Complete integrable systems with unconfined singularities

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Abstract

We prove that any globally periodic rational discrete system in $\mathbb{K}^k$ (where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$,) has unconfined singularities, zero algebraic entropy and it is complete integrable (that is, it has as many functionally independent first integrals as the dimension of the phase space). In fact, for some of these systems the unconfined singularities are the key to obtain first integrals using the Darboux-type method of integrability.

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Singularity confinement property in integrable discrete systems was first observed by Grammaticos, Ramani and Papageorgiou in [1], when studying the propagation of singularities in the lattice KdV equation $x_{j+1}^{i+1} = x_{j+1}^{i-1} + 1/x_{j+1}^i - 1/x_{j+1}^{i+1}$, and soon it was adopted as a detector of integrability, and a discrete analogous to the Painlevé property (see [2, 3] and references therein). It is well known that some celebrated discrete dynamical systems (DDS from now on) like the McMillan mapping and all the discrete Painlevé equations satisfy the singularity confinement property [1,4]. In [5, p. 152] the authors write: “Thus singularity confinement appeared as a necessary condition for discrete integrability. However the sufficiency of the criterion was not unambiguously established”. Indeed, numerical chaos has been detected in maps satisfying the singularity confinement property [6]. So it is common knowledge that singularity confinement is not a sufficient condition for integrability, and some complementary conditions, like the algebraic entropy criterion have been proposed to ensure sufficiency [7,8].

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On the other hand a DDS can have a first integral and do not satisfy the singularity confinement property, as shown in the following example given in [9]: Indeed, consider the DDS generated by the map $F(x, y) = (y, y^2/x)$ which has the first integral given by $I(x, y) = y/x$. Recall that a first integral for a map $F : \text{dom}(F) \in \mathbb{K}^k \to \mathbb{K}^k$, is a $\mathbb{K}$-valued function $H$ defined in $U$, an open subset of $\text{dom}(F) \in \mathbb{K}^k$, satisfying $H(F(x)) = H(x)$ for all $x \in U$.

The above example shows that singularity confinement is not a necessary condition for integrability if “integrability” means the existence of a first integral. The first objective of this letter is to point out that more strong examples can be constructed if there are considered globally periodic analytic maps. A map $F : U \subseteq \mathbb{K}^k \to U$ is globally $p$-periodic if $F^p \equiv \text{Id}$ in $U$. Global periodicity is a current issue of research see a large list of references in [10, 11].

Indeed there exist globally periodic maps with unconfined singularities, since global periodicity forces the singularity to emerge after a complete period. However from [10, Th.7] it is know that an analytic and injective map $F : U \subseteq \mathbb{K}^k \to U$ is globally periodic if and only if it is complete integrable, that is, there exist $k$ functionally independent analytic first in $U$). Note that there is a difference between the definition of complete integrable DDS and the definition of complete integrable continuous DS: For the later case the number of functionally independent first integrals has to be just $k - 1$, which is the maximum possible number; see [13]. This is because the foliation induced by the $k - 1$ functionally independent first integrals generically have dimension 1 (so this fully determines the orbits of the flow). Hence, to fully determine the orbits of a DDS, the foliation induced by the first integrals must have dimension 0, i.e. it has to be reduced to a set of points, so we need an extra first integral.

In this letter we only want to remark that there exist complete integrable rational maps with unconfined singularities and zero algebraic entropy (Proposition 1), and that these unconfined singularities and its pre–images (the forbidden set) in fact play a role in the construction of first integrals (Proposition 2) for some globally periodic rational maps. Prior to state this result we recall some definitions. In the following $F$ will denote a rational map.

