Abstract. This paper applies G. Lyubeznik’s notion of \( F \)-finite modules to describe in a very down-to-earth manner certain annihilator submodules of some top local cohomology modules over Gorenstein rings. As a consequence we obtain an explicit description of the test ideal of Gorenstein rings in terms of ideals in a regular ring.

1. Introduction

Throughout this paper \((R, m)\) will denote a regular local ring of characteristic \( p \), and \( A \) will be a surjective image of \( R \). We also denote the injective hull of \( R/m \) with \( E \) and for any \( R \)-module \( N \) we write \( \text{Hom}_R(N, E) \) as \( N^\vee \). We shall always denote with \( f : R \to R \) the Frobenius map, for which \( f(r) = r^p \) for all \( r \in R \) and we shall denote the \( e \)th iterated Frobenius functor over \( R \) with \( F^e_R(\cdot) \). As \( R \) is regular, \( F^e_R(\cdot) \) is exact (cf. Theorem 2.1 in [K]).

For any commutative ring \( S \) of characteristic \( p \), the skew polynomial ring \( S[T; f] \) associated to \( S \) and the Frobenius map \( f \) is a non-commutative ring which as a left \( R \)-module is freely generated by \( (T^i)_{i \geq 0} \), and so consists of all polynomials \( \sum_{i=0}^{n} s_i T^i \), where \( n \geq 0 \) and \( s_0, \ldots, s_n \in S \); however, its multiplication is subject to the rule

\[ Ts = f(s)T = s^pT \quad \text{for all } s \in S. \]

Any \( A[T; f] \)-module \( M \) is a \( R[T; f] \)-module in a natural way and, as \( R \)-modules, \( F^e_R(M) \cong RT^e \otimes_R M \).

It has been known for a long time that the local cohomology module \( H^{\dim A}_{m_A}(A) \) has the structure of an \( A[T; f] \)-module and this fact has been employed by many authors to study problems related to tight closure and to Frobenius closure. Recently R. Y. Sharp has described in [S] the parameter test ideal of \( F \)-injective rings in terms of certain \( A[T; f] \)-submodules of \( H^{\dim A}_{m_A}(A) \) and it is mainly this work which inspired us to look further into the structure of these \( A[T; f] \)-modules.

The main aim of this paper is to produce a description of the \( A[T; f] \)-submodules of \( H^{\dim A}_{m_A}(A) \) in terms of ideals of \( R \) with certain properties. We first do this when \( A \) is a
complete intersection. The \( F \)-injective case is described by Theorem 3.5 and as a corollary we obtain a description of the parameter test ideal of \( A \). Notice that for Gorenstein rings the test ideal the parameter test ideal coincide (cf. Proposition 8.23(d) in \[HH1\] and Proposition 4.4(ii) in \[Sm1\].) We then proceed to describe the parameter test ideal in the non-\( F \)-injective case (Theorem 5.3.) We generalise these results to Gorenstein rings in section 6.

2. Preliminaries: \( F \)-finite modules

The main tool used in this paper is the notion of \( F \)-modules, and in particular \( F \)-finite modules. These were introduced in G. Lyubeznik’s seminal work \[L\] and provide a very fruitful point of view of local cohomology modules in prime characteristic \( p \).

One of the tools introduced in \[L\] is a functor \( \mathcal{H}_{R, A} \) from the category of \( A[T; f] \)-modules which are Artinian as \( A \)-modules to the category of \( F \)-finite modules. For any \( A[T; f] \)-module \( M \) which is Artinian as an \( A \)-module the \( F \)-finite structure of \( \mathcal{H}_{R, A}(M) \) is obtained as follows. Let \( \gamma : RT \otimes_R M \to M \) be the \( R \)-linear map defined by \( \gamma(rT \otimes m) = rT m \); apply the functor \( \vee \) to obtain \( \gamma^\vee : M^\vee \to F_R(M)^\vee \). Using the isomorphism between \( F_R(M)^\vee \) and \( F_R(M^\vee) \) (Lemma 4.1 in \[L\]) we obtain a map \( \beta : M^\vee \to F_R(M^\vee) \) which we adopt as a generating morphism of \( \mathcal{H}_{R, A}(M) \).

We shall henceforth assume that the kernel of the surjection \( R \to A \) is minimally generated by \( u = (u_1, \ldots, u_n) \). We shall also assume until section 6 that \( A \) is a complete intersection. We shall write \( u = u_1 \cdot \ldots \cdot u_n \) and for all \( t \geq 1 \) we let \( u^t R \) be the ideal \( u_t R + u_{t+1} R + \cdots + u_n R \).

To obtain the results in this paper we shall need to understand the \( F \)-finite module structure of

\[
\mathcal{H}_{R, A} \left( \mathbb{H}^{\dim R}_{m A} A(R) \right) \cong \mathbb{H}^{\dim R - \dim A(R)}_{u R};
\]

this has generating root

\[
\begin{array}{ccc}
R & \overset{u^{p-1}}{\longrightarrow} & R \\
\downarrow & & \downarrow \\
u R & \overset{u^{p-1}}{\longrightarrow} & u^p R
\end{array}
\]

(cf. Remark 2.4 in \[L\].)

