Determination of QCD phase diagram from the imaginary chemical potential region

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We test the reliability of the the Polyakov-loop extended Nambu–Jona-Lasinio (PNJL) model, comparing the model result with the lattice data at nonzero imaginary chemical potential. The PNJL model with the vector-type four-quark and scalar-type eight-quark interactions reproduces the lattice data on the pseudocritical temperatures of the deconfinement and chiral phase transitions. The QCD phase diagram in the real chemical potential region is predicted by the PNJL model. The critical endpoint survives, even if the vector-type four-quark interaction is taken into account.

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I. INTRODUCTION

Quantum Chromodynamics (QCD) is a remarkable theory. It is renormalizable and essentially parameter free. QCD accounts for the rich phenomenology of hadronic and nuclear physics. Thermodynamics of QCD is also well defined. Nevertheless, it is not well known because of its nonperturbative nature. In particular, QCD phase diagram is essential for understanding not only natural phenomena such as compact stars and the early universe but also laboratory experiments such as relativistic heavy-ion collisions.

Unfortunately, quantitative calculations of the phase diagram from first-principle lattice QCD (LQCD) have the well known sign problem when the chemical potential ($\mu$) is real; for example, see Ref. [1] and references therein. So far, several approaches have been proposed to circumvent the difficulty; for example, the reweighting method [2], the Taylor expansion method [3] and the analytic continuation to real chemical potential ($\mu$) [4, 5, 6, 7, 8]. However, those are still far from perfection.

As an approach complementary to first-principle lattice QCD, we can consider effective models such as the Nambu–Jona-Lasinio (NJL) model [9, 10, 11, 12, 13, 14, 15, 16, 17] and the Polyakov-loop extended Nambu–Jona-Lasinio (PNJL) model [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. The NJL model describes the chiral symmetry breaking, but not the confinement mechanism. The PNJL model is designed [20] to make it possible to treat the Polyakov loop as well as the chiral symmetry breaking.

In the NJL-type models, the input parameters are determined at $\mu = 0$ and $T \geq 0$, where $T$ is temperature. It is then highly nontrivial whether the models predict properly dynamics of QCD at finite $\mu_R$. This should be tested from QCD.

Fortunately, this is possible in the $\mu_1$ region, since lattice QCD has no sign problem there. The canonical partition function $Z_C(n)$ with real quark number $n$ is the Fourier transform of the grand-canonical one $Z_{GC}(\theta)$ with $\theta = \mu_1/T$ [36]:

$$Z_C(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-in\theta} Z_{GC}(\theta).$$

Thus, the thermodynamic potential of QCD, $\Omega_{QCD}(\theta) = -T \ln(Z_{GC}(\theta))$, at finite $\theta$ includes all dynamics at real $\mu$ and hence at finite $\mu_1$. Therefore, the reliability of effective models at finite $\mu_R$ can be tested in the $\mu_1$ region.

Roberge and Weiss found [34] that QCD has a periodicity $\Omega_{QCD}(\theta) = \Omega_{QCD}(\theta + 2\pi k/3)$, showing that $\Omega_{QCD}(\theta + 2\pi k/3)$ is transformed into $\Omega_{QCD}(\theta)$ by the $Z_3$ transformation with integer $k$. This means that QCD is invariant under a combination of the $Z_3$ transformation and a parameter transformation $\theta \rightarrow \theta + 2\pi k/3$ [34, 35].

