New types of bialgebras arising from the Hopf equation

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Abstract

Let $M$ be a $k$-vector space and $R \in \text{End}_k(M \otimes M)$. In [10] we introduced and studied what we called the Hopf equation: $R^{12}R^{23} = R^{23}R^{13}R^{12}$. By means of a FRT type theorem, we have proven that the category $\mathcal{H}M^H$ of $H$-Hopf modules is deeply involved in solving this equation. In the present paper, we continue to study the Hopf equation from another perspective: having in mind the quantum Yang-Baxter equation, in the solution of which the co-quasitriangular (or braided) bialgebras play an important role (see [7]), we introduce and study what we call bialgebras with Hopf functions. The main theorem of this paper shows that, if $M$ is finite dimensional, any solution $R$ of the Hopf equation has the form $R = R_{\sigma}$, where $M$ is a right comodule over a bialgebra with a Hopf function $(B(R), C, \sigma)$ and $R_{\sigma}$ is the special map $R_{\sigma}(m \otimes n) = \sum \sigma(m_{<1>} \otimes n_{<1>})m_{<0>} \otimes n_{<0>}$. 

0 Introduction

Let $H$ be a Hopf algebra over a field $k$. The strong link between the category $\mathcal{H}M^H$ of Hopf modules and the category $\mathcal{H}YD^H$ of Yetter-Drinfel’d modules recently highlighted in [1] (namely, the fact that both are particular cases of the same general category $\mathcal{A}M(H)^C$ of Doi-Hopf modules, defined by Doi in [4]) led us in [2] and [3] to study the implication of the category $\mathcal{H}YD^H$ in the classic, non-quantic part of Hopf algebra theory. In [10] we called this technique ”quantisation”. For example, the theorem 4.2 of [2], stating that the forgetful functor $\mathcal{H}YD^H \rightarrow \mathcal{H}M$ is Frobenius if and only if $H$ is finite dimensional and unimodular, can be viewed as the ”quantum version” of the classical theorem saying that any finite dimensional Hopf algebra is Frobenius.

In [10] we start to study the reverse problem (it was called ”dequantisation”): that is, we study the category $\mathcal{H}M^H$ in connection with problems which so far were specific solely to the
category \( _H \mathcal{YD}^H \). The starting point is simple; it is enough to remember that the category \( _H \mathcal{YD}^H \) is deeply involved in the quantum Yang-Baxter equation:

\[
R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}
\]

where \( R \in \text{End}_k(M \otimes M) \), \( M \) being a \( k \)-vector space. We evidence the fact that the category \( _H \mathcal{M}^H \) can also be studied in connection with a certain non-linear equation. We call it the Hopf equation, and it is:

\[
R^{12} R^{23} = R^{23} R^{13} R^{12}
\]

The main result of [10] is a FRT type theorem which shows that in the finite dimensional case, any solution \( R \) of the Hopf equation has the form \( R = R(\mathcal{M}, \cdot, \rho) \), where \( (\mathcal{M}, \cdot, \rho) \) is an object in \( \mathcal{B}(R) \mathcal{M}(R) \), for some bialgebra \( B(R) \). In this paper we shall continue to study the Hopf equation from another perspective. To begin with, we remind that in the quantum Yang-Baxter equation another important role is played by the co-quasitriangular bialgebras. These can be viewed as bialgebras \( H \) with a \( k \)-bilinear map \( \sigma : H \otimes H \to k \) satisfying properties which ensure that the special map

\[
R_\sigma : M \otimes M \to M \otimes M, \quad R_\sigma(m \otimes n) = \sum \sigma(m_{<1>} \otimes n_{<1>}) m_{<0>} \otimes n_{<0>}
\]

is a solution for the quantum Yang-Baxter equation. Starting from here, we introduce new classes of bialgebras which will play for the Hopf equation the same role as the co-quasitriangular bialgebras do for the quantum Yang-Baxter equation. We called them \( (H, C, \sigma) \) bialgebras with a Hopf function \( \sigma : C \otimes H \to k \), where \( C \) is a subcoalgebra of \( H \). The reason why the map \( \sigma \) is not defined for the entire \( H \otimes H \), but only relative to a subcoalgebra \( C \) of \( H \) is explained in Remarks 2.2 and 2.9. The main result of this paper is theorem 2.8: if \( M \) is a finite dimensional vector space and \( R \) is a solution of the Hopf equation, then for a special subcoalgebra \( C \) of \( \mathcal{B}(R) \) there exists a unique Hopf function \( \sigma : C \otimes B(R) \to k \) such that \( R = R_\sigma \). We apply the above results by presenting several examples of Hopf functions on bialgebras. In the last section, as an appendix, we also introduced the concept corresponding to quasitriangular bialgebras.

1 Preliminaries

Throughout this paper, \( k \) will be a field. All vector spaces, algebras, coalgebras and bialgebras considered are over \( k \). \( \otimes \) and \( \text{Hom} \) will mean \( \otimes_k \) and \( \text{Hom}_k \). For a coalgebra \( C \), we will use Sweedler’s \( \Sigma \)-notation, that is, \( \Delta(c) = \sum c_{(1)} \otimes c_{(2)} \), \( (I \otimes \Delta) \Delta(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \), etc. We will also use Sweedler’s notation for right \( C \)-comodules: \( \rho_M(m) = \sum m_{<0>} \otimes m_{<1>} \), for any \( m \in M \) if \( (M, \rho) \) is a right \( C \)-comodule. \( \mathcal{M}^C \) will be the category of right \( C \)-comodules and \( \mathcal{C} \)-colinear maps and \( \mathcal{A} \mathcal{M} \) will be the category of left \( A \)-modules and \( A \)-linear maps, if \( A \) is a \( k \)-algebra.

From now on, \( H \) will be a bialgebra. An element \( T \in H^* \) is called a right integral on \( H \) (see [11]) if

\[
Tf = f(1_H)T
\]
for all $f \in H^*$. This is equivalent to

$$\sum T(h_{(1)})h_{(2)} = T(h)1_H$$

for all $h \in H$. Recall that a (left-right) $H$-Hopf module is a left $H$-module $(M, \cdot)$ which is also a right $H$-comodule $(M, \rho)$ such that

$$\rho(h \cdot m) = \sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)}m_{<1>}$$

(1) for all $h \in H$, $m \in M$. $_H\mathcal{M}$ will be the category of $H$-Hopf modules and $H$-linear $H$-colinear homomorphisms. If $(M, \cdot, \rho) \in _H\mathcal{M}$ we can define the special map

$$R_{(M, \cdot, \rho)} : M \otimes M \to M \otimes M, \quad R_{(M, \cdot, \rho)}(m \otimes n) = \sum n_{<1>} \cdot m \otimes n_{<0>}.$$

For a vector space $V$, $\tau : V \otimes V \to V \otimes V$ will denote the switch map, that is, $\tau(v \otimes w) = w \otimes v$ for all $v, w \in V$. If $R : V \otimes V \to V \otimes V$ is a linear map we denote by $R^{12}, R^{13}, R^{23}$ the maps of $\text{End}_k(V \otimes V)$ given by

$$R^{12} = R \otimes I, \quad R^{23} = I \otimes R, \quad R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau).$$

Using the notation $R(u \otimes v) = \sum u_1 \otimes v_1$, then

$$R^{12}(u \otimes v \otimes w) = \sum u_1 \otimes v_1 \otimes w_0$$

where the subscript $(0)$ means that $w$ is not affected by the application of $R^{12}$. Let $H$ be a bialgebra and $(M, \cdot)$ a left $H$-module which is also a right $H$-comodule $(M, \rho)$. We recall that $M(\cdot, \cdot, \rho)$ is a Yetter-Drinfel’d module if the following compatibility relation holds:

$$\sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)}m_{<1>} = \sum (h_{(2)} \cdot m)_{<0>} \otimes (h_{(2)} \cdot m)_{<1>},$$

for all $h \in H$, $m \in M$. $_H\mathcal{YD}$ will be the category of Yetter-Drinfel’d modules and $H$-linear $H$-colinear homomorphism. For a further study of the Yetter-Drinfel’d category we refer to [8], [12], [13], [14], or to the more recent [1], [2], [3], [4].