**Definition 2.1.** Define \( \mathcal{I}(R, u) \) to be the set of all ideals \( I \subseteq R \) containing \( (u_1, \ldots, u_n) R \) with the property that

\[
u^{p-1} (I + u R) \subseteq I^{[p]} + u^p R .
\]

**Lemma 2.2.** Consider the \( F_R \)-finite \( F \)-module \( M = \mathbb{H}^n_{u R}(R) \) with generating root

\[
\begin{array}{ccc}
R & \overset{u^{p-1}}{\longrightarrow} & R \\
\downarrow & & \downarrow \\
u R & \overset{u^{p-1}}{\longrightarrow} & u^p R
\end{array}
\]
(a) For any $I \in \mathcal{I}(R, u)$ the $F_R$-finite module with generating root
\[
\frac{I + uR}{uR} \xrightarrow{u^{p-1}} \frac{I[p] + u^pR}{u^pR} \cong F_R \left( \frac{I + uR}{uR} \right)
\]
is an $F$-submodule of $M$ and every $F_R$-finite $F$-submodule of $M$ arises in this way.

(b) For any $I \in \mathcal{I}(R, u)$ the $F_R$-finite module with generating morphism
\[
\frac{R}{I + uR} \xrightarrow{u^{p-1}} \frac{R}{I[p] + u^pR} \cong F_R \left( \frac{R}{I + uR} \right)
\]
is an $F$-module quotient of $M$ and every $F_R$-finite $F$-module quotient of $M$ arises in this way.

**Proof.** (a) For any $I \in \mathcal{I}(R, u)$, the map
\[
\frac{I + uR}{uR} \xrightarrow{u^{p-1}} \frac{I[p] + u^pR}{u^pR}
\]
is well defined and is injective; now the first statement follows from Proposition 2.5(a) in [L]. If $N$ is any $F_R$-finite $F$-submodule of $M$, the root of $N$ is a submodule of the root of $M$, i.e., the root of $N$ has the form $(I + uR)/uR$ for some ideal $I \subseteq R$ (cf. [L], Proposition 2.5(b)) and the structure morphism of $N$ is induced by that of $M$, i.e., by multiplication by $u^{p-1}$, so we must have $u^{p-1}I \subseteq I[p] + u^pR$, i.e., $I \in \mathcal{I}(R, u)$.

(b) For any $I \in \mathcal{I}(R, u)$, the map
\[
\frac{R}{I + uR} \xrightarrow{u^{p-1}} \frac{R}{I[p] + u^pR} \cong F_R \left( \frac{R}{I + uR} \right)
\]
is well defined and we have the following commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \rightarrow & I + uR & \rightarrow & R & \rightarrow & R & \rightarrow & 0 \\
\downarrow u^{p-1} & & \downarrow u^{p-1} & & \downarrow u^{p-1} & & \downarrow u^{p(p-1)} & & \downarrow u^{p(p-1)} & & \downarrow u^{p(p-1)} \\
0 & \rightarrow & F_R \left( \frac{I + uR}{uR} \right) & \rightarrow & F_R \left( \frac{R}{uR} \right) & \rightarrow & F_R \left( \frac{R}{I + uR} \right) & \rightarrow & 0 \\
\end{array}
\]
Taking direct limits of the vertical maps we obtain an exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ which establishes the first statement of (b).

Conversely, if $M''$ is a $F$-module quotient of $M$, say, $M'' \cong M/M'$ for some $F$-submodule $M'$ of $M$ use (a) to find a generating root of $M'$ of the form
\[
\frac{I + uR}{uR} \xrightarrow{u^{p-1}} \frac{I[p] + u^pR}{u^pR}
\]
for some $I \in \mathcal{I}(R, u)$. Looking again at the direct limits of the vertical maps in (1) we establish the second statement of (b).

\[ R \rightarrow \frac{R}{I + uR} \xrightarrow{\mu^p} \frac{R}{I^p + u^pR} \cong F_R \left( \frac{R}{I + uR} \right). \]

**Definition 2.3.** For all $I \in \mathcal{I}(R, u)$ we define $N(I)$ to be the $F$-module quotient of $H^p_\mathcal{I}(R)$ with generating morphism

\[ \frac{R}{I + uR} \xrightarrow{\mu^p} \frac{R}{I^p + u^pR} \]

**Lemma 2.4.** Assume that $R$ is complete. Let $H$ be an Artinian $A[T; f]$-module and write $M = \mathcal{H}_{R, A}(H)$. Let $N$ be a homomorphic image of $M$ with generating morphism $N_0$. Then $N_0^\vee$ is an $A[T; f]$-submodule of $H$ and $N \cong \mathcal{H}_{R, A}(N_0^\vee)$.

