Fermi points and topological quantum phase transitions in a multi-band superconductor

T O Puel\textsuperscript{1}, P D Sacramento\textsuperscript{1,2} and M A Continentino\textsuperscript{1}

\textsuperscript{1}Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150—Urca, Rio de Janeirom—RJ 22290-180, Brazil
\textsuperscript{2}CeFEEMA, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail: tharnier@me.com

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Abstract

The importance of models with an exact solution for the study of materials with non-trivial topological properties has been extensively demonstrated. The Kitaev model plays a guiding role in the search for Majorana modes in condensed matter systems. Also, the $sp$-chain with an anti-symmetric mixing among the $s$ and $p$ bands is a paradigmatic example of a topological insulator with well understood properties. Interestingly, these models share the same universality class for their topological quantum phase transitions. In this work we study a two-band model of spinless fermions with attractive inter-band interactions. We obtain its zero temperature phase diagram, which presents a rich variety of phases including a Weyl superconductor and a topological insulator. The transition from the topological to the trivial superconducting phase has critical exponents different from those of Kitaev’s model.

Keywords: topological quantum phase transitions, multi-band superconductor, Fermi points, hybridization, zero energy modes, Weyl superconductor

(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that the Kitaev model \cite{1–3}—anti-symmetric pairs of spinless fermions in 1D—exhibits a non-trivial topological phase with Majorana modes at the ends of a $p$-wave superconducting chain. The excitations at the ends of the chain depend on the quantum state of the system. The weak pairing phase presents Majorana fermions at its ends. Otherwise, in a strong coupling superconducting phase with trivial topological properties has no end states \cite{3}. The importance of the mixing of $sp$ bands for topological insulators has already been pointed out in different contexts \cite{4, 5}. The $sp$ mixing is in a special class that mixes orbitals with angular momenta that differ by an odd number. This implies the anti-symmetric property $V(-r) = -V(r)$ or in momentum space $V(-k) = -V(k)$ \cite{6}. In addition, there has recently been shown \cite{7–9} an intimate relation between a two band insulator with anti-symmetric hybridization and the Kitaev model, as regards the topological properties and their end states. By tuning the parameters of the 1D $sp$ chain, the system can be driven, through a topological quantum phase transition, from a trivial to a topological insulator. As a result they found two localized zero modes at the ends of an insulating chain. Since the first strong experimental evidence of Majorana fermions \cite{10} the search for exotic states supporting Majorana fermions has attracted increasing interest in condensed matter physics. Recent observations have reinforced the existence of Majorana \cite{11}. On the other hand, anomalous behaviour
indicates that the appearance of Majorana may have yet unknown sources [14, 15]. The running for the experimental discovery of Majorana is well described in [16].

Here we consider a different model of a $p$-wave superconducting chain. For a fixed small value of the mixing there is a topological quantum phase transition from a gapless (topologically non-trivial) to a gapped (trivial) superconducting phase. The phase diagram resembles that of Kitaev’s model, however, the nature of the topological phases, as well as, the topological transition are distinct. In our model the non-trivial superconducting phase has gapless Fermi points. These gapless points have the characteristic of Weyl fermions in 3D systems [17–19], as they have a non-degenerated linear dispersion relation and appear or disappear in pairs, only when two Fermi points unite. The conservation of the topological charge associated with these Fermi points [20] confers a non-trivial topological character for this phase. Increasing the hybridization the system is driven from the Weyl superconducting phase (WSC) to a topological insulator through a first-order quantum phase transition. Topological phase transitions are known to produce anomalies in thermodynamic quantities [21] and we obtain these here, with special emphasis on the behaviour of the compressibility.

