Algebraic aspects of the correlation functions of the integrable higher-spin XXZ spin chains with arbitrary entries

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We discuss some fundamental properties of the XXZ spin chain, which are important in the algebraic Bethe-ansatz derivation for the multiple-integral representations of the spin-s XXZ correlation function with an arbitrary product of elementary matrices. For instance, we construct Hermitian conjugate vectors in the massless regime and introduce the spin-s Hermitian elementary matrices.

Keywords: Correlation functions; XXZ spin chains; algebraic Bethe ansatz; quantum groups; multiple-integral representations.

1. Introduction

The correlation functions of the spin-1/2 XXZ spin chain have attracted much interest in mathematical physics through the last two decades. One of the most fundamental results is the exact derivation of their multiple-integral representations. The multiple-integral representations of the XXZ correlation functions were derived for the first time by making use of the q-vertex operators through the affine quantum-group symmetry in the massive regime for the infinite lattice at zero temperature. They were also derived in the massless regime by solving the q-KZ equations. Making use of algebraic Bethe-ansatz techniques such as scalar products, the multiple-integral representations were derived for the XXZ correlation func-
tions under a non-zero magnetic field. They were extended into those at finite temperatures, and even for a large finite chain. Interestingly, they are factorized in terms of single integrals. Furthermore, the asymptotic expansion of a correlation function of the XXZ model has been systematically discussed. Thus, the exact study of the XXZ correlation functions should play an important role not only in the mathematical physics of integrable models but also in many areas of theoretical physics.

The Hamiltonian of the spin-1/2 XXZ spin chain under the periodic boundary conditions is given by

$$H_{\text{XXZ}} = \frac{1}{2} \sum_{j=1}^{L} (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z).$$

(1.1)

Here $\sigma_j^a$ ($a = X, Y, Z$) are the Pauli matrices defined on the $j$th site and $\Delta$ denotes the XXZ coupling. We define parameter $q$ by

$$\Delta = (q + q^{-1})/2.$$  

(1.2)

We define $\eta$ by $q = \exp \eta$. In the massive regime: $\Delta > 1$, we put $\eta = \zeta$ with $\zeta > 0$. At $\Delta = 1$ (i.e. $q = 1$) the Hamiltonian (1.1) gives the antiferromagnetic Heisenberg (XXX) chain. In the massless regime: $-1 < \Delta \leq 1$, we set $\eta = i\zeta$, and we have $\Delta = \cos \zeta$ with $0 \leq \zeta < \pi$ for the spin-1/2 XXZ spin chain (1.1). In the paper we consider a massless region: $0 \leq \zeta < \pi/2$ for the ground-state of the integrable spin-s XXZ spin chain.

Recently, the correlation functions and form factors of the integrable higher-spin XXX and XXZ spin chains have been derived by the algebraic Bethe-ansatz method. The solvable higher-spin generalizations of the XXX and XXZ spin chains have been derived by the fusion method in several references. In the region: $0 \leq \zeta < \pi/2s$, the spin-s ground-state should be given by a set of string solutions. Furthermore, the critical behavior should be given by the SU(2) WZNW model of level $k = 2s$ with central charge $c = 3s/(s + 1)$. For the integrable higher-spin XXZ spin chain correlation functions have been discussed in the massive regime by the method of $q$-vertex operators.

In the present paper we discuss several important points in the algebraic Bethe-ansatz derivation of the correlation functions for the integrable spin-s XXZ spin chain where $s$ is an arbitrary integer or a half-integer. In particular, we briefly discuss a rigorous derivation of the finite-sum expression of correlation functions for the spin-s XXZ spin chain.

The content of the paper consists of the following. In section 2 we formulate the $R$-matrices in the homogeneous and principal gradings, respectively.
They are related to each other by a similarity transformation. In section 3 we introduce the Hermitian elementary matrices and construct conjugate basis vectors for the spin-$s$ Hilbert space in the massless regime. In section 4 we construct fusion monodromy matrices. In section 5, we first present formulas for expressing the Hermitian elementary matrices in terms of global operators. Then, we review the multiple-integral representations of the spin-$s$ XXZ correlation function for an arbitrary product of elementary matrices. In section 6 we briefly sketch the derivation of the finite-sum expression of correlation functions for the spin-$s$ XXZ spin chain, which leads to the multiple-integral representation in the thermodynamic limit. Here the spin-1/2 case corresponds to eq. (5.6) of Ref. 11.

2. Symmetric and asymmetric $R$-matrices

2.1. $R$-matrix and the monodromy matrix of type $(1, 1^\otimes L)$

Let us now define the $R$-matrix of the XXZ spin chain.\textsuperscript{7–9,11} For two-dimensional vector spaces $V_1$ and $V_2$, we define $R^\pm(\lambda_1 - \lambda_2)$ acting on $V_1 \otimes V_2$ by

$$R^\pm(\lambda_1 - \lambda_2) = \sum_{a,b,c,d=0,1} R^\pm(u)^{a,b}_{c,d} e^{a,c} \otimes e^{b,d} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c^\mp(u) & 0 \\ 0 & c^\pm(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(2.1)

where $u = \lambda_1 - \lambda_2$, $b(u) = \sinh u/\sinh(u+\eta)$ and $c^\pm(u) = \exp(\pm u)\sinh\eta/\sinh(u+\eta)$. We denote by $e^{a,b}$ a unit matrix that has only one nonzero element equal to 1 at entry $(a,b)$ where $a, b = 0, 1$.

The asymmetric $R$-matrix (2.1), $R^+(u)$, is compatible with the homogeneous grading of $U_q(\widehat{sl}_2)$.\textsuperscript{3,18} We denote by $R^{(p)}(u)$ or simply by $R(u)$ the symmetric $R$-matrix where $c^\pm(u)$ of (2.1) are replaced by $c(u) = \sinh\eta/\sinh(u+\eta)$.\textsuperscript{18} It is compatible with the affine quantum group $U_q(\widehat{sl}_2)$ of the principal grading.\textsuperscript{3,18} Hereafter, we denote them concisely by $R^{(w)}(u)$ with $w = \pm$ and $p$, where $w = +$ and $w = p$ in superscript show the homogeneous and the principal grading, respectively.

Let $s$ be an integer or a half-integer. We shall mainly consider the tensor product $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$ of $(2s + 1)$-dimensional vector spaces $V_j^{(2s)}$ with parameters $\xi_j$, where $L = 2sN_s$. Here $N_s$ denotes the lattice size of the spin-$s$ chain. In general, we may consider the tensor product $V_0^{(2s_0)} \otimes V_1^{(2s_1)} \otimes \cdots \otimes V_r^{(2s_r)}$ with $2s_1 + \cdots + 2s_r = L$, where $V_j^{(2s_j)}$ have parameters $\lambda_j$ or $\xi_j$ for $j = 1, 2, \ldots, r$. For a given set of matrix elements $A_{b,s}^{a,q}$ for
Let us introduce operators \( \Phi_j \) by

\[
A_{j,k} = \sum_{a,b=1}^{\ell} \sum_{\alpha,\beta} A^a_{k,\beta} I_0^{(2s_0)} \otimes I_1^{(2s_1)} \otimes \cdots \otimes I_{j-1}^{(2s_{j-1})} \otimes E^a_{j} \otimes \chi^\alpha_{j,k} \otimes E^\beta_{k} \otimes I_{k+1}^{(2s_{k+1})} \otimes \cdots \otimes I_{r}^{(2s_r)}. \tag{2.2}
\]

Here \( E^{a,b}_{j} \) denote the elementary matrices in the spin-\( s_j \) representation, each of which has nonzero matrix element only at entry \((a,b)\).

When \( s_0 = \ell/2 \) and \( s_1 = \cdots s_r = s \), we denote the type by \((\ell, (2s)^{s_r})\).

In particular, for \( s = 1/2 \), we denote it by \((\ell, 1^L)\).

### 2.2. Gauge transformations

Let us introduce operators \( \Phi_j \) with arbitrary parameters \( \phi_j \) for \( j = 0, 1, \ldots, L \) as follows:

\[
\Phi_j = \begin{pmatrix} 1 & 0 \\ 0 & e^{\phi_j} \end{pmatrix}_{[j]} = I^{\otimes(j)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{\phi_j} \end{pmatrix} \otimes I^{\otimes(L-j)}. \tag{2.3}
\]

In terms of \( \chi_{jk} = \Phi_j \Phi_k \), we define a similarity transformation on the \( R \)-matrix by

\[
R^{\chi}_{jk} = \chi_{jk} R_{jk} \chi^{-1}_{jk}. \tag{2.4}
\]

Explicitly, the following two matrix elements are transformed.

\[
\begin{align*}
(R^{\chi}_{jk})_{21}^{12} &= c(\lambda_j, \lambda_k) e^{\phi_j - \phi_k}, \\
(R^{\chi}_{jk})_{21}^{12} &= c(\lambda_j, \lambda_k) e^{\phi_j + \phi_k}.
\end{align*}
\tag{2.5}
\]

Putting \( \phi_j = \lambda_j \) for \( j = 0, 1, \ldots, L \) in eq. (2.3) we have

\[
R^\pm_{jk}(\lambda_j, \lambda_k) = (\chi_{jk})^\pm_1 R_{jk}(\lambda_j, \lambda_k) (\chi_{jk})^\pm_1 \quad (j, k = 0, 1, \ldots, L). \tag{2.6}
\]

Thus, the asymmetric \( R \)-matrices \( R^\pm_{jk}(\lambda_1, \lambda_2) \) are derived from the symmetric one through the gauge transformation \( \chi_{jk} \).

