Inter-Clique Influence Networks

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Abstract

Cliques have interesting properties that make them an ideal subject for social network analysis. Their members are close-knit and share common interests, which makes cliques a potent force for spreading influence. Since social networks often contain multiple cliques, it is important to understand how they can influence one another. In this study, we propose a model that describes the opinion dynamics of interconnected cliques. The model has two versions, one with randomized dynamics and the other with deterministic dynamics. We perform some analysis on their convergence properties and demonstrate their behaviors via simulations.

1 Introduction

Social network analysis enables researchers to examine the collective behaviors resulting from the relationships and interactions of social actors. The underlying structure of a social network, which is the combination of all the relationships of its members, makes it possible for different types of social phenomena to emerge when individuals perform their actions. This makes social network models a viable tool for understanding the aggregate behaviors of individuals.

Social groups can exhibit varying degrees of closeness. Of particular interest is the concept of cliques, which are a group of individuals who have strong influence over one another and interact with each other more frequently than they do with others [1]. Members of the same clique typically share similar views and interests which can influence others who may want to be part of their clique and can be a catalyst for trends. The presence of multiple cliques, then, can have a significant impact on the social network where they belong. This makes them an important subject in social network analysis and, in fact, graph theory has its own analogue for cliques which carries the same name [2].

A collection of interconnected cliques can be seen as a social network with multiple homogenous clusters, which is a situation that can be observed in real-world social networks, such as organizations within communities and the various groups in social media with shared interests. As such, a network of cliques can be a tool for understanding how influence can spread among groups with different views. This can be performed through opinion dynamics, which is the study of how opinions propagate within a social network. Models of opinion dynamics typically describe the patterns that emerge from the evolution of the opinions of individual agents [3], [4]. However, these approaches can also be applied on subgroups to examine how they can affect each other’s opinion profiles.

In this paper, we propose an opinion dynamics model for interconnected cliques. In particular, we explore how multiple cliques can influence one another in order to reach an agreement even if they may only have a few connections between them while their members are densely connected. We present two variations of the model, namely randomized and deterministic, and analyze their converge properties. Both versions are demonstrated via numerical examples.

Notation and Preliminaries. In this paper, we denote an undirected graph as $G = (V, E)$, where $V = \{1, \ldots, n\}$ is the set of nodes and $E \subseteq V \times V$ is the set of edges. The neighbors of agent $i$ is given by $N_i = \{j \mid (i, j) \in E\}$. A clique is a subgraph that is a complete graph, i.e. every pair of nodes in a clique has an edge between them. We use $e_i \in \mathbb{R}^n$ to denote a standard basis vector, where the $i$th element is 1 while the rest are zeroes. The vector of ones is given by $1$.

2 Inter-Clique Model

In this section, we present two variations of an opinion dynamics model for a network of interconnected cliques. We discuss the randomized version in section 2.1 and the deterministic version in section 2.2.

We start first by describing our representation of an inter-clique network. Let the social network $G = (V, E)$ be an undirected graph formed by connecting $m$ cliques $C^1, C^2, \ldots, C^m$. We assume that $m \geq 2$ to avoid trivialities. Each clique $C^k = (V^k, E^k)$, where $V^k \subseteq V$ and $E^k \subseteq E$, is a complete subgraph of $G$ that represents a close-knit group of agents. Additionally, each clique has at least one designated leader $i \in V^k$ that is connected to a member of another clique. The neighbors of a leader $i$ that also belong in $V^L$ is given by $N_i^L \subseteq N_i$. For our study, we assume that every pair of leaders are connected, thus the subgraph induced by $V^L$ is also complete. The total number of
agents in \( G \) is given by \( n = \sum_{s=1}^{m} |V^s| \) and the total number of leaders is given by \( l = |V^L| \).

