The Pfaff lattice, Matrix integrals and a map from Toda to Pfaff

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Contents

0 Introduction 3

1 Splitting theorems, as applied to the Toda and Pfaff Lattices 9

2 Wave functions and their bilinear equations for the Pfaff Lattice 15

3 Existence of the Pfaff \( \tau \)-function 21

4 Semi-infinite matrices \( m_\infty \), (skew-)orthogonal polynomials and matrix integrals 30

4.1 \( \partial m/\partial t_k = \Lambda^k m \), orthogonal polynomials and Hermitean matrix integrals. 30

4.2 \( \partial m/\partial t_k = \Lambda^k m + m \Lambda^k \), skew-orthogonal polynomials and symmetric and symplectic matrix integrals. 31

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| Chapter   | Title                                                | Page |
|-----------|------------------------------------------------------|------|
| 5         | A map from the Toda to the Pfaff lattice             | 37   |
| 6         | Example 1: From Hermitean to symmetric matrix integrals | 42   |
| 7         | Example 2: From Hermitean to symplectic matrix integrals | 46   |
| 8         | Appendix 1: Free parameter in the skew-Borel decompostion | 52   |
| 9         | Appendix 2: Simultaneous (skew) - symmetrization of $L$ and $N$ | 54   |
| 10        | Appendix 3: Proof of Lemma 3.4                        | 55   |
0 Introduction

Consider a weight on $\mathbb{R}$, depending on $t = (t_1, t_2, \ldots) \in \mathbb{C}^\infty$, 
\[
\rho_t(z) dz = e^{\sum_{i=1}^{\infty} t_i z^i} \rho(z) dz = e^{-V(z) + \sum_{i=1}^{\infty} t_i z^i} dz, \quad \text{with} \quad -\frac{\rho'(z)}{\rho(z)} = V'(z) = \frac{g(z)}{f(z)}.
\]

Hermitean matrix integrals (revisited) This weight leads to a $t$-dependent moment matrix 
\[
m_n(t) = (\mu_{k+\ell}(t))_{0 \leq k, \ell \leq n-1} = \left( \int_{\mathbb{R}} z^{k+\ell} \rho_t(z) dz \right)_{0 \leq k, \ell \leq n-1},
\]
with the semi-infinite moment matrix $m_\infty$, satisfying the commuting equations 
\[
\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty = m_\infty \Lambda^k,
\]
where $\Lambda$ is the customary shift matrix. Considering the lower- and upper-triangular matrix Borel decomposition 
\[
m_\infty = S^{-1} S^\top^{-1},
\]
which is determined by the following $t$-dependent matrix integrals\footnote{We set $\text{vol}(\mathcal{U}(n)) = 1$ for all $n.$} ($n \geq 0$) 
\[
\tau_n(t) := \int_{\mathcal{H}_n} e^{Tr(-V(X) + \sum t_i X^i)} dX = \det m_n, \quad \text{and} \quad \tau_0 = 1,
\]
with Haar measure $dX$ on the ensemble $\mathcal{H}_n = \{n \times n \text{ Hermitean matrices}\}$. 
As is well known, the integral (0.4) is a solution to the following two systems, 
(i) the KP-hierarchy 
\[
\left( p_{k+4}(\partial) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau_n \circ \tau_n = 0, \quad \text{for} \quad k, n = 0, 1, 2, \ldots
\]
(ii) the Toda lattice; i.e., the tridiagonal matrix 
\[
L(t) := SAS^{-1} = \begin{pmatrix}
\frac{\partial}{\partial t_1} \log \tau_1 & \frac{\tau_0}{\tau_1}^{1/2} & 0 \\
\frac{\tau_0}{\tau_1}^{1/2} & \frac{\partial}{\partial t_1} \log \tau_1 & \frac{\tau_1}{\tau_2}^{1/2} \\
0 & \frac{\tau_1}{\tau_2}^{1/2} & \frac{\partial}{\partial t_1} \log \tau_2 & \ddots
\end{pmatrix}
\]
satisfies the following commuting Toda equations

$$\frac{\partial L}{\partial t_n} = \left[ \frac{1}{2} (L^n)_{sk}, L \right],$$

where \((A)_{sk}\) denotes the skew-part of the matrix \(A\) for the Lie algebra splitting into skew and lower-triangular matrices. Moreover, the following \(t\)-dependent polynomials in \(z\), are defined by the \(S\)-matrix obtained from the Borel decomposition (0.3); it is also given, on the one hand, by a classic determinantal formula, and on the other hand, in terms of the functions \(\tau_n(t)\):

$$p_n(t, z) := \left( S(t) \chi(z) \right)_n = \frac{1}{\sqrt{T_n t_n + 1}} \det \begin{pmatrix} m_n(t) & 1 \\ \mu_{n,0}(t) & \vdots & \mu_{n,n-1}(t) \end{pmatrix} z^n \right).$$

The \(p_n\)'s are orthonormal with respect to the (symmetric) inner-product \(\langle \cdot, \cdot \rangle_{sy}\), defined by \(\langle z^i, z^j \rangle_{sy} = \mu_{ij}\), which is a restatement of the Borel decomposition (0.3). The vector \(p(t, z) = (p_n(t, z))_{n \geq 0}\) is an eigenvector of the matrix \(L(t)\) in (0.6):

$$L(t)p(t, z) = zp(t, z).$$

**Symmetric and symplectic matrix integrals.** Instead consider the following skew-symmetric matrix \(m_\infty = (\mu_{ij})_{i,j \geq 0}\) of moments\(^3\)

$$\mu_{ij}^{(1)}(t) = \int \int_{\mathbb{R}^2} x^i y^j \varepsilon(x - y) \rho_t(x) \rho_t(y) dx dy \quad \text{or} \quad \mu_{ij}^{(2)}(t) = \int_{\mathbb{R}} \{y^i, y^j\} \rho_t(y)^2 dy, \quad (0.7)$$

both satisfying the equations

$$\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^i. \quad (0.8)$$

In this paper, we consider symmetric matrix integrals\(^4\)

$$\tau_{2n}^{(1)}(t) := \frac{1}{(2n)!} \int_{\mathbb{S}_2n} e^{Tr (-V(X) + \sum_{i=1}^{2n} x^i x^i)} dX = pf(m_{2n}^{(1)}), \quad (0.9)$$

\(^2\varepsilon(x) = 1, \text{ for } x \geq 0 \text{ and } = -1, \text{ for } x < 0 \text{ and } \{f, g\} = f'g - fg'.\)

\(^3\text{where again we set the volume of the orthogonal and symplectic groups equal to 1.}\)
and symplectic matrix integrals
\[ \tau^{(1)}_{2n}(t) := \frac{1}{n!} \int_{T_{2n}} e^{2Tr(-V(X) + \sum_{i=1}^{\infty} t_i X_i)} dX = pf(m^{(2)}_{2n}), \quad (0.10) \]
both expressed in terms of the Pfaffian of the “moment” matrix \( m^{(i)}_{\infty} \), where
(1) in the first case, \( dX \) denotes Haar measure on the space \( S_{2n} \) of symmetric matrices and,
(2) in the second case, \( dX \) denotes Haar measure on the \( 2n \times 2n \) matrix realization \( T_{2n} \) of the space of self-dual \( n \times n \) Hermitian matrices, with quaternionic entries.

Since \( m_{\infty} \) is skew-symmetric, the Borel decomposition of \( m_{\infty} \) will require the interjection of a skew-symmetric matrix \( J \), used throughout this paper,

\[ J = \begin{pmatrix}
\ddots \\
0 & 1 \\
-1 & 0 \\
0 & 1 \\
-1 & 0 \\
\ddots & \ddots & \ddots \\
0 & 1 \\
-1 & 0
\end{pmatrix} \quad \text{with } J^2 = -I \quad (0.11) \]

and the order 2 involution on the space \( D \) of infinite matrices
\[ \mathcal{J} : D \rightarrow D : a \mapsto \mathcal{J}(a) := J a^\top J. \quad (0.12) \]

The skew-Borel decomposition
\[ m_{\infty}(t) = Q^{-1}(t) J Q^{-1\top}(t), \quad (0.13) \]
can entirely be expressed in terms of the integrals \( \tau_{2n}(t) \), (0.9) and (0.10) corresponding respectively to the first and second moment matrix (0.7). They satisfy both
(i) the Pfaffian KP-hierarchy for \( k, n = 0, 1, 2, \ldots, \)
\[ \left( p_{k+4}(\tilde{\partial}) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau_{2n} \circ \tau_{2n} = p_k(\tilde{\partial}) \tau_{2n+2} \circ \tau_{2n-2}, \quad (0.14) \]
(ii) the Pfaff lattice; i.e., the matrix, constructed by dressing up \( \Lambda \) with \( Q \) and which this time is full below the main diagonal,

\[
L = QA^{-1} = h^{-1/2} \begin{pmatrix}
\hat{L}_{00} & \hat{L}_{01} & 0 & 0 \\
\hat{L}_{10} & \hat{L}_{11} & \hat{L}_{12} & 0 \\
* & \hat{L}_{21} & \hat{L}_{22} & \hat{L}_{23} \\
* & * & \hat{L}_{32} & \hat{L}_{33} & \ldots
\end{pmatrix} h^{1/2},
\]

satisfies the Hamiltonian commuting equations

\[
\frac{\partial L}{\partial t_i} = \left[ \left( (L^i)_+ + J((L^i)_+) \right) + \frac{1}{2} \left( (L^i)_0 + J((L^i)_0) \right), L \right],
\]

with the entries \( \hat{L}_{ij} \) and the entries of \( h \), being \( 2 \times 2 \) matrices

\[
h = \text{diag}(h_0 I_2, h_2 I_2, h_4 I_2, \ldots), \ h_{2n} = \tau_{2n+2}/\tau_{2n},
\]

and \( (\cdot) = \frac{\partial}{\partial t} \)

\[
\hat{L}_{nn} := \begin{pmatrix}
-(\log \tau_{2n}) & 1 \\
\frac{s_2(\bar{\partial})\tau_{2n}}{\tau_{2n}} - \frac{s_2(-\bar{\partial})\tau_{2n+2}}{\tau_{2n+2}} & (\log \tau_{2n+2})
\end{pmatrix}, \quad \hat{L}_{n,n+1} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

\[
\hat{L}_{n+1,n} := \begin{pmatrix}
* & (\log \tau_{2n+2})^{-1} \\
* & *
\end{pmatrix}.
\]

The following \( t \)-dependent polynomials \( q_n(t, z) = (S\chi(z))_n \) in \( z \), defined by the \( S \) matrix of the skew-Borel decomposition (0.13), have determinantal and Pfaffian \( \tau \)-function expression, in analogy with the Hermitean case:

\[
q_{2n}(t, z) = \frac{1}{\sqrt{\tau_{2n}(t)\tau_{2n+2}(t)}} pf \begin{pmatrix}
1 & m_{2n+1}(t) & 1 \\
\vdots & -1 & \ddots & \vdots \\
& 0 & \cdots & -z^{2n}
\end{pmatrix} z^{2n}
\]

\[
= z^{2n} \frac{\tau_{2n}(t - [z^{-1}])}{\sqrt{\tau_{2n}(t)\tau_{2n+2}(t)}}
\]
and

\[
q_{2n+1}(t, z) = \frac{1}{\sqrt{\tau_{2n}\tau_{2n+2}(t)}} pf\left(\begin{array}{cc}
1 & \mu_{0,2n+1} \\
z & \mu_{1,2n+1} \\
& \vdots \\
& 0 \\
-1 & -z & \ldots & 0 \\
-\mu_{0,2n+1} & -\mu_{1,2n+1} & \ldots & -z^{2n+1} \\
z^{2n} & \mu_{2n,2n+1} & \ldots & 0
\end{array}\right) = \frac{1}{\sqrt{\tau_{2n}(t}\tau_{2n+2}(t)}} \left(z + \frac{\partial}{\partial t_1}\right) \sqrt{\tau_{2n}(t)\tau_{2n+2}(t)} q_{2n}(t, z)
\]

\[
= z^{2n} \frac{\left(z + \frac{\partial}{\partial t_1}\right)\tau_{2n}(t - [z^{-1}])}{\sqrt{\tau_{2n}(t)\tau_{2n+2}(t)}}
\]

(0.17)

form skew-orthonormal sequences with respect to the skew inner-product \(\langle \cdot, \cdot \rangle_{sk}\), defined by \(\langle y^i, z^j \rangle_{sk} = \mu_{ij}\), namely, we have the following restatement of the skew-Borel decomposition (0.13):

\[
(\langle q_i, q_j \rangle)_{0 \leq i, j < \infty} = J.
\]

(0.18)

Finally, the vector \(q(z) = (q_n(z))_{n \geq 0}\) forms a eigenvector for the matrix \(L\):

\[
L(t)q(t, z) = zq(t, z).
\]

(0.19)

In section 2, we show how a general skew-symmetric infinite matrix flowing via (0.8) and its skew-Borel decomposition (0.13), lead to wave vectors \(\Psi\), satisfying bilinear relations and differential equations. Section 3 deals with the existence, in the above general setting, of a so-called Pfaffian \(\tau\)-function, satisfying bilinear equations and a KP-type hierarchy. In [6], these results were obtained, by embedding the system in 2-Toda theory, while in this paper, they are obtained in an intrinsic fashion.

For \(k = 0\), the KP-like equation (0.14) has already appeared in the context of the charged BKP hierarchy, studied by V. Kac and van de Leur [12]; the precise relationship between the charged BKP hierarchy of Kac and van de Leur and the Pfaff Lattice, introduced here, deserves further investigation. See the recent paper of van de Leur [16].

**A remarkable map from Toda to Pfaff lattice:** Remembering the no-
On the $t$-dependent orthonormal polynomials $p_n(t, z)$ in $z$; in [2], we showed that the matrix $N$ defined by
\begin{equation}
C_n p(t, z) = \left(f(L)M - \frac{f'}{2} + g(L)\right)p(t, z) =: N p(t, z) \tag{0.21}
\end{equation}
is skew-symmetric. The $t$-dependent matrix $N$ is expressed in terms of $L$ and a new matrix $M$, defined by
\begin{equation}
z p = L p \quad \text{and} \quad e^{\frac{1}{4} \sum t_k z^k} \frac{d}{dz} e^{\frac{1}{2} \sum t_k z^k} p = M p. \tag{0.22}
\end{equation}
Consider now the skew-Borel decomposition of $N(2t)$ and its inverse $N(2t)^{-1}$, in terms of lower-triangular matrices $O_+(t)$ and $O_-(t)$ respectively:
\begin{equation}
N(2t) = -O_+^{-1}(t) JO_+^{-1}(t), \tag{0.23}
\end{equation}
Then, the lower-triangular matrices $O_+(t)$ map orthonormal into skew-orthonormal polynomials, and the tridiagonal $L$-matrix into an $\tilde{L}$-matrix:
\begin{equation}
p_n(t, z) \quad \mapsto \quad g_n^{(\pm)}(t, z) = O_{(\pm)}(t)p_n(t, z)
\quad \text{(Toda Lattice)}
\end{equation}
\begin{equation}
L(t) \quad \mapsto \quad \tilde{L}(t) = O_{(\pm)}(t)L(2t)O_{(\pm)}(t)^{-1}. \tag{0.24}
\quad \text{(Pfaff Lattice)}
\end{equation}
It also maps the weight into a new weight
\begin{equation}
\rho(z) = e^{-V(z)} \mapsto \tilde{\rho}_{\pm}(z) = e^{-\tilde{V}(z)} := e^{-\frac{1}{2}(V(z) \mp \log f(z))},
\end{equation}
and the corresponding string of $\tau$-functions into a new string of pfaffian $\tau$-functions: (remember $V_t(z) = V(z) - \sum_1^\infty t_i z^i$)
\begin{equation}
\tau_k(t) = \int_{H_k} e^{Tr (-V_t(X))} dX \mapsto \begin{cases}
\tau_{2n}^{(+)}(t) := \int_{T_{2n}} e^{Tr 2(-\tilde{V}_t(X))} dX, \quad (\beta = 4) \\
\tau_{2n}^{(-)}(t) := \int_{S_{2n}} e^{Tr (-\tilde{V}_t(X))} dX \quad (\beta = 1). \end{cases}
\end{equation}
For the classical orthogonal polynomials \( p_n(z) \), we have shown in \([2]\), that \( \mathcal{N}(0) \) is not only skew-symmetric, but also tridiagonal; i.e.,

\[
L = \begin{bmatrix} b_0 & a_0 & & & \\ a_0 & b_1 & a_1 & & \\ & a_1 & b_2 & \ddots & \\ & & \ddots & \ddots & \\ & & & \ddots & b_\ell \end{bmatrix}, \quad -\mathcal{N} = \begin{bmatrix} 0 & c_0 & & & \\ -c_0 & 0 & c_1 & & \\ & -c_1 & 0 & \ddots & \\ & & \ddots & \ddots & \\ & & & \ddots & 0 \end{bmatrix} .
\]

(0.25)

In section 6 and 7, we show that the maps \( O(-) \) and \( O(+) \) only involves three steps, in the following sense:

\[
q_{2n}^{-} (0, z) = \sqrt{\frac{c_{2n}}{a_{2n}}} p_{2n} (0, z)
\]

\[
q_{2n+1}^{-} (0, z) = \sqrt{\frac{a_{2n}}{c_{2n}}} 
\left(-c_{2n-1} p_{2n-1} (0, z) + \frac{c_{2n}}{a_{2n}} (\sum_{0}^{2n} b_i) p_{2n} (0, z) + c_{2n} p_{2n+1} (0, z)\right)
\]

(\( \beta = 1 \)) (0.26)

\[
p_{2n} (0, z) = -c_{2n-1} \sqrt{\frac{a_{2n-2}}{c_{2n-2}}} q_{2n-2}^{-} (0, z) + \sqrt{a_{2n} c_{2n}} q_{2n}^{-} (0, z)
\]

\[
p_{2n+1} (0, z) = -c_{2n} \sqrt{\frac{a_{2n-2}}{c_{2n-2}}} q_{2n-2}^{-} (0, z) - (\sum_{0}^{2n} b_i) \sqrt{\frac{c_{2n}}{a_{2n}}} q_{2n}^{-} (0, z) + \sqrt{\frac{c_{2n}}{a_{2n}}} q_{2n+1}^{-} (0, z),
\]

(\( \beta = 4 \)) (0.27)

The abstract map \( O(-) \) for \( t = 0 \) appears already in the work of E. Brézin and H. Neuberger \([7]\). This has been applied in \([14]\) to a problem in the theory of random matrices.

