Patterns of i.i.d. Sequences and Their Entropy - Part I: General Bounds

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Abstract

Tight bounds on the block entropy of patterns of sequences generated by independent and identically distributed (i.i.d.) sources are derived. A pattern of a sequence is a sequence of integer indices with each index representing the order of first occurrence of the respective symbol in the original sequence. Since a pattern is the result of data processing on the original sequence, its entropy cannot be larger. Bounds derived here describe the pattern entropy as function of the original i.i.d. source entropy, the alphabet size, the symbol probabilities, and their arrangement in the probability space. Matching upper and lower bounds derived provide a useful tool for very accurate approximations of pattern block entropies for various distributions, and for assessing the decrease of the pattern entropy from that of the original i.i.d. sequence.

Index Terms: patterns, index sequences, entropy.

1 Introduction

Several recent works (see, e.g., [1], [6], [7], [12], [15], [16]) have considered universal compression for patterns of independent and identically distributed (i.i.d.) sequences. The pattern of a sequence $x^n \triangleq (x_1, x_2, \ldots, x_n)$ is a sequence $\psi^n \triangleq \psi \triangleq \Psi (x^n)$ of pointers that point to the actual alphabet

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letters, where the alphabet letters are assigned *indices* in order of first occurrence. For example, the pattern of all sequences $x^n = \text{lossless}$, $x^n = \text{sellsoll}$, $x^n = 12331433$, and $x^n = 76887288$ is $\psi^n = \Psi(x^n) = 12331433$. Capital $\Psi(\cdot)$ is used to denote the operator of taking a pattern of a sequence. A pattern sequence thus contains all positive integers from 1 up to a maximum value in increasing order of first occurrence, and is also independent of the alphabet of the actual data.

Universal compression of patterns is interesting when compressing sequences generated by an initially unknown alphabet, such as a document in an unknown language. In such applications, separate dictionary and pattern compression can be performed. Most initial work on this topic focused on showing diminishing universal compression redundancy rates for the *individual sequence* case [6], [7], and for the average case [12], [15], [16]. However, since a pattern $\Psi(x^n)$ is the result of data processing on the original sequence $x^n$, its entropy must be no greater than that of the original sequence. Specifically, if $x^n$ is generated by an i.i.d. source of alphabet size $k$,

$$nH_\theta(X) - \log |k!| (\max \{0, k - n\})! \leq H_\theta(\Psi^n) \leq nH_\theta(X),$$

where capital letters denote random variables, and $\theta$ is the probability parameter vector governing the source. The lower bound is since $H_\theta(\Psi^n) = H_\theta(X^n, \Psi^n) - H_\theta(X^n | \Psi^n) = H_\theta(X^n) - H_\theta(X^n | \Psi^n)$, where the second equality is because there is no uncertainty about $\Psi^n$ given $X^n$. Finally, $H_\theta(X^n | \Psi^n)$ is bounded by logarithm of the total possible mappings from indices to symbols.

The bounds in (1) already show that for $k = o(n)$, the pattern entropy rate equals the i.i.d. one for non-diminishing $H_\theta(X)$. However, the bounds in (1) are usually loose. Specifically, the *description length* shown for sufficiently large alphabets in [12] (see also [16]) for a universal sequential compression method for patterns was significantly smaller than the block i.i.d. entropy. This indicates that not only is there an entropy decrease in patterns, but for large alphabets, this decrease is more significant than universal coding redundancy. Hence, it is essential to study the behavior of the pattern entropy. Pattern entropy is also important in learning applications. Consider all the new species an explorer observes. The explorer can identify these species with the first time each was seen. There is no difference if it sees specie $A$ or specie $B$ (and never sees the other). The next time the observed specie is seen, it is identified with its index. The entropy of patterns can model uncertainty of such processes. Its exponent gives an approximate count of the typical patterns one is likely to observe. If the uncertainty goes to 0, we are likely to observe only one pattern.

Initial results from this paper, first presented in [14], bounded the range of values within which the entropy of a pattern can be, depending on the alphabet size. Subsequently to our initial results

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1Logarithms are taken to base 2, here and elsewhere. The natural logarithm is denoted by $\ln$. 
pattern entropy rates were independently studied with a different view of the problem in \[14\] and \[8\]. The main result was that for discrete i.i.d. sources the pattern entropy rate is equal to that of the underlying i.i.d. process. This result was also extended to discrete finite entropy stationary processes. Some limiting order of magnitude bounds on block pattern entropies were also provided.

This paper extensively studies block entropy of patterns, providing tight upper and matching lower bounds on the block entropy. The bound pairs can be used together to provide very accurate approximations of the entropy of $\Psi^n$. Specific distributions are studied in \[13\]. The basic method partitions the probability space into a grid of points. Between each two points, we obtain a bin. Symbols whose probabilities lie in the same bin can be exchanged in a given $x^n$ to provide another sequence $x'^n$ with the same pattern and almost equal probability. Counting all these sequences leads to the bounds on the pattern entropy. Very low probabilities are combined into one point mass. A key factor in obtaining tight bounds is a proper choice of increased-spacing grids.

The outline of the paper is as follows. Section 2 defines some notation and presents some preliminaries. A summary of the main results in the paper is given in Section 3. Then, in Section 4, upper and lower bounds for pattern entropy of i.i.d. sources with sufficiently large probabilities are derived. Section 5 contains the derivations of more general upper and lower bounds, that do not require a condition on the letter probabilities. Finally, Section 6 shows the range of values that the pattern entropy can take for bounded probabilities, depending on the actual source distribution.

2 Preliminaries

Let $x^n$ be an $n$-tuple with components $x_i \in \Sigma \triangleq \{1, 2, \ldots, k\}$ (where the alphabet is defined without loss of generality). The asymptotic regime is that $n \to \infty$, but $k$ may also be greater then $n$. The vector $\theta \triangleq (\theta_1, \theta_2, \ldots, \theta_k)$ is the set of probabilities of all letters in $\Sigma$. Since the order of the probabilities does not affect the pattern, we assume, without loss of generality, that $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_k$. Boldface letters denote vectors, whose components are denoted by their indices. Capital letters will denote random variables. The probability of $\psi^n$ induced by an i.i.d. source is

$$P_\theta (\psi^n) = \sum_{y^n: \Psi(y^n) = \psi^n} P_\theta (y^n).$$

This probability can also be expressed by fixing the actual sequence and summing over all permutations of occurring symbols of the parameter vector, i.e.,

$$P_\theta [\Psi (x^n)] = \sum_{\sigma = \{\sigma_i: i \in x^n\}} P_{\theta(\sigma)} (x^n),$$

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where \( \sigma = \{ \sigma_1, \ldots, \sigma_k \} \) is a permutation set. For example, if \( \theta = (0.4, 0.1, 0.2, 0.3) \) and \( \sigma = (3, 1, 4, 2) \), then \( \theta(\sigma) = (0.2, 0.4, 0.3, 0.1) \) and \( \theta(\sigma_2) = \theta_1 = 0.4 \). The only relevant components of \( \sigma \) in (3) are those of occurring symbols. Thus if only \( m < k \) symbols occur in \( x^n \), there are only \( k!/(k-m)! \) elements in the sum in (3). The entropy rate of an i.i.d. source is \( H_\theta(X) \), and its sequence (block) entropy is \( H_\theta(X^n) = nH_\theta(X) \). The pattern sequence entropy of order \( n \) is

\[
H_\theta(\Psi^n) \triangleq - \sum_{\psi^n} P_\theta(\psi^n) \log P_\theta(\psi^n).
\]

(4)

To derive bounds on the pattern entropy, we define three different grids: \( \tau, \eta, \) and \( \xi \), the first two for upper bounding and the third for lower bounding. Spacing between grid points is motivated by the fact that two probability parameters \( \theta \) and \( \theta' \) separated by \( O(\sqrt{\delta}/\sqrt{n^{1+\delta}}) \); \( \delta > 0 \), are near enough to appear similar in \( x^n \). On the other hand, if \( |\theta - \theta'| > \sqrt{\delta}/\sqrt{n^{1-\delta}} \), the parameters are far enough to appear different. For simplicity of notation, we omit the dependence on \( n \) from definitions of grid points. For \( \varepsilon > 0 \), let \( \tau \triangleq (\tau_0, \tau_1, \tau_2, \ldots, \tau_0, \ldots, \tau_{B_\tau}) \) be a grid of \( B_\tau + 1 \) points defined by \( \tau_0 = 0, \) and

\[
\tau_b = \sum_{j=1}^{b} \frac{2(j - \frac{1}{2})}{n^{1+\varepsilon}} = \frac{b^2}{n^{1+\varepsilon}}, \quad b = 1, 2, \ldots, B_\tau.
\]

(5)

Let \( \eta' \) be defined almost like \( \tau \),

\[
\eta'_b \triangleq \sum_{j=1}^{b} \frac{2(j - \frac{1}{2})}{n^{1+2\varepsilon}} = \frac{b^2}{n^{1+2\varepsilon}},
\]

(6)

The grid \( \eta \triangleq (\eta_0, \eta_1, \ldots, \eta_{B_\eta}) \) is defined by \( \eta_0 = 0, \eta_1 = \eta_2 = \frac{1}{n^{1+\varepsilon}}, \) and

\[
\eta_b = \eta'_b - \lfloor n^{3\varepsilon/2} \rfloor - 2, \quad b = 3, 4, \ldots, B_\eta.
\]

(7)

Unlike \( \tau \) and \( \eta \), \( \xi \triangleq (\xi_0, \xi_1, \ldots, \xi_{B_\xi}) \) is defined for lower bounds purposes. It is defined in a similar manner as the others, where \( \xi_0 = 0 \), and for an arbitrarily small \( \varepsilon > 0 \),

\[
\xi_b \triangleq \sum_{j=1}^{b} \frac{2(j - 0.5)}{n^{1-\varepsilon}} = \frac{b^2}{n^{1-\varepsilon}}, \quad b = 1, 2, \ldots, B_\xi.
\]

(8)

For all grids, \( \tau_{B_\tau + 1} = \eta_{B_\eta + 1} = \xi_{B_\xi + 1} \triangleq 1 \). We thus have \( B_\tau = \lfloor \sqrt{n^{1+\varepsilon}} \rfloor, \) \( B_\eta = \lfloor \sqrt{n^{1+2\varepsilon}} \rfloor - \lfloor n^{3\varepsilon/2} \rfloor + 2, \) and \( B_\xi = \lfloor \sqrt{n^{1-\varepsilon}} \rfloor \). We also define the maximal indices \( A_\tau, A_\eta, \) and \( A_\xi \) whose grid points do not exceed 0.5 for \( \tau, \eta, \) and \( \xi \), respectively. Hence, \( A_\tau = \lfloor \sqrt{n^{1+\varepsilon}/\sqrt{2}} \rfloor, A_\eta = \lfloor \sqrt{n^{1+2\varepsilon}/\sqrt{2}} \rfloor - \lfloor n^{3\varepsilon/2} \rfloor + 2, \) and \( A_\xi = \lfloor \sqrt{n^{1-\varepsilon}/\sqrt{2}} \rfloor \).

By definition of \( \eta \), for every \( \theta \in [\eta_b, \eta_{b+1}] \) where \( b \geq 2 \),

\[
\eta_{b+1} - \eta_b = \frac{2[(b + d) + 0.5]}{n^{1+2\varepsilon}} \leq \frac{3(b + d)}{n^{1+2\varepsilon}} = \frac{3\sqrt{\eta_{b+d}}}{\sqrt{n^{1+2\varepsilon}}} = \frac{3\sqrt{\eta_{b}}}{\sqrt{n^{1+2\varepsilon}}} \leq \frac{3\sqrt{\theta}}{\sqrt{n^{1+2\varepsilon}}},
\]

(9)
where \( d \triangleq \lfloor n^{\varepsilon/2} \rfloor - 1 \). A similar bound applies to \( \tau_b \), \( b \geq 1 \), with \( \varepsilon \) in place of \( 2\varepsilon \). Similarly,

\[
\xi_{b+1} - \xi_b = \frac{2(b + 0.5)}{n^{1 - \varepsilon}} = \frac{2 \left( \sqrt{\xi_b} \sqrt{n^{1 - \varepsilon}} + 0.5 \right)}{n^{1 - \varepsilon}} \geq \frac{2\sqrt{\xi_b}}{\sqrt{n^{1 - \varepsilon}}}. \tag{10}
\]

We use \( c_b; b = 0, 1, \ldots, B_\tau \), \( k_b; b = 0, 1, \ldots, B_\eta \), and \( \kappa_b; b = 0, 1, \ldots, B_\xi \), to denote the number of symbols for which \( \theta_i \in (\tau_b, \tau_{b+1}] \), \( \theta_i \in (\eta_b, \eta_{b+1}] \), and \( \theta_i \in (\xi_b, \xi_{b+1}] \), respectively. Respective vectors containing all components are denoted by \( \mathbf{c} \), \( \mathbf{k} \), and \( \mathbf{\kappa} \). In addition, define \( \kappa'_b; b = 1, 2, \ldots, B_\xi \), as zero if \( \kappa_b \) is zero, and otherwise, as the number of symbols for which \( \theta_i \in (\xi_{b-1}, \xi_{b+2}] \), with the exception of \( \kappa'_1 \), which will only count letters for which \( \theta_i \in (\xi_1, \xi_3] \). (There is clearly an overlap between adjacent counters in \( \kappa' \), which is needed for derivation of a lower bound.)

