A NOTE ABOUT THE TOPOLOGICAL TYPE OF FAMILIES OF COMPLEX KLEINIAN GROUPS IN $\mathbb{P}_C^2$

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Abstract. We give a complete description of the topological type of the quotient space $\Omega/G$ of Complex Kleinian Groups with the maximum numbers of complex projectives lines in general position contained in its Kulkarni’s limit set is four.

1. Introduction

The complex Kleinian groups were introduced by José Seade and Alberto Verjosky [17] in order to study the discrete subgroups of the group of automorphisms of complex projective spaces. These groups are natural generalizations of Kleinian groups in the context of hyperbolic spaces. A complex Kleinian group $G$ is a subgroup of $\text{PSL}(n+1, \mathbb{C})$ for which exists a $G$-invariant open non empty set of $\mathbb{P}_C^n$ where the action of $G$ is properly discontinuous. We notice the group $G$ is a discrete subgroup of $\text{PSL}(n+1, \mathbb{C})$ but the converse is not necessarily true, for example the group $\text{PSL}(3, \mathbb{Z})$ is a discrete subgroup of $\text{PSL}(3, \mathbb{C})$ but this not a complex Kleinian group [4] and that give a difference with the kleinian groups of hyperbolic spaces. Another important difference with the classical Kleinian groups is the concept of limit set, in the case of complex Kleinian groups with do not have an standard definition, we have three notions: Kulkarni limit set, Myrberg limit set, complement of a maximal region of discontinuity which are discussed in detail in [], but by some additional hypotheses of all these concepts of limit set are equivalents [1]. In [9] Angel Cano and José Seade show that every infinite discrete subgroup of $\text{PSL}(3, \mathbb{C})$ has a complex projective line contained in the limit set, in consequence, the limit set of infinite subgroups of $\text{PSL}(3, \mathbb{C})$ is an uncountable subset of $\mathbb{P}_C^2$, this result give a big difference with the classical Kleinian groups because the most simple groups called elementary the cardinal of his limit set is 0, 1 or 2 but in the case of complex Kleinian groups the cardinal of the limit set is always zero or infinite.

An interesting problem, in analogy with the classical theory of Kleinian groups, is to define we understand as complex elementary Kleinian group. An alternative is to define an elementary complex Kleinian group, as that group whose limit set contains a finite number of complex projectives lines [6], this definition is good in some cases, but we can construct examples of groups whose limit set contained infinitely many complex projectives lines but only a finite number of them in general position, then we can define an elementary complex Kleinian group of type II of the last way.

In [3] the authors give a caracteriorization of complex Kleinian groups with the maximum numbers of complex projectives lines in general position contained in its Kulkarni’s limit set is four. In this article we described the topology of these groups a we obtain the following theorem

**Theorem 1.1.** Let $G$ be complex Kleinian group with the maximum numbers of complex projective lines in general position contains in it’s Kulkarni limit set is four, then we have the follows:
(1) The group \( G \) is isomorphic to a lattice of the group \( \text{Sol} \).
(2) Let \( \Omega_0 \) be a \( G \)-invariant connected component of the Kulkarni discontinuity region of \( G \), then \( \Omega_0/G \) is diffeomorphic to \((\text{Sol}/G) \times \mathbb{R}\).

**Corollary 1.2.** There is a countable number of complex Kleinian groups non isomorphic with the maximum numbers of complex projective lines in general position contains in it’s Kulkarni limit set is four.

**Corollary 1.3.** Under the hypotheses of Theorem 1.1, \( \Omega_0/\Gamma \) is a fiber bundle with base \( S^1 \times \mathbb{R} \) and fiber \( \mathbb{T}^2 \times \mathbb{R} \).

**Theorem 1.4.** Let \( G \) a lattice of three dimensional real Heisenberg group \( H \), then there exist a \( G \)-invariant open set \( \Omega \subset P^2_\mathbb{C} \) such that \( \Omega/G \) is diffeomorphic to \((H/G) \times \mathbb{R}\).

This article is organized in the following way: In section 1 we give some basic preliminaries about complex Kleinian groups, and we give a description of the groups with maximum numbers of complex lines contained its Kulkarni’s limit set equal four [3]. The next section we give the proof of Theorem 1.1 and its corollaries, in fact the strategy of the proof is the following:

Firstly we notice that \( G \) is naturally identified with the lattice of \( \text{Sol}, \mathbb{Z}^2 \times_A \mathbb{Z} \), where \( A \in \text{SL}(2, \mathbb{Z}) \) is an hyperbolic automorphism of torus. In the second section we defined an immersion of the group \( \text{Sol} \) to the bydisc \( \mathbb{H} \times \mathbb{H} \) and we show this immersion can be extended to a \( G \)-equivariant diffeomorphism of \( \text{Sol} \times \mathbb{R} \) to \( \mathbb{H} \times \mathbb{H} \) and we finish the proof. The corollaries are consequence of the Theorem 1.1 and proposition 30 [10]. For the Theorem 1.4, the procedure is similar to Theorem 1.1 except that we do not have a diffeomorphism \( G \)-equivariant between \( \mathbb{C} \times H \) and \( H \times \mathbb{R} \), nevertheless because the natural action of \( G \) in \( \mathbb{C} \times \mathbb{H} \) translated to \( H \times \mathbb{R} \) is the classical action in the first factor of \( G \) on \( H \) and in the second factor is the identity, we can establish the proof of this theorem.

2. Preliminaries

The purpose of this section is to provide some definitions and results about complex Kleinian groups that will be helpful to the reader, for more details see [6], [1] and [3].

2.1. Projective Geometry. We recall that the complex projective plane \( P^2_\mathbb{C} \) is defined as the orbit space of the usual scalar multiplication action of Lie group \( \mathbb{C}^* \) in \( \mathbb{C}^3 \setminus \{0\} \) and it is denoted by

\[
P^2_\mathbb{C} := (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*.
\]

