A new approach to infrared and ultraviolet divergences.

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Abstract

An interesting attempt for solving infrared divergence problems via the theory of generalized wave operators was made by P. Kulish and L. Faddeev. Our method of using the ideas from the theory of generalized wave operators is essentially different. The scattering operator commutes in our case with the unperturbed operator. (In the Kulish-Faddeev theory this property is not fulfilled.) We reduce the divergence problem to the scalar case, whereas Kulish and Faddeev considered the operator case. Finally, Kulish and Faddeev considered only infrared divergences but we consider both infrared and ultraviolet divergences.

1 Introduction

1. In the momentum space the unperturbed Dirac equation takes the form (see \[1\], Ch.IV):

\[ i \frac{\partial}{\partial t} \Phi(q,t) = H(q)\Phi(q,t), \quad q = (q_1, q_2, q_3), \]  

where \(H(q)\) and \(\Phi(q,t)\) are matrix functions of order 4×4 and 4×1 respectively. Here the matrix \(H(q)\) is defined by the relation \(2.1\). The unperturbed Dirac equation in the presence of electrodynamics field takes the form
(see [1], Ch.IV):

\[ i \frac{\partial}{\partial t} \tilde{\Phi}(q,t) = \tilde{H}(q)\tilde{\Phi}(q,t), \quad (1.2) \]

where \( \tilde{\Phi}(q,t) \) is the matrix functions of order 8\times1 and

\[ \tilde{H}(q) = \begin{pmatrix} H(q) & 0 \\ 0 & H(q) \end{pmatrix}. \quad (1.3) \]

The equations (1.1) and (1.2) can be rewritten in the forms

\[ i \frac{\partial}{\partial t} \Phi(q,t) = A_0 \Phi(q,t), \quad (1.4) \]

and

\[ i \frac{\partial}{\partial t} \tilde{\Phi}(q,t) = \tilde{A}_0 \tilde{\Phi}(q,t), \quad (1.5) \]

where the operators \( A_0 \) and \( \tilde{A}_0 \) are defined by the relations

\[ A_0 f(q) = H(q)f(q), \quad \tilde{A}_0 \tilde{f}(q) = \tilde{H}(q)\tilde{f}(q). \quad (1.6) \]

Here \( f(q) \in L^2_4(R^3), \quad \tilde{f}(q) \in L^2_8(R^3) \).

**Remark 1.1** We do not take into account the perturbed operators \( A \) and \( \tilde{A} \). We investigate only the corresponding scattering operators \( S(A,A_0) \) and \( S(\tilde{A},\tilde{A}_0) \). So, we partially follow the Heisenberg’s S-matrix program. In addition to this S-program we assume that the unperturbed operators \( A_0 \) and \( \tilde{A}_0 \) are known. This fact gives us an important information about the structure of the scattering operators.

It is well known that the scattering operators satisfy the following conditions:
1) The operators \( S(A,A_0) \) and \( S(\tilde{A},\tilde{A}_0) \) are unitary operators in the spaces \( L^2_4(R^3) \) and \( L^2_8(R^3) \) respectively.
2) The commutative relations

\[ A_0 S(A,A_0) = S(A,A_0) A_0, \quad \tilde{A}_0 S(\tilde{A},\tilde{A}_0) = S(\tilde{A},\tilde{A}_0) \tilde{A}_0 \quad (1.7) \]

are valid.