2. Defining the model

We consider a two-band problem with hybridization and triplet inter-band superconductivity in 1D, i.e. a chain with two orbitals per site, with angular momenta differing by an odd number, let’s say $p$ and $s$. The pairing between fermions on different bands (inter-band) is always $p$-wave kind, in the sense that the pairing of spinless fermions is anti-symmetric. The problem can be viewed as a generalization of Kitaev’s model to two orbitals and only interband pairing. We also have the anti-symmetric hybridization term that was shown to be responsible for topological phases [7–9]. The simplest Hamiltonian in the momentum space that describes those types of superconductivity and hybridization can be written as

\[
\mathcal{H} = \sum_k \left\{ -\mu \left( \epsilon_k^1 c_k^1 \epsilon_k^1 c_k^1 + \epsilon_k^2 c_k^2 \right) + 2\cos(k) \left( p_k^1 p_k^1 - \epsilon_k^1 c_k^1 \right) - i\Delta \sin(k) \epsilon_k^1 c_k^2 + iV \sin(k) \epsilon_k^1 c_k^2 + \text{h.c.} \right\},
\]

(1)

where $\mu$ is the chemical potential, $\Delta$ is the $sp$ pairing amplitude, and $V$ is the anti-symmetric hybridization amplitude. Note that the hopping amplitude $r$ has different sign in each band, representing particles for the orbital $s$ and holes for the orbital $p$. We can write the same Hamiltonian using the Bogoliubov–de Gennes (BdG) representation as $\mathcal{H} = \sum_k C_k^\dagger \mathcal{H}_k^c C_k$, with $C_k^c = (\epsilon_k^1 c_k^1, p_k^1)$ and $\mathcal{H}_k = -\mu \Gamma_0 - \epsilon_k \Gamma_z + \Delta_k \Gamma_{xy} - V_k \Gamma_y$, where $\Gamma_{ab} = \tau_a \otimes \tau_b$, $a, b = x, y, z$, and $\Gamma_{xz} = \Gamma_{zy}$ are the Pauli matrices acting on particle–hole/orbitals space, respectively, and $\tau_0 = \tau_0$ are the $2 \times 2$ identity matrix. We have defined $\epsilon_k = 2\cos(k)$, $V_k = V \sin(k)$, and $\Delta_k = \Delta \sin(k)$.

3. Energy spectrum

Since a topological phase transition only occurs when a gap closes, looking for gapless points on the energy spectrum may indicate this transition. The model considered here has the following energy dispersion relations, $E(k) = \pm \sqrt{Z_k^2 + 2|Z_k^2|}$, where $Z_k = A(k) + B(k)$ and $Z_k = A(k)B(k)$, with $A(k) = \epsilon_k^2 + V_k^2$ and $B(k) = \Delta_k^2 + \mu^2$. Looking for gapless points $(E(k) = 0)$ the possible solution is $A(k) = B(k)$, i.e. $\mu^2 = \epsilon_k^2 + V_k^2 - \Delta_k^2 = [(2\tau)^2 \cos^2(k) + (V^2 - \Delta^2) \sin^2(k)]$. We will analyze this equation more deeply in the next sections. First we would like to highlight the case with no hybridization, $V = 0$, in which the system is always gapless whenever $|\mu| < 2\tau$. The existence of these gapless modes represents a substantial difference between this and the Kitaev model. We will see in the next section that even in this non-gapped region the system shows superconductivity. On the other side, when $|\mu| > 2\tau$, the system is fully gapped but superconductivity is still present up to $|\mu| < 4\tau$. Deep inside the gapless phase the crossings between bands have a linear dispersion relation and define Dirac like nodes. Furthermore, we note that the bands are non-degenerate and the nodes appear and disappear only when two nodes are combined. This is a characteristic of Weyl fermions in 3D or 2D SC [19] and in topological superfluidity [17, 18]. In this sense, the model here presented can be called 1D Weyl SC.

4. Self-consistent equations for the superconductivity

We may calculate the self-consistent inter-band superconducting order parameter, $\Delta$, from the gap equation

\[
\Delta = -\frac{4g}{L} \sum_k i \sin(k) \langle \rho_{p,k} c_k \rangle,
\]

(2)

where $g$ is the attractive energy between the spinless fermions, $L$ is the length of the chain, and the correlation function $\langle \rho_{p,k} c_k \rangle$ is obtained from the fluctuation-dissipation theorem [22] (a similar calculation was recently done in [6]), such that

\[
\langle \rho_{p,k} c_k \rangle = \frac{1}{2\pi} \int f(\omega) \left\{ \langle \rho_{s,k} c_k \rangle^* - \langle \rho_{s,k} c_k \rangle \right\} d\omega;
\]