This is a compact connected complex 2-dimensional manifold. Let \( [\cdot] : \mathbb{C}^3 \setminus \{0\} \to P^2_\mathbb{C} \) be the quotient map. If \( \beta = \{e_1, e_2, e_3\} \) is the standard basis of \( \mathbb{C}^3 \), we write \( [e_j] = e_j \), for \( j = 1, 2, 3 \), and if \( z = (z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{0\} \) then we write \( [z] = [z_1 : z_2 : z_3] \). Also, \( \ell \subset P^2_\mathbb{C} \) is said to be a complex line if \( [\ell]^{-1} \cup \{0\} \) is a complex linear subspace of dimension 2. Given two distinct points \( [z], [w] \in P^2_\mathbb{C} \), there is a unique complex projective line passing through \( [z] \) and \( [w] \), such complex projective line is called a line, for short, and it is denoted by \( [z, w] \). Consider the action of \( \mathbb{C}^* \) on \( \text{GL}(3, \mathbb{C}) \) given by the usual scalar multiplication, then

\[
PGL(3, \mathbb{C}) = \text{GL}(3, \mathbb{C})/\mathbb{C}^*
\]

is a Lie group whose elements are called projective transformations. Let \( [[\cdot]] : \text{GL}(3, \mathbb{C}) \to PGL(3, \mathbb{C}) \) be the quotient map, \( g \in PGL(3, \mathbb{C}) \) and \( g \in \text{GL}(3, \mathbb{C}) \), we say that \( g \) is a lift of \( g \) if \( [[g]] = g \). One can show that \( PGL(3, \mathbb{C}) \) is a Lie group which acts transitively, effectively and by biholomorphisms on \( P^2_\mathbb{C} \) by \( [[g]][[w]] = [g(w)] \), where \( w \in \mathbb{C}^3 \setminus \{0\} \) and \( g \in \text{GL}(3, \mathbb{C}) \).
We could have considered the action of the cube roots of unity \( \{1, \omega, \omega^2\} \subset \mathbb{C}^* \) on \( \text{SL}(3, \mathbb{C}) \) given by the usual scalar multiplication, then

\[
\text{PSL}(3, \mathbb{C}) = \text{SL}(3, \mathbb{C})/\{1, \omega, \omega^2\} \cong \text{PGL}(3, \mathbb{C}).
\]

We denote by \( M_{3 \times 3}(\mathbb{C}) \) the space of all \( 3 \times 3 \) matrices with entries in \( \mathbb{C} \) equipped with the standard topology. The quotient space

\[
\text{SP}(3, \mathbb{C}) := (M_{3 \times 3}(\mathbb{C}) \setminus \{0\})/\mathbb{C}^*
\]

is called the space of pseudo-projective maps of \( \mathbb{P}_3^2 \) and it is naturally identified with the projective space \( \mathbb{P}_3^2 \). Since \( \text{GL}(3, \mathbb{C}) \) is an open, dense, \( \mathbb{C}^* \)-invariant set of \( M_{3 \times 3}(\mathbb{C}) \setminus \{0\} \), we obtain that the space of pseudo-projective maps of \( \mathbb{P}_3^2 \) is a compactification of \( \text{PGL}(3, \mathbb{C}) \) (or \( \text{PSL}(3, \mathbb{C}) \)). As in the case of projective maps, if \( s \) is an element in \( M_{3 \times 3}(\mathbb{C}) \setminus \{0\} \), then \([s]\) denotes the equivalence class of the matrix \( s \) in the space of pseudo-projective maps of \( \mathbb{P}_3^2 \). Also, we say that \( s \in M_{3 \times 3}(\mathbb{C}) \setminus \{0\} \) is a lift of the pseudo-projective map \( S \) whenever \([s] = S\).

Let \( S \) be an element in \( (M_{3 \times 3}(\mathbb{C}) \setminus \{0\})/\mathbb{C}^* \) and \( s \) a lift to \( M_{3 \times 3}(\mathbb{C}) \setminus \{0\} \) of \( S \). The matrix \( s \) induces a non-zero linear transformation \( s : \mathbb{C}^3 \to \mathbb{C}^3 \), which is not necessarily invertible. Let \( \text{Ker}(s) \subset \mathbb{C}^3 \) be its kernel and let \( \text{Ker}(S) \) denote its projectivization to \( \mathbb{P}_3^2 \), taking into account that \( \text{Ker}(S) := \emptyset \) whenever \( \text{Ker}(s) = \{(0,0,0)\} \).

### 2.2. Discontinuous actions on \( \mathbb{P}_3^2 \)

**Definition 2.1.** Let \( G \subset \text{PSL}(3, \mathbb{C}) \) be a group. We say that \( G \) is a **complex Kleinian group** if it acts properly and discontinuously on an open non-empty \( G \)-invariant set \( U \subset \mathbb{P}_3^2 \), that means, for each pair of compact subsets \( C, D \subset U \), the set

\[
\{g \in G : g(C) \cap D \neq \emptyset\}
\]

is **finite**.

One of the main difficulties in deciding whether a group \( G \) is Kleinian complex is to find an open set verifying the definition above. In order to give an answer to this problem we propose study two mathematical concepts: The **Equicontinuity set** of \( G \) and the Kulkarni discontinuity region of \( G \). Now, we discuss each one of these definitions.

### 2.3. The equicontinuity set

The concept of equicontinuity has long been studied in mathematics. For convenience to reader, we include the definition and notation that we use in this work.

**Definition 2.2.** The **equicontinuity set** for a family \( \mathcal{F} \) of endomorphisms of \( \mathbb{P}_3^2 \), denoted \( \text{Eq}(\mathcal{F}) \), is defined as the set of points \( z \in \mathbb{P}_3^2 \) for which there is an open neighbourhood \( U \) of \( z \) such that \( \{f|_U : f \in \mathcal{F}\} \) is a normal family.

This modern approach and ideas of this concept were studied by Angel Cano in his Ph.D thesis, however thanks to reference of Ravi Kulkarni about Myrberg works, we found that some of these results have already been discovered before, only in an arcane mathematical language; however it is fair to recognize Angel Cano for rediscovering these results and apply with success to theory of complex Kleinian groups.

**Definition 2.3.** Let \( G \subset \text{PSL}(3, \mathbb{C}) \) be a discrete group. If

\[ G' = \{ S \text{ is a pseudo-projective map of } \mathbb{P}_3^2 : S \text{ is a cluster point of } G \}; \]

then the **Myrberg limit set** (see [13]) is defined as the set

\[ \Lambda_{\text{Myr}}(G) = \bigcup_{S \in G'} \text{Ker}(S). \]
Myrberg [13] shows that $G$ acts properly and discontinuously on $\mathbb{P}_\mathbb{C}^2 \setminus \Lambda_{\text{Myr}}(G)$.

**Theorem 2.4.** (See [1]) If $G \subset \text{PSL}(3,\mathbb{C})$ is a discrete group, then:

1. The group $G$ acts properly and discontinuously on $\text{Eq}(G)$.
2. The equicontinuity set of $G$ satisfies:
   \[ \text{Eq}(G) = \mathbb{P}_\mathbb{C}^2 \setminus \Lambda_{\text{Myr}}(G) \]
3. If $U$ is an open $G$-invariant subset such that $\mathbb{P}_\mathbb{C}^2 \setminus U$ contains at least three complex lines in general position, then $U \subset \text{Eq}(G)$.

