Combinatorial Bandits without Total Order for Arms

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Abstract

We consider the combinatorial bandits problem, where at each time step, the online learner selects a size-$k$ subset $s$ from the arms set $A$, where $|A| = n$, and observes a stochastic reward of each arm in the selected set $s$. The goal of the online learner is to minimize the regret, induced by not selecting $s^*$ which maximizes the expected total reward. Specifically, we focus on a challenging setting where 1) the reward distribution of an arm depends on the set $s$ it is part of, and crucially 2) there is no total order for the arms in $A$.

In this paper, we formally present a reward model that captures set-dependent reward distribution and assumes no total order for arms. Correspondingly, we propose an Upper Confidence Bound (UCB) algorithm that maintains UCB for each individual arm and selects the arms with top-$k$ UCB. We develop a novel regret analysis and show an $O\left(\frac{k^2 n \log T}{\epsilon}\right)$ gap-dependent regret bound as well as an $O\left(k^2 \sqrt{nT \log T}\right)$ gap-independent regret bound. We also provide a lower bound for the proposed reward model, which shows our proposed algorithm is near-optimal for any constant $k$. Empirical results on various reward models demonstrate the broad applicability of our algorithm.

1 Introduction

Arising from various real-world applications (online advertisement, recommendation systems, etc.), combinatorial bandits [Chen et al., 2013] have become an important problem in the online learning. In this paper, we focus on the setting that for a given set of arms $A$ with size $n$ (e.g., $n$ products to be recommended), at every time step $t$, the online learner selects $k$ arms from $A$, and offers the selected set $s$ to the customer. The customer rewards each arm $a_i \in s$ with a set-dependent $X_{i,s}$, and the online learner observes the rewards of each arm. The goal of the online learner is to minimize the regret of not selecting $s^*$ which maximizes the expected reward.

It is observed that a human’s preference is typically constructed only when offered a set of alternatives, and the preference can be inconsistent across different sets [MacDonald et al., 2009]. For example, for 3 items $A, B, C$ offered in sets of two, a person can prefer $A$ over $B$, $B$ over $C$ and $C$ over $A$. The loops and reverses in preference motivate us to study the combinatorial bandits setting where the reward distribution of each arm is set-dependent, and crucially, without a total order (Definition 2) in $A$.

1.1 An old Algorithm, a weak assumption, and a key observation for regret analysis

Upper Confidence Bound (UCB) algorithm is the standard off-the-shelf choice for many bandit problems. Even in the presence of set-dependent reward, one can nevertheless ignore the set $s$ and maintain UCB

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estimations for the arms in $A$. In each time step, set $s$ is constructed with the $k$ arms with the highest UCB. The UCB of an arm $a_i \in A$ is defined in the usual way as $UCB_i(t) = C_i(t)/N_i(t) + \sqrt{\frac{\ln t}{N_i(t)}}$, where $C_i(t)$ is the cumulative reward of arm $a_i$, $N_i(t)$ is the number of times that arm $a_i$ is in the selected set $s$ up to time $t$, and $\alpha$ is a constant. This is the algorithm we study in this paper (Algorithm 1).

Empirically, even in the setting where the reward distribution is set-dependent, people still use the aforementioned UCB algorithm [e.g. the closely-related Sparring algorithm Ailon et al., 2014], however, only as a heuristic with little theoretical understanding. **Existing analysis of UCB does not provide a regret bound in this setting**, as there is no fixed expected reward associated with the arms. In particular, in general, it is impossible to prove any regret bound better than $O(n^k)$ without any additional assumption - since without any additional assumption, the feedback for one set does not give any indication about any other sets.

In this paper, we propose a new assumption for the reward model which we call **weak optimal set consistency** (Assumption 1), under which the UCB algorithm provably achieves small regret. Assumption 1 assumes that given the optimal set $s^*$, for any sub-optimal set $s$ and any arm $a$ that is common in $s$ and $s^*$, the reward expectation of $a$ is higher in $s$ than in $s^*$ (since other arms in $s$ are "less competitive"). As the assumption does not constrain the relationship between any two sub-optimal sets, it does not assume any total order for $A$. Examples (see Example 1 and Section 3.4) are constructed to show Assumption 1 can capture a wide range of set-dependent reward distribution with no total order. Moreover, many previously studied reward models (Multinomial Logit, Random Utility Model, etc.) are special cases of Assumption 1 (see discussion in Section 3.2).

To build intuition for how the UCB algorithm works under Assumption 1, we present an illustrative experiment. The imaginary environment considers offering suggestions from 6 candidates (shown in Figure 1) to customers looking for cameras, where 3 of them need to be offered each time. The reward experiment. The imaginary environment considers offering suggestions from 6 candidates (shown in Figure 1) to customers looking for cameras, where 3 of them need to be offered each time. The reward

![Figure 1: Evolution of UCB in the environment defined in Section 3.4](image-url)

**Observation:** the UCB of all arms decrease together initially, and the arms in the optimal set separate out later. $\rho(t)$ (defined later) precisely captures this dynamics. Note this happens without the arms having total order or fixed set-independent reward expectation. In fact, the Digital Camera has the highest reward expectation in most sets, while the optimal set is \{Nikon, Canon, Sony\}. Figure 1 shows the process.

Here we formalize the observation in Figure 1. We introduce $\rho(t)$, which helps to characterize the dynamics of UCB. Let $s(t)$ be the set of arms selected by the aforementioned UCB algorithm at time $t$, $\rho(t) = \min_{a_i \in s(t)} UCB_i(t)$, and $\rho(t) = \min_{\tau \leq t} \rho'(\tau)$. By definition, $\rho(t)$ is monotonically non-increasing, and $UCB_i(t) \geq \rho'(t) \geq \rho(t)$, $\forall a_i \in s(t)$, (i.e. $\rho(t)$ is a lower bound for the UCB of the arms in $s(t)$). The following lemma shows that, for the arms not in $s(t)$, $\rho(t)$ is always an upper bound, and soon a tight estimate of all their UCB (proof in Section 4).

**Lemma 1** (Dynamics of UCB). $\rho(t) \geq UCB_i(t) \geq \rho(t) \left( 1 - \frac{1}{N_i(t)} \right)$, $\forall a_i \notin s(t)$.

Notice that, under Assumption 1 once $\rho(t) \leq P(a_i|s^*)$ for some $a_i \in s^*$, all subsequent $s(t)$ will always contain $a_i$, due to the fact that with high probability, $UCB_i(t) \geq P(a_i|s^*)$ for all $t \in [T]$ (see Section 4). This matches the observation in Figure 1. Further, we can upper bound the time it takes
| Algorithm            | Regret                                      | Fixed $k$ | Set-Dep. Reward | No Total Order |
|----------------------|---------------------------------------------|-----------|-----------------|----------------|
| CUCB [Chen et al., 2013] | $O\left(\frac{k^2 n \log T}{\epsilon}\right)$ | ✓         | ✓               | ✓              |
| CombUCB1 [Kveton et al., 2015] | $O\left(\frac{kn \log T}{\epsilon}\right)$ | ✓         | ×               | ×              |
| ESCB [Combes et al., 2015] | $O\left(\sqrt{kn \log T}\right)$ | ✓         | ×               | ✓              |
| MNL-TS [Agrawal et al., 2017] | $O\left(\sqrt{NT \log TK}\right)$ | ✓         | ✓ (MNL)         | ×              |
| Explor.-Exploit. [Agrawal et al., 2019] | $O\left(\frac{kn \log T}{\epsilon}\right)$ | ✓         | ✓ (MNL)         | ×              |
| MaxMin-UCB [Saha and Gopalan, 2019] | $O\left(\frac{n \log T}{\epsilon}\right)$ | ×         | ✓ (MNL)         | ×              |
| Rec-MaxMin-UCB [Saha and Gopalan, 2019] | $O\left(\frac{n \log T}{k \epsilon}\right)$ | ✓         | ✓ (MNL)         | ×              |
| Choice Bandits [Agarwal et al., 2020] | $O\left(\frac{n^2 \log n}{\epsilon^2} + \frac{n \log T}{\epsilon}\right)$ | ×         | ✓               | ✓              |
| Algorithm 1 (Ours) | $O\left(\frac{k^2 n \log T}{\epsilon}\right)$ | ✓         | ✓               | ✓              |

Table 1: Regret upper bounds and settings for stochastic combinatorial bandits. The check marks in "Fixed $k$" mean the algorithms do not need to change the size of $s$ in different time $t$, while cross marks mean they need to change $k$ to achieve small regret. The check marks in "Set-Dep. Reward" mean the reward distribution of arms depends on the set they reside in, while cross marks mean the reward of the arms are generated independent of the set. The cross marks in "No Total Order" mean assuming individual arms to have intrinsic value, and a total order among the arms, while check marks mean the algorithm does not require such assumption.

for $\rho(t)$ to be smaller than $P(a_i|s^*)$, which can be converted into a finite time regret bound. We want to emphasize that all the analysis is done without requiring the arms to have set-independent reward expectation (or any notion of intrinsic value), which is drastically different from the standard UCB analysis.

As a summary, our **main contributions** are:

- We formalize the combinatorial bandits problem with **weak optimal set consistency** assumption (Assumption 1) which does not require a total order for arms. The new assumption covers many commonly adopted reward models (e.g. Multinomial Logit, and Random Utility Model, etc).

- We present a novel analysis of the UCB algorithm (Algorithm 1) when the arms do not have set-independent expected reward (or any notion of intrinsic value). Specifically, we prove Algorithm 1 has a gap-dependent $O(\frac{nk^2 \log T}{\epsilon})$ regret upper bound (Theorem 3), as well as a gap-independent $O(\frac{k^2 \sqrt{nT \log T}}{\epsilon})$ regret upper bound (Theorem 4). Here $n$ is the total number of arms, $k$ is the size of selected set $s$, $T$ is the time horizon and $\epsilon$ is the minimum gap between the optimal and sub-optimal set.

