Isotropic Lifshitz point in the $O(N)$ Theory

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(Dated: October 1, 2018)

The presence of an isotropic tricritical Lifshitz point for the $O(N)$ scalar theory is investigated at large $N$ in the improved Local Potential Approximation (LPA') by means of the Functional Renormalization Group equations. At leading order, the non-trivial Lifshitz point is observed if the number of dimensions $d$ is taken between $d = 4$ and $d = 8$, and the eigenvalue spectrum of the associated eigendirections is derived. At order $1/N$ of the LPA' the anomalous dimension $\eta_N$ is computed and it is found to vanish both in $d = 4$ and $d = 8$. The dependence of our findings on the infrared regulator is discussed.

Keywords: Functional Renormalization Group; Lifshitz point; $1/N$ expansion

Introduction – The description of a tricritical Lifshitz point by a Landau-Ginzburg $\phi^4$ model, where the derivatives of the field with respect to the coordinates of a $m$-dimensional subset of a $d$-dimensional space and those of the complementary $(d - m)$-dimensional subspace possess different scaling laws, was first presented in [1]. More specifically, in [1] the kinetic term with square gradient of the field, $O(\partial^2)$, is kept finite only for the second subset of coordinates, while the corresponding term of the $m$-dimensional subset is suppressed, so that the term with four powers of the gradient, $O(\partial^4)$, becomes the leading kinetic term of the $m$-dimensional subspace and this induces drastic changes in the scaling properties of the theory.

The Lifshitz points, which are related to the coexistence on the phase diagram of three phases, one with vanishing order parameter, another with finite constant order parameter and the third characterized by a modulated order parameter with finite wave vector, find application in various fields such as magnetic systems as well as polymer mixtures or high $T_C$ superconductors (for reviews see [2, 3]), but, recently, also in different contexts such as Lorentz symmetry violation, [4, 5], or emergent gravity theories [6-10]. In addition, an oscillating phase has been predicted for a very wide class of systems [11-15], and it is conceivable to expect that a Lifshitz point could be associated to these modulated phases. In this sense, a more complete understanding of the properties of the Lifshitz point is certainly desirable.

Rather than considering the general case with $0 < m < d$, where the different scaling properties in the two separate subspaces lead to a peculiar critical behaviour that involves two different anomalous dimensions and correlation lengths, we shall focus on the isotropic case with $m = d$. In fact, if $m < d$, due to the different behaviour of the two sets of coordinates, the isotropy of the problem is lost while, when $m = d$, all the space coordinates have the same critical behaviour and spatial isotropy is preserved. Clearly, in this latter case the critical scaling remains different from the standard one because, as explained before, the kinetic term in the action is quartic, rather than quadratic, in the field derivatives.

The critical properties of the Lifshitz point were studied in the $\varepsilon$-expansion [1] as well as in the $O(1/N)$ expansion [16]. The isotropic case $m = d$ was considered within an expansion around $d = 8$ and $\varepsilon = 8 - d$ [17] while, recently, a numerical Monte-Carlo study indicated a possible disappearance of the Lifshitz point, when fluctuations are properly included [18].

Furthermore, another non-perturbative technique already employed to study this problem is the Functional Renormalization Group (FRG) [19-21], which consists of a set of differential flow equations either for various operators entering the effective action of the theory, or for one or more $n$-point Green functions derived from the effective action. Fixed points correspond to stationary points of these equations and the critical exponents, that classify relevant, marginal and irrelevant operators, are extracted by determining the eigenvalue spectrum of the linear reduction of the differential equations around the fixed point solutions. Coming to the Lifshitz point, the FRG was applied to study this problem for a one component scalar theory, $N = 1$, [22], and for the $N = 3$ theory, [23], both in the uniaxial ($m = 1$) case. Finally, the isotropic case ($m = d$) with $N = 1$ was considered in [24] and, in this last case, the Proper Time version [25-27] of the FRG, which can be formally derived in the framework of the background field flows [28, 29], was used because it proved to be quite accurate and suitable for the numerical analysis of the critical properties of a theory at a fixed point [27, 30-32], and, in addition, the Proper Time flow equation of the $O(\partial^2)$ operator (coupled to the potential and to the $O(\partial^4)$ operator equations), that is necessary to treat a Lifshitz point, had been already derived in [32].

The numerical analysis performed in [24] for the $N = 1$ theory, shows at the lowest order (in the Local Potential Approximation - LPA), i.e. by considering the fixed potential equation only, that a non-trivial solution exists when the number of spatial dimensions is $4 < d < 8$, and for $d \geq 8$ the solution merges with the trivial, gaussian fixed point, while for $d \leq 4$ the asymptotic structure of the differential equation changes and no discrete set of non-trivial solution is available. Then, when going beyond the LPA and including the differential equations for the $O(\partial^2)$ and $O(\partial^4)$ operators, a solution was observed in the range $5.5 < d < 8$, but the numerical analysis for smaller $d$ becomes too demanding and it was not possible to establish whether the Lifshitz point survives down to $d = 4$ or, rather, the fluctuations associated with higher derivatives terms, $O(\partial^2)$ and $O(\partial^4)$, effectively destroy the critical behaviour when $d$ approaches 4.

