Dipole-Deformed Bound States and Heterotic Kodaira Surfaces

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Abstract
We study a particular $\mathcal{N} = 1$ confining gauge theory with fundamental flavors realised as seven branes in the background of wrapped five branes on a rigid two-cycle of a non-trivial global geometry. In parts of the moduli space, the five branes form bound states with the seven branes. We show that in this regime the local supergravity solution is surprisingly tractable, even though the background topology is non-trivial. New effects such as dipole deformations may be studied in detail, including the full backreactions. Performing the dipole deformations in other ways leads to different warped local geometries. In the dual heterotic picture, which is locally given by a $\mathbb{C}^*$ fibration over a Kodaira surface, we study details of the geometry and the construction of bundles. We also point out the existence of certain exotic bundles in our framework.

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1. Introduction

Recently, it has become apparent that in order to deal with flux compactifications the target manifold has to be generically non-Kähler \[^1\], although conformally Kähler manifolds may sometimes arise. Examples of compact complex manifolds have been explicitly constructed in heterotic theories \[^2\], \[^3\], \[^4\], \[^5\], \[^6\], \[^7\], \[^8\] and the classification of torsional manifolds \[^8\], \[^9\] has appeared in \[^10\], \[^11\], \[^12\]. These manifolds form the foundation on which string compactifications to lower dimension can be performed. In the absence of fluxes, and for heterotic theories, when the spin-connection is embedded in the gauge connection (via the so-called standard embedding) the original Calabi-Yau compactifications are valid \[^14\]. However, when spin-connection is not embedded in the gauge connection, the compactification manifolds are no longer Calabi-Yau but are the generic non-Kähler manifolds \[^15\], \[^16\], \[^17\].

On the other hand, non-compact non-Kähler manifolds are interesting from a different point of view. They sometimes appear as gravity duals to certain confining $N = 1$ gauge theories in type IIA \[^17\], \[^18\] and heterotic theories \[^19\], \[^20\], but may or may not be complex\[^2\]. In type IIB, the known gravity duals for confining gauge theories are generically Kähler (more appropriately conformally Kähler), which could be further dual to non-Kähler manifolds with both kinds of background three-forms.

\[^1\] Even when the fluxes are absent!

\[^2\] Examples of compact non-complex non-Kähler manifolds are constructed in \[^19\], \[^20\].
The conformally Kähler geometry we studied in our earlier works [17], [21], [16], [18] exhibits a globally integrable complex structure derived from an F-theory picture. As such, both bi–fundamental and fundamental matter appeared in the construction due to the presence of three, five and seven-branes. Unfortunately, the global metric was too involved to derive, mainly because the construction involved a patch by patch description. However, with some effort the local geometry near the origin (i.e the far IR in the dual gauge theory) was derived by keeping only the D5 branes and the seven branes [17], [16], [18] in the local neighborhood. The resulting metric turned out to be rather simple in the far IR, at least when the seven branes are kept away and the D5 branes wrap local patches of a $T^2$ rather than a two-sphere.

An underlying F-theory picture implies that special points in the moduli space of the solutions may exist. There are two possible consequences of being at such a point:

- The system could be governed by an orientifold model, and
- Bound states of D5 branes with the underlying seven branes could appear.

The second possibility does not imply the first, although once we are in an orientifold picture the existence of bound states is automatic: there will always be a point in the same moduli space where such states can appear. Elaborating on this will be the topic of sec. (2.1) and sec. (2.2).

Imagine we are away from the orientifold point. This is possible by considering any small perturbation to the orientifold picture. One may ask whether bound states can appear in this scenario. It turns out that bound states are very generic, and appear whether we are at the orientifold point or not. The question then is how to distinguish between the two scenarios? Herein lies the subtlety. When we are at the orientifold point in the moduli space, the bound-state configurations undergo a dipole deformation in addition to any other possible deformations. On the other hand, once we are away from the orientifold point there seems to be no constraint on the system to undergo a dipole deformation, and other deformations may take over.

Thus the crucial point seems to be the dipole deformation of bound states. Things become complicated however, because the underlying topology becomes non-trivial. Details on this will be presented in sec. 3. We will calculate the local backreactions due to the dipole deformations on the background geometry. We will show in sec. (3.1) and (3.2) that there are multiple ways to perform dipole deformations. In both cases, the final results nicely confirm our earlier predictions regarding the local geometry.
On the heterotic side, we analyse some more aspects of the local geometry given by a $\mathbb{C}^*$ fibration over a Kodaira surface. Recall that the heterotic geometry is not U-dual to the corresponding type IIB picture. In sec. 4 we study details of the bundle structure on the local geometry. We provide three different ways to construct a bundle on our manifold. In sec (4.1) we give an intrinsic construction on the local geometry. In sec. (4.2) we use the method of pulling back a bundle constructed on the base torus, and in sec. (4.3) we use the method of pulling back a bundle constructed on the primary Kodaira surface.

We conclude with a short summary of our results.

2. $\mathcal{N} = 1$ bound state metric

In our earlier paper [18], we discussed a set-up in type IIB where $N$ D5 branes and some local and non-local seven branes wrap a two-cycle of some non-trivial global geometry, giving rise to $\mathcal{N} = 1$ gauge theory with fundamental flavors. This is basically the gauge-gravity scenario, where the gravity dual is given by another geometry with at least one topological non-trivial three-cycle on which we allow three-form $H_{RR}$ fluxes and one non-compact three-cycle on which we have three-form $H_{NS}$ fluxes.

F-theory provides the full global geometry both before and after the geometric transition [17], [16], [21], [18]. The geometries are conformally Kähler on both sides, but full global metrics are not known. However, the precise local geometries on any given patch are determined in [17], [21], [16] and [18] up to possible subtleties mentioned therein.

We shall encounter several subtleties in our construction. Firstly, due to an inherent orientifold action only certain components of $B_{NS}$ can survive. These $B_{NS}$ fields give rise to the dipole deformation [22], [23] in our theory (see [18] for more details). Secondly, the metric involves non-trivial background topology, fluxes and branes (both D7 and D5). Thirdly, due to the existence of both kind of branes, there will be a point in the moduli space where the D5 branes form a bound state with the D7 branes. The dipole deformation backreacts on the $\mathbb{P}^1$-wrapped bound state, and in [18] we computed a precise local solution taking it into account for a D7 brane wrapping a $\mathbb{P}^1$. We argued in [18] that bound states of D5 branes can be constructed by allowing a non-trivial first Chern class on the D7 brane. However, we did not compute the backreactions from allowing $c_1 \neq 0$. Our conclusion then was that the backreactions would be small, and therefore the local geometry would be no different from the ones we examined earlier in [17], [21], and [16]. In this section, we aim to compute precisely the backreaction and see how far we are from the predicted local
geometry of [17], [21], [16], which was determined for a separated system of D5 and D7 branes in a non-trivial geometry. We will show that at this point in the moduli space, the local metric is surprisingly simple to determine even after we take into account all possible backreactions. However before we delve into the determination of the local metric, we first need some details on bound states of branes in a flat background.

2.1. Bound state of D5 branes and D7 branes: a first look

Following our earlier works [17], [21], [16], [18] we keep a D5 brane oriented along the $x^{0,1,2,3,8,9}$ directions and a D7 parallel to the D5 brane at some point in the $x^4, x^5$ directions. Using the harmonic function ansatz, it is easy to write down the metric for a system of parallel Dp-branes. We expect the metric to be given by

$$ds^2 = f_1^{-1} f_2^{-1} ds^2_{012389} + f_1 f_2^{-1} ds^2_{67} + f_1 f_2 ds^2_{45}. \tag{2.1}$$

where $f_1$ and $f_2$ are related respectively to the harmonic functions of the D5 brane and D7 brane. However, this configuration is clearly not supersymmetric and is therefore not stable. Strings stretching between these branes have a mass given by

$$m^2 = -\frac{1}{2} + \frac{d^2}{(\pi \alpha')^2}, \tag{2.2}$$

with $d$ being the distance in the $x^{4,5}$ plane. For $d < \frac{\alpha'}{\sqrt{2}}$ the string becomes tachyonic and therefore reduces the total energy of the system to its bound-state value (see for example [24] for more details).

One way to construct the metric of a bound state of D5 and D7 branes is to start directly with the metric (2.1) and calculate the change due to tachyon condensation from the D5-D7 strings. Alternatively, we can start with the metric of a D7 brane

$$ds^2 = f_2^{-1} ds^2_{01236789} + f_2 ds^2_{45}. \tag{2.3}$$

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3 Supersymmetry for this system was discussed in detail in [17], [16], [18] using an F-theory construction with primitive G-fluxes. Notice that the global geometry is not a Calabi-Yau resolved conifold, but is a Kähler geometry with at least one $\mathbb{P}^1$. Locally near $r \to 0$ the geometry somewhat resembles a resolved conifold with vanishing $\mathbb{P}^1$. Clearly this kind of geometry is expected for the IR description of our gauge theory to make sense.

4 The above results are strictly valid in the flat-space limit [24]. For a curved background simple mode expansions are not possible and the result can change as we saw earlier [17], [16], [18].
and switch on gauge fluxes along the $x^{6,7}$ directions. Finally, once we determine the backreactions of the fluxes on the geometry we will obtain the desired metric. Our ansatz for the final bound-state metric in a flat background can therefore be presented as

$$ds^2_{\text{bound}} = \tilde{f}_2^{-1}ds^2_{012389} + f_3\ ds^2_{67} + \tilde{f}_2\ ds^2_{45}, \quad (2.4)$$

where we have a new warp factor $f_3$ along the directions $x^{6,7}$ to take into account the gauge fluxes $F_{67}$ on the D7 brane. The $\tilde{f}_2$ factor signals any possible changes to the original warp factor $f_2$. In the following we will give a simple way to determine these warp factors.

To begin, first observe that in addition to the metric, we expect all the other type IIB background fields to have non-zero expectation values. For example, we will now have both axion and dilaton, respectively $(\chi, \phi)$, and also $H_{NS}$ and $H_{RR}$. To describe $SL(2, \mathbb{Z})$ invariant quantities in type IIB, we must define the following factor that depends on the asymptotic values of the axion-dilaton $(\chi_0, \phi_0)$:

$$\Delta = e^{-\phi_0} + e^{\tilde{\phi}_0} \quad (2.5)$$

with $\tilde{\phi}_0 = \phi_0 + 2 \ln (1 - \chi_0)$. The above relation (2.5) is valid only for a single D5 and a single D7 (which is our present interest). For a system with $m$ D5s and $n$ D7s we will need the following replacement:

$$\phi_0 \rightarrow \phi_0 - 2 \ln n, \quad \tilde{\phi}_0 \rightarrow \tilde{\phi}_0 + 2 \ln \left| \frac{m - n\chi_0}{1 - \chi_0} \right|. \quad (2.6)$$

Using (2.5), one can fix the seven brane central charge $Q_7$ uniquely as

$$Q_7 = \frac{\sqrt{\Delta}}{2\pi}, \quad (2.7)$$

where the effect of the bound D5 appears from the definition of (2.5). Fixing the D7 brane charge also implies that all subsequent charges in the $SL(2, \mathbb{Z})$ multiplets are completely fixed. This becomes important when we have to define non-local seven branes in our scenario (which we will encounter later).

