Stochastic Porous Media Equation on General Measure Spaces with Increasing Lipschitz Nonlinearities *

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Abstract. We prove the existence and uniqueness of probabilistically strong solutions to stochastic porous media equations driven by time-dependent multiplicative noise on a general measure space \((E, \mathcal{B}, \mu)\), and the Laplacian replaced by a negative definite self-adjoint operator \(L\). In the case of Lipschitz nonlinearities \(\Psi\), we in particular generalize previous results for open \(E \subset \mathbb{R}^d\) and \(L=\text{Laplacian}\) to fractional Laplacians. We also generalize known results on general measure spaces, where we succeeded in dropping the transience assumption on \(L\), in extending the set of allowed initial data and in avoiding the restriction to superlinear behavior of \(\Psi\) at infinity for \(L^2(\mu)\)-initial data.

Keywords: Wiener process; Porous media equation; Sub-Markovian contractive semigroup.

1 Introduction

In this paper, we consider stochastic porous media equations (SPMEs) of the following type:

\[
\begin{align*}
    dX(t) - L\Psi(X(t))dt &= B(t, X(t))dW(t), \quad \text{in } [0, T] \times E, \\
    X(0) &= x \text{ on } E \text{ (with } x \in F_{1,2}^{*} \text{ or } L^2(\mu)),
\end{align*}
\]

where \(T \in (0, \infty)\) is fixed, \(L\) is the negative definite self-adjoint generator of a sub-Markovian strongly continuous contraction semigroup \((P_t)_{t \geq 0}\) on \(L^2(\mu) := L^2(E, \mathcal{B}, \mu)\), and \((E, \mathcal{B}, \mu)\) is a standard measurable space \((\Omega, \mathcal{F}, \mathbb{P})\) with normal filtration \((\mathcal{F}_t)_{t \geq 0}\). \(\Psi(\cdot): \mathbb{R} \to \mathbb{R}\) is a monotonically nondecreasing Lipschitz continuous function, \(B\) is a progressively measurable process in the space of Hilbert-Schmidt operator from \(L^2(\mu)\) to \(F_{1,2}^{*}\), \(W(t)\) is an \(L^2(\mu)\)-valued cylindrical \(\mathcal{F}_t\)-adapted Wiener process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with normal filtration \((\mathcal{F}_t)_{t \geq 0}\).

For the definition of the Hilbert space \(F_{1,2}^{*}\) and the precise conditions on \(B\) we refer to the next section.

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In the special case when $E = \mathbb{R}^d$, $L$ is equal to the Laplace operator $\Delta$ and $B$ is time-independent linear multiplicative, equation (1.1) was recently analyzed in [2]. The aim of this paper is to prove analogous results as in [2] for the general case. The above framework is inspired by the work of Fukushima and Kaneko [1] (see also [3]).

The main motivation for this generality is that we would like to cover fractional powers of the Laplacian, i.e., $L = -(-\Delta)^{\alpha}$, $\alpha \in (0, 1)$, generalized Schrödinger operators, i.e., $L = \Delta + \frac{2 \rho }{\eta} \cdot \nabla$, and Laplacians on fractals (see Section 4 below).

Recently, there has been much work on stochastic versions of the porous media equations. Based on the variational approach and monotonicity assumptions on the coefficients, [13] presents a generalization of Krylov-Rozovskii’s result [2] on the existence and uniqueness of solutions to monotone stochastic differential equations, which applies to a large class of stochastic porous media equations. It should be said that in [13] (see also [14]), $\Psi$ is of solutions to monotone stochastic differential equations, which applies to a large class of stochastic porous media equations. It should be said that in [13] (see also [14]), $\Psi$ is assumed to be continuous such that $r \rightarrow \infty$ as $r \rightarrow \infty$. In this paper we show that for Lipschitz continuous $\Psi$ this condition can be dropped for initial data in $L^2(\mu)$, extending the corresponding result from [2] to general operators $L$ as above. We would also like to emphasize that in contrast to [13, 14], in this paper, we do not assume that $L$ is the generator of a transient Dirichlet form on $L^2(E, \mathcal{B}, \mu)$. In our case we can drop the transience assumption. In particular, in contrast to [13] (and [14]), we do not need any restriction on $d$ when $E = \mathbb{R}^d$ and $L = -(-\Delta)^{\alpha}$, $\alpha \in (0, 1)$. For more references on stochastic porous media equations we refer to [1]. In addition, we work in the state space $\mathcal{F}_{1,2}$ which is larger than the state space $\mathcal{F}_{r,2}$ considered in [13], hence we can allow more general initial conditions (as done in [14] under assumptions much stronger than transience).

Section 4 of [2] deals with the case where $\Psi$ is a maximal monotone multivalued function as above. We would also like to extend also this result to our more general equation (1.1). This will be the subject of our future work.

The paper is organized as follows: in Section 2, we recall some notions concerning sub-Markovian semi-groups and introduce a suitable Gelfand triple. Section 3 is devoted to verify the existence and uniqueness of strong solutions to (1.1). Note that the Riesz isomorphism $1 - L$, through which we identify $H := \mathcal{F}_{1,2}^*$ and $H^* := \mathcal{F}_{1,2}$, plays an essential role in the proof. In Section 4, we will apply our results to a number of examples.

## 2 Preliminaries

First of all, let us recall some basic definitions and spaces which will be used throughout the paper (see [3, 4, 5]).

Let $(E, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space. Let $\{P_t\}_{t \geq 0}$ be a strongly continuous contraction sub-Markovian semigroup on $L^2(\mu)$ with negative definite self-adjoint generator $(L, D(L))$.

The gamma-transform $V_r(r > 0)$ of $\{P_t\}_{t \geq 0}$ is defined by

$$V_r = \Gamma\left(\frac{r}{2}\right)^{-1} \int_0^{\infty} s^{\frac{r}{2} - 1} e^{-s} P_s ds.$$  

In this paper, we consider the Hilbert space $(\mathcal{F}_{1,2}, \| \cdot \|_{\mathcal{F}_{1,2}})$ defined by $\mathcal{F}_{1,2} = V_1(L^2(\mu))$, with norm $\|u\|_{\mathcal{F}_{1,2}} = |f|_{\mathcal{F}_{1,2}}$ for $u = V_1 f, \ f \in L^2(\mu)$, where the norm $| \cdot |_2$ is defined as $|f|_2 = (\int_E |f|^2 d\mu)^{\frac{1}{2}}$. In particular,

$$V_1 = (1 - L)^{-\frac{1}{2}},$$

so that $\|u\|_{\mathcal{F}_{1,2}} = |V_1^{-1}u|_2 = |(1 - L)^{\frac{1}{2}}u|_2$. 


The dual space of $F_{1,2}$ is denoted by $F^*_{1,2}$.

Let $H$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and $H^*$ its dual. Let $V$ be a reflexive Banach space, such that $V \subset H$ continuously and densely. Then for its dual space $V^*$ it follows that $H^* \subset V^*$ continuously and densely. Identifying $H$ and $H^*$ via the Riesz isomorphism we have that

$$V \subset H \subset V^*$$

continuously and densely and if $\nu^* \langle \cdot, \cdot \rangle_V$ denotes the dualization between $V^*$ and $V$ (i.e. $\nu^* \langle z, v \rangle_V := z(v)$ for $z \in V^*$, $v \in V$), it follows that

$$\nu^* \langle z, v \rangle_V = \langle z, v \rangle_H, \text{ for all } z \in H, \ v \in V.$$

$(V, H, V^*)$ is called a Gelfand triple.

In the following, we concentrate on finding a suitable Gelfand triple $V \subset H \equiv H^* \subset V^*$ with $H := F^*_{1,2}$. Let $F_{1,2} \langle \cdot, \cdot \rangle_{F_{1,2}}$ denote the duality between $F_{1,2}$ and $F^*_{1,2}$. Define $(1-L) : F_{1,2} \rightarrow F^*_{1,2}$ as follows, given $u \in F_{1,2}$,

$$F_{1,2} \langle (1-L)u, v \rangle_{F_{1,2}} := \int_E (1-L)^{\frac{1}{2}}u \cdot (1-L)^{\frac{1}{2}}v \, d\mu \text{ for all } v \in F_{1,2}. \tag{2.1}$$

To show that $(1-L) : F_{1,2} \rightarrow F^*_{1,2}$ is well-defined, we have to prove that the right-hand side of $(2.1)$ defines a linear continuous function on $v \in F_{1,2}$ with respect to $\| \cdot \|_{F_{1,2}}$. But for $u \in F_{1,2}$, we have for all $v \in F_{1,2}$,

$$|F_{1,2} \langle (1-L)u, v \rangle_{F_{1,2}}| = \left| \int_E (1-L)^{\frac{1}{2}}u \cdot (1-L)^{\frac{1}{2}}v \, d\mu \right|$$

$$= \left| \langle (1-L)^{\frac{1}{2}}u, (1-L)^{\frac{1}{2}}v \rangle_2 \right|$$

$$\leq \| (1-L)^{\frac{1}{2}}u \|_2 \cdot \| (1-L)^{\frac{1}{2}}v \|_2$$

$$= \|u\|_{F_{1,2}} \cdot \|v\|_{F_{1,2}}.$$

This implies

$$\|(1-L)u\|_{F^*_{1,2}} \leq \|u\|_{F_{1,2}}.$$

Now we would like to identify $F^*_{1,2}$ with its dual $F_{1,2}$ via the corresponding Riesz isomorphism $R : F^*_{1,2} \rightarrow F_{1,2}$ defined by $Rx = \langle x, \cdot \rangle_{F^*_{1,2}}, \ x \in F^*_{1,2}$.