- Under Assumption 1, we prove a regret lower bound $\Omega(\frac{n \log T}{k \epsilon})$ when only one of the arms in the selected set has non-zero reward; and a lower bound $\Omega(\frac{n \log T}{\epsilon})$ when multiple arms in the selected set can have non-zero reward (Theorem 9). It demonstrates the optimality of Algorithm 1 for any constant set size $k$.

## 2 Motivation and Related Work

### Set-dependent Reward without Arms’ Total Order.

The inconsistency of human preference [MacDonald et al., 2009] motivates us to study the combinatorial bandit where the reward distribution of each arm depends on the set it resides in, without a total order among the arms. Correspondingly, we propose the **weak optimal set consistency** reward model (Assumption 1), which covers various reward models adopted by many combinatorial bandits work.
The simplest reward model assumes the reward of each arm is generated independent of the selected set (see Section 3.3) and has been studied in [Chen et al., 2013, Kveton et al., 2015, Combes et al., 2015]. Other work adopt more complicated models to capture the set-dependent reward distribution. However, many of them, on the contrary of Assumption 1, assume a total order among the arms. For example, the Multinomial Logit Model (MNL) assumes a deterministic utility associated with each arm, which induces a total order [Abeliuk et al., 2016, Agrawal et al., 2019, Saha and Gopalan, 2019, Flores et al., 2019]. Désir et al. [2015], Blanchet et al. [2016] approximate the user’s choice as a random walk on a Markov chain. Berbeglia [2016] shows that the discrete choice model and the Markov chain model can be viewed as instances of a "random utility model" (RUM), which also assumes a total order of all the arms. We will show in Section 3.3 that MNL and RUM are both special cases of Assumption 1.

For related work that does not assume total order, Yue and Guestrin [2011] study linear bandits and assumed a submodular value function which is known to the algorithm. The Choice Bandits [Agarwal et al., 2020] assumes there exists a single best arm that has the largest expected reward in any set, which comes from a different perspective compared with our work.

**Fixed Set Size k.** Our setting requires the size of the selected set $s$ to be exactly $k$. In practice, $k$ represents the available "displaying slots", which should be fully utilized. One common alternative is to require the size of $s$ less than or equal to $k$. However, that alternative usually leads to algorithms that yield set with size strictly less than $k$ most of the time [Saha and Gopalan, 2019]. Other related settings [Chen et al., 2013, Kveton et al., 2015, Combes et al., 2015, Agrawal et al., 2019, 2017] do not allow the algorithm to freely change the size of $s$.

**Feedback Model.** There are two commonly studied feedback models. One assumes the online learner only observes the (stochastically) best arm within the set and its reward; the other one assumes each arm generates reward independently, conditioned on the set, and the online learner observes the reward of all arms in the set.

The first feedback model reflects the relative goodness of one arm when comparing with the rest of arms in the set. Such relative feedback has been studied in the dueling bandit problem [Yue et al., 2012], with the focus on relative feedback of 2 arms. Several algorithms have been proposed for the dueling bandits [Yue et al., 2012, Zoghi et al., 2013], while others reduce the dueling bandits to standard multi-arm bandits [Ailon et al., 2014]. Going beyond 2 arms, the multi-dueling bandits problem [Brost et al., 2016, Sui et al., 2017] focuses on the pairwise relative feedback which has strictly more information than the single best arm feedback. Saha and Gopalan [2018, 2019] consider the case where only the best arm in the set is revealed, but focus on recovering the single best arm, instead of the best set.

The second feedback model reveals absolute goodness of the arms within the set, which is more commonly adopted in the stochastic combinatorial bandit problem with semi-bandit feedback [Chen et al., 2013, Kveton et al., 2015, Combes et al., 2015]. Our assumption, algorithm and analysis cover both of the feedback models.

3 Problem Setup and the Weak Optimal Set Consistency Assumption

In this section, we first present the combinatorial bandit problem setup and introduce the weak optimal set consistency assumption (Assumption 1). We then formally define the "total order" for the arms, and show that many widely studied models (MNL, RUM, etc.) assume such total order and are covered by Assumption 1. We conclude the section with an illustrative example, showing Assumption 1 covers non-trivial cases, where there is no total order for $A$. 

4
3.1 Notations and Definitions

We consider the stochastic combinatorial multi-armed bandits problem. Given a fixed set of arms \( A = \{a_1, a_2, \cdots, a_n\} \), let \( S \) denote the all \( n \)-choose-\( k \) subsets of \( A \). At each time step \( t \), the online learner selects a \( a(t) \in S \) (|\( s(t) \)| = \( k \) by definition). The online player then observes the stochastic reward \( X_{i,s(t)} \) of all the arms in \( s(t) \). To remove ambiguity, we always refer the \( a \in A \) as arm, and the \( s \in S \) as set.

The total reward of set \( s(t) \) is defined as \( Q(s(t)) = \sum_{a_i \in s(t)} X_{i,s(t)} \). Let \( s^* \) be the optimal set, which maximizes the expected reward \( \arg \max_{s \in S} \mathbb{E}[Q(s)] \). The regret is then defined to be

\[
\text{reg}(t) = \mathbb{E}[Q(s^*) - Q(s(t))], \quad \text{and} \quad R(T) = \sum_{t=1}^{T} \text{reg}(t),
\]

where the \( \text{reg}(t) \) is the regret at step \( t \), and \( R(T) \) is the total regret up to \( T \). Our goal is to design algorithm for the online player to minimize \( R(T) \).

3.2 Weak Optimal Set Consistency Assumption

One important feature that distinguishes our setting with standard stochastic combinatorial bandits is the set-dependent reward distribution and not assuming a total order for the arms.

Here we focus on the binary reward with \( X_{i,s} \in \{0,1\} \) and let \( P(a_i|s) = \mathbb{E}[X_{i,s}] \), with extensions to any bounded reward distribution discussed in Section 6. Formally, we have the following assumption about \( P(a_i|s) \):

**Assumption 1 (Weak Optimal Set Consistency).** For any sub-optimal set \( s \) and any \( a \) that is common in \( s \), \( s^* \), we assume \( P(a|s) \geq P(a|s^*) \).

One salient feature of Assumption 1 is not assuming the arms \( a \in A \) to have total order at any time \( t \). We will present several examples that are allowed by our assumption but not other reward models, and formally discuss the "total order" in next subsection.

**Example 1.** For any \( k > 2 \), with out loss of generality, we take \( a_1 \in s^* \), \( a_2 \in s^* \) with \( P(a_1|s^*) \geq P(a_2|s^*) \), and take \( a_3, a_4 \notin s^* \). For some sub-optimal set \( s_4 \), Assumption 1 allows for:

1. **Reversed relative reward expectation:**
   \[
   P(a_1|s^*) \geq P(a_2|s^*), \quad P(a_2|s_1) > P(a_1|s_1), \quad \text{for some } s_1 \supset \{a_1, a_2\} \\
   P(a_3|s_2) > P(a_4|s_2), \quad P(a_4|s_3) > P(a_3|s_3), \quad \text{for some } s_2, s_3 \text{ both containing } a_3, a_4.
   \]

2. **Non-transitive relative reward expectation:** for some \( s_4 \supset \{a_2, a_3\}, s_5 \supset \{a_1, a_3\} \),

   \[
   P(a_1|s^*) > P(a_2|s^*), \quad P(a_2|s_4) > P(a_3|s_4), \quad P(a_3|s_5) > P(a_1|s_5).
   \]

Note that the \( s_5 \) in the "non-transitive" part of Example 1 also shows that Assumption 1 allows the arms not in \( s^* \) to be better than the arms belonging to \( s^* \) in some sub-optimal set.

3.3 Total Order for Arms and More Restrictive Existing Models

We start by formally defining the "total order" for arms.

**Definition 2** (Total order for the arms). Given a reward model \( P(a|s) \) and any two arms \( a_1, a_2 \in A \), we say \( a_1 \leq a_2 \) if \( P(a_1|s) \leq P(a_2|s) \) for every \( s \) containing \( a_1, a_2 \).

Further, a reward model \( P(a|s) \) assumes total order for \( A \) if: (1) comparability, for all \( a_1, a_2 \in A \), either \( a_1 \leq a_2 \) or \( a_2 \leq a_1 \); and (2) transitivity, \( a_1 \leq a_2, a_2 \leq a_3 \) implies \( a_1 \leq a_3 \).
From Example[1] we see that Assumption[1] needs not satisfy either *comparability* or *transitivity* and thus does not assume a total order for $A$. Further, we show that many existing models assume total order for $A$ according to Definition[2] and are special cases of Assumption[1].

**Multinomial Logit (MNL):** MNL assumes a deterministic utility $v_i$ associated with each $a_i$ and the probability of $a_i$ receiving non-zero reward in $s$ is $P(a_i|s) = \frac{e^{v_i}}{e^{v_i} + \sum_{a_j \in s} e^{v_j}}$, where $v_0$ is a constant. One can verify that the $v_i$s of MNL induce a total order for $A$, and the optimal set $s^*$ is composed by arms with highest $v_i$. Assumption[1] covers MNL as $\sum_{a_i \in s} e^{v_i} \leq \sum_{a_j \in s} e^{v_j}$ for any $s \neq s^*$.

