Dipoles at $\nu = 1$

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We consider the problem of Bosonic particles interacting repulsively in a strong magnetic field at the filling factor $\nu = 1$. We project the system in the Lowest Landau Level and map the dynamics into an interacting Fermion system. We study the resulting Hamiltonian in the Hartree–Fock approximation in the case of a $\delta$ repulsive potential. The physical picture which emerges is in agreement with the proposal of N. Read that the composite Fermions behave as a gas of dipoles. We argue that the consequence of this is that the composite Fermions interact with screened short range interactions. We develop a Landau theory which we also expect to describe the physical $\nu = 1/2$ Fermionic state. The Form factor, the effective mass and the conductivity are analyzed in this model.

1 Introduction

There has been a renewed interest in the quantum Hall effect when the filling factor is a fraction with an even denominator. Willets and his collaborators have observed an anomalous behavior in the surface acoustic wave propagations near $\nu = 1/2$ and $\nu = 1/4$. A remarkable outcome of their experiments is that they probe a longitudinal conductivity $\sigma_{xx} (q, \omega)$ increasing linearly with the wave vector $q$. Halperin, Lee and Read have suggested that the system exhibits a Fermi liquid behavior at this particular value. Another approach followed by Rezayi and Read and Haldane et al. consists in obtaining trial wave functions which enable to study numerically the properties of the system at this filling factor. In these studies the cyclotron frequency is supposed to be sufficiently large so that the only relevant excitations are confined to the lowest Landau level. Here we introduce a model which accounts for the success of these trial wave functions and enables to compute the physical properties of the system in the infinite cyclotron energy limit. This is a different approach than the Chern Simon field theory which mixes the Landau levels and requires the mass of the electron as a parameter of the theory. We do not address the physical problem of electrons in a magnetic field at $\nu = 1/2$. Instead we have considered the problem of Bosonic particles interacting repulsively in a magnetic field at a filling factor $\nu = 1$. Although it may at first look quite different, the problem of formation of a Fermi sea is essentially the same as in the $\nu = 1/2$ case. If one applies the analyses of the composite Fermions or the Chern Simon approach to such a system, one is essentially led to the same picture of Fermi sea formation as in the $\nu = 1/2$ case. We have also verified this hypothesis by performing a numerical simulation for a small system on a sphere (see figure 1). The main reason why this is simpler theoretical problem to look at than the $\nu = 1/2$ physical problem is that the wave function one needs to start from is the Slater determinant of the lowest Landau level one body wave functions which is a much simpler object to consider than the Laughlin $\nu = 2$ wave function which one would have to use in the $\nu = 1/2$ case.

Read has interpreted the fluxes attached to the electron as physical vortices bound to it. We believe that his proposal differs considerably from the mean field interpretation for the following reason. The mean field treats the composite electron as a charged particle which couples minimally to the electro-magnetic field. In Reeds picture, the vortices carry a charge equal to minus one half of that of the electron so that the bound state must be viewed as a neutral particle which propagates in a constant charge background. In this case the response to an electric field

\*This lecture is issued from an unpublished paper on $\nu = 1$ Bosons ("Composite fermions and confinement" March 1996). Although some results are outdated, some aspects considered here have not been discussed in the recent literature. I give a list of recent references on the subject for the interested reader [11-15].
depends on the internal structure of the composite object. We are led to this picture in the $\nu = 1$ case. The essential simplification is that there is a single vortex coupled to the Boson, this vortex is a Fermion carrying the opposite charge as the Boson and we can use a second quantized formalism to analyse the model. The bound state is then a dipole whose structure was discovered long ago.

Away from $\nu = 1/2$, the vortices no longer carry the opposite charge as the Fermion and we expect the dipoles to carry an electric charge proportional $\Delta B$ the difference of the magnetic field with $B_{1/2}$.

