Improved regularity for the parabolic normalized p-Laplace equation

Pêdra D. S. Andrade · Makson S. Santos

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Abstract
We derive regularity estimates for viscosity solutions to the parabolic normalized \( p \)-Laplace. By using approximation methods and scaling arguments for the normalized \( p \)-parabolic operator, we show that the gradient of bounded viscosity solutions is locally asymptotically Lipschitz continuous when \( p \) is sufficiently close to 2. In addition, we establish regularity estimates in Sobolev spaces.

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1 Introduction

We study the regularity of viscosity solutions to the parabolic normalized \( p \)-Laplace equation

\[
\frac{\partial u}{\partial t}(x, t) - \Delta^N_p u(x, t) = f(x, t) \quad \text{in} \quad Q_1,
\]

where \( p \in (1, \infty) \), \( f \in C(Q_1) \cap L^\infty(Q_1) \) and \( Q_1 := B_1 \times (-1, 0] \). We prove new and sharp regularity results for the viscosity solutions to (1), when the exponent \( p \) is close to 2. In particular, we obtain gains of regularity in Sobolev and Hölder spaces. We exploit the properties of the regularized operator

\[
\left( u_\varepsilon \right)_t - \Delta u_\varepsilon - (p - 2) \frac{D^2 u_\varepsilon D u_\varepsilon \cdot D u_\varepsilon}{|D u_\varepsilon|^2 + \varepsilon^2},
\]

to establish the integral estimates for the solutions of the homogeneous normalized \( p \)-Laplace equation. For the regularity in Hölder spaces, we argue by approximation methods importing regularity from the homogeneous heat equation to (1).

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\( \varepsilon \) Pêdra D. S. Andrade
pedra.andrade@icmc.usp.br
Makson S. Santos
makson.santos@icmc.usp.br

1 Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Centro, São Carlos, SP 13566-590, Brazil
The normalized $p$-Laplace operator is defined as
\[
\Delta_p^N u := |Du|^{2-p} \Delta_p u = \Delta u + (p-2)|Du|^p u,
\]
where $\Delta_p u := |Du|^{2-p} \text{div}(|Du|^{p-2} Du)$ is the $p$-Laplace operator. Hence, the Eq. (1) can be seen as a uniformly parabolic model in nondivergence form, with constants $\min\{p-1, 1\}$ and $\max\{(p-1), 1\}$. We also notice that this model has a singularity on the set $\{Du = 0\}$, which implies that we can not use directly the classic $C^{1,\alpha}$ regularity theory of viscosity solutions, as in [20, 25, 26, 36].

The normalized $p$-Laplace equation appears in many branches of mathematics, ranging from differential geometry to stochastic process. At the limit $p = 1$, the Eq. (1) is related to the level set formulation of the mean curvature flow, as in the work of Evans and Spruck [15]. For $1 < p < \infty$, Peres and Sheffield [29] introduced a stochastic approach for the normalized $p$-Laplace equation using the theory of tug-of-war games with noise. For the parabolic framework we refer to the work of Manfredi et al. [27]. As $p$ approaches infinity, the Eq. (1) finds an application in image processing; see [13, 22]. It is worth noticing that [14, 23, 28] address the game theoretical characterization for the borderlines cases $p = 1$ and $p = \infty$.

The normalized $p$-Laplace equation has been extensively studied by many authors. The existence and uniqueness was established in [6, 11]. In particular, Banerjee and Garofalo [6] and Does [11] proved Lipschitz regularity with respect to the space variable. Since the model (1) is uniformly parabolic, Hölder regularity of the solutions follows from the Krylov-Safonov theory, see [24, 26].

In [21] T. Jin and L. Silvestre showed that solutions to the homogeneous case are of class $C^{1+\alpha, \frac{1+\alpha}{2}}$. In that paper, the authors follow the strategy adopted in [19], for a class of fully nonlinear degenerate models. For the equation with a bounded right-hand side, Attouchi and Parviainen [3] also established that solutions are of class $C^{1+\alpha, \frac{1+\alpha}{2}}$. More recently, the variable exponent case was studied in [16], where Y. Fang and C. Zhang obtained Hölder regularity for the gradient of the solutions.

Besides the regularity in Hölder spaces, many authors have studied regularity estimates in Sobolev spaces. In the homogeneous setting Høeg and Lindqvist [17] proved $W^{2,1;2-\delta_n,p}_{\text{loc}}$ regularity for the viscosity solutions, when $p \in \left(\frac{6}{3}, \frac{14}{5}\right)$. This range for $p$ comes from their own method, and it is not clear if it can be extended to $p \in (1, \infty)$ for $d \geq 3$. This result were improved by Dong et al. [12], where they proved that solutions are of class $W^{2,1;2}_{\text{loc}}$ for $q < 2 + \delta_{n,p}$ with $\delta_{n,p} \in (0, 1)$, when $p \in (1, 2) \cup \left(2, 3 + \frac{2}{d-2}\right)$.

