Quantum Phases of Vortices in Rotating Bose-Einstein Condensates

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We investigate the groundstates of weakly interacting bosons in a rotating trap as a function of the number of bosons, $N$, and the average number of vortices, $N_{\nu}$. We identify the filling fraction $\nu \equiv N/N_{\nu}$ as the parameter controlling the nature of these states. We present results indicating that, as a function of $\nu$, there is a zero temperature phase transition between a triangular vortex lattice phase, and strongly-correlated vortex liquid phases. The vortex liquid phases appear to be the Read-Rezayi parafermion states.

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A fundamental characteristic of condensed Bose systems is their response to rotation [1]. A transition to a “normal” phase might be expected at sufficiently high angular velocities, $\omega$, of the container (or trap) by loose analogy with a superconductor in a magnetic field. At zero temperature this phase would constitute a novel uncondensed ground state. Such a regime is entered when the vortex cores start to overlap. The corresponding value of $\omega$ is unattainable with bulk $^3$He, but may be achievable in the very dilute degenerate atomic gases initially explored in Ref. [2], and studied extensively in Refs. [3,4]. Apart from the identification [3] of the Laughlin state as the ground state at sufficiently high $\omega$, work on the most interesting regime of large numbers of vortices has been restricted to either mean field theory [3] or exact diagonalisation [3,4]. These two approaches have exhibited apparently contradictory pictures. Within Gross-Pitaevskii (GP) mean-field theory, the groundstates are vortex lattices (distorted by the confinement), with broken rotational symmetry [3]. On the other hand, exact diagonalisations have identified groundstates which do not have crystalline correlations of vortex locations [3]; they are strongly correlated vortex liquids, closely related to incompressible liquid states responsible for the fractional quantum Hall effect [3,4].

Here we present results of extensive exact diagonalisations (EDs) that elucidate the relationship between these two pictures. By using a periodic geometry, we have been able to study systems containing many vortices up to boson densities far in excess of previous EDs. Our results indicate that both crystalline and liquid phases of vortices exist. A clean distinction between these phases can only be made for a large number of vortices. In this limit, we argue that there is a zero-temperature phase transition as a function of the “filling fraction”, $\nu \equiv N/N_{\nu}$, the ratio of the number of bosons, $N$, to the average number of vortices, $N_{\nu}$. For large $\nu$ the groundstate is a vortex lattice (characterised by broken translational/rotational symmetry). For small $\nu$ the groundstates are strongly-correlated vortex liquids. We find that the vortex-liquid groundstates are related to the Read-Rezayi “parafermion” states [10] that were introduced in the context of fractional quantum Hall systems.

In a frame of reference rotating with angular velocity $\omega \hat{z}$, the Hamiltonian for a particle of mass $m$ in an (isotropic) harmonic trap of natural frequency $\omega_0$ is

$$H_\omega = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2 - \omega \hat{z} \cdot \mathbf{r} \times \mathbf{p}$$

$$= \left[ (p - m\omega \hat{z} \times \mathbf{r})^2 \right] \frac{2m}{2m} + \frac{1}{2}m \left[ (\omega_0^2 - \omega^2)(x^2 + y^2) + \omega_0^2 z^2 \right].$$

The second form indicates the equivalence to the Hamiltonian of a particle of charge $q^*$ experiencing an effective magnetic field $\mathbf{B}^* = \nabla \times \left( \frac{m\omega \hat{z} \times \mathbf{r}}{q^*} \right) = \left( \frac{2m\omega}{q^*} \right) \hat{z}$ (the particle also feels a reduced $xy$-confinement). Of particular importance to our discussion is the average filling fraction, $\nu$, for the bosons in this effective magnetic field. For $N$ bosons spread over an area $A$, one finds

$$\nu \equiv \frac{N}{A} \frac{\hbar}{q^*B^*} = \frac{N}{A} \frac{\hbar}{2m\omega} = \frac{N}{N_{\nu}},$$

where $N_{\nu}$ is the average number of vortices. For large $N_{\nu}$ the vortex density is approximately uniform, and $N_{\nu} = (2m\omega A)/\hbar$, or (equivalently) $N_{\nu} = 2L/N$ [1], where $L$ is the total angular momentum in units of $\hbar$.