### 2.3. Monodromy matrices

Applying definition (2.2) for matrix elements \( R(u)^{ab}_{jk} \) of a given \( R \)-matrix, \( R(u)^{ab}_{jk} \) for \( w = \pm \) and \( p \), we define \( R \)-matrices \( R^w_{jk}(\lambda_j, \lambda_k) = R^w_{jk}(\lambda_j - \lambda_k) \) for integers \( j, k \) and \( 0 \leq j < k \leq L \). For integers \( j, k, \ell \) with \( 0 \leq j < k < \ell \leq L \), the \( R \)-matrices satisfy the Yang-Baxter equations

\[
\begin{align*}
R^w_{jk}(\lambda_j - \lambda_k) R^w_{j\ell}(\lambda_j - \lambda_\ell) R^w_{k\ell}(\lambda_k - \lambda_\ell) \\
= R^w_{k\ell}(\lambda_k - \lambda_\ell) R^w_{j\ell}(\lambda_j - \lambda_\ell) R^w_{jk}(\lambda_j - \lambda_k).
\end{align*}
\tag{2.7}
\]
Let us introduce notation for expressing products of $R$-matrices.

$$R_{1,2}^{(w)} = R_{1n}^{(w)} \cdots R_{13}^{(w)} R_{12}^{(w)} ,$$
$$R_{12}^{(w)} = R_{1n}^{(w)} R_{2n}^{(w)} \cdots R_{n-1,n}^{(w)} .$$

Here $R_{ab}^{(w)}$ denote the $R$-matrix $R_{ab}^{(w)} (\lambda_a - \lambda_b)$ for $a, b = 1, 2, \ldots, n$.

We now define the monodromy matrix of principal grading, $\Phi_{12 \cdots L}^{(1,1)} (\lambda; \{ w_j \}_L)$, of type $(1, 1^{\otimes L})$ with grading $w$. Expressing the symbol $(1, 1^{\otimes L})$ briefly as $(1, 1)$ in superscript we define it by

$$T_{0,12 \cdots L}^{(1,1)} (\lambda_0; \{ w_j \}_L) = R_{0L}^{(w)} (\lambda_0 - w_L) \cdots R_{02}^{(w)} (\lambda_0 - w_2) R_{01}^{(w)} (\lambda_0 - w_1)$$
$$= R_{0L}^{(w)} R_{0L-1}^{(w)} \cdots R_{01}^{(w)} = R_{0,12 \cdots L}^{(w)} (\lambda_0; \{ w_j \}_L) .$$

Here we have put $\lambda_j = w_j$ for $j = 1, 2, \ldots, L$. They are arbitrary. We call them inhomogeneous parameters. We express the operator-valued matrix elements of the monodromy matrix as follows.

$$T_{0,12 \cdots L}^{(1,1)} (\lambda; \{ w_j \}_L) = \begin{pmatrix} A_{12 \cdots L}^{(1+)} (\lambda; \{ w_j \}_L) & B_{12 \cdots L}^{(1+)} (\lambda; \{ w_j \}_L) \\ C_{12 \cdots L}^{(1+)} (\lambda; \{ w_j \}_L) & D_{12 \cdots L}^{(1+)} (\lambda; \{ w_j \}_L) \end{pmatrix} .$$

We also denote the operator-valued matrix elements by $[T_{0,12 \cdots L}^{(1,1)} (\lambda; \{ w_j \}_L)]_{a,b}$ for $a, b = 0, 1$. Here $\{ w_j \}_L$ denotes the inhomogeneous parameters $w_1, w_2, \ldots, w_L$. Hereafter we denote by $\{ \mu_j \}_N$ the set of $N$ numbers or parameters $\mu_1, \ldots, \mu_N$.

The monodromy matrix of principal grading, $T_{0,12 \cdots L}^{(1,1)} (\lambda; \{ w_j \}_L)$, is related to that of homogeneous grading via similarity transformation $\chi_{01 \cdots L} = \Phi_0 \Phi_1 \cdots \Phi_L$ as follows.

$$T_{0,12 \cdots L}^{(1,1)} (\lambda; \{ w_j \}_L) = \chi_{01 \cdots L} T_{0,12 \cdots L}^{(1,1)} (\lambda; \{ w_j \}_L) \chi_{01 \cdots L}^{-1}$$
$$= \begin{pmatrix} A_{12 \cdots L}^{(1+)} (\lambda; \{ w_j \}_L) & B_{12 \cdots L}^{(1+)} (\lambda; \{ w_j \}_L) \\ C_{12 \cdots L}^{(1+)} (\lambda; \{ w_j \}_L) & D_{12 \cdots L}^{(1+)} (\lambda; \{ w_j \}_L) \end{pmatrix} .$$

In Ref. operator $A^{(1+)} (\lambda)$ has been written as $A^+ (\lambda)$.

### 2.4. Operator $\tilde{R}$: Another form of the R-matrix

Let $V_1$ and $V_2$ be $(2s + 1)$-dimensional vector spaces. We define permutation operator $\Pi_{1,2}$ by

$$\Pi_{1,2} v_1 \otimes v_2 = v_2 \otimes v_1 , \quad v_1 \in V_1 , \quad v_2 \in V_2 .$$

In the spin-1/2 case, we define operator $\tilde{R}_{j,j+1}^{(w)} (u)$ by

$$\tilde{R}_{j,j+1}^{(w)} (u) = \Pi_{j,j+1} \tilde{R}_{j,j+1}^{(w)} .$$
3. The quantum group invariance

3.1. Quantum group \( U_q(sl_2) \)

The quantum algebra \( U_q(sl_2) \) is an associative algebra over \( \mathbb{C} \) generated by \( X^\pm, K^\pm \) with the following relations:\(^{45-47}\)

\[
KK^{-1} = KK^{-1} = 1, \quad KX^\pm K^{-1} = q^{\pm 2}X^\pm,
\]

\[
[X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}.
\]

(3.1)

The algebra \( U_q(sl_2) \) is also a Hopf algebra over \( \mathbb{C} \) with comultiplication

\[
\Delta(X^+) = X^+ \otimes 1 + K \otimes X^+,
\]

\[
\Delta(X^-) = X^- \otimes K^{-1} + 1 \otimes X^-,
\]

\[
\Delta(K) = K \otimes K,
\]

(3.2)

and antipode: \( S(K) = K^{-1}, S(X^+) = -K^{-1}X^+, S(X^-) = -X^-K \), and coproduct: \( \epsilon(X^\pm) = 0 \) and \( \epsilon(K) = 1 \).

It is easy to see that the asymmetric \( R \)-matrix gives an intertwiner of the spin-1/2 representation of \( U_q(sl_2) \):

\[
R^{12}_{12}(u)\Delta(x) = \tau \circ \Delta(x)R^{12}_{12}(u) \quad \text{for} \quad x = X^\pm, K.
\]

(3.3)

Here we remark that spectral parameter \( u \) is arbitrary and independent of \( X^\pm \) or \( K \).

3.2. Temperley-Lieb algebra

Operators \( \tilde{R}^{\pm}_{j,j+1}(u) \) are decomposed in terms of the generators of the Temperley-Lieb algebra as follows:\(^7\)

\[
\tilde{R}^{\pm}_{j,j+1}(u) = I - b(u)U^\pm_j.
\]

(3.4)

\( U^+_j \)'s \((U^-_j)'s\) satisfy the defining relations of the Temperley-Lieb algebra:\(^7\)

\[
U^\pm_j U^\pm_{j+1} U^\pm_j = U^\pm_j,
\]

\[
U^\pm_{j+1} U^\pm_j U^\pm_{j+1} = U^\pm_j, \quad \text{for} \quad j = 0, 1, \ldots, L - 2,
\]

\[
(U^\pm_j)^2 = (q + q^{-1})U^\pm_j \quad \text{for} \quad j = 0, 1, \ldots, L - 1,
\]

\[
U^\pm_j U^\pm_k = U^\pm_k U^\pm_j \quad \text{for} \quad |j - k| > 1.
\]

(3.5)

We remark that the asymmetric \( R \)-matrices \( \tilde{R}^{\pm}_{j,j+1}(u) \) derived from the symmetric \( R \)-matrix through the gauge transformation are related to the Jones polynomial.\(^{49}\)
3.3. Basis vectors of spin-ℓ/2 representation of $U_q(sl_2)$

