1 Introduction

The classic, global approach to black holes in terms of event horizons has accomplished much: One has a definition of black hole horizon which is unambiguous and general, depending only on the assumption that space-time is asymptotically flat or asymptotically (anti-)de Sitter. It is a definition valid for fully dynamical black holes that possess no symmetry. For event horizons, one can prove among other things Hawking’s very general area theorem, which implies that the areas of event horizons always grow or stay constant in time. If one furthermore restricts consideration to stationary black holes, then the ADM angular momentum and ADM energy can be interpreted as the angular momentum and energy of the black hole itself, giving a notion of these quantities for black holes that is well-rooted in their deeper meaning as generators of flows on phase space. By using the global stationary Killing field available for such space-times, one also arrives at well-defined notions of surface gravity and angular velocity of the horizon. These quantities, together with surface area, energy and angular momentum, satisfy the zeroth and first laws of black hole mechanics, part of the evidence suggesting that black holes are thermodynamic objects.

In spite of these successes, both of these notions of black hole are, from a physical perspective, not completely satisfactory:

- The event horizon definition requires knowledge of the entire space-time all the way to future null infinity. However, physically, one can never know the full history of the universe, nor can one measure quantities at infinity.

- The use of stationary space-times to derive black hole thermodynamics is also not ideal: In all other physical situations, in order to derive laws of equilibrium thermodynamics, it is only necessary to assume equilibrium of the system in question, not the entire universe.

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Furthermore, beyond such physical considerations, the global nature of event horizons and Killing horizons make them difficult to use in practice. In particular, in quantum theory, in order for a definition of black hole to make sense, one needs to be able to formulate it in terms of phase space functions which can be quantized. Event horizons are not amenable to such a characterization in any obvious or simple way. By contrast, the condition defining a Killing horizon can be formulated even in terms of local functions on space (if the Killing field is fixed). However, because the restriction is imposed on fields outside, as well as within, the horizon, the degrees of freedom outside the horizon are reduced and it is again no longer clear if canonical quantum gravity methods, in particular those of loop quantum gravity, can be applied reliably there.

Additionally, in numerical relativity, the global notions of ADM energy and ADM angular momentum are of limited use, because they do not distinguish the mass of black holes from the energy of surrounding gravitational radiation. If a quasi-local framework could enable a clear definition of angular momentum and energy of a black hole using standard canonical notions, this would be useful for interpreting numerical simulations in a gauge-invariant and systematic way.

For all of the above reasons it is both physically desirable and practical to seek a quasi-local alternative to the event horizon and Killing horizon frameworks.

One can ask: Is it possible to find a framework for describing black holes which is quasi-local yet nevertheless retains all of the desirable features of Killing horizons? It is not a priori obvious that this is possible, because part of the reason for the success of the Killing horizon framework is that one can use global stationarity to relate physics at the horizon to physics at infinity, where the asymptotic flat metric is available to aid in the definition of various quantities. In a quasi-local approach, such access to the structures at infinity will not be possible. In spite of this challenge, as we shall see, the isolated horizon framework allows one to answer the above question in the affirmative. Moreover, due to the fact that the isolated horizon conditions restrict only the intrinsic geometric structure of the horizon, they can be implemented in quantum theory, giving rise to a quantum description of black holes within loop quantum gravity. In this framework of quantum isolated horizons, one can account for the statistical mechanical origin of the Bekenstein-Hawking entropy of black holes in a large number of physically relevant situations.

The isolated horizon framework has been developed by many authors [1–14]. Furthermore, the framework is closely related to, and inspired by, not only the Killing horizons as hinted above, but also the trapping horizons of Hayward [15–17]. Roughly speaking, isolated horizons correspond to portions of trapping horizons that are null. Portions of trapping horizons that are space-like roughly correspond to
a complementary notion called ‘dynamical horizons’ [18]. A more broad review of both of these topics and their relation to trapping horizons has been given in [19].

In this chapter, after reviewing Killing horizons and black-hole thermodynamics for motivational purposes, we review the geometrical observations regarding null surfaces which naturally lead to the isolated horizon definition. We then review a covariant canonical framework for space-times with isolated horizons, leading to a canonical, quasi-local notion of angular momentum and mass which then satisfy a quasi-local version of the first law of black hole mechanics. Motivated by the expression for the angular momentum, quasi-local angular momentum and mass multipoles for isolated horizons are also reviewed, in passing. Finally, we review the quantization of this framework, using the methods of loop quantum gravity [20–22], leading to a statistical mechanical explanation of the Bekenstein-Hawking entropy of black holes.

2 Motivation from Killing horizons and black-hole thermodynamics

2.1 Black hole horizons and the four laws of black-hole mechanics

A black hole is by definition an object with a gravitational field so strong that no radiation can escape from it to the asymptotic region of space-time. Mathematically, a black hole region $B$ of a space-time $\mathcal{M}$ is that which excludes all events that belong to the causal past of future null infinity, here denoted $\mathcal{I}^+$: $B = \mathcal{M} \setminus J^-(\mathcal{I}^+)$, with $J^-(N)$ representing the union of the causal pasts of all events contained in $N$ [23]. The event horizon, $\mathcal{H}$, is the boundary of the black hole region: $\mathcal{H} = \partial B$. The two-dimensional cross section $S$ of $\mathcal{H}$ is the intersection of $\mathcal{H}$ with a three-dimensional spacelike (partial Cauchy) hypersurface $\mathcal{M}$: $S = \mathcal{H} \cap \mathcal{M}$.

A space-time is said to be stationary if the metric admits a Killing field $t^a (a, b, \cdots \in \{0, 1, 2, 3\})$ which approaches a time-translation at spatial infinity. A space-time is said to be axisymmetric if the metric admits a Killing field $\phi^a$ that generates an $SO(2)$ isometry. Examples of space-times that are both stationary and axisymmetric are the Schwarzschild solution and more generally the Kerr-Newman family of solutions. The stationary Killing field $t^a$ is normalized by requiring that it be unit at spatial infinity, while the axisymmetric Killing field $\phi^a$ is normalized by requiring that the affine length of its orbits be $2\pi$.

A space-time is said to contain a Killing horizon $\mathcal{K}$ if it possesses a Killing vector field $\xi^a$ that becomes null at $\mathcal{K}$ and generates $\mathcal{K}$. That is, $\mathcal{K}$ is a null hypersurface with $\xi^a$ as its null normal [24, 25]. For stationary space-times, under very general assumptions, the event horizon is also a Killing horizon [23, p.331]. As $\xi^a$ is by definition hypersurface-orthogonal at $\mathcal{K}$, by the Frobenius theorem it satisfies

$$\xi_{[a} \nabla_{b} \xi_{c]} \equiv 0$$
where \(\cong\) denotes equality on \(\mathcal{K}\). From this one can furthermore deduce that \(\xi\) is geodesic at \(\mathcal{K}\):

\[
\xi^a \nabla_a \xi^b \cong \kappa \xi^b .
\]  

(1)

This defines the surface gravity \(\kappa\) of the black hole. From this definition it is clear that the surface gravity rescales as \(\kappa \rightarrow c\kappa\) under rescalings of the Killing vector field \(\xi \rightarrow c\xi\), with \(c\) a constant. Equation (1) can be rewritten

\[
2\kappa \xi_a \cong \nabla_a (-\xi_b \xi^b) .
\]

Making use of the Frobenius theorem, geodesic equation and Killing equation for the vector \(\xi^a\), one obtains the following explicit expression for the surface gravity \(\kappa\):

\[
\kappa^2 = -\frac{1}{2} (\nabla_a \xi_b) (\nabla^a \xi^b) .
\]  

(2)

If the dominant energy condition is satisfied, then it follows that \(\kappa\) not only is constant along the null generators of \(\mathcal{H}\), but also it does not vary from generator to generator. This means that \(\kappa\) remains constant over all of \(\mathcal{H}\). This is the zeroth law of black-hole mechanics:

**Zeroth Law.** *If the dominant energy condition is satisfied, then the surface gravity \(\kappa\) is constant over the entire event horizon \(\mathcal{H}\).*

The black-hole uniqueness theorems [23, 26–30] state that there is a unique three-parameter set of solutions to the Einstein-Maxwell field equations that are stationary and asymptotically flat; these are the Kerr-Newman family of solutions parameterized by the ADM mass \(M\), ADM angular momentum \(J\) and electric charge \(Q\) of the space-time. For this family, the Killing vector field defining \(\mathcal{K}\) is given by \(\xi^a = t^a + \Omega \phi^a\), where \(\Omega\) is called the angular velocity of \(\mathcal{K}\). The electric potential at \(\mathcal{K}\) is furthermore defined as \(\Phi = -A_a \xi^a|_{\mathcal{K}}\) where and \(A_a\) is the 4-vector potential of the electromagnetic field. If the mass of a solution contained in the Kerr-Newman family is perturbed by an amount \(\delta M\), then the changes in the surface area \(a\) and asymptotic charges \((M,J,Q)\) of \(\mathcal{K}\) are governed by the following law.

**First Law.** *Let \(a, \kappa, \Omega, \Phi, \text{ and } (M,J,Q)\) denote the horizon area, surface gravity, angular velocity, electric potential, and asymptotic charges of a stationary black hole. For any variation \(\delta\) within the space of stationary black holes, one has

\[
\delta M = \frac{\kappa}{8\pi G} \delta a + \Phi \delta Q + \Omega \delta J .
\]  

(3)

This form of the first law is known as the “equilibrium” form. That is, the condition (3) describes the changes of the black-hole parameters from a solution to nearby solutions within the phase space. There is, however, a “physical process” interpretation: if one drops a small mass \(\delta M\) of matter into a black hole, the resulting changes \(\delta Q\) and \(\delta J\) in the charge and angular momentum of the black hole
Table 1: A summary of the four laws of black-hole mechanics and the corresponding laws of thermodynamics. Here, we make the identifications $U = M$, $T = \kappa/(2\pi)$ and $S = a/4$. [Adapted from [25].]

will be such that the first law equation is satisfied. Such an interpretation is valid if the space-time is quasi-stationary — i.e., approximately stationary at each point in time.

If some matter with stress energy $T_{ab}$ is dropped into a black hole, then the mass of black hole is going to change. From the first law, the surface area $a$ will have to change as well. A remarkable fact is that, if $T_{ab}\xi^a\xi^b \geq 0$, then this change cannot be a decrease. This is a statement of the second law of black-hole mechanics:

**Second Law** The surface area $a$ can never decrease in a physical process if the stress-energy tensor $T_{ab}$ satisfies the null energy condition $T_{ab}\xi^a\xi^b \geq 0$.

One final property of quasi-stationary horizons is that it is impossible for one to become extremal within finite advanced time (i.e., finite time as experienced by free-falling observers near the horizon) [31]. That is, the surface gravity cannot be reduced to zero in finite advanced time. This is the statement of the third law of black-hole mechanics:

**Third Law.** The surface gravity $\kappa$ of a quasi-stationary black hole cannot be reduced to zero by any physical process within finite advanced time.

### 2.2 Black-hole thermodynamics

At this point it is instructive to pause for a moment to summarize the established four laws of black-hole mechanics, and compare them to the corresponding four laws of thermodynamics. In the following Table 1, $T$ is the temperature of a system, $U$ its internal energy and $S$ its entropy. As one can see, there is a striking formal similarity between these two sets of laws. Motivated by this similarity, Bekenstein
conjectured that for a black hole [32, 33]:

\[ \kappa \propto T \quad \text{and} \quad a \propto S . \]

This identification of the surface area of the horizon with thermal entropy also offered a way to compensate for the apparent violation of the second law of thermodynamics which would seem to occur when matter falls into a black hole. The idea is that, because black holes are now endowed with entropy, a generalized second law of thermodynamics would still hold, even during such processes:

\[ \delta S_{\text{Universe}} + \delta S_{\text{Black-hole}} \geq 0 . \]

However, it turns out that when trying to match \( a \) with \( S \), and in order for \( S \) to remain dimensionless, we require a combination of the physical constants \( c, G \) and \( \hbar \). By convention we take temperature to be measured in units of energy so that the Boltzmann constant is unity, whence the unique combinations of fundamental constants which will fit into the proportionality relations are

\[ S = \text{constant} \times \frac{a}{4l_P^2} \quad \text{and} \quad T = \text{constant} \times \hbar\kappa , \]

with \( l_P = \sqrt{G\hbar/c^3} \) the Planck length.

In 1974, Hawking [34] discovered that black holes radiate a blackbody spectrum with a temperature of \( T = \hbar\kappa/(2\pi) \). This is known as the Hawking Effect. The result came from considering quantum field theory on a fixed black hole background space-time. Hawking was able to use this framework to fix the proportionality constant in Bekenstein’s surface-gravity/temperature relation by requiring that

\[ T\delta S = \frac{\kappa}{8\pi G}\delta a . \quad (4) \]

One finds that

\[ S \equiv \frac{a}{4l_P^2} \quad \text{(5)} \]

in Planck units with \( l_P = \sqrt{G\hbar/c^3} \) the Planck length, or

\[ S \equiv \frac{a k_B c^3}{4\hbar G} \quad \text{(6)} \]

in SI units with \( k \) Boltzmann’s constant.

