DYNAMICS NEAR COUETTE FLOW FOR THE $\beta$-PLANE EQUATION

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Abstract. In this paper, we study stationary structures near the planar Couette flow in Sobolev spaces on a channel $T \times [-1, 1]$, and asymptotic behavior of Couette flow in Gevrey spaces on $T \times \mathbb{R}$ for the $\beta$-plane equation. Let $T > 0$ be the horizontal period of the channel and $\alpha = \frac{2\pi}{T}$ be the wave number. We obtain a sharp region $O$ in the whole $(\alpha, \beta)$ half-plane such that non-parallel steadily traveling waves do not exist for $(\alpha, \beta) \in O$ and such traveling waves exist for $(\alpha, \beta)$ in the remaining regions, near Couette flow for $H^{\frac{5}{2}}$ velocity perturbation. The borderlines between the region $O$ and its remaining are determined by two curves of the principal eigenvalues of singular Rayleigh-Kuo operators. Our results reveal that there exists $\beta_* > 0$ such that if $|\beta| \leq \beta_*$, then non-parallel traveling waves do not exist for any $T > 0$, while if $|\beta| > \beta_*$, then there exists a critical period $T_\beta > 0$ so that such traveling waves exist for $T \in [T_\beta, \infty)$ and do not exist for $T \in (0, T_\beta)$, near Couette flow for $H^{\frac{5}{2}}$ velocity perturbation. This contrasting dynamics plays an important role in studying the long time dynamics near Couette flow with Coriolis effects. Moreover, for any $\beta \neq 0$ and $T > 0$, there exist no non-parallel traveling waves with speeds converging in $(-1, 1)$ near Couette flow for $H^{\frac{5}{2}}$ velocity perturbation, in contrast to this, we construct non-shear stationary solutions near Couette flow for $H^{\frac{5}{2}}$ velocity perturbation, which is a generalization of Theorem 1 in [22] but the construction is more difficult due to the $\beta$'s term. Finally, we prove nonlinear inviscid damping for Couette flow in some Gevrey spaces by extending the method of [4] to the $\beta$-plane equation on $T \times \mathbb{R}$.

1. Introduction

Dynamics of oceans and planetary atmospheres is one of the central topics in geophysical fluid dynamics. In the study of such large-scale motion in a rotating frame, it is reasonable to include Coriolis force to be geophysically relevant. A common model for large-scale motion is described by the $\beta$-plane equation

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla P - \beta y J \vec{v}, \quad \nabla \cdot \vec{v} = 0, \tag{1.1}$$

where $\vec{v} = (u, v)$ is the fluid velocity, $P$ is the pressure,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the rotation matrix, and $\beta$ is the Coriolis parameter. Then the vorticity $\omega = \text{curl} \vec{v} = \partial_x v - \partial_y u$ solves

$$\partial_t \omega + (\vec{v} \cdot \nabla) \omega + \beta v = 0. \tag{1.2}$$

We will work on the domain $D_T = T \times [-1, 1]$ with non-permeable boundary condition

$$v = 0 \quad \text{on} \quad y = \pm 1, \tag{1.3}$$

where $T = \mathbb{R}/(T\mathbb{Z})$.

A shear flow is a steady solution of (1.2). The Couette flow $(y, 0)$ is one of the simplest laminar flows. We are interested in the long time dynamics near Couette flow for the $\beta$-plane
In general, for \( \beta = 0 \), it is known that nonlinear inviscid damping is true if the perturbation is taken in a suitable Gevrey space \([4, 13]\). In the Sobolev space \( H^\frac{5}{2} \) for velocity perturbation, richer dynamics around Couette flow was found by constructing non-shear steady states (and traveling waves) near Couette, and on the other hand, relatively simpler dynamics for \( H^\frac{5}{2} \) velocity perturbation was obtained by proving non-existence of non-parallel steadily traveling waves \([22, 6]\).

In this paper, we study whether similar results are true in Gevrey and Sobolev spaces respectively for \( \beta \neq 0 \). Our main results roughly state that if the perturbation is taken in a suitable Gevrey space or in the (velocity) Sobolev space \( H^\frac{5}{2} \), similar results are true for \( \beta \neq 0 \), while if the velocity perturbation is considered in the Sobolev space \( H^5 \), it turns out that the situation is very different from the case \( \beta = 0 \). In fact, there exists \( \beta_* > 0 \) such that similar results are still true for \( 0 < |\beta| \leq \beta_* \). The difference is for the case \( |\beta| > \beta_* \), namely, there exists a critical period \( T_\beta > 0 \) such that traveling waves always exist for \( T \in [T_\beta, \infty) \) near Couette flow no matter how much regularity is required and traveling waves do not exist for \( T \in (0, T_\beta) \), where \( T_\beta = \frac{2\pi}{\alpha_\beta} \) is given in Theorem \([13]\).

In the \( \beta \)-plane model, the \( \beta \)'s term brings some fundamental changes to the internal structure of \((1.2)\) near a shear flow on \( \mathbb{T} \times [-1, 1] \). This induces new long time dynamical behavior around a shear flow, which is useful in understanding the various large-scale physical phenomena in atmospheres and oceans. Let us now explain how the \( \beta \)'s term in \((1.2)\) influences the spectrum of the linearized operator and dynamics around a shear flow. By the incompressible condition, we can introduce the stream function \( \psi \) such that \( \vec{v} = (\partial_y \psi, -\partial_x \psi) \). The linearized equation of \((1.2)\) around a shear flow \((u(y), 0)\) is

\[
\partial_t \Delta \psi + u \partial_x \Delta \psi + (\beta - u'') \partial_x \psi = 0.
\]

Taking Fourier transform in \( x \), we have \((\partial^2_y - \alpha^2)\partial_y \hat{\psi} = i\alpha((u'' - \beta) - u(\partial_y^2 - \alpha^2))\hat{\psi} \), where \( \alpha = \frac{2\pi}{T} \) is the wave number. For \( \beta \in \mathbb{R} \), the linearized operator is given by

\[
R_{\alpha, \beta} \hat{\psi} := -(\partial_y^2 - \alpha^2)^{-1}((u'' - \beta) - u(\partial_y^2 - \alpha^2))\hat{\psi}.
\]

The essential spectrum \( \sigma_e(R_{\alpha, \beta}) = \text{Ran}(u) \), and \( c \in \sigma_d(R_{\alpha, \beta}) \) (the discrete spectrum) if and only if its corresponding eigenfunction \( \psi_c \) satisfies the Rayleigh-Kuo boundary value problem (BVP):

\[
\mathcal{L}_{\alpha, \beta} \phi := -\phi'' + \frac{u'' - \beta}{u - c} \phi = \lambda \phi, \quad \phi(\pm 1) = 0,
\]

where \( \phi \in H^1_0 \cap H^2(-1, 1) \) and \( \lambda = -\alpha^2 \). An important difference is that

\[
\sigma_d(R_{\alpha, \beta}) \cap \mathbb{R} = \emptyset \quad \text{if} \quad \beta = 0,
\]

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\]

in general \([21]\). For example, if \( |\beta| \) is sufficiently large, then \( \sigma_d(R_{\alpha, \beta}) \cap \mathbb{R} \neq \emptyset \) for some wave numbers. Reflected on the dynamical behavior near the shear flow, this difference of the linearized operators’ spectrum brings new non-parallel steady traveling wave families, with traveling speeds converging outside the range of the flow, near the shear flow for \( \beta \neq 0 \), while no such traveling wave families exist for \( \beta = 0 \) \([20]\). This implies that the long time dynamics near a shear flow is richer for \( \beta \neq 0 \). In fact, the long time dynamics near a shear flow might be much complicated due to the \( \beta \)'s term. Taking Sinus flow for example, there are infinitely many such traveling wave families for \( \beta < -\frac{1}{2}\pi^2 \) or \( \beta > \frac{9}{10}\pi^2 \) and any horizontal period \([20]\).

Lyapunov stability is a classical issue in the context of hydrodynamics for general stationary flows. For shear flows, Rayleigh \([31]\) proved that a necessary condition for linear instability
is that $u$ has an inflection point for $\beta = 0$. Kuo [18] extended the necessary condition to $\beta \neq 0$ that $\beta - u''$ must change sign. Howard [12] proved that the unstable eigenvalues must lie in a semicircle region, which is called the Howard semicircle theorem for $\beta = 0$. Pedlosky [29] extended the radius of the semicircle by $|\beta|$ for $\beta \neq 0$. By introducing the energy-Casimir functional, Arnol’d proved nonlinear Lyapunov stability for a class of stationary flow by showing that it is a minimizer or a maximizer of the functional [1, 2]. The so-called energy-Casimir method has been developed in [3, 30] and also extended to many other physical models, such as the quasi-geostrophic equations for planetary-scale rotating flows [5, 32]. The index formula developed by Lin and Zeng [24] provides a useful tool to study the sufficient conditions for linear instability of stationary flows, as well as to count the number of unstable modes, if the linearized equation has a Hamiltonian structure.

Our concern is the long time dynamics around Couette flow, and this problem has become a topic of interest for $\beta = 0$ and attracted the attention of many mathematicians. In the earlier works, Kelvin [17] gave the construction of exact solutions to the linearized problem near Couette flow. Orr [28] observed the decay of velocity for the linearized equation around Couette flow. Lin and Zeng [22] confirmed the linear inviscid damping around Couette flow for $L^2$ vorticity perturbation. For the nonlinear equation, as mentioned above, they found Cat’s eyes flow near Couette for $H^{<3/2}$ vorticity perturbation, and proved that non-parallel steadily traveling waves do not exist for $H^{>3/2}$ perturbation. Similar results were obtained for the Vlasov-Poisson system [23], and instability in high Sobolev spaces was established in [3]. Castro and Lear [6] showed the existence of nontrivial and smooth traveling waves close to Couette flow for $H^{<3/2}$ vorticity perturbation with speed of order 1. As for the perturbation in Gevrey spaces, Bedrossian and Masmoudi [4] proved nonlinear inviscid damping around the Couette flow in Gevrey class $G^{3/2,1}$ on $\mathbb{T} \times \mathbb{R}$. Deng and Masmoudi [7] showed that this is the critical regularity by proving the instability in Gevrey class $G^{3/2,1}$. Ionescu and Jia [13] proved nonlinear inviscid damping in a channel $\mathbb{T} \times [-1,1]$ under the compacted support’s assumption on the initial vorticity perturbation. We also point out some important progress on linear inviscid damping for general monotone and non-monotone shear flows in [39, 40, 33, 34, 35, 15, 16, 10], on nonlinear inviscid damping for monotone shear flows in [14, 25], and on nontrivial invariant structures near Kolmogorov flow and Poiseuille flow in various domains in [38]. It is still challenging to prove nonlinear damping for non-monotone flows.

We now turn back to the case $\beta \neq 0$. For the linearized equation, Lin, Yang and the third author gave a method to study the linear Lyapunov instability for a class of shear flows based on Hamiltonian systems and spectral analysis of ODEs, see Subsection 3.3 in [21]. Then Wei and the last two authors gave the explicit decay rate of the velocity for a class of monotone shear flows based on the space-time estimate and the vector field method, as well as proved the linear damping for a class of general shear flows under some spectral conditions, see Theorems 1.1 and 1.2 in [36]. For the nonlinear equation, Lin, Wei and the last two authors found some richer dynamics near a class of shear flows based on asymptotic behavior of spectrum of Rayleigh-Kuo BVP and bifurcation theory of nonlinear maps. More precisely, we proved that if the flow $u$ has a critical point at which $u$ attains its minimal value, then there exists a unique $\beta_+$ in the positive half-line such that the number of traveling wave families near the shear flow changes suddenly from finite one to infinity when $\beta$ passes through it. On the other hand, if $u$ has no such critical points, then the number is always at most finite for positive $\beta$ values. A similar result holds true in the negative half-line, see Theorems 1.2 and 2.1 in [20]. Here, the traveling speeds lie outside the range of the flow. Elgindi,
Pusateri and Widmayer took the advantage of dispersive operator induced by the Coriolis effect and proved the stability of the zero solution for $\beta \neq 0$ in [8, 30].

In this paper, we are interested in the long time nonlinear dynamics around Couette flow for the case $\beta \neq 0$. At first, we consider the perturbation in the Sobolev spaces. By Corollaries 2.2 and 2.4 in [20], it is known that there are at most finitely many traveling wave families near Couette flow for $H^{\geq 3}$ velocity perturbation, where the traveling speeds converge outside the range of the flow. It is necessary to clarify whether “at most finitely” implies existence or not. Furthermore, if we try to understand the nontrivial stationary structures in any reference frame near Couette flow deeply, then two questions naturally arise:

**Q1.** Are there traveling waves with the traveling speeds $c$ lying inside the range of Couette flow $[-1, 1]$ near the flow for $H^{\geq 3}$ velocity perturbation? This is more delicate than the case that traveling speeds lie outside $[-1, 1]$, since we have to deal with the singularity in the terms involved with the factor like $\frac{y}{|y|}$. 

**Q2.** By the method of Theorem 2 in [22], a conclusion can be essentially obtained as follows: for any $\delta > 0$, there exists $\beta_\delta > 0$ small enough such that if $|\beta| < \beta_\delta$ and the horizontal period is arbitrary, then there exist no traveling waves near Couette flow for $H^{\geq 3}$ velocity perturbation. By Corollary 2.4 in [20], if $|\beta| > 1$, there exist traveling waves near Couette flow for $H^{\geq 3}$ velocity perturbation and for some period in $x$. Consider $H^{\geq s_0}$ velocity perturbation near Couette flow for some $s_0 \geq 3$. For any fixed $\beta \neq 0$, can we determine for which periods there exist traveling waves near Couette flow, and for other periods there exist no traveling waves near the flow? For any fixed horizontal period $T > 0$, can we determine for which $\beta$ values there exist traveling waves near Couette flow, and for other $\beta$ values there exist no traveling waves near the flow? Here, the traveling waves always mean the non-parallel steadily ones.

To answer Q1, we have the following result.

**Theorem 1.1.** Let $\beta \neq 0$. For any $T > 0$, $s \geq 5$ and $0 < \delta < 1$, there exists $\epsilon_\delta > 0$ such that any traveling wave solution $(u(x - ct, y), v(x - ct, y))$ to the $\beta$-plane equation (1.1)--(1.3) with $c \in [-1 + \delta, 1 - \delta]$, $x$-period $T$ and satisfying that

$$
\|(u, v) - (y, 0)\|_{H^s(D_T)} < \epsilon_\delta,
$$

must have $v(x, y) \equiv 0$, that is, $(u, v)$ is necessarily a shear flow.

**Remark 1.2.** $[-1 + \delta, 1 - \delta]$ in Theorem 1.1 can not be extended to the whole range $[-1, 1]$, since there exist traveling waves with traveling speeds converging to $\pm 1$, see Theorem 1.3 (2i) for $|\beta| > \beta_*$ and $T = T_\beta$.

Let

$$
\lambda_1(\beta, -1) = \inf_{\phi \in H^1([-1, 1]), \|\phi\|_{L^2([-1, 1])} = 1} \int_{-1}^{1} \left( |\phi'|^2 - \frac{\beta}{y + 1} |\phi|^2 \right) dy
$$

be the principal eigenvalue of the singular Rayleigh-Kuo BVP (1.5) with $u(y) = y$, $c = -1$. The properties of $\lambda_1(\beta, -1)$ is given in Section 3. Next, we give a positive answer to Q2.

**Theorem 1.3.** Let $s \geq 5$ and $\beta_* > 0$ be the unique point such that $\lambda_1(\beta, -1) = 0$, where $\lambda_1(\beta, -1)$ is defined in (1.6).

1. Let $0 < |\beta| \leq \beta_*$. For any $T > 0$, there exists $\epsilon_0 > 0$ such that any traveling wave solution $(u(x - ct, y), v(x - ct, y))$ to the $\beta$-plane equation (1.1)--(1.3) with $c \in \mathbb{R}$, $x$-period $T$ and satisfying that

$$
\|(u, v) - (y, 0)\|_{H^s(D_T)} < \epsilon_0,
$$

...
must have \( v(x,y) \equiv 0 \), that is, \((u,v)\) is necessarily a shear flow.

(2) Let \(|\beta| > \beta_s\), \(\alpha_\beta = \sqrt{-\lambda_1(|\beta|, -1)} > 0\) and \(T_\beta = \frac{2\pi}{\alpha_\beta}\).

(2i) Fix \(T \in [T_\beta, \infty)\). Then for any \(\varepsilon > 0\), there exists a traveling wave solution \((u_\varepsilon(x - c_\varepsilon t, y), v_\varepsilon(x - c_\varepsilon t, y))\) to the \(\beta\)-plane equation \((1.1)-(1.3)\) with \(x\)-period \(T\) and satisfying that

\[
\|(u_\varepsilon, v_\varepsilon) - (y,0)\|_{H^s(D_T)} < \varepsilon,
\]

but \(v_\varepsilon(x,y) \neq 0\). As \(\varepsilon \to 0\), we have \(c_\varepsilon \to -1\) for \(\beta > \beta_s\) and \(T = T_\beta\); \(c_\varepsilon \to c_0 \in (-\infty, -1)\) for \(\beta > \beta_s\) and \(T > T_\beta\); \(c_\varepsilon \to 1\) for \(\beta < -\beta_s\) and \(T = T_\beta\); \(c_\varepsilon \to c_0 \in (1, \infty)\) for \(\beta < -\beta_s\) and \(T > T_\beta\).

(2ii) Fix \(T \in (0, T_\beta)\). Then similar conclusion in (1) holds true.

Moreover, \(\alpha_\beta\) is continuous and increasing on \(\beta \in [\beta_s, \infty)\), and \(\alpha_\beta \to \infty\) as \(\beta \to \infty\).

Theorem [1.3] is illustrated in Figure 1. The whole half-plane is divided into three regions

\[
I_- = \{(\alpha, \beta) | \beta < -\beta_s, 0 < \alpha \leq \alpha_\beta\}, \quad I_+ = \{(\alpha, \beta) | \beta > \beta_s, 0 < \alpha \leq \alpha_\beta\},
\]
\[
O = \{(\alpha, \beta) | \beta < -\beta_s, \alpha > \alpha_\beta\} \cup \{(\alpha, \beta) | \beta > \beta_s, \alpha \geq 0\} \cup \{(\alpha, \beta) | \beta > \beta_s, \alpha > \alpha_\beta\}.
\]

Theorem [1.3] reveals contrasting dynamics near Couette flow between \((\alpha, \beta) \in O\) and \((\alpha, \beta) \in I_- \cup I_+\): non-parallel steadily traveling waves do not exist for \((\alpha, \beta) \in O\) and such traveling waves exist for \((\alpha, \beta) \in I_- \cup I_+\), near Couette flow for \(H^{2,5}\) velocity perturbation. Here, the conclusion for \(\beta = 0\) is proved in Theorem 2 of [22]. The borderlines between the region \(O\) and its remaining regions are the symmetry curves

\[
\Gamma_- = \{(\alpha, \beta) | \beta < -\beta_s, \alpha = \alpha_\beta\}, \quad \Gamma_+ = \{(\alpha, \beta) | \beta > \beta_s, \alpha = \alpha_\beta\},
\]

where \(-\alpha_\beta^2 = \lambda_1(\beta, 1)\) for \(\beta < -\beta_s\) and \(-\alpha_\beta^2 = \lambda_1(\beta, -1)\) for \(\beta > \beta_s\) are exactly the two curves of the principal eigenvalues of singular Rayleigh-Kuo BVP in (1.5) with \(u(y) = y\), \(c = \pm 1\). The symmetry of \(\Gamma_\pm\) and \(I_\pm\) with respect to the vertical axis is due to symmetry of the principal eigenvalues of Rayleigh-Kuo operators for Couette flow, see Lemma [3.11].

From the perspective of fixed horizontal period, we get the following restatement of Theorem [1.3].

Restatement of Theorem [1.3]. Let \(s \geq 5\), \(T > 0\) and \(\beta_T > 0\) be the unique point such that \(-\alpha_\beta^2 = \lambda_1(\beta_T, -1)\).
Remark 1.4. From the perspective of fixed horizontal period $T > 0$, then the traveling speeds of constructed traveling waves converge to an isolated real eigenvalue of $\mathcal{L}$ with $c \in \mathbb{R}$, $x$-period $T$ and satisfying that

$$\|(u,v) - (y,0)\|_{H^s(D_T)} < \varepsilon_0,$$

must have $v(x,y) \equiv 0$, that is, $(u,v)$ is necessarily a shear flow.

(2) Fix $\beta \in (-\infty,-\beta_T] \cup [\beta_T,\infty)$. Then for any $\varepsilon > 0$, there exists a traveling wave solution $(u_\varepsilon(x - c_\varepsilon t,y),v_\varepsilon(x - c_\varepsilon t,y))$ to the $\beta$-plane equation (1.1)-(1.3) with $x$-period $T$ and satisfying that

$$\|(u_\varepsilon,v_\varepsilon) - (y,0)\|_{H^s(D_T)} < \varepsilon,$$

but $v_\varepsilon(x,y) \not\equiv 0$. As $\varepsilon \to 0$, we have $c_\varepsilon \to -1$ for $\beta = \beta_T$; $c_\varepsilon \to 0 \in (-\infty,-1)$ for $\beta > \beta_T$; $c_\varepsilon \to 1$ for $\beta = -\beta_T$; $c_\varepsilon \to 0 \in (1,\infty)$ for $\beta < -\beta_T$.

Moreover, $\beta_T$ is continuous and decreasing on $T \in (0,\infty)$, $\beta_T \to \infty$ as $T \to 0$, and $\beta_T \to \beta_*$ as $T \to \infty$.

Theorem 1.3 and its restatement are briefly refined as follows. Consider $H^{\geq 5}$ velocity perturbation near Couette flow. From the perspective of fixed $\beta \in \mathbb{R}$, we have the conclusions as follows.

- Fix $\beta \in [-\beta_*,\beta_*]$. Then for any horizontal period $T > 0$, non-parallel traveling waves do not exist near Couette flow.
- Fix $\beta \in (-\infty,-\beta_*) \cup (\beta_*,\infty)$. Then
  - (1) non-parallel traveling waves exist for the horizontal period $T \in [T_\beta,\infty)$,
  - (2) non-parallel traveling waves do not exist for the horizontal period $T \in (0,T_\beta)$.

From the perspective of fixed horizontal period $T > 0$, we have

- non-parallel traveling waves do not exist for $\beta \in (-\beta_T,\beta_T),$
- non-parallel traveling waves exist for $\beta \in (-\infty,-\beta_T] \cup [\beta_T,\infty)$.

Remark 1.4. (1) By Theorem 1.3 (2i), if $(\alpha,\beta) \in (I_+ \cup I_-) \setminus \Gamma_+ \cup \Gamma_-$ (i.e. $T = \frac{2\pi}{\alpha} \in (T_\beta,\infty)$), then the traveling speeds of constructed traveling waves converge to an isolated real eigenvalue of $\mathcal{R}_{\alpha,\beta}$ if while if $(\alpha,\beta) \in \Gamma_+ \cup \Gamma_-$ (i.e. $T = T_\beta$), then the traveling speeds of constructed traveling waves converge to the embedding eigenvalues $\pm 1$ of $\mathcal{R}_{\alpha,\beta}$.

(2) For $(\alpha,\beta) \in I_+ \cup I_-$, the long time dynamics near Couette flow is richer due to the existence of non-parallel traveling waves, as the evolutionary velocity might tend asymptotically to some nontrivial (relative) equilibrium if the initial data is taken close to Couette flow for $H^{\geq 5}$ velocity perturbation. It is very challenging to give a complete description of asymptotic behavior for the solutions if the initial data is taken near Couette flow. On the other hand, for $(\alpha,\beta) \in O$ (including the case of no Coriolis effects), the dynamics near Couette flow is relatively simpler on account of the absence of non-parallel traveling waves.

(3) From the perspective of spectrum of $\mathcal{R}_{\alpha,\beta}$, we give the differences among $(\alpha,\beta) \in O, \Gamma_+, \Gamma_-, I_+ \setminus \Gamma_+ \text{ and } I_- \setminus \Gamma_-$, where $\mathcal{R}_{\alpha,\beta}$ always denotes the linearized operator in (1.4) with $u(y) = y$.

- For $(\alpha,\beta) \in O$, $\mathcal{R}_{\alpha,\beta}$ has no embedding eigenvalues or isolated real eigenvalues.
- For $(\alpha,\beta) \in \Gamma_+$, $\mathcal{R}_{\alpha,\beta}$ has a unique embedding eigenvalue $-1$ and no isolated real eigenvalues.
- For $(\alpha,\beta) \in \Gamma_-$, $\mathcal{R}_{\alpha,\beta}$ has a unique embedding eigenvalue $1$ and no isolated real eigenvalues.
- For $(\alpha,\beta) \in I_+ \setminus \Gamma_+$, $\mathcal{R}_{\alpha,\beta}$ has a unique isolated real eigenvalue $c_0 \in (-\infty,-1)$ and no embedding eigenvalues.
• For \((\alpha, \beta) \in I_- \setminus \Gamma_-, \mathcal{R}_{\alpha,\beta}\) has a unique isolated real eigenvalue \(c_0 \in (1, \infty)\) and no embedding eigenvalues.

The definition of the embedding eigenvalue of \(\mathcal{R}_{\alpha,\beta}\) is given in Definition 3.10 of [36]. By Remark 1.3 (3) in [36], the potential embedding eigenvalues of \(\mathcal{R}_{\alpha,\beta}\) can only be \(\pm 1\) for Couette flow and any \((\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}\). Note that in the case \(\beta = 0\), there exist no embedding eigenvalues or isolated real eigenvalues for the linearized Euler operator around Couette flow.

(4) For Sinus flow, which is a non-monotone shear flow, even though we only count the families of traveling waves with traveling speeds converging outside the range of the flow, the number is infinite for \(\beta < -\frac{1}{16} \pi^2\) or \(\beta > \frac{9}{10} \pi^2\), and any horizontal period, see Figure 1 in [20]. For Couette flow, which is a monotone shear flow, if we count the families of traveling waves with traveling speeds converging no matter inside or outside of the range of the flow, the number is zero for \((\alpha, \beta) \in O\) and finite for \((\alpha, \beta) \in I_{\pm}\). Thus, the long time dynamics near a non-monotone flow seems more complicated than a monotone flow for the \(\beta\)-plane equation.

(5) \(s \geq 5\) might be improved, as \(s \geq 3\) is sufficient for (2i), see its proof. However, as is shown in the next theorem, the optimal \(s\) value can not be less than \(\frac{5}{2}\).

The proof of Theorem 1.1 is to rule out the traveling waves with traveling speeds converging to \(c \in (-1, 1)\), while the main task in the proof of Theorem 1.3 (1) and (2ii) is to rule out the traveling waves with traveling speeds converging to \(c = \pm 1\). For the proof of Theorem 1.1, our approach is to prove that the sequence of \(L^2\) normalized vertical velocity \(\tilde{v}_n\) of the traveling waves is uniformly \(H^4\) bounded. This allows us to take limits at the equation for \(\tilde{v}_n\). The limit equation is exactly the Rayleigh-Kuo equation with \(c \in (-1, 1)\), which contradicts that \(c\) is not an embedding eigenvalue of \(\mathcal{R}_{\alpha,\beta}\) [21]. The difficulty is to prove the uniform \(H^4\) bound for \(\tilde{v}_n\), \(n \geq 1\). Thanks to the \(\beta\)'s term, we use the integral expression of the velocity to cancel the singularity induced by \(c \in (-1, 1)\), and apply the Gagliardo-Nirenberg interpolation inequality to close the estimates. For the proof of Theorem 1.3 (1) and (2ii), we have no singularity cancelation as above and could only prove the uniform \(H^2\) bound for \(\tilde{v}_n\), \(n \geq 1\). This turns out to be enough after we consider \(\tilde{v}_n\)'s equation in the weak sense and fully use the non-permeable boundary condition for both \(\tilde{v}_n\) and the test functions.