The bound state of $m$ D5s and $n$ D7s can now be determined using the techniques elaborated in [25]. The simplest way is to take $n$ coincident D3 branes with $m$ units of
electric flux and U-dualise the resulting background to get the D5/D7 bound state. The metric for this configuration is exactly as predicted in (2.4) with $f_3$ and $\tilde{f}_2$ given as

$$f_3 = \frac{e^{-\phi_0} \sqrt{h} \Delta}{\Delta - n^2 e^{-\phi_0} (1 - h)}$$

where $\Delta$ is now defined for an $(m, n)$ bound state, and $h$ is the modified harmonic function (notice the relative minus sign). Thus, we see that switching on $F_{67}$ fluxes on $n$ coincident D7 branes changes the metric along the $x^{6,7}$ directions from $\frac{1}{\sqrt{h}}$ to $a \sqrt{\Delta}$, with $a$ given by

$$a = \frac{h \sqrt{\Delta}}{\Delta e^{\phi_0} - n^2 (1 - h)},$$

and so tells us the backreaction of the gauge flux on the bulk geometry. In a non-trivial topology, such backreactions would be useful to evaluate the full IR geometry of the corresponding $\mathcal{N} = 1$ gauge theory.

S-dualising this background will give rise to coincident NS5 branes with magnetic seven branes. The behavior of string coupling near NS5 branes has been studied earlier in various papers. The magnetic seven brane could also contribute to the coupling constant. Therefore, in the original bound-state configuration we expect non-trivial behavior of the string coupling as we approach the region near the core of the bound state. The behavior is

$$g_s^2 = \frac{g_0^2}{(1 - Q_7 \ln r)(1 - Q_8 \ln r)},$$

with $Q_8 \equiv \frac{e^{2\phi_0} - 1}{2\pi \sqrt{\Delta}}$ and $Q_7$ as defined in (2.7). As we approach the core of the bound state, the theory becomes weakly coupled. The constant coupling $g_0^2$ is defined at a point where $\ln r = 0$, and is in fact given by $g_0^2 = \frac{4 \pi^2 Q_7 Q_8}{n^2}$. We also observe that the description is valid only in the regime $r < e^{1/Q_7}$ or $r > e^{1/Q_8}$, while in the regime

$$e^{1/Q_7} < r < e^{1/Q_8}$$

we must use a different description\textsuperscript{5}. That is also one of the reasons why our local description is particularly good when dealing with seven branes.

\textsuperscript{5} Note that $Q_7 > Q_8$ for our case.
One can show that \( n \) coincident D3 branes with \( m \) units of electric flux will exhibit exactly the same behavior for the bound-state metric. This is expected since the fluxes contribute equally. What would be different in this case is the spatial dependence of the warp factors. That is, the string coupling near the bound state will be different from the one that we found earlier in (2.10). The precise dependence is easy to work out, and is given by

\[
g_s = g_0 \sqrt{Q + \tilde{r}^4 \over Q + r^4},
\]

(2.12)

where \( Q \) is the charge and \( \tilde{r} \) is the scaled radius with the scale factor given by \( \left( \Delta e^{\phi_0 n^2} \right)^{1 \over 4} \).

We see that near \( r = 0 \), the string coupling is not necessarily small, and is given by \( g_s = g_0 = \Delta^{1 \over n} \). Indeed the coupling constant lies in the range

\[
e^{-\phi_0 \over 2} \leq g_s \leq \Delta^{1 \over n}
\]

(2.13)

defined for \( 0 \leq r \leq \infty \). \( g_s \) is nowhere divergent, and the metric therefore serves as a good description in the whole region. It turns out that the generic behavior of the dilaton for any \((m, n)\) bound state can always be put in the form

\[
\phi = \alpha \ln \Delta + \beta \ln h + \gamma \ln a,
\]

(2.14)

where \( a \) is defined in (2.9) and \((\alpha, \beta, \gamma)\) are constants\(^6\). As an example, one can check that in type IIB, the dilatons \( \phi_p \) of fundamental string/Dp-brane bound states are described in terms of \((\alpha, \beta, \gamma)\) as

\[
\phi_1 = \left( -{1 \over 2}, {1 \over 2}, -1 \right), \quad \phi_3 = \left( -{1 \over 4}, 0, -{1 \over 2} \right), \quad \phi_5 = \left( 0, -{1 \over 2}, 0 \right), \quad \phi_7 = \left( {1 \over 4}, -1, {1 \over 2} \right),
\]

(2.15)

with the values of \((a, h)\) changing accordingly for each bound state. One can now see that for our type IIB D5/D7 bound state, the dilaton is given by

\[
\phi = \left( {1 \over 4}, -1, {1 \over 2} \right),
\]

(2.16)

which reduces to (2.10) and is therefore not globally defined. As we observed above, this is not a matter of concern because we will only have a description on a given patch once we go to a more involved scenario [17], [16], [18].

\(^6\) One might be concerned by the fact that (2.14) has no apparent \( \phi_0 \) term. This is not an issue because \( a \) has the required powers of \( \phi_0 \) (see (2.9)).
Existence of this bound state can also be inferred from M-theory with a Taub-NUT background. Consider the $n = 1$ case with $m$ arbitrary. Then the harmonic form changes from its standard value to the one given in [26] (see for example equations (33) and (51) in the first reference of [26]), due to the backreactions of the G-fluxes. The $m$ five branes can be thought of as wrapping the degenerating cycle of our Taub-NUT space. Clearly the G-fluxes source these five-branes, so that we have a bound-state configuration.

Having determined the dilaton, our next goal is to find the axion $\chi$. The axion is expected because we have seven branes. For our bound-state configuration, the axion can easily be determined, and is given by

$$d\chi = \frac{1}{9!} \frac{1}{1 + b(h - 1)} \left[ f(x_4) \, dx_5 - f(x_5) \, dx_4 \right].$$

(2.17)

One can integrate this equation to determine $\chi$. The functions $f(x_i)$ are determined by using the relation $\frac{\partial h}{\partial x_i} = -\frac{\sqrt{\Delta}}{r_i}$ to get

$$f(x_i) = \frac{x_i}{\pi r_i^2} \left[ (m - n\chi_0) \left( 6h^{-1}g(h) - \chi_0h^{-2} \right) + ne^{-2\phi_0}h^{-2} \right],$$

(2.18)

where we have denoted the asymptotic value of the axion as $\chi_0$, and $b$ in (2.17) above is given by $b = \frac{n^2}{e^{\phi_0} \Delta} \equiv \tilde{h}^{-1} \frac{h - 1}{h - 1}$, with the latter equality serving as a definition of $\tilde{h}$. One can use this definition to write the function $g(h)$ in a compact form as

$$g(h) = \tilde{h}^{-2\Delta} e^{-\phi_0} \left\{ \tilde{h}h^{-1} \left[ mn(h - 1) + \chi_0\Delta e^{\phi_0} \right] - n(m - n\chi_0) \right\}.$$  

(2.19)

A non-trivial axion also exists for other bound states. For example, an electric flux on a D-string induces a non-trivial axion given by

$$\chi_1 = \frac{\alpha + \gamma_1 h}{\tilde{h}},$$

(2.20)

where $\alpha = \chi_0 - \gamma_1$ and $\gamma_1 = \frac{mn e^{-\phi_0}}{\Delta}$ with $h$ defined accordingly for the bound state. In fact, this will have important consequences when we embed our system in a non-trivial

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7 These G-fluxes have two legs in the internal TN space and two legs along seven-dimensional spacetime.

8 One should note that $dh = -\pi^{-1}r^{-2}\sqrt{\Delta}(x_4 \, dx_4 + x_5 \, dx_5)$ so $\chi$ is not linearly dependent on $h$.

9 Although electric fluxes on D3 branes do not switch on any axion.
topology and perform a dipole deformation. From the above analysis we can now define a complex coupling $\tau \equiv \chi + i e^{-\phi}$ for our background.

The next step is to find the $H_{RR}$ that forms the source of the bound D5 branes. There are various ways to do this. One simple way is to compute the $C_6$ sources from the D5 branes and then Hodge-dualise. This procedure yields

$$H_{RR} = \left( m - n\chi_0 \frac{7!}{3! \pi r^2} \right) h^{-\frac{3}{2}} \tilde{h}^{-2} e^{-\phi} \left[ x_4 dx_5 \wedge dx_6 \wedge dx_7 - x_5 dx_4 \wedge dx_6 \wedge dx_7 \right].$$

(2.21)

Given the RR field, supersymmetry requires us to have $H_{NS}$ fields also. An easy way to see this is to construct a three-form $G = H_{RR} + \tau H_{NS}$ and then consider the supersymmetry constraint $G = \pm \ast iG$ for a given non-trivial background, as in [27], [28]. Anticipating later generalisations, we see that this requires non-zero $H_{NS}$ as well. It turns out that the NS three-form is given by

$$H_{NS} = d\chi_1 \delta h_{1-6} \ln r$$

$$= \frac{n(m - n\chi_0)}{\Delta h^2 e^{\phi_0}} dh \wedge dx_6 \wedge dx_7.$$  

(2.22)

where the functional form of $\chi_1$ is as in (2.20). We see that the NS-form is switched on precisely because of the axion in the D-string–with–flux case. Once we have such a $B_{NS}$ field, the dipole deformation becomes particularly involved, as we shall soon see.

Since $H_{NS}$ is non-trivial, $B_{NS}$ cannot be gauged away. $B_{NS}$ has legs along the directions $x_6, x_7$. Recall that the bound D5 branes are oriented along the $x^{0,1,2,3,8,9}$ directions, and therefore the B-field measures the charge of these five-branes. Incidentally the $B_{RR}$ fields are also along the same directions. This makes sense precisely because of the orientations of the D5 branes.

One should note at this stage that the NS B-field (2.22) is not a priori related to the dipole deformation. It has both of its legs orthogonal to the D5 branes and parallel to the D7 brane. This is a unique case, where from the D5 point of view one would expect some kind of pinning effect as in [29], and from the D7 point of view a non-commutative effect [30]. We will discuss these possible generalisations elsewhere once we switch on the dipole deformation. It is also interesting to note that in lower–dimensional branes, for example

\footnote{Of course not in a flat background, but something like in a Taub-NUT space [29].}
for a D3 brane, once we switch on a magnetic flux the four-form charge of the D3 brane changes to a new value given by

\[ Q_4 = |b_2| \left( \chi_0 - 6|b_1| - \frac{ne^{-2\phi_0}}{m-n\chi_0} \right). \]  

(2.23)

Here, \(|b_1|\) and \(|b_2|\) are respectively the magnitudes of background NS and RR two-forms, which satisfy the constraint that \(b_1 \wedge b_2\) generates a Chern-Simons term on the D3 brane.

2.2. Bound states in a non-trivial topology

So far our analysis has concentrated on determining the metric of a bound state of \(m\) D5 and \(n\) D7 branes in a flat background. In the following we want to compute the backreactions induced by allowing a non-trivial background geometry with additional non-trivial topology.

The reason this is important has already been emphasised. Our earlier F-theory picture gave a supersymmetric configuration of D5 and D7 branes in a background that locally resembled a resolved conifold [17], [16], [18]. By moving the five or seven branes, we can reach a point in the moduli space where bound states exist. These branes would then wrap a non-trivial two-cycle in the local geometry, whose explicit form (given earlier in [18]) can be written as:

\[ ds^2 = A \, dr_1^2 + B \left( dz + f_1 \, dx + f_2 \, dy \right)^2 + (C \, d\theta_1^2 + D \, dx^2) + (E \, d\theta_2^2 + F \, dy^2). \]  

(2.24)

Here the warp factors are functions of the radial coordinate \(r_1\) and \(f_i = f_i(\theta_i)\). Recall that our coordinates \((r_1, z, x, y, \theta_i)\) are local, and thus the metric (2.24) is only for a local patch (see discussions in [18]).