**Random utility model (RUM):** RUM assumes a (random) utility associated for all $a_i \in A$, with $U_t = v_i + \epsilon_i$, where $v_i$ is a deterministic utility and $\epsilon_i$s are i.i.d. random variables drawn from distribution $D$ at every time step $t$. The probability of $a_i$ in $s$ receiving non-zero reward is given by $P(a_i|s) = P(U_t > U_j, \forall a_j \in s \land i \neq j)$. To model the event of no arm $a \in s$ receiving non-zero reward, $s$ can be augmented to $\hat{s} = s \cup \{a_0\}$, with random utility $U_0$ of $a_0$ defined similarly. When $U_0$ is the largest, no arm $a \in s$ receives non-zero reward. It can be verified that $v_i$s in RUM induce a total order for $A$, and the optimal set $s^*$ is composed by arms with highest $v_i$. For any arm $a \in s^*$, putting it to sub-optimal set $s$ leads to $a$ having a larger chance of receiving non-zero reward, as other arms have smaller $v_i$, thus satisfies Assumption[1].

**Independent reward:** Independent reward model assumes a deterministic reward expectation $v_i$ associated with arm $a_i$. For the arm $a_i$ in any set $s$, it assumes $P(a_i|s) = v_i$. The $v_i$s immediately induce a total order for $A$. The independent reward model is also covered by Assumption[1] as $P(a_i|s)$ does not change in different $s$.

### 3.4 An Illustrative Example

To further build intuition on Assumption[1] we present a synthetic example of providing suggestions to customers looking for cameras. There are 6 candidates \{Nikon, Sony, Canon, Digital Camera, Keyboard, Shoes\}. Every time we need to offer 3 suggestions and the customer picks at most one of them.

![Figure 2: Four representative sets of the example in Section 3.4.](image)

The set #1 is optimal, as it maximizes the sum of accepting probability of the suggestions. The *Digital Camera* has highest accepting in many sub-optimal sets (even when paired with the suggestions belonging to the optimal set. see set #3). Such instances break the total order, but are covered by Assumption[1].

For the accepting probability, we set $P(\text{Nikon}|\cdot) = 0.35$, $P(\text{Canon}|\cdot) = 0.3$, $P(\text{Sony}|\cdot) = 0.25$, $P(\text{Digital Camera}|s) = 0.85 - \sum_{a_i \in s \neq \text{Digital Camera}} P(a_i|s)$ and $P(\text{Shoes}|\cdot) = 0.01$. We show 4 representative sets in Figure[2] It can be verified that the optimal set is \{Nikon, Sony, Canon\} and this example satisfies Assumption[1] but cannot be covered by any model that assumes a total order (it violates both *comparability* and *transitivity*).

Notice that the existence of the *Digital Camera* suggestion makes the problem harder. We observe that the *Digital Camera* has the highest accepting probability in many sets. This makes *Digital Camera* seemingly the best single suggestion, but it is not part of the optimal set.
4 Algorithm and Regret Analysis

In this section, we formally describe the algorithm and present its regret bound (both gap-dependent and gap-independent). We also show the sketch of regret analysis, which presents a novel way of proof, without the arms having fixed reward expectation. The analysis follows by characterizing the dynamics of UCB, for whom the intuition has been discussed in Section 1.

4.1 Algorithm

Denote $N_i(t)$ to be the number of times that $a_i$ is included in the selected set $s$ up to time $t$, $C_i(t)$ to be the cumulative reward of arm $a_i$ at time $t$. We have the algorithm shown in Algorithm 1.

Algorithm 1: UCB for Combinatorial Bandits without Total Order for Arms

1: Task: Given $A$, minimize the regret of not selecting the best $n$-choose-$k$ subset of $A$
2: Input: arm set $A$, set size $k$, time horizon $T$
3: Parameter: A problem independent constant $\alpha$. Normally set to 2
4: Initialize: $UCB_i(1) = \infty$, $N_i(1) = 0$, $C_i(1) = 0$ for all arm $a_i \in A$
5: for $t = 1$ to $T$ do
6: Construct set $s(t)$ with arms that have top-$k$ $UCB_i(t)$, ties break randomly. For all $a_i \in s(t)$, Set $N_i(t + 1) = N_i(t) + 1$
7: Observe feedback. Set $C_i(t + 1) = C_i(t) + X_{i,s(t)}$
8: $UCB_i(t + 1) = C_i(t + 1)/N_i(t + 1) + \sqrt{\frac{\alpha \log T}{N_i(t + 1)}}$, for all arm $a_i \in s(t)$, and $UCB_i(t + 1) = UCB_i(t)$, for all other arms.

4.2 Regret Bound

We first present the gap-dependent regret bound. Let $\epsilon = \min_{s \neq s^*} \mathbb{E}[Q(s^*) - Q(s)]$ denote the minimum gap in expected reward between the optimal set $s^*$ and any sub-optimal set $s$. Recall that $k$ is the size of the selected set $s$, and $n$ is the size of $A$.

Theorem 3 (Gap-dependent regret upper bound). For combinatorial bandits problem under Assumption 7, run Algorithm 1 with parameter $\alpha \geq 2$, we have

$$R(T) \leq \frac{8\alpha k^2 n \log T}{\epsilon} + 30\alpha k^2 n \log T + n = O\left(\frac{\alpha k^2 n \log T}{\epsilon}\right).$$

Due to the combinatorial nature of $S$, we might see extremely small $\epsilon$. As complementary to Theorem 3, we present the following gap-independent regret bound which holds for any $\epsilon$.

Theorem 4 (Gap-independent regret upper bound). For combinatorial bandits problem under Assumption 7, run Algorithm 1 with parameter $\alpha \geq 2$, we have

$$R(T) \leq 2\sqrt{\alpha k n T \log T} + 15k^2 \sqrt{\alpha n T \log T} = O\left(k^2 \sqrt{\alpha n T \log T}\right).$$
4.3 Proof Sketch

We first prove Lemma 1. For any time step \( t \), Recall \( \rho'(t) = \min_{a_i \in s(t)} UCB_i(t) \), and \( \rho(t) = \min_{s \leq t} \rho'(s) \). Lemma 1 claims that

\[
\rho(t) \geq UCB_i(t) \geq \rho(t) \left( 1 - \frac{1}{N_i(T)} \right), \quad \forall a_i \notin s(t).
\]

**Proof.** For any arm \( a_i \notin s(t) \), let \( t' \leq t \) be the last time step that \( a_i \in s(t') \). We then have

\[
C_i(t') + \sqrt{\alpha N_i(t') \log T} \geq \rho'(t') N_i(t') \geq \rho(t) N_i(t') \geq \rho(t) N_i(t').
\]

The last step holds as \( \rho(t) \) is non-increasing. With \( C_i(t) \geq C_i(t') \) and \( N_i(t) = N_i(t') + 1 \), we have

\[
C_i(t) + \sqrt{\alpha N_i(t) \log T} \geq \rho(t) (N_i(t) - 1) .
\]

Dividing both side by \( N_i(t) \) gives the second inequality. It left to show \( \rho(t) \geq UCB_i(t) \), \( \forall a_i \notin s(t) \). Let \( t'' \leq t \) be the last time step \( \rho(t'') = \rho(t) \). It implies

\[
\rho'(\tau) > \rho(t'') = \rho(t) \geq UCB_i(t''), \quad \forall \tau \in (t'', t], a_i \notin s(t').
\]

Notice that \( UCB_i(\tau + 1) = UCB_i(\tau) \) if \( a_i \notin s(\tau) \). Therefore for any \( a_i \notin s(t'') \), it implies \( a_i \notin s(\tau) \), \( \forall \tau \in [t'', t] \). Since there are \( n - k \) arms not in \( s(t) \) and same number of arms not in \( s(t'') \), we have \( a_i \notin s(t'') \iff a_i \notin s(t) \). Thus

\[
UCB_i(t) = UCB_i(t'') \leq \rho'(t'') = \rho(t), \quad \forall a_i \notin s(t).
\]

This completes the proof. \( \square \)

With loss of generality, we assume \( s^* = \{a_1, a_2, \ldots, a_k\} \) with \( P(a_i|s^*) \geq P(a_2|s^*) \geq \cdots \geq P(a_k|s^*) \). Let time \( t_l \) be the last time we have \( \rho(t_l) \geq 1 \forall a_i \notin s(t_l) \), for \( l \leq k \), we have the following corollary of Lemma 1.

**Corollary 5.** For all time steps \( t \) after \( t_l \), we have \( \{a_1, a_2, \ldots, a_l\} \subset s(t) \).

Corollary 5 shows that after the time step \( t_l \), at which \( \rho(t) \) falls below \( P(a_i|s^*) \), then all subsequent \( s(t) \) will always include \( \{a_1, \cdots, a_l\} \). The next lemma shows the key to bound \( t_l \).

**Lemma 6.** For the time step \( t_l \), we have

\[
2 \sqrt{\alpha k n t_l \log T} \geq k t_l P(a_l|s^*) - \sum_{t=1}^{t_l} \sum_{i=1}^{n} P(a_i|s(t)) - n P(a_l|s^*). \tag{1}
\]

**Proof.** By Corollary 12, we have \( 2 \sqrt{\alpha N_i(t_l) \log T} \geq N_i(t_l) UCB_i(t_l) - \sum_{t=1}^{t_l} P(a_i|s(t)) \) for all \( a_i \in \mathcal{A} \) with high probability. Combining with Lemma 1 and summing for all \( i \in [n] \) give the desired inequality, with left-hand side follows from \( 2 \sqrt{\alpha k n t_l \log T} \geq \sum_{i=1}^{n} 2 \sqrt{\alpha N_i(t_l) \log T} \) by Cauchy-Schwarz inequality. \( \square \)

Intuitively, the left-hand side of Equation (1) scales as \( \Theta(\sqrt{t_l}) \) and the right-hand side scales as \( \Theta(t_l) \). Therefore it can be used to upper bound \( t_l \). However, the second term on the right-hand side of Equation (1) has minus sign before it, which requires a more careful analysis.

