Asymptotic Properties of Discrete Minimal $s$, $\log^t$-Energy Constants and Configurations

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Abstract

Combining the ideas of Riesz $s$-energy and log-energy, we introduce the so-called $s$, $\log^t$-energy. In this paper, we investigate the asymptotic behaviors for $N, t$ fixed and $s$ varying of minimal $N$-point $s$, $\log^t$-energy constants and configurations of an infinite compact metric space of diameter less than 1. In particular, we study certain continuity and differentiability properties of minimal $N$-point $s$, $\log^t$-energy constants in the variable $s$ and we show that in the limits as $s \to \infty$ and as $s \to s_0 > 0$, minimal $N$-point $s$, $\log^t$-energy configurations tend to an $N$-point best-packing configuration and a minimal $N$-point $s_0$, $\log^t$-energy configuration, respectively. Furthermore, the optimality of $N$ distinct equally spaced points on circles in $\mathbb{R}^2$ for some certain $s$, $\log^t$ energy problems was proved.

Keywords

discrete minimal energy; best-packing; Riesz energy; logarithmic energy.

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1 Introduction

The general setting of discrete minimal energy problem is the following. Let $(A, d)$ be an infinite compact metric space and $K: A \times A \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous kernel. For a fixed set of $N$ points $\omega_N \subset A$, we define the $K$-energy of $\omega_N$ as follows

$$E_K(\omega_N) := \sum_{x \neq y, x, y \in \omega_N} K(x, y).$$

The minimal $N$-point $K$-energy of the set $A$ is defined by

$$E_K(A, N) := \min_{\omega_N \subset A, \#\omega_N = N} E_K(\omega_N),$$

where $\#\omega_N$ stands for the cardinality of the set $\omega_N$. A minimal $N$-point $K$-energy configuration is a configuration $\omega^K_N$ of $N$ points in $A$ that minimizes such energy, namely

$$E_K(\omega^K_N) = \min_{\omega_N \subset A, \#\omega_N = N} E_K(\omega_N).$$

It is known that $\omega^K_N$ always exists and in general $\omega^K_N$ may not be unique.

Two important kernels in the theory on minimal energy are Riesz and logarithmic kernels. The (Riesz) $s$-kernel and log-kernel are defined by

$$K^s(x, y) := \frac{1}{d(x, y)^s}, \quad s \geq 0. \quad (1)$$

and

$$K_{\log}(x, y) := \log \frac{1}{d(x, y)},$$

for all $(x, y) \in A \times A$, respectively. It is not difficult to check that both kernels are lower semicontinuous on $A \times A$. The $s$-energy of $\omega_N$ and the minimal $N$-point $s$-energy of the set $A$ are

$$E^s(\omega_N) := E_K^s(\omega_N) \quad \text{and} \quad E^s(A, N) := \min_{\omega_N \subset A, \#\omega_N = N} E^s(\omega_N)$$

and we denote by $\omega^s_N := \omega^K_N$ and call this configuration a minimal $N$-point $s$-energy configuration. Similarly, the log-energy of $\omega_N$ and the $N$-point log-energy of the set $A$ are

$$E_{\log}(\omega_N) := E_K^\log(\omega_N) \quad \text{and} \quad E_{\log}(A, N) := \min_{\omega_N \subset A, \#\omega_N = N} E_{\log}(\omega_N).$$
and we denote by $\omega_N^{\log} := \omega_N^{K_{\log}}$ and call this configuration a \textit{minimal $N$-point log-energy configuration}.

Let us provide a short survey of these two energy problems.

The study of $s$-energy constants and configurations has a long history in physics, chemistry, and mathematics. Finding the arrangements of $\omega_N^s$ where the set $A$ is the unit sphere $S^2$ in the Euclidean space $\mathbb{R}^3$ has been an active area since the beginning of the 19th century. The problem is known as the generalized Thomson problem (see [1] and [2, Chapter 2.4]). Candidates for $\omega_N^s$ for several numbers of $N$ are available (see, e.g., [3]). However, the solutions (with rigorous proofs) are obtainable for handful values of $N$ (see, e.g., [4, 5, 6, Author1(year)]). For a general compact set $A$ in the Euclidean space $\mathbb{R}^m$, the study of the distribution of minimal $N$-point $s$-energy configurations of $A$ as $N \to \infty$ can be founded in [Author2(year)] and [Author3(year)]. In [Author3(year)], it was shown that when $s$ is any fixed number greater than the Hausdorff dimension of $A$, minimal $N$-point $s$-energy configurations of $A$ are “good points” to represent the set $A$ in the sense that they are asymptotically uniformly distributed over the set $A$ (see the precise statement in [Author3(year), Theorems 2.1 and 2.2]).

The log-energy problem has been heavily studied when $A$ is a subset of the Euclidean space $\mathbb{R}^2$ (or $\mathbb{C}$) because it has had a profound influence in approximation theory (see, e.g., [7, 8, 9, 10, 11]). For $A \subset \mathbb{C}$, the points in $\omega_N^{\log}$ are commonly known as Fekete points or Chebyshev points which can be used as interpolation points (see [12]). The log-energy problem received another special attention when Steven Smale posed Problem #7 in his book chapter under the title “Mathematical problems for the next century” [13]. The problem #7 asks for a construction of an algorithm which on input $N \geq 2$ outputs a configuration $\omega_N = \{x_1, \ldots, x_N\}$ of distinct points on $S^2$ embedded in $\mathbb{R}^3$ such that

$$E_{\log}(\omega_N) - E_{\log}(S^2, N) \leq c \log N$$

(where $c$ is a constant independent of $N$ and $\omega_N$) and requires that its running time grows at most polynomially in $N$. This arose from complexity theory in his joint work with Shub in [14]. In order to answer this question, it is natural to understand the asymptotic expansion of $E_{\log}(S^2, N)$ in the variable $N$ (see [15] for conjectures and the progress). The problem concerning the arrangements of $\omega_N^{\log}$ on the unit sphere $S^2$ in $\mathbb{R}^3$ is posed by Whyte [16] in 1952. The Whyte’s problem is also attractive and intractable. We refer to [17] for a glimpse of this problem.

