| Title       | Mackey-functor structure on the Brauer groups of a finite Galois covering of schemes |
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| Citation    | 代数幾何学シンポジウム記録 2007: 91-110                                               |
| Issue Date  | 2007                                                                               |
| URL         | http://hdl.handle.net/2433/214843                                                   |
| Type        | Departmental Bulletin Paper                                                       |
| Textversion | publisher                                                                        |

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MACKEY-FUNCTOR STRUCTURE ON THE BRAUER GROUPS
OF A FINITE GALOIS COVERING OF SCHEMES

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Abstract. For any finite étale covering of schemes, we can associate two
homomorphisms for Brauer groups, namely the pull-back and the norm map.
These homomorphisms make Brauer groups into a bivariant functor (a Mackey
functor). Restricting to a finite Galois covering of schemes, we obtain a coho-
mological Mackey functor on its Galois group. This is a generalization of the
result for rings by Ford [5]. Applying Bley and Boltje’s theorem [1], we can
derive certain isomorphisms for the Brauer groups of intermediate coverings.

1. Introduction

In this paper, a scheme $S$ is always assumed to be Noetherian, and $\pi(S)$ denotes
its étale fundamental group. Since we use Čech cohomology, we assume $S$ satisfies
the following:

Assumption 1.1. For any finite set $E$ of points of $S$, there exists an open set $U \subset S$,
such that $U$ contains every point in $E$.

As for the étale fundamental group and related notion, we follow the terminology
in [9]. For example a finite étale covering is just a finite étale morphism of schemes.

Our aim is to make the following generalization of the result for rings by Ford
[5].

Corollary (Corollary 4.2). Let $\pi: Y \to X$ be a finite Galois covering of schemes
with Galois group $G$. Then the correspondence

$$H \leq G \mapsto \text{Br}(Y/H)$$

forms a cohomological Mackey functor on $G$.

This follows from our main theorem;

Theorem (Theorem 3.5). Let $S$ be a connected Noetherian scheme. Let $(\text{Fet}/S)$
denote the category of finite étale coverings over $S$. Then, the Brauer group functor
$\text{Br}$ forms a cohomological Mackey functor on $(\text{Fet}/S)$.

As in Definition 3.1, a Mackey functor is a bivariant pair of functors $\text{Br} = (\text{Br}^*, \text{Br}_*)$. For any morphism $\pi: Y \to X$, the contravariant part $\text{Br}^*(\pi): \text{Br}(X) \to \text{Br}(Y)$ is the pull-back, and $\text{Br}_*(\pi): \text{Br}(Y) \to \text{Br}(X)$ is the norm map defined later.

By applying Bley and Boltje’s theorem (Fact 5.2) to Corollary 4.2, we can obtain
certain relations between Brauer groups of intermediate coverings:

The author wishes to thank Professor Kazuhiro Fujiwara for his useful comments.
Corollary (Corollary 5.3). Let \( X \) be a connected Noetherian scheme and \( \pi : Y \to X \) be a finite Galois covering with \( \text{Gal}(Y/X) = G \).

(i) Let \( \ell \) be a prime number. If \( H \leq G \) is not \( \ell \)-hypoelementary, then there is a natural isomorphism of \( \mathbb{Z}_\ell \)-modules

\[
\bigoplus_{U = H_0 < \cdots < H_n = H, \text{odd}} \text{Br}(Y/U)(\ell)^{[U]} \cong \bigoplus_{U = H_0 < \cdots < H_n = H, \text{even}} \text{Br}(Y/U)(\ell)^{[U]},
\]

(ii) If \( H \leq G \) is not hypoelementary, then there is a natural isomorphism of abelian groups

\[
\bigoplus_{U = H_0 < \cdots < H_n = H, \text{odd}} \text{Br}(Y/U)^{[U]} \cong \bigoplus_{U = H_0 < \cdots < H_n = H, \text{even}} \text{Br}(Y/U)^{[U]}.
\]

2. Restriction and corestriction

Remark 2.1. For any scheme \( X \), there exists a natural monomorphism

\[
\chi_X : \text{Br}(X) \hookrightarrow \text{Br}(X) = H^2_{\text{et}}(X, \mathbb{G}_m, X)_{\text{tor}},
\]

such that for any morphism \( \pi : Y \to X \),

\[
\begin{array}{ccc}
\text{Br}(X) & \xrightarrow{\pi^*} & \text{Br}(Y) \\
\downarrow{\chi_X} & & \downarrow{\chi_Y} \\
H^2_{\text{et}}(X, \mathbb{G}_m, X) & \xrightarrow{\pi^*} & H^2_{\text{et}}(Y, \mathbb{G}_m, Y)
\end{array}
\]

is a commutative diagram.