The four laws were first formulated for stationary space-times in four-dimensional Einstein-Maxwell theory [35], but later were extended using covariant phase space methods to include stationary black holes in arbitrary diffeomorphism-invariant theories [36–39]. This work revealed, remarkably, that the zeroth law holds for any stationary black hole space-time if the matter fields satisfy an appropriate energy condition. In addition, the surface-area term in the first law is modified only in cases when
gravity is supplemented with nonminimally coupled matter or higher-curvature interactions. For a general Lagrangian density \( L = L(g_{ab}, \nabla_c g_{ab}, R_{abcd}, \nabla_e R_{abcd}) \) (where \( L(\cdot, \cdot, \cdot, \cdot) \) involves no derivatives of its arguments), the entropy of a stationary black hole space-time is given by

\[
S = -2\pi \oint_S \frac{\delta L}{\delta R_{abcd}} n_{ab} n_{cd},
\]

with \( n_{ab} \) the binormal to the cross-section \( S \) of the horizon, with normalization \( n_{ab} n^{ab} = -2 \). The fact that higher-order terms in the curvature affect the entropy of black-hole space-times is a consequence of the fact that such terms modify the gravitational surface term in the symplectic structure. As an example, consider a modification to the Einstein-Hilbert Lagrangian consisting in adding the Euler density

\[
\mathcal{L}_\chi = R^2 - 4R_{ab} R^{ab} + R_{abcd} R^{abcd}.
\]

The resulting extra term in the action is a topological invariant of \( \mathcal{M} \), and thus only serves to shift the value of the Euclidean action by number which is locally constant in the space of histories. Nevertheless, it contributes to the gravitational surface term in the symplectic structure and therefore also shifts the value of the entropy by a number depending on the topology of \( \mathcal{M} \). Though usually non-dynamical, for black-hole mergers, this term will in general be dynamical, due to the fact that the topology of \( \mathcal{M} \) changes during the merging process [40, 41].

### 2.3 Global equilibrium: limitations

The standard definition of a black hole event horizon is teleological in the sense that we need to know the structure of the entire space-time in order to construct the event horizon. This is a major drawback. The usual way to resolve this is to consider solutions to the field equations that are stationary, as we have done above. These are solutions that admit a time translation Killing field everywhere, not just in a small neighborhood of the black hole region. While this simple idealization is a natural starting point, it seems to be overly restrictive.

Physically, it should be sufficient to impose boundary conditions at the horizon which ensure that only the black hole itself is in equilibrium. This viewpoint is consistent with what is usually done in thermodynamics: for the laws of equilibrium thermodynamics to hold in other situations, one need only assume the system in question is in equilibrium, not the whole universe. An approach to quasi-local black-hole horizons that achieves this is the isolated horizon (IH) framework. More precisely, an isolated horizon models a portion of an event horizon in which the intrinsic geometric structures are ‘time independent’, and in this sense are in ‘equilibrium’, while the geometry outside may be dynamical, even in an arbitrarily small neighborhood of the horizon. In terms of physical processes, an isolated horizon
Figure 1: A Penrose-Carter diagram of gravitational collapse of an object that forms a singularity $\mathcal{I}$ and a horizon $\Delta$, in the presence of external matter fields. The region of the space-time $\mathcal{M}$ being considered contains the quasi-local black-hole boundary $\Delta$ intersecting the partial Cauchy surfaces $M_1$ and $M_2$ that extend to spatial infinity $i^0$. Here, $\Delta$ is in equilibrium with dynamical radiation fields $\mathcal{R}$ in the exterior region $\mathcal{M} \setminus \Delta$; these fields are not just in the exterior but can exist within an arbitrarily small neighbourhood of $\Delta$.

can be characterized as having no flux of matter or gravitational energy through it. In realistic situations of gravitational collapse, such an assumption will be approximately valid only for certain intervals of time, as represented in figure 1. In the following sections, we define and review the isolated horizon framework in detail, and review a selection of its applications.

3 Isolated horizons

3.1 Null hypersurfaces in equilibrium: non-expanding horizons

A null surface $\mathcal{N}$ has a normal $\ell^a$, which, when raised with the space-time metric, is tangent to $\mathcal{N}$. Given $\mathcal{N}$, the null normal is of course not uniquely determined, but rather one has the freedom to rescale by a positive smooth function: $\ell_a \rightarrow \ell'_a = f \ell_a$. The intrinsic metric $q_{ab}$ on $\mathcal{N}$ is the pullback of the space-time metric; because $q_{ab}\ell^a \cong g_{ab}\ell^a \cong \ell^\flat \ell^\flat \cong 0$ (with “$\cong$” denoting equality restricted to $\mathcal{N}$ and “$\flat$” denoting pullback to $\mathcal{N}$) it follows that $q_{ab}$ is degenerate, i.e. $\ell^a$ is the degenerate direction of $q_{ab}$. The signature of $q_{ab}$ is then $(0,+,+)$ which means that the determinant of $q_{ab}$ is zero. As a result, $q_{ab}$ is non-invertable. However, an inverse metric $q^{ab}$ can be defined such that $q^{ab}q_{ac}q_{bd} \cong q_{cd}$; any tensor that satisfies this
identity is said to be an inverse of \( q_{ab} \).

If \( q^{ab} \) is an inverse, then so is \( \tilde{q}^{ab} = q^{ab} + \ell^a X^b \) for any \( X^b \) tangent to \( \mathcal{N} \). To see this, we observe that

\[
\tilde{q}^{ab} q_{am} q_{bn} \equiv (q^{ab} + \ell^a X^b) q_{am} q_{bn}
\]

\[
\equiv q^{ab} q_{am} q_{bn} + \frac{1}{2} \ell^a X^b q_{am} q_{bn} + \frac{1}{2} \ell^b X^a q_{am} q_{bn}
\]

\[
\equiv q_{mn} .
\]

In going from the second to the third line, we used the property that \( \ell^a q^{ab} = 0 \).

Because \( \ell_a \) is hypersurface orthogonal, it satisfies \( \ell_a [\nabla_b \ell_c] = 0 \). From this alone, one can furthermore deduces that \( \ell^a \) is geodesic, so that \( \ell^a \) generates a geodesic null congruence on \( \mathcal{N} \). The twist, expansion, and shear of this null congruence are given respectively by the anti-symmetric, trace, and symmetric parts of \( \nabla_a \ell_b \). Because \( \ell_a \) is surface-forming, it is twist free. This leaves the expansion and shear,

\[
\theta(\ell) = q^{ab} \nabla_a \ell_b \quad \text{and} \quad \sigma_{ab} = \frac{1}{2} q_{ac} q^{cd} \nabla_c \ell_d .
\]

These both vanish iff \( \nabla_a \ell_b \) vanishes.

An important property of the expansion and shear is that both are independent of the choice of inverse metric \( q^{ab} \). This follows from the following:

\[
(q^{ab} - q^{ab}) \nabla_a \ell_b \equiv \ell^a X^b \nabla_a \ell_b
\]

\[
\equiv \frac{1}{2} [\ell^a (\nabla_a \ell_b) X^b + (\ell^b \nabla_a \ell_b) X^a]
\]

\[
\equiv \frac{1}{2} \ell^a \ell_b \nabla_a X^b + \frac{1}{4} X^a \nabla_a (\ell_b \ell^b) = 0 ;
\]

the first term vanishes because \( \ell_a X^a = 0 \) and the second term vanishes because \( \ell_a \ell^a = 0 \).

A Killing horizon for stationary space-times is a null hypersurface, and so its null normal generates a twist-free geodesic null congruence on the horizon. This null congruence is furthermore expansion-free. The condition that a general null hypersurface \( \mathcal{N} \) be expansion free is in fact independent of the null normal used to define the expansion: If \( \ell' = f \ell \) are two null normals related by some positive, smooth function \( f \), we have

\[
\theta(\ell') = q^{ab} \nabla_a f \ell_b \equiv q^{ab} f \ell_b \nabla_a f + f \theta(\ell) \equiv f \theta(\ell) .
\]

So that \( \theta(\ell') = 0 \) iff \( \theta(\ell) = 0 \). The vanishing of the expansion is therefore an intrinsic property of a null surface \( \mathcal{N} \). This enables us to incorporate this local property of a Killing horizon into the following definition.

**Definition 1. (Non-Expanding Horizon).** A three-dimensional null hypersurface \( \Delta \subset \mathcal{M} \) of a space-time \( (\mathcal{M}, g_{ab}) \) is said to be a non-expanding horizon (NEH) if the following conditions hold: (i) \( \Delta \) is topologically \( R \times S \) with \( S \) a compact two-dimensional manifold; (ii) the expansion \( \theta(\ell) \) of any null normal \( \ell \) to \( \Delta \) vanishes;
(iii) the field equations hold at $\Delta$; and (iv) the stress-energy tensor $T_{ab}$ of external matter fields is such that, at $\Delta$, $-T^a_b\ell^b$ is a future-directed and causal vector for any future-directed null normal $\ell$.

For now we leave the horizon cross section $S$ arbitrary. In Section 3.4 we will prove that, once the definition is strengthened a bit more, for vanishing cosmological constant, $S$ has to be a two-sphere, thereby generalizing the Hawking Topology Theorem to non-stationary space-times.

The weakest notion of equilibrium for a NEH is the requirement that $£_\ell q_{ab} \equiv 0$. This means that the intrinsic geometry of $\Delta$ is invariant under time translations. The condition is equivalent to

$$£_\ell q_{ab} \equiv £_\ell g_{ab} \equiv 2\nabla_{(a}(\ell_{b)} \equiv 0. \quad (10)$$

If $£_\ell q_{ab} \equiv 0$ for one null normal $\ell$ then it is true for any other null normal $\ell' = f\ell$:

$$£_{\ell'} q_{ab} \equiv 2\nabla_{(a}(\ell'_{b)} \equiv 2\nabla_{(a}(f\ell_{b)} + 2f\nabla_{(a}\ell_{b)} \equiv 2\nabla_{(a}\ell_{b)} \equiv f£_\ell q_{ab}. \quad (11)$$

It follows that $£_{\ell'} q_{ab} = 0$ if $£_\ell q_{ab} = 0$.

### 3.2 Intrinsic geometry of non-expanding horizons

Let us now discuss the restrictions on the Riemann curvature tensor for space-times in the presence of a NEH $\Delta$. The Riemann tensor is defined by the condition $2\nabla_{[a}\nabla_{b]}X^c = -R_{abcd}X^d$; the tensor $R_{abcd}$ decomposes into a trace part determined by the Ricci tensor $R_{ab} = R_{abc}^c$ and a trace-free part $C_{abcd}$ such that:

$$R_{abcd} = C_{abcd} + \frac{2}{D-2} \left(g_a[cR_{d]b} - g_{b[cR_{d]a}}\right) - \frac{2}{(D-1)(D-2)}Rg_{a[cg_{d]b}}. \quad (11)$$

The tensor $C_{abcd}$ is called the Weyl tensor. The Ricci tensor is determined by the matter fields through the Einstein-Maxwell equations. The remaining trace-free part of $R_{abcd}$ therefore corresponds to the gravitational degrees of freedom.

Next, let us introduce a null basis adapted to $\Delta$. This can be done by partially gauge-fixing the tetrad so that $(e^a_0 + e^a_1)/\sqrt{2}$ is a null normal to $\Delta$; we then define

$$\ell^a := \frac{1}{\sqrt{2}}(e^a_0 + e^a_1) \quad n^a := \frac{1}{\sqrt{2}}(e^a_0 - e^a_1)$$

$$m^a := \frac{1}{\sqrt{2}}(e^a_2 + ie^a_3) \quad \overline{m}^a := \frac{1}{\sqrt{2}}(e^a_2 - ie^a_3). \quad (12)$$

These are all null, and satisfy the usual normalizations $\ell^a n_a = 1, m^a \overline{m}_a = 1$ for a complex Newman-Penrose tetrad. In terms of this basis, the Riemann tensor can
be decomposed into 15 scalar quantities:

\[ \Psi_0 = C_{abcd}a^b n^c n^d \quad \Phi_{00} = \frac{1}{2} R_{ab} a^a n^b \quad \Phi_{12} = \frac{1}{2} R_{ab} a^a \bar{n}^b \]

\[ \Psi_1 = C_{abcd}m^b n^c n^d \quad \Phi_{01} = \frac{1}{2} R_{ab} a^a m^b \quad \Phi_{20} = \frac{1}{2} R_{ab} m^a \bar{n}^b \]

\[ \Psi_2 = C_{abcd}m^b \bar{m}^c n^d \quad \Phi_{02} = \frac{1}{2} R_{ab} a^a m^b \quad \Phi_{21} = \frac{1}{2} R_{ab} m^a \bar{n}^b \]

\[ \Psi_3 = C_{abcd}a^b \bar{m}^c n^d \quad \Phi_{10} = \frac{1}{2} R_{ab} a^a \bar{m}^b \quad \Phi_{22} = \frac{1}{2} R_{ab} a^a n^b \]

\[ \Psi_4 = C_{abcd}a^b \bar{m}^c \bar{m}^d \quad \Phi_{11} = \frac{1}{3} R_{ab} (a^a n^b + m^a \bar{n}^b) \quad \Lambda = \frac{R}{24}. \]

The four real scalars \( (\Phi_{00}, \Phi_{11}, \Phi_{22}, \lambda) \) and three complex scalars \( (\Psi_{10}, \Phi_{20}, \Phi_{21}) \) correspond to the Ricci tensor, and the five complex scalars \( (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4) \) correspond to the Weyl tensor. Of these, \( \Phi_{00}, \Phi_{01}, \Phi_{10}, \Phi_{20}, \Phi_0 \) and \( \Phi_1 \) are all identically zero at \( \Delta \) by the boundary conditions and the Raychaudhuri equation; the last two of these conditions imply that the Weyl tensor is algebraically special at the horizon. The remaining components of \( R_{abcd} \) are unconstrained at \( \Delta \). Furthermore, one can show that \( \Psi_2 \) is independent of the null tetrad provided that \( \ell \) is a null normal to \( \Delta \).

