Note

Effective action in elliptic and hyperbolic spacetimes

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Abstract

The full one-loop effective action is computed for both hyperbolic and elliptic spaces.

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1. Introduction

There are not many examples of exact effective actions, even to one loop order, and even for scalar fields. The usual approach (cf [1] and references therein) only determines the ultraviolet divergent small proper-time DeWitt coefficients. This leaves undetermined the infrared behaviour.

There is however a theorem [2, 3] asserting that whenever the spacetime manifold is such that its Ricci curvature is non-negative, \( R_{\mu \nu} \geq 0 \), and the manifold has got maximal volume growth, then the heat kernel corresponding to the ordinary Laplacian obeys

\[
\lim_{\tau \to \infty} V \left( \sqrt{\tau} \right) K \left( x, x', \tau \right) = \frac{\Omega(n)}{(4\pi)^{n/2}},
\]

(1)

where \( \Omega(n) \) is the volume of the unit ball in \( \mathbb{R}^n \) and \( V \left( \sqrt{\tau} \right) \) is the volume of the geodesic ball centred at \( x' \) and radius \( \sqrt{\tau} \). The asymptotic volume ratio is defined as

\[
\lim_{r \to \infty} \frac{V(r)}{r^n} = \Theta > 0
\]

(2)

the fact that \( \Theta > 0 \) is what qualifies for the assertion that the manifold has maximal volume growth. In fact this is a generalization of a previous work by Li and Yau [4] (cf in particular theorem 4.1) asserting that with the same hypothesis there should exist a constant \( C(\epsilon) \) such

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that
\[ \frac{1}{C(\epsilon)V(\sqrt{\tau})} e^{-\frac{\sigma(x,x')}{(2-\epsilon/2)\tau}} \leq K(x,x';\tau) \leq \frac{C(\epsilon)}{V(\sqrt{\tau})} e^{-\frac{\sigma(x,x')}{(2+\epsilon/2)\tau}} \] (3)

the situation improves in spacetimes with special amounts of symmetry, where we can find exact expressions for the heat kernel corresponding to the ordinary Laplacian \(^1\).

In this work we shall precisely be concerned with maximally symmetric spacetimes and their Euclidean counterparts. The Riemann tensor obeys
\[ R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \] (4)
and the curvature can be positive or negative
\[ R = \pm \frac{n(n-1)}{L^2}. \] (5)

Elliptic spacetimes have got positive curvature. In our conventions, anti-de Sitter spacetime (AdS\(_n\)) is one such. In Poincaré coordinates
\[ ds^2_{AdS_n} = \sum_{i=1}^{n-1} \eta_{ij} dy^i dy^j - L^2 dz^2 / z^2, \] (6)
(where as usual, \(\eta_{ij} \equiv \text{diag}(1,-1,\ldots,-1)\)). Its Euclidean version is the sphere \(S_n\), which does not admit Poincaré coordinates \([7]\), although, as all other spacetimes considered here, does admit stereographic coordinates.

Hyperbolic spacetimes have got negative curvature. De Sitter (dS\(_n\)) spacetime falls in this category. In Poincaré coordinates
\[ ds^2_{dS_n} = -\sum_{i=1}^{n-1} \delta_{ij} dy^i dy^j + L^2 dz^2 / z^2 \] (7)
with \(z\) is a timelike coordinate, the Euclidean version reads
\[ ds^2_{EdS_n} = \sum_{i=1}^{n-1} \delta_{ij} dy^i dy^j + L^2 dz^2 / z^2 \] (8)

Correlators in this family of spaces, including the energy–momentum tensor, have been thoroughly analysed in \([8]\) under the hypothesis that those only depend on the invariant arc length, \(s\) and its derivatives. Physically this is equivalent to the assumption that the relevant vacuum enjoys all spacetime isometries.

For timelike geodesics the arc length coincides with the physical proper time. We shall refrain from using this notation though because we shall use it in Schwinger’s sense later.

