REMARKS ON THE ROSENHEAD APPROXIMATION FOR THE EVOLUTION OF VORTEX FILAMENTS

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Abstract. We study the regularity of weak solutions to the vorticity equation for incompressible viscous flows in $\mathbb{R}^3$. For a family of desingularised models for the vortex filament motions (including the classical Rosenhead approximation in [13]), we show that a weak solution is automatically strong, provided that the vortex filament remains smooth.

1. Introduction and Statement of the Main Result

We consider the regularity of the weak solutions to the Navier–Stokes vorticity equation of incompressible fluid dynamics in 3D:

$$\partial_t \omega + (u \cdot \nabla) \omega - \nu \Delta \omega = S \cdot \omega \quad \text{in } [0, T^*] \times \mathbb{R}^3. \quad \text{(1.1)}$$

Here and throughout $T^* \in ]0, \infty]$, $\omega : [0, T^*] \times \mathbb{R}^3 \to \mathbb{R}^3$ is the vorticity of a 3D incompressible viscous fluid, $u : [0, T^*] \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity, $\nu > 0$ is the viscosity, and the $3 \times 3$ matrix field $S : T^* \times \mathbb{R}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3$, defined by

$$S := \nabla u + \nabla^\top u, \quad \text{(1.2)}$$

is known as the rate-of-strain tensor. Eq. (1.1) is implemented by the initial data:

$$\omega|_{t=0} = \omega_0, \quad \text{(1.3)}$$

and the incompressibility condition:

$$\nabla \cdot u = 0 \quad \text{in } [0, T^*] \times \mathbb{R}^3. \quad \text{(1.4)}$$

Setting $\nu = 0$ in Eq. (1.1), we obtain the Euler vorticity equation for incompressible inviscid flows.

In this paper, Eq. (1.1) is studied in the context of evolution of singular vortex filaments in 3D incompressible viscous fluids. For each $t \in [0, T^*]$, suppose that $\gamma(t, \cdot) : [0, 1] \to \mathbb{R}^3$ parametrises a smoothly embedded $S^1$ in $\mathbb{R}^3$, i.e., a knot. $\gamma$ is identified with its image. Then, assume that the vorticity $\omega$ is supported on $\gamma$: formally we take

$$\omega(t, x) = \Gamma \int_{\mathbb{R}^3} \delta(x - \gamma(t, y)) \gamma_y(t, y) \, dy, \quad \text{(1.5)}$$

where $\delta$ is the Dirac measure, $\gamma_y = \partial \gamma / \partial y$ is the tangent vector field to $\gamma$, and $\Gamma > 0$ is a constant measuring the strength of the vortex filament. We further require that $\gamma$ is transported by the fluid:

$$\partial_t \gamma(t, y) = u(t, y). \quad \text{(1.6)}$$

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In this setting, if we adopt the usual definition for vorticity:

\[ \omega = \nabla \times u, \tag{1.7} \]

it then gives rise to the well-known Biot–Savart law:

\[ u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y) \, dy \tag{1.8} \]

as well as the following evolution equation for the curve \( \gamma \):

\[ \partial_t \gamma(t, x) = -\frac{\Gamma}{4\pi} \int_0^1 \frac{\gamma(t, x) - \gamma(t, y)}{|\gamma(t, x) - \gamma(t, y)|^3} \times \partial_y \gamma(t, y) \, dy. \tag{1.9} \]

See Saffman [14] for discussions on the vortex filament dynamics.

However, the above model for vortex ring evolution has singularities on \( \gamma \), where the magnitude of vorticity is infinite (cf. the formal identity (1.5)). To resolve this issue, in 1930 Rosenhead [13] proposed an approximate model for the above problem. It de-singularises the vorticity by modifying Eq. (1.9) as follows: For some constant \( \mu > 0 \), set

\[ \partial_t \gamma(t, x) = -\frac{\Gamma}{4\pi} \int_0^1 A_{\gamma}(t, x, y) \times \partial_y \gamma(t, y) \, dy \tag{1.10} \]

where

\[ A_{\gamma}(t, x, y) := \nabla \phi \left( \gamma(t, x) - \gamma(t, y) \right) \equiv \frac{\gamma(t, x) - \gamma(t, y)}{\left| \gamma(t, x) - \gamma(t, y) \right|^2 + \mu^2}^{3/2}. \tag{1.11} \]