Let us introduce the $q$-integer for an integer $n$ by $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. We define the $q$-factorial $[n]_q!$ for integers $n$ by

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q.$$  

For integers $m$ and $n$ satisfying $m \geq n \geq 0$ we define the $q$-binomial coefficients as follows

$$\binom{m}{n}_q = \frac{[m]_q!}{[m-n]_q! [n]_q!}.$$  

We now define the basis vectors of the $(\ell+1)$-dimensional irreducible representation of $U_q(sl_2)$, $|\ell,n\rangle$ for $n = 0, 1, \ldots, \ell$ as follows. We define $|\ell,0\rangle$ by

$$|\ell,0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_\ell.$$  

Here $|\alpha\rangle_j$ for $\alpha = 0, 1$ denote the basis vectors of the spin-1/2 representation defined on the $j$th position in the tensor product. We define $|\ell,n\rangle$ for $n \geq 1$ and evaluate them as follows:

$$|\ell,n\rangle = \left(\Delta^{(\ell-1)}(X^-)\right)^n |\ell,0\rangle \frac{1}{[n]_q!} \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma_{i_1}^+ \cdots \sigma_{i_n}^-(0) q^{i_1 + i_2 + \cdots + i_n - n\ell + n(n-1)/2}.$$  

We define the conjugate vectors explicitly by the following:

$$\langle \ell,n| = \left[\begin{array}{c} \ell \\ n \end{array}\right]_q^{-1} q^{n(n-\ell)} \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \langle 0|\sigma_{i_1}^+ \cdots \sigma_{i_n}^+ q^{i_1 + i_2 + \cdots + i_n - n\ell + n(n-1)/2}.$$  

It is easy to show the normalization conditions: $\langle \ell,n| |\ell,n\rangle = 1$. In the massive regime where $q = \exp \eta$ with real $\eta$, conjugate vectors $\langle \ell,n|$ are Hermitian conjugate to vectors $|\ell,n\rangle$.

3.4. Conjugate vectors

In order to construct Hermitian elementary matrices in the massless regime where $|q| = 1$, we now introduce another set of dual basis vectors. For a given nonzero integer $\ell$ we define $\langle \ell,n|$ for $n = 0, 1, \ldots, n$, by

$$\langle \ell,n| = \left[\begin{array}{c} \ell \\ n \end{array}\right]^{-1} \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \langle 0|\sigma_{i_1}^+ \cdots \sigma_{i_n}^+ q^{-(i_1 + \cdots + i_n) + n\ell - n(n-1)/2}.$$  

(3.11)
They are conjugate to $|\ell,m\rangle |\ell,n\rangle = \delta_{m,n}$. Here we have denoted the binomial coefficients for integers $\ell$ and $n$ with $0 \leq n \leq \ell$ as follows.

$$\binom{\ell}{n} = \frac{\ell!}{(\ell-n)!n!}. \quad (3.12)$$

We now introduce vectors $|\tilde{\ell},n\rangle$ which are Hermitian conjugate to $|\ell,n\rangle$ when $|q| = 1$ for positive integers $\ell$ with $n = 0, 1, \ldots, \ell$. Setting the norm of $|\tilde{\ell},n\rangle$ such that $\langle \tilde{\ell},n|\tilde{\ell},n\rangle = 1$, vectors $|\tilde{\ell},n\rangle$ are given by

$$\sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma_{i_1} \cdots \sigma_{i_n} |0\rangle q^{-(i_1 + \cdots + i_n) + n(\ell-n) - \ell(n-1)/2} \left[ \frac{\ell}{n} \right]_q q^{-n(n-1)/2} \binom{\ell}{n}.$$

$$\quad (3.13)$$

We have the following normalization condition:

$$\langle \tilde{\ell},n|\tilde{\ell},n\rangle = \left[ \frac{\ell}{n} \right]_q^2 \binom{\ell}{n}^{-2}. \quad (3.14)$$

### 3.5. Hermitian elementary matrices

In the massless regime we define elementary matrices $\tilde{E}^{m,n}(2s\pm)$ for $m,n = 0,1,\ldots,2s$ by

$$\tilde{E}^{m,n}(2s\pm) = |\tilde{2s},m\rangle \langle 2s,n| \quad (3.15)$$

In the massless regime where $|q| = 1$, matrix $|\tilde{\ell},n\rangle \langle \tilde{\ell},n|$ is Hermitian: $\langle \tilde{\ell},n|\tilde{\ell},n\rangle^\dagger = |\tilde{\ell},n\rangle \langle \tilde{\ell},n|$. However, in order to define projection operators $P$ such that $PP = P$, we have formulated vectors $|\tilde{\ell},n\rangle$.

### 3.6. Projection operators

We define projection operators acting on the 1st to the $\ell$th tensor-product spaces by

$$P_{1\cdots \ell}^{(\ell)} = \sum_{n=0}^{\ell} |\tilde{\ell},n\rangle \langle \tilde{\ell},n|. \quad (3.16)$$

Let us now introduce another set of projection operators $\tilde{P}_{1\cdots \ell}^{(\ell)}$ as follows.

$$\tilde{P}_{1\cdots \ell}^{(\ell)} = \sum_{n=0}^{\ell} |\tilde{\ell},n\rangle \langle \tilde{\ell},n|. \quad (3.17)$$
Projector $\tilde{P}^{(t)}_{1\cdots t}$ is idempotent: $(\tilde{P}^{(t)}_{1\cdots t})^2 = \tilde{P}^{(t)}_{1\cdots t}$. In the massless regime where $|q| = 1$, it is Hermitian: $(\tilde{P}^{(t)}_{1\cdots t})^\dagger = \tilde{P}^{(t)}_{1\cdots t}$. From (3.16) and (3.17), we show the following properties:

$$
\begin{align*}
\tilde{P}^{(t)}_{12\cdots t} \tilde{P}^{(t)}_{1\cdots t} &= \tilde{P}^{(t)}_{12\cdots t}, \\
\tilde{P}^{(t)}_{1\cdots t} \tilde{P}^{(t)}_{12\cdots t} &= \tilde{P}^{(t)}_{1\cdots t}.
\end{align*}
$$

(3.18) (3.19)

In the tensor product of quantum spaces, $V^{(2s)}_1 \otimes \cdots \otimes V^{(2s)}_{N_s}$, we define $\tilde{P}^{(2s)}_{12\cdots L}$ by

$$
\tilde{P}^{(2s)}_{12\cdots L} = \prod_{i=1}^{N_s} \tilde{P}^{(2s)}_{2s(i-1)+1}.
$$

(3.20)

Here we recall $L = 2sN_s$.

The projection operators are also constructed by the fusion method. For $\ell > 2$ we can construct projection operators inductively with respect to $\ell$ as follows.\cite{25,26,46}

$$
P^{(t)}_{12\cdots t} = P^{(\ell-1)}_{12\cdots t-1} \tilde{R}^{+}_{t-1,\ell-1} ((\ell - 1)\eta) P^{(\ell-1)}_{12\cdots t-1}.
$$

(3.21)

The projection operator $P^{(t)}_{12\cdots t}$ gives a $q$-analogue of the full symmetrizer of the Young operators for the Hecke algebra.\cite{46}

4. Fusion construction

4.1. Higher-spin monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$

We now set the inhomogeneous parameters $w_j$ for $j = 1, 2, \ldots, L$, as $N_s$ sets of complete 2s-strings.\cite{18} We define $w^{(2s)}_{(b-1)\ell + \beta}$ for $\beta = 1, \ldots, 2s$, as follows.

$$
w^{(2s)}_{2s(b-1)\ell + \beta} = \xi_b - (\beta - 1)\eta, \quad \text{for} \quad b = 1, 2, \ldots, N_s.
$$

(4.1)

We shall define the monodromy matrix of type $(1, (2s)^{\otimes N_s})$ associated with homogeneous grading. We first define the massless monodromy matrix by

$$
\begin{align*}
\tilde{T}^{(1,2s+)}_{0,12\cdots N_s} (\lambda_0; \{ \xi_b \}_{N_s}) &= \tilde{F}^{(2s)}_{12\cdots L} R^{(1,1+)}_{0,1\cdots L} (\lambda_0; \{ w^{(2s)}_{2s} \}_{L}) \tilde{F}^{(2s)}_{12\cdots L} \\
&= \begin{pmatrix} \tilde{A}^{(2s+)}(\lambda; \{ \xi_b \}_{N_s}) & \tilde{B}^{(2s+)}(\lambda; \{ \xi_b \}_{N_s}) \\
\tilde{C}^{(2s+)}(\lambda; \{ \xi_b \}_{N_s}) & \tilde{D}^{(2s+)}(\lambda; \{ \xi_b \}_{N_s}) \end{pmatrix}.
\end{align*}
$$

(4.2)

Let us introduce a set of 2s-strings with small deviations from the set of complete 2s-strings.