**Lemma 2.1** The map $(1-L) : F_{1,2} \rightarrow F^*_{1,2}$ is an isometric isomorphism. In particular,

$$\langle (1-L)u, (1-L)v \rangle_{F^*_{1,2}} = \langle u, v \rangle_{F_{1,2}} \text{ for all } u, v \in F_{1,2}. \tag{2.2}$$

Furthermore, $(1-L)^{-1} : F^*_{1,2} \rightarrow F_{1,2}$ is the Riesz isomorphism for $F^*_{1,2}$, i.e., for every $u \in F^*_{1,2}$,

$$\langle u, \cdot \rangle_{F^*_{1,2}} = F_{1,2} \langle (1-L)^{-1}u, \cdot \rangle_{F^*_{1,2}}. \tag{2.3}$$

**Proof** For all $u, v \in F_{1,2}$, by $(2.1)$ we know

$$F_{1,2} \langle (1-L)u, v \rangle_{F_{1,2}} = \langle (1-L)^{\frac{1}{2}}u, (1-L)^{\frac{1}{2}}v \rangle_2 = \langle u, v \rangle_{F_{1,2}},$$

i.e., $(1-L) : F_{1,2} \rightarrow F^*_{1,2}$ is the Riesz isomorphism for $F_{1,2}$. 

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In particular, for all $u, v \in F_{1,2}$, since the Riesz isomorphism is isometric,
\[ \langle (1 - L)u, (1 - L)v \rangle_{F_{1,2}^*} = \langle u, v \rangle_{F_{1,2}^*}. \]  
(2.4)

Furthermore, for all $u, v \in F_{1,2}^*$,
\[ \langle u, v \rangle_{F_{1,2}^*} = \langle (1 - L)^{-1}u, (1 - L)^{-1}v \rangle_{F_{1,2}^*} = \langle (1 - L)^{-1}u, v \rangle_{F_{1,2}^*}. \]

In this sense, we identify $F_{1,2}^*$ with $F_{1,2}$ via the Riesz map $(1 - L)^{-1} : F_{1,2}^* \to F_{1,2}$, thus $F_{1,2}^* \equiv F_{1,2}$. Note that $L^2(\mu)$ can be considered as a subset of $F_{1,2}^*$, since for $u \in L^2(\mu)$, the map
\[ v \mapsto \langle u, v \rangle_2, \quad v \in F_{1,2}, \]
belongs to $F_{1,2}^*$. Here $(\cdot, \cdot)_2$ denotes the usual inner product on $L^2(\mu)$. Obviously, in this sense $L^2(\mu) \subset F_{1,2}^*$ continuously and densely. Consequently, we get a Gelfand triple with $V := L^2(\mu)$, $H := F_{1,2}^*$,
\[ V = L^2(\mu) \subset F_{1,2}^* \subset (L^2(\mu))^*, \]
which satisfies
\[ v \cdot \langle u, v \rangle_V = \langle u, v \rangle_H, \quad \text{for all } u \in F_{1,2}^*, v \in L^2(\mu). \]  
(2.5)

**Lemma 2.2** The map
\[ 1 - L : F_{1,2} \to F_{1,2}^* \]
extends to a linear isometry
\[ 1 - L : L^2(\mu) \to (L^2(\mu))^*, \]
and for all $u, v \in L^2(\mu)$,
\[ (L^2(\mu))^* \langle (1 - L)u, v \rangle_{L^2(\mu)} = \int_E u \cdot v \, d\mu. \]  
(2.6)

**Proof** Let $u \in F_{1,2}$. Since $(1 - L)u \in F_{1,2}^*$, from (2.3) and (2.5) we obtain that for all $v \in L^2(\mu)$,
\[ (L^2(\mu))^* \langle (1 - L)u, v \rangle_{L^2(\mu)} = \langle (1 - L)u, v \rangle_{F_{1,2}^*} = \langle u, v \rangle_{F_{1,2}^*} = \langle u, v \rangle_2, \]  
(2.7)
the last equality holds since $F_{1,2} \subset L^2(\mu) \subset F_{1,2}^*$ densely and continuously. Therefore,
\[ \| (1 - L)u \|_{(L^2(\mu))^*} \leq |u|_2. \]

In this sense, $1 - L$ extends to a continuous linear map
\[ 1 - L : L^2(\mu) \to (L^2(\mu))^* \]
such that (2.7) holds for all $u \in L^2(\mu)$, i.e., (2.6) is proved.

So, applying it to $u \in L^2(\mu)$ and
\[ v := |u|^{-1}_2 u \in L^2(\mu), \]
by (2.7) we obtain that
\[ v \cdot \langle (1 - L)u, v \rangle_V = \langle u, v \rangle_2 = \langle u, |u|^{-1}_2 u \rangle_2 = |u|_2, \]
and \(|v|_2 = 1\), so \(|(1 - L)u|_{V^*} = |u|_V\) and the assertion is completely proved.

Consider the quadratic form \((\mathcal{E}, D(\mathcal{E}))\) on \(L^2(\mu)\) associated with \((L, D(L))\), i.e.

\[
D(\mathcal{E}) := F_{1,2}
\]

and

\[
\mathcal{E}(u, v) := \mu(\sqrt{-Lu}\sqrt{-Lv}); \quad u, \ v \in F_{1,2}.
\]

From \([10]\), we know \((L, D(L))\) is indeed the associated Dirichlet operator on \(L^2(\mu)\).

Throughout the paper, let \(L^2([0, T] \times \Omega; L^2(\mu))\) denote the space of all \(L^2(\mu)\)-valued square-integrable functions on \([0, T] \times \Omega\), and \(C([0, T]; F_{1,2}^*)\) the space of all continuous \(F_{1,2}^*\)-valued functions on \([0, T]\). For two Hilbert spaces \(H_1\) and \(H_2\), the space of Hilbert-Schmidt operators from \(H_1\) to \(H_2\) is denoted by \(L_2(H_1, H_2)\). For simplicity, the positive constants \(c, C, C_i, i = 1, 2, 3\) used in this paper may change from line to line. We would like to refer \([11]\) for more background information and results on SPMEs.

### 3 The Main Result

Consider \((1.1)\) under the following conditions:

**(H1)** \(\Psi(\cdot) : \mathbb{R} \rightarrow \mathbb{R}\) is a monotonically nondecreasing Lipschitz function with \(\Psi(0) = 0\).

**(H2)** \(B : [0, T] \times L^2(\mu) \times \Omega \rightarrow L_2(L^2(\mu), F_{1,2}^*)\) is progressively measurable, i.e. for any \(t \in [0, T]\), this mapping restricted to \([0, t] \times L^2(\mu) \times \Omega\) is measurable w.r.t. \(\mathcal{B}([0, t]) \times \mathcal{B}(L^2(\mu)) \times \mathcal{F}_t\), where \(\mathcal{B}(\cdot)\) is the Borel \(\sigma\)-field for a topological space. For simplicity, below we will write \(B(t, u)\) meaning the mapping \(\omega \mapsto B(t, u, \omega)\), and \(B(t, u)\) satisfies

1. \(\text{(i)}\) there exists \(C_1 \in [0, \infty)\) satisfying

\[
\|B(\cdot, u) - B(\cdot, v)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2 \leq C_1\|u - v\|_{F_{1,2}}^2 \quad \text{for all } u, v \in L^2(\mu) \text{ on } [0, T] \times \Omega;
\]

2. \(\text{(ii)}\) there exists \(C_2 \in (0, \infty)\) satisfying

\[
\|B(\cdot, u)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2 \leq C_2\|u\|_{F_{1,2}}^2 \quad \text{for all } u \in L^2(\mu) \text{ on } [0, T] \times \Omega.
\]

3. \(\text{(iii)}\) there exists \(C_3 \in (0, \infty)\) satisfying

\[
\|B(\cdot, u)\|_{L_2(L^2(\mu), L^2(\mu))}^2 \leq C_3\|u\|_2^2 + 1 \quad \text{for all } u \in L^2(\mu) \text{ on } [0, T] \times \Omega.
\]

**Remark 3.1** In the following, we cite one example from \([14\text{, Section 2}]\) of \(B\), which satisfies \((H2)(ii)\).

**(M)** Let \(N \in \mathbb{N} \cup \{+\infty\}\) and \(e_k \in L^2(\mu) \cap L^\infty(\mu), 1 \leq k \leq N\), be an orthonormal system in \(L^2(\mu)\) such that for every \(1 \leq k \leq N\) there exists \(\xi_k \in (0, \infty)\) such that for all \(a \in (0, \infty)\)

\[
|\left(F_{1,2} \langle x, e_k u \rangle F_{1,2}^*\right)| \leq \xi_k \|x\|_{H_a} \mathcal{E}_a(u, u)^{1/2}, \quad \text{for all } u \in D(\mathcal{E}),
\]

where \(\mathcal{E}_a := a \mathcal{E} + \langle \cdot, \cdot \rangle\) on \(D(\mathcal{E})\) and \(\|\cdot\|_{H_a}\) denotes the corresponding norm on the dual space of \(D(\mathcal{E})\).
Choose \( \mu_k \in (0, \infty) \) such that
\[
\sum_{k=1}^{\infty} \xi_k^2 \mu_k^2 < \infty,
\]
and define for \( x \in H, \ B(x) \in L_2(L^2(\mu); H) \) by
\[
B(x)h := \sum_{k=1}^{\infty} \mu_k(x_k, h) e_k, \ h \in L^2(\mu).
\]
Indeed, (extending \( \{e_k\} \) to an orthonormal basis of \( L^2(\mu) \)) by (M) we have for \( x \in H, a \in (0, \infty) \)
\[
\|B(x)\|_{L^2(L^2(\mu), H_a)}^2 = \sum_{k=1}^{\infty} \|B(x)e_k\|_{H_a}^2
\]
\[
= \sum_{k=1}^{\infty} \mu_k^2 \|x_k\|_{H_a}^2.
\]
and since \( x \to B(x) \) is linear and \( V \subset H \), condition (H2)(ii) follows. For more examples, we refer to [14, Section 2].

