Charging of a quantum dot coupled to Luttinger liquid leads

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Luttinger liquid behavior of one-dimensional correlated electron systems is characterized by power-law scaling of a variety of physical observables with exponents determined by a single interaction dependent parameter $K$. We suggest a setup to study Luttinger liquid behavior in quantum wires which allows to determine $K$ from two independent measurements: transport through a quantum dot embedded in the wire and the charge on the dot. Consistency of the two $K$’s for a single probe would provide strong experimental evidence for the Luttinger liquid paradigm.

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I. INTRODUCTION

Theoretically it is well established that the two-particle interaction $U$ in metallic, one-dimensional (1d) electron systems leads to Luttinger liquid (LL) physics. One of the characterizing properties is the power-law scaling of a variety of physical observables as functions of external parameters (e.g. the temperature $T$) with exponents which can be expressed in terms of a single, interaction dependent LL parameter $K < 1$ (for repulsive interactions, with $K = 1$ for $U = 0$). However, there are only a few experiments on quasi 1d systems which reveal clear indications of LL behavior. Even in these rare examples mostly only a single observable as a function of a single control parameter was measured. In this case it is very difficult to convincingly exclude any other source for the observed power-law scaling than LL physics. The situation becomes more complex as even for the same type of quantum wire (e.g. semi-conductor heterostructures, carbon nanotube, or a row of atoms on a surface) $K$ might vary from probe to probe since it depends not only on $U$ but also on other details such as the band structure and filling.

A more direct evidence for LL behavior could be achieved in the following way. Using a single probe one should measure two observables for which LL theory predicts power-law scaling with different exponents $\beta_1(K)$ and $\beta_2(K)$ – in the optimal case even as functions of two different control parameters. If the two exponents turn out to be consistent, that is $K(\beta_1) \approx K(\beta_2)$, strong evidence for LL physics is achieved. A step in this direction is the linear conductance $G(T)$ measurement by Yao et al. across an impurity free part of a metallic single-wall carbon nanotube as well as across a part of the same tube containing a single kink (impurity).

We propose the setup sketched in Fig. 1 in which one can measure $G$ through a quantum dot (QD) embedded in a 1d wire as a function of $T$ and, for the same probe, the charge $n$ accumulated on the dot as a function of the dot level position varied by an external gate voltage $V_g$. The QD is formed by two high barriers within the 1d system, e.g. realized by additional gates. The charge is detected by the current running through a nearby quantum point contact (QPC). While transport of 1d correlated electrons through double barriers has extensively been studied theoretically in recent years less is known about the charging of a small QD coupled to two LL leads. Here we investigate in detail how $n(V_g)$ is affected by LL physics using two approaches. First we consider a field-theoretical, effective low-energy model, the infinite Luttinger model and perturbation theory in the dot-LL coupling $\Gamma$. This can be done for arbitrary $0 < K < 1$, but is restricted to small $\Gamma$. In a complementary, second step we study an interacting microscopic lattice model of finite length coupled to noninteracting leads, a model being closer to experimental setups. To treat the correlations we use the functional renormalization group (fRG). This method can be applied for arbitrary $\Gamma$, but is restricted to small $1 - K$. Both approaches lead to consistent results and we show that $n(V_g)$ is governed by power-law scaling, which should be detectable in the suggested setup. We consider a dot with a large level spacing such that only a single level matters. Furthermore, we mainly consider spinless fermions and suppress the Kondo effect. Experimentally this can be achieved by a magnetic field lifting the spin degeneracy of the dot level or by measuring at $T > T_K$, with $T_K$ being the Kondo temperature. For transport through a dot showing the Kondo effect coupled to LL leads, see Ref.

From the linear conductance $G(V_g,T)$ through a double barrier $K$ can be extracted in several ways, some of them restricted to certain regimes of $K$ values or symmetric barriers. To be as general as possible we here present a prediction which holds for all $0 < K < 1$, symmetric as well as asymmetric barriers, and which does not
require any other fine tuning of parameters. For a fixed gate voltage away from resonance, which we assume to be at \( V_g = 0 \), one finds \( G \sim \max\{T, \delta\}^{2(1/K - 1)} \) at asymptotically small scales.\(^{27}\) Here \( \delta \) denotes an energy scale \( \sim 1/N \), with \( N \) being the length of the LL wire, which is eventually coupled to noninteracting leads.

