A dihedral Bott-type iteration formula and stability of symmetric periodic orbits

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Abstract

In 1956, Bott in his celebrated paper on closed geodesics and Sturm intersection theory, proved an Index Iteration Formula for closed geodesics on Riemannian manifolds. Some years later, Ekeland improved this formula in the case of convex Hamiltonians and, in 1999, Long generalized the Bott iteration formula by putting in its natural symplectic context and constructing a very effective Index Theory. The literature about this formula is quite broad and the dynamical implications in the Hamiltonian world (e.g. existence, multiplicity, linear stability etc.) are enormous.

Motivated by the recent discoveries on the stability properties of symmetric periodic solutions of singular Lagrangian systems, we establish a Bott-type iteration formula for dihedrally equivariant Lagrangian and Hamiltonian systems.

We finally apply our theory for computing the Morse indices of the celebrated Chenciner and Montgomery figure-eight orbit for the planar three body problem in different equivariant spaces. Our last dynamical consequence is an hyperbolicity criterion for reversible Lagrangian systems.

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1 Introduction and description of the main results

Symmetric periodic orbits of Lagrangian or more general Hamiltonian systems have been discovered in the last decades by developing suitable variational methods on a space of loops symmetric with respect to a chosen symmetry compact Lie group $G$. The well-known Palais principle of symmetric criticality (cf. [Pal79]) claims that critical points of the restriction of a $G$-invariant functional to the space fixed by $G$ are critical points of the functional. Variational minimization methods on the space of $G$-equivariant loops, were recently successfully employed by many authors for proving the existence of amazing symmetric periodic orbits; e.g. the Chenciner-Montgomery figure-eight solution of the planar three-body problem with equal masses. (Cf., for further details, [CM00, Che02a, Che02b, FT04] and references therein). In the aforementioned problem the idea of minimizing the Lagrangian action on a space of loops symmetric with respect to a chosen symmetry group plays a crucial role in order to prove the existence of collisionless periodic orbits.

The existence of $G$-symmetric periodic orbits of a Hamiltonian system is the first step in order to penetrate the intricate dynamics of the problem. The second step in this analysis is to investigate the stability and multiplicity properties of these solutions. Many techniques have been developed in the last years for tackling both these problems and among all of them a central role is essentially played by the index theory. An enormous contribution has been given in the last decades by Long and his collaborators in the construction of an index theory based on a symplectic invariant nowadays known in literature as Maslov index. A central device in order to investigate the aforementioned problems is based on the celebrated Bott-type iteration formula for the Maslov-type index, which directly provides an estimate of the elliptic height (i.e. the total algebraic multiplicity of all eigenvalues on the unit circle of the complex plane).

Bott-type iteration formula represents the starting point of our analysis and the initial motivations for further investigations. This formula was introduced by author in [Bot56] for his investigation on closed geodesics on Riemannian manifolds and some years later, Ekeland in [Eke90] and Long in [Lon02] put it into a symplectic context by generalizing such a formula to any linear $T$-periodic Hamiltonian system. The idea behind this formula is based on the fact that if $\gamma$ denotes the fundamental solution of a linear Hamiltonian $T$-periodic system, we can associate an integer, let’s say $\iota_1(\gamma; [0, T])$. It is very natural to single out for considerations also the indices $\iota_1(\gamma; [0, 2T])$, $\iota_1(\gamma; [0, 3T])$, etc. corresponding to the intervals which are multiples of the basic period $T$. Bott’s iteration formula, essentially allow us to relate the index $\iota_1(\gamma; [0, kT])$ to $\iota_1(\gamma; [0, T])$ and to the Floquet multipliers of the Hamiltonian system.

Let $z$ be a periodic solution of the Hamiltonian system corresponding to the Hamiltonian function $H$ of class $C^2$; namely

\[
\begin{align*}
\dot{z}(t) &= J\nabla H(t, z(t)), \quad t \in [0, T] \\
z(0) &= z(T)
\end{align*}
\]

with $J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ and where $I$ denotes the identity on $\mathbb{R}^m$. Its associated fundamental solution $\gamma$ is the matrix-valued solution of the linear initial value problem obtained by linearizing the Equation (1.1) along $z$; i.e.

\[
\begin{align*}
\dot{\gamma}(t) &= JD^2H(t, z(t))\gamma(t) \\
\gamma(0) &= I.
\end{align*}
\]
It is well-known that $\gamma$ is a path in the symplectic group of $(\mathbb{R}^{2m}, \omega)$ where $\omega$ denotes the standard symplectic structure. If $\mathbf{U}$ is the unit circle of the complex plane, we denote by $\iota_\omega(\gamma)$ the so-called $\omega$-Maslov-type index of $\gamma$ defined as intersection index of $\gamma$ with a transversally oriented 1-codimensional manifold in the symplectic group $\text{Sp}(2m)$. (We refer the interested reader to [Lon02] and references therein for further details).

Based on the index function $\iota_\omega$, Long established a Bott-type iteration formula to any continuous path $\gamma : [0, T] \to \text{Sp}(2m)$ such that $\gamma(0) = I$. More precisely, for any $T > 0$, $\gamma$ as before, $z \in \mathbf{U}$ and $n \in \mathbb{N}$,

$$\iota_z(\gamma, n) = \sum_{\omega^n = z} \iota_\omega(\gamma)$$

$$\nu_z(\gamma, n) = \sum_{\omega^n = z} \nu_\omega(\gamma)$$

where $\nu_\omega(\gamma)$ denotes the $\omega$-nullity of $\gamma$ defined as follows

$$\nu_\omega(\gamma) := \dim_{\mathbb{C}} \ker_{\mathbb{C}} (\gamma(T) - \omega I).$$

Motivated by some concrete problems in Celestial Mechanics, some years ago, in 2009 in fact, authors in [HS09, Theorem 1.1] generalized the Bott-type iteration formula given in Equation (1.2) in a form which is crucial for applying it, if we are in presence of a cyclic group $\mathbb{Z}_n$ acting on the orbit. The main idea in order to establish a $\mathbb{Z}_n$-invariant Bott-type iteration formula, is essentially based on a $\mathbb{Z}_n$-equivariant decomposition of the path space into isotypical components.

Few years ago, more precisely in 2006, authors in [LZZ06], working on the famous Seifert conjecture (claiming the existence of at least $n$ geometrically distinct brake orbits on a given regular compact hypersurface $\Sigma$), were able to prove the existence of two brake orbits on $\Sigma$ under an additional condition that $H$ is even. The proof of this result is essentially based on a new Maslov-type index theory for brake orbits and Morse theory applied to the Hamiltonian action functional. Some related formulas, although in a quite different context, were recently found by Liu and Zhang in [LZ14a] and [LZ14b]. In this paper, authors established some Bott-type iteration formulas for the $L$-index (namely the Maslov-type index of symplectic paths associated with a Lagrangian subspace). It is worth to quote the paper [LT15] where authors, proved a Bott-type iteration formula for the Maslov $P$-index which can be taken as a generalization of the standard Bott-type iteration formula where $P = I_{2n}$. Recently, the authors in [WZ16] gave a direct proof of the iteration formulae for the Maslov-type indices of symplectic paths by using a splitting of the nullity to yield a splitting formula for the Maslov-type indices of symplectic paths in weak symplectic Hilbert space.

Pushing further this analysis and motivated by the fact that many interesting periodic orbits are actually symmetric with respect to a more general group action (like the aforementioned figure-eight orbit which is $D_{2n}$-symmetric where $D_n$ denotes the dihedral group of order $2n$), in this paper we establish an abstract $D_n$-equivariant spectral flow formula and a new $D_n$-equivariant Bott-type iteration formulas for Hamiltonian (resp. for Lagrangian) systems, through a suitable isotypical decomposition of the path space of the phase (resp. configuration) space.

### 1.1 Equivariant set-up, description of the problem and main results

Let $\mathcal{E}$ be a complex separable Hilbert space and $G$ be a finite (abstract) group acting on $\mathcal{E}$. Denoting by $\mathbf{U}(\mathcal{E})$ the group of unitary operators, we assume that $G$ acts on $\mathcal{E}$ through its unitary representation

$$G \ni g \mapsto U_g \in \mathbf{U}(\mathcal{E}).$$

We start to consider the (finite) cyclic group of order $n$ presented as follows $\mathbb{C}_n := \langle \tau | \tau^n = 1 \rangle$, where we denoted by $\tau$ the identity of the group. It is well-known that it can be represented by the group of rotations through angles $2k\pi/n$ around an axis. We denote by $D_n$ the semi-direct
product of $C_n$ and $C_2$; in symbols $D_n = C_n \rtimes C_2$, where $C_n := \langle r | r^n = e \rangle$ and $C_2 := \langle s | s^2 = e \rangle$. The group $D_n$ is usually termed dihedral group of degree $n$ and order $2n$. For $n \geq 3$, $D_n$ is the group of symmetries of a regular $n$-gon in the plane, namely, the group of all the plane symmetries that preserves a regular $n$-gon. It contains $n$ rotations (which form a subgroup isomorphic to $C_n$) as well as $n$ reflections. From an algebraic viewpoint, we point out that, it is a metabelian group having the cyclic normal subgroup $C_n$ of index 2 and the following presentation

\[ D_n := \langle r, s | r^n = s^2 = 1, \ srs = r^{-1} \rangle. \]

Each element of $D_n$ can be uniquely written, either in the form $r^k$, with $0 \leq k \leq n - 1$ (if it belongs to $C_n$), or in the form $sr^k$, with $0 \leq k \leq n - 1$ (if it doesn’t belong to $C_n$). Observe that the relation $srs = r^{-1}$ implies that $sr^k s = r^{-k}$ and $(sr^k)^2 = 1$. It is worth noticing also that the irreducible representations of $D_n$ are one or two dimensional. More precisely, if $n$ is odd except the (trivial) identity and the minus identity representations, all others are two-dimensional; if $n$ is even we have other than the previous one-dimensional irreducible representations also the sign representations whilst all other are two-dimensional.

With a slight abuse of notation, we denote with the same symbol the dihedral group as well as its unitary representation in $E$. Thus, the group $D_n$ (actually a unitary representation of the dihedral group $D_n$) is presented as follows

\[ D_n := \langle \mathcal{R}, s \in \mathbf{U}(E) | \mathcal{R}^n = S^2 = (S\mathcal{R})^2 = I_E \rangle \subset \mathbf{U}(E). \]

As direct consequence of the spectral mapping theorem, it readily follows that the spectrum of $\mathcal{R}$ is given by $\sigma(\mathcal{R}) = \{ \zeta_n^k \in \mathbb{C} \mid k = 0, \ldots, n - 1 \}$. Furthermore, if $E_k := \ker(\mathcal{R} - \zeta_n^k I_E)$ denotes the spectral space corresponding to the eigenvalue $\zeta_n^k$, by the spectral theory of normal operators, it is well-known that $E_k$ are mutually orthogonal.

We define the following closed subspaces of $E$

\[ F_k := \begin{cases} E_0 & \text{if } k = 0 \\ E_k \oplus E_{-k} & \text{if } k = 1, \ldots, \lfloor (n - 1)/2 \rfloor \\ E_{n/2} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases} \]

if $k = n/2$

where we denoted by $\lfloor . \rfloor$ the integer part. (We refer the interested reader to Appendix A, for further details). For any $k$, he subspace $F_k$ defined in Equation (1.3) is a $D_n$-module with the action given by

\[ C_n \times F_k \to F_k : (\zeta_n^k, v) \mapsto \mathcal{R} v := \begin{bmatrix} \zeta_n^k I_E & 0 \\ 0 & \zeta_n^{-k} I_E \end{bmatrix} v \quad \text{and} \quad C_2 \times F_k \to F_k : (s, v) \mapsto S v := \begin{bmatrix} 0 & I_E \\ I_E & 0 \end{bmatrix} v. \]

Thus, we get a decomposition of the Hilbert space $E$ into mutually orthogonal $D_n$-stable modules, given by

\[ E = F_0 \oplus \cdots \oplus F_{\bar{n}} \]

where $\bar{n}$ denotes the largest integer not greater than $\lfloor n/2 \rfloor$.

Remark 1.1. We observe that the decomposition given in Equation (1.4) is the isotypic decomposition of $E$ induced by the unitary representation of the dihedral group. In particular each subspace $E_k$ is given by the direct sum of (infinitely many) one-dimensional irreducible representations of the cyclic group. (Cf. Appendix A and references therein).
We denote by \( \mathcal{F}^{sa}(E) \) the space of all bounded selfadjoint Fredholm operators on \( E \) and let \( A : [0, 1] \to \mathcal{F}^{sa}(E) \) be a continuous path of operators commuting with \( \mathcal{R} \) and \( \mathcal{S} \); namely
\[
\mathcal{A}(\lambda)\mathcal{R} = \mathcal{R}\mathcal{A}(\lambda) \quad \text{and} \quad \mathcal{A}(\lambda)\mathcal{S} = \mathcal{S}\mathcal{A}(\lambda) \quad \text{for all} \quad \lambda \in [0, 1].
\]
For each \( k = 0, \ldots, n \), let \( \mathcal{A}_k \) be the continuous path defined by \( \mathcal{A}_k := \mathcal{A}|_{F_k} : [0, 1] \to \mathcal{F}^{sa}(F_k) \).

With respect to this orthogonal decomposition, the path \( \mathcal{A} \) can be written as
\[
\mathcal{A}(\lambda) = \mathcal{A}_0(\lambda) \oplus \cdots \oplus \mathcal{A}_n(\lambda) \quad \text{for} \quad \lambda \in [0, 1].
\]

As direct consequence of the decomposition into (mutually orthogonal) \( D_n \)-stable modules given in Formula (1.4) as well as the additivity property of the spectral flow under direct sum, the following result holds.

**Proposition 1.** (An equivariant spectral flow formula) Under the previous notation, we have
\[
sf(\mathcal{A}; [0, 1]) = sf(\mathcal{A}_0; [0, 1]) + \cdots + sf(\mathcal{A}_n; [0, 1]).
\]

**Remark 1.2.** It is worth to observe that all the spectral flow formulas can be established for more general class of paths of selfadjoint Fredholm operators (e.g. for gap continuous path of selfadjoint Fredholm operators).

For each \( k = 0, \ldots, n \) and \( h = 0, \ldots, n - 1 \), we define the closed subspaces
\[
F_{k,h}^\pm = \left\{ u \in F_k \mid \mathcal{S}^h u = \pm u \right\} \quad \text{and} \quad F_h^\pm = \bigoplus_{k=1}^n F_{k,h}^\pm
\]

Since \( \mathcal{R}|_{E_0} = I \), then \( F_{0,h}^\pm = F_{0,0}^\pm = \left\{ u \in F_0 \mid \mathcal{S} u = \pm u \right\} \). If \( n \) is even, \( \mathcal{R}|_{E_{n/2}} = -I \), so we have \( F_{n/2,h}^\pm = \left\{ u \in F_{n/2} \mid (-1)^h \mathcal{S} u = \pm u \right\} \). It is worth to note that, for \( k = 1, \ldots, [(n - 1)/2] \), we get
\[
F_{k,h}^\pm = \left\{ \begin{bmatrix} I \\ \mathcal{S}^h \end{bmatrix} z \mid z \in E_k \right\}
\]
and by a straightforward calculation for any \( h = 0, \ldots, n - 1 \), also that
\[
(\mathcal{S}^h \mathcal{R}) \mathcal{K}_\lambda = \mathcal{K}_\lambda \quad \text{where} \quad \mathcal{K}_\lambda := \ker \mathcal{A}_\lambda \quad \text{and} \quad \lambda \in [0, 1].
\]

In the case of \( D_n \)-equivariant path of bounded selfadjoint Fredholm operators, the parity of the spectral flow actually depends only on the restriction of the path on the finite dimensional subspace \( E_0 \) (resp. \( E_0, E_{n/2} \)) if \( n \) is odd (resp. \( n \) is even).

**Proposition 2.** Let \( A \in \mathcal{C}^0([0, 1]; \mathcal{F}^{sa}(E)) \) a \( D_n \)-equivariant path. If

- \( n \) is odd, then
  \[
sf(\mathcal{A}; [0, 1]) \equiv sf\left(\mathcal{A}|_{E_0}; [0, 1]\right) \quad (\text{mod} \ 2)
  \]
- \( n \) is even, then
  \[
sf(\mathcal{A}; [0, 1]) \equiv sf\left(\mathcal{A}|_{E_0}; [0, 1]\right) + sf\left(\mathcal{A}|_{E_{n/2}}; [0, 1]\right) \quad (\text{mod} \ 2).
  \]

Let \( h \in \{1, \ldots, n - 1\} \) and let us consider a subgroup \( G_h \) in \( D_n \) generated by \( \mathcal{S}^h \mathcal{R} \); thus
\[
G_h := (\mathcal{S}^h \mathcal{R}; \mathcal{R}, \mathcal{S} \text{ are the generators of } D_n).
\]
We let $E^+_h$ and $E^-_h$ be the invariant subspaces respectively given by $E^+_h := \{ x \in E \mid x = \mathcal{S}^h x \}$ and $E^-_h := \{ x \in E \mid x = -\mathcal{S}^h x \}$. It is easy to check that

$$E^\pm_h = \bigoplus_{k=0}^n F^\pm_{k,h}.$$  

As direct consequence of the additivity properties of the spectral flow, the following spectral flow summation formulas hold.

**Proposition 3.** Let $A \in \mathcal{C}^0([0,1];\mathcal{F}^m(E))$ pointwise commuting with the generators $\mathcal{R}$ and $\mathcal{S}$ of the unitary representation of $D_n$ in $E$ and let $h = 0, \ldots, n - 1$. Thus we have

$$\text{sf}(A; [0,1]) = \text{sf}(A|_{E^+_h}; [0,1]) + \text{sf}(A|_{E^-_h}; [0,1])$$

and

$$\text{sf}(A|_{E^+_h}; [0,1]) = \sum_{k=0}^n \text{sf} \left( A|_{F^+_h}; [0,1] \right).$$

An interesting application of the spectral formula proved in Proposition 3 is the following. We assume that $A$ is a selfadjoint essentially positive Fredholm operator; thus, in particular, $\hat{A}$ has a finite Morse index and let us consider the (analytic) path $\lambda \mapsto \hat{A}_\lambda := A + \lambda \mathcal{K}$ where $\mathcal{K}$ denotes a compact and symmetric linear bounded operator. Denoting by $n_-(\hat{A})$ the Morse index of $\hat{A}$ (namely the dimension of the negative spectral space of $\hat{A}$), it is well-known that

$$n_-(\hat{A}) = \text{sf}(\hat{A}_\lambda; \lambda \in [0,1]).$$

(Cf. Appendix B.2 for further details). As direct consequence of Proposition 3, the following result holds.