The proof of Theorem 1.3 (2i) is to construct traveling waves by bifurcation at suitable shear flows near Couette flow. In the case \((\alpha, \beta) \in \Gamma_{\pm}\), since the eigenvalue \(\mp 1\) for \(\mathcal{R}_{\alpha,\beta}\) is embedded in \([-1, 1]\), it is difficult to get the \(C^2\) regularity of the nonlinear bifurcated map if we consider the bifurcation as Couette flow itself. Our approach is to consider the bifurcation at the scaled nearby shear flow \((ay, 0)\) with \(|a| < 1\), and the eigenvalue of the linearized operator around \((ay, 0)\) becomes an isolated one. Thus, we could use the bifurcation result in [20] at \((ay, 0)\). In the case \((\alpha, \beta) \in (I_+ \cup I_-) \setminus (\Gamma_+ \cup \Gamma_-)\), the traveling waves are constructed by bifurcation directly at Couette flow, since we prove the existence of isolated real eigenvalue of \(\mathcal{R}_{\alpha,\beta}\) in Section 3.

Our next result is to consider the dynamics near Couette flow for the \(\beta\)-plane equation in the Sobolev spaces with low regularity. We construct non-shear stationary solutions near Couette flow for \(H^{\frac{5}{2}}\) velocity perturbation. This result is a generalization of Theorem 1 in [22], but the bifurcation lemma and construction of the modified shear flow turn out to be more delicate due to the \(\beta\)'s term.

**Theorem 1.5.** (1) Let \(\beta \neq 0\), \(T > 0\) and \(0 \leq s < \frac{5}{2}\). Then for any \(\varepsilon > 0\), there exists a steady solution \((u_\varepsilon(x, y), v_\varepsilon(x, y))\) to the \(\beta\)-plane equation (1.12)-(1.13) with \(x\)-period \(T\) satisfying that \[\|(u_\varepsilon, v_\varepsilon) - (y, 0)\|_{H^s(D_T)} < \varepsilon,\]
but \( v_\epsilon(x, y) \neq 0 \).

(2) Let \( 0 < |\beta| < \frac{4\sqrt{2}}{3\pi} \), \( T > 0 \) and \( 0 \leq s < \frac{5}{2} \). Then the conclusion in (1) holds true, and moreover, \( T \) is the minimal period in \( x \) of the steady solution \((u_\epsilon(x, y), v_\epsilon(x, y))\).

The bifurcation lemma in \([22]\) can not be applied to the case \( \beta \neq 0 \), since there is a singularity at the middle point 0 of the linearized bifurcated map, which is difficult to deal with. We introduce a bifurcation lemma for \( \beta \neq 0 \) such that the potential term of the Rayleigh-Kuo operator is flat near 0. To adjust the flatness condition, we add a cut-off \( \beta \)'s term in the constructed shear flow, and to produce negative eigenvalues of the Rayleigh-Kuo BVP, we try to add the Gauss error function introduced in \([22]\). However, a direct addition of the error function induces new singularity at 0. Our method is to translate the cut-off Gauss error function such that its support does not intersect that of the \( \beta \)'s term and to make the translation sufficiently close to 0, besides the size of the cut-off function should be suitably small.

**Remark 1.6.** We summarize the modified shear flows at which the bifurcation could be used to construct traveling waves for the \( \beta \)-plane equation, which are technically important.

- Scaled modified shear flow \((ay, 0)\), which is used to construct traveling waves near Couette flow if \((\alpha, \beta)\) are chosen such that the linearized operator has an embedding eigenvalue 1 or \(-1\), see the proof of Theorem 1.3 (2).
- Couette flow + cut-off \( \beta \)'s term + cut-off Gauss error function (see \( (5.5) \)), which is used to construct steady states near Couette flow at low regularity.
- \((u + \nu u_1 + \tau u_2, 0)\), which is used to guarantee that the bifurcated solutions near the flow \((u, 0)\) is not a shear one in Lemma 2.3 of \([20]\) and Lemma 5.1. Here, \( u_1 \) and \( u_2 \) are added to obtain monotonicity of the eigenvalues of corresponding Rayleigh-Kuo operators, and \( \nu, \tau \in \mathbb{R} \) are sufficiently small.

Finally, we generalize the asymptotic stability of Couette flow in Gevrey spaces \([4]\) to the \( \beta \)-plane equation on \( \Omega = T_{2\pi} \times \mathbb{R} \). Consider the \( \beta \)-plane equation \((1.1)\) on \( \Omega \). Take \( \tilde{v} = (y, 0) + \tilde{U} = (y, 0) + (U^x, U^y) \), where \( \tilde{U} = (U^x, U^y) \) denotes the velocity perturbation. Then the equation reads

\[
\begin{align*}
\partial_t \tilde{U} + y \partial_y \tilde{U} + (U^y, 0) + \tilde{U} \cdot \nabla \tilde{U} + \nabla P &= -\beta y(-U^y, y + U^x), \\
\nabla \cdot \tilde{U} &= 0.
\end{align*}
\]

Let \( w = \text{curl} \tilde{U} = \partial_y U^y - \partial_x U^x \), then the total vorticity is \( \omega = -1 + w \) and the vorticity form \((1.2)\) becomes

\[
\begin{align*}
\partial_t w + y \partial_y w + \tilde{U} \cdot \nabla w + \beta U^y &= 0, \\
\tilde{U} &= \nabla^\perp (\Delta)^{-1} w, \\
w|t=0 &= w_{in}.
\end{align*}
\]

Here, \((x, y) \in T_{2\pi} \times \mathbb{R}, \nabla^\perp = (-\partial_y, \partial_x) \) and \((\tilde{U}, w)\) are periodic in the \( x \) variable with period normalized to \( 2\pi \). Denote \( \tilde{\psi} = \Delta^{-1} w \). We take an interest in the long time behavior of \((1.7)\) for small initial perturbations \( w_{in} \) and get the following result.

**Theorem 1.7.** For all \( \frac{1}{2} < s \leq 1, \lambda_0 > \lambda' > 0 \), there exists \( \epsilon_0 = \epsilon_0(\lambda_0, \lambda', s) \leq \frac{1}{2} \) such that for all \( \epsilon \leq \epsilon_0 \), if \( w_{in} \) satisfies \( \int_\Omega w_{in} \, dxdy = 0, \int_\Omega |yw_{in}| \, dxdy < \epsilon \) and

\[
\|w_{in}\|^2_{G^{\lambda_0}(\Omega)} := \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |\tilde{w}_{in}(k, \eta)|^2 e^{2\lambda_0 |k| \eta} \, d\eta \leq \epsilon^2,
\]


then there exists \( f_\infty \) with \( \int_\Omega f_\infty \, dx \, dy = 0 \) and \( \| f_\infty \|_{\mathcal{G}^\nu(\Omega)} \lesssim \epsilon \) such that
\[
\| w(t, x + ty + \Phi(t, y), y) - f_\infty(x, y) \|_{\mathcal{G}^\nu(\Omega)} \lesssim \frac{\epsilon^2}{(t)^2},
\]
where \( \Phi(t, y) \) is given by
\[
\Phi(t, y) = \frac{1}{2\pi} \int_0^t \int_{\pi x} U^x(\tau, x, y) \, dx \, d\tau = u_\infty(y) t + O(\epsilon),
\]
with \( u_\infty(y) = \partial_y \partial_y^{-1} \frac{1}{2\pi} \int_{\pi x} f_\infty(x, y) \, dx \) for \( y \in \mathbb{R} \). Moreover, the velocity field \( \tilde{U} \) satisfies
\[
\left\| \frac{1}{2\pi} \int_{\pi x} U^x(t, x, \cdot) \, dx - u_\infty(y) \right\|_{\mathcal{G}^\nu(\Omega)} \lesssim \frac{\epsilon^2}{(t)^2},
\]
\[
\left\| U^x - \frac{1}{2\pi} \int_{\pi x} U^x(t, x, \cdot) \, dx \right\|_{L^2(\Omega)} \lesssim \frac{\epsilon}{(t)^2},
\]
\[
\left\| U^y(t) \right\|_{L^2(\Omega)} \lesssim \frac{\epsilon}{(t)^2}.
\]

We use the same time-dependent norm and main energy as in [4], and prove the bootstrap proposition. The difference comes from the new term \( \int_{\pi x} AfA(\beta \partial_z \phi) \, dx \), where \( A \) is the multiplier in Subsection 2.3 of [4]. To treat this term, we improve the elliptic control (Proposition 2.4) in [4]. Notice that \( \int_{\pi x} AfA(\beta \partial_z \phi) \, dx = \beta \int_{\pi x} AfA(\partial_z \phi_1) \, dx \) for \( \phi_1 = P_{\neq 0}f - \Delta_L^{-1}P_{\neq 0}f \), where \( f \) is the vorticity under changed coordinates \((z, \psi)\), \( \Delta_L = \partial_z^2 + (\partial_\theta - t \partial_z)^2 \) and \( P_{\neq 0}f = f - \langle f \rangle = f - \int_{\pi x} f \, dx \). The new term has similar bound with Reaction (Proposition 2.3 in [4]), so the method of [4] works here.

The rest of this paper is organized as follows. In Section 2, we prove the non-existence of non-parallel traveling waves with traveling speeds converging inside \((-1, 1)\) near Couette flow for \( H^{\geq 5} \) velocity perturbation and \( \beta \neq 0 \). In Section 3, we study properties of the principal eigenvalues of Rayleigh-Kuo operators for Couette flow. In Section 4, we determine the sharp region such that no traveling waves exist for \((\alpha, \beta)\) in this region and traveling waves exist for \((\alpha, \beta)\) in the remaining regions near Couette flow for \( H^{\geq 5} \) velocity perturbation. In Section 5, we construct non-shear steady states near Couette flow in low Sobolev spaces for \( \beta \neq 0 \). Note that the domain in Sections 2-5 is a finite channel \( \mathbb{T} \times [-1, 1] \) and the perturbation is considered in Sobolev spaces. Finally, we prove nonlinear inviscid damping near Couette flow in suitable Gevrey spaces for \( \beta \neq 0 \) on \( \Omega = \mathbb{T}_{2\pi} \times \mathbb{R} \) in Section 6.

2. Non-existence of traveling waves with traveling speeds inside \((-1,1)\)

In this section, we prove that there are no non-parallel traveling waves with traveling speeds converging in \((-1, 1)\) near Couette flow for \( H^{\geq 5} \) velocity perturbation, \( \beta \neq 0 \) and any \( x \)-period \( T \), which is stated in Theorem 1.3.

The following two lemmas will be used, which are Hardy type inequality [11, 20] and Gagliardo-Nirenberg interpolation inequality [9, 26, 27].

**Lemma 2.1.** Let \( \phi \in H^1(a, b) \) and \( \phi(y_0) = 0 \) for some \( y_0 \in [a, b] \). Then
\[
\left\| \frac{\phi}{y - y_0} \right\|_{L^2(a, b)}^2 \leq C\| \phi' \|_{L^2(a, b)}^2.
\]
Lemma 2.2. Let $G$ be a bounded domain in $\mathbb{R}^n$ having the cone property. For $1 \leq q, r \leq \infty$, suppose $u \in L^q(G)$ and $D^m u \in L^r(G)$. Then for $0 \leq j < m$, the following inequalities hold (with constant $C_1, C_2$ depending only on $G, m, j, q, r$)

\begin{equation}
\|D^j u\|_{L^p(G)} \leq C_1 \|D^m u\|_{L^q(G)}^{\alpha} \|u\|_{L^r(G)}^{1-\alpha} + C_2 \|u\|_{L^r(G)},
\end{equation}

where

$$
\frac{1}{p} = \frac{j}{n} + \alpha \left(\frac{1}{r} - \frac{m}{n}\right) + (1 - \alpha)\frac{1}{q},
$$

for all $\alpha \in \left[\frac{m}{n}, 1\right]$, unless $1 < r < \infty$ and $m - j - n/r$ is a nonnegative integer, in which case (2.1) holds only for $\alpha$ satisfying $\alpha \in \left[\frac{m}{n}, 1\right]$.

Proof of Theorem 1.1 Suppose otherwise, there exist $\{\varepsilon_n\}_{n=1}^{\infty}$, $c_\varepsilon \in [-1+\delta, 1-\delta]$ and $(u_n(x-c_n t, y), v_n(x-c_n t, y))$ to the $\beta$-plane equation (1.1)-(1.3) such that $c_\varepsilon \to 0$, $(u_n, v_n)$ is $T$-periodic in $x$, $\|(u_n, v_n) - (y, 0)\|_{H^s(D_T)} \leq \varepsilon_n$ and $\|v_n\|_{L^2(D_T)} \not= 0$. Then

\begin{equation}
(u_n - c_\varepsilon)\partial_x \omega_n + v_n(\partial_y \omega_n + \beta) = 0,
\end{equation}

where $\omega_n = \partial_x v_n - \partial_y u_n$. Up to a subsequence, $c_\varepsilon \to c_0 \in [-1 + \delta, 1 - \delta]$. $s \geq 5$ implies

\begin{align}
&\|u_n - y\|_{C^3(D_T)} + \|v_n\|_{C^3(D_T)} \leq C\|(u_n, v_n) - (y, 0)\|_{H^s(D_T)} \leq C\varepsilon_n, \\
&\|\omega_n + 1\|_{C^2(D_T)} \leq C\|\omega_n + 1\|_{H^s(D_T)} \leq C\|(u_n, v_n) - (y, 0)\|_{H^s(D_T)} \leq C\varepsilon_n,
\end{align}

\begin{equation}
\|\partial_{xxx} \omega_n\|_{L^4(D_T)} + \|\partial_{xxy} \omega_n\|_{L^4(D_T)} + \|\partial_{xxy} \omega_n\|_{L^4(D_T)} + \|\partial_{yyx} \omega_n\|_{L^4(D_T)} \leq C\|\omega_n + 1\|_{H^s(D_T)} \leq C\varepsilon_n.
\end{equation}

Then

\begin{equation}
\|u_n\|_{C^3(D_T)} \leq C, \quad \|\omega_n\|_{C^2(D_T)} \leq C.
\end{equation}

Moreover,

\begin{equation}
\frac{1}{2} \partial_y u_n(x, y) < 3/2, \quad (x, y) \in D_T
\end{equation}

for $n$ sufficiently large. Then $u_n(x, \cdot)$ is increasing on $[-1, 1]$ for $x \in [0, T]$ and by taking $n$ larger, we have $\|u_n - y\|_{L^\infty(D_T)} \leq C\varepsilon_n < \frac{\delta}{2}$. For fixed $n$, we claim that $u_n(x, -1) < c_n < u_n(x, 1)$ for $x \in [0, T]$. Suppose otherwise, there exists $x_0 \in [0, T]$ such that $u_n(x_0, 1) \leq c_n$ or $c_n \leq u_n(x_0, -1)$. For the first case, we have

$$
\frac{\delta}{2} > C\varepsilon_n > \|u_n - y\|_{C^0(D_T)} \geq |u_n(x_0, c_n) - c_n| \geq |u_n(x_0, c_n) - u_n(x_0, 1)| > \frac{1}{2}|c_n - 1| > \frac{\delta}{2}.
$$

which is a contradiction. The latter case $c_n \leq u_n(x_0, -1)$ is similar since $\frac{\delta}{2} > C\varepsilon_n > \|u_n - y\|_{C^0(D_T)} \geq |u_n(x_0, c_n) - c_n| \geq |u_n(x_0, c_n) - u_n(x_0, -1)| > \frac{1}{2}|c_n + 1| > \frac{\delta}{2}.$

Thus, there exists a unique $y_n(x) \in (-1, 1)$ such that $u_n(x, y_n(x)) = c_n$ for $x \in [0, T]$ and $n$ sufficiently large. Since $\|\partial_y \omega_n\|_{C^0(D_T)} \leq \|\omega_n + 1\|_{C^2(D_T)} \leq C\varepsilon_n$ and $\beta \not= 0$, we have $|\partial_y \omega_n(x, y) + \beta| \not= 0$ for $(x, y) \in D_T$ and $n$ sufficiently large. This, along with $u_n(x, y_n(x)) = c_n$ and (2.2), implies that $v_n(x, y_n(x)) = 0$ for $x \in [0, T]$.

By the incompressible condition, we have $\partial_y \omega_n = \Delta v_n$. Let $\tilde{v}_n = v_n/\|v_n\|_{L^2(D_T)}$. Then $\tilde{v}_n(x, y_n(x)) = 0$ for $x \in [0, T]$. By (2.2), we have

\begin{equation}
\Delta \tilde{v}_n + (\partial_y \omega_n + \beta) \frac{\tilde{v}_n}{u_n - c_n} = 0.
\end{equation}

First, we prove the uniform $H^2$ bound for $\tilde{v}_n$, $n \geq 1$. It follows from (2.7) that $\left|\frac{y-y_n(x)}{u_n(x,y) - u_n(x,y_n(x))}\right| \leq 2$ for $(x, y) \in D_T$. Thus, $\|y-y_n(x)/u_n(x,y) - c_n\|_{L^\infty(D_T)} \leq C$. This, along with (2.8),
(2.6) and Lemma 2.1 gives
\[
\|\Delta \tilde{v}_n\|_{L^2(D_T)} \leq \left\| (\partial_y \omega_n + \beta) \frac{\tilde{v}_n}{u_n - c_n} \right\|_{L^2(D_T)} \leq C \left\| \frac{\tilde{v}_n}{u_n - c_n} \right\|_{L^2(D_T)}
\]
\[
\leq C \left\| \frac{\tilde{v}_n}{y - y_n(x)} \right\|_{L^2(D_T)} \left\| \frac{y - y_n(x)}{u_n - c_n} \right\|_{L^\infty(D_T)}
\]
\[
\leq C \left\| \frac{\tilde{v}_n}{y - y_n(x)} \right\|_{L^2(D_T)} \leq C \left( \int_0^T \| \tilde{v}_n(x, \cdot) \|^2 H^1_{(-1,1)} \, dx \right)^{\frac{1}{2}}
\]
(2.9)
\[
\leq C \| \tilde{v}_n \|_{H^1(D_T)} \leq C \| \tilde{v}_n \|_{H^2(D_T)} \| \tilde{v}_n \|_{L^2(D_T)} = C \| \tilde{v}_n \|_{H^2(D_T)}.
\]
Thus, by periodic boundary condition in \(x\) and Dirichlet boundary condition in \(y\) of \(\tilde{v}_n\), we have
\[
(2.10) \quad \| \tilde{v}_n \|_{H^2(D_T)} \leq C \| \Delta \tilde{v}_n \|_{L^2(D_T)} \leq C \| \tilde{v}_n \|_{H^2(D_T)} \implies \| \tilde{v}_n \|_{H^2(D_T)} \leq C.
\]
Next, we prove the uniform \(H^3\) bound for \(\tilde{v}_n, n \geq 1\). Taking derivative of (2.8) with respect to \(y\), we have
\[
\partial_y \Delta \tilde{v}_n + \frac{\partial_y^2 \omega_n}{u_n - c_n} \frac{\tilde{v}_n}{u_n - c_n} + (\partial_y \omega_n + \beta) \partial_y \left( \frac{\tilde{v}_n}{u_n - c_n} \right) = 0.
\]
Thus, by (2.6) and (2.9) we have
\[
\| \partial_y \Delta \tilde{v}_n \|_{L^2(D_T)} \leq \| \partial_y^2 \omega_n \|_{L^\infty(D_T)} \left\| \frac{\tilde{v}_n}{u_n - c_n} \right\|_{L^2(D_T)} + \| \partial_y \omega_n + \beta \|_{L^\infty(D_T)} \left\| \partial_y \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)}
\]
(2.11)
\[
\leq C \| \tilde{v}_n \|_{H^1(D_T)} + C \left\| \partial_y \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)}.
\]
Since \(\tilde{v}_n(x, y_n(x)) = u_n(x, y_n(x)) - c_n = 0\) for \(x \in [0, T]\) and \(n\) sufficiently large, we have
\[
\frac{\tilde{v}_n(x, y)}{u_n(x, y) - c_n} = \frac{\tilde{v}_n(x, y) - \tilde{v}_n(x, y_n(x))}{u_n(x, y) - u_n(x, y_n(x))} = \frac{\int_{y_n(x)}^y \partial_y \tilde{v}_n(x, s_n) \, ds_n}{\int_{y_n(x)}^y \partial_y u_n(x, s_n) \, ds_n}
\]
(2.12)
\[
= \frac{\int_0^1 \partial_y \tilde{v}_n(x, y_n(x) + t(y - y_n(x))) \, dt}{\int_0^1 \partial_y u_n(x, y_n(x) + t(y - y_n(x))) \, dt} = \frac{\int_0^1 \partial_y \tilde{v}_n(x, s_n) \, dt}{\int_0^1 \partial_y u_n(x, s_n) \, dt}
\]
where \(s_n = y_n(x) + t(y - y_n(x))\). Here, \(s_n\) is dependent on \(x\), and we always use \(s_n\) to avoid tedious notation. Then
\[
\partial_y \left( \frac{\tilde{v}_n(x, y)}{u_n(x, y) - c_n} \right) = \frac{\int_0^1 \partial_y^2 \tilde{v}_n(x, s_n) \, dt \int_0^1 \partial_y u_n(x, s_n) \, dt - \int_0^1 \partial_y \tilde{v}_n(x, s_n) \, dt \int_0^1 \partial_y^2 u_n(x, s_n) \, dt}{\left( \int_0^1 \partial_y u_n(x, s_n) \, dt \right)^2}.
\]
By (2.6) and (2.7), we have
\[
(2.13) \quad 1/2 < \int_0^1 \partial_y u_n(x, s_n) \, dt < 3/2 \quad \forall (x, y) \in D_T, \quad \left\| \int_0^1 \partial_y^2 u_n(x, s_n) \, dt \right\|_{L^\infty(D_T)} \leq C.
\]
Thus,
\[
\left\| \partial_y \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)} \leq C \left\| \int_0^1 \partial_y^2 \tilde{v}_n(x, s_n) \, dt \right\|_{L^2(D_T)} \left\| \int_0^1 \partial_y u_n(x, s_n) \, dt \right\|_{L^\infty(D_T)}
\]
A direct computation implies

\[
\int_0^1 |\partial_y \tilde{v}_n(x, s_n)| dt \leq C \left\| \int_0^1 \partial_y^2 \tilde{v}_n(x, s_n) dt \right\|_{L^2(D_T)} + C \left\| \int_0^1 \partial_y \tilde{v}_n(x, s_n) dt \right\|_{L^2(D_T)}
\]

(2.14)

A direct computation implies

\[
\left\| \int_0^1 |\partial_y \tilde{v}_n(x, s_n)| dt \right\|^2_{L^2(D_T)} = \int_0^T \int_{-1}^1 \int_{y_n(x)}^y |\partial_y \tilde{v}_n(x, s_n)|^2 ds_n \, dy \, dx
\]

\[
\leq \int_0^T \int_{-1}^1 \frac{1}{|y - y_n(x)|} \int_{y_n(x)}^y |\partial_y \tilde{v}_n(x, s_n)|^2 ds_n \, dy \, dx
\]

\[
= \int_0^T \int_{y_n(x)}^y \frac{1}{y_n(x) - y} \int_{y_n(x)}^y |\partial_y \tilde{v}_n(x, s_n)|^2 ds_n \, dy \, dx
\]

\[
+ \int_0^T \int_{y_n(x)}^y \frac{1}{y_n(x) - y} \int_{y_n(x)}^y |\partial_y \tilde{v}_n(x, s_n)|^2 ds_n \, dy \, dx
\]

\[
= - \int_0^T \left( \ln(y_n(x) - y) \int_{y_n(x)}^y |\partial_y \tilde{v}_n(x, s_n)|^2 ds_n \right) \left|_{y_n(x)}^{y_n(x)} \right| dx
\]

(2.15)

\[
= I_1(\partial_y \tilde{v}_n) + I_2(\partial_y \tilde{v}_n) + I_3(\partial_y \tilde{v}_n) + I_4(\partial_y \tilde{v}_n).
\]

Since \( \tilde{v}_n \in C^1(D_T) \), for fixed \( n \geq 1 \) and \( x \in [0, T] \) we have

\[
\lim_{y \to y_n(x)} \ln |y_n(x) - y| \int_{y_n(x)}^y |\partial_y \tilde{v}_n(x, s_n)|^2 ds_n
\]

\[
= \lim_{y \to y_n(x)} |y_n(x) - y| \ln |y_n(x) - y| \cdot \lim_{y \to y_n(x)} \int_{y_n(x)}^y |\partial_y \tilde{v}_n(x, s_n)|^2 ds_n
\]

\[
= 0 \cdot |\partial_y \tilde{v}_n(x, y_n(x))|^2 = 0.
\]

By (2.7) and (2.8), we have

\[
|y_n(x) - c_n| = |y_n(x) - u_n(x, y_n(x))| \leq \|u_n - y\|_{C^0(D_T)} \leq C \varepsilon_n,
\]

and thus, (2.16)

\[
|y_n(x) - c_0| \leq |y_n(x) - c_n| + |c_n - c_0| \to 0
\]

as \( n \to \infty \) uniformly for \( x \in [0, T] \). Then \( |y_n(x)| < \frac{1 + |c_0|}{2} \leq 1 - \frac{\varepsilon}{2} \) for \( x \in [0, T] \) and \( n \) sufficiently large. Thus,

\[
|I_1(\partial_y \tilde{v}_n)| + |I_3(\partial_y \tilde{v}_n)| \leq \int_0^T |\ln(y_n(x) + 1)| \int_{-1}^{y_n(x)} |\partial_y \tilde{v}_n(x, s_n)|^2 ds_n dx
\]
\[ + \int_0^T \left| \ln(1 - y_n(x)) \right| \int_{y_n(x)}^1 |\partial_y \tilde{v}_n(x, s_n)|^2 ds_n dx \]

(2.17)

\[ \leq C \| \partial_y \tilde{v}_n \|_{L^2(D_T)}^2 \leq C \| \tilde{v}_n \|_{H^2(D_T)}^2. \]

By (2.10) we have

\[ \| \partial_y \tilde{v}_n \|_{L^2(D_T)} \leq C \| \tilde{v}_n \|_{H^2(D_T)} \leq C, \]

and thus,

\[ |I_2(\partial_y \tilde{v}_n)| \leq \left( \int_0^T \int_{y_n(x)}^{y_n(x) + 1} |\ln(y_n(x) - y)|^2 dy dx \right)^{1/2} \left( \int_0^T \int_{y_n(x)}^{y_n(x) + 1} |\partial_y \tilde{v}_n(x, y)|^4 dy dx \right)^{1/2} \]

\[ \leq \left( \int_0^T \int_0^{y_n(x) + 1} |\ln \tau_n|^2 d\tau_n dx \right)^{1/2} \| \partial_y \tilde{v}_n \|_{L^2(D_T)}^2 \]

(2.18)

\[ \leq \left( \int_0^T \int_0^2 |\ln \tau|^2 d\tau dx \right)^{1/2} \| \tilde{v}_n \|_{H^2(D_T)}^2 \leq C \| \tilde{v}_n \|_{H^2(D_T)}^2 \leq C. \]

Similarly, we have

\[ |I_4(\partial_y \tilde{v}_n)| \leq \left( \int_0^T \int_0^2 |\ln \tau|^2 d\tau dx \right)^{1/2} \| \tilde{v}_n \|_{H^2(D_T)}^2 \leq C. \]

(2.19)

Combining (2.15), (2.17), (2.18) and (2.19), we have

\[ \left\| \int_0^1 |\partial_y \tilde{v}_n(x, s_n)| dt \right\|_{L^2(D_T)} \leq C \| \tilde{v}_n \|_{H^2(D_T)} \leq C. \]

(2.20)

Thanks to the factor \( t \), we can also prove that

\[ \left\| \int_0^1 \partial_y^2 \tilde{v}_n(x, s_n) t dt \right\|_{L^2(D_T)} \leq C \| \tilde{v}_n \|_{H^2(D_T)} \leq C. \]

(2.21)

In fact,

\[ \left\| \int_0^1 \partial_y^2 \tilde{v}_n(x, s_n) t dt \right\|_{L^2(D_T)}^2 \]

\[ = \int_0^T \int_{y_n(x)}^{y_n(x) + 1} \left| \partial_y^2 \tilde{v}_n(x, s_n) \frac{s_n - y_n(x)}{(y - y_n(x))^2} ds_n \right|^2 dy dx \]

\[ \leq \int_0^T \int_{y_n(x)}^{y_n(x) + 1} \frac{1}{|y - y_n(x)|^2} \left| \int_{y_n(x)}^{y} |\partial_y^2 \tilde{v}_n(x, s_n)|^2 (s_n - y_n(x))^2 ds_n \right| dy dx \]

\[ = \int_0^T \frac{1}{(y_n(x) - y)^2} \int_{y_n(x)}^{y} |\partial_y^2 \tilde{v}_n(x, s_n)|^2 (y_n(x) - s_n)^2 ds_n dy dx \]

\[ + \int_0^T \frac{1}{(y - y_n(x))^2} \int_{y_n(x)}^{y} |\partial_y^2 \tilde{v}_n(x, s_n)|^2 (s_n - y_n(x))^2 ds_n dy dx \]

\[ = \frac{1}{2} \int_0^T \left( \frac{1}{(y_n(x) - y)^2} \int_{y_n(x)}^{y} |\partial_y^2 \tilde{v}_n(x, s_n)|^2 (y_n(x) - s_n)^2 ds_n \right) \bigg|_{y=y_n(x)}^{y=y_n(x) + 1} dx \]
By (2.14), (2.20) and (2.21), we have
\[ v_n \leq \int_{\partial T} v_n(x) \, ds_n + \frac{1}{2} \int_{\Omega} (\partial_y^2 v_n(x, s_n))^2 (s_n - y_n(x))^2 \, ds_n \, dx \]
\[ + \frac{1}{2} \int_0^T \int_{y_n(x)}^{y_n(x)} |\partial_y^2 v_n(x, y)|^2 \, dy \, dx \]
\[ \leq \frac{1}{2} \int_{y_n(x)}^{y_n(x)} \frac{1}{(y_n(x) - x)^2} \int_{y_n(x)}^{y_n(x)} |\partial_y^2 v_n(x, s_n)|^2 (s_n - y_n(x))^2 \, ds_n \, dx + \frac{1}{2} \|\partial_y^2 v_n\|_{L^2(D_T)}^2 \]
\[ + \frac{1}{2} \int_{y_n(x)}^{y_n(x)} \frac{1}{(1 - x_n(x))^2} \int_{y_n(x)}^{y_n(x)} |\partial_y^2 v_n(x, s_n)|^2 (s_n - y_n(x))^2 \, ds_n \, dx + \frac{1}{2} \|\partial_y^2 v_n\|_{L^2(D_T)}^2 \]
\[ (2.22) \quad \leq 2\|\partial_y^2 v_n\|_{L^2(D_T)}^2 \leq 2\|\tilde{v}_n\|_{H^2(D_T)}^2 \leq C, \]
where we used \(\tilde{v}_n \in C^2(D_T)\) to deduce that
\[ \lim_{y \to y_n(x)} \int_{y_n(x)}^{y_n(x)} |\partial_y^2 \tilde{v}_n(x, s_n)|^2 (s_n - y_n(x))^2 \, ds_n = 0 \]
for fixed \(n\) and \(x \in [0, T]\).

