CONTACT FORMULAS FOR RATIONAL PLANE CURVES 
VIA STABLE MAPS

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Abstract. Extending techniques of [4], we use stable maps, and their stable lifts to the Semple bundle variety of second-order curvilinear data, to calculate certain characteristic numbers for rational plane curves. These characteristic numbers involve first-order (tangency) and second-order (inflectional) conditions. Although they may be virtual, they may be used as inputs in an enumeratively significant formula for the number of rational curves having a triple contact with a specified plane curve and passing through $3d - 3$ general points.

1. Introduction

Two plane curves are said to have a triple contact at a point if both curves are smooth at the point and if their intersection number there is 3. (For example, a curve has a triple contact with its tangent line at each flex.) For a specified curve $C$, one may ask how many rational plane curves of degree $d$ have a triple contact with $C$ and also pass through $3d - 3$ general points of the plane. Our formula (8.5) shows that this contact number is the inner product of two vectors, the first of which consists of the degree $c$ of the specified curve, its class $\check{c}$, and the number of cusps $\kappa$. The second vector consists of three second-order characteristic numbers of the family of rational plane curves of degree $d$. Thus to give a satisfactory answer to the question—and to questions of a similar nature—one needs a method of determining these characteristic numbers. In this paper we show how to compute (in every degree) thirteen characteristic numbers, including the three needed in the contact formula, by developing a recursive procedure for calculating the Gromov-Witten invariants of certain stacks of stable maps, and by showing that these invariants agree with the characteristic numbers.

In §4 we briefly discuss the Semple bundle construction of a complete variety of second-order curvilinear data of $P^2$. In §5 we introduce the notion of second-order stable lifts of maps to $P^2$ and their associated stacks, and present three key examples. We introduce the relevant Gromov-Witten invariants in §6, and prove that they agree with

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the characteristic numbers of [3]. In §5 we study the special boundary divisors on the stack of second-order stable lifts and describe in detail those components that actually matter in the subsequent computations. In §6 we construct various generating functions for Gromov-Witten invariants, called potentials, and show how they are related by a basic differential equation. Using this equation in §7, we describe a recursive process which determines, in every degree, the thirteen Gromov-Witten invariants; we give a complete table of values through degree 6. Finally, in §8, we present two “contact formulas” (generalizing Theorem 4 of [3]) in which the Gromov-Witten invariants are the inputs, and prove that the outputs are enumeratively significant. We also demonstrate the second formula by using it to calculate the answer to the question posed at the beginning.

To obtain explicit answers to related questions, we would need to know a larger set of characteristic numbers. Our recursion determines, in each degree, those thirteen characteristic numbers specified by at least $3d - 3$ “point conditions,” beginning (as input) with those in degree 1. For plane conics, there are 153 second-order characteristic numbers which are nontrivial, i.e., specified by a selection of conditions including neither divisors nor the fundamental class. If we knew the full set of values (and one could certainly hope to find them by ad hoc methods), our recursive scheme would then compute the corresponding 153 numbers in every degree, and our Theorem 8.3 would then yield an explicit formula for the number of rational plane curves having a triple contact with each of two specified plane curves and passing through $3d - 5$ points. Another possible generalization would be to discover and validate formulas for higher-order contact. Semple’s construction actually creates a tower of $\mathbb{P}^1$-bundles parametrizing higher-order curvilinear data, which could presumably be used in such a project. (See [2] or [3].) There seem to be, however, two obstructions. First, we do not know how to prove the analogue of Proposition 5.2, the essential tool in classifying the relevant components of the special boundary divisors on the stack of stable lifts. Second, the number of $PGL(2)$-orbits on the Semple bundle variety grows larger with the order of data; this makes it difficult to use transversality theory to establish enumerative significance. (In fact since the dimension of the variety of $n$th-order data is $n + 2$ there are infinitely many orbits when $n > 6$.) For similar reasons, it will probably be difficult to extend our methods to the study of higher-dimensional varieties or even to nonrational surfaces; even in the case of other rational surfaces we anticipate complications.

Several people have suggested the possibility of using gravitational descendants to deal with problems of higher-order contact. In discussing this approach, however, Gathmann shows by an example [6, p. 27] that it will have to deal with spurious contributions arising from singular curves. We believe that the stable lift method avoids this difficulty.

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2. The variety of second-order curvilinear data

We provide a quick summary of the salient features of $S_2$, the variety of second-order curvilinear data of $\mathbb{P}^2$, introduced by Semple in [9]. For additional details, see [2] or [3].

Let $S_1$ be the total space of the projectivized tangent bundle of $\mathbb{P}^2$; equivalently, $S_1$ is the incidence correspondence of points and lines. Let $f_1: S_1 \rightarrow \mathbb{P}^2$ be the tautological projection, so that a fiber represents the pencil of tangent directions at a point of the plane. The focal plane at a point $x \in S_1$ is defined to be the preimage, via the derivative $df_1$, of the line in $T_1(x)\mathbb{P}^2$ represented by $x$. This local construction gives rise to a global rank 2 focal plane bundle $\mathcal{F}_1$ and we define $S_2$, the Semple bundle variety of second-order data of $\mathbb{P}^2$, to be $\mathbb{P}_x \mathcal{F}_1$, the total space of the projectivized bundle. The Semple bundle variety $S_2$ is equipped with a tautological projection $f_2: S_2 \rightarrow S_1$. Note that $S_2$ is a subvariety of $PT(S_1)$, the projectivized tangent bundle of $S_1$. The relative tangent bundle $T(S_1/\mathbb{P}^2)$ is naturally a rank 1 subbundle of $\mathcal{F}_1$, so that $PT(S_1/\mathbb{P}^2)$ is naturally a section of $S_2$, which we call the divisor at infinity and denote by $I$.

If $C$ is a reduced curve in $\mathbb{P}^2$, there is a rational map from $C$ to $S_1$ that sends a nonsingular point $x$ of $C$ to the point of $S_1$ representing the tangent direction to $C$ at $x$. We call the closure of the image of this map the lift of $C$ and denote it by $C_1$. Similarly, $C_1$ may be lifted to a curve $C_2$ in $PT(S_1)$. It is not difficult to see that the tangent direction to $C_1$ at a nonsingular point must be in the focal plane. Consequently, the second-order lift $C_2$ is a curve not just in $PT(S_1)$, but in $S_2$. If $C_2$ passes through a point $x$ of $S_2$, we sometimes say that the germ of $C$ represents $x$. For example, the germ at the origin of $y^2 = x^3$ represents a point on the divisor at infinity. Intuitively, points on this divisor are represented by curve germs with “infinite curvature.”

Families of curves may also be lifted. If $C \subset \mathbb{P}^2 \times T$ is a family of plane curves with reduced general member, then we lift the reduced members of $C$ to $S_2$, obtaining a rational lifting map from $\mathcal{C}$ to $S_2 \times T$. We define $\mathcal{C}_2 \subset S_2 \times T$ to be the closure over $\mathcal{C}$ of the image of this lifting map.

In the construction of the Semple bundle variety we call $\mathbb{P}^2$ the base. One may also construct a Semple bundle variety using the dual projective plane $\mathbb{P}^2$ as base. Note that $PT(\mathbb{P}^2)$ and $PT(\mathbb{P}^2)$ are both isomorphic to the point-line incidence correspondence. Furthermore, if $(x,l)$ is a point of the incidence correspondence then the preimages of $x \in \mathbb{P}^2$ and $l \in \mathbb{P}^2$ are curves through $(x,l)$ and the focal plane at $(x,l)$ is the plane spanned by the tangent lines to these curves. Hence in both constructions the focal plane bundle is the same, and thus the two second-order Semple bundle varieties are the same. There are, however, two disjoint divisors at infinity. We shall continue to denote by $I$ the divisor at infinity that arises from using $\mathbb{P}^2$ as base, but denote by $Z$ the divisor at infinity that arises from using $\mathbb{P}^2$ as base. Note that the lift of a smooth curve meets $Z$ above $x \in \mathbb{P}^2$ if and only if $x$ is a flex.

Since $S_2 \rightarrow S_1 \rightarrow \mathbb{P}^2$ is a sequence of $\mathbb{P}^1$-bundles, we may regard $A^*(\mathbb{P}^2)$ and $A^*(S_1)$ as subrings of the intersection ring $A^*(S_2)$. Let $h$ and $\bar{h}$ denote the hyperplane classes on $\mathbb{P}^2$ and $\mathbb{P}^2$. Let $i$ denote the class of the divisor at infinity $I$, and $z$ the class of $Z$. 

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Then, according to §3 of [3], the Chow ring $A^*(S_2)$ is generated by $h$, $\bar{h}$, and $i$, subject to the relations
\[ h^3 = \bar{h}^3 = h^2 - \bar{h} + \bar{h}^2 = 0, \quad i^2 = 3(h - \bar{h})i, \]
and the class $z$ satisfies these equations:
\[ i - z = 3(h - \bar{h}), \quad iz = 0. \]

3. Stable lifts and their stacks

We assume that the reader is familiar with the basic notions of stable maps [3]. Throughout this paper we will consider only stable maps from curves of arithmetic genus zero; thus we will omit the subscript 0 from the standard notation for moduli stacks of stable maps. Given a stable map from a tree of $\mathbb{P}_1$’s, we will use the term twig to refer to one of the $\mathbb{P}_1$’s of the tree, reserving the term component to refer to an irreducible component of an appropriate moduli space or to a divisor on a moduli space in order to avoid confusion. We will call the nodes of the source curve attachment points.

If $\mu: (\mathbb{P}_1, p_1, \ldots, p_n) \to \mathbb{P}^2$ is a nonconstant map, then we define its strict lift $\mu_1^{\text{strict}}: (\mathbb{P}_1, p_1, \ldots, p_n) \to S_1$ by associating to each point $x$ the direction of $d\mu(x)$ at $\mu(x)$. Note that this map is defined even at singular points of $\mu$. Repeating the construction, we obtain a map $\mu_2^{\text{strict}}: (\mathbb{P}_1, p_1, \ldots, p_n) \to S_2$, called the second-order strict lift of $\mu$. (One needs to verify that the image of this map actually lies on $S_2$ inside of the projectivized tangent bundle $\mathbb{P}T(S_1)$.) If $\mu$ is an immersion, then its degree $d$ determines the homology class $\lambda_d$ of $\mu_2^{\text{strict}}(\mathbb{P}_1)$. Thus we have a rational lifting map from the moduli stack $\overline{\mathcal{M}}_n(\mathbb{P}^2, d)$ to $\overline{\mathcal{M}}_n(S_2, \lambda_d)$, and we define the stack of second-order stable lifts, denoted $\overline{\mathcal{M}}_n^2(\mathbb{P}^2, d)$, to be the closure of its image. The lifting map is birational onto $\overline{\mathcal{M}}_n^2(\mathbb{P}^2, d)$, and its inverse is inclusion followed by projection. Given a point $\mu$ of $\overline{\mathcal{M}}_n^2(\mathbb{P}^2, d)$, we will call the points of its preimage in $\overline{\mathcal{M}}_n(\mathbb{P}^2, d)$ its second-order stable lifts.

