Matter Density Perturbations in Modified Teleparallel Theories

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Abstract

We study the matter density perturbations in modified teleparallel gravity theories, where extra degrees of freedom arise from the local Lorentz violation in the tangent space. We formulate a vierbein perturbation with variables addressing all the 16 components of the vierbein field. By assuming the perfect fluid matter source, we examine the cosmological implication of the 6 unfamiliar new degrees of freedom in modified $f(T)$ gravity theories. We find that despite the new modes in the vierbein scenario provide no explicit significant effect in the small-scale regime, they exhibit some deviation from the standard general relativity results in super-horizon scales.

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I. INTRODUCTION

Theories constructed in “Teleparallelism” have been widely considered as an alternative origin to explain the acceleration of the cosmic expansion of the Universe. These models are mainly achieved by modifying the gravitational Lagrangian of the teleparallel equivalence of general relativity (TEGR) which reveals an equivalent formulation of classical gravity from general relativity \[1, 2\]. It has seen modifications in the favor of the non-linear generalization of TEGR, known as \(f(T)\) gravity theories \[3\], or by introducing a scalar with the non-minimal coupling to the gravity \[4\].

However, perhaps the most important feature for those modified teleparallel theories \[3, 4\] is the introduction of extra degrees of freedom (EDoFs) due to the lack of local Lorentz invariance. These EDoFs are unfamiliar to the usual metric scenario since they merely contribute as the total divergence in the simplest teleparallel construction, i.e. TEGR. To be more precisely, there are 6 components of the vierbein field released from the gauge freedom of the local Lorentz transformation in the tangent frame, which become physical modes to be determined by the field equations \[5\]. In a further analysis of \(f(T)\) theories \[6\], it has been found that modified teleparallel theories contain three additional physical degrees of freedom (ADoFs) to the usual massless spin-2 graviton. Via conformal transformations \[7\] or scalar-tensor formulations of the \(f(T)\) theories \[8\], one of the ADoFs reveals explicitly as the (conformal) scalar mode, which is related to the derivative of the arbitrary function \(f\). This scalar degree of freedom is also familiar in the ad hoc study of \(f(R)\) theories, thus does not belong to EDoFs. The result from the counting of degrees of freedom indicates that only two of the 6 EDoFs will turn into physical modes with dynamical importance beyond the metric perturbation scenario.

As discussed in \[9\], from a covariant and gauge invariant approach of \(f(T)\) gravity, these unfamiliar physical quantities can have virtual significant effects during the cosmological evolutions. In the present work, we study the matter density perturbations of modified teleparallel theories with regarding to all the dynamical variables of the vierbein field. First of all, we illustrate perturbation modes of the vierbein field including the usual variables well studied in the metric perturbations as well as the unfamiliar components induced from the lack of Lorentz symmetry. Our formulation of the perturbed variables is able to separate the analysis into scalar, vector and tensor modes, as the similar scenario in the metric
perturbations. Moreover, the 6 EDoFs in *vierbein perturbations* appear to be described by a spatial vector and a dynamical spatial antisymmetric tensor with each containing 3 independent components.

We specify our study to the matter density perturbations in $f(T)$ gravity by examining the behavior of the EDoFs. Under the perfect fluid assumption of the matter source, we find that the scalar modes of EDoFs perform with dynamical significance; while the new vector modes remain decaying modes only. Nevertheless, the perturbed equations indicate that these new scalar modes have a mere implicit contribution to the evolution of the density perturbations for sub-horizon scales $k \gg aH$, where $k$, $a$ and $H$ are the wave number, the scale factor and Hubble parameter respectively. In the super-horizon scales $k \ll aH$, it is the new mode of EDoFs that causes the deviation from general relativity. These results agree with the previous finding from both metrical or non-metrical approaches [9, 12, 15].

The paper is organized as follows. In Sec. II, we review modified teleparallel theories. In Sec. III, we illustrate the variables in vierbein perturbations. We apply our formulation to $f(T)$ gravity in Sec. IV. Finally, conclusions are given in Sec. V.

II. MODIFIED TELEPARALLEL THEORIES

The teleparallel formalism uses the vierbein field $\mathbf{e}_A(x^\mu)$ as the dynamical variables, which form an orthonormal basis of the tangent space: $\mathbf{e}_A \cdot \mathbf{e}_B \equiv \eta_{AB} = \text{diag}(1, -1, -1, -1)$. The vector $\mathbf{e}_A$ is commonly addressed by its components $e^\mu_A$ in a coordinate basis, that is $\mathbf{e}_A = e^\mu_A \partial_\mu$, while the metric tensor is obtained from the dual vierbein as $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$. Although the “distance” in teleparallel gravity is still determined by the metric tensor $g_{\mu\nu}(x)$, the gravitational effect is, instead of the concept of curvature in general relativity, geometrized purely by the torsion tensor,

$$T^\rho_{\mu\nu} = e^\rho_A (\partial_\nu e^A_\mu - \partial_\mu e^A_\nu),$$

which is composed by the subtraction of the curvatureless connection: $\Gamma^\rho_{\mu\nu} = e^\rho_A \partial_\nu e^A_\mu$. Such curvatureless connection defines an absolute parallel transportation of the objects on the manifold with only regard to the torsion effect. In order to arrive at some second order field