Using the notations of our local metric, we can see that the D7 branes are oriented along the \((z, r_1, y, \theta_2)\) directions and are located at a point on the torus described by the coordinates \((x, \theta_1)\). The D5 branes are located at a different point on the \((x, \theta_1)\) torus, although they wrap the other torus \((y, \theta_2)\) exactly as the D7 branes do (see figure 2 in [18]).

We should now relate the local coordinates used here to the coordinates used in the previous section. The \((x, \theta_1)\) torus is related to the \((x_4, x_5)\) cycle, and the \((y, \theta_2)\) torus is
related to the \((x_8, x_9)\) cycle. The radial coordinate used here (i.e. \(r_1\)) is proportional to the \(x_7\) coordinate used before. Finally, \(z\) is related to the compact \(x_6\) coordinate. Thus
\[
(r_1, z, x, \theta_1, y, \theta_2) \propto (x_7, x_6, x_4, x_5, x_8, x_9) \quad (2.25)
\]
which means that the radial coordinate we defined earlier, namely \(r = \sqrt{x_4^2 + x_5^2}\), is not the radial coordinate \(r_1\) used in the local geometry, but is related to the distances along the \((x, \theta_1)\) torus. In terms of the NS and RR fields, this means that both the B-fields have components parallel to the D7 branes in the \((z, r_1)\) directions but, as discussed before, this doesn’t lead to any pinning effects for the D5 branes.

How do we now embed a bound set-up of \(m\) D5 and \(n\) D7 branes in the local geometry \((2.24)\)? Our first observation is that the D5/D7 bound state cannot change the topology of the manifold. This is easy to understand, as local metric deformations do not change the topology of the underlying manifold. What could then change? There are two possibilities:

(a) The warp factors will change, but no additional terms will appear in the metric, or
(b) The warp factors will change, and additional terms will appear in the metric. These additional terms deform the metric without changing the topology.

To verify one of these cases, we make the following observations:

(a) If we begin with the metric \((2.24)\) and remove the bound–state configuration, the resulting metric should resemble the local solution given by
\[
\mathcal{A}(r_1) = C(r_1)^2, \quad \mathcal{B}(r_1) = C(r_1)^{-2}, \quad \mathcal{D}(r_1) = C(r_1), \quad \mathcal{E}(r_1) = \mathcal{F}(r_1) = C(r_1), \quad (2.26)
\]
with the warp factor \(C(r_1)\) defined as
\[
C = 1 + \left(\frac{1}{\mathcal{F}_3(r_0)\sqrt{\mathcal{F}_1(r_0)}} \frac{\partial \mathcal{F}_3}{\partial r_1} \bigg|_{r_1 = r_0}\right) r_1 = 1 + Q r_1. \quad (2.27)
\]
These terms are explained in \[18\] (see section 2 therein).

(b) If we begin with the metric \((2.24)\) and remove the bound D5 branes, i.e. make \((m, n) = (0, 1)\), then the solution \((2.26)\) changes according to
\[
\mathcal{A} \rightarrow k^{-1} \mathcal{A}, \quad \mathcal{B} \rightarrow k^{-1} \mathcal{B}, \quad (\mathcal{C}, \mathcal{D}) \rightarrow (k \mathcal{C}, k \mathcal{D}), \quad (\mathcal{E}, \mathcal{F}) \rightarrow (k^{-1} \mathcal{E}, k^{-1} \mathcal{F}), \quad (2.28)
\]
without generating an extra term in the metric \[18\]. Here \(k = k(r_1)\) is the harmonic function whose value was left undetermined in \[18\]. On the other hand, \((m, n) \geq (1, 0)\), the original case studied by \[31\], may not be supersymmetric \[32\], \[17\], \[16\], \[21\], \[18\].
The above set of observations might naively imply that embedding a bound state of D5/D7 in the local geometry (2.26) should generate no additional terms in the metric, other than the one that we already have; only the warp factors should change. Therefore our first ansatz for the metric of a bound state of m D5 and n D7 branes in the local geometry (2.26) is given by

\[
A \rightarrow \sqrt{h} \frac{p+q}{p+q} h, \quad B \rightarrow \frac{\sqrt{h}}{p+q} h
\]

\[
(C, D) \rightarrow (\sqrt{h} C, \sqrt{h} D), \quad (E, F) \rightarrow \left( \frac{E}{\sqrt{h}}, \frac{F}{\sqrt{h}} \right),
\]

(2.29)

where \((p, q)\) are integers defined in terms of the \(\Delta\) given in the previous section:

\[
p = e^{\phi_0} - q, \quad q = \frac{n^2}{\Delta}.
\]

(2.30)

A little thought will tell us that this cannot be the complete answer, since our method for constructing the bound state implies that the metric depends non-trivially on the \((x, \theta_1)\) directions instead of the \(r_1\) direction. Therefore, a simple linear superposition like (2.29) may not provide the full scenario, and we require corrections to the above ansatz.

To entertain possible corrections, we first observe that we can make the coefficients (2.26) constant if we take \(Q\) in (2.27) to be vanishingly small. This is the regime where the fibration described by (2.24) becomes trivial (at least to first order) [17], [16], [18]. Second, we observe that a \(U(n)\) gauge theory in 2 + 1 dimensions with \(m\) units of magnetic flux exhibits somewhat similar properties to those of the bound state that we are looking for, as long as in a certain regime the 2 + 1 dimensional theory can be described as the dual configuration of a \(U(m)\) gauge theory with \(n\) units of magnetic flux. This is nothing but a type IIA D2 brane configuration with magnetic fluxes allowing the bulk \(U(1)\) electric flux

\[
A_0 = -\frac{1}{\kappa} \cdot \frac{m - \gamma n}{h\sqrt{\Delta}}.
\]

(2.31)

Here \(\kappa\) and \(\gamma\) are constants that may be fixed by going to a type IIB theory, where \(\tau(r \rightarrow \infty) = \gamma + i\kappa\) implies that \((\phi_0, \chi_0) \equiv (-\log \kappa, \gamma)\) are the respective asymptotic values of the dilaton-axion we defined earlier. We have kept \(h\) as the 2d harmonic function.

This bulk electric field is affected by the worldvolume magnetic fluxes, as can be seen from (2.31). To determine the worldvolume fluxes, we need a D2 brane oriented along \(x^{0,6,7}\) and fluxes satisfying \(\int F_{07} = m\). Such a configuration affects the string coupling,
and therefore the behavior of the dilaton. We compute that in terms of the parameters in (2.14). The dilaton $\phi_2$ becomes

$$\phi_2 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right),$$

so that the full description of the near–core region can only be captured by M-theory, while a type IIA description is valid away from the core.

Once we lift the configuration to M-theory, there will be G-fluxes and globally–defined $C$-fields. For the present case, it is not too difficult to work out the three-form field. It is given by

$$C = \frac{1}{h} \left[ \alpha_1 - \alpha_2 \left( \frac{h + \beta_1}{h + \beta_2} \right) \right] \, dx_0 \wedge dx_6 \wedge dx_7 + \frac{m}{n} \left( \frac{h + \beta_1}{h + \beta_2} \right) \, dx_{11} \wedge dx_6 \wedge dx_7$$

with $G = dC$ as the G-flux. The constants $\alpha_i$ and $\beta_i$ are defined as:

$$\alpha_1 = \frac{\gamma (m - \gamma n) - n \kappa^2}{\kappa \sqrt{\Delta}}, \quad \beta_1 = \frac{\gamma \Delta}{mn \kappa} - 1,$$

$$\alpha_2 = \frac{6m(m - \gamma n)}{n \kappa \sqrt{\Delta}}, \quad \beta_2 = \frac{\Delta}{n^2 \kappa} - 1.$$ 

We see that both the G-fluxes and the C-fields are not necessarily sourced only by the M2 branes. Extra fluxes appear in the theory from the backreactions of the worldvolume $F$ fluxes.

The backreaction of the worldvolume $F$ fluxes on the geometry can also be worked out. If we are away from the core, then a type IIA description suffices. The backreaction is only felt along the directions of the $F$ fluxes, i.e along $x^{6,7}$, and is given by

$$\delta g_{66} = \delta g_{77} = \frac{1}{\sqrt{h}} \cdot \frac{1}{h + \beta_2} \left[ h \left( \frac{\Delta}{n^2} - 1 \right) - \beta_2 \right], \quad \delta g_{mn} = 0; \quad m, n \neq 6, 7,$$

where $\beta_2$ is defined in (2.34). We should also note that in M-theory all the metric components will change, and therefore the near–core description will be a little more complicated. The metric will involve the usual fibration structure due to the presence of the electric flux (2.31), and the warp factors will change due to the explicit dependence of the string coupling on internal magnetic fluxes.

Once we know the complete background of D2 branes with magnetic fluxes, we can use the following set of duality arguments to determine a candidate metric for our D5/D7

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\[11\] We are absorbing the unimportant constant $\kappa$ in the definition of the coordinates.
bound state on a non-trivial background. Clearly we will not be able to simulate completely the local geometry on which we want to have our bound state, but we can come very close.

The following are a set of steps that can potentially lead us to the required answer:

- The D2 brane is oriented along $x^{6,7}$ and is orthogonal to the other directions. Consider the $x^4$ direction. If we compactify the $(x^6, x^4)$ directions on a torus $T^2$, then generically the torus can have arbitrary complex structure $\tau_1$. We can parametrise the complex structure by a real coordinate $\sigma_1$ such that $\text{Im} \, \tau_1 = 0$. Such parametrisation is exactly of the form given earlier in [23], [22], [26].

- As we discussed earlier, the near–core region of this configuration is at strong coupling. The complete picture can therefore only be given via M-theory. Lifting this configuration to M-theory gives rise to a three–torus $T^3$, where one of the toroidal directions is the eleventh direction $x^{11}$ with radius $R_{11}$. We can pick a torus along $(x^6, x^{11})$ and shrink it to zero size, while keeping the $x^4$ cycle in $T^3$ invariant. This will take us to type IIB theory with an orthogonal set of D1 branes. Since we kept the $x^4$ cycle inside $T^3$ unchanged, the original non-trivial complex structure will generate an additional $B_{NS} = B_{64}$ field along with the bound state.

- We can keep the system at a point on a $T^4$ along $x^{1,2,3,9}$. Let the volume of the $T^4$ be $V = 16\pi^4 \, R_1 R_2 R_3 R_9$, where $R_i$ are the radii of the cycles as measured in the warped geometry. Shrink the volume of the torus to zero size. Observe that this doesn’t affect the $B_{64}$ field, as it is orthogonal to the torus.

- One can easily show that this configuration is dual to a configuration of two intersecting Taub-NUT spaces in M-theory where one of the TNs is along $x^{4,5,7}$ and the eleventh direction $x^{11}$ and the other TN is along $x^{4,5,6}$ and $x^{11}$ along with some G-fluxes. The existence of non-zero G-fluxes can be accounted for from the fact that the corresponding three–forms thread through the degenerating cycle of the two Taub-NUT spaces. For the present case there are at least two non-zero components of the C-field ($C_{4,6,11}$ and $C_{8,6,11}$) through which this duality can be explicitly analysed.

- It is easy to go to type IIB from M-theory by shrinking some two-torus to zero size. For the case in question, not all two-tori would give the kind of answer that we are looking for. In fact there is one non-trivial two-torus along the directions $(x^6, x^{11})$ that is particularly

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12 This duality, although not quite like AdS/CFT, is in the same spirit as gauge/gravity dualities, in that a gauge theory on an intersecting D-string configuration is mapped to a theory of gravity without branes.
suited for our purpose. In the limit where this torus shrinks to zero volume, we have an exact duality to a bound-state configuration of D5/D7 branes! More interestingly however, some components of the C-fields will dissolve completely in the geometry to give us a non-trivial local solution resembling our required local solution (2.24), at least in certain limits. The other surviving components will provide the necessary dipole deformation to the bound-state configuration.

