PERVERSE SHEAVES ON REAL LOOP GRASSMANNIANS

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ABSTRACT. The aim of this paper is to identify a certain tensor category of perverse sheaves on the loop Grassmannian $\text{Gr}_R$ of a real form $G_R$ of a connected reductive complex algebraic group $G$ with the category of finite-dimensional representations of a connected reductive complex algebraic subgroup $\hat{H}$ of the dual group $\hat{G}$. The root system of $\hat{H}$ is closely related to the restricted root system of $G_R$. The fact that $\hat{H}$ is reductive implies that an interesting family of real algebraic maps satisfies the conclusion of the Decomposition Theorem of Beilinson-Bernstein-Deligne.

1. Introduction

It is a general principle that the representation theory of a connected reductive complex algebraic group $G$ is reflected in the geometry of its dual group $\hat{G}$. Although it is simple to define $\hat{G}$ – it is the reductive group with based root datum dual to the based root datum of $G$ – the duality is nevertheless mysterious. One way to concretely obtain $\hat{G}$ from $G$ is to study perverse sheaves on the loop Grassmannian $\text{Gr}$ of $G$. A certain tensor category of perverse sheaves on $\text{Gr}$ is equivalent to the category $\text{Rep}(\hat{G})$ of finite-dimensional representations of $\hat{G}$ [Lus83, Gin96, BD, MV00, MV04]. (See [MV04, Theorem 7.3] for a final account.) The result is fundamental in the geometric Langlands program [BD, FGV01, BG02]. It also leads to a construction of canonical bases [MV00, BG01, MV04], and a deeper understanding of the Satake isomorphism [Gai01], among other applications [BFGM02]. It also may be interpreted as providing a down-to-earth perspective on the duality itself. For example, according to this approach, the dual group of $\text{GL}(V)$ is canonically $\text{GL}(H^*(\mathbb{P}(V)))$, where $\mathbb{P}(V)$ denotes the space of lines in $V$, and $H^*(\mathbb{P}(V))$ its cohomology.

In this paper, we study perverse sheaves on the loop Grassmannian $\text{Gr}_R$ of a real form $G_R$ of $G$. Although the theory of perverse sheaves as developed in [BBD82] is primarily for complex algebraic spaces, it is possible to consider certain perverse sheaves on $\text{Gr}_R$. The reason is that the natural finite-dimensional stratification of $\text{Gr}_R$ is real even-codimensional. It turns out that some statements about sheaves on complex algebraic spaces – the frequent vanishing of odd-dimensional intersection Betti numbers, the Decomposition Theorem [BBD82, Théorème 6.2.5] for pushforwards of intersection cohomology sheaves – also hold for certain sheaves on $\text{Gr}_R$ though it is only real algebraic. The topological results of this paper are all consequences of the main result and its proof. It states that a certain category of perverse sheaves on $\text{Gr}_R$ is a tensor category equivalent to the category $\text{Rep}(\hat{H})$ of finite-dimensional representations of a connected reductive complex algebraic subgroup $\hat{H} \subset \hat{G}$. 

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In the remainder of the introduction, we discuss the associated subgroup $\tilde{H} \subset \tilde{G}$, and then sketch some of the geometry involved in its construction.

1.1. The associated subgroup. In general, the subgroup $\tilde{H} \subset \tilde{G}$ associated to a real form $G_\mathbb{R}$ of $G$ may be realized as the identity component of the fixed points of an involution of a certain Levi subgroup $\tilde{L}_1 \subset \tilde{G}$. The description simplifies in the following circumstances. A real form $G_\mathbb{R}$ is called \textit{quasi-split} if the complexification $P \subset G$ of a minimal parabolic subgroup $P_\mathbb{R} \subset G_\mathbb{R}$ is a Borel subgroup of $G$. There is a canonical bijection from the set of conjugacy classes of quasi-split real forms to the set of involutions of the based root datum of $G$, and so also to the set of involutions of the based root datum of $\tilde{G}$. Suppose that the real form $G_\mathbb{R}$ is quasi-split. Then the Levi subgroup $\tilde{L}_1$ is the entire dual group $\tilde{G}$. Suppose in addition that the involution of the root datum of $\tilde{G}$ associated to the conjugacy class of $G_\mathbb{R}$ has the property that it fixes a node in each component of the Dynkin diagram of $\tilde{G}$ that it preserves. Then the subgroup $\tilde{H} \subset \tilde{G}$ is the identity component of the fixed points of a lift to $\tilde{G}$ of the involution of the based root datum of $\tilde{G}$ associated to the conjugacy class of $G_\mathbb{R}$.

Section 10.7 contains a concrete description of the subgroup $\tilde{H} \subset \tilde{G}$ in general. The discussion there is self-contained, and the interested reader may consult it independently.

We list here some basic properties of the subgroup $\tilde{H} \subset \tilde{G}$. The duality between the groups $G$ and $\tilde{G}$ descends to a duality between their Lie algebras $\mathfrak{g}$ and $\mathfrak{g}$. An isogeny of groups $G^1 \to G^2$ with Lie algebra $\mathfrak{g}$ is dual to an isogeny $\tilde{G}^1 \to \tilde{G}^2$ of groups with Lie algebra $\mathfrak{g}$. Similarly, the association of the subgroup $\tilde{H} \subset \tilde{G}$ to the real form $G_\mathbb{R}$ descends to an association of the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to the real form $\mathfrak{g}_\mathbb{R}$. An isogeny $G^1 \to G^2$ which commutes with conjugation leads to an isogeny $\tilde{G}^2 \to \tilde{G}^1$ which restricts to an isogeny $\tilde{H}^2 \to \tilde{H}^1$. In addition, the subgroup $\tilde{H} \subset \tilde{G}$ associated to a product of real forms is the product of the associated subgroups.

It is possible to read off some invariants of the root system of $\mathfrak{h}$ directly from the restricted root system of $\mathfrak{g}_\mathbb{R}$. For example, the rank of $\mathfrak{h}$ is equal to the real rank of $\mathfrak{g}_\mathbb{R}$, and the Weyl group of $\mathfrak{h}$ is isomorphic to the small Weyl group of $\mathfrak{g}_\mathbb{R}$. We refer the reader to Table 1 for a list of the associated Lie algebras $\mathfrak{h}$ for non-compact real forms $\mathfrak{g}_\mathbb{R}$ with simple complexification $\mathfrak{g}$. It is worth pointing out that when $\mathfrak{g}$ is simple, $\mathfrak{h}$ is simple as well.

When the real form $G_\mathbb{R}$ is compact, the real loop Grassmannian $\text{Gr}_\mathbb{R}$ is a single point, and the associated subgroup is the identity $\tilde{H} = (1) \subset \tilde{G}$. A real form $G_\mathbb{R}$ is called \textit{split} if there is a torus $T_\mathbb{R} \subset G_\mathbb{R}$ isomorphic to a product of copies of $\mathbb{R}^\times$ such that its complexification $T \subset G$ is a maximal torus of $G$. When the real form $G_\mathbb{R}$ is split, the associated subgroup is the entire dual group $\tilde{H} = \tilde{G}$. In this case, the fact that $\tilde{H}$ coincides with $\tilde{G}$ implies interesting topological statements. The case of a quasi-split real form $G_\mathbb{R}$ is also noteworthy for topological reasons, and we shall return to it shortly.

1.2. Sketch of geometry. Let $K = \mathbb{C}((t))$ be the field of formal Laurent series, and let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of formal power series. Let $G(K)$ be the group of $K$-valued points of $G$, and let $G(\mathcal{O})$ be the group of $\mathcal{O}$-valued points. The quotient set $G(K)/G(\mathcal{O})$ is the $\mathbb{C}$-points of a (not necessarily reduced) ind-finite type complex algebraic ind-scheme. In this paper, we shall only be interested in the space of $\mathbb{C}$-points of this ind-scheme.
**Table 1.** Associated Lie algebras $\mathfrak{h}$ for non-compact real Lie algebras $\mathfrak{g}_R$ with simple complexifications $\mathfrak{g}$. Notation following É. Cartan, and [He78].

| $\mathfrak{g}_R$ | $\mathfrak{g}$ | $\mathfrak{g}$ | $\mathfrak{h}$ | Remarks |
|------------------|----------------|----------------|----------------|---------|
| AI $\mathfrak{sl}_n(\mathbb{R})$ | $\mathfrak{sl}_n(\mathbb{C})$ | $\mathfrak{sl}_n(\mathbb{C})$ | $\mathfrak{sl}_n(\mathbb{C})$ | split |
| AII $\mathfrak{sl}_n(\mathbb{R})$ | $\mathfrak{sl}_n(\mathbb{C})$ | $\mathfrak{sl}_n(\mathbb{C})$ | $\mathfrak{sl}_n(\mathbb{C})$ | $p \leq q$ |
| AIII/AIV $\mathfrak{sp}(p, q)$ | $\mathfrak{sp}_p(\mathbb{C})$ | $\mathfrak{sp}_p(\mathbb{C})$ | $\mathfrak{sp}_p(\mathbb{C})$ | quasi-split if $q = p$ or $q = p + 1$ |
| BI/BII $\mathfrak{so}(n, n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| CI $\mathfrak{sp}(p, q)$ | $\mathfrak{sp}_p(\mathbb{C})$ | $\mathfrak{sp}_p(\mathbb{C})$ | $\mathfrak{sp}_p(\mathbb{C})$ | $p \leq q$ |
| CII $\mathfrak{so}(n, n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $p + q = n$ |
| DI/DII $\mathfrak{so}(n, n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| DIII $\mathfrak{so}(n, n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $p + q = 2n + 1$ |
| E | $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $p = n/2$ |
| EI $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| EII $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | quasi-split |
| EIII $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| EIV $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $p = n/2$ |
| EV $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| EVI $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| EVII $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| EVIII $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| EX $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| F | $\mathfrak{sp}(n, n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| FII $\mathfrak{sp}(n, n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |
| G $\mathfrak{so}(2n, 2n)$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | $\mathfrak{so}_n(\mathbb{C})$ | split |

equipped with its classical topology. We call this space the loop Grassmannian of $G$, and denote it by Gr. The filtration by order of pole exhibits Gr as an increasing union of projective varieties. As a topological space, Gr is homeomorphic to the space of based loops from a circle $S^1$ to a compact form $G_c$ of $G$ whose Fourier expansions are polynomial. It is homotopy equivalent to the space of continuous based loops from $S^1$ to $G$. It is called a Grassmannian since it may also be realized as a certain collection of subspaces in the infinite-dimensional $C$-vector space $\mathfrak{g}(K)$. See [Lus83, Section 11] and [PPS6] Chapter 8] for more details. Although the latter work in the context of
polynomial loops, the polynomial version of the loop Grassmannian is isomorphic to Gr.

The action of $G(\mathcal{O})$ on Gr by left-multiplication refines the filtration by order of pole. The orbits of the action are parameterized by the dominant weights $\lambda$ of the dual group $\hat{G}$. Lusztig [Lus83 Section 11, c)] discovered that the local invariants of the intersection cohomology sheaf $IC^\lambda$ of an orbit closure coincide with the weight multiplicities of the corresponding irreducible representation $V_\lambda$ of $\hat{G}$. In particular, he showed that the dimension $\dim \mathbb{H}(\text{Gr}, IC^\lambda)$ of the hypercohomology of the intersection cohomology sheaf is equal to the dimension $\dim V_\lambda$ of the corresponding representation. Furthermore, he showed that the decomposition of the convolution of such sheaves agrees with the decomposition of the tensor product of irreducible representations.

Thanks to further work by Ginzburg [Gin96], Beilinson-Drinfeld [BD], and Mirković-Vilonen [MV00] [MV04], we have the following.

**Theorem 1.2.1** ([MV04], Theorem 7.3, Proposition 6.3). The category $\mathcal{P}_{G(\mathcal{O})}(\text{Gr})$ of $G(\mathcal{O})$-equivariant perverse sheaves on the loop Grassmannian Gr of a connected reductive complex algebraic group $G$ is a tensor category equivalent to the category $\text{Rep}(\hat{G})$ of finite-dimensional representations of the dual group $\hat{G}$. Under the equivalence, the hypercohomology of a perverse sheaf corresponds to the underlying vector space of a representation.

**Remark 1.2.1.** In [MV04, Theorem 12.1], it is shown that for any commutative, unital, Noetherian ring $R$ of finite global dimension, the category of $G(\mathcal{O})$-equivariant perverse sheaves with $R$-coefficients on Gr is equivalent to the category of finite-dimensional $R$-representations of the canonical smooth, split, reductive group scheme $\hat{G}_R$ over $R$ whose root datum is dual to that of $G$. In this paper, we work only with sheaves with complex coefficients.

We mention here two main ingredients in the proof of the theorem. First, using the Beilinson-Drinfeld Grassmannian of $G$ over a curve $X$, it is possible to reformulate the convolution of perverse sheaves in a form which is clearly commutative [MV04, Section 5]. Second, using the perverse cells of Mirković-Vilonen, it is possible to decompose the hypercohomology of a perverse sheaf into weight spaces [MV04, Theorem 3.6]. The ideas involved in these two constructions are essential to the arguments of this paper. In an appendix, we explain why in the theorem one obtains the dual group.

It is also worth mentioning that the paper [MV04] cites this one for two technical points. Lest there appear to be a logical loop, we take a moment to comment on this. First, in Section 2 we discuss the formalism we have adopted in order to work with sheaves on infinite-dimensional spaces such as the loop Grassmannian Gr. The discussion is quite general and depends on no other results. The paper [MV04] uses these conventions. Second, in working with certain sheaves on the Beilinson-Drinfeld Grassmannian over a curve $X$, it is often useful to formalize the fact that they are constant along $X$. This is explained in Section 5.2. When working over $\mathbb{A}^1$, it may also be accomplished by choosing a global coordinate. This is the approach taken in [MV04] Sections 5 and 6]. They mention that to extend their results, one could adopt the formalism explained in this paper. In any case, this paper contains proofs of all assertions which are needed but which are not explicitly in [MV04].
Now let $G_{\mathbb{R}}$ be a real form of $G$. The conjugation of $G$ with respect to $G_{\mathbb{R}}$ induces a conjugation of $Gr$. We call the subspace of fixed points of the induced conjugation the real loop Grassmannian of $G_{\mathbb{R}}$, and denote it by $Gr_{\mathbb{R}}$. The conjugation of $G_{\mathbb{R}}$ with respect to $G_{\mathbb{R}}$ induces a conjugation of $Gr_{\mathbb{R}}$. We call the subspace of fixed points of the induced conjugation the real loop Grassmannian of $G_{\mathbb{R}}$, and denote it by $Gr_{\mathbb{R}}$. Let $\mathcal{O}_{\mathbb{R}} = \mathbb{R}[[t]]$ be the ring of real formal power series, and let $K_{\mathbb{R}} = \mathbb{R}((t))$ be the field of real formal Laurent series. We may identify $Gr_{\mathbb{R}}$ with the quotient $G_{\mathbb{R}}(K_{\mathbb{R}})/G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})$ where $G_{\mathbb{R}}(K_{\mathbb{R}})$ is the group of $K_{\mathbb{R}}$-valued points of $G_{\mathbb{R}}$, and $G(\mathcal{O}_{\mathbb{R}})$ is the group of $\mathcal{O}_{\mathbb{R}}$-valued points. Since the conjugation of $Gr_{\mathbb{R}}$ preserves the filtration by order of pole, $Gr_{\mathbb{R}}$ is an increasing union of real projective varieties. It was first recognized by Quillen, then proved in [Mit88, Theorems 5.1 and 5.2], that as a topological space, $Gr_{\mathbb{R}}$ is homotopy equivalent to the based loop space of the symmetric variety $G/K$ where $K$ is the complexification of a maximal compact subgroup of $G_{\mathbb{R}}$. It also may be realized as a certain collection of subspaces in the infinite-dimensional $\mathbb{R}$-vector space $g_{\mathbb{R}}(K_{\mathbb{R}})$.

Each $G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})$-orbit in $Gr_{\mathbb{R}}$ is finite-dimensional and real even-codimensional in the closure of any other. Thus the components of $Gr_{\mathbb{R}}$ are of two types: those containing only even-dimensional $G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})$-orbits, and those containing only odd-dimensional $G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})$-orbits. In this paper, we restrict our attention to sheaves supported on the union $Gr_{\mathbb{R}}^+$ of the components of $Gr_{\mathbb{R}}$ containing even-dimensional orbits. The topological aspects of the theory of perverse sheaves hold for such a stratified space. For example, there is a self-dual perversity, and the resulting category of perverse sheaves is abelian with simple objects the intersection cohomology sheaves with irreducible local system coefficients.

The category $P_{G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})}(Gr_{\mathbb{R}}^+)$ of $G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})$-equivariant perverse sheaves on $Gr_{\mathbb{R}}^+$ is more complicated than the category $P_{G(\mathcal{O})}(Gr)$ of $G(\mathcal{O})$-equivariant sheaves on $Gr$. For example, it follows from the results of Lusztig [Lus83, Section 11] that $P_{G(\mathcal{O})}(Gr)$ is semisimple, and the hypercohomology is an exact and faithful functor. Neither of these statements is true in general for $P_{G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})}(Gr_{\mathbb{R}}^+)$. In this paper, we restrict our attention to a certain strict full subcategory $Q(Gr_{\mathbb{R}})$. Roughly speaking, we shall only consider perverse sheaves built out of the intersection cohomology sheaves of strata with irreducible local system coefficients.

To define the category $Q(Gr_{\mathbb{R}})$, let $X$ be a smooth complex algebraic curve with $X_{\mathbb{R}}$ a nonempty real form of $X$. There is a real algebraic family over $X$ with fiber $Gr$ over points of $X \setminus X_{\mathbb{R}}$, and fiber $Gr_{\mathbb{R}}$ over points of $X_{\mathbb{R}}$. It is a real form of the Beilinson-Drinfeld Grassmannian of $G$. We define the specialization functor

$$R : P_{G(\mathcal{O})}(Gr) \to D_{G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})}(Gr_{\mathbb{R}})$$

to be the nearby cycles in this family. The specialization takes a sheaf on $Gr$ to a sheaf supported on the components $Gr_{\mathbb{R}}^+$. Therefore we may define the perverse specialization functor

$$p^R : P_{G(\mathcal{O})}(Gr) \to P_{G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})}(Gr_{\mathbb{R}}^+)$$

to be the direct sum of the perverse homology sheaves of the specialization

$$p^R = \sum_k p^H_k \circ R.$$
We take $Q(Gr_R)$ to be the strict full subcategory of $P_{G_R(O)}(Gr_R^+)$ consisting of objects isomorphic to subquotients of objects of the form $p_R(\mathcal{P})$, as $\mathcal{P}$ runs through all objects in $P_{G(O)}(Gr)$. (For the sake of simplicity, we have ignored here the natural $\mathbb{Z}$-grading on the perverse specialization $p_R$, but it plays an important role in the identification of the category $Q(Gr_R).$)

Our main result is the following.

**Theorem 1.2.2.** The category $Q(Gr_R)$ is a tensor category equivalent to the category $Rep(\tilde{H})$ of finite-dimensional representations of a connected reductive complex algebraic subgroup $\tilde{H} \subset \tilde{G}$. Under the equivalence, the hypercohomology of a perverse sheaf corresponds to the underlying vector space of a representation, and the perverse specialization corresponds to the restriction functor on representations.

As mentioned earlier, Section 10.7 contains a concrete description of the subgroup $\tilde{H} \subset \tilde{G}$.

The following topological corollary was one of our original motivations for establishing the equivalence of categories of the theorem. It asserts in particular that there is an interesting family of real algebraic maps that satisfy the conclusion of the Decomposition Theorem of Beilinson-Bernstein-Deligne [BBD82, Théorème 6.2.5].

**Corollary 1.2.1.** The category $Q(Gr_R)$ is semisimple. Each object is isomorphic to a direct sum of intersection cohomology sheaves with constant coefficients. In particular, the convolution and perverse specialization are semisimple.

Finally, we point out an interesting aspect of the construction of the category $Q(Gr_R)$. It is a fundamental result in the theory of perverse sheaves that the nearby cycles in a complex algebraic family over a curve are perverse. (See [BBD82, Section 4.4] or [GM83, Section 6.5].) But the nearby cycles in a real algebraic family are not in general perverse.

**Theorem 1.2.3.** The specialization

$$R : P_{G_R(O)}(Gr) \to D_{G_R(O)}(Gr_R^+)$$

is perverse if and only if the real form $G_R$ is quasi-split.

1.3. **Organization of paper.** We conclude the introduction with a brief summary of the contents of the other sections of the paper.

In Section 2 we collect notation used throughout the paper, and then establish the formalism we have adopted in order to work with infinite-dimensional spaces such as the loop Grassmannian $Gr$. The upshot is that for our needs we may discuss such spaces as if they were finite-dimensional. In Section 3 we give a brief account of perverse sheaves on the loop Grassmannian $Gr$, and then collect those results which extend easily to the real loop Grassmannian $Gr_R$. In Section 4 we describe the Beilinson-Drinfeld Grassmannian over a curve and its real forms. We then prove some basic results concerning their natural embeddings and stratifications. In Section 5 we define specialization functors which take sheaves on the loop Grassmannian $Gr$ to sheaves on its real form $Gr_R$. In Section 6 we prove that these specialization functors are monoidal. In Section 7 we introduce the category $Q(Gr_R)$, and its graded versions.
In Section 8, we recall the construction of weight functors due to Mirković-Vilonen. We describe their basic properties, then apply them to understand the specialization functors. In Section 9, we show that $\mathbb{Q}(Gr_R)$ is a neutral Tannakian category. Finally, in Section 10, we apply Tannakian formalism to the category $\mathbb{Q}(Gr_R)$ and identify the corresponding group $\tilde{H}$.

We have also included an appendix sketching how one identifies the Tannakian group of the category $\mathcal{P}_{G(O)}(Gr)$ of $G(O)$-equivariant perverse sheaves on Gr with the dual group $\tilde{G}$.

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2. Preliminaries

2.1. Notation. Throughout this paper, $G$ will be a connected reductive complex algebraic group, $\theta$ will be a conjugation of $G$, and $G_R$ will be the real form of $G$ with respect to $\theta$.

Choose once and for all a maximal split torus $S_R \subset G_R$, and a maximal torus $T_R \subset G_R$ such that $S_R \subset T_R$. By definition, the torus $S_R$ is isomorphic to $(\mathbb{R}^\times)^r$, for some $r$, and maximal among subgroups of $G_R$ with this property. Let $S \subset G$ be the complexification of $S_R$, and let $T \subset G$ be the complexification of $T_R$.

Choose a minimal parabolic subgroup $P_R \subset G_R$ such that $T_R \subset P_R$, and a Levi factor $M_R \subset P_R$ such that $T_R \subset M_R$. Let $P \subset G$ be the complexification of $P_R$, and let $M \subset G$ be the complexification of $M_R$.

Choose a Borel subgroup $B \subset G$ such that $T \subset B$, and $B \subset P$.

2.1.1. Notation for $G$. Let $\Lambda_T = \text{Hom}(\mathbb{C}^\times, T)$ be the lattice of coweights of $T$, and let $\Lambda_T^+ \subset \Lambda_T$ be the semigroup of coweights dominant with respect to $B$.

Let $R \subset \Lambda_T$ be the set of coroots of $G$ with respect to $T$, let $R^{\text{pos}} \subset R$ be the set of positive coroots with respect to $B$, and let $\Delta_{B,T} \subset R^{\text{pos}}$ be the set of simple coroots.

Let $Q \subset \Lambda_T$ be the lattice generated by $R$, and let $Q^{\text{pos}} \subset Q$ be the semigroup generated by $R^{\text{pos}}$.

The coweight lattice $\Lambda_T$ is naturally ordered: $\lambda, \mu \in \Lambda_T$ satisfy $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a non-negative integral linear combination of positive coroots of $G$.

Let $W_G = N_G(T)/T$ be the Weyl group of $G$, where $N_G(T) \subset G$ is the normalizer of $T$. It acts naturally on the coweight lattice $\Lambda_T$, and there is a unique dominant coweight in each orbit.

Let $2\rho$ be the sum of the positive roots of $G$, and let $\langle 2\rho, \lambda \rangle \in \mathbb{Z}$ be the natural pairing for a coweight $\lambda \in \Lambda_T$. 

2.1.2. Notation for $G_{\mathbb{R}}$. We call $\Lambda_S = \text{Hom}(\mathbb{C}^\times, S)$ the lattice of real coweights, and we call $\Lambda^+_S = \Lambda_S \cap \Lambda^+_T$ the semigroup of dominant real coweights.

The real coweight lattice $\Lambda_S$ is naturally ordered by the restriction of the ordering from the coweight lattice $\Lambda_T$: $\lambda, \mu \in \Lambda_S$ satisfy $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a non-negative integral linear combination of positive coroots of $G$.

Let $W_{G_{\mathbb{R}}} = N_{G_{\mathbb{R}}}(S_{\mathbb{R}})/Z_{G_{\mathbb{R}}}(S_{\mathbb{R}})$ be the small Weyl group of $G_{\mathbb{R}}$, where $N_{G_{\mathbb{R}}}(S_{\mathbb{R}}) \subset G_{\mathbb{R}}$ is the normalizer of $S_{\mathbb{R}}$, and $Z_{G_{\mathbb{R}}}(S_{\mathbb{R}}) \subset G_{\mathbb{R}}$ is the centralizer of $S_{\mathbb{R}}$. It acts naturally on the real coweight lattice $\Lambda_S$, and there is a unique dominant real coweight in each orbit.

The conjugation $\theta$ induces an involution of $\Lambda_T$ which we also denote by $\theta$. For a coweight $\lambda : \mathbb{C}^\times \to T$, the coweight $\theta(\lambda)$ is defined to be the composition $\theta(\lambda) : \mathbb{C}^\times \xrightarrow{c} \mathbb{C}^\times \xrightarrow{\lambda} \mathbb{R} \xrightarrow{\theta} T$ where $c$ is the standard conjugation of $\mathbb{C}^\times$ with respect to $\mathbb{R}^\times$.

The real coweight lattice $\Lambda_S$ is the fixed points of the involution $\theta$ of the coweight lattice $\Lambda_T$. Therefore we have the projection

$$\sigma : \Lambda_T \to \Lambda_S$$

$$\sigma(\lambda) = \theta(\lambda) + \lambda,$$

whose image we denote by $\sigma(\Lambda_T) \subset \Lambda_S$.

Let $2\rho_M$ be the sum of the positive roots of the Levi factor $M \subset P$. We have the projection

$$\Sigma : \Lambda_T \to \Lambda_S \times \mathbb{Z}$$

$$\Sigma(\lambda) = (\theta(\lambda) + \lambda, (2\rho_M, \lambda)),$$

whose image we denote by $\Sigma(\Lambda_T) \subset \Lambda_S \times \mathbb{Z}$.

The following is a useful characterization of the ordering on the real coweight lattice $\Lambda_S$.

Lemma 2.1.1. The intersection $\Lambda_S \cap Q^\text{pos}$ is generated by the elements $\alpha \in \Lambda_S \cap R^\text{pos}$, and $\sigma(\alpha) \in \Lambda_S$, for $\alpha \in R^\text{pos}$.

Proof. Since $\Lambda_S$ pairs trivially with $2\rho_M$, we may write $\beta \in \Lambda_S \cap Q^\text{pos}$ uniquely as a sum $\beta = \sum_i \alpha_i$ of simple coroots $\alpha_i \in \Delta_0$ of the Levi subgroup $L_0 \subset G$ that centralizes $2\rho_M$. Since $\theta$ preserves the set $\Delta_0$, and the sum $\beta = \sum_i \alpha_i$ is unique, $\theta$ also preserves the set $\{\alpha_i\}$ of simple coroots counted with multiplicities appearing in the sum. The assertion follows by induction on the size of this set. $\square$

2.1.3. Graded categories. Let $\text{Vect}$ be the category of finite-dimensional $\mathbb{C}$-vector spaces.

For a lattice $\Lambda$, we have the category $\text{Vect}_\Lambda$ of finite-dimensional $\Lambda$-graded vector spaces. It is canonically equivalent to the category $\text{Rep}(\hat{S}_\Lambda)$ of finite-dimensional representations of the torus $\hat{S}_\Lambda = \text{Spec}(\mathbb{C}[\Lambda])$.

For a lattice homomorphism $\tau : \Lambda_1 \to \Lambda_2$, we have the functor

$$\tau : \text{Vect}_{\Lambda_1} \to \text{Vect}_{\Lambda_2}$$

$$\tau(V)^\lambda = \sum_{\tau(\mu) = \lambda} V^\mu.$$
It is canonically isomorphic to the restriction functor $\text{Rep}(\tilde{S}_{\Lambda_2}) \to \text{Rep}(\tilde{S}_{\Lambda_1})$ coming from the group homomorphism $\tilde{S}_{\Lambda_2} \to \tilde{S}_{\Lambda_1}$ induced by the ring homomorphism $\mathbb{C}[\Lambda_1] \to \mathbb{C}[\Lambda_2]$.

More generally, for a lattice $\Lambda$, and a $\mathbb{C}$-linear abelian category $\mathcal{C}$, we have the $\mathbb{C}$-linear abelian category $\mathcal{C}_\Lambda = \mathcal{C} \otimes \text{Vect}_\Lambda$ whose objects are $\Lambda$-graded sums of objects of $\mathcal{C}$, and whose morphisms are $\Lambda$-graded sums of morphisms of $\mathcal{C}$.

For a lattice homomorphism $\tau : \Lambda_1 \to \Lambda_2$, we have the functor
\[
\tau : \mathcal{C}_{\Lambda_1} \to \mathcal{C}_{\Lambda_2}
\]
\[
\tau(X)^\Lambda = \sum_{\tau(\mu) = \lambda} X^\mu.
\]

When $\tau : \langle 0 \rangle \to \Lambda$ is the inclusion of zero, we obtain the fully faithful functor $e : \mathcal{C} \to \mathcal{C}_\Lambda$ that places objects and morphisms in degree zero, and when $\tau : \Lambda \to \langle 0 \rangle$ is the projection to zero, we obtain the forgetful functor $F : \mathcal{C}_\Lambda \to \mathcal{C}$ that forgets the grading on objects and morphisms.

2.2. Ind-schemes. Many of the spaces we discuss in this paper are infinite-dimensional. However, all of the geometry we study takes place in finite-dimensional approximations. The complication is that no single finite-dimensional approximation is sufficient. Instead, we consider compatible families of approximations. In this section, we describe the approach we have adopted to formalize this. All varieties and schemes are either real or complex, and not assumed to be irreducible.

Our basic object is a compatible family of varieties

\[
\begin{array}{cccc}
Z^\ell_0 & \to & Z^\ell_1 & \to & \cdots & \to & Z^\ell_k & \to & Z^\ell_{k+1} & \to & \cdots \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \cdots \\
Z^1_0 & \to & Z^1_1 & \to & \cdots & \to & Z^1_k & \to & Z^1_{k+1} & \to & \cdots \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \cdots \\
\vdots & & \vdots & & \cdots & & \vdots & & \vdots & & \cdots \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \cdots \\
Z^0_0 & \to & Z^0_1 & \to & \cdots & \to & Z^0_k & \to & Z^0_{k+1} & \to & \cdots
\end{array}
\]

satisfying the requirements:

1. The horizontal maps are closed embeddings.
2. The vertical maps are smooth fibrations such that their fibers are affine spaces for $\ell$ large relative to $k$.
3. All squares in the diagram are Cartesian.

Informally speaking, we are willing to throw away any bounded part of the diagram, and to consider the resulting diagram as equivalent. To formalize this, we consider the inverse limit $Z_k = \lim_{\leftarrow} Z^\ell_k$, and the direct limit $Z = \lim_{\to} Z_k$. We use the terms scheme and ind-scheme to refer to spaces which arise in this way.
We say that $Z$ is stratified if the varieties $Z^\ell_k$ are compatibly stratified in the sense that for any map in the diagram, the strata of the domain are the inverse images of the strata of the codomain. We only consider conjugations $\theta$ and real forms $Z_{\mathbb{R}}$ of complex $Z$ which are the limits of compatible families of conjugations and real forms. Moreover, the conjugations we consider always respect the stratifications considered, and the strata of $Z_{\mathbb{R}}$ are always taken to be the real forms of the strata of $Z$.

