COMPUTATION OF THE HOCHSCHILD COHOMOLOGY ALGEBRA OF THE DOWN-UP ALGEBRA $A(0, 1, 0)$

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Abstract. This document is supplementary material for the author’s paper The cap product on hochschild cohomology for localizations of filtered koszul algebras, arXiv:1304.0527 (2013). We give a detailed computation of the Hochschild cohomology algebra for the Down-Up algebra with parameters $(0, 1, 0)$. All notation is as in [1]. We take $\mathbf{k}$ to be a field of characteristic $\neq 2$. Let $A = A(0, 1, 0)$ be the so called Down-Up algebra $A = \mathbf{k}\langle x, y \rangle / ([x^2], [x, y^2])$. (Our “Down-Up” algebra is actually a particular member of a family of Down-Up algebras parametrized by $\mathbf{k}^3$. Whence the notation $A(0, 1, 0)$.) The algebra $A$ itself is 3 Koszul. However, we can introduce the redundant generator $z = xy + yx$ to express $A$ as a filtered algebra

$$A = \frac{\mathbf{k}\langle x, y, z \rangle}{([x, z], [y, z], xy + yx - z)}.$$ 

The algebra $A$ has a basis of monomials $\{x^{n_1}y^{n_2}z^{n_3}\}_{n_i \in \mathbb{Z} \geq 0}$ and grades to the skew polynomial ring

$$\text{gr}\, A = \mathbf{k}_q[x, y, z] := \frac{\mathbf{k}\langle x, y, z \rangle}{([x, z], [y, z], xy + yx)}.$$ 

The Koszul dual of $\mathbf{k}_q[x, y, z]$ is the skew exterior algebra

$$\Lambda = \bigwedge_q (t, u, v) := \frac{\mathbf{k}\langle t, u, v \rangle}{(t^2, u^2, v^2, tu - ut, tv + vt, uv + vu)},$$

where $t$, $u$, and $v$ are dual to $x$, $y$, and $z$ respectively. Monomials in $\Lambda$ will be denoted using the wedge notation, e.g. $t \wedge u$. The differential on $\Lambda$ takes $v$ to $t \wedge u$ and all other monomials to 0. Note that $A$ is a domain, since $\text{gr}\, A$ is a domain, and that $z$ is central in $A$.

The remainder of this document is dedicated to the proof of the following comprehensive result.

Proposition 0.1. Let $A$ be the Down-Up algebra $A = \mathbf{k}\langle x, y \rangle / ([x^2, y], [x, y^2])$.

i) $HH^0(A) = Z(A)$ is a free commutative algebra with generators $x, y$, and $xy + yx$.

ii) $HH^1(A)$ contains a rank 3 free $Z(A)$-module and is generated by 4 elements.

iii) $HH^2(A)$ contains a rank 3 free $Z(A)$-module and is also generated by 4 elements.

iv) $HH^3(A)$ is a direct sum of two cyclic modules. One of which is free, the other of which is annihilated by all the generators of $Z(A)$. 

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v) \( HH^\bullet(A) \) is a finitely generated \( Z(A) \)-algebra. It has four generators in degree 1, one in degree 2, and one in degree 3.

0.1. \( \mathcal{A}(A) \) and \( HH^0(A) \). The dg algebra \( \mathcal{A}(A) \) calculating \( HH^\bullet(A) \) is

\[
0 \to A \to \langle t, u, v \rangle \otimes A \to \langle t \wedge u, t \wedge v, u \wedge v \rangle \otimes A \to \langle t \wedge u \wedge v \rangle \otimes A \to 0
\]

It is clear from our original presentation of \( A \) that \( x^2 \) and \( y^2 \) are central. So we have \( k[x^2, y^2, z] \subset Z(A) \). Now, using the (multi-)grading on \( A \) one can check that \( Z(\text{gr}A) = k[x^2, y^2, z] \). The induced filtration on \( Z(A) \) is such that \( \text{gr}Z(A) \subset Z(\text{gr}A) = k[x^2, y^2, z] \). Whence we conclude

\[
HH^0(A) = Z(A) = k[x^2, y^2, z].
\]

Take \( S = Z(A) \). Note that \( S \) is already graded and that \( A \) is the finitely generated free \( S \)-module

\[
A = S \oplus (x \cdot S) \oplus (y \cdot S) \oplus (xy \cdot S). \tag{1}
\]

