Self-Gravitating Relativistic Models of Fermions with Anisotropy and Cutoff Energy in their Distribution Function

Marco Merafina

Department of Physics, University of Rome La Sapienza,
Piazzale Aldo Moro 2, I-00185 Rome, Italy

Giuseppe Alberti

Laboratoire de Physique Théorique (UMR 5152), IRSAMC,
Université Paul Sabatier, Route de Narbonne 118, F-31062 Toulouse, France

Abstract

In this paper we study the gravitational equilibrium of the systems of anisotropic self-gravitating fermions, by extending to General Relativity the solutions obtained in a previous paper. This treatment also generalizes to anisotropic systems the relativistic self-gravitating Fermi gas model, by considering different degrees of anisotropy, and solving the equilibrium equations to obtain the density profiles. We discuss some general characteristics of the models and generalize the relation between the anisotropy and the mass of particles in the relativistic regime.

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I. INTRODUCTION

The theory of the General Relativity (GR) is the rigorous way to describe properties and structure of systems kept bound by strong gravitational forces. Although a wide range of astrophysical objects can be analyzed only by using the classical mechanics (e.g., the globular clusters or anisotropic Newtonian systems, see [1]), there are some cases in which it is necessary to use Einstein’s field theory (e.g., the compact objects). However, a lot of phenomena do not require the full employment of the GR. In fact, it is sufficient to use an approximate method and consider two possibilities: the 1PN (first post-Newtonian) and the weak field approximations [2]. The 1PN approximation, where some examples of this application are in Refs. [3, 4], gives corrections up to order $v^2/c^2$ (with $v$ the typical velocity in the system being considered and $c$ the speed of light) working for non relativistic particles ($v \ll c$); the weak field approximation, where examples can be found in Refs. [5, 6], is instead related to problems considering the gravitational radiation.

Other possibilities, in order to analyze the properties of self-gravitating systems in GR, are represented by solving the relativistic versions of the collisional and noncollisional Boltzmann equation or by considering a statistical approach. Important results of the first method are represented by the relativistic stellar clusters ([7], hereafter BKMV10); for the second approach we have to mention the important papers of Fowler [8], Chandrasekhar [9] and Oppenheimer and Volkoff [10], that represent the first applications of the self-gravitating Fermi gas model in astrophysics.

More recent applications of the Fermi gas model in the framework of the GR have been proposed by Bilić and Viollier [11, 12], who studied the general relativistic version of the Thomas-Fermi model and applied it to galactic dark halos, by supposing the existence of fermions spheres made by massive neutrons ($m \sim 15$ keV). Nakajima and Morikawa [13] considered the equilibrium configurations of weakly interacting fully degenerate fermionic dark matter at various scales in the Universe, also finding a limiting mass in the range $2 - 30$ eV. Furthermore, Narain et al. [14] proposed different models of compact stars, constituted by fermionic dark matter, finding the typical values for the masses of these stars by considering all the fermionic candidates for dark matter, from the heaviest to the lightest ones.

More refined models can be obtained by considering the presence of anisotropies, for example, in the distribution function characterizing the system under investigation. Anisotropic solutions of the general relativistic noncollisional Boltzmann equation had been advanced by Bisnovatyi-Kogan and Zel’dovich [15, 16], while the papers of Bowers and Liang [17] and Heinzelmann and Hillebrandt [18] can be considered as the piononering works for the anisotropic neutron stars models (examples
of more recent works on the same topic are given in Refs. [19–21]). Extended polytropic models for anisotropic systems can be found in Refs. [22, 23], whereas examples of models for anisotropic general relativistic fluids are considered in Refs. [24–27]. Furthermore, some proposals connected to the galactic halos and the gravitational lensing of the dark matter can be found in Refs. [28, 29].

In this work we extend the Newtonian models of the collisionless semidegenerate Fermi gas, described in our previous paper (30, hereafter Paper I), to GR. In Sec. II we introduce the distribution function and define the thermodynamic quantities, as the tensor pressure and the density, to solve the equilibrium equations. In Sec. III we present the results of the numerical integration, by discussing the characteristics of the models studied. In Sec. IV we derive a relation (valid in the limit of full degeneracy) between the mass of the particles and the anisotropy in the distribution function. Finally, in Sec. V we draw some conclusions.

II. THE MODEL

A. Distribution Function and Useful Variables

The distribution function has the form (see Paper I)

\[ f = \frac{g}{h^3} \left(1 + \frac{L^2}{L_c^2}\right)^4 \frac{1 - e^{(\epsilon-\epsilon_c)/kT}}{e^{(\epsilon-\mu)/kT} + 1} \quad \text{for } \epsilon \leq \epsilon_c, \]
\[ f = 0 \quad \text{for } \epsilon > \epsilon_c, \quad (1) \]

where \( T \) (constant) is the total temperature of the system; \( L_c = mc r_a \) is a constant depending on the anisotropy radius \( r_a \), \( L = mv_t r = p r \) is the angular momentum of a single particle (\( m, r, \) and \( v_t \) are, respectively, mass, radius, and tangential velocity) while \( \epsilon \) and \( \epsilon_c \) are the kinetic and the cutoff kinetic energy of a single particle. Since we are considering a spherically symmetric configuration, we have to use the Schwarzschild metric

\[ ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 (d\psi^2 + \sin^2 \psi d\phi^2), \quad (2) \]

and the equations of the gravitational equilibrium are given by (15)

\[ \frac{dP_{rr}}{dr} = -\frac{G}{rc^2} \left( P_{rr} + \rho c^2 \right) \frac{(M_r c^2 + 4\pi P_{rr} r^3)}{rc^2 - 2GM_r} - \frac{2}{r} (P_{rr} - P_t), \]
\[ \frac{dM_r}{dr} = 4\pi \rho(r) r^2, \quad (3) \]
with the conditions $P_{rr}(0) = P_{rr0}$ and $M_r(0) = 0$. The metric coefficients are determined by

$$e^\lambda = \left(1 - \frac{2GM_r}{rc^2}\right)^{-1},$$ \hfill (4)

$$\frac{dv}{dr} = \frac{2G M_r c^2 + 4\pi P_{rr} c^2}{c^2 r (rc^2 - 2GM_r)},$$ \hfill (5)

$$e^{\nu_R} = e^{-\lambda_R} = 1 - \frac{2GM}{Rc^2}.$$ \hfill (6)

In order to solve Eq.(3) and evaluate the thermodynamic functions, let us introduce the variables

$$\epsilon = \sqrt{p^2 c^2 + m^2 c^4} - mc^2, \quad \epsilon_c = \sqrt{p^2 c^2 + m^2 c^4} - mc^2, \quad T_r = Te^{-\nu/2},$$

$$x = \frac{\epsilon}{kT_r}, \quad y = \frac{\epsilon}{mc^2}, \quad W = \frac{\epsilon_c}{kT_r}, \quad \theta = \frac{\mu}{kT_r}, \quad \beta = \frac{kT_R}{mc^2}.$$ \hfill (7)

Here, $\epsilon$ and $\epsilon_c$ are the variables appearing in the distribution function, $T_r$ is the local temperature, $x$, $y$, $W$, $\theta$ and $\beta$ are dimensionless variables. In particular, $\theta$ and $\beta$ are, respectively, the degeneracy and the relativistic temperature parameters. Moreover we have, on the basis of the energy conservation

$$(\epsilon + mc^2)e^{\nu/2} = \text{const.},$$

$$(\epsilon_c + mc^2)e^{\nu/2} = mc^2 e^{\nu_R/2},$$

$$(\mu + mc^2)e^{\nu/2} = (\mu_R + mc^2)e^{\nu_R/2}.$$ \hfill (8)

Using relations (7) and (8) we obtain also

$$\frac{mc^2}{kT_r} = \frac{1 - \beta W}{\beta},$$ \hfill (9)

that we can rewrite as

$$1 - \beta W = \frac{\beta mc^2}{kT_r} = \frac{T_R}{T_r} = \frac{Te^{-\nu_R/2}}{Te^{-\nu/2}} = e^{\nu - \nu_R}.$$ \hfill (10)
From Eq. (10) we have the constraint \( 0 \leq \beta W < 1 \) (see Refs. [32, 33]) and moreover, we get

\[
e^{\nu} = e^{\nu_R}(1 - \beta W)^2 \quad \text{and} \quad \frac{d\nu}{dr} = -\frac{2\beta}{1 - \beta W} \frac{dW}{dr}.
\] (11)

Substituting Eq. (5) into Eq. (11) we obtain, instead of the first of Eq. (3),

\[
\frac{dW}{dr} = -\frac{G}{c^2} \frac{1 - \beta W}{\beta} \frac{M_r c^2 + 4\pi P_{rr} r^3}{r(2c^2 - 2GM_r)}.
\] (12)

B. Thermodynamic Quantities and Gravitational Equilibrium

The thermodynamic variables are defined by relations (see BK MV10)

\[
n = \frac{2\pi g}{h^3} \sum_{k=0}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) \left( \frac{r}{r_a} \right)^{2k} \int_0^{\pi} (\sin \phi)^{2k+1} d\phi \int_0^{p_c} p^{2k+2} \frac{1 - e^{(\epsilon - \epsilon_c)/kT}}{e^{(\epsilon - \epsilon_c)/kT} - 1} dp,
\] (13)

\[
\rho c^2 = \frac{2\pi g}{h^3} \sum_{k=0}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) \left( \frac{r}{r_a} \right)^{2k} \int_0^{\pi} (\sin \phi)^{2k+1} d\phi \int_0^{p_c} p^{2k+2} \sqrt{p^2 c^2 + m^2 c^4} \frac{1 - e^{(\epsilon - \epsilon_c)/kT}}{e^{(\epsilon - \epsilon_c)/kT} - 1} dp,
\] (14)

\[
\rho_{rr} = \frac{2\pi g}{h^3} \sum_{k=0}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) \left( \frac{r}{r_a} \right)^{2k} \int_0^{\pi} (\sin \phi)^{2k+1} d\phi \int_0^{p_c} \frac{p^{2k+4}}{\sqrt{p^2 c^2 + m^2 c^4}} e^{(\epsilon - \epsilon_c)/kT} \frac{1 - e^{(\epsilon - \epsilon_c)/kT}}{e^{(\epsilon - \epsilon_c)/kT} - 1} dp,
\] (15)

\[
P_t = \frac{\pi g}{h^3} \sum_{k=0}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) \left( \frac{r}{r_a} \right)^{2k} \int_0^{\pi} (\sin \phi)^{2k+3} d\phi \int_0^{p_c} \frac{p^{2k+4}}{\sqrt{p^2 c^2 + m^2 c^4}} e^{(\epsilon - \epsilon_c)/kT} \frac{1 - e^{(\epsilon - \epsilon_c)/kT}}{e^{(\epsilon - \epsilon_c)/kT} - 1} dp,
\] (16)