$$
w^{(2s;\epsilon)}_{2s(b-1)\ell + \beta} = \xi_b - (\beta - 1)\eta + \epsilon^{(\beta)}_b, \quad \text{for} \quad b = 1, 2, \ldots, N_s, \quad \text{and} \quad \beta = 1, 2, \ldots, 2s.
$$

(4.3)
Here $\epsilon$ is very small and $r^{(\beta)}_b$ are generic parameters. We express the elements of the monodromy matrix $T^{(1,1)}$ with inhomogeneous parameters given by $w_j^{(2s; \epsilon)}$ for $j = 1, 2, \ldots, L$ as follows.

$$T^{(1,1)}_{0, 12 \ldots L}(\lambda; \{w_j^{(2s; \epsilon)}\}_L) = \begin{pmatrix}
A_{12 \ldots L}^{(2s; \epsilon)}(\lambda) & B_{12 \ldots L}^{(2s; \epsilon)}(\lambda) \\
C_{12 \ldots L}^{(2s; \epsilon)}(\lambda) & D_{12 \ldots L}^{(2s; \epsilon)}(\lambda)
\end{pmatrix}. \quad (4.4)
$$

Here $A_{12 \ldots L}^{(2s; \epsilon)}(\lambda)$ denotes $A_{12 \ldots L}(\lambda; \{w_j^{(2s; \epsilon)}\}_L)$.

$$\tilde{A}_{12 \ldots N_s}^{(2s; \epsilon)}(\lambda; \{\xi_p\}_{N_s}) = \lim_{\epsilon \to 0} \tilde{P}_{12 \ldots L}^{(2s)} A_{12 \ldots L}^{(2s; \epsilon)}(\lambda; \{w_j^{(2s; \epsilon)}\}_L) \tilde{P}_{12 \ldots L}^{(2s)} \quad (4.5)
$$

We define the massless monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$ by

$$\tilde{T}_{0, 12 \ldots N_s}^{(\ell, 2s)}(\lambda; \{\xi_p\}_{N_s}) = \tilde{P}_{a_1 a_2 \ldots a_\ell}^{(\ell)} \tilde{A}_{a_1, 12 \ldots N_s}^{(1, 2s; 2s)}(\lambda_{a_1}) \tilde{T}_{a_2, 12 \ldots N_s}^{(1, 2s; 2s)}(\lambda_{a_2} - \eta) \cdots \tilde{T}_{a_\ell, 12 \ldots N_s}^{(1, 2s; 2s)}(\lambda_{a_\ell} - (\ell - 1)\eta) \tilde{P}_{a_1 a_2 \ldots a_\ell}^{(\ell)}. \quad (4.6)
$$

### 4.2. Integrable spin-s Hamiltonians

We define the massless transfer matrix\(^1\) of type $(\ell, (2s)^{\otimes N_s})$ by

$$\tilde{T}_{12 \ldots N_s}^{(\ell, 2s)}(\lambda) = \text{tr}_{V_1^{(2s)}} \left( \tilde{T}_{12 \ldots N_s}^{(\ell, 2s)}(\lambda) \right) = \sum_{n=0}^{2s} a_{\ell, n} \left| T_{a_1, 12 \ldots N_s}^{(1, 2s; 2s)}(\lambda) \right| \times \tilde{T}_{a_2, 12 \ldots N_s}^{(1, 2s; 2s)}(\lambda - \eta) \cdots \tilde{T}_{a_\ell, 12 \ldots N_s}^{(1, 2s; 2s)}(\lambda - (\ell - 1)\eta) \left| \ell, n \right|_{a}. \quad (4.7)
$$

It follows from the Yang-Baxter equations that the higher-spin transfer matrices commute in the tensor product space $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$, which is derived by applying projection operator $P_{12 \ldots L}^{(2s)}$ to $V_1^{(1)} \otimes \cdots \otimes V_{N_s}^{(1)}$.

The massless spin-s $R$-matrix $\tilde{R}_{12 \ldots N_s}^{(2s; 2s; 2s)}(u)$ becomes the permutation operator at $u = 0$: $\tilde{R}_{12 \ldots 2s}^{(2s; 2s; 2s)}(0) = \Pi_{1, 2, \ldots, 2s}$. Therefore, putting inhomogeneous parameters $\xi_p = 0$ for $p = 1, 2, \ldots, N_s$, we show that the transfer matrix $\tilde{T}_{12 \ldots N_s}^{(2s; 2s; 2s)}(\lambda)$ becomes the shift operator at $\lambda = 0$. We derive the massless spin-s XXZ Hamiltonian by the logarithmic derivative of the massless spin-s transfer matrix.

$$\mathcal{H}_{XXZ}^{(2s)} = \frac{d}{d\lambda} \log \tilde{T}_{12 \ldots N_s}^{(2s; 2s; 2s)}(\lambda) \bigg|_{\lambda = 0, \xi = 0} = \sum_{i=1}^{N_s} \frac{1}{d} \left. \frac{d}{du} \tilde{R}_{i+1}^{(2s; 2s)}(u) \right|_{u = 0}. \quad (4.8)$$
5. Spin-$\ell/2$ massless XXZ correlation functions

5.1. Spin-$s$ local operators in terms of global operators

In the massless regime, we can express the Hermitian elementary matrices in terms of global operators as follows. For $m \geq n$ we have

$$\tilde{E}^{m, n}(\ell) = \begin{pmatrix} \ell \\ n \\ m \end{pmatrix} E^{(\ell)}_{1 \ldots \ell} \prod_{\alpha=1}^{(i-1)\ell} (A^{(1)+} + D^{(1)+})(w_\alpha)$$

$$\times \prod_{k=1}^{n} D^{(1)+}(w_{(i-1)\ell+k}) \prod_{k=m+1}^{\ell} A^{(1)+}(w_{(i-1)\ell+k})$$

$$\times \prod_{\alpha=\ell+1}^{\ell N_s} (A^{(1)+} + D^{(1)+})(w_\alpha) \tilde{P}^{(\ell)}_{1 \ldots \ell}. \quad (5.1)$$

For $m \leq n$ we have

$$\tilde{E}^{m, n}(\ell) = \begin{pmatrix} \ell \\ n \\ m \end{pmatrix} E^{(\ell)}_{1 \ldots \ell} \prod_{\alpha=1}^{(i-1)\ell} (A^{(1)+} + D^{(1)+})(w_\alpha)$$

$$\times \prod_{k=1}^{m} D^{(1)+}(w_{(i-1)\ell+k}) \prod_{k=m+1}^{n} C^{(1)+}(w_{(i-1)\ell+k}) \prod_{k=n+1}^{\ell} A^{(1)+}(w_{(i-1)\ell+k})$$

$$\times \prod_{\alpha=\ell+1}^{\ell N_s} (A^{(1)+} + D^{(1)+})(w_\alpha) \tilde{P}^{(\ell)}_{1 \ldots \ell}. \quad (5.2)$$

By the quantum inverse-scattering problem (QISP) of Ref. 10 the local spin operators are expressed in terms of global operators and the transfer matrices for the integrable spin-$s$ XXX spin chain. However, it is not clear how one can derive (5.1) and (5.2) by the QISP method even for $q = 1$.

5.2. Symbols for expressing sequences

Let us denote by $(a_j)_m$ a sequence of numbers $a_j$ for $j = 1, 2, \ldots, m$, i.e. $(a_j)_m = (a_1, a_2, \ldots, a_m)$.

**Definition 5.1.** We say that a sequence $(b_k)_n$ is a subsequence of $(a_j)_m$ if (i) $n \leq m$, (ii) $b_k \in \{a_1, \ldots, a_m\}$ for $k = 1, 2, \ldots, n$, (iii) for any pair of integers $j$ and $k$ satisfying $1 \leq j < k \leq n$, there exists a pair of integers $\ell(j)$ and $\ell(k)$ such that $a_j = b_{\ell(j)}$, $a_k = b_{\ell(k)}$ and $\ell(j) < \ell(k)$.

For a pair of sequences $(a_j)_m$ and $(b_k)_n$, we define the product $(a_j)_m \#(b_k)_n$ by a sequence $(c_j)_{m+n}$ such that $c_j = a_j$ for $j = 1, 2, \ldots, m$ and $c_j = b_j$ for $j = m+1, m+2, \ldots, m+n$. 
5.3. Conjecture of the spin-s Ground-state solution

Let us now introduce the conjecture that the ground state of the spin-s case $|\psi_g^{(2s+)}\rangle$ is given by $N_s/2$ sets of $2s$-strings:

$$\lambda_a^{(\alpha)} = \mu_a - (\alpha - 1/2)\eta + \epsilon_a^{(\alpha)}, \quad \text{for } a = 1, 2, \ldots, N_s/2 \text{ and } \alpha = 1, 2, \ldots, 2s. \quad (5.3)$$

Here we assume that string deviations $\epsilon_a^{(\alpha)}$ are very small when $N_s$ is very large. In terms of $\lambda_a^{(\alpha)}$, the spin-s ground state in the homogeneous grading is given by

$$|\psi_g^{(2s+)}\rangle = \prod_{a=1}^{N_s/2} \prod_{\alpha=1}^{2s} B^{(2s+)}(\lambda_a^{(\alpha)}; \{\xi_p\}_N) |0\rangle. \quad (5.4)$$

We denote by $M$ the number of Bethe roots: $M = 2s N_s/2 = sN_s$.