**Definition 3.1** Let \( x \in F_{1,2}^* \). A continuous \((\mathcal{F}_t)_{t \geq 0}\)-adapted process \( X : [0, T] \to F_{1,2}^* \) is called strong solution to (1.1) if the following conditions are satisfied:
\[
X \in L^2([0, T] \times \Omega; L^2(\mu)) \cap L^2(\Omega; C([0, T]; F_{1,2}^*)),
\tag{3.1}
\]
\[
\int_0^t \Psi(X(s))ds \in C([0, T]; F_{1,2}), \ \mathbb{P}\text{-a.s.}, \tag{3.2}
\]
\[
X(t) - L \int_0^t \Psi(X(s))ds = x + \int_0^t B(s, X(s))dW(s), \ \forall t \in [0, T], \ \mathbb{P}\text{-a.s.} \tag{3.3}
\]

**Theorem 3.1** Suppose \((H1)\) and \((H2)\) are satisfied. Then, for each \( x \in L^2(\mu) \), there is a unique strong solution \( X \) to (1.1) and exists \( C \in [0, \infty) \) satisfying
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} X(t)^2 \right] 
\leq 2|x|^2 e^{CT}.
\]
Assume further that
\[
\Psi(r)r \geq cr^2, \ \forall r \in \mathbb{R}, \tag{3.4}
\]
where \( c \in (0, \infty) \). Then, there is a unique strong solution \( X \) to (1.1) for all \( x \in F_{1,2}^* \).

For the proof of the above theorem, we firstly consider the approximating equations for (1.1):
\[
\begin{cases}
    dX^\nu(t) + (\nu - L)\Psi(X^\nu(t))dt = B(t, X^\nu(t))dW(t), \text{ in } (0, T) \times E, \\
    X^\nu(0) = x \text{ on } E,
\end{cases}
\tag{3.5}
\]
where \( \nu \in (0, 1) \). And we have the following result for (3.5).
Lemma 3.1 Suppose \((H1)\) and \((H2)\) are satisfied. Then, for each \(x \in L^2(\mu)\), there is a unique \((\mathcal{F}_t)_{t \geq 0}\)-adapted solution to \((3.3)\), denoted by \(X^\nu\), i.e., in particular it has the following properties,

\[
X^\nu \in L^2([0, T] \times \Omega; L^2(\mu)) \cap L^2(\Omega; C([0, T]; F_{1,2}^*)),
\]

\[
X^\nu(t) + (\nu - L) \int_0^t \Psi(X^\nu(s))ds = x + \int_0^t B(s, X^\nu(s))dW(s), \forall t \in [0, T], \mathbb{P} - a.s.. \tag{3.7}
\]

Furthermore, there exists \(C \in (0, \infty)\) such that for all \(\nu \in (0, 1)\),

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X^\nu(t)|^2 \right] \leq 2|x|_2^2 e^{CT}. \tag{3.8}
\]

In addition, if \((3.4)\) is satisfied, there is a unique solution \(X^\nu\) to \((3.3)\) satisfying \((3.6)\) and \((3.7)\) for all \(x \in F_{1,2}^*\).

**Proof** We proceed in two steps. In Step 1, we consider the case when the initial value \(x \in F_{1,2}^*\) and that \((3.4)\) is satisfied. In Step 2, we will prove the existence and uniqueness result when \(x \in L^2(\mu)\) and without assumption \((3.4)\), by replacing \(\Psi\) with \(\Psi + \lambda I\), \(\lambda \in (0, 1)\) and then letting \(\lambda \to 0\).

**Step 1:** Assume \(x \in F_{1,2}^*\) and that \((3.4)\) is satisfied. Set \(V := L^2(\mu)\), \(H := F_{1,2}^*\), \(Au := (L - \nu)\Psi(u)\) for \(u \in V\). The space \(F_{1,2}^*\) is equipped with the equivalent norm

\[
\|\eta\|_{F_{1,2}^*} := \langle \eta, (\nu - L)^{-1}\eta \rangle_V^\frac{1}{2}, \quad \eta \in F_{1,2}^*.
\]

Under the Gelfand triple \(V \subset H \subset V^*\), we shall prove the existence and uniqueness of the solution to \((3.5)\) by using [9] Theorem 4.2.4 (or [12] Theorem 4.2.4).

In the following, we shall verify the four conditions of the existence and uniqueness theorem in [9, 12].

(i) (Hemicontinuity)

Let \(u, v, w \in V = L^2(\mu)\). We have to show for \(\lambda \in \mathbb{R}, |\lambda| \leq 1\),

\[
\lim_{\lambda \to 0} \nu \cdot \langle A(u + \lambda v), w \rangle_V - \nu \cdot \langle Au, w \rangle_V = 0.
\]

By Lemma 2.2

\[
\nu \cdot \langle A(u + \lambda v), w \rangle_V = \nu \cdot \langle (L - \nu)\Psi(u + \lambda v), w \rangle_V
\]

\[
= -\nu \cdot \langle (1 - L)\Psi(u + \lambda v), w \rangle_V + (1 - \nu)\nu \cdot \langle (1 - L)(1 - L)^{-1}\Psi(u + \lambda v), w \rangle_V
\]

\[
= -\langle \Psi(u + \lambda v), w \rangle_V + (1 - \nu)\langle (1 - L)^{-1}\Psi(u + \lambda v), w \rangle_V
\]

\[
= -\int_E \Psi(u + \lambda v) \cdot wd\mu + (1 - \nu)\int_E (1 - L)^{-1}\Psi(u + \lambda v) \cdot wd\mu.
\]

By the Lipschitz continuity of \(\Psi\) and denoting \(k := \text{Lip}\Psi\), the first integrand in the right-hand side of the above equality is bounded by

\[
|\Psi(u + \lambda v)| \cdot |w| \leq k(|u| + |v|) \cdot |w|,
\]
which by Hölder’s inequality is in \( L^1(\mu) \). Since \((1 - L)^{-1}\) is a contraction on \( L^2(\mu) \) ([10, Chapter I]), in order to prove the convergence of \((1 - L)^{-1}\Psi(u + \lambda v) \cdot w\) in \( L^1(\mu) \), it is sufficient to show the convergence of \( \Psi(u + \lambda v) \) in \( L^2(\mu) \), which is obvious because \( \Psi \) is Lipschitz and
\[
|\Psi(u + \lambda v)| \leq k(|u| + |v|).
\]

(ii) (Weak Monotonicity)
Let \( u, v \in V = L^2(\mu) \), then by Lemma 2.2 and (2.5)
\[
2 \nu \cdot (Au - Av, u - v)_V + \|B(\cdot, u) - B(\cdot, v)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2
= 2 \nu \cdot ((1 - \nu)(\Psi(u) - \Psi(v)), u - v)_V + \|B(\cdot, u) - B(\cdot, v)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2
= -2 \nu \cdot ((1 - \nu)(\Psi(u) - \Psi(v)), u - v)_V + 2 (1 - \nu) \nu \cdot (\Psi(u) - \Psi(v), u - v)_V
+ \|B(\cdot, u) - B(\cdot, v)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2
= -2 \nu ||\Psi(u) - \Psi(v)||_V^2 + 2 (1 - \nu) \nu ||\Psi(u) - \Psi(v)||_V^2 + \|B(\cdot, u) - B(\cdot, v)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2.
\]
Set \( \tilde{\alpha} := (Lip \Psi + 1)^{-1} \). By assumption (H1) on \( \Psi \), we know that
\[
(\Psi(r) - \Psi(r'))(r - r') \geq \tilde{\alpha} ||\Psi(r) - \Psi(r')||^2, \quad \forall r, r' \in \mathbb{R}.
\]
Since \( L^2(\mu) \subset F_{1,2}^* \) continuously, by Young’s inequality
\[
||\Psi(u) - \Psi(v)||_F_{1,2}^* \leq ||\Psi(u) - \Psi(v)||_{F_{1,2}^*} \cdot ||u - v||_{F_{1,2}^*} \leq \frac{\tilde{\alpha}}{1 - \nu} ||\Psi(u) - \Psi(v)||_V^2 + \frac{1 - \nu}{\tilde{\alpha}} ||u - v||_{F_{1,2}^*}^2.
\]
By (H2) (i), and taking (3.10), (3.11) into account, (3.9) is dominated by
\[
-2 \tilde{\alpha} ||\Psi(u) - \Psi(v)||_V^2 + 2 \tilde{\alpha} ||\Psi(u) - \Psi(v)||_V^2 + \frac{2(1 - \nu)^2}{\tilde{\alpha}} ||u - v||_{F_{1,2}^*}^2 + C_1 ||u - v||_{F_{1,2}^*}^2
= \left[ \frac{2(1 - \nu)^2}{\tilde{\alpha}} + C_1 \right] \cdot ||u - v||_{F_{1,2}^*}^2.
\]
Hence weak monotonicity holds.

(iii) (Coercivity)
Let \( u \in L^2(\mu) \). By Lemma 2.2 and (2.5)
\[
2 \nu \cdot (Au, u)_V + \|B(\cdot, u)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2
= -2 \nu \cdot ((1 - \nu) \Psi(u), u)_V + 2 (1 - \nu) \nu \cdot (\Psi(u), u)_V + \|B(\cdot, u)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2
= -2 \nu ||\Psi(u)||_V^2 + 2 (1 - \nu) \nu ||\Psi(u)||_V^2 + \|B(\cdot, u)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2.
\]
By (3.4)
\[
-2 \nu ||\Psi(u)||_V^2 = -2 \int_E \Psi(u) \cdot u \, d\mu \leq -2c ||u||_{F_{1,2}^*}^2.
\]
Since \( L^2(\mu) \subset F_{1,2}^* \) continuously, by Young’s inequality for \( \varepsilon \in (0, 1) \)
\[
\langle \Psi(u), u \rangle_{F_{1,2}^*} \leq \|\Psi(u)\|_{F_{1,2}} \cdot \|u\|_{F_{1,2}}^2 \\
\leq \|\Psi(u)\|_2 \cdot \|u\|_{F_{1,2}}^2 \\
\leq \varepsilon^2 k^2 |u|_2^2 + \frac{1}{\varepsilon^2} \|u\|_{F_{1,2}}^2. \tag{3.14}
\]

By (H2) (ii), and taking (3.13) and (3.14) into account, (3.12) is dominated by
\[
\left[ -2c + 2\varepsilon^2 k^2 (1 - \nu) \right] |u|_2^2 + \left[ \frac{2(1 - \nu)}{\varepsilon^2} + C_2 \right] \cdot \|u\|_{F_{1,2}}^2.
\]

Choosing \( \varepsilon \) small enough, \(-2c + 2\varepsilon^2 k^2 (1 - \nu) \) becomes negative, which implies the coercivity.