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1 For the *coordinate and tangent frames*, Greek indices $\mu, \nu, ...$ and capital Latin indices $A, B, ...$ run over space and time, while Latin indices $i, j, ...$ and $a, b, ...$, represent the spatial part of 1, 2, 3, respectively.
equations, the teleparallel gravity Lagrangian appears the quadratic of the torsion tensor, while TEGR is found to be of the specific choice \[2\]:

\[
L_{TEGR} = \frac{1}{2} T = \frac{1}{8} T_{\rho \mu \nu} T^\rho_{\mu \nu} - \frac{1}{4} T^\rho_{\mu \nu} T^\rho_{\mu \nu} - \frac{1}{2} T^\rho_{\mu \nu} T^\nu_{\mu}.
\] (2.2)

Denoting the tensor \( S_{\rho \mu \nu} = \frac{1}{2}(K_{\rho \mu \nu} + \delta_{\rho}^{\mu} T_{\alpha \mu}^\nu - \delta_{\rho}^{\nu} T_{\alpha \mu}^\mu) \) where \( K_{\rho \mu \nu} = \frac{1}{2}(T_{\nu \rho}^\mu + T_{\mu \rho}^\nu - T_{\mu \nu}^\rho) \), the variation of the TEGR action \( S = \frac{1}{2} \int d^4 x e [T/\kappa^2 + L_m] \) with respect to vierbein gives the field equation

\[
e^{-1} \partial_{\mu}(e e^\rho_A S^\mu_{\rho \nu}) - e^\lambda_A T^\rho_{\mu \lambda} S^\nu_{\rho \mu} - \frac{1}{4} e^\nu_A T = \frac{\kappa^2}{2} e^\rho_A \Theta_{\rho \nu}.
\] (2.3)

where \( e^\rho_A \Theta_{\rho \nu} \equiv e^{-1} \delta L_m/\delta e^\rho_A \) is the energy-momentum tensor of matter. It is noteworthy that

\[
2e^{-1} \partial_{\mu}(e e^\rho_A S^\mu_{\rho \nu}) - 2 e^\lambda_A T^\rho_{\mu \lambda} S^\nu_{\rho \mu} - \frac{1}{2} e^\nu_A T = G^\nu_A,
\] (2.4)

is nothing but an equivalent mathematical manipulation of the Einstein tensor, thus (2.3) illustrates identically the geometrical formulation of general relativity.

The equivalence formulation TEGR has received many extensions for the cosmological purpose of the late time accelerating universe. One of the common modification is inspired from \( f(R) \) gravity to generalize the torsion scalar to become an arbitrary function \( T \rightarrow f(T) \).

For the interest of dark energy phenomena, it is usually to consider the action of the form

\[
S = \frac{1}{2\kappa^2} \int d^4 x e [T + f(T) + L_m],
\] (2.5)

where \( e = \det(e^A_\mu = \sqrt{-g}) \). The action (2.5) provides the field equation

\[
(1 + f_T) \left[ e^{-1} e^\rho_A \partial_\lambda(e e^\rho_A S^\lambda_{\rho \nu}) - T^\rho_{\lambda \mu} S^\nu_{\rho \mu} \right] - \frac{1}{4} \delta_{\nu}^\mu [T + f(T)]
\]

\[
+ S^\lambda_{\mu \nu} (\partial_\lambda f_T) = \frac{\kappa^2}{2} \Theta_{\mu \nu},
\] (2.6)

which is obtained by variation with respect to the vierbein and then transit the tangent frame indices to coordinate ones. We use the notation \( f_T \equiv \partial f(T)/\partial T \) and \( f_{TT} \equiv \partial^2 f(T)/\partial T^2 \) and so on. The critical issue in teleparallel gravity arises from the fact that the curvatureless connection \( \Gamma^\rho_{\mu \nu} = e^\rho_A \partial_\nu e^A_\mu \) is not an invariant quantity under the local Lorentz transformation in the tangent frame \( e^A_\mu = \Lambda^A_B(x) e^B_\mu \), where \( \eta_{CD} = \Lambda^A_C \Lambda^B_D \eta_{AB} \). Hence, both \( T^\rho_{\mu \nu} \) and \( T \) are Lorentz violation quantities as well [10]. It can be seen that the left hand side of (2.6) shows no symmetric property between the two indices \( \mu \) and \( \nu \); while, on the other hand, the energy-momentum tensor \( \Theta_{\mu \nu} \) has to be symmetric due to the invariance principle of the
local Lorentz transformation in the matter sector (see verifications in both [5] and [11]). By using the relation (2.4), the field equations (2.6) can be rewritten into the covariant version as

\[(1 + f_T) G_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} (f - T f_T) + 2 S_{\mu}^{\lambda \nu} (\partial_{\lambda} f_T) = \kappa^2 \Theta_{\mu}^{\nu}. \]  

