Differentiable Sphere Theorems whose Comparison Spaces are Standard Spheres or Exotic Ones*†

Kei KONDO‡ · Minoru TANAKA

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Abstract

We prove that for an arbitrarily given compact Riemannian manifold $M$ admitting a point $p \in M$ with a single cut point, every compact Riemannian manifold $N$ admitting a point $q \in N$ with a single cut point is diffeomorphic to $M$ if the radial curvatures of $N$ at $q$ are sufficiently close in the sense of $L^1$-norm to those of $M$ at $p$. Hence, our result produces a weak version of the Cartan–Ambrose–Hicks theorem in the case where underlying manifolds admit a point with a single cut point. In particular, that result generalizes one of theorems in Cheeger’s Ph.D. Thesis in that case. Remark that every exotic sphere of dimension $> 4$ admits a metric such that there is a point whose cut locus consists of a single point.

1 Introduction

In the global Riemannian geometry, the relationship between curvatures and structures, especially topology, of Riemannian manifolds has been studied from various kinds of viewpoint, and a great number of results concerning with such a relation has been gotten. It is the topological $\delta$-pinching sphere theorem that is counted among the masterpieces of such results from the geodesic theory’s standpoint. This masterpiece was very first proved by Rauch [28] for $\delta \sim 3/4$, and worked out by Berger [3] and Klingenberg [23] for $\delta = 1/4$ as the optimal constant.

Further, that masterpiece produced the 1/4-pinning race as the problem if “homeomorphic” in the statement could be replaced by “diffeomorphic”. There were a large number of entrant for the race, e.g., Gromoll [11], Calabi, Shikata [31], Sugimoto–Shiohama–Karcher [34], Grove–Karcher–Ruh [14, 15], Im Hof–Ruh [20], and Suyama [35], et al.

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If a compact simply connected Riemannian manifold admits a metric whose sectional curvature lies in $(1/4, 1]$, then the manifold is homeomorphic to a sphere.

For example, the complex projective space admits a metric whose sectional curvature lies in $[1/4, 1]$ and is not homeomorphic to a sphere.
Using the Ricci flow introduced by Hamilton [18], Brendle and Schoen [5] finally proved that the masterpiece can be reinforced into the differentiable 1/4-pinching sphere theorem, which implies that every exotic sphere does not admit a 1/4-pinched metric.

By remembering that the 1/4-pinching race (problem) had originated in Hopf’s curvature pinching conjecture, the solution to the problem by Brendle–Schoen asks the following natural question of us.

**Question.** Replacing the unit standard sphere in the Hopf conjecture by an arbitrary compact simply connected Riemannian manifold $X$, should a compact simply connected Riemannian manifold whose radial curvature is close to that of $X$ be diffeomorphic to $X$? That is, can we weaken the assumption of the Cartan–Ambrose–Hicks theorem [1, 6] to closeness of radial curvatures of the manifolds?

Here, the radial curvature is, by definition, the restriction of the sectional curvature of a pointed Riemannian manifold to all 2-dimensional planes which contain the unit tangent velocity vector, as one of its basis, of any minimal geodesic emanating from the base point.

The purpose of this article is to solve the question above by hypothesizing that underlying compact manifolds admit metrics such that there is a point whose cut locus consists of a single point. It is worthy of note that every homotopy $n$-sphere of dimension $n \geq 5$ admits such a metric, and so are all exotic $n$-spheres. This note follows from Smale’s $h$-cobordism theorem [32, 33] and Weinstein’s deformation technique [36] for metrics on twisted spheres (also see Proposition 7.19 in [4]).

Now, we are going to state our main theorem precisely. For each $k = 1, 2$, let $M_k$ be a compact $n$-manifold of dimension $n \geq 2$ admitting a point whose cut locus consists of a single point. Note that $M_k$ is homeomorphic to a sphere $S^n$ of dimension $n$. We take any point $p_k \in M_k$ satisfying $\text{Cut}(p_k) = \{q_k\}$, where $q_k \in M_k$, and fix it. Here, $\text{Cut}(p_k)$ denotes the cut locus of $p_k$. Normalizing the metric, we can assume here that

$$d_{M_k}(p_k, q_k) = \pi,$$

where $d_{M_k}$ is the distance function of $M_k$. We denote

$$S^{n-1}_{p_k} := \{ u \in T_{p_k}M_k \mid \|u\| = 1 \},$$

where $T_{p_k}M_k$ is the tangent space to $M_k$ at $p_k$. For each $u_k \in S^{n-1}_{p_k}$, let $\tau_{u_k} : [0, \pi] \rightarrow M_k$ denote a geodesic segment emanating from $p_k = \tau_{u_k}(0)$ to $q_k = \tau_{u_k}(\pi)$ in the direction $u_k = \dot{\tau}_{u_k}(0) := (d\tau_{u_k}/dt)(0)$, i.e.,

$$(1.1) \quad \tau_{u_k}(t) = \exp_{p_k}tu_k$$

for all $t \in [0, \pi]$.

Take any $u_1 \in S^{n-1}_{p_1}$, and fix it. We denote

$$S^{n-2}_{u_1} := \{ x \in T_{u_1}(S^{n-1}) \mid \|x\| = 1 \}.$$
Choose a linear isometry $I_{u_1}: T_{p_1} M_1 \to T_{p_2} M_2$. For the $u_1$, we set $u_2 := I_{p_1}(u_1)$. For $t \in [0, \pi]$, let $P_t^{(u_1)}$ denote the parallel translation along the geodesic $\tau_{u_1}$ from $p_1$ to $\tau_{u_1}(t)$. Define the linear isometry $\Psi_t^{(u_1)}: T_{p_1} M_1 \to T_{\tau_{u_2}(t)} M_2$ by

$$
\Psi_t^{(u_1)} := P_t^{(u_2)} \circ I_{p_1},
$$

where $P_t^{(u_2)}$ is the parallel translation along the geodesic $\tau_{u_2} = \tau_{I_{p_1}(u_1)}$ from $p_2$ to $\tau_{u_2}(t)$. Moreover, we define the function $\lambda : [0, \pi] \to \mathbb{R}$ by

$$
\lambda(t) := \max_{u_1 \in S^{n-1}_{p_1}} \left| K^{(1)}(P_t^{(u_1)}(x_1) \wedge P_t^{(u_1)}(u_1)) - K^{(2)}(\Psi_t^{(u_1)}(x_1) \wedge \Psi_t^{(u_1)}(u_1)) \right|,
$$

where $K^{(k)}(x \wedge y)$ ($k = 1, 2$) is the sectional curvature of the plane spanned by two linearly independent tangent vectors $x$ and $y$ at a point on $M_k$, i.e.,

$$
K^{(k)}(x \wedge y) = \frac{\langle R^{(k)}(x, y) x, y \rangle}{\|x\|^2 \|y\|^2 - \langle x, y \rangle^2}.
$$

Here, $R^{(k)}$ denotes the curvature tensor of $M_k$ given by

$$
R^{(k)}(X, Y) Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z
$$

for all vector fields $X, Y, Z$ on $M_k$, where $\nabla$ is the Levi–Civita connection on $M_k$.

With the notation above, our main theorem is stated as follows.

**Theorem 1.1 (Main Theorem)** There exists a positive number $\varepsilon_n(M_1)$ depending on $n$ and $M_1$ such that if

$$
\int_0^\pi \lambda(t) dt < \varepsilon_n(M_1),
$$

then $M_2$ is diffeomorphic to $M_1$.