By (2.14), (2.20) and (2.21), we have
\[ \left\| \partial_y \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)} \leq C, \]
and thus, by (2.11), (2.23) and (2.10) we have
\[ \|\partial_x \Delta \tilde{v}_n\|_{L^2(D_T)} \leq C \|\tilde{v}_n\|_{H^1(D_T)} + C \leq C. \]

Taking derivative of (2.23) with respect to \(x\), by (2.6) and (2.9) we have
\[ \|\partial_x \Delta \tilde{v}_n\|_{L^2(D_T)} \leq \|\partial_x \omega_\infty\|_{L^\infty(D_T)} \left\| \frac{\tilde{v}_n}{u_n - c_n} \right\|_{L^2(D_T)} + \|\partial_y \omega_\infty + \beta\|_{L^\infty(D_T)} \left\| \partial_x \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)} \]
\[ \leq C \|\tilde{v}_n\|_{H^1(D_T)} + C \left\| \partial_x \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)}. \]

Taking derivative of (2.12) with respect to \(x\), we have
\[ \partial_x \left( \frac{\tilde{v}_n(x, y)}{u_n(x, y) - c_n} \right) = \int_0^1 \left\{ \partial_{xy} \tilde{v}_n(x, s_n) + \partial_y^2 \tilde{v}_n(x, s_n)(1 - t) y'_n(x) \right\} \, dt \]
\[ - \int_0^1 \partial_y \tilde{v}_n(x, s_n) \, dt \int_0^1 \left( \partial_{x} u_n(x, s_n) + \partial_y^2 u_n(x, s_n)(1 - t) y'_n(x) \right) \, dt \]
\[ - \int_0^1 \partial_y \tilde{v}_n(x, s_n) \, dt \int_0^1 \left( \partial_{x} u_n(x, s_n) + \partial_y^2 u_n(x, s_n)(1 - t) y'_n(x) \right) \, dt \]
\[ \leq C \|y'_n\|_{L^\infty(D_T)} \leq C \|\partial_x \omega_\infty\|_{L^\infty(D_T)} \leq C. \]

This, along with (2.13) and (2.20), gives
\[ \left\| \partial_x \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)} \leq C \left\| \int_0^1 \partial_{xy} \tilde{v}_n(x, s_n) \, dt \right\|_{L^2(D_T)} + C \left\| \int_0^1 \partial_y^2 \tilde{v}_n(x, s_n)(1 - t) y'_n(x) \, dt \right\|_{L^2(D_T)} \]
+ C \left\| \int_0^1 \partial_y \tilde{v}_n(x, s_n) dt \right\|_{L^2(D_T)} 
abla x \tilde{v}_n(x, s_n) \right\|_{L^2(D_T)} + C \left\| \int_0^1 \partial_y^2 \tilde{v}_n(x, s_n) dt \right\|_{L^2(D_T)} + C.

(2.28)

\n
Let \( \Gamma_1 \tilde{v}_n = \partial_{x y} \tilde{v}_n \) and \( \Gamma_2 \tilde{v}_n = \partial_y^2 \tilde{v}_n \). We prove that for \( i = 1, 2 \),

\[
\left\| \int_0^1 |\Gamma_i \tilde{v}_n(x, s_n)| dt \right\|_{L^2(D_T)} \leq C \| \tilde{v}_n \|_{H^3(D_T)}^2 + C,
\]

(2.29)

where \( \alpha \in (0, 1) \). By the same computation as in (2.15), we have

\[
\left\| \int_0^1 |\Gamma_i \tilde{v}_n(x, s_n)| dt \right\|_{L^2(D_T)}^2 \leq \int_1 \left( \Gamma_i \tilde{v}_n(x, s_n) \right)_{\Gamma} \leq \int_1 \left( \Gamma_i \tilde{v}_n(x, s_n) \right)_{\Gamma} + I_3(\Gamma_i \tilde{v}_n) + I_4(\Gamma_i \tilde{v}_n)
\]

for \( i = 1, 2 \). Since \( \tilde{v}_n \in C^2(D_T) \), for fixed \( n \geq 1 \) and \( x \in [0, T] \) we have

\[
\lim_{y \rightarrow y_n(x)} \ln |y_n(x) - y| \left| \int_y^{y_n(x)} |\Gamma_i \tilde{v}_n(x, s_n)|^2 ds_n \right| = 0.
\]

Since \( |y_n(x)| < 1 - \frac{d}{2} \) for \( x \in [0, T] \) and \( n \) sufficiently large, by (2.10) we have

\[
|I_1(\Gamma_i \tilde{v}_n)| + |I_3(\Gamma_i \tilde{v}_n)| \leq C \| \Gamma_i \tilde{v}_n \|_{L^2(D_T)} \leq C \| \tilde{v}_n \|_{H^2(D_T)} \leq C.
\]

(2.31)

Letting \( q = 2, r = 2, m = 1, n = 2, j = 0, \alpha \in (0, 1) \), \( G = D_T \) and \( u = \Gamma_i \tilde{v}_n \) in Lemma (2.2) we have by (2.10) that

\[
\| \Gamma_i \tilde{v}_n \|_{L^p(D_T)} \leq C \| D^1(\Gamma_i \tilde{v}_n) \|_{L^2(D_T)} \leq C \| \tilde{v}_n \|_{H^2(D_T)} \leq C \| \tilde{v}_n \|_{H^3(D_T)} + C,
\]

(2.30)

where \( p = \frac{2}{1 - \alpha} \in (2, \infty) \). Let \( p_0 = \left( \frac{2}{1 - \alpha} \right) = \frac{p}{p-2} \geq 1 \). Then

\[
|I_2(\Gamma_i \tilde{v}_n)| \leq \left( \int_0^T \int_{y_n(x)}^{y_n(x)} \left| \ln(y_n(x) - y) \right|^{p_0} dy dx \right)^\frac{1}{p_0} \left( \int_0^T \int_{y_n(x)}^{y_n(x)} |\Gamma_i \tilde{v}_n(x, y)|^{p_0} dy dx \right)^\frac{2}{p}
\]

\[
\leq \left( \int_0^T \int_0^2 \left| \ln \tau \right|^{p_0} d\tau dx \right)^\frac{1}{p_0} \left\| \Gamma_i \tilde{v}_n \right\|_{L^p(D_T)} \leq C \| \tilde{v}_n \|_{H^3(D_T)} \leq C \| \tilde{v}_n \|_{H^3(D_T)}^2 + C.
\]

(2.32)

A similar argument gives

\[
|I_4(\Gamma_i \tilde{v}_n)| \leq C \| \tilde{v}_n \|_{H^3(D_T)}^2 + C.
\]

By (2.30), (2.31), (2.32) and (2.33), we have

\[
\left\| \int_0^1 |\Gamma_i \tilde{v}_n(x, s_n)| dt \right\|_{L^2(D_T)}^2 \leq C \| \tilde{v}_n \|_{H^3(D_T)}^2 + C,
\]

(2.34)

and thus, (2.29) holds for \( i = 1, 2 \). Then we deduce from (2.28) and (2.29) that

\[
\left\| \partial_x \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)} \leq C \| \tilde{v}_n \|_{H^3(D_T)} + C.
\]

(2.35)

By (2.25), (2.34) and (2.10), we have

\[
\| \partial_x \Delta \tilde{v}_n \|_{L^2(D_T)} \leq C \| \tilde{v}_n \|_{H^3(D_T)} + C \| \tilde{v}_n \|_{H^3(D_T)} + C \leq C \| \tilde{v}_n \|_{H^3(D_T)} + C.
\]
Thus, by the periodic boundary condition in \( x \) and Dirichlet boundary condition in \( y \) of \( \tilde{v}_n \), we infer from (2.24) and (2.35) that
\[
\| \tilde{v}_n \|_{H^4(D_T)} \leq C \| \partial_y \Delta \tilde{v}_n \|_{L^2(D_T)} + C \| \partial_x \Delta \tilde{v}_n \|_{L^2(D_T)} \leq C \| \tilde{v}_n^\alpha \|_{H^3(D_T)} + C,
\]
where \( \alpha \in (0, 1) \). Thus,
\[
(2.36) \quad \| \tilde{v}_n \|_{H^3(D_T)} \leq C, \quad \| \partial_x \left( \frac{\tilde{v}_n}{u_n-c_n} \right) \|_{L^2(D_T)} \leq C, \quad \left\| \int_0^1 [\Gamma_i \tilde{v}_n(x, s_n)] dt \right\|_{L^2(D_T)} \leq C
\]
for \( i = 1, 2 \).

Then we prove the uniform \( H^4 \) bound for \( \tilde{v}_n \), \( n \geq 1 \). Taking derivative of (2.8) with respect to \( x \) twice, we have
\[
\partial_x^2 \Delta \tilde{v}_n + \partial_{xxy} \omega_n \left( \frac{\tilde{v}_n}{u_n-c_n} \right) + 2 \partial_{xy} \omega_n \partial_x \left( \frac{\tilde{v}_n}{u_n-c_n} \right) + (\partial_y \omega_n + \beta) \partial_x^2 \left( \frac{\tilde{v}_n}{u_n-c_n} \right) = 0.
\]
By (2.9), (2.23) and (2.36), we have
\[
(2.37) \quad \left\| \frac{\tilde{v}_n}{u_n-c_n} \right\|_{H^3(D_T)} \leq C \implies \left\| \frac{\tilde{v}_n}{u_n-c_n} \right\|_{L^2(D_T)} \leq C \left\| \tilde{v}_n \right\|_{H^4(D_T)} \leq C.
\]
Then by (2.5), (2.6) and (2.37), we have
\[
\| \partial_x^2 \Delta \tilde{v}_n \|_{L^2(D_T)} \leq \| \partial_{xxy} \omega_n \|_{L^4(D_T)} \left\| \frac{\tilde{v}_n}{u_n-c_n} \right\|_{L^4(D_T)} + 2 \| \partial_{xy} \omega_n \|_{C^0(D_T)} \left\| \partial_x \left( \frac{\tilde{v}_n}{u_n-c_n} \right) \right\|_{L^2(D_T)}
\]
\[
+ \| \partial_y \omega_n + \beta \|_{C^0(D_T)} \left\| \partial_x^2 \left( \frac{\tilde{v}_n}{u_n-c_n} \right) \right\|_{L^2(D_T)}
\]
\[
(2.38) \quad \leq C + C \left\| \partial_x^2 \left( \frac{\tilde{v}_n}{u_n-c_n} \right) \right\|_{L^2(D_T)}.
\]
By (2.26), we have
\[
\partial_x^2 \left( \frac{\tilde{v}_n(x, y)}{u_n(x, y)-c_n} \right)
\]
\[
= \frac{\int_0^1 (\partial_{xxy} \tilde{v}_n + 2 \partial_{xyy} \tilde{v}_n(1-t)y'_n + \partial_y^2 \tilde{v}_n(1-t)^2 y''_n + \partial_y^2 \tilde{v}_n(1-t)y''_n) dt}{\int_0^1 \partial_y u_n dt}
\]
\[
- 2 \frac{\int_0^1 (\partial_{xxy} \tilde{v}_n + \partial_y^2 \tilde{v}_n(1-t)y'_n) dt}{\left( \int_0^1 \partial_y u_n dt \right)^2}
\]
\[
- \frac{\int_0^1 \partial_y \tilde{v}_n dt}{\int_0^1 \partial_y u_n dt} \frac{\int_0^1 (\partial_{xxy} u_n + 2 \partial_{xyy} u_n(1-t)y'_n + \partial_y^2 u_n(1-t)^2 y''_n + \partial_y^2 u_n(1-t)y''_n) dt}{\left( \int_0^1 \partial_y u_n dt \right)^2}
\]
\[
+ 2 \frac{\int_0^1 \partial_y \tilde{v}_n dt}{\left( \int_0^1 \partial_y u_n dt \right)^3}
\]
\[
(2.39) \quad = I_1 + I_2 + I_3 + I_4.
\]
Since \(u_n(x, y_n(x)) = c_n\) for \(x \in [0, T]\), we have
\[
y_n''(x) = -\frac{\partial^2_x u_n(x, y_n(x)) + 2 \partial_{xy} u_n(x, y_n(x)) y_n'(x) + \partial^2_y u_n(x, y_n(x))(y_n'(x))^2}{\partial_y u_n(x, y_n(x))},
\]
and thus, we deduce from (2.6), (2.7) and (2.27) that
\[
(4.0) \quad \|y_n''\|_{L^\infty(0, T)} \leq C,
\]
\[
(4.1) \quad \left\| \int_0^1 (\partial_{xy} u_n + 2 \partial_{xx} u_n(1-t)y_n' + \partial^2_x u_n(1-t)^2 y_n^2 + \partial^2_y u_n(1-t)y_n'') dt \right\|_{L^\infty(D_T)} \leq C.
\]
By (2.13), (2.27), (2.30) and (2.41), we have
\[
\begin{align*}
\|I_1\|_{L^2(D_T)} &\leq C \int_0^1 |\partial_{xy} \tilde{v}_n| dt \left\| _{L^2(D_T)} + C \int_0^1 |\partial_{xxy} \tilde{v}_n| dt \right\|_{L^2(D_T)} \\
&\quad + C \int_0^1 |\partial^2_x \tilde{v}_n| dt \left\| _{L^2(D_T)} + C \int_0^1 |\partial^2_y \tilde{v}_n| dt \right\|_{L^2(D_T)},
\end{align*}
\]
\[
\|I_2\|_{L^2(D_T)} \leq C \int_0^1 |\partial_{xy} \tilde{v}_n| dt \left\| _{L^2(D_T)} + C \int_0^1 |\partial^2_x \tilde{v}_n| dt \right\|_{L^2(D_T)},
\]
\[
(4.2) \quad \|I_j\|_{L^2(D_T)} \leq C \int_0^1 |\partial_{xy} \tilde{v}_n| dt \left\| _{L^2(D_T)}
\]
for \(j = 3, 4\). Let \(\tilde{\Gamma}_1 \tilde{v}_n = \partial_{xxy} \tilde{v}_n, \tilde{\Gamma}_2 \tilde{v}_n = \partial_{xxy} \tilde{v}_n\) and \(\tilde{\Gamma}_3 \tilde{v}_n = \partial^2_y \tilde{v}_n\). We prove that for \(i = 1, 2, 3,\)
\[
(4.3) \quad \left\| \int_0^1 |\tilde{\Gamma}_i \tilde{v}_n(x, s_n)| dt \right\|_{L^2(D_T)} \leq C \|\tilde{v}_n\|_{H^4(D_T)}^\alpha + C,
\]
where \(\alpha \in (0, 1)\). Based on the uniform \(H^3\) bound for \(\tilde{v}_n(n \geq 1)\), the proof is similar as (2.29), and we give it here for completeness. In fact, we have
\[
(4.4) \quad \left\| \int_0^1 |\tilde{\Gamma}_i \tilde{v}_n(x, s_n)| dt \right\|_{L^2(D_T)}^2 \leq I_1(\tilde{\Gamma}_i \tilde{v}_n) + I_2(\tilde{\Gamma}_i \tilde{v}_n) + I_3(\tilde{\Gamma}_i \tilde{v}_n) + I_4(\tilde{\Gamma}_i \tilde{v}_n),
\]
\[
(4.5) \quad |I_1(\tilde{\Gamma}_i \tilde{v}_n)| + |I_3(\tilde{\Gamma}_i \tilde{v}_n)| \leq C \|\tilde{\Gamma}_i \tilde{v}_n\|_{L^2(D_T)}^2 \leq C \|\tilde{v}_n\|_{H^3(D_T)}^2 \leq C,
\]
where we used \(\tilde{v}_n \in C^3(D_T)\). By Lemma 2.2, we deduce from (2.36) that
\[
\|\tilde{\Gamma}_i \tilde{v}_n\|_{L^p(D_T)} \leq C \|D^1(\tilde{\Gamma}_i \tilde{v}_n)\|_{L^2(D_T)} \|\tilde{\Gamma}_i \tilde{v}_n\|_{H^{1-\alpha}(D_T)} + C \|\tilde{\Gamma}_i \tilde{v}_n\|_{L^2(D_T)} \leq C \|\tilde{v}_n\|_{H^4(D_T)} \|\tilde{v}_n\|_{H^{4-\alpha}(D_T)} + C \|\tilde{v}_n\|_{H^4(D_T)} \leq C \|\tilde{v}_n\|_{H^4(D_T)}^\alpha + C,
\]
where \(p = \frac{2-\alpha}{1-\alpha} \in (2, \infty)\). Let \(p_0 = \left(\frac{p}{2}\right) = \frac{p}{p-2} > 1\). Then
\[
(4.6) \quad |I_j(\tilde{\Gamma}_i \tilde{v}_n)| \leq \left( \int_0^1 \int_0^1 |\ln \tau|^{p_0} d\tau dx \right)^{\frac{1}{p_0}} \|\tilde{\Gamma}_i \tilde{v}_n\|_{L^p(D_T)}^2 \leq C \|\tilde{v}_n\|_{H^4(D_T)}^{2\alpha} + C
\]
for \(j = 2, 4\). By (4.4), (4.5) and (4.6), we have
\[
\left\| \int_0^1 |\tilde{\Gamma}_i \tilde{v}_n(x, s_n)| dt \right\|_{L^2(D_T)}^2 \leq C \|\tilde{v}_n\|_{H^4(D_T)}^{2\alpha} + C;
\]
and thus, \((2.43)\) holds for \(i = 1, 2, 3\). By \((2.12)\), \((2.38)\), \((2.36)\) and \((2.20)\), we have
\[
\|I_1\|_{L^2(D_T)} \leq C\|\tilde{v}_n\|_{H^4(D_T)}^\alpha + C, \quad \|I_2\|_{L^2(D_T)} \leq C
\]
for \(j = 2, 3, 4\), and thus, it follows from \((2.39)\) and \((2.38)\) that
\[
(2.47) \quad \left\| \partial_y^2 \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)} \leq C\|\tilde{v}_n\|_{H^4(D_T)}^\alpha + C \quad \implies \quad \left\| \partial_y^2 \Delta \tilde{v}_n \right\|_{L^2(D_T)} \leq C\|\tilde{v}_n\|_{H^4(D_T)}^\alpha + C.
\]
Taking derivative of \((2.8)\) with respect to \(y\) twice, we have
\[
\partial_y^2 \Delta \tilde{v}_n + \partial_y^3 \omega_n \left( \frac{\tilde{v}_n}{u_n - c_n} \right) + 2\partial_y^2 \omega_n \partial_y \left( \frac{\tilde{v}_n}{u_n - c_n} \right) + (\partial_y \omega_n + \beta) \partial_y^2 \left( \frac{\tilde{v}_n}{u_n - c_n} \right) = 0.
\]
Then we deduce from \((2.5)\), \((2.6)\) and \((2.37)\) that
\[
\left\| \partial_y^2 \Delta \tilde{v}_n \right\|_{L^2(D_T)} \leq \left\| \partial_y^3 \omega_n \right\|_{L^4(D_T)} \left\| \frac{\tilde{v}_n}{u_n - c_n} \right\|_{L^4(D_T)} + 2\left\| \partial_y^2 \omega_n \right\|_{C^0(D_T)} \left\| \partial_y \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)}
\]
\[
\leq C + C \left\| \partial_y^2 \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)}.
\]
By \((2.12)\) and direct computation, we have
\[
\partial_y^2 \left( \frac{\tilde{v}_n(x, y)}{u_n(x, y) - c_n} \right) = \int_0^1 \frac{\partial_y^3 \tilde{v}_n}{u_n} dt \frac{\partial_y^2 u_n}{\partial_y u_n} \left( \int_0^1 \partial_y u_n dt \right)^2 - 2 \int_0^1 \frac{\partial_y^2 \tilde{v}_n}{u_n} dt \frac{\partial_y^2 u_n}{\partial_y u_n} \left( \int_0^1 \partial_y u_n dt \right)^2
\]
\[
- \int_0^1 \frac{\partial_y \tilde{v}_n}{u_n} dt \int_0^1 \partial_y^3 u_n dt \frac{\partial_y^2 u_n}{\partial_y u_n} \left( \int_0^1 \partial_y u_n dt \right)^2 + 2 \int_0^1 \frac{\partial_y \tilde{v}_n}{u_n} dt \left( \int_0^1 \partial_y^2 u_n dt \right)^2 \left( \int_0^1 \partial_y u_n dt \right)^3
\]
\[
= III_1 + III_2 + III_3 + III_4.
\]
By \((2.6)\), we have \(\left\| \int_0^1 \partial_y^3 u_n dt \right\|_{L^\infty(D_T)} \leq C\), and thus, we deduce from \((2.13)\), \((2.43)\), \((2.36)\) and \((2.20)\) that
\[
\|III_1\|_{L^2(D_T)} \leq C \left\| \int_0^1 \partial_y^3 \tilde{v}_n dt \right\|_{L^2(D_T)} \leq C\|\tilde{v}_n\|_{H^4(D_T)}^\alpha + C,
\]
\[
(2.50) \quad \|III_j\|_{L^2(D_T)} \leq C \left\| \int_0^1 \partial_y^2 \tilde{v}_n dt \right\|_{L^2(D_T)} + C \left\| \int_0^1 \partial_y \tilde{v}_n dt \right\|_{L^2(D_T)} \leq C
\]
for \(j = 2, 3, 4\). Here we omit the variables \((x, s_n)\) for the derivatives of \(\tilde{v}_n\), \(u_n\) and the variable \(x\) for the derivative of \(y_n\) in \((2.39)\), \((2.41)\), \((2.42)\), \((2.49)\) and \((2.50)\). By \((2.49)\), \((2.50)\) and \((2.48)\), we have
\[
(2.51) \quad \left\| \partial_y^2 \left( \frac{\tilde{v}_n}{u_n - c_n} \right) \right\|_{L^2(D_T)} \leq C\|\tilde{v}_n\|_{H^4(D_T)}^\alpha + C \quad \implies \quad \left\| \partial_y^2 \Delta \tilde{v}_n \right\|_{L^2(D_T)} \leq C\|\tilde{v}_n\|_{H^4(D_T)}^\alpha + C.
\]
In summary, by the periodic boundary condition in \(x\) and Dirichlet boundary condition in \(y\) of \(\tilde{v}_n\), we deduce from \((2.47)\) and \((2.51)\) that
\[
\|\tilde{v}_n\|_{H^4(D_T)} \leq C\|\partial_y^2 \Delta \tilde{v}_n\|_{L^2(D_T)} + C\|\partial_y^2 \Delta \tilde{v}_n\|_{L^2(D_T)} \leq C\|\tilde{v}_n\|_{H^4(D_T)}^\alpha + C,
\]
Remark 2.3. 

where \( \alpha \in (0, 1) \). Thus, we have

\[
(2.52) \quad \| \tilde{v}_n \|_{H^4(D_T)} \leq C.
\]

By (2.52), there exists \( \tilde{v}_0 \in H^4(D_T) \) such that up to a subsequence, \( \tilde{v}_n \to \tilde{v}_0 \) in \( H^4(D_T) \), \( \tilde{v}_n \to \tilde{v}_0 \) in \( C^2(D_T) \) and \( \| \tilde{v}_0 \|_{L^2(D_T)} = 1 \). Since \( |\tilde{v}_0(x, c_0)| = |\tilde{v}_n(x, y_n(x)) - \tilde{v}_0(x, c_0)| \leq |\tilde{v}_n(x, y_n(x)) - \tilde{v}_n(x, c_0)| + |\tilde{v}_n(x, c_0) - \tilde{v}_0(x, c_0)| \leq \| \partial_x \tilde{v}_n \|_{L^\infty(D_T)}|y_n(x) - c_0| + |\tilde{v}_n(x, c_0) - \tilde{v}_0(x, c_0)| \to 0 \), we have \( \tilde{v}_0(x, c_0) = 0 \) for \( x \in [0, T] \). We claim that \( \frac{\tilde{v}_0}{u_n - c_n} \to \frac{\tilde{v}_0}{y - c_0} \) in \( C^0(D_T) \). In fact, by (2.13), (2.16), (2.52), (2.3) and the fact that \( \tilde{v}_n \to \tilde{v}_0 \) in \( C^1(D_T) \), we have

\[
\begin{aligned}
&\left| \frac{\tilde{v}_n(x, y) - \tilde{v}_0(x, y)}{u_n(x, y) - c_n} - \frac{\tilde{v}_0(x, y)}{y - c_0} \right| = \left| \frac{1}{\int_0^1 \partial_y \tilde{v}_n(x, s_n) dt} - \int_0^1 \partial_y \tilde{v}_0(x, ty + (1 - t) c_0) dt \right| \\
&\leq \int_0^1 \left| \partial_y \tilde{v}_n(x, s_n) dt \right| \cdot \left| \frac{1}{\int_0^1 \partial_y u_n(x, s_n) dt} - 1 \right| + \int_0^1 \left| \partial_y \tilde{v}_n(x, s_n) - \partial_y \tilde{v}_0(x, ty + (1 - t) c_0) \right| dt \\
&\to 0, \quad n \to \infty
\end{aligned}
\]

uniformly for \((x, y) \in D_T\). By (2.4), \( \partial_y \omega_n \to 0 \) in \( C^0(D_T) \). Since \( \tilde{v}_n \to \tilde{v}_0 \) in \( C^2(D_T) \) and \( \frac{\tilde{v}_n}{u_n - c_n} \to \frac{\tilde{v}_0}{y - c_0} \) in \( C^0(D_T) \), we now send \( n \to \infty \) in (2.8) to get

\[
\Delta \tilde{v}_0 + \frac{\beta}{y - c_0} \tilde{v}_0 = 0.
\]

Since \( \tilde{v}_0 \) is \( T \)-periodic in \( x \), we have \( \tilde{v}_0(x, y) = \sum_{k \in \mathbb{Z}} \tilde{v}_{0,k}(y) e^{ik\alpha x} \neq 0 \), where \( \alpha = \frac{2\pi}{T} \). Since \( \tilde{v}_0 \in H^4(D_T) \) and \( \| \tilde{v}_0 \|_{L^2(D_T)} = 1 \), there exists \( k_0 \in \mathbb{Z} \) such that \( 0 \neq \tilde{v}_{0,k_0} \in H^4(-1, 1) \) solves

\[
-\tilde{v}_{0,k_0} - \frac{\beta}{y} \tilde{v}_{0,k_0} = -(k_0 \alpha)^2 \tilde{v}_{0,k_0}, \quad \tilde{v}_{0,k_0}(\pm 1) = 0.
\]

If \( k_0 \neq 0 \), this is a contradiction to Theorem 1 in [21]. If \( k_0 = 0 \), since \( \tilde{v}_n = v_n / \| v_n \|_{L^2(D_T)} = -\partial_x \psi_n / \| v_n \|_{L^2(D_T)} \), we have \( \frac{1}{T} \int_0^T \tilde{v}_n dx = 0 \). Taking limit as \( n \to \infty \), we have \( \frac{1}{T} \int_0^T \tilde{v}_0 dx = 0 \) and thus, \( \tilde{v}_{0,0}(y) = \frac{1}{T} \int_0^T \tilde{v}_0 dx = 0 \), which is also a contradiction. \( \square \)

Remark 2.3. (1) \( \frac{\tilde{v}_n}{u_n - c_n}, n \geq 1 \), have a uniform \( H^2 \) bound due to (2.37), (2.47), (2.51) and (2.52).

