Numerical simulation in nonlinear dynamic systems with retiming of motions of individual components

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Abstract. We study differential equations describing nonlinear processes in dynamical systems (macro-systems) that consist of a large number of components (micro-systems) and allow a synchronization in behavior of these components. In physics, these processes underlie the echo phenomena. An empirical solution of such equations is proposed. A solution is given in the form of power series of power series of the spectra of external perturbations acting on a macro-system. Numerical simulations of this solution give a good agreement with a number of experimentally observed echo phenomena.

1. Introduction

We investigate the behavior of dynamical systems (macro-systems) that contain a generous number of components (micro-systems) with nonlinear properties. In such systems, the echo phenomena can be observed, i.e., the appearance some response after a finite external influence on the time-axis.

According to Gould [1], the formation of an echo phenomenon is provided by the presence in micro-systems the nonlinearities of three types.

Nonlinearity of type I occurs only during the external influence to all micro-systems. As a result, we get a phase shift of the oscillations in every micro-system. The latter is ensured in view of the fact that the micro-systems prevent the external influence [1]. The nature of the nonlinearity of type I is different for various kind of micro-systems. However, all the cases with this nonlinearity can be described by the following equation

\[
\frac{\partial^2 U_j}{\partial t^2} + 2\Gamma (\chi_{\{t>0\}, s(t)}) \frac{\partial U_j}{\partial t} + \omega_j^2 (\chi_{\{t>0\}, s(t)}) U_j = \chi_{\{t=0\}, \omega_j} \sum_{n=0}^{\infty} \mu_n U_n^* (t).
\]  

Here \( U_j \) is an unknown function characterizing the state of the \( j^{th} \) micro-system, \( \Gamma \) is a given function determined by the properties of the system, \( s \) is an external signal acting to the system, \( \omega_j \) denotes the oscillation frequency of the \( j^{th} \) micro-system, \( \mu_n \) are given constants, and \( \chi_{\{t>0\}} \) denotes the characteristic function of the set \{\( t : s(t) > 0 \)\}, i.e., \( \chi_{\{t>0\}} = 1 \) if \( s(t) > 0 \) and \( \chi_{\{t>0\}} = 0 \) otherwise. Observe that the well-known Bloch equations describing the spin echo can be transformed into the form (1).

Nonlinearity of type II implies to the appearance of a nonlinear dispersion. This nonlinearity manifests in the absence of an external influence, and its nature is related only to the form of a micro-system. This type of nonlinearity occurs, for example, in a phonon echo from grains of powder [2-4] and in echo in plasma [5]. Nonlinearity of type III is related to the nonlinear damping of oscillations of a micro-system, i.e., to the dependence of the damping coefficient on the amplitude of oscillations. Such a nonlinearity arises, for instance, in plasma [6-8]. We note that for some macro-systems two or even all three types of nonlinearity may occur simultaneously [9].
Observe that the cases with nonlinearities of types II and III can be described with the help of the following equation:

\[
\frac{\partial^2 U_i}{\partial t^2} - \frac{\partial^2 U_i}{\partial x^2} + 2\Gamma(U_i) \frac{\partial U_i}{\partial t} = \chi_{[\varepsilon > 0]} s(t, x) + \sum_{n=0}^{N} \mu_n U_n^0(t, x). \tag{2}
\]

The functions and constants involving in (2) are understood analogously to those from (1).

The behavior of nonlinear systems exposed to external influences demonstrates a variety of motions. Among the most common motions are periodic oscillations and solitary waves. The type of the prevailing motion is determined by the properties of the system and by the type and intensity of an external action. It is known that a response of a nonlinear system to a uniform (with respect to the space) external influence can be obtained by approximate analytical methods in the form of a spectral representation of an external action \[10\].

The expansion of this response in powers of the spectra of external action allows to identify some rules applied to formation of echo signals \[11-12\].