This amounts to assuming an approximate Biot–Savart law:

\[ u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla \phi(x - y) \times \omega(y) \, dy, \tag{1.12} \]

with the potential \( \phi : \mathbb{R}^3 \to \mathbb{R} \) given by

\[ \phi(z) \equiv \phi_0(z) := \frac{\Gamma}{\sqrt{|z|^2 + \mu^2}}. \tag{1.13} \]

Let us briefly recall some recent contributions to the above model and its related problems. Using the fixed point arguments, Berselli–Bessaih [1] proved the local existence and uniqueness of the solution to Eq. (1.10) in the following space:

\[ \gamma \in W^{1,\infty}(0, T^*; L^2([0, 1]; \mathbb{R}^3)) \cap L^\infty(0, T^*; H^1_{\mu}(0, 1)), \]

where \( H^1_{\mu}(0, 1) := \{ \sigma : [0, 1] \to \mathbb{R}^3 : \sigma(0) = \sigma(1) \} \) and \( T^* > 0 \). In fact, it was proved for more general choices of \( A_{\gamma} \) (see Hypothesis A, p1733 in [1]). Then, Berselli–Gubinelli [3] established the global existence and uniqueness of Eq. (1.10), provided that \( \gamma \) remains a smoothly embedded \( S^1 \) before \( T^* \). The above result in [3] also holds for more general models than the Rosenhead approximation (1.13), provided that several symmetry and integrability conditions on the Fourier transform of \( \phi \) are verified (see Hypothesis A, p698 in [3]). The problem of vortex filament evolution has also been studied by various authors from the probabilistic perspectives, e.g., using ideas and methods from random walks, mean field equations and rough paths; cf. Chorin [5], Lions–Majda [12], Flandoli [10], Bessaih–Gubinelli–Russo [4] and the references cited therein. Moreover, let us remark that the above works are mainly for the Euler equations; nevertheless, the study of the vortex filament evolution in incompressible Navier–Stokes equations is also of central importance; one recent work in this direction is [9] by Enciso, Lucà and Peralta–Salas.
Our primary goal is to analyse the vorticity equation \(|\textbf{1.1}\) together with the approximate Biot–Savart law \(|\textbf{1.12}\), under the natural assumption of \textit{finite total circulation} (signed):

\[
\Sigma := \sup_{t \in [0, T^*]} \int_{\mathbb{R}^3} |\omega(t, x)| \, dx < \infty. \tag{1.14}
\]

In particular, Eq. \(|\textbf{1.12}\) expresses \(u\) in terms of \(\omega\), and the potential \(\phi\) is allowed to take the following more general form than the Rosenhead approximation:

\[
\phi(z) \equiv \phi_\delta(z) := \frac{\Gamma}{\sqrt{|z|^2 + \mu^2 |z|^6}} \quad \text{where } \Gamma, \mu > 0, \delta \in \left[0, \frac{4}{5}\right]. \tag{1.15}
\]

It contains the Rosenhead approximation as the special case \(\delta = 0\).

A vector field \(\omega\) is said to be a \textit{weak solution} to Eq. \(|\textbf{1.1}\) if the equation holds in the sense of distributions. The main contribution of this paper is the following regularity result:

\textbf{Theorem 1.1.} Let \(\omega \in L^\infty(0, T^*; L^1 \cap H^{-1}(\mathbb{R}^3, \mathbb{R}^3)) \cap L^2(0, T^*; L^2(\mathbb{R}^3, \mathbb{R}^3))\) be a weak solution to the vorticity equation \(|\textbf{1.1}\). Let \(u\) be related to \(\omega\) by the approximate Biot–Savart law \(|\textbf{1.12}| \textbf{1.15}\) with \(\phi = \phi_\delta\). Assume that \(\gamma\) supporting \(\omega\) is smooth before time \(T^*\). Then \(\omega\) is a strong solution to Eq. \(|\textbf{1.1}|\) in the sense that \(\omega \in L^\infty(0, T^*; H^1(\mathbb{R}^3, \mathbb{R}^3))\).

The above theorem aims to offer a new perspective for the regularity problem of the vortex filament evolutions. Instead of imposing differentiability conditions on \(\phi\), e.g., on the Fourier transform \(\hat{\phi}\) as in Berselli–Gubinelli \(|\textbf{3}|\), we approach the regularity problem by singular integral estimates and geometric arguments in the spirit of the classical work \(|\textbf{6}|\) by Constantin–Fefferman.

