Fock Spaces with Reflection Condition and Generalized Statistics

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Abstract

An oscillator algebra and the associated Fock space with reflecting boundary and generalized statistics are constructed and is generalized to the multicomponent case. The oscillator algebra depends manifestly on the reflection factor and the statistical (exchange) factor, and the corresponding Fock space can be obtained from that of the usual bosonic oscillator without reflection condition by certain projection operation.
1 Introduction

Oscillator algebras and their Fock representations play important basic roles in many branches of modern quantum field theory, condensed matter and solid state physics. Oscillator algebras are also the theoretical foundation of the second quantization method. In the recent years, as the theories of quantum groups and quantum algebras develop, various generalized oscillator algebras have been proposed and acquired much attentions. Among these are $q$-oscillator algebra [1, 2], quon-algebra [3, 4], $(p, q)$-deformed oscillator algebra [5]-[7] etc.

On the other hand, some hints show that the quasi-particles corresponding to the cluster excitations in solids, such as the anions that played some role in fractional quantum Hall effect, do not obey the usual Bose or Fermi statistics [8]-[11]. Following from this fact, the so-called Fock space and its associated oscillator algebra with generalized statistics are constructed [12, 13].

Recently, much progress has been made in the studies of massive field theoretical and lattice statistical models with integrable boundaries [14, 15]. One of the important progresses in these fields is the establishment of the concept of “boundary states” [14, 15], which correspond to the lowest-energy states of quantum integrable field theories and/or solvable lattice statistical models with integrable boundaries. Based on this concept, systematical approaches for calculating the boundary form factors and boundary correlation functions in some boundary solvable lattice statistical models [14] and boundary integrable quantum field theories [16] have been established. However, all these approaches use the technique of bosonization via (deformed) free fields and thus the generalizations to models with higher rank underlying (quantum) Lie algebras seems to be very complicated. Moreover, as the complete Fock space structure of boundary integrable field theories and solvable statistical models is not clear yet, the alternative approach (i.e. that does not depend on the technique of bosonization) is difficult to be established.

In this article an oscillator algebra and the associated Fock space satisfying reflection boundary condition and obeying generalized statistics will be constructed. The oscillator algebra depends crucially on the boundary reflection factor while the corresponding Fock space can be obtained from that of the usual harmonic oscillator algebra with continuous spectrum via simple projection. The creation and anihilation operators in this new oscillator algebra is considered to correspond to quasi-particles in systems with generalized statistics, and the results can be generalized to the cases with more than one kinds of quasi-particle excitations.

2 Oscillator algebra with reflection boundary condition

Let us consider the continuum limit of an $s$-dimensional condensed matter system. In such a system the creation and anihilation operators of fundamental excitations (denoted respectively $b^*$ and $b$) will depend on a continuous $s$-vector $x$ which corresponds to the wave vector or spectrum of the excitations. In the case when there is no reflection boundary condition the creation and anihilation operators satisfy the following usual harmonic oscillator algebra,
\[ [b(x), b(y)] = [b^*(x), b^*(y)] = 0, \]
\[ [b(x), b^*(y)] = \delta(x - y), \]  
(2.1)

where \( \delta(x - y) = \prod_{i=1}^{s} \delta(x^i - y^i) \) and \( x^i \) is the \( i \)-th component of \( x \).

Now suppose that the system is bounded in some direction, say, in the first coordinate direction, and the action of the creation operator, say \( a^*(x) \), on the lowest energy state \( \Omega_B \) (the boundary vacuum state) satisfy the “reflection boundary condition”

\[ a^*(x) \, \Omega_B = K(x) \, a^*(\sigma(x)) \, \Omega_B, \]  
(2.2)

where \( \sigma(x) \) is the reflection of the “wave vector” \( x \) which is defined as

\[ \sigma(x) \equiv \sigma(x^1, x^2, ..., x^s) = (-x^1, x^2, ..., x^s), \quad \sigma^2 = \text{Id}, \]  
(2.3)

\( K(x) \) is called reflection factor which specify the status of the boundary. Requiring that Eq.(2.2) be consistent, the following must hold for \( K(x) \),

\[ K(x) \, K(\sigma(x)) = 1. \]  
(2.4)

In what follows we shall prove that the creation operator satisfying the conditions (2.2-2.4) and the anihilation operator which kills the vacuum state \( \Omega_B \) satisfy an oscillator algebra which is different from eq.(2.1), and the corresponding Fock space can be obtained from that of eq.(2.1) via a simple projection operation.

