Hall effect between parallel quantum wires

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Abstract – We study theoretically the parallel quantum wires of the experiment by Auslaender et al. (Science, 308 (2005) 88) at low electron density. It is shown that a Hall effect as observed in two- or three-dimensional electron systems develops as the wires enter the spin-incoherent regime of small spin bandwidth. This together with magnetic-field–dependent tunneling exponents clearly identifies spin incoherence in such experiments and it serves to distinguish it from disorder effects.

Since its discovery over a century ago the Hall effect has helped to uncover a number of most fundamental physical effects. Among the most famous are a quantization of electrical conductance in the integer quantum Hall effect [1], a fractionalization of the electric charge in the fractional quantum Hall effect [2], and anomalous velocities due to Berry phases in ferromagnets [3,4].

In this letter we show that, contrary to what one may expect, Hall measurements are also a powerful probe of one-dimensional quantum wires. We predict clear signatures of “spin-incoherent” physics in Hall measurements on tunnel-coupled, parallel quantum wires. The spin-incoherent limit of the interacting one-dimensional electron gas is reached when the temperature $T$ becomes larger than the spin bandwidth $J$, $kT \gg J$ [5]. This regime is a generic property of interacting electrons at low densities, when a Wigner crystal with large inter-electron spacing is formed. As one of the few known regimes of one-dimensional conductors that displays physics qualitatively different from the conventional Luttinger liquid, this limit has drawn much recent theoretical attention [5–7]. Experimentally, however, it has not been identified conclusively, yet. One of the most promising candidate systems for reaching the low-density regime required for observing spin-incoherent physics are semiconductor quantum wires as those in the experiment by Auslaender et al., refs. [8,9]. The tunneling current in that measurement has shown a loss of momentum resolution at low electron densities. This finding was likely due to a breaking of translational invariance by disorder [9], but it is also the main previously known [10] signature of spin incoherence in the setup of refs. [8,9].

Further progress towards the conclusive observation of spin incoherence in this experimental setting thus hinges on the availability of probes that distinguish spin-incoherent physics from the effects of disorder. The Hall measurements proposed here are such a probe.$^1$

In the experiments of refs. [8,9] two parallel one-dimensional wires are close enough for electrons to tunnel between them and a magnetic field $B$ is applied perpendicular to the plane of the wires (see fig. 1). A Hall effect in this geometry should induce a voltage $V_H$ between the

\[ V_H = \frac{eBd}{2}\]

$^1$A spin-polarizing magnetic field that can reverse the loss of momentum resolution if caused by spin-incoherence would be another possibility. Since the orbital effects of that magnetic field are also able to reduce disorder effects, however, additional signatures are desirable.
two wires in response to a current $I$ flowing through them. For noninteracting electrons in a translationally invariant setup, however, no such voltage is expected. Tunneling then is momentum-resolved and occurs only between a few discrete momentum states. In the generic case that a change $\Delta I$ of the current $I$ that flows through the wires is not carried by any of the states that participate in the tunneling between them, the tunnel current, and correspondingly $V_{\text{H}}$, are independent of $\Delta I$. Nevertheless, a transverse voltage can be observed in such experiments if translational invariance is broken or through electron-electron interactions. We show that at $kT \ll J$ the breaking of translational invariance induces a transverse voltage $V_{\text{H}}$ that is nonlinear in $B$ and that strongly depends on the difference between the currents in the two wires. In contrast, in the spin-incoherent regime of $kT \gg J$ a Hall effect as known from higher-dimensional electron systems is found, with a Hall voltage linear in $B$. Here, a difference between the currents $dI_{xy}$ needed to cancel the tunneling current. When $I_{xy} = 0$ as the counter voltage $V_{\text{H}} = -V_T$ needed to cancel the tunneling current. At $I_{xy} = I$, mimicking the higher-dimensional case, we find a transverse resistance $R_{xy} = V_{xy}/I$, where $I = I_{xy} + I_{x}$.

\[
I_T \propto T^\alpha \sum_{\sigma=\pm 1} f (\sigma k^0_{\text{F}} + \sigma' k^0_{\text{F}} - q_{\text{B}}) \times \left( \frac{\pi \sigma' k^0_{\text{F}} + \sigma I_{xy}}{2e} - V_T \right)
\]