**Remark 1.2** We give here several remarks on Theorem 1.1 and related results to it:

- Theorem 1.1 is the very Cartan–Ambrose–Hicks theorem if $\lambda(t) \equiv 0$ on $[0, \pi]$, and hence produces a weak version of the theorem. Note here that sectional curvature and curvature tensor are equivalent (see, e.g., (2.7)). In particular, our theorem generalizes Cheeger’s theorem (Theorem 7.36 in [3]) in the case where underlying manifolds admit a point with a single cut point, because we do not assume either closeness of $\nabla R^{(1)}$ and $\nabla R^{(2)}$ along $\tau_{u_1}$ and $\tau_{u_2}$ or $\text{vol}(M_2) > \nu$ for some $\nu > 0$, where $\text{vol}(M_2)$ denotes the volume of $M_2$, that additionally he assumed in his theorem; besides, we need to look around such manifolds only at their base points $p_1$ and $p_2$. Moreover, it is apparent that our theorem extends and weakens (iii) of Theorem 3 in Katz and the first author’s [22] to a wider class of metrics than that of radially symmetric metrics in it.

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3 Note that the definition of $R^{(k)}$ differs from that of curvature tensor in literatures such as [8] and [29] by a sign.
The constant $\varepsilon_n(M_1)$ in (1.5) is obtained as the unique solution of the following equation

\begin{equation}
\eta_n(M_1) \cdot x \exp(2(n-1)x) = \frac{1}{2} \left[ \sqrt{1 + \left\{ \frac{8}{\pi} (n-1) \right\}^{\frac{1}{2}}} - 1 \right]
\end{equation}

for all $x \in [0, \infty)$, where $\eta_n(M_1)$, depending on $n$ and $M_1$, denotes some positive number concerning with Jacobi fields along $\tau_{u_1}$ (see (2.25) in Section 2 for more details). The constant $1 + \{(8/\pi)(n-1)\}^{-1/2}$ found in (1.6) is the same as Karcher [21] estimated in order to prove a sharper version of Shikata’s theorem in [30].

From the above note on the homotopy spheres, the class of compact manifolds in Theorem 1.1 includes all exotic $n$-spheres $\Sigma^n$. Moreover, while every $\Sigma^n$ does not admit a metric with non-positive sectional curvature, which is a direct consequence of the Cartan–Hadamard theorem [6, 17], the radial curvature of $M_1$ as a kind of model can wildly change its signs along $\tau_{u_1}$. As one of results concerning with sectional curvature of $\Sigma^n$, it is highly important that all Milnor’s exotic 7-spheres, which are associated to $SO(4)$-principal bundles over $S^3$ admitting cohomogeneity-one actions by $SO(4) \times SO(3)$ with codimension-two singular orbits, admit metrics with non-negative sectional curvature. This fact was proved by Grove and Ziller [16] applying their result, Theorem E in [16], on cohomogeneity-one manifolds with codimension-two singular orbits. Note that Petersen and Wilhelm [27] proposed that the Gromoll-Meyer exotic 7-sphere in [12], which is a bi-quotient of Lie groups, admits a metric with positive sectional curvature everywhere. See Ziller’s [37] for further information, details, references, and some mysterious history concerning with the study of Riemannian manifolds with positive sectional curvature.

The related results to Theorem 1.1 are the differentiable exotic sphere theorems I and II proved by authors [24]. In the theorem I, the hypothesis (1.5) can be replaced by either

\begin{equation}
\left\| \frac{d^2 c_\gamma}{dt^2} \right\|^2 - 2 \text{Lip}^b(\sigma)^2 \leq 2 \left\{ \frac{\sqrt{2} - 1}{2(e^\pi - 1)} \right\}^2 - 1
\end{equation}

for all unit speed geodesic segments $\gamma([0, \pi]) \subset S_{q_1}^{n-1} := \{ v \in T_{q_1}M_1 \mid \|v\| = 1 \}$, or

$$\text{Lip}^b(\sigma)^2 \leq 1 + \left\{ \frac{8}{\pi} (n-1) \right\}^{-\frac{1}{2}}.$$

Here, $\sigma : S_{q_1}^{n-1} \longrightarrow S_{q_2}^{n-1} := \{ v \in T_{q_2}M_2 \mid \|v\| = 1 \}$ is the diffeomorphism defined by (2.10), $\text{Lip}^b(\sigma)$ denotes the bi-Lipschitz constant of $\sigma$, i.e.,

$$\text{Lip}^b(\sigma) := \inf \{ a \mid a^{-1} \|u - v\| \leq \|\sigma(u) - \sigma(v)\| \leq a \|u - v\| \text{ for all } u, v \in S_{q_1}^{n-1} \},$$

and we denote $c_\gamma := \sigma \circ \gamma : [0, \pi] \longrightarrow S_{q_2}^{n-1}$ for each geodesic segment $\gamma([0, \pi]) \subset S_{q_1}^{n-1}$. In the theorem II, the hypothesis (1.7) is replaced by

$$\angle(\overline{e_\gamma(t)}, c_\gamma(t)) < \frac{\pi}{2}.$$
for all unit speed geodesic segments $\gamma([0, \pi]) \subset S^{n-1}_{p1}$, where $\overline{\gamma}$ is the smooth curve on $T_{p2}M_2$ given by $\overline{\gamma}(t) := \varsigma(t) + (dc_\gamma/dt)(0) \sin t$ ($t \in [0, \pi]$). To the proofs of the theorems I and II, we applied our approximation method in [24] for a Lipschitz map via immersions employing the notion of non-smooth analysis established by F.H. Clarke in [9, 10]. This method will be also applied to the proof of Theorem 1.1 (see Section 3).

## 2 Key lemma

The aim of this section is to prove Key Lemma (Lemma 2.2) to the proof of Theorem 1.1 which will be applied in Section 3. The integral form of the Grönwall inequality [2, 13] is the key tool in the proof of Lemma 2.2. Throughout this section, for each $k = 1, 2$, let $M_k$ be a compact $n$-manifold of dimension $n \geq 2$ admitting a point $p_k \in M_k$ such that $\text{Cut}(p_k) = \{q_k\}$, where $\text{Cut}(p_k)$ denotes the cut locus of $p_k$, and for any $u_k \in S^{n-1}_{p_k}$, let

$$\tau_{u_k} : [0, \pi] \rightarrow M_k$$

be the geodesic segment emanating from $p_k$ to $q_k$ defined by (1.1). Here, we assume $d_{M_k}(p_k, q_k) := \pi$ by normalizing the metric, where $d_{M_k}$ denotes the distance function on $M_k$. All other notations in the following are the same as those defined in Section 3.

