THE INVARIANCE OF KNOT LATTICE HOMOLOGY

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Abstract. Assume Γ is a negative-definite forest with exactly one unframed vertex, and \( M(\Gamma) \) is the resulting plumbed 3-manifold with a knot embedded. We show that the filtered lattice chain homotopy type of \( \Gamma \) is an invariant of the diffeomorphism type of \( M(\Gamma) \).

Introduction

Lattice (co)homology is a theory introduced by András Némethi \[3\] for negative definite plumbed 3-manifolds and is conjecturally isomorphic to Heegaard Floer homology. Oszváth, Stipsicz and Szabó \[1\] defined a knot refinement for lattice homology which is a filtration on the lattice chain complex. It is natural to ask whether this filtration is in fact an invariant of the knot (and thus if the “knot filtration” is well defined).

We now recall the setup. Let \( G \) be a negative definite framed forest. To define the resulting manifold \( M(G) \), see the graph as a link in the boundary of \( \mathbb{D}^4 \) (by replacing vertices by loops and edges by crossings) and plumb according to the framing on each vertex. This defines a new 4-manifold whose boundary is the 3-manifold \( M(G) \). Similarly in the knot lattice setup we let \( \Gamma \) be a negative definite plumbed forest with exactly one unframed vertex \( v_0 \) and let \( G := \Gamma - \{v_0\} \). The same construction yields \( M(G) \) and a knot \( K_{v_0} \) embedded in it, thus we will often refer to \( M(\Gamma) := (M(G), K_{v_0}) \). Oszváth, Stipsicz and Szabó \[1\] define the knot filtration combinatorially using the graph \( \Gamma \).

Our main result is:

**Theorem 0.1.** Let \( \Gamma \) be a negative definite forest with exactly one unframed vertex. The filtered lattice chain homotopy type of \( \Gamma \) is an invariant of the oriented diffeomorphism type of \( M(\Gamma) \).

To prove it, we will show the following propositions:

**Proposition 0.2.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be two negative definite forests with exactly one unframed vertex each, such that \( M(\Gamma_1) \) and \( M(\Gamma_2) \) are diffeomorphic. Then, blow-ups and blow-downs are sufficient to turn \( \Gamma_1 \) into \( \Gamma_2 \).

**Proposition 0.3.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be two negative definite forests that differ by a blow-up. Then the filtered lattice chain complexes \((\text{CF}^{-}(G_1), A)\) and \((\text{CF}^{-}(G_2), A')\) and chain homotopic.

We will prove Proposition 0.2 in Section 1 and Proposition 0.3 in Sections 2 and 3.

This topic was suggested by András Stipsicz for an internship I did under his supervision at the Alfréd Rényi Institute in Budapest. I would like to thank him for his precious help.
First some preliminary definitions:

**Definition 1.1.** Let $\Gamma$ be a graph with framings on all its vertices except for $v_0$ and $G$ the graph obtained from $\Gamma$ by deleting $v_0$ and all edges adjacent to it. $\Gamma$ is called *negative definite* if the intersection matrix of $G$ (with the framings on the diagonal) is negative definite.

We refer to [2] for the definitions of blow-down and blow-up (operation R1 and its inverse) and adapt them to graphs with an unframed vertex by considering it as having framing $-\infty$: the only forbidden operation is to blow-down the unframed vertex. We only allow blow-ups or blow-downs that stay in the class of graphs considered (negative definite trees with exactly one unframed vertex). Therefore there are three types of blow-ups (and blow-downs) to study: generic blow-up, the blow-up of a vertex and the blow-up of an edge.

**Definition 1.2.** Two negative definite forests with exactly one unframed vertex $\Gamma$ and $\Gamma'$ are said to be *equivalent* if they are related by a finite sequence of blow-ups and blow-downs. This defines an equivalence relation on the set of negative definite graphs with exactly one unframed vertex. Denote their equivalence class by $[\Gamma]$.

**Remark 1.3.** Notice that if two trees $\Gamma_1$ and $\Gamma_2$ are equivalent, then they define diffeomorphic 3-manifolds and knots $M(\Gamma_1) \cong M(\Gamma_2)$.

**Definition 1.4.** A graph $\Gamma$ with exactly one unframed vertex is said to be *reduced* if no blow-downs can be applied to $\Gamma$.

**Remark 1.5.** Let $\Gamma'$ denote the graph $\Gamma$ after a blow-down. A simple exercise in linear algebra shows that if $\Gamma$ is negative definite, then so is $\Gamma'$. This implies that any class $[\Gamma]$ contains reduced graphs.

**Proposition 1.6.** Let $\Gamma$ and $\Gamma'$ be negative definite trees with exactly one unframed vertex.

1. The equivalence class $[\Gamma]$ contains a unique reduced graph.
2. Suppose $\Gamma_1$ and $\Gamma_2$ are in reduced form and $M(\Gamma_1) \cong M(\Gamma_2)$. Then $\Gamma_1 \cong \Gamma_2$.

**Proof.** If $\Gamma$ is reduced, let $\Gamma_{\text{norm}} = \Gamma_{\text{norm}}(n)$ denote the graph $\Gamma$ with the following changes: choose $-n$ large enough such that $\Gamma(n)$ satisfies conditions N1, N2, N4, N5, and N6 of normal form (see [2]) and $n$ is the smallest framing on $\Gamma(n)$. On any component of $\Gamma(n)$ with same shape as figure 1 ($k \geq 0$ and $e \leq -1$) don’t apply any changes.
On all other components, if $e \leq -3$ or $e$ has at least 2 edges incident to it and $m \geq 0$, (we use the writing conventions from [2])

These changes make $\Gamma_{\text{norm}}$ satisfy condition N3 and respect the diffeomorphism type of $\Gamma(n)$. Therefore, $\Gamma_{\text{norm}}$ is the normal form of $\Gamma(n)$. Now suppose $\Gamma'$ is equivalent to $\Gamma$ and reduced. This implies that for $-m$ large enough $\Gamma'(m)$ also has normal form $\Gamma_{\text{norm}}$ (see [2] Theorem 4.2, one might need to increase $-n$).

