Hamiltonian mean curvature flow

Djidémè F. Houénou* and Léonard Todjihoundé*

Abstract

Let $(\Sigma, \omega)$ be a compact Riemann surface with constant curvature $c$. In this work, we proved that the mean curvature flow of a given Hamiltonian diffeomorphism on $\Sigma$ provides a smooth path in $Ham(\Sigma)$, the group of all Hamiltonian diffeomorphisms of $\Sigma$. This result gives a proof, in the case of graph of Hamiltonian diffeomorphisms to the conjecture of Thomas and Yau asserting that the mean curvature flow of a compact embedded Lagrangian submanifold $S$ with zero Maslov class in a Calabi-Yau manifolds $M$ exists for all time and converges smoothly to a special Lagrangian submanifold in the Hamiltonian isotopy class of $S$.

Keywords: Geometric evolution equations, Hamiltonian diffeomorphism group, Lagrangian submanifold ; Maslov class.

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1 Introduction

The deformation of maps between Riemannian manifolds has been intensively studied for a long time. The idea is to find a natural process to deform a map to a canonical one. The harmonic heat flow is probably the famous example although Ricci flow and mean curvature flow are also well used. The latter is an evolution process under which a

*Institut de Mathématiques et de Sciences-Physiques (IMSP)

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submanifold of a given manifold evolves in the direction of its mean curvature vector. From the first variation formula for the volume functional, one easily observes that the mean curvature flow represents the most effective way to decrease the volume of a submanifold such that it is very useful when minimal submanifolds or volume minimizer submanifolds are sorted for under suitable conditions. Many results have been found for mean curvature flow in codimension one while the higher codimension is still receiving attention of number of researchers.

The simplest case of higher codimension is the mean curvature flow of surface in 4-dimensional manifolds. It compounds two important classes known as symplectic mean curvature flow and Lagrangian mean curvature flow; since being symplectic or Lagrangian is preserved along the flow. In the last decades, several works in geometric analysis field research are devoted to these classes (see e.g. [4, 10, 12, 13]).

It is well known that the geometric structures of the ambient space plays a fundamental role when studying the existence and the properties of the mean curvature flow. For instance, to the authors knowledge, symplectic mean curvature flow exists only when the ambient manifold carries at least an (almost) Kähler-Einstein structure [13]; or particularly when it is Calabi-Yau (target spaces for superstring compactification). Moreover Lagrangian and special Lagrangian submanifolds of Calabi-Yau manifolds are considered as the cornerstones for understanding the mirror symmetry phenomenon between pairs of Calabi-Yau manifold both of the categorical point of view and from a physical-geometrical standpoint (see e.g [11]).

There exists a cohomology class attached to any given Lagrangian submanifold of a symplectic manifold: the Maslov class. This class can be represented by a closed 1-form expressed solely in term of the mean curvature of the submanifold and the symplectic form of the ambient manifold. Therefore one observes that minimal Lagrangian has zero Maslov class. In light of this fact we consider the deformation of Hamiltonian diffeomorphism by the mean curvature flow. We will call this flow, when it exists in the group of Hamiltonian
diffeomorphisms, a *Hamiltonian mean curvature flow*.

Let us recall that being graph and Lagrangian are preserved along the mean curvature flow (see e.g. [13], [14]) and these two properties together yield the mean curvature flow of symplectomorphism under the hypothesis that the universal covering of the ambient manifold is of the type $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{R}^2 \times \mathbb{R}^2$ or $\mathbb{H}^2 \times \mathbb{H}^2$.