In this letter we consider another aspect of the problem and
analyze the existence of a Lifshitz point for a scalar $O(N)$-symmetric theory, in order to find out whether the critical behaviour survives to the presence of the strong infrared fluctuations due to the transverse modes. To this aim, a numerical analysis would require the resolution of a very large number of differential equations that would probably present the same kind of problems observed for the simpler $N = 1$ case.

Therefore, we follow a different approach. We start by considering the procedure developed in [33,34], where the flow equation for the effective action is projected onto a set of flow equations for the $n$-point Green functions which is to be truncated at some specific $n$ and, for our purposes, we retain the three equations for the potential and the longitudinal and transverse two-point functions. Then, we treat these equations in the framework of the $1/N$ expansion to extract the corresponding $O(1/N)$ contribution to the anomalous dimension $\eta$.

Actually, this combined procedure neither amounts to a full resolution of the flow equations for the two point functions derived in [33,34], nor to a complete $O(1/N)$ computation, however it allows us to go one step beyond the leading order of the $1/N$ expansion (at which the full eigenvalue spectrum of the Lifshitz point is determined) and establish both the survival of the Lifshitz point and the first non-vanishing contribution to $\eta$ at criticality. In particular we find that the expression derived for $\eta$ according to this procedure reduces to the result that is obtained in the minimal improvement of the LPA [59], also known as LPA$'$.

**Flow equations** – In order to write down the fixed point equation, we start from the full FRG equation [21]:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int_q \partial_q R_k(q) \left[ \Gamma_k^{(2)}(q,-q;\phi) + R_k(q) \right]^{-1} \tag{1}$$

$\Gamma_k[\phi]$ being the running effective action at scale $k$, and $R_k(q)$ a suitable regulator that suppress the modes with $q \ll k$ and allows to integrate those with $q \gg k$. The specific choice of the regulator $R_k(q)$ is discussed below.

Rather than introducing the running parameters by means of an explicit form of the effective action, we proceed by displaying the second functional derivative of the effective action $\Gamma_k^{(2)}(p;\phi) \equiv \delta^2 \Gamma_k/(\delta \phi_a(p) \delta \phi_b(-p))$ that, according to the $O(N)$ symmetry, has the general form ($\phi \equiv \phi_\alpha / 2$):

$$\Gamma_k^{(2)}(p,\phi) = \Gamma_A(p,\rho) \delta_{ab} + \phi_a \phi_b \Gamma_B(p,\rho) \tag{2}$$

Then, we parametrize $\Gamma_A$ and $\Gamma_B$ in terms of the potential $V$ and of the renormalization functions $Z_A$, $Z_B$, $W_A$, $W_B$, i.e. the coefficients of the quadratic and quartic powers of the momentum $p$:

$$\Gamma_A(p^2,\rho) = W_A(\rho) p^4 + Z_A(\rho) p^2 + V$$

$$\Gamma_B(p^2,\rho) = N W_B(\rho) p^4 + N^2 Z_B(\rho) p^2 + V'$$

where prime indicates the derivative with respect to $\rho$ and $N$ is the number of field components.

The factor $N$ appearing in front of $W_B$ and $Z_B$ is due to a specific rescaling of the potential and of the field with respect to the standard definitions, $V \to NV$ and $\phi_a \to \sqrt{N} \phi_a$, which is made in order to derive a fixed point equation that is directly arranged in a $1/N$ expanded structure. Clearly, this rescaling has no effect on $V'$, while it changes $V'' \to V''/N$ as well as the factor $\phi_a \phi_b \to N \phi_a \phi_b$ in Eq. (2), these last two transformations being responsible for the factor $N$ appearing in the definition of $\Gamma_B$ in Eq. (4). Therefore, from Eqs. (3) and (4), it is easy to expect the parameters $W_B$, $Z_B$ to be $1/N$ suppressed with respect to $W_A$, $Z_A$, as it will be checked below.

Then, if we separate the longitudinal (L) and transverse (T) components in the inverse of $\Gamma^{(2)}_{\alpha \beta}(p,\phi)$, that is the propagator of the theory $G_{\alpha \beta}(p,\phi)$, according to:

$$G_{ab}(p,\phi) = \left( \delta_{ab} - \frac{\phi_a \phi_b}{2 \rho} \right) G_T(p,\rho) + \frac{\phi_a \phi_b}{2 \rho} G_L(p,\rho) \tag{5}$$

one finds

$$G_T^{-1}(p,\rho) = \Gamma_A(p,\rho) \tag{6}$$

and

$$G_L^{-1}(p,\rho) = \Gamma_A(p,\rho) + 2 \rho \Gamma_B(p,\rho). \tag{7}$$

It is understood that the field dependent parameters $V, W_A, W_B, Z_A, Z_B$ also depend on the running scale $k$ and, with these settings, we can rely on the derivation of the flow equations carried out in [34]. We define the integrals

$$J_n^{ab}(p,\rho) = \int_q \partial_q R_k(q) \tilde{G}^{n-1}_a(q,\rho) \tilde{G}_{\beta}(p+q,\rho), \tag{8}$$

and

$$I_n^{ab}(\rho) = J_n^{ab}(0,\rho), \tag{9}$$

where $n \geq 1$, $\int_q \equiv \int \frac{d^k q}{(2\pi)^k}$, $\alpha$ and $\beta$ stand either for $L$ or $T$, and

$$\left( \tilde{G}_a(q,\rho) \right)^{-1} \equiv \left( G_0^n(q,\rho) \right)^{-1} + R_k(q). \tag{10}$$