A special case of such a compatible family is when the vertical families are constant, and the diagram simplifies to

$$Z_0 \to Z_1 \to Z_2 \to \cdots \to Z_k \to \cdots.$$  

In this case, the direct limit $Z$ is an ind-scheme of ind-finite type.

Another special case is when the horizontal families are constant, and in addition each of the varieties in the family

$$L^0 \leftarrow L^1 \leftarrow L^2 \leftarrow \cdots \leftarrow L^\ell \leftarrow \cdots$$

is a linear algebraic group. In this case, the inverse limit $L$ is a group-scheme, although not usually of finite type. We only consider group-schemes of this form such that for large $\ell_1 < \ell_2$, the kernel of the projection $L^{\ell_1} \leftarrow L^{\ell_2}$ is unipotent. We call such group-schemes stable.

We only consider maps between ind-schemes which are the limit of compatible families of maps defined for $\ell$ large relative to $k$. By an action of a group-scheme $L = \lim \leftarrow L^\ell$ on an ind-scheme $Z = \lim \to \lim \gets Z^\ell_k$, we mean a compatible family of actions of the group-schemes $L^\ell$ on the varieties $Z^\ell_k$ defined for $\ell$ large relative to $k$. When the vertical maps $Z^\ell_k \leftarrow Z^\ell_{k+1}$ are the projections of $L^\ell$-torsors, we say the ind-scheme $Z$ is an $L$-torsor over the ind-scheme $Z_0$.

We next discuss categories of sheaves on ind-schemes. We work with sheaves of $\mathbb{C}$-vector spaces in the classical topology. For a variety $Z$ acted upon by a linear algebraic group $L$, we write $D_L(Z)$ for the derived category of $L$-equivariant sheaves on $Z$ with bounded cohomology sheaves. If $S$ is a locally-trivial stratification of $Z$, we write $D_{L,S}(Z)$ for the derived category of $L$-equivariant sheaves on $Z$ with bounded $S$-constructible cohomology sheaves. See [BL94a] for the definitions of these categories. Note that in the case when the orbits of $L$ in $Z$ coincide with the strata of $S$, then the forgetful functor $D_{L,S}(Z) \to D_L(Z)$ is an equivalence. We freely use the term sheaf to mean a complex of sheaves, and $S$-constructible to mean with $S$-constructible cohomology sheaves.

If an ind-scheme $Z = \lim \to Z_k$ of ind-finite type is acted upon by a stable group scheme $L = \lim \gets L^\ell$, we may construct a directed system of categories

$$\cdots \to D_L(Z_k) \to D_L(Z_{k+1}) \to \cdots$$

in which each of the maps is the direct image functor. We call the direct limit $D_L(Z) = \lim \to D_L(Z_k)$ the derived category of $L$-equivariant sheaves on $Z$. If $Z$ is stratified, we similarly define the derived category $D_{L,S}(Z)$ of $L$-equivariant, $S$-constructible sheaves on $Z$.

To work with sheaves on an ind-scheme $Z = \lim \to \lim \gets Z^\ell_k$ not necessarily of ind-finite type, we assume that $Z$ is stratified. If in addition $Z$ is acted upon by a stable
group-scheme \( L \) then we have a directed system of categories
\[
\cdots \to D_{L,t,S}(Z_k^\ell) \to D_{L,t+1,S}(Z_k^{\ell+1}) \to \cdots
\]
in which each of the maps is the pull-back functor. This sequence of categories stabilizes, because the projections \( Z_k^\ell \leftarrow Z_k^{\ell+1} \) are compatibly stratified and acyclic for \( \ell \) large relative to \( k \), and the kernels of the projections \( L^\ell \leftarrow L^{\ell+1} \) are unipotent for \( \ell_1 < \ell_2 \) large. We have the limit derived category \( D_{L,S}(Z_k) = \lim \to D_{L,t,S}(Z_k^\ell) \) of \( L \)-equivariant, \( S \)-constructible sheaves on \( Z_k \). Thanks to the stability, we have a fully faithful directed system of categories
\[
\cdots \to D_{L,S}(Z_k) \to D_{L,S}(Z_{k+1}) \to \cdots
\]
in which each of the maps is the direct image functor. We call the direct limit \( D_{L,S}(Z) = \lim \to D_{L,S}(Z_k) \) the derived category of \( L \)-equivariant, \( S \)-constructible sheaves on \( Z \).

We only consider functors between such categories of sheaves which are the limit of compatible families of functors defined for \( \ell \) large relative to \( k \). For a map \( f : Y \to Z \) which is the limit of maps \( f_k^\ell : Y_k^\ell \to Z_k^\ell \), to obtain well-defined limit functors \( f_* \), \( f_1 \), \( f^* \), \( f^! \), one must require certain properties of the maps \( f_k^\ell \) and the maps in the diagrams of \( Y \) and \( Z \). For example, for the pushforwards \( f_* \), \( f_! \), one must have a Cartesian diagram
\[
\begin{array}{ccc}
Y_k^\ell & \xrightarrow{f_k^\ell} & Z_k^\ell \\
\downarrow & & \downarrow \\
Y_k^\ell & \xrightarrow{f_k^!} & Z_k^! 
\end{array}
\]
Then by the standard identity for the composition of maps and the smooth base change isomorphism, the limit functors \( f_* \), \( f_! \) are well-defined. All of our maps satisfy the necessary requirements where we use such functors. The standard identities among such functors hold in this setting.

For an ind-scheme \( Z \) of ind-finite type, there is in general no dualizing object, but we do have the Verdier duality functor since the horizontal maps in the diagram of \( Z \) are closed embeddings.

We shall often make use of the following construction. Let \( X \) and \( Y \) be stratified ind-schemes of ind-finite type, let \( L \) and \( M \) be stable group-schemes, and let \( p : Z \to X \) be a stratified \( L \)-torsor. Assume that \( M \) acts on \( Z \) and \( X \) such that \( p : Z \to X \) is \( M \)-equivariant. Then the pull-back functor \( p^* : D_{M,S}(X) \to D_{M \times L,S}(Z) \) is an equivalence. If \( L \) also acts on \( Y \), then we may form the twisted product
\[
X \times_L Y = \lim \lim Z_k^\ell \times_L Y_k
\]
where the variety \( Z_k^\ell \times_L Y_k \) makes sense for \( \ell \) large relative to \( k \). The twisted product \( X \times_L Y \) is stratified by the twisted products of the strata of \( X \) and \( Y \). In this situation, we may define a functor
\[
\hat{\otimes} : D_{M,S}(X) \times D_{L,S}(Y) \to D_{M,S}(X \times_L Y)
\]
by the requirement that
\[
q^*(F_1 \hat{\otimes} F_2) = p^*(F_1) \otimes F_2
\]
where \( q : Z \times Y \to X \times_L Y \) is the natural projection.
Finally, suppose a real or complex ind-scheme $Z = \lim_{\rightarrow} Z_k$ of ind-finite type is stratified such that the strata are of even real dimension. If in addition $Z$ is acted upon by a stable group-scheme $L$, then we call the direct limit $P_{L,S}(Z) = \lim_{\rightarrow} P_{L,S}(Z_k)$ the category of $L$-equivariant, $S$-constructible perverse sheaves on $Z$. Here $P_{L,S}(Z_k)$ is the heart of $D_{L,S}(Z_k)$ with respect to the perverse $t$-structure defined by pulling back the perverse $t$-structure from $D_S(Z_k)$ via the forgetful functor $D_{L,S}(Z_k) \to D_S(Z_k)$.

Since the direct image functor is $t$-exact for a closed embedding, there is an induced $t$-structure on the limit category $D_{L,S}(Z)$, and $P_{L,S}(Z)$ is the heart of the category $D_{L,S}(Z)$ with respect to the induced $t$-structure. For any integer $k$, we have the functor $p^H_k : D_{L,S}(Z) \to P_{L,S}(Z)$ which assigns to a sheaf its $k$-th perverse homology sheaf.

3. Real loop Grassmannians

In this section, we recall some basic results about the loop Grassmannian $Gr$ of the connected reductive complex algebraic group $G$, then describe those results which extend readily to the real loop Grassmannian $Gr_R$ of the real form $G_R$.

3.1. Loop Grassmannians. Let $O = \mathbb{C}[[t]]$ be the ring of formal power series, and for $\ell \geq 0$, let $O^\ell \subset O$ be the ideal generated by $t^\ell \in O$, and let $J^\ell = O/O^\ell$ be the finite-dimensional quotient.

The group $\text{Aut}(O)$ of automorphisms of $O$ is the inverse limit of the linear algebraic groups $\text{Aut}(J^m)$, and for any $m \geq \ell > 0$, the kernel of the projection $\text{Aut}(J^m) \to \text{Aut}(J^\ell)$ is unipotent. Let $\text{Aut}(O^\ell) \subset \text{Aut}(O)$ be the kernel of the projection $\text{Aut}(O) \to \text{Aut}(J^\ell)$.

The group $G(O)$ of $O$-valued points of $G$ is the inverse limit of the linear algebraic groups $G(J^\ell)$, and for any $m \geq \ell > 0$, the kernel of the projection $G(J^m) \to G(J^\ell)$ is unipotent. Let $G(O^\ell) \subset G(O)$ be the kernel of the projection $G(O) \to G(J^\ell)$.

Let $K = \mathbb{C}((t))$ be the field of formal Laurent series, and for $k \geq 0$, let $K_k \subset K$ be the $O$-submodule generated by $t^{-k} \in K$.

The group $GL_N(K)$ of $K$-valued points of $GL_N$ is the direct limit of the schemes $GL_N(K)_k$ of matrices $g \in GL_N(K)$ such that the entries of $g$ and $g^{-1}$ lie in $K_k$.

Choose an embedding $G \subset GL_N$. The group $G(K)$ of $K$-valued points of $G$ is the direct limit of the schemes $G(K)_k = G(K) \cap GL_N(K)_k$, and the finite-dimensional varieties $G(k)_k = G(K)_k/G(O^\ell)$ are compatibly stratified by the orbits of $G(O)$ acting by multiplication on the left and right. This realizes $G(K)$ as a stratified ind-scheme, although not of ind-finite type.

The loop Grassmannian $Gr = G(K)/G(O)$ is the direct limit of the finite-dimensional projective varieties $Gr_k = G(K)_k/G(O^\ell)$ which are compatibly stratified by the orbits of $G(O)$ acting by left multiplication. This realizes $Gr$ as a stratified ind-scheme of ind-finite type.

3.2. Spherical perverse sheaves. To describe the $G(O)$-orbits in $Gr$, observe that each coweight $\lambda \in \Lambda_T$ defines a point $\lambda \in Gr$ via the embedding $\Lambda_T \subset G(K)$. Let $Gr^\lambda$ be the $G(O)$-orbit $G(O) \cdot \lambda \subset Gr$ through $\lambda \in Gr$. The Cartan decomposition [LM65, Corollary 2.17] states that each $G(O)$-orbit in $Gr$ is of the form $Gr^\lambda$ for some $\lambda \in \Lambda_T$, and two orbits $Gr^{\lambda_1}$ and $Gr^{\lambda_2}$ coincide if and only if $\lambda_1$ and $\lambda_2$ are in the same $W_G$-orbit.
in $\Lambda_T$. Note that the $G(\mathcal{O})$-orbit $\text{Gr}^\lambda$ contains the $G$-orbit $G \cdot \lambda$ which is isomorphic to the flag manifold $G/P^\lambda$ where the parabolic subgroup $P^\lambda \subset G$ is the stabilizer of $\lambda$. As $z \in \mathbb{C}^\times$ tends to 0, the automorphism of $\text{Gr}$ induced by the coordinate change $t \mapsto zt$ provides a retraction $\text{Gr}^\lambda \to G/P^\lambda$. The fibers of the retraction are the orbits of the pro-unipotent congruence subgroup $G(\mathcal{O}^1) \subset G(\mathcal{O})$. For more details and arguments for the other assertions of the following proposition, see for example [Lus83, Section 11, a) and b)] and [PS86, Section 8.6]. Although the latter work in the context of polynomial loops, the polynomial version of the loop Grassmannian is isomorphic to $\text{Gr}$.

**Proposition 3.2.1.** The loop Grassmannian $\text{Gr}$ is the disjoint union of the $G(\mathcal{O})$-orbits $\text{Gr}^\lambda$ through the dominant coweights $\lambda \in \Lambda_T^+$. The closure of the orbit $\text{Gr}^\lambda$ is the union of the orbits through $\mu \in \Lambda_T^+$ with $\mu \leq \lambda$. The orbit $\text{Gr}^\lambda$ is a vector bundle over the flag manifold $G/P^\lambda$, and its closure is a projective variety of dimension $2\langle \rho, \lambda \rangle$.

For $k, m \geq 0$, there exists $\ell \geq 0$ such that $G(\mathcal{O}^\ell) \rtimes \text{Aut}(\mathcal{O}^\ell)$ acts trivially on $G(\mathcal{K})_k^m$, and the action of $G(\mathcal{O}) \rtimes \text{Aut}(\mathcal{O})$ passes to $G(\mathcal{J}^\ell) \rtimes \text{Aut}(\mathcal{J}^\ell)$. In particular, for $k \geq 0$, there exists $\ell \geq 0$ such that $G(\mathcal{O}^\ell) \rtimes \text{Aut}(\mathcal{O}^\ell)$ acts trivially on $\text{Gr}_k$, and the action of $G(\mathcal{O}) \rtimes \text{Aut}(\mathcal{O})$ passes to $G(\mathcal{J}^\ell) \rtimes \text{Aut}(\mathcal{J}^\ell)$. Therefore $G(\mathcal{O}) \rtimes \text{Aut}(\mathcal{O})$ acts on $\text{Gr}$ according to our conventions.

We have the category $P_{G(\mathcal{O}) \rtimes \text{Aut}(\mathcal{O})}(\text{Gr})$ of $G(\mathcal{O}) \rtimes \text{Aut}(\mathcal{O})$-equivariant perverse sheaves on $\text{Gr}$, and the category $P_{G(\mathcal{O})}(\text{Gr})$ of $G(\mathcal{O})$-equivariant perverse sheaves on $\text{Gr}$. We also have the category $P_{S}(\text{Gr})$ of perverse sheaves on $\text{Gr}$ constructible with respect to the stratification by $G(\mathcal{O})$-orbits. In the following, the first equivalence is in [Gai01, Proposition 1], and both are in [MV04, Appendix A]. The proof given here shows that for complex coefficients the categories are in fact semisimple.

**Proposition 3.2.2.** The forgetful functors are equivalences

$$P_{G(\mathcal{O}) \rtimes \text{Aut}(\mathcal{O})}(\text{Gr}) \sim P_{G(\mathcal{O})}(\text{Gr}) \sim P_{S}(\text{Gr}).$$

**Proof.** By Proposition 3.2.1, each of the $G(\mathcal{O})$-orbits in $\text{Gr}$ is connected and simply-connected. The stabilizer in $G(\mathcal{O})$ of a coweight $\lambda \in \mathcal{G}$ is the parahoric subgroup $P^\lambda$ which is connected. The stabilizer in $G(\mathcal{O}) \rtimes \text{Aut}(\mathcal{O})$ is the semidirect product $P^\lambda \rtimes \text{Aut}(\mathcal{O})$ which is also connected. Therefore each of the categories have the same simple objects, and there are no self-extensions of simple objects. By Proposition 3.2.1, the $G(\mathcal{O})$-orbits in a given component of $\text{Gr}$ are either all even-dimensional or all odd-dimensional. By [Lus83, Section 11, c)], the stalks of the simple objects in the categories have the parity vanishing property: they are non-zero only in the parity of the dimension of their support. Therefore there are no other extensions, and we conclude that the categories are semisimple. 

3.3. **Convolution.** First, recall that from the $G(\mathcal{O})$-torsor $p : G(\mathcal{K}) \to \text{Gr}$ and the $G(\mathcal{O})$-action on $\text{Gr}$ we may construct the twisted product $\text{Gr} \times_G \text{Gr}$. To be concrete, it is the direct limit of the finite-dimensional projective varieties $\text{Gr}_{k_1} \times \text{Gr}_{k_2} = G(\mathcal{K})_{k_1} \times_{G(\mathcal{O})} \text{Gr}_{k_2}$ which are compatibly stratified by the twisted product of strata $G^\lambda_{k_1} \times G^\lambda_{k_2} = p^{-1}(G^\lambda_{k_1}) \times_{G(\mathcal{O})} (G^\lambda_{k_2})$. This realizes $\text{Gr} \times_G \text{Gr}$ as a stratified ind-scheme of ind-finite type on which $G(\mathcal{O})$ acts by left multiplication.
Next, we have the convolution diagram

\[
\text{Gr} \times \text{Gr} \xrightarrow{p} G(K) \times \text{Gr} \xrightarrow{q} \text{Gr} \times \text{Gr} \rightarrow \text{Gr}.
\]

The map \( p \) realizes \( G(K) \times \text{Gr} \) as a \( G(O) \)-torsor over \( \text{Gr} \times \text{Gr} \). The map \( q \) realizes \( G(K) \times \text{Gr} \) as a \( G(O) \)-torsor over \( \text{Gr} \times \text{Gr} \). (Note this confirms that the stratification of \( \text{Gr} \times \text{Gr} \) is locally-trivial, since the smooth map \( q \) preserves strata, and the strata of \( G(K) \times \text{Gr} \) are locally-trivial group orbits.) The map \( m \) is the multiplication, and is the direct limit of the maps \( m : G(K)_{k_1} \times G(O) \text{Gr}_{k_2} \rightarrow \text{Gr}_{k_1+k_2} \).

Now to define the convolution product

\[ \circ : \mathbf{P}_{G(O)}(\text{Gr}) \times \mathbf{P}_{G(O)}(\text{Gr}) \rightarrow \mathbf{P}_{G(O)}(\text{Gr}), \]

recall that for perverse sheaves \( \mathcal{P}_1, \mathcal{P}_2 \) in the category \( \mathbf{P}_{G(O)}(\text{Gr}) \), there is a unique perverse sheaf \( \mathcal{P}_1 \boxtimes \mathcal{P}_2 \) in the category \( \mathbf{P}_{G(O)}(\text{Gr} \times \text{Gr}) \) such that

\[ q^*(\mathcal{P}_1 \boxtimes \mathcal{P}_2) \simeq p^*(\mathcal{P}_1 \otimes \mathcal{P}_2). \]

The convolution is defined to be

\[ \mathcal{P}_1 \circ \mathcal{P}_2 = m_!(\mathcal{P}_1 \boxtimes \mathcal{P}_2). \]

Lusztig [Lus83, Section 11] first proved that the convolution takes perverse sheaves to perverse sheaves. The assertion follows from his calculations in the affine Hecke algebra. The following result of Mirković-Vilonen gives a direct geometric proof. See also [NP01, Section 9].

**Theorem 3.3.1** ([MV04], Lemma 4.3). The multiplication map \( m : G(K) \times_{G(O)} \text{Gr} \rightarrow \text{Gr} \) is a stratified semismall map.

The theorem refers to the stratification of \( \text{Gr} \) by the \( G(O) \)-orbits \( \text{Gr}^\lambda \), and the stratification of \( \text{Gr} \times \text{Gr} \) by the twisted products \( \text{Gr}^{\lambda_1} \times \text{Gr}^{\lambda_2} \). To be precise, it asserts that each of the maps \( m : G(K)_{k_1} \times_{G(O)} \text{Gr}_{k_2} \rightarrow \text{Gr}_{k_1+k_2} \) is stratified semismall with respect to these stratifications. See [MV04, Section 4] for the notion of a stratified semismall map.

### 3.4. Tannakian formalism

It is possible to place associativity and commutativity constraints on the category \( \mathbf{P}_{G(O)}(\text{Gr}) \) with respect to convolution, and then to check that it is rigid. The hypercohomology functor is an exact faithful tensor functor. The most delicate part is the commutativity constraint, for which see [MV04, Section 5] for the most straightforward approach, or [Gai01, Remark in Section 1.1 and Theorem 1(b)] for an equivalent approach. In the appendix, we sketch how one identifies the Tannakian group of the category \( \mathbf{P}_{G(O)}(\text{Gr}) \) with the dual group \( \hat{G} \).

**Theorem 3.4.1** ([MV04], Theorem 7.3, Proposition 6.3). The category \( \mathbf{P}_{G(O)}(\text{Gr}) \) is naturally a tensor category with respect to convolution. It is equivalent as a tensor category to the category \( \text{Rep}(\hat{G}) \) of finite-dimensional representations of the dual group \( \hat{G} \). Under the equivalence, the hypercohomology of a perverse sheaf corresponds to the underlying vector space of a representation.
Example 3.4.1. As a topological space, the loop Grassmannian $\text{Gr}_T$ of a torus $T$ is homeomorphic to the coweight lattice $\Lambda_T$. The category $\mathbf{P}^{(\mathcal{O})}_T(\text{Gr}_T)$ is equivalent as a tensor category to the category $\text{Rep}(\hat{T})$ of finite-dimensional representations of the dual torus $\hat{T}$.

3.5. Real forms. As mentioned in the preliminaries, we only consider conjugations and real forms of ind-schemes which are the limit of compatible families of conjugations and real forms. In addition, the conjugations always respect the stratifications considered, and the strata of a real form are always taken to be the real forms of the strata of its complexification. Thus once we have described the ind-structure or stratification of a complex ind-scheme, a real form of it inherits an ind-structure and stratification by restriction. In addition, we only consider the actions of real group-schemes which are the limit of compatible families of actions of real forms. Thus once we have described the action of a complex group-scheme on a complex ind-scheme, a real form of the group-scheme inherits an action on the real form of the ind-scheme.

Let $\mathcal{O}_R = R[[t]]$ be the ring of real formal power series, and let $\mathcal{K}_R = R((t))$ be the field of real formal Laurent series. Let $c$ be the conjugation of $\mathcal{K}$ with respect to $\mathcal{K}_R$. Let $\theta$ be the conjugation of the connected reductive complex algebraic group $G$ with respect to the real form $G_R$.

The conjugation $\theta$ extends from $G$ to a conjugation of $G(\mathcal{K})$ which we also denote by $\theta$. For $g \in G(\mathcal{K})$ thought of as a map $g : \text{Spec}(\mathcal{K}) \to G$, the extended conjugation $\theta$ takes $g$ to the composite map

$$\theta(g) : \text{Spec}(\mathcal{K}) \xrightarrow{\sim} \text{Spec}(\mathcal{K}) \overset{\theta}{\rightarrow} G \overset{\theta}{\rightarrow} G.$$ 

We may identify the resulting real form of $G(\mathcal{K})$ with the real stratified ind-scheme $G_R(\mathcal{K}_R)$.

The conjugation of $G(\mathcal{K})$ restricts to a conjugation of $G(\mathcal{O})$, and we may identify the resulting real form of $G(\mathcal{O})$ with the real stable group-scheme $G_R(\mathcal{O}_R)$.

Since the conjugation of $G(\mathcal{K})$ preserves $G(\mathcal{O})$, it induces a conjugation of $G$, and we may identify the resulting real form $G_R$ with the real stratified ind-scheme of ind-finite type $G_R(\mathcal{K}_R)/G_R(\mathcal{O}_R)$. We call it the real loop Grassmannian of $G_R$. As a topological space, $G_R$ is known [Mit88 Theorems 5.1 and 5.2] to be homotopy equivalent to the based loop space of the symmetric variety $G/K$ where $K$ is the complexification of a maximal compact subgroup of $G_R$.

3.6. Real spherical sheaves. The restriction of the stratification of $\text{Gr}$ to the real form $G_R$ coincides with the orbits of the action of $G_R(\mathcal{O}_R)$ by left-multiplication. The Cartan decomposition (which follows from the Bruhat decomposition of [Mit88 Theorem 5.3]) states that the $G_R(\mathcal{O}_R)$-orbit $G_R^\lambda \subset G_R$ is non-empty if and only if $\lambda$ is a real coweight $\lambda \in \Lambda_S$, and two orbits $G_R^\lambda_1$ and $G_R^\lambda_2$ coincide if and only if $\lambda_1$ and $\lambda_2$ are in the same $W_G$-orbit in $\Lambda_S$. Note that the $G_R(\mathcal{O}_R)$-orbit $G_R^\lambda$ contains the $G_R$-orbit $G_R \cdot \lambda$ which is isomorphic to the real flag manifold $G_R/P_R^\lambda$ where the parabolic subgroup $P_R^\lambda \subset G$ is the stabilizer of $\lambda$. 
Proposition 3.6.1. The real loop Grassmannian $\text{Gr}_R^\lambda$ is the disjoint union of the $G_R(\Omega_R^\lambda)$-orbits $\text{Gr}_R^\lambda$ through the dominant real coweights $\lambda \in \Lambda^+_S$. The closure of the orbit $\text{Gr}_R^\lambda$ is the union of the orbits through $\mu \in \Lambda^+_S$ with $\mu \leq \lambda$. Each orbit $\text{Gr}_R^\lambda$ is a real vector bundle over the real flag manifold $G_R/P^\lambda_R$, and its closure is a real projective variety of dimension $2(\hat{\rho}, \lambda)$.

Proof. Thanks to Proposition 3.2.1 and the discussion preceding it, it only remains to prove that if $\lambda - \mu$ is a non-negative integral linear combination of positive coroots of $G$, then $\text{Gr}_R^\mu \subset \text{Gr}_R^\lambda$. By Lemma 2.1.1, it suffices to prove this when $\lambda - \mu$ is a positive coroot $\alpha \in R^{\text{pos}}$, or when $\lambda - \mu$ is of the form $\theta(\alpha) + \alpha$, for a positive coroot $\alpha \in R^{\text{pos}}$, but $\lambda - \mu$ is not a multiple of a positive coroot. In the first case, we may find SL$_2(\mathbb{R}) \subset G_R$ such that $\alpha$ is its positive coroot. Then the orbit through $\lambda$ of the one parameter subgroup $U_\alpha(rt) \subset G_R(\Omega_R)$, for $r \in \mathbb{R}$, is isomorphic to $\mathbb{R}$, and its closure is isomorphic to $\mathbb{R}P^1$ with $\nu$ the point at infinity. In the second case, we may find SL$_2(\mathbb{C}) \subset G_R$ such that $\theta(\alpha) + \alpha$ is its positive coroot. Then the orbit through $\lambda$ of the one parameter subgroup $U_{\theta(\alpha)+\alpha}(ct) \subset G_R(\Omega_R)$, for $c \in \mathbb{C}$, is isomorphic to $\mathbb{C}$, and its closure is isomorphic to $\mathbb{C}P^1$ with $\nu$ the point at infinity.

The conjugation $c$ of $\mathcal{O}$ induces a conjugation of Aut$(\mathcal{O})$, and we may identify the resulting real form with the real stable group-scheme Aut$(\mathcal{O}_R)$ of automorphisms of $\mathcal{O}_R$. We define Aut$^0(\mathcal{O}_R)$ to be the connected component of the identity consisting of orientation-preserving automorphisms.

We have the derived category $D_{G_R(\Omega_R) \times \text{Aut}^0(\mathcal{O}_R)}(\text{Gr}_R)$ of $G_R(\Omega_R) \times \text{Aut}^0(\mathcal{O}_R)$-equivariant sheaves on $\text{Gr}_R$, and the derived category $D_{G_R(\mathcal{O}_R)}(\text{Gr}_R)$ of $G_R(\mathcal{O}_R)$-equivariant sheaves on $\text{Gr}_R$. We also have the derived category $D_{G_R \times S}(\text{Gr})$ of $G_R$-equivariant sheaves on $\text{Gr}_R$ constructible with respect to the stratification by $G_R(\mathcal{O}_R)$-orbits.

The following lemma is useful in constructing equivariant sheaves via functors which appear to only provide sheaves constructible with respect to the orbit stratification.

Lemma 3.6.1. The forgetful functors are equivalences

$$D_{G_R(\Omega_R) \times \text{Aut}^0(\mathcal{O}_R)}(\text{Gr}_R) \stackrel{\sim}{\to} D_{G_R(\mathcal{O}_R)}(\text{Gr}_R) \stackrel{\sim}{\to} D_{G_R \times S}(\text{Gr}_R).$$

Proof. For the first equivalence, the group scheme $\text{Aut}^0(\mathcal{O}_R)$ is contractible, and each $\text{Aut}^0(\mathcal{O}_R)$-orbit in $\text{Gr}_R$ is contained in a $G_R(\mathcal{O}_R)$-orbit. For the second, the kernel of the projection $G_R(\mathcal{O}_R) \to G_R$ is the inverse limit of unipotent groups.

3.7. Real convolution. To define the convolution product

$$\circ : D_{G_R(\mathcal{O}_R)}(\text{Gr}_R) \times D_{G_R(\mathcal{O}_R)}(\text{Gr}_R) \to D_{G_R(\mathcal{O}_R)}(\text{Gr}_R),$$

consider the real form of the convolution diagram

$$\text{Gr}_R \times \text{Gr}_R \xrightarrow{\text{pr}_1} G_R(K_R) \times \text{Gr}_R \xrightarrow{\text{pr}_2} \text{Gr}_R \times \text{Gr}_R \xrightarrow{m} \text{Gr}_R.$$

The product $G_R(K_R) \times \text{Gr}_R$ and twisted product $\text{Gr}_R \times \text{Gr}_R$ are real ind-schemes, stratified by the real forms of the strata of their complexifications.

For sheaves $\mathcal{F}_1, \mathcal{F}_2$ in the category $D_{G_R(\mathcal{O}_R)}(\text{Gr}_R)$, the convolution is defined to be

$$\mathcal{F}_1 \circ \mathcal{F}_2 = m_!(\mathcal{F}_1 \hat{\otimes} \mathcal{F}_2).$$
where $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ is the unique sheaf in the category $D_{G_\mathbb{R}(O_\mathbb{R})}(Gr_\mathbb{R})$ such that
\[ q^*(F_1 \boxtimes F_2) \simeq p^*(F_1 \boxtimes F_2). \]

### 3.8. Real spherical perverse sheaves.

In this paper, we restrict our attention to sheaves supported on certain connected components of $Gr_\mathbb{R}$. The following parameterization of $\pi_0(Gr_\mathbb{R})$ and the subsequent dimension assertion follow directly from Proposition 3.6.1.

**Lemma 3.8.1.** The inclusion $\Lambda_S \subset Gr_\mathbb{R}$ induces an isomorphism $\Lambda_S/(\Lambda_S \cap Q) \cong \pi_0(Gr_\mathbb{R})$ where $Q$ is the coroot lattice of $G$. The strata $Gr^\lambda_\mathbb{R}$ in a connected component of $Gr_\mathbb{R}$ are all even-dimensional or all odd-dimensional depending on whether $2(\tilde{\rho}, \lambda)$ is congruent to 0 or 1 mod 2.

We define $Gr^+_\mathbb{R}$ to be the union of the components of $Gr_\mathbb{R}$ containing the even-dimensional strata. Note by Proposition 3.6.1 the action of $G_\mathbb{R}$ on $Gr_\mathbb{R}$ is self-dual perverse $t$-structure on the category $D_{G_\mathbb{R}(O_\mathbb{R})}(Gr^+_\mathbb{R})$. Its heart is the category $P_{G_\mathbb{R}(O_\mathbb{R})}(Gr^+_\mathbb{R})$. All of the topological aspects of the theory of perverse sheaves hold in this setting. For example, the category $P_{G_\mathbb{R}(O_\mathbb{R})}(Gr^+_\mathbb{R})$ is abelian, and its simple objects are the intersection cohomology sheaves of strata with coefficients in irreducible $G_\mathbb{R}(O_\mathbb{R})$-equivariant local systems.