(iv) (Boundedness)
Let \( u \in L^2(\mu) \). Since
\[
|Au|_V = |(L - \nu)\Psi(u)|_V = \sup_{|v|_2 = 1} \nu \langle (L - \nu)\Psi(u), v \rangle_V,
\]
by Lemma 2.2 and since \((1 - L)^{-1}\) is a contraction on \( L^2(\mu) \), we deduce
\[
\nu \langle (L - \nu)\Psi(u), v \rangle_V \\
= -\nu \langle (1 - L)\Psi(u), v \rangle_V + (1 - \nu) \nu \langle (1 - L)(1 - L)^{-1}\Psi(u), v \rangle_V \\
= -\langle \Psi(u), v \rangle_2 + (1 - \nu) \langle (1 - L)^{-1}\Psi(u), v \rangle_2 \\
\leq |\Psi(u)|_2 \cdot |v|_2 + (1 - \nu) |\Psi(u)|_2 \cdot |v|_2.
\]
So
\[
|Au|_V \leq 2|\Psi(u)|_2 \leq 2k|u|_2.
\]
Hence the boundedness holds.

By [9, Theorem 4.2.4], there exists a unique solution to (3.15), denoted by \( X^\nu \), which takes values in \( F_{1,2}^* \) and satisfies (3.6) and (3.7).

**Step 2:** If \( \Psi \) does not satisfy (3.4), the above (i), (ii) and (iv) still hold, but (iii) not in general. In this case, we will approximate \( \Psi \) by \( \Psi + \lambda I \), \( \lambda \in (0, 1) \).

Consider the approximating equation:
\[
\begin{align*}
X^\nu_X(t) + (\nu - L)(\Psi(X^\nu_X(t)) + \lambda X^\nu_X(t)) dt &= B(t, X^\nu_X(t))dW(t), \text{ in } [0, T] \times E, \\
X^\nu_X(0) &= x \in F_{1,2}^* \text{ on } E.
\end{align*}
\tag{3.15}
\]

By [9, Theorem 4.2.4], it is easy to prove that there is a solution \( X^\nu_X \) to (3.15) which satisfies \( X^\nu_X \in L^2([-T, T] \times \Omega; L^2(\mu)) \cap L^2(\Omega; C([-T, T]; F_{1,2}^*)) \),
\[
X^\nu_X(t) + (\nu - L) \int_0^t \Psi(X^\nu_X(s)) + \lambda X^\nu_X(s) \, ds = x + \int_0^t B(s, X^\nu_X(s))dW(s), \quad \mathbb{P} - a.s.
\]
and
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|X^\nu_X(t)\|_{F_{1,2}}^2 \right] < \infty. \tag{3.16}
\]

In the following, we want to prove that the sequence \( \{X^\nu_X\} \) converges to the solution of (3.15) as \( \lambda \to 0 \). From now on, we assume that the initial value \( x \in L^2(\mu) \).
Claim 3.1

\[
\mathbb{E}\left[\sup_{s \in [0,T]} \left| X_\lambda^\nu(s) \right|^2 \right] + 4\lambda \mathbb{E} \int_0^t \left\| X_\lambda^\nu(s) \right\|_{F_{1,2}}^2 ds \leq C_T |x|^2 + 1,
\]

where \( C_T \) is independent of \( \nu, \lambda \in (0,1) \).

Proof. Rewrite (3.15), for \( t \in [0,T] \),

\[
X_\lambda^\nu(t) = x + \int_0^t (L - \nu)(\Psi(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s))ds + \int_0^t B(s, X_\lambda^\nu(s))dW(s). \tag{3.17}
\]

For \( \alpha > \nu \), applying the operator \((\alpha - L)^{-\frac{1}{2}} : F_{1,2}^\nu \to L^2(\mu)\) to both sides of the above equation, we get

\[
(\alpha - L)^{-\frac{1}{2}}X_\lambda^\nu(t) = (\alpha - L)^{-\frac{1}{2}}x + \int_0^t (L - \nu)(\alpha - L)^{-\frac{1}{2}}(\Psi(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s))ds
\]
\[
+ \int_0^t (\alpha - L)^{-\frac{1}{2}}B(s, X_\lambda^\nu(s))dW(s).
\]

Applying Itô’s formula ([9, Theorem 4.2.5]) with \( H = L^2(\mu) \), we obtain, for \( t \in [0,T] \),

\[
\left| (\alpha - L)^{-\frac{1}{2}}X_\lambda^\nu(t) \right|^2
\]
\[
= \left| (\alpha - L)^{-\frac{1}{2}}x \right|^2 + 2 \int_{F_{1,2}} \left\langle (L - \nu)(\alpha - L)^{-\frac{1}{2}}\Psi(X_\lambda^\nu(s)), (\alpha - L)^{-\frac{1}{2}}X_\lambda^\nu(s) \right\rangle_{F_{1,2}} ds
\]
\[
+ 2\lambda \int_{F_{1,2}} \left\langle (L - \nu)(\alpha - L)^{-\frac{1}{2}}X_\lambda^\nu(s), (\alpha - L)^{-\frac{1}{2}}X_\lambda^\nu(s) \right\rangle_{F_{1,2}} ds
\]
\[
+ \int_0^t \left\| (\alpha - L)^{-\frac{1}{2}}B(s, X_\lambda^\nu(s)) \right\|_{L^2(\mu)}^2 ds
\]
\[
+ 2 \int_0^t \left\langle (\alpha - L)^{-\frac{1}{2}}X_\lambda^\nu(s), (\alpha - L)^{-\frac{1}{2}}B(s, X_\lambda^\nu(s))dW(s) \right\rangle_{L^2}. \tag{3.18}
\]

To estimate the second term in the right hand side of (3.18), set \( P := (\alpha - \nu)(\alpha - L)^{-1} \).

For \( f \in L^2(\mu) \), we have

\[
(P - I)f = [(\alpha - L)^{-\frac{1}{2}}(\alpha - \nu)(\alpha - L)^{-\frac{1}{2}} - (\alpha - L)^{-\frac{1}{2}}(\alpha - L)(\alpha - L)^{-\frac{1}{2}}]f
\]
\[
= [(\alpha - L)^{-\frac{1}{2}}(L - \nu)(\alpha - L)^{-\frac{1}{2}}]f.
\]

Since \( L \) is the infinitesimal generator of a symmetric sub-Markovian strongly continuous contraction semigroup \((P_t)_{t \geq 0}\) on \( L^2(\mu) \), then, \( P \) is a symmetric sub-Markovian contraction on \( L^2(\mu) \). From [14] Lemma 5.1 (i), there exists a probability kernel \( p \) on \((E, \mathcal{B}, \mu)\) such that for all \( f \in L^2(\mu) \)

\[
P f(\xi) := \int_E f(\xi)p(\xi, d\xi), \quad \xi \in E.
\]

Applying [14] Lemma 5.1 (ii) (here the assumption that \((E, \mathcal{B}, \mu)\) is a standard measurable space is needed) with \( f := X_\lambda^\nu(s) \) and \( g := \Psi(X_\lambda^\nu(s)) \), since \( \Psi \) is monotone, \( \Psi(0) = 0 \) and
$P1 \leq 1$, one obtains
\[
2 \int_0^t \langle (L - \nu)(\alpha - L)^{-\frac{1}{2}}\Psi(X^\nu(s)), (\alpha - L)^{-\frac{1}{2}}X^\nu(s) \rangle_{F_{1,2}} ds
= 2 \int_0^t \langle \Psi(X^\nu(s)), (P - I)X^\nu(s) \rangle_{2} ds
= -\int_0^t \int_E \int_E \left[ \Psi(f(\xi)) - \Psi(f(\xi)) \right] [f(\xi) - f(\xi)] p(\xi, d\xi) d\xi ds
-2 \int_0^t \int_E (1 - P1) f(\xi) \cdot \Psi(f(\xi)) d\xi ds
\leq 0.
\] (3.19)

For the second integral on the right hand side of (3.18), rewrite $L - \nu = -(1 - L) + (1 - \nu)$, by (2.6) we have for all $\nu, \lambda \in (0, 1)$
\[
2\lambda \int_0^t \langle (L - \nu)(\alpha - L)^{-\frac{1}{2}}X^\nu(s), (\alpha - L)^{-\frac{1}{2}}X^\nu(s) \rangle_{F_{1,2}} ds
= -2\lambda \int_0^t \langle (1 - L)(\alpha - L)^{-\frac{1}{2}}X^\nu(s), (\alpha - L)^{-\frac{1}{2}}X^\nu(s) \rangle_{F_{1,2}} ds
+2\lambda(1 - \nu) \int_0^t \langle (\alpha - L)^{-\frac{1}{2}}X^\nu(s), (\alpha - L)^{-\frac{1}{2}}X^\nu(s) \rangle_{F_{1,2}} ds
= -2\lambda \int_0^t \| (\alpha - L)^{-\frac{1}{2}}X^\nu(s) \|_{F_{1,2}}^2 ds + 2\lambda(1 - \nu) \int_0^t \| (\alpha - L)^{-\frac{1}{2}}X^\nu(s) \|_{2}^2 ds
\leq -2\lambda \int_0^t \| (\alpha - L)^{-\frac{1}{2}}X^\nu(s) \|_{F_{1,2}}^2 ds + 2 \int_0^t \| (\alpha - L)^{-\frac{1}{2}}X^\nu(s) \|_{2}^2 ds.
\] (3.20)

Multiplying both sides of (3.18) by $\alpha$, taking (3.19) and (3.20) into account, we have for all $t \in [0, T],$
\[
\| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}}X^\nu(t) \|_{2}^2 + 2\lambda \int_0^t \| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}}X^\nu(s) \|_{F_{1,2}}^2 ds
\leq \| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}}x \|_{2}^2 + 2 \int_0^t \| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}}X^\nu(s) \|_{2}^2 ds
+ \int_0^t \| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}}B(s, X^\nu(s)) \|_{L_2(L^2(\mu), L^2(\mu))}^2 ds
+2 \int_0^t \langle \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}}X^\nu(s), \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}}B(s, X^\nu(s))dW(s) \rangle_{2}.
\] (3.21)

Before doing further estimates, we need to prove that
\[
\sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} is a contraction on $L^2(\mu),
\] (3.22)
and
\[
\| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}}u \|_{2}^2 \rightarrow \| u \|_{2}^2 \ in \ L^2(\mu) \ as \ \alpha \rightarrow \infty.
\] (3.23)
These are true because $\alpha(\alpha - L)^{-1}$ is a contraction on $L^2(\mu)$, then from [10] Page:8, Proposition 1.3 (ii), we know that $|\alpha(\alpha - L)^{-1}| \leq 1$ and for all $u \in L^2(\mu),$
\[
\alpha(\alpha - L)^{-1}u \rightarrow u \ in \ L^2(\mu) \ as \ \alpha \rightarrow \infty,
\] (3.24)
so,

\[
|\sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} u|^2 = (\sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} u, \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} u)_2
\]

\[
= F_{1,2} \langle \alpha(\alpha - L)^{-1} u, u \rangle_{F_{1,2}}
\]

\[
= (\alpha(\alpha - L)^{-1} u, u)_2
\]

\[
\leq |\alpha(\alpha - L)^{-1} u|_2 \cdot |u|_2
\]

\[
\leq |u|_2^2.
\]  

(3.25)

which indicates (3.22), from the third step of (3.25) and (3.24) we can get (3.23).