In the previous relations we have used the polar coordinates of the plane \((p_r, p_t)\), where \(p_c\) is the value of the momentum corresponding to the cutoff energy, and we have rewritten the part of the distribution function depending on the angular momentum by using the Newton binomial relation

\[
\left( 1 + \frac{L^2}{L_c^2} \right)^l = \sum_{k=0}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) \left( \frac{L}{L_c} \right)^{2k}, \quad \text{with} \quad \left( \begin{array}{c} l \\ k \end{array} \right) = \frac{l!}{k!(l-k)!}, \quad 0! = 1.
\] (17)

In order to transform the integrals of Eqs. (13)-(16) in a more suitable form, following BKMV10, it is more convenient to use the variables \(x\) and \(y\) [see Eq. (7)] instead of \(p/mc\)

\[
\frac{\sqrt{p^2 c^2 + m^2 c^4}}{mc^2} = \frac{\epsilon}{mc^2} + 1 = y + 1, \quad \text{where} \quad p = mc \sqrt{y(y+2)}.
\] (18)
By differentiating and using Eq. (7) we obtain:

\[
dp = mc \frac{y + 1}{\sqrt{y(y + 2)}} dy = \sqrt{\frac{\beta}{1 - \beta W}} \frac{1 + \frac{\beta x}{1 - \beta W}}{\sqrt{1 + \frac{\beta x}{2(1 - \beta W)}}} \sqrt{2x} \tag{19}
\]

and, substituting this result into the expressions of the thermodynamic functions we get:

\[
n = \frac{\pi gm^3 c^3}{h^3} \sum_{k=0}^{l} \left( \frac{1}{k} \right) \left( \frac{r}{r_a} \right)^{2k} A_k \left( \frac{2\beta}{1 - \beta W} \right)^{k+\frac{1}{2}} I_{nk}, \tag{20}
\]

\[
\rho = \frac{\pi gm^4 c^3}{h^3} \sum_{k=0}^{l} \left( \frac{1}{k} \right) \left( \frac{r}{r_a} \right)^{2k} A_k \left( \frac{2\beta}{1 - \beta W} \right)^{k+\frac{1}{2}} I_{\rho k}, \tag{21}
\]

\[
P_{\tau \tau} = \frac{\pi gm^4 c^5}{h^3} \sum_{k=0}^{l} \left( \frac{1}{k} \right) \left( \frac{r}{r_a} \right)^{2k} (A_k - A_{k+1}) \left( \frac{2\beta}{1 - \beta W} \right)^{k+\frac{1}{2}} I_{P k}, \tag{22}
\]

\[
P_t = \frac{\pi gm^4 c^5}{h^3} \sum_{k=0}^{l} \left( \frac{1}{k} \right) \left( \frac{r}{r_a} \right)^{2k} A_{k+1} \left( \frac{2\beta}{1 - \beta W} \right)^{k+\frac{1}{2}} I_{P k}. \tag{23}
\]

Here, the \( A_k \) coefficients \[34\] and the integrals \( I_{nk}, I_{\rho k} \) and \( I_{P k} \) are defined, respectively, by

\[
A_k = \int_0^\pi (\sin \phi)^{2k+1} d\phi = 2 \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{2i+1}, \tag{24}
\]

\[
I_{nk} = \int_0^W \left( 1 + \frac{\beta x}{1 - \beta W} \right) \left[ 1 + \frac{\beta x}{2(1 - \beta W)} \right]^{k+\frac{1}{2}} x^{k+\frac{1}{2}} g(x, W) dx, \tag{25}
\]

\[
I_{\rho k} = \int_0^W \left( 1 + \frac{\beta x}{1 - \beta W} \right)^2 \left[ 1 + \frac{\beta x}{2(1 - \beta W)} \right]^{k+\frac{1}{2}} x^{k+\frac{1}{2}} g(x, W) dx, \tag{26}
\]

\[
I_{P k} = \int_0^W \left[ 1 + \frac{\beta x}{2(1 - \beta W)} \right]^{k+\frac{1}{2}} x^{k+\frac{1}{2}} g(x, W) dx, \tag{27}
\]

and the \( g(x, W) \) function is given by (see Paper I)

\[
g(x, W) = \frac{1 - e^{x-W}}{e^{x-W-\theta R} + 1} = e^{\theta R} \frac{e^W - e^x}{e^x + e^{W+\theta R}}, \tag{28}
\]
with $\theta_R = \theta(R)$. In particular, the first three values of the $A_k$ coefficients $A_0, A_1,$ and $A_2$ are 2, 4/3, and 16/15, respectively. Thus, the equilibrium equations can be rewritten as

$$\frac{dW}{dr} = \frac{G}{rc^2} \left( \frac{1 - \beta W}{\beta} \right) M_r c^2 + \frac{4\pi P_{rr} r^3}{rc^2 - 2GM_r},$$

$$\frac{dM_r}{dr} = 4\pi \rho(r) r^2,$$

with the initial conditions $W(0) = W_0$ and $M_r(0) = 0$.

C. Nondimensional Variables

Following the same procedure of the Newtonian case, let us introduce the nondimensional variables

$$r = \xi \tilde{r}, \quad r_a = \xi \tilde{a}, \quad n = \frac{c^2 \tilde{n}}{Gm\xi^2}, \quad \rho = \frac{c^2 \tilde{\rho}}{G\xi^2}, \quad P_{rr} = \frac{c^4 \tilde{P}_{rr}}{G\xi^2}, \quad P_l = \frac{c^4 \tilde{P}_l}{G\xi^2}, \quad M_r = \frac{c^2 \tilde{M}_r \xi}{G},$$

where $\xi = (k^3/gcGm^4)^{1/2}$ and $a$ is the anisotropy parameter. The thermodynamic quantities in dimensionless form are:

$$\tilde{n} = \pi \sum_{k=0}^{l} \left( \frac{l}{k} \right) \frac{(\tilde{r})^{2k}}{(\tilde{a})^k} A_k \left( \frac{2\beta}{1 - \beta W} \right)^{k+\frac{3}{2}} I_{nk},$$

$$\tilde{\rho} = \pi \sum_{k=0}^{l} \left( \frac{l}{k} \right) \frac{(\tilde{r})^{2k}}{(\tilde{a})^k} A_k \left( \frac{2\beta}{1 - \beta W} \right)^{k+\frac{3}{2}} I_{\rho k},$$

$$\tilde{P}_{rr} = \pi \sum_{k=0}^{l} \left( \frac{l}{k} \right) \frac{(\tilde{r})^{2k}}{(\tilde{a})^k} (A_k - A_{k+1}) \left( \frac{2\beta}{1 - \beta W} \right)^{k+\frac{3}{2}} I_{Pk},$$

$$\tilde{P}_l = \pi \sum_{k=0}^{l} \left( \frac{l}{k} \right) \frac{(\tilde{r})^{2k}}{(\tilde{a})^k} A_{k+1} \left( \frac{2\beta}{1 - \beta W} \right)^{k+\frac{3}{2}} I_{Pk}.$$

The equilibrium equations then become:

$$\frac{dW}{d\tilde{r}} = -\frac{1 - \beta W}{\beta} \frac{\tilde{M}_r + 4\pi \tilde{P}_{rr} \tilde{r}^3}{\tilde{r} - 2\tilde{M}_r},$$

$$\frac{d\tilde{M}_r}{d\tilde{r}} = 4\pi \tilde{\rho}(\tilde{r}) \tilde{r}^2,$$

with the initial conditions $W(0) = W_0$ and $\tilde{M}_r(0) = 0$. 

III. RESULTS OF THE NUMERICAL INTEGRATION

A. Evidences of the Anisotropy

In order to explicitly analyze the effects of the presence of the anisotropy in the equilibrium configurations, following the Newtonian treatment, it is useful to define the parameter $\eta$

$$\eta = \frac{2\langle v_r^2 \rangle}{\langle v_t^2 \rangle} = \frac{2\rho_r}{P_t} = \frac{\tilde{P}_{rr}}{P_t} = \frac{1 + \frac{4}{3} (\frac{\beta}{1 - \beta W}) \left(\frac{\tilde{V}}{\tilde{E}}\right)^2 I_{P5/2}}{1 + \frac{8}{3} (\frac{\beta}{1 - \beta W}) \left(\frac{\tilde{V}}{\tilde{E}}\right)^2 I_{P5/2}}. \tag{36}$$

Since this definition is the same showed in BKMV10, we expect a similar trend, by analyzing the behavior of $\eta$ starting from the center towards the boundary of the equilibrium configurations. The Figs. 1, 2 and 3, in fact, confirm what we expect: both in the limit $r \to 0$ (at the center) and in the limit $r \to R$ (at the edge), we see that the ratio $P_{rr}/P_t \to 1$ and $\eta$ reaches its maximum value ($\eta_{\text{max}} = 1$), by showing a prevalence of isotropic motion of the particles. In the intermediate zones, we instead can note a decrease of $\eta$ until to its minimum value (which cannot be less than $\eta_{\text{min}} = 0.5$), clear gauge of the prevalence of tangential motion.