2 The Microscopic Model

We consider $N$ particles of identical charge interacting with a repulsive force in a domain of area $\Omega$ thread by a magnetic field $B$ so that the flux per unit area is equal to one. We take units where $\hbar = 1$ and the magnetic length $l = \sqrt{\hbar c/eB} = 1$. We assume that the dynamic can be restricted to the Lowest Landau Level. The one body Hamiltonian has $N$ degenerate eigenstates, thus in the case where the particles are Fermions the only accessible state is given by the Slater determinant of the one body wave functions. This state will define the vacuum of the theory. We now discuss the case where there are two sets of particles obeying distinct statistics. The first set contains $N_1$ Fermions and the second set contains $N_2$ Bosons. We keep the sum $N_1 + N_2 = N$ fixed so that the filling factor remains equal to one. We also keep the interaction equal between all the particles. It is instructive to first look at the case of 1 Boson interacting with $N - 1$ Fermions. By performing a particle hole transformation on the Fermions, we can equivalently regard this as a Boson interacting with a hole. This problem has been studied by Kallin and Halperin. A surprising outcome is that the wave functions which describe this two body state are independent of the potential and are given by the ground state eigen-functions of the free Hamiltonian. They describe a neutral dipole with a momentum perpendicular to its canonical momentum. Our hypothesis is that the system where several Bosons interact with the same number of holes reorganizes into neutral fermionic dipoles with a small residual interaction. In order to test this hypotheses we have also performed a numerical evaluation of the ground state energies for a mixed system consisting of $N_1$ Bosons and $N_2$ Fermions on a sphere keeping $N_1 + N_2 = N$ fixed. The sphere has $N - 1$ quantum fluxes and the interaction between the particles is a delta function interaction. The gross features of the spectrum are those of a system of $N_2$ free Fermions on a sphere with no magnetic field (see fig. 2).

In order to study the system we introduce a second quantization formalism by defining Bosonic ($a_s^+$) and Fermionic ($b_s^+$) creation operators which create the one body states in the Lowest Landau level. They obey the standard commutation relations $[a_s, a_{s'}^+] = \delta_{s,s'}, \{b_s, b_{s'}^+\} = \delta_{s,s'}$. The vacuum $|0\rangle$ is the filled Landau level state which is characterized by $a_s |0\rangle = b_s^+ |0\rangle = 0$. We also define the fields which create a Boson (a Fermion) at position $x$ in the Lowest Landau Level:

$$\Phi_b^+ (\vec{x}) = \sum_s \langle \vec{x}|s\rangle a_s^+$$
$$\Phi_f^+ (\vec{x}) = \sum_s \langle \vec{x}|s\rangle b_s^+$$

We define the field $A^+ (\vec{x})$ which creates the exciton (destroys a fermion and creates a boson) at position $\vec{x}$:

$$A^+ (\vec{x}) = 1/\sqrt{N} \Phi_b^+ (\vec{x}) \Phi_f (\vec{x})$$

The Fourier modes of $A^+ (x)$ are given by:

$$A_p^+ = e^{ipx_p/2}1/\sqrt{N} \sum_s e^{-ipx_s} a_{s-p}^+ b_{s-p}$$
In a similar way we also define the densities of Bosons and fermion operators:

\[
\rho^b(\vec{x}) = \Phi_b^+(\vec{x}) \Phi_b(\vec{x})
\]

\[
\rho^f(\vec{x}) = \Phi_f^+(\vec{x}) \Phi_f(\vec{x})
\]

and their Fourier transforms as in (8). One ends up with similar expressions:

\[
\rho^b_{\vec{p}} = e^{i\vec{p} \times \vec{q}/2} \sum_s e^{-i\vec{p} \times \vec{s}/2} a^+_s a_{s-p}
\]

\[
\rho^f_{\vec{p}} = e^{i\vec{p} \times \vec{q}/2} \sum_s e^{-i\vec{p} \times \vec{s}/2} b^+_s b_{s-p}
\]

(5)

The commutations between these fields lead to a generalization of the magnetic translation algebra. The relations between the \(\rho\) themselves are given by:

\[
[\rho^b_{\vec{p}}, \rho^b_{\vec{q}}] = e^{-i\vec{p} \times \vec{q}/2} A^+_{\vec{p}+\vec{q}}
\]

\[
[\rho^f_{\vec{p}}, \rho^f_{\vec{q}}] = e^{-i\vec{p} \times \vec{q}/2} A^+_{\vec{p}+\vec{q}}
\]

\[
[\rho^b_{\vec{p}}, \rho^f_{\vec{q}}] = 0
\]

(6)

where \(\vec{p} \times \vec{q} = p_x q_y - p_y q_x\). The relation between the \(\rho\) and the \(A^+\) are:

\[
[\rho^b_{\vec{p}}, A^+_{\vec{q}}] = e^{-i\vec{p} \times \vec{q}/2} A^+_{\vec{p}+\vec{q}}
\]

\[
[\rho^f_{\vec{p}}, A^+_{\vec{q}}] = e^{-i\vec{p} \times \vec{q}/2} A^+_{\vec{p}+\vec{q}}
\]

