Homogeneous variational problems: a minicourse

D. J. Saunders

April 28, 2013

Abstract

A Finsler geometry may be understood as a homogeneous variational problem, where the Finsler function is the Lagrangian. The extremals in Finsler geometry are curves, but in more general variational problems we might consider extremal submanifolds of dimension $m$. In this minicourse we discuss these problems from a geometric point of view.

MSC: 35A15, 58A10, 58A20
Keywords: calculus of variations, parametric problems

1 Introduction

This paper is a written-up version of the major part of a minicourse given at the sixth Bilateral Workshop on Differential Geometry and its Applications, held in Ostrava in May 2011. Much of the discussion at these workshops is on Finsler geometry, where the interest is in variational problems defined on tangent manifolds by a ‘Finsler function’, a smooth function defined on the slit tangent manifold (excluding the zero section) and satisfying certain homogeneity and nondegeneracy properties. The extremals of such problems are geometric curves in the original (base) manifold, without any particular parametrization but with an orientation.

For this particular workshop it was felt that it might be worthwhile to describe slightly more general problems, looking at variational problems where the extremals were submanifolds of dimension $m$, but where the action function still depended upon no more than the first derivatives of the submanifold [2],[4]; for example, minimal surface problems would be included in this description. This minicourse introduces a version of the geometric background needed to express such problems, in terms of velocity manifolds. There is an alternative approach to such problems involving manifolds of contact elements (quotients of velocity manifolds); we refer to this only briefly, when we consider the action of the jet group.

Although we consider only first order variational problems, we nevertheless need to use second order velocities: for instance, the Euler-Lagrange equations for first order variational problems are second-order differential equations. We do this in a slightly unusual way, looking at a particular submanifold of the double velocity manifold. Having done this, we look at some geometrical and cohomological constructions, before obtaining a version of the first variation formula for variational
problems with fixed boundary conditions. The final part of the minicourse, which considered various concepts of regularity, has been omitted from this paper for reasons of space; the concepts described may be found in a recent paper [1]. We give only a few other references: [3] provides extensive background material on various types of jet manifold and the actions of the jet groups; [5] introduces in a more general context the type of cohomological approach we use these types of variational problem; and [6], with a philosophy similar to that of the present paper, compares these problems with those defined on jets of sections of fibrations.

I should like to thank the organisers of the Workshop for inviting me to give this course. I acknowledge the support of grant no. 201/09/0981 for Global Analysis and its Applications from the Czech Science Foundation; grant no. MEB 041005 for Finsler structures and the Calculus of Variations; and also the joint IRSES project GEOMECH (EU FP7, nr 246981).

2 Velocities

In this section we see how to construct manifolds of first order and second order velocities, and also how certain groups, the jet groups, act on these manifolds.

2.1 First order velocities

Let $E$ be a connected, paracompact, Hausdorff manifold of class $C^\infty$ and of finite dimension $n$; let $O \subset \mathbb{R}^m$ (with $m < n$) be open and connected, with $0 \in O$. A map $\gamma : O \to E$ will be called an $m$-curve in $E$. The 1-jet $j_0^1 \gamma$ of $\gamma$ at zero will be called a velocity (or $m$-velocity), and the set $T_m E = \{ j_0^1 \gamma \}$ of velocities of all $m$-curves in $E$ will be called the velocity (or $m$-velocity) manifold of $E$. We map $T_m E$ to $E$ by

$\tau_{mE} : T_m E \to E$, $\tau_{mE}(j_0^1 \gamma) = \gamma(0)$.

We shall show that $T_m E$ really is a manifold (and is connected, paracompact and Hausdorff, and indeed is a vector bundle over $E$) by identifying it with the Whitney sum over $E$ of $m$ copies of the tangent manifold $T E$.

**Lemma 1.** There is a canonical identification $T_m E \cong \bigoplus^m T E$.

**Proof.** Let $i_k : \mathbb{R} \to \mathbb{R}^m$ be the inclusion $i_k(s) = (0, \ldots, 0, s, 0, \ldots, 0)$. Then each $\gamma \circ i_k$ is a curve in $E$, and the map

$j_0^1 \gamma \mapsto (j_0^1 (\gamma \circ i_1), \ldots, j_0^1 (\gamma \circ i_m))$

is a bijection $T_m E \to \bigoplus^m T E$ preserving the fibration over $E$. $\square$

**Corollary 2.** Let $\{ dt^i \}$ be the canonical basis of $\mathbb{R}^m\ast$; then

$T_m E \to T E \otimes \mathbb{R}^m\ast$, $(\xi_1, \ldots, \xi_m) \mapsto \xi_i \otimes dt^i$

is a vector bundle isomorphism. $\square$

If $(U; u^a)$ is a chart on $E$ then $(U^1; u^a, u^a_i)$ is a chart on $T_m E$, where

$U^1 = \tau_{mE}^{-1}(U)$, $u^a_i(j_0^1 \gamma) = D_i \gamma^a(0) = D_i(u^a \circ \gamma)(0)$.
If \( j_0^1 \gamma = (\xi_1, \ldots, \xi_m) \) then it is clear that \( u^a_i(j_0^1 \gamma) = \dot{w}^a(\xi_i) \). The rule for changing coordinates on \( T_mE \) is therefore

\[
v^b_i(j_0^1 \gamma) = \frac{\partial v^b_i}{\partial u^a}(\gamma(0)) u^a_i(j_0^1 \gamma).
\]

We can see from this that the superscript \( a \) labeling the coordinate function \( u^a_i \) depends on the original choice of chart \( u^a \) on \( E \), whereas the subscript \( i \) is independent of this choice and so is the index of a component of the velocity (namely, the tangent vector \( \xi_i \)). We call indices of this latter type \textit{counting indices} rather than \textit{coordinate indices}.

We shall be particularly interested in the subsets of \( T_mE \) containing those velocities \( j_0^1 \gamma \) where the \( m \)-curve \( \gamma \) has certain properties. Write \( \check{T}_mE \) for the subset

\[
\{ j_0^1 \gamma \in T_mE : \gamma \text{ is an immersion near zero} \};
\]

if \( j_0^1 \gamma = (\xi_1, \ldots, \xi_m) \) and \( j_0^1 \gamma \in \check{T}_mE \) then \( \{ \xi_1, \ldots, \xi_m \} \) will be linearly independent. An element of \( \check{T}_mE \subset T_mE \) will be called a \textit{regular velocity}.

**Proposition 3.** The regular velocities form an open-dense submanifold.

**Proof.** To show that \( \check{T}_mE \) is open in \( T_mE \), define the map \( \wedge : T_mE \to \wedge^m TE \) by \( (\xi_1, \ldots, \xi_m) \mapsto \xi_1 \wedge \cdots \wedge \xi_m \). Then

- The map \( \wedge \) is fibred over the identity on \( E \) and is continuous (it is polynomial in the fibre coordinates \( u^a_i \));
- \( j_0^1 \gamma \in \check{T}_mE \) exactly when \( \wedge(j_0^1 \gamma) \neq 0 \);
- the zero section of \( \wedge^m TE \) is closed.

To show that \( \check{T}_mE \) is dense in \( T_mE \), define the map \( f : U^1 \to \mathbb{R} \) by \( f(j_0^1 \gamma) = \det(u^a_i(j_0^1 \gamma)) \), where \( (u^a_i) \) is the \( m \times m \) submatrix containing the first \( m \) rows of the \( n \times m \) matrix \( u^a_i \). If \( j_0^1 \gamma \in O \subset U^1 \) where \( O \) is open and \( O \cap \check{T}_mE = \emptyset \) then \( f \) vanishes on \( O \). But

\[
\frac{\partial^m f}{\partial u^1 \partial u^2 \cdots \partial u^m}(j_0^1 \gamma) = 1.
\]

\[\square\]

### 2.2 Second order velocities

We define a \textit{second-order m-velocity} in the same way as a 2-jet at zero of an \( m \)-curve, and write

\[
T^2_mE = \{ j_0^2 \gamma \}, \quad \check{T}^2_mE = \{ j_0^2 \gamma : \gamma \text{ is an immersion near zero} \}.
\]

We also let \( \tau^2 \): \( T^2_mE \to E \), \( \tau^{2,1} \): \( T^{2,1}_mE \to T_mE \) be the projections

\[
\tau^2_mE(j_0^2 \gamma) = \gamma(0) \quad \tau^{2,1}_mE(j_0^1 \gamma) = j_0^1 \gamma.
\]

We take charts on \( T^2_mE \) to be \( (U^2; u^a\gamma, u^a_i\gamma, u^a_{ij}\gamma) \) where \( U^2 = (\tau^2_mE)^{-1}(U) \) and

\[
u^a_i(j_0^2 \gamma) = D\gamma^a(0), \quad u^a_{ij}(j_0^2 \gamma) = D_i D_j \gamma^a(0).
\]
so that \( u^a_{ij} = u^a_{ji} \) (this constraint will cause complications in certain coordinate formulæ). These charts form an atlas such that \( T^2_mE \) becomes a manifold with the standard properties. We shall not demonstrate this directly; we shall show instead that it may be identified with a closed submanifold of a larger manifold, the manifold of double velocities.

### 2.3 Double velocities

We know that \( T_mE \) is a manifold, so it has its own velocity manifold

\[
T_m'T_mE = \{ j^1_0 \tilde{\gamma} \}
\]

where \( \tilde{\gamma} \) is an \( m' \)-curve in \( T_mE \). This is the \((m', m)\) double velocity manifold. Charts on \( T_m'T_mE \) are therefore

\[
\{(U^1; u^a, u^a_i, u^a_{ij})\},
\]

where \( 1 \leq i \leq m \) and \( 1 \leq j \leq m' \), corresponding to the charts \( (U^1; u^a, u^a_0) \) on \( T_mE \). In most applications we have either \( m' = m \) or \( m' = 1 \). We shall be interested in a particular submanifold of double velocities, known as holonomic double velocities.

### 2.4 Holonomic double velocities

If \( \gamma \) is an \( m \)-curve in \( E \) then its prolongation is the \( m \)-curve \( j^1\gamma \) in \( T_mE \) where

\[
j^1\gamma(t) = j^1_0 (\gamma \circ T_t)
\]

and \( T_t : \mathbb{R}^m \to \mathbb{R}^m \) is the translation map \( T_t(s) = t + s \). Thus \( j^1_0 j^3\gamma \in T_mT_mE \).

We use the notation \( j^1\gamma \) rather than \( j^1\gamma \); the latter would be a map satisfying \( j^1\gamma(t) = j^1_0 \gamma \) whose codomain would be a set containing jets at arbitrary points of \( \mathbb{R}^m \) rather than just at zero.

**Proposition 4.** The map

\[
i : T^2_mE \to T_mT_mE, \quad i(j^2_0 \gamma) = j^1_0 j^3\gamma
\]

is an injection. Its image is the submanifold described in coordinates by

\[
u^a_i = u^a_{ij}, \quad u^a_{ij} = u^a_{ji}.
\]

The image of the chart \((U^2; u^a, u^a_i, u^a_{ij})\) under the injection is the restriction of the chart \((U^1; u^a, u^a_i, u^a_{ij})\) to the submanifold.

**Proof.** Suppose \( \gamma_1, \gamma_2 \) are two \( m \)-curves in \( E \) such that \( j^1_0 j^3\gamma_1 = j^1_0 j^3\gamma_2 \). Then for \( \gamma_1 \)

\[
u^a_i (j^0_0 j^3\gamma_1) = u^a_i (\gamma_1(0)) = u^a_i (j^1_0 j^3\gamma_1(0)) = u^a_i (j^0_0 j^1_0 j^3\gamma_1);
\]

\[
u^a_i (j^0_0 j^3\gamma_1) = D_i (u^a \circ \gamma_1)(0) = u^a_i (j^1_0 j^3\gamma_1(0)) = u^a_i (j^0_0 j^1_0 j^3\gamma_1);
\]

\[
u^a_{ij} (j^0_0 j^3\gamma_1) = D_i D_j (u^a \circ \gamma_1)(0) = D_i (u^a_j \circ j^3\gamma_1)(0) = u^a_{ij} (j^0_0 j^1_0 j^3\gamma_1)
\]

and similarly for \( \gamma_2 \), so that \( j^2_0 \gamma_1 = j^2_0 \gamma_2 \) and the map is an injection.

For any \( m \)-curve \( \gamma \) in \( E \)

\[
u^a_i (j^0_0 j^3\gamma) = D_i (u^a \circ j^3\gamma)(0) = D_i (u^a \circ \gamma)(0) = u^a_i (j^3\gamma(0)) = u^a_i (j^0_0 j^1_0 j^3\gamma)
\]

4
and
\[ u^a_{ij}(j_0^1 \gamma) = D_i(D_j(u^a \circ \gamma))(0) = D_i(D_j(u^a \circ \gamma))(0) \]
so that \( u_i^a = u_i^a \) and \( u_{ij}^a = u_{ij}^a \) when restricted to the image of the injection.

Furthermore, if \( \gamma \) is an \( m \)-curve in \( T_m E \) satisfying
\[ u_i^a(j_0^1 \gamma) = u_i^a(j_0^1 \gamma), \quad u_{ij}^a(j_0^1 \gamma) = u_{ij}^a(j_0^1 \gamma) \]
then the \( m \)-curve \( \gamma \) in \( E \) given in coordinates near \( \tau_m E(\gamma(0)) \) by
\[ \gamma^a(t) = u^a(j_0^1 \gamma) + u_i^a(j_0^1 \gamma) t^i + \frac{1}{2} u_{ij}^a(j_0^1 \gamma) t^i t^j \]
so that \( j_0^1 j_0^1 \gamma = j_0^1 \gamma \); thus the image of the injection is described locally by the equations \( u_i^a = u_j^a \), \( u_i^a = u_j^a \), and is therefore a submanifold of \( T_m T_m E \).