Note that the real convolution product takes sheaves supported on $Gr^+_\mathbb{R}$ to sheaves supported on $Gr^+_\mathbb{R}$ since the convolution of two even-dimensional strata is also even-dimensional. The real multiplication map $m : Gr_\mathbb{R} \times Gr_\mathbb{R} \to Gr_\mathbb{R}$ is a real form of the complex multiplication map, and the strata of its domain and codomain are real forms of the strata of their complexifications, thus it is a stratified semismall map by Theorem 3.3.1. We conclude that the restriction of the real convolution product to the category $D_{G_\mathbb{R}(O_\mathbb{R})}(Gr^+_\mathbb{R})$ is $t$-exact, and preserves the category $P_{G_\mathbb{R}(O_\mathbb{R})}(Gr^+_\mathbb{R})$.

### 3.9. Component refinement.

In this paper, we in fact restrict our attention to sheaves supported on only certain connected components of $Gr^+_\mathbb{R}$.

Recall that $Gr$ is homotopy equivalent to the based loop space of $G$, and $Gr_\mathbb{R}$ is homotopy equivalent to the based loop space of $G/K$ where $K$ is the complexification of a maximal compact subgroup of $G_\mathbb{R}$. The fibration $K \to G \to G/K$ gives rise to an exact sequence of component groups
\[ \pi_0(Gr) \to \pi_0(Gr_\mathbb{R}) \to \pi_0(G_\mathbb{R}), \]
which may be identified with the exact sequence
\[ \Lambda_T/Q \xrightarrow{\sigma} \Lambda_S/(\Lambda_S \cap Q) \xrightarrow{\partial} \pi_0(G_\mathbb{R}) \]
where $Q$ is the coroot lattice of $G$, $\sigma(\lambda) = \theta(\lambda) + \lambda$ is the projection, and $\partial(\lambda) = [\lambda(-1)]$ is the boundary map.

We define $Gr^0_\mathbb{R}$ be the union of the components of $Gr_\mathbb{R}$ in the image of $\sigma$, or equivalently in the kernel of $\partial$. 

}\end{document}
Lemma 3.9.1. The components $\text{Gr}_{R}^{0}$ are a subset of the components $\text{Gr}_{R}^{\pm}$.

Proof. Recall that $M \subset G$ is the Levi factor of the complexification $P \subset G$ of the minimal parabolic subgroup $P_{R} \subset G_{R}$. Using Lemma 3.8.1, we calculate the parity of the dimension of strata

$$\langle 2\rho, \sigma(\lambda) \rangle = \langle 2\rho - 2\rho_{M}, \theta(\lambda) + \lambda \rangle = 2(2\rho - 2\rho_{M}, \lambda) = 0 \mod 2.$$ 

In the above equation, the identity $\langle 2\rho_{M}, \sigma(\lambda) \rangle = 0$ gives the first equality, and the identity $\theta(2\rho - 2\rho_{M}) = 2\rho - 2\rho_{M}$ gives the second. □

4. Real Beilinson-Drinfeld Grassmannians

In this section, we recall some basic properties of Beilinson-Drinfeld Grassmannians, then describe similar properties of their real forms. We also collect some basic results about embeddings and stratifications. Throughout this section and later sections, we fix a smooth projective complex algebraic curve $X$ with non-empty real form $X_{R}$, and let $c$ denote the conjugation of $X$ with respect to $X_{R}$.

In what follows, we shall define certain ind-schemes by the functors from $\mathbb{C}$-algebras to sets which they represent. The sets shall always be taken up to the natural notion of equivalence.

4.1. Local loop Grassmannians. Fix $x \in X$. Let $\mathcal{O}_{x}$ be the completion of the local ring of $X$ at $x$, and let $K_{x}$ be its fraction field.

The local loop Grassmannian $\mathcal{O}_{x}\text{Gr} = G(K_{x})/G(\mathcal{O}_{x})$ is a stratified ind-scheme of ind-finite type isomorphic to the loop Grassmannian $\text{Gr}$. To see this, choose a formal coordinate at $x$. The identification $\mathcal{O}_{x} \simeq \mathcal{O}$ induces a bijection $\mathcal{O}_{x}\text{Gr} \sim \rightarrow \text{Gr}$ for every choice of real formal coordinate at $x$, and the strata $\mathcal{O}_{x}\text{Gr}^{\lambda} \subset \mathcal{O}_{x}\text{Gr}$ for dominant real coweights $\lambda \in \Lambda_{R}^{+}$. If we choose a different formal coordinate, the resulting bijection differs by the action on $\text{Gr}$ of an element of $\text{Aut}(\mathcal{O})$. Since each of the varieties in the family $\text{Gr}_{k}$ is $\text{Aut}(\mathcal{O})$-invariant, we see that $\mathcal{O}_{x}\text{Gr}$ is naturally an ind-scheme of ind-finite type, and the bijection is an isomorphism for any choice of local coordinate. Furthermore, the isomorphism takes each $G(\mathcal{O}_{x})$-orbit in $\mathcal{O}_{x}\text{Gr}$ to a $G(\mathcal{O})$-orbit in $\text{Gr}$. Since each $G(\mathcal{O})$-orbit in $\text{Gr}$ is $\text{Aut}(\mathcal{O})$-invariant, the orbit correspondence is independent of the choice of local coordinate.

We may therefore unambiguously index the $G(\mathcal{O}_{x})$-orbits $\mathcal{O}_{x}\text{Gr}^{\lambda} \subset \mathcal{O}_{x}\text{Gr}$ by dominant coweights $\lambda \in \Lambda_{R}^{+}$.

Similarly, for $x \in X_{R}$, we have the real local loop Grassmannian $x\text{Gr}_{R}$, an isomorphism $x\text{Gr}_{R} \sim \rightarrow \text{Gr}_{R}$ for every choice of real formal coordinate at $x$, and the strata $x\text{Gr}_{R}^{\lambda} \subset x\text{Gr}_{R}$, for dominant real coweights $\lambda \in \Lambda_{R}^{+}$.

The following proposition represents an important change in perspective. It was first proved for $G = \text{SL}_{n}$ in [BL94a Proposition 2.1]. For a $\mathbb{C}$-algebra $A$ and scheme $Z$, let $Z_{A}$ denote the product $Z \times \text{Spec}(A)$.

Proposition 4.1.1 ([LS97], Proposition 3.10). The local loop Grassmannian $\mathcal{O}_{x}\text{Gr}$ represents the functor $A \mapsto \{(\mathcal{F}, \nu)\}$, where $\mathcal{F}$ is a $G$-torsor on $X_{A}$, and $\nu$ is a trivialization of $\mathcal{F}$ over $(X \setminus \{ x \})_{A}$.
4.2. **Beilinson-Drinfeld Grassmannians.** Some of the material discussed here may be found in [BD, Sections 5.3.10, 5.3.11] and [MV04, Section 5]. The Beilinson-Drinfeld Grassmannian \( Gr^{(n)} \) of the group \( G \) is the ind-scheme of ind-finite type which represents the functor \( A \mapsto \{((x_1, \ldots, x_n), F, \nu)\} \), where \((x_1, \ldots, x_n) \in X^n(A)\), \( F \) is a \( G \)-torsor on \( X_A \), and \( \nu \) is a trivialization of \( F \) over \( X_A \setminus (x_1 \cup \cdots \cup x_n) \). Here we think of the points \( x_i : \text{Spec}(A) \to X \) as subschemes of \( X_A \) by taking their graphs. See Section [14] for confirmation that \( Gr^{(n)} \) is indeed an ind-scheme of ind-finite type.

One of the most important properties of \( Gr^{(n)} \) is its factorization with respect to the projection \( \pi : Gr^{(n)} \to X^n \). To describe this, we introduce the incidence stratification of \( X^n \). The strata are indexed by partitions of the set \( \{1, \ldots, n\} \). The stratum \( T_\tau \) indexed by the partition \( \tau \) consists of the points \((x_1, \ldots, x_n) \in X^n\) such that the coincidences specified by \( \tau \) occur among the points \( x_1, \ldots, x_n \in X \).

**Proposition 4.2.1.** For each stratum \( T_\tau \subset X^n \), there is a canonical isomorphism

\[
Gr^{(n)} | T_\tau \xrightarrow{\sim} (\prod_{i=1}^k Gr^{(1)}) | T_0,
\]

where \( k \) is the number of parts in the partition \( \tau \), and \( T_0 \) denotes the open stratum of distinct points \( y_1, \ldots, y_k \in X \).

**Proof.** The map is defined by \( ((x_1, \ldots, x_n), F, \nu) \mapsto \prod_{i=1}^k(y_i, F_i, \nu_i) \), where \( y_1, \ldots, y_k \in X \) are the distinct points such that there is an equality of sets \( \{y_1, \ldots, y_k\} = \{x_1, \ldots, x_n\} \), the torsor \( F_i \) coincides with \( F \) over \( X \setminus (y_1 \cup \cdots \cup y_i \cup \cdots \cup y_k) \), and the trivialization \( \nu_i \) coincides with \( \nu \) over \( X \setminus (y_1 \cup \cdots \cup y_k) \). We leave it to the reader to check that the map is an isomorphism.

We next describe the standard stratification of \( Gr^{(n)} \). The strata of \( Gr^{(1)} \) are indexed by the dominant coweights \( \lambda \in \Lambda_+^T \). The stratum of \( Gr^{(1)} \) indexed by \( \lambda \) consists of the union over all fibers \( Gr^{(1)}|x \) of the points which map to the \( G(\mathcal{O}_x) \)-orbit \( xGr^n \) in \( xGr \) via the canonical isomorphism \( Gr^{(1)}|x \xrightarrow{\sim} xGr \). In Section 5.2 we shall see that it is useful to realize \( Gr^{(1)} \) as the twisted product of \( Gr \) and the curve \( X \). From this perspective, each stratum of \( Gr^{(1)} \) is the twisted product of a stratum of \( Gr \) with \( X \).

In general, the strata of \( Gr^{(n)} \) are indexed by labeled partitions \( (\tau, \{\lambda_1, \ldots, \lambda_k\}) \), where \( \tau \) is a partition of \( \{1, \ldots, n\} \) into \( k \) parts, and \( \lambda_1, \ldots, \lambda_k \in \Lambda_+^T \) are dominant coweights each assigned to a part of the partition. The stratum of \( Gr^{(n)} \) indexed by \( (\tau, \{\lambda_1, \ldots, \lambda_k\}) \) consists of the points which project to the stratum \( T_\tau \subset X^n \), and which map via the isomorphism \( Gr^{(n)} | T_\tau \xrightarrow{\sim} (\prod_{i=1}^k Gr^{(1)}) | T_0 \) to the product stratum indexed by \( \{\lambda_1, \ldots, \lambda_k\} \).

4.3. **Real forms.** Recall that \( c \) denotes the conjugation of \( X \) with respect to \( X_R \), and \( \theta \) denotes the conjugation of \( G \) with respect to \( G_R \). The pair of conjugations induce a conjugation \( \theta \) of \( Gr^{(n)} \) as follows. For data \( ((x_1, \ldots, x_n), F, \nu) \), choose a cover \( \{U_i \subset X\}_i \) so that we may realize \( F \) as the \( G \)-torsor obtained from the disjoint union \( \sqcup_i(U_i \times G) \) via gluing maps \( \varphi_{ij} : U_i \cap U_j \to G \). Then we define the conjugation to be

\[
\theta((x_1, \ldots, x_n), F, \nu) = ((c(x_1), \ldots, c(x_n)), F^\theta, \nu^\theta).
\]
Here $F^\theta_c$ is the $G$-torsor obtained from the disjoint union $\sqcup_i(c(U_i) \times G)$ via gluing maps
$$c(U_i \cap U_j) \xrightarrow{\zeta} U_i \cap U_j \xrightarrow{\phi_{ij}} G \xrightarrow{\theta} G.$$ Thinking of the trivialization $\nu$ as a section of $F$ over $X \setminus (x_1 \cup \cdots \cup x_n)$, the trivialization $\nu^\theta_c$ is the composite section
$$\nu^\theta_c : X \setminus c(x_1 \cup \cdots \cup x_n) \xrightarrow{\zeta} X \setminus (x_1 \cup \cdots \cup x_n) \xrightarrow{\nu} F \xrightarrow{f^\theta_c} F^\theta_c,$$ where $f^\theta_c : F \to F^\theta_c$ is the map defined by $f^\theta_c((u_i, g)) = (c(u_i), \theta(g))$, for $u_i \in U_i$.

Now fix a permutation $\omega$ in the symmetric group $\Sigma_n$ such that $\omega^2 = e$. Then we obtain involutions of $X^n$ and of $\text{Gr}^{(n)}$ by permuting the labels of the points
$$(x_1, \ldots, x_n) \mapsto (x_{\omega(1)}, \ldots, x_{\omega(n)}).$$ The composition of a conjugation and an involution is another conjugation. Let $c_\omega$ be the conjugation of $X^n$ defined by
$$c_\omega(x_1, \ldots, x_n) = (c(x_{\omega(1)}), \ldots, c(x_{\omega(n)})),$$ and let $\theta_\omega$ be the conjugation of $\text{Gr}^{(n)}$ defined by
$$\theta_\omega((x_1, \ldots, x_n), F, \nu) = (c(x_{\omega(1)}), \ldots, c(x_{\omega(n)}), F^\theta_c, \nu^\theta_c).$$

Let $X_{\mathbb{R}}^{(\omega)}$ be the real form of $X^n$ with respect to the conjugation $c_\omega$, and let $\text{Gr}^{(\omega)}_{\mathbb{R}}$ be the real form of $\text{Gr}^{(n)}$ with respect to the conjugation $\theta_\omega$. When $\omega$ is the identity permutation, we write $X^n_{\mathbb{R}}$ in place of $X_{\mathbb{R}}^{(\omega)}$, and $\text{Gr}^{(n)}_{\mathbb{R}}$ in place of $\text{Gr}^{(\omega)}_{\mathbb{R}}$.

As usual we stratify the real forms by taking the real forms of the strata of their complexifications. The strata of $X_{\mathbb{R}}^{(\omega)}$ are indexed by those partitions $\tau$ of $\{1, \ldots, n\}$ such that the action of $\omega$ sends each part of $\tau$ to another part of $\tau$. The strata of $\text{Gr}^{(\omega)}_{\mathbb{R}}$ are indexed by labelled partitions $(\tau, \{\lambda_1, \ldots, \lambda_k\}, \{\mu_1, \ldots, \mu_\ell\})$, where $\tau$ is a partition of $\{1, \ldots, n\}$ such that the action of $\omega$ sends each part of $\tau$ to another part of $\tau$ with $k$ parts fixed and $\ell$ pairs of parts exchanged, $\lambda_1, \ldots, \lambda_k \in \Lambda_{\mathbb{R}}^+$ are dominant real coweights each assigned to a part of $\tau$ fixed by $\omega$, and $\mu_1, \ldots, \mu_\ell \in \Lambda_{\mathbb{R}}^+$ are dominant coweights each assigned to a pair of parts exchanged by $\omega$.

To describe the factorization of $\text{Gr}^{(\omega)}_{\mathbb{R}}$ with respect to the projection $\pi : \text{Gr}^{(\omega)}_{\mathbb{R}} \to X^{(\omega)}$, it is convenient to break the symmetry as assumed in the following.

**Proposition 4.3.1.** Let $\tau$ be a partition such that $\omega$ sends each of its parts to another. For each pair of parts of $\tau$ exchanged by $\omega$, choose one of the parts. Then there is a canonical strata-preserving isomorphism
$$\text{Gr}^{(\omega)}_{\mathbb{R}}|_{T_\tau} \simeq \left( \prod_{i=1}^k \text{Gr}^{(1)}_{\mathbb{R}}(y_i) \times \prod_{j=1}^\ell \text{Gr}^{(1)}_{\mathbb{R}}(z_j) \right)|_{T_0},$$
where $k$ is the number of parts in $\tau$ fixed by $\omega$, $\ell$ is the number of pairs of parts exchanged by $\omega$, and $T_0$ denotes the open stratum of distinct points $y_1, \ldots, y_k, z_1, \ldots, z_\ell \in X$. 
Proof. By Proposition 4.2.1 and the definitions, $\operatorname{Gr}^{(\omega)}_R | T_\tau$ is the real form of the product
\[
\left( \prod_{i=1}^k \operatorname{Gr}^{(1)} \times \prod_{j=1}^\ell (\operatorname{Gr}^{(1)} \times \operatorname{Gr}^{(1)}) \right) | T_0
\]
with respect to the standard conjugation on the factors of type $\operatorname{Gr}^{(1)}$, and the composition of the standard conjugation with the exchange involution on the factors of type $\operatorname{Gr}^{(1)} \times \operatorname{Gr}^{(1)}$. The choice of a part in each pair of parts exchanged by $\omega$ distinguishes one of the factors $\operatorname{Gr}^{(1)}$ in each of the products $\operatorname{Gr}^{(1)} \times \operatorname{Gr}^{(1)}$. Projection of the real form of each of the products $\operatorname{Gr}^{(1)} \times \operatorname{Gr}^{(1)}$ to its distinguished factor $\operatorname{Gr}^{(1)}$ is an isomorphism. □

If we fix an identification
\[
X^{(\omega)}_R \simeq X^r_R \times X^s
\]
where $r$ is the number of fixed points of $\omega$, $s$ is the number of pairs of points exchanged by $\omega$, and $n = r + 2s$, then by the proposition, for $(x_1, \ldots, x_r, z_1, \ldots, z_s) \in X^r_R \times X^s$, we have a canonical isomorphism
\[
\operatorname{Gr}^{(\omega)}_R | (x_1, \ldots, x_r, z_1, \ldots, z_s) \sim \prod_{i=1}^k y_i \operatorname{Gr}^{(1)} \times \prod_{j=1}^\ell w_j \operatorname{Gr}^{(1)}
\]
where $y_1, \ldots, y_k \in X_R$, $w_1, \ldots, w_\ell \in X \setminus X_R$ are the distinct points with an equality of sets
\[
\{y_1, \ldots, y_k, w_1, \ldots, w_\ell\} = \{x_1, \ldots, x_r, z_1, \ldots, z_s\}.
\]

4.4. Embeddings. It will be useful to have an embedding of the Beilinson-Drinfeld Grassmannian $\operatorname{Gr}^{(n)}$ in an ind-scheme that is the limit of smooth varieties. We describe one such construction here.

First, fix an embedding $G \subset \operatorname{GL}_N$ to obtain an embedding $\operatorname{Gr}^{(n)} \subset \operatorname{Gr}^{(n)}_N$ of the corresponding Beilinson-Drinfeld Grassmannians.

Next, let $\operatorname{Gr}^{(n)}_{N,k}$ be the variety that represents the functor $A \mapsto \{(x_1, \ldots, x_n, M)\}$, where $x_1, \ldots, x_n \in X(A)$, and
\[
\mathcal{M} \subset \mathcal{O}^{\oplus N}_{X_A}(k(x_1 \cup \cdots \cup x_n))
\]
is a subsheaf of $\mathcal{O}_{X_A}$-modules such that
\[
\mathcal{O}^{\oplus N}_{X_A}(-k(x_1 \cup \cdots \cup x_n)) \subset \mathcal{M},
\]
and $\mathcal{O}^{\oplus N}_{X_A}(k(x_1 \cup \cdots \cup x_n))/\mathcal{M}$ is $\operatorname{Spec}(A)$-flat.

The inclusions $\mathcal{O}^{\oplus N}_{X_A}(k(x_1 \cup \cdots \cup x_n)) \subset \mathcal{O}^{\oplus N}_{X_A}((k+1)(x_1 \cup \cdots \cup x_n))$ induce canonical closed embeddings $\operatorname{Gr}^{(n)}_{N,k} \to \operatorname{Gr}^{(n)}_{N,k+1}$.

**Proposition 4.4.1.** The direct limit of the varieties $\operatorname{Gr}^{(n)}_{N,k}$ is canonically isomorphic to the Beilinson-Drinfeld Grassmannian $\operatorname{Gr}^{(n)}_N$ of $\operatorname{GL}_N$. 
Proof. A $\text{GL}_N$-torsor over $X$ with a trivialization over $X \setminus (x_1 \cup \cdots \cup x_n)$ defines a vector bundle $V$ over $X$ and an isomorphism $\mathcal{O}(V) \sim \mathcal{O}_X^{\oplus N}$ over $X \setminus (x_1 \cup \cdots \cup x_n)$. For $k$ sufficiently large, the isomorphism extends to a map $\mathcal{O}(V) \to \mathcal{O}_X^{\oplus N}(k(x_1 \cup \cdots \cup x_n))$ over $X$. And again for $k$ sufficiently large, the image of the extended map will contain $\mathcal{O}_X^{\oplus N}(-k(x_1 \cup \cdots \cup x_n))$. We take the subsheaf $\mathcal{M} \subset \mathcal{O}_X^{\oplus N}(k(x_1 \cup \cdots \cup x_n))$ to be the image for $k$ large. We leave it to the reader to check that this gives an isomorphism. □

Finally, let $\text{adGr}_{N,k}$ be the variety that represents the functor $A \mapsto \{(x_1, \ldots, x_n, \mathcal{M})\}$, where $x_1, \ldots, x_n \in X(A)$, and

$$\mathcal{M} \subset \mathcal{O}_X^{\oplus N}(k(x_1 \cup \cdots \cup x_n))$$

is a subsheaf of $\mathbb{C}$-modules such that

$$\mathcal{O}_X^{\oplus N}(-k(x_1 \cup \cdots \cup x_n)) \subset \mathcal{M},$$

and $\mathcal{O}_X^{\oplus N}(k(x_1 \cup \cdots \cup x_n))/\mathcal{M}$ is $\text{Spec}(A)$-flat.

Lemma 4.4.1. The variety $\text{adGr}_{N,k}$ is smooth.

Proof. We may identify $\text{adGr}_{N,k}$ with the Grassmann bundle (of $\mathbb{C}$-subspaces of any dimension) of the vector bundle $H^0(X^n, \mathcal{M}_k^N) \to X^n$, for the quotient sheaf

$$\mathcal{M}_k^N = \mathcal{O}_X^{\oplus N}(k(x_1 \cup \cdots \cup x_n))/\mathcal{O}_X^{\oplus N}(-k(x_1 \cup \cdots \cup x_n)).$$

□

Define the ind-scheme $\text{adGr}_N^{(n)}$ to be the direct limit of the varieties $\text{adGr}_{N,k}$. We conclude that for the choice of an embedding $G \subset \text{GL}_N$, we obtain embeddings of ind-schemes

$$\text{Gr}^{(n)} \subset \text{Gr}_N^{(n)} \subset \text{adGr}_N^{(n)}$$

with $\text{adGr}_N^{(n)}$ the limit of a family of smooth varieties.

Remark 4.4.1. The above discussion confirms that the Beilinson-Drinfeld Grassmannian $\text{Gr}^{(n)}$ is an ind-scheme of ind-finite type.

Before continuing on, we note that there is a factorization of $\text{adGr}$ with respect to the projection $\pi : \text{adGr} \to X^n$ which restricts to give the factorization of $\text{Gr}^{(n)}$ of Proposition 4.2.2. For each stratum $T_\tau \subset X^n$, let $T^\tau \subset X^n$ be the union of the strata of $X^n$ containing $T_\tau$ in their closures.

Proposition 4.4.2. For each stratum $T_\tau \subset X^n$, there is a canonical isomorphism

$$\text{adGr}_N^{(n)}|T^\tau \sim \left(\prod_{i=1}^k \text{adGr}_N^{(n_i)}\right)|T^\tau$$

where $k$ is the number of parts of $\tau$, and $n_i$ is the size of the part $\tau_i$. 

Proof. If the sheaf $\mathcal{M} \in \text{adGr}^{(n)}_X$ is supported at points $x_1, \ldots, x_n \in X$ with $(x_1, \ldots, x_n) \in T^{\tau}$, then the coincidences among the points $x_1, \ldots, x_n$ are coarser than the coincidences specified by $\tau$. We may therefore define a map $\mathcal{M} \mapsto \prod_{i=1}^k \mathcal{M}_i$, requiring that $\mathcal{M} \simeq \sum_{i=1}^k \mathcal{M}_i$, and that each sheaf $\mathcal{M}_i$ is supported at those points contained in the part $\tau_i$ of the partition $\tau$. We leave it to the reader to check that this map is an isomorphism. \[\square\]

4.5. Stratifications. See [Mat70] or [GMSS, Section 1.2, and Section 2.A.1] for the notions of a Whitney stratification of a manifold and of a Thom stratified map between manifolds.

We say that a stratification of a variety $V$ is a Whitney stratification if for some embedding $V \subset M$ into a smooth manifold, the stratification of $M$ by the strata of $V$ and the complement $M \setminus V$ is a Whitney stratification.

We say that a stratification of an ind-variety $Z$ is a Whitney stratification if the closure of each stratum of $Z$ is Whitney stratified by the strata of $Z$ in the closure.

We say that a map $V \rightarrow N$, where $V$ is a stratified variety and $N$ is a smooth manifold, is a Thom stratified map if for some commutative diagram

$$
\begin{array}{ccc}
V & \subset & M \\
\downarrow & & \downarrow \\
N & = & N
\end{array}
$$

with $M$ smooth, the map $M \rightarrow N$ is a Thom stratified map with respect to the stratification of $M$ by the strata of $V$ and the complement $M \setminus V$.

We say that a map $Z \rightarrow N$, where $Z$ is a stratified ind-variety and $N$ is a smooth manifold, is a Thom stratified map if the restriction of the map to the closure of each stratum of $Z$ is a Thom stratified map with respect to the stratification of the closure by the strata of $Z$ in the closure.

**Proposition 4.5.1.** The stratifications of the Beilinson-Drinfeld Grassmannian $Gr^{(n)}$ and its real form $Gr^{(\omega)}_R$ are Whitney stratifications. The projection $\pi : Gr^{(n)} \rightarrow X^n$ and its restriction $\pi : Gr^{(\omega)}_R \rightarrow X^{(\omega)}_R$ are Thom stratified maps.

**Proof.** First, note that if we choose $G \subset GL_N$ to be equivariant with respect to conjugation, then the embeddings

$$
Gr^{(n)} \subset Gr^{(n)}_N \subset \text{adGr}^{(n)}_N
$$

of the previous section will also be equivariant with respect to conjugation. Therefore, by the following easily-verified lemma, it suffices to prove the proposition in the complex case.

**Lemma 4.5.1.** Let $M$ and $N$ be Whitney stratified manifolds, and let $K$ be a compact group acting smoothly on $M$ and $N$ such that the actions preserve the stratifications. Then the fixed point manifolds $M^K$ and $N^K$ are Whitney stratified by the fixed points of the strata. If $M \rightarrow N$ is a Thom stratified $K$-equivariant map, then $M^K \rightarrow N^K$ is Thom stratified as well.
Now, for the closure $V$ of a stratum of $Gr^{(n)}$, we may find $k$ large so that we have an embedding

$$V \subset Gr^{(n)}_{N,k} \subset \text{ad}Gr^{(n)}_{N,k}.$$ 

Recall that the factorization of $\text{ad}Gr_N^{(n)}$ restricts to give that of $Gr^{(n)}$, and the strata of $Gr^{(n)}$ are by definition products with respect to the factorization. Therefore it suffices to verify Whitney’s condition $B$ and Thom’s condition $A_\pi$ are satisfied at a point $M \in \text{ad}Gr^{(n)}_{N,k}$ such that $\pi(M) = (x, \ldots, x)$. To accomplish this, consider the group $G^\text{loc}_x$ of maps $X_x \to G$, where $X_x$ is the localization of $X$ at $x$.

**Lemma 4.5.2.** For $\ell \geq 0$, the composite homomorphism $G^\text{loc}_x \to G(O_x) \to G(J^\ell_x)$ is surjective. $\blacksquare$

**Proof.** In [Gai01, Lemma 4], the analogous assertion is proven for Iwahori subgroups which immediately implies this assertion. $\blacksquare$

Using the lemma, it is easy to confirm Thom’s condition $A_\pi$ at the point $M$.

To verify Whitney’s condition $B$ at the point $M$, let $M_i$ be a sequence converging to $M$ in the stratum $S$ containing $M$, let $L_i$ be a sequence also converging to $M$ in some stratum $R$, and suppose that the limits $\ell = \lim_i M_iL_i$ and $\tau = \lim_i T_{L_i}R$ exist. Since $\pi(M) = (x, \ldots, x)$, and $X$ is smooth, we may assume that the sequence $M_i$ is in the same fiber as $M$. Then using the lemma, we may assume that the sequence $M_i$ is in fact constant equal to $M$. Now the assertion that $\ell \subset \tau$ for such sequences follows from standard arguments such as the Curve Selection Lemma [Mil68, Chapter 3]. $\blacksquare$

## 5. Specialization

In this section, we define a functor that takes perverse sheaves on the loop Grassmannian $Gr$ to perverse sheaves on the subspace $Gr^0_R$ of the real form $Gr_R$.

Note that the complement of the real curve $X_R$ in its complexification $X$ is the union of two connected components. Throughout what follows, we distinguish one of the components and denote it by $X_+$. This also distinguishes an orientation of $X_R$ by the rule that the complex structure of $X$ takes a positively-oriented tangent vector to a tangent vector pointing into $X_+$.

### 5.1. Global specialization.

To define the global specialization functor

$$R_X : D_{G,S}(Gr^{(1)}) \to D_{Gr^0_S,Gr^0_R}(Gr^{(1)}_R),$$

we work with the real form $Gr^{(\sigma)}_R$ of the Beilinson-Drinfeld Grassmannian $Gr^{(2)}$, where $\sigma$ is the non-trivial element of the symmetric group $\Sigma_2$.

We identify the projection $Gr^{(\sigma)}_R \to X^{(\sigma)}_R$ with a map $Gr^{(\sigma)}_R \to X$ via the isomorphism $X^{(\sigma)}_R \cong X$ defined by $(z, c(z)) \mapsto z$. By Proposition 4.3.1, we have canonical identifications

$$Gr^{(\sigma)}_R | X_+ \cong Gr^{(1)}| X_+$$

$$Gr^{(\sigma)}_R | X_R \cong Gr^{(1)}_R.$$
Consider the diagram
\[
\begin{array}{c}
\text{Gr}^{(1)}|X_+ \simeq \text{Gr}^{(\sigma)}|X_+ \xrightarrow{j} \text{Gr}^{(\sigma)}|X \xleftarrow{i} \text{Gr}^{(\sigma)}|X_\mathbb{R} \simeq \text{Gr}^{(1)} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
X_+ \rightarrow X \leftarrow X_\mathbb{R}
\end{array}
\]

Define the global specialization by
\[
R_X(F) = i^*j_*(F|X_+).
\]
Informally speaking, it is the nearby cycles in the family \(\text{Gr}^{(\sigma)}|X_\mathbb{R} \rightarrow X\).

By Proposition 4.5.1, the stratification of \(\text{Gr}^{(\sigma)}|X_\mathbb{R}\) is a Whitney stratification and so locally-trivial. Thus \(R_X\) takes \(\mathcal{S}\)-constructible sheaves to \(\mathcal{S}\)-constructible sheaves. Since all of the maps in the above diagram are \(G_\mathbb{R}\)-equivariant, \(R_X\) takes \(G\)-equivariant sheaves to \(G_\mathbb{R}\)-equivariant sheaves.

5.2. Perverse sheaves constant along \(X\). Let \(\hat{X} \rightarrow X\) be the \(\text{Aut}(O)\)-torsor of smooth maps \(\text{Spec}(O) \rightarrow X\). It is the inverse limit of the \(\text{Aut}(J^\ell)\)-torsors \(\hat{X}_\ell \rightarrow X\) of smooth maps \(\text{Spec}(J^\ell) \rightarrow X\).