(7)

When we express the commutator between \(A^+_{\vec{p}}\) and \(A^+_{\vec{q}}\) in terms of \(\rho_{h}\) the normal ordered form of \(\rho_f\) the commutator takes form:

\[
\{A^+_{\vec{p}}, A^+_{\vec{q}}\} = \delta_{\vec{p}, \vec{q}} + 1/N \left( e^{-i\vec{p} \times \vec{q}/2} \rho^b_{\vec{p}-\vec{q}} - e^{i\vec{p} \times \vec{q}/2} \rho^b_{\vec{p}+\vec{q}} \right)
\]

(8)

Assuming that the coefficient of \(1/N\) is an operator of order one, up to a \(1/N\) correction this commutator is equal to the usual commutator between creation and annihilation operators. This will be our main approximation in the following.

There is a natural representation of the operators \(\rho^b_{\vec{p}}\rho^b_{\vec{p}}\) and \(A^+_{\vec{q}}\) in terms of creation and annihilation operators obeying Fermionic commutation relations: \(\{c_{\vec{p}}, c^+_{\vec{q}}\} = \delta_{\vec{p}, \vec{q}}\). It is given by:

\[
\rho^b_{\vec{p}} = \sum_r e^{-i\vec{p} \times r/2} c^+_{\vec{p}+r} c_r
\]

\[
\rho^b_{\vec{p}} = \sum_r e^{i\vec{p} \times r/2} c^+_{\vec{p}+r} c_r
\]

\[
A^+_{\vec{p}} = c^+_{\vec{p}}
\]

(9)

It is easy to verify that the relations (8, 7) are satisfied by this representation. We have made the assumption that the \(1/N\) correction can be neglected in the commutator (8) and replace the field \(A^+_{\vec{p}}\) by \(c^+_{\vec{p}}\) in the following.
The Hamiltonian which governs the dynamics of the model is given by the projection of the interaction potential energy on the lowest Landau level.

\[ H = \frac{1}{2 \Omega} \sum_{\vec{q}} \tilde{V}(\vec{q}) \rho^\dagger(\vec{q}) \rho(\vec{q}) \]  

where \( \tilde{V}(\vec{q}) = e^{-q^2/2} \int d^2xV(\vec{x})e^{i\vec{q}\cdot\vec{x}} \).

If we use the Fermionic representation of \( \rho^\dagger \), it can be expressed in a more conventional form of an interacting Fermion Hamiltonian. The ground state energy \( E_0 \) can be evaluated in a Hartree–Fock approximation. Denoting \( n(p) \) the ground state distribution \( n(p) = 1 \) if \( p < k_f, n(p) = 0 \) if \( p > k_f \) one obtains:

\[ E_0 = \frac{1}{2 \Omega} \sum_{\vec{p}, \vec{q}} \tilde{V}(\vec{q}) 4 \sin^2(\vec{p} \times \vec{q}/2) n(\vec{p}) (1 - n(\vec{p} - \vec{q})) \]  

Using this expression we can determine the appropriate Landau parameters in this approximation:\[ \] .

\[ \varepsilon(p) = 1/2 \Omega \sum_{\vec{q}} \tilde{V}(\vec{q}) 4 \sin^2(\vec{p} \times \vec{q}/2) (1 - n(\vec{p} - \vec{q}) - n(\vec{p} + \vec{q})) \]

\[ f(\vec{p}, \vec{q}) = -1/\Omega \tilde{V}(\vec{p} - \vec{q}) 4 \sin^2(\vec{p} \times \vec{q}/2) \]  

From this dispersion relation we can deduce the fermi velocity at the Fermi momentum \( k_f = \sqrt{2}, v_f = 7.5 \times 10^{-2} \). If we compare the effective mass \( m^* = k_f/v_f \) with the “bare mass” \( m_0 = 2\pi \) defined by the curvature of the dispersion relation at zero momentum \( \varepsilon(p) = p^2/2m_0 \) one has approximately \( m^*/m_0 \approx 3 \). Note that due to the lack of Galilean invariance of the theory, there is no relation between the mass and the Landau parameter \( F_1 \).

