same applies for Chern-Simons theory on the noncommutative plane. The theory becomes nontrivial when written on a commutative manifold with a boundary, and all the dynamics resides at the boundary.\cite{2},\cite{3} These are so-called edge states, a familiar phenomena in the FQHE. It is then natural to search for noncommutative manifolds which in some limit reduce to commutative manifolds with boundaries, and to demonstrate that edge states are recovered in the limit. This is of physical interest because of the proposal by Susskind that noncommutative $U(1)$ Chern-Simons theory may describe short distance effects in the FQHE.\cite{4} Another exciting possibility is the application to $2 + 1$ quantum gravity which can be written in terms of Chern-Simons theory based on the Poincaré gauge group.\cite{5}

The manifold we consider is one with a very simple boundary. It is the plane with one point removed, where the puncture possibly represents a quasiparticle in the FQHE or a conical singularity in $2 + 1$ gravity. In ref. \cite{6} we showed that the noncommutative analogue of this manifold is obtained after removing states from the infinite dimensional harmonic oscillator Hilbert space of the noncommutative plane. In ref. \cite{7} we wrote down Chern-Simons theory on this manifold, and showed that the gauge invariant observables form a nonlinear deformation of the $w_\infty$ algebra. The ‘undeformed’ $w_\infty$ algebra is associated with area preserving diffeomorphisms, and its relevance for FQHE phenomena is known.\cite{8} So if the proposal of ref. \cite{4} is correct the nonlinear deformation may provide a more accurate description for the FQHE. The ‘undeformed’ $w_\infty$ algebra is recovered in the commutative limit. Here we show that its spatial support is in a shrinking region near the puncture. Therefore we recover edge states in the limit.

There has been an effort to write Chern-Simons systems on the space of finite matrices for the purpose of describing a FQHE droplet.\cite{9} It was found that for these systems the question of taking the commutative limit and recovering systems with boundaries and edge states is subtle.\cite{10} Recent progress has been made in \cite{11} and \cite{12}. The approach taken here may also be adapted in this direction.\cite{7}

In section 2 we review the noncommutative plane, while defects are inserted in section 3. Chern-Simons theory is written on the noncommutative plane in sec. 4 and on the noncommutative plane with defect in sec. 5 where we recover edge states. Although this report relies heavily on calculations in refs. \cite{6}, \cite{7}, \cite{13} we make it as self-contained as space allows.
2 The noncommutative plane

The noncommutative plane $\mathcal{M}^{(0)}$ is generated by the Heisenberg algebra. Say operator $z$ and its hermitian conjugate $z^\dagger$ satisfy the commutation relation

$$[z, z^\dagger] = \Theta_0 ,$$

where $\Theta_0$ is a dimensionful nonzero central element, and is identified with the noncommutativity parameter. By rescaling $z$ and $z^\dagger$ one gets standard annihilation and creation operators $a$ and $a^\dagger$, $z = \sqrt{\Theta_0} \ a$ and $z^\dagger = \sqrt{\Theta_0} \ a^\dagger$, which can be realized on the harmonic oscillator Hilbert space $H^{(0)}$, having basis $\{|n> , \ n = 0, 1, 2, ...\}$, with $a|0> = 0$ and hence $z|0> = 0$. A field $\Phi$ on the noncommutative plane is a polynomial function of $z$ and $z^\dagger$. A pair of commuting inner derivatives $\nabla_z$ and $\nabla_{z^\dagger}$ can be defined

$$\nabla_z = -i[p, ] , \quad \nabla_{z^\dagger} = -i[p^\dagger, ]$$

The requirement that they commute means that the commutator of the operator $p$ with its hermitian conjugate $p^\dagger$ is a central element. We once again get a Heisenberg algebra, which expressable in terms of $a$ and $a^\dagger$. A convenient choice is

$$p = -i\Theta_0^{-1/2} \ a^\dagger , \quad p^\dagger = i\Theta_0^{-1/2} \ a ,$$

for then $\nabla_z$ and $\nabla_{z^\dagger}$ resemble ordinary partial derivatives: $\nabla_z z = \nabla_{z^\dagger} z^\dagger = 1$, $\nabla_z z^\dagger = \nabla_{z^\dagger} z = 0$.