Here \( \pi^* : \text{Br}(X) \to \text{Br}(Y) \) is the pull-back of Azumaya algebras, while \( \pi^* : H^2_{\text{et}}(X, \mathbb{G}_m, X) \to H^2_{\text{et}}(Y, \mathbb{G}_m, Y) \) is defined as the composition of the canonical morphism

\[
H^2_{\text{et}}(X, \pi_* \mathbb{G}_m, Y) \to H^2_{\text{et}}(Y, \mathbb{G}_m, Y)
\]

and

\[
H^2_{\text{et}}(\pi_\sharp) : H^2_{\text{et}}(X, \mathbb{G}_m, X) \to H^2_{\text{et}}(X, \pi_* \mathbb{G}_m, Y),
\]

where \( \pi_\sharp : \mathbb{G}_m, X \to \pi_* \mathbb{G}_m, Y \) is the canonical (structural) homomorphism of étale sheaves on \( X \). We call these \( \pi^* \) the restriction maps.

Remark 2.2. For any finite étale covering \( \pi : Y \to X \), there exists a homomorphism of étale sheaves on \( X \)

\[
N_{Y/X} : \pi_* \mathbb{G}_m, Y \to \mathbb{G}_m, X
\]

which induces the norm map for finite étale ring extensions.

When \( \pi : Y \to X \) is a finite étale covering, the canonical homomorphism

\[
H^2_{\text{et}}(X, \pi_* \mathbb{G}_m, Y) \to H^2_{\text{et}}(Y, \mathbb{G}_m, Y)
\]

becomes isomorphic (cf. [6]). By composing \( H^2_{\text{et}}(N_{Y/X}) \) with the inverse of this canonical isomorphism, we define the corestriction map for cohomology groups:

\[
\text{cor}_\pi : H^2_{\text{et}}(Y, \mathbb{G}_m, Y) \xrightarrow{\cong} H^2_{\text{et}}(X, \pi_* \mathbb{G}_m, Y) \xrightarrow{H^2_{\text{et}}(N_{Y/X})} H^2_{\text{et}}(X, \mathbb{G}_m, X).
\]
Proposition 2.3. Let $\pi : Y \to X$ as before. There exists a corestriction homomorphism for Brauer groups

$$\text{cor}_\pi : \text{Br}(Y) \to \text{Br}(X),$$

such that

$$\xymatrix{ \text{Br}(Y) \ar[r]^-{\text{cor}_\pi} \ar[d]_{\chi_Y} & \text{Br}(X) \ar[d]^{\chi_X} \\ H^2_{\text{et}}(Y, \mathbb{G}_m,Y) \ar[r]^-{\text{cor}_\pi} & H^2_{\text{et}}(X, \mathbb{G}_m,X) }$$

is commutative.

To construct $\text{cor} : \text{Br}(Y) \to \text{Br}(X)$, we define a monoidal functor

$$\mathcal{N}_{Y/X} : q\text{-Coh}(Y) \to q\text{-Coh}(X).$$

Lemma 2.4. Let $\pi : Y \to X$ be a finite étale covering of constant degree $d$. There exists a monoidal functor (unique up to a natural isomorphism)

$$\mathcal{N}_\pi = \mathcal{N}_{Y/X} : q\text{-Coh}(Y) \to q\text{-Coh}(X),$$

$(q\text{-Coh}(X) : \text{the category of quasi-coherent Zariski sheaves on } X).$ such that

(i) When $Y$ is isomorphic to a disjoint union of $d$-copies of $X$, i.e. when

$$Y = \bigsqcup_{1 \leq i \leq d} Y_i$$

and $\exists \eta_i : X \xrightarrow{\sim} Y_i$, then $\mathcal{N}_{Y/X}$ is defined by

$$\mathcal{N}_{Y/X}(F) := \eta_i^* (F|_{Y_i}) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \eta_d^* (F|_{Y_d}) \quad (\forall F \in q\text{-Coh}(Y)).$$