Based on a stronger version of Lemma 1 (see Lemma 14), we can bound the number of times that a sub-optimal \( s \) is selected before \( t_l \). Let \( t_l' \) be the number of times that \( s^* \) is selected before \( t_l \).
Lemma 7 (Bound the times of selecting sub-optimal set). We can bound $t_l - t'_l$ as,

$$t_l - t'_l \leq \frac{40\alpha kn \log T}{(\Delta_l + \epsilon)^2}, \text{ if } \Delta_l \geq \frac{\epsilon}{10}; \text{ and } t_l - t'_l \leq \frac{40\alpha kn \log T}{\epsilon^2}, \text{ otherwise},$$

where $\Delta_l := \sum_{i=l}^{k} [P(a_i|s^*) - P(a_i|s^*)]$.

The next lemma connects regret $R(T)$ to $t_l - t'_l$ for $l \leq k$.

Lemma 8 (Regret decomposition). For the regret at time $T$, we have

$$R(T) \leq 2\sqrt{\alpha kn (t_k - t'_k) \log T} + \sum_{i=1}^{k-1} \delta_{ik} (t_l - t'_l) + nP(a_k|s^*).$$

where $\delta_{ij} := P(a_i|s^*) - P(a_j|s^*)$.

Now we are ready to prove Theorem 3, which gives the gap-dependent regret bound.

Proof. Combining Lemmas 7 and 8, we have

$$R(T) \leq \frac{8\alpha k^{3.5} n \log T}{\epsilon} + \sum_{i=1}^{k-1} \delta_{ik} (t_l - t'_l) + nP(a_k|s^*).$$

Directly applying Lemma 7 to the summation leads to a $O(k^3)$ term. To obtain the $O(k^2)$, we can use the Lemma 13, which gives

$$\sum_{i=1}^{k-1} \delta_{ik} (t_l - t'_l) \leq \frac{30\alpha k^2 n \log T}{\epsilon}. $$

Combining the two inequalities gives the $O\left(\frac{\alpha k^2 n \log T}{\epsilon}\right)$ regret bound. □

The proof of Theorem 4 follows by discussing the relationship between $\Delta_i + \epsilon$ and $k \sqrt{\frac{\alpha n \log T}{T}}$.

Proof. Recall that $\delta_{ik} = P(a_i|s^*) - P(a_k|s^*)$, and $\Delta_i = \sum_{i=1}^{l} \delta_{ii}$. Let $m$ denote the largest $i \in [0, k]$ such that $\Delta_i + \epsilon \geq 10k \sqrt{\frac{\alpha n \log T}{T}}$. Further note that a trivial bound for all $t_l - t'_l$ is $T$. Combining Lemmas 7 and 8 we have

$$R(T) \leq 2\sqrt{\alpha kn T \log t} + \frac{50\alpha k^3 n \log T}{\Delta_m + \epsilon} + \sum_{i=m+1}^{k} \delta_{ik} T.$$ 

By definition of $\Delta_{m+1}$, we have $\delta_{ik} \leq \Delta_{m+1}, \forall i \geq m + 1$. Therefore, we have

$$R(T) \leq 2\sqrt{\alpha kn T \log t} + \frac{50\alpha k^3 n \log T}{\Delta_m + \epsilon} + (k-m)\Delta_{m+1} T.$$ 

With $\Delta_m + \epsilon \geq 10k \sqrt{\frac{\alpha n \log T}{T}} \geq \Delta_{m+1} + \epsilon$, we have the desired regret bound. □
5 Regret Lower Bound

We present the regret lower bound under Assumption 1. In particular, we distinguish two reward models with 1) $M_1$, that allows at most 1 of the arms in the selected set $s$ to have non-zero reward (this includes the RUM and MNL model); and 2) $M_2$, that allows multiple arms to have non-zero reward (this includes the independently generated reward). Both $M_1$, $M_2$ are covered by Assumption 1, but the lower bounds differ by a factor of $k$.

**Theorem 9** (Regret Lower Bound). For any online learning algorithm, there exists an environment instance with reward model $M_1$ and satisfies Assumption 1, such that the algorithm induces a regret of $R(T) = \Omega \left( \frac{n \log T}{k \epsilon} \right)$. There exists another environment instance with reward model $M_2$ and satisfies Assumption 1, such that the algorithm induces a regret of $R(T) = \Omega \left( \frac{n \log T}{\epsilon} \right)$.

**Proof.** We defer the detailed proof to Appendix C and highlight the reason for the difference in $k$ here. Intuitively, for two different environments $E_1, E_2$, one need to select the sets that have different reward distribution in $E_1, E_2$ to accumulate enough "information" (KL-Divergence) to distinguish the two environments.

Now consider two distributions $p, q \in \mathbb{R}_+^k$, which are the reward expectations of all arms in set $s$ under environment $E_1$ and $E_2$. Each element of $p, q$ corresponds to one arm in $s$. For simplicity, let $p_1$ and $p_2$ be the two smallest elements in $p$, and $q$ differs from $p$ as $q_1 = p_1 + \epsilon$ and $q_2 = p_2 - \epsilon$. One can show that $D_{KL}(p, q) = \frac{\epsilon^2}{p_1} + \frac{\epsilon^2}{p_2} + o(\epsilon^2)$.

Under feedback model $M_1$, as the rewards are mutually exclusive, we need $\sum_{i=1}^k p_i \leq 1$. It implies that $p_1$ and $p_2$ are smaller than $\frac{1}{k-1}$. Whereas for feedback model $M_2$, we can set $p_1 = p_2 = \frac{1}{2}$. Therefore playing one sub-optimal set in $M_1$ typically brings $k$-times larger "information" than in $M_2$, which means one can distinguish $E_1$ and $E_2$ by selecting $k$-times less sets in $M_1$. This brings the difference in the regret lower bound.

The dependency of $n, B, T, \epsilon$ in the lower bound matches the upper bound (Theorem 3). Algorithm 1 is thus near-optimal for constant set size $k$ for both $M_1$ and $M_2$, under Assumption 1.

There is a gap on $k$ for between Theorem 3 and Theorem 9. The gap on $k$ also shows up under the stronger MNL assumption [Agrawal et al., 2019]. There exists several stronger lower bounds in previous work. By allowing the size of set to change (instead of fixing the size to $k$ as ours), the lower bound can be improved to be $k$-independent for $M_1$ [Chen and Wang, 2017] with a differently defined $S$, a lower bound that linearly scales with $k$ can be obtained for $M_2$ [Kveton et al., 2015]. Those results are not directly comparable with ours for the difference in settings.

We believe our lower bound can potentially be improved, since the arms still have a total order in our environment construction for lower bound analysis, which implies that Theorem 9 does not fully capture the hardness of our setting (under Assumption 1).

6 Beyond Binary Reward

In previous sections, we focus on the setting with $X_{i,s} \in \{0, 1\}$. Here we extend the reward distribution to any bounded distribution. With a minor change in Algorithm 1, it achieves the same regret bound as in Theorems 3 and 4.

6.1 Extended Problem Setting and Assumption

We keep all previous settings but the reward distribution the same. For any set $s$, the reward $X_{i,s}$ are now generated from any bounded distribution with $X_{i,s} \in [0, B]$, and the online learner observes all rewards.
Figure 3: Synthetic experiments with different reward models. The curves are the average of 5 independent runs, with the shaded area representing the standard deviation. The "UCB w/o arms order" corresponds to Algorithm\ 1 with $\alpha = 2$. "E-E MNL-bandit" refers to the "Exploration-Exploitation algorithm for MNL-Bandit" [Agrawal et al., 2019]. "Stagewise Elimination" was proposed in [Simchowitz et al., 2016]. The parameters are specified as suggested in the original papers.

$X_{i,s}$. Correspondingly, we extend the \textit{weak optimal set consistency} assumption.

**Assumption 2 (Extended Weak Optimal Set Consistency).** For any sub-optimal set $s$ and any $a$ that is common in $s, s^*$, we assume $E[X_{i,s}] \geq E[X_{i,s^*}]$.

### 6.2 Algorithm and Regret Upper Bound

For the extended setting, we can simply modify the $UCB_i$ update of Algorithm\ 1 to

$$UCB_i(t + 1) = C_i(t + 1)/N_i(t + 1) + B \sqrt{\frac{\alpha \log T}{N_i(t + 1)}}.$$ 

The new $UCB_i$ update provides valid upper bound in the extended setting, as the new reward distributions conditioned on the set $s(t)$ are all sub-Gaussian with parameter $B$. As an immediate corollary of Theorems 3 and 4, we have

**Corollary 10.** For combinatorial bandits problem with feedback model under Assumption 2, run the modified Algorithm\ 1 with parameter $\alpha \geq 2$, we have

$$R(T) = O \left( \min \left( \frac{\alpha k^2 B^2 n \log T}{\epsilon}, k^2 B \sqrt{\alpha n T \log T} \right) \right).$$

### 7 Experiments

We empirically evaluate the performance of Algorithm\ 1 on environments with different reward models (see Figure 3), which shows the broad applicability of our proposed algorithm. We summarize the environments below, with details provided in Appendix D.

**Multinomial Logit:** Each arm $a_i$ has an intrinsic value $v_i$ and the MNL model is used to determine the reward probability. The total number of arms is set to $n = 20$ and the set size is set to $k = 10$. The number of possible sets is 184756.

**Random Utility Model:** Each arm $a_i$ has an intrinsic utility $v_i$. In every step, the random utility $U_i$ of all arms in the set $s$ are independently generated with mean $\mu_i$ and unit variance from Gaussian distribution. The arm with largest random utility $U_i$ receives the reward. The total number of arms is set to $n = 20$ and the set size is set to $k = 5$. The number of possible sets is 15504.