For an immersion, the only possible stable lift is the strict lift. For other sorts of stable maps, however, the strict lift will be only a subset of the twigs of a stable lift; for examples, see Propositions 3.2, 3.3, and 3.4 below. Furthermore, some stable maps have more than one stable lift. This happens, for example, with the triple cover of a line in $\mathbb{P}^2$ ramified at just two points; we will, however, make no use of this phenomenon.

Suppose that $\nu: \mathbb{P}_1 \to S_1$ is an isomorphism onto a fiber of $S_1$ over a point of $\mathbb{P}^2$. Then the strict lift from $\mathbb{P}_1$ to $\mathbb{P}T(S_1)$ actually lands inside $S_2$. Indeed, since the composite $f_1 \circ \nu$ is a constant map, the image of the strict lift is even inside the divisor at infinity $I$. As we will see below, such strict lifts occur as twigs of certain stable lifts.

**Proposition 3.1.** Suppose that $n \geq m$. Then the following diagram (in which the horizontal morphism are inclusions and the vertical morphisms are forgetful) is a fiber
\[
\begin{array}{ccc}
\overline{M}_m^2(\mathbb{P}^2, d) & \longrightarrow & \overline{M}_n(S_2, \lambda_d) \\
\downarrow & & \downarrow \\
\overline{M}_m^2(\mathbb{P}^2, d) & \longrightarrow & \overline{M}_m(S_2, \lambda_d)
\end{array}
\]

**Proof.** Over a point of \(\overline{M}_m^2(\mathbb{P}^2, d)\) representing the strict lift of an \(m\)-pointed immersion \(\mathbb{P}^1 \to \mathbb{P}^2\), the substack of \(\overline{M}_n^2(\mathbb{P}^2, d)\) representing the same immersion together with an additional \(m-n\) markings, is dense in the fiber of \(\overline{M}_n(S_2, \lambda_d)\). \(\square\)

The divisors on \(\overline{M}_n(\mathbb{P}^2, d)\) representing maps from reducible curves are called *boundary divisors*. There is one such divisor \(D(A_1, A_2; d_1, d_2)\) for each partition \(\{1, \ldots, n\} = A_1 \cup A_2\) and each partition \(d = d_1 + d_2\). (If \(d_1 = 0\) then \(A_1\) must have at least two elements; if \(d_2 = 0\) then \(A_2\) must have at least two elements.) A general point of \(D(A_1, A_2; d_1, d_2)\) represents a map \(\mu: C \to \mathbb{P}^2\) from a two-twig curve; the first twig carries the markings indexed by \(A_1\), and its image in \(\mathbb{P}^2\) is a curve of degree \(d_1\); the second twig carries the markings indexed by \(A_2\), and its image in \(\mathbb{P}^2\) is a curve of degree \(d_2\). (See [5, p. 51] for a drawing.) Let \(p \in \mathbb{P}^2\) be the image of the attachment point.

**Proposition 3.2.** Suppose that \(d_1\) and \(d_2\) are both positive. Then a general element \(\mu\) of \(D(\emptyset, \emptyset; d_1, d_2)\) has a unique second-order stable lift \(\mu_2\). The source curve is a chain of five twigs. On each peripheral twig, \(\mu_2\) is the second-order strict lift of one of the two twigs of \(\mu\). On the central twig, \(\mu_2\) is a map of degree two into the divisor at infinity \(I\) and onto the pencil of tangent directions at \(p\), with the points of attachment mapping to the tangent directions of the two image curves; the map is ramified at these attachment points. On each of the remaining two twigs, \(\mu_2\) is a map of degree three onto the fiber of \(S_2\) over the point of \(S_1\) representing the tangent direction at \(p\) of one of the two image curves. It is ramified only at the points of attachment. (See Figure 1.)

**Proof.** Since \(\overline{M}_0(\mathbb{P}^2, d)\) is smooth and birationally equivalent to \(\overline{M}_0^2(\mathbb{P}^2, d)\), the general point of the divisor \(D(\emptyset, \emptyset; d_1, d_2)\) has a unique preimage. To see the nature of the stable lift, we follow the semistable reduction recipe of Fulton and Pandharipande [5, pp. 64–65]. We may assume that the source curve of \(\mu\) has just two twigs, that on each twig the map is an immersion, and that the maps are transverse at the attachment point. We may create a family in which the source curves are the plane curves \(xy = \epsilon\) and in which the central member is \(\mu\). Away from the central member, the strict lifts piece together to give a map to \(S_1\), and we now proceed to compute the “limiting member” over \(\epsilon = 0\). (Since the moduli space is separated and proper, there is a unique such limit.) In fact the map to \(S_1\) extends to every point of \(xy = 0\) except the point of attachment, and we can remove the indeterminacy by one blowup. The central member will now have three twigs, and a calculation shows that on the central twig the map to \(S_1\) is an isomorphism onto a fiber over a point of \(\mathbb{P}^2\), while on each peripheral twig it is the strict lift of the
original map. The calculation also shows that the map to $S_2$ is indeterminate at the two attachment points. Again the indeterminacy is removed by blowing up the two points, and now the central member has five twigs arranged in a chain. The map to $S_2$ takes the newly introduced twigs onto fibers over points over $S_1$. On the central twig the map is the lift of the map at the previous stage; thus it is a map into the divisor at infinity. On each peripheral twig the map is the second-order strict lift of the original map.

The central member is not reduced, however: its central twig has multiplicity 2 and the adjacent twigs have multiplicity 3. Thus, following the recipe, we introduce the base change $\epsilon = \delta^0$ and then normalize the resulting surface. After these operations we

Figure 1. The second-order stable lift $\mu_2$ of the map $\mu$ in Proposition 3.2. The vertical twig of the source of $\mu_2$ maps into the divisor at infinity in $S_2$; its image is shown dashed.
find that the central member is still a chain of five twigs, that the map on the central
member is a degree 2 cover, that the maps on the adjacent twigs are degree 3 covers,
and that all ramification is concentrated at the points of attachment.

Now consider maps of degree \( d \geq 3 \) from \( \mathbb{P}^1 \) to \( \mathbb{P}^2 \) which are immersions except at
one point, at which the map onto the image curve is ramified. Let \( \overline{C}(\mathbb{P}^2, d) \) be the
 closure of the locus on \( \mathcal{M}_0(\mathbb{P}^2, d) \) representing such maps; it is a divisor. For such a
map, let \( p \in \mathbb{P}^2 \) indicate the image of the ramification point, and let \( q \in S_1 \) indicate its
image under the first-order strict lift (i.e, the point of \( S_1 \) representing the cusp tangent
direction at \( p \)).

**Proposition 3.3.** A general element \( \mu \) of \( \overline{C}(\mathbb{P}^2, d) \) has a unique second-order stable lift
\( \mu_2 \). The source curve has five twigs. On one of these twigs, which is attached to each of
the others, the map to \( S_2 \) is the constant map to the point of \( I \) representing the tangent
directions to \( \mathbb{P}^2 \) at \( p \). The other four twig maps are

- the strict lift of \( \mu \) (attached at a point labeled \( a \))
- a map of degree one into the divisor at infinity \( I \) and onto the pencil of tangent
directions at \( p \) (attached at a point \( b \))
- two maps of degree one onto the fiber of \( S_2 \) over \( q \) (attached at points \( c \) and \( d \))

The attachment points are arranged in such a way that there could exist a double cover
of \( \mathbb{P}^1 \) ramified at \( a \) and \( b \), with \( c \) and \( d \) as the points of a fiber. (In other words, the
cross-ratio \([a, b, c, d]\) is 2. See Figure 3.)

**Proof.** Again we know that the general point of the divisor \( \overline{C}(\mathbb{P}^2, d) \) has a unique preim-
age, and again we see the nature of the stable lift by following the recipe of Fulton and
Pandharipande. This time we may begin with a family whose source is the product of
\( \mathbb{P}^1 \) and a curve, with the map to \( \mathbb{P}^2 \) being an immersion except on the special member.
The maps to \( S_1 \) and \( S_2 \) obtained by piecing together the strict lifts are defined every-
where but at the ramification point of the special member. At this point and at \( p \) we
may choose local coordinates so that the special member is \( t \mapsto (t^2, t^3) \), then extend to
the family of maps

\[ (\epsilon, t) \mapsto (t^2, t^3 - \epsilon t). \]

The indeterminacy in the map to \( S_1 \) is resolved by one blowup. On one of the two twigs
of the new special member we have the strict lift; on the other we have an isomorphism
onto the fiber of \( S_1 \) over \( p \); the attachment point maps to \( q \); and the image curves are
tangent there. To resolve the indeterminacy in the map to \( S_2 \), we first blow up the new
attachment point. Then the new special member has a chain of three twigs, with the
map on the central one being constant. There is still, however, indeterminacy at a point
on this twig. One more blowup creates a fourth twig, mapping onto the fiber of \( S_2 \) over
\( q \). Both this twig and the central one occur in the special member with multiplicity 2.
Thus when we make the base change \( \epsilon = \delta^2 \) and then normalize the resulting surface we
obtain two inverse images of the fourth twig, and on the central twig the normalization map is a double cover ramified at the points of attachment to the other two twigs.

The case $d = 2$ is special. If $\mu : \mathbb{P}^1 \to \mathbb{P}^2$ is not an immersion, then it must be a double cover of a line. For such a map, let $a$ and $b$ denote the branch points in $\mathbb{P}^2$. Let $\overline{\mathcal{C}}(\mathbb{P}^2, 2)$ be the closure of the locus on $\overline{\mathcal{M}}_0(\mathbb{P}^2, d)$ representing such maps; it is a divisor.
Figure 3. The second-order stable lift of a double cover of a line in $\mathbb{P}^2$. Each of the two twigs of attached to the peripheral twigs is mapped to a point of $S_2$. The two peripheral twigs map into the divisor at infinity; their images are shown dashed.

Proposition 3.4. A double cover of a line has a unique stable lift. The source curve is a chain of seven twigs. On the central twig the map is the strict lift of the double cover. On each of the adjacent twigs the map is a triple cover of a fiber of $S_2$ over $S_1$, ramified only at the attachment points. On each peripheral twig we have a map into the divisor at infinity and onto the pencil of directions at $a$ or $b$. On each of the remaining twigs the map is constant. (See Figure 3.)