2.4. **Kulkarni discontinuity region.** In 1978 Ravi Kulkarni motivated by the study of classical theory of Kleinian groups defined a limit set for groups of homeomorphism acting on locally compact Hausdorff space. For convenience to reader, we explain this construction in the context of projective spaces

**Definition 2.5.** If $G \subset \text{PSL}(3,\mathbb{C})$ is a group, Kulkarni defines (see [11]):

- The set $L_0(G)$ as the closure of the set of points in $\mathbb{P}_\mathbb{C}^2$ with infinite isotropy group.
- The set $L_1(G)$ is the closure of the set of cluster points of the orbit $Gz$, where $z$ runs over $\mathbb{P}_\mathbb{C}^2 \setminus L_0(G)$.
- The set $L_2(G)$ is the closure of the set of cluster points of the family of compact sets \( \{ g(K) : g \in G \} \), where $K$ runs over all the compact subsets of $\mathbb{P}_\mathbb{C}^2 \setminus (L_0(G) \cup L_1(G))$.

The **Kulkarni limit set** of $G$ is defined as \( \Lambda_{\text{Kul}}(G) = L_0(G) \cup L_1(G) \cup L_2(G) \).

The **Kulkarni discontinuity region** of $G$ is defined as:
\[ \Omega(G) = \mathbb{P}_\mathbb{C}^2 \setminus \Lambda_{\text{Kul}}(G). \]

Kulkarni proves in [11] that $G$ acts properly and discontinuously on the set $\Omega(G)$. However, $\Omega(G)$ is not necessarily the maximal open subset of $\mathbb{P}_\mathbb{C}^2$ where $G$ acts properly and discontinuously.

We notice that the definition of Kulkarni’s limit set is a generalization of limit set of hyperbolic geometry. A difficulties of Kukarni’s approach it is very hard to give an explicit computations of these limit set. In the P.H.D thesis of Juan Navarrete ([14], [15]) we can find these computations for the cyclic subgroups of PSL(3, $\mathbb{C}$) and also for discrete subgroups of PU(2, 1).

A third possibility to see if a group is a complex Kleinian group consist to postulate the existence of an open maximal where the group acts properly and discontinuously, but this approach does not say how to build this $G$-invariant open set, however when we ensure their existence this open set has good properties [4].

2.5. **Four complex Kleinian groups.** This section is devoted of complex Kleinian groups of PSL(3, $\mathbb{C}$) with the maximum numbers of complex projective lines contained in its Kulkarni’s limit set is four. For simplicity we called this kind of group **four complex Kleinian groups.** In [3], the authors gave a caracherization of four complex Kleinian groups. Because in the present work, the article [3] is essential, we reproduce briefly the main ideas and the notation used.

Let $A \in \text{SL}(2, \mathbb{Z})$, with $|\text{tr}(A)| > 2$, we define the following discrete subgroup of PSL(3, $\mathbb{C}$), called hyperbolic toral group
\[ G_A = \left\{ \begin{pmatrix} A^k & b \\ 0 & 1 \end{pmatrix} : b \in M(2 \times 1, \mathbb{Z}), k \in \mathbb{Z} \right\}. \]

The group $G_A$ is a four complex Kleinian group and moreover if $G$ is a four complex Kleinian groups, then there exist $G_A$ such that $[G : G_A] \leq 8$.

It is possible to conjugate the group $G_A$ to a group, still denoted by $G_A$, where each element is of the form
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\[
\begin{pmatrix}
  \lambda^k & 0 & ny_0 + mx_0 \\
  0 & \lambda^{-k} & nx_0 + mz_0 \\
  0 & 0 & 1
\end{pmatrix}
\]

where $k, n$ and $m$ run over $\mathbb{Z}$ and $\lambda$ is one of the eigenvalues of $A$. At this point it is not hard to see the Kulkarni discontinuity region consist of four disjoint copies of $\mathbb{H}^+ \times \mathbb{H}^+$, where $\mathbb{H}^+$ is the upper half plane and $\mathbb{H}^-$ is the lower half plane.

2.6. The Sol geometries. Sol is one of the eight geometries defined por William Thurston in his famous program of geometrization of compact three manifolds. We define the group Sol as follows: given that the space $\mathbb{R}^2 \times \mathbb{R}$ we define the group operation:

\[
\begin{pmatrix}
  x_1 \\
  y_1
\end{pmatrix}
\cdot
\begin{pmatrix}
  x_2 \\
  y_2
\end{pmatrix}
\cdot
\begin{pmatrix}
  x_1 + e^{t_1}x_2 \\
  y_1 + e^{-t_1}y_2
\end{pmatrix}
= \begin{pmatrix}
  x_1 + t_1 + t_2 \\
  y_1 + t_2
\end{pmatrix}
\]

With the previous operation, we have that $\mathbb{R}^3$ is a Lie group and it is denoted by Sol. It is well know that Sol is 3-Riemannian manifold with metric $ds^2 = e^{2t}dx^2 + e^{-2t}dy^2 + dt^2$, when the group Sol is the isometry group. An interesting fact about the Sol geometries, is given by the following theorem of [10], that we state for convenience:

**Proposition 2.1.** Let $A, B$ in $GL(2, \mathbb{Z})$ be two matrices with traces of absolute value strictly larger than 2. The semi-direct product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_B \mathbb{Z}$ are

a) isomorphic if and only if $A$ is conjugate in $GL(2, \mathbb{Z})$ to $B$ or $B^{-1}$.

b) quasi-isometric in all cases.

Then Sol/$\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ gives examples of compact three manifolds where the topological type is determined by the fundamental group. For more details about this subject the reader we can see [16],[18] and [10].

3. FOLIATION OF $\mathbb{H} \times \mathbb{H}$ BY SOL

**Definition 3.1.** Given $\lambda > 0$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a nondegenerated real matrix. Sol is the group consisting of all matrices of the form

\[
\begin{pmatrix}
  \lambda^t & 0 & ax + by \\
  0 & \lambda^{-t} & cx + dy \\
  0 & 0 & 1
\end{pmatrix},
\]

with $t, x, y \in \mathbb{R}$ arbitrary.

**Theorem 3.2.** Sol is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$

**Proof.** See [10] for a proof.

**Definition 3.3.** Let $z_1, z_2 \in \mathbb{H}$. We define the action of Sol in $\mathbb{H} \times \mathbb{H}$ by

\[
\begin{pmatrix}
  \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
  \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}
\end{pmatrix}
\cdot
\begin{pmatrix}
  z_1 \\
  z_2
\end{pmatrix}
= \begin{pmatrix}
  \lambda^t z_1 + ax + by \\
  \lambda^{-t} z_2 + cx + dy
\end{pmatrix}.
\]

**Theorem 3.4.** The natural action of the group Sol on $\mathbb{H}$ verify the following statements

i) The action is free.

ii) For each $z = (z_1, z_2) \in \mathbb{H}$ the function $f_z : Sol \to \mathbb{H}$ defined by $f_z(g) = gz$ is a smooth embedding.
iii) Let \( e_1, e_2, e_3, e_4 \) the canonical basis of \( \mathbb{R}^4 \), then
\[
X = -\ln(\lambda)(\lambda^{-t}y_2e_2 + \lambda^ty_1e_4).
\]
is the normal field of the embedding \( f_z(\text{Sol}) \) in \( \mathbb{H} \), moreover this vector field is smooth.