The grid \( \tau \) is defined so that all letters \( \theta_i \leq 1/n^{1+\varepsilon} \) are grouped in the same bin. Grid \( \eta \) also groups probabilities in \( (1/n^{1+\varepsilon}, 1/n^{1-\varepsilon}] \) in bin 1. In particular, \( k_0 \) and \( k_1 \) denote the symbol counts of the two groups, respectively. We will also use \( k_{01} \triangleq k_0 + k_1 \) to denote the total letters with \( \theta_i \leq 1/n^{1-\varepsilon} \) (thus \( k - k_{01} \) denotes the count of symbols with \( \theta_i > 1/n^{1-\varepsilon} \)). Let

\[
\varphi_b \triangleq \sum_{\theta_i \in (\eta_b, \eta_{b+1}]} \theta_i \tag{11}
\]

be the total probability of letters in bin \( b \) of grid \( \eta \). Of particular importance will be \( \varphi_0, \varphi_1 \), defined with respect to (w.r.t.) bins 0, 1, respectively, and \( \varphi_{01} \triangleq \varphi_0 + \varphi_1 \). Define \( \ell_0, \ell_1 \), and \( \ell_{01} \), where \( \ell_b \triangleq \min(k_b, n) \).

The probability that letter \( i \) does not occur in \( X^n \) is

\[
P_{\theta} (i \notin X^n) = (1 - \theta_i)^n. \tag{12}
\]

Taking an exponent of the logarithm of \((12)\), using Taylor series expansion in the exponent,

\[
e^{-n(\theta_i + \theta_i^2)} \leq P_\theta (i \notin X^n) \leq e^{-n\theta_i}, \quad \text{if } \theta_i \leq 3/5. \tag{13}
\]

If \( \theta_i > 3/5 \), the upper bound is the same, but the lower bound is 0. Following \((13)\),

\[
1 - e^{-n\theta_i} \leq P_\theta (i \in X^n) \leq 1 - e^{-n(\theta_i + \theta_i^2)}, \tag{14}
\]

where the upper bound is replaced by 1 for \( \theta_i > 3/5 \).

The mean number of occurrences of letter \( i \) in \( X^n \) is given by \( E_\theta N_x (i) = n\theta_i \), where \( n_x (i) \) is the number of occurrences of \( i \) in \( x^n \), \( N_x (i) \) is its random variable, and \( E_\theta \) is expectation given \( \theta \). Then, the mean number of re-occurrences (beyond the first occurrence) of letter \( i \) in \( X^n \) is given by

\[
E_\theta N_x (i) - P_\theta (i \in X^n) = n\theta_i - 1 + (1 - \theta_i)^n. \tag{15}
\]
It is thus bounded by

\[ n\theta_i - 1 + e^{-n(\theta_i + \theta_i^2)} \leq E_{\theta} N_x (i) - P_\theta (i \in X^n) \leq n\theta_i - 1 + e^{-n\theta_i}, \] (16)

where, again, the last term of the lower bound is replaced by 0 for \( \theta_i > 3/5 \). Using the Binomial expansion on (12), the probability of an occurrence of letter \( i \) for \( \theta_i \leq 1/n \) can be bounded by

\[ n\theta_i - \left( \frac{n}{2} \right) \theta_i^2 \leq P_\theta (i \in X^n) \leq n\theta_i - \left( \frac{n}{2} \right) \theta_i^2 + \left( \frac{n}{3} \right) \theta_i^3, \] (17)

and then the mean number of re-occurrences of letter \( i \) is bounded by

\[ \left( \frac{n}{2} \right) \theta_i^2 - \left( \frac{n}{3} \right) \theta_i^3 \leq E_{\theta} N_x (i) - P_\theta (i \in X^n) \leq \left( \frac{n}{2} \right) \theta_i^2. \] (18)

Let \( K_b; C_b \) denote random variables counting the distinct symbols from bin \( b \) of \( \eta; \tau \), respectively, that occur in \( X^n \). Let \( K \) denote the total number of distinct letters occurring in \( X^n \). The mean number of distinct letters from bin \( b \) of \( \eta \) that occur in \( X^n \) is

\[ L_b \overset{\triangle}{=} E_{\theta} [K_b] = \sum_{\theta_i \in (\eta_b, \eta_{b+1})} [1 - (1 - \theta_i)^n] \] (19)

where \( L_0, L_1, L_{01} \) are of specific interest, and also \( L \overset{\triangle}{=} E_{\theta} [K] \) is computed in a similar manner. As in (14),

\[ k_b - \sum_{\theta_i \in (\eta_b, \eta_{b+1})} e^{-n\theta_i} \leq L_b \leq k_b - \sum_{\theta_i \in (\eta_b, \eta_{b+1}), \theta_i \leq 3/5} e^{-n(\theta_i + \theta_i^2)}. \] (20)

In particular, for bin 0, as in (17),

\[ n\varphi_0 - \left( \frac{n}{2} \right) \sum_{i=1}^{k_0} \theta_i^2 \leq L_0 \leq n\varphi_0 - \left( \frac{n}{2} \right) \sum_{i=1}^{k_0} \theta_i^2 \overset{k_0}{\sum_{i=1}^{k_0}} \theta_i^3. \] (21)

Packing lower bin(s) into single point masses, we can thus define,

\[ H_{\theta}^{(0)} (X) \overset{\triangle}{=} -\varphi_0 \log \varphi_0 - \sum_{i=k_0+1}^{k} \theta_i \log \theta_i, \] (22)

\[ H_{\theta}^{(01)} (X) \overset{\triangle}{=} -\varphi_{01} \log \varphi_{01} - \sum_{i=k_{01}+1}^{k} \theta_i \log \theta_i, \] (23)

\[ H_{\theta}^{(0,1)} (X) \overset{\triangle}{=} -\sum_{b=0}^{1} \varphi_{b} \log \varphi_{b} - \sum_{i=k_{01}+1}^{k} \theta_i \log \theta_i. \] (24)

The expressions in (22)-(24) will be used to express some of the bounds in the paper, where low probability letters are packed into one or two point masses. (These expressions also depend on the choice of \( \varepsilon \). This dependence is omitted for convenience.)
3 The Main Results

The main results in the paper are summarized below. First, if $\theta_i > 1/n^{1-\varepsilon}$, $\forall i$, the pattern entropy is bounded by

$$nH_\theta (X) - \sum_{b=1}^{A_\xi} \log (\kappa_b!) - k \log 3 - o(1) \leq H_\theta (\Psi^n) \leq nH_\theta (X) - (1-\varepsilon) \sum_{b=2}^{A_\eta} \log (k_b!) + o(k).$$

(25)

Namely, the pattern entropy decreases to first order from the i.i.d. block entropy by the logarithm of the product of permutations within all the bins of the probability space. The bounds in (25) depend on the arrangement of the letters in the probability space. However, even if we only know the number of letters in the alphabet, we can still bound the range that the pattern entropy can be in. The actual point in this range does depend on the arrangement of the letters in the probability space. However, if the alphabet is large enough, the pattern entropy must decrease w.r.t. the i.i.d. one regardless of this arrangement. In all, if $\theta_i > 1/n^{1-\varepsilon}$, $\forall i$, we have

$$nH_\theta (X) - \log (k!) \leq H_\theta (\Psi^n) \leq \begin{cases} nH_\theta (X), & \text{if } k < n^{1/3+\varepsilon}; \\ nH_\theta (X) - \frac{3}{2} k \log \frac{k}{e n^{1/3+\varepsilon/2}}, & \text{if } k \geq n^{1/3+\varepsilon}. \end{cases}$$

(26)

The bound above shows that the decrease in the pattern entropy w.r.t. the i.i.d. one for large alphabets is to first order between $\log k$ bits and $\log (k^{1.5}/\sqrt{n})$ bits for each alphabet letter.

If the alphabet contains letters with low probabilities, namely, with $\theta_i \leq 1/n^{1-\varepsilon}$ ($k_{01} > 0$), the pattern entropy is upper bounded by

$$H_\theta (\Psi^n) \leq nH_\theta^{(0,1)} (X) - \sum_{b=2}^{A_\eta} (1-\varepsilon) \log (k_b!) + (n\varphi_1 - L_1) \log \left[ \min \{ k_1, n \} \right] + n\varphi_1 h_2 \left( \frac{L_1}{n\varphi_1} \right) + \left( \frac{n^2}{2} \sum_{i=1}^{k_0} \theta_i^2 \right) \log \left\{ 2e \cdot \varphi_0 \cdot \min \{ k_0, n \} \right\},$$

(27)

where $h_2 (\alpha) \triangleq -\alpha \log \alpha - (1-\alpha) \log (1-\alpha)$. The third and forth terms contribute at most $O(n\varphi_1 \log n)$, and the last term $o(n)$. The bound in (27) implies that the source appears to contain a single letter for bin 0 and another single letter for bin 1, and its entropy decreases, again, by the logarithm of the number of permutations leading to typical sequences w.r.t. all other bins. In addition, there is a limited penalty reflected in the last three terms for packing all letters in bins 1 and 0 as two point masses. This penalty is higher for bin 1, which is the boundary between two different asymptotic behaviors. For non-diminishing i.i.d. entropies $H_\theta^{(0,1)} (X)$ the penalty of packing all letters in bin 0 into one point mass is negligible.
A lower bound of a similar nature is then obtained, showing that the pattern entropy satisfies

$$H_\theta (\Psi^n) \geq nH_\theta^{(01)} (X) - \sum_{b=1}^{A_\eta} \log (\kappa_b!) - (k - \kappa_0) \log 3$$

$$+ \sum_{i=1}^{k_{01} - 1} \left[ n\theta_i - 1 + e^{-n(\theta_i - \theta_i^2)} \right] \log \frac{\varphi_{01}^i}{\theta_{k_{01}}} + (n\theta_{k_{01}} - 1) \log \frac{\varphi_{01}^i}{\theta_{k_{01}}}$$

$$+ (\log e) \sum_{i=1}^{L_{01} - 1} (L_{01} - i) \frac{\theta_i}{\varphi_{01}} - \log \left( \frac{k_{\vartheta}^- + k_{\vartheta}^+}{k_{\vartheta}^+} \right) - o(1), \quad (28)$$

where $k_{\vartheta}^-$ denotes the number of letters with $\theta_i \in (\vartheta^- / n^{1-\varepsilon}, 1/n^{1-\varepsilon}]$ and $k_{\vartheta}^+$ the number of letters with $\theta_i \in (1/n^{1-\varepsilon}, \vartheta^+/n^{1-\varepsilon}]$, and $\vartheta^-$ and $\vartheta^+$ are constants, such that $\vartheta^+ > 1 > \vartheta^- > 0$. This bound illustrates similar behavior to that in (27), where the pattern entropy behaves like that of a source for which the low probabilities in bins 0 and 1 are packed into one point mass, and a similar behavior to that in (25) is shown for greater probabilities. Packing of bins 01 results in correction terms reflecting the increase in entropy due to repetitions and first occurrences, and another correction term (the seventh term) reflecting the unclear boundary between two different asymptotic behaviors. For many sources, variations of the last two bounds are very close to each other and lead to very accurate approximations of the pattern entropy [13].

4 Bounds for Small and Large Alphabets

This section studies pattern entropy with bounded letter probabilities $\theta_i > 1/n^{1-\varepsilon}, \forall i$ (i.e., $k_{01} = 0$). Upper and lower bounds are presented.

4.1 An Upper Bound

**Theorem 1** Fix $\delta > 0$. Let $n \to \infty$ and $\varepsilon \geq (1 + \delta)(\ln \ln n)/(\ln n)$. If $\theta_i > 1/n^{1-\varepsilon}, \forall i, 1 \leq i \leq k$,

$$H_\theta (\Psi^n) \leq nH_\theta (X) - (1 - \varepsilon) \sum_{b=2}^{A_\eta} \log (k_b!) + o(k). \quad (29)$$

The bound can be tightened by substituting $\varepsilon$ in the second term by $\exp \{- [0.1n^\varepsilon - 2 \ln n]\}$. The grid $\eta$, which is used for the proof, is defined with the same $\varepsilon$. Theorem 1 shows that letters whose probabilities are in the same bin of $\eta$ can be exchanged in a typical $x^n$ generating sequences $x^n$ with $P_\theta (x^n) \approx P_\theta (x^n)$ and $\Psi (x^n) = \Psi (x^n)$. This increases $P_\theta [\Psi (x^n)]$ by a factor of the total of such possible permutations, and decreases the entropy by its logarithm. Summation in the second term of (29) is only up to $A_\eta$ because larger index bins contain at most a single symbol probability.
**Proof:** The proof separates typical $x^n$ from unlikely (untypical) $x^n$. Then, $P_\theta(x^n)$ is lower bounded by the sum $P_\theta(x^n)$ of typical $x^n$ with $\Psi(x^n) = \psi^n$. For all such $x^n$, $P_\theta(x^n)$ is almost equal. The number of such sequences results in the entropy decrease and is equal to the number of possible permutations within the bins of $\eta$.