The relationship between the charts \((U^2; \nu^a, \nu^a_1)\) and \((U^1; \nu^a, \nu^a_1, \nu^a_2)\) is immediate. \( \square \)

The image of \( T_m^2 E \) in \( T_m T_m E \) is called the submanifold of holonomic double velocities. There is no canonical projection \( T_m T_m E \to T_m^2 E \); we may, however, consider a tubular neighbourhood \( \nu : N \to T_m^2 E \) of \( T_m^2 E \) in \( T_m T_m E \), and then the condition \( \nu \circ t = \text{id}_{T_m E} \) (where \( t : T_m^2 E \to T_m T_m E \) is the injection) gives rise to the constraints
\[
\frac{\partial u^a}{\partial \nu^p} = \delta^a_c, \quad \frac{\partial \nu^a}{\partial u^p_1} + \frac{\partial \nu^a}{\partial u^p_2} = 0, \quad \frac{\partial u^a}{\partial u^p_{1q}} + \frac{\partial u^a}{\partial u^p_{2q}} = 0
\]
for the coordinates of \( \nu \), and hence to the conditions
\[
d\nu^a = du^a + \frac{\partial u^a}{\partial u^p_1}(du^p_1 - du^p_2) + \frac{1}{2} \frac{\partial u^a}{\partial u^p_{1q}}(du^p_{1q} - du^p_{2q})
\]
\[
d\nu^a_1 = \frac{1}{2}(du^a_1 + du^a_2) + \frac{1}{2} \left( \frac{\partial u^a_1}{\partial u^p_1} - \frac{\partial u^a_1}{\partial u^p_2} \right)(du^p_1 - du^p_2) + \frac{1}{2} \frac{\partial u^a_1}{\partial u^p_{1q}}(du^p_{1q} - du^p_{2q})
\]
\[
d\nu^a_2 = \frac{1}{2}(du^a_1 + du^a_2) + \frac{\partial u^a_1}{\partial u^p_1}(du^p_1 - du^p_2) + \frac{1}{2} \frac{\partial u^a_1}{\partial u^p_{1q}}(du^p_{1q} - du^p_{2q}).
\]

We shall use these conditions later on.

### 2.5 The exchange map

There is another way of describing the submanifold of holonomic velocities.

A map \( \psi : O' \times O \to E \), where \( O \subset \mathbb{R}^m \), \( O' \subset \mathbb{R}^{m'} \) are open and connected, and where \( \theta_{0m} \in O \) and \( \theta_{0m'} \in O' \), is called a double \((m', m)\)-curve. For each \( s \in O' \)
\[
\psi_s : O \to E, \quad \psi_s(t) = \psi(s, t)
\]
is then an \( m \)-curve in \( E \), so that \( j_0^1 \psi_s \in T_m E \). Thus
\[
j_0^1(s \mapsto j_0^1 \psi_s) \in T_m T_m E.
\]
Lemma 5. The exchange map $e : T_m T_m E \to T_m T_m E$ is well-defined by $\psi \mapsto \dot{\psi}$ where $\dot{\psi}(t, s) = \psi(s, t)$ and is a smooth bijection.

Proof. The element of $T_m T_m E$ defined by $\psi$ satisfies

\[
\begin{align*}
\hat{u}^a(j_1^1(s \mapsto j_0^1 \psi_s)) &= \hat{u}^a(j_0^1 \psi_0) = \psi^a_0(0) = \psi^a(0, 0), \\
\hat{u}^i(j_1^1(s \mapsto j_0^1 \psi_s)) &= u^i_l(j_0^1 \psi_0) = D_i(u^a \circ \psi_0)(0) = D_{2:i} \psi^a(0, 0), \\
\hat{u}^a_l(j_1^1(s \mapsto j_0^1 \psi_s)) &= D_j(u^a \circ (s \mapsto j_0^1 \psi_s))(0) = D_j(s \mapsto \psi^a_0)(0) = D_{1:j} \psi^a(0, 0), \\
\hat{u}^a_{ij}(j_1^1(s \mapsto j_0^1 \psi_s)) &= D_j(u^a \circ (s \mapsto j_0^1 \psi_s))(0) = D_{1:j} D_{2:i} \psi^a(0, 0)
\end{align*}
\]

and carrying out the same calculation for $\dot{\psi}$ shows that $e$ is a well-defined injection. It is clearly an involution, and hence is a bijection. The coordinate formulæ

\[
u^a \circ e = u^a, \quad u^a_i \circ e = u^a_{i1}, \quad u^a_{ij} \circ e = u^a_{ij}, \quad u^a_{ij} \circ e = u^a_{ij},
\]

show that it is smooth. \hfill \Box

Proposition 6. The holonomic submanifold of $T_m T_m E$ is the fixed point set of the exchange map.

Proof. This is immediate from the coordinate formulæ for $e$. \hfill \Box

2.6 Jet groups

If we consider $m$-curves in $\mathbb{R}^m$ rather than in some other manifold, then we have the possibility of composing two such $m$-curves. If we insist that the origin must map to itself then the composition will always exist, although possibly with a smaller domain than the domains of the two original $m$-curves. We shall want the jets of these $m$-curves to have inverses, so that the curves themselves will need to be immersions near zero; it is convenient to assume that they are, in fact, diffeomorphisms onto their images.

So let $O \subset \mathbb{R}^m$ be open and connected with $0 \in O$, and let $\phi : O \to \phi(O) \subset \mathbb{R}^m$ be a diffeomorphism with $\phi(0) = 0$. The first and second order jet groups are

\[
L^1_m = \{j_0^1 \phi\}, \quad L^2_m = \{j_0^2 \phi\}
\]

The products for $L^1_m$ and $L^2_m$ are given by

\[
j_0^1 \phi_1 \cdot j_0^1 \phi_2 = j_0^1(\phi_1 \circ \phi_2), \quad j_0^2 \phi_1 \cdot j_0^2 \phi_2 = j_0^2(\phi_1 \circ \phi_2).
\]

Lemma 7. The product rules define group structures on $L^1_m$ and $L^2_m$.

Proof. The products are well-defined because the first (or second) derivatives of a composite depend only upon the first (or second) derivatives of the individual maps, by the first (or second) order chain rule; associativity of the products is inherited from that of composition. The diffeomorphism $\text{id}_{\mathbb{R}^m}$ satisfies

\[
j_0^1(\text{id}_{\mathbb{R}^m}) \cdot j_0^1 \phi = j_0^1(\text{id}_{\mathbb{R}^m} \circ \phi) = j_0^1 \phi;
\]

the map $\tilde{\phi} : \phi(O) \to O$ given by $\tilde{\phi} = \phi^{-1}$ satisfies $\tilde{\phi}(0) = 0$, and

\[
j_0^1 \tilde{\phi} \cdot j_0^1 \phi = j_0^1(\tilde{\phi} \circ \phi) = j_0^1(\text{id}_O) = j_0^1(\text{id}_{\mathbb{R}^m}).
\]

Similar formulæ hold for second-order jets. \hfill \Box
The map \( L^1_m \rightarrow \mathbb{R}^{m^2} \), \( j^1_0 \phi \mapsto (D_j \phi^i(0)) \) defines global coordinates on \( L^1_m \), and identifies it with \( \text{GL}(m, \mathbb{R}) \). The map \( L^2_m \rightarrow \mathbb{R}^{m^2(m+3)/2} \), \( j^2_0 \phi \mapsto (D_j \phi^i(0), D_j D_k \phi^i(0)) \) defines global coordinates on \( L^2_m \). Writing

\[
A^i_j = D_j \phi^i(0), \quad B^i_{jk} = D_j D_k \phi^i(0)
\]

where \( \det A^i_j \neq 0 \) because \( \phi \) is a diffeomorphism, the product rule in \( L^1_m \) is

\[
(A \hat{A})^i_j = A^i_h \hat{A}^h_j
\]

and the product rule in \( L^2_m \) is

\[
\begin{align*}
((A, B)(\hat{A}, \hat{B}))^i_j &= A^i_h \hat{A}^h_j, \\
((A, B)(\hat{A}, \hat{B}))^i_{jk} &= A^i_h \hat{B}^h_{jk} + B^i_{hl} \hat{A}^h_j \hat{A}^l_k,
\end{align*}
\]

the latter formula arising from the second order chain rule

\[
D_j D_k (\phi \hat{\phi})^i(0) = D_j (D_i \phi^k \circ \hat{\phi}) D_k \hat{\phi}^l(0)
= D_i \hat{\phi}^k(0) D_j D_k \hat{\phi}^l(0) + D_h D_i \phi^k(0) D_j \hat{\phi}^h(0) D_k \hat{\phi}^l(0)
\]

using \( \hat{\phi}(0) = \hat{\phi}(0) = 0. \)

**Corollary 8.** The groups \( L^1_m \) and \( L^2_m \) are Lie groups. \( \square \)

**Lemma 9.** The oriented subgroups \( L^1_m^+ \) and \( L^2_m^+ \), where \( \phi \) preserves orientation, are connected.

**Proof.** As \( L^1_m \) may be identified with \( \text{GL}(m, \mathbb{R}) \), the subgroup \( L^1_m \) where \( \phi \) preserves orientation may be identified with \( \text{GL}^+(m, \mathbb{R}) \), the subgroup of matrices satisfying \( \det A^i_j > 0 \), which is connected.

The map \( L^1_m \rightarrow L^2_m \) given by \( j^1_0 \phi \mapsto j^2_0 \hat{\phi} \), where \( \hat{\phi} \) is the linear map \( \hat{\phi}(t) = A^i_j t^j \) with \( (A^i_j) \) being the matrix corresponding to \( j^1_0 \phi \), is continuous; the coordinates of the image are \( (A^i_j, 0) \). The image of the subgroup \( L^1_m^+ \) under this map is therefore connected. But every element of \( L^2_m^+ \) may be joined to an element of this image by a path given in coordinates by

\[
s \mapsto (A^i_j, sB^i_{jk}), \quad s \in [0, 1]
\]

\( \square \)

### 2.7 Group actions

The jet groups \( L^1_m \) and \( L^2_m \) act on the velocity manifolds \( T_m E \) and \( T^2_m E \) by

\[
(j^1_0 \phi, j^1_0 \gamma) \mapsto j^1_0 (\gamma \circ \phi), \quad (j^2_0 \phi, j^2_0 \gamma) \mapsto j^2_0 (\gamma \circ \phi).
\]

These are right actions, and in coordinates they are

\[
\begin{align*}
&u^a \mapsto u^a \\
u^a_i \mapsto u^a_i A^h_j \\
u^a_{ij} \mapsto u^a_{ij} A^h_j + u^a_{jk} B^h_{jk}
\end{align*}
\]

where \( A^i_j \) and \( B^i_{jk} \) are the global coordinates of \( j^2_0 \phi \).
Lemma 10. The action of $L^1_m$ on $T_mE$ restricts to $\hat{T}_mE$, and the restricted action is free. The action of $L^2_m$ on $T_mE$ restricts to $T^2_mE$, and the restricted action is free.

Proof. The map $\phi$ is a diffeomorphism onto its image, so if $\gamma$ is an immersion near zero then so is $\gamma \circ \phi$.

We use coordinates to show that the restricted actions are free. Suppose first that $j^1_0(\gamma \circ \phi) = j^1_0\gamma$, so that

$$u^2_0 j^1_0 = u^1_0 j^1_0 A^1_j;$$

as $\gamma$ is an immersion near zero and $u^2_0 j^1_0 = Dj^1_0(0)$, it follows that the $m \times n$ matrix $u^2_0 j^1_0$ must have rank $m$, so that $A^1_j = \delta^1_j$ and hence $j^1_0 \phi = j^1_0 (\text{id}_m)$.

Now suppose that $j^2_0(\gamma \circ \phi) = j^2_0\gamma$, so that $u^2_0 j^2_0 = u^1_0 j^2_0 A^2_j$ and now also

$$u^2_0 j^2_0 = u^2_0 j^2_0 = u^1_0 j^2_0 A^2_j + u^2_0 j^2_0 B^2_{hk}.$$ 

As before we see that $A^1_j = \delta^1_j$, so that

$$u^2_0 j^2_0 = u^2_0 j^2_0 + u^2_0 j^2_0 B^2_{hk}$$

and therefore that $u^2_0 j^2_0 B^2_{hk} = 0$; the rank condition on $u^2_0 j^2_0$ now tells us that $B^2_{hk} = 0$.

2.8 Infinitesimal actions

Let $(a^1_j)$ be an element of the Lie algebra of $L^1_m$; the identification of the group with $GL(m,r)$ means that its Lie algebra may be identified with $\mathfrak{gl}(m,\mathbb{R})$ so that $(a^1_j)$ is an arbitrary $m \times m$ matrix.

Lemma 11. The vector field on $T_mE$ corresponding to $(a^1_j)$ is

$$a^1_j a^2_i \frac{\partial}{\partial u^2_j}.$$ 

Proof. The map $\sigma : (-\varepsilon, \varepsilon) \to GL(m,\mathbb{R})$, defined for sufficiently small $\varepsilon$ by $\sigma(s) = (\delta^1_j + sa^1_j)$, is a curve in $GL(m,\mathbb{R})$ whose tangent vector at the identity is $(a^1_j)$. If $j^1_0 \gamma \in T_mE$ then the corresponding curve through $j^1_0 \gamma$ is given in coordinates by

$$s \mapsto (u^1 h j^1_0 (\gamma), (\delta^1_j + sa^1_j) u^1 h j^1_0 (\gamma)).$$

The resulting tangent vector $\xi \in T_j^{\gamma} T_mE$ satisfies

$$u^1 h (\xi) = 0, \quad u^1 h (\xi) = a^1_j u^1 h j^1_0 (\gamma)$$

so that the vector field on $T_mE$ defined by the Lie algebra element $(a^1_j)$ is

$$a^1_j u^1 h \frac{\partial}{\partial u^2_j}.$$ 

We write $\Delta^1_j$ for the Lie derivative operation of the basis vector field $\Delta^1_j = u^2_i \partial/\partial u^2_j$. 

8
2.9 Second order infinitesimal actions

There is a similar result for the action of the Lie algebra of $L^2_m$.

**Lemma 12.** Let $(a^i_j, b^i_{jk})$ be an element of the Lie algebra of $L^2_m$. The corresponding vector field on $T_m^2 E$ is

\[ a^i_j u^a \frac{\partial}{\partial u^a_i} + \frac{1}{\#(jk)} (2a^i_j u^a_{ik} + b^i_{jk} u^a_i) \frac{\partial}{\partial u^a_{jk}}. \]

where $\#(jk)$ equals 1 if $j = k$ and equals 2 otherwise.