**Remark 1.2** The conditions 1) and 2) are fulfilled for classical and generalized scattering operators (see [10], [15], [17]).
In the section 2 we find the eigenvectors and the eigenvalues of the matrices $H(q)$ and $\tilde{H}(q)$. Hence we reduce the matrices $H(q)$ and $\tilde{H}(q)$ to the diagonal forms. It follows from conditions 1) and 2) matrices $S(A, A_0)$ and $S(\tilde{A}, \tilde{A}_0)$ can be reduced to the diagonal forms simultaneously with $H(q)$ and $\tilde{H}(q)$ respectively. The diagonal elements $d_k(A, A_0), (1 \leq k \leq 4)$ of $S(A, A_0)$ and the diagonal elements $\tilde{d}_k(\tilde{A}, \tilde{A}_0), (1 \leq k \leq 8)$ of $S(\tilde{A}, \tilde{A}_0)$ are such that

$$|d_k(A, A_0)| = 1, \quad (1 \leq k \leq 4) \quad (1.8)$$

$$|\tilde{d}_k(\tilde{A}, \tilde{A}_0)| = 1, \quad (1 \leq k \leq 8) \quad (1.9)$$

Thus, the investigation of the scattering operators is reduced to the scalar case, i.e. to the diagonal elements $d_k$ and $\tilde{d}_k$.

Using this fact we present a new approach to the divergence problems in quantum electrodynamics (QED). Let us explain the situation. In QED the higher order approximations of matrix elements of the scattering matrix contain integrals which diverge. We think that these divergences are result of the used scattering matrix representation: the series by a small parameter $e$. In the section 3 we try to answer J.R. Oppenheimer question [14]: "Can be procedure be freed of the expansion in $e$ and carried out rigorously?"

We introduce a new representation of the scattering matrix. We do not remove the divergences. Our aim is to prove that they are absent. In our approach we essentially use the ideas of the generalized wave operators theory [17], [19], [2]. The classical theory of divergences just rejects the divergent integrals. In our approach this divergent integrals get the physical sense. We construct the deviation factors with the help of these integrals. The deviation factors characterize a deviation of initial and final waves from free waves. So, we not only have received exact results in the theory of the divergences, but also have received the new facts about behavior of system. These facts, as it seems to us, can be checked by experiment.

2. The theory of generalized wave operators was used by P.P. Kulish and L.D. Faddeev [12] for solving infrared divergence problems. Interesting new results have been obtained in this field quite recently [3, 6, 9]. In particular, an interesting paper by C. Gomez and M. Panchenko [7] is dedicated to the development of the results of the Kulish-Faddeev theory. However, the natural condition of the commutativity of the scattering and unperturbed operators is missing in these works.

Our approach to the divergence problem is essentially different from the Kulish-Faddeev approach:
1) The scattering operator in our case is commutative with unperturbed operator. In the Kulish-Faddeev theory this property is not fulfilled.
2) We reduce the divergence problem to the scalar case. Kulish and Faddeev considered the operator case.
3) We consider infrared and ultraviolet divergences. Kulish and Faddeev considered only infrared divergences.

2 Spectral properties of the matrices \( H(q), S(A, A_0) \) and \( S(\widehat{A}, \widehat{A}_0) \)

Let us consider equation (1.1). The corresponding matrix \( H(q) \) has the form.

\[
H(q) = \begin{bmatrix}
m & 0 & q_3 & q_1 - iq_2 \\
0 & m & q_1 + iq_2 & -q_3 \\
q_3 & q_1 - iq_2 & -m & 0 \\
q_1 + iq_2 & -q_3 & 0 & -m
\end{bmatrix}. \quad (2.1)
\]

The eigenvalues \( \lambda_k \) and the corresponding eigenvectors \( g_k \) of \( H(q) \) are important, and we find them below:

\[
\lambda_{1,2} = -\sqrt{m^2 + |q|^2}, \quad \lambda_{3,4} = \sqrt{m^2 + |q|^2} \quad (|q|^2 := q_1^2 + q_2^2 + q_3^2); \quad (2.2)
\]

\[
g_1 = \begin{bmatrix}
(q_3 + i(q_1 + iq_2))/(m + \lambda_3) \\
0 \\
1
\end{bmatrix}, \quad g_2 = \begin{bmatrix}
-q_3/(m + \lambda_3) \\
(q_1 - i(q_1 - iq_2))/(m + \lambda_3) \\
1
\end{bmatrix}, \quad (2.3)
\]

\[
g_3 = \begin{bmatrix}
(q_3 + i(q_1 + iq_2))/(m - \lambda_3) \\
0 \\
1
\end{bmatrix}, \quad g_4 = \begin{bmatrix}
-q_3/(m - \lambda_3) \\
(q_1 - i(q_1 - iq_2))/(m - \lambda_3) \\
1
\end{bmatrix}. \quad (2.4)
\]