(2.7)

It becomes evident from this version that the antisymmetrization of the field equations leads to some non-trivial constraints of the vierbein:

\[(g^{\mu \alpha} S_{\mu}^{\lambda \beta} - g^{\nu \beta} S_{\nu}^{\lambda \alpha}) \partial_{\lambda} f_T = 0. \]  

(2.8)

These 6 additional equations imply the existence of the 6 e.d.o.f as the consequence of the lack of local Lorentz symmetry in the teleparallel formalism. Nevertheless, in the TEGR limit of \(f_T \rightarrow \text{const.}\) Eq. (2.8) automatically vanishes and the dynamical degrees of freedom in teleparallel gravity reduce to be the same as general relativity.

It is remarkable in the context of the dark energy interest that (2.8) disappears identically in the background vierbein choice for the flat FRW geometry:

\[e_A^\mu = \text{diag}(1, a, a, a). \]  

(2.9)

In the perturbation study of modified teleparallel theories, however, such additional constraints are virtually important as they govern the equations of motion for those new dynamical degrees of freedom.

III. DYNAMICAL VARIABLES IN VIERBEIN PERTURBATIONS

Although the metric perturbation is conventionally addressed by variables corresponding to the 10 degrees of freedom of the metric tensor, perturbations in teleparallel gravity, however, demand to find variables for all the 16 components of the vierbein field. It is noticeable that the part of the variables, depicting the 6 EDoFs, shall show no contribution in the specific limit of TEGR; while the metric scenario must be recovered by the other 10 degrees of freedom. This requirement can be seen by a decomposition of the vierbein as

\[e_A^\mu (x) = \bar{e}_A^\mu (x) + \alpha e_A^\mu (x), \]  

(3.1)

which satisfies the condition

\[g_{\mu \nu} (x) = \eta_{AB} e_A^\mu (x) e_B^\nu (x) = \eta_{AB} \bar{e}_A^\mu (x) \bar{e}_B^\nu (x), \]  

(3.2)
where $\bar{e}_\mu^A$ illustrates the part of vierbein quantities that are familiar in the metric perturbations. In fact, the perturbed variables of $\bar{e}_\mu^A$ have been considered in [12] of the form

$$
\bar{e}_\mu^0 = \delta_\mu^0 (1 + \psi) + a\delta_\mu^i (G_i + \partial_i F),
$$

(3.3)

$$
\bar{e}_\mu^a = a\delta_\mu^a (1 - \varphi) + a\delta_\mu^i (h_i^a + \partial_i \partial^a B + \partial^a C_i),
$$

(3.4)

which give rise to the usual perturbed metric

$$
g_{00} = 1 + 2\psi,
$$

$$
g_{0\alpha} = a(\partial_i F + G_i),
$$

$$
g_{ij} = -a^2 [(1 - 2\varphi)\delta_{ij} + h_{ij} + \partial_i \partial_j B + \partial_j C_i + \partial_i C_j],
$$

(3.5)

with scalar modes $\varphi$ and $\psi$, transverse vector modes $C_i$ and $G_i$ as well as the transverse traceless tensor mode $h_i^a$. Note that our notations are used only $\delta_{ab}$ to the upper and lower spatial indices with $\eta_{ab} \equiv -\delta_{ab}$, and also to the transition between frames $h_{ij} = \delta_{ai}h_j^a$.

It is evident from the decomposition (3.4) that all the unfamiliar part of the vierbein, $\bar{e}_\mu^A$, will not appear in any metrical quantities such as the Ricci scalar $R$ or the Einstein tensor $G_{\mu\nu}$. The specific formulation of teleparallel gravity, TEGR, involves no contribution from $\bar{e}_\mu^A$ as well, given that the torsion scalar $T$ differs from $R$ by a divergent term. Moreover, in the context following the flat FRW background $e_\mu^A = \text{diag}(1, a, a, a)$, $\bar{e}_\mu^A$ reveals as purely perturbed quantities. The contribution of this unfamiliar component, thus, appears only to the perturbation equations of modified teleparallel theories.