We define a typical set. Let $\hat{\theta}$ be the Maximum Likelihood (ML) estimator of $\theta$ from $x^n$. Then,

\begin{align*}
T_x &\triangleq \left\{ x^n : \forall i, |\hat{\theta}_i - \theta_i| < \frac{\sqrt{\theta_i}}{2\sqrt{n^{1-\varepsilon}}} \right\}, \\
\bar{T}_x &\triangleq \left\{ x^n : \exists i, |\hat{\theta}_i - \theta_i| \geq \frac{\sqrt{\theta_i}}{2\sqrt{n^{1-\varepsilon}}} \right\}.
\end{align*}

**Lemma 4.1**

\[ P_\theta(\bar{T}_x) \leq \exp\{-0.1n^{\varepsilon} + (2 - \varepsilon) \ln n\} \stackrel{\Delta}{=} \varepsilon_n. \]  

The proof of Lemma 4.1 is in Appendix A. For the choice of $\varepsilon$ in Theorem 1, $\varepsilon_n \to 0$.

Now, define $S$ as the set of all permutations $\sigma$ that permute symbols only within bins of $\eta$, i.e.,

\[ S \triangleq \{ \sigma : \theta_i \in (\eta_b, \eta_{b+1}] \Rightarrow \theta(\sigma_i) \in (\eta_b, \eta_{b+1}], \forall i = 1, 2, \ldots, k \}. \]  

The definition of $S$ is independent of $x^n$, and depends only on $\theta$.

**Lemma 4.2** Let $x^n \in T_x$ and $\sigma \in S$. Then,

\[ \ln \frac{P_\theta(x^n)}{P_{\theta(\sigma)}(x^n)} \leq \frac{6k}{n^{\varepsilon/2}} = o(k). \]  

Lemma 4.2 shows that the probability of a typical $x^n$ given by a permuted parameter vector in $S$ diverges only by a negligible factor from $P_\theta(x^n)$. Its proof is in Appendix B.

Let $M_{\theta, \eta}$ be the number of permutation vectors $\sigma$ in $S$. Using (34), for $x^n \in T_x$,

\[ \log P_\theta(\Psi(x^n)) \geq \log P_\theta(x^n) + \log M_{\theta, \eta} - o(k), \]  

Hence, applying (35) and Lemma 4.1 (step (a) below),

\[ H_\theta(\Psi^n) = - \sum_{x^n \in T_x} P_\theta(x^n) \log P_\theta(\Psi(x^n)) - \sum_{x^n \in \bar{T}_x} P_\theta(x^n) \log P_\theta(\Psi(x^n)) \leq H_\theta(\bar{T}_x) - (1 - \varepsilon_n) \log M_{\theta, \eta} + o(k) \]

\[ \leq nH_\theta(X^n) - (1 - \varepsilon_n) \sum_{b=2}^{A_\eta} \log (k_b!) + o(k). \]  

(37)
The proof of Theorem 1 is concluded. □

4.2 Lower Bounds

Theorem 2 Fix $\delta > 0$. Let $n \to \infty$ and $\varepsilon \geq (1 + \delta)(\ln \ln n)/(\ln n)$. If $\theta_i > 1/n^{1-\varepsilon}$, $\forall i, 1 \leq i \leq k$,

$$H_\theta (\Psi^n) \geq nH_\theta (X) - \sum_{b=1}^{A_\xi} \log (\kappa_b!) - k \log 3 - o(1), \quad (38)$$

and also

$$H_\theta (\Psi^n) \geq nH_\theta (X) - \sum_{b=1}^{A_\xi} \log (\kappa'_b!) - o(1). \quad (39)$$

The two bounds above are very close and except one step are proved similarly. The bound of (38) does not count occurrences in a given bin more than once (except the correction term of $k \log 3$). However, there exist distributions, such as the geometric distribution (see, e.g., [13]), where components of $\theta$ sparsely populate bins, for which the bound of (39) will be tighter. The last term of $o(1)$ decays at an exponential rate of $O(\varepsilon_n n \log n)$, where $\varepsilon_n$ is defined in (32). The pattern entropy is shown to decrease by logarithm of the number of permutations within bins of $\xi$.

Proof: Define the set of typical patterns as

$$T_\psi \triangleq \{ \psi^n : \exists x^n \in T_x, \psi^n = \Psi (x^n) \} \quad (40)$$

the set of patterns, each of at least one typical sequence as defined in (30). Now, for $\psi^n \in T_\psi$, let $M_{\theta,\xi} (\psi^n) \triangleq |y^n \in T_x : \psi^n = \Psi (y^n)|$ be the number of typical sequences that have the pattern $\psi^n$, and let $\bar{M}_{\theta,\xi}$ and $\bar{M}'_{\theta,\xi}$ denote upper bounds on $M_{\theta,\xi} (\psi^n)$ for $\psi^n \in T_\psi$.

Lemma 4.3 Let $\psi^n \in T_\psi$. Then,

$$M_{\theta,\xi} (\psi^n) \leq \bar{M}_{\theta,\xi} \triangleq 3^k \cdot \prod_{b=1}^{A_\xi} \kappa_b!, \quad (41)$$

$$M_{\theta,\xi} (\psi^n) \leq \bar{M}'_{\theta,\xi} \triangleq \prod_{b=1}^{A_\xi} \kappa'_b!. \quad (42)$$

The proof of Lemma 4.3 is in Appendix C. It now follows that

$$H_\theta (\Psi^n) = H_\theta (X^n) - H_\theta (X^n|\Psi^n) \quad (a)$$

$$\geq H_\theta (X^n) - P_\theta (T_x) H_\theta (X^n|\Psi^n, T_x) - P_\theta (\bar{T}_x) H_\theta (X^n|\Psi^n, \bar{T}_x) - h_2 [P_\theta (T_x)] \quad (b)$$

$$\geq H_\theta (X^n) - \log \bar{M}_{\theta,\xi} - \varepsilon_n \log k! - o(n\varepsilon_n)$$

$$= nH_\theta (X) - \log \bar{M}_{\theta,\xi} - o(1) \quad (43)$$
where (a) follows from the chain rule, namely,

\[ H_\theta(X^n | \Psi^n) = H_\theta(X^n, T | \Psi^n) = H_\theta(X^n | \Psi^n, T) + H_\theta[T | \Psi^n] \tag{44} \]

\[ = P_\theta(T_x) H_\theta(X^n | \Psi^n, T_x) + P_\theta(\bar{T}_x) H_\theta(X^n | \Psi^n, \bar{T}_x) + H_\theta[T | \Psi^n], \]

where \( T \) is a Bernoulli random variable, taking value 1 if \( T_x \) occurs, and the last term of step (a) of (43) follows since conditioning reduces entropy. Step (b) of (43) follows from \( P_\theta(T_x) \leq 1 \), \( H_\theta(X^n | \Psi^n, T_x) \leq \log M_{\theta, \xi} \), Lemma 4.1, \( H_\theta(X^n | \Psi^n, \bar{T}_x) \leq \log k \), and from \( h_2(P_\theta(T_x)) = o(n \varepsilon_n) \).

Substituting \( \bar{M}_{\theta, \xi} \) from (41) in (43) yields the bound of (38). Similarly, using \( \bar{M}_{\theta, \xi}' \) from (42) yields (39). □

5 Bounds for Very Large Alphabets

The more general case is now considered, where there exist alphabet letters with very small probabilities that may not occur in \( x^n \). Specifically, the effect of such letters on \( H_\theta(\Psi^n) \) is considered.

5.1 Upper Bounds

General upper bounds are derived by designing a low-complexity (non-universal) sequential probability assignment method for \( \psi^n \), whose average description length serves as an upper bound on \( H_\theta(\Psi^n) \). Instead of coding \( \psi^n \) by itself, the pair \( (\psi^n, \beta^n) \) is jointly coded, where \( \beta^n \) represents the sequence of bins corresponding to letters in \( x^n \). Different grids produce different bounds. Examples and study of pattern entropy for specific distributions in [13] demonstrate that tightness depends on the specific source distribution. One bound may be tighter for one and another for another.

**Theorem 3** Fix \( \delta > 0 \). Let \( n \rightarrow \infty \) and \( \varepsilon \geq (1 + \delta)(\ln \ln n)/(\ln n) \) (also for \( \eta \) in (6)). Then,

\[
H_\theta(\Psi^n) \leq n H_\theta^{(0,1)}(X) - (1 - \varepsilon) \sum_{b=2}^{A_n} \log (k_b!) \\
+ (n \varphi_1 - L_1) \log \{ \min \{ k_1, n \} \} + n \varphi_1 h_2 \left( \frac{L_1}{n \varphi_1} \right) \\
+ \left( \frac{n^2}{2} \sum_{i=1}^{k_0} \theta_i^2 \right) \log \left\{ \frac{2e \cdot \varphi_0 \cdot \min \{ k_0, n \}}{n \sum_{i=1}^{k_0} \theta_i^2} \right\}. \tag{45}
\]

The bound in (45) consists of four major components: 1) the i.i.d. entropy in which bins 0 and 1 of \( \eta \) are each packed into a point mass (the first term), 2) the gain in first occurrences of symbols
i with \( \theta_i > 1/n^{1-\varepsilon} \) (the second term), 3) the loss in packing bin 1 (the next two terms), and 4) the loss in packing bin 0 (the last term). The sum of the third and fourth terms in (45) decreases with \( L_1 \) for \( k_1 \geq (1 + \varepsilon)n^\varepsilon \), thus \( L_1 \) can be replaced by a lower bound as in (20).

If the symbols in bins 0 and 1 formed by \( \eta \) are packed into a single point mass, a simpler upper bound that uses \( H_\theta^{(01)} \) and \( \varphi_{01} \) instead of \( H_\theta^{(0,1)} \), and both \( \varphi_0 \) and \( \varphi_1 \), respectively, can be obtained. Using \( \tau \) instead of \( \eta \) produces other bounds.

**Corollary 1** Fix \( \delta > 0 \). Let \( n \to \infty \) and \( \varepsilon \geq (1 + \delta)(\ln \ln n)/((\ln n)) \) (also for \( \eta \) in (45)). Then,

\[
H_\theta(\Psi^n) \leq nH_\theta^{(01)}(X) - (1 - \varepsilon) \sum_{b=2}^{A_\eta} \log (k_b!) + (n\varphi_{01} - L_{01}) \log \min\{k_{01}, n\} + n\varphi_{01}h_2\left(\frac{L_{01}}{n\varphi_{01}}\right). \tag{46}
\]

Let \( n \to \infty \) and \( \varepsilon \geq 0 \). Then,

\[
H_\theta(\Psi^n) \leq nH_\theta^{(0)}(X) + \left(\frac{n^2}{2} \sum_{i=1}^{k_0} \theta_i^2\right) \log \left(\frac{2e \cdot \varphi_0 \cdot \min\{k_0, n\}}{n \sum_{i=1}^{k_0} \theta_i^2}\right), \tag{47}
\]

\[
H_\theta(\Psi^n) \leq nH_\theta^{(0)}(X) - \sum_{b=1}^{A_\tau} \sum_{m=0}^{c_b} P_\theta(C_b = m) \log \left(\frac{c_b!}{(c_b - m)!}\right) + \frac{9 \log e}{n^{\varepsilon}} \sum_{b \geq 1, c_b > 1} c_b + \left(\frac{n^2}{2} \sum_{i=1}^{k_0} \theta_i^2\right) \log \left(\frac{2e \cdot \varphi_0 \cdot \min\{k_0, n\}}{n \sum_{i=1}^{k_0} \theta_i^2}\right). \tag{48}
\]

The bound in (48) is in many cases the tightest but is harder to compute. It can be simplified using Stirling’s approximation,

\[
\sqrt{2\pi m} \left(\frac{m}{e}\right)^m \leq m! \leq \sqrt{2\pi m} \left(\frac{m}{e}\right)^m \cdot e^{1/(12m)}, \tag{49}
\]

and Jensen’s inequality, at the expense of loosening it, by replacing the inner sum in its second term by \( (E_\theta[C_b]) \log \{(E_\theta[C_b]) / e\} \), where \( E_\theta[C_b] \) is the expected distinct letter count in bin \( b \) of \( \tau \). The bounds in (45) and (46) trade off two costs: (45) has a larger first term, while (46) pays a higher penalty in its last two terms. The better trade off is distribution dependent. Roughly, if letters in bins 0 and 1 of \( \eta \) are better separated, (45) is tighter, while otherwise (46) is tighter. The bound in (47) is the simplest, but ignores gains of first occurrences of letters with large probabilities. Its best use is thus for fast decaying distributions. Both (47) and (48) may be tightened in certain cases by separating low probabilities (bin 0 of \( \eta \)) into two or more regions (see, e.g., [13]).
Example 1: For a uniform distribution with \( k = k_1 = n^{1-\nu} \) parameters \( \theta_i = 1/n^{1-\nu} \), where \( 0 < \nu \leq \varepsilon \),
\[
H_\theta (\Psi^n) \leq n \log n^{1-\nu} - n^{1-\nu} \log \frac{n^{1-2\nu}}{e} - \Theta \left( n^{1-2\nu} \right)
\] (50)
with (45) and (46). Bound (47) produces only the first term. Then, with the loosened (48),
\[
H_\theta (\Psi^n) \leq n \log n^{1-\nu} - n^{1-\nu} \log \frac{n^{1-\nu}}{e} + \Theta \left( n^{1-\varepsilon-\nu} \right).
\] (51)
The last bound from (48) is the tightest.