**Proof.** Let $\gamma$ be the curve in $L^2_m$ through the identity $j^2_0(\text{id})$ given in coordinates by

\[ s \mapsto (\delta^a_i j^2_0 u^a_i, s a^a_i j^2_0 u^a_i). \]

If $j^2_0 \gamma \in T_m^2 E$ then the corresponding curve through $j^2_0 \gamma$ is given in coordinates by

\[ s \mapsto (u^a_i (j^2_0 \gamma), u^a_i (j^2_0 \gamma) (\delta^a_i + sa^a_i j^2_0 \gamma), u^a_i (j^2_0 \gamma) (\delta^a_i + sa^a_i j^2_0 \gamma), u^a_i (j^2_0 \gamma) s b^a_{jk} j^2_0 \gamma). \]

The resulting tangent vector $\xi \in T_{j^2_0 \gamma} T_m^2 E$ satisfies

\[ \dot{u}^a_i (\xi) = 0, \]
\[ \dot{u}^a_j (\xi) = a^i_j u^a_i (j^2_0 \gamma), \]
\[ \dot{u}^a_{jk} (\xi) = a^i_k u^a_i (j^2_0 \gamma) + a^i_k u^a_i (j^2_0 \gamma) + b^i_{jk} u^a_i (j^2_0 \gamma). \]

so that the vector field on $T_m^2 E$ defined by the Lie algebra element corresponding to $(a^i_j, b^i_{jk})$ is

\[ a^i_j u^a \frac{\partial}{\partial u^a_i} + \frac{1}{\#(jk)} (2a^i_j u^a_{ik} + u^a_{jk} u^a_i) \frac{\partial}{\partial u^a_{jk}}. \]

We write $d^j_i$ and $d^i_{jk}$ for the Lie derivative operation of the basis vector fields

\[ \Delta^j_i = u^a_i \frac{\partial}{\partial u^a_j} + \frac{2}{\#(jk)} u^a_{ik} \frac{\partial}{\partial u^a_{jk}}, \quad \Delta^j_{ik} = \frac{1}{\#(jk)} u^a_{ik} \frac{\partial}{\partial u^a_{jk}}. \]

Note the use of the symbol $\#(jk)$ to compensate for the fact that the coordinate functions $u^a_{jk}$ and $u^a_{kj}$ are equal, so that summing over $j$ and $k$ could result in double-counting.

3 Geometric structures

The special structure of velocity manifolds manifests itself in the existence of certain differential operators (‘total derivatives’) and differential forms (‘contact forms’) which capture certain aspects of the structure. The total derivatives and contact forms may also be used to identify those maps between velocity manifolds, and vector fields on velocity manifolds, which have been constructed by a process known as prolongation. Finally, there is an algebraic method of lifting tangent vectors from a manifold to its velocity manifold called the vertical lift, and this gives rise to vertical endomorphisms.
3.1 Total derivatives

The identity map $T_mE \to T_mE$ defines a section of the pull-back bundle $\tau_{mE}^* T_mE \to T_mE$. Its components $d_t$ are the total derivatives, vector fields along $\tau_{mE}$. At a point $j^1_{\partial \gamma}$, the identification $T_mE \cong \bigoplus^m TE$ from Lemma 1 gives the $k$-th component of $j^1_{\partial \gamma}$ as

$$d_k|_{j^1_{\partial \gamma}} = j^1_0 (\gamma \circ \partial_k) = T\gamma (j^1_0 |_{\partial_k}).$$

Note that the subscript $k$ is a counting index, not a coordinate index. In coordinates, if $f$ is a function on $E$ then

$$d_k f|_{j^1_{\partial \gamma}} = d_k|_{j^1_{\partial \gamma}} f = T\gamma \left( \frac{\partial}{\partial u^a} \right) f = \frac{\partial (f \circ \gamma)}{\partial x^a} \bigg|_0$$

$$= \frac{\partial f}{\partial u^a} \gamma(0) = u^a_k (j^1_0 \gamma) \frac{\partial f}{\partial u^a} |_{\gamma(0)}$$

so that

$$d_k = u^a_k \frac{\partial}{\partial u^a}.$$

It is clear from this coordinate formula that the image of $(d_1, \ldots, d_m)$, a subspace of $T\gamma(0)E$ corresponding to each point $j^1_{\partial \gamma} \in T_mE$, does not have constant rank on $T_mE$. But its restriction to $\tilde{T}_mE$, where the $m \times n$ matrix $u^a_i$ has maximal rank, does have constant rank $m$.

3.2 Second order total derivatives

We take a similar approach to second order total derivatives. The inclusion map $T_m^2 E \to T_m T_m E$ defines a section of the pull-back bundle $\tau_{mE}^{2,1} T_m T_m E \to T^2_m E$; its components $d_t$ are the second order total derivatives, vector fields along $\tau_{mE}^{2,1}$. At a point $j^2_{\partial \gamma}$,

$$d_k|_{j^2_{\partial \gamma}} = T(j^1_{\partial \gamma}) \left( \frac{\partial}{\partial u^a} \right) ;$$

in coordinates

$$d_k = u^a_k \frac{\partial}{\partial u^a} + u^a_{kj} \frac{\partial}{\partial u^j}.$$

Once again the image of $(d_1, \ldots, d_m)$, a subspace of $T\gamma(0)E$ corresponding to each point $j^2_{\partial \gamma} \in T_m^2 E$, does not have constant rank on $T_m^2 E$, but its restriction to $\tilde{T}_m E$ does have constant rank $m$.

3.3 Contact 1-forms

Contact 1-forms on $T_m E$ or on $T_m^2 E$ are the horizontal 1-forms which annihilate total derivatives, so that $\theta$ is a contact 1-form exactly when

$$\langle \theta, d_k \rangle = 0.$$

Here, ‘horizontal’ means horizontal over $E$ for a 1-form on $T_m E$, and it means horizontal over $T_m E$ for a 1-form on $T_m^2 E$, so that it makes sense to evaluate such
forms on total derivatives; indeed, the modules of such horizontal 1-forms are dual to the modules of vector fields along $T_m E \to E$ or along $T_m^2 E \to T_m E$.

In fact we shall consider contact 1-forms, not on the whole of $T_m E$ or $T_m^2 E$, but on the submanifolds of regular velocities $\tilde{T}_m E$ and $\tilde{T}_m^2 E$. The reason is that, as mentioned previously, the image of the map $(d_1, \ldots, d_m)$ has constant rank $m$ only on the regular submanifolds; it is, for example, zero on the zero section of $T_m E$, and so every horizontal cotangent vector on that zero section is annihilated by all the total derivatives. If we were to include non-regular velocities then there would be ‘contact’ cotangent vectors which were not the values of any (smooth, and hence continuous) contact 1-form.

The important property of contact 1-forms is that they always pull back to zero under prolongations.

**Lemma 13.** If $\theta$ is a contact 1-form on $\tilde{T}_m E$ then $(\bar{j}^1)\gamma^* \theta = 0$. If it is a contact 1-form on $\tilde{T}_m^2 E$ then $(\bar{j}^2)\gamma^* \theta = 0$, where the prolonged $m$-curve $\bar{j}^\gamma \gamma$ is defined by $\bar{j}^\gamma \gamma(t) = j_0^\gamma (\gamma \circ T_t)$.

**Proof.** If $\theta$ is a contact 1-form on $\tilde{T}_m E$ then

$$
\left\langle (\bar{j}^1)\gamma^* \theta |_t, \frac{\partial}{\partial t^k} |_t \right\rangle = \left\langle (j^1 (\gamma \circ T_t))^* \theta |_t, \frac{\partial}{\partial t^k} |_t \right\rangle \\
= \left\langle (j^1 \gamma)^* \theta |_0, \frac{\partial}{\partial t^k} |_0 \right\rangle \\
= \theta \mid_{j_0^1 \gamma}, T^\gamma \left( \frac{\partial}{\partial t^k} |_0 \right) \\
= \theta \mid_{j_0^1 \gamma}, d_k j_0^1 \gamma = 0.
$$

The proof for a contact 1-form on $\tilde{T}_m^2 E$ is similar. □

**Proposition 14.** If $\theta$ is a 1-form on $\tilde{T}_m E$ satisfying $(\bar{j}^1)\gamma^* \theta = 0$ for every prolonged $m$-curve $\bar{j}^\gamma \gamma$ in $\tilde{T}_m E$ then $\theta$ is horizontal over $E$, and is a contact 1-form. A similar result holds for contact 1-forms on $\tilde{T}_m^2 E$.

**Proof.** We show first that $\theta$ is horizontal over $E$, by showing that it is horizontal at each point $j_0^1 \gamma \in \tilde{T}_m E$. Write $\theta$ in coordinates around such a point as

$$
\theta = \theta_a du^a + \theta_i^a du_i^a ;
$$

then if $\gamma$ is a representative $m$-curve for the velocity $j_0^1 \gamma$ we have

$$(\bar{j}^1)\gamma^* \theta = (\theta_a \circ j^1 \gamma )((\bar{j}^1)\gamma^* du^a) + (\theta_i^a \circ j^1 \gamma )((\bar{j}^1)\gamma^* du_i^a) .$$

But

$$(\bar{j}^1)\gamma^* du^a |_0 = d(u^a \circ j^1 \gamma) |_0 = d\gamma^a |_0 = \frac{\partial\gamma^a}{\partial t^i} |_0 dt^i |_0$$

$$(\bar{j}^1)\gamma^* du_i^a |_0 = d(u_i^a \circ j^1 \gamma) |_0 = d\left( \frac{\partial\gamma^a}{\partial t^i} \right) |_0 = \frac{\partial\gamma^a}{\partial t^i \partial t^j} |_0 dt^j |_0$$

11
Finally we observe that
\[ 0 = (j^1\gamma)^*\theta \bigg|_0 = \left( (\theta_a \circ j^1\gamma)(0) \frac{\partial \gamma^a}{\partial u^j} \bigg|_0 + (\theta^i_a \circ j^1\gamma)(0) \frac{\partial \gamma^a}{\partial u^i} \bigg|_0 \right) dt^j \bigg|_0 \]
and hence
\[ \theta_a(j^1\gamma) \frac{\partial \gamma^a}{\partial t^j} \bigg|_0 + \theta^i_a(j^1\gamma) \frac{\partial \gamma^a}{\partial u^i} \bigg|_0 = 0. \]
Choosing a different representative \( m \)-curve \( \tilde{\gamma} \) of \( j^1\gamma \) which differs in its second derivatives from \( \gamma \) (although necessarily having the same first derivatives) allows us to conclude that \( \theta^i_a(j^1\gamma) = 0 \), so that \( \theta \) is horizontal at \( j^1\gamma \) and hence is a horizontal 1-form. We also see from this argument that
\[ \theta_a(j^1\gamma) \frac{\partial \gamma^a}{\partial t^j} \bigg|_0 = 0. \]
Finally we observe that
\[ \langle \theta, d_k \rangle = \left< \theta_a du^a, u_k^b \frac{\partial}{\partial u^b} \right> = \theta_a u_k^a \]
so that
\[ \langle \theta, d_k \rangle |_{j^1\gamma} = \theta_a(j^1\gamma) \frac{\partial \gamma^a}{\partial u^k} \bigg|_0 = 0 \]
for each point \( j^1\gamma \in T_mE \), showing that \( \langle \theta, d_k \rangle = 0 \) and hence that \( \theta \) is a contact 1-form.

The proof for forms on \( T^*_mE \) is similar in principle but involves more complicated calculations.

The coordinate expressions for contact 1-forms on velocity manifolds are quite different from those on jet manifolds, and involve determinants: indeed, contact 1-forms on \( T^*_mE \) are sums of scalar multiples of \((m+1) \times (m+1)\) determinants

\[ \theta^{a_1a_2\cdots a_{m+1}} = \begin{vmatrix} u_1^{a_1} & u_2^{a_1} & \cdots & u_m^{a_1} & u_{m+1}^{a_1} \\ u_1^{a_2} & u_2^{a_2} & \cdots & u_m^{a_2} & u_{m+1}^{a_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_1^{a_m} & u_2^{a_m} & \cdots & u_m^{a_m} & u_{m+1}^{a_m} \\ du^{a_1} & du^{a_2} & \cdots & du^{a_m} & du^{a_{m+1}} \end{vmatrix}. \]

To see that such a determinant is indeed a contact 1-form, evaluate it on the total derivative \( d_k = u_k^b \partial/\partial u^b \) to give

\[ \langle \theta^{a_1a_2\cdots a_{m+1}}, d_k \rangle = \begin{vmatrix} u_1^{a_1} & u_2^{a_1} & \cdots & u_m^{a_1} & u_{m+1}^{a_1} \\ u_1^{a_2} & u_2^{a_2} & \cdots & u_m^{a_2} & u_{m+1}^{a_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_1^{a_m} & u_2^{a_m} & \cdots & u_m^{a_m} & u_{m+1}^{a_m} \\ u_k^{a_1} & u_k^{a_2} & \cdots & u_k^{a_m} & u_k^{a_{m+1}} \end{vmatrix} = 0. \]

To show that these forms span the local contact 1-forms, we show that their values at each point span the contact cotangent vectors at that point. Let the coordinate functions on the fibres of \( T^*\hat{T}_mE \) corresponding to the coordinates \((u^a, u_i^a)\) on \( \hat{T}_mE \)
be \((p_a, p'_a)\); then horizontal cotangent vectors satisfy the equations \(p'_a = 0\), and we have seen that the condition \(\langle \theta, d_k \rangle = 0\) corresponds to a coordinate condition which may now be written as \(u'_k p_a = 0\).

Now observe that at each point \(j_0^1 \gamma\) there is at least one set of \(m\) coordinates \((u^a_1, u^a_2, \ldots, u^a_m)\) such that the determinant \(\det u^a_j\) does not vanish at \(j_0^1 \gamma\); suppose, without loss of generality, that this set is \((u^a_1, u^a_2, \ldots, u^a_m)\), for we may always rearrange the order of the base coordinates \(u^a\) if necessary. It is clear that the cotangent vectors

\[
\theta^{12\cdots m+1}_{j^0 \gamma}, \theta^{12\cdots m+2}_{j^0 \gamma}, \ldots, \theta^{12\cdots m}_{j^0 \gamma}
\]

are linearly independent, so that the subspace of the space of contact cotangent vectors at \(j_0^1 \gamma\) spanned by them has dimension \(n - m\). But \(\dim T^*_m E = n\) and the \(m\) equations \(u'_k p_a\) characterising contact 1-forms are linearly independent for regular velocities, so that the dimension of the space of contact cotangent vectors at \(j_0^1 \gamma\) is \(n - m\).