Then, by following [35] (see also [33]), we get the flow equation for the potential $V$:

$$\partial_t V(p) = \frac{1}{2} \left[ \frac{I^{TT}}{1} + \frac{1}{N} \left[ I^{LL}_1 (\rho) - I^{LT}_1 (\rho) \right] \right] \tag{11}$$

and for the two-point functions, properly subtracted of the zero-momentum contribution:

$$\partial_t \left[ \Gamma_X(p^2,\rho) - \Gamma_X(0,\rho) \right] = F_X(p^2,\rho) - F_X(0,\rho) \tag{12}$$

where $X$ stands either for $A$ or $B$, and

$$F_A(p^2,\rho) = -\frac{1}{2} I^{TT}_2 \Gamma_A' + \frac{1}{N} \left[ 2 \rho \left( J^{LT}_1 \Gamma_A^2 + J^{LL}_1 \Gamma_B^2 \right) - I^{LL}_2 \left( \frac{\Gamma_A^2}{2} + \rho \Gamma_A' \right) - I^{TT}_1 \left( \frac{\Gamma_B^2}{2} \right) \right], \tag{13}$$

$$F_B(p^2,\rho) = 0.$$
\[ F_B(p^2, \rho) = J^{TT}_B r^2 B_T - \frac{1}{2} J^{TT}_B r^2 B_T + O \left( \frac{1}{N} \right) \] (14)

Eqs. (12), (13), (14) can be reduced to flow equations either for \( W_X \) or \( Z_X \), by selecting in \( F_X \) the terms proportional respectively to \( p^4 \) or \( p^2 \). Then, it is evident from Eqs. (4), (12) and (14) that, in order to avoid any inconsistency in the \( 1/N \) expansion, \( W_B \) and \( Z_B \) must be \( O(1/N) \) so that \( \Gamma_B \sim O(1) \). Accordingly, we are allowed to neglect \( O(1/N) \) corrections in Eq. (14), as they contribute to \( W_B \) and \( Z_B \) to order \( O(1/N^2) \).

Let us now consider the regulator \( R_k(q) \). A particularly useful regulator, that has the advantage of reducing the integrals to simple structures which can be analytically solved in most cases, was introduced in (16) and has the form:

\[ R_k^0(q) = (k^2 - q^2) \hat{Z}_k \theta(k^2 - q^2) \] (15)

where \( \theta \) is the Heaviside step function and a \( k \)-dependent (but field independent) normalization factor \( \hat{Z}_k \) is included. For the present problem the regulator in Eq. (15) should be modified into \( R_k^0(q) = (k^4 - q^4) \hat{W}_k \theta(k^2 - q^2) \) with \( \hat{W}_k \) taken equal to \( W_A \), evaluated at a particular value of \( \rho : \hat{W}_k = W_A(p) \), with \( p \) to be specified. However, due to the presence of the Heaviside function, the second and higher derivatives of \( R_k^0(q) \) with respect to \( q \), generate a singular behaviour of the integrals involved in this analysis. Therefore it is preferable to replace \( R_k^0(q) \) with a smooth, one-parameter \( (\alpha) \) regulator:

\[ R_k(q) = \frac{\hat{W}_k}{2} \left[ (k^4 - q^4) + \sqrt{(k^4 - q^4)^2 + (2\alpha)^2} \right] \] (16)

In fact, \( R_k(q) \) in Eq. (16) approaches \( R_k^0(q) \) in the limit \( \alpha \rightarrow 0 \), and, for values of the dimensionless parameter \( k^4 \alpha \sim 10^3 \) or larger, \( R_k(q) \) and its first derivative can be practically replaced by \( R_k^0(q) \) in the resolution of the integrals but, on the other hand, all its derivatives are regular so that it does not generate any singularity as long as \( \alpha \) is kept finite, i.e. \( \alpha \neq 0 \).

Incidentally, as the vanishing of the regulator at \( k = 0 \), \( R_{k=0}^0(q) = 0 \), is a necessary requirement of the flow equations, then \( \alpha \) must be a function of the scale \( k \) that vanishes at \( k = 0 \). This can be easily achieved e.g. by taking \( (2\alpha)^{-1} = \Lambda^4 \text{tanh}(k/\Lambda) \), where \( \Lambda \) is a fixed mass scale and \( \alpha \) and \( \mu \) two small dimensionless parameters that can be adjusted to set the size of \( (2\alpha)^{-1} \) and of its derivatives.

Due to the dependence of \( \alpha \) on \( k \), the fixed point equations do contain additional terms proportional to \( \mu (2\alpha)^{-1} / \partial k \) = \( 2 \mu (k/\Lambda)^4 (2\alpha)^{-1} \text{sinh}^{-1}(2(k/\Lambda)^4) \), but one easily realizes that even the largest contributions (proportional to \( \mu \)) encountered in the following calculations, when multiplied by this factor, for sufficiently small values of \( \mu \) turn out to be systematically suppressed with respect to the other terms appearing in the the fixed point equations. Therefore, we neglect the contributions proportional to \( \mu (2\alpha)^{-1} / \partial k \) and simply treat \( \alpha \) as a free parameter.