Later on, in [16, Theorem 1.1], the author proved the following:

**Theorem**

Let $(\Sigma_1, \omega_1)$ and $(\Sigma_2, \omega_2)$ be two homeomorphic compact Riemann surface of the same constant curvature $c = -1, 0, 1$. Suppose $\Sigma$ is the graph of a symplectomorphism $f : \Sigma_1 \rightarrow \Sigma_2$ as a Lagrangian submanifold of $M = (\Sigma_1 \times \Sigma_2, \omega_1 - \omega_2)$ and $\Sigma_t$ is the mean curvature flow with initial surface $\Sigma_0 = \Sigma$. Then $\Sigma_t$ remains the graph of a symplectomorphism $f_t$ along the mean curvature flow. The flow exists smoothly for all time and $\Sigma_t$ converges smoothly to a minimal Lagrangian submanifold as $t \rightarrow \infty$.

In account of this we proved that deforming Hamiltonian diffeomorphism by the mean curvature flow provides a path in the group of Hamiltonian diffeomorphisms. The result is stated as follows:

**Theorem 1.1**

Let $(\Sigma, \omega)$ be a compact connected Riemann surface with non-negative constant curvature. Then any Hamiltonian diffeomorphism on $\Sigma$ deforms through Hamiltonian diffeomorphisms under the mean curvature flow.

In the sequel we set some of the needed materials in the approach of Hamiltonian mean curvature flow and recall the technical tools used for the study of such geometric evolution problem. We then discuss the Hamiltonian property of the time slice of the flow.
2 Preliminaries

Throughout this exposition, all manifolds are smooth and closed (compact without boundary) unless it is stated otherwise.

Definition 2.1

Let $M$ be a differentiable manifold. The mean curvature motion of $S \subset M$ is a 1-parameter family of immersions of submanifolds $S_t$ in $M$ which admits a parametrization $F_t : S \hookrightarrow S_t \subset M$ over $S$ with normal velocity equal to the mean curvature vector i.e.

$$\left( \frac{\partial}{\partial t} F_t(x) \right)^\perp = H(F_t(x)), \quad x \in S,$$

$$F_0 = \text{id.}$$  \hspace{1cm} (1)

$$F_0 = \text{id.}$$

Remark 2.1

The mean curvature motion is a non linear weakly parabolic system for $F$ and is invariant under reparametrization of $S$. Indeed, by coupling with a diffeomorphism $\varphi$ of $S$, the flow can be made into a normal direction, i.e.

$$\frac{\partial}{\partial t} F_t(\varphi(x)) = H(F_t(\varphi(x))).$$  \hspace{1cm} (2)

For any smooth compact initial data, one can establish the short time existence solution for (2) and the uniqueness of the solution for suitable conditions on the initial data.

Let $(\Sigma, \omega)$ be a compact Riemann surface with a constant curvature $c$ and let $f \in \mathcal{D}iff(\Sigma, \omega)$ be a diffeomorphism of $\Sigma$. Put $M = \Sigma \times \Sigma$ and denote by $S$ the graph of $f$.

The mean curvature flow of $f$ is realized through the mean curvature motion of $S$. In fact, knowing that the mean curvature flow preserved the graph and Lagrangian properties, one obtains the flow in the group of symplectic diffeomorphisms $\text{Symp}(\Sigma, \omega)$.

Definition 2.2
A diffeomorphism $f$ on a symplectic manifold $(\Sigma, \omega)$ is said to be Hamiltonian if there exists a smooth function $G : \Sigma \to \mathbb{R}$ such that $f \in \{f_s\}_s$, where $\{f_s\}_s$ is the Hamiltonian flow of $X = X_G$, i.e. the family of diffeomorphisms obtained by solving the ordinary differential equation:

$$
\begin{align*}
\frac{\partial}{\partial s} f_s(x) &= X(f_s(x)) \\
 f_0(x) &= x.
\end{align*}
$$

The definition above is a classical definition of Hamiltonian diffeomorphism. In this definition, the vector field does not depend on time and is often refer to as autonomous vector field. The analog for time-dependent vector field is the characterization using the flux homomorphism. It is stated as in the following:

**Definition 2.3**

The time-one map of a symplectic isotopy (path to identity) with null flux is called a Hamiltonian diffeomorphism (see [1]).