**Proof.** Notice that $M$ (and hence $N$) are $F$-finite modules (cf. [L], Theorems 2.8 and 4.2). Let $N_0$ be root of $N$ and $M_0$ a root of $M$ so that we have a commutative diagram with exact rows

\[
\begin{array}{ccc}
M_0 & \xrightarrow{\mu} & N_0 \\
\downarrow{\mu} & & \downarrow{\nu} \\
F_R(M_0) & \xrightarrow{\nu} & F_R(N_0) \\
\end{array}
\]

where the vertical arrows are generating morphisms. Apply the functor $\text{Hom}(-, E)$ to the commutative diagram above to obtain the following commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & F_R(N_0)^\vee \\
\downarrow{\mu^\vee} & & \downarrow{\nu^\vee} \\
0 & \rightarrow & N_0^\vee \\
\end{array}
\]

and recall that $M_0$ is isomorphic to $H^\vee$ (cf. [L], Theorems 2.8 and 4.2). Since $R$ is complete, $(H^\vee)^\vee \cong H$ and we immediately see that $N_0^\vee$ is a $R$-submodule of $H$. We now show that $N_0^\vee$ is an $A[T; f]$-submodule of $H$ by showing that $T N_0^\vee \subseteq N_0^\vee$.

The construction of the functor $\mathcal{H}_{R, A}(-)$ is such that for any $h \in H \cong M_0^\vee$, $Th$ is the image of $T \otimes_R h$ under the map

\[ F_R(M_0)^\vee \xrightarrow{\mu^\vee} M_0^\vee \]

and so for $h \in N_0^\vee$, $Th$ is the image of $T \otimes_R h$ under the map

\[ F_R(N_0)^\vee \xrightarrow{\nu^\vee} N_0^\vee \]

and hence $Th \in N_0^\vee$.

Now the fact that $N \cong \mathcal{H}_{R, A}(N_0^\vee)$ follows the construction of the functor $\mathcal{H}_{R, A}(-)$. □
Notation 2.5. Let $M$ be a left $A[T, f]$-module. We shall write $AT^\alpha M$ for the $A$-module generated by $T^\alpha M$. Note that $AT^\alpha M$ is a left $A[T, f]$-module. We shall also write $M^* = \bigcap_{\alpha \geq 0} AT^\alpha M$.

Lemma 2.6. Assume that $R$ is complete. Let $H$ be an $A[T, f]$-module and assume that $H$ is $T$-torsion-free. Let $I, J \subseteq A$ be ideals. If, for some $\alpha \geq 0$,

$$AT^\alpha \text{ann}_H IA[T, f] = AT^\alpha \text{ann}_H JA[T, f]$$

then $\text{ann}_H IA[T, f] = \text{ann}_H JA[T, f]$.

Proof. Both $AT^\alpha \text{ann}_H IA[T, f]$ and $AT^\alpha \text{ann}_H JA[T, f]$ are left $A[T, f]$-submodules. Now for every $T$-torsion-free $A[T, f]$-module $M$, and every ideal $K \subseteq A$, if

$$\left( \bigoplus_{i \geq 0} KT^i \right) AT^\alpha M = \left( \bigoplus_{i \geq 0} KT^{i+\alpha} \right) M$$

vanishes then so does

$$\left( \bigoplus_{i \geq 0} K[p^\alpha]T^{i+\alpha} \right) M = \left( \bigoplus_{i \geq 0} T^\alpha KT^i \right) M = T^\alpha \left( \bigoplus_{i \geq 0} KT^i \right) M$$

and since $M$ is $T$-torsion-free,

$$\left( \bigoplus_{i \geq 0} KT^i \right) M = 0.$$

We deduce that gr-ann $AT^\alpha M = gr-ann M$. Now

$$gr-ann AT^\alpha(\text{ann}_H IA[T, f]) = gr-ann \text{ann}_H IA[T, f],$$

$$gr-ann AT^\alpha(\text{ann}_H JA[T, f]) = gr-ann \text{ann}_H JA[T, f]$$

and Lemma 1.7 in [S] shows that $\text{ann}_H IA[T, f] = \text{ann}_H JA[T, f]$. \ hfill \Box

3. The $A[T, f]$ Module Structure of Top Local Cohomology Modules of $F$-injective Gorenstein Rings

Definition 3.1. As in [Sm1] we say that an ideal $I \subseteq A$ is an $F$-ideal if $\text{ann}_{H^{\dim(A)}(A)} I$ is a left $A[T, f]$-module, i.e., if $\text{ann}_{H^{\dim(A)}(A)} I = \text{ann}_{H^{\dim(A)}(A)} IA[T, f]$.

Theorem 3.2. Assume that $R$ is complete. Consider the $F_R$-finite $F$-module $M = H^{\alpha}_{uR}(R)$ with generating root

$$R \xrightarrow{u^{p-1}} R \xrightarrow{wR}$$

and consider the Artinian $A[T, f]$ module $H = H^{\dim(A)}(A)$. Let $N$ be a homomorphic image of $M$. 


(a) $M = \mathfrak{K}_{R,A}(-)(H)$ and has generating root $H^\vee \cong R/uR \xrightarrow{u^{p-1}} R/u^pR \cong F_R(H^\vee)$.

