Monotone Riemannian metrics and dynamic structure factor in condensed matter physics

N. S. Tonchev

Institute of Solid State Physics, Bulgarian Academy of Sciences, 1784 Sofia, Bulgaria

An analytical approach is developed to the problem of computation of monotone Riemannian metrics (e.g. Bogoliubov-Kubo-Mori, Bures, Chernoff, etc.) on the set of quantum states. The obtained expressions originate from the Morozova, Čencov and Petz correspondence of monotone metrics to operator monotone functions. The used mathematical technique provides analytical expansions in terms of the thermodynamic mean values of iterated (nested) commutators of a model Hamiltonian $T$ with the operator $S$ involved through the control parameter $h$. Due to the sum rules for the frequency moments of the dynamic structure factor new presentations for the monotone Riemannian metrics are obtained. Particularly, relations between any monotone Riemannian metric and the usual thermodynamic susceptibility or the variance of the operator $S$ are discussed. If the symmetry properties of the Hamiltonian are given in terms of generators of some Lie algebra, the obtained expansions may be evaluated in a closed form. These issues are tested on a class of model systems studied in condensed matter physics.
I. INTRODUCTION

In recent decade different concepts emerging from quantum-information geometry \(^1,^4\) have been intensively incorporated in condensed matter physics \(^5,^17\). The underlying idea may be briefly reviewed as follows. The properties of macroscopic phases of matter should be encoded in the structure of the quantum state. It is defined by a density matrix \(\rho\) depending via the Hamiltonian \(\mathcal{H}\) on a set of parameters, e.g. coupling constants \(\lambda\) and external "fields" \(h\). If \(\mathcal{H}\) smoothly depends on the coupling constants and external fields one has a map between \(\lambda, h\), the inverse temperature \(\beta\) and the set of density matrices: \(\rho \equiv \rho(\beta, \lambda, h) := [Z(\beta, \lambda, h)]^{-1} \exp[-\beta \mathcal{H}(\lambda, h)]\), where \(Z(\beta, \lambda, h)\) is the partition function of the system. Further, the set of density matrices \(\rho\) should be endowed with a metric structure and thus explored as a Riemannian manifold. The realization of the set of density matrices \(\rho\) as a Riemannian manifold implies to define a relevant distance between two mixed quantum states \(^1,^3\). The notion of distance has found applications in different fields. For example: thermodynamic of small systems \(^5,^16\), geometrical description of phase transitions and quantum criticality \(^6,^15\), quantum estimation of Hamiltonian parameters \(^18,^19\) and hypothesis testing and discrimination of states \(^20,^23\), are only a part of them. If we consider the distance between two states obtained by an infinitesimal change in the values of the parameters that specify the quantum state, we come to the notion of a metric tensor, i.e. the set of coefficients of the linear element \(ds^2\) when written as a quadratic form in the differentials of these parameters. In the present study the infinitesimal variation of a single parameter, e.g. the field \(h\) is considered, and the notations of a linear element and metric will be used interchangeably.

For our purposes we shall recall some metrics which are interesting in the context of their physical applications. The metric \(d_{BK}^2\), after the name of Bogoliubov, Kubo and Mori \(^24,^20\), has a clear physical meaning since it describes the isothermal susceptibility of the system. Along the years the Bogoliubov-Kubo-Mori (BKM) metric (under different names: Kubo-Mori scalar product, Bogoliubov inner product, Duhamel two-point function or canonical correlation) has been intensively studied in the community of both physicists and mathematicians, see e.g. \(^17,^27–^33\) and references therein.

Notably, the Bures metric \(d_B^2\), depending on the fidelity between two density matrices has obtained increasing popularity in the information approach to physics, see e.g. \(^2,^10,^36,^38\) and references therein. The Bures metric \(d_B^2\) appears under different names: quantum Fisher information (apart from a numerical factor), SLD metric, fidelity susceptibility, etc \(^4,^6,^18,^19\).

Much as for quantification of the asymptotic behavior of the error in the quantum state discrimination problem, one may define the quantum Chernoff metric \(d_{QC}^2\), which is expressed in terms of the non-logarithmic variety of the quantum Chernoff bound \(^21,^28\). The quantum Chernoff metric also enjoys increasing popularity in the information approach in physics \(^39,^41\). It appears under several different names, among which are (up to a factor of one forth) the Hellinger metric \(^4\) or the Wigner-Yanase metric \(^21,^28\).

Another Riemannian metric in the physical setting, \(d_{MC}^2\) (the origin of the subscript MC will become clear in the subsequent consideration), is related to the quadratic fluctuations of a quantum observable (the variance of this observable). The metric \(d_{MC}^2\) was obtained \(^42\) as an approximated form of the dispersion of the Umegaki’s relative entropy or as the Hessian of the quasi-entropy considered in \(^28\).

Here we stress that all the mentioned above metrics \(d_{BK}^2, d_B^2, d_{QC}^2\) and \(d_{MC}^2\) belong to the large class of the monotone (or contractive) metrics. In the geometrical approach to statistics proposed by Morozova and Čencov \(^43\) the monotone metrics can be introduced and studied from a unified point of view due to the work of Petz \(^44\).

In view of Morozova, Čencov and Petz studies it is established that there exists one to one correspondence between monotone metrics \(d_f^2\) and Löwner operator monotone functions "f" (on the subject of the operator monotone functions see, e.g. the monograph \(^45\)). Each one of the metrics may be written as a sum of two contributions; the classical Fisher-Rao term and a quantum term that depends on the definition of the inner product that induces the monotone metric on the space of quantum states. The last is far to provide uniqueness. The consequences of this fact are subject of considerable interest, see, e.g. \(^1,^4\) along with a number of references therein. The relation between operator monotone functions and monotone metrics allows to explore in deep the relations between the different metrics in the context of quantum statistical mechanics and condensed matter physics.

The aim of the present study is to show that the isothermal susceptibility (respectively \(d_{BK}^2\)) and the variance of the operator involved through the control parameter (respectively \(d_{MC}^2\)) may serve as reference metrics for the whole class of monotone metrics. In other words, each one of the monotone metrics can be divided into two parts: the \(d_{BK}^2\) (or \(d_{MC}^2\)) and a constituent in the form of an infinite series of thermodynamic mean values of products of iterated commutators which are related with the noncommutativity of the problem. It is shown that this suggests a neat and deep interplay with the linear response theory through the relation with the moments of the Dynamical Structure factor (DSF). If the Hamiltonian is a linear form of the generators of a Lie algebra the final result may be presented in a closed form.

