A New Method to Construct Golay Complementary Set by Paraunitary Matrices and Hadamard Matrices

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Abstract
Golay complementary sequences have been put a high value on the applications in orthogonal frequency-division multiplexing (OFDM) systems since its good peak-to-mean envelope power ratio (PMEPR) properties. However, with the increase of the code length, the code rate of the standard Golay sequences suffer a dramatic decline. Even though a lot of efforts have been paid to solve the code rate problem for OFDM application, how to construct large classes of sequences with low PMEPR is still difficult and open now. In this paper, we propose a new method to construct $q$-ary Golay complementary set of size $N$ and length $N^n$ by $N \times N$ Hadamard Matrices where $n$ is arbitrary and $N$ is a power of 2. Every item of the constructed sequences can be presented as the product of the specific entries of the Hadamard Matrices. The previous works in [3] can be regarded as a special case of the constructions in this paper and we also obtained new quaternary Golay sets never reported in the literature.

Keywords: Golay sequences, Golay set, PMEPR, Hadamard matrix, Paraunitary matrix.

1 Introduction
In the fundamental paper written by M. Golay[10], binary complementary sequences were first introduced in the context of infrared spectrometry. A complementary pair has the useful property that the aperiodic autocorrelations of the two sequences sum up to zero for each nonzero shift. Golay complementary pairs and sequences have been found many applications in the fields of science and engineering, especially in wireless communication. One of the most important applications is in orthogonal frequency-division multiplexing (OFDM) [16][13]. The good property of $q$-ary Golay sequences, that their peak-to-mean
envelope power ratio (PMPPR) is at most 2, makes them appropriate to use as codewords in OFDM system.

In [10], Golay proposed recursive concatenation and interleaving algorithms to construct binary Golay complementary pairs of length $2^n$. Binary Golay complementary pairs were generalized to complementary sets in [23], and to polyphase sequences in [20]. In 1999, Davis and Jedwab’s milestone work showed that known $q$-ary sequences in Golay pair fill up specific second-order cosets of the generalized first-order Reed-Muller codes [6]. Such sequences are so-called Golay-Davis-Jedwab (GDJ) sequences or standard Golay sequences. However, with the increase of the code length, the code rate of the standard Golay sequences suffers a dramatic decline [6].

To solve the code rate problem of the standard Golay sequences for OFDM application, a lot of efforts have been paid in the literature where PMEPR is bounded by a finite value $c \geq 2$, such as finding more non-standard Golay sequences[11][8][7], design Golay complementary set [15][19], design near complementary sequences[18], design sequences with low PMEPR over QAM constellation[17][22][24], and design Golay complementary sequences over QAM constellation[5][12][22][14]. However, these coding methods did not solve the code rate problem in OFDM system. For this reason, how to construct large classes of sequences with low PMEPR is critical and still open now.

All the above Boolean-function-based constructions are proved by the property that the summation of the aperiodic autocorrelations of the two or more sequences equals zero for each nonzero shift. There exists related matrix-based approaches to complementary sequence design in the literature. For instance, the complete complementary code approach of [21] and the paraunitary matrix approach to filter banks for complementary sequences, as discussed in [1][2]. In particular, Budisin and Spasojević proposed a new method to construct Golay complementary pairs by paraunitary matrices over both PSK and QAM constellations in [3]. By this method, not only all the standard Golay pair over PSK constellation and known Golay pair over QAM constellation can be explained, but also some new QAM complementary pairs can be constructed. Here the word ‘paraunitary’ refers to a matrix polynomial in $z$ that is unitary for all $z$ on unit circle.

In this paper, inspired by the paraunitary-matrix-based method in [3], we propose a new method to construct $q$-ary Golay complementary set of size $N$ and length $N^n$ by $N \times N$ Hadamard matrices where $n$ is arbitrary and $N$ is a power of 2. Every item of the constructed sequences can be presented as the product of the specific entries of the Hadamard matrices. To make the results easier to understand for readers, we only show the proof for $N = 2^2 = 4$, and the case for other is $N$ similar to $N = 4$. Budisin and Spasojević’s algorithm for $q$-ary Golay pair in [3] can be regarded as a special case of the construction in this paper by setting $N = 2$ and selecting specific Hadamard matrices. We also show new quaternary Golay pair of size 4 and degree 3 never reported in the literature can be obtained by the method presented in this paper.