The above properties of a NEH furthermore ensure that the space-time derivative operator \( \nabla_a \) induces a natural intrinsic derivative operator on \( \Delta \). On a space-like or time-like hypersurface, an intrinsic derivative operator is always induced by \( \nabla_a \), but on a null hypersurface this is not always the case. Recall that a covariant derivative operator \( \mathcal{D} \) intrinsic to a manifold \( \Sigma \) can be specified by giving a map from each pair of vector fields \( X, Y \) on \( \Sigma \) to a vector field \( \mathcal{D}X \approx X^b \mathcal{D}Y^a \), also on \( \Sigma \), satisfying the axioms

1. \( \mathcal{D}_Z(X + Y) = D_Z X + D_Z Y \)
2. \( \mathcal{D}_Z(f X) = f D_Z X + X \ell_Z f \)

The action of \( \mathcal{D} \) on tensors of other types is then determined by linearity \( \mathcal{D}_X(T^A + S^A) = D_X T^A + \mathcal{D}_X S^A \) and the Leibnitz rule \( \mathcal{D}_X(T^{AC} S_{BC}) = (\mathcal{D}_X T^{AC}) S_{BC} + T^{AC}(\mathcal{D}_X S_{BC}) \), where \( A, B, C \) each denote any combination of tensor indices.

Given a space-like or time-like hypersurface \( \Sigma \), the derivative operator \( \mathcal{D} \) induced by \( \nabla_a \) is usually defined by

\[ (\mathcal{D}_X Y)^a = h^a_b (\nabla_X Y)^b \]

where \( h^b_a = \delta^b_a + sn^b n_a \) is the projector onto the tangent space of \( \Sigma \), with \( n^a \) a unit normal to \( \Sigma \) satisfying \( n^a n_a \equiv s = \pm 1 \). On a null hypersurface such as \( \Delta \), however, no such canonical projector exists, and so the above standard prescription fails. In order for \( \nabla_a \) to induce a natural derivative operator on \( \Delta \), one therefore needs the property that, if \( X \) and \( Y \) are vector fields tangent to \( \Delta \), then \( \nabla_X Y \) is already also tangent to \( \Delta \), so that no projector is needed. For a NEH, remarkably, this is in fact
the case: Given $X$ and $Y$ tangent to $\Delta$, we have

$$Y^b \nabla_b (X^a \ell_a) \equiv Y^b X^a \nabla_b \ell_a + Y^b \ell_a \nabla_b X^a$$

$$\equiv Y^b X^a \nabla_b \ell_a + \ell_a Y^b \nabla_b X^a .$$

On $\Delta$, however, $X^a \ell_a = 0$. In addition, $\nabla_a \ell_b \equiv 0$ and $\ell$ is twist-free so that $\nabla_a \ell_b \equiv 0$. Thus $\ell_a Y^b \nabla_b X^a \equiv 0$, so that $Y^b \nabla_b X^a$ is tangent to $\Delta$, as required. It follows that $\nabla_a$ induces a well-defined derivative operator on $\Delta$, which we denote by $D_a$. Because $\nabla_a$ is metric ($\nabla_a g_{bc} = 0$) and torsion-free, one deduces that $D_a$ is also metric ($D_a g_{bc} = 0$) and torsion-free. However, because $q_{ab}$ is degenerate, these two conditions do not uniquely determine $D_a$. Thus, in contrast to the situation on space-like or time-like hypersurfaces, the derivative operator $D_a$ contains more information than is contained in $q_{ab}$.

On a NEH, one can show that the vanishing of the expansion, shear and twist of $\ell^a$ implies that certain components of $D_a$ are reduced to a single intrinsic one-form $\omega_a$, known as the induced normal connection:

$$\nabla_a \ell_b \equiv D_a \ell_b \equiv \omega_a \ell_b .$$

This quantity is gauge-dependent under rescalings of the null normal on $\Delta$. Under transformations $\ell \rightarrow \ell' = f \ell$ with $f$ some smooth function, we find that

$$D_a \ell'_b \equiv f D_a \ell_b + (D_a f) \ell_b$$

$$\equiv \omega_a \ell'_b + \frac{(D_a f)}{f} \ell'_b$$

$$\equiv (\omega_a + D_a \ln f) \ell'_b .$$

The induced normal connection $\omega'_a$ associated to $\ell'_a$ is thus

$$\omega'_a = \omega_a + D_a \ln f .$$

From (13), an expression for the surface gravity $\kappa(\ell)$ can be isolated. Contracting both sides of equation (13) with $\ell^a$,

$$\ell^a \nabla_a \ell^b = (\ell^a \omega_a) \ell^b .$$

This equation is the geodesic equation for $\ell$ with non-zero acceleration. Therefore the surface gravity of $\Delta$ is

$$\kappa(\ell) = \ell^a \omega_a .$$

This quantity is likewise gauge-dependent under rescalings of $\ell$. For $\ell' = f \ell$, $\kappa(\ell)$ transforms as

$$\kappa(\ell') = f \ell^a (\omega_a + D_a \ln f) = f \kappa(\ell) + \mathcal{L}_\ell f .$$
However, from this, one sees that the curvature $d\omega$ of $\omega$ is \textit{gauge-invariant}. In fact one can find an explicit expression for it. We have the following.

**Proposition 1.** The curvature of the induced normal connection $\omega$ on $\Delta$ is given by

$$d\omega \equiv 2(\text{Im}[\Psi_2]) \bar{\ell} ,$$

with $\Psi_2$ the second Weyl scalar, and $\bar{\ell} \equiv \text{im} \wedge \mathbf{m}$ the area element on $\Delta$.

**Proof.** We follow the proof that is presented in [6]. Recall the definition of curvature, $\mathbf{2}\nabla [a \nabla_b] X^c = - R_{c a b d} X^d$. Applied to the case $X^a = \ell^a$, one has $\mathbf{2}\nabla [a \nabla_b] \ell^c = - R_{c a b d} \ell^d$. Pulling back the $a, b$ indices and using $\nabla_a \ell^b \equiv \omega_a \ell_b$ from (13), one obtains

$$2\ell^c D_{[a} \omega_{b]} \equiv - R_{c a b d}\ell^d .$$

Furthermore, from (11) one has $R_{c a b d}\ell^d \equiv C_{c a b d}\ell^d$ because $R_{c a b d} \ell^d \equiv 0$. Combining this with (19) and contracting with $n_c$ gives

$$2\ell^c D_{[a} \omega_{b]} \equiv C_{c a b d}n^d .$$

The Weyl tensor can be expanded in terms of the complex scalars $\{\Psi\}$ as

$$C_{a b c d}n^d \equiv 4(\text{Re}[\Psi_2]) n_{[a} \ell_{b]} + 2\Psi_3 \ell_{[a} m_{b]} + 2\bar{\Psi}_3 \ell_{[a} \bar{m}_{b]} - 2\Psi_1 n_{[a} \bar{m}_{b]} - 2\bar{\Psi}_1 n_{[a} m_{b]} + 4\text{Im}[\Psi_2] m_{[a} \bar{m}_{b]} .$$

Pulling back the $a, b$ indices and substituting in the curvature, the expression (18) follows.

### 3.3 Isolated horizons

It is clear from the transformation law (17) for surface gravity that, on a given NEH, the zeroth law of black hole mechanics cannot hold for all possible null normals. We now ask the question: for which, if any, null normals does it hold? The Cartan identity reads

$$\mathcal{L}_\ell \omega_b \equiv 2 \ell^a D_{[a} \omega_{b]} + D_b (\ell^a \omega_a) .$$

Combining this with (18), we have

$$0 \equiv 4\ell^a \text{Im}[\Psi_2] m_{[a} \bar{m}_{b]} \equiv \mathcal{L}_\ell \omega_b - D_b (\ell^a \omega_a) ,$$

so that $D_b n_{(\ell)} \equiv \mathcal{L}_\ell \omega_b$. It follows that the surface gravity is constant over the entire NEH iff $\mathcal{L}_\ell \omega_b \equiv 0$, i.e., iff $\omega_a$ is in ‘equilibrium’ with respect to $\ell$.

The condition $\mathcal{L}_\ell \omega_a \equiv 0$ can also be interpreted in terms of ‘extrinsic curvature’. Strictly speaking, because $\Delta$ is null, it does not have an extrinsic curvature in the usual sense. Nevertheless, one can define an analogue of extrinsic curvature by using the same formula that is used for space-like surfaces involving the Levi-Civita derivative operator and normal to the surface:

$$K_{a}^b \equiv \nabla_a \ell^b \equiv D_a \ell^b .$$

13
Note that, because $\Delta$ is a null surface, this analogue of extrinsic curvature has the curious property that it is fully determined by the intrinsic structures $D_a$ and $\ell_a$, so that the nomenclature “extrinsic geometry” is only appropriate by analogy, and should not be taken too literally. From (22), one sees that, on a NEH, fixing the extrinsic geometry of $\Delta$ to be time independent is equivalent to fixing the induced normal connection $\omega_a$ of $\Delta$ to be time independent, which in turn, as we saw above, is the necessary and sufficient condition for the (gravitational) zeroth law to hold.

Note furthermore that if this condition holds for a single null normal $\ell$, then it will hold for all other null normals $\ell' = c\ell$ related by a constant rescaling. That is, if we wish the zeroth law to hold, it is sufficient to restrict to an equivalence class of null normals, where two normals are equivalent if they are related by a constant rescaling. Note the similarity to Killing horizons, where one has the freedom to rescale the Killing field $\xi^a$ only by a constant.

In Einstein-Maxwell theory, one can establish a similar zeroth law for the electric potential $\Phi(\ell) := \ell^a A_a$ as follows. First, the energy condition imposed in Definition 1 implies that the Maxwell field satisfies $\ell \cdot F \equiv 0$.¹ Using this, together with the Cartan identity and the Bianchi identity ($d F = 0$), this implies $L_\ell F = d(\ell \cdot F) + \ell \cdot d F \equiv 0$. It follows that the electromagnetic potential $A$ can be partially gauge-fixed such that $L_\ell A \equiv 0$. This is referred to as a gauge adapted to the horizon. When this is satisfied, it follows that $0 = L_\ell A = d(\ell \cdot A) + \ell \cdot d A = d\Phi(\ell)$, where we used the condition $\ell \cdot F \equiv 0$. It follows that the electric potential is constant over the entire NEH, so that the zeroth law also holds for the electric potential $\Phi(\ell)$.

The above observations lead to the following definition.

**Definition 2. (Weakly Isolated Horizon).** A NEH $\Delta$ equipped with an equivalence class $[\ell]$ of future-directed null normals, with $\ell' \sim \ell$ if $\ell' = c\ell$ ($c > 0$ a constant), such that the $L_\ell \omega_a \equiv 0$ and $L_\ell A = 0$ for all $\ell \in [\ell]$, is said to be a weakly isolated horizon (WIH).

Note that because $c$ is now a constant, $\omega_a$ is uniquely determined by the equivalence class $[\ell]$. Under the re-scaling $\ell' = c\ell$, the surface gravity $\kappa(\ell)$ transforms as $\kappa_{\ell'} = c\kappa(\ell)$. However, the condition $\kappa(\ell) = 0$ is independent of the choice of $\ell \in [\ell]$. If $\kappa(\ell) = 0$, we say the WIH is extremal, in analogy with the nomenclature for Killing horizons. When discussing a WIH, it is convenient at this point to strengthen the partial gauge-fixing of the tetrad so that $\ell^a \in [\ell]$, and additionally $dn = 0$. Via the Cartan identity, the latter condition implies $L_\ell n_a = 0$.

Let us now ask: Given a NEH, under what conditions does there exist an equivalence class $[\ell]$ of null normals such that it becomes a WIH? The answer to this question turns out to be always [8]. Thus, as far as geometry is concerned, the WIH definition is not more restrictive than the NEH definition. Rather, the importance of the WIH definition, as compared to the NEH definition, lies in the selection of

¹Here and throughout this chapter $\cdot \cdot$ denotes contraction of a vector with the first index of a form.
an equivalence class \([\ell]\).

There is also a stronger notion of isolated horizon that one can introduce. On a WIH, while the normal connection \(\omega_a\) on a WIH is time independent, the other components of the connection \(D_a\) can in general vary. Thus, a stronger condition would be to impose that the entire connection \(D_a\) on \(\Delta\) possess \(\ell^a\) as a symmetry. This condition is equivalent to \([\mathcal{L}_\ell, D_a] = 0\). By contrast, the condition \(\mathcal{L}_\ell \omega_a = 0\) in Definition 2 is equivalent to the weaker condition \([\mathcal{L}_\ell, D_a] \ell^b = 0\). We have the following.

**Definition 3. (Strongly Isolated Horizon).** A NEH \(\Delta\) equipped with an equivalence class \([\ell]\) of future-directed null normals such that \([\mathcal{L}_\ell, D_a] X^b \equiv 0\) for every \(X^b\) tangent to \(\Delta\) is said to be a strongly isolated horizon (SIH).

In particular, every Killing horizon is also a SIH because all of the geometry at \(\Delta\) (including the connection) possesses \(\ell^a\) as a symmetry.

Given a NEH, the above stronger condition cannot always be met by simply choosing the equivalence class \([\ell]\) appropriately – to be a SIH involves a genuinely stronger restriction on the geometry of the horizon. When an equivalence class \([\ell]\) does exist making an NEH into a SIH, it is furthermore unique \([8]\). The sole remaining ambiguity in the choice of \(\ell^a\) is then to rescale it by a constant, similar to the ambiguity to rescale by a constant the Killing vector field in a stationary space-time. In the case of a stationary space-time, this rescaling freedom can be eliminated by requiring the Killing vector field to be unit at spatial infinity. In the case of a SIH in Einstein-Maxwell theory, as we shall see in the next section, it is also possible to remove the remaining constant rescaling ambiguity in \(\ell^a\). As we shall see in the next section, for a SIH in the context of Einstein-Maxwell theory, one can similarly remove the remaining constant rescaling ambiguity in \(\ell^a\), but this time in a way that is purely quasi-local. For the moment, we leave the freedom intact.