Recall that for \(x \in X\), we have an identification \(\text{Gr}^{(1)}|x \simeq_x \text{Gr}\), and for the choice of a formal coordinate at \(x\), we obtain an isomorphism \(\text{Gr}^{(1)}|x \simeq \text{Gr}\). It follows that \(\text{Gr}^{(1)}\) is the twisted product obtained from the \(\text{Aut}(O)\)-torsor \(\hat{X} \rightarrow X\) and the action of \(\text{Aut}(O)\) on \(\text{Gr}\). In other words, there is a canonical isomorphism
\[
\text{Gr}^{(1)} \simeq \hat{X} \times_{\text{Aut}(O) \text{ Gr}} \text{Gr}.
\]

Consider the diagram
\[
\begin{array}{c}
X \times \text{Gr} \xleftarrow{p} \hat{X} \times \text{Gr} \xrightarrow{q} \hat{X} \times_{\text{Aut}(O) \text{ Gr}} \text{Gr} \simeq \text{Gr}^{(1)}
\end{array}
\]

By Lemma 3.2.2, we may define a functor
\[
\rho : \mathcal{P} \mathcal{G}(O)(\text{Gr}) \rightarrow D_{G,S}(\text{Gr}^{(1)})
\]
by the formula
\[
\rho(\mathcal{P}) = \mathbb{C}_X \hat{\otimes} \mathcal{P}
\]
where
\[
q^*(\mathbb{C}_X \hat{\otimes} \mathcal{P}) = p^*(\mathbb{C}_X \otimes \mathcal{P}).
\]
(Although it may be more natural to shift here so that \(\rho\) is perverse, we have found it convenient to avoid all such shifts.)

Define the category \(\mathcal{P}_{G,S}(\text{Gr}^{(1)})\) to be the strict full subcategory of \(D_{G,S}(\text{Gr}^{(1)})\) whose objects are isomorphic to objects of the form \(\rho(\mathcal{P})\), where \(\mathcal{P}\) runs through all objects of \(\mathcal{P} \mathcal{G}(O)(\text{Gr})\). Note that objects in the subcategory \(\mathcal{P}_{G,S}(\text{Gr}^{(1)})\) are equivariant with respect to the groupoid of pairs of points \((x, y) \in X^2\), and an isomorphism between their formal neighborhoods.

For a choice of \(x \in X\), and formal coordinate at \(x\), the isomorphism \(\text{Gr}^{(1)}|x \simeq \text{Gr}\) provides an inverse
\[
\rho^{-1} : \mathcal{P}_{G,S}(\text{Gr}^{(1)}) \rightarrow \mathcal{P} \mathcal{G}(O)(\text{Gr})
\]
defined by restriction to the fiber
\[ \rho^{-1}(P) = P|_x. \]

By the definition of \( P_{G,\mathcal{S}}(\text{Gr}^{(1)}) \), the inverse functors for different choices of \( x \in X \), and formal coordinate at \( x \), are canonically isomorphic.

5.3. **Sheaves constant along** \( X_\mathbb{R} \). Here we work with the \( \text{Aut}^0(\mathcal{O}_{\mathbb{R}}) \)-torsor \( \hat{X}_\mathbb{R}^0 \to X_\mathbb{R} \) of orientation-preserving smooth maps \( \text{Spec}(\mathcal{O}_{\mathbb{R}}) \to \hat{X}_\mathbb{R} \), and the canonical isomorphism
\[ \hat{X}_\mathbb{R}^0 \times_{\text{Aut}^0(\mathcal{O}_{\mathbb{R}})} \text{Gr}_\mathbb{R} \simeq \text{Gr}_\mathbb{R}^{(1)}. \]

Consider the diagram
\[ X_\mathbb{R} \times \text{Gr}_\mathbb{R} \xleftarrow{p} \hat{X}_\mathbb{R}^0 \times \text{Gr}_\mathbb{R} \xrightarrow{q} \hat{X}_\mathbb{R}^0 \times_{\text{Aut}^0(\mathcal{O}_{\mathbb{R}})} \text{Gr}_\mathbb{R} \simeq \text{Gr}_\mathbb{R}^{(1)}. \]

By Lemma 3.6.1 we may define a functor
\[ \rho_\mathbb{R} : D_{G\mathbb{R}}(\mathcal{O}_{\mathbb{R}})(\text{Gr}_\mathbb{R}) \to D_{G\mathbb{R},\mathcal{S}}(\text{Gr}_\mathbb{R}^{(1)}) \]
by the formula
\[ \rho_\mathbb{R}(\mathcal{F}) = \mathcal{C}_{X_\mathbb{R}} \tilde{\boxtimes} \mathcal{F} \]
where
\[ q^*(\mathcal{C}_{X_\mathbb{R}} \tilde{\boxtimes} \mathcal{F}) = p^*(\mathcal{C}_{X_\mathbb{R}} \boxtimes \mathcal{F}). \]

Note that here we are able to define the functor on the entire derived category, not only on the category of perverse sheaves.

Define the category \( D_{G\mathbb{R},\mathcal{S}}(\text{Gr}^{(1)}_\mathbb{R}) \) to be the strict full subcategory of \( D_{G\mathbb{R},\mathcal{S}}(\text{Gr}^{(1)}_\mathbb{R}) \) whose objects are isomorphic to objects of the form \( \rho_\mathbb{R}(\mathcal{F}) \), where \( \mathcal{F} \) runs through all objects of \( D_{G\mathbb{R}}(\mathcal{O}_{\mathbb{R}})(\text{Gr}_\mathbb{R}) \). An object in the category \( D_{G\mathbb{R},\mathcal{S}}(\text{Gr}^{(1)}_\mathbb{R}) \) is in the subcategory \( D_{G\mathbb{R},\mathcal{S}}(\text{Gr}^{(1)}_\mathbb{R}) \) if and only if it is equivariant with respect to the groupoid of pairs of points \( (x, y) \in X_\mathbb{R}^2 \), and an orientation-preserving isomorphism between their formal neighborhoods. (Since the group \( \text{Aut}^0(\mathcal{O}_{\mathbb{R}}) \) is contractible, such equivariance is a property not a structure.)

For the choice of \( x \in X_\mathbb{R} \), and formal coordinate at \( x \), the resulting isomorphism \( \text{Gr}^{(1)}_\mathbb{R}|_x \simeq \text{Gr}_\mathbb{R} \) provides an inverse
\[ \rho^{-1}_\mathbb{R} : D_{G\mathbb{R},\mathcal{S}}(\text{Gr}^{(1)}_\mathbb{R}) \to D_{G\mathbb{R}}(\mathcal{O}_{\mathbb{R}})(\text{Gr}_\mathbb{R}) \]
defined by restriction to the fiber
\[ \rho^{-1}_\mathbb{R}(\mathcal{F}) = \mathcal{F}|_x. \]

By the definition of \( D_{G\mathbb{R},\mathcal{S}}(\text{Gr}^{(1)}_\mathbb{R}) \), the inverse functors for different choices of \( x \in X_\mathbb{R} \), and formal coordinate at \( x \), are canonically isomorphic.
5.4. **Local specialization.** To define the local specialization

$$R : P_G(O)(Gr) \to D_{G_R(O_R)}(Gr_R),$$

we use the following proposition.

**Proposition 5.4.1.** The global specialization descends to a functor

$$R_X : P_{G,S}(Gr^{(1)}) \to D_{G_R,S}(Gr_R^{(1)}).$$

**Proof.** A sheaf $P$ in the category $P_{G,S}(Gr^{(1)})$ is equivariant with respect to the groupoid of a pair of points $(x, y) \in X^2$, and an isomorphism between their formal neighborhoods. For points $x, y \in X_R$, we may find analytic disks $D(x), D(y) \subset X$, and a conjugation-equivariant isomorphism between them preserving the orientations of their real forms $D_R(x), D_R(y) \subset X_R$. It follows that the sheaf $R_X(P)$ is equivariant with respect to the groupoid of a pair of points $(x, y) \in \mathbb{X}_R^2$, and an isomorphism between their formal neighborhoods, and thus it is in the category $D_{G_R,S}(Gr_R^{(1)}).$ \hfill \Box

By the proposition, we may define the local specialization by

$$R(P) = \rho_R^{-1}(R_X(P)).$$

Here we use Lemma 3.6.1 to obtain a $G_R(O_R)$-equivariant sheaf from a $G_R$-equivariant, $S$-constructible sheaf.

5.5. **Perverse specialization.** Recall from Section 3.9 that $Gr_R^0$ is the union of certain components of $Gr_R$ defined as follows. The exact sequence of component groups

$$\pi_0(Gr) \rightarrow \pi_0(Gr_R) \rightarrow \pi_0(Gr),$$

may be identified with the exact sequence

$$\Lambda_T/Q \xrightarrow{\sigma} \Lambda_S/(\Lambda_S \cap Q) \xrightarrow{\partial} \pi_0(Gr)$$

where $Q$ is the coroot lattice of $G$, $\sigma(\lambda) = \theta(\lambda) + \lambda$ is the projection, and $\partial(\lambda) = [\lambda(-1)]$ is the boundary map. By definition, $Gr_R^0$ is the union of the components of $Gr_R$ in the image of $\sigma$, or equivalently in the kernel of $\partial$.

**Lemma 5.5.1.** For $P \in P_G(O)(Gr)$, the support of $R(P) \in D_{G_R(O_R)}(Gr_R)$ lies in $Gr_R^0$.

**Proof.** Consider the real form $Gr_T^{(r)} \subset Gr_R$ of the Beilinson-Drinfeld Grassmannian $Gr_T^{(2)} \subset Gr^{(2)}$ of the torus $T \subset G$. By Proposition 3.6.1, it suffices to understand the limits of points in the family $Gr_T^{(r)} \rightarrow X$. Note that the projection $\pi : Gr_T^{(r)} \rightarrow X$ has discrete fibers. It is easy to check that the limit of a point $\lambda \in \Lambda_T \simeq Gr_T^{(r)} | x_+$, for $x_+ \in X_+$, is the point $\sigma(\lambda) \in \Lambda_S \simeq Gr_T^{(r)} | x$, for $x \in X_R$. \hfill \Box
By Lemma 3.9.1, the components $\text{Gr}^0_R$ are in fact the union of certain components of $\text{Gr}_R^+$. Since there is a perverse t-structure on the category $\mathcal{D}_{\text{Gr}(\mathcal{O}_k)}(\text{Gr}^+_R)$, we may ask whether the specialization $R$ takes perverse sheaves to perverse sheaves. It is a fundamental result that the nearby cycles in a complex algebraic family are perverse. (See [BBD82, Section 4.4] or [GM83, Section 6.5].) In general this is not true for real algebraic families. The family $\text{Gr}_R^{(\sigma)} \to X$ is a good example of this: in general the specialization $R$ does not take perverse sheaves to perverse sheaves.

**Theorem 5.5.1.** The specialization $R : \mathcal{P}_{G(\mathcal{O})}(\text{Gr}) \to \mathcal{D}_{\text{Gr}(\mathcal{O}_k)}(\text{Gr}^+_R)$ is perverse if and only if $G^+_R$ is quasi-split.

To prove the theorem we need to know more about the specialization. The result is not used in what follows, and we postpone the proof. See Corollary 8.3.4.

We define $\mathcal{P}_{G(\mathcal{O})}(\text{Gr})$ to be the category $\mathcal{P}_{G(\mathcal{O})}(\text{Gr}) \otimes \text{Vect}_Z$, and identify it with the subcategory $\sum_k \mathcal{P}_{G(\mathcal{O})}(\text{Gr}^+_R)[k]$ of the derived category $\mathcal{D}_{\text{Gr}(\mathcal{O}_k)}(\text{Gr}^+_R)$ from which it inherits a convolution product.

We define the perverse specialization

$$\mathcal{P}^R : \mathcal{P}_{G(\mathcal{O})}(\text{Gr}) \to \mathcal{P}_{G(\mathcal{O}_k)}(\text{Gr}^+_R)$$

as the sum

$$\mathcal{P}^R = \sum_k \mathcal{P}^H(k) \circ R$$

of the perverse homology sheaves of the specialization where the $Z$-grading corresponds to the degree of perverse homology.

### 6. Monoidal structure for specialization

The aim of this section is to construct an isomorphism

$$\mathcal{P}_R : \mathcal{P}^R(- \otimes -) \xrightarrow{\sim} \mathcal{P}^R(-) \otimes \mathcal{P}^R(-)$$

for the perverse specialization

$$\mathcal{P}^R : \mathcal{P}_{G(\mathcal{O})}(\text{Gr}) \to \mathcal{P}_{G(\mathcal{O}_k)}(\text{Gr}^+_R).$$

We shall refer to such an isomorphism as a *monoidal structure* for the functor. Observe that the composition of two functors with monoidal structures inherits a monoidal structure.

#### 6.1. Monoidal structure for local specialization ⇒ monoidal structure for perverse specialization.

Recall that the convolution

$$\otimes : \mathcal{D}_{\text{Gr}(\mathcal{O}_k)}(\text{Gr}^+_R) \times \mathcal{D}_{\text{Gr}(\mathcal{O}_k)}(\text{Gr}^+_R) \to \mathcal{D}_{\text{Gr}(\mathcal{O}_k)}(\text{Gr}^+_R)$$

is t-exact since the multiplication map is stratified semismall. This provides isomorphisms

$$\mathcal{P}^H_k : \mathcal{P}^H(k \otimes -) \xrightarrow{\sim} \sum_{m+n=k} \mathcal{P}^H(m) \otimes \mathcal{P}^H(n(-))$$

for the perverse homology $\sum_k \mathcal{P}^H_k : \mathcal{D}_{\text{Gr}(\mathcal{O}_k)}(\text{Gr}^+_R) \to \mathcal{P}_{\text{Gr}(\mathcal{O}_k)}(\text{Gr}^+_R)$. 

If we have a monoidal structure

$$r : R(\cdot \odot \cdot) \xrightarrow{\sim} R(\cdot) \odot R(\cdot)$$

for the local specialization $R : P_{G(\mathcal{O})}(\text{Gr}) \to D_{Gr(\mathcal{O}_\theta)}(\text{Gr}^+_\mathbb{R})$, then we obtain a monoidal structure

$$p_R : pR(\cdot \odot \cdot) \simeq pR(\cdot) \odot pR(\cdot)$$

for the perverse specialization $pR = \sum_k p^*H^k \circ R : P_{G(\mathcal{O})}(\text{Gr}) \to P_{Gr(\mathcal{O}_\theta)}(\text{Gr}^+_\mathbb{R})$.

6.2. Global convolution. Following [MV04, Section 5], we recall the global version of the convolution product. For a $\mathbb{C}$-algebra $A$, and $x \in X(A)$, let $(X^A)_x$ denote the formal neighborhood of the graph of $x$ in the product $X_A = X \times \text{Spec}(A)$.

Consider the global convolution diagram

$$\text{Gr}^{(1)} \times \text{Gr}^{(1)} \xrightarrow{\sim} \text{Gr}^{(1)} \times \text{Gr}^{(1)} \xrightarrow{\sim} \text{Gr}^{(1)} \times \text{Gr}^{(1)} \xrightarrow{m} \text{Gr}^{(2)} \xleftarrow{d} \text{Gr}^{(1)}.$$

The ind-scheme $\text{Gr}^{(1)} \times \text{Gr}^{(1)}$ represents the functor $A \mapsto \{(x_1, x_2, F_1, F_2, \nu_1, \nu_2, \mu_1)\}$ where for $i = 1, 2$, $x_i \in X(A)$, $F_i$ is a $G$-torsor on $X_A$, and $\nu_i$ is a trivialization of $F_i$ over $X_A \setminus x_i$, and $\mu_1$ is a trivialization of $F_1$ over $(X^A)_x$. The ind-scheme $\text{Gr}^{(1)} \times \text{Gr}^{(1)}$ represents the functor $A \mapsto \{(x_1, x_2, F_1, F, \nu_1, \eta)\}$ where $x_1, x_2 \in X(A)$, $F_1, F$ are $G$-torsors on $X_A$, $\nu_1$ is a trivialization of $F_1$ over $X_A \setminus x_1$, and $\eta$ is an isomorphism from $F_1$ to $F$ over $X_A \setminus x_2$. The map $p$ forgets the trivialization $\mu_1$. The map $q$ is given by $(F_1, F_2, \nu_1, \nu_2, \mu_1) \mapsto (F_1, F, \nu_1, \eta)$ where $F$ is obtained by gluing $F_1$ over $X_A \setminus x_1$ with $F_2$ over $(X^A)_x$ via the isomorphism $\nu_2 \circ \mu_1^{-1} \mid (X_A \setminus x_2) \cap (X^A)_x$. The map $m$ is given by $(F_1, F, \nu_1, \eta) \mapsto (F, \nu)$ where $\nu = \eta \circ \nu_1 \mid (X \setminus (x_1 \cup x_2))$. The map $d$ is the inclusion $\text{Gr}^{(1)} \xrightarrow{\sim} \text{Gr}^{(2)} \Delta \subset \text{Gr}^{(2)}$ where $\Delta$ is the diagonal in $X^2$.

The global convolution

$$\odot_X : P_{G, \mathcal{S}}(\text{Gr}^{(1)}) \times P_{G, \mathcal{S}}(\text{Gr}^{(1)}) \to P_{G, \mathcal{S}}(\text{Gr}^{(1)})$$

is defined to be

$$F_1 \odot_X F_2 = d^*m_!(F_1 \boxtimes F_2)$$

where $F_1 \boxtimes F_2$ is the unique sheaf with the required property since $p$ and $q$ are not the projections of torsors for a group-scheme which is the limit of linear algebraic groups. Fortunately, we may extend the diagram to obtain maps with this property. Consider the diagram

$$\begin{array}{ccc}
\text{Gr}^{(1)} \times \hat{X} \times \text{Gr} & \xleftarrow{\phi} & \text{Gr}^{(1)} \times \text{Gr}^{(1)} \\
\downarrow & & \downarrow q \\
\text{Gr}^{(1)} \times \text{Gr}^{(1)} & \xleftarrow{p} & \text{Gr}^{(1)} \times \text{Gr}^{(1)}.
\end{array}$$

The ind-scheme $\text{Gr}^{(1)} \times \hat{X}$ represents the functor $A \mapsto \{(x_1, x_2, F_1, \nu_1, \mu_1, \phi)\}$ where $x_1, x_2 \in X(A)$, $F_1$ is a $G$-torsor on $X_A$, $\nu_1$ is a trivialization of $F_1$ over $X_A \setminus x_1$, $\mu_1$ is a trivialization of $F_1$ over $(X^A)_x$, and $\phi$ is an isomorphism $\text{Spec}(A[[t]]) \to (X^A)_x$. 


The vertical map is the projection of an Aut(O)-torsor, and the diagonal maps are the projections of G(O) × Aut(O)-torsors.

6.3. **Real global convolution.** Consider the real global convolution diagram

\[
\text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \xrightarrow{\rho} \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \xrightarrow{q} \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \xrightarrow{m} \text{Gr}^{(2)}_\mathbb{R} \xleftarrow{d} \text{Gr}^{(1)}_\mathbb{R}.
\]

The real global convolution

\[
\text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \xrightarrow{\rho} \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \xrightarrow{q} \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \xrightarrow{m} \text{Gr}^{(2)}_\mathbb{R} \xleftarrow{d} \text{Gr}^{(1)}_\mathbb{R}
\]

is defined to be

\[\text{F}_1 \circ \text{F}_2 = \text{d}^* m_!(\text{F}_1 \boxtimes \text{F}_2)\]

where \(\text{F}_1 \boxtimes \text{F}_2\) is the unique sheaf such that \(q^*(\text{F}_1 \boxtimes \text{F}_2) = p^*(\text{F}_1 \boxtimes \text{F}_2)\). Here we may use the extended diagram

\[
\begin{array}{ccc}
\text{Gr}^{(1)}_\mathbb{R} \times \hat{X}^0 \times \text{Gr}^\mathbb{R} & \xrightarrow{\rho} & \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \xrightarrow{q} \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \\
\downarrow & & \downarrow \\
\text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} & \xrightarrow{\rho} & \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \xrightarrow{q} \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R}
\end{array}
\]

to see that there exists a unique sheaf \(\text{F}_1 \boxtimes \text{F}_2\) with the required property.

6.4. **Monoidal structure for global specialization \(\Rightarrow\) Monoidal structure for local specialization.**

**Lemma 6.4.1.** There is a canonical isomorphism

\[
\rho(\cdot) \circ_X \rho(\cdot) \simeq \rho(\cdot \circ \cdot)
\]

for \(\rho : \text{P}_{G(O)}(\text{Gr}) \to \text{P}_{G,S}(\text{Gr}^{(1)})\).

**Proof.** Consider the diagram

\[
\begin{array}{cccccc}
\text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} & \xrightarrow{\rho} & \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} & \xrightarrow{q} & \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} & \xrightarrow{m} & \text{Gr}^{(2)}_\mathbb{R} \\
d \uparrow & & d \uparrow & & d \uparrow & & d \uparrow \\
\text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \times \Delta & \xrightarrow{\rho} & \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \times \Delta & \xrightarrow{q} & \text{Gr}^{(1)}_\mathbb{R} \times \text{Gr}^{(1)}_\mathbb{R} \times \Delta & \xrightarrow{m} & \text{Gr}^{(2)}_\mathbb{R} \times \Delta \\
\text{X} \times \text{Gr} \times \text{Gr} & \xrightarrow{\rho} & \text{X} \times G(K) \times \text{Gr} & \xrightarrow{q} & \text{X} \times G(K) \times G(O) \text{Gr} & \xrightarrow{m} & \text{X} \times \text{Gr}
\end{array}
\]

The first row is the global convolution diagram, and the second is its restriction to the diagonal. The third row is the product of the local convolution diagram with the enhanced curve \(\hat{X}\). The arrows \(\text{d}\) are the inclusions, and the arrows \(s\Delta\) are the projections of Aut(O)-torsors.

Let \(r\) be the composition \(d \circ s\Delta\). The proof is a diagram chase using the standard identity \(r^* p^* \simeq p^* r^*\), \(r^* q^* \simeq q^* r^*\) for the composition of maps, the base change isomorphism \(r^* m_! \simeq m_! r^*\), and the canonical isomorphism \(r^*(\rho(\cdot) \boxtimes \rho(\cdot)) \simeq \text{C}_\text{X} \boxtimes (\cdot) \boxtimes (\cdot)\).
which results from a chase in the diagram
\[
\begin{array}{ccc}
\text{Gr}^{(1)} \times \text{Gr}^{(1)} & \xrightarrow{d} & \text{Gr}^{(1)} \times \text{Gr}^{(1)} \\
\downarrow s \Delta & & \uparrow s \times s \\
\hat{X} \times \text{Gr} \times \text{Gr} & \xrightarrow{d} & \hat{X} \times \hat{X} \times \text{Gr} \times \text{Gr} \\
\downarrow & & \downarrow \\
\text{Gr} \times \text{Gr} & = & \text{Gr} \times \text{Gr}.
\end{array}
\]

\[\square\]

Lemma 6.4.2. There is a canonical isomorphism
\[
\rho_{\mathbb{R}}^{-1}(\cdot) \circ \rho_{\mathbb{R}}^{-1}(\cdot) \simeq \rho_{\mathbb{R}}^{-1}(\cdot \circ x_{\mathbb{R}} \cdot)
\]
for \(\rho_{\mathbb{R}}^{-1} : D_{G_{\mathbb{R}}, s}(\text{Gr}(1)) \to D_{G_{\mathbb{R}}(\mathcal{O})}(\text{Gr}_{\mathbb{R}})\).

Proof. Fix \(x \in X_{\mathbb{R}}\), and a formal coordinate at \(x\). Consider the diagram
\[
\begin{array}{ccc}
\text{Gr}^{(1)}_{\mathbb{R}} \times \text{Gr}^{(1)}_{\mathbb{R}} & \overset{p}{\leftarrow} & \text{Gr}^{(1)}_{\mathbb{R}} \times \text{Gr}^{(1)}_{\mathbb{R}} \\
\downarrow l \uparrow r & & \downarrow l \uparrow r \\
\text{Gr}^{(1)}_{\mathbb{R}} \times \text{Gr}^{(1)}_{\mathbb{R}}(x, x) & \overset{q}{\leftarrow} & \text{Gr}^{(1)}_{\mathbb{R}} \times \text{Gr}^{(1)}_{\mathbb{R}}(x, x) \\
\downarrow m \uparrow l & & \downarrow m \uparrow l \\
\text{Gr}_{\mathbb{R}} \times \text{Gr}_{\mathbb{R}} & \overset{q}{\leftarrow} & \text{Gr}_{\mathbb{R}} \times \text{Gr}_{\mathbb{R}}
\end{array}
\]
The first row is the real global convolution diagram, and the second is its restriction to \((x, x) \in X^{2}_{\mathbb{R}}\). The third row is the real convolution diagram. The arrows \(r\) are the inclusions, and the isomorphisms are provided by the formal coordinate at \(x\).

The proof is a diagram chase using the standard identity \(r^*p^* \simeq p^*r^*,\ r^*q^* \simeq q^*r^*\) for the composition of maps, and the base change isomorphism \(r^*m_! \simeq m_!r^*\). \[\square\]

Now if we have a monoidal structure
\[
r_X : R_X(\cdot \circ \cdot) \xrightarrow{\sim} R_X(\cdot) \circ R_X(\cdot)
\]
for the global specialization \(R_X : D_{G_{\mathbb{R}}, s}(\text{Gr}(1)) \to D_{G_{\mathbb{R}}, s}(\text{Gr}_{\mathbb{R}})\), then we obtain a monoidal structure
\[
r : R(\cdot \circ \cdot) \xrightarrow{\sim} R(\cdot) \circ R(\cdot)
\]
for the local specialization \(R = \rho_{\mathbb{R}}^{-1} \circ R_X \circ \rho : P_{G(\mathcal{O})}(\text{Gr}) \to D_{G_{\mathbb{R}}(\mathcal{O})}(\text{Gr}_{\mathbb{R}})\).

6.5. Monoidal structure for global specialization.

Proposition 6.5.1. There is a canonical isomorphism
\[
r_X : R_X(\cdot \circ_{x} \cdot) \simeq R_X(\cdot) \circ_{x_{\mathbb{R}}} R_X(\cdot)
\]
for the global specialization \(R_X : P_{G, s}(\text{Gr}(1)) \to D_{G_{\mathbb{R}}, s}(\text{Gr}^{(1)}_{\mathbb{R}})\).

Proof. Consider the intertwining diagram
\[
\begin{array}{ccc}
\text{Gr}^{(2)} \times \text{Gr}^{(2)} & \overset{p}{\leftarrow} & \text{Gr}^{(2)} \times \text{Gr}^{(2)} \\
\downarrow q \uparrow m & & \downarrow q \uparrow m \\
\text{Gr}^{(2)} \times \text{Gr}^{(2)} & \overset{d}{\leftarrow} & \text{Gr}^{(2)}
\end{array}
\]
The ind-scheme $\widetilde{\text{Gr}}_{\mathbb{R}}^{(2)} \times \text{Gr}_{\mathbb{R}}^{(2)}$ represents the functor $A \mapsto \{(x_1, x_2, x_3, x_4, F_1, F_2, \nu_1, \nu_2, \mu_1)\}$ where $x_1, x_2, x_3, x_4 \in X(A)$, $F_1, F_2$ are $G$-torsors on $X_A$, $\nu_1$ is a trivialization of $F_1$ over $X_A \setminus (x_1 \cup x_2)$, $\nu_2$ is a trivialization of $F_2$ over $X_A \setminus (x_3 \cup x_4)$, and $\mu_1$ is a trivialization of $F_1$ over $(X_A)_{x_3 \cup x_4}$. The ind-scheme $\widetilde{\text{Gr}}_{\mathbb{R}}^{(2)} \times \text{Gr}_{\mathbb{R}}^{(2)}$ represents the functor $A \mapsto \{(x_1, x_2, x_3, x_4, F_1, F, \nu_1, \eta)\}$ where $x_1, x_2, x_3, x_4 \in X(A)$, $F_1, F$ are $G$-torsors on $X_A$, $\nu_1$ is a trivialization of $F_1$ over $X_A \setminus (x_1 \cup x_2)$, $\eta$ is an isomorphism from $F_1$ to $F$ over $X_A \setminus (x_3 \cup x_4)$. The maps are straightforward generalizations of the maps in the global convolution diagram.

To construct the isomorphism of the proposition, we work with the real intertwining diagram

$$\text{Gr}_{\mathbb{R}}^{(\sigma)} \times \text{Gr}_{\mathbb{R}}^{(\sigma)} \xrightarrow{p} \text{Gr}_{\mathbb{R}}^{(\sigma)} \times \text{Gr}_{\mathbb{R}}^{(\sigma)} \xrightarrow{q} \text{Gr}_{\mathbb{R}}^{(\sigma)} \times \text{Gr}_{\mathbb{R}}^{(\sigma)} \xrightarrow{m} \text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)} \xrightarrow{d} \text{Gr}_{\mathbb{R}}^{(\sigma)}$$

which is the real form of the intertwining diagram with respect to the conjugation associated to the product element $\sigma \times \sigma \in \Sigma_2 \times \Sigma_2 \subset \Sigma_4$.

To simplify the notation in what follows, write $Y_+ = X_+ \times X_+$ and $Y_\mathbb{R} = X_\mathbb{R} \times X_\mathbb{R}$ for the products, and $Y = Y_+ \cup Y_\mathbb{R}$ for their union. Let $\Delta \subset Y$, $\Delta^+ \subset Y^+$, and $\Delta_\mathbb{R} \subset Y_\mathbb{R}$ denote the diagonals, and $j : Y^+ \to Y$, $i : Y_\mathbb{R} \to Y$, and $d : \Delta \to Y$ the inclusions.

Observe that the restriction of the real intertwining diagram to $Y_\mathbb{R}$ is canonically identified with the real global convolution diagram. Thus we may use it to calculate the real global convolution. The restriction of the real intertwining diagram to $Y^+$ is canonically identified with the restriction of two copies of the global convolution diagram to $X_+$. The following lemma confirms that we may use this to calculate the global convolution.

**Lemma 6.5.1.** There is a canonical isomorphism $(\cdot \odot_X \cdot)|X_+ \simeq (\cdot|X_+ \odot_X (\cdot|X_+))$.

**Proof.** We break the global convolution into two steps.

First, consider the diagram

$$\text{Gr}^{(1)} \times \text{Gr}^{(1)} \xrightarrow{j} \text{Gr}^{(1)} \times \text{Gr}^{(1)} \xrightarrow{j} \text{Gr}^{(1)} \times \text{Gr}^{(1)}$$

We have the standard identity $j^*p^* \simeq p^*j^*$, $j^*q^* \simeq q^*j^*$ for the composition of maps. It is a diagram chase to see that this provides an isomorphism $(\cdot \odot \cdot)|Y^+ \simeq (\cdot|X_+ \odot \cdot|X_+)$. Next, consider the diagram

$$\text{Gr}^{(1)} \times \text{Gr}^{(1)} \xrightarrow{j} \text{Gr}^{(2)} \xrightarrow{j} \text{Gr}^{(1)}$$

We have the base change isomorphism $j^*m_1 \simeq m_1j^*$, and the standard identity $j^*d^* \simeq d^*j^*$ for the composition of maps. The required isomorphism is the composition

$$(\cdot \odot_X \cdot)|X_+ = j^*d^*m_1(\cdot \odot \cdot) \simeq d^*m_1j^*(\cdot \odot \cdot) \simeq d^*m_1((\cdot|X_+ \odot \cdot)|X_+) = (\cdot|X_+ \odot_X (\cdot|X_+))$$
Lemma 6.5.2. There is a canonical isomorphism $i^* j_*(\tilde{\mathcal{X}}) \simeq (i^* j_* \tilde{\mathcal{X}})$.

Proof. This is a diagram chase using the standard identity $i^* p^* \simeq p^* i^*$, $i^* q^* \simeq q^* i^*$ for the composition of maps, and the smooth base change isomorphism $j_* p^* \simeq p^* j_*$, $j_* q^* \simeq q^* j_*$. The maps $p$ and $q$ are smooth since they are the projections of torsors for the smooth relative group-scheme $G^R(\mathcal{O}_R) \to X^R$ which is a real form of the smooth relative group-scheme $G(2)(\mathcal{O}) \to X^2$ that represents the functor $A \mapsto \{(x_1, x_2, \mu)\}$ where $x_1, x_2 \in X(A)$ and $\mu$ is a trivialization of the trivial $G$-torsor over $(X \cup x_1, X \cup x_2)$.

