Shrinkage Methods for Treatment Choice *

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Abstract

This study examines the problem of determining whether to treat individuals based on observed covariates. The most common decision rule is the conditional empirical success (CES) rule proposed by Manski (2004), which assigns individuals to treatments that yield the best experimental outcomes conditional on the observed covariates. Conversely, using shrinkage estimators, which shrink unbiased but noisy preliminary estimates toward the average of these estimates, is a common approach in statistical estimation problems because it is well-known that shrinkage estimators have smaller mean squared errors than unshrunk estimators. Inspired by this idea, we propose a computationally tractable shrinkage rule that selects the shrinkage factor by minimizing the upper bound of the maximum regret. Then, we compare the maximum regret of the proposed shrinkage rule with that of CES and pooling rules when the parameter space is correctly specified or misspecified. Our theoretical results demonstrate that the shrinkage rule performs well in many cases and these findings are further supported by numerical experiments. Specifically, we show that the maximum regret of the shrinkage rule can be strictly

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smaller than that of the CES and pooling rules in certain cases when the parameter space is correctly specified. In addition, we find that the shrinkage rule is robust against misspecifications of the parameter space. Finally, we apply our method to experimental data from the National Job Training Partnership Act Study.

1 Introduction

This study examines the problem of determining whether to treat individuals based on observed covariates. The most common decision rule is the conditional empirical success (CES) rule proposed by Manski (2004), which is a rule assigning individuals to treatments that yield the best experimental outcomes conditional on the observed covariates. The CES rule uses only the average treatment effect (ATE) estimate conditional on each covariate value. By contrast, a common method in statistical estimation problem is to shrink unbiased but noisy preliminary estimates toward the average of these estimates. It is well known that shrinkage estimators have smaller mean squared error than unshrunk estimators. This study assumes that the dispersion of conditional ATEs (CATEs) is bounded and proposes a shrinkage rule that assigns individuals to treatments based on shrinkage estimators. We also propose a method to select the shrinkage factor by minimizing an upper bound of the maximum regret. By considering the treatment rules for individuals that are based not only on each CATE but also on the CATEs of others, it is possible to incorporate information across individuals. This allows the proposed shrinkage rule to perform as well as or better than existing treatment rules in the sense of maximum regret and to be more flexible to the heterogeneity of individuals. In addition, we compare the shrinkage rule with other rules when the parameter space is correctly specified or misspecified.

The contributions of this study are as follows. First, our approach is attractive from a computational perspective. The computation of the exact minimax regret rule is often challenging in the context of statistical treatment choice. Indeed, when the parameter space is restricted and the number of possible covariate values is large, it is difficult to obtain a shrinkage rule that minimizes the maximum regret. To overcome this problem, we propose a shrinkage rule that minimizes a tractable upper bound of the maximum regret. In this approach, each shrinkage factor is obtained by optimizing a single parameter and hence the proposed shrinkage rule is easy to compute.
Second, we compare the maximum regret of the shrinkage rule with those of alternative rules when the parameter space is correctly specified. As an alternative to the CES and shrinkage rules, one could consider using the pooling rule that determines whether to treat the individuals based on the average of the CATE estimates. As the CES and pooling rules are special cases of shrinkage rules, the proposed shrinkage rule is expected to outperform these two rules. However, because the proposed shrinkage rule does not minimize the exact maximum regret, its maximum regret may be larger than those of the CES and pooling rules. Therefore, it is essential to compare the maximum regrets. If the dispersion of the standard errors of the estimated CATEs is small compared with the dispersion of the CATEs, then the proposed shrinkage rule has a smaller maximum regret than the CES rule. Detailed definitions of the dispersion of both CATEs and the standard errors of the estimated CATEs are provided in Section 4. We also demonstrate that the shrinkage rule is no worse than the pooling rule when the dispersion of the CATEs is sufficiently small or large. Furthermore, combined with these results, we show that the proposed shrinkage rule outperforms the CES and pooling rules when the dispersion is moderate.

Third, we evaluate the maximum regret of the shrinkage rule when the parameter space is misspecified. The choice of the parameter space is important in practice because the minimax decision rule depends on the parameter space. For example, Armstrong and Kolesár (2018) and Armstrong and Kolesár (2021) consider the minimax estimation and inference problem for treatment effects and show that it is not possible to choose the parameter space automatically in a data-driven manner. Hence, it is crucial to analyze the decision rule under the misspecification of the parameter space. We investigate the performance of the shrinkage rule and show that our results are robust to the misspecification of the parameter space. To the best of our knowledge, this is the first study to consider the misspecification of the parameter space in the treatment choice problem.

Consequently, this study contributes to the growing literature on statistical treatment choice initiated by Manski (2000, 2004). Following Manski (2004, 2007), Hirano and Porter (2009), Stoye (2009, 2012), and Tetenov (2012), we focus on the maximum regrets of statistical treatment rules. Similar to Stoye (2012), Tetenov (2012), Ishihara and Kitagawa (2021), and Yata (2021), we assume that CATE estimates are normally distributed. This assumption can be viewed as an asymptotic approximation and is consistent with the argument in Hirano and
Porter (2009). The authors show that a simple Gaussian statistical model can approximate a statistical model for analyzing treatment rules in large samples.

The analysis in this study most closely relates to that of Stoye (2012), who also considers Gaussian experiments for CATEs and investigates the properties of the minimax regret treatment rule when CATEs depend on covariates with bounded variations. Stoye (2012) shows that the CES rule achieves minimax regret when the dispersion of CATEs is sufficiently large and the pooling rule achieves minimax regret when the dispersion is sufficiently small. However, we do not know the minimax regret rule when dispersion is moderate. By contrast, we propose a treatment rule and compare the shrinkage rule with the CES and pooling rules for any dispersion value.

The remainder of this paper is organized as follows. Section 2 explains the decision problem and introduces the shrinkage rules. Section 3 proposes a shrinkage rule that selects the shrinkage factor by minimizing the maximum regret’s upper bound and analyzes the shrinkage rule’s asymptotic behavior. Section 4 compares the maximum regrets of shrinkage, CES, and pooling rules when the parameter space is correctly specified and misspecified. Section 5 presents numerical analyses to compare the shrinkage rule with CES and pooling rules. As an illustration, we apply our method to experimental data from the National Job Training Partnership Act (JTPA) Study in Section 6. Finally, Section 7 concludes the paper.

2 The decision problem

2.1 Settings

Suppose that we have experimental data \( \{(Y_i, D_i, X_i)\}_{i=1}^n \), where \( X_i \in X \) is a vector of observable pre-treatment covariates, \( D_i \in \{0, 1\} \) is a binary indicator of the treatment, and \( Y_i \) is a post-treatment outcome. We assume that the support of covariates \( X \) is finite, that is, \( X = \{x_1, \ldots, x_K\} \), and consider the case in which we want to determine whether to treat individuals with \( x_k \) based on the data. This setting is similar to that of Manski (2004), who proposes the CES rule.

The CES rule assigns persons to treatments that yield the best experimental outcomes
conditional on covariates. For \( k \in 1, \ldots, K \), we define

\[
\theta_k \equiv E[Y|D = 1, X = x_k] - E[Y|D = 0, X = x_k],
\]

\[
\hat{\theta}_k \equiv \frac{1}{n_{1,k}} \sum_{i: D_i = 1, X_i = x_k} Y_i - \frac{1}{n_{0,k}} \sum_{i: D_i = 0, X_i = x_k} Y_i,
\]

where \( n_{d,k} \equiv \sum_{i=1}^n 1\{D_i = d, X_i = x_k\} \). That is, \( \theta_k \) is the ATE conditional on \( x_k \) and \( \hat{\theta}_k \) is a natural estimator of \( \theta_k \). The CES rule determines whether to treat individuals with \( x_k \) based on the sign of \( \hat{\theta}_k \). Then, the treatment rules can be viewed as a map from estimates \( \hat{\theta} \equiv (\hat{\theta}_1, \ldots, \hat{\theta}_K)' \) to the binary decisions of the treatment choice. Hence, the CES rule can be expressed as follows:

\[
\hat{\delta}^{\text{CES}}(\hat{\theta}) \equiv (\hat{\delta}_1^{\text{CES}}(\hat{\theta}), \ldots, \hat{\delta}_K^{\text{CES}}(\hat{\theta})), \quad \text{where } \hat{\delta}_k^{\text{CES}}(\hat{\theta}) \equiv 1\{\hat{\theta}_k \geq 0\}.
\]

Because \( \hat{\theta}_k \) is consistent and asymptotically normal under some weak conditions, we assume that \( \hat{\theta}_1, \ldots, \hat{\theta}_K \) are independently distributed and

\[
\hat{\theta}_k \sim N(\theta_k, \sigma_k^2), \quad k = 1, \ldots, K,
\]

where \( \sigma_k \) is the standard deviation of \( \hat{\theta}_k \). We assume that \( \sigma_k \) is known. In practice, we can only construct a consistent estimator for \( \sigma_k \). This assumption can be viewed as an asymptotic approximation and is consistent with the argument in Hirano and Porter (2009). The authors show that a simple Gaussian statistical model can approximate a statistical model for analyzing treatment rules in large samples. Because the treatment effect can vary with observable individual characteristics, we allow \( \theta_k \) to vary across the covariates.

