Energy of electroweak sphalerons in 1-loop selfconsistent calculations

Wolfram Schroers*,1
Ingo Börnig*,1 Christian Schulzky**,3 and Klaus Goeke*4

* Institut für Theor. Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany
** Institut für Physik, TU-Chemnitz, D-09107 Chemnitz, Germany

Abstract

We calculate the energy of electroweak sphalerons including one-loop fermionic corrections. This calculation has previously been done by adding the correction to the tree-level sphaleron and interpreting the resulting energy as the 1-loop energy. However, in this paper we calculate the energy by doing a full renormalisation of the parameters on 1-loop-level and redetermining the sphaleron configuration using the full energy functional. When comparing the final result with the tree-level solution we find that the 1-loop calculation will only cause small deviations of the sphaleron energy.

1 Wolfram.Schroers@tp2.ruhr-uni-bochum.de
2 Ingo.Boernig@tp2.ruhr-uni-bochum.de
3 c.Schulzky@physik.tu-chemnitz.de
4 goeke@hadron.tp2.ruhr-uni-bochum.de
1 Introduction

In the study of non-abelian gauge theories it has been found that the non-trivial structure of the gauge fields can lead to baryon number violating processes due to the anomaly of baryon and lepton currents (see [1]). This fascinating property of Yang-Mills theory was discovered by Faddeev [2] and by Jackiw and Rebbi [3] who showed that different field configurations of the gauge-fields exist that are topologically distinct but are physically all vacuum configurations. Thus the field energy of these configurations will be zero. These configurations are numbered by a certain functional of the fields, the Chern-Simons number $N_{CS}$ (which is, mathematically speaking, the winding number). They can be continuously deformed into each other but this process will violate the boundary condition that the energy is zero. However, there exists a way of deforming the fields that will make the required energy as low as possible. The peak of this path is known as the sphaleron and its energy is of vital interest for the whole process. The energy of the sphaleron is of the order of $m_W/\alpha$ where $m_W$ is the mass of the $W$ boson and $\alpha$ the $SU(2)$ coupling constant.

The classical sphaleron energy (the tree-level result) has been known for some time already. Naturally the question arises, how accurate the tree-level result is. Quantum corrections to the sphaleron have been computed by Bokharev and Shaposhnikov; they have included boson fluctuations through an effective potential of the Higgs field (see [4, 5]). A direct computation of the bosonic determinant over nonzero modes has been performed by Carson and McLerran [6, 7] and Baacke and Junker [8].

In [9] for the first time fermionic quantum corrections both at zero and at finite temperature were examined and their influence turned out to be significant. However, these calculations performed the renormalisation only approximately and they did not take into account that the gauge fields might look different on 1-loop level than on tree-level. The generalisation of these result to the case including bosonic contributions was done in [10]. However, the conclusions drawn there were based on the above-mentioned simplifications in the renormalization and differ from the results given by Moore in [11] and Shaposhnikov [12].

The computation of the one-loop sphaleron configuration by using a self-consistent approach is the primary aim of our paper. We further perform a correct renormalisation of all parameters of the theory and discuss the effects on the results given in [10].

In section 2 we discuss the background of sphaleron configurations and fermionic fluctuations. Section 3 discusses the methods and results of the self-consistent calculation. In the last chapter we draw the conclusions and relate this work to other papers.
2 Sphalerons and fermionic fluctuations

The particle model under examination is the minimal standard model with one Higgs doublet. In [13] it has been analysed, how large the influence of the $U(1)$-sector to the sphaleron energy is. As its influence is only of the order of $\approx 1\%$ it seems to be justified to neglect the $U(1)$-sector in the model (by setting the Weinberg angle $\theta_W = 0$) and work only with the $SU(2)$-symmetric Yang-Mills theory with a single Higgs doublet. The Lagrangian is thus (for one fermion species)

$$
\mathcal{L} = \bar{\psi}_L \gamma^\mu D_\mu \psi_L + \bar{\psi}_R \gamma^\mu \partial_\mu \psi_R - \bar{\psi}_L M \psi_R - \bar{\psi}_R M^\dagger \psi_L - \frac{1}{4g^2} F_{\mu\nu}^a F_a^{\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{\lambda^2}{2} \left( \Phi^\dagger \Phi - \frac{v^2}{2} \right)^2
$$

where we introduced the covariant derivative $D_\mu = \partial_\mu - iA_\mu$ with $A_\mu = \frac{1}{2}A^a_\mu \tau^a$ and the field strength tensor $F_{\mu\nu} = \frac{1}{2}F_a^{\mu\nu} \tau^a$, where $F_a^{\mu\nu} = \partial_\mu A_a^\nu - \partial_\nu A_a^\mu + \epsilon^{abc} A_b^\mu A_c^\nu$. The mass matrix $M$ consists of the components of the Higgs field $\Phi = \left( \Phi^+ \Phi^- \right)$ and the Yukawa couplings $h_u$ and $h_d$.