**Corollary 1.** We assume that the path $A$ is $D_n$-equivariant. Then

$$n_-(\hat{A}) = n_-(\hat{A}|_{E^+_h}) + n_-(\hat{A}|_{E^-_h}),$$

and

$$n_-(\hat{A}|_{E^+_h}) = \sum_{k=0}^n n_-(\hat{A}|_{F^+_h}).$$

Given $T > 0$ we denote by $T \mathbb{Z}$ the lattice generated by $T \in \mathbb{R}$ and we set $T := \mathbb{R}/(T \mathbb{Z}) \subset \mathbb{R}^2$ be a circle in $\mathbb{R}^2$ of length $T = |T|$. Given $Q \in \mathcal{U}(2m)$, let $E$ denote the $W^{1,2}$ closure of the set of smooth maps $z: T \to \mathbb{C}^{2m}$ such that $z(t) = Qz(t + T)$; thus

$$E := \left\{ z \in W^{1,2}(T; \mathbb{C}^{2m}) \mid z(t) = Qz(t + T) \right\}.$$  

Let $M \in \mathcal{U}(2m)$ and $N \in \mathcal{U}(2m)$ be such that the following commutativity properties holds

$$M^n = Q, \quad N^2 = I, \quad NM^* = MN.$$  

For any $k = 0, \ldots, n - 1$, we denote by $E_k$ the closed subspace of $E$ given by

$$E_k = \left\{ z \in E \mid Mz \left( t + \frac{T}{n} \right) = \zeta_n^k z(t) \right\}.$$  

and for $k = 1, \ldots, [(n - 1)/2]$, we define $\tilde{M}_k$, $\tilde{N}_k$ the following block diagonal matrices

$$\tilde{M}_k := \begin{bmatrix} c^{-k} M & 0 \\ 0 & \zeta_n^k M \end{bmatrix} \quad \text{and} \quad \tilde{N}_k := \begin{bmatrix} 0 & \zeta_n^{-k} N \\ \zeta_n^k N & 0 \end{bmatrix}.$$
As above, we set $F_0 = E_0$, $F_{n/2} = E_{n/2}$ if $n$ is even, $F_{n/2} = 0$ if $n$ is odd and finally $F_k := E_k \oplus E_{-k}$ for $k = 1, \cdots, [(n - 1)/2]$. Thus, we have

\begin{equation}
F_k = \left\{ u \in W^{1,2}\left(\frac{T}{n}; \mathbb{C}^{2m} \oplus \mathbb{C}^{2m}\right) \mid u \equiv \begin{bmatrix} z \\ w \end{bmatrix}, z \in E_k, w \in E_{-k} \text{ and } u(0) = \hat{M}_k u \left(\frac{T}{n}\right) \right\}.
\end{equation}

We now define the two unitary operators on $E$ as follows

\begin{equation}
M : E \ni z \mapsto (Mz)(\cdot) := M z (\cdot + T/n) \in E \text{ and } \quad \mathcal{N} : E \ni z \mapsto (Nz)(\cdot) := N z (T/n - t) \in E.
\end{equation}

By using the properties given in Equation (1.7), it is immediate to check that $E$ equipped by the action defined in Formula (1.9) turns out a $D_n$-equivariant space. Moreover, as direct consequence of the relations on the generators of the dihedral group $D_n$, we also get that, for every $h = 0, \ldots, n - 1$, $(N M^h)^2 = L$. In this way we decompose $E$ defined in Equation (1.6) into a direct sum of $D_n$-closed stable subspaces; namely $E = F_0 \oplus \cdots \oplus F_n$ where $F_k$ were defined in Formula (1.8).

Now, let $Q \in \text{Sp}(2m, \mathbb{R}) \cap \mathbb{O}(2m)$ be such that

\[MJ = JM, \quad M^n = Q, \quad N^2 = I, NJ = -JN\]

and we assume that $H \in \mathcal{C}^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ is a Hamiltonian function such that

\[H(t - T/n, Mz) = H(t, z) \quad \text{and} \quad H(T - t, Nz) = H(t, z),^2\]

Given a Lagrangian subspace $L$, we denote by $z$ a solution of the Hamiltonian system

\begin{equation}
\begin{cases}
z'(t) = \nabla H(t, z(t)), & t \in [0, T] \\
(z(0), z(T)) \in L.
\end{cases}
\end{equation}

To any solution $z$, we associate $\gamma : [0, T] \to \text{Sp}(2m)$ which is the fundamental solution of the linearized Hamiltonian system along it

\begin{equation}
\begin{cases}
\gamma'(t) = JB(t) \gamma(t), & t \in [0, T] \\
\gamma(0) = I
\end{cases}
\end{equation}

where $B(t) := D^2H(t, z(t))$. Now, to each $z$ we will associate an integer called the \textbf{geometrical index} and defined in terms of the Maslov-type index $\mu_{CLM}$ introduced by authors in [CLM94].

\textbf{Notation 1.3.} We will denote by $V_\pm(*)$ the positive and negative spectral space of the operator $*$.

\textbf{Definition 1.4.} We define the geometric index of a solution $z$ of the Hamiltonian System given in Equation (1.10) as

\[\iota_{geo}(z) := \mu_{CLM}(L, \text{Gr}(\gamma); [0, T])\]

where $L = \text{Gr}(Q)$.

Analogously, by setting $L^\pm = V_\pm(MN) \times V_\pm(NM^{n-1})$, we define the positive and negative geometric indices, as follows

\[\iota^\pm_{geo}(z) := \iota_{geo}(L^\pm, \text{Gr}(\gamma); t \in [0, T/2]).\]

\footnotetext{In the autonomous case we assume that $H(Mz) = H(z)$ and $H(Nz) = H(z)$.}
Remark 1.5. Being $L^\pm$ Lagrangian subspaces (cf. Lemma 3.4), it readily follows that Definition 1.4 is well-given.

**Theorem 1. (A $D_n$-equivariant Bott-type formula for Hamiltonian Systems)** Let $\gamma$ be the (fundamental) solution of the Hamiltonian system given in Equation (1.11) and let $\bar{\gamma} := \text{diag}(\gamma, \gamma)$. For $k = 0, \ldots, n$, we denote by $P$ and $Q_k$ the matrices respectively defined by $P := \begin{bmatrix} 0 & MN \\ MN & 0 \end{bmatrix}$ and by $Q_k := \begin{bmatrix} 0 & \zeta_n^{-k(n-1)}N \\ \zeta_n^{k(n-1)}N & 0 \end{bmatrix}$. Then, we have

\[
t_{\text{geo}}(z) = t_{\text{geo}}^0(z) + t_{\text{geo}}^- (z).
\]

Moreover, if

(i) $n$ odd

\[
i_{\text{geo}}^+(z) = \mu_{\text{CLM}}^+(V_\pm(N), \gamma(t)V_\pm(MN); t \in \left[0, \frac{T}{2n}\right])
\]

\[
+ \sum_{k=1}^{n-1} \mu_{\text{CLM}}^+(V_\pm(Q_k), \bar{\gamma}(t)V_\pm(P); t \in \left[0, \frac{T}{2n}\right]).
\]

(ii) $n$ even

\[
i_{\text{geo}}^-(z) = \mu_{\text{CLM}}^-(V_\pm((-1)^{n-1}N), \gamma(t)V_\pm((-1)^nMN); t \in \left[0, \frac{T}{2n}\right])
\]

\[
+ \mu_{\text{CLM}}^-(V_\pm(N), \gamma(t)V_\pm(MN); t \in \left[0, \frac{T}{2n}\right])
\]

\[
+ \sum_{k=1}^{n-1} \mu_{\text{CLM}}^+(V_\pm(Q_k), \bar{\gamma}(t)V_\pm(P); t \in \left[0, \frac{T}{2n}\right]).
\]

We now assume that $Q,S,N \in O(m)$ be such that $S^n = Q$, $N^2 = I_m$, $SN = NS^T$ and we define the Hilbert spaces:

\[
E_R = \{ x \in W^{1,2}(R/(TZ), R^m) \mid x(t) = Qx(t + T) \} \quad \text{and} \quad E = E_R \otimes C.
\]

Given the Lagrangian function $L \in \mathcal{C}^2([0,T] \times R^n, R)$ satisfying the following two properties

\[
L(t, x, v) = L(t - T/n, Sx, Sv) \quad \text{and} \quad L(t, u, v) = L(T/n - t, Nx, Nv),
\]

we associate the Lagrangian action functional $\mathcal{J}_L : E_R \to R$, defined by

\[
\mathcal{J}_L(x) := \int_0^T L(t, x(t), \dot{x}(t)) \, dt.
\]

We observe that the relations given in Equation (1.12) insure that $\mathcal{J}_L$ is $D_n$-equivariant and up to standard regularity arguments $x$ is a critical point of $\mathcal{J}_L$ if and only if it is a classical solution of the following boundary value problem

\[
\begin{cases}
\frac{d}{dt} \frac{\partial L}{\partial t}(t, x, \dot{x}) - \frac{\partial L}{\partial x}(t, x, \dot{x}) = 0, & t \in [0, T] \\
x(0) = Qx(T) \quad \text{and} \quad \dot{x}(0) = Q\dot{x}(T).
\end{cases}
\]

Let $x$ be a $D_n$-equivariant critical point; thus $x \in E^{D_n}$. By the second variation of the Lagrangian action, we get that the index form of $x$ is given by

\[
\mathcal{J}(u, v) = \int_0^T \left\{(P\ddot{u} + Qu), \dot{u}\right\} + \left\{Q^T\ddot{u}, v\right\} + \left\{Ru, v\right\} dt, \quad u, v \in E
\]
where $P(t) := \frac{\partial^2 L}{\partial t^2}(t, x(t), \dot{x}(t))$, $Q(t) := \frac{\partial^2 L}{\partial t \partial \dot{v}^0}(t, x(t), \dot{x}(t))$ and $R(t) := \frac{\partial^2 L}{\partial t^2}(t, x(t), \dot{x}(t))$. We set
\[
\hat{\mathcal{A}} = -\frac{d}{dt} \left( P(t) \frac{d}{dt} + Q(t) \right) + Q(t) \frac{d}{dt} + R(t)
\]
and we observe that, since $x \in E_D$, then $\hat{\mathcal{A}}$ is $D_n$-equivariant. By the $D_n$-equivariance of $\mathcal{J}_L$, we immediately get
\[
SP(t + \frac{T}{n}) = P(t)S, \quad SQ(t + \frac{T}{n}) = Q(t)S, \quad SR(t + \frac{T}{n}) = R(t)S, \quad NP(t) = P(t)N, \quad NQ(t) = Q(t)N, \quad NR(t) = R(t)N.
\]

**Theorem 2.** (A $D_n$-equivariant Bott-type formula for Lagrangian systems) Under the previous notation, for any $h = 0, \ldots, n - 1$, we have
\[
n_-(x) = n_-(\langle \mathcal{J} \rangle_{E^+_h}) + n_-(\langle \mathcal{J} \rangle_{E^-_h}).
\]
Moreover,
\[
n_-(\langle \mathcal{J} \rangle_{E^+_h}) = \sum_{k=0}^{n} n_-(\langle \mathcal{J} \rangle_{F^+_{k,h}}).
\]

**Morse indices of the Figure-eight orbit for the planar 3BP.** As promised in the beginning the first application of the Bott dihedral invariant formulas is the computation of the $G$-equivariant Morse index for the figure-eight orbit in the planar three body problem. For, we decompose the path space of the collisionless configuration space of the planar three body problem with equal masses into $D_n$-invariant closed stable subspaces and we compute the Morse index of the restriction of the second variation of the Lagrangian action onto these subspaces.

We consider three point-masses particles $x_i \in \mathbb{R}^2$ (thus, $(x_1, x_2, x_3) \in (\mathbb{R}^2)^3$) self-interacting with the Newtonian gravitational potential $U$ and having unit masses. Newton equations are given by
\[
\frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i}(x_1, x_2, x_3), \quad i = 1, 2, 3
\]
where
\[
U(x_1, x_2, x_3) := \frac{1}{\sum_{i<j} |x_i - x_j|}
\]
is the potential function. The *configuration space* of the system having centre of mass in 0 is given by
\[
\mathcal{X} := \{ x = (x_1, x_2, x_3) \in (\mathbb{R}^2)^3 | x_1 + x_2 + x_3 = 0 \}.
\]
For each pair of indices $i, j \in \{1, 2, 3\}$, let $\Delta_{i,j} := \{ x \in \mathcal{X} | x_i = x_j \}$ be the collision set of the $i$-th and $j$-th particle and let
\[
\Delta := \bigcup_{i,j=1 \atop i \neq j}^3 \Delta_{i,j}
\]
be the collision set in $\mathcal{X}$ (actually an arrangement of hyperplanes). We also define the collisionless configuration space as $\tilde{\mathcal{X}} := \mathcal{X} \setminus \Delta$. For a fixed period $T$, we consider on the Sobolev space $E := W^{1,2}(\mathbb{R}/(T\mathbb{Z}), \tilde{\mathcal{X}})$ the Lagrangian action functional
\[
\Phi : W^{1,2}(\mathbb{R}/T\mathbb{Z}, \tilde{\mathcal{X}}) \to \mathbb{R} \text{ defined by } \Phi(x) := \int_0^T L(x(t), \dot{x}(t))dt
\]
where $L(x(t), \dot{x}(t)) = \frac{1}{2} |\dot{x}(t)|^2 + U(x(t))$ is the Lagrangian function. It’s well-known that the figure-eight orbit can be seen as the minimizer of the action functional on the $D_{0}$-equivariant loop space (cf. [Che02a, Che02b, CM00] and references therein), where

$$D_{0} = \langle g_{1}, g_{2} \rangle g_{0}^{t} = I_{0}, g_{2}^{2} = I_{0}, g_{1}g_{2} = g_{2}g_{1}^{-1}$$

For any $x = (x_{1}, x_{2}, x_{3}) \in W^{1,2}(\mathbb{R}/(T\mathbb{Z}), (\mathbb{R}^{2})^3)$, we assume that the generators $g_{1}, g_{2}$ of $D_{0}$ act on $W^{1,2}(\mathbb{R}/(T\mathbb{Z}), (\mathbb{R}^{2})^3)$ as follows:

$$(g_{1}x)(t) := -RX(t + \frac{T}{6}) \quad (g_{2}x)(t) := RNx(\frac{T}{6} - t)$$

where $S = \begin{bmatrix}
0 & 0 & I_{2} \\
I_{2} & 0 & 0 \\
0 & I_{2} & 0
\end{bmatrix}$, $N = \begin{bmatrix}
I_{2} & 0 & 0 \\
0 & I_{2} & 0 \\
0 & 0 & I_{2}
\end{bmatrix}$ and $R$ is given by $R := \begin{bmatrix}
R_{2} & 0 & 0 \\
0 & R_{2} & 0 \\
0 & 0 & R_{2}
\end{bmatrix}$ with $R_{2} := \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}$. (For further details, we refer the interested reader to [HS09, Section 5]).

For any $h = 0, \ldots, 5$ the following orthogonal direct sum decomposition of the $G_{h}$-equivariant space $E$ holds

$$E_{h} = E_{h}^{+} \bigoplus E_{h}^{-}$$

where

$$E_{h}^{+} := \bigoplus_{k=1}^{2} F_{k,h}^{+} \oplus F_{0,h}^{+} \oplus F_{3,h}^{+} \quad \text{and} \quad E_{h}^{-} := \bigoplus_{k=1}^{2} F_{k,h}^{-} \oplus F_{0,h}^{-} \oplus F_{3,h}^{-}.$$

Being the figure-eight orbit $x$ a $D_{0}$-equivariant critical point and by invoking the Palais principle of symmetric criticality, it follows that $x$ is a critical point for the restriction of $\Phi$ to the space fixed by the action of $G_{h}$ (that in shorthand notation will be denoted by $\Phi^{h}$), where $G_{h} \subset D_{n}$ is the subgroup generated by $S \mathbb{R}^{h}$, where $\mathcal{R}$ and $S$ are the generators of $D_{n}$.

**Notation 1.6.** In what follows, we denote by $n^{h}(x)$ the Morse index of $x$ as critical point of the restricted map $\Phi^{h}$; thus the Morse index of the second variation of $\Phi^{h}$ on the space $E^{G_{h}}$.

**Theorem 3.** Under the notation above, the Morse index $n^{h}(x)$ of the Figure-eight orbit in the $G_{h}$-equivariant space $E_{h}$ is given by

$$n^{h}(x) = \sum_{k=0}^{3} \left[ n_{-}(F_{k,h}^{+}) + n_{-}(F_{k,h}^{-}) \right],$$

where we denoted by the symbol $n_{-}(F_{k,h}^{\pm})$ the Morse index of the restriction of the second variation of $\Phi$ on $F_{k,h}^{\pm}$ defined in Equation (1.5). Furthermore, we have

$$n_{-}(F_{0,h}^{\pm}) = n_{-}(F_{3,h}^{\pm}) = n_{-}(F_{2,h}^{\pm}) = 0, \quad n_{-}(F_{1,h}^{\pm}) = 1 \quad \text{for } h = 0, \ldots, 5 \quad \text{and}$$

$$n_{-}(E_{2}) = n_{-}(E_{4}) = n_{-}(E_{5}) = 0, \quad n_{-}(E_{1}) = n_{-}(E_{3}) = 1.$$

**Remark 1.7.** Being a minimizer on the $D_{0}$-equivariant loop space of the collision manifold, this in particular implies that $n_{-}(F_{0,h}^{\pm}) = 0$ (In fact, the restriction of $\mathcal{R}$ on $E_{0}$ is the identity as well as the restriction of $S$ on $F_{0,h}^{\pm}$). We point out that the claims $n_{-}(x) = 2, n_{-}^{2}(x) = n_{-}^{3}(x) = 0$, follows directly from [HS09, Remark 5.12]; it is worth noticing that these results were numerically obtained by using Matlab.

