FETI-DP preconditioners for 2D Biot model with discontinuous Galerkin discretization

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Abstract Dual-primal FETI (FETI-DP) preconditioners are developed for a 2D Biot model. The model is formulated with mixed-finite elements as a saddle-point problem. The displacement $u$ and the Darcy flux flow $z$ are represented with $P_1$ piecewise continuous elements and pore-pressure $p$ with $P_0$ piecewise constant elements, i.e., overall three fields with a stabilizing term. We have tested the functionality of FETI-DP with and without Dirichlet preconditioners. Numerical experiments show a signature of scalability of the resulting parallel algorithm in the compressible elasticity with permeable Darcy flow as well as almost incompressible elasticity.

1 Introduction

Poroelasticity, i.e., elasticity of porous media with permeated Darcy flow, pioneered by Biot [1, 2] has been used broadly in geoscience [3] and biomechanics [4, 5, 6] among many others. The difficulties for solving the linear elasticity and incompressible flow problems also arise in solving the poroelastic problem, and there have been diverse mathematical formulations and discretizations. When a continuous Galerkin approach was formulated with mixed finite elements with three-fields of displacement, Darcy flow flux, and pressure [7], the main numerical difficulties are elastic locking and non-physical oscillatory pressure profiles. There have been some new methods for dealing with these difficulties; for example, continuous Galerkin with non-standard three-fields of displacement, pressure, and volumetric stress [8], discontinuous Galerkin formulations [9][10] with standard three-fields as well as non-conforming mixed finite elements [10][11]. When lowest-order finite elements are applied, stabilizing terms should be added [12][13][14] to satisfy the inf-sup condition [15][16].

In this paper, we propose a numerical scheme for solving the Biot model with three-fields linear poroelasticity. We consider a discontinuous Galerkin discretization, i.e., the displacement and Darcy flow flux discretized as piecewise continuous in $P_1$ elements, and the pore pressure as piecewise constant in the $P_0$ space with a stabilizing term. The emerging formulation is a saddle-point problem, and more specifically, a twofold saddle-point problem. This indefinite system is computational challenging with slow convergence in iterative methods. It is necessary to incorporate relevant preconditioners for saddle-point problems [17][18].
FETI-DP preconditioners transform indefinite problems to positive definite interface problems of Lagrangian multipliers for subdomains and a primal problem for the coarse space [19]. They have been applied for linear elasticity [22, 25] and incompressible Stokes flows [20, 21, 23, 24] as saddle-point problems. There are theoretical boundedness of condition numbers in the preconditioned systems independent to partitioned subdomains. We show numerical scalability of GMRES preconditioned by FETI-DP preconditioners for the three-fields Biot model discretized with stabilized $P_1 - P_1 - P_0$.

2 Linear poroelastic model

Poroelastic models describe the interaction of fluid flows and deformable elastic porous media saturated in the fluid. Let $u$ be the elastic displacement, $u_f$ be the fluid velocity, $p$ be the pore-pressure. We assume that the permeability is homogeneous: $K = \kappa I$. Denote $z = \phi(u_f - \partial_t u)$ as the Darcy volumetric fluid flux. The quasi-static Biot model reads as:

$$
-(\lambda + \mu) \nabla (\nabla \cdot u) - \mu \nabla^2 u + \alpha \nabla p = f \tag{1}
$$

$$
K^{-1} z + \nabla p = 0, \tag{2}
$$

$$
\frac{\partial}{\partial t} (\alpha \nabla \cdot u + c_0 p) + \nabla \cdot z = g. \tag{3}
$$

The first equation is the moment conservation. The second equation is Darcy’s law. The third equations is the mass conservation equation. For simplicity, we neglect the effects of gravity acceleration. In the above equations, $f$ is the body force, $g$ is a source or sink term, $c_0 > 0$ is the constrained specific storage coefficient, $\alpha$ is the Biot-Willis constant which is close to 1. $\lambda$ and $\mu$ are the first and second Lamé parameters, respectively.