Since $n$ is the smallest framing of $\Gamma_{\text{norm}}$, it is possible to identify the unframed vertices of $\Gamma$ and $\Gamma'$, and figures 2 and 3 define an injective map, therefore $\Gamma' \cong \Gamma$.

This shows (1). (2) is a consequence of [2] Theorem 4.2, of the injectivity of the above construction and of the fact that $\Gamma_{\text{norm}}$ is in normal form implies $\Gamma(n)$ is reduced.

Proposition 0.2 is now a direct consequence of Proposition 1.6.
2. Blow-up and blow-down of a vertex

Let \( \Gamma \) be a negative definite tree with an unframed vertex \( v_0 \) and \( G := \Gamma - \{v_0\} \). Notice that we can consider each component of \( \Gamma - \{v_0\} \) independently and then apply the connected sum formula ([1], Theorem 4.8) in order to check the invariance of the filtered lattice chain homotopy type. Therefore we can suppose \( v_0 \) is a leaf. Let \( \Gamma' \) denote the graph obtained by blowing up \( v \). Thus \( v \in G' := \Gamma' - \{v_0\} \) has framing \( m_v - 1 \), \( G' \) has a new vertex \( w \) with framing \(-1\), and \( v \) and \( w \) are connected by an edge. Let \( \mathbb{C}F^-(G) \) denote the lattice chain complex of \( G \) and let \( A \) denote the knot filtration. As in [1] we define the blow-down map:

\[
P: \mathbb{C}F^-(G') \to \mathbb{C}F^-(G) \quad \begin{cases} U^*([(K, p + j), E]) & \text{if } w \notin E \\ 0 & \text{if } w \in E \end{cases}
\]

with

\[
s := g([(K, p + j), E]) - g([(K, p, j), E]) + \frac{j^2 - 1}{8}
\]

and the blow-up map:

\[
R: \mathbb{C}F^-(G) \to \mathbb{C}F^-(G') \quad [(K, p), E] \mapsto C_w([K, p + 1, -1], E)
\]

The appendix of [1] (on arXiv) shows that \( P \) and \( R \) are both graded chain maps and that they are graded homotopy equivalences. Let’s check they respect the knot filtrations.

**Lemma 2.1.** \( P \) is a filtered chain map

**Proof.** We already know from [1] Lemma 7.10 that \( P \) is a chain map. Let’s prove it respects the knot filtrations. Let \( \Sigma \in H_2(X_G; \mathbb{Q}) \) be the homology class satisfying:

\[\Sigma = v_0 + \sum_{j=1}^{n} a_j.v_j + a_v.v \quad \text{and} \quad v_i.\Sigma = 0 \quad \forall i \in \{1, \ldots, n\}\]

where \( X_G \) is the plumbed 4-manifold defined by \( G \). \( \Sigma \) exists and is unique because \( G \) is assumed to be negative definite. Similarly a quick calculation gives \( \Sigma' \in H_2(X_G'; \mathbb{Q}) \):

\[\Sigma' = v_0 + \sum_{j=1}^{n} a_j.v_j + a_v.v + a_v.w\]

Therefore,

\[
\Sigma^2 = v_0^2 + \left( \sum_{j=1}^{n} a_j.v_j \right)^2 + a_v^2m_v + 2v_0.\sum_{j=1}^{n} a_j.v_j + 2a_v.v_0.v + 2a_v.v.\sum_{j=1}^{n} a_j.v_j
\]

\[= \Sigma'^2\]

Recall that

\[
A(U^j[K, E]) = -j + \frac{1}{2} (L_{[K, E]}(\Sigma) + \Sigma^2)
\]

As in [1] the set of characteristic elements is \( \text{Char}(G) := \{L : H_2(X_G, \mathbb{Z}) \to \mathbb{Z} \mid K(x) = x.x \mod 2\} \). In the following \((K, p) \in \text{Char}(G)\) will denote the characteristic element worth \( p \in \mathbb{Z} \) on \( v \) and whose restriction to \( G - \{v\} \) is \( K \). Similarly
\[(K, p, j) \in \text{Char}(G')\] will denote the characteristic element worth \(p\) on \(v\), \(j\) on \(w\) and whose restriction to \(G' - \{v, w\}\) is \(K\). We calculate,

\[
L_{[(K, p, j), E]}(\Sigma) = L_{[(K, p, j), E]}(v_0) + K\left(\sum_{j=1}^{n} a_j v_j\right) + (p + j) a_v
\]

\[
L_{[(K, p + j), E]}(\Sigma') = L_{[(K, p + j), E]}(v_0) + K\left(\sum_{j=1}^{n} a_j v_j\right) + (p + j) a_v
\]

and

\[
L_{[(K, p), E]}(v_0) = -v_0^2 + 2g[(K, p, j), E] - 2g[(K, p, j) + 2v_0^*, E]
\]

\[
L_{[(K, p + j), E]}(v_0) = -v_0^2 + 2g[(K, p + j), E] - 2g[(K, p + j) + 2v_0^*, E]
\]

Therefore, if \(w \notin E\),

\[
A(P[(K, p, j), E]) - A'[[(K, p, j), E]] = -\frac{j^2 - 1}{8} + g[(K, p, j) + 2v_0^*, E] - g[(K, p + j) + 2v_0^*, E]
\]

Notice that if \(v \notin I \subset E\) and \(w \notin E\),

\[
2f[(K, p, j) + 2v_0^*, I \cup v] = 2f[(K, p + j) + 2v_0^*, I \cup v] - j - 1
\]

\[
2f[(K, p, j) + 2v_0^*, I] = 2f[(K, p + j) + 2v_0^*, I]
\]

Therefore,

\[
g[(K, p, j) + 2v_0^*, E] - g[(K, p, j) + 2v_0^*, E] \leq \max(0, -\frac{j - 1}{2})
\]

