Analytic solution of the Domain Wall non-equilibrium stationary state

Mario Collura,1 Andrea De Luca,1 and Jacopo Viti2
1The Rudolf Peierls Centre for Theoretical Physics, Oxford University, Oxford, OX1 3NP, United Kingdom.
2ECT & Instituto Internacional de Fisica, UFRN, Lagoa Nova 59078-970 Natal, Brazil
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We consider the out-of-equilibrium dynamics generated by joining two domains with arbitrary opposite magnetisations. We study the stationary state which emerges by the unitary evolution via the spin 1/2 XXZ Hamiltonian, in the gapless regime, where the system develops a stationary spin current. Using the generalized hydrodynamic approach, we present a simple formula for the space-time profile of the spin current and the magnetisation exact in the limit of large times. As a remarkable effect, we show that the stationary state has a strongly discontinuous dependence on the strength of interaction. This feature allows us to give a qualitative estimation for the transient behavior of the current which is compared with numerical simulations. Moreover, we analyse the behavior around the edge of the magnetisation profile and we argue that, unlike the XX free-fermionic point, interactions always prevent the emergence of a Tracy-Widom scaling.

Introduction.— Recent experimental developments with cold atoms [1] have given a new perspective to the study of non-equilibrium transport under coherent evolution. As an example, the measurement of conductances well beyond the regime of linear response has provided clear examples of the thermoelectric effect [2, 3]. The simplest protocol to induce an out-of-equilibrium behavior is the one of quantum quenches, in which the system is prepared in an equilibrium state of the initial Hamiltonian $H_0$, which is suddenly switched to $H$, thus inducing a non-trivial time-evolution [4–7]. Then, in describing the long-time dynamics, a fundamental role is played by the conserved quantities of $H$, i.e. the set of local (or quasi-local [8]) operators $\{Q_k\}$ satisfying $[Q_k, H] = 0$. As the system is isolated, the expectation value of these conserved quantities remains constant during the evolution. For homogeneous systems, these conditions are sufficient to predict the exact behavior of any local observable at long times: this is based on assuming equilibration to the generalized Gibbs ensemble (GGE), which results from the maximization of entropy given the constraints imposed by conserved quantities [9]. This principle suggests a dichotomy between generic models, for which a finite number of conserved quantities exist (i.e. the Hamiltonian and few others) and integrable ones, which instead present an infinite number of them [10]. Nowadays, the GGE scheme has been validated by several experiments [11–18] and theoretical works, employing free theories [19–21], integrability [22–27] and numerical methods [9, 28–30].

However, the study of transport requires considering more generic situations, where for instance, an initial spatial inhomogeneity is used to induce particle or energy flow. The simplest examples are local quenches where $H_0$ and $H$ differ only in a finite region of space, due, for instance, to the presence of a localized defect [31–37]. In particular, in the partitioned protocol, the initial density matrix is factorized into two halves, i.e. $\rho_0 = \rho_L \otimes \rho_R$, which are suddenly connected, inducing an out-of-equilibrium dynamics around the junction [38, 54]. This problem was well-understood in non-interacting theories [40–51] (with even mathematically rigorous treatments [52, 53]) and field theories [38, 54–64], even in higher dimensions [65–67]. However, for interacting models, only numerical approaches [68–76] and approximate results were available [77–81]. While at extremely long times $v_0 t \gg L$ (with $L$ the system length and $v_0$ the maximal velocity [82–84], one expects the system to become homogeneous, the most interesting regime is the one where $a \ll v_0 t \ll L$ (with $a$ the typical microscopic length). In this regime, conserved quantities are dynamically exchanged between different portions of space and therefore the simple knowledge of their initial value is not enough to characterize the local behavior of the steady state. Nevertheless, conserved quantities must still satisfy a continuity equation $\partial_t Q_k(x,t) + \partial_j J_k(x,t) = 0$, with $Q_k(x,t)$ the local density of $\dot{Q}_k$ and $J_k(x,t)$ the corresponding current. This condition was recently employed [85, 86] to derive a generalized hydrodynamic description (GHD) applicable to a large class of one-dimensional integrable models [87–97]. For the partitioned protocol, this description becomes exact and at large times, a Local quasi-stationary state (LQSS) emerges, in which local observables only depend on the scaling variable $\zeta = x/t$.

In this letter, we consider the XXZ spin 1/2 in the gapless regime, prepared in the partitioned initial state composed by two domains of arbitrary opposite magnetisations. We solve the hydrodynamic equations, obtaining simple analytic expressions for the magnetisation and spin current profiles. To the best of our knowledge, this represents a unique example of an out-of-equilibrium steady state of an interacting quantum system, admitting an explicit exact solution. Remarkably, the stationary spin current exhibits a strongly discontinuous behavior as a function of the anisotropy, as a result of the peculiar structure of bound states in the model.
The Model.— We consider the XXZ Hamiltonian

$$H = \sum_{i=-L}^{L} \left[ s^x_i s^x_{i+1} + s^y_i s^y_{i+1} + \Delta \left( s^z_i s^z_{i+1} - \frac{1}{4} \right) \right] , \quad (1)$$