Given $F : U \subseteq \mathbb{K}^k \to U$, with $F = (F_1, \ldots, F_k)$, a rational map, denote by

$$S(F) = \{ x \in \mathbb{K}^k \text{ such that } \text{den}(F_i) = 0 \text{ for some } i \in \{1, \ldots, k\} \},$$

the singular set of $F$. A singularity for the discrete system $x_{n+1} = F(x_n)$ is a point $x_* \in S(F)$. The set

$$\Lambda(F) = \{ x \in \mathbb{K}^k \text{ such that there exists } n = n(x) \geq 1 \text{ for which } F^n(x) \in S(x) \},$$
is called the forbidden set of $F$, and it is conformed by the set of the preimages of the singular set. If $F$ is globally periodic, then it is bijective on the good set of $F$, that is $\mathcal{G} = \mathbb{K}^k \setminus \Lambda(F)$ (see [11] for instance). Moreover $\mathcal{G}$ is an open full measured set ([12]).

A singularity is said to be confined if there exists $n_0 = n_0(x_s) \in \mathbb{N}$ such that $\lim_{x \to x_s} F^{n_0}(x)$ exists and does not belong to $\Lambda(F)$. This last conditions is sometimes skipped in the literature, but if it is not included the “confined” singularity could re-emerge after some steps, thus really being unconfined.

Rational maps on $\mathbb{K}^k$ extend to homogeneous polynomial maps on $\mathbb{K}P^k$, acting on homogeneous coordinates. For instance, the Lyness’ Map $F(x, y) = (y, (a + y)/x)$, associated to celebrated Lyness’ difference equation $x_{n+2} = (a + x_{n+1})/x_n$, extends to $\mathbb{K}P^2$ by $F_p[x, y, z] = [xy, az^2 + yz, xz]$. Let $d_n$ denote the degree of the $n$–th iterate of the extended map once all common factors have been removed. According to [7], the algebraic entropy of $F$ is defined by $E(F) = \lim_{n \to \infty} \log (d_n)/n$.

The first result of the paper is

**Proposition 1.** Let $F : \mathcal{G} \subseteq \mathbb{K}^k \to \mathcal{G}$ be a globally $p$–periodic periodic rational map. Then the following statements hold.

(a) $F$ has $k$ functionally independent rational first integrals (complete integrability).

(b) All the singularities are unconfined.

(c) The algebraic entropy of $F$ is zero.

**Proof.** Statement (a) is a direct consequence of [10] Th.7 whose proof indicates how to construct $k$ rational first integrals using symmetric polynomials as generating functions.

(b) Let $x_s \in S(F)$, be a confined singularity of $F$ (that is, there exists $n_0 \in \mathbb{N}$ such that $x_{\{n_0, s\}} := \lim_{x \to x_s} F^{n_0}(x_s)$ exists and $x_{\{n_0, s\}} \notin \Lambda(F)$). Consider $\epsilon \simeq 0 \in \mathbb{K}^k$, such that $x_s + \epsilon \notin \Lambda(F)$ (so that it’s periodic orbit is well defined). Set $x_{\{n_0, s, \epsilon\}} := F^{n_0}(x_s + \epsilon)$. The global periodicity in $\mathbb{K}^k \setminus \Lambda(F)$ implies that there exists $l \in \mathbb{N}$ such that $F^{lp-n_0}(x_{\{n_0, s, \epsilon\}}) = F^{lp}(x_s + \epsilon) = x_s + \epsilon$, hence

$$\lim_{\epsilon \to 0} F^{lp-n_0}(x_{\{n_0, s, \epsilon\}}) = \lim_{\epsilon \to 0} x_s + \epsilon = x_s.$$ 

But on the other hand

$$\lim_{\epsilon \to 0} F^{lp-n_0}(x_{\{n_0, s, \epsilon\}}) = F^{lp-n_0}(x_{\{n_0, s\}}).$$

Therefore $x_{\{n_0, s\}} \in \Lambda(F)$, which is a contradiction.

