Fundamental Constraints on Multicast Capacity Regions

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November 24, 2009

Abstract

Much of the existing work on the broadcast channel focuses only on the sending of private messages. In this work we examine the scenario where the sender also wishes to transmit common messages to subsets of receivers. For an $L$-user broadcast channel there are $2^L - 1$ subsets of receivers and correspondingly $2^L - 1$ independent messages. The set of achievable rates for this channel is a $2^L - 1$-dimensional region. There are fundamental constraints on the geometry of this region. For example, observe that if the transmitter is able to simultaneously send $L$ rate-one private messages, errorfree to all receivers, then by sending the same information in each message, it must be able to send a single rate-one common message, errorfree to all receivers. This swapping of private and common messages illustrates that for any broadcast channel, the inclusion of a point $R^*$ in the achievable rate region implies the achievability of a set of other points that are not merely componentwise less than $R^*$. We formerly define this set and characterize it for $L = 2$ and $L = 3$. Whereas for $L = 2$ all the points in the set arise only from operations relating to swapping private and common messages, for $L = 3$ a form of network coding is required.

1 Introduction

The broadcast channel has predominantly been studied in the context of unicast messaging, where the transmitter wishes to send one private message to each of the $L$ receivers (see [1] for example). We refer to this as unicasting. The transmitter may however wish to send different messages to different subsets of receivers. We refer to this as multicasting. The most general multicast structure comprises of $2^L - 1$ messages (the powerset). For $L = 2$ there are three messages, one required only by the first receiver, one required only by the second receiver, and one required by both receivers.

The multicast capacity region for a broadcast channel is the set of $2^L - 1$-dimensional rate vectors that are achievable. For $L = 2$ this is the set of achievable rate vectors...
\((R_1, R_2, R_{12})\), where \(R_{12}\) denotes the rate of the common message. One question of interest is, can the multicast capacity region be inferred from the unicast capacity region? That is, can we always compute the multicast capacity region from the unicast capacity region, i.e. without knowing the structure of the channel? For certain broadcast channels this is true, although it is not true in general. Thus the multicast capacity region provides additional information about the communication limits of the channel beyond that of the unicast capacity region.

Multicasting has received significant attention in the network-coding literature. In [2] and [3] the maximum rate at which a common message can be sent from a source node through a network of directed noiseless links to a collection of sink nodes, is shown to equal the minimum-cut of the associated graph. In [4] and [5] the multicast capacity region for one-source-two-sink networks is fully characterized. It is again given by the minimum-cuts of the associated graph. For three or more sinks this is not the case and the problem is open. In this exposition we shed light on it by characterizing some of the structure for three-sink networks.

There is an oddity to multicasting. Suppose we have a two-user broadcast channel that can support a rate vector \((1, 1, 1)\). That is the transmitter can simultaneously deliver one bit of private information to the first receiver, one bit of private information to the second receiver, and one bit of common information to both receivers. An important point to clarify is that there is no secrecy requirement – “private” information sent to the first receiver may or may not be decodable by the second receiver and vice versa. Then the channel can also support a rate vector \((2, 1, 0)\). The transmitter merely uses the common bit to send private information to the first receiver. Of course the second receiver is capable of decoding this bit too, but the information is of no interest to it. By symmetry the achievability of rate vector \((1, 1, 1)\) also implies the achievability of rate vector \((1, 2, 0)\). There is one more implication in this vein: the achievability of \((1, 1, 1)\) implies the achievability of \((0, 0, 2)\). The reasoning is similar. The transmitter sends the same information on the two private bits. In this way the first user receives the same private bit as the second user, in addition to the same common bit. Thus two common bits have been sent. These three manipulations are summarized in figure as extremal rays stemming from \((1, 1, 1)\) and represent three distinct encoding/decoding operations that can always be performed, regardless of the structure of the broadcast channel. In this sense they are universal. By time-sharing one can achieve any point in the polytope indicated in figure. To summarize: if a rate-vector \((1, 1, 1)\) is achievable, so must be the region illustrated, regardless of the channel. Is this set of operations complete? Put in reverse, are there any rate vectors outside the polytope in figure that are achievable on for all broadcast channels for which \((1, 1, 1)\) is achievable? The answer is that there are not – there exists a broadcast channel where the rate vector \((1, 1, 1)\) is achievable, but no rate vector outside the polytope in figure is. Thus for the two-user broadcast channels the three operations discussed form a complete set - they are the only distinct universal encoding/decoding operations.

It is straightforward to generalize these operations to broadcast channels with an arbi-

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1To be more precise, we define the multicast capacity region of a network as the convex-hull of the union of all multicast capacity regions of broadcast channels that arise from specifying the encoding and decoding operations at intermediate nodes in the network.
trary number of users. Consider for example the three user broadcast channel. There are seven messages. Suppose a rate vector \((R_1, R_2, R_3, R_{12}, R_{13}, R_{23}, R_{123}) = (1, 1, 1, 1, 1, 1, 1)\) is achievable (for example, \(R_{13}\) represents the rate of the message intended for receivers 1 and 3). Then for any two subsets of receivers \(\mathcal{I} \subset \mathcal{J}\) we can perform the operation \(R_{\mathcal{I}} \rightarrow R_{\mathcal{I}} + 1, R_{\mathcal{J}} \rightarrow R_{\mathcal{J}} - 1\), and for any two subsets of receivers \(\mathcal{I} \neq \mathcal{J}\) we can perform the operation \(R_{\mathcal{I}} \rightarrow R_{\mathcal{I}} - 1, R_{\mathcal{J}} \rightarrow R_{\mathcal{J}} - 1, R_{\mathcal{I} \cup \mathcal{J}} \rightarrow R_{\mathcal{I} \cup \mathcal{J}} + 1\). For instance we may swap the first and second receivers’ private bits for a common bit that is sent to the pair, so that the rate vector \((0, 0, 1, 2, 1, 1, 1)\) is achieved. Similarly the rate vector \((0, 1, 1, 1, 0, 2)\) can be achieved by using the first receivers private bit and the bit common to the second and third receivers, to send information common to all three receivers. It can be shown that the number of distinct operations of this form is 15. That is, if the rate vector \((1, 1, 1, 1, 1, 1, 1)\) is achievable, so is the set of points contained within a 15-edged polytope, which is the generalization to \(L = 3\) of the polytope in figure 1.

Again we ask the question, is this set of operations complete? Are there any points outside this 15-edged polytope that are universally achievable on any three-user broadcast channel? The answer, perhaps surprisingly, is yes. There exists a sixteenth distinct universal encoding/decoding operation. It does not involve a mere relabeling of common and private bits. It enables the rate vector \((1, 1, 1, 0, 0, 0, 3)\) to be achieved. This new operation together with the fifteen trivial ones forms the complete set of distinct universal encoding/decoding operations for \(L = 3\). That is, all other rate vectors universally achievable from \((1, 1, 1, 1, 1, 1)\) can be achieved by time sharing between these 16 distinct universal encoding/decoding operations.

Now we turn to the multiple access channel (MAC) with \(L\) users. The MAC has also typically been studied in the context of unicast messaging where it’s capacity region has in many cases been completely characterized. For multicasting the capacity of the discrete memoryless MAC is computed in [6] and a conjecture regarding the generalization of this result to an arbitrary number of users is given.