We shall also assume that hypothesis in the present work; this amounts to demand invariance (or proper behaviour) under all conformal isometries \([9]\). When working with Lorentzian signature the square of the arc length is not positive semidefinite; it can become zero or even negative. This is in fact the reason why Synge introduced the world function \([10]\), which is essentially the square of the invariant arc length. Our formulas however remain formally valid with appropriate analytic continuation.

1 Related computations have been done for the Dirac operator by Camporesi \([5]\), see also \([6]\).
2. Hyperbolic heat kernel

Acting on functions of the geodesic arc length [8] the Laplacian in $\mathbb{H}_n$ reads

$$\Box = \frac{\partial^2}{\partial s^2} + \frac{n - 1}{L \tanh \frac{s}{L}} \frac{\partial}{\partial s}$$

the corresponding heat equation reads

$$\frac{\partial}{\partial \tau} K_{p+1}(\tau, s) = D_p K_{p+1}(\tau, s),$$

where

$$D_p K_{p+1}(\tau, s) = \left( \frac{\partial^2}{\partial s^2} + \frac{p}{L \tanh \frac{s}{L}} \frac{\partial}{\partial s} \right) K_{p+1}(\tau, s)$$

in the following we shall often work with in terms of a dimensionless proper time and dimensionless arc length, $\frac{s}{L}$.

- In the flat limit $s \to 0$, then $\tanh \frac{s}{L} \approx \frac{s}{L}$, and $D_p K_{p+1}(\tau, s)$ reduces to

$$D_p K_{p+1}(\tau, s) = \left( \frac{\partial^2}{\partial s^2} + \frac{ps}{s \partial s} \right) K_{p+1}(\tau, s)$$

which together with the heat equation, equation (10), yields

$$\frac{\partial^2}{\partial s^2} K(\tau, s) + \frac{p}{s} \frac{\partial}{\partial s} K(\tau, s) = \frac{\partial}{\partial \tau} K(\tau, s).$$

This is just the flat space heat equation in arc length variables. The well-known canonical solution is

$$K(\tau, s) = (4\pi \tau)^{-(p+1)/2} \exp \left( -\frac{s^2}{4\tau} \right)$$

this obeys the correct flat boundary condition

$$\lim_{\tau \to \infty} K_0(\tau, s) = \delta^n(s - x') \neq \delta(s).$$

Please note that the $n$-dimensional delta is much more singular than the one-dimensional delta of solely the arc.

- In the opposite limit, $s \to \infty$, i.e. $\tanh \frac{s}{L} \approx 1$, the heat equation reduces to

$$\frac{\partial^2}{\partial s^2} K(\tau, s) + \frac{p}{L} \frac{\partial}{\partial s} K(\tau, s) = \frac{\partial}{\partial \tau} K(\tau, s)$$

whose general solution is a wave packet composed out of

$$K(\tau, s) = \sqrt{\frac{\pi}{\tau}} \exp \left( -\frac{s^2}{4\tau} - \frac{ps}{2L} \frac{p^2 \tau}{4L^2} \right)$$

this does not satisfy the boundary condition at $\tau = 0$, but this is presumably natural because our approximation is valid for large values of $s$ only.
• Let us find a recurrence relation in flat spacetime. This recurrence relation is known in the mathematical literature [11], but our proof stems directly from the heat equation. Apply the lineal operator

\[ D^p_L = \frac{\partial^2}{\partial s^2} + \frac{p}{s} \frac{\partial}{\partial s} \]

with \( p \) completely arbitrary

\[ D^p_L \left( \frac{K'_{p-1}}{s} \right) = \frac{2 - p}{s^3} K'_{p-1} + \frac{p - 2}{s^2} K''_{p-1} + \frac{1}{s} K'''_{p-1}, \]  

(19)

where \( K' = \frac{\partial K}{\partial s} \). On the other hand, let us assume that the function \( K \) obeys the heat kernel equation on \( p - 2 \) dimension, and take one more derivative

\[ \left( \frac{\partial^3}{\partial s^3} + \frac{p - 2}{s} \frac{\partial^2}{\partial s^2} - \frac{p - 2}{s^2} \frac{\partial}{\partial s} \right) K_{p-1} = \frac{\partial}{\partial \tau} K'_{p-1} \]