In an important work Furusaki and Matveev analyzed \( n(V_g) \) for strongly interacting systems with \( K < 1/2 \) within the infinite Luttinger model using perturbation theory in \( \Gamma \) and the mapping to related problems.\(^5\) They showed that for sufficiently small \( \Gamma \), \( n(V_g) \) is discontinuous at \( V_g = 0 \). For \( 1/3 < K < 1/2 \) the finite \( V_g \) behavior adjacent to the jump shows scaling with the exponent \( 1/K - 2 \), while for even smaller \( K \) the deviations are linear in \( V_g \). The perturbation theory in \( \Gamma \) for the Green function – not for the self-energy, as used by us – breaks down for \( 1/2 < K < 1 \). In an attempt to investigate LLs characterized by such \( K \)'s a numerical method was used for systems of up to 150 sites in Ref.\(^3\). The authors concluded that \( n(V_g) \) is continuous and does not show LL physics. Below we confirm the first statement but show that the second is incorrect as finite size corrections completely mask the power-law behavior.

II. PERTURBATION THEORY IN THE LEVEL-LEAD COUPLING FOR THE SEMI-INFINITE LUTTINGER MODEL

We first consider a QD coupled to two LLs via tunnel barriers with hopping amplitudes \( t_{1,2} \). For simplicity the LLs are assumed to be equal and described by the semi-infinite Luttinger model (with an open boundary on the side coupled to the dot). To leading order in \( \Gamma = t_{1}^{2} + t_{r}^{2} \) the dot self-energy is given by \( \Sigma_{d}(z) = \Gamma \mathcal{G}(z) \), with the single-particle Green function \( \mathcal{G} \) of the disconnected semi-infinite LL at the boundary. The low-energy behavior of the imaginary part of \( \mathcal{G} \) for \( z = \omega + i0 \), that is the spectral function \( \rho \), is known exactly from bosonization.\(^{28}\) It is given by \( \rho(\omega) \sim |\omega|^{1/K - 1} \). To be specific we assume that \( \rho(\omega) \) has support \([-\omega_c, \omega_c]\).

\[
\omega_c \rho(\omega) = \theta(\omega_c - |\omega|) |\omega/\omega_c|^{1/K - 1}/(2K) .
\]

It is then straightforward to compute \( \text{Re} \mathcal{G}^{R}(\omega) \) by Hilbert transformation. The leading behavior at \( |\omega/\omega_c| \ll 1 \) is given by

\[
\omega_c \text{Re} \mathcal{G}^{R}(\omega) \sim \begin{cases} 
- \text{sign}(\omega) \frac{|\omega_c|^{1/K - 1}}{\omega_c} & \text{for } \frac{1}{2} < K < 1 \\
\frac{|\omega_c|}{\omega_c} \ln |\omega/\omega_c| & \text{for } K = \frac{1}{2} \\
- \frac{|\omega_c|}{\omega_c} & \text{for } K < \frac{1}{2} .
\end{cases}
\]

Using the Dyson equation the dot spectral function \( \rho_{d} \) follows from the perturbative \( \Sigma_{d} \) as

\[
\rho_{d}(\omega) = \frac{\Gamma \rho(\omega)}{|\omega - V_g - \Gamma \text{Re} \mathcal{G}^{R}(\omega)|^{2} + [\pi \Gamma \rho(\omega)]^{2}} .
\]

The dot charge is

\[
n(V_g) = \int_{-\omega_c}^{\omega_c} d\omega \rho_{d}(\omega) ,
\]

with the chemical potential \( \mu = 0 \). Because of the particle-hole symmetry it obeys \( n(V_g) = - n(-V_g) \) and from now on we focus on \( V_g \geq 0 \). In contrast to the perturbation theory in \( \Gamma \) for the dot Green function itself used in Ref.\(^8\) which is restricted to \( K < 1/2 \), our approach can be applied for all \( 0 < K \leq 1 \).