Let us apply the reasoning we applied above for the broadcast channel, to the MAC. Consider a two user MAC. Each transmitter wishes to send a private message of rate \(R_i\) to the receiver for \(i \in \{1, 2\}\). In addition there is a common message of rate \(R_{12}\) that both transmitters share, and desire to be sent to the receiver. Suppose for a given MAC a rate vector \((R_1, R_2, R_{12}) = (1, 1, 1)\) is achievable. Then the first transmitter could just label its rate-one bit stream as common information and send it to the transmitter. Thus the rate vector \((0, 1, 2)\) is also achievable. By symmetry the second transmitter could do the same so \((1, 0, 2)\) is achievable too. Are there any other operations that tradeoff between elements of the rate vector?\(^2\) The answer is no. For the broadcast channel we could swap common information for private, but not so for the MAC. More specifically we cannot relabel common information as private, as a common bitstream may require both transmitters have access to it in order for it to be passed to the receiver. A private bitstream assumes only a single transmitter has access to it. The \((1, 1, 1)\)-multicast region for the two-user MAC is plotted in figure 2. There are three extremal rays and correspondingly three distinct universal/encoding decoding operations. The first two are stated above and

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\(^2\)We could combine these two arguments to conclude \((0, 0, 3)\) is achievable but we will not be interested in this operation as it can be expressed as a linear combination of others.
the third consists of merely lowering the common rate so as to arrive at the point $(1, 1, 0)$.

Unlike the broadcast channel, this structure directly generalizes to $L$ users. For three users there are ten universal encoding/decoding operations. Six result from relabeling private information as pairwise. Three result from relabeling pairwise as common and the last results from lowering the common rate. Thus the multicast capacity region of the multiple access channel has a less intricate structure than that of the broadcast.

In this paper we characterize the complete set of distinct universal encoding/decoding operations and the associated region of achievable rate vectors, for both the broadcast channel and the MAC channel, for $L = 3$. In essence this is a characterization of the universal constraints on the multicast capacity region of these channels.

Section II describes the notation we use. In section III we describe the problem in detail. Section IV presents the results and section V and VI the proofs.

## 2 Preliminary Notation

We briefly describe some of the notation that will be used. Typically $\mathcal{I}$ and $\mathcal{J}$ will be used to denote subsets of $\{1, 2, 3\}$. For example we may have $\mathcal{I} = \{2, 3\}$, which would imply $R_{\mathcal{I}} \equiv R_{\{2,3\}} \equiv R_{23}$. Rates in bold font represent tuples, for example we may have $\mathbf{R} = (R_1, R_2, R_{12})$. Elements of time series are indicated by a index in parentheses following the variable, for example $Y(i)$. An entire time series is represented by bold font, for example $\mathbf{W}_1 = [W_1(1), \ldots, W_1(n)]$ If $\mathcal{S}$ is a set then $2^\mathcal{S}$ denotes the powerset (the set of all subsets of $\mathcal{S}$) excluding the nullset, e.g. if $\mathcal{S} = \{1, 2\}$ then $2^\mathcal{S} \equiv \{\{1\}, \{2\}, \{1, 2\}\}$. We denote the
Figure 2: The \((1, 1, 1)\)-multicast region for the multiple access channel, \(L = 2\).

nullset by \(\phi\). The symbol \(\preceq\) denotes element-wise inequality.

3 Problem Setup

Consider a broadcast channel with three receivers. The input alphabet is denoted \(\mathcal{X}\) and the output alphabets \(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\). The probability transition function is \(p(y_1, y_2, y_3|x)\). The message vector is

\[
(W_1, W_2, W_3, W_{12}, W_{13}, W_{23}, W_{123}).
\]

The subscript denotes the subset of receivers for which the message in intended, for example message \(W_{23}\) is intended for receivers 2 and 3. Denote the rate vector \(\mathbf{R} = (R_1, R_2, R_3, R_{12}, R_{13}, R_{23}, R_{123})\). A \((2^n\mathbf{R}, n)\) code consists of an encoder

\[
x^n : \prod_{\mathcal{I} \subseteq \{1, 2, 3\}} \{1, \ldots, 2^{nR_\mathcal{I}}\} \to \mathcal{X}^n
\]

and twelve decoders

\[
\hat{W}_{i, \mathcal{I}} : \mathcal{Y}_i^n \to 2^{nR_\mathcal{I}}
\]

where \(i \in \{1, 2, 3\}\) denotes the receiver and \(\mathcal{I} \subseteq \{1, 2, 3\}\) with \(i \in \mathcal{I}\) denotes the message index. Thus each receiver decodes four messages (the first receiver decodes \(W_1, W_{12}, W_{13}, W_{123}\), etc...). The probability of error \(P_e^{(n)}\) is defined to be the probability that at least one of
the decoded messages is not equal to the transmitted message, i.e.

$$P_{e}^{(n)} = P \left( \bigcup_{\mathcal{I} \subseteq \{1,2,3\}} \left\{ \hat{W}_{i,\mathcal{I}}(Y_{i}^{n}) \neq W_{i,\mathcal{I}} \right\} \right).$$

where the seven messages are assumed to be mutually independent and uniformly distributed over $\prod_{\mathcal{I} \subseteq \{1,2,3\}} \{1, \ldots, 2^{nR_{\mathcal{I}}}\}$.

**Definition 3.1.** A multicast rate vector $\mathbf{R}$ is said to be achievable for the broadcast channel if there exists a sequence of $(2^{nR}, n)$ codes with $P_{e}^{(n)} \to 0$.

**Definition 3.2.** The multicast capacity region of the broadcast channel is the closure of the set of achievable multicast rate vectors. It is denoted $C_{p(y_{1},y_{2},y_{3}|x)}$ or $\mathcal{C}$ for short.

Often we will omit the adjective ‘multicast’.

We now give a definition that makes precise the operation of swapping common and private messages, and quantifies the change in the rate vector. Let $\mathbf{R}^{W}$ and $\mathbf{R}^{M}$ be two rate vectors.
Definition 3.3. A \((d\mathbf{R}, n)\)-universal encoding/decoding operation is a pair of mappings

\[
W_{J} : \prod_{I \subseteq \{1,2,3\}} \{1, \ldots, 2^{nR_{I}^{W}}\} \rightarrow \{1, \ldots, 2^{nR_{I}^{W}}\},
\]

\[
\hat{M}_{i,J} : \prod_{J \subseteq \{1,2,3\}} \{1, \ldots, 2^{nR_{i,J}^{W}}\} \rightarrow \{1, \ldots, 2^{nR_{i,J}^{W}}\}
\]

for all \(J \subseteq \{1,2,3\}\) and all \(i \in \{1,2,3\}\) such that \(i \in I\), with the properties \(M_{i,J} = \hat{M}_{i,J}\) for all \(i, j \in \{1,2,3\}\), \(\mathbf{R}^{M} \neq \mathbf{R}^{W}\) and

\[
\frac{\mathbf{R}^{M} - \mathbf{R}^{W}}{\|\mathbf{R}^{M} - \mathbf{R}^{W}\|} = d\mathbf{R},
\]

\(W(M)\) being the universal encoder output and \(\hat{M}(\hat{W})\) being the universal decoder output. The vector \(d\mathbf{R}\) is referred to as the ‘normalized difference vector’.

The property \(\hat{M}_{i,J} = \hat{M}_{j,J}\) for all \(i, j \in \{1,2,3\}\) ensures that all users agree on the common messages they decode. See figure 3 for a system diagram that illustrates the relationship between \(M, W, \hat{M}\) and \(\hat{W}\).

Example 3.4. Suppose \(\mathbf{R}^{W} = (1,0,0,1,0,0,0)\) and \(\mathbf{R}^{M} = (2,0,0,0,0,0,0)\). Let \(n = 1\). Then the mapping \(W_{1}(M) = M_{1}(1)\), \(W_{12}(M) = M_{1}(2)\) is a universal encoding operation with \(d\mathbf{R} = (1,0,0, -1,0,0,0)\). The universal decoding operation is the inverse mapping given by \(\hat{M}_{1}(\hat{W}) = [\hat{W}_{1}, \hat{W}_{12}]\).