Next, consider the middle portion of the real intertwining diagram

$\begin{array}{cc}
\text{Gr}_R^{(s)} & \times \text{Gr}_R^{(s)} \mid Y^+ \\
\downarrow j & \\
\text{Gr}_R^{(s)} & \times \text{Gr}_R^{(s)} \mid Y \\
\downarrow i & \\
\text{Gr}_R^{(s)} & \times \text{Gr}_R^{(s)} \mid Y_R \\
\end{array}
\xleftarrow{m} 
\begin{array}{cc}
\text{Gr}_R^{(s)} & \times \text{Gr}_R^{(s)} \mid Y^+ \\
\downarrow j & \\
\text{Gr}_R^{(s)} & \times \text{Gr}_R^{(s)} \mid Y \\
\downarrow i & \\
\text{Gr}_R^{(s)} & \times \text{Gr}_R^{(s)} \mid Y_R \\
\end{array}
\xrightarrow{m}
\begin{array}{cc}
\text{Gr}_R^{(s)} & \times \text{Gr}_R^{(s)} \mid Y^+ \\
\downarrow j & \\
\text{Gr}_R^{(s)} & \times \text{Gr}_R^{(s)} \mid Y \\
\downarrow i & \\
\text{Gr}_R^{(s)} & \times \text{Gr}_R^{(s)} \mid Y_R \\
\end{array}

Lemma 6.5.3. There is a canonical isomorphism $i^* j_* m_1 \simeq m_1 i^* j_*$.

Proof. This follows from the base change isomorphism $m i^* \simeq i^* m$, the standard identity $m_1 j_* \simeq j_* m_1$ for the composition of maps, and the fact that $m_1 = m_1$ for the sheaves under consideration since $m$ is proper on their supports.

Finally, consider the right-hand portion of the real intertwining diagram

$\begin{array}{cc}
\text{Gr}_R^{(s \times s)} \mid Y^+ & \xleftarrow{d} \\
\downarrow j & \\
\text{Gr}_R^{(s \times s)} \mid \Delta^+ \\
\downarrow j & \\
\text{Gr}_R^{(s \times s)} \mid Y^2 & \xleftarrow{d} \\
\downarrow i & \\
\text{Gr}_R^{(s \times s)} \mid \Delta \\
\downarrow i & \\
\text{Gr}_R^{(s \times s)} \mid Y_R & \xleftarrow{d} \\
\downarrow j & \\
\text{Gr}_R^{(s \times s)} \mid \Delta_R \\
\end{array}

Lemma 6.5.4. There is a canonical isomorphism $d^* i^* j_* \simeq i^* j_* d^*$.

Proof. To obtain the candidate for the isomorphism, we apply $d^* j_*$ to the adjunction morphism $\text{Id} \to d_! d^*$. We obtain the morphism $d^* j_* \to d^* j_* d_* d^* \simeq d^* d_* j_* d^* \simeq j_* d^*$, where the first isomorphism is the standard identity for the composition of maps, and the second is the isomorphism obtained from the standard identity for the composition of maps.
and the second follows from the fact that the adjunction morphism $d^*d_* \to \text{Id}$ is an isomorphism since $d$ is an embedding. Now applying $i^*$ to the above composite morphism, we obtain

$$d^*i^*j_* \sim i^*d^*j_* \to i^*j_*d^*,$$

where the initial isomorphism is the standard identity for the composition of maps. We shall prove this morphism is an isomorphism.

Write $W^+ = Y^+ \setminus \Delta^+$ for the complement, and $k : \text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}|Y^+ \to \text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}|Y$ for the inclusion. The adjunction morphism $\text{Id} \to d_*d^*$ fits into a distinguished triangle $k!k^* \to \text{Id} \to d_*d^*$ \cite{[1]}. Applying $i^*d^*j_*$ and using the isomorphism $d^*j_*d_*d^* \sim j_*d^*$, we obtain the distinguished triangle $i^*d^*j_*k!k^* \to i^*d^*j_* \to i^*j_*d^*$ \cite{[1]}. We shall prove

$$i^*d^*j_*k!k^* = 0,$$

which immediately implies the composition $d^*i^*j_* \sim i^*d^*j_* \to i^*j_*d^*$ is an isomorphism.

To establish this, we show that for any sheaf $\mathcal{F} \in D_{G_{\mathbb{R},S}(\text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}|Y^+)}$ the stalk cohomology vanishes $H(j_*k!k^*\mathcal{F})_x = 0$, for $x \in \text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}|\Delta_{\mathbb{R}}$. To be specific, we show that for any open neighborhood $N \subset \text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}|Y$ containing $x$ there is an open neighborhood $N' \subset N$ containing $x$ such that the hypercohomology vanishes

$$\mathbb{H}(N', j_*k!k^*\mathcal{F}) = 0.$$

For the remainder of the proof, we fix a point $x \in \text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}|\Delta_{\mathbb{R}}$ and open neighborhood $N \subset \text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}|Y$ containing $x$.

**Lemma 6.5.5.** There is a closed ball $B \subset N$ with $x \in B^0$, and an open ball $D \subset Y$ with $\pi(x) \in D$ such that $\pi : B|D \to D$ is a proper, stratified submersion of Whitney stratified sets.

**Proof.** The argument is standard in the theory of nearby cycles. The essential point is that the projection $\pi : \text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}|Y \to Y$ is Thom stratified by Proposition 4.5.1. This implies that if $B$ is chosen so that its boundary $\partial B$ is transverse to all the strata in $\text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}|X_{\mathbb{R}}$, then $\partial B$ will also be transverse to all the strata in the nearby fibers of $\pi$. Thus one can find $D$ such that $B|D$ is Whitney stratified by the restriction of the strata of $\text{Gr}_{\mathbb{R}}^{(\sigma \times \sigma)}$. $\square$

Fix a closed ball $B$ and an open ball $D$ as provided by the lemma. We first assert that it is possible to choose an open ball $D' \subset D$ containing $x$ so that there is a complete vector field on $W^+ \cap D'$ whose flow $f_t$ satisfies $\lim_{t \to 0} f_t(w) \in \Delta^+$ and $\lim_{t \to \infty} f_t(w) \notin \Delta^+$, for all $w \in W^+ \cap D'$. This is not difficult to see by examining the stratification of $Y$.

We next assert that for the neighborhood $N' = B^0|D'$, we have the vanishing $\mathbb{H}(N', j_*k!k^*\mathcal{F}) = 0$. To see this, first note $\mathbb{H}(N', j_*k!k^*\mathcal{F}) \simeq \mathbb{H}(N'|Y^+, k!k^*\mathcal{F})$. Now by the lemma and the first Thom-Mather isotopy lemma (see \cite{[Mat70]} or \cite{[GM88]} Section 1.5), we may lift the flow $f_t$ on $W^+ \cap D'$ to a flow $F_t$ on $B|(W^+ \cap D')$. Then by restriction we obtain a lift to $N'|W^+$. It follows that $\mathbb{H}(N'|Y^+, k!k^*\mathcal{F}) = 0$, since any
cycle in this group must be supported away from $N'|\Delta^+$, and hence may be realized as a boundary using the flow $F_t$ on $N'|W^+$. □

Finally, to complete the proof of the proposition, the monoidal structure

$$r_X : R_X(\cdot \circ_X \cdot) \simeq R_X(\cdot) \circ_X R_X(\cdot)$$

for the global specialization $R_X : P_{G,\mathbb{S}}(Gr^{(1)}) \to D_{G,\mathbb{S}}(Gr^{(1)}_R)$ is obtained from the isomorphisms of the lemmas via a diagram chase. □

7. Image category $I$

First, we define $Q(Gr_R)$ to be the strict full subcategory of $P_{G(\mathcal{O})}(Gr)$ whose objects are isomorphic to subquotients of objects of the form $p_R(\mathcal{P})$, where $\mathcal{P}$ runs through all objects of $P_{G(\mathcal{O})}(Gr)$, and $p_R$ is the perverse specialization $p_R : P_{G(\mathcal{O})}(Gr) \to P_{G(\mathcal{O})}(Gr^+_R)$.

Next, we define $Q(Gr_R)$ to be the strict full subcategory of $P_{G(\mathcal{O})}(Gr)$ whose objects are isomorphic to objects of the form $F(Q)$, where $Q$ runs through all objects of $Q(Gr_R)$, and $F$ is the forgetful functor $F : P_{G(\mathcal{O})}(Gr^+_R) \to P_{G(\mathcal{O})}(Gr^+_R)$.

Finally, we define $Q(Gr_R)_Z$ to be the category $Q(Gr_R) \otimes \text{Vect}_Z$, and identify it with the strict full subcategory $\sum_k Q(Gr_R)[k]$ of the category $P_{G(\mathcal{O})}(Gr^+_R)_Z$.

Each of the three categories $Q(Gr_R), Q(Gr_R)_R,$ and $Q(Gr_R)_Z$ inherits a convolution product from $P_{G(\mathcal{O})}(Gr^+_R)_Z$. We may organize the categories into a commutative diagram of functors with monoidal structure

$$Q(Gr_R)_R \xrightarrow{e} Q(Gr_R)_Z \xrightarrow{F} Q(Gr_R).$$

Here the arrows labelled $e$ are the obvious fully faithful functors, and the arrows labelled $F$ are the obvious forgetful functors.

8. Character functors

In this section, we discuss the weight functors first introduced in [MV00] and studied in [MV04]. See also [NP01].

8.1. Semi-infinite orbits. Let $U$ be the unipotent radical of the Borel subgroup $B \subset G$.

The group $U(\mathcal{K})$ acts on the loop Grassmannian $Gr$ by left-multiplication. Recall that each coweight $\nu \in \Lambda_T$ defines a point $\nu \in Gr$. Let $S^\nu$ be the $U(\mathcal{K})$-orbit $U(\mathcal{K}) \cdot \nu \subset Gr$ through $\nu$. By the Iwasawa decomposition [IM65 Proposition 2.33], each $U(\mathcal{K})$-orbit in $Gr$ is of the form $S^\nu$, for some coweight $\nu \in \Lambda_T$, and the orbits are distinct for distinct coweights. For the rest of the following, see [MV04 Proposition 3.1].
Proposition 8.1.1. The loop Grassmannian $\text{Gr}$ is the disjoint union of the $U(K)$-orbits $S^\nu$ through $\nu \in \Lambda_T$. The closure of the orbit $S^\nu$ is the union of the orbits $S^\mu$ with $\mu \leq \nu$.

Note that there is a similar statement for the orbits $T^\nu = U^0(K) \cdot \nu \subset \text{Gr}$, where $U^0 \subset G$ is the unipotent subgroup opposite to $U$.

8.2. Real semi-infinite orbits. Let $U_{P_R}$ be the unipotent radical of the minimal parabolic subgroup $P_R \subset G_R$.

The group $U_{P_R}(K_R)$ acts on the real loop Grassmannian $\text{Gr}_R$ by left-multiplication. Recall that each real coweight $\nu \in \Lambda_S$ defines a point $\nu \in \text{Gr}_R$. Let $S^\nu_R$ be the $U_{P_R}(K_R)$-orbit $U_{P_R}(K_R) \cdot \nu \subset \text{Gr}_R$ through $\nu$.

Lemma 8.2.1. For $\nu \in \Lambda_S$, the $U_{P_R}(K_R)$-orbit $S^\nu_R$ equals the intersection $S^\nu \cap \text{Gr}_R$, and for $\nu \in \Lambda_T \setminus \Lambda_S$, the intersection $S^\nu \cap \text{Gr}_R$ is empty.

Proof. We show that if a $U_P(K)$-orbit in $\text{Gr}$ intersects $\text{Gr}_R$, then it must contain an element of $\Lambda_S$. By Proposition 8.1.1, the group $P(K)$ acts transitively on $\text{Gr}$, and thus each $U_P(K)$-orbit intersects the loop Grassmannian $\text{Gr}_M$ of the Levi factor $M \subset P$. Therefore it suffices to show that the real loop Grassmannian of the Levi factor $M_R \subset P_R$ is equal to $\Lambda_S$, or equivalently, that the real loop Grassmannian of the derived group $M'_R \subset M_R$ is equal to a single point. Since $M'_R$ is compact, this follows immediately from Proposition 8.6.1.

We have the following Iwasawa decomposition.

Proposition 8.2.1. The real loop Grassmannian $\text{Gr}_R$ is the disjoint union of the $U_{P_R}(K_R)$-orbits $S^\nu_R$ through $\nu \in \Lambda_S$. The closure of the orbit $S^\nu_R$ is the union of the orbits $S^\mu_R$ with $\mu \leq \nu$.

Proof. Thanks to the previous lemma and Proposition 8.1.1, it only remains to prove that if $\nu - \mu$ is a non-negative integral linear combination of positive coroots of $G$, then $S^\mu_R \subset S^\nu_R$. The action on $\text{Gr}_R$ of elements of the real coweight lattice $\Lambda_S$ translate the orbits, so we may assume that $\nu$ is the trivial coweight 0. By Lemma 2.1.1, it suffices to prove the assertion when $-\mu$ is a positive coroot $\alpha \in R^\text{pos}$, or when $-\mu$ is of the form $\theta(\alpha) + \alpha$, for a positive coroot $\alpha \in R^\text{pos}$, but $-\mu$ is not a multiple of a positive coroot. In the first case, we may find $\text{SL}_2(\mathbb{R}) \subset G_R$ such that $\alpha$ is its positive coroot. Then the orbit through 0 of the one parameter subgroup $U_\alpha(rt^{-1}) \subset U_{P_R}(K_R)$, for $r \in \mathbb{R}$, is isomorphic to $\mathbb{R}$, and its closure is isomorphic to $\mathbb{RP}^1$ with $\mu$ the point at infinity. In the second case, we may find $\text{SL}_2(\mathbb{C}) \subset G_R$ such that $\theta(\alpha) + \alpha$ is its positive coroot. Then the orbit through 0 of the one parameter subgroup $U_{\theta(\alpha)+\alpha}(ct^{-1}) \subset U_{P_R}(K_R)$, for $c \in \mathbb{C}$, is isomorphic to $\mathbb{CP}^1$, and its closure is isomorphic to $\mathbb{CP}^1$ with $\mu$ the point at infinity.

Note that there are similar results for the orbits $T^\nu_R = U_{P_R}(K_R) \cdot \nu \subset \text{Gr}_R$, where $U_{P_R} \subset G_R$ is the unipotent radical of the parabolic subgroup $P_R \subset G_R$ opposite to $P_R$. 
8.3. Flows. For $\xi \in \mathfrak{g}$, we have the flow
\[ \varphi^t_\xi : \mathbb{R} \times \text{Gr} \to \text{Gr} \]
\[ \varphi^t_\xi(x) = \exp(t\xi) \cdot x \]
generated by $\xi$ via the action of $G$ on $\text{Gr}$.

Choose a Cartan involution of $G$ which commutes with the conjugation $\theta$, and let $G_c \subset G$ be the resulting maximal compact subgroup.

For the following, see [MV04, Sections 3 and 6] or [NP01, Section 4] for more details.

**Lemma 8.3.1.** Let $\xi \in \mathfrak{t} \cap i\mathfrak{g}_c$ be in the interior of the dominant Weyl chamber. Then the fixed points in $\text{Gr}$ of the flow $\varphi^t_\xi$ are the coweights $\nu \in \mathbf{\Lambda}_T$, and the orbits $S_\nu'$ are the ascending sets
\[ S_\nu'^t = \{ x \in \text{Gr} | \lim_{t \to +\infty} \varphi^t_\xi(x) = \nu \}, \]
and the orbits $T_\nu'$ are the descending sets
\[ T_\nu'^t = \{ x \in \text{Gr} | \lim_{t \to -\infty} \varphi^t_\xi(x) = \nu \}. \]

**Proof.** The torus $T$ centralizes $\xi$, and the weights of $\xi$ in $\mathfrak{u}$ are positive, and the weights in $\mathfrak{u}$ are negative. \qed

For $\xi \in \mathfrak{g}_R$, the flow generated by $\xi$ preserves the real form $\text{Gr}_R$ since the action of $G_R$ preserves $\text{Gr}_R$. Recall that the torus $S_R \subset G_R$ is split, so that $s_R \subset i\mathfrak{g}_c$.

**Lemma 8.3.2.** Let $\xi \in \mathfrak{s}_R$ be in the interior of the dominant Weyl chamber for $W_{\text{Gr}}$. Then the fixed points in $\text{Gr}_R$ of the flow $\varphi^t_\xi$ are the real coweights $\nu \in \mathbf{\Lambda}_S$, and the orbits $S_{\nu}'_R$ are the ascending sets
\[ S_{\nu}'_R = \{ x \in \text{Gr}_R | \lim_{t \to +\infty} \varphi^t_\xi(x) = \nu \}, \]
and the orbits $T_{\nu}'_R$ are the descending sets
\[ T_{\nu}'_R = \{ x \in \text{Gr}_R | \lim_{t \to -\infty} \varphi^t_\xi(x) = \nu \}. \]

**Proof.** The torus $S_R$ centralizes $\xi$, and the weights of $\xi$ in $\mathfrak{u}_{P_R}$ are positive, and the weights in $\mathfrak{u}$ are negative. \qed

Let $P' \subset P$ be the derived group of the parabolic subgroup $P \subset G$. Let $Q_M \subset \mathbf{\Lambda}_T$ be the coroot lattice of the Levi factor $M \subset P$.

For $\bar{\nu} \in \Lambda_T/Q_M$, choose $\nu \in \Lambda_T$ projecting to $\bar{\nu}$. Let $S_\nu^0_P$ be the connected component of the orbit $P'(K) \cdot \nu \subset \text{Gr}$ through $\nu$, and let $\text{Gr}_{M,\bar{\nu}}$ be the connected component of the loop Grassmannian $\text{Gr}_M$ containing $\nu$. The notation is justified by the fact that for $\bar{\nu} \in \Lambda_T/Q_M$, and $\nu \in \Lambda_T$ projecting to $\bar{\nu}$, the connected component $S_\nu^0_P$ is the disjoint union of the $U(K)$-orbits $S_\mu^0$, for $\mu = \nu$ mod $Q_M$, and the connected component $\text{Gr}_{M,\bar{\nu}}$ contains $\mu \in \Lambda_T$ if and only if $\mu = \nu$ mod $Q_M$.

Similarly, for $\bar{\nu} \in \Lambda_T/Q_M$, we have the connected components $T_\nu^0_P$ of the orbits $P^0(P'(K) \cdot \nu \subset \text{Gr}$ where $P^0 \subset G$ is the parabolic subgroup opposite to $P$, and $P^0$ is its derived group.
Lemma 8.3.3. Let $\xi \in s_\mathbb{R}$ be in the interior of the dominant Weyl chamber for $W_G$. Then the fixed points in $Gr$ of the flow $\varphi_\xi$ are the components $Gr_{M,\rho}$, and the components $S^\rho_P$ are the ascending sets

$$S^\rho_P = \{ x \in Gr | \lim_{t \to +\infty} \varphi_\xi^t(x) \in Gr_{M,\rho} \},$$

and the components $T^\rho_P$ are the descending sets

$$T^\rho_P = \{ x \in Gr_\mathbb{R} | \lim_{t \to -\infty} \varphi_\xi^t(x) \in Gr_{M,\rho} \}.$$

Proof. The Levi factor $M$ centralizes $\xi$, the weights of $\xi$ in $u_P$ are positive, and the weights in $u_P^o$ are negative. □

8.4. Weight functors. Although in general the $U(K)$-orbits $S^\nu \subset Gr$ are neither finite-dimensional nor finite-codimensional, their intersections with the strata provide perverse cell decompositions of the strata. The results of this section are all from [MV04, Section 3].

Proposition 8.4.1 ([MV04], Theorem 3.2). For $\lambda \in \Lambda^+_T$, and $\nu \in \Lambda_T$, the orbit $S^\nu$ meets the stratum $Gr^\lambda$ if and only if $\nu \in Gr^\lambda$. In this case, we have

$$\dim_c(Gr^\lambda \cap S^\nu) = \langle \check{\rho}, \lambda + \nu \rangle.$$

This and the subsequent proposition are the primary ingredients in the proof of the following.

Theorem 8.4.1 ([MV04], Theorem 3.5). For $\nu \in \Lambda_T$, and $P \in P_{G(O)}(Gr)$, we have the vanishing

$$H^k_c(S^\nu, P) = 0 \text{ if } k \neq 2\langle \check{\rho}, \nu \rangle.$$

Proposition 8.4.2 ([MV04], Theorem 3.5). For $P \in P_{G(O)}(Gr)$, there is a canonical isomorphism

$$H^k_c(S^\nu, P) \simeq H^k_T(Gr, P), \text{ for all } k.$$

For $\nu \in \Lambda_T$, we call the functor

$$F^\nu : P_{G(O)}(Gr) \to \text{Vect}$$

$$F^\nu(\cdot) = H^2_c(\check{\rho}, \nu)(S^\nu, \cdot)$$

a weight functor.

Corollary 8.4.1 ([MV04], Theorem 3.5). For $\nu \in \Lambda_T$, the weight functor $F^\nu : P_{G(O)}(Gr) \to \text{Vect}$ is exact.

Recall that $\text{Vect}_{\Lambda_T}$ is the category of finite-dimensional $\Lambda_T$-graded vector spaces. We collect the weight functors into a functor

$$\text{Ch} : D_{G(O)}(Gr) \to \text{Vect}_{\Lambda_T}$$

$$\text{Ch} = \sum_{\nu \in \Lambda_T} F^\nu$$

we call the character functor.

Recall that $H^c$ is the hypercohomology functor, and $F$ the forgetful functor which forgets the grading of a vector space.
Corollary 8.4.2 ([MV04], Theorem 3.6). There is a canonical isomorphism
\[ \mathbb{H} \simeq F \circ Ch : P_{G(O)}(Gr) \to Vect. \]
More generally, for any set \( Y \subset \Lambda_T \) of coweights, there is a canonical isomorphism
\[ H^\ast_c \left( \bigcup_{\nu \in Y} S^\nu, \cdot \right) \simeq \sum_{\nu \in Y} H^\ast_c (S^\nu, \cdot) : P_{G(O)}(Gr) \to Vect. \]

Corollary 8.4.3 ([MV04], Corollary 3.7). The hypercohomology functor \( \mathbb{H} : P_{G(O)}(Gr) \to Vect \) is exact and faithful.

8.5. Real weight functors. In the real case, the basic dimension estimate takes the following form.

Proposition 8.5.1. For \( \lambda \in \Lambda^+_S \), and \( \nu \in \Lambda_S \), the orbit \( S^\nu \) meets the stratum \( \text{Gr}_R^\lambda \) if and only if \( \nu \in \overline{\text{Gr}_R^\lambda} \). In this case, we have
\[ \dim_R (\text{Gr}_R^\lambda \cap S^\nu) \leq \langle \bar{\rho}, \lambda + \nu \rangle. \]

Proof. By Lemma 8.2.1, \( S^\nu \) equals the intersection \( S^\nu \cap \text{Gr}_R^\lambda \), and so \( S^\nu \cap \overline{\text{Gr}_R^\lambda} \) equals the intersection \( (S^\nu \cap \text{Gr}_R^\lambda) \cap \text{Gr}_R^\lambda \). Thus the result follows immediately from Proposition 8.4.1.

Theorem 8.5.1. For \( \nu \in \Lambda_S \), and \( P \in P_{G_k(O_k)}(\text{Gr}_R^+) \), we have the vanishing
\[ H^k_c (S^\nu, P) = 0 \text{ if } k \neq \langle \bar{\rho}, \nu \rangle. \]

Proof. The dimension estimate implies the vanishings
\[ H^k_c (S^\nu, P) = 0 \text{ for } k > \langle \bar{\rho}, \nu \rangle, \]
\[ H^k_{T_R}(\text{Gr}_R, P) = 0 \text{ for } k < \langle \bar{\rho}, \nu \rangle. \]
Therefore the following proposition implies the theorem.

Proposition 8.5.2. For \( F \in D_{G_k(O_k)}(\text{Gr}_R) \), there is a canonical isomorphism
\[ H^k_c (S^\nu, F) \simeq H^k_{T_R}(\text{Gr}_R, F), \text{ for all } k. \]

Proof. For \( \xi \in s_R \) in the interior of the dominant Weyl chamber for \( W_{G_k} \), the map \( \varphi_\xi : \text{Gr}_R \to \text{Gr}_R \) defined by \( \varphi_\xi (x) = \exp(\xi) \cdot x \) is weakly hyperbolic in the sense of [GM93, Section 1.3]. Therefore by Lemma 8.3.2 and [GM93, Proposition 9.2], we have the asserted isomorphism.

For \( \nu \in \Lambda_S, z \in \mathbb{Z} \), we call the functor
\[ F^{\nu,z}_R : D_{G_k(O_k)}(\text{Gr}_R) \to Vect \]
\[ F^{\nu,z}_R (\cdot) = H^c_c (S^\nu, \cdot) \]
a real weight functor. When \( F^{\nu,z}_R \) is applied to perverse sheaves, the integer \( z \) becomes superfluous by the theorem. We sometimes then forget it and simply write \( F^\nu_R : P_{G_k(O_k)}(\text{Gr}_R^+) \to Vect_{\Lambda_S} \) for the functor \( F^\nu_R (\cdot) = H^c_c (S^\nu, \cdot) \).

Corollary 8.5.1. For \( \nu \in \Lambda_S \), the weight functor \( F^\nu_R : P_{G_k(O_k)}(\text{Gr}_R^+) \to Vect \) is exact.
We collect the real weight functors into a functor
\[ \text{Ch}_\mathbb{R} : \mathcal{D}_{G_\mathbb{R}(\mathcal{O}_\mathbb{R})}(\text{Gr}_\mathbb{R}) \to \text{Vect}_{S \times \mathbb{Z}} \]
\[ \text{Ch}_\mathbb{R} = \sum_{\nu \in \Lambda_S} \sum_{z \in \mathbb{Z}} F_{\nu, z}^{\mathbb{R}} \]
we call the real character functor.

**Remark 8.5.1.** In the real setting, the critical degree \( k \) for the cohomology group \( H^k_c(S^\nu_\mathbb{R}, \mathcal{P}) \), for \( \mathcal{P} \in \mathcal{P}_{G_\mathbb{R}(\mathcal{O}_\mathbb{R})}(\text{Gr}_\mathbb{R}^+) \), is half of what it is in the complex case. In particular, it is not always of even parity.

8.6. **Two useful facts.** In contrast to the complex case, the intersections of the \( U_{P_\mathbb{R}}(K_\mathbb{R}) \)-orbits \( S^\nu_\mathbb{R} \subset \text{Gr}_\mathbb{R} \) with the strata are in general not of pure dimension. In addition, in some cases their dimension is strictly less than the upper bound seen in Proposition 8.5.1. We note here the further vanishing this implies.

**Proposition 8.6.1.** For \( \lambda \in \Lambda_\mathbb{T}^+ \), and \( \nu \in \Lambda_\mathbb{S} \), if we have
\[ \dim_\mathbb{R}(\text{Gr}_\mathbb{R}^\lambda \cap S^\nu_\mathbb{R}) < \langle \check{\rho}, \lambda + \nu \rangle, \]
then for \( \mathcal{P} \in \mathcal{P}_{G_\mathbb{R}(\mathcal{O}_\mathbb{R})}(\text{Gr}_\mathbb{R}^+) \), we have the vanishing
\[ H^k_c(S^\nu_\mathbb{R}, \mathcal{P}) = 0 \text{ for all } k. \]

**Proof.** The dimension assumption implies the vanishing
\[ H^k_c(S^\nu_\mathbb{R}, \mathcal{P}) = 0 \text{ for } k \geq \langle \check{\rho}, \nu \rangle, \]
and the dimension estimate of Proposition 8.5.1 implies the vanishing
\[ H^k_c(\text{Gr}_\mathbb{R}, \mathcal{P}) = 0 \text{ for } k < \langle \check{\rho}, \nu \rangle, \]
so by Proposition 8.5.2 we have the asserted vanishing. \( \square \)

We have the following symmetry of the real weight functors analogous to [NP01, Section 5]. Recall that the small Weyl group \( \mathcal{W}_{G_\mathbb{R}} \) acts on the real coweights \( \Lambda_\mathbb{S} \).

**Proposition 8.6.2.** For \( \nu \in \Lambda_\mathbb{S}^+ \), and \( w \in \mathcal{W}_{G_\mathbb{R}} \), we have
\[ F_{\nu}(\text{IC}^\nu) \simeq \mathbb{C} \]
where \( \text{IC}^\nu \) is the intersection cohomology sheaf of the closure of the stratum \( \text{Gr}_\mathbb{R}^\nu \).

**Proof.** The intersection \( \text{Gr}_\mathbb{R}^\nu \cap \text{IC}^\nu \) is a real affine space, and by Proposition 8.2.1, the intersection \( \text{Gr}_\mathbb{R}^\nu \cap \text{IC}^\nu \) is empty for any \( \mu \) in the closure of \( \text{Gr}_\mathbb{R}^\nu \). \( \square \)

8.7. **Global preliminaries.** For \( x \in X \), and a choice of formal coordinate at \( x \), the \( U(K_\mathbb{R}) \)-orbits in the local loop Grassmannian \( _x\text{Gr} \) are taken to the \( U(K) \)-orbits in \( \text{Gr} \) under the isomorphism \( _x\text{Gr} \cong \text{Gr} \). Since the \( U(K) \)-orbits in \( \text{Gr} \) are \( \text{Aut}(\mathcal{O}) \)-invariant, we may unambiguously index the \( U(K_\mathbb{R}) \)-orbits \( _xS^\nu \subset _x\text{Gr} \) by coweights \( \nu \in \Lambda_\mathbb{T} \).

Similarly, for \( x \in X_{\mathbb{R}} \), we may unambiguously index the \( U_{P_\mathbb{R}}(K_{\mathbb{R}x}) \)-orbits \( _xS^\nu_\mathbb{R} \subset _x\text{Gr}_\mathbb{R} \) by real coweights \( \nu \in \Lambda_\mathbb{S} \).

Recall the factorization of the real form \( \text{Gr}_\mathbb{R}^{(\omega)} \) with respect to the projection
\[ \pi : \text{Gr}_\mathbb{R}^{(\omega)} \to X^{(\omega)}. \]
Fixing an identification $X^{(\omega)} \simeq X^*_R \times X^s$, for $(x_1, \ldots, x_r, z_1, \ldots, z_s) \in X^*_R \times X^s$, we have a canonical isomorphism

$$\text{Gr}^{(\omega)}_{\mathcal{R}}(x_1, \ldots, x_r, z_1, \ldots, z_s) \simeq \prod_{i=1}^k y_i \text{Gr}_{\mathcal{R}} \times \prod_{j=1}^\ell w_j \text{Gr}.$$ 

where $y_1, \ldots, y_k \in X_R, w_1, \ldots, w_\ell \in X$ are the distinct points with an equality of sets

$$\{y_1, \ldots, y_k, w_1, \ldots, w_\ell\} = \{x_1, \ldots, x_r, z_1, \ldots, z_s\}.$$ 

For $\nu \in \Lambda_S$, let $S^{(\omega)}_\nu \subset \text{Gr}^{(\omega)}_{\mathcal{R}}$ be the union over all fibers of the subspaces

$$\prod_{i=1}^k y_i^{(\nu)} \text{Gr}_{\mathcal{R}} \times \prod_{j=1}^\ell w_j^{(\nu)} \text{Gr} \subset \prod_{i=1}^k y_i \text{Gr}_{\mathcal{R}} \times \prod_{j=1}^\ell w_j \text{Gr} \text{ with } \nu = \sum_{i=1}^k \nu_i + \sum_{j=1}^\ell \sigma(\mu_j),$$

and let $s_\nu : S^{(\omega)}_\nu \to \text{Gr}^{(\omega)}_{\mathcal{R}}$ be the inclusion. Here as usual $\sigma : \Lambda_T \to \Lambda_S$ is the projection $\sigma(\lambda) = \theta(\lambda) + \lambda$.