(2)

\[
R_{xy} = \frac{\pi q_{\text{B}}}{e^2 (2k_{\text{F}}^0)^3} \frac{\prod_{\sigma=\pm 1} \left| (\Delta k_{\text{F}} - q_{\text{B}})^2 + b_{\text{br}}^2 \right|}{\Delta k_{\text{F}}^2 + q_{\text{B}}^2 + b_{\text{br}}^2}
\]

(3)

at $|\Delta k_{\text{F}}|, q_{\text{B}}, b_{\text{br}} \ll k_{\text{F}}^0$ ($\Delta k_{\text{F}} = k_{\text{F}}^0 - k_{\text{F}}^0$). We make two observations: i) $R_{xy}$ is nonlinear in $B$ on the scale $\Delta B \approx \max(|\Delta k_{\text{F}}|/e, (ed_{\text{br}})^{-1})$, as illustrated in fig. 2; ii) the “differential Hall coefficient” $dR_{xy}/dB|_{B=0} = R_{\text{H}}^{(0)} \times (\Delta k_{\text{F}}^2 + 1/b_{\text{br}}^2)/(2k_{\text{F}}^0)^2$ is suppressed below the Hall coefficient $R_{\text{H}}^{(0)} = -1/\epsilon_{\text{2D}}$ that one would expect in a two-dimensional electron gas. Here, $\epsilon_{\text{2D}} = (n_{\text{F}}^0 + n_{\text{F}})/d$ is an effective two-dimensional electron density between the two wires with one-dimensional densities $n_{\text{F}} = 2k_{\text{F}}^0/\pi$. Also the Hall response $R_{xy}^{-1}$ to the current difference $I_{xy}^\alpha = I^\alpha - I^\beta$, $V_{xy} = R_{xy}I_{xy} + R_{xy}^{-1} I_{xy}^{-1}$, is nonlinear in $B$ on the scale $\Delta B$: $R_{xy}^{-1} = -q_{\text{B}}/e^2 (\Delta k_{\text{F}}^2 + q_{\text{B}}^2 + b_{\text{br}}^2)$ (again at $|\Delta k_{\text{F}}|, q_{\text{B}}, b_{\text{br}} \ll k_{\text{F}}^0$). The differential Hall response to a difference between the currents $dR_{xy}^{-1}/dB|_{B=0} = -\delta (\Delta k_{\text{F}}^2 + b_{\text{br}}^2) \times dR_{xy}/dB|_{B=0}$, however, is strongly enhanced. Other mechanisms for the lifting of momentum conservation, such as disorder, are described by eq. (2) with a (possibly) different $f(k)$. Both of our main conclusions hold for any kind of translational invariance breaking and also in the regime $v_{\text{F}}/b_{\text{br}} \gg \nu \gg kT$.

Spin incoherence. – In the opposite, spin-incoherent limit $kT \gg J$ the ordering of the electron spins along the
wires becomes effectively static. The conduction electrons are thus described by static spin backgrounds and spinless Luttinger liquids of fermions $c_\mu$ [5,15,16]. The microscopic electron fields $\psi_{\mu\sigma}$ are expressed in terms of these fermions $c_\mu$ and operators $S_{\mu\sigma}(x)$ that remove a spin $\sigma$ from the spin background of wire $\mu$ at position $x$ as $\psi_{\mu\sigma}(x) = c_\mu^\dagger(x)S_{\mu\sigma}(x)$. To evaluate eq. (1) we need

$$G_{\mu\sigma}^>(x,x',\tau) = -i\langle\psi_{\mu\sigma}(x,\tau)\psi_{\mu\sigma}^\dagger(x',0)\rangle$$

and $G^<$, similarly defined. The spin expectation value in eq. (4) is nonvanishing only if adding a spin $\sigma$ at position $x'$ and time $\tau$ do not alter the spin background. This holds if all $N_{x\mu}(\tau) - N_{x'\mu}(0)$ background spins between the added and the removed spin have orientation $\sigma$, which occurs with probability $p_\sigma[N_{x\mu}(\tau) - N_{x\mu}(0)]$ ($N_x$ is the number of electrons to the left of point $x$). Here we used that at $x < kT$ the spins become independent of each other, each found in state $\sigma$ with probability $p_\sigma$. Consequently the spin and the charge expectation values in eq. (4) do not factorize and

$$G_{\mu\sigma}^>(x,x',\tau) = -i \int \frac{dk}{2\pi} \langle\psi_{\mu\sigma}(x,\tau)e^{ikx}\rangle$$

and

$$G_{\mu\sigma}^<(x,x',\tau) = \int \frac{dk}{2\pi} \langle\psi_{\mu\sigma}(x,\tau)e^{ikx}\rangle$$