A **hyperbolicity criterion for reversible Lagrangian systems.** The last application of the theory developed is related to study the strongly instability and hyperbolicity of dihedral
equivariant critical points. For, let \( L \in \mathcal{C}^2([0, T] \times \mathbb{R}^{2m}, \mathbb{R}) \) be a Lagrangian function satisfying the Legendre convexity condition and, as before, we consider the Lagrangian action functional

\[
\mathcal{J}_L : E_{\mathbb{R}} \to \mathbb{R} \text{ defined by } \mathcal{J}_L(x) = \int_0^T L(t, x(t), \dot{x}(t)) \, dt,
\]

where \( E_{\mathbb{R}} := W^{1,2}([0, T]; \mathbb{R}^m) \). If \( x \) is a critical point, by the second variation of \( \mathcal{J}_L \) we get the associated index form given by

\[
\mathcal{J}_\omega(x, \dot{x}) = \int_0^T \left[ \langle (P \dot{x} + Qx), \dot{x} \rangle + \langle Q^T \dot{x}, \dot{x} \rangle + \langle R \dot{x} + Q^* \dot{x}, \dot{x} \rangle \right] \, dt, \quad x, \dot{x} \in E_{\mathbb{R}}.
\]

For any \( \omega \in U \), we let

\[
E^\omega(\omega) := \{ u \in E \mid u(0) = \omega u(T) \}
\]

where we denoted by \( E \) the complexification of \( E_{\mathbb{R}} \); i.e. \( E := E_{\mathbb{R}} \otimes \mathbb{C} \). The corresponding \( \omega \)-index form is given by

\[
\mathcal{J}_\omega(x, \dot{x}) = \int_0^T \left[ \langle (P \dot{x} + Qx), \dot{x} \rangle + \langle Q^T \dot{x}, \dot{x} \rangle + \langle R \dot{x} + Q^* \dot{x}, \dot{x} \rangle \right] \, dt, \quad x, \dot{x} \in E
\]

where we extended the Euclidean product in \( \mathbb{R}^m \), \( \langle \cdot , \cdot \rangle \) to the standard Hermitian product in \( \mathbb{C}^m \).

In what follows we will denote by \( n_{-}(\omega, x) \) the Morse index of \( \mathcal{J}_\omega \) on \( E^\omega(\omega) \). Let us consider the linear Sturm differential operator

\[
(1.13) \quad A_\xi := -\frac{d}{dt} (P(t) \dot{\xi} + Q(t) \xi) + Q^T(t) \dot{\xi} + R(t) \xi, \quad t \in [0, T].
\]

By using the Legendre transform we reduce the second order system \( A_\xi z = 0 \) to the linear Hamiltonian system

\[
(1.14) \quad \dot{z} = JB(t)z, \quad t \in [0, T]
\]

where

\[
B(t) := \begin{bmatrix}
P^{-1}(t) & -P^{-1}(t)Q(t) \\
-Q^T(t)P^{-1}(t) & Q^T(t)P^{-1}(t)Q(t) - R(t)
\end{bmatrix}.
\]

Let \( \gamma \) be the fundamental solution of the Hamiltonian system given in Equation (1.14) and let \( M_T := \gamma(T) \) be the induced monodromy matrix. Let \( \varepsilon > 0 \) and let us denote by \( \mathcal{M}_\varepsilon(M_T) \) the \( \varepsilon \) neighborhood of \( M_T \) in \( \text{Sp}(2m) \) (with respect to the operator topology induced by \( \mathcal{L}(\mathbb{R}^m) \)).

**Definition 1.8.** A \( T \)-periodic orbit of the Hamiltonian system given in Equation (1.14) is termed strongly stable if there exists \( \varepsilon > 0 \) such that

\[
\text{for every } M \in \mathcal{M}_\varepsilon(M_T) \Rightarrow M \text{ is linearly stable.}
\]

The next result put on evidence the prominent role played by the Neumann boundary condition with respect to the other selfadjoint boundary conditions on the Sturm operator \( A \) for controlling the strong stability and the hyperbolicity.

**Theorem 4.** Let \( S, N \in O(m) \) be such that \( S^n = I_m \) and \( N^2 = (NS)^2 = I_m \) acting on \( E_{\mathbb{R}} \) dihedrally through the action given by

\[
S : E \ni z \rightarrow (Sz)(\cdot) := S z (\cdot + T/n) \in E_{\mathbb{R}} \text{ and }
N : E \ni z \rightarrow (Nz)(\cdot) := N z (T/n - t) \in E_{\mathbb{R}}.
\]

Under the above notation, if \( f \) the restriction of \( A \) given in Equation (1.13) on

\[
E^2(T/(2n)) := \{ u \in W^{2,2}([0, T/(2n)], \mathbb{C}^m) \mid \dot{u}(0) = \dot{u}(T/(2n)) = 0 \}
\]

is positive semi-definite, then \( x \) can not be strongly stable.
Corollary 2. Under the same assumptions of Theorem 4 and if $A$ is positive definite on $E^2(T/(2n))$, then $x$ is hyperbolic.

We close this section by observing that as a direct consequence of Corollary 2, we provide a completely new proof of [Off92, Pag.627, Theorem 2.9]. More precisely, let us consider the second order linear differential operator
\[ \mathcal{L} = -\frac{d^2}{dt^2} + Q(t) \]
where $Q$ is a $T$-periodic path of symmetric matrices. We recall that the operator $\mathcal{L}$ is termed reversible if
\[ Q(-t) = Q(t) \quad t \in [0,T]. \]
(Cf., for further details, [Off92] and references therein).

Corollary 3 (Offin 1992). We assume that the operator $\mathcal{L}$ is reversible, non-degenerate (meaning that there are no symmetric $T$-periodic solutions of the equation $\mathcal{L}u = 0$) and have a vanishing Morse index. Then the associated monodromy matrix $M_T$ is hyperbolic.

2 Equivariant preliminaries and symmetry constraints

This section is devoted to introduce the basic definitions and notation and to describe the results that we will need in the sequel. Our basic references for the material contained in this section are [Ser77, Pal79, Kow, Mac89] and references therein.

2.1 Equivariant preliminaries

Let $G$ be a finite group acting on a (representation) space $X$. The space $X$ is called $G$-equivariant space. We recall that the isotropy group or the fixer of $x$ in $G$ is defined as $G_x := \{ g \in G \mid gx = x \}$. If $H \subset G$ is a subgroup, the space $X^H \subset X$ consists of all points $x \in X$ which are fixed by $H$. Given two $G$-equivariant spaces $X$ and $Y$, an equivariant map $f : X \to Y$ is a map having the property that $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and for all $x \in X$. We observe that an equivariant map $f : X \to Y$ induces by restriction to the $X^H$ fixed by the subgroup $H \subset G$ a map $f^H : X^H \to Y^H$. Let $E$ be a separable complex Hilbert space, with identity $I_E$ and let $G$ be a topological compact group. By a unitary representation $U$ of such a group $G$ we shall mean a mapping $G \ni g \mapsto U_g \in U(E)$ where $U(E)$ denotes the group of unitary operators of $E$, which has the following two properties:

(i) $U_{g_1 g_2} = U_{g_1} U_{g_2}$ for every $g_1, g_2 \in G$;

(ii) $G \ni g \mapsto U_g(x) \in H$ is a continuous function from $G$ to $H$ for every $u \in H$.

Remark 2.1. It is worth noticing that any arbitrary group $G$ equipped with the discrete topology can be turned into a topological group. Moreover in this case the continuity is automatic. Furthermore if $G$ is finite, it is clearly compact.

For the sake of the reader we briefly recall in a very special situation which is sufficient for the theory developed in this paper, the classical Palais principle of symmetric criticality proved by author in [Pal79].

Let $G$ be a finite group acting on $E$ through unitary transformations, $f : E \to \mathbb{R}$ be a $G$-invariant functional of $\mathcal{C}^2$-class and let us denote by $f^G$ the map induced by restriction of $f$ on the (closed) subspace $E^G$. Let us denote by $f^G$ the map induced by the action of $G$.

Lemma 2.2. (Palais Principle of Symmetric Criticality) If $f$ is $G$-invariant then a critical point of $f^G$ in $E^G$ is a critical point of $f$ in $E$. 

Proof. Since \( f \) is \( G \)-invariant then, for every \( g \in G \) we have \( f(U_g x) = f(x) \) for every \( x \in E \), where \( U_g \) denotes the unitary representation of \( g \). Denoting by \( \nabla \) the gradient and by \( d \) the differential, we get

\[
 df(x)[v] = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} f(x + \varepsilon v) = \langle \nabla f(x), v \rangle
\]

\[
 df(U_g x)[v] = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} f(U_g(x + \varepsilon v)) = \langle \nabla f(U_g x), U_g v \rangle = \langle U_g^* \nabla f(U_g x), v \rangle, \quad \forall v \in E
\]

where we denoted by \( U_g^* \) the conjugate transpose (or Hermitian transpose) of \( U_g \). In particular

\[
 U_g^* \nabla f(U_g x) = \nabla f(x) \quad x \in E.
\]

Thus

\begin{equation}
 U_g \nabla f(x) = \nabla f(U_g x) \quad g \in G, \quad x \in E
\end{equation}

and by this it readily follows that if \( x \in E^G \) then \( \nabla f(x) \in E^G \) or which is the same \( \nabla f(x) = \nabla f^G(x) \).

(\( \Rightarrow \)). In order to prove the if part, we assume that \( x \in E^G \) is a critical point of \( f^G \). Thus, we have \( U_g x = x \) and \( \nabla f^G(x) = 0 \). Thus, by Equation (2.1), we have

\[
 0 = \nabla f^G(x) = \nabla f^G(U_g x) = \nabla f(U_g x) = U_g \nabla f(x)
\]

which implies that \( x \) is a critical point of \( E \).

(\( \Leftarrow \)). The proof of the only if part, readily follows by the previous arguments. (For further details, we refer the interested reader to [Pal79]). This conclude the proof.

The next result put on evidence a commutativity property enjoyed by the Hessian of a \( G \)-invariant functional \( f \) at a critical point for \( f^G \).

Let \( x \in E^G \) be a critical point of \( f^G \). If \( D^2 f(x) \) denotes the Hessian matrix of \( f \) at \( x \), the following commutativity property holds

\[
 U_g D^2 f(x) = D^2 f(x) U_g, \quad \forall U_g \in U(E)
\]

Proof. By the very same calculations as above, we have for all \( v, w \in E \)

\[
 d^2 f(x)[v, w] = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \langle \nabla f(x + \varepsilon w), v \rangle = \langle D^2 f(x) w, v \rangle
\]

\[
 d^2 f(U_g x)[U_g v, U_g w] = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \langle \nabla f(U_g(x + \varepsilon w)), U_g v \rangle
\]

\[
 = \langle D^2 f(U_g x) U_g w, U_g v \rangle = \langle (D^2 f(x) U_g) U_g w, (U_g^* D^2 f(x) U_g) v \rangle.
\]

Since, by the chain rule \( \forall v, w \in E, \; d^2 f(x)[v, w] = d^2 f(U_g x)[U_g v, U_g w] \) by the previous computation, we immediately get \( D^2 f(x) = U_g^* D^2 f(x) U_g \) and this conclude the proof.

We are now ready to prove an abstract \( D_n \)-equivariant spectral flow formula for \( D_n \)-equivariant paths of linear and bounded selfadjoint Fredholm operators on the (complex) Hilbert space \( E \).

With a slight abuse of notation, we denote with the same symbol the dihedral group as well as its unitary representation in \( E \). Thus, the group \( D_n \) (actually a unitary representation of the dihedral group \( D_n \)) is presented as follows

\[
 D_n := \langle \mathcal{R}, S \in U(E) | \mathcal{R}^n = S^2 = (8\mathcal{R})^2 = I_E \rangle \subset U(E).
\]

Let \( \mathcal{A} : [0, 1] \to \mathcal{F}^w(E) \) be a continuous path of closed selfadjoint Fredholm operators commuting with \( \mathcal{R} \) and \( \mathcal{S} \); in symbols, we have

\[
 \mathcal{A}(\lambda) \mathcal{R} = \mathcal{R} \mathcal{A}(\lambda) \quad \text{and} \quad \mathcal{A}(\lambda) \mathcal{S} = \mathcal{S} \mathcal{A}(\lambda) \quad \text{for all} \; \lambda \in [0, 1].
\]

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As already showed in Section 1, we can decompose the Hilbert space $E$ into mutually orthogonal $D_{\nu}$-stable modules, given by

\[(2.2) \quad E = F_0 \oplus \cdots \oplus F_n\]

where $F_k$ were defined in Equation (1.3). Now, for each $k = 0, \ldots, n$, let $A_k$ be the continuous path defined by $A_k := A|_{F_k} : [0, 1] \to \mathcal{F}^n(F_k)$. Thus, we have $A(\lambda) = A_0(\lambda) \oplus \cdots \oplus A_n(\lambda)$, for $\lambda \in [0, 1]$.

Moreover it is immediate to check that

[Proof.]

We start by observing that for each $k = 0, \ldots, n$, let $A_k$ be the continuous path defined by $A_k := A|_{F_k} : [0, 1] \to \mathcal{F}^n(F_k)$. Thus, we have $A(\lambda) = A_0(\lambda) \oplus \cdots \oplus A_n(\lambda)$, for $\lambda \in [0, 1]$.

For $k = 0, \ldots, n$ and $h = 0, \ldots, n - 1$, we define the closed subspaces

$$F_{k,h}^\pm = \{ u \in F_k \mid S^{h} u = \pm u \}.$$  

Since $R|_{E_k} = I$, we get that $F_{0,0}^\pm = F_{0,0}^\pm = \{ u \in F_0 \mid S u = \pm u \}$. In the case $n$ is even, $R|_{E_n/2} = -I$, so we have $F_{n/2,0}^\pm = \{ u \in F_{n/2} \mid (-1)^h S u = \pm u \}$. It is worth noticing that, for $k = 1, \ldots, [(n - 1)/2]$, we get $F_{k,h}^\pm = \{ z \in F_k \mid z \in E_k \}$. We also observe that for any $h = 0, \ldots, n - 1$ we have

$$(S^h )K(\lambda) = K(\lambda), \quad \text{where } K(\lambda) := \ker A(\lambda), \quad \lambda \in [0, 1].$$

By the direct sum property of the spectral flow, we have

$$\sf(A|_{F_k} : [0, 1]) = \sf(A|_{E_k} : [0, 1]) + \sf(A|_{E_{-k}} : [0, 1])$$

$$= \sf(A|_{F_{k,0}^\pm} : [0, 1]) + \sf(A|_{F_{0,h}^\pm} : [0, 1]).$$

**Proposition 2.4.** For every $k = 1, \ldots, [(n - 1)/2]$ and $h = 0, \ldots, n - 1$, we have

$$\sf(A|_{F_k} : [0, 1]) = 2 \sf(A|_{E_k} : [0, 1]) = 2 \sf(A|_{E_{-k}} : [0, 1])$$

$$= 2 \sf(A|_{F_{k,0}^\pm} : [0, 1]) = 2 \sf(A|_{F_{0,h}^\pm} : [0, 1]).$$

In particular

$$\sf(A|_{F_k} : [0, 1]) \equiv 0 \pmod{2} \quad \text{for every } k = 1, \ldots, [(n - 1)/2].$$

**Proof.** We start by observing that for each $\lambda \in [0, 1]$, it holds

$$A_{\lambda}|_{F_k} = \begin{pmatrix} A_{\lambda}|_{E_k} & 0 \\ 0 & A_{\lambda}|_{E_{-k}} \end{pmatrix}. $$

Moreover it is immediate to check that

$$z \in E_k, \quad A_{\lambda} z = 0 \iff z \in E_{-k}, \quad A_{\lambda} z = 0$$

or otherwise stated $\ker A_{\lambda}|_{E_k} = \ker A_{\lambda}|_{E_{-k}}$. In particular, they have the same dimension. Moreover, if $\lambda_0 \in [0, 1]$ is a crossing instant, then, for each $k = 0, \ldots, [(n - 1)/2]$, it is immediate to verify that the crossing operators across $\lambda_0$ both coincide; thus, we have

$$\Gamma(A_{\lambda}|_{E_k}, \lambda_0) = \Gamma(A_{\lambda}|_{E_{-k}}, \lambda_0).$$

By taking into account the perturbation result (cfr. [Wat15, Theorem 2.6]), we don’t lead in generalities in assuming that the path $A$ is regular. By taking the sum over all crossing points of the signature of the crossing forms (since we are assuming that the path $A$ is regular), we immediately get

$$\sf(A|_{E_k} : [0, 1]) = \sf(A|_{E_{-k}} : [0, 1]).$$

In order to conclude the proof, we start to observe that if $z \in \ker A$ then $S^h z \in \ker A$. So

$$\tilde{A}_{\lambda} := \begin{pmatrix} A_{\lambda} & 0 \\ 0 & A_{\lambda} \end{pmatrix} \begin{pmatrix} z \\ 8 S^h z \end{pmatrix} = 0 \iff \begin{pmatrix} A_{\lambda} & 0 \\ 0 & A_{\lambda} \end{pmatrix} \begin{pmatrix} z \\ -8 S^h z \end{pmatrix} = 0.$$
meaning that \( \ker (A\lambda|_{F_{k,h}^+}) = \ker (A\lambda|_{F_{k,h}^-}) \) for all \( \lambda \in [0,1] \). We denote by \( \tilde{\Gamma} \left( A\lambda|_{F_{k,h}^+}, \lambda_0 \right) \) the crossing operator onto \( \ker (A\lambda_0|_{F_{k,h}^+}) \), namely

\[
\tilde{\Gamma} \left( A\lambda|_{F_{k,h}^+}, \lambda_0 \right) : \ker (A\lambda|_{F_{k,h}^+}) \longrightarrow \mathbb{R}
\]
given by

\[
\tilde{\Gamma} \left( A\lambda|_{F_{k,h}^+}, \lambda_0 \right) \left[ \frac{z}{z^*} \right] = \left\langle \hat{A}_{\lambda_0} z, z \right\rangle_E + \left\langle \left( S \mathbb{R}^k \right)^* \hat{A}_{\lambda_0} \left( S \mathbb{R}^k \right) z, z \right\rangle_E.
\]

Summing over all (regular) crossing instants, we get

\[
\text{sf} \left( A|_{F_{k,h}^+} : [0,1] \right) = \text{sf} \left( A|_{F_{k,h}^-} : [0,1] \right).
\]

The second claim is a direct consequence of the previous computations. This conclude the proof.

\[\Box\]

**Proof of Proposition 2.** By invoking Proposition 2.4, we already know that for every \( k = 1, \ldots, \lfloor (n-1)/2 \rfloor \), \( \text{sf}(A|_{F_k} : [0,1]) \) is even. Now the result easily follows by taking into account the decomposition given in Equation (2.2). This conclude the proof.

\[\Box\]

### 3 A dihedral-equivariant decomposition of the path space

The aim of this paragraph is to construct an equivariant decomposition of the loop space of the configuration manifold and to establish a dihedral equivariant Bott-type iteration formula in terms of the Maslov index of suitable induced continuous paths of Lagrangian subspaces.