(b) If $N$ has generating morphism
\[
\frac{R}{I + uR} \xrightarrow{u^{p-1}} \frac{R}{I[p^p] + u^pR}
\]
then $IA$ is an $F$-ideal, $N \cong \mathfrak{K}_{R,A}(\text{ann}_H IA[T; f])$. If, in addition, $H$ is $T$-torsion free then $\text{gr-ann} \, \text{ann}_H IA[T; f] = IA[T; f]$ and $I$ is radical.

(c) Assume that $H$ is $T$-torsion free (i.e., $H_r = H$ in the terminology of [L]). For any ideal $J \subset R$, the $F$-finite module $\mathfrak{K}_{R,A}(\text{ann}_H JA[T; f])$ has generating morphism
\[
\frac{R}{I + uR} \xrightarrow{u^{p-1}} \frac{R}{I[p^p] + u^pR}
\]
for some ideal $I \in \mathfrak{I}(R, u)$ with $\text{ann}_H IA[T; f] = \text{ann}_H JA[T; f]$.

Proof. The first statement is a restatement of the discussion at the beginning of section 2.

Notice that Lemma 2.2 implies that $N$ must have a generating morphism of the form given in (b) for some $I \in \mathfrak{I}(R, u)$.

Since $A$ is Gorenstein, $H$ is an injective hull of $A/mA$ which we denote $E$. Lemma 2.4 implies that $N \cong \mathfrak{K}_{R,A}(L)$ where $L = \left(\frac{R}{I + uR}\right)^\vee$ is a $A[T; f]$-submodule of $H = E$. But
\[
\left(\frac{R}{I + uR}\right)^\vee = \text{ann}_E (I + uR) = \text{ann}_{(\text{ann}_u R)} I = \text{ann}_E I.
\]

But $L$ is a $A[T; f]$-submodule of $E$ and so $IA$ is an $F$-ideal and $L = \text{ann}_E IA[T; f]$. Also,
\[
(0 :_R \text{ann}_E IA[T; f]) = (0 :_R \text{ann}_E I) = (0 :_R (R/I)^\vee) = (0 :_R (R/I)) = I
\]
(where the third equality follows from 10.2.2 in [BS]) If $H$ is $T$-torsion free, Proposition 1.11 in [S] implies that $I = \text{gr-ann} \, \text{ann}_E IA[T; f]$ and Lemma 1.9 in [S] implies that $I$ is radical.

To prove part (c) we recall Lemma 2.2 which states that $\mathfrak{K}_{R,A}(\text{ann}_H JA[T; f])$ has generating morphism
\[
\frac{R}{I + uR} \xrightarrow{u^{p-1}} \frac{R}{I[p] + u^pR}
\]
for some $I \in \mathfrak{I}(R, u)$ and we need only show that $\text{ann}_H IA[T; f] = \text{ann}_H JA[T; f]$. 

Part (b) implies that \( \mathcal{H}_{R,A}( \text{ann}_H JA[T; f] ) = \mathcal{H}_{R,A}( \text{ann}_H IA[T; f] ) \) for some \( I \in \mathcal{I}(R, u) \) and Theorem 4.2 (iv) in [L] implies
\[
\bigcap_{i=0}^{\infty} AT^i( \text{ann}_H JA[T; f] ) = \bigcap_{i=0}^{\infty} AT^i( \text{ann}_H IA[T; f] )
\]
and since \( H \) is Artinian there exists an \( \alpha \geq 0 \) for which \( AT^\alpha( \text{ann}_H JA[T; f] ) = AT^\alpha( \text{ann}_H IA[T; f] ) \) and the result follows from Lemma 2.6.

**Remark 3.3.** Theorem 3.2 can provide an easy way to show that \( H = H_{\text{dim}(A)} \) is not \( T \)-torsion free. As an example consider \( R = \mathbb{K}[x, y, a, b] \), \( u = x^2a - y^2b \) and \( A = R/uR \). It is easy to verify that \( (x, y, a^2)R \in \mathcal{I}(R, u) \) when \( \mathbb{K} \) has characteristic 2, and we deduce that \( H^3_{(x, y, a^2)}(A) \) is not \( T \)-torsion free.

**Theorem 3.4.** Assume that \( R \) is complete and that \( H_{\text{dim}(A)} \) is \( T \)-torsion free.

(a) For all \( A[T; f] \)-submodules \( L \) of \( H_{\text{dim}(A)} \),
\[
L^\ast = \bigcap_{i=0}^{\infty} AT^i L
\]
has the form \( AT^\alpha M \) where \( \alpha \geq 0 \) and \( M \) is a special annihilator submodule in the terminology of [S].

(b) The set \( \{ N(I) \mid I \in \mathcal{I}(R, u) \} \) is finite.