where $L$ is the length of the chain and $s_i^q$ are spin-1/2 operators each acting on the local Hilbert space at site $i$. We focus on the gapless phase, thus specializing $\Delta = \cos(\gamma)$ with $\gamma = \pi Q/P$, where $Q$ and $P$ are two coprime integers with $1 \leq Q < P$. The ratio $Q/P$ admits a finite continued fraction representation $Q/P = [0; \nu_1, \nu_2, \ldots, \nu_k]$ with length $\delta$. For any finite $L$ such model is exactly solvable via Bethe-ansatz method [98–100]. In the thermodynamic limit (i.e., when $L \to \infty$ with fixed particle density), a generic thermodynamic state can be fully characterized by a set of functions $\{\rho_j(\lambda), \rho_j^h(\lambda)\}$ with $j \in \{1, \ldots, \ell\}$, $\ell = \sum_{j=1}^{\delta} \nu_j$ and $\lambda \in [-\infty, \infty]$. These functions, also known as “root densities”, describe different species of quasiparticles (different “strings”) and are solutions of a system of coupled nonlinear integral equations [98–100]. We can associate to each string $j$ a given parity $\nu_j \in \{-1, 1\}$, length $n_j \in \{1, \ldots, P-1\}$ and sign $\sigma_j \in \{-1, 1\}$. The filling factors are introduced as the ratios $\vartheta_j(\lambda) \equiv \rho_j(\lambda)/\rho_j^h(\lambda)$; we refer the reader to the Supplemental Material [101] for further details.

The Local Quasi Stationary State Redux.— Starting from a partitioned initial state $\varrho_0 = \varrho_L \otimes \varrho_R$, and unitarily evolving this state under the Hamiltonian (1), the general formal solution of the LQSS on the space-time coordinates $x, t$ reads [86]

$$\varrho_{j,\zeta}(\lambda) = \varrho^L_j(\lambda) \Theta(v_{j,\zeta}(\lambda) - \zeta) + \varrho^R_j(\lambda) \Theta(\zeta - v_{j,\zeta}(\lambda)), \quad (2)$$

in terms of the scaling variable $\zeta = x/t$, with $\Theta(z)$ being the Heaviside step function.

The functions $\varrho^L/R_j(\lambda)$ are the filling factors which describe the homogeneous stationary state emerging on the very far left/right part of the system. Eq. (2) formally identifies, for each type of quasiparticles, the related stationary distribution function. This solution admits a very simple geometrical interpretation: for any value of $j$ and $\lambda$, starting from $\zeta = -\infty$, the left bulk stationary description extends up to $\zeta^*_j(\lambda)$, such that $v_{j,\zeta^*_j}(\lambda) = \zeta^*$, thereafter, $\varrho_{j,\zeta_j}(\lambda)$ suddenly jumps to the right bulk stationary description. In practice this formal solution explicitly depends on the dressed velocity

$$v_{j,\zeta}(\lambda) = e_{j,\zeta}(\lambda)/p_{j,\zeta}(\lambda), \quad (3)$$

eigenvalues $q_j(\lambda)$, the dressing is obtained solving

$$q_j^*(\lambda) = q_j(\lambda) - \sum_k \int d\mu T_{j,k}(\lambda - \mu) \sigma_k \varrho(\mu) q_k(\mu) , \quad (4)$$

where we chose the convention to use calligraphic notation for bare quantities $q_j(\lambda)$. Introducing the function

$$a_n^{(v)}(\lambda) = \frac{v}{\cosh(2\lambda) - v \cos(\gamma n)} , \quad (5)$$

the kernel $T_{j,k}(\lambda)$ assumes the form

$$T_{j,k}(\lambda) = (1 - \delta_{n_j,n_k}) a_n^{(v)}(\lambda) + 2 e_{n_j,n_k}(\lambda) + 2 a_{n_j+n_k}(\lambda) \quad (6)$$

while the bare eigenvalues for the energy and the momentum derivative are

$$e_j(\lambda) = -\pi \sin(\gamma) a_j(\lambda) , \quad p_j(\lambda) = 2 \pi a_j(\lambda) \quad (7)$$

where we defined $a_j(\lambda) \equiv a_n^{(v)}(\lambda)$. Note that, as the dressing operation (4) is performed over the state $\varrho_{j}(\lambda) = \varrho_{j,\zeta}(\lambda)$, the solution for the LQSS has to be found self consistently in such a way that it keeps the form in Eq. (2) with its own dressed velocity (3). Therefore, in general, the dressed velocity will depend on the scaling variable $\zeta$, via the state $\varrho_{j,\zeta}(\lambda)$.