Proof. Again we know that the general point of the divisor $\mathcal{C}(\mathbb{P}^2, 2)$ has a unique preimage, and again we see the nature of the stable lift by following the recipe. We may choose local coordinates so that the family of maps to $\mathbb{P}^1$ is

$$(\epsilon, t) \mapsto (t^2, \epsilon t).$$

Over the origin, the indeterminacy of the map to $S_1$ is resolved by one blowup. The map to $S_2$ is still indeterminate at the new attachment point, and two further blowups are required. (The first blowup creates a twig on which the map is constant, and the second one creates a twig mapping to a fiber of $S_2$.) A similar sequence of blowups over the other branch point creates a chain or seven twigs. Two of these twigs (those on which the map is constant) occur in the special member with multiplicity 2, and two others (those created by the last blowups) occur with multiplicity 3. Thus we make the base change $\epsilon = \delta^6$ and then normalize, creating double and triple covers with ramification concentrated at the points of attachment.

\[\square\]

4. GROMOV-WITTEN INVARIANTS AND CHARACTERISTIC NUMBERS

Here we define the Gromov-Witten invariants and characteristic numbers of the family of rational plane curves of degree $d$, and show that they agree.

As before, let $\overline{M}_n^2(\mathbb{P}^2, d)$ be the stack of second-order stable lifts. Let $e_k$ denote evaluation at the $k$th point. Suppose that $\gamma_1, \ldots, \gamma_n$ are classes in $A^*(S_2)$. Then the
associated Gromov-Witten invariant is the number
\[
\left\langle \prod_{k=1}^{n} \gamma_k \right\rangle_d = \int \prod_{k=1}^{n} e_k^*(\gamma_k) \cap \mathcal{M}_n^2(\mathbb{P}^2, d).
\]

Here are two basic properties:

1. If \( \gamma_n = 1 \) then \( \left\langle \prod_{k=1}^{n} \gamma_k \right\rangle_d = 0. \)

2. If \( \gamma_n \) is a divisor class, then
\[
\left\langle \prod_{k=1}^{n} \gamma_k \right\rangle_d = (\int \gamma_n \cap \lambda_d) \left\langle \prod_{k=1}^{n-1} \gamma_k \right\rangle_d
\]
where \( \lambda_d \in A_1(S_2) \) is the class of the image of the second-order strict lift of an immersion of degree \( d \). In particular,
\[
\left\langle \prod_{k=1}^{n} \gamma_k \right\rangle_d = \begin{cases} 
\frac{d}{2} \left\langle \prod_{k=1}^{n-1} \gamma_k \right\rangle_d & \text{if } \gamma_n = h, \\
(2d - 2) \left\langle \prod_{k=1}^{n-1} \gamma_k \right\rangle_d & \text{if } \gamma_n = \check{h}, \\
0 & \text{if } \gamma_n = i, \\
(3d - 6) \left\langle \prod_{k=1}^{n-1} \gamma_k \right\rangle_d & \text{if } \gamma_n = z.
\end{cases}
\]

(The lift of an immersion misses the divisor at infinity; the last formula is a consequence of the others and the identity \( z = i - 3h - 3\check{h} \).)

Now let \( \mathcal{R}(d) \subset \mathbb{P}^N \times \mathbb{P}^2 \) denote the total space of the family of rational plane curves of degree \( d \), over the parameter space \( T(d) \subset \mathbb{P}^N \), where \( N = \binom{d+2}{2} - 1 \). Let \( \overline{\mathcal{R}}(d) \) and \( \overline{T}(d) \) denote the closures. Let
\[
\overline{\mathcal{R}}^2_n(d) = (\overline{\mathcal{R}}(d))_2 \times_{\overline{T}(d)} (\overline{\mathcal{R}}(d))_2 \times \overline{T}(d) \cdots \times \overline{T}(d) (\overline{\mathcal{R}}(d))_2,
\]
the \( n \)-fold fiber product of the second-order lift of this family. It is a subvariety of \( S_2 \times S_2 \times \cdots \times S_2 \times \overline{T}(d) \). A general point of \( \overline{T}(d) \) represents a curve with a unique second-order lift, and the fiber of \( \overline{\mathcal{R}}^2_n(d) \) over this point is a variety of dimension \( n \) representing \( n \)-tuples of points on the lift. Thus there is a unique component of \( \overline{\mathcal{R}}^2_n(d) \) dominating
\[
\mathcal{R}_n(d) = \mathcal{R}(d) \times_{\overline{T}(d)} \mathcal{R}(d) \times_{\overline{T}(d)} \cdots \times_{\overline{T}(d)} \mathcal{R}(d),
\]
which we call the join, and denote by \( J\overline{\mathcal{R}}^2_n(d) \). (There may be other components. For example, a double line has a 2-parameter family of second-order lifts. Thus if \( d = 2 \) and \( n \geq 3 \) the fiber product has another component supported over the locus of double lines; if \( n \geq 4 \) this component even has a larger dimension than the join.)

Let \( e_k: J\overline{\mathcal{R}}^2_n(d) \rightarrow S_2 \) be projection onto the \( k \)th factor. Then we define the characteristic numbers by
\[
\left\langle \prod_{k=1}^{n} \gamma_k \right\rangle_d = \int \prod_{k=1}^{n} e_k^*(\gamma_k) \cap [J\overline{\mathcal{R}}^2_n(d)].
\]
These characteristic numbers need not have enumerative significance. For example,
\[ \{h^2 \cdot h^2 \cdot z\}_1 = -3, \]
as one can see by an excess intersection calculation or by replacing \( z \) by \( i - 3h + 3\hat{h} \).
Nevertheless these numbers are the appropriate inputs for the contact formulas of §8, whose output does have enumerative significance.

**Theorem 4.1.** For each \( d, n \), and classes \( \gamma_1, \ldots, \gamma_n \),
\[ \left\langle \prod_{k=1}^{n} \gamma_k \right\rangle_d = \left\langle \prod_{k=1}^{n} \gamma_k \right\rangle_d. \]

**Proof.** There is a birational morphism from \( \overline{M}_n^2(\mathbb{P}^2, d) \) to \( \mathcal{J}\mathcal{R}_n^2(d) \) which associates to the stable lift of an immersion its image curve in \( S_2 \) together with the images of the \( n \) markings. Denote the graph of this morphism by \( \mathcal{G} \). Then two applications of the projection formula show that the Gromov-Witten invariant and the characteristic number both equal
\[ \int \prod_{k=1}^{n} e_k^*(\gamma_k) \cap [\mathcal{G}]. \]

\[ \square \]

5. **Special boundary divisors**

Suppose that \( n \geq 4 \). Consider the “forgetful” morphism from \( \overline{M}_n(S_2, \lambda_d) \) to \( \overline{M}_4 \), the moduli space of 4-pointed stable genus 0 curves, which associates to a stable map its source curve with only the first four markings retained. (Unstable twigs are contracted as necessary.) Composing with inclusion, we have a morphism \( \overline{M}_n^2(\mathbb{P}^2, d) \rightarrow \overline{M}_4 \). The space \( \overline{M}_4 \) is isomorphic to \( \mathbb{P}^1 \). It has a distinguished point \( P(12 \mid 34) \) representing the two-twig curve having the first two markings on one twig and the latter two on the other; similarly there are two other distinguished points \( P(13 \mid 24) \) and \( P(14 \mid 23) \).

Thus on \( \overline{M}_n^2(\mathbb{P}^2, d) \) there are three linearly equivalent divisors \( D(12 \mid 34) \), \( D(13 \mid 24) \), and \( D(14 \mid 23) \), which we call the special boundary divisors. (We will also use this term to refer to individual components of \( D(12 \mid 34) \), etc.)

To obtain the Gromov-Witten invariants defined in §4, we will not need to analyze these divisors completely; it will be enough to identify just those components that affect our calculations. A divisor \( D \) on \( \overline{M}_n^2(\mathbb{P}^2, d) \) is called numerically irrelevant if
\[ (5.1) \quad \int e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n) \cap [D] = 0 \]
for all \( \gamma_1, \ldots, \gamma_n \in A^*(S_2) \). Divisors that are not numerically irrelevant are said to be numerically relevant. Similarly, for \( n \geq 4 \), a divisor is called irrelevant with respect to
A divisor become unstable. In analogy with Proposition 5.4 of [4], we have the following result.

Let \( \pi: \overline{M}_n^2(P^2, d) \rightarrow \overline{M}_0(P^2, d) \) be the morphism that composes a stable map \( \mu: C \rightarrow S_2 \) with the projection \( S_2 \rightarrow P^2 \), forgets all markings, and contracts any twigs that have become unstable. In analogy with Proposition 5.4 of [4], we have the following result.

**Proposition 5.2.** A divisor \( D \) on the stack \( \overline{M}_n^2(P^2, d) \) is numerically irrelevant if \( \text{codim} \tau(D) \geq 2 \).

**Proof.** Without loss of generality, we may assume that \( D \) is irreducible. By linearity we may assume that the \( \gamma_j \) in (5.1) come from the following basis for \( A^*(S_2) \):

\[
\{1, h, \bar{h}, h^2, \bar{h}^2, h^2\bar{h}, i, \bar{h}i, h^2i, \bar{h}^2i, h^2\bar{h}i\}.
\]

Note first that if any \( \gamma_j \), say \( \gamma_n \), is the identity element then, by the projection formula applied to the forgetful morphism \( \pi: \overline{M}_n^2(P^2, d) \rightarrow \overline{M}_{n-1}(P^2, d) \), we have

\[
\int e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n-1) \cup e_1^*(1) \cap [D] = \int e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n-1) \cap \pi_*[D],
\]

since the evaluation maps \( e_1, \ldots, e_n \) factor through \( \pi \). But \( \pi_*[D] \) is zero unless \( \dim \pi(D) = \dim D = \dim \overline{M}_{n-1}(P^2, d) \). By the irreducibility of the moduli stack, we therefore have \( \pi(D) = \overline{M}_{n-1}(P^2, d) \) and hence that \( \tau(D) = \overline{M}_0(P^2, d) \). Given that \( \text{codim} \tau(D) \) is at least 2, we see that the proposition holds in this instance.

Next, consider, for \( k \leq n \), the class

\[
\alpha_k = e_1^*(\gamma_1) \cup \cdots \cup e_k^*(\gamma_k) \cap [D]
\]

where none of the \( \gamma_j \)'s is the identity. We claim that if \( \alpha_k \) is nonzero, then it can be represented by a cycle \( D_k \) for which

\[
\text{codim} \tau(D_k) \geq 2 + \sum_{j=1}^{k} (\text{codim} \gamma_j - 1).
\]

To establish the claim, we use induction on \( k \), the case \( k = 0 \) being trivial. In view of the remarks above, we need only consider the cases where \( \text{codim} \gamma_k \geq 2 \). For \( \gamma_k = h^2, \bar{h}^2 \), or \( h^2\bar{h} \), the argument is given in the proof of Proposition 5.4 of [4]. Thus we prove the result when \( \gamma_k = hi, h^2i, \bar{h}i, \bar{h}^2i, \) or \( h^2\bar{h}i \).