\section{2. Proof of Theorem \textbf{1.1}}

Our proof is based on the energy estimate for Eq. \(|\textbf{1.1}|\). Multiplying \(\omega\) to both sides and integrating over \(\mathbb{R}^3\), we get for any \(t \in [0, T^*]\)

\[
\frac{d}{dt} E(t) + \nu \int_{\mathbb{R}^3} |\nabla \omega(t, x)|^2 \, dx = \int_{\mathbb{R}^3} S(t, x) : \omega(t, x) \otimes \omega(t, x) \, dx. \tag{2.1}
\]

Write \(E(t)\) for the \textit{enstrophy} of the fluid:

\[
E(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\omega(t, x)|^2 \, dx. \tag{2.2}
\]

It thus remains to estimate the right-hand side of \(|\textbf{2.1}|\), known as the \textit{vorticity stretching term}.

The main idea of the proof is adapted from \(|\textbf{6}|\) by Constantin–Fefferman. We represent the rate-of-strain tensor \(S\) by a singular integral of \(\omega\) via the approximate Biot–Savart law \(|\textbf{1.12}|\), and show that the vorticity stretching term can be controlled by the enstrophy \(D\). Theorem \textbf{1.1} thus follows from the Grönwall lemma. To control the term \((S : \omega \otimes \omega)\), recall that in \(|\textbf{6}|\) Constantin–Fefferman proved the following: If the geometric condition

\[
\left| \sin \angle \left( \omega(t, x), \omega(t, y) \right) \mathbb{1}_{\{\omega(t,x) \geq \Lambda, |\omega(t,y)| \geq \Lambda \}} \right| \leq c |x - y|^{\beta}
\]

for \(\beta = 1\) and constants \(c, \Lambda > 0\) independent of \(t\) \(|\textbf{2.3}|\), is satisfied — in physical terms, in regions where the vorticity magnitude is large, the vortex directions are nearly aligned or anti-aligned — then a weak solution is automatically a strong solution; here \(\mathbb{1}_E\) denotes the indicator function of the set \(E\). Beirão da Veiga–Berselli generalised it to \(\beta \in [1/2, 1]\), which is the optimal lower bound up to date; see the note \(|\textbf{11}|\) by the author.
In view of integration by parts and the divergence theorem, it equals

take an arbitrary test function

By the choice of \( \beta \), the right-hand side is controlled by \( C_{\varepsilon}e^{-\frac{3\mu}{2}} \| \chi \|_{L^\infty(\mathbb{R}^3)} \) for \( C \)
depending only on \( \mu, \Gamma \). For \( \delta \in [0, 4/5] \) it tends to zero as \( \varepsilon \to 0^+ \); therefore, Eq. (2.6) is proved.

Next, taking \( \nabla \) in Eq. (2.7), we compute in local coordinates:

\[
\nabla_i \nabla_j \phi(z) = -\frac{\Gamma}{2} \left\{ \frac{2r^2 + \mu^2 |z|^2}{A^{3/2}(|z|)} \right\}
\]

\[
= -\frac{\Gamma}{2} A^{-3}(|z|) \left\{ A^{3/2}(|z|) \left[ 2\delta_{ij} + \delta \mu^2 \delta_{ij} |z|^{\delta-2} + \delta(\delta - 2) \mu^2 |z|^{\delta-4}z^i z^j \right] \right\}
\]

for an \( L^q \) variant, as well as the many references contained therein. Notice that if the range of \( \beta \) can be relaxed to \( \beta \in [0, 1] \), then one may deduce the regularity of the 3D incompressible Navier–Stokes equations. Our method in the current paper, in effect, shows that \( \beta = 0 \) can be achieved for the approximate Biot–Savart law.