**Preference Matrix:** We set the total number of arms to $n = 10$ and the set size to $k = 2$, then directly specify a 10-by-10 preference matrix $M$ to determine the probability of an arm receiving reward. In particular, we set the matrix such that there is no total order for the arms.

**Random Weak Optimal Set Consistency:** We randomly generate the environment that satisfies Assumption 1 via rejection sampling. We set the total number of arms to $n = 10$ and the set size to $k = 5$. 
Notice that, these randomly generated environments need not to satisfy the assumption of MNL model (or RUM) other than Assumption 1.

Along with Algorithm 1, we also take "E-E for MNL-bandit" (Exploration-Exploitation algorithm for MNL, [Agrawal et al., 2019]) and "Stagewise Elimination" [Simchowitz et al., 2016] for comparisons, which are designed for "Multinomial Logit" and "Random Utility Model" environment. The algorithms are tested in the environments listed above, with results shown in Figure 3.

"E-E for MNL-bandit" and "Stagewise Elim" perform relatively good in the environments that they are designed for. Note that in the "Preference Matrix" environment and "Random Weak Optimal Set Consistency" environment, there is no total order among the arms. The "Stagewise Elimination" falsely eliminates an arm that belongs to the optimal set (due to model mis-specification), and therefore suffers from linear regret. Algorithm 1 performs better in all the testing environments.

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A  Technical Results

Lemma 11 (Validity of Upper Confidence Bound). Denote \( P_i(t) = P(\pi_i|s(t)) \). For the probability measure generated by all sequences of assortments and reward up to time \( T \), we have

\[
P \left( \left| C_i(t) - \sum_{c=1}^{t} P_i(c) \right| \geq \sqrt{\alpha N_i(t) \log T} \right) \leq \frac{2}{T^{2\alpha}}, \quad \forall t \leq T, \forall i \in [n].
\]

Proof. Consider the quantity

\[D_i(t) = C_i(t) - \sum_{c=1}^{t} P_i(c)\]

It is not hard to see that \( D_i(0) \) to \( D_i(T) \) is a martingale. By Azuma’s inequality, we have

\[P(D_i(t) \geq d) \leq \exp(-2d^2/N_i(t)) \quad P(D_i(t) \leq -d) \leq \exp(-2d^2/N_i(t))
\]

This comes from the fact that at each time step, if \( i \) is selected, the corresponding difference is bounded by 1. Equivalently, we have

\[P(D_i(t) \geq \sqrt{\alpha N_i(t) \log T}) \leq \left( \frac{1}{T} \right)^{2\alpha} \quad P(D_i(t) \leq -\sqrt{\alpha N_i(t) \log T}) \leq \left( \frac{1}{T} \right)^{2\alpha}
\]

Therefore, we conclude that

\[\forall t \leq T, \forall i \in [n] \quad P \left( \left| C_i(t) - \sum_{c=1}^{t} P_i(c) \right| \geq \sqrt{\alpha N_i(t) \log T} \right) \leq \frac{2}{T^{2\alpha}}\]

Corollary 12 (Corollary of Lemma 11). For all time step \( t \in [T] \), and all arm \( \pi_i \in A \), we have

\[2\sqrt{\alpha N_i(t) \log T} \geq N_i(t) UCB_i(t) - \sum_{c=1}^{t} P(\pi_i|s(c)).\]

Lemma 13. Recall that we assumed \( \pi_1, \ldots, \pi_k \) all belong to \( s^* \), with \( P_i^* = P(\pi_i|s^*) \) for \( i \in [k] \), and \( P_1^* \geq P_2^* \geq \cdots \geq P_k^* \). Recall \( \delta_{ij} = P_i^* - P_j^* \) and \( \Delta_l = \sum_{i=l}^{k} \delta_{ii} \). Let \( t_l \) be the last time with \( \rho(t) \geq P_l^* \), and \( t'_l \) be the number of times that the optimal set \( s^* \) is played. For any \( l \leq k \), we have

\[
\sum_{i=1}^{l-1} \delta_u(t_i - t'_l) \leq \frac{30\alpha lkn \log T}{\Delta_l + \epsilon}.
\]

Proof. Denote \( l' \) to be the largest \( i \) with \( \Delta_i \geq \epsilon/10 \). Using Lemma 7 for \( l \leq l' \), we have

\[
\sum_{i=1}^{l-1} \delta_u(t_i - t'_l) \leq \frac{10\alpha lkn \log T}{\Delta_l + \epsilon} \cdot \sum_{i=1}^{l-1} \frac{4\delta_u(\Delta_l + \epsilon)}{(\Delta_i + \epsilon)^2} \leq \frac{20\alpha lkn \log T}{\Delta_l + \epsilon},
\]

where the last inequality follows from Lemma 15. For \( l > l' \), we have

\[
\sum_{i=1}^{l-1} \delta_u(t_i - t'_l) \leq \frac{10\alpha lkn \log T}{\Delta_l + \epsilon} \cdot \left( \sum_{i=1}^{l'} \frac{4\delta_u(\Delta_l + \epsilon)}{(\Delta_i + \epsilon)^2} + \sum_{i=l'+1}^{l-1} \frac{4\delta_u(\Delta_l + \epsilon)}{\epsilon^2} \right)
\leq \frac{15\alpha lkn \log T}{\Delta_l + \epsilon} \cdot \left( \sum_{i=1}^{l'} \frac{4\delta_u(\Delta_l + \epsilon)}{(\Delta_i + \epsilon)^2} + \sum_{i=l'+1}^{l-1} \frac{4\delta_u(\Delta_l + \epsilon)}{(\Delta_i + \epsilon)^2} \right)
\leq \frac{30\alpha lkn \log T}{\Delta_l + \epsilon}.
\]
The second inequality follows from $\Delta_l \leq \epsilon/10$ for $l > l'$, and the last inequality follows from Lemma 13. \hfill \Box

## B Proof for Section 4

### B.1 Supporting Lemmas

**Lemma 14** (Stronger version of Lemma 6). For simplicity, denote $P_1^* = P(a_1|s^*)$, $\cdots$ $P_k^* = P(a_k|s^*)$, and $P(a_i|s(t)) = P_i(t)$. Let $\delta_{ij} = P(a_i|s^*) - P(a_j|s^*)$. For any $t \in [T]$ and any $l \leq k$, recall that $t_l$ is the last time step with $t_l \geq P^*_l$, we have

\[
\sqrt{4\alpha n \ln T} \left(t_l - \frac{l}{k} t'_l\right) \geq \sum_{i=1}^{l} P_i^* t_i + (k-l)P_l^* t_l - \sum_{i=1}^{n} \sum_{c=1}^{t_i} P_i(c) - \sum_{i=1}^{n} \delta_{il}(t_i - t'_i) - nP_l^*.
\]

**Proof.** By Corollary 12 and Lemma 1 at time $t_l$, we have

\[
2\sqrt{\alpha N_i(t_l)} \geq N_i(t)P_t^* - \sum_{i=1}^{n} P(i|s(t)) - P_l^*.
\]

Summing up for $i \geq l + 1$, we have

\[
2 \sum_{i=l+1}^{n} \sqrt{\alpha \ln T N_i(t_l)} \geq \sum_{i=1}^{l} P_i^* N_i(t_l) + \sum_{i=1}^{n} P_i^* N_i(t_l) - \sum_{i=1}^{n} \sum_{c=1}^{t_i} P_i(c) - nP_l^* - \sum_{i=1}^{n} \delta_{il}(t_i - t'_i) - nP_l^*. \tag{1}
\]

The first inequality follows from $P_i(c) \geq P_i^*$ for any $c$ and $i \leq l$, by Assumption 1. The second inequality follows from $t_i - N_i(t_l) \leq t_i - t'_i$, by Corollary 5. The desired inequality follows by Cauchy-Schwart inequality

\[
\sqrt{4\alpha kn \ln T} \left(k_l - \frac{l}{k} t'_l\right) \geq 2 \sum_{i=l+1}^{n} \sqrt{\alpha \log T N_i(t_l)}.
\]

\[
\Box
\]

**Lemma 15.** Recall that we assumed $a_1, \cdots, a_k$ all belong to $s^*$, with $P_i^* = P(a_i|s^*)$ for $i \in [k]$, and $P_1^* > P_2^* > \cdots > P_k^*$. Recall $\delta_{ij} = P_i^* - P_j^*$ and $\Delta_l = \sum_{i=l}^{k} \delta_{il}$. Let $\sigma_{ij} = \frac{4\delta_{ij}(\Delta_j + \epsilon)}{(\Delta_i + \epsilon)^2}$, we have

\[
\sum_{j=i}^{k} \sigma_{ij} \leq 2, \forall i \leq k, \forall \epsilon \geq 0.
\]

**Proof.** Expanding the summation, we have

\[
\sum_{j=i}^{k} \sigma_{ij} = \sum_{j=i}^{k} \frac{4\delta_{ij}(\Delta_j + \epsilon)}{(\Delta_i + \epsilon)^2} = 4 \sum_{j=i}^{k} \frac{\delta_{ij}}{\Delta_i + \epsilon} \left( \sum_{m=j}^{k} \frac{\delta_{mk}}{\Delta_i + \epsilon} + \frac{\epsilon}{\Delta_i + \epsilon} \right).
\]

