A remark on quantum key distribution with two way communication: the classical complexity in decoding the CSS code can be removed

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Abstract

So far all the proven unconditionally secure prepare and measure protocols for the quantum key distribution (QKD) must solve the very complex problem of decoding the classical CSS code. In the decoding stage, Bob has to compare his string with an exponentially large number of all the strings in certain code space to find out the closest one. Here we have spotted that, in an entanglement purification protocol (EPP), the random basis in the state preparation stage is only necessary to those check qubits, but unnecessary to the code qubits. In our modified two way communication EPP (2-EPP) protocol, Alice and Bob may first take all the parity checks on Z basis to reduce the bit flip error to strictly zero with a high probability, e.g., $1 - 2^{-30}$, and then use the CSS code to obtain the final key. We show that, this type of 2-EPP protocol can be reduced to an equivalent prepare and measure protocol. In our protocol, the huge complexity of decoding the classical CSS code is totally removed.

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Due to the Hesenberg uncertain principle, quantum key distribution is different from classical cryptography in that an unknown quantum state is in principle not known to Eve unless it is disturbed, rather than the conjectured difficulty of computing certain functions. The first published protocol, proposed in 1984 [1], is called BB84 after its inventors (C. H. Bennett and G. Brassard.) For a history of the subject, one may see e.g. [2]. In this protocol, the participants (Alice and Bob) wish to agree on a secret key about which no eavesdropper (Eve) can obtain significant information. Alice sends each bit of the secret key in one of a set of conjugate bases which Eve does not know, and this key is protected by the impossibility of measuring the state of a quantum system simultaneously in two conjugate bases. Since then, studies on QKD are extensive. In particular, the strict proof on the unconditional security have abstracted much attentions. The original papers proposing quantum key distribution [1] proved it secure against certain attacks, including those feasible using current experimental techniques. However, for many years, it was not rigorously proven secure against an adversary able to perform any physical operation permitted by quantum mechanics.

The first general although rather complex proof of unconditional security was given by Mayers [3], which was followed by a number of other proofs [4,5]. Building on the quantum privacy amplification idea of Deutsch et al. [6], Lo and Chau [7], proposed a conceptually simpler proof of security. This protocol, although has a drawback of requiring a quantum computer, opens the possibility of finding simple proofs on a prepare and measure protocol. Later on, Shor and Preskill [8] unified the techniques in [3] and [7] and provided a simple proof of security of standard BB84. (See also [9] for a detailed exposition of this proof.) Shor-Preskill’s proof is a reduction from the purification scheme to the quantum error correction with CSS code [10] and finally to the BB84 scheme of decoding the classical CSS code with one way classical communication. Very recently, motivated for higher bit error rate tolerance and higher efficiency, Gottesman and Lo [11] studied the two way communication entanglement purification protocol (2-EPP) and proposed a 4 state prepare and measure protocol with the highest bit error tolerance among all modified BB84 protocols so far. They
also significantly increased the previous bit error tolerance rate for the six state protocol. The tolerable bit error rate for six state protocol is then further improved by Chau [12]. A general theorem on the sufficient condition to convert a 2-EPP protocol to a classical one is also given in [11]. However, it has not been studied there on how to remove the complexity of decoding the CSS code in their prepare and measure protocol. So far in all those protocols based on CSS code, in the decoding stage, Bob has to compare his string with an exponentially large number of all the strings in certain code space to find out the one with the shortest distance with his string. The complexity of such a comparison can be huge without any preshared secret string. For example, if we try to distill a final key of 300 bits, the complexity will be far beyond the power of any existing classical computer. Studies towards the removal of the decoding complexity are rare. To the best of our knowledge the only report on this topic is given by H.K. Lo [13]. However, Lo’s scheme requires that Alice and Bob have a pre-shared secret string. Here we take a further study on the 2-EPP QKD [11] and we find that besides the advantage of a higher bit error tolerance as reported in [11], the 2-EPP protocol has another advantage, it can be used to remove the huge complexity in decoding the classical CSS code. We will construct a specific prepare and measure protocol without the decoding complexity. That means, in our protocol, even a large classical computer is unnecessary. Before going into details of our protocol, we first make some mathematical notations and some background presumptions for the quantum key distribution.