For most of the rest of this chapter, the results we consider will require only weakly isolated horizons. For this reason, hence forth, the term ‘isolated horizon’(IH), when not otherwise qualified, shall refer specifically to a weakly isolated horizon.

### 3.4 The topology of strongly isolated horizons

Let us make a couple of further definitions. If the expansion of \(n^a\) is negative on \(\Delta\), \(\theta_n := q^{ab}\nabla_a n_b < 0\), let us call \(\Delta\) a future isolated horizon. In this case \(\Delta\) describes a black hole horizon and not a white hole horizon. Furthermore, recall that, although \(\kappa(\ell)\) in general depends on the choice of \(\ell\) in the equivalence class \([\ell]\), the sign of \(\kappa(\ell)\) does not. We have already defined \(\Delta\) to be extremal if \(\kappa(\ell) = 0\). Furthermore, for future isolated horizons, if \(\kappa(\ell) > 0\), we call \(\Delta\) sub-extremal, and if \(\kappa(\ell) < 0\) we call \(\Delta\) super-extremal. We then have the following result:

**Proposition 2.** Suppose \(\Delta\) is a future SIH in a space-time with zero cosmological constant. If \(\Delta\) is sub-extremal, then its cross-sections have 2-sphere topology. If
\( \Delta \) is extremal, then its cross-sections can have either 2-sphere or 2-torus topology, with the 2-torus topology occurring iff both \( \tilde{\omega} = 0 \) and \( T_{ab}\ell^an^b \equiv 0 \).

**Proof.**

Consider the evolution equation for the expansion of the auxiliary null normal \( n^a \) (see [42], or equivalently (3.7) in [8]):

\[
\mathcal{L}_\ell \theta_{(n)} + \kappa \theta_{(n)} + \frac{1}{2} \mathcal{R} = \mathcal{D}_a \tilde{\omega}^a + \|\tilde{\omega}\|^2 + (\Lambda - 8\pi G T_{ab}\ell^an^b). \tag{23}
\]

Here, \( \mathcal{R} \) is the scalar curvature of \( \mathcal{S} \), \( \mathcal{D}_a \) is the covariant derivative operator that is compatible with the metric \( \tilde{g}^{ab} = g^{ab} + \ell_an_b + \ell_bn_a \) on \( \mathcal{S} \), and \( \|\tilde{\omega}\|^2 = \tilde{\omega}_a \tilde{\omega}^a \) with \( \tilde{\omega}_a = \tilde{g}_a^b \omega_b = \omega_a + \kappa(\ell)n_a \) the projection of \( \omega \) onto \( \mathcal{S} \). From strong isolation, and \( \mathcal{L}_\ell n = 0 \), one has \( \mathcal{L}_\ell \theta_{(n)} = \mathcal{L}_\ell q^{ab}D_an_b = q^{ab}D_a \mathcal{L}_\ell n_b = 0 \). Using this fact with \( \Lambda = 0 \), and \( \theta_{(n)} < 0 \), (23) becomes

\[
-\kappa|\theta_{(n)}| = -\frac{1}{2} \mathcal{R} + \mathcal{D}_a \tilde{\omega}^a + \|\tilde{\omega}\|^2 + T_{ab}\ell^an^b. \tag{24}
\]

Integrating both sides over the surface \( \mathcal{S} \), and finally using that \( \oint_\mathcal{S} \bar{\epsilon} \mathcal{D}_a \tilde{\omega}^a = 0 \), one has

\[
\oint_\mathcal{S} \bar{\epsilon} \mathcal{R} \geq 2 \oint_\mathcal{S} \bar{\epsilon}(T_{ab}\ell^an^b + \kappa|\theta_{(n)}| + \|\tilde{\omega}\|^2). \tag{25}
\]

The dominant energy condition requires that \( T_{ab}\ell^an^b \geq 0 \). In addition, because we have excluded the super-extremal case, the second term is non-negative. Lastly, \( \|\tilde{\omega}\|^2 \) is manifestly non-negative. It follows that \( \oint_\mathcal{S} \bar{\epsilon} \mathcal{R} \geq 0 \) with equality iff \( \kappa(\ell) = 0 \), \( T_{ab}\ell^an^b = 0 \) and \( \tilde{\omega} = 0 \). On the other hand, the Gauss-Bonnet theorem gives

\[
\oint_\mathcal{S} \bar{\epsilon} \mathcal{R} = 8\pi (1 - g)
\]

where \( g \) is the genus of \( \mathcal{S} \), so that \( \oint_\mathcal{S} \bar{\epsilon} \mathcal{R} > 0 \) iff \( \mathcal{S} \) is a 2-sphere, and \( \oint_\mathcal{S} \bar{\epsilon} \mathcal{R} = 0 \) iff \( \mathcal{S} \) is a 2-torus. Thus, if \( \Delta \) is sub-extremal \( (\kappa(\ell) > 0) \), one has \( \oint_\mathcal{S} \bar{\epsilon} \mathcal{R} > 0 \), so that \( \mathcal{S} \) is a 2-sphere, whereas if \( \Delta \) is extremal, both 2-sphere and 2-torus topologies are possible, with the latter occurring iff \( T_{ab}\ell^an^b = 0 \) and \( \tilde{\omega} = 0 \). □

### 3.5 Existence of Killing spinors

The extremal Kerr-Newman black hole is a solution to the \( N = 2 \) supergravity field equations with the fermion fields set to zero. The condition for this solution to have positive energy is that [43]

\[
\mathfrak{M} = |\Omega|, \tag{26}
\]

relating the mass \( \mathfrak{M} \) and total charge \( \Omega \equiv \sqrt{q_e^2 + q_m^2} \) (with \( q_e \) and \( q_m \) the electric and magnetic charges); this is the extremality condition for the Kerr-Newman black hole. This is also the saturated Bogomol'ny-Prasad-Sommerfeld (BPS) inequality [43].
This leads to an interesting question: If we try to define a supersymmetric *isolated horizon*, do we have a similar restriction to extremality?

Generally, Einstein-Maxwell theory (in four dimensions) can be viewed as the bosonic sector of $N = 2$ supergravity — that is, the sector in which the spin-3/2 gravitino field and its complex conjugate vanish in vacuum. In this sector, the defining property of supersymmetric configurations is the existence of a *Killing spinor*, whence we look for a restricted space of solutions having this property. In the presence of zero cosmological constant, a Killing spinor can be defined as a (Dirac) spinor $\zeta$ satisfying

$$\left[ \nabla_a + \frac{i}{4} F_{bc} \gamma^{bc} \gamma_a \right] \zeta = 0 .$$

(27)

Here, $\gamma^a$ are a set of gamma matrices that satisfy the usual anticommutation rule

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2 g^{ab}$$

(28)

and the antisymmetry product

$$\gamma_{abcd} = \epsilon_{abcd} .$$

(29)

$\gamma_{a_1 \ldots a_D}$ denotes the antisymmetrized product of $D$ gamma matrices. The complex conjugate $\bar{\zeta}$ of $\zeta$ is defined as

$$\bar{\zeta} = i (\zeta) \dagger \gamma_0 ;$$

(30)

with $\dagger$ denoting Hermitian conjugation.

From $\zeta$ and $\bar{\zeta}$ one can construct five (real) bosonic bilinear covariants

$$f = \bar{\zeta} \zeta , \quad g = i \bar{\zeta} \gamma^5 \zeta , \quad V^a = \bar{\zeta} \gamma^a \zeta , \quad W^a = i \bar{\zeta} \gamma^5 \gamma^a \zeta , \quad \Psi^{ab} = \bar{\zeta} \gamma^{ab} \zeta .$$

(31)

These bilinear covariants are all related to each other via several algebraic conditions (from the Fierz identity) and differential equations (from the Killing spinor equation (27)). In particular, the vector $V$ satisfies the equations

$$V_a V^a = -(f^2 + g^2) ,$$

(32)

$$\nabla_a V_b = - f F_{ab} + \frac{g}{2} \epsilon_{abcd} F^{cd} .$$

(33)

The above reviews the standard notion of a Killing spinor. What we now wish to consider is a Killing spinor which exists on the horizon $\Delta$ only (which we assume to be strongly isolated). Then, in place of (27), we can at most impose a version with the derivative pulled back to $\Delta$:

$$\left[ \nabla_a + \frac{i}{4} F_{bc} \gamma^{bc} \gamma_a \right] \zeta = 0 .$$

(34)

which, in place of (33), leads to

$$\nabla_a V_b = - f F_{ab} + \frac{g}{2} \epsilon_{abcd} F^{cd} .$$
If one additionally pulls back $b$ to $\Delta$ and symmetrizes, one sees that
\[
\mathcal{L}_V q_{ab} = 2\nabla_{(a} V_{b)} = 0,
\]
which is the same as the condition (10) satisfied by $\ell$ on $\Delta$ as a NEH. If we furthermore stipulate that $V$ be equal to a null normal in the equivalence class $[\ell]$ making $\Delta$ a SIH, we have the notion of a supersymmetric isolated horizon.

**Definition 4. (Supersymmetric Isolated Horizon).** A SIH $\Delta$ equipped with a Killing spinor $\zeta$ and complex conjugate $\bar{\zeta} = i(\zeta)\gamma_0$, such that $V := \bar{\zeta}^a \zeta \in [\ell]$, is said to be a supersymmetric isolated horizon (SSIH).

For SSIHs, the following can be proved.

**Proposition 3.** An SSIH of Einstein-Maxwell theory with zero cosmological constant is necessarily extremal and non-rotating.

**Proof.** For a SSIH, $V^a$ and $\ell^a$ are identified. Hence, at $\Delta$, $V^a$ is null so that from (32) we have $f = g = 0$. Equation (13) together with (33) then gives
\[
\nabla_{\ell} \ell^b = \omega_a \ell^b = 0, \quad (35)
\]
so that $\omega \equiv 0$. This condition implies that the gravitational angular momentum is identically zero. This follows from (52) below. However, in general $A$ is non-zero, which means that there may be non-zero angular momentum stored in the electromagnetic fields. The condition also implies that $\kappa(\ell) = \ell \cdot \omega = 0$. Therefore, SSIHs are extremal and non-rotating. □

## 4 Hamiltonian mechanics

Up until now we have been considering geometric properties of isolated horizons. At this point we show how the isolated horizon boundary conditions lead to a consistent variational and canonical framework. Specifically, we start with an action principle, derive from this the covariant phase space, and then finally discuss the definition of angular momentum and mass of isolated horizons as generators of rotations and time translations. This will lead directly to a derivation of the first law of black hole mechanics involving only quantities which are quasi-locally defined in terms of fields intrinsic to the horizon.

For conceptual clarity, in this presentation we focus on Einstein-Maxwell theory. Furthermore, we use the Palatini formulation of gravity in terms of a co-tetrad $e^I_a$ and associated internal Lorentz connection $A^I_a$, where $I, J = 0, 1, 2, 3$ are internal indices. This will greatly simplify the necessary formulae, and allow an easier transition to the brief discussion on quantum theory at the end of the chapter. In this formulation, the space-time metric is constructed from the co-tetrad as $g_{ab} = \eta_{IJ} e^I_a e^J_b$ where $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$ is the internal Lorentz metric, and the curvature $F^{IJ}_{ab}$ of $A^I_a$ determines the Riemann tensor via $F^{IJ}_{ab} = (dA^I_a + A^I_K \wedge A^K_J)_{ab} = R_{abc} e^c e^I_a$. Internal indices are raised and lowered with
Figure 2: The region of space-time $\mathcal{M}$ considered has an internal boundary $\Delta$ which will be the isolated horizon, and is bounded to the past and future by two three-dimensional Cauchy surfaces $M_1$ and $M_2$. $M$ is a partial Cauchy surface that intersects $\Delta$ and $\mathcal{I}$ each in a two-sphere.

$\eta^{IJ}, \eta_{IJ}$. We denote the Maxwell vector potential by $A$, so that the field strength is $F = dA$.

### 4.1 Action principle

Let us consider the action for Einstein-Maxwell theory in the Palatini formulation on a four-dimensional manifold $\mathcal{M}$ with boundary $\partial \mathcal{M} \cong M_1 \cup M_2 \cup \Delta \cup \mathcal{I}$. Here $\mathcal{I}$ represents the two-sphere at spatial infinity crossed with time, where appropriate asymptotically flat boundary conditions are imposed. $\Delta$ is a three-dimensional manifold with topology $S^2 \times [0, 1]$ constrained to be an isolated horizon equipped with a fixed equivalence class of null normals $[\ell_a]$, with all equations of motion holding at $\Delta$. $M_1, M_2$ play the role of partial Cauchy surfaces. As before, the isolated horizon boundary conditions are understood to include the requirement that the Maxwell potential be in a gauge adapted to the horizon, $\mathcal{L}_{\ell_a}A = 0$. The space-time region $\mathcal{M}$ thus described is shown in Figure 2.

The action is then given by

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} \Sigma_{IJ} \wedge F^{IJ} - \frac{1}{16\pi G} \int_{\mathcal{I}} \Sigma_{IJ} \wedge A^{IJ} - \frac{1}{8\pi} \int_{\partial \mathcal{M}} F \wedge \ast F. \tag{36}$$

where $F = dA$ is the field strength of the Maxwell field and $\Sigma^{IJ} := \frac{1}{2} \epsilon^{IJ}_{KL} e^K \wedge e^L$ with $\epsilon_{IJKL}$ the internal alternating tensor. The boundary term at $\mathcal{I}$ is necessary to make the action differentiable. By contrast, as we will show below, no term at $\Delta$ is necessary for the action to be differentiable [44–46].