(ii) For any pull-back by a morphism $f : X' \to X$

$$\xymatrix{ Y' \ar[r]^-{\pi'} \ar[d]_g & X' \ar[d]^f \\ Y \ar[r]_-{\pi} & X }$$

there exists a natural isomorphism of monoidal functors

$$\mathcal{N}_{Y'/X'} \circ g^* \cong f^* \circ \mathcal{N}_{Y/X}.$$

Proof. When $Y$ is isomorphic to a disjoint union of $d$-copies of $X$, then $\mathcal{N}_{Y/X}$ is defined by as in (i).

For a general case, remark that

Remark 2.5. For any finite étale covering $\pi : Y \to X$ of constant degree $d$, there exists a fpqc morphism $f : X' \to X$ such that $Y \times_X X'$ is isomorphic to a disjoint union of $d$-copies of $X'$.

For any $F \in q\text{-Coh}(Y)$, put $\mathcal{F} := \mathcal{N}_{Y'/X'}(g^*(F))$. Then $\mathcal{F}$ descends to yield $\mathcal{N}_{Y/X}(F) \in q\text{-Coh}(X)$. Thus we obtain a monoidal functor $\mathcal{N}_{Y/X}$. This construction does not depend on the choice of $f$, up to an isomorphism of monoidal functors. By the reduction to the disjoint-union case as above, we can show (ii).
While this $N_{Y/X}$ is a generalization of the norm functor for a finite étale ring extension (Knus-Ojanguren [8], Ferrand [4]), it is also possible to define $N_{Y/X}$ by gluing those for affines.

**Lemma 2.6.** Let $\pi : Y \to X$ be a finite étale covering of constant degree $d$. $N_{Y/X}$ has the following properties:

(0) $N_{Y/X}$ is monoidal.

(1) For any $F, G \in \text{q-Coh}(Y)$, there exists a functorial morphism

$$\theta_{Y/X} : N_{Y/X}(\text{Hom}_{O_Y}(F, G)) \to \text{Hom}_{O_X}(N_{Y/X}(F), N_{Y/X}(G)).$$

(1') Moreover if $G$ is locally free of finite rank, this is an isomorphism.

(2) There exists a natural isomorphism

$$N_{Y/X}(O_Y^\oplus n) \cong O_X^\oplus dn.$$  

(2') More generally, if $F$ is locally free $O_Y$-module of finite rank $n$, then $N_{Y/X}(F)$ becomes locally free $O_X$-module of rank $nd$.

For a general (non-constant degree) $\pi : Y \to X$, we can define the norm functor on each connected component of $X$ as above, and glue them to obtain the norm functor $N_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X)$.

**Proof.** Conditions (0) and (2) follow from the definition of $N_{Y/X}$. By taking an affine cover $X = \bigcup_{i \in I} U_i$, (2') reduces to the case where $X, Y$ are affine, shown by Ferrand [4]. As for condition (1), existence of $\theta_{Y/X}$ simply follows from the fact that $N_{Y/X}$ is a monoidal functor between closed symmetric monoidal categories. (1') is shown by a reduction to the affine case. \[ \square \]

**Proof.** (proof of Proposition) By the above lemma, especially we have an isomorphism

$$N_{Y/X}(M_n(O_Y)) \cong M_{nd}(O_X)$$

of $O_X$-algebras, for any finite étale covering $Y/X$ of constant degree $d$.

Remark that for any $O_Y$-algebra $A$ of finite type, $A$ is an Azumaya algebra if and only if $A$ is étale locally isomorphic to $M_n(O_Y)$. Thus for any Azumaya algebra $A$, there exists a covering $V := \{ V_i \xrightarrow{g_i} Y \}_{i \in I}$ of $Y$ in the étale topology (simply written ‘$V \in \text{Cov}_{\text{et}}(Y)$’) such that

$$g_i^* A \cong M_{n_i}(O_{V_i}) \quad (\exists n_i \in \mathbb{N}).$$

Replacing $V$ by its refinement, we may assume that there exists a covering $U = \{ U_i \xrightarrow{f_i} X \}_{i \in I} \subset \text{Cov}_{\text{et}}(X)$ such that

$$V = \pi^* U := \{ Y \times_X U_i \xrightarrow{pr_Y} Y \}_{i \in I}.$$ 

Then we have $f_i^* N_{Y/X}(A) \cong N_{Y_i/U_i}(g_i^* A) \cong M_{n_i^2}(O_{U_i})$. Thus $N_{Y/X}(A)$ also becomes an Azumaya algebra.