From the thermodynamic Bethe ansatz (TBA) description of the local quasi stationary state we can easily evaluate the expectation value of a generic charge density $q = Q/L$

$$\langle q \rangle_{\zeta} = \sum_k \int \frac{d\lambda}{2\pi} q_k(\lambda) \sigma_k p_{k,\zeta}(\lambda) \varrho_{k,\zeta}(\lambda) , \quad (8)$$

and the associated current density

$$\langle j_q \rangle_{\zeta} = \sum_k \int \frac{d\lambda}{2\pi} q_k(\lambda) v_{k,\zeta}(\lambda) \sigma_k p_{k,\zeta}(\lambda) \varrho_{k,\zeta}(\lambda) . \quad (9)$$

Opposite magnetisation domains.— The system is initially prepared into two halves with infinite temperature and opposite values of magnetic field $h$ in the $\hat{z}$ direction, namely

$$\varrho_0 \equiv \varrho_L(h) \otimes \varrho_R(-h) = e^{2h S_z^L}/Z_L \otimes e^{-2h S_z^R}/Z_R , \quad (10)$$

where $S_z^{L/R} = \sum_{i \in L/R} s_i^z$ is the $\hat{z}$-component of the total spin in the left/right part of the system.

A generic thermodynamic state $\varrho_{L/R}(h)$ is stationary under the unitary evolution induced by its own XXZ Hamiltonian. It admits a TBA description in terms of constant filling factors $\varrho^{(h)}_j$ (i.e. independent of the rapidity $\lambda$), which satisfy the major properties (see [101]
for the complete definition of $\vartheta_j^{(h)}$, $\forall j \in \{1, \ldots, \ell\}$

$$\vartheta_j^{(h)} = \vartheta_j^{(-h)} \quad j < \ell - 1, \quad \vartheta_{\ell-1}^{(h)} = 1 - \vartheta_{\ell}^{(-h)}.$$  \hfill (11)

In the limit $h \to \infty$ the state $\varphi_0$ reduces to the Domain Wall (DW) $|\uparrow\rangle \otimes |\downarrow\rangle$ product state, with $\vartheta_j^{(0)} = 0$ and $\vartheta_j^{(h)} = \delta_{j,\ell} + \delta_{j,\ell-1}$, for $j = 1, \ldots, \ell$.

The full analytic solution.— Now if we consider the protocol generated attaching two states with $h$ (left) and $-h$ (right), the $\zeta \to \pm \infty$ boundary conditions in Eq. (2) read $\vartheta_j^{(h)}(\zeta) = \vartheta_j^{(-h)}(\zeta)$. Thanks to the symmetries (11) of the boundary filling factors, when constructing the LQSS, only the filling factors $\vartheta_{j,\zeta}(\lambda)$ corresponding to the last two strings $j = \ell - 1$ and $j = \ell$ may depend on $\zeta$. In order to fix them, we need to determine the dressed velocities. Using that $T_{\ell,k}(\lambda) = -T_{\ell-1,k}(\lambda)$ and $a_{\ell}(\lambda) = -a_{\ell-1}(\lambda)$, we have

$$p'_{\ell,\zeta}(\lambda) = -p'_{\ell-1,\zeta}(\lambda), \quad e'_{\ell,\zeta}(\lambda) = -e'_{\ell-1,\zeta}(\lambda),$$  \hfill (12)

which implies $v_{\ell,\zeta}(\lambda) = v_{\ell-1,\zeta}(\lambda)$. As the last two strings always have opposite sign, i.e. $\sigma_{\ell-1} = -\sigma_{\ell}$, we can reduce Eq. (4) for the dressed momentum derivative to

$$p_j^{\prime}(\lambda) = p_j^{\prime}(\lambda) - \sum_{k < j} \sigma_k \vartheta_j^{(h)} \int d\mu T_{j,k}(\lambda - \mu) p_k^{\prime}(\mu)$$

$$- \sigma_{\ell} \int d\mu [\vartheta_{\ell-1}(\mu) - \vartheta_{\ell-1}(\mu)] T_{\ell,\ell}(\lambda - \mu) p_{\ell}(\mu),$$

which does not depend on the space-time scaling variable $\zeta$, since $\vartheta_{\ell,\zeta}(\mu) - \vartheta_{\ell-1,\zeta}(\mu) = \vartheta_{\ell}^{(h)} - \vartheta_{\ell-1}^{(h)}$. From now on we discard the subscript $\zeta$ whenever it will be superfluous. As a consequence of the last result, we can calculate the dressed momentum derivative solving

$$p_j^{\prime}(\lambda) = p_j^{\prime}(\lambda) - \sum_{k} \sigma_k \vartheta_j^{(h)} \int d\mu T_{j,k}(\lambda - \mu) p_k^{\prime}(\mu),$$  \hfill (13)

which correspond to evaluate the dressing on the left thermodynamic state $\varphi_L(h)$. Note that the dressing can be equivalently evaluated in the right part of the system, as it is even in sign of the magnetic field. Eq. (13) can be solved in Fourier transform, reducing to an algebraic system of linear equations. For the last two strings the dressing operation reduces to a simple rescaling of the bare quantities, i.e.