In [2], Borodachov, Hardin, and Saff investigated asymptotic properties of minimal $N$-point $s$-energy constants and configurations for fixed $N$ and varying $s$. Because this will be our main interest in this paper, we will state these results below.

The first theorem [2, Theorems 2.7.1 and Theorem 2.7.3] concerns the continuity
and differentiability of the function
\[ f(s) := \mathcal{E}^s(A, N), \quad s \geq 0. \] (2)

In order to state such theorem, let us define a set
\[ G^s_{\log}(A, N) := \left\{ \sum_{x \neq y, x, y \in \omega_N} K^s(x, y)K_{\log}(x, y) : \omega_N \subset A \text{ and } E^s(\omega_N) = \mathcal{E}^s(A, N) \right\}, \] (3)

for \( s \geq 0 \).

**Theorem A.** Let \((A, d)\) be an infinite compact metric space and let \( N \geq 2 \) be fixed. Then,

(a) the function \( f(s) \) defined in (2) is continuous on \([0, \infty)\).

(b) the function \( f(s) \) is right differentiable on \([0, \infty)\) and left differentiable on \((0, \infty)\) with
\[ f'_+(s) := \lim_{r \to s^+} \frac{f(r) - f(s)}{r - s} = \inf G^s(A, N), \quad s \geq 0, \]
and
\[ f'_-(s) := \lim_{r \to s^-} \frac{f(r) - f(s)}{r - s} = \sup G^s(A, N), \quad s > 0. \]

We will see in Theorems B and C below that there are certain relations between minimal \( s \)-energy problems, as \( s \to 0^+ \), and best-packing problem defined as follows.

The \( N \)-point best-packing distance of the set \( A \) is defined
\[ \delta_N(A) := \max\{\delta(\omega_N) : \omega_N \subset A\}, \] (4)
where
\[ \delta(\omega_N) := \min_{1 \leq i \neq j \leq N} d(x_i, x_j) \]
denotes the separation distance of an \( N \)-point configuration \( \omega_N = \{x_1, \ldots, x_N\} \), and \( N \)-point best-packing configurations are \( N \)-point configurations attaining the maximum in (4).

The following theorem [2, Corollary 2.7.5 and Proposition 3.1.2] explains the behavior of \( \mathcal{E}^s(A, N) \) as \( s \to 0^+ \) and \( s \to \infty \).

**Theorem B.** For \( N \geq 2 \) and an infinite compact metric space \((A, d)\),
\[ \lim_{s \to 0^+} \frac{\mathcal{E}^s(A, N) - N(N - 1)}{s} = \mathcal{E}_{\log}(A, N) \]
\[
\lim_{s \to \infty} (\mathcal{E}^s(A, N))^{1/s} = \frac{1}{\delta_N(A)}.
\]

Before we state more results, let us define a cluster configuration. Let \(s_0 \in [0, \infty]\) We say that

- an \(N\)-point configuration \(\omega_N \subset A\) is a cluster configuration of \(\omega_N^s\) as \(s \to s_0^+\) if there is a sequence \(\{s_k\}_{k=1}^{\infty} \subset (s_0, \infty)\) such that \(\lim_{k \to \infty} s_k = s_0\) and \(\lim_{k \to \infty} \omega_N^{s_k} = \omega_N\) in the topology of \(A^N\) induced by the metric \(d\).

- an \(N\)-point configuration \(\omega_N \subset A\) is a cluster configuration of \(\omega_N^s\) as \(s \to s_0^-\) if there is a sequence \(\{s_k\}_{k=1}^{\infty} \subset [0, s_0)\) such that \(\lim_{k \to \infty} s_k = s_0\) and \(\lim_{k \to \infty} \omega_N^{s_k} = \omega_N\) in the topology of \(A^N\) induced by the metric \(d\).

- an \(N\)-point configuration \(\omega_N \subset A\) is a cluster configuration of \(\omega_N^s\) as \(s \to \infty\) if there is a sequence \(\{s_k\}_{k=1}^{\infty} \subset (0, \infty)\) such that \(\lim_{k \to \infty} s_k = \infty\) and \(\lim_{k \to \infty} \omega_N^{s_k} = \omega_N\) in the topology of \(A^N\) induced by the metric \(d\).

The properties of cluster configurations of minimal \(N\)-point \(s\)-energy configurations as \(s\) varies (see [2, Theorem 2.7.1 and Proposition 3.1.2]) are in

**Theorem C.** Let \((A, d)\) be an infinite compact metric space and, for \(s \geq 0\) and \(N \geq 2\), let \(\omega_N^s\) denote a minimal \(N\)-point \(s\)-energy configuration on \(A\). Then,

(a) for \(s_0 > 0\), any cluster configuration of \(\omega_N^s\) as \(s \to s_0\) is a minimal \(N\)-point \(s_0\)-energy configuration;

(b) any cluster configuration of \(\omega_N^s\) as \(s \to 0^+\) is a minimal \(N\)-point log-energy configuration;

(c) any cluster configuration of \(\omega_N^s\) as \(s \to \infty\) is a \(N\)-point best-packing configuration.

In this paper, we consider the following \(s, \log^t\)-kernel

\[
K^{s, t}_{\log}(x, y) = \frac{1}{d(x, y)^s} \left(\log \frac{1}{d(x, y)}\right)^t, \quad s \geq 0, \quad t \geq 0.
\]  

with corresponding \(s, \log^t\)-energy of \(\omega_N\) and minimal \(N\)-point \(s, \log^t\)-energy of the set \(A\)

\[
E^{s, t}_{\log}(\omega_N) := E_{K^{s, t}_{\log}}(\omega_N) \quad \text{and} \quad \mathcal{E}^{s, t}_{\log}(A, N) := \min_{\omega_N \subset A} E^{s, t}_{\log}(\omega_N),
\]
respectively. We set
\[ \omega_N^{s,\log t} := \omega_N^{K_s,\log t}, \]
and call it a \textit{minimal N-point s, log t-energy configuration}. Note that the kernel \( K_s^{\log t}(x, y) \) is lower semicontinuous on \( A \times A \) and this \( s, \log t \)-energy can be viewed as a generalization of both \( s \)-energy and log-energy. The kernel in (5) was first appeared in the study of the differentiability of the function \( f(s) \) in [2, Theorem 2.7.3]. To the authors’ knowledge, no study involving \( s, \log t \)-energy constants and configurations appears in the literature.