1. Suppose \( \gamma_k = hi \). If \( L \subset P^2 \) is a general line, then \( \alpha_k \) may be represented by a cycle \( D_k \) for which \( \tau(D_k) \) is contained in the intersection of \( \tau(D_{k-1}) \) and the set of curves with a singularity on \( L \). Hence \( \text{codim} \tau(D_k) > \text{codim} \tau(D_{k-1}) \) and (5.3) holds by induction.
2. Suppose $\gamma_k = h^2i$. If $P \in \mathbf{P}^2$ is a general point, then $\alpha_k$ may be represented by a cycle $D_k$ for which $\tau(D_k)$ is contained in the intersection of $\tau(D_{k-1})$ and the set of curves with a singularity at $P$. Hence $\text{codim} \tau(D_k) \geq 2 + \text{codim} \tau(D_{k-1})$ and \((\ref{3})\) holds by induction.

3. Suppose $\gamma_k = \bar{h}i$. Let $P \in \mathbf{P}^2$ be a general point. Since $i = z + 3(h - \bar{h})$, we have that $\alpha_k$ is represented by a cycle $D_k$ for which $\tau(D_k)$ is in the intersection of $\tau(D_{k-1})$ and the union of the set of curves having a flex tangent that passes through $P$ and the set of curves whose tangent cone at a singular point passes through $P$. Thus $\text{codim} \tau(D_k) > \text{codim} \tau(D_{k-1})$.

4. Suppose $\gamma_k = \bar{h}^2i$. Let $L \subset \mathbf{P}^2$ be a general line. Using $z$ in place of $i$ as in case 3, we have that $\alpha_k$ is represented by a cycle $D_k$ for which $\tau(D_k)$ is in the intersection of $\tau(D_{k-1})$ and the union of the set of curves having $L$ as flex tangent and the set of curves whose tangent cone at a singular point contains $L$. Thus $\text{codim} \tau(D_k) \geq 2 + \text{codim} \tau(D_{k-1})$.

5. Suppose $\gamma_k = \bar{h}^2\bar{h}i$. Let $(P, L)$ be a general flag. Using $z$ in place of $i$, we have that $\alpha_k$ is represented by a cycle $D_k$ for which $\tau(D_k)$ is in the intersection of $\tau(D_{k-1})$ and the union of the set of curves with a flex at $P$ and flex tangent equal to $L$ and the set of curves singular at $P$ whose tangent cone contains $L$. Thus $\text{codim} \tau(D_k) \geq 3 + \text{codim} \tau(D_{k-1})$.

To finish the proof, note that $\int e_1^z(\gamma_1) \cup \cdots \cup e_n^z(\gamma_n) \cap [D]$ must be zero unless $\sum_{j=1}^n \text{codim} \gamma_j = 3d - 2 + n$, i.e., unless $2 + \sum_{j=1}^n (\text{codim} \gamma_j - 1) = 3d$. But in this case \((\ref{3})\) implies that $\text{codim} \tau(D_n) \geq 3d > \dim \overline{M}_0(\mathbf{P}^2, d)$. Hence $D_n$ must be zero, as desired.

\[\square\]

**Corollary 5.4.** If $D$ is a numerically relevant special boundary divisor on the stack $\overline{M}_0^g(\mathbf{P}^2, d)$, then $\tau(D)$ is one of the following:

1. all of $\overline{M}_0(\mathbf{P}^2, d)$,
2. a divisor $D(\emptyset; d_1, d_2)$, where $d_1$, $d_2$ are positive and $d_1 + d_2 = d$,
3. a divisor $\overline{C}(\mathbf{P}^2, d)$, for $d \geq 2$.

**Proof.** An immersion $\mathbf{P}^1 \rightarrow \mathbf{P}^2$ has a unique (unmarked) second-order stable lift, whose source is again $\mathbf{P}^1$. Thus the general $n$-marked stable lift of an immersion likewise has $\mathbf{P}^1$ as source. Thus the image in $\overline{M}_0(\mathbf{P}^2, d)$ of a numerically relevant divisor is either the entire stack or a divisor representing non-immersive maps. Every non-immersion either has a reducible source or is singular at one or more points; thus it is represented by a point of one of the listed divisors.

Suppose that $\mu$ is an element of $\overline{M}_0^g(\mathbf{P}^2, d)$ for which $\tau(\mu)$ is an immersion. According to Proposition \([3.1]\), one of the twigs is the strict lift of $\tau(\mu)$. Other twigs, if any, must be mapped to single points of $S_2$, and thus must carry at least two markings. Such a
Figure 4. A second-order stable lift over an (unmarked) immersion.

stable lift is represented by a point of $D(12 \mid 34)$ if and only if the markings 1 and 2 are separated from the markings 3 and 4 by at least one attachment point. Thus there is a component

$$ \overline{M}^2_{A_1 \cup \{\ast\}}(\mathbb{P}^2, d) \times_{S_2} \overline{M}^2_{A_2 \cup \{\ast\}}(S_2, 0) $$

of $D(12 \mid 34)$ corresponding to every partition in which 1, 2 belong to one of the subsets and 3, 4 to the other. Here $\ast$ indicates the marking used to create the fiber product; $A_1$ indexes the markings on the strict lift twig and $A_2$ those on the constant twig. (A typical member is shown in Figure 4.)

In Proposition 3.2, we have described the unique (unmarked) second-order stable lift of a general element of $D(\emptyset, \emptyset; d_1, d_2)$. (Also see Figure 1.) According to Proposition 3.1 on an $n$-marked stable lift, the markings may be distributed on this five-twig curve in any way (with additional twigs arising as markings are brought together). Thus there is a component $D^2(A_1, A_2, A_3, A_4, A_5; d_1, d_2)$ of $D(12 \mid 34)$ corresponding to each valid partition, i.e., in which marks 1 and 2 occur are separated from marks 3 and 4 by at least one attachment point. Here $A_1$ indexes the markings on the peripheral twig carrying the strict lift of the map of degree $d_1$, and $A_2$ indexes the markings on the adjacent twig, etc. To avoid redundancy, we may assume that marks 1 and 2 occur before marks 3 and 4. The divisor $D^2(A_1, A_2, A_3, A_4, A_5; d_1, d_2)$ may be expressed as a fiber product

$$ \overline{M}^2_{A_1 \cup \{\bullet\}}(\mathbb{P}^2, d_1) \times_{S_2} \overline{R}_{A_2 \cup \{\bullet, \triangledown\}}(2, 3) \times_{S_2} \overline{R}_{A_3 \cup \{\circ, \heartsuit\}}(1, 2) \times_{S_2} \overline{R}_{A_4 \cup \{\bigtriangleup, \bigstar\}}(2, 3) \times_{S_2} \overline{M}^2_{A_5 \cup \{\bullet\}}(\mathbb{P}^2, d_2) $$

in which the first and last factors are stacks of stable lifts. The middle factor is the stack whose general member represents a double cover of the lift to $S_2$ of a fiber of $S_1$ over a point of $\mathbb{P}^2$, ramified at the two markings $\bigtriangleup$ and $\bigstar$ and carrying additional markings indexed by $A_3$. Similarly, the second and fourth factors are stacks whose general member represents a triple cover of a fiber of $S_2$ over a point of $S_1$, ramified only at the two special markings. The fiber products are created using the evaluation maps at $\bigtriangleup$, $\bigstar$, $\bigcirc$, and $\bullet$. 
Similarly, in Proposition 3.3, for \( d \geq 3 \), we have described the unique unmarked second-order lift of a general element of \( \overline{C}(\mathbb{P}^2, d) \), and on an \( n \)-marked stable lift the markings may be distributed in any way. (Also see Figure 4.) Thus there is a component \( C^2(A_1, A_2, A_3, A_4, A_5; d) \) of \( D(12 \mid 34) \) corresponding to each valid partition. Likewise, there is a component \( C^2(A_1, A_2, A_3, A_4, A_5, A_6, A_7; 2) \) for each valid partition.

In view of these remarks and Corollary 5.4, the following theorem is now clear.

**Theorem 5.5.** If \( D \) is a numerically relevant component of \( D(12 \mid 34) \), then it is one of the following:

1. a divisor \( \overline{M}^2_{A_1 \cup \{ \star \}}(\mathbb{P}^2, d) \times S_2 \overline{M}^2_{A_2 \cup \{ \star \}}(S_2, 0) \) where \( |A_2| \geq 2 \);
2. a divisor \( D^2(A_1, A_2, A_3, A_4, A_5; d_1, d_2) \);
3. a divisor \( C^2(A_1, A_2, A_3, A_4, A_5; d) \) with \( d \geq 3 \);
4. a divisor \( C^2(A_1, A_2, A_3, A_4, A_5, A_6, A_7; 2) \).

**Theorem 5.6.** The numerically relevant components of type (3) or (4) are irrelevant with respect to the base. Among those of type (2), only those in which markings 1 and 2 belong to \( A_1 \), and markings 3 and 4 belong to \( A_5 \), are relevant with respect to the base.

**Proof.** Consider a component \( D \) of type (2) in which the first marking belongs to \( A_2 \). Then the evaluation map \( \epsilon_1 \) factors through projection of the fiber product \( D \) onto its second factor, the stack \( \overline{R}_{A_2 \cup \{ \star \}}(2, 3) \). Since this stack represents maps to fibers of \( S_2 \) over \( S_1 \), there is a morphism which picks out the image of the fiber and thus a composite morphism \( \epsilon: D \to \mathbb{P}^2 \). If \( \gamma_1 \) is an element of \( A^*(\mathbb{P}^2) \), the classes \( \epsilon_1^*(\gamma_1) \) and \( \epsilon^*(\gamma_1) \) agree. Let \( D' \) denote \( D^2(A_1, A_2 \setminus \{ 1 \}, A_3, A_4, A_5; d_1, d_2) \), and let \( \epsilon': D' \to \mathbb{P}^2 \) be the analogous morphism. Then \( \epsilon = \epsilon' \circ \varphi \), where \( \varphi: D \to D' \) is the forgetful morphism. If \( \gamma_2, \ldots, \gamma_n \) are elements of \( A^*(S_2) \), then by the projection formula

\[
\int e_1^*(\gamma_1) \cup e_2^*(\gamma_2) \cup \cdots \cup e_n^*(\gamma_n) \cap [D] = \int e^*(\gamma_1) \cup e_2^*(\gamma_2) \cup \cdots \cup e_n^*(\gamma_n) \cap [D]
\]

\[
= \int e'^*(\gamma_1) \cup e_2^*(\gamma_2) \cup \cdots \cup e_n^*(\gamma_n) \cap \varphi_*[D].
\]

Since the relative dimension of \( D \) over \( D' \) is 1, this class vanishes. Hence \( D \) is irrelevant with respect to the base.

