ON TIME DECAY FOR THE SPHERICALLY SYMMETRIC VLASOV-POISSON SYSTEM

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In memory of Robert Glassey

ABSTRACT. A collisionless plasma is modeled by the Vlasov-Poisson system. Solutions in three space dimensions that have smooth, compactly supported initial data with spherical symmetry are considered. An improved field estimate is presented that is based on decay estimates obtained by Illner and Rein. Then some estimates are presented that ensure only particles with sufficiently small velocity can be found within a certain (time dependent) ball.

1. Introduction. Consider the Vlasov-Poisson system:

\[
\begin{aligned}
\partial_t f_\beta + \frac{1}{m_\beta} v \cdot \nabla_x f_\beta + e_\beta E \cdot \nabla_v f_\beta &= 0 & \beta = 1, \ldots, N \\
\rho(t,x) &= \sum_\beta e_\beta \int f_\beta(t,x,v) dv \\
\nabla \cdot E &= 0 \quad \text{as } |x| \to \infty
\end{aligned}
\]

where \( t \geq 0 \) is time, \( x \in \mathbb{R}^3 \) is position, and \( v \in \mathbb{R}^3 \) is momentum. \( f_\beta \) is the number density in phase space of particles of the \( \beta^{th} \) species which have mass \( m_\beta > 0 \) and charge \( e_\beta \). Collisional effects are neglected. The initial condition

\[ f_\beta(0,x,v) = f_{\beta 0}(x,v) \geq 0, \]

for \( (x,v) \in \mathbb{R}^6 \) is given for each \( \beta \) where it is assumed that \( f_{\beta 0} \in C^1_0(\mathbb{R}^6) \) is nonnegative and compactly supported. It is known that solutions remain smooth for all time ( [15], [12]). We will be interested in the case of spherical symmetry and in this case the existence and uniqueness of smooth solutions was established earlier in [2].

By a spherically symmetric solution we mean a solution for which

\[ f_\beta(t,Ox,OV) = f_\beta(t,x,v) \]

for every orthogonal transformation, \( O \), from \( \mathbb{R}^3 \) to itself. Such a solution is a function of \( t,r,u,\alpha \) where

\[
\begin{aligned}
r &= \sqrt{x_1^2 + x_2^2 + x_3^2}, \\
u &= \sqrt{v_1^2 + v_2^2 + v_3^2},
\end{aligned}
\]

and \( \alpha \in [0,\pi] \) is defined by

\[ ru\cos(\alpha) = x \cdot v. \]
It is known from [1] that for small initial data all particles scatter freely and the field decays like \( t^{-2} \). This decay of \( E(t,x) \) is also established in [10] for spherically symmetric solutions, but only for a plasma with a single species of charge. See also [13] for a more complete examination of this situation. It is not known if this same decay persists for large data solutions, even symmetric ones, with multiple species. It is interest in this question that motivated this work. We will restrict attention to spherically symmetric solutions. Unfortunately, even in this restricted context, the estimates presented here are not strong enough to answer this question. It is hoped that this work will spur interest in this question and provide a starting point for further work.

This work relys on an identity from [11] (see also [14]) which is valid for the above system with multiple species of charge. Hence the estimates of this paper apply to the multiple species case. To streamline the presentation we will henceforth consider the case of a single species, thus we will drop \( \beta \) and the sums over \( \beta \) in the above equations. Similarly we will take \( m = 1 \) and \( e = 1 \).

Define the characteristics of the Vlasov equation, \( (X(s,t,x,v),V(s,t,x,v)) \), by

\[
\begin{aligned}
\frac{dX}{ds} &= V \\
X(t,t,x,v) &= x \\
\frac{dV}{ds} &= E(s,X) \\
V(t,t,x,v) &= v.
\end{aligned}
\]

Then

\[
f(s,X(s,t,x,v),V(s,t,x,v)) = f(t,x,v).
\]

Further introduce the notation

\[
R = \sqrt{X_1^2 + X_2^2 + X_3^2},
\]

\[
U = \sqrt{V_1^2 + V_2^2 + V_3^2}
\]

and \( A \in [0,\pi] \) defined by

\[
RU\cos A = X \cdot V.
\]

For spherically symmetric solutions

\[
|X \times V| = RU\sin(A)
\]

is constant in \( s \) (conservation of angular momentum). This follows since \( E \) is the gradient of a function of \( t \) and \( r \). Define

\[
\ell(x,v) = |x \times v| = rusin(\alpha).
\]

It will be assumed that the initial data, \( f_0 \), is continuously differentiable, compactly supported, and vanishes in a neighborhood of \( \ell(x,v) = 0 \). Since angular momentum is conserved, this implies that at later times the solution still vanishes in this same neighborhood of \( \ell(x,v) = 0 \).