The paper is organized as follows: In Sec.II, using the fact that all monotone Riemannian metrics are characterized
II. MONOTONE RIEMANNIAN METRICS: GENERIC FORMULA

Following the work of Petz \cite{petz} (see also \cite{davies1, davies2}) let us recall that a Riemannian metric on the manifold $D$ of the density matrices $\rho$ can be written in the form

$$g_\rho(X,Y) = \langle X, m_f(L_\rho, R_\rho)^{-1}(Y) \rangle_{HS},$$

(1)

where $X, Y$ belong to the tangent space $T_\rho D$ at $\rho$ of the manifold $D$, $L_\rho(A) := \rho A$ and $R_\rho(A) := A \rho$ stand for the left and right multiplication by $\rho$ for $A$ belonging to $T_\rho D$,

$$m_f(A,B) := A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

(2)

is the Kubo-Ando operator means and $\langle AB \rangle_{HS} := \text{Tr}(A^* B)$ is the Hilbert-Schmidt inner product. Hereafter the subscript $f$ stands for a monotone metric which depends on the operator monotone function $f(x) : f(x^{-1}) = f(x)/x$ with $f(1) = 1$. A Riemannian metric $g_\rho$ defines a square distance $d_f^2$ between two infinitesimally close states $\rho$ and $\rho + d\rho$ which in the basis independent form is given by

$$d_f^2 = g_\rho(d\rho, d\rho).$$

(3)

Note that since the operators $L_\rho$ and $R_\rho$ commute, the following relation holds $m_f(L_\rho, R_\rho)^{-1} = c_f(L_\rho, R_\rho)$, where the symmetric function $c_f(x,y) = c_f(y,x)$, is called the Morozova-Čencov function. Recall that the Morozova-Čencov function has the following presentation

$$c_f(x,y) = \frac{1}{xf(y/x)}.$$  

(4)

The function $c_f(\rho_m, \rho_n)$ obeys the equation $c_f(tx, ty) = t^{-1} c_f(x,y)$ for any $t \in \mathbb{R}$.

From now for concreteness we shall consider one-parameter family of density matrices

$$\rho(h) = [Z_N(h)]^{-1} \exp[-\beta \mathcal{H}(h)],$$

(5)

defined on the family of $N$-particles Hamiltonians of the form

$$\mathcal{H}(h) = T - hS,$$

(6)

where the Hermitian operators $T$ and $S$ do not commute in the general case. Here, $h$ is a real (control) parameter, $Z_N(h) = \text{Tr} \exp[-\beta \mathcal{H}(h)]$ is the corresponding partition function and $\beta = (K_B T)^{-1}$ is the inverse temperature. We assume that the Hermitian operator $T$ has a complete orthonormal set of eigenvectors $|m\rangle$ with a non-degenerate spectrum \{$T_m$; $T|m\rangle = T_m |m\rangle$\}, where $m = 1,2,\ldots$. In this basis the zero-field density matrix $\rho := \rho(0)$ is diagonal:

$$\langle m | \rho(0) | n \rangle = \rho_m \delta_{m,n}, \quad \rho_m := e^{-\beta T_m}/Z_N(0), \quad m, n = 1, 2, \ldots.$$  

(7)

Now, let us consider two nearby states $\rho_1 = \rho$ and $\rho_2 = \rho + d\rho$ produced by infinitesimal changing of the control parameter $h$. We choose $\rho = \rho(0)$ and consider the matrix elements of $d\rho$, e.g. $\langle m | d\rho | n \rangle$, in the basis where $\rho$ is diagonal. Thus, for $\rho = \rho(0)$ with matrix elements (7) through (3) any monotone Riemannian metric on the set of quantum states (up to a proportionality constant) is presented explicitly as (see, e.g. \cite{petz, davies1, davies2, morozova}):

$$d_f^2 = \frac{1}{4} \sum_m \frac{d\rho^2_m}{\rho_m} + \frac{1}{4} \sum_{m,n,m\neq n} c_f(\rho_m, \rho_n)|\langle m | d\rho | n \rangle|^2.$$  

(8)
It is however possible to give a more convenient (for calculations) presentation of \(d^2_f\) by adopting the statistical mechanics viewpoint which will be presented below.

Following [6] (see also [36, 37]) for the first term in Eq.(8) we have

\[
\frac{1}{4} \sum_m \frac{d \rho^2_m}{\rho_m} = \frac{\beta^2}{4} \langle \left(\delta S^d \right)^2 \rangle_T, (9)
\]

where \(S^d\) is the diagonal part of the operator \(S\) and

\[
\langle \cdots \rangle_T = \left[ Z(T) \right]^{-1} \text{Tr} \{ \exp[\beta T] \cdots \}
\]

denotes the thermodynamic mean value. A next step consists in using the following relations (see, e.g.[6, 13]):

\[
\langle m | d \rho | n \rangle = \langle m | \delta S \rangle \rho_n - \rho_m, \quad m \neq n,
\]

(11)

and

\[
\langle m | d n \rangle = \left( \langle m | \partial \mathcal{H}(h) | n \rangle \right) T_n - T_m, \quad \partial \mathcal{H}(h) \equiv \frac{\partial \mathcal{H}(h)}{\partial h} = -S.
\]

(12)

By means of Eqs.(9), (11) and (12), Eq.(8) takes the more convenient form

\[
d^2_f = \frac{\beta^2}{4} \left\{ \langle \left(\delta S^d \right)^2 \rangle_T + \sum_{m,n,m \neq n} c_f(\rho_m, \rho_n) \left( \frac{\rho_n - \rho_m}{\rho_n - \ln \rho_m} \right)^2 \right\}. \quad (13)
\]

This expression is a basic element in our further calculations.

There are so many monotone metrics \(d^2_f\) as \(f\)’s are. Ones that are mostly proposed in the literature operator monotone functions are as follows [44]:

\[
\begin{align*}
\text{fHar}(x) &= \frac{2x}{x+1}, \\
\text{fBKM}(x) &= \frac{x-1}{\ln x}, \\
\text{fWY}(x) &= \frac{(1+\sqrt{x})^2}{4}, \\
\text{fB}(x) &= \frac{x+1}{2}, \\
\text{fMC}(x) &= \left( \frac{x-1}{\ln x} \right)^2 \frac{2}{1+x}.
\end{align*}
\]

(14)

The function \(f_{\text{Har}}(x)\) is the minimal, while \(f_{\text{B}}(x)\) is the maximal operator monotone functions on \([0, +\infty)\). The former defines a metric known as RLD metric while the last one gives rise to the SLD metric (named also Bures metric or fidelity susceptibility). The function \(f_{\text{BKM}}(x)\) leads to the Bogoliubov-Kubo-Mori metric. The function \(f_{\text{WY}}(x)\) is associated to the Wigner-Yanase metric (or an equal ground the n ame quantum Chernoff metric is reasonable). The function \(f_{\text{MC}}(x)\) was first conjectured in the paper [43], which explains the subscript \(MC\). Its matrix monotonicity was proved in [44].