$$p_j^{\prime}(\lambda) = \mathcal{R}(h) p_j^{\prime}(\lambda), \quad p_{\ell-1}(\lambda) = \mathcal{R}(h) p_{\ell-1}^{\prime}(\lambda),$$  \hfill (14)

with the following rescaling factor

$$\mathcal{R}(h) \equiv \frac{\tanh(h) \sinh((n_\ell + n_{\ell-1})h)}{2 \sinh(n_\ell h) \sinh(n_{\ell-1} h)}$$  \hfill (15)

where in the last line we used the relation $n_\ell + n_{\ell-1} = P$, and as expected $\mathcal{R}(-h) = \mathcal{R}(h)$. As a consequence, the quasiparticle velocity of the last two strings is not changed by the dressing operation. It can therefore be expressed in terms of the undressed momentum as follow

$$v_{\ell} = \frac{\nu_{\ell} \sin(\gamma)}{\sin(n_{\ell} h)} \sinh(p_{\ell}) = \zeta_0 \sin(\sigma_{\ell} p_{\ell}),$$

with $\zeta_0 = \sin(\gamma)/\sin(n_{\ell}/P)$ and $\sigma_{\ell} p_{\ell}(\lambda)$ a strictly increasing function in $[\pi/P, \pi/P]$. Therefore, the velocity $v_{\ell}(\lambda) \in [-\sin(\gamma), \sin(\gamma)]$. The explicit form of the LQSS for the last two strings thus reads (for $j \in \{\ell - 1, \ell\}$)

$$\vartheta_j^{\prime}(\lambda) = \vartheta_j^{(h)} \Theta(\sigma_j p_j - p_j^*) + \vartheta_j^{(-h)} \Theta(p_j^* - \sigma_j p_j),$$

where $p_j^* \equiv \arcsin([\zeta/\zeta_0]$. From this, using $\text{Tr}[s_j^+ \varphi_L(h)] = \tanh(h)/2$ and $\mathcal{R}(h) = \tanh(h)/(1 - \vartheta_{\ell}^{(h)} - \vartheta_{\ell-1}^{(h)})$, we can easily evaluate the magnetisation and spin current profile inside the light-cone $\zeta \in [-\sin(\gamma), \sin(\gamma)]$,

$$\langle s_j^+ \rangle = \frac{\tanh(h)}{2\pi/P} \arcsin \left( \frac{\zeta_0}{\zeta} \right),$$

$$\langle j_{s_j^+} \rangle = \frac{\tanh(h)}{2\pi/P} \zeta_0 \left[ \frac{1 - \zeta_0^2}{\zeta_0^2} - \cos \left( \frac{\pi}{P} \right) \right],$$

which are simply related one another via the continuity equation $\vartheta_{\ell,\zeta}(\mathcal{R}(s_j^+)) = \partial_t \langle j_{s_j^+} \rangle$. Interestingly, the way in which the magnetic field $h$ enters in the stationary solutions is almost trivial: indeed, Eqs. (18) coincide with the DW solutions ($h \to \infty$) simply rescaled by the factor $\tanh(h)$. Moreover in this limit $\mathcal{R}(h) \to 1$, showing that for the DW initial state, no dressing occurs.

The anisotropy dependence.— It is interesting to investigate how the interaction strength $\Delta$ affects the stationary state. Both current and magnetisation profiles have an explicit dependence on the denominator $P$ of $\pi/\gamma$: as one can pick two arbitrarily close values $\gamma = \pi Q/P$ and $\gamma = \pi Q/4 P$, with very different values of $P$ and $P$, the magnetisation and current profiles exhibit jumps in correspondence of any rational $\gamma/\pi$, corresponding to a dense subset of $\Delta \in [-1, 1]$. Nevertheless, the continuation to irrational values is well defined taking $P \to \infty$ with $\gamma$ finite. In such limit, the current profile reduces to (for $\gamma/\pi \in \mathbb{R}/\mathbb{Q}$)

$$\langle j_{s_j^+} \rangle = \frac{\tanh(h)}{4} \left[ \sin(\gamma) - \frac{\zeta_0^2}{\sin(\gamma)} \right],$$

and the magnetisation behaves linearly in $\zeta$. For any irrational number $\gamma/\pi$, although the large time limit will be characterised by the stationary values in (19), we expect the relaxation dynamics to spend long times on the rational approximations of such an irrational, i.e. the truncated continued fractions $[0; \nu_1, \ldots, \nu_n]$. The ideal case to verify this hypothesis corresponds to all $\nu_k = 1$, i.e. $\gamma = \pi/\varphi$, with $\varphi \equiv (1 + \sqrt{5})/2$, the golden ratio. Its $n$-th order rational approximation is given
by $F_n/F_{n+1}$, where $F_n$ are the Fibonacci numbers and $1/\varphi = \lim_{n \to \infty} F_n/F_{n+1}$. In Fig. 1, the numerical data for the spin current clearly oscillate in time between different stationary values associated to different orders of approximation of the golden ratio. The curve remains close to the $n$-th rational approximation for an exponentially long time, $t \propto F_n^2 \sim \varphi^{2n}$.

Remarkably, our exact result definitively gives analytical confirmation to the tightness of the bound in [102] for the spin Drude weight $D_{ss}$, numerically corroborated in [94]. In the linear response regime indeed, the spin Drude weight gives the magnitude of the singular part of the spin conductivity, therefore signaling ballistic transport [103–108]. Following [94], we integrate the current (18b) over $\zeta$ to obtain for $\beta \to 0$:

$$
\frac{16}{\beta} D_{ss} = \zeta_0^2 \left[ 1 - \frac{\sin(2\pi/P)}{2\pi/P} \right].
$$

This result exactly coincides with the lower bound obtained in [102], confirming that it is in fact saturated.