Note that

\[
\sum_{m=j}^{k} \delta_{mk} + \sum_{m=i}^{j} \delta_{im} \leq \Delta_i + \epsilon \implies \sum_{m=j}^{k} \frac{\delta_{mk}}{\Delta_i + \epsilon} \leq 1 - \sum_{m=i}^{j} \frac{\delta_{im}}{\Delta_i + \epsilon}.
\]
For brevity, let $x_m = \frac{\delta_m}{\Delta + \epsilon}$, we have $\sum_{m=1}^{k} x_m \leq 1$ and

$$\sum_{j=i}^{k} \sigma_{ij} \leq 4 \sum_{j=i}^{k} x_j (1 - \sum_{m=i}^{j} x_m) \leq 2.$$  

The last inequality holds for any $\sum_{m=1}^{k} x_m \leq 1$. \hfill \Box

**Lemma 16.** For any $1 \leq i < j \leq k$, define function $f(i, j) = 0.4\sigma_{ij} + \sum_{m=i+1}^{j-1} 0.4f(m, j)$. We have

1. $f(i, j) = 0.4\sigma_{ij} + \sum_{m=i+1}^{j-1} 0.4f(i, m)\sigma_{mj}$
2. $f(i, j) \leq 1$

**Proof.** We first prove the first part. Let $\Pi(i, j)$ be the power set of $\{i, i + 1, \ldots, j - 1, j\}$. Let $\Gamma(i, j) = \{x | x \in \Pi(i, j), i \in x, j \in x\}$. Further, for $x \in \Gamma(i, j)$ defining

$$g(x) = \sigma_{x_1, x_2} \cdot \sigma_{x_2, x_3} \cdot \ldots \cdot \sigma_{x_{|x|-1}, x_{|x|}}.$$

For example, for $x = \{2, 3, 5, 7\}$, we have $g(x) = \sigma_{23} \cdot \sigma_{35} \cdot \sigma_{57}$. By definition, it can be shown via induction that

$$f(i, j) = \sum_{x \in \Gamma(i, j)} 0.4|x|g(x),$$

which is equivalent to the first equation in Lemma 16. For the second part of the proof, we prove by induction. It can be easily verified that for any $i, j$ such that $j - i = 1$, we have $f(i, j) = 0.4\sigma_{ij} \leq 1$. Now, suppose that the inequality holds for any $i, j$ with $j - i = l - 1$, then for any $j', i'$ with $j' - i' = l$, we have

$$f(i, j) \leq 0.4\sigma_{ij} + \sum_{m=i+1}^{j-1} 0.4\sigma_{im} = 0.4 \sum_{m=i+1}^{j} \sigma_{im} \leq 0.8$$

The last inequality follows from Lemma 15. \hfill \Box

### B.2 Proof of Lemma 7

**Proof.** Recall that $P^*_1 = P(a_1 | s^*), \ldots, P^*_k = P(a_k | s^*)$. Define $\delta_{ij} = P^*_{i} - P^*_{j}, \Delta_i = \sum_{i=1}^{k} \delta_{li}$. Define $t_l$ to be the last time step with $\rho(t_l) \geq P^*_l$. Denote $t'_l$ to be the number of times $s(t) = s^*$ for $t \leq t_l$.

**Case I:** $\Delta_i \geq \frac{\epsilon}{10}$.

By Lemma 14 we have

$$\sqrt{4\alpha kn \ln T \left( t_l - \frac{l}{k} t'_l \right)} \geq \sum_{i=1}^{l} P^*_i t_l + (k - l)P^*_i t_l - \sum_{i=1}^{n} \sum_{c=1}^{t_l} P_i(c) - \sum_{i=1}^{l-1} \delta_{il}(t_i - t'_l) - nP^*_l.$$ 

Note that

$$\sum_{i=1}^{l} P^*_i t_l + (k - l)P^*_i t_l - \sum_{i=1}^{k} P^*_i = \sum_{i=l}^{k} \delta_{li} = \Delta_i.$$ 

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By the fact \( \sum_{i=1}^{k} P_i(t) \leq \sum_{i=1}^{k} P_i^* - \epsilon \) for suboptimal assortment, we have
\[
\sqrt{4\alpha kn \ln T} \left( t_l - \frac{l}{k} t_l' \right) \geq \Delta_l t_l + \epsilon (t_l - t_l') - \sum_{i=1}^{l-1} \delta_{il} (t_i - t_i') - nP_i^*
\]
\[
\geq (\Delta_l + \epsilon) \left( t_l - \frac{l}{k} t_l' \right) - \frac{\epsilon (k - l) - \Delta_l t_l'}{k} - \sum_{i=1}^{l-1} \delta_{il} (t_i - t_i') - nP_i^*.
\]

For \( \ln T \geq 5 \), with the fact \( k \geq \Delta_l, 1 \geq P_i^*, \alpha \geq 1 \), we have \( 4n(\Delta_l + \epsilon)P_i^* \leq 0.8\alpha kn \ln T \), we therefore have the following bound for \( \sqrt{t_l - \frac{l}{k} t_l'} \),
\[
\sqrt{t_l - \frac{l}{k} t_l'} \leq \frac{\sqrt{4\alpha kn \ln T} \left( 1 + \sqrt{1.2 + \sum_{i=1}^{l-1} \frac{4\delta_{il}(\Delta_l + \epsilon)}{(\Delta_l + \epsilon)^2} c_i + \frac{4(\Delta_l + \epsilon) \frac{\epsilon (k - l) - \Delta_l t_l'}{k} t_l'}{4\alpha kn \ln T} \right)}{2 (\Delta_l + \epsilon)}.
\]

For simplicity, we write \( t_l - t_l' \) in the following form
\[
t_l - t_l' = c_l \frac{4\alpha kn \ln T}{(\Delta_l + \epsilon)^2}.
\]

Equation (2) can then be rewritten as
\[
\sqrt{t_l - \frac{l}{k} t_l'} \leq \frac{1}{2} \left( 1 + \sqrt{1.2 + \sum_{i=1}^{l-1} \frac{\sigma_{il} c_i}{(\Delta_l + \epsilon)^2} + \frac{4(\Delta_l + \epsilon) \frac{\epsilon (k - l) - \Delta_l t_l'}{k(\Delta_l + \epsilon)} t_l'}{4\alpha kn \ln T} \right) \frac{\sqrt{4\alpha kn \ln T}}{\Delta_l + \epsilon}.
\]

Further, define \( \sigma_{i,l} = \frac{4\delta_{il}(\Delta_l + \epsilon)}{(\Delta_l + \epsilon)^2} \), we have
\[
\sqrt{t_l - \frac{l}{k} t_l'} \leq \frac{1}{2} \left( 1 + \sqrt{1.2 + \sum_{i=1}^{l-1} \frac{\sigma_{il} c_i}{(\Delta_l + \epsilon)^2} + \frac{4(\Delta_l + \epsilon) \frac{\epsilon (k - l) - \Delta_l t_l'}{k(\Delta_l + \epsilon)} t_l'}{4\alpha kn \ln T} \right) \frac{\sqrt{4\alpha kn \ln T}}{\Delta_l + \epsilon}.
\]

By the fact \( (1 + a)^2 \leq 1.1a^2 + 11 \) for any real number \( a \), we have
\[
t_l - \frac{l}{k} t_l' \leq \frac{1}{4} \left( 11 + 1.32 + 1.1 \sum_{i=1}^{l-1} \frac{\sigma_{il} c_i}{(\Delta_l + \epsilon)^2} \frac{4\alpha kn \ln T}{(\Delta_l + \epsilon)^2} + 1.1 \frac{\epsilon (k - l) - \Delta_l t_l'}{k(\Delta_l + \epsilon)} t_l' \right).
\]

Since \( \Delta_l \geq \frac{\epsilon}{10} \), we have \( \frac{\epsilon}{3} (k - l) \leq \frac{3}{5} \Delta_l (k + \frac{l}{2}) \), which implies \( \epsilon (k - l) - \Delta_l t_l' \leq \frac{3}{5} (k - l)(\Delta_l + \epsilon) \).

Therefore we have
\[
t_l - \frac{l}{k} t_l' = \left( 3.08 + 0.275 \sum_{i=1}^{l-1} \frac{\sigma_{il} c_i}{(\Delta_l + \epsilon)^2} + \frac{1}{k} \frac{t_l'}{t_l'} \right) \frac{4\alpha kn \ln T}{(\Delta_l + \epsilon)^2} + \frac{k - l}{k} t_l'
\]
\[
\implies t_l - t_l' \leq \left( 3.08 + 0.275 \sum_{i=1}^{l-1} \frac{\sigma_{il} c_i}{(\Delta_l + \epsilon)^2} \right) \frac{4\alpha kn \ln T}{(\Delta_l + \epsilon)^2}.
\]

Plug in the convention of \( c_l \), we have
\[
c_l \leq 3.08 + 0.275 \sum_{i=1}^{l-1} \sigma_{il} c_i.
\]
With Lemma 16, we can use $f(i, j)$ to upper bound $c_l$. First define $c'_l = \frac{c_l}{10}$, which implies that

$$c'_l \leq 0.308 + 0.275 \sum_{i=1}^{l-1} \sigma_{il} c'_i.$$

Next we proceed to show that

$$c'_l \leq 0.308 + \sum_{i=1}^{l-1} f(i, l).$$

We prove Equation (3) by induction. For $l = 1, 2$, we have

$$c'_1 \leq 0.308, \quad c'_2 \leq 0.308 + 0.275\sigma_{12} c'_1 \leq 0.308 + 0.275\sigma_{12} = 0.308 + f(1, 2).$$

Suppose Equation (3) holds for $c'_{l-1}$, then we have

$$c'_l \leq 0.308 + 0.275 \sum_{i=1}^{l-1} \sigma_{il} c'_i \leq 0.308 + 0.275 \sum_{i=1}^{l-1} \left[ 0.308 + \sum_{j=1}^{i-1} f(j, i) \right] \sigma_{il}$$

$$\leq 0.308 + \sum_{i=1}^{l-1} \left[ 0.275\sigma_{il} + 0.275 \sum_{j=1}^{i-1} f(j, i) \sigma_{il} \right]$$

$$\leq 0.308 + \sum_{j=1}^{l-1} \sigma_{jl} + \sum_{i=j+1}^{l-1} 0.275 f(j, i) \sigma_{il}$$

$$\leq 0.308 + \sum_{j=1}^{l-1} \left[ 0.275 \delta_{il} + \sum_{i=j+1}^{l-1} 0.275 f(j, i) \sigma_{il} \right]$$

$$\leq 0.308 + \sum_{i=1}^{l-1} f(i, l).$$

The last inequality follows from the first equation in Lemma 16. Combining with the second inequality in Lemma 16, we have $c_l \leq 10l$. This completes the proof of the first case in Lemma 7.