1 Splitting theorems, as applied to the Toda and Pfaff Lattices

In this section, we show how each of the equations

\[
\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty \quad \text{and} \quad \frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^{T_i},
\]

(1.1)
lead to commuting Hamiltonian vector fields related to a Lie algebra splitting. First recall the splitting theorem, due to Adler-Kostant-Symes [1] and the R-version to Reiman and Semenov-Tian-Shansky [15]. The R-version allows for more general initial conditions.

**Proposition 1.1** Let $g = k + n$ be a (vector space) direct sum of a Lie algebra $g$ in terms of Lie subalgebras $k$ and $n$, with $g$ paired with itself via a non-degenerate $\text{ad}$-invariant inner product $\langle , \rangle$; this in turn induces a decomposition $g = k^\perp + n^\perp$ and isomorphisms $g \simeq g^*$, $k^\perp \simeq n^*$, $n^\perp \simeq k^*$. $\pi_k$ and $\pi_n$ are projective onto $k$ and $n$ respectively. Let $G$, $G_k$ and $G_n$ be the groups associated with the Lie algebras $g$, $k$ and $n$. Let $\mathcal{I}(g)$ be the $\text{Ad}^*$-invariant functions on $g^* \simeq g$.

(i) Then, given an element $\epsilon \in g : [\epsilon, k] \subset k^\perp$ and $[\epsilon, n] \subset n^\perp,$

the functions

$\varphi(\varepsilon + \xi')|_{k^\perp}$ with $\varphi \in \mathcal{I}(g)$ and $\xi' \in k^\perp$, (1.2)

respectively Poisson commute for the respective Kostant-Kirillov symplectic structures of $n^* \simeq k^\perp$; the associated Hamiltonian flows are expressed in terms of the Lax pairs\(6\)

$\dot{\xi} = [-\pi_k \nabla \varphi(\xi), \xi] = [\pi_n \nabla \varphi(\xi), \xi]$ for $\xi \equiv \varepsilon + \xi', \xi' \in k^\perp$ (1.3)

(ii) The splitting also leads to a second Lie algebra $g_R$, derived from $g$, such that $g^*_R \simeq g_R$, namely:

$g_R : [x, y]_R = \frac{1}{2} [Rx, y] + \frac{1}{2} [x, Ry] = [\pi_k x, \pi_k y] - [\pi_n x, \pi_n y], \quad (1.4)$

with $R = \pi_k - \pi_n$. The functions

$\varphi(\xi)|_{g_R}$ with $\varphi \in \mathcal{I}(g)$ and $\xi \in g_R$

respectively Poisson commute for the respective Kostant-Kirillov symplectic structures of $g^*_R \simeq g_R$, with the same associated (Hamiltonian) Lax pairs\(7\)

$\dot{\xi} = [-\pi_k \nabla \varphi(\xi), \xi] = [\pi_n \nabla \varphi(\xi), \xi]$ for $\xi \in g_R$. (1.5)

\(6\)$(\text{Ad}_g X; Y) = \langle X, \text{Ad}_{g^{-1}} Y \rangle$, $g \in G$, and thus $\langle [z, x], y \rangle = \langle x, [-z, y] \rangle$.

\(7\)$\nabla \varphi$ is defined as the element in $g^*$ such that $d\varphi(\xi) = \langle \nabla \varphi, d\xi \rangle$, $\xi \in g$. **
Each of the equations (1.3) and (1.5) has the same solution expressible in two different ways:\footnote{Naively written $\text{Ad}_{K(t)}\xi_0 = K(t)\xi_0 K(t)^{-1}$, $\text{Ad}_{S^{-1}(t)}\xi_0 = S^{-1}(t)\xi_0 S(t)$.}

\[ \xi(t) = \text{Ad}_{K(t)}\xi_0 = \text{Ad}_{S^{-1}(t)}\xi_0, \]

with\footnote{With regard to the group factorization $A = \pi_{G_k}A\pi_{G_n}A$.}$K(t) = \pi_{G_k}e^{t\nabla \varphi(\xi_0)}$, and $S(t) = \pi_{G_n}e^{t\nabla \varphi(\xi_0)}$.

**Example 1:** The standard Toda lattice and the equations $\frac{\partial m}{\partial t_i} = \Lambda^i m$ for the Hänkel matrix $m_\infty$. Since, in particular, the matrix $m_\infty$ is symmetric, the Borel decomposition into lower- times upper-triangular matrix must be done with the same lower-triangular matrix $S$:

\[ m_\infty = S^{-1}S^\top^{-1}. \]  

In turn, the matrix $S$ defines a wave vector $\Psi$, and operators\footnote{In the formulas below $\chi(z) = (z^0, z, z^2, \ldots)$ and $\partial$ is the matrix such that $\frac{d}{dz}\chi(z) = \partial \chi(z)$.}$L$ and $M$, the same as the ones defined in (0.22),

\[ \Psi(t, z) := e^\frac{1}{2} \sum_{i=1}^{\infty} i z^i S\chi, \quad L := SAS^{-1}, \quad M := S(\partial + \frac{1}{2} \sum_{i=1}^{\infty} it_i \Lambda_i^{-1})S^{-1}, \]

satisfying the following well-known equations\footnote{Where the $(\cdot)_{sk}$ and $(\cdot)_{bo}$ refers to the skew-part and the lower-triangular (Borel) part respectively; i.e., projection onto $k$ and $n$ respectively.}

\[ L\Psi = z\Psi \quad M\Psi = \frac{\partial}{\partial z} \Psi, \quad \text{with} \quad [L, M] = 1, \]

\[
\begin{align*}
\frac{\partial S}{\partial t_n} &= -\frac{1}{2} (L^n)_{bo} S \\
\frac{\partial \Psi}{\partial t_n} &= \frac{1}{2} (L^n)_{sk} \Psi \\
\frac{\partial L}{\partial t_n} &= \frac{1}{2} [(L^n)_{sk}, L] \\
\frac{\partial M}{\partial t_n} &= \frac{1}{2} [(L^n)_{sk}, M].
\end{align*}
\]  

The wave vector $\Psi$ can then be expressed in terms of a sequence of $\tau$-functions $\tau_n(t) = \text{det} m_n(t)$, but also has the simple expression in terms of orthonormal
polynomials, with respect to the moment matrix $m_\infty$:

$$
\Psi(t, z) = e^{\frac{1}{2} \sum t_i z_i} \left( \sum_{n \geq 0} \tau_n(t - [z^{-1}]) \sqrt{\tau_n(t) \tau_{n+1}(t)} \right)_{n \geq 0} = e^{\frac{1}{2} \sum t_i z_i} (p_n(t,z))_{n \geq 0}.
$$

The vector fields (1.9) on $L$ are commuting Hamiltonian vector fields, in view of the Adler-Kostant-Symes splitting theorem (version (i)),

$$
\frac{\partial L}{\partial t_i} = [-\pi_k \nabla \mathcal{H}_i, L] = [\pi_n \nabla \mathcal{H}_i, L], \quad \mathcal{H}_i = \frac{tr L_{i+1}^i}{i+1}, \quad \nabla \mathcal{H}_i = L_i,
$$

for the splitting of the Lie algebra of semi-infinite matrices

$$
\mathcal{D} = gl_\infty = k + n := \{\text{skew-symmetric}\} + \{\text{lower-triangular}\}
= k^\perp + n^\perp := \{\text{symmetric}\} + \{\text{strictly upper-triangular}\},
$$

with the form (1.12) of $L$ being preserved in time. Note that the solution (1.6) to (1.5) in the AKS theorem is nothing but the factorization of $m_\infty$ followed by the dressing up of $\Lambda$.

**Example 2: The Pfaff lattice and the equations**

Throughout this paper the Lie algebra $\mathcal{D} = gl_\infty$ of semi-infinite matrices is viewed as composed of $2 \times 2$ blocks. It admits the natural decomposition into subalgebras:

$$
\mathcal{D} = \mathcal{D}_- \oplus \mathcal{D}_0 \oplus \mathcal{D}_+ = \mathcal{D}_- \oplus \mathcal{D}_-^\perp \oplus \mathcal{D}_0^\perp \oplus \mathcal{D}_+^\perp
$$

where $\mathcal{D}_0$ has $2 \times 2$ blocks along the diagonal with zeroes everywhere else and where $\mathcal{D}_+$ (resp. $\mathcal{D}_-$) is the subalgebra of upper-triangular (resp. lower-triangular) matrices with $2 \times 2$ zero matrices along $\mathcal{D}_0$ and zero below (resp. above). As pointed out in (1.14), $\mathcal{D}_0$ can further be decomposed into two Lie subalgebras:

$$
\mathcal{D}_0^- = \{\text{all } 2 \times 2 \text{ blocks } \in \mathcal{D}_0 \text{ are proportional to } \text{Id}\}
\mathcal{D}_0^+ = \{\text{all } 2 \times 2 \text{ blocks } \in \mathcal{D}_0 \text{ have trace } 0\}.
$$
Remember from (0.10) and (0.11) in the introduction, the matrix $J$ and the associated Lie algebra order 2 involution $\mathcal{J}$. The splitting into two Lie subalgebras

$$\mathcal{D} = \mathfrak{k} + \mathfrak{n},$$

with

$$\mathfrak{k} = \mathcal{D}_- + \mathcal{D}_0^-$$

$$\mathfrak{n} = \{a \in \mathcal{D}, \text{ such that } \mathcal{J}a = a\} = \{b + \mathcal{J}b, \ b \in \mathcal{D}\} = \text{sp}(\infty),$$

with corresponding Lie groups $\mathcal{G}_k$ and $\mathcal{G}_n = Sp(\infty)$, will play a crucial role here. Let $\pi_k$ and $\pi_n$ be the projections onto $\mathfrak{k}$ and $\mathfrak{n}$. Notice that $\mathfrak{n} = \text{sp}(\infty)$ and $\mathcal{G}_n = Sp(\infty)$ stand for the infinite rank affine symplectic algebra and group; e.g. see [11]. Any element $a \in \mathcal{D}$ decomposes uniquely into its projections onto $\mathfrak{k}$ and $\mathfrak{n}$, as follows:

$$a = \pi_k a + \pi_n a$$

$$= \left\{ (a_- - \mathcal{J}a_+) + \frac{1}{2} (a_0 - \mathcal{J}a_0) \right\} + \left\{ (a_+ + \mathcal{J}a_+) + \frac{1}{2} (a_0 + \mathcal{J}a_0) \right\}.$$

The following splitting, with

$$\mathfrak{k}_+ = \mathcal{D}_+ + \mathcal{D}_0^- \quad \text{and} \quad \mathfrak{n}_+ = \mathfrak{n},$$

will also be used in section 2; the projections take on the following form,

$$a = \pi_{k+} a + \pi_{n+} a$$

$$= \left\{ (a_- - \mathcal{J}a_-) + \frac{1}{2} (a_0 - \mathcal{J}a_0) \right\} + \left\{ (a_+ + \mathcal{J}a_-) + \frac{1}{2} (a_0 + \mathcal{J}a_0) \right\}.$$

---

12 Note $\mathfrak{n}$ is the fixed point set of $\mathcal{J}$.
13 $\mathcal{G}_k$ is the group of invertible elements in $\mathfrak{k}$, i.e., invertible lower-triangular matrices, with non-zero $2 \times 2$ blocks proportional to Id along the diagonal.
Note $\mathcal{J}$ intertwines $\pi_k$ and $\pi_{k^+}$:

$$\mathcal{J}\pi_k = \pi_{k^+}\mathcal{J}. \quad (1.20)$$

For a skew-symmetric semi-infinite matrix $m_\infty$, the skew-Borel decomposition

$$m_\infty := Q^{-1}JQ^{-1\top} \text{ with } Q \in \mathcal{G}_k, \quad (1.21)$$

is unique, as was shown in [5]. Here we may assume $m_\infty$ to be bi-infinite, as long as the factorization (1.21) is unique, upon imposing a suitable normalization. Then we use $Q$ to dress up $\Lambda$:

$$L = QAQ^{-1}.$$

Then letting $m_\infty$ run according to the equations $\partial m/\partial t_i = \Lambda^i m + m\Lambda^{\top}_i$, we show in the next proposition and corollary that $L$ evolves according to a system of commuting equations, which by virtue of the AKS theorem are Hamiltonian vector fields; for details, see [5].

**Proposition 1.2** For the matrices

$$m_\infty := Q^{-1}JQ^{-1\top} \text{ and } L := QAQ^{-1} \text{, with } Q \in \mathcal{G}_k,$$

the following three statements are equivalent

(i) $\frac{\partial Q}{\partial t_i}Q^{-1} = -\pi_kL^i$

(ii) $L^i + \frac{\partial Q}{\partial t_i}Q^{-1} \in \mathfrak{n}$

(iii) $\frac{\partial m}{\partial t_i} = \Lambda^i m + m\Lambda^{\top}_i$.

Whenever the vector fields on $Q$ or $m$ satisfy (i), (ii) or (iii), then the matrix $L = QAQ^{-1}$ is a solution of the AKS-Lax pair

$$\frac{\partial L}{\partial t_i} = [-\pi_k L^i, L] = [\pi_n L^i, L].$$

**Proof:** Written out and using (1.18), proposition 1.2 amounts to showing the equivalence of the three formulas:

(I) $\frac{\partial Q}{\partial t_i}Q^{-1} + \left( (L^i)_- - J(L^i)^\top J \right) + \frac{1}{2} \left[ (L^i)_0 - J((L^i)_0)^\top J \right] = 0$
(II) \( \left( L^i + \frac{\partial Q}{\partial t_i} Q^{-1} \right) - J \left( L^i + \frac{\partial Q}{\partial t_i} Q^{-1} \right)^T J = 0 \)

(III) \( \Lambda^i m + m \Lambda^T - \frac{\partial m}{\partial t_i} = 0. \)

The point is to show that

\[
(I)_{+} = 0, \quad (I)_{-} = (II)_{-} = - J (II)_{+}^T J, \quad (I)_0 = \frac{1}{2} (II)_0, \]

\[
Q^{-1}(II)JQ^{-1T} = (III). \quad (1.22)
\]

The reader will find the details of this proof in [5].

2 Wave functions and their bilinear equations for the Pfaff Lattice

Consider the commuting vector fields

\[
\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^T i
\]

on the skew-symmetric matrix \( m_\infty(t) \) and the skew-Borel decomposition

\[
m_\infty(t) = Q^{-1}(t) J Q^{-1T}(t), \quad Q(t) \in \mathcal{G}_k; \quad (2.1)
\]

remember from (1.17), \( Q(t) \in \mathcal{G}_k \) means: \( Q(t) \) is lower-triangular, with along the “diagonal” \( 2 \times 2 \) matrices \( c_{2n} I \), with \( c_{2n} \neq 0 \).

In this section, we give the properties of the wave vectors and their bilinear relations. Upon setting

\[
Q_1 = Q(t) \quad \text{and} \quad Q_2 = JQ^{-1T}(t), \quad (2.2)
\]

the matrix \( Q(t) \) defines wave operators

\[
W_1(t) = Q_1(t)e^{\sum_{i=1}^{\infty} t_i \Lambda^i}, \quad W_2(t) = Q_2(t)e^{-\sum_{i=1}^{\infty} t_i \Lambda^T i} = JW_1^{-1T}(t), \quad (2.3)
\]

\( L \)-matrices

\[
L := L_1 := Q_1 \Lambda Q_1^{-1}, \quad L_2 := -J(L_1) = Q_2 \Lambda^T Q_2^{-1}, \quad (2.4)
\]
and wave and dual wave vectors
\[
\begin{align*}
\Psi_1(t, z) &= W_1(t)\chi(z) \quad \Psi_1^\dagger(t, z) = W_1^{-1}(t)^\top \chi(z^{-1}) = -J\Psi_2(t, z^{-1}) \\
\Psi_2(t, z) &= W_2(t)\chi(z) \quad \Psi_2^\dagger(t, z) = W_2^{-1}(t)^\top \chi(z^{-1}) = J\Psi_1(t, z^{-1}).
\end{align*}
\] (2.5)

From the definition, it follows that the wave functions \(\Psi_1\) have the following asymptotics
\[
\begin{align*}
\Psi_{1,2n}(t, z) &= e^{\sum t_k z^k z^{2n} c_{2n}(t)} \psi_{1,2n}(t, z), \quad \psi_{1,2n} = 1 + O(z^{-1}) \\
\Psi_{1,2n+1}(t, z) &= e^{\sum t_k z^k z^{2n+1} c_{2n+1}(t)} \psi_{1,2n+1}(t, z), \quad \psi_{1,2n+1} = 1 + O(z^{-2})
\end{align*}
\]
where the \(c_i\) are the elements of the diagonal part of \(Q\).