By the isomorphism

$$N_{Y/X}(\text{End}(\mathcal{E})) \cong \text{End}(N_{Y/X}(\mathcal{E})) \quad (\forall \mathcal{E} : \text{locally free of finite rank})$$

and the monoidality of $N_{Y/X}$, we obtain a well-defined homomorphism

$$(\text{cor}_\pi) : \text{Br}(Y) \xrightarrow{\Psi} \text{Br}(X) \xrightarrow{\Psi} N_{Y/X}(A).$$
By using Čech cohomology, we can show the commutativity of

\[
\begin{array}{ccc}
Br(Y) & \xrightarrow{\text{cor}_\pi} & Br(X) \\
\downarrow_{\chi_Y} & \circ & \downarrow_{\chi_X} \\
H^2_{\text{et}}(Y, \mathbb{G}_m, Y) & \xrightarrow{\text{cor}_\pi} & H^2_{\text{et}}(X, \mathbb{G}_m, X).
\end{array}
\]

\[\square\]

3. BRAUER-GROUP MACKEY FUNCTOR

For any profinite group \( G \), let (fin. \( G \)-space) denote the category of finite discrete \( G \)-spaces and equivariant \( G \)-maps.

**Definition 3.1.** Let \( \mathcal{C} \) be a Galois category, with fundamental functor \( F \) (i.e. there exists a profinite group \( \pi(\mathcal{C}) \) such that \( F \) gives an equivalence from \( \mathcal{C} \) to (fin. \( \pi(\mathcal{C}) \)-space)).

A cohomological Mackey functor on \( \mathcal{C} \) is a pair of functors \( M = (M^*, M_* ) \) from \( \mathcal{C} \) to Ab, where \( M^* \) is contravariant and \( M_* \) is covariant, satisfying the following conditions:

1. (Additivity) For each coproduct \( X \sqcup Y \rightarrow X \sqcup Y \sqcup Z \) in \( \mathcal{C} \), the canonical morphism

\[
(M^*(i_X), M^*(i_Y)) : M(X \sqcup Y) \rightarrow M(X) \oplus M(Y)
\]

is an isomorphism.

2. (Mackey condition) For any pull-back diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{\pi'} & Y \\
\downarrow_{\pi} & \circ & \downarrow_{\pi'} \\
X' & \xrightarrow{\mathcal{C}} & X,
\end{array}
\]

the following diagram is commutative:

\[
\begin{array}{ccc}
M(Y) & \xrightarrow{M^*(\pi')} & M(Y') \\
\downarrow_{M_*(\pi)} & \circ & \downarrow_{M_*(\pi')} \\
M(X) & \xrightarrow{M^*(\pi)} & M(X').
\end{array}
\]

3. (Cohomological condition) For any morphism \( \pi : X \rightarrow Y \) in \( \mathcal{C} \) with \( X \) and \( Y \) connected, we have

\[
M_*(\pi) \circ M^*(\pi) = \text{multiplication by } \deg(\pi)
\]

where \( \deg(\pi) := \sharp F(Y)/\sharp F(X) \).
A standard example is the cohomological Mackey functor on a profinite group $G$ (in terminology of [1], a cohomological Mackey functor on the finite natural Mackey system on $G$):

**Definition 3.2.** Let $G$ be a profinite group, and put $C := \text{fin. } G$-space), $F := \text{id.}$ A cohomological Mackey functor on $C$ is simply called a cohomological Mackey functor on $G$, and their category is denoted by $\text{Mack}_c(G)$.

