On Drinfel’d associators

G. H. E. Duchamp‡, V. Hoang Ngoc Minh◊, K. A. Penson♭

‡Université Paris XIII, 99 Jean-Baptiste Clément, 93430 Villetaneuse, France.
◊Université Lille II, 1, Place Déliot, 59024 Lille, France.
♭Université Paris VI, 75252 Paris Cedex 05, France.

1 Knizhnik-Zamolodchikov differential equations and coefficients of Drinfel’d associators

In 1986 [6], in order to study the linear representations of the braid group $B_n$ coming from the monodromy of the Knizhnik-Zamolodchikov differential equations, Drinfel’d introduced a class of formal power series $\Phi$ on noncommutative variables over the finite alphabet $\{x_0, x_1\}$. Such a power series $\Phi$ is called an associator. For $n = 3$, it leads to the following fuchsian noncommutative differential equation with three regular singularities in $\{0, 1, +\infty\}$:

$$\frac{dG}{dz} = \left( x_0 \frac{dz}{z} + x_1 \frac{dz}{1-z} \right) G(z).$$

Solutions of $(DE)$ are power series, with coefficients which are mono-valued functions on the simply connex domain $\Omega = \mathbb{C} - \{0, 1, +\infty\}$ and can be seen as multi-valued over $1\mathbb{C} - \{0, 1\}$. Drinfel’d proved that $(DE)$ admits two particular mono-valued solutions on $\Omega$, $G_0(z) \sim 0 \exp\left[ x_0 \log(z) \right]$ and $G_1(z) \sim 1 \exp\left[ -x_1 \log(1-z) \right]$ [7, 8].

After that, via representations of the chord diagram algebras, Lê and Murakami [17] expressed the coefficients of $\Phi_{KZ} \in \mathbb{R}\langle\langle X \rangle\rangle$ such that $G_0 = G_1 \Phi_{KZ}$ [7, 8].

More precisely, if $(s_1, \ldots, s_r) \in \mathcal{H}_r$, one has two ways of thinking $\zeta_r(s_1, \ldots, s_r)$ as limits, fulfilling identities [14, 13, 1]:

$$\lim_{z \to 1} \text{Li}_{s_1, \ldots, s_k}(z) = \lim_{n \to \infty} H_{s_1, \ldots, s_k}(n) =: \zeta_r(s_1, \ldots, s_k)$$

$$1 \mathbb{C} - \{0, 1\}.\]
else it does not hold, for \((s_1, \ldots, s_r) \notin \mathcal{H}_r\), while \(\text{Li}_{s_1, \ldots, s_k}\) is well defined over \(\{z \in \mathbb{C}, |z| < 1\}\) and so is \(\mathcal{H}_{s_1, \ldots, s_k}\), as Taylor coefficients of the following function

\[
(1 - z)^{-1}\text{Li}_{s_1, \ldots, s_k}(z) = \sum_{n \geq 1} \mathcal{H}_{s_1, \ldots, s_k}(n) z^n, \quad \text{for } z \in \mathbb{C}, |z| < 1.
\]  

(4)

Note also that, for \(r = 1\), \(\zeta_1\) is nothing else but the famous Riemann zeta function and, for \(r = 0\), it is convenient to set \(\zeta_0\) to the constant function \(1\).

In all the sequel, for simplification, we will adopt the notation \(\zeta\) for \(\zeta_r, r \in \mathbb{N}\).

In this work, we will describe the regularized solutions of \((DE)\).

For that, we are considering the alphabets \(X = \{x_0, x_1\}\) and \(Y_0 = \{y_s\}_{s \geq 0}\) equipped of the total ordering \(x_0 < x_1\) and \(y_0 > y_1 > y_2 > \ldots\), respectively. Let \(Y = Y_0 - \{y_0\}\). The free monoid generated by \(X\) (resp. \(Y, Y_0\)) is denoted by \(X^*\) (resp. \(Y^*, Y_0^*\)) and admits \(1_X\) (resp. \(1_Y, 1_{Y_0}\)) as unit.

The sets of, respectively, polynomials and formal power series, with coefficients in a commutative \(A\) \(-\)algebra \(A\), over \(X^*\) (resp. \(Y^*, Y_0^*\)) are denoted by \(A(X)\) (resp. \(A(Y), A(Y_0)\)) and \(A\langle X \rangle\) (resp. \(A\langle Y \rangle, A\langle Y_0 \rangle\)). The sets of polynomials are the \(A\) -modules and endowed with the associative concatenation, the associative commutative shuffle (resp. quasi-shuffle) product, over \(A(X)\) (resp. \(A(Y), A(Y_0)\)). Their associated coproducts are denoted, respectively, \(\Delta_{\text{uni}}\) and \(\Delta_{\text{shuf}}\). The algebras \((A(X), \omega, 1_X^*)\) and \((A(Y), \omega, 1_Y^*)\) admit the sets of Lyndon words denoted, respectively, by \(\mathcal{L}X\) and \(\mathcal{L}Y\), as transcendance bases \(\mathbb{L}\) (resp. \(\mathbb{L}_8\) \([15] [16]\)).

