Solving one dimensional time-space fractional vibration string equation

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Abstract. The article presents a solution for the one-dimensional space-time fractional vibration equation (FVE) by the separation of variables method (Fourier method). We describe the fractional derivatives in the sense of Caputo and Riemann-Liouville operators. Our method performs in the extreme well in terms of simplicity and efficiency. A sample of the problem of structural mechanics has been considered. This sample allows the demonstration of some advantages of the application of the suggested approach to solve the fractional vibration equation.

1. Introduction
Recently, fractional partial differential equations play an increasingly important role in many fields of science and engineering [1, 2] such as physics, fluid dynamics, signal processing, viscoelasticity, mechanical engineering, electrochemistry focus, mathematical biology and so on. The advantages of the analytical methods in explaining the basics of physical connotation and mechanical engineering problem, by which it is easier to study a different physical and mechanical engineering problem, as well as, it is less time consuming compared with the numerical method. However as is known that it is difficult to obtain the exact solutions of space-time fractional partial differential equations FPDEs by using the existing analytical methods [3, 4]. Thus, different numerical methods have been used to solve space-time FPDEs, see [4, 5, 6]. Also, many numerical methods, such as homotopy perturbation method (HPM), homotopy analysis method (HAM), finite difference method, finite element method, Adomian decomposition method (ADM), etc. are constructed and analyzed to obtain the numerical solutions for fractional partial differential equations.

In this paper, we will consider in the region $D = \{0 < x < L , 0 < t < T\}$ the following one-dimensional time-space fractional vibration string equation

$$a \frac{\partial^2 V(x,t)}{\partial t^2} = \frac{\partial^2 V(x,t)}{\partial x^2} + c_0 D_{0t}^{\beta} V(x,t) + c_1 D_{0x}^{\alpha} V(x,t),$$

constrained by boundary conditions

$$V(0,t) = V(L,t) = 0,$$
and initial conditions

\[
V(x, 0) = \varphi(x), \quad V_t(x, 0) = \psi(x), \tag{3}
\]

here, \(1 < \alpha, \beta < 2\), are the viscoelasticity parameters of the medium \(a, c_1\) and \(c_0\)-are the mass of the granule and the viscosity modulus of resin respectively. \(D^\beta_{t} \) and \(D^\alpha_{x}\) denote the temporal Caputo derivative with order \(\beta\) and spatial Riemann-Liouville derivative of order \(\alpha\) respectively, \(V(x,t)\) is the displacement.

Where the Caputo fractional derivative of \(h \in C^m_{m_{-1}}\) with order \(\beta > 0\), and the Riemann-Liouville fractional derivative of \(f \in C^m_{m_{-1}}\) with order \(\alpha > 0\), for \(m \in N \cup \{0\}\) are defined respectively as (see [7]),

\[
D^\beta_{t} h(t) = \frac{1}{\Gamma(m - \beta)} \int_0^t (t - \tau)^{m - \beta - 1} h^{(m)}(\tau) d\tau, \quad m - 1 < \beta < m, m \in N, \tag{4}
\]

\[
D^\alpha_{x} f(t) = \frac{1}{\Gamma(m - \beta)} \frac{d^m}{dx^m} \int_0^x (x - \xi)^{m - \alpha - 1} f(\xi) d\xi, \quad m - 1 < \alpha < m, m \in N, \tag{5}
\]

which, In particular, equation (1) is used to characterize the vibration of a string taking into account the friction in a fractal geometry medium. This equation is used in this work to model changes in polymer concrete's deformation-strength characteristics under loading. Intrinsically, equation (1) is a particular form of multi-term space-time fractional wave equations. Many works on analytical and numerical methods for space-time fractional wave equations have been existed, see [4, 5, 8]. The rest of the paper is organized as follows. In section 2, we apply the separation of variables scheme based on Fourier method for solving equation (1). Section 3, is devoted to illustrate some numerical example on mentioned method. In section 4, we give the conclusion.