We will use two level quantum states as our qubits. For example, spin half particles or linearly polarized photons. A quantum state can be prepared or measured in different basis. We define the spin up, down or polarization of horizontal, vertical as the $Z$ basis, i.e., the basis of $\{|0\rangle, |1\rangle\}$. We define the spin right, left or polarization of $\pi/4, 3\pi/4$ as the $X$ basis, i.e., the basis of $\{|+\rangle, |-\rangle\}$. These basis are related by $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|−\rangle = \frac{1}{\sqrt{2}}(|0\rangle − |1\rangle)$.

There are four maximally entangled states (Bell basis)
\[ \Psi^\pm = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \quad \Phi^\pm = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \]

which form an orthonormal basis for the quantum state space of two qubits.

There are three Pauli matrices:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The matrix \( \sigma_x \) applies a bit flip error to a qubit, while \( \sigma_z \) applies a phase flip error. We denote the Pauli matrix \( \sigma_a \) acting on the \( l \)th bit of the CSS code by \( \sigma_{a(l)} \) for \( a \in \{x, y, z\} \).

For a binary vector \( \gamma \), we let

\[
\sigma_a^{[\gamma]} = \sigma_{a(1)}^{\gamma_1} \otimes \sigma_{a(2)}^{\gamma_2} \otimes \sigma_{a(3)}^{\gamma_3} \otimes \ldots \otimes \sigma_{a(n)}^{\gamma_n}
\]

where \( \sigma_a^0 \) is the identity matrix and \( \gamma_i \) is the \( i \)th bit of \( \gamma \). The matrices \( \sigma_x^{[\gamma]} \) (\( \sigma_z^{[\gamma]} \)) have all eigenvalues \( \pm 1 \).

We also need a short review the properties of CSS code [10]. Here we directly borrow the review materials given in ref. [8]. Quantum error-correcting codes are subspaces of the Hilbert space \( \mathbb{C}^{2^n} \) which are protected from errors in a small number of these qubits, so that any such error can be measured and subsequently corrected without disturbing the encoded state. A quantum CSS code \( Q \) on \( n \) qubits comes from two binary codes on \( n \) bits, \( C_1 \) and \( C_2 \), one contained in the other:

\[
\{0\} \subset C_2 \subset C_1 \subset \mathbb{F}_2^n,
\]

where \( \mathbb{F}_2^n \) is the binary vector space on \( n \) bits [10].

A set of basis states (which we call codewords) for the CSS code subspace can be obtained from vectors \( v \in C_1 \) as follows:

\[
v \longrightarrow \frac{1}{|C_2|^{1/2}} \sum_{w \in C_2} |v + w\rangle.
\]  \hspace{1cm} (1)

If \( v_1 - v_2 \in C_2 \), then the codewords corresponding to \( v_1 \) and \( v_2 \) are the same. Hence these codewords correspond to cosets of \( C_2 \) in \( C_1 \), and this code protects a Hilbert space
of dimension $2^{\dim C_1 - \dim C_2}$. Moreover, there is a class of quantum error correcting codes equivalent to $Q$, and parameterized by two $n$-bit binary vectors $x$ and $z$. Suppose that $Q$ is determined as above by $C_1$ and $C_2$. Then $Q_{x,z}$ has basis vectors indexed by cosets of $C_2$ in $C_1$, and for $v \in C_1$, the corresponding codeword is

$$v \longrightarrow |\xi_{v,z,x}\rangle = \frac{1}{|C_2|^{1/2}} \sum_{w \in C_2} (-1)^{z \cdot w} |x + v + w\rangle. \quad (2)$$

We now make some presumptions. Without any loss of generality, we assume a Pauli channel between Alice and Bob. All Eve’s action can be regarded as (part of) channel noise. A pauli channel is a channel acts independently on each qubit by the Pauli matrices with classical probability. We shall only consider two independent errors which are $\sigma_x$ (error(bit flip error)) and $\sigma_z$ (phase flip) error. All $\sigma_y$ error can be regarded as the joint error of $\sigma_x$ and $\sigma_z$. Although the channel is noisy, we assume all qubits stored by Alice are never corrupted. Moreover, we assume the classical communication between Alice and Bob is noiseless.