$$
M = \begin{pmatrix}
    h_u \Phi^0 & h_d \Phi^+ \\
    -h_u \Phi^{\dagger 0} & h_d \Phi^0
\end{pmatrix}.
$$

The $SU(2)$ fermion doublets are defined by

$$
\psi_L = \frac{1}{2} (1 - \gamma_5) \psi = \begin{pmatrix}
    \psi^u_L \\
    \psi^d_L
\end{pmatrix},
$$

$$
\psi_R = \frac{1}{2} (1 + \gamma_5) \psi = \begin{pmatrix}
    \psi^u_R \\
    \psi^d_R
\end{pmatrix}.
$$

Masses are generated by the non-vanishing vacuum expectation value of the Higgs field $\langle 0 | \Phi | 0 \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This mechanism yields on tree-level $m_W = \frac{g v}{\sqrt{2}}, m_{u,d} = \frac{h_u v}{\sqrt{2}}$ and $m_H = \lambda v$. The renormalised masses on 1-loop level are explicitly given in appendix A and will be needed later for obtaining the selfconsistent solution. In order to be consistent with previous works we choose to scale the quantities in the following way:

$$
x^\mu \rightarrow m_W^{-1} x^\mu, \quad A_\mu^a \rightarrow m_W A_\mu^a, \quad \Phi \rightarrow \frac{m_W}{\sqrt{2g}} \Phi.
$$

Furthermore we use the following representation of the Dirac matrices

$$
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

which allows us to reduce the fermion fields to two components:

$$
\psi_L^{u,d} \rightarrow m_W^{3/2} \begin{pmatrix} \psi_L^{u,d} \\ 0 \end{pmatrix}, \quad \psi_R^{u,d} \rightarrow m_W^{3/2} \begin{pmatrix} 0 \\ \psi_R^{u,d} \end{pmatrix}.
$$
With these replacements the new Lagrangian becomes
\[ L = m_4^2 W(\bar{i\psi}_L(D_0 - \sigma_i D_i)\psi_L + \bar{i\psi}_R(\partial_0 + \sigma_i \partial_i)\psi_R - \bar{\psi}_L M \psi_R - \bar{\psi}_R M^\dagger \psi_L + \frac{1}{g^2} \left( -\frac{1}{4} F_{\mu \nu}^a F^{a\mu\nu} + \frac{1}{2} (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{1}{32} \nu_H^2 (\Phi^\dagger \Phi - 4)^2 \right) \), (4) with the mass matrix
\[ M = \frac{1}{2} \begin{pmatrix} \nu_u \Phi^{0*} & \nu_d \Phi^+ \\ -\nu_u \Phi^+ & \nu_d \Phi^0 \end{pmatrix}. (5) \]
The masses have to be taken from experiment. However, when comparing the masses of the different fermions, one finds that the top quark largely dominates the whole fermionic mass spectrum. Its energy of \( \simeq 175 \text{GeV} \) is much larger than that of any other quark, so the other fermion doublets can be considered as massless. In [9] the authors find that the influence of massless fermion doublets is negligibly small when compared to a massive doublet, so only the top/bottom doublet needs to be considered. However, when applying the usual framework to compute the energy spectrum, only doublets that are degenerate in mass can be computed. Consequently the top/bottom doublet was approximated by 1.5 fermionic doublets with a degenerate mass of \( m_T \) and a corresponding bare value of \( \nu_T = \nu_u = \nu_d \). To estimate the effect of nondegenerate fermion masses we developed a perturbative method, which can be used to justify this approach. In Appendix [3] and [14] we present these results. The parameters in this work are chosen to be \( g = 0.67, \quad m_W = 83 \text{GeV}, \quad m_T = 175 \text{GeV}. \)
The Higgs mass will be considered as a free parameter. One aim of this work is to express the sphaleron energy as a function of the Higgs mass. The relation of the bare parameters \( \nu_T \) and \( \nu_H \) to these physical parameters is discussed below.
Now we will focus our attention to static solutions of the gauge and Higgs-fields. In this case it is useful to use the temporal gauge with \( A_0 \equiv 0 \). In this gauge (4) yields the following energy functional:
\[ E_{\text{class}} = \frac{m_W}{g^2} \int d^3 r \left( \frac{1}{4} F_{ij}^a F^{aij} + \frac{1}{2} (D_i \Phi)^\dagger (D^i \Phi) + \frac{1}{32} \nu_H^2 (\Phi^\dagger \Phi - 4)^2 \right). (6) \]
Configurations with \( A_0^a = 0 \) and \( \Phi^\dagger \Phi = 4 \) have \( E_{\text{class}} = 0 \) which means they are vacuum configurations. Due to the transformation properties of (4) one can transform this configuration to \( A_i^a = i U(x) \partial_i U^\dagger(x) \) (with \( U(x) \in SU(2) \)) where the new configuration must still have \( E_{\text{class}} = 0 \), due to gauge invariance.
The field configurations can furthermore be characterised by the following functional (called Chern-Simons-number):
\[ N_{\text{CS}} = \frac{1}{16\pi^2} \int d^3 r \epsilon^{ijk} \left( A_i^a \partial_j A_k^a + \frac{1}{3} \epsilon^{abc} A_i^a A_j^b A_k^c \right). (7) \]
One can show that for configurations of the form $A_i = i U(x) \partial_i U^\dagger(x)$ the Chern-Simons-number is an integer value $N_{CS} \in \mathbb{Z}$. If $N_{CS}$ is not an integer then Eq. (1) will yield an energy larger than 0. Thus it is impossible to deform vacuum configurations with different $N_{CS}$ continuously into each other without violating the boundary condition $E_{\text{class}} = 0$. Of special interest is the path of lowest energy going from $N_{CS} = 0$ to $N_{CS} = 1$. It can be obtained by minimising the energy functional (6) for a given and fixed $N_{CS}$. This path has been constructed in [16]. From symmetry arguments the point at $N_{CS} = 1/2$ is the highest point and is called sphaleron. It is a saddle point in the space of Higgs- and gauge fields. There is only one negative eigenmode, which is lying in the direction of changing $N_{CS}$. As this point already has a high symmetry in the space of Higgs- and gauge-fields it appears logical to choose a spherically symmetric ansatz for the spatial dependence of the fields. This ansatz is called hedgehog and has been used in a variety of works on this subject [16, 14, 17]. The Higgs- and gauge-fields are represented by 5 functions $A(r), B(r), C(r), G(r)$ and $H(r)$ in the following way:

$$
A_i^a(r) = \epsilon_{aij} n_j \left[ 1 - \frac{A(r)}{r} \right] + \left( \delta_{ai} - n_a n_i \right) \frac{B(r)}{r} + n_a n_i \frac{C(r)}{r},
$$

$$
\Phi(r) = 2 \left( H(r) + i G(r) n \cdot \tau \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

In this ansatz the energy (6) has the following form:

$$
E_{\text{class}} = \frac{4 \pi m_W}{g^2} \int_0^R dr \left( \left( A' + \frac{B C'}{r} \right)^2 + \left( B' - \frac{A C'}{r} \right)^2 \right)
$$

$$
+ \frac{1}{2} \left( \frac{A^2 + B^2 - 1}{r} \right)^2 + 2 r^2 \left( H'^2 + G'^2 \right)
$$

$$
+ 2 r \left( H' G - G' H \right) C - 4 B G H + 2 A \left( G^2 - H^2 \right)
$$

$$
+ \left( 1 + A^2 + B^2 + \frac{1}{2} C^2 \right) \left( G^2 + H^2 \right) + \frac{v^2}{2} r^2 \left( G^2 + H^2 - 1 \right)^2. \tag{9}
$$

For $N_{CS}$ one obtains the following expression

$$
N_{CS} = \frac{1}{2 \pi} \int_0^\infty dr \left( \frac{(A^2 + B^2 - 1) C}{r} + (A'B - B'A) + B' \right). \tag{10}
$$