(3)

the Green’s functions $\langle \rho_{s,k} c_k \rangle$ (retarded and advanced) are obtained from the Greenian operator [23–25], i.e. $\langle \rho_{s,k} c_k \rangle = (\omega \mathbb{1}_{4 \times 4} - \mathcal{H}_k)^{-1}$, and $f(\omega)$ is the Fermi distribution. If we proceed with the calculations the gap equation at $T = 0$ becomes

\[
\frac{1}{g} = \frac{1}{L} \sum_k \frac{4 \sin^2(k)}{(\omega_1 + \omega_2)} \delta_k,
\]

(4)

where $\delta_k = 1$ if $B(k) > A(k)$ and $\delta_k = 0$ otherwise. Also, $\omega_1$ and $\omega_2$ are the eigenvalues of the Hamiltonian, such that, $\omega_{1,2}(k) \equiv E(k), \omega_{+} = \sqrt{Z_k^2 + 2|Z_k^2|}$. We can verify the stability of the superconducting phase calculating the parameters $\Delta$ and $\mu$ self-consistently from the gap and the occupation number.
equation given by $n = n_s + n_p = \frac{1}{2} \sum_k \left[ (\epsilon^s_k) + (\epsilon^p_k) \right]$, with $n_s$ and $n_p$ the occupation numbers for the $s$ and $p$ bands, respectively. For the model considered here, this equation is $n = \frac{1}{2} \sum_k \left( \frac{\mu}{\omega} \right) \delta^0 + \frac{1}{2}$, where $0 < n < 2$ is the total occupation number per site of the chain.

5. Phase diagram

The solution of the coupled self-consistent equations for the gap and the chemical potential is complicated by the constraints ($\delta^0$) of the sums in momentum space, equation (4). In this section we present the phase diagram of our model system obtained directly from a numerical solution of the BdG equations fixing the chemical potential. The self-consistent solution implies that the pairing can be obtained using equation (2). In figure 1(a) we show the numerical results for the order parameter $\Delta$ as a function of the chemical potential and hybridization for a fixed value of the attractive interaction $g = 1.7$. All quantities are normalized by the hopping term $t$. In figure 1(c) we show the gap for excitations for the same range of parameters. The results in these figures allow us to obtain the zero temperature phase diagram of the system shown in figure 1(b).

In agreement with our previous discussions we find a gapless superconducting phase for $(|\mu|/2t) < 1$ and $V/2t < V_c(\mu)/2t \equiv [(\mu/2t)^2 + (\Delta/2t)^2]^{1/2}$ named Weyl SC (WSC) in the phase diagram. We note that at the transition ($V = V_c(\mu)$) the gapless points always occur at $k = \pm \pi/2$. The quantity $\Delta_0 = \Delta_0(\mu)$ is the value of the order parameter at $V = 0$ for a given chemical potential value $\mu$. For $(|\mu|/2t) > 1$ and $V < V_c(\mu)$ the system presents a gapped superconducting phase with trivial topological properties similar to the strong coupling superconducting phase of Kitaev’s model. In this phase, named SC in the phase diagram, the order parameter vanishes continuously as the chemical potential increases. For a fixed $V < V_c(\mu)$, the range of this phase for increasing $\mu/2t$ depends on the strength of the attractive interaction $g$.

On the other hand for $(|\mu|/2t) < 1$, but for $V > V_c(\mu)$, there is a gapped non-superconducting phase, that corresponds to a topological insulator (TI), as will be discussed below. This phase is characterized by zero energy modes localized at the ends of the chain for $\mu = 0$. There are also localized modes if $\mu \neq 0$ that have finite subgap energy. We notice that the conditions for the existence of a gap are given by $\mu^2 = \epsilon^2_k + V_k^2 = \Delta^2_k$, as discussed above. For instance, in the case of strong hybridization and weak or no superconductivity, such that $1 + (\Delta/2t)^2 < (V/2t)^2$, the system becomes gapless whenever $(\mu/2t) \geq 1$ and $V > V_c(\mu)$. This corresponds to the phase $M$ in the phase diagram of figure 1(b) which is a normal metallic or insulating phase (not shown in the figure) depending on the occupation number.