Let \( T \subset \mathbb{R}^2 \) be a circle in \( \mathbb{R}^2 \) of length \( T = |T| \) which can be identified with \( \mathbb{R}/(T \mathbb{Z}) \), where \( T \mathbb{Z} \) denotes the lattice generated by \( T \in \mathbb{R} \). Given \( Q \in U(2m) \) let \( E \) be denote the Sobolev completion of the set of smooth maps \( z : T \rightarrow \mathbb{C}^{2m} \) such that \( z(t) = Q z(t + T) \); thus

\[
E := \{ z \in W^{1,2}(T; \mathbb{C}^{2m}) \mid z(t) = Q z(t + T) \}.
\]

Let \( M \in U(2m) \) and \( N \in U(2m) \) be such that the following commutativity properties holds

\[
M^n = Q, \quad N^2 = I, \quad NM^* = MN.
\]

For any \( k = 0, \ldots, n-1 \), we denote by \( E_k \) the closed subspace of \( E \) given by

\[
E_k = \left\{ z \in E \left| M z \left( t + \frac{T}{n} \right) = \zeta^k_n z(t) \right. \right\}.
\]

and for \( k = 1, \cdots, \lfloor (n-1)/2 \rfloor \), we define \( \tilde{M}_k, \tilde{N}_k \) the following block diagonal matrices

\[
\tilde{M}_k = \begin{bmatrix} c_{-k} M & 0 \\ 0 & c_k M \end{bmatrix} \quad \text{and} \quad \tilde{N}_k = \begin{bmatrix} 0 & \zeta_{-k}^n N \\ \zeta_k^n N & 0 \end{bmatrix}.
\]

As above, we set \( F_0 = E_0, F_{n/2} = E_{n/2} \) if \( n \) is even, \( F_k := E_k \oplus E_{-k} \) for \( k = 1, \cdots, \lfloor (n-1)/2 \rfloor \) and \( F_{n/2} = 0 \) if \( n \) is odd. Thus, we have

\[
F_k = \left\{ u \in W^{1,2} \left( \frac{T}{n}; \mathbb{C}^{2m} \oplus \mathbb{C}^{2m} \right) \left| u := \begin{bmatrix} z \\ w \end{bmatrix} \right. \right\}, \quad \text{where} \quad z \in E_k, \quad w \in E_{-k} \text{ and } u(0) = \tilde{M}_k u \left( \frac{T}{n} \right).
\]

We now define the two unitary operators on \( E \) as follows

\[
M : E \ni z \mapsto (M z)(\cdot) := M z (\cdot + T/n) \in E \quad \text{and} \quad N : E \ni z \mapsto (N z)(\cdot) := N z (T/n - t) \in E.
\]
Lemma 3.1. For each of the relations on the generators of the dihedral group \(D_n\), we also get that, for every \(h = 0, \ldots, n-1\), \((NM^h)^2 = I\). In this way we decompose the Hilbert space \(E\) defined in Equation (3.1) into a direct sum of \(D_n\)-closed stable subspaces; namely \(E = F_0 \oplus \cdots \oplus F_n\), where each \(F_k\) is defined in Formula (3.3). By restriction, we get the unitary operators

\[
\mathcal{N}_k := \mathcal{M}|_{E_k} \in \mathbf{U}(E_k), \quad \mathcal{N}_k := \mathcal{M}|_{E_k} \in \mathbf{U}(E_k), \quad \hat{N}_k := \begin{bmatrix} M_k & 0 \\ 0 & M_k \end{bmatrix} \in \mathbf{U}(F_k)
\]

and finally \(\hat{N}_k := \begin{bmatrix} 0 & N_k \\ N_k & 0 \end{bmatrix} \in \mathbf{U}(F_k)\).

Lemma 3.2. For each \(h = 0, \ldots, n-1\), the spectrum of \(\mathcal{N}M^h\) is \([-1,1]\).

Proof. This fact readily follows by observing that for every \(h = 0, \ldots, n-1\), \((NM^h)^2 = I\).

As direct consequence of Lemma 3.1, the spectrum of

\[
\hat{N} \hat{M}^h := \begin{bmatrix} 0 & NM^h \\ NM^h & 0 \end{bmatrix}
\]

as well as \(\hat{N}_k \hat{M}^h_k\) is \([-1,1]\) for every \(h = 0, \ldots, n-1\) and \(k = 0, \ldots, \bar{n}\). Furthermore, for each \(u \in F_k\) and \(k = 1, \ldots, [(n-1)/2]\), we have

\[
(\hat{N}_k \hat{M}^h_k u)(t) = \begin{bmatrix} 0 & N_k M^h_k \\ N_k M^h_k & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}(t) = \begin{bmatrix} N_k M^h_k w \\ N_k M^h_k z \end{bmatrix}(t) = \begin{bmatrix} \zeta_n^{-kh} N_k w \\ \zeta_n^{kh} N_k z \end{bmatrix}(t)
\]

(3.5)

\[
= \hat{N}_k \hat{M}^h \begin{bmatrix} z(T/n-t) \\ w(T/n-t) \end{bmatrix}, \quad t \in [0,T].
\]

We denote by \(V_+(\hat{N}_k \hat{M}^h_k), V_-(\hat{N}_k \hat{M}^h_k)\) be the positive and negative spectral spaces of \(\hat{N}_k \hat{M}^h_k\) and we set \(F_{k,h} \pm := V_\pm(\hat{N}_k \hat{M}^h_k)\).

Lemma 3.2. For \(h = 0, \ldots, n-1\) and \(k = 1, \ldots, [(n-1)/2]\), we let \(u \in V_\pm(\hat{N}_k \hat{M}^h_k)\). Thus, we have

- \(\bar{u}(0) \in V_\pm(\hat{N}_k \hat{M}^h_k) = \left\{ \begin{bmatrix} x \bar{M}x \\ x \bar{N}x \end{bmatrix} \mid x \in \mathbf{C}^{2m} \right\}\)

Furthermore if

- if \(u \in F_{0,h}^\pm\), then we have
  \(\bar{u}(0) \in V_\pm(MN)\) and \(u(T/(2n)) \in V_\pm(N)\).

- if \(u \in F_{n/2,h}^\pm\), then we have
  \(\bar{u}(0) \in V_\pm((-1)^{(h+1)}MN)\) and \(u(T/(2n)) \in V_\pm((-1)^hN)\)
Proof. By Lemma 3.1 we know that, for each $h = 0, \ldots, n - 1$, the spectrum of $\hat{N}_k \hat{M}_k^h$ is \{+1, -1\}. Let $u \in V_+(\hat{N}_k \hat{M}_k^h)$; namely $\hat{N}_k \hat{M}_k^h u = u$. By Formula (3.5) it follows that pointwise it holds that

$$
\begin{bmatrix}
    z(t) \\
    w(t)
\end{bmatrix} =
\begin{bmatrix}
    0 & \zeta_n^{-k} N \\
    \zeta_n^{k} N & 0
\end{bmatrix}
\begin{bmatrix}
    z(T/n - t) \\
    w(T/n - t)
\end{bmatrix}.
$$

In particular

$$
\begin{bmatrix}
    z(0) \\
    w(0)
\end{bmatrix} =
\begin{bmatrix}
    0 & \zeta_n^{-k} N \\
    \zeta_n^{k} N & 0
\end{bmatrix}
\begin{bmatrix}
    z(T/n) \\
    w(T/n)
\end{bmatrix} =
\begin{bmatrix}
    0 & \zeta_n^{-k} N \\
    \zeta_n^{k} N & 0
\end{bmatrix}
\begin{bmatrix}
    \zeta_n^{k} M^* z(0) \\
    \zeta_n^{-k} M^* w(0)
\end{bmatrix} =
\begin{bmatrix}
    0 & \zeta_n^{-k} N \\
    \zeta_n^{k} N & 0
\end{bmatrix}
\begin{bmatrix}
    z(0) \\
    w(0)
\end{bmatrix}.
$$

Thus $\begin{bmatrix}
    z(0) \\
    w(0)
\end{bmatrix} \in \text{Fix}
\begin{bmatrix}
    0 & \zeta_n^{k(h+1)} M^* N \\
    \zeta_n^{-k} N & 0
\end{bmatrix} = \left\{ \left( \zeta_n^{k(h+1)} M N, x \right), x \in \mathbb{C}^{2m} \right\}.$

For $t = T/(2n)$, we have

$$
\begin{bmatrix}
    z(T/(2n)) \\
    w(T/(2n))
\end{bmatrix} =
\begin{bmatrix}
    0 & \zeta_n^{-k} N \\
    \zeta_n^{k} N & 0
\end{bmatrix}
\begin{bmatrix}
    z(T/(2n)) \\
    w(T/(2n))
\end{bmatrix}.
$$

So

$$
\begin{bmatrix}
    z(T/(2n)) \\
    w(T/(2n))
\end{bmatrix} \in \text{Fix}
\begin{bmatrix}
    0 & \zeta_n^{-k} N \\
    \zeta_n^{k} N & 0
\end{bmatrix} = \left\{ \left( \zeta_n^{k(h+1)} N, x \right), x \in \mathbb{C}^{2m} \right\}.
$$

The proof in the remaining case is completely analogous and the details are left to the reader.

In the case $k = 0$, we start to observe that, for $z \in F^\pm_0 = F^\pm_{0,h}$, we have

$$
\begin{align*}
    \begin{cases}
    M z = z \\
    N M^h z = \pm z
    \end{cases} \iff \begin{cases}
    M z \left( t + \frac{T}{n} \right) = z(t) \\
    N z(T/n - t) = \pm z(t)
    \end{cases} \quad \forall t \in [0, T].
\end{align*}
$$

Thus

$$
z(0) = M z(T/n), N z(T/n) = \pm z(0) \Rightarrow M N z(0) = \pm z(0), \text{ i.e. } z(0) \in V_\pm(MN)
$$

and

$$
\pm z \left( \frac{T}{2n} \right) = N z \left( \frac{T}{2n} \right),
$$

and by this it follows that $z \left( \frac{T}{2n} \right) \in V_\pm(N)$. If $n$ is even, $k = n/2$ and if $z \in F^\pm_{n/2,h}$, then we have

$$
\begin{align*}
    \begin{cases}
    M z = -z \\
    N M^h z = \pm z
    \end{cases} \iff \begin{cases}
    M z \left( t + \frac{T}{n} \right) = -z(t) \\
    (-1)^h N z(T/n - t) = \pm z(t)
    \end{cases} \quad \forall t \in \mathbb{T}.
\end{align*}
$$

Thus $M z(T/n) = -z(0), (-1)^h N z(T/n) = \pm z(0) \Rightarrow M N z(0) = (-1)^{h+1} z(0)$ and hence $z(0) \in V_\pm((-1)^{h+1} MN)$ as well as $\pm z \left( \frac{T}{2n} \right) = (-1)^h N z \left( \frac{T}{2n} \right)$. By this it readily follows that $z \left( \frac{T}{2n} \right) \in V_\pm((-1)^h N)$. This conclude the proof. \qed
The quaternionic unitary group Let us consider the complex symplectic space \((\mathbb{C}^{2m}, \omega)\) where \(\omega\) is the standard symplectic space defined by \(\omega(x, y) = (Jx, y)\). The compact symplectic group \(\text{Sp}(m)\) is isomorphic to the group of unitary and symplectic matrices; i.e.

\[
\text{Sp}(m) \cong \text{Sp}(2m, \mathbb{C}) \cap \text{U}(2m).
\]

Remark 3.3. The group \(\text{Sp}(m)\) is the subgroup of the invertible quaternionic matrices \(\text{GL}(m, \mathbb{H})\) that preserves the Hermitian form on \(\mathbb{H}^m\)

\[
\langle x, y \rangle := \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n.
\]

We observe that \(\text{Sp}(m)\) is just the quaternionic unitary group \(\text{U}(m, \mathbb{H})\) and for this reason is sometimes also termed hyperunitary.

We assume that \(Q, M \in \text{Sp}(m)\) and \(N \in \text{U}(2m)\) be such that the following commutativity properties holds:

\[
M^n = Q, \quad N^2 = I, \quad M J = J M, \quad N J = -J N, \quad N = N^*, \quad N M^* = M N.
\]

Lemma 3.4. For any \(k = 0, \ldots, n\), the subspaces \(V_\pm(M^k N)\) are Lagrangian subspaces of \((\mathbb{C}^{2m}, \omega)\).

Proof. We only prove that \(V_+(N), V_-(N) \in \Lambda(\mathbb{C}^{2m}, \omega)\) and we leave to the interested reader the proof that \(V_\pm(M^k N) \in \Lambda(\mathbb{C}^{2m}, \omega)\), being completely similar.

Let \(N \in \text{U}(2m)\), \(N^2 = I, \ N J = -J N\) and \(N = N^*\); so \(N^* N J = -N^* N J = -J\). For all \(x, y \in V_+(N), \ N x = x\) and \(N y = y\). Then

\[
\langle J x, y \rangle = \langle J N x, N y \rangle = \langle N^* J N x, y \rangle = -\langle J x, y \rangle
\]

so \(\langle J x, y \rangle = 0\). This computation immediately shows that \(V_+(N)\) is an isotropic subspace. By the very same arguments it is possible to conclude also that \(V_-(N)\) is an isotropic subspace. Since \(\mathbb{C}^{2m} = V_+(N) \oplus V_-(N)\), so \(V_\pm(N)\) are maximal isotropic subspaces and hence Lagrangian subspaces. This conclude the proof. \(\square\)

We let \(\hat{J} = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}\) and we observe that the pair \((\mathbb{C}^{4m}, \hat{\omega})\) with \(\hat{\omega}(x, y) = \langle \hat{J} x, y \rangle\) is a (complex) symplectic space. With a slight abuse of notation, we’ll denote by the same symbol \(\hat{J}\) the operator on \(F_k\) induced by \(J\).

Lemma 3.5. The following relations hold

\[
\hat{M}_k \hat{J} = \hat{J} \hat{M}_k, \quad (\hat{N}_k^h)^2 = I, \quad \hat{N}_k^h \hat{J} = -\hat{J} \hat{N}_k^h, \quad \hat{N}_k^h = (\hat{N}_k^h)^*, \quad \hat{N}_k^h (\hat{M}_k^h)^* = \hat{M}_k \hat{N}_k^h.
\]

Proof. The proof immediately follows by a straightforward calculation. \(\square\)

Arguing precisely as in Lemma 3.4 and as a direct consequence of the relations given in Equation (3.2), it readily follows that \(V_\pm(\hat{M}_k^h \hat{N}_k), V_\pm(\hat{M}_k^h) \in \Lambda(\mathbb{C}^{4m}, \hat{\omega})\), for every \(h = 0, \ldots, n - 1\) and \(k = 1, \cdots, [(n - 1)/2]\).

4 A dihedral equivariant Bott-type iteration Formulas

The scope of this section, is to prove a Bott-type iteration formula for

- \(D_h\)-equivariant solutions of a Hamiltonian System under Lagrangian boundary conditions;
- \(D_h\)-equivariant solutions of a Lagrangian System under selfadjoint boundary conditions.

The general formula in the case of Hamiltonian systems will be derived in Subsection 4.1 whilst the case of Lagrangian systems will be given in Subsection 4.2.
4.1 A dihedral-equivariant Bott-type formula for Hamiltonian systems

Let $H \in \mathcal{C}^2 \left([0, T] \times \mathbb{R}^{2m}, \mathbb{R}\right)$ be a time-dependent Hamiltonian function and let $L$ be a Lagrangian subspace of the symplectic space $(\mathbb{R}^{2m} \oplus \mathbb{R}^{2m}, -\omega \oplus \omega)$. We define the closed (in $L^2$) subspace $\mathcal{D}(T, L) := \{ z \in W^{1,2}([0, T], \mathbb{R}^{2m}) \mid (z(0), z(T)) \in L \}$. We denote by $\overline{\mathcal{D}(T, L)}$ the closure in the $W^{1/2,2}$-norm topology of $\mathcal{D}(T, L)$ and we consider the symplectic action functional

$$\mathcal{A}_H : \overline{\mathcal{D}(T, L)} \to \mathbb{R}$$

defined by $\mathcal{A}_H(z) := \int_0^T \left[ -J \frac{dz(t)}{dt}, z(t) \right] - H(t, z(t)) \right] \, dt.$

By standard regularity arguments, it follows that a critical point of $\mathcal{A}_H$ is weak (in the Sobolev sense)-solution of the boundary value problem

$$\begin{cases}
\dot{z}(t) = J\nabla H (t, z(t)), & t \in [0, T] \\
(z(0), z(T)) \in L.
\end{cases}$$

(4.1)

Remark 4.1. We observe that the periodic solutions can be obtained by setting $L = \Delta$ where $\Delta$ denotes the diagonal subspace in the product space $\mathbb{R}^{2m} \oplus \mathbb{R}^{2m}$.