### 3.4 Contact r-forms

We define contact r-forms using the pull-back condition, so that an r-form \(\omega\) on \(\hat{T}_m E\) is a contact r-form if \((j^1 \gamma)^* \omega = 0\), and an r-form \(\omega\) on \(\hat{T}_m E\) is a contact r-form if \((j^2 \gamma)^* \omega = 0\). Note that contact r-forms need not be horizontal if \(r > 1\).

We now see another important difference between contact forms on velocity manifolds and contact forms on jet manifolds. In the latter context, the contact r-forms are generated by the contact 1-forms and their exterior derivatives; but this is not the case on velocity manifolds. For example, on \(\hat{T}_2^1 \mathbb{R}^3\) the contact 1-forms are generated by the single 1-form

\[
\theta = \begin{vmatrix}
  u^1_1 & u^2_1 & u^3_1 \\
  u^1_2 & u^2_2 & u^3_2 \\
  du^1 & du^2 & du^3
\end{vmatrix};
\]

but \((u^1_1 du^2 - u^2_1 du^1) \wedge du^3 - (u^1_2 du^2 - u^2_2 du^1) \wedge du^3\) is a contact 2-form which cannot be written in terms of \(\theta\) and \(d\theta\).

### 3.5 Prolongations of maps

Let \(E_1, E_2\) be manifolds, and let \(f : E_1 \to E_2\) a map. The prolongation of \(f\) to \(T_m E_1\) is the map

\[
T_m f : T_m E_1 \to T_m E_2
\]

defined by

\[
T_m f (j^0_0 \gamma) = j^0_0 (f \circ \gamma).
\]

It is immediate from this definition that \(T_m (f \circ g) = T_m f \circ T_m g\) and that \(T_m (\text{id}_E) = \text{id}_{T_m E}\), so that \(T_m\) is a covariant functor. In coordinates,

\[
u^a \circ T_m f = f^a, \quad u^a_i \circ T_m f = d_i f^a.
\]

It is important to note that \(T_m f\) might not restrict to a map \(\hat{T}_m E_1 \to \hat{T}_m E_2\), because \(f \circ \gamma\) might not be an immersion, even though \(\gamma\) is an immersion.
3.6 Prolongations and the exchange map

As a particular example, the prolongation of the vector bundle projection $\tau_{mE} : T_mE \to E$ to $T_mT_mE$ is

$$T_m\tau_{mE} : T_mT_mE \to T_mE.$$  

**Lemma 15.** The exchange map $e : T_mE \to E$ satisfies

$$T_m\tau_{mE} \circ e = \tau_m(E).$$

**Proof.** From Lemma 5, $e$ may be expressed in coordinates as

$$u^a \circ e = u^a, \quad u^a_i \circ e = u^a_i, \quad u^a_{i;\j} \circ e = u^a_{i;\j}.$$  

Thus

$$u^a \circ \tau_{mE} \circ e = u^a, \quad u^a_i \circ \tau_{mE} \circ e = u^a_i,$$

whereas

$$u^a \circ T_m\tau_{mE} \circ e = u^a \circ e = u^a, \quad u^a_i \circ T_m\tau_{mE} \circ e = u^a_i \circ e = u^a_i.$$

In other words, the exchange map interchanges these two diagrams.

3.7 Prolongations of vector fields

A vector field $X$ on $E$ is a map $E \to TE$, and so its prolongation (as a map) is $T_mX : T_mE \to T_mE$.

**Lemma 16.** The composition $X^1_m = e \circ T_mE$, where $e : T_mE \to TT_mE$ is the exchange map, is a vector field on $T_mE$

**Proof.** From Lemma 15,

$$\tau_{mE} \circ e \circ T_mE = T_m\tau_E \circ T_mE$$

$$= T_m(T_mE \circ X)$$

$$= T_m(id_E)$$

$$= id_{T_mE}.$$

The vector field $X^1_m$ is called the prolongation of $X$ to $T_mE$.

**Proposition 17.** If $\psi_s$ is the flow of $X$ then $T_m\psi_s$ is the flow of $X^1_m$.  

14
Proof. We first compute a coordinate formula for the vector field whose flow is $T_m^\psi_s$.

Choose a point $j^0_0 \gamma \in T_mE$ and let $\varphi$ be the flow of $X$ in a neighbourhood of $\gamma(0)$. Let $(U, y)$ be a chart around $\gamma(0)$ so that, if

$$X = X^a \frac{\partial}{\partial u^a},$$

$\varphi$ satisfies

$$\left. \frac{\partial \varphi^a}{\partial s} \right|_{(0, \cdot)} = X^a.$$

Let $\tilde{\varphi}$ denote the map $(s, q) \mapsto T_m^\varphi_s(q)$, so that

$$\tilde{\varphi}^a = \varphi^a, \quad \tilde{\varphi}^a_i = d_i \varphi^a$$

where we define $(d_i \varphi^a)(s, q) = (d_i \varphi^a_s)(q)$. Then

$$\left. \frac{\partial \tilde{\varphi}^a_i}{\partial s} \right|_{(0, \cdot)} = \frac{\partial (d_i \varphi^a)}{\partial s} \left|_{(0, \cdot)} \right. = \frac{\partial}{\partial s} \left|_{(0, \cdot)} \right. (u^b \frac{\partial \varphi^a}{\partial u^b})$$

$$= u^b \left. \frac{\partial \varphi^a}{\partial u^b} \right|_{(0, \cdot)} = d_i \left. \frac{\partial \varphi^a}{\partial s}(0, \cdot) \right|_{(0, \cdot)} = d_i X^a,$$

so that, in coordinates, the vector field whose flow is $T_m^\psi_s$ is

$$X^a \frac{\partial}{\partial u^a} + (d_i X^a) \frac{\partial}{\partial u^a_i}.$$

On the other hand, regarding $X$ as a map $E \to TE$, and writing $\dot{u}^a$ as $u^a_1$,

$$u^a \circ X = u^a, \quad u^a_1 \circ X = X^a$$

so that

$$u^a \circ T_mX = u^a, \quad u^a_1 \circ T_mX = X^a, \quad u^a_i \circ T_mX = u^a_i, \quad u^a_1 \circ T_mX = d_i X^a;$$

thus

$$u^a \circ e \circ T_mX = u^a, \quad u^a_1 \circ e \circ T_mX = X^a, \quad u^a_i \circ e \circ T_mX = u^a_i, \quad u^a_1 \circ e \circ T_mX = d_i X^a$$

so that

$$X^a_1 = e \circ T_mX = X^a \frac{\partial}{\partial u^a} + (d_i X^a) \frac{\partial}{\partial u^a_i}.$$

Unlike prolongations of maps, prolongations of vector fields do restrict to $\dot{T}_mE$. 

\[ \square \]
3.8 Second prolongations

By extending the first order approach, maps \( f : E_1 \to E_2 \) may be prolonged to maps \( T^2_m f : T^2_mE_1 \to T^2_mE_2 \), and vector fields \( X \) on \( E \) may be prolonged to vector fields \( X^2_m \) on \( T^2_mE \). In coordinates,

\[
  u^a \circ T^2_m f = f^a, \quad u_i^a \circ T^2_m f = d_i f^a, \quad u_{ij}^a \circ T^2_m f = d_id_j f^a
\]

and if \( X = X^a \partial/\partial u^a \) then

\[
  X^2 = X^a \frac{\partial}{\partial u^a} + (d_i X^a) \frac{\partial}{\partial u_i^a} + \frac{1}{\#(ij)}(d_id_j X^a) \frac{\partial}{\partial u_{ij}^a}.
\]

The calculations are similar in principle to those given for the first order case, but more complicated in detail. Again \( T^2_m f \) might not restrict to a map \( \tilde{T}^2_mE_1 \to \tilde{T}^2_mE_2 \), whereas \( X^2_m \) does restrict to \( \tilde{T}^2_mE \).

3.9 Prolongations, contact forms, and total derivatives

Let \( f : E_1 \to E_2 \) be a map. If \( \theta \) is a contact form on \( \tilde{T}^2_mE_2 \) and if \( T^2_m f \) restricts to \( \tilde{T}^2_mE_1 \) then \( (T^2_m f)^* \theta \) is a contact form on \( \tilde{T}^2_mE_1 \), because

\[
  (j^1\gamma)^* (T^2_m f)^* \theta = (T^2_m f \circ j^1\gamma)^* \theta = (j^1(f \circ \gamma))^* \theta = 0.
\]

If \( X \) is a vector field on \( E \) and \( \theta \) is a contact form on \( \tilde{T}^2_mE \) then the Lie derivative \( \mathcal{L}_X^2 \theta \) by the prolongation of \( X \) is also a contact form, because the flow of \( X^2_m \) is the prolongation of the flow of \( X \). These results, using the characterisation of a contact form by vanishing pullback, apply to both 1-forms and to \( r \)-forms with \( r > 1 \). They also hold for contact forms on \( \tilde{T}^2_mE \).

The corresponding result for total derivatives is more complicated, as these operators are vector fields along a map rather than on a manifold.

**Lemma 18.** Prolongations and basis total derivatives commute, so that

\[
  d_i \circ \mathcal{L}_X = \mathcal{L}_{X_i^m} \circ d_i, \quad d_i \circ \mathcal{L}_{X_i^m} = \mathcal{L}_{X_i^m} \circ d_i.
\]

**Proof.** We check this using coordinates. In the first order case, if \( f \) is a function on \( E \) then

\[
  d_i(\mathcal{L}_X f) = d_i\left( X^a \frac{\partial f}{\partial u^a} \right) = u_i^b \left( \frac{\partial X^a}{\partial u^b} \frac{\partial f}{\partial u^a} + X^a \frac{\partial f}{\partial u^b \partial u^a} \right)
\]

whereas

\[
  \mathcal{L}_{X_i^m}(d_i f) = \mathcal{L}_{X_i^m}(u_i^b \frac{\partial f}{\partial u^b}) = (d_i X^b) \frac{\partial f}{\partial u^b} + u_i^b X^a \frac{\partial f}{\partial u^b \partial u^a}.
\]

A similar but slightly more lengthy calculation is used in the second order case. \( \square \)

3.10 Vertical endomorphisms

We have seen that \( T_mE \to E \) is a vector bundle and so, as with every vector bundle, it has a canonical vertical lift operator. Denote the vertical lift to \( (\eta_i) \in \bigoplus^n TE \cong T_mE \) by

\[
  T_{m|E}(\eta_i) \to T_{(\eta_i)} T_mE, \quad (\xi_k) \mapsto (\xi_k)^{\gamma(\eta_i)};
\]

16
in coordinates this is
\[(\xi_k)_{\gamma}^{(\eta_i)} = u_j^{\gamma}(\xi_k) \frac{\partial}{\partial u_j^{(\eta_i)}}.\]

For each vector \(\zeta \in T_{(\eta_i)}T_mE\) define the vector \(S^j\zeta \in T_{(\eta_i)}T_mE\) by
\[S^j\zeta = (0, \ldots, 0, T\tau_m(\zeta), 0, \ldots, 0)_{\gamma}^{(\eta_i)}\]
where the non-zero vector \(T\tau_m(\zeta)\) is in the \(j\)-th position. It is evident that \(S^j\) is a vector bundle map \(TT_mE \to TT_mE\), or alternatively a type \((1, 1)\) tensor field on \(T_mE\), called a \textit{vertical endomorphism}. Note that the superscript \(j\) is a counting index, not a coordinate index. In coordinates
\[S^j = du^a \otimes \frac{\partial}{\partial u_j}.\]

There is a close relationship between vertical endomorphisms and total derivatives.

**Lemma 19.** If \(\omega\) is an \(r\)-form on \(E\) then
\[S^j d_k \omega = r\delta_k^j (\tau^*_m \omega).\]

**Proof.** Suppose first that \(\theta\) is a 1-form; we shall give a proof in coordinates, omitting explicit mention of the pullback map. If \(\theta = \theta_a du^a\) then
\[S^j d_k \theta = S^j((d_\kappa \theta_a) du^a + \theta_a du^a_k) = \delta_k^j \theta_a du^a = \delta_k^j \theta.\]

We now use induction on \(r\). Suppose \(\omega\) is an \(r\)-form and that \(S^j d_k \omega = r\delta_k^j (\tau^*_m \omega)\); then
\[S^j d_k (\theta \wedge \omega) = S^j (d_\kappa \theta \wedge \tau^*_m \omega + \tau^*_m \theta \wedge d_\kappa \omega) = S^j d_k \theta \wedge \tau^*_m \omega + \tau^*_m \theta \wedge S^j d_k \omega = \delta_k^j (\tau^*_m \theta \wedge \tau^*_m \omega) + r\delta_k^j (\tau^*_m \theta \wedge \tau^*_m \omega) = (r + 1)\delta_k^j \tau^*_m (\theta \wedge \omega)\]
using the fact that \(\tau^*_m \theta\) and \(\tau^*_m \omega\) are horizontal over \(E\). The result now follows by linearity.

### 3.11 Second order vertical endomorphisms

There is also a version of the vertical endomorphism defined on second order velocity manifolds. This cannot be constructed in the same way as the first order vertical endomorphism, as \(T^2_mE \to T_mE\) is not a vector bundle but is instead an affine sub-bundle of \(T_mT_mE \to T_mE\). We shall establish our construction by modifying the first-order vertical endomorphism on \(T_mE\). There is an alternative method, based on the construction of vertical lifts using double \((1, m)\)-curves, which may be used in both first and second order cases, but we shall not describe that here.

So let \(\nu : T^2_mE \to T_mE\) be some tubular neighbourhood of \(T^2_mE\) in \(T_mE\); and let \(\iota : T^2_mE \to T_mE\) be the inclusion from Proposition 4. As before, let \(e : T_mE \to T_mE\) be the exchange map.
Proposition 20. Let $\theta$ be a 1-form on $T_m^2 E$; then the operation

$$\theta \mapsto \iota^* \left(S^k \downarrow (\nu^* \theta + e^* \nu^* \theta)\right),$$

where $S^k$ is the vertical endomorphism on $T_n T_m E$), does not depend on the choice of tubular neighbourhood map $\nu$ and hence defines a vertical endomorphism on $T_m^2 E$.