Theorem 2.1 The \(Q_i, L_i\) and \(\Psi_i\) satisfy the equations
\[
\begin{align*}
\frac{\partial Q_1}{\partial t_i} &= -(\pi_k L_i^i) Q_1 \quad \frac{\partial Q_2}{\partial t_i} = -(J(\pi_k L_i^i)) Q_2 = (\pi_{k+} L_i^i) Q_2 \quad (2.7) \\
\frac{\partial L_1}{\partial t_i} &= -[\pi_k L_1^i, L_1] \quad \frac{\partial L_2}{\partial t_i} = [\pi_k L_2^i, L_2] \quad (2.8) \\
L_1 \Psi_1 &= z \Psi_1 \quad L_2 \Psi_2 = z^{-1} \Psi_2, \quad (2.9) \\
\frac{\partial \Psi_1}{\partial t_i} &= (\pi_n L_i^i) \Psi_1 \quad \frac{\partial \Psi_2}{\partial t_i} = -(L_2^i - \pi_{k+} L_i^i) \Psi_2 = -(\pi_{n+} L_i^i) \Psi_2, \quad (2.10)
\end{align*}
\]
with \(\Psi_i\) satisfying the following bilinear identity for all \(n, m \in \mathbb{Z}\),
\[
\oint_0^\infty \Psi_{1,n}(t, z) \Psi_{2,m}(t', z^{-1}) \frac{dz}{2\pi i z} + \oint_0^\infty \Psi_{2,n}(t, z) \Psi_{1,m}(t', z^{-1}) \frac{dz}{2\pi i z} = 0. \quad (2.11)
\]

For later use, we shall also consider the “monic” wave functions, with the factors \(c_{2n}(t)\) removed, i.e.,
\[
\hat{\Psi}_1(t, z) := Q_0^{-1} \Psi_1 \quad \text{and} \quad \hat{\Psi}_2(t, z) := Q_0 \Psi_2 \quad (2.12)
\]
and the matrix \(\hat{L}_1\), normalized so as to have 1’s above the main diagonal, with \(Q := Q_0^{-1} Q\).
\[ \hat{L}_1 = Q_0^{-1}L_1Q_0 = (Q_0^{-1}Q)\Lambda(Q_0^{-1}Q)^{-1} = \hat{Q}\Lambda\hat{Q}^{-1}, \]
\[ \hat{L}_2 = Q_0L_2Q_0^{-1} = -Q_0\mathcal{J}(L_1)Q_0^{-1} = -\mathcal{J}(\hat{L}_1) \] (2.13)

Then, in terms of the elements \( \hat{q}_{ij} \) of the matrix \( \hat{Q} := Q_0^{-1}Q \), one easily computes by conjugation, that \( \hat{L}_1 \) has the following block structure:

\[ \hat{L}_1 = Q_0^{-1}L_1Q_0 = (Q_0^{-1}Q)\Lambda(Q_0^{-1}Q)^{-1} = \begin{pmatrix} \vdots & \hat{L}_{00} & \hat{L}_{01} & 0 & 0 \\ \hat{L}_{10} & \hat{L}_{11} & \hat{L}_{12} & 0 \\ \ast & \hat{L}_{21} & \hat{L}_{22} & \hat{L}_{23} \\ \ast & \ast & \hat{L}_{32} & \hat{L}_{33} & \vdots \end{pmatrix}, \]

with

\[ \hat{L}_{ii} := \begin{pmatrix} \hat{q}_{2i,2i-1} & 1 \\ \hat{q}_{2i+1,2i-1} - \hat{q}_{2i+2,2i} & -\hat{q}_{2i+2,2i+1} \end{pmatrix}, \quad \hat{L}_{i,i+1} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

\[ \hat{L}_{i+1,i} := \begin{pmatrix} \ast & -\hat{q}_{2i+2,2i+1} - \hat{q}_{2i+3,2i+1} + \hat{q}_{2i+2,2i} \\ \ast & \ast \end{pmatrix}. \] (2.14)

**Theorem 2.2** \( \hat{L}_i, \hat{Q}, \hat{\Psi}_1, \hat{\Psi}_2 \) satisfy the following equations:

\[ \frac{\partial \hat{Q}}{\partial t_n} = -((\hat{L}_1^n)_- - Q_0^{-2}\mathcal{J}((\hat{L}_1^n)_+)^\top JQ_0^2) \hat{Q}, \] (2.15)

and

\[ \hat{L}_1\hat{\Psi}_1 = z\hat{\Psi}_1 \quad \hat{L}_2\hat{\Psi}_2 = z^{-1}\hat{\Psi}_2, \] (2.16)

with

\[ \frac{\partial}{\partial t_n} \hat{\Psi}_1(t,z) = ((\hat{L}_1^n)_+ + (\hat{L}_1^n)_0 + Q_0^{-2}\mathcal{J}((\hat{L}_1^n)_+)Q_0^2) \hat{\Psi}_1(t,z), \]

\[ \frac{\partial}{\partial t_n} \hat{\Psi}_2(t,z) = \mathcal{J}((\hat{L}_1^n)_+ + (\hat{L}_1^n)_0 + Q_0^{-2}\mathcal{J}((\hat{L}_1^n)_+)Q_0^2) \hat{\Psi}_2(t,z) \]

\[ = -((\hat{L}_2^n)_- + (\hat{L}_2^n)_0 + Q_0^2\mathcal{J}((\hat{L}_2^n)_-)Q_0^{-2}) \hat{\Psi}_2(t,z). \]
The proof of Theorem 2.1 hinges on the following matrix version of the bilinear identities:

**Lemma 2.3** The matrices $W_1(t)$ and $W_2(t)$, defined in (2.3), satisfy

$$W_1(t)W_1(t')^{-1} = W_2(t)W_2(t')^{-1}. \tag{2.17}$$

**Proof:** The solution to the equation (2.0) is given by

$$m_{\infty}(t) = e^{\sum t_k A_k} m_{\infty}(0) e^{\sum t_k A_k^T}.$$  

Therefore skew-Borel decomposing $m_{\infty}(t)$ and $m_{\infty}(0)$, we find

$$Q^{-1}(0)JQ^{-1}(t)JQ^{-1}(t) = e^{-\sum_{t_i}(t)\Lambda_i}$$

and so, from the definition of $W_1$ and $W_2$,

$$W_1^{-1}(0)W_2(0) = Q^{-1}(0)JQ^{-1}(0)$$

implies the independence in $t$ of the right hand side of (2.19). Therefore, we have

$$W_1(t)^{-1}W_2(t) = W_1(t')^{-1}W_2(t'), \quad \text{for all } t, t' \in C^\infty,$$

and so

$$W_1(t)W_1^{-1}(t') = W_2(t)W_2^{-1}(t').$$

**Proof of Theorem 2.1:** The proof of equation (2.7) for $Q_1$, namely

$$\frac{\partial Q_1}{\partial t_i} = -(\pi_k L_i)Q_1,$$

follows at once from Proposition 1.2.
The proof of (2.7) for $Q_2 = JQ_1^\top - 1$ is based on the identity $\mathcal{J}\pi_k a = \pi_k \mathcal{J} a$. Indeed, we compute
\[
\frac{\partial Q_2}{\partial t_i} Q_2^{-1} = -JQ_1^\top Q_1^\top Q_2^{-1} = -JQ_1^\top Q_1^\top (\pi_k L_i^1) Q_1^\top Q_2^{-1} = -J(\pi_k L_i^1),
\]
using (2.4),
\[
\frac{\partial Q_2}{\partial t_i} = -\pi_k \mathcal{J} L_i^1,
\]
\[
\frac{\partial Q_2}{\partial t_i} = -\pi_k \mathcal{J} (J L_2^i)^i, \text{ using (2.4)},
\]
\[
\frac{\partial Q_2}{\partial t_i} = -\pi_k \mathcal{J} (-1)^i (J L_2^i)^i,
\]
\[
\frac{\partial Q_2}{\partial t_i} = \pi_k L_i^2.
\]
Equations (2.8) and (2.10) for $L_1$, $L_2$ and $\Psi_1$, $\Psi_2$ are then straightforward.

Finally, the proof of the bilinear identity (2.11) proceeds as follows: By a well-known lemma (see [?]),
\[
W_1 W_1 (t) W_1 (t')^{-1} = \int_0^\infty \Psi_1 (t, z) \otimes \Psi_2 (t', z) \frac{dz}{2\pi i z}
\]
and so the statement of Lemma 2.3 yields
\[
\int_0^\infty \Psi_1 (t, z) \otimes \Psi_2 (t', z) \frac{dz}{2\pi i z} = \int_0^\infty \Psi_2 (t, z) \otimes \Psi_2 (t', z) \frac{dz}{2\pi i z},
\]
whose $(m, n)$th component is
\[
\int_0^\infty \psi_{1,n} (t, z) \psi_{1,m} (t', z) \frac{dz}{2\pi i z} - \int_0^\infty \psi_{2,n} (t, z) \psi_{2,m} (t', z) \frac{dz}{2\pi i z} = 0.
\]
Next we use the relations $\Psi_1^* (t, z) = -J\Psi_2 (t, z)$ and $\Psi_2^* (t, z) = J\Psi_1 (t, z)$, to yield
\[
\int_0^\infty \Psi_1 (t, z) \otimes J\Psi_2 (t', z^{-1}) \frac{dz}{2\pi i z} + \int_0^\infty \Psi_2 (t, z) \otimes J\Psi_1 (t', z^{-1}) \frac{dz}{2\pi i z} = 0,
\]
which again componentwise leads to (2.11).
Proof of Theorem 2.2: To prove (2.15), remember from Theorem 2.1, 
\[ \frac{\partial Q}{\partial t_n} Q^{-1} = -\pi_k L^n = -((L^n)_+ + J(L^n_+)^\top J) - \frac{1}{2}((L^n)_0 - J((L^n)_0)^\top J); \]

hence, taking the \((0)\)-part of this expression, yields

\[ \frac{\partial \log Q_0}{\partial t_n} = \left( \frac{\partial Q}{\partial t_n} Q^{-1} \right)_0 = -\pi_k (L^n)_0 = -\frac{1}{2}((L^n)_0 + J(L^n)^\top J). \]

Using the fact that \(Q_0, L^n, Q_0\), \(D_{0,\pm}, D_{\pm,0} \subset D_{\pm}\), we compute for \(Q = Q_0^{-1}L^nQ_0, \hat{L}_1 = Q_0^{-1}L_1Q_0, \) (see (2.13))

\[ \frac{\partial \hat{Q}}{\partial t_n} \hat{Q}^{-1} = -Q_0^{-1} \hat{L}_1 Q_0^{-1} L^n Q_0^{-1} Q_0 + Q_0^{-1} \hat{Q} Q^{-1} Q_0 \]
\[ = -Q_0^{-1} \hat{L}_1 Q_0^{-1} Q_0 + Q_0^{-1} \hat{Q} Q^{-1} Q_0 \]
\[ = Q_0^{-1} \left( -Q_0^{-1} \hat{Q} Q^{-1} \right) Q_0 \]
\[ = Q_0^{-1} \left( -(L^n)_+ + J(L^n_+)^\top J \right) Q_0 \]
\[ = -(Q_0^{-1}L^nQ_0)_+ + Q_0^{-1} J \left( Q_0(Q_0^{-1}L^nQ_0)_+Q_0^{-1} \right)^\top J Q_0 \]
\[ = -(\hat{L}_1)_+ + Q_0^{-2} J((\hat{L}_1)_+) Q_0. \]

Using this result and \(L_1 \hat{\Psi}_1(t,z) = z \hat{\Psi}_1(t,z)\), we find

\[ \frac{\partial \hat{\Psi}_1(t,z)}{\partial t_n} \]
\[ = \frac{\partial}{\partial t_n} e^{\sum t_i z_i} \hat{Q} \chi(z) \]
\[ = z^n e^{\sum t_i z_i} \hat{Q} \chi(z) + e^{\sum t_i z_i} \left( -(L^n)_+ + Q_0^{-2} J((L^n)_+) Q_0^2 \right) \hat{Q} \chi(z) \]
\[ = (\hat{L}_1)_+ + Q_0^{-2} J((\hat{L}_1)_+) Q_0^2 \hat{\Psi}_1(t,z) \]
\[ = (\hat{L}_1)_+ + (\hat{L}_1)_0 + Q_0^{-2} J((\hat{L}_1)_+) Q_0^2 \hat{\Psi}_1(t,z). \] (2.20)

But, we also have that \(\hat{\Psi}_1 = Q_0^{-1} \hat{\Psi}_1(t,z)\) and \(\hat{\Psi}_2 = Q_0 \hat{\Psi}_2(t,z)\) satisfy, using \(W_2 = J W_1^{-1}, W_1^{\top}\),

\[ \frac{\partial \hat{\Psi}_1(t,z)}{\partial t_n} = (Q_0^{-1}W_1) \chi(z) = (Q_0^{-1}W_1)(Q_0^{-1}W_1)^{-1} \hat{\Psi}_1(t,z) \] (2.21)
\[
\frac{\partial \hat{\Psi}_2(t,z)}{\partial t_n} = (Q_0W_2) \chi(z) = (Q_0W_2)(Q_0W_2)^{-1}(Q_0\Psi_2)
\]
\[
= (\dot{Q_0}W_2 + Q_0\dot{W}_2)W_2^{-1}Q_0^{-1}(Q_0\Psi_2)
\]
\[
= (\dot{Q_0}Q_0^{-1} + Q_0\dot{W}_2W_2^{-1}Q_0^{-1})Q_0\Psi_2
\]
\[
= (\dot{Q_0}Q_0^{-1} + Q_0\mathcal{J}(W_1W_1^{-1})Q_0^{-1})Q_0\Psi_2
\]
\[
= (\dot{Q_0}Q_0^{-1} + Q_0\mathcal{J}(W_1W_1^{-1})^\top JQ_0^{-1})Q_0\Psi_2
\]
\[
= (\mathcal{J}(Q_0^{-1}W_1)(Q_0^{-1}W_1)^{-1})(Q_0\Psi_2). \quad (2.22)
\]

Comparing (2.21), (2.22) and (2.23), and using
\[-\mathcal{J}(\hat{L}^n_1) = \hat{L}^n_2,
\]
and so, in particular,
\[-\mathcal{J}(\hat{L}^n_1)_\pm = (\hat{L}^n_2)_\mp \quad \text{and} \quad -\mathcal{J}(\hat{L}^n_1)_0 = (\hat{L}^n_2)_0,
\]
\[
\frac{\partial \hat{\Psi}_2(t,z)}{\partial t_n} = -((\hat{L}^n_2)_- + (\hat{L}^n_2)_0 + Q_0^2\mathcal{J}(L_2)_-Q_0^{-2})\hat{\Psi}_2(t,z),
\]
which establishes theorem 2.2. \hfill \blacksquare

## 3 Existence of the Pfaff \(\tau\)-function

The point of this section is to show that the solution of the Pfaff Lattice can be expressed in terms of a sequence of functions \(\tau\), which are not \(\tau\)-functions in the usual sense, but enjoys a different set of bilinear identities and partial differential equations.