$q \rightarrow U q, \ A_\nu \rightarrow U A_\nu U^{-1} - i/g(\partial_\nu U) U^{-1}$,

$$\theta \rightarrow \theta + 2\pi k/3,$$

where $U(x, \tau)$ are elements of SU(3) with $U(x, \beta = 1/T) = \exp(-2\pi k/3)U(x, 0)$ and $q$ is the quark field. We call this combination the extended $Z_3$ transformation. Thus, $\Omega_{QCD}(\theta)$ has the extended $Z_3$ symmetry, and hence quantities invariant under the extended $Z_3$ transformation have the $Z_3$ periodicity [34, 35]. At the present stage, the PNJL model is only a realistic effective model that possesses both the extended $Z_3$ symmetry and chiral symmetry [34, 35]. This property makes it possible to compare PNJL with lattice QCD quantitatively in the $\mu_1$ region. If the PNJL model succeeds in reproducing the lattice data, we may think that the PNJL model will predict, with high reliability, the QCD phase structure in the $\mu_1$ region.

The extended $Z_3$ symmetry in QCD is a remnant of the $Z_3$ symmetry, namely the confinement mechanism, in the pure gauge system. The extended $Z_3$ symmetry appears as the RW periodicity in the $\mu_1$ region and implicitly affects dynamics in the $\mu_R$ region. Actually, the mechanism largely shifts the critical endpoint [14, 16, 24, 31] toward higher $T$ and lower $\mu$ than the NJL model predicts [21, 25, 31]. In contrast, the vector-type four-quark interaction $G_v(q^\gamma q)^2$ largely moves the critical endpoint in the opposite direction [14, 16, 24, 31], if it is newly
added to the NJL and PNJL models. Thus, it is essential to
determine the strength of the coupling \( G_v \) of the vector-type in-
teraction, although the interaction is often ignored in the NJL
and PNJL calculations.

In the relativistic meson-nucleon theory \([37]\), the repulsive
force mediated by vector mesons is essential to account for the
saturation property of nuclear matter. Using the auxiliary field
method, one can convert quark-quark interactions to meson-
quark interactions; for example, see Refs. \([17,38,39]\) and
references therein. In the hadron phase, quarks have a large ef-
effective mass as a result of spontaneous chiral symmetry break-
ing, and then nucleons can be considered to be formed by such
tree heavy quarks, i.e. three constituent quarks. It is then nat-
ural to think that there exists the correspondence between the
meson-nucleon interactions and the quark-quark interactions.
In this sense, it is very likely that the vector-type four-quark
interaction \([10,14,16,35]\) and the scalar-type eight-quark in-
teraction \([10,14,16,35]\) and the scalar-type eight-quark in-
teraction can reproduce the latter data. The primary result
of the lattice simulations is that

\[
\Phi = \frac{1}{N_c} \text{Tr} L, \quad \Phi^* = \frac{1}{N_c} \text{Tr} L^\dagger, \tag{4}
\]

with

\[
L(x) = \mathcal{P} \exp \left[ i \int_0^\beta d\tau A_4(x, \tau) \right], \tag{5}
\]

where \( \mathcal{P} \) is the path ordering and \( A_4 = iA_0 \). In the chiral limit
\((m_0 = 0)\), the Lagrangian density has the exact \( SU(N_f)_L \times
SU(N_f)_R \times U(1)_c \times SU(3)_c \) symmetry.

The temporal component of the gauge field is diagonal in the
flavor space, because the color and the flavor space are
completely separated out in the present case. In the Polyakov
gauge, \( L \) can be written in a diagonal form in the color space \([20]\):

\[
L = e^{i\beta(\phi_3 + \phi_8 + 2\phi_3 \phi_8)} = \text{diag}(e^{i\beta\phi_a}, e^{i\beta\phi_b}, e^{i\beta\phi_c}),
\tag{6}
\]

where \( \phi_a = \phi_3 + \phi_8/\sqrt{3}, \phi_b = -\phi_3 + \phi_8/\sqrt{3} \) and
\( \phi_c = -\phi_3 + \phi_8/\sqrt{3}. \) The Polyakov loop \( \Phi \) is an exact
order parameter of the spontaneous \( Z_3 \) symmetry breaking in
the pure gauge theory. Although the \( Z_3 \) symmetry is not an
exact one in the system with dynamical quarks, it still seems
to be a good indicator of the deconfinement phase transition.
Therefore, we use \( \Phi \) to define the deconfinement phase transition.