The main goal of this paper is to prove analogues of Theorems A, B, and C for \( s, \log t \)-energy constants and configurations. We would like to emphasize that we will limit our interest to the sets \( A \) with \( \text{diam}(A) < 1 \), where
\[ \text{diam}(A) := \sup_{x,y \in A} d(x, y) \]
denotes the diameter of \( A \). For the cases where \( \text{diam}(A) \geq 1 \), the values of the kernel \( K_s^{\log t}(x, y) \) can be 0 or negative and the analysis becomes laborious. Furthermore, we investigate the arrangement of \( \omega_N^{s,\log t} \) on circles in \( \mathbb{R}^2 \) for certain values of \( s \) and \( t \).

An outline of this paper is as follows. The main results in this paper are stated in Section 2. We keep all auxiliary lemmas in Section 3. The proofs of the main results are in Section 4.

\section{Main Results}

Asymptotic behavior of minimal \( N \)-point \( s, \log t \)-energy constants and configurations as \( s \to \infty \) can be explained in the following theorem.

\textbf{Theorem 2.1.} Let \( N \geq 2 \) and \( t \geq 0 \) be fixed. Assume that \((A, d)\) is an infinite compact metric space with \( \text{diam}(A) < 1 \). Then,
\[ \lim_{s \to \infty} \left( \mathcal{E}_N^{s}(A, N) \right)^{1/s} = \frac{1}{\delta_N(A)}. \]

Furthermore, every cluster configuration of \( \omega_N^{s,\log t} \) as \( s \to \infty \) is an \( N \)-point best-packing configuration on \( A \).

For a fixed \( t \geq 0 \), we define
\[ g(s) := \mathcal{E}_N^{s}(A, N), \quad s \geq 0. \]
The continuity of \( g(s) \) is stated below.
Theorem 2.2. Let $N \geq 2$ and $t \geq 0$ be fixed. Assume that $(A,d)$ is an infinite compact metric space with $\text{diam}(A) < 1$. Then, the function $g(s)$ is continuous on $[0, \infty)$.

Analysis of cluster configurations of $\omega_N^{s, \log^t}$ as $s \to s_0 > 0$ is in the following theorem.

Theorem 2.3. Let $N \geq 2$ and $t \geq 0$ be fixed. Assume that $(A,d)$ is an infinite compact metric space with $\text{diam}(A) < 1$. Denote by $\omega_N^{s, \log^t}$ a minimal $N$-point $s, \log^t$-energy configuration on $A$. Then, for any $s_0 > 0$, any cluster configuration of $\omega_N^{s, \log^t}$, as $s \to s_0$, is a minimal $N$-point $s_0, \log^t$-energy configuration on $A$.

For $s \geq 0$ and $t \geq 0$, we set
\[ G_{\log^t+1}(A, N) := \{ E_{\log^t+1}(\omega_N) : \omega_N \subset A \text{ and } E_{\log^t}(\omega_N) = E_{\log^t+1}(A, N) \}. \]

The differentiability properties of $g(s)$ are in Theorems 2.4 and 2.5.

Theorem 2.4. Let $N \geq 2$ and $t \geq 0$ be fixed. Assume that $(A,d)$ is an infinite compact metric space with $\text{diam}(A) < 1$. Then, the function $g(s)$ is right differentiable on $[0, \infty)$ and left differentiable on $(0, \infty)$ with
\[ g'_+(s) := \lim_{r \to s^+} \frac{g(r) - g(s)}{r - s} = \inf G_{\log^t+1}(A, N), \quad s \geq 0, \] \[ g'_-(s) := \lim_{r \to s^-} \frac{g(r) - g(s)}{r - s} = \sup G_{\log^t+1}(A, N), \quad s > 0. \]

Theorem 2.5. Let $N \geq 2$ and $t \geq 0$ be fixed. Assume that $(A,d)$ is an infinite compact metric space with $\text{diam}(A) < 1$. Then,

(a) the function $g(s)$ is differentiable at $s = s_0 > 0$ if and only if
\[ \inf G_{\log^t}(A, N) = \sup G_{\log^t}(A, N); \]

(b) if $\omega_N^*$ is a cluster point of $\omega_N^{s, \log^t}$ as $s \to s_0^+ \geq 0$, then
\[ E_{\log^t+1}(\omega_N^*) = \inf G_{\log^t+1}(A, N) = g'_+(s_0); \]

(c) if $\omega_N^{**}$ is a cluster point of $\omega_N^{s, \log^t}$ as $s \to s_0^- > 0$, then
\[ E_{\log^t+1}(\omega_N^{**}) = \sup G_{\log^t+1}(A, N) = g'_-(s_0); \]
(d) For $s_0 > 0$, if there exists a configuration $\omega^*_N$ that is both cluster configurations of $\omega^*_{N,s} \log t$ as $s \to s^+_0$ and $s \to s^-_0$, then the function $g(s)$ is differentiable at $s = s_0$ with

$$E^s_{\log t+1}(\omega^*_N) = g'(s_0).$$

Let $d_u$ be the 2-dimensional Euclidean metric of $\mathbb{R}^2$. For $\alpha > 0$, we denote by

$$S^1_\alpha := \{x \in \mathbb{R}^2 : d_u(0, x) = \alpha\}$$

the circle centered at 0 of radius $\alpha$. We let $L(x, y)$ be the geodesic distance between the points $x$ and $y$ on $S^1_\alpha$; that is, the length of the shorter arc of $S^1_\alpha$ connecting the points $x$ and $y$.

The optimality of $N$ distinct equally spaced points on $S^1_\alpha$ with the Euclidean metric $d_u$ or the geodesic distance $L$ for the certain $s, \log t$-energy problems is stated in Propositions 2.1-2.3.

**Proposition 2.1.** Let $N \geq 2$, $s \geq 0$, $t \geq 1$, and $0 < \alpha < 1/2$. Then, $\omega_N$ is a minimal $N$-point $s, \log t$-energy configuration on $S^1_\alpha$ with the geodesic distance $L$ if and only if $\omega_N$ is a configuration of $N$ distinct equally spaced points on $S^1_\alpha$.