We introduce the following linear spans:

\[
M_1(q) = \text{Span}\{g_k(q), k = 1, 2\}, \quad M_2(q) = \text{Span}\{g_k(q), k = 3, 4\}. \quad (2.5)
\]

According to condition 2) the subspaces \( M_1(q) \) and \( M_2(q) \) are invariant subspaces of \( H(q) \) and \( S(A, A_0) \). Then there exist common eigenvectors \( h_k(q) \) of \( H(q) \) and \( S(A, A_0) \) such that \( h_k(q) \in M_1(q), (k = 1, 2) \) and \( h_k(q) \in M_2(q), (k = 3, 4) \). Hence \( H(q) \) and \( S(A, A_0) \) can be reduced to the diagonal forms simultaneously. In the same way we prove that \( \tilde{H}(q) \) and \( S(\tilde{A}, \tilde{A}_0) \) can be reduced to the diagonal forms simultaneously.
3 New approach to the ultraviolet divergence problems: power series

1. Let the diagonal element \( d(q) \) of the scattering matrix either \( S(A, A_0) \) or \( S(\tilde{A}, \tilde{A}_0) \) be represented in the form of the power series

\[
d(q) = 1 + e a_1(q) + e^2 a_2(q) + ... \tag{3.1}
\]

where \( e \) is elementary charge. We assume that

\[
a_2 = \lim_{L \to \infty} \int_{\Omega} F(P, Q) d^4 P. \tag{3.2}
\]

Here \( P = [-ip_0, p_1, p_2, p_3], \quad Q = [-iq_0, q_1, q_2, q_3] \). In a number of concrete examples the functions \( F(P, Q) \) are rational \([1]\). The invariant region of integration \( \Omega \) is four dimensional sphere with radius \( L \).

We shall investigate the cases when the limit in the right hand side of (3.2) does not exist. In this case we have ultraviolet divergence. It is known that the corresponding divergences can be removed by mass and charge renormalization \([1]\). F.J. Dyson \([5]\) stressed that it is important "to prove the convergence in the frame of the theory". We shall do it for a broad class of examples.

**Example 3.1** Let the relation

\[
a_2(q, L) = \int_{\Omega} F(p, q) d^4 p = i[\phi(q)lnL + \psi(q) + O(1/L)], \quad L \to + \infty. \tag{3.3}
\]

is valid. Here \( \phi(q) = \overline{\phi(q)}, \quad \psi(q) = \overline{\psi(q)} \).

Thus, the corresponding integral (see (3.3)) diverges logarithmic. Hence the second term of power series (3.1) is equal to infinity.

*Let us use a new representation of \( d(q) \).*

To do it we introduce \( d(q, L) \):

\[
d(q, L) = 1 + e a_1(q) + e^2 a_2(q, L) + ... \tag{3.4}
\]

We write

\[
d(q, L) = L^{e^2 \phi(q)} \tilde{d}(q, L), \tag{3.5}
\]
where Using (3.4) and (3.5) we have
\[
\tilde{d}(q, L) = 1 + ea_1(q) + e^2[a_2(q, L) - i\phi(q)lnL] + ...
\] (3.6)

It follows from (3.3) that the second term
\[
\tilde{a}_2(q, L) = a_2(q, L) - i\phi(q)lnL
\] (3.7)
of power series (3.6) converges when \( L \to \infty \).

**Remark 3.2** The factor \( U_0(L, q) = L^{ie^2\phi(q)} \) is an analogue of deviation factor \( W_0(t) \) in the theory of generalized wave and scattering operators \cite{20}.