By (3.25) and (H2)(iii), we have

\[
\int_0^t \|\sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} B(s, X^\nu(s))\|_{L_2(L^2(\mu), L^2(\mu))}^2 ds
\]

\[
\leq \int_0^t \|B(s, X^\nu(s))\|_{L_2(L^2(\mu), L^2(\mu))}^2 ds
\]

\[
\leq C_3 \int_0^t (|X^\nu(s)|_2^2 + 1) ds.
\]  

(3.26)

Taking (3.26) into (3.21), by (3.25), and using the Burkholder-Davis-Gundy (BDG) inequality (with \( p = 1 \)), we obtain

\[
E \left[ \sup_{s \in [0, t]} \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right|^2 \right] + 2\lambda E \int_0^t \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right|^2_{F_{1,2}} ds
\]

\[
\leq \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} x \right|^2_2 + 2E \int_0^t |X^\nu(s)|_2^2 ds + C_3 \int_0^t (|X^\nu(s)|_2^2 + 1) ds
\]

\[
+ 6E \left[ \int_0^t \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right|^2 \cdot \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} B(s, X^\nu(s)) \right|^2_{L_2(L^2(\mu), L^2(\mu))} ds \right]^{\frac{1}{2}}.
\]  

(3.27)

The last term of the right hand side of the above inequality can be estimated by

\[
6E \left[ \sup_{s \in [0, t]} \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right|^2 \right] \cdot \int_0^t \|\sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} B(s, X^\nu(s))\|_{L_2(\mu), L_2(\mu))}^2 ds
\]

\[
\leq \frac{1}{2} E \left[ \sup_{s \in [0, t]} \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right|^2 \right] + C_3 E \int_0^t (|X^\nu(s)|_2^2 + 1) ds.
\]  

(3.28)

Taking (3.28) into (3.27), we obtain that for \( t \in [0, T] \),

\[
E \left[ \sup_{s \in [0, t]} \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right|^2 \right] + 2\lambda E \int_0^t \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right|^2_{F_{1,2}} ds
\]

\[
\leq \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} x \right|^2_2 + C\lambda E \int_0^t (|X^\nu(s)|_2^2 + 1) ds
\]

\[
+ \frac{1}{2} E \left[ \sup_{s \in [0, t]} \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right|^2 \right].
\]  

(3.29)

Note that the first summand of the left hand side of the above inequality is finite by (3.16).

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since $|\sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}}|_2$ is equivalent to $\| \cdot \|_{F_1^2}$. (3.29) shows that

$$
\mathbb{E} \left[ \sup_{s \in [0,t]} \left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right|^2 \right] + 4\lambda \mathbb{E} \int_0^t \left\| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} X^\nu(s) \right\|_{F_1^2}^2 ds 
\leq 2\left| \sqrt{\alpha}(\alpha - L)^{-\frac{1}{2}} x \right|^2 + C^E \int_0^t (|X^\nu(s)|^2 + 1) ds.
$$

(3.30)

Now we want to take $\alpha \to \infty$, by (3.29) we can easily estimate the first term in both sides of (3.30), to estimate the second term in the left hand-side, we introduce

$$\mathcal{E}^{ex}(u, u) := \begin{cases} \mathcal{E}(u, u), & \text{if } u \in F_1^2; \\ +\infty, & \text{if } u \in L^2(\mu) \setminus F_1^2, \end{cases}$$

and prove that $L^2(\mu) \ni u \mapsto \mathcal{E}^{ex}(u, u)$ is lower semicontinuous on $L^2(\mu)$. To clarify this, recall from 3 that

$$\|u\|_{F_1^2}^2 := \mathcal{E}_1(u, u) = \mathcal{E}(u, u) + |u|^2_2, \quad \forall u \in F_1^2 (= D(\mathcal{E})).$$

Without lose of generality, let us consider a sequence $\{u_n\}_{n \in \mathbb{N}} \subset F_1^2$ such that

$$\liminf_{n \to \infty} \mathcal{E}(u_n, u_n) < \infty,$$

and with an element $u \in L^2(\mu)$ such that $u_n \to u$ in $L^2(\mu)$ as $n \to \infty$. Then there exists a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \to \infty} \mathcal{E}(u_{n_k}, u_{n_k}) = \liminf_{n \to \infty} \mathcal{E}(u_n, u_n) := C$$

and

$$\mathcal{E}_1(u_{n_k}, u_{n_k}) \to C + |u|^2_2, \quad \text{as } k \to \infty.$$

Hence $\{\|u_{n_k}\|_{F_1^2}^2\}_{k \in \mathbb{N}}$ is bounded, again there exists a subsequence $\{u_{n_{l_k}}\}_{l \in \mathbb{N}}$ and $u_0 \in F_1^2$ such that $u_{n_{l_k}} \rightharpoonup u_0$ in $(F_1^2, \| \cdot \|_{F_1^2})$ as $l \to \infty$, from [10] Page:184, Theorem 2.2 we know that there exists a subsequence (for simplicity here we use the same notation) such that the Cesaro mean

$$\frac{1}{N} \sum_{l=1}^N u_{n_{l_k}} \rightharpoonup u_0 \quad \text{strongly in } (F_1^2, \| \cdot \|_{F_1^2}), \quad \text{as } N \to \infty,$$

hence also in $L^2(\mu)$. So $u = u_0$, $u \in F_1^2$, $u_{n_{l_k}} \rightharpoonup u$ in $(F_1^2, \| \cdot \|_{F_1^2})$ as $l \to \infty$, and because $(F_1^2, \| \cdot \|_{F_1^2})$ is a Hilbert space, thus by the weakly lower semicontinuity of norms we have

$$\mathcal{E}(u, u) + |u|^2_2 = \|u\|^2_{F_1^2} \leq \liminf_{l \to \infty} \|u_{n_{l_k}}\|_{F_1^2}^2 \leq \liminf_{l \to \infty} \left( \mathcal{E}(u_{n_{l_k}}, u_{n_{l_k}}) + |u_{n_{l_k}}|^2_2 \right) = \liminf_{l \to \infty} \mathcal{E}(u_{n_{l_k}}, u_{n_{l_k}}) + |u|^2_2.$$

Therefore, we have

$$\mathcal{E}^{ex}(u, u) \leq \liminf_{n \to \infty} \mathcal{E}(u_n, u_n), \quad \forall u \in L^2(\mu),$$

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and furthermore,
\[
\mathcal{E}^{ex}(u, u) := \mathcal{E}^{ex}(u, u) + |u|^2 \leq \liminf_{n \to \infty} \mathcal{E}_1(u_n, u_n) = \liminf_{n \to \infty} \|u_n\|^2_{F_{1,2}}, \quad \forall u \in L^2(\mu). \quad (3.31)
\]

Let us continue to estimate (3.30), letting \( \alpha \to \infty \), by (3.22), (3.31) and Fatou’s lemma,
\[
\mathbb{E}\left[ \sup_{s \in [0,T]} \left| X^\alpha_X(s) \right|^2 \right] + 4\lambda \mathbb{E} \int_0^t \left\| X^\alpha_X(s) \right\|^2_{F_{1,2}} \, ds \\
\leq \mathbb{E}\left[ \sup_{s \in [0,T]} \liminf_{\alpha \to \infty} \left| \sqrt{\alpha} (\alpha - L)^{-\frac{1}{2}} X^\alpha_X(s) \right|^2 \right] + 4\lambda \mathbb{E} \int_0^t \liminf_{\alpha \to \infty} \left\| \sqrt{\alpha} (\alpha - L)^{-\frac{1}{2}} X^\alpha_X(s) \right\|^2_{F_{1,2}} \, ds \\
\leq \liminf_{\alpha \to \infty} \left\{ \mathbb{E}\left[ \sup_{s \in [0,T]} \left| \sqrt{\alpha} (\alpha - L)^{-\frac{1}{2}} X^\alpha_X(s) \right|^2 \right] + 4\lambda \mathbb{E} \int_0^t \left\| \sqrt{\alpha} (\alpha - L)^{-\frac{1}{2}} X^\alpha_X(s) \right\|^2_{F_{1,2}} \, ds \right\} \\
= 2|x|^2 + C \mathbb{E} \int_0^t \left( \left| X^\alpha_X(s) \right|^2 + 1 \right) ds. \quad (3.32)
\]

Then Gronwall’s inequality yields the result. \( \square \)

**Claim 3.2** \( \{ X^\lambda \}_{\lambda \in (0,1)} \) converges to an element \( X^\nu \in L^2([0,T] \times \Omega; L^2(\mu)) \) as \( \lambda \to 0 \).