However, as discussed below, even the regulator in Eq. (16) is not sufficient to get rid of all potentially large (divergent in the limit \( \alpha \rightarrow 0 \)) terms and therefore at some point we find convenient to analyze our equations by adopting the smoother exponential regulator

\[ R_k^b(q) = \frac{b \hat{W}_k q^4}{e^{q^2/\kappa} - 1} \] (17)

where \( b \) is a dimensionless adjustable parameter.

**Leading order of the \( 1/N \) expansion** – As anticipated, Eqs. (11), (12), (13) and (14), are already arranged in a \( 1/N \) expansion structure and we can straightforwardly extract the leading \( (1/N) \) flow equations for the suitably rescaled parameters, and also the associated fixed point equations, which are obtained by requiring the rescaled parameters to be \( t \)-independent. The rescaled parameters, relevant for our analysis, are \( \rho = k^{-d+4-\eta} \rho \), \( v = k^{-d} \), \( w^A = k^\eta W_A \), \( w^B = k^{d-4+2\eta} W_B \), \( z^A = k^{-\eta} Z_A \), \( z^B = k^{d-6+2\eta} Z_B \), where the scaling dimensions, i.e. the exponents in the powers of the scale \( k \), are given in (23, 24), and the fixed point equations for \( w^A \) and \( z^A \) at \( 1/N = 0 \) are:

\[ -\eta_0 w_0^A + (d - 4 + \eta_0) q w_0^A = -\frac{1}{2} J^{TT}_2 w_0^A \] (18)

\[ (2 - \eta_0) z_0^A + (d - 4 + \eta_0) q z_0^A = -\frac{1}{2} J^{TT}_2 z_0^A . \] (19)

In Eqs. (18) and (19) the prime indicates derivation with respect to \( \rho \) and the subscript 0 indicates the lowest order of the \( 1/N \) expansion. It is easy to check that a field independent \( w_0^A \) (and therefore \( w_0^A = 0 \)) together with \( \eta_0 = z_0^A = 0 \) is a solution of this set of equations. Therefore, we can take \( w_0^A = 1 \) to set the overall normalization of the effective action.

Then, we turn to the fixed point equation for the potential, Eq. (11), and, after setting \( \hat{W}_k = 1 \) in Eq. (16), the integral \( J^{TT}_1 \) can be solved and Eq. (11) conveniently written as:

\[ (x + f(x))^2 \text{d}x - 1 \] (20)

with the following definitions \( x = \sqrt{2} \rho \): \( f(x) = dv/dx; f_0(x) = df/dx; d_0 = (d + 4)/(2\tau) \) and finally \( \tau = 2/[(4\pi)^d/2\Gamma(1 + d/2)] \) is the factor coming from the resolution of the integral \( J^{TT}_1 \).

Eq. (20) can be easily attacked numerically, but all the essential features can be deduced by simple inspection. In fact we immediately see that the constant function \( f_0(x) = 0 \) is a solution of Eq. (20), that plays the same role of the gaussian fixed point for the standard scaling. In addition, we observe that a viable non-trivial Lifshitz solution \( f_L(x) \) must vanish at the origin \( f_L(0) = 0 \) due to the symmetry of the problem and, in addition, another zero of \( f_L \) must occur at

\[ \tau = \frac{2\tau}{d - 4} \] (21)

i.e. \( f_L(\tau) = 0 \) with non-vanishing derivative \( f_L(\tau) \neq 0 \). By expanding Eq. (20) around \( \tau \), one finds from the linear terms:

\[ f_L(\tau) = \frac{8 - d}{d - 4} . \] (22)
Eq. (21) loses meaning when \( d \leq 4 \), while the vanishing of \( f_{LT, x}(x) \) from Eq. (22) at \( d = 8 \) indicates a flattening of the solution \( f_t \) onto the trivial solution \( f_G \). The latter result accords with the numerical analysis of [24] with \( N = 1 \), which indicates that the two solutions merge at \( d = 8 \) and only the trivial solution survives for \( d \geq 8 \). Therefore we limit the study of Eq. (20) to the range \( 4 \leq d \leq 8 \).