(c) Let $\bar{F}$ denote the extension of $F$ to $\mathbb{K}P^k$. $\bar{F}$ is $p$–periodic except on the set of pre-images of $[0, \ldots, 0]$ (which is not a point of $\mathbb{K}P^k$), hence $d_{n+p} = d_n$ for all $n \in \mathbb{N}$ (where $d_n$ denote the degree of the $n$–th iterate once all factors have been removed). Therefore $E(F) = \lim_{n \to \infty} \log (d_n)/n = 0$.  

\[\square\]
As an example, consider for instance the globally 5–periodic map \( F(x, y) = (y, (1+y)/x) \), associated to the Lyness’ difference equation \( x_{n+2} = (1 + x_{n+1})/x_n \), which is posses the unconfined singularity pattern \( \{0, 1\infty, \infty, 1\} \). Indeed, consider an initial condition \( x_0 = (\varepsilon, y) \) with \( |\varepsilon| \ll 1 \), and \( y \neq -1, y \neq 0 \) and \( 1 + y + \varepsilon \neq 0 \) (that is, close enough to the singularity, but neither in the \( \mathcal{S}(F) \) nor in \( \Lambda(F) \)). Then \( x_1 = F(x_0) = (y, (1+y)/\varepsilon) \), \( x_2 = F(x_1) = ((1+y)/\varepsilon, (1+y+\varepsilon)/(\varepsilon y)) \), \( x_3 = F(x_2) = ((1+y+\varepsilon)/(\varepsilon y), (1+\varepsilon)/y) \), and \( x_4 = F(x_3) = ((1+\varepsilon)/y, \varepsilon) \), and finally \( x_5 = F(x_4) = x_0 \). Therefore the singularity is unconfined since it propagates indefinitely.

But the Lyness’ equation is complete integrable since it has the following two functionally independent first integrals [10]:

\[
\begin{align*}
H(x, y) &= xy^4 + p_3(x)y^3 + p_2(x)y^2 + p_1(x)y + p_0(x), \\
I(x, y) &= \frac{(1 + x)(1 + y)(1 + x + y)}{xy},
\end{align*}
\]

Where \( p_0(x) = x^3 + 2x^2 + x \), \( p_1(x) = x^4 + 2x^3 + 3x^2 + 3x + 1 \), \( p_2(x) = x^3 + 5x^2 + 3x + 2 \), \( p_3(x) = x^3 + 2x^2 + 2x + 1 \). The extension of \( F \) to \( \mathbb{C}P_2 \) is given by \( \tilde{F}[x, y, z] = [xy, z(y+z), xz] \), which is again 5–periodic, hence \( d_n = d_{n+5} \) for all \( n \in \mathbb{N} \), and the algebraic entropy is \( E(F) = \lim_{n\to\infty} \log (d_n)/n = 0 \).

More examples of systems with complete integrability, zero algebraic entropy and unconfined singularities, together with the complete set of first integrals can be found in [10].

The second objective of this letter is to notice that the unconfined singularities can even play an essential role in order to construct a Darbouxian–type first integral of some DDS, since they can help to obtain a closed set of functions for their associated maps. This is the case of some rational globally periodic difference equations, for instance the ones given by

\[
\begin{align*}
x_{n+2} &= \frac{1 + x_{n+1}}{x_n}, \\
x_{n+3} &= \frac{1 + x_{n+1} + x_{n+2}}{x_n}, \text{ and } \ x_{n+3} = \frac{-1 + x_{n+1} - x_{n+2}}{x_n},
\end{align*}
\]

To show this role we apply the Darboux–type method of integrability for DDS (developed in [14] and [15] Appendix) to find first integrals for maps.

Set \( F : \mathcal{G} \subseteq \mathbb{K}^k \to \mathbb{K}^k \). Recall that a set of functions \( \mathcal{R} = \{ R_i \}_{i \in \{1, \ldots, m\}} \) is closed under \( F \) if for all \( i \in \{1, \ldots, m\} \), there exist functions \( K_i \) and constants \( \alpha_{i,j} \), such that

\[
R_i(F) = K_i \left( \prod_{j=1}^{m} R_j^{\alpha_{i,j}} \right),
\]

with \( \prod_{j=1}^{m} R_j^{\alpha_{i,j}} \neq 1 \). Each function \( K_i \) is called the cofactor of \( R_i \). Very briefly, the method works as follows: If there exist a closed set of functions for \( F \), say \( \mathcal{R} = \{ R_i \}_{i \in \{1, \ldots, m\}} \), it can be tested if the function \( H(x) = \prod_{i=1}^{m} R_i^{\beta_i}(x) \) gives a first integral for some values \( \beta_i \), just imposing \( H(F) = H \).
In this letter, we will use the unconfined singularities of the maps associated to equations in \([1]\) and its pre–images to generate closed set of functions.