Let us start by recalling the Fock space of the system \([1, 2, 13]\). According to standard quantum field theories, the space of single particle states is a Hilbert space which is now identified with the space \( H \equiv L^2(\mathbb{R}^s, d^s x) \) of square integrable functions on the Euclidean space \( \mathbb{R}^s \) with respect to the measure \( d^s x \). The \( n \)-particle space \( H^n \) is identified with \( H \otimes_n H \), and the Fock space \( F(H) \) of the system \([2.1]\) is just the direct sum of all the multi-particle subspaces, namely

\[ F(H) = \bigotimes_{n=0}^{\infty} H^n, \quad H^0 \equiv \mathbb{C}. \]  
(2.5)

Points of the Fock space \( F(H) \) are denoted by their “coordinates”, \( \varphi = (\varphi^{(0)}, \varphi^{(1)}, ..., \varphi^{(n)}, ...), \) where \( \varphi^{(n)} \in H^n \). The finite particle subspace \( F^0(H) \) of \( F(H) \) is consisted of those points for which \( \varphi^{(n)} \) is zero for \( n \) large enough.

Consider the subset \( D^n \) of decomposable vectors in \( H^n \),

\[ D^n = \{ f_1 \otimes f_2 \otimes ... \otimes f_n | f_i \in H \} \subset H^n. \]

Define the operations

\[ b(f) : D^n \to D^{n-1}, \quad (n \geq 1); \quad b^*(f) : D^n \to D^{n+1}, \quad (n \geq 0), \quad f \in H \]
on \( D^n \) such that

\[ b(f) \, f_1 \otimes ... \otimes f_n = \sqrt{n} \, (f, f_1) \, f_2 \otimes ... \otimes f_n, \]
\[ b^*(f) \, f_1 \otimes ... \otimes f_n = \sqrt{n+1} \, f \otimes f_1 \otimes ... \otimes f_n, \]  
(2.6)
where \((f, g) \equiv \int d^x \bar{f}(x) g(x), \bar{f}(x)\) represent the conjugation of \(f(x)\).

The actions of \(b(f)\) and \(b^*(f)\) can be extended onto the dense subspace \(L(\mathcal{D}^n)\) of \(\mathcal{H}\) which consists of all the finite linear combinations of the elements of \(\mathcal{D}^n\) by linearity. Thus for any \(\varphi^{(n)} \in L(\mathcal{D}^n)\), we have

\[
[b(f) \varphi^{(n)}](x_1, x_2, \ldots, x_n) = \sqrt{n + 1} \int d^x \bar{f}(x) \varphi^{(n+1)}(x, x_1, \ldots, x_n),
\]

\[
[b^*(f) \varphi^{(n)}](x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(x_k) \varphi^{(n-1)}(x_1, \ldots, \hat{x}_k, \ldots, x_n),
\]

(2.7)

where \(\hat{x}_k\) indicates where \(x_k\) should not appear. One can introduce distributions corresponding to the operators \(b(f)\) and \(b^*(f)\) respectively as

\[
b(f) = \int d^x \bar{f}(x) b(x), \quad b^*(f) = \int d^x f(x) b^*(x).
\]

(2.8)

It can then be shown using eq.(2.7) that the distributions \(b(f)\) and \(b^*(f)\) satisfy eq.(2.1).

In order to construct an oscillator algebra which satisfy the reflection boundary conditions (2.2-2.4) let us define the operators \(\pi_n\) \((n \geq 1)\) acting on \(L(\mathcal{D}^n)\) as

\[
[\pi_n \varphi^{(n)}](x_1, x_2, \ldots, x_n) = K(x_n) \varphi^{(n)}(x_1, \ldots, x_{n-1}, \sigma(x_n)).
\]

(2.9)

Thanks to (2.4), we have

\[
\pi_2^n = Id,
\]

(2.10)

and in order that \(\pi_n\) be Hermitian we also require

\[
\bar{K}(x) = K(\sigma(x)).
\]

(2.11)

It is obvious that

\[
P_B^{(n)} = \frac{1}{2}(Id + \pi_n), \quad \bar{P}_B^{(n)} = \frac{1}{2}(Id - \pi_n)
\]