(20)

obtain

\[ D^p_L \left( \frac{K'_{p-1}}{s} \right) = \frac{1}{s} \frac{\partial}{\partial \tau} K'_{p-1} \]

(21)

in conclusion, this implies

\[ K_{p+2}(\tau, s) = -\frac{1}{2\pi s} \frac{\partial}{\partial s} K_p(\tau, s). \]

(22)

Of course, in flat spacetime, where we know the full dependence of the heat kernel with the spacetime dimension

\[ K_n(s) = \frac{1}{(4\pi \tau)^{n/2}} e^{-s^2/4\tau} \]

(23)

there is another trivial recurrence relation

\[ K_{n+1}(s) = -\frac{1}{s} \sqrt{\frac{\tau}{\pi}} \frac{\partial K_n(s)}{\partial s}. \]

(24)

• Let us try to generalize this to the hyperbolic case. Consider the expression (which is independent of the dimension \( p \))

\[ D^p \left( \frac{K'_{p-1}}{\sinh s} \right) = \frac{1 - p}{\sinh s} K'_{p-1} + \frac{2 - p}{\sinh^2 s} K''_{p-1} + \frac{(p - 2) \cosh s}{\sinh^3 s} K'''_{p-1} + \frac{1}{\sinh s} K''''_{p-1} \]

(25)

taking one more derivative into the heat equation for \( p - 2 \) dimension

\[ \left( \frac{\partial^3}{\partial s^3} + \frac{p - 2}{\tanh s} \frac{\partial^2}{\partial s^2} - \frac{p - 2}{\sinh^2 s} \frac{\partial}{\partial s} \right) K_{p-1} = \frac{\partial}{\partial \tau} K'_{p-1} \]

(26)

obtain
\[
D_p \left( \frac{K'_{p-1}}{\sinh s} \right) = \frac{1 - p}{\sinh s} K'_{p-1} + \frac{1}{\sinh s} \frac{\partial}{\partial \tau} K'_{p-1} + e^{-(1-p)\tau} \frac{\partial}{\partial \tau} \left( e^{(1-p)\tau} \frac{K'_{p-1}}{\sinh s} \right)
\]

this implies
\[
\frac{K'_{p-1}}{\sinh s} = e^{-(1-p)\tau} K_{p+1}
\]

because with the heat equation
\[
D_p \left( e^{-(1-p)\tau} K_{p+1} \right) = e^{-(1-p)\tau} \frac{\partial}{\partial \tau} K_{p+1}
\]

finally we have the recurrence relationship
\[
K_{p+1}(\tau, s) = -\frac{e^{-p\tau}}{2\pi \sinh s} \frac{\partial}{\partial s} K_p(\tau, s)
\]

it is to be stressed that this relationship is a consequence of the heat equation exclusively, independently of any boundary conditions.

2.1. Dimensional reduction from odd to even dimensions

The starting point in the recurrence relationship is the formal one-dimensional case. It can be easily checked that
\[
K_1(\tau, s) = \frac{e^{-\tau^2}}{(4\pi \tau)^{1/2}}
\]

it obeys
\[
\frac{\partial^2}{\partial s^2} K_1(\tau, s) = \frac{\partial}{\partial \tau} K_1(\tau, s)
\]

the engineering dimensions of the heat kernel are determined by its behaviour when \( \tau = 0 \).

From here on we can determine via the recurrence all odd dimension heat kernels.

