Dynamics of a Dirac Fermion in the presence of spin noncommutativity

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Recently, it has been proposed a spacetime noncommutativity that involves spin degrees of freedom, here called “spin noncommutativity”. One of the motivations for such a construction is that it preserves Lorentz invariance, which is deformed or simply broken in other approaches to spacetime noncommutativity. In this work, we gain further insight in the physical aspects of the spin noncommutativity. The noncommutative Dirac equation is derived from an action principle, and it is found to lead to the conservation of a modified current, which involves the background electromagnetic field. Finally, we study the Landau problem in the presence of spin noncommutativity. For this scenario of a constant magnetic field, we are able to derive a simple Hermitean non-commutative correction to the Hamiltonian operator, and show that the degeneracy of the excited states is lifted by the noncommutativity at the second order or perturbation theory.

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I. INTRODUCTION

The idea that spacetime may be noncommutative at very small scales has its roots in semiclassical arguments stating that the principles of quantum mechanics and general relativity together imply in an absolute limit in the localization of events near the Planck scale [1]. The various instances of noncommutative spaces that have been studied in the literature in the last decades, therefore, represent an effort in uncovering some of the properties of spacetime at very small length scales, so gaining some understanding about quantum gravity. One usually expects physical effects related to quantum gravity to appear only in very high-energy processes, where quantum field theory is the most adequate theoretical tool. However, the study of relativistic or even nonrelativistic quantum mechanics with noncommutative coordinates has the advantage of exploring the noncommutativity of coordinates in a simpler setting, better clarifying its physical consequences.

In this context, various possibilities may arise, the simpler one being when the noncommutativity is parametrized by some constant matrix. This so-called “canonical noncommutativity” became quite popular since the discovery of its connection with string theory [2]. Quantum mechanics with canonical noncommutativity is defined by the commutation relations

\[ [\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \quad (1) \]

and it is implemented by means of the change of variables

\[ \hat{x}_i = x_i - \frac{1}{2} \sum_j \theta_{ij} p_j, \quad \hat{p}_i = p_i, \quad (2) \]

where \( x_i \) and \( p_i \) are operators satisfying the standard commutation relations of quantum mechanics, and \( i, j = 1, 2, 3 \). In this way, the Schroedinger equation in noncommutative space has the standard form, but involves the modified potential

\[ V \left( x_i - \frac{1}{2} \sum_j \theta_{ij} p_j \right). \quad (3) \]

Specific quantum mechanical potentials may then be studied using standard perturbation theory [3-5] or \( 1/N \) expansion [6], for example. One shortcoming of this approach is that Lorentz – or rotational, in the non-relativistic case – symmetry is generally lost since the constant \( \theta_{ij} \) may define a preferred direction in space. For other aspects of noncommutative quantum mechanics, see for example [7-13].
One may find in the literature several alternative approaches which does not suffer from this symmetry loss – such as, for example, Snyder’s work of 1947 [14], usually referred to as the first proposal of a noncommutative spacetime. There, the commutator of two coordinates is proportional to the Lorentz generator,

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\Lambda^2} M_{\mu\nu},$$

where $\Lambda$ is some large UV scale. The Snyder’s algebra preserves Lorentz invariance as it involves only covariant objects. For some recent developments regarding Snyder’s noncommutativity, see for example [15–18]. Closer to the canonical noncommutativity proposal is the idea that relation (1) is compatible with a twisted Lorentz symmetry, understood as a Hopf algebraic symmetry with a non-trivial coproduct [19–25]. For other ways to conciliate Lorentz Symmetry with noncommutativity of spacetime see for example [26–30].

Another point of view is to understand Eq. (1) as a first approximation to a more general setting,

$$[\hat{x}_i, \hat{x}_j] = i \hat{\theta}_{ij} (\hat{x}),$$

where the commutator of coordinates may itself be a non-constant operator, a function of the coordinates themselves [8, 10, 31–33]. One interesting aspect of this possibility is that one often faces the appearance of non-Hermitean operators [34, 35], a feature that will somehow appear in our work (for a general discussion of quantum mechanics with non-Hermitean operators, see the review [36]).

Recently, another idea was put forward in [37], involving a kind of noncommutativity with mixed spatial and spin degrees of freedom and a non-relativistic dynamics – to be hereafter referred as “spin noncommutativity”. Such a mixture could be theoretically understood as a non-relativistic analog of the Snyder’s proposal [4], where instead of the angular momentum, the commutator of coordinates is proportional to the spin. Subsequently, this idea was applied to the study of the Aharonov-Bohm scattering, which for small distances unveiled a strong anisotropy [38]. In [39], the spin noncommutativity was obtained by means of a consistent deformation of the Berezin-Marinov pseudoclassical model for the spinning particle [40]. Besides that, it was extended to the relativistic situation, and in this context the spin noncommutativity exhibits at least one advantage over the canonical one, which is the preservation of the Lorentz symmetry. Also, a modified Dirac equation for a fermion living in a space with spin noncommutativity was proposed.
The aim of the present work is to pursue further the study of the physical implications of this type of noncommutativity. This work is organized as follows: our starting point is the noncommutative Dirac equation defined in [39], which is discussed in Sec. II. The action from which such equation can be derived is presented and discussed in Sec. III. By applying Noether’s theorem and also by a direct manipulation of the equation of motion, we obtain a current which is conserved in Sec. IV. Sec. V contains an investigation of the effects of the noncommutativity in a simple quantum mechanical problem, a particle in the presence of a constant magnetic field. Here we find that the noncommutative modification in the theory is embodied in an Hermitian term added to the standard Dirac Hamiltonian, which is studied perturbatively up to the second order in the noncommutativity parameter. Finally, Sec. VI contains our conclusions and perspectives.