(2) The uniform bound of the left hand side of (2.21) can be also obtained from (2.36).

Here, we use the factor \( t \) in (2.22) to give a direct proof of (2.21) since the left hand side of (2.21) can be bounded by the \( H^2 \) bound of \( \tilde{v}_n, n \geq 1 \).

3. The principal eigenvalues of the singular Rayleigh-Kuo BVP

In this section, we study the principal eigenvalues of the singular Rayleigh-Kuo BVP for Couette flow, which determine the borderlines \( \Gamma_- \cup \Gamma_+ \) in Figure 1. Dynamics near Couette flow is different if \((\alpha, \beta)\) passes through the borderlines (see Theorem 1.3).

First, we prove the existence, monotonicity and continuity of the principal eigenvalues \( \lambda_1(\beta, c) \) of the singular Rayleigh-Kuo BVP (1.5) with \( u(y) = y, c = \pm 1 \). We mainly discuss
$\beta \geq 0$ and $c = -1$, and give the analog result by symmetry for $\beta \leq 0$ and $c = 1$. Recall that $\lambda_1(\beta, -1)$ is precisely defined in (1.6). Then we have the following basic property of $\lambda_1(\beta, -1)$.

**Lemma 3.1.** Let $\beta \geq 0$. Then the infimum in (1.3) is attained by some function $\phi_\beta \in H_0^1(\mathbb{R})$, $\|\phi_\beta\|_{L^2(\mathbb{R})} = 1$. Moreover, there exists $m_\beta > -\infty$ such that $m_\beta \leq \lambda_1(\beta, -1) \leq \frac{\pi^2}{4}$.

**Proof.** Let $\{\phi_n\}$ be a minimizing sequence with $\|\phi_n\|_{L^2(\mathbb{R})} = 1$ and

$$
\int_{-1}^{1} \left| \phi_n' \right|^2 \, dy = \beta \left( \ln(y + 1) \right)_{y=-1}^{1} - \frac{\beta}{y+1} \left| \phi_n \right|^2 \, dy \rightarrow \lambda_1(\beta, -1) \quad \text{as} \quad n \rightarrow \infty.
$$

Let $q = 2$, $m = 1$, $r = 2$, $j = 0$, $p = 4$, $\alpha = \frac{1}{4}$, $G = (-1, 1)$ and $u = \phi_n$ in Lemma 2.2 we have

$$
\|\phi_n\|_{L^q(\mathbb{R})} \leq C_1 \|\phi_n\|_{L^2(\mathbb{R})}^\frac{3}{2} \|\phi_n'\|_{L^2(\mathbb{R})}^\frac{1}{2} + C_2 \|\phi_n\|_{L^2(\mathbb{R})},
$$

Since $|\ln(y+1)||\phi_n(y)|^2 \leq (y+1)|\ln(y+1)||\phi_n'\|_{L^2(\mathbb{R})} \rightarrow 0$ as $y \rightarrow -1^+$ for fixed $n \geq 1$, we have by (3.2) that

$$
\int_{-1}^{1} \frac{\beta}{y+1} \left| \phi_n \right|^2 \, dy = \beta \left( \ln(y + 1) \right)_{y=-1}^{1} - \frac{\beta}{y+1} \left| \phi_n \right|^2 \, dy \leq 2\beta \left( \ln(y + 1) \right)_{y=-1}^{1} - \frac{\beta}{y+1} \left| \phi_n \right|^2 \, dy \leq C(1) \|\phi_n\|_{L^2(\mathbb{R})}^\frac{3}{2} \|\phi_n'\|_{L^2(\mathbb{R})}^\frac{1}{2} + C_2 \|\phi_n\|_{L^2(\mathbb{R})}.
$$

Moreover, there exists $1 = \|\phi_n\|_{L^2(\mathbb{R})} \leq C\|\phi_n'\|_{L^2(\mathbb{R})}$. By (3.1), (3.3) and the fact that $\lambda_1(\beta, -1) \leq \frac{\pi^2}{4}$, there exists $M > 0$ such that

$$
M \geq \|\phi_n'\|_{L^2(\mathbb{R})}^2 - \int_{-1}^{1} \frac{\beta}{y+1} \left| \phi_n \right|^2 \, dy \geq \|\phi_n'\|_{L^2(\mathbb{R})}^2 - C\|\phi_n'\|_{L^2(\mathbb{R})}^\frac{5}{2},
$$

which implies

$$
\|\phi_n'\|_{L^2(\mathbb{R})}^2 \leq C \implies \|\phi_n\|_{H^1(\mathbb{R})}^2 \leq C.
$$

Thus, there exists $\phi_\beta \in H_0^1(\mathbb{R})$ such that up to a subsequence, $\phi_n \rightharpoonup \phi_\beta$ in $H_0^1(\mathbb{R})$ and $\phi_n \rightarrow \phi_\beta$ in $L^4(\mathbb{R})$, and

$$
\|\phi_\beta\|_{H^1(\mathbb{R})}^2 \leq \limsup_{n \rightarrow \infty} \|\phi_n\|_{H^1(\mathbb{R})}^2 \implies \|\phi_\beta\|_{L^2(\mathbb{R})}^2 \leq \liminf_{n \rightarrow \infty} \|\phi_n'\|_{L^2(\mathbb{R})}^2.
$$

Note that

$$
\lim_{n \rightarrow \infty} \int_{-1}^{1} \frac{1}{y+1} \left| \phi_n \right|^2 \, dy = \int_{-1}^{1} \frac{1}{y+1} \left| \phi_\beta \right|^2 \, dy.
$$

In fact, since $|\ln(y+1)||\phi_n(y)|^2 - |\phi_\beta(y)|^2 | \leq (y+1)|\ln(y+1)||\phi_n'\|_{L^2(\mathbb{R})}^2 + |\phi_\beta'\|_{L^2(\mathbb{R})}^2 \rightarrow 0$ as $y \rightarrow -1^+$, we have

$$
\int_{-1}^{1} \frac{1}{y+1} \left( |\phi_n|^2 - |\phi_\beta|^2 \right) \, dy = \ln(y+1)(|\phi_n|^2 - |\phi_\beta|^2) \bigg|_{y=-1}^{1} - \int_{-1}^{1} 2\ln(y+1)(\phi_n\phi_n' - \phi_\beta\phi_\beta') \, dy = - \int_{-1}^{1} 2\ln(y+1)(\phi_n - \phi_\beta)\phi_n' \, dy - \int_{-1}^{1} 2\ln(y+1)(\phi_\beta(\phi_n' - \phi_\beta')) \, dy.
$$


Note that
\[ |I_1| \leq \| \ln(y + 1) \|_{L^1((-1, 1))} \| \phi_n - \phi_\beta \|_{L^1((-1, 1))} \| \phi'_n \|_{L^2((-1, 1))} \leq C \| \phi_n - \phi_\beta \|_{L^1((-1, 1))} \to 0. \]

Define a functional \( f \) on \( H^1_0(-1, 1) \) by
\[ \langle f, \varphi \rangle = \int_{-1}^{1} \ln(y + 1) \phi_\beta \varphi' \, dy, \quad \varphi \in H^1_0(-1, 1). \]

Then
\[ \langle f, \varphi \rangle \leq \| \ln(y + 1) \|_{L^1((-1, 1))} \| \phi_\beta \|_{L^1((-1, 1))} \| \varphi' \|_{L^2((-1, 1))} \leq C \| \phi_\beta \|_{H^1((-1, 1))} \| \varphi \|_{H^1((-1, 1))} \leq C \| \varphi \|_{H^1((-1, 1))}. \]

Thus, \( f \) is a bounded functional on \( H^1_0(-1, 1) \) and \( f \in H^{-1}(-1, 1) \). Since \( \phi_n \to \phi_\beta \) in \( H^1_0(-1, 1) \), we have
\[ I_2 = -2 \langle f, \phi_n - \phi_\beta \rangle \to 0 \quad \text{as} \quad n \to \infty. \]

This proves (3.6). By (3.5), (3.6) and (3.1), we have
\[ \int_{-1}^{1} \left( |\phi'_\beta|^2 - \frac{\beta}{y + 1} |\phi_\beta|^2 \right) \, dy \leq \liminf_{n \to \infty} \int_{-1}^{1} \left( |\phi'_n|^2 - \frac{\beta}{y + 1} |\phi_n|^2 \right) \, dy = \lambda_1(\beta, -1). \]

Thus,
\[ \lambda_1(\beta, -1) = \int_{-1}^{1} \left( |\phi'_\beta|^2 - \frac{\beta}{y + 1} |\phi_\beta|^2 \right) \, dy. \]

\( \phi_\beta \) is called an eigenfunction of \( \lambda_1(\beta, -1) \) in the sequel.

For any \( \phi \in H^1_0(-1, 1) \) with \( \| \phi \|_{L^2((-1, 1))} = 1 \), similar to (3.4), there exists \( m_\beta > -\infty \) such that
\[ \int_{-1}^{1} \left( |\phi'|^2 - \frac{\beta}{y + 1} |\phi|^2 \right) \, dy \geq \| \phi' \|^2_{L^2((-1, 1))} - C_\beta \| \phi' \|^2_{L^2((-1, 1))} \geq m_\beta. \]

For \( \beta > 0 \), we have \( \lambda_1(\beta, -1) \leq \lambda_1(0, -1) = \frac{\pi^2}{4} \). Thus,
\[ (3.7) \quad m_\beta \leq \lambda_1(\beta, -1) \leq \frac{\pi^2}{4}, \]
which implies that \( \lambda_1(\beta, -1) \) is finite for \( \beta > 0 \).

Then we get the monotonicity of \( \lambda_1(\cdot, -1) \).

**Corollary 3.2.** \( \lambda_1(\cdot, -1) \) is decreasing on \([0, \infty)\).

**Proof.** Let \( 0 \leq \beta_1 < \beta_2 < \infty \). By Lemma 3.1 \( \lambda_1(\beta_1, -1) \) is attained by \( \phi_{\beta_1} \in H^1_0(-1, 1) \) with \( \| \phi_{\beta_1} \|_{L^2} = 1 \). Then
\[ \lambda_1(\beta_1, -1) = \int_{-1}^{1} \left( |\phi'_{\beta_1}|^2 - \frac{\beta_1}{y + 1} |\phi_{\beta_1}|^2 \right) \, dy \geq \int_{-1}^{1} \left( |\phi'_{\beta_1}|^2 - \frac{\beta_2}{y + 1} |\phi_{\beta_1}|^2 \right) \, dy \geq \lambda_1(\beta_2, -1). \]

Next, we consider the continuity of \( \lambda_1(\cdot, -1) \).

**Lemma 3.3.** \( \lambda_1(\cdot, -1) \) is continuous on \([0, \infty)\).
Proof. First, we consider the left continuity of $\lambda_1(\cdot, -1)$ at $\beta_0 \in (0, \infty)$. By Lemma 3.1, $\lambda_1(\beta_0, -1)$ is attained by $\phi_{\beta_0} \in H^1_0(-1, 1)$. Since $|\phi_{\beta_0}(y)|^2 \leq |\phi_{\beta_0}'|^2_{L^2(-1, 1)}(y + 1)$, we have
$$\int_{-1}^{1} \frac{1}{y+1}|\phi_{\beta_0}'|^2 dy \leq 2|\phi_{\beta_0}'|^2_{L^2(-1, 1)} < \infty.$$ For $\varepsilon > 0$, there exists $\delta > 0$ such that
$$\delta \int_{-1}^{1} \frac{1}{y+1}|\phi_{\beta_0}'|^2 dy < \varepsilon.$$ By the monotonicity of $\lambda_1(\cdot, -1)$, we have
$$0 \leq \lambda_1(\beta, -1) - \lambda_1(\beta_0, -1) \leq \int_{-1}^{1} \left( |\phi_{\beta}'|^2 - \frac{\beta}{y+1}|\phi_{\beta}'|^2 \right) dy - \int_{-1}^{1} \left( |\phi_{\beta_0}'|^2 - \frac{\beta_0}{y+1}|\phi_{\beta_0}'|^2 \right) dy = (\beta_0 - \beta) \int_{-1}^{1} \frac{1}{y+1}|\phi_{\beta_0}'|^2 dy < \delta \int_{-1}^{1} \frac{1}{y+1}|\phi_{\beta_0}'|^2 dy < \varepsilon$$ for $0 < \beta_0 - \beta < \delta$.

Next, we consider the right continuity of $\lambda_1(\cdot, -1)$ at $\beta_0 \in [0, \infty)$. We claim that
$$\lambda_1(\cdot, -1) \leq C \text{ for } \beta \in [\beta_0, \beta_0 + 1].$$

In fact, since $\lim_{y \to -1+} \ln(y + 1)|\phi_{\beta}(y)|^2 = 0$, by Lemma 2.2 and $1 = |\phi_{\beta}|_{L^2(-1, 1)} \leq C|\phi_{\beta}'|_{L^2(-1, 1)}$ we have
$$\int_{-1}^{1} \frac{1}{y+1}|\phi_{\beta}'|^2 dy = -\int_{-1}^{1} \ln(y + 1)|\phi_{\beta}'|^2 dy \leq \ln(y + 1)|||\phi_{\beta}'|||_{L^2(-1, 1)}|||\phi_{\beta}'|||_{L^2(-1, 1)} \leq C(C_1|||\phi_{\beta}'|||_{L^2(-1, 1)}^2 + C_2|||\phi_{\beta}'|||_{L^2(-1, 1)}^2) \leq C|||\phi_{\beta}'|||_{L^2(-1, 1)}^2.$$ This, along with (3.7), gives
$$\frac{\pi^2}{4} \geq \frac{\pi^2}{4} - \beta C|||\phi_{\beta}'|||_{L^2(-1, 1)}^2 \geq \frac{\pi^2}{4} - (\beta_0 + 1)C|||\phi_{\beta}'|||_{L^2(-1, 1)}^2$$ for $\beta \in [\beta_0, \beta_0 + 1]$. This proves (3.8). By (3.8), we have $\int_{-1}^{1} \frac{1}{y+1}|\phi_{\beta}'|^2 dy \leq 2|||\phi_{\beta}'|||_{L^2(-1, 1)}^2 < 2C$ uniformly for $\beta \in [\beta_0, \beta_0 + 1]$. For $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that $\delta \int_{-1}^{1} \frac{1}{y+1}|\phi_{\beta}'|^2 dy < 2\delta C < \varepsilon$ for $\beta \in [\beta_0, \beta_0 + 1]$. By the monotonicity of $\lambda_1(\cdot, -1)$, we have
$$0 \leq \lambda_1(\beta_0, -1) - \lambda_1(\beta, -1) \leq \int_{-1}^{1} \left( |\phi_{\beta}'|^2 - \frac{\beta_0}{y+1}|\phi_{\beta}'|^2 \right) dy - \int_{-1}^{1} \left( |\phi_{\beta_0}'|^2 - \frac{\beta_0}{y+1}|\phi_{\beta_0}'|^2 \right) dy = (\beta_0 - \beta) \int_{-1}^{1} \frac{1}{y+1}|\phi_{\beta_0}'|^2 dy < \delta \int_{-1}^{1} \frac{1}{y+1}|\phi_{\beta_0}'|^2 dy < \varepsilon$$ for $0 < \beta - \beta_0 < \delta$. 

For $\beta \geq 0$ and $c < -1$,
$$\lambda_1(\beta, c) = \inf_{\phi \in H^1_0(-1, 1), ||\phi||_{L^2(-1, 1)} = 1} \int_{-1}^{1} \left( |\phi'|^2 - \frac{\beta}{y-c}|\phi|^2 \right) dy$$ is the principal eigenvalue of the regular Rayleigh-Kuo BVP
$$-\phi'' - \frac{\beta}{y-c}\phi = \lambda\phi, \quad \phi(\pm 1) = 0,$$ where $\phi \in H^2 \cap H^1_0(-1, 1)$. Then we have the following result.
Lemma 3.4. (1) $\lambda_1(\beta, \cdot)$ is decreasing on $(-\infty, -1]$ for fixed $\beta > 0$.
(2) $\lim_{\beta \to \infty} \lambda_1(\beta, c) = -\infty$ for fixed $c \leq -1$.

Proof. (1) For $c_1 < c_2 \leq -1$, we have

$$\lambda_1(\beta, c_1) = \int_{-1}^{1} \left( |\phi'_{\beta,c_1}|^2 - \frac{\beta}{y-c_1} |\phi_{\beta,c_1}|^2 \right) dy > \int_{-1}^{1} \left( |\phi'_{\beta,c_2}|^2 - \frac{\beta}{y-c_2} |\phi_{\beta,c_2}|^2 \right) dy \geq \lambda_1(\beta, c_2),$$

where $\phi_{\beta,c_1} \in H^1_0(-1, 1)$ is an eigenfunction of $\lambda_1(\beta, c_1)$ with $\|\phi_{\beta,c_1}\|_{L^2(-1,1)} = 1$.

(2) Let $\varphi$ be a smooth function such that $\text{supp}(\varphi) \subset \left(\frac{1}{4}, \frac{3}{4}\right)$ and $\|\varphi\|_{L^2(-1,1)} = 1$. Then (2) can be obtained by

$$\lambda_1(\beta, c) \leq \int_{\frac{1}{4}}^{\frac{3}{4}} \left( |\varphi|^2 - \frac{\beta}{y-c} |\varphi|^2 \right) dy \to -\infty \text{ as } \beta \to \infty.$$  

□

We list some properties of $\lambda_1(\beta, c)$ for $\beta \geq 0$ and $c \leq -1$.

• For fixed $c < -1$, $\lambda_1(\cdot, c)$ is smooth and decreasing on $[0, \infty)$, see Lemma 11 in [21].
• For $c = -1$, $\lambda_1(\cdot, c)$ is continuous and decreasing on $[0, \infty)$, see Corollary 3.2 and Lemma 3.3.
• For fixed $c \leq -1$, $\lambda_1(0, c) = \frac{\pi^2}{4}$ and $\lim_{\beta \to \infty} \lambda_1(\beta, c) = -\infty$, see Lemma 3.4 (2).
• For fixed $\beta > 0$, $\lambda_1(\beta, \cdot)$ is continuous and decreasing on $(-\infty, -1]$, see Lemma 3.4 (1) and Lemma 3.6.

Thus, there exists a unique $\beta(c) > 0$ such that $\lambda_1(\beta(c), c) = 0$ for $c \leq -1$. Now, we define the transitional value as

$$\beta_* = \inf_{c \in (-\infty, -1]} \beta(c).$$

Then we claim that $\beta = \beta_*$ is the unique point such that $\lambda_1(\beta, -1) = 0$.

Lemma 3.5.

$$\beta_* = \beta(-1) \in (0, \infty).$$

Proof. By Lemma 3.4 (1), we have

$$\lambda_1(\beta(-1), c) > \lambda_1(\beta(-1), -1) = 0 = \lambda_1(\beta(c), c)$$

for $c < -1$, which implies that $\beta(-1) < \beta(c)$ by Lemma 11 in [21]. □

Then we study the asymptotic behavior of the principal eigenvalues for regular approximations of singular Rayleigh-Kuo BVP with $c \to -1^-$. 

Lemma 3.6. $\lim_{c \to -1^-} \lambda_1(\beta, c) = \lambda_1(\beta, -1)$ for fixed $\beta > 0$.

Proof. By Lemma 3.4 $\lambda_1(\beta, -1) \leq \lim_{c \to -1^-} \lambda_1(\beta, c)$. Conversely, let $\phi_{\beta}$ be an eigenfunction of $\lambda_1(\beta, -1)$ with $\|\phi_{\beta}\|_{L^2(-1,1)} = 1$. For $c < -1$, we have

$$\lim_{c \to -1^-} \int_{-1}^{1} \frac{1}{y-c} |\phi_{\beta}|^2 dy = \int_{-1}^{1} \frac{1}{y+1} |\phi_{\beta}|^2 dy.$$  

In fact,

$$\left| \int_{-1}^{1} \left( \frac{1}{y-c} - \frac{1}{y+1} \right) |\phi_{\beta}|^2 dy \right| = \left| \int_{-1}^{1} \left( \frac{1+c}{(y-c)(y+1)} \right) |\phi_{\beta}|^2 dy \right|.$$
\[ \leq |1 + c|^{-1} \| \rho' \|_{L^2(-1,1)}^2 \int_{-1}^{1} \frac{1}{y - c} dy = |1 + c|^{-1} \| \rho' \|_{L^2(-1,1)}^2 (\ln(1 - c) - \ln(-1 - c)) \to 0 \]
as \( c \to -1^- \). Taking limit in
\[ \lambda_1(\beta, c) \leq \int_{-1}^{1} \left( |\rho'|^2 - \frac{\beta}{y - c} |\phi| \right) dy, \]
by (3.12) we have
\[ \lim_{c \to -1^-} \lambda_1(\beta, c) \leq \int_{-1}^{1} \left( |\rho'|^2 - \frac{\beta}{y + 1} |\phi| \right) dy = \lambda_1(\beta, -1). \]

Then we point out the difference of the principal eigenvalues of the Rayleigh-Kuo BVP before and after \( \beta^* \).

**Proposition 3.7.** (1) Let \( \beta > \beta^* \). Then there exists \( c_{\beta} < -1 \) such that \( \lambda_1(\beta, c_{\beta}) = 0 \) and \( \{\lambda_1(\beta, c) | c \in (c_{\beta}, -1)\} = (\lambda_1(\beta, -1), 0) \).
(2) Let \( 0 < \beta \leq \beta^* \). Then \( \lambda_1(\beta, c) \geq 0 \) for \( c \leq -1 \).

**Remark 3.8.** We get more precise conclusions for the range of the principal eigenvalue.
(1) Fix \( \beta > 0 \). Since \( \lim_{c \to -\infty} \lambda_1(\beta, c) = \frac{\pi^2}{4} \) and \( \lim_{c \to -1^-} \lambda_1(\beta, c) = \lambda_1(\beta, -1) \) by Lemma 3.6, we infer from Lemma 3.4 (1) that \( \{\lambda_1(\beta, c) | c \in (-\infty, -1]\} = \left[ \lambda_1(\beta, -1), \frac{\pi^2}{4} \right) \).
(2) For any \( (\lambda_0, \beta_0) \in I := \{(\lambda, \beta) | \beta \in (0, \infty), \lambda \in \left( \lambda_1(\beta, -1), \frac{\pi^2}{4} \right)\} \), there exists a unique \( c_0 \in (-\infty, -1]\) such that \( \lambda_1(\beta_0, c_0) = \lambda_0 \). In particular, for any \( (\lambda_0, \beta_0) \in \tilde{I} := \{(\lambda, \beta) | \beta \in (\beta^*, \infty), \lambda \in (\lambda_1(\beta, -1), 0)\} \), there exists a unique \( c_0 \in (c_{\beta_0}, -1) \) such that \( \lambda_1(\beta_0, c_0) = \lambda_0 \). See Figure 2.

**Figure 2.**

**Proof.** (1) Since \( \beta > \beta^* \), by Corollary 3.2 we have \( \lambda_1(\beta, -1) < 0 \). The conclusion then follows from Lemma 3.4 (1), Lemma 3.6 and the fact that \( \lim_{c \to -\infty} \lambda_1(\beta, c) = \frac{\pi^2}{4} \).
Similarly, we have \( \lambda_1(\beta, -1) \geq 0 \). By Lemma 3.3 (1), we have \( \lambda_1(\beta, c) \geq \lambda(\beta, -1) \geq 0 \) for \( c \leq -1 \).

Similarly, for \( \beta \leq 0 \), we can define

\[
\lambda_1(\beta, 1) = \inf_{\phi \in H^1_0((-1,1), \|\phi\|_{L^2((-1,1))}=1} \int_{-1}^{1} \left( |\phi'|^2 - \frac{\beta}{y - 1} |\phi|^2 \right) dy.
\]

For \( \beta \leq 0 \) and \( c > 1 \), \( \lambda_1(\beta, c) \) defined in (3.13) is the principal eigenvalue of the regular Rayleigh-Kuo BVP (3.10). We list some properties of \( \lambda_1(\beta, c) \) for \( \beta \leq 0 \) and \( c \geq 1 \), whose proof is similar as above, and thus, omitted here.

**Lemma 3.9.** (1) Let \( \beta \leq 0 \). Then the infimum in (3.13) is attained by some function \( \phi_\beta \in H^1_0((-1,1), \|\phi\|_{L^2((-1,1))}=1 \). Moreover, there exists \( m_\beta > -\infty \) such that \( m_\beta \leq \lambda_1(\beta, 1) \leq \frac{\pi^2}{4} \).

(2) For fixed \( c > 1 \), \( \lambda_1(\cdot, c) \) is smooth and increasing on \( \beta \in (-\infty, 0] \).

(3) For \( c = 1 \), \( \lambda_1(\cdot, c) \) is continuous and increasing on \( \beta \in (-\infty, 0] \).

(4) For fixed \( c \geq 1 \), \( \lambda_1(0, c) = \frac{\pi^2}{4} \) and \( \lim_{\beta \to -\infty} \lambda_1(\beta, c) = -\infty \).

(5) For fixed \( \beta < 0 \), \( \lim_{c \to 1^+} \lambda_1(\beta, c) = \lambda_1(\beta, 1) \), and \( \lambda_1(\beta, \cdot) \) is continuous and increasing on \( c \in [1, \infty) \).

By Lemma 3.9 (2)-(4), there exists a unique \( \beta^- (c) < 0 \) such that \( \lambda_1(\beta^-(c), c) = 0 \) for \( c \geq 1 \). Define

\[
\beta_\beta^- = \sup_{c \in [1, +\infty)} \beta^-(c).
\]

Then \( \beta^-_\beta \) has the following analog properties as \( \beta_\beta^- \), whose proof is similar to Lemma 3.5 and Proposition 3.7.

**Proposition 3.10.** (1) \( \beta^-_\beta = \beta(1) \in (-\infty, 0) \).

(2) Let \( \beta < \beta^-_\beta \). Then there exists \( c_\beta > 1 \) such that \( \lambda_1(\beta, c_\beta) = 0 \) and \( \{ \lambda_1(\beta, c) | c \in (1, c_\beta) \} = (\lambda_1(\beta, 1), 0) \).

(3) Let \( \beta^-_\beta \leq \beta < 0 \). Then \( \lambda_1(\beta, c) \geq 0 \) for \( c \geq 1 \).

Symmetry of the principal eigenvalue \( \lambda_1(\beta, c) \) could be deduced by that of Couette flow.