The recurrence relationship easily leads to
\[
K_{2p+1}(\tau, s) = e^{-p^2 \tau} \left( \frac{1}{2\pi \tau} - \frac{1}{\sinh s} \frac{\partial}{\partial s} \right)^p K_1(\tau, s)
\]

here we can see why dimensional reduction from \( n = 2p + 1 \) toward \( n = 2p \) (what Jacques Hadamard [12] dubs ‘the method of descent’) does not naively work in this case. The reason is clearly that
\[
p \equiv \frac{n - 1}{2}
\]

is a fractional number for even \( n \), so that we have to take a fractional power of the operator
\[
D \equiv -\frac{1}{\sinh s} \frac{\partial}{\partial s}
\]
which has been worked out for example in [13] in the framework of Schrödinger’s equation, with the result for hyperbolic spaces,

\[
\left( -\frac{1}{\sinh s} \frac{\partial}{\partial s} \right)^{\frac{n}{2}} f(s) = \frac{1}{\sqrt{\pi}} \int_s^\infty dx \left( -\frac{1}{\sinh x} \frac{\partial}{\partial x} \right)^{n/2} f(x) \frac{\sinh x}{\sqrt{\cosh x - \cosh s}}
\]

(36)

let us examine in detail a couple of examples.

3. Four dimensional hyperbolic space $\mathcal{H}_4$

Our recurrence relation equation (33), leads to

\[
K_4(\tau, s) = \sqrt{2} e^{-\frac{m^2 L^2}{4} \tau} \left( \frac{2}{4\pi \tau} \right)^{3/2} \int_s^\infty dx \frac{x^2 - 2\tau + 2x \cosh x}{\sinh^2 x} \frac{\sinh x}{(\cosh x - \cosh s)^{1/2}} e^{-\frac{x^2}{4\tau}}
\]

(37)

the effective action (please cf to related work found in e.g. [14]) is

\[
V_{\text{eff}}[\bar{\phi}] = \int_0^\infty \frac{d\tau}{\tau} K(\tau, s)
\]

(38)

for a $\frac{g}{4!} \phi^4$ self-interaction is then given in the effective potential approximation by

\[
V_{\text{eff}}[\bar{\phi}] = \sqrt{2} \int_s^\infty dx \frac{1}{x^3 \sinh^2 x (\cosh x - \cosh s)^{1/2}} \frac{\sinh x}{(\cosh x - \cosh s)^{1/2}} \left[ 8 + 4x \sqrt{9 + 4M^2 L^2} + x^3 (9 + 4M^2 L^2) \right]
\]

\[
+ 2x \left( 2 + x \sqrt{9 + 4M^2 L^2} \right) \coth x \] e^{-x \sqrt{9 + 4M^2 L^2}} + \frac{g}{4!} \bar{\phi}^4
\]

(39)

where

\[
M^2 \equiv m^2 + \frac{g}{2} \bar{\phi}^2
\]

(40)

the minimum is still at $\bar{\phi} = 0$ as long as $m^2 \geq 0$.

4. Five dimensional hyperbolic space $\mathcal{H}_5$

Working with our recurrence equation (33) again we are lead to the five-dimensional heat-kernel

\[
K_5(\tau, s) = \frac{e^{-4\tau - M^2 L^2 \tau - \frac{s^2}{2}}}{(4\pi \tau)^{5/2}} \frac{\sinh^2 s}{\sinh^2 s \sqrt{\cosh x - \cosh s}}
\]

(41)

There is no polynomial interaction of the type $g\phi^n$ that enjoys dimensionless coupling constant in five dimensions, since the field itself has $[\phi] = \frac{3}{2}$.

The only potential (besides the mass term) with a positive dimension coupling constant corresponds to a $g\phi^3$ interaction with $[g] = \frac{1}{2}$, which we will assume henceforth.
The effective potential, in turn, reads

\[ V_{\text{eff}}[\bar{\phi}] = \frac{1}{(2\pi)^2} \frac{e^{-\sqrt{M^2L^2+4}}}{s^3 \sinh^2 s} \left[ 2 + 2s \sqrt{M^2L^2 + 4} + s^2(M^2L^2 + 4) \right] + s \left( 1 + s \sqrt{M^2L^2 + 4} \right) \coth s \right] + \frac{g}{31} \bar{\phi}^3, \tag{42} \]

where the effective mass in order to compute the effective potential is

\[ M^2 \rightarrow m^2 + g \bar{\phi} \tag{43} \]

and we are making the approximation that \( \bar{\phi} \) is constant

\[ \partial_\mu \bar{\phi} = 0 \tag{44} \]

the minimum is still at the origin \( \bar{\phi} = 0 \) as long as \( m^2 \geq 0 \).

