The fluctuating $\alpha$-effect and Waldmeier relations in the nonlinear dynamo models

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Abstract
We study the possibility of reproducing the statistical relations of the sunspot activity cycle, such as the so-called Waldmeier relations, the cycle period–amplitude and the cycle rise rate–amplitude relations, by means of the mean-field dynamo models with fluctuating $\alpha$-effect. The dynamo model takes into account the long-term fluctuations of the $\alpha$-effect and two types of nonlinear feedback of the mean field on the $\alpha$-effect, namely, the algebraic quenching and the dynamic quenching due to the magnetic helicity generation. We found that the models are able to reproduce qualitatively and quantitatively the inclination and dispersion across the Waldmeier relations with 20% fluctuations of the $\alpha$-effect. The models with dynamic quenching are in better agreement with observations than the models with algebraic $\alpha$-quenching. We compare the statistical distributions of the modelled parameters, such as the amplitude, period, and the rise and decay rates of the sunspot cycles with observations.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction
It is observed that sunspot activity is organized in time and latitude and forms large-scale patterns that are called the Maunder butterfly diagram. This pattern is believed to be produced by the large-scale toroidal magnetic field generated in the convection zone. Another component of the solar activity is represented by the global poloidal magnetic field extending outside the Sun and shaping the solar corona. Both components synchronously evolve as the solar 11-year cycle progresses. The global poloidal field reverses sign in the polar regions near the time of maximum sunspot activity.

A remarkable feature of cyclic solar activity is that it is far from being just a cycle. Cycle amplitude and shape varies from one cycle to the other. It is very attractive to use the statistical properties of these variations to shed light on the dynamo mechanisms operating inside the solar convective zone. Solar activity observations give many hints that various tracers of solar activity that are exploited to quantify the phenomenon demonstrate some relation to each other, which opens the possibility of predicting the future evolution of solar activity based on available observations of other indices. Waldmeier [37] pointed out at first this option (an inverse correlation between the length of the ascending phase of a cycle, or its ‘rise time’, and the peak sunspot number of that cycle) and applied it in [38] to give a prediction of the following cycle. The latter paper is, in practice, the first accessible (at least for German-speaking readers) paper in the area. Later, another relation of this type was suggested and summarized as Waldmeier relations. This development was clearly summarized in [36] and recently in [13]. The nature of the physical processes that are manifested in the Waldmeier relations is not clear; see discussion, e.g., in [7, 9, 11]. It seems to be remarkable, however, that these statistical properties of magnetic activity also exist for the other tracers related to sunspot activity (e.g. sunspot group and squares of sunspot groups; see [9, 13, 36]) and even for the other kinds of the solar and stellar activity indices, e.g. for the Ca II index [32]. The Waldmeier relations are considered as a valuable test of the dynamo models [9, 16, 30].
A natural way of pushing the understanding of the problem forward is to clarify the physics underlying the Waldmeier relations. It is more or less accepted that cyclic solar activity is driven by a dynamo, i.e. a mechanism that transforms the kinetic energy of hydrodynamical motions into a magnetic one. Most of the current solar dynamo models suggest that the toroidal magnetic field that emerges on the surface and forms sunspots is generated near the bottom of the convection zone, in the tachocline or just beneath it in a convection overshoot layer (see, e.g., [33]). This kind of dynamo can be approximated by Parker’s surface dynamo waves [27]. The direction of the dynamo waves propagation is defined by the Parker–Yoshimura rule [39]. It states that for the \( \alpha \Omega \) kind of dynamo, the wave propagate along iso-surfaces of the angular velocity. The propagation process can be modified by the turbulent transport (associated with the mean drift of magnetic activity in the turbulent media by means turbulent mechanisms), by the anisotropic turbulent diffusivity (see [15]) and by meridional circulation [8]. A viewpoint that is an alternative to Parker’s surface dynamo waves is presented by the distributed dynamo with subsurface shear, e.g. [3]. The dynamo waves here propagate along the radius in the main part of the solar convection zone [15]. The near-surface activity is shaped by the subsurface shear. One more option is the flux-transport dynamo, e.g. [8, 10].