**Proof** By Itô’s formula we get that, for \( \lambda, \lambda' \in (0,1) \) and \( t \in [0,T] \),
\[
\left\| X^\lambda_X(t) - X^{\lambda'}_X(t) \right\|^2_{F_{1,2,\nu}} \\
+ 2 \int_0^t \langle \Psi(X^\nu_X(s)) - \Psi(X^{\lambda'}_X(s) + \lambda X^\nu_X(s) - \lambda' X^{\lambda'}_X(s), X^\nu_X(s) - X^{\lambda'}_X(s) \rangle ds \\
= \int_0^t \left\| B(s, X^\nu_X(s)) - B(s, X^{\lambda'}_X(s)) \right\|^2_{L_2(L^2(\mu), F^*_{1,2,\nu})} ds \\
+ 2 \int_0^t \left\langle X^\nu_X(s) - X^{\lambda'}_X(s), (B(s, X^\nu_X(s)) - B(s, X^{\lambda'}_X(s))) \, dW(s) \right\rangle_{F^*_{1,2,\nu}}. \quad (3.33)
\]

(3.10) implies that for the second term on the left hand side in (3.33) we have
\[
2 \int_0^t \left\langle \Psi(X^\nu_X(s)) - \Psi(X^{\lambda'}_X(s) + \lambda X^\nu_X(s) - \lambda' X^{\lambda'}_X(s), X^\nu_X(s) - X^{\lambda'}_X(s) \right\rangle ds \\
\geq 2\lambda \int_0^t \left\| \Psi(X^\nu_X(s)) - \Psi(X^{\lambda'}_X(s)) \right\|^2 ds \\
+ 2 \int_0^t \left\langle \Lambda X^\nu_X(s) - \Lambda X^{\lambda'}_X(s), X^\nu_X(s) - X^{\lambda'}_X(s) \right\rangle. \quad (3.34)
\]

The assumption (H2)(i) yields
\[
\int_0^t \left\| B(s, X^\nu_X(s)) - B(s, X^{\lambda'}_X(s)) \right\|^2_{L_2(L^2(\mu), F^*_{1,2,\nu})} ds \leq C_1 \int_0^t \left\| X^\nu_X(s) - X^{\lambda'}_X(s) \right\|^2_{F_{1,2,\nu}}. \quad (3.35)
\]
Using the BDG inequality and Young’s inequality, for \( t \in [0, T] \), (3.33)-(3.35) imply
\[
\mathbb{E} \left[ \sup_{s \in [0, t]} \| X^\nu_s(t) - X^\nu_s(s) \|_{F_{1,2,s}^*}^2 \right] + 2\alpha \mathbb{E} \int_0^t |\Psi(X^\nu_s(s)) - \Psi(X^\nu_s(s))|^2 ds \\
\leq C_1 \mathbb{E} \int_0^t \| X^\nu_s(t) - X^\nu_s(s) \|_{F_{1,2,s}^*}^2 ds \\
- 2 \mathbb{E} \int_0^t \langle \lambda X^\nu_s(s) - \lambda X^\nu_s(s), X^\nu_s(s) - X^\nu_s(s) \rangle ds \\
+ 2 \mathbb{E} \left[ \int_0^t \| X^\nu_s(s) - X^\nu_s(s) \|_{F_{1,2,s}^*}^2 \cdot \| B(s, X^\nu_s(s)) - B(s, X^\nu_s(s)) \|_{L^2(F_{1,2,s}^*, \mathcal{F}_s^* \mu)}^2 \right]^\frac{1}{2} \\
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{s \in [0, t]} \| X^\nu_s(s) - X^\nu_s(s) \|_{F_{1,2,s}^*}^2 \right] + C \mathbb{E} \int_0^t \| X^\nu_s(s) - X^\nu_s(s) \|_{F_{1,2,s}^*}^2 ds \\
+ 4(\lambda + \lambda') \mathbb{E} \int_0^t \left( \| X^\nu_s(s) \|_{L^2(\Omega)}^2 + \| X^\nu_s(s) \|_{L^2(\Omega)}^2 \right) ds. 
\]  
(3.36)

Since \( x \in L^2(\mu) \), Gronwall’s lemma and Claim 3.1 imply for some constant \( C \in (0, \infty) \) independent of \( \lambda, \lambda' \) such that
\[
\mathbb{E} \left[ \sup_{s \in [0, T]} \| X^\nu_s(s) - X^\nu_s(s) \|_{F_{1,2,s}^*}^2 \right] + \mathbb{E} \int_0^T \| \Psi(X^\nu_s(s)) - \Psi(X^\nu_s(s)) \|_{L^2(\Omega; \mathcal{F}_s^* \mu)}^2 ds \leq C(\lambda + \lambda'). 
\]  
(3.37)

implies that there exists an \( \mathcal{F}_s \)-adapted continuous \( F_{1,2,s}^* \)-valued process \( \{ X^\nu(t) \}_{t \in [0, T]} \) such that \( X^\nu \in L^2(\Omega; C([0, T], F_{1,2,s}^*)) \). This together with Claim 3.1 implies that \( X^\nu \in L^2([0, T] \times \Omega; L^2(\mu)) \).

\[\text{□}\]

Claim 3.3 \( X^\nu \) satisfies (3.4).

\[\text{Proof}\] From Claim 3.2, we know that
\[
X^\nu_s \to X^\nu \text{ and } \int_0^t B(s, X^\nu_s(s))dW(s) \to \int_0^t B(s, X^\nu(s))dW(s), \quad \lambda \to 0
\]  
(3.38)
in \( L^2(\Omega; C([0, T], F_{1,2,s}^*)) \). (3.17), (3.38) yield that
\[
\int_0^t \left( \Psi(X^\nu_s(s)) + \lambda X^\nu_s(s) \right) ds, \quad \lambda > 0,
\]
converges to some element in \( L^2(\Omega; C([0, T], F_{1,2,s}^*)) \) as \( \lambda \to 0 \). In addition, by Claim 3.1, we have that, as \( \lambda \to 0 \),
\[
\int_0^t \left( \Psi(X^\nu_s(s)) + \lambda X^\nu_s(s) \right) ds \to \int_0^t \Psi(X^\nu(s)) ds
\]
in \( L^2(\Omega; L^2([0, T]; L^2(\mu))) \). This and (3.38) imply the claim. \[\text{□}\]

By lower semi-continuity of norms, (3.38) follows immediately from Claim 3.1.

Uniqueness

If \( X^\nu_1, X^\nu_2 \) are two solutions to (3.7), we have \( \mathbb{P} - a.s. \)
\[
X^\nu_1(t) - X^\nu_2(t) + (\nu - L) \int_0^t \Psi(X^\nu_1(s)) - \Psi(X^\nu_2(s)) ds \\
= \int_0^t (B(s, X^\nu_1(s)) - B(s, X^\nu_2(s)))dW(s), \quad \forall \, t \in [0, T]
\]  
(3.39)
in $\Omega \times [0, T] \times E$. Apply Itô’s formula in $F_{1, 2}^{\nu}$ to $\|X_1^{\nu}(t) - X_2^{\nu}(t)\|^2_{F_{1, 2}^{\nu}}$, we get

$$\|X_1^{\nu}(t) - X_2^{\nu}(t)\|^2_{F_{1, 2}^{\nu}} + 2 \int_0^t \langle \Psi(X_1^{\nu}(s)) - \Psi(X_2^{\nu}(s)), X_1^{\nu}(s) - X_2^{\nu}(s) \rangle_2 ds$$

$$= \int_0^t \|B(s, X_1^{\nu}(s)) - B(s, X_2^{\nu}(s))\|^2_{L^2(L^2(\mu), F_{1, 2}^{\nu})} ds$$

$$+ 2 \int_0^t \langle X_1^{\nu}(s) - X_2^{\nu}(s), (B(s, X_1^{\nu}(s)) - B(s, X_2^{\nu}(s)))dW(s) \rangle_{F_{1, 2}^{\nu}}. \tag{3.40}$$

The Lipschitz assumption (H1) on $\Psi$ implies that

$$(\Psi(r) - \Psi(r'))(r - r') \geq (\text{Lip}\Psi + 1)^{-1}|\Psi(r) - \Psi(r')|^2, \text{ for } r, r' \in \mathbb{R}. \tag{3.41}$$

Taking expectation of both sides to (3.40), then taking (3.41) and (H2)(i) into account, we obtain

$$\mathbb{E}\|X_1^{\nu}(t) - X_2^{\nu}(t)\|^2_{F_{1, 2}^{\nu}} + 2(\text{Lip}\Psi + 1)^{-1}\mathbb{E} \int_0^t |\Psi(X_1^{\nu}(s)) - \Psi(X_2^{\nu}(s))|^2_2 ds \tag{3.42}$$

$$\leq C_1 \mathbb{E} \int_0^t \|X_1^{\nu}(s) - X_2^{\nu}(s)\|^2_{F_{1, 2}^{\nu}} ds. \tag{3.43}$$

The second term in the left-hand side of (3.42) is positive, thus we have

$$\mathbb{E}\|X_1^{\nu}(t) - X_2^{\nu}(t)\|^2_{F_{1, 2}^{\nu}} \leq C_1 \mathbb{E} \int_0^t \|X_1^{\nu}(s) - X_2^{\nu}(s)\|^2_{F_{1, 2}^{\nu}} ds.$$

By Gronwall’s inequality, we get $X_1^{\nu} = X_2^{\nu}, \mathbb{P} - \text{a.s.}$, which indicates the uniqueness.

Hence the proof of Lemma 3.1 is complete. \hfill \square

Based on Lemma 3.1, we shall now give the proof of our main result Theorem 3.1. The idea is to prove that the sequence $\{X^{\nu}\}_{\nu \in (0, 1)}$ converges to the solution of (1.1) as $\nu \to 0$. The method that we use here is similar to that in Lemma 3.1.