**Lemma 3.11.** For \( \beta \geq 0 \) and \( c \leq -1 \), we have

\[
\lambda_1(\beta, c) = \lambda_1(-\beta, -c).
\]

**Proof.** Let \( \phi_0 \in H^1_0((-1,1)) \) be an eigenfunction of \( \lambda_1(\beta, c) \) with \( \|\phi_0\|_{L^2((-1,1))}=1 \), and define \( \tilde{\phi}_0(y) = \phi_0(-y) \) for \( y \in [-1, 1] \). Then

\[
\lambda_1(\beta, c) = \int_{-1}^{1} \left( |\phi_0'(y)|^2 - \frac{\beta}{y - c} |\phi_0(y)|^2 \right) dy
\]

\[
= \int_{-1}^{1} \left( |\phi_0'(-z)|^2 - \frac{\beta}{-z - c} |\phi_0(-z)|^2 \right) dz
\]

\[
= \int_{-1}^{1} \left( |\tilde{\phi}_0'(|z|)|^2 - \frac{\beta}{z - (c)} |\tilde{\phi}_0(|z|)|^2 \right) dz
\]

\[
\geq \inf_{\phi \in H^1_0((-1,1), \|\phi\|_{L^2((-1,1))}=1} \int_{-1}^{1} \left( |\phi'|^2 - \frac{-\beta}{y - (c)} |\phi|^2 \right) dy
\]

\[
= \lambda_1(-\beta, -c).
\]

Similarly, we have \( \lambda_1(-\beta, -c) \geq \lambda_1(\beta, c) \).
Lemma 3.12. \( \beta^- = -\beta^+ \).

Proof. Note that \( \lambda_1(\beta(1),1) = 0 = \lambda_1(\beta(-1),-1) \). By Corollary 3.2 and Lemma 3.9 (3), we infer from Lemma 3.11 that \( -\beta(1) = \beta(-1) \). Thus, \( \beta^- = \beta(1) = -\beta(-1) = -\beta^+ \) by Proposition 3.10 (1) and Lemma 3.5.

Combining Propositions 3.7 and 3.10 (2)-(3), we get the following result.

Proposition 3.13. (1) Let \( |\beta| > \beta^+ \). Then there exists \( |c_\beta| > 1 \) such that \( \beta c_\beta < 0 \), \( \lambda_1(\beta,c_\beta) = 0 \) and \( \{\lambda_1(\beta,c)||c|| \in (1,|c_\beta|), \beta c < 0\} = (\lambda_1(|\beta|,-1),0) \).

(2) Let \( 0 < |\beta| \leq \beta^+ \). Then \( \lambda_1(\beta,c) \geq 0 \) for \( |c| \geq 1 \).

Proof. It suffices to prove for the case \( \beta c > 0 \) in (2). Since \( -\frac{\beta}{y-c} > 0 \) for \( y \in (-1,1) \), the conclusion then follows from (3.9).

4. Contrasting dynamics near Couette flow for \( 0 < |\beta| \leq \beta^+ \) and \( |\beta| > \beta^+ \).

In this section, we prove Theorem 1.13 which gives contrasting dynamics near Couette flow between \((\alpha,\beta) \in O \) and \((\alpha,\beta) \in I_\pm \), see Figure 1.

Proof of Theorem 1.13 First, we give the proof of (1) and (2ii) for \( \beta > 0 \), and the proof for \( \beta < 0 \) is similar. Suppose otherwise, there exist \( \{\varepsilon_n\}_{n=1}^\infty \), \( c_\beta \in \mathbb{R} \) and traveling wave solutions \((u_n(x-c_\beta t,y),v_n(x-c_\beta t,y))\) to the \( \beta \)-plane equation (1.1)-(1.3) such that \( \varepsilon_n \to 0 \), \((u_n,v_n)\) is \( T \)-periodic in \( x \), \( \| (u_n,v_n) - (y,0) \|_{H^s(D_T)} \leq \varepsilon_n \) and \( \|v_n\|_{L^2(D_T)} \neq 0 \). Then \((u_n,v_n)\) solves (2.2). \( s \geq 5 \) implies (2.3)-(2.7) holds for \( n \) sufficiently large. Let \( \tilde{v}_n = v_n/\|v_n\|_{L^2(D_T)} \). By (2.2) and the fact that \( \partial_x \omega_n = \Delta v_n \), we have

\[
\Delta \tilde{v}_n + (\partial_y \omega_n + \beta) \frac{\tilde{v}_n}{u_n - c_\beta} = 0.
\]

Then up to a subsequence, \( c_n \to c_0 \in \mathbb{R} \cup \{\pm \infty\} \), and we divide the proof into four cases in terms of \( c_0 \).

Case 1. \( |c_0| = \infty \).

Up to a subsequence, we can assume that \( |c_0| > M_n + 2 \) with \( 0 < M_n \to \infty \). By (2.3)-(2.4), we have \( \|u_n\|_{C^0(D_T)} < \frac{\sqrt{2}}{2} \) and \( \|\partial_y \omega_n\|_{C^0(D_T)} < 1 \) for \( n \) large enough, and thus,

\[
\|\Delta \tilde{v}_n\|_{L^2(D_T)} \leq \frac{|\beta| + 1}{M_n} \|\tilde{v}_n\|_{L^2(D_T)} \to 0.
\]

However, by periodic boundary condition in \( x \) and Dirichlet boundary condition in \( y \) of \( \tilde{v}_n \), we have

\[
1 \leq \|\tilde{v}_n\|_{H^2(D_T)} \leq C\|\Delta \tilde{v}_n\|_{L^2(D_T)},
\]

which is a contradiction.

Case 2. \( |c_0| > 1 \).

By Lemma 2.5 in [20] and \( |c_0| > 1 \), \( c_0 \in \cup_{k \geq 1} (\sigma_d(R_{k\alpha,\beta}) \cap \mathbb{R}) \). Thus, \( \lambda_1(\beta,c_0) = -(k_0\alpha)^2 < 0 \) for some \( k_0 \geq 1 \), where \( \lambda_1(\beta,c_0) \) is the principal eigenvalue of the regular Rayleigh-Kuo BVP (3.10) with \( c = c_0 \).

For (1), by Proposition 3.13 (2), we have \( \lambda_1(\beta,c_0) \geq 0 \) for \( 0 < \beta \leq \beta^+ \), which is a contradiction.

For (2ii), since \( k_0\alpha = \frac{2\alpha\pi}{y} > \alpha_\beta = \sqrt{-\lambda_1(\beta,-1)} \), we have \( \lambda_1(\beta,c_0) = -(k_0\alpha)^2 < \lambda_1(\beta,-1) \). If \( c_0 > 1 \), then \( \lambda_1(\beta,c_0) \geq 0 \) by (3.9), which contradicts \( \lambda_1(\beta,c_0) < 0 \). If \( c_0 < -1 \), then \( \lambda_1(\beta,c_0) > \lambda_1(\beta,-1) \) by Lemma 3.4 (1), which contradicts \( \lambda_1(\beta,c_0) < \lambda_1(\beta,-1) \).

Case 3. \( c_0 = \pm 1 \).
We only give the proof for $c_0 = -1$ and the other is similar. By (2.7), $u_n(x, \cdot)$ is increasing on $[-1, 1]$ for $n$ large enough and $x \in [0, T]$. Divide $[0, T]$ into three subsets

$$P_n = \{x|c_n \leq u_n(x, -1)\}, \quad Q_n = \{x|c_n \geq u_n(x, 1)\}, \quad S_n = \{x|u_n(x, -1) < c_n < u_n(x, 1)\}.$$ 

For $c_0 = -1$, we have $Q_n = \emptyset$ for $n$ sufficiently large. For fixed $n$ and $x \in [0, T]$, let $d_n(x)$ be the point closest to $c_n$ in $\text{Ran}(u_n(x, \cdot))$, i.e.

$$d_n(x) = \begin{cases} 
    u_n(x, -1), & x \in P_n, \\
    u_n(x, 1), & x \in Q_n, \\
    c_n, & x \in S_n.
\end{cases}$$

Then there exists a unique $z_n(x) \in [-1, 1]$ such that $u_n(x, z_n(x)) = d_n(x)$ for $x \in [0, T]$. For $x \in S_n$, we have $u_n(x, z_n(x)) = d_n(x) = c_n$, and it follows from (2.2) and (2.4) that $\tilde{v}_n(x, z_n(x)) = 0$. For $x \in P_n$, $z_n(x) = -1$ and by the non-permeable boundary condition, we also have $\tilde{v}_n(x, z_n(x)) = 0$. Thus, $\tilde{v}_n(x, z_n(x)) = 0$ for $x \in [0, T]$. Moreover, $|u_n(x, y) - c_n| \geq |u_n(x, y) - d_n(x)|$ for $(x, y) \in D_T$.

Now we can compute the uniform $H^2$ bound for $\tilde{v}_n$, $n \geq 1$. By (2.7),

$$\left|\frac{y-z_n(x)}{u_n(x,y)-u_n(x,z_n(x))}\right| \leq 2 \text{ for } (x, y) \in D_T \text{ and } n \text{ large enough.} \quad \text{Thus, } \left\|\frac{y-z_n(x)}{u_n-d_n(x)}\right\|_{L^\infty(D_T)} \leq C.$$ 

This, along with (4.1), (2.6) and Lemma (2.1) gives

$$\|\Delta \tilde{v}_n\|_{L^2(D_T)} \leq \left\| (\partial_y \omega_n + \beta) \frac{\tilde{v}_n}{u_n-c_n} \right\|_{L^2(D_T)} \leq C \left\| \frac{\tilde{v}_n}{u_n-c_n} \right\|_{L^2(D_T)} \leq C \left\| \frac{\tilde{v}_n}{u_n-d_n(x)} \right\|_{L^2(D_T)}$$

$$\leq C \left\| \frac{\tilde{v}_n}{y-z_n(x)} \right\|_{L^2(D_T)} \leq C \left( \int_0^T \left\| \tilde{v}_n(x, \cdot) \right\|_{H^1(-1, 1)}^2 dx \right)^{1/2}$$

$$\leq C \|\tilde{v}_n\|_{H^2(D_T)} \leq C \|\tilde{v}_n\|_{H^2(D_T)}^{1/2} \|\tilde{v}_n\|_{L^2(D_T)}^{1/2} = C \|\tilde{v}_n\|_{H^2(D_T)}^{1/2}.$$ 

Thus, by periodic boundary condition in $x$ and Dirichlet boundary condition in $y$ of $\tilde{v}_n$, we have

$$\|\tilde{v}_n\|_{H^2(D_T)} \leq C \|\Delta \tilde{v}_n\|_{L^2(D_T)} \leq C \|\tilde{v}_n\|_{H^2(D_T)}^{1/2} \|\tilde{v}_n\|_{L^2(D_T)}^{1/2} = C \|\tilde{v}_n\|_{H^2(D_T)}.$$ 

Then there exists $\tilde{v}_0 \in H^2(D_T)$ such that up to a subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}_0$ in $H^2(D_T)$, $\tilde{v}_n \rightarrow \tilde{v}_0$ in $H^1(D_T)$ and $C^0(D_T)$. By (4.1), we have

$$\int_{D_T} \left(-\nabla \tilde{v}_n \cdot \nabla \phi + (\partial_y \omega_n + \beta) \frac{\tilde{v}_n \phi}{u_n-c_n} \right) dxdy = 0$$

for any $\phi \in H^1(D_T)$ satisfying periodic boundary condition in $x$ and Dirichlet boundary condition in $y$. We prove that

$$\int_{D_T} \left(-\nabla \tilde{v}_0 \cdot \nabla \phi + \tilde{v}_0 \phi \frac{y}{y+1} \right) dxdy = 0.$$ 

By (2.4), (2.2) and (4.1), we have

$$\left| \int_{D_T} \partial_y \omega_n \frac{\tilde{v}_n \phi}{u_n-c_n} \right| \leq \|\partial_y \omega_n\|_{C^0(D_T)} \left\| \frac{\tilde{v}_n}{u_n-c_n} \right\|_{L^2(D_T)} \|\phi\|_{L^2(D_T)} \leq C \|\tilde{v}_n\|_{H^1(D_T)} \|\phi\|_{L^2(D_T)} \rightarrow 0.$$
Moreover,
\[
\left| \beta \int_{D_T} \left( \frac{\tilde{v}_n}{u_n - c_n} - \frac{\tilde{v}_0}{y + 1} \right) \phi dxdy \right| 
\leq \left| \beta \int_{D_T} \left( \frac{\tilde{v}_n}{u_n - c_n} - \frac{\tilde{v}_p}{y + 1} \right) \phi dxdy \right| + \left| \beta \int_{D_T} \left( \frac{\tilde{v}_n}{y + 1} - \frac{\tilde{v}_0}{y + 1} \right) \phi dxdy \right| = I + II.
\]

For \( II \), since \( \phi(x, \pm 1) = 0 \) for \( x \in [0, T] \), by Lemma 2.1, we have
\[
II \leq \beta \left| \int_{D_T} (\tilde{v}_n - \tilde{v}_0) \frac{\phi}{y + 1} dxdy \right| \leq C \|\tilde{v}_n - \tilde{v}_0\|_{C^0(D_T)} \|\phi\|_{H^1(D_T)} \to 0.
\]

For \( I \), we decompose it into two parts:
\[
I \leq |\beta| \int_{S_n} \int_{-1}^{1} \left( \frac{\tilde{v}_n}{u_n - c_n} (y + 1) - \tilde{v}_n \right) \frac{\phi}{y + 1} dydx + \left| \beta \int_{P_n} \int_{-1}^{1} (y + 1) \left( \frac{y + 1}{u_n - c_n} - \frac{\tilde{v}_p}{y + 1} \right) \phi dxdy \right| = I_1 + I_2.
\]

For \( x \in S_n \), we have \( u_n(x, z_n(x)) = c_n \), and thus,
\[
\frac{\tilde{v}_n}{u_n - c_n} (y + 1) = \frac{\tilde{v}_n}{u_n - c_n} (y - z_n(x) + z_n(x) + 1) = \frac{\tilde{v}_n}{u_n - c_n} + \frac{\tilde{v}_n}{u_n - c_n} (z_n(x) + 1),
\]
where \( \tau_n = ty + (1 - t)z_n(x) \). By (2.3), we have
\[
\left| \frac{1}{\int_{0}^{1} \partial_y u_n(x, \tau_n)dt} - 1 \right| \leq C \|\partial_y u_n - 1\|_{C^0(D_T)} \leq C \varepsilon_n \to 0.
\]
and
\[
|z_n(x) + 1| \leq |z_n(x) - c_n| + |c_n + 1| \leq |z_n(x) - u_n(x, z_n(x))| + |c_n + 1|
\]
for \( x \in [0, T] \). So by (4.2)-(4.3), Lemma 2.1 and the fact that \( \phi(x, \pm 1) = 0 \) for \( x \in [0, T] \),
\[
I_1 \leq |\beta| \int_{S_n} \int_{-1}^{1} \left( \frac{\tilde{v}_n}{u_n - c_n} (y + 1) - \tilde{v}_n \right) \frac{\phi}{y + 1} dydx + \left| \beta \int_{S_n} \int_{-1}^{1} \frac{\tilde{v}_n}{u_n - c_n} (z_n(x) + 1) \frac{\phi}{y + 1} dydx \right|
\leq C \varepsilon_n \int_{D_T} |\tilde{v}_n| \frac{\phi}{y + 1} dydx + C(C \varepsilon_n + |c_n + 1|) \int_{D_T} \frac{|\tilde{v}_n|}{u_n - c_n} \frac{|\phi|}{y + 1} dydx
\leq C \varepsilon_n \left\| \tilde{v}_n \right\|_{L^2(D_T)} \left\| \phi \right\|_{H^1(D_T)} + C(C \varepsilon_n + |c_n + 1|) \left\| \tilde{v}_n \right\|_{H^1(D_T)} \left\| \phi \right\|_{H^1(D_T)} \to 0.
\]

By (2.7), we have \(|u_n(x, y) - c_n| \geq |u_n(x, y) - u_n(x, -1)| \geq \frac{1}{2}(y + 1)\) for \( x \in P_n \) and \( y \in (-1, 1) \),
and thus,
\[
\left| \frac{y + 1}{u_n(x, y) - c_n} \right| \leq 2|y + 1 - (u_n(x, y) - c_n)| \leq 2|u_n - y|_{C^0(D_T)} + |c_n + 1| \to 0.
\]

Then by the fact that \( \tilde{v}_n(x, \pm 1) = 0, \phi(x, \pm 1) = 0 \) for \( x \in [0, T] \), Lemma 2.1 and (4.3), we have
\[
I_2 \leq C(2|u_n - y|_{C^0(D_T)} + |c_n + 1|) \left\| \tilde{v}_n \right\|_{H^1(D_T)} \left\| \phi \right\|_{H^1(D_T)} \to 0.
\]
Taking (4.10)-(4.11) into account, we have

\[(4.12) \quad \left| \beta \int_{D_T} \left( \frac{\tilde{v}_n}{u_n - c_n} - \frac{\tilde{v}_0}{y + 1} \right) \phi dx \right| \rightarrow 0.\]

By (4.6), (4.12) and the fact that \(\tilde{v}_n \rightarrow \tilde{v}_0\) in \(H^1(D_T)\), we obtain (4.5) by sending \(n \rightarrow \infty\) in (4.4).

Since \(\tilde{v}_n = -\partial_x \tilde{\psi}_n/\|v_n\|_{L^2(D_T)}\) is \(T\)-periodic in \(x\), we have \(\tilde{v}_0(x,y) = \sum_{k \neq 0} \tilde{v}_{0,k}(y)e^{i k \alpha x}\).

It follows from \(\|\tilde{v}_0\|_{L^2(D_T)} = 1\) that there exists \(k_0 \neq 0\) such that \(\tilde{v}_{0,k_0} \neq 0\), where \(\alpha = \frac{2\pi}{T}\).

Notice that for \(j = 0, 1, 2\),

\[\|\tilde{v}_{0,k_0}(j)\|_{L^2(-1,1)}^2 = \int_{-1}^1 \left| \int_0^T \partial_y \tilde{v}_0(x,y)e^{-ik_0 \alpha x} dx \right|^2 dy \leq C \int_{-1}^1 \int_0^T |\partial_y \tilde{v}_0(x,y)|^2 dx dy < \infty\]

and \(\tilde{v}_{0,k_0}( \pm 1) = \frac{1}{T} \int_0^T \tilde{v}_0(x, \pm 1) e^{-ik_0 \alpha x} dx = 0\). Thus, we can take the test function \(\phi(x,y) = \tilde{v}_{0,k_0}(y)e^{ik_0 \alpha x}\) in (4.5), and use the fact that \(\tilde{v}_{0,k_0} = \tilde{v}_{0,-k_0}\) and \(\int_0^T e^{i(k-k_0)\alpha x} dx = 0\) for \(k \neq k_0\) to obtain

\[\int_{-1}^1 \left( -|\partial_y \tilde{v}_{0,k_0}(y)|^2 - (k_0 \alpha)^2 |\tilde{v}_{0,k_0}(y)|^2 + \frac{\beta}{y + 1} |\tilde{v}_{0,k_0}(y)|^2 \right) dy = 0.\]

Thus,

\[(4.13) \quad \int_{-1}^1 \left( |\partial_y \tilde{v}_{0,k_0}(y)|^2 - \frac{\beta}{y + 1} |\tilde{v}_{0,k_0}(y)|^2 \right) dy = -\int_{-1}^1 (k_0 \alpha)^2 |\tilde{v}_{0,k_0}(y)|^2 dy < 0.\]

For (1), since \(0 < \beta < \beta_* = \beta(-1)\), by Corollary 3.2 we have \(\lambda_1(\beta, -1) > \lambda_1(\beta(-1), -1) = 0\), and thus,

\[\inf_{\phi \in H^1_0((-1,1), \|\phi\|_{L^2((-1,1))} = 1} \int_{-1}^1 \left( |\phi'|^2 - \frac{\beta}{y + 1} |\phi|^2 \right) dy > 0,\]

which is a contradiction.

For (2ii), by (4.13) we have \(\lambda_1(\beta, -1) \leq -(k_0 \alpha)^2 < -\alpha_\beta^2 = \lambda_1(\beta, -1)\), which is a contradiction.

**Case 4.** \(|c_0| < 1\).

Let \(\delta_0 = 1 - \frac{1}{|c_0|} > 0\). Then \([c_0 - \delta_0, c_0 + \delta_0] \subset [-1 + \delta_0, 1 - \delta_0]\). By Theorem 1.1 there exists \(\varepsilon_{\delta_0} > 0\) such that any traveling wave solution \((u(x - c t, y), v(x - c t, y))\) to the \(\beta\)-plane equation with \(c \in [-1 + \delta_0, 1 - \delta_0]\), \(x\)-period \(T\) and satisfying that \(\|(u, v) - (y, 0)\|_{H^s(D_T)} < \varepsilon_{\delta_0}\), must have \(v(x, y) \equiv 0\). Since \(c_n \rightarrow c_0\) and \(\varepsilon_n \rightarrow 0\), we have \(c_n \in [c_0 - \delta_0, c_0 + \delta_0]\) and \(\|(u_n, v_n) - (y, 0)\|_{H^s(D_T)} \leq \varepsilon_n \leq \varepsilon_{\delta_0}\) for \(n\) large enough. Then \(v_n \equiv 0\) on \(D_T\) for \(n\) large enough, which contradicts \(\|v_n\|_{L^2(D_T)} \neq 0\).

Finally, we prove (2i). We only prove it for \(\beta > \beta_*\), and the proof for \(\beta < -\beta_*\) is similar. We divide the discussion in terms of \(\alpha\).

**Case I.** \(\alpha = \frac{2\pi}{T} = \sqrt{-\lambda_1(\beta, -1)}\).

We first modify Couette flow to the nearby shear flow \((ay, 0)\) with \(a \in (0, 1)\), and then construct traveling waves by bifurcation at \((ay, 0)\).

For the shear flow \((ay, 0)\), let \(\tilde{\lambda}(\beta, c)\) be the principal eigenvalue of the Rayleigh-Kuo BVP

\[(4.14) \quad -\phi'' - \frac{\beta}{ay - c} \phi = \lambda \phi, \quad \phi(\pm 1) = 0,\]
where $\phi \in H^1_0 \cap H^2(-1, 1)$ and $c \leq -1 < -a$. Recall that $\lambda_1(\beta, c)$ is the principal eigenvalue of the Rayleigh-Kuo BVP (3.11) for Couette flow. Since $-\phi'' - \frac{\beta}{ay-c}\phi = -\phi'' - \frac{\beta/a}{y-c/a}\phi$, we have

$$\lambda_1(\beta, c) = \lambda_1 \left( \frac{\beta}{a}, \frac{c}{a} \right)$$

for $c \leq -a$. Since $a \in (0, 1)$, we infer from Corollary 3.2 that $0 > -\alpha^2 = \lambda_1(\beta, -1) > \lambda_1 \left( \frac{\beta}{a}, -1 \right)$. It follows from Proposition 3.7 (1) and Lemma 3.4 (1) that there exists a unique $c_a \in (-\infty, -1)$ such that $-\alpha^2 = \lambda_1 \left( \frac{\beta}{a}, c_a \right)$. This, along with Lemma 3.6, implies that $c_a \to -1$ as $a \to 1^-$. By (4.15), we have $-\alpha^2 = \lambda_1 \left( \frac{\beta}{a}, c_a \right) = \tilde{\lambda}_1(\beta, ac_a)$. Note that $ac_a < -a$, and thus, $ac_a \notin \text{Ran}(ay) = [-a, a]$. This implies that $ac_a \in \sigma_d(R_{\alpha, \beta})$ with $u(y) = ay$ in (1.4).

Let $s \geq 3$. For any $\varepsilon > 0$, there exists $a = a_\varepsilon \in (0, 1)$ such that

$$\| (e_x y, 0) - (y, 0) \|_{H^s(D_T)} \leq C|a_\varepsilon - 1| < \frac{\varepsilon}{2}$$

and $|a_\varepsilon c_{a_\varepsilon} + 1| < \frac{\varepsilon}{2}$.

By Corollary 2.4 in [20] and $e_x c_{a_\varepsilon} \in \sigma_d(R_{\alpha, \beta})$ with $u(y) = ay$ in (1.4), there exists a traveling wave solution $(u_\varepsilon(x - c_\varepsilon t, y), v_\varepsilon(x - c_\varepsilon t, y))$ to (1.1) - (1.3) which has period $T = 2\pi/\alpha$ in $x$,

$$\| (u_\varepsilon, v_\varepsilon) - (e_x y, 0) \|_{H^s(D_T)} \leq \frac{\varepsilon}{2}$$

and $|c_\varepsilon - a_\varepsilon c_{a_\varepsilon}| < \frac{\varepsilon}{2}$. Then by (4.16) - (4.17), we have $\| (u_\varepsilon, v_\varepsilon) - (y, 0) \|_{H^s(D_T)} < \varepsilon$ and $|c_\varepsilon + 1| < \varepsilon$.

**Case II.** $\alpha = \frac{\pi}{\sqrt{2}} \in (0, \sqrt{-\lambda_1(\beta, -1)})$.

By Proposition 3.7 (1), there exists $c_{\alpha, \beta} < -1$ such that $\lambda_1(\beta, c_{\alpha, \beta}) = -\alpha^2$. Thus, $c_{\alpha, \beta} \in \sigma_d(R_{\alpha, \beta})$. Let $s \geq 3$. Then the conclusion follows from Corollary 2.4 in [20].

5. Existence of non-shear steady state in $H^{s/2}$

In this section, we prove the existence of non-shear steady states near Couette flow in (velocity) $H^{s/2}$ space. The method is to construct non-shear steady states by bifurcation at modified shear flows near Couette. First, we give a bifurcation lemma for $\beta \neq 0$. Note that the bifurcation lemma for $\beta = 0$ in [22] cannot be applied or extended to the case $\beta \neq 0$, since the extension cause a singularity at the middle point $y = 0$ for the Rayleigh-Kuo BVP, which is difficult to deal with. In the following bifurcation lemma, we require that $u'' - \beta = 0$ near 0, and the price is that a similar construction of modified shear flow in [22] cannot be applied here, since the Gauss error function is an odd function. We introduce the new modified shear flow latter. Another difference is to deal with the degeneracy of the Rayleigh-Kuo BVP.

**Lemma 5.1.** Consider a shear flow $u \in C^4([-1, 1])$, $u(0) = 0$, $u' > 0$ on $[-1, 1]$, $\beta \in \mathbb{R}$ and $-u''(y) + \beta y \equiv K$ on $[-\delta_0, \delta_0]$ for some $\delta_0 \in (0, 1)$ and $K \in \mathbb{R}$. Define the Rayleigh-Kuo operator

$$\mathcal{L}\phi := -\phi'' + \frac{u'' - \beta}{u}\phi, \quad H^2 \cap H^1_0(-1, 1) \to L^2(-1, 1).$$

If the principal eigenvalue of $\mathcal{L}$ satisfies $\mu_1 = -a_0^2 < 0$, then there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, there exists a steady solution $(u_\varepsilon(x, y), v_\varepsilon(x, y))$ to the $\beta$-plane equation with minimal period $T_\varepsilon$ in $x$ such that

$$\| (u_\varepsilon, v_\varepsilon) - (u, 0) \|_{H^s(D_{T_\varepsilon})} < \varepsilon.$$
and $v_\varepsilon \neq 0$, where $D T_\varepsilon = [0, T_\varepsilon] \times [-1, 1]$. Furthermore, $T_\varepsilon \to \frac{2\pi}{\alpha_0}$ as $\varepsilon \to 0^+$.

Proof. $(u, v)$ is a solution of the dynamical system if and only if $\partial_x \omega \partial_y \psi - \partial_x \psi (\partial_x \omega + \beta) = 0$ and $\psi$ takes constant values on $\{y = \pm 1\}$, where $\omega = \partial_x v - \partial_y u$ and $(u, v) = (\partial_y \psi, -\partial_x \psi)$. Let $\mu_n$ be the $n$-th eigenvalue of $\mathcal{L}$ for $n \geq 1$. The proof is divided into two cases.

Case 1. $\mu_n \neq 0$ for $n \geq 2$.