Definition 3.5. A \(d\mathbf{R}\)-universal encoding/decoding operation is called ‘distinct’ if the vector \(d\mathbf{R}\) cannot be expressed as positive linear combination of vectors \(\{d\mathbf{R}_{i}\} \neq d\mathbf{R}\) for which there exist \(d\mathbf{R}_{i}\)-universal encoding/decoding operations for \(i = 1,2,\ldots\). The (rays associated with the) normalized difference vectors corresponding to distinct \(d\mathbf{R}\)-universal encoding/decoding operations are called ‘extremal rays’.

By positive linear combination we mean a weighted linear sum with non-negative coefficients.

Example 3.6. It will be evident later that the universal encoding/decoding operation given in example 3.4 is distinct and thus \((1,0,0, -1,0,0,0)/\sqrt{2}\) is an extremal ray. By symmetry \((0,1,0, -1,0,0)/\sqrt{2}\) is also an extremal ray. Note distinctness does not imply uniqueness—the universal encoding/decoding operation that moves from rate vector \(\mathbf{R}^{W} = (1,0,0,1,0,0,0)\) to rate vector \(\mathbf{R}^{M} = (1.5,0,0,0.5,0,0,0)\) is also classified as distinct, but it has the same normalized difference vector. An example of a universal encoding/decoding operation that is not distinct is one that moves from rate vector \(\mathbf{R}^{W} = (1,0,0,1,0,0,0)\) to rate vector \(\mathbf{R}^{M} = (1.5,0.5,0,0,0,0,0)\). Denote the corresponding normalized difference vector is \(d\mathbf{R}_{A} \triangleq (0.5,0.5,0, -1,0,0,0)/\sqrt{1.5}\). The universal encoding/decoding operations that achieve this shift correspond to time-sharing between two operations, one with normalized difference vector \(d\mathbf{R}_{B} \triangleq (1,0,0, -1,0,0,0)/\sqrt{2}\), the other with normalized difference vector \(d\mathbf{R}_{C} \triangleq (0,1,0, -1,0,0,0)/\sqrt{2}\). Indeed we have

\[
d\mathbf{R}_{A} = \frac{1}{\sqrt{3}}d\mathbf{R}_{B} + \frac{1}{\sqrt{3}}d\mathbf{R}_{C}.
\]
\[ G_{BC,2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad H_{BC,2} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \]

\[ G_{MAC,2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad H_{MAC,2} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \]

Figure 4: Results for \( L = 2 \).

We now give a formal definition of the region alluded to in figure 1. Let

\[ R^* = (R_1^*, R_2^*, R_3^*, R_{12}^*, R_{13}^*, R_{23}^*, R_{123}^*) \]

be a parameter.

**Definition 3.7.** The ‘\( R^* \)-multicast region’ is the intersection of the capacity regions of all broadcast channels for which the rate vector \( R^* \) is achievable, i.e.

\[ \bigcap_{p(y_1, y_2, y_3 | x)} C_{p(y_1, y_2, y_3 | x)} \cap_{R^* \in C_{p(y_1, y_2, y_3 | x)}} \]

See figures 1 for examples of this region.

As the problem setup for the multiple access channel is entirely analogous to the aforementioned setup for the broadcast channel, we do not explicitly describe it. An example of the \( R^* \)-multicast region is given in figure 2.

The aim of this paper is to characterize the \( R^* \)-multicast cones for both the broadcast and multiple access channels.

## 4 Results

**Theorem 4.1.** For \( L = 3 \) the \( R^* \)-multicast region of the broadcast channel is the set of all \( R \in \mathbb{R}_+^7 \) satisfying

\[ G_{BC,3}^T (R - R^*) \preceq 0 \]  \hspace{1cm} (1)

where \( G_{BC,3} \) is given in figure 4. This region is a polytope, characterized by the cone \( \{ R \in \mathbb{R}_+^7 : G_{BC,3}^T R \preceq 0 \} \). We refer to this cone as the \( L = 3 \) ‘multicast cone’. The sixteen extremal rays of this cone are given by the columns of the matrix \( H_{BC,3} \) in figure 4. Thus there are 16 distinct universal encoding/decoding operations for \( L = 3 \).

The \((1, 1, 1)\)-multicast region for the broadcast channel for \( L = 2 \) is illustrated in figure 1. For \( L = 2 \) there are 3 distinct universal encoding/decoding operations. The \( G_{BC,2} \) and \( H_{BC,2} \) matrices are given in figure 4. The columns of \( G_{BC,2} \) are the normal vectors to the three hyperplanes bounding the region. The columns of \( H_{BC,2} \) are the three extremal rays (see figure 1).
\[ G_{BC,3} = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 2
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2
1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2
1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2
\end{bmatrix} \]

\[ H_{BC,3} = \begin{bmatrix}
-1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
-1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1
0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & -1
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & -1
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 2
\end{bmatrix} \]

\[ G_{MAC,3} = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \]

\[ H_{MAC,3} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0
\end{bmatrix} \]

Figure 5: Results for \( L = 3 \).
Theorem 4.2. For $L = 3$ the $R^*$-multicast region of the multiple access channel is the set of all $R \in \mathbb{R}^7_+$ satisfying

$$G^{T}_{MAC,3}(R - R^*) \preceq 0 \quad (2)$$

where $G_{MAC,3}$ is given in figure 3. This region is also a polytope characterized by the cone

$\{R \in \mathbb{R}^7 : G^{T}_{MAC,3}R \preceq 0\}$. The 10 extremal rays of this cone are given by the columns of the matrix $H_{MAC,3}$ in figure 4. Thus there are 10 distinct universal encoding/decoding operations for $L = 3$.

The $(1,1,1)$-multicast region for the MAC for $L = 2$ is illustrated in figure 2. There are 3 distinct universal encoding/decoding operations. The $G_{MAC,2}$ and $H_{MAC,2}$ matrices are given in figure 4.

An alternative interpretation of theorem 4.1 is the following (the same interpretation applies for 4.2). For notational simplicity we denote the capacity region of an arbitrary broadcast channel by $C$. Let

$$R^*(\alpha) = \arg \max_{R \in C} \alpha^T R$$

$R^*(\alpha)$ is the rate vector lying on the boundary of the capacity region in the direction of $\alpha$. Let

$$C^*(\alpha) = \{R \in \mathbb{R}^7_+ | \alpha^T R \leq \alpha^T R^*(\alpha)\}.$$  

$C^*(\alpha)$ is the halfspace of all rate vectors lying underneath the hyperplane $\alpha^T R = \alpha^T R^*(\alpha)$. The region $C$ is convex and thus we can characterize it by its support function $C^*(\alpha)$, i.e.

$$C = \bigcap_{\alpha \in \mathbb{R}^7_+} C(\alpha).$$

However this is not the minimal dual representation of $C$. Let

$$H^*_3 \triangleq \{\alpha \in \mathbb{R}^7_+ | \alpha^T H_{BC,3} \preceq 0\}$$

Corollary 4.3. The multicast capacity region of an any broadcast channel with three receivers can be expressed as

$$C = \bigcap_{\alpha \in H} C(\alpha).$$

if and only if

$$H \supseteq H^*_3.$$  

This says the following: when computing the multicast capacity region of a broadcast channel by maximizing the weighted sum-rate, the smallest set that one need vary the weighting coefficients $\alpha$ over is $H^*_3$. Put another way, the normal vector $\alpha$ to any point on the boundary of the multicast capacity region is always contained in the set $H^*_3$. See figure 0.
Figure 6: The normal vector $\alpha$ of the broadcast channel capacity region satisfies $\alpha^T \in \mathcal{H}^*$.  
(a) A capacity region that cannot occur.  (b) A capacity region that can occur.
5 Proof of Theorem 4.1

The direct part of the proof consists of showing that for any broadcast channel, if a rate vector $R^*$ is achievable then all rate vectors in the region given by equation (1) are achievable. This establishes that the $R^*$-multicast region is ‘at least as large’ as the region given by equation (1). The converse part of the proof consists of illustrating, for each $R^* \in \mathbb{R}_+^7$, a broadcast channel for which no rate vector outside the region given by equation (1) is achievable. This establishes that the $R^*$-multicast region is ‘at least as small’ as the region given by equation (1). We start with the direct part. For notational simplicity we drop the broadcast channel (BC) subscript.