Take any $u_1 \in S^{n-1}_{p_1}$, and fix it. Choose an orthonormal basis $e^{(u_1)}_1, e^{(u_1)}_2, \ldots, e^{(u_1)}_n$ of $T_{p_1}M_1$ with $e^{(u_1)}_n = u_1$. Setting

$$u_2 := I_{p_1}(u_1),$$

we have the orthonormal basis $e^{(u_2)}_1, e^{(u_2)}_2, \ldots, e^{(u_2)}_n$ of $T_{p_2}M_2$ defined by

$$e^{(u_2)}_i := I_{p_2}(e^{(u_1)}_i), \quad i = 1, 2, \ldots, n,$

which satisfies $e^{(u_2)}_n = u_2$. For each $k = 1, 2$, we denote

$$E^{(k)}_i(t) := P^{(u_k)}_t(e^{(u_k)}_i), \quad i = 1, 2, \ldots, n.$$

Then, $E^{(1)}_1, E^{(2)}_1, \ldots, E^{(k)}_n$ are the parallel orthonormal fields along $\tau_{u_k}$. In particular,

$$E^{(1)}_i(t) = P^{(u_1)}_t(e^{(u_1)}_i), \quad E^{(2)}_i(t) = P^{(u_2)}_t(I_{p_1}(e^{(u_1)}_i)) = \Psi^{(u_1)}_t(e^{(u_1)}_i)$$

for each $i = 1, 2, \ldots, n - 1$, and

$$\dot{\tau}_{u_1}(t) = E^{(1)}_n(t) = P^{(u_1)}_t(u_1), \quad \dot{\tau}_{u_2}(t) = E^{(2)}_n(t) = \Psi^{(u_1)}_t(u_1).$$

Moreover, we denote

$$a^{(k)}_{ij}(t) := \left\langle R^{(k)}(E^{(k)}_i(t), \dot{\tau}_{u_k}(t)) \dot{\tau}_{u_k}(t), E^{(k)}_j(t) \right\rangle$$

for all $t \in [0, \pi]$ and all $i, j = 1, 2, \ldots, n - 1$. Further, we define the square matrix $A(t; u_k)$ of order $2(n - 1)$ by

$$A(t; u_k) := \begin{pmatrix} 0 & I_{n-1} \\ \left( a^{(k)}_{ij}(t) \right) & 0 \end{pmatrix},$$


where \( I_{n-1} \) is the \((n-1)\)-th unit matrix. Note that \( (a_{ij}^{(k)}(t)) \) is the symmetric matrix of order \( n-1 \).

**Lemma 2.1** For any \( t \in [0, \pi] \), we have

\[
\| A(t; u_1) - A(t; u_2) \| \leq 2(n-1)\lambda(t),
\]

where \( \| \cdot \| \) denotes the linear operator norm. In particular,

\[
\| A(t; u_2) \| \leq c_1(n, p_1) + 2(n-1)\lambda(t)
\]

holds for all \( t \in [0, \pi] \), where we denote

\[
c_1(n, p_1) := \max \{ \| A(t; u_1) \| \mid u_1 \in S_{p_1}^{n-1}, t \in [0, \pi] \} (\geq 0).
\]

**Proof.** Let \( k = 1, 2 \). Take any \( t \in [0, \pi] \), and fix it. Since

\[(2.4)\quad a_{ij}^{(k)}(t) = - \left \langle R^{(k)}(E_i^{(k)}(t), \hat{\tau}_{uk}(t))E_j^{(k)}(t), \hat{\tau}_{uk}(t) \right \rangle,
\]

we have, by \((1.4)\),

\[(2.5)\quad K^{(k)}(E_i^{(k)}(t) \wedge \hat{\tau}_{uk}(t)) = \left \langle R^{(k)}(E_i^{(k)}(t), \hat{\tau}_{uk}(t))E_i^{(k)}(t), \hat{\tau}_{uk}(t) \right \rangle = -a_{ii}^{(k)}(t).
\]

Since \( E_i^{(k)}(t) + E_j^{(k)}(t) \) is orthonormal to \( \hat{\tau}_{uk}(t) \), and since \( \| E_i^{(k)}(t) + E_j^{(k)}(t) \| = \sqrt{2} \), we see, by \((1.4), (2.4)\), and \((2.5)\),

\[(2.6)\quad K^{(k)}((E_i^{(k)}(t) + E_j^{(k)}(t)) \wedge \hat{\tau}_{uk}(t)) = \frac{1}{2} \left( -a_{ii}^{(k)}(t) - 2a_{ij}^{(k)}(t) - a_{jj}^{(k)}(t) \right).
\]

Then, it is obvious from \((2.5)\) and \((2.6)\) that

\[(2.7)\quad a_{ij}^{(k)}(t) = \frac{1}{2} \left\{ K^{(k)}(E_i^{(k)}(t) \wedge \hat{\tau}_{uk}(t)) + K^{(k)}(E_j^{(k)}(t) \wedge \hat{\tau}_{uk}(t)) + K^{(k)}((E_i^{(k)}(t) + E_j^{(k)}(t)) \wedge \hat{\tau}_{uk}(t)) \right\}
\]

\[
- K^{(k)}((E_i^{(k)}(t) + E_j^{(k)}(t)) \wedge \hat{\tau}_{uk}(t)).
\]

Since \( K^{(k)}(x \wedge y) \) does not depend on the choice of the spanning vectors, we have, by \((2.2), (2.3), (2.7)\) and the triangle inequality,

\[(2.8)\quad \left\| a_{ij}^{(1)}(t) - a_{ij}^{(2)}(t) \right\| \leq 2\lambda(t).
\]

Therefore, by \((2.8)\), we have

\[
\| A(t; u_1) - A(t; u_2) \| = \left\| \left( a_{ij}^{(1)}(t) - a_{ij}^{(2)}(t) \right) \right\|
\]

\[
\leq (n-1) \max_{i,j=1,2,\ldots,n-1} \left\| a_{ij}^{(1)}(t) - a_{ij}^{(2)}(t) \right\|
\]

\[
\leq 2(n-1)\lambda(t),
\]

which is the first assertion. Since

\[
\| A(t; u_2) \| = \| A(t; u_2) - A(t; u_1) + A(t; u_1) \| \leq \| A(t; u_2) - A(t; u_1) \| + \| A(t; u_1) \|,
\]

the second assertion follows from the first one. \(\square\)
Let $k = 1, 2$. Since $\text{Cut}(p_k) = \{q_k\}$, we have the diffeomorphism $\sigma_{q_k}^p$ from $S_{p_k}^{n-1}$ onto $S_{q_k}^{n-1} := \{v \in T_{q_k}M_k \mid \|v\| = 1\}$ given by

$$\sigma_{q_k}^p(u_k) := -\tau_{u_k}(\pi),$$

for all $u_k \in S_{p_k}^{n-1}$. Thus, the map $\sigma$ from $S_{q_1}^{n-1}$ onto $S_{q_2}^{n-1}$ defined by

$$\sigma := \sigma_{q_2}^p \circ I_{p_1} \circ \sigma_{q_1}^p$$

is a diffeomorphism, where we denote $\sigma_{q_1}^p := (\sigma_{q_1}^{p_1})^{-1}$. Moreover, for each $u_1 \in S_{p_1}^{n-1}$ we define the linear isometry $I_{q_1}^{(u_1)} : T_{q_1}M_1 \rightarrow T_{q_2}M_2$ by

$$I_{q_1}^{(u_1)} := \Psi_{\pi}^{(u_1)} \circ (\pi_{(u_1)})^{-1}.$$

**Lemma 2.2 (Key Lemma)** Set

$$\delta_n := \sqrt{1 + \left\{ \frac{8}{\pi} (n-1) \right\}^\frac{1}{2}} - 1.$$

Then, there exists a positive number $\varepsilon_n(M_1)$ such that if

$$\int_0^\pi \lambda(t)dt < \varepsilon_n(M_1),$$

then for any $u_1 \in S_{p_1}^{n-1}$, the differential $d\sigma_{q_1}^p$ of $\sigma$ at $v_1 := \sigma_{q_1}^p(u_1) \in S_{q_1}^{n-1}$ and $I_{q_1}^{(u_1)}$ are $\delta_n/2$-close with respect to the linear operator norm.