In order to present a further study, we denote each part of $\bar{e}_\mu^A$ as

$$
\bar{e}_\mu^0 = \delta_\mu^0 \bar{e} + \delta_\mu^i \bar{e}_i,
$$

(3.5)

$$
\bar{e}_\mu^a = \delta_\mu^0 A^a + \delta_\mu^i B^a_i.
$$

These components are not independent from each other as, to the linear order, the condition (3.2) leads to

$$
\bar{e} = 0, \quad \bar{e}_i = aA_i, \quad \text{and } B_{ij} + B_{ji} = 0.
$$

(3.6)

As a result, the dynamical variables of $\bar{e}_\mu^A$ are described by a vector $A^i$ and a spatial antisymmetric tensor $B_{ij}$, which contain overall $3 + 3 = 6$ degrees of freedom. These degrees of freedom address completely of those of EDoFs released from the local Lorentz violation in the teleparallel formalism. A similar decomposition analogous to the metric variables is
also treated to these new quantities as

\[ A^i = \partial^i \alpha + \alpha^i ; \partial_j B^j = \partial^i \beta + \beta^i, \]  

(3.7)

where \( \alpha^i \) and \( \beta^i \) are transverse vectors, which satisfy \( \partial_i \alpha^i = \partial_i \beta^i = 0 \), and hence, \( \partial_i \partial_j B^j = \partial^2 \beta = 0 \).

In the perturbation theory for modified teleparallel gravity models, both variables of \( \bar{e}^A_{\mu} \) and \( \alpha^A_{\mu} \) dynamically contribute to the cosmological evolution. Namely, in the following we will consider the scalar part of the perturbed vierbein

\[ e_0^a = \delta_0^a (1 + \psi) + a \delta^i_\mu \partial_i (F + \alpha), \]  

(3.8)

\[ e_0^a = a \delta_\mu^a (1 - \varphi) + a \delta^i_\mu (\partial_i \partial^a B + B^a_i) + \delta^0_\mu \partial^a \alpha, \]

as well as the vector one

\[ e_0^a = \delta_0^a + a \delta^i_\mu (G_i + \alpha_i), \]  

(3.9)

\[ e_0^a = a \delta_\mu^a + a \delta^i_\mu (\partial^a C_i + B^a_i) + \delta^0_\mu \alpha^a. \]

We remark that the 16 degrees of freedom in the vierbein perturbation are composed separately by 6 scalar modes: \( \psi, \phi, B, F, \alpha \) and \( \beta \); and 4 vector modes: \( C_i, G_i, \alpha_i \) and \( \beta_i \); as well as a transverse traceless tensor mode \( h_{ij} \).

Before proceeding the calculations to the theories of our concern, we shall review some gauge issues similar to the scenario of metric perturbations. Even though teleparallel gravity is formulated via the vierbein with an explicit reference to the tangent frame, the theories are still described by covariant tensors of the spacetime so that are invariant under the general coordinate transformations \( x^\mu \rightarrow x^\mu + \epsilon^\mu (x) \). This transformation changes the vierbein field by \( \delta e^A_\mu = -\tilde{e}^A_\lambda \partial_\mu e^\lambda - e^\lambda \partial_\lambda \tilde{e}^A_\mu \) to the linear order, and provides some gauge choices for us to eliminate some part of the perturbed variables. To proceed any gauge choice, \( \epsilon^\mu \) is separately treated by its temporal part \( \epsilon_0 \) and spatial vector one \( \epsilon^i \), while the spatial part can be decomposed into a spatial scalar plus a transverse vector: \( \epsilon_i = \partial_i \epsilon^S + \epsilon^V_i, \partial_i \epsilon^V_i = 0 \). The varied vierbein \( \delta e^A_\mu \) then gives

\[ \delta e_0^a = -\delta_0^0 \partial_0 \epsilon^0 - \delta_\mu^i \partial_i \epsilon^0 \]  

(3.10)

\[ \delta e_0^a = -a \delta^i_\mu (\partial_0 \partial^a \epsilon^S + \partial_0 \epsilon^V_i) - a \delta_\mu^i (\partial^a \partial_i \epsilon^S + \partial_i \epsilon^V_i + \epsilon^0 \delta^a_i), \]

2 There is no reference for the decomposition of an antisymmetric tensor in cosmological perturbations.

We treat \( \partial_j B_{ji} \) as a spatial vector for that \( B_{ij} \) only presents in the perturbed \( \dot{e}^{ij} \) of this form, see the discussion in Sec. IV.
and this transformation changes the metric tensor by

$$\delta g_{\mu\nu} = -\bar{g}_{\lambda\mu} \partial_\nu \epsilon^\lambda - \bar{g}_{\lambda\nu} \partial_\mu \epsilon^\lambda - \epsilon^\lambda \partial_\lambda \bar{g}_{\mu\nu}. \quad (3.11)$$

It becomes straightforwardly to proceed any gauge choice by eliminating the variables in vierbein perturbations from (3.10). For instance, the choice for the Longitudinal gauge and synchronous gauge has been considered in [12].