A similar argument applies if the second, third, or fourth marking belongs to \( A_2 \). The argument also works if any of these markings belong to \( A_3 \) or \( A_4 \), since again the corresponding evaluation map factors through projection onto a factor of \( D \) representing maps to fibers of \( S_2 \) over \( S_1 \), or representing maps to fibers of \( S_1 \) over \( \mathbb{P}^2 \). For a component of type (3) or (4) we may also employ the same sort of argument, since for any such component at least one (in fact two) of the four special markings must lie on a twig which either maps to a fiber of \( S_2 \) over \( S_1 \), or maps to a fiber of \( S_1 \) over \( \mathbb{P}^2 \), or is contracted to a point. \( \Box \)
Having identified the components of $D(12 \mid 34)$ which are relevant with respect to the base, we must investigate their multiplicities. Here it is important to distinguish between the stack $\overline{M}_n^2(P^2, d)$ and its coarse moduli space. A general point of the closed substack $D^2(A_1, \emptyset, \emptyset, A_5; d_1, d_2)$ represents a stable map with an automorphism group of order 18; thus the map from this substack to its image on the coarse moduli space—a divisor—has degree $1/18$. Under the birational morphism from the coarse moduli space of $\overline{M}_n^2(P^2, d)$ to that of $\overline{M}_n(P^2, d)$, this divisor is mapped onto a divisor. Now on the coarse moduli space of $\overline{M}_n^2(P^2, d)$, all components of $D(12 \mid 34)$ appear with multiplicity 1. Hence on the stack $\overline{M}_n^2(P^2, d)$, the fundamental class of the substack $D^2(A_1, \emptyset, \emptyset, A_5; d_1, d_2)$ appears in $D(12 \mid 34)$ with multiplicity 18. By Proposition 3.1, a similar statement holds when there are markings on the middle twigs: $D^2(A_1, A_2, A_3, A_4, A_5; d_1, d_2)$ appears in $D(12 \mid 34)$ with multiplicity 18. The same sort of argument shows that a divisor of type (1) (the general point of which represents an automorphism-free stable map) occurs in $D(12 \mid 34)$ with multiplicity 1. (See [10] for a general discussion of intersection theory on stacks.)

6. Potentials and partial differential equations

We now present potential functions associated to each of the special boundary divisors that are relevant with respect to the base; these are the potentials that will be needed in order to recursively calculate the second-order Gromov-Witten invariants. Since many of the intermediate expressions are exceedingly long, we will describe them rather than write them out explicitly. In these calculations we employ two ordered bases for $A^*(S_2)$, which are dual under the intersection pairing. Here is the “$z$-basis”:

| $Y_{000}$ | $Y_{100}$ | $Y_{200}$ | $Y_{010}$ | $Y_{020}$ | $Y_{210}$ | $Y_{001}$ | $Y_{101}$ | $Y_{201}$ | $Y_{011}$ | $Y_{021}$ | $Y_{211}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1         | $h$       | $h^2$     | $\hat{h}$ | $\hat{h}^2$ | $\hat{h}^2\hat{h}$ | $z$       | $h$       | $h^2$     | $h$       | $h^2$     | $h^2$     |

This is the “$i$-basis”:

| $Y_{211}^*$ | $Y_{021}^*$ | $Y_{011}^*$ | $Y_{201}^*$ | $Y_{101}^*$ | $Y_{210}^*$ | $Y_{020}^*$ | $Y_{010}^*$ | $Y_{200}^*$ | $Y_{100}^*$ | $Y_{000}^*$ |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $h^2$       | $\hat{h}$   | $\hat{h}$   | $\hat{h}$   | $\hat{h}$   | $\hat{h}$   | $\hat{h}$   | $\hat{h}$   | $\hat{h}$   | $\hat{h}$   | $\hat{h}$   |

Here $z$ is the class of $Z$ and $i$ is the class of the divisor at infinity $I$. Note that the subscript $k$ on $Y_k$ is a vector having three integer components, and that the basis element dual to $Y_k$ is $Y_k^*$, where $k + k^* = 211$ or 121. We will write a general element of $A^*(S_2)$ as $\sum y_k Y_k$ or as $\sum y_i^* Y_i^*$.
We write the classical potential $P$ of $S_2$ in a somewhat nonstandard way by using both bases. Let $\gamma = \sum y_k Y_k$ and $\delta = \sum y^*_l Y^*_l$. Then
\[
P = \sum_{n \geq 2} \frac{1}{n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cup e_{n+1}^*(\delta) \cap [M_{n+1}(S_2, 0)] \]
\[
= \frac{1}{2} \int \gamma \cup \gamma \cup \delta \cap [S_2].
\]

The quantum potential of second-order stable lifts is defined by
\[
N = \sum_{d \geq 1} \frac{1}{n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cap [M_d^1(P^2, d)].
\]

Using the basic properties of Gromov-Witten invariants given in §4, we may express $N$ as
\[
N = \sum_{d \geq 1} N^{(d)} E^{(d)}
\]
where $E^{(d)} = \exp(dy_{100} + (2d - 2)y_{010} + (3d - 6)y_{001})$ and $N^{(d)}$ is a polynomial in the variables $y_k$ corresponding to the eight elements of the $z$-basis other than the identity and the divisors. The polynomial $N^{(d)}$ is weighted homogeneous, where weight($y_k$) = codim($Y_k$) - 1; it is the generating series of Gromov-Witten invariants, without divisor conditions, of degree $d$. To be more explicit, let
\[
a = (a_{200}, a_{020}, a_{210}, a_{101}, a_{201}, a_{011}, a_{021}, a_{211})
\]
be a vector of nonnegative integers. If $k$ is any of the possible subscripts appearing in $a$, let $|k|$ denote the sum of its entries, and set
\[
\|a\| = \sum_k (|k| - 1)a_k.
\]
Let $y^a = \prod_k y_k^{a_k}$ and $a! = \prod_k a_k!$. Set $N_d(a) = \langle \prod_i \gamma_i \rangle_d$ where $a_{200}$ of the $\gamma_i$ are $Y_{200}$, $a_{020}$ are $Y_{020}$, etc. Then
\[
N^{(d)} = \sum_{\|a\|=3d-1} N_d(a) \frac{y^a}{a!}
\]

Next we introduce a potential $T$ for the maps on the three middle twigs of the divisor $D^2(A_1, A_2, A_3, A_4, A_5; d_1, d_2)$:
\[
T = \sum_{n \geq 0} \frac{1}{2n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cup e_{n+1}^*(\delta) \cup e_{n+2}^*(\delta) \cap [T_n]
\]
Here $T_n$ is the stack whose general point represents a stable map into $S_2$ with a three-twig source: the central twig maps as a double cover of the lift of a fiber of $S_1$ over $P^2$ with ramification occurring at the attachment points, and the peripheral twigs each map as a triple cover of a fiber of $S_2$ over $S_1$ with ramification concentrated at two points,
one of which is an attachment point. In the definition above, we express both \( \gamma \) and \( \delta \) in terms of the \( z \)-basis, and we use the fundamental class of the stack rather than the fundamental class of its associated coarse moduli space. (As explained at the end of §3, the two classes differ by a factor of 18.)

The stack \( \mathcal{T}_n \) is the fiber product of three simpler stacks, each with an associated potential, which we now describe. Let \( \mathcal{R}_{n,2}(1, 2) \) denote the stack of stable maps whose general member represents an \((n + 2)\)-marked map \( \mathbf{P}^1 \to S_2 \) satisfying these conditions:

- the map is a double cover of a fiber of the lift to \( S_2 \) of a fiber of \( S_1 \) over \( \mathbf{P}^2 \);
- the last two markings always occur at the ramification points.

In other words, this stack represents central twigs of members of \( \mathcal{T}_n \), with two extra markings for “gluing.” Again we emphasize that we are employing the fundamental class of the stack rather than its associated coarse moduli space. (Here the two classes differ by a factor of 2.) Let \( \mathcal{R}(1, 2) \) denote the associated potential:

\[
\mathcal{R}(1, 2) = \sum_{n \geq 0} \frac{1}{2n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cup e_{n+1}^*(\delta) \cup e_{n+2}^*(\delta) \cap [\mathcal{R}_{n,2}(1, 2)].
\]

We will express both \( \gamma = \sum y_k Y_k \) and \( \delta = \sum z_k Y_k \) in terms of the \( z \)-basis.

Similarly, we define \( \mathcal{R}_{n,2}(2, 3) \) to be the stack for the peripheral twigs. The general member represents an \((n + 2)\)-marked map \( \mathbf{P}^1 \to S_2 \) that is a triple cover of a fiber of \( S_2 \) over \( S_1 \), and again we demand that the last two markings occur at the ramification points. The associated potential is

\[
\mathcal{R}(2, 3) = \sum_{n \geq 0} \frac{1}{2n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cup e_{n+1}^*(\delta) \cup e_{n+2}^*(\delta) \cap [\mathcal{R}_{n,2}(2, 3)].
\]

In \( \mathcal{R}(2, 3) \) we will express \( \gamma = \sum y_k Y_k \) in terms of the \( z \)-basis and \( \delta = \sum z_i^* Y_i^* \) in terms of the \( i \)-basis.

**Lemma 6.1.** Let \( y_{110} = y_{020} \) and let \( z_{110} = z_{200} \). The potential \( \mathcal{R}(1, 2) \) is the sum of terms in

\[
\frac{1}{2} \exp(2y_{010}) \exp(2y_{110}) \exp(2y_{210}) \exp(z_{010}) \exp(z_{110}) \exp(z_{210})
\]

that are quadratic in \( z_{010}, z_{110}, z_{210} \) and for which the (vector) sum of subscripts is \((2, n + 2, 0)\) for some \( n \).

**Proof.** First use the alternative basis to the \( z \)-basis where \( \tilde{h}^2 \) replaces \( \tilde{h}^2 \) and \( \tilde{h} \tilde{z} \) replaces \( \tilde{h}^2 \). Denote the coefficients of \( \mathcal{R}(1, 2) \) with respect to this alternative basis by \( \tilde{y}_k \)'s and \( \tilde{z}_i \)'s. Each basis element may be written in the form \( h^a \gamma \), where \( a = 0, 1, \) or \( 2, \) and \( \gamma \) does not involve \( h \). The Gromov-Witten invariant

\[
\int e_1^*(h^{a_1} \gamma_1) \cup \cdots \cup e_n^*(h^{a_n} \gamma_n) \cup e_{n+1}^*(h^{a_{n+1}} \gamma_{n+1}) \cup e_{n+2}^*(h^{a_{n+2}} \gamma_{n+2}) \cap [\mathcal{R}_{n,2}(1, 2)]
\]
is zero unless \( a_1 + \cdots + a_{n+2} = 2 \), and otherwise equals
\[
\int e^*_1(\gamma_1) \cup \cdots \cup e^*_n(\gamma_n) \cup e^*_{n+1}(\gamma_{n+1}) \cup e^*_{n+2}(\gamma_{n+2}) \cap [L]
\]
where \([L]\) denotes the fundamental class of the lift of a fiber of \( S_1 \) over \( \mathbb{P}^2 \). This in turn must be zero if \( z \) appears in any \( \gamma_i \) (since the divisor at infinity \( \emptyset \) is disjoint from the dual divisor \( Z \)). It also vanishes unless all \( \gamma_i \)'s equal \( \hat{h} \), in which case its value is \( 2^{n-1} \).