**Proof.** Let $k = 1, 2$. Take any $u_1 \in S_{p_1}^{n-1}$, and fix it. Set $u_2 := I_{p_1}(u_1) \in S_{p_2}^{n-1}$. For any fixed $x_1 \in S_{q_1}^{n-2}$, we denote

$$x_2 := I_{p_1}(x_1) \in S_{q_2}^{n-2} := \{x \in T_{u_2}(S_{p_2}^{n-1}) \mid \|x\| = 1\}.$$

where we identify $T_{u_1}(T_{p_1}M_k)$ with $T_{p_1}M_k$. Let $J_{u_k}^{(x_k)}$ be the Jacobi field along $\tau_{u_k}$ given by $J_{u_k}^{(x_k)}(0) = 0$ and $\langle (DJ_{u_k}^{(x_k)})/dt)(0) = x_k \in S_{u_k}^{n-2}$, where $DJ_{u_k}^{(x_k)}/dt$ denotes the covariant derivative of $J_{u_k}^{(x_k)}$ along $\tau_{u_k}$. By the definition of $J_{u_k}^{(x_k)}$ and the Gauss lemma, we can write

$$J_{u_k}^{(x_k)}(t) = \sum_{i=1}^{n-1} f_i^{(k)}(t) E_i^{(k)}(t).$$

Here, each $f_i^{(k)}(t) := \langle J_{u_k}^{(x_k)}(t), E_i^{(k)}(t) \rangle$ is the smooth function, where the inner product $\langle \cdot, \cdot \rangle$ denotes the one induced from the Riemannian metric of $M_k$, and each $E_i^{(k)}$ is the parallel orthonormal field, defined by (2.11), along $\tau_{u_k}$. In what follows, we denote

$$\dot{f}_i^{(k)}(t) := \frac{df_i^{(k)}}{dt}(t), \quad \ddot{f}_i^{(k)}(t) := \frac{d^2 f_i^{(k)}}{dt^2}(t).$$
for all \( i = 1, 2, \ldots, n - 1 \). Since
\[
(\text{d} \sigma_{q_k}^{p_k})_{u_k}(x_k) = (\text{d} \sigma_{q_k}^{p_k})_{u_k} \left( \frac{D J_{u_k}^{(x_k)}}{dt}(0) \right) = -\frac{D J_{u_k}^{(x_k)}}{dt}(\pi),
\]
we have, by the definition \((2.10)\) of \( \sigma \),
\[
(2.13)
\text{d} \sigma_{v_1} \left( \frac{D J_{u_1}^{(x_1)}}{dt}(\pi) \right) = (\text{d} \sigma_{q_2}^{p_2} \circ I_{p_1})_{u_1}(-x_1) = -(\text{d} \sigma_{q_2}^{p_2})_{u_2}(x_2)
\]
\[
= \frac{D J_{u_2}^{(x_2)}}{dt}(\pi) = \sum_{i=1}^{n-1} f_i^{(2)}(\pi) E_i^{(2)}(\pi).
\]
By \((2.1)\) and the second equation of \((2.2)\), we see
\[
I_{q_1}^{(u_1)}(E_i^{(1)}(\pi)) = \Psi_{\pi}^{(u_1)}(e_i^{(u_1)}) = E_i^{(2)}(\pi).
\]
Then, we have
\[
(2.14)
I_{q_1}^{(u_1)} \left( \frac{D J_{u_1}^{(x_1)}}{dt}(\pi) \right) = I_{q_1}^{(u_1)} \left( \sum_{i=1}^{n-1} f_i^{(1)}(\pi) E_i^{(1)}(\pi) \right) = \sum_{i=1}^{n-1} f_i^{(1)}(\pi) E_i^{(2)}(\pi).
\]
Thus, it follows from \((2.13)\) and \((2.14)\) that
\[
(2.15)
\left\| \text{d} \sigma_{v_1} \left( \frac{D J_{u_1}^{(x_1)}}{dt}(\pi) \right) - I_{q_1}^{(u_1)} \left( \frac{D J_{u_1}^{(x_1)}}{dt}(\pi) \right) \right\| = \left\| \sum_{i=1}^{n-1} (f_i^{(2)}(\pi) - f_i^{(1)}(\pi)) E_i^{(2)}(\pi) \right\|.
\]
Let
\[
\tilde{J}_{u_k}^{(x_k)}(t) := t(f_1^{(k)}(t), \ldots, f_{n-1}^{(k)}(t), f_1^{(2)}(t), \ldots, f_{n-1}^{(2)}(t)) \in \mathbb{R}^{2(n-1)}.
\]
Define the smooth function \( \varphi : [0, \pi] \to \mathbb{R} \) by
\[
\varphi(t) := \left\| \tilde{J}_{u_1}^{(x_1)}(t) - \tilde{J}_{u_2}^{(x_2)}(t) \right\|.
\]
The case where \( \varphi(t) \equiv 0 \) holds on \([0, \pi]\) implies
\[
f_i^{(1)}(\pi) = f_i^{(2)}(\pi), \quad i = 1, 2, \ldots, n - 1.
\]
By \((2.15)\), \( \text{d} \sigma_{v_1} \) and \( I_{q_1}^{(u_1)} \) are, of course, \( \delta_n/2 \)-close. From this argument, we need to consider the case where there exists an interval \([a, b] \subset [0, \pi]\) such that \( \varphi(t) > 0 \) on \((a, b)\) with \( \varphi(a) = 0 \). Since \( J_{u_k}^{(x_k)} \) satisfies the Jacobi equation
\[
\frac{D^2 J_{u_k}^{(x_k)}}{dt^2}(t) + R^{(k)}(\tilde{\tau}_{u_k}(t), J_{u_k}^{(x_k)}(t)) \tilde{\tau}_{u_k}(t) = 0, \quad t \in [0, \pi],
\]
we observe
\[
(2.16)
f_{\tilde{\tau}}^{(k)}(t) - \sum_{i=1}^{n-1} a_{i,j}^{(k)}(t) f_i^{(k)}(t) = 0
\]
for all $t \in [0, \pi]$ and $j = 1, 2, \ldots, n - 1$. Plugging (2.16) into the following
\[ \frac{d\mathcal{J}^{(xk)}_{u_k}}{dt}(t) = t(\dot{j}_1^{(k)}(t), \ldots, \dot{j}_{n-1}^{(k)}(t), \ddot{j}_1^{(k)}(t), \ldots, \ddot{j}_{n-1}^{(k)}(t)), \]
we see
\[ (2.17) \quad \frac{d\mathcal{J}^{(xk)}_{u_k}}{dt}(t) = A(t; u_k)\mathcal{J}^{(xk)}_{u_k}(t) \in \mathbb{R}^{2(n-1)} \]
for all $t \in [0, \pi]$. Using the Cauchy–Schwarz inequality and the triangle one, we see, by (2.17), that for any $t \in (a, b)$,
\[ (2.18) \quad \varphi'(t) = \frac{1}{\|\mathcal{J}^{(x1)}_{u_1}(t) - \mathcal{J}^{(x2)}_{u_2}(t)\|} \left( \|d\mathcal{J}^{(x1)}_{u_1}/dt - d\mathcal{J}^{(x2)}_{u_2}/dt\| \right) \]
\[ \leq \|A(t; u_1)\mathcal{J}^{(x1)}_{u_1}(t) - A(t; u_2)\mathcal{J}^{(x2)}_{u_2}(t)\| \]
\[ = \|(A(t; u_1) - A(t; u_2))\mathcal{J}^{(x1)}_{u_1}(t) + A(t; u_2)\mathcal{J}^{(x1)}_{u_1}(t) - \mathcal{J}^{(x2)}_{u_2}(t)\| \]
\[ \leq \|A(t; u_1) - A(t; u_2)\| \cdot \|\mathcal{J}^{(x1)}_{u_1}(t)\| + \|A(t; u_2)\| \cdot \varphi(t). \]
Set
\[ h_a(t) := \int_a^t \|A(s; u_1) - A(s; u_2)\| \cdot \|\mathcal{J}^{(x1)}_{u_1}(s)\| ds, \quad t \in (a, b). \]
Since $\varphi(a) = 0$, the integration of (2.18) from $a$ to $t$ yields the inequality
\[ \varphi(t) \leq h_a(t) + \int_a^t \|A(s; u_2)\| \cdot \varphi(s) ds. \]
Since $\varphi(t)$, $h_a(t)$, and $\|A(t; u_2)\|$ are continuous on $(a, b)$, and since $\|A(t; u_2)\| \geq 0$ on $(a, b)$, it follows from the integral form [2] of Grönwall’s inequality [13] that
\[ (2.19) \quad \varphi(t) \leq h_a(t) + \int_a^t h_a(s) \|A(s; u_2)\| \exp \left( \int_s^t \|A(r; u_2)\| dr \right) ds, \quad t \in (a, b). \]
Noting that $h_a(t)$ is non-decreasing on $(a, b)$, we see, by (2.19),
\[ (2.20) \quad \varphi(t) \leq h_a(t) + h_a(t) \int_a^t \|A(s; u_2)\| \exp \left( \int_s^t \|A(r; u_2)\| dr \right) ds \]
\[ = h_a(t) + h_a(t) \int_a^t \frac{\partial}{\partial s} \left\{ - \exp \left( - \int_s^t \|A(r; u_2)\| dr \right) \right\} ds \]
\[ = h_a(t) + h_a(t) \left[ - \exp \left( - \int_t^s \|A(r; u_2)\| dr \right) \right]_{s=a}^{s=t} \]
\[ = h_a(t) + h_a(t) \left\{ -1 + \exp \left( \int_t^a \|A(r; u_2)\| dr \right) \right\} \]
\[ = h_a(t) \exp \left( \int_a^t \|A(r; u_2)\| dr \right), \quad t \in (a, b). \]
Since the functions $\|A(t; u_1) - A(t; u_2)\| \cdot \|\tilde{J}^{(x_1)}(t)\|$ and $\|A(t; u_2)\|$ are well-defined on $[0, b)$ and are integrable on $[0, b)$, it is clear that