Similarly, for $\nu \in \Lambda_S$, let $T^{(\omega)}_\nu \subset \text{Gr}^{(\omega)}_{\mathcal{R}}$ be the union over all fibers of the subspaces

$$\prod_{i=1}^k y_i^{(\nu)} T_{\mathcal{R}} \times \prod_{j=1}^\ell w_j^{(\nu)} T_{\mathcal{R}} \subset \prod_{i=1}^k y_i \text{Gr}_{\mathcal{R}} \times \prod_{j=1}^\ell w_j \text{Gr} \text{ with } \nu = \sum_{i=1}^k \nu_i + \sum_{j=1}^\ell \sigma(\mu_j),$$

and let $t_\nu : T^{(\omega)}_\nu \to \text{Gr}^{(\omega)}_{\mathcal{R}}$ be the inclusion.

Since the conjugation $\theta$ restricts to a Cartan involution of the derived group $M'$ of the Levi factor $M \subset P$, each connected component of the real form $\text{Gr}_{M,\mathcal{R}}$ is a single real coweight $\nu \in \Lambda_S$. Note that the connected components of the real form $\text{Gr}^{(\omega)}_{M,\mathcal{R}}$ may be identified with the connected components of $\text{Gr}_{M,\mathcal{R}}$ by collecting together points in the base $X^{(\omega)}$. Therefore each connected component $\text{Gr}^{(\omega)}_{M,\nu,\mathcal{R}}$ of the real form $\text{Gr}^{(\omega)}_{\mathcal{R}}$ may be indexed by $\nu \in \Lambda_S$.

**Lemma 8.7.1.** Let $\xi \in \mathfrak{g}_R$ be in the interior of the dominant Weyl chamber for $\mathcal{W}_{\mathcal{R}}$. Then the fixed points in $\text{Gr}^{(\omega)}_{\mathcal{R}}$ of the flow $\varphi_{\xi}$ are the components $\text{Gr}^{(\omega)}_{M,\nu,\mathcal{R}}$, and the sets $S^{(\omega)}_\nu$ are the ascending sets

$$S^{(\omega)}_\nu = \{ x \in \text{Gr}^{(\omega)}_{\mathcal{R}} \mid \lim_{t \to +\infty} \varphi_{\xi}^t(x) \in \text{Gr}^{(\omega)}_{M,\nu,\mathcal{R}} \},$$

and the sets $T^{(\omega)}_\nu$ are the descending sets

$$T^{(\omega)}_\nu = \{ x \in \text{Gr}^{(\omega)}_{\mathcal{R}} \mid \lim_{t \to -\infty} \varphi_{\xi}^t(x) \in \text{Gr}^{(\omega)}_{M,\nu,\mathcal{R}} \}.$$ 

**Proof.** The flow $\varphi_{\xi}$ respects the factorization, so the statement reduces to the local Lemmas 8.3.2 and 8.3.3. \qed

We have the following global version of Proposition 8.3.2

**Proposition 8.7.1.** For $\nu \in \Lambda_S$, there is a canonical isomorphism

$$\pi_! s^{(\omega)}_\nu \simeq \pi_* t^{(\omega)}_\nu : D_{\mathcal{S}}(\text{Gr}^{(\omega)}_{\mathcal{R}}) \to D(X^{(\omega)}_{\mathcal{R}}).$$
Proof. Consider the contractions
\[ p : S_\nu^{(\omega)} \to \text{Gr}_{M,\nu,R}^{(\omega)} \quad \text{and} \quad q : T_\nu^{(\omega)} \to \text{Gr}_{M,\nu,R}^{(\omega)} \]
provided by the previous lemma. For any sheaf \( F \in \mathcal{D}_S(\text{Gr}_R^{(\omega)}) \), the supports of \( p! s_\nu^* F \) and \( q^* t_\nu^! F \) are compact. Thus by the standard identity for the composition of maps, it suffices to construct a canonical isomorphism
\[ p! s_\nu^* F \simeq q^* t_\nu^! F. \]
For \( \xi \in s_\nu \) in the interior of the dominant Weyl chamber for \( \mathcal{W}_G \), the map \( \varphi_\xi : \text{Gr}_R^{(\omega)} \to \text{Gr}_R^{(\omega)} \) defined by \( \varphi_\xi(x) = \exp(\xi) \cdot x \) is weakly hyperbolic in the sense of [GM93, Section 1.3]. Therefore by the previous lemma, the asserted isomorphism may be deduced from [GM93, Proposition 9.2]. \( \square \)

8.8. Relation to specialization I. Recall that we have the tensor functor
\[ \Sigma : \text{Vect}_{\Lambda_T} \to \text{Vect}_{\Lambda_S \times Z} \]
\[ \Sigma(V)^{(\nu,z)} = \sum_{\Sigma(\lambda) = (\nu,z)} V^\lambda \]
where \( \Sigma : \Lambda_T \to \Lambda_S \times Z \) is the projection \( \Sigma(\lambda) = (\theta(\lambda) + \lambda, \langle 2\check{\rho}_M, \lambda \rangle) \).

**Theorem 8.8.1.** There is a canonical isomorphism
\[ \text{Ch}_R \circ p^R \simeq \Sigma \circ \text{Ch} : P_{G(O)}(\text{Gr}) \to \text{Vect}_{\Lambda_S \times Z}. \]

**Proof.** By the following lemma, it suffices to prove a similar statement for the local specialization.

**Lemma 8.8.1.** There is a canonical isomorphism
\[ \text{Ch}_R \circ p^H \simeq \text{Ch}_R : D_{G_0(\mathcal{O})}(\text{Gr}) \to \text{Vect}_{\Lambda_S \times Z}. \]

**Proof.** By Corollary 8.5.1, the real weight functors are \( t \)-exact, so the real character functor is \( t \)-exact. \( \square \)

By the lemma, it suffices to construct an isomorphism
\[ \text{Ch}_R \circ \circ^{(\nu,z)} \simeq \Sigma \circ \text{Ch} : P_{G(O)}(\text{Gr}) \to \text{Vect}_{\Lambda_S \times Z}, \]
or in other words, to construct isomorphisms
\[ F_{\nu,z}^R \circ R \simeq \sum_{\Sigma(\lambda) = (\nu,z)} F^\lambda : P_{G(O)}(\text{Gr}) \to \text{Vect}, \text{ for } \nu \in \Lambda_S, z \in \mathbb{Z}. \]

Recall the specialization diagram
\[
\begin{array}{ccc}
\text{Gr}^{(1)}|X_+ & \simeq & \text{Gr}_R^{(\sigma)}|X_+ \\
\downarrow \simeq & \downarrow \simeq & \downarrow \simeq \\
\text{Gr}_R^{(\sigma)}|X & \overset{j}{\to} & \text{Gr}_R^{(\sigma)}|X \\
\downarrow \simeq & \downarrow \simeq & \downarrow \simeq \\
X_+ & \to & X \\
\end{array}
\]

Define a functor
\[ \mathcal{L}^\nu : P_{G(O)}(\text{Gr}) \to D(X_+ \cup X_R) \]
\[ \mathcal{L}^\nu(P) = \pi s_\nu^* j_! (\rho(P)|X_+) \]
where \( s_\nu : S_\nu^{(\sigma)} \to \text{Gr}_R^{(\sigma)} \) is the inclusion, and \( \pi : \text{Gr}_R^{(\sigma)} \to X \) is the projection.
Lemma 8.8.2. The sheaf $\mathcal{L}^\nu(\mathcal{P})$ is constant.

Proof. First, observe that the sheaf $\mathcal{L}^\nu(\mathcal{P})|_{X_+}$ is the local system associated to the $\text{Aut}(\mathcal{O})$-torsor $\tilde{X}_+ \to X_+$ and the discrete $\text{Aut}(\mathcal{O})$ representation $H^*_c(\mathcal{S}^\nu, \mathcal{P})$. Since $\text{Aut}(\mathcal{O})$ is connected, the sheaf $\mathcal{L}^\nu(\mathcal{P})|_{X_+}$ is constant.

Next, note the following chain of isomorphisms

$$\mathcal{L}^\nu(\mathcal{P}) = \pi_* s^\nu_* j_*(\mathcal{P}) \simeq \pi_* s^\nu_* t^\nu_*(\mathcal{P}) \simeq j_* \pi_* s^\nu_*(\mathcal{P}) \simeq j_*(\mathcal{L}^\nu(\mathcal{P})|_{X_+}).$$

Here we used Proposition 8.7.1, the base change isomorphism, and the standard identity for the composition of maps.

Since each $x \in X_\mathbb{R}$ has a neighborhood that intersects $X_+$ in a contractible set, the pushforward $j_*(\mathcal{L}^\nu(\mathcal{P})|_{X_+})$ is constant. □

Lemma 8.8.3. We have canonical identifications of cohomology stalks

$$H^{(\rho,\nu) + z}(\mathcal{L}^\nu(\mathcal{P}))_x \simeq F^\nu_{\mathbb{R}}(\mathbb{R}(\mathcal{P})), \text{ for } x \in X_\mathbb{R},$$

$$H^{(\rho,\nu) + z}(\mathcal{L}^\nu(\mathcal{P}))_x \simeq \sum_{\Sigma(\mu) = (\nu, z)} F^\mu(\mathcal{P}), \text{ for } x \in X_+. $$

Proof. For $x \in X$, we have

$$H^*(\mathcal{L}^\nu(\mathcal{P}))_x \simeq H^*_c(S^\nu(\mathcal{P}^\sigma)|_x, (j_* \rho(\mathcal{P})|_{X_+})_x).$$

By definition, we have identifications

$$S^\nu(\mathcal{P}^\sigma)|_x \simeq x S^\nu_{\mathbb{R}}, \text{ for } x \in X_\mathbb{R},$$

$$S^\nu(\mathcal{P}^\sigma)|_x \simeq \bigcup_{\sigma(\mu) = \nu} x S^\mu, \text{ for } x \in X_+. $$

By choosing formal coordinates, we obtain isomorphisms

$$H^*_c(x S^\nu_{\mathbb{R}}, j_*(\mathcal{P})_x) \simeq H^*_c(S^\nu_{\mathbb{R}}, \mathbb{R}(\mathcal{P})), \text{ for } x \in X_\mathbb{R},$$

$$H^*_c(x S^\mu, (\mathcal{P})_x) \simeq H^*_c(S^\mu, \mathcal{P}), \text{ for } x \in X_+. $$

Since $\text{Aut}_0(\mathcal{O}_\mathbb{R})$, respectively $\text{Aut}(\mathcal{O})$, is connected, the first, respectively second, isomorphism is independent of the choice of formal coordinate.

This proves the first assertion of the lemma. For the second, thanks to Corollary 8.4.2, it only remains to check that the $\mathbb{Z}$-gradings agree. We must check that if $\nu = \theta(\mu) + \mu$, then we have

$$\langle 2\tilde{\rho}, \mu \rangle = \langle \tilde{\rho}, \nu \rangle + \langle 2\tilde{\rho}_M, \mu \rangle.$$

Using the identities $\tilde{\theta}(\tilde{\rho} - \tilde{\rho}_M) = \tilde{\rho} - \tilde{\rho}_M$, and $\langle 2\tilde{\rho}_M, \nu \rangle = 0$, we have

$$\langle \tilde{\rho}, \nu \rangle = \langle \tilde{\rho} - \tilde{\rho}_M, \theta(\mu) - \mu \rangle = \langle 2(\tilde{\rho} - \tilde{\rho}_M), \mu \rangle.$$

□

The identification of the stalks of the constant cohomology sheaf $H^z(\mathcal{L}_c^\nu(\mathcal{P}))$, for $\nu \in \Lambda_\mathcal{S}$, fixes $z \in \mathbb{Z}$, provides the sought-after isomorphism. □

We have the following corollaries. Recall that $\Sigma(\Lambda_T) \subset \Lambda_\mathcal{S} \times \mathbb{Z}$ is the image of the projection $\Sigma : \Lambda_T \to \Lambda_\mathcal{S} \times \mathbb{Z}$. 
Corollary 8.8.1. The real character functor descends to a functor
\[ \text{Ch}_R : Q(\text{Gr}_R)_{\mathbb{R}} \to \text{Vect}_{\Sigma(\Lambda_T)}. \]

Proof. By definition, an object \( Q \in Q(\text{Gr}_R)_{\mathbb{R}} \) is isomorphic to a subquotient of an object of the form \( p^R(\mathcal{P}) \), for \( \mathcal{P} \in \mathcal{P}_{G(O)}(\text{Gr}) \). Since each real weight functor is exact, the real character functor is exact, and so \( \text{Ch}_R(Q) \) is a subquotient of \( \text{Ch}_R(p^R(\mathcal{P})) \). By the theorem, we have \( \text{Ch}_R(p^R(\mathcal{P})) \simeq \Sigma(\text{Ch}(\mathcal{P})) \), and so \( \text{Ch}_R(Q) \) is a subquotient of \( \Sigma(\text{Ch}(\mathcal{P})) \).

Recall that \( H \) denotes the hypercohomology functor, and \( F \) the forgetful functor which forgets the grading of a vector space.

Corollary 8.8.2. There are canonical isomorphisms
\[ H \simeq F \circ \text{Ch}_R : Q(\text{Gr}_R)_{\mathbb{R}} \to \text{Vect} \]
\[ H \simeq H \circ p^R : \mathcal{P}_{G(O)}(\text{Gr}) \to \text{Vect}. \]

Proof. By definition, an object \( Q \in Q(\text{Gr}_R)_{\mathbb{R}} \) is isomorphic to a subquotient of an object of the form \( p^R(\mathcal{P}) \), for \( \mathcal{P} \in \mathcal{P}_{G(O)}(\text{Gr}) \). We may assume without loss of generality that \( \mathcal{P} \) is supported on a single component of \( \text{Gr} \), and that \( Q \) is a subquotient of \( \text{p}^H(p^R(\mathcal{P})) \), for some \( k \).

By Theorem 8.4.1 and Proposition 3.2.1, for \( \mathcal{P} \in \mathcal{P}_{G(O)}(\text{Gr}) \) supported on a single component of \( \text{Gr} \), \( \text{Ch}(\mathcal{P}) \) is non-zero in only even or odd degrees. Thus by the theorem, \( \text{Ch}_R(p^H(p^R(\mathcal{P}))) \) is non-zero in only even or odd degrees, and since \( \text{Ch}_R \) is exact, \( \text{Ch}_R(Q) \) is non-zero in only even or odd degrees. Therefore the compactly supported cohomology along the orbits \( S^\nu_{\mathbb{R}} \) provides a filtration of the hypercohomology functor. Similarly, the local cohomology along the orbits \( T^\nu_{\mathbb{R}} \) also provides a filtration. The isomorphism of Proposition 8.5.2 implies that the two filtrations split each other.

This proves the first assertion, and now the second is obtained from the theorem by applying the forgetful functor and using the first assertion and Corollary 8.4.2.

Corollary 8.8.3. The hypercohomology \( H : Q(\text{Gr}_R)_{\mathbb{R}} \to \text{Vect} \) is exact and faithful.

Proof. The real weight functors are exact, and so the real character functor is exact. By the previous corollary, the hypercohomology functor is exact as well. Since the hypercohomology functor is exact, to see that it is faithful, it suffices to check that it does not vanish on any \( Q \in Q(\text{Gr}_R)_{\mathbb{R}} \). But for any \( Q \in Q(\text{Gr}_R)_{\mathbb{R}} \), some weight functor will not vanish on \( Q \). For example, we could take the weight functor \( F^\nu_{\mathbb{R}} \) associated to the dominant real coweight \( \nu \) in an open stratum in the support of \( Q \).

Corollary 8.8.4. The specialization \( R : \mathcal{P}_{G(O)}(\text{Gr}) \to D_{G_\mathbb{R}(O_b)}(\text{Gr}_\mathbb{R}) \) is perverse if and only if \( G_\mathbb{R} \) is quasi-split.

Proof. By the theorem, the \( \mathbb{Z} \)-grading on the perverse specialization is trivial if and only \( G_\mathbb{R} \) is quasi-split.

8.9. Monoidal structure for real character functor. In this section, we equip the real character functor \( \text{Ch}_R : Q(\text{Gr}_\mathbb{R})_{\mathbb{R}} \to \text{Vect}_{\Sigma(\Lambda_T)} \), and the hypercohomology \( H : Q(\text{Gr}_\mathbb{R})_{\mathbb{R}} \to \text{Vect} \) with monoidal structures.
Theorem 8.9.1. There is a canonical isomorphism
\[ \mathbb{H}(\cdot \circ \cdot) \simeq \mathbb{H}(\cdot) \otimes \mathbb{H}(\cdot) \]
for the hypercohomology functor \( \mathbb{H} : \mathcal{Q}(\text{Gr}_R)_{\mathbb{R}} \to \text{Vect} \).

Proof. Recall the real global convolution diagram
\[
\text{Gr}_R^{(1)} \times \text{Gr}_R^{(1)} \xymatrix{ \nLeftarrow \ar[r] & \text{Gr}_R^{(1)} } \ar[r] & \text{Gr}_R^{(1)} \times \text{Gr}_R^{(1)} \xymatrix{ \nLeftarrow \ar[r] & \text{Gr}_R^{(1)} } \ar[r] & \text{Gr}_R^{(1)}.
\]
Define a functor
\[
\mathcal{L}_R : \mathcal{Q}(\text{Gr}_R)_{\mathbb{R}} \times \mathcal{Q}(\text{Gr}_R)_{\mathbb{R}} \to \mathcal{D}(X^2_{\mathbb{R}})
\]
\[
\mathcal{L}_R(Q_1, Q_2) = \pi_!(\rho_!(Q_1) \otimes \rho_!(Q_2))
\]
where \( \pi : \text{Gr}_R^{(1)} \times \text{Gr}_R^{(1)} \to X^2_{\mathbb{R}} \) is the projection.

Proposition 8.9.1. The sheaf \( \mathcal{L}_R(Q_1, Q_2) \) is constant with cohomology stalk
\[
H^*(\mathcal{L}_R(Q_1, Q_2))_x \simeq \mathbb{H}(\text{Gr}_R, Q_1) \otimes \mathbb{H}(\text{Gr}_R, Q_2), \text{ for } x \in X^2_{\mathbb{R}}.
\]

Proof. Recall that \( \text{Gr}_R^{(1)} \) classifies data \( (x, \mathcal{F}, \nu) \), where \( x \in X_{\mathbb{R}} \), \( \mathcal{F} \) is a \( G_R \)-torsor on \( X_{\mathbb{R}} \), and \( \nu \) is a trivialization of \( \mathcal{F} \) over \( X_{\mathbb{R}} \setminus x \). By Lemma 5.5.1 a sheaf \( \mathcal{Q}_1 \in \mathcal{Q}(\text{Gr}_R)_{\mathbb{R}} \) is supported on the components \( \text{Gr}_R^0 \subset \text{Gr}_R \) that are in the kernel of the map \( \partial : \pi_0(\text{Gr}_R) \to \pi_0(\hat{\text{Gr}}_R) \). Thus the sheaf \( \rho_!(\mathcal{Q}_1) \) is supported on the components of \( \text{Gr}_R^{(1)} \) for which the trivialization \( \nu \) extends to a trivialization of the torsor \( \mathcal{F}/G^0_{\mathbb{R}} \to X_{\mathbb{R}} \) for the component group \( G_{\mathbb{R}}/G^0_{\mathbb{R}} \), or in other words, for which the torsor \( \mathcal{F} \) is induced from a connected group \( G^0_{\mathbb{R}} \).

Recall that \( \text{Gr}_R^{(1)} \times \text{Gr}_R^{(1)} \) classifies data \( (x_1, x_2, \mathcal{F}_1, \mathcal{F}, \nu_1, \eta) \) where \( x_1, x_2 \in X_{\mathbb{R}} \), \( \mathcal{F}_1, \mathcal{F} \) are \( G_{\mathbb{R}} \)-torsors on \( X_{\mathbb{R}} \), \( \nu_1 \) is a trivialization of \( \mathcal{F}_1 \) over \( X_{\mathbb{R}} \setminus x_1 \), and \( \eta \) is an isomorphism from \( \mathcal{F}_1 \) to \( \mathcal{F} \) over \( X_{\mathbb{R}} \setminus x_2 \). We have also seen that it is the twisted product constructed from the action of \( G_{\mathbb{R}}(O_{\mathbb{R}}) \ltimes \text{Aut}^0(O_{\mathbb{R}}) \) on \( \text{Gr}_{\mathbb{R}} \), and the \( G_{\mathbb{R}}(O_{\mathbb{R}}) \ltimes \text{Aut}^0(O_{\mathbb{R}}) \)-torsor
\[
\text{Gr}_R^{(1)} \times \hat{X}^0_{\mathbb{R}} \to \text{Gr}_R^{(1)}
\]
that classifies data \( (x_1, x_2, \mathcal{F}_1, \nu_1, \mu_1, \phi) \), where \( x_1, x_2 \in X_{\mathbb{R}} \), \( \mathcal{F}_1 \) is a \( G \)-torsor on \( X_{\mathbb{R}} \), \( \nu_1 \) is a trivialization of \( \mathcal{F}_1 \) over \( X_{\mathbb{R}} \setminus x_1 \), \( \mu_1 \) is a trivialization of \( \mathcal{F}_1 \) over the formal neighborhood of \( x_2 \), and \( \phi \) is an orientation-preserving isomorphism from the abstract formal disk to the formal neighborhood of \( x_2 \).

Consider the projection
\[
r : \text{Gr}_R^{(1)} \times \text{Gr}_R^{(1)} \to \text{Gr}_R^{(1)}
\]
defined by \( (x_1, x_2, \mathcal{F}_1, \mathcal{F}, \nu_1, \eta) \mapsto (x_1, \mathcal{F}_1, \nu_1) \), with fiber the \( G_{\mathbb{R}}(O_{\mathbb{R}}) \ltimes \text{Aut}^0(O_{\mathbb{R}}) \)-twist of \( \text{Gr}_{\mathbb{R}} \). We claim that there is a canonical isomorphism
\[
r_!(\rho!_R(Q_1) \otimes \rho!_R(Q_2)) \simeq \rho_!(Q_1) \otimes \mathbb{H}(\text{Gr}_R, Q_2).
\]
Assuming this isomorphism for the moment, we shall then have
\[
\mathcal{L}_R(Q_1, Q_2) \simeq \pi n_!(\rho!_R(Q_1) \otimes \rho!(Q_2)) \simeq \pi_!(\rho_!(Q_1) \otimes \mathbb{H}(\text{Gr}_R, Q_2)) \simeq \mathbb{H}(\text{Gr}_R, Q_1) \otimes \mathbb{H}(\text{Gr}_R, Q_2).
\]
Here the last isomorphism \( \pi_1(\rho(\mathcal{Q}_1)) \cong \mathbb{H}(\text{Gr}_R, \mathcal{Q}_1) \) follows from the fact that \( \text{Aut}^0(\mathcal{O}_R) \) is connected. Thus to prove the lemma, we are left to establish the asserted isomorphism.

Observe that the restriction of the \( G_R (\mathcal{O}_R) \times \text{Aut}^0(\mathcal{O}_R) \)-torsor

\[
\text{Gr}^{(1)}_R \times \hat{X}^0_\mathbb{R} \to \text{Gr}^{(1)}_R
\]

to the components of \( \text{Gr}^{(1)}_R \) that classify data \((x, \mathcal{F}, \nu)\) where \( \mathcal{F} \) is induced from a \( G^0_\mathbb{R} \)-torsor, is itself induced from a torsor for the connected group \( G_R (\mathcal{O}_R)^0 \times \text{Aut}^0(\mathcal{O}_R) \). The sheaf \( n(\rho(\mathcal{Q}_1) \otimes \rho(\mathcal{Q}_2)) \) is the product of \( \rho(\mathcal{Q}_1) \) with the \( G_R (\mathcal{O}_R)^0 \times \text{Aut}^0(\mathcal{O}_R) \)-twist of \( \mathbb{H}(\text{Gr}_R, \mathcal{Q}_2) \), and since \( G_R (\mathcal{O}_R)^0 \times \text{Aut}^0(\mathcal{O}_R) \) is connected, the twist must be trivial. □

The identification of the stalks of the constant cohomology sheaf \( H^* (\mathcal{Q}_R(\mathcal{Q}_1, \mathcal{Q}_2)) \) provides the sought-after isomorphism. □

**Theorem 8.9.2.** There is a canonical isomorphism

\[
\text{Ch}_R (\cdot \odot \cdot) \simeq \text{Ch}_R (\cdot) \otimes \text{Ch}_R (\cdot)
\]

for the real character functor \( \text{Ch}_R : \mathcal{Q}(\text{Gr}_R)_R \to \text{Vect}_{\Sigma(\Lambda)} \).

**Proof.** For \( \nu \in \Lambda_S \), define functors

\[
\mathcal{Q}_R^{(\nu)} : \mathcal{Q}(\text{Gr}_R)_R \times \mathcal{Q}(\text{Gr}_R)_R \to \mathcal{D}(X^2_R)
\]

\[
\mathcal{Q}_R^{(\nu)}(\mathcal{Q}_1, \mathcal{Q}_2) = \pi_1 s^{(2)}_\nu m_1 (\rho(\mathcal{Q}_1) \otimes \rho(\mathcal{Q}_2))
\]

where \( m : \text{Gr}^{(1)}_R \times \text{Gr}^{(1)}_R \to \text{Gr}^{(2)}_R \) is the multiplication, \( s^{(2)}_\nu : \text{Gr}^{(2)}_R \to \text{Gr}^{(2)}_R \) is the inclusion, and \( \pi : \text{Gr}^{(2)}_R \to X^2_R \) is the projection.

**Lemma 8.9.1.** There are canonical identifications of cohomology stalks

\[
H^* (\mathcal{Q}_R^{(\nu)}(\mathcal{Q}_1, \mathcal{Q}_2))_x \simeq F^\nu(\mathcal{Q}_1 \odot \mathcal{Q}_2), \text{ for } x \in \Delta_R,
\]

\[
H^* (\mathcal{Q}_R^{(\nu)}(\mathcal{Q}_1, \mathcal{Q}_2))_x \simeq \sum_{\lambda + \mu = \nu} F^\lambda(\mathcal{Q}_1) \otimes F^\mu(\mathcal{Q}_2), \text{ for } x \in X^2_R \setminus \Delta_R.
\]

**Proof.** For \( x \in X^2_R \), we have

\[
H^* (\mathcal{Q}_R^{(\nu)}(\mathcal{Q}_1, \mathcal{Q}_2))_x = H^*_c (S^{(2)}_\nu | x, m_1 (\rho(\mathcal{Q}_1) \otimes \rho(\mathcal{Q}_2))_x).
\]

By definition, we have identifications

\[
S^{(2)}_\nu | x \simeq \bigcup_{\lambda + \mu = \nu} x_1 S^{(2)}_\lambda \times x_2 S^{(2)}_\mu, \text{ for } (x_1, x_2) \in X^2_R \setminus \Delta_R.
\]

By choosing formal coordinates, we obtain isomorphisms

\[
H^*_c (x_1 S^{(2)}_\lambda \times x_2 S^{(2)}_\mu, m_1 (\rho(\mathcal{Q}_1) \otimes \rho(\mathcal{Q}_2))_x) \simeq H^*_c (S^{(2)}_\nu, \mathcal{Q}_1 \odot \mathcal{Q}_2), \text{ for } x \in \Delta_R,
\]

\[
H^*_c (x_1 S^{(2)}_\lambda \times x_2 S^{(2)}_\mu, m_1 (\rho(\mathcal{Q}_1) \otimes \rho(\mathcal{Q}_2))(x_1, x_2)) \simeq H^*_c (S^{(2)}_\lambda \mathcal{Q}_1) \otimes H^*_c (S^{(2)}_\mu \mathcal{Q}_2), \text{ for } (x_1, x_2) \in X^2_R \setminus \Delta_R.
\]

Since \( \text{Aut}^0(\mathcal{O}_R) \) is connected, the isomorphisms are independent of the choice of formal coordinates. □
Lemma 8.9.2. The sheaf $L^\nu_R(Q_1, Q_2)$ is constant.

Proof. For $i = 1, 2$, an object $Q_i \in \mathcal{Q}(\text{Gr}_R)_R$ is isomorphic to a subquotient of an object of the form $p^R(P_i)$, for $P_i \in \mathcal{P}_{G(O)}(\text{Gr})$. We may assume without loss of generality that $P_i$ is supported on a single component of Gr, so that the hypercohomology of $P_i$ is possibly non-zero only in degrees congruent to some $d_i \mod 2$. We may also assume that $Q_i$ is a subquotient of $p^R_h(R(P_i))$, for some $k_i$, so that by the proof of Corollary 8.8.2, $F^\nu_R(Q_i)$ is possibly non-zero only in degrees congruent to $d_i + k_i \mod 2$. The product $Q_1 \odot Q_2$ is isomorphic to a subquotient of the sheaf $p^R_h(R(P_1)) \odot p^R_h(R(P_2))$, so that by the proof of Corollary 8.8.2, $F^\nu_R(Q_1 \odot Q_2)$ is possibly non-zero only in degrees congruent to $d_1 + k_1 + d_2 + k_2 \mod 2$. Therefore by the previous lemma, the sheaves $L^\nu_R(Q_1, Q_2)$ provide a filtration of the sheaf $L^\nu_R(Q_1, Q_2)$, and similarly the sheaves constructed for the opposite minimal parabolic subgroup also provide a filtration. The isomorphism of Proposition 8.7.1 implies that the two filtrations split each other. The lemma now follows from the fact that a direct summand of a constant sheaf must itself be constant.

The identification of the stalks of the constant cohomology sheaf $H^\nu(L^\nu_R(Q_1, Q_2))$, for $\nu \in \Lambda_S$, provides the sought-after isomorphism.

Remark 8.9.1. Note that by construction, the monoidal structure for the hypercohomology functor $H : \mathcal{Q}(\text{Gr}_R)_R \to \text{Vect}$ is obtained from that of the real character functor $Ch_R : \mathcal{Q}(\text{Gr}_R)_R \to \text{Vect}_{\Sigma(\Lambda_T)}$ by applying the forgetful functor.

8.10. Relation to specialization II. In Theorem 8.8.1, we constructed a canonical isomorphism

$$(	ext{Ch}_R \circ p^R) \simeq \Sigma \circ \text{Ch} : \mathcal{P}_{G(O)}(\text{Gr}) \to \text{Vect}_{\Sigma(\Lambda_T)}$$

intertwining the perverse specialization and the character functors. In Section 6, we constructed a monoidal structure for the perverse specialization $p^R$, and in the previous section, we constructed a monoidal structure for the real character functor $Ch_R$. Therefore the functor on the left hand side has a monoidal structure. One may give the character functor $Ch$ a monoidal structure in the same way as we did the real character functor $Ch_R$ in the previous section. (See [MV04, Proposition 6.4].) Therefore the functor on the right hand side has a monoidal structure as well. The following proposition confirms that the monoidal structures agree.

Proposition 8.10.1. We have a commutative square

$$
\begin{array}{ccc}
\text{Ch}_R(p^R(\cdot \circ \cdot)) & \xrightarrow{\approx} & \Sigma(\text{Ch}(\cdot \circ \cdot)) \\
\downarrow & & \downarrow \\
\text{Ch}_R(p^R(\cdot)) \otimes \text{Ch}_R(p^R(\cdot)) & \xrightarrow{\approx} & \Sigma(\text{Ch}(\cdot)) \otimes \Sigma(\text{Ch}(\cdot))
\end{array}
$$

The proof is an elaboration of the techniques of previous sections, and we leave it to the reader. Applying the forgetful functor, and using Corollaries 8.4.2 and 8.8.2, we obtain the following corollary confirming that under the isomorphism

$$H \circ p^R \simeq H : \mathcal{P}_{G(O)}(\text{Gr}) \to \text{Vect},$$

the monoidal structures on each side agree.
Corollary 8.10.1. We have a commutative square

\[
\begin{array}{ccc}
\mathbb{H}(\rho \mathbb{R}(\cdot \circ \cdot)) & \overset{\sim}{\to} & \mathbb{H}(\cdot \circ \cdot) \\
\downarrow & & \downarrow \\
\mathbb{H}(\rho \mathbb{R}(\cdot)) \otimes \mathbb{H}(\rho \mathbb{R}(\cdot)) & \overset{\sim}{\to} & \mathbb{H}(\cdot) \otimes \mathbb{H}(\cdot)
\end{array}
\]

9. The image category II

9.1. Tannakian dictionary. We review the dictionary between monoidal categories and bi-algebras developed in [SR72] and [DM82]. The results are valid over any field \(K\) which we will take to be \(\mathbb{C}\).