Based on Eqs.\(^{11-14}\) the leading small \( V_g \) behavior of \( 1/2 - n(V_g) \) can be determined analytically. For \( 1/2 < K \leq 1 \), \( n(V_g) \) is a continuous function with \( n(V_g = 0) = 1/2 \). This implies that the width \( \delta \) over which \( n(V_g) \) changes from 1 to 0 is finite.\(^2\) The function \( n(V_g) \) contains regular terms proportional to \( V_{g}^{2l+1} \), with \( l \in \mathbb{N}_0 \), as well as anomalous terms with exponents containing \( K \). The leading anomalous term is \( \sim V_{g}^{(2K-1)/(1-K)} \). Depending on \( K \) either the linear term or the anomalous term dominates. A special situation is reached at \( K = 2/3 \), where logarithmic corrections appear. The leading \( V_g \) dependence is given by

\[
\frac{1}{2} - n(V_g) \sim \begin{cases} 
\frac{V_g}{\omega_c} & \text{for } \frac{2}{3} < K \leq 1 \\
\frac{V_g}{\omega_c} \ln \left( \frac{V_g}{\omega_c} \right) & \text{for } K = \frac{2}{3} ,
\end{cases}
\text{for } \frac{1}{2} < K < \frac{2}{3} .
\]

At \( K = 1/2 \), \( n(V_g) \) is still continuous and for \( V_g \to 0 \) approaches \( 1/2 \) with corrections \( \sim 1/|\ln(V_g/\omega_c)| \).

For \( K < 1/2 \) and small \( \Gamma \), \( n(V_g) \) shows a jump at \( V_g = 0 \), that is \( \lim_{V_g \to 0} n(V_g) = \Delta < 1/2 \). In this regime our perturbative approach for the self-energy, which guarantees the correct analytical structure of the dot Green function, gives the same results as the perturbation theory for the Green function itself used in Ref.\(^8\). This follows from two observations. According to Eq.\(^2\) the real part of \( \mathcal{G}^{R} \) becomes linear at small \( \omega \) and can thus be absorbed in the first term in the denominator of Eq.\(^3\). In addition, for small \( V_g \) the contribution of \( \rho \) in the denominator of Eq.\(^3\) can be neglected compared to the term linear in \( \omega \). For the small \( V_g \) analysis and to leading order in \( \Gamma \) the integrand in Eq.\(^4\) becomes equivalent to the one obtained in Ref.\(^8\).

\[
n(V_g) \sim \Gamma \int_{0}^{\omega_c} d\omega \frac{\omega^{1/K - 1}}{(\omega + V_g)^2} .
\]

The jump at \( V_g \to 0 \) is given by \( \Delta = \Gamma/[(2 - 4K)\omega_c^{2}] \) which is nonuniversal as it depends on the cutoff \( \omega_c \). Evidently, for \( K \) close to 1/2 this expression only holds for sufficiently small \( \Gamma \). In Ref.\(^8\) it is argued that increasing \( \Gamma \) beyond the perturbative regime \( \Delta \) decreases, approaches the minimal value \( \Delta_{0} = \sqrt{K/2} \) at a certain \( \Gamma_{0} \), and for \( \Gamma > \Gamma_{0} \), \( n \) becomes a continuous function of \( V_g \) even for \( K < 1/2 \). The finite \( V_g \) corrections of \( n \) for small
These results show that for $1/3 < K < 2/3$, that is for sufficiently strong, but not too strong interactions, the LL parameter $K$ can be extracted from a measurement of $n(V_g)$ for gate voltages close to the resonance value.

A second way to extract the LL parameter in the regime in which $n(V_g)$ is continuous, that is for $1/2 < K < 1$, is given by the $\Gamma$ dependence of the characteristic width $w$ over which the charge changes from 1 to 0. In particular, this includes weak interactions with $2/3 < K < 1$ for which $1/2 - n(V_g)$ itself is linear in $V_g$ and cannot directly be used to determine $K$. The width can e.g. be defined by $w = 2V_g^0$ with $n(V_g^0) = 1/4$. In experimental setups in which the two barriers are realized by gates, $\Gamma$ can be tuned by varying the applied voltages and $w(\Gamma)$ can be extracted. For $\Gamma \to 0$, $w(\Gamma)$ follows from Eq. (6) and scales as

$$
\frac{w(\Gamma)}{\omega_c} \sim \left( \frac{\Gamma}{\omega_c^2} \right)^{K/(2K-1)} \quad \text{for} \quad 1/2 < K \leq 1. \quad (8)
$$

On first glance the appearance of an anomalous exponent in $w$ might be at odds with the linear $V_g$ dependence of $1/2 - n(V_g)$ for $2/3 < K < 1$. In fact, both results are consistent as the regime over which $n(V_g)$ goes linearly through 1/2 around $V_g \approx 0$ shrinks with decreasing $\Gamma$ and decreasing $K$. To experimentally observe the predicted power-law scaling the temperature has to be sufficiently smaller than the width $w$.