**Proposition 2.** Consider the maps \(F_1(x,y) = (y, (1 + y)/x), F_2(x,y,z) = (y,z,(1 + y + z)/x),\) and \(F_3(x,y,z) = (y,z,(-1 + y - z)/x)\) associated to equations in \([1]\) respectively. The following statements hold:

(i) The globally 5–periodic map \(F_1\) has the closed set of functions \(\mathcal{R}_1 = \{x, y, 1 + y, 1 + x + y, 1 + x\}\), which describe \(\Lambda(F_1)\), and generates the first integral
\[
I_1(x,y) = \frac{(1+x)(1+y)(1+x+y)}{xy}.
\]

(ii) The globally 8–periodic map \(F_2\) has the closed set of functions \(\mathcal{R}_2 = \{x, y, z, 1 + y + z, 1 + x + y + z + xz, 1 + x + y\}\), which describe \(\Lambda(F_2)\), and generates the first integral
\[
I_2(x,y,z) = \frac{(1+y+z)(1+x+y)(1+x+y+z+xz)}{xyz}.
\]

(iii) The map \(F_3\) has the closed set of functions \(\mathcal{R}_3 = \{x, y, z, -1 + y - z, 1 - x - y + z + xz, -1 + x - z - xy - xz + y^2 - yz, 1 - x + y + z + xz, -1 + x - y\}\), which describe \(\Lambda(F_3)\), and generates the first integral
\[
I_3(x,y,z) = (-1 + y - z)(1-x-y+z+xz)(1-x+y+z+xz)(-1+x-z-xy-xz+y^2-yz)(x-y-1)/(x^2y^2z^2).
\]

**Proof.** We only proof statement (ii) since statements (i) and (iii) can be obtained in the same way. Indeed, observe that \(\{x = 0\}\) is the singular set of \(F_2\). We start the process of characterizing the pre–images of the singular set by setting \(R_1 = x\) as a “candidate” to be a factor of a possible first integral. \(R_1(F_2) = y\), so \(\{y = 0\}\) is a pre–image of the singular set \(\{R_1 = 0\}\). Set \(R_2 = y\), then \(R_2(F_2) = z\) in this way we can keep track of the candidates to be factors of \(I_4\). In summary:

\[
\begin{align*}
R_1 &:= x &\Rightarrow R_1(F_2) &= y, \\
R_2 &:= y &\Rightarrow R_2(F_2) &= z, \\
R_3 &:= z &\Rightarrow R_3(F_2) &= (1+y+z)/x = (1+y+z)/R_1, \\
R_4 &:= 1+y+z &\Rightarrow R_4(F_2) &= (1+x+y+z+xz)/x = (1+x+y+z+xz)/R_1, \\
R_5 &:= 1+x+y+z+xz &\Rightarrow R_5(F_2) &= (1+y+z)(1+x+y)/x = R_4(1+x+y)/R_1, \\
R_6 &:= 1+x+y &\Rightarrow R_6(F_2) &= 1+y+z = R_4.
\end{align*}
\]

From this computations we can observe that \(\mathcal{R}_2 = \{R_i\}_{i=1,..,6}\) is a closed set under \(F_2\).

Hence a natural candidate to be a first integral is
\[
I(x,y) = x^\alpha y^\beta z^\delta (1 + y + z)^\gamma (1 + x + y)^\sigma (1 + x + y + z + xz)^\tau
\]
Imposing $I(F_2) = I$, we get that $I$ is a first integral if $\alpha = -\tau$, $\beta = -\tau$, $\delta = -\tau$, $\gamma = \tau$, and $\sigma = \tau$. Taking $\tau = 1$, we obtain $I_2$.