**Theorem 1.1.** Assume that \( f_0 \geq 0 \) is continuously differentiable, compactly supported, and vanishes in a neighborhood of \( \ell(x,v) = 0 \). Then there is a positive constant, \( C_1 \), determined by the initial condition, such that

\[
|E(t,x)| \leq C_1 \min((1+t)^{-1},(1+t)^{-5/9}r^{-10/9},r^{-2})
\]

for all \( t \geq 1 \) and \( x \in \mathbb{R}^3 \).
Theorem 1.2. Assume that $f_0 \geq 0$ is continuously differentiable, compactly supported, and vanishes in a neighborhood of $\ell(x,v) = 0$. Then there are positive constants, $C_2$ and $C_3$, determined by the initial condition, such that $f(t,x,v) = 0$ for all $t \geq 1$ and $x \in \mathbb{R}^3$ with

$$|v| \geq C_2 t^{-3/10}$$

and

$$|x| \leq C_3 t^{7/10}.$$  

Consider the set on which (1.3) does not hold, that is that $|x| > C t^{7/10}$.

Then by the compact support of the initial condition and the conservation of angular momentum we have

$$C \geq \tau \sin(\alpha) \geq C t^{7/10} \sin(\alpha).$$

From [2] it is known, using symmetry, that the velocity support of $f(t,x,v)$ is uniformly bounded in all variables. Hence the set \( \{ v : f(t,x,v) \neq 0 \} \) is contained within a cylinder of length $C$ and radius $C t^{-7/10}$. It follows that

$$|\rho(t,x)| \leq C \pi (C t^{-7/10})^2 \leq C t^{-7/5}$$

on the set where (1.3) does not hold.

Some decay estimates are derived in [11]. The present work is based on these decay estimates and they are stated in the beginning of section 2. We mention some time decay results in low dimension. For the Vlasov Poisson system in one space dimension see [3], [4], and [17]. For the Vlasov Maxwell system in one space dimension see [6], [7], and [8].

For general background on the rigorous treatment of collisionless plasma the reader is referred to [5] and [16].

The letter, $C$, is used to denote a generic positive constant, which changes from line to line and is determined by the initial condition. When reference to a specific constant is desired, it is denoted by $C$ with a numerical subscript. For example $C_1$ appearing in Theorem 1.1. Also we choose $C_0$ so that

$$f_0(x,v) = 0$$

if $|x| > C_0$ or $|v| > C_0$. We will use the notation, for example,

$$\|\rho(t)\|_p = \left( \int |\rho(t,x)|^p \right)^{1/p}$$

for $p \geq 1$. The proof of Theorem 1.1 is in section 2 and the proof of Theorem 1.2 is in section 3.