We now introduce repulsive interactions [12]

$$V = \sum_{i<j=1}^N \delta(\mathbf{r}_i - \mathbf{r}_j),$$

with $g = 4\pi\hbar^2 a/m$, chosen to give the correct $s$-wave scattering length $a$. Throughout this work, we make use of the limit of weak interactions formulated in Ref. [3]. For $g \ll \hbar \omega_0 a^3$, with $a$ the interparticle spacing, the bosons are restricted to single particle states in the lowest Landau level, and lowest oscillator state of $z$. For $\omega \sim \omega_0$, the repulsive interactions give rise to the appearance of rotating (vortex) groundstates [3,4].

GP theory [3] takes account of interactions by finding the fully-condensed state that minimises the total energy.
In EDs \( 1\), the groundstate is found by diagonalising \( 2\) within the set of all states of \( N \) bosons with fixed total angular momentum \( L \) (\( L \) is conserved by interactions). An important distinction between these two approaches is that the GP groundstates exhibit broken rotational symmetry \( 3\), while the ED groundstate is necessarily an eigenstate of angular momentum, \( L \). However, by performing EDs on large numbers of bosons (up to \( N = 30 \) at \( L \sim 2N \)), we find that, as \( N \) becomes large at fixed \( L/N \), a macroscopic number of quasi-degeneracies appear between states with different \( L \). This signals the emergence of broken rotational symmetry. Indeed, it appears from these and other \( 4\) studies that as \( N \) becomes large for fixed \( L/N \), there is a crossover to a regime in which GP theory is essentially correct. We believe that this crossover is related to the phase transition, discussed in detail below, between vortex liquids at small \( \nu \), and a vortex lattice at large \( \nu \). Applying a simple Lindemann criterion \( 5\), one finds that a triangular vortex lattice is unstable to quantum fluctuations for \( \nu \lesssim 14 \). The crossover to GP behaviour for increasing \( N \) at fixed \( L/N \) is the remnant of this phase transition in a system with a finite number of vortices \( N_V = 2L/N \).

To investigate in detail the dependence of the groundstate on \( \nu \), we have conducted extensive (Lanczos) diagonalisations in a toroidal geometry \( 6\). This periodic geometry represents the bulk of a system containing a large number of vortices. We consider a torus of sides \( a \) and \( b \). There are then \( N_V = (2\pi ab)/\hbar = ab(q^*B^*/\hbar) \) vortices, which is the number of single-particle states on the torus in the lowest Landau level \( 4\), and hence an integer. Thus, both \( N \) and \( N_V \) are integers, and \( \nu \equiv N/N_V \) is a rational fraction. Finally, we classify all states by the Haldane momentum \( 1\), which runs over a Brillouin zone containing \( N^2 \) points, where \( N \) is the greatest common divisor of \( N \) and \( N_V \). In the following we shall refer to the \( x \) and \( y \) momenta by the dimensionless vector, \( (K_x, K_y) \), using units of \( (2\pi\hbar/a) \) and \( (2\pi\hbar/b) \). We report only positive values of \( K_x, K_y \) up to the Brillouin zone boundary [states at \( (\pm K_x, \pm K_y) \) are degenerate by symmetry]. We also choose to measure energies in units of \( g/(\sqrt{4\pi\ell^3}) \), where \( \ell \equiv \hbar/(q^*B^*) = \sqrt{\hbar/(2m\omega_0)} \) is the magnetic length at \( \omega = \omega_0 \).