**Proof of Theorem 3.1 (continued)**

First, we rewrite (3.3) as

$$dX^{\nu}(t) + (1 - L)\Psi(X^{\nu}(t))dt = (1 - \nu)\Psi(X^{\nu}(t))dt + B(t, X^{\nu}(t))dW(t).$$

For the function $\varphi(x) = \frac{1}{2}\|x\|^2_{F_{1, 2}^{\nu}}$ with $x \in F_{1, 2}^{\nu}$, Itô’s formula yields

$$\frac{1}{2} \mathbb{E}\|X^{\nu}(t)\|^2_{F_{1, 2}^{\nu}} + \int_0^t \langle \Psi(X^{\nu}(s)), X^{\nu}(s) \rangle_2 ds$$

$$= \frac{1}{2} \mathbb{E}\|x\|^2_{F_{1, 2}^{\nu}} + (1 - \nu)\mathbb{E} \int_0^t \langle \Psi(X^{\nu}(s)), X^{\nu}(s) \rangle_{F_{1, 2}^{\nu}} ds$$

$$+ \frac{1}{2} \mathbb{E} \int_0^t \|B(s, X^{\nu}(s))\|^2_{L^2(L^2(\mu), F_{1, 2}^{\nu})} ds. \tag{3.44}$$

The condition (H1) implies

$$\Psi(r)r \geq \hat{\alpha} \cdot |\Psi(r)|^2, \quad r \in \mathbb{R}. \tag{3.45}$$
By (3.44) and (3.45), we have
\[
\begin{align*}
\frac{1}{2} & \mathbb{E}\|X^\nu(t)\|^2_{F^*_{1,2}} + \bar{\alpha} \cdot \mathbb{E} \int_0^t |\Psi(X^\nu(s))|^2 ds \\
\leq & \frac{1}{2} \|x\|^2_{F^*_{1,2}} + \mathbb{E} \int_0^t \|\Psi(X^\nu(s))\|_{F^*_{1,2}} \cdot \|X^\nu(s)\|_{F^*_{1,2}} ds \\
+ & \frac{1}{2} C_2 \mathbb{E} \int_0^t \|X^\nu(s)\|^2_{F^*_{1,2}} ds.
\end{align*}
\]

Since \(L^2(\mu)\) is continuously embedded into \(F^*_{1,2}\), Young’s inequality and the Gronwall’s inequality yield that there exists a constant \(C \in (0, \infty)\) such that, for \(t \in [0, T]\) and \(\nu \in (0, 1)\),
\[
\mathbb{E}\|X^\nu(t)\|^2_{F^*_{1,2}} \leq C \|x\|^2_{F^*_{1,2}}. \quad (3.46)
\]

In the following, we will prove the convergence of \(\{X^\nu\}_{\nu \in (0, 1)}\). Applying Itô’s formula to \(\|X^\nu(t) - X^{\nu'}(t)\|^2_{F^*_{1,2}}\), we get that, for all \(t \in [0, T]\),
\[
\begin{align*}
\|X^\nu(t) - X^{\nu'}(t)\|^2_{F^*_{1,2}} + 2 \int_0^t \langle \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s) \rangle_{F^*_{1,2}} ds \\
= & 2 \int_0^t \langle \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s) \rangle_{F^*_{1,2}} ds \\
- & 2 \int_0^t \langle \nu \Psi(X^\nu(s)) - \nu' \Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s) \rangle_{F^*_{1,2}} ds \\
+ & 2 \int_0^t \|B(s, X^\nu(s)) - B(s, X^{\nu'}(s))\|^2_{L^2(L^2(\mu), F^*_{1,2})} ds \\
+ & 2 \int_0^t \langle X^\nu(s) - X^{\nu'}(s), (B(s, X^\nu(s)) - B(s, X^{\nu'}(s))dW(s) \rangle_{F^*_{1,2}} ds. \quad (3.47)
\end{align*}
\]

The second term on the right hand side of (3.47) can be dominated by
\[
-2 \int_0^t \langle \nu \Psi(X^\nu(s)) - \nu' \Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s) \rangle_{F^*_{1,2}} ds \\
\leq 2C \int_0^t (\nu|\Psi(X^\nu(s))|_2 + \nu'|\Psi(X^{\nu'}(s))|_2) \cdot \|X^\nu(s) - X^{\nu'}(s)\|_{F^*_{1,2}} ds. \quad (3.48)
\]

By assumption (H1) on \(\Psi\) and (3.45), we obtain
\[
\begin{align*}
2 \int_0^t \langle \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s) \rangle_{F^*_{1,2}} ds \\
= & 2 \int_0^t \int_E \left( \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)) \right) \cdot (X^\nu(s) - X^{\nu'}(s)) d\mu ds \\
\geq & 2 \int_0^t \int_E \tilde{\alpha} |\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))|^2 d\mu ds \\
= & 2\tilde{\alpha} \int_0^t \|\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))\|^2_{L^2(ds}. \quad (3.49)
\end{align*}
\]
\(\text{(3.47) - (3.49)}\) imply
\[
\left\| X^\nu(t) - X^{\nu'}(t) \right\|_{F_{t,2}}^2 + 2\alpha \int_0^t \left| \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)) \right|^2 ds \\
\leq C_1 \int_0^t \left| \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)) \right| \cdot \left\| X^\nu(s) - X^{\nu'}(s) \right\|_{F_{t,2}} ds \\
+ C_2 \int_0^t \left( \nu |\Psi(X^\nu(s))| + \nu' |\Psi(X^{\nu'}(s))| \right) \cdot \left\| X^\nu(s) - X^{\nu'}(s) \right\|_{F_{t,2}} ds \\
+ C_3 \int_0^t \left\| X^\nu(s) - X^{\nu'}(s) \right\|^2_{F_{t,2}} ds \\
+ 2 \int_0^t \langle X^\nu(s) - X^{\nu'}(s), (B(s, X^\nu(s)) - B(s, X^{\nu'}(s))) dW(s) \rangle_{F_{t,2}} ds.
\]

Taking expectation to both sides of the above inequality and using Young’s and the BDG inequality for \(p = 1\), we obtain, for all \(t \in [0, T]\),
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} \left\| X^\nu(s) - X^{\nu'}(s) \right\|_{F_{t,2}}^2 \right] + 2\alpha \mathbb{E} \int_0^t \left| \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)) \right|^2 ds \\
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{s \in [0,t]} \left\| X^\nu(s) - X^{\nu'}(s) \right\|_{F_{t,2}}^2 \right] + \alpha \mathbb{E} \int_0^t \left| \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)) \right|^2 ds \\
+ C_1 \mathbb{E} \int_0^t \left\| X^\nu(s) - X^{\nu'}(s) \right\|_{F_{t,2}} ds + C_2 \mathbb{E} \left[ \nu \left| \Psi(X^\nu(s)) \right|^2 + \nu' \left| \Psi(X^{\nu'}(s)) \right|^2 \right] ds.
\]

This yields
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} \left\| X^\nu(s) - X^{\nu'}(s) \right\|_{F_{t,2}}^2 \right] + 2\alpha \mathbb{E} \int_0^t \left| \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)) \right|^2 ds \\
\leq C_1 \mathbb{E} \int_0^t \left\| X^\nu(s) - X^{\nu'}(s) \right\|_{F_{t,2}} ds + C_2 (\nu + \nu') \mathbb{E} \int_0^t \left( \left| \Psi(X^\nu(s)) \right|^2 + \left| \Psi(X^{\nu'}(s)) \right|^2 \right) ds.
\]

Note that if the initial value \(x \in F_{1,2}\) and (3.4) is satisfied, we have (3.40). If \(x \in L^2(\mu)\), we have (3.8). Hence, Gronwall’s inequality and Young’s inequality yields that there exists a positive constant \(C \in (0, \infty)\) which is independent of \(\nu, \nu'\) such that
\[
\mathbb{E} \left[ \sup_{s \in [0,T]} \left\| X^\nu(s) - X^{\nu'}(s) \right\|_{F_{t,2}}^2 \right] + \mathbb{E} \int_0^T \left| \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)) \right|^2 ds \\
\leq C (\nu + \nu').
\]

Hence, there exists an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process \(X \in L^2(\Omega; C([0, T], F_{1,2}^*)) \cap L^2([0, T] \times \Omega; L^2(\mu))\) such that \(X^\nu \to X\) in \(L^2(\Omega; C([0, T], F_{1,2}^*))\) as \(\nu \to 0\).

Next, we will prove \(X\) satisfies (3.3), the proof is similar to that in Claim 3.3.

From above we know that
\[
X^\nu \to X \text{ and } \int_0^t B(s, X^\nu(s)) dW(s) \to \int_0^t B(s, X(s)) dW(s), \nu \to 0
\]

(3.52)
in $L^2(\Omega; C([0, T]; F_{1,2}^*))$. (3.7) and (3.52) yield that
\[ \int_0^\bullet \Psi(X^\nu(s))ds, \ \nu > 0, \]
converges to some element in $L^2(\Omega; C([0, T], F_{1,2}))$ as $\nu \to 0$, and from (3.51) we know as $\nu \to 0$,
\[ \int_0^\bullet \Psi(X^\nu(s))ds \to \int_0^\bullet \Psi(X(s))ds \]
in $L^2(\Omega; L^2([0, T]; L^2(\mu)))$. Hence $X$ satisfies (3.3). This completes the existence proof for Theorem 3.1.

**Uniqueness**

If $X_1$ and $X_2$ are two solutions to (1.1), we have $\mathbb{P} - a.s.$
\[ X_1(t) - X_2(t) - L \int_0^t (\Psi(X_1(s)) - \Psi(X_2(s)))ds \]
\[ = \int_0^t (B(s, X_1(s)) - B(s, X_2(s)))dW(s), \ t \in [0, T] \] (3.53)
in $\Omega \times [0, T] \times E$.