2 Preliminaries

2.1 Peak Power Control in OFDM and Golay Set

Let $a = (a_0, a_1, \cdots, a_{L-1})$ be a $q$-ary sequence of length $L$ where $a_i$ is a $q$th roots of unity. In an OFDM system with $L$ subcarriers, the transmitted signal by employing sequence $a$ can be modeled as the real part of

$$s_a(t) = \sum_{i=0}^{L-1} a_i e^{2\pi \sqrt{-1} (f_0 + i\Delta f)t}, \quad t \in \left[0, \frac{1}{\Delta f}\right),$$

where $\Delta f$ is the frequency separation between adjacent subcarrier pairs and $f_0$ is the base frequency.

The sequence $a$ can be associated with the polynomial

$$A(z) = \sum_{i=0}^{L-1} a_i z^i. \quad (1)$$

in indeterminate $z$. We use $A(z)$ instead of sequence $a$ in the rest of the paper for convenience.

By restricting $z$ to lie on the unit circle in the complex plane, i.e., $z = e^{2\pi \sqrt{-1} \Delta ft}$. We have

$$|s_a(t)| = |A(e^{2\pi \sqrt{-1} \Delta ft})|.$$

Then the instantaneous envelope power of the transmitted signal is determined by $A(z)\overline{A}(z^{-1})$ where the conjugate polynomial $\overline{A}(z^{-1}) = \sum_{i=0}^{L-1} \overline{a}_i z^{-i}$, and the peak-to-mean envelope power ratio (PMEPR) is determined by

$$\text{PMEPR}(a) = \frac{1}{L} \sup_{|z|=1} A(z)\overline{A}(z^{-1}). \quad (2)$$

If a set of sequences $A_i(z)$ for $1 \leq i \leq N$ satisfy

$$\sum_{i=1}^{N} A_i(z)\overline{A}_i(z^{-1}) = LN, \quad (3)$$

$\{A_i(z) : 1 \leq i \leq N\}$ is called a Golay set of size $N$. It is straightforward that the PMEPR of each sequence $A_i(z)$ in Golay set is upper bounded by $N$ by definition in (2).

2.2 Butson Type Hadamard matrix

A complex Hadamard matrix is a complex $N \times N$ matrix $H$ satisfying $|H_{ij}| = 1$ ($i, j = 1, 2, \cdots, N$) and $HH^\dagger = N \cdot I_N$, where $H^\dagger$ denotes the Hermitian transpose of $H$ and $I_N$ is the identity matrix. A complex Hadamard matrix of size $N$ is called of Butson type $H(q, N)$ [4] if all the entries of $H$ are $q$th roots of unity. We are only interested in Butson type Hadamard matrices in the rest of the paper.
Two Butson type \((q, N)\) Hadamard matrices \(H_1\) and \(H_2\) are called equivalent, written as \(H_1 \simeq H_2\), if there exist diagonal unitary matrices \(D_1, D_2\) where each diagonal entry of \(D_i\) is \(q\)th root of unity and permutation matrices \(P_1, P_2\) such that:

\[
H_1 = D_1 P_1 H_2 P_2 D_2.
\]  

(4)

For example, all binary Hadamard matrices in \(H(2,4)\) are equivalent to the \(4 \times 4\) Hadamard matrix:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix},
\]

so every \((2,4)\) Hadamard matrices can be generated by the above matrix. And all quaternary Hadamard matrices in \(H(4,4)\) are equivalent to one of the following \(4 \times 4\) Hadamard matrix:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}, \text{ and } \\
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & i & -1 & -i \\
1 & -1 & 1 & -1 \\
1 & -i & -1 & i
\end{pmatrix},
\]

so every \((4,4)\) Hadamard matrices can be generated by one of the above matrices by using formula (4).