The purpose of this paper is to study gains of the regularity of solutions to (1) when $p$ is close to 2. In this scenario, our model can be seen as a perturbation of the heat equation, as in [31], and then we can implement the so-called approximation methods. The approximation methods were introduced by L. Caffarelli in the seminal paper [8] and has been applied in more general settings in the works of E. Teixeira, J.M. Urbano and their collaborators. We refer the reader to [1, 10, 32–34], just to cite a few. See also the survey [30].

We combine the approximation methods and scaling arguments to produce the desired estimate in Hölder spaces. This is the content of our first result:

**Theorem 1.1** Let $u \in C(Q_1)$ be a viscosity solution to (1), with $f \in C(Q_1) \cap L^\infty(Q_1)$. Given $\alpha \in (0, 1)$, there exists $\varepsilon > 0$, to be determined later, such that if $|p-2| \leq \varepsilon$,
then \( u \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_{1/2}) \) and there exists a constant \( C > 0 \) depending only on the dimension \( d \) and \( \alpha \) such that

\[
\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_{1/2})} \leq C \left( \|u\|_{L^\infty(Q_1)} + \|f\|_{L^\infty(Q_1)} \right).
\]

Theorem 1.1 ensures that although solutions to (1) are of class \( C^{1+\beta,\frac{1+\beta}{2}} \) (\( \beta \) can be very small), the gradient of the solutions become almost Lipschitz continuous when \( p \) is sufficiently close to 2.

**Remark 1.1** It is not clear if it is possible to extend Theorem 1.1 to the class of equations

\[
 u_t(x, t) - |D\nabla u|^\gamma \Delta_p u(x, t) = f(x, t) \quad \text{in} \quad Q_1,
\]

under the assumption

\[ |\gamma| + |p - 2| \leq \varepsilon. \]

We refer the reader to [2, 4, 5, 18] for more details.

Our next result concerns regularity in Sobolev spaces in the homogeneous setting

\[
 u_t(x, t) - \Delta_p^n u(x, t) = 0 \quad \text{in} \quad Q_1.
\]

Here, we make use of the regularized operator \( (2) \) and the \( W^{2,1;q} \)-regularity theory developed by Wang [35] to prove our estimates, see also [7] for the elliptic counterpart. More precisely, we prove the following:

**Theorem 1.2** Let \( u \in C(Q_1) \) be a viscosity solution to (4). There exists \( \varepsilon_1 > 0 \), to be determined later, such that if

\[ |p - 2| \leq \varepsilon_1, \]

then \( u \in W^{2,1;q}(Q_{1/2}) \) for every \( 1 < q < \infty \). In addition, there exists a constant \( C > 0 \) depending only on the dimension \( d \) and \( p \) such that

\[
\|u\|_{W^{2,1;q}(Q_{1/2})} \leq C \|u\|_{L^\infty(Q_1)}.
\]

Theorem 1.2 provides higher integrability for the viscosity solutions of (4), with the trade off of losing the precise range on the values of \( p \) for which the estimate holds true.

**Remark 1.2** By Sobolev embeddings we observe that Theorem 1.2 is a stronger version of Theorem 1.1 for the case \( f \equiv 0 \), since \( \varepsilon_1 \) would not depend on the Hölder exponent. See Corollary 4.1 below.

The remainder of this paper is organized as follows: In Sect. 2 we fix some notations and gather a few facts used throughout the paper. The regularity in Hölder spaces is the subject of Sect. 3. We conclude this article with the regularity in Sobolev spaces in Sect. 4.

## 2 Notations and preliminary results

This section puts forward elementary notations and gathers a few results used throughout the paper.
2.1 Elementary notation

In what follows $B_r \subset \mathbb{R}^d$ denotes the open ball of radius $r$ and centered at the origin. The parabolic domain is given by

$$Q_r := B_r \times (-r^2, 0] \subset \mathbb{R}^{d+1}.$$ 

We define the parabolic distance between the points $(x_1, t_1)$ and $(x_2, t_2)$ by

$$d((x_1, t_1), (x_2, t_2)) := \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}.$$ 

Similarly, the distance between the sets $U$ and $V$ stands for

$$\text{dist}(U, V) := \inf\{d((x_1, t_1), (x_2, t_2)) : (x_1, t_1) \in U, (x_2, t_2) \in V\},$$

where $U, V$ are subsets in $\mathbb{R}^{d+1}$.

