The non-existence of a $[[13, 5, 4]]$-quantum stabilizer code

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Abstract
We solve one of the oldest problems in the theory of quantum stabilizer codes by proving the non-existence of quantum $[[13, 5, 4]]$-codes.

1 Introduction

After the determination of the parameter spectrum of additive quantum codes of distance 3 (see [2]) the oldest open existence problem for quantum stabilizer codes concerns the parameters $[[13, 5, 4]]$. We give a negative answer:

Theorem 1. There is no $[[13, 5, 4]]$-quantum stabilizer code.

The reduction of the problem of quantum error-correction to codes in symplectic geometry essentially is in [7]. For a geometric approach see also [6]. We use the following definitions:
Definition 1. Let \( k \) be such that \( 2k \) is a positive integer. An additive quaternary \([n,k]\)-code \( C \) (length \( n \), dimension \( k \)) is a \(2k\)-dimensional subspace of \( \mathbb{F}_2^{2n} \), where the coordinates come in pairs of two. We view the codewords as \( n \)-tuples where the coordinate entries are elements of \( \mathbb{F}_2^2 \).

A generator matrix of \( C \) is a binary \((2k,2n)\)-matrix whose rows form a basis of the binary vector space \( C \).

In the case of quantum stabilizer codes we view the ambient space \( \mathbb{F}_2^{2n} \) as a binary symplectic space, where each of the \( n \) parameter sections corresponds to a hyperbolic plane, equivalently a 2-dimensional symplectic space. Each codeword is therefore a vector in the \( 2n \)-dimensional symplectic geometry over \( \mathbb{F}_2 \).

Definition 2. A quaternary quantum stabilizer code is an additive quaternary code \( C \) which is contained in its dual, where duality is with respect to the symplectic form.

Describe \( C \) by a generator matrix \( M \). Each of the \( n \) coordinate sections contains 2 columns which we view as points in binary projective space. The geometric description of the quantum code is in terms of the system of \( n \) lines (the codelines) generated by those \( n \) pairs of points.

Definition 3. Let \( C \) be a quaternary additive code of length \( n \), with generator matrix \( M \). The strength of \( C \) is the largest number \( t \) such that any \( t \) codelines are in general position.

Observe that the strength \( t(C) \) is one less than the dual distance.

Definition 4. An \([n,m,d]\)-code \( C \) where \( m > 0 \) is a quaternary quantum stabilizer code of binary dimension \( n - m \) satisfying the following: any codeword of \( C^\perp \) having weight at most \( d - 1 \) is in \( C \).

The code is pure if \( C^\perp \) does not contain codewords of weight \( \leq d - 1 \), equivalently if \( C \) has strength \( t \geq d - 1 \).

An \([n,0,d]\)-code \( C \) is a self-dual quaternary quantum stabilizer code of strength \( t = d - 1 \).

The optimal parameters of quantum stabilizer codes of length \( \leq 13 \) are known, with the sole exception of parameters \([13,5,4]\) (see the database in [9]). The remainder of the paper is dedicated to a proof of Theorem 1. Assume \( C \) is a \([13,5,4]\)-quantum code. In the next section we show that \( C \) is necessarily pure.
2 The purity of the code

Proposition 1. Let $C$ be a $[[13, 5, 4]]$-quantum code. Then $C$ is pure.

In general the geometric objects defined by the column pairs of a generator matrix (which we called codelines) may be lines, points or even the empty set (if the corresponding pair of columns has all entries = 0). The following basic fact follows from the definition:

Lemma 1. Whenever some $\leq d-1$ codelines of a quantum code of distance $d$ are not in general position there is a hyperplane containing all the remaining codelines.

In the remainder of this section we prove Proposition 1. It follows from Proposition 3.1 of [6] that the codeobjects of $C$ are indeed lines and that no line occurs more than once. Quantum code $C$ is therefore described by a set of 13 different lines in $PG(7, 2)$. Observe that the (quaternary) minimum weight of nonzero words in $C^\perp$ therefore is $\geq 2$. As we are assuming that $C$ is not pure there are three codelines $L_1, L_2, L_3$ contained in a subspace $PG(4, 2)$.

Lemma 2. Let $L_i, L_j, L_k$ be three codelines not in general position. Let $v(\{L_i, L_j, L_k\}) \in C$ a nonzero codeword with support in coordinates $i, j, k$.