(2.12)

are both projection operators which are orthogonal to each other. When supplied with \(P_B^{(0)} = Id\), one can extend the set \(\{P_B^{(n)} \mid (n \geq 0)\}\) onto the whole Fock space \(\mathcal{F}(\mathcal{H})\),

\[
\mathcal{F}_B(\mathcal{H}) \equiv P_B \mathcal{F}(\mathcal{H}), \quad P_B|\mathcal{H}^n = P_B^{(n)}.
\]

(2.13)

The projected Fock space \(\mathcal{F}_B(\mathcal{H})\) has a natural decomposition \(\mathcal{F}_B(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n_B\), where \(\mathcal{H}^n_B = P_B^{(n)} \mathcal{H}^n\). Correspondingly, the projected creation and annihilation operators read

\[
a^{\#}(f) = P_B b^{\#}(f) P_B,
\]

(2.14)

where \(b^{\#}\) stands for both \(b\) and \(b^*\).

By straightforward calculations using eqs. (2.7), (2.9), (2.12) and (2.14) we have, for arbitrary \(\varphi^{(n)} \in L(\mathcal{D}^n_B)\),
\[ [a(f) \varphi^{(n)}(x_1, ..., x_n) = \sqrt{n+1} \int d^nx \bar{f}(x) \varphi^{(n+1)}(x, x_1, ..., x_n), \quad (2.15) \]
\[ [a^*(f) \varphi^{(n)}(x_1, ..., x_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} [f(x_k) + K(x_k)f(\sigma(x_k))]] \times \varphi^{(n-1)}(x_1, ..., \hat{x}_k, ..., x_n). \quad (2.16) \]

Now define
\[ a(f) = \int d^nx \bar{f}(x) a(x), \quad a^*(f) = \int d^nx f(x) a^*(x) \]
in analogy to (2.8), we have, from (2.15-2.16),
\[ [a(x), a(y)] = [a^*(x), a^*(y)] = 0, \]
\[ [a(x), a^*(y)] = \delta(y-x) + K(x)\delta(y-\sigma(x)). \quad (2.17) \]

This algebra is just what we have called the oscillator algebra satisfying the reflection boundary condition. Since the vacuum state \( \Omega_B \) of this algebra is just the image of that of the algebra (2.1) under the projection (2.13), and the vacuum state \( \Omega \) of (2.1) belongs to \( \mathcal{H}^0 \) and \( P_B^{(0)} = \text{Id} \), we have the conclusion \( \Omega_B = \Omega \).

Let us end this section by proving that the algebra (2.17) indeed satisfy the reflection boundary conditions (2.2-2.4). Since \( a^*(f)\Omega_B \in \mathcal{H}_B = P_B^{(1)}\mathcal{H} \), we have
\[ \bar{P}_B^{(1)}(a^*(f)\Omega_B) = 0, \quad (2.18) \]
or written as
\[ a^*(f)\Omega_B - \pi_1 a^*(f)\Omega_B = 0. \quad (2.19) \]
Using the definition of \( \pi_1 \) and rewriting \( a^*(f) \) in terms of the distribution \( a^*(x) \), one can easily see that eq. (2.19) is equivalent to (2.2), and we thus have obtained the oscillator algebra (2.17) satisfying the reflection boundary conditions (2.2-2.4) and its Fock space \( \mathcal{F}_B(\mathcal{H}) \).

### 3 Oscillator algebra with reflection boundary condition and generalized statistics

In the last section we obtained the oscillator algebra (2.17) which satisfy the reflection boundary conditions (2.2-2.4). However, this algebra still obeys the usual bose statistics, i.e. \( a(x) \) and \( a(y) \) exchange in the way the usual bosons do. In this section we shall generalize the construction to include the generalized statistics. For this purpose we adopt the technique introduced in Ref. [12, 13]. On arbitrary \( \varphi^{(n)} \in \mathcal{H}^n \) we introduce the following operations,
\[ s_i \varphi^{(n)}(x_1, ..., x_n) = R(x_i, x_{i+1}) \varphi^{(n)}(x_1, ..., x_{i+1}, x_i, ..., x_n), \]
\[ R(x_i, x_{i+1})R(x_{i+1}, x_i) = 1, \quad \bar{R}(x_i, x_{i+1}) = R(x_{i+1}, x_i), \quad (i = 1, ..., n-1). \tag{3.1} \]