2. Vibration string equation and Fourier method

In this section, by using the method of separating variables, we solve the vibration string equation. (1). Assume \(V(x, t) = \chi(x)T(t)\) and replacing \(V(x, t)\) in equation (1), we obtain on the following differential equation for \(\chi(x)\):

\[
\chi''(x) + c_1 D^0_{0x} \chi(x) + \lambda \chi(x) = 0; \chi(0) = 0, \chi(L) = 0, \tag{6}
\]

the parameter \(\lambda\) is an eigenvalue of the equation (6), if and only if \(\lambda\) is the zero of the function

\[
\omega(\lambda) = \chi + \sum_{n=1}^{\infty} (-1)^n \frac{n}{k} \frac{c^k \lambda^{n-k}}{\Gamma(2n + 2 - k\alpha)} \tag{7}
\]

We find the first five eigenfunctions of the boundary value problem (6) numerically using the high-level technical computing language Wolfram Mathematica, and plot their graphs, taking \(\alpha = 1.5\) and \(c_1 = 0.5\), where the first five eigenvalues [8] are written in table 1.

Now to seek a solution of equation (6), substitute from equation (5) in (6) and integrate equation (6) from 0 to \(x\), we have

\[
\int_0^x \chi''(t) dt + \int_0^x c_1 D^0_{0x} \chi(t) dt = \int_0^x \lambda \chi(t) dt = 0; \tag{8}
\]

\[
\chi'(x) + \frac{c_1}{\Gamma(1 - \alpha)} \int_0^x \chi(t) (x - t)^{-\alpha} dt = \lambda \int_0^x \chi(t) dt + \chi'(0); \tag{9}
\]
integrate both sides again from 0 to $x$,

$$\chi(x) + \int_0^x \frac{c_1}{\Gamma(2-\alpha)} (x-t)^{1-\alpha} - \lambda(x-t)\chi(x)\,dt = x.A + B,$$

(10)

where, $\chi'(0) = A$, and $\chi(0) = B$. Notice that the solution of equation (10) is

$$\chi(x) = x_0(x) + \ldots + x_n(x),$$

(11)

such that,

$$x_0(x) = x.A + B, \quad x_n(x) = \int_0^x \left[ \frac{-c_1}{\Gamma(2-\alpha)} (x-t)^{1-\alpha} + \lambda(x-t) \right] x_{n-1}(t)\,dt,$$

(12)

integrating equation (12) and substituting in equation (11), then we get a solution of equation (6)

$$\chi_j(x) = A \left[ x + \sum_{n=1}^{\infty} (-1)^n \sum_{k=0}^{n} \frac{(n)_k \lambda ^n}{(2n + 2 - k\alpha) \Gamma(2n + 2 - k\alpha)} x^{2n+1-kn} \right] ; \ j = 1, 2, \ldots, 5.$$

(13)

The graphs of the first five eigenfunctions $\chi(x)$ of equation (6) are presented in figure 2 at $c_1 = 0.5$, $\alpha = 1.5$. 

Figure 1. The graph of the function $\omega(\lambda)$ for $c_1 = 0.5$, at different $\alpha = 1.25, 1.5, 1.75$. 

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|-------------|-------------|-------------|-------------|-------------|
| 11.63       | 44.93       | 99.01       | 173.73      | 268.93      |

Table 1. The first five eigenvalues of equation (6) at $c_1 = 0.5$ and $\alpha = 1.5$

Figure 2. The graphs of the eigenfunctions $\chi_j(x); \ j = 1, 2, \ldots, 5$ at $c_1 = 0.5$, $\alpha = 1.5$. 


The system of eigenfunctions (13) is complete [9] anyhow, not orthogonal. Thus, we arise a system that will be biorthogonal to the system (13); therefore, we consider the operator \( A(f) = f'' + c_1 D^\alpha_{0x} f \), so assume that the operator \( A^* \) is conjugated with the operator \( A \), such that \( \langle A(f), h \rangle = \langle f, A^*(h) \rangle \), where we have the scalar product \( \langle f, h \rangle = \int_0^L f(x) h(x) dx \).