Therefore, the flux is an obstruction for symplectomorphism which is isotopic to identity to be a Hamiltonian diffeomorphism. Let us recall the definition of the flux homomorphism. For more details we refer to [2] where a comprehensive exposition is made. The flux homomorphism is defined as follows, [Theorem 3.1.1, [2]]:

$$
\overline{\text{Flux}} : \widetilde{\text{Symp}}_0(\Sigma, \omega) \to H^1(\Sigma, \mathbb{R})
$$

$$
\{\tilde{f}_s\} \mapsto \left[ \int_0^1 f_s^*(i_{X_s} \omega) \, ds \right]
$$

(3)

where $\widetilde{\text{Symp}}_0(\Sigma, \omega)$ is the universal covering of the connected component of the identity in the group of symplectomorphisms, $\{\tilde{f}_s\}$ is a homotopy class of an isotopy $f_s$ generated by $X_s$, and $f_s^*(i_{X_s} \omega)$ stands for the pull-back of the form $i_{X_s} \omega$ (interior product of $\omega$ by $X_s$). Notice that the flux homomorphism descends to the group $\text{Symp}_0(\Sigma, \omega)$; for convenience, we will recall this definition in the following section.
Let \( \{f_s\}_{0 \leq s \leq 1} \) be a symplectic isotopy to \( f \), generated by the time-dependent vector field \( X_s \) and denote \( \mathcal{F} \), its flux form i.e.

\[
\mathcal{F} = \int_0^1 f_s^* i_{X_s} \omega \, ds.
\]

The cohomology class \([\mathcal{F}]\) depends only on the homotopy classes of the isotopy \( \{f_s\}_{0 \leq s \leq 1} \) relatively to fixed ends. The behavior of \( \mathcal{F} \) under the mean curvature flow will be the major ingredient for the purpose of preserving Hamiltonian condition by the mean curvature flow since it is well known that the mean curvature flow of symplectomorphism exists smoothly for all time and converges (see e.g. \([10, 13, 14, 15]\)).

In the sequel, we compute the evolution equation of \( \mathcal{F} \) and use it to find out under which hypothesis the Hamiltonian property is preserved along the flow.

### 3 Hamiltonian property along the flow

Recall that the group of Hamiltonian diffeomorphisms \( Ham(\Sigma, \omega) \) is the kernel of an onto homomorphism

\[
Flux : \text{Symp}_0(\Sigma, \omega) \longrightarrow H^1(\Sigma, \mathbb{R})/\Gamma_{\omega}
\]

where the flux group \( \Gamma_{\omega} = Flux \left( \pi_1 \left( \text{Symp}_0(\Sigma, \omega) \right) \right) \) is finitely generated but is not known to be discrete in all cases. Hence the most one can say in general is that \( Ham(\Sigma, \omega) \) sits inside the identity component \( \text{Symp}_0(\Sigma, \omega) \) as the leaf of a foliation. Therefore, we do not use the topology on \( Ham(\Sigma, \omega) \) induced from \( \text{Symp}_0(\Sigma, \omega) \) but instead use the topology on \( Ham(\Sigma, \omega) \) induced from the \( C^2 \)-topology on the Lie algebra of Hamiltonian functions with zero mean. Thus a neighborhood of the identity consists of all time 1-maps of Hamiltonian flows generated by Hamiltonians \( G_s \) that are sufficiently small in the \( C^2 \)-topology.