The fermionic fluctuations have been investigated in [9]. Here we use only the expression for the renormalised fermionic field energy, which is a function of some numerical parameters and the physical quantities $\nu_F, \nu_H$ and $m_W$. An additional parameter has to be introduced, the renormalisation point $\nu_{\text{ren}}$. Of course, the final results should be independent of $\nu_{\text{ren}}$. Note that for the $\nu_F$ and $\nu_H$ one has to use the renormalised quantities on the 1-loop-level (see appendix A).
The numerical parameters, as introduced in [9] are the box size $R$ and the cut-off $\Lambda$. Thus the whole fermionic energy at zero temperature becomes

$$E_{\text{ferm}}^{\text{ren}}(\Lambda, R, \nu_F, \nu_H, \nu_{\text{ren}}) = E_{\text{sea}}(\ldots) - E_{\text{div}}(\ldots),$$

where the sea energy and the divergent energy are

$$E_{\text{sea}}(\Lambda) = \frac{m_W}{4\sqrt{\pi}} \int_{\Lambda^{-2}}^{\infty} \frac{dt}{t^{3/2}} \text{Tr} \left( e^{-t\mathcal{H}^2} - e^{-t\mathcal{H}^{(0)}^2} \right),$$

$$E_{\text{div}}(\Lambda) = \frac{m_W}{32\pi^2} \int d^3r \left( \nu_{\text{ren}}^2 - \Lambda^2 \right) \nu_F^2 \left( \Phi^\dagger \Phi - 4 \right)$$

$$+ \ln \frac{\Lambda^2}{\nu_{\text{ren}}^2} \left( \frac{1}{6} \left( F_{\mu}^\mu \right)^2 + \nu_F^2 \left( D_i \Phi \right)^\dagger \left( D_i \Phi \right) \right)$$

$$+ \frac{1}{4} \nu_{\text{ren}}^2 \left( \left( \Phi^\dagger \Phi - 4 \right)^2 + 8 \left( \Phi^\dagger \Phi - 4 \right) \right).$$

The fermionic Hamiltonian is defined to be

$$\mathcal{H} = \left( \begin{array}{cc} i\sigma_i D_i & M \\ M^\dagger & -i\sigma_i \partial_i \end{array} \right).$$

The task of evaluating the energy contribution (12) by diagonalising (14) in an appropriate basis has been described thoroughly in [9] and shall not be repeated here. Before we further investigate the 1-loop energy expressions we shall examine the parameters of the Lagrangian on 1-loop, $\nu_F, \nu_H$ and $\nu_{\text{ren}}$. On tree level, the relations between these parameters and the physical particle masses are trivial and have been given above. But on 1-loop-level these relations become extremely complicated. One gets a set of three equations depending on the three parameters. One equation can be eliminated to define the mass scale of the system. It appears natural to choose the equation for $m_W$. Thus, finally

$$m_H = m_H(\nu_{\text{ren}}, \nu_F, \nu_H),$$

$$m_F = m_F(\nu_{\text{ren}}, \nu_F, \nu_H).$$

These system of equations is overdetermined and we have the freedom to choose one parameter arbitrarily. Usually one takes $\nu_{\text{ren}}$ as a free parameter and examines if the final results are independent of $\nu_{\text{ren}}$. The full set of equations (13) is given in appendix A.

Furthermore we need the vacuum expectation value of the Higgs field in the 1-loop case. To obtain this value we have to examine the value of the fermionic energy in the vacuum which is given by (\Gamma is Euler’s constant):

$$E_{\text{ferm}}^{\text{ren}}[F_{ij} = 0, \Phi = \text{const}] = \frac{m_W N_c}{32\pi^2} \int d^3r \left( -\nu_F^2 \nu_{\text{ren}}^2 \Phi^\dagger \Phi \right)$$

$$+ \frac{\nu_F^2 \left( \Phi^\dagger \Phi \right)^2}{8} \left( \frac{3}{2} - \Gamma - \ln \left( \frac{\nu_F^2 \Phi^\dagger \Phi}{4\nu_{\text{ren}}^2} \right) \right).$$

(16)
By performing $\frac{\delta E_{\text{ren}}}{\delta (\Phi^\dagger \Phi)} = 0$ one gets the equation for $v_1$:

$$v_1^2 - 4 = -\frac{N_c g^2}{2\nu_H^2 \pi^2} \left(-\nu_F^2 \nu_{\text{ren}}^2 + \frac{\nu_F^2 v_1^2}{4} \left(1 - \Gamma - \ln \left(\frac{\nu_F^2 v_1^2}{4\nu_{\text{ren}}^2}\right)\right)\right).$$

(17)

This equation is to be solved numerically for given $\nu_F$, $\nu_H$ and $\nu_{\text{ren}}$. It should be noted that there is an apparent dependence of $v_1$ on $\nu_{\text{ren}}$, but by renormalising the mass-scale (see above) this dependence is removed. Furthermore it should be noted that massless fermions do not influence the value of $v_1$; since the energy in (16) is $\propto \nu_F^2$ the influence of massless fermions is 0.