### 2.3 Paraunitary Matrix and Golay Set

An \(N \times N\) complex Matrix is called unitary if \(MM^\dagger = I_N\). An \(N \times N\) polynomial matrix \(M(z)\), where every entry \(M_{ij}(z)\) is a polynomial in indeterminate \(z\) with complex coefficients, is called paraunitary if

\[
M(z)M^\dagger(z^{-1}) = c \cdot I_N
\]  

(5)

for real constant \(c\). We are only interested in the paraunitary matrix \(M(z)\) where all the coefficients of every entry \(M_{ij}(z)\) are \(q\)th roots of unity in the rest of the paper.

In the next section, we present a new method to design \(N \times N\) paraunitary matrix \(M(z)\) satisfying

\[
M(z)M^\dagger(z^{-1}) = L \cdot I_N,
\]  

(6)

where every entry \(M_{ij}(z)\) is a polynomial of degree \(L - 1\) with \(q\)th root coefficients, i.e., every entry corresponds to a \(q\)-ary sequence of length \(L\). From the definition of the Golay set, the sequences in every row or every column of the matrix \(M(z)\) satisfying (6) form a \(q\)-ary Golay set of size \(N\) and length \(L\).

### 3 New Construction

In this section, by using paraunitary matrix and Butson type \((q, N)\) Hadamard matrices, we propose a new method to construct Golay set.
Construction 1. Let $N$ be a power of 2, $n$ a non-negative integer, and $H^{(i)}$ an arbitrary Butson type $(q,N)$ Hadamard matrix for $0 \leq i \leq n$ if it exists. Let $D(z)$ be a diagonal paraunitary matrix with the form $D(z) = \text{diag}\{1,z,z^2,\ldots,z^{N-1}\}$, and $\pi$ a permutation of numbers $\{0,1,2,\ldots,n-1\}$. Define $N \times N$ matrix $M^{(n)}(z)$ as follows:

\[
M^{(n)}(z) = H^{(0)} \cdot (D(z))^{N^n(0)} \cdot H^{(1)} \cdots (D(z))^{N^n(n-1)} \cdot H^{(n)} \\
= H^{(0)} \cdot \prod_{t=1}^{n} \left( (D(z))^{N^n(t-1)} \cdot H^{(t)} \right) \\
= H^{(0)} \cdot \prod_{t=1}^{n} \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & z^{N^n(t)} & 0 & 0 & \ldots & 0 \\
0 & 0 & z^{2.N^n(t)} & 0 & \ldots & 0 \\
0 & 0 & 0 & z^{3.N^n(t)} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & z^{(N-1).N^n(t)} \\
\end{pmatrix} \cdot H^{(t)}.
\] (7)

Theorem 2. The sequences in every row or every column of the matrix $M^{(n)}(z)$ defined above form a $q$-ary Golay set of size $N$ and length $N^n$.

To verify the Theorem 2, we need to prove the following properties by Section 2.3.

- (1) $M^{(n)}(z)$ is a paraunitary matrix satisfying $M^{(n)}(z)M^{(n)}(z^{-1}) = N^{n+1} \cdot I_N$

- (2) The degree of every entry of $M^{(n)}_{ij}(z)$ is $N^n - 1$

- (3) All the coefficients of every entry $M^{(n)}_{ij}(z)$ are $q$th roots of unity

The first property is very easy to check, since $H^{(i)}H^{(i)^\dagger} = N \cdot I_N$ for $0 \leq i \leq n$, and $D(z)D(z)^\dagger = I_N$. We prove the second and third properties by verifying every coefficient of $M^{(n)}_{ij}(z)$ presented by the product of the entries of the involved Butson type Hadamard matrices.

For $N = 2$, the proof for Theorem 2 can be easily derived from Budisin and Spasojević’s algorithm in [3], which can generate all $q$-ary standard Golay pairs. We will show the proof for the case $N = 4$ in the next section, in which new Golay set of size 4 can be obtained. For general $N$, the proof is similar to the case $N = 4$.