Fix $1 \le q \le \infty$, the parabolic Sobolev space $W^{2,1,q}(Q_r)$ is defined as follows

$$W^{2,1,q}(Q_r) := \{u \in L^q(Q_r) : u_t, \; Du, \; D^2u \in L^q(Q_r)\}.$$ 

If $u \in W^{2,1,q}(Q_r)$, we define its norm to be

$$\|u\|_{W^{2,1,q}(Q_r)} = \left[ \|u\|_{L^q(Q_r)}^q + \|u_t\|_{L^q(Q_r)}^q + \|Du\|_{L^q(Q_r)}^q + \|D^2u\|_{L^q(Q_r)}^q \right]^\frac{1}{q}.$$ 

We say that $u$ belongs to $W^{2,1,q}_{loc}(Q_r)$, if $u \in W^{2,1,q}(Q_r)$ for every $Q' \Subset Q_r$, where $Q' \Subset Q_r$ means dist$(Q', \partial Q_r) > 0$, where $\partial_p$ denotes the parabolic boundary of $Q_r$.

For $0 < \alpha < 1$, the parabolic Hölder space stands for $C^{\alpha, \frac{\alpha}{2}}(Q_r)$. We define its norm to be

$$\|u\|_{C^{\alpha, \frac{\alpha}{2}}(Q_r)} := \|u\|_{L^\infty(Q_r)} + \left[u\right]_{C^{\alpha, \frac{\alpha}{2}}(Q_r)},$$

where $\left[u\right]_{C^{\alpha, \frac{\alpha}{2}}(Q_r)}$ denotes the semi-norm

$$\left[u\right]_{C^{\alpha, \frac{\alpha}{2}}(Q_r)} := \sup_{(x_1, t_1), (x_2, t_2) \in Q_r, (x_1, t_1) \neq (x_2, t_2)} \frac{|u(x_1, t_1) - u(x_2, t_2)|}{d((x_1, t_1), (x_2, t_2))^{\alpha}}.$$ 

We say that $u$ is $\alpha$-Hölder continuous with respect to the spatial variable and $\frac{\alpha}{2}$-Hölder continuous with respect to the temporal variable if its norm is finite. Similarly, we say that $u \in C^{1+\alpha, \frac{\alpha}{2}}(Q_r)$ if there exists the spatial gradient $Du(x, t)$ for every $(x, t)$ in $Q_r$ in the classical sense and its norm

$$\|u\|_{C^{1+\alpha, \frac{\alpha}{2}}(Q_r)} := \|u\|_{L^\infty(Q_r)} + \|Du\|_{L^\infty(Q_r)} + \sup_{(x_1, t_1), (x_2, t_2) \in Q_r, (x_1, t_1) \neq (x_2, t_2)} \frac{|u(x_1, t_1) - u(x_2, t_2) - Du(x_1, t_1) \cdot (x_1 - x_2)|}{d((x_1, t_1), (x_2, t_2))^{1+\alpha}}.$$ 

is finite. This means that $Du$ is $\alpha$-Hölder continuous and $u$ is $\frac{1+\alpha}{2}$-Hölder continuous with respect to the temporal variable. For more details see [9, 10] and the references therein.

Finally, we define the oscillation of the function $u$ in $Q_1$ as

$$\text{osc}_{Q_1} u = \sup_{Q_1} u - \inf_{Q_1} u.$$
2.2 Preliminary notions

We start with the definition of viscosity solutions to (1). Let $S(d)$ be the space of real $d \times d$ symmetric matrices and $M \in S(d)$. We define the smallest and the greatest eigenvalues associated with $M$, as follows:

$$\lambda_{\min}(M) := \min_{|\xi| = 1} \langle M\xi, \xi \rangle \quad \text{and} \quad \lambda_{\max} := \max_{|\xi| = 1} \langle M\xi, \xi \rangle.$$ 

**Definition 2.1** Let $1 < p < \infty$ and $f$ be a continuous function in $Q_1$. We say that $u \in C(Q_1)$ is a viscosity subsolution to (1), if for every $(x_0, t_0) \in Q_1$ and $\varphi \in C^2(Q_1)$ such that $u - \varphi$ has a local maximum at $(x_0, t_0)$, we have

$$\begin{cases}
\varphi_t(x_0, t_0) - \Delta^N_p \varphi(x_0, t_0) \leq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) \neq 0, \\
\varphi_t(x_0, t_0) - \Delta\varphi(x_0, t_0) - (p - 2)\lambda_{\max}(D^2\varphi(x_0, t_0)) \leq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) = 0 \text{ and } p \geq 2,
\end{cases}$$

where $\Delta^N_p$ is a normalized Laplacian, and $\lambda_{\max}$ is the greatest eigenvalue of $M$. Conversely, we say that $u \in C(Q_1)$ is a viscosity supersolution to (1), if $x_0 \in Q_1$ and $\varphi \in C^2(Q_1)$ such that $u - \varphi$ has a local minimum at $(x_0, t_0)$, we have

$$\begin{cases}
\varphi_t(x_0, t_0) - \Delta^N_p \varphi(x_0, t_0) \geq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) \neq 0, \\
\varphi_t(x_0, t_0) - \Delta\varphi(x_0, t_0) - (p - 2)\lambda_{\min}(D^2\varphi(x_0, t_0)) \geq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) = 0 \text{ and } p \geq 2,
\end{cases}$$

Whenever $u$ is a viscosity subsolution and supersolution to (1) in $Q_1$, we say that $u$ is a viscosity solution to the Eq. (1). In addition, we say that a viscosity solution $u$ is a normalized viscosity solution if $\sup_{Q_1} |u| \leq 1$.