From Eq. (36) we can also study the behavior of $\eta$ as a function of the temperature parameter $\beta$, both in the limit $\beta \to 0$ and in the limit $\beta \to \infty$. In the first case, we note that $\beta/(1 - \beta W) \to 0$ and $\eta \to 1$ (furthermore the expressions of $I_{P3/2}$ and $I_{P5/2}$ in Eq. (36) tend to the corresponding ones in the Newtonian limit). In the second case, we see that $\beta/(1 - \beta W) \gg 1$ and thus $\eta \to 0.5$ (see Figs. 2 and 3).

B. Mass - Central Density Diagrams

In this section we aim at describing the mass - central density diagrams. In Fig. 4 we represented the effects of the anisotropy with changing the parameter $a$ at fixed values of $\beta$. We may note that a larger degree of anisotropy requires the existence of configurations with smaller masses. If indeed we fix the value of $a$ and vary the value of $\beta$ (see Figs. 5 and 6), we may note that configurations with high values of $\beta$ have a total mass generally larger than configurations with small ones (i.e., the Newtonian limit). In particular, it is interesting to note how the value of $\beta$ affects the level of degeneracy in the equilibrium configurations, due to the constraints [10] and the condition $\theta \leq W$ that implies $\theta_R \leq 0$ (for more details see Paper I).

In Fig. 7 we show the behavior of the equilibrium configurations in the isotropic limit, which is recovered for $a \to \infty$ (but $a = 1$ already constitutes an excellent approximation of this limit). In this
diagram we have pointed our attention to the influence of the degeneracy level on the equilibrium configurations. To do this we have constructed some curves for $\beta = 10^{-5}$ and $\beta = 10^3$ choosing four values of $\theta_R$: 0, $-2.31$, $-5$, $-10$. Considering the curves at $\beta = 10^{-5}$, in the degenerate limit and for large values of the central density, we observe the four curves follow the same behavior whereas, for smaller values of $\rho_0$, we note a split of them in correspondence of the bifurcation point. In particular, the curve with $\theta_R = -10$ reaches higher values of the mass than the other three and present a local maximum before the bifurcation point. When we refer to the semidegenerate limit we do not see an overlapping of the curves and, like the previous case, the curve with $\theta_R = -10$ reaches again the highest values of the mass. For $\beta = 10^3$ we observe a more regular behavior where the curves at smaller values of $\theta_R$ show higher values of the mass.

C. Density Profiles

By considering $l = 1$ in Eq. (1) and integrating Eq. (35), we can construct the density profiles of the configurations. Expressing in terms of nondimensional quantities, we have

$$\tilde{\rho} = 2\pi \left( \frac{2\beta}{1 - \beta W} \right)^{3/2} \left\{ I_{\rho^1/2} + \frac{4}{3} \left( \frac{2\beta}{1 - \beta W} \right) \left( \frac{\tilde{r}}{a} \right)^2 I_{\rho^3/2} \right\}. \tag{37}$$

The variation of the anisotropy parameter remarkably influences the behavior of the density function. From Eq. (37), in the limit $a \to 0$, it follows immediately that the second addend prevails, implying a general increase of the density $\tilde{\rho}$ and, in particular, of its maximum value. In Figs. 8, 9, and 10 we have represented the quantity $\rho/\rho_0$ as a function of the dimensionless radial coordinate $r/\xi$, for different values of $a$, $\beta$, $W_0$ and $\theta_0$.

In Fig. 8 we show the behavior of the density profiles, once fixed the values of $a$ and $\beta$ by varying the value of the central degeneracy parameter $\theta_0$. One can see how the value of the maximum increases by decreasing $\theta_0$ and its position tends to move in the direction of the periphery of the configuration. The trend of the density profiles, also in relativistic regime, shows the existence of hollow configurations and confirm the results obtained by Nguyen and Pedraza [27] and by Ralston and Smith [35], giving a clear indication that the presence of the anisotropy is the reason of this kind of configurations.

Figs. 9 and 10 show the influence of the temperature parameter $\beta$ on the density profiles. If we look at Eq. (37) we see that, in the limit $\beta \to 0$, the second term in the sum becomes negligible and we recover the behavior typical of the isotropic systems. On the contrary, in the limit $\beta \to \infty$,
the curve comes back to the behavior of the hollow systems. In Tables I, II and III we summarize some results of the numerical integration for particular values of $a$, $\beta W_0$, $\beta$ and $W_0$.

IV. LIMITS ON THE PARTICLE MASS

In Paper I, we derived, in the limit of full degeneracy, an analytical expression relating the mass of particles with the anisotropy in the momentum distribution. According to the relation obtained, an increase of the anisotropy within the distribution of velocities provoked a decrease of the lower limit of the particle mass. On the contrary, when the system recovered the isotropy, we noted an increase of the limiting mass. In order to extend this result to the GR regime, we have to consider a new parameter not considered in the Newtonian regime, i.e. the temperature parameter $\beta$. Let us rewrite the definition of the density $\rho$ as

$$
\rho = \frac{\pi g m^4 c^3}{h^3} \sum_{k=0}^{l} \left( \frac{l}{k} \right) \left( \frac{r}{r_a} \right)^{2k} A_k \left( \frac{2\beta}{1 - \beta W} \right)^{\frac{2k+1}{2}} I_{\rho k}.
$$

(38)

In the limit of full degeneracy [i.e., $\theta_R \to 0$ and $g(x, W) \to 1$] Eq.(38) becomes

$$
\rho \leq \frac{2\pi g m^4 c^3 \beta^{3/2}}{\alpha^4 h^3} \sum_{k=0}^{l} \left( \frac{l}{k} \right) \left( \frac{r}{r_a} \right)^{2k} A_k \beta^k \int_{0}^{W} (\beta x + \alpha)^2 [x(\beta x + 2\alpha)]^{\frac{2k+1}{2}} dx,
$$

(39)

with $\alpha = 1 - \beta W$. Then, we transform the integral as (see Appendix)

$$
\rho \leq \frac{2\pi g m^4 c^3 \beta^{3/2}}{\alpha^4 h^3} \sum_{k=0}^{l} \left( \frac{l}{k} \right) \left( \frac{r}{r_a} \right)^{2k} A_k \beta^k \times
$$

$$
\frac{\alpha^4}{2} \left( 2(k+1) - (2k+1)\alpha^2 \right) - (2k+1)\alpha^{2k+4} \int_{0}^{\frac{1+\sqrt{1-\alpha^2}}{\alpha}} \left( \frac{1}{\alpha^2} \right) \left( \beta r^2 r_a^2 - 3 \right) dy.
$$

(40)

Equation (40) is a general expression corresponding to any values of $l$. In our case $l = 1$, then

$$
\rho \leq \frac{2\pi g m^4 c^3 \beta^{3/2}}{\alpha^4 h^3} \left\{ \sqrt{1 - \alpha^2} \left[ 2 - \alpha^2 + \frac{8 - 14\alpha^2 + 3\alpha^4}{9\alpha^2 r_a^2} \right] + \alpha^4 \frac{3}{\alpha} \ln \left( \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \right) \left( \beta r^2 r_a^2 - 3 \right) \right\}.
$$

(41)

and, solving for the mass, we have

$$
m \geq \left( \frac{\rho h^3}{2\pi g c^3 \beta^{3/2}} \right) \frac{\alpha}{\left\{ \sqrt{1 - \alpha^2} \left[ 2 - \alpha^2 + \frac{8 - 14\alpha^2 + 3\alpha^4}{9\alpha^2 r_a^2} \right] + \alpha^4 \frac{3}{\alpha} \ln \left( \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \right) \left( \beta r^2 r_a^2 - 3 \right) \right\}^{\frac{1}{3}}.
$$

(42)
We could compute Eq. (42) at the center (with $\alpha_0 = 1 - \beta W_0$). We get

$$m \geq \left( \frac{\rho_0 h^3}{2 \pi g c^3 \beta^3/2} \right)^{1/4} \frac{\alpha_0}{\sqrt{1 - \alpha_0^2 (2 - \alpha_0^2) - \alpha_0^4 \ln \left( \frac{1 + \sqrt{1 - \alpha_0^2}}{\alpha_0} \right)}},$$

(43)
even if this expression (that for $\rho_0 \sim 10^{15}$ g/cm$^3$, $g = 2$, $\beta W_0 = 0.295$ and $\beta = 0.1$ can give $m \geq 1.68 \times 10^{-24}$ g $\approx 939.5$ MeV) does not take into account the effects of the anisotropy. To have an evidence of this effect, we follow the same approach of Paper I, by computing Eq. (42) at the core radius $r = r_c$, where $\alpha_c = 1 - \beta W_c$ is the value corresponding to $r = r_c$ and $\rho = \rho(r_c) \equiv \rho_c$. Thus

$$m \geq \left( \frac{\rho_c h^3}{2 \pi g c^3 \beta^3/2} \right)^{1/4} \frac{\alpha_c}{\sqrt{1 - \alpha_c^2 \left[ 2 - \alpha_c^2 + \frac{(8 - 14 \alpha_c^2 + 3 \alpha_c^4) \beta^2 r_c^2}{9 \alpha_c^2 r_c^2} \right] + \alpha_c^4/3 \ln \left( \frac{1 + \sqrt{1 - \alpha_c^2}}{\alpha_c} \right) \left( \frac{\beta^2 r_c^2}{r_c^2 a} - 3 \right)}}. \quad (44)$$

We may also write Eq. (44) as $m \geq m_* F(a, r, \beta)$, where we defined

$$m_* = \left( \frac{\rho_c h^3}{2 \pi g c^3} \right)^{1/4} \quad (45)$$
as the “dimensional” mass and $F(a, r, \beta)$ by

$$F(a, r, \beta) = \frac{\alpha}{\beta^{3/8} \left\{ \sqrt{1 - \alpha^2 \left[ 2 - \alpha^2 + \frac{(8 - 14 \alpha^2 + 3 \alpha^4) \beta^2 r_c^2}{9 \alpha_c^2 r_c^2} \right] + \alpha^4/3 \ln \left( \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \right) \left( \frac{\beta^2 r_c^2}{r_c^2 a} - 3 \right) \right\}^{1/4}} \quad (46)$$
The Figures 11 and 12 show the behavior of the function $F$.