It is an easy practice to prove that \( s_i \) \( (i = 1, ..., n-1) \) and \( s_n \equiv \pi_n \) satisfy the following relations

\[
(s_i s_j)^{m_{ij}} = 1,
\]
\[
m_{ij} = \begin{cases} 
1, & |i - j| = 0 \\
2, & |i - j| > 1 \\
3, & |i - j| = 1 \text{ and } i, j \neq n \\
4, & |i - j| = 1 \text{ and } \max(i, j) = n
\end{cases} \tag{3.2}
\]

provided the “exchange factor” \( R(x_i, x_{i+1}) \) satisfy the following condition

\[ R(x, y)R(y, \sigma(x)) = R(x, \sigma(y))R(\sigma(x), y). \tag{3.3} \]

Eq. (3.2) is exactly the generating relation of the Weyl group \( B_n \) which corresponds to the Lie algebra \( B_n \) and \( C_n \) \([17]\). Therefore, eqs. (2.9) and (3.1) actually define an action of the Weyl group \( B_n \) on \( H^n \). Eq. (3.3) may seem to be misterious at a first glance. However, it will be clear in the next section that this is actually the most degenerated case of the (generalized) boundary Yang-Baxter equation.

Our purpose is to find a subspace \( \mathcal{F}_R(\mathcal{H}) \) of \( \mathcal{F}(\mathcal{H}) \) on which the exchange of any two neighboring particles sitting for example at cites \( i \) and \( i+1 \) in an arbitrary \( n \)-particle state give rise to the “exchange factor” \( R(x_i, x_{i+1}) \), and the single particle state still satisfy the conditions (2.2-2.4). To achieve this we now define the projection operators

\[
P^{(n)}_R = \frac{1}{\dim B_n} \sum_{g \in B_n} g, \quad n \geq 2 \tag{3.4}
\]

and also let \( P^{(0)}_R = \text{Id} \), \( P^{(1)}_R = P^{(1)}_B \). It follows from the fact \( \pi_n \in B_n \) and the rearrangement theorem that \( (\text{Id} - \pi_n)P^{(n)}_R = 0 \), which implies that \( P^{(n)}_R \) and \( P^{(n)}_B \) are orthogonal to each other. Similar to the case of \( P^{(n)}_B \), one can also extend the action of \( P^{(n)}_R \) to the whole Fock space \( \mathcal{F}(\mathcal{H}) \),

\[
\mathcal{F}_R(\mathcal{H}) = P_R \mathcal{F}(\mathcal{H}), \quad P_R|_{\mathcal{H}^n} = P^{(n)}_R.
\]

Correspondingly, the operators \( b^\# \) are projected to \( a^\#_R \),

\[
a^\#_R = P_R b^\# P_R.
\]

It can be proved that the action of \( a^\#_R \) on arbitrary \( \varphi^{(n)} \in \mathcal{L}(\mathcal{D}^R_R) \) gives rise to the following results,
After rather tedious calculations, one can get

\[ [a_R(f)\varphi^{(n)}(x_1, ..., x_n)] = \sqrt{n + 1} \int \, \, d^n x \, f(x) \, \varphi^{(n+1)}(x, x_1, ..., x_n), \]

\[ [a_R^*(f)\varphi^{(n)}(x_1, ..., x_n)] = \frac{1}{\sqrt{n}} [R(x_{n-1}, x_n)...R(x_1, x_n)f(x_n)
+ K(x_n)R(x_{n-1}, \sigma(x_n))...R(x_1, \sigma(x_n))f(\sigma(x_n))] \varphi^{(n-1)}(x_1, ..., x_{n-1})
+ \frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} R(x_k, x_{k+1})...R(x_1, x_n)[R(x_n, x_k)...R(x_{k+1}, x_k)
\times R(x_{k-1}, x_k)...R(x_1, x_k)f(x_k) + K(x_k)R(x_{n-1}, \sigma(x_k))...R(x_{k+1}, \sigma(x_k))
\times R(x_{k-1}, \sigma(x_k))...R(x_1, \sigma(x_k))f(\sigma(x_k))] \varphi^{(n-1)}(x_1, ..., x_{n-1}). \] (3.5)

After rather tedious calculations, one can get

\[ a_R(x)a_R(y) - R(y, x)a_R(y)a_R(x) = 0, \]
\[ a_R^*(x)a_R^*(y) - R(y, x)a_R^*(y)a_R^*(x) = 0, \] (3.6)
\[ a_R(x)a_R^*(y) - R(x, y)a_R^*(y)a_R(x) = \delta(y - x) + F(x)\delta(y - \sigma(x)), \]

where \( F(x) \) is an Hermitian operator acting on \( F_R(H) \) as

\[ [F(x)\varphi^{(n)}(x_1, ..., x_n)] = R(x, x_1)...R(x, x_n)K(x)R(x_n, \sigma(x))...R(x_1, \sigma(x))\varphi^{(n)}(x_1, ..., x_n). \] (3.7)