II. THE NONCOMMUTATIVE DIRAC EQUATION

The spin noncommutativity for a relativistic system may be implemented through the following deformation of the standard position and momentum operators,

\[ x^\mu \rightarrow \hat{x}^\mu = x^\mu I + \theta W^\mu, \quad p^\mu \rightarrow \hat{p}^\mu = p^\mu, \]

where \( W^\mu \) is the Pauli-Lubanski vector

\[ W^\mu = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} p^\rho S_{\sigma} = \frac{1}{2} \gamma^5 \sigma^{\mu \nu} \partial_\nu, \]

and \( S_{\rho \sigma} \) is the spin operator. Our conventions are the following: the flat spacetime metric satisfies \( \eta^{00} = -\eta^{ii} = 1 \), the Dirac gamma matrices are

\[ \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \]

in terms of the Pauli matrices \( \sigma^i \), and also,

\[ \sigma^{\mu \nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu), \quad \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \]

A direct consequence of Eq. (6) is the noncommutativity among spacetime coordinates,

\[ [\hat{x}^\mu, \hat{x}^\nu] = -i \theta \epsilon^{\mu \nu \rho \sigma} S_{\rho \sigma} + \frac{i}{2} \theta^2 \epsilon^{\mu \nu \rho \sigma} W_{\rho \sigma}. \]
As in Snyder’s proposal [4], only covariant objects appear in this last equation, so Lorentz symmetry is preserved by this construction.

In standard quantum mechanics, \( \hat{x} \) is an observable and therefore it should necessarily be an Hermitean operator. In our model the position operator has a non-trivial matrix structure in spinor space, and it satisfies

\[
(\hat{x}^\mu)^\dagger = \gamma^0 \hat{x}^\mu \gamma^0.
\]

The fact that \( \hat{x}^\mu \) is not Hermitean poses a difficulty in its interpretation as an observable. One might investigate the possibility that we are dealing with a \( PT \) symmetric system with a real spectrum for the \( \hat{x}^\mu \) operator, in which case a proper redefinition of variables could fix this problem [34, 35]. However, it is far from obvious whether the standard physical interpretation of the spectra of the coordinate operators applies in a noncommutative scenario, where exact localization of events in spacetime points is impossible. In this work, we adopt a more pragmatic point of view, and we shall consider the commuting coordinate \( x^\mu \) that will appear in the noncommutative Dirac equation as a label, in the spirit of quantum field theory. Besides, we observe that Eq. (11) is actually a natural requirement for an operator in spinor space, which will help to obtain the conjugate Dirac equation and a real Lagrangian density for our model.

The noncommutative Dirac equation for spin noncommutativity was introduced in [39] as

\[
\{i\gamma^\mu \left[ \partial_\mu + ieA_\mu (\hat{x}) \right] - m \} \psi (x) = 0,
\]

where the operator \( A_\mu (\hat{x}) \) is constructed from \( \hat{x} \) via the Weyl (symmetric) ordering,

\[
f (\hat{x}) = \int \frac{d^4k}{(2\pi)^4} \tilde{f} (k) e^{-ik_\mu \hat{x}^\mu}.
\]

It should be noted that the operator \( A_\mu (x^\mu \mathbf{1} + \theta \gamma^5 \sigma^{\mu\nu} \partial_\nu) \) has a nontrivial matrix structure which does not commute with \( \gamma^\mu \), so we face an ordering ambiguity in the noncommutative generalization for the matrix product \( \gamma^\mu A_\mu \). By introducing both left and right orderings with arbitrary coefficients \( a_1 \) and \( a_2 \) such that \( a_1 + a_2 = 1 \), we may define a deformed Dirac equation

\[
\hat{O} \psi = [i\gamma^\mu \partial_\mu - m - e (a_1 \gamma^\mu A_\mu (\hat{x}) + a_2 A_\mu (\hat{x}) \gamma^\mu)] \psi (x) = 0.
\]

For the ordinary commutative spacetime, the property

\[
[i\gamma^\mu \partial_\mu - m - e\gamma^\mu A_\mu]^\dagger = \gamma^0 [i\gamma^\mu \partial_\mu - m - e\gamma^\mu A_\mu] \gamma^0,
\]
may be used to prove that the Hamiltonian operator one finds when writing the ordinary Dirac equation in the form $i\partial_t \psi = H\psi$ is Hermitean. It is therefore natural to require that the operator $\hat{O}$ appearing in Eq. (14) also satisfies

$$\hat{O}^+ = \gamma^0 \hat{O} \gamma^0.$$  

(16)