V. CONCLUSIONS

The goal of this paper is a complete understanding of the properties of the systems composed by anisotropic self-gravitating fermions in GR regime. In order to make an analysis of the various aspects, we considered the effects of the anisotropy on the motion of the particles (via the $\eta$ parameter) and on the distribution of the matter (via the density profiles and the mass - central density diagrams). In addition, we have schematized the configurations constructed through the parameters $\beta$, $\theta_0$ and $a$, by establishing, in Sections IIIA, IIIB and IIIC a “standing of their influence” according to the order $\beta$, $a$ and $\theta_0$. By referring to the kind of motion of the particles, we have recovered the same behavior found in the Paper I and, in the classical limit, the same
results obtained in BKMV10. In particular, Figs. 2 and 3 show the importance of the influence of the β parameter; configurations with an higher “degree of relativity”, described by the β parameter, present a strong prevalence of tangential motion.

If we now consider the density profiles, we see clearly the behavior typical of the hollow systems [27, 35] and, in the Paper I, we have argued the possibility to set the value $a = 0.1$ as the critical threshold for the triggering of the hollowness. However, in the relativistic regime, the things appear quite different, due to the presence of β. If we look at Fig. 10, we do not see evidences of hollowness; on the contrary, the trend of the profile is very similar to one obtained by Ruffini and Stella [36], by using the distribution function (1) in the isotropic limit $L_c \to \infty$. By looking at Fig. 9, we can say that the hollowness appears for $a = 0.1$ but with $\beta \geq 0.1$, at least.

The more interesting situations are obtained by considering the mass - central density diagrams. In the Figure 4, we can observe how a larger degree of anisotropy reduces the values of the masses of the equilibrium configurations, as well we can further note how, the curve with $a \geq 0.1$ approaches the typical curve of the isotropic systems, indicating, in the relativistic regime, that $a = 0.1$ can be viewed as a good approximation of the isotropic limit for the equilibrium configurations. If we now change the incidence of β (see Figs. 5 and 6), we can note that the configurations tend to a universal curve for large value of the central density while, for decreasing central densities, the degeneracy appears and the curves split, according to the degeneracy level. We have also to mention that, for small values of the central density, the configurations become more and more classical, by gradually leaving their quantum behavior.

Nevertheless, a representation of the influence of the degeneracy level is showed in Fig. 7, where we have chosen different values of $\theta_R$, in order to show the gradual passage from the degenerate configurations to the semidegenerate and the classical ones. We have also chosen the two extremal values of β, representing the Newtonian ($\beta = 10^{-5}$) and the full relativistic ($\beta = 10^3$) limit. Newtonian configurations split up at $\tilde{\rho}_0 \sim 10^{-5}$, showing different behaviors. For $\theta_R \geq -2.31$, the curves have an absolute minimum that is missing for $\theta_R < -2.31$ and, furthermore, the curve with $\theta_R = -10$, corresponding to classical configurations, shows, before the point of minimum, the presence of a local maximum. If we look at the fully relativistic configurations, we see, instead, a regular behavior of the curves, indicating the effects of relativity are much stronger than ones due to the degeneracy level.

Moreover, in the limit of full degeneracy we found an expression [see Eq.(44)] that relates the anisotropy to the mass $m$ of the particles. We have found values in the GeV scale. The Figs. 13 and 14 show a similar behavior, independently of the value of β. We observe an initial decrease
of \( m \) for \( 0.1 < \beta W_0 < 0.5 \), then an increase until the achievement of a maximum for \( \beta W_0 \sim 0.7 \). It should be noted, in Fig. 14, that the maximum value reached for \( a = 0.1 \) is smaller than the corresponding ones obtained for \( a = 10^{-5} \) and \( 10^{-3} \).

It is also necessary to study the dynamic stability of the equilibrium configurations constructed in this paper and in the Paper I. In spite of the presence of the anisotropy in Eq. (1), we could draw conclusions about the dynamical stability by advancing a general criterion for the anisotropic systems in terms of the polytropic exponent \( \gamma \). This problem will be addressed to a forthcoming publication.

**Appendix A: Integral of Eq. (39)**

We give the proof of the following formula [see Eqs. (39) - (40)]

\[
J_{\rho k} = \int_W^0 (\beta x + \alpha)^2 [x(\beta x + 2\alpha)]^{2k+1} dx =
\]

\[
= (1 - \alpha^2) \frac{2k+1}{2} [2(k + 1) - (2k + 1)\alpha^2] - (2k + 1)\alpha^{2k+4} \int_0^1 \frac{\ln\left(\frac{1+\sqrt{1-\alpha^2}}{\alpha}\right)}{(\sinh y)^{2k}} dy,
\]

(A1)

The first step is to change the variable, by defining \( z = \beta x + \alpha \). The integral in Eq. (A1) becomes

\[
J_{\rho k} = \int_0^1 z^2 \left[ \frac{z - \alpha}{\beta} (z + \alpha) \right]^{2k+1} \frac{dz}{\beta} = \frac{1}{\beta^{2k+3}} \int_0^1 z^2 (z^2 - \alpha^2)^{2k+1} \frac{dz}{\beta}.
\]

(A2)

Integrating by parts we obtain

\[
J_{\rho k} = \frac{1}{\beta^{2k+3}} \int_0^1 z^2 (z^2 - \alpha^2)^{2k+1} \frac{dz}{\beta} = \frac{z(z^2 - \alpha^2)^{2k+1}}{(2k + 3)\beta^{2k+3}} \bigg|_0^1 - \frac{1}{\beta^{2k+3}} \int_0^1 (z^2 - \alpha^2)^{2k+3} \frac{dz}{\beta}
\]

\[
= \frac{1}{(2k + 3)\beta^{2k+3}} \left\{ (1 - \alpha^2)^{2k+3} - \int_0^1 (z^2 - \alpha^2)^{2k+3} \frac{dz}{\beta} \right\}.
\]

(A3)

Then, making the substitution \( z = \alpha \cosh(y) \) in the integral of Eq. (A3) and integrating by parts, we have

\[
\int_0^1 (z^2 - \alpha^2)^{2k+3} \frac{dz}{\beta} = \alpha^{2k+4} \int_0^1 \frac{\ln\left(\frac{1+\sqrt{1-\alpha^2}}{\alpha}\right)}{(\sinh y)^{2k+4}} (\sinh y)^{2k+4} dy =
\]

\[
= \frac{\alpha^{2k+4}}{2k + 4} \left\{ \cosh y (\sinh y)^{2k+3} \frac{1}{\alpha} \ln\left(\frac{1+\sqrt{1-\alpha^2}}{\alpha}\right) - (2k + 3) \int_0^1 \frac{\ln\left(\frac{1+\sqrt{1-\alpha^2}}{\alpha}\right)}{(\sinh y)^{2k+2}} (\sinh y)^{2k+2} dy \right\}. \tag{A4}
\]
The last term, integrating by parts, becomes

\[
\int_{0}^{1} \ln \left( \frac{1+\sqrt{1-\alpha^2}}{\alpha} \right) (\sinh y)^{2k+2} \, dy = \frac{\cosh y (\sinh y)^{2k+1}}{2k+2} \bigg|_{0}^{\infty} \left( \frac{1+\sqrt{1-\alpha^2}}{\alpha} \right) + \\
- \frac{2k+1}{2k+2} \int_{0}^{\alpha} \ln \left( \frac{1+\sqrt{1-\alpha^2}}{\alpha} \right) (\sinh y)^{2k} \, dy
\]

and, substituting it in Eq. (A4), we get

\[
\int_{\alpha}^{1} (z^2 - \alpha^2)^{\frac{2k+3}{2}} dz = \frac{\alpha^{2k+4}}{2k+4} \left\{ \cosh y (\sinh y)^{2k+3} \bigg|_{0}^{\infty} \left( \frac{1+\sqrt{1-\alpha^2}}{\alpha} \right) + \\
- \frac{2k+3}{2k+2} \cosh y (\sinh y)^{2k+1} \bigg|_{0}^{\infty} \log \left( \frac{1+\sqrt{1-\alpha^2}}{\alpha} \right) - (2k+1) \int_{0}^{\alpha} \ln \left( \frac{1+\sqrt{1-\alpha^2}}{\alpha} \right) (\sinh y)^{2k} \, dy \right\} = (A6)
\]

\[
= \frac{(1-\alpha^2)^{2k+3}}{2k+2} - \frac{2k+3}{2k+2} \left( \alpha^2 (1-\alpha^2)^{\frac{2k+1}{2}} - (2k+1)\alpha^{2k+4} \int_{0}^{\alpha} \ln \left( \frac{1+\sqrt{1-\alpha^2}}{\alpha} \right) (\sinh y)^{2k} \, dy \right)
\]

Finally, by inserting this expression in Eq. (A3) and simplifying, we arrive to

\[
J_{\rho k} = \frac{(1-\alpha^2)^{2k+1}}{4(k+1)(k+2)\beta^{\frac{2k+3}{2}}} \left[ 2(k+1) - (2k+1)\alpha^2 \right] - (2k+1)\alpha^{2k+4} \int_{0}^{\alpha} \ln \left( \frac{1+\sqrt{1-\alpha^2}}{\alpha} \right) (\sinh y)^{2k} \, dy, \quad (A7)
\]

that is the expression of Eq. (A1).