We start by applying Gross-Pitaevskii theory \( 3\) on the torus. In general, the GP groundstate is a vortex lattice, with broken translational symmetry: the wavefunction is not an eigenstate of the Haldane momentum, but has weight at a set of reciprocal lattice vectors (RLVs). While the symmetry of the lattice depends, in general, on \( N_V \) and the aspect ratio \( a/b \), the absolute minimum of energy is always obtained for a triangular vortex lattice \( 16\).

In ED studies, the groundstate is necessarily an eigenstate of the Haldane momentum. The signature of translational symmetry breaking is the development of quasi-degenerate levels at the set of momenta given by the RLVs of the broken symmetry lattice \( 1\). To search for such degeneracies, we show in Fig. 1 the evolution with \( \nu \) of the excitation energies for \( N_V = 8 \) vortices at an aspect ratio \( a/b = \sqrt{3}/4 \) for which the GP groundstate is a triangular lattice.

A collapse of the excitation energies at the RLVs of a triangular lattice is observed at \( \nu \sim 6 \). Similar plots for \( N_V = 4, 6 \) indicate that the excitation energies at RLVs fall exponentially with \( \nu \) for \( \nu \gtrsim 6 \) (shown in Fig. 1 for one RLV for \( N_V = 6 \)). For \( \nu = 15 \) at \( N_V = 6 \) the excitation energies are 6 orders of magnitude smaller at the RLVs than at any other momentum. This strong quasi-degeneracy at the reciprocal lattice vectors of the lattice formed in GP theory indicates a strong tendency to broken translational symmetry \( 17\). GP theory accurately describes the states at large values of \( \nu \).

We view the collapse of the excitation gaps at \( \nu \sim 6 \) as an indication, in this finite-size system, of a true phase-transition from translationally-invariant “vortex-liquid” phases, to a (triangular) vortex lattice. The phase transition is rounded due to the finite number of vortices, and becomes sharper for larger \( N_V \) (over the range of \( N_V \) we can study). Note that we have chosen aspect ratios that are commensurate with a triangular lattice, which is likely to help stabilise the vortex lattice. Similar plots at other aspect ratios show transitions to a vortex lattice at larger values of \( \nu \) (up to \( \nu_c \sim 15 \) for \( N_V = 4 \)). One should therefore view \( \nu_c \sim 6 \) as a lower bound on the critical value of \( \nu \) at which the transition occurs. Our

![FIG. 1. Solid lines: Excitation energies at momenta measured relative to the groundstate, for \( N_V = 8 \), \( a/b = \sqrt{3}/4 \) (inset shows the GP groundstate: dark = low boson density). The excitation energies at the RLVs of the triangular lattice (filled symbols) collapse at \( \nu \sim 6 \), signalling the onset of a groundstate quasi-degeneracy; all other momenta retain non-zero excitation energies (two such momenta are shown as open symbols). Dashed Line: The excitation energy at one RLV \((2,0)\) for \( N_V = 6 \) and \( a/b = 1/\sqrt{3} \), showing that the collapse at \( \nu \sim 6 \) initiates an exponential decrease with \( \nu \).](image-url)
numerical results are consistent with a transition in the vicinity of \( \nu \sim 10 \pm 5 \).

We now turn to discuss the vortex liquids at \( \nu \lesssim 6 \). In this regime, we find incompressible liquid states similar to those in fractional quantum Hall systems. Some of these incompressible states can be accounted for by the use of a composite fermion construction that has previously been shown to describe accurately ED results on small systems in the disk geometry \([6]\). In the present uniform geometry, this theory predicts a sequence of incompressible states at \( \nu = \frac{\nu_{c}/+1}{\nu_{c}/+1} = \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{1}{2}, \infty \), which is a bosonic version of the Jain sequence of fractional quantum Hall states \([18]\). Many of the strongest incompressible states we find cannot be accounted for in this way. In particular, the largest (finite) value in the composite fermion sequence is \( \nu = 2 \), while the transition to a vortex lattice does not occur until \( \nu = 6 \). To investigate the liquid states in this regime, we plot in Fig. 2 the energy gaps as a function of \( \nu \) for \( \nu \nu = 6 \) vortices. The energy gap is related to the discontinuity in the chemical potential. To minimise finite size effects, we define the gap \( \Delta \) by

\[
\Delta(N) = N \left[ \frac{E(N+1)}{N+1} + \frac{E(N-1)}{N-1} - \frac{2E(N)}{N} \right],
\]

which reduces to the standard definition as \( N \to \infty \).