**Proposition 2.2.** Let $N \geq 2$, $0 < \alpha < (e\pi)^{-1}$, and $s, t$ satisfy $s > 0$, $t \geq 0$ or $s = 0$, $t > 0$. Then, $\omega_N$ is a minimal $N$-point $s, \log t$-energy configuration on $S^1_\alpha$ with the geodesic distance $L$ if and only if $\omega_N$ is a configuration of $N$ distinct equally spaced points on $S^1_\alpha$.

**Proposition 2.3.** Let $N \geq 2$, $s \geq 0$, $t \geq 1$, and $0 < \alpha < 1/2$. Then, $\omega_N$ is a minimal $N$-point $s, \log t$-energy configuration on $S^1_\alpha$ with the Euclidean metric $d_u$ if and only if $\omega_N$ is a configuration of $N$ distinct equally spaced points on $S^1_\alpha$.

Note that the conditions $0 < \alpha < 1/2$ in Proposition 2.1 and $0 < \alpha < 1/2$ in Proposition 2.3 are needed to make sure that diam($S^1_\alpha$) < 1 corresponding to the Euclidean metric $d_u$ and the geodesic distance $L$, respectively.

### 3 Auxiliary Lemmas

**Lemma 3.1.** Let $\beta \geq 0$ and $h : (0, 1) \to (0, \infty)$ be a function defined by

$$h(x) := x \left(\log \frac{1}{x}\right)^{-\beta} \quad \text{for all } x \in (0, 1).$$

Then, $h(x)$ is strictly increasing on $(0, 1)$.
**Proof of Lemma 3.1.** Because
\[ h'(x) = \beta \left( \log \frac{1}{x} \right)^{-(\beta+1)} + \left( \log \frac{1}{x} \right)^{-\beta} \]
and \((\log(1/x))^{-\beta} > 0\) for all \(x \in (0, 1)\) and \(\beta \geq 0\), \(h'(x) > 0\) for all \(x \in (0, 1)\). Therefore, \(h(x)\) is strictly increasing on \((0, 1)\).

**Lemma 3.2.** Let \((s, t) \in [0, \infty) \times [0, \infty) \setminus \{(0,0)\}\) and \(p : (0,1) \to (0,\infty)\) be a function defined by
\[ p(x) := \frac{1}{x^s} \left( \log \frac{1}{x} \right)^t \quad \text{for all } x \in (0,1). \]
Then, \(p(x)\) is strictly decreasing on \((0,1)\).

**Proof of Lemma 3.2.** Using Lemma 3.1, we set \(\beta = t/s\) and
\[ p(x) = \left( \frac{1}{h(x)} \right)^s = \frac{1}{x^s} \left( \log \frac{1}{x} \right)^t \]
is strictly decreasing on \((0,1)\). \(\square\)

**Lemma 3.3.** Let \((A, d)\) be an infinite compact metric space with \(\text{diam}(A) < 1\) and \(s, t \geq 0\). Then, for all \(N\)-point configurations \(\omega_N \subset A\),
\[ E_{\log^s}^r(\omega_N) \geq \frac{N(N-1)}{(\text{diam}(A))^s} \left( \log \frac{1}{\text{diam}(A)} \right)^t. \]

**Proof of Lemma 3.3.** The proof relies on the fact that \(p(x)\) in Lemma 3.2 is strictly decreasing on \((0,1)\). \(\square\)

**Lemma 3.4.** Let \((A, d)\) be an infinite compact metric space with \(\text{diam}(A) < 1\) and \(\omega_N = \{x_1, \ldots, x_N\}\) be any configuration of \(N\) distinct points of \(A\). Then, for any \(s > r \geq 0\) and \(t \geq 0\),
\[ E_{\log^s}^{r+1}(\omega_N) \leq \frac{E_{\log^s}^r(\omega_N) - E_{\log^r}^r(\omega_N)}{s-r} \leq E_{\log^{s+1}}^r(\omega_N). \]

**Proof of Lemma 3.4.** Let \(x_i, x_j \in \omega_N\) where \(1 \leq i \neq j \leq N\), let \(s > r \geq 0\), and let \(t \geq 0\). Then,
\[ \frac{1}{d(x_i, x_j)^r} \log \frac{1}{d(x_i, x_j)} \leq \frac{1}{d(x_i, x_j)^s} - \frac{1}{d(x_i, x_j)^r} \leq \frac{1}{d(x_i, x_j)^s} \log \frac{1}{d(x_i, x_j)}. \]
Since \((\log \frac{1}{d(x_i, x_j)})^t > 0,\)

\[ \frac{1}{d(x_i, x_j)^s} \left( \log \frac{1}{d(x_i, x_j)} \right)^{t+1} \leq \frac{1}{d(x_i, x_j)^{s-r}} \left( \log \frac{1}{d(x_i, x_j)} \right)^t - \frac{1}{d(x_i, x_j)^r} \left( \log \frac{1}{d(x_i, x_j)} \right)^t \]

\[ \leq \frac{1}{d(x_i, x_j)^{s-r}} \left( \log \frac{1}{d(x_i, x_j)} \right)^{t+1}. \]

It follows that

\[ E_{s \log^{t+1}}^r(\omega_N) \leq \frac{E_s^r(\omega_N) - E_{s \log^t}(\omega_N)}{s-r} \leq E_{s \log^t(\omega_N)}. \]
Then,
\[
\frac{1}{\delta_N(A)} \left( \log \frac{1}{c} \right)^{t/s} \leq \frac{1}{\delta(\omega^s_{\log^t})} \left( \log \frac{1}{c} \right)^{t/s} \leq \frac{1}{\delta(\omega^s_{\log^t})} \left( \log \frac{1}{\delta(\omega^s_{\log^t})} \right)^{t/s}
\]
\[
\leq \left( E^s_{\log^t}(\omega^s_{\log^t}) \right)^{1/s} = \left( E^s_{\log^t}(A, N) \right)^{1/s} \leq \left( E^s_{\log^t}(\omega^\infty_N) \right)^{1/s} \leq \frac{1}{\delta_N(A)} \left( E^s_{\log^t}(\omega^\infty_N) \right)^{1/s}.
\]  
(8)