Since \((K, p, j)\) is a characteristic element and \(w^2 = -1\), \(j\) must be odd. Therefore, \(\frac{j - 1}{2} - \frac{j^2 - 1}{8} \leq 0\). We conclude:

\[
A(P[(K, p, j), E]) - A'[[(K, p, j), E]] \leq 0
\]

As in [1] Definition 7.5, a generator \([K, E] \in CF^-(G')\) (with \(w \notin E\)) is of type-a if \(a_w[K + 2v_0^*w^*, E \cup w] = 0\) and of type-b otherwise. Notice that \(w\) is a good vertex and \(-w^2 = 1\). Let \(T = T_{[K, E]}\) be equal to 1 if \([K, E]\) is of type-a and \(-1\) if \([K, E]\) is of type-b. Now consider the map \(H_0 : CF^-(G') \rightarrow CF^-(G')\) defined as:

\[
H_0[K, E] = \begin{cases} 
0 & \text{if } w \in E \text{ or } (T - 2) \leq K(w) < T \\
[K, E \cup w] & \text{if } w \notin E \text{ and } K(w) \geq T \\
[K - 2w^*, E \cup w] & \text{if } w \notin E \text{ and } K(w) < (T - 2)
\end{cases}
\]

As in [1], \(H_0\) increases the Maslov grading by 1. Define \(C_0 = \text{Id} + \partial \circ H_0 + H_0 \circ \partial\). For each \([K, E]\) there is an \(N = N_{[K, E]}\) such that the \(N\)-th iterate of \(C_0\) stabilizes. Denote \(C_w := C_0^\infty\). By [1] Theorem 7.8,

\[
C_w[K, E] = 0 \text{ if } w \in E
\]

and

\[
C_w[K, E] = [K, E] + \partial \circ H_w[K, E] + H_w \circ \partial[K, E]
\]

By [1] Example 7.9,

\[
C_w[(K, p, j), E] = C_w[(K, p + j + 1, -1), E]
\]

\textbf{Lemma 2.5.} \(H_0\) is filtered.
Proof. For \([K, E]\) a generator of \(\text{CF}^{-} (G')\) with \(w \notin E\), consider \(H_0[K, E]\).

- If \(K(w) \geq 1\), then by \([1]\) Lemma 7.3, \(a_w[K, E \cup w] = 0\). Similarly, \((K + 2v_0^*)_0(w) = K_0(w) \geq 0\), therefore \(a_w[K + 2v_0^*, E \cup w] = 0\). Thus, by \([1]\) formula (3.2), \(A'[K, E \cup w] - A'[K, E] = 0\).
- If \(K(w) \leq -3\) then \((K - 2w^* + 2v_0^*)_0(w) = K_0(w) + 2 \leq -1\). Therefore, by \([2]\) Lemma 7.3 and formula (3.3), \(A'[K - 2w^*, E \cup w] - A'[K, E] = 0\).
- As \(K\) is a characteristic element, \(K(w)\) must be odd, therefore \(K(w) \neq 0\).

Suppose now that \(K(w) = -1\). If \([K, E]\) is of type-b then \(H_0[K, E] = 0\). If \([K, E]\) is of type-a, then \(H_0[K, E] = [K, E \cup w]\) and
\[
a_w[K, E \cup w] = g[K, E] - g[K, E \cup w] = 0
\]
Therefore, by \([1]\) formula (3.2), \(A'[K, E] \geq A'[K, E \cup w]\) is equivalent to
\[
a_w[K + 2v_0^*, E \cup w] := g[K + 2v_0^*, E] - g[K + 2v_0^*, E \cup w] = 0
\]
By minimality, \(g[K + 2v_0^*, E \cup w] \leq g[K + 2v_0^*, E]\), and since \(v_0\) is a leaf, we have
\[
(2.6) \quad g[K, E] \leq g[K + 2v_0^*, E \cup w] = g[K + 2v_0^*, E] \leq g[K, E] + 1
\]
There are a certain number of subsets \(I \subset E\) such that \(g[K, E] = f[K, I]\).
First, suppose that there is such a subset \(I_0\) satisfying \(v_0. \sum_{u \in I_0} u = 0\).
Then
\[
2f[K + 2v_0^*, I_0] = K \sum_{u \in I_0} u + 2v_0. \sum_{u \in I_0} u + (\sum_{u \in I_0} u)^2
= 2f[K, I]
\]
Therefore, \(g[K + 2v_0^*, E] = g[K, E] = g[K + 2v_0^*, E \cup w]\). Now suppose that
\[
(2.7) \quad g[K, E] = f[K, I] \Rightarrow v_0. \sum_{u \in I} u = 1
\]
This implies that \(f[K + 2v_0^*, I] = f[K, I] + 1\) for any such \(I\). If \(f[K + 2v_0^*, I] = g[K + 2v_0^*, E \cup w]\) then \(g[K + 2v_0^*, E \cup w] = g[K + 2v_0^*, E]\) by \([2.6]\). If \(g[K + 2v_0^*, E \cup w] = f[K + 2v_0^*, I] - 1 = f[K, I]\), then there is a subset \(J \subset E \cup w\) verifying \(f[K + 2v_0^*, J] = g[K + 2v_0^*, E \cup w]\). Since \(f[K + 2v_0^*, J] \geq f[K, J], f[K, J]\) and \(f[K, I]\) must be equal. Moreover, the fact that \(K(w) = -1\) guarantees that \(f[K, J] = f[K, J - w]\), therefore \(v_0. \sum_{u \in J - w} u = 1\) by \([2.7]\), thus \(f[K + 2v_0^*, J] = f[K, I] + 1\) which is absurd because \(f[K + 2v_0^*, J] = g[K + 2v_0^*, E \cup w] = f[K, I]\).

\[\square\]

As an immediate corollary, \(C_w\) is a filtered chain map and \(H_w\) is filtered.