Using standard coherent states the product between functions on the noncommutative plane can be mapped to an associative star product between functions on the commutative plane. The standard coherent states $\{|\zeta> , \zeta \in \mathbb{C}\}$ form an overcomplete basis of unit vectors with a resolution of unity which diagonalize $a$, and hence $z$

$$z|\zeta> = \zeta|\zeta>$$

They can be obtained by acting on the ground state $|0>$ with the unitary operator $U(\zeta, \bar{\zeta}) = \exp i(\zeta p + \bar{\zeta} p^\dagger)$. A field $\Phi$ on the noncommutative plane is mapped to a function $\phi$ on the complex plane (called the ‘covariant symbol’ of $\Phi$) according to

$$\Phi \rightarrow \phi(\zeta, \bar{\zeta}) = <\zeta|\Phi|\zeta> ,$$

while the operator product between two fields $\Phi$ and $\Psi$ is mapped to the star product

$$\Phi \Psi \rightarrow [\phi \ast \psi](\zeta, \bar{\zeta}) = <\zeta|\Phi \Psi|\zeta>$$

It can be expressed compactly in terms of an infinite number of derivatives

$$\ast = \exp \Theta_0 \frac{\partial}{\partial \zeta} \frac{\partial^\dagger}{\partial \bar{\zeta}} ,$$

and is called the Voros star product[14], which is equivalent to the Moyal star product.[15] The lowest order terms in (7) define the commutative limit $\Theta_0 \rightarrow 0$. In this limit, the star product
between functions reduces to the point-wise product, while the star commutator goes to the Poisson bracket:

\[ [\phi \star \psi](\zeta, \bar{\zeta}) \rightarrow \phi(\zeta, \bar{\zeta}) \psi(\zeta, \bar{\zeta}) \]

\[ [\phi \star \psi - \psi \star \phi](\zeta, \bar{\zeta}) \rightarrow i\Theta_0 \{\phi, \psi\}(\zeta, \bar{\zeta}) , \tag{8} \]

where

\[ \{\phi, \psi\}(\zeta, \bar{\zeta}) = -i \left( \frac{\partial \phi}{\partial \zeta} \frac{\partial \psi}{\partial \bar{\zeta}} - \frac{\partial \phi}{\partial \bar{\zeta}} \frac{\partial \psi}{\partial \zeta} \right) \tag{9} \]

3 The noncommutative plane with defect

In ref. [6] defects were inserted on the noncommutative plane by removing states from \( H^{(0)} \). For example, consider projecting out the first \( n_0 \) states \( |0>, |1>, ..., |n_0 - 1> \), and call the result \( H^{(n_0)} \). This should be the Hilbert space on which generators of the new noncommuting manifold \( \mathcal{M}^{(n_0)} \) act. For convenience we again denote generators by \( z \) and \( z^\dagger \), and assume they behave as lowering and raising operators, respectively. So we now need that \( z|n_0 >= 0 \). If we consider \( H^{(n_0)} \) embedded in \( H^{(0)} \), then the lowering and raising operators \( a \) and \( a^\dagger \) acting on the latter cannot be expressed in terms of the lowering and raising operators \( z \) and \( z^\dagger \) acting on the former. Moreover, (1) cannot be realized on \( H^{(n_0)} \). We can still define a pair of commuting inner derivatives as in (2), with \( p \) and \( p^\dagger \) written again in terms of \( a \), \( a^\dagger \) and the noncommutativity parameter \( \Theta_0 \), as in (3), even though \( p \) is not well defined on \( H^{(n_0)} \). In order that \( \nabla_z \) and \( \nabla_{z^\dagger} \) are well defined on \( \mathcal{M}^{(n_0)} \) we have to impose appropriate boundary conditions on the fields. A field \( \Phi \) is once again a polynomial function of \( z \) and \( z^\dagger \). It has a well defined action on \( H^{(n_0)} \), or alternatively we can set \( \Phi|n_0 - 1 >= \Phi|n_0 - 2 >= ... = \Phi|0 >= 0 \). If \( \nabla_z \Phi \) is to have a well defined action on \( H^{(n_0)} \) we also need \( \Phi|n_0 >= 0 \). (Similarly, if \( \nabla_{z^\dagger} \Phi \) is to have a well defined action on the dual of \( H^{(n_0)} \) we need \( <n_0 |\Phi = 0 \).) Stronger boundary conditions are needed for higher derivatives to be defined.