Let \( f : \Sigma \longrightarrow \Sigma \) be a Hamiltonian diffeomorphism on a compact Riemann surface with
constant curvature. As asserted above, we have two different ways to regard $f$. Let us consider a symplectic isotopy viewpoint, meaning $f$ is the end point of some symplectic isotopy $\{f_s\}_{0 \leq s \leq 1}$ with zero flux. The mean curvature flow of $f$ gives rise to a 2-parameter family of symplectomorphisms $\{f_{s,t}\}$ satisfying:

\[
\begin{cases}
  f_{1,0} = f \\
  f_{0,t} = id, \\
  f_{1,t} = f_t \\
  \text{for each } t \\

\end{cases}
\]

\[
\text{Flux}\{f_{s,0}\} = \left[ \int_0^1 f_{s,0}^* (i_{X_{s,0}} \omega) ds \right] = [\mathcal{F}_0] = 0
\]

\[
\frac{\partial}{\partial s} f_{s,t} = X_{s,t}
\]

\[
\frac{\partial}{\partial t} f_{s,t} = H_{s,t}
\]

\[
\frac{\partial}{\partial t} X_{s,t} = \frac{\partial}{\partial s} H_{s,t} - [H_{s,t}, X_{s,t}]
\]

where $X_{s,t}$ is the isotopy vector field and $H_{s,t}$ the mean curvature vector field; $s$ is the isotopy parameter and $t$ is the one of the mean curvature flow.

**Definition 3.1**

*Let $X$ be a vector field with a local 1-parameter group $(\varphi_t)$ of local diffeomorphisms and $S$ a tensor field on a differentiable manifold $M$. The Lie derivative of $S$ in the direction $X$ is defined as*

\[
\varphi_t^* L_X S := \left( \frac{d}{dt} \varphi_t^* S \right) \bigg|_{t=0}
\]

Let $M$ and $N$ be two differentiable manifolds. Assume $\varphi_t : M \rightarrow N$ is a smooth 1-parameter family of maps between $M$ and $N$ and $\omega_t$ is a smooth family of forms on $N$. Then $\varphi_t^* \omega_t$ is a smooth family of forms on $M$ and the basic formula of differential calculus
of forms gives (see e.g [7]):

$$\frac{d}{dt} \varphi_t^* \omega_t = \varphi_t^* L_{X_t} \omega_t + \varphi_t^* \frac{d}{dt} \omega_t,$$

(5)

where $X_t$ is the tangent vector field along $\varphi_t$.

**Lemma 3.1**

The flux form $\mathcal{F}_t$ of the isotopy $f_{s,t}$ satisfies the following equation:

$$\frac{\partial}{\partial t} \mathcal{F}_t = f_t^* i_{H_t} \omega + dK_t$$

(6)

where $K_t = \int_0^1 f_{s,t}^* \omega(X_{s,t}, H_{s,t}) ds$ and $f_t$ (the time $t$-slice of the flow) is the time-one map of the isotopy $\{f_{s,t}\}_{0 \leq s \leq 1}$.

**Proof:**

Taking into account the fact that $X_{s,t}$ and $H_{s,t}$ are symplectic vector fields, and using Definition 2.2, a direct computation yields

$$\frac{\partial}{\partial t} \mathcal{F}_t = \int_0^1 \frac{\partial}{\partial t} \left( f_{s,t}^* i_{X_{s,t}} \omega \right) ds$$

(7)

$$= \int_0^1 \left( f_{s,t}^* L_{H_{s,t}} i_{X_{s,t}} \omega + f_{s,t}^* \frac{\partial}{\partial t} i_{X_{s,t}} \omega \right) ds$$

$$= \int_0^1 \left( f_{s,t}^* d i_{H_{s,t}} i_{X_{s,t}} \omega + f_{s,t}^* \frac{\partial}{\partial s} i_{H_{s,t}} \omega - f_{s,t}^* i_{[H_{s,t}, X_{s,t}]} \omega \right) ds$$

$$= \int_0^1 \left( f_{s,t}^* d i_{H_{s,t}} i_{X_{s,t}} \omega + f_{s,t}^* \frac{\partial}{\partial s} i_{H_{s,t}} \omega + f_{s,t}^* L_{X_{s,t}} i_{H_{s,t}} \omega \right) ds$$

$$= \int_0^1 \left( f_{s,t}^* d i_{H_{s,t}} i_{X_{s,t}} \omega + \frac{\partial}{\partial s} \left( f_{s,t}^* i_{X_{s,t}} \omega \right) \right) ds$$