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TABLE I: Some numerical characteristics of fermions for $a = 10^{-1}$ and different values of $\beta W_0$, $\beta$, $W_0$ and $\theta_0$. $\bar{R}$ and $\bar{M}$ are the dimensionless radius and mass of the equilibrium configurations, respectively.

| $\beta W_0$ | $\beta$    | $W_0$   | $\theta_0$ | $\bar{R}$       | $\bar{M}$   |
|-------------|------------|---------|-------------|-----------------|-------------|
| 0.1         | $10^{-5}$  | $10^4$  | $10^4$      | $4.13 \times 10^{-1}$ | $2.73 \times 10^{-2}$ |
|             | $10^{-4}$  | $10^3$  | $10^3$      | $4.14 \times 10^{-1}$ | $2.73 \times 10^{-2}$ |
|             | $10^{-3}$  | $10^2$  | $10^2$      | $4.29 \times 10^{-1}$ | $2.72 \times 10^{-2}$ |
|             | $10^{-2}$  | $10$    | $10$        | $5.22 \times 10^{-1}$ | $2.75 \times 10^{-2}$ |
|             | $10^{-1}$  | $1$     | $1$         | $1.09 \times 10^0$   | $5.09 \times 10^{-2}$ |
|             | $1$        | $10^{-1}$| $10^{-1}$   | $2.01 \times 10^0$   | $1.17 \times 10^{-1}$ |
|             | $10$       | $10^{-2}$| $10^{-2}$   | $3.56 \times 10^0$   | $2.31 \times 10^{-1}$ |
|             | $10^2$     | $10^{-3}$| $0$         | $6.31 \times 10^0$   | $4.24 \times 10^{-1}$ |
|             | $10^3$     | $10^{-4}$| $0$         | $1.12 \times 10^1$   | $7.62 \times 10^{-1}$ |
| 0.2         | $10^{-5}$  | $2 \times 10^4$ | $2 \times 10^4$ | $2.94 \times 10^{-1}$ | $3.25 \times 10^{-2}$ |
|             | $10^{-4}$  | $2 \times 10^3$ | $2 \times 10^3$ | $2.94 \times 10^{-1}$ | $3.25 \times 10^{-2}$ |
|             | $10^{-3}$  | $2 \times 10^2$ | $2 \times 10^2$ | $2.97 \times 10^{-1}$ | $3.25 \times 10^{-2}$ |
|             | $10^{-2}$  | $2 \times 10^1$ | $2 \times 10^1$ | $3.34 \times 10^{-1}$ | $3.29 \times 10^{-2}$ |
|             | $10^{-1}$  | $2 \times 10^0$ | $2 \times 10^0$ | $6.27 \times 10^{-1}$ | $4.87 \times 10^{-2}$ |
|             | $1$        | $2 \times 10^{-1}$ | $2 \times 10^{-1}$ | $1.19 \times 10^0$   | $1.11 \times 10^{-1}$ |
|             | $10$       | $2 \times 10^{-2}$ | $0$         | $2.09 \times 10^0$   | $2.21 \times 10^{-1}$ |
|             | $10^2$     | $2 \times 10^{-3}$ | $0$         | $3.68 \times 10^0$   | $4.06 \times 10^{-1}$ |
|             | $10^3$     | $2 \times 10^{-4}$ | $0$         | $6.53 \times 10^0$   | $7.30 \times 10^{-1}$ |
| 0.3         | $10^{-5}$  | $3 \times 10^4$ | $3 \times 10^4$ | $2.29 \times 10^{-1}$ | $3.16 \times 10^{-2}$ |
|             | $10^{-4}$  | $3 \times 10^3$ | $3 \times 10^3$ | $2.29 \times 10^{-1}$ | $3.16 \times 10^{-2}$ |
|             | $10^{-3}$  | $3 \times 10^2$ | $3 \times 10^2$ | $2.31 \times 10^{-1}$ | $3.17 \times 10^{-2}$ |
|             | $10^{-2}$  | $3 \times 10^1$ | $3 \times 10^1$ | $2.52 \times 10^{-1}$ | $3.21 \times 10^{-2}$ |
|             | $10^{-1}$  | $3 \times 10^0$ | $3 \times 10^0$ | $4.41 \times 10^{-1}$ | $4.31 \times 10^{-2}$ |
|             | $1$        | $3 \times 10^{-1}$ | $3 \times 10^{-1}$ | $8.54 \times 10^{-1}$ | $9.64 \times 10^{-2}$ |
|             | $10$       | $3 \times 10^{-2}$ | $0$         | $1.49 \times 10^0$   | $1.91 \times 10^{-1}$ |
|             | $10^2$     | $3 \times 10^{-3}$ | $0$         | $2.61 \times 10^0$   | $3.51 \times 10^{-1}$ |
|             | $10^3$     | $3 \times 10^{-4}$ | $0$         | $4.62 \times 10^0$   | $6.30 \times 10^{-1}$ |
| 0.4         | $10^{-5}$  | $4 \times 10^4$ | $4 \times 10^4$ | $1.87 \times 10^{-1}$ | $2.82 \times 10^{-2}$ |
|             | $10^{-4}$  | $4 \times 10^3$ | $4 \times 10^3$ | $1.87 \times 10^{-1}$ | $2.82 \times 10^{-2}$ |
|             | $10^{-3}$  | $4 \times 10^2$ | $4 \times 10^2$ | $1.88 \times 10^{-1}$ | $2.83 \times 10^{-2}$ |
|             | $10^{-2}$  | $4 \times 10^1$ | $4 \times 10^1$ | $2.02 \times 10^{-1}$ | $2.87 \times 10^{-2}$ |
| \( \beta W_0 \) | \( \beta \) | \( W_0 \) | \( \theta_0 \) | \( \hat{R} \) | \( \hat{M} \) |
|---|---|---|---|---|---|
| 10^{-1} | 4 \times 10^{0} | 4 \times 10^{0} | 3.42 \times 10^{-1} | 3.67 \times 10^{-2} |
| 1 | 4 \times 10^{-1} | 4 \times 10^{-1} | 6.63 \times 10^{-1} | 8.01 \times 10^{-2} |
| 10 | 4 \times 10^{-2} | 0 | 1.18 \times 10^{0} | 1.59 \times 10^{-1} |
| 10^2 | 4 \times 10^{-3} | 0 | 2.05 \times 10^{0} | 2.89 \times 10^{-1} |
| 10^3 | 4 \times 10^{-4} | 0 | 3.62 \times 10^{0} | 5.19 \times 10^{-1} |
| 0.5 | 10^{-5} | 5 \times 10^{4} | 5 \times 10^{4} | 1.61 \times 10^{-1} | 2.40 \times 10^{-2} |
| 10^{-4} | 5 \times 10^{3} | 5 \times 10^{3} | 1.61 \times 10^{-1} | 2.40 \times 10^{-2} |
| 10^{-3} | 5 \times 10^{2} | 5 \times 10^{2} | 1.62 \times 10^{-1} | 2.40 \times 10^{-2} |
| 10^{-2} | 5 \times 10^{1} | 5 \times 10^{1} | 1.73 \times 10^{-1} | 2.43 \times 10^{-2} |
| 10^{-1} | 5 \times 10^{0} | 5 \times 10^{0} | 2.91 \times 10^{-1} | 3.03 \times 10^{-2} |
| 1 | 5 \times 10^{-1} | 5 \times 10^{-1} | 6.08 \times 10^{-1} | 6.50 \times 10^{-2} |
| 10 | 5 \times 10^{-2} | 0 | 1.04 \times 10^{0} | 1.28 \times 10^{-1} |
| 10^2 | 5 \times 10^{-3} | 0 | 1.79 \times 10^{0} | 2.31 \times 10^{-1} |
| 10^3 | 5 \times 10^{-4} | 0 | 3.15 \times 10^{0} | 4.13 \times 10^{-1} |
| 0.6 | 10^{-5} | 6 \times 10^{4} | 6 \times 10^{4} | 1.49 \times 10^{-1} | 1.98 \times 10^{-2} |
| 10^{-4} | 6 \times 10^{3} | 6 \times 10^{3} | 1.49 \times 10^{-1} | 1.98 \times 10^{-2} |
| 10^{-3} | 6 \times 10^{2} | 6 \times 10^{2} | 1.50 \times 10^{-1} | 1.99 \times 10^{-2} |
| 10^{-2} | 6 \times 10^{1} | 6 \times 10^{1} | 1.61 \times 10^{-1} | 2.01 \times 10^{-2} |
| 10^{-1} | 6 \times 10^{0} | 6 \times 10^{0} | 2.86 \times 10^{-1} | 2.47 \times 10^{-2} |
| 1 | 6 \times 10^{-1} | 6 \times 10^{-1} | 6.54 \times 10^{-1} | 5.26 \times 10^{-2} |
| 10 | 6 \times 10^{-2} | 0 | 1.12 \times 10^{0} | 1.01 \times 10^{-1} |
| 10^2 | 6 \times 10^{-3} | 0 | 1.94 \times 10^{0} | 1.81 \times 10^{-1} |
| 10^3 | 6 \times 10^{-4} | 0 | 3.42 \times 10^{0} | 3.22 \times 10^{-1} |
| 0.7 | 10^{-5} | 7 \times 10^{4} | 7 \times 10^{4} | 1.63 \times 10^{-1} | 1.71 \times 10^{-2} |
| 10^{-4} | 7 \times 10^{3} | 7 \times 10^{3} | 1.63 \times 10^{-1} | 1.71 \times 10^{-2} |
| 10^{-3} | 7 \times 10^{2} | 7 \times 10^{2} | 1.64 \times 10^{-1} | 1.71 \times 10^{-2} |
| 10^{-2} | 7 \times 10^{1} | 7 \times 10^{1} | 1.80 \times 10^{-1} | 1.72 \times 10^{-2} |
| 10^{-1} | 7 \times 10^{0} | 7 \times 10^{0} | 3.78 \times 10^{-1} | 2.14 \times 10^{-2} |
| 1 | 7 \times 10^{-1} | 7 \times 10^{-1} | 1.03 \times 10^{0} | 4.77 \times 10^{-2} |
| 10 | 7 \times 10^{-2} | 0 | 2.06 \times 10^{0} | 9.08 \times 10^{-2} |
| 10^2 | 7 \times 10^{-3} | 0 | 4.04 \times 10^{0} | 1.61 \times 10^{-1} |
| 10^3 | 7 \times 10^{-4} | 0 | 7.56 \times 10^{0} | 2.88 \times 10^{-1} |
| 0.8 | 10^{-5} | 8 \times 10^{4} | 8 \times 10^{4} | 2.00 \times 10^{-1} | 1.84 \times 10^{-2} |
| 10^{-4} | 8 \times 10^{3} | 8 \times 10^{3} | 2.00 \times 10^{-1} | 1.84 \times 10^{-2} |
Table I (continued).