Observe that $v(\{L_i, L_j, L_k\})$ in Lemma 2 has weight 2 or 3.

The 10 remaining codelines are in a hyperplane $H$. In the sequel we use basic facts concerning additive quaternary codes, see [4]. The nonexistence of a quaternary additive $[10, 6, 5, 4]$ and its dual shows that the family of remaining codelines cannot have strength 3. It follows that $L_4, L_5, L_6$ are in a subspace $PG(4, 2)$. By Lemma 1 there is a hyperplane containing all codelines $\not\in \{L_4, L_5, L_6\}$. This shows that the 7 codelines $\not\in \{L_1, \ldots, L_6\}$ are contained in a secundum $S$ (a $PG(5, 2)$). The non-existence of a quaternary $[7, 4, 4]$-code and its dual shows that three of the seven remaining lines ($L_7, L_8, L_9$, say) are not in general position. It follows from Lemma 1 that $L_{10}, \ldots, L_{13}$ are contained in a subspace $PG(4, 2)$.

We start from the information that some four lines which we now call $L_1, L_2, L_3, L_4$ are in a subspace $PG(4, 2)$. The codewords $v(\{L_1, L_2, L_3\})$ and $v(\{L_2, L_3, L_4\})$ show that there is a secundum $S$ (a $PG(5, 2)$) containing the remaining 9 codelines. The usual argument, based on the non-existence
of a quaternary $[9,6,4]$-code, shows that there is a $PG(4,2)$ containing 6 codelines.

Start again and use the knowledge that some six codelines $L_1, \ldots, L_6$ are contained in a subspace $PG(4,2)$. Applying our argument to subsets of three codelines shows that the remaining 7 codelines are contained in a $PG(4,2)$.

Finally we use the fact some seven codelines $L_1, \ldots, L_7$ are contained in a $PG(4,2)$. Apply our argument to the following triples of codelines:

- $\{L_1, L_2, L_3\}$ yielding $v(\{L_1, L_2, L_3\})$ which we can choose to have nonzero entries in coordinates 1, 2 (and possibly 3),
- $\{L_2, L_3, L_4\}$ where we choose notation such that 2 is in the support of $v(\{L_2, L_3, L_4\})$, and
- $\{L_3, L_4, L_5\}$

This yields the contradiction $L_6 = L_7$. Proposition \( \square \) has been proved.

3 The structure of the proof

Let $C$ be a $[[13,5,4]]$ quantum code, described by a set of lines $L_1, \ldots, L_{13}$ in the ambient space $U$ (a $PG(7,2)$). We know that the strength is 3. Let $e_1, \ldots, e_8$ be a basis of the underlying vector space $V$ and choose $L_1 = \langle e_1, e_2 \rangle$, $L_2 = \langle e_3, e_4 \rangle$. Consider the factor space $V/\langle e_1, e_2, e_3, e_4 \rangle$ and the corresponding $PG(3,2)$ which we call $\Pi$. We work in $U$ and in the factor space $\Pi$. Because of strength 3 each codeline $L_i, i > 2$ defines a line in $\Pi$.

Definition 5. Let $g$ be a line of $\Pi$ (a $PG(3,2)$). Define the weight $w(g)$ of $g$ as 2 less than the number of codelines contained in the preimage of $g$. For points $P$ and planes $E$ of $\Pi$ define

$$w(P) = \sum_{g \in P} w(g), \quad w(E) = \sum_{g \in E} w(g).$$

The geometric meaning of $w(P)$ and $w(E)$ is as follows: $w(P) + 2$ is the number of codelines which meet the preimage of $P$ (a $PG(4,2)$) nontrivially, $w(E) + 2$ is the number of codelines contained in the preimage of $E$ (a hyperplane $PG(6,2)$).
Proposition 2. We have $\sum_{g} w(g) = 11$ where the sum is over all lines $g$ of $\Pi$. For each line $h$ of $\Pi$ the number of lines of our multiset which intersect $h$ nontrivially is odd.

Proof. We think of the multiplicities $w(g)$ as defining a multiset, clearly of 11 lines. Let $h$ be a line of $\Pi$. Its preimage under the canonical mapping onto $\Pi$ is a secundum of the ambient space $U$. The orthogonality condition of Definition 2 translates as follows in geometric terms: for each secundum $S$ of $U$ the number of codelines meeting $S$ nontrivially is odd (see also [6]). Applying this to the preimage of line $h$ yields our claim.

