ONE FORM OF SUCCESSIVE APPROXIMATION METHOD AND CHOICE PROBLEM

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1. INTRODUCTION

A mathematical model of Subject behaviour choice is proposed. The background of the model is the concept of two preference relations determining Subject behaviour. These are an "internal" or subjective preference relation and an "external" or objective preference relation.

The first (internal) preference relation is defined by some partial order on a set of states of the Subject. The second (external) preference relation on the state set is defined by a mapping from the state set to another partially ordered set. The mapping will be called evaluation mapping (function).

We research the process of external preference maximization in a fashion that uses the external preference as little as possible. On the contrary, Subject may use the internal preference without any restriction.

The complexity of a maximization procedure depends on the disagreement between these preferences. To solve the problem we apply some kind of the successive approximations methods. In terms of evaluation mappings this method operates on a decomposition of the mapping into a superposition of several standard operations and "easy" mappings (see the details below). Obtained in such way superpositions are called approximating forms.

We construct several such forms and present two applications. One of them is concerned with a hypothetic origin of logic. The other application provides a new interpretation of the well known model of human choice by Lefebvre [4, 5]. The interpretation seems to suggest a justification different from the one proposed by Lefebvre himself.

2. SCHEME OF BEHAVIOUR CHOICE BASED ON TWO PREFERENCES

We consider a Subject faced with a choice among a set of states in the environment. Some of them may be better than others and some states are incomparable. Subject’s goal is to reach a satisfactory state (generally, a set of states).

One fundamental feature of many real-world behaviour problems is the difference between the evaluations of a state before and after the state is arrived at. We try to describe this by introducing two preference relations on the state set. One relation describes "internal" system of values based on the Subject’s internal representation (or model) of the world. The other, "external" relation, is based on consequences of
chosen states and reflects the actual nature of the interaction between the environment and the Subject.

Unlike the objective external relation, the internal preference relation is intrinsic to the Subject’s mentality for the Subject perceives the world in terms of it. Contradictions between the internal and external preference relations create problems for the Subject. Besides, in general, there is a cost associated with obtaining information on the true external preference relation. Here we refer not only to the cost of accessing the information but also the cost of changing of Subject’s behaviour patterns. (However, we abstract from the issue of what the costs may actually be).

Thus, informally the problem is to find a maximum in the external preference under given restrictions on access to the information about the external preference.

However, there are no restrictions on the internal preference usage. Therefore, the Subject has to seek a maximum using the internal preference as much as possible. Naturally, to get any value of using the internal preference, the Subject somehow needs to approximate the external (“leading”) preference with its internal one.

It seems to us that such interpretation of the choice problem corresponds to a certain conservatism on the side of the Subject when it is necessary to follow some external pressure. Indeed, even if the Subject is aware of its incomplete and/or incorrect representation it often might not be able to correct it instantaneously. Therefore, it will need to refine/reconstruct its representation starting with what is available.

Thus, we turn to the successive approximation principle that will guide our further investigations. The underlying idea is as follows. The Subject needs to follow some part of its internal preference for as long as possible. Then, on the basis of accessible information on the external preference the Subject finds the next part of the internal preference and uses it to proceed further. The process then repeats. So the Subject needs a scheme to select current parts of its intrinsic (internal) preference. The following section is devoted to a theory of such schemes.

3. AN EXPLICATION OF SUCCESSIVE APPROXIMATION METHOD

Let $S$ be a set of Subject states, $(M, \leq_m), (L, \leq_l)$ be partially ordered sets of internal and external estimates correspondingly. Let $\varphi : S \to M, \psi : S \to L$ be mappings that define internal and external preference relations correspondingly.

It is possible to simplify this description by introducing an order $\leq_S$ on $S$ through the mapping $\varphi$ and the poset $(M, \leq_m)$ as follows. Let us set $s \leq_S s' \iff \varphi(s) <_m \varphi(s') \lor s = s'$. Thus, we move from an initial description to the description $\langle (M, \leq_m$
of \((S, \leq_S)\) above and \(\psi = \) is called the evaluation mapping (function).

It is worth noting that generally in each of the choice problems the Subject gets corresponding internal and external preferences and evaluation mappings. These three objects can depend functionally on some parameters of the choice problem. First, we consider the case of a single problem of choice. Later on, in the section related to Lefebvre’s model, we will generalize to a family of choice problems.

If the evaluation function \(\psi\) appears to be a monotonic mapping (i.e., the condition \((\forall m_1, m_2 \in M)[m_1 \leq_m m_2 \implies \psi(m_1) \leq \psi(m_2)]\) is met), then both preference relations \((M, \leq_m)\) and \((L, \leq_l)\) are compatible (coordinated) and de facto the Subject may follow its internal preference to reach the target state (that is, a state with the maximum value).

Otherwise, it is natural to represent \(\psi\) as a superposition of monotonic evaluation mappings from \((M, \leq_m)\) to \((L, \leq_l)\) and several connecting operations (connectives). We seek to obtain representations that can be used as instructions for successive approximations. In finding a representation of this kind such that uses as few monotonic evaluation mappings as possible (apart from the connectives), we will use the external preference relation as little as possible.

So our next goal is to propose some collections of connectives such that it will be possible to prove the existence of a corresponding representation with required features (we call it approximating form henceforth). Then, we will demonstrate their utility for certain applications.

### 3.1. Approximating forms

The suggested version of the principle deals with some special but yet fairly general representation of the evaluation function \(\psi\) (operator \(\psi\)) in the so-called ”approximating form”. It uses three axiomatically defined operations \(\Box, \exists, \odot\) based only on general properties of the posets \((M, \leq_m), (L, \leq_l)\) as follows.