It is instructive to consider the generic formula Eq. (13) in the concrete cases of the operator monotone functions [14].

\[\text{A. Bogoliubov - Kubo - Mori metric}\]

The derivation of the BKM metric and its physical meaning as susceptibility were discussed earlier in [17, 28–35]. In order to introduce the needed definitions we shall give our slightly different derivation based on the notion of the Bogoliubov-Duhamel (or Kubo-Mori) inner product (see the Appendix). The corresponding Morozova-Cencov function (everywhere below we shall use the subscript of the function \(f\) instead of the very function) is

\[
c_{\text{BKM}}(\rho_m, \rho_n) = \left( \frac{\rho_n - \rho_m}{\ln \rho_n - \ln \rho_m} \right).
\]

(15)

From the other hand, let us consider the spectral representation of the Bogoliubov-Duhamel inner product [26, 29, 31, 32] (see also the Appendix):

\[
F_0(S; S) := \frac{1}{2} \sum_{m,n,m \neq n} |\langle n | S | n \rangle|^2 \frac{\rho_n - \rho_m}{X_{mn}} + \sum_n \rho_n |\langle n | S | n \rangle|^2,
\]

(16)
where for convenience the quantity
\[ X_{mn} = \frac{1}{2}(\ln \rho_n - \ln \rho_m) \equiv \frac{\beta}{2}(T_m - T_n) \]
is introduced. It can be shown by simple algebra that
\[ F_0(\delta S; \delta S) = F_0(S; S) - |\langle S \rangle_T|^2, \]
where \( \delta S = S - \langle S \rangle_T \). Since by definition
\[ \langle (\delta S^d)^2 \rangle_T := \sum_m \rho_m |\langle m | S | n \rangle|^2 - |\langle S \rangle_T|^2, \]
where \( \delta S^d = S^d - \langle S^d \rangle_T \), we can present Eq. (16) in the form
\[ \frac{1}{2} \sum_{m,n,m \neq n} |\langle m | S | n \rangle|^2 \frac{\rho_m - \rho_n}{X_{nm}} = F_0(\delta S; \delta S) - \langle (\delta S^d)^2 \rangle_T. \]

In view of Eqs. (15) and (20) from Eq. (13), we obtain
\[ d^2_{BKM} = \frac{\beta^2}{4} F_0(\delta S; \delta S). \]
Recall that alternatively we have the well known relation (see, e.g. [47])
\[ F_0(\delta S; \delta S) = \int_0^1 d\tau \langle [\delta S(\tau) \delta S] \rangle_T = \frac{1}{\beta^2} \frac{\partial^2 \ln Z(h)}{\partial^2 h} |_{h=0}. \]
Using the definition of the isothermal susceptibility with respect to the field \( h \):
\[ \chi^T_{h=0} := \frac{1}{\beta} \frac{\partial^2 \ln Z(h)}{\partial h^2} |_{h=0}, \]
the following result emerges
\[ d^2_{BKM} = \frac{\beta}{4} \chi^T_{h=0}. \]
We shall follow the above recipe in order to study the relations between different metrics.

**B. Morozova-Čencov metric**

We shall show that the operator monotone function \( f_{MC}(x) = \left( \frac{x-1}{\ln x} \right)^2 \frac{2}{1+x} \) makes sense in physical applications. The corresponding Morozova-Čencov function
\[ c_{MC}(\rho_m, \rho_n) = \left( \frac{\ln \rho_n - \ln \rho_m}{\rho_n - \rho_m} \right)^2 \frac{\rho_n + \rho_m}{2} \]
yields the metric \( d^2_{MC} \) already discussed in the Introduction. Since to our knowledge appropriate citation of this fact is unknown a simply deviation is presented. By inserting (25) into (13), we obtain
\[ d^2_{MC} = \frac{1}{4} \sum_m \frac{d\rho_m}{\rho_m} + \frac{\beta^2}{8} \sum_{m,n,m \neq n} |\langle m | S | n \rangle|^2 (\rho_n + \rho_m). \]
Now, using the relation
\[ \langle S^2 \rangle_T = \frac{1}{2} \sum_{m,n} |\langle m | S | n \rangle|^2 (\rho_m + \rho_n) \]
and Eqs. (9) and (11) we get the final result
\[ d^2_{MC} = \frac{\beta^2}{4} \langle (S - \langle S \rangle_T)^2 \rangle_T. \]
Note that in the case of commuting operators \( T \) and \( S \), according to Eq. (22) both metrics \( d^2_{BKM} \) and \( d^2_{MC} \) coincide, as it must be.
C. Bures metric

There are different ways to obtain the analytical expression of the Bures metric $d_B^2$. It may be obtained from the infinitesimally close form of the Bures distance. For any two states $\rho_1$ and $\rho_2$ the Bures distance can be expressed in terms of the Uhlmann-Jozsa fidelity:

$$f(\rho_1, \rho_2) = \text{Tr} \sqrt{\rho_1^{1/2} \rho_2 \rho_1^{1/2}}, \quad (29)$$

as

$$D_B(\rho_1, \rho_2) = \sqrt{2 - 2f(\rho_1, \rho_2)}. \quad (30)$$

Thus $d_B^2$ is obtained as the leading term in the expansion of Eq.(30). Among the operator monotone functions introduced by Morozova Čencov and Petz it is associated with the maximal operator monotone function $f_B(x) = x + 1/2$. By setting it in Eq.(13) we obtain the result

$$d_B^2 = \frac{1}{4} \sum_m \frac{d\rho_m^2}{\rho_m} + \frac{\beta^2}{2} \sum_{m,n,m \neq n} \frac{|\langle n | S | m \rangle|^2}{\ln \rho_m - \ln \rho_n^2} \frac{\rho_n - \rho_m}{\rho_n + \rho_m}, \quad (31)$$

(see, e.g. [6, 36–38] and comments therein).