6. Nature of the transitions

6.1. WSC-TI half-filling

Let us first consider the case of half-filling bands ($n = 1$), where the chemical potential is fixed at $\mu = 0$. The constraint ($\delta^0 = 1$) in equation (4) now reads $\Delta_0^2 > (\epsilon^2_k + V_k^2)$. Important points in momentum space correspond to those wave-vectors where this inequality becomes an equality, i.e. $\tan^2(k_0) = \frac{(2t)^2}{\Delta^2 - V^2}$. At these points the system becomes gapless and they characterize the Weyl points. From this result, we can immediately see that there are no gapless nodes when $V > \Delta$. It is easy to see also that with increasing hybridization the Weyl points collapse at $k_0 = \pi/2 \equiv k_F$ for $V = \Delta$ at a discontinuous quantum phase transition from the WSC to the TI phase where the superconducting order parameter drops to zero. This collapse of superconductivity is associated with the appearance of zero energy modes exactly at the Fermi surface $k_F$, of the half-filled system. In the superconducting side before the transition, the order parameter (when $\mu = 0$) is given by the gap equation, $(1/g) = \frac{4}{\Delta^2} \sum_{k=0}^{k_0} \sin(k)$, where $k_0$ is the largest momentum value that contributes to the superconductivity. Notice that this equation has no trivial analytic solution since $k_0$ depends on $\Delta$. At the transition, for $V = \Delta$ the momenta $k_0 = \pm \pi/2$ and $\Delta_0 = 4g/2\pi$ before dropping abruptly to zero at the TI phase.

At this point, for completeness, we discuss the influence of the strength of the interaction $g$ in the zero temperature phase diagram. Figure 2 shows the superconducting order parameter $\Delta$, for fixed $\mu = 0$, calculated self-consistently for three values of the hybridization $V$. When $V = 0$ even for small values of $g$ the system is a zero temperature superconductor, although
with small values of $\Delta$. As $V$ increases ($V = 0.5$) the system requires a minimum value of the interaction $g$ to become superconductor. In this case superconductivity sets in abruptly in a first-order quantum phase transition. For still larger hybridization, $V = 1$, the values of $g$ shown in figure 2 are not sufficiently strong to produce a superconducting ground state. Notice from the curve for $V = 0.5$ that the instability of the superconducting phase occurs whenever $\Delta = 2V$. Finally, the inset of figure 2 shows the gap in the spectra of excitations. These turn out to be always zero in the superconducting phase. This is a peculiar behaviour of our system where the superconducting phase (large $g$) is gapless, whereas the non-superconducting phase (small $g$) is gapped.

6.2. WSC-SC

We now investigate the transition from the non-trivial topological superconductor to the trivial one by increasing the chemical potential at fixed hybridization. Let us for simplicity consider the case of $V = 0$. The WSC-SC transition occurs for $(\mu/2t)_c = 1$ as shown in figure 1(c). It is associated, as can be easily checked with the collapse of two Weyl points at the center of the Brillouin zone ($k = 0$) and at its extremities ($k = \pm \pi$). Expanding the dispersion relation of the excitations close to $k = 0$ and $(\mu/2t)_c = 1$, we get

$$\omega_2(k) = 2t\sqrt{\left(1 - \frac{\mu}{2t}\right)^2 + \left(\frac{\Delta}{2t}\right)^2}k^2.$$  (5)

We have omitted the $k^2$ term, since its coefficient is proportional to $(1 - \mu/2t)_c$ and vanishes at the quantum topological phase transition at $(\mu/2t)_c = 1$. Then, at the quantum critical point, the spectrum of excitations $\omega_2(k) \propto k^2$, which allows one to identify the dynamical exponent $z = 2$ for this transition. On the other hand, at $k = 0$, the gap $\omega_2(k = 0) = (\mu/2t)_c - (\mu/2t)$, vanishes linearly at the quantum critical point with a gap exponent $\nu = 1$. The critical exponents $\nu = 1/2$ and $z = 2$ show that the quantum phase transition from the topological to the trivial superconducting phase in the inter-band model is in a different universality class from that of the Kitaev model [8].