Proof. We use coordinates to show that the result is independent of $\nu$. Let $\theta = \theta_a du^a + \theta^i d\nu^i + \theta^{ij} d\nu^i d\nu^j$; then

$$\nu^* \theta = (\nu^* \theta_a) du^a + (\nu^* \theta^i_a) d\nu^i_a + (\nu^* \theta^{ij}_a) d\nu^i_a d\nu^i_j$$

$$= (\nu^* \theta_a) \left( du^a + \partial \nu_a \partial u_p (du^c_p - du^c_p) + \frac{1}{2} \partial \nu_a \partial u_{pq} (du^c_p - du^c_q) \right)$$

$$+ (\nu^* \theta^i_a) \left( \frac{1}{2} (du^i_p + du^i_p) + \frac{1}{2} \left( \partial \nu^i_a \partial u_p - \partial \nu^i_a \partial u_{p}^{c} \right)(du^c_p - du^c_p) \right)$$

$$+ \frac{1}{2} \partial \nu_a \partial u_{pq} (du^c_p - du^c_q)$$

$$+ (\nu^* \theta^{ij}_a) \left( \frac{1}{2} (du^i_p + du^i_p) + \partial \nu^i_a \partial u^c_p - \partial \nu^i_a \partial u^c_p \right)$$

using the coordinate formulæ for the tubular neighbourhood map given in Section 2.

Thus

$$S^k \downarrow \nu^* \theta = (\nu^* \theta_a) \left( - \partial \nu^a \partial u^c_k du^c_k + \frac{1}{2} \left( \partial \nu^a \partial u_{pc} - \partial \nu^a \partial u_{ck} \right) du^c_p \right)$$

$$+ (\nu^* \theta^i_a) \left( \delta^i_k \frac{1}{2} du^a - \frac{1}{2} \left( \partial \nu^i_a \partial u^c_p - \partial \nu^i_a \partial u^c_k \right) du^c_p \right)$$

$$+ \frac{1}{2} \left( \partial \nu^a \partial u^c_p - \partial \nu^a \partial u^c_k \right) du^c_p$$

so that

$$\iota^* \left(S^k \downarrow \nu^* \theta\right) = \theta_a \left( - \iota^* \left( \partial \nu^a \partial u^c_k \right) du^c_k + \frac{1}{2} \iota^* \left( \partial \nu^a \partial u_{pc} - \partial \nu^a \partial u_{ck} \right) du^c_p \right)$$

$$+ \theta^i_a \left( \delta^i_k \frac{1}{2} du^a - \frac{1}{2} \iota^* \left( \partial \nu^i_a \partial u^c_p - \partial \nu^i_a \partial u^c_k \right) du^c_p \right)$$

$$+ \frac{1}{2} \iota^* \left( \partial \nu^a \partial u^c_p - \partial \nu^a \partial u^c_k \right) du^c_p$$

$$+ \theta^{ij}_a \left( \frac{1}{2} (\delta^i_k du^a + \delta^i_k du^a) - \iota^* \partial \nu^a \partial u^c_k \right) du^c_p$$

$$+ \frac{1}{2} \iota^* \left( \partial \nu^a \partial u^c_p - \partial \nu^a \partial u^c_k \right) du^c_p.$$
derivatives may now be extended to a kind of homotopy formula. The relationship given in Lemma 19 between vertical endomorphisms and total

\[ \theta \]
using

\[ \frac{1}{2} \frac{\partial}{\partial u^c_k} - \frac{\partial}{\partial u^c_{kp}} \] du^c_p \]

and similarly

\[ S^k \mathcal{J} e^* \nu^* \theta = \left( e^* \nu^* \theta \right) \left( \frac{\partial}{\partial u^c_k} \right) du^c + \frac{1}{2} e^* \left( \frac{\partial}{\partial u^c_{kp}} - \frac{\partial}{\partial u^c_{k}^p} \right) du^c_p \]

so that, using \( \iota^* e^* = \iota^* \),

\[ \iota^* \left( S^k \mathcal{J} e^* \nu^* \theta \right) = \theta \left( \frac{\partial}{\partial u^c_k} \right) du^c + \frac{1}{2} \iota^* \left( \frac{\partial}{\partial u^c_{kp}} - \frac{\partial}{\partial u^c_{k}^p} \right) du^c_p \]

Thus, adding, we obtain

\[ \iota^* \left( S^k \mathcal{J} \left( \nu^* \theta + e^* \nu^* \theta \right) \right) = \theta^a_k du^a + 2 \theta^a_k du^a \]

using \( \theta^a_k = \theta^a_k \).

In coordinates, therefore, the second order vertical endomorphisms may be written as tensor fields

\[ S^k = du^a \otimes \frac{\partial}{\partial u^a_k} + \frac{2}{\#(ik)} du^a \otimes \frac{\partial}{\partial u^a_{ik}} \]

the factor \( 1/\#(ik) \) arises here because the contraction of \( \partial/\partial u^a_{ik} \) with \( du^a_{pq} \) equals \( \frac{1}{2} \#(ik) \delta^c_a (\delta^i_p \delta^k_q + \delta^i_q \delta^k_p) \), so that

\[
\frac{\partial}{\partial u^a_{ik}} \mathcal{J} (\theta^a_p du^c_{pq}) = \frac{\#(ik)}{2} \delta^c_a \left( \delta^i_p \delta^k_q + \delta^i_q \delta^k_p \right) \theta^a_p = \#(ik) \theta^a_{ik} .
\]

The relationship given in Lemma 19 between vertical endomorphisms and total derivatives may now be extended to a kind of homotopy formula.

Lemma 21. If \( \omega \) is an \( r \)-form on \( T_m E \) then

\[ S^j d_k \omega - d_k S^j \omega = r \delta^j_k \left( r_{m}^{-1} \right) \omega . \]
Proof. Suppose first that $\theta$ is a 1-form; we shall give a proof in coordinates, omitting explicit mention of the pullback map. If $\theta = \theta_a du^a + \theta^i du^i$ then

$$d_k \theta = (d_k \theta_a) du^a + \theta_a du_k^a + (d_k \theta^i) du^a_i + \theta^i du_k^a$$

so that

$$S^j d_k \theta = (\delta^j_k \theta_a + (d_k \theta^i_j)) du^a + \delta^j_k \theta^i_a du^a_i + \theta^i_k du_k^a.$$

On the other hand, $S^j \theta = \theta^j_a du^a$, so that

$$d_k S^j \theta = (d_k \theta^j_a) du^a + \theta^j_a du_k^a$$

and hence

$$S^j d_k \theta - d_k S^j \theta = \delta^j_k \theta_a du^a + \delta^j_k \theta^i_a du^a_i = \delta^j_k \theta.$$

We now use induction on $r$. Suppose $\omega$ is an $r$-form and that $S^j d_k \omega - d_k S^j \omega = r \delta^j_k (\tau_{mE}^2 \omega)$; then, as both $S^j$ and $d_k$ are derivations of degree zero, their commutator is a derivation of degree zero, and so

$$(S^j d_k - d_k S^j)(\theta \wedge \omega) = (S^j d_k - d_k S^j) \theta \wedge \tau_{mE}^2 \omega + \tau_{mE}^2 \theta \wedge (S^j d_k - d_k S^j) \omega$$

$$= r \delta^j_k \tau_{mE}^2 \theta \wedge \tau_{mE}^2 \omega + \delta^j_k \tau_{mE}^2 \theta \wedge \tau_{mE}^2 \omega$$

$$= (r + 1) \delta^j_k \tau_{mE}^2 (\theta \wedge \omega).$$

The result now follows by linearity. \hfill \qed

4 Vector forms

We often use vectors of operators, tensors, forms, and so on. For instance, we have defined the total derivatives $d_k$ and the vertical endomorphisms $S^j$, where $j$ and $k$ are counting indices rather than coordinate indices. These operators fit into a framework of vector forms, to which we can associate a cohomology theory. Although the full cohomology theory requires the use of higher-order velocity manifolds, we can see some aspects of the theory in the first and second order cases.

4.1 Vector forms

We consider differential forms on $E$, $\dot{T}_m E$ and $\ddot{T}_m E$ taking values in the vector space $\mathbb{R}^{m*}$ and its exterior powers. Write $\dot{T}_m E$ with $k = 0, 1, 2$ and put

$$\Omega^r_{k,s} = \left( \Omega^r \dot{T}_m E \right) \otimes \left( \Lambda^s \mathbb{R}^{m*} \right).$$

Then a typical element of $\Omega^r_{k,s}$ is

$$\Xi = \chi_{i_1 \ldots i_r} \otimes dt^{i_1} \wedge \ldots \wedge dt^{i_r} \in \Omega^r_{k,s}$$

where the scalar forms $\chi_{i_1 \ldots i_r}$ are skew-symmetric in their indices, and where, as in Corollary 2, $\{dt^i\}$ is the canonical basis of $\mathbb{R}^{m*}$. It is clear that $\Omega^r_{k,s}$ is a module over the algebra of functions on $\dot{T}_m E$. 

20
4.2 Operations on vector forms

Define the operators $d$ and $d_T$ on the modules of vector forms by their actions on decomposable forms,

$$d : \Omega^r_s \rightarrow \Omega^{r+1}_s, \quad d(\chi \otimes \omega) = d\chi \otimes \omega$$

$$d_T : \Omega^r_s \rightarrow \Omega^{r+1}_{k+1}, \quad d_T(\chi \otimes \omega) = d_i\chi \otimes (dt^i \wedge \omega),$$

so that

$$dd_T(\chi \otimes \omega) = d(d_i\chi \otimes (dt^i \wedge \omega)) = d_i d\chi \otimes (dt^i \wedge \omega) = dT(d\chi \otimes \omega) = d_T d(\chi \otimes \omega)$$

and

$$d^2_T(\chi \otimes \omega) = d_j d_i\chi \otimes (dt^j \wedge dt^i \wedge \omega) = 0,$$

showing that $dd_T = d_T d$ and $d^2_T = 0$. We say that $d\Xi$ is the differential of the vector form $\Xi$, and that $d_T\Xi$ is its total derivative.

The total derivative of a vector form is a type of Lie derivative, and so we can also define the corresponding contraction operation. Put

$$i_T : \Omega^r_s \rightarrow \Omega^{r-1,s+1}_{k+1}, \quad i_T(\chi \otimes \omega) = (d_i \chi) \otimes dt^i \wedge \omega,$$

where $d_i \chi$ denotes the contraction of the ‘vector field along a map’ $d_i$ with the scalar form $\chi$, so that

$$d_T = di_T + i_T d.$$

4.3 Equivariant vector forms

Let $\alpha j^{1}_{\phi} \circ \phi \in L^{1+}_{m}$ on $T_m E$ by

$$\alpha j^{1}_{\phi}(j^{1}_{\phi}(\gamma)) = j^{1}_{\phi}(\gamma \circ \phi);$$

also, let $A_{j^{1}_{\phi}} : \mathbb{R}^{m*} \rightarrow \mathbb{R}^{m*}$ denote the linear map

$$A_{j^{1}_{\phi}}(dt^i) = (D_j \phi^i(0)) dt^j,$$

and extend this by multilinearity to $A_{j^{1}_{\phi}} : \wedge^* \mathbb{R}^{m*} \rightarrow \wedge^* \mathbb{R}^{m*}$. The vector form $\chi_{i_1 \cdots i_s} \otimes (dt^{i_1} \wedge \cdots \wedge dt^{i_s}) \in \Omega^r_1$ is said to be equivariant if, for every $j^{1}_{\phi}$,

$$a^{*}_{j^{1}_{\phi}}(\chi_{i_1 \cdots i_s}) \otimes (dt^{i_1} \wedge \cdots \wedge dt^{i_s}) = \chi_{i_1 \cdots i_s} \otimes A_{j^{1}_{\phi}}(dt^{i_1} \wedge \cdots \wedge dt^{i_s}).$$

Thus an equivariant form, regarded as a map from objects defined on a velocity manifold to elements of a vector space, commutes with the action of the jet group on the manifold and the vector space. We use the oriented jet group in our definition, as our application will be to problems in the calculus of variations where we need to integrate the forms.

We shall be particularly interested in equivariant elements of $\Omega^0_{0,m}$, namely $0$-forms (functions) taking their values in the one-dimensional vector space $\wedge^* \mathbb{R}^{m*}$.

Then

$$A_{j^{1}_{\phi}}(dt^1 \wedge \cdots \wedge dt^m) = J \phi(0)(dt^1 \wedge \cdots \wedge dt^m)$$
where $\mathcal{J} \phi = \det(Dj\phi^i)$ is the Jacobian of $\phi$, and so, writing $d^m t$ for $dt^1 \wedge \cdots \wedge dt^m$, an element $\Lambda = L d^m t$ is equivariant when

$$(L \circ c_{j\phi}) d^m t = \det(Dj\phi^i(0)) L d^m t.$$ 

Thus, writing an element of $T_m E \cong \bigoplus^m TE$ as $(\xi_1, \ldots, \xi_m)$, $\Lambda$ is equivariant when for each matrix $A \in \text{GL}^+(m, \mathbb{R})$,

$$L(\xi_i A_j) = (\det A) L(\xi_j).$$

As the oriented jet group $L^1_+ m$ is connected, there is an infinitesimal condition for equivariance. For a vector form $\chi_{i_1} \cdots \chi_{i_s} \otimes (dt^{i_1} \wedge \cdots \wedge dt^{i_s}) \in \Omega^{r,s}_{1}$, we require

$$d^j_i (\chi_{i_1} \cdots \chi_{i_s}) \otimes (dt^{i_1} \wedge \cdots \wedge dt^{i_s}) = \chi_{i_1} \cdots \chi_{i_s} \otimes \mathcal{L}_{\partial^{i_1} \partial^{i_s}} (dt^{i_1} \wedge \cdots \wedge dt^{i_s})$$

In the particular case where $s = m$ we have $\mathcal{L}_{\partial^{i_1} \partial^{i_s}} d^m t = \delta^j_i d^m t$, so the condition simplifies to

$$d^j_i \chi = \delta^j_i \chi.$$

### 4.4 The bicomplex

It is clear that for $-1 \leq s \leq m - 2$ we can use the operators $d$ and $d_T$ to construct a bicomplex:

```
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^0_{s} & \longrightarrow & \Omega^1_{s} & \longrightarrow & \Omega^2_{s} & \longrightarrow & \Omega^3_{s} & \longrightarrow & 0 \\
\downarrow & d_T & \downarrow & d_T & \downarrow & d_T & \downarrow & d_T & \downarrow & d_T & \downarrow \\
0 & \longrightarrow & \Omega^0_{s+1} & \longrightarrow & \Omega^1_{s+1} & \longrightarrow & \Omega^2_{s+1} & \longrightarrow & \Omega^3_{s+1} & \longrightarrow & 0 \\
\downarrow & d_T & \downarrow & d_T & \downarrow & d_T & \downarrow & d_T & \downarrow & d_T & \downarrow \\
0 & \longrightarrow & \Omega^0_{s+2} & \longrightarrow & \Omega^1_{s+2} & \longrightarrow & \Omega^2_{s+2} & \longrightarrow & \Omega^3_{s+2} & \longrightarrow & 0 \\
\end{array}
```

where if $s = -1$ then $\Omega^*_s = 0$. In this bicomplex $\Omega^*_s$ means ‘modulo constant functions’, and is used instead of the usual beginning $0 \rightarrow \mathbb{R} \rightarrow \Omega^0 \rightarrow \ldots$ of the de Rham sequence.