Let $H$ be a bialgebra over $k$ and $\sigma : H \otimes H \to k$ be a $k$-bilinear map. Recall that the pair $(H, \sigma)$ is a co-quasitriangular (or braided) bialgebra if

1. $\sum \sigma(x_{(1)} \otimes y_{(1)})y_{(2)}x_{(2)} = \sum \sigma(x_{(2)} \otimes y_{(2)})x_{(1)}y_{(1)}$ \hspace{1cm} (B1)
2. $\sigma(x \otimes 1) = \varepsilon(x)$ \hspace{1cm} (B2)
3. $\sigma(x \otimes yz) = \sum \sigma(x_{(1)} \otimes y)\sigma(x_{(2)} \otimes z)$ \hspace{1cm} (B3)
4. $\sigma(1 \otimes x) = \varepsilon(x)$ \hspace{1cm} (B4)
5. $\sigma(xy \otimes z) = \sum \sigma(y \otimes z_{(1)})\sigma(x \otimes z_{(2)})$ \hspace{1cm} (B5)

for all $x, y, z \in H$. If $(H, \sigma)$ is a co-quasitriangular bialgebra and $(M, \rho)$ is a right $H$-comodule, then the special map

$$R_\sigma : M \otimes M \to M \otimes M, \quad R_\sigma(m \otimes n) = \sum \sigma(m_{<1>} \otimes n_{<1>})m_{<0>} \otimes n_{<0>}.$$
is a solution for the quantum Yang-Baxter equation

\[ R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}. \]

Conversely, if \( M \) is a finite dimensional vector space and \( R \) is a solution of the quantum Yang-Baxter equation, then there exists a bialgebra \( A(R) \) and a unique \( k \)-bilinear map \( \sigma : A(R) \otimes A(R) \to k \) such that \( (A(R), \sigma) \) is co-quasitriangular, \( M \in M^{A(R)} \) and \( R = R_{\sigma} \).

A quasitriangular bialgebra is a pair \((H, R)\), where \( H \) is a bialgebra and \( R \in H \otimes H \) such that the following conditions are fulfilled:

\[(QT1) \quad \sum \Delta(R^{1}) \otimes R^{2} = R^{13}R^{23} \]
\[(QT2) \quad \sum \varepsilon(R^{1})R^{2} = 1 \]
\[(QT3) \quad \sum R^{1} \otimes \Delta(R^{2}) = R^{13}R^{12} \]
\[(QT4) \quad \sum R^{1} \varepsilon(R^{2}) = 1 \]
\[(QT5) \quad \Delta^{\text{cop}}(h)R = R\Delta(h), \text{ for all } h \in H. \]

Recall from [10] the following:

**Definition 1.1** Let \( M \) be a vector space and \( R \in \text{End}_{k}(M \otimes M) \). We say that \( R \) is a solution for the Hopf equation if

\[ R^{23}R^{13}R^{12} = R^{12}R^{23} \] \hspace{1cm} (2)

The following lemma will be important

**Lemma 1.2** Let \( M \) be a finite dimensional vector space and \( \{m_{1}, \ldots, m_{n}\} \) a basis of \( M \). Let \( R, S \in \text{End}_{k}(M \otimes M) \) given by

\[ R(m_{v} \otimes m_{u}) = \sum_{i,j} x_{ij}^{uv} m_{i} \otimes m_{j}, \quad S(m_{v} \otimes m_{u}) = \sum_{i,j} y_{ij}^{uv} m_{i} \otimes m_{j}, \]

for all \( u, v = 1, \ldots, n \), where \( (x_{ij}^{uv})_{i,j,u,v}, (y_{ij}^{uv})_{i,j,u,v} \) are two families of scalars of \( k \). Then

\[ R^{23}S^{13}S^{12} = S^{12}R^{23} \]

if and only if

\[ \sum_{u,v,\beta} x_{ij}^{uv} y_{ki}^{\alpha} y_{lq}^{\beta} = \sum_{\alpha} x_{ki}^{ij} y_{\alpha q} \]

for all \( i, j, k, l, p, q = 1, \ldots, n \). In particular, \( R \) is a solution for the Hopf equation if and only if

\[ \sum_{u,v,\beta} x_{ij}^{uv} y_{ki}^{\alpha} y_{lq}^{\beta} = \sum_{\alpha} x_{ki}^{ij} x_{\alpha q} \] \hspace{1cm} (3)

for all \( i, j, k, l, p, q = 1, \ldots, n \).
Proof For \( k, l, q = 1, \ldots, n \) we have:

\[
R^{23}S^{13}S^{12}(m_q \otimes m_l \otimes m_k) = R^{23}S^{13}\left(\sum_{\beta, v} y_{lq}^{\beta} m_\beta \otimes m_v \otimes m_k\right)
\]

\[
= R^{23}\left(\sum_{\beta, v, u, p} y_{k\beta}^u y_{lq}^{\beta} m_p \otimes m_v \otimes m_u\right)
\]

\[
= \sum_{i, j, p, u, v, \beta} (\sum_{u, v, \beta} x_{ui}^{ji} x_{lq}^{up} y_{k\beta}^{uv}) m_p \otimes m_i \otimes m_j
\]

and

\[
S^{12}R^{23}(m_q \otimes m_l \otimes m_k) = S^{12}\left(\sum_{j, \alpha} x_{kl}^{j\alpha} m_q \otimes m_\alpha \otimes m_j\right)
\]

\[
= \sum_{j, \alpha, p, i} x_{kl}^{j\alpha} y_{\alpha q}^{ip} m_p \otimes m_i \otimes m_j
\]

\[
= \sum_{i, j, p, \alpha} (\sum_{\alpha} x_{kl}^{j\alpha} y_{\alpha q}^{ip}) m_p \otimes m_i \otimes m_j
\]

Hence, the conclusion follows. \(\Box\)

Recall now the main results of [10].

**Theorem 1.3** Let \( M \) be a finite dimensional vector space and \( R \in \text{End}_k(M \otimes M) \) be a solution of the Hopf equation. Then

1. There exists a bialgebra \( B(R) \) such that \( M \) has a structure of \( B(R) \)-Hopf module \((M, \cdot, \rho)\) and \( R = R_{(M, \cdot, \rho)} \).

2. The bialgebra \( B(R) \) is a universal object with this property: if \( H \) is a bialgebra such that \((M, \cdot', \rho') \in H \mathcal{M}^H\) and \( R = R_{(M, \cdot', \rho')}) \) then there exists a unique bialgebra map \( f : B(R) \rightarrow H \) such that \( \rho' = (I \otimes f)\rho \). Furthermore, \( a \cdot m = f(a) \cdot' m \), for all \( a \in B(R) \), \( m \in M \).