### 2.2 Welfare and regret

Given a treatment-choice action \( \delta \equiv (\delta_1, \ldots, \delta_K)' \in \{0, 1\}^K \), we define the welfare attained at \( \delta \) as follows:

\[
W(\theta, \delta) \equiv \sum_{k=1}^K p_k \cdot \{\theta_k \cdot \delta_k + \mu_{0,k}\},
\]

where \( \theta = (\theta_1, \ldots, \theta_K)' \), \( p_k \equiv P(X = x_k) \) and \( \mu_{d,k} \equiv E[Y|D = d, X = x_k] \) for \( d \in \{0, 1\} \) and \( k = 1, \ldots, K \). Note that \( \theta_k \) is written as \( \theta_k = \mu_{1,k} - \mu_{0,k} \). If we know the true value of \( \theta \), then the optimal treatment choice action is given by

\[
\delta^* \equiv (\delta^*_1, \ldots, \delta^*_K)' \equiv (1\{\theta_1 \geq 0\}, \ldots, 1\{\theta_K \geq 0\})'.
\]
However, the treatment-choice action $\delta^*$ is infeasible because the true value of $\theta$ is unknown.

Let $\hat{\delta} : \mathbb{R}^K \to \{0, 1\}^K$ be a treatment rule that maps the estimates $\hat{\theta}$ to the binary decisions of treatment choice. The welfare regret of $\hat{\delta}(\hat{\theta}) \equiv \left( \hat{\delta}_1(\hat{\theta}), \ldots, \hat{\delta}_K(\hat{\theta}) \right)'$ is defined as

$$R(\theta, \hat{\delta}) \equiv E_{\theta} \left[ W(\theta, \delta^*) - W(\theta, \hat{\delta}) \right] = \sum_{k=1}^K p_k \cdot \left[ \theta_k \cdot \left\{ \delta_k^* - E_{\theta}[\hat{\delta}_k(\hat{\theta})] \right\} \right], \quad (4)$$

where $E_{\theta}$ is the expectation with respect to the sampling distribution of estimates $\hat{\theta}$ given the parameters $\theta$. Following existing studies, we evaluate the treatment rule $\hat{\delta}$ using the maximum regret

$$\max_{\theta \in \Theta} R(\theta, \hat{\delta}),$$

where $\Theta$ is the parameter space of $\theta$. The minimax regret criterion selects the statistical treatment rule that minimizes the maximum regret.

### 2.3 Shrinkage rules

The CES rule does not use $\hat{\theta}_l$ for $l \neq k$ to determine whether or not to treat individuals with $x_k$. However, in the problem of estimating $\theta \equiv (\theta_1, \ldots, \theta_K)'$, a common method is to shrink $\hat{\theta}_k$ toward the average of estimates $\text{ave}(\hat{\theta}) \equiv \frac{1}{K} \sum_{k=1}^K \hat{\theta}_k$ and it is well known that shrinkage estimators have smaller mean squared errors than unshrunk estimators. Hence, we propose the following shrinkage rules $\hat{\delta}^w : \mathbb{R}^K \to \{0, 1\}^K$ for $w \equiv (w_1, \ldots, w_K)' \in [0, 1]^K$.

$$\hat{\delta}^w(\hat{\theta}) \equiv \left( \hat{\delta}^w_1(\hat{\theta}), \ldots, \hat{\delta}^w_K(\hat{\theta}) \right)' \quad (5)$$

where

$$\hat{\delta}^w_k(\hat{\theta}) \equiv \left\{ w_k \cdot \hat{\theta}_k + (1 - w_k) \cdot \text{ave}(\hat{\theta}) \geq 0 \right\}.$$

When the vector of shrinkage factors $w$ is $1 \equiv (1, \ldots, 1)'$, the shrinkage rule $\hat{\delta}^w$ becomes the CES rule $\hat{\delta}^{\text{CES}}$ defined in (1). Hence, the class of shrinkage rules contains the CES rule as a special case. Furthermore, when $w$ is $0 \equiv (0, \ldots, 0)'$, this rule becomes pooling rule $\hat{\delta}^{\text{pool}}(\hat{\theta}) \equiv \hat{\delta}^0(\hat{\theta})$.

From (2), we observe that

$$w_k \cdot \hat{\theta}_k + (1 - w_k) \cdot \text{ave}(\hat{\theta}) \sim N \left( w_k \cdot \theta_k + (1 - w_k) \cdot \bar{\theta}, s^2_k(w_k) \right),$$

where
where \( \bar{\theta} \equiv K^{-1} \sum_{k=1}^{K} \theta_k \) and \( s_k^2(w_k) \) is the variance of \( w_k \cdot \hat{\theta}_k + (1 - w_k) \cdot \text{ave}(\hat{\theta}) \), that is,

\[
s_k^2(w_k) = \left\{ w_k^2 + 2w_k(1 - w_k) / K \right\} \sigma_k^2 + (1 - w_k)^2 \left\{ K^{-2} \sum_{k=1}^{K} \sigma_k^2 \right\}.
\]

Hence, from (4), the welfare regret of shrinkage treatment rule \( \hat{\delta}^w(\hat{\theta}) \) can be written as follows:

\[
R(\theta, \hat{\delta}^w) = \sum_{k=1}^{K} p_k \cdot \left\{ \theta_k \cdot \left\{ 1\{\theta_k \geq 0\} - \Phi\left( \frac{w_k \cdot \theta_k + (1 - w_k) \cdot \bar{\theta}}{s_k(w_k)} \right) \right\} \right\}
\]

\[
= \sum_{k=1}^{K} p_k \cdot \left\{ |\theta_k| \cdot \Phi\left( -\text{sgn}(\theta_k) \cdot \frac{\theta_k - (1 - w_k)(\bar{\theta} - \theta_k)}{s_k(w_k)} \right) \right\}
\]

\[
= \sum_{k=1}^{K} p_k \cdot \left\{ |\theta_k| \cdot \Phi\left( -\frac{|\theta_k| - (1 - w_k) \cdot \text{sgn}(\theta_k)(\bar{\theta} - \theta_k)}{s_k(w_k)} \right) \right\}, \quad (6)
\]

where \( \text{sgn}(x) \equiv 1\{x > 0\} - 1\{x < 0\} \), \( \Phi(\cdot) \) is the distribution function of \( N(0, 1) \), and the second equality follows from the symmetry of the normal distributions.

### 3 Shrinkage methods

#### 3.1 Choice of the shrinkage factors

In this section, we consider how to choose the shrinkage factors \( w \) under the following assumption.

**Assumption 1.** For a positive constant \( \kappa > 0 \), the parameter \( \theta \) satisfies the following condition:

\[
|\theta_k - \bar{\theta}| \leq \kappa, \quad k = 1, \ldots, K.
\]

Under Assumption 1, the parameter space of \( \theta \) becomes

\[
\Theta(\kappa) \equiv \left\{ \theta = (\theta_1, \ldots, \theta_K)' \in \mathbb{R}^K : |\theta_k - \bar{\theta}| \leq \kappa \right\}, \quad (7)
\]

where the constant \( \kappa \) can be interpreted as controlling the dispersion of parameters \( \theta_k \) around the mean \( \bar{\theta} \). This assumption is similar to Assumption 1 in Stoye (2012). Stoye (2012) assumes that \( |\mu_{d,k} - \mu_{d,l}| \leq \kappa \) for all \( d \in \{0, 1\} \) and \( k, l \in \{1, \ldots, K\} \), where \( \mu_{d,k} = E[Y | D = d, X = x_k] \).

Because \( \theta_k = \mu_{1,k} - \mu_{0,k} \), this assumption implies that

\[
|\theta_k - \theta_l| \leq 2\kappa, \quad k, l \in \{1, \ldots, K\}. \quad (8)
\]
If Assumption 1 holds, then we have $|\theta_k - \theta_l| \leq |\theta_k - \bar{\theta}| + |\theta_l - \bar{\theta}| \leq 2\kappa$; thus (8) is satisfied. Conversely, if (8) holds, then Assumption 1 is satisfied by replacing $\kappa$ with $2\kappa$.