Let us denote the dynamical variables $(\epsilon, A, A)$ collectively by $\Psi$. Application of an arbitrary variation $\delta$ to the action then yields the form

$$\delta S = \int_{\mathcal{M}} E[\Psi] \cdot \delta \Psi - \int_{\partial \mathcal{M}} \theta(\delta)[\Psi]. \tag{37}$$
Here $E[\Psi] = 0$ denotes the equations of motion. Specifically, these are:

$$
\epsilon_{IJKL} e^J \wedge F^{KL} = T_I ,
$$

(38)

$$
d\Sigma_{IJ} - A_J^K \wedge \Sigma_{IK} - A_J^K \wedge \Sigma_{IK} = 0 ,
$$

(39)

$$
d \star F = 0 .
$$

(40)

where in the first of these equations, $T_I$ denotes the electromagnetic stress-energy three-form. The second equation imposes that $A_{IJ}^a$ be the unique torsion-free connection compatible with $e^a_I$. Together the above equations are equivalent to the Einstein-Maxwell equations in metric variables, with the components of $T_I$ identified with appropriate components the electromagnetic stress-energy tensor.

The integrand $\theta(\delta)$ of the surface term in (37) is given by

$$
\theta(\delta) = \theta_{\text{Grav}}(\delta) + \theta_{\text{EM}}(\theta),
$$

(41)

where we have defined $\theta_{\text{Grav}}(\delta) := (1/16\pi G) \Sigma_{IJ} \wedge \delta A_{IJ}^a$ and $\theta_{\text{EM}}(\delta) := (1/16\pi G) \star F \wedge \delta A$, and where we suppress explicit representation of the dependence on $\Psi = (e, A, A)$.

The action $S$ is said to be differentiable if, when the configuration fields $e, A, A$ are fixed on $M_1$ and $M_2$, the boundary term in (37) vanishes. Let us show that this is the case. Because $A$ and $A$ are held fixed at $M_1, M_2$, $\theta(\delta)$ vanishes there. In addition, the boundary term in the action (36) is constructed precisely such that the boundary terms at $I$ coming from the variation of the action cancel. Therefore it remains only to show that the integral of $\theta(\delta)$ over $\Delta$ vanishes.

To see that this is true, let us define an internal Newman-Penrose basis $\ell^I = (1,1,0,0)/\sqrt{2}$, $n^I = (1,-1,0,0)/\sqrt{2}$, $m^I = (0,0,1,i)/\sqrt{2}$, $\overline{m} = (0,0,1,-i)/\sqrt{2}$. The relations (12) then become

$$
\ell^a = e^a_I \ell_I , \quad n^a = e^a_I n_I , \quad m^a = e^a_I m_I , \quad \overline{m}^a = e^a_I \overline{m}_I .
$$

From the definition of $\omega$ (13), one then deduces the equation

$$
A_{\ell IJ} \ell^I \equiv \omega_\ell \ell_I .
$$

By expanding the pull-backs to $\Delta$ of $A_{IJ}^a$ and $\Sigma_{IJ}$ in terms of the Newman-Penrose basis, using the above identity, and simplifying, one obtains

$$
\Sigma_{IJ} \wedge \delta A_{IJ}^a \equiv 2\hat{\epsilon} \wedge \delta \omega ,
$$

with $\hat{\epsilon} = m \wedge \overline{m}$ the area two-form on $\Delta$. One thus has

$$
\theta_{\text{Grav}}(\delta) \equiv \frac{1}{16\pi G} \hat{\epsilon} \wedge \delta \omega .
$$

(42)

Throughout the space of histories considered, $\mathcal{L}_\epsilon \omega = 0$, so that $\mathcal{L}_\epsilon \delta \omega = 0$. This, combined with the fact that $A_{\ell IJ}^a$, $e^I_a$, and hence $\omega$, is fixed on $S_1 := M_1 \cap \Delta$ and
\[ S_2 := M_2 \cap \Delta, \] implies \(\delta \omega = 0\) on all of \(\Delta\), so that \(\overline{\theta}_{\text{Grav}}(\delta) = 0\). The argument for the electromagnetic part \(\overline{\theta}_{\text{EM}}(\delta)\) is similar: Because \(A\) is in a gauge adapted to the horizon, so that \(L_\ell A = 0\) throughout the space of histories, we have \(L_\ell \delta A = 0\). This combined with the fact that \(A\) is fixed on \(S_1\) and \(S_2\) implies \(\delta A = 0\), implying \(\overline{\theta}_{\text{EM}}(\delta) = 0\). This shows that, when the configuration variables \((e, A, A)\) are fixed on \(M_1\) and \(M_2\), the boundary term in the variation of the action (37) vanishes, so that \(\delta S = 0\) implies the equations of motion, as required.

4.2 Covariant phase space

The covariant phase space of a theory is the space of solutions of the equations of motion. Because it is not formulated in terms of initial data, such a formulation of phase space enables a space-time covariant version of the canonical framework. Furthermore, in this approach, one can derive the symplectic structure directly through (anti-symmetrized) second variations of the action, as we shall review. The notion of the covariant phase space as the space of solutions, and the corresponding method of determining the symplectic structure can be traced back to the work of Lagrange himself (see [47, 48]).

The covariant phase space \(\Gamma_{\text{Cov}}\) and its symplectic structure \(\Omega_{\text{Cov}}\) are directly related to the more standard canonical phase space \(\Gamma\) and its symplectic structure \(\Omega\), through the fact that the space of initial data is in one-to-one correspondence with the space of solutions. This isomorphism furthermore maps the covariant symplectic structure into the canonical symplectic structure, so that the two frameworks are completely equivalent.

In the present context, the situation is more subtle, for two reasons: First, general relativity is a constrained theory, and usually one works with the phase space of unconstrained initial data. By contrast, in the covariant phase space, all equations of motion are by definition satisfied. Second, the space-times under consideration are not globally hyperbolic, but rather only admit partial Cauchy surfaces, with inner boundary at the isolated horizon \(\Delta\). Consequently, solutions in the covariant phase space have more information than is present in the initial data on any given spatial hypersurface — solutions in the covariant phase space “know” what fell into the black hole in the past, whereas the initial data does not. Nevertheless, the two frameworks can be related. If we let \(\overline{\Gamma}\) denote the space of constrained initial data, one has a projector \(\pi_M : \Gamma_{\text{Cov}} \to \overline{\Gamma}\) mapping a given solution to the initial data which it induces on a given fixed hypersurface \(M\). Using this projector, and the inclusion map \(\iota : \overline{\Gamma} \hookrightarrow \Gamma\), one can show that

\[
(\iota \circ \pi_M)^*\Omega = \Omega_{\text{Cov}}
\]

so that the symplectic structure on the usual unconstrained phase space of initial data is exactly mapped into that on the covariant phase space computed using the second variations of the action.
In the following we will use the covariant phase space framework. This will not only make the space-time geometry more transparent, but will also simplify many calculations. More specifically, let \( \Gamma_{\text{IH}} \) denote the covariant phase space of possible fields \((e, A, A)\) on \(M\) (1.) satisfying the Einstein-Maxwell equations, (2.) possessing appropriate asymptotically flat fall off conditions at infinity, and (3.) such that \(\Delta\) is an isolated horizon with fixed equivalence class of null normals \([\ell]\).

Symplectic structure on \(\Gamma_{\text{IH}}\)

The boundary term \(\theta(\delta)\) in the variation of the action (37) also determines the symplectic structure. Though \(\theta(\delta)\) is a 3-form on space-time, as it is linear function of a single variation \(\delta\), it is also a 1-form on phase space. One can therefore take its exterior derivative to obtain a 2-form, called the symplectic current, \(\omega \coloneqq d\theta\), where \(d\) denotes the exterior derivative on (the infinite dimensional space) \(\Gamma_{\text{IH}}\). Explicitly, one obtains

\[
\omega(\delta_1, \delta_2) = \frac{1}{16\pi G} \left[ \delta_1 A^{IJ} \wedge \delta_2 \Sigma_{IJ} - \delta_2 A^{IJ} \wedge \delta_1 \Sigma_{IJ} \right] + \frac{1}{4\pi} \left[ \delta_1 A \wedge \delta_2 (\star F) - \delta_2 A \wedge \delta_1 (\star F) \right].
\]

(43)

The symplectic current has the property that for any two variations \(\delta_1, \delta_2\) tangent to the space of solutions, \(\omega(\delta_1, \delta_2)\) is closed: \(d\omega(\delta_1, \delta_2) = 0\). Normally one would then define the symplectic structure for a given Cauchy hypersurface \(M\) to be

\[
\Omega_B^M(\delta_1, \delta_2) := \int_M \omega(\delta_1, \delta_2).
\]

However, in the present case, because of the presence of the inner boundary \(\Delta\), this symplectic structure is not conserved in time. That is: for two different partial Cauchy surfaces \(M_1, M_2\), \(\Omega_B^M_{M_1} \neq \Omega_B^M_{M_2}\). The reason for this can be seen in figure 3: symplectic current is escaping across the horizon \(\Delta\). More precisely, because the symplectic current is closed (\(d\omega(\delta_1, \delta_2) = 0\)), Stokes theorem implies \(\oint_{\partial M} \omega(\delta_1, \delta_2) = 0\), so that

\[
\int_{M_2} \omega(\delta_1, \delta_2) = \int_{M_1} \omega(\delta_1, \delta_2) - \int_{\Delta} \omega(\delta_1, \delta_2).
\]

(The part of the boundary integral at spatial infinity \(I\) vanishes due to the imposition of fall-off conditions.) The solution to this problem is to use the IH boundary conditions to rewrite the integral over \(\Delta\) as an integral over the two-sphere intersections \(S_1\) and \(S_2\) of \(\Delta\) with \(M_1\) and \(M_2\):

\[
\int_{\Delta} \omega(\delta_1, \delta_2) = \left( \oint_{S_2} - \oint_{S_1} \right) \lambda(\delta_1, \delta_2)
\]

(44)

for some two-form \(\lambda(\delta_1, \delta_2)\) on \(\Delta\). If one then defines the full symplectic structure on a spatial slice \(M\) to be

\[
\Omega_M(\delta_2, \delta_1) = \int_M \omega(\delta_1, \delta_2) + \oint_S \lambda(\delta_1, \delta_2),
\]

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Figure 3: Symplectic current escaping across the horizon.

with $S = M \cap \Delta$, then conservation of the symplectic structure is restored, giving

$$\Omega_{M_1}(\delta_2, \delta_2) = \Omega_{M_2}(\delta_1, \delta_2).$$

To carry this out for the present case, one defines potentials $\psi$ and $\chi$ for the surface gravity $\kappa(\ell)$ and electric potential $\Phi(\ell)$ such that

$$\mathcal{L}_\ell \psi \equiv \mathcal{L}_\ell \omega = \kappa(\ell) \quad \text{and} \quad \mathcal{L}_\ell \chi \equiv \mathcal{L}_\ell A = -\Phi(\ell). \quad (45)$$

The pullback to $\Delta$ of the symplectic current will then be exact [6], so that, using Stokes theorem, the integral over $\Delta$ decomposes as in (44). The final symplectic structure turns out to be

$$\Omega_{\text{Cov}}(\delta_1, \delta_2) = \frac{1}{16\pi G} \int_M \left[ \delta_1 A^{IJ} \wedge \delta_2 \Sigma_{IJ} - \delta_2 A^{IJ} \wedge \delta_1 \Sigma_{IJ} \right]$$

$$+ \frac{1}{4\pi} \int_M \left[ \delta_1 A \wedge \delta_2 (\ast F) - \delta_2 A \wedge \delta_1 (\ast F) \right]$$

$$- \frac{1}{8\pi G} \oint_S \left[ \delta_1 \psi \delta_2 \tilde{\epsilon} - \delta_2 \psi \delta_1 \tilde{\epsilon} \right] - \frac{1}{4\pi} \oint_S \left[ \delta_1 \chi \delta_2 (\ast F) - \delta_2 \chi \delta_1 (\ast F) \right]. \quad (46)$$

### Symmetry classes and the phase space of rigidly rotating horizons

For the purposes of black hole mechanics and the quantum theory of black holes, it is useful to categorize different black hole geometries according to their symmetry groups. A symmetry of an isolated horizon is an infinitesimal diffeomorphism on $\Delta$ which preserves $q_{ab}$ and $D$, and at most rescales $\ell$ by a positive constant. (We restrict consideration to infinitesimal symmetries because the symmetry vector fields we consider will not always be complete.) By definition of an isolated horizon, diffeomorphisms generated by $\ell$ are symmetries, so that the symmetry group $G_\Delta$ of $\Delta$ is at least 1-dimensional. But beyond this, the group of symmetries of an isolated horizon, unlike the symmetries of the asymptotic-flat metric at spatial infinity, is not universal. The symmetry group can be classified according to its dimension into three categories:
1. Type I: \((q, D)\) is spherically symmetric, and \(G\Delta\) is four-dimensional.

2. Type II: \((q, D)\) is axisymmetric, and \(G\Delta\) is two-dimensional.

3. Type III: \((q, D)\) has no symmetry other than \(\ell\), and \(G\Delta\) is one-dimensional.

Note that these symmetries refer only to geometric structures intrinsic to the horizon. No assumption is made about symmetries of fields outside the horizon, even in an arbitrarily small neighborhood of \(\Delta\).