According to analytic and numerical studies, we may assume the following properties of string deviations $\epsilon_a^{(\alpha)}$'s. For very large $N_s$, the deviations are given by $\epsilon_a^{(\alpha)} = i \delta_a^{(\alpha)}$, where $i$ denotes $\sqrt{-1}$ and $\delta_a^{(\alpha)}$ are real. Moreover, $\delta_a^{(\alpha)} - \delta_a^{(\alpha+1)} > 0$ for $\alpha = 1, 2, \ldots, 2s - 1$, and $|\delta_a^{(\alpha)}| > |\delta_a^{(\alpha+1)}|$ for $\alpha < s$, while $|\delta_a^{(\alpha)}| < |\delta_a^{(\alpha+1)}|$ for $\alpha \geq s$.

In the limit: $N_s \to \infty$, the density of string centers, $\rho_{tot}(\mu)$, is given by

$$\rho_{tot}(\mu) = \frac{1}{N_s} \sum_{p=1}^{N_s} \frac{1}{2\zeta \cosh(\pi(\mu - \xi_p)/\zeta)}. \quad (5.5)$$

For the homogeneous chain where $\xi_p = 0$ for $p = 1, 2, \ldots, N_s$, we denote the density of string centers by $\rho(\lambda)$.

$$\rho(\lambda) = \frac{1}{2\zeta \cosh(\pi\lambda/\zeta)}. \quad (5.6)$$

Let us introduce useful notation of the suffix of rapidities. For rapidities $\lambda_a^{(\alpha)} = \lambda_{(a,\alpha)}$ we define integers $A$ by $A = 2s(a - 1) + \alpha$ for $a = 1, 2, \ldots, N_s/2$ and for $\alpha = 1, 2, \ldots, 2s$. We thus denote $\lambda_{(a,\alpha)}$ also by $\lambda_A$ for $A = 1, 2, \ldots, sN_s$, and put $\lambda_{(a,\alpha)}$ in increasing order with respect to $A = 2s(a - 1) + \alpha$ such as $\lambda_{(1,1)} = \lambda_1, \lambda_{(1,2)} = \lambda_2, \ldots, \lambda_{(N_s/2,2s)} = \lambda_{sN_s}$.

In the ground state, rapidities $\lambda_A$ for $A = 1, 2, \ldots, M$, are expressed by

$$\lambda_{2s(a-1)+\alpha} = \mu_a - (\alpha - 1/2)\eta + \epsilon_a^{(\alpha)} \quad (1 \leq a \leq N_s/2; \ 1 \leq \alpha \leq 2s). \quad (5.7)$$

For $A = 2s(a - 1) + \alpha$ with $1 \leq \alpha \leq 2s$, integer $a$ is given by $a = [(A - 1)/2s] + 1$, and integer $\alpha$ is given by $\alpha = A - 2s[(A - 1)/2s]$.

For a real number $x$ we define $[x]$ by the greatest integer less than or equal to $x$. We define $a(j)$ and $\alpha(j)$ for $j = 1, 2, \ldots, M$ as follows.

$$a(j) = [(j - 1)/2s] + 1, \quad \alpha(j) = j - 2s[(j - 1)/2s]. \quad (5.8)$$
5.4. Correlation functions of the integrable spin-s XXZ model on a long finite chain

We define the correlation function of the integrable spin-2s XXZ spin chain for a given product of \((2s + 1) \times (2s + 1)\) elementary matrices such as \(\tilde{E}_1^{i_1}, \tilde{J}_1^{(2s + 1)} \ldots \tilde{E}_m^{i_m}, \tilde{J}_m^{(2s + 1)}\) on the spin-s ground state, \(|\psi_g^{(2s + 1)}\rangle\), as follows.

\[
F_m^{(2s + 1)}(\{i_k, j_k\}) = \langle \psi_g^{(2s + 1)} | \prod_{k=1}^{m} \tilde{E}_{k}^{i_k, j_k}^{(2s + 1)} | \psi_g^{(2s + 1)} \rangle / \langle \psi_g^{(2s + 1)} | \psi_g^{(2s + 1)} \rangle.
\]

(5.9)

By formulas (5.1) and (5.2) we express the \(m\)th product of \((2s + 1) \times (2s + 1)\) elementary matrices in terms of a \(2m\)th product of \(2 \times 2\) elementary matrices with entries \(\{\epsilon_j, \epsilon_j'\}\) as follows.

\[
\prod_{b=1}^{m} \tilde{E}_{b}^{i_b, j_b}^{(2s + 1)} = C(\{i_b, j_b\}) \tilde{P}^{(2s)}_{12 \ldots L} \prod_{k=1}^{2sm} \epsilon_k^{\epsilon_k'} \cdot \tilde{P}^{(2s)}_{12 \ldots L}. \tag{5.10}
\]

By making use of (5.1) and (5.2), \(C(\{i_b, j_b\})\) is given by

\[
C(\{i_k, j_k\}) = \prod_{b=1}^{m} \left\{ \begin{array}{cc} 2s & 2s \\ i_b & j_b \end{array} \right\} \left[ \begin{array}{cc} 2s & 2s \\ q & q \end{array} \right]^{-1}. \tag{5.11}
\]

Here \(\epsilon_{2s(b-1)+\beta}^2\) and \(\epsilon_{2s(b-1)+\beta}'\) \((b = 1, \ldots, N_s; \beta = 1, \ldots, 2s)\) are given by

\[
\epsilon_{2s(b-1)+\beta} = \begin{cases} 1 & (1 \leq \beta \leq j_b) \\ 0 & (j_b < \beta \leq 2s) \end{cases}, \quad \epsilon_{2s(b-1)+\beta}' = \begin{cases} 1 & (1 \leq \beta \leq i_b) \\ 0 & (i_b < \beta \leq 2s) \end{cases}. \tag{5.12}
\]

We evaluate the spin-2s XXZ correlation function \(F_m^{(2s + 1)}(\{i_k, j_k\})\) by

\[
F_m^{(2s + 1)}(\{i_k, j_k\}) = C(\{i_k, j_k\}) \langle \psi_g^{(2s + 1)} | \tilde{P}^{(2s)}_{12 \ldots L} \times \prod_{j=1}^{2sm} \epsilon_j^{\epsilon_j'} \cdot \tilde{P}^{(2s)}_{12 \ldots L} | \psi_g^{(2s + 1)} \rangle / \langle \psi_g^{(2s + 1)} | \psi_g^{(2s + 1)} \rangle \tag{5.13}
\]

Let \(\alpha^+\) be the set of \(j\) with \(\epsilon_j = 0\), and \(\alpha^-\) the set of \(j\) with \(\epsilon_j' = 1\):

\[
\alpha^+ = \{j; \epsilon_j = 0\}, \quad \alpha^- = \{j; \epsilon_j' = 1\}. \tag{5.14}
\]

We denote by \(r\) and \(r'\) the number of elements of the set \(\alpha^-\) and \(\alpha^+\), respectively. Due to charge conservation, we have

\[
r + r' = 2sm. \tag{5.15}
\]

We denote by \(j_{\min}\) and \(j_{\max}\) the smallest element and the largest element of \(\alpha^-\), respectively. We also denote by \(j'_{\min}\) and \(j'_{\max}\) the smallest element and the largest element of \(\alpha^+\), respectively.
Recall that the ground state $|\psi^{(2s)}\rangle$ has $M$ Bethe roots with $M = sN_s$. Let $c_j$ ($j \in \alpha^-$) and $c'_j$ ($j \in \alpha^+$) be integers such that $1 \leq c_j \leq M$ for $j \in \alpha^-$ and $1 \leq c'_j \leq M + j$ for $j \in \alpha^+$. We define sequence $(b_{\ell})_{2sm}$ by

$$
(b_1, b_2, \ldots, b_{2sm}) = (c'_{j_{\text{max}}}, \ldots, c'_{j_{\text{min}}}, \ldots, c_{j_{\text{max}}}).
$$

Here sequence $(c'_{j_{\text{max}}}, \ldots, c'_{j_{\text{min}}}, \ldots, c_{j_{\text{max}}})$ is given by the composite sequence of $c'_j$'s in decreasing order with respect to suffix $j$, and $c_j$'s in increasing order with respect to suffix $j$. We introduce the following symbols:

$$
\prod_{j \in \alpha^-} \left( \sum_{c_j=1}^{M} \right) \prod_{j \in \alpha^+} \left( \sum_{c'_j=1}^{M+j} \right) = \sum_{c_{j_{\text{min}}}=1}^{M} \cdots \sum_{c_{j_{\text{max}}}=1}^{M} \sum_{c'_{j_{\text{min}}}=1}^{M+c'_{j_{\text{max}}}} \cdots \sum_{c'_{j_{\text{max}}}=1}^{M+c'_{j_{\text{max}}}}.
$$