We refer to the condition of Proposition 2 as the quantum condition. Observe that in the quantum condition the sum is over all lines, including $h$ itself: each of the 35 lines of $\Pi$ gives a condition, and the sum is over all $g$.

As $C$ is pure sets of strength 3 play an important role.

4 Sets of strength 3

Definition 6. A set of objects in a projective space has strength 3 if any subset of three of those objects are in general position. An $(n, m)$-set is a set of strength 3 consisting of $n$ lines and $m$ points.

Proposition 3. Assume $H$ is a hyperplane in $U$ containing precisely $n$ codelines. Then $H$ meets the union of the codelines in an $(n, 13 - n)$-set whose points meet each hyperplane $S$ of $H$ in a cardinality whose parity is different from $n$.

Proof. Each of the 13 codelines either is contained in $H$ or it meets $H$ in a point. This proves the first part. Let $S$ be a hyperplane of $H$. Then $S$ is a secundum of $U$ and therefore meets an odd number of codelines. As $S$ does meet the $n$ codelines contained in $H$ the second statement follows.

In order to obtain bounds on $w(P), w(g), w(E)$ consider the corresponding preimage spaces ($PG(4, 2), PG(5, 2), VPG(6, 2)$, respectively) with their $(n, m)$-sets formed by the intersection with codelines.

Lemma 3. A $(2, m)$-set of strength 3 in $PG(4, 2)$ has $m \leq 4$. All these sets are embedded in a uniquely determined $(2, 4)$-set.
Proof. The lines are without restriction \( L_1 = \langle e_1, e_2 \rangle, L_2 = \langle e_3, e_4 \rangle \), the points of the \((2, 4)\)-set of strength 3 can be chosen as

\[ e_5, \ e_1 + e_3 + e_5, \ e_2 + e_4 + e_5, \ e_1 + e_2 + e_3 + e_4 + e_5. \]

\( \square \)

Of particular importance are the hyperoval in \( PG(2, 4) \) and the \([7, 3, 5, 4]\)-codes.

**Lemma 4.** An \((n, 0)\)-set in \( PG(5, 2) \) has \( n \leq 6 \). For each \( n \) it is uniquely determined. They are all embedded in the uniquely determined \((6, 0)\)-set, which we call the **binary hyperoval**. Consider \((n, m)\)-sets in \( V_6 \). If \( n = 6 \), then \( m = 0 \). If \( n = 5 \), then \( m \leq 2 \). If \( n = 4 \), then \( m \leq 4 \).

**Proof.** The first 3 lines can be chosen as usual:

\[ L_1 = \langle e_1, e_2 \rangle, L_2 = \langle e_3, e_4 \rangle, L_3 = \langle e_5, e_6 \rangle. \]

There are exactly 27 points, the transversal points, each forming a \((3, 1)\)-set with \( \{L_1, L_2, L_3\} \). Then \( L_4 = \langle e_1 + e_3 + e_5, e_2 + e_4 + e_6 \rangle \) is the essentially unique fourth line. There remain 6 points each forming a \((4, 1)\)-set together with \( L_1, \ldots, L_4 \). These are exactly the six points on the remaining lines

\[ L_5 = \langle e_1 + (e_3 + e_4) + e_6, e_2 + e_3 + (e_5 + e_6) \rangle, \quad L_6 = \langle e_1 + e_4 + (e_5 + e_6), e_2 + (e_3 + e_4) + e_5 \rangle \]

of the binary hyperoval. The uniqueness statement follows. \( \square \)

We chose the term **binary hyperoval** as the \((6, 0)\)-set in \( PG(5, 2) \) is the binary image of the hyperoval in \( PG(2, 4) \). It is well known that the hyperoval has the symmetric group \( S_6 \) as its group of automorphisms. The automorphism group of the binary hyperoval has order \( 3 \times 6! \) where the additional factor 3 stems from the multiplicative group of the field.

As for the case of \((n, m)\)-sets in \( PG(6, 2) \) we use earlier work in relation to additive \([7, 3, 5, 4]\)-codes, see [3, 5].