We start from recalling Lo and Chau’s protocol based on the entanglement purification. Suppose initially Alice and Bob share some impure EPR pairs. They randomly select a subset of them to check the bit flip error rate and the phase flip error rate. They then distill a small number of almost perfect EPR pairs from the remained pairs. They obtain the final key by measuring them in each side in $Z$ basis. Note that here the only thing that is important is to distill some almost perfect EPR pairs, it does not matter on how Alice prepares the initial state. Actually, the random Hadamard transform on the code qubits in Shor-Prekskill protocol is unnecessary. Note that every qubit in transmission has the same density operator. In intercepting the qubits from Alice, Eve has neither classical information nor quantum information to distinguish which ones are check bits and which ones are code bits. Eve cannot treat them differently. Therefore the bit flip error rate and the phase flip error rate in the code bits must be close to that in the check bits, given a large number of check bits and code bits. All these properties are not unchanged no matter whether Alice takes random Hadamard transformation to the code bits which are sent to Bob. We therefore have the following modified Lo-Chau-Shor-Prekskill scheme:
Protocol 1: Modified Lo-Chau-Shor- Preskill protocol

1: Alice creates $2n$ EPR pairs in the state $(\Phi^+)^\otimes n$.

2: Alice sends the second half of each EPR pair to Bob.

3: Bob receives the qubits and publicly announces this fact.

4: Alice selects $n$ of the $2n$ encoded EPR pairs to serve as check bits to test for Eve’s interference. In using the check bits, she just randomly chooses the Z or X basis to measure and tells Bob does the same measurement to his halves on the same basis. They compare the measurement result on each check qubits. If too many of these measurements outcomes disagree, they abort the protocol.

5: Alice and Bob make the measurements on their code qubits of $\sigma^z[r]$ for each row $r \in H_1$ and $\sigma^x[r]$ for each row $r \in H_2$. Alice and Bob share the results, compute the syndromes for bit and phase flips, and then transform their state so as to obtain some nearly perfect EPR pairs.

6: Alice and Bob measure the EPR pairs in the $|0\rangle, |1\rangle$ basis to obtain a shared secret key.

Different from that in [8], here Alice does not take any random Hadamard transform to the code qubits sent to Bob. Actually, step 5 can be replaced by a two way communication purification scheme satisfying certain restrictions [11]. Moreover, it can be divided into two steps, i.e., correcting all the bit flip error first and then correcting the phase flip errors. Now we show how to correct all bit flip errors. We shall call a purification protocol using the above two steps as the extremely unsymmetric protocol in comparison with the normal protocols correcting bit flips and phase flips alternatively. This includes two stages:

1. **Crude bit flip error correction correction**: Sharing a large number( say, $n$) of imperfect EPR pairs with known upper bound of bit flip error rate, Alice and Bob may just randomly pick out two pairs($ j$ and $k$) and compare the parity. More specifically, they take a controlled-not operation $U_c$ on each side(they use qubit $j$ as the control qubit and qubit $k$ as the target qubit). They each measure the target bit, $k$ in $Z$ basis and compare the
value (see figure 1). $U_c$ here is defined as

$$U_c|x_j, x_k\rangle = |x_j, x_j \oplus x_k\rangle$$

where $|x_j, x_k\rangle$ is any possible quantum state for qubits $j$ and $k$, expressed in Z basis. If the values on each side are same, they drop the target qubit $k$ and keep the control qubit in a new set $d_1$. If the values are different, they drop both qubits. They then randomly pick out another two pairs from the remained $n - 2$ pairs and check the parity again by the controlled-not gate and measurement on the target qubits in Z basis in each side. They can repeatedly do so until they have picked out all $n$ imperfect pairs. If the original bit error rate for the $n$ imperfect pairs is $\epsilon_b$, the new bit error rate in the set $d_1$ is now reduced to a little bit higher than $\epsilon_b^2$. They can take the same parity check action to the qubits in the new set $d_1$. They can take the similar action iteratively until they believe that the bit flip error rate in the remained qubits have been decreased to a very small value, e.g., $10^{-3}$ (or $10^{-4}$). They then divide their qubits into a number of subset $\{S_i\}$, e.g., each subset includes 100 (or 1000) qubits. There must be some subsets where the bit flip errors have been all corrected. Now they have to find out those subsets whose bit errors have been all corrected.