| $\beta W_0$ | $\beta$ | $W_0$ | $\theta_0$ | $\hat{R}$ | $\hat{M}$ |
|--------------|--------|-------|------------|-----------|-----------|
| $10^{-3}$    | $8 \times 10^2$ | $8 \times 10^2$ | $2.03 \times 10^{-1}$ | $1.84 \times 10^{-2}$ |           |
| $10^{-2}$    | $8 \times 10^1$ | $8 \times 10^1$ | $2.26 \times 10^{-1}$ | $1.85 \times 10^{-2}$ |           |
| $10^{-1}$    | $8 \times 10^0$ | $8 \times 10^0$ | $5.33 \times 10^{-1}$ | $2.46 \times 10^{-2}$ |           |
| 1            | $8 \times 10^{-1}$ | $8 \times 10^{-1}$ | $1.26 \times 10^0$ | $6.01 \times 10^{-2}$ |           |
| 10           | $8 \times 10^{-2}$ | 0 | $2.34 \times 10^0$ | $1.25 \times 10^{-1}$ |           |
| $10^2$       | $8 \times 10^{-3}$ | 0 | $4.05 \times 10^0$ | $2.37 \times 10^{-1}$ |           |
| $10^3$       | $8 \times 10^{-4}$ | 0 | $7.07 \times 10^0$ | $4.34 \times 10^{-1}$ |           |

$\beta W_0 = 0.9$
| $10^{-5}$ | $9 \times 10^4$ | $9 \times 10^4$ | $1.90 \times 10^{-1}$ | $2.03 \times 10^{-2}$ |           |
| $10^{-4}$ | $9 \times 10^3$ | $9 \times 10^3$ | $1.90 \times 10^{-1}$ | $2.03 \times 10^{-2}$ |           |
| $10^{-3}$ | $9 \times 10^2$ | $9 \times 10^2$ | $1.92 \times 10^{-1}$ | $2.03 \times 10^{-2}$ |           |
| $10^{-2}$ | $9 \times 10^1$ | $9 \times 10^1$ | $2.11 \times 10^{-1}$ | $2.04 \times 10^{-2}$ |           |
| $10^{-1}$ | $9 \times 10^0$ | $9 \times 10^0$ | $4.37 \times 10^{-1}$ | $2.63 \times 10^{-2}$ |           |
| 1          | $9 \times 10^{-1}$ | $9 \times 10^{-1}$ | $9.95 \times 10^{-1}$ | $6.00 \times 10^{-2}$ |           |
| 10         | $9 \times 10^{-2}$ | 0 | $1.82 \times 10^0$ | $1.18 \times 10^{-1}$ |           |
| $10^2$     | $9 \times 10^{-3}$ | 0 | $3.17 \times 10^0$ | $2.11 \times 10^{-1}$ |           |
| $10^3$     | $9 \times 10^{-4}$ | 0 | $5.63 \times 10^0$ | $3.73 \times 10^{-1}$ |           |

TABLE II: The same as Table I, for $a = 10^{-3}$.

| $\beta W_0$ | $\beta$ | $W_0$ | $\theta_0$ | $\hat{R}$ | $\hat{M}$ |
|--------------|--------|-------|------------|-----------|-----------|
| 0.1          | $10^{-5}$ | $10^4$ | $10^4$ | $5.70 \times 10^{-2}$ | $5.30 \times 10^{-3}$ |           |
| $10^{-4}$    | $10^3$ | $10^3$ | $5.71 \times 10^{-2}$ | $5.29 \times 10^{-3}$ |           |
| $10^{-3}$    | $10^2$ | $10^2$ | $5.80 \times 10^{-2}$ | $5.28 \times 10^{-3}$ |           |
| $10^{-2}$    | 10     | 10    | $6.74 \times 10^{-2}$ | $5.34 \times 10^{-3}$ |           |
| $10^{-1}$    | 1      | 1     | $1.12 \times 10^{-1}$ | $7.67 \times 10^{-3}$ |           |
| 1            | $10^{-1}$ | 1     | $1.99 \times 10^{-1}$ | $1.36 \times 10^{-2}$ |           |
| 10           | $10^{-2}$ | 1     | $3.54 \times 10^{-1}$ | $2.42 \times 10^{-2}$ |           |
| $10^2$       | $10^{-3}$ | 0     | $6.29 \times 10^{-1}$ | $4.31 \times 10^{-2}$ |           |
| $10^3$       | $10^{-4}$ | 0     | $1.12 \times 10^0$ | $7.66 \times 10^{-2}$ |           |
| 0.2          | $10^{-5}$ | $2 \times 10^4$ | $2 \times 10^4$ | $3.87 \times 10^{-2}$ | $5.73 \times 10^{-3}$ |           |
| $10^{-4}$    | $2 \times 10^3$ | $2 \times 10^3$ | $3.87 \times 10^{-2}$ | $5.73 \times 10^{-3}$ |           |
| $10^{-3}$    | $2 \times 10^2$ | $2 \times 10^2$ | $3.90 \times 10^{-2}$ | $5.73 \times 10^{-3}$ |           |
| $10^{-2}$    | $2 \times 10^1$ | $2 \times 10^1$ | $4.25 \times 10^{-2}$ | $5.80 \times 10^{-3}$ |           |
| $10^{-1}$    | $2 \times 10^0$ | $2 \times 10^0$ | $6.54 \times 10^{-2}$ | $7.44 \times 10^{-3}$ |           |
| 1            | $2 \times 10^{-1}$ | $2 \times 10^{-1}$ | $1.16 \times 10^{-1}$ | $1.30 \times 10^{-2}$ |           |
Table II (continued).

| $\beta W_0$ | $\beta$ | $W_0$   | $\theta_0$ | $\tilde{R}$ | $\tilde{M}$ |
|--------------|---------|---------|------------|-------------|-------------|
| 10           | $2 \times 10^{-2}$ | 0       | $2.07 \times 10^{-1}$ | $2.32 \times 10^{-2}$ |             |
| 10^2         | $2 \times 10^{-3}$ | 0       | $3.67 \times 10^{-1}$ | $4.12 \times 10^{-2}$ |             |
| 10^3         | $2 \times 10^{-4}$ | 0       | $6.52 \times 10^{-1}$ | $7.33 \times 10^{-2}$ |             |
| 0.3          | $10^{-5}$    | $3 \times 10^4$ | $3 \times 10^4$ | $2.95 \times 10^{-2}$ | $5.26 \times 10^{-3}$ |
| 0.4          | $10^{-5}$    | $4 \times 10^4$ | $4 \times 10^4$ | $2.39 \times 10^{-2}$ | $4.50 \times 10^{-3}$ |
| 0.5          | $10^{-5}$    | $5 \times 10^4$ | $5 \times 10^4$ | $2.07 \times 10^{-2}$ | $3.67 \times 10^{-3}$ |
| 0.6          | $10^{-5}$    | $6 \times 10^4$ | $6 \times 10^4$ | $2.04 \times 10^{-2}$ | $2.92 \times 10^{-3}$ |
| $\beta W_0$ | $\beta$ | $W_0$  | $\theta_0$ | $\hat{R}$  | $\hat{M}$  |
|---------|--------|--------|------------|------------|------------|
| $10^{-1}$ | 6 $\times 10^0$ | 6 $\times 10^0$ | 3.22 $\times 10^{-2}$ | 3.44 $\times 10^{-3}$ |
| 1       | 6 $\times 10^{-1}$ | 6 $\times 10^{-1}$ | 6.05 $\times 10^{-2}$ | 5.73 $\times 10^{-3}$ |
| 10      | 6 $\times 10^{-2}$ | 0 | 1.09 $\times 10^{-1}$ | 1.03 $\times 10^{-2}$ |
| 10$^2$  | 6 $\times 10^{-3}$ | 0 | 1.92 $\times 10^{-1}$ | 1.81 $\times 10^{-2}$ |
| 10$^3$  | 6 $\times 10^{-4}$ | 0 | 3.40 $\times 10^{-1}$ | 3.22 $\times 10^{-2}$ |
| 0.7     | $10^{-5}$ | 7 $\times 10^4$ | 7 $\times 10^4$ | 2.90 $\times 10^{-2}$ | 2.48 $\times 10^{-3}$ |
|         | $10^{-4}$ | 7 $\times 10^3$ | 7 $\times 10^3$ | 2.90 $\times 10^{-2}$ | 2.48 $\times 10^{-3}$ |
|         | $10^{-3}$ | 7 $\times 10^2$ | 7 $\times 10^2$ | 2.93 $\times 10^{-2}$ | 2.48 $\times 10^{-3}$ |
|         | $10^{-2}$ | 7 $\times 10^1$ | 7 $\times 10^1$ | 3.23 $\times 10^{-2}$ | 2.49 $\times 10^{-3}$ |
|         | $10^{-1}$ | 7 $\times 10^0$ | 7 $\times 10^0$ | 6.23 $\times 10^{-2}$ | 2.94 $\times 10^{-3}$ |
| 1       | $10^{-1}$ | 7 $\times 10^{-1}$ | 7 $\times 10^{-1}$ | 1.37 $\times 10^{-1}$ | 5.12 $\times 10^{-3}$ |
| 10      | 7 $\times 10^{-2}$ | 0 | 2.48 $\times 10^{-1}$ | 9.20 $\times 10^{-3}$ |
| 10$^2$  | 7 $\times 10^{-3}$ | 0 | 4.37 $\times 10^{-1}$ | 1.62 $\times 10^{-2}$ |
| 10$^3$  | 7 $\times 10^{-4}$ | 0 | 7.78 $\times 10^{-1}$ | 2.89 $\times 10^{-2}$ |
| 0.8     | $10^{-5}$ | 8 $\times 10^4$ | 8 $\times 10^4$ | 3.51 $\times 10^{-2}$ | 3.44 $\times 10^{-3}$ |
|         | $10^{-4}$ | 8 $\times 10^3$ | 8 $\times 10^3$ | 3.52 $\times 10^{-2}$ | 3.44 $\times 10^{-3}$ |
|         | $10^{-3}$ | 8 $\times 10^2$ | 8 $\times 10^2$ | 3.55 $\times 10^{-2}$ | 3.44 $\times 10^{-3}$ |
|         | $10^{-2}$ | 8 $\times 10^1$ | 8 $\times 10^1$ | 3.95 $\times 10^{-2}$ | 3.45 $\times 10^{-3}$ |
|         | $10^{-1}$ | 8 $\times 10^0$ | 8 $\times 10^0$ | 6.91 $\times 10^{-2}$ | 4.35 $\times 10^{-3}$ |
| 1       | $10^{-1}$ | 8 $\times 10^{-1}$ | 8 $\times 10^{-1}$ | 1.25 $\times 10^{-1}$ | 7.78 $\times 10^{-3}$ |
| 10      | 8 $\times 10^{-2}$ | 0 | 2.23 $\times 10^{-1}$ | 1.40 $\times 10^{-2}$ |
| 10$^2$  | 8 $\times 10^{-3}$ | 0 | 3.93 $\times 10^{-1}$ | 2.47 $\times 10^{-2}$ |
| 10$^3$  | 8 $\times 10^{-4}$ | 0 | 6.98 $\times 10^{-1}$ | 4.39 $\times 10^{-2}$ |
| 0.9     | $10^{-5}$ | 9 $\times 10^4$ | 9 $\times 10^4$ | 2.84 $\times 10^{-2}$ | 3.22 $\times 10^{-3}$ |
|         | $10^{-4}$ | 9 $\times 10^3$ | 9 $\times 10^3$ | 2.84 $\times 10^{-2}$ | 3.22 $\times 10^{-3}$ |
|         | $10^{-3}$ | 9 $\times 10^2$ | 9 $\times 10^2$ | 2.87 $\times 10^{-2}$ | 3.22 $\times 10^{-3}$ |
|         | $10^{-2}$ | 9 $\times 10^1$ | 9 $\times 10^1$ | 3.11 $\times 10^{-2}$ | 3.24 $\times 10^{-3}$ |
|         | $10^{-1}$ | 9 $\times 10^0$ | 9 $\times 10^0$ | 5.21 $\times 10^{-2}$ | 3.85 $\times 10^{-3}$ |
| 1       | $10^{-1}$ | 9 $\times 10^{-1}$ | 9 $\times 10^{-1}$ | 1.79 $\times 10^{-1}$ | 1.17 $\times 10^{-2}$ |
| 10      | 9 $\times 10^{-2}$ | 0 | 1.80 $\times 10^{-1}$ | 1.18 $\times 10^{-2}$ |
| 10$^2$  | 9 $\times 10^{-3}$ | 0 | 3.18 $\times 10^{-1}$ | 2.09 $\times 10^{-2}$ |
| 10$^3$  | 9 $\times 10^{-4}$ | 0 | 5.65 $\times 10^{-1}$ | 3.71 $\times 10^{-2}$ |
TABLE III: The same as Table I, for $a = 10^{-5}$.