**Proposition 4.** There is no \((7, 0)\)-set in \( PG(5, 2) \) and no \((8, 0)\)-set in \( PG(6, 2) \). There are precisely three non-equivalent \((7, 0)\)-sets in \( PG(6, 2) \). Exactly one of them defines a self-dual code with respect to the Euclidean form (the dot product).
Proof. A \((7,0)\)-set in \(V_6\) would define an additive \([7,4,4]_4\)-code. In the same way an \((8,0)\)-set in \(V_7\) would lead to an \([8,4,5,4]_4\)-code. Those codes do not exist.

The classification of \((7,0)\)-sets in \(PG(6,2)\) has been carried out independently several times, most recently in Danielsen-Parker [8] and Han-Kim [10].

**Proposition 5.** Consider the three \((7,0)\)-sets in \(PG(6,2)\). The number \(c\) of points that complete them to a \((7,1)\)-set is \(c = 1\), \(c = 2\) and \(c = 8\), respectively. The case of 8 extension points occurs when the code generated by the \((7,0)\)-set is self-dual. This \((7,0)\)-set can be extended to a uniquely determined \((7,7)\)-set and to a \((7,6)\)-set which is uniquely determined up to projectivity.

**Proof.** This is a computer result. The self-dual code is the one with 8 extension points. Here it is:

\[
\begin{pmatrix}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 \\
00 & 00 & 01 & 00 & 01 & 01 & 01 \\
01 & 00 & 00 & 01 & 00 & 01 & 01 \\
01 & 01 & 00 & 00 & 01 & 00 & 01 \\
00 & 00 & 10 & 10 & 10 & 00 & 10 \\
10 & 00 & 00 & 10 & 10 & 10 & 00 \\
00 & 10 & 00 & 00 & 10 & 10 & 10 \\
11 & 11 & 11 & 11 & 11 & 11 & 11
\end{pmatrix}
\]

The eight extension points are \(P_0 = (0 : 0 : 0 : 0 : 0 : 0 : 1)\) and the columns of

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

forming a set \(E\). The \((7,0)\)-set has an automorphism group \(G\) of order 42 which fixes \(P_0\), preserves the hyperplane \(H\) with equation \(x_7 = 0\) and acts transitively on \(E\). Consider the cone with vertex \(P_0\) consisting of the lines
from \( P_0 \) to the points \( L_i \cap H \). The third points on those lines make up \( E \). It follows that the uniquely determined \((7,7)\)-set is defined by point set \( E \) and the essentially uniquely determined \((7,6)\)-set is obtained by omitting one point from \( E \).

\[ \square \]

5 **The weights in the factor space \( PG(3,2) \)**

Consider the weights \( w(g) \) of lines in \( \Pi = PG(3,2) \) and the induced weights \( w(P), w(E) \) on points and planes.

**Lemma 5.** For points, lines, planes of \( \Pi \) we have \( w(P) \leq 4, w(g) \leq 3, w(E) \leq 4 \).

**Proof.** The statement on points follows from Lemma 3. Proposition 4 shows that \( w(E) \leq 5 \). The hyperplane \( H \) corresponding to a plane \( E \) of weight \( n \) yields an \((n+2,11-n)\)-set. Assume \( n = 5 \). Then there is a \((7,6)\)-set in \( V_7 \). By Proposition 5 the 7 lines are uniquely determined as only the self-dual cyclic example has more than 2 extension points. There is a uniquely determined \((7,6)\)-set in \( PG(6,2) \) (see Proposition 1), but it does not satisfy the quantum condition of Proposition 3. It follows \( w(E) \leq 4 \). Assume now \( w(g) = 4 \). The quantum condition shows that it is contained in a plane of weight 5, contradiction. \( \square \)

We can improve on Lemma 5:

**Proposition 6.** \( w(E) \leq 3 \) for each plane \( E \) of \( \Pi \). Each hyperplane \( H \) of \( U \) contains at most 5 codelines. The codelines define a quaternary \([13, 4, 8]\)-code.

**Proof.** All three statements of the proposition are equivalent. Assume \( w(E) = 4 \). Assume at first \( E \) contains a line \( g \) such that \( w(g) = 3 \). Then in the \( PG(5,2) \) corresponding to \( g \) we have the lines \( L_1, \ldots, L_5 \) corresponding to an oval in \( PG(2,4) \) and \( L_6 = \langle e_1+e_4+e_5+e_6, e_7 \rangle \) in the hyperplane corresponding to \( E \). Those 6 lines must be completable to a \((6,7)\)-system in \( PG(6,2) \) which satisfies the quantum condition: each hyperplane of the \( PG(6,2) \) must meet the set of 7 extension points in odd cardinality. A computer search shows that this problem has no solution.