5.1 Direct Part

Suppose that $R^*$ is achievable for a particular broadcast channel. We show that any rate-vector $R \in \mathbb{R}_+^7$ satisfying

$$R \preceq R^* + H_{BC,3}\Delta$$

for $\Delta \in \mathbb{R}_+^{16}$ is also achievable. We then show that this region is precisely the one given in equation (1). Let $\Delta_i$ denote the $i$th element of $\Delta$ and $H_{BC,3}(i)$ denote the $i$th column of $H_{BC,3}$. To show that any rate-vector satisfying equation (2) is achievable, we show that each of the 16 rate-vectors given by

$$R^{(i)} = R^* + H_{BC,3}(i)\Delta^*_i, \quad i = 1, \ldots, 16$$

are achievable where

$$\Delta^*_i = \max_{H_{BC,3}(i)\Delta \preceq R^*} \Delta$$

By time sharing between these vectors the entire boundary region \{ $R^* - H_{BC,3}\Delta | \Delta \in \mathbb{R}_+^{16}$ \} is achieved and hence any point within it (i.e. satisfying equation (2)) can also be achieved.

Let $M, \hat{M}$ correspond to the binary message vector and estimate of the message vector, respectively, that the transmitter wishes to send at rate vector $R^{(i)}$. We illustrate the achievability of equation (3) for $i = 3$.

To universally encode for $i = 3$, assume without loss of generality that $R^*_1 \leq R^*_2$. In what follows we ignore rounding effects as it will be clear that in the limit $n \to \infty$ they are negligible. Set

$$W_1^n = [M_{12}(1), \ldots, M_{12}(nR^*_1)]$$

$$W_2^n = [M_{12}(1), \ldots, M_{12}(nR^*_1), M_2(1), \ldots, M_2(nR^*_2 - nR^*_1)]$$

$$W_{12}^n = [M_{12}(nR^*_1 + 1), \ldots, M_{12}(nR^*_1 + nR^*_2)]$$

In words, the information common to receivers 1 and 2 is split into two parts. The first part is replicated and sent separately down both receiver 1 and receiver 2’s private channels. The second part is sent down the channel common to both receivers. As receiver 2’s private channel can accommodate a higher bit-rate than receiver 1’s, there is some bandwidth left over. This is allocated to sending some of receiver 2’s private information.
For all other subsets $\mathcal{I}$ of $\{1, 2, 3\}$ set $W^n_{\mathcal{I}} = M^n_{\mathcal{I}}$ and $\hat{M}^n_{\mathcal{I}} = \hat{W}^n_{\mathcal{I}}$. Universal decoding is straightforward. The first receiver sets 

$$
\hat{M}^n_{1,12} = [\hat{W}^n_1, \hat{W}^n_{12}] \\
\hat{M}^n_{1,13} = \hat{W}^n_{13} \\
\hat{M}^n_{1,123} = \hat{W}^n_{123}
$$

and in this way successfully recovers its message, as the achievability of $R^*$ implies that $W$ was decoded correctly. The second receiver sets 

$$
\hat{M}^n_{2,2} = [\hat{W}_2(nR^*_1 + 1), \ldots, \hat{W}_2(nR^*_2)] \\
\hat{M}^n_{2,12} = [\hat{W}_2(1), \ldots, \hat{W}_2(nR^*_1), \hat{W}^n_{12}] \\
\hat{M}^n_{2,23} = \hat{W}^n_{23} \\
\hat{M}^n_{2,123} = \hat{W}^n_{123}
$$

and is similarly successful in decoding. The third receivers sets $\hat{M}^n_{\mathcal{I}} = \hat{W}^n_{\mathcal{I}}$ for all of its messages. Then we have achieved a rate vector of 

$$
R^{(3)} = R^* + \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} R^*_1
$$

$$
= R^* + H_{BC,3}(3) \Delta_3.
$$

with $\Delta_3 = R^*_1$. The universal encoding and decoding procedures for all other $i \in \{1, \ldots, 15\}$ are similar and follow from the structure of the columns of the matrix $H_{BC,3}$.

Universal encoding and decoding for $i = 16$ is different. Assume without loss of generality that $R^*_1 \leq R^*_2 \leq R^*_3$. To encode, set $W^n_i = M^n_i$ for $i = 1, 2, 3$ and

$$
W^n_{12} = [M_{123}(1), \ldots, M_{123}(nR^*_1)] \\
W^n_{13} = [M_{123}(nR^*_1 + 1), \ldots, M_{123}(2nR^*_1), M_{13}(1), M_{13}(nR^*_2 - nR^*_1)] \\
W^n_{23} = [M_{123}(1) \oplus M_{123}(nR^*_2 + 1), \ldots, M_{123}(2nR^*_2) \oplus M_{123}(2nR^*_1), M_{23}(1), \ldots, M_{23}(nR^*_3 - nR^*_2)] \\
W^n_{123} = [M^n_{1,23}(2nR^*_1), \ldots, M^n_{1,23}(2nR^*_1 + nR^*_2)]
$$

In words, the information common to all receivers is split into three streams. The first and second are sent at rate $R^*_1$ using the three pairwise links. The third stream is sent at rate $R^*_1$ across the link common to all receivers.

The first receiver decodes by setting $M^n_{1,1} = \hat{W}^n_1$ and 

$$
M^n_{13} = [W_{13}(nR^*_1 + 1), \ldots, W_{13}(nR^*_2)] \\
M^n_{123} = [W^n_{12}, W^n_{13}, W^n_{123}].
$$
The second receiver decodes by setting \( \hat{M}_{1,2}^n = \hat{W}_2^n \) and

\[
M_{23}^n = [W_{23}(nR_{12}^* + 1), \ldots, W_{23}(nR_{23}^*)]
\]

\[
M_{123}^n = [W_{12}^n, W_{12}^n \oplus W_{23}^n, W_{123}^n]
\]

The third receiver decodes by setting \( \hat{M}_{1,3}^n = \hat{W}_3^n \) and

\[
M_{13}^n = [W_{13}(nR_{12}^* + 1), \ldots, W_{13}(nR_{13}^*)]
\]

\[
M_{23}^n = [W_{23}(nR_{12}^* + 1), \ldots, W_{23}(nR_{23}^*)]
\]

\[
M_{123}^n = [W_{13}^n, W_{13}^n \oplus W_{23}^n, W_{123}^n]
\]

Then we have achieved a rate vector of

\[
R^{(3)} = R^* + \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 2 \end{bmatrix} R_{12}^*
\]

\[
= R^* + H_{BC,3}(16) \Delta_{16}
\]

with \( \Delta_{16} = R_{12}^* \). Thus the 16 rate vectors satisfying equation (3) are achievable and by time sharing between them, all rate vectors in the region given by equation (2) are achievable.

It remains to show that this region is equivalent to the one in equation (1), i.e. that for any \( R^* \in \mathbb{R}_+^7 \)

\[
\{ R \in \mathbb{R}_+^7 \mid G_{BC,3}^T R \preceq G_{BC,3}^T R \} \equiv \{ R \in \mathbb{R}_+^7 \mid R \preceq R^* + H_{BC,3} \Delta, \forall \Delta \in \mathbb{R}_+^{16} \}
\]

On the left is the characterization of the polytope in terms of the hyperplanes bounding it. On the right is the dual characterization in terms of the edges of the polytope (1-dimensional facets). This equivalence can be demonstrated using computer software such as polymake.