**Proof.**

i) Let \( z = (z_1, z_2) \in \mathbb{H} \times \mathbb{H} \), and assume that \( \gamma \in \text{Sol} \) is such that \( \gamma \cdot z = z \). Let \( z_k = x_k + i y_k \). Taking imaginary parts in the action, we get
\[
\lambda^ty_1 = y_1, \quad \lambda^{-t}y_2 = y_2,
\]
so, \( \lambda^t = \lambda^{-t} = 1 \), since each \( y_k > 0 \). Taking real parts,
\[
\begin{align*}
x_1 + ax + by &= x_1, \\
x_2 + cx + dy &= x_2,
\end{align*}
\]
and since \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), is non degenerated, \( x = y = 0 \).

ii) From the definition of the action, it is clear that \( f_z \) is smooth in \( t, x, y \), which parametrizes \( \text{Sol} \). By straightforward computations we have that \( df_z \) has Jacobian matrix given by,
\[
[df_z] = \begin{pmatrix}
\ln(\lambda) \lambda^t x_1 & a \\
\ln(\lambda) \lambda^t y_1 & 0 \\
-\ln(\lambda) \lambda^{-t} x_2 & c \\
-\ln(\lambda) \lambda^{-t} y_2 & 0
\end{pmatrix}.
\]
The vectors \( (a, 0, c, d)^t \), \( (b, 0, d, 0)^t \) are linearly independent by the condition in the coefficients \( a, b, c, d \). Since \( y_1 > 0 \), the first column in the Jacobian matrix is linearly independent with the previous vectors. Therefore, \( f_z \) in an immersion.

If \( z_k = x_k + iy_k \), and \( z' = (x'_1, y'_1, x'_2, y'_2) \in \mathbb{H} \times \mathbb{H} \), define \( t \) such that \( \lambda^t = y'_1/y_1 \), and \( (x', y') \) such that
\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} x'_1 - \frac{y'_1x_2}{y_1} \\ x'_2 - \frac{y'_2y_1}{y_1} \end{pmatrix}.
\]

These values for \( (t, x', y') \) define a mapping \( F \), from \( \mathbb{H} \times \mathbb{H} \) to \( \mathbb{R}^3 \cong \text{Sol} \), such that \( F \circ f_z = Id \). Note that \( F \) is a left continuous inverse for \( f_z \), and hence, \( f_z \) is an homeomorphism.

iii) Since the \( e_k \) form an orthonormal basis for the euclidean metric, the vector product in this basis can be calculated by the method of the determinate
\[
X = \ln(\lambda) \begin{vmatrix}
e_1 & e_2 & e_3 & e_4 \\
\lambda^t x_1 & \lambda^ty_1 & -\lambda^{-t} x_2 & -\lambda^{-t} y_2 \\
a & 0 & c & 0 \\
b & 0 & d & 0
\end{vmatrix}.
\]
The result follows. Finally, let \( z \in \mathbb{H} \times \mathbb{H} \) be arbitrary, then trivially \( z \) belongs to the leaf given by \( f_z \). Since \( \text{Sol} \) acts freely, \( z \) is the image of the identity element by \( f_z \). By the expression given in theorem ??,
\[
X = -\ln(\lambda)(y_2e_2 + y_1e_4).
\]

Therefore, \( X \) varies smoothly.

\[\square\]

By this theorem, if we vary \( z \), we obtain a foliation \( f_z(\text{Sol}) \) of \( \mathbb{H} \times \mathbb{H} \) by copies of \( \text{Sol} \). We proceed to show that this foliation is globally rectifiable. In the sequel, we will use the Euclidean metric in \( \mathbb{H} \times \mathbb{H} \) given by the natural identification with a subspace of \( \mathbb{R}^4 \).
Definition 3.5. For \( z \in H \times H \) fixed, define the normal vector field to \( f_z(Sol) \), \( X \), by the triple vector product \( \partial_t f_z \wedge \partial_x f_z \wedge \partial_y f_z \).

4. Geometry of the leaves

In the previous section we described how \( Sol \) foliates the space \( H \times H \) and gave a closed expression for a smooth vector field \( X \) normal to any leaf in the foliation. In this section, we will study the dynamics of the integral curves. Let \( \psi(t) \) be an integral curve of the field, with components \( \psi = (z_1(t), z_2(t)) \), and \( z_k = x_k + y_ki \) as before. From the definition of the field, it follows that the integral curves satisfy the following set of equations:

\[
\begin{align*}
\dot{x}_k &= 0, \\
\dot{y}_k &= y_k.
\end{align*}
\]

These equations can be readily solved to get constant solutions in the real part of each copy of the hyperbolic and exponentials in the imaginary parts. The flow of the normal field defines a one parameter family of diffeomorphisms in \( H \times H \). From now on, we will denote it by \( \psi_t(z_1, z_2) \), where

\[
\psi_t(z_1, z_2) = (x_1, e^{ty_1}, x_2, e^{ty_2}),
\]

where the parameter \( t \) is completely inextensible, i.e. \( t \in \mathbb{R} \).

Theorem 4.1. The flow \( \psi_t \) rules \( H \times H \) by geodesics.

Proof. Every factor \( (x_k, e^{ty_k}) \) in \( \psi_t \) corresponds to the parametrization of a vertical geodesic in \( H \) with the hyperbolic metric. Since the metric we consider is homotetic to the standard hyperbolic metric, with a constant factor, \( (x_k, e^{ty_k}) \) will be the parametrization of a geodesic with this metric as well. Since the metric in \( H \times H \) is a product, the result follows.

Theorem 4.2. The action of \( f_z \) is equivariant with the action of the one parameter family given by the flow, that is,

\[
\psi_s \circ f_z = f_{\psi_s(z)}.
\]

Proof. \[
\psi_s \circ f_z(t, x, y) = (e^{s}x_1 + ax + by, e^{s}y_1, e^{-t}x_2 + cy, e^{-t}e^{s}y_2),
\]

which is the same expression obtained calculating \( f_{\psi_s(z)} \).