For every poset \((R, \leq_r)\) the standard mappings \((\cdot)^\Delta, (\cdot)^\nabla : R \to 2^R\) are defined by
\[
t^\Delta = \{t' \in R | t' \leq_r t\}, t^\nabla = \{t' \in R | t \leq_r t'\}.
\]

Let us suppose a binary operation \(\Box : L \times L \to L\) and unary operations \(\exists : 2^L \to L\), \(\odot : L \to L\) are defined in such a way that the following system \(\mathcal{A}\) of axioms holds.

\[
\begin{align*}
\mathcal{A}_1: \quad & (\forall s \in M)(\exists \tilde{s} \subseteq S)[(\forall s' \in S')[(\exists \tilde{s}' \in \tilde{S})[\tilde{s} \leq_m s'] \& (\forall \tilde{s}, \tilde{s}' \in \tilde{S})[\tilde{s} \not\subseteq_m \tilde{s}']]]. \\
\mathcal{A}_2: \quad & (\forall l', L'' \subseteq L)(\forall x \in L')[(x \leq_l \odot(L'))\&(L' \subseteq L'' \implies \Box(L') \leq_l \Box(L''))]. \\
\mathcal{A}_3: \quad & (\forall l, l' \in L)[\Box(l, \odot(l)) = l \& (l \leq_l l' \implies \odot(l) \leq_l \odot(l'))]. \\
\mathcal{A}_4: \quad & (\forall l, l' \in L)[l \leq_l l' \implies (\exists l'' \in L)[\Box(l', l'') = l \& \odot(l') \leq_l l'']].
\end{align*}
\]
For every operator $\nu : M \rightarrow L$ we call set $n(\nu) = \{(m, m') | (m \leq m', \nu(m) \leq L \nu(m'))\}$ non-monotonicity domain of $\nu$. If $n(\nu) = \emptyset$ then $\nu$ is called monotonic operator.

**Theorem 1.** Let all axioms of the system $A$ be satisfied for $(M, \leq m), (L, \leq t)$ and lengths of all increasing chains in $(M, \leq m)$ not exceed some integer $D$. Then for every $\psi : M \rightarrow L$ there exists a representation $\psi = \sqcup(\varphi_1, \psi_1)$ and lengths of all increasing chains $\psi$ $a$ simpler operator $D$.

*Proof.* Let us reduce the problem for a given operator $\psi$ to the same problem for a simpler operator $\psi_1$ such that the following holds $\psi = \sqcup(\varphi_1, \psi_1)$. For every operator $\varphi_1$ we now have $z \in M$ such that the following holds $\varphi(x) \leq \psi(x)$ and, therefore, $y \in M_1 \cap M_1 \cap M_1 = \emptyset$ which leads to a contradiction.

Second, $\varphi_1$ maps $(M_1, \leq m)$ into $(L, \leq t)$ monotonically in accordance with $A_2$.

Finally, let us consider the ”mixed” case when $x \in M_1, y \in M_1$, and all elements of $M$ are comparable with respect to $\leq m$. It is clear that $y \leq m x$ is impossible since condition $z \in M_1 \implies z \not\subseteq M_1$ immediately follows from the definition of $M_1$. Thus, it remains to consider the possibility of $x \leq m y$. In such a case $\varphi_1(y) = \sqcup(\varphi(y^\nu)) \geq \psi(x)$ in accordance to $A_2$. On the other hand, $\psi(x) = \varphi_1(x)$ on $M^1$ follows from the definition of $\varphi_1$. Hence, operator $\varphi_1$ is monotonic.

We are now ready to prove the last assertion of the theorem. For that it is sufficient to demonstrate the inclusion $M_1 \cup \tilde{M}_1 \subseteq M^2$. Here $M^2, M_2$ are defined for $\psi$ in the same way as $M_1, \tilde{M}_1$ were defined for $\psi$ above. $\tilde{M}_1$ is the set of all minimal elements of set $M_1$, see $A_1$. Namely: $M^2 = \tilde{M}_2$ and $M_2 = \{x \in M | n(\psi_1) \cap (x \otimes \psi^\nu) \neq \emptyset\}$.

We now have $M_2 \subseteq (M_1 \setminus \tilde{M}_1)$ and $n(\psi_1) \subseteq n(\psi) \setminus \tilde{M}_1 \times M_1$. So the sequence $M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq \ldots$ ends at a step with the number that can not be greater
than the highest of lengths of the increasing chains in poset \((M, \leq_m)\). Indeed, since 
\(\bar{M}_2 \subseteq M_1 \setminus \bar{M}_1\) then in accordance with \(A_1\) for every element \(y \in \bar{M}_2\) there exists some \(x \in \bar{M}_1\) such that \(x \leq_l y\). Therefore, one can choose an increasing chain of representatives of sets \(\bar{M}_1, \bar{M}_2, \bar{M}_3, \ldots\) which are mutually disjoint sets.

We will now prove that \(M^1 \cup \bar{M}_1 \subseteq M^2\). First, \(\varphi_1(x) = \psi(x)\) holds for every \(x \in M^1\). From here \(\psi_1(x) = \circ(\psi(x))\). However, mapping \(\psi_1\) is monotonic on \(M^1\) in view of \(A_3\) and since \(\psi\) is monotonic on \(M^1\). So \((M^1 \times M^1) \cap \mathfrak{n}(\psi_1) = \emptyset\) and therefore \(M^1 \subseteq M^2\).