4 Proof for $N = 4$

Suppose that $N = 4$ and $N^n = 4^m = L$, $m$ is an integer and $m \in [0, L-1]$. Note that the binary expansion of $m$ is:

\[
m = \sum_{k=0}^{n-1} (\delta_{2k}(m)2^{2k} + \delta_{2k+1}(m)2^{2k+1}).
\] (8)

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Lemma 3. Let \( L = 4^n \), where \( n \) is a non-negative integer. Then we have
\[
\prod_{k=0}^{n-1} (F_k(0,0) + F_k(1,0) + F_k(0,1) + F_k(1,1)) = \sum_{m=0}^{L-1} \left\{ \prod_{k=0}^{n-1} F_k(\delta_{2k}(m),\delta_{2k+1}(m)) \right\},
\]
(9)
where \( F_k(x_0,x_1) \) are matrix complex Boolean functions, i.e. \( F_k(x_0,x_1) = M_1 \cdot \overline{x_0} \cdot \overline{x_1} + M_2 \cdot x_0 \cdot \overline{x_1} + M_3 \cdot \overline{x_0} \cdot x_1 + M_4 \cdot x_0 \cdot x_1 \) where \( M_1, M_2, M_3, M_4 \) are arbitrary complex matrices and \( \overline{0} = 1, \overline{1} = 0 \).

Proof.
\[
\begin{align*}
\prod_{k=0}^{n-1} (F_k(0,0) + F_k(1,0) + F_k(0,1) + F_k(1,1)) &= (F_0(0,0) + F_0(1,0) + F_0(0,1) + F_0(1,1)) \\
&\quad \times (F_1(0,0) + F_1(1,0) + F_1(0,1) + F_1(1,1)) \times \cdots \times \\
&\quad (F_{n-2}(0,0) + F_{n-2}(1,0) + F_{n-2}(0,1) + F_{n-2}(1,1)) \\
&\quad \times (F_{n-1}(0,0) + F_{n-1}(1,0) + F_{n-1}(0,1) + F_{n-1}(1,1)) \\
&= F_{n-1}(0,0) \cdots F_1(0,0) F_0(0,0) + F_{n-1}(0,0) \cdots F_1(0,0) F_0(0,1) + \\
&\quad \cdots + F_{n-1}(1,1) \cdots F_1(1,1) F_0(1,1) \\
&= \sum_{\delta_0=0}^{1} \sum_{\delta_1=0}^{1} \cdots \sum_{\delta_{n-1}=0}^{1} \left\{ \prod_{k=0}^{n-1} F_k(\delta_{2k},\delta_{2k+1}) \right\} \\
&= \sum_{m=0}^{L-1} \left\{ \prod_{k=0}^{n-1} F_k(\delta_{2k}(m),\delta_{2k+1}(m)) \right\}.
\end{align*}
\]
(10)

\[\square\]

Corollary 4. Let \( \pi \) be a permutation of numbers \( \{0,1,2, \cdots, n-1\} \), then the equation in Lemma 3 can be rewritten as:
\[
\prod_{t=0}^{n-1} (F_{\pi(t)}(0,0) + F_{\pi(t)}(1,0) + F_{\pi(t)}(0,1) + F_{\pi(t)}(1,1)) = \sum_{m=0}^{L-1} \left\{ \prod_{t=0}^{n-1} F_{\pi(t)}(\delta_{2\pi(t)}(m),\delta_{2\pi(t)+1}(m)) \right\}.
\]
(11)