It will be convenient at times to replace \( xy \) with the more symmetric generator \( xy - yx = 2xy - z \), so that

\[
A = S \oplus (x \cdot S) \oplus (y \cdot S) \oplus ((xy - yx) \cdot S).
\]

0.2. Calculating \( HH^1(A) \). As usual, we let \( Z^\bullet \) denote the subalgebra of cycles in \( HH^\bullet(A) \) and \( B^\bullet \) denote the ideal of boundaries in \( Z^\bullet \). The decomposition (1) gives \( B^1 \) as the subspace

\[
t \otimes [x, y] \cdot S + u \otimes [y, x] \cdot S + (t \otimes [x, xy] + u \otimes [y, xy]) \cdot S.
\]

Since \( [x, y] = 2xy - z = xy - yx \) this sum becomes

\[
B^1 = (t \otimes (xy - yx)S) \oplus (u \otimes (xy - yx)S) \oplus ((t \otimes (2x^2y - xz) - u \otimes (2xy^2 - yz))S).
\]

Now we seek a description of the space of cycles \( Z^1 \). On the \( S \)-generators for \( A \), the operation \( \xi_y := a \mapsto (ya + ay) \) is such that

\[
d|t \otimes A = id \otimes \xi_y : \begin{align*}
t \otimes 1 & \mapsto t \otimes 2y \\
t \otimes x & \mapsto t \otimes z \\
t \otimes y & \mapsto t \otimes 2y^2 \\
t \otimes (xy - yx) & \mapsto 0.
\end{align*} \tag{2}
\]
and the operation $\xi_x := a \mapsto (xa + ax)$ is such that

\[
d|u \otimes A = id \otimes \xi_x : \\
\begin{array}{ll}
u \otimes 1 & \mapsto u \otimes 2x \\
u \otimes x & \mapsto u \otimes 2x^2 \\
u \otimes y & \mapsto u \otimes z \\
u \otimes (xy - yx) & \mapsto 0.
\end{array}
\]

(3)

One should note that $d(t \otimes A + u \otimes A) \subset t \wedge u \otimes A$ while an element $v \otimes a$ maps to something in $t \wedge u \otimes A$ if and only if $a$ is central.

From our decomposition of $A$, and subsequent decomposition of $(t \wedge u, t \wedge v, u \wedge v) \otimes A$, it is clear that if an element

\[t \otimes a_1 + u \otimes a_2 - v \otimes a_3\]

is a cycle then $\xi_y(a_1), \xi_z(a_2)$, and $a_3$ are all in central. Furthermore, for any $t \otimes a_1 + u \otimes a_2$ mapping into $t \wedge u \otimes S$, there exists a unique $a_3$ such that

\[t \otimes a_1 + u \otimes a_2 - v \otimes a_3\]

is a cycle. Namely, $a_3 = \xi_y(a_1) + \xi_z(a_2)$. So we see

\[Z^1 = \oplus (t \otimes x - v \otimes z)S \oplus (t \otimes y - v \otimes 2y^2)S \oplus (u \otimes y - v \otimes z)S \oplus (u \otimes x - v \otimes 2x^2)S\]

Let us exchange the generators $(t \otimes x - v \otimes z)$ and $(u \otimes y - v \otimes z)$ for the generators $(t \otimes x - u \otimes y)$ and $(t \otimes x + u \otimes y - v \otimes 2z)$. Then, if we take

\[
M_0 = (t \otimes x - u \otimes y)k[x^2, y^2] \\
M_1 = (t \otimes x + u \otimes y - v \otimes 2z)S \\
T_y = (t \otimes y - v \otimes 2y^2)S \\
U_x = (u \otimes x - v \otimes 2x^2)S
\]

we will have

\[HH^1(A) = M_0 \oplus M_1 \oplus T_y \oplus U_x\]

as a vector space. As a module, $S$ will act freely on $M_1, T_x, and U_y$, the subalgebra $k[x^2, y^2]$ will act freely on $T_y$. The generator $z$ will act on $T_y$ as the $k[x^2, y^2]$-linear map

\[M_0 \rightarrow T_y \oplus U_x \\
(t \otimes x - u \otimes y) \mapsto (t \otimes y - v \otimes 2y^2)2x^2 - (u \otimes x - v \otimes 2x^2)2y^2.
\]

This action takes account of the final summand in the above decomposition of $B^1$.