2. Experiment / Calculation / Theory

Consider a macro-system consisting of the micro-systems which are described either by Eq. (1) or by Eq. (2), and subject all the micro-systems to the same external action \(s(t)\) given by the sequence of \(M\) real non-overlapping pulses, i.e.,

\[ s(t) = \sum_{n=1}^{M} s_n(t - \tau_n). \]

Here \(\tau_n\) stands for the beginning of the \(m\)-th pulse. Let us redefine the function \(s(t)\) along negative time-axis by the even reflection. It is easy to see that the spectrum of \(s(t)\) can be written in the following form:

\[
\hat{S}(\omega) = \sum_{n=1}^{M} [\hat{s}_n(\omega)e^{-i\omega\tau_n} + \hat{s}_n^*(\omega)e^{i\omega\tau_n}], \tag{3}
\]

where \(i\) is the imaginary unit, \(\hat{s}_n(\omega) = \int_{0}^{\infty} s_n(t)e^{-i\omega t}dt\), and \(\hat{s}_n^*(\omega)\) is the complex conjugation of \(\hat{s}_n(\omega)\).

Further, denoting the spectrum of \(U_i(t)\) by \(\hat{U}_i(\omega)\) and applying the same arguments as in \[11-12\] one can show that

\[
\hat{U}_i(\omega) = \sum_{n=0}^{N} [a_n(\hat{S}(\omega))]^n, \tag{4}
\]

where \(a_n\) are factors determining the level of the corresponding components. Thus, \(U_i(t)\) follows from (4) with a use of the inverse Fourier transform \(F^{-1}\).

It is obvious that there is no echo if we set \(N \leq 2\) in formula (4). Consider, for simplicity, two cases \(N = 3, M = 1\) and \(N = 3, M = 2\). For \(M = 1\), we have one external impulse \(s_1(t)\) starting at the moment \(\tau_1 = 0\) and ending up at the moment \(t = \Delta t_1\). Substituting representation (3) in (4), applying the inverse Fourier transform, and taking into account the moments of the phases reversal, we conclude that the echo signal arising after \(\Delta t_1\) is expressed only by the term

\[
F^{-1}[a_1(\hat{s}_1(\omega))]^3]. \tag{5}
\]

For \(M = 2\), there are two external pulses, \(s_1(t)\) starting at the moment \(\tau_1 = 0\) and \(s_2(t)\) starting at the moment \(\tau_2\). Repeating the above arguments, we deduce that the echo signal arising during the doubled time interval between the first and second pulses is expressed by the term

\[
F^{-1}[3a_2(\hat{s}_2(\omega))(\hat{s}_1(\omega))^2 e^{-2i\omega\tau_2}]. \tag{6}
\]

For the detailed justification of formulas (5) and (6) we refer the reader to \[11-12\]. Observe also that the cases \(N \geq 4\) and \(M \geq 3\) are treated in the similar manner.
Using the above described algorithms, we perform the numeric simulations for long and complex external signals, respectively. In the first case, we simulate a single-pulse spin echo observed in thin ferromagnetic cobalt films from a pulse with harmonic filling of 10 µs duration. In the second case, we model a two-pulses echo observed in the powder high temperature superconductor (HTSC). It is easy to see that the first case corresponds to Eq. (1), while the second one relates to Eq. (2). The corresponding results are shown in figures 1 and 2. The experiments and simulations are in good agreement.

Figure 1. Experimentally observed the single-pulse spin echo in thin ferromagnetic cobalt films from a pulse with harmonic filling of 10 µs duration and its simulation on Maple 14.

Figure 2. Experimentally observed the two-pulses echo in the powder high temperature superconductor (HTSC) and its simulation on Maple 14. The first pulse has the duration of 2 µs and noise coverage, the second pulse has the duration of 0.2 µs and harmonic filling. The delay between pulses is equal to 6 µs.

3. Discussion

Comparison of experimental and numerical results allows us to conclude that the proposed algorithms for representing the solutions of equations (1) and (2) provide the correct results for distinct types of external signals and echoes of a different nature. It would be of special interest to generalize the suggested method to spatially multidimensional cases as well as to provide the corresponding simulations and compare them with experimental data.

As in the case of simple external actions considered in [12], one can conclude that in the case of complex external influences with equal carrier frequencies, the minimum value of parameter $N$ in formula (4) providing the existing of an echo signal is $N = 3$. It should be also noted that considered nonlinear phenomena are manifested only at the level of a macro-system, while nonlinearities take places in the corresponding micro-systems.
4. Conclusions

The proposed algorithms are especially attractive for analysis of echo-signals upon application of complicated and noise external actions since the obtaining of approximate solutions of nonlinear equations is extremely difficult.

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