An important property of the bicomplex is that all columns (apart from the first) are globally exact, we show this by obtaining a homotopy formula for $d_T$. Strictly speaking the homotopy formula involves third order forms which are horizontal over $E$, because the operator $P_2$ defined in the statement of the theorem involves applying a total derivative to (scalar) second-order forms which are horizontal over $E$; but if $d_T \Xi = 0$ then the operator $P_2$ is not involved and the formula is genuinely second order. We feel, nevertheless, that it is worthwhile giving the more general statement, on the understanding that the definition of the total derivative of a second order form, and the consequent generalisation of Lemma 21, follow exactly the same pattern as before. We also use the operator $P_2$ when studying equivalents of first-order Lagrangians, although in that context the image of $P_2$ is always second-order rather than third-order.
Theorem 22. If $\Xi \in \Omega^{r,s+1}_{1}$ with $r > 0$ then, to within a pullback,

$$P_2 d_T \Xi + d_T P_1 \Xi = \Xi,$$

where

$$P_1 (\chi_{i_1 \cdots i_{s+1}} \otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}) = \frac{1}{r(m-s)} S^j \chi_{i_1 \cdots i_{s+1}} \otimes \left( \frac{\partial}{\partial \nu^j} dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}} \right)$$

for first-order $r$-forms $\chi_{i_1 \cdots i_{s+1}}$, and

$$P_2 (\eta_{i_1 \cdots i_{s+2}} \otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+2}})$$

$$= \left( \frac{1}{r(m-s-1)} S^j \eta_{i_1 \cdots i_{s+2}} - \frac{1}{r^2(m-s)(m-s-1)} d_i S^l S^j \eta_{i_1 \cdots i_{s+2}} \right) \otimes \left( \frac{\partial}{\partial \nu^j} dt^{i_1} \wedge \cdots \wedge dt^{i_{s+2}} \right)$$

for second-order $r$-forms $\eta_{i_1 \cdots i_{s+2}}$.

Proof. This is a consequence of Lemma 21. Put

$$P_1^j = \frac{1}{r(m-s)} S^j$$

$$P_2^j = \frac{1}{r(m-s-1)} S^j - \frac{1}{r^2(m-s)(m-s-1)} d_i S^l S^j ;$$

then

$$P_2 d_T \Xi = P_2 (d_k \chi_{i_1 \cdots i_{s+1}} \otimes dt^{k} \wedge dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}})$$

$$= P_1^j d_k \chi_{i_1 \cdots i_{s+1}} \otimes \left( \frac{\partial}{\partial \nu^j} dt^{k} \wedge dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}} \right)$$

$$= P_1^j d_k \chi_{i_1 \cdots i_{s+1}} \otimes \left( \delta^j_k dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}} \right)$$

$$- P_2^j d_k \chi_{i_1 \cdots i_{s+1}} \otimes dt^{i_1} \wedge \left( \frac{\partial}{\partial \nu^j} dt^{i_2} \wedge \cdots \wedge dt^{i_{s+1}} \right)$$

$$= P_2^j d_k \chi_{i_1 \cdots i_{s+1}} \otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}$$

$$- (s+1) P_2^j d_k \chi_{j_1 \cdots j_{s+1}} \otimes dt^{i_1} \wedge dt^{i_2} \wedge \cdots \wedge dt^{i_{s+1}}$$

whereas

$$d_T P_1 \Xi = d_T \left( P_1^j \chi_{i_1 \cdots i_{s+1}} \otimes \left( \frac{\partial}{\partial \nu^j} dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}} \right) \right)$$

$$= d_k P_1^j \chi_{i_1 \cdots i_{s+1}} \otimes dt^{k} \wedge \left( \frac{\partial}{\partial \nu^j} dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}} \right)$$

$$= (s+1) d_k P_1^j \chi_{j_1 \cdots j_{s+1}} \otimes dt^{i_1} \wedge dt^{i_2} \wedge \cdots \wedge dt^{i_{s+1}}$$

so that

$$P_2 d_T \Xi + d_T P_1 \Xi = P_2^j d_k \chi_{i_1 \cdots i_{s+1}} \otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}$$

$$- (s+1) P_2^j d_k \chi_{j_1 \cdots j_{s+1}} \otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}$$

$$+ (s+1) d_k P_1^j \chi_{j_1 \cdots j_{s+1}} \otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}.$$
But, using Lemma 21, the operators acting on $\chi_{i_1 \cdots i_{s+1}}$ satisfy
\[
\delta_{i_1}^j P_{2}^{k} d_k = \frac{1}{r(m-s-1)} \delta_{i_1}^j S^k d_k - \frac{1}{r^2(m-s)(m-s-1)} \delta_{i_1}^j d_i S^l S^k d_l \\
= \frac{1}{r(m-s-1)} \delta_{i_1}^j (d_k S^k + m r) \\
- \frac{1}{r^2(m-s)(m-s-1)} \delta_{i_1}^j (d_k d_i S^l S^k + (m+1) r d_i S^l) \\
= \frac{m}{m-s-1} \delta_{i_1}^j - \frac{s+1}{r(m-s)(m-s-1)} \delta_{i_1}^j d_k S^k,
\]
using the fact that $S^l S^k \chi_{i_1 \cdots i_{s+1}} = 0$ because the $\chi_{i_1 \cdots i_{s+1}}$ are first-order forms. Similarly
\[
-(s+1) P_{2}^{j} d_i = -\frac{s+1}{r(m-s-1)} S^j d_i + \frac{s+1}{r^2(m-s)(m-s-1)} d_i S^l S^j d_l \\
= -\frac{s+1}{r(m-s-1)} (d_i S^j + r \delta_{i_1}^i) \\
+ \frac{s+1}{r^2(m-s)(m-s-1)} (d_i d_i S^l S^j + r d_i S^j + r \delta_{i_1}^i d_i S^j) \\
= -\frac{s+1}{(m-s-1)} \delta_{i_1}^j - \frac{(s+1)}{r(m-s)} d_i S^j + \frac{s+1}{r(m-s)(m-s-1)} \delta_{i_1}^j d_k S^k
\]
and
\[
(s+1) d_i P_{1}^{j} = \frac{s+1}{r(m-s)} d_i S^j,
\]
from which we see that
\[
\delta_{i_1}^j P_{2}^{k} d_k - (s+1) P_{2}^{j} d_i + (s+1) d_i P_{1}^{j} = \delta_{i_1}^j
\]
and the result follows.

### 4.5 The bottom left corner

The part of the bicomplex which holds the major interest for the calculus of variations is in the bottom left-hand corner; we shall repeat it, with a pull-back map shown explicitly where appropriate.

Take $[\Lambda] \in \mathfrak{M}_{1,m}$, so that, for some function $L$ on $T_m E$, we have for any representative
\[
\Lambda = L dt^1 \wedge \cdots \wedge dt^m = L d^m t.
\]
Here, $L$ will play the role of a (first order) Lagrangian function in the calculus of variations, and the vector-valued function $\Lambda$ will have the capability of being
integrated along $m$-curves in $T_{m}E$ (and, in particular, along prolongations to $\dot{T}_{m}E$ of $m$-curves in $E$). So, given the equivalence class $[\Lambda]$, define

$$\Theta_{1} = Sd\Lambda, \quad \varepsilon_{0} = \tau_{mE}^{2,1}d\Lambda - d_{T}\Theta_{1},$$

where the choice of representative in the equivalence class is immaterial as we consider only $d\Lambda$ in the definition. We may compute $\Theta_{1}$ and $\varepsilon_{0}$ in coordinates; they are

$$\Theta_{1} = S^{j}\left(\frac{\partial L}{\partial u^{a}_{j}}du^{a} + \frac{\partial L}{\partial u^{a}_{k}}du^{a}_{k}\right) \otimes \left(\frac{\partial}{\partial u^{j}}j(dt^{1} \wedge \cdots \wedge dt^{m})\right)$$

and

$$\varepsilon_{0} = \left(\frac{\partial L}{\partial u^{a}}du^{a} + \frac{\partial L}{\partial u^{a}_{i}}du^{a}_{i}\right) \otimes (dt^{1} \wedge \cdots \wedge dt^{m})$$

$$- d_{k}\left(\frac{\partial L}{\partial u^{a}_{j}}du^{a}_{j}\right) \otimes dt^{k} \wedge \left(\frac{\partial}{\partial u^{j}}j(dt^{1} \wedge \cdots \wedge dt^{m})\right)$$

$$= \left(\frac{\partial L}{\partial u^{a}} - d_{k}\left(\frac{\partial L}{\partial u^{a}_{k}}\right)\right)du^{a} \otimes (dt^{1} \wedge \cdots \wedge dt^{m}).$$

5 Variational problems

Our main application of the theory of vector forms, and their associated cohomology, will be to problems in the calculus of variations. These will be parametric problems: that is, problems where the solutions are submanifolds without a given parametrization (although with a particular orientation). In the one-dimensional case, as exemplified by Finsler geometry, all the vector forms are essentially scalar forms, and so this theory only provides further insight in the case where the submanifolds have dimension two or more.

5.1 Homogeneous variational problems

We now study $m$-dimensional variational problems on $E$, with fixed boundary conditions. As before, a vector function $\Lambda = L_{m}dt^{m} \in \Omega_{1}^{0,m}$ will be called a Lagrangian for a variational problem. It will be called homogeneous if it is equivariant with respect to the action of the oriented jet group $L_{m}^{1}$. Thus $\Lambda$ is homogeneous when the scalar function $L$ satisfies the infinitesimal condition

$$d_{j}^{i}L = \delta_{j}^{i}L$$

or, equivalently, the finite condition

$$L \circ \alpha_{j}^{i}(\phi) = (\det D_{j}\phi^{i}(0))L$$

for every every $j_{i}^{i} \phi \in L_{m}^{1}$. We now consider submanifolds of $E$ of the form $\gamma(C)$ where $\gamma : \mathbb{R}^{m} \to E$ is an immersion and $C \subset \mathbb{R}^{m}$ is a connected compact $m$-dimensional submanifold with
boundary $\partial C$. The fixed-boundary variational problem defined by $\Lambda$ is the search for extremal submanifolds $\gamma(C) \subset E$ satisfying

$$
\int_C ((\bar{j}^3(\gamma))^*L_{X^m_j}L)d^m t = 0
$$

for every variation field $X$ on $E$ satisfying $X|_{\gamma(\partial C)} = 0$.

**Theorem 23.** If $\Lambda$ is homogeneous and $\gamma(C)$ is an extremal submanifold then $\gamma \circ \phi$ is also an extremal submanifold, for any orientation-preserving reparametrization $\phi$ whose image contains $C$.

**Proof.** We shall show that if $\Lambda$ is homogeneous then, for any immersion $\gamma$,

$$
\int_{\phi^{-1}(C)} (L \circ \bar{j}^3(\gamma \circ \phi))d^m t = \int_C (L \circ \bar{j}^3(\gamma))d^m t
$$

so that the integral itself is invariant under reparametrization; hence extremals will be invariant under reparametrization. As

$$
\int_{\phi^{-1}(C)} (L \circ \bar{j}^3(\gamma \circ \phi))d^m t = \int_C (\phi^{-1})^*((L \circ \bar{j}^3(\gamma \circ \phi))d^m t)
$$

$$
= \int_C (L \circ \bar{j}^3(\gamma \circ \phi) \circ \phi^{-1})(\phi^{-1})^*d^m t,
$$

it will be sufficient to show that

$$
(L \circ \bar{j}^3(\gamma \circ \phi) \circ \phi^{-1})(\phi^{-1})^*d^m t = (L \circ \bar{j}^3(\gamma))d^m t.
$$

Now for any $s \in \mathbb{R}^m$

$$
d^m t_s = (\mathcal{J} \phi \circ \phi^{-1})(s) (\phi^{-1})^*d^m t|_s,
$$

and so it will be sufficient to show that, for each $s$,

$$
(L \circ \bar{j}^3(\gamma \circ \phi) \circ \phi^{-1})(s) = (L \circ \bar{j}^3(\gamma))(\mathcal{J} \phi \circ \phi^{-1})(s).
$$

Note that we do not require the diffeomorphism $\phi$ to satisfy the condition $\phi(0) = 0$.

To see how this can be obtained from the homogeneity condition, write the latter as

$$
L \circ \alpha_{\mathcal{J} \phi \varphi} = (\mathcal{J} \varphi)(0) L
$$

where $\varphi$ is a diffeomorphism which does satisfy $\varphi(0) = 0$; then, for any immersion $\gamma : \mathbb{R}^m \to E$,

$$
(\mathcal{J} \varphi)(0) L(j^3_0(\gamma \circ T_s)) = L(\alpha_{\mathcal{J} \phi \varphi}(j^3_0(\gamma \circ T_s)))
$$

$$
= L(j^3_0(\gamma \circ T_s \circ \varphi)).
$$

Now put $\varphi = T_{-s} \circ \phi \circ T_{\phi^{-1}(s)}$, and note that $\varphi(0) = 0$; also

$$
(\gamma \circ T_s) \circ \varphi = \gamma \circ \phi \circ T_{\phi^{-1}(s)}
$$

and

$$
(\mathcal{J} \varphi)(0) = (\mathcal{J} \phi)(\phi^{-1}(s)),
$$

so that

$$
(\mathcal{J} \phi)(\phi^{-1}(s)) L(j^3_0(\gamma \circ T_s)) = L(j^3_0(\gamma \circ \phi \circ T_{\phi^{-1}(s)}))
$$

and hence

$$
(\mathcal{J} \phi)(\phi^{-1}(s)) L(j^3_0(\gamma)) = L(j^3(\gamma \circ \phi) \circ \phi^{-1}(s)).$$
5.2 Equivalents of Lagrangians

Let $\Lambda \in \Omega_{\nabla}^0$ be a homogeneous Lagrangian. Any scalar $m$-form $\Theta_m \in \Omega^m_1$ which is horizontal over $E$ will be called an integral equivalent of $\Lambda$ if

$$\Lambda = \left(\frac{(-1)^m(m-1)/2}{m!}\right)i_T^m \Theta_m;$$

any vector $r$-form $\Theta_r \in \Omega^r_1$ which is horizontal over $E$ will be called an intermediate equivalent if

$$\Lambda = \left(\frac{(-1)^r(r-1)/2}{m!}\right)i_T^r \Theta_r, \quad 0 \leq r \leq m - 1.$$

**Lemma 24.** If $\Theta_{r+1}$ is an equivalent of $\Lambda$ then

$$\Theta_r = \left(\frac{(-1)^r}{m-r}\right)i_T \Theta_{r+1}$$

is also an equivalent.