With the help of the expression for the classical energy (the tree level functional) (6) and the 1-loop fluctuations in (11) the correct expression for the 1-loop energy functional can be obtained (now we have to use the 1-loop values for $\nu_F$, $\nu_H$ and $v_1$):

$$E[A, \Phi] = E_{\text{class}} + E_{\text{ren}}^\text{ferm} - (E_{\text{class}} + E_{\text{ren}}^\text{ferm}) |_{\Phi^\dagger \Phi = v_1^2}$$

$$= \frac{m_W}{g^2} \int d^3r \left(\frac{1}{4} \left(F_{ij}^a\right)^2 + \frac{1}{2} \left(D_i \Phi^\dagger (D_i \Phi) + \frac{\nu_H^2}{32} \left(\Phi^\dagger \Phi - v_1^2\right)^2\right)\right)$$

$$+ E_{\text{ren}}^\text{ferm} |_{\Phi^\dagger \Phi^2 = v_1^2} + \frac{m_W \nu_F^2}{16 g^2} (v_1^2 - 4) \int d^3r \left(\Phi^\dagger \Phi - v_1^2\right).$$

(18)

After the following replacement

$$\Phi \rightarrow \Phi \cdot \frac{v_1}{2}, \quad A \rightarrow A \cdot \frac{v_1}{2}, \quad r \rightarrow r \cdot \frac{2}{v_1}, \quad \nu_{\text{ren}} \rightarrow \nu_{\text{ren}} \cdot \frac{2}{v_1},$$

one finally gets the following form of (18)

$$E[A, \Phi] = \frac{v_1}{2} \left(E_{\text{class}} + E_{\text{ren}}^\text{ferm} |_{\nu_{\text{ren}} v_1^2} + \frac{m_W \nu_F^2}{4 g^2} \left(\frac{1}{v_1^2} - \frac{4}{v_1^4}\right) \int d^3r \left(\Phi^\dagger \Phi - 4\right)\right).$$

(19)

The task of finding the sphaleron configuration now requires minimising the functional (19) at a given value for $N_{CS}$. The question is if this value is bound to be $N_{CS} = 1/2$ as it was in the classical case. Clearly the answer is no; since the whole path is not symmetric (if the vacuum at $N_{CS} = 0$ has an energy of $E = 0$ then the vacuum at $N_{CS} = 1$ will have the energy of the created valence fermions which is clearly $> 0$) we would expect the highest point to lie at $N_{CS} > 1/2$, but still very close to it. In principle, the difference in energy should be less than the energy of the valence fermions which is $\simeq 1\%$ of the sphaleron energy. So it appears to be justified to take the minimal value of (19) at $N_{CS} = 1/2$ as an approximation for the sphaleron energy without introducing larger errors.
3 Numerical results

Now we wish to minimise the functional (19) at the fixed value for $N_{CS} = 1/2$. In the 1-loop case with the very complicated expression (11) this is a very complex task especially since the condition for $N_{CS}$ cannot be cast into some explicit form for the hedgehog-fields. The problem arises from the fact that the sphaleron configuration is rather a saddle point than a minimum. So we have to modify the functional in a way that allows us to apply some minimisation procedure.

The most natural modification involving the boundary condition $N_{CS} = 1/2$ is to add a term of the form

$$a \left( N_{CS} - 1/2 \right)^b$$

with some parameters $a$ and $b$. It has turned out that the choices of $b = 2$ (higher values would make the minimum too large) and $a = 1000 m_W$ (which is one magnitude higher than the sphaleron energy) give good results. So finally one ends up with a new functional

$$E_X = E_X[A, \Phi] + a \left( N_{CS}[A, \Phi] - 1/2 \right)^2.$$ (20)

$E_X$ may either be (8) or (19) since the procedure may also be applied to the classical case.

At this moment we still have a gauge freedom we can use to fix the shape of the $C$-field appearing in the hedgehog ansatz in (8). We choose the following form which turned out to be useful for our purposes

$$\bar{C}(x) = -280 \pi \left( \frac{x}{s} \right)^4 \left( \frac{x}{s} - 1 \right)^4.$$  

The quantity $s$ has been set to $s = 3 m_W$ which is the size of the other hedgehog fields, too.

In order to do the actual minimisation, the fields $A$, $B$, $G$ and $H$ have been discretised.