Since
\[
\lim_{s \to \infty} \frac{1}{\delta_N(A)} \left( \log \frac{1}{c} \right)^{t/s} = \frac{1}{\delta_N(A)}
\]
and
\[
\lim_{s \to \infty} \frac{1}{\delta_N(A)} \left( E^s_{\log^t}(\omega^\infty_N) \right)^{1/s} = \frac{1}{\delta_N(A)}
\]

it follows that
\[
\lim_{s \to \infty} \left( E^s_{\log^t}(A, N) \right)^{1/s} = \frac{1}{\delta_N(A)}.
\]

Let \( \omega^*_N \) be a cluster configuration of \( \omega^s_{\log^t} \) as \( s \to \infty \). This implies that there is a sequence \( \{ s_k \}_{k=1}^\infty \subset \mathbb{R} \) such that \( s_k \to \infty \) and \( \omega^s_{\log^t} \to \omega^*_N \) as \( k \to \infty \). Arguing as in (8), we have
\[
\frac{1}{\delta(\omega^s_{\log^t})} \left( \log \frac{1}{c} \right)^{t/s_k} \leq \left( E^s_{\log^t}(\omega^s_{\log^t}) \right)^{1/s_k} = \left( E^s_{\log^t}(A, N) \right)^{1/s_k} \leq \left( E^s_{\log^t}(\omega^\infty_N) \right)^{1/s_k}
\]
\[
\leq \frac{1}{\delta(\omega^\infty_N)} \left( E^s_{\log^t}(\omega^\infty_N) \right)^{1/s_k}.
\]

Taking \( k \to \infty \), we obtain
\[
\delta_N(A) = \delta(\omega^\infty_N) \leq \delta(\omega^*_N).
\]

This means that \( \omega^*_N \) is also an \( N \)-point best-packing configuration on \( A \).

\[\square\]

**Proof of Theorem 2.2.** First of all, we show that \( g(s) \) is continuous on \((0, \infty)\). Let \( s > 0 \) and let \( \omega^s_{\log^t} \) be a minimal \( N \)-point \( s, \log^t \)-energy configuration on \( A \). Using Lemma 3.4 we obtain for any \( \omega^s_{\log^t} \),
\[
\liminf_{r \to s^-} \frac{g(r) - g(s)}{r - s} \geq \liminf_{r \to s^-} \frac{E^r_{\log^t}(\omega^s_{\log^t}) - E^s_{\log^t}(\omega^s_{\log^t})}{r - s}
\]
\[
\geq \lim_{r \to s^-} E_{\log^{t+1}}^{r, \log^t}(\omega_N^{s, \log^t}) = E_{\log^{t+1}}^{s, \log^t}(\omega_N^{s, \log^t}) \geq \sup_{r \to s^-} G_{\log^{t+1}}^{s}(A, N) > 0, \tag{9}
\]

and

\[
\limsup_{r \to s^-} \frac{g(r) - g(s)}{r - s} \leq \limsup_{r \to s^-} \frac{E_{\log^{t}}^{r, \log^t}(\omega_N^{r, \log^t}) - E_{\log^{t}}^{s, \log^t}(\omega_N^{r, \log^t})}{r - s} \leq \limsup_{r \to s^-} E_{\log^{t+1}}^{s, \log^t}(\omega_N^{r, \log^t}), \tag{10}
\]

where the second inequality in (9) follows from the arbitrariness of \( \omega_N^{s, \log^t} \) and the last inequality in (9) follows from Lemma 3.3.

Let \( \omega_N \) be a fixed configuration of \( N \) distinct points of \( A \). Note that \( 0 < \delta(\omega_N) < 1 \). For all \( r \in (s/2, s) \), we have

\[
\left( \frac{1}{\delta(\omega_N^{r, \log^t})} \right)^{s/2} \left( \log \frac{1}{\delta(\omega_N^{r, \log^t})} \right)^t \leq \left( \frac{1}{\delta(\omega_N^{r, \log^t})} \right)^r \left( \log \frac{1}{\delta(\omega_N^{r, \log^t})} \right)^t \leq E_{\log^{t}}^{r, \log^t}(\omega_N^{r, \log^t}) \leq E_{\log^{t}}^{r}(\omega_N^{r, \log^t}) \leq \left( \frac{1}{\delta(\omega_N)} \right)^r \left( \log \frac{1}{\delta(\omega_N)} \right)^t N(N - 1) \leq \left( \frac{1}{\delta(\omega_N)} \right)^s \left( \log \frac{1}{\delta(\omega_N)} \right)^t N(N - 1).
\]

That is,

\[
(\delta(\omega_N^{r, \log^t}))^{s/2} \left( \log \frac{1}{\delta(\omega_N^{r, \log^t})} \right)^{-t} \geq (\delta(\omega_N))^s \left( \log \frac{1}{\delta(\omega_N)} \right)^{-t} (N(N - 1))^{-1}.
\]

This implies that for all \( r \in (s/2, s) \),

\[
\delta(\omega_N^{r, \log^t}) \left( \log \frac{1}{\delta(\omega_N^{r, \log^t})} \right)^{-2t/s} \geq (\delta(\omega_N))^2 \left( \log \frac{1}{\delta(\omega_N)} \right)^{-2t/s} (N(N - 1))^{-2/s} =: c_1 > 0.
\]

Since by Lemma 3.1,

\[
h(x) := x \left( \log \frac{1}{x} \right)^{-\beta}, \quad \beta > 0,
\]

is a strictly increasing function on \((0, 1)\), there exists a constant \( c_2 > 0 \) such that for all \( r \in (s/2, s) \),

\[
\delta(\omega_N^{r, \log^t}) \geq c_2 > 0.
\]
Therefore, \( E_{\log^{t+1}}^{s} (\omega_N^{\log^t}) \) are bounded above where \( r \in (s/2, s) \). From this and (10),

\[
\limsup_{r \to s^-} \frac{g(r) - g(s)}{r - s} \leq \limsup_{r \to s^-} E_{\log^{t+1}}^{s} (\omega_N^{r, \log^t}) < \infty. \tag{11}
\]