D. Chernoff/Wigner-Yanase metric

The quantum Chernoff metric is the infinitesimally close form of the Chernoff distance:

$$D_{qc}(\rho_1, \rho_2) = 1 - Q(\rho_1, \rho_2), \quad (32)$$

where

$$Q(\rho_1, \rho_2) = \min_{0 \leq s \leq 1} \text{Tr}(\rho_1^{1-s} \rho_2^s) \quad (33)$$

is known as the nonlogarithmic variety of the quantum Chernoff bound [20]. It can be written as [6, 20, 21, 23, 39]

$$d_{qc}^2 = \frac{1}{8} \sum_m \frac{d\rho_m^2}{\rho_m} + \frac{\beta^2}{2} \sum_{m,n,m \neq n} \frac{|\langle n | S | m \rangle|^2}{\ln \rho_m - \ln \rho_n^2} (\rho_n^{1/2} - \rho_m^{1/2})^2. \quad (34)$$

The metric (34) coincides up to a factor 1/2 with the Wigner-Yanase metric [48], i.e. $d_{qc}^2 = (1/2)d_{wy}^2$, which may be obtained from Eq.(13) with the help of the operator monotone function $f_W(x) = (1+\sqrt{x})^2$.

Here, the following comment is in order. In terms of Green’s functions a finite-temperature generalization of the fidelity susceptibility was proposed in [36]. This quantity is quite different from the Bures-Uhlmann fidelity susceptibility (i.e. $d_B^2$) and we denoted it as $\chi_T^{\text{Fid}}$. The only similarity between both metrics is that they have the same $T = 0$ limit. The fidelity susceptibility $\chi_T^{\text{Fid}}$ is obtained as the first nonvanishing term in the expansion of the “fidelity”

$$F_T(\rho_1, \rho_2) = \sqrt{\text{Tr}(\rho_1^{1/2} \rho_2^{1/2})}. \quad (35)$$

For for more details see ref. [12]. It is worth noting, that the spectral representation of $\chi_T^{\text{Fid}}$ is presented by Eq. (14) in ref. [36]. It is simply to check that, in our notations, it emerges exactly as the quantum Chernoff metric $d_{qc}^2$, Eq.(34).

Since, the computational problems in studying $\chi_T^{\text{Fid}}$ can be efficiently tackled by the quantum Monte Carlo approach [36] this fact transfers new computational possibilities in the field concerning the quantum Chernoff metric.

III. BKM AND MC METRICS AS REFERENCE METRICS

In order to quantify the deviation of the one of monotone Riemannian metric $d_f^2$ from the $d_{BKM}^2$ or $d_{MC}^2$ we shall recast Eq. (13) in a form suitable for further elaborations.
A. BKM metric

In order to discuss the deviation of a metric from the BKM metric, we need to rewrite Eq. (13) in the form:

\[
d_f^2 = \frac{1}{4} \sum_m \frac{d\rho_m^2}{\rho_m} + \frac{\beta^2}{4} \sum_{m,n,m\neq n} g_f(X_{mn}) \frac{\rho_n - \rho_m}{\ln \rho_n - \ln \rho_m} |\langle m|S|n \rangle|^2,
\]

(36)

where we introduced the family of functions

\[
g_f(x) := \frac{e^{2x} - 1}{2xf(e^{2x})}.
\]

(37)

These functions satisfy (due to the symmetry \(xf(x^{-1}) = f(x)\)) the condition \(g_f(x) = g_f(-x)\) and \(g_f(0) = 1\). In the considered particular cases of \(f\)’s one is led to

\[
g_{H\alpha r}(x) = \frac{\sinh 2x}{2x}, \ g_{BKM}(x) = 1, \ g_{WY}(x) = \frac{\tanh \frac{1}{4}x}{\frac{1}{4}x}, \ g_B(x) = \frac{\tanh x}{x}, \ g_{MC}(x) = \frac{x}{\tanh x}.
\]

(38)

(We use argument \(X_{mn}\) in the function \(g_f(\cdot)\) instead of \(\frac{\rho_n - \rho_m}{\ln \rho_n - \ln \rho_m}\) in the rhs of (36), remembering the relation Eq. (17).) The presentation (36) is convenient for two reasons.

First, using the well known inequalities between hyperbolic functions one obtains

\[
g_{H\alpha r}(X_{mn}) \geq g_{MC}(X_{mn}) \geq g_{BKM}(X_{mn})(= 1) \geq g_{WY}(X_{mn}) \geq g_B(X_{mn}).
\]

(39)

From the above inequalities immediately follow the inequalities between different metrics.

Second, since functions \(g_f(X_{mn})\) are even functions they have series expression with only even degrees of \(X_{mn}\). Thus, functions \(g_f(X_{mn})\) can be expressed in a unified fashion as a formal series

\[
g_f(X_{mn}) = 1 + \sum_{l=1}^{\infty} a_{2l-1}(f)(X_{mn})^{2l}.
\]

(40)

We hope no confusion will arise by using the subscript \(2l-1\) instead of \(2l\) in order to enumerate the coefficients \(a_l\)’s in the above expansion. Particular, in the case of Bures metric from the series expansion of the function \(x^{-1}\tanh x\) it is easy to obtain

\[
a_{2l-1}(B) = \frac{2^{2l+2}(2^{2l+2} - 1)}{(2l + 2)!} B_{2l+2},
\]

(41)

where \(B_{2l+2}\) are the Bernoulli numbers. The separation in coefficients \(a_l(f)\) the very dependence on functions \(f\) allows to introduce the terms

\[
2^{-2l} a_{2l-1}(f) |\ln \rho_n - \ln \rho_m|^{2l-1} (\rho_n - \rho_m) |\langle m|S|n \rangle|^2
\]

(42)

in the expansion Eq. (42). Remarkably, we shall show that these terms generate well known thermodynamic mean values. The presentation (42) provides the role of Bogoliubo-Kubo-Mory metric as a reference metric. Recall that Eq. (36) simply presents the definition of \(d_f^2\) if we choose \(g_f(X_{mn}) = g_{BKM}(X_{mn}) = 1\). Thus \(g_f(X_{mn}) \neq 1\) measures the deviation of the corresponding metric \(d_f^2\) from the Bogoliubo-Kubo-Mory metric.