Further, let \(x, y \in M^1 \cup \bar{M}_1\) and \(x \leq m y\). Then we can show that \(\psi_1(x) \leq_l \psi_1(y)\). Indeed, the case \(x, y \in M^1\) was considered above. The case \(x, y \in \bar{M}_1\) is impossible since all elements of \(\bar{M}_1\) are incomparable by the definition. Above, we saw that \(x \in M_1 \land x \leq m y \implies y \in M_1\). Besides \(M^1 \cap M_1 = \emptyset\). Therefore, \(x \in M^1, y \in \bar{M}_1\) is the only case remaining to consider. By definition \(\psi_1(x) = \circ(\psi(x))\) and relation \(\boxdot(\varphi_1(y), \psi_1(y)) = \psi(y)\) holds. Moreover, \(\psi(y) \leq_l \varphi(y)\). In accordance with \(A_4\) we have \(\circ(\varphi_1(y)) \leq y \psi_1(y)\). Hence, \(\psi_1(x) \leq_l \psi_1(y)\) takes place since \(\circ\) is a monotonic operation in view of \(A_3\) and \(\psi(z) \leq_l \varphi_1(z), z \in M\) in accordance to \(A_2\) and the construction.

Let us denote by \(\mathcal{M}\) the class of all monotonic mappings from \((M, \leq_M)\) to \((L, \leq_L)\). Also let \(S(\varphi) = \{x|\varphi(x) >_L \circ(x)\}, \varphi \in \mathcal{M}\).

**Corollary 1.** Under the conditions of theorem 1 for every \(\psi : M \to L\) there exists a substitution \(p : \{z_1, \ldots, z_{D+1}\} \to \mathcal{M}\) such that \(S(p(z_n)) \subseteq S(p(z_{n+1})), n = 1, D, \) and \(\psi = Sb_{z_1}^{z_{D+1}} \boxdot (z_1, \boxdot(z_2, \boxdot(\ldots \boxdot(z_D, z_{D+1})), \ldots))\).

**Proof.** Let us fix a formula \(\Phi(z_1, \ldots, z_{D+1}) = \boxdot(z_1, \boxdot(z_2, \ldots \boxdot(z_D, z_{D+1}), \ldots))\) and consider substitutions of monotonic functions instead of variables \(z_1, \ldots, z_{D+1}\) when their results are determined.

According to theorem 1 for every \(\psi : M \to L\) there exists representation \(\psi = \boxdot(\varphi_1, \boxdot(\varphi_2, \boxdot(\varphi_3, \ldots)))\) where all \(\varphi_i, i = 1, 2, 3, \ldots\) are monotonic mappings from \((M, \leq_M)\) into \((L, \leq_L)\). The number of occurrences of operation \(\boxdot\) in this representation does not exceed \(D\).

Now, if for a given mapping \(\psi\) we obtain representation \(\psi = \boxdot(\varphi_1, \boxdot(\varphi_2, \boxdot(\varphi_3, \ldots \boxdot(\varphi_k, \varphi_{k+1}))))\) with \(k < D\), then one can always continue the expression on the right side of the representation till \(\boxdot(\varphi_1, \boxdot(\varphi_2, \boxdot(\varphi_3, \ldots), \varphi_{D+1}))\). For that it is sufficient to set
\[\varphi_i(x) = \circ(\varphi_{i-1}(x)), i = k+2, D+1.\]
In accordance with axiom $\mathcal{A}_3$ the obtained functions are monotonic and

$$\psi = \Box (\varphi_1, \Box (\varphi_2, \Box (\varphi_3, \ldots \Box (\varphi_D, \varphi_{D+1}))))$$

□

For the application below we will need some special monotonic functions in approximating forms. To define the functions we come up with the following auxiliary construction. Based on axiom $\mathcal{A}_1$ let us split set $M$:

$$M_1 = M^\perp;$$

$$M_{n+1} = (M \setminus \bigcup_{j \leq n} M_j)^\perp.$$

Here $M_j$ consists of the elements that are incomparable in $(M, \leq_M)$ for any $j$.

We denote by $\theta$-function of rank $i$ any monotonic mapping $\theta : M \to L$ such that $\theta(x) = \ominus (x)$ for all $x \in \bigcup_{j<i} M_j$ as well as $\theta(x) \in \max(L, \leq_L)$ for all $x \in \bigcup_{j>i} M_j$. The rank of a given function $\theta$ is denoted as $\rho(\theta)$. Let $\Theta$ be the class of all $\theta$-functions.

**Corollary 2.** Let conditions of theorem 1 be fulfilled, $D$ be the exact upper bound of lengths of increasing chains in $(M, \leq_M)$ and $(L, \leq_L)$ contain the greatest element $\gamma$. Then for any mapping $\psi : M \to L$ there exists a substitution $p : \{z_1, \ldots, z_{D+1}\} \to \Theta^\mu$, such that $\rho(p(z_i)) = i, i = 1, D + 1$ and $\psi = \text{Sb}^{z_1} \ldots \text{Sb}^{z_{D+1}} \Box (z_1, \Box (\gamma, \psi_1(x))) = \psi(x)$.

**Proof.** We use induction on $D$. In the case of $D = 0$ the statement is obvious since there are no restrictions for $\theta$-functions. Therefore, $\psi \in \Theta$.

**Induction step:** Let us define $\theta_1$ of rank 1 in the following manner:

$$\theta_1(x) = \begin{cases} 
\psi(x), & \text{if } x \in M_1, \\
\gamma, & \text{else.}
\end{cases}$$

Then we may write $\psi = \Box (\theta_1, \psi_1)$ where for $\psi_1$ we have $\psi_1(x) = \ominus (x)$ if $x \in M_1$ else $\psi_1(x)$ satisfies $\Box (\gamma, \psi_1(x)) = \psi(x)$.