![FIG. 2. Energy gap (1) as a function of \( \nu \) for \( \nu \nu = 6 \) vortices, at \( a/b = 1/\sqrt{3} \). Upward spikes signal values of \( \nu \) for which the groundstate is incompressible. The collapse of the gaps at \( \nu \sim 6 \) indicates the transition to the vortex lattice phase. (Inset shows the density of the GP groundstate.)](image)

As well as the Laughlin state at \( \nu = \frac{1}{3} \), \( \frac{1}{2} \), incompressible states appear clearly in Fig. 2 at \( \nu = 1, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, 5, 6 \) (the loss of gaps for \( \nu \gtrsim 6 \) is another indication of the transition to the vortex lattice, perhaps re-entrant around \( \nu = 6 \)). It is not immediately apparent how to construct incompressible states for this sequence of \( \nu \). One possibility is that the vortices themselves are forming Laughlin states. This would provide a set of states with vortex filling fraction \( \nu_{V} = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots \), and hence \( \nu = 1/\nu_{V} = 2, 4, 6 \ldots \). However, this construction does not account for states at \( \nu = 1, \frac{3}{2}, \frac{5}{2}, 3, \frac{7}{2}, \frac{9}{2}, 5 \) \([21]\). Moreover, the trial wavefunctions of this form that we have tested (on a disc) have high interaction energies: while they keep the vortices apart, they do not introduce favourable correlations between the bosons. States of this type are likely to describe systems in which the underlying interactions can be described as repulsive two-body forces between vortices \([19,21,13]\). They do not provide an accurate description in the present situation, where the interactions cannot be represented by pairwise vortex interactions \([22]\).

A clue to the nature of the incompressible vortex liquid states lies in the existence of an incompressible state at \( \nu = 1 \), which also cannot be accounted for in terms of non-interacting composite fermions. Rather, this state is well-described \([23]\) by the Moore-Read (“Pfaffian”) wavefunction \([24]\). We find that the exact groundstate has large overlap with the Moore-Read state at the Haldane momenta for which it can be constructed on a torus \([25]\).

Motivated by this success, we have compared the incompressible states at higher integer and half-integer \( \nu \) with “parafermion” wavefunctions introduced by Read and Rezayi \([10]\) as generalisations of the Moore-Read state. These states may be represented \([26]\) as a (symmetrised) product of \( k \) Laughlin states via

\[
\Psi^{(k)}(\{z_i\}) = S \left[ \prod_{i<j \in A} (z_i - z_j)^2 \prod_{l<m \in B} (z_l - z_m)^2 \ldots \right]
\]

where \( z = x + iy \), and we omit the exponential factor of lowest Landau level states as usual. The symbol \( S \) indicates symmetrisation over all partitions of \( N \) particles into sets \( A, B, \ldots \) of \( N/k \) particles (we assume that \( N \) is divisible by \( k \)). The cases \( k = 1 \) and \( k = 2 \) correspond to the Laughlin and Moore-Read wavefunctions. For general \( k \), the wavefunction \([10]\) describes a system with filling fraction \( \nu = N^2/2L = k/2 \), and is a zero energy eigenstate of a \((k+1)\)-body version of the repulsion \([2]\).