For a general definition of this notion, we refer the reader to [3]. In our arguments the scaled functions satisfy a variant of (1), namely:

$$u_t - \Delta u - (p - 2) \left( D^2 u \frac{D u + \xi}{|D u + \xi|}, \frac{D u + \xi}{|D u + \xi|} \right) = f \quad \text{in} \quad Q_1,$$

where $\xi \in \mathbb{R}^d$ is arbitrary. On account of completeness, we proceed by stating a local compactness result used in this paper that ensures the convergence property for the sequences in Lemma 3.1.

**Lemma 2.1** (Compactness) Let $u \in C(Q_1)$ be a normalized viscosity solution to (5). Then for all $r \in (0, 1)$, there exist constants $\beta \in (0, 1)$ and $C > 0$ such that if $\|f\|_{L^\infty(Q_1)} \leq 1$ then we have

$$\|u\|_{C^{\beta, \beta/2}(Q_1)} \leq C.$$

For a proof of this result, we refer the reader to [3, Lemma 3.1]. We close this section with the scaling properties of our model. Throughout the paper, we require

$$\|u\|_{L^\infty(Q_1)} \leq 1 \quad \text{and} \quad \|f\|_{L^\infty(Q_1)} \leq \varepsilon.$$
for some $\epsilon$ to be determined. These conditions in (6) are not restrictive. In fact, consider the function
\[ v(x, t) = \frac{u(\rho x, \rho^2 t)}{K}, \]
with $0 < \rho \ll 1$ and $K > 0$. Notice that $v$ is also a viscosity solution to (1) in $Q_1$, with the right-hand side
\[ \tilde{f}(x, t) = \frac{\rho^2}{K} f(\rho x, \rho^2 t). \]
Hence, by choosing
\[ K = \|u\|_{L^\infty(Q_1)} + \epsilon^{-1} \|f\|_{L^\infty(Q_1)}, \]
we can assume (6) without loss of generality.

3 Hölder estimates

This section is devoted to the Proof of Theorem 1.1. As usual, constants stand for $C$ may change from line to line, and depend only on the appropriate quantities.

3.1 Geometric tangential path

First, we provide an approximation lemma relating our model with the heat equation. This lemma plays a pivotal role in the paper.

Lemma 3.1 (Approximation Lemma) Let $u \in C(Q_1)$ be a normalized viscosity solution to (5) with $f \in L^\infty(Q_1) \cap C(Q_1)$. Given $\delta > 0$ there exists $\epsilon > 0$ such that, if
\[ \|f\|_{L^\infty(Q_1)} + |p - 2| < \epsilon, \]
then we can find $h \in C^{2,1}(Q_{7/9})$ such that
\[ \sup_{Q_{7/9}} |u(x, t) - h(x, t)| < \delta. \]

Proof We argue by contradiction. Suppose that the statement does not hold, then there are $\delta_0 > 0$ and sequences $(u_j)_{j \in \mathbb{N}}$, $(\xi_j)_{j \in \mathbb{N}}$, $(p_j)_{j \in \mathbb{N}}$ and $(f_j)_{j \in \mathbb{N}}$ satisfying
\[ \|f_j\|_{L^\infty(Q_1)} + |p_j - 2| < \frac{1}{j}, \]
\[ (u_j)_t - \Delta u_j - (p_j - 2) \left( D^2 u_j \frac{Du_j + \xi_j}{|Du_j + \xi_j|}, \frac{Du_j + \xi_j}{|Du_j + \xi_j|} \right) = f_j \text{ in } Q_1, \]
and for every $h \in C^{2,1}(Q_{7/9})$ and for all $j \in \mathbb{N},$
\[ \sup_{(x, t) \in Q_{7/9}} |u_j(x, t) - h(x, t)| > \delta_0. \]
From Lemma 2.1, we have
\[ u_j \in C^{\beta, \frac{\gamma}{2}}(Q_{8/9}) \text{ and } \|u_j\|_{C^{\beta, \frac{\gamma}{2}}(Q_{8/9})} \leq C. \]
where $0 < \beta < 1$ and $C > 0$ are constants that do not depend on $j \in \mathbb{N}$. Applying the Arzelâ–Ascoli Theorem, there exist a subsequence $(u_j)_{j \in \mathbb{N}}$ and a continuous function $u_\infty$ so that $(u_j)_{j \in \mathbb{N}}$ converges uniformly to $u_\infty$ in $Q_{8/9}$.