(5.17)

Recall that $a(j)$ are defined in (5.8). We define $\beta(j)$ by

$$
\beta(j) = j - 2s[(j - 1)/2s] \quad (1 \leq j \leq M).
$$

(5.18)

For $\ell, k = 1, 2, \ldots, 2sm$, we define the $(\ell, k)$ element of $M^{(2sm)}((b_{\ell})_{2sm})$ by

$$
\left( M^{(2sm)}((b_{\ell})_{2sm}) \right)_{\ell, k} =
\begin{cases}
-\delta_{b_\ell, M, k} & (b_\ell > M) \\
\delta_{\beta(b_\ell), \beta(k)} \cdot \rho(\lambda_{b_k} - w^{(2s)}_k + \eta/2)/(N_s \rho_{\text{tot}}(\mu_{a(b_{\ell})})) & (b_\ell \leq M)
\end{cases}
$$

(5.19)

Here, continuous variable $\mu$, which is the argument of density $\rho_{\text{tot}}(\mu)$, is evaluated at $\mu_{a(b_{\ell})}$, one of the “string centers” $\mu_{a}$ of $2s$-strings (5.7).

We can rigorously derive a concise expression of correlation functions of the spin-$s$ XXZ spin chain in the massless region: $0 \leq \zeta < \pi/2s$ for a large finite chain. Introducing $\varphi(\lambda) = \sinh \lambda$ we have

$$
P_{m}^{(2s)}(\{i_k, j_k\}) = C(\{i_k, j_k\}) \prod_{j \in \alpha^-} \left( \sum_{c_j=1}^{M} \right) \prod_{j \in \alpha^+} \left( \sum_{c'_j=1}^{M+j} \right) \det M^{(2sm)}((b_{\ell})_{2sm})
\times (-1)^{n} \prod_{j \in \alpha^-} \left( \prod_{k=1}^{j-1} \varphi(\lambda_{c_j} - w^{(2s)}_k + \eta) \prod_{k=j+1}^{2sm} \varphi(\lambda_{c_j} - w^{(2s)}_k) \right)
\times \prod_{j \in \alpha^+} \left( \prod_{k=1}^{j-1} \varphi(\lambda'_{c'_j} - w^{(2s)}_k - \eta) \prod_{k=j+1}^{2sm} \varphi(\lambda'_{c'_j} - w^{(2s)}_k) \right)
\times \prod_{1 \leq k \leq \ell \leq 2sm} \varphi(\lambda_{b_k} - \lambda_{b_k} + \eta)
\times O(1/N_s).
$$

(5.20)
We remark that we derive (5.20) sending $\epsilon$ to zero. Before taking the limit, inhomogeneous parameters $w_j$'s are generic due to small parameter $\epsilon$, and the sums over variables $c_j$ in (5.20) are restricted up to $M$ for all $j$.

### 5.5. Multiple-integral representations of spin-$s$ XXZ correlation function for arbitrary matrix elements

In the thermodynamic limit: $N_s \to \infty$, rapidities $\lambda_\ell$ with $b_\ell$ defined in (5.16), correspond to integral variables $\lambda_\ell$ for $\ell = 1, 2, \ldots, 2sm$. For $1 \leq b_\ell \leq M$ they are given by the Bethe roots of $2s$-strings (5.7), while for $b_\ell > M$ they are given by complete $2s$-strings $w_j^{(2s)}$ defined by (4.1).

We define $\alpha(\lambda_j)$ by $\alpha(\lambda_j) = \gamma$ for an integer $\gamma$ with $1 \leq \gamma \leq 2s$, if $\lambda_j$ is related to integral variable $\mu_j$ by $\lambda_j = \mu_j - (\gamma - 1/2)\eta$, or if $\lambda_j$ takes a value close to $w_j^{(2s)}$ with $\beta(k) = \gamma$, where $w_j^{(2s)}$ are part of complete $2s$-strings (4.1). Here, variables $\mu_j$ correspond to “string centers” of variables $\lambda_j$.

We define the $(j, k)$ element of matrix $S = S\left((\lambda_j)_{2sm}; (w_j^{(2s)})_{2sm}\right)$ by

$$S_{j,k} = \rho(\lambda_j - w_j^{(2s)} + \eta/2) \delta(\alpha(\lambda_j), \beta(k)), \quad \text{for } j, k = 1, 2, \ldots, 2sm.$$  

(5.21)

Here $\delta(\alpha, \beta)$ denotes the Kronecker delta, and we recall (5.18) for $\beta(k)$.

Let $\Gamma_j$ be a small contour rotating counterclockwise around $\lambda = w_j^{(2s)}$. Since $\det S$ has simple poles at $\lambda = w_j^{(2s)}$ with residue $1/2\pi i$, we have

$$\int_{-\infty+ie}^{\infty+ie} \det S((\lambda_k)_{2sm}) d\lambda_1 = \int_{-\infty-ie}^{\infty-ie} \det S((\lambda_k)_{2sm}) d\lambda_1 - \oint_{\Gamma_1} \det S((\lambda_k)_{2sm}) d\lambda_1.$$  

(5.22)

For sets $\alpha^-$ and $\alpha^+$ with relation (5.16), we define integral variables $\check{\lambda}_j$ for $j \in \alpha^-$ and $\tilde{\lambda}_j$ for $j \in \alpha^+$, respectively, by the following:

$$\left(\check{\lambda}_{j_{\max}'}, \ldots, \check{\lambda}_{j_{\min}'}, \tilde{\lambda}_{j_{\min}}, \tilde{\lambda}_{j_{\max}}\right) = (\lambda_1, \ldots, \lambda_{2sm}).$$  

(5.23)

Thus, from expression (5.20) of the correlation function in terms of a finite sum, we derive the multiple-integral representation as follows.

$$F_{m}^{(2s+)}(\{i_k, j_k\}) = C(\{i_k, j_k\}) \times$$

$$\times \left(\int_{-\infty+ie}^{\infty+ie} + \ldots + \int_{-\infty-i\tilde{\zeta}+ie}^{\infty-i\tilde{\zeta}+ie}\right) d\lambda_1 \cdots \left(\int_{-\infty+ie}^{\infty+ie} + \ldots + \int_{-\infty-i\tilde{\zeta}+ie}^{\infty-i\tilde{\zeta}+ie}\right) d\lambda_{\ell'}$$

$$\times \left(\int_{-\infty-ie}^{\infty-ie} + \ldots + \int_{-\infty-i\tilde{\zeta}-ie}^{\infty-i\tilde{\zeta}-ie}\right) d\lambda_{\ell} \cdots \left(\int_{-\infty-ie}^{\infty-ie} + \ldots + \int_{-\infty-i\tilde{\zeta}-ie}^{\infty-i\tilde{\zeta}-ie}\right) d\lambda_{2sm}$$

$$\times Q(\{\epsilon_j, \epsilon_j'; \lambda_1, \ldots, \lambda_{2sm}\}) \det S(\lambda_1, \ldots, \lambda_{2sm}).$$  

(5.24)
Here $\tilde{\zeta}_s = (2s - 1)\zeta$, $\tilde{r} = r' + 1$, and $Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_{2sm})$ is given by

$$Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_{2sm})$$

$$= (-1)^{r'} \frac{\prod_{j \in \alpha}^r \left( \prod_{b=1}^{j-1} \varphi(\tilde{\lambda}_b - w_k^{(2s)}) + \eta \right) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}_k - w_k^{(2s)}/\zeta)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(\lambda_k - \lambda_\ell + \eta + \epsilon_\ell, \epsilon_k)}$$

$$\times \frac{\prod_{j \in \alpha'} \left( \prod_{k=1}^{j-1} \varphi(\tilde{\lambda}'_k - w_k^{(2s)}) - \eta \right) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}'_k - w_k^{(2s)}/\zeta)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(\tilde{w}_k^{(2s)} - \tilde{w}_\ell^{(2s)})/\zeta).} \quad (5.25)$$

In the denominator we set $\epsilon_{k,\ell} = i\epsilon$ for $Im(\lambda_k - \lambda_\ell) > 0$ and $\epsilon_{k,\ell} = -i\epsilon$ for $Im(\lambda_k - \lambda_\ell) < 0$, where $\epsilon$ is infinitesimally small: $|\epsilon| \ll 1$. Here, $Im(a + ib) = b$ for real numbers $a$ and $b$. Here, for $\alpha^\pm$, we recall (5.14).

We evaluate $\alpha(\lambda_j)$ in (5.24), replacing paths $(-\infty - i(\gamma - 1)\zeta \pm i\epsilon, \infty - i(\gamma - 1)\zeta \pm i\epsilon)$ by $(-\infty - i(\gamma - 1/2)\zeta, \infty - i(\gamma - 1/2)\zeta)$ for $\gamma = 1, 2, \ldots, 2s$, respectively. The integrals over $\lambda_j$ for $j \geq \tilde{r}$ do not change when $\epsilon \to \zeta/2$.