Fig. 2: Transverse resistance $R_{xy}$ of two coupled quantum wires at $I^a = I^r$. At $kT < J$ (solid lines) the dependence on $B$ is nonlinear. In the spin-incoherent case $kT \gg J$ (broken line), in contrast, $R_{xy}$ is linear in $B$ with a slope greatly exceeding $dR_{xy}/dB|_{B=0}$ at $kT < J$ (solid line; $\Delta k_B l_{th} \gg 1$; broken line: for identical wires).

temperatures, $e_F^\mu/L \ll kT \ll \ln p_1/\delta$ with $\delta \sim 1/v_F k_F^\mu$, we obtain

$$G_{\mu\sigma}^>(x,x',\tau) \simeq \frac{\eta_\mu^e e^{i\pi T(x-x')/x_F^\mu} e^{i\phi^e_{\mu\sigma}(x,\tau)}}{\sqrt{2\pi g_\mu^e \ln s(\tau)}}$$

\begin{equation}
\times \int dk' |p'_{\mu}| \cos \pi k \ e^{-\pi^2 e^2 \ln s(\tau)} \end{equation}

(6)

(at $\tau \sim 1/\kappa T \gg \delta/\ln p_1$), with $s(\tau) = [\sinh \pi k T(\tau - i\delta)]/\pi \kappa k T \delta$ and the density of the now spinless fermions $n_F = k_F^\mu/\pi$. As a consequence of spin incoherence, $G_{\mu\sigma}$ decays quickly as a function of $x - x'$. Assuming that this is the dominant mechanism for the lifting of momentum conservation, $\max(1/k_F^\mu \ln p_1, \sqrt{-G_{\mu\sigma}^e(k_F^\mu)/k_F^\mu} \ll l_{th}$, we find from eq. (1) with eq. (6) and a similar expression for $G^<$ that at low temperatures $|\ln(kT\delta)| \gg \pi^2/k_F^\mu (\ln p_1)^2$

$$I_T \sim T^a \sum_{\sigma^\alpha,\sigma^\beta = \pm 1} \ln p_1 \frac{\alpha}{\sigma^\alpha + \eta_\mu^G [g^\mu(k_F^\mu)]^2}$$

\begin{equation}
	imes \left[ \left( V - V_T \right) + g_\mu^G \left( \frac{\pi l_{th}^\mu}{e^2 n_F k_F^\mu} + \frac{\pi I_{th}}{e^2 g_F(k_F^\mu)^2} \right) ^2 \right] \end{equation}

(7)

at $eV \ll kT$, with $\eta_\mu^G = g_\mu^G n_F^\mu n_F^\mu/[g^\mu(n_F^\mu)^2 + g_\mu^G (n_F^\mu)^2]$ and $\alpha = 1/2g^G + 1/2g_\mu^G + \eta_\mu^G /2k_F^\mu k_F^\mu - 2$.

We first note that, in contrast with the conventional Luttinger liquid, eq. (2), the scaling exponent $\alpha$ of the tunneling current now depends on the magnetic field $B$. This is the first distinctive signature of the physics of quantum-mechanical amplitudes for tunneling between the two wires. The tunneling rate between the wires is obtained from amplitudes for processes involving the addition of an electron to wire $\mu$ multiplied by complex conjugated amplitudes, describing the removal of an electron from $\mu$. Again because of the effectively static ordering of the electron spins along the wires, these pairs of amplitudes are constrained to add and remove a spin at the same site of the spin configuration in wire $\mu$ (for infinitely many spin states $\sigma$, when $p_\sigma \rightarrow 0$ — the argument readily generalizes to $p_\sigma \approx 1$). Let us suppose that an electron in wire $\mu$ crosses the point of tunneling during the time between the addition and the removal of the above tunneling electron. This shifts the spin background in wire $\mu$ by one lattice site. The above constraint can thus only be satisfied if the locations in space for the addition and the removal of the tunneling electron differ by one inter-electron distance. Consequently, the phases of these two amplitudes are different and a magnetic field shifts their difference $\delta \phi$ by $q_\sigma n_F^\mu$ (see eq. (1)). This translates into a $B$-dependent phase shift $\delta \phi$ for the electron that crossed the point of tunneling in wire $\mu$. The associated Fermi

With our bosonization approach we access only length scales $\gg k_F^{-1}$.