**Lemma 2.8.** \(R\) is a filtered chain map

**Proof.** We know \(R\) is a chain map and \(C_w\) is filtered. Therefore, it is sufficient to show that \(A[(K, p), E] - A'[(K, p + 1, -1), E] \geq 0\). By definition:
\[
A[(K, p), E] - A'[(K, p + 1, -1), E] = \frac{1}{2} \left( L_{(K,p),E}(v_0) - L_{(K,p+1,-1),E}(v_0) \right)
= g[(K, p), E] - g[(K, p) + 2v_0^*, E] + g[(K, p + 1, -1) + 2v_0^*, E] - g[(K, p + 1, -1), E]
\]
If \( w \in E \) then \( R[(K,p),E] = C_w[(K,p+1,-1),E] = 0 \). Suppose \( w \notin E \). Notice that for \( I \subset E \), such that \( v \notin I \),

\[
2f[(K,p+1,-1),I] = 2f[(K,p),I] \\
2f[(K,p+1,-1),I \cup v] = 2f[(K,p),I \cup v] \\
2f[(K,p+1,-1) + 2v^*_0, I] = 2f[(K,p+1,-1) + 2v^*_0, I \cup v] \\
2f[(K,p+1,-1) + 2v^*_0, I \cup v] = 2f[(K,p) + 2v^*_0, I \cup v]
\]

Thus

\[
g[(K,p),E] = g[(K,p+1,-1),E] \\
g[(K,p) + 2v^*_0, E] = g[(K,p+1,-1) + 2v^*_0, E]
\]

Therefore, \( A[(K,p),E] - A'[(K,p+1,-1),E] = 0 \)

\[ \square \]

**Proposition 2.9.** \( P \) and \( R \) are filtered graded chain homotopy equivalences

**Proof.** \( P \) and \( R \) are filtered graded chain maps. Using the fact that \( P \) is a chain map that preserves the Maslov grading and that \( P \circ H_w = 0 \),

\[
P \circ R[(K,p),E] = P[(K,p+1,-1),E] + P \circ \partial \circ H_w[(K,p+1,-1),E] + P \circ H_w \circ \partial[(K,p+1,-1),E] = U^0[(K,p),E] + 0 + 0
\]

Therefore, \( P \circ R = \text{Id}_{\mathcal{CF}^-(G)} \).

Let’s show that \( R \circ P = C_w \). If \( w \in E \) both equal 0. If \( w \notin E \), then using equation (2.4) and the fact that \( P, R \) and \( C_w \) preserve Maslov grading,

\[
R \circ P[(K,p,j),E] = C_w[(K,p+j+1,-1),E] = C_w[(K,p,j),E]
\]

And \( C_w \) is filtered chain homotopic to \( \text{Id}_{\mathcal{CF}^-(G)} \) (because \( H_w \) is filtered).

\[ \square \]

3. Blow-up and Blow-down of an Edge

Let \( \Gamma \) be a negative definite tree with an unframed vertex \( v_0 \) and \( G := \Gamma - \{v_0\} \). By the connected sum formula (1, Theorem 4.8) we can suppose \( v_0 \) is a leaf. Let \( v \) and \( w \) be two vertices of \( G \) connected by an edge with respective framings \( m_v \) and \( m_w \). Notice that \( v \) or \( w \) can be connected to \( v_0 \), and that they can be good or bad. Let \( \Gamma' \) denote the tree obtained by blowing-up the edge between \( v \) and \( w \), and suppose \( G' := \Gamma' - \{v_0\} \) is negative definite. \( \Gamma' \) has a new vertex \( e \) with framing \(-1\) connected to the vertices \( v \) and \( w \) with framings \( m_v - 1 \) and \( m_w - 1 \). \( \mathcal{CF}^-(G) \) will denote the lattice chain complex of \( G \) and \( A \) will denote the knot filtration.

As in [1], consider the map \( H_0 \) defined as:

\[
H_0 : \mathcal{CF}^-(G') \rightarrow \mathcal{CF}^-(G') \\
[K,E] \mapsto \begin{cases} 0 & \text{if } e \in E \text{ or } K(e) = T - 2 \\ [K,E \cup e] & \text{if } e \notin E \text{ and } K(e) \geq T \\ [K - 2e^*, E \cup e] & \text{if } e \notin E \text{ and } K(e) \leq T - 4 
\end{cases}
\]

where \( T = -1 \) if \( [K,E] \) is of type-a and \( T = 1 \) otherwise (see [1] Definition 7.5).

**Lemma 3.1.** If \( e \notin E \) and \( K(e) \leq -3 \), then \( b_e[K,E \cup e] = 0 \).
Proof. Equivalently, we will show that \( A_e[K, E \cup e] \geq B_e[K, E \cup e] \). Recall that:

\[
A_e[K, E \cup e] := \min\{ f[K, I] \mid I \subset E \}
\]

\[
B_e[K, E \cup e] := \min\{ f[K, I \cup e] \mid I \subset E \}
\]

By definition,

\[
2f[K, I \cup e] = 2f[K, I] + K(e) + e^2 + 2e. \sum_{u \in I} u
\]

and

\[
K(e) + e^2 + 2e. \sum_{u \in I} u \leq -3 - 1 + 2 \times 2 \leq 0
\]

Therefore, \( B_e[K, E \cup e] \leq A_e[K, E \cup e] \).

\[\square\]

Lemma 3.2. \( H_0 \) increases Maslov grading by 1.

Proof. Let \( \partial_a \) denote the terms of the differential where the vertex \( u \) is deleted. We will check that in \( \partial_e \circ H_0[K, E] \) the term \( [K, E] \) always has exponent 0, and since \( \partial \) decreases the Maslov grading by 1, \( H_0 \) must increase it by 1. We only need to consider the case where \( e \notin E \).