Making the mean field approximation and performing the
path integral over quark field, one can obtain the thermody-
namic potential \( \Omega \) (per volume),

\[
\Omega = -2N_f \int \frac{d^3p}{(2\pi)^3} \left[ 3E(p) \right. \\
+ \frac{1}{\beta} \ln \left[ 1 + 3(\Phi + \Phi^* e^{-\beta E^-(p)}) e^{-\beta E^+(p)} + e^{-3\beta E^-(p)} \right] \\
+ \frac{1}{\beta} \ln \left[ 1 + 3(\Phi + \Phi^* e^{-\beta E^+(p)}) e^{-\beta E^-(p)} + e^{-3\beta E^+(p)} \right] \\
+ U_M + \mathcal{U}. \tag{7}
\]

where, \( \sigma = \langle \bar{q}q \rangle, \Sigma = -2G_s\sigma, M = m_0 + \Sigma, U_M = G_\sigma \sigma^2, \)
\( E(p) = \sqrt{p^2 + M^2} \) and \( E^\pm(p) = E(p) \pm \mu = E(p) \pm i\delta/\beta. \)
In \( \Omega \), only the first term of the right-hand side diverges. It is
then regularized by the three-dimensional momentum cutoff
\( \Lambda \) \([20,24]\). We use \( \mathcal{U} \) of Ref. \([25]\) that is fitted to a lattice
QCD simulation in the pure gauge theory at finite \( T \) \([43,44]\):

\[
\mathcal{U} = T^4 \left[ \frac{a(T)}{2} \Phi^* \Phi \right. \\
+ b(T) \ln \left( 1 - 6\Phi^* \Phi + 4(\Phi^3 + \Phi^*^3) - 3(\Phi\Phi^*^2) \right), \tag{8}
\]

\[
a(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2, \quad b(T) = b_3 \left( \frac{T_0}{T} \right)^3 \tag{9}
\]
where parameters are summarized in Table I. The Polyakov potential yields a first-order deconfinement phase transition at $T = T_0$ in the pure gauge theory. The original value of $T_0$ is 270 MeV evaluated by the pure gauge lattice QCD calculation. However, the PNJL model with this value of $T_0$ yields somewhat larger value of the transition temperature at zero chemical potential than the full LQCD simulation [40, 41, 42] predicts. Therefore, we rescale $T_0$ to 212 MeV; the detail will be shown in subsection III A.


table: Summary of the parameter set in the Polyakov sector used in Ref. [25]. All parameters are dimensionless.

| $a_0$ | $a_1$ | $a_2$ | $b_3$ |
|-------|-------|-------|-------|
| 3.51  | -2.47 | 15.2  | -1.75 |

TABLE I: Summary of the parameter set in the Polyakov sector used in Ref. [25]. All parameters are dimensionless.

The variables $X = \Phi, \Phi^*$ and $\sigma$ satisfy the stationary conditions,

$$\partial \Omega / \partial X = 0. \quad (10)$$

The solutions of the stationary conditions do not give the global minimum $\Omega$ necessarily. There is a possibility that they yield a local minimum or even a maximum. We then have checked that the solutions yield the global minimum when the solutions $X(\theta)$ are inserted into $\Omega$.

The thermodynamic potential $\Omega$ of Eq. (7) is not invariant under the $Z_3$ transformation,

$$\Phi(\theta) \rightarrow \Phi(\theta) e^{-i2\pi k/3}, \quad \Phi(\theta)^* \rightarrow \Phi(\theta)^* e^{i2\pi k/3}, \quad (11)$$

although $U$ of (8) is invariant. Instead of the $Z_3$ symmetry, however, $\Omega$ is invariant under the extended $Z_3$ transformation,

$$e^{\pm i \theta} \rightarrow e^{\pm i \theta} e^{\pm i2\pi k/3}, \quad \Phi(\theta) \rightarrow \Phi(\theta) e^{-i2\pi k/3}, \quad \Phi(\theta)^* \rightarrow \Phi(\theta)^* e^{i2\pi k/3}. \quad (12)$$