As stated above, (1) cannot be realized on \( H^{(n_0)} \). Rather one can write

\[ [z, z^\dagger] = \Theta(zz^\dagger) \tag{10} \]

The function \( \Theta \) is assumed to be nonsingular on \( H^{(n_0)} \), and can be determined after reexpressing \( p \) and \( p^\dagger \) in terms of generators \( z \) and \( z^\dagger \). A natural choice is

\[ p = -iz^\dagger \Theta(zz^\dagger)^{-1} , \quad p^\dagger = i\Theta(zz^\dagger)^{-1} z , \tag{11} \]

as it reduces correctly to the case of the noncommutative plane when \( \Theta = \Theta_0 \). More generally, after identifying\(^\dagger\) (3) and (11) a recursion relation was found for \( \Theta \) in ref. [6]. From dimensional

\(^\dagger\)There is a subtlety due to the fact that the action of \( p^\dagger \) on the ground state \( |n_0 > \) differs for (3) and (11), as \( a \) takes \( |n_0 > \) out of \( H^{(n_0)} \). However, due to the boundary conditions on the fields, the derivatives obtained with either (3) or (11) can be identified.
arguments $\Theta$ is linear in $\Theta_0$. Also it has the feature that for states with increasing eigenvalue of the number operator one approaches a constant value of $\Theta$, and thus the noncommutative plane. The behavior is

$$\Theta(z z^\dagger)^{-1}|n> \rightarrow \Theta_0^{-1} \left(1 + \frac{r_0}{\sqrt{\Theta_0} n} \right)|n> , \quad \text{as} \quad n \rightarrow \infty ,$$

(12)

where $r_0$ is a dimensionful constant depending on $n_0$.

The large $n$ limit corresponds to the limit of large distances. This statement can be made more precise after again introducing coherent states. In this regard, the standard coherent states used in the previous section are not convenient for this purpose because they no longer diagonalize $z$. An alternative set of coherent states (which for convenience we again denote by \{|$\zeta>$\}) were developed in ref. [16] which preserve (4). Like the standard coherent states, they form an overcomplete basis of unit vectors with a resolution of unity. On the other hand, they are not obtained by acting on the ground state (now $|n_0>$) with a unitary operator. In ref. [13] we used this set of coherent states to construct a star product, and showed that it reduces to the Voros star product in the limit that $\Theta$ goes to a central element. The construction is the same as in (6), although now this does not result in a compact expression for $\star$ like in (7).

The lowest order terms in a derivative expansion are

$$1 + \frac{\partial}{\partial \zeta} \theta(|\zeta|^2) \frac{\partial}{\partial \bar{\zeta}} + \frac{1}{4} \left[ \frac{\partial^2}{\partial \zeta^2} \frac{\partial}{\partial \zeta} \theta(|\zeta|^2)^2 \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}} \theta(|\zeta|^2)^2 \frac{\partial}{\partial \bar{\zeta}} \right] + \cdots$$

(13)

This can also be regarded as an expansion in $\theta(|\zeta|^2)$, which is the covariant symbol of $\Theta$, $\theta(|\zeta|^2) = \langle \zeta |\Theta(z z^\dagger) |\zeta \rangle$. Since $\theta(|\zeta|^2) \rightarrow 0$ in the limit of vanishing noncommutativity parameter $\Theta_0$, the commutative limit is also the limit of small derivatives. So for $\Theta_0 \rightarrow 0$ we again get (8), but now with the Poisson bracket

$$\{\phi, \psi\}(\zeta, \bar{\zeta}) = -i \theta_{cl}(\zeta^2) \left( \frac{\partial \phi}{\partial \zeta} \frac{\partial \psi}{\partial \bar{\zeta}} - \frac{\partial \phi}{\partial \bar{\zeta}} \frac{\partial \psi}{\partial \zeta} \right) ,$$