0.3. Calculating $HH^2(A)$. From the expressions (2) and (3) it is clear that

\[d(t \otimes A) + d(u \otimes A) = t \wedge u \otimes (z, x^2, y^2) + t \wedge u \otimes yS + t \wedge u \otimes xS,
\]

where $(z, x^2, y^2)$ is the maximal ideal generated by $z, x^2$ and $y^2$ in $S$. Since the differential maps $v \otimes s$ to $t \wedge u \otimes s$, for any $s \in S$, it is also clear that

\[d(v \otimes A) \supset t \wedge u \otimes S.
\]

The fact that $d(v \otimes a) \not\in t \wedge u \otimes A$ whenever $a$ is not central then implies

\[(t \wedge u \otimes S) \oplus (t \wedge u \otimes xS) \oplus (t \wedge u \otimes yS) = B^2 \cap (t \wedge u \otimes A).
\]
Checking the value of the differential on \( v \otimes x, v \otimes y, \) and \( v \otimes xy \) independently gives

\[
d(v \otimes A) = (t \wedge u \otimes S) + (t \wedge u \otimes y + t \wedge v \otimes (xy - yx))S + (t \wedge u \otimes x - u \wedge v \otimes (xy - yx))S + (t \wedge u \otimes xy + t \wedge v \otimes (2yx^2 - xz) - u \wedge v \otimes (2xy^2 - yz))S.
\]

Taking this all together we have

\[
B^2 = (t \wedge u \otimes (S \oplus xS \oplus yS)) + (t \wedge v \otimes (xy - yx) \cdot S) + (u \wedge v \otimes (xy - yx) \cdot S) + (t \wedge v \otimes (2yx^2 - xz) - u \wedge v \otimes (2xy^2 - yz) + t \wedge u \otimes xy)S.
\]

As for the cycles, clearly all of \( t \wedge v \otimes A \) is contained in \( Z^2 \). We will use the functions \( \xi_x \) and \( \xi_y \) to analyse the differential \( d^3 \) in the same way as was done in the previous section. Since

\[
d(t \wedge v \otimes s) = t \wedge u \wedge v \otimes 2ys, \quad d(u \wedge v \otimes s) = t \wedge v \wedge u \otimes 2xs,
\]

for all \( s \in S \), and since \( A \) is a domain, we know the only cycle in \( t \wedge u \otimes S + u \wedge v \otimes S \) is 0. So, for an element \( t \wedge v \otimes a_1 - u \wedge v \otimes a_2 \) to be a cycle requires

\[
a_1 = xs_1 + ys_1' + (xy - yx)s_1''
\]

and

\[
a_2 = xs_2 + ys_2' + (xy - yx)s_2''.
\]

Of course, we need not concern ourselves with the summands \((xy - yx)s_1''\), since they correspond to cycles. So we can replace \( a_1 \) and \( a_2 \) with \( a_1 = xs_1 + ys_1' \) and \( a_2 = ys_2 + xs_2' \). In this case

\[
t \wedge v \otimes a_1 - u \wedge v \otimes a_2 \mapsto (t \wedge u \wedge v) \otimes (zs_1 + 2yx^2s_1' - zsz_2 - 2x^2s_2'). \quad (4)
\]

Using the above formula one can check that the elements

\[
(t \wedge v \otimes x - u \wedge v \otimes y), \ (u \wedge v \otimes 2yx^2 - u \wedge v \otimes xz), \\
(t \wedge v \otimes yx^2 - u \wedge v \otimes xy^2), \ (t \wedge v \otimes 2xy^2 - t \wedge v \otimes yz), \quad (5)
\]

are all cycles, for example.