In the noncommutative case, one has to demand that $a_1 = a_2$, i.e., *symmetric ordering*, to have this property. The symmetric ordering will also ensure the reality of the Lagrangean density corresponding to Eq. (14); finally it simplifies considerably the derivation of the noncommutative Hamiltonian we will discuss in Sec. V. These facts are enough for us to choose hereafter the ordering defined by $a_1 = a_2 = 1/2$, which fixes the noncommutative form of the Dirac equation as

$$\left[ i\gamma^\mu \partial_\mu - m - \frac{e}{2} \left( \gamma^\mu A_\mu (\hat{x}) + A_\mu (\hat{x}) \gamma^\mu \right) \right] \psi (x) = 0.$$ \hspace{1cm} (17)

An interesting feature of this model is that, in spite of the presence of noncommutativity and nonlocality, it is Lorentz invariant, in the sense that the deformed Dirac equation in Eq. (17) is Lorentz covariant, and the noncommutative parameter $\theta$ is a Lorentz scalar. Of course, the defining map in Eq. (6) was devised for this to happen, since it only contains covariant objects.

The action of Weyl ordered operator $f (x^\mu I + \theta \gamma^5 \sigma^{\mu\nu} \partial_\nu)$ on a spinor $\psi (x)$ can be represented by means of a “star operation” $\star$ as follows,

$$f (x^\mu I + \theta \gamma^5 \sigma^{\mu\nu} \partial_\nu) \psi = f \star \psi = f \exp \left( \theta \gamma^5 \sigma^{\mu\nu} \partial_\nu \right) \psi,$$

(18)

or, more explicitly,

$$f \star \psi = f \psi + \theta \partial_\mu f \gamma^5 \sigma^{\mu\nu} \partial_\nu \psi + \frac{\theta^2}{2} \partial_\mu_1 \partial_\nu_2 f \gamma^5 \sigma^{\mu_1 \nu_1} \gamma^5 \sigma^{\mu_2 \nu_2} \partial_{\nu_1} \partial_{\nu_2} \psi + \ldots.$$ \hspace{1cm} (19)

We note that the star operation defined above involves a regular (scalar) function $f$ and a Dirac spinor (column vector): it is not a “star product” in the usual sense since it does not map two similar objects in the same class of objects, therefore we cannot even discuss associativity of this operation. We will shortly define what we mean by a “star operation” involving other objects such as conjugate spinors, and then discuss some of its properties.

The relevant fact at this point is that the noncommutative Dirac equation (17) can be cast in terms of the star operation as

$$\left[ -i\gamma^\mu \partial_\mu + m + \frac{e}{2} (A_\mu (x) \gamma^\mu \star + A_\mu (x) \star \gamma^\mu) \right] \psi (x) = 0,$$ \hspace{1cm} (20)
which, taking into account Eq. (19), turns out to be

\[
\left[-i\gamma^\mu \partial_\mu + m + e\gamma^\mu A_\mu (x) + \frac{e\theta}{2} \partial_\alpha A_\mu \left(\gamma^\mu \gamma^5 \sigma^{\alpha_1 \beta_1} + \gamma^5 \sigma^{\alpha_1 \beta_1} \gamma^\mu\right) \partial_{\beta_1}
\right.
\]
\[
+ \frac{e\theta^2}{4} \partial_\alpha \partial_\beta A_\mu \left(\gamma^\mu \sigma^{\alpha_1 \beta_1} \sigma^{\alpha_2 \beta_2} + \sigma^{\alpha_1 \beta_1} \sigma^{\alpha_2 \beta_2} \gamma^\mu\right) \partial_{\beta_1} \partial_{\beta_2} + \ldots \right] \psi (x) = 0. \quad (21)
\]

The noncommutative Dirac equation has in general an infinite tower of time derivatives, so the usual Hamiltonian interpretation of quantum mechanics – based on an Hermitean Hamiltonian, which ensures unitarity of time evolution and conservation of probability – is not possible. In spite of that, we shall demonstrate in Sec. IV that a conserved charge current can be defined in general, and for a particular choice of \( A_\mu \), we shall be able to derive a consistent Hamiltonian formulation in Sec. V.

III. THE ACTION

The star operation and its properties are useful in deriving the noncommutative Dirac equation (17) from an action principle. We start by obtaining the conjugate Dirac equation, and for that end we define a star operation between the usual Dirac conjugate spinor \( \bar{\psi} = \psi^\dagger \gamma^0 \) and a function \( f \) by the following rule,

\[
\bar{\psi} \star f \equiv (f \star \psi)^\dagger \gamma^0, \quad (22)
\]

or, more explicitly,

\[
\bar{\psi} \star f = \bar{\psi} \exp \left( \theta \partial_\mu \gamma^5 \sigma^{\mu \nu} \partial_\nu \right) f
\]
\[
= \bar{\psi} f + \theta \partial_\mu \bar{\psi} \gamma^5 \sigma^{\mu \nu} \partial_\nu f + O (\theta^2) . \quad (23)
\]