Let us assume that the functions \( f \) and \( g \) satisfy the boundary conditions (6), consider

\[
\langle f, h \rangle = \langle f'' + c_1 D^\alpha_{0x} f, h \rangle,
\]

we use the self-adjoint operator and linear differential operator

\[
\langle f'' + c_1 D^\alpha_{0x} f, h \rangle = \langle h'', f \rangle + c_1 \langle D^\alpha_{0x} f, h \rangle,
\]

further,

\[
\langle D^\alpha_{0x} f, h \rangle = \int_0^L \frac{1}{\Gamma(2-\alpha)} \left( \frac{d^\alpha}{dx^\alpha} \right) \int_0^x (x-\xi)^{1-\alpha} f(\xi) d\xi \rangle h(x) dx,
\]

let

\[
V(x) = \int_0^x (x-\xi)^{1-\alpha} f(\xi) d\xi,
\]

then

\[
\int_0^L V''(x) h(x) dx = V'(x) h(x)|_0^L - \int_0^L V'(x) h'(x) dx
\]

\[
= -V(x) h'(x)|_0^L + \int_0^L V(x) h''(x) dx
\]

\[
= -V(L) h'(L) + V(0) h'(0) + \int_0^L V(x) h''(x) dx,
\]

hence, equation (16) can be written as,

\[
\langle D^\alpha_{0x} f, h \rangle = \frac{1}{\Gamma(2-\alpha)} \left( \int_0^L V(x) h''(x) dx - V(L) h'(L) \right)
\]

\[
= \frac{1}{\Gamma(2-\alpha)} \int_0^L \left\{ \int_0^x (x-\xi)^{1-\alpha} f(\xi) d\xi \right\} h''(x) dx
\]

\[
- \frac{h'(L)}{\Gamma(2-\alpha)} \int_0^L (L - \xi)^{1-\alpha} h(\xi) d\xi,
\]

by change the limits of integration in the double integral, so equation(18), can be written as,

\[
\langle D^\alpha_{0x} f, h \rangle = \frac{1}{\Gamma(2-\alpha)} \int_0^L \left\{ \int_\xi^L (x-\xi)^{1-\alpha} h''(x) dx - h'(L) (L - \xi)^{1-\alpha} \right\} f(\xi) d\xi,
\]

by using the relationship between Caputo fractional derivative, and Riemann-Liouville fractional derivative, we obtain:

\[
\frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{ds^n} \left( \frac{h(s)}{(s-t)^{n-\alpha}} \right) ds = \frac{1}{\Gamma(n-\alpha)} \int_\xi^L \frac{h^{(n)}(s)}{(s-t)^{n-\alpha}} ds + \sum_{k=0}^{n-1} \frac{(-1)^k h^{(k)}(L)}{\Gamma(k-\alpha+1)} (L-t)^{k-\alpha},
\]

therefore,

\[
\langle D^\alpha_{0x} f, h \rangle = \frac{1}{\Gamma(2-\alpha)} \int_0^L \frac{d^2}{d\xi^2} \left\{ \int_\xi^L (x-\xi)^{1-\alpha} h(x) dx \right\} f(\xi) d\xi = \langle f, D^\alpha_{xx} h \rangle,
\]
hence,
\[ D_{xL}^\alpha h = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_0^L (x-\xi)^{1-\alpha} h(x) dx. \] (22)

Consequently, the adjoint operator
\[ A^*(h) = h'' + c_1 D_{xL}^\alpha h, \]
therefore, we consider the following conjugated problem associated with problem (6)
\[ \tilde{\chi}''(x) + c_1 D_{0x}^\alpha \tilde{\chi}(x) + \lambda \tilde{\chi}(x) = 0, \]
\[ \tilde{\chi}(0) = \tilde{\chi}(L) = 0; \quad \alpha \in (1, 2), \] (23)

where, \( \tilde{\chi}_j(x) \) be the eigenfunctions of problem (23), then
\[ \tilde{\chi}_j(x) = \chi_j(1-x). \] (24)

Hence,
\[ \tilde{\chi}_j(x) = \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n} \binom{n}{k} c_1 \lambda_j^{n-k} \lambda_j (1-x)^{2n+1-k\alpha}; \quad j = 1, 2, \ldots, 5. \] (25)

The graphs of the first four eigenfunctions of \( \chi_j(x) \) and \( \tilde{\chi}_j(x) \) are presented in figure 3.

![Graphs of eigenfunctions](image)

**Figure 3.** The graphs of the eigenfunctions \( \chi_j(x), \tilde{\chi}_j(x) \); \( j = 1, \ldots, 4 \) at \( c_1 = 0.5, \alpha = 1.5 \).