**Remark 3.3.** Since any object $X$ in (fin. $G$-space) is a direct sum of transitive $G$-sets of the form $G/H$ where $H$ is an open subgroup of $G$, a Mackey functor on $G$ is equivalent to the following datum:

- An abelian group $M(H)$ for each open $H \leq G$, with structure maps
  - a homomorphism $\text{res}_H^K : M(H) \to M(K)$ for each open $K \leq H \leq G$,
  - a homomorphism $\text{cor}_H^K : M(K) \to M(H)$ for each open $K \leq H \leq G$,
  - a homomorphism $c_{g,H} : M(H) \to M(gH)$ for each open $H \leq G$ and $g \in G$, where $gH := gHg^{-1}$, satisfying certain compatibilities (cf. [1]). Here $M(G/H)$ is abbreviated to $M(H)$ for any open $H \leq G$.

**Example 3.4.** In this notation, for any $G$-module $A$ and any $n \geq 0$, the group cohomology

$$H \mapsto H^n(H, A) \ (\forall H \leq G \text{ open})$$

becomes a cohomological Mackey functor on $G$, with appropriate structure maps.

For any finite étale covering $\pi : Y \to X$, put $\text{Br}^*(\pi) := \text{res}_\pi$ and $\text{Br}_*(\pi) := \text{cor}_\pi$. Then we obtain a cohomological Mackey functor $\text{Br}$ (and similarly $\text{Br}', H^2_{\text{et}}(-, G_m)$):

**Theorem 3.5.** For any connected Noetherian scheme $S$, we have a sequence of cohomological Mackey functors on $(\text{FEt}/S)$

$$\text{Br} \leftrightarrow \text{Br}' \leftrightarrow H^2_{\text{et}}(-, G_m).$$

**Proof.** We only show Mackey and cohomological conditions. Since restrictions and corestrictions are compatible with inclusions

$$\text{Br}(X) \leftrightarrow \text{Br}'(X) \leftrightarrow H^2_{\text{et}}(X, G_m, X),$$

it suffices to show for $H^2_{\text{et}}(-, G_m)$.

(Mackey condition) For any pull-back diagram

$$X \xleftarrow{\varpi} Y \xrightarrow{\pi} Y'$$

in $(\text{FEt}/S)$, we have a commutative diagram

$$
\begin{array}{ccc}
\pi_*G_m, Y & \xrightarrow{\pi_*\varpi'_{|Y'}} & \pi_*G_m, Y' \\
N_{Y'/X} & \circlearrowright & \pi_*\varpi'_{|Y'/X'} \\
\varpi & \mapsto & \varpi_{|Y'/X'}
\end{array}
$$

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This yields a commutative diagram

\[
\begin{array}{ccc}
H^2_{\text{et}}(Y, G_{m,Y}) & \xrightarrow{\text{res}_{\pi'}} & H^2_{\text{et}}(Y', G_{m,Y'}) \\
\text{cor}_{\pi} & \circlearrowright & \text{cor}_{\pi'} \\
H^2_{\text{et}}(X, G_{m,X}) & \xrightarrow{\text{res}_{\pi}} & H^2_{\text{et}}(X', G_{m,X'})
\end{array}
\]

(Cohomological condition) For any morphism \( \pi : Y \to X \) in \( \text{FEt}/S \) with \( X \) and \( Y \) connected, since

\[ N_{Y/X} \circ \pi_2 : G_{m,X} \to G_{m,X} \]

is equal to the multiplication by \( d = \deg(\pi) \)

we obtain \( \text{cor}_{\pi} \circ \text{res}_{\pi} = d \).

\[
\begin{array}{ccc}
& \xrightarrow{\text{res}_{\pi}} & \xrightarrow{\pi_2^*} \\
H^2_{\text{et}}(Y, G_{m,Y}) & \xrightarrow{\pi_2} & H^2_{\text{et}}(Y', G_{m,Y'}) \\
\downarrow \circlearrowright & \text{can.} & \downarrow \circlearrowright \\
H^2_{\text{et}}(X, G_{m,X}) & \xrightarrow{\text{res}_{\pi}} & H^2_{\text{et}}(X', G_{m,X'})
\end{array}
\]

4. Restriction to a finite Galois covering

Thus we have obtained a Mackey functor \( \text{Br} \) on \( \text{FEt}/S \). By pulling back by a quasi-inverse \( S \) of the fundamental functor

\[ F : (\text{FEt}/S) \xrightarrow{\sim} (\text{fin. } \pi(S)-\text{space}), \]

we can obtain a Mackey functor on \( \pi(S) \):

**Corollary 4.1.** There is a sequence of cohomological Mackey functors

\[ \text{Br} \circ S \hookrightarrow \text{Br}' \circ S \hookrightarrow H^2_{\text{et}}(-, G_{m}) \circ S \]

on \( \pi(S) \), where \( \text{Br} \circ S := (\text{Br}^* \circ S, \text{Br}_* \circ S) \) and so on.