2. **Verification of zero bit flip error:** The task now is to find out which subsets have been corrected perfectly on bit flip errors. We can use the verification scheme by asking the fair questions used in [4]. Let’s consider an arbitrary subset $S_i$. Suppose there are $n_s$ qubits in this subset. Zero bit flip error on this subset means that, if Alice and Bob measured each of them in Z basis, they would share a common string

$$s_i = s_iA = s_iB.$$ 

Here $s_iA$ and $s_iB$ are the strings for bit values at Alice’s side and Bob’s side, respectively. Suppose they each had measured their qubits of $S_i$. To verify $s_iA = s_iB$ is equivalent to verify that $s_{i0} = s_{iA} \oplus \bar{s}_{iB} = r_0$, where $r_0$ is a string with all elements 1. i.e. $r_0 = 111 \cdots 1$ and $\bar{s}_{iB} = r_0 \oplus s_{iB}$. To verify a classical string $s_{i0} = r_0$, Alice may generate $m$ random strings $\{R_j\}$ in the same length with $s_{i0}$, where each bit value in the random strings $\{R_j\}$ are determined by a coin tossing. One can calculate the value $s_{i0} \cdot R_j$. If all $R_i$ satisfies

$$s_{i0} \cdot R_j (mod 2) = P(R_j)$$

(4)
$s_{i0}$ must be identical to $r_0$ with a probability $1 - 2^{-m}$. Here $P(R_i)$ is the parity of string $R_i$.

In our EPP protocol, we have to verify that there is no bit flip error for the $n_s$ pairs in the subset \{$S_i$\}. It is easy to see that

$$s_{i0} \cdot R_j \pmod{2} = (s_{iA} \cdot R_j \oplus s_{iB} \cdot R_j \oplus r_0 \cdot R_j) \pmod{2}. \quad (5)$$

Therefore the condition that $s_{i0} \cdot R_j \pmod{2} = P(R_j)$ is equivalent to

$$s_{iA} \cdot R_j \pmod{2} = s_{iB} \cdot R_j \pmod{2}, \quad (6)$$

where we have used the fact that $r_0 \cdot R_j \pmod{2} = P(R_j)$. To verify the above formula, Alice and Bob actually need not measure each of the qubits in $Z$ basis. As we are showing now, they can first take the controlled not operations in each side and gather the information of $s_{iA} \cdot R_j \pmod{2}$ and $s_{iB} \cdot R_j \pmod{2}$ to one qubit in each side therefore the measurement is only done on one qubit in each side. Alice may first create $m$ classical random string \{$R_j$\} and announce them. The length of $R_j$ are $n_s, n_s - 1 \cdots n_s - m$ respectively. They first use the random string $R_1$. Suppose all those bits in $R_1$ with bit value 1 are on the position $p_1, p_2 \cdots p_k$ (normally $k$ is around $n_s/2$), Alice and Bob each do a controlled unitary transformation $U'_c$ on qubits at the position $p_1, p_2 \cdots p_k$ in $S_j$. They use qubit $p_k$ in each side as the target qubit (see figure(2)). The unitary operator $U'_c$ is defined by

$$U'_c| x_{p_1}, x_{p_2} \cdots x_{p_k} \rangle = | x_{p_1}, x_{p_2} \cdots x_{p_{k-1}}, x'_{p_k} \rangle. \quad (7)$$

and

$$x'_{p_k} = \sum_{j=1}^{k} x_{p_j}. \quad (8)$$

Here $| x_{p_1}, x_{p_2} \cdots x_{p_k} \rangle$ is a quantum state in $Z \otimes Z \cdots Z$ basis. Unitary transformation $U'_c$ replaces the state of $k$th qubit by the parity of all the qubits of $p_1, p_2 \cdots p_k$ in $Z$ basis and keep all other qubits unchanged. Alice and Bob then measure the qubit at the position $p_k$ in each side in $Z$ basis. The outcomes are just $s_{iA} \cdot R_j \pmod{2}$ and $s_{iB} \cdot R_j \pmod{2}$, respectively. If they are different, they discard all qubits which are originally in $S_i$. If they are identical,
they discard qubit $p_k$ in $S_i$ and change the qubit index $l$ into $l-1$ for any $l > p_k$ in $S_i$. Now the qubit index is from 1 to $n_s - 1$. They use random string $R_2$ to redo the similar operation as that with string $R_1$. They take the operations repeatedly until they have exhausted all $R_j$ (or discard all qubits which are originally in $S_j$ whenever they find the values of the target bits in the two sides are different). If the target bit values in two sides are always identical, they accept the remained $n_s - m$ qubits in subset $S_i$. Now the probability of no bit flip error for the survived qubits in $S_i$ is $1 - 2^{-m}$. Suppose after the crude bit flip error correction the bit flip error is $\epsilon_b^c$ and $n_s \epsilon_b^c << 1$, the probability of discarding $S_i$ is a little bit larger than $n_s \epsilon_b^c (1 - \epsilon_b^c)^{n_s-1}$ after the verification stage. Note that after this bit flip error correction, the phase flip error for the remained qubits is increased. We denote the new phase error rate by $\epsilon'_p$. Suppose before any error correction, the bit flip error rate is $\epsilon_b$ and the phase flip error rate is $\epsilon_p$ and the joint error($\sigma_y$ type error) rate is $\epsilon_{bp}$. The prior probability for a qubit carrying a phase flip error but no bit flip error is $\epsilon_p - \epsilon_{bp}$. After all bit flip errors are corrected (i.e., all r qubits carrying only a bit flip error and all qubits carrying both errors are removed), the post probability for a qubit carrying a phase flip error is