The minimax regret criterion selects the shrinkage factors that minimize the maximum regret. From (6), the optimal shrinkage factors are obtained by minimizing the following:

$$
\max_{\theta \in \Theta(\kappa)} \sum_{k=1}^{K} p_k \cdot \left\{ |\theta_k| \cdot \Phi \left( -\frac{|\theta_k| - (1 - w^*_k) \cdot \kappa}{s_k(w^*_k)} \right) \right\}. 
$$

However, obtaining the optimal shrinkage factors becomes computationally challenging when $K$ is large. To overcome this problem, we propose selecting shrinkage factors that minimize an upper bound of the maximum regret. If $\theta$ is contained in $\Theta(\kappa)$, then the regret of shrinkage rule $\hat{\delta}^w(\theta)$ is bounded above by

$$
\sum_{k=1}^{K} p_k \cdot \left\{ |\theta_k| \cdot \Phi \left( -\frac{|\theta_k| - (1 - w^*_k) \cdot \kappa}{s_k(w^*_k)} \right) \right\}. 
$$

Using this upper bound, we obtain

$$
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^w) \leq \max_{\theta \in \Theta(\kappa)} \sum_{k=1}^{K} p_k \cdot \left\{ |\theta_k| \cdot \Phi \left( -\frac{|\theta_k| - (1 - w^*_k) \cdot \kappa}{s_k(w^*_k)} \right) \right\}
\leq \sum_{k=1}^{K} p_k \cdot \max_{\theta_k \in \mathbb{R}} \left\{ |\theta_k| \cdot \Phi \left( -\frac{|\theta_k| - (1 - w^*_k) \cdot \kappa}{s_k(w^*_k)} \right) \right\}
= \sum_{k=1}^{K} p_k \cdot s_k(w^*_k) \eta \left( \frac{(1 - w^*_k) \cdot \kappa}{s_k(w^*_k)} \right),
$$

(10)

where $\eta(a) \equiv \max_{t \geq 0} \{ t \cdot \Phi(-t + a) \} = \max_{t \in \mathbb{R}} \{ t \cdot \Phi(-t + a) \}$ for $a \in \mathbb{R}$. Tetenov (2012) and Ishihara and Kitagawa (2021) show that function $\eta(\cdot)$ is strictly increasing and convex. Figure 1 displays the shape of $\eta(a)$. Using function $\eta(\cdot)$, we propose the following shrinkage factors:

$$
w^*(\kappa) \equiv (w^*_1(\kappa), \ldots, w^*_K(\kappa))^t, \quad w^*_k(\kappa) \equiv \arg \min_{w_k \in [0,1]} \{ \psi_k(w_k; \kappa) \} \text{ for } k = 1, \ldots, K, \quad (11)
$$

where $\psi_k(w_k; \kappa) \equiv s_k(w_k) \eta \left( (1 - w_k) \cdot \kappa / s_k(w_k) \right)$ for $k = 1, \ldots, K$. If the right-hand side of (10) is a good approximation of the maximum regret, this rule is expected to be close to the minimax regret rule.

Our approach is attractive from a computational perspective. Indeed, the proposed shrinkage factors are easy to compute because (11) is obtained by optimizing the objective function over a single parameter while it is difficult to obtain the shrinkage factors that minimize the exact maximum regret when the number of possible covariate values is large.
Figure 1: Functional form of $\eta(a)$. The function $\eta(a)$ is strictly increasing and convex and $\eta(0)$ is approximately equal to 0.17.

Remark 1. Our results can be extended to the following randomized statistical treatment rules:

$$
\delta_k^{w_k,v_k}(\hat{\theta}) \equiv \begin{cases} 
1 & \text{with probability } \Phi_{v_k}(w_k \cdot \hat{\theta}_k + (1-w_k) \cdot \text{ave}(\hat{\theta}) + v_k Z_k) \\
0 & \text{with probability } 1 - \Phi_{v_k}(w_k \cdot \hat{\theta}_k + (1-w_k) \cdot \text{ave}(\hat{\theta}))
\end{cases},
$$

where $Z_k \sim N(0,1)$ is independent of $\hat{\theta}$ and $v_k \geq 0$ is the randomization factor. Then, conditional on $\hat{\theta}$, we obtain

$$
\hat{\delta}_k^{w_k,v_k}(\hat{\theta}) = \begin{cases} 
1, & \text{with probability } \Phi_{v_k}(w_k \cdot \hat{\theta}_k + (1-w_k) \cdot \text{ave}(\hat{\theta}) + v_k Z_k) \\
0, & \text{with probability } 1 - \Phi_{v_k}(w_k \cdot \hat{\theta}_k + (1-w_k) \cdot \text{ave}(\hat{\theta}))
\end{cases},
$$

where $\Phi_v$ denotes the distribution function of $N(0,v^2)$. When $v_k = 0$, this rule becomes a non-randomized rule.

Let $v \equiv (v_1, \ldots, v_K)'$ and $\hat{\delta}^{w,v}(\hat{\theta}) \equiv (\hat{\delta}_1^{w_1,v_1}(\hat{\theta}), \ldots, \hat{\delta}_K^{w_K,v_K}(\hat{\theta}))(\hat{\theta}) \equiv (\hat{\delta}_1^{w_1,v_1}(\hat{\theta}), \ldots, \hat{\delta}_K^{w_K,v_K}(\hat{\theta}))(\hat{\theta})$. Because we have

$$
w_k \cdot \hat{\theta}_k + (1-w_k) \cdot \text{ave}(\hat{\theta}) + v_k Z_k \sim N\left(w_k \cdot \theta_k + (1-w_k) \cdot \text{ave}(\hat{\theta}), s_k^2(w_k) + v_k^2\right),
$$

we obtain the following upper bound of the maximum regret:

$$
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w,v}) \leq \sum_{k=1}^K p_k \cdot \sqrt{s_k^2(w_k) + v_k^2} \cdot \eta\left(\frac{(1-w_k) \cdot \kappa}{\sqrt{s_k^2(w_k) + v_k^2}}\right).
$$
Similar to the non-randomized shrinkage rule, the shrinkage and randomization factors can be easily obtained. Yata (2021) and Montiel Olea, Qiu, and Stoye (2023) show that the minimax regret rule can be a randomized rule under partial identification. However, in our setting, $\theta$ is point-identified. Hence, in this study, we focus on non-randomized treatment rules.

3.2 Asymptotic behavior of $w^*(\kappa)$

In this section, we discuss the asymptotic behavior of the proposed shrinkage rule. We consider the following three asymptotic situations: (i) the dispersion of the parameters becomes larger, that is, $\kappa \to \infty$, (ii) the dispersion of the parameters becomes smaller, that is, $\kappa \to 0$, and (iii) the number of subgroups becomes larger, that is, $K \to \infty$.

First, we consider the case where $\kappa$ is large. Because $\eta(a)$ is convex, we have $\eta(a) \geq \eta(0) + \eta'(0)a$ for $a \geq 0$. This implies

$$\psi_{ik} \geq \eta(0) \cdot s_k(w_k) + (1 - w_k) \cdot \eta'(0)\kappa,$$

where $\eta'(0) \approx 0.226$ and the equality holds when $w_k = 1$. Hence, if the right-hand side is minimized at $w_k = 1$, we obtain $w_k^*(\kappa) = 1$. Because the derivative of the right-hand side becomes

$$\eta(0) \cdot \{s_k(w_k)\}' - \eta'(0)\kappa,$$

the shrinkage factor $w_k^*(\kappa)$ becomes one when $\eta(0) \cdot \{s_k(w_k)\}' \leq \eta'(0)\kappa$ holds for all $w_k \in [0, 1]$. For $w_k \in [0, 1]$, we obtain

$$\{s_k(w_k)\}' = \left\{\sqrt{s_k^2(w_k)}\right\}' = \frac{\{s_k^2(w_k)\}'}{2 s_k(w_k)}\leq \frac{2w_k + \frac{2}{K}(1 - 2w_k)}{\min_{w_k \in [0,1]} s_k(w_k)} s_k^2 \leq \frac{\sigma_k^2}{\min_{w_k \in [0,1]} s_k(w_k)}.$$ 

As $s_k(w_k)$ is bounded away from zero, we obtain $w_k^*(\kappa) = 1$ for a sufficiently large $\kappa$. Hence, the proposed shrinkage rule becomes a CES rule when $\kappa$ is sufficiently large.

Next, we consider the case where $\kappa$ decreases. As $\kappa \to 0$, we have that

$$\psi_{ik} \to s_k(w_k)\eta(0), \text{ for } w_k \in [0, 1].$$
Thus, the limit of \( \psi_k(w_k; \kappa) \) is minimized at \( w_k = \arg \min_{w \in [0,1]} s_k^2(w) \). Hence, if the homoscedasticity assumption holds, that is, \( \sigma_k = \sigma \) for all \( k \); then the limit of \( \psi_k(w_k; \kappa) \) is minimized at \( w_k = 0 \). Hence, if the dispersion of the parameters decreases, the proposed shrinkage rule approaches the pooling rule.

If we consider \( w_k \cdot \hat{\theta}_k + (1-w_k) \cdot \text{ave} (\hat{\theta}) \) to be an estimator of \( \theta_k \), then this estimator becomes unbiased when \( w_k = 1 \). Hence, these two asymptotic situations imply that the shrinkage factor \( w_k^*(\kappa) \) chooses a less biased estimator when \( \kappa \) is large and a small variance estimator when \( \kappa \) is small. This result can be seen as a type of bias-variance trade-off.

Finally, we consider the case where \( K \) increases. We assume \( \frac{1}{K^2} \sum_{k=1}^{K} \sigma_k^2 \rightarrow 0 \) as \( K \rightarrow \infty \). Under this condition, \( s_k(w_k) \) can be approximated as \( w_k \sigma_k \). Hence, in this situation, we have

\[
\psi_k(w_k; \kappa) \rightarrow \tilde{\psi}_k(w_k; \kappa) \equiv \sigma_k w_k \eta \left( (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) \right).
\]

By letting \( \tilde{w}_k^*(\kappa) \equiv \arg \min_{w_k \in [0,1]} \tilde{\psi}_k(w_k; \kappa) \), \( w_k^*(\kappa) \) can be approximated by \( \tilde{w}_k^*(\kappa) \) when \( K \) is large. Hence, when \( K \) is sufficiently large, \( \tilde{w}_k^*(\kappa) \) is useful to understand the properties of \( w_k^*(\kappa) \).