Finally, we introduce a star operation between two spinors by the formula

\[
\bar{\varphi} \star \psi = \bar{\varphi} \exp \left( \theta \partial_\mu \gamma^5 \sigma^{\mu \nu} \partial_\nu \right) \psi
\]
\[
= \bar{\varphi} \psi + \theta \partial_\mu \bar{\varphi} \gamma^5 \sigma^{\mu \nu} \partial_\nu \psi + O (\theta^2) . \quad (24)
\]

We may now quote some useful properties that can be proved regarding the star operation defined in Eqs. (18), (23) and (24). First of all, we find the exact equality

\[
(\bar{\varphi} \star \psi)^* = \bar{\psi} \star \varphi . \quad (25)
\]
Next, integration by parts and the antisymmetry of $\sigma^{\mu\nu}$ leads to

$$ \int d^4x \bar{\varphi} \star \psi = \int d^4x \bar{\varphi} \psi + \text{surface terms}, \quad (26) $$

a property that is well known from the studies involving canonical noncommutativity and its associated star (Moyal) product.

We will also need to manipulate expressions of the general form $\int d^4x \bar{\varphi} (f \star \psi)$, involving two arbitrary spinors $\varphi$ and $\psi$ and a function $f$. Starting with

$$ \int d^4x \bar{\varphi} (f \star \psi) = \int d^4x \left( \bar{\varphi} f \psi + \theta \bar{\varphi} \partial_\mu f \gamma^5 \sigma^{\mu\nu}\partial_\nu \psi + \frac{\theta^2}{2} \bar{\varphi} \partial_\mu \partial_\nu f \gamma^5 \sigma^{\mu_1\nu_1} \gamma^5 \sigma^{\mu_2\nu_2} \partial_{\nu_1} \partial_{\nu_2} \psi + \ldots \right), \quad (27) $$

one integrates by parts all derivatives acting on $\psi$, taking care of the antisymmetry of $\sigma^{\alpha\beta}$, obtaining

$$ \int d^4x \bar{\varphi} (f \star \psi) = \int d^4x \left[ \bar{\varphi} f \psi + \theta \bar{\varphi} \partial_\mu \bar{\varphi} \partial_\nu f \gamma^5 \sigma^{\mu\nu} \psi + \frac{\theta^2}{2} \bar{\varphi} \partial_\mu \partial_\nu \bar{\varphi} \partial_{\mu_1} \partial_{\mu_2} f \gamma^5 \sigma^{\mu_1\nu_1} \gamma^5 \sigma^{\mu_2\nu_2} \partial_{\nu_1} \partial_{\nu_2} \psi + \ldots + \partial_\mu E^\mu \right], \quad (28) $$

where

$$ E^\mu = \theta \bar{\varphi} \partial_\nu f \gamma^5 \sigma^{\nu\mu} \psi + \frac{\theta^2}{2} \bar{\varphi} \partial_{\mu_1} \partial_{\mu_2} f \gamma^5 \sigma^{\mu_1\nu_1} \gamma^5 \sigma^{\mu_2\nu_2} \partial_{\nu_1} \partial_{\nu_2} \psi + \mathcal{O} (\theta^3). \quad (29) $$

Then one recognizes in the right hand side of (28) the expansion of $(\bar{\varphi} \star f) \psi$, i.e.,

$$ \int d^4x \bar{\varphi} (f \star \psi) = \int d^4x [ (\bar{\varphi} \star f) \psi + \partial_\mu E^\mu \psi]. \quad (30) $$

It should be stressed that, while we have only explicitly written $E^\mu$ up to the second order in $\theta$, the fact that Eq. (30) holds (i.e., the difference between the two integrals is a surface term) actually is true for any order of $\theta$, as it is clear from this derivation.

In particular, expressions like the one in Eq. (30) will appear in which $f$ is the electromagnetic potential $A_\mu$, which always appears contracted with a $\gamma^\mu$. In this case, one should be careful with the order of the star operation and the $\gamma^\mu$ since they do not commute. In
any case, it can be shown that,

\[
\int d^4 x \tilde{\varphi} (A_\mu \star \gamma^\mu \psi) = \int d^4 x \left[ (\tilde{\varphi} \star A_\mu \gamma^\mu) \psi + \partial_\mu F^\mu \right],
\]

(31a)

\[
\int d^4 x \tilde{\varphi} (A_\mu \gamma^\mu \star \psi) = \int d^4 x \left[ (\tilde{\varphi} \gamma^\mu \star A_\mu) \psi + \partial_\mu G^\mu \right],
\]

(31b)

where

\[
G^\mu = \theta \bar{\varphi} \partial_\alpha A_\nu \gamma^5 \sigma^{\alpha\mu} \gamma^\nu \psi + O (\theta^2),
\]

(32a)

\[
H^\mu = \theta \bar{\varphi} \partial_\alpha A_\nu \gamma^\nu \gamma^5 \sigma^{\alpha\mu} \psi + O (\theta^2).
\]

(32b)

Finally, we can write an action describing the interaction of a Dirac fermion with an electromagnetic potential \(A_\mu\) in a spacetime with spin noncommutativity,

\[
S [\psi, A] = \int d^4 x \bar{\psi} (x) \left[ -i \gamma^\mu \partial_\mu \psi (x) + m \psi (x) + \frac{e}{2} (A_\mu (x) \gamma^\mu \star A_\mu (x) \star \gamma^\mu) \psi (x) \right].
\]