We stress that
\[
|U_0(L, q)| = 1.
\] (3.8)

**Example 3.3** Let the relation
\[
a_2(q, L) = i[\phi(q)L^2 + \psi(q)L + \nu(q)lnL + \mu(q) + O(1/L)],
\] (3.9)
is valid. Here \( \phi(q) = \phi(q), \psi(q) = \psi(q), \nu(q) = \nu(q), \mu(q) = \mu(q) \) and \( L \to + \infty \).

In this case the factor \( U_0(L, q) \) has the form
\[
U_0(L, q) = \exp[ie^2(\phi(q)L^2 + \psi(q)L)]L^{ie^2\nu q}.
\] (3.10)

We use (3.4) and write the formulas
\[
d(q, L) = U_0(L, q)[\tilde{d}(q, L)],
\] (3.11)
where
\[
\tilde{d}(q, L) = [U_0^{-1}(L, q)d(q, L)].
\] (3.12)

Relations (3.6) in case (3.9) takes the forms
\[
\tilde{d}(q, L) = 1 + ea_1(q) + e^2\tilde{a}_2(q, L) + ..., \quad (3.13)
\]
where term
\[
\tilde{a}_2(q, L) = a_2(q, L) - i[\phi(q)L^2 + \psi(q)L + \nu(q)lnL]
\] (3.14)
of power series (3.13) converges when \( L \to \infty \).
Remark 3.4  It follows from (3.3) that the necessary condition for using our method is the equality:
\[ \Re a_2(q, L) = 0. \] (3.15)

Remark 3.5  Many concrete problems of collision of particles satisfy the condition (3.15). In particular, all divergences in irreducible diagrams belong to the class (3.9) and satisfy the condition (3.15) (see [7], sections 46 and 47).

The simplest case of Example 3.4 we obtain when
\[ \phi(q) = 0, \nu(q) = 0, \psi(q) = 1. \] (3.16)

In this case we have
\[ U_0(L, q) = \exp[i e^2 L]. \] (3.17)

2. Now we assume that the coefficients \( a_m(q, L) \) has the form
\[ a_m(q, L) = \sum_{p=0}^{\infty} [\phi_{p,m}(q) \ln^p L + O(1/L)], \quad L \to \infty, \quad (1 \leq m \leq N). \] (3.18)

It is proved (see review [4]), that in many cases the Feynman amplitudes have the poly-logarithmic structure (3.18). The integrals \( a_m(q, L) \) which corresponds to the terms of series \( a_m(q) \), \((1 \leq m \leq N)\) diverges. It was proved in the paper [18] that the corresponding deviation factor \( U_0(L, q) \) has the form
\[ U_0(L, q) = \exp[i \sum_{p=1}^{N} (\ln^p L) \phi(q, p, e)], \] (3.19)

where
\[ \phi(q, p, e) = \sum_{m=2}^{N} e^m \psi(q, p, m), \quad \psi(q, p, m) = \overline{\psi(q, p, m)}. \] (3.20)

Let us consider the interesting model example.

Example 3.6 We assume that the terms \( a_m(q, L) \) of series (2.1) are given by formulas
\[ a_m(q, L) = \sum_{k=0}^{m} \psi_{m-k}(q) \frac{(i \phi(q) \ln L)^k}{k!}. \] (3.21)
It is easy to see that
\[ \tilde{d}(q,L) = L^{-i\phi(q)}d(q, L) = 1 + e\psi_1(q) + e^2\psi_2(q) + ... \] (3.22)
So, in this case we obtain the regular scattering function \( \tilde{d}(q, L) \). We note that Coulomb potentials (see [16] and [17]) have the properties of type (3.21).

**Remark 3.7** Deviation factors are not uniquely defined. If \( U_0(L, q) \) is the deviation factor, then \( C(q)U_0(L, q) \) (\( |C(q)| = 1 \)) is the deviation factor too. The choice of multipliers \( C(q) \) depends on the particular physical problem under consideration.