$$= f_t^* i_{H_t} \omega + d \int_0^1 f_{s,t}^* \omega(X_{s,t}, H_{s,t}) ds$$

□

It was discovered in 1965 by V. P. Maslov that there is a cohomology class which appears naturally in the resolution by the Hamilton-Jacobi method of the Schrödinger
equation of quantum physics. In mathematics, it is a cohomology class attached to a given Lagrangian submanifold a symplectic manifold. This class is an important cohomology invariant and is called the Maslov class. Since J.-M. Morvan’s work, it has been found possible to express this class solely in terms of the Riemmanian structure of the Lagrangian immersion associated to the Kähler metric on a symplectic manifold (see e.g [9]).

Definition 3.2

Let \((M, \omega)\) be a Kähler-Einstein \(2n\)-dimensional manifold and \(L \hookrightarrow M\) be an immersed Lagrangian submanifold of \(M\). Then the Maslov class of \(L\) is defined by:

\[
\frac{n}{\pi} [i_H \omega],
\]

where \(H\) is the mean curvature vector along \(L\).

Therefore we obtain the following:

Lemma 3.2

Let \((f_s)_{0 \leq s \leq 1}\) be a symplectic isotopy to a Hamiltonian diffeomorphism \(f\). Then the cohomology class of the flux form of \(f_s\) deforms to the Maslov class of \(S\) by the mean curvature flow.

Proof:

From equation (6) one gets:

\[
\frac{\partial}{\partial t} [\mathcal{F}_t] = \left[ \frac{\partial}{\partial t} \right] F_t = [f_t^* i_H \omega].
\]

\(\square\)

As an immediate consequence to Lemma 3.2, the following holds:

Proposition 3.1
Let $\Sigma$ be a compact connected Riemann surface with constant curvature and $f \in \text{Ham}(\Sigma)$. Suppose $S = \text{graph} f$ has zero Maslov class, then the flux of any symplectic isotopy to $f$ is preserved along the mean curvature flow.

**Proof:**

Let $\{f_s\}_{0 \leq s \leq 1}$ be a symplectic isotopy to $f$ generated by the vector field $X_s$. The mean curvature flow of $f$ is a 2-parameter family $\{f_{s,t}\}_{s,t}$ of symplectomorphisms. So using Lemma 3.2 and taking into account the fact that $\Sigma$ is connected, one concludes that the cohomology class of the flux form is constant along the flow. □

We now state the main results of this work.

**Theorem 3.1**

Let $\Sigma$ be a compact connected Riemann surface with constant curvature and $f \in \text{Ham}(\Sigma)$. Assume that $S = \text{graph} f$ has zero Maslov class. Then any Hamiltonian diffeomorphism on $\Sigma$ deforms through Hamiltonian diffeomorphisms by the mean curvature flow.

**Proof:**

Let $f \in \text{Ham}(M, \omega)$. There exists a symplectic isotopy $\{f_s\}$ to $f$ such that the mean curvature flow of $f$ is a 2-parameter as in system (A). We know that the flow exists (see e.g. [14]). So using the Proposition 3.1, for each time $t$, the flux of the isotopy $\{f_{s,t}\}$ to $f_t$ is zero. Therefore $f_t$ is a Hamiltonian diffeomorphism. □

Thus we call Hamiltonian mean curvature flow a mean curvature flow for which any time slice of the flow is Hamiltonian or equivalently a mean curvature flow of Lagrangian graphs Hamiltonian isotopic to the diagonal.

### 3.1 Non-negative curvature case

In this section we assume that $\Sigma$ has non-negative constant curvature and observe that the assumption of zero Maslov class can be removed. We have the following:
**Theorem 3.2**

Let $(\Sigma, \omega)$ be a compact connected Riemann surface with constant curvature $c$ and $f \in \text{Ham}(\Sigma)$. If $c$ is non-negative, then $f$ deforms through Hamiltonian diffeomorphisms under the mean curvature flow.