Absence of Tracy-Widom distribution and diffusion.---

The profiles in (18) exhibit a smooth dependence on the scaling variable $\zeta$, apart from the edges of the light-cone, i.e. $\zeta = \pm \sin(\gamma)$, where the derivatives are non-analytic. In particular, one has

$$
\frac{\partial_t \langle j_{ss} \rangle_{\zeta = \sin(\gamma)}}{\tanh(h)} = -\frac{\tan(\pi/P)}{2\pi/P}
$$

which remains finite for any value of $\gamma$ but $\pi/2$, i.e. the free-fermion point, where it diverges indicating a square root singularity. The absence of such a singularity in the magnetisation and current profiles for $|\zeta| = \sin(\gamma)$ is a strong hint that the edges of the front cannot be described by a Tracy-Widom scaling [109, 110] as soon as $\Delta \neq 0$ and the model is interacting. Given the absence of dressing for the DW initial conditions, it is tempting to re-interpret Eqs. (18) in terms of free fermions. In the simplest case of principal roots of unity, i.e. $\gamma = \pi/P$, the magnetisation profile (18a) can be seen as the density of the XXZ spin chain requires extremely large simulation times. Nevertheless, the dependence on the scaling variable $X$ of the magnetisation profiles at the edges is visible in the Fig. 2 and rules out for $\Delta \neq 0$ the $t^{1/3}$ scaling characteristic of the Tracy-Widom behavior. Finally, we observe that, within this picture, in the isotropic limit $\gamma \to 0$, i.e. $\Delta \to 1^-$, the magnetisation profile is expected to be a scaling function of the ratio $\varphi/\sqrt{X}$, for all values of $h$, thus signaling a diffusive behavior [68, 112–114]. Similar conclusions are suggested by the return probability, indicating diffusive scaling but with slow corrections [115], providing a possible justification for the anomalous scaling observed in [116].

Conclusions.--- We considered the emblematic nonequilibrium protocol generated by joining two domains with opposite magnetisation. Exploiting the properties of the XXZ spin-1/2 chain, we were able to find a full analytic solution for the LQSS. We consequently obtained closed expression for both the magnetisation and spin current stationary profiles. Interestingly, our analytic results show a strongly discontinuous behavior as a function of the interaction $\Delta$, confirming the predictions obtained via the Drude weight. Moreover, for the DW initial case we took advantage of a free-fermion analogy to fully characterise the scaling of the stationary profiles at the edges of the light-cone. Such analysis has been supported by numerical DMRG simulations and, it gave evidence of the absence of a Tracy-Widom scaling a part for the noninteracting point $\Delta = 0$.

Our simple solution is a promising framework to de-
derive a continuous field theory description of the LQSS, thus extending the results of [47, 63, 64] in the presence of interactions. We are extremely grateful to J. Dubail and J-M. Stéphan for many stimulating conversations. M.C. acknowledges support by the Marie Sklodowska-Curie Grant No. 701221 NET4IQ. This work was supported by EPSRC Quantum Matter in and out of Equilibrium Ref. EP/N01930X/1 (A.D.L.).