Remark 4.3. If we pull back \( f_z \) with the mapping \( \Phi^{-1} \) defined previously, the expression in the theorem acquires the simpler and equivalent form

\[
\psi_s \circ f_z(t, x, y) = (e^{s}x_1 + x, e^{s}y_1, e^{-t}x_2 + y, e^{-t+s}y_2).
\]

Theorem 4.4. Let \( z = (y_1 i, y_2 i) \). The induced metric in the leaf \( f_z \circ \Phi^{-1}(Sol) \) is

\[
dt^2 + \frac{e^{-2t}}{2y_1^2} dx^2 + \frac{e^{2t}}{2y_2^2} dy^2.
\]

In particular, if \( y_1 = y_2 = 1/\sqrt{2} \), \( f_z \) is an isometric embedding of \( Sol \) into \( H \times H \).

Proof. Given the definition of \( \Phi \),

\[
f_z \circ \Phi^{-1}(t, x, y) = (x, e^{t}y_1, y, e^{-t}y_2).
\]
Therefore, the jacobian matrix of $f_z \circ \Phi^{-1}$ is
\[
\begin{pmatrix}
0 & 1 & 0 \\
\varepsilon^t y_1 & 0 & 0 \\
0 & 0 & 1 \\
-e^{-t} y_2 & 0 & 0
\end{pmatrix}.
\]
Applying the product metric to the basis vectors $e^t y_1 e_2 - e^{-t} y_2 e_4$, $e_1$, $e_3$ we arrive to the result.

In the sequel, unless otherwise stated, $z_0$ will denote the special point $1/\sqrt{2} (i, i)$.

**Corollary 4.5.** The leaves $f_{\psi(z_0)}(\text{Sol})$ can be identified with $\text{Sol}$, up to a homotecy in the direction spanned by $xy$ coordinates.

**Proof.** By virtue of equation 4.1, we have
\[
\psi_s \circ f_{z_0}(t, x, y) = \left( x, \frac{1}{\sqrt{2}} e^{-t+s} y, \frac{1}{\sqrt{2}} e^{-t+s} x \right).
\]
Proceeding as in the proof of theorem 4.4, if we pull back the induced metric to $\text{Sol}$, we get
\[
dt^2 + e^{-2(t+s)} dx^2 + e^{2(t-s)} dy^2.
\]
By a simple algebraic manipulation, the previous metric is
\[
dt^2 + e^{-2t} e^{-2s} dx^2 + e^{2t} e^{-2s} dy^2.
\]
Define $F_s : \text{Sol} \to \text{Sol}$ by $F_s(t, x, y) = (t, e^s x, e^s y)$. Another pullback, this time with $F_s$, turns the metric into
\[
dt^2 + e^{-2t} dx^2 + e^{2t} dy^2.
\]

**Theorem 4.6.** The foliation is globally rectifiable: there is a diffeomorphism $\Psi : \mathbb{R}^3 \times \mathbb{R} \to H \times H$, such that each hyperplane $\mathbb{R}^3 \times \{c\}$ is diffeomorphic to a leaf.

**Proof.** $\text{Sol}$ is diffeomorphic to $\mathbb{R}^3$ in a natural way. Any $\gamma \in \text{Sol}$ is uniquely determined by a triplet $(t, x, y)$. Define $\Psi : \mathbb{R}^3 \times \mathbb{R} \to H \times H$ by $\Psi(t, x, y, s) = \psi_s \circ f_{z_0}(\gamma)$. $\Psi$ is injective because the action is free. Given $z' \in H \times H$, there is a leaf going through it, and since $\psi_s(z_0)$ traverses all the leaves, there is $s$, such that $\psi_{-s}(z')$ is in the leave passing through $z_0$. Let $\gamma \in \text{Sol}$ be such that $\psi_{-s}(z') = f_{z_0}(\gamma)$. Therefore,
\[
z' = \psi_s \circ f_{z_0}(\gamma),
\]
which implies that $\Psi$ is also surjective. Finally, a direct calculation yields $\Psi$’s jacobian to be
\[
[d\Psi] = \begin{pmatrix}
0 & 1 & 0 \\
e^{t+s}/\sqrt{2} & 0 & 0 \\
0 & 0 & 1 \\
-e^{-t+s}/\sqrt{2} & 0 & 0
\end{pmatrix},
\]
which is nondegenerated in the whole domain. By the inverse function theorem, $\Psi$ is a diffeomorphism. The last claim follows from the fact that $\psi_s$ maps leaves onto leaves.

**Theorem 4.7.** The previous diffeomorphism can be modified, such that not only maps the foliation to a cartesian product globally, but also maps each leaf in the foliation isometrically to $\text{Sol}$. 

\[
\]
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Proof. By pulling back the metric in $\mathbb{H} \times \mathbb{H}$ with the previous diffeomorphism, we arrive to

$$dt^2 + e^{-2(t+s)} dx^2 + e^{2(t-s)} dy^2 + ds^2,$$

which is analogous to the expression in corollary 4.5. Let

$$\tilde{\Psi}(t, x, y, s) = \Psi(t, e^s x, e^s y, s),$$

$\Psi$ is a leave preserving diffeomorphism such that, for fixed $s$, maps isometrically $\text{Sol}$ into the leave $\mathbb{R}^3 \times \{s\}$. □

Remark 4.8. It doesn’t look like it could be possible to find an even more rigid diffeomorphism, i.e. a global isometry $\text{Sol} \times \mathbb{R} \cong \mathbb{H} \times \mathbb{H}$ preserving leaves. However, a proof is pending in this regard.

5. Extrinsic geometry

Recall the metric we endowed in $\mathbb{H} \times \mathbb{H}$ is homotetic to the product of hyperbolic metrics:

$$\frac{dx_1^2 + dy_1^2}{2y_1^2} + \frac{dx_2^2 + dy_2^2}{2y_2^2}.$$

Theorem 5.1. Integral curves of the normal field $X$ are geodesics.

Proof. We had previously found that $X$’s integral curves are given by $\gamma(t) = (x_1, e^t y_1, x_2, e^t y_2)$. Let $\phi(t)$ be a smooth curve in $\mathbb{H}$ with the homotetic metric. Then,

$$||\dot{\phi}(t)||^2 = \frac{\dot{x}^2 + \dot{y}^2}{2y^2},$$

which is half the standard hyperbolic square length. Therefore, a curve minimizes hyperbolic arclength if and only if minimizes the homotetic metric arclength, i.e. geodesics in both cases are the same. It is a well known fact that the vertical curves $(x_k, e^t y_k)$ are geodesics in hyperbolic space. Finally, since $\gamma$ can be projected in two geodesics and the metric is a product, $\gamma$ is a geodesic in $\mathbb{H} \times \mathbb{H}$. (see 3.15 in [?]). □

Theorem 5.2. There are isometries in $\mathbb{H} \times \mathbb{H}$ acting transitively and sending leaves onto leaves.