(33)

Clearly, Eq. (20) is obtained by variation of Eq. (33), \(\delta S / \delta \bar{\psi} (x) = 0\). We split, as usual, this action in free and interaction part,

\[
S = S_0 + S_I.
\]

(34)

The usual free Dirac action

\[
S_0 = \int d^4 x \bar{\psi} (-i \gamma^\mu \partial_\mu \psi + m \psi),
\]

(35)

could also be written in a more symmetrical form involving the star operation due to Eq. (26),

\[
S_0 = \int d^4 x \left( \frac{-i}{2} \bar{\psi} \star \gamma^\mu \partial_\mu \psi + \frac{i}{2} \partial_\mu \bar{\psi} \gamma^\mu \star \psi + m \bar{\psi} \star \psi \right).
\]

On the other hand, for the interaction part we write

\[
S_I = \frac{e}{2} \int d^4 x \bar{\psi} (\gamma^\mu A_\mu \star \psi + A_\mu \star \gamma^\mu \psi)
\]

(36)

\[
= \frac{e}{4} \int d^4 x \left( \bar{\psi} (\gamma^\mu A_\mu \star \psi) + (\bar{\psi} \gamma^\mu \star A_\mu) \psi + \bar{\psi} (A_\mu \star \gamma^\mu \psi) + (\bar{\psi} \gamma^\mu A_\mu) \psi \right),
\]

(37)

where we have used Eq. (31), and finally

\[
S_I = \frac{e}{4} \int d^4 x \left[ \bar{\psi} \star (\gamma^\mu A_\mu \star \psi) + (\bar{\psi} \gamma^\mu \star A_\mu) \star \psi \\
+ \bar{\psi} \star (A_\mu \star \gamma^\mu \psi) + (\bar{\psi} \gamma^\mu A_\mu) \star \psi \right],
\]

(38)
after using Eq. (26).

One can verify that the action in Eq. (33) is real. This property is a consequence of Eq. (16), which is only valid if we adopt the symmetric ordering as in Eq. (17).

Finally, one might use the properties of the star operation quoted in this Section to show that the equation satisfied by the Dirac conjugate $\bar{\psi}$ reads

$$\bar{\psi}(x)\left[i\gamma^\mu \partial_\mu + m + \frac{e}{2} \gamma^\mu A_\mu(x) + \frac{e}{2} \gamma^\mu * A_\mu(x)\right] = 0. \tag{39}$$

Then, Eqs. (26) and (31) are used to rewrite Eq. (33) in the form

$$S = \int d^4x \left[i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} + \frac{e}{2}(\bar{\psi} * A_\mu \gamma^\mu + \bar{\psi}\gamma^\mu * A_\mu)\right] \psi, \tag{40}$$

and now the variation over $\psi(x)$ gives the conjugate Dirac equation in Eq. (39), as it should be.

## IV. CONSERVATION OF THE ELECTRICAL CURRENT

In this section, we want to find an expression for the conserved electric current $j^\mu$ in our theory, since the existence of such a current is crucial for the physical meaning of the model. The action (33) has global phase invariance, so Noether’s theorem provides a general formula for the associated conserved current. Due to the appearance of arbitrary high-order derivatives in $\psi$, one would need to generalize the well-known formula for the Noether current (see for example [41]). Expanding Eq. (36) in the first order of $\theta$, however, one finds

$$S_I = e \int d^4x A_\mu \bar{\psi}\gamma^\mu \psi - e\frac{\theta}{2} \int d^4x \bar{\psi}\gamma^\delta [\sigma^{\mu\nu}, \gamma^\alpha] \partial_\mu A_\alpha \partial_\nu \psi + O(\theta^2), \tag{41}$$

which, with the help of the identities

$$\gamma^\mu \sigma^{\rho\sigma} = + (\eta^{\rho\mu} \gamma^\sigma - \eta^{\sigma\mu} \gamma^\rho) - i\epsilon^{\mu\nu\rho\sigma} \gamma^5 \gamma_\nu, \tag{42a}$$
$$\sigma^{\rho\sigma} \gamma^\mu = - (\eta^{\rho\mu} \gamma^\sigma - \eta^{\sigma\mu} \gamma^\rho) - i\epsilon^{\mu\nu\rho\sigma} \gamma^5 \gamma_\nu, \tag{42b}$$

can be cast as

$$S_I = \int d^4x e A_\mu \bar{\psi}\gamma^\mu \psi - ie\theta \int d^4x \bar{\psi}\gamma^\delta (\partial_\mu A^\nu \gamma^\mu \partial_\nu \psi - \partial_\mu A^\nu \gamma^\nu \partial_\mu \psi) + O(\theta^2). \tag{43}$$
Since, in this approximation, there are no higher-order derivatives acting on $\psi$, one may use the standard formula for the Noether current associated to the phase symmetry $\delta \psi = -i\alpha \psi$,

$$j^\mu = -i \frac{\partial L}{\partial (\partial_\mu \psi)} \psi = \bar{\psi} \gamma^\mu \psi + e \theta \bar{\psi} \gamma^5 (\partial_\nu A^\nu \gamma^\mu - \partial_\mu A^\nu \gamma^\nu) \psi + O(\theta^2). \quad (44)$$