We consider $\Omega \in \mathbb{R}^2$ as a bounded domain. For the ease of presentation, we consider mixed partial Neumann and partial Dirichlet boundary conditions in this paper. Specifically, the boundaries for $u$ and $p$ are divided into

$$
\partial \Omega = \Gamma_d \cup \Gamma_t \quad \text{and} \quad \partial \Omega = \Gamma_p \cup \Gamma_f.
$$

Here, $\Gamma_d$ and $\Gamma_t$ are the Dirichlet boundary and the Neumann boundary for elastic part, respectively; $\Gamma_p$ and $\Gamma_f$ are the Dirichlet boundary and the Neumann boundary for fluid, respectively. The boundary conditions are the following:

$$
\begin{cases}
  u = 0 & \text{on } \Gamma_d, \\
  (\sigma(u) - \alpha p I) \cdot n = t & \text{on } \Gamma_t, \\
  p = 0 & \text{on } \Gamma_p, \\
  z \cdot n = g_2 & \text{on } \Gamma_f.
\end{cases} \tag{4}
$$

For simplicity, the Dirichlet conditions are assumed to be homogeneous.
3 Formulation of the Biot model as a saddle-point problem

3.1 Weak formulation

We define the following spaces for displacement, Darcy fluid flux, and pore pressure:

\[ W^u(\Omega) := \{ u \in (H^1(\Omega))^3 : u = 0 \text{ on } \Gamma_D \}, \]
\[ W^z(\Omega) := \{ z \in (H^1_0(\Omega))^3 : z \cdot n = 0 \text{ on } \Gamma_F \}, \]
\[ \mathcal{L}(\Omega) := \begin{cases} L^2(\Omega) & \text{if } \Gamma_t \cup \Gamma_p \neq \emptyset, \\ L^2_{0}(\Omega) & \text{if } \Gamma_t \cup \Gamma_p = \emptyset, \end{cases} \]

The continuous weak problem is to find \( u \in W^u(\Omega), z \in W^z(\Omega), \) and \( p \in \mathcal{L}(\Omega) \) for \( t \in [0, T] \) such that

\[
\begin{align*}
 a(u, v) - (\alpha p, \nabla \cdot v) &= (f, v) + (t, v)_{\Gamma_t}, \quad \forall v \in W^u(\Omega), \\
 (K^{-1} z, w) - (p, \nabla \cdot w) &= (b, w) \quad \forall w \in W^z(\Omega), \\
 (\alpha \nabla \cdot u, q) + (\nabla \cdot z, q) &= (g, q) \quad \forall q \in \mathcal{L}(\Omega),
\end{align*}
\]

where

\[
 a(u, v) = \int_{\Omega} 2\mu(\varepsilon(u) : \varepsilon(v)) + \lambda(\nabla \cdot u)(\nabla \cdot v) \, dx.
\]

3.2 Discrete formulation: \( \mathbf{P}_1 - \mathbf{P}_1 - P_0 \)

We apply finite element method with domains normally shaped as triangles in \( \mathbb{R}^2 \). Let \( T_h \) be a partition of \( \Omega \) into non-overlapping elements \( K \). We denote by \( h \) the size of the largest element in \( T_h \). On the given partition \( T_h \) we apply the following finite element spaces [12]:

\[ V_h := \{ u_h \in (C^0(\Omega))^3 : u_h|_K \in \mathbf{P}_1(K) \forall K \in T_h, u_h = 0 \text{ on } \Gamma_D \}, \]
\[ W_h := \{ z_h \in (C^0(\Omega))^3 : z_h|_K \in \mathbf{P}_1(K) \forall K \in T_h, z_h \cdot n = 0 \text{ on } \Gamma_F \}, \]
\[ Q_h := \{ p_h : p_h|_K \in \mathbf{P}_0(K) \forall K \in T_h \}. \]