For \(Z = X\) or \(Y\), denoting \(\text{Lie}_{A}(Z)\) and \(\text{Lie}_{A}\langle Z \rangle\) the sets of, respectively, Lie polynomials and Lie series, the enveloping algebra \(\mathcal{U}(\text{Lie}_{A}(Z))\) is isomorphic to the Hopf algebra \((A(X), 1_{Z^*}, \Delta_{\text{uni}}, \theta)\). We get also \(\mathcal{H}_{\text{uni}} := (A(Y), 1_{Y^*}, \Delta_{\text{uni}}, \theta) \cong \mathcal{U}(\mathcal{L}X)\), where \(\mathcal{L}X\) \([15] [16]\).

\[
\text{Prim}(\mathcal{H}_{\text{uni}}) = \text{span}_A\{\pi_1(w)|w \in Y^*\},
\]

\[
\pi_1(w) = \sum_{k=1}^{(w)} (-1)^{k-1} \frac{1}{k} \sum_{u_1, \ldots, u_k \in Y^*} \langle w | u_1 \ldots u_k \rangle u_1 \ldots u_k. \quad (6)
\]

2 Indexing polylogarithms and harmonic sums by words and their generating series

For any \(r \in \mathbb{N}\), since any combinatorial composition \((s_1, \ldots, s_r) \in \mathbb{N}^r_+\) can be associated with words \(x_0^{s_0-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1\) and \(y_1 \ldots y_{s_r} \in Y^*\). Similarly, any multi-indiced \((s_1, \ldots, s_r) \in \mathbb{N}^r_+\) can be associated with words \(y_{s_1} \ldots y_{s_r} \in Y_0^*\). Then let \(\text{Li}_{x_0^r}(z) := (\log(z))^r/r!\), and \(\text{Li}_{y_{s_1} \ldots y_{s_r}}\) and \(\mathcal{H}_{s_1, \ldots, s_k}\) be indexed by words \(\mathbb{L}\) : \(\text{Li}_{x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1} := \text{Li}_{s_1, \ldots, s_k}\) and \(\mathcal{H}_{y_{s_1} \ldots y_{s_r}} := \mathcal{H}_{s_1, \ldots, s_k}\).

Similarly, \(\text{Li}_{-s_1, \ldots, -s_k}\) and \(\mathcal{H}_{-s_1, \ldots, -s_k}\) be indexed by words \(\mathbb{L}\) : \(\text{Li}_{-y_{s_1} \ldots y_{s_r}} := \mathcal{H}_{-s_1, \ldots, -s_k}\).

\footnote{The weight of \((s_1, \ldots, s_r) \in \mathbb{N}^r_+\) (resp. \( \mathbb{N}^r\)) is defined as the integer \(s_1 + \ldots + s_r\) which corresponds to the weight, denoted \(\|w\|\), of its associated word \(w \in Y^*\) (resp. \(Y_0^*\)) and corresponds also to the length, denoted by \(|w|\), of its associated word \(w \in X^*\).}

\footnote{Note that, all these \(\{\text{Li}_{w}^r\}_{w \in Y_0^*}\) and \(\{\mathcal{H}_{w}^r\}_{w \in Y_0^*}\) are divergent at their singularities.
where the PBW basis read then \( [2, 14] \) are no longer linearly independent and the values \( \{H, \psi\} \) are group-like, for \( \Delta \). Moreover, the families \( \{H^j\}_{j \geq 0} \) and \( \{w_j\}_{j \geq 0} \) are \( \mathbb{Q} \)-linearly independent.

On the other hand, the following morphisms of algebras are **surjective**

\[
\begin{align*}
H^\bullet_\psi &: (\mathbb{Q}\langle Y \rangle, \psi, 1_{Y^*}) \to (\mathbb{Q}\{w\}_{w \in Y^*}, \times, 1), \quad w \mapsto H^\psi, \\
Li^\bullet_\psi &: (\mathbb{Q}\langle Y \rangle, \psi, 1_{Y^*}) \to (\mathbb{Q}\{w\}_{w \in Y^*}, \times, 1), \quad w \mapsto Li^\psi
\end{align*}
\]

and \( \text{ker } H^\bullet_\psi = \text{ker } Li^\bullet_\psi = \mathbb{Q}\{w - w^1 Y^* | w \in Y^* \} \) [4]. Moreover, the families \( \{H^j\}_{j \geq 0} \) and \( \{w_j\}_{j \geq 0} \) are \( \mathbb{Q} \)-linearly independent.

On the other hand, the following morphisms of algebras are **injective**

\[
\begin{align*}
H_\bullet &: (\mathbb{Q}\langle Y \rangle, \psi, 1_{Y^*}) \to (\mathbb{Q}\{w\}_{w \in Y^*}, \times, 1), \quad w \mapsto H_\psi, \\
Li_\bullet &: (\mathbb{Q}\langle X \rangle, \psi, 1_{X^*}) \to (\mathbb{Q}\{w\}_{w \in X^*}, \times, 1), \quad w \mapsto Li_\psi
\end{align*}
\]

Moreover, the families \( \{H^j\}_{j \geq 0} \) and \( \{w_j\}_{j \geq 0} \) are \( \mathbb{Q} \)-linearly independent and the families \( \{H^j\}_{j \in \mathbb{Z}_{\geq 0}} \) and \( \{w_j\}_{j \in \mathbb{Z}_{\geq 0}} \) are \( \mathbb{Q} \)-algebraically independent. But at singularities of \( \{H^j\}_{j \in \mathbb{Z}_{\geq 0}} \) and \( \{w_j\}_{j \in \mathbb{Z}_{\geq 0}} \), the following convergent values

\[
\forall u \in Y^* - y^* \mathbb{Z}, x(u) := H_u(+\infty) \quad \text{and} \quad \forall u \in x_0 X^* x_1, \zeta(v) := Li_u(1)(11)
\]

are no longer linearly independent and the values \( \{H_i(+\infty)\}_{i \in L \cap Y^* - (y_1)} \) (resp. \( \{Li_i(1)\}_{i \in L \cap X^* - x_1} \)) are no longer algebraically independent [14, 19].