The different values of the critical exponents imply distinct behaviour for the compressibility of the two models at the topological quantum phase transition inside the superconducting phase. The compressibility close to this transition is given by, $\chi_c = \partial^2 F/\partial J^2 \propto ((\mu/2t)_c - (\mu/2t)^{1-\alpha}$ where $f$ is the free energy density. The exponent $\alpha$ is related to the correlation length and dynamical exponents by the quantum hyperscaling relation [26–28], $2 - \alpha = \nu(d + z)$. It can be easily verified that for the inter-band model $\alpha = 1/2$ implying a strong singularity for the compressibility at the topological transition [29]. Indeed in our model the topological transition is in the universality of the Lifshitz transition [20]. Notice that this is a purely topological quantum phase transition, since both phases are characterised by the same order parameter. In spite of this, they have singularities described by critical exponents which obey the quantum hyperscaling relation [26]. Although the usual Landau approach of expanding the free energy in terms of order parameters that become small close to a continuous phase transition is of no use here, the renormalisation group still provides an adequate description of this critical phenomenon [28].

7. Energy spectrum in real space

In order to find the energy spectrum in real space through the BdG transformation, we write the Hamiltonian in the form, $\mathcal{H} = CHC^\dagger$, where $C = \left(c_1 p_1 c_1^{\dagger} p_1^{\dagger} \cdots c_n p_n c_n^{\dagger} p_n^{\dagger}\right)^T$ and the matrix $H$ is comprised by the following $(4 \times 4)$ interaction matrices

$$\begin{pmatrix}
H_{r,r} & -\mu \Gamma_{0r} \\
H_{r,r+1} & -i\Delta_{0r} - i\Delta_{yr} + iV \Gamma_{yr} \\
H_{r,r-1} & -i\Delta_{0r} + i\Delta_{yr} - iV \Gamma_{yr} \\
H_{r,r'} & = 0 & \forall r' \neq \{r, r+1 \text{ or } r-1\}.
\end{pmatrix}$$  (6)

The BdG transformation diagonalizes the Hamiltonian, $\mathcal{H} = E_0 + \sum\gamma E^{\gamma} \gamma^{\dagger} \gamma$, such that, $U^\dagger H U = E$, where $U$ is formed by all the BdG coefficients $u_\alpha, v_\alpha, u_p$ and $v_p$, and has the property to be unitary $U^\dagger U = I$. The matrix $E$ is diagonal and contains the energy spectrum ($E_\alpha$) of the system. We have calculated the energy spectrum for a chain of $L = 100$ sites with two-orbitals per site and inter-band interactions in the presence of hybridization. The spectrum consists of 4L energies. We have checked that this size is large enough to prevent finite size effects. We set the chemical potential to zero $\mu = 0$, and take the hybridization strong enough ($V > V_c$ or $\Delta = 0$), such that the system is in the TI phase of the phase diagram in figure 1(b). We can see in figure 3(a) the appearance of zero energy fermionic modes (four-fold degenerated). We also checked that these zero energy modes are localized at the ends of the chain. This gapped insulating phase share the same properties of the topological insulating phase found in a normal $sp$ chain [7–9]. In this situation the fermionic modes resemble the Majorana zero modes, as will be discussed in the next section. Next, we remove the chemical potential from zero and keep ($V > V_c$), such that the system remains in the TI

![Figure 2](image.png)

Figure 2. The superconducting order parameter $\Delta$ as function of the inverse of the coupling strength ($1/g$) for three different values of hybridization $V$ (see text). The inset shows the gap in the spectra of excitations versus $1/g$ for the same values of $V$. 

[Image 63x625 to 279x777]
phase, specifically $\mu = 0.5$ and $\Delta = 0$, see figure 3(b). We find that there are two (plus two particle–hole symmetric) energies in the spectrum constituting a subgap, that also correspond to localized edge states but with longer spatial extent. A more intriguing situation happens when we induce superconductivity on this TI phase and other energies displaced from the spectrum appear, see figure 3(c). Accurately, we set $\mu = 0.7$ and $V = 1.4$. The system now has 4 localized states (plus 4 particle–hole symmetric) consisting of two double-degenerate states. We also show that all those particular energies are localized in the end of the chain, but with lower occupation number each. We notice that for large values of $\Delta$, specifically when $\Delta > V$ and $V$ ceases to be greater than $V_c$, the system becomes gapless, where the transition to the WSC phase happens.