- If \( K(e) \geq 1 \) then by [1] Lemma 7.3, \( a_e[K, E \cup e] = 0 \).
- If \( K(e) \leq -5 \) then \( K(e) - 2e^2 \leq -3 \). Therefore, by Lemma 3.1, \( b_e[K - 2e^*, E \cup e] = 0 \).
- If \( K(e) = -1 \) and \( [K, E] \) is of type-a, then \( i_0 = 0 \). Therefore, \( a_e[K, E \cup e] = 0 \).
- If \( K(e) = -3 \) and \( [K, E] \) is of type-b, ie \( a_e[K + 2i_0e^*, E \cup e] > 0 \), then \( i_0 = -1 \). Therefore \( b_e[K + 2i_0e^*, E \cup e] = b_e[K - 2e^*, E \cup e] = 0 \).

\[\square\]

As in [1], we now define

\[
H : \mathbb{CF}^-(G') \rightarrow \mathbb{CF}^-(G')
\]

\[
[K, E] \mapsto \begin{cases} 0 & \text{if } e \in E \text{ or } K(e) = T - 2 \\ \sum_{i=0}^{t} U^i[K + 2i e^*, E \cup e] & \text{if } e \notin E \text{ and } K(e) = T + 2t \text{ with } t \geq 0 \\ \sum_{i=0}^{t-2} U^i[K - 2(i+1)e^*, E \cup e] & \text{if } e \notin E \text{ and } K(e) = T + 2t \text{ with } t \leq -2 \end{cases}
\]

Define \( C := \text{Id}_{\mathbb{CF}^-(G')} + \partial \circ H + H \circ \partial \) and \( C_0 := \text{Id}_{\mathbb{CF}^-(G')} + \partial \circ H_0 + H_0 \circ \partial \). Notice that for any \( [K, E], C_0^0[K, E] \) (\( C_0 \) composed with itself \( n \) times applied to \( [K, E] \)) eventually stabilizes for \( n \) big enough and \( C_0^\infty = C \). Similarly, there is an integer \( k = k_{[K, E]} \) such that \( (H_0 \circ \partial)^k \circ H_0[K, E] = H[K, E] \).

\( (K, p, q, l) \) will denote the characteristic cohomology class taking values \( p \) on \( v \), \( q \) on \( w \), \( l \) on \( e \), and whose restriction to \( G' - \{ v, w, e \} \) is \( K \).

Lemma 3.3. \( C[(K, p, q, l), E \cup e] = 0 \)

Proof.

\[
C[L, E \cup e] = [L, E \cup e] + \partial \circ H[L, E \cup e] + H \circ \partial[L, E \cup e]
\]

\[
= [L, E \cup e] + 0 + U^x H[L, E] + U^y H[L + 2e^*, E]
\]

Using the fact that \( H \) increases the Maslov grading by exactly 1, all terms in \( U^x H[L, E] + U^y H[L + 2e^*, E] \) will cancel except for \( [L, E \cup e] \).

\[\square\]
Lemma 3.4. \( C[(K, p, q, l), E] = C[(K, p + l + 1, q + l + 1, -1), E] \)

Proof. We will just give an idea of the proof as there are too many terms to provide a closed formula (a similar technique is used in [1], Example 7.9). Let \((K, p, q, l)\) be a characteristic cohomology class. We want to compute \(\partial \circ H\) and \(H \circ \partial\) to see when they cancel. For \(\partial_e\) and \(\partial_u\) with \(u \neq v\) and \(u \neq w\), the case is the same as in [1] Example 7.9. As for \(\partial_v \circ H\) and \(H \circ \partial_v\), they are sums of terms of the form \(U^*[L, (E \cup e) - v]\). Since \(H\) increases the Maslov grading by exactly 1, two identical generators must always have same \(U\) exponent. Moreover, the contributions of terms of the form \([L, E - v]\) and \([L, E]\) cancel if and only if they are of same type (similarly for \([L, E - w]\) and \([L, E]\)). Notice \([L, H]\) and \([L + 2ne^*, H]\) are always of same type. Using the fact that if \(n := \frac{l+1}{2}\), then \((K, p, q, l) + 2ne^* = (K, p + l + 1, q + l + 1, -1)\), we get
\[
C[(K, p, q, l), E] = C[(K, p + l + 1, q + l + 1, -1), E]
\]
\(\square\)

We define the blow-down map:
\[
S : \mathbb{C}F^-(G') \rightarrow \mathbb{C}F^-(G)
\]
\[
[(K, p, q, l), E] \mapsto \begin{cases} 0 & \text{if } e \in E \\ U^*[K, p + l, q + l], E & \text{if } e \notin E \end{cases}
\]
where
\[
s := \frac{l^2 - 1}{2} + g[(K, p + l, q + l), E] - g[(K, p, q, l), E]
\]

Lemma 3.5. \(s\) is always positive.

Proof. Suppose \(I \subset E\) such that \(v, w \notin I\) and \(e \notin E\). We have
\[
f[(K, p + l, q + l), I] = f[(K, p, q, l), I]
\]
\[
f[(K, p + l, q + l), I \cup v] = f[(K, p, q, l), I \cup v] + \frac{l + 1}{2}
\]
\[
f[(K, p + l, q + l), I \cup w] = f[(K, p, q, l), I \cup w] + \frac{l + 1}{2}
\]
\[
f[(K, p + l, q + l), I \cup v \cup w] = f[(K, p, q, l), I \cup v \cup w] + l + 2
\]
Therefore,
\[
g[(K, p + l, q + l), E] - g[(K, p, q, l), E] \geq \min\{0, l + 2, \frac{l + 1}{2}\}
\]
And if \(l\) is odd,
\[
\frac{l^2 - 1}{8} + \min\{0, l + 2, \frac{l + 1}{2}\} \geq 0
\]
\(\square\)

Remark 3.6. Lemma 3.5 guarantees that \(S\) is well defined.