(14)

where $\theta_{cl}(\zeta^2) \equiv \lim_{\Theta_0 \rightarrow 0} \theta(\zeta^2) / \Theta_0$ . From (12) the limit corresponds to large $n$ where $\Theta$ is slowly varying. Then\footnote{For $r_0$ nonvanishing in this limit, we also need that $n_0 \rightarrow \infty$, keeping $\Theta_0 n_0$ constant.}

$$\theta_{cl}(\zeta^2) = \frac{1}{1 + \frac{r_0}{\zeta}} ,$$

(15)

which is defined away from the origin. The result shows that the commutative limit is a punctured plane. At large distances $|\zeta| >> r_0$ from the puncture one approaches the Poisson structure associated with the noncommutative plane.

Now consider the operator $P_{n_0} = |n_0> <n_0|$ which projects out the ground state. Its associated covariant symbol is $|<\zeta|n_0>|^2$, which was identified in ref. [6] as a normalization constant in the expansion of the coherent states in terms of the $|n>$ eigenstates. Its asymptotic
form was computed for small $\Theta_0$ in ref. [6] and found to be a Gaussian with support in a tiny region around the puncture:

$$| < \zeta | n_0 > |^2 \rightarrow \exp \left\{ -\frac{ |\zeta|^2 + 2\sqrt{2} r_0 |\zeta|}{\Theta_0} \right\}, \quad \text{as} \quad \Theta_0 \rightarrow 0 \quad (16)$$

The covariant symbol of $P_{n_0}$ is thus defined at a point in the commutative limit. In section 5 we shall see that all gauge invariant observables of Chern-Simons theory on this noncommutative manifold have this feature.

4 Chern-Simons theory on the noncommutative plane

We first consider Chern-Simons theory written on $\mathcal{M}^{(0)} \times \mathbb{R}$, where $\mathbb{R}$ corresponds to time, and show that this is an empty theory. The degrees of freedom for noncommutative $U(1)$ Chern-Simons theory can be taken to be a conjugate pair of potentials $A$ and $A^\dagger$. Under gauge transformations:

$$A \rightarrow iU^\dagger \nabla_z U + U^\dagger AU \quad A^\dagger \rightarrow iU^\dagger \nabla_{z^\dagger} U + U^\dagger A^\dagger U, \quad (17)$$

where $U$ is unitary function. It is convenient to introduce $X = p + A$ and $X^\dagger = p^\dagger + A^\dagger$ for they transform covariantly: $X \rightarrow U^\dagger X U$, $X^\dagger \rightarrow U^\dagger X^\dagger U$. The field strength is

$$F = i\nabla_z A^\dagger - i\nabla_{z^\dagger} A + [A, A^\dagger] = [X, X^\dagger] - [p, p^\dagger], \quad (18)$$

which then also transforms covariantly. The Chern-Simons Lagrangian can be written

$$L_{cs} = k \text{Tr} \left[ \frac{i}{2} \Theta_0 \left( D_t X X^\dagger - X(D_t X)^\dagger \right) + A_0 \right], \quad (19)$$

where $D_t X = \dot{X} - i[A_0, X]$ , the dot denotes a time derivative and Tr is the trace over basis states in $H^{(0)}$. $A_0$ plays the role of a Lagrange multiplier. It is assumed to be hermitian and gauge transform as $A_0 \rightarrow iU^\dagger \dot{U} + U^\dagger A_0 U$ and so $D_t X$ and its hermitian conjugate $(D_t X)^\dagger$ transform covariantly. The constant $k$ is called the level. Gauge invariance of $\exp i \int_{\mathbb{R}} dt \ L_{cs}$ was shown in refs. [17],[18] to lead to level quantization $k = \text{integer} \times \hbar$, and the integer was identified in refs. [4],[9] with the inverse of the filling fraction $\nu$ in the FQHE.