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[1] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[2] J.-P. Brantut et al. Science 337, 1069 (2012).
[3] J.-P. Brantut et al. Science 342, 713 (2013).
[4] P. Calabrese and J. Cardy, Phys. Rev. Lett. 96, 136801 (2006);
P. Calabrese and J. Cardy, J. Stat. Mech. (2007) P06008.
[5] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. 83, 863 (2011).
[6] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, Rev. Mod. Phys. 83, 1405 (2011).
[7] P. Calabrese, F. H. L. Essler, and G. Mussardo, J. Stat. Mech. (2016) 64001.
[8] E. Ilievski, M. Medenjak, T. Prosen and L. Zadnik, J. Stat. Mech. (2016) 064008.
[9] L. Vidmar and M. Rigol, J. Stat. Mech. (2016) 64007.
[10] R. J. Baxter, Exactly Solvable Models in Statistical Mechanics, Academic Press (1982); B. Sutherland, Beautiful Models World Scientific (2004).
[11] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature 440, 900 (2006).
[12] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, Nature 449, 324 (2007).
[13] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, Science 337, 1318 (2012).
[14] T. Fukuhara, P. Schauß, M. Endres, S. Hild, M. Chemoux, I. Bloch, and C. Gross, Nature 502, 76 (2013).
[15] T. Langen, R. Geiger, M. Kuhnert, B. Rauer, and J. Schmiedmayer, Nature Phys. 9 640 (2013).
[16] R. Geiger, T. Langen, I. E. Mazets, and J. Schmiedmayer, New J. Phys. 16, 053034 (2014).
[17] T. Langen, S. Erne, R. Geiger, B. Rauer, T. Schweigler, M. Kuhnert, W. Rohringer, I. E. Mazets, T. Gasenzer, and J. Schmiedmayer, Science 348, 207 (2015).
[18] T. Langen, T. Gasenzer, and J. Schmiedmayer, J. Stat. Mech. (2016) 64009.
[19] P. Calabrese, F. Essler and M. Fagotti, Phys. Rev. Lett. 106, 227203 (2011).
[20] P. Calabrese, F. Essler and M. Fagotti, J. Stat. Mech. (2012) 07016; J. Stat. Mech. (2012) 07022.
[21] F. H. L. Essler and M. Fagotti, J. Stat. Mech. (2016) 64002.
[22] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Phys. Rev. Lett. 115, 157201 (2015).
[23] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, T. Prosen, Phys. Rev. Lett. 115, 157201 (2015).
[24] E. Ilievski, E. Quinn, J. De Nardis, M. Brockmann, J. Stat. Mech. (2016) 063101.
[25] L. Piroli, E. Vernier, and P. Calabrese, Phys. Rev. B 94, 54313 (2016).
[26] A. De Luca and G. Mussardo, J. Stat. Mech. (2016) 064011.
[27] M. Mestyán, B. Bertini, L. Piroli, and P. Calabrese, arXiv:1705.00851 (2017).
[28] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. 98, 050405 (2007).
[29] G.P. Brandino et al., Phys. Rev. B 85, 214435 (2012).
[30] L. Vidmar, D. Iyer, and M. Rigol, Phys. Rev. X 7, 21012 (2017).
[31] P. Calabrese, and J. Cardy, J. Stat. Mech. (2007) P10004.
[32] J.-M. Stéphan, J. Dubail, J. Stat. Mech. (2011) P08019.
[33] A. De Luca, Phys. Rev. B 90, 081403 (2014).
[34] M. Fagotti, J. Phys. A: Math. Theor. 50 034005 (2017).
[35] B. Bertini and M. Fagotti, Phys. Rev. Lett. 117, 130402 (2016).
[36] A. Bastianello and A. De Luca, arXiv:1705.09270 (2017).
[37] A. L. de Paula, H. Bragana, R. G. Pereira, R. C. Drumond, and M. C. O. Aguiar, Phys. Rev. B 95, 45125 (2017).
[38] D. Bernard and B. Doyon, J. Phys. A: Math. Theor. 45, 362001 (2012).
[39] D. Bernard and B. Doyon, Ann. Henri Poincaré 16, 113
Meisner, Phys. Rev. B 84, 205115 (2011).
[108] M. Žnidarič, Phys. Rev. Lett. 106, 220601 (2011).
[109] H. Widom and C. Tracy, Phys. Lett. B, 305, 115-118 (1994).
[110] V. Eisler and Z. Rácz, Phys. Rev. Lett., 110 060602, (2013).
[111] R. Wong, Asymptotics Approximation of Integrals, SIAM Classics in Applied Mathematics.
[112] K. Fabricius and B. M. McCoy, Phys. Rev. B 57 8340, (1998).
[113] R. Steinigeweg and W. Brenig, Phys. Rev. Lett. 107, 250602 (2011).
[114] M. Medenjak, C. Karrasch, T. Prosen, arXiv:1702.04677 (2017).
[115] J-M. Stéphan, to appear (2017).
[116] M. Ljubotina, M. Žnidarič, T. Prosen, arXiv:1702.04210 (2017).
[117] V. Alba and P. Calabrese, arXiv:1608.00614 (2016); V. Alba arXiv:1706.00020 (2017)
STRING PROPERTIES

Here we summarize the general rule to determine the parity $\nu_j$, length $n_j$ and sign $\sigma_j$ of a specific string. Let us recall that we defined $\Delta = \cos(\gamma)$ with $\gamma = \pi Q/P = [0; \nu_1, \nu_2, \ldots, \nu_8]$. Following Ref. 98, let’s start by introducing the two series of numbers $\{y_{-1}, y_0, \ldots, y_\delta\}$ and $\{m_0, m_1, \ldots, m_\delta\}$,

\begin{align*}
y_i &= \nu_i y_{i-1} + y_{i-2}, \quad y_0 = 1, \quad y_{-1} = 0, \\
m_i &= \sum_{j=1}^i \nu_j, \quad m_0 = 0,
\end{align*}

(S1) (S2)
in terms of which we have the following relation for the length $n_j$:

\begin{equation}
n_j = y_{i-1} + (j - m_i)y_i \quad \text{for} \quad m_i \leq j < m_{i+1},
\end{equation}

(S3)

defining the parity $\nu_j$:

\begin{equation}
\nu_{m_1} = -1, \quad \nu_j = (-1)^{\left\lfloor (n_j - 1)\frac{P}{P} \right\rfloor} \quad \text{for} \quad j \neq m_1,
\end{equation}

(S4)

and the sign $\sigma_j$:

\begin{equation}
\sigma_j = (-1)^i \quad \text{for} \quad m_i \leq j < m_{i+1}.
\end{equation}

(S5)

Finally, let us collect some useful relations involving the last strings (where we used the definition $\ell = m_\delta$ and the fact that $y_\delta = P$):

\begin{equation}
n_\ell = y_{\delta-1}, \quad n_\ell + n_{\ell-1} = P, \quad n_{\ell-1} - n_\ell = n_{\ell-2}, \quad \sigma_{\ell-1} = -\sigma_\ell, \quad \sigma_\ell \sin(\pi/P) = v_\ell \sin(n_\ell \gamma).
\end{equation}