The problem is to find \((u^n_h, z^n_h, p^n_h) \in V_h \times W_h \times Q_h\) such that

\[
\begin{align*}
 a(u^n_h, v_h) - (p^n_h, \nabla \cdot v_h) &= (f^n, v_h) + (t^n, v_h)_{\Gamma_t}, \quad \forall v_h \in V_h \\
 (K^{-1} z^n_h, w_h) - (p^n_h, \nabla \cdot w_h) &= (b^n, w_h), \quad \forall w_h \in W_h \\
 (\nabla \cdot u^n_h, q_h) + (\nabla \cdot z^n_h, q_h) + \delta_{\text{STAB}} (p^n_h, q_h) + J(p^n_{\Delta t, h}, q_h) &= \frac{1}{\alpha} (g^n, q_h), \quad \forall q_h \in Q_h
\end{align*}
\]

where

\[
 J(p, q) = \delta_{\text{STAB}} \sum_K \int_{\partial K \cap \partial \Omega} h_{\partial K} |p| q \, ds
\]

is a stabilizing term [13], and \( p^n_{\Delta t, h} = (p^n_h - p^{n-1}_h)/\Delta t \). The finite element discretization will lead to a twofold saddle-point problem of the following form:
Let us represent dual interface spaces\

\[ \begin{bmatrix} A & 0 & B_1^T \\ 0 & A & B_2^T \\ B_1 & B_2 & -A_p \end{bmatrix} \begin{bmatrix} u_h \\ z_h \\ p_h \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}. \]  (16)

4 FETI-DP formulation for Biot model with discontinuous pressure field

4.1 FETI-DP algorithm for Biot model: interior and interface spaces

We decompose the discrete displacement space \( V \), Darcy flux space \( W \) into interior and interface spaces, and the discontinuous pressure space \( Q \) into interior and constant spaces:

\[
\begin{align*}
V &= V_1 \oplus V_I, \\
W &= W_1 \oplus W_I, \\
Q &= Q_I \oplus Q_0,
\end{align*}
\]  (17-19)

where \( V_1, \ W_1, \ Q_I \) are the direct sums of subdomain interior spaces:

\[
\begin{align*}
V_1 &= \oplus_{i=1}^N V_i^1, \\
W_1 &= \oplus_{i=1}^N W_i^1, \\
Q_I &= \oplus_{i=1}^N Q_i^I,
\end{align*}
\]  (20-22)

and \( Q_0 \) is from the subdomain constant pressure parts.

4.2 FETI-DP algorithm for Biot model: primal and dual variables

\[
V_I = V_\Delta \oplus V_{II} = (\oplus_{i=1}^N V_\Delta^i) \oplus V_{II},
\]  (23)

where \( V_{II} \) is the continuous, coarse level, and primal space. \( V_\Delta \) is the direct sum of independent subdomain dual interface spaces \( V_\Delta^i \) \((23)\). Similarly \( W_I \) is decomposed to \( W_\Delta \) and \( W_{II} \).

Let us represent \( u \) and \( z \) together as \( U = (u, z) \in V \times W \). The problem turns out to find \((u_I, z_I, p_I, u_{II}, z_{II}, u_\Delta, z_\Delta, p_0)\) \(\in (V_1, W_1, Q_I, V_{II}, W_{II}, V_\Delta, W_\Delta, Q_0)\) such that

\[
\begin{align*}
\begin{bmatrix}
A_{II} & B_{II}^T & A_{III}^T & A_{II}^T \\
B_{II} & 0 & B_{III} & B_{II} \\
A_{III} & B_{II}^T & A_{III} & A_{III}^T \\
A_{II} & B_{II}^T & A_{II} & B_{II}^T \\
0 & 0 & B_{III} & B_{II} \\
A_{III} & B_{II}^T & A_{III} & A_{III}^T \\
0 & 0 & B_{III} & B_{II} \\
0 & 0 & B_{III} & B_{II} \\
0 & 0 & B_{III} & B_{II}
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_{II} \\
U_\Delta
\end{bmatrix}
= \begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}
\end{align*}
\]  (24)
4.3 FETI-DP algorithm for Biot model: Schur complement