Proof. We work in the $\mathbb{R}^3 \times \mathbb{R}$ picture with the $\tilde{\Psi}$ isometry. A straightforward calculation shows that the mappings

$$(t, x, y, s) \rightarrow (t + t', e^{t'} x + x', e^{-t'} y + y', s + s')$$

are isometries. The first claim comes from the fact that given a pair of points $(t_k, x_k, y_k, s_k)$, there exists exactly one such isometry sending one onto another. That this isometry sends leaves onto leaves is obvious, since under this diffeomorphism, they correspond to hypersurfaces $s = \text{constant}$. □

We aim to calculate the distance between any pair of leaves. Recall in any metric space, the distance from a point $p$ to a set $S \neq \emptyset$ is given by the expression

$$d(p, S) = \inf \{d(p, x) : x \in S\}.$$ 

Theorem 5.3. The separation between two leaves in $\mathbb{H} \times \mathbb{H}$ is constant. Moreover, if leaves are parametrized with the normal field affine parameter, then leaves separation is given by the difference $|s - s'|$ between the parameter corresponding to any leaf.
Proof. A point in a leaf can be parametrized as
\[
\left( x, \frac{e^{s+t}}{\sqrt{2}}, y, \frac{e^{s-t}}{\sqrt{2}} \right),
\]
where \(x, y, t\) are arbitrary, and \(s\) is the parameter corresponding to the leaf. Given a second point in another leaf, say \(\left( x', \frac{e^{s'+t'}}{\sqrt{2}}, y', \frac{e^{s'-t'}}{\sqrt{2}} \right)\), and since the metric is a product, we can find a geodesic minimizing arc length in \(\mathbb{H} \times \mathbb{H}\), such that, in each factor \(\mathbb{H}\), the distance is also minimized. On the other hand, the metric we use in each factor of \(\mathbb{H} \times \mathbb{H}\) is half the hyperbolic distance, for which a well known formula gives us the distance. Let \(p_k\) denote the distance in each factor with our metric, then,
\[
\cosh \left( \sqrt{2} p_1 \right) = 1 + \frac{2 (x - x')^2 + \left( e^{s+t} - e^{s'+t'} \right)^2}{2 e^{s+t} e^{s'+t'}},
\]
\[
\cosh \left( \sqrt{2} p_2 \right) = 1 + \frac{2 (y - y')^2 + \left( e^{s-t} - e^{s'-t'} \right)^2}{2 e^{s-t} e^{s'-t'}},
\]
where the \(\sqrt{2}\) factor within the hyperbolic cosine is due to the factor relating standard hyperbolic metric with ours. The previous expression shows that, in order to get the minimum distance, \(x'\) must be equal to \(x\) and \(y'\) to \(y\). Simplifying the previous expressions for such values of \(x'\) and \(y'\), we find
\[
\cosh \left( \sqrt{2} p_1 \right) = \cosh (s - s' + t - t'),
\]
\[
\cosh \left( \sqrt{2} p_2 \right) = \cosh (s - s' + t' - t).
\]
Therefore,
\[
p_1 = \frac{|s - s' + t - t'|}{\sqrt{2}}, \quad p_2 = \frac{|s - s' + t' - t|}{\sqrt{2}},
\]
and the distance in the product metric is given by \(\sqrt{p_1^2 + p_2^2}\). In order for this distance to be a minimum, a short analysis shows that one must take \(t' = t\), and the theorem statement follows. 

\[\square\]

**Theorem 5.4.** The principal curvatures of each leaf are \(-1\) with multiplicity two, and \(0\). The principal directions are determined by the integral curves of the vectors \(\partial_x, \partial_y, \partial_t\) respectively.

Proof. Recall the principal curvatures and directions for an orientable submanifold \(M\) are determined by the shape operator, \(S\), which in codimension one, can be regarded as the mapping \(TM \to TM\) given by \(v_x \mapsto \nabla v_x X\), where \(X\) is the normal field to the manifold, compatible with orientation. Here, the principal directions and curvatures are the shape operator eigenvectors, and eigenvalues. Consider a leaf embedded in \(\mathbb{H} \times \mathbb{H}\),
\[
\left( x, \frac{e^{-t-s}}{\sqrt{2}}, y, \frac{e^{t-s}}{\sqrt{2}} \right),
\]
with normal field \(X = x_2 \partial_2 + x_3 \partial_3\), where \(x_2 = e^{-t-s}/\sqrt{2}, x_4 = e^{t-s}/\sqrt{2}\). A straightforward calculation shows that \(\nabla X = -dx_1 \partial_1 - dx_2 \partial_2 \partial_3\), i.e., the shape operator is diagonal, once expressed in the base for the tangent space to the leaf, spanned by the coordinate vectors \(\partial_1, \partial_3,\) and the vector \(-x_2 \partial_2 + x_3 \partial_4\). Moreover, the eigenvalues are precisely \([-1, -1, 0]\). \[\square\]
6. The Heisenberg group

Given a symplectic vector space, $V$, with symplectic form $\omega$. Recall the Heisenberg group, $\text{Heis}$, is the space $V \times \mathbb{R}$, with the product operation given by

$$(v, t) \ast (w, s) = (v + w, t + s + \omega(v, w)).$$

If $V$ is of dimension 2, and $\{\partial_p, \partial_q\}$ is a symplectic base for $V$, that is, $\omega(\partial_p, \partial_q) = 1$, a well known fact from Lie group theory is that there is a faithful representation $\text{Heis} \rightarrow \text{SL}(3, \mathbb{R})$ given by

$$((p \partial_p + q \partial_q, t) \mapsto \begin{bmatrix} 1 & p & t + \frac{1}{2}pq \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}.$$

We will use this representation and identify $\text{Heis}$ with $\text{SL}(3, \mathbb{R})$. Therefore, we will identify $\text{Heis}$ with $\mathbb{R}^3$, with group structure,

$$(a, b, c) \ast (a', b', c') = (a + a', b + b', c + c' + ab'),$$

which corresponds to the matrix product

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix}.$$

With these identifications, there is a natural action $\text{Heis} \acts C \times \mathbb{H}$:

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ w \\ 1 \end{bmatrix} = \begin{bmatrix} z + aw + c \\ w + b \\ 1 \end{bmatrix},$$

which we will denote $(a, b, c) \ast (z, w)$.

**Theorem 6.1.** The action of $\text{Heis}$ in $\mathbb{C} \times \mathbb{H}$ is free.

*Proof.* If $(a, b, c) \ast (z, w) = (z, w)$, then

$$z + aw + c = z,$$

$$w + b = w.$$

From this linear system, one deduces that $a = b = c = 0$. \square

**Theorem 6.2.** For fixed $(z, w) \in \mathbb{C} \times \mathbb{H}$, the orbit $h \in \text{Heis} \mapsto h \ast (z, w)$, defines a differentiable embedding $\text{Heis} \hookrightarrow \mathbb{C} \times \mathbb{H}$.