**Case II:** $\Delta_i < \frac{\epsilon}{10}$.

Denote $l'$ to be the largest $i$ with $\Delta_i \geq \epsilon/10$. By definition, we know $l > l'$. Applying Lemma 14 to all arms, we have that

$$2\sqrt{\alpha kn (t_l - t_{l'}) \ln T} \geq \sum_{i=k+1}^{n} P_i^* N_i(t_l) + \sum_{i=1}^{k} P_i^* N_i(t_l) - \sum_{i=1}^{n} \sum_{c=1}^{t_l} P_i(c) - nP_l^*$$

$$\geq \epsilon (t_l - t_{l'}) - \sum_{i=1}^{l-1} \delta_{ik}(t_i - t_{i'}) - nP_l^*.$$

Solving for $t_l - t_{l'}$, we have

$$\sqrt{t_l - t_{l'}} \leq \sqrt{4\alpha kn \ln T} + \sqrt{4.8\alpha kn \ln T} + \sum_{i=1}^{l-1} 4\delta_{ik} \epsilon (t_i - t_{i'}) \epsilon.$$

Similarly as $c_l$, we write $t_l$ for $l > l'$ as

$$t_l - t_{l'} = d_n \frac{4\alpha kn \ln T}{\epsilon^2}.$$
Therefore
\[
\sqrt{t_l - t'_l} \leq \frac{\sqrt{4\alpha kn \ln T \left( 1 + \sqrt{1.2 + \sum_{i=1}^{l'} \frac{4\Delta_i\epsilon}{(\Delta_i + \epsilon)^2} c_i + \sum_{i=l'+1}^{l-1} \frac{4\Delta_i\epsilon}{\epsilon^2} d_i} \right)}}{\epsilon}
\]

\[
\leq \frac{\sqrt{4\alpha kn \ln T \left( 1 + \sqrt{1.2 + \sum_{i=1}^{l'} \frac{4\Delta_i(\Delta_i + \epsilon)}{(\Delta_i + \epsilon)^2} c_i + 1.21 \sum_{i=l'+1}^{l-1} \frac{4\Delta_i(\Delta_i + \epsilon)}{(\Delta_i + \epsilon)^2} d_i} \right)}}{\epsilon}.
\]

The second inequality follows from \( \frac{(\Delta_i + \epsilon)^2}{\epsilon^2} \leq 1.21 \) as \( \Delta_i < \frac{\epsilon}{10} \) for all \( i > l' \). Simplify the inequality, we have
\[
\sqrt{d_l} \leq \frac{1}{4} \left( 1 + \sqrt{1.2 + \sum_{i=1}^{l'} \sigma_{il} c_i + 1.21 \sum_{i=l'+1}^{l-1} \sigma_{il} d_i} \right).
\]

Again use the fact that \( (a + b)^2 \leq 11 + 1.1a^2 \), we have
\[
d_l \leq \frac{1}{4} \left( 11 + 1.32 + 1.1 \sum_{i=1}^{l'} \sigma_{il} c_i + 1.331 \sum_{i=l'+1}^{l-1} \sigma_{il} d_i \right)
\]
\[
\leq 3.08 + 0.275 \sum_{i=1}^{l'} \sigma_{il} c_i + 0.34 \sum_{i=l'+1}^{l-1} \sigma_{il} d_i.
\]

Recall that we’ve defined \( f(i, j) = 0.4\sigma_{ij} + \sum_{k=i+1}^{j-1} 0.4\sigma_{ik} f(k, j) \). Similar to showing \( c_l \leq 3.08 + \sum_{i=1}^{l-1} f(i, l) \), we can define \( d'_l = d_l/10 \) and have
\[
d'_l \leq 0.308 + 0.275 \sum_{i=1}^{l'} \sigma_{il} c'_i + 0.34 \sum_{i=l'+1}^{l-1} \sigma_{il} d'_i
\]
\[
\leq 0.308 + 0.275 \sum_{i=1}^{l'} \left[ 0.308 + \sum_{j=1}^{i-1} f(j, i) \right] \sigma_{il} + 0.34 \sum_{i=l'+1}^{l-1} \left[ 0.308 + \sum_{j=1}^{i-1} f(j, i) \right] \sigma_{il}
\]
\[
\leq 0.308 + \sum_{i=1}^{l-1} \left[ 0.34\sigma_{il} + 0.34 \sum_{j=1}^{i-1} f(j, i)\sigma_{il} \right]
\]
\[
\leq 0.308 + \sum_{j=1}^{l-1} \sum_{i=j+1}^{l-1} 0.34 f(j, i)\sigma_{il} + \sum_{i=1}^{l-1} 0.34\sigma_{il}
\]
\[
\leq 0.308 + \sum_{j=1}^{l-1} \left[ 0.34\sigma_{jl} + \sum_{i=j+1}^{l-1} 0.34 f(j, i)\sigma_{il} \right]
\]
\[
\leq 0.308 + \sum_{i=1}^{l-1} f(i, l).
\]

Therefore we have \( d_l \leq 3.08 + 10 \sum_{i=1}^{l-1} f(i, l) \leq 10l \), which completes the proof for the second case. \( \square \)
B.3 Proof of Lemma 8

Proof. Note that by Assumption 1 we have \( \rho(T) \geq P(a_k|s^*) \) which implies \( R(T) \leq R(t_k) \). Plug in Lemma 14 with \( l = k \), we have

\[
\sqrt{4\alpha kn \ln T} (t_k - t'_k) \geq \sum_{i=1}^{k} P(a_i|s^*)t_k - \sum_{i=1}^{n} \sum_{c=1}^{t_k} P_i(c) - \sum_{i=1}^{k-1} \delta_{ik}(t_i - t'_i) - nP(a_k|s^*).
\]

Note that \( R(t_k) = \sum_{i=1}^{k} P(a_i|s^*)t_k - \sum_{i=1}^{n} \sum_{c=1}^{t_k} P_i(c), \) Rearranging the terms, we have

\[
R(T) \leq R(t_k) \leq \sqrt{4\alpha kn \ln T} (t_k - t'_k) + \sum_{i=1}^{k-1} \delta_{ik}(t_i - t'_i) + nP(a_k|s^*)
\]

\[\square\]

C Proof for Section 5

C.1 Regret Lower bound for Feedback Model \( M1 \)

We prove the lower bound for the feedback model \( M1 \) with mutually exclusive rewards. By constructing a family of environments \( \mathcal{E}_i, i \in [n] \). We define the arm set as \( \mathcal{A} = \{a_1, \ldots, a_{n+k-1}\} \).

In environment \( \mathcal{E}_i \), the optimal set is \( \{a_i, a_{n+1}, a_{n+2} \cdots, a_{n+k-1}\} \). We assume those arms to have \( \frac{1}{k} \) probability of receiving positive reward in any set. All other arms not belonging to the optimal set have \( \frac{1}{k} - \epsilon \) probability of receiving positive reward in any set. It’s easy to verify that all environments \( \mathcal{E}_i \) satisfies Assumption 1 and the minimum gap between optimal and sub-optimal set is \( \epsilon \). We then have the following regret lower bound.

Denote \( q_i \) to be the distribution of \( T \)-step history induced by \( \mathcal{E}_i \). We then have the following Lemma:

Lemma 17 (Lower Bound for Each Arm). Under feedback model \( M1 \), let \( \phi \) be an algorithm for the combinatorial bandits problem with Assumption 2 such that the regret is \( R_\phi(T) = o(T^a) \) for all \( a > 0 \). Then for the environment \( \mathcal{E}_1 \) we have \( \mathbb{E}_{q_1}(N_j(T)) = \Omega \left( \frac{\log T}{k \epsilon^2} \right) \) for all arm \( a_j \).