Let $z$ be a solution of the Hamiltonian System given in Equation (4.1) and let us denote by $\gamma$ the fundamental solution of its linearisation along $z$; namely $\gamma$ solves

$$\begin{cases}
\dot{\gamma}(t) = JD^2H (t, z(t)) \gamma(t), & t \in [0, T] \\
\gamma(0) = I_{2m}.
\end{cases}$$

We set $B(t) := D^2H (t, z(t))$ and let us define the closed self-adjoint Fredholm operators in $L^2$ having domain $\mathcal{D}(T, L) := \{ z \in W^{1,2}([0, T], \mathbb{R}^{2m}) \mid (z(0), z(T)) \in L \}$ and given by

$$A_1 := -J \frac{d}{dt} - B(t) \quad \text{and} \quad A_0 := -J \frac{d}{dt}.$$

(4.2)

Following authors in [HS09, Definition 2.1], we define the relative Morse index of $z$ as follows

$$\iota_{\text{spec}}(z) := I(A_0, A_1) = -\text{sf}(A_1; [0, 1])$$

where $A : [0, 1] \to \mathcal{C}^2 \mathfrak{F}^{2m}(E)$ is a continuous path of closed self-adjoint Fredholm operators defined by $A_s := A_0 + B_s$ where $s \mapsto B(s)$ is such that $B_0 = 0$ and $B_1 := B$ on the $s$-independent domain $\mathcal{D}(T, L)$. We define the geometrical index of the solution $z$ of the Hamiltonian system given in Equation (4.1) as

$$\iota_{\text{geo}}(z) := \mu^{CM}(L, \text{Gr}(\gamma); [0, T]).$$

We observe that $z_s \in \ker(A(s)|_{\mathcal{D}(T, L)})$ if and only if $z_s$ is a solution of the linear Hamiltonian boundary value problem

$$\begin{cases}
\dot{z}_s = JB_s(t) z_s(t), & t \in [0, T] \\
(z_s(0), z_s(T)) \in L \cap \text{Gr}(\gamma_s(T))
\end{cases}$$

(4.3)

where $\gamma_s$ is the fundamental solution of the Hamiltonian system given in Equation (4.3).

Proposition 4.2. (A spectral flow formula) Under the previous notations, we have

$$\iota_{\text{geo}}(z) = \iota_{\text{spec}}(z).$$

Proof. For the proof of this result we refer the interested reader to [HS09, Theorem 2.5].
Remark 4.3. It is worth noticing that if \( L = L_1 \oplus L_2 \in \Lambda(\mathbb{R}^{2m} \oplus \mathbb{R}^{2m}, -\omega \oplus \omega) \), where \( L_i \in \Lambda(\mathbb{R}^{2m}, \omega) \), for \( i = 1, 2 \), then we have \( \mu_{\text{CLM}}(L_1 \oplus L_2, \text{Gr}(\gamma); [0, T]) = \mu_{\text{CLM}}(L_2, \ell_1; [0, T]) \) where \( \ell_1(\cdot) := \gamma(\cdot) L_1 \).

Let us consider the operator \( N M^{n-1} \) on \( E \) which is given by \( (N M^{n-1})z(t) := N M^{n-1}z(T-t) \). By taking into account Lemma 3.1, we can decompose \( E \) into the following orthogonal direct sum

\[
E = V_+(N M^{n-1}) \oplus V_-(N M^{n-1})
\]

By a direct calculation it follows that if \( z \in V_+(N M^{n-1}) \), then \( z(t) = N M^{n-1}z(T-t) \) for all \( t \in [0, T] \) and hence \( z(0) = N M^{n-1}z(T) \). By multiplying the last equation on the left by \( M N \), we get \( M N z(0) = M^{m} z(T) = Q z(T) = z(0) \), namely \( z(0) \in V_+(M N) \). Analogously, \( z(T/2) = N M^{n-1}z(T/2) \), or which is equivalent \( z(T/2) \in V_+(N M^{n-1}) \). In conclusion \( V_\pm \) are the closed \( D_\nu \)-invariant subspaces defined as

\[
V_\pm(N M^{n-1}) := \left\{ z \in E \mid N M^{n-1}z = \pm z, \ (z(0), z(T/2)) \in V_\pm(M N) \times V_\pm(N M^{n-1}) \right\}.
\]

For \( k = 1, \ldots, [(n-1)/2] \), we denote by \( F_k^{\pm} \) the closed \( D_\nu \)-invariant subspaces defined by

\[
F_k^{\pm}(N M^{n-1}) := \left\{ u \in F_k \left| N M^{n-1} u = \pm u, \ u(0) \in V_{\pm}(M N) \right\}
\]

and we set

\[
E_0^{\pm}(N M^{n-1}) := \{ z \in E_0 \mid N M^{n-1} u = \pm u, \ (u(0), u(T/(2n))) \in V_{\pm}(M N) \times V_{\pm}(N) \}.
\]

By this, the following orthogonal direct sum decomposition holds

\[
E^{\pm}(N M^{n-1}) = \bigoplus_{k=0}^{n} F_k^{\pm}(N M^{n-1})
\]

where \( F_0^{\pm} = E_0 \) and, for \( n \) even \( F_{n/2}^{\pm} = E_{n/2}^{\pm} \) where

\[
E_{n/2}^{\pm} := \{ z \in E_{n/2} \mid N M^{n-1} u = \pm u, \ (u(0), u(T/(2n))) \in V_{\pm}(M N) \times V_{\pm}(N) \}.
\]

Proof of Theorem 1. We start define the analytic path \( \lambda \mapsto A(\lambda) := A_0 - \lambda B \) where \( A_0 \) and \( B \) were defined in Equation (4.2). Since both these operators commutes with \( M \) and \( N \), then the whole path \( A(\lambda) \) too. In order to conclude the proof it is enough to invoke Proposition 3 and Proposition 4.2 and the symplectic additivity properties of the geometrical indices. This conclude the proof.

4.2 A dihedral-equivariant Bott-type formula for Lagrangian systems

Let \( L \in \mathcal{C}^2([0, T] \times \mathbb{R}^{2m}, \mathbb{R}) \) be a Lagrangian function and let \( \mathcal{L} : W^{1,2}([0, T]; \mathbb{R}^m) \to \mathbb{R} \) be the Lagrangian action functional defined as

\[
\mathcal{L}(x) := \int_0^T L(t, x(t), \dot{x}(t)) \ dt.
\]

We assume that the function \( L \) satisfying the Legendre convexity condition:

\[
\left\langle D_{wv}, L(t, q, v) \right\rangle w, w > 0 \text{ for } t \in [0, T], \ w \in \mathbb{R}^m \setminus \{0\}, \ (q, v) \in \mathbb{R}^m \times \mathbb{R}^m
\]

20
and let \( V \subset \mathbb{R}^m \oplus \mathbb{R}^m \) be a fixed subspace. A weak (in a Sobolev sense) solution of the Euler-Lagrange Equation with the boundary condition \((\gamma(0), \gamma(T)) \in V\) is a critical point of \(\mathcal{J}_L\) in the space

\[
E_V := \{ x \in W^{1,2}([0,T], \mathbb{R}^m) \mid (x(0), x(T)) \in V \}.
\]

More precisely, \( x \) is a solution of the following second order system

\[
\begin{aligned}
\frac{d}{dt} \partial_v L(t,x(t),\dot{x}(t)) - \partial_q L(t,x(t),\dot{x}(t)) &= 0, \quad t \in [0,T] \\
(x(0), x(T)) \in V, \quad (D_v L(0,x(0),\dot{x}(0)), -D_v L(T,x(T),\dot{x}(T))) \in V^\perp,
\end{aligned}
\]

where \( V^\perp \) is the orthogonal complement of \( V \) in \( \mathbb{R}^m \oplus \mathbb{R}^m \). By using the Legendre transformation \( p = D_v L(t,q,v) \) and setting \( H(t,p,q) = \langle p,v \rangle - L(t,q,v) \) the Euler-Lagrange equation given in Equation (4.4) can be converted into the following Hamiltonian System

\[
\dot{z}(t) = J \nabla H(t,z(t))
\]

with \( z(t) := (y(t), x(t)) = (D_v L(t,x(t),y(t)), x(t)) \). (Cf., for instance to [APS08] for further details). We let \( \bar{J} = -J + J \) and we observe that the subspace \( L_V := JV^\perp \oplus V \subset \mathbb{R}^{2m} \oplus \mathbb{R}^{2m} \) is a Lagrangian subspace of \( (\mathbb{R}^{2m} \oplus \mathbb{R}^{2m}, -\omega \oplus \omega) \). Thus \( x \in \mathcal{C}^2([0,T] \times \mathbb{R}^m, \mathbb{R}) \) solves the boundary value problem given in Equation (4.4) if and only if \( z \in \mathcal{C}^2([0,T] \times \mathbb{R}^{2m}, \mathbb{R}) \) is a solution of the Equation (4.5) under the following Lagrangian boundary condition

\[
(z(0), z(T)) \in L_V.
\]

**Proposition 4.4. (Morse-type Index Theorem)** Let \( x \) be a critical point of \( \mathcal{J}_L \) and we assume that the Legendre convexity condition holds. Then the Morse index of \( x \) is finite and we have

\[
n_-(x) + i(L) = \text{spec}_v(z) = \text{spec}_e(z)
\]

where \( i(L) = I \begin{pmatrix} -J & C \end{pmatrix} \) and \( C \) is the operator pointwise induced by the matrix

\[
C = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}
\]

on the domain \( \mathcal{D}(T,L) \).

**Proof.** For the proof of this result we refer the interested reader to [HS09, Theorem 3.4]. \( \square \)

**Remark 4.5.** We observe that for Dirichlet boundary condition, the corresponding Lagrangian subspace is the horizontal Lagrangian subspace \( L_D := \mathbb{R}^m \oplus \{0\} \) of the phase space and in this case it is easy to compute that \( i(L_D) = m \). We observe that in the case of periodic boundary condition the corresponding Lagrangian subspace is the diagonal \( \Delta \) and in this case \( i(\Delta) = m \). By a direct calculation it is possible to check that in the case of Neumann boundary condition the corresponding Lagrangian subspace \( L_N \) in the phase space is the horizontal and \( i(L_N) = 0 \). For further details we refer the interested reader to [HS09, Remark 3.6].

**Proof of Theorem 2.** By a direct linearisation of the Euler-Lagrange equation along the solution \( x \), we get

\[
-\frac{d}{dt} (P(t)\dot{y} + Q(t)y) + Q^T(t)\dot{y} + R(t)y = 0 \quad t \in [0,T],
\]

We observe that \( y \) is solution of Equation (4.6) if and only if \( y \in \ker(J) \); moreover the associated linear Hamiltonian system is given by

\[
\dot{z}(t) = JB(t)z(t), \quad t \in [0,T]
\]
we investigate the hyperbolicity and the strongly instability of symmetric Lagrangian systems.

Let $S_d := \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$ and $N_d := \begin{bmatrix} -N & 0 \\ 0 & N \end{bmatrix}$. Then we have

$$B(t)S_d = S_dB(t + \frac{T}{n}) \quad \text{and} \quad B(t)N_d = N_dB(t + \frac{T}{n} - t).$$

Let us define the unitary operators pointwise given by

$$\tilde{g}_1 x := S_d x(t + T/n) \quad \text{and} \quad \tilde{g}_2 x = N_d x(T/n - t)$$

and we observe that $-J\frac{d}{dt}$ as well as $B$ both commute with $\tilde{g}_1$ and $\tilde{g}_2$.

By invoking Proposition 4.4, we get the relation of Maslov index and Morse index

$$\mu_{\text{Mas}}[x] + \iota(L) = \iota_{\text{spec}}(z) = \iota_{\text{geo}}(z)$$

where $\iota(L) = I \left( -J\frac{d}{dt} - J\frac{d}{dt} - C \right)$ and $C = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}$. Since the operator pointwise induced by $C$ commute with $\tilde{g}_1, \tilde{g}_2$, then get

$$I \left( -J\frac{d}{dt} - J\frac{d}{dt} - C \right) = I \left( -J\frac{d}{dt} I_{E^+}, -J\frac{d}{dt} - C I_{E^+} \right) + I \left( -J\frac{d}{dt} I_{E^-}, -J\frac{d}{dt} - C I_{E^-} \right)$$

and

$$I \left( -J\frac{d}{dt} I_{E^+}, -J\frac{d}{dt} - C I_{E^+} \right) = \sum_{k=0}^{n} I \left( -J\frac{d}{dt} I_{E^+}, -J\frac{d}{dt} - C I_{E^+} \right).$$

The proof readily follows by invoking Theorem 1. This conclude the proof.

5 Some dynamical and variational consequences

The aims of this section is to derive some dynamical consequences of the theory developed in the previous sections. More precisely, in Subsection 5.1, we apply the dihedral equivariant Bott-type formula to the celebrated figure-eight orbit for the planar three-body problem whilst Subsection 5.2 we investigate the hyperbolicity and the strongly instability of symmetric Lagrangian systems.

5.1 Decomposition of the figure-eight orbit

We start to consider three point particles in the Euclidean plane, namely $(x_1, x_2, x_3) \in (\mathbb{R}^2)^3$ self-interacting with the Newtonian gravitational potential $U$ and having unitary mass. We introduce the Jacobian coordinates through the canonical transformation

$$\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} K^{-T} & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix},$$

where $K^{-T}$ denotes the transpose of $K^{-1}$, $u = (u_1, u_2, u_3)^T, v = (v_1, v_2, v_3)^T \in (\mathbb{R}^2)^3$ and

$$K = \begin{bmatrix} 0_2 & -\frac{1}{\sqrt{2}}I_2 & \frac{1}{\sqrt{2}}I_2 \\ \frac{1}{\sqrt{3}}I_2 & \frac{1}{\sqrt{6}}I_2 & -\frac{1}{\sqrt{6}}I_2 \\ \frac{1}{3}I_2 & \frac{1}{3}I_2 & \frac{1}{3}I_2 \end{bmatrix}.$$
and by a straightforward calculation, we readily get \( K^{-1} = \begin{bmatrix} 0_2 & \frac{\sqrt{6}}{3} I_2 & I_2 \\ \frac{\sqrt{2}}{I_2} & -\frac{\sqrt{2}}{2} I_2 & \frac{\sqrt{6}}{6} I_2 \\ \frac{\sqrt{2}}{I_2} & -\frac{\sqrt{2}}{2} I_2 & -\frac{\sqrt{6}}{6} I_2 \end{bmatrix} \). By a direct computation, with respect to the Jacobian coordinates the configuration space transforms into the following

\[ \tilde{X} := \{ u = (u_1, u_2, u_3) \in (\mathbb{R}^2)^3 \mid u_3 = 0 \} . \]

Similarly the collision set fits into the following

\[ \tilde{\Delta} := \{ u \in \tilde{X} \mid u_1 = \sqrt{3}u_2 \text{ or } u_1 = -\sqrt{3}u_2 \text{ or } u_1 = 0 \} . \]

Then the transformed configuration space is \( \tilde{\mathbb{X}} := \tilde{X} \setminus \tilde{\Delta} \). In these new coordinates \( u = (u_1, u_2) \), if we denote by \( \tilde{g}_1, \tilde{g}_2 \) the generators of the group acting on \( W^{1,2}(\mathbb{R}/(T \mathbb{Z}), \mathbb{X}) \), then

\[ (\tilde{g}_1 \circ u)(t) = \tilde{S}u(t + \frac{T}{6}) \quad \text{and} \quad (\tilde{g}_2 \circ u)(t) = \tilde{N}u(t + \frac{T}{6} - t) . \]

Here \( \tilde{S} = \begin{bmatrix} 1 & \frac{\sqrt{2}}{2} R_2 & \frac{\sqrt{2}}{2} R_2 \\ \frac{\sqrt{2}}{2} R_2 & 1 & \frac{\sqrt{2}}{2} R_2 \\ \frac{\sqrt{2}}{2} R_2 & \frac{\sqrt{2}}{2} R_2 & 1 \end{bmatrix} \) and \( \tilde{N} = \begin{bmatrix} -R_2 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_2 \end{bmatrix} \). It’s easy to check that \( \tilde{S} \) and \( \tilde{N} \) satisfy the following properties

1. \( \tilde{S} \in O(4) \) and \( \tilde{S}^6 = I \)
2. \( \tilde{N} \in O(4) \), \( \tilde{N} = \tilde{N}^T \) and \( \tilde{N}^2 = I \)
3. \( \tilde{N}\tilde{S}^T = \tilde{S}\tilde{N} \).

Let \( E = W^{1,2}(\mathbb{R}/(T \mathbb{Z}), \mathbb{C}^4) \). The eigenvalues of \( \tilde{g}_1 \) are \( \omega_i = e^{i \pi \sqrt{-1}/3} \) for \( i = 0, 1, \ldots, 5 \). Denoting by \( E_i \) the eigenspace corresponding to \( \omega_i \), then we have

\[ E = \bigoplus_{i=0}^{5} E_i . \]

By taking into account the properties of the matrices \( \tilde{S} \) and \( \tilde{N} \), it follows that \( \tilde{g}_1, \tilde{g}_2 \) defined in Equation (5.1) defined an action of the \( D_6 \)-group on \( E \). Now, for any \( u \in E_i \), we have \( \tilde{g}_1 u = \omega_i u \).

Moreover, since

\[ \tilde{g}_1 \tilde{g}_2 u = \tilde{g}_2 \tilde{g}_1^{-1} u = \tilde{N} \tilde{g}_2 u , \]

so, we get \( \tilde{g}_2 u \in \tilde{E}_i \) and hence for any \( h = 0, 1, \ldots, 5 \)

\[ \tilde{g}_2 \tilde{g}_1^h : E_i \rightarrow \tilde{E}_i , \]

where we denoted by \( \tilde{E}_i \) the eigenspace corresponding to the eigenvalue \( \tilde{\omega}_i \). We now define the following subspaces

\[ F_0 = E_0, \quad F_3 = E_3 \quad \text{and finally} \quad F_i = E_i \bigoplus E_{6-i}, \quad i = 1, 2 \]

and we observe that

\[ \begin{bmatrix} 0 & \tilde{g}_2 \tilde{g}_1^h \\ \tilde{g}_2 \tilde{g}_1^h \end{bmatrix} F_i = F_i \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad \tilde{g}_2 \tilde{g}_1^h F_i = F_i \quad \text{for} \quad i = 0, 3 . \]
By a direct calculation, for $i = 1, 2$ we get

$$F_i = \left\{ \begin{array}{l}
\left( \begin{array}{c}
 u_1 \\
 u_2
\end{array} \right) \in W^{1,2}([0, T/6], C^4 \bigoplus C^4), \\
\left( \begin{array}{cc}
 \tilde{\omega}_i S \\
 0
\end{array} \right) \left( \begin{array}{c}
 u_1(T/6) \\
 u_2(T/6)
\end{array} \right) = \left( \begin{array}{c}
 u_1(0) \\
 u_2(0)
\end{array} \right)
\end{array} \right\}$$

and $F_i = \{ u \in W^{1,2}([0, T/6], C^4), u(0) = (-1)^i u(T/6), \} \text{ for } i = 0, 3$. From Lemma 3.2, we also have

$$F_{i,h}^\pm = \left\{ \begin{array}{l}
\left( \begin{array}{c}
 u_1 \\
 u_2
\end{array} \right) \in W^{1,2}([0, T/12], C^4 \bigoplus C^4), \\
\left( \begin{array}{c}
 u_1(0) \\
 u_2(0)
\end{array} \right) \in V_\pm \left( \begin{array}{cc}
 0 & \omega^h S N \\
 \omega^h S N & 0
\end{array} \right) \\
\left( \begin{array}{c}
 u_1(T/12) \\
 u_2(T/12)
\end{array} \right) \in V_\pm \left( \begin{array}{cc}
 0 & \omega^h S N \\
 \omega^h S N & 0
\end{array} \right)
\end{array} \right\}$$

$$F_{0,h}^\pm = \{ u \in W^{1,2}([0, T/12], C^4), u(0) \in V_\pm (S N), u(T/12) \in V_\pm (N) \}$$

$$F_{3,h}^\pm = \{ u \in W^{1,2}([0, T/12], C^4), u(0) \in V_\pm (-1)^h S N), u(T/12) \in V_\pm (-1)^h N) \}.$$  

So, $E$ can be decomposed into a direct sum of orthogonal $D_0$-invariant closed subspaces

$$E = E_h^+ \bigoplus E_h^-$$

where

$$E_h^+ = \bigoplus_{k=1}^2 F_{h,k}^+ \bigoplus F_{0,h}^+ \bigoplus F_{3,h}^+$$

and

$$E_h^- = \bigoplus_{k=1}^2 F_{h,k}^- \bigoplus F_{0,h}^- \bigoplus F_{3,h}^-.$$  

Thus the Morse index $n_-(x)$ of the Figure-eight orbit can be expressed as:

$$n_-(x) = \sum_{k=0}^3 \left[ n_-(F_{k,h}^+) + n_-(F_{k,h}^-) \right].$$

Since the figure-eight orbit is a minimizer on the $D_0$-equivariant loop space of the collision manifold, this in particular implies that $n_-(F_{0,h}^0) = 0$.