B. MC metric

One may use an alternative presentation of \(d_f^2\) instead of Eq. (36):

\[
d_f^2 = \frac{1}{4} \sum_m \frac{d\rho_m^2}{\rho_m} + \frac{\beta^2}{4} \sum_{m,n,m\neq n} \hat{g}_f(X_{mn}) \frac{\rho_n + \rho_m}{2} |\langle m|S|n \rangle|^2,
\]

(43)

where

\[
\hat{g}_f(x) := g_f(x) \frac{\tanh x}{x}.
\]

(44)
Similarly, the Eq. (43) reads
\[ \text{we shall show that} \]
d\[ \text{These terms also can be expressed as some thermodynamical mean values (c.f. with Eq. (42)). In the next subsection} \]
\[ \text{Now, as a reference metric one can use} d_{MC}^2. \text{If we set} f = f_{MC} \text{ in Eq. (44) then} \hat{g}_{MC} (X_{mn}) = 1 \text{ and Eq. (43) reduces} \]
to the definition of \( d_{MC}^2 \). Hence, functions \( \hat{g}_f \) serve as a measure of deviation of the corresponding \( d_f^2 \)'s from \( d_{MC}^2 \).

For the functions \( f \) presented in (14), the application of definitions (38) in (44) yields
\[ \hat{g}_{Har}(x) = \frac{2(\cosh 2x - 1)}{(2x)^2}, \quad \hat{g}_{BKM}(x) = \frac{\tanh x}{x}, \]
\[ \hat{g}_{MC}(x) = 1, \quad \hat{g}_{WY}(x) = \frac{2[1 - (\cosh x)^{-1}]}{x^2}, \]
\[ \hat{g}_B(x) = \left( \frac{\tanh x}{x} \right)^2. \] (45)

The even functions Eq. (44) can be presented as a formal series
\[ \hat{g}_f (X_{mn}) = 1 + \sum_{l=1}^{\infty} a_{2l}(f)(X_{mn})^{2l}, \] (46)
which introduces in the summand of the expansion Eq. (43) the terms
\[ 2^{-(2l+1)} a_{2l}(f)(\ln \rho_n - \ln \rho_m)^2(\rho_n + \rho_m)|⟨m|S|n⟩|^2. \] (47)
These terms also can be expressed as some thermodynamical mean values (c.f. with Eq. (42)). In the next subsection we shall show that \( d_{MC}^2 \) appears as a first term of the obtained series representation of the monotone Riemannian metrics.

C. Series presentations in terms of iterated commutators

Let us consider the functionals
\[ F_n(S; S) := \sum_{m} |⟨m|S|l⟩|^2 \frac{[\rho_l - (-1)^n \rho_m]}{|\ln \rho_l - \ln \rho_m|^{n-1}}, \quad n = 0, 1, 2, \ldots. \] (48)
These functionals have been introduced in (37) (see also the Appendix) as an useful tool to obtain different thermo-
dynamic inequalities. By using the definition (48), one can present the terms (42) and (47) in the form
\[ 2^{-2l} a_{2l-1}(f) F_{2l}(S; S) \] (49)
and
\[ 2^{-(2l+1)} a_{2l}(f) F_{2l+1}(S; S), \] (50)
respectively. Now, the Eq. (50) reads
\[ d_f^2 = d_{BKM}^2 + \frac{\beta^2}{4} \sum_{l=1}^{\infty} 2^{-2l} a_{2l-1}(f) F_{2l}(S; S). \] (51)
Similarly, the Eq. (43) reads
\[ d_f^2 = d_{MC}^2 + \frac{\beta^2}{4} \sum_{l=1}^{\infty} 2^{-(2l+1)} a_{2l}(f) F_{2l+1}(S; S). \] (52)
Indeed, the first terms in Eqs. (51) and (52) are immediate consequences of the definitions (21) and (28), while the summands follow from the terms (49) and (50), respectively. The key advantage of both formulas is that in the basis independent form functionals \( F_n(S; S) \) have the following presentations (see the Appendix)
\[ F_n(S; S) = 2(-1)^{n+1} \beta^{n-1} ⟨R_{n-1} R_0⟩_T, \] (53)
where the notion of iterated commutators
\[ R_0 ≡ R_0(S) ≡ S, \quad R_1 ≡ R_1(S) := [T, S], \ldots, \]
\[ R_n ≡ R_n(S) := [T, R_{n-1}(S)], \quad n = 0, 1, 2, \ldots, \] (54)
is introduced. (Note that nested commutators is also frequently used term.) It is worthwhile to emphasize the relation of Eq. \(63\) with the moments of the dynamical structure factor \(M_p(S)\) (see the Appendix)

\[
F_n(S; S) = 2\beta^{n-1}M_{n-1}(S), \quad n = 0, 1, 2, ...
\]

(55)

With the help of Eq.\(55\) one obtains from Eqs. \(51\) and \(52\) the result

\[
d_i^2 = d_{BK}^2 + \frac{\beta^2}{4} \sum_{l=1}^{\infty} \left( \frac{\beta}{2} \right)^{2l-1} a_{2l-1}(f)M_{2l-1}(S).
\]

(56)

and

\[
d_i^2 = d_{MC}^2 + \frac{\beta^2}{4} \sum_{l=1}^{\infty} \left( \frac{\beta}{2} \right)^{2l} a_{2l}(f)M_{2l}(S).
\]

(57)

This result attributes thermodynamical meaning to pure information theory ingredients.

The series representations \(51\) and \(52\) yield a proper definition of a monotone Riemannian metric provided the corresponding convergence condition is fulfilled. In order to remove the restrictions imposed by the convergence conditions one needs to perform an analytic extension of the obtained formulas. This issue in the particular case of the fidelity susceptibility (Bures metric) has been examined in the ref.\(38\) by the examples of several popular models.

The appearance of the iterated commutators (terms with \(n > 1\)) is a reminiscence of the well known Feynman’s disentangling procedure \(47\), \(49\).

In the next Section we shall demonstrate that this allows the underlaying symmetry of the Hamiltonian to be efficiently explored and a closed form of the functionals \(48\) to be obtained.

IV. APPLICATION TO A MODEL

Let us consider the Hamiltonian \(50\), \(51\):

\[
\mathcal{H}(h) = k\omega \left( Q_k^0 - \frac{1}{k^2} \right) + h\sqrt{k^2}(Q_k^+ + Q_k^-), \quad k = 1, 2, ..., \quad (58)
\]

where \(Q_k^\pm\) are operators obeying the commutation relations

\[
[Q_k^\pm, Q_k^\pm] = \pm Q_k^\pm, \quad [Q_k^+, Q_k^-] = \Phi_k(Q_k^0) - \Phi_k(Q_k^0 - 1), \quad (59)
\]

with the structure function

\[
\Phi_k(Q_k^0) = -\Pi_{i=1}^k Q_k^0 + \frac{i}{k} - \frac{1}{k^2}
\]

(60)

being a \(k\)th-order polynomial in \(k\). The Hamiltonian \(58\) is employed in various physical problems (for definitions and a partial list of references, see \(50\), \(51\)).