Since cliques are formed by individuals with similar views, every agent in \( C^C \) gives equal importance to the opinions of all the members of the same clique, including itself. This is represented by the matrix \( W^s = \begin{bmatrix} 1/|V^s|, & i \in V^s, j \in V^s \\ 1, & i = j \in V^L \\ 0, & otherwise \end{bmatrix} \), where \( w_{ij}^s \) is the weight given by agent \( i \) on the opinion on agent \( j \). For the entire social network \( G \), the matrix of interpersonal influence within each clique is given by the block diagonal matrix \( W \in \mathbb{R}^{n \times n} \), defined as

\[
W = \begin{bmatrix}
W^1 & 0 & \cdots & 0 \\
0 & W^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & W^m
\end{bmatrix}.
\]

The leaders in \( G \) have their own corresponding weight matrix \( H \in \mathbb{R}^{n \times n} \), defined as

\[
H = [h_{ij}], \quad h_{ij} = \begin{cases}
1/|V^L|, & i \in V^L, j \in V^L \\
1, & i = j \in V^L \\
0, & otherwise
\end{cases}.
\]

Alternatively, \( H \) can be computed as

\[
H = I - \frac{1}{|V|} \sum_{i \in V^L} \sum_{j \in N_i} e_i (e_i - e_j)^T.
\]

The vector \( x(k) \in \mathbb{R}^n \) stores the opinions of all the agents at time \( k \in \mathbb{Z}_{\geq 0} \) and \( x(0) \) contains the initial opinions.

### 2.1 Randomized Dynamics

We describe here a randomized opinion dynamics model for the inter-clique networks we previously described. In this model, either the opinions of all agents or the opinions of all leaders are updated synchronously based on a random process.

Let \( M(k) \in \mathbb{R}^{n \times n} \) be a Bernoulli random matrix such that

\[
\mathbb{P}[M(k) = A] = \begin{cases}
1 - \alpha, & A = W \\
\alpha, & A = H
\end{cases},
\]

where \( \alpha \in [0,1] \). The opinion profile of the agents in \( G \) are iteratively updated as

\[
x(k + 1) = M(k) x(k),
\]

where time \( k \geq 0 \). The model (1) corresponds to a sequence of opinion changes that depend on whether the interactions that occurred are within cliques or among leaders. If \( M(k) = W \), the opinions of all \( i \in V \) are updated as

\[
x_i(k + 1) = \sum_{j \in N_i} w_{ij} x_j(k),
\]

which results to each clique having an opinion profile that is the average of the previous opinions of its members. On the other hand, if \( M(k) = H \), the opinions of all \( i \in V^L \) are updated as

\[
x_i(k + 1) = \sum_{j \in N_i} h_{ij} x_j(k),
\]

while the opinions of the other agents remain the same. This step enables cliques to influence one another through the interactions of their leaders. By combining the opinion updates (2) and (3), the model (1) achieves consensus, as stated in the following theorem.

**Theorem 1.** If \( \alpha \in (0,1) \), the model (1) converges almost surely to the limit given by

\[
\lim_{k \to \infty} x(k) = n^{-1}1^T x(0)
\]

which a consensus based on the average of the initial opinions of the agents in \( G \).

**Proof:** The sequence \( \{M(k)\}_{k=0}^{\infty} \) is independent and identically distributed (i.i.d.). When \( 0 < \alpha < 1 \), the sequence includes both \( W \) and \( H \), which are double-stochastic matrices with a positive diagonal. Also, combining the graphs included by \( W \) and \( H \) produces a connected graph. Based on these properties, the limit above can be obtained using the proofs in [5].

It can be easily seen that when \( \alpha = 0 \), clusters will converge to the initial opinions of their members, and when \( \alpha = 1 \), the leaders will converge to the average of their initial opinions while the opinions of the other agents will remain unchanged.

### 2.2 Deterministic Dynamics

Another way of expressing the opinion dynamics of interconnected cliques is by using a deterministic process that combines intra-clique and inter-clique interactions at every iteration. This can be achieved by utilizing the expected dynamics of (1).

The expected value of \( M(k) \) can be directly computed as

\[
\mathbb{E}[M(k)] = (1 - \alpha) W + \alpha H.
\]

If we define \( M := \mathbb{E}[M(k)] \), then the deterministic opinion dynamics model for the inter-clique networks under consideration is given by

\[
x(k + 1) = M x(k).
\]

Starting with the initial opinions \( x(0) \), the opinion of each agent \( i \) is updated at each time \( k \geq 0 \) as

\[
x_i(k + 1) = (1 - \alpha) \sum_{j \in N_i} w_{ij} x_j(k) + \alpha \sum_{j \in N_i} h_{ij} x_j(k)
\]
In (1), \( \alpha \) denotes the probability that cluster leaders exchange their views, which is essentially the probability that the different clusters influence one another. In (4), \( \alpha \) represents how strongly cliques influence one another. When \( \alpha = 0 \), only the agents belonging in the same clique can influence one another. On the other hand, \( \alpha = 1 \) means that only the leaders can affect each other’s opinions.