\[(2.21) \quad \varphi(t) \leq h_a(t) \exp \left( \int_a^t \|A(r; u_2)\| \, dr \right) \leq h_0(t) \exp \left( \int_0^t \|A(r; u_2)\| \, dr \right)\]

for all $t \in (a, b)$. Since the function

\[t \rightarrow h_0(t) \exp \left( \int_0^t \|A(r; u_2)\| \, dr \right)\]

is increasing on $[0, \pi]$, we have, by (2.21),

\[(2.22) \quad \varphi(t) \leq h_0(\pi) \exp \left( \int_0^\pi \|A(r; u_2)\| \, dr \right), \quad t \in (a, b).\]

Since (2.22) still holds for some $t_0 \in [0, \pi]$ with $\varphi(t_0) = 0$, we finally get

\[(2.23) \quad \varphi(t) \leq h_0(\pi) \exp \left( \int_0^\pi \|A(r; u_2)\| \, dr \right)\]

for all $t \in [0, \pi]$.

We denote

\[c_2(n, p_1) := \max\{\|\tilde{J}^{(x_1)}(t)\| \mid u_1 \in S^{n-1}_{p_1}, x_1 \in S^{n-2}_{u_1}, t \in [0, \pi]\}.\]

Applying Lemma 2.1 to (2.23), we observe

\[(2.24) \quad \varphi(t) \leq h_0(\pi) \exp \left( \int_0^\pi \|A(r; u_2)\| \, dr \right) \]

\[= \int_0^\pi \|A(r; u_1) - A(r; u_2)\| \cdot \|\tilde{J}^{(x_1)}(r)\| \, dr \cdot \exp \left( \int_0^\pi \|A(r; u_2)\| \, dr \right) \]

\[\leq 2(n - 1) \cdot c_2(n, p_1) \cdot \exp(\pi c_1(n, p_1)) \int_0^\pi \lambda(r) \, dr \cdot \exp \left( \int_0^\pi 2(n - 1)\lambda(r) \, dr \right) \]

\[= c(n, M_1) \int_0^\pi \lambda(r) \, dr \cdot \exp \left( 2(n - 1) \int_0^\pi \lambda(r) \, dr \right).\]

for all $t \in [0, \pi]$, where we set

\[c(n, M_1) := 2(n - 1) \cdot c_2(n, p_1) \cdot \exp(\pi c_1(n, p_1)).\]

Let $\varepsilon_n(M_1) > 0$ be the unique solution of the following equation

\[(2.25) \quad \frac{c(n, M_1) \cdot x \exp(2(n - 1)x)}{c_3(n, p_1)} = \frac{1}{2} \delta_n\]
for all \( x \in [0, \infty) \), where we denote
\[
c_3(n, p_1) := \min \left\{ \left\| \frac{DJ_{u_1}(x_1)}{dt}(\pi) \right\| \mid u_1 \in S_{p_1}^{n-1}, x_1 \in S_{u_1}^{n-2} \right\}.
\]
Now, we assume that
\[
\int_0^\pi \lambda(s)ds < \varepsilon(M_1).
\]
Then, (2.24) yields
\[
(2.26) \quad \varphi(\pi) < c(n, M_1) \cdot \varepsilon(M_1) \cdot \exp(2(n - 1)\varepsilon(M_1)) = \frac{c_3(n, M_1)}{2} \delta_n.
\]
Thus, it follows from (2.15) and (2.26) that
\[
(2.27) \quad \left\| d\sigma_{v_1} \left( \frac{DJ_{u_1}(x_1)}{dt}(\pi) \right) - I_{q_1}^{(u_1)} \left( \frac{DJ_{u_1}(x_1)}{dt}(\pi) \right) \right\| \leq \frac{1}{c_3(n, p_1)} \sum_{i=1}^{n-1} \left( \dot{f}_i^{(2)}(\pi) - \dot{f}_i^{(1)}(\pi) \right)^2
\]
\[
\leq \frac{\varphi(\pi)}{c_3(n, p_1)} < \frac{1}{2} \delta_n.
\]
From the arbitrariness of \( x_1 \), (2.27) implies that
\[
\left\| d\sigma_{v_1} - I_{q_1}^{(u_1)} \right\| = \sup_{w \in T_{v_1}(S_{q_1}^{n-1})} \sup_{w \neq 0} \left\| d\sigma_{v_1}(w) - I_{q_1}^{(u_1)}(w) \right\| \leq \frac{1}{2} \delta_n.
\]
Therefore, \( d\sigma_{v_1} \) and \( I_{q_1}^{(u_1)} \) are \( \delta_n/2 \)-close. \( \Box \)