Example 2: Let \( \theta \) consist of two sets of probabilities: \( k_0 = \varphi_0 n^{1+\mu} \); \( \mu \geq \varepsilon \), probabilities of \( 1/n^{1+\mu} \), and \( k_1 = \varphi_1 n^{1-\nu} \); \( 0 < \nu < \varepsilon \), probabilities of \( 1/n^{1-\nu} \), where \( \varphi_0 + \varphi_1 = 1 \). Then,
\[
H_\theta (\Psi^n) \leq \begin{cases} 
(1-\nu) n \varphi_1 \log n - n \varphi_0 \log \varphi_0 - \Theta \left( n^{1-\nu} \log n \right), & \text{using (45) and (48)} \\
n \varphi_1 \log n + n h_2 (\varphi_0) - O \left( n^{1-\nu} \log n \right), & \text{using (46)} \\
(1-\nu) n \varphi_1 \log n - n \varphi_0 \log \varphi_0 + \Theta \left( n^{1-\mu} \log n \right), & \text{using (47)}.
\end{cases}
\] (52)
The bound from (46) is looser because it ignores the clear separation between bins 0 and 1. The gain ignored in (47) also slightly loosens the resulting bound. The greater \( \varphi_0 \) is, the smaller \( H_\theta (\Psi^n) \) is from \( nH_\theta (X) \).

Example 3: For a given \( \varepsilon > 0 \), let \( \theta \) consist of two sets of probabilities: \( k_0 = \varphi_0 n^{1+\mu} \); \( \mu \geq \varepsilon \), probabilities of \( 1/n^{1+\mu} \), and \( k_1 = \varphi_1 n^{1-\nu} \); \( 0 < \nu < \varepsilon \), probabilities of \( 1/n^{1-\nu} \), where \( \varphi_0 + \varphi_1 = 1 \). Here, (45) results in a bound of \( n h_2 (\varphi_0) + O \left( n^{1-\nu} \log n \right) \). A much tighter bound of \( \Theta \left( n^{1-\nu} \log n \right) \) is produced by (46). This is because the two sets here are of “small” probabilities. Looser bounds of \( \Theta \left( n \log n \right) \) are produced by (47) and the loosened (48), with a smaller coefficient for the second. However, since \( \varepsilon \geq 0 \) for these two bounds, \( \varepsilon < \nu \) can be used to produce similar bounds to that of (46). Such flexibility is limited with the other bounds that have positive lower limits on \( \varepsilon \).

Example 4: Let \( \theta \) consist of two sets of probabilities: \( k_0 = \varphi_0 n^{1+\mu} \); \( \mu \geq \varepsilon \), probabilities of \( 1/n^{1+\mu} \), and \( k_1 = \varphi_1 n \) probabilities of \( 1/n \), where \( \varphi_0 + \varphi_1 = 1 \). Then,
\[
H_\theta (\Psi^n) \leq \begin{cases} 
\frac{n \varphi_1}{e} \log n + n \left[ h_2 (\varphi_0) + \varphi_1 h_2 \left( \frac{1}{e} \right) + \frac{\varphi_1}{e} \log \varphi_1 \right] + O \left( n^{1-\mu} \log n \right), & \text{with (45)} \\
\frac{n \varphi_1}{e} \log n + n h_2 \left( \frac{\varphi_1}{e} \right) + O \left( n^{1-\mu} \log n \right), & \text{with (46)} \\
n \varphi_1 \log n - n \varphi_0 \log \varphi_0 + O \left( n^{1-\mu} \log n \right), & \text{with (47)} \\
\frac{n \varphi_1}{e} \log n + n \left[ h_2 (\varphi_0) + \frac{\varphi_1}{e} \log \frac{\varphi_1 (1-\frac{1}{e})}{\varepsilon} + \frac{\varphi_1}{1-\frac{1}{e}} \log \frac{e}{1-\frac{1}{e}} \right] + O \left( n^{1-\mu} \log n \right), & \text{with (48)}.
\end{cases}
\] (53)
Again, the tightest bound is from (46), implying that all probabilities here are still “small”. The next is that of (45). Unlike the other examples, (48) and (47) lead to the loosest bounds. If (48) is used with \( \varepsilon = 0 \), an even weaker bound with first term 0.5\( \varphi_1 n \log n \) will result because of the use of the upper bound of (48) for mean re-occurrence count, which is looser than that of (46). Using (46) instead for the last term of (47)-(48) yields the bound of (46) for this case. \( \square \)

In Theorem 3, Corollary 1 and the examples above, contributions of small probabilities influence the pattern entropy. The next corollary shows the limits of these contributions.

**Corollary 2** I. The total combined contribution of all letters with \( \theta \leq 1/n^{1-\varepsilon} \) to \( H_\theta (\Psi^n) \) beyond the term \(-n\varphi_0 \log \varphi_0 \) of \( nH_\theta^{(01)}(X) \) is upper bounded by the maximum between \( O \left( n^{2\varepsilon} \log n \right) \) and

\[
n\varphi_0 \log \ell_0 + \varphi_0 n^{-\varepsilon} \log \frac{en^{\varepsilon}}{\ell_0} + \Theta \left( \varphi_0 n^{-1-\varepsilon} \right) = O \left( n\varphi_0 \log n \right),
\]

Similarly, the sum of the third and fourth term in (45) is \( O \left( \max \{ n\varphi_1, n^{2\varepsilon} \} \log n \right) \).

II. The total combined contribution of all letters with \( \theta_i \leq 1/n^{i+\varepsilon} \), for any \( i \geq 1 \), beyond the term \(-n\varphi_0 \log \varphi_0 \) of \( nH_\theta^{(0,1)}(X) \) is \( O \left( n^{2-\mu-\varepsilon} \log n \right) \). Similarly, the last term of (48) is upper bounded by

\[
\frac{\varphi_0 \cdot n^{-1-\varepsilon}}{2} \log (2en^{1+\varepsilon}) = O \left( n^{-1-\varepsilon} \varphi_0 \log n \right) = o(n).
\]

Corollary 2 is proved in Appendix E. It shows that the per-symbol (normalized by \( n \)) contribution of bin 0 of \( \eta \) beyond a single point mass is diminishing. Furthermore, any letter with \( \theta_i \leq 1/n^{2+\varepsilon} \) has diminishing contribution to the block entropy beyond that of the single point mass of bin 0.

The subsection is concluded with the proof of Theorem 3 and Corollary 1.

**Proof of Theorem 3 and Corollary 1** For some \( x^n \), let \( \psi^n = \Psi(x^n) \), and define \( \beta^n = (\beta_1, \beta_2, \ldots, \beta_n) \) by \( \beta_j = b \) if \( \theta_{x_j} \in (\eta_b, \eta_{b+1}] \). The joint sequence \( (\psi^n, \beta^n) \) is sequentially assigned probability

\[
Q \left[ (\psi^n, \beta^n) \right] \overset{\triangle}{=} \prod_{j=1}^{n} Q \left[ \psi_j, \beta_j \mid (\psi^{j-1}, \beta^{j-1}) \right],
\]

where

\[
Q \left[ \psi_j, \beta_j \mid (\psi^{j-1}, \beta^{j-1}) \right] = \begin{cases} 
\rho_{\beta_j} & \text{if } (\psi_j, \beta_j) \in (\psi^{j-1}, \beta^{j-1}) \\
\varphi_{\beta_j} - k_{\beta_j} \left( (\psi^{j-1}, \beta^{j-1}) \right) \cdot \rho_{\beta_j} & \text{if } \psi_j = \max \{ \psi_1, \ldots, \psi_{j-1} \} + 1, \\
0 & \text{otherwise},
\end{cases}
\]

where \( \rho_b \overset{\triangle}{=} \varphi_b / k_b \) for \( b \geq 2 \), \( \rho_0 \) and \( \rho_1 \) are optimized later, and \( k_{\beta_j} \left( (\psi^{j-1}, \beta^{j-1}) \right) \) is the number of distinct indices that jointly occurred with bin index \( \beta_j \) in \( (\psi^{j-1}, \beta^{j-1}) \) (e.g., if \( \psi^{j-1} = 1232345 \) and
The optimal choice of $\beta_j = 122242$ then $k_{\beta_j} \left[ \left( \psi^7, \beta^7 \right) \right]$ is 3 for $\beta_j = 2$, 1 for $\beta_j = 1$ and $\beta_j = 4$, and is 0 for any other value of $\beta_j$. Initially, every bin $b$ is assigned its total probability $\varphi_b$. Each new index occurring with a letter in bin $b$ is assigned the remaining probability in bin $b$ for its first occurrence. For any re-occurrence, it is assigned the average symbol probability in $b$; $\rho_b$, unless $b \leq 1$, where a different (smaller) value which favors first occurrences is used for $\rho_b$. After a new occurrence of a symbol in bin $b$, $\rho_b$ is subtracted from the remaining bin probability.

Since joint entropy is not smaller than the entropy of one of the components,

$$ H_\theta (\Psi^n) \leq H_\theta (\Psi^n, B^n) \leq -E \log Q (\Psi^n, B^n) $$

$$ = - \sum_{x^n \in \Sigma^n} P_\theta (x^n) \sum_{j=1}^{n} \log Q \left\{ \Psi (x_j), \beta (x_j) \mid [\Psi (x_j^{-1}), \beta (x_j^{-1})] \right\} $$

$$ \equiv -n \sum_{i=k_{01}+1}^{k} \theta_i \log \rho_b (\theta_i) - \sum_{b=2}^{B_n} \sum_{m=0}^{k_b} P_\theta (K_b = m) \log \frac{k_b!}{(k_b - m)!} $$

$$ - \frac{1}{R_b} \left\{ (n \varphi_b - E_\theta [K_b]) \log \rho_b + \sum_{m=0}^{\min \{ n, k_b \}} P_\theta (K_b = m) \sum_{l=0}^{m-1} \log (\varphi_b - l \rho_b) \right\}, \quad (56) $$

where $\rho_b (\theta_i)$ is the mean symbol probability in bin $b$, where $\theta_i \in \{ \eta_b, \eta_{b+1} \}$. Equality (a) is obtained as follows: The first term is the coding cost of “large” probability letters. The second term describes the gain of first occurrences of these letters. The first symbol occurring in a bin is assigned probability $k_b \rho_b$ at first occurrence, the second ($k_b - 1) \rho_b$, and so on. The remaining terms $R_b$ describe similar costs for bins 0 and 1. The first element for each is the re-occurrence cost. The second is the first occurrence cost. Bounds on all terms are summarized below.

**Lemma 5.1**

$$ -n \sum_{i=k_{01}+1}^{k} \theta_i \log \rho_b (\theta_i) \leq n H_\theta^{(0,1)} (X) + n \sum_{b=0}^{1} \varphi_b \log \varphi_b + \frac{9 \log e}{n^2} \cdot \sum_{b \geq 2, k_b > 1} \sum_{b=2}^{b} k_b $$

$$ - \sum_{b=2}^{B_n} \sum_{m=0}^{k_b} P_\theta (K_b = m) \log \frac{k_b!}{(k_b - m)!} \leq - \sum_{b=2}^{A\eta} [1 - k_b \exp \{-n \eta_b \}] \log (k_b!) $$

The optimal choice of $\rho_b; b = 0, 1$, is

$$ \rho_b = \frac{(n \varphi_b - L_b) \varphi_b}{n \varphi_b \ell_b} = \frac{(n \varphi_b - L_b)}{n \cdot \min \{ k_b, n \}} $$

With this choice,

$$ R_b \leq -n \varphi_b \log \varphi_b + (n \varphi_b - L_b) \log \min \{ k_b, n \} + n \varphi_b \cdot h_2 \left( \frac{L_b}{n \varphi_b} \right), \quad b = 0, 1, \quad (60) $$
which decreases with $L_b$ for $b = 0$ and also for $b = 1$ if $k_1 \geq (1 + \varepsilon) n^\varepsilon$. Specifically,

$$R_0 \leq -n\varphi_0 \log \varphi_0 + \left( \frac{n^2}{2} \sum_{i=1}^{k_0} \theta_i^2 \right) \log \frac{2e \cdot \varphi_0 \cdot \min \{k_0, n\}}{n \sum_{i=1}^{k_0} \theta_i^2}. \quad (61)$$

Lemma 5.1 is proved in Appendix D. Summing (57), (58), (60) for $b = 1$, and (61) yields (45), where the last term of (57) and the decaying terms in (58), which decay at least as fast as $k_b \exp \{-n^\varepsilon\}$ each, are absorbed by the leading $\varepsilon$ of the second term in (45). The decrease of (60) with $L_1$ implies that the lower bound on $L_1$ of (20) can be used in (45) as long as $k_1 \geq (1 + \varepsilon) n^\varepsilon$.

Corollary 1 follows from similar steps. To prove (46), bins 0 and 1 are grouped to one point mass. Then, (57) and (60) are adjusted with $H_{\theta}(01)(X)$, $k_{01}$, $L_{01}$, and $\varphi_{01}$, and summed together with (58) to produce (46). Bound (47) is obtained by packing bin 0 of $\tau$ into a point mass, but coding each “large” probability symbol as an independent bin. If, in addition, the “large” probability bins of $\tau$ are coded as in proving (45), an additional gain as the left hand side of (58) w.r.t. $\tau$ is achieved. Using $\tau$, the denominator of the last term of (57) is $n^\varepsilon$ (as can be seen in (D.1)). □

5.2 Lower Bounds

The main difficulty in deriving a general lower bound on $H_\theta(\Psi^n)$ is separating between “small” probabilities $\theta_i \leq 1/n^{1-\varepsilon}$, whose symbols $i$ may or may not occur in $X^n$, and “large” probabilities, for which the results of Theorem 2 can be used. The key idea is to use an auxiliary Bernoulli indicator random sequence $Z^n$ to aid in the separation.