Using eqs. (3.3) and (3.7) one can also obtain the exchange relations

\[ R(x, y)F(x)a_R(y)R(y, \sigma(x)) - a_R(y)F(x) = 0, \]
\[ F(x)a_R^*(y) - R(y, x)a_R^*(y)F(x)R(\sigma(x), y) = 0, \] (3.8)
\[ F(x)F(y) - F(y)F(x) = 0. \]

Eqs. (3.6) and (3.8) together form an oscillator algebra satisfying the reflection boundary conditions (2.2) and obeying the \( R \)-generalized statistics. Notice that if in eq. (1.2) we did not include \( \pi_n \), the corresponding generating relations would become that of the permutation group, and the resulting algebra will turn out to be the oscillator algebra without the reflection condition [12, 13].

### 4 Multicomponent generalization (summary)

In this section we shall generalize the construction even further to the case when more than one kinds of creation and annihilation operators are involved. Since the multicomponent case is far more complicated comparing to the single component case and such case is believed to be very important in the context of quantum factorizable scattering theory, we shall give the detailed calculations elsewhere and present here only with the brief summary.

Suppose there are \( N \)-kinds of elementary excitations in the system. Then the space of single-particle states before the projection will be
\[ \mathcal{H} = \bigoplus_{\alpha=1}^{N} L^{2}(\mathbb{R}^n, d^2x). \]  

(4.1)

Correspondingly the Fock space reads

\[ \mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{n}. \]  

(4.2)

In \( \mathcal{H} \) the inner product is defined as

\[ (f, g) = \int d^2x \ f^{*}(x)g(x) = \sum_{\alpha=1}^{N} \int d^2x \ \bar{f}_{\alpha}(x)g_{\alpha}(x), \]  

(4.3)

where \( f = (f_1, \ldots, f_N)^T \in \mathcal{H}. \) In the above system we introduce the \( N^2 \times N^2 \) matrix \( (R^{\alpha_1,\beta_2}_{\alpha_2,\beta_1}(x_1, x_2)) \) as exchange factor and the \( N \times N \) matrix \( (K^\alpha_{\beta}(x)) \) as reflection factor, which satisfy the following constraints,

\[ R^{\alpha_1,\beta_2}_{\alpha_2,\beta_1}(x_1, x_2)R^{\gamma_3,\delta_4}_{\delta_3,\gamma_2}(x_1, x_3)R^{\beta_4,\alpha_2}_{\alpha_1,\beta_3}(x_2, x_3) = R^{\alpha_1,\beta_2}_{\alpha_2,\beta_1}(x_2, x_3)R^{\gamma_3,\delta_4}_{\delta_3,\gamma_2}(x_1, x_3)R^{\beta_4,\alpha_2}_{\alpha_1,\beta_3}(x_1, x_2), \]  

(4.4)

\[ R^{\alpha_1,\beta_2}_{\alpha_2,\beta_1}(x_1, x_2)R^{\delta_4,\beta_3}_{\beta_3,\delta_2}(x_2, x_1) = \delta^{\beta_1}_{\alpha_1} \delta^{\beta_2}_{\alpha_2}, \]  

(4.5)

\[ R^{\alpha_1,\beta_2}_{\alpha_2,\beta_1}(x_1, x_2) = R^{\alpha_1,\beta_2}_{\alpha_2,\beta_1}(x_2, x_1), \]  

(4.6)

\[ R^{\alpha_1,\beta_2}_{\alpha_2,\beta_1}(x_1, x_2)K^{\alpha_3,\gamma_4}_{\gamma_2,\alpha_2}(x_1)R^{\beta_4,\delta_3}_{\delta_3,\beta_2}(x_2, \sigma(x_1))K^{\beta_3,\gamma_4}_{\gamma_2,\beta_2}(x_2) \]  

\[ = K^{\alpha_3,\gamma_4}_{\gamma_2,\alpha_2}(x_2)R^{\beta_4,\delta_3}_{\delta_3,\beta_2}(x_1, \sigma(x_2))K^{\beta_3,\gamma_4}_{\gamma_2,\beta_2}(x_1)R^{\alpha_1,\beta_2}_{\alpha_2,\beta_1}(\sigma(x_2), \sigma(x_1)), \]  