Let \(C\) be a \(K\)-linear abelian category equipped with a \(K\)-linear exact faithful functor \(\omega : C \to \text{Vect}\). For an object \(X \in \text{Ob}(C)\), let \(\langle X \rangle\) be the strict full subcategory of \(C\) whose objects are isomorphic to subquotients of the objects \(X^{\otimes n}\), for some \(n\). Define the \(K\)-algebra

\[A(X) = \text{End}(\omega(\langle X \rangle))\]

to be the algebra of endomorphisms of the functor \(\omega\) restricted to the subcategory \(\langle X \rangle\).

Define the \(K\)-coalgebra

\[B(X) = \text{Hom}(A(X), K)\]

to be the coalgebra dual to the algebra \(A(X)\). Let \(B(C)\) be the inverse limit of the coalgebras \(B(X)\) over all objects \(X \in \text{Ob}(C)\).

Proposition 9.1.1 ([DM82], Proposition 2.14). There is an equivalence of categories \(C \to \text{Comod}_{B(C)}\) under which \(\omega\) corresponds to the forgetful functor.

A homomorphism \(f : B \to B'\) defines the functor

\[\phi^f : \text{Comod}_B \to \text{Comod}_{B'}\]

that takes a \(B\)-comodule \(X\) with coaction \(a_X : X \to X \otimes B\) to the \(B'\)-comodule \(X\) with coaction

\[a'_X : X \xrightarrow{\phi^f} X \otimes B \xrightarrow{1 \otimes f} X \otimes B'.\]

Proposition 9.1.2 ([DM82], p 135). The map \(f \mapsto \phi^f\) defines a bijection from the set of homomorphisms \(f : B \to B'\) to the set of functors \(\phi : \text{Comod}_B \to \text{Comod}_{B'}\) which cover the identity on the underlying vector spaces.

For a \(K\)-coalgebra \(B\), a homomorphism \(u : B \otimes B \to B\) defines the functor

\[\phi^u : \text{Comod}_B \times \text{Comod}_B \to \text{Comod}_B\]

that takes a pair of \(B\)-comodules \(X,Y\), with coactions \(a_X : X \to X \otimes B\), \(a_Y : Y \to Y \otimes B\), to the \(B\)-comodule \(X \otimes Y\) with coaction

\[a_{X \otimes Y} : X \otimes Y \xrightarrow{a_X \otimes a_Y} X \otimes B \otimes Y \otimes B \xrightarrow{1 \otimes u} X \otimes Y \otimes B.\]

Proposition 9.1.3 ([DM82], Proposition 2.16). The map \(u \mapsto \phi^u\) defines a bijection from the set of homomorphisms \(u : B \otimes B \to B\) to the set of functors \(\phi : \text{Comod}_B \times \text{Comod}_B \to \text{Comod}_B\) which cover the ordinary tensor product on the underlying vector spaces.
The natural associativity and commutativity constraints on \( \text{Vect} \) induce similar constraints on \( \text{Comod}_B \) equipped with the product \( \phi^u \) if and only if \( u \) is associative and commutative. There is an identity object for \( \phi^u \) in \( \text{Comod}_B \) with underlying vector space \( \mathbb{K} \) if and only if there is an identity element for \( u \) in \( B \).

When there is an associative and commutative homomorphism \( u : B \otimes B \to B \) and an identity element for \( u \) in \( B \), then \( \text{Spec}(B) \) is an affine monoid-scheme and there is an equivalence

\[
\text{Rep} (\text{Spec}(B)) \simeq \text{Comod}_B.
\]

To obtain \( \text{Spec}(B) \) directly from the category \( \text{Comod}_B \), one checks that it is the monoid of tensor endomorphisms of the forgetful functor.

**Proposition 9.1.4 (\cite{Ul90}).** Suppose the homorphism \( u : B \otimes B \to B \) is associative and there is an identity element for \( u \) in \( B \). Then the tensor category \( \text{Comod}_B \) is rigid if and only if there is an antipode \( S : B \to B \) making \( B \) a Hopf algebra.

When the multiplication \( u : B \otimes B \to B \) is commutative and there is an antipode \( S : B \to B \) making \( B \) a commutative Hopf algebra, then \( \text{Spec}(B) \) is an affine group-scheme. One checks that in this case every tensor endomorphism of the forgetful functor is in fact an automorphism.

### 9.2. Coherence constraints.

The category \( \mathbf{P}_{G(\mathcal{O})}(\text{Gr}) \) equipped with the convolution product is a rigid tensor category [\cite{MV04} Theorem 7.3], and the hypercohomology \( \mathbb{H} : \mathbf{P}_{G(\mathcal{O})}(\text{Gr}) \to \text{Vect} \) is an exact faithful tensor functor [\cite{MV04} Corollary 3.7, Proposition 6.3].

We have seen that the category \( \mathbf{Q}(\text{Gr}_R)_\mathbb{R} \) is equipped with a convolution product, and that the perverse specialization \( \mathbf{p}_R : \mathbf{P}_{G(\mathcal{O})}(\text{Gr}) \to \mathbf{Q}(\text{Gr}_R)_\mathbb{R} \) is equipped with a monoidal structure such that there is a canonical isomorphism of functors \( \mathbb{H} \simeq \mathbb{H} \circ \mathbf{p}_R : \mathbf{P}_{G(\mathcal{O})}(\text{Gr}) \to \text{Vect} \) respecting the monoidal structures.

**Proposition 9.2.1.** There exist unique associativity and commutativity constraints for the category \( \mathbf{Q}(\text{Gr}_R)_\mathbb{R} \) equipped with the convolution product such that the perverse specialization \( \mathbf{p}_R : \mathbf{P}_{G(\mathcal{O})}(\text{Gr}) \to \mathbf{Q}(\text{Gr}_R)_\mathbb{R} \) and hypercohomology \( \mathbb{H} : \mathbf{Q}(\text{Gr}_R)_\mathbb{R} \to \text{Vect} \) equipped with their monoidal structures respect the constraints.

**Proof.** First, since the canonical isomorphism \( \mathbb{H} \simeq \mathbb{H} \circ \mathbf{p}_R \) respects monoidal structures, the composite functor \( \mathbb{H} \circ \mathbf{p}_R \) is an exact faithful tensor functor. We may apply the Tannakian dictionary to the category \( \mathbf{P}_{G(\mathcal{O})}(\text{Gr}) \) and the functor \( \mathbb{H} \circ \mathbf{p}_R \) to obtain a commutative Hopf algebra \( B(\mathbf{P}_{G(\mathcal{O})}(\text{Gr})) \) with identity such that \( \mathbf{P}_{G(\mathcal{O})}(\text{Gr}) \) is equivalent to the category of \( B(\mathbf{P}_{G(\mathcal{O})}(\text{Gr})) \)-comodules.

Next, we may apply the Tannakian dictionary to the category \( \mathbf{Q}(\text{Gr}_R)_\mathbb{R} \) and the functor \( \mathbb{H} \) to obtain a coalgebra \( B(\mathbf{Q}(\text{Gr}_R)_\mathbb{R}) \) with a multiplication such that \( \mathbf{Q}(\text{Gr}_R)_\mathbb{R} \) is equivalent to the category of \( B(\mathbf{Q}(\text{Gr}_R)_\mathbb{R}) \)-comodules. By the Tannakian dictionary, the perverse specialization \( \mathbf{p}_R \) provides a coalgebra homomorphism

\[
B(\mathbf{P}_{G(\mathcal{O})}(\text{Gr})) \to B(\mathbf{Q}(\text{Gr}_R)_\mathbb{R})
\]

which is surjective since every object of \( \mathbf{Q}(\text{Gr}_R)_\mathbb{R} \) is isomorphic to a subquotient of an object of the form \( \mathbf{p}_R(\mathcal{P}) \), for some object \( \mathcal{P} \) in the category \( \mathbf{P}_{G(\mathcal{O})}(\text{Gr}) \).
Lemma 9.2.1. For coalgebras $B, B'$, given a homomorphism $f : B \to B'$ inducing the functor $\phi^f : \text{Comod}_B \to \text{Comod}_{B'}$, and given homomorphisms $u : B \otimes B \to B, u' : B' \otimes B' \to B'$ inducing the functors $\phi^u : \text{Comod}_B \otimes \text{Comod}_B \to \text{Comod}_B, \phi^{u'} : \text{Comod}_{B'} \otimes \text{Comod}_{B'} \to \text{Comod}_{B'}$, there is an identity of homomorphisms

$$f \circ u = u' \circ (f \otimes f)$$

if and only if the identity isomorphism in $\text{Vect}$ induces an isomorphism of functors

$$\phi^f \circ \phi^u \simeq \phi^{u'} \circ (\phi^f \times \phi^f).$$

Proof. Follows directly from the definitions. \hfill \Box

Applying the lemma to the perverse specialization $^pR$, we see that the coalgebra homomorphism $B(\mathbf{P}_{G(\mathcal{O})}(\text{Gr})) \to B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})_R)$ respects multiplication. Since it is surjective, we conclude that the multiplication of $B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})_R)$ is associative and commutative. By the Tannakian dictionary, we obtain unique associativity and commutativity constraints on the category $\mathbf{Q}(\text{Gr}_{\mathbb{R}})_R$ equipped with the convolution product such that the perverse specialization $^pR$ and hypercohomology $\mathcal{H}$ respect the constraints. \hfill \Box

9.3. Rigidity. The following is a result of Waterhouse.

Proposition 9.3.1 (Nic78, Theorem 0). Let $B$ be a finitely-generated Hopf algebra over a field $\mathbb{K}$ with comultiplication $\Delta : B \to B \otimes B$, counit $\varepsilon : B \to \mathbb{C}$, and antipode $S : B \to B$. Let $I \subset B$ be an ideal such that $\Delta(I) \subset I \otimes B + B \otimes I$. Then $\varepsilon(I) = \{0\}$, and $S(I) = I$.

When $B$ is commutative, the proposition implies that a closed sub-semigroup $\text{Spec}(B/I)$ of an affine algebraic group $\text{Spec}(B)$ automatically contains the identity element and is closed under taking inverses, and therefore is itself a group.

Applying the proposition to the kernel of the surjective homomorphism $B(\mathbf{P}_{G(\mathcal{O})}(\text{Gr})) \to B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})_R)$, we conclude that the unit and antipode of $B(\mathbf{P}_{G(\mathcal{O})}(\text{Gr}))$ descend to the quotient $B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})_R)$ making it a Hopf algebra. By the Tannakian dictionary, this confirms that $\mathbf{Q}(\text{Gr}_{\mathbb{R}})_R$ has an identity object and is rigid.

9.4. Other categories. By the Tannakian dictionary, from the commutative diagram of Section 7, we obtain a commutative diagram of coalgebras with multiplication

$$
\begin{array}{ccc}
B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})_R) & \longrightarrow & B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})_Z) \\
\downarrow & & \downarrow \\
B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})) & \longrightarrow & B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})).
\end{array}
$$

By definition, every object of $\mathbf{Q}(\text{Gr}_{\mathbb{R}})$ is isomorphic to a subquotient of an object of the form $F(Q)$, for some object $Q$ in the category $\mathbf{Q}(\text{Gr}_{\mathbb{R}})_R$, where $F : \mathbf{Q}(\text{Gr}_{\mathbb{R}})_R \to \mathbf{Q}(\text{Gr}_{\mathbb{R}})$ is the forgetful functor. Therefore the diagonal arrow is surjective, and we conclude that $B(\mathbf{Q}(\text{Gr}_{\mathbb{R}}))$ is a commutative Hopf algebra.

Since the category $\mathbf{Q}(\text{Gr}_{\mathbb{R}})_Z$ is the product category $\mathbf{Q}(\text{Gr}_{\mathbb{R}}) \otimes \mathbf{Vect}_Z$ with the product convolution product, the coalgebra $B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})_Z)$ is the product coalgebra $B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})) \otimes B(\mathbf{Vect}_Z)$ with the product multiplication. Noting that $B(Z)$ is nothing but the group algebra $\mathbb{C}[Z]$, we conclude that $B(\mathbf{Q}(\text{Gr}_{\mathbb{R}})_Z)$ is a commutative Hopf algebra.
The vertical arrow and first horizontal arrow are the obvious inclusions, and the second horizontal arrow is the obvious projection.

10. The associated subgroup

In this section, we identify the Tannakian group associated to the category \( \mathbf{Q}(\text{Gr}_\mathbb{R}) \) with the fiber functor \( \mathbb{H} : \mathbf{Q}(\text{Gr}_\mathbb{R}) \to \text{Vect} \).

10.1. Root data. A based root datum \( \Psi \) consists of a quadruple

\[
\Psi = (\mathring{\Lambda}, \mathring{\Delta}, \Lambda, \Delta)
\]

where \( \mathring{\Lambda} \) and \( \Lambda \) are lattices in duality with respect to a fixed perfect pairing

\[
\langle \cdot, \cdot \rangle : \mathring{\Lambda} \times \Lambda \to \mathbb{Z},
\]

and \( \mathring{\Delta} \subset \mathring{\Lambda} \) and \( \Delta \subset \Lambda \) are subsets with a fixed bijection

\[
\mathring{\Delta} \simeq \Delta,
\]

such that the quadruple satisfies certain requirements \[\text{Spr}98\text{ Sections 7.4.1, 16.2.1}\]. The requirements are symmetric, and the dual based root datum \( \check{\Psi} \) is the quadruple

\[
\check{\Psi} = (\Lambda, \Delta, \mathring{\Lambda}, \mathring{\Delta}),
\]

where \( \Lambda \) and \( \mathring{\Lambda} \) are in duality with respect to the same perfect pairing, and \( \Delta \) and \( \mathring{\Delta} \) are related by the same bijection.

The based root datum \( \Psi(G, B, T) \) of a connected reductive complex algebraic group \( G \) with respect to a Borel subgroup \( B \subset G \), and maximal torus \( T \subset B \), is the quadruple

\[
\Psi(G, B, T) = (\mathring{\Lambda}_T, \mathring{\Delta}_{B,T}, \Lambda_T, \Delta_{B,T}).
\]

For any other choice of Borel subgroup \( B' \subset G \), and maximal torus \( T' \subset B \), there is a canonical isomorphism

\[
\Psi(G, B, T) \simeq \Psi(G, B', T').
\]

The based root datum \( \Psi(G) \) is defined to be the projective limit of the based root data \( \Psi(G, B, T) \) with respect to the canonical isomorphisms over the set of choices of Borel subgroups \( B \subset G \), and maximal tori \( T \subset B \).

There is a short exact sequence

\[
1 \to \text{Inn}(G) \to \text{Aut}(G) \xrightarrow{\psi} \text{Aut}(\Psi(G)) \to 1
\]

where \( \text{Inn}(G) \) is the group of inner automorphisms of \( G \). We may split the sequence as follows. Choose a Borel subgroup \( B \subset G \), a maximal torus \( T \subset B \), and for each simple root \( \hat{\alpha} \in \mathring{\Delta}_{B,T} \), a basis vector \( X_{\hat{\alpha}} \) in the corresponding simple root space for \( T \) acting on \( \mathfrak{b} \). Define the subgroup \( \text{Aut}(G, B, T, \{X_{\hat{\alpha}}\}) \subset \text{Aut}(G) \) to consist of the automorphisms that preserve \( B, T \), and the set \( \{X_{\hat{\alpha}}\} \). Then the restriction of the homomorphism \( \psi \) to \( \text{Aut}(G, B, T, \{X_{\hat{\alpha}}\}) \) is an isomorphism. See \[\text{Spr}79\text{ Sections 1 and 2}\] for more details.
10.2. The dual group. A dual group for a connected reductive complex algebraic group $G$ is a connected reductive complex algebraic group $\hat{G}$ together with a fixed isomorphism

$$\Psi(\hat{G}) \simeq \hat{\Psi}(G).$$

By the above short exact sequence, any two dual groups are isomorphic, and the isomorphism is canonical up to composition with inner automorphisms. Note that there is a canonical isomorphism

$$\text{Aut}(\Psi(G)) \simeq \text{Aut}(\Psi(\hat{G})),\]$$

and for the torus $T$, the dual torus $\hat{T}$ is canonically a dual group.

Recall that the category $P_{G(\mathcal{O})}(\text{Gr})$ of $G(\mathcal{O})$-equivariant perverse sheaves on $\text{Gr}$ is equivalent as a tensor category to the category $\text{Rep}(\hat{G})$ of finite-dimensional representations of the dual group $\hat{G}$ so that the hypercohomology $\mathbb{H} : P_{G(\mathcal{O})}(\text{Gr}) \to \text{Vect}$ corresponds to the forgetful functor $[\text{MV04}, \text{Theorem 7.3, Corollary 3.7, Proposition 6.3}].$ In other words, the group of tensor automorphisms $\text{Aut}^\otimes(\mathbb{H})$ is a connected reductive complex algebraic group which we denote by $\hat{\mathbb{G}}$, and there is a canonical isomorphism of based root data

$$\hat{\Psi}(\hat{G}) \simeq \hat{\Psi}(\hat{\mathbb{G}}).$$

To see the root datum of $\hat{G}$, first recall that the character functor $\text{Ch} : P_{G(\mathcal{O})}(\text{Gr}) \to \text{Vect}_A$ is a tensor functor $[\text{MV04}, \text{Proposition 6.4}],$ and there is a canonical isomorphism

$$\mathbb{H} \simeq F \circ \text{Ch} : P_{G(\mathcal{O})}(\text{Gr}) \to \text{Vect}$$

where $F : \text{Vect}_A \to \text{Vect}$ is the forgetful functor $[\text{MV04}, \text{Theorem 3.6}].$ Therefore we may identify the group $\hat{G}$ with the group of tensor automorphisms of the composite functor $F \circ \text{Ch}$, and we have a canonical homomorphism

$$\text{Aut}^\otimes(F) \to \hat{G}.$$ 

This corresponds to an embedding of the dual torus

$$\hat{T} = \text{Aut}^\otimes(F)$$

as a maximal torus. (See $[\text{MV04}, \text{Theorem 7.3}],$ or Proposition $\text{11.0.2}$ of the appendix.)

Next, recall that the simple objects in the category $P_{G(\mathcal{O})}(\text{Gr})$ are the intersection cohomology sheaves $IC^\lambda$ of the closures of the $G(\mathcal{O})$-orbits $Gr^\lambda \subset \text{Gr}$, with coefficients in trivial one-dimensional local systems. (See $[\text{MV04}, \text{Lemma 7.1}],$ or Proposition $\text{11.0.1}$ of the appendix.) The weight functor $F^\lambda : P_{G(\mathcal{O})}(\text{Gr}) \to \text{Vect}$ evaluated on $IC^\lambda$ provides a canonical line in the graded vector space $\text{Ch}(IC^\lambda)$. By forgetting the grading, for each simple object $IC^\lambda$, we obtain a canonical line $F(F^\lambda(IC^\lambda))$ in the vector space $F(\text{Ch}(IC^\lambda)).$ The subgroup

$$\hat{B} \subset \hat{G}$$

of those tensor automorphisms of $F \circ \text{Ch}$ which preserve the collection of lines $F(F^\lambda(IC^\lambda))$ is a Borel subgroup. By construction, it contains the maximal torus $\hat{T}$.

Finally, there is a canonical isomorphism of based root data

$$\hat{\Psi}(\hat{G}, \hat{B}, \hat{T}) \simeq \hat{\Psi}(G, B, T).$$

See $[\text{MV04}, \text{Theorem 7.3}]$ or the appendix for more details.
10.3. **Associated subgroups.** Via the Tannakian dictionary, we have seen that the category $\mathcal{Q}(\text{Gr}_R)_R$ is equivalent as a tensor category to the category $\text{Rep}(\tilde{H}_R)$ of finite-dimensional representations of a group-scheme $\tilde{H}_R$ so that the hypercohomology $\mathbb{H} : \mathcal{Q}(\text{Gr}_R)_R \to \text{Vect}$ corresponds to the forgetful functor.

We have also seen that the perverse specialization $^pR : \mathbf{P}_{G(O)}(\text{Gr}) \to \mathcal{Q}(\text{Gr}_R)_R$ is a tensor functor, and that there is a canonical isomorphism of tensor functors $H \cong H \circ ^pR : \mathbf{P}_{G(O)}(\text{Gr}) \to \text{Vect}$. Therefore we may identify $\tilde{G}$ with the group of tensor automorphisms of the composite functor $H \circ ^pR$, and we have a canonical homomorphism $\tilde{H}_R \to \tilde{G}$.

Since every object of $\mathcal{Q}(\text{Gr}_R)_R$ is isomorphic to a subquotient of an object of the form $^pR(\mathcal{P})$, where $\mathcal{P}$ runs through all objects of $\mathbf{P}_{G(O)}(\text{Gr})$, the homomorphism is an embedding.

Via the Tannakian dictionary, we have seen that the category $\mathcal{Q}(\text{Gr}_R)_R$ is equivalent as a tensor category to the category $\text{Rep}(\tilde{H})$ of finite-dimensional representations of a group-scheme $\tilde{H}$ so that the hypercohomology $\mathbb{H} : \mathcal{Q}(\text{Gr}_R) \to \text{Vect}$ corresponds to the forgetful functor. From the commutative diagram of Section 9.3, we obtain a commutative diagram of groups

\[
\begin{array}{ccc}
\tilde{H}_R & \cong & \tilde{H} \\
\uparrow & & \searrow \\
\tilde{H} \quad \tilde{H} \times \mathbb{C}^\times \quad \tilde{H} & \hookleftarrow & \tilde{H}
\end{array}
\]

where the horizontal arrows are the obvious projection and inclusion, the vertical arrow is a surjection, and the diagonal arrow is an inclusion.

10.4. **Embedding into Levi subgroup.** Recall that $2\tilde{\rho}_M$ is the sum of the positive roots of the Levi factor $M \subset P$ of the complexification $P \subset G$ of the minimal parabolic subgroup $P_R \subset G_R$. We define the Levi subgroup $L_0 \subset \tilde{G}$ to be the centralizer of $2\tilde{\rho}_M$ under the adjoint representation.

**Proposition 10.4.1.** The embedding $\tilde{H}_R \to \tilde{G}$ factors through $L_0$.

**Proof.** The composite functor

\[
\mathbf{P}_{G(O)}(\text{Gr}) \overset{^pR}{\to} \mathcal{Q}(\text{Gr}_R)_R \overset{\mathbb{H}}{\to} \mathcal{Q}(\text{Gr}_R)_Z \overset{\mathbb{F}}{\to} \text{Vect}_\mathbb{Z}
\]

provides a composite homomorphism

\[
\langle 1 \rangle \times \mathbb{C}^\times \to \tilde{H} \times \mathbb{C}^\times \to \tilde{H}_R \to \tilde{G}
\]

in which the homomorphism $\tilde{H} \times \mathbb{C}^\times \to \tilde{H}_R$ is surjective.

The image of the composite homomorphism is the subgroup $\langle \exp(2c\tilde{\rho}_M) | c \in \mathbb{C} \rangle \subset \tilde{G}$ since the isomorphism $\text{Ch}_R \circ ^pR \simeq \Sigma \circ \text{Ch} : \mathbf{P}_{G(O)}(\text{Gr}) \to \text{Vect}_{\mathbb{A} \times \mathbb{Z}}$ implies that the $\mathbb{Z}$-grading on the perverse specialization $^pR$ corresponds to the $\mathbb{Z}$-grading on $\text{Ch}$ which was defined by pairing with $2\tilde{\rho}_M$. \qed
10.5. **Maximal torus.** We have seen that the real character functor \( \text{Ch}_R : \mathcal{Q}(\text{Gr}_R)_{pR} \to \text{Vect}_{\Sigma(\Lambda_T)} \) is a tensor functor, and there is a canonical isomorphism of tensor functors

\[
\mathcal{H} \simeq F \circ \text{Ch}_R : \mathcal{Q}(\text{Gr}_R)_{pR} \to \text{Vect}
\]

where \( F : \text{Vect}_{\Sigma(\Lambda_T)} \to \text{Vect} \) is the forgetful functor. Therefore we may identify the group \( \tilde{\mathcal{H}}_{pR} \) with the group of tensor automorphisms of the composite functor \( F \circ \text{Ch}_R \), and we have a homomorphism

\[
\tilde{\mathcal{S}}_{\Sigma(\Lambda_T)} \to \tilde{\mathcal{H}}_{pR}
\]

where \( \tilde{\mathcal{S}}_{\Sigma(\Lambda_T)} \) is the torus

\[
\tilde{\mathcal{S}}_{\Sigma(\Lambda_T)} = \text{Spec}(\mathbb{C}[\Sigma(\Lambda_T)])
\]

where \( \Sigma(\Lambda_T) \) is the image of the projection \( \Sigma : \Lambda_T \to \Lambda_S \times \mathbb{Z} \) defined by \( \Sigma(\lambda) = (\lambda + \theta(\lambda), (2\rho_M, \lambda)) \).

We have also seen that there is a canonical isomorphism of tensor functors

\[
\text{Ch}_R \circ p_R \simeq \Sigma \circ \text{Ch} : \mathcal{P}_{G(O)}(\text{Gr}) \to \text{Vect}_{\Sigma(\Lambda_T)}
\]

Therefore we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{H}}_{pR} & \to & \tilde{G} \\
\uparrow & & \uparrow \\
\tilde{\mathcal{S}}_{\Sigma(\Lambda_T)} & \to & \tilde{T}
\end{array}
\]

where the embedding \( \tilde{\mathcal{S}}_{\Sigma(\Lambda_T)} \to \tilde{T} \) corresponds to the tensor functor \( \Sigma : \text{Vect}_{\Lambda_T} \to \text{Vect}_{\Sigma(\Lambda_T)} \). Since the homomorphism \( \tilde{\mathcal{S}}_{\Sigma(\Lambda_T)} \to \tilde{T} \to \tilde{G} \) is an embedding, we conclude that the homomorphism \( \tilde{\mathcal{S}}_{\Sigma(\Lambda_T)} \to \tilde{\mathcal{H}}_{pR} \) is an embedding.

Our aim is to show this is the embedding of a maximal torus.

**Lemma 10.5.1.** If \( \mathcal{Q} \) is a simple object in the category \( \mathcal{Q}(\text{Gr}_R)_{pR} \), then it is isomorphic to the shifted intersection cohomology sheaf \( \text{IC}^\lambda[z] \) of the closure of a stratum \( \text{Gr}^\lambda_{pR} \), for a pair \( (\lambda, z) \in \Sigma(\Lambda_T) \), with coefficients in a one-dimensional trivial local system.

**Proof.** If \( \mathcal{Q} \) is a simple object in the category \( \mathcal{Q}(\text{Gr}_R)_{pR} \), then it must be the shifted intersection cohomology sheaf \( \text{IC}^\lambda(\mathcal{L})[z] \) of the closure of a stratum \( \text{Gr}^\lambda_{pR} \) with coefficients in an irreducible \( G_{pR} \)-equivariant local systems \( \mathcal{L} \). Thanks to the isomorphism

\[
F^\lambda_{pR}(\text{IC}^\lambda(\mathcal{L})) \simeq \mathcal{L}|\lambda,
\]

the real character functor \( \text{Ch}_R \) does not vanish on \( \text{IC}^\lambda(\mathcal{L}) \). By Corollary 3.8.1, we conclude that \( (\lambda, z) \in \Sigma(\Lambda_T) \).

To prove the local system \( \mathcal{L} \) is trivial, it suffices by Lemma 3.6.1 to show that the stabilizer \( P^\lambda_{pR} \subset G_{pR} \) acts trivially on the stalk \( \mathcal{L}|\lambda \). As a representation of \( P^\lambda_{pR} \), the stalk \( \mathcal{L}|\lambda \) is isomorphic to the weight space \( F^\lambda_{pR}(\text{IC}^\lambda(\mathcal{L})) \). The weight space \( F^\lambda_{pR}(\text{IC}^\lambda(\mathcal{L})) \) is a direct summand in the hypercohomology \( \mathbb{H}(\text{Gr}_R, \text{IC}^\lambda_{pR}(\mathcal{L})) \). The hypercohomology \( \mathbb{H}(\text{Gr}_R, \text{IC}^\lambda_{pR}(\mathcal{L})) \) is a subquotient of the hypercohomology \( \mathbb{H}(\text{Gr}_R, p^R_R(\mathcal{P})) \), for some object \( \mathcal{P} \) in the category \( \mathcal{P}_{G(O)}(\text{Gr}) \). The hypercohomology \( \mathbb{H}(\text{Gr}_R, p^R_R(\mathcal{P})) \) is isomorphic to the hypercohomology \( \mathbb{H}(\text{Gr}, \mathcal{P}) \). Since \( G \) is connected, it acts trivially on \( \mathbb{H}(\text{Gr}, \mathcal{P}) \), and so the subgroup \( P^\lambda_{pR} \) does as well. \( \square \)
Proposition 10.5.1. The group $H_{\text{IR}}$ is connected, and the embedding $\tilde{S}_{\Sigma(\Lambda_T)} \to \tilde{H}_{\text{IR}}$ is that of a maximal torus.

Proof. For any dominant real coweight $\lambda \in \Lambda_S^+$, the support of the $n$-fold convolution of the intersection cohomology sheaf $IC^\lambda$ of the closure of the stratum $G_{\text{IR}}^\lambda$ contains the coweight $n\lambda$, and so by the lemma and the criterion of [DMS2, Corollary 2.22], the group $\tilde{H}_{\text{IR}}$ is connected.

The lemma also implies that the rank of $\tilde{H}_{\text{IR}}$ is less than or equal to the rank of the embedded torus $\tilde{S}_{\Sigma(\Lambda_T)}$. Thus they are the same, and $\tilde{S}_{\Sigma(\Lambda_T)}$ is a maximal torus. □

10.6. Root system. In this section, we identify the root system of the subgroup $\tilde{H} \subset \tilde{G}$. Note that due to the surjective homomorphism $\tilde{H} \times \mathbb{C}^\times \to \tilde{H}_{\text{IR}}$, the root system of $\tilde{H}$ is equal to that of $\tilde{H}_{\text{IR}}$.

By forgetting the $\mathbb{Z}$-grading, from the embedding $\tilde{S}_{\Sigma(\Lambda_T)} \to \tilde{H}_{\text{IR}}$, we obtain a commutative diagram of embeddings

$$
\begin{array}{ccc}
\tilde{S}_{\Sigma(\Lambda_T)} & \rightarrow & \tilde{H}_{\text{IR}} \\
\uparrow & & \uparrow \\
\tilde{S}_{\sigma(\Lambda_T)} & \rightarrow & \tilde{H}
\end{array}
$$

where the torus $\tilde{S}_{\sigma(\Lambda_T)}$ is defined to be

$$
\tilde{S}_{\sigma(\Lambda_T)} = \text{Spec}(\mathbb{C}[\sigma(\Lambda_T)])
$$

where $\sigma(\Lambda_T)$ is the image of the projection $\sigma : \Lambda_T \to \Lambda_S$ defined by $\sigma(\lambda) = \lambda + \theta(\lambda)$.

Proposition 10.6.1. The group $\tilde{H}$ is connected, and the embedding $\tilde{S}_{\sigma(\Lambda_T)} \to \tilde{H}_{\text{IR}}$ is that of a maximal torus. The rank of $\tilde{H}$ equals the split-rank of $G_{\text{IR}}$.