Proof. For a fixed \( j \notin \{1, n\} \), we define the event \( B_j = \{N_j(T) \leq \log T/\epsilon^2\} \). If \( q_1(B_j) < 1/3 \), we have

\[
\mathbb{E}_{q_1}(N_j(T)) \geq q_1(B_j^c) \log T/\epsilon^2 = \Omega(\log T/\epsilon^2)
\]

Now suppose \( q_1(B_j) \geq 1/3 \). Note that in environment \( \mathcal{E}_j \), the algorithm will incur at least \( \epsilon \) regret if not selecting \( a_j \). Therefore we have \( \mathbb{E}_{q_j}(T - N_j(T)) = o(T^a) \). By Markov’s inequality, we have

\[
q_j(B_j) = q_j \left( \{T - N_j(T) > T - \log T/\epsilon^2\} \right) \leq \frac{\mathbb{E}_{q_j}(T - N_j(T))}{T - \log T/\epsilon^2} = o(T^{a-1})
\]

From [Karp and Kleinberg 2007], we know that for any event \( B \) and two distributions \( p, q \) with \( p(B) > 1/3 \) and \( q(B) < 1/3 \), we have

\[
D_{KL}(p; q) \geq \frac{1}{3} \ln \left( \frac{1}{3q(B)} \right) - \frac{1}{\epsilon}
\]

Putting \( q_1, q_j \) and \( B_j \) into the inequality above, we have

\[
D_{KL}(q_1; q_j) \geq \frac{1}{3} \ln \left( \frac{1}{3o(T^{a-1})} \right) - \frac{1}{\epsilon} = \Omega(\ln T)
\]
On the other hand, since the only different arm between \( E_1 \) and \( E_j \) is arm \( a_j \). We need to bound the KL-divergence by playing any set containing \( a_j \). Suppose \( p \) is a categorical distribution with parameters \( p_1, \ldots, p_k \) for \( k \) items and \( p' \) is another categorical distribution with parameters \( p_1 - \epsilon_1, \ldots, p_k - \epsilon_k \). Then we have

\[
D_{\text{KL}}(p, p') = \sum_{i=1}^{k} (p'_i + \epsilon_i) \log \frac{p'_i + \epsilon_i}{p'_i} \leq \sum_{i=1}^{k} \frac{\epsilon_i}{p'_i} = \sum_{i=1}^{k} \frac{\epsilon_i^2}{p'_i},
\]

where the last inequality holds because \( \sum_{i=1}^{k} \epsilon_i = 0 \). Therefore we can directly bound the KL-divergence of \( q_1 \) and \( q_j \) by

\[
D_{\text{KL}}(q_1; q_j) \leq C \mathbb{E}(N_j(T)) k \epsilon^2,
\]

where \( C \) is a problem-independent constant. It then directly implies that \( C \mathbb{E}(N_j(T)) k \epsilon^2 = \Omega(\log T) \Rightarrow \mathbb{E}_{q_i}(N_j(T)) = \Omega \left( \frac{\log T}{k \epsilon^2} \right) \)

which completes the proof.

From Lemma 18 we know that in \( E_1 \) each arm will be played for \( \Omega(\log T/k \epsilon^2) \), and each time a sub-optimal arm is played, it induces at least \( \epsilon \) regret. Since we have \( n + k - 1 \) arms in \( A \), it immediately implies that the regret is lower bounded by \( \Omega(n \log T/k \epsilon) \). For the algorithm that doesn’t satisfy the assumption in Lemma 18 (i.e. for some \( a > 0 \), the \( o(T^a) \) regret bound doesn’t hold), the lower bound holds directly. As a summary, we have Theorem 9.

C.2 Regret Lower bound for Feedback Model \( M_2 \)

The environment construction is similar to the one for \( M_1 \). The only difference is to replace all \( \frac{1}{k+1} \) with \( \frac{1}{2} \). Accordingly, we have

**Lemma 18 (Lower Bound for Each Arm).** Under feedback model \( M_2 \), let \( \phi \) be an algorithm for the combinatorial bandits problem with Assumption 7 such that the regret is \( R_\phi(T) = o(T^a) \) for all \( a > 0 \). Then for the environment \( E_1 \) we have

\[
\mathbb{E}_{q_i}(N_j(T)) = \Omega \left( \frac{\log T}{\epsilon^2} \right)
\]

for all arm \( a_j \).

Similar to previous subsection, it implies a \( \Omega(n \log T/\epsilon) \) lower bound.

D Experiment Setup

D.1 Multinomial Logit

In this environment, the reward is generated according to a multinomial logit model

\[
\mathbb{P}(a_i|s(t)) = \frac{v_i}{1 + \sum_{a_i \in s(t)} v_i}, \quad \mathbb{P}(a_0|s(t)) = \frac{1}{1 + \sum_{a_i \in s(t)} v_i}
\]

where \( v_i \) is the value associated with each arm \( a_i \), determining the reward probability. In this experiment, we set \( v_i = 1 - 0.04i \) with \( i \in [20] \). The size of set is set to \( k = 10 \), and the optimal set is \( s^* \) is composed by arms from \( a_1 \) to \( a_{10} \). The regret of set \( s(t) \) is given by

\[
\text{reg}(s(t)) = \frac{1}{1 + \sum_{a_i \in s(t)} v_i} - \frac{1}{1 + \sum_{a_i \in s^*} v_i}
\]
D.2 Random Utility Model

In this environment, for an set $s(t)$ at time step $t$, each arm $a_i \in s(t)$ will independently draw a Gaussian distributed random variable $x_i \sim \mathcal{N}(\mu_i, 1)$, where $\mu_i$ is the mean associated with each arm $a_i$. Along with that, $a_0$ will draw a $x_0 \sim \mathcal{N}(2, 1)$. The arm $a_i$ (including $a_0$) with highest $x_i$ will receive reward. Thus we have the probability of $a_i$ getting reward as

$$P(a_i|s(t)) = P(x_i = \max_{a_j \in s(t) \cup \{a_0\}} x_j)$$

Here, we set $\mu_i = 1 - 0.04i$ with $i \in [20]$. The size of set is set to $k = 5$, and the optimal set $s^*$ is composed by the arms from $a_1$ to $a_5$. For the convenience of computation, the regret of set $s(t)$ is defined slightly different as

$$reg(s(t)) = \sum_{a_i \in s^*} \mu_i - \sum_{a_i \in s(t)} \mu_i$$

Once $s(t)$ recovers the optimal set $s^*$, which maximizes the probability of $s(t)$ receiving reward, we will have this regret $reg(s(t)) = 0$.

D.3 Preference Matrix

In this environment, the probability of one arm $a_i$ getting reward is fully specified by a preference matrix. For ease of representation, we set the number of arms to $n = 10$ and the size of set to $k = 2$. Th total number of sets is 45, much lesser than the previous two environments. However, with a specially designed preference matrix (including the loop in preference, etc), the environment turns out to be the hardest.

We set $M$ to be the preference matrix with $M_{i,j} = P(a_i|s(t) = \{a_i, a_j\}) - P(a_j|s(t) = \{a_i, a_j\})$. We set the optimal set to be $s^* = \{a_1, a_2\}$ with $P(a_0|s^*) = 0.08$. For all other sets $s$ which are sub-optimal, we set $P(a_0|s) = 0.1$. The preference matrix $M$ is given in Table 2.

|     | a\_1 | a\_2 | a\_3 | a\_4 | a\_5 | a\_6 | a\_7 | a\_8 | a\_9 | a\_10 |
|-----|------|------|------|------|------|------|------|------|------|-------|
| a\_1| -    | 0.02 | 0.05 | 0.1  | 0.1  | 0.2  | 0.25 | 0.3  | 0.3  | 0.3   |
| a\_2| -0.02| -    | 0.05 | 0.1  | 0.1  | 0.2  | 0.25 | 0.3  | 0.3  | 0.3   |
| a\_3| -0.05| -0.05| -    | 0.45 | 0.45 | 0.45 | 0.45 | 0.45 | 0.45 | 0.45  |
| a\_4| -0.1 | -0.1 | -0.45| -    | -0.3 | 0.3  | 0    | 0    | 0    | 0     |
| a\_5| -0.1 | -0.1 | -0.45| 0.3  | -    | -0.3 | 0    | 0    | 0    | 0     |
| a\_6| -0.2 | -0.2 | -0.45| -0.3 | 0.3  | -    | 0    | 0    | 0    | 0     |
| a\_7| -0.25| -0.25| -0.45| 0    | 0    | 0    | -    | 0    | 0    | 0     |
| a\_8| -0.3 | -0.3 | -0.45| 0    | 0    | 0    | 0    | -    | 0    | 0     |
| a\_9| -0.3 | -0.3 | -0.45| 0    | 0    | 0    | 0    | -    | 0    | 0     |
| a\_10| -0.3| -0.3 | -0.45| 0    | 0    | 0    | 0    | 0    | -    | 0     |

Table 2: Preference Matrix $M$

We can see that when $a_3$ pairs with any other sub-optimal arm, it will have a higher chance of getting reward than $a_1$ and $a_2$. It makes $a_3$ the seemingly best single arm. Also note that when $a_4$ pairs with $a_5$, $a_5$ will have a higher chance of getting reward. Similarly, $a_6$ will win over $a_5$ and $a_4$ will win over $a_6$. The preference therefore forms a loop among $a_4, a_5, a_6$.

The regret of $s(t)$ is given by

$$reg(s(t)) = P(a_0|s(t)) - P(a_0|s^*)$$
D.4 Random Weak Optimal Set Consistency

In this environment, we randomly generate the environment with Algorithm 2 that satisfies the Assumption 1.

**Algorithm 2**

**Generating Environment Satisfies Assumption 1**

1: **Input:** Number of Arms $n$, set Size $k$.
2: Set set $s^* = \{1, 2, \cdots, k\}$ be the optimal set. Randomly Sample $P(a|s^*) \sim \text{Uniform}(0, \frac{1}{k})$.
3: **for** set $s \neq s^*$ **do**
4: **while** $\sum_{a \in s} P(a|s) > \sum_{a^* \in s^*} P(a^*|s^*)$ **do**
5: **for** $a \in s$ **do**
6: **if** $a \in s^*$ **then**
7: Sample $P(a|s) \sim \text{Uniform}(P(a|s^*), \frac{1}{k})$.
8: **else**
9: Sample $P(a|s) \sim \text{Uniform}(0, \frac{1}{k})$.

By construction, the environment satisfies Assumption 1. Moreover, as we randomly sample the feedback for each set randomly, it’s not necessary for the generated environment to satisfy more stronger Assumption, e.g. the strict preference order. The regret of set $s(t)$ is given by

$$\text{reg}(s(t)) = \sum_{a \in s(t)} P(a|s_t) - \sum_{a^* \in s^*} P(a^*|s^*).$$