**Corollary 4.2.** Let \( X \) be a connected Noetherian scheme. For any finite Galois covering \( \pi : Y \to X \) with \( \text{Gal}(Y/X) = G \), there exists a cohomological Mackey functor \( \text{Br} \) on \( G \) which satisfies

\[ \text{Br}(H) \cong \text{Br}(Y/H) \quad (\forall H \leq G), \]

with structure maps induced from restrictions and corestrictions of Brauer groups. (We abbreviate \( \text{Br}(G/H) \) to \( \text{Br}(H) \).)
Proof. By the previous corollary, we have a cohomological Mackey functor $\text{Br} \circ S$ on $\pi(X)$. Since there is a projection $\text{pr} : \pi(X) \rightarrow G^\text{op}$, we can regard any finite $G^\text{op}$-set naturally as a finite $\pi(X)$-space, to obtain a functor

$$\text{fin. } G^\text{op}\text{-space} \rightarrow \text{fin. } \pi(X)\text{-space}.$$ 

Pulling back by this functor, and taking the opposite Mackey functor, we obtain

$$\text{Mack}_c(\pi(X)) \rightarrow \text{Mack}_c(G) \rightarrow \text{Mack}_c(G^\text{op}) \rightarrow \text{Mack}_c(G).$$

In terms of subgroups of $G$, $M_G$ satisfies

$$M_G(H) = M(\text{pr}^{-1}(H^\text{op})) \quad (\forall H \leq G).$$

Applying this to $\text{Br} \circ S$, we obtain $B_r := (\text{Br} \circ S)_G \in \text{Mack}_c(G)$. Since the equivalence $S : (\text{fin. } \pi(X)\text{-space}) \rightarrow (\text{FEt}/X)$ satisfies

$$S(\pi(X)/\text{pr}^{-1}(H^\text{op})) \cong Y/H,$$

we have

$$B_r(H) \cong B_r(Y/H).$$

Similarly we can define $B_r'$ (and also $(H_2^\text{et}(-, \mathbb{G}_m) \circ S)_G$). Since $\text{Mack}_c(G)$ is an abelian category with objectwise (co-)kernels (see for example [3]), we can take the quotient Mackey functor $B_r'/B_r \in \text{Mack}_c(G)$, which satisfies

$$(B_r'/B_r)(H) \cong (B_r'(Y/H))/(B_r(Y/H)).$$

5. Application of Bley and Boltje’s theorem

Let $\ell$ be a prime number. For any abelian group $A$, let

$$A(\ell) := \{ m \in A \mid \exists e \in \mathbb{N}_{\geq 0}, \ell^e m = 0 \}$$

be the $\ell$-primary part. This is a $\mathbb{Z}_\ell$-module.

Definition 5.1 ([1]). For any finite group $H$, $H$ is $\ell$-hypoelementary $\iff$ $H$ has a normal $\ell$-subgroup with a cyclic quotient.

$H$ is hypoelementary $\iff$ $H$ is $\ell$-hypoelementary for some prime $\ell$.

Fact 5.2 ([1]). Let $M$ be a cohomological Mackey functor on a finite group $G$.

(i) Let $\ell$ be a prime number. If $H \leq G$ is not $\ell$-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\bigoplus_{U=H_0 \prec \cdots \prec H_n=H} \bigoplus_{n \text{ even}} M(U)(\ell)^{[U]} \cong \bigoplus_{U=H_0 \prec \cdots \prec H_n=H} M(U)(\ell)^{[U]}.$$

(ii) If $H \leq G$ is not hypoelementary and $M(U)$ is torsion for any subgroup $U \leq H$, then there is a natural isomorphism of abelian groups

$$\bigoplus_{U=H_0 \prec \cdots \prec H_n=H} M(U)^{[U]} \cong \bigoplus_{U=H_0 \prec \cdots \prec H_n=H} M(U)^{[U]}.$$

Here, $|U|$ denotes the order of $U$. 