The graphs of the isomorphisms of algebras, \( Li_\bullet \) and \( H_\bullet \), as generating series, read then [2, 13]

\[
L := \sum_{w \in X^*} Li_\psi w = \prod_{l \in L \cap Y} e^{Li_{y_l} P_l}, \quad H := \sum_{w \in Y^*} H_\psi w = \prod_{l \in L \cap Y} e^{H_{y_l} P_l}(12)
\]

where the PBW basis \( \{P_w\}_{w \in X^*} \) (resp. \( \{P_w\}_{w \in Y^*}\)) is expanded over the basis of \( \mathbb{L} \) of the algebras \( \mathcal{H}(\psi) \) (resp. \( \mathcal{H}(\psi) \)), \( \{P_l\}_{l \in L \cap Y^*} \) (resp. \( \{P_l\}_{l \in L \cap Y^*} \)), and \( \{S_w\}_{w \in X^*} \) (resp. \( \{S_w\}_{w \in Y^*}\)) is a basis of \( \mathcal{H}(\psi) \) (resp. \( \mathcal{H}(\psi) \)), containing the transcendence basis \( \{L_l\}_{l \in L \cap Y^*} \) (resp. \( \{L_l\}_{l \in L \cap Y^*} \)).

By termwise differentiation, \( L \) satisfies the noncommutative differential equation (DE) with the boundary condition \( L(z) = e^{x_0 \log(z)} \). It is immediate that the power series \( H \) and \( L \) are group-like, for \( \Delta \psi \) and \( \Delta \psi \), respectively. Hence, the following noncommutative generating series are well defined and are group-like, for \( \Delta \psi \) and \( \Delta \psi \), respectively [13, 19, 16]:

\[
Z_\psi := \prod_{l \in L \cap Y - (y_1)} e^{H_{y_l} (+\infty) P_l} \quad \text{and} \quad Z_\psi := \prod_{l \in L \cap Y} e^{Li_{y_l}(1) P_l}(13)
\]

Definitions [13] and [11] lead then to the following surjective poly-morphism

\[
\zeta : (\mathbb{Q}^1_{X^*} \oplus x_0 \mathbb{Q}<X>X_1 \psi, 1_{X^*}) \to (Z, \times, 1), (14)
\]

\[
\begin{align*}
x_0 x_1^{r_1-1} \ldots x_0 x_1^{r_k-1} y_{s_1} \ldots y_{s_k} \mapsto \sum_{n_1 > \ldots > n_k \geq 0} n_1^{-s_1} \ldots n_k^{-s_k}(15)
\end{align*}
\]
These suggest to extend the morphism $L_i\in \mathcal{C}$ with $z \in \mathcal{C}$ where $i, j \geq 1$. For any $2$, let $\mathcal{V}$.

In particular, for $s$, $\mathcal{X}_i(x) \subseteq \mathcal{C}(\mathcal{X}, \mathbb{C})$, via Lazard’s elimination, as follows (subjected to be convergent)

$$\text{LiS}(z) = \sum_{n > 0} \langle S \mid x^a \rangle \frac{\log^n(z)}{n!} + \sum_{k > 1 \in \mathbb{C}} \sum_{w \in (x^a x^b)^k \in \mathbb{C} \mathcal{X}_i(x)} \langle S \mid w \rangle L_i(L_i(z)) \log((1 - z)^{-1})$$

with $\mathcal{C}(\mathcal{X}, \mathbb{C}) \subseteq \mathcal{C}(\mathcal{X}, \mathbb{C}) \subseteq \mathcal{C}(\mathcal{X}_i(x) \subseteq \mathcal{C}(\mathcal{X}, \mathbb{C})$ and $\mathcal{C}(\mathcal{X}, \mathbb{C})$ denotes the closure, of $\mathcal{C}(\mathcal{X}_i(x) \subseteq \mathcal{C}(\mathcal{X}, \mathbb{C})$, by $\{+\}$, for example $L_i$.

1. For any $i, j \in \mathbb{N}_+$ and $x \in \mathcal{X}_i$, since $$(t_0 x_0 + t_1 x_1)^*(z) = (t_0 x_0)^*(z) \omega(t_1 x_1)^*(z)$$ and $(x^a)^*k = (ix)^*$ then $L_i(x^{\ast}k) = \omega_{\mathbb{X}_i(x)}(z) = z^k(1 - z)^{-j}$.

2. For $a \in \mathbb{C}, x \in \mathcal{X}_i, i \in \mathbb{N}_+$, since $(ax)^i = (ax)^i \omega(1 + ax)^i - 1$ then

$$\text{Li}(ax^i\omega^i(z)) = z^a \sum_{k=0}^{\infty} \binom{i-1}{k} (a \log(z))^k k!,$$

$$\text{Li}(ax^i\omega^i(z)) = \frac{1}{(1 - z)^a} \sum_{k=0}^{\infty} \binom{i-1}{k} \log((1 - z)^{-1})^k k!$$

3. Let $V = (t_1 x_0)^{\ast s_1} x_0^{a_1 - 1} x_1 \ldots (t_r x_0)^{\ast s_r} x_0^{s_r - 1} x_1$, for $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$. Then

$$\text{LiV}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{(n_1 - 1)^{s_1} \ldots (n_r - 1)^{s_r}},$$

In particular, for $s_1 = \ldots = s_r = 1$, then one has

$$\text{LiV}(z) = \sum_{n_1, \ldots, n_r > 0} \text{Li}_{x_0^{a_1 - 1} x_0^{s_r - 1} x_1} \frac{z^{n_1}}{(n_1 - 1)^{s_1} \ldots (n_r - 1)^{s_r}}$$

4. From the previous points, one has

$$\{\text{LiS}\}_{S \in \mathcal{C}(\mathcal{X}_i \subseteq \mathbb{C}) \subseteq \mathbb{C}(\mathcal{X}_i \subseteq \mathbb{C}) \subseteq \mathcal{C}(\mathcal{X}_i \subseteq \mathbb{C})} = \text{span}_{\mathbb{C}} \left\{ \frac{z^a}{(1 - z)^b} \text{Li}_{w}(z) \right\} \in \mathbb{C} \omega_{\mathbb{X}^*},$$

$$\text{span}_{\mathbb{C}} \left\{ \text{Li}_{s_1, \ldots, s_r} \right\} \in \mathbb{C} \omega_{\mathbb{X}^*},$$

$$\text{span}_{\mathbb{C}} \left\{ z^a \text{Li}_{s_1, \ldots, s_r} \in \mathbb{C} \omega_{\mathbb{X}^*} + \text{span}_{\mathbb{C}} \{ z^a | a \in \mathbb{C} \} \right\}.$$
3 Noncommutative evolution equations

As we said previously Drinfel’d proved that \((DE)\) admits two particular solutions on \(\Omega\). These new tools and results can be considered as pertaining to the domain of noncommutative evolution equations. We will, here, only mention what is relevant for our needs.