In this case the proposed approach is very effective since the iterative commutation between \(T = k\omega (Q_k^0 - \frac{1}{k^2})\) and \(S = \sqrt{k^2}(Q_k^+ + Q_k^-)\) implies some periodic operator structures after a finite number of steps \(38\)

\[
R_n = (-1)^n R_n^+ = \alpha^n [Q_k^+ + (-1)^n Q_k^-], \quad (Q_k^-)^+ = Q_k^+.
\]

(61)

indicating an analytical expression as a function of \(n\). The parameters \(k\) and \(\omega\) enter in the c-number \(\alpha = (k\omega)^k \sqrt{k^2}\) . Thus, the obtained series expansions Eqs. \(51\) and \(52\) can be used in a rather simple way to obtain closed-form expressions.

The polynomial algebra of degree \(k - 1\) defined by Eqs. \(59\) has the following one-mode boson realization \(50\):

\[
Q_k^+ = \frac{1}{(\sqrt{k})^k} (b^+)^k, \quad Q_k^- = \frac{1}{(\sqrt{k})^k} b^k.
\]

(62)

In terms of Eqs. \(52\) the Hamiltonian of the model takes the more familiar form \(51\)

\[
\mathcal{H}(h) = \omega b^+ b + h[(b^+)^k + b^k], \quad \omega > 0, \quad k = 1, 2, 3, ...
\]

(63)
where bosonic operators $b, b^+$ obey the canonical commutation relations.

It is worse noting that the $k = 1$ and $k = 2$ cases of \[ (63) \] give the Hamiltonians of the displaced and single-mode squeezed harmonic oscillators, respectively. The Hamiltonian \[ (63) \] for $k = 2$ is also known as Lipkin-Meshkov-Glick (LMG) model in the Holstein-Primakoff single boson representation (see e.g. \[ 14 \] and refs. therein) and all the result obtained here can be related to this field.

Inserting the expressions $R_0$ and $R_{2n-1}$ in \[ (53) \], we obtain
\[
F_{2n}(S; S) = -2(k\beta\omega)^{2n-1}\mathcal{K}(k), \quad n = 0, 1, 2, ..., \quad k = 1, 2, ..., \tag{64}
\]
where
\[
\mathcal{K}(k) = k^k\langle [Q_k^+, Q_k^-][Q_k^+, Q_k^-] \rangle_T, \quad k = 1, 2, ... \tag{65}
\]
Inserting expressions of $R_0$ and $R_{2n}$ in \[ (53) \], we obtain
\[
F_{2n+1}(S; S) = 2(k\beta\omega)^{2n}\mathcal{L}(k), \quad n = 0, 1, 2, ..., \quad k = 1, 2, ..., \tag{66}
\]
where
\[
\mathcal{L}(k) = k^k\langle [Q_k^+ + Q_k^-]^2 \rangle_T, \quad k = 1, 2, ... \tag{67}
\]
Evaluation of the correlation functions Eqs. \[ (65) \] and \[ (67) \] with the quadratic Hamiltonian $T$ is now straightforward. The results for $k = 1$ and $k = 2$ are:
\[
\begin{align*}
\mathcal{K}(1) &= -1, \quad \mathcal{L}(1) = 2n + 1, \\
\mathcal{K}(2) &= -2(2n + 1), \quad \mathcal{L}(2) = 4n^2,
\end{align*} \tag{68}
\]
where $n = (e^{\beta\omega} - 1)^{-1}$. With the help of Eqs.\[ (64) \] and \[ (66) \], Eqs. \[ (51) \] and \[ (52) \] may be recast in the form
\[
d_f^2 = d_{BKM}^2 + \frac{\beta^2}{4} \left( \frac{k\beta\omega}{2} \right)^{-1} \left[ 1 - g_f \left( \frac{k\beta\omega}{2} \right) \right] \mathcal{K}(k) \tag{69}
\]
and
\[
d_f^2 = d_{MC}^2 - \frac{\beta^2}{4} \left[ 1 - \hat{g}_f \left( \frac{k\beta\omega}{2} \right) \right] \mathcal{L}(k), \tag{70}
\]
respectively. The relation \[ (69) \] on the particular example of $f_B$ was found earlier in \[ 38 \]. Here we are able to consider the whole class of the monotone Riemannian metric.

**V. SUMMARY AND CONCLUSIONS**

In this paper, we study model systems described by a Hamiltonian comprising the non commuting operators $T$ and $S$ via monotone Riemannian metrics. The formulas Eqs. \[ (51) \] and \[ (52) \] are the key results of our study. They present the monotone Riemannian metric as a series in terms of the earlier introduced \[ 35 \] functionals $F_n(S; S)$ (see also the Appendix). These are defined as a rather complicated double sum which however may be written in a basis independent form as thermodynamic mean values of $n$-times iterated commutators between $T$ and $S$. The last provides significant computational advantage if the Hamiltonian is a linear form of the generators of some Lie algebra. In this case $F_n(S; S)$ should be obtained in a closed-form expression as a function of $n$. Recall that the lowest-ordered functionals $F_0(S; S)$ and $F_1(S; S)$ are the Bogoliubov-Duhamel inner product and the symmetrized thermodynamic mean value of the operator $S$, respectively. Then $F_0(\delta S; \delta S)$ is proportional to the BKM metric given by Eq. \[ (24) \], while $F_1(\delta S; \delta S)$ is proportional to the MC metric given by Eq. \[ (28) \]. These quantities are the starting point in our expansions Eqs. \[ (51) \] and \[ (52) \].

From the linear response theory it is well known that the iterated commutators of an observable $S$ with the Hamiltonian are related to the moments of the dynamic structure factor (DSF) through some sum rules \[ 52 \]. This allows to present our results in an alternative form, Eqs. \[ (56) \] and \[ (57) \]. A look at the above formulas shows that the dependence on the operator monotone function $f$ is only in the coefficients in front of the moments. If one aims to characterize a monotone metric on this setting, then moments of all orders (odd or even) should be considered together.
Formulas Eqs. \((56)\) and \((57)\) do not provide computational benefits per se. However they are very useful and informative because they transfer the information geometry problems into the realm of condensed matter physics with its wealth of methods for computing the DSF and its moments. For example, an instructive illustration may present the study \([54]\), where an application of appropriate (physical) approximations to Hamiltonian described one-component Coulomb plasma in thermodynamic equilibrium leads to explicit formulas for the arbitrary integer moments of the DSF expressed in terms of simple functions.