*Proof.* The map is injective, since the action is free. Let $w = p + qi$, the Jacobian matrix of the mapping in $(a, b, c) \in \text{Heis}$ is given by

$$\begin{bmatrix} p & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since the Jacobian has rank 3, the action defines a local diffeomorphism, hence an embedding. \square

Therefore, the action of $\text{Heis}$ defines a foliation of $\mathbb{C} \times \mathbb{H}$, in analogy with the foliation of $\mathbb{H} \times \mathbb{H}$ generated by $\text{Sol}$. 
that

It can be shown that $\Psi$ is bijective. It is a diffeomorphism, since an explicit calculation shows

$$C \times \{z, w\} \text{ to the orbit of } (\cdot, \cdot).$$

If $\Gamma$ acts properly discontinuous in $C \times \mathbb{H}$, then the metric in $\mathbb{H}$ is conformal to the euclidean, and therefore $X$ is perpedicular to $e_4$. Finally, $q_e 4$ is unitary in the hyperbolic metric.

\[ \square \]

**Theorem 6.4.** Let $(z, w) \in C \times \mathbb{H}$, $z = x + y, w = p + qi$. The integral curves of $X$ are geodesics.

\[ \square \]

Theorem 6.4. Let $X$ and $Y$ be two locally compact spaces. If $\Gamma \cap X \times Y$, and the action of $g \in \Gamma$ can be decomposed as $g \cdot (x, y) = (g \cdot x, y)$ then $\Gamma$ acts properly discontinuous in $X$ if it acts properly discontinuous in $X \times Y$.

**Proof.** Let $K \subset X$ be a compact set. Fix $y \in Y$. With the product topology, $K \times \{y\}$ is a compact set in $X \times Y$. One can easily verify the equality

$$\{g \in \Gamma : g \cdot K \cap K \neq \emptyset\} = \{g \in \Gamma : g \times 1 \cdot K \times \{y\} \cap K \times \{y\} \neq \emptyset\}. $$

If $\Gamma$ acts properly discontinuous in $X \times Y$, the previous equality implies that it acts properly discontinuous in $X$. On the other hand, if $K \subset X \times Y$ is compact, the product topology together with the local compacity implies that we can find an open set $U \times V$, with $U \in X$ and $V \in Y$, such that $\bar{U}$ is compact in $X$, $\bar{V}$ is compact in $Y$, and $K \subset U \times V$. We have the contention

$$\{g \in \Gamma : gK \cap K \neq \emptyset\} \subset \{g \in \Gamma : g \cdot (\bar{U} \times \bar{V}) \cap \bar{U} \times \bar{V} \neq \emptyset\}. $$

Take $g \in \Gamma$, $(x, y) \in \bar{U} \times \bar{V}$, such that $g \cdot (x, y) \in \bar{U} \times \bar{V}$. Since $g \cdot (x, y) = (g \cdot x, y)$, it follows that $g \cdot x \in \bar{U}$. Therefore, the second set in the previous contention is at the same time contained in

$$\{g \in \Gamma : g \cdot \bar{U} \cap \bar{U} \neq \emptyset\}. $$

If $\Gamma$ acts properly discontinuous in $X$, this set has to be finite, and the same must be true for the set of intersections in $X \times Y$. i.e. $\Gamma$ acts properly discontinuous in $X$.

\[ \square \]

**Theorem 6.5.** $C \times \mathbb{H}$ is diffeomorphic to Heis $\times \mathbb{R}$, where, up to diffeomorphism, Heis acts on the first factor only.

**Proof.** Let $\gamma = (a, b, c) \in \text{Heis}$. And take $(0, qi) \in C \times \mathbb{H}$. We can describe the orbits $\gamma \cdot (0, qi)$:

$$\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 \\
qi \\
1
\end{pmatrix} = \begin{pmatrix}
aqi + c \\
qi + b \\
1
\end{pmatrix}. $$

Therefore, there is exactly one $(0, qi)$ in each orbit of the group action. Define $\Psi : \text{Heis} \times \mathbb{R} \to C \times \mathbb{H}$ as $\Psi(\gamma, q) = \gamma \cdot (0, qi)$.

It can be show that $\Psi$ is bijective. It is a diffeomorphism, since an explicit calculation shows that $d\Psi$ maps the canonical vectors $T_{(\gamma, q)\text{Heis}} \times \mathbb{R} \cong \mathbb{R}^4 \to T_{(0, qi)C} \times \mathbb{H} \cong \mathbb{R}^4$: \[ \square \]
\{\partial_1, \ldots, \partial_4\} \mapsto \{q\partial_2, \partial_3, \partial_1, a\partial_2 + \partial_4\}.

The last assertion follows since the action is associative, i.e. \(\gamma' \cdot (\gamma \cdot (0, qj)) = (\gamma' \cdot \gamma) \cdot (0, qj)\), and therefore, preserves the imaginary part on the second factor. \(\square\)

**Corollary 6.7.** If \(\Gamma < \text{Heis}\) is a discrete subgroup acting properly discontinuous in \(\mathbb{C} \times \mathbb{H}\), up to diffeomorphism, \(\mathbb{C} \times \mathbb{H}/\Gamma \cong \text{Heis}/\Gamma \times \mathbb{R}\), and the quotient \(\text{Heis}/\Gamma\) is a manifold, whose fundamental group is \(\pi^1(\text{Heis}/\Gamma) \cong \Gamma\).

**Example 6.8.** Let \(\text{Heis}_\mathbb{Z} < \text{Heis}\) be the discrete subgroup of Heisenberg matrices with integer coefficients. It can be shown that the unit cube \(K_C = [0,1]^3 \subset \text{Heis}\) is a fundamental region for the action of \(\text{Heis}_\mathbb{Z}\). The quotient \(\text{Heis}_\mathbb{Z}/\text{Heis}\) is an example of nilmanifold, whose fundamental group is

\[\text{Heis}_\mathbb{Z} \cong \langle m, n, k : [m, n] = k^4 \rangle.\]

In view of the previous results, \(\text{Heis}_\mathbb{Z}\) acts properly discontinuous in \(\mathbb{C} \times \mathbb{H}\), and the quotient \(\mathbb{C} \times \mathbb{H}/\text{Heis}_\mathbb{Z}\) is a product of nilmanifold times \(\mathbb{R}\), whose fundamental group has the previous presentation.

### 6.1. Metric properties.

Let \((p, q, t)\) denote local coordinates in \(\text{Heis}\). In this coordinates, the standard metric is defined to be

\[dp^2 + (1 + p^2) dq^2 + dt^2 - pdq \cdot dt.\]

However, we found no relation between the standard metric and the metric induced by the family of embeddings \(\text{Heis} \to \mathbb{C} \times \mathbb{H}\) induced by the action.