Let $\psi_0$ be a stream function associated with the shear flow $(u, 0)$, i.e., $\psi_0(y) = u(y)$. Since $u < 0$ on $[-1, 0)$ and $u > 0$ on $(0, 1]$, $\psi_0$ is decreasing on $[-1, 0)$ and increasing on $(0, 1]$. Let $\psi_0^{-1}$ and $\psi_0^{-1}$ be the inverse maps of $\psi_0\vert_{[0, 1]}$ and $\psi_0\vert_{[-1, 0]}$, respectively. Then we can define two functions $f_-(\psi_0'' + \beta(\cdot)) \circ \psi_0^{-1}$ on $[\psi_0(0), \psi_0(-1)]$ and $f_+(\psi_0'' + \beta(\cdot)) \circ \psi_0^{-1}$ on $[\psi_0(0), \psi_0(1)]$. Then $f_\pm \equiv K$ on $[\psi_0(0), \psi_0(-\delta_0)]$ and $\psi_0(\psi_0(1)]$. Clearly, $f_\pm \in C^3(\psi_0(-\delta_0/2), \psi_0(-1))$ and $f_\pm \in C^3(\psi_0(\delta_0/2), \psi_0(1))$. Thus, $f_\pm \in C^3(\psi_0(0), \psi_0(-1))$ and $f_\pm \in C^3(\psi_0(0), \psi_0(1))$. Then we extend $f_\pm$ to $f_\pm$ such that $f_\pm \in C^3(\mathbb{R})$, $f_- = f_\pm \circ \psi_0(0), \psi_0(-\delta_1, \psi_0(\delta_0))$, $f_+ = f_\pm \circ \psi_0 \equiv K$ on $[\psi_0(0), \psi_0(1)]$ and $f_\pm \equiv K$ on $[\psi_0(0), \psi_0(1)]$ for a given $\delta_1 \in (0, \frac{\pi}{\delta_0})$. Then we construct steady solutions near $(u, 0)$ by solving the elliptic equation

$$
-\Delta \psi + \beta y = 1_{[-1, 0]}(y)f_-(\psi(x, y)) + 1_{[0, 1]}(y)f_+(\psi(x, y)) = \begin{cases}
    f_-(\psi(x, y)) & \text{if } y < 0, \\
    f_+(\psi(x, y)) & \text{if } y \geq 0.
\end{cases}
$$

Let $\xi = \alpha x$ and $\psi(x, y) = \tilde{\psi}(\xi, y)$, where $\tilde{\psi}(\xi, y)$ is $2\pi$-periodic in $\xi$. We use $\alpha^2$ as the bifurcation parameter. The equation for $\tilde{\psi}(\xi, y)$ becomes

\begin{equation}
(5.1) \quad -\alpha^2 \frac{\partial^2 \tilde{\psi}}{\partial \xi^2} - \frac{\partial^2 \tilde{\psi}}{\partial y^2} + \beta y - \left(1_{[-1, 0]}(y)f_-(\tilde{\psi}(\xi, y)) + 1_{[0, 1]}(y)f_+(\tilde{\psi}(\xi, y))\right) = 0
\end{equation}

with the boundary conditions that $\tilde{\psi}$ takes constant values on $\{y = \pm 1\}$. Define a map $\mathcal{F} : H^4(T_{2\pi} \times [-1, 1]) \to H^2(T_{2\pi} \times [-1, 1])$ by

$$
\mathcal{F}(\psi)(x, y) = 1_{[-1, 0]}(y)f_-(\psi(x, y)) + 1_{[0, 1]}(y)f_+(\psi(x, y)) = \begin{cases}
    f_-(\psi(x, y)) & \text{if } y < 0, \\
    f_+(\psi(x, y)) & \text{if } y \geq 0.
\end{cases}
$$

Then $\mathcal{F} \in C^2(V)$, where $V = \{\psi \in H^4(T_{2\pi} \times [-1, 1]) \mid \|\psi - \psi_0\|_{H^4} \leq \delta_2\}$ for $\delta_2 > 0$ sufficiently small. In fact, since $|\psi(x, 0) - \psi_0(0)| \leq \|\psi - \psi_0\|_{C^0(T_{2\pi} \times [-1, 1])} \leq C\|\psi - \psi_0\|_{H^4(T_{2\pi} \times [-1, 1])} \leq C\delta_2$ for $x \in T_{2\pi}$ and $\psi \in V$, we have $\psi(x, 0) \in \left(\psi_0(0) - \frac{\delta_2}{2}, \psi_0(0) + \frac{\delta_2}{2}\right)$ by taking $\delta_2 \in \left(0, \frac{\delta_0}{2}\right)$. Thus, $f_-(s) = f_+(s) = K$ for $s \in \left(\psi(x, 0) - \frac{\delta_2}{2}, \psi(x, 0) + \frac{\delta_2}{2}\right)$. Since $f_+, f_- \in C^3_0(\mathbb{R})$, we have $\mathcal{F}(\psi) \in C^2(T_{2\pi} \times [-1, 1])$ and $\mathcal{F} \in C^2(V)$. Note that

\begin{equation}
(5.2) \quad -\alpha^2 \frac{\partial^2 \phi}{\partial \xi^2} - \frac{\partial^2 \phi}{\partial y^2} - (\mathcal{F}(\phi) + \psi_0) = -\frac{u''(y) - \beta}{u(y)}.
\end{equation}

Define the perturbation of the stream function by

$$
\phi(\xi, y) = \tilde{\psi}(\xi, y) - \psi_0(y).
$$

Then we reduce the equation (5.1) to

\begin{equation}
-\alpha^2 \frac{\partial^2 \phi}{\partial \xi^2} - \frac{\partial^2 \phi}{\partial y^2} - (\mathcal{F}(\phi + \psi_0) - \mathcal{F}(\psi_0)) = 0.
\end{equation}
Define the spaces
\[ B = \{ \phi(\xi, y) \in H^4(T_{2\pi} \times [-1, 1]) : \phi(\xi, \pm 1) = 0 \text{ and even in } \xi \} \]
and
\[ C = \{ \phi(\xi, y) \in H^2(T_{2\pi} \times [-1, 1]) : \text{ even in } \xi \} . \]
Consider the mapping
\[ F : B \times \mathbb{R}^+ \rightarrow C \]
defined by
\[ F(\phi, \alpha^2) = -\alpha^2 \frac{\partial^2 \phi}{\partial \xi^2} - \frac{\partial^2 \phi}{\partial y^2} - \left( \mathcal{F}(\phi + \psi_0) - \mathcal{F}(\psi_0) \right) . \]
We study the bifurcation near the trivial solution \( \phi = 0 \) of the equation \( F(\phi, \alpha^2) = 0 \) in \( B \). First, \( F \in C^2(V \times \mathbb{R}^+) \). By [5,2], the linearized operator of \( F \) around \((0, \alpha_0^2)\) has the form
\[ \mathcal{G} := F'(0, \alpha_0^2) = -\alpha_0^2 \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial y^2} + \frac{u'' - \beta}{u} . \]
Since \( \mu_1 = -\alpha_0^2 < 0 \) is the principal eigenvalue of the operator \( \mathcal{L} \) and \( \mu_n \neq 0 \) for \( n \geq 2 \), the kernel of \( \mathcal{G} : B \rightarrow C \) is given by
\[ \ker(\mathcal{G}) = \{ \phi_0(y) \cos \xi \} , \]
where \( \phi_0 \) is an eigenfunction corresponding to \( \mu_1 \). Thus, \( \dim(\ker(\mathcal{G})) = 1 \). Since \( \mathcal{G} \) is self-adjoint, \( \phi_0(y) \cos \xi \notin \text{Ran}(\mathcal{G}) \). Note that \( \partial_{\alpha^2} \partial_{\alpha} F(\phi, \alpha^2) \) is continuous and
\[ \partial_{\alpha^2} \partial_{\alpha} F(0, \alpha_0^2)(\phi_0(y) \cos \xi) = -\frac{\partial^2}{\partial \xi^2} [\phi_0(y) \cos \xi] = \phi_0(y) \cos \xi \notin \text{Ran}(\mathcal{G}) . \]
By the Crandall-Rabinowitz local bifurcation theorem, there exists a local bifurcating curve \((\phi_\kappa, \alpha(\kappa)^2), \kappa \in [-\kappa_0, \kappa_0] \), of the equation \( F(\phi, \alpha^2) = 0 \), which intersects the trivial curve \((0, \alpha^2)\) at \( \alpha^2 = \alpha_0^2 \), such that
\[ \phi_\kappa(\xi, y) = \kappa \phi_0(y) \cos \xi + o(|\kappa|) , \]
\( \alpha(\kappa)^2 \) is a continuous function of \( \kappa \), and \( \alpha(0)^2 = \alpha_0^2 \). So the stream functions of the perturbed steady flows in \((\xi, y)\) coordinates take the form
\[ \psi_\kappa(\xi, y) = \psi_0(y) + \kappa \phi_0(y) \cos \xi + o(|\kappa|) . \]
Let the velocity \( \bar{u}_\kappa(x) = (u_\kappa(x), v_\kappa(x)) = (\partial_y \psi_\kappa, -\partial_x \psi_\kappa) \). Then
\[ u_\kappa(x, y) = u(y) + \kappa \phi_0(y) \cos(\alpha x) + o(|\kappa|) , \]
\[ v_\kappa(x, y) = -\alpha \kappa \phi_0(y) \sin(\alpha x) + o(|\kappa|) \neq 0 \]
for \( \kappa \in [-\kappa_0, \kappa_0] \). Then \( \| (u_\kappa(x), v_\kappa(x)) - (u, 0) \|_{H^3([0, 2\pi/\alpha(\kappa)] \times [-1, 1])} \leq C_0 \kappa \) for some constant \( C_0 > 0 \).

Then we can choose \( \kappa_0 \) smaller and \( \varepsilon_0 = C_0 \kappa_0 > 0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \), \( (u_\varepsilon, v_\varepsilon) := (u_\kappa, v_\kappa) \)|\( \kappa=\varepsilon/C_0 \) satisfies that \( \| (u_\varepsilon, v_\varepsilon) - (u, 0) \|_{H^3(D_{T_\varepsilon})} \leq \varepsilon \) and \( \| v_\varepsilon \|_{L^2(D_{T_\varepsilon})} \neq 0 \), where \( D_{T_\varepsilon} = [0, 2\pi/\alpha(\varepsilon/C_0)] \times [-1, 1] \) and \( T_\varepsilon = 2\pi/\alpha(\varepsilon/C_0) \rightarrow 2\pi/\alpha_0 \) as \( \varepsilon \rightarrow 0^+ \).

**Case 2.** There exists \( n_0 \geq 2 \) such that \( \mu_{n_0} = 0 \).

Choose \( \zeta \in C^4([-1, 1]) \) such that \( \zeta \equiv 0 \) on \([-\delta_0, 0] \) and \( \zeta < 0 \) on \([-1, -\delta_0) \cup (\delta_0, 1] \). Let \( u_1 \) solve the regular ODE \( u_1'' \beta u_1' - (u_1'' - \beta) u_1 = \zeta \) on \([-1, 1] \). Then \( u_1 \in C^4([-1, 1]) \) and \( u_1'' \equiv 0 \) on \([-\delta_0, 0] \). Thus, \( u'' + \nu u_1'' - \beta \equiv 0 \) on \([-\delta_0, \delta_0] \) for \( \nu \in \mathbb{R} \). Let \( \mu_n(\nu) \) be the \( n \)-th eigenvalue of the regular BVP
\[ \mathcal{L}_n \phi = \frac{d^2}{dy^2} \phi + \frac{u'' + \nu u_1'' - \beta}{u + \nu u_1} \phi = \mu \phi, \quad \phi \in H^2 \cap H^0_0(-1, 1) \]
for \( \nu \in [-\nu_0, \nu_0] \), where \( n \geq 1 \), and \( \nu_0 > 0 \) satisfies that there exists \( \delta_3 > 0 \) such that 
\[
|u(y) + \nu u_1(y)| \geq \delta_3
\]
for \( y \in [-1, -\delta_0] \cup [\delta_0, 1] \) and \( \nu \in [-\nu_0, \nu_0] \). Then
\[
\mu'_n(0) = \int_{-1}^{1} \frac{u'' u - (u'' - \beta)u_1}{u^2} \phi_n^2 dy = \int_{-1}^{1} \frac{\zeta}{u^2} \phi_n^2 dy < 0,
\]
where \( \phi_n \) is a real \( L^2 \) normalized eigenfunction of \( \mu_n(0) \in \sigma(L(0)) \). Then \( \mu_1(\cdot) \) and \( \mu_{\nu_0}(\cdot) \) are decreasing on \([-\nu_0, \nu_0]\) if we take \( \nu_0 > 0 \) smaller.

Note that \( \phi_1 \) is linearly independent of \( \phi_{\nu_0} \). Choose \( \xi_1, \xi_2 \in C^4([-1, 1]) \) such that \( \xi_1, \xi_2 \) are supported on \([-1, -\delta_0] \cup [\delta_0, 1] \) and
\[
\int_{-1}^{1} \frac{\xi_1}{u^2} \phi_1^2 dy = \int_{-1}^{1} \frac{\xi_2}{u^2} \phi_{\nu_0}^2 dy = -\int_{-1}^{1} \frac{\xi_1}{u^2} \phi_1^2 dy = \int_{-1}^{1} \frac{\xi_1}{u^2} \phi_{\nu_0}^2 dy \neq 0.
\]
Then there exist \( k_1, k_2 \in \mathbb{R} \), which are not both zero, such that \( \xi = k_1 \xi_1 + k_2 \xi_2 \) satisfies
\[
\int_{-1}^{1} \frac{\xi}{u^2} \phi_1^2 dy = 1 \quad \text{and} \quad \int_{-1}^{1} \frac{\xi}{u^2} \phi_{\nu_0}^2 dy = -1.
\]
Let \( u_2 \) solve the regular ODE \( u''_2 u - (u'' - \beta)u_2 = \xi \) on \([-1, 1]\). Then \( u_2 \in C^4([-1, 1]) \) and \( u''_2 \equiv 0 \) on \([-\delta_0, \delta_0] \). Thus, \( u'' + \nu u'_1 + \tau u'_2 - \beta \equiv 0 \) on \([-\delta_0, \delta_0] \). Let \( \mu_n(\nu, \tau) \) be the \( n \)-th eigenvalue of the regular BVP
\[
\mathcal{L}(\nu, \tau) \phi = -\frac{d^2}{dy^2} \phi + \frac{u'' + \nu u'_1 + \tau u'_2 - \beta}{u + \nu u_1 + \tau u_2} \phi = \mu \phi, \quad \phi \in H^2 \cap H^1_{0}(-1, 1)
\]
for \( \nu \in [-\nu_0, \nu_0] \) and \( \tau \in [-\tau_0, \tau_0] \), where \( n \geq 1, \tau_0 > 0 \) and \( \nu_0 > 0 \) can be taken smaller such that there exists \( \delta_4 > 0 \) such that \( |u(y) + \nu u_1(y) + \tau u_2(y)| > \delta_4 \) for \( y \in [-1, -\delta_0] \cup [\delta_0, 1] \), \( \nu \in [-\nu_0, \nu_0] \) and \( \tau \in [-\tau_0, \tau_0] \). Then
\[
\partial_\tau \mu_1(0, 0) = \int_{-1}^{1} \frac{u''_2 u - (u'' - \beta)u_2}{u^2} \phi_1^2 dy = \int_{-1}^{1} \frac{\xi}{u^2} \phi_1^2 dy = 1 > 0,
\]
\[
\partial_\tau \mu_{\nu_0}(0, 0) = \int_{-1}^{1} \frac{\xi}{u^2} \phi_{\nu_0}^2 dy = -1 < 0.
\]
Take \( \nu_0, \tau_0 > 0 \) smaller such that \( \partial_\tau \mu_1(\nu, \tau) > 0 \) for \( (\nu, \tau) \in [0, \nu_0] \times [-\tau_0, \tau_0] \). Note that \( \mu_{\nu_0-1}(0, 0) < \mu_{\nu_0}(0, 0) = 0 < \mu_{\nu_0+1}(0, 0) \). By the continuity of \( \partial_\tau \mu_{\nu_0}, \partial_\nu \mu_{\nu_0-1} \) and \( \mu_{\nu_0+1}, \) we can choose \( \nu_0 > 0 \) and \( \tau_0 > 0 \) smaller such that \( \partial_\tau \mu_{\nu_0}(\nu, \tau) < 0 \) and \( \mu_{\nu_0-1}(\nu, \tau) < 0 < \mu_{\nu_0+1}(\nu, \tau) \) for \( (\nu, \tau) \in [0, \nu_0] \times [-\tau_0, \tau_0] \). Since \( \partial_\tau \mu_1(0, 0) > 0 \) and \( \mu_1(0, 0) < \mu_1(0, 0) \), we can choose \( \tau \in (0, \tau_0) \) such that \( \mu_1(\nu_0, \tau) < \mu_1(0, 0) = -\alpha_0 \). Then there exists \( \nu_\tau \equiv (0, 0) \) such that \( \mu_1(\nu_\tau, \tau) = -\alpha_0 \). On the other hand, \( \partial_\tau \mu_{\nu_0}(\nu, \tau) < 0 \) and \( \mu_{\nu_0}(\cdot, 0) \) is decreasing on \( [0, \nu_0] \), we have \( \mu_{\nu_0}(\nu, \tau) < \mu_{\nu_0}(\nu, 0) < \mu_{\nu_0}(0, 0) = 0 \), which implies that \( \mu_{\nu_0+1}(\nu, \tau) > 0 > \mu_{\nu_0}(\nu, \tau) \) for \( (\nu, \tau) \in (0, \nu_0) \times (0, \tau_0) \). Since \( \nu_\tau, \tilde{\tau} \equiv (0, 0) \times (0, \tau_0) \), we have \( \nu_\tau(\nu_\tau, \tilde{\tau}) \neq 0 \) for \( n \geq 2 \). Fix \( \epsilon \in (0, \epsilon_0) \). Then we can choose \( \nu_0, \tau_0 > 0 \) smaller such that for \( \nu_\tau \equiv (0, 0) \) and \( \tilde{\tau} \equiv (0, \tau_0) \),
\[
\|u + \nu_\tau u_1 + \tilde{\tau} u_2, 0\|_H^2(-1, 1) < \frac{\epsilon}{2}.
\]
Since \( \mu_1(\nu_\tau, \tilde{\tau}) = -\alpha_0^2 \) and \( \mu_{\nu_\tau}(\nu_\tau, \tilde{\tau}) \neq 0 \) for \( n \geq 2 \), we can apply Case 1 to the shear flow \( (u + \nu_\tau u_1 + \tilde{\tau} u_2, 0) \) to obtain that there exists a nonparallel steady solution \( (u_\epsilon, v_\epsilon) \) which has minimal period \( T_\epsilon \) in \( x \),
\[
\|(u, v_\epsilon) - (u + \nu_\tau u_1 + \tilde{\tau} u_2, 0)\|_{H^2(D_{T_\epsilon})} \leq \frac{\epsilon}{2}
\]
and \( \|v_\epsilon\|_{L^2(D_{T_\epsilon})} \neq 0 \). Then by \( (5.3)-(5.4) \), we have \( \|(u, v_\epsilon) - (u, 0)\|_{H^2(D_{T_\epsilon})} < \epsilon. \) \( \square \)
Construction of the modified shear flow

Fix $\beta \neq 0$. Let $\gamma < \min\{\frac{1}{2}, \frac{1}{10|\beta|}\}$. Recall that
\[
erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds, \quad x \in \mathbb{R}
\]
is the Gauss error function introduced in [22]. Let
\[
\alpha(y) = \begin{cases}
0, & x \leq 0, \\
 e^{\frac{1}{2}}, & x > 0,
\end{cases}
\]
and
\[
\tilde{\eta}(x) = \alpha(x-1) \cdot \alpha(2-x),
\]
and
\[
\tilde{I}(x) = \frac{\int_{|x|/\gamma}^2 \tilde{\eta}(t) dt}{\int_1^2 \tilde{\eta}(t) dt}.
\]
Then $\tilde{I}(x) \in C^\infty(\mathbb{R})$, $\tilde{I}(x) = 1$ for $|x| \leq 1$ and $\tilde{I}(x) = 0$ for $|x| \geq 2$. $\tilde{I}'$ and $\tilde{I}''$ are smooth functions with compact support. Thus, there exist $M_1, M_2 > 0$ such that $|\tilde{I}'(x)| \leq M_1$ and $|\tilde{I}''(x)| \leq M_2$ for $x \in \mathbb{R}$. Define the cut-off function by
\[
I_\gamma(y) = \tilde{I}\left(\frac{y}{\gamma}\right) = \frac{\int_{|y|/\gamma}^2 \tilde{\eta}(t) dt}{\int_1^2 \tilde{\eta}(t) dt}.
\]
Then
\[
I_\gamma(y) = \begin{cases}
1, & y \in [-\gamma, \gamma], \\
\text{smooth}, & y \in [-2\gamma, -\gamma] \cup [\gamma, 2\gamma], \\
0, & y \in [-1, -2\gamma] \cup [2\gamma, 1].
\end{cases}
\]
We define the modified shear profile
\[
(5.5) \quad U_{\gamma,a}(y) = y + \frac{1}{2} \beta y^2 I_\gamma(y) + a \gamma^2 erf\left(\frac{y-5\gamma}{\gamma}\right) I_\gamma(y-5\gamma), \quad y \in [-1, 1].
\]
Here, the second term is supported on $[-2\gamma, 2\gamma]$, and the last term is supported on $[3\gamma, 7\gamma]$. The second term is added such that $U''_{\gamma,a} - \beta \equiv 0$ on $[-2\gamma, 2\gamma]$, which cancels the singularity near $y = 0$. To apply the bifurcation lemma for $\beta \neq 0$, we have to add an error function to produce negative eigenvalues of the Rayleigh-Kuo operator. A direct addition of the Gauss error function in [22] induces new singularity near $y = 0$. Our novelty is to translate a cut-off Gauss error function away from 0 but not very far, and to choose the size of the cut-off function suitably small.

Denote
\[
Q_{\gamma,a}(y) = \frac{U''_{\gamma,a}(y) - \beta}{U_{\gamma,a}(y)},
\]
and define the Rayleigh-Kuo operator by
\[
\mathcal{L}_{[\gamma,a]} = -\frac{d^2}{dy^2} + Q_{\gamma,a}(y) : H^2 \cap H^1_0(-1,1) \to L^2(-1,1).
\]
Then $\mathcal{L}_{[\gamma,a]} \phi = \lambda \phi$ is a regular BVP for $\gamma > 0$ small enough and fixed $a \geq 0$. Let $\lambda_{n,\gamma,a}$ be the $n$-th eigenvalue of $\mathcal{L}_{[\lambda,a]}$, and consider it as a function of $(a, \gamma)$ on $[0, +\infty) \times (0, +\infty)$. Then we have the following estimates of the principal eigenvalue $\lambda_{1,\gamma,a}$ of $\mathcal{L}_{[\gamma,a]}$. 
Lemma 5.2. Fix $\beta \neq 0$ and $a > 0$. Then

\begin{equation}
\limsup_{\gamma \to 0^+} \lambda_{1, \gamma, a} \leq 3 + \frac{3}{2} b_0 a,
\end{equation}

where $b_0 = 2 \int_0^2 \left( \frac{1}{(x+5)^2} - \frac{1}{(x-5)^2} \right) \text{erf}(x) dx$ is a negative constant. In particular, we have the following conclusions.

1. Fix $a > -\frac{2}{5}$. Then $\lambda_{1, \gamma, a} < 0$ for $\gamma > 0$ sufficiently small.
2. Fix $d < 0$ and $a_d > \frac{2d-6}{3b_0}$. Then $\lambda_{1, \gamma, d} < 0$ for $\gamma > 0$ sufficiently small.

Proof. The principal eigenvalue $\lambda_{1, \gamma, a}$ of $L_{[\gamma, a]}$ satisfies

\begin{equation}
\lambda_{1, \gamma, a} = \inf_{\phi \in H^1_0(-1,1), \|\phi\|_{L^2(-1,1)} = 1} \langle L_{[\gamma, a]} \phi, \phi \rangle = \inf_{\phi \in H^1_0(-1,1), \|\phi\|_{L^2(-1,1)} = 1} \left( \|\phi'\|_{L^2(-1,1)}^2 + \int_{-1}^1 Q_{\gamma, a}(y)|\phi(y)|^2 dy \right).
\end{equation}

Let $\phi_1(y) = 1 - |y|$ for $y \in [-1, 1]$. Then $\|\phi_1\|_{L^2(-1,1)}^2 = \frac{2}{3}$ and

\begin{align*}
\langle L_{[\gamma, a]} \phi, \phi \rangle &= \|\phi_1'\|_{L^2(-1,1)}^2 + \int_{-1}^1 Q_{\gamma, a}(y)\phi_1(y)^2 dy \\
&= 2 + \int_{-1}^1 \left( a \gamma^2 \text{erf} \left( \frac{y - 5\gamma}{\gamma} \right) \right)'' \left( 1 - |y| \right) dy + \int_{-1}^1 \frac{1}{U_{\gamma, a}(y)} \left( 1 - |y| \right) dy \\
&= 2 + B(\gamma, a) + C(\gamma, a).
\end{align*}

Note that

\begin{equation}
\frac{1}{U_{\gamma, a}} = \frac{1}{y + \frac{1}{2} \beta \gamma \frac{\gamma}{y} \tilde{I} \left( \frac{y}{\gamma} \right) + \gamma a \Lambda \left( \frac{y - 5\gamma}{\gamma} \right) \frac{y - 5\gamma}{\gamma} \tilde{I} \left( \frac{y - 5\gamma}{\gamma} \right)},
\end{equation}

where $\Lambda(y) = \frac{\text{erf}(y)}{y}$. Since $|x \tilde{I}(x)| \leq C$ and $|\Lambda(x)| \leq C$ for $x \in \mathbb{R}$, $\frac{2}{3} \in \left[ \frac{1}{2}, \frac{1}{3} \right]$ for $y \in [3\gamma, 7\gamma]$, we have

\begin{equation}
D(\gamma) := \frac{1}{1 + \frac{1}{2} \beta \gamma \frac{\gamma}{y} \tilde{I} \left( \frac{y}{\gamma} \right) + \gamma a \Lambda \left( \frac{y - 5\gamma}{\gamma} \right) \frac{y - 5\gamma}{\gamma} \tilde{I} \left( \frac{y - 5\gamma}{\gamma} \right)} \to 1
\end{equation}

as $\gamma \to 0^+$. Since $\tilde{I}(\pm 2) = \tilde{I}'(\pm 2) = 0$ and $\text{erf}(\cdot)$ is an odd function, we have

\begin{align*}
\int_{-1}^1 \frac{1}{y} \left( a \gamma^2 \text{erf} \left( \frac{y - 5\gamma}{\gamma} \right) \right)'' dy &= \int_{-3\gamma}^{3\gamma} \frac{1}{y} \left( a \gamma^2 \text{erf} \left( \frac{y - 5\gamma}{\gamma} \right) \right)'' dy \\
&= a \int_{-2}^{2} \frac{1}{x + 5} \left( \text{erf}(x) \tilde{I}(x) \right)'' dx \\
&= a \int_{-2}^{2} \left( \frac{1}{x + 5} \right)'' \left( \text{erf}(x) \tilde{I}(x) \right) dx \\
&= 2a \int_{-2}^{2} \frac{1}{(x + 5)^3} \left( \text{erf}(x) \tilde{I}(x) \right) dx
\end{align*}
\[= 2a \int_0^\gamma \left( \frac{1}{(x + 5)^3} - \frac{1}{(-x + 5)^3} \right) \left( \text{erf}(x)I(x) \right) \, dx = b_0 a, \]

where \(b_0 = 2 \int_0^\gamma \left( \frac{1}{(x + 5)^3} - \frac{1}{(-x + 5)^3} \right) \text{erf}(x)I(x) \, dx \) is a negative constant. We claim that

\[
(5.8) \quad B(\gamma, a) \rightarrow b_0 a \quad \text{as} \quad \gamma \rightarrow 0^+. 
\]

Note that

\[
|B(\gamma, a) - (b_0 a)| 
\leq \int_{3\gamma}^{7\gamma} \left| \frac{a\gamma^2 \text{erf} \left( \frac{y-5\gamma}{\gamma} \right)}{y} \right|^2 |D_\gamma(y) - 1| |\phi_1(y)|^2 \, dy 
\]

\[
+ \int_{3\gamma}^{7\gamma} \left| \frac{a\gamma^2 \text{erf} \left( \frac{y-5\gamma}{\gamma} \right)}{y} \right|^2 (\phi_1(y)^2 - 1) \, dy = B_1(\gamma, a) + B_2(\gamma, a). 
\]

For any \(\varepsilon > 0\), there exists \(\delta_1 > 0\) such that \(|D_\gamma(y) - 1| < \varepsilon\) for \(y \in [3\gamma, 7\gamma]\) and \(0 < \gamma < \delta_1\).