The series representation \((51)\) (or \((52)\)) may be regarded as a proper definition of the considered monotone Riemannian metric provided the corresponding convergence condition is fulfilled which can be checked on the framework of concrete models.

We demonstrated our approach in an example with a Hamiltonian, expressed in terms of the generators of a polynomial deformation Lie algebra, Eq.\((58)\), employed in various physical problems. It is shown that in this case, the infinite set of the moments of all orders (odd and even) can be written in a closed form, Eqs.\((64)\) and Eq.\((66)\), for the all monotone Riemannian metrics. Besides being of interest for its own sake the presented result may also be considered as a contribution to the linear response theory.

At the end the following comment is in order. The implication of the immanent relations between the zero-temperature fidelity susceptibility and the moments of the DSF recently has been demonstrated in ref. \([55]\). However the elucidation of the role of the moments \(M_p(S)\) (named as p-order generalized fidelity susceptibility) has been considered in quite different context.

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APPENDIX: THE BOGOLIOBOV-DUHAMEL INNER PRODUCT, MOMENTS OF THE DYNAMIC STRUCTURE FACTOR AND SUM RULES

Let us \(S\) is an arbitrary operator. The Bogoliubov- Duhamel inner product is defined us \([26, 29–31, 35]\):

\[
(S; S)_T = \int_0^1 d\lambda K_{SS}(\lambda),
\]

where

\[
K_{SS}(\lambda) := \langle S^+(\lambda) S \rangle_T, \quad S^+(\lambda) := e^{\lambda \beta T} S e^{-\lambda \beta T},
\]

and

\[
\langle \cdots \rangle_T := [Z(T)]^{-1} \text{Tr}\{\exp[-\beta T] \cdots\}
\]

denotes the thermodynamic mean value. We warn the reader that definitions differ from \((71)\) by factor \(\beta\) and/or by involving on the first place in the Eq.\((72)\) the operator \(S\) instead of its adjoint \(S^+\) exist in the literature.

With the help of the Kubo identity (see, e.g. \([47]\))

\[
[S, e^{-\beta T}] = -\beta \int_0^1 d\tau e^{-\beta(1-\tau)T}[S, T] e^{-\beta \tau T},
\]

the following useful formula can be verified:

\[
\beta (S; [T, S])_T = \langle [S^+, S] \rangle_T,
\]

where \(S\) and \(T\) are arbitrary operators.

The definition of the Bogoliubov- Duhamel inner product [Eq. \((71)\)] can be presented in terms of the spectral representation in which \(T\) is diagonal. Let us assume that the Hermitian operator \(T\) has a complete orthonormal set of eigenstates \(|l\rangle\) and eigenvalues \(T_l\); \(T|l\rangle = T_l|l\rangle\). Then, in the basis of the eigenvectors of the Hamiltonian \(T\) we have the alternative form

\[
(S; S)_T := \frac{1}{2} \sum_{m,l,m \neq l} \frac{\rho_l - \rho_m}{X_{ml}} |\langle m | S | l \rangle|^2 + \sum_l \rho_l |\langle l | S | l \rangle|^2,
\]

\[(75)\]
where the notations \(|l⟩|ρ|l⟩ = ρ_l\) and \(X_{ml} = 2^{-1} β(T_m - T_l)\) are introduced.

It is convenient to consider the spectral representation (for more details and history remarks, see e.g.\textsuperscript{[34]})

\[ K_{SS}(0) = \int_{-∞}^{∞} dω Q_S(ω), \]  

(76)

where

\[ Q_S(ω) = [Z(T)]^{-1} ∑_{m,n} e^{-βT_m} |⟨n|S|m⟩|^2 δ(ω - ω_{nm}) ≥ 0, \]  

(77)

is the dynamic structure factor (DSF) relative to the operator \(S\), where \(ω\) is a real frequency and \(ω_{nm} = T_n - T_m\) \( (ℏ = 1)\). DSF obeys the relation

\[ Q_S(ω) = e^{βω} Q_S(-ω), \]  

(78)

known as the principle of detailed balancing.

Correspondingly the spectral representation of \(K_{SS}(λ)\) is given by Luttinger \textsuperscript{[56]}

\[ K_{SS}(λ) = \int_{-∞}^{∞} dω Q_S(ω; λ), \quad Q_S(ω; λ) = Q_S(ω)e^{λβω}. \]  

(79)

Hence, as follows from Eqs. \(71\) and \(79\) we get

\[ (S; S)_T = \int_{-∞}^{∞} dω \frac{e^{λβω} - 1}{βω} Q_S(ω), \]  

(80)

which is the original expression obtained by Bogoliubov\textsuperscript{[26]}. For the further consideration it is useful to introduce the moments of the DSF

\[ M_p(S) := \int_{-∞}^{+∞} dω^p Q_S(ω), \quad p = -1, 0, 1, 2, ... \]  

(81)

Then another alternative form of Eq.\(71\) is

\[ (S; S)_T = β^{-1} [M_{-1}(S) + M_{-1}(S^+)]. \]  

(82)

Eq.\(82\) directly follows from Eqs. \(77\), \(78\) and \(81\).

In our paper \textsuperscript{[35]} the functionals \(F_n(S; S), n = 0, 1, 2, ...\)

\[ F_n(S; S) := 2^{n-1} ∑_{ml} |⟨m|S|l⟩|^2 |ρ_l - (-1)^n ρ_m|.|X_{ml}|^{n-1}, \]  

(83)

have been introduced as a generalization of the Bogoliubov-Duhamel inner product [Eq.\(75\)]. Some applications of these functionals have been demonstrated in \textsuperscript{[32, 57, 38]}. Since \(F_0(S; S) = (S; S)_T\) hereafter in the text we shall use this notation for the Bogoliubov-Duhamel inner product. Recall that \(F_1(S; S) = (S^+S + SS^+)_T.\) Using the notion of iterated commutators

\[ R_0 = R_0(S) := S, \quad R_1 = R_1(S) := [T, S], \quad \ldots, \quad R_l = R_l(S) := [T, R_{l-1}(S)]. \]  

(84)

one has for even \(n = 2l, \quad l = 0, 1, 2, 3, ...\)

\[ F_{2l}(S; S) = β^{2l}(R_l; R_l)_T = β^{2l-1} ([R_l^+ R_{l-1}^+ - R_{l-1} R_l^+])_T, \]  

(85)

where \(R_l^+\) denotes the Hermitian conjugate of \(R_l\) and by definition \(R_{-1} = X_{ST}\) is a solution of the operator equation

\[ S = [T, X_{ST}]. \]  

(86)

In view of relations \(85\), the functional \(F_{2l}(S; S)\) may be called \"Bogoliubov-Duhamel inner product of order \(l\\".