| $\beta W_0$ | $\beta$ | $W_0$ | $\theta_0$ | $\bar{R}$      | $\bar{M}$      |
|-------------|--------|-------|------------|----------------|----------------|
| 0.1         | $10^{-5}$ | $10^4$ | $10^4$     | $5.70 \times 10^{-3}$ | $5.34 \times 10^{-4}$ |
|             | $10^{-4}$ | $10^3$ | $10^3$     | $5.70 \times 10^{-3}$ | $5.34 \times 10^{-4}$ |
|             | $10^{-3}$ | $10^2$ | $10^2$     | $5.80 \times 10^{-3}$ | $5.33 \times 10^{-4}$ |
|             | $10^{-2}$ | 10    | 10         | $6.70 \times 10^{-3}$ | $5.39 \times 10^{-4}$ |
|             | $10^{-1}$ | 1     | 1          | $1.11 \times 10^{-2}$ | $7.70 \times 10^{-4}$ |
|             | 1       | $10^{-1}$ | $10^{-1}$ | $1.98 \times 10^{-2}$ | $1.36 \times 10^{-3}$ |
|             | 10      | $10^{-2}$ | $10^{-2}$ | $3.53 \times 10^{-2}$ | $2.42 \times 10^{-3}$ |
|             | $10^2$   | $10^{-3}$ | 0         | $6.29 \times 10^{-2}$ | $4.31 \times 10^{-3}$ |
|             | $10^3$   | $10^{-4}$ | 0         | $1.12 \times 10^{-1}$ | $7.66 \times 10^{-3}$ |
| 0.2         | $10^{-5}$ | $2 \times 10^4$ | $2 \times 10^4$ | $3.87 \times 10^{-3}$ | $5.77 \times 10^{-4}$ |
|             | $10^{-4}$ | $2 \times 10^3$ | $2 \times 10^3$ | $3.87 \times 10^{-3}$ | $5.77 \times 10^{-4}$ |
|             | $10^{-3}$ | $2 \times 10^2$ | $2 \times 10^2$ | $3.91 \times 10^{-3}$ | $5.77 \times 10^{-4}$ |
|             | $10^{-2}$ | $2 \times 10^1$ | $2 \times 10^1$ | $4.26 \times 10^{-3}$ | $5.84 \times 10^{-4}$ |
|             | $10^{-1}$ | $2 \times 10^0$ | $2 \times 10^0$ | $6.50 \times 10^{-3}$ | $7.48 \times 10^{-4}$ |
|             | 1       | $2 \times 10^{-1}$ | $2 \times 10^{-1}$ | $1.15 \times 10^{-2}$ | $1.30 \times 10^{-3}$ |
|             | 10      | $2 \times 10^{-2}$ | 0         | $2.06 \times 10^{-2}$ | $2.32 \times 10^{-3}$ |
|             | $10^2$   | $2 \times 10^{-3}$ | 0         | $3.66 \times 10^{-2}$ | $4.13 \times 10^{-3}$ |
|             | $10^3$   | $2 \times 10^{-4}$ | 0         | $6.52 \times 10^{-2}$ | $7.33 \times 10^{-3}$ |
| 0.3         | $10^{-5}$ | $3 \times 10^4$ | $3 \times 10^4$ | $2.95 \times 10^{-3}$ | $5.29 \times 10^{-4}$ |
|             | $10^{-4}$ | $3 \times 10^3$ | $3 \times 10^3$ | $2.95 \times 10^{-3}$ | $5.29 \times 10^{-4}$ |
|             | $10^{-3}$ | $3 \times 10^2$ | $3 \times 10^2$ | $2.97 \times 10^{-3}$ | $5.29 \times 10^{-4}$ |
|             | $10^{-2}$ | $3 \times 10^1$ | $3 \times 10^1$ | $3.16 \times 10^{-3}$ | $5.37 \times 10^{-4}$ |
|             | $10^{-1}$ | $3 \times 10^0$ | $3 \times 10^0$ | $4.62 \times 10^{-3}$ | $6.56 \times 10^{-4}$ |
|             | 1       | $3 \times 10^{-1}$ | $3 \times 10^{-1}$ | $8.20 \times 10^{-3}$ | $1.13 \times 10^{-3}$ |
|             | 10      | $3 \times 10^{-2}$ | 0         | $1.46 \times 10^{-2}$ | $2.00 \times 10^{-3}$ |
|             | $10^2$   | $3 \times 10^{-3}$ | 0         | $2.59 \times 10^{-2}$ | $3.56 \times 10^{-3}$ |
|             | $10^3$   | $3 \times 10^{-4}$ | 0         | $4.61 \times 10^{-2}$ | $6.33 \times 10^{-3}$ |
| 0.4         | $10^{-5}$ | $4 \times 10^4$ | $4 \times 10^4$ | $2.39 \times 10^{-3}$ | $4.52 \times 10^{-4}$ |
|             | $10^{-4}$ | $4 \times 10^3$ | $4 \times 10^3$ | $2.39 \times 10^{-3}$ | $4.52 \times 10^{-4}$ |
|             | $10^{-3}$ | $4 \times 10^2$ | $4 \times 10^2$ | $2.41 \times 10^{-3}$ | $4.53 \times 10^{-4}$ |
|             | $10^{-2}$ | $4 \times 10^1$ | $4 \times 10^1$ | $2.54 \times 10^{-3}$ | $4.60 \times 10^{-4}$ |
|             | $10^{-1}$ | $4 \times 10^0$ | $4 \times 10^0$ | $3.60 \times 10^{-3}$ | $5.48 \times 10^{-4}$ |
|             | 1       | $4 \times 10^{-1}$ | $4 \times 10^{-1}$ | $6.40 \times 10^{-3}$ | $9.27 \times 10^{-4}$ |
|             | 10      | $4 \times 10^{-2}$ | 0         | $1.14 \times 10^{-2}$ | $1.66 \times 10^{-3}$ |
Table III (continued).