**Proposition 3.1** There exists functions \(\tau_{2n}(t)\) such that
\[
\psi_{1,2n}(t,z) = \frac{\tau_{2n}(t - [z^{-1}])}{\tau_{2n}(t)} \quad \text{and} \quad \psi_{2,2n}(t,z) = \frac{\tau_{2n+2}(t + [z])}{\tau_{2n+2}(t)}. \quad (3.1)
\]

The proof of proposition 3.1 will be postponed until later. For future use, we define the diagonal matrix
\[
h = \text{diag}(..., h_{-2}, h_{-2}, h_0, h_0, h_2, h_2, ...) \in \mathcal{D}_0^-, \quad \text{with} \quad h_{2n} = \frac{\tau_{2n+2}}{\tau_{2n}}. \quad (3.2)
\]
Theorem 3.2

\[
\begin{align*}
\Psi_{1,2n}(t, z) &= e^{\sum t_i z^i 2^n} \frac{\tau_{2n}(t - [z^{-1}])}{\sqrt{\tau_2(t)\tau_{2n+2}(t)}} \\
\Psi_{1,2n+1}(t, z) &= e^{\sum t_i z^i 2^n (z + \partial/\partial t_1) \tau_{2n}(t - [z^{-1}])} \\
\Psi_{2,2n}(t, z) &= e^{-\sum t_i z^{-i} 2^n+1} \frac{\tau_{2n+2}(t + [z])}{\sqrt{\tau_2(t)\tau_{2n+2}(t)}} \\
\Psi_{2,2n+1}(t, z) &= e^{-\sum t_i z^{-i} 2^n+1} \frac{(z^{-1} - \partial/\partial t_1) \tau_{2n+2}(t + [z])}{\sqrt{\tau_2(t)\tau_{2n+2}(t)}}
\end{align*}
\]

(3.3)

with the \(\tau_{2n}(t)\) satisfying the following bilinear identity

\[
\oint_{z=\infty} \tau_{2n}(t - [z^{-1}]) \tau_{2n+2}(t' + [z^{-1}]) e^{\sum (t_i - t'_i) z^i 2^n-2m-2} \frac{dz}{2\pi i} + \oint_{z=0} \tau_{2n+2}(t + [z]) \tau_{2m}(t' - [z]) e^{\sum (t'_i - t_i) z^{-i} 2^n-2m} \frac{dz}{2\pi i} = 0.
\]

(3.4)

Then \(L\) has the following representation in terms of the Pfaffian \(\tau\)-functions:

\[
h^{1/2} L h^{-1/2} = \begin{pmatrix}
\vdots & \hat{L}_{00} & 0 & 0 \\
\hat{L}_{10} & \hat{L}_{11} & \hat{L}_{12} & 0 \\
* & \hat{L}_{21} & \hat{L}_{22} & \hat{L}_{23} \\
* & * & \hat{L}_{32} & \hat{L}_{33} & \ddots
\end{pmatrix}
\]

with \((= \frac{\partial}{\partial t_1})\)

\[
\hat{L}_{nn} := \begin{pmatrix}
-(\log \tau_{2n}) & 1 \\
\frac{s(\hat{t})\tau_{2n}}{\tau_{2n}} & \frac{s(\hat{-t})\tau_{2n+2}}{\tau_{2n+2}}
\end{pmatrix} \quad \hat{L}_{n,n+1} := \begin{pmatrix} 0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\]

(3.5)
The following bilinear relations are due to [6]:

**Corollary 3.3** The functions \( \tau_{2n}(t) \) satisfy the following “differential Fay identity”\[14\]

\[
\{\tau_{2n}(t - [u]), \tau_{2n}(t - [v])\} \\
+ (u^{-1} - v^{-1})(\tau_{2n}(t - [u])\tau_{2n}(t - [v]) - \tau_{2n}(t)\tau_{2n}(t - [u] - [v])) \\
= uv(u - v)\tau_{2n-2}(t - [u] - [v])\tau_{2n+2}(t), \tag{3.6}
\]

and Hirota type bilinear equations, always involving nearest neighbours:

\[
\left( p_{k+4}(\bar{\partial}) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau_{2n} \circ \tau_{2n} = p_k(\bar{\partial}) \tau_{2n+2} \circ \tau_{2n-2} \tag{3.7}
\]

\[k, n = 0, 1, 2, \ldots .\]

**Lemma 3.4** Consider an arbitrary function \( \varphi(t, z) \) depending on \( t \in \mathbb{C}^\infty, \ z \in \mathbb{C} \), having the asymptotics \( \varphi(t, z) = 1 + O\left(\frac{1}{z}\right) \) for \( z \rightarrow \infty \) and satisfying the functional relation

\[
\frac{\varphi(t - [z^{-1}], z_1)}{\varphi(t, z_1)} = \frac{\varphi(t - [z^{-1}], z_2)}{\varphi(t, z_2)}, \quad t \in \mathbb{C}^\infty, z \in \mathbb{C}.
\]

Then there exists a function \( \tau(t) \) such that

\[
\varphi(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)}.
\]

**Proof:** See appendix (Section 10)

**Lemma 3.5** The following holds for the Pfaffian wave function \( \Psi_1 \) and \( \Psi_2 \), as in (2.6),

\[
\frac{\psi_{1,2n}(t - [z_2^{-1}], z_1)}{\psi_{1,2n}(t, z_1)} = \frac{\psi_{1,2n}(t - [z_1^{-1}], z_2)}{\psi_{1,2n}(t, z_2)} \tag{3.8}
\]

and

\[
\psi_{2,2n-2}(t - [z^{-1}], z^{-1})\psi_{1,2n}(t, z) = 1. \tag{3.9}
\]

\[14\{f, g\} = f'g - fg', \text{ where } f' = \partial / \partial t_1.\]
Proof: Setting (2.6) in the bilinear equation (2.11), with \( n \mapsto 2n, \ m \mapsto 2n - 2 \), yields

\[
\frac{c_{2n}(t)}{c_{2n-2}(t)} \int_{-\infty}^{\infty} e^{\sum (t_{i} - t'_{j})z} \psi_{1,2n}(t, z) \psi_{2,2n-2}(t', z^{-1}) \frac{dz}{2\pi i} \\
+ \frac{c_{2n-2}(t)}{c_{2n}(t)} \int_{0}^{\infty} e^{\sum (t' - t)z} \psi_{2,2n}(t, z) \psi_{1,2n-2}(t', z^{-1}) \frac{dz}{2\pi i} = 0.
\]

Setting

\[
t - t' = [z_{1}^{-1}] + [z_{2}^{-1}]
\]

in the above and using \( e^{\sum_{i=1}^{\infty} x^i/i} = 1/(1 - x) \) yields

\[
\frac{c_{2n}}{c_{2n-2}} \int_{-\infty}^{\infty} \psi_{1,2n}(t, z) \psi_{2,2n-2}(t', z^{-1}) \frac{dz}{(1 - z_{1}^{-1})(1 - z_{2}^{-1})/2\pi i} \\
= -\frac{c_{2n-2}}{c_{2n}} \int_{z=0}^{\infty} z^{2} \left( \frac{1 - z}{z_{1}} \right) \left( \frac{1 - z}{z_{2}} \right) \psi_{2,2n}(t, z) \psi_{1,2n-2}(t', z^{-1}) \frac{dz}{2\pi i}.
\]

Since the integrand on the right hand side is holomorphic, it suffices to evaluate the integral on the left hand side, which can be viewed as an integral along a contour encompassing \( \infty \) and the points \( z_{1} \) and \( z_{2} \), thus leading to

\[
\psi_{1,2n}(t, z_{1}) \psi_{2,2n-2}(t - [z_{1}^{-1}] - [z_{2}^{-1}], z_{1}^{-1}) = \psi_{1,2n}(t, z_{2}) \psi_{2,2n-2}(t - [z_{1}^{-1}] - [z_{2}^{-1}], z_{2}^{-1})
\]

(3.10)

with

\[
\psi_{1,2n}(t, z) = 1 + O\left(z^{-1}\right), \quad \psi_{2,2n-2}(t - [z_{1}^{-1}] - [z_{2}^{-1}], z^{-1}) = 1 + O(z^{-2}).
\]

Therefore, letting \( z_{2} \rightarrow \infty \), one finds

\[
\psi_{1,2n}(t, z_{1}) \psi_{2,2n-2}(t - [z_{1}^{-1}], z_{1}^{-1}) = 1,
\]

(3.11)
yielding (3.9), and so, upon shifting \( t \mapsto t - [z_{2}^{-1}] \),

\[
\psi_{2,2n-2}(t - [z_{1}^{-1}] - [z_{2}^{-1}], z_{1}^{-1}) = \frac{1}{\psi_{1,2n}(t - [z_{2}^{-1}], z_{1})};
\]

similarly

\[
\psi_{2,2n-2}(t - [z_{1}^{-1}] - [z_{2}^{-1}], z_{2}^{-1}) = \frac{1}{\psi_{1,2n}(t - [z_{1}^{-1}], z_{2})}.
\]

(3.12)

Setting the two expressions (3.12) in (3.10) yields

\[
\frac{\psi_{1,2n}(t - [z_{2}^{-1}], z_{1})}{\psi_{1,2n}(t, z_{1})} = \frac{\psi_{1,2n}(t - [z_{1}^{-1}], z_{2})}{\psi_{1,2n}(t, z_{2})}.
\]

24
Proof of Proposition 3.1: From Lemmas 3.4 and 3.5, there exists, for each $2n$, a function $\tau_{2n}$ such that the first relation of (3.1) is satisfied, i.e.,

$$\psi_{1,2n}(t, z) = \frac{\tau_{2n}(t - [z^{-1}])}{\tau_{2n}(t)},$$

and so from (3.9)

$$\psi_{2,2n-2}(t - [z^{-1}], z^{-1}) = \frac{1}{\psi_{1,2n}(t, z)} = \frac{\tau_{2n}(t)}{\tau_{2n}(t - [z^{-1}]},$$

thus leading to

$$\psi_{2,2n-2}(t, z) = \frac{\tau_{2n}(t + [z])}{\tau_{2n}(t)},$$

which is the second relation of (3.1).

Proof of Theorem 3.2: At first, remembering that $\hat{Q} = Q_0^{-1}Q$, observe that

$$e^{\sum t_i z^i ((\hat{Q}) \chi(z))_{2n}} = (Q_0^{-1} \Psi_1(t, z))_{2n}$$

$$= e^{\sum t_i z^i 2^n \psi_{1,2n}(t, z)}$$

$$= e^{\sum t_i z^i 2^n \frac{\tau_{2n}(t - [z^{-1}])}{\tau_{2n}(t)}}$$

$$= e^{\sum t_i z^i \frac{2^n \left(1 + \sum_{n=1}^{\infty} S_k(-\tilde{\partial}) \tau_{2n}(t)\right)}{\tau_{2n}(t)}},$$

showing that a few subdiagonals of the matrix $\hat{Q}$ are given by

$$\hat{Q} = \begin{pmatrix}
\ddots & \ddots \\
& 1 & 0 \\
0 & \ddots & 1 \\
& \hat{q}_{2n,2n-2} & \hat{q}_{2n,2n-1} \\
& \hat{q}_{2n+1,2n-2} & \hat{q}_{2n+1,2n-1} & \ddots & \ddots \\
\end{pmatrix},$$

with

$$\hat{q}_{2n,2n-1} = -\frac{\partial}{\partial t_1} \log \tau_{2n}, \quad \hat{q}_{2n,2n-2} = \frac{S_2(-\tilde{\partial}) \tau_{2n}}{\tau_{2n}}.$$

(3.13)
Remembering that

\[
\hat{\Psi}_{1,2n}(t, z) = e^{t \sum t_k z^k} \psi_{1,2n}(t, z), \quad \psi_{1,2n} = 1 + O(z^{-1})
\]

\[
\hat{\Psi}_{1,2n+1}(t, z) = e^{t \sum t_k z^k} \psi_{1,2n+1}(t, z), \quad \psi_{1,2n+1} = 1 + O(z^{-2})
\]  

(3.14)

\[
\hat{\Psi}_{2,2n}(t, z) = e^{-t \sum t_k z^{-k}} \psi_{2,2n}(t, z), \quad \psi_{2,2n} = 1 + O(z)
\]

\[
\hat{\Psi}_{2,2n+1}(t, z) = e^{-t \sum t_k z^{-k}} \psi_{2,2n+1}(t, z), \quad \psi_{2,2n+1} = 1 + O(z^2),
\]  

(3.15)

we now compute, using theorem 2.2,

\[
e^{t \sum t_i z^i} \left( \frac{\partial}{\partial t_1} + z \right) z^{2n} \psi_{1,2n}(t, z)
\]

\[
= \left( \frac{\partial}{\partial t_1} \hat{\Psi}_{1}(t, z) \right)_{2n}
\]

\[
= \left( \left( (\hat{L}_{1})_+ + (\hat{L}_{1})_0 + Q_0^{-2} J(\hat{L}_{1+})^\top J Q_0^2 \right) \hat{\Psi}_{1}(t, z) \right)_{2n} \tag{3.16}
\]

and

\[
-e^{t \sum t_i z^{-i}} \left( \frac{\partial}{\partial t_1} - \frac{1}{z} \right) z^{2n+1} \psi_{2,2n}(t, z)
\]

\[
= \frac{\partial}{\partial t_1} (\hat{\Psi}_{2}(t, z))_{2n}
\]

\[
= \left( \left( J((\hat{L}_{1})_+ + (\hat{L}_{1})_0 + Q_0^{-2} J(\hat{L}_{1+})^\top J Q_0^2) \right) \hat{\Psi}_{2}(t, z) \right)_{2n}. \tag{3.17}
\]

In this expression, the matrix equals, according to (2.13),

\[
(\hat{L}_{1})_+ + (\hat{L}_{1})_0 + Q_0^{-2} J(\hat{L}_{1+})^\top J Q_0^2 =
\]

\[
\begin{pmatrix}
\vdots \\
\hat{q}_{0,-1} \\
\hat{q}_{1,-1} - \hat{q}_{20} & 1 & 0 & 0 & 0 \\
0 & \hat{q}_{21} & 1 & 0 & 0 \\
\hat{q}_{31} - \hat{q}_{42} & -\hat{q}_{43} & 1 & 0 & \hat{q}_{53} - \hat{q}_{64} & -\hat{q}_{65} & \ldots \\
0 & 0 & \hat{q}_{53} - \hat{q}_{64} & -\hat{q}_{65} & \ldots & \vdots
\end{pmatrix}
\]

and, acting with \( J \) on this matrix,
\[ \mathcal{J} \left( (\hat{L}_1)_+ + (\hat{L}_1)_0 + Q_0^2 J(\hat{L}_1_+)^\top JQ_0^2 \right) = \]
\[
\begin{pmatrix}
\vdots \\
-\hat{q}_{21} & 1 & 0 & 0 & 0 & 0 \\
\hat{q}_{1,-1} - \hat{q}_{20} & \hat{q}_{0,-1} & c_n^2/c_2 & 0 & 0 & 0 \\
0 & 0 & -\hat{q}_{43} & 1 & 0 & 0 \\
1 & 0 & \hat{q}_{31} - \hat{q}_{42} & \hat{q}_{21} & c_2^2/c_2 & 0 \\
0 & 0 & 0 & 0 & -\hat{q}_{53} & \hat{q}_{43} \\
0 & 0 & 0 & 1 & 0 & \hat{q}_{53} - \hat{q}_{64} & \hat{q}_{43} \\
\vdots 
\end{pmatrix}
\]

using the fact that
\[
\mathcal{J} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Therefore the 2n-th rows of both matrices respectively have the form
\[
(0, \ldots, 0, \hat{q}_{2n,2n-1}(t), 1, 0, 0, \ldots)_{\uparrow 2n}
\]

and thus from (3.16) and (3.17), and the expansions (3.14) and (3.15), we have
\[
\left( \frac{\partial}{\partial t_1} + z \right) \psi_{1,2n}(t, z) = \hat{q}_{2n,2n-1}(t) \psi_{1,2n} + z \psi_{1,2n+1}
\]

\[
\left( \frac{\partial}{\partial t_1} - z^{-1} \right) \psi_{2,2n}(t, z) = \hat{q}_{2n+2,2n+1}(t) \psi_{2,2n} + z^{-1} \psi_{2,2n+1} \quad (3.18)
\]

and so, using the expression (3.13) for \( \hat{q}_{2n,2n-1} \) and the first expression (3.1),
\[
z^{2n+1} \psi_{1,2n+1}(t, z)
\]

\[
= \left( z + \frac{\partial}{\partial t_1} \right) z^{2n} \psi_{1,2n}(t, z) - \hat{q}_{2n,2n-1}(t) z^{2n} \psi_{1,2n}(t, z)
\]

\[
= \left( z + \frac{\partial}{\partial t_1} \right) z^{2n} \psi_{1,2n}(t, z) + \left( \frac{\partial}{\partial t_1} \log \tau_{2n}(t) \right) z^{2n} \psi_{1,2n}(t, z)
\]

27
\[
\begin{align*}
\psi_{1,2n+1}(t,z) &= 1 + \frac{1}{\tau_{2n}(t)} \left( S_2(-\bar{\partial}) \tau_{2n} z^{-2} + O(z^{-3}) \right) \\
\end{align*}
\]

thus
\[
\begin{align*}
\dot{q}_{2n+1,2n} = 0, \quad \dot{q}_{2n+1,2n-1} = \frac{1}{\tau_{2n}} \left( S_2(-\bar{\partial}) - \frac{\partial^2}{\partial t_1^2} \right) \tau_{2n} = -\frac{S_2(\bar{\partial}) \tau_{2n}}{\tau_{2n}}. \quad (3.21)
\end{align*}
\]

Setting \( n \mapsto 2n \) and \( m \mapsto 2n \) in the bilinear relation (2.11) and substituting, using (2.6) and the expressions for \( \psi_{1,2n}(t,z) \) and \( \psi_{2,2n}(t,z) \) in the proof of proposition 3.1,
\[
\Psi_{1,2n}(t,z) = e^{\sum t_k z^{-k}} \frac{\tau_{2n}(t - [z^{-1}])}{\tau_{2n}(t)}
\]
and
\[
\Psi_{2,2n}(t',z) = e^{-\sum t_k z^{-k}} \frac{\tau_{2n+2}(t + [z])}{\tau_{2n+2}(t)}
\]
into
\[
\int_{\infty}^{0} \Psi_{1,2n}(t,z)\Psi_{2,2n}(t',z^{-1}) \frac{dz}{2\pi iz} + \int_{0}^{\infty} \Psi_{2,2n}(t,z)\Psi_{1,2n}(t',z^{-1}) \frac{dz}{2\pi iz} = 0
\]
yields
\[
\begin{align*}
&\int_{\infty}^{0} \frac{c_{2n}(t)}{c_{2n}(t')} e^{\sum (t_k - t'_k) z^{-k}} \frac{\tau_{2n}(t - [z^{-1}]) \tau_{2n+2}(t' + [z^{-1}])}{\tau_{2n}(t) \tau_{2n+2}(t')} \frac{dz}{2\pi iz^2} = 0 \\
&= 0
\end{align*}
\]
\[
\frac{c_{2n}(t')}{c_{2n}(t)} \int_0^\infty e^{\sum (t_k'-t_k)z^{-k}} \frac{\tau_{2n+2}(t+[z])\tau_{2n}(t'-[z])}{\tau_{2n+2}(t)\tau_{2n}(t')} \frac{dz}{2\pi i}.
\]

Setting \( t' = t + [\alpha] \) amounts to replacing the exponential:
\[
e^{\sum (t_k'-t_k)z^{-k}} = 1 - \alpha z, \quad e^{\sum (t_k'-t_k)(z^{-k})} = \frac{1}{1 - \alpha/z},
\]
so that the first integral has a simple pole at \( z = \infty \) and the second integral one at \( z = \alpha \). Evaluating the integrals yield
\[
-\alpha \frac{c_{2n}^2(t)}{c_{2n}^2(t')} \frac{\tau_{2n+2}(t)}{\tau_{2n}(t)} + \alpha = 0;
\]
i.e.,
\[
\left( e^{\sum \frac{\alpha}{1 + \alpha} \frac{\partial}{\partial t_i}} - 1 \right) c_{2n}^2(t) \frac{\tau_{2n+2}(t)}{\tau_{2n}(t)} = 0
\]
yielding the following relation, which involves a constant \( c_n \), independent of time,
\[
c_{2n}^2(t) = c_n \frac{\tau_{2n}(t)}{\tau_{2n+2}(t)} = c_n \cdot h_{2n}^{-1}(t). \quad (3.22)
\]

Rescaling \( \tau_{2n} \mapsto \tau_{2n}/(c_1 c_2 \cdots c_{n-1}) \) yields (3.3). Using the expressions for \( \psi_{1,2n}(t,z) \) and \( \psi_{2,2n}(t,z) \) (see the proof of proposition 3.1), (3.19), (3.20), (3.21), (2.6) and substituting (3.3) into (2.11) yields (3.4).