3 Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1, where we apply Lemma 2.2 and our approximation method in [24] for a Lipschitz map via immersions to the proof. Throughout the section, for each \( k = 1, 2 \), let \( M_k \) be a compact \( n \)-manifold of dimension \( n \geq 2 \) admitting a point \( p_k \in M_k \) such that \( \text{Cut}(p_k) = \{q_k\} \). Additionally, we assume
\[
\int_0^\pi \lambda(t)dt < \varepsilon(M_1),
\]
where \( \lambda(t) \) is the function defined by (1.3), and \( \varepsilon(M_1) > 0 \) is the unique solution of the equation (2.25).
Let \( F \) be the bi-Lipschitz homeomorphism from \( M_1 \) onto \( M_2 \) given by
\[
F(\exp_{p_1} tu_1) := \exp_{p_2} (tI_{p_1}(u_1))
\]
for all \((t, u_1) \in [0, \pi] \times S_{p_1}^{n-1}\). Note that \(F\) is not differentiable only at \(q_1\). Moreover, we define the map \(\tilde{F} : B_\pi(o_{q_1}) \rightarrow B_\pi(o_{q_2})\) by

\[
\tilde{F} := \exp_{q_2}^{-1} \circ F \circ \exp_{q_1}.
\]

Here, we denote \(B_\pi(o_{q_k}) := \{x \in T_{q_k}M_k \mid \|x\| < \pi\}\) \((k = 1, 2)\), where \(o_{q_k}\) is the origin of \(T_{q_k}M_k\). By the very same argument as Section 3.3 in [24], we see that

\[
\tilde{F}(x) = \begin{cases} \|x\|\sigma \left( \frac{x}{\|x\|} \right) & \text{for all } x \in B_\pi(o_{q_1}) \setminus \{o_{q_1}\}, \\ o_{q_2} & \text{for } x = o_{q_1}, \end{cases}
\]

where \(\sigma : S_{q_1}^{n-1} \rightarrow S_{q_2}^{n-1}\) is the diffeomorphism defined by (2.10). Note that \(\tilde{F}\) is a bi-Lipschitz homeomorphism (see Lemma 3.13 in [24] for more details).

**Lemma 3.1** For any \(x \in B_\pi(o_{q_1}) \setminus \{o_{q_1}\}\) and any \(X \in B_\pi(o_{q_1}) \setminus \{o_{q_1}\}\) with \(\|X\| = 1\), we have

\[
1 - \frac{1}{2} \delta_n \leq \|d\tilde{F}_x(X)\| \leq 1 + \frac{1}{2} \delta_n,
\]

where \(\delta_n\) is the positive number given by (2.12).

**Proof.** Take any \(x \in B_\pi(o_{q_1}) \setminus \{o_{q_1}\}\), and fix it. Then, there exist \(v \in S_{q_1}^{n-1}\) and \(\ell > 0\) such that \(x = \ell v\). We identify \(T_x(T_{q_1}M_1)\) with \(T_{q_1}M_1\). Then, by the proof of Lemma 3.7 in [24], we have

\[
d\tilde{F}_x(av) = d\tilde{F}_{\ell v}(av) = a\sigma(v)
\]

for all \(a \in \mathbb{R}\), and have

\[
d\tilde{F}_x(w) = d\tilde{F}_{\ell v}(w) = d\sigma_v(w)
\]

for all \(w \in S_{q_1}^{n-2} := \{w \in T_v(S_{q_1}^{n-1}) \mid \|w\| = 1\}\). Since

\[
\|d\sigma_v(w)\| = \|I_{\sigma_{q_1}^{p_1}(v)}(w) + d\sigma_v(w) - I_{\sigma_{q_1}^{p_1}(v)}(w)\|,
\]

we see, by the triangle inequality and Lemma 2.2

\[
1 - \frac{1}{2} \delta_n \leq \|d\sigma_v(w)\| \leq 1 + \frac{1}{2} \delta_n
\]

for all \(w \in S_{q_1}^{n-2}\), where \(I_{\sigma_{q_1}^{p_1}(v)} : T_{q_1}M_1 \rightarrow T_{q_2}M_2\) is the linear isometry given by (2.11), and \(\sigma_{q_1}^{p_1}\) is the inverse of the diffeomorphism \(\sigma_{q_1}^{p_1} : S_{p_1}^{n-1} \rightarrow S_{q_1}^{n-1}\) defined by (2.9). Take any \(X \in T_x(B_\pi(o_{q_1}))\) with \(\|X\| = 1\), and fix it. Then, we can write \(X = \alpha v + \beta w_0\) for some \(w_0 \in S_{q_1}^{n-2}\) and some \(\alpha, \beta \in \mathbb{R}\). Moreover, by (3.2) and (3.3), we have

\[
\|d\tilde{F}_x(X)\|^2 = \alpha^2 + \beta^2\|d\sigma_v(w_0)\|^2.
\]
Since
\[(3.6) \quad \left(1 - \frac{\delta_n}{2}\right)^2 = (\alpha^2 + \beta^2) \left(1 - \frac{\delta_n}{2}\right)^2 \leq \alpha^2 + \beta^2 \|d\sigma_v(w_0)\|^2 \leq (\alpha^2 + \beta^2) \left(1 + \frac{\delta_n}{2}\right)^2 = \left(1 + \frac{\delta_n}{2}\right)^2\]
from (3.4), where note that \(\delta_n \in (0, 1)\), we see, by substituting (3.5) into (3.6), that our assertion (3.1) holds. \(\square\)

**Lemma 3.2** For any \(y, z \in B_\pi(o_{q_1})\),
\[(3.7) \quad \left(1 - \frac{1}{2}\delta_n\right) \|y - z\| \leq \|\tilde{F}(y) - \tilde{F}(z)\| \leq \left(1 + \frac{1}{2}\delta_n\right) \|y - z\|\]
holds. In particular, we have
\[(3.8) \quad \text{Lip}^b(\tilde{F}) \leq 1 + \delta_n,\]
where \(\text{Lip}^b(\tilde{F})\) denotes the bi-Lipschitz constant of \(\tilde{F}\) defined by
\[\text{Lip}^b(\tilde{F}) := \inf \{L \mid L^{-1}\|y - z\| \leq \|\tilde{F}(y) - \tilde{F}(z)\| \leq L \|y - z\| \text{ for all } y, z \in B_\pi(o_{q_1})\}.\]

**Proof.** Take any \(x \in B_\pi(o_{q_1}) \setminus \{o_{q_1}\}\), and fix it. Let \(Y \in T_{\tilde{F}(x)}(B_\pi(o_{q_2}))\) with \(Y \neq 0\). Since \(d\tilde{F}_x\) is bijective, there exists \(X \in T_x(B_\pi(o_{q_1}))\) with \(X \neq 0\) such that \(d\tilde{F}_x(X) = Y\). Since \(\|d\tilde{F}_x(X)\| = \|Y\|\), we have, by the left side inequality of (3.4),
\[\left(1 - \frac{1}{2}\delta_n\right) \|X\| \leq \|d\tilde{F}_x(X)\| = \|Y\|,\]
and hence
\[(3.9) \quad \|X\| \leq \left(1 - \frac{1}{2}\delta_n\right)^{-1} \|Y\|\]
holds. Since \((d\tilde{F}_x)^{-1} = (d\tilde{F}^{-1})_{\tilde{F}(x)}\), we have \((d\tilde{F}^{-1})_{\tilde{F}(x)}(Y) = X\). Thus, by (3.9), we get
\[(3.10) \quad \|(d\tilde{F}^{-1})_{\tilde{F}(x)}(Y)\| = \|X\| \leq \left(1 - \frac{1}{2}\delta_n\right)^{-1} \|Y\|\]