Proof. The first two statements are proved similarly to those of Proposition 10.5.1. To see the last, note that the lattice $\sigma(\Lambda_T)$ is a finite index sublattice in the real coweight lattice $\Lambda_S$, and $S \subset G$ is the complexification of a maximal split torus $S_\text{R} \subset G_\text{R}$. □

By construction, the projection $\sigma : \Lambda_T \to \sigma(\Lambda_T)$ is the restriction of weights induced by the inclusion $\tilde{S}_{\sigma(\Lambda_T)} \to \tilde{T}$. Therefore the roots of $\tilde{H}$ with respect to the maximal torus $\tilde{S}_{\sigma(\Lambda_T)}$ are a subset of the projection $\sigma(R_0)$ of the roots $R_0$ of the Levi subgroup $\tilde{L}_0$ with respect to the maximal torus $\tilde{T}$. Thanks to the identity $\tilde{\theta}(2\tilde{\rho}_M) = -2\tilde{\rho}_M$ for the dual involution $\tilde{\theta}$, the involution $\theta$ preserves the set of simple roots $\Delta_0$ of the Levi subgroup $\tilde{L}_0$ with respect to the Borel subgroup $\tilde{B} \cap \tilde{L}_0$. Therefore the projection $\sigma(R_0)$ forms a possibly non-reduced root system in $\sigma(\Lambda_T)$ in which the multiplicity of a projected root is one or two. (A non-reduced root system is one in which there is a root $\alpha$ such that $2\alpha$ is also a root. The multiplicity of a projected root is the number of roots which project to it.) The classification in [Ara62] immediately implies the following.

Lemma 10.6.1. The root system $\sigma(R_0)$ is non-reduced if and only if the based root system $\Delta_0$ contains a copy $\mathfrak{a}_2$ of the based root system of $\mathfrak{sl}_3$ such that the involution $\theta$ exchanges the simple roots of $\mathfrak{a}_2$. To be precise, for a projected root $\alpha \in \sigma(R_0)$, twice it is also a projected root $2\alpha \in \sigma(R_0)$ if and only if there are roots $\beta_1, \beta_2 \in R_0$ such that $\alpha = \sigma(\beta_1) = \sigma(\beta_2)$ and the roots $\beta_1, \beta_2$ are the simple roots of a copy of the root system of $\mathfrak{sl}_3$ in the root system $R_0$. 
Remark 10.6.1. By the classification, for \( \mathfrak{g} \) simple, the root system \( \sigma(R_0) \) is non-reduced if and only if \( \mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \) and \( \mathfrak{g}_\mathbb{R} = \mathfrak{su}(p,q) \) with \( p + q = n \) odd.

Now the obvious candidate for the root system of \( \hat{H} \) with respect to the maximal torus \( \hat{S}_{\sigma(\Lambda_T)} \) would be the projected root system \( \sigma(R_0) \subset \sigma(\Lambda_T) \), but we have seen that in general it is not reduced. We define the root system
\[
\Xi \subset \sigma(\Lambda_T)
\]
to be all of the projected roots \( \alpha \in \sigma(R_0) \) satisfying \( 2\alpha \notin \sigma(R_0) \). In other words, if \( \sigma(R_0) \) is reduced, then we take \( \Xi \) to be all of the projected roots \( \sigma(R_0) \), but if \( \sigma(R_0) \) is non-reduced, then we take \( \Xi \) to be all of the projected roots \( \sigma(R_0) \) except we discard the shorter root of any pair along a single ray.

Theorem 10.6.1. The group \( \hat{H} \) is reductive with root system \( \Xi \subset \sigma(\Lambda_T) \).

Proof. We first identify the root system of the maximal reductive quotient \( \hat{H}_{\text{red}} \) with respect to the image of the maximal torus \( \hat{S}_{\sigma(\Lambda_T)} \) under the natural projection \( \hat{H} \to \hat{H}_{\text{red}} \).

Proposition 10.6.2. The root system of \( \hat{H}_{\text{red}} \) contains a root in every direction in which the root system \( \Xi \) contains a root.

Proof. By definition, the root directions of the root systems \( \Xi \) and \( \sigma(R_0) \) coincide. Since the roots of \( \hat{H}_{\text{red}} \) are a subset of \( \sigma(R_0) \), the Weyl group of \( \hat{H}_{\text{red}} \) is a subgroup of the Weyl group of \( \sigma(R_0) \). By Proposition 8.6.2, the Weyl group of \( \hat{H}_{\text{red}} \) is isomorphic to the small Weyl group \( W_{\mathcal{G}_R} \) so that their actions on \( \sigma(\Lambda_T) \) agree. It is a standard fact that \( W_{\mathcal{G}_R} \) is the Weyl group of the root system \( \sigma(R) \), and so the Weyl group of \( \sigma(R_0) \) is a subgroup of \( W_{\mathcal{G}_R} \). We conclude that the Weyl group of \( \hat{H}_{\text{red}} \), the Weyl group of \( \sigma(R_0) \), and the small Weyl group \( W_{\mathcal{G}_R} \) must all coincide. Since the root directions of a root system are the negative eigenspaces of the simple reflections of its Weyl group, the assertion follows.

Since the roots of \( \hat{H}_{\text{red}} \) are a subset of \( \sigma(R_0) \), if \( \sigma(R_0) \) is reduced, so that \( \Xi \) coincides with it, then the proposition immediately implies that \( \Xi \) is the root system of \( \hat{H}_{\text{red}} \).

If \( \sigma(R_0) \) is non-reduced, then it contains roots \( \alpha \) and \( 2\alpha \), and by the proposition, the root system of \( \hat{H}_{\text{red}} \) contains either \( \alpha \) or \( 2\alpha \). By definition, the root system \( \Xi \) contains 2\( \alpha \) not \( \alpha \). To arrive at a contradiction, suppose the root system of \( \hat{H}_{\text{red}} \) contains \( \alpha \) not 2\( \alpha \). Then the zero weight space of any representation of \( \hat{H}_{\text{red}} \) with non-empty \( \alpha \) weight space must also be non-empty. In particular, the irreducible representation of highest weight \( \alpha_0 \) a Weyl group translate of \( \alpha \) must have non-empty zero weight space. It must correspond to the intersection cohomology sheaf \( IC^{\alpha_0} \) of the stratum \( Gr^{\alpha_0}_R \).

Lemma 10.6.2. Assume the root system \( \sigma(R_0) \) is non-reduced and contains both \( \alpha \) and 2\( \alpha \). Then the intersection of the orbit \( S^0_R \) through the zero coweight with the stratum \( Gr^{\alpha_0}_R \) satisfies
\[
\dim_{\mathbb{R}}(S^0_R \cap Gr^{\alpha_0}_R) < (\check{\rho}, \alpha_0).
\]

Proof. It suffices to show that none of the components of the intersection \( S^0_R \cap Gr^{\alpha_0}_R \) is preserved by the conjugation \( \theta \) of \( Gr \). By the explicit description provided by
Lemma 10.6.1, the coweight $\alpha_0$ is in fact a coroot of $G$, and the intersection cohomology sheaf $IC^{\alpha_0}$ of the stratum $Gr^{\alpha_0}$ corresponds to a summand in the adjoint representation of $\tilde{G}$. It is easy to check that each irreducible component of the intersection $S^0 \cap Gr^{\alpha_0}$ contains a unique simple coroot of $G$ in its closure. But by the explicit description provided by Lemma 10.6.1 none of the simple coroots in the stratum $Gr^{\alpha_0}$ are fixed by the conjugation $\theta$. □

By Proposition 8.6.1, the real weight functor $F^0_\mathbb{R}$ for the zero weight space vanishes on $IC^{\alpha_0}$, and we arrive at a contradiction. Therefore by Proposition 10.6.2, the root system of $\tilde{H}_{\text{red}}$ contains the root $2\alpha$, not $\alpha$, and thus coincides with $\Xi$.

It remains to show that $\tilde{H}$ is reductive, or in other words, that its unipotent radical $\tilde{H}_u$ is trivial. Since $\tilde{H}_u$ is connected, the following suffices.

Lemma 10.6.3. The Lie algebra $\tilde{h}_u$ is trivial.

Proof. Consider the Lie algebra $\tilde{L}_0$ of the Levi subgroup $\tilde{L}_0$ as a representation of $\tilde{H}$. It contains $\tilde{h}$ as a subrepresentation which in turn contains $\tilde{h}_u$ as a subrepresentation. In what follows, references to the weights of these representations are with respect to the torus $S_{\sigma(\Lambda_T)}$.

The zero weight space of $\tilde{L}_0$ is the Cartan subalgebra $\tilde{t}$. Since the intersection $\tilde{h}_u \cap \tilde{t}$ is trivial, the zero weight space of $\tilde{h}_u$ is empty. Therefore the weights of $\tilde{h}_u$ lie in the projection $\sigma(R_0)$ of the roots of $\tilde{L}_0$. Furthermore, if $\alpha$ is a weight of $\tilde{h}_u$, then $\alpha$ is not a root of $\tilde{H}_{\text{red}}$ since otherwise the zero weight space of $\tilde{h}_u$ would be non-empty. In summary, we see that the weights of $\tilde{h}_u$ lie in the complement of the roots of $\tilde{H}_{\text{red}}$ within the projection $\sigma(R_0)$ of the roots of $\tilde{L}_0$. Therefore if the root system $\sigma(R_0)$ is reduced, so that $\Xi$ coincides with it, then we are done.

It remains to consider the case when the root system $\sigma(R_0)$ is non-reduced. We may assume that $G$ is simple and adjoint, so that by the classification we have $G = \text{PGL}_n \mathbb{C}$ and $G_{\mathbb{R}} = \text{PU}(p,q)$ with $p + q = n$ odd. The standard representation of $\tilde{G} = \text{SL}_n \mathbb{C}$ corresponds to the intersection cohomology sheaf of $\mathbb{P}(\mathbb{C}^n)$. The specialization of this stratum is topologically equivalent to the map that collapses the disjoint union of the linear subvarieties $\mathbb{P}(\mathbb{C}^p)$ and $\mathbb{P}(\mathbb{C}^q)$ to a point. Thus the specialization in this case is clearly semisimple. But if $\tilde{h}_u$ were not trivial, then the restriction of this representation to $\tilde{H}$ would not be a direct sum of irreducibles. □

This completes the proof of the theorem. □

Corollary 10.6.1. The Weyl group of $\tilde{H}$ is isomorphic to the small Weyl group of $G_{\mathbb{R}}$.

Proof. Follows from Proposition 8.6.2 as was noted in the proof of Proposition 10.6.2. □

Corollary 10.6.2. The category $\mathcal{Q}(\text{Gr}_{\mathbb{R}})$ is semisimple. Its object are sheaves isomorphic to a direct sum of intersection cohomology sheaves $IC^\lambda$ of the closures of strata $\text{Gr}_{\mathbb{R}}^\lambda$, for $\lambda \in \sigma(\Lambda_T)$, with coefficients in trivial local systems. In particular, convolution and perverse specialization are semisimple.
10.7. **Fixed points of automorphisms.** In this section, we explain how the subgroup \( \hat{H} \subset \hat{G} \) may be realized as the identity component of the fixed points of an involution of a certain Levi subgroup \( \hat{L}_1 \subset \hat{G} \). To make the discussion self-contained, we begin by collecting notation and results from previous sections.

We have fixed a minimal parabolic subgroup \( P_{\mathbb{R}} \subset G_{\mathbb{R}} \), with Levi factor \( M_{\mathbb{R}} \subset P_{\mathbb{R}} \), maximal torus \( T_{\mathbb{R}} \subset M_{\mathbb{R}} \), and maximal split torus \( S_{\mathbb{R}} \subset T_{\mathbb{R}} \). We write \( P \) for the complexification of \( P_{\mathbb{R}} \), \( M \) for that of \( M_{\mathbb{R}} \), \( T \) for that of \( T_{\mathbb{R}} \), and \( S \) for that of \( S_{\mathbb{R}} \).

In Section 2.1, we have seen that the conjugation of \( G \) with respect to \( G_{\mathbb{R}} \) induces an involution \( \theta \) of the coweight lattice \( \Lambda_T \). The real coweight lattice \( \Lambda_S \) is precisely the fixed points of the involution \( \theta \), and we have a projection \( \sigma : \Lambda_T \to \Lambda_S \) defined by \( \sigma(\lambda) = \theta(\lambda) + \lambda \).

In Section 10.2, we have seen that the dual group \( \hat{G} \) comes equipped with a distinguished Borel subgroup \( \hat{B} \subset \hat{G} \), and maximal torus \( \hat{T} \subset \hat{G} \) which is identified with the torus dual to \( T \). We write \( \Delta_{B,\hat{T}} \subset \Lambda_T \) for the simple roots of \( G \) with respect to \( \hat{B} \) and \( \hat{T} \).

In Section 10.4, we have seen that the subgroup \( \hat{H} \subset \hat{G} \) is in fact a subgroup of the Levi subgroup \( \hat{L}_0 \subset \hat{G} \) which is the centralizer of \( 2\hat{\rho}_M \) under the adjoint representation. Here as usual \( 2\hat{\rho}_M \) is the sum of the positive roots of the Levi subgroup \( M \subset G \).

The simple roots \( \Delta_0 \subset \Delta_{\hat{B},\hat{T}} \) of the Levi subgroup \( \hat{L}_0 \subset \hat{G} \) with respect to the Borel subgroup \( \hat{B} \cap \hat{L}_0 \), and maximal torus \( \hat{T} \), satisfy

\[
\alpha \in \Delta_0 \text{ if and only if } \langle 2\hat{\rho}_M, \alpha \rangle = 0.
\]

Note that we have \( \theta(\Delta_0) = \Delta_0 \) thanks to the identity \( \hat{\theta}(2\hat{\rho}_M) = -2\hat{\rho}_M \), where \( \hat{\theta} \) is the dual involution.

To define the Levi subgroup \( \hat{L}_1 \subset \hat{G} \), for each simple root \( \alpha \in \Delta_0 \), choose a basis vector \( X_\alpha \) in the simple root space \( (\mathfrak{b} \cap \mathfrak{l}_0)_\alpha \). Define elements \( \tau(\alpha) \in \Lambda_T \) to be

\[
\tau(\alpha) = \alpha - \theta(\alpha), \text{ if } [X_\alpha, X_{\theta(\alpha)}] \neq 0
\]

\[
\tau(\alpha) = 0, \text{ if } [X_\alpha, X_{\theta(\alpha)}] = 0.
\]

Clearly the elements \( \tau(\alpha) \) are independent of the choice of basis vectors \( X_\alpha \).

Define the Levi subgroup \( \hat{L}_1 \subset \hat{G} \) to be the centralizer of the span of \( 2\rho_M \) and the elements \( \tau(\alpha) \), for all simple roots \( \alpha \in \Delta_0 \), under the coadjoint representation. Here as usual \( 2\rho_M \) is the sum of the positive coroots of the Levi subgroup \( M \subset G \).

The simple roots \( \Delta_1 \subset \Delta_{\hat{B},\hat{T}} \) of the Levi subgroup \( \hat{L}_1 \subset \hat{G} \) with respect to the Borel subgroup \( \hat{B} \cap \hat{L}_1 \), and maximal torus \( \hat{T} \), are of two types

\[
\alpha \in \Delta_1, \text{ for } \alpha \in \Delta_0 \text{ with } [X_\alpha, X_{\theta(\alpha)}] = 0, \text{ and}
\]

\[
\alpha + \theta(\alpha) \in \Delta_1, \text{ for } \alpha \in \Delta_0 \text{ with } [X_\alpha, X_{\theta(\alpha)}] \neq 0.
\]

Note that we have a similar description for the simple coroots \( \Delta_1 \).

The involution \( \theta \) preserves the set \( \Delta_1 \), and similarly the dual involution \( \hat{\theta} \) preserves the set \( \Delta_1 \). Therefore they provide an involution of the based root datum

\[
\Psi(\hat{L}_1, \hat{B} \cap \hat{L}_1, \hat{T}) = (\Lambda_T, \Delta_1, \hat{\Lambda}_T, \hat{\Delta}_1)
\]

of the Levi subgroup \( \hat{L}_1 \subset \hat{G} \). To lift the involution to an involution of \( \hat{L}_1 \), we must choose for each simple root \( \beta \in \Delta_1 \), a basis vector \( X_\beta \) in the corresponding simple root
space \((\mathfrak{b} \cap \mathfrak{l}_1)_{\beta}\). Then there is a unique lift of the involution to an involution of \(\tilde{L}_1\) which we denote by \(\tilde{\theta}_1\) such that
\[
\tilde{\theta}_1(\mathfrak{B} \cap \mathfrak{L}_1) = \mathfrak{B} \cap \mathfrak{L}_1 \quad \tilde{\theta}_1(\mathfrak{T}) = \mathfrak{T} \quad \tilde{\theta}_1(\mathfrak{X}_\beta) = \mathfrak{X}_{\theta(\beta)}.
\]
For any other choice of basis vectors, the resulting lift will be conjugate to \(\tilde{\theta}_1\) via an inner automorphism. Moreover, this inner automorphism may be realized as conjugation by a unique element of \(\mathfrak{T}/\mathfrak{Z}(\mathfrak{L}_1)\), where \(\mathfrak{Z}(\mathfrak{L}_1)\) is the center of \(\mathfrak{L}_1\).

**Remark 10.7.1.** We mention here a criterion for \(\tilde{L}_1\) to equal \(\tilde{L}_0\). For all simple roots \(\alpha \in \Delta_0\), we have
\[
[X_\alpha, X_{\theta(\alpha)}] = 0
\]
if and only if for each connected component \(C\) in the Dynkin diagram of \(\tilde{L}_0\) with \(\tilde{\theta}_0(C) = C\), there is a node \(c \in C\) such that \(\theta(c) = c\).

Alternatively, one can realize the Levi subgroup \(\tilde{L}_1\) as the fixed points of an inner automorphism \(\tilde{\eta}_0\) of the Levi subgroup \(\tilde{L}_0\). For each simple root \(\alpha \in \Delta_0\), consider the new basis vectors \(X'_\alpha\) defined by
\[
X'_\alpha = X_\alpha, \text{ if } [X_\alpha, X_{\theta(\alpha)}] = 0,
\]
\[
X'_\alpha = k_\alpha X_\alpha, \text{ if } [X_\alpha, X_{\theta(\alpha)}] \neq 0,
\]
for any non-zero numbers \(k_\alpha\). Then there is a unique inner automorphism \(\tilde{\eta}_0\) of the group \(\tilde{L}_0\) such that
\[
\tilde{\eta}_0(\mathfrak{B} \cap \mathfrak{L}_0) = \mathfrak{B} \cap \mathfrak{L}_0 \quad \tilde{\eta}_0(\mathfrak{T}) = \mathfrak{T} \quad \tilde{\eta}_0(\mathfrak{X}_\alpha) = X'_\alpha,
\]
and it does not depend on the choice of the basis vectors \(X_\alpha\). If one takes the numbers \(k_\alpha\) to be different from 1, then the Levi subgroup \(\tilde{L}_1\) is the fixed points in the Levi subgroup \(\tilde{L}_0\) of the inner automorphism \(\tilde{\eta}_0\).

**Theorem 10.7.1.** There is a choice of basis for the simple root spaces of the Levi subgroup \(\tilde{L}_1 \subset \hat{G}\) such that the subgroup \(\mathcal{H} \subset \hat{G}\) associated to the real form \(G_{\mathbb{R}}\) is the identity component of the fixed points of the involution \(\tilde{\theta}_1\).

**Proof.** In Section 10.5, we have seen that the torus \(\tilde{S}_{\sigma(\Lambda_T)} = \text{Spec}(\mathbb{C}[\sigma(\Lambda_T)])\) is a maximal torus of \(\hat{H}\). Here as usual \(\sigma(\Lambda_T)\) denotes the image of the projection \(\sigma : \Lambda_T \rightarrow \Lambda_S\).

**Lemma 10.7.1.** The identity component of the fixed points of the restriction of the involution \(\tilde{\theta}_1\) to the torus \(\tilde{T}\) is equal to the torus \(\tilde{S}_{\sigma(\Lambda_T)}\).

**Proof.** By definition, the restriction of the involution \(\tilde{\theta}_1\) to the torus \(\tilde{T}\) is induced by the involution \(\theta\) of the lattice \(\Lambda_T\). \(\square\)

We write \(R_0 \subset \Lambda_T\) for the roots of \(\tilde{L}_0\) with respect to \(\tilde{T}\), and \(R_1 \subset \Lambda_T\) for the roots of \(\tilde{L}_1\) with respect to \(\tilde{T}\). In Section 10.6, we have seen that the roots \(\Xi \subset \sigma(\Lambda_T)\) of the group \(\hat{H}\) with respect to \(\tilde{S}_{\sigma(\Lambda_T)}\) consist of those projected roots \(\sigma(\alpha) \in \sigma(\Lambda_0)\) such that \(2\sigma(\alpha) \notin \sigma(\Lambda_0)\).

**Lemma 10.7.2.** For a root \(\alpha \in R_0\), we have \(\sigma(\alpha) \in \Xi\) if and only if \(\alpha \in R_1\). Therefore we have the inclusion \(\hat{H} \subset \tilde{L}_1\).
Proof. Follows from the definition of $\tilde{L}_1$, and Lemma \ref{lemma:1.6.1} $\Box$

It remains to show that there is a choice of basis vectors $X_\beta$ for the simple root spaces of $\tilde{L}_1$ such that the simple root spaces of $\tilde{H}$ are the fixed subspaces of the involution $\tilde{\theta}_1$. For a simple root in $\Xi$, there are two possibilities: one simple root $\beta \in \Delta_1$ projects to it, or two simple roots $\beta_1, \beta_2 \in \Delta_1$ project to it. If only one simple root $\beta$ projects to a root in $\Xi$, we take $X_\beta$ to be any non-zero vector in the corresponding root space.

Lemma 10.7.3. If two simple roots $\beta_1, \beta_2$ project to a root in $\Xi$, then the intersection of $\tilde{h}$ with the span of the corresponding simple root spaces is exactly one-dimensional, and the projection of the intersection to either of the two root spaces is an isomorphism.

Proof. In Section \ref{section:10.6}, we have seen that the intersection is at least one-dimensional. Since $\tilde{S}_{\sigma(\Lambda^T)}$ is a maximal torus of $\tilde{H}$, the intersection is no greater than one-dimensional, and cannot lie entirely in either simple root space. $\Box$

If two simple roots $\beta_1, \beta_2$ project to a root in $\Xi$, we choose any non-zero vector in the intersection, and take $X_{\beta_1}, X_{\beta_2}$ to be its images under the projections.

It is straightforward to check that the subalgebra $\tilde{h}$ is exactly the fixed points in $\tilde{L}_1$ of the involution $\tilde{\theta}_1$ constructed with respect to this basis. $\Box$

Alternatively, one can define an automorphism of the Levi subgroup $\tilde{L}_0 \subset \tilde{G}$ such that the subgroup $\tilde{H} \subset \tilde{G}$ is the identity component of its fixed points. In the definition of the inner automorphism $\tilde{\eta}_0$, for each pair of simple roots $\alpha, \theta(\alpha) \in \Delta_0$ such that $[X_\alpha, X_{\theta(\alpha)}] \neq 0$, if one takes the numbers $k_\alpha, k_{\theta(\alpha)}$ to be any pair such that $k_\alpha k_{\theta(\alpha)} = -1$, then the subgroup $\tilde{H} \subset \tilde{G}$ is the identity component of the fixed points of the automorphism $\tilde{\theta}_0 \circ \tilde{\eta}_0$ of the Levi subgroup $L_0$. For example, for the choice $k_\alpha = 1$ and $k_{\theta(\alpha)} = -1$, the automorphism $\tilde{\theta}_0 \circ \tilde{\eta}_0$ is of order 4.

11. APPENDIX: IDENTIFICATION OF THE DUAL GROUP

Let $G$ be a connected reductive complex algebraic group with fixed Borel subgroup $B \subset G$, and maximal torus $T \subset B$. We sketch here how to show the Tannakian group $\text{Aut}^\otimes(\mathbb{H})$ of the category $\mathcal{P}_{G(O)}(\text{Gr})$ with fiber functor $\mathbb{H} : \mathcal{P}_{G(O)}(\text{Gr}) \rightarrow \text{Vect}$ is a dual group for $G$. We must show that $\text{Aut}^\otimes(\mathbb{H})$ is a connected reductive complex algebraic group, and find a canonical isomorphism of based root data

$$\Psi(\text{Aut}^\otimes(\mathbb{H})) \simeq \tilde{\Psi}(G).$$

In what follows, for $\lambda \in \Lambda^+_T$, we write $\text{IC}^\lambda$ for the intersection cohomology sheaf of the closure of the $G(O)$-orbit $\text{Gr}^\lambda \subset \text{Gr}$, with coefficients in the trivial one-dimensional local system.

Proposition 11.0.1. The category $\mathcal{P}_{G(O)}(\text{Gr})$ is semisimple, with simple objects isomorphic to $\text{IC}^\lambda$, for $\lambda \in \Lambda^+_T$.

Proof. The stabilizer in $G(O)$ of a coweight $\lambda \in \text{Gr}$ is the parahoric subgroup $\mathfrak{p}^\lambda$ which is connected. Therefore the simple objects of the category $\mathcal{P}_{G(O)}(\text{Gr})$ are as asserted, and there are no self-extensions of simple objects. By Proposition \ref{proposition:8.2.1}, the strata $\text{Gr}^\lambda$ in a given component of $\text{Gr}$ are either all even-dimensional or all odd-dimensional.
By [Lus83, Section 11, c)], the stalks of IC\(^{\lambda}\) have the parity vanishing property: they are non-zero only in the parity of the dimension of Gr\(^{\lambda}\). Therefore there are no other extensions, and we conclude that the category \(P_{G(O)}(\text{Gr})\) is semisimple.

**Corollary 11.0.1.** The Tannakian group \(\text{Aut}^\otimes(\mathbb{H})\) is a connected reductive complex algebraic group.

**Proof.** Choose generators \(\lambda_1, \ldots, \lambda_r \in \Lambda_T^+\), and for \(\lambda \in \Lambda_T^+\), write \(\lambda = \sum n_i \lambda_i\). Then the open part of the support of the convolution \((\text{IC}^{\lambda_i}) \otimes n_1 \circ \cdots \circ (\text{IC}^{\lambda_r}) \otimes n_r\) contains \(\lambda\), and so IC\(^{\lambda}\) appears as a summand in the convolution. Therefore by the proposition, the direct sum \(\oplus_i \text{IC}^{\lambda_i}\) is a tensor generator for the category \(P_{G(O)}(\text{Gr})\), and so by [DMS2, Proposition 2.20(b)], the group \(\text{Aut}^\otimes(\mathbb{H})\) is algebraic.

For \(\lambda \in \Lambda_T^+\), the open part of the support of the convolution \((\text{IC}^{\lambda}) \otimes n\) contains \(n\lambda\), and so IC\(^{n\lambda}\) appears as a summand in the convolution. Therefore by the proposition and [DMS2 Corollary 2.22], the group \(\text{Aut}^\otimes(\mathbb{H})\) is connected.

Finally, by the proposition and [DMS2 Proposition 2.23], the group \(\text{Aut}^\otimes(\mathbb{H})\) is reductive. \(\square\)

The character functor \(\text{Ch} : P_{G(O)}(\text{Gr}) \to \text{Vect}_{\Lambda_T}\), which is by definition the sum \(\text{Ch} = \sum_{\lambda \in \Lambda_T} F^\lambda\) of the weight functors \(F^\lambda : P_{G(O)}(\text{Gr}) \to \text{Vect}\), and the canonical isomorphism \(\mathbb{H} \simeq F \circ \text{Ch} : P_{G(O)}(\text{Gr}) \to \text{Vect}_{\Lambda_T}\), where \(F : \text{Vect}_{\Lambda_T} \to \text{Vect}\) is the forgetful functor, provide a canonical homomorphism from the dual torus \(\hat{T} = \text{Aut}^\otimes(F)\) to \(\text{Aut}^\otimes(\mathbb{H})\).

**Proposition 11.0.2.** The homomorphism \(\hat{T} \to \text{Aut}^\otimes(\mathbb{H})\) is the embedding of a maximal torus.

**Proof.** For \(\lambda \in \Lambda_T^+\), we clearly have \(F^\lambda(\text{IC}^{\lambda}) \simeq C\), so by [DMS2 Proposition 2.21 (b)], the homomorphism is injective. The rank of \(\text{Aut}^\otimes(\mathbb{H})\) equals the rank of \(P_{G(O)}(\text{Gr})\), and by the previous proposition, the rank of \(P_{G(O)}(\text{Gr})\) equals the rank of \(G\). This in turn equals the rank of \(\hat{T}\), and so the image of the homomorphism is a maximal torus. \(\square\)

A Borel subgroup \(\hat{B} \subset \text{Aut}^\otimes(\mathbb{H})\) containing \(\hat{T}\) is equivalent to the choice of a \(\hat{T}\)-invariant line \(L^\lambda\) in each irreducible representation \(V^\lambda\), for \(\lambda \in \Lambda_T^+\) such that \(L^\lambda \otimes L^\mu = L^{\lambda+\mu}\) under the projection \(V^\lambda \otimes V^\mu \to V^{\lambda+\mu}\). The Borel subgroup \(\hat{B} \subset \text{Aut}^\otimes(\mathbb{H})\) is the stabilizer of the lines, or conversely the lines are the highest weight lines of the Borel subgroup. Using the canonical isomorphism \(\mathbb{H} \simeq F \circ \text{Ch}\), we take \(L^\lambda = F^\lambda(\text{IC}^{\lambda})\), for \(\lambda \in \Lambda_T^+\).

It remains to show there is a canonical isomorphism of based root data

\[\Psi(\text{Aut}^\otimes(\mathbb{H}), \hat{B}, \hat{T}) \simeq \Psi(G, B, T).\]

**Proposition 11.0.3.** With respect to the given Borel subgroups and maximal tori, (i) the set of dominant weights of \(\text{Aut}^\otimes(\mathbb{H})\) coincides with the set of dominant coweights of \(G\) as subsets of \(\Lambda_T\), and (ii) the set of simple roots of \(\text{Aut}^\otimes(\mathbb{H})\) coincides with the set of simple coroots of \(G\) as subsets of \(\Lambda_T\).
Proof. Assertion (i) is immediate: the weights of the lines $L^\lambda \subset \mathbb{H}(\text{Gr}, IC^\lambda)$ coincide with the dominant coweights of $G$.

For assertion (ii), it suffices to show that for $\lambda \in \Lambda^+_T$, and $\mu \in \Lambda_T$, the weights $F^\mu(\text{IC}^\lambda)$ vanish if $\lambda - \mu$ is not a non-negative integral linear combination of positive coroots $\alpha \in R^\text{pos}$, and for a simple coroot $\alpha \in \Delta_{B,T}$, the weights $F^{\lambda - \alpha}(\text{IC}^\lambda)$ do not vanish. Proposition 8.1.1 implies the vanishing, and Propositions 8.1.1 and 8.4.1, and the explicit description of the weights given in [MV00, Section 5] implies the non-vanishing. □

Now we have the asserted isomorphism of based root data. The simple coroot directions of $\text{Aut}^\otimes(\mathbb{H})$ are the extremal rays of the positive semigroup dual of the dominant weights of $\text{Aut}^\otimes(\mathbb{H})$. Similarly, the simple root directions of $G$ are the extremal rays of the positive semigroup dual of the dominant coweights of $G$. By part (i) of the proposition, these rays coincide. For a root $\alpha$ and the corresponding coroot $\dot{\alpha}$, one has $\langle \dot{\alpha}, \alpha \rangle = 2$. Therefore the simple coroots of $\text{Aut}^\otimes(\mathbb{H})$ coincide with the simple roots of $G$ since they are in the same rays and since, by part (ii) of the proposition, the simple roots of $\text{Aut}^\otimes(\mathbb{H})$ coincide with the simple coroots of $G$.

Remark 11.0.2. For Levi subgroups $L \subset G$ other than the torus $T$, one may generalize the character functor $\text{Ch} : \mathbf{P}_{G(O)}(\text{Gr}) \to \text{Vect}_{\Lambda_T}$ to tensor functors $\text{Ch}_L : \mathbf{P}_{G(O)}(\text{Gr}) \to \mathbf{P}_{L(O)}(\text{Gr}_L)$. In particular, for each simple coroot of $G$, this provides an explicit embedding of an $\mathfrak{sl}_2$-triple into the Lie algebra of $\text{Aut}^\otimes(\mathbb{H})$. For more details, see for example [BG02 Section 4.3.1]. This is used in [BG01] Theorem 2.2] to construct crystals for $\hat{g}$.

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