For that reason we assume here that a magnetic field is applied in the plane of the wires that polarizes the spins, $1 - p_\sigma \ll 1$, such that $(k_F^\mu \ln p_1)^{-1} \gg (k_F^\mu)^{-1}$. We expect, however, all results to remain qualitatively valid also at $p_\sigma \approx p_1$. We only evaluate $G_{\mu\sigma}$ here since the minority spin tunnel current is expected to be negligible.

\footnote{Via the $x$-dependence of $N_F^\mu$ the integrations in eq. (5) generate a space dependence of fermionic amplitudes on the scale $(k_F^\mu \ln p_1)^{-1}$.}
edge singularity [19,20] has a magnetic-field-dependent tunneling exponent, as predicted in eq. (7).

When the Wigner crystals slide at velocities \( v^u = l^u / e v^u \) the point of tunneling and the phase \( \delta \phi \) become time dependent. This induces a (Hall) voltage between the wires, found from eq. (7) with the condition \( I_T = 0 \),

\[
V_H = B \left[ R_H I + R_H^{(-)} I^{(-)} \right]
\]

(see footnote 4). Remarkably, eq. (8) predicts an (almost) conventional Hall effect for wires at \( kT \gg J \), as a second distinguishing signature of spin incoherence (see fig. 2). The Hall voltage \( V_H \) is linear in \( B \) at least up to a scale of the order of \( \min\{k_F^u, k_F^l\} / e d \), beyond which our bosonization approach loses its validity. At \( kT \ll J \), in contrast, \( V_H \) has been found above to be nonlinear in \( B \) on the typically much smaller scale \( \Delta k_F / e d \). The Hall coefficient \( R_H = -(g d / 2 e) (1 / g^u n^u + 1 / g^l n^l) \) in eq. (8) is of the order of the classically expected one, whereas it is strongly suppressed in the conventional Luttinger liquid. In addition, the Hall response to the difference between the currents through the wires \( R_H^{(-)} = -(g d / 2 e) (1 / g^u n^u - 1 / g^l n^l) \) is in the spin-incoherent regime weaker than \( R_H \), while it is strongly enhanced in the absence of spin incoherence.

Counterintuitively, the Hall response to currents in the wire with the lower electron density (found as \( R_H \pm R_H^{(-)} \) with the positive sign if the upper wire has smaller density than the lower wire) is smaller than the one in the wire with higher density —although the lower-density crystal slides faster and experiences a stronger Lorentz force at \( I^u = I^l \). The conventional relation \( V_H = R_H^{(0)} I \) holds only if both crystals slide at the same velocity. Also these features are readily understood by analyzing the rate of tunneling between the wires. The addition and the removal of an electron in each contributing pair of amplitudes typically occur within a time \( \tau_T \sim 1 / e U_T \). Spin incoherence again constrains the amplitudes for adding and removing a spin to act on the same sites of the spin configurations of each of the wires. If \( v^u \neq v^l \), however, the tunneling sites in these configurations are diverging in space at the average speed \( v^u - v^l \). After the time \( \tau_T \) they are aligned only if the two crystals are deformed by amounts \( \Delta x^u \) and \( \Delta x^l \) with \( \Delta x^u - \Delta x^l = -(v^u - v^l) \tau_T \). This costs an elastic energy \( \varepsilon_{\text{elastic}} \propto (n^u \Delta x^u)^2 / g^u + (n^l \Delta x^l)^2 / g^l \). Maximizing the probability \( \exp (-S) \) (where \( S \propto \varepsilon_{\text{elastic}} \), under the constraint \( \Delta x^u - \Delta x^l = -(v^u - v^l) \tau_T \) we find \( \Delta x^u = -\tau_T (v^u - v^l) g^u / n^l g^l \). This crystal deformation changes the velocity of the tunneling electron during \( \tau_T \) from the averages \( v^u \) to \( v^l = -(R_H I + R_H^{(-)} I^{(-)}) / d \), which results in the mentioned suppression of the Hall coefficient of the low-density wire: because the electron configuration in the low-density wire is deformed more easily, \( v_{\text{T}} \) (and thus \( V_H \)) is predominantly deter-

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