### 5.2 Converse Part

To establish the converse we now present, for each \( R^* \), a particular (deterministic) broadcast channel and show its capacity region is equal to (1). Let the input alphabet \( \mathcal{X} = \prod_{I \subseteq \{1,2,3\}} \{0, \ldots, 2^{nR_{ij}^*} - 1 \} \) with the \( i \)th channel input

\[
X(i) = [X_1(i), X_2(i), X_3(i), X_{12}(i), X_{13}(i), X_{23}(i), X_{123}(i)]
\]

so that each \( X_I \in \{0, \ldots, 2^{nR_{ij}^*} - 1 \} \), and let \( \mathcal{Y}_i \in \prod_{I \subseteq \{1,2,3\}, i \in I} \{0, \ldots, 2^{nR_{ij}^*} - 1 \} \) for \( i = 1, 2, 3 \) with

\[
Y_1(i) = [X_1(i), X_{12}(i), X_{13}(i), X_{123}(i)]
\]

\[
Y_2(i) = [X_2(i), X_{12}(i), X_{23}(i), X_{123}(i)]
\]

\[
Y_3(i) = [X_3(i), X_{13}(i), X_{23}(i), X_{123}(i)]
\]
Figure 7: Illustration of the deterministic broadcast channel used in converse.

See figure 7 for an illustration of the channel. Suppose the channel is used \( n \) times. The messages to be transmitted are \( W \sim U(\{1, \ldots, 2^n R_I\}) \) and mutually independent. Denote the length-\( n \) vector of channel inputs by \( X \) and the length-\( n \) vectors of channel outputs by \( Y_1, Y_2 \) and \( Y_3 \). Let \( G_{BC,3}(i) \) denote the \( i \)th column of \( G_{BC,3} \). We wish to show

\[
G_{BC,3}(i)^T R \leq G_{BC,3}(i)^T R^*
\]

for \( i = 1, \ldots, 15 \). Before this we introduce some notation. Suppose \( A \) is a collection of subsets of \( \{1, 2, 3\} \), for example \( A = \{1, 2, 12, 13, 123\} \). The collection \( A \) should be thought as the indices of a subset of the seven channel links (see figure 7), for example \( A = \{1, 12\} \) corresponds to two links, the private one from \( X_1 \) to \( Y_{1,1} \) and the common one from \( X_{123} \) to \( Y_{1,123}, Y_{2,123}, Y_{3,123} \). By \( \lfloor A \rfloor \) we denote the indices of the messages intended for those receivers cut by \( A \). For example if \( A = \{1, 2, 12, 13, 123\} \) then all links to the first receiver are cut, but not all links to the second or the third. As the first receiver is sent the messages \( W_1, W_{12}, W_{13} \) and \( W_{123} \), we have \( \lfloor A \rfloor = \{1, 12, 13, 123\} \). As another example let \( A = \{2, 3, 12, 13, 23, 123\} \). Then all links to both the second and third receivers are cut and \( \lfloor A \rfloor = \{2, 3, 12, 13, 23, 123\} = A \). As a final example if \( A = \{1, 12, 23, 123\} \) no receivers are completely cut, and thus \( \lfloor A \rfloor = \emptyset \).

**Lemma 5.1.** Let \( A_1, A_2 \) and \( A_3 \) be three collections of subsets of \( \{1, 2, 3\} \) such that either \( A_1 \subseteq A_2 \cup A_3, A_2 \subseteq A_1 \cup A_3 \) or \( A_3 \subseteq A_1 \cup A_2 \). Then

\[
\sum_{I \in [A_1 \cup A_2 \cup A_3]} R_I + \sum_{I \in [A_1 \cup A_2] \cap [A_1 \cup A_3]} R_I + \sum_{I \in [A_1] \cap [A_2] \cap [A_3]} R_I \leq \sum_{I \in A_1} R_I^* + \sum_{I \in A_2} R_I^* + \sum_{I \in A_3} R_I^*
\]

This lemma is a generalization of the cutset bounds to multiple subsets of cuts. Indeed if
we set $A_2 = \phi$ and $A_3 = \phi$ we are left with

$$\sum_{I \in |A_1|} R_I \leq \sum_{I \in A_1} R_I^*$$

which are precisely the cutset bounds.

Proof.

$$n \left( \sum_{I \in A_1} R_I^* + \sum_{I \in A_2} R_I^* + \sum_{I \in A_3} R_I^* \right)$$

$$\geq \sum_{I \in A_1} H(X_I) + \sum_{I \in A_2} H(X_I) + \sum_{I \in A_3} H(X_I)$$

$$\geq H(\bigcup_{I \in A_1} X_I) + H(\bigcup_{I \in A_2} X_I) + H(\bigcup_{I \in A_3} X_I)$$

$$= H(\bigcup_{I \in A_1 \cup A_2 \cup A_3} X_I) + I(\bigcup_{I \in A_1 \cup A_2} X_I; \bigcup_{I \in A_1} X_I; \bigcup_{I \in A_2} X_I; \bigcup_{I \in A_3} X_I)$$

$$+ I(\bigcup_{I \in A_1} X_I; \bigcup_{I \in A_2} X_I; \bigcup_{I \in A_3} X_I)$$

$$\geq H(\bigcup_{I \in |A_1 \cup A_2 \cup A_3|} W_I) + H(\bigcup_{I \in |A_1 \cup A_2| \cap |A_1 \cup A_3| \cap |A_2 \cup A_3|} W_I)$$

$$+ H(\bigcup_{I \in |A_1| \cap |A_2| \cap |A_3|} W_I) + \epsilon_n$$

$$= n \left( \sum_{I \in |A_1 \cup A_2 \cup A_3|} R_I + \sum_{I \in |A_1 \cup A_2| \cap |A_1 \cup A_3| \cap |A_2 \cup A_3|} R_I + \sum_{I \in |A_1| \cap |A_2| \cap |A_3|} R_I \right)$$

where the third step follows from lemma 7.2 in the appendix and the fourth from the requirement $P_{e^{(n)}} \to 0$ (Fano’s inequality) and lemma 7.1 in the appendix.

Applying lemma 5.1 to the sets of indices in table 1 establishes equation (11) for columns $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15$ of $G_{BC,3}$. Unfortunately for column $i = 11$ the condition that either $A_1 \subseteq A_2 \cup A_3$, $A_2 \subseteq A_1 \cup A_3$ or $A_3 \subseteq A_1 \cup A_2$ must hold, is violated. Consequently the 11th converse bound is established in a different fashion.

Let $A_1, A_2, A_3$ be defined by the 11th row of table 1. Then
\begin{align*}
&n \left( \sum_{I \in \mathcal{A}_1} R^*_I + \sum_{I \in \mathcal{A}_2} R^*_I + \sum_{I \in \mathcal{A}_3} R^*_I \right) \\
&\geq \sum_{I \in \mathcal{A}_1} H(X_I) + \sum_{I \in \mathcal{A}_2} H(X_I) + \sum_{I \in \mathcal{A}_3} H(X_I) \\
&\geq H(\bigcup_{I \in \mathcal{A}_1} X_I) + H(\bigcup_{I \in \mathcal{A}_2} X_I) + H(\bigcup_{I \in \mathcal{A}_3} X_I) \\
&\geq H(\bigcup_{I \in \mathcal{A}_1} X_I) + H(\bigcup_{I \in \mathcal{A}_2} X_I) + H(\bigcup_{I \in \mathcal{A}_3} X_I) - H(X_{123}) \\
&\geq H(X_{123} | W_1, W_{12}, W_{13}, W_{123}) + H(W_1, W_{12}, W_{13}, W_{123}) \\
&+ H(X_{123} | W_2, W_{12}, W_{23}, W_{123}) + H(W_2, W_{12}, W_{23}, W_{123}) \\
&+ H(X_{123} | W_3, W_{13}, W_{23}, W_{123}) + H(W_3, W_{13}, W_{23}, W_{123}) - H(X_{123}) \\
&\geq n (R_1 + R_2 + R_3 + 2R_{12} + 2R_{13} + 2R_{23} + 2R_{123}) \\
\end{align*}

where the fourth step follows from the requirement $P_e(n) \to 0$ (Fano’s inequality), the sixth from lemma 7.3 and the seventh from the independence of the messages.