The homogeneous transport equation which follows from the Landau theory is given by:\[ \]

\[ (-s + \cos(\theta)) \hat{n}(\theta) + 1/2\pi \cos(\theta) \int_0^{2\pi} d\theta F(\theta - \theta') \hat{n}(\theta') = 0 \]  

where the fluctuation of the quasiparticle distribution takes the form

\[ \delta n(\vec{p}, \vec{r}, t) = \delta(\varepsilon(\vec{p}) - \mu) \hat{n}(\theta) e^{i\vec{q}\vec{r} - \omega t} \]  

\[ s = \omega/qv_f \] and \( \theta \) labels a point on the Fermi surface. The coefficient \( F(\theta) = \Omega m^*/f(\vec{p}, \vec{p}')/2\pi \), where \( \vec{p}, \vec{p}' \) are two momenta on the Fermi surface making an angle \( \theta \) with each other.

The two first Fourier modes \( (F_n = \int d\theta/2\pi F(\theta)) \) of \( F(\theta) \) are both less than zero, furthermore, when \( p_f/\sqrt{2} = .985 \) \( F_0 \) becomes less than \(-1 \) and the system becomes unstable (The compressibility is negative) and \( F_1 = -.5 \).

Let us compare the expression of the static form factor we obtain with some theoretical predictions. The form factor \( S(q) \) is defined as:

\[ S(q) = \langle \rho^\dagger(\vec{q}) \rho^\dagger(-\vec{q}) \rangle \]  

We can obtain it by setting \( \tilde{V} = 2\delta^2 \) in (10). For \( q \) small \( S(q) \) behaves as \( 2k_fq^2/3\pi \) instead of \( 2q/\sqrt{2}k_f \) in a Fermi liquid. The \( q^2 \) behavior of the form factor agrees with the numerical predictions. As for the quantum Hall effect form factor\[ \] there is a \( q^2 \) reduction with respect to the normal Fermi liquid behavior at small \( q \).

For \( q > 2k_f, 2\pi S(q) \) goes to \( k_f^2 \) as \( k_f^2 (1 - 2j_1(qk_f)/qk_f) \) where \( j_1 \) denotes the Bessel function (we use a normalization of \( \rho_0 \) for which there is no exponential behavior of \( S(q) \) at large \( q \) unlike in[8]). The limiting value \( k_f^2 \) is an exact result related to the Casimir operator of the \( \rho \) algebra.
This prediction also indicates the limitation of the present model to the $\nu = 1$ Bosonic case since in the physical $\nu = 1/2$ Fermionic case the limiting value should be $1/2$ (The general limiting value of is $1 + \nu \langle P_{ij} \rangle$ where $\langle P_{ij} \rangle$ denotes the expectation value of the permutation operator $+1$ for Bosons, $-1$ for Fermions). This $S(q)$ does not reproduce a cusp singularity at $2k_f$ which is indicated by the simulations.

We can gain a better understanding of the preceding result by comparing the expression of the form factor with the response function evaluated in the quasistatic limit: $(s = \omega/v_f q \ll 1)$.

In this limit one can interpret the system as a 2D Fermi liquid consisting of dipoles. At the Fermi surface the dipole vector of a quasiparticle with a momentum equal to $k_f$ is given by $d_i = \varepsilon_{ij} k_j$. Let $\rho_q = \sum_k c^+_k q c_k$ denote the Fourier modes of the dipole density. A scalar potential $\phi(\vec{r}, t)$ acts on the system through an interacting Hamiltonian:

$$H_e = \sum_{\vec{q}} \int d\omega \rho_{-\vec{q}} \vec{d} \phi(\vec{q}, \omega) e^{-i\omega t} \quad (16)$$

where $\phi(\vec{q}, \omega)$ is the Fourier transform in space and time of $\phi(\vec{r}, t)$. In the long wavelength limit ($q \ll k_f$) the dipole-vector can be replaced by its value at the Fermi surface and the net effect is to replace the interaction Hamiltonian by the usual coupling to a scalar potential:

$$H_e = \sum_{\vec{q}} \int d\omega \rho'_{-\vec{q}} \phi(\vec{q}, \omega) e^{-i\omega t} \quad (17)$$

where $\rho'_q = \sum_{\vec{k}} \left( \vec{k} \times \vec{q} \right) c^+_{k+\vec{q}} c_{\vec{q}}$ denotes the long wavelength limit of the total density operator.