Thus
\[
\mathcal{R}(1, 2) = \sum_{n, k} \frac{1}{2n!} C \hat{y}_{k_1} \cdots \hat{y}_{k_n} \hat{z}_{k_{n+1}} \hat{z}_{k_{n+2}}
\]
where the coefficient \( C \) is \( 2^{n-1} \) if the first entry of \( k_1 + \cdots + k_{n+2} \) is 2, the last entry of this sum is 0, and each second entry is 1; otherwise the coefficient is zero. Thus \( \mathcal{R}(1, 2) \) is the sum of terms in
\[
\frac{1}{2} \exp(2\hat{y}_{010}) \exp(2\hat{y}_{110}) \exp(2\hat{y}_{210}) \exp(\hat{z}_{010}) \exp(\hat{z}_{110}) \exp(\hat{z}_{210})
\]
that are quadratic in \( \hat{z}_{010}, \hat{z}_{110}, \hat{z}_{210} \) and for which the sum of subscripts is \( (2, n + 2, 0) \) for some \( n \). To obtain the lemma, note that \( y_k = \hat{y}_k \) for all subscripts \( k \) appearing in the formula (which do not include the subscripts 200 or 201).

By a similar argument, we establish the following lemma. Recall that we use both the \( z \)-basis and the \( i \)-basis to write the potential \( \mathcal{R}(2, 3) \).

**Lemma 6.2.** The potential \( \mathcal{R}(2, 3) \) is the sum of terms in
\[
\frac{1}{3} \prod_{k_3=1} \exp(3y_{k_3}) \prod_{l_3=1} \exp(y^*_l)
\]
that are quadratic in the \( y^*_l \)'s and for which the (vector) sum of subscripts is \( (2, 1, n+2) \) or \( (1, 2, n+2) \) for some \( n \).

**Proof.** The Gromov-Witten invariant
\[
\int e^*_1(h^{a_1} h^{b_1} \gamma_1) \cup \cdots \cup e^*_n(h^{a_n} h^{b_n} \gamma_n) \\
\cup e^*_{n+1}(h^{a_{n+1}} h^{b_{n+1}} \delta_{n+1}) \cup e^*_{n+2}(h^{a_{n+2}} h^{b_{n+2}} \delta_{n+2}) \cap [\mathcal{R}_{n,2}(2, 3)]
\]
is zero unless \( a_1 + \cdots + a_{n+2} = 1 \) or 2, \( b_1 + \cdots + b_{n+2} = 2 \) or 1 (respectively), \( \gamma_1 = \cdots = \gamma_n = z \), and \( \delta_{n+1} = \delta_{n+2} = i \), in which case it has value \( 3^{n-1} \). Thus
\[
\mathcal{R}(2, 3) = \sum_{n, k} \frac{1}{2n!} C y_{k_1} \cdots y_{k_n} y^*_{k_{n+1}} y^*_{k_{n+2}}
\]
where the coefficient \( C \) is \( 3^{n-1} \) if the first entry of \( k_1 + \cdots + k_{n+2} \) is 1 or 2, the second entry is 2 or 1 (respectively), and the last entry of each \( k_j \) is 1; it is zero otherwise. Hence the lemma follows. \( \square \)
To put the potentials $R(1,2)$ and $R(2,3)$ together to obtain an expression for the potential $T$, we appeal to some general results. We will need the following easy consequence of the chain rule.

**Proposition 6.3.** Suppose $\{Y_0, \ldots, Y_m\}$ and $\{Z_0, \ldots, Z_m\}$ are two ordered bases for a vector space. Let $\gamma = \sum_{k=0}^{m} y_k Y_k$ and $\delta = \sum_{l=0}^{m} z_l Z_l$. Suppose $K$ is a power series in $y_0, \ldots, y_m, z_0, \ldots, z_m$ that can be written as

$$K = \sum_{a,b\geq 0} \frac{K_{a,b}(\gamma^a; \delta^b)}{a!b!}$$

where each $K_{a,b}$ is multilinear and symmetric in both the first $a$ and last $b$ arguments.

Let $k_1, \ldots, k_r$ and $l_1, \ldots, l_s$ have values in $\{0, \ldots, m\}$. Then

$$\frac{\partial^{r+s} K}{\partial y_{k_1} \cdots \partial y_{k_r} \partial z_{l_1} \cdots \partial z_{l_s}} = \sum_{a,b\geq 0} \frac{K_{a+r,b+s}(\gamma^a Y_{k_1} \cdots Y_{k_r}; \delta^b Z_{l_1} \cdots Z_{l_s})}{a!b!}.$$ 

Now suppose that $X$ is an arbitrary scheme, that $\beta_1 + \beta_2$ is a partition of the element $\beta \in A_1X$, and that $A_1 \cup A_2$ is a partition of the index set $\{1, \ldots, n\}$. Suppose that $M_1$ is a substack of $\overline{M}_{A_1 \cup \{\ast\}}(X, \beta_1)$ and $M_2$ is a substack of $\overline{M}_{A_2 \cup \{\ast\}}(X, \beta_2)$. Then the fiber product $M_1 \times_X M_2$, formed using the evaluation maps at $\ast$, is a substack of $\overline{M}_n(X, \beta)$.

We have a fiber square

$$\begin{array}{ccc}
M_1 \times_X M_2 & \xrightarrow{\iota} & M_1 \times M_2 \\
\downarrow & & \downarrow \\
X^{n+1} & \xrightarrow{D} & X^{n+2}
\end{array}$$

where $\iota$ is inclusion, $D$ is the diagonal inclusion that repeats the last coordinate, and the coordinates of the vertical maps are evaluation maps to $X$. In particular the evaluation maps $e_{n+1}, e_{n+2}$ on the right are at the two markings labeled by $\ast$; note that they agree when restricted to the fiber product.

Let $\{Y_0, \ldots, Y_m\}$ and $\{Y^*_m, \ldots, Y^*_0\}$ be ordered bases for $A^*(X)$ that are dual with respect to the intersection pairing. Then the fundamental class of the diagonal $\Delta$ in $X \times X$ is

$$[\Delta] = \sum_{k=0}^{m} Y_k \times Y^*_m - k.$$ 

Thus

$$\iota_* (e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n)) = \sum_{k=0}^{m} e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n) \cup e_{n+1}^*(Y_k) \cup e_{n+2}^*(Y^*_m - k).$$

Taking degrees, we obtain the following proposition.
Proposition 6.4. In this situation
\[
\int e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n) \cap [M_1 \times X M_2] = \\
\sum_{k=0}^m \left( \int \bigcup_{t \in A_1} e_t^*(\gamma_t) \cup e_{n+1}^*(Y_k) \cap [M_1] \right) \left( \int \bigcup_{t \in A_2} e_t^*(\gamma_t) \cup e_{n+2}^*(Y_{m-k}) \cap [M_2] \right).
\]

Let \( \gamma = \sum_{k=0}^m y_k Y_k \). Suppose, for \( i = 1, 2 \) that the stack \( M_i \) has potential \( M_i \), i.e.,
\[
M_i = \sum_{n \geq 0} \frac{1}{n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cap [M_i].
\]

Proposition 6.5. The potential \( M \) for the fiber product \( M_1 \times X M_2 \) is
\[
M = \sum_{k=0}^m \frac{\partial M_1}{\partial y_k} \frac{\partial M_2}{\partial y_{m-k}}.
\]

Proof. Set \( \gamma_1 = \ldots = \gamma_n = \gamma = \sum y_k Y_k \). Apply Proposition 6.4 and sum over all \( n \geq 0 \). Then apply Proposition 6.3. \( \square \)

Using Proposition 6.5 twice, we deduce the following result.

Proposition 6.6. The potential \( T \) is given by
\[
T = \frac{1}{2} \sum_{s,t} \frac{\partial R(2, 3)}{\partial y_s^*} \frac{\partial^2 R(1, 2)}{\partial y_s \partial y_t} \frac{\partial R(2, 3)}{\partial y_t^*}.
\]

Proposition 6.6, together with Lemmas 6.1 and 6.2, is sufficient to give an explicit expression for \( T \).

Next we derive partial differential equations in a manner analogous to the derivation of the WDVV-equation of quantum cohomology (equation (58) of [3]). Fix an ordered quadruple \( Y_i, Y_j, Y_k, Y_l \) of elements from the \( z \)-basis, and set
\[
G(ij | kl) = \sum_{n \geq 4} \frac{1}{n!} \sum_{d \geq 0} e_1^*(Y_i) \cup e_2^*(Y_j) \cup e_3^*(Y_k) \cup e_4^*(Y_l) \cup e_5^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cap [D(12 | 34)],
\]
where \( D(12 | 34) \) denotes the special boundary divisor on \( \overline{M_n(P^2, d)} \). Note that the same notation \( D(12 | 34) \) is used for divisors in the various stacks \( \overline{M_n(P^2, d)} \).

The linear equivalence \( D(12 | 34) \sim D(14 | 23) \) implies the following equality.

Theorem 6.7. For all subscripts \( i, j, k, \) and \( l \),
\[
G(ij | kl) = G(il | jk).
\]
In general this will be a complicated identity, since it will involve contributions from all numerically relevant components of $D(12 \ | \ 34)$ and $D(14 \ | \ 23)$. However, if each subscript is either $100$ or $200$ (i.e., if the classes $Y_i$, $Y_j$, $Y_k$, $Y_l$ are pullbacks to $S_2$ of classes from $\mathbf{P}^2$), then we need to know only the components which are relevant with respect to the base. In these cases, Theorem 5.6 and Proposition 6.5 imply the following formula:

$$G(ij \ | \ kl) = \sum_s \left( \frac{\partial^3 N}{\partial y_i \partial y_j \partial y_s} \frac{\partial^3 P}{\partial y_k \partial y_l \partial y_s^*} + \frac{\partial^3 N}{\partial y_k \partial y_l \partial y_s} \frac{\partial^3 P}{\partial y_i \partial y_j \partial y_s^*} \right)$$

$$+ 18 \sum_{s,t} \frac{\partial^3 N}{\partial y_i \partial y_j \partial y_s} \frac{\partial^2 T}{\partial y_t^*} \frac{\partial^3 N}{\partial y_k \partial y_l \partial y_t}.$$  \hfill (6.8)

Thus the equation of Theorem 6.7 is a third-order partial differential equation.