Let \(\delta_2 = \frac{\pi}{7}\). Then \(|\phi_1(y)^2 - 1| < 14\gamma < \varepsilon\) for \(3\gamma < y < 7\gamma\) and \(0 < \gamma < \delta_2\). Since \(\left| \frac{y - 5\gamma}{\gamma} \right| \leq C\),

\[
|\gamma I'_\gamma(y - 5\gamma)| \leq C \quad \text{and} \quad |\gamma^2 I''_\gamma(y - 5\gamma)| \leq C 
\]

for \(y \in [3\gamma, 7\gamma]\), we have

\[
(5.9) \quad \left( \gamma^2 \text{erf} \left( \frac{y-5\gamma}{\gamma} \right) I_\gamma(y - 5\gamma) \right)'' 
= \frac{4}{\sqrt{\pi}} e^{-\left( \frac{y-5\gamma}{\gamma} \right)^2} \left( \frac{-y - 5\gamma}{\gamma} I_\gamma(y - 5\gamma) + \gamma I'_\gamma(y - 5\gamma) \right) + \gamma^2 \text{erf} \left( \frac{y-5\gamma}{\gamma} \right) I''_\gamma(y - 5\gamma) 
\leq C, 
\]

and thus,

\[
\int_{3\gamma}^{7\gamma} \left| \frac{\gamma^2 \text{erf} \left( \frac{y-5\gamma}{\gamma} \right) I_\gamma(y - 5\gamma)}{y} \right|^2 \, dy \leq C \int_{3\gamma}^{7\gamma} \frac{1}{y^2} \, dy = C \ln(7/3) \leq C. 
\]

Then

\[
|B(\gamma, a) - (b_0 a)| \leq |B_1(\gamma, a)| + |B_2(\gamma, a)| \leq C\varepsilon 
\]

for \(0 < \gamma < \delta = \min\{\delta_1, \delta_2\}\). This proves (5.8).

For \(C(\gamma, a)\), let \(g_\gamma(y) = \left( \frac{1}{2} \beta y^2 I_\gamma(y) \right)'' - \beta\) for \(y \in [-1, 1]\), which is an even function. Note that \(g_\gamma(y) = 0\) for \(y \in [0, \gamma]\) and \(g_\gamma(y) = -\beta\) for \(y \in [2\gamma, 1]\). For \(y \in [\gamma, 2\gamma]\), we get

\[
g_\gamma(y) = \left( \frac{1}{2} \beta y^2 I_\gamma(y) \right)'' - \beta 
= \beta I_\gamma(y) + 2\beta y I'_\gamma(y) + \frac{1}{2} \beta y^2 I''_\gamma(y) - \beta 
= \beta(I_\gamma(y) - 1) + 2\beta y I' \left( \frac{y}{\gamma} \right) + \frac{1}{2} \beta \left( \frac{y}{\gamma} \right)^2 I'' \left( \frac{y}{\gamma} \right). 
\]

Thus,

\[
(5.10) \quad |g_\gamma(y)| \leq (1 + 4M_1 + 2M_2)|\beta| 
\]
for \( y \in [\gamma, 2\gamma] \). Then

\[
C(\gamma, a) = \int_{-\gamma}^{\gamma} g_\gamma(y) (1 - |y|)^2 dy
\]

\[
= \int_{0}^{\gamma} \left( \frac{1}{U_{\gamma,a}(y)} + \frac{1}{U_{\gamma,a}(-y)} \right) g_\gamma(y)(1 - |y|)^2 dy
\]

\[
= \int_{\gamma}^{2\gamma} -g_\gamma(y)(1 - |y|)^2 \beta y^2 I_\gamma(y) \left( \frac{1}{y^2} - \left( \frac{1}{2} \beta y I_\gamma(y) \right)^2 \right) dy
\]

\[
+ \int_{3\gamma}^{2\gamma} \left( \frac{1}{y + a\gamma^2 \Lambda \left( \frac{u-5\gamma}{\gamma} \right) \frac{u-5\gamma}{\gamma} I \left( \frac{u-5\gamma}{\gamma} \right)} - \frac{1}{y} \right) g_\gamma(y)(1 - |y|)^2 dy
\]

\[
= C_1(\gamma, a) + C_2(\gamma, a).
\]

Since \( |g_\gamma(y)| \leq C|\gamma|, \left( 1 - |y| \right)^2 I_\gamma(y) \leq 1, \beta \left( \frac{y}{\gamma} \right)^2 \leq 4|\beta| \) and \( \left( \frac{y}{\gamma} \right)^2 - \left( \frac{1}{2} \beta \gamma \left( \frac{y}{\gamma} \right)^2 I_\gamma(y) \right)^2 \geq \frac{24}{25} \)

for \( y \in [\gamma, 2\gamma] \) and \( \gamma < \min \left\{ \frac{1}{2}, \frac{1}{10|\beta|} \right\} \), we have

\[
|C_1(\gamma, a)| \leq C\beta^2 \gamma \to 0
\]
as \( \gamma \to 0^+ \). For any \( \varepsilon > 0 \), there exists \( \delta_3 > 0 \) such that

\[
\left| \frac{1}{1 + \gamma a\gamma^2 \Lambda \left( \frac{u-5\gamma}{\gamma} \right) \frac{u-5\gamma}{\gamma} I \left( \frac{u-5\gamma}{\gamma} \right)} - 1 \right| < \varepsilon
\]

for \( y \in [3\gamma, 7\gamma] \) and \( 0 < \gamma < \delta_3 \), and thus,

\[
|C_2(\gamma, a)| \leq C|\gamma| \ln(7/3) \varepsilon \to 0
\]
as \( \gamma \to 0^+ \). In summary, for fixed \( a > 0 \) we have

\[
\lambda_{1,\gamma,a} \leq \frac{3}{2} C_{[\gamma,a]} \phi_1, \phi_1 = \frac{3}{2} (2 + B(\gamma, a) + C(\gamma, a)) \to 3 + \frac{3}{2} b_0 a
\]
as \( \gamma \to 0^+ \). This proves (5.6). The conclusions (1)-(2) are direct applications of (5.6). \( \Box \)

For \( \beta \neq 0 \), we study the range of the principal eigenvalue \( \lambda_{1,\gamma,a} \) with respect to \( a \) for \( \gamma > 0 \) small enough. Roughly speaking, the range could cover any interval \([a, b]\) for some \( b < 0 \) and any \( a < b \). Note that there exist \( M, M_0 > 0 \) such that \( \left| \left( x^2 I(x) \right)' \right| \leq M \) and

\[
\left| \left( x^2 I(x) \right)' \right| \leq M_0 \text{ for } x \in \mathbb{R}.
\]
Lemma 5.3. Let $\beta \neq 0$ and $C_\beta := \inf_{0 < \gamma \leq \delta_*} \lambda_{1, \gamma, 0}$, where $\delta_* \in \left(0, \frac{2}{|\beta|_M}\right)$. Then $C_\beta > -\infty$, and for any $d < C_\beta$ and $a_d > \frac{2d - \delta^2}{\delta_0}$, there exists $\delta = \delta(a_d) > 0$ such that $[d, C_\beta) \subset \{\lambda_{1, \gamma, a} : a \in [0, a_d]\}$ for any fixed $\gamma \in (0, \delta)$.

Proof. Since $\delta_* \in \left(0, \frac{2}{|\beta|_M}\right)$, we have $|U'_{\gamma,0}(y)| \geq 1 - \frac{1}{2} \beta y \gamma \left((\cdot)^2 I_{\gamma}(\cdot)\right)' \circ \left(\frac{y}{y} \right) > 0$ for $0 < \gamma \leq \delta_*$ and $y \in [-1, 1]$. Thus, $\lambda_{1, \gamma, 0}$ is continuous with respect to $\gamma \in (0, \delta_*]$. It suffices to show that $\lambda_{1, \gamma, 0}$ has lower bound as $\gamma \to 0^+$. For any real function $\phi \in H^1_0(-1,1)$ with $\|\phi\|_{L^2(-1,1)} = 1$, we have:

$$
\langle L_{\gamma,0} \phi, \phi \rangle = \|\phi\|_{L^2(-1,1)}^2 + \int_{-1}^{1} Q_{\gamma,0}(y) \phi(y)^2 dy \\
= \|\phi\|_{L^2(-1,1)}^2 + \left(\int_{-2\gamma}^{-\gamma} + \int_{\gamma}^{2\gamma}\right) \frac{g_{\gamma}(y) \gamma}{y + \frac{1}{2} \beta y^2 I_{\gamma}(y)} \phi(y)^2 dy + \left(\int_{-1}^{-\gamma} + \int_{\gamma}^{1}\right) - \frac{\beta}{y} \phi(y)^2 dy
$$

(5.11) $\|\phi\|_{L^2(-1,1)}^2 + B(\gamma) + C(\gamma)$.

For $B(\gamma)$, we have

$$
B(\gamma) = \int_{-\gamma}^{\gamma} g_{\gamma}(y) \left(\frac{\phi(y)^2}{y + \frac{1}{2} \beta y^2 I_{\gamma}(y)} - \frac{\phi(-y)^2}{y - \frac{1}{2} \beta y^2 I_{\gamma}(y)}\right) dy \\
= \int_{-\gamma}^{\gamma} \frac{1}{y} g_{\gamma}(y) \left(\frac{\phi(y)^2 - \phi(-y)^2}{y + \frac{1}{2} \beta y^2 I_{\gamma}(y)} - \frac{\phi(y)^2}{y - \frac{1}{2} \beta y^2 I_{\gamma}(y)}\right) dy.
$$

(5.12)

Note that $|g_{\gamma}(y)| < C|\beta| < C$, $|\phi(y) - \phi(-y)| \leq \|\phi\|_{L^2(-1,1)} \sqrt{2y}$, $|\phi(y) + \phi(-y)| \leq \|\phi\|_{L^2(-1,1)} \sqrt{2y} + 1 \leq 2\sqrt{2} \|\phi\|_{L^2(-1,1)}$ and $\phi(y)^2 + \phi(-y)^2 \leq \|\phi\|_{L^2(-1,1)}^2 (y + 1) + \|\phi\|_{L^2(-1,1)}^2 (y - 1) \leq 4 \|\phi\|_{L^2(-1,1)}^2$ for $y \in [-\gamma, \gamma]$. For any $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that $\left|1 - \left(\frac{1}{2} \beta y I_{\gamma}(y)\right)^2\right| \geq \frac{1}{2}$ and $\left|\frac{1}{2} \beta y I_{\gamma}(y)\right| + \sqrt{y} \leq \frac{\varepsilon_0}{8C\ln 2}$ for $y \in [\gamma, 2\gamma]$, where $\gamma \in (0, \delta_0)$. Then

$$
|B(\gamma)| \leq \frac{\varepsilon_0}{8C\ln 2} 8C \|\phi\|_{L^2(-1,1)}^2 \int_{-\gamma}^{\gamma} \frac{1}{y} dy \leq \|\phi\|_{L^2(-1,1)}^2 \varepsilon_0
$$

(5.13) for $\gamma \in (0, \delta_0)$. For $C(\gamma)$, we have

$$
C(\gamma) = -\beta \left(\int_{-1}^{-\gamma} + \int_{\gamma}^{1}\right) \frac{1}{y} \phi(y)^2 dy \\
= -\beta \left(\int_{-\gamma}^{\gamma} \frac{\phi(y)^2 \ln y}{2\gamma} - \int_{-\gamma}^{\gamma} 2\phi(y) \phi'(y) \ln ydy + \phi(y)^2 \ln y \left|\int_{-1}^{-\gamma} - \int_{-1}^{-\gamma} 2\phi(y) \phi'(y) \ln y |dy\right|\right) \\
= -\beta \left(\ln(2\gamma) \left(\phi(-2\gamma)^2 - \phi(2\gamma)^2\right) - \left(\int_{-1}^{-\gamma} + \int_{\gamma}^{1}\right) 2\phi(y) \phi'(y) \ln y |dy\right) \\
= -\beta (I_{\gamma} + II_{\gamma}).
$$

Choose $\delta_0 > 0$ smaller such that

$$
|I_{\gamma}| = |\ln(2\gamma)| \cdot |\phi(-2\gamma) - \phi(-1)| + |\phi(2\gamma) - \phi(1)| \cdot |\phi(-2\gamma) - \phi(2\gamma)| \\
\leq |\ln(2\gamma)| (4\gamma)^{\frac{1}{2}} \|\phi\|_{L^2(-1,1)}^2 \leq \varepsilon_0 \|\phi\|_{L^2(-1,1)}^2.
$$
For any $\varepsilon_1, \varepsilon_2 > 0$, we have
\[
|II_\gamma| \leq \left( \int_{-1}^{0} + \int_{0}^{1} \right) \left( \frac{1}{\varepsilon_1} (\phi(y) \ln |y|)^2 + \varepsilon_1 |\phi(y)|^2 \right) dy
\leq \left( \int_{-1}^{0} + \int_{0}^{1} \right) \left( \frac{1}{2\varepsilon_1^2} |\ln |y||^4 + \frac{\varepsilon_2}{2\varepsilon_1} |\phi(y)|^4 + \varepsilon_1 |\phi'(y)|^2 \right) dy
\leq \frac{1}{2\varepsilon_1^2} \left( \int_{-1}^{0} |\ln |y||^4 dy + \frac{\varepsilon_2}{2\varepsilon_1} \int_{0}^{1} |\phi(y)|^4 dy + \varepsilon_1 \int_{-1}^{0} |\phi'(y)|^2 dy \right)
\leq \frac{C_0}{2\varepsilon_1^2} + \frac{\varepsilon_2}{2\varepsilon_1} \left( C_1 \int_{-1}^{0} |\phi'(y)|^2 dy + C_2 \right) + \varepsilon_1 \int_{-1}^{0} |\phi'(y)|^2 dy,
\]
where we used Gagliardo-Nirenberg interpolation inequality (2.14) in the last inequality. Take $\varepsilon_0, \varepsilon_1$ and $\varepsilon_2$ smaller such that $\varepsilon_0 + |\beta| \left( \varepsilon_0 + \varepsilon_1 + \frac{C_1 \varepsilon_2}{2\varepsilon_1} \right) < 1$. Then we have
\[
\langle L_{(\gamma, \phi)}, \phi \rangle \geq \left( 1 - \varepsilon_0 - |\beta| \left( \varepsilon_0 + \varepsilon_1 + \frac{C_1 \varepsilon_2}{2\varepsilon_1} \right) \right) \|\phi\|_{L^2(-1,1)}^2 - \left( \frac{C_0}{2\varepsilon_1^2} + \frac{C_2 \varepsilon_2}{2\varepsilon_1} \right) |\beta| \geq -\infty
\]
for $\gamma \in (0, \delta)$. This proves that $\lambda_{1,\gamma,0}$ has lower bound as $\gamma \to 0^+$, and thus, $C_\beta < -\infty$.

For any $d < C_\beta$ and $a_d > \frac{2d + 6}{M_0}$, we infer from Lemma 5.2 (2) that there exists $\delta_{d,a} > 0$ such that $\lambda_{1,\gamma,a} < d$ for $0 < \gamma < \delta_{d,a}$. By the definition of $C_\beta$, we have $\lambda_{1,\gamma,0} \geq C_\beta$ for $0 < \gamma < \delta_s$.

For fixed $0 < \gamma < \delta = \delta(a_d) = \min \left\{ \delta_s, \delta_{a_d}, \frac{1}{\|M + a_d M_0\|} \right\}$, we claim that $\lambda_{1,\gamma,a}$ is continuous with respect to $a \in [0, a_d]$. In fact, since $0 < \gamma < \frac{1}{\|M + a_d M_0\|}$ and
\[
U_{\gamma,a}(y) = y + \frac{1}{2} \beta \gamma^2 \left( (\cdot)^2 I(\cdot) \right) \circ \left( \frac{y}{\gamma} \right) + a \gamma^2 \left( erf(\cdot) I(\cdot) \right) \circ \left( \frac{y - 5\gamma}{\gamma} \right),
\]
we have
\[
U'_{\gamma,a}(y) = 1 + \frac{1}{2} \beta \gamma \left( (\cdot)^2 I(\cdot) \right) \circ \left( \frac{y}{\gamma} \right) + a \gamma \left( erf(\cdot) I(\cdot) \right) \circ \left( \frac{y - 5\gamma}{\gamma} \right) > 0
\]
for $y \in [-1, 1]$, which implies that there is no singularity for $Q_{\gamma,a}$ and thus, $\lambda_{1,\gamma,a}$ is continuous on $a \in [0, a_d]$. This, along with the fact that $\lambda_{1,\gamma,a} < d$ and $\lambda_{1,\gamma,0} \geq C_\beta$, implies that $[d, C_\beta] \subset \{ \lambda_{1,\gamma,a} : a \in [0, a_d] \}$ for any fixed $\gamma \in (0, \delta)$.

**Remark 5.4.** (1) For $|\beta| \gg 1$, $\lambda_{1,\gamma,0} < 0$ for $\gamma > 0$ small enough, and thus, we can not expect $C_\beta > 0$ in general. In fact, for any fixed $n \geq 1$, if $|\beta|$ is large enough, then $\lambda_{n,\gamma,a}(\beta) < 0$ uniformly for $a \in [0, 1]$ and $\gamma \in \left( 0, \frac{1}{\|M + a_d M_0\|} \right)$, where we write $\lambda_{n,\gamma,a} = \lambda_{n,\gamma,a}(\beta)$ to indicate its dependence on $\beta$.

We only give the proof for $\beta \gg 1$. Let $\varphi_i, 1 \leq i \leq n$, be $n$ smooth functions such that $\text{supp}(\varphi_i) = \left( \frac{1}{2} + \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n} \right)$ and $\|\varphi_i\|_{L^2(-1,1)} = 1$. Thus, $\varphi_i \perp \varphi_j$ in the sense of $L^2$ for $i \neq j$. Let $V_n = \text{span}\{\varphi_1, \cdots, \varphi_n\}$. Then $V_n \subset H_0^1(-1, 1)$. Moreover, $C_n := \max_{1 \leq i \leq n} \int_{-1}^{1} \left( |\varphi_i|^2 - \frac{\beta}{y} |\varphi_i|^2 \right) dy < 0$ for $\beta > 0$ sufficiently large. Since $\gamma \in \left( 0, \frac{1}{\|M + a_d M_0\|} \right)$,
there is no singularity on $Q_{\gamma,a}$ by (5.14), and thus,

\begin{equation}
\lambda_{n,\gamma,a}(\beta) = \inf_{\dim V_n = n} \sup_{\phi \in H_0^1((-1,1), \phi \in V_n, \|\phi\|_{L^2((-1,1))} = 1} \int_{-1}^{1} \left( |\phi'|^2 + \frac{U_n'' - \beta}{U_n' |\phi|^2} \right) dy.
\end{equation}

Then there exist $b_{i,\beta} \in \mathbb{R}$, $i = 1, \cdots, n$, with $\sum_{i=1}^{n} |b_{i,\beta}|^2 = 1$ such that $\varphi_{\beta} = \sum_{i=1}^{n} b_{i,\beta} \varphi_{i} \in V_{n}$ with $\|\varphi_{\beta}\|_{L^2}^2 = 1$, and

\begin{align*}
\lambda_{n,\gamma,a}(\beta) &\leq \sup_{\|\varphi\|_{L^2} = 1, \varphi \in V_{n}} \int_{-1}^{1} \left( |\phi'|^2 + \frac{U_n'' - \beta}{U_n' |\phi|^2} \right) dy = \int_{-1}^{1} \left( |\varphi'|^2 + \frac{U_n'' - \beta}{U_n' |\varphi|^2} \right) dy \\
&= \int_{\frac{1}{2}}^{1} \left( |\varphi'|^2 + \frac{\beta}{|\varphi|^2} \right) \frac{dy}{\phi} = \sum_{i=1}^{n} |b_{i,\beta}|^2 \int_{\frac{1}{2}}^{1} \left( |\varphi'|^2 + \frac{\beta}{|\varphi|^2} \right) dy \leq C_n < 0
\end{align*}

uniformly for $a \in [0,1]$ and $\gamma \in \left( 0, \frac{1}{2M+M_0} \right)$.

(2) As is indicated in Remark 5.3 (1), the number of negative eigenvalues of the Rayleigh-Kuo operator $\mathcal{L}_{\gamma,a}$ could take any positive integer as long as $|\beta|$ is taken appropriately large. Then we prove that the number is exactly 1 for some $(\gamma,a)$ when $|\beta|$ is small. Precisely, let $|\beta| < 1$ and $a_0 > 0$. Then there exists $\delta = \delta(a_0) > 0$ such that $\lambda_{2,\gamma,a} > 0$ for $a \in [0,a_0]$ and $\gamma \in (0,\delta)$.

For any 2-dimensional space $V_2 = \text{span} \{\varphi_1, \varphi_2\} \subset H_0^1((-1,1)$, there exists $0 \neq (\xi_1, \xi_2) \in \mathbb{R}^2$ such that $\xi_1 \varphi_1(0) + \xi_2 \varphi_2(0) = 0$. Define $\phi_* = \xi_1 \varphi_1 + \xi_2 \varphi_2$. We normalize $\phi_*$ such that $\|\phi_*\|_{L^2((-1,1))} = 1$. Then $\phi_* \in V$, $\phi_*(0) = 0$ and

\begin{align*}
\langle \mathcal{L}_{\gamma,a} \phi_*, \phi_* \rangle &= \|\phi_*\|_{L^2((-1,1))}^2 + \int_{-1}^{1} \left( a\gamma^2 e_{\gamma} f \left( \frac{y-5\gamma}{7} \right) I_{\gamma}(y-5\gamma) \right)'' \phi_*^2 dy \\
&\quad + \int_{-1}^{1} \frac{1}{U_{\gamma,a}(y)} \left( \frac{y}{2} I_{\gamma}(y) \right)'' - \frac{\beta}{U_{\gamma,a}(y)} \phi_*^2 dy = \|\phi_*\|_{L^2((-1,1))}^2 + B_\gamma(\gamma,a) + C_\gamma(\gamma,a).
\end{align*}

For $B_\gamma(\gamma,a)$, we have

\begin{equation}
|B_\gamma(\gamma,a)| = \left| \int_{3\gamma}^{7\gamma} \frac{1}{y} \frac{a\gamma^2 e_{\gamma} f \left( \frac{y-5\gamma}{7} \right) I_{\gamma}(y-5\gamma)}{1 + a\gamma^2 e_{\gamma} f \left( \frac{y-5\gamma}{7} \right) I_{\gamma}(y-5\gamma)} \phi_*^2 dy \right|.
\end{equation}

Choose $\delta_1(a_0) > 0$ such that $\left| 1 + \frac{a\gamma^2 e_{\gamma} f \left( \frac{y-5\gamma}{7} \right) I_{\gamma}(y-5\gamma)}{1 + a\gamma^2 e_{\gamma} f \left( \frac{y-5\gamma}{7} \right) I_{\gamma}(y-5\gamma)} \right| > \frac{1}{2}$ for $y \in [3\gamma, 7\gamma]$, where $\gamma \in (0,\delta_1(a_0))$. By (5.19), we have

\begin{equation}
\left| \frac{\gamma^2 e_{\gamma} f \left( \frac{y-5\gamma}{7} \right) I_{\gamma}(y-5\gamma)}{1 + a\gamma^2 e_{\gamma} f \left( \frac{y-5\gamma}{7} \right) I_{\gamma}(y-5\gamma)} \right| \leq C_1
\end{equation}

for $y \in [3\gamma, 7\gamma]$. For $C_\gamma(\gamma,a)$, we have

\begin{align*}
|C_\gamma(\gamma,a)| &\leq \left( \int_{-1}^{-\gamma} + \int_{-\gamma}^{1} + \int_{-\gamma}^{-2\gamma} + \int_{-2\gamma}^{3\gamma} \right) |\beta| |\phi_*|^2 dy \\
&\quad + \left( \int_{-\gamma}^{-2\gamma} + \int_{-2\gamma}^{2\gamma} \right) \frac{1}{|y|} \frac{|g_{\gamma}(y)|}{1 + \frac{1}{2}\beta y I_{\gamma}(y)} \phi_*^2 dy
\end{align*}
Choose \( \delta_2(a_0) > 0 \) such that 
\[
\left| 1 + a\gamma \left( \frac{y}{y_1} \right) \right| \geq \frac{1}{2} \text{ for } y \in [3\gamma, 7\gamma] \text{ and } \\
\left| 1 + \frac{1}{2} \beta y L_1(y) \right| \geq \frac{1}{2} \text{ for } y \in [-2\gamma, -\gamma] \cup [\gamma, 2\gamma], \text{ where } \gamma \in (0, \delta_2(a_0)).
\]
This, along with (5.10), gives
\[
\frac{|\beta|}{1 + a\gamma \left( \frac{y}{y_1} \right) \left| I_1(y_1 - 5\gamma) \right|} \leq C_2 |\beta|
\]
for \( y_1 \in [3\gamma, 7\gamma] \) and \( y \in [-2\gamma, -\gamma] \cup [\gamma, 2\gamma] \). Let \( \varepsilon_0 > 0 \) and \( \varepsilon_1 > 0 \) such that \( \varepsilon_0 + \varepsilon_1 + |\beta| = 1 \) and set \( \delta(a_0) = \min \left\{ \left( \frac{2}{\ln^2 14\gamma} \right) \varepsilon_0, \delta_1(a_0), \delta_2(a_0) \right\} > 0 \). Since \( \phi_*(0) = 0 \), we have
\[
|\phi_*(y)|^2 \leq |\phi_*(y)|^2_{L^2(-1,1)} |y| \text{ for } y \in [-1, 1].
\]
Thus, by (5.10), (5.17), (5.18) and (5.19) we have
\[
|B_*(\gamma, a)| + |C_*(\gamma, a)|
\]

Choose \( \varepsilon_0 > 0 \) and \( \varepsilon_1 > 0 \) such that \( \frac{3\sqrt{2|\beta|}}{8} + \varepsilon_0 + \varepsilon_1 = 1 \). Choose \( \delta_0 > 0 \) such that 
\[
1 - \left( \frac{1}{2} \beta y L_1(y) \right)^2 \geq \frac{1}{2}, \quad \frac{1}{2} \beta y L_1(y) + y \leq \frac{\varepsilon_0}{8 \ln 2}
\]
for \( y \in [\gamma, 2\gamma] \), where \( \gamma \in (0, \delta_0) \). By a similar argument as in (5.12)–(5.13), we have
\[
|B(\gamma)| \leq |\phi_*(y)|^2_{L^2(-1,1)} \varepsilon_0
\]
for $\gamma \in (0, \delta_0)$. Since $|\phi(y) - \phi(-y)| \leq \|\phi''\|_{L^2(-y,y)} \sqrt{2y}$, and

$$|\phi(y) + \phi(-y)| = |\phi(y) - \phi(1) + \phi(-y) - \phi(-1)| \leq \sqrt{1 - y \left(\|\phi'\|_{L^2(\gamma,1)} + \|\phi'\|_{L^2(-1,\gamma)}\right)},$$

we have

$$|C(\gamma)| \leq |\beta| \int_{2\gamma}^{1} \frac{1}{y} |\phi(y) - \phi(-y)||\phi(y) + \phi(-y)|dy$$

$$\leq |\beta| \sqrt{2} \int_{2\gamma}^{1} \frac{1}{\sqrt{y}} \sqrt{1 - y \left(\|\phi'\|_{L^2(\gamma,1)} + \|\phi'\|_{L^2(-1,\gamma)}\right)} dy$$

$$\leq |\beta| \sqrt{2} \int_{2\gamma}^{1} \frac{1}{\sqrt{y}} \sqrt{1 - y \left(\frac{1}{4} \left(\|\phi'\|_{L^2(\gamma,1)} + \|\phi'\|_{L^2(-1,\gamma)} + \|\phi'\|_{L^2(-1,\gamma)}\right)^2\right)} dy$$

$$(5.23) \quad \leq |\beta| \frac{3\sqrt{2}}{4} \|\phi''\|_{L^2(-1,1)} \int_{0}^{1} \frac{1}{\sqrt{y}} \sqrt{1 - y} dy = \frac{3\sqrt{2\pi} |\beta|}{8} \|\phi''\|_{L^2(-1,1)}^2.$$

Let $\tilde{\lambda}_1$ be the principal eigenvalue of $-\phi'' + \tilde{\lambda} \phi$, $\phi(\pm 1) = 0$. Then $\tilde{\lambda}_1 = \frac{\pi^2}{4}$ and

$$(5.24) \quad \frac{\pi^2}{4} = \frac{\pi^2}{4} \|\phi''\|_{L^2(-1,1)}^2 \leq \|\phi''\|_{L^2(-1,1)}^2.$$

Then by (5.22), (5.23) and (5.24), we have

$$\langle C_{[\gamma,0]} \phi, \phi \rangle = \|\phi''\|_{L^2(-1,1)}^2 + B(\gamma) + C(\gamma) \geq \left(1 - \varepsilon_0 - \frac{3\sqrt{2\pi} |\beta|}{8}\right) \|\phi''\|_{L^2(-1,1)}^2$$

$$= \varepsilon_1 \|\phi''\|_{L^2(-1,1)}^2 \geq \frac{\pi^2}{4} \varepsilon_1.$$

This, along with (5.7), implies that $\lambda_{1,\gamma,0} \geq \frac{\pi^2}{4} \varepsilon_1 > 0$ for $0 < \gamma < \delta_0$. \qed

By means of the above lemma, we get better conclusion about the range of the principal eigenvalue $\lambda_{1,\gamma,a}$ with respect to $a$ for $|\beta| < \frac{\pi \sqrt{2}}{3\delta}$.