**Proof.** (a) Let \( L \) be a \( A[T; f] \)-submodule of \( H_{\text{dim}(A)} \). Pick a \( I \in \mathcal{I}(R, u) \) such that \( N(I) = \mathcal{H}_{R,A}(L) \). Now use part (b) of Theorem 3.2 and deduce that \( N(I) \cong \mathcal{H}_{R,A}(\text{ann}_H IA[T; f]) \).

Now the result follows from Theorem 4.2 (iv) in [L].

(b) Theorem 3.2(b) implies that
\[
\{ N(I) \mid I \in \mathcal{I}(R, u) \} = \{ \mathcal{H}_{R,A}(\text{ann}_{H^i_{\text{dim}(A)}}(A) IA[T; f]) \mid I \in \mathcal{I}(R, u) \};
\]
now Corollary 3.11 and Proposition 1.11 in [S] imply that the set on the right is finite.

The following Theorem reduces the problem of classifying all \( F \)-ideals of \( A \) (in the terminology of [Sm1]) or all special \( H_{\text{dim}(A)} \)(-ideals (in the terminology of [S]) in the case where \( A \) is an \( F \)-injective complete intersection, to problem of determining the set \( \mathcal{I}(R, u) \).

**Theorem 3.5.** Assume \( H := H_{\text{dim}(A)} \) is \( T \)-torsion free and let \( \mathcal{B} \) be the set of all \( H \)-special \( A \)-ideals (cf. §6 in [S]).

(a) The map \( \Psi : \mathcal{I}(R, u) \to \mathcal{B} \) given by \( \Psi(I) = IA \) is a bijection.
(b) There exists a unique minimal element \( \tau \) in \( \{ I \mid I \in \mathcal{I}(R, u), \text{ht} IA > 0 \} \) and that \( \tau \) is a parameter-test-ideal for \( A \).

(c) \( A \) is \( F \)-rational if and only if \( \mathcal{I}(R, u) = \{0, R\} \).

**Proof.** (a) Assume first that \( R \) is complete. Theorem \( \mathbb{S} \) (b) implies that \( \Psi \) is well defined, i.e., \( \Psi(I) \in \mathcal{B} \) for all \( I \in \mathcal{I}(R, u) \), and, clearly, \( \Psi \) is injective. The surjectivity of \( \Psi \) is a consequence of Theorem \( \mathbb{S} \) (c).

Assume now that \( R \) is not complete, denote completions with \( \hat{\ } \) and write \( \hat{\mathcal{H}} = H_{m\mathcal{A}}^{\dim(\mathcal{A})} \). If \( I \) is a \( \hat{\mathcal{H}} \)-special \( \hat{\mathcal{A}} \)-ideal, i.e., if there exists an \( \hat{\mathcal{A}}[T; f] \)-submodule \( N \subseteq \hat{\mathcal{H}} \) such that gr-ann \( N = IA[T; f] \) then \( I = (0 :_\mathcal{A} N) \) (cf. Definition 1.10 in \( \mathbb{S} \)). But recall that \( \hat{\mathcal{H}} = \mathcal{H} \) and \( N \) is an \( A[T; f] \)-submodule of \( \mathcal{H} \); now \( I = (0 :_\mathcal{A} N) = (0 :_A N)\hat{\mathcal{A}} \). If we let \( \hat{\mathcal{B}} \) be the set of \( H_{m\mathcal{A}}^{\dim(\mathcal{A})} \)-special \( \hat{\mathcal{A}} \)-ideals, we have a bijection \( \Upsilon : \mathcal{B} \rightarrow \hat{\mathcal{B}} \) mapping \( I \) to \( I\hat{\mathcal{A}} \). This also shows that all ideals in \( \mathcal{I}(\hat{R}, u) \) are expanded from \( R \), and now since \( \hat{R} \) is faithfully flat over \( R \), we deduce that all ideals in \( \mathcal{I}(\hat{R}, u) \) have the form \( I\hat{R} \) for some \( I \in \mathcal{I}(R, u) \). We now obtain a chain of bijections

\[
\mathcal{I}(R, u) \longleftrightarrow \mathcal{I}(\hat{R}, u) \longleftrightarrow \hat{\mathcal{B}} \longleftrightarrow \mathcal{B}.
\]

(b) This is immediate from (a) and Corollary 4.7 in \( \mathbb{S} \).

(c) If \( A \) is \( F \)-rational, \( H_{m\mathcal{A}}^{\dim(\mathcal{A})}(A) \) is a simple \( A[T; f] \)-module (cf. Theorem 2.6 in \( \mathbb{S} \)) and the only \( H \)-special \( A \)-ideals must be 0 and \( A \). The bijection established in (a) implies now \( \mathcal{I}(R, u) = \{0, R\} \).

Conversely, if \( \mathcal{I}(R, u) = \{0, R\} \), part (b) of the Theorem implies that 1 \( \in A \) is a parameter-test-ideal, i.e., for all systems of parameters \( x = (x_1, \ldots, x_d) \) of \( A \), \( (xA)^* = (xA)^F = xA \) where the second equality follows from the fact that \( F_{m\mathcal{A}}^{\dim(\mathcal{A})}(A) \) is \( T \)-torsion free.