Finally to derive the form (3.5) of the matrix \( L \), set (3.13) and (3.20) in the elements just below the main diagonal of matrix (2.14), to yield \((\cdot = \partial/\partial t_1)\)
\[
-\dot{q}_{2n,2n-1} - \dot{q}_{2n+1,2n-1} + \dot{q}_{2n,2n-2} = -\left( \frac{\ddot{\tau}_{2n}}{\tau_{2n}} \right)^2 - \frac{S_2(-\ddot{\partial})}{\tau_{2n}} \frac{\tau_{2n}}{\tau_{2n}} + \frac{S_2(-\ddot{\partial})}{\tau_{2n}} \frac{\tau_{2n}}{\tau_{2n}}
\]
\[
= \frac{\ddot{\tau}_{2n}}{\tau_{2n}} - \left( \frac{\dot{\tau}_{2n}}{\tau_{2n}} \right)^2
\]
\[
= (\log \tau_{2n})^{-2}
\]
and
\[
\dot{q}_{2n+1,2n-1} - \dot{q}_{2n+2,2n} = \frac{(S_2(-\ddot{\partial})}{\tau_{2n}} \frac{\tau_{2n}}{\tau_{2n}} + \frac{S_2(-\ddot{\partial})}{\tau_{2n+2}} \frac{\tau_{2n+2}}{\tau_{2n+2}}
\]
\[
= -\frac{S_2(\ddot{\partial})}{\tau_{2n}} - \frac{S_2(-\ddot{\partial})}{\tau_{2n+2}},
\]
concluding the proof of theorem 3.2, upon substituting these relations into (2.14).

4 Semi-infinite matrices \( m_\infty \), (skew-)orthogonal polynomials and matrix integrals

4.1 \( \partial m/\partial t_k = \Lambda^k m \), orthogonal polynomials and Hermitean matrix integrals.

For the sake of completeness and analogy, we add this subsection, which summarizes some of [2]. Consider a \( t \)-dependent weight \( \rho_t(dz) := e^{-V_t(z)}dz := e^{-\sum t_i z_i} \rho(dz) \) on \( \mathbb{R} \), as in (0.0) and the induced \( t \)-dependent measure

\[
e^{Tr(-V(X)+\sum t_i X^i)}dX,
\]

(4.1)

on the ensemble \( \mathcal{H}_n \) of Hermitean matrices, with Haar measure \( dX \); the latter can be decomposed into a spectral part (radial part) and an angular part:

\[
dX := \prod_{i=1}^n dX_{ii} \prod_{1 \leq i < j \leq n} (dR X_{ij} d\mathbb{R} X_{ij}) = \Delta^2(z) dz_1 \cdots dz_n \ dU,
\]

(4.2)

where \( \Delta(z) = \prod_{1 \leq i < j \leq n} (z_i - z_j) \) is the Vandermonde determinant. Here we form the following matrix integral

\[
\int_{\mathcal{H}_n} e^{Tr(-V(X)+\sum t_i X^i)}dX = c_n \int_{\mathbb{R}^n} \Delta^2(z) \prod_{i=1}^n \rho_t(dz_i).
\]

(4.3)

The weight \( \rho_t(dz) \) defines a (symmetric) inner product

\[
\langle f, g \rangle^s_y = \int f(z) g(z) \rho_t(dz)
\]

and so, the moments

\[
\mu_{ij}(t) := \langle z^i, z^j \rangle^s_y = \int_{\mathbb{R}} z^{k+\ell} e^{\sum t_i z^i} \rho(dz) = \mu_{i+\ell,j}(t)
\]

satisfy

\[
\frac{\partial \mu_{ij}}{\partial t_\ell} = \int_{\mathbb{R}} z^{i+j+\ell} e^{\sum t_k z^k} \rho(dz).
\]
Therefore the semi-infinite moment matrix $m_\infty = (\mu_{ij})_{i,j \geq 0}$ satisfies
\[
\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty = m_\infty \Lambda^i.
\] (4.4)

The point now is that the following integral can be expressed as a determinant of moments, namely
\[
n! \tau_n(t) = \int_{H_n} e^{-\text{Tr} V_i(X)} dX = \int_{R^n} \Delta^2(z) \prod_{k=1}^n \rho_t(dz_k)
\]
\[
= \int_{R^n} \sum_{\sigma \in S_n} \det(z_{\sigma(k)}^{\ell-1} z_{\sigma(k)}^{k-1}) \prod_{k=1}^n \rho_t(dz_k)
\]
\[
= \int_{R^n} \sum_{\sigma \in S_n} \det(z_{\sigma(k)}^{\ell+k-2}) \prod_{k=1}^n \rho_t(dz_{\sigma(k)})
\]
\[
= \sum_{\sigma \in S_n} \det(\int_R z_{\sigma(k)}^{\ell+k-2} \rho_t(dz_{\sigma(k)}))
\]
\[
= n! \det(\int_R z_{\sigma(k)}^{\ell+k-2} \rho_t(dz_{\sigma(k)}))
\]
\[
= n! \det(\mu_{ij})_{0 \leq i,j \leq n-1}
\]
is a $\tau$-function for the KP-equation; also in view of (4.4) and the upper-lower Borel decomposition (0.3) of $m_\infty$, the integrals form a vector of $\tau$-functions for the Toda lattice.

4.2 $\partial m/\partial t_k = \Lambda^k m + m \Lambda^{\top k}$, skew-orthogonal polynomials and symmetric and symplectic matrix integrals.

Consider a skew-symmetric semi-infinite matrix
\[m_\infty(t) = (\mu_{ij}(t))_{i,j \geq 0}, \text{ with } m_n(t) = (\mu_{ij}(t))_{0 \leq i,j \leq n-1},\]
satisfying
\[
\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty + m_\infty \Lambda^{\top n}.
\] (4.5)

Then we have shown in previous sections that, upon skew-Borel decomposing $m_\infty$, these equations ultimately imply the existence of functions $\tau(t)$ satisfying the bilinear equations (3.4). Remember also
\[
h(t) = \text{diag}(\ldots, h_{-2}, h_{-2}, h_0, h_0, h_2, h_2, \ldots) \in D_0^-, \text{ with } h_{2n}(t) = \frac{\tau_{2n+2}(t)}{\tau_{2n}(t)}.
\]

31
Here, we need the Pfaffian $pf(A)$ of a skew-symmetric matrix $A = (a_{ij})_{0 \leq i,j \leq n-1}$ for even $n$:

$$pf(A)dx_0 \wedge \cdots \wedge dx_{n-1} = \frac{1}{n!} \left( \sum_{0 \leq i < j \leq n-1} a_{ij} dx_i \wedge dx_j \right)^n,$$

$$= \frac{1}{2^{n/2}(n/2)!} \left( \sum_{\sigma} \varepsilon(\sigma)a_{i_0,i_1}a_{i_2,i_3} \cdots a_{i_{n-2},i_{n-1}} \right) dx_0 \wedge \cdots \wedge dx_{n-1},$$

(4.6)

so that $pf(A)^2 = \det A$. We now state the following theorem due to Adler-Horozov-van Moerbeke:

**Theorem 4.1** Consider a semi-infinite skew-symmetric matrix $m_{\infty}$, evolving according to (4.5); setting

$$\tau_{2n}(t) = pf(m_{2n}(t)), \quad \text{and} \quad h_{2n} = \frac{pf(m_{2n+2}(t))}{pf(m_{2n}(t))},$$

(4.7)

then the wave vector $\Psi_{1}$, defined by (3.3) is a sequence of polynomials, except for the exponential,

$$\Psi_{1,k}(t,z) = e^{\sum t_i z_i^i q_k(t,z)},$$

(4.8)

where the $q_k$’s are skew-orthonormal polynomials of the form (0.17) satisfying

$$\left( \langle q_i,q_j \rangle^s \right)_{0 \leq i,j < \infty} = J, \quad \text{with} \quad \langle y^i,z^j \rangle^s := \mu_{ij}.$$  \hspace{1cm} (4.9)

The matrix $Q$ defined by $q(z) = Q \chi(z)$ is the unique solution (modulo signs) to the skew-Borel decomposition of $m_{\infty}$:

$$m_{\infty}(t) = Q^{-1}JQ^{\top}-1, \quad \text{with} \quad Q \in k.$$  \hspace{1cm} (4.10)

The matrix $L = Q \Lambda Q^{-1}$, also defined by

$$zq(t,z) = Lq(t,z),$$

and the diagonal matrix $h$ satisfy the equations

$$\frac{\partial L}{\partial t_i} = [-\pi_k L^i, L], \quad \text{and} \quad h^{-1} \frac{\partial h}{\partial t_i} = 2\pi_k (L^i)_0.$$  \hspace{1cm} (4.11)

\footnote{In the formula below $(i_0,i_1,\ldots,i_{n-2},i_{n-1}) = \sigma(0,1,\ldots,n-1)$, where $\sigma$ is a permutation and $\varepsilon(\sigma)$ its parity.}
Sketch of proof: at first note that looking for skew-orthogonal polynomials is tantamount to the skew-Borel decomposition of $m_\infty$, so that (4.9) and (4.10) are equivalent. The skew-orthogonality of the polynomials (0.17) follows from expanding the determinants explicitly in terms of $z$-columns, upon using the expression for the pfaffian in terms of a column
\[
\sum_{0 \leq k \leq \ell - 1} (-1)^k a_{ki} pf(0, \ldots, \hat{k}, \ldots, \ell - 1) = pf(0, \ldots, \ell - 1, i).
\]
For details, see [5]. On the other hand, Theorem 3.2 gives $\Psi(t,z)$ and hence $Q$ in terms of $\tau_n(t)$ of (4.7). By the uniqueness of the decomposition (4.10), the two ways of arriving at $Q$, (0.16) and (3.3) must coincide.

Important remark: The polynomials (0.16) provide an explicit algorithm to perform the skew-Borel decomposition of the skew-symmetric matrix $m_\infty$. Namely, the coefficients of the polynomials $q_i$ provide the entries of the matrix $Q$. This fact will be used later in the examples.

Symmetric matrix integrals Here we shall focus on integrals of the type
\[
\int_{S_{2n}} e^{Tr (-V(X) + \sum_{i=1}^\infty t_i X_i)} dX,
\]
where $dX$ denotes Haar measure
\[
dX := \prod_{1 \leq i \leq j \leq n} d\mathcal{R}X_{ij} = |\Delta(z)| dz_1 \cdots dz_n \, dU,
\]
over the space $S_{2n}$ of symmetric matrices. As will appear below, the integral (4.12) leads to a skew-inner-product with weight $\rho_t(z)dz := e^{-V_t(z)}dz := e^{-V(z) + \sum_i t_i z^i}dz = e^{\sum_i t_i z^i} \rho(z)dz$ on an interval $\subseteq \mathbb{R}$, as in (0.1),
\[
\langle f(x), g(y) \rangle := \int_{\mathbb{R}^2} f(x)g(y) \varepsilon(x-y) \rho_t(dx) \rho_t(dy)
\]
and therefore skew-symmetric moments\(^16\)
\[
\mu_{ij}(t) = \int_{\mathbb{R}^2} x^i y^j \varepsilon(x-y) \rho_t(x) \rho_t(y) \, dx \, dy
\]
\(^{16}\varepsilon(x) = 1, \text{ for } x \geq 0 \text{ and } = -1, \text{ for } x < 0.\)
\[
\int \int_{x \geq y} (x^i y^j - x^j y^i) \rho_t(x) \rho_t(y) \, dx \, dy
\]
\[
= \int_R (F_j(x) G_i(x) - F_i(x) G_j(x)) \, dx. \tag{4.15}
\]

where
\[
F_i(x) := \int_{-\infty}^x y^i \rho_t(y) \, dy \quad \text{and} \quad G_i(x) := F'_i(x) = x^i \rho_t(x).
\]

By simple inspection, the moments \( \mu_{k\ell}(t) \) satisfy
\[
\frac{\partial \mu_{k\ell}}{\partial t} = \int \int_{\mathbb{R}^2} (x^{k+i} y^\ell + x^k y^{\ell+i}) \varepsilon(x - y) \rho_t(x) \rho_t(y) \, dx \, dy
\]
\[
= \mu_{k+i, \ell} + \mu_{k, \ell+i},
\]
and so \( m_\infty \) satisfies (4.5).

According to Mehta [13], the symmetric matrix integral can now be expressed in terms of the pfaffian, as follows, taking into account a constant \( c_{2n} \), coming from integrating the orthogonal group:

\[
\frac{1}{(2n)!} \int_{S_{2n}(E)} e^{Tr(-V(X)+\sum t_i X^i)} \, dX
\]
\[
= \frac{1}{(2n)!} \int_{\mathbb{R}^{2n}} |\Delta_{2n}(z)| \prod_{i=1}^{2n} \rho_t(z_i) \, dz_i
\]
\[
= \int_{-\infty < z_1 < z_2 < \cdots < z_{2n} < \infty} \det \begin{pmatrix} z^i_{j+1} \rho_t(z_{j+1}) \end{pmatrix}_{0 \leq i, j \leq 2n-1} \prod_{i=1}^{2n} \prod_{j=1}^{2n} \rho_t(z_{2j}) \, dz_{2j}
\]
\[
= \int_{-\infty < z_1 < z_2 < \cdots < z_{2n} < \infty} \prod_{k=1}^{n} \rho_t(z_{2k}) \, dz_{2k}
\]
\[
\det \begin{pmatrix} \int_{-\infty}^{z_{2j}} z^{i}_{1} \rho_t(z_1) \, dz_1, & z_{2j}^i, & \ldots, & \int_{z_{2n-2}}^{z_{2j}} z_{2n-1}^i \rho_t(z_{2n-1}) \, dz_{2n-1}, & z_{2n}^i \end{pmatrix}_{0 \leq i \leq 2n-1}
\]
\[
= \int_{-\infty < z_1 < z_2 < \cdots < z_{2n} < \infty} \prod_{k=1}^{n} \rho_t(z_{2k}) \, dz_{2k}
\]
\[
\det \begin{pmatrix} F_i(z_2), & z_{2}^i, & F_i(z_4) - F_i(z_2), & z_{4}^i, & \ldots, & F_i(z_{2n}) - F_i(z_{2n-2}), & z_{2n}^i \end{pmatrix}_{0 \leq i \leq 2n-1}
\]
\[
= \int_{-\infty < z_1 < z_2 < \cdots < z_{2n} < \infty} \prod_{i=1}^{n} \, dz_i \, \det \begin{pmatrix} F_i(z_2), & G_i(z_2), & \ldots, & F_i(z_{2n}), & G_i(z_{2n}) \end{pmatrix}_{0 \leq i \leq 2n-1},
\]

34
\[
\begin{align*}
&= \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{i=1}^{n} dy_i \ \det (F_i(y_1), G_i(y_1), \ldots, F_i(y_n), G_i(y_n))_{0 \leq i \leq 2n-1} \\
&= \det^{1/2} \left( \int_{\mathbb{R}} (G_i(y)F_j(y) - F_i(y)G_j(y)) dy \right)_{0 \leq i,j \leq 2n-1} \\
&= p_f(\mu_{ij})_{0 \leq i,j \leq 2n-1} \\
&= \tau_{2n}(t),
\end{align*}
\]

which is a Pfaffian \(\tau\)-function.

**Symplectic matrix integrals:** Here we shall concentrate on integrals of the type
\[
\int_{\mathcal{T}_{2n}} e^{2Tr (-V(X) + \sum_{i=1}^{\infty} t_i X^i)} dX,
\]
where \(dX\) denotes Haar measure\(^\text{17}\)

\[
dX = \prod_{1}^{N} dX_k \prod_{k \leq \ell} dX_{k\ell}^{(0)} dX_{k\ell}^{-(0)} dX_{k\ell}^{(1)} dX_{k\ell}^{-(1)},
\]
on the space \(\mathcal{T}_{2N}\) of self-dual \(N \times N\) Hermitean matrices, with quaternionic entries; the latter can be realized as the space of \(2N \times 2N\) matrices with entries \(X_{k\ell}^{(i)} \in \mathbb{C}\)

\[
\mathcal{T}_{2N} = \left\{ X = (X_{k\ell})_{1 \leq k,\ell \leq N}, \ X_{k\ell} = \begin{pmatrix} X_{k\ell}^{(0)} & X_{k\ell}^{(1)} \\ -X_{k\ell}^{(1)} & X_{k\ell}^{(0)} \end{pmatrix} \text{ with } X_{tk} = X_{kt}^\dagger \right\},
\]

A more exotic skew-symmetric matrix \(m_\infty\) satisfying (4.5) is given by the moments, with \(V(y,t) = e^{2(-V(y) + \sum t_\alpha y^\alpha)}\),

\[
\mu_{ij}(t) = \int_{\mathbb{R}} \{y^i, y^j\} e^{2(-V(y) + \sum t_\alpha y^\alpha)} I_E(y) dy = \int_{\mathbb{R}} \{y^i e^{V(y,t)}, y^j e^{V(y,t)}\} I_E(y) dy = \int_{\mathbb{R}} (G_i(y)F_j(y) - F_i(y)G_j(y)) dy,
\]

upon setting

\[
F_j(x) = x^j e^{V(x,t)} \quad \text{and} \quad G_j(x) := F_j'(x) = (x^j e^{V(x,t)})'.
\]

\(^{17}\) \(\bar{X}\) means the usual complex conjugate. The condition on the \(2 \times 2\) matrices \(X_{k\ell}\) implies that \(X_{kk} = X_k I\), with \(X_k \in \mathbb{R}\) and \(I\) the identity.