**Proposition 1.** Let \( t^*(a) \equiv \arg \max_{t \geq 0} \{ t \Phi(-t+a) \} \). We have \( \tilde{w}_k^*(\kappa) = 1 \) when \( \kappa/\sigma_k > t^*(0) \simeq 0.75 \) and \( \tilde{w}_k^*(\kappa) = 0 \) when \( \kappa = 0 \).

Proposition 1 implies that we obtain results similar to above two situations even when \( K \) is large. Specifically, the proposed shrinkage rule becomes the CES rule when \( \kappa/\sigma_k \) is larger than approximately 3/4 and \( K \) is large, and the proposed shrinkage rule becomes the pooling rule when \( \kappa = 0 \) and \( K \) is large.

**Remark 2.** To illustrate the proposed shrinkage rule, we consider a simple case in which \( \sigma_1 = \cdots = \sigma_K = \sigma \). Then, we observe that

\[
\psi'_k(w_k; \kappa) = s'_k(w_k) \eta \left( \frac{(1-w_k) \cdot \kappa}{s_k(w_k)} \right) + s_k(w_k) \eta' \left( \frac{(1-w_k) \cdot \kappa}{s_k(w_k)} \right) \left\{ \frac{(1-w_k) \cdot \kappa}{s_k(w_k)} \right\}'
\]

\[
= s'_k(w_k) \eta \left( \frac{(1-w_k) \cdot \kappa}{s_k(w_k)} \right) - \left\{ \frac{\kappa s_k(w_k) + \kappa (1-w_k) s'_k(w_k)}{s_k(w_k)} \right\} \eta' \left( \frac{(1-w_k) \cdot \kappa}{s_k(w_k)} \right),
\]

11
where $s_k(w_k) = \sqrt{w_k^2\sigma^2 + (1 - w_k^2)\sigma^2/K}$ and $s'_k(w_k) = \frac{\sigma^2(1-1/K)w_k}{s_k(w_k)}$. This implies that

$$
\psi'_k(1; \kappa) = \left(1 - \frac{1}{K}\right)\sigma\eta'(0) - \kappa\eta''(0),
$$

$$
\psi'_k(0; \kappa) = -\kappa \cdot \frac{\eta'(0)}{\sigma}.
$$

Therefore, in this setting, we have $\hat{\delta}^{w^*(\kappa)} \neq \hat{\delta}^{\text{CES}}$ when $\kappa < \frac{(1-1/K)\sigma\eta'(0)}{\eta''(0)} \simeq 0.75 \left(1 - 1/K\right)\sigma$, and $\hat{\delta}^{w^*(\kappa)} \neq \hat{\delta}^\text{pool}$ for any $\kappa > 0$.

### 3.3 Shrinkage toward the regression estimates

In the previous sections, we consider the shrinkage rules that shrink toward the average of the estimates. However, this section considers shrinkage rules that shrink toward the regression estimates and proposes the choice of shrinkage factors under the following assumption:

**Assumption 2.** Suppose that $\theta_k$ can be expressed as

$$
\theta_k = x'_k\beta + \xi_k, \quad k = 1, \ldots, K.
$$

Additionally, for a positive constant $\kappa > 0$, $\xi_1, \ldots, \xi_K$ satisfy

$$
\sum_{k=1}^{K} x_k \xi_k = 0 \quad \text{and} \quad |\xi_k| \leq \kappa, \quad k = 1, \ldots, K.
$$

If $x'_k\beta$ is the projection of $\theta_k$ onto $x_k$, $\xi_k$ satisfies $\sum_{k=1}^{K} x_k \xi_k = 0$. Hence, Assumption 2 implies that the residuals from the regression of $\theta_k$ on $x_k$ are bounded. If $x_k = 1$ for all $k$, then Assumption 2 is identical to Assumption 1. Therefore, this assumption generalizes Assumption 1.

We consider the following shrinkage rules that shrink toward the regression estimates:

$$
\tilde{\delta}^w(\hat{\theta}) = \left(\tilde{\delta}^{w_1}(\hat{\theta}), \ldots, \tilde{\delta}^{w_K}(\hat{\theta})\right),
$$

where $\tilde{\delta}^{w_k}(\hat{\theta}) \equiv 1\left\{ w_k \cdot \hat{\theta}_k + (1 - w_k) \cdot x'_k\hat{\beta} \right\}$ and $\hat{\beta} \equiv \left(\sum_{k=1}^{K} x_k x'_k\right)^{-1} \left(\sum_{k=1}^{K} x_k \hat{\theta}_k\right)$. Under Assumption 2, because $E_{\theta}[\hat{\beta}] = \left(\sum_{k=1}^{K} x_k x'_k\right)^{-1} \left(\sum_{k=1}^{K} x_k (x'_k\beta + \xi_k)\right) = \beta$, we have

$$
w_k \cdot \hat{\theta}_k + (1 - w_k) \cdot x'_k\hat{\beta} \sim N\left( w_k \cdot \theta_k + (1 - w_k) \cdot x'_k\beta, s^2_k(w_k) \right),
$$
where $\tilde{s}^2_k(w_k)$ denotes the variance of $w_k \cdot \hat{\theta}_k + (1 - w_k) \cdot x_k^t \hat{\beta}$. Let $\Theta_{\text{reg}}(\kappa)$ be the set of $\theta$ satisfying Assumption 2. Then, as in (10), we obtain
\[
\max_{\theta \in \Theta_{\text{reg}}(\kappa)} R(\theta, \hat{\delta}^w) \leq \sum_{k=1}^K p_k \cdot \tilde{s}_k(w_k) \eta \left( \frac{(1 - w_k) \cdot \kappa}{\tilde{s}_k(w_k)} \right).
\]
Hence, similar to $w^*(\kappa)$, we propose the following shrinkage factors:
\[
w^*_{\text{reg}}(\kappa) \equiv (w^*_{\text{reg},1}(\kappa), \ldots, w^*_{\text{reg},K}(\kappa))^t,
\]
where $w^*_{\text{reg},k}(\kappa) \equiv \arg \min_{w_k \in [0,1]} \left\{ \tilde{s}_k(w_k) \eta \left( \frac{(1 - w_k) \cdot \kappa}{\tilde{s}_k(w_k)} \right) \right\}$ for all $k$.

We can also consider other shrinkage rules, such those that shrink toward the weighted average of the estimates. However, since analyzing the maximum regrets of such shrinkage rules is burdensome, we focus on the shrinkage rule proposed in Section 3.1 in subsequent sections.

4 Main results

The proposed shrinkage rule does not minimize the maximum regret because $w^*(\kappa)$ minimizes an upper bound of the maximum regret. Hence, it is not known precisely whether the maximum regret of $\hat{\delta}^{w^*(\kappa)}(\hat{\theta})$ is smaller than that of $\hat{\delta}^{\text{CES}}(\hat{\theta})$ and $\hat{\delta}^{\text{pool}}(\hat{\theta})$. This section compares the maximum regret of the proposed shrinkage rule with the CES and pooling rules when $\kappa$ is correctly specified or misspecified.

4.1 Comparison with the CES and pooling rules when $\kappa$ is correctly specified

We compare the maximum regret of the proposed shrinkage rule with that of the CES and pooling rules when we know the true parameter space $\Theta(\kappa)$. First, we compare the maximum regrets of $\hat{\delta}^{w^*(\kappa)}(\hat{\theta})$ and $\hat{\delta}^{\text{CES}}(\hat{\theta})$.

**Theorem 1.** Suppose that Assumption 1 holds. Let $\underline{\sigma} \equiv \min_k \{\sigma_k\}$, $\overline{\sigma} \equiv \max_k \{\sigma_k\}$, and $\mathcal{K}(\kappa) \equiv \{k : \sigma_k \leq \underline{\sigma} + \kappa/t^*(0)\}$. Then, we have
\[
\frac{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w^*(\kappa)})}{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{\text{ CES}})} \leq \frac{\sum_{k=1}^K p_k \cdot \psi_k (w^*_k(\kappa) ; \kappa)}{\eta(0) \left( \sum_{k \in \mathcal{K}(\kappa)} p_k \sigma_k + \sum_{k \notin \mathcal{K}(\kappa)} p_k \underline{\sigma} \right)}. \tag{13}
\]
In particular, if \( \sigma_1, \ldots, \sigma_K \) and \( \kappa \) satisfy \( \bar{\sigma} - \sigma \leq \kappa/t^*(0) \), then \( \max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\kappa}^{\text{w}}(\kappa)) \leq \max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\kappa}^{\text{CES}}) \).

When \( \bar{\sigma} - \sigma \leq \kappa/t^*(0) \approx 1.33\kappa \) holds, the denominator on the right-hand side of (13) becomes \( \eta(0) \left( \sum_{k=1}^{K} p_k \sigma_k \right) \). Because \( \psi_k \left( w_k^*(\kappa); \kappa \right) \leq \psi_k(1; \kappa) = \eta(0) \sigma_k \), the maximum regret of \( \hat{\delta}_{\kappa}^{\text{w}}(\kappa) \) is not larger than that of \( \hat{\delta}_{\kappa}^{\text{CES}} \) when \( \bar{\sigma} - \sigma \leq 1.33\kappa \). Hence, if \( \kappa \) is correctly specified, the dispersion of \( \sigma_k \) is small compared to 1.33\( \kappa \), the proposed shrinkage rule has a smaller maximum regret than the CES rule. Under the homoscedasticity assumption, the maximum regret of the proposed shrinkage rule is always lower than that of the CES rule.