To see the existence of a conserved current $j^\mu$ at arbitrary order in $\theta$, we shall employ the following trick: using Eqs. (17,39) one may write the identity,

$$\left( i \partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} + \frac{e}{2} \bar{\psi} \gamma^\mu A_\mu + \frac{e}{2} \bar{\psi} \gamma^\mu \star A_\mu \right) \psi \nonumber$$

$$- \bar{\psi} \left( -i \gamma^\mu \partial_\mu \psi + m \psi + \frac{e}{2} \gamma^\mu \star \psi + \frac{e}{2} A_\mu \gamma^\mu \psi \right) = 0. \quad (46)$$

In the usual case (without the star operation), all that would remain would be $i \partial_\mu \bar{\psi} \gamma^\mu \psi + i \bar{\psi} \gamma^\mu \partial_\mu \psi = \partial_\mu (i \bar{\psi} \gamma^\mu \psi) = 0$, giving the conservation of the usual electric current. In our case, Eq. (46) can be written as

$$\partial_\mu \left( \bar{\psi} \gamma^\mu \psi \right) + \frac{i e}{2} \left[ \left( \bar{\psi} \star \gamma^\mu A_\mu \right) \psi - \bar{\psi} \left( A_\mu \star \gamma^\mu \psi \right) \right] \nonumber$$

$$+ \frac{i e}{2} \left[ \left( \bar{\psi} \gamma^\mu \star A_\mu \right) \psi - \bar{\psi} \left( \gamma^\mu A_\mu \star \psi \right) \right] = 0, \quad (47)$$

which, by virtue of Eqs. (31,32), leads to

$$\partial_\mu \left[ \bar{\psi} \gamma^\mu \psi + \frac{ie\theta}{2} \bar{\sigma} \partial_\alpha A_\mu \left( \gamma^5 \sigma^{\alpha\mu\gamma^\nu} + \gamma^\nu \gamma^5 \sigma^{\alpha\mu} \right) \psi + O(\theta^2) \right] = 0. \quad (48)$$

Finally, from Eqs. (42),

$$\partial_\mu \left[ \bar{\psi} \gamma^\mu \psi + e \theta \bar{\psi} \gamma^5 \left( \partial_\nu A^\nu \gamma^\mu - \partial_\mu A^\nu \gamma^\nu \right) \psi + O(\theta^2) \right] = 0 \quad (49)$$

For the reasons commented in the paragraph containing Eq. (30), this last equation holds for any order of $\theta$, which ensures the existence of the conserved current $j^\mu$.

It is noteworthy that the conserved current depends, already at first order in $\theta$, on the electromagnetic potential, which we have treated as a fixed background field. The same feature appears in canonical noncommutativity [7], and it makes interesting the problem of incorporating a dynamical potential $A_\mu$ in a consistent way.

**V. LANDAU PROBLEM IN THE PRESENCE OF SPIN NONCOMMUTATIVITY**

Having further explored the formal aspects of the spin noncommutativity, in this section we want to gain some insight into its possible observable consequences in a particular physical
problem. We consider the bound state problem for a charged particle subject to a constant magnetic field, known as the Landau problem.

When the noncommutativity is not present, the Landau problem is described in many textbooks such as [42]. The gauge potential corresponding to a constant magnetic field perpendicular to the $xy$ plane can be chosen as

$$A^2 = Bx^1 \quad \text{and} \quad A^0 = A^1 = A^3 = 0,$$

and the Dirac’s Hamiltonian

$$H_0 = -i\gamma^0\gamma^i\partial_i + m\gamma^0 + eBx_1\gamma^0\gamma^2,$$

has energy levels

$$E_{n,\alpha} = \sqrt{p_3^2 + m^2 + eB(2n + 1 - \alpha)},$$

with $\alpha = \pm 1$ for spin up and down, respectively. All energy levels exhibit an infinite-degeneracy relative to $p_1$ and $p_2$. Besides that, except for the (unique) ground-state $|0\rangle = |0, +1\rangle$, the excited energy levels are two-fold degenerate, since $|n, +1\rangle$ and $|n - 1, -1\rangle$ have the same energy. The energy eigenfunctions of $H_0$ can be cast in terms of the two-components eigenvectors $\chi_\alpha$ of the Pauli matrix $\sigma^3$,

$$\sigma^3 \chi_\alpha = \alpha \chi_\alpha,$$

as follows,

$$|n, \alpha\rangle = c_{n,\alpha} \begin{pmatrix} |n\rangle \chi_\alpha \\ \frac{\sigma_3}{E_{n,\alpha} + m} |n\rangle \chi_\alpha \end{pmatrix}.$$ (53)