Let \( s \geq 0 \). Using Lemma 3.3, we also obtain for any \( \omega_N^{s, \log^t} \),

\[
\limsup_{r \to s^+} \frac{g(r) - g(s)}{r - s} \leq \limsup_{r \to s^+} \frac{E_{\log^{t+1}}^{r} (\omega_N^{s, \log^t}) - E_{\log^{t+1}}^{s} (\omega_N^{s, \log^t})}{r - s} \leq \lim_{r \to s^+} E_{\log^{t+1}}^{s} (\omega_N^{s, \log^t}) = E_{\log^{t+1}}^{s} (\omega_N^{s, \log^t}) \leq \inf G_{\log^{t+1}}^{s} (A, N) < \infty, \tag{12}
\]

and

\[
\liminf_{r \to s^+} \frac{g(r) - g(s)}{r - s} \geq \liminf_{r \to s^+} \frac{E_{\log^{t+1}}^{r} (\omega_N^{s, \log^t}) - E_{\log^{t+1}}^{s} (\omega_N^{s, \log^t})}{r - s} \geq \liminf_{r \to s^+} E_{\log^{t+1}}^{s} (\omega_N^{r, \log^t}) > 0, \tag{13}
\]

where the second inequality in (12) follows from rom the arbitrariness of \( \omega_N^{s, \log^t} \) and the last inequality in (13) follows from Lemma 3.3.

The inequalities (9), (11), (12), and (13) imply that for all \( s > 0 \),

\[
0 < \liminf_{r \to s^-} \frac{g(r) - g(s)}{r - s} \leq \limsup_{r \to s^-} \frac{g(r) - g(s)}{r - s} < \infty \tag{14}
\]

and for all \( s \geq 0 \)

\[
0 < \liminf_{r \to s^+} \frac{g(r) - g(s)}{r - s} \leq \limsup_{r \to s^+} \frac{g(r) - g(s)}{r - s} < \infty. \tag{15}
\]

The inequalities in (14) and (15) further imply that \( g(s) \) is continuous for all \( s > 0 \) and is right continuous at \( s = 0 \).

\[ \square \]

**Proof of Theorem 2.3**  Let \( s_0 > 0 \). In order to show Theorem 2.3 it suffices to show that any cluster configuration of \( \omega_N^{s, \log^t} \) as \( s \to s_0^+ \) or as \( s \to s_0^- \) is a minimal \( N \)-point \( s_0, \log^t \)-energy configuration on \( A \).

Let \( \omega_N^{*} \) be a cluster configuration of \( \omega_N^{s, \log^t} \), as \( s \to s_0^+ \). Then, there is a sequence \( \{s_k\}_{k=1}^{\infty} \subset (s_0, \infty) \) such that \( s_k \to s_0 \) and \( \omega_N^{s_k, \log^t} \to \omega_N^{*} \) as \( k \to \infty \). Let \( \alpha = \text{diam}(A) \). For any configuration of \( N \) distinct points \( \omega_N \) on \( A \), notice that \( \alpha^s E_{\log^t}^{s} (\omega_N) \) is an increasing function of \( s \). Applying the continuity of \( g(s) := E_{\log^t}^{s} (A, N) \) at \( s_0 \), we have

\[
\alpha^{s_0} E_{\log^t}^{s_0} (\omega_N^*) = \lim_{k \to \infty} \alpha^{s_0} E_{\log^t}^{s_0} (\omega_N^{s_k, \log^t}) \leq \lim_{k \to \infty} \alpha^{s_k} E_{\log^t}^{s_k} (\omega_N^{s_k, \log^t})
\]
Then, using Lemma 3.1, there is a constant $c > 0$.

Proof of Theorem 2.4. Firstly, we show (6). Let $s \geq 0$ be fixed and $\{r_k\}_{k=1}^{\infty} \subset (s, \infty)$ be a sequence such that $r_k \to s$ as $k \to \infty$ and

$$\lim_{k \to \infty} E^{s}_{\log^i}(\omega_{r_k, log^i}) = \lim_{r \to s^+} E^{s}_{\log^i}(\omega_{r, log^i}).$$

(16)
Since $A^N$ is compact, there exists a subsequence $\{s_\ell\}_{\ell=1}^\infty \subset \{r_k\}_{k=1}^\infty$ such that
\[ \lim_{\ell \to \infty} \omega_{s_\ell}^{r_k, \log^t} = \omega_N^* \] (17)
and $\omega_N^*$ is a minimal $N$-point $s, \log^t$-energy configuration by Theorem 2.3. By
\[ \lim_{k \to \infty} E_{s \log^t+1}^s(\omega_{r_k}^{r_k, \log^t}) = \lim_{\ell \to \infty} E_{\log^t+1}^s(\omega_{s_\ell}^{s_\ell, \log^t}), \]
(12), (13), (16), and (17), we get
\[ \liminf_{r \to s^+} \frac{g(r) - g(s)}{r - s} \geq \liminf_{r \to s^+} E_{s \log^t+1}^s(\omega_{r}^{r, \log^t}) = \lim_{\ell \to \infty} E_{\log^t+1}^s(\omega_{s_\ell}^{s_\ell, \log^t}) \]
(18).

Then,
\[ g'_+(s) = \inf G_{\log^t+1}(A, N). \] (19)

It is easy to check that from Lemma 3.3, the constant $\inf G_{\log^t+1}(A, N)$ in (19) is
finite. This verifies (6).