Let \( s_0^2 = \frac{1}{K^2} \sum_{k=1}^{K} \sigma_k^2 \). As \( \psi_k \left( w_k^*(\kappa); \kappa \right) \leq \psi_k(0; \kappa) = s_0 \eta(\kappa/s_0) \), (13) also provides the following upper bounds:

\[
\frac{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\kappa}^{\text{w}}(\kappa))}{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\kappa}^{\text{CES}})} \leq \frac{s_0 \eta(\kappa/s_0)}{\eta(0) \bar{\sigma}} \leq \frac{\eta(0) s_0 + \kappa}{\eta(0) \bar{\sigma}},
\]

where the last inequality follows from Lemma 2 in Appendix 1. Even when \( \bar{\sigma} - \sigma \leq 1.33\kappa \) does not hold, if \( \kappa \leq \eta(0) (\bar{\sigma} - s_0) \), we have

\[
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\kappa}^{\text{w}}(\kappa)) \leq \max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\kappa}^{\text{CES}}).
\]

This implies that if \( \kappa \) is correctly specified and \( \kappa \) is sufficiently small, then the proposed shrinkage rule has a smaller maximum regret than the CES rule.

Next, we compare the maximum regrets of \( \hat{\delta}_{\kappa}^{\text{w}}(\theta) \) and \( \hat{\delta}_{\kappa}^{\text{pool}}(\theta) \).

**Theorem 2.** Suppose that Assumption 1 holds and \( K \) is even. Then, we have

\[
\frac{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\kappa}^{\text{w}}(\kappa))}{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\kappa}^{\text{pool}})} \leq \frac{\sum_{k=1}^{K} p_k \cdot \psi_k \left( w_k^*(\kappa); \kappa \right)}{\max \left\{ \frac{1}{2} s_0 \eta(\kappa/s_0), s_0 \eta(\kappa/s_0) - \kappa \right\}} \leq \min \left\{ 2, \frac{s_0 \eta(\kappa/s_0)}{s_0 \eta(\kappa/s_0) - \kappa}, \frac{2 \eta(0) \left( \sum_{k=1}^{K} p_k \sigma_k \right)}{s_0 \eta(\kappa/s_0)} \right\}, \tag{14}
\]

The first bound of (14) implies that the maximum regret of the proposed shrinkage rule is less than twice that of the pooling rule. Because \( \eta(0) > 0 \), the second bound of (14) implies that the ratio of the maximum regrets gets closer to one as \( \kappa \to 0 \). This result implies that the maximum regret of \( \hat{\delta}_{\kappa}^{\text{w}}(\theta) \) is almost smaller than that of \( \hat{\delta}_{\kappa}^{\text{pool}}(\theta) \) when \( \kappa \) is sufficiently small.
We observe that

\[ s_0\eta(\kappa/s_0) = \left( \frac{\eta(\kappa/s_0)}{\kappa/s_0} \right) \kappa \geq \frac{\kappa}{2}, \]

where we use \( \eta(a) \geq a/2 \) from Lemma 2 in Appendix 1. Hence, the third bound of (14) implies that the maximum regret of the proposed shrinkage rule is smaller than that of the pooling rule when

\[ \kappa \geq 4\eta(0) \left( \sum_{k=1}^{K} p_k\sigma_k \right) \simeq 0.68 \left( \sum_{k=1}^{K} p_k\sigma_k \right). \]

Furthermore, because the third bound of (14) approaches zero as \( \kappa \to \infty \), the upper bound of the ratio is not bounded, which implies that the maximum regret of the pooling rule can be significantly larger than that of the proposed shrinkage rule.

**Remark 3.** By combining Theorems 1 and 2, we show that the proposed shrinkage rule dominates the CES and pooling rules when \( \kappa \) is within a certain range. For simplicity, we consider the case where \( \sigma_1 = \cdots = \sigma_K = \sigma \). Then, it follows from Theorem 1 that the maximum regret of \( \hat{\delta}_w(\kappa) \) is not larger than that of \( \hat{\delta}_{\text{CES}} \). In addition, Remark 2 shows that \( \hat{\delta}_w(\kappa) \neq \hat{\delta}_{\text{CES}} \) when \( \kappa < 0.75 (1 - 1/K) \sigma \). This implies that the maximum regret of \( \hat{\delta}_w(\kappa) \) is strictly smaller than that of \( \hat{\delta}_{\text{CES}} \) when \( \kappa < 0.75 (1 - 1/K) \sigma \). As discussed previously, the maximum regret of \( \hat{\delta}_w(\kappa) \) is strictly smaller than that of \( \hat{\delta}_{\text{pool}} \) when \( \kappa > 0.68\sigma \). Therefore, when

\[ 0.68\sigma < \kappa < 0.75 (1 - 1/K) \sigma, \]  

the proposed shrinkage rule strictly dominates the CES and pooling rules.

When \( K \) is sufficiently large, we can broaden the range of (15). As \( K \to \infty \), we have \( s_0 = \sigma/\sqrt{K} \to 0 \). Because Lemma 2 implies that \( \eta(a)/a \to 1 \) as \( a \to \infty \), we obtain

\[ s_0\eta(\kappa/s_0) = \left( \frac{\eta(\kappa/s_0)}{\kappa/s_0} \right) \kappa \to \kappa \quad \text{as } K \to \infty. \]

This implies that the third bound of (14) converges to \( 2\eta(0)/\kappa \simeq 0.34/\kappa \). Hence, if \( K = \infty \), then the proposed shrinkage rule strictly dominates the CES and pooling rules when

\[ 0.34\sigma < \kappa < 0.75\sigma. \]

Therefore, the proposed shrinkage rule outperforms the CES and pooling rules when the dispersion of parameters \( \kappa \) is moderate.
Remark 4. In Theorem 2, it is assumed that $K$ is even. This assumption is unnecessary to obtain the upper bound of the ratio of the maximum regrets. However, when $K$ is even, we can easily obtain the lower bound of $\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\text{pool}})$. Because we have $\theta_t(\kappa) \equiv (t+\kappa, \ldots, t+\kappa, t-\kappa, \ldots, t-\kappa)' \in \Theta(\kappa)$ for all $t \in \mathbb{R}$ when $K$ is even, the following lower bound is obtained:

$$\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\text{pool}}) \geq \max_{t \in \mathbb{R}} R(\theta_t(\kappa), \hat{\delta}_{\text{pool}}).$$

The upper bound of Theorem 2 is derived using this result. Even if $K$ is odd, we can obtain similar results because $(t+\kappa, \ldots, t+\kappa, t, t-\kappa, \ldots, t-\kappa)' \in \Theta(\kappa)$ holds for all $t \in \mathbb{R}$.

4.2 Comparison with the CES and pooling rules when $\kappa$ is misspecified

In the previous section, we assume that parameter space $\Theta(\kappa)$ is known. However, in practice, it may be challenging to select a reasonable $\kappa$. In this section, we consider the case in which the researcher’s choice of the parameter space $\Theta(\kappa')$ is different from the true parameter space $\Theta(\kappa)$, that is, $\kappa$ is misspecified.

First, we compare the maximum regret of the proposed shrinkage rule with that of the CES rule when $\kappa$ is misspecified.

Theorem 3. Suppose that Assumption 1 holds. Then, we have

$$\frac{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{w^*(\kappa')})}{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\text{CES}})} \leq \frac{\sum_{k=1}^{K} p_k \cdot \psi_k \left( \frac{1}{s_k \left( \frac{w^*_k(\kappa')}{s_k(\kappa')} \right)} \right) \cdot \left( \frac{1}{s_k(\kappa')} \right) \cdot \left( \frac{1}{s_k(\kappa')} \right)}{\eta(0) \left( \sum_{k \in \kappa(\kappa)} p_k \sigma_k + \sum_{k \notin \kappa(\kappa)} p_k \sigma_k \right) \cdot \left( \frac{1}{s_k(\kappa')} \right) \cdot \left( \frac{1}{s_k(\kappa')} \right) \cdot \left( \frac{1}{s_k(\kappa')} \right)} \cdot \left( \frac{1}{s_k(\kappa')} \right), \quad (16)$$

where $H(a) \equiv a/\eta(a)$. In particular, if $\sigma_1, \ldots, \sigma_K$ and $\kappa$ satisfy $\sigma - \sigma \leq \kappa/t^*(0)$, then

$$\frac{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{w^*(\kappa')})}{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}_{\text{CES}})} \leq 1 + \max_{k} \left\{ \frac{1}{s_k \left( \frac{w^*_k(\kappa')}{s_k(\kappa')} \right)} \cdot \left( \frac{1}{s_k(\kappa')} \right) \right\} \cdot \left( \frac{1}{s_k(\kappa')} \right).$$

In Theorem 3, we use the parameter space $\Theta(\kappa')$, which may differ from the true parameter space $\Theta(\kappa)$, to determine the shrinkage factors. Hence, the upper bound of (16) differs from (13). However, when $\kappa' = \kappa$, Theorems 1 and 3 are equivalent.
Theorem 3 implies that we obtain an upper bound similar to that of Theorem 1 even when \( \kappa \) is misspecified. When \( \sigma - \sigma \leq \kappa/t^*(0) \) holds, the upper bound becomes one when \( \kappa' \geq \kappa \). This implies that the maximum regret of \( \hat{\delta}^{w^*(\kappa')} \) is not grater than that of \( \delta^{\text{CES}} \) when \( \kappa' \geq \kappa \). Because \( \eta(a) \geq a/2 \) from Lemma 2 in Appendix 1, we obtain \( H(a) \equiv a/\eta(a) \leq 2 \). This implies that when \( \kappa' < \kappa \) and \( \sigma - \sigma \leq \kappa/t^*(0) \), we have

\[
\frac{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w^*(\kappa')})}{\max_{\theta \in \Theta(\kappa)} R(\theta, \delta^{\text{CES}})} \leq 1 + 2 \left( \frac{\kappa - \kappa'}{\kappa'} \right).
\]

Hence, the upper bound is close to one if \( (\kappa - \kappa')/\kappa' \) is close to zero. As discussed in Section 3.2, the shrinkage factor \( w^*_k(\kappa') \) becomes one for a sufficiently large \( \kappa' \). Hence, if \( \sigma - \sigma \leq \kappa/t^*(0) \) holds and \( \kappa' \) is sufficiently large, the upper bound becomes

\[
1 + H(0) \cdot \left( \frac{|\kappa - \kappa'|}{\kappa'} \right) = 1.
\]

This is because the proposed shrinkage rule \( \hat{\delta}^{w^*(\kappa')} \) becomes the CES rule when \( \kappa' \) is sufficiently large.