With a slight abuse of notation we denote by $Z_2$, $Z_3$ the group generated by $\tilde{g}_1$, $\tilde{g}_2$, respectively (actually we are identifying the abstract cyclic groups $Z_2$ and $Z_3$ with their unitary representation on $E$) and we set $n_2$ and $n_3$ be the Morse indices of the figure eight orbit as the critical point of the action functional restricted on the spaces fixed by the action of $Z_2$ and $Z_3$. Since the $Z_2$ fixed space is $F_0 \oplus F_2$, and the $Z_3$ fixed space is $F_0 \oplus F_3$, we get

$$n_2(x) = n_-(F_0) + n_-(F_2) = n_-(F_0) + n_-(F_3).$$

By invoking [HS09, Remark 5.12] we already know that $n_-(x) = 2$, $n_2 = n_3 = 0$. Thus, we get

$$n_-(F_{0,h}) = n_-(F_{3,h}) = n_-(F_{2,h}) = n_-(E_1) = n_-(E_2) = n_-(E_3) = 0$$

for $h = 0, \cdots, 5$ and

$$n_-(F_{1,h}) = n_-(E_4) = n_-(E_5) = 1.$$  

**Proof of Theorem 3.** The proof of this result readily follows by summing up all the previous computations performed in Subsection 5.1.  

$\square$
5.2 Hyperbolicity and strong instability of Lagrangian systems

The aim of this paragraph is to prove some instability results. To do so we briefly recall the definition of splitting numbers and we fix our notations.

Let \( L \in \mathcal{C}^2([0,T] \times \mathbb{R}^{2m}, \mathbb{R}) \) be a Lagrangian function satisfying the Legendre convexity condition and, as before, we consider the Lagrangian action functional

\[
\mathcal{A}_L : \mathcal{E}_r \rightarrow \mathbb{R} \text{ defined by } \mathcal{A}_L(x) = \int_0^T L(t, x(t), \dot{x}(t)) \, dt,
\]

where \( \mathcal{E}_r := W^{1,2}([0,T]; \mathbb{R}^m) \). We assume that \( x \) is a critical point; by the second variation of \( \mathcal{A}_L \), we get the associated index form given by

\[
\mathcal{J}_R(\xi, \eta) = \int_0^T \left[ \langle (P \dot{\xi} + Q \xi), \dot{\eta} \rangle + \langle Q^T \dot{\xi}, \eta \rangle + \langle R \xi, \eta \rangle \right] \, dt \quad \xi, \eta \in E.
\]

For any \( \omega \in U \), we let

\[
E^1(\omega) := \{ u \in E \mid u(0) = \omega u(T) \}
\]

where we denoted by \( E \) the complexification of \( E_R \); i.e. \( E := E_R \otimes \mathbb{C} \). The corresponding \( \omega \)-index form is given by

\[
\mathcal{J}_\omega(\xi, \eta) = \int_0^T \left[ \langle (P \dot{\xi} + Q \xi), \dot{\eta} \rangle + \langle Q^T \dot{\xi}, \eta \rangle + \langle R \xi, \eta \rangle \right] \, dt \quad \xi, \eta \in E
\]

where we have extended the Euclidean product \( \langle \cdot, \cdot \rangle \) to the standard Hermitian product. In what follows we denote by \( n_-(\omega, x) \) the Morse index of \( \mathcal{J}_\omega \) on \( E^1(\omega) \).

**Definition 5.1.** The splitting number of \( x \) at \( \omega \in U \) is given by

\[
S^\pm(\omega, x) = \lim_{\theta \to \pm 0} \left[ n_-(\omega e^{\sqrt{-1} \theta}, x) - n_-(\omega, x) \right].
\]

Integrating by parts in Equation (5.2), we get that linear Sturm system

\[
A\xi := -\frac{d}{dt} \left( P(t) \dot{\xi} + Q(t) \xi \right) + Q^T(t) \dot{\xi} + R(t) \xi = 0 \quad t \in [0,T]
\]

which reduces to the linear Hamiltonian system

\[
\dot{z} = JB(t)z, \quad t \in [0,T]
\]

where

\[
B(t) := \begin{bmatrix}
P^{-1}(t) & -P^{-1}(t)Q(t) 
-Q^T(t)P^{-1}(t) & Q^T(t)P^{-1}(t)Q(t) - R(t)
\end{bmatrix}.
\]

Let \( \gamma \) be the fundamental solution of the Hamiltonian system given in Equation (5.4) and let \( M_T := \gamma(T) \) be the induced monodromy matrix.

**Lemma 5.2.** Under the previous notation, we have

\[
S^\pm(\omega, x) = S^\pm_{M_T}(\omega), \quad \omega \in U,
\]

where \( S^\pm_{M_T}(\omega) \) denotes the splitting numbers defined in [Lon02, pag. 191, Equation (4)].

**Proof.** For the proof of this result we refer the interested reader to [Lon02, pag.252, Corollary 4] or [HPY17] and references therein. This conclude the proof.

**Definition 5.3.** The Morse-Sturm system given in Equation (5.3), is termed index hyperbolic if

\[
\begin{aligned}
\left( S^+_{M_T}(e^{i2\pi \theta}), S^-_{M_T}(e^{i2\pi \theta}) \right) &= \begin{cases} 
(0,0) & \text{if } \theta \in \mathbb{Q} \\
(p,p) & \text{if } \theta \notin \mathbb{Q}
\end{cases}
\end{aligned}
\]
It is worth to observe that $\mathcal{A}$ is degenerate on

$$E^2(\omega) := \{ u \in W^{2,2}([0, T], C^m) \mid u(0) = \omega u(T) \text{ and } \dot{u}(0) = \omega \dot{u}(T) \}$$

if and only if $\omega$ is an eigenvalue of $M_T$; i.e. in symbols $\omega \in \mathfrak{sp}(M_T)$.

The next result gives a sufficient condition in terms of the linearized operator $\mathcal{A}$ in order that a $D_n$-invariant critical point is index hyperbolic.

**Theorem 5.4.** Let $S, N \in O(m)$ be such that $S^m = I_m$ and $N^2 = (NS)^2 = I_m$ acting on $E_\mathbb{R}$ dually through the action given by

$$S: E \ni u \mapsto (Su)(\cdot) := S(u \cdot + T/n) \in E_\mathbb{R} \quad \text{and} \quad N: E \ni u \mapsto (Nu)(\cdot) := N(u(T/n - t)) \in E_\mathbb{R}.$$ 

Under the above notation, if the restriction of $\mathcal{A}$ given in Equation (1.13) on

$$E^2(T/n) := \{ u \in W^{2,2} \left( [0, T/n], C^m \right) \mid \dot{u}(0) = 0 \}$$

is positive semi-definite, then $x$ is index hyperbolic.

**Proof.** We start to observe that by the group action $(Nu)(t) = Nu(T/n - t)$ on $E^2(T/n)$, we have the following decomposition

$$E^2 \left( \frac{T}{n} \right) \supset E^2_\pm \left( \frac{T}{2n} \right) \oplus E^2 \left( \frac{T}{2n} \right)$$

where

$$E^2_\pm \left( \frac{T}{2n} \right) := \{ u \in W^{2,2} \left( [0, T/2n], C^m \right) \mid u \left( \frac{T}{2n} \right) \in V_+ (N), \dot{u}(0) = 0 \}$$

and

$$E^2 \left( \frac{T}{2n} \right) := \{ u \in W^{2,2} \left( [0, T/2n], C^m \right) \mid u \left( \frac{T}{2n} \right) \in V_-(N), \dot{u}(0) = 0 \}.$$ 

Since the spectrum of $N$ is $\{+1, -1\}$, if $u \in V_1 (N)$, then we have

$$(Nu)(t) = Nu \left( \frac{T}{n} - t \right) = u(t) \Rightarrow Nu \left( \frac{T}{n} \right) = u(0) \Rightarrow u(0) \in C^m$$

and

$$(Nu) \left( \frac{T}{2n} \right) = Nu \left( \frac{T}{n} - \frac{T}{2n} \right) = u \left( \frac{T}{2n} \right) \Rightarrow u \left( \frac{T}{2n} \right) \in V_+ (N).$$

Analogously, we have

$$(Nu)(t) = Nu \left( \frac{T}{n} - t \right) = u(t) \Rightarrow -Nu \left( \frac{T}{n} - t \right) = \dot{u}(t) \Rightarrow -Nu \left( \frac{T}{n} \right) = \dot{u}(0) \Rightarrow \dot{u}(0) = 0$$

and

$$-Nu \left( \frac{T}{2n} \right) = \dot{u} \left( \frac{T}{2n} \right) \Rightarrow \dot{u} \left( \frac{T}{2n} \right) \in V_- (N).$$

By this calculation, we get the definition of $E^2_\pm \left( \frac{T}{2n} \right)$ in which we omit the trivial condition $u(0) \in C^m$. If $u \in V_- (N)$, by the very same computation we get the subspace $E^2 \left( \frac{T}{2n} \right)$.

Since $\mathcal{A}$ is positive semi-definite on $E^2 \left( \frac{T}{2n} \right)$, then the index form $J$ is positive semi-definite on $E^1 \left( \frac{T}{2n} \right) = W^{1,2} \left( \left[ 0, \frac{T}{2n} \right], C^m \right)$. Clearly, we have

$$E^1_\pm \left( \frac{T}{2n} \right) := \{ u \in W^{1,2} \left( \left[ 0, \frac{T}{n} \right], C^m \right) \mid u \left( \frac{T}{2n} \right) \in V_\pm (N) \} \subset E^1 \left( \frac{T}{2n} \right)$$
and we have the decomposition
\[
E_1\left(\frac{T}{n}\right) := W^{1,2}\left([0, \left(\frac{T}{n}\right)], C^m\right) = E_1^+\left(\frac{T}{2n}\right) \oplus E_1^-\left(\frac{T}{2n}\right),
\]
then the index form \(J\) is positive semi-definite on \(E_1(T/n)\). We let
\[
E_1\left(\frac{T}{n}, \omega\right) := \{ u \in W^{1,2}\left([0, \left(\frac{T}{n}\right)], C^m\right) \mid u(0) = \omega u(T/n) \} \subset E_1\left(\frac{T}{n}\right),
\]
and we observe that, precisely as before, the index form \(J\) is positive semi-definite on \(E_1\left(\frac{T}{n}, \omega\right)\) for every \(\omega \in U\). Equivalently, \(A\) is positive semi-definite on \(E_2\left(\frac{T}{2n}, \omega\right)\) for every \(\omega \in U\).

We also observe that the (closed) subspace \(E_1\left(\frac{T}{n}, \omega\right)\) can be decomposed as follows
\[
E_2(T, \omega) = \bigoplus_{\omega} E_2\left(\frac{T}{n}, \omega\right)
\]
and by the same arguments as before, we infer that \(A\) is semi-positive on \(E_1\left(\frac{T}{n}, \omega\right)\) for every \(\omega \in U\). By invoking [Lon02, pag. 172, Theorem 4] it holds that
\[
n_-(\omega) = i_{\omega}(\gamma(t); t \in [0, T])
\]
and by this equality we infer that \(i_{\omega}(\gamma(t); t \in [0, T]) \equiv 0\) for every \(\omega \in U\). By the definition of splitting numbers, it readily follows that
\[
\left( S_{M_T}^+, S_{M_T}^- \right) = (0, 0) \quad \text{for every } \omega \in U.
\]
This conclude the proof.

Proof of Theorem 4. The proof of this result follows directly from Theorem 5.4 by invoking [Eke90, pag. 11, Theorem 10]. (Cf. [HPY17] for further details). This conclude the proof.

Proof of Corollary 2. We start to observe that \(\omega \in U\) is an eigenvalue of \(M_T\) if and only if \(A\) is degenerate on \(E_2(T, \omega)\). Arguing precisely as in the proof of Theorem 5.4 we infer that since \(A\) is positive definite on \(E_2(T, \omega)\) for every \(\omega \in U\) then \(M_T\) has no eigenvalues on \(U\). Otherwise stated \(M_T\) is hyperbolic. This conclude the proof.

Proof of Corollary 3. Since \(\mathcal{L}\) is reversible and \(Q\) is \(T\)-periodic by assumption, we infer that
\[
Q(t) = Q(T - t) \quad t \in [0, T].
\]
Now, let us consider the index form
\[
J : E \to \mathbb{R} \text{ defined by } J(u, v) = \int_0^T \langle \dot{u}, \dot{v} \rangle + \langle Q(t)u, v \rangle \, dt
\]
where \(E \coloneqq \{ u \in W^{1,2}([0, T], \mathbb{R}^m) \mid u(0) = u(T) \} \). As already observed \(u \in \ker J\) if and only if \(u\) is a \(T\)-periodic solution of \(\mathcal{L}u = 0\). Now, let us consider the \(\mathbb{Z}_2\)-action on \(E\) defined by
\[
g : E \to E \text{ defined by } (gu)(t) := u(T - t) \quad t \in [0, T]
\]
Wedderburn-Artin decomposition), e.g.

\[ E_2^k(T/2) := \{ u \in W^{2,2}([0, T/2], \mathbb{R}^m) \mid \dot{u}(0) = \dot{u}(T/2) = 0 \} \]

\[ E_2^2(T/2) = \{ u \in W^{2,2}([0, T/2], \mathbb{R}^m) \mid u(T/2) = 0, \dot{u}(0) = 0 \} \]

By assumption that the Morse index is 0 and there are no symmetric \( T \)-periodic solutions; thus \( \mathcal{L} \) is non-degenerate, too. Then it readily follows that \( \mathcal{L} \) is positive definite on \( E_2^k(T/2) \). The thesis readily follows by invoking Corollary 2. This conclude the proof. \( \square \)

A The Cyclic and Dihedral group Algebras

Let \( G \) be a finite group with \( n \) elements, namely \( \text{ord}(G) = n \). We denote by \( C[G] \) the group algebra of \( G \) over the complex field, namely the set of all formal complex linear combinations of element of \( G \) with coefficients in the complex field. It is well-know that \( C[G] \) is a \( \mathbb{C} \)-vector space of dimension \( \text{ord}(G) \) and it is a non-commutative algebra unless the group is commutative. In what follows we are interested in \( C[G] \)-modules. An important example of \( C[G] \)-module is \( C[G] \) itself viewed as a \( C[G] \)-module and it is termed regular \( C[G] \)-module. As a direct consequence of the Maschke’s Theorem if \( V \) is a \( C[G] \)-module then it is semisimple; thus there exist simple \( C[G] \)-modules \( U_1, \ldots, U_k \) such that

\[ V = U_1 \oplus \cdots \oplus U_k. \]

In particular \( C[G] \) is semisimple. Moreover by the Schur’s Lemma, if \( V, W \) are simple \( C[G] \)-modules either \( \phi : V \to W \) is a \( C[G] \)-isomorphism or \( \phi \) is the trivial homomorphism. In the first case, \( \phi \) is a scalar multiple of the identity map \( Iv \). As direct consequence of Schur’s Lemma, if \( G \) is a finite abelian group, every simple \( C[G] \)-module \( V \) is of length one. For, let \( x, g \in G \) and \( v \in V \); thus we have

\[ xgv = g(xv) \quad g, x \in G, \quad v \in V. \]

Therefore, \( V \ni v \mapsto xv \in V \) is a \( C[G] \)-homomorphism. Since \( V \) is irreducible (being simple), Schur’s Lemma implies that there exists \( \lambda \in \mathbb{C} \) such that \( xv = \lambda v \), for all \( v \in V \). In particular, this implies that every subspace of \( V \) is a \( C[G] \)-module and since \( V \) is simple, \( \text{dim} V = 1 \). (Actually, also the converse is true). As a direct consequence of the Wedderburn-Artin Theory, we have that \( C[G] \) is isomorphic to the direct sum of matrix algebras over division rings (since we are working over an algebraic closed field, it turns out that each division rings is actually isomorphic to \( \mathbb{C} \)), there exists an isomorphism

\[ \Phi : C[G] \longrightarrow \text{Mat}(n_1; \mathbb{C}) \oplus \cdots \oplus \text{Mat}(n_k; \mathbb{C}), \]

such that \( \text{ord}(G) = \sum_{j=1}^k n_j^2 \).