**Proof**:

$M = \Sigma \times \Sigma$ is compact and its universal covering is either $S^2 \times S^2$ or $\mathbb{R}^2 \times \mathbb{R}^2$. The submanifold $S$ (graph of $f$) is Lagrangian w.r.t $\omega' = \omega \oplus \omega$ and symplectic w.r.t $\omega' \oplus \omega$. Then $f$ deforms through symplectomorphism [14]. What is left is to prove that each slice $f_t$ of the flow is Hamiltonian, i.e a time one map of some symplectic isotopy $\{f_{s,t}\}_{0 \leq s \leq 1}$ with zero flux.

1. Suppose $\Sigma$ is elliptic ($c > 0$), then $H^1(\Sigma, \mathbb{R})$ is trivial and so is $H^1(S, \mathbb{R})$; thus the flux is preserved. Since its initial value is zero, then each $t$-slice of the flow is Hamiltonian.

2. If $c = 0$, then $M$ is Calabi-Yau. Let $\theta_t$ be the Lagrangian angle of $S_t$. The mean curvature form satisfies

$$i_{H_t} \omega = d\theta_t$$

which implies that the Maslov class vanishes. Thus, the flux is preserved along the flow and since its initial value is zero, we deduce that each $f_t$ is Hamiltonian. $\square$

**Definition 3.3**

A diffeomorphism on $\Sigma$ is called a minimal diffeomorphism if its graph is a minimal embedding in $\Sigma \times \Sigma$.

As a consequence of Theorem 3.2, we obtain the following corollaries:

**Corollary 3.1**

Let $(\Sigma, \omega)$ be a compact connected Riemann surface with non-negative constant curvature $c$ and $f \in \text{Ham}(\Sigma)$. As $t \rightarrow \infty$, a sequence of the mean curvature flow of the graph of $f$ converges to a smooth minimal Hamiltonian graph.
Corollary 3.2

Let $(\Sigma, \omega)$ be a compact connected Riemann surface with strictly positive constant curvature $c$ and $f \in \text{Ham}(\Sigma)$. Then the Hamiltonian mean curvature flow of $S = \text{graph} f$ exists for all time $t$, each $S_t$ can be written as a graph of a Hamiltonian diffeomorphism $f_t$. The sequence of submanifolds $S_t$ converges to the diagonal as $t$ goes to infinity.

The proof of these corollaries are the same as in [14]; in addition with the preserving Hamiltonian property from Theorem 3.2.

A particular example of calibrated submanifolds was first introduced by Harvey and Lawson. These submanifolds are known as special Lagrangian (see definition below). It is not hard to check that calibrated submanifolds are volume minimizer in their homology class so special Lagrangian are minimal submanifolds.

Definition 3.4

A Lagrangian submanifold in a Calabi-Yau manifold $(M, \Omega)$ is called special if it has constant Lagrangian angle.

The Theorem 3.2 and its Corollary 3.1 give the proof in the case of graph of Hamiltonian diffeomorphism to the conjecture of Thomas and Yau asserting that the mean curvature flows of a Lagrangian submanifold $S$ with zero Maslov class exits for all time and converges to a special Lagrangian submanifold in the Hamiltonian isotopy class of $S$. We proved that the Hamiltonian isotopy is nothing else but the path obtained by the mean curvature flow.

Theorem 3.3

Let $S$ be a graph of some Hamiltonian diffeomorphism $f$ on a flat torus $T^2$. Then the Hamiltonian mean curvature flow of $S$ exists for all time and converges to a special Hamiltonian graph isotopic to $S$.

Proof:
Ham(T^2) is contractible, so every f ∈ Ham(T^2) flows through Hamiltonian diffeomorphisms to the identity which graph (the diagonal in T^2 × T^2) is a minimal surface. Then the mean curvature form is exact which infers that the Lagrangian angle is constant. Thus the limit is a special Lagrangian submanifold. □

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