(4.7)

\[ K^\alpha_{\beta}(x)K^\beta_{\gamma}(\sigma(x)) = \delta^\alpha_{\gamma}, \]  

(4.8)

\[ K^\alpha_{\beta}(x) = K^\alpha_{\beta}(\sigma(x)). \]  

(4.9)

These constraints are generalizations of the well-known (braid type) Yang-Baxter equation, the unitarity and cross unitarity conditions for the \( R \)-matrix, the boundary Yang-Baxter equation and the unitarity and cross unitarity conditions of the \( K \)-matrix. Since we did not introduce an explicit conjugation matrix, the presentation of eqs. (4.6) and (4.9) are slightly different from the form of the usual cross unitarity conditions for the \( R \)- and \( K \)-matrices. Notice that the counterpart of eq. (4.4) did not appear in the last section because it holds trivially there. Notice also that eq. (4.4) reduces to eq. (2.4) in the one component case.

Using the above equations one can show that the operators \( s_i, \ (i = 1, \ldots, n-1) \) and \( s_n \equiv \pi_n \), defined as follows, satisfy eq. (3.2),

\[ [s_i\varphi]^{(n)}_{\alpha_1 \ldots \alpha_n}(x_1, \ldots, x_n) \]  

\[ = [R_{i+1,i}(x_i, x_{i+1})]^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n} \varphi^{(n)}_{\beta_1 \ldots \beta_n}(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n), \quad (n \geq 2) \]  

\[ [\pi_n\varphi]^{(n)}_{\alpha_1 \ldots \alpha_n}(x_1, \ldots, x_n) = [K_n(x_n)]^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n} \varphi^{(n)}_{\beta_1 \ldots \beta_n}(x_1, \ldots, x_{n-1}, \sigma(x_n)), \quad (n \geq 1), \]  

\[ [R_{i,j}(x_i, x_j)]^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n} \equiv \delta^{\beta_1}_{\alpha_1} \delta^{\beta_2}_{\alpha_2} \ldots \delta^{\beta_n}_{\alpha_n} R^{\alpha_1 \beta_1}_{\beta_1 \alpha_1}(x_i, x_j), \]  

\[ [K_j(x_j)]^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n} \equiv \delta^{\beta_1}_{\alpha_1} \delta^{\beta_2}_{\alpha_2} \ldots \delta^{\beta_n}_{\alpha_n} K_{\beta_1 \alpha_1}(x_j). \]
Therefore, following exactly the parallel procedure as in the last section, one can define the projection operations and obtain the projected Fock space. Finally, in the projected Fock space, one can obtain a complete set of creation operators and annihilation operators which obey the following algebra,

\[ a_\alpha(x) a_\beta(y) - R^\delta_{\beta \gamma}(y, x) a_\gamma(y) a_\delta(x) = 0, \]  
\[ a^*\alpha(x) a^*\beta(y) - a^*\gamma(y) a^*\delta(x) R^\alpha_{\gamma \delta}(y, x) = 0, \]  
\[ a_\alpha(x) a^*\beta(y) - a^*\gamma(y) R^\alpha_{\gamma \beta}(x, y) a_\delta(x) = \delta^\delta_\alpha \delta(y - x) + F_\alpha^\beta(x) \delta(y - \sigma(x)), \]  
\[ R^\alpha_{\gamma \delta}(x, y) F^\beta_{\alpha}(x) a_\gamma(y) R^\delta_{\gamma \beta}(y, \sigma(x)) - a_\gamma(y) F^\beta_{\alpha}(x) = 0, \]  
\[ F^\beta_{\alpha}(x) a^*\gamma(y) - R^\beta_{\gamma \delta}(x, y) a^*\gamma(y) F^\gamma_{\beta}(x) R^\gamma_{\alpha \delta}(\sigma(x), y) = 0, \]  
\[ R^\gamma_{\alpha \beta}(x, y) F^\beta_{\alpha}(x) R^\gamma_{\beta \gamma}(y, \sigma(x)) F^\gamma_{\alpha}(x) \]
\[ = F^\gamma_{\alpha \beta}(x) R^\gamma_{\alpha \beta}(x, \sigma(y)) F^\gamma_{\alpha}(x) R^\gamma_{\alpha \beta}(\sigma(y)), \]  
where \( F^\beta_{\alpha}(x) \) is defined as

\[ [F^\beta_{\alpha}(x) \phi^{(n)}_{\beta_1 \ldots \beta_n}(x_1, \ldots, x_n)] = [R_{01}(x, x_1) \ldots R_{n-1}(x, x_n) K_{\alpha}(x) R_{n-1}(n, \sigma(x)) \ldots R_{01}(x_1, \sigma(x))]^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n} \times \phi^{(n)}_{\beta_1 \ldots \beta_n}(x_1, \ldots, x_n). \]