Type II horizons are the most interesting ones, because they allow rotation and distortion, but still have enough structure to admit a clear notion of (quasi-local) angular momentum, as we shall see. They include the Kerr-Newman family of horizons as well as generalizations possessing distortion due to external matter, other black holes, or even ‘hair’ due to matter such as Yang-Mills for which the black hole uniqueness theorem does not apply. In the rest of this section, we will focus on the type II case.

Let us fix an axial vector field \(\phi\) on \(\Delta\) such that it commutes with any one and hence all members of \([\ell]\). Define \(\Gamma^\phi\) to be the set of all data \((e, A, A^a)\) in \(\Gamma\) such that (1.) the induced \(q_{ab}\) and \(D\) on \(\Delta\) possess \(\phi\) as a symmetry: \(\mathcal{L}_\phi q_{ab} = 0\), and \([\mathcal{L}_\phi, D_a] = 0\), and (2.) \(\mathcal{L}_\phi \ast \mathcal{F} = \mathcal{L}_\phi \mathcal{F} = 0\). Endow \(\Gamma^\phi\) with the pull-back, via inclusion, of the symplectic structure on \(\Gamma\) derived above — we denote the resulting symplectic structure by \(\Omega^\phi\). The resulting phase space \((\Gamma^\phi, \Omega^\phi)\) is called the phase space of rigidly rotating horizons. This will form the basis of the discussion for the rest of this section. Furthermore, whenever a partial Cauchy surface \(M\) is used in the following, we assume it is chosen such that its intersection \(S\Delta\) with \(\Delta\) is everywhere tangent to \(\phi\).

### 4.3 Conserved charges and the first law

#### Symmetries and Hamiltonian flow.

In \(\Gamma^\phi\), all space-times possess as isolated horizon symmetries the fixed vector fields \(\phi\) and \(\ell\). A general symmetry at \(\Delta\) thus takes the form

\[
W^a \equiv B_{(W, \ell)} \ell^a + A_{(W)} \phi^a,
\]

where \(B_{(W, \ell)}\) depends on which \(\ell \in [\ell]\) is used. Both \(B_{(W, \ell)}\) and \(A_{(W)}\) are constants on \(\Delta\), but may be ‘q-numbers’ — i.e., may vary on phase space. Suppose we extend \(W^a\) arbitrarily to the rest of the space-time, such that at spatial infinity it approaches a symmetry of the asymptotic flat metric there. One can then ask: Is the flow generated by \(W^a\) Hamiltonian? Because the flow generated by \(W^a\) preserves the boundary conditions at \(\Delta\) and at infinity involved in the definition of \(\Gamma^\phi\), there exists a corresponding well-defined variation \(\delta_W\) on \(\Gamma^\phi\). The flow will be Hamiltonian if and only if there exists a Hamiltonian \(H\) on \(\Gamma^\phi\) such that
\[ \delta H = \Omega^\phi(\delta, \delta W) \text{ for all variations } \delta. \] Using (46), explicitly this becomes

\[
\delta H = \Omega^\phi(\delta, \delta W) = -\frac{1}{8\pi G} \oint_{S_\Delta} A(W) \delta[(\phi \cdot \omega) \hat{\epsilon}] + \kappa(W) \delta(\hat{\epsilon}) 
- \frac{1}{4\pi} \oint_{S_\Delta} A(W) \delta[(\phi \cdot A) \star F] - \Phi(W) \delta(\star F) 
+ \frac{1}{16\pi G} \int_{S_\infty} \text{tr}[\delta A \wedge (W \cdot \Sigma) + (W \cdot A) \delta \Sigma] 
- \frac{1}{4\pi} \int_{S_\Delta} \delta A \wedge (W \cdot F) + (W \cdot A) \delta(\star F)] ,
\]

where \( \kappa(W) = W^a \omega^a \) and \( \Phi(W) := -W^a A_a \) respectively denote the surface gravity and electric potential ‘in the frame defined by \( W \)’.

In the following, we will find such an \( H \) first for \( W \) a pure rotation, and then an appropriate time translation. This will lead to the definition of the horizon angular momentum, horizon mass, and a proof of a version of the first law of black hole mechanics involving only quasi-locally defined quantities. In passing, the definition of angular momentum multipoles and the associated mass multipoles will also be introduced.

**Horizon angular momentum as generator of rotations**

Angular momentum is the generator of rotations. In terms of equation (47), the case of spatial rotations corresponds to \( B_{(W,\ell)} = 0 \) and \( A_W = 1 \), in which case equation (48) reduces to

\[ \Omega^\phi(\delta, \delta W) = -\frac{1}{8\pi G} \oint_{S_\Delta} \delta[(\phi \cdot \omega) \hat{\epsilon}] - \frac{1}{4\pi} \oint_{S_\Delta} \delta[(\phi \cdot A) \star F] - \delta J_{\text{ADM}} , \]

so that the boundary integral at infinity is exact, equal to the variation of the ADM angular momentum. One can see that the terms associated with \( S_\Delta \) also form an exact variation. If we set

\[ J_\Delta := -\frac{1}{8\pi G} \oint_{S_\Delta} (\phi \cdot \omega) \hat{\epsilon} - \frac{1}{4\pi} \oint_{S_\Delta} (\phi \cdot A) \star F , \] (49)

then

\[ \Omega^\phi(\delta, \delta W) = \delta(J_\Delta - J_{\text{ADM}}) ; \] (50)

this means that \( J_\Delta - J_{\text{ADM}} \) is the Hamiltonian generating rotations in \( \Gamma^\phi \). Whereas \( J_{\text{ADM}} \) represents the angular momentum of the entire space-time, \( J_\Delta \) can be interpreted as the angular momentum of the black hole itself, so that \( J_{\text{ADM}} - J_\Delta \) is then the angular momentum outside of the horizon. Note how the extension of \( \phi^a \) in the bulk between \( \Delta \) and infinity did not matter precisely because the expression for the symplectic structure consists only in boundary terms.

One can show that the gravitational horizon angular momentum is equivalent to the Komar integral. To show that this statement is true, note that the normalization
\[\ell^a n_a = -1\] implies that the rotation one-form is given by \[\omega_a = -\ell^b \nabla_a \ell_b = \ell^b \nabla_b n_a.\]

Substituting this in the gravitational angular momentum expression, integrating by parts and using the Killing property of \(\phi\), we have:

\[
J_{\text{Grav}} = \frac{1}{8\pi G} \oint_{S_{\Delta}} (\phi \nabla \ell n) \hat{\epsilon} = \frac{1}{8\pi G} \oint_{S_{\Delta}} (\nabla_{\ell} \phi n) \hat{\epsilon} = -\frac{1}{16\pi G} \oint_{S_{\Delta}} (\ell_\phi \nabla n) \hat{\epsilon} = -\frac{1}{8\pi G} \oint_{S_{\Delta}} \ast d\phi . \tag{51}
\]

This is the Komar integral for the gravitational contribution to the angular momentum of \(\Delta\), evaluated at \(S_{\Delta}\). Note that \(J_{\text{Grav}}\) is equivalent to the Komar integral even in the presence of Maxwell fields.

**Multipoles**

The gravitational angular momentum can also be written as a particular moment of the imaginary part of \(\Psi_2\). This then leads to a way to define higher order angular momentum *multipoles* [11]. Let us see how this comes about. \(\phi\) is a symmetry of the intrinsic geometry of \(\Delta\) and therefore also a symmetry of \(\hat{\epsilon}\). This implies \(\mathcal{L}_{\phi} \hat{\epsilon} \equiv d(\phi \hat{\epsilon}) \equiv 0\) so that \(\phi_\phi \hat{\epsilon} = dg\) for some smooth function \(g\). Fix the freedom to add a constant to \(g\) by imposing \(\oint_{S_{\Delta}} g \hat{\epsilon} = 0\), so that \(g\) is unique. Some manipulation then gives

\[
J_{\text{Grav}} = -\frac{1}{8\pi G} \oint_{S_{\Delta}} (\phi_\phi \omega) \hat{\epsilon} = -\frac{1}{8\pi G} \oint_{S_{\Delta}} (d\omega) g = -\frac{1}{4\pi G} \oint_{S_{\Delta}} g \text{Im}\Psi_2 \hat{\epsilon} , \tag{52}
\]

where (18) has been used in the last line. One can show that \(g\) always has the range \((-R_\Delta^2, R_\Delta^2)\) where \(4\pi R_\Delta^2 := a_\Delta\) is the areal radius of the horizon. In fact, there exists a canonical round metric \(q^{ab}\) determined by the axisymmetric metric \(q_{ab}\), sharing the same area element and Killing field \(\phi\). In terms of the standard spherical coordinates \((\theta, \phi)\) adapted to \(q_{ab}\) (with \(\phi^a = \left(\frac{\partial}{\partial \phi}\right)^a\)), one has \(g = R_\Delta^2 \cos \theta\), and the angular momentum can be written

\[
J_{\text{Grav}} = -\frac{R_\Delta^4}{4\pi G} \oint_{S_{\Delta}} (\cos \theta) \Psi_2 d\Omega , \tag{53}
\]

where \(d\Omega = \sin \theta d\theta \wedge d\phi\). This leads to a more general definition of angular momentum *multipoles*

\[
J_n := -\frac{R_\Delta^{n+3}}{4\pi G} \oint_{S_{\Delta}} P_n(\cos \theta) \text{Im}\Psi_2 d\Omega , \tag{54}
\]

where \(P_n\) are the Legendre polynomials, and where the power of \(R_\Delta\) is chosen to give the correct dimensions. One can similarly take moments of the scalar curvature \(\mathcal{R}\) of the 2-metric on any slice of the horizon to give the *mass* multipoles

\[
M_n := \frac{M_\Delta R_\Delta^{n+2}}{8\pi G} \int_{S_{\Delta}} P_n(\cos \theta) \mathcal{R} d\Omega , \tag{55}
\]
where \( M_\Delta \) is the horizon mass, to be derived in the next section. These two sets of multipoles satisfy \( J_0 = 0 \) (no angular momentum monopole), \( J_1 = J_\Delta \), \( M_0 = M_\Delta \), and \( M_1 = 0 \) (one is ‘automatically in the center of mass frame’). More importantly, the multipoles are differomorphism invariant and together uniquely determine both \( q_{ab} \) and the horizon derivative operator \( D_a \) up to differomorphism \([11]\). They have been used both in numerical relativity \([49–51]\), as well as to extend the black hole entropy calculation in loop quantum gravity (to be reviewed later in this chapter) to include rotation and distortion compatible with axisymmetry \([52]\).

**Horizon energy and the first law**

Energy is the generator of time-translations. To derive a notion of energy, we therefore seek a linear combination of the symmetry vector fields \( \ell \) and \( \phi \)

\[
t \equiv B_{(t, \ell)} \ell - \Omega_{(t)} \phi
\]

(56)

to play the role of time translation, such that the corresponding flow is Hamiltonian. Here, as in the case of the Kerr-Newman family of solutions, \( \Omega_{(t)} \) is interpreted as the angular velocity of the horizon relative to \( t \). The generator of this flow will then provide us with the horizon energy. In turns out that, in order to accomplish this, unlike for the case of rotational symmetry considered above, the coefficients \( B_{(t, \ell)} \), \( \Omega_{(t)} \), and hence the vector field \( t^a \), must be allowed to vary from point to point in phase space. One can see a hint that this would be necessary already in the Kerr-Newman family of solutions: the stationary Killing vector field \( t \), determined by the condition that it approach a fixed unit time-translation at infinity, as a linear combination of \( \ell \) and \( \phi \), is not constant over the family. For example, in the Reissner-Nordström sub-family of solutions, \( \Omega_{(t)} \) is zero, and otherwise it is not.

Fix a unit time translation field of the asymptotic flat metric. At each point of \( \Gamma^\phi \), we introduce a vector field \( t \) on the entire space-time such that (1.) \( t \equiv B_{(t, \ell)} \ell - \Omega_{(t)} \phi \) at \( \Delta \) and (2.) \( t \) approaches the fixed unit time translation at infinity. Evaluating the symplectic structure (48) at \((\delta, \delta_t)\), one obtains

\[
\Omega^\phi(\delta, \delta_t) = \Omega^\Delta(\delta, \delta_t) + \delta E_{ADM},
\]

(57)

where \( E_{ADM} \) is the usual ADM energy, given by an integral at spatial infinity \( S_\infty = M \cap I^0 \). The integral at \( \Delta \) is given by

\[
\Omega_{\Delta}(\delta, \delta_t) = \frac{\kappa_{(t)}}{8\pi G} \delta \int_{S_{\Delta}} \epsilon + \frac{\Phi_{(t)}}{8\pi G} \delta \int_{S_{\Delta}} *F + \frac{\Omega_{(t)}}{8\pi G} \delta \int_{S_{\Delta}} [(\phi \omega)\epsilon + (\phi \omega)A] *F]
\]

(58)

where we used \( \kappa_{(t)} = \ell_t \psi = t_\omega \omega \) and \( \Phi_{(t)} = L_t \chi = t_\omega A \). The flow determined by \( t \) will be Hamiltonian iff \( \Omega^\phi(\delta, \delta_t) \) is an exact variation. From (57) and (58), this will be the case iff there exists a phase space function \( E_{\Delta} \) such that for all variations \( \delta \),

\[
\delta E_{\Delta}^{(t)} = \frac{\kappa_{(t)}}{8\pi G} \delta \int_{S_{\Delta}} \epsilon - \frac{\Phi_{(t)}}{8\pi G} \delta \int_{S_{\Delta}} *F - \frac{\Omega_{(t)}}{8\pi G} \delta \int_{S_{\Delta}} [(\phi \omega)\epsilon + (\phi \omega)A] *F]
\]

(59)
If this is true, one has
\[ \Omega^\phi(\delta, \delta_t) = \delta(E_{\text{ADM}} - E_\Delta). \] (60)

The above equation tells us that, if \( E_\Delta \) exists, \( E_{\text{ADM}} - E_\Delta \) will be the hamiltonian generating the flow determined by \( t \). \( E_{\text{ADM}} \) is interpreted as the energy of the entire space-time, whereas \( E_\Delta \) will have the interpretation of the energy of the black hole proper. \( E_{\text{ADM}} - E_\Delta \) is therefore the energy of the gravitational radiation and matter present in the entire intervening region between the horizon and infinity.