5. Elliptic spacetimes

Acting on functions of the geodesic arc length [8] the Laplacian in \( \mathbb{E}_n \) reads

\[ \Box = \frac{\partial^2}{\partial s^2} + \frac{n-1}{L \tan \frac{\theta}{2}} \frac{\partial}{\partial s} \tag{45} \]

there is a quite similar recurrence in the case of positive curvature, with trigonometric functions taking the place of hyperbolic ones. We shall be brief here. Again we start by acting with the operator of heat kernel over \( K \)

\[ \mathcal{D}_p \left( \frac{K'_{p-1}}{\sin s} \right) = \frac{p-1}{\sin s} K'_{p-1} + \frac{2 - p}{\sin^2 s} K''_{p-1} \tag{46} \]

on other hand if derive again the heat equation for \( p-2 \) dimension

\[ \left( \frac{\partial^3}{\partial s^3} + \frac{p-2}{\sin s} \frac{\partial^2}{\partial s^2} + \frac{p-2}{\sin^2 s} \frac{\partial}{\partial s} \right) K_{p-1} = \frac{\partial}{\partial \tau} K'_{p-1} \tag{47} \]

we obtain

\[ \mathcal{D}_p \left( \frac{K'_{p-1}}{\sin s} \right) = \frac{p-1}{\sin s} K'_{p-1} + \frac{1}{\sin s} \frac{\partial}{\partial s} K_{p-1} = e^{-(p-1)\tau} \frac{\partial}{\partial \tau} \left( e^{(p-1)\tau} \frac{K'_{p-1}}{\sin s} \right) \tag{48} \]

which implies

\[ \frac{K'_{p-1}}{\sin s} = e^{-(p-1)\tau} K_{p+1} \tag{49} \]

because with the heat equation

\[ \mathcal{D}_p \left( e^{-(p-1)\tau} K_{p+1} \right) = e^{-(p-1)\tau} \frac{\partial}{\partial \tau} K_{p+1} \tag{50} \]

finally we get the recurrence relationship

\[ K_{p+2}(\tau, s) = -\frac{e^{p\tau}}{2\pi \sin s} \frac{\partial}{\partial s} K_p(\tau, s) \tag{51} \]
again the recurrence relationship for all odd dimension reads

\[ K_{2p+1}(\tau, s) = \frac{e^{p\tau}}{(2\pi)^p} \left( -\frac{1}{\sin s \partial s} \right)^p K_1(\tau, s) \]  

(52)

with

\[ \left( -\frac{1}{\sin s \partial s} \right)^{n+1} f(s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \left( -\frac{1}{\sin x \partial x} \right)^{n/2} f(x) \frac{\sin x}{\sqrt{\cos x - \cos s}}. \]  

(53)

Let us work out an example in some detail.

6. Elliptic four dimensional space, \( \mathcal{E}_4 \)

The starting point, as well as in the hyperbolic case, is the \( n = 1 \) heat kernel which is common to both cases

\[ K_1 = \frac{e^{-\frac{s^2}{4}}}{(4\pi \tau)^{1/2}}. \]  

(54)

The recurrence (52) leads to the four-dimensional heat-kernel

\[ K_4(\tau, s) = \frac{\sqrt{2} e^{\frac{M^2L^2}{4\pi}}}{(4\pi \tau)^{5/2}} \int_{-\infty}^{\infty} dx \frac{x^2 - 2\tau + 2x \cot x}{\sin^2 x} \frac{\sin x}{(\cos x - \cos s)^{1/2}} e^{-x^2/4\pi}. \]  

(55)