Lemma 3.7. \(S\) respects the Maslov grading
Proof. Recall that $gr[K, E] = 2g[K, E] + |E| + \frac{1}{4}(K^2 + |Vert(G)|)$ and notice that
\begin{equation}
(K, p, q, l)^2 = (K, p + l, q + l)^2 - l^2
\end{equation}
If $e \in E$ then $S[(K, p, q, l), E] = 0$. Suppose now that $e \notin E$. We have:
\[
gr(S[(K, p, q, l), E]) = gr(U^s[(K, p + l, q + l), E])
\]
\[
= -2s + 2g[(K, p + l, q + l), E] + |E| + \frac{1}{4}((K, p + l, q + l)^2 + |Vert(G)|)
\]
\[
= 2g[(K, p, q, l), E] + |E| + \frac{1}{4}((K, p, q, l)^2 + |Vert(G)|) + l^2 - 1) - \frac{l^2 - 1}{4}
\]
\[
= gr[(K, p, q, l), E]
\]
\[
\square
\]
Lemma 3.9. $S$ is a chain map.

Proof.  
\begin{itemize}
    \item If $e \in E$, then $\partial \circ S[(K, p, q, l), E] = 0$. Using the fact that $(K, p, q, l) + 2e^* = (K, p + 2, q + 2, l - 2)$, we have:
    \[S \circ \partial'[(K, p, q, l), E] = U^x S[(K, p, q, l), E - e] + U^y S[(K, p + 2, q + 2, l - 2), E - e]\]
    \[= U^x [(K, p + l, q + l), E - e] + U^y [(K, p + l, q + l), E - e]\]
    and since $S$ preserves Maslov gradings, $x = y$. Therefore $S \circ \partial' = \partial \circ S = 0$.
    \item If $e \notin E$ and $v, w \notin E$, then using the fact that for $u \neq v$ and $u \neq w$,
    \[S[(K, p, q, l) + 2u^*, E - u] = U^x [(K, p + l, q + l) + 2u^*, E - u]\]
    and using the fact that $S$ preserves Maslov gradings, we have:
    \[\partial \circ S[(K, p, q, l), E] = U^x \partial [(K, p + l, q + l), E]\]
    \[= \sum_{u \in E} U^{x + u} [(K, p + l, q + l), E - u] + U^{x + u} [(K, p + l, q + l) + 2u^*, E - u]\]
    \[= S \left( \sum_{u \in E} U^{x + u} [(K, p, q, l), E - u] + U^{y + u} [(K, p, q, l) + 2u^*, E - u] \right)\]
    \[= S \circ \partial [(K, p, q, l), E]\]
    \item If $e \notin E$ and $v \in E$ (or $w \in E$), we can look at the $\partial_v$ (or $\partial_w$) term:
    \[\partial_v \circ S[(K, p, q, l), E] = U^x [(K, p + l, q + l), E - v] + U^y [(K, p + l, q + l) + 2v^*, E - v]\]
    \[S \circ \partial'_v [(K, p, q, l), E] = U^x [(K, p + l, q + l), E - v] + U^y S[(K, p, q, l) + 2v^*, E - v]\]
    Moreover,
    \[(K, p + l, q + l) + 2v^* = (K', p + 2m_v - 2, q, l + 2)\]
    \[(K', p + l, q + l) + 2v^* = (K', p + l + 2m_v, q + l + 2)\]
    Since $S$ preserves Maslov gradings, $x = x'$ and
    \[U^v S[(K, p, q, l) + 2u^*, E - v] = U^y [(K, p + l, q + l) + 2v^*, E - v]\]
    A similar argument holds for $w \in E$.
\end{itemize}
\[
\square
\]
We define the blow-up map:
\[
T : \mathbb{CF}^- (G) \rightarrow \mathbb{CF}^- (G')
\]
\[
[(K, p, q), E] \mapsto C[(K, p + 1, q + 1, -1), E]
\]
**Lemma 3.10.** $T$ preserves the Maslov grading.

**Proof.** Thanks to Lemma 3.3 we only need to consider the case where $e \notin E$. Lemma 3.4 tells us that

\[
T[(K, p, q), E] = C[(K, p + 1, q + 1, -1), E] = C[(K, p - 1, q - 1, 1), E]
\]

and since $C$ preserves Maslov grading,

\[
gr[(K, p + 1, q + 1, -1), E] = gr[(K, p - 1, q - 1, 1), E]
\]

Moreover, equation (3.8) tells us that

\[
(K, p + 1, q + 1, -1)^2 = (K, p, q)^2 - 1 = (K, p - 1, q - 1, 1)^2
\]

Therefore,

\[
g[(K, p + 1, q + 1, -1), E] = g[(K, p - 1, q - 1, 1), E]
\]

If $e, v, w \notin I$, then

\[
f[(K, p + 1, q + 1), I] = f[(K, p - 1, q - 1), I]
\]

\[
f[(K, p + 1, q + 1), I \cup v] = f[(K, p - 1, q - 1), I \cup v] + 1
\]

\[
f[(K, p + 1, q + 1), I \cup w] = f[(K, p - 1, q - 1), I \cup w] + 1
\]

\[
f[(K, p + 1, q + 1), I \cup v \cup w] = f[(K, p - 1, q - 1), I \cup v \cup w] + 2
\]

Therefore,

\[
g[(K, p + 1, q + 1, -1), E] = f[(K, p + 1, q + 1, -1), I] \Rightarrow v, w \notin I
\]

Since

\[
v, w \notin I \Rightarrow f[(K, p + 1, q + 1, -1), I] = f[(K, p, q), I]
\]

we now know that $g[(K, p + 1, q + 1, -1), E] \geq g[(K, p, q), E]$. Moreover the proof of Lemma 3.5 shows that

\[
g[(K, p + 1, q + 1, -1), E] \leq g[(K, p, q), E]
\]