At this point, we are interested in verifying that $u_\infty$ is a viscosity solution to

$$(u_\infty)_t - \Delta u_\infty = 0 \quad \text{in} \quad Q_{8/9}.$$  \hfill (9)

For that, we examine two cases. We start by considering the case in which the sequence $(\xi_j)_{j \in \mathbb{N}}$ is bounded. Using the local compactness of $\mathbb{R}^d$, up to a subsequence $\xi_j$ converges to $\xi_\infty$. In addition, $p_j - 2$ goes to zero and $f_j$ goes to zero in the $L^\infty$-topology. Consider $\phi$ be a quadratic polynomial touching $u_\infty$ from below in $(x_0, t_0) \in Q_{8/9}$ denoted by

$$\phi(x, t) = \frac{1}{2} \langle M(x-x_0), (x-x_0) \rangle + a(t-t_0) + b \cdot (x-x_0) + u_\infty(x_0, t_0).$$

For simplicity, we can assume $|(x_0, t_0)| = u_\infty(x_0, t_0) = 0$. As a consequence that $u_j$ converges uniformly to $u_\infty$, we define

$$\phi_j(x, t) := \frac{1}{2} \langle M(x-x_j), (x-x_j) \rangle + a(t-t_j) + b \cdot (x-x_j) + u_j(x_j, t_j),$$

for $r \in (0, 1)$ fixed and $(x_j, t_j)$ such that

$$\phi_j(x_j, t_j) - u_j(x_j, t_j) := \max_{(x, t) \in Q_r} (\phi_j(x, t) - u_j(x, t)).$$

Since $u_j$ solves (7) and $\phi_j$ is a smooth test function, we have that

$$
\begin{align*}
& a - \text{Tr}(M) - (p_j - 2) \left( M \frac{b+\xi_j}{|b+\xi_j|^1}, \frac{b+\xi_j}{|b+\xi_j|^1} \right) \leq f_j \quad \text{if} \quad b + \xi_j \neq 0, \\
& a - \text{Tr}(M) - (p_j - 2) \lambda_{\max} M \leq f_j, \quad \text{if} \quad b + \xi_j = 0 \quad \text{and} \quad p \geq 2, \\
& a - \text{Tr}(M) - (p_j - 2) \lambda_{\min} M \leq f_j, \quad \text{if} \quad b + \xi_j = 0, \quad 1 < p < 2.
\end{align*}
$$  \hfill (10)

In the case $b + \xi_j \neq 0$, we can estimate as follows:

$$a - \text{Tr}(M) - |p_j - 2||M|| \leq a - \text{Tr}(M) - |p_j - 2| \left| \left( M \frac{b+\xi_j}{|b+\xi_j|^1}, \frac{b+\xi_j}{|b+\xi_j|^1} \right) \right| \leq f_j.$$

With this in mind, we evaluate the limit in (10) as $j$ approaches infinity and, using the fact that $p_j - 2 \to 0$, $\xi_j \to \xi_\infty$ and the definition of $\phi$, we deduce

$$\phi_t - \Delta \phi \leq 0.$$

Therefore $u_\infty$ solves in the viscosity sense the equation

$$(u_\infty)_t - \Delta u_\infty \leq 0 \quad \text{in} \quad Q_{8/9}.$$  \hfill (11)

On the other hand, if the sequence $(\xi_j)_{j \in \mathbb{N}}$ is unbounded, we choose a subsequence $\xi_j$ such that $|\xi_j|$ goes to infinity for every $j$. By taking $e_j := \frac{\xi_j}{|\xi_j|}$ we have $e_j \to e_\infty$, for some $e_\infty \in \mathbb{R}^d$. Hence, we may rewrite the Eq. (7) as follows:

$$(u_j)_t - \Delta u_j - (p_j - 2) \left( D^2 u_j \frac{D u_j |\xi_j|^{-1} + e_j}{|D u_j |\xi_j|^{-1} + e_j}, \frac{D u_j |\xi_j|^{-1} + e_j}{|D u_j |\xi_j|^{-1} + e_j} \right) = f_j \quad \text{in} \quad Q_1.$$
Here we obtain the following equation from the fact \( u_j \) is a viscosity solution of the above equation.