2. Field estimates. From theorem 2 of [11] we have

$$\int |E(t,x)|^2 dx \leq C(1+t)^{-1}, \quad (2.4)$$

$$\int \int f |x-tv|^2 dvdx \leq C(1+t)^{-1}, \quad (2.5)$$

and

$$\|\rho(t)\|_{5/3} \leq C(1+t)^{-3/5} \quad (2.6)$$

for all $t \geq 0$. Letting

$$m(t,x) = m(t,r) = \int_0^r \rho(t,\lambda) 4 \pi \lambda^2 d\lambda$$

then

$$\int m(t,x) \rho(t,x) dx \leq C(1+t)^{-1}.$$
we have (using Hölder’s inequality) that
\[
|m(t,r)|^{9/5} \leq \frac{9}{5} \int_0^r |m(t,\lambda)|^{4/5} |\rho(t,\lambda)|^{4/5} \lambda^2 d\lambda
\]
\[
\leq C \int_0^r |E(t,\lambda)|^{4/5} (\lambda^{8/5} |\rho(t,\lambda)|) \lambda^2 d\lambda
\]
\[
\leq C r^{8/5} \int_0^r |E(t,\lambda)|^{4/5} |\rho(t,\lambda)| \lambda^2 d\lambda
\]
\[
\leq C r^{8/5} (\int_0^\infty |E(t,\lambda)|^2 \lambda^2 d\lambda)^{2/5} (\int_0^\infty |\rho(t,\lambda)|^{5/3} \lambda^2 d\lambda)^{3/5}.
\]
Hence, (2.4) and (2.6) yield
\[
|m(t,r)|^{9/5} \leq C r^{8/5} (1 + t)^{-1}
\]
and hence
\[
|E(t,r)| \leq C r^{-2} (r^{8/5} (1 + t)^{-1})^{5/9} = C r^{-10/9} (1 + t)^{-5/9}.
\]
(2.8)
We point out that (2.8) does not require the assumption that \(f_0\) vanish in a neighborhood of \(\ell(x,v) = 0\), but that this next estimate does. From (2.5) and for \(t \geq 1\) we have
\[
C(1 + t)^{-1} \geq \int \int f|v - x/t|^2 dvdx
\]
\[
= \int \int f(u^2 \sin^2(\alpha) + (uc(\alpha) - r/t)^2) dvdx
\]
\[
\geq \int \int fr^{-2}(rusin(\alpha))^2 dvdx
\]
\[
\geq C \int \int fr^{-2} dvdx
\]
where the lower bound on angular momentum was used in the last step above. Hence for any \(R > 0\) we have
\[
|E(t,R)| \leq CR^{-2} \int_{|x|<R} |\rho(t,x)| dx
\]
\[
\leq C \int_{|x|<R} \int fr^{-2} dvdx \leq C(1 + t)^{-1}.
\]
(2.10)
Combining (2.8) and (2.10) we have
\[
|E(t,r)| \leq C \min((1 + t)^{-1}, (1 + t)^{-5/9} r^{-10/9}).
\]
(2.11)
Finally by mass conservation
\[
|E(t,R)| \leq R^{-2} \int \int f(t,x,v) dvdx = CR^{-2}
\]
and (1.1) follows.
It is interesting to compare (2.11) with the following more standard estimate. By (2.6) and Hölder’s inequality we have for any \(R > 0\)
\[
|E(t,R)| \leq CR^{-2} \int_{|x|<R} |\rho(t,x)| dx
\]
\[
\leq CR^{-2} |\rho(t)|_{5/3} (\frac{4\pi}{3} R^3)^{2/5}
\]
\[
\leq C(1 + t)^{-3/5} R^{-4/5}
\]
(2.12)
and hence
\[
|E(t,r)| \leq C \min((1 + t)^{-1}, (1 + t)^{-3/5} r^{-4/5}).
\]
(2.13)
Note that for \(r > (1 + t)^{2/5}\), (2.11) is strictly better than (2.13).
3. **Escape estimates.** Let us consider \((t,x,v)\) with \(t \geq 1\). Assume that
\[
|v| \geq C_2 t^{-3/10}
\] (3.14)
and
\[
|x| \leq C_3 t^{7/10}
\] (3.15)
where
\[
C_3 = C_0 + 1
\] (3.16)
and \(C_2\) will be chosen later. Define
\[
N = |v|^{-1} v
\] (3.17)
and write
\[
X(s) = X(s,t,x,v), V(s) = V(s,t,x,v).
\]
First consider \(\bar{\tau} \in [\frac{1}{2} t, t]\). On this interval we use an estimate from [9]. Define
\[
\bar{\tau} = \inf \{ s \geq \tau : V \cdot N \text{ is of one sign on } [s,t] \}.
\]
Note that by (1.1) and since \(V \cdot N\) is of one sign on \([\bar{\tau}, t]\), we have
\[
|(v \cdot N)^2 - (V(\bar{\tau}) \cdot N)^2| = |2 \int_{\bar{\tau}}^{t} V(s) \cdot N E(s, X(s)) \cdot N ds|
\]
\[
\leq 2 C_1 \int_{\bar{\tau}}^{t} |V(s) \cdot N| \min(s^{-1}, s^{-5/9}|X(s)|^{-10/9}) ds
\]
\[
\leq 2 C_1 \int_{\bar{\tau}}^{t} |V(s) \cdot N| \min((\frac{1}{2} t)^{-1}, (\frac{1}{2} t)^{-5/9}|X(s) \cdot N|^{-10/9}) ds
\]
\[
= 2 C_1 \int_{\bar{\tau}}^{t} |V(s) \cdot N| \min((\frac{1}{2} t)^{-1}, (\frac{1}{2} t)^{-5/9}|X(s) \cdot N|^{-10/9}) ds
\]
\[
\leq 2 C_1 \int_{-\infty}^{\infty} \min((\frac{1}{2} t)^{-1}, (\frac{1}{2} t)^{-5/9}|\lambda|^{-10/9}) d\lambda
\]
\[
= 40 C_1 (\frac{1}{2} t)^{-3/5} \leq 80 C_1 t^{-3/5}.
\] (3.18)
Then, using (3.14) in (3.18) yields
\[
(V(\bar{\tau}) \cdot N)^2 \geq |v \cdot N|^2 - 80 C_1 t^{-3/5}
\]
\[
\geq (C_2 t^{-3/10})^2 - 80 C_1 t^{-3/5}
\] (3.19)
\[
= (C_2^2 - 80 C_1) t^{-3/5}.
\]
We will require
\[
C_2 \geq \sqrt{\frac{320 C_1}{3}}
\] (3.20)
so that (3.19) yields
\[
(V(\bar{\tau}) \cdot N)^2 \geq \left( \frac{1}{4} C_2^2 + \frac{3}{4} \frac{320 C_1}{3} - 80 C_1 \right) t^{-3/5} = \frac{1}{4} C_2^2 t^{-3/5}.
\]
By the definition of \(\bar{\tau}\) it follows that \(\bar{\tau} = \tau\) and that
\[
|V(\tau) \cdot N| \geq \frac{1}{2} C_2 t^{-3/10}.
\] (3.21)
To address the interval \([0, \frac{1}{2} t]\) define
\[
\tau_0 = \inf \{ \tau \in [0, t] : |V \cdot N| \geq \frac{1}{4} C_2 t^{-3/10} \text{ on } [\tau, t] \}.
\]
Note that by (3.21) we have $0 \leq \tau_0 \leq \frac{1}{2} t$. Consider any $\tau \in [\tau_0, \frac{1}{2} t]$. Then
\[
|X(\tau)| \geq |X(\tau) \cdot N|
\]
\[
\geq |\int_\tau^t V(s) \cdot N ds| - |x \cdot N|.
\] (3.22)