The response function is defined as:

$$\chi(q, \omega) = \langle \rho'_{\vec{q}}(\vec{q}, \omega) \rangle / \phi(\vec{q}, \omega). \quad (18)$$

To evaluate it we use the transport equation in the presence of the external force due to the scalar potential. It reads:

$$(-s + \cos(\theta)) \hat{n}(\theta) + \cos(\theta) \hat{n}(\theta') d\theta'/2\pi = (qk_f) \sin(\theta) \cos(\theta) \phi(\vec{q}, \omega) \quad (19)$$

We expand the solution of this equation in powers of $s$ and make use of the fact that: $\langle \rho'_{\vec{q}}(\vec{q}, \omega) \rangle = \sum_{\vec{p}} (\vec{p} \times \vec{q}) \delta n(\vec{p})$. Form the first term we deduce the static response function:

$$\chi(q, 0) = - (qk_f)^2 \nu(0) / 2 (1 + F_1) \quad (20)$$

where $\nu(0) = m^* \Omega/2\pi$ is the density of states on the Fermi surface.

Note the $q^2$ dependence which is different from the usual Fermi liquid behavior and that $F_1$ appears instead of $F_0$. This result does not satisfy the usual compressibility sum rule because the scalar potential does not see the quasiparticles as elementary but rather as dipoles. As a result, it does not deform the Fermi sea symmetrically.

The next order gives an imaginary contribution which is related to the dynamical Form factor. One deduces the following expressions:

$$S(\vec{q}, \omega) = \left( 2m^* \omega / (2\pi)^2 v_f q \right) (k_f q/1 + F_1)^2 \quad (21)$$

The first factor is the free fermion result. The factor $(k_f q)^2$ was predicted at the Hartree–Fock approximation and originates from the fact that dipoles couple much more weakly to the scalar
potential as ordinary quasiparticles. Finally the many-body effects renormalize the Hartree–Fock contribution by $(1 + F_1)^{-2} \approx 4$ (instead of $(1 + F_0)^{-2}$) in the usual Fermi liquid case.

Let $J$ denote the quasiparticle current at the Fermi surface. In the dipole theory we define a modified current $\tilde{J}$ in such a way that the variation of the kinetic energy with the time is proportional to $\tilde{J}(\vec{x}) = \vec{d} \cdot \nabla J(\vec{x})$ where $\vec{d}$ denotes the dipole vector. The computation of the conductivity then proceeds as for the response function and the net effect of this redefinition is to renormalize the Fermi liquid result by the factor $(k_f q)^2$. The physical interpretation is simple: The only quasiparticles which can radiate must travel with the group velocity of the electric field. When the group velocity is small compared to $v_f$ they lie at the two points of the Fermi surface with a Fermi momentum perpendicular to $\vec{q}$. Since the electric field is not constant in the direction of $\vec{q}$, it couples to the dipole vector of the quasiparticles and the system radiates when $\vec{E}$ is parallel to their velocity. Unfortunately, this argument does not produce the longitudinal conductivity ($\vec{E}$ and $\vec{q}$ in the $x$ direction) which is observed in the experiments but a transverse conductivity $\sigma_{\perp}(\vec{q}) = e^2 \sqrt{2q}/\pi$.

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Fig. 1. We plot the ground state energy of Bosons on a sphere at filling factor $\nu = 1$. The Bosons interact through a $\delta$ repulsive potential.

We plot the energy as a function of the number of bosons $n$ ($1 \leq n \leq 12$). As a function of $n$, the energy is roughly linear by pieces with a slope breaking at each perfect square ($n = 4, 9$). This indicates that the bosons have a similar ground state energy as a system of $n$ free fermions on a sphere without magnetic field.
Fig. 2. We consider a system of 10 particles interacting on a sphere with a magnetic field at \( \nu = 1 \). The interaction potential is a \( \delta \) function as in fig. 1.

These particles are split into \( n \) bosons and \( 10 - n \) fermions and we plot the ground state energy as a function of \( n \). We see that even more convincingly as in figure 1 the energy behaves as if there were \( n \) free fermions without magnetic field. The interpretation is that the boson binds to the fermionic hole to form a quasi-free bound state. The striking fact is that this feature remains true even where there is no fermion left \((n = 10)\).