Example A.1. (The cyclic group) Let \( C_n := \langle r \rangle \langle r^n = 1 \rangle \) be the cyclic group of order \( n \). It can be realized as the group of rotations through angles \( 2k\pi/n \) around an axis. Denoting by \( C[z] \) the ring of complex polynomials, it readily follows that the group algebra \( C[C_n] \) is isomorphic to the quotient of \( C[z] \) by the ideal generated by the \( n \)-th cyclotomic polynomial \( z^n - 1 \); i.e.

\[ C[C_n] \cong C[z]/(z^n - 1). \]

Since we may factor the polynomial \( z^n - 1 \) over the complex field as \( z^n - 1 = \prod_{k=0}^{n-1} (z - \zeta_n^k) \), where \( \zeta_n \) denotes a \( n \)-th root of unit, the group algebra splits as a product of \( n \) copies of \( \mathbb{C} \) (exactly the Wedderburn-Artin decomposition), e.g.

\[ C[C_n] = \prod_{k=0}^{n-1} C[z]/(z - \zeta_n^k). \]
Thus $\mathbb{C}[C_n]$ decomposes as the direct sum of $n$ one-dimensional (complex) subspaces, i.e.

$$\mathbb{C}[C_n] = \bigoplus_{0 \leq k < n} L_k$$

where $\dim \mathbb{C} L_j = 1$ for each $j = 1, \ldots, n$. The generator $t$ acts, in the corresponding factor, as multiplication by $\zeta_n^k$, meaning that $L_k$ is a $C_n$-module with respect to the following action $C_n \times L_k \to L_k : (t, w) \mapsto \zeta_n^k w$ and the group algebra $\mathbb{C}[C_n]$ splits into the sum of $n$, $C_n$-modules.

**Remark A.2.** It is worth noticing that, by Example A.1 and by invoking the structure theorem of finite abelian groups, it readily follows the group algebra’s decomposition of any abelian group.

**Example A.3. (The Dihedral group)** Let $D_n$ be the dihedral group of degree $n$ and order $2n$. For $n \geq 3$, $D_n$ is the group of symmetries of a regular $n$-gon in the plane, namely, the group of all the plane symmetries that preserves a regular $n$-gon. It contains $n$ rotations, which form a subgroup isomorphic to $C_n$ and $n$ reflections. From an algebraic viewpoint it is a metabelian group having the cyclic normal subgroup $C_n$ of index 2 and the following presentation

$$D_n := \langle r, s \vert r^n = s^2 = 1, srs = r^{-1} \rangle.$$ 

In conclusion $D_n$ is a semi-direct product of $C_n$ and $C_2$: in symbols $D_n = C_n \times C_2$, where $C_n := \langle r \rangle$ and $C_2 := \langle s \rangle$. Each element of $D_n$ can be uniquely written, either in the form $r^k$, with $0 \leq k \leq n - 1$ (if it belongs to $C_n$), or in the form $sr^k$, with $0 \leq k \leq n - 1$ (if it doesn’t belong to $C_n$). Observe that the relation $srs = r^{-1}$ implies that $sr^ks = r^{-k}$ and $(sr^k)^2 = 1$. By the Wedderburn-Artin decomposition theorem, the complex dihedral group algebra $\mathbb{C}[D_n]$ splits as a product of complex matrices as follows

$$\mathbb{C}[D_n] \cong \bigoplus_{d|n} M_d$$

where $M_d \cong \mathbb{C} \oplus \mathbb{C}$ if $d = 1, 2$ and $M_d \cong \text{Mat}(2; \mathbb{C})$ if $d > 2$. Denoting by $L_k$ the one-dimensional complex subspace defined in Example A.1, we set $\widetilde{L}_k := L_k \oplus L_{-k}$ and we identify $L_k$ with the horizontal subspace $L_k \times \{0\} \subset \widetilde{L}_k$ and $L_{-k}$ with the vertical, namely with $\{0\} \times L_{-k} \subset \widetilde{L}_k$. With the action $\phi : D_n \times \widetilde{L}_k \to \widetilde{L}_k$ defined by

$$\phi(r, (x, y)) = (\zeta_n x, \zeta_n^{-1} y), \quad \phi(s, (x, y)) = (y, x)$$

$\widetilde{L}_k$ is a $D_n$-module. As long as $k \not\equiv -k \mod n$, the module $\widetilde{L}_k$ is irreducible. On the contrary, if $n = 2k$, then $\widetilde{L}_k = L_k \oplus L_{-k}$ decomposes. Indeed, let $v_+ \equiv (1, 1)$ and $v_- \equiv (1, -1)$, then the subspaces $L_+ := C v_+$ and $L_- := C v_-$ are invariant, so $\widetilde{L}_k$ decomposes as $L_+ \oplus L_-$. On $L_+$, the element $s$ acts as the identity and $r$ as the multiplication by $-1$, and on $L_-$ both act as multiplication by $-1$. Thus the group algebra decomposes in a direct sum of the $D_n$-modules as follows

$$\mathbb{C}[D_n] = \bigoplus_{k=1}^{n/2} 2 \widetilde{L}_k \oplus \widetilde{L}_0 \oplus \delta \widetilde{L}_{n/2}$$

where $\delta := 0$ if $n$ is odd and $1$ if $n$ is even.

Let $E$ be a separable complex Hilbert space and let $I_E$ be denote the identity operator in $E$. Given the unitary operator $R \in \text{U}(E)$, let $C_n$ be the finite subgroup defined by $C_n := \langle R \in \text{U}(E) \rangle \subset \text{U}(E)$. Thus $C_n$ is isomorphic to the cyclic group $C_n$ and it is nothing but a unitary representation of the cyclic group $C_n$. By using the spectral mapping theorem it readily follows that the spectrum of $R$ is respectively given by $\sigma(R) = \{ \zeta_n^k \in \mathbb{C} \mid k = 0, \ldots, n - 1 \}$. Let $E_k := \ker(R - \zeta_n^k I_E)$ be the eigenspace corresponding to the eigenvalue $\zeta_n^k$. By the spectral theory

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\[^3\] We observe that $D_1$ and $D_2$ are atypical groups and more precisely $D_1$ is the cyclic group of order 2 whilst $D_2$ is the Klein four-group. It is worth noticing that for $n > 3$, $D_n$ is non-abelian.
of normal operators, it is well-known that $E_k$ are mutually orthogonal and, clearly $C_n$ invariant, the action being given the multiplication by $\zeta_n^k$

$$C_n \times E_k \to E_k : (\zeta_n^k, v) \mapsto Rv = \zeta_n^k v.$$ 

Thus, we get an orthogonal decomposition of the Hilbert space $E$ into $C_n$-closed stable subspaces as follows

\begin{equation}
E = E_1 \oplus \cdots \oplus E_n.
\end{equation}

**Remark A.4.** We observe that the decomposition given in Equation (A.1) is the isotypic decomposition induced by the unitary representation of the cyclic group. In particular each subspace $E_k$ is given by the direct sum of (infinitely many) one-dimensional irreducible representations described in Example A.1.

We denote by $D_n$ the finite subgroup (actually a unitary representation of the dihedral group $D_n$) of the unitary group of $E$, presented by

$$D_n := \langle R, S \in U(E) | R^n = S^2 = (R S)^2 = I_E \rangle \subset U(E).$$

For $k = 0, \ldots, n - 1$, we define the closed linear subspaces $E_k := \ker(R - \zeta_n^k I_E)$ and $E_{-k} := \ker(R - \zeta_n^{-k} I_E)$ and for $k = 1, \ldots, n$, we let

\begin{equation}
F_k := E_k \oplus E_{-k}.
\end{equation}

We observe that $F_k$ defined in Equation (A.2) is a $D_n$-module with the action given by

$$C_n \times F_k \to F_k : (\zeta_n^k, v) \mapsto Rv = \begin{bmatrix} \zeta_n^k I_E & 0 \\ 0 & \zeta_n^{-k} I_E \end{bmatrix} v$$

and

$$C_2 \times F_k \to F_k : (s, v) \mapsto Sv = \begin{bmatrix} 0 & I_E \\ I_E & 0 \end{bmatrix} v.$$

Thus, we get a decomposition of the Hilbert space $E$ into mutually orthogonal $D_n$-stable modules, given by

\begin{equation}
E = \bigoplus_{k=0}^n F_k.
\end{equation}

**Remark A.5.** We observe that the decomposition given in Equation (A.3), is the isotypic decomposition of $E$ with respect to the irreducible representations of $D_n$.

### B Maslov index, Spectral Flow and Index Theorems

The goal of this Section is to briefly recall the Definitions and the main Properties of the *Maslov index* (and its friends) and the Spectral flow. In order to make the presentation as smooth as possible, we split this Section into two different Subsections. In the first Subsection B.1 we start to recalling the differentiable structure of the Lagrangian Grassmannian of a real and complex symplectic space as well as some well-known facts about the *(relative)* $L_0$-Maslov index or more generally the *Maslov index for paths of Lagrangian subspaces* both on real and complex symplectic spaces. In order to make the Notations and Definitions as uniform as possible we start by recalling the differentiable structure of the Lagrangian Grassmannian which plays a crucial role in the description of the Maslov index through an intersection theory. Our basic references for the material contained in the first Subsection are [Arn67, RS93, CLM94, LZ00b, GPP03, HS09] and references therein. In Subsection B.2, starting on some useful functional analytic preliminaries and we continue to provide the construction as well as useful properties of the spectral flow for paths of closed selfadjoint Fredholm operators which are continuous in the gap topology, which we’ll need in the later Sections. Our basic references are [AS69, APS76, RS95, BLP05, Les05, LZ00a, HP17].
B.1 The geometry of the Lagrangian Grassmannian and the Maslov index

Let \((V, \omega)\) be (finite) \(2m\)-dimensional real symplectic vector space, \(\mathcal{L}(V)\) be the vector space of all bounded and linear operators of \(V\) and let \(J \in \mathcal{L}(V)\) be a complex structure compatible with the symplectic form \(\omega\), meaning that \(\omega(\cdot, J\cdot)\) is an inner product on \(V\). Let us consider the Lagrangian Grassmannian of \((V, \omega)\), namely the set \(\Lambda(V, \omega)\) and we recall that it is a real compact and connected analytic embedded \(m(m + 1)/2\)-dimensional submanifold of the Grassmannian manifold of \(V\). Moreover for any \(L \in \Lambda(V, \omega)\) the tangent space \(T_L\Lambda(V, \omega)\) is canonically isomorphic to the space \(B_{\text{sym}}(L)\) of all symmetric bilinear forms on \(L\). Given \(L_0 \in \Lambda(V, \omega)\) and any non-negative integer \(j \in \{0, \ldots, m\}\), we define the sets \(\Lambda^j(L_0; V) := \{L \in \Lambda(V, \omega) : \dim(L \cap L_0) = k\}\) and we observe that \(\Lambda(V, \omega) := \bigcup_{j=0}^m \Lambda^j(L_0; V)\). It is well-known that \(\Lambda^j(L_0; V)\) is a connected embedded analytic submanifold of \(\Lambda(V, \omega)\) (being locally closed orbit of the Lie group \(\text{Sp}(V, L_0)\) of all symplectomorphism of \((V, \omega)\) which preserve \(L_0\) (cfr. [Var74, Theorem 2.9.7]), and connected) having codimension equal to \(j(j + 1)/2\). In particular \(\Lambda^1(L_0; V)\) has codimension 1 and for \(k \geq 2\) the codimension of \(\Lambda^j(L_0; V)\) in \(\Lambda(V, \omega)\) is bigger or equal to 3. Its tangent space is canonically isomorphic to the space of all symmetric bilinear forms over \(V\) vanishing on \(L \cap L_0\). A central object in our discussion is played by the universal Maslov (singular) cycle with vertex at \(L_0\), being an algebraic (actually a determinantal) variety defined by

\[
\Sigma(L_0; V) := \bigcup_{j=1}^m \Lambda^j(L_0; V).
\]

We observe that the Maslov cycle is the (topological) closure of lowest codimensional stratum \(\Lambda^1(L_0; V)\). In particular, \(\Lambda^0(L_0; V)\), the set of all Lagrangian subspaces that are transversal to \(L_0\), is an open and dense subset of \(\Lambda(V, \omega)\). The (top stratum) codimensional 1-submanifold \(\Lambda^1(L_0; V)\) in \(\Lambda(V, \omega)\) is co-oriented or otherwise stated it carries a transverse orientation. In fact given \(\varepsilon > 0\), for each \(L \in \Lambda^1(L_0; V)\), the smooth path of Lagrangian subspaces \(\ell : (-\varepsilon, \varepsilon) \to \Lambda(V, \omega)\) defined by \(\ell(t) := \exp(tJ)\) crosses \(\Lambda^1(L_0; V)\) transversally. The desired transverse orientation is given by the direction along the path when the parameter runs between \((-\varepsilon, \varepsilon)\). In an equivalent way, the co-orientation or transverse orientation is meant in the following sense; we first observe that the mapping

\[
T_L\Lambda(V, \omega) \cong B_{\text{sym}}(L) \ni B \mapsto B|_{(L_0 \cap L) \times (L_0 \cap L)} \in B_{\text{sym}}(L_0 \cap L)
\]

passes to the quotient \(T_L\Lambda(V, \omega)/T_L\Lambda^1(L_0; V) \to B_{\text{sym}}(L_0 \cap L)\). The hypersurface \(\Lambda^1(L_0; V)\) carries a canonical transverse orientation which is defined by declaring that a vector \(B \in T_L\Lambda(V, \omega), B \notin T_L\Lambda^1(L_0; V)\) is positively oriented if the non-zero symmetric bilinear form \(B|_{(L \cap L_0) \times (L \cap L_0)}\) on the line \(L \cap L_0\) is positive definite. Thus the Maslov cycle is two-sidedly embedded in \(\Lambda(V, \omega)\). Based on these properties, Arnol’d in [Arn67], defined an intersection index for closed loops in \(\Lambda(V, \omega)\) (actually in \((\mathbb{R}^{2m}, \omega)\), but the treatment in this more general situation presents no difficulties) via transversality arguments. This general position arguments can be generalised to Lagrangian paths (not only closed) with endpoints out of the Maslov cycle. Following authors in [CLM94, HS09] we introduce the following Definition.

Definition B.1. Let \(L_0 \in \Lambda(V, \omega)\) and, for \(a < b\), let \(\ell \in \mathcal{C}^0([a, b], \Lambda(V, \omega))\). We define the (relative) Maslov index of \(\ell\) with respect to \(L_0\) as the integer given by

\[
\mu_{\text{CLM}}(L_0, \ell) := |\exp(-\varepsilon J)\ell(a)\exp(-\varepsilon J)\ell(b)| \in \Sigma(L_0; V)
\]

where \(\varepsilon \in (0, 1)\) is sufficiently small and where the right-hand side denotes the intersection number.

Remark B.2. A few Remarks on the Definition B.1 are in order. By the basic geometric observation given in [CLM94, Lemma 2.1], it readily follows that there exists \(\varepsilon > 0\) sufficiently small such that \(\exp(-\varepsilon J)\ell(a), \exp(-\varepsilon J)\ell(b)\) doesn’t lie on \(\Sigma(L_0; V)\). By [RS93, Step 2, Proof of Theorem 2.3], there exists a perturbed path \(\tilde{\ell}\) having only simple crossings (namely the path \(\ell\) intersects the Maslov cycle transversally and in the top stratum. Since, simple crossings are isolated, on a
compact interval are in a finite number. To each crossing instant \( t_i \in (a, b) \) we associate the number \( s(t_i) = 1 \) (resp. \( s(t_i) = -1 \)) according to the fact that, in a sufficiently small neighbourhood of \( t_i \), \( \ell \) have the same (resp. opposite) direction of \( \exp(tJ)\ell(t_i) \). Then the intersection number given in Formula (B.1) is equal to the summation of \( s(t_i) \), where the sum runs over all crossing instants \( s(t_i) \).

The Maslov index given in Definition B.1 have many important properties (cfr. [RS93, CLM94] for further details). Below we list only a few of them that we’ll use in the sequel for computing the Maslov index and, for further details, we refer the interested reader to the aforementioned paper and references therein.

**Property I (Reparametrisation Invariance)** Let \( \psi : [a, b] \to [c, d] \) be a continuous function with \( \psi(a) = c \) and \( \psi(b) = d \). Then \( \mu_{\text{CLM}}(L_0, \ell) = \mu_{\text{CLM}}(L_0, \ell \circ \psi) \).

**Property II (Homotopy Invariance Relative to the Ends)** Let \( \mathcal{T} : [0, 1] \times [a, b] \to \Lambda(V, \omega) : (s, t) \mapsto \mathcal{T}(s, t) \)

be a continuous two-parameter family of Lagrangian subspaces such that \( \dim(L_0 \cap \mathcal{T}(s, a)) \) and \( \dim(L_0 \cap \mathcal{T}(s, b)) \) are independent on \( s \). Then \( \mu_{\text{CLM}}(L_0, \mathcal{T}_0) = \mu_{\text{CLM}}(L_0, \mathcal{T}_1) \) where \( \mathcal{T}_0(\cdot) := \mathcal{T}(0, \cdot) \) and \( \mathcal{T}_1(\cdot) := \mathcal{T}(1, \cdot) \).