Since $V \cdot N$ can not change sign on the interval $[\tau, t]$ we have
\[
|X(\tau)| \geq \int_\tau^t |V(s) \cdot N| ds - |x|
\]
and the definition of $\tau_0$ yields
\[
|X(\tau)| \geq \int_\tau^t C_2 t^{-3/10} ds - |x|
\]
\[
\geq (\frac{1}{2} t)(\frac{1}{2} C_2 t^{-3/10}) - |x|
\]
\[
= \frac{1}{8} C_2 t^{7/10} - |x|.
\] (3.23)

Using (3.15) we have
\[
|X(\tau)| \geq \frac{1}{8} C_2 t^{7/10} - C_3 t^{7/10}.
\]

We will require
\[
C_2 \geq 16 C_3
\] (3.24)
so that
\[
|X(\tau)| \geq C_3 t^{7/10}
\] (3.25)
follows. Also, using (3.21) and (1.1), we have
\[
|V(\tau) \cdot N| \geq |V(\frac{1}{2} t) \cdot N| - |\int_\tau^{t/2} E(s,X(s)) \cdot N ds|
\]
\[
\geq \frac{1}{2} C_2 t^{-3/10} - \int_\tau^{t/2} C_1 |X(s)|^{-2} ds.
\] (3.26)

Using (3.25) in this yields
\[
|V(\tau) \cdot N| \geq \frac{1}{2} C_2 t^{-3/10} - \int_\tau^{t/2} C_1 (C_3 t^{7/10})^{-2} ds
\]
\[
\geq \frac{1}{2} C_2 t^{-3/10} - (\frac{1}{2} t) C_1 (C_3 t^{7/10})^{-2}
\]
\[
= \frac{1}{2} C_2 t^{-3/10} - \frac{1}{2} C_1 C_3^{-2} t^{-4/10}
\]
\[
\geq \frac{1}{2} (C_2 - C_1 C_3^{-2}) t^{-3/10},
\] (3.27)
since $t \geq 1$. We will require
\[
C_2 > 2C_1 C_3^{-2}
\] (3.28)
so that
\[
|V(\tau) \cdot N| > \frac{1}{4} C_2 t^{-3/10}.
\]

Now it follows from the definition of $\tau_0$ that $\tau_0 = 0$ and hence (3.25) and (3.16) yield
\[
|X(0)| \geq C_3 t^{7/10} \geq C_3 > C_0
\]
and therefore
\[
f(t,x,v) = f_0(0,X(0),V(0)) = 0.
\]

Taking
\[
C_2 = \max(\sqrt{\frac{320C_1}{3}},16 C_3,2C_1 C_3^{-2} + 1),
\]
we find that (3.20), (3.24), and (3.28) hold. Theorem 1.2 now follows.

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