As the preliminary step, let us prove:

**Lemma 2.1.** In the setting of Theorem 1.1, the rate-of-strain tensor \( \mathbf{S} \) can be represented as follows:

\[
\mathbf{S}(x) = \text{p.v.} \int_{\mathbb{R}^3} \left\{ \frac{1}{4} \Gamma \delta(2 - \delta) \mu^2 A^{-\frac{\delta}{2}} |x - y| |x - y|^{\delta-4} + \frac{3}{8} \Gamma B^2 |x - y| A^{-\frac{\delta}{2}} |x - y| \right\} \chi(x) \nabla \nabla (x - y) \times \omega(y) \, dy.
\]

(2.4)

Here \( \text{p.v.} \) denotes the principal value integral, the \( t \) variable is suppressed, and \( A, B \) are given by

\[
A(r) := r^2 + \mu^2 r^\delta, \quad B(r) := 2 + \delta \mu^2 r^{\delta-2} \quad \text{for } r > 0.
\]

(2.5)

**Proof.** First, let us show that

\[
\nabla u(x) = \text{p.v.} \int_{\mathbb{R}^3} \nabla \nabla (x - y) \times \omega(y) \, dy,
\]

where the \( 3 \times 3 \) tensor

\[
(\nabla \nabla \times \omega)_i^j := \nabla_i (\nabla \omega \times \omega)_j
\]

for \( i, j \in \{1, 2, 3\} \). Indeed, as

\[
\nabla \omega(z) = -\frac{\Gamma}{2} \nabla \left( |z|^2 + \mu^2 |z|^{\delta} \right) \left( |z|^2 + \mu^2 |z|^{\delta} \right)^{-3/2}
\]

\[
= -\frac{\Gamma}{2} \left( 2z + \mu^2 z^{\delta-2} \right) \left( |z|^2 + \mu^2 |z|^{\delta} \right)^{-3/2}
\]

(2.7)

is locally integrable on \( \mathbb{R}^3 \), the weak Hessian \( \nabla \nabla \phi \) can be computed by the dominated convergence theorem as follows: Take an arbitrary test function \( \chi \in C_c^\infty(\mathbb{R}^3) \); then

\[
-\langle \chi, \nabla \nabla \phi \rangle = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \nabla \phi(x) \otimes \nabla \chi(x) \, dx.
\]

In view of integration by parts and the divergence theorem, it equals

\[
\lim_{\varepsilon \to 0^+} \left\{ - \int_{|x| \geq \varepsilon} \nabla \nabla \phi(x) \chi(x) \, dx + \int_{|x|=\varepsilon} \chi(x) \nabla \phi(x) \otimes \frac{x}{|x|} \, d\mathcal{H}^2(x) \right\}.
\]

Rescaling the second term, we get

\[
\int_{|x|=\varepsilon} \chi(x) \nabla \phi(x) \otimes \frac{x}{|x|} \, d\mathcal{H}^2(x) = \varepsilon^2 \int_{|x|=1} \chi(\varepsilon x) \nabla \phi(\varepsilon x) \otimes x \, d\mathcal{H}^2(x).
\]

By the choice of \( \phi \) in Eq. (1.15), the right-hand side is controlled by \( C_{\varepsilon}e^{-\frac{3\mu}{2}} \| \chi \|_{L^\infty(\mathbb{R}^3)} \) for \( C \)
depending only on \( \mu, \Gamma \). For \( \delta \in [0, 4/5] \) it tends to zero as \( \varepsilon \to 0^+ \); therefore, Eq. (2.6) is proved.
Thanks to $\nabla A(|z|) = z t B(|z|)$, we have

$$\nabla \nabla \phi(z) = -\frac{\Gamma}{2} A^{-3/2}(|z|) \left[ \kappa \delta i j + \delta (\delta - 2) \mu^2 |z|^{\delta - 4} z \otimes z \right] + \frac{3\Gamma}{4} A^{-5/2}(|z|) B^2(|z|) z \otimes z.$$  

It implies that

$$\nabla \nabla (x - y) \times \omega(y) = \sigma + \left[ -\frac{\Gamma}{2} \delta (\delta - 2) \mu^2 A^{-3/2}(|x - y|)|x - y|^{\delta - 4}
+ \frac{3\Gamma}{4} A^{-5/2}(|x - y|) B^2(|x - y|) \right] \left\{ (x - y) \times \omega(y) \otimes (x - y) \right\}, \quad (2.8)$$

where $\sigma$ is an anti-symmetric $3 \times 3$ matrix. Therefore, taking the symmetrisation of $\nabla u$ and using Eqs. (2.6) and (2.8), we conclude the proof.