Next, we prove (7). Let $s > 0$ be fixed and $\{r_k\}_{k=1}^\infty \subset [0, s)$ be a sequence such
that $r_k \to s$ as $k \to \infty$ and
\[ \lim_{k \to \infty} E_{s \log^t+1}^s(\omega_{r_k}^{r_k, \log^t}) = \limsup_{r \to s^-} E_{s \log^t+1}^s(\omega_{r}^{r, \log^t}). \] (20)

Because $A^N$ is compact, there exists a subsequence $\{s_\ell\}_{\ell=1}^\infty \subset \{r_k\}_{k=1}^\infty$ such that
\[ \lim_{\ell \to \infty} \omega_{s_\ell}^{r_k, \log^t} = \omega_{s_\ell}^* \]
and $\omega_{s_\ell}^*$ is a minimal $N$-point $s, \log^t$-energy configuration by Theorem 2.3. Then, we get
\[ \lim_{k \to \infty} E_{s \log^t+1}^s(\omega_{r_k}^{r_k, \log^t}) = \lim_{\ell \to \infty} E_{\log^t+1}^s(\omega_{s_\ell}^{s_\ell, \log^t}). \] (21)

Using (9), (10), (20), and (21), we obtain
\[ \liminf_{r \to s^-} \frac{g(r) - g(s)}{r - s} \geq \sup G_{s \log^t+1}(A, N) \geq E_{s \log^t+1}^s(\omega_{s_\ell}^{s_\ell, \log^t}) \]
\[ = \lim_{\ell \to \infty} E_{s \log^t+1}^s(\omega_{s_\ell}^{s_\ell, \log^t}) = \limsup_{r \to s^-} E_{s \log^t+1}^s(\omega_{r}^{r, \log^t}) \geq \limsup_{r \to s^-} \frac{g(r) - g(s)}{r - s}. \]
Then,

\[ g'(s) = \sup \mathcal{G}^{s}_{t+1}(A, N). \tag{22} \]

Next, we want to show that \( \sup \mathcal{G}^{s}_{t+1}(A, N) \) is finite. Let \( \omega_N \) be a fixed configuration of \( N \) distinct points on \( A \) and let \( \omega^{s, \log^t}_N \) be any minimal \( N \)-point configurations. Then,

\[
(\delta(\omega^{s, \log^t}_N))^{-s} \left( \log \frac{1}{\delta(\omega^{s, \log^t}_N)} \right)^t \leq E^{s}_{\log^t}(\omega^{s, \log^t}_N) \\
\leq E^{s}_{\log^t}(\omega_N) \leq (\delta(\omega_N))^{-s} \left( \log \frac{1}{\delta(\omega_N)} \right)^t N(N-1).
\]

That is,

\[
\delta(\omega^{s, \log^t}_N) \left( \log \frac{1}{\delta(\omega^{s, \log^t}_N)} \right)^{-t/s} \geq \delta(\omega_N) \left( \log \frac{1}{\delta(\omega_N)} \right)^{-t/s} (N(N-1))^{-1/s} =: c_1 > 0.
\]

It follows from Lemma 3.1 that there is a constant \( c_2 > 0 \) such that for any \( \omega^{s, \log^t}_N \),

\[
\delta(\omega^{s, \log^t}_N) \geq c_2 > 0.
\]

Since by Lemma 3.2

\[
p(x) := \frac{1}{x^s} \left( \log \frac{1}{x} \right)^{t+1},
\]

is a strictly decreasing function on \((0, 1)\), the set \( \mathcal{G}^{s}_{t+1}(A, N) \) is bounded above. This implies that \( \sup \mathcal{G}^{s}_{t+1}(A, N) \) in (22) is finite. Hence, (7) is proved. \( \square \)

**Proof of Theorem 2.5.** (a): This is a direct consequence of Theorem 2.4. (b): Let \( s_0 \geq 0 \) and \( \omega^{s_0}_N \) be a cluster configuration of \( \{\omega^{s, \log^t}_N\} \) as \( s \to s_0^+ \). Then, there exists a sequence \( \{s_k\}_{k=1}^{\infty} \subset (s_0, \infty) \) such that \( \lim_{k \to \infty} s_k = s_0 \) and \( \lim_{k \to \infty} \omega^{s_k, \log^t}_N = \omega^*_N \). Then, \( \omega^*_N \) is a minimal \( N \)-point \( s_0, \log^t \)-energy configuration by Theorem 2.3. Using (6) and the similar argument used to show (13), we have

\[
E^{s_0}_{\log^t+1}(\omega^*_N) = \lim_{k \to \infty} E^{s_0}_{\log^t+1}(\omega^{s_k, \log^t}_N) \leq \lim_{k \to \infty} \frac{g(s_k) - g(s_0)}{s_k - s_0} = g'_+(s_0) = \inf \mathcal{G}^{s_0}_{\log^t+1}(A, N).
\]

Since \( \inf \mathcal{G}^{s_0}_{\log^t+1}(A, N) \leq E^{s_0}_{\log^t+1}(\omega^*_N) \),

\[
E^{s_0}_{\log^t+1}(\omega^*_N) = \inf \mathcal{G}^{s_0}_{\log^t+1}(A, N) = g'_+(s_0).
\]
(c): Let $s_0 > 0$ and $\omega_N^{*\ast}$ be a cluster configuration of $\{\omega_N^{s,\log^t}\}$ as $s \to s_0^-$. Then, there exists a sequence $\{s_k\}_{k=1}^{\infty} \subset [0, s_0)$ such that $\lim_{k \to \infty} s_k = s_0$ and $\lim_{k \to \infty} \omega_N^{s_k,\log^t} = \omega_N^{*\ast}$. Then, $\omega_N^{*\ast}$ is a minimal $N$-point $s_0, \log^t$-energy configuration by Theorem 2.3. Using (7) and the similar argument used to show (11), we have

$$E_{\log^t+1}^{s_0}(\omega_N^{*\ast}) = \lim_{k \to \infty} E_{\log^t+1}^{s_0}(\omega_N^{s_k,\log^t}) \geq \lim_{k \to \infty} \frac{g(s_k) - g(s_0)}{s_k - s_0} = g'_-(s_0) = \sup G_{\log^t+1}^{s_0}(A, N).$$

Since $E_{\log^t+1}^{s_0}(\omega_N^{*\ast}) \leq \sup G_{\log^t+1}^{s_0}(A, N)$,

$$E_{\log^t+1}^{s_0}(\omega_N^{*\ast}) = \sup G_{\log^t+1}^{s_0}(A, N) = g'_-(s_0).$$

(d): This is a direct consequence of (b) and (c).