Thus, correlation functions (5.9) are expressed in the form of a single term of multiple integrals (5.24).

We can derive the symmetric expression for the multiple-integral representations of the spin-$s$ correlation function $F_m^{(2s+)}(\{i_k, j_k\})$ as follows:

$$F_m^{(2s+)}(\{i_k, j_k\}) = C(\{i_b, j_b\}) \times$$

$$\times \frac{1}{\prod_{1 \leq \alpha < \beta \leq 2s} \sinh^{2s}(\beta - \alpha)\eta} \prod_{1 \leq k < \ell \leq m} \sinh^{2s}(\pi(\xi_k - \xi_\ell)/\zeta)$$

$$\times \sum_{\sigma \in S_{2sm}/(S_m)^{2s}} (\text{sgn } \sigma) \prod_{j=1}^{r'} \left( \int_{-\infty + i\epsilon}^{\infty + i\epsilon} + \cdots + \int_{-\infty - i(2s-1)\zeta + i\epsilon}^{\infty - i(2s-1)\zeta + i\epsilon} \right) d\mu_{\sigma_j}$$

$$\times \prod_{j=r' + 1}^{2sm} \left( \int_{-\infty - i\epsilon}^{\infty - i\epsilon} + \cdots + \int_{-\infty - i(2s-1)\zeta - i\epsilon}^{\infty - i(2s-1)\zeta - i\epsilon} \right) d\mu_{\sigma_j}$$

$$\times Q(\{\epsilon_j, \epsilon'_j\}; \lambda_\sigma_1, \ldots, \lambda_\sigma_{2sm}) \left( \prod_{j=1}^{2sm} \prod_{b=1}^{m} \prod_{b'=1}^{m} \sinh(\lambda_j - \xi_b + \beta\eta) \prod_{b=1}^{m} \cosh(\pi(\mu_j - \xi_b)/\zeta) \right)$$

$$\times \frac{1}{(2i\zeta)^{2sm}} \prod_{\gamma=1}^{2s} \prod_{1 \leq b < a \leq m} \sinh(\pi(\mu_{2s(a-1)+\gamma} - \mu_{2s(b-1)+\gamma})/\zeta). \quad (5.26)$$

Here $\lambda_j$ are given by $\lambda_j = \mu_j - (\beta(j) - 1/2)\eta$ for $j = 1, \ldots, 2sm$.

It is straightforward to take the homogeneous limit: $\xi_k \to 0$. Here (sgn $\sigma$) denotes the sign of permutation $\sigma \in S_{2sm}/(S_m)^{2s}$. 
6. Derivation of finite-sum expression of spin-s XXZ correlation functions with arbitrary entries

6.1. Fundamental commutation relations

We now discuss briefly the derivation of (5.20), which expresses the spin-s XXZ correlation functions with arbitrary entries in terms of the product of finite sums over the Bethe roots.

Let $\Sigma_N$ be the set of integers $1, 2, \ldots, N$, i.e. $\Sigma_N = \{1, 2, \ldots, N\}$. Recall definition (5.14) of $\mathbf{a}^\pm$ and that of integers $c_j$ and $c_j'$. For a given set of $c_j, c_j'$, we introduce $A_j$ and $A_j'$ by

$$A_j = \{b; 1 \leq b \leq M + 2sm, b \neq c_k, c_k' \text{ for } k < j\},$$

$$A_j' = \{b; 1 \leq b \leq M + 2sm, b \neq c_k \text{ for } k \leq j, b \neq c_k' \text{ for } k < j\}. \quad (6.1)$$

We define sets $\mathbf{a}_j^\pm$ and $c(\mathbf{a}_j^\pm)$ as follows.

$$\mathbf{a}_j^- = \{k; k < j, k \in \mathbf{a}^-\}, \quad \mathbf{a}_j^+ = \{k; k < j, k \in \mathbf{a}^+\}, \quad (6.2)$$

$$c(\mathbf{a}_j^-) = \{c_k; k \in \mathbf{a}_j^-\}, \quad c(\mathbf{a}_j^+) = \{c_k'; k \in \mathbf{a}_j^+\}. \quad (6.3)$$

We have

$$A_j = \Sigma_{M+2sm} \setminus (c(\mathbf{a}_j^-) \cup c(\mathbf{a}_j^+)), \quad A_j' = \Sigma_{M+2sm} \setminus (c(\mathbf{a}_{j+1}^-) \cup c(\mathbf{a}_j^+)).$$

Let us denote by $t$ the number of $c_j (j \in \mathbf{a}^-)$ and $c_j' (j \in \mathbf{a}^+)$ such that $c_j, c_j' \leq M$, for a given set of $c_j$ and $c_j'$. We express (5.17) as follows.

$$\sum_{t=r}^{2sm} \sum_{\{c_j, c_j'\}} = \prod_{j \in \mathbf{a}^-} \left( \sum_{c_j=1}^{M} \right) \prod_{j \in \mathbf{a}^+} \left( \sum_{c_j'=1}^{M+j} \right). \quad (6.4)$$

Here the sum over $\{c_j, c_j'\}_{t}$ denotes the sums over $c_j$ and $c_j'$ such that the number of $c_j \leq M$ is fixed by $t-r$.

Suppose that $\lambda_a$ for $\alpha = 1, 2, \ldots, M$ give a set of solutions of the Bethe ansatz equations in the spin-1/2 case with $w_j = w_j^{(2s; \epsilon)}$ for $j = 1, 2, \ldots, L$. Here $w_j$ are inhomogeneous parameters. We set rapidities $\lambda_{M+j}$ by

$$\lambda_{M+j} = w_j, \quad j = 1, 2, \ldots, 2sm. \quad (6.5)$$

We can show the fundamental commutation relations as follows.\textsuperscript{11}

$$\langle 0 \rangle \left( \prod_{\alpha=1}^{M} C(\lambda_{\alpha}) \right) T_{t_1, t_1'} (\lambda_{M+1}) \cdots T_{t_{2sm}, t_{2sm}'} (\lambda_{M+2sm})$$

$$= \sum_{t=r}^{2sm} \sum_{\{c_j, c_j'\}} G_{\{c_j, c_j'\}} (\lambda_{1}, \ldots, \lambda_{M+2sm}) \langle 0 \rangle \prod_{k \in A_{2sm+1} (\{c_j, c_j'\})} C(\lambda_{k}).$$
where \( d(\lambda; \{w_k^{(2s;\epsilon)}\}_L) \) and \( G_{\{\epsilon_j, \epsilon'_j\}}((\lambda_\alpha)_M+2sm) \) are given by

\[
d(\lambda; \{w_k^{(2s;\epsilon)}\}_L) = \prod_{k=1}^L b(\lambda - w_k^{(2s;\epsilon)}),
\]

\[
G_{\{\epsilon_j, \epsilon'_j\}}((\lambda_\alpha)_M+2sm) = \prod_{j \in \alpha^+} \left( \frac{\prod_{b=1}^{M+j-1} a_j \varphi(\lambda_b - \lambda_j + \eta)}{\prod_{b=1}^{M+j} a_j \varphi(\lambda_b - \lambda_j' + \eta)} \right) \times \prod_{j \in \alpha^-} \left( \frac{d(\lambda_j; \{w_k^{(2s;\epsilon)}\}_L) \prod_{b=1}^{M+j-1} a_j \varphi(\lambda_j - \lambda_b + \eta)}{\prod_{b=1}^{M+j} a_j \varphi(\lambda_j - \lambda_b' - \eta)} \right).
\]

(6.6)

6.2. Finite-sum expression of correlation functions for a finite chain

We introduce disjoint subsets of \( \alpha^+, \alpha^+_j \) and \( \alpha^+_K \), as follows.

\[
\alpha^+_j = \{ j : j \in \alpha^+, 1 \leq \epsilon_j \leq M \}, \quad \alpha^+_K = \{ j : j \in \alpha^+, \epsilon_j' > M \}. \quad (6.7)
\]

We define sets \( c(\alpha^-), c(\alpha^+_j) \) and \( c(\alpha^+_K) \) as follows.

\[
c(\alpha^-) = \{ c_k : k \in \alpha^- \}, \quad c(\alpha^+_j) = \{ c_k : k \in \alpha^+_j \}, \quad c(\alpha^+_K) = \{ c_k : k \in \alpha^+_K \}.
\]

We define a sequence \( (b_k)_t \) by a subsequence of \( (b_k)_2sm \) such that \( b_k \leq M \) for \( k = 1, 2, \ldots, t \). We denote sequence \( (b_k)_2sm \) and \( (b_k)_t \) as sets by \( b \) and \( \bar{b} \), respectively, i.e. \( b = \{ b_1, b_2, \ldots, b_{2sm} \} \) and \( \bar{b} = \{ \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_t \} \). Here we note \( \bar{b}_i = c(\alpha^-) \cup c(\alpha^+_j) \). We define sequence \( (b'_k)_2sm-t \) by a subsequence of \( (b_k)_2sm \) such that \( b'_k > M \) for \( k = 1, 2, \ldots, 2sm-t \). We denote it as a set by \( b'_{2sm-t} \). Here we note \( b'_{2sm-t} = c(\alpha^+_K) \).