IV. PERTURBATION EQUATIONS IN $f(T)$ GRAVITY

In this section, we study the matter density perturbations of $f(T)$ gravity base on the vierbein perturbations (3.8) and (3.9). The background matter source is assumed to be a perfect fluid of the form

$$\Theta_{\mu\nu} = pg_{\mu\nu} - (\rho + p)u_\mu u_\nu, \quad (4.1)$$

where $u^\mu$ is the fluid 4-velocity. Using the background choice (2.9), we have $T = -6H^2$, and the field equations (2.6) then read

$$3H^2 = \kappa^2 \rho - \frac{f(T)}{2} - 6f_T H^2 \quad (4.2)$$

$$2\dot{H} = -\frac{\kappa^2 (\rho + p)}{1 + f_T - 12H^2 f_{TT}}, \quad (4.3)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. The matter density $\rho$ includes such as pressureless dust-like matter $\rho_m$ and radiation $\rho_r$, and satisfies the continuity equation $\dot{\rho} + 3H(\rho + p) = 0$. In what follows we separate our discussion into scalar and vector parts. The results of scalar perturbations are investigated in both sub-horizon and super-horizon scales.

A. Scalar Perturbations

We consider in the following to eliminate the variables $F$, $B$ and $C_i$ in (3.4) by properly choosing $\epsilon^0$, $\epsilon^S$ and $\epsilon^V$, respectively. Namely, we proceed the calculation with the scalar part of the vierbein perturbations given as

$$\epsilon^0_\mu = \delta^0_\mu (1 + \psi) + a\delta_{i\mu} \partial_i \alpha \quad (4.4)$$

$$\epsilon^a_\mu = a\delta^a_\mu (1 - \varphi) + a\delta_{i\mu} B^a_i + \delta^0_\mu \partial^a \alpha. \quad (4.5)$$
This provides the metric in terms of the Longitudinal gauge:

\[ ds^2 = (1 + 2\psi)dt^2 - a^2(1 - 2\phi)\delta_{ij}dx^i dx^j, \]  

(4.6)

which is commonly used for the matter density perturbations in modified gravity theories.

1. the effective gravitational coupling

We shall begin the discussion with the matter source as this involves no difference between the metric and vierbein scenarios. We consider the perturbed energy-momentum tensor \( \Theta^{\mu \nu} \), given by

\[ \Theta^0_0 = -(\rho + \delta\rho), \quad \Theta^0_i = -(\rho + p)\delta u_i, \quad \Theta^i_j = (p + \delta p)\delta^i_j + \partial^i \partial_j \pi, \]  

(4.7)

where \( \delta u_i \) characterizes the velocity perturbation of the fluid and \( \pi \) is the so-called anisotropic stress. In the same manner, \( \delta u_i \) shall be decomposed into a scalar vector potential \( \delta u \) and a transverse vector \( \delta u_i^V \). For a pressureless matter, i.e. \( p_m = 0 \), the standard continuity equation becomes

\[ \dot{\rho}_m + 3H\rho_m = 0. \]  

(4.8)

The conservation of energy-momentum \( \nabla^\mu \Theta_{\mu\nu} = 0 \) thus gives the equations of motion in the Fourier space as \[ \ref{4.8} \]:

\[ \delta\dot{\rho}_m + 3H\delta\rho_m = -\rho_m \left( 3\dot{\phi} + \frac{k^2}{a^2} \delta u_m \right), \]  

\[ \delta\dot{u} = -\psi, \]  

(4.9)

(4.10)

where \( k \) is a co-moving wave number and \( \delta u \equiv a\delta u_m \). It is conventional to define the gauge invariant variable

\[ \delta_m = \frac{\delta\rho_m}{\rho_m} + 3H\delta u \]  

(4.11)

so that under the sub-horizon approximation, \( k \gg aH \), we obtain the evolution equation of \( \delta_m \) in the Longitudinal gauge as

\[ \ddot{\delta}_m + 2H\dot{\delta}_m - \frac{k^2}{a^2} \psi \simeq 0. \]  

(4.12)

This expression is convenient to compare with the standard matter perturbation equation:

\[ \ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{eff}\rho_m \delta_m = 0, \]  

(4.13)

This is conserved with respect to the metric covariant derivative, which can be derived from the invariance under both general coordinate and local Lorentz transformations of the matter action. See also \cite{5,11}.
with $G_{\text{eff}}$ is the effective Newton’s gravitational constant, which is equal to $G$ in general relativity.

For the gravity sector of the vierbein perturbations in $f(T)$ theories, the torsion tensors from the perturbed vierbein (4.4) become

$$T^0_{0i} = -\partial_i \psi + a \partial_0 \partial_i \alpha,$$

$$T^i_{0j} = (H - \dot{\varphi}) \delta^i_j + \partial_0 B^i_j - a^{-1} \partial_j \partial^i \alpha,$$

$$T^i_{jk} = \partial_k (\delta^i_j \varphi - B^i_j) - \partial_j (\delta^i_k \varphi - B^i_k),$$

and the torsion scalar is given by

$$T = -6H^2 + 12H(\dot{\varphi} + H\psi) + 4a^{-2}\partial^2 \alpha_m,$$ (4.15)