**Proof.** If $\Theta_{r+1}$ is an equivalent of $\Lambda$ then by definition

$$\Lambda = \left(\frac{(-1)^r(r+1)/2}{m!}\right)i_T^{r+1} \Theta_{r+1},$$

so that

$$\left(\frac{(-1)^r(r-1)/2}{m!}\right)i_T^r \Theta_r = \left(\frac{(-1)^r(r+1)/2}{m!}\right)i_T^{r+1} \Theta_{r+1} \left(\frac{(-1)^r}{m-r}i_T \Theta_{r+1}\right) = \Lambda.$$

$\square$

In the case $r = m$ we use the term ‘integral equivalent’ for the following reason.

**Lemma 25.** If $\gamma$ is an $m$-curve in $E$ then $(\bar{j}^1 \gamma)^* \Lambda = (\bar{j}^1 \gamma)^* \Theta_m$, so that

$$\int_C (\bar{j}^1 \gamma)^* \Lambda = \int_C (\bar{j}^1 \gamma)^* \Theta_m.$$

It follows that $\Lambda = \Theta_0$ and $\Theta_m$ have the same extremals.

**Proof.** Suppose $\Theta \in \Omega^{r,m-r}$ may be written in coordinates in the particular form

$$\Theta = \Theta_{a_1 \ldots a_m} u^{a_{r+1}} \ldots u^{a_{m}} du^{a_1} \wedge \ldots \wedge du^{a_r} \otimes dt^{k_{r+1}} \wedge \ldots \wedge dt^{km},$$

where the functions $\Theta_{a_1 \ldots a_m}$ are skew-symmetric in their indices; then

$$i_T \Theta = \sum_{p=1}^r (-1)^{p-1} \Theta_{a_1 \ldots a_m} u^{a_{r+1}} \ldots u^{a_{m}} \left( u_k^a \frac{\partial}{\partial u^a} \right) \otimes dt^{k_{r+1}} \wedge \ldots \wedge dt^{km}$$

$$= r(-1)^{r-1} \Theta_{a_1 \ldots a_m} u^{a_{r+1}} \ldots u^{a_{m}} \left( u_k^a du^{a_1} \wedge \ldots \wedge du^{a_r} \wedge du^{a_{r-1}} \right) \otimes dt^{k_{r+1}} \wedge \ldots \wedge dt^{km}.$$
Thus if $\Theta \in \Omega^{m,0}$, we see that
\[
v_T^{m} \Theta = m!(-1)^{m(m-1)/2} \Theta_{a_1 \ldots a_m} \sum_{k=1}^{m} u^a_k \cdot \Theta_{a_1 \ldots a_m} dt^1 \wedge \cdots \wedge dt^m
\]
so that
\[
(j^1 \gamma)^* i_T^{m} \Theta = m!(-1)^{m(m-1)/2} \Theta_{a_1 \ldots a_m} \det(u^a_k) dt^1 \wedge \cdots \wedge dt^m.
\]

On the other hand,
\[
(j^1 \gamma)^* \Theta = (\Theta_{a_1 \ldots a_m} \circ j^1 \gamma) \det \left( \frac{\partial \gamma a_i}{\partial t^k} \right) dt^1 \wedge \cdots \wedge dt^m.
\]

\[ \Box \]

### 5.3 Euler forms

Let $\Theta_m$ be an integral equivalent of $\Lambda$. Define the scalar $(m+1)$-form $E_m \in \Omega^{m+1,0}$ by
\[
E_m = d \Theta_m
\]
and the vector forms $E_r \in \Omega^{r, m-r}$ by
\[
E_r = \iota_m^{2,1} \iota^* d \Theta_r - (-1)^r d_T \Theta_{r+1} \quad 0 \leq r \leq m - 1.
\]

The forms $E_r$ are called the Euler forms of $\Theta_m$.

**Lemma 26.** The Euler forms satisfy the recurrence relation
\[
E_r = \frac{(-1)^{r+1}}{m-r} i_T E_{r+1} \quad 0 \leq r \leq m - 1;
\]
consequently if $d \Theta_m = E_m = 0$ then $E = 0$.

**Proof.** This follows from the definition and Lemma 24. We have, omitting the pull-back maps,
\[
i_T E_{r+1} = i_T d \Theta_{r+1} (-1)^{r+1} i_T d_T \Theta_{r+2}
\]
\[
= d_T \Theta_{r+1} - d_T \Theta_{r+1} - (-1)^r d_T i_T \Theta_{r+2}
\]
\[
= d_T \Theta_{r+1} - (-1)^r d_T \Theta_{r+1} + (m-r-1)d_T \Theta_{r+1}
\]
\[
= (m-r)(d_T \Theta_{r+1} - (-1)^r d_T \Theta_{r})
\]
when $r+1 < m$, so that
\[
\frac{(-1)^{r+1}}{m-r} i_T E_{r+1} = (-1)^{r+1} d_T \Theta_{r+1} + d \Theta^r = E_r.
\]

Similarly,
\[
i_T E_m = i_T d \Theta_m
\]
\[
= d_T \Theta_m - d_T \Theta_m
\]
\[
= d_T \Theta_m - (-1)^{m-1} d \Theta_{m-1}
\]
so that

\((-1)^m i_T \mathcal{E}_m = (-1)^m d_T \Theta_m + d\Theta_{m-1} = \mathcal{E}_{m-1}.\)

\[ \square \]

The different spaces containing the various equivalents and Euler forms may be seen in this diagonal part of the bicomplex.

\[
\begin{align*}
\Theta_r = \frac{(-1)^r}{m-r} i_T \Theta_{r+1}, & \quad \Theta_0 = \Lambda \\
\Theta_1 \in \Omega^{1,m-1}_1 & \quad \Theta_2 \in \Omega^{2,m-2}_1 \\
\Theta_2 \in \Omega^{2,m-2}_1 & \quad \Theta_1 \in \Omega^{1,m-1}_1 \\
\Theta_1 \in \Omega^{1,m-1}_1 & \quad \Theta_0 \in \Omega^{0,m}_1 \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}_r = d\Theta_r - (-1)^r d_T \Theta_{r+1} & \quad \mathcal{E}_0 = \lambda \mathcal{E}_1 \\
\end{align*}
\]

\[ \lambda = \tau^{2,1}_{mE} \circ d \]

### 5.4 Lepagian forms

Let \(\Lambda\) be a homogeneous Lagrangian, and let \(\Theta_r\) be an equivalent of \(\Lambda\) \((1 \leq r \leq m)\). We shall say that \(\Theta_r\) is **Lepagian** if the corresponding Euler form \(\mathcal{E}_0 \in \Omega^{1,m}_2\) satisfies

\[ S\mathcal{E}_0 = 0, \]

so that \(\mathcal{E}_0\) is horizontal over \(E\).

**Theorem 27.** *The vector 1-form*

\[ \Theta_1 = Sd\Lambda \]

*is an integral equivalent of \(\Lambda\) \((m = 1)\) or an intermediate equivalent \((m \geq 2)\), and is Lepagian. It is called the Hilbert equivalent of \(\Lambda = L d^m t\).*

**Proof.** From the definition of \(S,*

\[ S\Xi = S^i \chi \otimes d^{m-1} t_j, \]

so that

\[
\begin{align*}
i_T Sd\Lambda &= i_T S(dL \otimes d^m t) \\
&= i_T (S^i dL \otimes d^{m-1} t_j) \\
&= i_k S^i dL \otimes dt^k \wedge d^{m-1} t_j \\
&= i_j S^i dL \otimes d^m t. \\
\end{align*}
\]
But for any 1-form \( \theta \) on \( \tilde{T}_mE \), if in coordinates \( \theta = \theta_a du^a + \theta_i^a du^i \) then
\[
i_j S^j \theta = i_j (\theta^a_i du^a) = u^a_j \theta^a_i,
\]
so that
\[
i_j S^j dL = u^a_j \frac{\partial L}{\partial u^a_j} = d^j_i L = mL
\]
using the homogeneity of the Lagrangian.

To show that \( \Theta_1 \) is Lagrangian, note that
\[
Sd_T \Theta_1 = Sd_T Sd \Lambda = S d (d^i_j \theta^j_i dL \otimes d^m_i t_i - d^i_j \theta^j_i dL \otimes d^m_i t_i)
\]
using Lemma 21 and the fact that \( L \) is defined on \( \tilde{T}_mE \) so that \( S^i S^j dL = 0 \); thus \( S_\mathcal{E}_0 = 0 \), as required.

**Theorem 28.** If \( \tilde{\Theta}_1 \) is another Lagrangian vector 1-form equivalent to \( \Lambda \), with corresponding Euler form \( \tilde{\mathcal{E}}_0 \), then
\[
\tilde{\mathcal{E}}_0 = \mathcal{E}_0 , \quad \tilde{\Theta}_1 - \Theta_1 = d_T \Phi \quad (\Phi \in \Omega^{1,m-2}_0).
\]

**Proof.** It follows straightforwardly from the Lagrangian condition \( S\tilde{\mathcal{E}}_0 = 0 \) that \( P_2 \tilde{\mathcal{E}}_0 = 0 \), so that we may use the homotopy condition of Theorem 22 to see that
\[
0 = P_2 \tilde{\mathcal{E}}_0 = P_2 (d\Lambda - d_T \tilde{\Theta}_1) = \Theta_1 - P_2 d_T \tilde{\Theta}_1 = \Theta_1 - (1 - d_T P_1) \tilde{\Theta}_1,
\]
giving \( \tilde{\Theta}_1 - \Theta_1 = d_T P_1 \tilde{\Theta}_1 \) (or \( \tilde{\Theta}_1 = \Theta_1 \) if \( m = 1 \)). Thus
\[
\tilde{\mathcal{E}}_0 - \mathcal{E}_0 = (d\Lambda - d_T \tilde{\Theta}_1) - (d\Lambda - d_T \Theta_1) = -d_T P_1 \tilde{\Theta}_1 = 0 .
\]
(Note that, as \( d\Lambda \) is a first-order vector 1-form, \( P_2 d\Lambda = Sd\Lambda = \Theta_1 \).)

**5.5 The First Variation Formula**

**Theorem 29.** Let \( C \) be a compact connected \( m \)-dimensional submanifold of \( \mathbb{R}^m \) with boundary \( \partial \mathcal{C} \), let \( \gamma \) be an \( m \)-curve in \( E \) whose domain contains \( C \), and let \( X \) be a variation field on \( E \) vanishing on \( \gamma(\partial \mathcal{C}) \) with prolongation \( X^m_\mathcal{C} \) on \( \tilde{T}_mE \). Then
\[
\int_C (\tilde{\mathcal{E}}_0)^* \mathcal{L}_{X^m_\mathcal{C}} \Lambda = \int_C (\tilde{\mathcal{E}}_0)^* i_X \mathcal{E}_0 ;
\]
consequently \( \gamma \) is an extremal of \( \Lambda \) precisely when \( \mathcal{E}_0 \) vanishes along the image of \( \tilde{\mathcal{E}}_0 \).
Proof. We note first that

\[ \int_C (\bar{\gamma}^1) \ast L_{X_1^m} \Lambda = \int_C (\bar{\gamma}^2) \ast i_{X_2^m} d\Lambda \]

\[ = \int_C (\bar{\gamma}^2) \ast i_{X_2^m} \tau_{mE} d\Lambda \]

\[ = \int_C (\bar{\gamma}^2) \ast i_{X_2^m} \mathcal{E}_0 + \int_C (\bar{\gamma}^2) \ast i_{X_2^m} d\Theta_1, \]

using the definition of the Euler form \( \mathcal{E}_0 \). But prolongations commute with basis total derivatives and \( \Theta_1 \) is horizontal over \( E \), so that

\[ \int_C (\bar{\gamma}^2) \ast i_{X_2^m} d\Theta_1 = \int_C (\bar{\gamma}^2) \ast i_{X_2^m} \mathcal{E}_0 = \int_C (\bar{\gamma}^2) \ast i_X \mathcal{E}_0 \]

and we see that the second integral vanishes; thus

\[ \int_C (\bar{\gamma}^1) \ast L_{X_m} \Lambda = \int_C (\bar{\gamma}^2) \ast i_{X_2^m} \mathcal{E}_0 = \int_C (\bar{\gamma}^2) \ast i_X \mathcal{E}_0 \]

because \( \mathcal{E}_0 \) is horizontal over \( E \).

Now let \( \gamma \) be an immersion. If \( \mathcal{E}_0 = 0 \) at every point in the image of \( \bar{\gamma}^2 \), then for any vector field \( X \) on \( E \) and any \( t \in C \) we will have \( (\bar{\gamma}^2) \ast i_X \mathcal{E}_0 \bigg|_t = 0 \), so that the integral over \( C \) will vanish and \( \gamma \) will be an extremal.