Next, we compare the maximum regrets of the proposed shrinkage rule with the pooling rule when \( \kappa \) is misspecified.

**Theorem 4.** Suppose that Assumption 1 holds and \( K \) is even. Then, we have

\[
\frac{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w^*(\kappa')})}{\max_{\theta \in \Theta(\kappa)} R(\theta, \delta^{\text{pool}})} \leq \max_{\theta \in \Theta(\kappa)} \frac{\sum_{k=1}^{K} p_k \cdot \psi_k (w^*_k(\kappa'); \kappa')}{\frac{1}{2} s_0 \eta(\kappa/s_0), s_0 \eta(\kappa/s_0) - \kappa} \times \left[ 1 + \max_{k} \left\{ H \left( \frac{1 - w^*_k(\kappa')}{s_k (w^*_k(\kappa'))} \right), \left( \frac{|\kappa - \kappa'|}{\kappa'} \right) \right\} \right].
\]

As in Theorem 3, when \( \kappa' = \kappa \), the upper bound of Theorem 4 is the same as that of Theorem 2. When \( \kappa' \geq \kappa \), the bound of (17) implies that:

\[
\frac{\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w^*(\kappa')})}{\max_{\theta \in \Theta(\kappa)} R(\theta, \delta^{\text{pool}})} \leq \min \left\{ \frac{2 \eta(\kappa'/s_0)}{\eta(\kappa/s_0)}, \frac{s_0 \eta(\kappa'/s_0)}{s_0 \eta(\kappa/s_0) - \kappa}, \frac{2 \eta(0)}{s_0 \eta(\kappa/s_0) - \kappa} \right\}.
\]

Because \( \eta'(a) \leq 1 \) from Lemma 1 in Appendix 1, we have \( \eta(a) \leq \eta(a') + (a - a') \) for \( a \geq a' \). Hence, when \( \kappa' \geq \kappa \), the first bound is bounded by

\[
\frac{2 \eta(\kappa'/s_0)}{\eta(\kappa/s_0)} \leq 2 \left\{ \frac{\eta(\kappa/s_0) + (\kappa' - \kappa)/s_0}{\eta(\kappa/s_0)} \right\} \leq 2 \left\{ 1 + H(\kappa/s_0) \left( \frac{\kappa' - \kappa}{\kappa} \right) \right\}.
\]
The upper bound of (17) approaches two if \((\kappa' - \kappa) / \kappa\) is close to zero even when \(\kappa\) is misspecified. Similarly, the second bound is bounded by the following:

\[
\frac{s_0 \eta(\kappa')}{s_0 \eta(\kappa)} - \kappa \leq \frac{s_0 \eta(\kappa) + (\kappa' - \kappa)}{s_0 \eta(\kappa) - \kappa}.
\]

Hence, the upper bound of (17) approaches one as \(\kappa'\) and \(\kappa\) approach zero. The third bound of (17) is identical to that in Theorem 2, implying that the maximum regret of the shrinkage rule is smaller than that of the pooling rule when \(\kappa \geq 0\).

5 Numerical examples

We present numerical examples to illustrate the results obtained in the previous sections. We demonstrate the relationship between \(\kappa\) and \(w^*_k(\kappa)\) under the homoscedasticity assumption. When homoscedasticity holds, that is, \(\sigma_1 = \cdots = \sigma_K = 1\), we obtain \(w^*_1(\kappa) = \cdots = w^*_K(\kappa) = w^*(\kappa)\). We calculate the shrinkage factor \(w^*(\kappa)\) numerically for each \(\kappa \in [0, 1]\). Figure 2 shows the relationship between \(\kappa\) and \(w^*(\kappa)\) for \(K = 2, 5, 100\). The proposed shrinkage rule becomes the CES rule when \(\kappa\) is sufficiently large and approaches the pooling rule when \(\kappa\) approaches zero in all settings. This result is consistent with the results presented in Section 3. Furthermore, as seen in Proposition 1, when the number of subgroups increases \((K = 100)\), the shrinkage factor becomes one if \(\kappa\) is larger than \(t^*(0) \simeq 0.752\).

We consider simple settings where \(K = 2\), \((\sigma_1, \sigma_2) = (1, 1), (0.75, 1.25)\), and \((p_1, p_2) = (0.5, 0.5), (0.75, 0.25)\) to compare the maximum regrets of the shrinkage, CES, and pooling.
**Figure 2**: The relationship between $\kappa$ and $w^*(\kappa)$ when $K = 2, 5, 100$. The solid, dashed, dotted lines denote the shrinkage factors when $K = 2, 5, 100$, respectively.

rules. Figures 3a-3d show the maximum regrets of the shrinkage, CES, and pooling rules when $\kappa$ is correctly specified. If $p_1 > p_2$, then the number of units with the covariate $x_1$ is expected to be larger than the number of units with $x_2$. Hence, the standard deviation of $\hat{\theta}_1$ is expected to be smaller than that of $\hat{\theta}_2$.

As expected from Theorem 1, the maximum regret of the shrinkage rule is always less than or equal to that of the CES rule in all settings. Additionally, the maximum regret of the shrinkage rule is equal to that of the CES rule when $\kappa$ is large. This is because the shrinkage rule becomes a CES rule when $\kappa$ is sufficiently large. Although the pooling rule is better than the shrinkage rule for some $\kappa$, as expected from Theorem 2, the shrinkage rule is not worse than the pooling rule when $\kappa$ is small. Additionally, as $\kappa$ increases, the maximum regret of the pooling rule increases.

Next, we calculate the maximum regret of the shrinkage rule when $\kappa$ is misspecified, that is, we calculate $\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\theta}^{w*(\kappa')})$. We consider two cases: (1) $\kappa' = 1.2\kappa$ and (2) $\kappa' = 0.8\kappa$. In case (1), the researcher’s choice of parameter space is larger than the true parameter space. In case (2), the researcher’s choice of parameter space is smaller. Figures 4a-4d show the maximum regrets of the shrinkage, CES, and pooling rules in case (1) and Figures 5a-5d show
Figure 3: The solid, dotted, and dashed lines denote the maximum regrets of the shrinkage, CES, and pooling rules when $\kappa$ is correctly specified.
the maximum regrets in case (2).

Figure 4: The solid, dotted, and dashed lines denote the maximum regrets of the shrinkage, CES, and pooling rules when $\kappa' = 1.2\kappa$.

Even when $\kappa$ is misspecified, we obtain results similar to those shown in Figures 3a–3d. These results imply that our shrinkage rule is robust to the misspecification of $\kappa$. In all settings, the maximum regret of the shrinkage rule is always less than or equal to that of the CES rule. Theorem 3 implies that the shrinkage rule is superior to the CES rule when $\kappa' \geq \kappa$. However, Figures 5a-5d show that similar results are obtained in these settings, even when $\kappa' \leq \kappa$. In these numerical examples, the maximum regret of the shrinkage rule decreases when $\kappa'$ is smaller than $\kappa$. This implies that the proposed shrinkage factors might be too large in some settings, as the choice of shrinkage factor minimizes the upper bound of the maximum regret.
Figure 5: The solid, dotted, and dashed lines denote the maximum regrets of the shrinkage, CES, and pooling rules when $\kappa' = 0.8\kappa$. 

(a) $(\sigma_1, \sigma_2) = (1, 1), (p_1, p_2) = (0.5, 0.5)$.  

(b) $(\sigma_1, \sigma_2) = (1, 1), (p_1, p_2) = (0.75, 0.25)$.  

(c) $(\sigma_1, \sigma_2) = (0.75, 1.25), (p_1, p_2) = (0.5, 0.5)$.  

(d) $(\sigma_1, \sigma_2) = (0.75, 1.25), (p_1, p_2) = (0.75, 0.25)$.  
6 Empirical application

We illustrate the proposed method by applying it to experimental data from the National Job Training Partnership Act (JTPA) Study and using the dataset in Abadie, Angrist, and Imbens (2002). The JTPA study is a randomized controlled trial whose purpose was to measure the impact of a training program on earnings. It also collected background information on the applicants prior to the random assignment and obtained data on their earnings for 30 months following the assignment.