Assume next \( E \) contains a line \( g \) of weight 2. We have the usual lines \( L_1, \ldots, L_4 \) in \( PG(5,2) \) and two more lines in the hyperplane which are not in the secundum. By Lemma 4 one of those lines can be chosen as \( L = \ldots \)
\[ \langle e_1 + e_3 + e_4 + e_6, e_7 \rangle \]. It remains to find the one remaining line and the system of 7 points in \( \text{PG}(6, 2) \) completing it to a \((6, 7)\)-system that satisfies the quantum condition. A computer search shows that there is no solution. At this point we have shown the following:

- Each hyperplane \( H \) of \( U \) which contains 6 codelines is generated by each 4 of its codelines.

This follows directly from the fact that for each plane \( E \) of weight 4 of \( \Pi \) we have \( w(g) \leq 1 \) for each line \( g \subset E \). Observe that we could have started from any pair of codelines instead of \( L_1, L_2 \) and considered the hyperplane corresponding to a plane of weight 4 in the factor space.

A computer search showed that there are exactly four families of 6 lines in \( \text{PG}(6, 2) \) satisfying the following:

- Any three of the lines are in general position.
- Any four of the lines generate the ambient space \( \text{PG}(6, 2) \).

Here they are:

\[
\begin{pmatrix}
 L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\
10 & 00 & 00 & 10 & 00 & 01 \\
01 & 00 & 00 & 10 & 10 & 01 \\
00 & 10 & 00 & 10 & 10 & 10 \\
00 & 01 & 00 & 01 & 10 & 10 \\
00 & 00 & 10 & 10 & 01 & 11 \\
00 & 00 & 01 & 10 & 01 & 01 \\
00 & 00 & 00 & 01 & 01 & 01
\end{pmatrix}
\]

\[
\begin{pmatrix}
 L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\
10 & 00 & 00 & 10 & 00 & 11 \\
01 & 00 & 00 & 10 & 10 & 11 \\
00 & 10 & 00 & 10 & 10 & 10 \\
00 & 01 & 00 & 00 & 01 & 11 \\
00 & 00 & 10 & 10 & 01 & 11 \\
00 & 00 & 01 & 10 & 11 & 11 \\
00 & 00 & 00 & 01 & 01 & 11
\end{pmatrix}
\]
In each of those cases another computer program shows that the corresponding family $F$ of codelines cannot be completed by a set $S$ of 7 points in $H = PG(6, 2)$ which together with the codelines form a $(6, 7)$-set of strength 3 and such that the quantum condition is satisfied.

\[ \begin{pmatrix}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\
10 & 00 & 00 & 10 & 00 & 11 \\
01 & 00 & 00 & 00 & 10 & 10 \\
00 & 10 & 00 & 10 & 01 & 10 \\
00 & 01 & 00 & 00 & 10 & 11 \\
00 & 00 & 10 & 10 & 10 & 11 \\
00 & 00 & 01 & 00 & 01 & 11 \\
00 & 00 & 00 & 01 & 01 & 01 
\end{pmatrix} \]

\[ \begin{pmatrix}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\
10 & 00 & 00 & 10 & 10 & 01 \\
01 & 00 & 00 & 00 & 11 & 10 \\
00 & 10 & 00 & 11 & 10 & 10 \\
00 & 01 & 00 & 00 & 01 & 10 \\
00 & 00 & 10 & 10 & 11 & 11 \\
00 & 00 & 01 & 00 & 11 & 01 \\
00 & 00 & 00 & 01 & 01 & 01 
\end{pmatrix} \]

6 Excluding a special configuration

In this section we show the following:

**Proposition 7.** Any five codelines generate either the ambient space $U$ or a hyperplane.

Assume this is not the case. If some five codelines were in a $PG(4, 2)$ then some hyperplane would contain six codelines, contradicting Proposition 6. Assume therefore some five codelines generate a secundum $S$. In terms of the factor space $\Pi$ this means there is some line $g_0$ of weight 3. As $w(E) \leq 3$ for each plane $E$ of $\Pi$ this implies $w(g) = 0$ for each line $g \neq g_0$ intersecting $g_0$ nontrivially.