Let
\[
M^{(n)}(z) = \sum_{m=0}^{L-1} M^{(n)}(m) \cdot z^m,
\]
where \( M^{(n)}(m) \) is the coefficient matrix of the polynomial matrix \( M^{(n)}(Z) \) of item \( z^m \). We show that every entry of \( M^{(n)}(m) \) is \( q \)th root of unity in the following theorem, which complete the proof of Theorem 2.
Theorem 5. Let \( N = 4 \) in Construction 1. The coefficient matrix of the polynomial matrix \( M^{(n)}(z) \) of item \( z^m \) can be presented by

\[
M^{(n)}(m) = H^{(0)} \cdot \prod_{t=1}^{n} (A_t \cdot H^{(t)}) ,
\]

where

\[
A_{t+1} = \begin{bmatrix}
\delta_{2\pi(t)}(m) \cdot \delta_{2\pi(t)+1}(m) & 0 & 0 \\
0 & \delta_{2\pi(t)}(m) \cdot \delta_{2\pi(t)+1}(m) & 0 \\
0 & 0 & \delta_{2\pi(t)}(m) \cdot \delta_{2\pi(t)+1}(m)
\end{bmatrix}
\]

Remark 6. Only one entry of matrix \( A_t \) is 1, and the others are all 0.

Proof. It is sufficient to prove that the univariate polynomial with coefficients in (12) equals to the expression in Construction 1 with \( N = 4 \), i.e.,

\[
\sum_{m=0}^{L-1} H^{(0)} \cdot \prod_{t=1}^{n} (A_t \cdot H^{(t)}) z^m = M^{(n)}(z).
\]

Here we substitute \( m \) with its binary expansion:

\[
m = \sum_{k=0}^{n-1} (\delta_{2k} 2^{2k} + \delta_{2k+1} 2^{2k+1})
\]

\[
= \sum_{t=0}^{n-1} (\delta_{2\pi(t)} 2^{2\pi(t)} + \delta_{2\pi(t)+1} 2^{2\pi(t)+1}).
\]

Then we have

\[
\sum_{m=0}^{L-1} H^{(0)} \cdot \prod_{t=1}^{n} (A_t \cdot H^{(t)}) z^m = H^{(0)} \cdot \sum_{m=0}^{L-1} \left\{ \prod_{t=1}^{n} (A_t \cdot H^{(t)}) \cdot z^{2^{2\pi(t)} + 2^{2\pi(t)+1}} \right\}
\]

\[
= H^{(0)} \cdot \sum_{m=0}^{L-1} \left\{ \prod_{t=0}^{n-1} (A_{t+1} \cdot z^{2^{2\pi(t)} + 2^{2\pi(t)+1}} H^{(t+1)}) \right\}
\]

Note that

\[
\delta_{2\pi(t)}(m) \cdot \delta_{2\pi(t)+1}(m) \cdot z^{2^{2\pi(t)} + 2^{2\pi(t)+1}} \cdot H^{(t+1)} = \delta_{2\pi(t)}(m) \cdot \delta_{2\pi(t)+1}(m) \cdot H^{(t+1)}
\]

\[
\delta_{2\pi(t)}(m) \cdot \delta_{2\pi(t)+1}(m) \cdot z^{2^{2\pi(t)} + 2^{2\pi(t)+1}} \cdot H^{(t+1)} = \delta_{2\pi(t)}(m) \cdot \delta_{2\pi(t)+1}(m) \cdot z^{2^{2\pi(t)} \cdot H^{(t+1)}}.
\]
\[
\delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} \cdot z^{\delta_{2^{\pi(t)}(2^{\pi(t)}+1)} + \delta_{2^{\pi(t)+1}(2^{\pi(t)}+1)}} \cdot H^{(t+1)} = \delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} \cdot z^{2^{\pi(t)+1}} \cdot H^{(t+1)},
\]

and

\[
\delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} \cdot z^{\delta_{2^{\pi(t)}(2^{\pi(t)}+1)} + \delta_{2^{\pi(t)+1}(2^{\pi(t)}+1)}} \cdot H^{(t+1)} = \delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} \cdot z^{2^{\pi(t)+1} + 2^{\pi(t)+1}} \cdot H^{(t+1)},
\]

we have

\[
\prod_{t=0}^{n-1} \left( A_t \cdot z^{\delta_{2^{\pi(t)}(2^{\pi(t)}+1)} + \delta_{2^{\pi(t)+1}(2^{\pi(t)}+1)}} \right) H^{(t+1)} = \prod_{t=0}^{n-1} \left( \frac{1}{0 0 0 0} \right) \delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} + \frac{0 0 0 0}{0 1 0 0} \cdot \delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} z^{2^{\pi(t)}} + \frac{0 0 0 0}{0 0 0 0} \cdot \delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} z^{2^{2\pi(t)}} H^{(t+1)}.
\]