For the minimisation procedure, the Powell method (see e.g. [18]) was used and the fields are allowed to be modified in the range from $r = 0$ to $r = R_{sphal}$. $R_{sphal}$ is thus the maximum size of the sphaleron that can be found with this method. Obviously $R_{sphal}$ has to be chosen such that

$$R_{sphal} \leq R,$$ (21)

where $R$ is the radius of the box for the Kahana-Ripka basis (see [9]). The problem is that the symmetries of the Lagrangian (4) are destroyed when one comes close to the point $r = R$; so a minimisation procedure that searches for a sphaleron in the free space (where $R \to \infty$) will encounter instabilities when it tries to manipulate the fields too close to the boundary. This problem is very severe if one uses a minimisation procedure which involves computing the functional derivatives of the fields (see [19]).
Table 1: The dependence of the self-consistent solution on the number of lattice points.

| $n_a$ | $n_b$ | $n_a$ | $n_b$ | $E_\Sigma$/GeV |
|-------|-------|-------|-------|---------------|
| 31    | 21    | 21    | 35    | 12353        |
| 59    | 43    | 43    | 59    | 12356        |
| 149   | 109   | 109   | 149   | 12356        |

Table 2: The self-consistent solution at different Higgs masses. The second column shows the initial configuration (the non-selfconsistent Sphaleron) and the third column the final configuration after the self-consistent minimisation. One can see that the differences are minimal and the total energy is well approximated by the initial configuration.

| $m_H$/GeV | $E_{\Sigma,0}$/GeV | $E_{\Sigma,\text{mini}}$/GeV |
|-----------|--------------------|-----------------------------|
| 83        | 8804               | 8746                        |
| 124.5     | 9244               | 9189                        |
| 166       | 9497               | 9356                        |
| 249       | 9842               | 9839                        |
| 415       | 10538              | 10534                       |

However, since our method does not involve derivatives it should be stable and reliable if we ensure that (21) is satisfied.

Before performing this examination the lattice spacing should be examined first; the number of lattice points of the different fields are presented in table [1]. The bare parameters of the Lagrangian have been chosen to be $\nu_H = 0.8$, $\nu_F = 2.1$ and $\nu_{\text{ren}} = 2.0$. We find that the first row in the table is fine enough to represent the fields.

In order to determine an appropriate value for $R_{\text{sphal}}$, we examined the energy functional (13). We found that the total energy does not depend on the choice of $R_{\text{sphal}}/R$, and it is convenient to use $R_{\text{sphal}} = 0.70R$ as a good compromise between accuracy and computational effort of the problem.

Now we can examine three different solutions: the tree-level sphaleron (the sphaleron solution of (3) considering only the classical energy of the boson field), the one-loop sphaleron (the field configuration of the tree-level sphaleron evaluated with (13)) and the selfconsistent sphaleron (the sphaleron solution of (13)).

By computing the energy of these solutions and comparing the deviations one can make a statement about the significance of self-consistency in the one-loop approximation of the sphaleron. The actual results for different values of the Higgs mass are listed in table [3].
One should keep in mind that the errors of the selfconsistent calculation are about 1% due to the inaccuracy of the minimisation process. However the influence of the self-consistency is negligible, and the simplest way to compute the one-loop sphaleron energy seems to be to minimise the classical part of (19) and then add the fermionic contribution. Of course one has to use the one-loop renormalised parameters in evaluating (19) and in computing the final energy.

\[
\begin{array}{cccccccc}
\text{Sphaleron energy} \\
E_{\text{sphal}} & \text{GeV} \\
\hline
8.0 & 8.5 & 9.0 & 9.5 & 10.0 & 10.5 & 11.0 & 11.5 & 12.0 \\
\end{array}
\]

Figure 1: Comparison of the sphaleron energies for different values of the Higgs mass. The dots represent the energies of the classical solution, the crosses represent the energies of the one-loop solutions. The circles represent the results based on the calculations in [9] using the renormalization from [10].

With this recipe at hand we can compare the one-loop and the tree-level energies of the sphaleron for a wider range of Higgs masses. This has been done in figure 1. One should remember that the vacuum configuration is no longer stable when the Higgs mass becomes too small; this phenomenon has also been used to find a lower bound for the Higgs mass (see e.g. [20]). However, in this work it only causes that equation (17) cannot be solved numerically and we cannot compute the one-loop solution. It is clearly visible that the shape of the configuration is maintained in comparison to the classical curve. But the overall energy has increased by an amount of about 2% – 3%. From figure 1 we can thus conclude that the influence of the one-loop fluctuations is only a few percent in comparison to the tree level result.