By defining a Schur complement operator $\tilde{S}$ as

$$
\begin{bmatrix}
A_{II} & B_{II}^T & A_{MM}^T & 0 \\
B_{II} & 0 & B_{II}^T & 0 \\
A_{III} & B_{III}^T & A_{MM}^T & B_{III}^T \\
0 & 0 & B_{II}^T & 0 \\
A_{MII} & B_{MII}^T & A_{MM} & B_{MII}^T
\end{bmatrix}
\begin{bmatrix}
U_1 \\
p_1 \\
U_{II} \\
p_0 \\
U_\Lambda
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
SU_\Lambda
\end{bmatrix}.
$$

(25)

We introduce Lagrange multiplier $\lambda$ and the jump operator $B_\Delta$ to enforce the continuity of $U_\Lambda$ across $\Gamma$ [20]:

$$
\begin{bmatrix}
\tilde{S} & B_\Delta^T \\
B_\Delta & 0
\end{bmatrix}
\begin{bmatrix}
U_\Delta \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
f_\Delta^* \\
0
\end{bmatrix}
$$

(26)

The problem is reduced to find $\lambda \in \Lambda = B_\Delta U_\Lambda$ such that

$$
B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta \tilde{S}^{-1} f_\Delta^*.
$$

(27)

This is solved by Preconditioned Conjugate Gradient (PCG).

4.4 Dirichlet preconditioner

We define a Schur complement operator, the discrete Harmonic $H_\Delta^{(i)}$ on $\Omega_i$ as follows:

$$
\begin{bmatrix}
A_{II}^{(i)} & A_{MI}^{(i)} \\
A_{MI}^{(i)} & A_{MM}^{(i)}
\end{bmatrix}
\begin{bmatrix}
U_1^{(i)} \\
U_\Delta^{(i)}
\end{bmatrix} =
\begin{bmatrix}
0 \\
H_\Delta^{(i)} U_\Lambda^{(i)}
\end{bmatrix}.
$$

(28)

The Dirichlet preconditioner is formulated in the following:

$$
M_{\Delta,D}^{-1} = B_{\Delta,D} H_{\Delta,D} B_{\Delta,D}^T,
$$

(29)

where $B_{\Delta,D}$ is scaled operator from $B_\Delta$ with the number of subdomains sharing each node in the interface $\Gamma$, and $H_{\Delta}$ is the direct sum of $H_\Delta^{(i)}$ [24].

5 Numerical experiments

A test problem is formulated with $\alpha = 1$, $c_0 = 0$, and $K = kI$. $\Omega = [0,1]^2$ and $t \in [0,0.25]$:

$$
-(\lambda + \mu) \nabla (\nabla \cdot u) - \mu \nabla^2 u + \nabla p = 0,
K^{-1} z + \nabla p = 0,
\nabla \cdot (u_t + z) = g_1.
$$

(30)
The involving initial and boundary conditions are the following:

\[
\begin{align*}
\mathbf{u} &= 0 \quad \text{on } \partial \Omega = \Gamma_D, \\
\mathbf{z} \cdot \mathbf{n} &= g_2 \quad \text{on } \partial \Omega = \Gamma_F, \\
\mathbf{u}(x, 0) &= 0, \quad x \in \Omega, \\
\mathbf{p}(x, 0) &= 0, \quad x \in \Omega.
\end{align*}
\]  

(31)

We consider the following analytic solution:

\[
\begin{align*}
\mathbf{u} &= \frac{-1}{4\pi(\lambda+2\mu)} \begin{bmatrix} 
\cos(2\pi x) \sin(2\pi y) \\
\sin(2\pi x) \cos(2\pi y)
\end{bmatrix}, \\
\mathbf{z} &= -2\pi k \begin{bmatrix} 
\cos(2\pi x) \sin(2\pi y) \\
\sin(2\pi x) \cos(2\pi y)
\end{bmatrix}, \\
\mathbf{p} &= \sin(2\pi x) \sin(2\pi y),
\end{align*}
\]  