6 Proof of Theorem 4.2

The direct part of this proof is entirely analogous to the direct part for the broadcast channel. This establishes the universal achievability of the $R^*$-multicast region. The converse
part is different. For each $R^*$ we present a sequence of channels. The limiting intersection of the capacity regions of these channels is the region in equation (2). The capacity regions of these channels are not precisely computed, but only outer bounded in a manner sufficient to establish their limiting intersection.

6.1 Direct part

As this part of the proof is trivial and entirely analogous section 5.1 we only provide a sketch. In essence we need to establish that each of the columns of $H_{MAC,3}$ are achievable in the sense of section 5.1. The first column is achieved by transmitting additional $M_{12}$ bits on the $W_1$ channel, the second column is achieved by transmitting additional $M_{13}$ bits on the $W_1$ channel, the third column is achieved by transmitting additional $M_{12}$ bits on the $W_2$ channel, and so on. The last column is achieved by lowering the rate of the $M_{123}$ message.

6.2 Converse part

For each $R^*$ we present a sequence of deterministic channels with capacity region tending to the region in equation (2). The capacity regions of these channels are not explicitly computed, only outer bounded, but we show the limiting outer bound is tight. The sequence is parameterized by the integer $k$.

Let $R^*$ be given and assume its elements are rational. Denote their numerators and denominators by $N_I$ and $D_I$, for $I \subseteq \{1, 2, 3\}$ so that $R^* = (N_1/D_1, \ldots, N_{123}/D_{123})$. Let $l = \text{LCM}(D_1, \ldots, D_{123})$. The $k$th channel is defined as follows. See figure 10 for a pictorial representation. Every $k \times l$ time steps the channel takes in a triple of inputs and outputs one symbol. The input alphabet is $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ where

$\mathcal{X}_1 = \{0, 1\}^{kN_1} \times \{0, 1\}^{kN_{12}} \times \{0, 1\}^{kN_{13}} \times \{0, 1\}^{kN_{123}}$

$\mathcal{X}_2 = \{0, 1\}^{kN_2} \times \{0, 1\}^{kN_{12}} \times \{0, 1\}^{kN_{13}} \times \{0, 1\}^{kN_{123}}$

$\mathcal{X}_3 = \{0, 1\}^{kN_3} \times \{0, 1\}^{kN_{13}} \times \{0, 1\}^{kN_{23}} \times \{0, 1\}^{kN_{123}}$

The output alphabet is

$\mathcal{Y} = \{0, 1\}^{kN_1} \times \{0, 1\}^{kN_2} \times \{0, 1\}^{kN_3}$

$\times (\{0, 1\}^{kN_{12}} \cup \{e\}) \times (\{0, 1\}^{kN_{13}} \cup \{e\}) \times (\{0, 1\}^{kN_{23}} \cup \{e\}) \times (\{0, 1\}^{kN_{123}} \cup \{e\})$

where $e$ is an output symbol that can be thought of as an erasure. The channel thus decomposes into one with $4 \times 3 = 12$ inputs and 7 outputs. The outputs at time $i$ are related deterministically to the inputs at time $i$ via

$Y_1(i) = X_{1,1}(i)$

$Y_2(i) = X_{2,1}(i)$

$Y_3(i) = X_{3,1}(i)$
The input streams thus consist of blocks of $kN_i$ bits. The output streams $Y_1(i), Y_2(i), Y_3(i)$ match their associated input streams. The output stream $Y_{12}(i)$ matches its associated input streams if and only if the input streams match at each bit, otherwise the erasure symbol is outputted. Likewise for the other output streams. For this reason the boxes inside the channel in figure 9 are labeled 'coordination channel'. See figure 10 for a pictorial example of one such coordination channel. The idea of the coordination channels is that in the limit of large $k$, they only let common information through. This should be intuitive from their definition and from the figure.

We now bound the capacity region of this channel. It is clear that we can further decompose the channel into seven parallel channels, one linking $X_{1,1}$ and $Y_1$, one linking $X_{2,1}$ and $Y_2$, one linking $X_{3,1}$ and $Y_3$, one linking $(X_{1,12}, X_{2,12})$ and $Y_{12}$, one linking $(X_{1,13}, X_{3,13})$ and $Y_{13}$, one linking $(X_{2,23}, X_{3,23})$ and $Y_{23}$, and one linking $(X_{1,123}, X_{2,123}, X_{3,123})$ and $Y_{123}$. The capacity region of the channel in question is thus the Minkowski sum of the capacity regions of these seven channels. Denote these seven capacity regions by $C^k_I$ for $I \subseteq \{1, 2, 3\}$. Then
| X_{1}(i) | X_{2}(i) | Y(i) |
|----------|----------|------|
| 0        | 0        | 0    |
| 0        | 1        | e    |
| 1        | 0        |      |
| 1        | 1        | 1    |

(a)

| X_{1}(2i), X_{1}(2i+1) | X_{2}(2i), X_{2}(2i+1) | Y(2i), Y(2i+1) |
|------------------------|------------------------|-----------------|
| 0,0                    | 0,0                    | 0,0             |
| 0,0                    | 0,1                    |                 |
| 0,0                    | 1,0                    |                 |
| 0,0                    | 1,1                    |                 |
| 0,1                    | 0,0                    |                 |
| 0,1                    | 0,1                    |                 |
| 0,1                    | 1,0                    |                 |
| 0,1                    | 1,1                    |                 |
| 1,0                    | 0,0                    |                 |
| 1,0                    | 0,1                    |                 |
| 1,0                    | 1,0                    |                 |
| 1,0                    | 1,1                    |                 |
| 1,1                    | 0,0                    |                 |
| 1,1                    | 0,1                    |                 |
| 1,1                    | 1,0                    |                 |
| 1,1                    | 1,1                    |                 |

(b)

Figure 10: (a) a coordination channel for \(k = 1\). (b) a coordination channel for \(k = 2\).
the capacity region of our channel is given by

\[ C^k = \sum_{\mathcal{I} \subseteq \{1,2,3\}} \mathcal{C}^k_{\mathcal{I}}. \]

where sigma denotes the Minkowski sum. In particular we wish to compute the limiting intersection of these regions

\[ \mathcal{C} = \lim_{K \to \infty} \bigcap_{k=1}^{K} \mathcal{C}^k = \sum_{\mathcal{I} \subseteq \{1,2,3\}} \lim_{K \to \infty} \bigcap_{k=1}^{K} \mathcal{C}^k_{\mathcal{I}} = \sum_{\mathcal{I} \subseteq \{1,2,3\}} \mathcal{C}_{\mathcal{I}}. \]

Lemma 6.1.