If, instead, \( q = j_0^2(\gamma \circ t_1) \) is some point in the image of \( \bar{\gamma}^2 \) where \( \mathcal{E}_0 \big|_q \) is non-zero, then there must be a vector field \( X \) on \( E \) such that the vector-valued function \( i_X \mathcal{E}_0 \) gives a strictly positive multiple of \( dm \) when evaluated at \( q \), and hence when evaluated in some neighbourhood \( U \) of \( q \). Let \( b \) be a positive bump function on \( E \) whose support lies in the interior of \( U \) and which satisfies \( b(q) = 1 \). Then

\[ \int_C (\bar{\gamma}^1) \ast L_{bX_1} \Lambda = \int_C (\bar{\gamma}^2) \ast i_{bX} \mathcal{E}_0 > 0, \]

so that \( \gamma \) cannot be an extremal. \( \square \)

5.6 Integral equivalents for \( m \geq 2 \)

Let \( \Lambda = L dm t \) be a homogeneous Lagrangian with \( m \geq 2 \), and write its Hilbert equivalent \( \Theta_1 \) as

\[ \Theta_1 = \bar{\psi}^i \otimes dm^{-1} t_i; \]

the scalar 1-forms \( \bar{\psi}_i \) are called the Hilbert forms of \( \Lambda \). If \( \Lambda \) never vanishes, define the Carathéodory equivalent \( \tilde{\Theta}_m \in \Omega_1^{m,0} \) by

\[ \tilde{\Theta}_m = \frac{1}{L_{m^{-1}}} \bigwedge_{i=1}^m \bar{\psi}^i. \]

Theorem 30. The Carathéodory equivalent \( \tilde{\Theta}_m \) is an integral equivalent of \( \Lambda \).

Proof. We must show that \( i_{\tilde{\Theta}_m} \mathcal{E}_0 = (-1)^{m(m-1)/2} m! \Lambda \), so rewrite \( \Theta_m \) as

\[ \Theta_m = \frac{1}{m! L_{m^{-1}}} \sum_{\sigma \in S_m} (-1)^\sigma \bar{\psi}^{(1)} \wedge \cdots \wedge \bar{\psi}^{(m)}, \]

31
where $\mathfrak{S}_m$ is the permutation group, and use induction. The calculation uses $d_j \mathcal{J} \vartheta^i = \delta_j^i L$, the proof of which is similar to that used to show that $i_T \Theta_1 = m \Lambda$; we also define $\tau_{r,s} \in \mathfrak{S}_m$ by

$$
\tau_{r,s}(i) = \begin{cases} 
m - s & (i = r) 
1 & (r + 1 \leq i \leq m - s) 
i & \text{otherwise}, \end{cases}
$$

Now

$$i_T \left( \sum_{\sigma \in \mathfrak{S}_m} (-1)^{\sigma} \vartheta^{\sigma(1)} \land \ldots \land \vartheta^{\sigma(m-s)} \otimes dt^{\sigma(m-s+1)} \land \ldots \land dt^{\sigma(m)} \right)$$

$$= \sum_{\sigma \in \mathfrak{S}_m} (-1)^{\sigma} d_j \mathcal{J} (\vartheta^{\sigma(1)} \land \ldots \land \vartheta^{\sigma(m-s)}) \otimes dt^j \land dt^{\sigma(m-s+1)} \land \ldots \land dt^{\sigma(m)}$$

$$= \sum_{r=1}^{m-s} \sum_{\sigma \in \mathfrak{S}_m} (-1)^{\sigma} (-1)^{r-1} (\vartheta^{\sigma(1)} \land \ldots \land (d_j \mathcal{J} \vartheta^{\sigma(r)}) \land \ldots \land \vartheta^{\sigma(m-s)}) \otimes$$

$$\otimes dt^j \land dt^{\sigma(m-s+1)} \land \ldots \land dt^{\sigma(m)}$$

$$= L \sum_{r=1}^{m-s} \sum_{\sigma \in \mathfrak{S}_m} (-1)^{\sigma} (-1)^{r-1} (\vartheta^{\sigma(1)} \land \ldots \land \vartheta^{\sigma(r-1)} \land \vartheta^{\sigma(r+1)} \land \ldots \land \vartheta^{\sigma(m-s)}) \otimes$$

$$\otimes dt^{\sigma(r)} \land dt^{\sigma(m-s+1)} \land \ldots \land dt^{\sigma(m)}$$

$$= L \sum_{r=1}^{m-s} \sum_{\sigma \in \mathfrak{S}_m} (-1)^{\sigma} (-1)^{r-1} (\vartheta^{\sigma(r-1)} \land \ldots \land \vartheta^{\sigma(r)} \land \vartheta^{\sigma(r+1)} \land \ldots \land \vartheta^{\sigma(m-s)} \otimes$$

$$\otimes dt^{\sigma(r)} \land dt^{\sigma(m-s+1)} \land \ldots \land dt^{\sigma(m)}$$

$$= (-1)^{m-s-1} L \sum_{r=1}^{m-s} (-1)^{\sigma} (\vartheta^{\sigma(1)} \land \ldots \land \vartheta^{\sigma(m-s-1)}) \otimes$$

$$\otimes dt^{\sigma(m-s)} \land dt^{\sigma(m-s+1)} \land \ldots \land dt^{\sigma(m)}$$

$$= (-1)^{m-s-1} (m-s) L \sum_{\sigma \in \mathfrak{S}_m} (-1)^{\sigma} (\vartheta^{\sigma(1)} \land \ldots \land \vartheta^{\sigma(m-s-1)}) \otimes$$

$$\otimes dt^{\sigma(m-s)} \land dt^{\sigma(m-s+1)} \land \ldots \land dt^{\sigma(m)}$$

so if

$$i_T \Theta_m = \frac{(-1)^s(2m-s-1)/2}{(m-s)!L^{m-s-1}} \left\{ \sum_{\sigma \in \mathfrak{S}_m} (-1)^{\sigma} \vartheta^{\sigma(1)} \land \ldots \land \vartheta^{\sigma(m-s)} \otimes dt^{\sigma(m-s+1)} \land \ldots \land dt^{\sigma(m)} \right\}$$

32
\[ i_T^{s+1} \Theta_m = \frac{(-1)^s(2m-s-1)/2}{(m-s)! L^{m-s-1}} \left\{ \right. \\
\left. (-1)^{m-s-1}(m-s)L \sum_{\sigma \in \mathcal{E}_m} (-1)^\sigma \left( \vartheta^{\sigma(1)} \land \cdots \land \vartheta^{\sigma(m-s-1)} \right) \right\} \\
\otimes dt^{\sigma(m-s)} \land dt^{\sigma(m-s+1)} \land \cdots \land dt^{\sigma(m)} \right\} \\
= \frac{(-1)^{s+1}(2m-s-2)/2}{(m-s-1)! L^{m-s-2}} \sum_{\sigma \in \mathcal{E}_m} (-1)^\sigma \left( \vartheta^{\sigma(1)} \land \cdots \land \vartheta^{\sigma(m-s-1)} \right) \otimes dt^{\sigma(m-s)} \land dt^{\sigma(m-s+1)} \land \cdots \land dt^{\sigma(m)} \right\} 
\]

as required. Hence
\[ i_T^m \Theta_m = \frac{(-1)^m(m-1)/2}{L^{-1}} \sum_{\sigma \in \mathcal{E}_m} (-1)^\sigma dt^{\sigma(1)} \land \cdots \land dt^{\sigma(m)} \]
\[ = (-1)^m(m-1)/2 \ m! L \ dt^1 \land \cdots \land dt^m \]
\[ = (-1)^m(m-1)/2 \ m! \Lambda. \]

\[ \square \]

We see also from the induction formula that
\[ i_T^{m-1} \Theta_m = (-1)^m(m-1)/2 \ (m-1)! \Theta_1 \]
where \( \Theta_1 \) is the Hilbert equivalent; consequently \( \Theta_m \) is Lepagian. Then, as \( d\Theta_m = E_m \),
\[ \int_C (j^1 \gamma)^* L_{X^1_m} \Theta_m = \int_C (j^1 \gamma)^* i_{X^1_m} E_m \]
\[ = (-1)^m(m-1)/2 \ m! \int_C (j^2 \gamma)^* i_X E_0 \]
for any vector field \( X \) on \( E \) vanishing on \( \gamma(\partial C) \), because contractions by vector fields anticommute, so that \( i_T^m i_{X^1_m} E_m = (-1)^m i_{X^1_m} i_T^m E_m \).

5.7 Another integral equivalent

When \( m = 1 \) then the only Lepagian integral equivalent of a Lagrangian is the Hilbert equivalent. But when \( m > 1 \) there may be other integral equivalents. Put
\[ \Theta_{r+1} = \frac{(-1)^r}{(r+1)^2} \ Sd\Theta_r \quad (1 \leq r < m) \]
where, as usual, \( \Theta_0 = \Lambda. \)

Lemma 31. Each \( \Theta_r \) is a first-order vector form, an element of \( \Omega_{1,m-r}^r \), horizontal over \( E \).
Proof. Each $\Theta_r$ is first-order because neither $S$ nor $d$ increases the order of a vector form. By definition $\Theta_0$ is horizontal over $E$, and if $\Theta_r$ is horizontal over $E$ then the contraction of $d\Theta_r$ with any vector field on $T_mE$ will again be horizontal over $E$; thus $\Theta_{r+1}$ will also be horizontal over $E$.

**Theorem 32.** The scalar $m$-form $\Theta_m$ is a LePagian integral equivalent of $\Lambda$ called the fundamental equivalent of $\Lambda$.

Proof. We first show that, in coordinates,

$$\Theta_r = \frac{1}{(r!)^2} \frac{\partial^r L}{\partial u_{i_1}^{a_1} \cdots \partial u_{i_r}^{a_r}} du^{a_1} \wedge \cdots \wedge du^{a_r} \otimes \left( \frac{\partial}{\partial t^1} \cdots \frac{\partial}{\partial t^m} dt^m \right).$$

This formula clearly holds for $r = 1$ (and, indeed, for $r = 0$); so suppose that it holds for a given value of $r$. Then

$$\Theta_{r+1} = \frac{1}{(r+1)!} \frac{\partial^{r+1} L}{\partial u_{i_1}^{a_1} \cdots \partial u_{i_{r+1}}^{a_{r+1}}} du^{a_1} \wedge \cdots \wedge du^{a_{r+1}} \otimes \left( \frac{\partial}{\partial t^1} \cdots \frac{\partial}{\partial t^m} dt^m \right)$$

so that the formula also holds for the case $r + 1$. In particular, therefore, we have

$$\Theta_m = \frac{1}{(m!)^2} \frac{\partial^m L}{\partial u_{i_1}^{a_1} \cdots \partial u_{i_m}^{a_m}} du^{a_1} \wedge \cdots \wedge du^{a_m} \times \left( \frac{\partial}{\partial t^1} \cdots \frac{\partial}{\partial t^m} dt^m \right),$$

Thus, using the calculation in the proof of Lemma 25,

$$i^n T \Theta_m = m!(1)^{m(m-1)/2} \left( \frac{1}{m!} \frac{\partial^m L}{\partial u_{i_1}^{a_1} \cdots \partial u_{i_m}^{a_m}} \det(u_{k_j}^{a_i}) \right) dt^1 \wedge \cdots \wedge dt^m = (-1)^{m(m-1)/2} m! L dt^1 \wedge \cdots \wedge dt^m = (-1)^{m(m-1)/2} m! \Lambda.$$
Theorem 33. The fundamental equivalent $\Theta_m$ of a homogeneous Lagrangian $\Lambda$ has
the property that $d\Theta_m = \mathcal{E}_m = 0$ if, and only if, $\mathcal{E}_0 = 0$.

Proof. If $\mathcal{E}_m = 0$ then $\mathcal{E}_0 = 0$ by the recurrence relation of Lemma 26. So show the
converse, we use the definition

$$\Theta_{r+1} = \frac{(-1)^r}{(r+1)^2} S d\Theta_r$$

and the fact that $d\Theta_r \in \Omega^{r+1,m-r}_1$ to see that the homotopy operator $P_1$ from
Theorem 22 takes the form

$$P_1(\chi_{i_1\ldots i_{m-r}} \otimes dt_{i_1} \wedge \ldots \wedge dt_{i_{m-r}}) = \frac{1}{(r+1)^2} S^j \chi_{i_1\ldots i_{m-r}} \otimes \left( \frac{\partial}{\partial t^j} \right) dt_{i_1} \wedge \ldots \wedge dt_{i_{m-r}}$$

(the formula in the proof of Theorem 22 was for an element of $\Omega^{r,s+1}_1$); thus we may
rewrite the definition of $\Theta_{r+1}$ as

$$\Theta_{r+1} = (-1)^r P d\Theta_r .$$

Now from

$$\mathcal{E}_r = d\Theta_r - (-1)^r d_T \Theta_{r+1}$$

we obtain

$$P_2 d\mathcal{E}_r = (-1)^r P_2 d_T d\Theta_{r+1}$$

$$= (-1)^r (d_T P_1 d\Theta_{r+1} - d\Theta_{r+1})$$

so that

$$(-1)^{r+1} P_2 d\mathcal{E}_r = d\Theta_{r+1} - d_T P_1 d\Theta_{r+1}$$

using the homotopy formula of Theorem 22; but

$$\mathcal{E}_{r+1} = d\Theta_{r+1} - (-1)^{r+1} d_T \Theta_{r+2}$$

$$= d\Theta_{r+1} - d_T P_1 d\Theta_{r+1}$$

so that

$$\mathcal{E}_{r+1} = (-1)^{r+1} P_2 d\mathcal{E}_r .$$

Similarly,

$$P_2 d\mathcal{E}_{m-1} = (-1)^{m-1} P_2 d_T d\Theta_m$$

$$= (-1)^m d\Theta_m$$

$$= (-1)^m \mathcal{E}_m .$$

It follows that if $\mathcal{E}_0 = 0$ then $\mathcal{E}_m = 0$. \qed
References

[1] M. Crampin, D.J. Saunders: Some concepts of regularity for parametric multiple-integral problems in the calculus of variations Czech Math. J. 59 (3) (2009) 741–758
[2] M. Giaquinta, S. Hildebrandt: Calculus of Variations II (Springer, 1996)
[3] I. Kolář, P.W. Michor, J. Slovák: Natural Operations in Differential Geometry (Springer, 1993)
[4] H. Rund: The Hamilton-Jacobi Equation in the Calculus of Variations (Krieger, 1973)
[5] D.J. Saunders: Homogeneous variational complexes and bicomplexes J. Geom. Phys. 59 (2009) 727–739
[6] D.J. Saunders: Some geometric aspects of the calculus of variations in several independent variables Comm. Math. 18 (1) (2010) 3–19