3. Now we introduce the following notion.

**Definition 3.8** We say say that the deviation factor \( U_0(L, q) \) belongs to the class \( \mathcal{A} \) if
\[ U_0(L + L_0, q)U_0^{-1}(L, q) \to 1, \; L \to + \infty. \] (3.23)
Let us compare the introduced deviation factor \( U_0(L, q) \) with deviation factor \( W_0(t) \) of the generalized wave operators theory [20]. The relation (3.23) for \( W_0(t) \) has the form
\[ W_0(t + \tau)W_0^{-1}(t) \to 1, \; t \to \pm \infty. \] (3.24)

**Example 3.9** If the relation (3.18) holds, then the corresponding deviation factors \( U_0(L, q) \) belong to the class \( \mathcal{A} \).

### 4 New approach to the infrared divergence problems: power series

1. We again assume (see section 3) that the diagonal element \( d(q) \) of the scattering matrix either \( S(A, A_0) \) or \( S(\tilde{A}, \tilde{A}_0) \) is represented in the form of the power series
\[ d(q) = 1 + e\alpha_1(q) + e^2\alpha_2(q) + ..., \] (4.1)
where
\[ \alpha_2 = \lim_{t \to +\infty, \omega \to -\infty} \int_{\Omega} F(P, Q)d^4P. \] (4.2)
We note, that in the infrared case we have $t \to +\infty$, $t_0 \to -\infty$ instead of $L \to \infty$ in the ultraviolet case (see [12] and [8]).

**Example 4.1** Let us consider the radial Schrödinger operator

$$\mathcal{L} f = -\frac{d^2}{dr^2} f + \left(\frac{\ell(\ell + 1)}{r^2} - \frac{2ez}{r} + eq(r)\right)f, \quad z = \bar{z} \quad (4.3)$$

with Coulomb type potentials $2ez/r - eq(r)$, where $q(r)$ satisfies the conditions

$$\int_0^\infty |q(r)|rdr < \infty, \quad q(r) = \bar{q}(r). \quad (4.4)$$

We introduce the operator $\mathcal{L}_0$:

$$\mathcal{L}_0 f = -\frac{d^2}{dr^2} f. \quad (4.5)$$

We consider the following boundary condition

$$f(0) = 0. \quad (4.6)$$

We consider the case, when $q(r)$ is the superposition of the Yukawa potentials

$$q(r) = \frac{1}{r} \int_m^\infty \exp(-r\beta)d\sigma(\beta), \quad \int_m^\infty d|\sigma(\beta)|, \quad m > 0. \quad (4.7)$$

Using Levy results (see [13]) we write the operators $\mathcal{L}_0$ and $\mathcal{L}$ in the momentum representation.

$$\mathcal{L}_0 f = k^2 f, \quad f \in L^2(0, \infty), \quad (4.8)$$

$$\mathcal{L} f = k^2 f + e \int_0^\infty f(p)pkR_\ell(k,p)dp, \quad (4.9)$$

where

$$R_\ell(k, p) = -\frac{2z}{\pi kp}Q_\ell\left(\frac{k^2 + p^2}{2kp}\right) + \frac{2}{\pi} \int_m^\infty \int_{-1}^1 P_\ell(x)\frac{dx\sigma(\beta)}{k^2 + p^2 - 2pkx + \beta^2} \quad (4.10)$$

Here $P_\ell(x)$ are Legendre polynomials, $Q_\ell(x)$ are Legendre polynomials of the second kind.