Through this set of duality transformations, we hope to get a handle on our background so that we can analyse all the field components that constitute the supergravity solution for the system. We will begin by analysing our duality chain from an intermediate stage that involves Taub-NUT spaces. The duality arguments leading to that configuration are easy, though tedious, to reproduce: we will leave them for the reader to derive.

Our starting point is to observe that the $B_{NS}$ field we obtain from the non-trivial complex structure (parametrised by $\sigma_1$) in the first step of our duality chain is a little involved. If we denote the asymptotic value of $B_{64}$ as $b_{\infty}$, then for $\tan \sigma_1 \equiv x$ we have

$$x^3 - cb_{\infty}x^2 + x - \kappa cb_{\infty} = 0,$$

(2.36)

where $\kappa$ is as defined earlier and $c$ is a multiplicative constant. The constant $c$ will in general be identity, but because of our duality to Taub-NUT spaces, it turns out to be a non-trivial constant which has interesting physical consequences. For the time being we will parametrise $c$ by another angular coordinate $\sigma_2$ as

$$c \equiv \sec \sigma_2.$$

(2.37)

With this description of $c$, one can show that the dual description of a system of orthogonal D-strings at a point on $T^4$ when $\text{vol}(T^4) \to 0$ is an intersecting warped Taub-NUT background with the following three-form fields:

$$C = \frac{1}{\kappa_1} \left[ \tan \frac{\sigma_2}{\kappa_2} dx_6 \wedge dx_8 - \frac{\chi_\alpha \cos \sigma_1 \sin \sigma_2}{\kappa_2} dx_7 \wedge dx_8 + \sqrt{h} \frac{\tan \sigma_1}{\cos \sigma_2} Q_- dx_4 \wedge dx_6 + \right.$$

$$- \sqrt{h} \sin \sigma_1 \chi_\alpha Q_- dx_4 \wedge dx_7 \left. \right] \wedge dx_{11},$$

(2.38)

with the corresponding G-fluxes given by $G = dC$. In the above, $\tilde{h}$ is as defined earlier. The other coefficients are defined in the following way:

$$\kappa_1 = \tilde{h}^{-1} \sqrt{h} (\kappa \cos^2 \sigma_1 + \tilde{h} \sin^2 \sigma_1) \equiv \tilde{h}^{-1} \sqrt{h} \kappa_1, \quad \chi_\alpha = \frac{\gamma}{\tilde{h}} + \frac{mn\kappa(h-1)}{\tilde{h}\Delta},$$

$$\kappa_2 = \sqrt{h} \cos^2 \sigma_2 + \kappa_1^{-1} \sin^2 \sigma_2 \equiv \sqrt{h} \kappa_2, \quad Q_\pm = 1 \pm \kappa_1^{-1} \kappa_2^{-1} \sin^2 \sigma_2,$$

(2.39)
where \( h \) is the corresponding warp factor in the intersecting Taub-NUT metric. This warp factor’s behavior can be traced through the duality chain explicitly\(^{13}\). We should also note that the coordinates \( x^i \) are the \textit{dual} coordinates defined for this space, and are related to the original coordinates (i.e. the coordinates of the intersecting system of D-strings) by coordinate transformations.

The G-fluxes we constructed from \((2.38)\) each have one of their components along the Taub-NUT circle \( x^{11} \). Such a choice of G-fluxes cannot completely specify the dual picture. We need more components of G-fluxes that are orthogonal to the Taub-NUT circle. In other words, if we specify the additional G-flux as \( G_2 \), then we require \( G_2 \wedge dC = 0 \). For our specific case, \( G_2 \) turns out to be

\[
G_2 = \frac{(m - \gamma n) \sin \sigma_2}{\kappa \, h^2} \sqrt{\Delta} \ast (dh \wedge dV_0 \wedge dx_{11}), \tag{2.40}
\]

with \( \ast \) the Hodge star for the warped metric (to be determined below) and \( dV_0 \equiv dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_9 \) a constant form.

Looking at the three-form that we get in \((2.38)\), we see that in some limits it can have constant pieces. Does this mean that we can gauge them away? For a flat background such components can be gauged away, but not for the present case. Since the constant pieces have one leg along the Taub-NUT circle \( dx^{11} \) and a normalisable harmonic \((1,1)\) form lives on the Taub-NUT, gauging away such components switches on other components (see \(^{33}\) for more details). Therefore such constant pieces survive.

All we now need is to specify the metric for our case. Since we expect an intersecting Taub-NUT solution, the metric will typically be a five-dimensional warped metric. Both the TNs share one degenerating cycle along the eleventh direction, and therefore the fibration structure will be non-trivial in the \( x^{11} \) cycle. The precise metric turns out to be

\[
ds^2 = \left( \frac{\kappa_1 \kappa_2 \hat{h}}{\kappa h} \right)^\frac{1}{3} \left[ ds^2_{01239} + \frac{\tilde{h} Q_{-}}{\kappa_1} (\sec \sigma_2 \, dx_6 - \chi_\alpha \cos \sigma_1 \, dx_7)^2 + \frac{1}{\kappa_2} (dx_8 + \frac{\tilde{h}}{\kappa_1} \tan \sigma_1 \sin \sigma_2 \, dx_4)^2 \right] + \left( \frac{\kappa_1 \kappa_2 \hat{h} h^2}{\kappa} \right)^\frac{1}{3} \left[ dx_5^2 + \frac{\kappa}{\hat{h}} \, dx_7^2 + dx_9^2 \right]+ \left( \frac{\sec^2 \sigma_1 - \frac{\tilde{h}}{\kappa_1} \tan^2 \sigma_1 }{\kappa_1} \right) \left( dx_11 + \omega \cdot dx \right)^2.
\tag{2.41}
\]

\(^{13}\) The value of \( \chi_\alpha \) above is same as the one we computed for D-strings, i.e \( \chi_1 \) in \((2.20)\). It would be an interesting exercise to see if this follows from our duality chain.
Here we see that the non-trivial fibration is via the one-form fields $\omega_\mu$, the precise form of which will be derived below. The two TN spaces can now be seen to be along $x^{4,5,6,11}$ and $x^{4,5,7,11}$ and therefore span a five-dimensional surface.

To determine the one-form $\omega_\mu$ we start by choosing a hypersurface in our space or-thogonal to the two intersecting TNs. Let $dV_1$ be a constant form on the surface. Using this we can define the following form on the manifold:

$$dS_1 \equiv -\frac{m-\gamma n}{\kappa \sqrt{\Delta} \sec \sigma_1} \left[ \gamma dh^{-1} - 5 d \left( \frac{X_\alpha}{h} \right) - \sin^2 \sigma_2 d \left( \frac{X_\alpha}{h \kappa_1 \kappa_2} \right) \right] \wedge dx_7 \wedge dV_1 \wedge dx_{11} + \frac{n \kappa}{\sqrt{\Delta} \sec \sigma_1} dh^{-1} \wedge dx_7 \wedge dV_1 \wedge dx_{11} + \frac{m-\gamma n}{\kappa \sqrt{\Delta}} \left[ \sec \sigma_2 d(h^{-1} Q_-) \wedge dx_6 \right] \wedge dV_1 \wedge dx_{11}. \quad (2.42)$$

This is a nine-form that cannot be gauged away. It turns out that this is not the only nine-form we can define for our case. As in the case of the original nine-form (2.42), we can use another hypersurface along $x^4$ that intersects the original hypersurface used to define (2.42) in a five-dimensional space and intersects the $x_8$ line at a point. Let $dV_2$ be a constant form on this hypersurface; we can then construct another nine-form

$$dS_2 \equiv -\frac{m-\gamma n}{\kappa \sqrt{\Delta}} \left\{ \sin \sigma_1 \sin \sigma_2 \left[ \gamma d(\kappa_1^{-1} h^{-1/2}) - 5 d \left( \frac{X_\alpha}{\kappa_1 \sqrt{h}} \right) + d \left( \frac{X_\alpha}{\kappa_1^2 \kappa_2 \sqrt{h}} \right) \sin^2 \sigma_2 \right] + \frac{n \kappa^2}{m-\gamma n} \sin \sigma_1 \sin \sigma_2 d(\kappa_1^{-1} h^{-1/2}) \right\} \wedge dx_7 \wedge dV_2 \wedge dx_{11} + \frac{m-\gamma n}{\kappa \sqrt{\Delta}} \tan \sigma_1 \tan \sigma_2 d \left( \frac{Q_+}{\kappa_1 \sqrt{h}} \right) \wedge dx_6 \wedge dV_2 \wedge dx_{11}. \quad (2.43)$$

Using the above equations (2.42) and (2.43), the one form $\omega_\mu$ determining the fibrations for both the TN spaces is defined as

$$d\omega = \lim_{R_{11} \to 0} * (dS_1 + dS_2), \quad (2.44)$$

where the Hodge dual is on the warped metric in the above limit. The reason we have to take a limit is that in M-theory there are no fundamental one-forms, which only exist in the type IIA limit. Therefore, plugging in the value of (2.44) into (2.41) yields the complete fibration. Along with the three-form (2.38) and G-fluxes (2.40) this specifies the full M-theory background. Thus, this configuration is dual to the intersecting D-string configuration in the limit when $\text{vol}(T^4) = 0$. 

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Our last step is to shrink the \((x^{11}, x^6)\) torus to zero volume in order to reach type IIB theory. To see this explicitly, we have to put holomorphic coordinates on a small patch of the torus. Let \(dz = dx_{11} + \tau_m \, dx_6\), where \(\tau_m\) is the complex structure. If the one-form \(\omega_6 \neq 0\), one can show that

\[
\text{Re} \, \tau_m = \omega_6, \quad \text{Im} \, \tau_m = \sqrt{\frac{\kappa_2 \tilde{h} \sqrt{h} Q - \sec^2 \sigma}{\kappa}}, \quad (2.45)
\]

where \(\kappa_2\) was defined in (2.39). Using this, the metric on the small patch can be written as

\[
d s_{\text{patch}}^2 = \left(\frac{\kappa}{\kappa_1 \kappa_2 \tilde{h} \sqrt{h}}\right)^{\frac{3}{4}} |dz|^2. \quad (2.46)
\]

The above choice of holomorphic coordinate does not imply the existence of an integrable complex structure for our case. Nevertheless, the full metric can be written down using real coordinates as we saw above. Shrinking the volume of the \(T^2\) to zero size implies that there will be a non-trivial dilaton given by

\[
\phi_b = \frac{1}{4} \ln \Delta - \ln h + \frac{1}{2} \ln a, \quad (2.47)
\]

where \(a\) was defined in (2.9). We see that (2.47) is exactly the flat-space limit of the dilaton given in (2.14). In fact, as we mentioned briefly before, the ansatz (2.14) – although suitable for bound states in a flat background – works also for the curved background with non-trivial topology as in our case.