When comparing these results with the results from previous computations [3, 10], we find that the previous results deviate significantly from the current results. The old curve (represented by circles in 1) is much too large for small values of the Higgs mass and even exhibits a local minimum at about \( m_H \approx m_W \). This is a strong indicator
for the incomplete nature of the renormalization scheme used.

4 Conclusion

In the last section we have found that the correction to the energy of the sphaleron configuration due to fermionic contributions is very small, only about 2% – 3%. Furthermore in computing this effect the influence of self-consistency turned out to be negligibly small. Since the fermionic contribution is only small in principle the shape of the dependency of the sphaleron energy on the Higgs mass did not change - it is only shifted by a small amount upwards. This puts the results in [9] into doubt, since they found due to the incomplete renormalization a strong increase in the sphaleron energy even at zero temperature.

Still, an interesting question is whether the results of this paper can be generalised to the case of non-zero temperature. In this case the correction generated by the fermion determinant is not much smaller than the classical sphaleron energy. Although it appears possible that now self-consistency becomes important, it has been shown in [9] that the fermionic contribution can be split up in two parts: the major part has the same shape as the classical energy and the remaining part can be treated as a small correction. Now one can include the first part of the fermionic energy into the classical one and compute the sphaleron energy again (practically this means one has to adjust the bare-parameters again and then proceed in a similar way as in this paper). But it was shown in [10] that then the remaining fermionic contribution again has the effect of a small correction to the overall energy. So in this case the energy of the whole configuration should not be significantly influenced by self-consistency. So we argue that also in the case of finite temperature it is unlikely that self-consistency introduces any changes in the energy.

However, the results obtained in [10] appear to suffer from the same shortcomings (incomplete renormalization) as those in [9] when it comes to the absolute value of the energy. Thus the conclusions drawn for the upper limit of the Higgs mass seem to require a correction that would lower the value and thus move it closer to the one cited by Shaposhnikov in [12].

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A Mass renormalisation

Here we want to present the shape of the equations (15). These were calculated in [21].

On tree level we have

\[ m_W = 83 \text{GeV}, \]
\[ m_H = \nu_H m_W, \]
\[ m_F = \nu_F m_W. \]  \hspace{1cm} (22)

In the case of one-loop fluctuations we find the following set of equations to determine the bare parameters:

\[ 0 \equiv -m_H^2 + \frac{\nu_H^2}{8} (3v_1^2 - 4) \]
\[ + \frac{g^2}{16\pi^2} \frac{N_c}{2} \left( 2\nu_F^2 \nu_{\text{ren}}^2 F' \left( \frac{v_1^2 \nu_F^2}{4\nu_{\text{ren}}^2} \right) \right. \]
\[ \nu_F^2 \left( v_1^2 \nu_F^2 - m_H^2 \left( -C - G \left( \frac{m_H^2}{\nu_{\text{ren}}^2}, \frac{v_1^2 \nu_F^2}{4\nu_{\text{ren}}^2} \right) \right) \right) \]
\[ 0 \equiv -m_W^2 + \frac{v_1^2}{4} \]
\[ + \frac{g^2}{16\pi^2} \frac{N_c}{2} \left( \nu_{\text{ren}}^2 F' \left( \frac{v_1^2 \nu_F^2}{4\nu_{\text{ren}}^2} \right) + \frac{v_1^2 \nu_F^2}{2} - \frac{m_W^2}{2} \right. \times \]
\[ \times \left( -C - G \left( \frac{m_W^2}{\nu_{\text{ren}}^2}, \frac{v_1^2 \nu_F^2}{4\nu_{\text{ren}}^2} \right) - \nu_{\text{ren}}^2 \left( -1 + (1 - C) \left( \frac{v_1^2 \nu_F^2}{4\nu_{\text{ren}}^2} - \frac{m_W^2}{6\nu_{\text{ren}}^2} \right) \right) \right) \]
\[ - \frac{m_W^2}{\nu_{\text{ren}}^2} \times \left( -C - G \left( \frac{m_W^2}{\nu_{\text{ren}}^2}, \frac{v_1^2 \nu_F^2}{4\nu_{\text{ren}}^2} \right) \right) \]
\[ + N_c \left( N_g + \frac{1}{2} \right) \left( -\nu_{\text{ren}}^2 - \frac{m_W^2}{2} \right) \left( -C - G \left( \frac{m_W^2}{\nu_{\text{ren}}^2}, 0 \right) \right) \]
\[ - \nu_{\text{ren}}^2 \left( -1 + (C - 1) \left( \frac{m_W^2}{6\nu_{\text{ren}}^2} - \frac{m_W^2}{\nu_{\text{ren}}^2} \right) \right) G' \left( \frac{m_W^2}{\nu_{\text{ren}}^2}, 0 \right) \left) \right) \]
\[ 0 \equiv \frac{1}{4} \nu_F^2 v_1^2 - m_F^2, \]  \hspace{1cm} (23)