This is easily understood as follows. It is convenient to introduce the modified Polyakov loop $\Psi \equiv e^{i\theta} \Phi$ and $\Psi^* \equiv e^{-i\theta} \Phi^*$ invariant under the transformation (12). The extended $Z_3$ transformation is then rewritten into

$$e^{\pm i \theta} \rightarrow e^{\pm i \theta} e^{\pm i2\pi k/3}, \quad \Psi(\theta) \rightarrow \Psi(\theta), \quad \Psi(\theta)^* \rightarrow \Psi(\theta)^*, \quad (13)$$

and $\Omega$ is also into

$$\Omega = -2N_f \int \frac{d^3p}{(2\pi)^3} \left[ 3E(p) + \frac{1}{\beta} \ln \left[ 1 + 3e^{-\beta E(p)} + 3\Psi e^{-2\beta E(p)} e^{-\beta \mu B} + e^{-3\beta E(p)} e^{-\beta \mu B} \right] \right. \left. + 3\Psi^* e^{-2\beta E(p)} e^{\beta \mu B} + e^{-3\beta E(p)} e^{\beta \mu B} \right] + U_M + U, \quad (14)$$

where $\beta \mu_B = 3/\beta \mu = 3i \theta$. Obviously, $\Omega$ is invariant under the extended $Z_3$ transformation (13), since it is a function of only extended $Z_3$ invariant quantities, $e^{3i\theta}$ and $\bar{X} = \Psi, \Psi^*, \sigma$.

The explicit $\theta$ dependence appears only through the factor $e^{3i\theta}$ in (14). Hence, the stationary conditions (10) show that $\bar{X} = \bar{X}(e^{3i\theta})$. Inserting the solutions back to (14), one can see that $\Omega = \Omega(e^{3i\theta})$. Thus, $\bar{X}$ and $\Omega$ have the RW periodicity,

$$\bar{X}(\theta + \frac{2\pi k}{3}) = \bar{X}(\theta), \quad \bar{X}(\theta + \frac{2\pi k}{3}) = \bar{X}(\theta), \quad (15)$$

while the Polyakov loop $\Phi$ and its Hermitian conjugate $\Phi^*$ have the properties

$$\Phi(\theta), \quad \Phi(\theta)^* \quad (16)$$

III. NUMERICAL RESULTS

A. Thermal system with no chemical potential

First, we consider the thermal system with no chemical potential to determine the parameters, $m_0, G_s, \Lambda$ and $T_0$ of the PNJL model. In the lattice calculations [40, 41, 42], the pseudocritical temperature $T_c(\sigma)$ of the crossover chiral phase transition coincides with that $T_c(\phi)$ of the crossover deconfinement one within 10% error: $T_c(\sigma) \approx T_c(\phi) \approx 173 \pm 8$ MeV [41].

The parameter set, $\Lambda = 631.5$ MeV, $G_s = 5.498$ [GeV$^{-2}$] and $m_0 = 5.5$ MeV, can reproduce the pion decay constant $f_\pi = 93.3$ MeV and the pion mass $m_\pi = 138$ MeV at $T = \mu = 0$ [16], and keeps a good reproduction also at finite $T$ [25]. We then adopt these values for $\Lambda, G_s$ and $m_0$. We adjust $T_0$ so that the PNJL calculation can reproduce the lattice result $T_c(\phi) = 173$ MeV; the value is $T_0 = 212$ MeV. The parameter set thus determined is shown as set A in Table II.

| set   | $G_s$    | $G_s$ | $G_V$ |
|-------|---------|-------|-------|
| A     | 5.498   | 0     | 0     |
| B     | 4.673   | 452.12 | 8     |

TABLE II: Summary of the parameter sets in the PNJL calculations. The parameters $\Lambda, m_0$ and $T_0$ are common among the three sets; $\Lambda = 631.5$ MeV, $m_0 = 5.5$ MeV and $T_0 = 212$ MeV.