8. Topological invariants and edge modes

8.1. WSC phase

The winding number is a proper topological invariant that classifies the topological phase of a gapped 1D system. In the gapless superconducting phase, it is not possible to calculate this by conventional methods since there are zero energy points that cannot be avoided in one-dimensional systems, or the sum over the Brillouin zone gives a vanishing winding number since the Fermi points appear in pairs and their contributions cancel out [30]. On the other hand, let us look closer to one of the linear dispersion relations that crosses the zero energy at some point $k = k_0$. In this region the Hamiltonian with Weyl nodes in 1D can be reduced to describe the two Bogoliubov bands that cross zero energy. The reduced low energy part of the Hamiltonian may be expanded in terms of Pauli matrix bands that cross zero energy. The reduced low energy part of the Hamiltonian may be expanded in terms of Pauli matrix bands that cross zero energy. The reduced low energy part of the Hamiltonian may be expanded in terms of Pauli matrix bands that cross zero energy. The reduced low energy part of the Hamiltonian may be expanded in terms of Pauli matrix bands that cross zero energy.

8.2. Majorana modes in TI phase

In order to clarify the existence of Majorana modes in our model, we write the Hamiltonian, equation (1), in real space. This is given by

$$\mathcal{H} = \sum_i \left\{ -p_i c_i^\dagger p_i c_i + \delta_{\alpha} p_i c_i^\dagger c_{i+1} - c_i^\dagger p_{i+1} - \delta_{\beta} (c_i^\dagger p_{i+1} + c_{i+1}^\dagger p_i) \right\},$$

This can be written in terms of Majorana operators, $\alpha_{A',r}$, $\alpha_{B',r}$, $\beta_{A',r}$ and $\beta_{B',r}$, via the relations, $c_i = \frac{1}{2}(\alpha_{A',r} + i\alpha_{B',r})$ and $c_i^\dagger = \frac{1}{2}(\alpha_{A',r} - i\alpha_{B',r})$.

3 Private discussions with Andrei Bernevig.
\[ p_r = \frac{i}{2}(\partial_{B_r} + i\partial_{A_r}). \] Now, we perform a second transformation on Majorana fermions—we call them unconventional hybridized Majorana fermions—such that \( \gamma_{A,r}^\pm = \alpha_{A,r} \sqrt{(V - \Delta)} \pm \beta_{A,r} \sqrt{(V + \Delta)} \) and \( \gamma_{B,r}^\pm = \alpha_{B,r} \sqrt{(V + \Delta)} \pm \beta_{B,r} \sqrt{(V - \Delta)} \). The result is the following

\[ \mathcal{H}' = \frac{i}{4} \sum_r \left\{ (-1 + C_r)(\gamma_{B,r}^+ \gamma_{A,r+1}^+ - \gamma_{A,r}^+ \gamma_{B,r+1}^+) + \right. \\
- (-1 + C_r)(\gamma_{B,r}^- \gamma_{A,r+1}^- - \gamma_{A,r}^- \gamma_{B,r+1}^-) + \right. \\
- \left( C_r \gamma_{B,r}^+ \gamma_{A,r} + \gamma_{B,r}^- \gamma_{A,r}^- \right), \tag{8} \]

where \( C_r \equiv \mu \sqrt{(V^2 - \Delta^2)} \) and \( C_r \equiv \mu \sqrt{(V^2 - \Delta^2)} \). If we go to the limit \( \mu = 0 \) and take \( C_r = 1 \), such that the system is in the TI phase in figure 1(b) \( (V > \Delta) \), the Hamiltonian reads,

\[ \mathcal{H}' = -\frac{i}{4} \sum_r \left( \gamma_{B,r}^+ \gamma_{A,r+1}^- - \gamma_{A,r}^+ \gamma_{B,r+1}^- \right), \tag{9} \]