We construct 24 subgroups using the following characteristics: race (black, Hispanic, or other), sex (male or female), marital status (married or unmarried), and working status prior to random assignment (worked for at least 12 weeks in the 12 months preceding random assignment or not). For each subgroup, we calculate the treatment effect of the training program on earnings for 30 months following the assignment and its standard error. Following Kitagawa and Tetenov (2018), we set the treatment cost as $774. Hence, we use the CATE estimate minus $774 as $\hat{\theta}_k$. Figure 6 shows that the benefits of the training program vary across subgroups but are statistically insignificant for all subgroups.

Figure 6: The black dots denote the CATE estimates minus $774 and the bars denote the 95% confidence intervals.
As the average of $\hat{\theta}$ is approximately $541$, the pooling rule determines to treat the individuals in all subgroups. However, Figure 6 indicates that the decisions of the CES rule vary across the subgroups. Hence, the shrinkage rule may differ from the CES rule depending on the values of the shrinkage factors.

We calculate the shrinkage factors $w^*(\kappa)$ for $\kappa = 500, 1,000$. Figure 7 shows the relationship between the shrinkage factor and standard error. As expected, the shrinkage factor decreases as the standard error increases. For $\kappa = 1,000$, the shrinkage rule becomes the CES rule when $\sigma_k$ is less than around 1,300. Proposition 1 indicates that $w^*_k(\kappa)$ approaches 1 asymptotically when $\kappa/\sigma_k \leq t^*(0) \simeq 0.75$, which is equivalent to $\sigma_k < \kappa/t^*(0) \simeq 1.33\kappa$. Hence, this result indicates that the approximation in Proposition 1 is useful.

Figures 8 and 9 illustrate the shrinkage estimates $w^*_k(\kappa) \cdot \hat{\theta}_k + (1 - w^*_k(\kappa)) \cdot \text{ave}(\hat{\theta})$ for $\kappa = 500, 1,000$. As the red line denotes the average of $\hat{\theta}$, the shrinkage estimate (white circle) is closer to the red line than $\hat{\theta}_k$ (black circle). As the average of $\hat{\theta}$ is positive, the decision of the shrinkage rule is the same as that of the CES rule when $\hat{\theta}_k$ is positive. Whereas, the decision regarding the shrinkage rule can differ from that regarding the CES rule when $\hat{\theta}_k$ is negative. Figure 8 shows that the decision of the shrinkage rule is not identical to that of the CES rule in certain subgroups. However, the shrinkage rule makes the same decisions as the CES rule when $\kappa = 1,000$.

This analysis focuses on the treatment choice problem when $\kappa = 500, 1,000$. As $w^*_k(\kappa)$ is increasing with respect to $\kappa$, the shrinkage rule makes the same decisions as the CES rule
Figure 8: The shrinkage rule when $\kappa = 500$. The black circles denote $\hat{\theta}_k$, the white circles denote the shrinkage estimates $w^*_k(\kappa) \cdot \hat{\theta}_k + (1 - w^*_k(\kappa)) \cdot \text{ave}(\hat{\theta})$, and the blue and red lines denote 0 and the average of $\hat{\theta}$, respectively.

when $\kappa$ is greater than 1,000. Hence, the decision regarding the shrinkage rule differs from that regarding the CES rule only when $\kappa$ is small. However, the choice of $\kappa = 500$ is not unrealistic. Letting $Z_k \sim N(0, \sigma_k^2)$, the median of $\max_{1 \leq k \leq K} \{Z_k\} - \min_{1 \leq k \leq K} \{Z_k\}$ is about 10,000. Whereas, the realized value of $\max_{1 \leq k \leq K} \{\hat{\theta}_k\} - \min_{1 \leq k \leq K} \{\hat{\theta}_k\}$ is 7,489. This implies that any value of $\kappa$ is not sufficiently small to be inconsistent with the actual data. Therefore, this empirical application shows that the decision of the shrinkage rule can differ from that of the CES rule even if $\kappa$ is a realistic value.

7 Conclusion

This study examined the problem of determining whether to treat individuals based on observed covariates. Particularly, we proposed a computationally tractable shrinkage rule that selects the shrinkage factor by minimizing the upper bound of the maximum regret. We also provided upper bounds of the ratio of the maximum regret of the shrinkage rule to those of the CES and pooling rules when the parameter space was correctly specified or misspecified. The
Figure 9: The shrinkage rule when $\kappa = 1,000$. The black circles denote $\hat{\theta}_k$, the white circles denote the shrinkage estimates $w^*_k(\kappa) \cdot \hat{\theta}_k + (1 - w^*_k(\kappa)) \cdot \text{ave}(\hat{\theta})$, and the blue and red lines denote 0 and the average of $\hat{\theta}$, respectively.

Theoretical and numerical results show that our shrinkage rule performs better than the CES and pooling rules in many cases when the parameter space is correctly specified. In addition, the results were robust to the misspecifications of the parameter space. Particularly, we found that the maximum regret of the shrinkage rule can be strictly smaller than that of the CES and pooling rules when the dispersion of the CATEs is moderate. Finally, we applied our method to experimental data from the JTPA study and showed that the decision of the shrinkage rule can differ from that of the CES rule even if $\kappa$ is a realistic value.
Appendix 1: Proofs and lemmas

Lemma 1. For any $a \geq 0$, we have
\[
\eta'(a) \equiv \frac{d}{da} \eta(a) = \Phi(-t^*(a) + a).
\]
In addition, $-t^*(a) + a$ is strictly increasing in $a$.

Proof. This lemma follows from the proof of Lemma 2 in Ishihara and Kitagawa (2021).

Lemma 2. For any $a \geq 0$ and $v \in \mathbb{R}$, we have
\[
v \Phi(-v) + \Phi(-v)a \leq \eta(a) \leq \eta(0) + a,
\]
\[
\eta(0) \sqrt{1 + a^2} \leq \eta(a) \leq \sqrt{1 + a^2}.
\]

Proof. This lemma follows from Lemmas 1 and 2 in Ishihara and Kitagawa (2021).

Proof of Proposition 1. Using Lemma 1, we obtain
\[
\frac{d}{d w_k} \tilde{\psi}_k(w_k; \kappa) = \sigma_k \eta \left( (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) \right)
\]
\[
-\kappa w_k^{-1} \Phi \left( t^* \left( (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) \right) + (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) \right).
\]
Because we have $\eta(a) = t^*(a) \cdot \Phi(-t^*(a) + a)$, we have
\[
\frac{d}{d w_k} \tilde{\psi}_k(w_k; \kappa) = \left\{ \sigma_k - \frac{\kappa w_k^{-1}}{t^* \left( (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) \right)} \right\} \eta \left( (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) \right)
\]
Hence, we obtain
\[
\text{sgn} \left( \frac{d}{d w_k} \tilde{\psi}_k(w_k; \kappa) \right) = \text{sgn} \left( t^* \left( (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) \right) - (\kappa/\sigma_k) w_k^{-1} \right). \quad (A.1)
\]
Because $t^*(0) > 0$, the sign of the derivative becomes strictly positive when $\kappa = 0$. Hence, we obtain $\tilde{w}_k^*(0) = 0$.

Because $-t^*(a) + a + t^*(0)$ is strictly positive for $a > 0$ by Lemma 1, we have
\[
t^* \left( (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) \right) - (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) - t^*(0) < 0 \quad \text{for all } w_k \in (0, 1).
\]
This implies that for all \( w_k \in (0, 1) \), we have
\[
t^* \left( (w_k^{-1} - 1) \cdot (\kappa/\sigma_k) \right) - (\kappa/\sigma_k)w_k^{-1} < t^*(0) - \kappa/\sigma_k.
\]
Therefore, if \( \kappa/\sigma_k > t^*(0) \), then \( \frac{d}{dw_k} \tilde{\psi}_k(w_k) \) is negative for all \( w_k \in (0, 1) \). As a result, if \( \kappa/\sigma_k > t^*(0) \), then we have \( \tilde{w}_k^*(\kappa) = 1 \).