**Theorem 4** Fix $\delta > 0$. Let $n \to \infty$ and $\varepsilon \geq (1 + \delta)(\ln \ln n)/(\ln n)$, define $\mathbf{1}$ with (5). Define $Z^n \triangleq (Z_1, Z_2, \ldots, Z_n)$ by $Z_j = 0$ if $\theta_{X_j} \leq 1/n^{1-\varepsilon}$, and 1 otherwise. Let $k_\theta^-$ be the count of letters $i$ such that $\theta_i \in (\vartheta^{-}/n^{1-\varepsilon}, 1/n^{1-\varepsilon})$ and $k_\theta^+$ the count of letters $i$ with $\theta_i \in (1/n^{1-\varepsilon}, \vartheta^{+}/n^{1-\varepsilon}]$, where $\vartheta^-, \vartheta^+$ are constants that satisfy $\vartheta^+ > 1 > \vartheta^- > 0$. Then,

$$H_\theta(\Psi^n) \geq nH_\theta^{(01)}(X) - S_1 + S_2 + S_3 - S_4 - o(1), \quad (62)$$

where

$$S_1 \leq \sum_{b=1}^{A_1} \log (\kappa_b!) + (k - \kappa_0) \log 3, \quad (63)$$

$$S_1 \leq \sum_{b=1}^{A_1} \log (\kappa_b!), \quad (64)$$

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\[ S_2 = \sum_{i=1}^{k_{01}} E \left[ N_x(i) - P_\theta(i \in X^n) \right] \log \frac{\varphi_{01}}{\theta_i}, \quad (65) \]

\[ S_2 \geq \sum_{i=1}^{k_{01}-1} \left[ n\theta_i - 1 + e^{-n(\theta_i + \theta_i^2)} \right] \log \frac{\varphi_{01}}{\theta_i} + (n\theta_{k_{01}} - 1) \log \frac{\varphi_{01}}{\theta_{k_{01}}}, \quad (66) \]

\[ S_2 \geq \left( 1 - \frac{1}{n^\varepsilon} \right) \frac{n^2}{2} \sum_{i=1}^{k_0} \theta_i^2 \log \frac{\varphi_{01}}{\theta_i} + \sum_{i=k_0+1}^{k_{01}-1} \left[ n\theta_i - 1 + e^{-n(\theta_i + \theta_i^2)} \right] \log \frac{\varphi_{01}}{\theta_i}, \quad (67) \]

\[ S_3 \geq \left( \log e \right) \sum_{i=1}^{L_{01}-1} (L_{01} - i) \frac{\theta_i}{\varphi_{01}}, \quad (68) \]

\[ S_4 \leq \log \left( \frac{k^-_\theta + k^+_\theta}{k^-_\theta} \right). \quad (69) \]

Theorem 4 lower bounds \( H_\theta (\Psi^n) \) in terms of \( H_\theta^{(01)} (X) \) with several correction terms, two of which are provided more than one bound. Term \( S_1 \) shows the decrease in \( H_\theta (\Psi^n) \) due to first occurrences of letters \( i \) with \( \theta_i > 1/n^{1-\varepsilon} \). Its bounds are similar to the correction term in Theorem 2. Term \( S_2 \) is the cost of re-occurrences of letters with “small” probabilities. Separation of the last term in (66) from the sum is only necessary if \( \theta_{k_{01}} > 3/5 \) (see (16) and discussion following it). Equation (67) separates the sum of (65) into bins 0 and 1, where the additional term of (66) can be added to tighten the bound. Term \( S_3 \) is the penalty in first occurrences of “small” probability symbols beyond the single point mass they are packed to. Its bound in (68) is obtained under a worst case assumption and may be tightened. Term \( S_4 \) is the correction from separation to “small” and “large” probabilities. The last term of \( -o(1) \) absorbs all the lower order terms. By proper equalities, (62) can be brought into several other forms including forms in terms of \( H_\theta^{(0)} (X) \) and \( H_\theta^{(01)} (X) \).

**Proof:** Using \( Z^n \),

\[ H_\theta (\Psi^n) = H_\theta (\Psi^n | Z^n) + H_\theta (Z^n) - H_\theta (Z^n | \Psi^n). \quad (70) \]

By definition of \( Z^n \),

\[ H_\theta (Z^n) = nh_2 (\varphi_{01}) \triangleq -\varphi_{01} n \log \varphi_{01} - (1 - \varphi_{01}) n \log (1 - \varphi_{01}). \quad (71) \]

The third term is bounded in the following lemma, which is proved in Appendix F.

**Lemma 5.2**

\[ S_4 \triangleq H_\theta (Z^n | \Psi^n) \leq \log \left( \frac{k^-_\theta + k^+_\theta}{k^-_\theta} \right) + o(1). \quad (72) \]
To bound the first term of (70), define two new pattern sequences $\dot{\psi}^n$ and $\ddot{\psi}^n$. The first is defined as $\dot{\psi}_j = \phi$ if $z_j = 1$, and for the second $\ddot{\psi}_j = \phi$ if $z_j = 0$, where $\phi$ is a do not care symbol. The other components of both $\dot{\psi}^n$ and $\ddot{\psi}^n$ are the patterns of the remaining symbols in $x^n$, respectively, i.e., $\dot{\psi}^n$ and $\ddot{\psi}^n$ are the patterns of low and high probability symbols occurring in $x^n$, respectively. In a similar manner, define $\dot{x}^n$ and $\ddot{x}^n$, where $\dot{x}_j = \phi$ if $z_j = 1$, $\dot{x}_j = x_j$, otherwise, and $\ddot{x}_j = \phi$ if $z_j = 0$, $\ddot{x}_j = x_j$, otherwise. Now,

$$H_\theta (\Psi^n \mid Z^n) = H_\theta (\dot{\psi}^n \mid Z^n) + H_\theta (\ddot{\psi}^n \mid Z^n)$$  \hspace{1em} (73)

because up to deterministic labeling of pattern indices, the uncertainty on both sides is equal.

Following the same steps in (43),

$$H_\theta (\dot{\psi}^n \mid Z^n) = H_\theta (\dot{\tilde{X}}^n \mid Z^n) - E_\theta \left\{ P_\theta (T_x \mid Z^n) H_\theta (\dot{\tilde{X}}^n \mid \Psi^n, Z^n, T_x) \mid Z^n \right\} - E_\theta \left\{ P_\theta (\ddot{T}_x \mid Z^n) H_\theta (\dot{\tilde{X}}^n \mid \Psi^n, Z^n, \ddot{T}_x) \mid Z^n \right\} - H_\theta (T \mid \Psi^n, Z^n)
\geq (1 - \varphi_{01}) nH_\theta (X \mid Z = 1) - S_1 - \varepsilon_n \log (k - \kappa_0)! - o (n\varepsilon_n)$$  \hspace{1em} (74)

where the external expectation is on $Z^n$, and $T_x$ and $T$ are as defined in Section 4. Now, $S_1$ can be upper bounded by either (63) or (64) following bounds similar to (41) and (42), respectively, $H_\theta (X \mid Z = 1)$ is the i.i.d. source entropy given only letters with $\theta_i > 1/n^{1-\varepsilon}$ are drawn, and (a) follows from $P_\theta (T_x \mid Z^n) \leq 1$, $H_\theta (\dot{\tilde{X}}^n \mid \Psi^n, Z^n, T_x) \leq S_1$, $H_\theta (\dot{\tilde{X}}^n \mid \Psi^n, Z^n, \ddot{T}_x) \leq \log (k - \kappa_0)!$, and then $E_\theta \left\{ P_\theta (T_x \mid Z^n) \mid Z^n \right\} = E_\theta \left\{ P_\theta (\ddot{T}_x \mid Z^n) \right\} = P_\theta (\ddot{T}_x) \leq \varepsilon_n$. Finally, $H_\theta (T \mid \Psi^n, Z^n) \leq h_2 [P_\theta (T_x)] = o (n\varepsilon_n)$. Summing (71) and (74),

$$(1 - \varphi_{01}) nH_\theta (X \mid Z = 1) - S_1 - o(1) + nh_2 (\varphi_{01}) = nH_\theta^{(01)} (X) - S_1 - o(1).$$  \hspace{1em} (75)

With the chain rule, and data processing,

$$H_\theta (\Psi^n \mid Z^n) = \sum_{j=1}^{n} H_\theta (\dot{\psi}_j \mid \dot{\psi}^{j-1}, Z^n) \geq \sum_{j=1}^{n} H_\theta (\dot{\psi}_j \mid X^{j-1}, Z^n)
\geq - \sum_{i=1}^{k_{01}} E_\theta \left\{ E_\theta \left[ N_x (i) - P_\theta (i \in X^n) \mid Z^n \right] \} \log P_\theta (i \mid Z = 0) + S_3
\geq \sum_{i=1}^{k_{01}} E_\theta \left[ N_x (i) - P_\theta (i \in X^n) \right] \log \frac{\varphi_{01}}{\theta_1} + S_3$$  \hspace{1em} (76)

where $S_2$ is the average cost of re-occurrence of letters $i$ with $\theta_i \leq 1/n^{1-\varepsilon}$, and $S_3$ is the average cost of first occurrences of such letters. Step (a) follows from rearranging the sum into re-occurrences.
and first occurrences, where each is expressed over all (small probability) alphabet symbols, (b) follows from \( E \{ E[U | V] \} = E[U] \), for random variables \( U \) and \( V \), and since \( -\log P_\theta (i | Z = 0) = \log (\varphi_{01}/\theta_i) \). This yields (65). Then, (66) and (67) follow from (16) and (18), respectively, where the preceding \( 1/n \) in the first sum of (67) follows from the lower bound in (18). Now, \[
S_3 \overset{(a)}{=} -E_\theta \left\{ E_\theta \left[ \sum_{i=0}^{K_01-1} \log \left( 1 - \sum_{j=1}^i \frac{\theta_i}{\varphi_{01}} \right) | Z^n \right] \right\} \overset{(b)}{=} \left( \log e \right) E_\theta \left\{ E_\theta \left[ \sum_{i=0}^{K_01-1} \sum_{j=1}^i \frac{\theta_i}{\varphi_{01}} | Z^n \right] \right\}
\] \[
\overset{(c)}{=} \left( \log e \right) \cdot \sum_{i=0}^{L_{01}-1} \sum_{j=1}^{i} \frac{\theta_i}{\varphi_{01}} \overset{(d)}{=} \left( \log e \right) \cdot \sum_{i=0}^{L_{01}-1} \left( L_{01} - i \right) \frac{\theta_i}{\varphi_{01}}.
\] (77)

where (a) follows because each new occurrence of an index is allocated the remaining total probability, where in the worst case, letters occur in ascending order of probabilities, (b) follows from \(-\log(1 - x) \geq x \log e\), (c) follows from Jensen’s inequality, where the function is convex in \( K_{01} \) because each increase in \( K_{01} \) results in no smaller increase of the expression than the previous increase in \( K_{01} \). Then, \( E_\theta K_{01} = L_{01} \) is used. Finally, (d) follows rearrangement of the double sum. The proof is concluded by combining (75), (76), and (72) to obtain all components of (70), where the bounds on all terms are provided in (63)-(69).

\[\square\]

6 Entropy Range

Bounds presented so far depend on the arrangement of probability parameters in the probability space. However, can we say more than (1) about the pattern entropy without knowledge of this arrangement? The answer is yes for large enough \( k \) and sufficiently large letter probabilities. There are \( O \left( \sqrt{n^{1+\varepsilon}} \right) \) bins in \( \tau \). Due to the constraint \( \sum \theta_i = 1 \), very few of the larger parameter bins are populated, essentially leading to \( O \left( n^{1/3+\varepsilon} \right) \) populated bins. If \( k \) is greater, this forces more than a single letter probability to populate a single bin, thus decreasing the pattern entropy. The range of values the entropy can take is bounded below.

**Theorem 5** Fix \( \delta > 0 \). Let \( n \to \infty \), and \( \varepsilon, \varepsilon_1 \geq (1 + \delta)(\ln \ln k)/(\ln n) \). Let \( \theta_i > 1/n^{1-\varepsilon_1} \), \( \forall i, 1 \leq i \leq k \), and let \( k \geq n^{1/3+\varepsilon} \). Then,

\[
nH_\theta (X) - \log (k!) \leq H_\theta (\Psi^n) \leq nH_\theta (X) - \frac{3}{2} k \log \frac{k}{en^{1/3+\varepsilon/2}}.
\] (78)

The bounds of Theorem 5 give a range within which the pattern entropy must be. For alphabets with \( k \geq n^{1/3+\varepsilon} \), the entropy must decrease essentially by at least \( 1.5 \log (k/n^{1/3}) \) bits per alphabet.
Figure 1: Region of decrease from i.i.d. to pattern entropy vs. \( k \) for \( n = 10^6 \) bits, \( \varepsilon = 0.2, n^{\varepsilon_1} = 20 \) (left), and for \( n = 10^{50} \) bits, \( \varepsilon = 0.1, n^{\varepsilon_1} = 1000 \) (right). The solid white curve on the left describes the asymptotic decrease expression in (78). On the right, it overlaps the non-asymptotic upper bound from (86).

symbol. All low order terms can be absorbed in the denominator \( \varepsilon/2 \) exponent. Alternatively, a term of \( O(k \log \log n) \) can be included, and the exponent is reduced to \( \varepsilon/3 \). Asymptotically, \( \varepsilon_1 \) and \( \varepsilon \) can be equal. However, for practical \( n \), different values may be required to guarantee occurrence of all letters, and that low order terms do not overwhelm the decrease in entropy.