**Property III (Path Additivity)** If \( c \in (a, b) \), then

\[
\mu_{\text{CLM}}(L_0, \ell) = \mu_{\text{CLM}}(L_0, \mathcal{T}|_{[a, c]}) + \mu_{\text{CLM}}(L_0, \mathcal{T}|_{[c, b]}).
\]

**Property IV (Symplectic Invariance)** Let \( \phi \in \mathcal{C}^0([a, b], \text{Sp}(V, \omega)) \) be a continuous path in the (closed) symplectic group \( \text{Sp}(V, \omega) \) of all symplectomorphisms of \( (V, \omega) \). Then

\[
\mu_{\text{CLM}}(L_0, \ell) = \mu_{\text{CLM}}(\phi(t) L_0, \phi(t)(\ell(t))), \quad t \in [0, 1].
\]

**Property V (Symplectic Additivity)** For \( i = 1, 2 \) let \( (V_i, \omega_i) \) be symplectic vector spaces, \( L_i \in \Lambda(V_i, \omega_i) \) and let \( \ell_i \in \mathcal{C}^0([a, b], \Lambda(V_i, \omega_i)) \). Then

\[
\mu_{\text{CLM}}(\ell_1 \oplus \ell_2, L_1 \oplus L_2) = \mu_{\text{CLM}}(\ell_1, L_1) + \mu_{\text{CLM}}(\ell_2, L_2).
\]

Although the Definition of the Maslov index given in Formula B.1 is apparently simple, the computation of this homotopy invariant is, in general, quite involved. One efficient technique for computing this invariant, was introduced (in the non-degenerate case) by the authors in [RS93] through the so-called *crossing forms* and, generalised (in the degenerate situation) by authors in [GPP03, GPP04]. For \( \varepsilon > 0 \) let \( \ell^\varepsilon : (-\varepsilon, \varepsilon) \to \Lambda(V, \omega) \) be a \( \mathcal{C}^1 \)-path such that \( \ell^\varepsilon(0) = L \). Let \( L_1 \) be a fixed Lagrangian complement of \( L \) and, for \( v \in L \) and for sufficiently small \( t \) we define \( w(t) \in L_1 \) such that \( v + w(t) \in \ell^\varepsilon(t) \). Then the form

\[
Q[v] = \frac{d}{dt}_{t=0} \omega(v, w(t))
\]

is independent of the choice of \( L_1 \). A *crossing instant* \( t_0 \) for the continuous curve \( \ell : [a, b] \to \Lambda(V, \omega) \) is an instant such that \( \ell(t_0) \in \Sigma(L_0; V) \). If the curve is \( \mathcal{C}^1 \), at each crossing, we define the *crossing form* as the quadratic form on \( \ell(t_0) \cap L_0 \) given by

\[
\Gamma(\ell, L_0, t_0) = Q(\ell(t_0), \dot{\ell}(t_0))|_{t(t_0) = L_0}
\]

where \( Q \) was defined in Formula (B.2). A crossing \( t_0 \) is called *regular* if the crossing form is non-degenerate; moreover if the curve \( \ell \) has only regular crossings we shall refer as a *regular path*. (Heuristically, \( \ell \) has only regular crossings if and only if it is transverse to \( \Sigma(L_0) \)). Following
authors in [LZ00b], if $\ell : [a, b] \to \Lambda(V, \omega)$ is a regular $C^1$-path, then the crossing instants are in a finite number and the Maslov index is given by:

$$\mu^{CLM}(L_0, \ell) = n_+ [\Gamma(\ell(a), L_0, a)] + \sum_{t_0 \in \ell^{-1}(\Sigma(L_0, V))} \text{sgn} [\Gamma(\ell(t_0), L_0, t_0)] - n_- [\Gamma(\ell, L_0, b)],$$

where $n_+, n_-$ denotes respectively the number of positive (coindex), negative eigenvalues (index) in the Sylvester’s Inertia Theorem and where $\text{sgn} := n_+ - n_-$ denotes the (signature). We observe that any $C^1$-path is homotopic through a fixed endpoints homotopy to a path having only regular crossings.

In order to prove a dihedral equivariant Bott-type iteration formula by using the Maslov-type index, we need to review the Maslov index theory of the complex Lagrangian subspaces. For, let $(C^{2m}, \omega)$ be the complex symplectic vector space with the symplectic form $\omega(x, y) = \langle Jx, y \rangle$, for all $x, y \in C^{2m}$, where $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian product in $C^{2m}$, $J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}$ and $I_m$ is the identity matrix. If no confusion is possible we’ll omit the subindex $m$. We recall that a complex subspace $L$ is Lagrangian if $|L| = 0$ and the dim$_C L = m$. We let $L^\pm = \ker(iJ \mp \|_{2m})$ and we observe that $L$ is a (complex) Lagrangian subspace $L$ if and only if it can be seen as the graph of a unitary operator $U : L^+ \to L^-$. Otherwise stated the complex Lagrangian Grassmannian is homeomorphic to the unitary group of $C^{2m}$:

$$(B.3) \quad \Lambda(C^{2m}, \omega) \cong U(m).$$

(Cfr. [Edw64, Arn00, Zhu06, Por10] and references therein, for further details). We denote by $\mathbb{F}$ the homeomorphism defined in Formula (B.3) and we observe that

$$\dim (L_1 \cap L_2) = \dim \ker (\mathbb{F}(L_2)^{-1}\mathbb{F}(L_1) - I_m).$$

For any fixed $U \in U(m)$, we let $\Sigma(U) := \{ U_0 \in U(m) \mid \det (U^{-1} U_0 - I_m) = 0 \}$. We refer to $\Sigma(U)$ as the singular cycle of $U$. We observe that, for any $U_0 \in \Sigma(U)$, there exists $\varepsilon > 0$ sufficiently small such that $e^{\varepsilon U_0}$ is transversal to $\Sigma(U)$. Let now $\mathcal{U} : [a, b] \to U(m)$ be a continuous path. For $\varepsilon > 0$ small enough, $e^{-\varepsilon} \mathcal{U}(a)$ and $e^{-\varepsilon} \mathcal{U}(b)$ are out of the singular cycle of $U$ and the intersection number of the perturbed path $e^{-\varepsilon} \mathcal{U}$ with the singular cycle $\Sigma(U)$, is well-defined. For this reason we are entitled to introduce the following Definition.

**Definition B.3.** Let $L \in \Lambda(C^{2m}, \omega)$ be fixed and let $\ell : [a, b] \to \Lambda(C^{2m}, \omega)$ be a continuous path. We define the (complex) Maslov index as $\mu^{CLM}(L, \ell ; [a, b]) := [e^{-\varepsilon} \mathbb{F}(\ell) : \Sigma(\mathbb{F}(L))]$.

Given $L \in \Lambda(R^{2m}, \omega)$ be a real Lagrangian subspace, then $L^C := L \otimes C \in \Lambda(C^{2m}, \omega)$. We define the manifold $\Lambda_R(C^{2m}, \omega) := \{ L = \mathbb{F} \mid L \in \Lambda(C^{2m}, \omega) \}$ and we observe that $\Lambda_R(C^{2m}, \omega)$ is isomorphic to $\Lambda(R^{2m}, \omega) \otimes C$. It is worth noticing that, given $L \in \Lambda(C^{2m}, \omega)$, we have $\mathbb{F}(e^{-\varepsilon J} L) = e^{-2\varepsilon J} \mathbb{F}(L)$. Thus the (real) Maslov index given in Definition B.1 coincides with the (complex) Maslov index given in Definition B.3 when the path consists of real Lagrangian subspaces. So the Maslov index of a path of real Lagrangian subspaces is the same as the path of complex Lagrangian subspaces. We conclude by observing that the Definition of the crossing form as well as the computation of the Maslov index for a $C^1$ regular path through crossing forms is the same as in the real case and it also fulfill Properties I-V given above.

**B.2 On the spectral flow for paths of closed self-adjoint Fredholm operators**

The aim of this Subsection is to briefly recall the Definition and the main properties of the spectral flow for a continuous path of closed self-adjoint Fredholm operator. It is well-known that this topological invariant was introduced by authors in [APS76] in order to develop an Index
Theory on manifolds with boundary. Since then, it has been extensively applied and investigated extensively.

Let $E$ be a separable complex Hilbert space and let $T : \mathcal{D}(T) \subset E \to E$ be a self-adjoint Fredholm operator. By the Spectral decomposition Theorem (cf., for instance, [Kat80, Chapter III, Theorem 6.17]), there is an orthogonal decomposition $E = E_+(T) \oplus E_0(T) \oplus E_-(T)$, that reduces the operator $T$ and has the property that

$$\sigma(T) \cap (-\infty, 0) = \sigma(T_{E_-(T)}), \quad \sigma(T) \cap \{0\} = \sigma(T_{E_0(T)}), \quad \sigma(T) \cap (0, +\infty) = \sigma(T_{E_+(T)}).$$

**Definition B.4.** Let $T \in \mathcal{C}^Fsa(E)$. We term $T$ essentially positive if $\sigma_{ess}(T) \subset (0, +\infty)$, essentially negative if $\sigma_{ess}(T) \subset (-\infty, 0)$ and finally strongly indefinite respectively if $\sigma_{ess}(T) \cap (-\infty, 0) \neq \emptyset$ and $\sigma_{ess}(T) \cap (0, +\infty) \neq \emptyset$.

If $\dim E_-(T) < \infty$, we define its Morse index as the integer denoted by $\mu_{\text{Morse}}[T]$ and defined as $\mu_{\text{Morse}}[T] := \dim E_-(T)$.

Given $T \in \mathcal{C}^Fsa(E)$, for $a, b \notin \sigma(T)$ we set $\mathcal{P}_{[a,b]}(T) := \text{Re} \left( \frac{1}{2\pi i} \int_{\gamma} (\lambda - T)^{-1} d\lambda \right)$ where $\gamma$ is the circle of radius $\frac{b - a}{2}$ around the point $\frac{a + b}{2}$. We recall that if $[a, b] \subset \sigma(T)$ consists of isolated eigenvalues of finite type then

$$\operatorname{rg} \mathcal{P}_{[a,b]}(T) = E_{[a,b]}(T) := \bigoplus_{\lambda \in [a,b]} \ker(\lambda - T);$$

(cf. [GGK90, Section XV.2], for instance) and 0 either belongs in the resolvent set of $T$ or it is an isolated eigenvalue of finite multiplicity. Let us now consider the graph distance topology which is the topology induced by the gap metric $d_G(T_1, T_2) := ||P_1 - P_2||$ where $P_i$ is the projection onto the graph of $T_i$ in the product space $E \times E$. The next result allow us to define the spectral flow for gap continuous paths in $\mathcal{C}^Fsa(E)$.

**Proposition B.5.** Let $T_0 \in \mathcal{C}^Fsa(E)$ be fixed.

(i) There exists a positive real number $a \notin \sigma(T_0)$ and an open neighborhood $\mathcal{N} \subset \mathcal{C}^Fsa(E)$ of $T_0$ in the gap topology such that $\pm a \notin \sigma(T)$ for all $T \in \mathcal{N}$ and the map

$$\mathcal{N} \ni T \mapsto \mathcal{P}_{[-a,a]}(T) \in \mathcal{L}(E)$$

is continuous and the projection $\mathcal{P}_{[-a,a]}(T)$ has constant infinite rank for all $T \in \mathcal{N}$.

(ii) If $\mathcal{N}$ is a neighborhood as in (i) and $-a \leq c \leq d \leq a$ are such that $c, d \notin \sigma(T_0)$ for all $T \in \mathcal{N}$, then $T \mapsto \mathcal{P}_{[c,d]}(T)$ is continuous on $\mathcal{N}$. Moreover the range of $\mathcal{P}_{[c,d]}(T) \in \mathcal{N}$ is finite and constant.

**Proof.** For the proof of this result we refer the interested reader to [BLP05, Proposition 2.10].

Let $\mathcal{A} : [a, b] \to \mathcal{C}^Fsa(E)$ be a gap continuous path. As consequence of Proposition B.5, for every $t \in [a, b]$ there exists $a > 0$ and an open connected neighborhood $\mathcal{N}_t \subset \mathcal{C}^Fsa(E)$ of $\mathcal{A}(t)$ such that $\pm a \notin \sigma(T)$ for all $T \in \mathcal{N}_t$ and the map $\mathcal{N}_t \ni T \mapsto \mathcal{P}_{[-a,a]}(T) \in \mathcal{B}$ is continuous and hence rank $\{\mathcal{P}_{[-a,a]}(T)\}$ does not depends on $T \in \mathcal{N}_t$. Let us consider the open covering of the interval $[a, b]$ given by the pre-images of the neighborhoods $\mathcal{N}_t$ through $\mathcal{A}$ and, by choosing a sufficiently fine partition of the interval $[a, b]$ having diameter less than the Lebesgue number of the covering, we can find $a =: t_0 < t_1 < \cdots < t_n := b$, operators $T_i \in \mathcal{C}^Fsa(E)$ and positive real numbers $a_i, i = 1, \ldots, n$ in such a way the restriction of the path $\mathcal{A}$ on the interval $[t_{i-1}, t_i]$ lies in the neighborhood $\mathcal{N}_{t_i, a_i}$ and hence the $\dim E_{[-a_i,a_i]}(\mathcal{A}_{t_i})$ is constant for $t \in [t_{i-1}, t_i], i = 1, \ldots, n$.

**Definition B.6.** The spectral flow of $\mathcal{A}$ (on the interval $[a, b]$) is defined by

$$\text{sf}(\mathcal{A}, [a, b]) := \sum_{i=1}^{N} \dim E_{[0,a_i]}(\mathcal{A}_{t_i}) - \dim E_{[0,a_i]}(\mathcal{A}_{t_{i-1}}) \in \mathbb{Z}. $$
(In shorthand Notation we denote $\text{sf}(\mathcal{A}, [a, b])$ simply by $\text{sf}(\mathcal{A})$ if no confusion is possible). The spectral flow as given in Definition B.6 is well-defined (in the sense that it is independent either on the partition or on the $a_i$) and only depends on the continuous path $\mathcal{A}$. (Cfr. [BLP05, Proposition 2.13] and references therein). We list some useful properties of the spectral flow and we refer to [BLP05] for further details.

**Property I (Path Additivity)** If $A_1 : [a, b] \to \mathcal{C}F^n(E), A_1, A_2 : [c, d] \to \mathcal{C}F^n(E)$ are two continuous path such that $A_1(b) = A_2(c)$, then

$$\text{sf}(A_1 \circ A_2) = \text{sf}(A_1) + \text{sf}(A_2).$$

**Property II (Homotopy Relative to the Ends)** If $h : [0, 1] \times [a, b] \to \mathcal{C}F^n(E) : (s, t) \mapsto h(s, t)$ is a continuous map such that $h_a : [0, 1] \ni s \mapsto \dim \ker h(s, a) \in \mathcal{C}F^n(E)$ and $h_b : [0, 1] \ni t \mapsto \dim \ker h(s, b) \in \mathcal{C}F^n(E)$ are independent on $s$, then

$$\text{sf}(h_0, [a, b]) = \text{sf}(h_1, [a, b]),$$

where $h_0(\cdot) := h(0, \cdot)$ and $h_1(\cdot) = h(1, \cdot)$.

**Property III (Direct sum)** If for $i = 1, 2, E_i$ are Hilbert spaces and if $h_i : [a, b] \to \mathcal{C}F^n(E_i)$ are two gap-continuous paths of self-adjoint Fredholm operators, the

$$\text{sf}(h_1 \oplus h_2, [a, b]) = \text{sf}(h_1, [a, b]) + \text{sf}(h_2, [a, b]).$$

As already observed, the spectral flow, in general, depends on the whole path and not just on the ends. However, if the path has a special form, it actually depends on the end-points. More precisely, let $\mathcal{A}, \mathcal{B} \in \mathcal{C}F^n(E)$ and let $\tilde{\mathcal{A}} : [a, b] \to \mathcal{C}F^n(E)$ be the path pointwise defined by $
\tilde{\mathcal{A}}(t) := \mathcal{A} + \mathcal{B}(t)$
where $\mathcal{B}$ is any continuous curve of $\mathcal{A}$-compact operators parametrised on $[0, 1]$ such that $\mathcal{B}(0) := 0$ and $\mathcal{B}(1) := \mathcal{B}_0$. In this case, the spectral flow depends of the path $\tilde{\mathcal{A}}$, only on the endpoints (cfr. [LZ00a] and reference therein).

**Remark B.7.** It is worth noticing that, since every operator $\tilde{\mathcal{A}}(t)$ is a compact perturbation of a a fixed one, the path $\tilde{\mathcal{A}}$ is actually a continuous path into $\mathscr{L}(W, E)$, where $W := \mathcal{D}(\mathcal{A})$.

**Definition B.8.** ([LZ00a, Definition 2.8]). Let $\mathcal{A}, \mathcal{B} \in \mathcal{C}F^n(E)$ and we assume that $\mathcal{B}$ is $\mathcal{A}$-compact (in the sense specified above). Then the **relative Morse index of the pair** $\mathcal{A}, \mathcal{A} + \mathcal{B}$ is defined by

$$I(\mathcal{A}, \mathcal{A} + \mathcal{B}) = -\text{sf}(\tilde{\mathcal{A}}; [a, b])$$

where $\tilde{\mathcal{A}} := \mathcal{A} + \mathcal{B}(t)$ and where $\mathcal{B}$ is any continuous curve parametrised on $[0, 1]$ of $\mathcal{A}$-compact operators such that $\mathcal{B}(0) := 0$ and $\mathcal{B}(1) := \mathcal{B}_0$.

In the special case in which the Morse index of both operators $\mathcal{A}$ and $\mathcal{A} + \mathcal{B}$ are finite, then

$$I(\mathcal{A}, \mathcal{A} + \mathcal{B}) = \mu_{\text{Morse}}(\mathcal{A} + \mathcal{B}) - \mu_{\text{Morse}}(\mathcal{A}).$$

Let $W, E$ be separable Hilbert spaces with a dense and continuous inclusion $W \hookrightarrow E$ and let $\mathcal{A} : [0, 1] \to \mathcal{C}F^n(E)$ having fixed domain $W$. We assume that $\mathcal{A}$ is a continuously differentiable path $\mathcal{A} : [0, 1] \to \mathcal{C}F^n(E)$ and we denote by $\mathcal{A}_{\lambda_0}$ the derivative of $\mathcal{A}_\lambda$ with respect to the parameter $\lambda \in [0, 1]$ at $\lambda_0$.

**Definition B.9.** An instant $\lambda_0 \in [0, 1]$ is called a **crossing instant** if $\ker \mathcal{A}_{\lambda_0} \neq 0$. The crossing form at $\lambda_0$ is the quadratic form defined by

$$\Gamma(\mathcal{A}, \lambda_0) : \ker \mathcal{A}_{\lambda_0} \to \mathbb{R}, \quad \Gamma(\mathcal{A}, \lambda_0)[u] = \langle \hat{\mathcal{A}}_{\lambda_0} u, u \rangle_E.$$  

Moreover a crossing $\lambda_0$ is called **regular**, if $\Gamma(\mathcal{A}, \lambda_0)$ is non-degenerate.

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We recall that there exists \( \varepsilon > 0 \) such that \( A + \delta I_E \) has only regular crossings for almost every \( \delta \in (-\varepsilon, \varepsilon) \). (Cfr., for instance [Wat15, Theorem 2.6] and references therein). In the special case in which all crossings are regular, then the spectral flow can be easily computed through the crossing forms. More precisely the following result holds.

**Proposition B.10.** If \( A : [0, 1] \to \mathcal{F}_{sa}(W, E) \) has only regular crossings then they are in a finite number and

\[
\text{sf}(A, [0, 1]) = -n_- \left[ \Gamma(A, 0) \right] + \sum_{t_0 \in (0, 1)} \text{sgn} \left[ \Gamma(A, t_0) \right] + n_+ \left[ \Gamma(A, 1) \right]
\]

where the sum runs over all the crossing instants.

**Proof.** The proof of this result follows by arguing as in [RS95]. This conclude the proof. \( \square \)

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