1. The region \( \mathcal{C}_1 \) is the set of all \( \mathbf{R} \in \mathbb{R}^+_7 \) satisfying \( R_1 + R_{12} + R_{13} + R_{123} \leq R^*_1 \) and \( R_{\mathcal{I}} = 0 \) for \( \mathcal{I} \in \{2,3,23\} \),

2. The region \( \mathcal{C}_2 \) is the set of all \( \mathbf{R} \in \mathbb{R}^+_7 \) satisfying \( R_2 + R_{12} + R_{23} + R_{123} \leq R^*_2 \) and \( R_{\mathcal{I}} = 0 \) for \( \mathcal{I} \in \{1,3,13\} \),

3. The region \( \mathcal{C}_3 \) is the set of all \( \mathbf{R} \in \mathbb{R}^+_7 \) satisfying \( R_3 + R_{13} + R_{23} + R_{123} \leq R^*_3 \) and \( R_{\mathcal{I}} = 0 \) for \( \mathcal{I} \in \{1,2,12\} \),

4. The region \( \mathcal{C}_{12} \) is the set of all \( \mathbf{R} \in \mathbb{R}^+_7 \) satisfying \( R_{12} + R_{123} \leq R^*_{12} \) and \( R_{\mathcal{I}} = 0 \) for \( \mathcal{I} \in \{1,2,3,13,23\} \),

5. The region \( \mathcal{C}_{13} \) is the set of all \( \mathbf{R} \in \mathbb{R}^+_7 \) satisfying \( R_{13} + R_{123} \leq R^*_{13} \) and \( R_{\mathcal{I}} = 0 \) for \( \mathcal{I} \in \{1,2,3,12,23\} \),

6. The region \( \mathcal{C}_{23} \) is the set of all \( \mathbf{R} \in \mathbb{R}^+_7 \) satisfying \( R_{23} + R_{123} \leq R^*_{23} \) and \( R_{\mathcal{I}} = 0 \) for \( \mathcal{I} \in \{1,2,3,12,13\} \),

7. The region \( \mathcal{C}_{123} \) is the set of all \( \mathbf{R} \in \mathbb{R}^+_7 \) satisfying \( R_{123} \leq R^*_{123} \) and \( R_{\mathcal{I}} = 0 \) for \( \mathcal{I} \in \{1,2,3,12,13,23\} \).

Proof. The first three regions are trivial. We establish the fourth. The messages \( \mathbf{W}_{\mathcal{I}} \) are uniformly distributed on \( \{1, \ldots, 2^{nR_{\mathcal{I}}}\} \) and mutually independent for fixed \( n \). Denote the \( n \)-length output sequence by \( \mathbf{Y}_{12} \). By Fano’s inequality we must have \( H(\cup_{\mathcal{I} \subseteq \{1,2,3\}} \mathbf{W}_{\mathcal{I}} \mid \mathbf{Y}_{12}) \leq \epsilon_n \) with \( \epsilon_n \to 0 \) as \( n \to \infty \) in order for the error probability to be made arbitrarily small. Thus by the mutual independence of the messages we have

\[
n(R_1 + R_2 + R_3 + R_{13} + R_{23}) = H(\mathbf{W}_1) + H(\mathbf{W}_2) + H(\mathbf{W}_{13}) + H(\mathbf{W}_{23}) \\
\leq H(\cup_{\mathcal{I} \subseteq \{1,2,3\}} \mathbf{W}_{\mathcal{I}}, \mathbf{Y}_{12}) - H(\mathbf{W}_{12}, \mathbf{W}_{123}) \\
\leq H(\mathbf{Y}_{12}) - H(\mathbf{W}_{12}, \mathbf{W}_{123}) + \epsilon_n \\
= H(\mathbf{Y}_{12} \mid \mathbf{W}_{12}, \mathbf{W}_{123}) + \epsilon_n
\]

Proof. The first three regions are trivial. We establish the fourth. The messages \( \mathbf{W}_{\mathcal{I}} \) are uniformly distributed on \( \{1, \ldots, 2^{nR_{\mathcal{I}}}\} \) and mutually independent for fixed \( n \). Denote the \( n \)-length output sequence by \( \mathbf{Y}_{12} \). By Fano’s inequality we must have \( H(\cup_{\mathcal{I} \subseteq \{1,2,3\}} \mathbf{W}_{\mathcal{I}} \mid \mathbf{Y}_{12}) \leq \epsilon_n \) with \( \epsilon_n \to 0 \) as \( n \to \infty \) in order for the error probability to be made arbitrarily small. Thus by the mutual independence of the messages we have

\[
n(R_1 + R_2 + R_3 + R_{13} + R_{23}) = H(\mathbf{W}_1) + H(\mathbf{W}_2) + H(\mathbf{W}_{13}) + H(\mathbf{W}_{23}) \\
\leq H(\cup_{\mathcal{I} \subseteq \{1,2,3\}} \mathbf{W}_{\mathcal{I}}, \mathbf{Y}_{12}) - H(\mathbf{W}_{12}, \mathbf{W}_{123}) \\
\leq H(\mathbf{Y}_{12}) - H(\mathbf{W}_{12}, \mathbf{W}_{123}) + \epsilon_n \\
= H(\mathbf{Y}_{12} \mid \mathbf{W}_{12}, \mathbf{W}_{123}) + \epsilon_n
\]
Assume for simplicity that \( n = mk \) where \( m \) is an integer. We write \( Y_{12} = [Y_{12}^1, \ldots, Y_{12}^m] \) where \( Y_{12}^i \) represents the \( i \)th block of \( k \) symbols in \( Y \). Similarly \( X^i \) represents the \( i \)th block of \( k \) symbols in \( X \). We also use the shorthand \( W \equiv \{W_{12}, W_{123}\} \). We proceed to show that \( H(Y_{12}|W) \) is sufficiently small.

\[
H(Y_{12}|W) \leq \sum_{i=1}^{m} H(Y_{12}^i|W) 
= -\sum_{i=1}^{m} \sum_{w} P(W = w) \sum_{x} P(Y_{12}^i = x|W = w) \log P(Y_{12}^i = x|W = w)
\]

From the channel definition we have

\[
P(Y_{12}^i = x|W = w) = \begin{cases} P(X_{1,12}^i = x, X_{2,12}^i = x|W = w) & x \neq c; \\ P(X_{1,12}^i \neq X_{2,12}^i|W = w) & x = c. \end{cases}
\]

Using this expression and the conditional independence of \( X_{1,12}^i \) and \( X_{2,12}^i \) given \( W \) we have

\[
-\sum_{x} P(Y_{12}^i = x|W = w) \log P(Y_{12}^i = x|W = w) \\
= -P(X_{1,12}^i \neq X_{2,12}^i|W = w) \log P(X_{1,12}^i \neq X_{2,12}^i|W = w) \\
- \sum_{x} P(X_{1,12}^i = x|W = w)P(X_{2,12}^i = x|W = w) \log P(X_{1,12}^i = x|W = w) \\
- \sum_{x} P(X_{1,12}^i = x|W = w)P(X_{2,12}^i = x|W = w) \log P(X_{2,12}^i = x|W = w).
\]

The first term can be upper bounded by 1 (as \(-x \log x < 1\) for all \( x \in \mathbb{R} \)). The second term can also be upper bounded by 1. To see this, maximize first over the distribution \( P(X_{2,12}^i|W = w) \) and then over the distribution \( P(X_{1,12}^i|W = w) \),

\[
\max_{P(X_{1,12}^i|W = w)} \sum_{x} P(X_{1,12}^i = x|W = w)P(X_{2,12}^i = x|W = w) \log P(X_{1,12}^i = x|W = w)
\]

\[
= \max_{P(X_{1,12}^i|W = w)} \left[ \max_{x} P(X_{1,12}^i = x|W = w) \right] \log \left[ \max_{x} P(X_{1,12}^i = x|W = w) \right] 
\leq 1
\]

Likewise the third term can be upper bounded by 1. Thus putting this all together we have

\[
H(Y_{12}|W) < 3 \sum_{i=1}^{m} \sum_{w} P(W = w) \\
= 3m
\]

and so

\[
R_1 + R_2 + R_3 + R_{13} + R_{23} < 3m/n \\
= 3/k
\]
Then by letting $k \to \infty$ we have $R_I = 0$ for $I \in \{1, 2, 3, 13, 23\}$. From the structure of the coordination channel it is clear that we can achieve points $(R_{12}, R_{123}) = (R_{12}^*, 0)$ and $(R_{12}, R_{123}) = (0, R_{12}^*)$. By time-sharing we can achieve all points in the region $R_{12} + R_{123} \leq R_{12}^*$. Conversely from Fano’s inequality we have

$$n(R_{12} + R_{123}) = H(W_{12}) + H(W_{123}) \leq H(Y) \leq \log (2^{kR_{12}} + 1)^m = n(R_{12}^* \pm \delta_k)$$

where $\delta_k \to 0$ as $k \to \infty$. This establishes the fourth component of the lemma. The remaining components are established in the same manner. We omit the details.