In the context of dynamo theory, the Waldmeier relations have to be explained by some mechanism that varies the amplitude and shape of the activity cycle, and fluctuations of the \( \alpha \)-effect are considered below as such a mechanism. This idea extends the approach proposed in [30] to explain these relations by changing the magnitude of the \( \alpha \)-effect.

The physical idea underlying this mechanism can be presented as follows. The \( \alpha \)-coefficient is a mean quantity taken over an ensemble of convective vortices. The number \( N \) of the vortices in a solar convective shell is large, but much smaller than, say, the Avogadro number, and so the fluctuations proportional to \( N^{-1/2} \) may not be negligible. A particular choice of \( N \) is obviously model dependent; however, if we take just for orientation \( N = 10^4 \), then \( N^{-1/2} \approx 0.01 \). Taking into account that \( \alpha \) is usually about 1/10 of turbulent velocity, we consider a dozen per cent of \( \alpha \)-fluctuations as a comfortable estimate. On the other hand, governing equations for a large-scale solar magnetic field deal with spatial averaging and have to include the contribution of \( \alpha \)-fluctuations [14].

A straightforward application of the idea with vortex turnover time and vortex size as the correlation time and length for \( \alpha \)-fluctuations needs fluctuations much larger than mean \( \alpha \). The works [25, 35] based on experiences in direct numerical simulations, e.g. [4], and the results of current helicity (related to \( \alpha \)) observation in solar active regions, e.g. [40], considered \( \alpha \)-fluctuations with correlation time comparable with cycle length and correlation length comparable with the extent of the latitudinal belts where sunspots occur to conclude that a reasonable \( \alpha \)-noise of the order of a few dozen per cent is sufficient to explain Grand minima of solar activity. The aim of this paper is to apply this idea to explain Waldmeier relations.

2. Basic equations

2.1. Two-dimensional (2D) model

The dynamo model is based on the standard mean-field induction equation in perfectly conductive media [20]:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{E} + \mathbf{U} \times \mathbf{B}),
\]

where \( \mathbf{E} = \mathbf{u} \times \mathbf{b} \) is the mean electromotive force, with \( \mathbf{u} \) being the turbulent fluctuating velocity and magnetic field, respectively; \( \mathbf{U} \) is the mean velocity (differential rotation). The axisymmetric magnetic field

\[
\mathbf{B} = \mathbf{e}_\theta B + \nabla \times \frac{\mathbf{A} e_\theta}{r \sin \theta},
\]

where \( \theta \) is the polar angle. We have used the expression for \( \mathbf{E} \) obtained in [28] (hereafter P08) and write it as follows:

\[
\mathbf{E}_i = (\alpha_{ij} + \gamma_{ij}) \nabla_j - \eta_{ij} \nabla \cdot \mathbf{B}_k.
\]

The tensor \( \alpha_{ij} \) represents the \( \alpha \)-effect, including the hydrodynamic and magnetic helicity contributions,

\[
\alpha_{ij} = C_u (1 + \xi) \psi_u(\beta) \sin^2 \theta \alpha_{ij}^{(H)} + \alpha_{ij}^{(M)},
\]

where \( \xi \) is the noise, the hydrodynamical part of the \( \alpha \)-effect, \( \alpha_{ij}^{(H)} \), and the quenching function, \( \psi_u \), are given in the appendix (see also [29]; hereafter PK11). The hydrodynamic \( \alpha \)-effect term is multiplied by \( \sin^2 \theta \) (\( \theta \) is co-latitude) to prevent the turbulent generation of a magnetic field at the poles. The contribution of the small-scale magnetic helicity \( \chi = \mathbf{a} \cdot \mathbf{b} \) (\( \mathbf{a} \) is a fluctuating vector-potential of magnetic field) to the \( \alpha \)-effect is defined as \( \alpha_{ij}^{(M)} = C_{ij}^{(M,\chi)} \chi \), where the coefficient \( C_{ij}^{(M,\chi)} \) depends on the turbulent properties and rotation and is given in the appendix. The other parts of equation (1) represent the effects of turbulent pumping, \( \gamma_{ij} \), and turbulent diffusion, \( \eta_{ij} \). They are the same as in PK11. We describe them in the appendix.