Even for one sided differential equations, in order to cope with limit initial conditions (see applications below), one needs the two sided version.

Let then \(\Omega \subset \mathbb{C}\) be simply connected and open and \(\mathcal{H}(\Omega)\) denote the algebra of holomorphic functions on \(\Omega\). We suppose given two series (called multipliers) \(M_1, M_2 \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle\) (\(X\) is an alphabet and the subscript indicates that the series have no constant term). Let then

\[
(DE_2) \quad S = M_1 S + SM_2.
\]

be our equation.

3.1 The main theorem

**Theorem 1.** Let

\[
(DE_2) \quad S = M_1 S + SM_2.
\]  \hspace{1cm} (24)

(i) Solutions of \((DE_2)\) form a \(\mathbb{C}\)-vector space.

(ii) Solutions of \((DE_2)\) have their constant term (as coefficient of \(1_X^*\)) which are constant functions (on \(\Omega\)); there exists solutions with constant coefficient \(1_{\Omega}\) (hence invertible).

(iii) If two solutions coincide at one point \(z_0 \in \Omega\), they coincide everywhere.

(iv) Let be the following one-sided equations

\[
(DE^{(1)}) \quad S = M_1 S \quad (DE^{(2)}) \quad S = SM_2.
\]  \hspace{1cm} (25)

and let \(S_i, i = 1, 2\) a solution of \((DE^{(i)})\), then \(S_1 S_2\) is a solution of \((DE_2)\).

Conversely, every solution of \((DE_2)\) can be constructed so.

(v) If \(M_i, i = 1, 2\) are primitive and if \(S\), a solution of \((DE_2)\), is group-like at one point, (or, even at one limit point) it is globally group-like.

**Proof.** Omitted.

**Remark 1.**

- Every holomorphic series \(S(z) \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle\) which is group-like \((\Delta(S) = S \otimes S\) and \(\langle S | 1_X^*\rangle\)) is a solution of a left-sided dynamics with primitive multiplier (take \(M_1 = (S)S^{-1}\) and \(M_2 = 0\)).

\(^4\)As the left \((DE)\) for instance.
Invertible solutions of an equation of type \( S' = M_1 S \) are on the same orbit by multiplication on the right by invertible constant series i.e. let \( S_i, \ i = 1, 2 \) be invertible solutions of \((DE^{(1)})\), then there exists an unique invertible \( T \in \mathbb{C}[[X]] \) such that \( S_2 = S_1 T \). From this and point (iv) of the theorem, one can parametrize the set of invertible solutions of \((DE_2)\).

### 3.2 Application: Unicity of solutions with asymptotic conditions.

In a previous work \[3\], we proved that asymptotic group-likeness, for a series, implies \[5\] that the series in question is group-like everywhere. The process above (theorem \[1\], Picard’s process) can be performed, under certain conditions with improper integrals we then construct the series \( L \) recursively as

\[
(L \mid w) = \begin{cases} 
\frac{\log^n(z)}{n!} & \text{if } w = x_0^n \\
\int_{0}^{z} \left( \frac{z}{L} \right) \langle L \mid u \rangle [s] \, ds & \text{if } w = x_1 u \\
\int_{0}^{z} \left( \frac{z}{L} \right) \langle L \mid u x_1 x_0^n \rangle [s] \, ds & \text{if } w = x_0 u x_1 x_0^n.
\end{cases}
\]

one can check that

- this process is well defined at each step and computes the series \( L \) as below.
- \( L \) is solution of \((DE)\), is exactly \( G_0 \) and is group-like

We here only prove that \( G_0 \) is unique using the theorem above. Consider the series \( T = L e^{-x_0 \log(z)} \). Then \( T \) is solution of an equation of the type \((DE_2)\)

\[
T' = \left( \frac{x_0}{z} + \frac{x_1}{1 - z} \right) T + T \left( \frac{x_0}{z} \right)
\]

but \( \lim_{z \to z_0} G_0 e^{-x_0 \log(z)} = 1 \) so, by the point (iii) of theorem \[1\] one has \( G_0 e^{-x_0 \log(z)} = L e^{-x_0 \log(z)} \) and then \( G_0 = L \).

A similar (and symmetric) argument can be performed for \( G_1 \) and then, in this interpretation and context, \( \Phi_{KZ} \) is unique.

### 4 Double global regularization of associators

Global singularities analysis leads to to the following global renormalization \[2\]

\[
\lim_{z \to 1} \exp \left( -y_1 \log \frac{1}{1 - z} \right) \pi_Y(L(z)) = \lim_{n \to \infty} \exp \left( \sum_{k \geq 1} H_{y_k}(n) \left( \frac{-y_1}{k} \right)^k \right) H(n) = \pi_Y(Z_{\text{uni}}).
\]

\[5\] Under the condition that the multiplier be primitive, result extended as point (v) of the theorem above.