**Theorem 6.9.** Let \((0, y_0 \cdot i) \in \mathbb{C} \times \mathbb{H}\) be the base point whose orbit generates a diffeomorphic copy of \(\text{Heis}\). Then the pullback metric in \(\text{Heis}\) is

\[y_0^2 dp^2 + \frac{1}{y_0} dq^2 + dt^2.\]

**Proof.** Denote by \(\psi : \text{Heis} \to \mathbb{C} \times \mathbb{H}\) the map defined by the action of \(\text{Heis}\) in the given point, then

\[\psi(p, q, t) = (t + p y_0 i, q + y_0 i).\]

For fixed \(h = (p, q, t) \in \text{Heis}\), the derivative \(d\psi_h\) induces a linear map \(T_h \text{Heis} \to T_{\psi(h)} \mathbb{C} \times \mathbb{H}\), such that the basic tangent vectors \(\partial_p, \partial_q, \partial_t\) are send to \(y_0 \partial_2, \partial_3, \partial_1\) respectively.

Upon identifying the basic vectors in \(\text{Heis}\) with its images, the local expression for the induced metric is obtained. \(\square\)

**Remark 6.10.** Note how the local expression for the metric resembles that of \(\text{Sol}\). In fact, under the diffeomorphism \(\text{Heis} \times \mathbb{R} \to \mathbb{C} \times \mathbb{H}\) \(y_0\) turns out to be an expression of the form \(e^s\), for \(s \in \mathbb{R}\). For \(s\) fixed at least, the induced metric becomes

\[e^{2s} dp^2 + e^{-2s} dq^2 + dt^2,\]

which looks analogous to what would be obtained in a section of \(\text{Sol} \times \mathbb{R}\).

**Remark 6.11.** Although the diffeomorphism \(\text{Heis} \times \mathbb{R} \cong \mathbb{C} \times \mathbb{H}\) lacks a metric relation, we can calculate nevertheless the separation between two foliation leaves, in analogy to what we did with \(\text{Sol}\).

**Theorem 6.12.** Let \(\Psi : \text{Heis} \times \mathbb{R} \to \mathbb{C} \times \mathbb{H}\) be the diffeomorphism induced by the action of \(\text{Heis}\) in \((0, e^s \cdot i)\), for \(s \in \mathbb{R}\). Then the separation between the hyperplanes (leaves) \(\text{Heis} \times \{s_0\}\) and \(\text{Heis} \times \{s_1\}\) is precisely \(|s_1 - s_0|\).
Proof. Given coordinates \((p, q, t, s)\) for \(\text{Heis} \times \mathbb{R}\), the expression for \(\Psi\) is \(\Psi(p, q, t, s) = (t + p e^{si}, q + e^{si})\).

Note that the metric in \(\mathbb{C} \times \mathbb{H}\) is invariant under “horizontal translations”: \((p, q, t, s) \mapsto (p + p_0, q + q_0, t + t_0, s)\). Given a path \(\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \mathbb{C} \times \mathbb{H}\) connecting a point in the first leave to a point in the second. Since the metric is a product, the length of the second component \(\gamma_2\), has to be a lower bound for the length of the total path:

\[
\ell(\gamma_2) \leq \int_0^1 ||\dot{\gamma}_2|| \, dt \leq \int_0^1 \sqrt{||\dot{\gamma}_1||^2 + ||\dot{\gamma}_2||^2} = \ell(\gamma).
\]

Consider the path \(\Psi(0, 0, 0, e^{(s(s_1 - s_0) + s_0)})\), joining \((0, e^{s_0} i)\) to \((0, e^{s_1} i)\). Since the projection to \(\mathbb{H}\) of any other path has to connect this points, the previous bound shows that its length has to be lesser than that of this path. However, in \(\mathbb{H}\), the minimum separation between the lines \(\mathbb{R} \times \{e^{s_0}\}\) and \(\mathbb{R} \times \{e^{s_1}\}\) is precisely given by the path \(e^{(s(s_1 - s_0))} i\), whose length is \(|s_1 - s_0|\).

\[\square\]

7. The topological type

In this paper we described two diffeomorphisms: \(\mathbb{H} \times \mathbb{H} \cong \text{Sol} \times \mathbb{R}\) and \(\mathbb{C} \times \mathbb{H} \cong \text{Heis} \times \mathbb{R}\). In the first case, the action of the group was equivariant with respect to the flow of the normal field to the orbits. In the second case, there isn’t such equivariance. However, in both cases, the action of the group can be factored out, that is to say, in both cases, if \(G\) denotes the corresponding group and \(X\) the target space, there is a diffeomorphism,

\[X \cong G \times \mathbb{R},\]

such that the action of \(G\) preserves the leaves \(G \times \{t\}\). Moreover, up to diffeomorphism, the action can be described as \(\gamma' \cdot (\gamma, x) = (\gamma' \gamma, x)\).

Let \(\Gamma < G\) be a discrete subgroup. The previous discussion shows that we can describre the quotient: \(\Gamma/X \cong (\Gamma/G) \times \mathbb{R}\). In particular, since the second factor is contractible, we must have isomorphisms \(\pi_1(\Gamma/X) \cong \pi_1(\Gamma/G)\), so that we can describe both quotients \(\Gamma/\mathbb{H} \times \mathbb{H}\), and \(\Gamma \times (\mathbb{C} \times \mathbb{H})\), where, for an abuse in notation, \(\Gamma\) denotes distinct discrete subgroups of \(\text{Sol}\) and \(\text{Heis}\).

7.1. Proof of main theorem 1. Let \(G\) be a complex kleinian group with maximum number of lines in general position contained in its limit set is four, then \(G\) acts properly and discontinuously in four copies disjoints of \(\mathbb{H} \times \mathbb{H}\). Without loss of generality we assume that \(\mathbb{H} \times \mathbb{H}\) is \(G\)-invariant. By the theorem, we have

\[\psi : \text{Sol} \times \mathbb{R} \to \mathbb{H} \times \mathbb{H}\]

is a diffeomorphism \(G\)-equivariant, then \(\mathbb{H} \times \mathbb{H}/G\) is diffeomorphic to \((\text{Sol}/G) \times \mathbb{R}\).

We notice the topological type is perfectly determinde by the group \(G\). In fact, the group \(G\) is the fundamental group of the manifold \(\mathbb{H} \times \mathbb{H}/G\). We remember the Kulkarni discontinuity region is equal a four copies disjoint of \(\mathbb{H} \times \mathbb{H}\), Hence \(\Omega/G\) is equal to four disjoint copies of \(\mathbb{H} \times \mathbb{H}/G\). We remark \(G\) represented a lattice of the Lie group \(\text{Sol}\), then \(\text{Sol}/G\) is a compact 3 manifold. This last statement implies in some sense \(\text{Sol}/G\) is the compact heart of \(\mathbb{H} \times \mathbb{H}/G\).

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