The nonlinear feedback of the large-scale magnetic field to the \( \alpha \)-effect is described as a combination of an ‘algebraic’ quenching by the function \( \psi_u(\beta) \) (see the appendix and [30]) and a dynamical quenching due to the magnetic helicity conservation constraint. The magnetic helicity, \( \chi \), subject to a conservation law, is described by the following equation [5, 19, 34]:

\[
\frac{\partial \chi}{\partial t} = -2(\mathbf{E} \cdot \mathbf{B}) - \frac{\chi}{R_x \tau_c} + \nabla \cdot (\eta_{ij} \nabla \chi),
\]

where \( \tau_c \) is a typical convection turnover time. The parameter \( R_x \) controls the helicity dissipation rate without specifying the nature of the loss. It seems to be reasonable that the helicity dissipation is most efficient in the near-surface layers because of the strong decrease of \( \tau_c \) (see figure 1(b)). The last term in equation (3) describes the diffusive flux of magnetic helicity [21]. We use the solar convection zone model computed in [33], in which the mixing length is defined as \( \ell = \alpha_{\text{MLT}} |\Delta^{(p)}|^{-1} \), where \( \Delta^{(p)} = \nabla \log B \) is the pressure variation scale, and \( \alpha_{\text{MLT}} = 2 \). The turbulent diffusivity is
shows a typical time–latitude χ profiles with a step of $0.025\Omega_0$, $\Omega_0 = 2.86 \times 10^7 \text{s}^{-1}$; (b) turnover convection time $\tau_c$, and the rms convective velocity $u'_c$ and the background turbulent diffusivity $\eta^{(0)}_T$ profiles; (c) the radial profiles of the $\alpha$-effect tensor components.

Figure 2. Typical time–latitude and time–radius (at 30$^\circ$ latitude) diagrams of the toroidal field (grey scale), the radial field (contours on the left panel) and the poloidal magnetic field (contours on the right panel) evolution in the 2D1 model (see table 1). The toroidal field is averaged over the subsurface layers in the range $0.9–0.99 R_\odot$; the radial field is taken at the top of the convection zone.

parameterized in the form $\eta_T = C_\eta \eta^{(0)}_T$, where $\eta^{(0)}_T = \eta^{(0)}$ is the characteristic mixing-length turbulent diffusivity, $\ell$ and $u'$ are the typical correlation length and rms convective velocity of turbulent flows, respectively, and $C_\eta$ is a constant to control the intensity of turbulent mixing. In this paper, we use $C_\eta = 0.05$. The model uses a modified version of an analytical approximation to helioseismology data of the angular velocity distribution proposed in [2]; see figure 1(a).

We use the standard boundary conditions to match the potential field outside and the perfect conductivity at the bottom boundary. As discussed above, the penetration of the toroidal magnetic field into the near-surface layers is controlled by the turbulent diffusivity and pumping effect. For magnetic helicity, similarly to [12, 22], we use the time-dependent conditions provided by equation (3) and the helicity flux conservation condition $V_r \chi = 0$ is applied at the bottom and the top of the domain. The latter gives a smooth transfer for solutions with and without the diffusive helicity flux.

The left panel in figure 2 shows a typical time–latitude diagram for the toroidal magnetic field averaged over the subsurface layers $0.9–0.99 R_\odot$ and the radial magnetic at the top of the integration domain. The right panel shows the time–radius diagram for the toroidal and poloidal magnetic field evolution at 30$^\circ$ latitude.