With the help of Lemma 2.1, the right-hand side of Eq. (2.1) can now be readily estimated. Throughout the paper we adopt the notation

$$\hat{z} := \frac{z}{|z|} \quad \text{for any } z \in \mathbb{R}^3.$$  

Thus, the vorticity stretching term equals

$$\int_{\mathbb{R}^3} S(x) : \omega(x) \otimes \omega(x) \, dx $$

$$= \int_{\mathbb{R}^3} |\omega(x)|^2 \left\{ \text{p.v.} \int_{\mathbb{R}^3} K(|x - y|)|\omega(y)| \left[ \frac{x - y \times \hat{\omega}(y) \otimes x - y}{x - y \otimes x - y \times \hat{\omega}(y)} \right] + \frac{\hat{\omega}(x) \otimes \hat{\omega}(x)}{\hat{\omega}(x) \times \hat{\omega}(x)} \right\} \, dy \right\} \, dx, \quad (2.9)$$

where

$$K(|z|) = \frac{1}{4} \Gamma \delta (2 - \delta) \mu^2 |z|^{\delta - 2} A^{-3/2}(|z|) + \frac{3}{8} \Gamma |z|^{2} B^2(|z|) A^{-5/2}(|z|). \quad (2.10)$$

Furthermore, we shall utilise a crucial observation by Constantin–Fefferman [6]:

$$\left[ \frac{x - y \times \hat{\omega}(y) \otimes x - y}{x - y \otimes x - y \times \hat{\omega}(y)} \right] + \left[ \hat{\omega}(x) \otimes \hat{\omega}(x) \right] = \mathcal{D} \left( x - y, \hat{\omega}(x), \hat{\omega}(y) \right),$$

where

$$\mathcal{D}(e_1, e_2, e_3) = e_1 \cdot e_3 \det(e_1, e_2, e_3)$$

for arbitrary unit (column) vectors $e_1, e_2, e_3$. This geometric quantity is controlled by the angle between the vorticity directions at $x$ and $y$:

$$\left| \mathcal{D} \left( x - y, \hat{\omega}(x), \hat{\omega}(y) \right) \right| \leq \left| \sin \left( \omega(t, x), \omega(t, y) \right) \right|. \quad (2.11)$$

The following bound for $K$ may be verified by a tedious yet straightforward computation, for which we only need the Cauchy–Schwarz inequality:

$$K(|z|) \leq \frac{1}{4} \Gamma \delta (2 - \delta) \mu^2 \left( |z|^{10 - 2\delta} + \mu^2 |z|^{4 + \delta} \right)^{-3/2} + 3\Gamma \left( |z|^{6/5} + \mu^2 |z|^{\delta - 4/5} \right)^{-5/2}$$

for arbitrary unit (column) vectors $e_1, e_2, e_3$. This geometric quantity is controlled by the angle between the vorticity directions at $x$ and $y$:

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\[ + \frac{3}{4} \Gamma \delta^2 \mu^4 \left( |z|^{\frac{4+4}{2}} + \mu^2 |z|^{\frac{4+4}{2}} \right)^{-5/2}. \] 

Then, let us consider separately the cases \(|x - y| = |z| \geq \eta\) and \(|z| < \eta\), where \(\eta > 0\) is some positive constant to be specified. Since \((c_1 + c_2)^\alpha \leq c_1^\alpha + c_2^\alpha\) for \(c_1, c_2 \geq 0\) and \(\alpha < -1\), we have

\[
K(|z|) \leq \frac{1}{4} \Gamma \delta (2 - \delta) \mu^2 \eta^{-2+\delta} + \left( \frac{1}{2} \Gamma \delta (1 - \delta) \right) \mu^{-1} \eta^{-2-\frac{\delta}{2}}
+ 3 \Gamma \eta^{-3} + \frac{3}{4} \Gamma \delta^2 \mu^4 \eta^{2\delta-7}
\]

\[=: \kappa_1(\Gamma, \delta, \mu, \eta) \quad \text{whenever } |z| \geq \eta. \] 