2 Hopf function on a bialgebra

First of all we will introduce the following key definition of this paper:

**Definition 2.1** Let \( H \) be a bialgebra and \( C \) be a subcoalgebra of \( H \). A \( k \)-bilinear map \( \sigma : C \otimes H \rightarrow k \) is called a Hopf function if:

\[
\begin{align*}
(H1) & \quad \sum \sigma(c_{(1)} \otimes h_{(1)})h_{(2)}c_{(2)} = \sum \sigma(c_{(2)} \otimes h)c_{(1)} \\
(H2) & \quad \sigma(c \otimes 1) = \varepsilon(c) \\
(H3) & \quad \sigma(c \otimes hk) = \sum \sigma(c_{(1)} \otimes h)\sigma(c_{(2)} \otimes k)
\end{align*}
\]

for all \( c \in C, h, k \in H \). In this case we shall say that \((H, C, \sigma)\) is a bialgebra with a Hopf function.
Remarks 2.2 1. The first question which arises is why we have not defined the map \( \sigma \) on the entire \( H \otimes H \) and we have instead presented a definition relative to a subcoalgebra \( C \) of \( H \). This choice was driven by the compatibility condition (H1): in the case \( \sigma : H \otimes H \to k \) and setting \( c = 1_H \), this would result in the integral type relation

\[
\sigma(1_H \otimes h)1_H = \sum \sigma(1_H \otimes h_{(1)})h_{(2)}
\]

(4)

for all \( h \in H \). Hence, from (H3), the map \( T_\sigma : H \to k \), \( T_\sigma(h) := \sigma(1_H \otimes h) \), is an algebra map, and from (4), a right integral on \( H \). Using lemma 2.2 of [9], we are led to the trivial \( H = k \). This is why defining \( \sigma \) relative to a subcoalgebra of \( H \) becomes mandatory.

2. The conditions of compatibility (H2) and (H3) are exactly (B2) and (B3), respecting the definition relative to \( C \). The left hand side of (H1) is the same with the left hand side of (B1), while the right hand side has suffered, as we expect, considerable changes.

3. Let \((H, C, \sigma)\) be a bialgebra with a Hopf function. If \( \sigma \) is right invertible in the convolution algebra \( \text{Hom}_k(C \otimes H, k) \), then (H2) follows from (H3). Indeed, for \( c \in C \) we have:

\[
\sigma(c \otimes 1) = \sum \sigma(c_{(1)} \otimes 1)\epsilon(c_{(2)})
= \sum \sigma(c_{(1)} \otimes 1)\sigma(c_{(2)} \otimes 1)\sigma^{-1}(c_{(3)} \otimes 1)
= \sum \sigma(c_{(1)} \otimes 1)\sigma^{-1}(c_{(2)} \otimes 1)\epsilon(c)
\]

If \( H \) has an antipode \( S \), then \( \sigma \) is invertible and \( \sigma^{-1}(c \otimes h) = \sigma(c \otimes S(h)) \), for all \( c \in C \), \( h \in H \).

We shall point out the link existing between the (H1) compatibility condition and the concept of right integral on \( H \).

Let \( H \) be a bialgebra and \( C \) a subcoalgebra of \( H \). If \( T \in H^* \) is a right integral on \( H \) then the map

\[
\sigma_T : C \otimes H \to k, \quad \sigma_T(c \otimes h) := \epsilon(c)T(h), \quad \forall c \in C, h \in H
\]

satisfies (H1).

Conversely, if \( 1_H \in C \) and \( \sigma : C \otimes H \to k \) satisfies (H1) then the map

\[
T_\sigma : H \to k, \quad T_\sigma(h) := \sigma(1_H \otimes h), \quad \forall h \in H
\]

is a right integral on \( H \). In addition, we suppose that \( H \) has an antipode and (H2) also holds. Then, \( T_\sigma(1_H) = 1_k \); so, using the classical dual Maschke theorem for Hopf algebras (see [11]) we obtain that \( H \) is cosemisimple. As \( T_{\sigma_T} = T \), we obtain that the map

\[
\{ \sigma : C \otimes H \to k \mid \sigma \text{ satisfies (H1)} \} \to \int^{r}_{H^*}, \quad \sigma \to T_\sigma
\]

is surjective, and the map

\[
\int^{r}_{H^*} \to \{ \sigma : C \otimes H \to k \mid \sigma \text{ satisfies (H1)} \}, \quad T \to \sigma_T
\]

is injective. We record these observation in the following:
Theorem 2.3 Let $H$ be a bialgebra and $C$ a subcoalgebra of $H$. Then:

1. if $T \in H^*$ is a right integral on $H$, then the map $\sigma_T : C \otimes H \to k$ satisfies (H1).

2. if $1_{H} \in C$ and $\sigma : C \otimes H \to k$ satisfies (H1), then the map $T_\sigma : H \to k$ is a right integral on $H$. Furthermore, if (H2) holds and $H$ has an antipode, then $H$ is cosemisimple.

In the next proposition we shall prove that, if a $k$-bilinear map $\sigma : C \otimes H \to k$ satisfies (H3) and (H1) holds for a basis of $C$ and a sistem of generators of $H$ as an algebra, then (H1) holds for any $c \in C$ and $h \in H$.

Proposition 2.4 Let $H$ be a bialgebra, $C$ be a subcoalgebra of $H$ and $\sigma : C \otimes H \to k$ a $k$-bilinear map which satisfies (H3). Suppose that (H1) holds for a basis of $C$ and a sistem of generators of $H$ as an algebra. Then (H1) holds for any $c \in C$ and $h \in H$.

Proof Let $c \in C$ be an element of the given basis and $x, y \in H$ two elements between the generators of $H$. It is enough to prove that (H1) holds for $(c, xy)$. We have:

$$\sum \sigma(c(2) \otimes xy)c(1) = \sum \sigma(c(2)(1) \otimes x)\sigma(c(2)(2) \otimes y)c(1)$$

$$= \sum \sigma(c(2) \otimes y)\sigma(c(1)(2) \otimes x)c(1(2))$$

((H1) holds for $x$ )

$$= \sum \sigma(c(1) \otimes x(1))x(2)\sigma(c(2)(2) \otimes y)c(2(1))$$

((H1) holds for $y$ )

$$= \sum \sigma(c(1) \otimes x(1))x(2)\sigma(c(2)(1) \otimes y(1))y(2)c(2(2))$$

(using (H3) )

$$= \sum \sigma(c(1) \otimes x(1))y(1)x(2)\sigma(c(2)(2) \otimes y(2))c(3)$$

and we are done. \qed

To give examples of Hopf functions on the classical examples of bialgebras seems to be a difficult problem (see examples 1) and 2) below). More examples will be given after the main results of this paper.

Examples 2.5 1. Let $G$ be a nontrivial group and $H = k[G]$ be the groupal Hopf algebra. Let $C$ be an arbitrary subcoalgebra of $H$. Then there exists no Hopf function $\sigma : C \otimes k[G] \to k$.

Indeed, any subcoalgebra of $k[G]$ has the form $k[F]$, where $F$ is a subset of $G$. Suppose that there exists $\sigma : k[F] \otimes k[G] \to k$ a Hopf function. From (H2) we get that $\sigma(f \otimes 1) = 1$ for all $f \in F$. Now let $g \in G$, $g \neq 1$ and $f \in F$. From (H1) obtain $\sigma(f \otimes g)f g = \sigma(f \otimes g)f$ i.e. $\sigma(f \otimes g) = 0$ for all $f \in F$. But then, using (H3), we get

$$1 = \sigma(f \otimes 1) = \sigma(f \otimes g g^{-1}) = \sigma(f \otimes g)\sigma(f \otimes g^{-1}) = 0,$$
2. Let \( H = k[X, X^{-1}] \), which is a bialgebra with \( \Delta(X) = X \otimes X \), \( \varepsilon(X) = 1 \). Let \( C \) be an arbitrary subcoalgebra of \( H \). Then there exists no Hopf function \( \sigma : C \otimes H \rightarrow k \).