35
That $m_\infty$ satisfies (4.5) follows at once from the first expression (4.17) above.

\[
\mu_{k\ell}(t) = \int \{y^k, y^\ell\} \rho_t(y)^2 dy \\
= \int (k - \ell) y^{k+\ell-1} \rho_t(y)^2 dy \\
\frac{\partial \mu_{k\ell}}{\partial t_i} = 2 \int \{y^k, y^\ell\} y^i e^{2(-V(y) + \sum t_i y^i)} dy \\
= \int ((k + i - \ell) y^{k+i+\ell-1} + (k - \ell - i) y^{k+i+\ell-1}) \rho_t(y)^2 dy \\
= \mu_{k+i, \ell} + \mu_{k, \ell+i},
\]

thus leading to (4.5). Using the relation

\[
\prod_{1 \leq i, j \leq n} (x_i - x_j)^4 = \det \left( x_1^i \ (x_1^i)' \ x_2^i \ (x_2^i)' \ \ldots \ x_n^i \ (x_n^i)' \right)_{0 \leq i \leq 2n-1},
\]

one computes, using again de Bruijn’s Lemma,

\[
\frac{1}{(n)!} \int_{T_2n} e^{2 Tr(-V(X) + \sum t_i X^i)} dX
\]

\[
= \frac{1}{n!} \int_{R^n} \prod_{1 \leq i, j \leq n} (x_i - x_j)^4 \prod_{i=1}^n \left( e^{-2V(x_i)} dx_i \right) \\
= \frac{1}{n!} \int_{R^n} \prod_{k=1}^n \left( dx_k e^{-2V(x_k, t)} dx_k \right) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \det \left( x_1^i \ (x_1^i)' \ x_2^i \ (x_2^i)' \ \ldots \ x_n^i \ (x_n^i)' \right)_{0 \leq i \leq 2n-1} \\
= \frac{1}{n!} \int_{R^n} \prod_{i=1}^n dy_i \ \det \left( F_i(y_1) \ G_i(y_1) \ \ldots \ F_i(y_n) \ G_i(y_n) \right)_{0 \leq i \leq 2n-1}, \\
= \det^{1/2} \left( \int_{R} (G_i(y) F_j(y) - F_i(y) G_j(y)) dy \right)_{0 \leq i, j \leq 2n-1} \\
= pf (\mu_{ij})_{0 \leq i, j \leq 2n-1} \\
= \tau_{2n(t)},
\]

which is a Pfaffian $\tau$-function as well.
5 A map from the Toda to the Pfaff lattice

Remember from (0.1), the notations $\rho_t(z) = \rho(z)e^{\sum t_k z^k}$, and $\rho'/\rho = -g/f$. Assuming, in addition, $f(z)\rho(z)$ vanishes at the endpoints of the interval under consideration (which could be finite, infinite or semi-infinite), one checks the $t$-dependent operator in $z$,

$$
\begin{align*}
\mathbf{n}_t & := \sqrt{\frac{f}{\rho_t}} \frac{d}{dz} \sqrt{f \rho_t} \\
& = e^{-\frac{1}{2} \sum t_k z^k} \left( \frac{d}{dz} f(z) - \frac{f' + g}{2} (z) \right) e^{\frac{1}{2} \sum t_k z^k} \\
& = \frac{d}{dz} f(z) - \frac{f' + g_t}{2} (z), \quad \text{with } g_t(z) = g(z) - f(z) \sum_{k=1}^{\infty} k t_k z^{k-1},
\end{align*}
$$

(5.1)

maintains $\mathcal{H}_+ = \{1, z, z^2, ...\}$ and is skew-symmetric with respect to the $t$-dependent inner-product $\langle \cdot, \cdot \rangle^g_t$, defined by the weight $\rho_t(z)dz$,

$$
\langle \mathbf{n}_t \varphi, \psi \rangle^g_t = \int_E (\mathbf{n}_t \varphi)(z) \psi(z) \rho_t(z)dz = -\int_E \varphi(\mathbf{n}_t \psi) \rho_t dz = -\langle \varphi, \mathbf{n}_t \psi \rangle^g_t.
$$

The orthonormality of the $t$-dependent polynomials $p_n(t, z)$ in $z$ imply

$$
\langle p_n(t, z), p_m(t, z) \rangle^g_t = \delta_{mn}.
$$

The matrices $L$ and $M$ are defined by

$$
z \mathbf{p} = L \mathbf{p} \quad \text{and} \quad e^{-\frac{1}{2} \sum t_k z^k} \frac{d}{dz} e^{\frac{1}{2} \sum t_k z^k} \mathbf{p} = M \mathbf{p}.
$$

The skewness of $\mathbf{n}_t$ implies the skew-symmetry of the matrix

$$
\mathcal{N}(t) = f(L)M - \frac{f' + g}{2} (L), \quad \text{such that } \mathbf{n}_t p(t, z) = \mathcal{N} p(t, z);
$$

(5.2)

so $\mathcal{N}(t)$ can be viewed as the operator $\mathbf{n}_t$, expressed in the polynomial basis $(p_0(t, z), p_1(t, z), ...)$. In the next theorem, we shall consider functions $F$ of two (non-commutative) variables $z$ and $\mathbf{n}_t$ so that the (pseudo)-differential operator in $z$ and the matrix

$$
\mathbf{u}_t := F(z, \mathbf{n}_t) \quad \text{and} \quad \mathcal{U} := F(L, \mathcal{N}),
$$

37
related by
\[ F(z, n_t)p(t, z) = F(L, N)p(t, z), \]
are skew-symmetric as well. Examples of \( F \)'s are
\[ F(z, n_t) := n_t, \quad n_t^{-1} \text{ or } \{ z^\ell, n_t^{2k+1} \}^\dagger, \]
corresponding to
\[ F(L, N) = N, \quad N^{-1} \text{ or } \{ N^{2k+1}, L \}^\dagger. \]

**Theorem 5.1** Any Hänkel matrix \( m_\infty \) evolving according to the vector fields
\[ \frac{\partial m_\infty(t)}{\partial t_k} = \Lambda^k m_\infty \]
leads to matrices \( L \) and \( M \), evolving according to the Toda lattice (1.9). Consider a function \( F \) of two variables, such that the operator \( u_t := F(z, n_t) \)
is skew-symmetric with respect to \( \langle \cdot, \cdot \rangle^{sy}_{t} \) and so the matrix
\[ U(t) = F(L(t), N(t)), \quad \text{defined by } u_t p(t, z) = U p(t, z) \]
is skew-symmetric. This induces a natural lower-triangular matrix \( O(t) \),
mapping the Toda lattice into the Pfaff lattice:

\[
\begin{align*}
\text{Toda lattice} & \quad \left\{ \begin{array}{l}
p_n(t, z) = (S(t)\chi(z))_n \quad \text{orthonormal with respect to} \\
m_\infty(t) = ((z^i, z^j)^{sy}_{t})_{0\leq i, j \leq \infty} = S^{-1} S^{T-1}
\end{array} \right. \\
L(t) = S A S^{-1} & \quad \text{satisfies} \quad \frac{\partial L}{\partial t_j} = \left[ -\frac{1}{2} \pi_{bo} L^j, L \right], \quad j = 1, 2, ... 
\end{align*}
\]

\footnote{with the understanding that \( F(L, N) \) reverses the order of \( z, u \) in \( F(z, u) \).}

\footnote{\[ \{ A, B \}^\dagger = AB + BA. \]}
map $O(2t)$ such that

$$\begin{cases}
  -U(2t) = O^{-1}(2t)JO^T(2t) \\
  O(2t) \text{ is lower-triangular} \\
  O(2t)S(2t) \in \mathcal{G}_k
\end{cases}$$

$\text{Pfaff lattice}$

$$q_n(t, z) = (O(2t)p(2t, z))_n, \text{ skew-orthonormal with regard to}$$

$$\tilde{m}_\infty(t) := -S^{-1}(2t)U(2t)S^T(2t) = Q^{-1}(t)JQ^{-1}(t)$$

$$= \left(\langle z^i, z^j \rangle_t^{sk}\right)_{0 \leq i, j \leq \infty}$$

$$= \left(\langle z^i, u_{2t}z^j \rangle_{2t}^{sy}\right)_{0 \leq i, j \leq \infty}$$

$$\bar{L}(t) := O(2t)L(2t)O(2t)^{-1} \text{ satisfies } \frac{\partial \bar{L}}{\partial t_j} = [-\pi_k \bar{L}^j, \bar{L}], j = 1, ...$$

**Proof:** Since $U(t)$ is skew-symmetric, it admits a skew-Borel decomposition

$$-U(t) = O^{-1}(t)JO^T(2t), \text{ with lower-triangular } O(t). \quad (5.3)$$

But the new matrix, defined by

$$\tilde{m}_\infty(t) := -S^{-1}(2t)U(2t)S^T(2t), \quad (5.4)$$

is skew-symmetric and thus admits a unique skew-Borel decomposition

$$\tilde{m}_\infty(t) = \tilde{Q}^{-1}(t)\tilde{Q}(t)^T, \text{ with } \tilde{Q}(t) \in \mathcal{G}_k. \quad (5.5)$$

Comparing (5.3), (5.4) and (5.5) leads to a unique choice of matrix $O(t)$, skew-Borel decomposing $-U(2t)$, as in (5.3), such that

$$O(2t)S(2t) = \tilde{Q}(t) \in \mathcal{G}_k. \quad (5.6)$$
Using
\[
\frac{\partial U}{\partial t_k}(2t) = [\pi_{sy} L^k(2t), U(2t)]
\]
and
\[
\frac{\partial S}{\partial t_k}(2t) = -(\pi_{bo} L^k(2t)) S(2t),
\]
we compute
\[
\frac{\partial \tilde{m}_\infty}{\partial t_k}(t) = -S^{-1}(\pi_{bo} L^k(2t)) S(2t) - S^{-1}(2t) \frac{\partial U}{\partial t_k}(2t) S^{-1}(2t)
\]
\[
= -S^{-1}(\pi_{bo} L^k(2t)) U S^{-1} - S^{-1}[\pi_{sy} L^k, U] S^{-1} - S^{-1}U(\pi_{bo} L^k)^\top S^{-1}
\]
\[
= -S^{-1}(\pi_{bo} L^k + \pi_{sy} L^k) U S^{-1} - S^{-1}U((\pi_{bo} L^k)^\top - \pi_{sy} L^k) S^{-1}
\]
\[
= -S^{-1}L^k U S^{-1} - S^{-1}UL^\top k S^{-1}, \quad \text{using (5.6) below}
\]
\[
= -\Lambda^k S^{-1}U S^{-1} - S^{-1}UL^\top k S^{-1}, \quad \text{using } L^k = S\Lambda^k S^{-1},
\]
\[
= \Lambda^k \tilde{m}_\infty(t) + \tilde{m}_\infty(t) \Lambda^\top k.
\]

For an arbitrary matrix \(A\), we have
\[
A = A^\top \iff A = (A_{bo})^\top - A_{sy}.
\]

Indeed, remembering that \(A_{bo} = 2A_+ + A_0\) and \(A_{sy} = A_+ - A_-\), one checks \((A_{bo})^\top - A_{sy} - A = 2(A_-)^\top + A_0 - (A_+ - A_-) - A_+ - A_- - A_0 = -2(A_+ - (A_-)^\top)\).

so that the left hand side vanishes, if the right hand side does; the latter means \(A\) is symmetric.

We now define \(\tilde{L}(t)\) by conjugation of \(L(2t)\) by \(O(2t)\):
\[
\tilde{L}(t) := O(2t)L(2t)O(2t)^{-1} = O(2t)S(2t)\Lambda S^{-1}(2t)O(2t)^{-1} = \tilde{Q}(t)\Lambda \tilde{Q}^{-1}(t).
\]

Therefore the sequence of polynomials
\[
q(t, z) := O(2t)p(2t, z) = O(2t)S(2t)\chi(z) = \tilde{Q}(t)\chi(z)
\]

\(^{20}\)\(A_{\pm}\) means the usual strictly upper(lower)-triangular part and \(A_0\) the diagonal part in the common sense.
is skew-orthonormal
\[ \langle q_i(t, z), q_j(t, z) \rangle^{sk} = J_{ij} \]
with regard to the skew inner-product specified by the matrix \( \tilde{m}_\infty \):
\[ \langle z^i, z^j \rangle^s_t = \tilde{\mu}_{ij}(t). \]
In the last step, we show that \( \langle \varphi, \psi \rangle^{sk} = \langle \varphi, u\psi \rangle^{sy} \). Since
\[ U(2t) = -O^{-1}(2t)JO^\top^{-1}(2t), \]
we compute
\[
\langle q_i(t, z), (u_{2t}q)_j(t, z) \rangle^{sy}_{2t} = \langle (Op)_i(2t), (uOp)_j(2t) \rangle^{sy}_{2t} \\
= \langle (Op)_i(2t), (Ou)p_j(2t) \rangle^{sy}_{2t} \\
= \langle (Op)_i(2t), (Ou)p_j(2t) \rangle^{sy}_{2t} \\
= (O(2t) \langle p_k(2t), p_\ell(2t) \rangle^{sy}_{k, \ell \geq 0} (OU)^\top(2t))_{ij} \\
= (O(2t)I(OU)^\top(2t))_{ij} \\
= (O(2t)U^\top(2t)O^\top(2t))_{ij} \\
= -J_{ij}, \text{ using (5.8).} \]
Therefore, defining a new skew inner-product \( \langle , \rangle^{sk'} \)
\[ \langle \varphi, \psi \rangle^{sk'} := \langle \varphi, u\psi \rangle^{sy}, \]
we have shown
\[ \langle q_i, q_j \rangle^{sk'}_t = \langle q_i, q_j \rangle^{sk}_t = J_{ij}, \]
and so by completeness of the basis \( q_i \), we have
\[ \langle , \rangle^{sk'}_t = \langle , \rangle^{sk}_t, \]
thus ending the proof of Theorem 4.1. \( \blacksquare \)
6 Example 1: From Hermitean to symmetric matrix integrals

Striking examples are given by using the map $O(t)$ obtained from skew-borel decomposing $\mathcal{N}^{-1}(t)$ and $\mathcal{N}(t)$; see (5.2). This section deals with $\mathcal{N}^{-1}(t)$, whereas the next will deal with $\mathcal{N}(t)$.