**Proof of Theorem 1.** The upper bound of (13) is obtained from the proof of Theorem 3.

**Proof of Theorem 2.** The upper bound of (14) is obtained from the proof of Theorem 4.

**Proof of Theorem 3.** Because \( \hat{\delta}^1(\hat{\theta}) = \hat{\delta}^{\text{CES}}(\hat{\theta}) \), it follows from (4) that we have
\[
R(\theta, \hat{\delta}^{\text{CES}}) = \sum_{k=1}^{K} p_k \cdot \left\{ |\theta_k| \cdot \Phi\left( -\frac{|\theta_k|}{\sigma_k} \right) \right\} 
= \sum_{k=1}^{K} p_k \cdot \left\{ \sigma_k \cdot \left( \frac{|\theta_k|}{\sigma_k} \right) \cdot \Phi\left( -\frac{|\theta_k|}{\sigma_k} \right) \right\}.
\]
For any \( t \), a hyper-rectangle \([t, t + \kappa]^K\) is included in \( \Theta(\kappa) \). Hence, for any \( t > 0 \), we obtain
\[
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{\text{CES}}) \geq \max_{\theta \in [t, t + \kappa]^K} R(\theta, \hat{\delta}^{\text{CES}}) = \sum_{k=1}^{K} p_k \sigma_k \cdot \max_{\theta_k \in [t, t + \kappa]} \{ (\theta_k/\sigma_k) \Phi(-\theta_k/\sigma_k) \}
= \sum_{k=1}^{K} p_k \sigma_k \cdot \max_{s \in [t/\sigma_k, (t + \kappa)/\sigma_k]} \{ s \Phi(-s) \}.
\]
When \( t = t^*(0) \cdot \sigma \), we have \( t/\sigma_k \leq t^*(0) \). Because \( s \Phi(-s) \) is single-peaked and maximized at \( s = t^*(0) \), for \( t = t^*(0) \cdot \sigma \) we obtain
\[
\max_{s \in [t/\sigma_k, (t + \kappa)/\sigma_k]} \{ s \Phi(-s) \} = \min \left\{ t^*(0), \frac{t^*(0) \sigma + \kappa}{\sigma_k} \right\} \cdot \Phi\left( -\min \left\{ t^*(0), \frac{t^*(0) \sigma + \kappa}{\sigma_k} \right\} \right)
\geq \min \left\{ t^*(0), \frac{t^*(0) \sigma + \kappa}{\sigma_k} \right\} \cdot \Phi(-t^*(0))
\geq \min \left\{ \eta(0), \frac{\eta(0) \sigma + \kappa \Phi(-t^*(0))}{\sigma_k} \right\}.
\]
Because \( \eta(0) = t^*(0) \Phi(-t^*(0)) \), we obtain

\[
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{\text{CES}}) \geq \sum_{k=1}^{K} p_k \cdot \min \left\{ \eta(0) \sigma_k, \frac{\eta(0) \sigma_k + \kappa \eta(0)}{t^*(0)} \right\} \\
= \eta(0) \cdot \left[ \sum_{k=1}^{K} p_k \cdot \left\{ \sigma_k - \left| \sigma_k - \sigma \right| - \frac{\kappa}{t^*(0)} \right\} \right] \\
\geq \eta(0) \cdot \left( \sum_{k \in K(\kappa)} p_k \sigma_k + \sum_{k \not\in K(\kappa)} p_k \sigma \right),
\]

where \( |a|_+ \equiv \max\{0, a\} \). Hence, if \( \sigma - \sigma \leq \kappa/t^*(0) \), the lower bound of the maximum regret of the CES rule becomes \( \eta(0) \left( \sum_{k=1}^{K} p_k \sigma_k \right) \).

Next, we derive the upper bound of \( \max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w^*(\kappa')}) \). We observe that

\[
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w^*(\kappa')}) \leq \sum_{k=1}^{K} p_k \cdot \psi_k \left( u_{k}^*(\kappa'); \kappa \right) \\
= \sum_{k=1}^{K} p_k \cdot \psi_k \left( u_{k}^*(\kappa'); \kappa \right) \cdot \left( \frac{\psi_k \left( u_{k}^*(\kappa'); \kappa \right)}{\psi_k \left( u_{k}^*(\kappa'); \kappa' \right)} \right) \\
= \sum_{k=1}^{K} p_k \cdot \psi_k \left( u_{k}^*(\kappa'); \kappa \right) \cdot \left\{ \eta \left( \frac{1-w_{k}^*(\kappa') \kappa}{s_{k} \left( w_{k}^*(\kappa') \right)} \right) \right\} \\
\]

This implies that \( \max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w^*(\kappa')}) \leq \sum_{k=1}^{K} p_k \cdot \psi_k \left( u_{k}^*(\kappa'); \kappa' \right) \) when \( \kappa \leq \kappa' \). Because \( \eta'(a) \leq 1 \) for all \( a \geq 0 \), we have \( \eta(a) \leq \eta(a') + (a - a') \) for \( a \geq a' \). Hence, when \( \kappa \geq \kappa' \), we obtain

\[
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w^*(\kappa')}) \\
\leq \sum_{k=1}^{K} p_k \cdot \psi_k \left( u_{k}^*(\kappa'); \kappa' \right) \cdot \left\{ 1 + \frac{\left( \frac{1-w_{k}^*(\kappa') \kappa}{s_{k} \left( w_{k}^*(\kappa') \right)} \right) \cdot \frac{\kappa - \kappa'}{\kappa'}}{\eta \left( \frac{1-w_{k}^*(\kappa') \kappa}{s_{k} \left( w_{k}^*(\kappa') \right)} \right)} \right\} \\
= \sum_{k=1}^{K} p_k \cdot \psi_k \left( u_{k}^*(\kappa'); \kappa' \right) \cdot \left\{ 1 + H \left( \frac{1-w_{k}^*(\kappa') \kappa}{s_{k} \left( w_{k}^*(\kappa') \right)} \right) \cdot \frac{\kappa - \kappa'}{\kappa'} \right\},
\]

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where $H(a) \equiv a/\eta(a)$ and $\max_{a \geq 0} H(a) \simeq 2$. Hence, we obtain the following upper bound:

$$
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{w^*(\kappa')}) \\
\leq \left\{ \sum_{k=1}^{K} p_k \cdot \psi_k \left( w_k^*(\kappa'); \kappa' \right) \right\} \cdot \left[ 1 + \max_{k} \left\{ H \left( \frac{1 - w_k^*(\kappa')}{s_k^*(\kappa')} \right) \cdot \frac{\kappa - \kappa'}{\kappa'} \right\} \right]. \tag{A.3}
$$

From (A.2) and (A.3), we obtain the upper bound of (16).

\[\square\]

**Proof of Theorem 4.** Without loss of generality, we assume $p_1 \geq \cdots \geq p_K$. Because $\hat{\delta}^{\text{pool}}(\hat{\theta}) = \hat{\delta}^0(\hat{\theta})$ and $(t + \kappa, \ldots, t + \kappa, t - \kappa, \ldots, t - \kappa)' \in \Theta(\kappa)$ for all $t \in \mathbb{R}$, we observe that

$$
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{\text{pool}}) = \max_{\theta \in \Theta(\kappa)} \sum_{k=1}^{K} p_k \cdot \left\{ |\theta_k| \cdot \Phi \left( -\text{sgn}(\theta_k) \cdot \frac{\theta}{s_0} \right) \right\}
$$

$$
\geq \max_{t \geq -\kappa} \sum_{k=1}^{K/2} p_k \cdot \left\{ (t + \kappa) \cdot \Phi \left( -\frac{(t + \kappa) - \kappa}{s_0} \right) \right\} + \sum_{k=K/2+1}^{K} p_k \cdot \left\{ |t - \kappa| \cdot \Phi \left( -\text{sgn}(t - \kappa) \cdot \frac{t}{s_0} \right) \right\}.
$$

Substituting $t = s_0 \cdot t^*(\kappa/s_0) - \kappa$, we obtain

$$
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{\text{pool}}) \geq \left( \sum_{k=1}^{K/2} p_k \right) \cdot s_0 \eta(\kappa/s_0) \geq \frac{1}{2} s_0 \eta(\kappa/s_0), \tag{A.4}
$$

where $\sum_{k=1}^{K/2} p_k \geq 1/2$ because $p_1 \geq \cdots \geq p_K$.

Next, we derive another lower bound of $\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{\text{pool}})$. Similar to the above argument, we observe that

$$
\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{\text{pool}}) \geq \max_{t \geq -\kappa} \sum_{k=1}^{K/2} p_k \cdot \left\{ (t + \kappa) \cdot \Phi \left( -\frac{(t + \kappa) - \kappa}{s_0} \right) \right\} + \sum_{k=K/2+1}^{K} p_k \cdot \left\{ |t - \kappa| \cdot \Phi \left( -\text{sgn}(t - \kappa) \cdot \frac{t}{s_0} \right) \right\}
$$

$$
\geq \max_{t \geq -\kappa} \sum_{k=1}^{K/2} p_k \cdot \left\{ (t + \kappa) \cdot \Phi \left( -\frac{(t + \kappa) - \kappa}{s_0} \right) \right\} + \sum_{k=K/2+1}^{K} p_k \cdot \left\{ (t - \kappa) \cdot \Phi \left( -\frac{t}{s_0} \right) \right\}.
$$
Substituting $t = s_0 \cdot t^* (\kappa/s_0) - \kappa$, we obtain

$$\max_{\theta \in \Theta(\kappa)} R(\theta, \hat{\delta}^{\text{pool}}) \geq \sum_{k=1}^{K/2} p_k \cdot s_0 \eta \left( \frac{\kappa}{s_0} \right) + \sum_{k=K/2+1}^{K} p_k \cdot s_0 \left\{ \left( t^* \left( \frac{\kappa}{s_0} \right) - \frac{2\kappa}{s_0} \right) \cdot \Phi \left( -t^* \left( \frac{\kappa}{s_0} \right) + \frac{\kappa}{s_0} \right) \right\}$$

$$\geq s_0 \eta \left( \frac{\kappa}{s_0} \right) - \left( \sum_{k=K/2+1}^{K} p_k \right) \cdot 2\kappa \cdot \Phi \left( -t^* \left( \frac{\kappa}{s_0} \right) + \frac{\kappa}{s_0} \right) \geq s_0 \eta (\kappa/s_0) - \kappa. \quad (A.5)$$

From (A.3), (A.4), and (A.5), we obtain the upper bound of (17). \qed
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