Let us suppose that \( C \) is a subcoalgebra of \( H \) and \( \sigma : C \otimes H \rightarrow k \) is a Hopf function. Then there exists \( t \in \mathbb{Z}^+ \) such that \( X^t \in C \). Then, by (H2), \( \sigma(X^t \otimes 1) = 1 \) and from (H1) we obtain that \( \sigma(X^t \otimes X)X^{t+1} = \sigma(X^t \otimes X)X^t \), i.e. \( \sigma(X^t \otimes X) = 0 \). But then, using (H3) we get
\[
1 = \sigma(X^t \otimes 1) = \sigma(X^t \otimes XX^{-1}) = \sigma(X^t \otimes X)\sigma(X^t \otimes X^{-1}) = 0,
\]
contradiction.

3. Let \( H = k_q < x, y \mid xy = qyx > \) be the quantum plane:
\[
\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes 1 + x \otimes y, \quad \varepsilon(x) = 1, \quad \varepsilon(y) = 0.
\]

Let \( C := kx \) be the one dimensional subcoalgebra of \( H \) with \( \{ x \} \) a \( k \)-basis. Let \( a \in k \) and \( \sigma_a : C \otimes H \rightarrow k \) given by
\[
\sigma_a(x \otimes 1) = 1, \quad \sigma_a(x \otimes x) = 0, \quad \sigma_a(x \otimes y) = a,
\]
and extend \( \sigma_a \) to the entire \( C \otimes H \) with (H3). Then, \( \sigma_a \) is a Hopf function.

Using proposition (2.4), it is enough to check that (H1) holds for \( h \in \{ x, y \} \). For \( h = x \), (H1) is
\[
\sigma_a(x \otimes x)x^2 = \sigma_a(x \otimes x)x
\]
which holds if and only if \( \sigma_a(x \otimes x) = 0 \). For \( h = y \), (H1) has the form
\[
\sigma_a(x \otimes y)x + \sigma_a(x \otimes x)yx = \sigma_a(x \otimes y)x,
\]
which is true, as \( \sigma_a(x \otimes x) = 0 \). In fact, we also prove the converse: if \( \sigma : C \otimes H \rightarrow k \) is a Hopf function, then there exists \( a \in k \) such that \( \sigma = \sigma_a \).

4. Let \( T(k) \) be the three dimensional noncommutative bialgebra constructed in [10], i.e.
- As a vector space, \( T(k) \) is three dimensional with \( \{ 1, x, z \} \) a \( k \)-basis.
- The multiplication rule is given by:
\[
x^2 = x, \quad xz = zx = z^2 = 0.
\]
- The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given by
\[
\Delta(x) = x \otimes x, \quad \Delta(z) = z \otimes 1 + z \otimes z, \quad \varepsilon(x) = 1, \quad \varepsilon(z) = 0.
\]

Let \( C := kx \) be the one dimensional subcoalgebra of \( T(k) \) with \( \{ x \} \) a \( k \)-basis. Then it is easy to see that the \( k \)-bilinear map
\[
\sigma : C \otimes T(k) \rightarrow k, \quad \sigma(x \otimes 1) = \sigma(x \otimes x) = 1, \quad \sigma(x \otimes z) = 0,
\]
defines a Hopf function.

5. Let \( M \) be a monoid and \( N := \{ n \in M \mid xn = n, \forall x \in M \} \). Let \( H = k[M] \) and \( C := k[F] \), where \( F \) is a subset of \( N \). Let \( \sigma : k[F] \otimes k[M] \rightarrow k \) such that
We shall give such an example. Let 

\[ \sigma(f \otimes 1) = 1 \]

\[ \sigma(f, \bullet) : M \to (k, \cdot) \] is a morphism of monoids

for all \( f \in F \). Then \( \sigma \) is a Hopf function.

We shall give such an example. Let \( a \in k \) and \( \mathcal{F}_a(k) = \{ u : k \to k \mid u(a) = a \} \) be the monoid (with the usual composition of functions), of all function \( u \) for all \( f \) \( \mathcal{F}_a(k) \).

Let \( H \) be a bialgebra, \( C \) a subcoalgebra of \( H \) and \( \sigma : C \otimes H \to k \) a \( k \)-bilinear map. We denote by \( \sigma_{12}, \sigma_{13}, \sigma_{23} \) the maps of Hom \( k(C \otimes C \otimes H, k) \) given by:

\[ \sigma_{12}(c \otimes d \otimes x) := \varepsilon(x)\sigma(c \otimes d), \quad \sigma_{13}(c \otimes d \otimes x) := \varepsilon(d)\sigma(c \otimes x), \quad \sigma_{23}(c \otimes d \otimes x) := \varepsilon(c)\sigma(d \otimes x) \]

for all \( c, d \in C, x \in H \).

Proposition 2.6 Let \((H, C, \sigma)\) be a bialgebra with a Hopf function \( \sigma : C \otimes H \to k \). Then:

1. in the convolution algebra Hom \( k(C \otimes C \otimes H, k) \) the following identity holds:

\[ \sigma_{23} * \sigma_{13} * \sigma_{12} = \sigma_{12} * \sigma_{23} \quad (5) \]

2. if \((M, \rho)\) is a right \( C \)-comodule, then the special map

\[ R_\sigma : M \otimes M \to M \otimes M, \quad R_\sigma(m \otimes n) = \sum \sigma(m_{<1>} \otimes n_{<1>})m_{<0>} \otimes n_{<0>} \]

is a solution for the Hopf equation.

Proof 1. Let \( c, d \in C \) and \( x \in H \). We have:

\[ (\sigma_{12} * \sigma_{23})(c \otimes d \otimes x) = \sum \varepsilon(x_{(1)})\sigma(c_{(1)} \otimes d_{(1)})\varepsilon(c_{(2)})\sigma(d_{(2)} \otimes x_{(2)}) = \sum \sigma(c \otimes d_{(1)})\sigma(d_{(2)} \otimes x) \]

and

\[ (\sigma_{23} * \sigma_{13} * \sigma_{12})(c \otimes d \otimes x) = \]

\[ = \sum \varepsilon(c_{(1)})\sigma(d_{(1)} \otimes x_{(1)})\varepsilon(d_{(2)})\sigma(c_{(2)} \otimes x_{(2)})\varepsilon(x_{(3)})\sigma(c_{(3)} \otimes d_{(3)}) \]

\[ = \sum \sigma(c_{(1)} \otimes x_{(2)})\sigma(c_{(2)} \otimes d_{(2)})\sigma(d_{(1)} \otimes x_{(1)}) \]

(using (H3))

\[ = \sum \sigma(c \otimes c_{(2)}d_{(2)})\sigma(d_{(1)} \otimes x_{(1)}) \]

\[ = \sum \sigma(c \otimes d_{(1)} \otimes x_{(1)}x_{(2)}d_{(2)}) \]

(using (H1))

\[ = \sum \sigma(c \otimes (d_{(2)} \otimes x)d_{(1)}) \]

\[ = \sum \sigma(c \otimes d_{(1)})\sigma(d_{(2)} \otimes x) \]
i.e. the formula (3) holds.