It remains to obtain the desirable representation of $\psi_1$ on the set $M \setminus M_1$ with the partial order induced by $\leq_M$. Since the length of the longest increasing chain in $(M \setminus M_1, \leq_M)$ is $D - 1$ we may use the induction supposition. □

**Remark.** Length of the representation obtained in theorem 1 can be essentially lower than lengths of the representations suggested by the corollaries. From our initial standpoint the lower the length is the better. However, sometimes we will need special forms of approximating functions.

Let us suppose a binary operation $\Box, \uplus : L \times L \to L$ and unary operations $\ominus : L \to L$ are defined in such a way that the system $\mathcal{B}_1 - \mathcal{B}_4$ of axioms takes place. Here $\mathcal{B}_i$ coincides with $\mathcal{A}_i$ for $i = 1, 3, 4$. Also
\[ B_2: (\forall x, y \in L)[x, y \leq_l \exists!(x, y)]. \]

**Theorem 2.** Let all axioms of the system \( B \) be satisfied for \((M, \leq_m), (L, \leq_l)\) and \((M, \leq_m)\) and lengths of all increasing chains in \((M, \leq_m)\) do not exceed some integer \( D \). Then for every \( \psi : M \to L \) there exists a representation \( \psi = \exists!(\varphi_1, \exists!(\varphi_2, \exists!(\varphi_3, \ldots))) \) where all \( \varphi_i, i = 1, 2, 3 \ldots \) are monotonic mappings from \((M, \leq_m)\) to \((L, \leq_l)\).

The number of occurrences of the operation \( \exists! \) in this representation does not exceed \( D \).

**Proof.** Firstly, in the case when \((\forall x \in M)[|x| < \infty \) is true we can prove this theorem using theorem 1. For that we will only need to note that in this case it is possible to replace \( \exists!(\psi(x)) \) with any expression of the kind \( \psi(\psi(z_1, \psi(z_2), \psi(z_3), \ldots)) \). Here \( z_1, \ldots, z_n \) is an enumeration of the finite set \( x \). Indeed, in the proof of theorem 1 we used axiom \( A_2 \) only for subsets of \( L \) of the form \( \psi(x) \). Thus, it is sufficient to check that axiom \( A_2 \) is met for sets of the kind \( \psi(x) \). This check is trivial on the basis of axiom \( B_2 \) for operation \( \exists! \).

Otherwise, when there are infinite sets \( x \) we can make use of the same scheme for the connective \( \exists! \) based on the condition of increasing chain finiteness in \((M, \leq_m)\). For that let us enumerate elements \( z_1, z_2, \ldots, z_n, \ldots \) of set \( x \) for a given \( x \in M \). In parallel we will enumerate expressions \( \psi(\psi(z_1, \psi(z_2), \psi(z_3), \ldots, \psi(z)) \), \( \psi(\psi(z_1, \psi(z_2), \psi(z_3), \ldots), z_n) \), \( \ldots \).

By axiom \( B_2 \) the values of these expressions do not decrease in \((L, \leq_l)\). In view of the finiteness supposition for increasing chains in \((L, \leq_l)\) the sequence of computed values becomes stable from some place. We set \( \varphi_1(x) \) to this final value.

Thus, \( \varphi_1 \) is monotonic mapping and \( \psi(x) \leq_l \varphi_1(x), x \in M \). The last part of the proof is analogous to the corresponding part of theorem 1. 

In place of or together with \( A \) the dual axiom system \( A^* \) can be fulfilled. This can be shown by replacing \( \leq \) with \( \geq \) and \( \exists! \) with \( \exists^* \), \( \exists^* \), \( \circ \) correspondingly:

\[
A_1^*: (\forall S \subseteq M)(\exists S \subseteq S)[(\forall s \in S)(\exists s' \in S)[s \geq_m s'] \& (\forall s, s' \in S)[s \not< m s']]].
\]

\[
A_2^*: (\forall L', L'' \subseteq L)(\forall x \in L'')[x \geq_l \exists^*(L') \& (L' \subseteq L'' \implies \exists^*(L') \geq_l \exists^*(L''))].
\]

\[
A_3^*: (\forall l, l' \in L)[\exists^*(\circ^*(l'), l) = l \& (l \geq_l l' \implies \circ^*(l') \geq_l \circ^*(l'))].
\]

\[
A_4^*: (\forall l, l' \in L)(l \geq_l l' \implies (\exists l'' \in L)[\exists^*(l'', l') = l \& \circ^*(l') \geq_l l'']).
\]

Then the dual theorem holds:

**Theorem 1'.** Let all axioms of the system \( A^* \) be fulfilled for posets \((M, \leq_m)\), \((L, \leq_l)\), operators \( \exists^*, \exists^*, \circ^* \); and the lengths of all increasing chains in \((M, \leq_m)\) do not exceed some integer \( D \). Then for every operator \( \psi : M \to L \) there exists repre-
sentation \( \psi = \bigcirc^* (\cdots \bigcirc^* (\bigcirc^* (\varphi_{n+1}, \varphi_{n}), \varphi_{n-1}, \ldots, \varphi_1) \) where all \( \varphi_i, i = 1, \ldots, n, n+1, \) are monotonic mappings from \((M, \leq_m)\) to \((L, \leq_l)\).

The number of occurrences of the operation \( \bigcirc \) in this representation does not exceed \( D \).