Define

\[
F_{\pi(t)}(\delta_{2^{\pi(t)}(m)}, \delta_{2^{\pi(t)+1}(m)}) = \left( \left[ \frac{1}{0 0 0 0} \right] \delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} + \frac{0 0 0 0}{0 0 0 0} \cdot \delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} \cdot z^{2^{\pi(t)}} + \frac{0 0 0 0}{0 0 0 0} \cdot \delta_{2^{\pi(t)}(m)} \cdot \delta_{2^{\pi(t)+1}(m)} \cdot z^{2^{2\pi(t)}} \right) H^{(t+1)}.
\]

Then it is easy to see that

\[
F_{\pi(t)}(0, 0) = \left[ \frac{1}{0 0 0 0} \right] \cdot H^{(t+1)},
\]

\[
F_{\pi(t)}(1, 0) = \left[ \frac{0 0 0 0}{0 0 0 0} \right] \cdot z^{2^{\pi(t)}} \cdot H^{(t+1)},
\]

\[
F_{\pi(t)}(0, 1) = \left[ \frac{0 0 0 0}{0 0 0 0} \right] \cdot z^{2^{2\pi(t)}} \cdot H^{(t+1)},
\]

and

\[
F_{\pi(t)}(1, 1) = \left[ \frac{0 0 0 0}{0 0 0 0} \right] \cdot z^{3 \cdot 2^{\pi(t)}} \cdot H^{(t+1)}.
\]
By using Corollary 4, we have
\[
\sum_{m=0}^{L-1} H^{(0)} \cdot \prod_{t=1}^{n} \left( A_t \cdot H^{(t)} \right) z^m
\]
\[
= H^{(0)} \cdot \left\{ \prod_{t=0}^{n-1} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right) \cdot H^{(t+1)} + \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right) \cdot z^{2^t} \cdot H^{(t+1)} \right\} + \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right) \cdot H^{(t+1)}
\]
\[
= H^{(0)} \cdot \left\{ \prod_{t=1}^{n} \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right) \cdot z^{4^t} \cdot H^{(t+1)} + \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right) \cdot H^{(t+1)} \right\}
\]
\[
= M^{(n)}(z).
\]  
(14)

This completes the proof.

Since each \( A_t \) has only one nonzero element, only one row of product \( A_t \cdot H^{(t)} \) is nonzero. Therefore, every entry of \( M^{(n)}(m) \) is a product of the specific entries of \( H^{(t)} \) for \( t \in 0, 1, 2, \ldots, n \), which must be a \( q \)-th root of unity.

**Example 7.** In Construction 1, let \( H^{(0)} = \left[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & i & -1 & -i \\
1 & -i & 1 & i \\
1 & -1 & 1 & -1 \\
\end{array} \right] \), \( H^{(1)} = \left[ \begin{array}{cccc}
1 & i & -1 & -i \\
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
\end{array} \right] \)

and \( H^{(2)} = \left[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & i & -1 & -i \\
1 & -1 & 1 & -1 \\
1 & -i & 1 & i \\
\end{array} \right] \) be Butson type \((4,4)\) Hadamard matrices. The Boolean functions corresponding to the elements of \( M^{(3)}(z) = H^{(0)} \cdot D(z)^{41} \cdot H^{(1)} \cdot D(z)^{40} \cdot H^{(2)} \) were calculated as follows (Suppose \( Boolfunc_{r,s} = f_{r,s}(x_0, x_1, x_2, x_3) \) be the Boolean function corresponding to \( M_{r,s}^{(3)}(z) \), where \( x_0 \) is the most significant bit).