$$
\epsilon'_p = \frac{\epsilon_p - \epsilon_{bp}}{1 - \epsilon_b}
$$

Obviously, the worst case $\epsilon_{bp} = 0$ leads to the highest value of $\epsilon'_p$. Therefore the upper bound for the new phase flip error rate after the bit flip error correction is

$$
\epsilon'_p = \frac{\epsilon_p}{1 - \epsilon_b}.
$$

(9)

Note that once all bit flip errors are corrected, the bit flip error will not increase any more by the subsequent phase flip error correction. Protocol 1 is now reduced to the following protocol

**Protocol 2: Extremely unsymmetric distillation protocol**

1: Alice creates $2n$ EPR pairs in the state $(\Phi^+)^{\otimes n}$.

2: Alice sends the second half of each EPR pair to Bob.

3: Bob receives the qubits and publicly announces this fact.
4: Alice selects $n$ of the $2n$ encoded EPR pairs to serve as check bits to test for Eve’s interference. In using the check bits, she just randomly chooses the Z or X basis to measure and tells Bob does the same measurement to his halves on the same basis. They compare the measurement result on each check qubits. They find the detected bit flip error rate is $\epsilon_b$ and the phase flip error rate is $\epsilon_p$. If these values exceed certain threshold set in advance, they abort the protocol.

5: Alice and Bob first use the crude bit flip error correction to reduce the bit flip error rate to $\epsilon_b^c$ and then divide the remained qubits into $q$ subsets, suppose there are $n_s$ qubits in each subset. They then use the verification of zero bit flip scheme as described above to distill a number of qubits where bit flip error is zero. Suppose $g$ subsets have passed the verification, Alice and Bob is now sharing $g(n_s - m)$ qubits whose bit flip error rate is strictly 0 with a probability of $1 - g \cdot 2^{-m}$ and phase flip error rate is $\epsilon'_p$.

6: Alice and Bob make the measurements on their code qubits of $\sigma_z^{[r]}$ for each row $r \in H_1$ and $\sigma_x^{[r]}$ for each row $r \in H_2$. Alice and Bob share the results, compute the syndromes for bit and phase flips, and then transform their state so as to obtain $m$ nearly perfect EPR pairs.

7: Alice and Bob measure the EPR pairs in the $|0\rangle, |1\rangle$ basis to obtain a shared secret key.

Protocol 2 is a CSS like protocol \[11\]. In particular, all operations including the controlled unitary transformations and measurements in step 5 only are done only in Z basis therefore the protocol satisfies the main theorem in \[11\]. Consequently, this protocol can be converted to the prepare and measure protocol, i.e. BB84 protocol.

In particular, using the arguments in Ref. \[8\], step 6 and 7 in protocol 2 can be reduced to the encoding and decoding of quantum CSS code and can be further reduced to a prepare and measure protocol followed by decoding a CSS code with one way classical communication. Step 6 and 7 are equivalent to the case that Alice starts with $g(n_s - m)$ perfect EPR pairs and send the second halves to Bob through an unsymmetric noisy channel causing no bit
flip error and a phase flip error rate bounded by $\epsilon'_p$. After Bob received the qubits from Alice they measure the syndromes and then distill a small number of perfect EPR pairs. As argued in Ref. [8], such a process is equivalent to the process that Alice measures each of her qubits in $Z$ basis at any time and then Alice and Bob obtain the final key by decoding a classical CSS code with one way classical communication. Specifically, step 6 and step 7 are equivalent to the following steps:

6': Alice measures all her qubits in $Z$ basis and obtain a $g(n_s - m)$-bit state $|x\rangle$. She randomly pick out a binary vector $v$ in code space $C_1$. She sends the binary classical string $x + v$ to Bob.