2. Let $R = R_s$ and $u, v, w \in M$. Then, the fact that $R$ is a solution of the Hopf equation will follow from equation (3) and from the formulas:

$$R^{12}R^{23} (u \otimes v \otimes w) = \sum (\sigma_{12} * \sigma_{23}) (u_{<1>} \otimes v_{<1>} \otimes w_{<1>}) u_{<0>} \otimes v_{<0>} \otimes w_{<0>}$$

and

$$R^{23}R^{12} (u \otimes v \otimes w) = \sum (\sigma_{23} * \sigma_{13} * \sigma_{12}) (u_{<1>} \otimes v_{<1>} \otimes w_{<1>}) u_{<0>} \otimes v_{<0>} \otimes w_{<0>}$$

Indeed, we have:

$$R^{12}R^{23} (u \otimes v \otimes w) = R^{12} \left( \sum \sigma (u_{<1>} \otimes w_{<1>}) u \otimes v_{<0>} \otimes w_{<0>} \right)$$

$$= \sum \sigma (u_{<1>} \otimes v_{<0,<1>}) \sigma (v_{<1>} \otimes w_{<1>}) u_{<0>} \otimes v_{<0>} \otimes w_{<0>}$$

$$= \sum \sigma (u_{<1>} \otimes v_{<1,(1)}) \sigma (v_{<1>(2)} \otimes w_{<1>}) u_{<0>} \otimes v_{<0>} \otimes w_{<0>}$$

$$= \sum (\sigma_{12} * \sigma_{23}) (u_{<1>} \otimes v_{<1>} \otimes w_{<1>}) u_{<0>} \otimes v_{<0>} \otimes w_{<0>}$$

On the other hand

$$R^{23}R^{12} (u \otimes v \otimes w) =$$

$$= R^{23} \left( \sum \sigma (u_{<1>} \otimes v_{<1>}) u_{<0>} \otimes v_{<0>} \otimes w \right)$$

$$= R^{23} \left( \sum \sigma (u_{<0,<1>} \otimes w_{<1>}) \sigma (u_{<1>} \otimes v_{<1>}) u_{<0>} \otimes v_{<0>} \otimes w_{<0>} \right)$$

$$= \sum \sigma (v_{<0,<1>} \otimes w_{<0,<1>}) \sigma (u_{<1>} \otimes v_{<1>}) \sigma (v_{<1> \otimes w_{<1>}}) u_{<0>} \otimes v_{<0>} \otimes w_{<0>}$$

$$= \sum \sigma (u_{<1,(1)} \otimes w_{<1,(2)}) \sigma (u_{<1,(2)} \otimes v_{<1,(2)}) \sigma (v_{<1,(1)} \otimes w_{<1,(1)}) u_{<0>} \otimes v_{<0>} \otimes w_{<0>}$$

$$= \sum (\sigma_{23} * \sigma_{13} * \sigma_{12}) (u_{<1>} \otimes v_{<1>} \otimes w_{<1>}) u_{<0>} \otimes v_{<0>} \otimes w_{<0>}$$

and the proof is complete now.

\[ \square \]

**Remark 2.7** There is another proof of the second statement of the above proposition. Let $(H, C, \sigma)$ be a bialgebra with a Hopf function $\sigma : C \otimes H \to k$. Then, we can construct a functor

$$F_\sigma : \mathcal{M}^C \to \mathcal{M}^H$$

given as follows: if $(M, \rho)$ is a right $C$-comodule then $F_\sigma (M) := M$ has a structure of right $H$-comodule via

$$M \overset{\rho}{\to} M \otimes C \overset{i \otimes \epsilon}{\to} M \otimes H$$

where $i : C \to H$ is the inclusion; $M$ has also a left $H$-module structure given by

$$h \cdot m := \sum \sigma (m_{<1>} \otimes h) m_{<0>}$$

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for all \( h \in H \) and \( m \in M \). Furthermore, \((M, \cdot, \rho) \in \mathcal{H} \mathcal{M}^{H}\) and \(R_\sigma = R_{(M, \cdot, \rho)}\). Now the fact that \(R_\sigma\) is a solution for the Hopf equation follows from proposition (2.6) of [10].

Indeed, (H2) and (H3) give us that \((M, \cdot)\) is a left \(H\)-module. We shall prove that \((M, \cdot, \rho)\) is an \(H\)-Hopf module using (H1). For \(h \in H\) and \(m \in M\) we have:

\[
\sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>} = \sum \sigma(m_{<0><1>} \otimes h_{(1)}) m_{<0><0>} \otimes h_{(2)} m_{<1>(2)}
\]

(using (H1) )

\[
= \sum m_{<0>} \otimes \sigma(m_{<1>(2)} \otimes h) m_{<1>(1)}
\]

\[
= \sum \sigma(m_{<1>} \otimes h) m_{<0><0>} \otimes m_{<0><1>}
\]

\[
= \rho \left( \sum \sigma(m_{<1>} \otimes h) m_{<0>} \right)
\]

\[
= \rho (h \cdot m)
\]

On the other hand

\[
R_{(M, \cdot, \rho)}(m \otimes n) = \sum n_{<1>} \cdot m \otimes n_{<0>}
\]

\[
= \sum \sigma(m_{<1>} \otimes n_{<1>}) m_{<0>} \otimes n_{<0>}
\]

\[
= R_\sigma(m \otimes n)
\]

for all \( m, n \in M \). We conclude that \((M, \cdot, \rho) \in \mathcal{H} \mathcal{M}^{H}\) and \(R_\sigma = R_{(M, \cdot, \rho)}\).

We recall from [10] that if \(M\) is a finite dimensional vector space and \(R \in \text{End}_k(M \otimes M)\) is a solution for the Hopf equation, then there exists a bialgebra \(B(R)\) such that \(M\) has a structure of \(B(R)\)-Hopf module \((M, \cdot, \rho)\) and \(R = R_{(M, \cdot, \rho)}\). We recall the construction of \(B(R)\):

Let \(\{m_1, \cdots, m_n\}\) be a basis for \(M\) and \((x_{uv}^{ji})_{i,j,u,v}\) a family of scalars of \(k\) such that

\[
R(m_v \otimes m_u) = \sum_{i,j} x_{uv}^{ji} m_i \otimes m_j
\]

(6)

for all \(u, v = 1, \cdots, n\).

Let \((C, \Delta, \varepsilon) = \mathcal{M}^n(k)\), be the comatrix coalgebra of order \(n\), i.e. \(C\) is the coalgebra with the basis \(\{c_{ij} \mid i, j = 1, \cdots, n\}\) such that

\[
\Delta(c_{jk}) = \sum_{u=1}^{n} c_{ju} \otimes c_{uk}, \quad \varepsilon(c_{jk}) = \delta_{jk}
\]

(7)

for all \(j, k = 1, \cdots, n\). Then \(B(R)\) is the free algebra generated by \((c_{ij})\) with the relations:

\[
\chi(i, j, k, l) = 0
\]

where

\[
\chi(i, j, k, l) := \sum_{u,v} x_{uv}^{ji} c_{uk} c_{vl} - \sum_{\alpha} x_{kl}^{ja} c_{i\alpha}
\]

(8)
for all \(i, j, k, l = 1, \cdots, n\). \(M\) has a right \(B(R)\)-comodule structure which extends the right \(C\)-comodule structure given by

\[
\rho(m_i) = \sum_{v=1}^{n} m_v \otimes c_{vl}
\]

for all \(l = 1, \cdots, n\).

We shall prove now the main result of the paper.

**Theorem 2.8** Let \(M\) be a finite dimensional vector space and \(R \in \text{End}_k(M \otimes M)\) be a solution for the Hopf equation. Let \(C\) be the subcoalgebra of \(B(R)\) with \((c_{ij})\) a \(k\)-sistem of generators of \(C\). Then:

1. There exists a unique Hopf function \(\sigma : C \otimes B(R) \to k\) such that \(R = R_\sigma\).

2. If \(R\) is bijective and \(R^{12}R^{13} = R^{13}R^{12}\), then \(\sigma\) is invertible in the convolution algebra \(\text{Hom}_k(C \otimes B(R), k)\).