Since \( A \) is a domain, none of the \( zs_1, 2y^2s_1', zs_2, 2x^2s_2' \) appearing in (4) are themselves zero, unless the \( s_1 \) or \( s_1' \) are zero. So each monomial \( xy^{2n_1}y^{2n_2}z^{n_3} \) from \( z_1 \) must cancel with a monomial from one of the \( 2y^2s_1', zs_2, 2x^2s_2' \), and each monomial \( 2y^x^{2n_1}y^{2n_2}z^{n_3} \) from \( 2y^2s_1' \) must cancel with a monomial from one of the \( zs_1, zs_2, 2x^2s_2' \), etc. One can conclude, by the above reasoning, that the \( S \)-module generated by the elements in (5) contains all the cycles \( t \wedge v \otimes a_1 - u \wedge v \otimes a_2 \), where \( a_1 \) and \( a_2 \) are as described above.

**Remark 0.2.** One can include the generators \((t \wedge v \otimes 2x^3 - u \wedge v \otimes xz)\) and \((t \wedge v \otimes yz - u \wedge v \otimes 2y^3)\) in (5). These generators are, however, redundant.

Note that the sum

\[
(t \wedge v \otimes 2xy^2 - t \wedge v \otimes yz)x^2 - (t \wedge v \otimes x - u \wedge v \otimes y)2x^2y^2 \\
+ (u \wedge v \otimes 2yx^2 - u \wedge v \otimes xz)y^2 - (t \wedge v \otimes yx^2 - u \wedge v \otimes xy^2)z = 0. \quad (6)
\]
Therefore, the sum of the cyclic $S$-modules generated by the elements at (5) is not direct. The submodule generated by these elements is, however, expressible as the direct sum

$$(t \wedge v \otimes yx^2 - u \wedge v \otimes xy^2)k[x^2, y^2] \oplus (t \wedge v \otimes x - u \wedge v \otimes y)S$$

The element $z$ acts on $(t \wedge v \otimes yx^2 - u \wedge v \otimes xy^2)$ according to the above equation. Taking all of this information together gives

$$Z^2 = (t \wedge u \otimes (xy - yx) \cdot S) \oplus (u \wedge v \otimes (xy - yx) \cdot S)$$

The space of boundaries annihilates the second and third summands, and identifies the first summand with a submodule of the third in homology.

Take

$$V_0 = (t \wedge v \otimes yx^2 - u \wedge v \otimes xy^2)k[x^2, y^2]$$
$$V_1 = (t \wedge v \otimes x - u \wedge v \otimes y)S$$
$$V_2 = (t \wedge v \otimes 2xy^2 - t \wedge v \otimes yz)S$$
$$V_3 = (u \wedge v \otimes 2yx^2 - u \wedge v \otimes xz)S.$$ 

Then we have the decomposition

$$HH^2(A) = V_0 \oplus V_1 \oplus V_2 \oplus V_3.$$ 

For $i > 0$, $S$ acts freely on $V_i$. The subalgebra $k[x^2, y^2]$ acts freely on $V_0$ and, using the equation (6), we see that $z$ acts as the $k[x^2, y^2]$-linear map

$$V_0 \rightarrow V_1 \oplus V_2 \oplus V_3$$

$$(t \wedge v \otimes yx^2 - u \wedge v \otimes xy^2) \mapsto -(t \wedge v \otimes x - u \wedge v \otimes y)2xy^2 + (t \wedge v \otimes 2xy^2 - t \wedge v \otimes yz)x^2 + (u \wedge v \otimes 2yx^2 - u \wedge v \otimes xz)y^2$$

0.4. $HH^3(A)$ and the $HH^0(A)$-algebra structure on $HH^*(A)$. Recall that $S = HH^0(A) = k[x^2, y^2, z]$. By the formulas for $\xi_y$ and $\xi_z$ given at (2) and (3) we have

$$B^3 = (t \wedge u \wedge v) \otimes ((z, x^2, y^2) \otimes yS \otimes xS),$$

and therefore

$$HH^3(A) = (t \wedge u \wedge v \otimes k) \oplus (t \wedge u \wedge v \otimes xyS).$$

Take $Q_1 = t \wedge u \wedge v \otimes k$ and $Q_{xy} = t \wedge u \wedge v \otimes xyS$. The decomposition $HH^3(A) = Q_1 \oplus Q_{xy}$ is one of $S$-modules. The generators $x^2, y^2, z \in S$ all annihilate $Q_1$ and $Q_{xy}$ is free over $S$. Taking all our findings together gives

$$HH^*(A) = S \oplus \left( \begin{array}{c} M_0 \\ \oplus M_1 \\ \oplus T_y \\ \oplus U_x \end{array} \right) \oplus \left( \begin{array}{c} V_0 \\ \oplus V_1 \\ \oplus V_2 \\ \oplus V_3 \end{array} \right) \oplus \left( \begin{array}{c} Q_1 \\ \oplus Q_{xy} \end{array} \right).$$