As with Chern-Simons theory on commutative $\mathbb{R}^3$, the above theory is empty. This is easily seen in the canonical formalism. The time derivative terms in $(19)$ define the Poisson structure. The phase space is spanned by matrix elements $\chi^m_n = < n | X | m >$ and $\bar{\chi}^m_n = < n | X^\dagger | m >$, with Poisson brackets

$$\{ \chi^m_n, \bar{\chi}^s_r \} = -\frac{i}{k \Theta_0} \delta^m_r \delta^s_n \quad (20)$$

The remaining terms in the trace in $(19)$ give the Gauss law constraints

$$G^m_n = < n | [X, X^\dagger] | m > + \Theta_0^{-1} \delta^m_n = \chi^r_n \bar{\chi}^m_r - \bar{\chi}^r_n \chi^m_r + \Theta_0^{-1} \delta^m_n \approx 0. \quad (21)$$
They are first class, and from

\[ ik \Theta_0 \{ \chi^m_n, G^s_r \} = \chi_r^m \delta^s_n - \chi^s_n \delta^m_r \]

\[ ik \Theta_0 \{ \tilde{\chi}^m_n, G^s_r \} = \tilde{\chi}_r^m \delta^s_n - \tilde{\chi}^s_n \delta^m_r \]  \hspace{1cm} (22)

generate gauge transformations. Since every first class constraint eliminates two phase space variables, no degrees of freedom remain after projecting to the reduced phase space.

5 Chern-Simons theory on the noncommutative plane with defect

We now consider Chern-Simons theory on \( M^{(n_0)} \times \mathbb{R} \). The field strength \( F \) involves first order derivatives \( \nabla z^A \) and \( \nabla z^A \). For them to be well defined on \( M^{(n_0)} \) we should impose the boundary conditions:

\[ < n_0 | A | n > = < n | A^\dagger | n_0 > = 0 \hspace{.5cm} , \hspace{.5cm} \forall \ n \geq n_0 \]

Since \( p \) and \( p^\dagger \) are proportional to raising and lowering operators, respectively, we can also write

\[ < n_0 | X | n > = < n | X^\dagger | n_0 > = 0 \hspace{.5cm} , \hspace{.5cm} \forall \ n \geq n_0 \]  \hspace{1cm} (23)

In order that these boundary conditions are preserved under gauge transformations we need the unitary matrices to satisfy

\[ < n_0 | U^\dagger | a >= < a | U | n_0 > = 0 \hspace{.5cm} , \hspace{.5cm} \forall \ a, b, ... \geq n_0 + 1 \]  \hspace{1cm} (24)

Since gauge transformations are thereby restricted, not all phase space degrees of freedom in Chern-Simons theory can be gauged away, as was the case previously.

For the Chern-Simons Lagrangian we once again assume (19), only now the trace is over a basis in \( H^{(n_0)} \). Returning to the Hamiltonian formulation, and now imposing the boundary conditions (23), one is left with the following phase space variables:

\[ \chi^b_a = \Theta_0 < a | X | b > \hspace{1cm} \tilde{\chi}^b_a = \Theta_0 < a | X^\dagger | b > , \]

\[ \psi_a = < a | X | n_0 > \hspace{1cm} \bar{\psi}^a = < n_0 | X^\dagger | a > , \]  \hspace{1cm} (25)

where again \( a, b, ... > n_0 \), and we have re-scaled \( \chi \) and \( \tilde{\chi} \) in order to later obtain the desired commutative limit. The nonzero Poisson brackets are

\[ \{ \chi^b_a, \tilde{\chi}^d_c \} = -\frac{i}{k} \Theta_0 \delta^b_c \delta^d_a \hspace{1cm} \{ \psi_a, \bar{\psi}^b \} = -\frac{i}{k \Theta_0} \delta^b_a \]  \hspace{1cm} (26)