\( \square \)

4. Examples

Throughout this section \( \mathbb{K} \) will denote a field of prime characteristic.

**Example 4.1.** Let \( R \) be the localization of \( \mathbb{K}[x, y] \) at \( (x, y) \), \( u = xy \) and \( A = R/uR \). Then \( H_{xyR}(R) = \mathcal{I}_{R, A}(H_{xA+yA}^1(A)) \) ought to have four proper \( F \)-finite \( F \)-submodules corresponding to the elements 0, \( xR \), \( yR \) and \( xR + yR \) of \( \mathcal{I}(R, xy) \).

We verify this by giving an explicit description the \( A[T; f] \)-module structure of

\[
H := H_{xA+yA}^1(A) \cong \lim_{\longrightarrow} \left( \begin{array}{c}
A/(x-y)A \\
A/(x-y)^2A \\
A/(x-y)^3A \\
\end{array} \right) \rightarrow \left( \begin{array}{c}
x-y \\
x-y/A \\
x-y/(x-y)^2A \\
\end{array} \right) \rightarrow \left( \begin{array}{c}
x-y/(x-y)^3A \\
x-y/(x-y)^4A \\
\end{array} \right) \rightarrow \cdots
\]
First notice that in \( H \), for all \( n \geq 1 \) and \( 0 < \alpha \leq n \), \( x^\alpha + (x-y)^nA = x + (x-y)^{n-\alpha+1} \)
and \( y^\alpha + (x-y)^nA = y + (x-y)^{n-\alpha+1} \) so \( H \) is the \( \mathbb{K} \)-span of \( \{x+(x-y)A\} \cup X \cup Y \cup U \)
where
\[
X = \{x+(x-y)^nA | n \geq 2\}, \\
Y = \{y+(x-y)^nA | n \geq 2\}, \\
U = \{1+(x-y)^nA | n \geq 1\}
\]
and notice also that the action of the Frobenius map \( f \) on \( H \) is such that \( T(x^\alpha + (x-y)^nA) = x^{\alpha p} + (x-y)^{np}A \) and \( T(y^\alpha + (x-y)^nA) = y^{\alpha p} + (x-y)^{np}A \) for all \( \alpha \geq 0 \).

Next notice that any \( A[T,f]-\)submodule \( M \) of \( H \) which contains an element \( 1 + (x-y)^nA \in U \) must coincide with \( H \): for \( 1 \leq m < n \) we have \( (x-y)^{n-m}(1+(x-y)^nA) = (x-y)^{n-m} + (x-y)^nA = 1 + (x-y)^mA \), whereas for \( m > n \), pick an \( e \geq 0 \) such that \( np^e > m \), write
\[
T^e(1+(x-y)^nA) = 1 + (x-y)^{np^e}A \in M
\]
and use the previous case \( (m < n) \) to deduce that \( 1 + (x-y)^mA \in M \). Since now \( U \subseteq M \), we see that \( M = H \).

We now show that there are only three non-trivial \( A[T,f]-\)submodules of \( H \), namely \( \text{Span}_K X \) and \( \text{Span}_K Y \), and \( \text{Span}_K \{x+(x-y)A\} \cup X \). By symmetry, it is enough to show that, if \( M \) is an \( A[T,f]-\)submodule of \( H \) and \( x+(x-y)A \in M \) for some \( n \geq 2 \), then \( X \subseteq M \). If \( 1 \leq m < n \),
\[
x^{n-m}(x+y)^nA = x^{n-m+1} + (x-y)^nA = x + (x-y)^{(n-m)}A = x + (x-y)^mA
\]
whereas, if \( m > n \geq 2 \), pick an \( e \geq 0 \) such that \( np^e - p^e + 1 > m \) and write
\[
T^e(x+(x-y)^nA) = x^{p^e} + (x-y)^{np^e}A = x + (x-y)^{np^e-p^e+1}A \in M
\]
and using the previous case \( (m < n) \) we deduce that \( x + (x-y)^mA \in M \).

**Example 4.2.** Let \( R \) be the localization of \( \mathbb{K}[x,y,z] \) at \( \mathfrak{m} = (x,y,z) \), \( u = x^2y + xyz + z^3 \) and \( A = R/uR \). Fedder’s criterion (cf. Proposition 2.1 in [F]) implies that \( A \) is \( F \)-pure, and Lemma 3.3 in [F] implies that the \( A[T; f] - \)module \( H_{\mathfrak{m}A}^1(A) \) is \( T \)-torsion-free.

Here \( J(R, u) \) contains the ideals 0, \( xR + zR \) and \( xR + yR + zR \). We deduce that \( A \) is not \( F \)-rational and that its parameter-test-ideal is \( xR + zR \). Also, Theorem 3.5(b) implies that the only proper ideals in \( J(R, u) \) are the ones listed above.