The dual theorem is related to the dual axiom system \( B^* \).

\[ B_1^*: (\forall S \subseteq M)(\exists \tilde{s} \subseteq S)[(\forall s \in S)(\exists s' \in \tilde{S})[s \geq_m \tilde{s} \& (\forall s, s' \in \tilde{S})[\tilde{s} \not<_{m} s']]]. \]
\[ B_2^*: (\forall x, y \in L)[x, y \geq_l \bigcirc^*(x, y)]. \]
\[ B_3^*: (\forall l, l' \in L)[\bigcirc^*(\bigcirc^*(l), l) = l & (l \geq_l l' \implies \bigcirc^*(l) \geq_l \bigcirc^*(l'))]. \]
\[ B_4^*: (\forall l, l' \in L)[l \geq_l l' \implies (\exists l'' \in L)[\bigcirc^*(l'', l') = l & \bigcirc^*(l') \geq_l l'']]. \]

Then the dual theorem holds:

**Theorem 2**. Let all axioms of the system \( A^* \) be fulfilled for posets \((M, \leq_m), (L, \leq_l)\), operators \( \bigodot^*, \bigoplus^*, \@^* \) and the lengths of all increasing chains in \((M, \leq_m)\) do not exceed some integer \( D \). Then for every operator \( \psi : M \rightarrow L \) there exists representation \( \psi = \bigcirc^*(\cdots \bigcirc^* (\bigcirc^* (\varphi_{n+1}, \varphi_{n}), \varphi_{n-1}, \ldots, \varphi_1) \) where all \( \varphi_i, i = 1, \ldots, n, n+1, \) are monotonic mappings from \((M, \leq_m)\) to \((L, \leq_l)\).

The number of occurrences of the operation \( \bigcirc \) in this representation does not exceed \( D \).

Below we refer to all these representations as approximating forms.

4. Possible Origin of Logic

It is easy to arrive at the classical two-valued propositional logic now. For that it is sufficient to choose \((\{0, 1\}, 0 \leq 1)\) as \((L, \leq_l)\) and the standard poset \((B^n, \preceq)\) on boolean cube \(B^n\) as poset \((M, \leq_m)\). It is well known that every finite poset can be isotonically included into \((B^n, \preceq)\) for the appropriate \( n \).

It is also well known that poset \((B^n, \preceq)\) is a self-dual poset for any \( n \). Therefore, both above introduced representations take place in this case.

**Lemma.** 1) The system of posets \((B^n, \preceq), (B, \preceq)\) as \((M, \leq_m), (L, \leq_l)\) correspondingly and operation \( \rightarrow \) as \( \bigcirc^* \), operation \( 1 : B^n \rightarrow \{1\} \) as \( \bigodot^* \), and operation \( \& \) as \( \@^* \) as \( \bigcirc^*(\beta^\top) \) fulfill the axiom set \( A^* \).

2) The system of posets \((B^n, \preceq), (B, \preceq)\) as \((M, \leq_m), (L, \leq_l)\) correspondingly and operation \( \rightarrow \) as \( \bigcirc^* \), operation \( 1 : B^n \rightarrow \{1\} \) as \( \bigodot^* \), and operation \( \& \) as \( \bigoplus^* \) obeys the axiom set \( B^{++} \).

**Proof.** This can be shown via a routine check of the axioms.

The direct corollary of this lemma and theorems above is the following
Theorem 3. In the special case of finite "internal" orders \((M, \leq_m)\) and linear "external" orders \((L, \leq_l)\), \(|L| = 2\), approximating forms from each of theorems 1 and 2 and their dual ones generate all formulae of the classical propositional logic (within logical equivalence).

As a result, this interesting statement follows.

Corollary 3. Every \(n\)-argument logical (boolean) function \(f\) can be represented by the implicative normal form \(f = P_k \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_1\), where \(k \leq n\), and \(P_i, i = 1, k\), are monotonic boolean function.

It is remarkable that just the dual approximating forms present the usual propositional implication. One may then wonder why the operation \(\rightarrow^*\) is not present in natural languages? In our opinion, the main reason is that the dual approximating forms of theorems 1\(^*\), 2\(^*\) begin with a given operator \(\psi\) and approximate it by means of successive simplifications: \(\psi_1 = \Box^*(\psi, \varphi_1), \psi_2 = \Box^*(\psi_1, \varphi_2), \ldots\) while \(\psi_i\) is not a monotonic operator (i.e., not an "easy" one). Thus, the approximation begins with a target unlike in the case of the approximating forms in theorems 1, 2.

Now one can consider the classical two-valued propositional logic merely as a realization of the above-mentioned principle of successive approximations for the problem of decision-making within Subject-environment survival framework.

Thus, from this viewpoint, the classical propositional logic can take its beginning from the survival problem. It is also important that this hypothetical origin of logic appears quite natural.

5. What stands behind Lefebvre's model

Lefebvre suggested a model of Subject facing a choice of an alternative out of a set. In his model the Subject is represented by the function \(X_1 = f(x_1, x_2, x_3)\) where \(X_1, x_1, x_2, x_3\) run over the \([0, 1]\) segment. As \([4, 5]\) presents it: the value of \(X_1\) is interpreted as the readiness to choose a positive pole with probability \(X_1\), and the value of \(x_3\) - as the Subject’s plan or intention to choose a positive pole with probability \(x_3\). Variables \(x_1\) and \(x_2\) represent the world influence on the subject.

This function \(f\) is required to obey the following axioms introduced by Lefebvre:

\[ L_1: (\forall x_3 \in [0, 1])(f(0, 0, x_3) = x_3) - "\text{the axiom of free choice}"; \]
\[ L_2: (\forall x_3 \in [0, 1])(f(0, 1, x_3) = 0) - "\text{the axiom of credulity}"; \]
\[ L_3: (\forall x_2, x_3 \in [0, 1])(f(1, x_2, x_3) = 1) - "\text{the axiom of non-evil-inclinations}"; \]
\[ L_4: (\forall i, j, k)[\{i, j, k\} = \{1, 2, 3\}] \implies (\forall x_j, x_k \in [0, 1])(\exists c, c' \in \mathbb{R})(\forall x_i \in [0, 1])[f(x_1, x_2, x_3) = cx_i + c'] - "\text{the postulate of simplicity}". \]
By means of the model Lefebvre gave explanations of several psychological experiments thusly putting his model under the spotlight (e.g., see bibliography in [5]).