6
Thus, the coefficients \(\{\langle Z_{w}\rangle | u \in X^*\) (i.e. \(\{\xi_{w}(u)\} | u \in X^*\) and \(\{\langle Z_{w}\rangle | v \in Y^*\) (i.e. \(\zeta_{w}(v)\} | v \in Y^*\)) represent the finite part of the asymptotic expansions, in \(\{(1 - z)^{-n} \log^k((1 - z))\} | a, b \in \mathbb{N}\) (resp. \(\{n^{-a}H_{y}^{b}(n)\} | a, b \in \mathbb{N}\) of \(\{\langle L_{w}\rangle | u \in X^*\) (resp. \(\{\langle H_{w}\rangle | v \in Y^*\). On the other way, by a transfer theorem \[10\], let \(\{\gamma_{w}\} | v \in Y^*\) be the finite parts of \(\{H_{w}\} | v \in Y^*\), in \(\{n^{-a} \log^k(n)\} | a, b \in \mathbb{N}\), and let \(Z_{\gamma}\) be their noncommutative generating series. The map \(\gamma_{\bullet} : (\mathbb{Q}/\langle Y\rangle, \mathbb{W}, \mathbb{Y}^*) \to (\mathbb{Z}, x, 1)\), mapping \(w\) to \(\gamma_{w}\) is then a character and \(Z_{\gamma}\) is group-like, for \(\Delta_{\mathbb{W}}\). Moreover \[15\] \[10\],

\[
Z_{\gamma} = \exp(\gamma y_{1}) \prod_{l \in \mathcal{L}_{\gamma}Y - \{y_{1}\}} \exp(\zeta(\Sigma_{l})Pi_{l}) = \exp(\gamma y_{1})Z_{\mathbb{W}}.
\] (28)

The asymptotic behavior leads to the bridge\[4\] equation \[2\] \[15\] \[16\]

\[
Z_{\gamma} = B(y_{1})\pi_{Y}(Z_{\mathbb{W}}) \quad \text{or equivalently} \quad Z_{\mathbb{W}} = B'(y_{1})\pi_{Y}(Z_{\mathbb{W}})
\] (29)

where \(B(y_{1}) = \exp(-\gamma y_{1} - \sum_{k \geq 2}(1 - y_{1})^k \zeta(k)/k)\) and \(B'(y_{1}) = \exp(-\gamma y_{1})B(y_{1})\).

Similarly, there is \(C_{w}^{-} \in \mathbb{Q}\) and \(B_{w}^{-} \in \mathbb{N}\), such that \(H_{w}^{-}(N) = N^{-\infty}(C_{w})^{-1}\) and \(L_{w}^{-}(z) = (1 - z)^{-n - |w|}B_{w}^{-}\) [2]. Moreover,

\[
C_{w}^{-} = \prod_{w = w, v \neq 1} ((v) + |v|)^{-1} \quad \text{and} \quad B_{w}^{-} = ((w) + |w|)!C_{w}^{-}.
\] (30)

Now, one can then consider the following noncommutative generating series:

\[
L^{-} := \sum_{w \in Y_{0}^*} L_{w}^{-} w, \quad H^{-} := \sum_{w \in Y_{0}^*} H_{w}^{-} w, \quad C^{-} := \sum_{w \in Y_{0}^*} C_{w}^{-} w.
\] (31)

Then \(H^{-}\) and \(C^{-}\) are group-like for, respectively, \(\Delta_{\mathbb{W}}\) and \(\Delta_{\mathbb{W}}\) and \[4\]

\[
\lim_{z \to 1} h^{-1}(1 - z)^{-1} \otimes L^{-}(z) = \lim_{N \to +\infty} g^{-1}(N) \otimes H^{-}(N) = C^{-},
\] (32)

\[
h(t) = \sum_{w \in Y_{0}^*} ((w) + |w|)t(w) + |w|, \quad g(t) = \left(\sum_{y \in Y_{0}} t(y) + 1\right)^{y}.
\] (33)

Next, for any \(w \in Y_{0}^*\), there exists then a unique polynomial \(p \in (\mathbb{Z}[t], x, 1)\) of degree \((w) + |w|\) such that \[4\]

\[
L_{w}^{-}(z) = \sum_{k=0}^{(w) + |w|} \frac{p_{k}}{1 - z + (w) + |w| - 1} \in (\mathbb{Z}[t], x, 1),
\] (34)

\[
H_{w}^{-}(n) = \sum_{k=0}^{(w) + |w|} \frac{p_{k}}{k!} (n + k - 1) \in (\mathbb{Q}[t], x, 1),
\] (35)

\[6\] This equation is different from Jean Écalle’s one \[9\].
where \( (n) \mapsto \mathbb{Q} \mapsto \mathbb{Q} \) mapping \( i \) to \( (n)_i = n(n-1) \ldots (n-i+1) \). In other terms, for any \( w \in Y^e_0, k \in \mathbb{N}, 0 \leq k \leq (w) + |w|, \) one has \( \langle \text{Li}_w, (1-z)^{-k} \rangle = k!(\text{H}_w | (n)_k) \).

Hence, denoting \( \tilde{p} \) the exponential transformed of the polynomial \( p \), one has \( \text{Li}_w(z) = p((1-z)^{-1}) \) and \( \text{H}_w(n) = \tilde{p}(n)_* \) with

\[
p(t) = \sum_{k=0}^{(w) + |w|} p_k t^k \in (\mathbb{Z}[t], x, 1) \quad \text{and} \quad \tilde{p}(t) = \sum_{k=0}^{(w) + |w|} \frac{p_k}{k!} t^k \in (\mathbb{Q}[t], x, 1).
\]

Let us then associate \( p \) and \( \tilde{p} \) with the polynomial \( \tilde{p} \) obtained as follows

\[
\tilde{p}(t) = \sum_{k=0}^{(w) + |w|} k! p_k t^k = \sum_{k=0}^{(w) + |w|} p_k t^k \in (\mathbb{Z}[t], x, 1).
\]

