Abstract

It was a difficult problem to determine the Gaussian fixed line from the numerical data, because close to the Berezinskii-Kosterlitz-Thouless multicritical point the divergence of the correlation length becomes very slow. Considering the renormalization group behavior, we find an efficient method to determine the Gaussian fixed line.

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The two dimensional (2D) sine-Gordon model, which is defined with the Lagrangian

\[ \mathcal{L} = \frac{1}{2\pi K} (\nabla \phi)^2 + \frac{y_1}{2\pi \alpha^2} \cos \sqrt{2} \phi, \]  

(1)

(with the identification \( \phi \equiv \phi + 2\pi/\sqrt{2} \)) and the dual field \( \theta \)

\[ \partial_x \phi = -\partial_y (iK \theta), \partial_y \phi = \partial_x (iK \theta), \]  

(2)

plays an important role in 2D classical and 1D quantum systems, such as the 2D XY model, 2D helium, 1D quantum spin models and 1D electron models. It is also related with the \( U(1) \times Z_2 \times Z_2 \) conformal field theory or the Tomonaga-Luttinger liquid. The sine-Gordon model has the Berezinskii-Kosterlitz-Thouless (BKT) \([1, 2, 3]\) phase boundaries and the Gaussian fixed line.

However, it was a difficult problem to determine the phase boundary and the universality class of the sine-Gordon model from the numerical calculation. Finite-size scaling technique, which is an efficient method to analyze the ordinary second order transition, leads to false conclusions for the BKT transition \([4, 5, 6, 7]\), since the divergence of the correlation length is very slow \( \xi \sim \exp(\text{const.}/\sqrt{T-T_c}) \) and there are logarithmic corrections caused by the marginally irrelevant field. One of us \([8]\) has proposed a remedy for these problems, the level spectroscopy, based on the renormalization analysis and the \( SU(2)/Z_2 \) symmetry of the BKT transition.

About the Gaussian fixed line, in the neighborhood of the BKT multicritical point, since the divergence of the correlation length is very slow, it is difficult to determine the critical line with the finite-size scaling technique. In addition, from the simple finite-size scaling analysis of the Gaussian model, two (pseudo) fixed points or no fixed point are obtained, which causes a finger-like massless region around the true Gaussian fixed line \([9, 10, 11, 5]\). In this letter we propose an improved method to obtain the Gaussian fixed line.

The renormalization group equations for (1) are \([8]\)

\[ \frac{dK^{-1}(l)}{dl} = \frac{1}{8} y_1^2(l), \]  

\[ \frac{dy_1(l)}{dl} = \left( 2 - \frac{K(l)}{2} \right) y_1(l). \]  

(3)
For the finite system, $l$ is related to $L$ by $e^l = L$. Close to the BKT multicritical point $K \approx 4, |y_1| \ll 1$, the divergence of the correlation length close to the Gaussian line becomes very slow, as mentioned above.

On the Gaussian fixed line ($y_1 = 0$), the operators $O_{n,m} \equiv \exp(in\sqrt{2}\phi + im\sqrt{2}\theta)$ has a critical dimension $x_{n,m}$ and a spin $l_{n,m}$ given by

$$x_{n,m} = \frac{1}{2} \left(n^2 K + \frac{m^2}{K}\right), \quad l_{n,m} = nm. \tag{4}$$

When we denote the transfer matrix of a strip of width $L$ with periodic boundary condition by $\exp(-H)$, then the energy gaps $\Delta E_{n,m}$ are related to the scaling dimension as

$$\Delta E_{n,m}(L) = \frac{2\pi vx_{n,m}}{L}, \tag{5}$$

where $v$ is the “velocity of light”. Therefore, we obtain

$$\Delta E_{0,2}(L)/\Delta E_{0,1}(L) = 4. \tag{6}$$

On the other hand, the off-critical behavior $y_1 \neq 0, K < 4$ is considered as follows. In this case eq. (3) is infrared unstable and the correlation length $\xi$ becomes finite. In the $L \gg \xi$ limit, it is considered that the excitation of $m = 2$ is a scattering state of two excitations of $m = 1$, which means

$$\Delta E_{0,2}(L)/\Delta E_{0,1}(L) = 2. \tag{7}$$

Therefore, we expect that the ratio $\Delta E_{0,2}(L)/\Delta E_{0,1}(L)$ becomes maximum at $y_1 = 0$ and decreases as increasing $|y_1|$ (see Figure 1).