In the absence of the Kondo effect (see above) including the spin degree of freedom does not lead to new physics. The perturbative analysis can be repeated after replacing the exponent $1/K - 1$ in the spectral function of Eq. (11) by the exponent for LLs with spin $(1/K - 1)/2$.1

**III. WEAK TO INTERMEDIATE INTERACTIONS IN A MICROSCOPIC LATTICE MODEL**

We next replace the LLs described by the semi-infinite Luttinger model by the microscopic lattice model with nearest-neighbor hopping $t > 0$ and nearest-neighbor interaction $U$. On both sides of the QD the LLs are assumed to be finite, each having $N/2$ sites, and adiabatically coupled to noninteracting 1d tight-binding leads.2 The interaction is treated by an approximation scheme that is based on the fRG and which was shown to be reliable for weak to intermediate interactions.3 In contrast to the perturbation theory in $\Gamma$, which is restricted to small $\Gamma$, this method can be applied for all $\Gamma$ and is thus complementary to the above approach. The Hamiltonian is given by

$$
H = -t \sum_{j=-\infty}^{N-1} \left( c_{j+1}^\dagger c_j + \text{H.c.} \right) + V_g n_{jd}
$$

$$
+ \sum_{j=1}^{N-1} U_{j,j+1} \left( n_j - \frac{1}{2} \right) \left( n_{j+1} - \frac{1}{2} \right) - (t_l - t) c_{jd}^\dagger c_{jd-1} - (t_r - t) c_{jd+1}^\dagger c_{jd} - \text{H.c.} \quad (9)
$$

in standard second quantized notation, with $n_j = c_{jd}^\dagger c_j$. To prevent any backscattering from the contacts to the noninteracting leads around $j \approx 1$ and $N$ the interaction is turned on and off smoothly over a few lattice sites, with a bulk value $U$, as described in Ref. 3. The dot is located at lattice site $jd$ somewhere close to $N/2$ (the results are insensitive to the exact position). The prime at the sum in the second line indicates, that the interaction across the barriers defining the QD is set to zero. We also studied the case in which the interaction on these bonds takes the bulk value $U$ and found that our conclusions are valid also for this setup. We focus on half-filling of the band. In this case the bulk model is a LL for $|U| < 2t$ and a closed expression for $K$ in terms of the model parameters can be given as $K^{-1} = 2 \arccos [-U/(2t)]/\pi$.

Within the fRG one introduces an energy cutoff $\Lambda$ into the noninteracting propagator. Taking the derivative of the generating functional of the one-particle irreducible vertices with respect to $\Lambda$ and neglecting higher order corrections one derives a set of $O(N)$ coupled differential equations for the $\Lambda$-flow of the self-energy and a renormalized nearest-neighbor interaction. It can be solved numerically for up to $10^7$ sites, resulting in an approximate expression for the dot Green function.
The dominating feature of $\rho_d \approx 100$ explains why in Ref. 9 it was possible to fit the dip region. The absence of this LL feature at small $\omega$ to $\omega \approx 10$ leads to a power-law suppression of $n(V_g)$ over which $n(V_g)$ changes from 1 to 0 as a function of the dot-LL coupling $\Gamma$ for $N = 10^5$ and different $U$.

In Fig. 2 $\rho_d(\omega)$ is shown for $U = 0.5$, symmetric barriers $t_D/t = t_s/t = \sqrt{0.5}$, $V_g/t = 1$, and different $N$ (note the log-scale of the y-axis in the main part). The upper inset shows $n(V_g)$ for $N = 10^5$. On the scale of the plot $n(V_g)$ does not change if one further increases $N$. The dominating feature of $\rho_d$ is the Lorentzian-like peak at $\omega \approx V_g$. Although a fermion occupying the dot is assumed to be noninteracting with the fermions in the leads, increasing $N$ the coupling to the LL wires clearly leads to a power-law suppression $\rho_d(\omega) \sim \omega^{1/K-1}$ close to $\omega = 0$, as also given by the perturbative expression Eq. (3). The lower inset of Fig. 2 shows a zoom-in of the dip region. The absence of this LL feature at small $N$ of order 100 explains why in Ref. 9 it was possible to fit $n(V_g)$ by a Fermi liquid form.