Let us recall also that, for any \( c \in \mathbb{C} \), one has \( (n)_c \sim n^c = e^{c \log(n)} \) and, with the respective scales of comparison, one has the following finite parts

\[
f.p. -1 e^{c \log(1-z)} = (1-z)^b \log((1-z)^{-1})) \rangle = \{(1-z)^b \log((1-z)^{-1}))  \}_{a \in \mathbb{Z}, k \in \mathbb{N}}.
\]

\[
f.p. -\infty e^{c \log(n)} = n^b \log(n) \rangle = \{n^b \log(n)) \}_{a \in \mathbb{Z}, k \in \mathbb{N}}.
\]

Hence, using the notations given in \( (34) \) and \( (35) \), one can see, from \( (38) \) and \( (39) \), that the values \( p(1) \) and \( \tilde{p}(1) \) obtained in \( (36) \) represent

\[
f.p. -1 \text{Li}_w(z) = f.p. -1 \text{Li}_w(z) = p(1) \in \mathbb{Z},
\]

\[
f.p. -\infty \text{H}_w(n) = f.p. -\infty \text{H}_w(n) = \tilde{p}(1) \in \mathbb{Q}.
\]

One can use then these values \( p(1) \) and \( \tilde{p}(1) \), instead of the values \( B_w^- \text{ and } C_w^- \), to regularize, respectively, \( \zeta_{\mathbb{Q}^e}(R_w) \) and \( \zeta_{\mathbb{Q}^e}(\pi_y(R_w)) \) as showed Theorem \( 2 \) below because, essentially, \( B_w^- \) and \( C_w^- \) do not realize characters for, respectively, \( (\mathbb{Q}, \mathbb{Z}), (\mathbb{Q}, \mathbb{Z}) \) and \( (\mathbb{Q}, \mathbb{Z}), (\mathbb{Q}, \mathbb{Z}) \).

Now, in virtue of the extension of \( \text{Li}_w \), defined as in \( (16) \) and \( (17) \), and of the Taylor coefficients, the previous polynomials \( p, \tilde{p} \) and \( \tilde{p} \) given in \( (36) \) can be determined explicitly thanks to

**Proposition 1.** 1. The following morphisms of algebras are bijective

\[
\lambda : (\mathbb{Z}[x^1], x^1, 1_{X}) \rightarrow (\mathbb{Z}((1-z)^{-1}), x, 1), \quad R \mapsto \text{Li}_R,
\]

\[
\eta : (\mathbb{Q}[y^1], y^1, 1_{Y}) \rightarrow (\mathbb{Q}[[n]_*], x, 1), \quad S \mapsto \text{H}_S.
\]

2. For any \( w = y_{s_1}, \ldots y_{s_e} \in Y^e_0 \), there exists a unique polynomial \( R_w \) belonging to \( (\mathbb{Z}[x^1], x^1, 1_{X}) \) of degree \( (w) + |w| \), such that

\[
\text{Li}_{R_w}(z) = \text{Li}_{w}(z) = p((1-z)^{-1}) \in (\mathbb{Z}((1-z)^{-1}), x, 1),
\]

\[
\text{H}_{\pi_{y}(R_w)}(n) = \text{H}_{w}(n) = p(n)_* \in (\mathbb{Q}[[n]_*], x, 1).
\]

\[
\text{Here, it is also convenient to denote } \mathbb{Q}[[n]_*] \text{ the set of ”polynomials” expanded as follows}
\]

\[
\forall p \in \mathbb{Q}[[n]_*], \quad p = \sum_{k=0}^{d} p_k(n)_k, \quad \text{deg}(p) = d.
\]
In particular, via the extension, by linearity, of \( R_\bullet \) over \( \mathbb{Q}(Y_0) \) and via the linear independent family \( \{ \text{Li}_{y_k}^- \}_{k \geq 0} \) in \( \mathbb{Q}(\text{Li}_{y_1}^-) \), \( w \in Y_0^* \), one has
\[
\forall k, l \in \mathbb{N}, \quad \text{Li}_{R_{yk}} \circ \text{Li}_{R_{yl}} = \text{Li}_{R_{y_{k+l}}} \quad \text{Li}_{R_{y_{k+l}}} = \text{Li}_{y_{k+l}} \circ \text{Li}_{y_{k+l}} = \text{Li}_{R_{y_{k+l}}},
\]

3. For any \( w \), one has \( \hat{p}(x_1^+) = R_w \).

4. More explicitly, for any \( w = y_{s_1}, \ldots, y_{s_r} \in Y_0^* \), there exists a unique polynomial \( R_w \) belonging to \( (\mathbb{Z}[x_1^*], \omega, 1_{X^*}) \) of degree \( (w)+|w| \), given by
\[
R_{y_{s_1} \cdots y_{s_r}} = \sum_{k_1=0}^{s_1} \sum_{k_2=0}^{s_1+s_2-k_1} \cdots \sum_{k_r=0}^{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}} \binom{s_1}{k_1} \binom{s_1+s_2-k_1}{k_2} \cdots \binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r} \rho_{k_1} \cdots \rho_{k_r},
\]
where, for any \( i = 1, \ldots, r \), if \( k_i = 0 \) then \( \rho_{k_i} = x_1^* - 1_{X^*} \), else, for \( k_i > 0 \), denoting the Stirling numbers of second kind by \( S_2(k, j) \)’s, one has
\[
\rho_{k_i} = \sum_{j=1}^{k_i} S_2(k_i, j) (j!)^2 \sum_{l=0}^{j} \frac{(-1)^l}{l!} \frac{(x_1^*)^l (j-t+1)}{(j-l)!}.
\]

Proposition 2 ([13] [16]). With notations of [13], similar to the character \( \gamma_\bullet \), the poly-morphism \( \gamma_\bullet \) can be extended as follows
\[
\zeta_{\omega}: (\mathbb{Q}(X), \omega, 1_{X^*}) \rightarrow (\mathbb{Z}, \times, 1), \quad \zeta_{\omega}: (\mathbb{Q}(Y), \omega, 1_{Y^*}) \rightarrow (\mathbb{Z}, \times, 1)
\]
satisfying, for any \( l \in \text{Lyn}Y - \{ y_l \} \), \( \zeta_{\omega}(\pi_X(l)) = \zeta_{\omega}(\pi_Y(l)) = \gamma_l = \zeta(l) \) and, for the generators of length (resp. weight) one, for \( X^* \) (resp. \( Y^* \)), \( \gamma_{y_1} = \gamma \) and \( \zeta_{\omega}(x_0) = \zeta_{\omega}(x_1) = \zeta_{\omega}(y_1) = 0 \).