Figure [1] shows the chiral condensate $\sigma$ normalized by $\sigma_0 = \sigma|_{T=0, \mu=0}$ and the absolute value of the Polyakov loop $|\Phi|$ as a function of $T/T_c$. In this paper $T_c$ is always taken to be 173 MeV. The green curves represent the PNJL results of parameter set A, where $\sigma_0 = -0.0302$ [GeV$^3$] in this case. Lattice QCD data [40, 41, 42] are also plotted by cross symbols with 10% error bar; $\sigma$ and $|\Phi|$ measured as a function of $T/T_c$ in Refs. [40, 41, 42] have only small errors, but we have added 10% error that the lattice calculation [41] has in determining $T_c$. For $|\Phi|$ the PNJL result (green solid curve) reasonably agrees with the lattice one ($\times$). For $\sigma$, however,
the PNJL result (green dashed curve) considerably overshoots the lattice data (+).

![Graph](image)

Fig. 1: Chiral condensate \( \sigma \) normalized by \( \sigma(T = 0, \mu = 0) \) and the absolute value of the Polyakov loop \( \Phi \). The blue (green) curve represents the PNJL result of parameter set B (A) with (without) the scalar-type eight-quark interaction; \( \sigma(|\Phi|) \) is denoted by the dashed (solid) curve. Lattice data (+) on \( \sigma \) are taken from Ref. [20] and those (×) on |\( \Phi \)| are from Ref. [42]. The lattice data are plotted with 10 % error bar, since lattice calculations have 10 % error in chiral and Polyakov-loop susceptibilities, \( \chi \) and \( \Phi \), respectively. The PNJL results (green curves) of parameter set B well reproduce the lattice data for both the chiral condensate and the Polyakov loop.

![Graph](image)

Fig. 2: \( T \) dependence of chiral and Polyakov-loop susceptibilities, \( \chi_\sigma \) (right scale) and \( \chi_\Phi \) (left scale). The blue (green) curve represents the PNJL result of parameter set B (A) with (without) the scalar-type eight-quark interaction; \( \chi_\sigma \) (\( \chi_\Phi \)) is denoted by the dashed (solid) curve. The region between two vertical gray lines \( T = (1 \pm 0.05)T_c \) is the prediction of lattice calculations [41].

Now, we introduce the scalar-type eight-quark interaction [16],

\[
G_{s8}[(\bar{q}q)^2 + (\bar{q}i\gamma_5\tau q)^2]^2, \tag{17}
\]

since the difference \( T_c(\sigma) - T_c(\Phi) \) is reduced by the interaction [34].

Since \( f_\pi \) and \( M_\pi \) calculated with PNJL depend on the strength of \( G_{s8} \), for each value of \( G_{s8} \) the strength of \( G_s \) is re-adjusted so as to reproduce the measured values \( f_\pi = 93.3 \) MeV and \( M_\pi = 138 \) MeV. As \( G_{s8} \) increases from zero, \( T_c(\sigma) \) calculated with PNJL decreases toward \( T_c = 173 \) MeV. When \( G_{s8} = 452.12 \) GeV\(^{-2}\), the ratio \( T_c(\sigma)/T_c \) becomes 1.05 and hence consistent with the corresponding lattice result within 10 % error. We adopt this strength. This parameter set is shown as set B in Table II. As shown in Fig. 1 the PNJL results (blue curves) of parameter set B well reproduce the lattice results for both the chiral condensate and the Polyakov loop.