\[
\begin{align*}
\begin{cases}
    a - \text{Tr}(M) - (p_j - 2) \left\{ M^{b|\xi_j|^{-1} + e_j} \frac{b|\xi_j|^{-1} + e_j}{|b| |\xi_j|^{-1} + e_j|} \right\} \leq f_j \\
    \quad \text{if } b|\xi_j|^{-1} + e_j \neq 0, \\
    a - \text{Tr}(M) - (p_j - 2) \lambda_{\text{max}} M \leq f_j, \\
    \quad \text{if } b|\xi_j|^{-1} + e_j = 0 \text{ and } p \geq 2, \\
    a - \text{Tr}(M) - (p_j - 2) \lambda_{\text{min}} M \leq f_j, \\
    \quad \text{if } b|\xi_j|^{-1} + e_j = 0, \ 1 < p < 2.
\end{cases}
\end{align*}
\]

(12)

Similarly to the bounded case, we apply the limit in the Eq. (12) and get that \( u_\infty \) solves (11) in the viscosity sense. Therefore, we can conclude that \( u_\infty \) is a viscosity subsolution to (9) in \( Q_{8/9} \). Similarly, we can show that \( u_\infty \) is a viscosity supersolution to (9). Hence, we have that \( u_\infty \) belongs to \( C^{2,1}(Q_{7/9}) \). Finally, by taking \( h = u_\infty \), we reach a contradiction. \( \square \)

### 3.2 Regularity in space

We proceed with the regularity in the spatial variable.

**Proposition 3.1** Let \( u \in C(Q_1) \) be a normalized viscosity solution of

\[
u_t - \Delta u - (p - 2) \left\{ D^2 u \frac{Du + \xi}{|Du + \xi|} \frac{Du + \xi}{|Du + \xi|} \right\} = f \quad \text{in } Q_1,
\]

for any arbitrary vector \( \xi \in \mathbb{R}^d \) and \( f \in L^\infty(Q_1) \cap C(Q_1) \). Given \( \alpha \in (0, 1) \), there exists \( \varepsilon > 0 \) such that if

\[
\| f \|_{L^\infty(Q_1)} + |p - 2| \leq \varepsilon,
\]

then we can find a constant \( 0 < \rho < 1 \) and a sequence of affine functions \( (\ell_n)_{n\in\mathbb{N}} \) of the form \( \ell_n(x, t) := a_n + b_n \cdot x \) satisfying

\[
\sup_{Q_\rho^n} |u(x, t) - \ell_n(x, t)| \leq \rho^n (1 + \alpha),
\]

\[
|a_{n+1} - a_n| \leq C \rho^n (1 + \alpha)
\]

and

\[
|b_{n+1} - b_n| \leq C \rho^{n\alpha}
\]

for a constant \( C > 0 \) and for every \( n \in \mathbb{N} \).

**Proof** The result follows by induction argument. For simplicity, we split the proof into two steps.

**Step 1.** Take \( \delta > 0 \) to be determined later. Lemma 3.1 implies that there exists a function \( h \in C^{2,1}(Q_{8/9}) \) satisfying

\[
\sup_{Q_{7/9}} |u(x, t) - h(x, t)| \leq \delta.
\]

Set

\[
\ell(x, t) := h(0, 0) + Dh(0, 0) \cdot x.
\]
Recall that \( h(0, 0) \) and \( Dh(0, 0) \) are uniformly bounded by a constant. Since \( h \in C^{2,1}(Q_{8/9}) \), we have that there exists a constant \( C > 0 \) such that

\[
\sup_{Q_\rho} |h(x, t) - \ell(x, t)| \leq C \rho^2,
\]

where \( \rho \) is the parabolic distance between the points \((x, t)\) and \((0, 0)\). As a consequence from the triangular inequality, we obtain

\[
\sup_{Q_\rho} |u(x, t) - \ell(x, t)| \leq \sup_{Q_\rho} |u(x, t) - h(x, t)| + \sup_{Q_\rho} |h(x, t) - \ell(x, t)| \\
\leq \delta + C \rho^2.
\]

Making universal choices, we define

\[
\rho := \left( \frac{1}{2C} \right)^{\frac{1}{2\alpha}} \quad \text{and} \quad \delta := \frac{\rho^{1+\alpha}}{2}.
\]

Notice that the universal choice of \( \delta \) determines the value of \( \varepsilon > 0 \) through the Lemma 3.1. Therefore

\[
\sup_{Q_\rho} |u(x, t) - \ell(x, t)| \leq \rho^{1+\alpha}.
\]

This completes the case \( n = 1 \).