With the information collected above, we are able to determine the eigenvalues \( \lambda_L \) of the flow equation, linearized around the fixed point solution. To this aim we follow the procedure originally worked out in [37, 38] for the standard Wilson-Fisher (WF) fixed point, (see also [31]) and, by writing the \( t \)-dependent function \( f(t, x) = f_L(x) + e^{\eta t} h(x) \), as the sum of the fixed point solution \( f_L(x) \) and a perturbation \( h(x) \), we get the following linear (in \( h(x) \) ) equation:

\[
\frac{\lambda h}{\tau} = \left[ d_+ - (x + f_L)^{-2} \right] h - \left[ d_- - (x + f_L)^{-2} \right] x h_x + 2(x + f_L)^{-3} \left( f_L h - x f_{LT, x} h \right)
\]  

(23)

The function \( h \) is supposed to be regular at any finite \( x \) and can be expanded around \( \eta \):

\[
h(x) = \sum_{i=0}^{\infty} a_i(x - \eta)^i
\]  

(24)

where the lowest power \( n \) must be a non-negative integer, \( n \geq 0 \). At \( x = \eta \) the coefficient of \( x h_x \) in square brackets vanishes and therefore \( h_x(x)/h(x) \) is either singular at \( x = \eta \) (with a simple pole singularity) or finite, the former case corresponding to \( n > 0 \) and the latter to \( n = 0 \) in Eq. (24). In both cases, after dividing both members of Eq. (23) by \( h \), one can make the replacement \( h_x(x)/h(x) = n/(x - \eta) \) in order to compute the linear corrections in the expansion of Eq. (23) around the point \( \eta \). This expansion, with the help of Eq. (22), yields the following eigenvalue spectrum (we recall \( 4 \leq d \leq 8 \) ):

\[
\lambda_L = d - 4 - 4n
\]  

(25)

parameterized by the non-negative integer \( n \geq 0 \). By following the same procedure, one derives from Eq. (23) the eigenvalues associated to \( f_G \) (again with integer \( n \geq 0 \)):

\[
\lambda_G = 4 - (d - 4)n
\]  

(26)

In particular one can determine those values of \( n \) that correspond to relevant (positive) eigenvalues, namely \( 0 \leq n < (d - 4)/4 \) from Eq. (25), and \( 0 \leq n < 4/(d - 4) \) from Eq. (26). In addition, we observe that in \( d = 8 \) the two spectra in Eq. (25) and (26) are equal, as the two fixed point solutions become coincident.

In conclusion, the solutions found at \( 1/N = 0 \) with this particular scaling, clearly resemble those obtained with standard scaling where, aside from the constant gaussian solution with eigenvalue spectrum \( \lambda_G = 2 - (d - 2)n \), one has the WF fixed point with \( \lambda_{WF} = d - 2 - 2n \). One clearly sees that the difference, at this order, is only in the range spanned by \( d \) which, in this case, goes from \( d = 2 \) to \( d = 4 \), while, in the analysis of the tricritical Lifshitz point, from \( d = 4 \) to \( d = 8 \). In fact, even the number of relevant directions is the same in the two cases, once the proper change in \( d \) is taken into account.

1/N corrections – At the leading order \( 1/N = 0 \), the equations for the momentum dependent parts admit the elementary field-independent solutions \( w_A^0 = 1 \) and \( z_A^0 = 0 \; \text{together with} \; \eta_0 = 0 \), while the equations for \( w^B \) and \( z^B \) at this order decouple from the other equations and one is left with the fixed point equation for the potential only.

For the next step, we consider the potential expansion \( v = v_0 + v_N/N + O(1/N^2) \) and the analogous expansions for \( \eta, w^A, z^A, w^B, z^B \), and insert them into the fixed point equations in order to analyze the 1/N corrections. We start by observing that Eq. (11) for \( v_N \) involves the \((1/N)\) corrections of all the above variables (we recall here that \( w_0^B = z_0^B = 0 \), but the first non-vanishing terms of the expansion of \( w^B \) and \( z^B \), which are \( O(1/N) \), contribute to the leading order \((1/N = 0)\) longitudinal propagator \( G_L \), because of the factor \( N \) in Eq. (5)). Therefore, a full determination of the 1/N corrections requires the resolution of five coupled equations.

However, it is possible to determine \( \eta_0, u \) without solving the whole set of equations. To this aim, a direct inspection of equations (12), (13), (14) shows that the vanishing field-independent solution \( w_A^0 = z_A^0 = w_0^B = z_0^B = 0 \) is not allowed because of the non-vanishing coefficients of the integrals \( J_j \) in Eqs. (13), (14), respectively \( \Gamma_A^2 \) and \( \Gamma_B^2 \), which are finite and field dependent due to their dependence on \( V' \) and \( V'' \), as shown in Eqs. (3), (4).

Nevertheless, at least for one particular value of the field \( \varphi = \bar{\varphi} \), we can extend at \( 1/N \) the normalization of the propagators, already fixed by the leading order solution \( w_0^A = 1; z_0^A = w_0^B = z_0^B = 0 \). This immediately implies \( w^A_N(\bar{\varphi}) = z^A_N(\bar{\varphi}) = w^B_N(\bar{\varphi}) = z^B_N(\bar{\varphi}) = 0 \) and it is natural to take \( \bar{\varphi} \) as the point where the derivative of the leading order potential vanishes, i.e. \( \bar{\varphi} = \bar{\varphi} = \bar{\varphi} = \bar{\varphi} \), with \( \bar{\varphi} \) defined in Eq. (21). Finally, we extract from Eq. (13) the two equations for \( w^A_N, z^A_N \) directly computed at \( \bar{\varphi} \):

\[
-\eta_N - (d - 4)\frac{\partial}{\partial \bar{\varphi}} w^A_N(\bar{\varphi}) = -\frac{1}{2}(2)_{0}^J \left( J^L_{\bar{\varphi}} \right)_{0} + 2\bar{\varphi} \left( J^L_{\bar{\varphi}} \right)_{0}
\]  