Here, $|n\rangle$ are essentially eigenstates of the harmonic oscillator,

$$\varphi_n(x) = \langle x|n\rangle = e^{i(p_2x_2 + ip_3x_3)}e^{-\xi^2/2}H_n(\xi),$$

where

$$\xi = \sqrt{eB\left(x_2 - \frac{p_2}{eB}\right)},$$

and $c_{n,\alpha}$ is a normalization factor,

$$c_{n,\alpha} = \frac{(eB)^{1/4}}{2\pi} \sqrt{\frac{E_{n,\alpha} + m}{E_{n,\alpha}} \frac{1}{\sqrt{\pi}2^{n+1}n!}}.$$ (56)
Our choice of \( c_{n,\alpha} \) is slightly different from the usual one, but it has the advantage that the eigenfunctions in Eq. (53) are orthonormal in the simplest sense, i.e.,

\[
\langle n', \alpha' | n, \alpha \rangle = \delta_{n',n} \delta_{\alpha',\alpha} .
\] (57)

The canonical momentum \( \vec{\pi} \) is

\[
\vec{\pi} \equiv (-i\partial_1, p_2 - eBx_1, p_3) = \sqrt{eB} \left( -i\partial_\xi, -\xi, \frac{p_3}{\sqrt{eB}} \right),
\] (58)

and we quote some formulae that are useful in calculating matrix elements of the deformed Hamiltonian,

\[
\begin{align*}
\pi^1 (c_{n,\alpha} | n \rangle) &= i\sqrt{eB} \left( \frac{1}{2} \frac{c_{n,\alpha}}{c_{n+1,\alpha}} | n + 1 \rangle - \frac{n c_{n,\alpha}}{c_{n-1,\alpha}} | n - 1 \rangle \right), \\
\pi^2 (c_{n,\alpha} | n \rangle) &= \sqrt{eB} \left( \frac{1}{2} \frac{c_{n,\alpha}}{c_{n+1,\alpha}} | n + 1 \rangle + \frac{n c_{n,\alpha}}{c_{n-1,\alpha}} | n - 1 \rangle \right), \\
\pi^3 | n \rangle &= p_3 | n \rangle .
\end{align*}
\] (59a, 59b, 59c)

The fact that the noncommutative Dirac equation (20) has in general higher orders in time derivatives precludes the definition of a Dirac Hamiltonian in the standard way. It is actually a consequence of Lorentz invariance that the non-locality in space introduced by noncommutativity should also extend to the time variable. This difficulty is circumvented in the particular problem studied in this section because the linearity of the electromagnetic potential makes the noncommutativity modification of the Dirac equation local both in time and space, and all higher orders corrections in Eq. (21) vanish. We end up with with the simple Hamiltonian

\[
H = H_0 + H_I ,
\] (60)

where

\[
H_I = \frac{i}{2} \theta eB p_2 \gamma^2 \gamma^3 = -\theta eB \frac{p_2}{2} \left( \begin{array}{cc} \sigma^1 & 0 \\ 0 & \sigma^1 \end{array} \right).
\] (61)

It should be stressed that Eq. (60) contains the exact modification of the Hamiltonian for the present problem. This observation is necessary since we will use the \( O(\theta) \) correction in Eq. (60) to calculate the corrections to the energy levels up to \( O(\theta^2) \) in the sequel. Another
remark is that the symmetric ordering adopted in Eq. (17) is also essential in keeping the noncommutative modification to the Hamiltonian exactly of first order in $\theta$: if we had $a_1 - a_2 \neq 0$, the calculation of the noncommutative Hamiltonian would involve a multiplicative factor $\left[\gamma^0 - \frac{i}{2} (a_1 - a_2) \theta e B \gamma^3\right]^{-1}$, which would introduce higher orders corrections. One may also quickly verify that Eq. (61) is Hermitean, so it maintains the reality of the energy spectrum.

First order corrections to the energy of the ground-state $|0\rangle = |0, +1\rangle$ are given by

$$\delta E_0^{(1)} = \langle 0 | H_I | 0 \rangle = -\frac{eB\theta}{2} p_2 \langle 0 \left| \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \right| 0 \rangle.$$  

(62)

Using the orthogonality relations (59), as well standard Dirac matrices manipulations, one can show that

$$\delta E_0^{(1)} = 0.$$  

(63)

For the correction to the degenerate energy levels,

$$E_n = \sqrt{p_3^2 + m^2 + 2neB},$$  

(64)

with $n \geq 1$, one has to solve the secular equation

$$\det (W_{ij} - \delta E_n^{(1)} \delta_{ij}) = 0,$$  

(65)

where

$$W_{ij} = \langle n, i | H_I | n, j \rangle.$$  

(66)

Here, $|n, 1\rangle = |n, +1\rangle$ and $|n, 2\rangle = |n - 1, -1\rangle$ is a basis for each degenerate level. Calculation of the matrix elements $W_{ij}$ is also straightforward, and one can show that $W_{ij} = 0$, so that

$$\delta E_n^{(1)} = 0.$$  

(67)

In summary, the spin noncommutativity does not change the spectrum of the Landau problem in the first order in $\theta$.

We found non-trivial corrections to the energy levels in the second order of perturbation theory. For the ground-state energy, one has to calculate

$$\delta E_0^{(2)} = \sum_{n,i} \frac{|W_{n,i}|^2}{E_0 - E_n} \quad n \geq 1 \text{ and } i = 1, 2,$$  

(68)
where $W_{n,i}$ are matrix elements of $H_I$ between the ground-state and the excited state $|n,i\rangle$.