It remains to show that the region $\sum_{I \subseteq \{1, 2, 3\}} C_I$, corresponds to the region in equation (2).

7 Appendix

**Lemma 7.1.** Let $X_1, X_2$ and $X_3$ be three sets of random variables satisfying at least one of the properties $X_1 \subseteq X_2 \cup X_3$, $X_2 \subseteq X_1 \cup X_3$ or $X_3 \subseteq X_1 \cup X_2$. Let $W$ be a random variable that satisfies $H(W|X_i) = 0$ for $i = 1, 2, 3$. Then

$$I(X_1; X_2; X_3) \geq H(W)$$

**Proof.**

$$I(X_1; X_2; X_3) = I(W, X_1; W, X_2; W, X_3)$$

$$= H(W, X_1) + H(W, X_2) + H(W, X_3)$$

$$- H(W, X_1, X_2) - H(W, X_1, X_3) - H(W, X_2, X_3) + H(W, X_1, X_2, X_3)$$

$$= H(W) + H(X_1|W) + H(X_2|W) + H(X_3|W)$$

$$- H(X_1, X_2|W) - H(X_1, X_3|W) - H(X_2, X_3|W) + H(X_1, X_2, X_3|W)$$

$$= H(W) + I(X_1; X_2; X_3|W)$$

$$\geq H(W)$$

where the last step follows from lemma 7.4.

**Lemma 7.2.**

$$H(A) + H(B) + H(C) = H(A, B, C) + I(A, B; A, C; B, C) + I(A; B; C)$$

**Proof.** ITIP

**Lemma 7.3.** Let $X_1, \ldots, X_n$ be a set of mutually independent r.v’s. Let $X_1, X_2$ and $X_3$ be three subsets of these r.v’s with the property $X_1 \cap X_2 \cap X_3 = \phi$. Then for any r.v $Y$

$$H(Y|X_1) + H(Y|X_2) + H(Y|X_3) \geq H(Y).$$
Proof.

\[ H(Y|X_1) + H(Y|X_2) + H(Y|X_3) \]
\[ \geq H(Y|X_1, X_2^c) + H(Y|X_2, X_3^c) + H(Y|X_3, X_1^c) \]
\[ = H(Y, X_1, X_2^c) + H(Y, X_2, X_3^c) + H(Y, X_3, X_1^c) - H(X_1, X_2^c) - H(X_2, X_3^c) - H(X_3, X_1^c) \]
\[ = H(Y, X_1, X_2, X_3) + I(Y, X_1, X_2^c; Y, X_2, X_3^c; Y, X_3, X_1^c) \]
\[ + I(Y, X_1, X_2, X_3; Y, X_1, X_2, X_3) \]
\[ - H(X_1, X_2) - H(X_2, X_3) - H(X_3, X_1) \]
\[ = 2H(Y, X_1, X_2, X_3) + I(Y, X_1, X_2^c; Y, X_2, X_3^c; Y, X_3, X_1^c) \]
\[ - 2H(X_1 \setminus X_2 \cup X_3) - 2H(X_2 \setminus X_1 \cup X_3) - 2H(X_3 \setminus X_1 \cup X_2) \]
\[ - 2H(X_1 \cap X_2 \setminus X_3) - 2H(X_1 \cap X_3 \setminus X_2) - 2H(X_2 \cap X_3 \setminus X_1) - 3H(X_1 \cap X_2 \cap X_3) \]
\[ = 2H(Y, X_1, X_2, X_3) + I(Y, X_1, X_2^c; Y, X_2, X_3^c; Y, X_3, X_1^c) \]
\[ - 2H(X_1 \setminus X_2 \cup X_3) - 2H(X_2 \setminus X_1 \cup X_3) - 2H(X_3 \setminus X_1 \cup X_2) \]
\[ - 2H(X_1 \cap X_2 \setminus X_3) - 2H(X_1 \cap X_3 \setminus X_2) - 2H(X_2 \cap X_3 \setminus X_1) - 2H(X_1 \cap X_2 \cap X_3) \]
\[ = 2H(Y, X_1, X_2, X_3) + I(Y, X_1, X_2^c; Y, X_2, X_3^c; Y, X_3, X_1^c) - 2H(X_1, X_2, X_3) \]
\[ \geq I(Y, X_1, X_2^c; Y, X_2, X_3^c; Y, X_3, X_1^c) \]
\[ \geq H(Y) \]

where the third step follows from lemma 7.2, the fourth from a set expansion made possible by the mutual independence of the underlying r.v.’s \( X_1, \ldots, X_n \), the fifth from the property \( X_1 \cap X_2 \cap X_3 = \phi \), the sixth by a set relationship, and the eighth by lemma 7.1. \( \square \)

**Lemma 7.4.** Let \( X_1, X_2 \) and \( X_3 \) be sets of random variables. If either \( X_1 \subseteq X_2 \cup X_3 \), \( X_2 \subseteq X_1 \cup X_3 \) or \( X_3 \subseteq X_1 \cup X_2 \) then for any r.v. \( W \),

\[ I(X_1; X_2; X_3|W) \geq 0. \]

**Proof.** Assume without loss of generality that the first containment property \( X_3 \subsetneq X_1 \cup X_2 \) holds. Then

\[ I(X_1, X_2, X_3|W) \]
\[ = H(X_1|W) + H(X_2|W) + H(X_3|W) \]
\[ - H(X_1, X_2|W) - H(X_1, X_3|W) - H(X_2, X_3|W) + H(X_1, X_2, X_3|W) \]
\[ = H(X_1|W) + H(X_2|W) + H(X_3|W) - H(X_1, X_2|W) - H(X_1, X_3|W) \]
\[ = I(X_1; X_3|W) + I(X_2; X_3|W) - H(X_3|W) \]
\[ \geq H(X_1 \cap X_3|W) + H(X_2 \cap X_3|W) - H(X_1 \cap X_3, X_2 \cap X_3|W) \]
\[ = I(X_1 \cap X_3; X_2 \cap X_3|W) \]
\[ \geq 0. \]

The second step follows from the containment property \( X_1 \subsetneq X_2 \cup X_3 \). The first term in the fourth step follows by applying lemma 7.6 with \( W = W, X = X_1, Y = X_3 \) and \( Z = X_1 \cap X_3 \), the second term by applying the same lemma with \( W = W, X = X_2, Y = X_3 \) and \( Z = X_2 \cap X_3 \). The third term in the third and fourth steps are equal by the containment property. \( \square \)
Lemma 7.5. If $H(Z|X) = 0$ and $H(Z|Y) = 0$ then $I(X;Y|W) \geq H(Z|W)$.

Proof.

\[
I(X;Y|W) = I(X, Z; Y, Z|W) \\
= H(X, Z|W) + H(Y, Z|W) - H(X, Y, Z|W) \\
= H(Z|W) + H(X|W, Z) + H(Z|W) + H(Y|W, Z) - H(Z|W) - H(X, Y|W, Z) \\
= H(Z|W) + I(X;Y|W, Z) \\
\geq H(Z|W).
\]

Lemma 7.6. If $H(Z|X) = 0$ and $H(Z|Y) = 0$ then $I(X;Y|W) \geq H(Z|W)$.

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