Proposition 6.1 The special transformation

$$U(t) = \mathcal{N}^{-1}(t) = \left( f(L)M - \frac{f' + g(L)}{2} \right)^{-1}(2t)$$

maps the Toda lattice $\tau$-functions with initial weight $\rho = e^{-V}$, $V' = -g/f$ (Hermitean matrix integral) to the Pfaff lattice $\tau$-functions (symmetric matrix integral), with initial weight

$$\tilde{\rho}_t(z) := \left( \frac{\rho_{2t}(z)}{f(z)} \right)^{\frac{1}{2}} = e^{-\frac{1}{2}(V(z) + \log f(z) - 2\sum_1^\infty t_iz^i)} = e^{-\tilde{V}(z) + \sum_1^\infty t_iz^i}.$$ 

To be precise:

\[
\begin{aligned}
\text{Toda lattice} & \quad \left\{ \begin{array}{l} 
p_n(t, z) \quad \text{orthonormal polynomials in } z \text{ for the inner-product} \\
\langle \varphi, \psi \rangle_t^{sy} = = \int \varphi(z)\psi(z)\rho_t(z)\,dz, \\
\mu_{ij}(t) = \langle z^i, z^j \rangle_t^{sy} \quad \text{and} \quad m_n = (\mu_{ij})_{0 \leq i, j \leq n-1}, \\
\tau_n(t) = \det m_n = \frac{1}{n!} \int_{\mathfrak{H}_n} e^{Tr(-V(X) + \sum_1^\infty t_iz^i)} dX \\
\end{array} \right. \\
\text{map } O(2t) \text{ such that} & \quad \left\{ \begin{array}{l} 
-\mathcal{N}^{-1}(2t) = O^{-1}(2t)JO^{\top-1}(2t) \\
O(2t) \text{ is lower-triangular} \\
O(2t)S(2t) \in \mathcal{G}_k \\
\end{array} \right.
\end{aligned}
\]

\[\text{Remember } \rho'/\rho = -V' = -g/f.\]
\[
q_n(t, z) = O(2t)p_n(2t, z) \quad \text{skew-orthonormal polynomials in } z \text{ for the skew-inner-product (weight } \tilde{\rho}) ,
\]

\[
\langle \varphi, \psi \rangle^\text{sk}_{t} := \left\langle \varphi, \left( n_{2t}^{-1} \psi \right)_{2t} \right\rangle
\]

\[
= \frac{1}{2} \int \int_{\mathbb{R}^2} \varphi(x)\psi(y)\varepsilon(x - y)\tilde{\rho}_t(x)\tilde{\rho}_t(y)dx\,dy
\]

\[
\tilde{\mu}_{ij}(t) = (x^i, y^j)_{t}^{sk} \quad \text{and} \quad \tilde{m}_n = (\tilde{\mu}_{ij})_{0 \leq i, j \leq n-1}
\]

\[
\tilde{\tau}_{2n}(t) = pf(\tilde{m}_{2n}) = \frac{1}{(2n)!} \int_{S_{2n}} e^{Tr(-\bar{V}(X) + \sum_{i=1}^{\infty} u_i X^i)} dX.
\]

In the first integral defining \( \tau_n(t) \), \( dX \) denotes Haar measure on Hermitean matrices (see section 4.1), whereas the second integral \( \tilde{\tau}_{2n}(t) \) involves Haar measure on symmetric matrices (see section 4.2)

**Proof:** At first, check that

\[
\left( \frac{d}{dx} \right)^{-1} \varphi(x) = \frac{1}{2} \int \varepsilon(x - y)\varphi(y)dy. \quad (6.1)
\]

Indeed,

\[
\frac{d}{dx} \left( \frac{d}{dx} \right)^{-1} \varphi(x) = \int \frac{1}{2} \frac{\partial}{\partial x} \varepsilon(x - y)\varphi(y)dy
\]

\[
= \int \delta(x - y)\varphi(y)dy \quad \text{using} \quad \frac{\partial}{\partial x} \varepsilon(x) = 2\delta(x)
\]

\[
= \varphi(x).
\]

Consider now the operator

\[
u_t = n_t^{-1} = \left( \sqrt{\frac{\rho_t}{\rho_t\,dz}} \sqrt{f\rho_t} \right)^{-1}, \quad \text{so that} \quad \nu_t p = n_t^{-1} p = N^{-1} p,
\]

according to (5.2). Let it act on a function \( \varphi(x) \):

\[
n_t^{-1} \varphi(x) = \left( \frac{1}{\sqrt{f(x)\rho_t(x)}} \left( \frac{d}{dx} \right)^{-1} \sqrt{\frac{\rho_t(x)}{f(x)}} \right) \varphi(x)
\]

\[
= \int_{\mathbb{R}} \frac{1}{\sqrt{f(x)\rho_t(x)}} \frac{\varepsilon(x - y)}{2} \sqrt{\frac{\rho_t(y)}{f(y)}} \varphi(y)dy, \quad \text{using } (6.1).
\]
One computes
\[ \langle \varphi, \psi \rangle_t^{sk} = \langle \varphi, u_{2t} \psi \rangle_{2t}^{sy} = \langle \varphi, n_{2t}^{-1} \psi \rangle_{2t}^{sy} = \frac{1}{2} \int \int_{\mathbb{R}^2} \rho(x) \rho(y) \varepsilon(x-y) \varphi(x) \psi(y) dx \, dy \]

\[ = \frac{1}{2} \int \int_{\mathbb{R}^2} \tilde{\rho}(x) \tilde{\rho}(y) e^{\sum_{i=1}^{\infty} t_i (x^k+y^k)} \varepsilon(x-y) \varphi(x) \psi(y) dx \, dy. \]

So, finally setting \( \tilde{V}(x) = \frac{1}{2} (V(x) + \log f(x)) \) yields
\[ \tilde{\tau}_{2n}(t) = pf(\tilde{m}_{2n}) = \frac{1}{(2n)!} \int_{S_2} e^{Tr(-\tilde{V}(x)+\sum_{i=1}^{\infty} t_i x^i)} dX. \]

The map \( O \) for the classical orthogonal polynomials at \( t = 0 \): Then, the matrix \( O \), mapping orthonormal \( p_k \) into skew-orthonormal polynomials \( q_k \), is given by a lower-triangular three-step relation:
\[ q_{2n}(0, z) = \sqrt{\frac{c_{2n}}{a_{2n}}} p_{2n}(0, z) \]
\[ q_{2n+1}(0, z) = \sqrt{\frac{a_{2n}}{c_{2n}}} \left( -c_{2n-1} p_{2n-1}(0, z) + \frac{c_{2n}}{a_{2n}} \sum_{i=0}^{2n} b_i p_{2n}(0, z) + c_{2n} p_{2n+1}(0, z) \right) \]

(6.2)

where the \( a_i \) and \( b_i \) are the entries in the tridiagonal matrix defining the orthonormal polynomials, and the \( c_i \)'s are the entries of the skew-symmetric matrix \( N \).

In [2], we showed that then \( N \) is tridiagonal, at the same time as \( L \), (see Appendix 2)
\[ L = \begin{bmatrix} b_0 & a_0 & a_1 & \cdots \\ a_0 & b_1 & a_1 & \cdots \\ a_1 & b_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \quad -N = \begin{bmatrix} 0 & c_0 & 0 & \cdots \\ -c_0 & 0 & c_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \quad (6.3) \]

with the following precise entries.
Hermite: $\rho(z) = e^{-z^2}$, $a_{n-1} = \sqrt{\frac{n}{2}}$, $b_n = 0$, $c_n = a_n$

Laguerre: $\rho(z) = e^{-z^a I_{[0, \infty)}(z)}$, $a_{n-1} = \sqrt{n(n+\alpha)}$, $b_n = 2n + \alpha + 1$, $c_n = a_n/2$

Jacobi: $\rho(z) = (1 - z)^\alpha (1 + z)^\beta I_{[-1, 1]}(z)$

$$a_{n-1} = \left( \frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)} \right)^{1/2}$$

$$b_n = \frac{\alpha^2 - \beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta-2)}$$

$$c_n = a_n \left( \frac{\alpha + \beta}{2} + n + 1 \right)$$

If the skew-symmetric matrix $N$ has the tridiagonal form above, then one checks its inverse has the following form:

$$-N^{-1} = \begin{pmatrix}
0 & -\frac{1}{c_0} & 0 & -\frac{c_1}{c_0 c_2} & 0 & -\frac{c_1 c_3}{c_0 c_2 c_4} & 0 & -\frac{c_1 c_3 c_5}{c_0 c_2 c_4 c_6} \\
\frac{1}{c_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{c_2} & 0 & -\frac{c_3}{c_2 c_4} & 0 & -\frac{c_3 c_5}{c_2 c_4 c_6} \\
\frac{c_1}{c_0 c_2} & 0 & \frac{1}{c_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{c_4} & 0 & \frac{c_5}{c_4 c_6} \\
\frac{c_1 c_3}{c_0 c_2 c_4} & 0 & \frac{c_3}{c_2 c_4} & 0 & \frac{1}{c_4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{c_6} \\
\frac{c_1 c_3 c_5}{c_0 c_2 c_4 c_6} & 0 & \frac{c_3 c_5}{c_2 c_4 c_6} & 0 & \frac{c_5}{c_4 c_6} & 0 & \frac{1}{c_6} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}.$$

In order to find the matrix $O$, we must perform the skew-Borel decomposition of the matrix $-U$:

$$-U = -N^{-1} = O^{-1}JO^{-1}. $$

The recipe for doing so was given in theorem 4.1 (see also the important remark, following that theorem). It suffices to form the pfaffians (0.17), by appropriately bordering the matrix $-N^{-1}$, as in (0.17), with rows and columns of powers of $z$, yielding monic skew-orthogonal polynomials; we choose to call them $r$'s, instead of the $q$'s of theorem 4.1, with $O\chi(z) = r(z)$. They turn out to be the following simple polynomials, with $1/\tilde{\tau}_2(n = \ldots$
\[ r_{2n}(z) = \frac{1}{\sqrt{\tau_{2n} \tilde{\tau}_{2n+2}}} \frac{c_{2n} z^{2n}}{c_0 c_2 \cdots c_{2n}} = \frac{1}{\sqrt{c_{2n}}} c_{2n} z^{2n} \]
\[ r_{2n+1}(z) = \frac{1}{\sqrt{\tau_{2n} \tilde{\tau}_{2n+2}}} \frac{c_{2n} z^{2n+1} - c_{2n-1} z^{2n-1}}{c_0 c_2 \cdots c_{2n}} = \frac{1}{\sqrt{c_{2n}}} (c_{2n} z^{2n+1} - c_{2n-1} z^{2n-1}). \]

Then, also from appendix 1, in order to get \( O \to \hat{O} \) in the correct form, we compute the skew-orthonormal polynomials \( \hat{r}_k \), with \( \hat{O} \chi(z) = \hat{r}(z) \):

\[ \hat{r}_{2n}(z) = \frac{1}{\sqrt{a_{2n}}} r_{2n}(z) = \sqrt{\frac{c_{2n}}{a_{2n}}} z^{2n} \]
\[ \hat{r}_{2n+1}(z) = \sum_{i=0}^{2n} b_i \sqrt{a_{2n}} r_{2n}(z) + \sqrt{a_{2n}} r_{2n+1}(z) \]
\[ = \sqrt{\frac{a_{2n}}{c_{2n}}} \left( -c_{2n-1} z^{2n-1} + \frac{c_{2n}}{a_{2n}} (\sum_{i=0}^{2n} b_i) z^{2n} + c_{2n} z^{2n+1} \right). \]

(6.5)

From the coefficients of the polynomial \( \hat{r}_k \), one reads off the transformation matrix from orthonormal to skew-orthonormal polynomials; it is given by the matrix \( \hat{O} \), such that \( \hat{O} \chi(z) = \hat{r}(z) \). Therefore \( q(t, z) = \hat{O}(2t) p(2t, z) \) yields, after setting \( t = 0 \),

\[ q_{2n}(0, z) = \sqrt{\frac{c_{2n}}{a_{2n}}} p_{2n}(0, z) \]
\[ q_{2n+1}(0, z) = \sqrt{\frac{a_{2n}}{c_{2n}}} \left( -c_{2n-1} P_{2n-1}(0, z) + \frac{c_{2n}}{a_{2n}} (\sum_{i=0}^{2n} b_i) P_{2n}(0, z) + c_{2n} P_{2n+1}(0, z) \right), \]

(6.6)

confirming (6.2).

7 Example 2: From Hermitean to symplectic matrix integrals
Proposition 7.1 The matrix transformation

\[ N = f(L)M - \frac{f' + g}{2}(L), \]

maps the Toda lattice \( \tau \)-functions with \( t \)-dependent weight

\[ \rho_t(z) = e^{-V(z) + \sum t_i z^i}, \quad V' = g/f \]

(Hermitean matrix integral) to the Pfaff lattice \( \tau \)-functions (Symplectic matrix integral), with \( t \)-dependent weight

\[ \tilde{\rho}_t(z) := (\rho_{2t}(z) f(z))^{1/2} = e^{-\frac{1}{2}V(z) - \log f(z) - 2 \sum t_i z^i} = e^{-\tilde{V}(z) + \sum t_i z^i}. \]

To be precise:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
p_n(t, z) \text{ orthonormal polynomials in } z \text{ for the inner-product } \\
\langle \varphi, \psi \rangle_t = \int \varphi(z) \psi(z) \rho_t(z) dz
\end{array} \right.
\end{aligned}
\]

Toda lattice

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\mu_{ij}(t) = \langle z^i, z^j \rangle_t, \text{ and } m_n = (\mu_{ij})_{0 \leq i,j \leq n-1}, \\
\tau_n(t) = \det m_n(t) = \frac{1}{n!} \int_{\mathbb{H}_n} e^{Tr(-V(X) + \sum t_i X^i)} dX
\end{array} \right.
\end{aligned}
\]

map \( O(2t) \) such that

\[
\begin{aligned}
&\left\{ \begin{array}{l}
-N(2t) = O^{-1}(2t)JO^{-1}(2t) \\
O(2t) \text{ is lower-triangular} \\
O(2t)S(2t) \in G_k
\end{array} \right.
\end{aligned}
\]
\[
\begin{align*}
q_n(t, z) &= O(2t)p_n(2t, z) \quad \text{skew-orthonormal polynomials in } z \text{ for the skew-inner-product (weight } \tilde{\rho}_t), \\
\langle \varphi, \psi \rangle_t^{sk} &= \langle \varphi, n_{2t}\psi \rangle_{2t}^{sy} \\
&= -\frac{1}{2} \int \int_{\mathbb{R}^3} \{\varphi(z), \psi(z)\} \tilde{\rho}_t^2(z) dz \\
\tilde{\mu}_{ij}(t) &= \langle z^i, z^j \rangle_t^{sk}, \text{ and } \tilde{m}_n = \det(\tilde{\mu}_{ij})_{0 \leq i, j \leq n-1} \\
\tilde{\tau}_n(t) &= pf(\tilde{m}_{2n}(t)) = \frac{1}{(-2)^nn!} \int_{\mathbb{T}_n} e^{2Tr(-\tilde{V}(X)+\sum t_iX^i)} dX.
\end{align*}
\]

**Proof:** Representing \( \frac{d}{dx} \) as an integral operator
\[
\frac{d}{dx}\varphi(x) = \int_{\mathbb{R}} \delta(x-y)\varphi'(y)dy = -\int_{\mathbb{R}} \frac{\partial}{\partial y} \delta(x-y)\varphi(y)dy = \int_{\mathbb{R}} \delta'(x-y)\varphi(y)dy,
\]
compute
\[
u_t = n_t = \sqrt{f} \frac{d}{dx} \sqrt{f} \rho_t, \quad \text{so that } n_t p(t, z) = N p(t, z);
\]
remember \( N \) from (5.2). Let it act on a function \( \varphi(x) \):
\[
u_t \varphi(x) = \left( \sqrt{\frac{f}{\rho_t} \frac{d}{dx} \sqrt{f} \rho_t} \right) \varphi(x)
\]
\[
= \int_{\mathbb{R}} \sqrt{\frac{f(x)}{\rho_t(x)}} \delta'(x-y) \sqrt{f(y)\rho_t(y)\varphi(y)}dy.
\]
Then
\[
\langle \varphi, \psi \rangle_t^{sk} = \langle \varphi, \nu_{2t}\psi \rangle_{2t}^{sy} = \langle \varphi, n_{2t}\psi \rangle_{2t}^{sy}
\]
\[
= \int \int_{\mathbb{R}^3} \rho_{2t}(x) \varphi(x) \sqrt{\frac{f(x)}{\rho_{2t}(x)}} \delta'(x-y) \sqrt{f(y)\rho_{2t}(y)\psi(y)} dx dy
\]
\[
= \int \int_{\mathbb{R}^3} \sqrt{f(x)\rho_{2t}(x)} \varphi(x) \delta'(x-y) \sqrt{f(y)\rho_{2t}(y)\psi(y)} dx dy
\]
\[
= -\int \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial x} \sqrt{f(x)\rho_{2t}(x)} \varphi(x) \right) \delta(x-y) \sqrt{f(y)\rho_{2t}(y)\psi(y)} dx dy
\]

48
\[-\int_{\mathbb{R}} \left( \frac{\partial}{\partial x} \sqrt{f(x)\rho_2(x)\varphi(x)} \right) \sqrt{f(x)\rho_2(x)\psi(x)} \, dx \]

\[= -\frac{1}{2} \int_{\mathbb{R}} \left( \frac{\partial}{\partial x} \sqrt{f(x)\rho_2(x)\varphi(x)} \right) \sqrt{f(x)\rho_2(x)\psi(x)} \, dx \]

\[\quad + \frac{1}{2} \int_{\mathbb{R}} \sqrt{f(x)\rho_2(x)\varphi(x)} \left( \frac{\partial}{\partial x} \sqrt{f(x)\rho_2(x)\psi(x)} \right) \, dx \]

\[= -\frac{1}{2} \int_{\mathbb{R}} \{ \sqrt{f(x)\rho_2(x)\varphi(x)}, \sqrt{f(x)\rho_2(x)\psi(x)} \} \, dx \]

\[= -\frac{1}{2} \int_{\mathbb{R}} \{ \varphi(x), \psi(x) \} \rho_0(x) e^{2 \sum_{t=1}^{\infty} t x^t} \, dx, \]

using the notation in the statement of this proposition. Setting \( \tilde{\rho}(x) = e^{-\tilde{V}(x)} \), with \( \tilde{V}(x) = \frac{1}{2} (V(x) - \log f) \)

\[
\langle x^i, x^j \rangle_{sk} = -\frac{1}{2} \int_{\mathbb{R}} \{ x^i, x^j \} \rho_0^2(x) e^{2 \sum_{t=1}^{\infty} t x^t} \, dx
\]

\[= -\frac{1}{2} \int_{\mathbb{R}} \{ x^i, x^j \} e^{-2(\tilde{V}(x)-\sum t x^t)} \, dx, \]

and so

\[\tau_{2n}(t) = pf(\tilde{m}_{2n}(t)) = \frac{1}{(-2)^{n!}} \int_{\tau_{2n}} e^{2Tr(-\tilde{V}(x)+\sum t x^t)} \, dx.\]

The map \( O^{-1} \) for the classical orthogonal polynomials at \( t = 0 \): Then, the matrix \( O \), mapping orthonormal \( p_k \) into skew-orthonormal polynomials \( q_k \), is given by a lower-triangular three-step relation:

\[p_{2n}(0, z) = -c_{2n-1} \sqrt{\frac{a_{2n-2}}{c_{2n-2}}} q_{2n-2}(0, z) + \sqrt{a_{2n} c_{2n}} \, q_{2n}(0, z)\]

\[p_{2n+1}(0, z) = -c_{2n} \sqrt{\frac{a_{2n-2}}{c_{2n-2}}} q_{2n-2}(0, z) - \left( \sum_{i=0}^{2n} b_i \right) \sqrt{\frac{c_{2n}}{a_{2n}}} q_{2n}(0, z) + \sqrt{\frac{c_{2n}}{a_{2n}}} q_{2n+1}(0, z),\]

where the \( a_i \) and \( b_i \) are the entries in the tridiagonal matrix defining the orthonormal polynomials, and the \( c_i \) the entries in the skew-symmetric matrix.