where we have used the following definitions:

\[ G(x, y) = \int_{-1/2}^{1/2} d\alpha \ln \left( x \left( \alpha^2 - \frac{1}{4} \right) + y - i\varepsilon \right) \]
\[ G'(x, y) = \int_{-1/2}^{1/2} d\alpha \alpha^2 \ln \left( x \left( \alpha^2 - \frac{1}{4} \right) + y - i\varepsilon \right). \]
B Nondegenerate Fermion Masses

As mentioned in the main text, it is not possible in our approach to treat nondegenerate fermion masses exactly. So we introduced a common fermion mass \( \nu_F \) and a degenerate mass matrix

\[
M_0 = \frac{1}{2} \nu_F \begin{pmatrix} \Phi^0 & \Phi^+ \\ -\Phi^+ & \Phi^0 \end{pmatrix}
\]

(24)

to compute the results presented here. In this appendix we try to estimate the error of that approximation by treating nondegenerate fermion masses perturbatively. In order to restore the original mass matrix \( M, (B) \), one can write

\[
M = M_0 \left( 1 + \frac{\Delta \nu}{\nu_F \tau_3} \right),
\]

(25)

where we introduced the parameters \( \nu_F = \frac{\nu_u + \nu_d}{2} \) and \( \Delta \nu = \frac{\nu_u - \nu_d}{2} \). One can see that the matrix \( \tau_3 \) spoils the spherical symmetry of our fermionic Hamiltonian (14) so that a direct diagonalisation can not be performed. But for small mass differences one can now expand \( M \) in a power series with respect to \( \Delta \nu/\nu_F \). Of course, in nature this parameter for the massive fermion doublet is of order \( O(1) \), so it is clear, that it is not possible to compute good estimates with this ansatz. On the other hand, if we are only interested in the qualitative behaviour of the fermionic energy, it can be used to see how the energy changes if one allows different fermion masses within one doublet. The fermionic 1-loop energy can then be evaluated as a Taylor series in the parameter \( \Delta \nu \):

\[
E_{\text{Ferm}}(\Delta \nu) = E_{\text{Ferm}}(0) + \frac{\partial E_{\text{Ferm}}}{\partial \Delta \nu} \bigg|_{\Delta \nu=0} \cdot \Delta \nu + \frac{1}{2} \frac{\partial^2 E_{\text{Ferm}}}{\partial \Delta \nu^2} \bigg|_{\Delta \nu=0} \cdot \Delta \nu^2 + O(\Delta \nu^3).
\]

(26)

In this series the linear term vanishes, because in a pure \( SU(2) \) gauge theory up- and down-components in one doublet cannot be distinguished if they are degenerate in mass. Therefore we compute numerically the first and third term of this series. To compare our result to the approximation we proceed in the following way: We start with degenerate fermion masses with \( \nu_F = \nu_t \) and count only \( n = 3/2 \) massive fermion doublets, so that in the end three top quarks masses are counted. This is exactly the approximation which was used before. Then we introduce small mass differences \( \Delta \nu \), but keep the value of \( \nu_t \) fixed. To ensure that we have the correct “mass content” we count

\[
n = \frac{3}{2} \frac{\nu_t}{\nu_t - \Delta \nu}
\]

(27)

massive fermion doublets, in order to preserve the sum of three top quark masses in the fermionic energy.
By increasing $\Delta \nu$ we eventually reach the “physical” point with $\Delta \nu = \nu_s/2$ and $n = 3$. This is plotted in figure 2. This method will work well in the region of small $\Delta \nu$, since perturbation theory is well justified here. We can clearly see, how the fermionic energy drops if one introduces mass splittings. From this point of view we can at least say that our approximation is a good upper limit for the energy arising from fermionic fluctuations.

![Figure 2](image.png)

Figure 2: In this picture one can see the fermionic energy as a function of the mass splitting $\Delta \nu$. The dashed line is the first term of the series (26) and the dotted line shows the third term. The solid line is the sum of the two contributions and therefore the whole fermionic energy up to $\Delta \nu^4$, since all odd powers vanish. One can see that even a small mass difference lowers the fermionic energy significantly.
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