(32)

and derive the compatible source term as follows:

\[
g_1 = \frac{2\pi}{\lambda+2\mu} \sin(2\pi x) \cos(2\pi y) + 8\pi^2 k \sin(2\pi x) \sin(2\pi y).
\]  

(33)

5.1 Numerical implementation

In the implementation of finite elements, we use a finite element library, libMesh \[27\]. We apply triangular element with 3 nodes. For domain partitioning, we apply ParMETIS \[31\]. Krylov subspace iterative main solver of GMRES and FETI-DP preconditioners are based on PETSc \[28, 29, 30\] and KSPFETIDP and PCBDDC classes within PETSc \[26\]. The initial guess is zero and the stopping criterion is set to be \(10^{-8}\), the reduction of the residual norm. The stabilizing factor \(\delta_{\text{STAB}} = 100\), the time-stepping is \(dt = 0.00625\), and Young’s modulus \(E = 1000\) Pa. In each test, we count the iteration of the GMRES solver.

5.2 Scalability of FETI-DP preconditioners

Scalability of FETI-DP preconditioning for the Biot model is tested with increasing number of subdomains \(N\). The subdomain size is set with \(H/h = 8\). In the first case, \((\nu = 0.3, k = 10^{-7})\) of compressible elasticity and permeable Darcy flow is tested without Dirichlet preconditioner. As shown in the first column of Table 1, GMRES iteration numbers are bounded when subdomains were increased from \(2 \times 2 \times 8 \times 8\). In the second case, \((\nu = 0.4999, k = 10^{-7})\) of almost incompressible elasticity and less permeable Darcy flow is tested without Dirichlet preconditioner. GMRES iteration number are larger than the first case, but still bounded while subdomains are increased from \(2 \times 2 \times 8 \times 8\), showing no issues of elastic locking (the second column in Table 1). This is consistent with a theoretical scalability of FETI-DP for almost incompressible elasticity \[32\]. In the third case, the same poroelastic properties of Poisson ratio and permeability is given with Dirichlet preconditioner. As shown in the third column of Table 1, GMRES iteration number were a bit larger than the case without Dirichlet preconditioner. More details needs to be analyzed, but one possibility...
is that $A_\nu$ is elliptic, but $A_z$ is isomorphic to identity matrix with scaling. The harmonic extension $H\Lambda$ in Eq. (28) is supposed to be ineffective for the block diagonal matrix structured with $A_\nu$ and $A_z$.

Table 1 Scalability of the FETI-DP preconditioning for the saddle-point problem of Biot model. Iteration counts for increasing number of subdomains $N$. The first two columns are without Dirichlet preconditioner and the third one is with Dirichlet preconditioner. Fixed $H/h = 8, \delta_{\text{STAB}} = 100, dt = 0.00625$, and $E = 1000$.

| $N$ | $\nu = 0.3$ | $\nu = 0.4999$ | $\nu = 0.4999, k = 10^{-7}$ |
|-----|--------------|----------------|--------------------------|
|     | k = $10^{-2}$ | k = $10^{-7}$ | Dirichlet preconditioner |
| 2 x 2 | 4 | 8 | 11 |
| 3 x 3 | 5 | 21 | 21 |
| 4 x 4 | 5 | 11 | 15 |
| 5 x 5 | 6 | 23 | 24 |
| 6 x 6 | 6 | 23 | 23 |
| 7 x 7 | 6 | 24 | 24 |
| 8 x 8 | 5 | 14 | 16 |

6 Conclusion

We have explored the scalability of FETI-DP for the 2D Biot model. Upon numerical scalabilities of compressible elasticity with Darcy’s flow as well as almost incompressible elasticity with limited Darcy’s flow, it remains to test parameter robustness, possibly in the presence of heterogeneity of parameters. Overall, the numerical results are a foundation for further advancement of scalable FETI-DP / BDDC preconditioners for poroelastic large deformation.

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