Therefore $T$ preserves the Maslov grading. \hfill $\square$

**Lemma 3.11.** $T$ is a chain map.

**Proof.** Consider $[(K, p, q), E]$ a generator of $\mathbb{C}F^-(G)$.

- If $v, w \notin E$

\[
T \circ \partial[(K, p, q), E] = T \left( \sum_{u \in E} U^x[(K, p, q), E - u] + U^y[(K, p, q) + 2u^*, E - u] \right)
\]

And since $u \neq v$ and $u \neq w$,

\[
T[(K, p, q) + 2u^*, E - u] = C[(K, p + 1, q + 1) + 2u^*, E - u]
\]

Therefore, since $T$ preserves Maslov gradings,

\[
T \circ \partial[(K, p, q), E] = \partial' \circ T[(K, p, q), E]
\]
• Now consider the \( v \) term:

\[
(K, p, q) + 2v^* = (K', p + 2m_v, q + 2) \\
(K, p + 1, q + 1, -1) + 2v^* = (K', p + 1 + 2m_v, -2, q + 1, 1)
\]

Therefore, using Lemma \[3.4\]

\[
T[(K, p, q) + 2v^*, E - v] = C[(K', p + 2m_v + 1, q + 3, -1), E - v] \\
= C[(K', p + 2m_v - 1, q + 1, 1), E - v] \\
= C[(K, p + 1, q + 1, -1) + 2v^*, E - v]
\]

Since \( T \) preserves Maslov gradings, this guarantees that

\[
\partial_v' \circ T = T \circ \partial_v
\]

A similar argument holds for the \( w \) term.

\[\square\]

**Lemma 3.12.** \( H_0 \) is filtered.

**Proof.** Let \([K, E]\) be a generator of \( \mathbb{C}F^-(G') \) with \( e \notin E \).

- If \( K(e) \geq 1 \) then \( K(e) + 2v_0.e = K(e) \geq 1 \). By \[1\] Lemma 7.3,

\[
a_v[K, E \cup e] = a_v[K + 2v_0^*, E \cup e] = 0
\]

Therefore, by \[1\] formula (3.2),

\[
A'[K, E] - A'[K, E \cup e] = 0
\]

- If \( K(e) \leq -5 \) then by Lemma \[3.1\]

\[
b_v[K - 2e^*, E \cup e] = b_v[K - 2e^* + 2v_0^*, E \cup e] = 0
\]

Therefore, by \[1\] formula (3.3),

\[
A'[K - 2e^*, E \cup e] - A'[K, E] = 0
\]

- If \( K(e) = -1 \) and \([K, E]\) is of type-a, then since \( v_0 \) is a leaf,

(3.13) \[
g[K, E] = g[K, E \cup e] \leq g[K + 2v_0^*, E \cup e] \leq g[K + 2v_0^*, E] \leq g[K, E] + 1
\]

- Suppose there exists \( I_0 \subset E \) such that \( g[K, E] = f[K, I_0] \) and \( v_0 \cdot \sum_{u \in I_0} u = 0 \). Then

\[
f[K + 2v_0^*, I_0] = f[K, I_0] = g[K, E]
\]

Therefore, (3.13) implies that \( g[K + 2v_0^*, E] = g[K + 2v_0^*, E \cup e] \). Using \[1\] formula (3.2) we see that \( A'[K, E] - A'[K, E \cup e] = 0 \).

- Suppose that for any \( I \subset E \)

\[
g[K, E] = f[K, I] \Rightarrow v_0 \cdot \sum_{u \in I} u = 1
\]

If there is an \( I \subset E \) such that \( f[K + 2v_0^*, I] \neq g[K + 2v_0^*, E \cup e] \) then obviously \( a_v[K + 2v_0^*, E \cup e] = 0 \). If for every \( I \subset E \) satisfying \( g[K, E] = f[K, I, J], g[K + 2v_0^*, E \cup e] = f[K + 2v_0^*, I] - 1 \), then consider \( J \subset E \cup e \) such that \( g[K + 2v_0^*, E \cup e] = f[K + 2v_0^*, J] \). Necessarily \( f[K + 2v_0^*, J] \neq f[K, J] \), therefore \( f[K, J] = f[K, I] \) by minimality of \( f[K, I] \). This implies that \( v_0 \cdot \sum_{u \in J} u = 1 \), so \( f[K + 2v_0^*, J] = f[K, J] + 1 \) which is absurd.
Suppose $K(e) = -3$ and $[K, E]$ is of type-b, then Lemma 3.1 tells us that $b_c[K - 2\varepsilon^*, E \cup e] = 0$. For any $I \subset E$ with $v, w \notin I$,

$$f[K - 2\varepsilon^* + 2v_0^*, I \cup e] = f[K - 2\varepsilon^* + 2v_0^*, I] - 1$$
$$f[K - 2\varepsilon^* + 2v_0^*, I \cup v \cup e] = f[K - 2\varepsilon^* + 2v_0^*, I \cup v]$$
$$f[K - 2\varepsilon^* + 2v_0^*, I \cup w \cup e] = f[K - 2\varepsilon^* + 2v_0^*, I \cup w]$$
$$f[K - 2\varepsilon^* + 2v_0^*, I \cup v \cup w \cup e] = f[K - 2\varepsilon^* + 2v_0^*, I \cup v \cup w] + 1$$

Therefore, the only problematic case is when $\forall I \subset E \cup e$,

$$g[K - 2\varepsilon^* + 2v_0^*, E \cup e] = f[K - 2\varepsilon^* + 2v_0^*, I] \Rightarrow v, w \in I$$

Suppose it is the case. We then know that $g[K - 2\varepsilon^* + 2v_0^*, E \cup e] = g[K - 2\varepsilon^* + 2v_0^*, E]$. Since $g[K - 2\varepsilon^*, E \cup e] < g[K - 2\varepsilon^*, E]$, we know that there exists $I_0 \subset E$ with $v, w \notin I_0$ such that $g[K - 2\varepsilon^*, E \cup e] = f[K - 2\varepsilon^*, I_0 \cup e]$. Therefore,

$$f[K - 2\varepsilon^* + 2v_0^*, I_0 \cup e] \leq f[K - 2\varepsilon^*, I_0 \cup e] + 1$$
$$= g[K - 2\varepsilon^*, E \cup e] + 1$$
$$= g[K - 2\varepsilon^*, E]$$
$$\leq g[K - 2\varepsilon^* + 2v_0^*, E] = g[K - 2\varepsilon^* + 2v_0^*, E \cup e]$$

which contradicts condition (3.14).