(S6)

FILLING FACTORS FOR INFINITE TEMPERATURE AND FINITE MAGNETIC FIELD STATE

The thermodynamic state

\begin{equation}
\varrho(h) = \frac{\exp(2hS^z)}{Z}
\end{equation}

(S7)

admits a thermodynamic Bethe ansatz description in terms of the following filling factors

\begin{align*}
\varrho_j^{(h)} &= \left[ \frac{\sinh(y_j h)}{\sinh((n_j + y_j)h)} \right]^2 \quad \text{for} \quad m_i \leq j < m_{i+1} \quad \text{and} \quad j < \ell - 1, \\
\varrho_{\ell-1}^{(h)} &= \frac{1}{1 + \kappa e^{h P}}, \quad \varrho_\ell^{(h)} = \frac{\kappa}{\kappa + e^{h P}}, \quad \kappa \equiv \frac{\sinh(n_{\ell-1} h)}{\sinh(n_\ell h)}.
\end{align*}

(S8) (S9)

Notice that, in the limit $h \to \infty$ we get the trivial TBA description of the reference state $|\uparrow\rangle \equiv |\uparrow \cdots \uparrow\rangle$, which obviously reads

\begin{equation}
\varrho_j^{(\uparrow)} = 0, \quad \forall \ j \in \{1, \ldots, \ell\}.
\end{equation}

(S10)

Otherwise, in the opposite limit $h \to -\infty$, we obtain the representation of the completely full state $|\downarrow\rangle \equiv |\downarrow \cdots \downarrow\rangle$,

\begin{equation}
\varrho_j^{(\downarrow)} = \delta_{j,\ell} + \delta_{j,\ell-1}, \quad \forall \ j \in \{1, \ldots, \ell\}.
\end{equation}

(S11)
FIG. S1. **Left.** Hydrodynamic interpretation of the interacting DW quench with anisotropy $\Delta = \cos(\gamma)$ and $\gamma = \frac{\pi}{7}$, $\ell = 2, 3, \ldots$. The last and second-to-last Bethe Ansatz strings are effectively two non-interacting fermions with momenta that cannot occupy two symmetric intervals centered around $\pm \pi/2$ and of width $\pi/2 - \gamma$. **Right.** The diagonal part of the error function kernel $K_b$ plotted against the scaling variable $X = \frac{x - t \sin(\gamma)}{[t \cos(\gamma)]^{1/2}}$. Notice that for large and negative $X$ the error function kernel behaves linearly, alike the magnetisation and current profiles in the interacting DW quench.

**EDGE BEHAVIOR AT $\Delta$ ROOT OF UNITY**

We consider the case $Q = 1$ and $P \equiv \ell$, i.e. $\gamma = \pi/\ell$, and the corresponding values of $\Delta$ are called roots of unity. Moreover we focus on the limit $h \to \infty$ that describes the Domain Wall (DW) initial state. As discussed in the previous section the fillings for the initial states $| \uparrow \rangle$ and $| \downarrow \rangle$ are trivial, in particular $\vartheta_j(\lambda) = 0$ and $\vartheta_j(\lambda) = \delta_{j,\ell-1} + \delta_{j,\ell}$. The index $j$ takes values $1, \ldots, \ell$ according to the string content when $\Delta$ is a root of unity. The LQSS is described by the fillings $\vartheta_{j,\zeta}(\lambda) = \vartheta_j(\lambda)\Theta(-\varphi_j(\lambda) + \zeta)$, being $\varphi_j,\zeta$ the dressed velocity (3). Now Eq. (4) in the main text implies that the only non-trivial fillings in the LQSS are the ones of the last two strings. Moreover from the properties of the kernels it also follows that the momentum and the energy derivatives do not dress for $j = \ell - 1, \ell$, as of course it is implied by the finite-$h$ solution in (14). The magnetisation profile is given by

$$\langle s^z \rangle_c = \frac{1}{2} - \sum_{j=\ell-1,\ell} n_j \int_{-\infty}^{\infty} d\lambda \rho_j(\lambda), \quad (S12)$$

that can be rewritten, passing to the bare momentum variable of the last string $p_\ell(\lambda) = -2 \arctan[\tan(\gamma/2) \tanh(\lambda)]$, as

$$\langle s^z \rangle_c = -\frac{1}{2} + \frac{1}{2\gamma} \int_{-\gamma}^{\gamma} dp \Theta(\hat{v}(p) - \zeta). \quad (S13)$$

It is important to observe that the integration variable in (S13) is bounded in the interval $p \in [-\gamma, \gamma]$, because of the analytic properties of $p_\ell(\lambda)$ for real $\lambda$. The function $\hat{v}(p)$ is nothing but the free-fermion velocity $\hat{v}(p) = \sin(p)$. The determination of the magnetisation profile in the DW quench seems formally analogous to the determination of the density profile in a free-fermion problem where two strips of width $\pi/2 - \gamma$ centered around the the points $p = \pm \pi/2$ are removed from the Brillouin zone. This is illustrated in Fig. S1 on the left. The two allowed bands for the fermions correspond to the possible values of the bare momenta for the last and second-to-last string in the Bethe Ansatz solution. The contribution of the two bands are identical when calculating the fermion density and we can focus only on one of the two.