Figure 1 demonstrates the region of decrease in the pattern entropy w.r.t. the i.i.d. one. The upper region bound shown is the non-asymptotic one given in (86) when proving (78). Smaller order terms influence the region for practical \( n \). The tightness of (78) depends on the particular source. For uniform sources, the lower bound gives the true behavior. For sources with monotonic parameters, the upper bound gives a more accurate behavior, as demonstrated in the following example.

**Example 5:** Let \( k = d\beta \geq n^{(1+\varepsilon)/3} \), where for \( b = 1, 2, \ldots, \beta \) there are \( d \) letters with probability \( \xi_b \). Hence, \( \sum_i \theta_i = \sum_{b=1}^\beta db^2/n^{1-\varepsilon} = 1 \). For \( n^{1-\varepsilon} \gg k \), since \( d\beta = k, \beta \geq \sqrt{3n^{1-\varepsilon}/k(1-o(1))} \).

This leads to

\[
\sum_{b=1}^{A_\xi} \log (\kappa_b!) = \log \left( \left( \frac{k}{\beta} \right)! \right)^{\beta} \leq (1+o(1)) k \log \frac{k}{\beta e} = (1+o(1)) \frac{3}{2} k \log \frac{k}{e^{2/3} n^{(1-\varepsilon)/3} 3^{1/3}}. \tag{79}
\]

Substituting (79) in (88) (with the third term of (88) omitted because the letters in adjacent bins are sufficiently spaced), the resulting lower bound asymptotically achieves the upper limit in (78).
Proof of Theorem 5: The upper bound is proved by deriving an upper bound on the second term of (48) in Corollary 1, which is determined by a lower bound on \( M_{\theta,\tau} \), following a similar bound w.r.t. \( \tau \) to that in (58). Since \( \theta_i > 1/n^{1-\varepsilon} > 1/n^{1+\varepsilon} \), only the first three terms of (48) exist. The first equals \( nH_\theta(X) \) because \( k_0 = 0 \). Now, for an arbitrary \( \beta < B_\tau \), the set of bins formed by \( \tau \) is partitioned into two parts: all bins up to \( \beta \) and all others. The maximal possible number of components of \( \theta \) is allocated to the second group, and the remaining components are distributed in the first group so that \( M_{\theta,\tau} \) is minimized. Then, since this holds for every \( \beta, \beta \) that maximizes the lower bound on \( M_{\theta,\tau} \) is chosen. For convenience, denote \( A \triangleq n^{1+\varepsilon} \). For \( \theta_i \in (\tau_\beta, \tau_{\beta+1}], \theta_i > \tau_\beta \). From (5) and since \( \sum_{i} \theta_i = 1 \), it follows that \( \sum_{b=\beta}^{\infty} c_b = k - \sum_{b=1}^{\beta-1} c_b < A/\beta^2 \). Hence,

\[
\sum_{b=1}^{\beta-1} c_b > k - \frac{A}{\beta^2}.
\]

(80)

An infimum on \( M_{\theta,\tau} \) is obtained by uniformly distributing \( k - A/\beta^2 \) symbol probabilities in \( \beta \) bins, where the remaining symbol probabilities are uniformly placed in all bins of \( \tau \) (this is a lower bound because it may violate \( \sum_{i} \theta_i = 1 \)). Following an equation similar to (36) w.r.t. \( \tau \),

\[
M_{\theta,\tau} \geq \left( \frac{k - A/\beta^2}{\beta} \right)^\beta
\]

(81)

for every \( \beta \). Applying Stirling’s approximation (19),

\[
\ln M_{\theta,\tau} \geq \left( k - \frac{A}{\beta^2} \right) \ln k - A/\beta^2 + \frac{\beta}{2} \ln 2\pi \left( k - A/\beta^2 \right).
\]

(82)

By differentiation, (82) is shown to be maximized by \( \beta = \sqrt{\gamma A/k} \), where \( \gamma \geq 2 \) satisfies

\[
\gamma = \ln \frac{(\gamma - 1)^2}{\gamma^3} + \ln \frac{e k^3}{A}.
\]

(83)

For \( k \geq n^{1/3+\varepsilon} \), this implies that \( \gamma \) must increase at \( O(\ln n) \). Thus, to first order,

\[
\beta_{\text{opt}} = \sqrt{\frac{\alpha A}{k \ln \frac{k^3}{A}}},
\]

(84)

where \( \alpha \) is a constant, asymptotically optimized slightly below \( \alpha = 1 \). (The exact value of \( \alpha \) only affects second order terms.) Plugging (84) with \( \alpha = 1 \) in (82),

\[
\ln M_{\theta,\tau} \geq \frac{3}{2} k \ln \frac{k}{e n^{1/3+\varepsilon/3}} - \frac{k}{2} \left( 1 - \frac{1}{\ln \frac{k^3}{n^{1+\varepsilon}}} \right) \ln \frac{k^3}{n^{1+\varepsilon}},
\]

(85)

as long as \( k \geq n^{1/3+\varepsilon} \). Plugging (85) in the second term of (48), using the upper bound of (13) on the probability of no occurrence of any letter,

\[
H_\theta (\Psi^n) \leq nH_\theta(X) - (1 - ke^{-n^0}) \left[ \frac{3}{2} k \log \frac{k}{e n^{1/3+\varepsilon/3}} - \frac{k}{2} \left( 1 - \frac{1}{\ln \frac{k^3}{n^{1+\varepsilon}}} \right) \log \frac{k^3}{n^{1+\varepsilon}} \right] + \frac{9k \log e}{n^\varepsilon}.
\]

(86)
With the valid choices of $\varepsilon$ and $\varepsilon_1$, all lower order terms can be absorbed in a term of $0.25\varepsilon k \log n$ for some sufficiently large $n$, and the upper bound of (78) follows. \hfill \square

### 7 Summary and Conclusions

The entropy of patterns of i.i.d. sequences was studied. Tight upper and lower bounds as function of an i.i.d. source entropy, the alphabet size, the letter probabilities, and their arrangement in the probability space were derived first for distributions with bounded probabilities, and then for the general case. The bounds demonstrated the range of values the pattern entropy can take, and showed that in many cases it must decrease substantially from the original i.i.d. sequence entropy. It was shown that low probability symbols contribute mostly as a single point mass to the pattern entropy. However, an additional correction term is necessary. Very low probability symbols contribute negligibly over the contribution of a single point mass. The bounds obtained can be used to provide very accurate approximations of the pattern block entropies for various distributions as shown in a followup paper [13].

### Appendix A – Proof of Lemma 4.1

The set $\bar{T}_x = \bigcup_i F_i$, where

$$F_i = \left\{ x^n : \left| \hat{\theta}_i - \theta_i \right| \geq \frac{\sqrt{\theta_i}}{2 \sqrt{n^{1-\varepsilon}}} \right\}.$$  \hfill (A.1)

Using large deviations analysis of typical sets [2], [3],

$$P_{\theta_i}(F_i) \leq n \cdot 2^{-n \min_{\theta_i} D(\hat{\theta}_i||\theta_i) \leq n \cdot 2^{-n \min[D(\theta_i+d_i||\theta_i), D(\theta_i-d_i||\theta_i)]} \leq \hfill (A.2)\leq 2^{-n \min[D(\theta_i+d_i||\theta_i), D(\theta_i-d_i||\theta_i)]}$$

where $d_i \triangleq \sqrt{\theta_i/(2 \sqrt{n^{1-\varepsilon})}}$ and $D(\hat{\theta}_i||\theta_i)$ is the divergence (relative entropy) between the two Bernoulli distributions given by $\hat{\theta}_i$ and $\theta_i$, respectively. The coefficient $n$ is a bound on the number of types. Using Taylor series expansions, for $n^{-(1-\varepsilon)} < \theta_i \leq 0.5$,

$$D(\theta_i \pm d_i||\theta_i) = (\theta_i \pm d_i) \log \left( \frac{1 + d_i}{\theta_i} \right) + (1 - \theta_i \mp d_i) \log \left( 1 \mp \frac{d_i}{1 - \theta_i} \right) \geq \log e \cdot \left\{ \frac{d_i^2}{2 \theta_i} \left( 1 \mp \frac{d_i}{3 \theta_i} \right) + \frac{d_i^2}{2 (1 - \theta_i)} \left( 1 \mp \frac{d_i}{3 (1 - \theta_i)} \right) \right\} \geq \frac{5 \log e}{48 n^{1-\varepsilon}} > \frac{\log e}{10 n^{1-\varepsilon}} \hfill (A.3)$$
where $\pm$ and $\mp$ are used respectively to compactly describe both cases. Step (a) is obtained by combining the first three terms of the expansions for each of the two logarithmic expressions. The first terms from both expansions cancel each other. Under the assumptions bounding $\theta_i$, the remaining terms are all nonnegative, yielding a lower bound. Plugging in the value of $d_i$, bounding $\theta_i$, the second term is now nonnegative negligible. For the worst case, $1 - d_i/(3\theta_i) \geq 5/6$, leading to (b). Using the relation between divergence and $L_1$ distance (see, e.g., \cite{2}), for $\theta_i > 0.5$,

$$D(\theta_i \pm d_i||\theta_i) \geq \frac{1}{2\ln 2} \left\| (\theta_i \pm d_i) - \theta_i \right\|^2 = \frac{\log e}{4n^{1-\varepsilon}} > \frac{\log e}{10n^{1-\varepsilon}}$$  \hspace{1cm} (A.4)

where $\left\| (\theta_i \pm d_i) - \theta_i \right\|$ is the $L_1$ distance between the Bernoulli distributions defined by $\theta_i \pm d_i$ and $\theta_i$, respectively. Applying the union bound on the bounds in (A.3) and (A.4) plugged in (A.2),

$$P_\theta (\hat{T}_x) \leq k \cdot n \cdot 2^{-(\log e)n^\varepsilon} \leq \exp \left\{ -0.1n^\varepsilon + (2 - \varepsilon) \ln n \right\},$$  \hspace{1cm} (A.5)

where the second inequality follows from the limit on $\theta_i$ implying $k \leq n^{1-\varepsilon}$. The bound is meaningful for $\varepsilon > (\ln \ln n + \ln 20)/\ln n$ and diminishes for $\varepsilon \geq (1 + \delta)(\ln \ln n)/\ln n$. 

\square

Appendix B – Proof of Lemma 4.2

For a source $\theta$, a permutation vector $\sigma$, and a sequence $x^n$, define

$$\delta_i \triangleq \theta_i - \theta(\sigma_i) \hspace{1cm} (B.1)$$

$$\hat{\delta}_i \triangleq \hat{\theta}_i - \theta(\sigma_i). \hspace{1cm} (B.2)$$

Then, by the conditions of the lemma, the definition of $S$ in (33), and by (9),

$$|\delta_i| \leq \frac{3\sqrt{\theta(\sigma_i)}}{\sqrt{n^{1+2\varepsilon}}}. \hspace{1cm} (B.3)$$

By the triangle inequality,

$$|\hat{\delta}_i| = |\hat{\theta}_i - \theta(\sigma_i)| \leq |\hat{\theta}_i - \theta_i| + |\theta_i - \theta(\sigma_i)|$$

\hspace{1cm} (a) \hspace{1cm} \leq \frac{\sqrt{\theta(\sigma_i)}}{2\sqrt{n^{1-\varepsilon}}} + \frac{3\sqrt{\theta(\sigma_i)}}{\sqrt{n^{1+2\varepsilon}}} = \frac{\sqrt{\theta(\sigma_i)}}{\sqrt{n^{1-\varepsilon}}} + \frac{3\sqrt{\theta(\sigma_i)}}{\sqrt{n^{1+2\varepsilon}}}

\hspace{1cm} (b) \hspace{1cm} \leq \frac{\sqrt{\theta(\sigma_i)}}{2\sqrt{n^{1-\varepsilon}}} + \frac{\sqrt{\delta_i}}{2\sqrt{n^{1-\varepsilon}}} + \frac{3\sqrt{\theta(\sigma_i)}}{\sqrt{n^{1+2\varepsilon}}}

\hspace{1cm} (c) \hspace{1cm} \leq \frac{\sqrt{\theta(\sigma_i)}}{\sqrt{n^{1-\varepsilon}}} + \frac{\sqrt{3\theta(\sigma_i)1/4}}{2n^{3/4}} \hspace{1cm} (d) \hspace{1cm} \leq \frac{2\sqrt{\theta(\sigma_i)}}{\sqrt{n^{1-\varepsilon}}}, \hspace{1cm} (B.4)$$

(a) is obtained from (30) and (B.3), (b) is since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$, (c) is from combining the first and last term and from (B.3), and (d) results from $\theta_i > 1/n^{1-\varepsilon} \Rightarrow 1/n^{1/4} < \theta(\sigma_i)^{1/4}$. 