### 7. Determining the characteristic numbers

We now use Theorem 6.7 and formula (6.8), applied with $Y_i = Y_j = h$, $Y_k = Y_l = h^2$ to give a recursive scheme for determining the Gromov-Witten invariants $\langle (h^2)^{3d-3} \gamma_1 \rangle_d$ and $\langle (h^2)^{3d-3} \gamma_2 \cdot \gamma_3 \rangle_d$ where $\gamma_1 \in A^3(S_2)$ and $\gamma_2, \gamma_3 \in A^2(S_2)$. To shorten expressions, we will employ the following conventions: If $M$ is a potential, $M_i$ will denote the partial derivative $\partial M/\partial y_i$ where $y_i$ is a coefficient with respect to the $z$-basis, whereas $M_{i*}$ will denote $\partial M/\partial y_i^*$ where $y_i^*$ is a coefficient with respect to the $i$-basis. For example, (6.8) abbreviates to

$$G(ij \ | \ kl) = \sum_s (N_{ijs} P_{kl*} + N_{kl*} P_{ijs*}) + 18 \sum_{s,t} N_{ij*} T_{s*|t*} N_{kl*}.$$  \hfill (7.1)

First we note that with $Y_i = Y_j = h$, $Y_k = Y_l = h^2$, the classical potential $P$ contributes just a single term to the first sum in (7.1), namely a 1 that arises when $i = j = 100$ and that corresponds to the triple product

$$h \cdot h \cdot \hat{h} = Y_{100} \cdot Y_{100} \cdot Y_{011}^*$$

in $A^*(S_2)$. Thus Theorem 5.7 yields

$$N_{(200)^3} = 18 \sum_{s,t} (N_{100,200,s} T_{s*|t*} N_{100,200,t} - N_{100,100,s} T_{s*|t*} N_{200,200,t}).$$

Expressed in terms of $N^{(d)}$ and $E^{(d)}$, this gives a formula in each degree:

$$N_{(200)^3}^{(d)} E^{(d)} = 18 \sum_{d_1 + d_2 = d, d_1, d_2 > 0} \left\{ \left( N^{(d_1)} E^{(d_2)} \right)_{100,200,s} T_{s*|t*} \left( N^{(d_2)} E^{(d_1)} \right)_{100,200,t} - \left( N^{(d_1)} E^{(d_2)} \right)_{100,100,s} T_{s*|t*} \left( N^{(d_2)} E^{(d_1)} \right)_{200,200,t} \right\}.$$
We simplify to obtain
\[
N^{(d)}_{(200)} = \frac{18}{E(d)} \sum_{d_1, d_2 = 0} \left\{ d_1 d_2 \left( N^{(d_1)}_{200} E^{(d_1)} \right)_s T^{* t^*} \left( N^{(d_2)}_{200} E^{(d_2)} \right)_t \right. \\
\left. - d_1 \left( N^{(d_1)}_{200} E^{(d_1)} \right)_s T^{* t^*} \left( N^{(d_2)}_{200,200} E^{(d_2)} \right)_t \right\}.
\] (7.2)

**Theorem 7.3.** Given \(N^{(1)}\), equation (7.2) uniquely determines \(N^{(d)}_{(200)^3d-3}\) for all \(d > 0\).

**Proof.** Apply \(\partial^{3d-6}/\partial y_{200}^{3d-6}\) to equation (7.2). From the first term on the right we obtain terms of the form

\[
d_1 d_2 \frac{\partial a_1}{\partial y_{200}^a} \left( N^{(d_1)}_{200} E^{(d_1)} \right)_s T^{* t^*} \frac{\partial a_2}{\partial y_{200}^b} \left( N^{(d_2)}_{200} E^{(d_2)} \right)_t
\]

and from the second term we obtain terms of the form

\[
d_1^2 \frac{\partial a_1}{\partial y_{200}^a} \left( N^{(d_1)}_{200} E^{(d_1)} \right)_s T^{* t^*} \frac{\partial a_2}{\partial y_{200}^b} \left( N^{(d_2)}_{200,200} E^{(d_2)} \right)_t,
\]

where \(a_1 + a_2 = b_1 + b_2 = 3d - 6\). Since

\[
\frac{\partial^{3d} N^{(d)}}{\partial y_{200}^{3d}} = 0,
\]

any nonzero terms must have \(a_i \leq 3d_i - 2\) for \(i = 1, 2\). Thus \(a_1 = 3d - 6 - a_2 \geq 3d_1 - 4\) and \(a_2 \geq 3d_2 - 4\). Similarly, \(b_1 \leq 3d_1 - 1\), so \(b_2 = 3d - 6 - b_1 \geq 3d_2 - 5\) and \(b_2 + 2 \leq 3d_2 - 1\), so \(b_1 = 3d - 6 - b_2 \geq 3d_1 - 3\). To summarize, \(N^{(d)}_{(200)^3d-3}\) is determined by \(N^{(d')}_{(200)^a}\) for \(d' < d\) and \(a \geq 3d' - 3\). The theorem follows by induction. 

**Corollary 7.4.** Let \(\gamma_1 \in A^3(S_2)\) and \(\gamma_2, \gamma_3 \in A^2(S_2)\). Then equation (7.3) determines the Gromov-Witten invariants \(\langle (h^2)^{3d-3} \gamma_1 \rangle_d\) and \(\langle (h^2)^{3d-3} \gamma_2 \cdot \gamma_3 \rangle_d\) for all \(d > 0\), given the invariants in the case \(d = 1\).

**Proof.** Recall that

\[
N^{(d)} = \sum_{|a|=3d-1} N_d(a) \frac{y^a}{a!}
\]

where \(N_d(a) = \langle \prod_i \gamma_i \rangle_d\) (See the beginning of §3 for notation.) Thus

\[
N^{(d)}_{(200)^3d-3} = \sum_{|b|=2} N_d(a) \frac{y^b}{b!}
\]

where \(b = a - (3d - 3, 0, \ldots, 0)\). Hence if \(a_{200} \geq 3d - 3\) we obtain the Gromov-Witten invariant \(N_d(a)\) by computing \(b!\) times the coefficient of \(y^b\) in \(N^{(d)}_{(200)^3d-3}\).
The Gromov-Witten invariants in degree 1 are

\[ \langle h^2 \hat{h} \rangle_1 = 1; \quad \langle h^2 z \rangle_1 = 3; \quad \langle \hat{h}^2 \rangle_1 = -3; \]
\[ \langle h^2 \cdot h^2 \rangle_1 = 1; \quad \langle h^2 \cdot \hat{h}^2 \rangle_1 = 0; \quad \langle h^2 \cdot h z \rangle_1 = 0; \quad \langle \hat{h}^2 \cdot \hat{h}^2 \rangle_1 = 0; \]
\[ \langle \hat{h}^2 \cdot h z \rangle_1 = 0; \quad \langle \hat{h}^2 \cdot \hat{h} z \rangle_1 = 0; \quad \langle h z \cdot h z \rangle_1 = 0; \quad \langle h z \cdot \hat{h} z \rangle_1 = 0; \quad \langle \hat{h} z \cdot \hat{h} z \rangle_1 = 9. \]

These numbers are easily obtained by replacing \( z \) by \( i - 3h + 3\hat{h} \) and noting that the lift of a line to \( S_2 \) does not meet the divisor at infinity \( I \). Thus

\[ N^{(1)} = y_{210} + 3y_{201} - 3y_{021} + \frac{1}{2}y_{200}^2 - 3y_{200}y_{011} + \frac{9}{2}y_{011}^2. \]

Using (7.2), we have obtained the corresponding thirteen Gromov-Witten invariants through degree 6; the values are shown in Table 1. The characteristic numbers that do not involve \( z \) were also obtained in [4]. There are also several cases overlapping those of Caporaso and Harris [1] and we have checked that our values agree with theirs.

Note that in five instances the entries of a row of Table 1 are three times as large as those in a preceding row; for example

\[ \langle (h^2)^{3d-3} \cdot h z \cdot \hat{h} z \rangle_d = 3\langle (h^2)^{3d-3} \cdot \hat{h}^2 \cdot \hat{h} z \rangle_d. \]

This is easily explained by using \( h z - 3\hat{h}^2 = hi \) and then employing an \textit{ad hoc} argument to show that

\[ \langle (h^2)^{3d-3} \cdot h \cdot \hat{h} z \rangle_d = 0. \]

Here is a rough outline of the argument. The class is represented by a cycle supported on two loci, the first of which is the locus of curves which are unions of curves of lower degree, passing through \( 3d - 3 \) specified points, meeting at a point on a specified line, and having a flex tangent line through a specified point. The other is the locus of curves having a cusp on a specified line, passing through \( 3d - 3 \) points, and having a flex tangent line through a specified point. For general data, these loci are empty. Similar arguments apply in the other four instances.

8. Contact formulas

In this final section, we present two enumerative formulas for simultaneous triple contacts between fixed plane curves and members of a family of plane curves. The first formula is a generalization of Theorem 4 of [3] to arbitrarily many fixed curves. (In that theorem we said we were using the fiber product of two copies of the second-order stable lift of the family of curves, but in fact an examination of the proof shows that we were using the join, i.e., the unique component dominating the fiber product of two copies of the family of curves.)

Before we state the result, we note that the action of the projective general linear group \( PGL(2) \) on \( \mathbf{P}^2 \) lifts to compatible actions on \( S_1 \) and \( S_2 \). This is well-known for \( S_1 \), the incidence correspondence for \( \mathbf{P}^2 \). Moreover, the action of \( PGL(2) \) on \( S_1 \) respects
fibers and hence the induced action on $\mathbf{P}(T S_1)$ must send a focal plane to another focal plane. Thus $PGL(2)$ acts naturally on $S_2$.

To set notation, let $C_1, \ldots, C_n$ denote fixed curves in $\mathbf{P}^2$ and $(C_1)_2, \ldots, (C_n)_2$ their lifts to $S_2$. Suppose $\mathcal{X}$ is a family of plane curves over $T$, a complete parameter space of dimension $2n$. As in §4, let $\mathcal{J} \mathcal{X}_n^2(T)$ denote the join over $T$ of $n$ copies of $\mathcal{X}_2$. Let $(S_2)^n$ be the product of $n$ copies of the Semple bundle variety. Let $\pi_i: (S_2)^n \to S_2$ be projection onto the $i$th factor and let $\sigma: (S_2)^n \times T \to (S_2)^n$ also denote projection.