We now ask: What are the conditions which \( t \) must satisfy in order for \( E_\Delta \) to exist? How many such vector fields \( t \) are there? This will be clarified in the following proposition. We shall see that there are an infinite number of evolution vectors leading to a Hamiltonian flow, and hence for which \( E_\Delta \) exists.

**Proposition 4.** There exists a function \( E_\Delta \) such that (59) holds if and only if \( \kappa(t), \Phi(t) \) and \( \Omega(t) \) can be expressed as functions of the ‘charges’ \( a_\Delta, Q_\Delta, J_\Delta \) defined by
\[
a_\Delta = \oint_{S_\Delta} \tilde{\epsilon}, \quad Q_\Delta = \frac{1}{8\pi G} \oint_{S_\Delta} \ast F, \quad J_\Delta = \frac{1}{8\pi G} \oint_{S_\Delta} [ (\phi \lrcorner \omega) \tilde{\epsilon} + (\phi \lrcorner A) \ast F],
\] (61)
and satisfy the integrability conditions
\[
\frac{1}{8\pi G} \frac{\partial \kappa(t)}{\partial J_\Delta} = \frac{\partial \Omega(t)}{\partial a_\Delta}, \quad \frac{1}{8\pi G} \frac{\partial \kappa(t)}{\partial Q_\Delta} = \frac{\partial \Phi(t)}{\partial a_\Delta}, \quad \frac{\partial \Omega(t)}{\partial Q_\Delta} = \frac{\partial \Phi(t)}{\partial J_\Delta}. \] (64)

**Proof.** Let \( \mathcal{d} \) and \( \wedge \) denote exterior derivative and exterior product on the infinite dimensional space \( \Gamma^\phi \). Suppose there exists a phase space function \( E_\Delta \) such that (59) holds. Equation (59) is equivalent to
\[
\mathcal{d} E_\Delta = \frac{\kappa(t)}{8\pi G} \mathcal{d} a_\Delta + \Phi(t) \mathcal{d} Q_\Delta + \Omega(t) \mathcal{d} J_\Delta.
\] (65)

Because the gradient of \( E_\Delta \) is a linear combination of the gradients of \( a_\Delta, Q_\Delta, J_\Delta \), \( E_\Delta \) is a function only of these parameters, \( E_\Delta = E_\Delta(a_\Delta, Q_\Delta, J_\Delta) \). The chain rule then implies
\[
\mathcal{d} E_\Delta = \frac{\partial E_\Delta}{\partial a_\Delta} \mathcal{d} a_\Delta + \frac{\partial E_\Delta}{\partial Q_\Delta} \mathcal{d} Q_\Delta + \frac{\partial E_\Delta}{\partial J_\Delta} \mathcal{d} J_\Delta.
\] (66)

From, for example, the Kerr-Newman family of solutions, we know \( a_\Delta, Q_\Delta \) and \( J_\Delta \) are independent quantities and hence \( \mathcal{d} a_\Delta, \mathcal{d} Q_\Delta, \mathcal{d} J_\Delta \) are linearly independent. From (65) and (66) one then has
\[
\frac{\kappa(t)}{8\pi G} = \frac{\partial E_\Delta}{\partial a_\Delta}, \quad \Phi(t) = \frac{\partial E_\Delta}{\partial Q_\Delta}, \quad \Omega(t) = \frac{\partial E_\Delta}{\partial J_\Delta}.
\] (67)
which imply in turn that \( \kappa(t), \Phi(t), \) and \( \Omega(t) \) similarly depend only on \( a_\Delta, Q_\Delta, \) and \( J_\Delta. \) Commutativity of partials together with (67) then imply the integrability conditions (64).

Conversely, if \( \kappa(t), \Phi(t), \Omega(t) \) depend on \( a_\Delta, Q_\Delta, \) and \( J_\Delta \) alone and satisfy the integrability conditions (64), there exists \( E_{\text{ADM}} \) such that (67), and hence such that (65) and (59) hold.

If we replace the surface gravity \( \kappa(t) \) and the area \( a_\Delta \) with the Hawking temperature \( T(t) = \kappa(t)/2\pi \) and the Bekenstein-Hawking entropy \( S_\Delta = a_\Delta/4G, \) (59) becomes

\[
\delta E_\Delta = T(t) \delta S_\Delta + \Phi(t) \delta Q_\Delta + \Omega \delta J_\Delta .
\]

which is none other than the first law of black hole mechanics. One thus sees that the necessary and sufficient condition for \( t \) to be Hamiltonian is that there exist a phase space function \( E_{\Delta} \) such that the first law holds.

The foregoing proposition shows us that there are an infinite number of such evolution vector fields \( t. \) Each of these vector fields is associated with functions \( \kappa(t), \Phi(t), \Omega(t) \), and each has a corresponding notion of horizon energy \( E_\Delta \) satisfying the first law. This is not so surprising: In the context of stationary space-times, the only thing which allows one to isolate a single time-translation vector field \( t \) at the horizon is the global stationarity of the space-time, which rigidly connects the choice of \( t \) at the horizon to the choice of \( t \) at infinity, where it can be fixed by requiring it to be a time translation of the asymptotic Minkowski metric. In the present context of isolated horizons, by contrast, one does not have global stationarity. Thus, physically, one expects an ambiguity in \( t \) and hence in the definition of horizon energy. What is remarkable is that all of these horizon energies satisfy the first law.

Furthermore, the foregoing proposition gives one tight control over this infinity of possible evolutions and corresponding energies. In the following subsection, we will see how this control can be exploited to select a unique, canonical \( t \) on the entire phase space, and hence a unique, canonical notion of horizon energy, which one calls the horizon mass.

**Mass**

For practical applications, such as in numerical relativity, it is useful to isolate a canonical notion of horizon mass. In the context of Einstein-Maxwell theory, the uniqueness theorem provides a way to do this. Specifically, one takes advantage of the fact that, although the phase space \( \Gamma^\phi \) is far larger than the Kerr-Newman family of solutions, the Kerr-Newman family nevertheless forms a subset of \( \Gamma^\phi, \) and on this subset we can stipulate that \( t \) be equal to the standard stationary Killing field \( t \) for the Kerr-Newman space-time in question, determined in the usual way by requiring \( t \) to approach a unit time translation at infinity. Combined with the results of Proposition 4, this suffices to uniquely determine \( t \) on the entire phase
Let us see how this comes about. From Proposition 4, $\kappa(t)$ must be a function of $a_{\Delta}, Q_{\Delta}, J_{\Delta}$ alone. The fact that we have stipulated that $t$ and hence $\kappa(t)$ take their standard canonical values on the Kerr-Newman family uniquely fixes this function to be

$$\kappa(t) = \kappa_0(a_{\Delta}, Q_{\Delta}, J_{\Delta}) := \frac{R_{\Delta}^3}{2R_{\Delta}^2 \sqrt{(R_{\Delta}^2 + GQ_{\Delta}^2)^2 + 4G^2J_{\Delta}^2}},$$ \hspace{1cm} (69)

where $R_{\Delta} = \sqrt{a_{\Delta}/(4\pi)}$ is areal radius of $S_{\Delta}$. With $\kappa(t)$ uniquely determined, the integrability conditions (64) can be used to uniquely determine the associated $\Phi(t)$ and $\Omega(t)$ as well. One finds that these are given by

$$\Phi(t) = \frac{Q_{\Delta}(R_{\Delta}^2 + GQ_{\Delta}^2)}{R_{\Delta} \sqrt{(R_{\Delta}^2 + GQ_{\Delta}^2)^2 + 4G^2J_{\Delta}^2}},$$ \hspace{1cm} (70)

$$\Omega(t) = \frac{2GJ_{\Delta}}{R_{\Delta} \sqrt{(R_{\Delta}^2 + GQ_{\Delta}^2)^2 + 4G^2J_{\Delta}^2}}.$$ \hspace{1cm} (71)

Finally, equation (67) can be integrated to uniquely determine the horizon energy up to an additive constant. This additive constant can be fixed by requiring the horizon energy to be equal to the usual value of the energy when evaluated on members of the Kerr-Newman family. The resulting horizon energy we denote $M_{\Delta}$ and is called the horizon mass. It is given by

$$M_{\Delta} = \frac{\sqrt{(R_{\Delta}^2 + GQ_{\Delta}^2)^2 + 4G^2J_{\Delta}^2}}{2GR_{\Delta}}.$$ \hspace{1cm} (72)

Note that this definition of the horizon mass involves only quantities intrinsic to the horizon. This is in contrast to the ADM or Komar definitions of mass, in which reference to infinity is required.

Finally, note that this strategy for selecting a canonical notion of mass works in Einstein-Maxwell theory only because the uniqueness theorem holds. If we consider theories in which the uniqueness theorem fails to hold, such as Einstein-Yang-Mills theory, it is no longer possible to consistently stipulate that $t^\alpha$ (and hence $\kappa(t)$, $\Phi(t)$, and $\Omega(t)$) be equal to its canonical value when evaluated on stationary solutions: Because each triple $(a_{\Delta}, J_{\Delta}, Q_{\Delta})$ corresponds to multiple stationary solutions, such a prescription now would contradict the requirement of Proposition 4 that $\kappa(t)$, $\Phi(t)$, and $\Omega(t)$ must be functions of $a_{\Delta}, J_{\Delta}, Q_{\Delta}$ alone. In such theories, one still has an infinite family of first laws. It is just that one cannot select a single one as canonical.

5 Quantum isolated horizons and black hole entropy

As first observed by Bekenstein [32], if we identify the surface gravity $\kappa$ of a black hole with its temperature and its area $a$ with its entropy, then the first law of black
hole mechanics takes the form of the standard first law of thermodynamics. With this identification, the zeroeth and second laws of black hole mechanics, reviewed in the introduction, furthermore take the form of the standard zeroeth and second laws of thermodynamics and furthermore provide a way to preserve the second law of thermodynamics in the presence of a black hole. These observations led Bekenstein to hypothesize that a black hole is a thermodynamic object, its area is its entropy, and its surface gravity is its temperature. This hypothesis was impressively confirmed by Hawking’s calculation showing that black holes radiate thermally at a temperature equal to \( T = \hbar \kappa / 2\pi \), a calculation which also fixed the coefficient relating \( T \) and \( \kappa \), which in turn allowed one, via the first law, to fix the coefficient relating the entropy \( S \) to the area \( a \): \[
S = \frac{a}{4\ell^2_{Pl}}.
\]
This is the celebrated Bekenstein-Hawking entropy. It provides tantalizing evidence that black holes are thermodynamic objects with an entropy whose statistical mechanical origin lay in quantum theory. To account for the statistical mechanical origin of black hole entropy has become one of the challenges for any quantum theory of gravity.

In this section we will review such an account provided by quantum isolated horizons using the approach to quantum gravity known as loop quantum gravity. The key principal which guides loop quantum gravity is that of background independence, which is equivalent to the symmetry principle of diffeomorphism covariance at the foundation of classical general relativity.

For simplicity, we here review the more recent \( SU(2) \) covariant account of this derivation of the entropy \([14,53]\), which built to a great extent on the original work \([3,54,55]\) which used a \( U(1) \) partial gauge-fixing to aide calculations. Furthermore, we will only present the type I case — i.e., the case in which the intrinsic geometry of the horizon is spherically symmetric. The type II and type III horizons are handled in \([52,56]\) (see also \([57]\)). Lastly, again in order to focus on the key ideas, we will handle only pure gravity. Inclusion of Maxwell and Yang-Mills fields, and even non-minimally coupled scalar fields can also be handled, and can be found in the references \([10,55,58]\).

### 5.1 Phase space and symplectic structure

In loop quantum gravity, one uses the Ashtekar-Barbero variables \([59]\), which we use in the form \((\gamma, \Sigma)\) consisting in the Ashtekar-Barbero connection \(\gamma^A_i\), and the ‘flux-2-form’ \(\Sigma_{ab}^i\), where \(i = 1, 2, 3\). These are related to the space-time variables used in the prior sections via

\[
\gamma^A_i = (\star^\Sigma A + \beta^a A)^{i0} \\
\Sigma^i = \varepsilon^{ijk} \xi_j \wedge \xi^k,
\]

31
where the arrows ∼ denote pull-back to the partial Cauchy surface \( M \), and where \( \beta \in \mathbb{R}^+ \) is the Barbero-Immirzi parameter, a quantization ambiguity consisting in a single real number. Furthermore, in the above equation, it is understood that the space-time variables \( A^I_a, e^a_i \) are in the time-gauge in which \( e^a_0 \) is normal to the partial Cauchy surface \( M \), reducing the internal local gauge group from \( SO(1,3) \) to \( SO(3) \). To give a sense of the meaning of these variables, in terms of generalized ADM variables, one has \( \Sigma^i_{ab} = \varepsilon_{abc} (\det e^d_j) e^c_i \) where \( e^a_i \) is the triad, and the two terms in the expression for \( \gamma A \) are given by \((\star_A)^{i0} = \Gamma^i_a\), the spin connection determined by the triad, and \( A^{i0} = K^i_a = K_{ab} e^{bi} \) where \( K_{ab} \) the extrinsic curvature of \( M \).