$p$
Applying (B.3)-(B.4), for \( x^n \in \mathcal{T}_x \) and \( \sigma \in \mathcal{S} \),

\[
\ln \frac{P_\theta (x^n)}{P_{\theta(\sigma)} (x^n)} = \sum_{i=1}^{k} n \hat{\theta}_i \ln \frac{\theta_i}{\theta(\sigma_i)} = n \sum_{i=1}^{k} \hat{\theta}_i \ln \left( 1 + \frac{\delta_i}{\theta(\sigma_i)} \right) \leq n \sum_{i=1}^{k} \frac{\hat{\theta}_i \delta_i}{\theta(\sigma_i)} = \frac{6k}{n^{\varepsilon/2}} = o(k)
\]  

(B.5)

where (a) follows from \( \ln(1+x) \leq x \), (b) follows from \( \hat{\theta}_i = \theta(\sigma_i) + \delta_i \). (c) is because all displacements sum to 0, and (d) follows by applying (B.3)-(B.4).

\[\square\]

Appendix C – Proof of Lemma 4.3

Let \( \sigma = \{\sigma_j\}_{j=1}^{k} \) be a permutation vector. Then, \( x^n \) is permuted by \( \sigma \) to \( w^n = \sigma(x^n) \triangleq (\sigma_{x_1}, \sigma_{x_2}, \ldots, \sigma_{x_n}) \). For example, let \( x^n = 3331122222222 \) and \( \sigma = (3,1,2) \), then, \( w^n = 222331111111 \), i.e., if \( \sigma_j = i \), letter \( j \) in \( x^n \) is replaced by \( i \) in \( w^n \). In the example, \( \sigma_2 = 1 \), and \( j = 2 \) is replaced by \( i = 1 \). We show that if \( x^n, w^n \in \mathcal{T}_x \), then letter \( i \) can replace only letters \( j \) whose probability parameters are in the same bin as \( \theta_i \) or in the two surrounding bins of \( \xi \). Then, the total number of such permutation vectors is upper bounded.

Lemma C.1 Let \( \psi^n \in \mathcal{T}_\psi \), and \( x^n \in \mathcal{T}_x \) such that \( \psi^n = \Psi(x^n) \). Let \( w^n = \sigma(x^n) \) such that \( w^n \in \mathcal{T}_x \). For \( i; 1 \leq i \leq k \), let \( \theta_i \in (\xi_b, \xi_{b+1}] \) and let \( j \) be such that \( \sigma_j = i \). Then, \( \theta_j \in (\xi_{b-1}, \xi_{b+2}] \).

Proof: The proof is by contradiction, \( \theta_j \notin (\xi_{b-1}, \xi_{b+2}] \) contradicts \( x^n, w^n \in \mathcal{T}_x \). By \( x^n, w^n \in \mathcal{T}_x \),

\[
\left| \hat{\theta}_j (x^n) - \theta_j \right| < \frac{\sqrt{\theta_j}}{2\sqrt{n^{1-\varepsilon}}} \tag{C.1} \\
\left| \hat{\theta}_i (w^n) - \theta_i \right| < \frac{\sqrt{\theta_i}}{2\sqrt{n^{1-\varepsilon}}} \tag{C.2}
\]

where \( \hat{\theta}_j (x^n) \) and \( \hat{\theta}_i (w^n) \) are the ML estimates of \( \theta_j \) and \( \theta_i \) from \( x^n \) and \( w^n \), respectively. By definition of \( j \), \( \hat{\theta}_i (w^n) = \hat{\theta}_j (x^n) \). By the triangle inequality, (C.1), and (C.2),

\[
|\theta_i - \theta_j| \leq \left| \theta_i - \hat{\theta}_j (x^n) \right| + \left| \hat{\theta}_j (x^n) - \theta_j \right| = \left| \theta_i - \hat{\theta}_i (w^n) \right| + \left| \hat{\theta}_j (x^n) - \theta_j \right| < \frac{\sqrt{\theta_i}}{2\sqrt{n^{1-\varepsilon}}} + \frac{\sqrt{\theta_j}}{2\sqrt{n^{1-\varepsilon}}} = \frac{\sqrt{\theta_i} + \sqrt{\theta_j}}{2\sqrt{n^{1-\varepsilon}}} \tag{C.3}
\]

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If \( \theta_j \not\in (\xi_{b-1}, \xi_{b+2}] \), it must satisfy \( \theta_j \in (\xi_\beta, \xi_{\beta+1}] \) for some \( \beta \geq b + 2 \) or \( \beta \leq b - 2 \). In the first case,

\[
\theta_j - \theta_i \geq \xi_\beta - \xi_{\beta-1} = \frac{2(\beta + 1 - 1.5)}{n^{1-\varepsilon}} \leq \frac{2\sqrt{\xi_{\beta+1}}}{\sqrt{n^{1-\varepsilon}}}, \tag{C.4}
\]

(a) follows from (10) with \( b = \beta - 1 \), (b) from (8) with \( b = \beta + 1 \), (c) is because \( \beta + 1 \geq b + 3 \geq 4 \) and thus \( \sqrt{\xi_{\beta+1}} \geq 4/\sqrt{n^{1-\varepsilon}} \), (d) is by the assumed range of \( \theta_j \), and (e) is again by the assumption that \( \theta_j > \theta_i \). For \( \theta_i > \theta_j \), where \( \beta \leq b - 2 \), in a similar manner,

\[
\theta_i - \theta_j \geq \xi_b - \xi_{b-1} = \frac{2(\beta + 1 - 1.5)}{n^{1-\varepsilon}} \geq \frac{\sqrt{\theta_j} + \sqrt{\theta_i}}{2\sqrt{n^{1-\varepsilon}}}. \tag{C.5}
\]

The last inequality is obtained as (C.4) by exchanging the roles of \( b \) and \( \beta \). Equations (C.4) and (C.5) contradict (C.3). Hence, \( \theta_j \in (\xi_{b-1}, \xi_{b+2}] \). \( \square \)

Equation (12) follows directly from Lemma C.1 since every letter \( i \) can only permute into \((\xi_{b-1}, \xi_{b+2}] \). To prove (11), a permutation replacing letter \( j \) with \( \theta_j \in (\xi_{b-1}, \xi_{b+2}] \) by letter \( i \) with \( \theta_i \in (\xi_b, \xi_{b+1}] \) can be done in the following steps: For each two adjacent bins select how many and which letters are exchanged between the two bins and exchange the occurrences of these letters. Then, permute only within the letters in a bin for all bins. Then,

\[
M_{\theta, \xi} (\psi^n) \leq \prod_{b=1}^{\beta} \left\{ \min_{u_b=0}^{\kappa_b} \left( \frac{k_b - u_b - 1}{u_b} \right) \right\} \cdot \prod_{\beta=1}^{A_\xi} \kappa_\beta!.
\]

Inequality (a) follows from the definition of the process above. Some permutations within adjacent bins lead to untypical sequences, yielding an inequality. There are up to \( \min \{\kappa_1, \kappa_2\} \) choices of exchanging letters between bins 1 and 2. (By definition \( u_0 = v_0 \triangleq 0 \).) For bin \( b \), there are up to at most the number of letters in the bin not exchanged with bin \( b - 1 \) to exchange with bin \( b + 1 \).
The last product represents permutations within bins after exchanges. Inequality (b) follows from $\sum a_i b_i \leq \sum a_i \sum b_i$ for $a_i, b_i \geq 0$, and from increasing the limit of one of the sums, (c) is a binomial series equality, (d) results from reorganization of terms such that the new general term originates from the second term at index $b-1$ and the first term with index $b$. Binomial series relations (since $3^{\kappa_b} = (2 + 1)^{\kappa_b}$) lead to (e), and upper bounding the sum of all $\kappa_b$ by $k$ leads to (f).

□

Appendix D  —  Proof of Lemma 5.1

First, define $\delta_i \triangleq \theta_i - \rho_b (\theta_i)$. By definition of the $\rho_b (\theta_i)$, $\theta_i$ and $\rho_b (\theta_i)$ must be in the same bin. Hence, by (9), $|\delta_i| \leq 3\sqrt{\rho_b (\theta_i)/n^{1+2\varepsilon}}$. Then,

\[-n \sum_{i=k_01+1}^{k_0} \theta_i \log \rho_b (\theta_i) = -n \sum_{i=k_01+1}^{k_0} \theta_i \log \theta_i - n \sum_{i=k_01+1}^{k_0} \theta_i \log \frac{\rho_b (\theta_i)}{\theta_i} \]

\[= nH_b^{(0,1)} (X) + n \sum_{b=0}^{1} \varphi_b \log \varphi_b + n \sum_{i=k_01+1}^{k_0} (\rho_b (\theta_i) + \delta_i) \log \left(1 + \frac{\delta_i}{\rho_b (\theta_i)}\right) \]

\[\leq nH_b^{(0,1)} (X) + n \sum_{b=0}^{1} \varphi_b \log \varphi_b + n \sum_{i=k_01+1}^{k_0} (\rho_b (\theta_i) + \delta_i) \frac{\delta_i}{\rho_b (\theta_i)} \log e \]

\[\leq nH_b^{(0,1)} (X) + n \sum_{b=0}^{1} \varphi_b \log \varphi_b + \frac{9 \log e}{n^{2\varepsilon}} \cdot \sum_{b \geq 2, k_b > 1} k_b \]  \hspace{1cm} (D.1)

where (a) follows from $\ln(1+x) \leq x$, and (b) is because the total divergence from the average in any bin is 0, the bound in (2), and since $\delta_i \neq 0$ only when $k_b > 1$. Equation (57) is proved. Following the upper bound in (13) and the union bound for each bin of $\eta$, and since $k_b \leq 1$ for $b > A_\eta$, (58) is obtained.

Recall that $\ell_b \triangleq \min \{k_b, n\}$. Then, assuming that the maximum of $\ell_b$ symbols in bin $b$ occurred prior to any new occurrence, thus reducing the allocation to any new symbol by $\ell_b \rho_b$,

\[R_b \leq - (n \varphi_b - L_b) \log \rho_b - L_b \log (\varphi_b - \ell_b \rho_b) ; \quad b = 0, 1. \]  \hspace{1cm} (D.2)

While the bound is loose, it serves its purpose well because low probability letters are unlikely to reoccur. The minimum for (D.2) is attained with $\rho_b$ in (59). It is a valid choice of $\rho_b$ because it leaves positive first occurrence probability after $\ell_b - 1$ first occurrences. Plugging (59) in (D.2) yields (60). The following lemma is now required.

Lemma D.1  The bound in (61) is decreasing in $L_b$ for $b = 0$ and also for $b = 1$ if $k_1 \geq (1 + \varepsilon) n^\varepsilon$. 
As a result of Lemma [D.1] an upper bound on \( R_0 \) can be derived from (60) by lower bounding \( L_0 \) using (21). Substituting (21), using Taylor series expansion of \( \log(1 - x) \) leads to (61).

**Proof of Lemma [D.1]**: The derivative of the expression in (60) w.r.t. \( L_b \) is log \((n \varphi_b - L_b) / (L_b \ell_b)\). It is thus negative and the function is decreasing if \( n \varphi_b - L_b < L_b \ell_b \). If \( k_b \geq n \) (for either \( b = 0 \) or \( b = 1 \)), this means that \( n \varphi_b - L_b < L_b n \), which is satisfied if \( L_b > \varphi_b \). Hence, we need to show that \( L_b - \varphi_b > 0 \). Using the lower bound on \( L_b \) from (21) and the definition of \( \varphi_b \),

\[
L_b - \varphi_b \geq k_b - \sum_{\theta_i \in (n_b, n_{b+1}]} e^{-n \theta_i} + \theta_i = \sum_{\theta_i \in (n_b, n_{b+1}]} \left[ 1 - e^{-n \theta_i} - \theta_i \right].
\]  
(D.3)

The function \( 1 - e^{-nx} - x \) is 0 for \( x = 0 \). It increases until \( x = \ln n \) and then starts decreasing. However, at the end of the bin 1 region, \( x = 1/n^{1-\varepsilon} \), it still attains a positive value which goes to 1. Hence, since all elements of the sum in (D.3) are positive, it must be greater than 0.

If \( n > k_b \) for \( b = 0 \), i.e., \( \theta_i \leq 1/n^{1+\varepsilon} \), we need to prove that \((k_0 + 1) L_0 - n \varphi_0 > 0 \). Using the lower bound in (21) on \( L_0 \),

\[
(k_0 + 1) L_0 - n \varphi_0 \geq k_0 n \varphi_0 - (k_0 + 1) \left( \frac{n}{2} \right) \sum_{i=1}^{k_0} \theta_i^2 \geq (1 - \varepsilon) k_0 n \varphi_0 > 0,
\]  
(D.4)

where the middle inequality is since \( \left( \frac{n}{2} \right) \sum \theta_i^2 = o(n \varphi_0) \). This can be shown as follows: Let \( \theta_i \triangleq \alpha_i/n^{1+\varepsilon} \) for a probability in bin 0, where \( \alpha_i \leq 1 \). Then, \( \sum_{i=1}^{k_0} \alpha_i = \varphi_0 n^{1+\varepsilon} \). Now,

\[
\left( \frac{n}{2} \right) \sum_{i=1}^{k_0} \theta_i^2 \leq \frac{1}{2 n^{2\varepsilon}} \sum_{i=1}^{k_0} \alpha_i^2 \leq \frac{1}{2 n^{2\varepsilon}} \sum_{i=1}^{k_0} \alpha_i = \frac{\varphi_0 n^{1-\varepsilon}}{2} = o(n \varphi_0).
\]  
(D.5)

The second inequality is since \( \alpha_i \leq 1 \).