**Proof** 1. First we prove the uniqueness. Let \(\sigma : C \otimes B(R) \to k\) be a Hopf function such that \(R = R_\sigma\). Let \(u, v = 1, \cdots, n\). Then

\[
R_\sigma(m_v \otimes m_u) = \sum_i \sigma((m_v)_{<1>} \otimes (m_u)_{<1>})(m_v)_{<0>} \otimes (m_u)_{<0>}
= \sum_{i,j} \sigma(c_{iv} \otimes c_{ju})m_i \otimes m_j
\]

Hence \(R_\sigma(m_v \otimes m_u) = R(m_v \otimes m_u)\) gives us

\[
\sigma(c_{iv} \otimes c_{ju}) = x_{uv}^{ji}
\]

for all \(i, j, u, v = 1, \cdots, n\). As \(B(R)\) is generated as an algebra by \((c_{ij})\), the relations (\(\text{III}\)) with (H2) and (H3) ensure the uniqueness of \(\sigma\).

Now we shall prove the existence of \(\sigma\). First we define \(\sigma_0 : C \otimes C \to k\) by the formulas (\(\text{III}\)). Then we extend \(\sigma_0\) to a map \(\sigma_1 : C \otimes T(C) \to k\) such that (H2) and (H3) hold. In order to prove that \(\sigma_1\) factorizes to a map \(\sigma : C \otimes B(R) \to k\), we have to show that \(\sigma_1(C \otimes I) = 0\), where \(I\) is the two-sided ideal of \(T(C)\) generated by all \(\chi(i, j, k, l)\). It is enough to prove that

\[
\sigma_1(c_{pq} \otimes \chi(i, j, k, l)) = 0
\]

for all \(i, j, k, l, p, q = 1, \cdots, n\). We have:

\[
\sigma_1(c_{pq} \otimes \chi(i, j, k, l)) = \sum_{u,v} x_{uv}^{ji} \sigma_1(c_{pq} \otimes c_{uk}c_{vl}) - \sum_{\alpha} x_{kl}^{j\alpha} \sigma_1(c_{pq} \otimes c_{\alpha l})
= \sum_{u,v,\beta} x_{uv}^{ji} \sigma_1(c_{pq} \otimes c_{\beta u}c_{\alpha v}) \sigma_1(c_{\beta q} \otimes c_{\alpha l}) - \sum_{\alpha} x_{kl}^{j\alpha} x_{\alpha q}^{ip}
= \sum_{u,v,\beta} x_{uv}^{ji} x_{k\beta l\alpha}^{up} x_{\beta q \alpha l}^{ip} - \sum_{\alpha} x_{kl}^{j\alpha} x_{\alpha q}^{ip}
\]

(from (\(\text{III}\))) = 0
We conclude that we have constructed $\sigma : C \otimes B(R) \to k$ such that (H2) and (H3) hold and $R = R_\sigma$. It remains to prove that (H1) also holds. Using proposition (2.4), it is enough to check (H1) for $c = c_{il}$ and $h = c_{jk}$, for all $i, j, k, l = 1, \ldots, n$. We have:

$$\sum \sigma(c_{(1)} \otimes h_{(1)})h_{(2)}c_{(2)} = \sum_{u,v} \sigma(c_{iu} \otimes c_{ju})c_{uk}c_{vl}$$

and

$$\sum \sigma(c_{(2)} \otimes h)c_{(1)} = \sum_{\alpha} \sigma(c_{\alpha l} \otimes c_{jk})c_{i\alpha}$$

Hence

$$\sum \sigma(c_{(1)} \otimes h_{(1)})h_{(2)}c_{(2)} - \sum \sigma(c_{(2)} \otimes h)c_{(1)} = \chi(i, j, k, l) = 0$$

i.e. (H1) also holds.

2. Suppose now that $R$ is bijective and let $S = R^{-1}$. Let $(y_{uv}^{ji})$ be a family of scalars of $k$ such that

$$S(m_v \otimes m_u) = \sum_{i,j} y_{uv}^{ji} m_i \otimes m_j,$$

for all $u, v = 1, \ldots, n$. As $RS = SR = Id_{M \otimes M}$ we have

$$\sum_{\alpha, \beta} x_{\alpha j}^{i\beta} y_{ij}^{\alpha} = \delta_{ij}\delta_{pq}, \quad \sum_{\alpha, \beta} y_{\alpha q}^{i\beta} x_{ij}^{\alpha} = \delta_{ij}\delta_{pq}$$

for all $i, j, p, q = 1, \ldots, n$. We define

$$\sigma_0' : C \otimes C \to k, \quad \sigma_0'(c_{iu} \otimes c_{ju}) := y_{uv}^{ji}$$

for all $i, j, u, v = 1, \ldots, n$. Now we extend $\sigma_0'$ to a map $\sigma_1' : C \otimes T(C) \to k$ in such a way that $\sigma_1'$ satisfies (H2) and (H3). First we prove that $\sigma_1'$ is an inverse in the convolution algebra $\text{Hom}_k(C \otimes T(C), k)$ of $\sigma_1$. Let $p, q, i, j = 1, \ldots, n$. We have:

$$\sum \sigma_1((c_{pq(1)} \otimes (c_{ij}(1)))\sigma_1'(c_{pq(2)} \otimes (c_{ij}(2))) = \sum_{\alpha, \beta} \sigma_1(c_{pa} \otimes c_{i\beta})\sigma_1'(c_{aq} \otimes c_{\beta j})$$

and

$$\sum \sigma_1'(c_{pq(1)} \otimes (c_{ij}(1)))\sigma_1(c_{pq(2)} \otimes (c_{ij}(2))) = \sum_{\alpha, \beta} \sigma_1'(c_{pa} \otimes c_{i\beta})\sigma_1(c_{aq} \otimes c_{\beta j})$$
Hence, $\sigma_1 \in \text{Hom}_k(C \otimes T(C), k)$ is invertible in convolution. In order to prove that $\sigma \in \text{Hom}_k(C \otimes B(R), k)$ remains invertible in the convolution, it is enough to prove that $\sigma'_1$ factorizes to a map $\sigma' : C \otimes B(R) \to k$. We will prove now the following:

$\sigma'_1 : C \otimes T(C) \to k$ factorizes to a map $\sigma' : C \otimes B(R) \to k$ if and only if $R^{12}R^{13} = R^{13}R^{12}$.

Indeed, $\sigma'_1$ factorizes to a map $\sigma' : C \otimes B(R) \to k$ if and only if, for any $i, j, k, l, p, q = 1, \cdots, n$, we have

$$\sigma'_1(c_{pq} \otimes \chi(i, j, l, k)) = 0,$$

which means

$$\sum_{u,v} x^{ji}_{uv} \sigma'_1(c_{pq} \otimes c_{uk}c_{vl}) = \sum_{\alpha} x^{ij}_{kl} \sigma'_1(c_{pq} \otimes c_{i\alpha})$$

which is equivalent to

$$\sum_{u,v,\beta} x^{ji}_{uv} \sigma'_1(c_{p\beta} \otimes c_{uk}) \sigma'_1(c_{\beta q} \otimes c_{vl}) = \sum_{\alpha} x^{j\alpha}_{kl} y^{ip}_{aq}$$

i.e.

$$\sum_{u,v,\beta} x^{ji}_{uv} y^{i\beta}_{k\beta} y^{\beta q}_{lq} = \sum_{\alpha} x^{j\alpha}_{kl} y^{ip}_{aq}.$$  

Now, from lemma (1.2) this equation is equivalent to

$$R^{23} S^{13} S^{12} = S^{12} R^{23}.$$

But $S = R^{-1}$, so the last equation turns into

$$R^{12} R^{23} = R^{23} R^{12} R^{13}. \quad (11)$$

$R$ is a bijective solution of the Hopf equation, i.e. $R^{12} R^{23} = R^{23} R^{13} R^{12}$. Hence, the equation (11) holds if and only if $R^{12} R^{13} = R^{13} R^{12}$.