where $\alpha_m \equiv aH\alpha$. We can denote $T = \bar{T} + \delta T$, where $\bar{T}$ and $\delta T$ are the background and perturbed parts, given by $\bar{T} = -6H^2$ and $\delta T \equiv 12H(\dot{\varphi} + H\psi) + 4a^{-2}\partial^2 \alpha_m$, respectively. As a result, the perturbations of $f$ and $f_T$ are decomposed into $f = \bar{f} + \delta f$ and $f_T = \bar{f}_T + \delta f_T$ with $\delta f = f_T \delta T$ and $\delta f_T = f_{TT} \delta T$. The bar of the background component will be omitted in the following discussion for simplicity. The scalar perturbations of the field equations (2.6) with the matter source $p_m = \delta p = 0$ are

$$(1 + f_T) [6H(\dot{\varphi} + H\psi) - 2\frac{k^2}{a^2} \varphi] - 6H^2 \delta f_T = -\kappa^2 \delta \rho_m,$$ (4.16)

$$2(1 + f_T)(\dot{\varphi} + H\psi) + 2f_{TT} \dot{T} \left( \frac{\varphi + \frac{1}{2} \beta}{2} \right) = -\kappa^2 \rho_m \delta u,$$ (4.17)

$$-2(1 + f_T)(\dot{\varphi} + H\psi) + 2H \delta f_T = \kappa^2 \rho_m \delta u,$$ (4.18)

$$(1 + f_T) \left[ 12H(\dot{\varphi} + H\psi) + 2(\dot{H} \psi + \dot{\varphi} + 2H \psi) \right]$$

$$+ 2f_{TT} \dot{T} (\dot{\varphi} + 2H\psi) - 3H^2 \delta f_T - 2H \delta \dot{f}_T = 0,$$ (4.19)

where $\partial^2 \equiv \delta_{ij} \partial_i \partial_j$, while the zero anisotropic stress assumption ($\pi = 0$) leads to

$$\psi = \varphi + \frac{12H f_{TT}}{1 + f_T} \alpha_m.$$ (4.20)

Note that $\delta \dot{f}_T$ is the brief for $\partial_0 (\delta f_T) = \dot{f}_{TT} \delta T + f_{TT} \partial_0 (\delta T)$. The trace equation corresponding to $\Theta^\mu_\mu$ is

$$(1 + f_T) \left[ 24H(\dot{\varphi} + H\psi) + 3\partial_0 (\dot{\varphi} + H\psi) + 3 \dot{H} \psi + \frac{k^2}{a^2} (\psi - 2\varphi + 4H \alpha_m) \right]$$

$$- 3(\dot{H} + 4H^2) \delta f_T - (1 + f_T) \delta T + 3f_{TT} \dot{T}(\dot{\varphi} + 2H\psi) - 3H \delta \dot{f}_T = \frac{\kappa^2}{2} \delta \rho_m,$$ (4.21)
derived from (4.16) and (4.19).

Nonetheless, to complete the perturbations in $f(T)$ gravity, we still need one more equation from the constraint (2.8) as

$$f_{TT} \dot{T}(\partial^i \varphi + \frac{1}{2} \partial^i \beta) + H \partial^i \delta f_T = 0.$$  (4.22)

This automatically makes (4.17) to be equal to (4.18), which shows the consistency to the matter source with $\Theta_0 = \Theta_\varphi$. Although both $\alpha_m$ and $\beta$ are involved in Eq. (4.22), the contribution of $\beta$ is in fact separable. Since $\partial^2 \beta = 0$, by taking the gradient of (4.22) we get $f_{TT} \dot{T} \varphi = -H \delta f_T$. Hence, we have three equations (4.16), (4.20) and (4.22) to illustrate the variable $\psi$ in terms of $\delta \rho_m$.

In order to simplify the density perturbation equations on sub-horizon scales, we also use the quasi-static approximations for those perturbed equations (4.16)-(4.19). To be more accuracy, these approximations are corresponding to [14]:

$$\frac{k^2}{a^2} |X| \gg H^2 |X|; \quad |\dot{X}| \lesssim |HX|,$$  (4.23)

where $X = \psi, \varphi, \alpha$ and $\beta$. It is explicit that, under these approximations, Eq. (4.22) indicates nothing but

$$\frac{k^2}{a^2} \alpha_m \simeq 0.$$  (4.24)

Consequently, we find from (4.16) with the substitution of (4.20) that

$$(1 + f_T) \frac{k^2}{a^2} \psi \simeq \frac{k^2}{2} \delta \rho_m.$$  (4.25)

This implies from (1.12) and (1.13) that the effective gravitational constant is given by

$$G_{eff} \simeq \frac{1}{1 + f_T} G,$$  (4.26)

where in the TEGR limit of $f_T = \text{const.}$, i.e. $f(T)$ being a linear function of $T$, we can see that the evolution of $\delta m$ follows that of general relativity. The effective gravitational constant (4.26) is identical to the results obtained from the purely metrical approach [12, 15].