We first prove the left side inequality of (3.7). Take any two points \(\tilde{y}, \tilde{z} \in B_\pi(o_{q_2})\), and fix them. We can assume \(\tilde{y} \neq \tilde{z}\) in this aim. Set \(\tilde{v} := (\tilde{z} - \tilde{y})/\|\tilde{z} - \tilde{y}\|\) and \(a := \|\tilde{z} - \tilde{y}\|\). Then, the geodesic segment \(\tilde{\gamma} : [0, a] \rightarrow B_\pi(o_{q_2})\) emanating from \(\tilde{y}\) to \(\tilde{z}\) is given by \(\tilde{\gamma}(t) := \tilde{y} + t\tilde{v}\). Since \(\tilde{F}^{-1}\) is Lipschitzian, and since \(\tilde{\gamma}(t) = \tilde{v}\), we see, by (3.10),
\[(3.11) \quad \|\tilde{F}^{-1}(\tilde{z}) - \tilde{F}^{-1}(\tilde{y})\| = \left\|\int_0^a \frac{d(\tilde{F}^{-1} \circ \tilde{\gamma})}{dt}(t) \, dt\right\| = \left\|\int_0^a (d\tilde{F}^{-1})_{\tilde{\gamma}(t)}(\tilde{v}) \, dt\right\|
\leq \int_0^a \|(d\tilde{F}^{-1})_{\tilde{\gamma}(t)}(\tilde{v})\| \, dt \leq \left(1 - \frac{1}{2}\delta_n\right)^{-1} \int_0^a \|\tilde{v}\| \, dt
\leq \left(1 - \frac{1}{2}\delta_n\right)^{-1} \int_0^a \|\tilde{v}\| \, dt = \left(1 - \frac{1}{2}\delta_n\right)^{-1} \|	ilde{z} - \tilde{y}\|.\]
Set $y := \tilde{F}^{-1}(\tilde{g})$ and $z := \tilde{F}^{-1}(\tilde{f})$. Then, (3.11) implies the left side inequality of (3.7). By an analogous argument, we also have the right side inequality of (3.7).

Finally, we prove (3.8). Since $\delta_n \in (0, 1)$, we see

$$\frac{1}{2} \delta_n - \frac{1}{1 + \delta_n} = \frac{\delta_n (1 - \delta_n)}{2(1 + \delta_n)} > 0,$$

and hence $(1 + \delta_n)^{-1} < (1 - \delta_n)/2$ holds. Since $1 + \delta_n/2 < 1 + \delta_n$, it follows from (3.7) that

$$(3.12) \quad (1 + \delta_n)^{-1} \|y - z\| \leq \|\tilde{F}(y) - \tilde{F}(z)\| \leq (1 + \delta_n) \|y - z\|.$$

Therefore, we obtain (3.8) from (3.12).

By applying the Nash embedding theorem [20], let $M_2$ be isometrically embedded into the Euclidean space $\mathbb{R}^m$ with the canonical Riemannian metric $\langle \cdot, \cdot \rangle$ where $m \geq n + 1$. Then, $F$ becomes a Lipschitz map from $M_1$ to $M_2 \subset \mathbb{R}^m$.

For any sufficiently small $\varepsilon > 0$, let $\tilde{F}_\varepsilon$ be the standard convolution of $\tilde{F}$ and the mollifier $\rho_\varepsilon$ near $a_{q_1}$, i.e., $\tilde{F}_\varepsilon(y) := \int_{\mathbb{R}^n} \tilde{F}(x) \rho_\varepsilon(x - y) dx$, where we identify $T_{q_1} M_1$ with $\mathbb{R}^n$. Substituting (2.12) for $\delta_\varepsilon$ in (3.8), we have

$$(3.13) \quad \text{Lip}^b(\tilde{F})^2 \leq 1 + \left\{ \frac{8}{\pi} (n - 1) \right\}^{-\frac{1}{2}}.$$

Thanks to (3.13), we can apply the proof of Theorem 5.1 in [21] to $\tilde{F}_\varepsilon$. According to the proof, we see that $\tilde{F}_\varepsilon$ is an immersion from some open ball $B_\varepsilon(a_{q_1}) \subset B_\varepsilon(a_{q_1})$ into $B_\varepsilon(a_{q_1})$. By the definition of $\tilde{F}$, the map $F_\varepsilon(q_1) := \exp_{q_2} \tilde{F}_\varepsilon \circ \exp_{q_1}$ from an open ball $B_\varepsilon(q_1) = \exp_{q_1} B_\varepsilon(a_{q_1})$ into $M_2$ is a local smooth approximation of $F$. It is clear that $F_\varepsilon(q_1)$ is an immersion.

Define the map $F_\varepsilon : M_1 \rightarrow \mathbb{R}^m$ by $F_\varepsilon := (1 - G) F + g F_\varepsilon(q_1)$. Here, $g : M_1 \rightarrow \mathbb{R}$ denotes a smooth function satisfying $0 \leq g \leq 1$ on $M_1$, $g \equiv 1$ on $B_\varepsilon(q_1)$, and $\text{supp } g \subset B_R(q_1)$, where $0 < r < R < a$.

Lemma 3.3 $F_\varepsilon$ is a smooth immersion for all sufficiently small $\varepsilon > 0$.

Proof. The definition of $F_\varepsilon$ implies $F_\varepsilon = F_\varepsilon(q_1)$ on $B_\varepsilon(q_1)$ and $F_\varepsilon = F$ on $M_1 \setminus \text{supp } g$, and hence $F_\varepsilon$ is a local diffeomorphism on $B_\varepsilon(q_1) \cup (M_1 \setminus \text{supp } g)$. Since $\tilde{F}$ is smooth on $B_\varepsilon(a_{q_1}) \setminus \{ a_{q_1} \}$, we easily see, by the definition of the differential of a smooth map, that $\tilde{F}_\varepsilon$ uniformly converges to $\tilde{F}$ on $B_R(a_{q_1}) \setminus B_\varepsilon(a_{q_1})$ as $\varepsilon \downarrow 0$ in the $C^1$-topology. From this argument, we observe that $F_\varepsilon(q_1)$ uniformly converges to $F$ on $B_R(q_1) \setminus B_\varepsilon(q_1)$ as $\varepsilon \downarrow 0$ in the $C^1$-topology, for $dF_\varepsilon(q_1) = d \exp_{q_2} \circ d\tilde{F}_\varepsilon \circ d \exp_{q_1}$ and $dF = d \exp_{q_2} \circ d\tilde{F} \circ d \exp_{q_1}$. Since

$$F_\varepsilon - F = g(F_\varepsilon(q_1) - F) = g(\exp_{q_2} \circ \tilde{F}_\varepsilon \circ \exp_{q_1} - \exp_{q_2} \circ \tilde{F} \circ \exp_{q_1})$$

on $M_1$, and since

$$(dF_\varepsilon)_x(v) - dF_x(v) = dg_x(v)(F_\varepsilon(q_1)(x) - F(x)) + g(x) \{(dF_\varepsilon(q_1))_x(v) - dF_x(v)\}$$

for $x \in B_\varepsilon(q_1)$, we get

$$dF_\varepsilon(q_1)_x(v) - dF_x(v) = d(g \circ \exp_{q_2})_x(v)(F_\varepsilon(q_1)(x) - F(x)) + g(x) \{(dF_\varepsilon(q_1))_x(v) - dF_x(v)\}$$

for $x \in B_\varepsilon(q_1)$. Therefore, 

$$\|dF_\varepsilon(q_1)_x(v) - dF_x(v)\| \leq \|d(g \circ \exp_{q_2})_x(v)\| \|F_\varepsilon(q_1)(x) - F(x)\| + g(x) \|\{(dF_\varepsilon(q_1))_x(v) - dF_x(v)\}\|$$

for $x \in B_\varepsilon(q_1)$. Thus, $F_\varepsilon$ is a smooth immersion.
for all \( v \in T_x M_1 \ (x \in M_1 \setminus \{ q_1 \}) \), \( F_\varepsilon \) uniformly converges to \( F \) on \( \overline{B_R(q_1)} \setminus B_\varepsilon(q_1) \) as \( \varepsilon \downarrow 0 \) in the \( C^1 \)-topology. Since \( F \) is diffeomorphic on \( M_1 \setminus \{ q_1 \} \), we see, by the above argument, that \( F_\varepsilon \) is a smooth immersion for any sufficiently small \( \varepsilon > 0 \). \( \square \)