Once we have the dilaton, we should look for the corresponding axion. Our earlier analysis for a flat background yielded the value given in (2.17). To determine the precise correction to our earlier result (2.17), we can use the constant form \(dV_1\) defined in (2.42). With this one can show that the axionic field undergoes a simple modification given by

\[
d \chi_b = d \chi \cos \sigma_1. \quad (2.48)
\]

The axion so obtained sources the D7 branes. From the bound-state analysis, we should then expect a source for the D5 branes also. This is indeed the case, and is given by

\[
H_{RR} = \frac{m - \gamma n}{\kappa \sqrt{\Delta}} \sec \sigma_2 * \{dh^{-1} \wedge dV_1\}. \quad (2.49)
\]
The above two fields (2.48) and (2.49) are not the only sources of the axion and RR three-forms. There are in fact sources of these fields that do not exist in the flat-space limit. For example there could be an axion source from $dV_2$ defined in (2.42) as:

$$d\chi_b^{(2)} = -\frac{m - \gamma n}{\kappa \sqrt{\Delta}} \sin \sigma_1 \ast \left\{ d\left( \frac{\gamma - 6 \chi_\alpha}{\kappa_1 \sqrt{h}} \right) \land DM \right\} + \frac{n \kappa}{\sqrt{\Delta}} \sin \sigma_1 \ast \left\{ d\left( \frac{1}{\kappa_1 \sqrt{h}} \right) \land DM \right\}$$

(2.50)

where we have defined $DM$ as

$$DM = dV_2 \land dx_7 \land (\sec \sigma_2 \, dx_8 - \sin \sigma_2 \, dx_6).$$

(2.51)

It is interesting to note that our background also has additional sources of $H_{RR}$, much like the additional sources of axion computed above in (2.50). These sources, as one might expect, do not come from the D5 branes only. They are given by

$$H_{RR}^{(2)} = \frac{m - \gamma n}{\kappa \sqrt{\Delta}} \left[ 2 \tan \sigma_1 \tan \sigma_2 \sin^2 \sigma_2 \ast \left\{ d\left( \kappa_1^{-2} \kappa_2^{-1} h^{-1/2} \right) \land dV_2 \right\} \right.$$  

$$+ \sin \sigma_1 \ast \left( dh^{-1} \land dV_0 \land dx_6 \right) \right],$$

(2.52)

with $dV_0$ defined in (2.40). We see that an equivalent term like $\ast (dh^{-1} \land dV_0)$ for (2.49) contributes only to $O(\sigma_1)$ to the $H_{RR}$ (similarly the other contribution goes like $O(\sigma_1 \sigma_3^2)$) and therefore in the limit

$$\sigma_{1,2} \to 0$$

(2.53)

the above contributions are suppressed. It turns out that all the new contributions to the axion and the RR three-form, other than (2.48) and (2.49), are suppressed in the limit of small $\sigma_{1,2}$.

Such a limit will provide us with tremendous simplifications. Therefore, in our notation, the flat-space results can be determined simply by setting $\sigma_1 = \sigma_2 = 0$. In the flat-space limit we expect a $B_{NS}$ field that provides the Chern class of the D7 brane gauge bundle. The corresponding $H_{NS}$ is therefore orthogonal to the D5 branes, and is given by

$$H_{NS} = \cos \sigma_1 \cos \sigma_2 \, d\chi_\alpha \land dx_6 \land dx_7.$$  

(2.54)

Thus it is related to the RR form (2.49) by supersymmetry. As before there could be new sources of $H_{NS}$ fields that are suppressed in the limit (2.53) as

$$H_{NS}^{(2)} = -\sin \sigma_1 \, d\left( \frac{\chi_\alpha Q}{\kappa_1 \sqrt{h}} \right) \land dx_4 \land dx_7$$

(2.55)
With the knowledge of NS and RR three-forms as well as the axion-dilaton, we can construct $G = H_{RR} + \tau_b H_{NS}$ with $\tau_b = \chi_b + i e^{-\phi_b}$. With the help of $G$, a superpotential for our background can then be easily constructed.

Our final venture is to determine the metric for the D5/D7 bound state. From the intersecting Taub-NUT solution (2.41) the result follows from shrinking the torus (2.46) to zero size. This gives us:

$$ds^2_{IIB} = \frac{ds^2_{0123}}{\sqrt{h}} + \frac{1}{\sqrt{h}} \left( dx_6^2 + \frac{Dx_8^2}{\kappa_2} \right) + \sqrt{h} \left( \frac{\kappa dx_4^2}{\kappa_1} + dx_5^2 \right) + \frac{\kappa \sqrt{h}}{h} \ dx_7^2 + \frac{\sqrt{h \kappa_1}}{h Q_-} \ cos^2 \sigma_2 (dx_6 + f_1 \ dx_4 + f_2 \ Dx_8)^2, \quad (2.56)$$

where the various coefficients are defined as:

$$f_1 = \sqrt{h \kappa_1}^{-1} \ tan \ \sigma_1 \ sec \ \sigma_2, \quad f_2 = -\kappa_1^{-1} \kappa_2^{-1} \ tan \ \sigma_2, \quad D x_8 = dx_8 + \sqrt{h \kappa_1}^{-1} \ tan \ \sigma_1 \ sin \ \sigma_2 \ dx_4. \quad (2.57)$$

At this stage we can impose the coordinate redefinitions (2.25) on our metric (2.56). The resulting background looks almost like (2.24) if we replace $dx_8$ in (2.24) by $D x_8$. Therefore the metric of D5/D7 bound states has almost the predicted form of (2.24) as we have been expecting. The metric (2.56) however looks exactly like (2.24) only in the limit (2.53). In the following we will discuss this limit, and also determine the changes to the metric after we make a dipole deformation.

3. Dipole-deformed bound states

To study the effect of the limit (2.53) on the metric (2.56) we have to make a small expansion about the angular terms $\sigma_{1,2}$. This way we will also be able to compare our result with our earlier proposal (2.29). Our claim is that the metric (2.56) resembles (2.24). To verify this for the two tori $T^2$ we find:

$$D = \sqrt{h(1 - s \sigma_1^2)}, \quad C = \sqrt{h},$$

$$F = \frac{1}{\sqrt{h}} \left\{ 1 + \sigma_2^2 \left[ 1 - \frac{h(1 - s \sigma_1^2)}{\kappa h} \right] \right\}, \quad E = \frac{1}{\sqrt{h}}, \quad (3.1)$$
where \( s \equiv \frac{\hat{h}}{\kappa} - 1 \). Thus we see that the corrections to (2.29) go like \( \mathcal{O}(\sigma_i^2) \) and are therefore suppressed in the limit (2.53). This continues to hold for the other coefficients of the metric (2.56) because

\[
A = \frac{\sqrt{h}}{p + qh}, \quad B = \frac{\sqrt{h}}{p + qh} \left( 1 + s\sigma_1^2 + \frac{\hat{h}}{h\kappa} \sigma_2^2 \right) (1 - \sigma_2^2),
\]  

(3.2)

which are again of the form (2.29). Finally the fibrations in the metric (2.56) take the following form:

\[
f_1 = \frac{\sigma_1 \hat{h}}{\kappa} \left[ 1 - s\sigma_1^2 + \frac{\sigma_2^2}{2} \right], \quad f_2 = -\frac{\sigma_2 \hat{h}}{h\kappa} \left[ 1 - s\sigma_1^2 + \sigma_2^2 \left( 1 - \frac{\hat{h}}{h\kappa} \right) \right],
\]  

(3.3)

which implies that the \( f_i \)s are determined by the angular coordinates \( \sigma_i \) linearly as \( f_i \propto \sigma_i \).

This will be crucial later.

The upshot of the above discussion is that the metric (2.56) follows the ansätze (2.29) in the limit \( Dx_8 \approx dx_8 \) and (2.53) with the fibration terms \( f_i \) given as (3.3). On the other hand, the NS three-form field is of the form

\[
B_{NS} = \left( \chi_\alpha \cos \sigma_1 \cos \sigma_2 \, dx_6 - \frac{\chi_\alpha \sin \sigma_1}{\kappa_1 \sqrt{h}} \, Q \frac{dx_4}{d\tau} \right) \wedge dx_7
\]

\[
\approx \chi_\alpha \left( 1 - \frac{\sigma_1^2}{2} \right) \left( 1 - \frac{\sigma_2^2}{2} \right) dx_6 \wedge dx_7 - \frac{\sigma_1 \chi_\alpha}{h\kappa} \left( 1 - s\sigma_1^2 - \frac{\hat{h}\sigma_2^2}{h\kappa} \right) dx_4 \wedge dx_7
\]  

(3.4)

as can be extracted from (2.54) and (2.55). We see that the second term is suppressed by \( \sigma_1 \).

Among the RR fields we will have the axion and the RR three-form \( dB_{RR} \). All of them will have pieces that remain finite in the limit (2.53). This means that we have both \( B_{NS} \) and \( B_{RR} \) along \( x^{6, 7} \) that remain finite, with additional components that are of \( \mathcal{O}(\sigma_i) \). Therefore our approximate background will be

\[
ds^2 \approx \frac{\sqrt{h}}{p + qh} \, dr_1^2 + \frac{\sqrt{h}}{p + qh} \left( dz + f_1 \, dx + f_2 \, dy \right)^2 + \sqrt{h} |dz_1|^2 + \frac{|dz_2|^2}{\sqrt{h}};
\]

\[
B_{NS} \approx b_{67}, \quad B_{RR} \approx \hat{b}_{67}, \quad h = h(r, \sigma_i), \quad r^2 = x^2 + \theta_1^2,
\]  

(3.5)

with \( z_i \) defined exactly as in [17], [21], [16], [18], namely \( z_1 = x + i\theta_1 = x_4 + ix_5 \) and \( z_2 = y + i\theta_2 = x_8 + ix_9 \); and \( r_1 = x_7 \) as before. There is also an axion-dilaton \( (\chi_b, \phi_b) \) that resembles the flat-space result.

We would like to make the following comments:
• The metric (3.5) and the original metric (2.56) resemble the metric of wrapped D5 branes when \( p >> q \) in (3.5). This is the expected case when \( m >> n \). On the other hand, the metric (2.56) or (3.5) resembles the metric of D7 branes when \( n >> m \) as one might expect. However in both the above limits the metrics do not resemble the metric of D3 branes at a point on our local geometry. That this is not inconsistent with our previous analysis on [17], [16], [21] and [18] is because we haven’t put in the required fluxes. Once the necessary fluxes are put in, the metric will resemble the metric of D3 branes at a point in our local geometry.