Proof of Proposition 2.1. Let $N \geq 2$, $s \geq 0$, $t \geq 1$, and $0 < \alpha < \pi^{-1}$. We prove this proposition using Lemma 3.5. The function $k : (0, 1) :\to \mathbb{R}$ in the lemma is

$$k(x) = \frac{1}{x^s} \left( \log \frac{1}{x} \right)^t.$$ 

By Lemma 3.2 $k(x)$ is strictly decreasing on $(0, 1)$. Since for all $x \in (0, 1)$,

$$k''(x) = \frac{1}{x^{s+2}} \left( \log \frac{1}{x} \right)^{-2+t} \left[ (-1 + t)t + (t + 2st) \log \frac{1}{x} + s(1 + s) \log^2 \frac{1}{x} \right] > 0, \quad (23)$$

$k(x)$ is strictly convex on $(0, 1)$. Hence, because the function $k(x)$ satisfies all required properties in Lemma 3.5, all minimal $N$-point $K$-energy configurations on $S^1_\alpha$ are configurations of $N$ distinct equally spaced points on $S^1_\alpha$ with respect to the arc length and vice versa.

Proof of Proposition 2.2. Let $N \geq 2$, $0 < \alpha < (e\pi)^{-1}$, and $s, t$ satisfy $s > 0, t \geq 0$ or $s = 0, t > 0$. We can use the same lines of reasoning as in the proof of Proposition 2.1 except the function $k$ is considered on $(0, 1/e)$ and for all $x \in (0, 1/e)$,

$$k''(x) = \frac{1}{x^{s+2}} \left( \log \frac{1}{x} \right)^{-2+t} \left[ (-1 + t)t + (t + 2st) \log \frac{1}{x} + s(1 + s) \log^2 \frac{1}{x} \right] \geq \frac{1}{x^{s+2}} \left( \log \frac{1}{x} \right)^{-2+t} \left[ t^2 + 2st \log \frac{1}{x} + s(1 + s) \log^2 \frac{1}{x} + \left( \log \frac{1}{x} - 1 \right) t \right] > 0.$$

Hence, because the function $k(x)$ satisfies all required properties in Lemma 3.5, Proposition 2.2 is proved.
Proof of Proposition 2.3. Let $N \geq 2$, $s \geq 0$, $t \geq 1$, and $0 < \alpha < 1/2$. Again, we want to use Lemma 3.5. The function $k : (0, \pi\alpha] \rightarrow \mathbb{R}$ in the lemma is

$$k(x) = \left(\frac{1}{2\alpha \sin(x/2\alpha)}\right)^s \left(\log\frac{1}{2\alpha \sin(x/2\alpha)}\right)^t.$$  

Since $2\alpha \sin(x/2\alpha)$ is strictly increasing on $(0, \pi\alpha]$ and $(1/x^s)(\log(1/x))^t$ is strictly decreasing on $(0, 1)$, $k(x)$ is strictly decreasing on $(0, \pi\alpha]$. Next, we want to show that $k(x)$ is strictly convex on $(0, \pi\alpha]$, i.e.

$$k''(x) > 0 \text{ for all } x \in (0, \pi\alpha). \quad (24)$$

To show (24), it suffices to show that $q''(x) > 0$ for all $x \in (0, \pi/2)$, where

$$q(x) := \left(\frac{1}{2\alpha \sin x}\right)^s \left(\log\frac{1}{2\alpha \sin x}\right)^t.$$  

Because for all $x \in (0, \pi/2)$,

$$q''(x) = s(cot^2 x)(2\alpha \sin x)^{-s} \left(\log\frac{1}{2\alpha \sin x}\right)^{t-1} + (t - 1)(cot^2 x)(2\alpha \sin x)^{-s} \left(\log\frac{1}{2\alpha \sin x}\right)^{t-2} \left(s \log\frac{1}{2\alpha \sin x} + t\right) + (csc^2 x + s cot^2 x)(2\alpha \sin x)^{-s} \left(\log\frac{1}{2\alpha \sin x}\right)^{t-1} \left(s \log\frac{1}{2\alpha \sin x} + t\right) > 0,$$

$k(x)$ is strictly convex on $(0, \pi\alpha]$. Hence, the function $k(x)$ satisfies all required properties in Lemma 3.5. This completes the proof. \qed

5 Discussion and Conclusions

We introduce minimal $N$-point $s, \log^t$-energy constants and configurations of an infinite compact metric space $(A, d)$. Such constants and configurations are generated using the kernel

$$K_{log^t}^s(x, y) = \frac{1}{d(x, y)^s} \left(\log\frac{1}{d(x, y)}\right)^t, \quad s \geq 0, \quad t \geq 0.$$  

In this paper, we study the asymptotic properties of minimal $N$-point $s, \log^t$-energy constants and configurations of $A$ with $\text{diam}(A) < 1$, and $N \geq 2$ and $t \geq 0$ are fixed. We show that the $s, \log^t$-energy

$$g(s) := \mathcal{E}_{log^t}^s(A, N)$$

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is continuous and right differentiable on $[0, \infty)$ and is left differentiable on $(0, \infty)$ in Theorems 2.2 and 2.4. The further analysis on the differentiability of $g(s)$ can be found in Theorem 2.5. In Theorem 2.1, we show that

$$
\lim_{s \to \infty} \left( \frac{E_s^{\log t}(A,N)}{s} \right)^{1/s} = \frac{1}{\delta_N(A)}.
$$

and every cluster configuration of $\omega_{s}^{s, \log t}$ as $s \to \infty$ is an $N$-point best-packing configuration on $A$. Furthermore, we show in Theorem 2.3 that for any $s_0 > 0$, any cluster configuration of $\omega_{s}^{s, \log t}$, as $s \to s_0$, is a minimal $N$-point $s_0, \log t$-energy configuration on $A$. When $\text{diam}(A) < 1$, our theorems generalize Theorems A, B, and C. The natural question would be “Do Theorems 2.1-2.5 hold true for $\text{diam}(A) \geq 1$?”

Investigation on arrangements of $\omega_{s}^{s, \log t}$ on circles in $\mathbb{R}^2$ is in Propositions 2.1-2.3. In these propositions, we show that for certain values of $s$ and $t$, all minimal $N$-point $\log t$-energy configurations on $S^1_\alpha$ with $\text{diam}(S^1_\alpha) < 1$ (corresponding to the Euclidean and geodesic distances) are the configurations of $N$ distinct equally spaced points. We would like to report that the Lemma 3.5 does not allow us to say something when $\text{diam}(S^1_\alpha) \geq 1$. It would be very interesting to develop a new tool to attack the case when $\text{diam}(S^1_\alpha) \geq 1$.

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