**Example 4.3.** Let \( R \) be the localization of \( \mathbb{K}[x,y,z] \) at \( \mathfrak{m} = (x,y,z) \) and assume that \( \mathbb{K} \) has characteristic 2. Let \( u = x^3 + y^3 + z^3 + xyz \) and \( A = R/uR \). Notice that we can factor
$u = (x + y + z)(x^2 + y^2 + z^2 + xy + xz + yz)$. Fedder’s criterion implies that $A$ is $F$-pure, and Lemma 3.3 in [F] implies that the $A[T; f]$ module $H_{mA}^1(A)$ is $T$-torsion-free.

Here

$$J(R, u) \supseteq \{0, (x + y + z)R, (x^2 + y^2 + z^2 + xy + xz + yz)R, (x + z, y + z)R, (x + y + z, y^2 + yz + z^2)R, (x, y, z)R\}.$$  

The images in $A$ of the first three ideals have height zero while the images in $A$ of the fourth and fifth ideals have height 1. Using 3.5(b) we conclude that the parameter test-ideal of $A$ is a sub-ideal of

$$J = (x + z, y + z)A \cap (x + y + z, y^2 + yz + z^2)A = (x^2 + yx, y^2 + xz, z^2 + xy)A.$$  

But this ideal defines the singular locus of $A$ and Theorem 6.2 in [HH2] implies that the parameter test-element of $A$ contains $J$, so $J$ is the parameter test-ideal of $A$.

5. The non-$F$-injective case

In this section we extend the results of the previous section to the case where $A$ is not $F$-injective. First we produce a criterion for the $F$-injectivity of $A$.

Definition 5.1. Define

$$J_0(R, u) = \{L \in J(R, u) \mid u^{(p-1)(1+p^{e-1})} \in L^p + u^e R \text{ for some } e \geq 1\}.$$  

Proposition 5.2. (a) For any $L \in J(R, u)$, $N(L) = 0$ if and only if $L \in J_0(R, u)$.

(b) $H_{mA}^{\dim(A)}(A)$ is $T$-torsion free if and only if $J_0(R, u) = \{R\}$.

Proof. (a) Recall that the $F$-finite module $N(L)$ has generating morphism

$$\frac{R}{L + uR} \xrightarrow{u^{p-1}} \frac{R}{L^p + u^e R} \cong F_R\left(\frac{R}{L + uR}\right).$$  

Proposition 2.3 in [L] implies that $N(L) = 0$ if and only if for some $e \geq 1$ the composition

$$\frac{R}{L + uR} \xrightarrow{u^{p-1}} \frac{R}{L^p + u^e R} \xrightarrow{u^{(p-1)p}} \frac{R}{L^{p^2} + u^e R} \cdots \xrightarrow{u^{(p-1)p^{e-1}}} \frac{R}{L^{p^{e-1}} + u^e R}$$  

vanishes, i.e., if and only if $u^{(p-1)(1+p^{e-1})} \in L^p + u^e R$ for some $e \geq 1$.

(b) Write $H = H_{mA}^{\dim(A)}(A)$. If $H$ is $T$-torsion free, the existence of the bijection described in Theorem 3.5(a) implies that for any non-unit $L \in J_0(R, u)$, $\text{ann}_H LA[T; f] \neq \text{ann}_H A[T; f] = 0$. Theorem 3.2(b) implies $N(L) \cong \mathcal{H}_{R,A}(\text{ann}_H LA[T; f])$ so $\mathcal{H}_{R,A}(\text{ann}_H LA[T; f]) = 0$. But Theorem 4.2(ii) in [L] now implies that $\text{ann}_H LA[T; f]$ is nilpotent, a contradiction.
Assume now that $H$ is not $T$-torsion free, i.e., $H_n \neq 0$. The short exact sequence

$$0 \rightarrow H_n \rightarrow H \rightarrow H/H_n \rightarrow 0$$

yields the short exact sequence

$$0 \rightarrow (H/H_n)^\vee \rightarrow \frac{R}{uR} \rightarrow H_n^\vee \rightarrow 0.$$  

Notice that as the functor $\text{Hom}(\cdot, E)$ is faithful, $H_n^\vee \neq 0$, and so $H_n^\vee \cong R/I$ for some ideal $uR \subseteq I \subset R$. Now $\mathcal{K}_{R,A}(H_n)$ is the $F$-finite quotient of $H$ with generating morphism

$$\frac{R}{T} \xrightarrow{u^{p-1}} \frac{R}{T[p]}$$

and this vanishes because of Theorem 4.2(ii) in [L], i.e., $I \in \mathcal{J}_0(R, u)$. □

We now describe the parameter test ideal of $A$. Henceforth we shall always denote $H_{m_A}^{\text{dim}(A)}(A)$ with $H$.