The last region is that in which \((1 + \varepsilon) n^{\varepsilon} \leq k_1 < n \). Since we consider bin 1, \( \theta_i \leq 1/n^{1-\varepsilon} \). Following the same steps as (D.3) and using the bound in (20),

\[
k_1 L_1 - n \varphi_1 \geq k_1 \cdot \left\{ \sum_{i=k_0+1}^{k_0} \left( 1 - e^{-n \theta_i} - \frac{n \theta_i}{k_1} \right) \right\}.
\]  
(D.6)

The function \( 1 - e^{-nx} - nx/k_1 \) is 0 for \( x = 0 \). It increases until \( x = \ln k_1 / n \), and then starts decreasing. However, at \( x = 1/n^{1-\varepsilon} \), it still approaches at least \( \varepsilon/(1 + \varepsilon) > 0 \) if \( k_1 \geq (1 + \varepsilon) n^{\varepsilon} \). Thus, \( n \varphi_1 - L_1 < k_1 L_1 = \ell_1 L_1 \), and the expression in (60) is decreasing in \( L_1 \). \(\Box\)
Appendix E – Proof of Corollary \[2\]

The contributions of all \(\theta_i\) such that \(\theta_i \leq 1/n^{1-\varepsilon}\), \(1/n^{1+\varepsilon} < \theta_i \leq 1/n^{1-\varepsilon}\) (third and fourth terms of \(45\)), and \(\theta_i \leq 1/n^{1+\varepsilon}\) (last term of \(45\)) considered in the two parts of Corollary \(2\) are bounded by the last two terms of \(60\) for \(b = 01\), \(b = 1\), and \(b = 0\), respectively (recall that \(60\) also holds for \(b = 01\)). Applying Lemma \(13.1\) to bin \(b\), this expression is decreasing in \(L_b\), where for \(b = 1\) and \(b = 01\) this is provided that \(k_b \geq (1 + \varepsilon)n^\varepsilon\). Thus a lower bound on \(L_b\) yields an upper bound on these two terms. For \(k_b < (1 + \varepsilon)n^\varepsilon\) in either case, the last two terms of \(46\) are \(O(n^{2\varepsilon}\log n)\) because \(\varphi_b \leq k_b/n^{1-\varepsilon} < (1 + \varepsilon)/n^{1-2\varepsilon}\).

Now, \(L_b\) is lower bounded similarly for \(b = 0, 1, 01\). Let \(\theta_M\) denote the maximal probability in bin \(b\), and denote the probability of letter \(i\) in bin \(b\) by \(\theta_i = \alpha_i \theta_M\), \(\alpha_i \leq 1\). Using \(20\),

\[
L_b \geq k_b - \sum_i e^{-n\theta_i} = \sum_i \left(1 - e^{-n\theta_i\alpha_i}\right) \\
\geq \sum_i \alpha_i \left(1 - e^{-n\theta_M}\right) \frac{\varphi_b}{\theta_M} \left(1 - e^{-n\theta_M}\right) \tag{E.1}
\]

where \((a)\) follows from \(1 - x^\alpha \geq \alpha(1 - x)\) for \(0 < \alpha \leq 1\) and \(0 < x \leq 1\), because \(1 - x^\alpha = \alpha + \alpha x\) equals \(1 - \alpha\) for \(x = 0\), \(0 \leq x \leq 1\), and is decreasing between \(x = 0\) and \(x = 1\). Equality \((b)\) follows from \(\varphi_b = \sum \theta_i = \sum \alpha_i \theta_M \Rightarrow \sum \alpha_i = \varphi_b/\theta_M\). Using Taylor series expansion,

\[
(n\varphi_b - L_b) \log \ell_b + n\varphi_b \cdot h_2 \left(\frac{L_b}{n\varphi_b}\right) \leq n\varphi_b \log \ell_b + L_b \log \frac{n\varphi_b \epsilon}{\ell_b L_b}. \tag{E.2}
\]

Substituting \(E.1\) to lower bound \(L_b\) for \(b = 1\) and \(b = 01\) \((\theta_M = 1/n^{1-\varepsilon})\),

\[
(n\varphi_b - L_b) \log \ell_b + n\varphi_b \cdot h_2 \left(\frac{L_b}{n\varphi_b}\right) \\
\leq n\varphi_b \log \ell_b + \varphi_b n^{1-\varepsilon} \log \frac{en^\varepsilon}{\ell_b} + \Theta \left(\varphi_b n^{1-\varepsilon} e^{-n^\varepsilon}\right) = O \left(n\varphi_b \log n\right), \tag{E.3}
\]

concluding the proof for Part I.

For \(b = 0\), the second term of \(61\) is increasing in \(\sum \theta_i^2\). Thus, using \(D.5\),

\[
\left(\frac{n^2}{2} \sum_{i=1}^{k_0} \theta_i^2\right) \log \frac{2e \cdot \varphi_0 \cdot \min \{k_0, n\}}{n \sum_{i=1}^{k_0} \theta_i^2} \leq \frac{\varphi_0 n^{1-\varepsilon}}{2} \log \left(2en^{1+\varepsilon}\right). \tag{E.4}
\]

For the other statement of Part II, first, if \(\forall \theta_i \leq 1/n^{1+\varepsilon}\), also \(\theta_i \leq 1/n^{\mu+\varepsilon}\), then,

\[
\left(\frac{n^2}{2} \sum_{i=1}^{k_0} \theta_i^2\right) \log \frac{2e \varphi_0 \ell_0}{n \sum_{i=1}^{k_0} \theta_i^2} \leq \frac{\varphi_0}{2} n^{2-\mu-\varepsilon} \log \left(2en^{\mu+\varepsilon}\right) = O \left(n^{2-\mu-\varepsilon} \log n\right) \tag{E.5}
\]

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by the same arguments of (E.4). Otherwise,

\[
\frac{n^2}{2} \sum_{\theta_i \leq 1/n^{\mu+\varepsilon}} \theta_i^2 \leq \frac{\varphi_0 \mu}{2} n^{2-\mu-\varepsilon}, \tag{E.6}
\]

where \(\varphi_0 \triangleq \sum_{\theta_i \leq 1/n^{\mu+\varepsilon}} \theta_i\), and also \(\sum_{i=1}^{k_0} \theta_i^2 > 1/n^{2\mu+2\varepsilon}\), because \(\exists \theta_i > 1/n^{\mu+\varepsilon}\) in bin 0. Therefore,

\[
\left(\frac{n^2}{2} \sum_{\theta_i \leq 1/n^{\mu+\varepsilon}} \theta_i^2 \right) \log \frac{2e\varphi_0 \ell_0}{n \sum_{j=1}^{k_0} \theta_j^2} \leq \frac{\varphi_0 \mu}{2} n^{2-\mu-\varepsilon} \log (2en^{2\mu+2\varepsilon}) = O \left(n^{2-\mu-\varepsilon} \log n\right), \tag{E.7}
\]

where \(\ell_0 \leq n\) and \(\varphi_0 \leq 1\) are used.

## Appendix F – Proof of Lemma 5.2

Four regions of \(\theta_i\) are considered: \(\theta_i \leq \mu_j/n^{1-\varepsilon}, j = 1, 2\), and \(\theta_i > \mu_j/n^{1-\varepsilon}, j = 3, 4\), where \(\{\mu_j\} = (\vartheta^-, 1, 1, \vartheta^+)\), respectively. Let \(\{\nu_j\} = (\gamma^-, \gamma^+, \gamma^-, \gamma^+)\), respectively, where \(\vartheta^- < \gamma^- < 1\) and \(1 < \gamma^+ < \vartheta^+\). Now, let

\[
\mathcal{F} = \left\{x^n : \exists \hat{\theta}_i; \hat{\theta}_i > \frac{\nu_j}{n^{1-\varepsilon}} \text{ for } \theta_i \leq \frac{\mu_j}{n^{1-\varepsilon}}; j = 1, 2, \text{ or } \hat{\theta}_i \leq \frac{\nu_j}{n^{1-\varepsilon}}, \text{ for } \theta_i > \frac{\mu_j}{n^{1-\varepsilon}}; j = 3, 4 \right\}. \tag{F.1}
\]

be the event that for \(\theta_i\) in one of the four regions defined above there exists an empirical ML estimate on the other side of the probability interval, that is separated from the boundary of the region of \(\theta_i\) by at least a complete interval between points in \((\vartheta^-, \gamma^-, 1, \vartheta^+, \gamma^+) / n^{1-\varepsilon}\). By typicality arguments and the union bound

\[
P_\theta(\mathcal{F}) \leq n \cdot k_{\theta_i > n^{-3}} \cdot 2^{-n \min_{\mathcal{F}} D(\hat{\theta}_i || \theta_i)} + \frac{1}{2\gamma^- n^{1+\varepsilon}}, \tag{F.2}
\]

where the additional term bounds the probability of re-occurrence \(\gamma^- n^\varepsilon\) or more times of any letter with \(\theta_i \leq 1/n^3\) using (I8), the bound in (E.6) with \(\mu + \varepsilon = 3\), and Markov’s inequality. Then, the union bound on the number of remaining letters (where \(k_{\theta_i > n^{-3}}\) denotes the total letters with \(\theta_i > 1/n^3\)) and the number of types (at most \(n\)) produces the first term. If \(\mathcal{F}\) occurs in region \(j\),

\[
D(\hat{\theta}_i || \theta_i) \geq \frac{\nu_j}{n^{1-\varepsilon}} \log \frac{\nu_j}{\mu_j} + \left(1 - \frac{\nu_j}{n^{1-\varepsilon}} \right) \log \frac{n^{1-\varepsilon} - \nu_j}{n^{1-\varepsilon} - \mu_j} \geq \frac{1}{n^{1-\varepsilon}} \left[\nu_j \log \frac{\nu_j}{\mu_j} + (\mu_j - \nu_j) \log \varepsilon\right] \tag{F.3}
\]

where the second inequality follows Taylor expansion. The values of \(\gamma^-\) and \(\gamma^+\) can be optimized to maximize the divergence in (F.3) by trading off between \(j = 1\) and \(j = 3\) for \(\gamma^-\) and between \(j = 2\) and \(j = 4\) for \(\gamma^+\). This yields

\[
\gamma^\pm = \frac{\vartheta^\pm - 1}{\ln \vartheta^\pm}, \tag{F.4}
\]
where ± is used to denote both cases. Plugging these choices of \( \gamma^\pm \), if \( F \) occurs

\[
D\left( \hat{\theta}_i \| \theta_i \right) \geq \frac{1}{n^{1-\varepsilon}} \left[ \min \left\{ \frac{\theta^\pm - 1}{\ln \theta^\pm} \log \frac{\theta^\pm - 1}{e \cdot \ln \theta^\pm} + \log e \right\} \right]
\]

where the minimum is taken between the value of the expression for \( \vartheta^- \) and for \( \vartheta^+ \). Hence,

\[
P_{\theta} (F) \leq \varepsilon'_n \triangleq n \cdot k_{\theta_i > n^{-3}} \cdot e^{-f(\vartheta^-, \vartheta^+)} n^\varepsilon + \frac{\ln \vartheta^-}{2(\vartheta^- - 1)n^{1+\varepsilon}}, \text{ where } \ (F.6)
\]

\[
f(\vartheta^-, \vartheta^+) \triangleq \min \left\{ \frac{\vartheta^\pm - 1}{\ln \vartheta^\pm} \ln \frac{\vartheta^\pm - 1}{e \cdot \ln \vartheta^\pm} + 1 \right\}. \quad \ (F.7)
\]

Specifically, choices of \( \vartheta^- = e^{-5.5} \approx 0.004 \) and \( \vartheta^+ = e^{1.4} \approx 4.06 \) result in \( \gamma^- \approx 0.18, \gamma^+ \approx 2.18, \)

\( f(\vartheta^-, \vartheta^+) > 0.5 \), and an upper bound of \( 2.77/n^{1+\varepsilon} \) on the last term of (F.6).

Let \( F \) denote the Bernoulli event of whether event \( F \) occurs. Then,

\[
H_{\theta} (Z^n | \Psi^n) \leq H_{\theta} (Z^n, F | \Psi^n) = H_{\theta} (Z^n | \Psi^n, F) + H_{\theta} (F | \Psi^n)
\]

\[
\leq P_{\theta} (\bar{F}) H_{\theta} (Z^n | \Psi^n, \bar{F}) + P_{\theta} (F) H_{\theta} (Z^n | \Psi^n, F) + H_{\theta} (F)
\]

\[
\leq \log \left( \frac{k^-_{\vartheta} + k^+_{\vartheta}}{k^-_{\vartheta} + k^+_{\vartheta}} \right) + \varepsilon'_n n + o(\varepsilon'_n n), \quad \ (F.8)
\]

where (a) follows since given \( \bar{F} \), the only uncertainty about \( Z^n \) is for indices for which \( \hat{\theta}_i \in (\gamma^-/n^{1-\varepsilon}, \gamma^+/n^{1-\varepsilon}] \), because in all other regions it is guaranteed that \( \hat{\theta}_i \) is on the correct side of \( 1/n^{1-\varepsilon} \), thus there is no uncertainty about the value of \( z_\ell \) corresponding to such \( \psi_\ell \). The only symbols for which it is possible to have \( \hat{\theta}_i \in (\gamma^-/n^{1-\varepsilon}, \gamma^+/n^{1-\varepsilon}] \) are the \( k^-_{\vartheta} + k^+_{\vartheta} \) letters with \( \theta_i \in (\vartheta^-/n^{1-\varepsilon}, \vartheta^+/n^{1-\varepsilon}] \). The uncertainty in \( Z^n \) is choosing which such symbols correspond to \( z = 1 \), and the worst case is when the total possible choices of \( k^+_{\vartheta} \) out of \( k^-_{\vartheta} + k^+_{\vartheta} \) are uniformly distributed. The second term is since \( H_{\theta} (Z^n | \Psi^n, F) \leq n \) for the Bernoulli process \( Z^n \).

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