### 2.2. 1D model

For comparison with the previous studies and also to study how the additional dimension affects the statistical properties of the dynamo, we consider a 1D model similar to that studied in [25]:

$$\frac{\partial A}{\partial t} = \sin \theta \left( (1 + \xi) \cos \theta \psi_B (B) + \chi \right) B + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial A}{\partial \theta} \right) - \eta_{CZ} A,$$

$$\frac{\partial B}{\partial t} = -D \tilde{\Omega} (\theta) \frac{\partial A}{\partial \theta} + \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial A}{\partial \theta} \right) - \eta_{CZ} B,$$

where the large-scale radial shear $\tilde{\Omega} (\theta) = \partial \Omega / \partial r$. The 1D model employs two possibilities for the shear profile. In one case, we put $\tilde{\Omega} (\theta) = 1$, which gives us the model explored in [25]. In another case, we use

$$\tilde{\Omega} = \frac{1}{m} (5 \sin^2 \theta - 4),$$

which is suggested in [17]. In agreement with the helioseismology results for the bottom of the convection zone, this profile is positive in equatorial regions and negative near the poles. The magnetic field strength in equation (5) is measured in the units of the equipartition magnetic field strength and the time is normalized to the typical diffusive time, $R_\odot^2 / \eta^{(0)}_T$. The evolution of the magnetic helicity for the 1D model is governed by the equation

$$\frac{\partial \chi}{\partial t} = -2 \left( (1 + \xi) \cos \theta \psi_B (B) + \chi \right) B^2 - 2B \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial A}{\partial \theta} \right) + \frac{2}{\sin^2 \theta} \frac{\partial A}{\partial \theta} \sin \theta \frac{\partial B}{\partial \theta} - \frac{\chi}{R_x} \eta_x \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \chi}{\partial \theta} \right)$$

(7)
In what follows, we will discuss the 1D models with constant shear, because they are more relevant to compare with observations. The differences in results for the 1D models with the variable shear given by equation (6) will be briefly mentioned in subsequent sections.

Summarizing, we exploit much more detailed and realistic dynamo models than the works [25, 35]. Our point is that the Waldmeier relations are a much more delicate phenomenon than Grand minima and the bulk of our knowledge concerning recent solar cycles is much richer than that for the remote past when Grand minima took place.

### 2.3. Noise model

The noise, $\xi$, contributes to the hydrodynamic part of the $\alpha$-effect (see equations (2) and (4)). Following [35], the models employ the long-term Gaussian fluctuating $\xi$ of small amplitude with rms deviation given in table 1 (last column). The time of renewal of $\xi$ is equal to the period of the model. The random numbers were generated with the help of the standard F90 subroutine and the quality of contemporary standard noise generator subroutine is shown to be sufficient for this kind of modelling, see e.g. [1]. It would be more realistic to consider the renewal time as the fluctuating quantity as well, but we would like to leave this effect for a different study. Also, we found that models that employ the magnetic helicity effect show very intermittent long-term behaviour. This makes the analysis procedure (e.g. division into subsequent cycles) more complicated. We isolate ourselves from these phenomena by considering noise models with lower rms.\(^5\)

| Model | $\eta_{\text{CZ}}$ | $\chi$ | $\eta_{\chi}/\eta_{\chi}$ | $R_s$ | $B_0$ | $C_w$ | $\sigma$ |
|-------|-----------------|-------|--------------------------|-------|-------|-------|-------|
| 1D1   | 1               | 0     | 0                        | 10    | 3     | 1200  | 0.15  |
| 2D1   | Equation (2)    | $10^{-3}$ | 200                       | 800   | 1     | 0.15  |
| 2D2   | Equation (2)    | 0.3   | $10^6$                    | 200   | 1     | 0.15  |