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Applying this theorem to $\text{Br}$, we obtain the following relations for the Brauer groups of intermediate étale coverings:

**Corollary 5.3.** Let $X$ be a connected Noetherian scheme and $\pi : Y \to X$ be a finite Galois covering with $\text{Gal}(Y/X) = G$.

(i) Let $\ell$ be a prime number. If $H \leq G$ is not $\ell$-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)(\ell)^{|U|} \cong \bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)(\ell)^{|U|}.$$ 

(ii) If $H \leq G$ is not hypoelementary, then there is a natural isomorphism of abelian groups

$$\bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)^{|U|} \cong \bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)^{|U|}.$$ 

Finally, we derive some numerical equations related to Brauer groups from Corollary 5.3.

**Definition 5.4.** Let $G$ be a finite group. For any subgroups $U \leq H \leq G$, put

$$\mu(U, H) := \sum_{U = H_0 < \cdots < H_n = H} (-1)^n, \quad \text{Möbius function.}$$

If $m$ (resp. $m_\ell$) is an additive invariant of abelian groups (resp. $\mathbb{Z}_\ell$-modules) which is finite on Brauer groups, we obtain the following equations:

**Corollary 5.5.** Let $\pi : Y \to X$ as before, $G = \text{Gal}(Y/X)$.

(i) If $H \leq G$ is not $\ell$-hypoelementary,

$$\sum_{U \leq H} |U| \cdot \mu(U, H) \cdot m_\ell(\text{Br}(Y/U)(\ell)) = 0.$$ 

(ii) If $H \leq G$ is not hypoelementary,

$$\sum_{U \leq H} |U| \cdot \mu(U, H) \cdot m(\text{Br}(Y/U)) = 0.$$ 

For a prime $\ell$ and an abelian group $A$, its corank is defined as $\text{rank}_{\mathbb{Z}_\ell}(T_\ell(A))$, where $T_\ell(A) = \lim \leftarrow \text{Ker}(\ell^n : A \to A)$. In this note, we denote this by

$$\text{rk}_\ell(A) := \text{rank}_{\mathbb{Z}_\ell}(T_\ell(A)).$$

$\text{Br}(X)(\ell)$ is known to be of finite corank, for example in the following cases ([7]):

- **(C1) $k$:** a separably closed or finite field, $X$: of finite type $/k$, and proper or smooth $/k$, or char($k$) = 0 or dim $X \leq 2$.

- **(C2) $X$:** of finite type $/\text{Spec}(\mathbb{Z})$, and smooth $/\text{Spec}(\mathbb{Z})$ or proper over $\exists$ open $\subset \text{Spec}(\mathbb{Z})$.

Remark that if $Y/X$ is a finite étale covering and $X$ satisfies (C1) or (C2), then so does $Y$.

**Example 5.6.** Assume $X$ satisfies (C1) or (C2). For any non-$\ell$-hypoelementary subgroup $H \leq G$, we have an equation

$$\sum_{U \leq H} |U| \mu(U, H) \cdot \text{rk}_\ell(\text{Br}(Y/H)(\ell)) = 0.$$
Another example is related with the comparison of Br and Br'. By Gabber’s lemma, for any finite étale covering \( Y/X \), we have
\[
\text{Br}'(X)/\text{Br}(X) \hookrightarrow \text{Br}'(Y)/\text{Br}(Y).
\]
In particular, if \( \text{Br}(Y) \subset \text{Br}(Y)' \) is of finite index, then so is \( \text{Br}(X) \subset \text{Br}(X)' \).

**Example 5.7.** Assume \( X \) satisfies \( [\text{Br}'(Y) : \text{Br}(Y)] < \infty \). Then for any non-hypoelementary subgroup \( H \trianglelefteq G \), we have an equation
\[
\sum_{U \leq H} |U| \mu(U, H) \cdot [\text{Br}'(Y/U) : \text{Br}(Y/U)] = 0.
\]

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