The following question is still open: Is the model only a compact representation (i.e., a “roll-up”) of certain experimental data or it describes some fundamental structure governing human behavior?

In order to substantiate his model, Lefebvre used, in particular, known ”anthropic principle” [5]. In our opinion, the justification presented by Lefebvre while being appealing does not appear entirely sound and bullet-proof. The specific comments are presented in [6]. In the following we suggest an alternative justification for the model. Namely, we develop the approach mentioned in [6] using the above constructed theory of approximating forms.

First, we show how it is possible to eliminate "the postulate of simplicity" introducing the notion of a pure L-ensemble. The last concept reduces the general case to the boolean case. This step leads to the boolean order for the external preference relation $(L, \leq_l), L = \{0, 1\}$. Second, we will show that the system of the first three axioms by Lefebvre can be replaced with a postulate of special poset $(M, \leq_m)$. Namely, this poset can be chosen in the form of a linear ordered three-element set. We suggest a natural interpretation of this form of poset $(L, \leq_l)$. Then Lefebvre’s function $f$ follows from one of our approximating forms.

5.1. Lefebvre’s ensembles. It is easy to check that in the boolean case $X_1, x_1, x_2, x_3 \in \{0, 1\}$ the axioms $L_1-L_3$ completely define $f$. Namely, in this case $f(x_1, x_2, x_3) = (x_3 \rightarrow x_2) \rightarrow x_1$. (The ”postulate of simplicity” $L_4$ sets $f$ on the interior of the three-dimensional cube $[0, 1]^3$ in the real-valued case. A methodological criticism of the postulate is expounded in [6]).

Let us consider a set $Q$ of Subjects $s_i$ with each being described by the probabilistic collection $\tilde{\alpha}_i$ of values of the boolean variables $(n_1, n_2, n_3)$. Let us assume that the probability of encountering a Subject with a collection $\tilde{\alpha}$ of the variable values in $Q$ is equal to $p_{\tilde{\alpha}}$.

If behavior $z_i$ each $s_i \in Q$ is described with the function $n_3 \rightarrow n_2 \rightarrow n_1$ then we refer to $Q$ as the Lefebvre’s ensemble (L-ensemble or simply ensemble) $\langle Q, P \rangle$ with characteristic $P = (p_0, \ldots, p_7)$. Besides, we call elements of the L-ensemble L-Subjects. (Here $p_k$ denotes $p_{\tilde{\alpha}}$ and $k$ is the decimal representation of the binary sequence $\tilde{\alpha}$).

Ensemble $\langle Q, P \rangle$ averaging Boolean variables $n_1, n_2, n_3, z_i$ yields real numbers $x_1, x_2, x_3, z \in [0, 1]$. Given the truth table of the Boolean function $n_3 \rightarrow n_2 \rightarrow n_1$
elementary probabilistic considerations lead to the following equalities:

\[
(1) \quad 1 = \sum_{k=0}^{7} p_k, \\
(2) \quad x_1 = p_4 + p_5 + p_6 + p_7, \\
(3) \quad x_2 = p_2 + p_3 + p_6 + p_7, \\
(4) \quad x_3 = p_1 + p_3 + p_5 + p_7, \\
(5) \quad z = p_1 + p_4 + p_5 + p_6 + p_7.
\]

It is therefore reasonable to ask for which \( L \)-ensembles \((Q, P)\) values of \(x_1, x_2, x_3, z\) satisfy Lefebvre’s equation

\[
z = x_1 + (1 - x_1 - x_2 + x_2 x_3) x_3.
\]

The following examples show that, generally speaking, \( z \neq f(x_1, x_2, x_3) \). Indeed, let us set \( p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = 0, 1 \). Then \( x_1 = x_2 = x_3 = 0.4 \) and \( f(x_1, x_2, x_3) = 0.544 \). However, the ensemble average \( z \) equals 0.5. Interestingly enough, the difference can be quite substantial as the following example demonstrates. Namely, \( p_0 = p_1 = p_2 = p_4 = p_6 = p_7 = 0, p_3 = p_5 = 0.5 \) correspond to \( x_1 = x_2 = 0.5, x_3 = 1 \). Then \( z = 0.5 \) but \( f(0.5, 0.5, 1) = 0.75 \). Thus, the error is at least 30%.

On the other hand, the equality \( z = f(x_1, x_2, x_3) \) is met for all possible (i.e., obeying equations (1)-(4)) characteristics \( P \) when \( (x_1, x_2, x_3) \in \{(x_1, x_2, x_3)|x_1 = 1\} \cup \{(x_1, x_2, x_3)|x_2 = 0\} \cup \{(x_1, x_2, x_3)|x_2 = 1\} \cup \{(x_1, x_2, x_3)|x_3 = 0\} \).

**Theorem 4.** For every collection \( x_1, x_2, x_3 \in [0, 1] \) there exists \( L \)-ensemble \((Q, P(x_1, x_2, x_3))\) with characteristic \( P(x_1, x_2, x_3)\) such that \( z = f(x_1, x_2, x_3) \).