The only nonvanishing of these matrix elements are

\begin{align}
W_{1,1} &= \frac{i\theta (Be)^{3/2} p_2 p_3}{2(E_0 + m)(E_1 + m)} \frac{c_{0,+1}}{c_{1,+1}}, \\
W_{1,2} &= \frac{\theta eB p_2}{2} \frac{c_{1,-1} c_{1,+1} (2eB + p_3^2) + c_{0,-1} c_{0,+1} eB}{c_{1,-1} c_{1,+1} (E_0 + m)(E_1 + m)}, \\
W_{3,2} &= \frac{3\theta (eB)^2 p_2}{(E_0 + m)(E_3 + m)} \frac{c_{0,+1} c_{2,-1}}{c_{1,-1} c_{1,+1}},
\end{align}

from which the final (nonvanishing) expression for $\delta E_{0}^{(2)}$ can be calculated,

\begin{align}
\delta E_{0}^{(2)} &= -\frac{\theta^2 (eB)^2 p_2^2}{4(E_0 + m)^2} \left[ \frac{eB p_3^2}{(E_1 - E_0)(E_1 + m)^2} \left( \frac{c_{0,+1}}{c_{1,+1}} \right)^2 + \frac{1}{(E_1 - E_0)(E_1 + m)^2} \left( eB \left( \frac{c_{0,+1} c_{0,-1}}{c_{1,-1} c_{1,+1}} - 2 \right) + p_3 \right)^2 + \frac{(eB)^2 p_3^2}{(E_3 - E_0)(E_3 + m)^2} \left( \frac{c_{0,+1} c_{2,-1}}{c_{1,+1} c_{1,-1}} \right)^2 \right] + (70)
\end{align}

More interesting is the calculation of the second-order energy corrections to the degenerate levels, since there we can investigate whether the degeneracy is broken by the noncommutativity. Physically, when the perturbation breaks the degeneracy, that means some symmetry is broken; in our problem, it is the constant magnetic field which breaks part of the rotational symmetry. In the commutative case, one still has the two-fold degeneracy of the excited levels $|n,i\rangle$. Since the noncommutative correction to the Hamiltonian $H_I$ depends on the magnetic field, it might be that this degeneracy is broken, even if the noncommutativity itself does not break further symmetries.

Second order corrections to the energy of degenerate levels are found by solving the secular equation \[43\]

\[ \det \left( W_{ij} + \sum_{m,\ell} W_{n,i,m,\ell} W_{m,\ell,m,j} \frac{E_n - E_m}{E_n - E_m} - \delta E_{n}^{(2)} \delta_{ij} \right) = 0, \]

where the sum is for $m \geq 1$ and $m \neq n$, and $\ell = 1, 2$, and $W_{n,i,m,\ell}$ is the matrix element of $H_I$ between two degenerate states. This calculation is straightforward but quite involved, so it was done using a Computer Algebra System (CAS) \[44\]. The resulting expressions are too long and not particularly informative to be quoted here, but the relevant fact is that Eq. \[71\] usually has two different solutions $\delta E_{n}^{(2)}$, what means degeneracy is indeed broken at the second order.
VI. CONCLUSIONS AND PERSPECTIVES

In this work, we gained further insight into the spin noncommutativity proposed in [39]. We have shown that the noncommutative Dirac equation can be derived from an action principle, involving a Lagrangean which is real and has global phase invariance. This implies, by Noether’s theorem, the existence of a conserved current. The existence of this current is encouraging because it is important for the physical interpretation of the model.

We also investigated a very simple quantum mechanical system – the Landau problem – and verified the physical effects of the introduction of the spin noncommutativity. In this simple setting, it was possible to derive a Hermitean Hamiltonian from the noncommutative Dirac equation, which consisted on the standard Dirac Hamiltonian plus a noncommutative correction of order $\theta$. By using standard perturbation theory, we shown that there is no correction to the energy levels at first order in $\theta$. The corrections to the spectrum appear at the second order in $\theta$, and they break the degeneracy of the excited states, despite the fact that the noncommutativity does not introduce further preferred directions in the problem.

These results are potentially interesting from the phenomenological point of view. In most treatments of similar problems in noncommutative quantum mechanics, both in relativistic and non-relativistic regimes, corrections to the spectra are found already at the first order in $\theta$ [3–5, 45], which can pose very stringent constraints on the noncommutativity parameters. In our relativistic model, corrections only appear at order $\theta^2$, so the noncommutativity parameters could be less constrained by existing experimental bounds.

Many questions are still open, however, regarding further developments in this line of research. Instead of a fixed background field, the dynamics of the electromagnetic field should be consistently incorporated in this scenario. More complicated potentials could be investigated, such as the Coulomb potential, and a particular interesting question is whether the physical effects of the noncommutativity appear only at order $\theta^2$, as in the Landau problem. Finally, since noncommutativity is expected to be a very high energy effect, one might investigate whether a quantum field theory could be defined based on this type of noncommutativity. The definition of a novel type of noncommutative quantum field theories, which preserves Lorentz invariance by construction, would certainly be a very interesting problem.
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