7': Bob measures his qubits in $Z$ basis and obtain $|x\rangle$ which is exactly identical to Alice’s measurement outcome with a probability $1 - g \cdot 2^{-m}$. With such a high probability that his state is identical to Alice’s, he simply always assumes that there is no deviation between his measurement result and Alice’s result. Using the information $x + v$ from Alice, he has a new string $v$ in code space $C_1$.

8: Alice and Bob use the coset of $v + C_2$ as their final key.

Prior to step 6’, all bit flip error had been removed, we only require our CSS code used there to correct $\epsilon'_p$ phase flip error and 0 bit flip error. Therefore we can safely set $\text{dim}(C_1) = g(n_s - m)$ in the CSS code.

Furthermore, step 5 is now followed immediately by Alice’s measurement in $Z$ basis to all of her qubits. Since all operations in step 5 are in $Z$ basis, we can change the order of all these operations. In particular, Alice may choose to measure all of her qubits in the beginning of step 5. This is equivalent to take measurement in $Z$ basis to all her code qubits in the beginning of the whole protocol. If she does so, The controlled unitary transformation and all the parity checks can be done classically as the following:

1. **Classical crude bit flip error correction**: Bob measures all his code qubits in $Z$ basis and obtain a classical string $s$. Alice and Bob randomly pick out two bits($x_j, x_k$) in the string and compare parity. If the values on each side are same, they drop $x_k$ and keep $x_j$ in a new set $d_1$. If the values are different, they drop both bits. They then
randomly pick out another two bits from the remained \( n - 2 \) bits in string \( s \) and check the parity. If the parity is same, they drop one and place another one in set \( d_1 \). If the parity is different, they drop both bits. They can repeatedly do so until they have picked out all bits in string \( s \). If the original bit error rate in string \( s \) is \( \epsilon_b \), the new bit error rate in the set \( d_1 \) is now reduced to a little bit higher than \( \epsilon_b^2 \). They can take the same parity check action to the bits in the new set \( d_1 \) and place all distilled bits in another set \( d_2 \). They can take the similar action iteratively until they believe that the bit flip error rate in the remained bits have been decreased to a very small value, e.g., \( 10^{-3} \) (or \( 10^{-4} \)). They then divide their bits into a number of substrings \( \{S_i\} \), e.g., each substring includes 100 bits (or 1000 bits). There must be some substrings where the bit flip errors have been all corrected. Now they start to find out those substrings whose bit flip errors have been all corrected.

2. **Classical verification of zero bit flip error:** Let's consider substring \( S_i \). Suppose there are \( n_s \) bits in this substring. Suppose \( s_{iA} \) and \( s_{iB} \) are the classical strings at Alice’s side and Bob’s side, respectively. Alice creates \( m \) classical random string \( \{R_j\} \) and announces them. The length of \( R_1, R_2 \cdots R_m \) are \( n_s, n_s - 1 \cdots n_s - m \), respectively. They first use the random string \( R_1 \). Suppose the last non-zero bit in \( R_1 \) is at position \( p_k \). They each calculate the value \( s_{iA} \cdot R_1(\mod 2) \) and \( s_{iB} \cdot R_1(\mod 2) \) respectively. If they get the same result, they discard bit \( p_k \) in \( S_i \) and keep all the others and change the bit index of \( l \) into \( l - 1 \) for any \( l > p_k \) in \( S_i \). Now there are only \( n_s - 1 \) bits remained in string \( S_i \). If they get a different result, they discard the whole \( S_i \). They take the operation repeatedly until they exhaust all \( R_j \) (or discard \( S_i \) once the have got the different value). If \( s_{iA} \cdot R_j(\mod 2) = s_{iB} \cdot R_j(\mod 2) \) for all \( R_j \), they accept the remaining \( n_s - m \) bits in substring \( S_i \). Now the probability of **no bit flip error** is \( 1 - 2^{-m} \) for the survived bits in \( S_i \). Suppose after the classical crude bit flip error correction the bit flip error rate is \( \epsilon_b^c \) and \( n_s \epsilon_b^c << 1 \), the probability of discarding \( S_i \) is a little bit larger than \( n_s \epsilon_b^c (1 - \epsilon_b^c)^{n_s - 1} \) after the classical verification stage. There must be a significant number of substrings that can pass the classical verification check provided the total bit flip error rate is rather small after the classical crude bit flip error correction. Again, as it was argued in [8], Alice may also chooses to measure all her check qubits
on $Z$ basis in the beginning of the protocol. If she does so, protocol 2 is equivalent to BB84 protocol, with a post selection on which ones are check qubits, which ones are code qubits and which ones are qubits measured in wrong basis which should be discarded immediately. Therefore we have the following final prepare and measure protocol:

**Protocol 3: Simplified BB84**

1: Alice generates a classical set $W = \{1, 2, 3, 4\}$. She randomly picks out one value from this set. If she gets 2, 3 or 4, she prepares a state in basis $\{|0\rangle, |1\rangle\}$. If she gets 1, she prepares a state in basis $\{|+\rangle, |-\rangle\}$. Alice creates $(4 + \delta)n$ states in this way.

2: Alice sends the resulting qubits to Bob.

3: Bob receives the $(4 + \delta)n$ qubits, measuring each of them in a basis randomly chosen from $X, Z$ by a coin tossing.

4: Alice announces the basis information for each qubit.

5: Bob discards any results where he measured in a different basis than Alice prepared. With high probability, there are at least $2n$ bits left (if not, abort the protocol). Bob chooses all those remained qubits measured in the $|+\rangle, |-\rangle$ basis and randomly chooses the same number of qubits measured in $Z$ basis as the check bits. They discard a few qubits and use the rest $n$ qubits as the code bits.

6: Alice and Bob announce the values of their check bits. If too few of these values agree, they abort the protocol. They find the bit flip error rate and the phase flip error rate on the checked bits are $\epsilon_b$ and $\epsilon_p$ respectively.

7: They use the classical crude bit flip error correction scheme and the classical verification of zero bit flip error scheme to distill $g(n_s - m)$ bits. There are strictly no bit flip error for these $g(n_s - m)$ bits with a probability $1 - g \cdot 2^{-m}$. The new phase flip error is bounded by $\epsilon_1 = \epsilon_p' + \eta$ with a probability larger than $1 - \exp(-\frac{1}{2} \eta^2 n/(\epsilon_p - \epsilon_p^2))$.

8: Alice announces $x + v$, where $x$ is a $g(n_s - m)$-bit binary string consisting of the measurement outcome for the remaining bits, and $v$ is a random binary string of $g(n_s - m)$ bits.
9: Bob subtracts $x + v$ from his code qubits, $x$, and obtains $v$.

10: Alice and Bob use the coset of $v + C_2$ as the final key.

This is a modified BB84 protocol. Here Bob measures \textit{all} the code bits in $Z$ basis instead of in the random basis $Z$ or $X$ used in the original BB84. In using the above protocol, the succeeding probability is larger than $(1 - 2^{-m} q)[1 - \exp(-\frac{1}{4} \eta^2 n / (\epsilon_p - \epsilon_p^2))]$. In making the crude error correction to bit flip error, the number of qubits in set $d_1$ will be less than $n/2$, that in $d_2$ will be less than $n/4$. The method of crude distillation plus verification is not necessarily the most efficient one. It should be interesting to find out the most efficient scheme to make the quantum key distribution without classical complexity.

In summary, we have spotted that the random Hadamard transformation on the code qubits sent to Bob is unnecessary in Alice’s state preparation in an EPP protocol for quantum key distribution. Based on this fact, we have taken a further study on the 2-EPP QKD protocol and we have constructed a prepare and measure QKD protocol where the bit flip correction and the phase flip error correction (privacy amplification) is totally decoupled therefore the complexity of CSS code decoding is totally avoided.

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Alice’s operation

FIG. 1. Controlled not operation used for the crude bit flip error correction. The horizontal lines marked by j and k are qubit j and k respectively. Alice and Bob compare the measurement outcomes of the target qubit k.

Bob’s operation

Alice’s operations

Bob’s operations

FIG. 2. Controlled unitary operation used for the verification of zero bit flip error. The horizontal lines marked by $p_i$s are qubits at position $p_i$s in set $S_j$. Alice and Bob compare the measurement outcomes of the target qubit $p_k$. 

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