2. The growth index

It is straightforward to examine the evolution of the matter density perturbation $\delta m$ following the results from the scalar perturbations. The growth of these small perturbations can
provide structures discriminated from general relativity. We adopt the parametrization in [17] with the growth index \( \gamma \) given by

\[
G(a) = \Omega_m(a)^\gamma - 1,
\]

where \( G \equiv d \ln(\delta/a)/d \ln a \). The matter dominance epoch gives \( G(a \ll 1) = 0 \). This approach suffices for the analysis beyond general relativity, provided that the effective gravitational constant is obtained. We can see that the perturbation equation (4.13) becomes

\[
\frac{d G}{d \ln a} + \left( 2 + \frac{\dot{H}}{H^2} \right) (G + 1) + (G + 1)^2 = \frac{3}{2} Q \Omega_m,
\]

where \( Q \equiv G_{\text{eff}}/G = 1/(1 + f_T) \) with the effective gravitational constant given by Eq. (4.26). The quantity \( Q \) illustrates the deviation from the standard general relativistic case with \( Q = 1 \) indicating the \( \Lambda \)CDM limit. As a practical simple example, we consider the power law form of \( f(T) \) gravity [18]:

\[
f(T) = \lambda(-T)^n = \lambda(6H^2)^n,
\]

where \( \lambda = (6H_0^2)^{1-n}(1 - \Omega_m^0)/(2n - 1) \) is given by the present Hubble parameter \( H_0 \) and the present matter density parameter \( \Omega_m^0 \). The condition \( n \ll 1 \) is required to fit the current observational data. It is convenient to define \( h \equiv H/H_0 \) so that we have

\[
\frac{\dot{H}}{H^2} = \frac{3}{2} \frac{1 + f/6H^2 + 2f_T}{1 + f_T - 12H^2f_T} = \frac{3}{2} \frac{1 - h^{2n-2}(1 - \Omega_m^0)}{1 - nh^{2n-2}(1 - \Omega_m^0)},
\]

Given that the evolution of \( h \) with \( h(z = 0) = 1 \) is given by

\[
\frac{dh^2}{d \ln a} = \frac{-3h^2 + 3h^2(1 - \Omega_m^0)}{1 - nh^{2n-2}(1 - \Omega_m^0)},
\]

we are able to solve the evolution of \( G \) numerically from (4.28) by adopting some fixed values of \( n \) and \( \Omega_m^0 \). Note that the \( G^2 \) term in (4.28) is neglected since the magnitude of today’s \( G \) is found of order \(-1/2\) even for \( n = 0.1 \) [15]. The result in Fig. 1 reveals the comparison with \( \Lambda \)CDM model (\( n = 0 \)) which has the asymptotic growth index \( \gamma_\infty \simeq 0.5454 \) when \( z \to \infty \) [17, 19].

B. Vector Perturbations

It is also noteworthy to examine the behavior of the two unfamiliar vector modes \( \alpha^i \) and \( \beta^i \) during the cosmological evolutions. In the Longitudinal gauge, the torsion tensors from
FIG. 1. Growth index ($\gamma$) as a function of the redshift ($z$) in the power law gravity of $f(T) = \lambda(-T)^n$ with $\Omega_m^0 = 0.28$, where the solid, dotted and dashed curves represent $n = 0, 0.05$ and $0.1$, respectively.

The equations (3.9) are given by

$$
T^0_{0i} = a\partial_0(G_i + \alpha_i),
$$

$$
T^0_{ij} = a[\partial_i(G_j + \alpha_j) - \partial_j(G_i + \alpha_i)],
$$

$$
T^i_{0j} = H\delta^i_j + \partial_0B^i_j - a^{-1}\partial_j\alpha^i,
$$

$$
T^i_{jk} = \partial_jB^i_k - \partial_kB^i_j. \quad (4.32)
$$

Given that the scalar functions $\delta T$, $f$ and $f_T$ are merely background quantities in the vector vierbein perturbations, the $\Theta^0_i$ equation of (2.6) then directly yields

$$
\frac{1}{2}(1 + f_T)\partial^2G_i = a\kappa^{-2}\rho_m\delta u^\gamma_i. \quad (4.33)
$$

Similar to metric perturbations, the energy momentum conservation $\nabla^\mu\Theta_{\mu\nu} = 0$ indicates that the evolution of $\delta u^\gamma_i$ decays as $1/a^3 \quad [11]$, while (4.33) implies that the vector mode $G_i$ behaves as $1/a^2$. Meanwhile, the constraints (2.8) provide

$$
3H\alpha^i = a^{-1}\beta^i, \quad (4.34)
$$

$$
\partial^i(G^j + \alpha^j) = \partial^j(G^i + \alpha^i), \quad (4.35)
$$

and it follows that in the Fourier space the two vector modes $\alpha^i$ and $\beta^i$ are proportional to $G^i$, and thus will also decay as $1/a^2$. 