Consider the other type ratio $\Delta E_{0,2}(2L)/\Delta E_{0,1}(L)$. It takes 2 at the Gaussian line, and in the massive region it also takes 2, therefore it is expected

$$\Delta E_{0,2}(2L)/\Delta E_{0,1}(L) = 2 \tag{8}$$

independently of $y_1$ (see Figure 2).

We shall explain these behaviors with the renormalization group argument. The renormalized critical dimensions are

$$x_{0,m} = \frac{m^2}{2} K^{-1}(l) \tag{9}$$
However, with only these relations, we cannot explain the behavior $\Delta E_{0,2}(L)/\Delta E_{0,1}(L)$ close to the Gaussian line, since the ratio of $x_{0,m}$ is independent of $y_1$. Considering the renormalization constant $L_0$, which is included in the definition $l$ as $l \equiv \log(L/L_0)$, we can understand these features (6),(7),(8). In order to satisfy the relation (8), the renormalization constant for $\Delta E_{0,2}$ should be $2L_0$, where $L_0$ is the renormalization constant for $\Delta E_{0,1}$. Then, we obtain

$$\frac{\Delta E_{0,2}(L)}{\Delta E_{0,1}(L)} = 4 \left( \frac{K^{-1}(l - \log 2)}{K^{-1}(l)} \right)_{l=\log L/L_0} \approx \frac{4}{K^{-1}(l)} \left[ K^{-1}(l) - \log 2 \frac{dK^{-1}(l)}{dl} \right]_{l=\log L/L_0} = 4 \left[ 1 - K(l) \frac{\log 2}{8} y_1^2(l) \right]_{l=\log L/L_0}, \tag{10}$$

by using eqs.(3). Consequently we can determine the Gaussian fixed line from the extremum point of the $\Delta E_{0,2}(L)/\Delta E_{0,1}(L)$.

As a physical model, we study the $S=1$ bond-alternating XXZ model \cite{15}

$$H = \sum_{j=1}^{L} (1 + \delta(-1)^j)(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z). \tag{11}$$

In this case $\Delta E_{0,m}(L)$ corresponds to $\Delta E(S_j^z = m, L)$, where $S_j^z = \sum S_j^z$.

In Figure 1, we show the ratio $\Delta E(2, L)/\Delta E(1, L)$. And in Figure 2, we show the ratio $\Delta E(2, 2L)/\Delta E(1, L)$. Their behaviors are consistent with the renormalization discussion. The remaining correction $1/L^2$ can be explained by the irrelevant operator $L_{-2}L_{-2}1$ with the critical dimension $x = 4$.

Here we analyze the corrections caused by the irrelevant operator $L_{-2}L_{-2}1$. For simplicity we treat the case $|y_1| \ll |K - 4|$. In this case $y_1 = y_1(0)L^{2-K}/2$.

And we assume the correction from the $x = 4$ term as $(c_1 + c_2(\delta - \delta_c))L^{-2}$ and the dependence of $y_1$ on the parameter $\delta$ as $y_1(0) = c_3(\delta - \delta_c)$. Then the extremum of the $\Delta E_{0,2}(L)/\Delta E_{0,1}(L)$ is situated at

$$\delta - \delta_c = \frac{1}{K \log 2} c_2 \frac{c_2}{c_3} L^{K-6}, \tag{12}$$

which rapidly converges to 0 in the $L \to \infty$ limit.

Finally we criticize the method proposed in \cite{8}. In this method, there appears the term $y_1^2(l)/(K(l) - 4)$. However, close to the BKT multicritical
point, the sign of $K(l) - 4$ changes, therefore it becomes difficult to determine the maximum.

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Figure 1: Ratio of excitation energies $\Delta E(2, L)/\Delta E(1, L)$ for $L = 10$ (×), $L = 12$ (□), $L = 14$ (+) and $L = 16$ (◇). (a) $\Delta = 0$ where the critical value of $\delta$ is $\delta = 0.23$. (b) $\Delta = 0.5$ where the critical value of $\delta$ is $\delta = 0.25$.

Figure 2: Ratio of excitation energies $\Delta E(2, 2L)/\Delta E(1, L)$ for $L = 4$ (×), $L = 6$ (□) and $L = 8$ (+). ◇ is the extrapolated value. (a) $\Delta = 0$. (b) $\Delta = 0.5$. 