**Step 2.** Assume that the case \( n = k \) has been verified. We shall prove the case \( n = k + 1 \). First, let us introduce an auxiliary function \( v_k : Q_1 \to \mathbb{R} \) defined by

\[
v_k(x, t) := \frac{u(\rho^k x, \rho^{-2k} t) - \ell_k(\rho^k x, \rho^{-2k} t)}{\rho^{k(1+\alpha)}}.
\]

Notice that \( v_k \) solves the following equation:

\[
(v_k)_t - \Delta v_k - (p - 2) \left( D^2 v_k \frac{Dv_k + \rho^{-k\alpha} b_k}{|Dv_k + \rho^{-k\alpha} b_k|}, \frac{Dv_k + \rho^{-k\alpha} b_k}{|Dv_k + \rho^{-k\alpha} b_k|} \right) = f_k \quad \text{in} \quad Q_1,
\]

where \( f_k := \frac{1}{\rho^{k(\alpha - 1)}} f \). That means \( f_k \in L^\infty(Q_1) \), if and only if, \( \alpha < 1 \). By induction hypothesis, we have that \( v_k \) is a normalized function in \( Q_{8/9} \). Hence \( v_k \) satisfies the assumptions of Lemma 3.1, which ensures the existence of \( \tilde{h} \in C^{2,1}(Q_{7/9}) \), so that

\[
\sup_{Q_{7/9}} |u(x, t) - \tilde{h}(x, t)| < \delta.
\]

As a consequence of Step 1, there exists an affine function \( \tilde{\ell} \) such that

\[
\sup_{Q_\rho} |v_k(x, t) - \tilde{\ell}(x, t)| \leq \rho^{1+\alpha}.
\]

Defining \( \ell_{k+1}(x, t) := \ell_k(x, t) + \rho^{k(1+\alpha)} \tilde{\ell}(\rho^{-k} x, \rho^{-2k} t) \) yields

\[
\sup_{Q_{\rho^{k+1}}} |u(x, t) - \ell_{k+1}(x, t)| \leq \rho^{(k+1)(1+\alpha)}.
\]

Also, the coefficients satisfy

\[
|a_{k+1} - a_k| \leq C \rho^{k(1+\alpha)} \quad (13)
\]
and
\[ |b_{k+1} - b_k| \leq C \rho^{k\alpha} \tag{14} \]
for every \( k \in \mathbb{N} \), and the proposition is concluded.

**Proof of Theorem 1.1** From (13) and (14), we conclude that the sequences \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) are Cauchy sequences, consequently there are constants \( a_\infty \) and \( b_\infty \) such that
\[ \lim_{n \to \infty} a_n = a_\infty \quad \text{and} \quad \lim_{n \to \infty} b_n = b_\infty. \]
Moreover, we have the estimates
\[ |a_k - a_\infty| \leq C \rho^{k(1+\alpha)} \quad \text{and} \quad |b_k - b_\infty| \leq C \rho^{k\alpha}. \]

Given any \( 0 < \rho \ll 1 \), let \( k \) be a natural number such that \( \rho^{k+1} < r < \rho^k \). Thus, we estimate from the previous computations
\[
\sup_{Q_r} |u(x, t) - \ell_\infty(x, t)| \leq \sup_{Q_{r\rho^k}} |u(x, t) - \ell_k(x, t)| + \sup_{Q_{r\rho^k}} |\ell_k(x, t) - \ell_\infty(x, t)| \\
\leq \rho^{k(1+\alpha)} + |a_k - a_\infty| + \rho^k |b_k - b_\infty| \\
\leq \rho^{k(1+\alpha)} + C \rho^{k(1+\alpha)} + C \rho^{k \cdot \rho^k} \\
\leq C \left( \frac{1}{\rho} \right)^{(1+\alpha)} \rho^{(k+1)(1+\alpha)} \\
\leq C r^{(1+\alpha)}. 
\]

To conclude the proof, we characterize the coefficients \( a_\infty \) and \( b_\infty \). In fact, evaluating the limit in the following inequality
\[ \sup_{Q_{\rho^n}} |u(x, t) - \ell_n(x, t)| \leq C \rho^{n(1+\alpha)}, \tag{15} \]
as \( n \) approaches infinity at \( (x, t) = (0, 0) \), we obtain that \( a_\infty = u(0, 0) \). From [3, Lemma A.1], we can conclude that \( b_\infty = Du(0, 0) \). This finishes the proof.

### 3.3 Regularity in time

For account of completeness we present the proof of the regularity in the time variable, see also [3, Lemma 3.5].

**Lemma 3.2** (Regularity in time) Let \( u \in C(Q_1) \) be a normalized viscosity solution to (1). Assume that \( \rho \) is as in Proposition 3.1. Given \( \alpha \in (0, 1) \), there exists \( \varepsilon \) such that if
\[ \|f\|_{L^\infty(Q_1)} + |p - 2| < \varepsilon, \]
then for every \( t \in (-r^2, 0) \)
\[ |u(0, t) - u(0, 0)| \leq C |t|^{\frac{1+\alpha}{2}}, \]
where \( C > 0 \) is a constant depending only on the dimension \( d \) and \( \alpha \).