| $\beta W_0$ | $\beta$ | $W_0$   | $\theta_0$ | $\hat{R}$    | $\hat{M}$    |
|------------|--------|---------|------------|--------------|--------------|
| $10^2$     | $4 \times 10^{-3}$ | 0       | $2.02 \times 10^{-2}$ | $2.93 \times 10^{-3}$ |
| $10^3$     | $4 \times 10^{-4}$  | 0       | $3.60 \times 10^{-2}$ | $5.21 \times 10^{-3}$ |
| 0.5        | $10^{-5}$ | $5 \times 10^4$ | $5 \times 10^4$ | $2.07 \times 10^{-3}$ | $3.69 \times 10^{-4}$ |
|            | $10^{-4}$ | $5 \times 10^3$ | $5 \times 10^3$ | $2.07 \times 10^{-3}$ | $3.70 \times 10^{-4}$ |
|            | $10^{-3}$ | $5 \times 10^2$ | $5 \times 10^2$ | $2.08 \times 10^{-3}$ | $3.70 \times 10^{-4}$ |
|            | $10^{-2}$ | $5 \times 10^1$ | $5 \times 10^1$ | $2.19 \times 10^{-3}$ | $3.76 \times 10^{-4}$ |
|            | $10^{-1}$ | $5 \times 10^0$ | $5 \times 10^0$ | $3.09 \times 10^{-3}$ | $4.41 \times 10^{-4}$ |
|            | 1       | $5 \times 10^{-1}$ | $5 \times 10^{-1}$ | $5.56 \times 10^{-2}$ | $7.39 \times 10^{-4}$ |
|            | 10      | $5 \times 10^{-2}$ | 0       | $9.97 \times 10^{-3}$ | $1.32 \times 10^{-3}$ |
|            | $10^2$  | $5 \times 10^{-3}$ | 0       | $1.76 \times 10^{-2}$ | $2.33 \times 10^{-3}$ |
|            | $10^3$  | $5 \times 10^{-4}$ | 0       | $3.13 \times 10^{-2}$ | $4.15 \times 10^{-3}$ |
| 0.6        | $10^{-5}$ | $6 \times 10^4$ | $6 \times 10^4$ | $2.04 \times 10^{-3}$ | $2.92 \times 10^{-4}$ |
|            | $10^{-4}$ | $6 \times 10^3$ | $6 \times 10^3$ | $2.04 \times 10^{-3}$ | $2.92 \times 10^{-4}$ |
|            | $10^{-3}$ | $6 \times 10^2$ | $6 \times 10^2$ | $2.06 \times 10^{-3}$ | $2.93 \times 10^{-4}$ |
|            | $10^{-2}$ | $6 \times 10^1$ | $6 \times 10^1$ | $2.17 \times 10^{-3}$ | $2.96 \times 10^{-4}$ |
|            | $10^{-1}$ | $6 \times 10^0$ | $6 \times 10^0$ | $3.21 \times 10^{-3}$ | $3.44 \times 10^{-4}$ |
|            | 1       | $6 \times 10^{-1}$ | $6 \times 10^{-1}$ | $6.04 \times 10^{-3}$ | $5.73 \times 10^{-4}$ |
|            | 10      | $6 \times 10^{-2}$ | 0       | $1.08 \times 10^{-2}$ | $1.03 \times 10^{-3}$ |
|            | $10^2$  | $6 \times 10^{-3}$ | 0       | $1.91 \times 10^{-2}$ | $1.81 \times 10^{-3}$ |
|            | $10^3$  | $6 \times 10^{-4}$ | 0       | $3.40 \times 10^{-2}$ | $3.22 \times 10^{-3}$ |
| 0.7        | $10^{-5}$ | $7 \times 10^4$ | $7 \times 10^4$ | $2.95 \times 10^{-3}$ | $2.48 \times 10^{-4}$ |
|            | $10^{-4}$ | $7 \times 10^3$ | $7 \times 10^3$ | $2.96 \times 10^{-3}$ | $2.48 \times 10^{-4}$ |
|            | $10^{-3}$ | $7 \times 10^2$ | $7 \times 10^2$ | $2.99 \times 10^{-3}$ | $2.48 \times 10^{-4}$ |
|            | $10^{-2}$ | $7 \times 10^1$ | $7 \times 10^1$ | $3.29 \times 10^{-3}$ | $2.49 \times 10^{-4}$ |
|            | $10^{-1}$ | $7 \times 10^0$ | $7 \times 10^0$ | $6.38 \times 10^{-3}$ | $2.94 \times 10^{-4}$ |
|            | 1       | $7 \times 10^{-1}$ | $7 \times 10^{-1}$ | $1.38 \times 10^{-2}$ | $5.13 \times 10^{-4}$ |
|            | 10      | $7 \times 10^{-2}$ | 0       | $2.49 \times 10^{-2}$ | $9.19 \times 10^{-4}$ |
|            | $10^2$  | $7 \times 10^{-3}$ | 0       | $4.38 \times 10^{-2}$ | $1.62 \times 10^{-3}$ |
|            | $10^3$  | $7 \times 10^{-4}$ | 0       | $7.78 \times 10^{-1}$ | $2.89 \times 10^{-3}$ |
| 0.8        | $10^{-5}$ | $8 \times 10^4$ | $8 \times 10^4$ | $3.52 \times 10^{-3}$ | $3.49 \times 10^{-4}$ |
|            | $10^{-4}$ | $8 \times 10^3$ | $8 \times 10^3$ | $3.53 \times 10^{-3}$ | $3.49 \times 10^{-4}$ |
|            | $10^{-3}$ | $8 \times 10^2$ | $8 \times 10^2$ | $3.56 \times 10^{-3}$ | $3.49 \times 10^{-4}$ |
|            | $10^{-2}$ | $8 \times 10^1$ | $8 \times 10^1$ | $3.96 \times 10^{-3}$ | $3.51 \times 10^{-4}$ |
|            | $10^{-1}$ | $8 \times 10^0$ | $8 \times 10^0$ | $6.86 \times 10^{-3}$ | $4.41 \times 10^{-4}$ |
Table III (continued).

| $\beta W_0$ | $\beta$  | $W_0$  | $\theta_0$ | $\tilde{R}$  | $\tilde{M}$    |
|------------|---------|--------|------------|-------------|----------------|
| 1          | $8 \times 10^{-1}$ | $8 \times 10^{-1}$ | $1.24 \times 10^{-2}$ | $7.81 \times 10^{-4}$ |
| 10         | $8 \times 10^{-2}$ | 0      | $2.25 \times 10^{-2}$ | $1.41 \times 10^{-3}$ |
| $10^2$     | $8 \times 10^{-3}$ | 0      | $3.92 \times 10^{-2}$ | $2.48 \times 10^{-3}$ |
| $10^3$     | $8 \times 10^{-4}$ | 0      | $6.98 \times 10^{-2}$ | $4.39 \times 10^{-3}$ |
| 0.9 $10^{-5}$ | $9 \times 10^4$ | $9 \times 10^4$ | $2.85 \times 10^{-3}$ | $3.21 \times 10^{-4}$ |
| $10^{-4}$  | $9 \times 10^3$ | $9 \times 10^3$ | $2.85 \times 10^{-3}$ | $3.21 \times 10^{-4}$ |
| $10^{-3}$  | $9 \times 10^2$ | $9 \times 10^2$ | $2.88 \times 10^{-3}$ | $3.21 \times 10^{-4}$ |
| $10^{-2}$  | $9 \times 10^1$ | $9 \times 10^1$ | $3.12 \times 10^{-3}$ | $3.24 \times 10^{-4}$ |
| $10^{-1}$  | $9 \times 10^0$ | $9 \times 10^0$ | $5.23 \times 10^{-3}$ | $3.84 \times 10^{-4}$ |
| 1          | $9 \times 10^{-1}$ | $9 \times 10^{-1}$ | $1.00 \times 10^{-2}$ | $6.59 \times 10^{-4}$ |
| 10         | $9 \times 10^{-2}$ | 0      | $1.81 \times 10^{-2}$ | $1.19 \times 10^{-3}$ |
| $10^2$     | $9 \times 10^{-3}$ | 0      | $3.16 \times 10^{-2}$ | $2.08 \times 10^{-3}$ |
| $10^3$     | $9 \times 10^{-4}$ | 0      | $5.65 \times 10^{-2}$ | $3.71 \times 10^{-3}$ |
FIG. 1. Values of the ratio of the velocities $\eta$ as a function of the relative radius $r/R$ for different values of $\theta_0$, with $a = 10^{-3}$, $\beta = 10^{-2}$ and $W_0 = 20$.

FIG. 2. Values of the ratio of the velocities $\eta$ for values of $\beta \geq 1$, with $a = 10^{-1}$ and $\beta W_0 = 0.3$. 
FIG. 3. The same as in Fig. 2 at small values of $\beta$, for $a = 10^{-1}$ and $\beta W_0 = 0.6$.

FIG. 4. Mass of the configurations as a function of the central density at different values of $a$, for $\beta = 10^{-5}$. The quantities are dimensionless.
FIG. 5. Mass of the configurations as a function of the central density at different values of $\beta$, for $a = 1$.

The quantities are dimensionless.

FIG. 6. The same as in Fig. 5, for $a = 10^{-5}$. 
FIG. 7. The same as in Fig. 5 at different values of $\theta_R$, in the isotropic limit $a \to \infty$.

FIG. 8. Relative density $\rho/\rho_0$ as a function of the dimensionless radial coordinate $r/\xi$ at different values of $\theta_0$, for $a = 10^{-3}$, $\beta W_0 = 0.5$ and $\beta = 10^{-3}$. 
FIG. 9. Relative density $\rho/\rho_0$ for values of $\beta \geq 1$, with $a = 10^{-3}$ and $\beta W_0 = 0.6$.

FIG. 10. The same as in Fig. 9 at small values of $\beta$, for $a = 10^{-1}$ and $\beta W_0 = 0.1$. 
FIG. 11. Behavior of the function $F$ as a function of the relative radius $r/R$, for $\beta W_0 = 0.4$ and $a = 10^{-5}$. The values of $\beta$, from the bottom to the top, are $10^{-2}$ (solid line), $10^{-3}$ (dashed line), $10^{-4}$ (dotted line) and $10^{-5}$ (dash-dotted line), respectively.

FIG. 12. The same as in Fig. 11, for $\beta W_0 = 0.75$, $\beta = 10^{-4}$ and different values of $a$. 
FIG. 13. Mass of the particles, according to Eq. (44), for $\rho_c = 10^{15} \text{ g/cm}^3$, $a = 10^{-5}$ and $\beta = 10^{-5}, 10^{-4}, 10^{-3}$. The masses are in GeV.

FIG. 14. The same as in Fig. 13, for $\rho_c = 10^9 \text{ g/cm}^3$, $\beta = 10^{-3}$ and $a = 10^{-1}, 10^{-3}, 10^{-5}$.