Lemma 5.6. Let $|\beta| < \frac{\pi \sqrt{2}}{3\delta}$. For any $d < 0$ and $a_d > \frac{2d-6}{3\delta_0}$, there exists $\delta = \delta(d_d) > 0$ such that $[d,0] \subset \{\lambda_{1,\gamma,a} : a \in [0,a_d]\}$ for any fixed $\gamma \in (0, \delta)$.

Proof. By Lemma 5.2 (2), there exists $\delta_d > 0$ such that $\lambda_{1,\gamma,a_d} < d$ for $0 < \gamma < \delta_d$. By Lemma 5.5 there exists $\delta_0 > 0$ such that $\lambda_{1,\gamma,0} > 0$ for $0 < \gamma < \delta_0$. Recall that

$$\left|x^2 \tilde{I}(x)\right| \leq M$$

and

$$\left|\text{erf}(x) \tilde{I}(x)\right| \leq M_0$$

for $x \in \mathbb{R}$. For fixed $0 < \gamma < \delta = \delta(d_d) = \frac{1}{M_0 \lambda_{1,\gamma,0} M_0}$, $\lambda_{1,\gamma,a}$ is continuous on $a \in [0,a_d]$ by a similar argument with (5.14). Since $\lambda_{1,\gamma,0} > 0$, $\lambda_{1,\gamma,a_d} < d$ and $\lambda_{1,\gamma,a}$ is continuous on $a \in [0,a_d]$, we have $[d,0] \subset \{\lambda_{1,\gamma,a} : a \in [0,a_d]\}$ for fixed $\gamma \in (0, \delta)$. \qed

Now, we are in the position to prove Theorem 1.5.

Proof. First, we prove (1). Let $T > 0$ and $a = \frac{2d}{T}$. Choose $k \in \mathbb{Z}^+$ such that $d_k := -(k\alpha)^2 < -|C_{\beta}|$. Set $a_{2d_k} = \frac{4d_k-6}{3\delta_0} + 1$. Taking $d = 2d_k$ in Lemmas 5.2 (2) and 5.3, there exists $\delta(a_{2d_k}) > 0$ small enough such that $\lambda_{1,\gamma,a_{2d_k}} < 2d_k$ and $[2d_k, C_{\beta}] \subset \{\lambda_{1,\gamma,a} : a \in [0,a_{2d_k}]\}$ for any $\gamma \in (0, \delta(a_{2d_k}))$. 

For any \( \tilde{d} \in [2d_k, C\beta] \) and \( \gamma \in (0, \delta(a_{2d_k})) \), we can define
\[
 f_{\tilde{d}}(\gamma) = \inf_{0 < a < a_{2d_k}} \{ a | \lambda_{1, \gamma, a} = \tilde{d} \}.
\]

Thus,
\[
 (5.25) \quad \sqrt{-\lambda_{1, \gamma, f_{\tilde{d}}(\gamma)}} = \sqrt{-\tilde{d}_k} < k\alpha < \sqrt{-\lambda_{1, \gamma, f_{3d_k/2}(\gamma)}},
\]
where \( \tilde{d}_k \in (d_k, -|C\beta|) \). Now let us check that for \( a \in [0, a_{2d_k}] \) and \( s \in [0, \frac{5}{2}) \),
\[
 (5.26) \quad \| (U_{\gamma, a}, 0) - (y, 0) \|_{H^s(D_T)} \to 0
\]
as \( \gamma \to 0^+ \). In fact, using the Fourier transform, we have
\[
 \left\| f \left( \frac{y}{\gamma} \right) \right\|_{H^s(\mathbb{R})}^2 = \int_{\mathbb{R}} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\eta|^{2s} |\hat{f}(\eta)|^2 d\eta = \gamma^{-2s+1} \| f(x) \|_{H^s(\mathbb{R})}^2,
\]
\[
 \left\| f \left( \frac{y}{\gamma} \right) \right\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\hat{f}(\eta)|^2 d\eta = \gamma \| f(x) \|_{L^2(\mathbb{R})}^2,
\]
\[
 \left\| f \left( \frac{y}{\gamma} \right) \right\|_{H^s(\mathbb{R})}^2 \leq C \left( \left\| f \left( \frac{y}{\gamma} \right) \right\|_{H^s(\mathbb{R})}^2 + \left\| f \left( \frac{y}{\gamma} \right) \right\|_{L^2(\mathbb{R})}^2 \right) \leq C \gamma^{-2s+1} \| f(x) \|_{H^s(\mathbb{R})}^2
\]
for any \( f \in H^s(\mathbb{R}) \) and \( \gamma < 1 \). Then
\[
 \left\| U_{\gamma, a} - y \right\|_{H^s(-1, 1)} \leq a_{2d_k} \gamma^2 e r f \left( \frac{y - 5\gamma}{\gamma} \right) L_y(y - 5\gamma) \left\| H^s(\mathbb{R}) \right\| + \frac{1}{2} \beta \left\| y^2 L_y(y) \right\|_{H^s(\mathbb{R})}
\]
\[
 = \gamma^2 \left( a_{2d_k} \left\| e r f \left( \frac{y - 5\gamma}{\gamma} \right) \right\|_{H^s(\mathbb{R})} \left\| L_y(y - 5\gamma) \right\|_{H^s(\mathbb{R})} + \frac{1}{2} \beta \left\| \left( \frac{y}{\gamma} \right)^2 L_y(y) \right\|_{H^s(\mathbb{R})} \right)
\]
\[
 = \gamma^2 \frac{2\alpha-1}{s} \left( C a_{2d_k} \left\| e r f (x - 5) I(x - 5) \right\|_{H^s(\mathbb{R})} + C \beta \left\| x^2 I(x) \right\|_{H^s(\mathbb{R})} \right)
\]
\[
 = C \gamma^{\frac{5}{2} - s},
\]
which implies (5.26) by the assumption that \( s < \frac{5}{2} \). Here, \( e r f (\cdot - 5) I(\cdot - 5) \) and \( (\cdot)^2 I(\cdot) \) are smooth functions with compact support and thus, belong to \( H^s(\mathbb{R}) \).

For any \( \varepsilon > 0 \), by taking \( \delta(2d_{a, k}) > 0 \) smaller, we have
\[
 (5.27) \quad \| (U_{\gamma, a}, 0) - (y, 0) \|_{H^s(D_T)} \leq \frac{\varepsilon}{2} \quad \text{for} \quad (\gamma, a) \in \Omega_{d_k},
\]
where \( \Omega_{d_k} = \{(\gamma, a) | 0 < \gamma \leq \delta(a_{2d_k}), f_{d_k}(\gamma) \leq a \leq f_{3d_k/2}(\gamma) \} \).

By Lemma 5.1, for any \( (\gamma, a) \in \Omega_{d_k} \), there exists non-shear steady flows of the \( \beta \)-plane equation near the shear flow \( (U_{\gamma, a}, y, 0) \). For fixed \( 0 < \gamma < \delta(a_{2d_k}) \), there exists \( r_0 > 0 \) (by compactness, independent of \( a \in (f_{d_k}(\gamma), f_{3d_k/2}(\gamma)) \)) such that for any \( 0 < r < r_0 \), there exists a non-shear steady solution
\[
 (u_{\gamma, a;r}(x, y), v_{\gamma, a;r}(x, y)),
\]
which has \( x \)-period \( T(\gamma, a; r) \) and
\[
 \| (u_{\gamma, a;r}, v_{\gamma, a;r}) - (U_{\gamma, a}, 0) \|_{H^s(D_T(\gamma, a; r))} \leq r,
\]
where $D_{T(\gamma,a;r)} = [0,T(\gamma,a;r)] \times [-1,1]$. Moreover, for $a \in (f_{d_{k}'}(\gamma), f_{3d_{k}/2}(\gamma))$, we have

$$\frac{2\pi}{T(\gamma,a;r)} \to \sqrt{-\lambda_{1,\gamma,a}} \text{ as } r \to 0^+.$$ 

By (5.43), when $r_0$ is small enough,

$$T(\gamma, f_{3d_{k}/2}(\gamma); r) < \frac{T}{k} < T(\gamma, f_{d_{k}'}(\gamma); r) \quad \text{for } 0 < r < r_0.$$ 

Since $T(\gamma,a;r)$ is continuous with respect to $a$ (by Crandall-Rabinowitz Theorem) for each $\gamma \in (0,\gamma_0)$ and $r > 0$ small enough, there exists $a_T = a_T(\gamma,r) \in (f_{d_{k}'}(\gamma), f_{3d_{k}/2}(\gamma))$ such that $T(\gamma,a_T;r) = \frac{T}{k}$. Then

$$(u_{\gamma,r}(x,y), v_{\gamma,r}(x,y)) := (u_{\gamma,a_T,r}(x,y), v_{\gamma,a_T,r}(x,y))$$

is a non-shear steady solution to (1.1)-(1.3) with minimal $x$-period $\frac{T}{k}$ (and thus $x$-period $T$), and

$$\|(u_{\gamma,r}, v_{\gamma,r}) - (U_{\gamma,a_T,0})\|_{H^3(D_T)} = \sqrt{k}\|(u_{\gamma,r}, v_{\gamma,r}) - (U_{\gamma,a_T,0})\|_{H^3(D_T)} \leq \sqrt{kr}.$$ 

Thus, for any $0 < r < \frac{1}{\sqrt{k}} \min\{r_0, \frac{\varepsilon}{2}\}$, combining with (5.27) we have

$$\|(u_{\gamma,r}, v_{\gamma,r}) - (y,0)\|_{H^3(D_T)} \leq \varepsilon.$$ 

Next, we prove (2). We replace $C_\beta$ by 0, and choose $k = 1$ in (1). Let $d_1 := -\alpha^2 < 0$. Set $a_{2d_1} = \frac{4d_1^2 - 6}{3d_0} + 1$. Taking $d = 2d_1$ in Lemmas 5.2 (2) and 5.6 there exists $\delta(a_{2d_1}) > 0$ small enough such that $\lambda_{1,\gamma,a_{2d_1}} < 2d_1$ and $[2d_1,0] \subset \{\lambda_{1,\gamma,a} : a \in [0,a_{2d_1}]\}$ for any $\gamma \in (0,\delta(a_{2d_1}))$. The rest of the proof is a repeated process of (1). $\square$

6. ASYMPTOTIC STABILITY OF SHEAR FLOWS NEAR COUETTE FOR $\beta$-PLANE EQUATION ON $\mathbb{T} \times \mathbb{R}$

This is a generalization of Theorem 1 in [4]. We use the same notation, conventions, coordinate transform and time-dependent norm. So we omit the same details and only point out the differences. To avoid confusion with former sections, we use $w, \vartheta$ to denote the vorticity perturbation and changed vertical coordinate here, instead of $\omega, v$ in [4], respectively.

The change of coordinates is $(t, x, y) \to (t, z, \vartheta)$, where

$$z(t, x, y) = x - t\vartheta,$$

$$\vartheta(t, y) = y + \frac{1}{t} \int_0^t \langle U^z \rangle(\tau, y) d\tau.$$ 

Here, $\langle U^z \rangle = \frac{1}{2\pi} \int_{2\pi} U^z dx$. Define $f(t, z, \vartheta) = w(t,x,y)$ and $\phi(t,z,\vartheta) = \tilde{\psi}(t,x,y)$. We denote $[\partial_t\vartheta](t, \vartheta) = \partial_z\vartheta(t,y)$, $\partial'\vartheta(t, \vartheta)$, $\partial''\vartheta(t, \vartheta)$, and $\partial^\alpha\vartheta(t, \vartheta)$ where $\alpha \geq 1$. Then we get the evolution equation for $f$,

$$\partial_t f + [\partial_t\vartheta] \partial_0 f + \partial_z f = -y \partial_z f + \vartheta' \partial_0 \phi \partial_z f - \vartheta' \partial_z \phi \partial_0 f - \beta \partial_z \phi.$$

Notice that $\partial_t z = -y - \langle U^z \rangle(t, y)$. Using the Biot-Savart law, we can transform $\langle U^z \rangle$ to $-\vartheta' \partial_0 \phi$ in the new variables. Then the equation becomes

$$\partial_t f - (\vartheta' \partial_0 (\phi - \langle \phi \rangle)) \partial_z f + (\partial_t \vartheta) \partial_z \phi \partial_0 f + \beta \partial_z \phi = 0.$$ 

The Biot-Savart law in the new variables reads

$$f = \partial_z \phi + (\vartheta')^2 (\partial_0 - t \partial_z)^2 \phi + \vartheta'' \partial_0 - t \partial_z \phi := \Delta_t \phi.$$

Thus, for any $0 < r < \frac{1}{\sqrt{k}} \min\{r_0, \frac{\varepsilon}{2}\}$, combining with (5.27) we have

$$\|(u_{\gamma,r}, v_{\gamma,r}) - (y,0)\|_{H^3(D_T)} \leq \varepsilon.$$ 

Next, we prove (2). We replace $C_\beta$ by 0, and choose $k = 1$ in (1). Let $d_1 := -\alpha^2 < 0$. Set $a_{2d_1} = \frac{4d_1^2 - 6}{3d_0} + 1$. Taking $d = 2d_1$ in Lemmas 5.2 (2) and 5.6 there exists $\delta(a_{2d_1}) > 0$ small enough such that $\lambda_{1,\gamma,a_{2d_1}} < 2d_1$ and $[2d_1,0] \subset \{\lambda_{1,\gamma,a} : a \in [0,a_{2d_1}]\}$ for any $\gamma \in (0,\delta(a_{2d_1}))$. The rest of the proof is a repeated process of (1). $\square$
The β-plane equation (1.2) becomes
\[
\begin{aligned}
\partial_t f + u \cdot \nabla_{z,\vartheta} f + \beta \partial_z \phi &= 0, \\
u &= (0, [\partial_t \vartheta]) + \vartheta' \partial_z \nabla_{z,\vartheta} P \neq 0 \phi, \\
\phi &= \Delta^{-1} f.
\end{aligned}
\]
Without confusion we write \(\nabla_{z,\vartheta} = \nabla\) in the following. Let \(\bar{u}(t, z, \vartheta) = U^x(t, x, y)\) and \(p(t, z, \vartheta) = P(t, x, y)\). Then we have the equation for \(\bar{u}\),
\[
\partial_t \bar{u} + [\partial_t \vartheta] \partial_{\vartheta} \bar{u} + \partial_z P \neq 0 \phi + \vartheta' (\nabla_{z,\vartheta} P \neq 0 \phi) \cdot \nabla \bar{u} = -\partial_z p + \beta y U_y.
\]
Isolating the zero mode of the velocity field by taking average in \(z\), we have
\[
\partial_t \bar{u}_0 + [\partial_t \vartheta] \partial_{\vartheta} \bar{u}_0 + \vartheta' (\nabla_{z,\vartheta} P \neq 0 \phi) \cdot \nabla \bar{u} = 0.
\]
Here, the term \(\beta y U_y\) brings nothing new since its average in \(z\) is zero. Finally, \(\vartheta'\) and \([\partial_t \vartheta]\) are solutions to (2.13) in [4] coupled to (6.1). Define the same time-dependent norm and main energy as in Subsection 2.3 of [4] by
\[
\| A(t) f(t) \|_{L^2(\Omega)}^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |A_k(t, \eta) \hat{f}_k(t, \eta)|^2 d\eta,
\]
\[
A_k(t, \eta) = e^{\lambda(t)[k, \eta]} (k, \eta)^\sigma J_k(t, \eta); \\
\hat{A}_k(t, \eta) = e^{\lambda(t)[k, \eta]} (k, \eta)^\sigma \hat{J}_k(t, \eta); \\
E(t) = \frac{1}{2} \| A(t) f(t) \|_{L^2(\Omega)}^2 + E_\vartheta(t).
\]
See (2.15)-(2.18) and (2.22) in [4] for more details of \(\lambda(t), J_k(t, \eta), \hat{J}_k(t, \eta)\) and \(E_\vartheta(t)\). The goal is to prove that the energy \(E(t)\) is uniformly bounded for all time, as long as \(\epsilon\) is small enough. The local well-posedness theory for the β-plane equation in Gevrey spaces is similar to that for 2D Euler equation, so we have the same result as Lemma 2.1 in [4], and thus we can focus on times \(t \geq 1\). Then we will prove the same bootstrap proposition as Proposition 2.1 in [4], under the same bootstrap hypotheses for \(t \geq 1\),
\[
\begin{align}
& \text{(B1) } E(t) \leq 4\epsilon^2; \\
& \text{(B2) } \| \vartheta' - 1 \|_{L^\infty} \leq \frac{2}{\epsilon}; \\
& \text{(B3) } \text{‘CK’ integral estimates (for ‘Cauchy-Kovalevskaya’)}:
\end{align}
\]
\[
\int_1^t \left[ CK_\lambda(\tau) + CK_w(\tau) + CK_{w,2}(\tau) + CK_{w,2}(\tau) + K^{-1}\lambda \left( CK_{w,1}(\tau) + CK_{\lambda,1}(\tau) \right) \\
+ K^{-1}\sigma \sum_{i=1}^2 (CK_{w,i}(\tau) + CK_{\lambda,i}(\tau)) \right] d\tau \lesssim \epsilon^2.
\]
The definitions of \(CK\) terms are same as (2.21), (2.29) and (2.31) in [4]. Let \(I_E\) be the connect set of times \(t \geq 1\), on which the bootstrap hypotheses (B1-B3) hold. To be rigorous, we only consider solutions with regularized initial data while calculating and finally perform a passage to the limit. So \(E(t)\) is a continuous function of \(t\), and thus \(I_E\) is a closed interval \([1, T^*]\) with \(T^* > 1\). If we can prove that \(I_E\) is also open (under the subspace topology of \([1, +\infty)\), then \(I_E = [1, +\infty)\).

**Proposition 6.1** (Bootstrap). There exists \(\epsilon_0 \in (0, \frac{1}{2})\) depending only on \(\lambda_0, \lambda', s\) and \(\sigma\) such that if \(\epsilon < \epsilon_0\), and the bootstrap hypotheses (B1-B3) hold on \([1, T^*]\), then for any \(t \in [1, T^*]\),

1. \(E(t) < 2\epsilon^2\);
(2) $\|\vartheta' - 1\|_{L^\infty} < \frac{5}{8}$;
(3) ‘CK’ controls satisfy:
\[
\int_1^t \left[ CK\lambda(\tau) + CK_w(\tau) + CK_{w,2}^\theta(\tau) + CK_{w,1}^\theta(\tau) + K_\theta^{-1} \left( CK_{w,1}^\theta(\tau) + CK_{w,1}^\theta(\tau) \right) + K_\theta^{-1} \sum_{i=1}^2 \left( CCK_{i}^\theta(\tau) + CCK_{i}^\theta(\tau) \right) \right] d\tau \leq 6\epsilon^2.
\]

We prove the bootstrap proposition in the same way as [4]. The first step is to prove (2.19) in [4]. So it is natural to compute the derivative of $E(t)$. The first difference comes from
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |Af|^2 dx = -CK\lambda - CK_w - \int_{T_\pi} AfA(u \cdot \nabla f) dx - \int_{T_\pi} AfA(\beta \partial_\varphi \phi) dx.
\]
Compared with (2.20) in [4], there is an additional term $- \int AfA(\beta \partial_\varphi \phi) dx$ on the right hand side, so we need to control this term besides three old contributions: Transport, Reaction and Remainder (see (2.26) in [4] for details). Controls of old contributions have no difference with Propositions 2.2, 2.3 and 2.6 in [4]. For Proposition 2.4 in [4], in fact their proof can get a more precise elliptic control (see the treatments of $T^1$ and $T^2$ in Subsection 4.2 of [4] for details), which will be useful in dealing with our new term.

**Proposition 6.2** (Precision elliptic control). **Under the bootstrap hypotheses** (B1-B3),
\[
\left\| \left( \frac{\partial_\theta}{t \partial_\varphi} \right)^{-1} (\partial_\varphi^2 + (\partial_\theta - t \partial_\varphi)^2) \left( \frac{\nabla \varphi}{\langle \varphi \rangle^2} A + \sqrt{\frac{\partial_\varphi w}{w}} \tilde{A} \right) \phi_1 \right\|_{L^2(\Omega)}^2 \lesssim \epsilon^2 CK\lambda + \epsilon^2 CK_w + \epsilon^2 \sum_{i=1}^2 (CCK_{\lambda}^i + CCK_{w}^i),
\]
where $\phi_1 = P_{\varphi \neq 0} f - \Delta_L^{-1} P_{\varphi \neq 0} f$, $\Delta_L = \partial_\varphi^2 + (\partial_\theta - t \partial_\varphi)^2$ and $P_{\varphi \neq 0} f = f - \langle f \rangle = f - \int_{\Omega} f dx$.

Proposition 2.5 in [4] can be proved in the same way with a little change since equation (6.1) is slightly different with (2.11) in [4]. Denote $\tilde{h}(t, \varphi) = \vartheta'(t, \varphi) - 1$ and write
\[
(6.2) \quad \partial_t h + [\partial_t \varphi] \partial_\varphi h = \frac{1}{t} (-f_0 - h) = \vartheta' \partial_\varphi [\partial_t \varphi] := \tilde{h}(t, \varphi).
\]
From (6.2) and (6.1), we derive
\[
(6.3) \quad \partial_t \tilde{h} = -\frac{\tilde{h}}{t} - \frac{2}{t} (\partial_t f_0 + \partial_t h) = -\frac{2}{t} \tilde{h} - [\partial_t \varphi] \partial_\varphi \tilde{h} + \frac{1}{t} (\vartheta' \nabla \varphi P_{\varphi \neq 0} \nabla f) + \frac{1}{t} (\beta \partial_\varphi \phi),
\]
which has nothing different with (8.9) in [4] since $\langle \beta \partial_\varphi \phi \rangle = 0$. So the same argument is valid.

The only thing left is to treat the new term $\int_{T_\pi} AfA(\beta \partial_\varphi \phi) dx$. Since $\int_{T_\pi} AfA(\partial_\varphi \phi_1) dx = 0$, we have
\[
\int_{T_\pi} AfA(\beta \partial_\varphi \phi) dx = \beta \int_{T_\pi} AfA(\partial_\varphi \phi_1) dx.
\]
We get the following estimate to control this term.
Proposition 6.3 (New term). Under the bootstrap hypotheses (B1-B3), we have
\[
\left| \int_{T_2} AfA\partial_2\phi_1dx \right| \lesssim \varepsilon CK_\lambda + \varepsilon CK_w + \frac{\varepsilon^3}{\langle t \rangle^{2-\kappa_D/2}} + \varepsilon CK_{\theta,1} + \varepsilon CK_{\theta,1}.
\]

Proof. Divide the new term into two parts:
\[
\int_{T_2} AfA\partial_2\phi_1dx = \frac{1}{2\pi} \sum_{k\neq 0} \int \overline{A_k^2(t,\eta)\hat{f}_k(\eta)}(\chi^R + \chi^{NR})d\eta = R_1 + R_2,
\]
where \( \chi^R = I_{t\in I_k,\eta} \) and \( \chi^{NR} = I_{t\notin I_k,\eta} \). For \( R_1 \) and \( R_2 \), we have
\[
|R_1| \lesssim \sum_{k\neq 0} \int \overline{A_k^2(t,\eta)\hat{f}_k(\eta)}(\chi^R + \chi^{NR})d\eta = \sum_{k\neq 0} \int \overline{\partial_t w}\frac{\hat{w}}{w}A\partial_2\phi_0f
\]
\[
\lesssim \frac{1}{\varepsilon} \left( \frac{\varepsilon}{\langle t \rangle^{2s}} \right)^2 + \varepsilon \left( \frac{\varepsilon}{\langle t \rangle^{2s}} \right)^2 + \frac{\varepsilon}{\langle t \rangle^{2s}} \lesssim \varepsilon CK_\lambda + \varepsilon CK_w + \frac{\varepsilon^3}{\langle t \rangle^{2-\kappa_D/2}} + \varepsilon CK_{\theta,1} + \varepsilon CK_{\theta,1}.
\]

By Cauchy-Schwarz inequality, we have
\[
\left| \int_{T_2} AfA\partial_2\phi_1dx \right| \lesssim \varepsilon CK_\lambda + \varepsilon CK_w + \frac{\varepsilon^3}{\langle t \rangle^{2-\kappa_D/2}} + \varepsilon CK_{\theta,1} + \varepsilon CK_{\theta,1}.
\]

The conclusion then follows from the estimates
\[
\langle t \rangle^{2s} \left\| \nabla A\partial_2\phi_1 \right\|_{L^2(\Omega)}^2 \lesssim \left| \frac{\partial_\theta}{\langle t \rangle^s} \right|^{-1} \Delta A\phi_1 \right\|_{L^2(\Omega)}^2,
\]
\[
\left\| \frac{\partial_t w}{w} \partial_2 |\chi^R A\phi_1| \right\|_{L^2(\Omega)}^2 \lesssim \left| \frac{\partial_\theta}{\langle t \rangle^s} \right|^{-1} \Delta A\phi_1 \right\|_{L^2(\Omega)}^2.
\]

Here, the proof of (6.4) is similar to that of Lemma 6.1 in [4], and we thus omit it. \( \square \)

Now, we give the proof of \( \text{Theorem 1.7} \).

Proof of Theorem 1.7. Controls of Transport, Reaction and Remainder contributions are the same as Propositions 2.2, 2.3 and 2.6 in [4]. By Propositions 6.2 and 6.3, we have
\[
\left| \int_{T_2} AfA\partial_2\phi_1dx \right| \lesssim \varepsilon CK_\lambda + \varepsilon CK_w + \frac{\varepsilon^3}{\langle t \rangle^{2-\kappa_D/2}} + \varepsilon CK_{\theta,1} + \varepsilon CK_{\theta,1}
\]
which shows that the new term has similar estimates with Reaction, and thus can be absorbed into Reaction. The method in the proof of Proposition 2.1 in [4] is still valid. Meanwhile, the change of coordinates cause no difference due to \( \langle \beta \partial_z \phi \rangle = 0 \) in (6.3). Therefore, Theorem 1 in [4] is also true for the \( \beta \)-plane equation on \( \Omega = T_{2\pi} \times \mathbb{R} \).

\[ \square \]

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