Rewrite (3.53) as
\[ X_1(t) - X_2(t) + (1 - L) \int_0^t (\Psi(X_1(s)) - \Psi(X_2(s)))ds \]
\[ = \int_0^t (\Psi(X_1(s)) - \Psi(X_2(s)))ds + \int_0^t (B(s, X_1(s)) - B(s, X_2(s)))dW(s) \] (3.54)

Apply Itô’s formula to $\|X_1(t) - X_2(t)\|_{F_{1,2}^*}^2$ in $F_{1,2}^*$, it follows
\[ \|X_1(t) - X_2(t)\|_{F_{1,2}^*}^2 + 2 \int_0^t \langle \Psi(X_1(s)) - \Psi(X_2(s)), X_1(s) - X_2(s) \rangle_{F_{1,2}^*}ds \]
\[ = 2 \int_0^t \langle \Psi(X_1(s)) - \Psi(X_2(s)), X_1(s) - X_2(s) \rangle_{F_{1,2}^*}ds \]
\[ + 2 \int_0^t \langle X_1(s) - X_2(s), (B(s, X_1(s)) - B(s, X_2(s)))dW(s) \rangle_{F_{1,2}^*} \]
\[ + \int_0^t \|B(s, X_1(s)) - B(s, X_2(s))\|_{L^2(L^2(\mu), F_{1,2}^*)}^2ds. \] (3.55)

Taking expectation of both sides, (3.10) and (H2)(i) yield that
\[ \mathbb{E}\|X_1(t) - X_2(t)\|_{F_{1,2}^*}^2 + 2\alpha \mathbb{E} \int_0^t \|\Psi(X_1(s)) - \Psi(X_2(s))\|_{F_{1,2}^*}^2ds \]
\[ \leq 2\mathbb{E} \int_0^t \|\Psi(X_1(s)) - \Psi(X_2(s))\|_{F_{1,2}^*} \cdot \|X_1(s) - X_2(s)\|_{F_{1,2}^*}ds \]
\[ + C_1 \mathbb{E} \int_0^t \|X_1(s) - X_2(s)\|_{F_{1,2}^*}^2ds. \]
Using Young’s inequality to the above inequality, and since \( L^2(\mu) \subset F_{1,2}^* \) continuously and densely, we obtain

\[
\mathbb{E}\|X_1(t) - X_2(t)\|_{F_{1,2}^*}^2 + 2\alpha \mathbb{E} \int_0^t \|\Psi(X_1(s)) - \Psi(X_2(s))\|_2^2 ds
\leq 2\alpha \mathbb{E} \int_0^t \|\Psi(X_1(s)) - \Psi(X_2(s))\|_2^2 ds + C \mathbb{E} \int_0^t \|X_1(s) - X_2(s)\|_{F_{1,2}^*}^2 ds

+ C_1 \mathbb{E} \int_0^t \|X_1(s) - X_2(s)\|_{F_{1,2}^*}^2 ds.
\]

Therefore,

\[
\mathbb{E}\|X_1(t) - X_2(t)\|_{F_{1,2}^*}^2 \leq (C + C_1) \mathbb{E} \int_0^t \|X_1(s) - X_2(s)\|_{F_{1,2}^*}^2 ds.
\]

By Gronwall’s lemma, we get \( X_1 = X_2 \mathbb{P} - a.s. \). Consequently, Theorem 3.1 is completely proved. \( \square \)

4 Some Examples

4.1 Classical Dirichlet forms with densities

We apply Theorem 3.1 to the Friedrichs extension of the operator

\[
L_0u = \Delta u + 2\frac{\nabla \rho}{\rho} \cdot \nabla u, \quad u \in C_0^\infty(\mathbb{R}^d),
\]

(4.1)

on \( L^2(\mathbb{R}^d, \rho^2 dx) \), where \( dx \) denotes Lebesgue measure and \( \rho \in H^1(\mathbb{R}^d) \). Here \( H^1 \) is the usual Sobolev space and \( H^{-1} \) denotes its dual space.

Clearly, \( \Delta u \in L^2(\mathbb{R}^d, \rho^2 dx) \), since \( u \in C_0^\infty(\mathbb{R}^d) \), and \( \frac{\nabla \rho}{\rho} \cdot \nabla u \in L^2(\mathbb{R}^d, \rho^2 dx) \), since

\[
\int |\frac{\nabla \rho}{\rho} \cdot \nabla u|^2 \rho^2 dx = \int |\nabla \rho|^2 |\nabla u|^2 dx \leq C \int |\nabla \rho|^2 dx < \infty.
\]

Hence \( L_0 \) is a well-defined linear operator from \( C_0^\infty(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^d, \rho^2 dx) \). In addition, by definition (4.1), for all \( u, v \in C_0^\infty(\mathbb{R}^d) \), we have

\[
\int L_0 u \cdot v \rho^2 dx = \int (\Delta u + 2\frac{\nabla \rho}{\rho} \cdot \nabla u) v \rho^2 dx
\]

\[
= \int \Delta u v \rho^2 dx + 2 \int \frac{\nabla \rho}{\rho} \cdot \nabla u v \rho^2 dx
\]

\[
= \int \text{div} \nabla u v \rho^2 dx + 2 \int \frac{\nabla \rho}{\rho} \cdot \nabla u v \rho^2 dx
\]

\[
= - \int \nabla u \cdot \nabla (v \rho^2) dx + 2 \int \frac{\nabla \rho}{\rho} \cdot \nabla u v \rho^2 dx
\]

\[
= - \int \nabla u \cdot \nabla v \rho^2 dx - \int \nabla u \cdot 2 v \rho \nabla \rho dx + 2 \int \frac{\nabla \rho}{\rho} \cdot \nabla u v \rho^2 dx
\]

\[
= - \int \nabla u \cdot \nabla v \rho^2 dx = \int u \cdot L_0 v \rho^2 dx,
\]

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which implies both that $L_0$ is a symmetric operator and negative definite. Hence $(L_0, C_0^\infty(\mathbb{R}^d))$ is closable in $L^2(\mathbb{R}^d, \rho^2 dx)$ (10). Let $(L, D(L))$ be its closure. According to [10] Proposition 3.3, we hence know that there exists a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mathbb{R}^d, \rho^2 dx)$, which is in fact the closure of

$$\mathcal{E}(u, v) = \int (\nabla u, \nabla v)_{\mathbb{R}^d} \rho^2 dx, \quad \text{for all } u, v \in C_0^\infty(\mathbb{R}^d),$$

on $L^2(\mathbb{R}^d, \rho^2 dx)$ with generator $(L, D(L))$. $(L, D(L))$ is thus the Friedrichs extension of $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho^2 dx)$. As a result, we know $T_t = e^{Lt}$ is the strongly continuous contraction sub-Markovian semigroup on $L^2(\mathbb{R}^d, \rho^2 dx)$ with generator $L$, which is negative definite and self-adjoint.

Consider the following equation

$$\begin{cases}
    dX(t) - L\Psi(X(t))dt = B(t, X(t))dW(t), \text{ in } [0, T] \times \mathbb{R}^d, \\
    X(0) = x \in L^2(\mathbb{R}^d, \rho^2 dx) \text{ or } x \in F_{1,2}^*, \text{ respectively if in addition (3.4) holds.}
\end{cases} \quad (4.2)$$

where $(L, D(L))$ is the Friedrichs extension of $(L_0, C_0^\infty(\mathbb{R}^d))$. $\Psi$ and $B$ satisfy (H1) and (H2) respectively, whereas the corresponding spaces are $E := \mathbb{R}^d$, $L^2(\mu) := L^2(\mathbb{R}^d, \rho^2 dx)$, $F_{1,2} := D(\mathcal{E})$ and $F_{1,2}^* := (D(\mathcal{E}))^*$.

As a consequence, we can apply Theorem 3.1.

### 4.2 General regular symmetric case

The example in Section 4.1 is a special case of the examples in [10] Chapter 2: Let $E := U \subset \mathbb{R}^d$, $U$ open, and $m$ a positive Radon measure on $U$ such that $\text{supp}[m] = U$. For $u, v \in C_0^\infty(U)$, define

$$\mathcal{E}(u, v) := \sum_{i,j=1}^d \int \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d\nu_{ij} + \int_{U \times U \setminus \Delta} (u(x) - u(y))(v(x) - v(y)) J(dx, dy) + \int uv \, dk. \quad (4.3)$$

Here $k$ is a positive Radon measure on $U$ and $J$ is a symmetric positive Radon measure on $U \times U \setminus \Delta$, where $\Delta := \{(x, x)|x \in U\}$, such that for all $u \in C_0^\infty(U)$

$$\int |u(x) - u(y)|^2 J(dx dy) < \infty. \quad (4.4)$$

For $1 \leq i, j \leq d$, $\nu_{ij}$ is a Radon measure on $U$ such that for every $K \subset U$, $K$ compact, $\nu_{ij}(K) = \nu_{ji}(K)$ and $\sum_{i,j=1}^d \sum_{i,j=1}^d \xi_i \xi_j \nu_{ij}(K) \geq 0$ for all $\xi_1, \ldots, \xi_d \in \mathbb{R}^d$.

Then $(\mathcal{E}, C_0^\infty(U))$ is a densely defined symmetric positive definite bilinear form on $L^2(U; m)$.

Suppose that $(\mathcal{E}, C_0^\infty(U))$ is closable on $L^2(U; m)$ and let $(\mathcal{E}, D(\mathcal{E}))$ be its closure, then $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form. Hence by [10] we know there exists a self-adjoint negative definite linear operator $(L, D(L))$ on $L^2(U; m)$ defined by

$$D(L) := \{u \in D(\mathcal{E})| \exists Lu \in L^2(U; m), \ s.t. \ \mathcal{E}(u, v) = (-Lu, v), \forall v \in D(\mathcal{E})\}. \quad (4.5)$$
Hence \((L, D(L))\) is the generator of a sub-Markovian strongly continuous contraction semigroup \((T_t)_{t>0}\) on \(L^2(U; m)\) given by
\[
T_t := e^{tL}, \ t > 0.
\] (4.6)

Consider the following equation
\[
\begin{cases}
  dX(t) - L\Psi(X(t))dt = B(t, X(t))dW(t), \text{ in } [0, T] \times U, \\
  X(0) = x \in L^2(U, m) \text{ or } x \in F^*_{1,2} \text{ respectively if in addition (3.4) holds.}
\end{cases}
\] (4.7)

where \((L, D(L))\) (defined in (4.5)) is the generator of a sub-Markovian strongly continuous contraction semigroup \((T_t)_{t>0}\) on \(L^2(U; m)\). \(\Psi\) and \(B\) satisfy (H1) and (H2) respectively, whereas the corresponding spaces are \(E := U, L^2(\mu) := L^2(U, m), F_{1,2} := D(\mathcal{E})\) and \(F^*_{1,2} := (D(\mathcal{E}))^*\).

Consequently, from Theorem 3.1 we know there exists a unique strong solution to (4.7).

**Remark 4.1:**

(i) Our result thus in particular applies to the case when \(L\) is the fractional Laplace operator
\[
L := -(-\Delta)^\alpha, \ \alpha \in (0, 1],
\]
since it is just a special case of the above (see [10, Chapter 2]).

(ii) Using Dirichlet form theory on fractals, Theorem 3.1 also applies to the case when \(L\) is the Laplace operator on a fractal, and the corresponding state space \(E\) is this fractal, (see [9, 8] for details).

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