All of the subspaces $M_j, T_y, U_x, V_1, V_2, V_3$ and $Q_4$ are cyclic $S$-modules of rank 1. Among the cyclic modules, $Q_1$ is the only one that is not free. Another two
generators are required to introduce \( M_0 \) and \( V_0 \). So, in particular, \( HH^\bullet(A) \) is a finitely generated \( S \)-module, and therefore a finitely generated \( S \)-algebra.

**Remark 0.3.** It is clear from our computations that the finiteness of \( HH^\bullet(A) \) over \( S = Z(A) \) is directly related to the finiteness of \( A \) over its center. This is, more directly, a reflection of the fact that \( S \) is Noetherian and that \( \mathcal{A}(A) \) is a finitely generated \( S \)-module.

One can compare this situation with our previous example of the universal enveloping algebra \( U(h) \) of the Heisenberg Lie algebra. The algebra \( U(h) \) is an infinite free module over its center and we saw that \( HH^\bullet(U(h)) \) is an infinitely generated algebra over the center of \( U(h) \).

Let \( m_1, \tau_y, \mu_x, \nu_z \), and \( q_* \) denote the generators for the cyclic modules \( M_1 \), \( T_y \), \( U_x \), \( V_* \), and \( Q_* \) respectively. Take \( m_0 = (t \otimes y - u \otimes x) \in M_0 \) and \( \nu_0 = (t \wedge v \otimes yx^2 - u \wedge v \otimes xy^2) \in V_0 \). Recall, once again, that the Hochschild cohomology ring is a graded commutative algebra over the center \( S \) of \( A \). The following multiplication table specifies the multiplication \( HH^1(A)HH^1(A) \):

\[
\begin{align*}
m_0m_1 &= \nu_02 \quad & m_1\tau_y &= \nu_12y^2 - \nu_22 \\
m_0\tau_y &= -\nu_12y^2 \quad & m_1\mu_x &= -\nu_12x^2 - \nu_32 \\
m_0\mu_x &= -\nu_22x^2 \quad & \tau_y\mu_x &= -\nu_1z \\
\end{align*}
\]  

(7)

The multiplication \( HH^1(A)HH^2(A) \) is given by the table:

\[
\begin{align*}
m_0\nu_0 &= 0 \quad & m_1\nu_0 &= 0 \quad & \tau_y\nu_0 &= q_{xy}y^2 \quad & \mu_x\nu_0 &= -q_{xy}x^2 \\
m_0\nu_1 &= 0 \quad & m_1\nu_1 &= -q_{xy}2 \quad & \tau_y\nu_1 &= 0 \quad & \mu_x\nu_1 &= 0 \\
m_0\nu_2 &= q_{xy}2y^2 \quad & m_1\nu_2 &= -q_{xy}2y^2 \quad & \tau_y\nu_2 &= 0 \quad & \mu_x\nu_2 &= -q_{xy}z \\
m_0\nu_3 &= q_{xy}2x^2 \quad & m_1\nu_3 &= q_{xy}2x^2 \quad & \tau_y\nu_3 &= q_{xy}z \quad & \mu_x\nu_3 &= 0. \\
\end{align*}
\]  

(8)

For simple grading reasons, all other products \( HH^i(A)HH^j(A) \), with \( i, j > 0 \), are 0. So the above tables, along with the \( S \)-module structure, completely specify the \( S \)-algebra structure on \( HH^\bullet(A) \). One should note, from (7), that \( \nu_0, \nu_2, \nu_3 \in HH^1(A)HH^1(A) \), while \( \nu_1 \notin HH^1(A)HH^1(A) \). Also, as can be seen from (8), \( q_{xy} \in HH^1(A)HH^2(A) \) while \( q_1 \notin HH^1(A)HH^2(A) \). This verifies the final point of Proposition 0.1.

**References**

[1] C. Negron, *The cup product on hochschild cohomology for localizations of filtered koszul algebras*, preprint arXiv:1304.0527 (2013).