(27)

\[
-(d - 4)\frac{\partial}{\partial \bar{\varphi}} z^A_N(\bar{\varphi}) = -\frac{1}{2}(2)_{0}^J \left( J^L_{\bar{\varphi}} \right)_{0} + 2\bar{\varphi} \left( J^L_{\bar{\varphi}} \right)_{0}
\]  

(28)

where the subscript \( 0 \) of the various integrals indicates that they must be computed by using the leading order \((1/N = 0)\) solution of the various parameters, while the subscript \( p^d \) in Eq. (27) and \( p^d \) in Eq. (28) of the integrals \( J_j \), indicates that only the the coefficient of that particular power of the momentum \( p \) in the expansion of the addressed integral is to be retained. We find that the 1/N correction to the anomalous dimension \( \eta_N \) does not appear in Eq. (28), but it is directly
obtained from Eq. (27), if one neglects the terms proportional to \( w_N^\alpha (\tilde{\mathcal{V}}) \). At the same time, the \( O(1/N) \) corrections to the fixed point potential in Eq. (11) are under control and easily computable by numerical integration.

As anticipated, we notice that the procedure adopted to compute \( \eta_N \) essentially coincides with the scheme introduced in [39], which leads to the improved Local Potential Approximation \( \text{LPA}' \). This can be straightforwardly checked by replacing Eq. (27) with the equation obtained by repeating the previous steps for the case of the anomalous dimension \( \eta_{WF} \) at the WF fixed point, which gives

\[
\eta_{WF} = -2 \tilde{\mathcal{V}}_0'' (\tilde{\mathcal{V}}) (J_{LT}^T \tilde{p}_2 + J_{LL}^T \tilde{p}_2) \delta_0.
\]

In this case the expansion is to be taken to order \( p^2 \) and the regulator in [15] can be safely chosen, because it does not generate any singularity. The corresponding integrals can be analytically computed, as shown in [39], and one finds \( (J_{LT}^T \tilde{p}_2 + J_{LL}^T \tilde{p}_2) \delta_0 = -\tau/(1+2\tilde{\mathcal{V}}_0'' (\tilde{\mathcal{V}}))^2 \) and, therefore, \( \eta_{WF} = 2\tau \tilde{\mathcal{V}}_0'' (\tilde{\mathcal{V}})^2 / (1+2\tilde{\mathcal{V}}_0'' (\tilde{\mathcal{V}}))^2 \). This is exactly the expression of the anomalous dimension which is used in the \( \text{LPA}' [21, 39, 40] \).

In order to test the reliability of this procedure, we can go one step further and replace in \( \eta_{WF} \), the particular value of \( \tilde{\mathcal{V}} \) and \( \tilde{\mathcal{V}}_0'' (\tilde{\mathcal{V}}) \) that are obtained from the leading order analysis \((1/N = 0)\) for the WF fixed point. Then, instead of Eq. (21), one has \( \tilde{\mathcal{V}} = \tau/(d-2) \) and Eq. (22) becomes

\[
\eta_{WF, k} (\tau) = 2\tau \tilde{\mathcal{V}}_0'' (\tilde{\mathcal{V}}) = (4-d)/(d-2) \quad \text{(these changes are due to the different scaling of the various quantities in the two cases and also to the different dimension of the regulator \( R_k \) that, when derived with respect to the scale \( k, \partial_x R_k \) produces a different factor). Thus, one finds the following \( 1/N \) correction to the anomalous dimension at the WF fixed point:

\[
\eta_{WF} = \frac{(d-2)(4-d)^2}{4} \quad \text{(29)}
\]

that is to be compared to the full result directly obtained in the \( 1/N \) expansion, [41] \((\epsilon = 4-d \) and \( \Gamma \) indicates the Gamma function):

\[
\eta = \frac{4\pi}{(4-\epsilon)\pi} \frac{\sin(\pi/2 \Gamma(2-\epsilon))}{\Gamma(1-\epsilon/2) \Gamma(2-\epsilon/2)} \quad \text{(30)}
\]

Remarkably, Eqs. (29) and (30) have the same behaviour both for \( d = 2 + \delta \) (with \( \delta \geq 0 \)), i.e. \( \eta_{WF} = \eta = \delta \), and for \( d \leq 4 \) (with \( \epsilon \geq 0 \)), i.e. \( \eta_{WF} = \eta = \epsilon^2/2 \). Instead, in \( d = 3 \), where the difference between Eq. (29) and Eq. (30) is largest, one finds \( \eta_{WF} = 1/4 \) and \( \eta = 8/(3\pi^2) = \Gamma(1/3.7) \). We take this small discrepancy as the measure of the reliability of the \( \text{LPA}' \) here considered even in the case of the Lifshitz critical behaviour.