Utilizing the high-level technical computing language Wolfram Mathematica, we calculate the values of the scalar products \( \langle \chi_m(x), \tilde{\chi}_k(x) \rangle \), which are written in table 2.

Also, for \( T(t) \) we obtain on the following equation of Caputo fractional derivative:
\[ aT''(t) - c_0 D_{0t}^\beta T(t) + \lambda T(t) = 0. \] (26)

Now, in order to seek a solution of equation (26), let us take
\[ T''(t) = Y(t), \] (27)
therefore, the solution is substituting by integrating both sides from 0 to $t$

$$T'(t) = \int_0^t Y(s)ds + T'(0);$$

$$T(t) = \int_0^t (t - s)Y(s)ds + T'(0)t + T(0),$$

(28)

substituting from equations (4), (27) and (28) in (26) and put $T'(0) = A, T(0) = B,$ so

$$Y(t) + \int_0^t \left[ -\frac{c_0}{a\Gamma(2 - \beta)}(t - \zeta)^{1-\beta} + \frac{\lambda}{a}(t - \zeta) \right] Y(\zeta)d\zeta = -\frac{\lambda}{a}At - \frac{\lambda}{a}B,$$

(29)

therefore, the solution is

$$Y(t) = \frac{\lambda}{a}A \left[ -t + \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{k=0}^{m} \frac{(m_k)(-c_0)^k m^{-k}}{a^m\Gamma(2m + 2 - k\beta)} t^{2m+1-k\beta} \right]$$

$$+ \frac{\lambda}{a}B \left[ -1 + \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{k=0}^{m} \frac{(m_k)(-c_0)^k m^{-k}}{a^m\Gamma(2m + 1 - k\beta)} t^{2m-k\beta} \right],$$

(30)

hence,

$$T'(t) = A \left[ 1 - \frac{\lambda t^2}{2a} + \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{k=0}^{m} \frac{(m_k)(-c_0)^k m^{1-k}}{a^{m+1}\Gamma(2m + 3 - k\beta)} t^{2m+2-k\beta} \right]$$

$$+ B \left[ -\frac{\lambda t}{a} + \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{k=0}^{m} \frac{(m_k)(-c_0)^k m^{1-k}}{a^{m+1}\Gamma(2m + 2 - k\beta)} t^{2m+1-k\beta} \right],$$

(31)

and

$$T(t) = A \left[ t - \frac{\lambda t^3}{6a} + \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{k=0}^{m} \frac{(m_k)(-c_0)^k m^{1-k}}{a^{m+1}\Gamma(2m + 4 - k\beta)} t^{2m+3-k\beta} \right];$$

$$+ B \left[ 1 - \frac{\lambda t^2}{2a} + \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{k=0}^{m} \frac{(m_k)(-c_0)^k m^{1-k}}{a^{m+1}\Gamma(2m + 3 - k\beta)} t^{2m+2-k\beta} \right],$$

(32)

put,

$$Z(t) = t - \frac{\lambda t^3}{6a} + \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{k=0}^{m} \frac{(m_k)(-c_0)^k m^{1-k}}{a^{m+1}\Gamma(2m + 4 - k\beta)} t^{2m+3-k\beta},$$

(33)
\[
\tilde{Z}(t) = 1 - \frac{\lambda t^2}{2a} + \sum_{m=1}^{\infty} \sum_{k=0}^{m} \frac{\binom{m}{k} (-c_0)^k \chi^{m+1-k}}{a^{m+1} \Gamma(2m+3-k\beta)} t^{2m+2-k\beta},
\]

which, deduce that the solution of equation (26) can be written as

\[
T(t) = A.Z(t) + B.\tilde{Z}(t).
\]

Then, from equation (13) and equation (35) yields that the solution of the problem (1)-(3) has the following standard form as

\[
V(x, t) = \sum_{m=1}^{\infty} \chi_m(x) T_m(t) = \sum_{m=1}^{\infty} \chi_m(x) \left[ A_m.Z_m(t) + B_m.\tilde{Z}_m(t) \right].
\]