A complete set of first integrals for the above maps can be found in [10].

As a corollary of both the method and Proposition 2, we re-obtain the recently discovered second first integral of the third–order Lyness’ equations (also named Todd’s equation). This “second” invariant was already obtained independently in [10] and [16], with other methods. The knowledge of this second first integral has allowed some progress in the study of the dynamics of the third order Lyness’ equation [17].

**Proposition 3.** The set of functions $R = \{x, y, z, 1 + y + z, 1 + x + y, a + x + y + z + xz\}$ is closed under the map $F_a(x, y, z) = (y, z, (a + y + z)/x)$ with $a \in \mathbb{R}$, which is associated to the third order Lyness’ equation $x_{n+3} = (a + x_{n+1} + x_{n+2})/x_n$. And gives the first integral

$$H_a(x, y, z) = \frac{(1 + y + z)(1 + x + y)(a + x + y + z + xz)}{xyz}.$$

**Proof.** Taking into account that from Proposition 2 (ii) when $a = 1$, $I_2$ is a first integral for $F_{\{a=1\}}(x, y, z)$, it seem that a natural candidate to be a first integrals could be

$$H_{\alpha,\beta,\gamma}(x, y, z) = \frac{(\alpha + y + z)(\beta + x + y)(\gamma + x + y + z + xz)}{xyz}$$

for some constants $\alpha$, $\beta$ and $\gamma$. Observe that

$$R_1 := x \quad \Rightarrow \quad R_1(F_a) = y,$$
$$R_2 := y \quad \Rightarrow \quad R_2(F_a) = z,$$
$$R_3 := z \quad \Rightarrow \quad R_3(F_a) = (a + y + z)/x = K_3/R_1,$$

where $K_3 = a + y + z$, at this point we stop the pursuit of the pre-images of the singularities because they grow indefinitely, and this way doesn’t seem to be a good way to obtain a family of functions closed under $F_a$. But we can keep track of the rest of factors in $H_{\alpha,\beta,\gamma}$.

$$R_4 := \alpha + y + z \quad \Rightarrow \quad R_4(F_a) = (a + \alpha x + y + z + xz)/x,$$
$$R_5 := \beta + x + y \quad \Rightarrow \quad R_5(F_a) = \beta + y + z,$$
$$R_6 := \gamma + x + y + z + xz \quad \Rightarrow \quad R_6(F_a) = (\gamma + y + z)x + (a + y + z)(1 + y)/x.$$

Observe that if we take $\alpha = 1$, $\beta = 1$, and $\gamma = a$, we obtain $R_4(F_a) = R_6/R_1$, $R_5(F_a) = R_4$ and $R_6(F_a) = K_3(R_5/R_1)$. Therefore $\{R_i\}_{i=1,...,6}$ is closed under $F_a$, furthermore $H_a = (R_4R_5R_6)/(R_1R_2R_3)$ is such that $H_a(F_a) = H_a$. 

In conclusion, singularity confinement is a feature which is present in many integrable discrete systems but the existence of complete integrable discrete systems with unconfined singularities evidences that is not a necessary condition for integrability (at least when
“integrability” means existence of at least an invariant of motion, a first integral). However it is true that globally periodic systems are themselves “singular” in the sense that they are sparse, typically non–generic when significant classes of DDS (like the rational ones) are considered.

Thus, the large number of integrable examples satisfying the singularity confinement property together with the result in [9, p.1207] (where an extended, an not usual, notion of the singularity confinement property must be introduced in order to avoid the periodic singularity propagation phenomenon reported in this letter -see the definition of periodic singularities in p. 1204-) evidences that singularity confinement still can be considered as a good heuristic indicator of “integrability” and that perhaps there exists an interesting geometric interpretation linking both properties. However, although some alternative directions have been started (see [18] for instance), still a lot of research must to be done in order to understand the role of singularities of discrete systems, their structure and properties in relation with the integrability issues.

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