Since \( M_2 \) is isometrically embedded into \( \mathbb{R}^m \), it follows from the tubular neighborhood theorem (cf. \cite{19}, \cite{25}) via the normal exponential map \( \exp^\perp : TM_2^\perp \rightarrow \mathbb{R}^m \) that there is a constant \( \mu > 0 \) such that \( \exp^\perp \) is a diffeomorphism from an open neighborhood \( U_\mu(O(TM_2^\perp)) \) of a set \( O(TM_2^\perp) \) in \( \mathbb{R}^{2m} \) onto an open one \( U_\mu(M_2) \) of \( M_2 \) in \( \mathbb{R}^m \), which is called the tubular neighborhood of \( M_2 \), where the three sets are given by

\[
U_\mu(O(TM_2^\perp)) := \{ X \in TM_2^\perp \mid \|X\| < \mu \}, \quad O(TM_2^\perp) := \{ o_x \in T_x M_2^\perp \mid x \in M_2 \},
\]

where \( o_x \) denotes the origin of \( T_x M_2^\perp \), and

\[
U_\mu(M_2) := \exp^\perp(U_\mu(O(TM_2^\perp))).
\]

Since \( \exp^\perp \mid _{U_\mu(O(TM_2^\perp))} \) is bijective, for any \( y \in U_\mu(M_2) \) there is a unique point \( (x, v) \in TM_2^\perp \) such that

\[
y = \exp^\perp(x, v) = \exp_x v.
\]

For such a pair \((y, (x, v))\) we thus have the smooth projection \( \pi_{M_2} : U_\mu(M_2) \rightarrow M_2 \) given by

\[
\pi_{M_2}(y) = \pi_{M_2}(\exp^\perp(x, v)) = x.
\]

From the definition of \( \pi_{M_2} \),

\[
(3.14) \quad \text{Ker}(d\pi_{M_2})_x \perp T_x M_2
\]

holds for all \( x \in M_2 \), and hence the first variation formula yields

\[
\|y - \pi_{M_2}(y)\| = \inf_{z \in M_2} \|y - z\|
\]

for all \( y \in U_\mu(M_2) \).

Since \( M_1 \) is compact, we see, by the definition of \( F_\varepsilon \) and the proof of Lemma \ref{lem:3.3} that \( \lim_{\varepsilon \downarrow 0} \|F_\varepsilon(p) - F(p)\| = 0 \) for all \( p \in M_1 \), which implies \( F_\varepsilon(M_1) \subset U_\mu(M_2) \) for any sufficiently small \( \varepsilon > 0 \). Thus, we can define the smooth map \( \psi_\varepsilon : M_1 \rightarrow M_2 \) by

\[
\psi_\varepsilon := \pi_{M_2} \circ F_\varepsilon
\]

for any sufficiently small \( \varepsilon > 0 \).

**Lemma 3.4** \( \psi_\varepsilon \) is a smooth immersion for any sufficiently small \( \varepsilon > 0 \).

**Proof.** Fix \( \varepsilon > 0 \) sufficiently small, so as \( \psi_\varepsilon \) can be defined. Since \( (dF_\varepsilon)_p \) is injective for all \( p \in M_1 \) by Lemma \ref{lem:3.3}, we need to show that for each \( p \in M_1 \),

\[
(3.15) \quad \text{rank}((d\pi_{M_2})_{(dF_\varepsilon)_p}) = n
\]

holds. As we have noted in the proof of Lemma \ref{lem:3.3} \( F_\varepsilon = F_\varepsilon(q_1) \) on \( \overline{B_\varepsilon(q_1)} \) and \( F_\varepsilon = F \) on \( M_1 \setminus \text{supp } g \). In particular,
\[ \psi_\varepsilon(\overline{B_r(q_1)}) = F_\varepsilon(B_r(q_1)) \subset M_2 \quad \text{and} \quad \psi_\varepsilon(M_1 \setminus \text{supp } g) = F_\varepsilon(M_1 \setminus \text{supp } g) \subset M_2. \]

So, it is sufficient to prove that (3.15) holds for all \( p \in B_R(q_1) \setminus \overline{B_r(q_1)} \). Indeed, as we have seen in the proof of Lemma 3.3, \( F_\varepsilon \) uniformly converges to \( F \) on \( B_R(q_1) \setminus B_r(q_1) \) as \( \varepsilon \downarrow 0 \) in the \( C^1 \)-topology, and hence we see, by (3.14), that for any sufficiently small \( \varepsilon > 0 \),

(3.16) \[ \text{Ker}(d\pi_{M_2})_{F_\varepsilon(p)} \cap \text{Im}(dF_\varepsilon)_p = \{ \partial_{F_\varepsilon(p)} \} \]

holds for all \( p \in B_R(q_1) \setminus \overline{B_r(q_1)} \), where \( \partial_{F_\varepsilon(p)} \) denotes the origin of \( T_{F_\varepsilon(p)} \mathbb{R}^m \). (3.16) shows that (3.15) holds for all \( p \in B_R(q_1) \setminus \overline{B_r(q_1)} \), which completes the proof.

Finally, we will prove that \( \psi_\varepsilon \) is a global diffeomorphism from \( M_1 \) onto \( M_2 \) for any sufficiently small \( \varepsilon > 0 \). Indeed, fix \( \varepsilon > 0 \) sufficiently small, so as \( \psi_\varepsilon \) can be a smooth immersion. Since \( \psi_\varepsilon(M_1) \subset M_2 \) is compact, and since \( M_2 \) is Hausdorff, \( \psi_\varepsilon(M_1) \) is closed in \( M_2 \). Since \( \psi_\varepsilon \) is a local homeomorphism on \( M_1 \) by Lemma 3.4, \( \psi_\varepsilon(M_1) \) is an open set in \( M_2 \). Thus, \( \psi_\varepsilon(M_1) \) is open and closed in \( M_2 \), and hence \( \psi_\varepsilon(M_1) = M_2 \), i.e., \( \psi_\varepsilon \) is surjective. Since \( \psi_\varepsilon^{-1}(V) \) is closed in \( M_1 \) for all closed sets \( V \) in \( M_2 \), \( \psi_\varepsilon^{-1}(V) \subset M_1 \) and \( V \subset M_2 \) are compact by the compactness of \( M_i \) \( (i = 1, 2) \), which implies that \( \psi_\varepsilon \) is a proper map, in particular, a covering map. Since \( M_2 \) is simply connected, \( \psi_\varepsilon \) is injective. Therefore, \( \psi_\varepsilon \) is a global diffeomorphism from \( M_1 \) onto \( M_2 \).

\[ \square \]

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K. Kondo
Department of Mathematical Sciences, Yamaguchi University
Yamaguchi City, Yamaguchi Pref. 753-8512, Japan
e-mail: keikondo@yamaguchi-u.ac.jp

M. Tanaka
Department of Mathematics, Tokai University
Hiratsuka City, Kanagawa Pref. 259-1292, Japan
e-mail: tanaka@tokai-u.jp