• The limit (2.53) that we used above to write the type IIB solution (2.56) in the form (3.5) may not always hold. There could be some regime (or a different patch) where (2.53) cannot be applied consistently. That would mean that (3.5) will not always capture the dynamics in certain patches of the full global geometry. On the other hand, if we demand

\[
h \equiv h(\text{Re} \ z_1) \tag{3.6}
\]

then there are certain definite advantages over (2.53):

1. \( D x_8 \) defined in (2.57) becomes a total derivative such that \( dD = D^2 = 0 \) and therefore forms a cohomology. In fact \( D x_8 = d x_8^+ \equiv d(x_8 + x_4^+) \) where

\[
x_4^+ = \int^{\text{Re} \ z_1} d(\text{Re} \ z_1) \frac{\hat{h} \tan \sigma_1 \sin \sigma_2}{\kappa \cos^2 \sigma_1 + \hat{h} \sin^2 \sigma_1} \tag{3.7}
\]

2. The harmonic function will become linear in \( \text{Re} \ z_1 \) and so can be approximated as \( h = 1 + c \text{Re} \ z_1 \) where \( c \) is a constant. This means that \( V_0 \) and \( V_2 \) are not independent forms, but are related as

\[
dV_2 = c \ dh \wedge dV_0 \tag{3.8}
\]

3. The following unnecessary components will vanish:

\[
dS_2 = d\chi_{b}^{(2)} = H_{NS}^{(2)} = H_{RR}^{(2)+} = 0 \tag{3.9}
\]

where these components are defined above and the superscript + implies that there are a few surviving components\(^{14}\). The fact that \( H_{NS} \) and \( H_{RR}^{(2)+} \) vanish also means that

\(^{14}\) In our notation, \( H_{RR}^{(2)} \) defined in (2.52) can be written as \( H_{RR}^{(2)} = H_{RR}^{(2)+} + H_{RR}^{(2)-} \) with \( \pm \) forming the first and the second components of (2.52) respectively.
the corresponding two–form fields can be gauged away. The ungauged components are precisely the ones that appear in (3.5) above along with an additional one

\[ H_{RR}^{(2)} = - \frac{c(m - \gamma n) \sin \sigma_1}{\kappa h^2 \sqrt{\Delta}} \star (dV_2 \wedge dx_6) \]  

(3.10)

(4) The final metric becomes

\[ ds_{\text{IIB}}^2 = \frac{ds_{0123}^2}{\sqrt{h}} + \frac{1}{\sqrt{h}} \left[ dx_5^2 + \bar{\kappa}_2^{-1} (dx_8^+)^2 \right] + \sqrt{h} \left[ (dx_4^-)^2 + dx_5^2 \right] + \frac{\kappa \sqrt{h}}{h} dx_7^2 + \frac{\sqrt{h \bar{\kappa}_1}}{hQ_-} \cos^2 \sigma_2 (dx_6 + f_1 dx_4 + f_2 dx_8^+)^2, \]  

(3.11)

which is the closest we get in realising the precise local geometry of our earlier papers. It is indeed remarkable to see that our earlier predictions fit perfectly with the above derivation; (3.11) should then be regarded as a derivation of our local geometry. With \( h \) defined as above and

\[ x_4 = \int dx_4^+ \sqrt{\bar{\kappa} \cos^2 \sigma_1} + x_4^+ \frac{\sin 2\sigma_1}{2\sin \sigma_2}, \]  

(3.12)

we can easily argue for the form (2.24) with \( C(r_1) \) inserted in the local limit. The final background therefore consists of the metric (3.11) with \( f_i = f_i(\theta_i) \) as derived in \[18\]. The other fields are the NS fields:

- Dilaton \( \phi \) and two-form \( b_{67} \)

which appear from (2.47) and (2.54) respectively. The RR fields are the three-forms coming from (2.49) and (2.52). However some of these components are gauged away. Similar things happen with the axions (2.48) and (2.50). One can show that the ungauged component here is only (2.48). We can dualise these forms and write everything in terms of the six-forms and eight-form as

- \( C_1 = C_{012369}^{(6)}, \quad C_2 = C_{012369}^{(6)}, \quad dC_3 = \ast d\chi_b \)

with \( C_3 \) being the required eight-form. With this configuration at hand we can now make dipole deformations to our background. Due to the existence of \( b_{67} \) there could be multiple ways to perform dipole deformations here. In the following we will analyse these aspects in detail.
3.1. First dipole deformation

The multiple ways of doing dipole deformations that we alluded to above are related to the fact that we can either consider the \( b_{67} \) field while performing the dipole deformation or not. The deformation without an intervening \( b_{67} \) field is predictably much easier to apply. In this section we will try this approach.

The dipole deformation – which we will call the first dipole deformation – can be parametrised by an angular coordinate \( \sigma_3 \). Our starting point is the metric (3.11) along with the NS and RR backgrounds discussed above. We define the following variables:

\[
\alpha_0 = \frac{\sqrt{\hbar \kappa_1}}{h Q_-} \cos^2 \sigma_2, \quad \alpha_2 = \frac{1}{\kappa_2} + \alpha_0 f_2^2 \sqrt{\hbar}, \quad j_0 = \frac{\cos^2 \sigma_3}{\sqrt{\hbar}} (1 + \alpha_2 \tan^2 \sigma_3)
\]

(3.13)

where all the parameters appearing above have been described earlier. We should also remember that under a dipole deformation both the metric and the coordinates describing the underlying space change. Let the new coordinates be \( y_i, i = 4, \ldots, 9 \). The dipole deformation then changes the metric (3.11) to

\[
d s_1^2 = \frac{d s_{123}^2}{\sqrt{\hbar}} + \left[ \sqrt{\hbar} (dy_4) (dy_5) \right] + \left[ \frac{dy_5^2}{\sqrt{\hbar}} + \frac{\sec^2 \sigma_3}{\sqrt{\hbar \kappa_2}} dy_8^2 \right] + \frac{\kappa \sqrt{\hbar}}{\hbar} dy_7^2 + \\
+ \alpha_0 (dy_6 + f_1 dy_4 + f_2 \sec \sigma_3 dy_8)^2 - \frac{\alpha_0^2 f_2^2 \sin^2 \sigma_3}{j_0} [dy_6 + f_1 dy_4 + (f_2 \sec \sigma_3 + x)dy_8]^2.
\]

(3.14)

Looking at the above metric we see that this is almost of the form of the initial starting metric (3.11) except for the \( y_6 \) fibration part because of the presence of an extra term. This term is defined as:

\[
x = \frac{1}{f_2} \cdot \frac{Q_-}{\kappa_1 \kappa_2} \cdot \sec^2 \sigma_2 \sec \sigma_3.
\]

(3.15)

Can this term be made smaller? To see this we first need to figure out whether the NS two-form performing the dipole deformation is affected or not. It turns out that the \( B_{NS} \) field is also affected in the following way

\[
B_{NS} = \frac{\alpha_0 f_2 \sin \sigma_3}{j_0} [dy_6 + f_1 dy_4 + (f_2 \sec \sigma_3 + x)dy_8] \wedge dy_5 + b_{67} dy_6 \wedge dy_7.
\]

(3.16)

This means that the \( B_{NS} \) fields do not follow the standard fibration structure of the background expected from the initial metric (3.11). This can be rectified by making \( x \ll 1 \). To allow this, observe that the free adjustable parameters in our problem are \( (\sigma_{1,2,3}) \) of which \( \sigma_3 \) is generically small. If we are also in the limit (2.53) then \( \sigma_{1,2} \) will also
be small. This will make $x \sim 1$. On the other hand we could be be in the limit (3.6) but not in the limit (2.53). For this case we can have

$$Q_- << 1 \Rightarrow x << 1$$

(3.17)

where $Q_-$ is defined in (2.39). With this choice the dipole deformation takes the following nice form

$$ds_1^2 = \frac{ds_{123}}{\sqrt{h}} + \sqrt{h} \left[ (dy_{\overline{1}})^2 + \frac{dy_5^2}{\cos^2 \sigma_3 + \alpha_2 \sin^2 \sigma_3} + \frac{1}{\sqrt{h}} \left( dy_5^2 + \frac{\sec^2 \sigma_3}{\kappa_2} \ dy_8^2 \right) + \kappa \sqrt{h} \ dy_7^2 + \alpha_0 \left( 1 - \frac{\beta_2 \ tan^2 \sigma_3}{1 + \alpha_2 \ tan^2 \sigma_3} \right) (dy_6 + f_1 \ dy_4 + f_2 \ sec \sigma_3 \ dy_8)^2, \right]$$

(3.18)

which is exactly as we had predicted in [18] if we take $f_i = \frac{f_i(\theta_i)}{1+\delta_{i2}(\sec \sigma_3-1)}$ and $C(r_1) \to 1$! Here $\beta_2 = \alpha_0 f_2^2 \sqrt{h}$ and denoting the $y_6$ fibration structure as $Dy_6$ the dipole B-field is given by

$$B = \frac{\beta_2}{f_2} \left( \frac{\tan \sigma_3 \ sec \sigma_3}{1 + \alpha_2 \ tan^2 \sigma_3} \right) \ Dy_6 \wedge dy_5 + b_{67} \ dy_6 \wedge dy_7$$

(3.19)

with appropriate field strength. It is also interesting to see that a component of the B-field (3.19) can provide a dipole deformation to the D7-brane gauge theory although not necessarily the component that makes a dipole deformation to the D5-brane gauge theory.

We will not evaluate the RR fields in detail because they can be easily worked out from the dipole deformations and following earlier works [22], but go directly into comparing the volumes of the two-cycles $\Sigma_2$ before and after the deformation. Before dipole deformation the volume of the two-cycle on which we have wrapped D5 branes is given by

$$V_{\text{initial}} = \int_{\Sigma_2} \frac{1}{\sqrt{h}} \left( \frac{1}{\kappa_2 \sqrt{h}} + \alpha_0 \ f_2^2 \right),$$

(3.20)

which is the volume of a $T^2$ in the geometry (3.11). After dipole deformation the volume of the two-cycle changes to

$$V_{\text{final}} = \int_{\Sigma_2} \ sec^2 \sigma_3 \left[ \frac{1}{\kappa_2 \sqrt{h}} + \alpha_0 \ f_2^2 (1 - z) \right],$$

(3.21)

with $z$ given as $z = \frac{\beta_2 \ tan^2 \sigma_3}{1 + \alpha_2 \ tan^2 \sigma_3}$. We see that for small enough deformation the volume of the two cycle shrinks making the KK states heavier. This is again consistent with our earlier conclusion [18].

25
3.2. Second dipole deformation

The second kind of dipole deformation will take into account the presence of the background \( b_{67} \) field. We will again parametrise the dipole deformation by the angular coordinate \( \sigma_3 \). We start by defining the quantity

\[
j_1 = \alpha_0^{-1} \cos^2 \sigma_3 (1 + \alpha_0 \sqrt{\hbar} \tan^2 \sigma_3)
\]  

which is similar to \( j_0 \) defined in (3.13). As before the coordinates of the dipole-deformed background will be different from the coordinates used in (3.11). For the sake of simplicity we shall again use \( y^i, i = 4, \ldots, 9 \) for the new background.

The background after the dipole deformation is different from the one that we discussed in the previous subsection. It is now given by:

\[
ds_{II}^2 = \frac{ds_{0123}^2}{\sqrt{\hbar}} + \sqrt{\hbar} [(dy_4^-)^2 + \sec^2 \sigma_3 dy_5^2] + \frac{1}{\sqrt{\hbar}} \left[ dy_9^2 + \frac{dy_8^2}{\kappa_2} \right] + \left( \frac{\kappa \sqrt{\hbar}}{\hbar} + \frac{b_{67}^2}{\alpha_0} \right) dy_7^2 + j_1^{-1} (dy_6 + f_1 \cos \sigma_3 dy_4 + f_2 \cos \sigma_3 dy_8)^2 - j_1^{-1} \left( \sqrt{\hbar} \tan \sigma_3 dy_5 - \frac{b_{67} \cos \sigma_3}{\alpha_0} dy_7 \right)^2.
\]  

(3.23)

The above metric differs from the previous one in many key respects. The \( y_6 \) fibration is consistent with expectation, whereas our earlier metric (3.14) had a different \( y_6 \) fibration structure. On the other hand we now have a cross term \( dy_5 dy_7 \), but this is suppressed by \( \tan \sigma_3 \).