Since the dynamics (1) always converges to the same consensus value when \( \alpha \in (0,1) \), it follows that the deterministic version also converges to the same limit under the same condition. While (1) and (4) converges to the same limit, the deterministic version allows us to characterize the eigenvalues of \( M \) in terms of \( \alpha \) and the number of cliques, which we state in the following theorem.

**Theorem 2.** Suppose that \( m \leq n - l + 1 \). Let the eigenvalues of \( M \) be ordered as \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). Then

\[
0 \leq \lambda_i \leq \min(1 - \alpha, \alpha) \quad i = 1, 2, \ldots, l - 1
\]

\[
\lambda_i = \alpha \quad i = l, \ldots, n - m
\]

\[
\max(1 - \alpha, \alpha) \leq \lambda_i \leq 1 \quad i = n - m + 1, \ldots, n.
\]

**Proof:** Note that \( W \) has \( m \) dominant eigenvalues while \( H \) has \( n - l + 1 \) dominant eigenvalues. Both matrices are stochastic, thus their dominant eigenvalues have a value of 1. Additionally, both are symmetric matrices. Given these properties, we can use the eigenvalue inequalities of the sum of two Hermitian matrices stated in Theorem 4.3.1 of [6] in order to get the result above.

The previous theorem implies that \( M \) has \( m \) eigenvalues that are equal to or almost equal to 1 when \( \alpha \) is sufficiently small. Thus, that the number of cliques is a significant indicator of the rate by which the dynamics (4) reaches consensus.

### 3 Numerical Examples

In this section, we demonstrate the behavior of both the randomized and the deterministic versions of our proposed opinion dynamics model for interconnected cliques.

Consider two social networks, both with 16 agents. The first network is equally divided into four cliques where each clique has one leader, shown in Figure 1, while the second network is equally divided into two cliques with one leader each, as shown in Figure 2. For both social networks, let the vector of initial opinions be

\[
x(0) = \begin{bmatrix} 0.1665 & 0.0079 & 0.9769 & 0.9361 \\ 0.8484 & 0.7603 & 0.5218 & 0.2386 \\ 0.4865 & 0.3282 & 0.5186 & 0.2323 \\ 0.0251 & 0.4666 & 0.3126 & 0.1604 \end{bmatrix}
\]

and let \( \alpha = 0.1 \).

Figure 3 and 4 shows the resulting dynamics of (1) and (4), respectively, on the social network with four cliques. In both figures, members of the same clique are given the same color to show how the opinion profile of each clique evolves. While Figures 3 and 4 shows that the opinion profile of the given social network evolves differently in the different versions of our models, both converged to the consensus value of 0.4367, which is the average of the initial opinions.

When we apply the dynamics of (1) and (4) on the social network with two cliques, convergence is significantly slower even if they also reached the same consensus value. The results are shown in Figures 5 and 6. This observation confirms our findings in Theorem 2 regarding the effect of the number of groups on the convergence of our model.

![Fig. 1: Social network with four interconnected cliques with four members each](image1)

![Fig. 2: Social network with two interconnected cliques with eight members each](image2)

### 4 Conclusion

In this paper, we proposed a model that describes the opinion dynamics of interconnected cliques. The model has two versions: randomized and deterministic. While they have different behaviors, they are able to reach the same consensus when there are agents that interact with members of other cliques. Additionally, we have shown that the number of cliques can affect the rate by which our proposed model reaches consensus. Finally, we demonstrated the behavior of both versions of our model using numerical examples. A generalization of the model which considers groups that are not necessarily cliques will be presented in a future work.
Fig. 3: Randomized dynamics applied on the social network in Fig. 1

Fig. 4: Deterministic dynamics applied on the social network in Fig. 1

Fig. 5: Randomized dynamics applied on the social network in Fig. 2

Fig. 6: Deterministic dynamics applied on the social network in Fig. 2

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