Now, to regularize \( \{ \zeta(s_1, \ldots, s_r) \}_{s_1, \ldots, s_r} \in \mathbb{C}^r \), we use

Lemma 1 ([3]).
1. The power series \( x_0^* \) and \( x_1^* \) are transcendent over \( \mathbb{C}(X) \).
2. The family \( \{ x_0^*, x_1^* \} \) is algebraically independent over \( \mathbb{C}(X), \omega, 1_{X^*} \) within \( \mathbb{C}(\langle X \rangle), \omega, 1_{X^*} \).
3. The module \( \langle \mathbb{C}(X), \omega, 1_{X^*} \rangle[\{ x_0^*, x_1^*, (-x_0)^* \} \) is \( \mathbb{C}(X) \)-free and the family \( \{ x_0^*, x_1^*, (-x_0)^* \} \in \mathbb{Z} \times \mathbb{N} \) forms a \( \mathbb{C}(X) \)-basis of it.

Hence, \( \{ w \cdot (x_0^*)^{w} \cdot (x_1^*)^{w} \}_{w \in X^*} \) is a \( \mathbb{C} \)-basis of it.
4. One has, for any \( x_i \in X, \mathbb{C}^{\mathfrak{at}} \{ x_i \} = \mathbb{C}^{\mathfrak{at}} \{ x_i \} = \mathbb{C}^{\mathfrak{at}} \{ x_i \} = \mathbb{C}(X) \).

Since, for any \( t \in \mathbb{C}, |t| < 1 \), one has \( L_{(tx_i),\bullet}(z) = (1-z)^{-t} \) and
\[
H_{x_1^*} (tx_i)^k = \exp \left( - \sum_{k \geq 1} H_{y_1^*} (\frac{-t}{k}) \right)
\]
then, with the notations of Proposition 2 we extend extend the characters \( \zeta_{\omega} \) and \( \gamma_\bullet \), defined in Proposition 2 over \( \mathbb{C}(X) \) \( \mathbb{C}[x_1^*] \) and \( \mathbb{C}(Y) \) \( \mathbb{C}[y_1^*] \), respectively, as follows
Proposition 3 (4). The characters $\zeta_{\omega}$ and $\gamma_{\bullet}$ can be extended as follows

$$
\zeta_{\omega} : (C(X) \oplus C[x^*_1], \omega, 1_X) \longrightarrow (C, x, 1_C),
$$
$$
\forall t \in C, |t| < 1, \quad (tx_1)^* \longrightarrow 1_C.
$$

$$
\gamma_{\bullet} : (C(Y) \oplus C[y^*_1], \omega, 1_Y) \longrightarrow (C, x, 1_C),
$$
$$
\forall t \in C, |t| < 1, \quad (ty_1)^* \longrightarrow \exp\left(\gamma t - \sum_{n \geq 2} \zeta(n) \frac{(-t)^n}{n}\right) = \frac{1}{\Gamma(1 + t)}.
$$

Therefore, in virtue of Propositions 1 and 3 we obtain then

Theorem 2. 1. For any $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$ associated with $w \in Y^*$, there exists a unique polynomial $p \in \mathbb{Z}[t]$ of valuation 1 and of degree $(w) + |w|$ such that

$$
p(x^*_1) = R_w \quad \in \quad (\mathbb{Z}[x^*_1], \omega, 1_X),
$$

$$
\tilde{p}(t) = H_{\gamma_Y(R_w)}(n) \quad \in \quad (\mathbb{Q}(n)^*_1, x, 1),
$$

$$
\zeta_{\omega}(-s_1, \ldots, -s_r) = p(1) \quad \in \quad \zeta_{\omega}(R_w) \quad \in \quad (\mathbb{Z}, x, 1),
$$

$$
\gamma_{-s_1, \ldots, -s_r} = \tilde{p}(1) \quad \in \quad \gamma_{\pi_Y(R_w)} \quad \in \quad (\mathbb{Q}, x, 1).
$$

2. Let $Y(n) \in \mathbb{Q}[\langle n \rangle] \langle Y \rangle$ and $\Lambda(z) \in \mathbb{Q}[\langle (1 - z)^{-1} \rangle] \langle X \rangle$ be the non-commutative generating series of $\{H_{\pi_Y(R_w)}\}_{w \in Y^*}$ and $\{\text{Li}_{\gamma_Y(R_w)}\}_{w \in X^*}:

$$
Y := \sum_{w \in Y^*} H_{\gamma_Y}(R_w) w \quad \text{and} \quad \Lambda := \sum_{w \in X^*} \text{Li}_{\gamma_Y}(R_w) w, \quad \text{with} \quad \langle \Lambda(z) \mid x_0 \rangle = \log(z).
$$

Then $Y$ and $\Lambda$ are group-like, for respectively $\Delta_{\omega}$ and $\Delta_{\omega}$, and:

$$
Y = \prod_{l \in \mathbb{L}_Y} e^{H_{\gamma_Y}(R_{S_l})} P_l \quad \text{and} \quad \Lambda = \prod_{l \in \mathbb{L}_X} e^{\text{Li}_{\gamma_Y}(S_l)} P_l.
$$