The Read-Rezayi states provide a consistent interpretation of the incompressible states in Fig. 2; they identify the sequence of incompressible states observed in the EDs \( \nu = \frac{k}{2} \) with integer \( k \); they have large overlaps with the exact wavefunctions, at least up to \( \nu = 3 \) (the largest \( \nu \) for which we have made the comparison). We construct the Read-Rezayi states on the torus by diagonalising the \((k+1)\)-body force-law directly to find the zero energy eigenstates. In general, we find more than one zero energy eigenstate, and recover a total ground-state degeneracy on a torus of \( k+1 \), consistent with Ref.
The overlap of the exact groundstates of the two-body force (3) with the Read-Rezayi states are given in Table 1 for \( \nu = \frac{5}{2} \). For comparison, the overlaps with the GP groundstate are also shown.

In conclusion, we have shown that the groundstates of weakly interacting bosons in a rotating trap exhibit both vortex lattices and incompressible vortex liquids. A clear distinction between these phases appears for a large number of bosons, \( N \), and vortices, \( N_V \), and is controlled by the filling fraction \( \nu \equiv N/N_V \). Vortex liquid phases appear for \( \nu \lesssim \nu_c \) and vortex lattice appear for \( \nu \gtrsim \nu_c \). A Lindemann criterion suggests \( \nu_c \sim 14 \), while exact diagonalisations indicate \( \nu_c \gtrsim 6 \). Current experiments [27] with \( N \sim 10^5 \) and \( N_V \sim 10 \) are deep in the regime in which the groundstate is a vortex lattice. Experiments that access the quantum-melted vortex liquid phases will require specific attention to small system sizes and high angular momentum. Our results indicate that novel correlated states emerge in this regime, which are well-described by the Read-Rezayi parafermion states whose excitations obey non-abelian statistics [10].

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[11] For small \( N_V \) there is a departure from this relation, such that a single vortex occurs when \( L/N = 1 \).

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| \( k \) | \( \nu \) | \( (K_x, K_y) \times \text{degeneracy} \) | \(|\Psi^{(k)}\rangle|\langle\Psi^{(k)}|\) | \(|\Psi^{(GP)}\rangle|\langle\Psi^{(GP)}|\) |
|-------|-----|----------------|----------------|----------------|
| 1/2   | (Laughlin) | (0,0) \( \times \) 2 | 1.000 | 0.555 |
| 1 (Moore-Read) | (3,3) \( \times \) 1 | 0.982 | N/W |
| 1 (Moore-Read) | (3,0) \( \times \) 1 | 0.982 | 0.408 |
| 1 (Moore-Read) | (0,3) \( \times \) 1 | 0.981 | 0.493 |
| 3/2   | (0,0) \( \times \) 4 | 0.967 | 0.234 |
| 4     | (0,0) \( \times \) 2 | 0.956 | 0.242 |
| 4     | (0,3) \( \times \) 1 | 0.966 | N/W |
| 4     | (0,3) \( \times \) 1 | 0.935 | N/W |
| 4     | (3,3) \( \times \) 1 | 0.844 | 0.547 |
| 5/2   | (0,0) \( \times \) 6 | 0.955 | 0.163 |
| 6     | (3,3) \( \times \) 2 | 0.960 | N/W |
| 6     | (3,0) \( \times \) 2 | 0.944 | 0.198 |
| 6     | (0,3) \( \times \) 1 | 0.744 | 0.534 |
| 6     | (0,3) \( \times \) 1 | 0.852 | N/W |

TABLE I. Wavefunction comparisons of the exact groundstates of the 2-body force law at \( \nu = k/2 \), for \( N_V = 6 \) and \( a/b = 1/\sqrt{3} \). In each case we report: the Haldane momenta at which Read-Rezayi states exist, with degeneracies; the overlap of the exact groundstate with the Read-Rezayi state (where there is more than one such state, we report the total overlap within this set); the overlap of the exact groundstate with the GP groundstate (we first project the GP state onto each component of momentum; N/W indicates that the GP groundstate has no weight at this momentum).