In the limit where the dipole deformation is small all the \( \tan \sigma_3 \)-dependent terms can be dropped from the metric, and we get the following metric:

\[
ds_{II}^2 = \frac{ds_{0123}^2}{\sqrt{\hbar}} + \sqrt{\hbar} [(dy_4^-)^2 + \sec^2 \sigma_3 dy_5^2] + \frac{1}{\sqrt{\hbar}} \left[ dy_9^2 + \frac{dy_8^2}{\kappa_2} \right] + \frac{\kappa \sqrt{\hbar}}{\hbar} dy_7^2 + \alpha_0 \sec^2 \sigma_3 (dy_6 + f_1 \cos \sigma_3 dy_4 + f_2 \cos \sigma_3 dy_8)^2,
\]  

(3.24)
As before, the B field has components that perform dipole deformations to the D7-brane gauge theory also. In addition to that the RR fields are also affected by the dipole deformations. We will not discuss them here as they are easy to work out. Instead we will concentrate on the volume of the two-cycle on which we have wrapped branes. The original volume is given by (3.20). The final volume after dipole deformation will be

$$V_{\text{final}} = \int_{\Sigma_2} \frac{1}{\sqrt{h}} \left( \frac{1}{k_2 \sqrt{h}} + \frac{\alpha_0 f_2^2}{1 + \alpha_0 \sqrt{h} \tan^2 \sigma_3} \right),$$

(3.26)

which is clearly smaller than (3.20) – confirming again our earlier arguments. Notice that, compared to the previous case, this deformation does not distinguish between the limits (2.53) and (3.6). If we are in the limit (2.53) then $\sigma_i \to 0$ and the harmonic function will be logarithmic as in [34]. Otherwise it will be a function as in (3.6).

More details of these calculations can also be worked out easily following the analysis of [35]. However we will not do so here, and instead turn to the heterotic picture where the story is equally interesting.

4. Heterotic Kodaira surfaces

The discussion so far about type IIB theory suggested that even when we are at a point in the moduli space where bound states can appear, the background metric on a local patch is of the form

$$ds^2_{M} \sim dr_1^2 + (dz + f_1 \, dx + f_2 \, dy)^2 + |dz_1|^2 + |dz_2|^2,$$

(4.1)

with $dz_i$ being the two tori with complex coordinates defined above. In the limit where the D7 branes are far from the D5 branes, the $dz$ fibration is defined with

$$f_i(\theta_i) = \cot \theta_i, \quad \theta_i \neq 0$$

(4.2)

In the limit where we expect bound states, the functions $f_i$ are in general more complicated than (4.2) as described above. For the delocalised harmonic function that we took i.e. $h(z_1)$ – which could be linear or logarithmic depending on the limits (2.53) or (3.6) chosen – $f_i$ could be brought in the expected form if we also make $h$ a function of $r_1$. This is like inserting correct prefactors of $C(r_1)$ for every term as mentioned above.

Once such a starting point is made precise, the rest of the steps are straight-forward (up to possible subtleties mentioned in [17]). The duality chain gives rise to local solutions
in Type II and M-theories whose global metrics could possibly be constructed by joining all the local patches.

The heterotic story, on the other hand, is equally interesting. The conjectured local metric after geometric transition proposed in [21] was shown in [16] to have a global description that resembled the MN type metric [36] in some limits! Clearly this would mean that the global metric is the gravity dual to the theory on wrapped five-branes which, here, would mean the theory on wrapped heterotic NS5 branes.

Our next step was to look for possible metrics before geometric transition. Unfortunately there was no known duality chain (like the one that we had for the Type II to M-theory) that could help us here. A duality to F-theory via an orientifold corner of the moduli space was not very helpful to give us the complete global picture, and for the derivation used in [16] to go to the full global metric we had to rely on various correlated ideas and connections (see [16] for details). In addition to that, although there was no a priori reason to justify that there is a geometric transition in the heterotic theory, the existence of a “dual” metric resembling the MN type metric gave us a hint that maybe the theory on wrapped NS5 branes could also be described by a dual gravity theory.

That brought us to the next stage of determining the metric before geometric transition. This time however there was no known way to determine the global metric. The local metric was determined in [18] to be of the form:

\[
\begin{align*}
    ds^2 &= \mathcal{H}(r)^2 \, dr^2 + \mathcal{H}(r)^{-2} \left( dz + F_1 \, dx + F_2 \, dy \right)^2 + \mathcal{H}_1(r) \left( 1 - \sigma_0 \right) |dx + \tau_6 \, dy|^2 \\
    &\quad + \mathcal{H}_2(r) \, d_6 \left( \sec^2 \theta \left[ d\theta_1 + \sin 2\theta \left( a \, dx - b \, dy \right) \right]^2 + |\tau_2|^2 \sec^2 \tilde{\theta} \left[ d\theta_2 - \sin 2\tilde{\theta} \left( \tilde{a} \, dx - \tilde{b} \, dy \right) \right]^2 \right)
\end{align*}
\]

where the \( F_i \) are functions of \( \theta_i \); and the new complex structure \( \tau_6 \) of the \((x, y)\)–torus is a function of \( \tau_3, \tau_5 \) and \( \sigma_0 \) and determined by:

\[
\begin{align*}
    \Re \tau_6 &= \frac{\Re \tau_3 - \sigma_0 \Re \tau_5}{1 - \sigma_0}, & |\tau_6|^2 &= \frac{|\tau_3|^2 - \sigma_0 |\tau_5|^2}{1 - \sigma_0}.
\end{align*}
\]

The \((\theta_1, \theta_2)\)–torus is non–trivially fibered over the \((x, y)\)–torus, forming a specific family of Kodaira surfaces. The local manifold is therefore a \( \mathbb{C}^* \) fibration over Kodaira surfaces.

Once we determine the local geometry, the next question is to study the bundle structure. This is related to the fact that in the analysis of the geometry after geometric

\[\text{15 All the other coefficients are defined in [18].}\]
transition \[16\], the study of vector bundles showed us a non-singular way to pull the bundle across a conifold transition. This transition would give us a two-fold result: the full global geometry and the bundle structure. However our analysis in \[18\], as discussed above, only provided us with a local metric. Therefore it is important to study the vector bundle on this local geometry. The full analysis of bundle structure with torsion on a non-Kähler manifold is particularly involved, so we will go to the Calabi-Yau limit of the background, and study the bundles there\[16\].

In the ensuing sections, we shall construct rank 2 bundles \(\mathcal{E}\) on our Calabi-Yau three-fold \(X\), utilizing several techniques. Recall that \(X\) consists of a \(\mathbb{C}^*\) fibration over a primary Kodaira surface \(S\), a non-trivial holomorphic \(T^2\) fibration over a base \(T^2\). We denote the base \(T^2\) as \(B\), as in the diagram:

\[
\begin{array}{c}
\mathbb{C}^* \quad X \\
\downarrow \pi_1 \\
T^2 \quad S \\
\downarrow \pi_2 \\
B
\end{array}
\]

We shall consider three methods for constructing a bundle on \(X\); an intrinsic construction on \(X\), pulling back a bundle constructed on \(B\), and pulling back a bundle constructed on \(S\). The bundles produced by each construction have differing Chern classes, so that the appropriate method of construction differs from compactification to compactification.

4.1. The Atiyah bundle

We construct the first bundle by finding a rank 2 bundle \(E\) on the elliptic curve \(B\), and then pulling back by \(\pi_2 \circ \pi_1\). We shall require that the bundle satisfies the anomaly cancellation requirements \(c_1(X) = c_1(\mathcal{E})\) and \(c_2(X) = c_2(\mathcal{E})\).

\[16\] This is possible in special cases where both anomaly and non-Kählerity are canceled by switching on fluxes and gluino-condensates. See sec. 5 of \[16\] (and also \[37\]). This works because both the condensate term and the non-Kählerity term come with the same powers of \(\alpha'\). However, this cancellation is checked only to some small orders in \(\alpha'\), and the generic result to all orders has not been analysed.
Since $X$ is Calabi-Yau, we must have $c_1(E) = 0$. To deduce $c_2(X)$, we use the multiplicative property of Chern classes under exact sequences: if

$$0 \to E' \to E \to E'' \to 0 \quad (4.5)$$

is exact and $c(E)$ denotes the total Chern class of $E$,

$$c(E) = c(E') \cdot c(E''). \quad (4.6)$$

To compute the second Chern class of $X$, consider the exact sequence

$$0 \to T_{\pi_1} \to T_X \to \pi_1^* T_S \to 0. \quad (4.7)$$

Here $T_X$ and $T_S$ respectively denote the tangent bundles of $X$ and $S$, while $T_{\pi_1}$ designates directions tangent to the fibres of $\pi_1$, which are isomorphic to $\mathbb{C}^*$.

Recall that in [18], the threefold $X$ was constructed by specifying a torsion class $c \in H^2(S, \mathbb{Z})$ as the image under the coboundary map $\delta: H^1(S, \mathcal{O}_S^*) \to H^2(S, \mathbb{Z})$. As a torsional element, its image in $H^2(S, \mathbb{C})$ vanishes and therefore $c_1(T_{\pi_1}) = 0$. It immediately follows from (4.6) and (4.7) that $c_1(S) = 0$ and $c_2(X) = \pi_1^* c_2(S)$. However, $c_2(S)$ is the Euler characteristic of the Kodaira surface $S$, which is zero. It follows that $c_2(X) = 0$.

We turn now to the construction of a bundle on $B$ satisfying these constraints. For a given elliptic curve and for each natural number $n$, Atiyah [38] constructs a rank $n$ degree 0 indecomposable vector bundle $A_n$ with $h^0(B, A_n) = 1$. The bundles are defined recursively, with $A_n$ the unique (up to isomorphism) non-trivial extension of $\mathcal{O}_B$ by $A_{n-1}$:

$$0 \to \mathcal{O}_B \to A_n \to A_{n-1} \to 0. \quad (4.8)$$

Recursion begins with $A_1$ defined as $\mathcal{O}_B$. Indecomposable bundles over elliptic curves are semistable [39], so it follows that each $A_n$ is semistable. Since our interest lies with rank two bundles, we concentrate on $A_2$. Explicitly, $A_2$ comprises the unique extension of $\mathcal{O}_B$ by itself:

$$0 \to \mathcal{O}_B \to A_2 \to \mathcal{O}_B \to 0. \quad (4.9)$$

Uniqueness follows from the fact that $\text{Ext}^1(\mathcal{O}_B, \mathcal{O}_B) \cong H^1(B, \mathcal{O}_B) \cong \mathbb{C}$.

To compute the Chern classes of $A_2$, we exploit the exact sequence (4.9). Clearly, $c_i(A_2) = 0$ for $i \geq 2$ since $\dim \mathbb{C}B = 1$. Using the sequence (4.9), we compute

$$c_1(A_2) = 2c_1(\mathcal{O}_B) \quad (4.10)$$

and deduce that $c_1(A_2) = 0$. 

30
4.2. The Serre construction

In the previous section we considered a model with \( c_2(X) = c_2(\mathcal{E}) \), which doesn’t necessarily indicate a lack of \( H \)-torsion. Switching on \( H \)-torsion will imply that \( dH \neq 0 \), and that the spin connection is not embedded in the gauge connection \([10], [11], [3], [5]\).

The detailed analysis of switching on a \( H \) torsion and studying the bundle structure on a non-Kähler manifold is particularly involved\(^{17}\). Let us therefore consider a toy model in which \( c_2(X) \neq c_2(\mathcal{E}) \). In heterotic string theory, such a choice would generically lead to an anomaly. This anomaly could possibly be cancelled by non-local terms contributing to \( H \) at \( \mathcal{O}(\alpha') \). However, we haven’t analysed this situation for heterotic theories, and therefore we will only mention the following analysis as a toy example\(^{18}\). We will also assume that \( X \) is still the Calabi-Yau manifold described above.

In the following we would therefore like to utilize the Serre construction as outlined in sec. (4.1) of \([16]\). The construction requires an elliptic curve \( \mathcal{C} \) inside the Calabi-Yau threefold \( X \). In this section, the fibre \( \mathbb{T}^2 \) shall be referred to as \( E \).