Using the initial condition (2), that is lead to substituting in equation (36) about \( t = 0 \)

\[
V(x, 0) = \sum_{m=1}^{\infty} \chi_m(x) \left[ A_m.Z_m(0) + B_m.\tilde{Z}_m(0) \right] = \varphi(x);
\]

\[
V_t(x, 0) = \sum_{m=1}^{\infty} \chi_m(x) \left[ A_m.Z'_m(0) + B_m.\tilde{Z}'_m(0) \right] = \psi(x),
\]

multiply both sides of the equation (37) with a scalar system \( \tilde{\chi}_m(x) \), so

\[
\sum_{m=1}^{\infty} \langle \chi_m(x), \tilde{\chi}_m(x) \rangle \left[ A_m.Z_m(0) + B_m.\tilde{Z}_m(0) \right] = \langle \varphi(x), \tilde{\chi}_m(x) \rangle;
\]

\[
\sum_{m=1}^{\infty} \langle \chi_m(x), \tilde{\chi}_m(x) \rangle \left[ A_m.Z'_m(0) + B_m.\tilde{Z}'_m(0) \right] = \langle \psi(x), \tilde{\chi}_m(x) \rangle,
\]

since, \( \{\chi_m(x)\}_{m=1,2,...} \) and \( \{\tilde{\chi}_m(x)\}_{m=1,2,...} \) are biorthogonal systems, then equation (38) can be written as

\[
A_m.Z_m(0) + B_m.\tilde{Z}_m(0) = \frac{\langle \varphi(x), \tilde{\chi}_m(x) \rangle}{\langle \chi_m(x), \tilde{\chi}_m(x) \rangle};
\]

\[
A_m.Z'_m(0) + B_m.\tilde{Z}'_m(0) = \frac{\langle \psi(x), \tilde{\chi}_m(x) \rangle}{\langle \chi_m(x), \tilde{\chi}_m(x) \rangle},
\]

then, the solution of the problem (1)-(3) has the following form

\[
V(x, t) = \sum_{m=1}^{\infty} \frac{\chi_m(x)}{\langle \chi_m(x), \tilde{\chi}_m(x) \rangle} \left[ Z_m(t) \langle \psi(x), \tilde{\chi}_m(x) \rangle + \tilde{Z}_m(t) \langle \varphi(x), \tilde{\chi}_m(x) \rangle \right].
\]

3. Numerical Construction of the Solution

The numerical solution to the problem (1)-(3) can be written as:

\[
V(x, t) \approx V_5(x, t) = \sum_{m=1}^{100} \chi_m(x) \left[ A_m.Z_m(t) + B_m.\tilde{Z}_m(t) \right].
\]
Consider the following example for illustrating the efficiency of the method. \((a = 1, c_1 = 0.5, c_0 = -1.8\) at \(\alpha = 1.5, \beta = 1.47\),

\[
a \frac{\partial^2 V(x,t)}{\partial t^2} = \frac{\partial^2 V(x,t)}{\partial x^2} + c_0 D_{0+}^\beta V(x,t) + c_1 D_{0+}^\alpha V(x,t), \quad 0 < x, t < 1,
\]

constrained by the conditions,

\[
\begin{align*}
V(0,t) &= V(1,t) = 0, \\
V(x,0) &= 0.1x(1-x), \\
V_t(x,0) &= -x^3.
\end{align*}
\]

We could construct graphs for a solution by formula (41) numerically by using the high-level technical calculation language Wolfram Mathematica, where figure 4, and figure 5 show the solution at \(a = 1, c_1 = 0.5, c_0 = -1.8\) when \(\alpha = 1.5, \beta = 1.47\).

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**Figure 4.** Graph the solution \(V(5)(x,t)\) of equation (41) at \(c_1=0.5, c_0=-1.8\) when \(\alpha=1.5, \beta=1.47\).

**Figure 5.** Graph the solution at \(t = 1, c_1 = 0.5, c_0 = -1.8\) when \(\alpha = 1.5, \beta = 1.47\).

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4. Conclusion

In this paper, the vibration string equation of a fractional derivative with respect to time and space has been characterized and displayed. We obtained the solution of the vibration string equation by Fourier method. The example shows some advantages of the suggested approach to solve the fractional vibration string equation.

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