**Theorem 8.1.** Suppose that the reduced plane curves $C_1, \ldots, C_n$ each contain no line and that the general member of $\mathcal{X}$ is reduced and contains no line. If $C_1, \ldots, C_n$ are in general position with respect to the action of $(PGL(2))^n$ on $(S_2)^n$, then the number of simultaneous triple contacts between $C_1, \ldots, C_n$, and members of $\mathcal{X}$ is given by

$$
\int_{(S_2)^n} \pi_1^*[(C_1)_2] \cup \cdots \cup \pi_n^*[(C_n)_2] \cap \sigma_*[\mathcal{J} \mathcal{X}_2^2(T)].
$$

Equivalently, by the projection formula, this number is

$$
\int_{(S_2)^n \times T} \sigma^* \pi_1^*[(C_1)_2] \cup \cdots \cup \sigma^* \pi_n^*[(C_n)_2] \cap [\mathcal{J} \mathcal{X}_2^2(T)].
$$

Table 1. The Gromov-Witten invariants $\langle (h_2^d \gamma_1)^3 \rangle_d$ and $\langle (h_2^d \gamma_2 \cdot \gamma_3)^3 \rangle_d$ where $\gamma_1 \in A^3(S_2)$ and $\gamma_2, \gamma_3 \in A^2(S_2)$ for $1 \leq d \leq 6$.

| $d$ | $\gamma_1$ $h^2 \bar{h}$ | $\gamma_1$ $h^2 z$ | $\gamma_1$ $\bar{h}^2 z$ | $\gamma_2 \cdot \gamma_3$ $h^2 \cdot \gamma_2$ | $\gamma_2 \cdot \gamma_3$ $h^2 \cdot \gamma_3$ | $\gamma_2 \cdot \gamma_3$ $h^2 \cdot h^2$ |
|-----|------------------|------------------|------------------|-------------------|-------------------|-------------------|
| 1   | 1                | 3                | -3               | 1                 | 0                 | 9                |
| 2   | 1                | 3                | 0                | 1                 | 0                 | 0                |
| 3   | 1                | 3                | 0                | 1                 | 0                 | 0                |
| 4   | 1                | 3                | 0                | 1                 | 0                 | 0                |
| 5   | 1                | 3                | 0                | 1                 | 0                 | 0                |
| 6   | 1                | 3                | 0                | 1                 | 0                 | 0                |

Table 1.
Proof. It was shown in [3] and [2, Theorem 1] that $S_2$ has three orbits under the action of $PGL(2)$: a dense orbit $\mathcal{O}(-)$, represented by the germ at the origin of a nonsingular curve without a flex; a three-dimensional orbit $\mathcal{O}(0) = \mathbb{Z}$, represented by the germ at the origin of a line; and another three-dimensional orbit $\mathcal{O}(\infty) = I$, represented by the germ at the origin of a curve with an ordinary cusp. Thus $(S_2)^n$ has $3^n$ orbits of the form $\mathcal{O} = \mathcal{O}(a_1) \times \cdots \times \mathcal{O}(a_n)$ where each $a_i$ can be one of the symbols $-\infty, 0, \infty$.

We will show that for every nondense orbit $\mathcal{O}$ on $(S_2)^n$

\begin{equation}
\dim ((C_1)_2 \times \cdots \times (C_n)_2 \cap \mathcal{O}) + \dim (\mathcal{J}\mathcal{X}_n^2(T) \cap (\mathcal{O} \times T)) < \dim \mathcal{O}.
\end{equation}

Hence, since $C_1, \ldots, C_n$ are assumed to be in general position with respect to the action of $(PGL(2))^n$, the transversality theory of [7] shows that

\[(C_1)_2 \times \cdots \times (C_n)_2 \cap \sigma(\mathcal{J}\mathcal{X}_n^2(T)) \cap \mathcal{O} = \emptyset.\]

Thus $(C_1)_2 \times \cdots \times (C_n)_2$ and $\sigma(\mathcal{J}\mathcal{X}_n^2(T))$ intersect transversely and all intersections must occur in the dense orbit $\mathcal{O}(-) \times \cdots \times \mathcal{O}(-)$. The product of lifts $(C_1)_2 \times \cdots \times (C_n)_2$ is the closure of the graph of function defined on a dense subset of $C_1 \times \cdots \times C_n$; the join $\mathcal{J}\mathcal{X}_n^2(T)$ is likewise the closure of the graph of a function defined on a dense subset of $\mathcal{X}_n(T)$, the fiber product over $T$ of $n$ copies of $\mathcal{X}$. Thus, by general position, all intersections between $(C_1)_2 \times \cdots \times (C_n)_2$ and $\sigma(\mathcal{J}\mathcal{X}_n^2(T))$ are intersections between the graphs. Therefore every intersection point is an $n$-tuple $(x_1, \ldots, x_n)$ in which $x_1$ lies over a nonsingular point of $C_1$ and some member $X_i$ of $\mathcal{X}$, $x_2$ lies over a nonsingular point of $C_2$ and $X_i$, etc. Since each $x_i$ is a point of $S_2$, we see that the second-order data of $C_i$ and $X_i$ at $x_i$ must be identical for $i = 1, \ldots, n$. Thus each intersection point is a simultaneous triple contact.

Establishing (8.2) is quite easy. First, if $\mathcal{O} = \mathcal{O}(a_1) \times \cdots \times \mathcal{O}(a_n)$, then

$$\dim \mathcal{O} = 4n - q$$

where $q$ is the number of $a_i$'s which are not the symbol "−". Next, since none of the curves $C_i$ contains a line, we have

$$\dim ((C_1)_2 \times \cdots \times (C_1)_2 \cap \mathcal{O}) \leq r$$

where $r$ is the number of $a_i$'s which are "−". Finally, since the general member of $\mathcal{X}$ is reduced and contains no line,

$$\dim (\mathcal{J}\mathcal{X}_n^2(T) \cap (\mathcal{O} \times T)) < \dim (\mathcal{J}\mathcal{X}_n^2(T)) = 3n,$$

from which (8.2) follows.

There is a natural generalization of Theorem 8.1 allowing a mixture of single, double and triple contact conditions. The proof is by a transversality argument just as above and is therefore omitted.

**Theorem 8.3.** Let $P_1, \ldots, P_r$ be points in $\mathbb{P}^2$; let $C_1, \ldots, C_s$ be any reduced plane curves; let $D_1, \ldots, D_t$ be reduced curves containing no lines. Assume that all these data are in general position with respect to the action of $(PGL(2))^n$ on $(\mathbb{P}^2)^n$, where...
n = r + s + t. Let $\mathcal{X}$ be a complete family of plane curves whose general member is reduced and contains no lines, over an $m$-dimensional parameter space where $m = r + s + 2t$. Then the number of members of $\mathcal{X}$ passing through the points $P_i$, tangent to the curves $C_i$ and making a triple contact with each of the curves $D_i$ is equal to

$$
\int_{(S^2)^n \times T} \prod_{i=1}^{r} \sigma^* \pi^*_i((f^*_2 f^*_1)[P_i]) \prod_{i=1}^{s} \sigma^* \pi^*_{r+i}(f^*_2[(C_i)_1]) \prod_{i=1}^{t} \sigma^* \pi^*_{r+s+i}[(D_i)_2] \cap [\mathcal{J} \mathcal{A}^2_n(T)].
$$

We now apply Theorem 8.3 to the (completed) family $\overline{\mathcal{R}}(d)$ of all rational plane curves of degree $d$. Using Theorem 4.1, the number of members of this family satisfying the conditions of Theorem 8.3, with $m = 3d - 1$, is equal to the Gromov-Witten invariant

$$
(8.4) \quad \left\langle \prod_{i=1}^{r} f^*_2 f^*_1 [P_i] \cdot \prod_{i=1}^{s} f^*_2 [(C_i)_1] \cdot \prod_{i=1}^{t} [(D_i)_2] \right\rangle_d.
$$

From [3, p. 182] we recall the identities

$$
[(C_i)_1] = d(C_i)\hat{h}^2 + \bar{d}(C_i)h^2
$$

and

$$
[(D_i)_2] = d(D_i)\hat{h}^2z + \bar{d}(D_i)h^2z + \kappa(D_i)h^2\bar{h}
$$

where $d(C) = \int h \cap [C]$ is the degree, $\bar{d}(C) = \int \bar{h} \cap [(C)_1]$ is the class, and $\kappa(C) = \int i \cap [(C)_2]$ is the number of cusps (assuming no worse singularities than ordinary nodes and cusps). Plugging these identities into (8.4), we get an expression in terms of the degree $d$ Gromov-Witten invariants of the family, and the three invariants of each specified curve.

In particular, we obtain an explicit formula for the number $N_d(C)$ of rational curves of degree $d$ passing through $3d - 3$ points and making a triple contact with one specified curve $C$ (assumed to be reduced and containing no lines). Strictly speaking, the formula applies to the completed family $\overline{\mathcal{R}}(d)$. But, as we now argue, all of the curves counted by the formula are honest rational curves; there are no contributions from reducible curves. Indeed, suppose that $d = d_1 + d_2$. If $d_1 > 1$, then according to Theorem 8.3 there are only finitely members of $\overline{\mathcal{R}}(d_1)$ making a triple contact with $C$ and passing through $3d_1 - 3$ points, and there are only finitely many members of $\overline{\mathcal{R}}(d_2)$ passing through $3d_2 - 1$ points. Since $(3d_1 - 3) + (3d_2 - 1) < 3d - 3$, there is no union of two such curves satisfying all the specified conditions. Similarly, there are finitely many flex tangent lines to $C$, and finitely many members of $\overline{\mathcal{R}}(d-1)$ passing through $3d - 4$ points. Thus there is no union of a line and another curve satisfying all the specified conditions.

To avoid a clash of notations we use $c$ and $\bar{c}$ for the degree and class of $C$. Then

$$
(8.5) \quad N_d(C) = c \langle (h^2)^{3d-3} \cdot \bar{h}^2z \rangle + \bar{c} \langle (h^2)^{3d-3} \cdot h^2\bar{z} \rangle + \kappa \langle (h^2)^{3d-3} \cdot \bar{h}^2\bar{h} \rangle.
$$

Using inputs from Table 1, we obtain the explicit formulas shown in Table 2 (The formula for $d = 1$, which can also be derived from the classical Plücker formulas, counts the number of flexes on $C$.).
\[
N_d(C) = -3c + 3\hat{c} + \kappa \\
3\hat{c} + \kappa \\
21c + 30\hat{c} + 10\kappa \\
1452c + 1284\hat{c} + 428\kappa \\
216180c + 153120\hat{c} + 51040\kappa \\
64150200c + 39900528\hat{c} + 13300176\kappa
\]

Table 2. The number of rational plane curves of degree \(d\) passing through \(3d - 3\) points and making a triple contact with a curve \(C\).

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