We therefore analyse the DW quench in a free fermion problem with dispersion relation $\varepsilon(p) = -\cos(p)$ and momenta restricted to $[-\gamma, \gamma]$. Notice that we need however to correctly normalise the fermion density $\langle \rho \rangle_c$, since in a bona fide free-fermion model we would have for $\zeta < -1$, $\langle \rho \rangle_c = \frac{2\zeta}{\pi}$, whereas in the DW quench $\langle \rho \rangle_c = 1$ for $\zeta < -1$. 


Letting aside this issue we consider the fermion propagator \[47\]

\[
\mathcal{G}_{x,y}(t) = \frac{2\pi}{\gamma} \int_{-\gamma}^{\gamma} \frac{dk}{2\pi} \int_{-\gamma}^{\gamma} \frac{dq}{2\pi} \frac{e^{-i(t \cos k + x \cos k - t \cos q - yq)}}{1 - e^{i(q - k + i0)}};
\]

(S14)

the density profile obtained from such an integral is \( \langle \rho \rangle_\zeta = \langle s^z \rangle_\zeta + 1/2 \), being \( \langle s^z \rangle_\zeta \) as in (S12). The stationary points of the integral satisfy, for instance in the variable \( q \), the equation \( \ddot{v}(q) = \zeta \) and therefore the light-cone boundaries are obtained from the condition \( \zeta_\pm = \pm \max_{q \in [-\gamma, \gamma]} \ddot{v}(q) \). If \( \gamma > \pi/2 \) the light-cone boundaries are at \( \zeta_\pm = \pm 1 \) whereas if \( \gamma < \pi/2 \) they are located at \( \zeta_\pm = \pm \sin(\gamma) \). These two cases lead to different scalings for the fermion propagator (S14). Indeed for \( \gamma > \pi/2 \), the uniform asymptotic in a neighborhood of \( \zeta \to \zeta_\pm \) is obtained by a cubic polynomial approximation of the phase, due to the coalescence of two stationary points [47]. The result of the stationary phase approximation shows that the fermion propagator is proportional to the Airy kernel [109]. However when \( \gamma < \pi/2 \), as in the fermion model associated to the interacting DW quench, the change in asymptotic for \( \zeta > |\zeta_\pm| \) is consequence of a stationary point leaving the domain of integration. As discussed for instance in [111], a uniform asymptotics is obtained through a quadratic approximation of the phase. For instance, in a neighborhood of \( \zeta_+ = \sin(\gamma) \), one gets

\[
\mathcal{G}_{x,y}(t) = \frac{2\pi e^{-\gamma(s-y)}}{\gamma [t \cos(\gamma)]^{1/2}} \mathfrak{R}_b(X,Y) + o(t^{1/2})
\]

(S15)

where we defined the error function kernel \( \mathfrak{R}_b \)

\[
\mathfrak{R}_b(X,Y) = \int_{0}^{\infty} \frac{dK}{2\pi} \int_{0}^{\infty} \frac{dQ}{2\pi} \frac{e^{iKX+i\frac{Q^2}{2}-iQY-i\frac{Q^2}{2}}}{i(Q-K-i0)};
\]

(S16)

and the scaling variable \( X = \frac{x-t\sin(\gamma)}{[t \cos(\gamma)]^{1/2}} \). For convenience we also introduce here the function

\[
U(X) = \int_{0}^{\infty} \frac{dQ}{2\pi} e^{i(QY+i\frac{Q^2}{2})},
\]

(S17)

in terms of which the error function kernel satisfies \( -(\partial_X + \partial_Y) \mathfrak{R}_b(X,Y) = U(X)U(Y) \) and \( \mathfrak{R}_b(X,Y) = \mathfrak{R}_b(Y,X) \). It follows therefore that \( \mathfrak{R}_b(X,X) \) is real and monotonically decreasing. We can determine exactly the diagonal part of the kernel integrating the differential equation \( -\frac{d}{dx} \mathfrak{R}_b(X,X) = |U(X)|^2 \) with the boundary condition \( \mathfrak{R}_b(X,X) = 0 \) for \( X \to \infty \); one finds

\[
\mathfrak{R}_b(X,X) = -X|U(X)|^2 + \frac{\text{Im}[U(X)]}{\pi}.
\]

(S18)

Notice that if we expand for large and negative \( X \) the diagonal part of the kernel we obtain the asymptotic expansion \( \mathfrak{R}_b(X) = -\frac{X}{2\pi} + O(1/X) \), that is we recover the expected linear behaviour near the light-cone of the density profile from (S12) (see also Fig. S1 on the right)

\[
\langle \rho \rangle_\zeta \simeq -\frac{1}{\gamma \cos(\gamma)}[\zeta - \sin(\gamma)].
\]

(S19)

Expanding for large and negative \( X \) the Airy kernel we would find instead at leading order \( \frac{1}{\pi} \sqrt{-X} \); namely a square root singularity in the fermion density near the edge of the light-cone. We remind that this case the correct scaling variable is however \( X = \frac{x-t}{(t/2)^{1/2}} \).