3. Let $Z^{-}_\gamma \in \mathbb{Q}[\langle Y \rangle]$ and $Z^{-}_{\omega} \in \mathbb{Z}[\langle X \rangle]$ be the noncommutative generating series of $\{\gamma_{\pi_Y(R_w)}\}_{w \in Y^*}$ and $\zeta_{\omega}(R_{\pi_Y(w)})$, respectively:

$$
Z^{-}_\gamma := \sum_{w \in Y^*} \gamma_{\pi_Y(R_w)} w \quad \text{and} \quad Z^{-}_{\omega} := \sum_{w \in X^*} \zeta_{\omega}(R_{\pi_Y(w)}) w.
$$

Then $Z^{-}_\gamma$ and $Z^{-}_{\omega}$ are group-like, for respectively $\Delta_{\omega}$ and $\Delta_{\omega}$, and:

$$
Z^{-}_\gamma = \prod_{l \in \mathbb{L}_Y} e^{\gamma_{\pi_Y(R_{S_l})}} P_l \quad \text{and} \quad Z^{-}_{\omega} = \prod_{l \in \mathbb{L}_X} e^{\zeta_{\omega}(S_l)} P_l.
$$

Moreover, F.P.$_{\gamma \rightarrow \infty} Y(n) = Z^{-}_\gamma$ and F.P.$_{\gamma \rightarrow 1} \Lambda(z) = Z^{-}_{\omega}$ meaning that, for any $v \in Y^*$ and $u \in X^*$, one has

$$
f.p._{\gamma \rightarrow \infty} \langle Y(n) \mid v \rangle = \langle Z^{-}_\gamma \mid v \rangle \quad \text{and} \quad f.p._{\gamma \rightarrow 1} \langle \Lambda(z) \mid u \rangle = \langle Z^{-}_{\omega} \mid u \rangle. \quad (43)
$$

---

On the one hand, by Proposition 2 one has $\langle Z^{-}_{\omega} \mid x_0 \rangle = \zeta_{\omega}(x_0) = 0$.

On the other hand, since $R_{y_1} = (2x_1)^* - x_1^*$ then $\text{Li}_{R_{y_1}}(z) = (1 - z)^{-2} - (1 - z)^{-1}$ and $H_{\gamma_Y(R_{y_1})}(n) = (\gamma_{y_1}^n - \gamma_{y_1}^0)$. Hence, one also has $\langle Z^{-}_{\omega} \mid x_1 \rangle = \zeta_{\omega}(R_{\pi_Y(y_1)}) = 0$ and $\langle Z^{-}_\gamma \mid x_1 \rangle = \gamma_{\pi_Y(R_{y_1})} = -1/2$. 10
References

[1] V.C. Bui, G.H.E. Duchamp, Hoang Ngoc Minh.– Structure of Polyzetas and Explicit Representation on Transcendence Bases of Shuffle and Stuffle Algebras, J. of Sym. Comp. (2016).

[2] Costermans C., Hoang Ngoc Minh.– Noncommutative algebra, multiple harmonic sums and applications in discrete probability, J. of Sym. Comp. (2009), 801-817.

[3] M. Deneufchâtel, G.H.E. Duchamp, Hoang Ngoc Minh, A.I. Solomon.– Independence of hyperlogarithms over function fields via algebraic combinatorics, dans Lecture Notes in Computer Science (2011), Volume 6742/2011, 127-139.

[4] G.H.E. Duchamp, Hoang Ngoc Minh, Q.H. Ngo– Harmonic sums and polylogarithms at negative multi-indices, J. of Sym. Comp. (2016).

[5] G.H.E. Duchamp, Hoang Ngoc Minh, Q.H. Ngo– Double regularization of polyzetas at negative multiindices and rational extensions, en préparation.

[6] V. Drinfel’d– Quantum group, Proc. Int. Cong. Math., Berkeley, 1986.

[7] V. Drinfel’d– Quasi-Hopf Algebras, Len. Math. J., 1, 1419-1457, 1990.

[8] V. Drinfel’d– On quasitriangular quasi-hopf algebra and a group closely connected with gal(¯q/¯q), Len. Math. J., 4, 829-860, 1991.

[9] Jean Écalle– L’équation du pont et la classification analytique des objets locaux, in Les fonctions résurgentes, 3, Publications de l’Université de Paris-Sud, Département de Mathématique (1985).

[10] Philippe Flajolet & Andrew Odlyzko– Singularity Analysis of Generating Functions, SIAM J. Discrete Math., 3(2), pp. 216240, 1982.

[11] Hoang Ngoc Minh– Summations of Polylogarithms via Evaluation Transform, Math. & Computers in Simulations, 1336, pp 707-728, 1996.

[12] Hoang Ngoc Minh & Jacob G.– Symbolic Integration of meromorphic differential equation via Dirichlet functions, Disc. Math. 210, pp. 87-116, 2000.

[13] Hoang Ngoc Minh, Jacob G., N.E. Oussous, M. Petitot– De l’algèbre des ζ de Riemann multivariées à l’algèbre des ζ de Hurwitz multivariées, journal électronique du Séminaire Lotharingien de Combinatoire, 44, (2001).

[14] Hoang Ngoc Minh & M. Petitot– Lyndon words, polylogarithmic functions and the Riemann ζ function, Disc. Math., 217, 2000, pp. 273-292.

[15] Hoang Ngoc Minh– On a conjecture by Pierre Cartier about a group of associators, Acta Math. Vietnamica (2013), 38, Issue 3, 339-398.
[16] Hoang Ngoc Minh– *Structure of polyzetas and Lyndon words*, Vietnamese Math. J. (2013), 41, Issue 4, 409-450.

[17] T.Q.T. Lê & J. Murakami– *Kontsevich’s integral for Kauffman polynomial*, Nagoya Math., pp 39-65, 1996.

[18] Reutenauer C.– *Free Lie Algebras*, London Math. Soc. Monographs (1993).

[19] D. Zagier– *Values of zeta functions and their applications*, in “First European Congress of Mathematics”, vol. 2, Birkhäuser, pp. 497-512, 1994.