The effective action in turn, is given by

\[ V_{\text{eff}}[\bar{\phi}] = \frac{\sqrt{2}}{(4\pi)^2} \int_{-\infty}^{\infty} dx \frac{1}{x^3 \sin^2 x} \frac{\sin x}{(\cos x - \cos s)^{1/2}} \times \left[ 8 + 4x \sqrt{-9 + 4M^2L^2} + x^2(-9 + 4M^2L^2) + 2x \left( 2 + x \sqrt{-9 + 4M^2L^2} \right) \cot x \right] e^{-x^2/9 + 4M^2L^2/2} + \frac{g^4}{4! \bar{\phi}^4}, \]  

(56)

where the effective mass is given by

\[ M^2 \to m^2 + g\bar{\phi}. \]  

(57)

In the conclusions we explain some of the implications of this expression.

7. Elliptic five dimensional space, \( \mathcal{E}_5 \)

Again the recurrence (52) leads to the five-dimensional heat-kernel

\[ K_5(\tau, s) = \frac{e^{4\tau - M^2L^2 \tau}}{(4\pi \tau)^{3/2}} \frac{s^2 - 2\tau + 2s \cot s}{\sin^2 s}. \]  

(58)
The effective action is defined by

\[
V_{\text{eff}}[\bar{\phi}] = \frac{1}{(2\pi)^2} e^{-s\sqrt{M^2L^2}} \left[ 2 + 2s \sqrt{M^2L^2 - 4} + s^2(M^2L^2 - 4) \right] + s \left( 1 + s \sqrt{M^2L^2 - 4} \right) \cot s + \frac{g}{3!} \bar{\phi}^3,
\]

where the effective mass in order to compute the effective potential is given by

\[M^2 \to m^2 + g\bar{\phi}.\] (59)

Also here we shall explain the physical meaning of this formula in the conclusions.

8. Conclusions

It is remarkable that the effective action for massless scalars in elliptic spacetimes (which are compact when Euclidean) is infrared (IR) divergent, whereas in (noncompact even when Euclidean) hyperbolic spacetimes it is not.

Actually, there is always a minimal value for the effective mass above which the effective action does converge.

Consider first the hyperbolic case. In four dimensions, \(H_4\), the effective potential is well defined and positive provided the effective mass obeys

\[M^2 \geq -\frac{9}{4L^2}.\] (60)

In five dimensions, \(H_5\), the same is true whenever

\[M^2 \geq -\frac{4}{L^2}.\] (61)

Turning now to elliptic spaces, in four dimensions \(E_4\) as indicated in (56), whenever

\[M^2 \equiv m^2 + g\bar{\phi} \geq \frac{9}{4L^2}\] (62)

the exponential term vanishes in the deep infrared \((s \to \infty)\) and the effective action converges in this limit. On the other hand, for smaller values of the effective mass the effective potential gets an imaginary part.

In the five dimensional elliptic space, \(E_5\) equation (59), the corresponding bound is given by

\[M^2 \equiv m^2 + g\bar{\phi} \geq \frac{4}{L^2}.\] (63)

The physical reason for the divergence in the massless case \(m = 0\) might be related to the presence of antipodal points. The origin of those stems from the fact that the Laplacian in elliptic spaces

\[\Box = \frac{\partial^2}{\partial s^2} + \frac{n-1}{L} \tan \frac{L}{s} \frac{\partial}{\partial s} + \frac{\partial^2}{\partial \bar{s}^2},\] (64)
is singular not only at coincidence (that is, when $s = 0$), but also at fine separation, namely whenever

$$s = \frac{(2n + 1)\pi L}{2}$$

for any admissible integer $n$.

This does not happen in hyperbolic spaces, where the hyperbolic tangent only vanishes when $s = 0$. This is the reason why Schrödinger [15] proposed already in 1957 the elliptic interpretation, where all antipodal points are identified. While suggestive, the fact that the spacetime $dS_n/\mathbb{Z}_2$ is not orientable poses many physical problems.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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