Going back to the Lifshitz fixed point problem, we have to compute \( \eta_N \) from Eq. (27) by neglecting the terms proportional to \( w^\alpha N (\tilde{\mathcal{V}}) \). However, as anticipated, this time a strong dependence on the regulator is observed. In particular, the parameter \( \alpha \) introduced in Eq. (16) explicitly shows up in the resolution of the integrals, because \( \partial^2 R_k (q^4) / \partial q^8 \rvert_{q^4 = k^4} = \alpha. \)

Namely, we get

\[
\eta_N = \frac{4\pi \tilde{\mathcal{V}}_0'' (\tilde{\mathcal{V}})^2}{D^2} \left[ 4\pi - \frac{24\tau + d + 8}{(d+2)D} + \frac{6}{(d+2)2D^2} \right] \quad \text{(31)}
\]

where we introduced the dimensionless parameter \( \bar{\alpha} = k^4 \alpha \) and \( D = (1+2\tilde{\mathcal{V}}_0'' (\tilde{\mathcal{V}})) \). Then, with the help of Eqs. (21) and (22) one gets the analogous of Eq. (29) for the Lifshitz case, with no need to solve the fixed point equation for the \( O(1/N) \) corrections to the potential, or the wave function renormalizations:

\[
\eta_N = \frac{(d-4)(8-d)^2}{16} \left\{ 4\bar{\alpha} - \frac{(d-4)}{4(d+2)} \left( \frac{24\bar{\alpha} + d + 8}{(d+2)} - \frac{3(d-4)^2}{8(d+2)} \right) \right\} \quad \text{(32)}
\]

We observe the explicit dependence on the parameter \( \bar{\alpha} \) in Eq. (32) and it is evident that the alternative use of the Heavy-side cutoff \( R_k \), associated to the limit \( 1/\bar{\alpha} \to 0 \) would produce a singular behaviour of \( \eta_N \). Instead, for finite values of \( \bar{\alpha}, \eta_N \) is finite in the whole range \( 4 < d < 8 \).

However the \( \bar{\alpha} \)-dependence in Eq. (32) has strong drawbacks: for instance in \( d = 6 \), one can take \( \bar{\alpha} \) sufficiently large that the term \( \eta_N / (1/N) \) in the \( 1/N \) expansion of the anomalous dimension is so big, even with \( N \gg 1 \), that the expansion itself become questionable. The only two cases in which the \( \bar{\alpha} \)-dependence becomes irrelevant are the two limits of \( \eta_N \) for \( d \to 4^+ \) and \( d \to 8^- \), that vanish for any fixed value of \( \bar{\alpha} \), due to the factor in front of the curly bracket in the right hand side of Eq. (32).

| \( d = 4.1 \) | \( d = 5 \) |
|-----|-----|
| \( b \) | \( \eta_N \) | \( b \) | \( \eta_N \) |
| 10^{-4} | 0.081 | 10^{-4} | 0.137 |
| 10^{-3} | 0.086 | 10^{-3} | 0.230 |
| 10^{-2} | 0.089 | 10^{-2} | 0.313 |
| 1.5 10^{-2} | \*0.0894 | 1.2 10^{-2} | \*0.3144 |
| 10^{-1} | 0.085 | 10^{-1} | 0.147 |
| 2 10^{-1} | 0.080 | 2 10^{-1} | 0.023 |
| 20 | -0.036 | 20 | -0.720 |

TABLE I: \( \eta_N \) as obtained for different values of \( b \) in Eq. (17).

In order to to collect further indications on the effect of the regulator in the computation of \( \eta_N \), we solve the integrals in Eq. (27) with the exponential regulator defined in Eq. (17), which has the advantage of being essentially smoother than the one in Eq. (16) and free of additional dimensionful parameters but, on the other hand, no analytical expression for \( \eta_N \) can be derived.

Therefore, we report in Table II the values of \( \eta_N \) obtained with different values of the parameter \( b \) of the regulator in Eq. (17) for two values of the dimension \( d \), namely \( d = 4.1 \) and \( d = 5 \). In both cases \( \eta_N \) shows the same qualitative behaviour, by reaching a maximum value (indicated by a star) around \( b \approx 10^{-2} \), and then systematically decreasing down to large negative values. Unlike the result in Eq. (32) that shows a linear dependence on \( \bar{\alpha} \), in this case we can invoke the min-
imal sensitivity criterion to select the maximal values as estimates of the anomalous dimension $\eta_N$.

However when $d$ is increased to 6 or to larger values, a more cumbersome picture shows up. In fact, already at $d = 6$ the simple $b$-dependence of Table I is lost and one finds three different extrema in $\eta_N$ when $b$ grows (two maxima with a minimum in between), namely $\eta_N = 0.154, -0.282, -0.214$, respectively for $b = 6 \times 10^{-3}, 0.41, 2.5$. We notice that $\eta_N < 0$ both at the minimum and at the second maximum and the same pattern of three extrema is observed for larger $d$.

| $b$ = 0.015 | $b$ = 1 |
|------------|-------|
| $d$ | $\eta_N/(d-4)$ | $d$ | $-8\eta_N/(8-d)$ |
| 4.1 | 0.8939 | 7.9 | 0.9762 |
| 4.05 | 0.9454 | 7.95 | 0.9880 |
| 4.01 | 0.9888 | 7.99 | 0.9976 |
| 4.005 | 0.9944 | 7.995 | 0.9988 |
| 4.001 | 0.9989 | 7.999 | 0.9997 |

TABLE II: $\eta_N$ as obtained with $d$ approaching 4 and 8.