13
We may now give a brief remark on the behavior of those 6 EDoFs: $\alpha$, $\beta$, $\alpha^i$ and $\beta^i$ in the cosmological perturbations of $f(T)$ gravity. It is obvious in the TEGR limit that we have the familiar equation of $\psi = \phi$. Hence, we find from (4.20) and (4.22) that $\alpha$ and $\beta$ are two new modes in $f(T)$ gravity when comparing with metric perturbations, while, on the other hand, $\alpha^i$ and $\beta^i$ are determined by $G^i$ which give no significance in the cosmological evolution. In addition to the transverse traceless tensor mode $h_{ij}$, which is beyond our main concern in the present work, the vierbein perturbation virtually provides three dynamical scalar modes $\psi$ (or $\phi$), $\alpha$ and $\beta$. This result exactly matches the number of dynamical degrees of freedom found in [6].

C. Super-horizon Scales

Although the effect of the EDoFs seems implicit in sub-horizon scales, in the large-scale limit of $k \ll aH$, the cosmological evolution of the perturbed variables can be much different. In general relativity, one can obtain a simple solution $\psi = \varphi = \text{const.}$ in super-horizon scales, which implies the matter perturbation $\delta \simeq \text{const.}$ [16]. For convenience, we denote a new variable $\Phi \equiv (\dot{\varphi} + H\psi)/H$ here, so that we have $\delta T \simeq 12H^2\Phi$ in $k \ll aH$. (4.16), (4.17) and (4.22) are given by

\begin{align*}
6H^2 \left(1 + f_T - 12H^2 f_{TT}\right) \Phi &\simeq -\kappa^2 \rho_m \delta_m, \quad (4.36) \\
-2H \left(1 + f_T - 12H^2 f_{TT}\right) \Phi &\simeq \kappa^2 \rho_m \delta_u, \quad (4.37) \\
\dot{H} \varphi &\simeq H^2 \Phi, \quad (4.38)
\end{align*}

respectively, while (4.12) becomes

\begin{align*}
\ddot{\delta}_m + 2H \dot{\delta}_m &\simeq 3\ddot{\Psi} + 6H \dot{\Psi}, \quad (4.39)
\end{align*}

where $\Psi \equiv \psi - H\delta u$. It is easy to obtain $3H \dot{\delta}_m = \delta_m$ by comparing (4.36) with (4.37). Consequently, (4.39) indicates $\ddot{\psi} + 2H \dot{\psi} = 0$. Therefore, we find a simple solution $\psi = \psi_s = \text{const.}$ similar to the case in general relativity.

Substituting (4.38) into (4.36) and using the background relations (4.12), we finally arrive at

\begin{align*}
\ddot{\delta}_m &\simeq 3\varphi = 3\psi_s - 36 \frac{\dot{H} f_{TT}}{1 + f_T} \alpha_m, \quad (4.40)
\end{align*}
where (4.20) has also been used. It is evident that in the TEGR limit we obtain the expected result $\delta_m \simeq \text{const}$. The last term in (4.40) addresses the deviation from general relativity as the impact of the new degree of freedom, $\alpha_m$. It has been demonstrated numerically that the deviation in $f(T)$ gravity from $\Lambda\text{CDM}$ becomes inevitably apparent at some rather larger scale $k \sim 10^{-4} \, h \, \text{Mpc}^{-1}$ [9].

V. CONCLUSIONS

We have investigated the matter density perturbations for modified teleparallel gravity models of dark energy. The lack of local Lorentz symmetry in the teleparallel formulation introduces dynamical degrees of freedom beyond the metric scenario and hence, a complete formula for vierbein perturbations is required. We have found from a specific decomposition of the vierbein field that the 6 extra degrees of freedom can be illustrated by two scalar and two transverse vector modes.

We have also applied the vierbein perturbation in $f(T)$ gravity to examine particularly the cosmological implication of those unfamiliar variables, i.e. $\alpha$, $\beta$, $\alpha^i$ and $\beta^i$. Our study has been simplified by imposing the perfect fluid assumption to the matter source on both background and perturbation levels. Although the result in sub-horizon scales indicates no significant contribution from the EDoFs, we have shown that the density perturbation in super-horizon scales is indeed affected by the new scale modes.

In summary, given that the two transverse vectors $\alpha^i$ and $\beta^i$ are mere decaying modes, we have matched the number of physical degrees of freedom found in modified teleparallel theories. Namely, the three degrees of freedom other than the transverse-traceless tensor are addressed by one usual scalar plus the two new scalar modes of $\alpha$ and $\beta$. Nevertheless, it remains interesting that only the scalar $\alpha$ tends to show up the physical importance in the present study, while the mode $\beta$ involves in the results nowhere. The significance of such a scalar mode could be worthy of the further investigation in teleparallel gravity theories.

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