Eqs. (4.10-4.14) form an multicomponent generalization of the algebra obtained in the last section. Notice that if \( F^\beta_{\alpha}(x) = 0 \), then the above algebra will become the well-known Faddeev-Zamolodchikov algebra appeared in the context of quantum factorizable scattering theory. In the present case \( a^*\alpha(x) \) obeys the reflection boundary condition

\[ a^*\alpha(x) \Omega_B = K^\alpha_{\beta}(x) a^*\beta(\sigma(x)) \Omega_B. \]

This observation may imply the possible application of the algebra \( 4.10 \) in the boundary factorizable scattering theories.

5 Discussions

In the above we have obtained the oscillator algebra and the corresponding Fock space which satisfy the reflection boundary condition and obey the generalized statistics. These results already enables us to calculate the \( n \)-particle correlation functions in the corresponding system. For example, in the one component case, the \( n \)-particle states are nothing but the linear combinations of the vectors of the form \( a_R(x_1) \ldots a_R(x_n) \Omega_B \). Therefore the \( n \)-point correlation function can be expressed as

\[ \omega_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = \frac{(a^*_R(x_1) \ldots a^*_R(x_n) \Omega_B, a^*_R(y_1) \ldots a^*_R(y_n) \Omega_B)}{(\Omega_B, \Omega_B)}. \]

Actually one can write out an iterative relation for \( \omega_n \),
\[
\omega_n(x_1, ..., x_n; y_1, ..., y_n) \\
= \left[ \delta(y_1 - x_1) + R(x_1, y_2)R(x_1, y_n)K(x_1) \right. \\
\times R(y_n, \sigma(x_1))...R(y_2, \sigma(x_1))\delta(y_1 - \sigma(x_1)) \\
\left. \times \omega_{n-1}(x_2, ..., x_n; y_2, ..., y_n) \right] \\
+ \sum_{k=2}^{n} R(x_1, y_1)...R(x_1, y_{k-1})[\delta(y_k - x_1) + R(x_1, y_{k+1})...R(x_1, y_n)K(x_1) \\
\times R(y_n, \sigma(x_1))...R(y_{k+1}, \sigma(x_1))\delta(y_k - \sigma(x_1))] \\
\times \omega_{n-1}(x_2, ..., x_n; y_1, ..., \hat{y}_k, ..., y_n).
\]

The first few of the \( \omega_n \) read

\[ \omega_0 = 1, \quad \omega_1(x, y) = \delta(y - x) + K(x)\delta(y - \sigma(x)), \]
\[ \omega_2(x_1, x_2; y_1, y_2) = \left[ \delta(y_1 - x_1) + R(x_1, y_2)R(x_1, \sigma(x_1))\delta(y_1 - \sigma(x_1)) \right. \\
\times [\delta(y_2 - x_2) + K(x_2)\delta(y_2 - \sigma(x_2))] \\
+ R(x_1, y_1)[\delta(y_2 - x_1) + K(x_1)\delta(y_2 - \sigma(x_1))] \\
\left. \times [\delta(y_2 - x_2) + K(x_2)\delta(y_2 - \sigma(x_2))] \right]. \]

Many problems related to the construction in this article are still left for study. For examples, the second quantization based on the oscillator algebra of this article, the coherent states, the partition function of the multi-particle system and the properties of the ideal gas consisted of the (quasi-)particles all deserve to be further studied. Particularly, for the second quantization problem, Liguori and Mintchev \[12, 13\] have considered the possibility of obtaining multi-particle Hamiltonian with nontrivial interactions starting from a single free particle Hamiltonian in the case when there is no boundary reflection and successfully applied the results to the Leinaas-Myrheim anion system \[18, 19\]. It is desirable that the similar construction can be carried out in the boundary case and the resulting Hamiltonians will include not only interactions but also self-interactions. We would like to come back to this point at a later time.

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