**Proof.** Let us consider three independent Boolean random variable \( \zeta, \eta, \theta : N \rightarrow \{0, 1\} \) with the mean values \( x_1, x_2, x_3 \) correspondingly. Then random variable \( (\zeta, \eta, \theta) : N \rightarrow \{0, 1\}^3 \) runs over the desired ensemble \((Q, P(x_1, x_2, x_3))\). For the \( i \)-th component of the characteristic \( p_i(x_1, x_2, x_3) = \prod_{j=1,2,3} (1 - \sigma_j + (-1)^{x_j} x_j) \) is true where \( i = \sum_{j=1,2,3} 2^\sigma j \). The verification by substitution shows that the interrelations (1)-(4) are fulfilled and if \( z \) satisfies (5), then \( z = f(x_1, x_2, x_3) \). \( \square \)

We call the ensembles described in this theorem pure Lefebvre’s ensembles (PL-ensembles). Thus, a PL-ensemble is a collection of \( L \)-Subjects with random parameters \((n_1, n_2, n_3)\) distributed independently in such a way that the probability \( P\{n_i = 1\} \) equals the given number \( x_i \in [0, 1], i = 1, 2, 3 \).

The descriptions of behaviour constructed by means of \( L \)-ensembles can be thinner than the descriptions ”smoothed” by using Lefebvre’s function \( f \) for some aspects. For example, let us consider how ”golden section” for categorization of stimuli with-
out measurable intensity can be explained in terms of Lefebvre’s theory ([5], p.51) and in terms of \( PL \)-ensembles.

In this case Lefebvre completes his ”Realist condition” \( x_3 = f(x_1, x_2, x_3) \) with equations \( x_1 = x_2, x_1 = 1 - x_3 \). (A justification is given in [5], p.51). In turn, that yields the equation \( x_3^3 - 2x_3 + 1 = 0 \) for the choice of \( x_3 \). One possible solution is the well known ”golden section” \( x_3 = \frac{\sqrt{5} - 1}{2} \).

Following the alternative approach suggested in this paper, we construct the desired \( PL \)-ensemble by first postulating the Boolean ”Realist condition” \( n_3 \rightarrow n_2 \rightarrow n_1 = n_3 \). Then considering the truth area \( R = \{000, 001, 010, 101, 111\} \) of the condition we form the ensemble with the help of Boolean random variables \( \zeta, \eta, \theta \) in the following fashion. The variables \( \zeta, \eta \) are independent with the mean value \( 1 - x_3 \), and the value of the random variable \( \theta \) depends on the values of \( \zeta, \eta \) in accordance with the table:

| \( \zeta \) | \( \eta \) | \( \theta \) |
|-----|-----|-----|
| 0   | 0   | 0, 1|
| 0   | 1   | 0   |
| 1   | 0   | 1   |
| 1   | 1   | 1   |

It is important that in the first line of the table value 1 is chosen with the probability of \( x_3 \). Then if \( x_3 \) satisfies \( x_3^3 - 2x_3 + 1 = 0 \) then we obtain the desired \( PL \)-ensemble. Indeed, every element of the ensemble is a ”Realist” and the probability to encounter an \( L \)-Subject with parameters \( (n_1, n_2, 1) \) is determined by solutions to the equation \( x_3^3 - 2x_3 + 1 = 0 \). Finally, we arrive at the ”golden section” choosing the corresponding solution exactly as it was done by Lefebvre.

We believe that the \( L \)-ensemble tool introduced in this paper opens new opportunities for Lefebvre’s theory and its applications. Indeed, the ensemble structure is a new powerful parameter for modeling. It is possible to explain some deviations of the actual values of variable \( X_1 \) in real-world experiments by means of the corresponding deviations of the real \( L \)-ensembles from the \( PL \)-ensembles. Thus, the dynamics of this parameter open a new research avenue.

5.2. Application of approximating forms. We will now show how one can arrive at Lefebvre’s model on the basis of the theory presented earlier in this paper.

First, we determine appropriate internal and external preferences. Because of the binary choice in Lefebvre model it is naturally to take \( (\{0, 1\}, \leq) \) as the external order. (Here \( \leq \) is the usual order on the set of integers).
Second, according to the interpretation of variables $x_1, x_2, x_3$ given by Lefebvre, the values of these variables describe directions of impulses (motivations) pushing the Subject to the positive or the negative pole. Indeed, $x_1$ corresponds to an impulse exerted by the external world, $x_2$ corresponds to an impulse exerted by Subject’s experience, and, finally, $x_3$ corresponds to Subject’s will.

So on one hand, $x_1, x_2, x_3$ are connected to the motivations. On the other hand, at any decision node these variables have boolean values. Furthermore, the choice of some of these values represents the result of the decision node.

In our approach these two sides of variables work simultaneously. We describe the impulses ("pure motivations") by partial orders (whereas results of Subject’s choice are numbers 0 or 1). Two possible values of a variable present two possible pure motivations for this variable. Our choice of domain of these partial orders is based on the following reasons.

These six (two specific pure motivations for every variable of $\{x_1, x_2, x_3\}$) partial orders are basic and their interaction would determine Subject’s choices within our frame of two-preference decision-making. The decision making is done in two stages. At the first stage some of the given pure motivations (i.e., some variables) are chosen. At the second stage the Subject proceeds to the pole associated by Lefebvre’s interpretation with the given value of the variable. This means that Lefebvre’s state set $M_*$ has to be $\{x_1, x_2, x_3\}$. In our scheme the chosen state has to maximize external value that is computed with the current evaluation mapping $\psi$. Hence, $\psi$ sets the external preference and the latter, in turn, determines Subject’s choices (decisions).

We now seem to come to the conclusion that it is the interaction of pure motivations that produces these external preferences or equivalently $\psi$. So the external preferences have to be some sort of "mixture" of pure motivations. (Here one can notice a vague analogy with quantum mechanics.)