The LL suppression of $\rho_d$ around $\omega = 0$ will manifest itself also in the charging of the dot. To illustrate this we confirm the prediction of Eq. (3) for the $\Gamma$ dependence of $w$. We extract $w$ from $n(V_g)$ (for an example of $n(V_g)$ see the inset of Fig. 2) for $N = 10^5$, a variety of $\Gamma$ (for simplicity assuming symmetric barriers), and different $U$. The results for $w(\Gamma)$ are shown in Fig. 3 on a log-log scale. At small $\Gamma$, $w$ shows power-law scaling. In Fig. 4 the exponent as a function of $U$, obtained by fitting the data of Fig. 3 and additional data sets, is compared to $K/(2K-1)$ determined in perturbation theory in $\Gamma$ [see Eq. (8)]. We used the exact relation between $K$ and $U$ mentioned above. The results agree quite well for $0 \leq U/t \leq 1/2$. For larger $U$ higher order corrections neglected in our truncated RG scheme become important. For sufficiently large $U$ the exponent $K/(2K-1)$ becomes large (it diverges for $K \downarrow 1/2$) and should experimentally be clearly distinguishable from the noninteracting value 1.

IV. SUMMARY

Using two different models and methods we have investigated the charge $n(V_g)$ accumulated on a QD coupled to two LL wires when the dot level position is varied by an external gate voltage. Depending on the strength of the two-particle interaction $U$, LL physics manifests itself in power-law scaling of $n(V_g)$ close to the resonance at $V_g = 0$ and the width $w(\Gamma)$ over which $n(V_g)$ changes from 1 to 0. The corresponding exponents can be expressed in terms of the LL parameter $K$. We proposed a setup which simultaneously allows to measure $n(V_g)$, and thus $w(\Gamma)$, as well as the temperature dependence of the linear conductance $G(T)$ through the QD. Off-resonance the latter is also governed by power-law scaling with an exponent which can be expressed in terms of $K$. Consistency of the extracted $K$’s would provide strong evidence for the experimental observation of LL physics.

Acknowledgments

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1 For recent reviews on the theoretical and experimental status see the articles by K. Schönhammer and by A. Ya-
coby in Strong Interactions in Low Dimensions, Eds.: D. Baeriswyl and L. Degiorgi, Kluwer Academic Publishers, Dordrecht (2005).
2 Z. Yao, H. Postma, L. Balents, and C. Dekker, Nature 402, 273 (1999).
3 C.L. Kane and M.P.A. Fisher, Phys. Rev. B 46, 7268 (1992); A. Furusaki, ibid. 57, 7141 (1998); D.G. Polyakov and I.V. Gornyi, Phys. Rev. Lett. 91, 126804 (2003); Y.V. Nazarov, Phys. Rev. B 71, 041302(R) (2005).
4 Large dots coupled to LLs were e.g. discussed in:
E.B. Kolomeisky, R.M. Konik, and X. Qi, Phys. Rev. B 66, 075318 (2002); P. Kakashvili and H. Johannesson, Phys. Rev. Lett. 91, 186403 (2003).

T. Enss, V. Meden, S. Andergassen, X. Barnabé-Thériault, W. Metzner, and K. Schönhammer, Phys. Rev. B 71, 155401 (2005) and references therein.

S. Andergassen, T. Enss, and V. Meden, Phys. Rev. B 73, 153308 (2006).

For asymmetric barriers \( t^2_l \neq t^2_r \), \( 0 < K < 1 \), and in the infrared limit \( T \to 0 \) as well as \( \delta \to 0 \), the linear conductance through a double barrier vanishes for all \( V_g \). This holds even at \( V_g = 0 \) where for \( K = 1 \) a resonance peak of transmission \( 4t^2_l t^2_r / (t^2_l + t^2_r) \) is located. At finite \( T \) or \( \delta \) a transmission peak of nonzero height remains at \( V_g = 0 \). For symmetric barriers \( t^2_l = t^2_r \), a resonance of unitary transmission is located at \( V_g = 0 \), but the resonance width \( \sigma \) vanishes \( \sim \max\{T, \delta\}^{1-K} \).

A. Furusaki and K.A. Matveev, Phys. Rev. Lett. 88, 226404 (2002).

M. Sade, Y. Weiss, M. Goldstein, and R. Berkovits, Phys. Rev. B 71, 153301 (2005).

C.L. Kane and M.P. A. Fisher, Phys. Rev. B 46, 15233 (1992).

F. D. M. Haldane, Phys. Rev. Lett. 45, 1358 (1980).

We here (for the fRG data) defined the width \( w \) of \( n(V_g) \) as the full width at half maximum of \( -dn(V_g)/dV_g \). Concerning the scaling properties of \( w(\Gamma) \) this definition is equivalent to the one used in the derivation of Eq. (8).