B. Thermal system with imaginary chemical potential

In this subsection, we consider the thermal system with finite imaginary chemical potential and compare the PNJL result with the lattice data [41][8] in which the lattice size is \( 8^3 \times 4 \) and the two-flavor KS(Wilson) fermion is considered. First, we analyze the deconfinement phase transition. Since the eight-quark interaction hardly changes the Polyakov loop, we do the PNJL calculation with parameter set A. Figure 3 presents \( T \) dependence of the Polyakov-loop susceptibility \( \chi_\Phi \) in three cases of \( \theta = 0, 0.56 \) and 0.96; each case is distinguished by using different colors. For each \( \theta \), the PNJL result (solid curve) reproduces the corresponding lattice result (crosses) in its peak position. Thus, the PNJL results are consistent with the lattice ones for the pseudocritical temperature of the crossover deconfinement phase transition.

![Graph](image)

Fig. 3: \( T \) dependence of the Polyakov-loop susceptibilities in three cases of \( \theta = 0, 0.56 \) and 0.96; each case is distinguished by using different colors. The solid curves represent the PNJL results of set A (right scale). Lattice data shown by crosses (left scale) are taken from Ref. [8].

Figure 4 presents the phase diagram of the deconfinement phase transition in the \( \theta \)-\( T \) plane, where \( \theta \) is divided by \( \pi/3 \) and \( T \) is normalized by \( T_c = 173 \) MeV. Lattice data [8] measured as a function of \( T/T_c \) have only small errors, as shown by thick error bars in Fig. 4. This is an error bar in the case
that lattice calculations have no error in $T_c$. However, the lattice calculation [41] has about 10% error in determining $T_c$, as mentioned in subsection III A. This 10% error should be added to the original small error; this 10% error will be shown later in Fig. 6. The PNJL result (solid curve) of set A agrees with the lattice one (crosses) within the error bars. The phase diagram has a periodicity of $2\pi/3$ in $\theta$. This is called the Roberge and Weiss (RW) periodicity [36]. The phase diagram is also $\theta$ even, because so is $\chi_\sigma$. On the dot-dashed line going up from an endpoint $(\theta_{RW}, T_{RW}) = (\pi/3, 1.09T_c)$, the quark number density $n$ and the phase $\phi$ of the Polyakov loop are discontinuous in the PNJL calculations [34, 35]. This is called the RW phase transition line. The lattice data [4, 8] on $\phi$ are also discontinuous on the line, as shown later in Fig. 7. Thus, the PNJL result is consistent with the lattice results [4, 8] also for the location of the RW phase transition line.

![Fig. 4: Phase diagram on the $\theta$–$T$ plane. The solid curve represents the deconfinement phase transition, while the dot-dashed line does the RW phase transition predicted by the PNJL calculation with set A. Lattice data are taken from Ref. [8].](image1)

The lattice simulations [4, 8] point out that $T_c(\sigma)$ agrees with $T_c(\chi)$ within numerical errors in the entire region $0 \leq \theta \leq \pi/3$. We then take the case of $\theta = \pi/3$ to consider this point. It is predicted by the lattice simulations that $T_c(\sigma)$ and $T_c(\chi)$ are located in the region between two vertical gray lines of Fig. 5. Panel (a) shows $\sigma$ and $|\phi|$ as a function $T/T_c$ and panel (b) does $\chi_\sigma$ and $\chi_\phi$ as a function $T/T_c$. The green (blue) curves represent results of the PNJL calculations with set A (B). The eight-quark interaction hardly shifts the peak position of $\chi_\sigma$, i.e. $T_c(\sigma)$, from the value $1.09T_c$. The peak position is consistent with the lattice result shown by the region between two vertical gray lines. In contrast, the eight-quark interaction largely shifts the peak position of $\chi_\phi$, i.e. $T_c(\chi)$, from $1.53T_c$ to $1.24T_c$, but the shifted value still deviates from $T_c(\chi) = (1.1 \pm 0.05)T_c$, that is, the lattice data near $\theta = \pi/3$ [4, 8] shown by the region between two vertical gray lines.