Maximizing Subject’s adaptation abilities leads to the best survival chances. Therefore, one seeks a universal "mixing" procedure. Corollary 2 tells us that such a procedure can be attained using the universality of the corresponding approximating form (in our case $D = 2$ because two is the upper bound of lengths of the longest increasing chains possible in posets of three elements):

$$\psi = Sb_{p(z_1),p(z_2),p(z_3)}^{z_1,z_2,z_3}(z_1, \Box z_2, z_3).$$

Here $p$ runs over the class of special functions $\Theta$. Given the chosen external order we can set $\Box = \rightarrow$ (see lemma 1 above. $\rightarrow$ is the connective dual for implication).
Hence it follows that $\Theta$ is the set of pure motivations (impulses) in this case. So $|\Theta| = 6$. Therefore, the sought order on $M_*$ has to be a linear.

It is natural to deem that this internal linear poset reflects the common division of the time axis in three periods: "past", "present" and "future". Then the current representation of the world (variable $x_2$ in Levebvre’s model) corresponds to the point "present" and the Subject’s intention ($x_3$) corresponds to the point "future". Thus, the remaining variable $x_1$ ought to correspond to the point "past". Such assignment appears natural because the pressure put on the Subject by the environment is the background of the decision-making problem itself. Thus, we arrive at the internal preference relation $(M_*, \leq_*)$, where $M_* = \{x_1, x_2, x_3\}$, $x_1 <_* x_2 <_* x_3$.

It may seem that $x_2 <_* x_1$ ought to hold since we interpret $x_2$ as the "past experience" and $x_1$ as the "current pressure of the environment". However, we should keep in mind that we are currently dealing with the internal order on states in the process of decision making. In that process "past experience" $x_2$ serves a role of Subject’s "current base" and it is $x_1$ that initiates the decision making. $x_3$ is merely a means to produce a solution and as such is most likely related to the future. (Note that such crude models often cover several various factors with one parameter).

Every decision making act done by the Subject can be characterized by a given boolean 3-tuple $x_1, x_2, x_3$ of values of variables $x_1, x_2, x_3$. On the other hand, as pointed out above, we associate a pure motivation $\theta_{x_i} \in \Theta$ with any $x_i, i = 1, 2, 3$ when $x_i = x_i \in \{0, 1\}$. Here

$$\theta_{x_i}^{x_k}(x_k) = \begin{cases} 1, & \text{if } i < k, \\ x_i, & \text{if } i = k, \\ 0, & \text{if } k < i. \end{cases}$$

The general external order for 3-tuple $x_1, x_2, x_3$ is determined by formula

$$\psi_{x_1, x_2, x_3} = \theta_{x_1} \implies (\theta_{x_2} \implies \theta_{x_3}).$$

Any obtained motivation $\psi : M_* \to \{0, 1\}$ determines Subject’s decision choice $x_i, i = 1, 2, 3$, for a given decision making act. In order to find the solution we use the following local extremization algorithm for $\psi_{x_1, x_2, x_3}$:

1) Starting at the state $x_1$ in order $(M_*, \leq_*)$ proceed to the nearest extremum of $\theta_{x_1}^{x_k}$.
2) Then continue from the found state to the nearest extremum of the inverted function $\theta_{x_2}^{x_k}$ (due to its place in the approximating form).
3) Finally, repeat starting from the found state this time using the function $\theta_3 x_3$.

(Note, that the last step of the algorithm uses a double inversion of the motivation $\theta_3 x_3$.)

It is easy to check that the algorithm computes an element of $\arg \max \psi_{x_1, x_2, x_3} \subseteq \{x_1, x_2, x_3\}$. At the second step we turn a chosen $x_i$ into $x_i$. This number is Subject’s choice in the decision making situation defined via $x_1, x_2, x_3$.

It turns out that for all boolean 3-tuples $(x_1, x_2, x_3)$ the boolean value computed with the aforementioned scheme coincides with the value computed with the formula $x_3 \rightarrow (x_2 \rightarrow x_1)$. Thus, Lefebvre’s latter formula of behaviour can be derived from our model of behaviour. In our opinion, Lefebvre’s subjects are distinguished merely by a particular internal preference: the state order $(M_*, \leq_*)$. (The binarity of the external poset is presumed in Lefebvre’s problem statement.)

Perhaps the approach introduced in this section can, in principle, replace Lefebvre’s axioms. It has no need for such presumptions as ”Anthropic Principle”, ”Principle of Freedom”, and ”Simplicity postulate”. In our opinion, such a difference is advantageous since it appears extremely difficult to find a solid justification for these presumptions.

Indeed, instead of seeking a body of philosophical support we can apply the theory of approximating forms. Additionally, the restriction in this section by these orders does not need any special justification.

In the continuous case the toolbox of $L$-ensembles not only reduces it to the Boolean case but also extends the theory’s capacity.

6. Conclusion

As this paper demonstrates, the classical two-valued propositional logic can be viewed merely as a realization of the principle of successive approximations for the decision-making problem within the framework of Subject-environment survival.

From this viewpoint, the classical propositional logic can take its beginning from the survival problem. It is also important that such hypothetical origin of Logics appears quite natural.

Furthermore, this approach can serve as a background for consideration of other families of mappings from one poset to another with a chosen notion of simplicity of mapping. Any such case generates a corresponding logic.

Later in the paper we demonstrated how the effects explained by Lefebvre’s model can be viewed merely as implications of choosing the binary linear external order
and the three-element linear internal preference. Such choice reflects on the ordinary
division of the time axis into three parts: the past, the present, and the future.
Taking into account the established connection between logic and approximating
forms one may say that the psychological effects described via Lefebvre’s model can
be interpreted as logic of evaluation operators of the kind ψ : (\{1, 2, 3\}, \{1 < 2, 1 < 3, 2 < 3\}) → (\{0, 1\}, \{0 < 1\}). This fact can explain prevalence of the effects and
partially of the “golden section” method.

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