Let \( \Gamma^{(I)} \) denote the space of possible fields \((\gamma A, \Sigma)\) on \( M \) satisfying Einstein’s equations, such that the inner boundary \( S := \Delta \cap M \) corresponds to a type I isolated horizon of a fixed area \( a_0 \), and such that appropriate asymptotically flat boundary conditions are satisfied at infinity. Thus, the intrinsic geometry of \( S \) is restricted to be round in \( \Gamma^{(I)} \). If one starts from the symplectic structure (46), imposes the above conditions, and rewrites the result using the Ashtekar-Barbero variables \((A^i, \Sigma^i)\), one obtains the following symplectic structure:

\[
\Omega^{(I)}(\delta_1, \delta_2) = \frac{1}{8\pi G \beta} \int_M \left[ \delta_1 \Sigma^i \wedge \delta_2 \gamma A_i - \delta_2 \Sigma^i \wedge \delta_1 \gamma A_i \right] - \frac{1}{8\pi G \beta \pi (1 - \beta^2)} \int_S \delta_1 A_i \wedge \delta_2 A^i . \tag{73}
\]

In deriving this, one makes key use of the fact that, in terms of the Ashtekar-Barbero variables, the isolated horizon boundary conditions take the form

\[
\Sigma^i = -\frac{a_0}{\pi (1 - \beta^2)} F^i(\gamma A) . \tag{74}
\]

This is referred to as the quantum horizon boundary condition. Also note that the symplectic structure (73) consists in two terms — a bulk term equal to the standard symplectic structure used in loop quantum gravity, and a surface term at \( S \), equal to the symplectic structure of an \( SU(2) \) Chern-Simons theory. This observation, together with the horizon boundary condition, are the keys to the quantization of this system.

### 5.2 Quantization

As just remarked, the symplectic structure (73) consists in two terms, a bulk term identical to that used in LQG, and a horizon, \( SU(2) \) Chern-Simons theory term. This motivates one to quantize the bulk and horizon degrees of freedom separately. After they have been separately quantized, the horizon boundary condition (74) will be used to couple them again. Let us start with the bulk. The bulk Hilbert space of states \( \mathcal{H}^B \) is the standard one in loop quantum gravity, spanned by spin-network states. A spin-network state \(|\gamma, \{j_e\}, \{i_v\}\rangle\) is labelled by a collection of
edges $\gamma$ called a graph with each edge labelled by a half-integer spin $j$, and each point at the end of an edge, called a vertex, labelled by an intertwiner among the irreducible representations on the edges meeting at the vertex. Spin-network states are eigenstates of area: For any 2-surface $T$, one classically has a corresponding area $a_T$, and hence in quantum theory a corresponding area operator $\hat{a}_T$, and one has
\[ \hat{a}_T|\gamma, \{j_e\}, \{i_v\} \rangle = \left( 8\pi G \beta \sum_{p \in T \cap \gamma} \sqrt{j_p(j_p + 1)} \right) |\gamma, \{j_e\}, \{i_v\} \rangle . \]
(75)

In the present case, because the spatial manifold $M$ has boundary $S$, $\gamma$ may have edges which intersect this boundary. We call such intersections punctures. Suppose we are given a finite set of points $P$, and assignment of a spin $j_p$ to each point. Let $H_{B,P,\{j_p\}}$ denote the span of all spin-network states whose punctures are at the points $P$ and the spins on the corresponding edges are $\{j_p\}$. In terms of these spaces, the full bulk Hilbert space can be expressed as a direct sum over spins and a direct limit over all finite subsets of $S$:
\[ H_B = \lim_{\longrightarrow} \bigoplus_{P \subset S} \bigoplus_{\{j_p\} \in P} H_{B,P,\{j_p\}} . \]
(76)

Consider one of these spaces $H_{B,P,\{j_p\}}$. On this space, the action of the operator corresponding to the two form $\Sigma_{ab}$ pulled back to $S$ reduces to
\[ \epsilon^{ab} \hat{\Sigma}_{ab}(x) = 8\pi G \beta \sum_{p \in P} \delta(x, x_p) \hat{J}^i(p) , \]
(77)
where at each puncture $p$ the operators $\hat{J}^i(p)$ satisfy the usual angular momentum algebra $[\hat{J}^i(p), \hat{J}^j(p)] = \epsilon^{ij}_{\ k} \hat{J}^k(p)$. Substituting this into (74), we get
\[ -\frac{a_o}{\pi(1 - \beta^2)} \epsilon^{ab} \hat{F}_{ab}^i = 8\pi G \beta \sum_{p \in P} \delta(x, x_p) \hat{J}^i(p) . \]
(78)
Here $\hat{F}_{ab}$ is the quantization of the curvature of $\gamma A^i_a$ pulled back to $S$, and hence the curvature of the $SU(2)$ Chern-Simons connection. This shows us that the generators $\hat{J}^i(p)$ act as point sources for the $SU(2)$ Chern-Simons theory. The quantization of $SU(2)$ Chern-Simons theory with such point sources is well-understood — see for example Witten [60]. In the end, for each fixed set of spins $j_p$ associated to the point sources, one obtains a Hilbert space of states $H_{CS}^k(\{j_p\})$, where $k := a_o/(2\pi \beta(1 - \beta^2) \ell_P^2)$ is the ‘level’ of the Chern-Simons theory, appearing in the coefficient of the surface symplectic structure in (73). This Hilbert space can be viewed as a subset of the tensor product of carrying spaces associated to the $SU(2)$ representations labeled by the spins $j_p$
\[ H_{CS}^k(\{j_p\}) \subset \otimes_{p \in P} V_{j_p} , \]
where $\hat{J}^i(p)$ acts on each $V_{jp}$ irreducibly. $\mathcal{H}_{\mathcal{P},\{jp\}}$ can similarly be decomposed in a way that makes the action of the generators apparent

$$\mathcal{H}_{\mathcal{P},\{jp\}}^B = \mathcal{H}_{\{jp\}} \otimes \left( \otimes_{p \in \mathcal{P}} V_{jp} \right),$$

where again $V_{jp}$ denotes the spin $jp$ carrying space on which $\hat{J}^i(p)$ acts irreducibly. For a given set of punctures $\mathcal{P}$ and spins $\{jp\}$, one is therefore led to the following space of states satisfying the quantum version of the horizon boundary condition (74),

$$\mathcal{H}_{\mathcal{P},\{jp\}}^{\text{Kin}} = \mathcal{H}_{\{jp\}} \otimes \mathcal{H}_{k}^{CS}(\{jp\}). \tag{79}$$

Upon reassembling these spaces using a direct sum over spins and direct limit over sets of punctures, one obtains the full space of states solving the horizon boundary condition (78):

$$\mathcal{H}^{\text{Kin}} = \lim_{\mathcal{P} \subset \mathcal{S}} \bigoplus_{\{jp\} \in \mathcal{P}} \mathcal{H}_{\{jp\}} \otimes \mathcal{H}_{k}^{CS}(\{jp\}). \tag{80}$$

Here ‘Kin’ indicates that the diffeomorphism and Hamiltonian constraints have not yet been imposed. Imposition of the diffeomorphism constraint roughly speaking leads to replacement of quantum states with their diffeomorphism equivalence class. See [55] for further details. The solution to the diffeomorphism constraint then takes the form

$$\mathcal{H}^{\text{Diff}} = \bigoplus_n \bigoplus_{(jp)_n^\mathcal{P}} \mathcal{H}_{(jp)_n^\mathcal{P}} \otimes \mathcal{H}_{k}^{CS}(\{jp\}), \tag{81}$$

where now one is no longer summing over possible positions of punctures, but only over the total number of punctures and corresponding spins. Because lapse is restricted to vanish on the horizon [3, 14], the Hamiltonian constraint is imposed only in bulk, resulting in replacement of $\mathcal{H}_{(jp)_n^\mathcal{P}}$ with an appropriate subspace $\mathcal{H}_{(jp)_n^\mathcal{P}}$, yielding the final space of physical states

$$\mathcal{H}^{\text{Phys}} = \bigoplus_n \bigoplus_{(jp)_n^\mathcal{P}} \mathcal{H}_{(jp)_n^\mathcal{P}} \otimes \mathcal{H}_{k}^{CS}(\{jp\}). \tag{82}$$

For further details, we refer to [14, 55].

### 5.3 Ensemble and entropy

We are interested in the ensemble of physical states in $\mathcal{H}^{\text{phys}}$ consisting in eigenstates of the horizon area $\hat{a}_S$ with eigenvalue in the range $(a_o - \delta, a_o + \delta)$ for some tolerance $\delta$. Let $\mathcal{H}^{\text{bh}}$ denote the span of such states. One would like to define the entropy via the standard Boltzmann formula as the logarithm of the dimension of this space. However, the dimension of this space is infinite, for the simple reason that all of bulk degrees of freedom are represented in it — degrees of freedom which are irrelevant for the question which interests us.
To eliminate this infinity, one needs to talk more precisely about the horizon degrees of freedom. To this end, we define

$$\mathcal{H}^S := \bigoplus_{n} \bigoplus_{(j \rho)^n} \mathcal{H}^{CS}_k(\{j \rho\}).$$

(83)

We say that a given horizon state $\psi^S \in \mathcal{H}^{CS}_k(\{j \rho\}) \subset \mathcal{H}^S$ is ‘compatible’ with $\mathcal{H}^{bh}$ if there exists $\psi^B \in \mathcal{H}^{(j \rho)^n}$ such that $\psi^B \otimes \psi^S \in \mathcal{H}^{bh}$. Let $\mathcal{H}^{bh}_S$ denote the span of all such compatible horizon states. The entropy is then given by the usual Boltzmann formula

$$S := \log \dim \mathcal{H}^{bh}_S.$$ 

One finds [61, 62] the final entropy to be

$$S = \frac{\beta_0 a_0}{4\beta \ell^2_P}, \quad \beta_0 = 0.274067 \ldots$$

(84)

so that if we choose $\beta = \beta_0 = 0.274067 \ldots$, the Bekenstein-Hawking entropy formula results. Note the role played by $\beta$: it is a single quantization ambiguity, which in principle can be fixed by a single experiment. By requiring the Bekenstein-Hawking entropy law to hold for any black hole with spherical intrinsic geometry (type I) and a single value of the area, $\beta$ becomes fixed. The fact that the entropy law then continues to be satisfied for black holes of other areas is already a non-trivial test. But, in fact, the present framework has been shown to pass much stronger tests as well: if one extends the framework to include arbitrary Maxwell and Yang-Mills fields [55], or type II (axisymmetric) and type III (generic) isolated horizons [52,56], one again obtains an entropy equal to one quarter times the area, for the same value of the Barbero-Immirzi parameter.\(^2\) One can even include a scalar field which is non-minimally coupled to gravity [10,58]. As mentioned in the introduction, this leads to a modified entropy law. The present framework, when extended to include a non-minimally coupled scalar field, has been shown to lead precisely to the required modified entropy law, again for the same value of $\beta$.

6 Summary and discussion

We have reviewed in detail the mathematical foundations of IHs in classical and quantum gravity. Let us briefly summarize the main features of this framework.

An IH is defined to be a null hypersurface with compact spatial cross-sections, whose outgoing null normal is non-expanding, and on which certain components of the Levi-Civita connection are Lie dragged by the null normal. We have seen that a well-defined first-order action principle can be given for space-times possessing an IH as an inner boundary, with the IH boundary conditions ensuring that the action

\(^2\)As long as one identifies the horizon degrees of freedom in the same way as has been done here, counting the $j$ labels as distinguishing horizon states. See [63,64] for a discussion.
is functionally differentiable. From second variations of this action, a conserved symplectic current can be identified on the covariant phase space of solutions to the field equations. Using this, we have seen how one can construct a well-defined canonical framework. In this framework, local rotations and time translations at the horizon are generated by clear notions of angular momentum and energy of the IH. These quantities are independent of the ADM charges at infinity, and satisfy a quasi-local version of the first law of black hole mechanics. Furthermore, the expression for the horizon angular momentum can be naturally generalized to give a quasi-local, diffeomorphism-invariant notion of angular momentum and mass multipoles.

When the symplectic structure of this canonical framework is recast in terms of real $SU(2)$ Ashtekar connections and triads, the boundary term in the symplectic structure at the horizon becomes that of $SU(2)$ Chern-Simons theory. This leads one to quantize the bulk degrees of freedom using loop quantum gravity methods, and the horizon degrees of freedom using a well-understood quantization of $SU(2)$ Chern-Simons theory. The bulk and surface degrees of freedom are then coupled through a quantum version of a ‘horizon boundary condition’ embodying the fact that the inner boundary is an isolated horizon. The statistical entropy of an ensemble of states with area in a small window of values is found to be equal to 1/4 times the horizon area for a single, fixed value of the Barbero-Immirzi parameter, a single quantization ambiguity present in the loop quantization of gravity.

There are some key features of the IH framework that we would like to highlight at this point. First, we stress that, in the IH framework, there is no need to make reference to asymptotic infinity at all. This paradigm shift is essential in order that we may understand situations in which a black hole is in equilibrium with dynamical fields in the exterior region that are in an arbitrarily small neighbourhood of the horizon. Moreover, we note that the isolated horizon definition places a restriction only geometric structures intrinsic to the horizon. Not only does this make the definition simpler to check in practice, but it ensures that the full complement of degrees of freedom outside the horizon remains intact. This is important in quantum theory, and allows the methods of loop quantum gravity to be used outside the horizon without change.

Lastly, perhaps the most notable feature of the IH framework is that it provides a unified mathematical construction for understanding equilibrium black holes in both classical and quantum gravity. Indeed, the IH framework can be used to study the geometry, topology, supersymmetry, mechanics and quantum statistical mechanics of quasi-local black holes in general relativity (and beyond).

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