We define sets \( Z \) and \( K \) by \( Z = \Sigma_M \setminus \bar{b} \) and \( K = \Sigma_{2sm} \setminus b'_{2sm-t} \), respectively. We define a sequence \( (z(\alpha))_{m-t} \) by putting the elements of \( Z \) in increasing order: \( z(1) < z(2) < \cdots < z(M-t) \) where \( Z = \{ z(i) : i = 1, 2, \ldots, M-t \} \), and a sequence \( (\kappa_j)_t \) by putting the elements of \( K \) in increasing order: \( \kappa_1 < \kappa_2 < \cdots < \kappa_t \) where \( K = \{ \kappa_j : j = 1, 2, \ldots, t \} \).

We derive the spin-s correlation functions from those of the spin-1/2 case sending \( \epsilon \) to zero:

\[
F_m^{(2s;+)}(\{i_k, j_k\}; (w_j^{(2s;\epsilon)})_L) = C(\{i_k, j_k\}) \lim_{\epsilon \to 0} F_{2sm}^{(1+)}(\{\epsilon_j, \epsilon'_j\}; (w_j^{(2s;\epsilon)})_L) \).
\]

(6.8)
Applying (6.6) to (5.9) (or (5.13)) we have

\[ F_{2sm}^{(1)} \{ (\epsilon_j, \epsilon_j') \}; (w_j(2^s; r_t))_L = \sum_{l=r} G_{(c_j, \epsilon_j')} (\lambda_1, \cdots, \lambda_{M+2sm}) \]

\[ \times \phi_{2sm} (\{ \lambda_\alpha \}_M) \frac{\langle 0 | \prod_{\alpha=1}^{M+t} C(\lambda_{z(\alpha)}) \prod_{\gamma=1}^{t} C(w_{\kappa_\gamma}) \prod_{\beta=0}^{M-t} B(\lambda_{z(\beta)}) \prod_{r=1}^{t} B(\lambda_{b_r}) | 0 \rangle}{\langle 0 | \prod_{\alpha=1}^{M+t} C(\lambda_{z(\alpha)}) \prod_{\gamma=1}^{t} C(w_{b_{\gamma}}) \prod_{\beta=0}^{M-t} B(\lambda_{z(\beta)}) \prod_{r=1}^{t} B(\lambda_{b_r}) | 0 \rangle} \]

\[ = \sum_{l=r} \sum_{c_j, c_j'} \prod_{a=1}^{M} \prod_{\alpha=1}^{t} \frac{\varphi(\lambda_a - w_j) \varphi(\lambda_a - w_j + \eta)}{\varphi(\lambda_{z(a)} - \lambda_{b_j}) \varphi(\lambda_{z(a)} - w_{\kappa_j} + \eta)} \prod_{j \in \alpha^t} \frac{\prod_{b=1}^{M+t} \varphi(\lambda_a - \lambda_{b_j}) \varphi(\lambda_a - \lambda_{b_j})}{\prod_{b=1}^{M+t} \varphi(\lambda_a - \lambda_{b_j}) \varphi(\lambda_a - \lambda_{b_j})} \prod_{1 \leq k < r \leq t} \varphi(\lambda_a - \lambda_{b_j}) \]

\[ \times \prod_{a=1}^{M-t} \prod_{\epsilon=1}^{t} \varphi(\lambda_{z(\alpha)} - \lambda_{b_j}) \prod_{a=1}^{t} \varphi(\lambda_a - \lambda_{b_j}) \prod_{a=1}^{t} \varphi(\lambda_a - \lambda_{b_j}) \]

\[ \times \det \left( \Phi^{-1} (\lambda_{z(a)})_{M-t} \# (\lambda_{b_j})_t \times \right. \]

\[ \left. \times \Psi' (\lambda_{z(a)})_{M-t} \# (w_{\kappa_j})_t, (\lambda_{z(a)})_{M-t} \# (\lambda_{b_j})_t \right) \]

(6.9)

Here, \( \phi_m (\{ \lambda_\alpha \}) = \prod_{j=1}^{n} \prod_{a=1}^{M} b(\lambda_a - w_j) \), and matrix elements \( \Psi_{ab} \) for \( a, b = 1, 2, \ldots, M \) are given by

\[ \begin{cases} \Phi_{a, b} (\{ \lambda_{z(a)} \}_{M-t} \# (\lambda_{b_j}^t))_{r \alpha}, (\lambda_{z(a)})_{M-t} \# (w_{\kappa_j}^t) ; (w_k)_L) \end{cases} \]

(6.10)

The matrix elements of the Gaudin matrix are given as follows.

\[ \Phi'_{a, b} (\{ \lambda_{z(a)} \}_{M-t} \# (\lambda_{b_j}^t) ; (w_k)_L) = \Phi'_{z(a), z(b)} (\{ \lambda_\alpha \}_M ; (w_k)_L) \]

(6.11)
Proposition 6.1.

\[
F_{2sm}^{(1^+)}(\{\epsilon_j, \epsilon'_j\}; (w_j^{(2s; \epsilon)})_L) = \sum_{t=r}^{2sm} \sum_{\{\epsilon_j, \epsilon'_j\}} \left( \prod_{j,k \in \alpha_K^\ast, \epsilon_j' < \epsilon_k, j < k} (-1) \right) \times (-1)^{2sm-t} \prod_{j \in \alpha_K^\ast} \left( \prod_{\ell \in \alpha_j^\ast, \ell > j} (-1) \cdot \prod_{\kappa \in K, \kappa + M < \epsilon'_j} (-1) \right) \times \det(\Phi')^{-1} \Psi'((\lambda_{z(\alpha)})_{M-t} # (\xi_{\ell})_t, (\lambda_{z(\alpha)})_{M-t} # (\lambda_{c'}_{\chi})_t) \times \prod_{j \in \alpha^\ast} \left( \prod_{k=1}^{2sm} \varphi(\lambda_{c_j'} - w_k^{(2s; \epsilon)}) + \eta \right) \prod_{k=1}^{2sm} \varphi(\lambda_{c_j'} - w_k^{(2s; \epsilon)}) \prod_{1 \leq k < 2 sm} \varphi(w_k^{(2s; \epsilon)} - w_l^{(2s; \epsilon)}) \right),
\]

(6.12)

We define matrix elements \((j, k)\) of \(\phi_M^{(2sm)}((b_l)_{2sm})\) \((1 \leq j \leq 2sm)\).

If \(b_j > M\), \(\left(\phi_M^{(2sm)}((b_l)_{2sm})\right)_{j,k} = -\delta_{b_j-M, k}\) for \(k = 1, 2, \ldots, 2sm\),

if \(b_j \leq M\), there is an integer \(i\) such that \(b_j = \tilde{b}_i\)

\[
\left(\phi_M^{(2sm)}((b_l)_{2sm})\right)_{j,k} = (\Phi')^{-1} \Psi'((\lambda_{z(\alpha)})_{M-t} # (\lambda_{c'}_{\chi})_t, (\lambda_{z(\alpha)})_{M-t} # (\xi_{\ell})_t + M - t, k + M - t),\]

for \(k = 1, 2, \ldots, t\),

and \(\phi_M^{(2sm)}((b_l)_{2sm})_{j,b'_k} = 0\) for \(k = 1, 2, \ldots, 2sm - t\).

(6.13)

We can show the following proposition.

Proposition 6.2.

\[
\det(\Phi')^{-1} \Psi'((\lambda_{z(\alpha)})_{M-t} # (w_{\kappa})_t, (\lambda_{z(\alpha)})_{M-t} # (\lambda_{\chi})_t)) = \det \phi_M^{(2sm)}((b_l)_{2sm}) (-1)^{2sm-t} \left( \prod_{j,k \in \alpha_K^\ast, \epsilon_j' < \epsilon_k, j < k} (-1) \right) \times \prod_{j \in \alpha_K^\ast} \left( \prod_{\ell \in \alpha_j^\ast, \ell > j} (-1) \cdot \prod_{\kappa \in K, \kappa + M < \epsilon'_j} (-1) \right),
\]

(6.14)
When $N_s$ is large enough, solving the integral equations for $\phi_m^{(2s_m)}((b_t)_{2s_m})$, we can show

$$\det \phi_m^{(2s_m)}((b_t)_{2s_m}) = \det M^{(2s_m)}((b_t)_{2s_m}) + O(1/N_s).$$

(6.15)

We thus obtain the finite-size spin-$s$ XXZ correlation functions with arbitrary entries (5.20).

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