In order to solve this problem, we introduce the vector-type four-quark interaction

$$-G_v(q^2, \mu^2) \chi_\sigma$$

and add it to the PNJL Lagrangian $\mathcal{L}$; see Ref. [35] for the detail of this formulation. As mentioned in Ref. [35], the phase structure in the real chemical potential region is quite sensitive to the strength of the coupling $G_v$. It is then important to determine the strength, but it has not been done yet. Since the vector-type interaction does not change the pion mass and the pion decay constant at $T = \mu = 0$ and the chiral condensate and the Polyakov loop at $T \geq 0$ and $\mu = 0$, we can simply add the interaction to set B. As $G_v$ increases from zero, $T_c(\sigma)$ goes down toward $T_c(\chi)$, while $T_c(\phi)$ moves little. When $G_v = 4.673 \text{GeV}^{-2}$, $T_c(\sigma)$ gets into the region between the vertical gray lines. We adopt this strength of $G_v$. This set is shown as set C in Table II.

![Fig. 5: $T$ dependence of (a) the normalized chiral condensate and the absolute value of the Polyakov loop and (b) the susceptibilities $\chi_\sigma$ (right scale) and $\chi_\phi$ (left scale) at $\theta = \pi/3$. In panel (a), $\sigma$ ($|\phi|$) is denoted by the dashed (solid) curve. In panel (b), $\chi_\sigma$ ($\chi_\phi$) is denoted by the dashed (solid) curve. The PNJL calculations are done with three parameter sets of A, B and C and these are distinguished by using different colors, green, blue and red, respectively.](image2)

Figure 6 shows the phase diagram of the chiral phase transition determined by $T_c(\sigma)$. Green, blue and red curves are results of the PNJL calculations with sets A, B and C, respectively. In the entire region $0 \leq \theta \leq 2\pi/3$, the eight-quark interaction moves $T_c(\sigma)$ down from the green dashed curve (set A) to the blue one (set B). However, the blue dashed curve still overshoots the lattice result (symbols) with 10% error near $\theta = \pi/3$. The vector-type interaction makes the blue dashed curve go down to the red one (set C) that is consistent with the lattice result [8]. Thus, the PNJL calculations with set C can reproduce the lattice result [4, 8] that $T_c(\sigma)$ coincides with $T_c(\chi)$ within numerical errors in the entire region $0 \leq \theta \leq 2\pi/3$. 

![Fig. 6: Phase diagram of the chiral phase transition determined by $T_c(\sigma)$. Green, blue and red curves are results of the PNJL calculations with sets A, B and C, respectively. In the entire region $0 \leq \theta \leq 2\pi/3$, the eight-quark interaction moves $T_c(\sigma)$ down from the green dashed curve (set A) to the blue one (set B). However, the blue dashed curve still overshoots the lattice result (symbols) with 10% error near $\theta = \pi/3$. The vector-type interaction makes the blue dashed curve go down to the red one (set C) that is consistent with the lattice result [8]. Thus, the PNJL calculations with set C can reproduce the lattice result [4, 8] that $T_c(\sigma)$ coincides with $T_c(\chi)$ within numerical errors in the entire region $0 \leq \theta \leq 2\pi/3$.](image3)
The imaginary chemical potential region calculated with three parameter sets data [8] are shown with 10% error that are presented by dashed curves; green, blue and red ones are results of the PNJL calculations with set A, B and C, respectively. Lattice data [8] are plotted by symbols. Curves represent results of PNJL calculations with set A. In panel (b), five cases (red, green, blue, pink and light blue) from top to bottom represent results of PNJL calculations with sets A, B and C, respectively. Panel (c) is the most reliable result, since the PNJL result of set C is consistent with the lattice result [4,8] in the imaginary chemical potential region. Comparing the three panels, we find that the vector-type four-quark interaction and the scalar-type eight-quark interaction give sizable effects on the phase structure. In particular for the critical endpoint E, the eight-quark interaction shifts point E to larger $T$ and smaller $\mu$, and the vector-type interaction moves it in the opposite direction. On the red solid curve between point E and point D both the first-order chiral and deconfinement phase transitions take place simultaneously. The light-blue dot-dashed curve moving up from point I represents the RW phase transition of first order, and point I is the critical endpoint. The green dashed curve between point I and point E does the crossover deconfinement transition. Point F (G) is a crossing point between the dashed (solid) curve and the $\phi/\pi$ line. Positions of points D–I are summarized in Table III. In panel (c), the pink dotted curve represents the lower bound of the location $\mu_E/T_E$ of the critical endpoint E that the LQCD analyses of Ref. [45] predict. The position of point E in the case of parameter set C is consistent with the results of the LQCD analyses.