Consider a point \( p \in B \), and denote the fibre over \( p \) as \( E_p = \pi_{2}^{-1}(p) \). We shall consider this fibre as a submanifold of \( S \), so that we may restrict the \( \mathbb{C}^* \) bundle structure of \( X \) to \( E_p \subset S \). If \( X|_{E_p} \) admits a global section \( s \), the image \( E' \) of \( E_p \) under \( s \) forms an elliptic curve in \( X|_{E_p} \), and thus one in \( X \) by inclusion. The diagram below succinctly captures this data:

\[
\begin{array}{c}
E' \xleftarrow{s} X \xrightarrow{\pi} \mathcal{C}^* \\
| \quad | \\
\pi \quad \pi \\
E_p \xleftarrow{s} S
\end{array}
\]

Because \( X \) comprises a principal \( \mathbb{C}^* \) bundle, sections over any set exist only when the bundle restricted to that set is trivial. Thus, we must verify that a bundle on \( S \) can

\(^{17}\) See \([7], [8]\) where bundle structure was addressed for the kind of models studied in \([2], [3], [3], [12]\).

\(^{18}\) Observe however that, if we pull a bundle through a geometric transition, the chern classes of the bundle before and after the transition will differ by the class of the curve on which the geometric transition is based. Before the transition, branes wrapping the curve allow cancellation of anomalies with \( c_2(E) \neq c_2(X) \). After the transition, we can then have \( c_2(X) = c_2(E) \), as required by the disappearance of the branes. More details on this will be presented elsewhere. We thank Ron Donagi and Eric Sharpe for discussion.
restrict to a trivial bundle on \( E_p \). We recall that a non-trivial \( \mathbb{C}^\ast \) bundle may be thought of as a trivial \( \mathbb{R} \) bundle over a non-trivial \( U(1) \) bundle. Let \( c_1 \in H^2(S, \mathbb{Z}) \) be the first Chern class of \( X \) as a principal \( U(1) \) bundle. Then, triviality of \( X \) on \( E_p \) is equivalent to the vanishing of \( c_1 \) when restricted to \( E_p \):

\[
c_1|_{E_p} \equiv c_1 \cdot [E_p] = 0. \tag{4.11}
\]

Our question as to the existence of a section is thus reduced to whether the structure of \( S \) affords enough freedom to select a \( c_1 \) to satisfy (4.11). Consider \( E_p \) as map \( H^2(S, \mathbb{Z}) \xrightarrow{E_p} H^4(S, \mathbb{Z}) \). It is known that \( H^2(S, \mathbb{Z}) \cong \mathbb{Z}^4 \oplus \mathbb{Z}_m \), and that \( H^4(S, \mathbb{Z}) \cong \mathbb{Z} \). Since \( E_p \) is a group homomorphism, all the torsion elements are automatically sent to zero. If \( S \) is torsionless, \( E_p \) sends \( \mathbb{Z}^4 \to \mathbb{Z} \), so it must have a non-trivial kernel. Thus for any \( p \), we can find a \( c_1 \in H^2(S, \mathbb{Z}) \) which vanishes on \( E_p \).

We can then apply the Serre construction to the elliptic curve \( E' \) to arrive at a rank 2 vector bundle \( V \) on \( X \) satisfying \( c_1(V) = 0 \) and \( c_2(V) = [E'] \neq 0 \).

4.3. Families of bundles

In this section, we will discuss how to obtain a family of rank-2 bundles on a Kodaira surface. The construction requires a preexisting holomorphic rank-2 bundle satisfying a stability condition. The idea stems from a series of papers (\[43\],\[44\],\[45\],\[46\]) that explored the existence and classification of stable rank-2 holomorphic vector bundles on non-Kähler elliptic surfaces. We will also relate a method for obtaining a bundle of arbitrary rank \( r \geq 2 \) on \( S \), as detailed in \[43\].

We first present the method of obtaining a bundle on \( S \). Consider a genus 2 curve \( C \), and let \( f : C \to B \) be a ramified covering of degree \( r \). Next, construct the fibre product of \( S \) and \( C \) over \( B \); \( Y \equiv S \times_B C \), and note that the projection \( \pi : Y \to S \) forms an \( r \)-fold cover. If we push a line bundle \( L \to Y \) forward to a sheaf \( \pi_*L \) on \( S \), we actually obtain a rank \( r \) vector bundle on \( S \) with Chern classes given by equation (2.1) of \[43\].

When searching for bundles with specified Chern classes, it is helpful to know whether or not they exist. The main result of \[43\] was to show that on primary Kodaira surfaces, holomorphic rank 2 vector bundles exist whenever

\[
\frac{1}{2} \left( c_2 - \frac{1}{4} c_1^2 \right) \geq 0. \tag{4.12}
\]
Observe that when constructing bundles on $S$ for use in Calabi-Yau compactifications with non-vanishing $H$-torsion, $c_2$ must be positive.

Before embarking on our construction of families, we reproduce the definitions of degree and stability for non-Kähler manifolds from [45]. Gauduchon [47] defines a metric conformally equivalent to any hermitian metric on a compact complex manifold $M$; a metric whose associated $(1,1)$ form $\omega$ satisfies $\partial \bar{\partial} \omega^{d-1} = 0$. Using this form, we define the degree of a line bundle $\mathcal{L}$ with curvature $\mathcal{R}$ to be

$$\deg \mathcal{L} = \int_M \mathcal{R} \wedge \omega^{d-1}. \quad (4.13)$$

The degree of any torsion-free coherent sheaf follows from the degree of its associated determinant bundle, and the slope is defined as its degree divided by its rank. A stable torsion-free coherent sheaf on $M$ is one for which every coherent subsheaf with lesser rank has lesser slope.

Primary Kodaira surfaces possess a naturally associated surface called the relative Jacobian of $S$: $J(S) = B \times T^\vee$, with $T^\vee$ the torus dual to the fibre torus of the surface. When we fix a bundle $\mathcal{E}$ on $S$, it picks out a special divisor $S_{\mathcal{E}}$ in the Jacobian called the spectral curve. This divisor is defined using only the bundle and intrinsic data of the surface: see section 2.2 of [46] for details.

Now, we follow the construction from section 4.2 of [46]. Fixing a bundle $\mathcal{E}$, we obtain the double cover $S_{\mathcal{E}} \to B$. Note that Kodaira surfaces are not multiply fibred, so the normalisation $W \cong S \times_B S_{\mathcal{E}}$, and the maps $\bar{\gamma}$ and $\rho$ are just projections:

$$\begin{array}{ccc}
S \times_B S_{\mathcal{E}} & \xrightarrow{\bar{\gamma}} & S \\
\rho \downarrow & & \downarrow \\
S_{\mathcal{E}} & \to & B
\end{array}$$

The cover $S_{\mathcal{E}} \to B$ tells us that $W \to S$ is a double cover as well. Furthermore, $S_{\mathcal{E}}$ naturally induces a line bundle $L$ on $W$. If we push this line bundle forward to a sheaf on $S$, we obtain a bundle $\delta \to S$ by taking its determinant: $\delta = \text{det}(\bar{\gamma}_* L)$. Defining $\bar{\iota}$ as the involution on the Picard group of $W$ induced by exchanging the two sheets of the covering $W \to X$, we pull the Picard group of $S_{\mathcal{E}}$ back to $W$, $\rho^* \text{Pic}(S_{\mathcal{E}}) \subset \text{Pic}(W)$ and take the following subgroup:

$$P := \left\{ \lambda \in \rho^* \text{Pic}(S_{\mathcal{E}}) \mid \bar{\iota}^* \lambda \otimes \lambda = \mathcal{O}_W \text{ and } \bar{\gamma}_*(c_1(\lambda)) = 0 \text{ in } H^2(S, \mathbb{Z}) \right\}. \quad (4.14)$$
Then, every rank 2 vector bundle on $X$ with determinant $\delta$ (that is, fixed $c_1 = c_1(\delta)$) and spectral cover $S_\mathcal{E}$ is obtained as $\bar{\gamma}_* (L \otimes \lambda)$, $\lambda \in P$. One can show that $P$ is isomorphic to the Prym variety $\text{Prym}(S_\mathcal{E}/B)$ associated to the covering $S_\mathcal{E} \to B$. By Proposition 3.2 of [45], if $\mathcal{E}$ is stable, then all bundles in the family are stable.

5. Summary and discussion

Our main aim in this paper was to analyse the metric of D5 branes wrapped on some two-cycle of a local geometry in the regime where the D5 branes form a bound state with the seven branes. Our earlier study of the local metric done in [17], [21] and [18] were always away from the D7 brane flavors. We did make some attempt in [18] to determine the full metric using an order-by-order expansion, but could only analyse the effects without incorporating the backreactions from the full bound-state configuration. Accordingly, the dipole deformation was also approximate. Nevertheless we predicted in [18] that the local metric with all possible backreactions and dipole deformation would resemble (2.24).

A direct study of this using equations of motion seemed more difficult this time because the backreactions involve, among other things, brane worldvolume terms. To solve this problem we devised a set of duality transformations that used aspects of U-dualities, gauge-gravity dualities and certain strong-coupling dynamics. Using these, the resulting analysis of the actual configuration turned out to be richer than expected. Studying various limiting procedures gave us an indication that:

- There are multiple ways to perform dipole deformations here, resulting in different warped geometries like eqns (3.14) and (3.24). These metrics respectively differ from (2.24) precisely by the limits (2.53) and (3.6). Once such limits are applied, the metrics take the conjectured form (2.24), giving us the final:

- Dipole-deformed metrics given by eqns (3.18) and (3.25).

We see that any of these metrics could be taken as the starting point in the duality cycle of [17], in the regime where we would be interested in considering the flavors together. The non-Kählerity in type IIA for each of these cases could be easily determined from the

- Dipole-deforming $B_{NS}$ fields given by eqns (3.16) and (3.26)

respectively along with the RR fields (which we left for the reader to derive). The above choices of background anti-symmetric fields differ from the choices that we took in [17].
and [21], because we are in the regime where the deformations are done by the dipoles. We could easily go away from this regime and study the theory with non-commutative deformations, for example. All these analyses are easy to perform now, because we have described a standard way to derive the metric configurations. Of course, one definite advantage of dipole deformations, as emphasised earlier in [18], is to observe that

- The volumes of the two-cycle shrink for both kinds of dipole deformations.

The type II story is now more or less complete, although the heterotic side is far from clear. The bundle structure and global metric forming a possible \( d \)ual to the wrapped NS5 branes in the heterotic theory have already been evaluated in [16]. In [18], we studied the local geometry before geometric transition and found that the metric is a particular \( \mathbb{C}^* \) fibration over Kodaira surfaces. The heterotic NS5 branes wrap two-cycles of this geometry giving rise to non-trivial torsion classes (see sec. (3.2) of [18]). In this paper we

- Construct vector bundles on \( \mathbb{C}^* \) fibration over Kodaira surfaces,

thus confirming that such a local solution may indeed be a candidate manifold for gauge/gravity dualities, although a direct calculation still needs to be done\(^{19}\). The last link of the story is to see whether the heterotic manifold has a global completion in which the base tori are deformed to one or more non-singular \( \mathbb{P}^1 \)s. This and other issues will be addressed elsewhere.

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\(^{19}\) It may be helpful to use string-string dualities to study such issues. A recent paper dealing with Type IIA/Heterotic theory in the presence of torsion is [18]. Although our heterotic geometry is not U-dual to any type IIB background, it may still be dual to some type IIA geometry. It will be interesting to exploit this angle.
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