This completes the proof of the theorem.

\[ \square \]

**Remark 2.9** There exists a major difference between the second point of our theorem and the corresponding case for the quantum Yang-Baxter equation. In this latter case, if $R$ is a bijective solution for the quantum Yang-Baxter equation, then the map $\sigma : A(R) \otimes A(R) \to k$, which makes the bialgebra $A(R)$ co-quasitriangular, is invertible in convolution. Behind this lies the elementary observation that, if $R$ is a solution for the quantum Young-Baxter equation, then $R^{-1}$ is also a solution. Now, if $R$ is a bijective solution of the Hopf equation, then $R^{-1} = S$ is not a solution for the Hopf equation; more precisely, $S$ is a solution for the equation $S^{12} S^{13} S^{23} = S^{23} S^{12}$.

Now we shall apply our theorem in order to give more examples of Hopf functions on bialgebras.

**Examples 2.10** 1. Let $q$ be a scalar of $k$ and $f_q : k^2 \to k^2$, $f_q((x, y)) := (x + qy, 0)$ for all $(x, y) \in k^2$, i.e., if $k = \mathbb{R}$, $f_q$ sends all the points of the euclidian plane on the Ox coordinate axis under an angle $\arctg(q)$ with respect to the Ox axis. In [10] we constructed three bialgebras associated to this map.
1. the first one, $B^2_q(k)$, corresponds to the solution of the Hopf equation $f_q \otimes (Id_{k^2} - f_q)$. $B^2_q(k)$ can be described as follows:

(a) as an algebra, $B^2_q(k)$ is generated by $x, y, z$ with the relations

$$yx = x, \quad yz + qy^2 = qx$$

(b) the comultiplication $\Delta$ and the counity $\varepsilon$ are given by:

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = x \otimes z + z \otimes y$$

and

$$\varepsilon(x) = \varepsilon(y) = 1, \quad \varepsilon(z) = 0.$$  \hspace{1cm} (12)

2. the second, $D^2_q(k)$, corresponds to $f_q \otimes Id_{k^2}$ and can be described as follows:

(a) as an algebra, $D^2_q(k)$ is generated by $x, y, z$ with the relations

$$x^2 = x = yx, \quad zx = 0, \quad z^2 + qzy = 0, \quad xz + qx y = yz + qy^2 = qx.$$  \hspace{1cm} (13)

(b) the comultiplication $\Delta$ and the counity $\varepsilon$ are given by equations (12) and (13).

3. the third, $E^2_q(k)$, corresponds to $f_q \otimes f_q$ and can be described as follows:

(a) as an algebra, $E^2_q(k)$ is generated by $x, y, z$ with the relations

$$x^2 = x, \quad xz + qxy = qx, \quad zx + qyx = qx, \quad z^2 + qyz + qzy + q^2 y^2 = q^2 x.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (13)

(b) the comultiplication $\Delta$ and the counity $\varepsilon$ are given by equations (12) and (13).

Let $C$ be the three dimensional subcoalgebra of $B^2_q(k)$ (resp. $D^2_q(k), E^2_q(k)$) with \{x, y, z\} a $k$-basis. Then:

1. there exists a Hopf function

$$\sigma : C \otimes B^2_q(k) \to k$$

such that

$$\sigma(x \otimes 1) = 1, \quad \sigma(x \otimes x) = 0, \quad \sigma(x \otimes y) = 1, \quad \sigma(x \otimes z) = -q,$$

$$\sigma(y \otimes 1) = 1, \quad \sigma(y \otimes x) = 0, \quad \sigma(y \otimes y) = 0, \quad \sigma(y \otimes z) = 0,$$

$$\sigma(z \otimes 1) = 0, \quad \sigma(z \otimes x) = 0, \quad \sigma(z \otimes y) = q, \quad \sigma(z \otimes z) = -q^2.$$  \hspace{1cm} (12)

2. there exists a Hopf function

$$\sigma : C \otimes D^2_q(k) \to k$$

such that

$$\sigma(x \otimes 1) = 1, \quad \sigma(x \otimes x) = 1, \quad \sigma(x \otimes y) = 1, \quad \sigma(x \otimes z) = 0,$$

$$\sigma(y \otimes 1) = 1, \quad \sigma(y \otimes x) = 0, \quad \sigma(y \otimes y) = 0, \quad \sigma(y \otimes z) = 0,$$

$$\sigma(z \otimes 1) = 0, \quad \sigma(z \otimes x) = q, \quad \sigma(z \otimes y) = q, \quad \sigma(z \otimes z) = 0.$$  \hspace{1cm} (13)
3. there exists a Hopf function

\[ \sigma : C \otimes E_q^2(k) \to k \]

such that

\[ \sigma(x \otimes 1) = 1, \quad \sigma(x \otimes x) = 1, \quad \sigma(x \otimes y) = 0, \quad \sigma(x \otimes z) = q, \]
\[ \sigma(y \otimes 1) = 1, \quad \sigma(y \otimes x) = 0, \quad \sigma(y \otimes y) = 0, \quad \sigma(y \otimes z) = 0, \]
\[ \sigma(z \otimes 1) = 0, \quad \sigma(z \otimes x) = q, \quad \sigma(z \otimes y) = 0, \quad \sigma(z \otimes z) = q^2. \]

2. Let \( k \) be a field of characteristic two and \( \mathcal{F}(k) \) the five dimensional noncommutative and noncocommutative bialgebra constructed in [10], corresponding to

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

which is a solution of the Hopf equation. \( \mathcal{F}(k) \) is described by the following:

- as a vector space, \( \mathcal{F}(k) \) is five dimensional with \( \{1, x, y, z, t\} \) a \( k \)-basis.
- the multiplication rule is given by:
  \[
x^2 = x, \quad y^2 = z^2 = 0, \quad t^2 = t,
  \]
  \[
  xy = y, \quad yx = 0, \quad xz = zx = z, \quad xt = t, \quad tx = x,
  \]
  \[
  yz = 0, \quad zy = x + t, \quad yt = 0, \quad ty = y, \quad zt = tz = z.
  \]
- the comultiplication \( \Delta \) and the counity \( \varepsilon \) are given in such way that the matrix

\[
\begin{pmatrix}
x & y \\
z & t
\end{pmatrix}
\]

is comultiplicative. Let \( C \) be the four dimensional subcoalgebra of \( \mathcal{F}(k) \) with \( \{x, y, z, t\} \) a \( k \)-basis. Then there exists a Hopf function

\[ \sigma : C \otimes \mathcal{F}(k) \to k \]

such that

\[ \sigma(x \otimes 1) = 1, \quad \sigma(x \otimes x) = 1, \quad \sigma(x \otimes y) = 0, \quad \sigma(x \otimes z) = 0, \quad \sigma(x \otimes t) = 1, \]
\[ \sigma(y \otimes 1) = 0, \quad \sigma(y \otimes x) = 0, \quad \sigma(y \otimes y) = 0, \quad \sigma(y \otimes z) = 1, \quad \sigma(y \otimes t) = 0, \]
\[ \sigma(z \otimes 1) = 0, \quad \sigma(z \otimes x) = 0, \quad \sigma(z \otimes y) = 0, \quad \sigma(z \otimes z) = 1, \quad \sigma(z \otimes t) = 0, \]
\[ \sigma(t \otimes 1) = 1, \quad \sigma(t \otimes x) = 1, \quad \sigma(t \otimes y) = 0, \quad \sigma(t \otimes z) = 0, \quad \sigma(t \otimes t) = 0, \]

As \( R \) is bijective and

\[
R^{12}R^{13} = R^{13}R^{12} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
we obtain that $\sigma$ is invertible in convolution. Since $k$ has the characteristic two, $R^{-1} = R$, hence $\sigma^{-1} = \sigma$.