For later convenience we also re-scale the Gauss law constraints:

\[ G^b_a = \Theta_0^2 < a | [X, X^\dagger] | b > + \Theta_0 \delta^b_a = \chi^b_a \chi^a_c - \tilde{\chi}^b_a \tilde{\chi}^a_c + \Theta_0 \delta^b_a + \Theta_0^2 \psi_a \bar{\psi}^b \approx 0 \]  \hspace{1cm} (27)
They generate gauge transformations which are consistent with (24):

\[ ik\Theta_0^{-1} \left\{ \chi^b_a, G^d_c \right\} = \chi^b_a \delta^d_c - \chi^d_a \delta^b_c \]

\[ ik\Theta_0^{-1} \left\{ \bar{\chi}^b_a, G^d_c \right\} = \bar{\chi}^b_a \delta^d_c - \bar{\chi}^d_a \delta^b_c \]

\[ ik\Theta_0^{-1} \left\{ \psi^a_b, G^c_d \right\} = \psi^b_c \delta^d_a - \psi^d_a \delta^b_c \]

\[ ik\Theta_0^{-1} \left\{ \bar{\psi}^a_b, G^c_d \right\} = -\bar{\psi}^b_c \delta^d_a - \bar{\psi}^d_c \delta^b_a \]  

(28)

From a counting argument alone the variables \( \chi^b_a \) and \( \bar{\chi}^b_a \) can be gauged away, leaving only \( \psi^a_b \) and \( \bar{\psi}^a_b \). But the latter are not gauge invariant. Instead they transform as a vector and conjugate vector, while \( \chi^b_a \) and \( \bar{\chi}^b_a \) transform as tensors. We can construct gauge invariant observables of the form⁴

\[ M_{(\alpha, \beta)} = -k \bar{\psi}(\bar{\chi})^\alpha(\chi)^\beta \psi \]  

(29)

Their Poisson bracket algebra, and corresponding quantum algebra, were computed in ref. [7] and found to be a nonlinear deformation of the classical \( w_\infty \) algebra. The classical \( w_\infty \) algebra which generates area preserving diffeomorphisms is recovered in the commutative limit \( \Theta_0 \to 0 \),

\[ \{ M_{(\alpha, \beta)}, M_{(\rho, \sigma)} \} \to -i(\beta \rho - \alpha \sigma) \ M_{(\alpha + \rho - 1, \beta + \sigma - 1)} \]  

(30)

Finally we consider mapping the observables to the complex plane, and taking the commutative limit. For this we once again utilize coherent states of section 3, and construct covariant symbols of the relevant operators. For example, the covariant symbols associated with matrix elements \( \psi_a = \langle a | X | n_0 \rangle \) and \( \bar{\psi}^a = \langle n_0 | X^\dagger | a \rangle \) are

\[ \psi(\zeta, \bar{\zeta}) = \langle \zeta | a > \psi_a < \zeta | n_0 \rangle >, \quad \bar{\psi}(\zeta, \bar{\zeta}) = \langle \zeta | n_0 > \bar{\psi}^a < a | \zeta >, \]  

(31)

respectively. Since the gauge invariant observables \( M_{(\alpha, \beta)} \) involve products of \( \psi_a \) and \( \bar{\psi}^a \), the covariant symbol associated with \( M_{(\alpha, \beta)} \) will involve star products of \( \psi(\zeta, \bar{\zeta}) \) and \( \bar{\psi}(\zeta, \bar{\zeta}) \). They are therefore determined by \( \langle \zeta | n_0 \rangle \to | \zeta \rangle \) and its derivatives. From (16) we then conclude that in the commutative limit, all of the \( w_\infty \) observables are concentrated near the point singularity. The area preserving diffeomorphism generators are then edge states.

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⁴Another set of gauge invariants are \( \text{Tr} \ (\bar{\chi})^\alpha(\chi)^\beta \) but their Poisson bracket algebra is ill-defined due to problems in defining the trace.
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