\( Boolfunc_{0,0} = 3x_0 + x_1 + 2x_2 + x_3 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 1 \)

\( Boolfunc_{0,1} = 3x_0 + x_1 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 \)

\( Boolfunc_{0,2} = 3x_0 + x_1 + 2x_2 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 3 \)

\( Boolfunc_{0,3} = 3x_0 + x_1 + 2x_3 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 2 \)

\( Boolfunc_{1,0} = x_0 + 2x_2 + x_3 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 \)

\( Boolfunc_{1,1} = x_0 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 3 \)

\( Boolfunc_{1,2} = x_0 + 2x_2 + 2x_3 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 2 \)

\( Boolfunc_{1,3} = x_0 + 2x_3 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 1 \)

\( Boolfunc_{2,0} = x_0 + 2x_3 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 1 \)
$\text{Boolfunc}_{2,1} = x_0 + 2x_1 + 2x_0x_1 + 2x_0x_2 + 3x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 1$
$\text{Boolfunc}_{2,2} = x_0 + 2x_1 + 2x_2 + 3x_3 + 2x_0x_1 + 2x_0x_2 + 3x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3$
$\text{Boolfunc}_{2,3} = x_0 + 2x_1 + 2x_2 + 3x_0x_1 + 2x_0x_2 + 3x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 3$
$\text{Boolfunc}_{3,0} = 3x_0 + 3x_1 + 2x_2 + x_3 + 2x_0x_1 + 2x_0x_2 + 3x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 3$
$\text{Boolfunc}_{3,1} = 3x_0 + 3x_1 + 2x_0x_1 + 2x_0x_2 + 3x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 2$
$\text{Boolfunc}_{3,2} = 3x_0 + 3x_1 + 2x_2 + 3x_3 + 2x_0x_1 + 2x_0x_2 + 3x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3 + 1$
$\text{Boolfunc}_{3,3} = 3x_0 + 3x_1 + 2x_3 + 2x_0x_1 + 2x_0x_2 + 3x_0x_3 + 2x_1x_2 + x_1x_3 + 2x_0x_1x_3$

In the above example, any set $\{\text{Boolfunc}_{i,j}|j = 0, 1, 2, 3\}$ or $\{\text{Boolfunc}_{i,j}|i = 0, 1, 2, 3\}$ is a complementary set of size 4. It can be seen that any sequences in the complementary set have algebraic degree 3. These complementary set cannot be obtained by the known methods given in [15] and [19].

Example 8. In Construction 1, let $N = 2$ and choose $H^{(t)} = \begin{bmatrix} 1 & \theta^{c_1} \\ -\theta^{c_1} & 1 \end{bmatrix}$, where $\theta$ is a primitive $q$th root and $c_1 \in \{0, 1, \cdots, q - 1\}$. Then all the $q$-ary standard Golay pairs can be presented as the first row of $M^{(n)}(z)$. This results have been proved by Budisin and Spasojevic’s in [3].

5 Conclusion

Inspired by the paraunitary-matrix-based method in [3], we propose a new method to construct $q$-ary Golay complementary set of size $N$ and length $N^n$ by $N \times N$ Hadamard matrices where $n$ is arbitrary and $N$ is a power of 2. Each item of the constructed sequences can be presented as the product of the specific entries of the Hadamard matrices. To make the results easier to understand for readers, we only show the proof for $N = 4$. Budisin and Spasojevic’s algorithm for $q$-ary Golay pair in [3] can be regarded as a special case of the construction in this paper by setting $N = 2$ and choosing specific Hadamard matrices. We also show new quaternary Golay pair of size 4 and degree 3 never reported in the literature can be obtained by the method presented in this paper.

Since any Butson type Hadamard matrix can be used in our construction, and in general, matrix multiplication is not commutative, a large families of sequences with PMEPR upper bounded by constant can be obtained by choosing different Hadamard matrix in Construction 1. The details will be considered in the extended paper.

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