In this case, we need to perform the following skew-Borel decomposition at \( t = 0 \),

\[-\mathcal{U} = -\mathcal{N} = O^{-1} J O^\top^{-1},\]
where $\mathcal{N}$ is the matrix (6.3). Here again, in order to find $O$, we use the recipe given in theorem 4.1, namely writing down the corresponding skew-orthogonal polynomials (0.17), but where the $\mu_{ij}$ are the entries of $-U = -\mathcal{N}$: consider the pfaffians of the bordered matrices (0.17); they have leading term

$$\tilde{\tau}_{2n} = \prod_{0}^{n-1} c_{2j}.$$  

Then one computes

$$r_{2n} = \frac{1}{\sqrt{\tilde{\tau}_{2n} \tilde{\tau}_{2n+2}}} \sum_{i=0}^{n} z^{2n-2i} \left( \prod_{0}^{i-1} c_{2j} \right) \left( \prod_{0}^{i} c_{2n-2j-1} \right)$$

$$r_{2n+1} = \frac{1}{\sqrt{\tilde{\tau}_{2n} \tilde{\tau}_{2n+2}}} \left( z^{2n+1} \prod_{0}^{n-1} c_{2j} + \sum_{i=1}^{n} z^{2n-2i} \left( \prod_{0}^{i-1} c_{2j} \right) \left( \prod_{0}^{i} c_{2n-2j-1} \right) \right)$$

with

$$\sqrt{\tilde{\tau}_{2n} \tilde{\tau}_{2n+2}} = c_{0} c_{2} c_{2n-2} \sqrt{c_{2n}} \quad \sqrt{\tilde{\tau}_{0} \tilde{\tau}_{2}} = \sqrt{c_{0}}.$$  

Setting

$$D := \text{diag}(\sqrt{\tilde{\tau}_{0} \tilde{\tau}_{2}}, \sqrt{\tilde{\tau}_{2} \tilde{\tau}_{4}}, \sqrt{\tilde{\tau}_{2} \tilde{\tau}_{4}}, ...)$$

the matrix $O$ is the set of coefficients of the polynomials above, i.e.,

$$O = D^{-1} = D^{-1} R$$

As before, in order to get the skew-symmetric polynomials in the right form, from the orthogonal ones, one needs to multiply to the left with the matrix $E$, defined in (8.2) in the appendix:

$$\hat{O} = EO = ED^{-1} R,$$

50
and so,
\[ \hat{O}^{-1} = R^{-1}DE^{-1}; \]  \hspace{1cm} (7.5)

it turns out the matrix \( \hat{O} \) is complicated, but its inverse is simple. Namely, compute

\[
R^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_1 c_0 & 0 & -\frac{1}{c_0} & 0 & 0 & 0 & 0 & 0 \\
-c_2 c_0 & 0 & 0 & -\frac{1}{c_0} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{c_2}{c_0 c_2} & 0 & -\frac{1}{c_0 c_2} & 0 & 0 & 0 \\
0 & 0 & -\frac{c_2}{c_0 c_2} & 0 & 0 & -\frac{1}{c_0 c_2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{c_5}{c_0 c_2 c_4} & 0 & -\frac{1}{c_0 c_2 c_4} & 0 \\
0 & 0 & 0 & 0 & -\frac{c_5}{c_0 c_2 c_4} & 0 & 0 & -\frac{1}{c_0 c_2 c_4}
\end{pmatrix},
\]

and

\[
E^{-1} = \begin{pmatrix}
\alpha_0 & 0 \\
-\beta_0 & \frac{1}{\alpha_0} \\
0 & 0 \\
\alpha_2 & 0 \\
-\beta_2 & \frac{1}{\alpha_2} \\
0 & 0 \\
\alpha_4 & 0 \\
-\beta_4 & \frac{1}{\alpha_4}
\end{pmatrix}. \hspace{1cm} (7.6)
\]

with \( \alpha_{2n} \) and \( \beta_{2n} \) as in (8.5). Carrying out the multiplication (7.5) leads to the matrix \( \hat{O}^{-1} \), with a few non-zero bands, yielding the map (7.1).
8 Appendix 1: Free parameter in the skew-Borel decomposition

If the Borel decomposition of \(-H = O^{-1}JO^{\top}^{-1}\) is given by a matrix \(O \in G_k\), with the diagonal part of \(O\) being

\[
(O)_0 = \begin{pmatrix}
\sigma_0 & 0 \\
0 & \sigma_0 \\
\sigma_2 & 0 \\
0 & \sigma_2 \\
\sigma_4 & 0 \\
0 & \sigma_4 \\
& & & & & & \\
& & & & & & \\
0 & & & & & & \\
\end{pmatrix},
\] (8.1)

then the new matrix

\[
\hat{O} := \begin{pmatrix}
1/\alpha_0 & 0 \\
\beta_0 & \alpha_0 \\
& & & & & & \\
1/\alpha_2 & 0 \\
\beta_2 & \alpha_2 \\
& & & & & & \\
1/\alpha_4 & 0 \\
\beta_4 & \alpha_4 \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
0 & & & & & & \\
\end{pmatrix}
\]

with free parameters \(\alpha_{2n}, \beta_{2n}\), is a solution of the Borel decomposition \(-H = \hat{O}^{-1}J\hat{O}^{\top}^{-1}\), as well. The diagonal part of \(\hat{O}\) consists of \(2 \times 2\) blocks

\[
\left(\begin{array}{cc}
1/\alpha_{2n} & 0 \\
\beta_{2n} & \alpha_{2n}
\end{array}\right) \left(\begin{array}{cc}
\sigma_{2n} & 0 \\
0 & \sigma_{2n}
\end{array}\right) = \left(\begin{array}{cc}
\sigma_{2n}/\alpha_{2n} & 0 \\
\beta_{2n}\sigma_{2n} & \alpha_{2n}\sigma_{2n}
\end{array}\right).
\]

Imposing the condition that

\[
q(z) = \hat{O}p(z), \quad \text{with } p_k(z) = \sum_{i=0}^{k} p_{ki} z^i
\]

has the required form, i.e., the same leading term for \(q_{2n}\) and \(q_{2n+1}\) and no \(z^{2n}\)-term in \(q_{2n+1}\),

\[
q_{2n}(z) = q_{2n,2n} z^{2n} + \cdots \\
q_{2n+1}(z) = q_{2n,2n} z^{2n+1} + q_{2n,2n-1} z^{2n-1} + \cdots
\] (8.3)
implies
\[ \frac{\sigma_{2n}}{\alpha_{2n}} p_{2n,2n} = \sigma_{2n} \alpha_{2n} p_{2n+1,2n+1} \]
\[ \sigma_{2n} \beta_{2n} p_{2n,2n} + \sigma_{2n} \alpha_{2n} p_{2n+1,2n+1} = 0 \]
yielding, upon using the explicit form of the coefficients \( p_{k\ell} \) of the polynomials \( p_k \), associated with three step relations (see next lemma),
\[ \alpha_{2n}^2 = \frac{p_{2n,2n}}{p_{2n+1,2n+1}} = a_{2n} \]
\[ \beta_{2n} = \frac{p_{2n+1,2n}}{p_{2n,2n}} = \frac{\sum_{0}^{2n} b_i}{a_{2n}}. \] (8.4)

Hence
\[ \alpha_{2n} = \sqrt{a_{2n}} \text{ and } \beta_{2n} = \frac{1}{\sqrt{a_{2n}}} \sum_{0}^{2n} b_i. \] (8.5)

So, if \( r(z) = O\chi(z) \),
then
\[ \hat{r}(z) := \hat{O}\chi(z) = \begin{pmatrix}
\frac{1}{\alpha_0} & 0 & \cdots \\
0 & \alpha_0 & \cdots \\
\beta_0 & \alpha_0 & \cdots \\
1/\alpha_2 & 0 & \cdots \\
\beta_2 & \alpha_2 & \cdots \\
1/\alpha_4 & 0 & \cdots \\
\beta_4 & \alpha_4 & \cdots \\
0 & & & & & \\
\end{pmatrix} \]
\[ \hat{r}(z) = E r(z), \]
and thus
\[ \hat{r}_{2n}(z) = \frac{1}{\sqrt{a_{2n}}} r_{2n}(z) \] (8.6)
\[ \hat{r}_{2n+1}(z) = \frac{\sum_{0}^{2n} b_i}{\sqrt{a_{2n}}} r_{2n} + \sqrt{a_{2n}} r_{2n+1}(z) \] (8.7)
Lemma 8.1 A sequence of polynomials $p_n(z) = \sum_{i=0}^{n} p_{ni} z^i$ of degree $n$ satisfying three-step recursion relation

$$zp_n = a_{n-1}p_{n-1} + b_n p_n + a_n p_{n+1}, \quad n = 0, 1, \ldots, \quad (8.8)$$

has the form

$$p_{n+1}(z) = \frac{p_{n,n}}{a_n} \left( z^{n+1} - \left( \sum_{i=0}^{n} b_i \right) z^n + \cdots \right).$$

Proof: Equating the $z^{n+1}$ and $z^n$ coefficients of (8.8) divided by $p_{n,n}$ yields

$$\frac{p_{n+1,n+1}}{p_{n,n}} = \frac{1}{a_n}$$

and

$$\frac{p_{n,n-1}}{p_{n,n}} = a_n \frac{p_{n+1,n}}{p_{n,n}} + b_n.$$  

Combining both equations leads to

$$a_n \frac{p_{n+1,n}}{p_{n,n}} - a_{n-1} \frac{p_{n,n-1}}{p_{n-1,n-1}} = b_n,$$

yielding

$$a_n \frac{p_{n+1,n}}{p_{n,n}} = - \sum_{i=0}^{n} b_i, \quad \text{using} \, a_{-1} = 0.$$

9 Appendix 2: Simultaneous (skew) - symmetrization of $L$ and $N$.

For the classical polynomials, the matrices $L$ and $N$ can be simultaneously symmetrized and skew-symmetrized.

We sketch the proof of this statement, which has been established by us in [4]. Given the monic orthogonal polynomials $\tilde{p}_n$ with respect to the weight $\rho$, with $\rho'/\rho = -g/f$, we have that the operators $z$ and

$$n = \sqrt{f} \frac{d}{d z} \sqrt{f \rho} = f \frac{d}{d z} + \frac{f' - g}{2}.$$  

\[\text{with } a_{-1} = 0.\]
acting on the polynomials \( \tilde{p}_n \)'s have the following form:

\[
\begin{align*}
 z \tilde{p}_n &= a_{n-1}^2 \tilde{p}_{n-1} + b_n \tilde{p}_n + \tilde{p}_{n+1} \\
 n \tilde{p}_n &= \ldots - \gamma_n \tilde{p}_{n+1},
\end{align*}
\]  

(9.1)
in view of the fact that for the classical orthogonal polynomials

\[
\begin{align*}
\text{Hermite:} & \quad n = \frac{d}{dz} - z \\
\text{Laguerre:} & \quad n = z \frac{d}{dz} - \frac{1}{2}(z - \alpha - 1) \\
\text{Jacobi:} & \quad n = (1 - z^2) \frac{d}{dz} - \frac{1}{2}((\alpha + \beta + 2)z + (\alpha - \beta)).
\end{align*}
\]

For the orthonormal polynomials, the matrices \( L \) and \( -N \) are symmetric and skew-symmetric respectively. Therefore the right hand side of these expressions must have the form:

\[
\begin{align*}
 z \tilde{p}_n &= a_{n-1}^2 \tilde{p}_{n-1} + b_n \tilde{p}_n + \tilde{p}_{n+1} \\
 n \tilde{p}_n &= a_{n-1}^2 \gamma_{n-1} \tilde{p}_{n-1} - \gamma_n \tilde{p}_{n+1}.
\end{align*}
\]

Therefore, upon rescaling the \( \tilde{p}_n \)'s, to make them orthonormal, we have

\[
\begin{align*}
 z p_n &= (Lp)_n = a_{n-1} p_{n-1} + b_n p_n + a_n p_{n+1} \\
 n p_n &= (Np)_n = a_{n-1} \gamma_{n-1} p_{n-1} - a_n \gamma_n p_{n+1},
\end{align*}
\]

from which it follows that

\[
-N = \begin{bmatrix}
0 & c_0 \\
-c_0 & 0 & c_1 \\
& -c_1 & 0 & \ddots \\
& & & \ddots 
\end{bmatrix}, \quad \text{with} \quad c_n = -a_n \gamma_n,
\]

where \(-\gamma_n\) is the leading term in the expression (9.1).

### 10 Appendix 3: Proof of Lemma 3.4

For future use, consider the first order differential operators

\[
\eta(t, z) = \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial t_j} \quad \text{and} \quad B(z) = -\frac{\partial}{\partial z} + \sum_{j=1}^{\infty} z^{-j-1} \frac{\partial}{\partial t_j}
\]

(10.1)
having the property

\[
B(z)e^{-\eta(z)} f(t) = B(z)f(t - [z^{-1}]) = 0.
\]

(10.2)

\[\text{23 with respective weights } \rho = e^{-z}, \quad \rho = e^{-z} z^\alpha, \quad \rho = (1 - z)^\alpha (1 + z)^\beta.\]
Lemma 10.1 Consider an arbitrary function $\varphi(t,z)$ depending on $t \in \mathbb{C}^\infty$, $z \in \mathbb{C}$, having the asymptotics $\varphi(t,z) = 1 + O\left(\frac{1}{z}\right)$ for $z \to \infty$ and satisfying the functional relation

$$\frac{\varphi(t - [z^{-1}], z_1)}{\varphi(t, z_1)} = \frac{\varphi(t - [z^{-1}], z_2)}{\varphi(t, z_2)}, \quad t \in \mathbb{C}^\infty, z \in \mathbb{C}.$$  \hspace{1cm} (10.3)

Then there exists a function $\tau(t)$ such that

$$\varphi(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)}.$$  \hspace{1cm} (10.4)

Proof: Applying $B_1 := B(z_1)$ to the logarithm of (10.3) and using (10.1) and (10.3) yields

$$\left(e^{-\eta(z_2)} - 1\right)B_1 \log \varphi(t; z_1) = -B_1 \log \varphi(t, z_2) = \sum_{j=1}^{\infty} z_1^{-j-1} \frac{\partial}{\partial t_j} \log \varphi(t, z_2),$$

which, upon setting

$$f_j(t) = \text{Res}_{z_1 = \infty} z_1^j B_1 \log \varphi(t, z_2),$$

yields termwise in $z_1$,

$$\left(e^{-\eta(z_2)} - 1\right)f_j(t) = -\frac{\partial}{\partial t_j} \log \varphi(t, z_2).$$  \hspace{1cm} (10.5)

Acting with $\frac{\partial}{\partial t_i}$ on the latter expression and with $\frac{\partial}{\partial t_j}$ on the same expression with $j$ replaced by $i$, and subtracting\(^2^4\), one finds

$$\left(e^{-\eta(z_2)} - 1\right) \left( \frac{\partial f_i}{\partial t_j} - \frac{\partial f_j}{\partial t_i} \right) = 0,$$

yielding

$$\frac{\partial f_i}{\partial t_j} - \frac{\partial f_j}{\partial t_i} = 0;$$

the constant vanishes, because $\frac{\partial f_i}{\partial t_j}$ never contains constant terms.

\(^{24}\)It is obvious that $\left[ \frac{\partial}{\partial t_i}, e^{-\eta(z)} \right] = 0$. 

56
Therefore there exists a function $\log \tau(t_1, t_2, \ldots)$ such that
\[
- \frac{\partial}{\partial t_j} \log \tau = f_j(t) = \text{Res}_{z=\infty} z^j B \log \varphi
\]
and hence, using (10.5)
\[
\frac{\partial}{\partial t_j} \log \varphi(t, z) = (e^{-\eta(z)} - 1) \frac{\partial}{\partial t_j} \log \tau
\]
or, what is the same,
\[
\frac{\partial}{\partial t_j} (\log \varphi - (e^{-\eta} - 1) \log \tau) = 0,
\]
from which it follows that
\[
\log \varphi - (e^{-\eta} - 1) \log \tau = - \sum_{i=1}^{\infty} \frac{b_i}{i} z^{-i}
\]
is, at worst, a holomorphic series in $z^{-1}$ with constant coefficients, which we call $-b_i/i$. Hence
\[
\varphi(t, z) = \frac{\tau(t - [z^{-1}] e^{-\sum_{i=1}^{\infty} \frac{b_i}{i} z^{-i}})}{\tau(t)} = \frac{\tau(t - [z^{-1}] e^{\sum_{i=1}^{\infty} b_i(t_i - \frac{z^{-i}}{i})})}{\tau(t) e^{\sum_{i=1}^{\infty} b_i t_i}},
\]
i.e.
\[
\varphi(t, z) = \frac{\tilde{\tau}(t - [z^{-1}])}{\tilde{\tau}(t)},
\]
where
\[
\tilde{\tau} = \tau(t) e^{\sum_{i=1}^{\infty} b_i t_i}.
\]

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