We conclude the analysis with the regulator in Eq. (17), by showing in Table II the behaviour of $\eta_N$ when $d$ approaches the two extremal values $d = 4$ and $d = 8$. In the former case, $b$ is obviously selected by the presence of a single maximum in $\eta_N$, while in the latter case we took $b = 1$ that corresponds to a rather stable ($d = 8$), negative value of $\eta_N$, very close to its second maximum (note that for the remaining two extrema, the effect shown in Table II is not observed). Remarkably, Table II shows that the power law behaviour already seen in Eq. (29) for the WF fixed point, and in Eq. (32) for the Lifshitz fixed point with the other regulator, is in fact recovered in this case both when $d \to 4^+$ and when $d \to 8^-$. 

Discussion – We investigated the existence of the isotropic tricritical Lifshitz point for the $O(N)$ theory in the $1/N$ expansion and explicitly computed the associated anomalous dimension in the LPA’. More specifically, instead of directly implementing the LPA’ to the Lifshitz case, our analysis started from a set of coupled flow equations for the potential and the two point functions derived in [33, 34], which were then evaluated at the next to leading order in the $1/N$ expansion and under further assumptions. This procedure produced an equation for the anomalous dimension $\eta$ that turned out to be equivalent to the equation for $\eta$ derived in the LPA’. 

It must be remarked that $\eta_N$ determined in the LPA’ does not include the full $O(1/N)$ corrections to the anomalous dimension and, therefore, an indication of the difference between the two determinations was obtained in the case of the WF fixed point, where the maximum discrepancy amounts to about 8% at $d = 3$ while, close to the extremal values, $d = 2$ and $d = 4$, the two calculations coincide.

We find that, already at leading order, the non-trivial Lifshitz point is observed only between $4 < d < 8$. At order $1/N$ of the LPA’, the presence of the Lifshitz point is confirmed and the anomalous dimension $\eta_N$ vanishes both at $d = 4$ and $d = 8$. This is in agreement with the conjecture that these two values respectively represent the lower and upper critical dimension for the Lifshitz point of the $O(N)$ theory.

In particular, in [23], it is argued for the Lifshitz point of the $N = 1$ theory, that the lower critical dimension could be associated to the large field behaviour of the fixed potential, corresponding to the particular value of $d$ below which the potential does no longer diverge as a power law for large values of the field $\phi$ but, instead, a continuous set of solutions (constant at large $\phi$) of the fixed potential equation is found. This value of $d$ is related to the change of sign of the scaling dimension of $\phi$, that for the case considered is $D_\phi = (d - 4 + \eta)/2$ and therefore, if the anomalous dimension vanishes or is neglected, $d = 4$ is the requested value. For the Lifshitz point of the $O(N)$ theory where, as shown above, $\eta_N = 0$ in $d = 4$, it is natural to accept it as the lower critical dimension. Needless to say, this argument is the restatement of what occurs for the scaling of the $O(N)$ theory at the lower critical dimension of the WF fixed point, $d = 2$.

In addition, one can focus on the leading order potential equation, Eq. (20), directly in $d = 4$. Actually, this equation can be solved analytically and, as for the case with $d < 4$, one ends up with a continuous set of solutions, parameterized by one real parameter.

Finally we comment on the dependence on the regulator of the result obtained for $\eta_N$. While we checked that the regulator (15) is well behaved for the computation of $\eta_{WF}$ in the WF case, even its smoothed version in Eq. (16) produces potentially dangerous terms in the Lifshitz point case; terms that become irrelevant only in the limits $d \to 4^+$ and $d \to 8^-$. Then, the use of the smoother regulator (17) on the one hand confirms the behaviour of $\eta_N$ in the region close to $d = 4$ and $d = 8$ but on the other hand, still produces the undesired effect, at least only for more than six dimensions, of generating multiple spurious extrema in $\eta_N$, regarded as a function of $b$. Therefore, away from the extremal points $d = 4$ and $d = 8$, no firm statement can be made on $\eta_N$ in the LPA’. Conceivably, this is due to the modified two point functions (3), (4) with a leading $O(p^4)$ term, which provide the major difference between the Lifshitz point and the standard fixed point case where, conversely, the LPA’ provides reliable results.

We conclude by observing that the tricritical Lifshitz point which, when looking at the eigenvalue spectrum at the leading order of the $1/N$ expansion, could appear as a trivial duplicate of the WF fixed point with a suitable redefinition of the scaling dimensions of the various operators, does actually show original features. In fact, not only rather different properties of the anomalous dimension (with respect to the WF case) show up at order $1/N$, but it must also be noticed that, as soon as the wave function renormalizations are explicitly included in the fixed point equations, the coefficient of $(\partial\phi)^2$, $Z$, has positive scaling dimension $2 - \eta$, which indicates the existence of a relevant direction that has no correspondence at the WF critical point. On the other hand the similarities in the two cases could be a hint that the structure observed around $d = 2$, such as the presence of multi-critical solutions [40, 42, 43], or the relation...
with phase transitions of different nature \cite{46,47}, could have a counterpart in the Lifshitz scaling around $d = 4$.

Acknowledgments. This work has been carried out within the INFN project QFT-HEP.

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