**Proof** For \((x, t) \in Q_r\), we set
\[ v(x, t) := u(x, t) - u(0, 0) - Du(0, 0) \cdot x. \]
Proposition 3.1 implies that
\[ |v(x_1, t) - v(x_2, t)| \leq C r^{1+\alpha}, \]
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for \(x_1, x_2 \in B_r, t \in [-r^2, 0]\), and consequently we have

\[
\text{osc}_{B_r} v(\cdot, t) \leq C r^{1+\alpha} =: A.
\]

We claim that \(\text{osc}_{Q} v \leq C A + 4r^2 \| f \|_{L^\infty(Q_1)}\). In fact, notice that \(v\) solves

\[
\partial_t v - \Delta v - (p - 2) \left( D^2 v \frac{Dv + b}{|Dv + b|} \cdot \frac{Dv + b}{|Dv + b|} \right) = f \quad \text{in} \quad Q_r,
\]

where \(b := Du(0, 0)\). Employing the Lemma 2.2 in [3], we obtain

\[
\text{osc}_{Q} v \leq C r^{1+\alpha} + 4r^2 \| f \|_{L^\infty(Q_1)}.
\]

Therefore,

\[
|u(0, t) - u(0, 0)| = |v(0, t)| \leq C |t|^{1+\alpha}.
\]

\[\square\]

4 Regularity in Sobolev spaces

In this section, we present the Proof of Theorem 1.2. Here, we examine the homogeneous problem

\[
\frac{\partial u}{\partial t} - \Delta^N u(x, t) = 0 \quad \text{in} \quad Q_1.
\]

(16)

Let \(u \in C(Q_1)\) be a viscosity solution to (16). We consider the following regularized Dirichlet problem:

\[
\begin{aligned}
(v_{\varepsilon})_t - \Delta v_{\varepsilon} - (p - 2) \frac{D^2 v_{\varepsilon} Dv_{\varepsilon} \cdot Dv_{\varepsilon}}{|Dv_{\varepsilon}|^2 + \varepsilon^2} &= 0 \quad \text{in} \quad Q_{3/4} \\
v_{\varepsilon} &= u \quad \text{on} \quad \partial Q_{3/4}.
\end{aligned}
\]

(17)

It is well known that \(v_{\varepsilon}\) is a classical solution (in the interior), and the gradient of \(v_{\varepsilon}\) is uniformly bounded with respect to \(\varepsilon\), see for instance [11].

**Proof of Theorem 1.2** Consider the operator

\[
F(D^2 u, x, t) := -\Delta u - (p - 2) \frac{D^2 u Dv_{\varepsilon} \cdot Dv_{\varepsilon}}{|Dv_{\varepsilon}|^2 + \varepsilon^2}.
\]

First, notice that \(F\) is a uniformly elliptic operator with constants \(\lambda = \min(1, p - 1)\) and \(\Lambda = \max(1, p - 1)\). In addition, the oscillation of \(F\) given by

\[
\theta_F(x, t) := \sup_M \frac{|F(M, x, t) - F(M, 0, 0)|}{|M| + 1},
\]

satisfies

\[
\theta_F(x, t) \leq 2|p - 2|.
\]

It follows from (17) that \(v_{\varepsilon}\) solves

\[
(v_{\varepsilon})_t + F(D^2 v_{\varepsilon}, x, t) = 0.
\]

Hence, there exists a positive constant \(\varepsilon_1\), such that if

\[
|p - 2| \leq \varepsilon_1,
\]
then the oscillation \( \theta_F \) is sufficiently small and we can apply the \( W^{2,q} \)-estimates in [35, Theorem 5.7] to assure that \( v_\varepsilon \in W^{2,1/q}_{\text{loc}}(Q_1) \) for all \( q \in [1, \infty) \). Moreover, there is a constant \( C > 0 \) independent of \( \varepsilon_1 \) such that

\[
\|v_\varepsilon\|_{W^{2,1/q}} \leq C. \tag{18}
\]

Since \( v_\varepsilon \to u \) uniformly in compact sets, see [17, Lemma 3.1], we obtain that \( u \) also belongs to \( W^{2,1/q}_{\text{loc}}(Q_1) \) and satisfies the estimate (18). This finishes the proof.

The following result is a direct consequence of the Theorem 1.2.

**Corollary 4.1** Let \( u \in C(Q_1) \) be a normalized viscosity solution of (16). Then \( u \in C^{1+\alpha, \frac{1+\alpha}{2}}_{\text{loc}}(Q_1) \) for every \( \alpha \in (0, 1) \), with the estimate

\[
\|u\|_{C^{1+\alpha, \frac{1+\alpha}{2}}_{\text{loc}}(Q_1)} \leq C\|u\|_{L^\infty(Q_1)},
\]

where \( C \) is a positive constant that depends only on the dimension \( d \).

**Proof** The corollary follows immediately from Theorem 1.2 with general Sobolev inequalities in Sobolev spaces.

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