The corresponding deviation factor has the form (see [18]).

$$W_0(t, k)f(k) = (2kt)^{izsign/k} f(k), \quad f(k) \in L^2(0, \infty). \quad (4.11)$$
Then the generalized scattering operator \( S(\mathcal{L}, \mathcal{L}_0) \) exists and can be written in the form

\[
S(\mathcal{L}, \mathcal{L}_0) = \lim_{W_0(t,k)S(t,\tau)W_0^{-1}(\tau, k), \ t \to + \infty, \ \tau \to - \infty} (4.12)
\]

where

\[
S(t, \tau) = \exp(it\mathcal{L}_0)\exp(-it\mathcal{L})\exp(i\tau\mathcal{L})\exp(-i\tau\mathcal{L}_0). \quad (4.13)
\]

Expanding the right hand side of (4.12) in powers of \( e \) we obtain the regular perturbation series

\[
S(\mathcal{L}, \mathcal{L}_0) = I + \sum_{n=1}^{\infty} S_n(k, \ell)e^n. \quad (4.14)
\]

It was proved (see [18]), that

\[
S_1(k, \ell) = \frac{-2i}{k\Gamma(\ell + 1)} - \frac{2i}{k} \int_{-1}^{1} \int_{0}^{\infty} P_t(x) k \frac{k}{2k^2(1 - x) + \beta^2} dxd\sigma(\beta). \quad (4.15)
\]

**Example 4.2** Let the diagonal element \( d(q) \) of the scattering matrix either \( S(A, A_0) \) or \( S(\tilde{A}, \tilde{A}_0) \) be represented in the form of the power series

\[
d(q) = 1 + ea_1(q) + e^2a_2(q) + \ldots \quad (4.16)
\]

where \( e \) is elementary charge.

Now we assume that

\[
a_1(q) = \lim a_1(t, \tau), \quad t \to + \infty, \quad \tau \to - \infty. \quad (4.17)
\]

If

\[
a_1(q, t, \tau) = i\phi(q)ln|t\tau| + O(1), \ \phi(q) = \overline{\phi(q)}, \quad t \to + \infty, \ \tau \to - \infty, \quad (4.18)
\]

then the right hand side of (4.17) diverges logarithmic.

**Definition 4.3** We say that the coefficient \( a_1(q, t, \tau) \) has Coulomb type divergence, if relation (4.17) is valid.
We note that the Coulomb interaction is the typical example of infrared catastrophe \[11\]. The relation (4.17) is carried out in case of the Coulomb type potential (the radial Schrödinger equation (see \[18\]) and the radial Dirac equation (see \[16\])).

Let us use a new representation of \(d(q)\). To do it we introduce \(d(q,t,\tau)\):

\[
d(q,t,\tau) = 1 + e a_1(q,t,\tau) + e^2 a_2(q,t,\tau) + \ldots \quad (4.19)
\]

We write

\[
d(q,t,\tau) = |t\tau|^{ie\phi(q)} \tilde{d}(q,t,\tau),
\]

where Using (4.19) and (4.20) we have

\[
\tilde{d}(q,t,\tau) = 1 + e [a_1(q,t,\tau) - i\phi(q)ln|t\tau|] + \ldots \quad (4.21)
\]

It follows from (4.18) that the first term

\[
\tilde{a}_1(q,t,\tau) = a_1(q,t,\tau) - i\phi(q)ln|t\tau|
\]

of power series (4.21) converges when \(t \to +\infty\) and \(\tau \to -\infty\).

\section{Conclusion}

Let \(d(q)\) is the matrix element of the scattering operator \(S(q)\). We assume that this element has the property

\[
|d(q)| = 1.
\]

Here we want to give reasons in favour of the formulated assumptions (5.1).

1. It follows from (3.1), (4.1) and (5.1) that

\[
\Re a_1(q) = 0.
\]

In concrete examples condition (5.2) is fulfilled (see \[1\] and \[5\]). Hence our approach is correct in the of the first approximation.

2. The condition (5.1) is fulfilled for the diagonal elements of scattering operator (see (1.8) and (1.9)).

3. The next necessary condition for equality (5.1) is the relation

\[
\Re a_2(q) = a_1^2(q).
\]

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4. The conditions (5.1) and (5.3) are fulfilled for the following important cases: the radial Schrödinger equation (see [18]) and the radial Dirac equation (see [16]).

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