**Theorem 5.3.** Assume that $R$ is complete. The parameter test ideal of $A$ is given by

$$\bigcap \{ I \in \mathcal{J}(R, u) \mid \text{ht} IA > 0 \}.$$

**Proof.** Write $\tau$ for the parameter test ideal of $A$ and let $\tau$ be its pre-image in $R$. Recall that $\tau$ is an $F$-ideal (Proposition 4.5 in [Sm1]), i.e., $\text{ann}_H \tau$ is an $A[T; f]$-submodule of $H$, and $\mathcal{K}_{R,A}(\text{ann}_H \tau)$ has generating morphism

$$(\text{ann}_H \tau)^\vee \xrightarrow{u^{p-1}} F_R ((\text{ann}_H \tau)^\vee).$$

But

$$(\text{ann}_H \tau)^\vee \cong ((A/\tau)^\vee)^\vee \cong R/(\tau + uR)$$

so the generating morphism of $\mathcal{K}_{R,A}(\text{ann}_H \tau)$ is

$$R/(\tau + uR) \xrightarrow{u^{p-1}} R/(\tau[p] + u^p R)$$

and so we must have $\tau \in \mathcal{J}(R, u)$.

As $A$ is Cohen-Macaulay, $\tau = (0 :_A 0_H)$ (cf. Proposition 4.4 in [Sm1].)

By Theorem 3.2(b), for each $I \in \mathcal{J}(R, u)$, the ideal $IA$ is an $F$-ideal and, if $\text{ht} I > 0$, $\text{ann}_H IA = \text{ann}_H IA[T; f] \subseteq 0_H$ and so

$$\tau = (0 :_A 0_H) \subseteq \bigcap \{ (0 :_A \text{ann}_H IA) \mid IA \in \mathcal{J}(R, u), \text{ht} IA > 0 \}.$$

But $H$ is an injective hull of $A/mA$ so

$$(0 :_A \text{ann}_H IA) = (0 :_A \text{Hom}(A/IA, H)) = (0 :_A A/IA) = IA$$
and
\[ \tau \subseteq \bigcap \{ IA \mid IA \in \mathcal{I}(R, u), \text{ht} IA > 0 \}. \]

But as \( \tau \) is one of the ideals in this intersection, we obtain \( \tau = \bigcap \{ IA \in \mathcal{I}(R, u) \mid \text{ht} IA > 0 \} \).

6. The Gorenstein case

In this section we generalise the results so far to the case where \( A \) is Gorenstein.

Write \( \delta = \dim R - \dim A \) and \( \mathcal{E} = E_A(A/\mathfrak{m}A) \). Local duality implies \( \text{Ext}^k_R(A, R) = H^{\dim A}(A)^\vee \cong \text{Hom}(H^{\dim A}(A), \mathcal{E}) \) and since \( A \) is Gorenstein this is just \( A = R/\mathfrak{u}R \).

Now \( \text{Ext}^k_R(R/\mathfrak{u}R, A) \cong R/\mathfrak{u}R, \text{Ext}^k_R(R/\mathfrak{u}^pR, A) \cong R/\mathfrak{u}^pR \) and \( H_{R,A}(H^{\dim A}) = H^0_{R}(R) \) has generating morphism \( R/\mathfrak{u} \rightarrow R/\mathfrak{u}^pR \) given by multiplication by some element of \( R \) which we denote \( \varepsilon(u) \) (this is unique up to multiplication by a unit.) Unlike the complete intersection case, the map \( R/\mathfrak{u} \overset{\varepsilon(u)}{\rightarrow} R/\mathfrak{u}^pR \) may not be injective, i.e., this generating morphism of \( H^0_{R}(R) \) is not a root. However, if define
\[ K_u := \bigcup_{e \geq 0} (\mathfrak{u}^{e+1} R : R \varepsilon(u)^{e+p+\cdots+p^e}) \]
we obtain a root \( R/K_u \overset{\varepsilon(u)}{\rightarrow} R/K_u^{[p]} \) (cf. Proposition 2.3 in [4].)

We now extend naturally our definition of \( \mathcal{I}(R, u) \) when \( A \) is Gorenstein as follows.

**Definition 6.1.** If \( A = R/\mathfrak{u}R \) is Gorenstein we define \( \mathcal{I}(R, u) \) to be the set of all ideals \( I \) of \( R \) containing \( K_u \) for which \( \varepsilon(u)I \subseteq I^{[p]} \).

Now a routine modification of the proofs of the previous sections gives the following two theorems.

**Theorem 6.2.** Assume \( A \) is Gorenstein and that \( H^{\dim A}_{mA}(A) \) is \( T \)-torsion-free.

(a) The map \( I \mapsto IA \) is a bijection between \( \mathcal{I}(R, u) \) and the \( A \)-special \( H^{\dim A}_{mA}(A) \)-ideals.

(b) There exists a unique minimal element \( \tau \) in \( \{ I \mid I \in \mathcal{I}(R, u), \text{ht} IA > 0 \} \) and that \( \tau \) is a parameter-test-ideal for \( A \).

(c) \( A \) is \( F \)-rational if and only if \( \mathcal{I}(R, u) = \{ 0, R \} \).

**Theorem 6.3.** Assume that \( R \) is complete and that \( A \) is Gorenstein. The parameter test ideal of \( A \) is given by
\[ \bigcap \{ I \in \mathcal{I}(R, u) \mid \text{ht} IA > 0 \}. \]
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