On the other hand, if we choose

\[
\eta \equiv \eta(\mu, \delta) \geq \max \left\{ \mu^{-\frac{6}{\tau + \tau}}, \mu^{-\frac{10}{\tau + \tau}} \right\},
\]

then \(\mu^2 \eta^2 \geq 1\) and \(\mu^2 \eta^{4+4+\delta} \geq 1\), thus we have

\[
K(|z|) \leq \frac{1}{4} \Gamma \delta (2 - \delta) \mu^2 + \frac{3}{4} \Gamma \delta^2 \mu^4 + 3 \Gamma \mu^{-5} \eta^{2-\frac{5\delta}{2}}
\]

\[=: \kappa_2(\Gamma, \delta, \mu, \eta) \quad \text{whenever } |z| \leq \eta. \] 

To conclude the proof, we deduce from Eqs. \((2.19), (2.11)\) and the above bounds for \(K\) the following estimate of the vorticity stretching term:

\[
\left| \int_{\mathbb{R}^3} S(t, x) : \omega(t, x) \otimes \omega(t, x) \, dx \right| \leq \max \{ \kappa_1, \kappa_2 \} \Sigma E(t). \] 

In this above we make use of the naïve bound \(|\sin \angle(\omega(t, x), \omega(t, y))| \leq 1\), which amounts to taking \(\beta = 0\) in Constantin and Fefferman’s geometric condition \((2.3)\). Here \(\kappa_1, \kappa_2\) are constants depending on \(\delta, \mu, \Gamma\) and \(\eta\) (cf. Eqs. \((2.13), (2.15)\)), and \(\eta\) is chosen according to Eq. \((2.14)\). The constant \(\Sigma\) is the finite total signed circulation. The energy estimate \((2.11)\) now becomes

\[
\frac{d}{dt} E(t) + \nu \int_{\mathbb{R}^3} |\nabla \omega(t, x)|^2 \, dx \leq \max \{ \kappa_1, \kappa_2 \} \Sigma E(t) \quad \text{for all } t \in [0, T^*].
\]

An application of the Grönewall’s inequality gives us

\[
\sup_{t \in [0, T^*]} E(t) \leq E(0) e^{\max \{ \kappa_1, \kappa_2 \} \Sigma};
\]

\[
\sup_{t \in [0, T^*]} \int_{\mathbb{R}^3} |\nabla \omega(t, x)|^2 \, dx \leq \nu^{-1} \max \{ \kappa_1, \kappa_2 \} \Sigma E(0) e^{\max \{ \kappa_1, \kappa_2 \} \Sigma}. \]

Therefore, \(\omega \in L^\infty(0, T; H^1(\mathbb{R}^3, \mathbb{R}^3))\) is a strong solution to the vorticity equation \((1.1)\). The proof of Theorem \((1.1)\) is now complete.

We conclude the paper by three remarks:

**Remark 2.2.** In view of the definition of \(\kappa_1, \kappa_2\) in Eqs. \((2.13), (2.15)\), Theorem \((7.7)\) clearly fails in the singular limit \(\mu \to 0^+\). In particular, the choice for \(\eta\) is invalid in Eq. \((2.14)\). Moreover, in the special case \(\delta = 0\) (i.e., the Rosenhead approximation in \((13)\)), the method in our paper shows that

\[
\left| \int_{\mathbb{R}^3} S(t, x) : \omega(t, x) \otimes \omega(t, x) \, dx \right| \leq \frac{3}{4} \max \{ \eta^{-3}, \eta^2 \mu^{-5} \} \Sigma E(t) \quad \text{for any } \eta > 0.
\]

Thus, the best we can get from the preceding proof is \(\|\omega\|_{L^\infty_t H^1_x} \lesssim O(\mu^{-3})\).
Remark 2.3. Taylor expansion suggests that Theorem 1.1 remains valid for potentials of the slightly more general form

\[ \phi(z) = \frac{\Gamma}{\sqrt{|z|^2 + p(z)}}, \]

(2.19)

where \( p(z) \) is a perturbation such that \( |p(z)| \lesssim O(|z|^\delta) \) as \(|z| \to 0\) for sufficiently small \( \delta \geq 0 \), and that \( p(z) \neq 0 \) on \( \mathbb{R}^3 \setminus \{0\} \). Moreover, our choice of the potentials \( \phi = \phi_\delta \) satisfies weaker conditions than those considered in [1, 3].

Remark 2.4. The optimal range for \( \delta \) in Theorem 1.1 and Remark 2.3, say \([0, \delta^*]\), is unknown. By now we only know that \( \delta^* \in [\frac{4}{5}, 2] \).

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