### C. Thermal system with real chemical potential

In this subsection, we predict the phase diagram in the real $\mu$ region by using the PNJL model. In Fig. 8 panels (a)-(c) represent results of the PNJL calculations with sets A, B and C, respectively. Panel (c) is the most reliable result, since the PNJL result of set C is consistent with the lattice result [4,8] in the imaginary chemical potential region. Comparing the three panels, we find that the vector-type four-quark interaction and the scalar-type eight-quark interaction give sizable effects on the phase structure. In particular for the critical endpoint E, the eight-quark interaction shifts point E to larger $T$ and smaller $\mu$, and the vector-type interaction moves it in the opposite direction. On the red solid curve between point E and point D both the first-order chiral and deconfinement phase transitions take place simultaneously. The light-blue dot-dashed curve moving up from point I represents the RW phase transition of first order, and point I is the critical endpoint. The green dashed curve between point H and point E in the case of $\phi/\pi$ is another $\mu$ image of the region (I).

### IV. SUMMARY

We have tested the reliability of the PNJL model, comparing the model result with lattice data in the imaginary ch...
TABLE III: Positions of points D–I in $\mu$-$T$ plane. The positions of these points are normalized as $(\mu/T_c, T/T_c)$ with $T_c = 173$ MeV.

| set | D    | E    | F    | G    | H    | I    |
|-----|------|------|------|------|------|------|
| A   | $(2.02, 0.00)$ | $(1.84, 0.72)$ | $(0.00, 1.25)$ | $(0.00, 1.00)$ | $i\pi/3 \times 1.53$ | $1.53$ |
|     |      |      |      |      |      |      |
|     |      |      |      |      | $i\pi/3 \times 1.09, 1.09$ |      |
| B   | $(1.68, 0.00)$ | $(1.02, 0.87)$ | $(0.00, 1.05)$ | $(0.00, 1.00)$ | $i\pi/3 \times 1.24, 1.24$ |      |
|     |      |      |      |      |      |      |
|     |      |      |      |      | $i\pi/3 \times 1.09, 1.09$ |      |
| C   | $(1.80, 0.00)$ | $(1.51, 0.72)$ | $(0.00, 1.05)$ | $(0.00, 1.00)$ | $i\pi/3 \times 1.13, 1.13$ |      |
|     |      |      |      |      |      |      |
|     |      |      |      |      | $i\pi/3 \times 1.07, 1.07$ |      |

Fig. 8: Phase diagram in the real chemical potential region. (a), (b), and (c) are calculated with the parameter set A, B, and C, respectively. Cross symbols with error bars indicate the lattice data taken from Ref. [8]. Points D–I are explained in the text.

Finally, we quantitatively predict the phase diagram in the $\mu_R$ region by using the PNJL model with the parameter set mentioned above. The critical endpoint does not disappear in virtue of the eight-quark interaction, even if the vector-type interaction is taken into account. This is the primary result of the present work. The lattice calculations at nonzero $\mu$ have small lattice size ($8^3 \times 4$) [4,8]. Therefore, it is highly expected that lattice simulations with larger size will be done in the $\mu_1$ region.

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