In the case of odd \(n = 2l + 1, \quad l = 0, 1, 2, 3, ...\), one has

\[ F_{2l+1}(S; S) = β^{2l} ([R_l^+ R_l + R_l R_l^+])_T. \]  

(87)
It is remarkable that both formulas (85) and (87) may be written as one formula in the form
\[ F_n(S; S) = \beta^{n-1}[(-1)^{n-1}\langle R_{n-1}S^+\rangle_T + \langle R_{n-1}^+S\rangle_T], \quad n = 0, 1, 2, ... \] (88)
which is more useful in some cases. Eq. (88) is obtained from Eq. (85) and Eq. (87) with the successively using of the following identities
\[ \langle R_{i-1}^+R_i \rangle_T = \langle R_{i-1}^+R_i \rangle_T, \quad \langle R_{i-1}R_i^+ \rangle_T = \langle R_iR_{i-1}^+ \rangle_T, \]
\[ \langle R_i^+R_i \rangle_T = \langle R_{i-1}R_{i+1} \rangle_T, \quad \langle R_iR_i^+ \rangle_T = \langle R_{i+1}R_{i-1}^+ \rangle_T, \] (89)
which are simple consequence of the cyclic property of the trace and the definition of the iterated commutators (84).

Setting \( n = 0 \) in Eq. (88) one gets the relation
\[ F_0(S, S) = -\beta^{-1}[(\langle R-1S^+ \rangle_T - \langle R^+1S \rangle_T). \] (90)

It presents the Bogoliubov-Duhamel inner product as a thermodynamic mean value of the solution of Eq. (86) and \( S \).

From the other side the functionals \( F_n(S; S) \) defined by Eq. (88) may be obtained using the definition of the moments of the DSF Eq. (81) and the relation (78), i.e.
\[ F_n(S; S) = \beta^{n-1}[M_{n-1}(S) + M_{n-1}(S^+)], \quad n = 0, 1, 2, ... \] (91)

Now, it is easy to check that
\[ M_{n-1}(S) = (-1)^{n-1}\langle R_{n-1}S^+ \rangle_T, \quad M_{n-1}(S^+) = \langle R_{n-1}^+S \rangle_T, \] (92)
which are not but the well known sum rules for the moments of the DSF in the linear response theory (see, e.g. [52] and [53], where some sum rules for the lowest moments are presented). The expressions Eq. (92) provide an algebraic way to evaluate the moments of the DSF.

The expressions (85) and (87) take a simpler form in the case \( S = S^+ \) that is required for the calculation in the text of the paper. From Eq. (88) with the observation that \( R_{n, n}^+ = (-1)^{n}R_{n} \), \( n = 0, 1, 2, ... \) we obtain
\[ F_n(S; S) = 2(-1)^{n+1}\beta^{n-1}\langle R_{n-1}S \rangle_T, \quad n = 0, 1, 2, ... \] (93)
or alternatively from Eq. (91)
\[ F_n(S; S) = 2\beta^{n-1}M_{n-1}(S), \quad n = 0, 1, 2, ... \] (94)

We note that \( K_{SS}(\lambda) \) has the properties of the scalar product and has been studied separately [32, 34] because of its relation with the Wigner, Yanase and Dyson (WYD) skew information
\[ I_{S,S}(\lambda) = -\frac{1}{2}[Z(T)]^{-1}Tr \left( e^{-\beta\lambda T}[e^{-\beta(1-\lambda)T}S][e^{-\beta(1-\lambda)T}, S] \right), \quad 0 \leq \lambda \leq 1. \] (95)
via the relation
\[ K_{SS}(\lambda) = K_{SS}(0) - I_{SS}(\lambda). \] (96)
The Eq. (96) allows to emphasize the relation of \( F_n(S; S) \) with the WYD information (95). In our further consideration we shall follow ref. [34]. Using the derivatives
\[ \frac{d}{d\lambda}e^{-\beta\lambda T} = -\beta e^{-\beta\lambda T}, \quad \frac{d}{d\lambda}e^{-\beta(1-\lambda)T} = \beta e^{-\beta(1-\lambda)T}, \] (97)
after some simple algebra one obtains
\[ \frac{d^n}{d\lambda^n}K_{SS}(\lambda) = (-1)^{n}\beta^nK_{SR_n}(\lambda), \quad n = 0, 1, 2, ..., \] (98)
where \( R_n = [T, R_{n-1}], \) and \( R_0 = S \). The properties of these derivatives are studied in details in [34] emphasizing the relation with the Bogoliubov and Tyablicov Green’s function method. This seems to provide an useful method for their calculation. Our finding is that (if \( S^+ = S \))
\[ F_n(S; S) = 2\beta^{-1}\frac{d^{(n-1)}}{d\lambda^{(n-1)}}K_{SS}(\lambda)|_{\lambda=0}, \quad n = 1, 2, ... \] (99)
or equivalently

\[ F_n(S; S) = -2\beta^{-1} \frac{d^{(n-1)}}{d\lambda^{(n-1)}} I_{SS}(\lambda)|_{\lambda=0}, \quad n = 1, 2, \ldots \]  

(100)
due to Eq. (95). Differentiating and integrating Eq. (79) with respect to \( \lambda \), one obtains [34]:

\[ \frac{d^n}{d\lambda^n} K_{SS}(\lambda) = \beta^n \int_{-\infty}^{\infty} d\omega \omega^n Q_S(\omega; \lambda), \quad n = 0, 1, 2, \ldots \]  

(101)
and

\[ \int d\lambda K_{SS}(\lambda) = \beta^{-1} \int_{-\infty}^{\infty} d\omega \omega^{-1} Q_S(\omega; \lambda), \]  

(102)
respectively. At \( \lambda = 0 \), these are proportional to the moments of the DSF

\[ M_n(S) := \int_{-\infty}^{\infty} d\omega \omega^n Q_S(\omega), \quad n = -1, 0, 1, 2, \ldots \]  

(103)
Finally, let us note that from Eqs. (99) and (101) we obtain Eq. (94) as it must be.

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