The codelines in $S$ can be chosen as $L_1, \ldots, L_5$ according to Lemma 4. Let now $H \supseteq S$ be a hyperplane and $M = \{M_0, \ldots, M_7\}$ the points of
intersection with the eight remaining codelines. Then \( M_i \notin S \). Without restriction \( M_0 = e_7 \). Write \( M_i = e_7 + w_i \). Then the following conditions must be satisfied:

1. \( w_i \notin L_1 \cup \ldots L_5 \) for \( i = 1, \ldots, 7 \).

2. \( w_i + w_j \notin L_1 \cup \ldots L_5 \) for \( i \neq j \).

3. Let \( W \) be the \((7,8)\)-matrix with the elements of \( \mathcal{M} \) as columns. Then all codewords of the code generated by \( W \) have even weights.

Here the last condition represents the quantum condition: each hyperplane of \( H \) meets \( \mathcal{M} \) in even cardinality.

A computer search showed that up to equivalence there are 12 systems \( \mathcal{M} \) satisfying the conditions above.

Here is the structure of the generator matrix that far:

\[
\begin{pmatrix}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & L_{11} & L_{12} & L_{13} \\
10 & 00 & 00 & 10 & 10 & 00 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
01 & 00 & 00 & 01 & 01 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 10 & 00 & 10 & 11 & 00 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
00 & 01 & 00 & 01 & 10 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 00 & 10 & 10 & 01 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 00 & 01 & 01 & 11 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 00 & 00 & 00 & 00 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
00 & 00 & 00 & 00 & 00 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 \\
\end{pmatrix}
\]

For each choice of \( \mathcal{M} \) we need to determine the solutions of the problem in \( \text{PG}(3,2) \) (the last four rows of the generator matrix). Finally the generator matrix needs to be completed. The computer showed that this completion is impossible.

### 7 Completing the proof

Let \( L_1, \ldots, L_5 \) be codelines not generating the ambient space. They generate a hyperplane \( H \). Consider the corresponding \((5,8)\)-set in \( H \). The lines define an additive \([5,3.5]_{4}\)-code of strength 3. As its dual, a \([5,1.5,4]_{4}\)-code,
is uniquely determined (corresponding to a set of 5 lines in the Fano plane), the same is true of the code itself. We can therefore choose

\[ L_1 = \langle e_1, e_2 \rangle, L_2 = \langle e_3, e_4 \rangle, L_3 = \langle e_5, e_6 \rangle, \]

\[ L_4 = \langle e_1 + e_3 + e_5, e_7 \rangle, L_5 = \langle e_1 + e_4 + e_6, e_2 + e_3 + e_7 \rangle. \]

No four of those are on a hyperplane. How many points complete them to a \((5, 1)\)-set of strength 3? There are 15 points on the lines, 10 \times 3/2 = 15 in the intersection of the two spaces generated by two lines and 10 \times 6 further points on spaces generated by two lines. This leaves space for 127 − 90 = 37 extension points. Within this set of 37 points we have to find a subset \(\mathcal{M}\) of eight points which satisfy the conditions

- \(\mathcal{M}\) is a cap.
- Secants of \(\mathcal{M}\) do not meet any of the lines \(L_i\).
- Let \(W\) be the \((7, 8)\)-matrix with the elements of \(\mathcal{M}\) as columns. Then all codewords of the code generated by \(W\) have even weights.

The general form of the generator matrix is

\[
\begin{pmatrix}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_{10} & L_{11} & L_{12} & L_{13} \\
10 & 00 & 00 & 10 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
01 & 00 & 00 & 00 & 01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 10 & 00 & 10 & 01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 01 & 00 & 00 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 00 & 10 & 10 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 00 & 01 & 00 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 00 & 00 & 01 & 01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 00 & 00 & 00 & 00 & 01 & 01 & 01 & 01 & 01 & 01 & 01 \\
\end{pmatrix}
\]

A computer program did the following:

- Determine the solutions \(\mathcal{M}\).
- For each solution \(\mathcal{M}\) determine the 8 lines in \(\Pi\) completing the projections of the eight points of \(\mathcal{M}\) such that the orthogonality condition on the last four rows of the generator matrix are satisfied.
• Complete the generator matrix.

Observe that in the second step the projection to \( \Pi \) may lead to repeated points. This has to be taken into account when adapting the lines in \( \Pi \) to the points of \( \mathcal{M} \). The computer search showed that there are no solutions. This completes the proof of Theorem I.

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