which couples Majorana fermions only at adjacent lattice sites. Proceeding with the same analysis as [3], we may easily see that the Majorana modes \( \gamma_1 = \gamma_{A,1}^+ \), \( \gamma_2 = \gamma_{B,1}^+ \), \( \gamma_3 = \gamma_{B,1}^- \), and \( \gamma_4 = \gamma_{A,1}^- \) are not present in the above Hamiltonian, it means that they have no cost of energy to be added to the system; they are called Majorana zero modes. These modes persist out of the fine-tuned \( C_r = 1 \) provided that \( \mu = 0 \), since we know that there is no gap closing for this range of parameters. When \( \mu \neq 0 \), the topological character is preserved, in the sense that we still have localized states at the ends of the chain, but the Majorana zero modes are not robust such that they acquire a finite energy, the subgap energy, see figure 3. In this situation, or when generally referring to them, we call them fermionic modes, instead of Majorana ones.

### 8.3. TI phase

The non-trivial topological character of the TI phase can be shown by calculating the winding number for the special case \( \mu = 0 \). In this region of the phase diagram we have \( \Delta = 0 \) and the \( 4 \times 4 \) Hamiltonian can be decoupled in two \( 2 \times 2 \) Hamiltonians, such as \( \mathcal{H} = -V_c \sigma_y - \varepsilon_4 \sigma_x \). This equation can be rewritten as the Hamiltonian of a spin 1/2 in a \( k \)-dependent magnetic field, \( \mathcal{H} = -\mathbf{h}(k) \cdot \sigma \), where \( \mathbf{h}(k) = (h_x(k), h_y(k), h_z(k)) = (0, -V_c, -\varepsilon_4) \) with the properties \( h_x(k) = -h_x(-k) \) and \( h_z(k) = h_z(-k) \). The winding number \( \nu \) is obtained as the product of the signs of the magnetic field on the center and at the extreme of the Brillouin zone, i.e., \( \nu = \text{sgn}(\mathbf{h}(k = 0)) \text{sgn}(\mathbf{h}(k = \pi)) \). Since \( \mathbf{h}(k = 0) = (0, 0, 2t) \) and \( \mathbf{h}(k = \pi) = (0, 0, -2t) \), we get \( \nu = -1 \), which characterizes the non-trivial topological character of the TI phase along the line \( \mu = 0 \). Furthermore, the system belongs to the BDI class of Hamiltonians according to [32], and is characterized by the \( \mathbb{Z} \) topological invariant. In this case the chiral symmetry protects the topology. For instance, the system may lose the time-reversal and particle–hole symmetries and remains in the same TI phase, but now it belongs to the AIII class of Hamiltonians.

The topological nature of this phase is associated with the existence of zero energy modes at the ends of the chain, as discussed above. Those zero modes form pairwise charged fermions, which may be present or not without cost of energy. In the TI phase there is only one of these fermions at each edge, while in the superconducting state there are two of them. If one calculates the winding number (by usual methods) for the whole system, the \( 4 \times 4 \) Hamiltonian including \( \Delta \), it shows itself to be trivial. The topology is hidden by the charge conjugation (or particle–hole) symmetry imposed to the system. Some attempts to calculate the winding number using new methods were proposed to uncover this kind of topology [33]. On the other hand, the topological character of the whole TI phase is guaranteed since it is adiabatically connected with the topological case just shown (when \( \Delta = 0 \)).

### 9. Conclusions

In this work we studied a 1D spin-chain with attractive interband interactions and anti-symmetric hybridization. The model presents a rich phase diagram including non-trivial topological phases. It presents a weak coupling superconducting phase containing Fermi points with gapless excitations. We studied the quantum topological phase transition from this phase to the, trivial, strong coupling one and found that it is in a different universality class from that of Kitaev’s model. We obtained the critical exponents including that characterizing the singular behavior of the compressibility. We have also shown the existence of a discontinuous transition from a Weyl superconductor to a topological insulator. This is caused by the appearance of zero energy modes exactly at the Fermi surface of the normal non-interacting system. The topological insulating phase has been characterized through its winding number and by the presence of zero energy fermionic modes at the ends of the chain.

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