This implies that $C$ and $H$ are also filtered.

**Lemma 3.15.** $S$ is filtered.

**Proof.** Suppose $e \notin E$.

$$A'(\{(K, p, q, l), E\} - A(U^*[(K, p + l, q + l), E])$$

$$= s + \frac{1}{2} \left(L_{\{(K, p, q, l), E\}}(\Sigma^l) - L_{\{(K, p + l, q + l), E\}}(\Sigma)\right)$$

$$= \frac{l^2 - 1}{8} + g[(K, p + l, q + l) + 2v_0^*, E] - g[(K, p, q, l) + 2v_0^*, E]$$

And similarly to the proof of Lemma 3.5

$$g[(K, p + l, q + l) + 2v_0^*, E] - g[(K, p, q, l) + 2v_0^*, E] \geq \min\{0, l + \frac{l + 1}{2}\}$$

Therefore,

$$A'(\{(K, p, q, l), E\} - A(U^*[(K, p + l, q + l), E]) \geq 0$$

**Lemma 3.16.** $T$ is filtered.

**Proof.** If $e \in E$ then $T[K, E] = 0$. Suppose $e \notin E$. Since $C$ is filtered, it suffices to show

$$(*) := A'(\{(K, p + 1, q + 1, -1), E\} - A[(K, p, q), E] \leq 0$$

Moreover, the proof of Lemma 3.10 tells us that

$$g[(K, p + 1, q + 1, -1), E] = g[(K, p, q), E]$$
Therefore,

\[ S[(K, p + 1, q + 1, -1), E] = U^0[(K, p, q), E] \]

and as a consequence,

\[ (\ast) = A'[(K, p + 1, q + 1, -1), E] - A(S[(K, p + 1, q + 1, -1), E]) \]

Using the proof of Lemma 3.15,

\[ (\ast) = 0 + g[(K, p, q) + 2v^*_0, E] - g[(K, p + 1, q + 1, -1) + 2v^*_0, E] \]

If \( v, w \notin I \subset E \), then

\[
\begin{align*}
  f[(K, p, q) + 2v^*_0, I] &= f[(K, p + 1, q + 1, -1) + 2v^*_0, I] \\
  f[(K, p, q) + 2v^*_0, I \cup v] &= f[(K, p + 1, q + 1, -1) + 2v^*_0, I \cup v] \\
  f[(K, p, q) + 2v^*_0, I \cup w] &= f[(K, p + 1, q + 1, -1) + 2v^*_0, I \cup w] \\
  f[(K, p, q) + 2v^*_0, I \cup v \cup w] &= f[(K, p + 1, q + 1, -1) + 2v^*_0, I \cup v \cup w] + 1
\end{align*}
\]

Therefore \((\ast) \leq 0 \) unless \( \forall I \subset E \),

\[ (3.17) \quad g[(K, p + 1, q + 1, -1) + 2v^*_0, E] = f[(K, p + 1, q + 1, -1) + 2v^*_0, I] \Rightarrow v, w \in I \]

From the proof of Lemma \( 3.10 \) we know that

\[ (3.18) \quad g[(K, p + 1, q + 1, -1), E] = f[(K, p + 1, q + 1, -1), I] \Rightarrow v, w \notin I \]

Suppose \( I \) satisfies \( (3.17) \) and \( J \) satisfies \( (3.18) \). We have

\[
\begin{align*}
  f[(K, p+1, q+1, -1), J] \leq f[(K, p+1, q+1, -1), I] \leq f[(K, p+1, q+1, -1) + 2v^*_0, I]
\end{align*}
\]

and

\[
\begin{align*}
  f[(K, p+1, q+1, -1) + 2v^*_0, I] \leq f[(K, p+1, q+1, -1) + 2v^*_0, J] \leq f[(K, p+1, q+1, -1), J] + 1
\end{align*}
\]

Therefore either

\[
\begin{align*}
  f[(K, p + 1, q + 1, -1) + 2v^*_0, I] &= f[(K, p + 1, q + 1, -1) + 2v^*_0, J] \\
  f[(K, p + 1, q + 1, -1), I] &= f[(K, p + 1, q + 1, -1), J]
\end{align*}
\]

Both cases are absurd. \( \Box \)

**Proposition 3.19.** \((\mathcal{CF}^-(G), A)\) and \((\mathcal{CF}^-(G'), A')\) are filtered graded chain homotopic.

**Proof.** \( S \) and \( T \) are filtered graded chain maps. Notice that \( S \circ T = \text{Id}_{\mathcal{CF}^-(G)} \) (because \( S \) and \( T \) preserve Maslov gradings). Let’s show that \( T \circ S = C \).

- If \( e \in E \) then \( T \circ S[(K, p, q, l), E] = 0 = C[(K, p, q), E] \).
- If \( e \notin E \), then

\[
T \circ S[(K, p, q, l), E] = U^*C[(K, p + l + 1, q + l + 1, -1), E] = U^*C[(K, p, q), E]
\]

and since \( T, S \) and \( C \) preserve Maslov gradings, \( s = 0 \).

Therefore \( T \circ S = \text{Id}_{\mathcal{CF}^-(G')} + \partial \circ H + H \circ \partial \) and \( H \) is filtered. \( \Box \)
4. Conclusion

Ozsváth, Stipsicz and Szabó show that the other types of blow-ups and blow-downs preserve the filtered lattice chain homotopy type ([1] Corollary 4.9 and Theorem 7.13). We deduce Theorem 0.1 from this and from the previous sections.

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