3 Appendix: the Hopf elements

In the preceding section we introduced the concept of bialgebra $(H, C, \sigma)$ with a Hopf function $\sigma : C \otimes H \to k$, which, for the Hopf equation, plays the same role as the co-quasitriangular bialgebra for the quantum Yang-Baxter equation. Now we shall define the correspondent of the concept of quasitriangular bialgebra, which is also involved in the quantum Yang-Baxter equation (see [4]). For the reasons presented in the preceding section, the definition of this concept relative to a subalgebra becomes mandatory.

Definition 3.1 Let $H$ be a bialgebra and $A$ be a subalgebra of $H$. An element $R = \sum R^1 \otimes R^2 \in A \otimes H$ is called a Hopf element if:

$$(HE 1) \quad \sum \Delta(R^1) \otimes R^2 = R^{13} R^{23}$$

$$(HE 2) \quad \sum \varepsilon(R^1) R^2 = 1$$

$$(HE 3) \quad \Delta^{cop}(a) R = R(1 \otimes a)$$

for all $a \in A$. In this case, we shall say that $(H, A, R)$ is a bialgebra with a Hopf element.

Remarks 3.2 1. (HE 1) and (HE 2) are (QT 1) and (QT 2), respecting the definition relative to the subalgebra $A$. (HE 3) is obtained by modifying the right hand side of (QT 5), such that an integral type condition is obtained. We shall detail:

Let $t := \sum R^1 \varepsilon(R^2) \in A$. (HE 3) can be written:

$$\sum a^{(2)} R^1 \otimes a^{(1)} R^2 = \sum R^1 \otimes R^2 a$$

for all $a \in A$. Applying $I \otimes \varepsilon$ in this equation, we get

$$at = \varepsilon(a)t$$

for all $a \in A$. Hence, if $A$ is a subbialgebra of $H$, then $t$ is a left integral in $A$. Now, if we apply $I \otimes \varepsilon \otimes \varepsilon$ to (HE 1) we get $t^2 = t$. It follows that $t = tt = \varepsilon(t)t$, hence $\varepsilon(t) = 1$. Using the Maschke theorem for Hopf algebras, we conclude that: if $(H, A, R)$ is a bialgebra with a Hopf element and $A$ is a finite dimensional subbialgebra of $H$ with an antipode, then $A$ is semisimple.

Conversely, if $t$ is a left integral in $A$, then $R := t \otimes 1$ satisfies (HE 3).

2. Let $H$ be a bialgebra and $A$ be a subalgebra of $H$. Then $R = 1 \otimes 1$ is a Hopf element if and only if $A = k$.

Indeed, if $R = 1 \otimes 1$ then (HE 3) becomes $\Delta^{cop}(a) = 1 \otimes a$, for all $a \in A$. Hence, $a = \varepsilon(a)1_H$, for all $a \in A$, i.e. $A = k$. 

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3. Let \((H, A, R)\) be a bialgebra with a Hopf element. Suppose that \(H\) has an antipode \(S\). Then \(R\) is invertible and \(R^{-1} = \sum S(R^1) \otimes R^2\). Moreover, if we denote \(u := \sum S(R^2)R^1 \in H\) then

\[ S(a)u = \varepsilon(a)u, \]

for all \(a \in A\). This formula is obtained if we apply \(m_H \tau(I \otimes S)\) in the equation \((14)\). We observe that if \(A \neq k\) then \(u\) is not invertible (if \(u\) is invertible, then \(R^{-1} = \sum S(R^1) \otimes R^2 = \sum \varepsilon(R^1) \otimes R^2 = 1 \otimes 1\), i.e. \(A = k\)).

**Proposition 3.3** Let \((H, A, R)\) be a bialgebra with a Hopf element \(R \in A \otimes H\). Then:

1. in the tensor product algebra \(A \otimes H \otimes H\), the following identity holds

\[ R^{23}R^{13}R^{12} = R^{12}R^{23} \]  \hspace{1cm} (15)

2. if \((M, \cdot)\) is a left \(H\)-module, then the map

\[ R : M \otimes M \rightarrow M \otimes M, \hspace{0.5cm} R(m \otimes n) = \sum R^1 \cdot m \otimes R^2 \cdot n \]

is a solution of the Hopf equation.

**Proof** 1. (HE 1) is equivalent to

\((HE 1')\) \[ \sum \Delta^{\text{cop}}(R^1) \otimes R^2 = R^{23}R^{13} \]

Now, for \(r = R\), we have:

\[ R^{23}R^{13}R^{12} = \sum (\Delta^{\text{cop}}(R^1) \otimes R^2)(r^1 \otimes r^2 \otimes 1) \]
\[ = \sum \Delta^{\text{cop}}(R^1)R \otimes R^2 \]
\[ = \sum R(1 \otimes R^1) \otimes R^2 \]
\[ = \sum r^1 \otimes r^2 R^1 \otimes R^2 \]
\[ = R^{12}R^{23} \]

2. Follows from equation \((15)\), as

\[ R^{23}R^{13}R^{12}(l \otimes m \otimes n) = R^{23}R^{13}R^{12} \cdot (l \otimes m \otimes n) \]

and

\[ R^{12}R^{23}(l \otimes m \otimes n) = R^{12}R^{23} \cdot (l \otimes m \otimes n) \]

for all \(l, m, n \in M\). \(\square\)

Before presenting a few examples, we note that, if the compatibility condition \((HE 3)\) holds for a set of generators of \(A\) as an algebra, then it holds for any \(a \in A\).
Examples 3.4 1. Let $H = \mathcal{T}(k)$ be the three dimensional bialgebra from the previous section and $A$ be the two dimensional subalgebra of $\mathcal{T}(k)$ generated by $x$. Let $R := x \otimes 1$. Then $R$ is a Hopf element.

Indeed, it is enough to check that (HE 3) holds for $a = x$. We have:

$$\Delta^{\text{cop}}(x)R = (x \otimes x)(x \otimes 1) = x^2 \otimes x = x \otimes x = (x \otimes 1)(1 \otimes x),$$

hence, $R$ is a Hopf element.

2. Let $H = E_q^2(k)$ and $A$ be the two dimensional subalgebra of $E_q^2(k)$ generated by $x$. Then $R := x \otimes 1$ is a Hopf element. The proof is the same as in the previous example.

3. If in the above examples we have constructed bialgebras with Hopf elements in which $A$ is finite dimensional, now we shall give an example in which $A$ is infinite dimensional.

Let $H = D_q^2(k)$ and $A$ be the subalgebra of $D_q^2(k)$ generated by $x$ and $y$. Then $R := x \otimes 1$ is a Hopf element. We only verify for $a = y$ that (HE 3) holds (for $a = x$, the proof is the one presented in the previous example). We have:

$$\Delta^{\text{cop}}(y)(x \otimes 1) = (y \otimes y)(x \otimes 1) = y x \otimes y = x \otimes y = (x \otimes 1)(1 \otimes y)$$

i.e. (HE 3) holds for $y$.

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