2.4. The sunspot cycle model and the Waldmeier relations

In the paper, we define the Waldmeier relations as the set of mean properties of the sunspot cycle. We will deal with the following properties of the Wolf sunspot number (which is taken either from the observational database or simulated from the model): the relation between the period and the amplitude of the same cycle, the relation between the rise rate and the amplitude of the cycle and the shape of the sunspot cycle, characterized by the ratio between the decay rate and the rise rate in the cycle. The other kinds of relations, such as the link between the rise time and the amplitude of the cycle, can be considered as derivatives from the above relation. For comparison with other analyses of the observational data and also with the results of the dynamo models presented by Karak and Choudhuri [9], we show the results for the rise time of the cycles as well, the relation between the rise time and the amplitude of the cycle and the relation between the cycle amplitude and period of the preceding cycle (see [13, 36]). The amplitude of the cycles is defined by the difference between the maximum sunspot number and the sunspot number in the preceding minimum. Even for the harmonic cycles the latter differs from zero due to the spatial overlap in subsequent cycles. The period of the cycle is equal to the time between the subsequent minima, and the rise time of the cycle is defined by the difference between the moment of the cycle maximum and the moment of the preceding minimum of the cycle. The rise rate is defined as the ratio between the difference of the sunspot number amplitude during the maximum and minimum of the cycle and the rise time of the cycle. There is a similar definition for the decay rate of the cycle.

\(^5\) In part, the given problem is likely due to the very rough model for the Wolf number, see equation (8).
Remember that sunspots are not directly presented in dynamo models and we have to relate its number to a quantity involved in a dynamo model under consideration. We assume that the sunspots are produced from toroidal magnetic fields by means of nonlinear instability and we avoid considering the instability in detail. To model the sunspot number \( W \) produced by the dynamo, we use the following ansatz:

\[
W(t) = C_W \frac{B_{\text{max}}}{B_0} \exp\left(-\frac{B_0}{B_{\text{max}}}ight),
\]

where for the 2D models, \( B_{\text{max}} \) is the maximum of the toroidal magnetic field strength over latitudes averaged over the subsurface layers in the range of 0.9–0.99\( R_\odot \), and for the 1D models, \( B_{\text{max}} \) is simply the maximum of the toroidal magnetic field strength over latitudes; \( B_0 \) is the typical strength of the toroidal magnetic field that is enough to produce the sunspot; \( C_W \) is the parameter to calibrate the modelled sunspot number relative to observations. All the parameters that were employed in the different models are listed in table 1.

In the dynamo models, we explore the effect of Gaussian fluctuations of the \( \alpha \)-effect, or the parameter \( C_\alpha \) with the typical time equal to the period of the cycle and standard deviations lower than 0.2\( C_\alpha \). In the models presented here, we fix the standard deviation at 0.15\( C_\alpha \).

For comparison with simulation, we use the smoothed data set from [31] that starts at 1750. Choosing this data set, we appreciate that in principle the Waldmeier relations can be valid for the normal cycle only and their applicability to epochs of Grand minima of solar activity must be addressed separately. Available instrumental data concerning solar activity in the 17th—early 18th centuries give only a limited possibility to address this important point, which obviously is outside the scope of this paper. On the other hand, there are various indirect (mainly isotopic) tracers of solar activity which give limited information concerning its shape over a much longer time interval than instrumental data. Our point is that Waldmeier relations and the regularities of such long-term time series (see, e.g., [23, 24]) have to be discussed in a separate paper and here we use as an illustrative example the extended time series of the sunspot data proposed in [26] (referred to hereafter as NIMV). These data sets are shown in figure 4. Table 2 contains linear fits and correlations between different parameters of the cycles for observational data sets and for the dynamo models. In particular, the parameters of the relation between rise time and amplitude and parameters of the amplitude–period effect (a) and (b) (associated with the period of the preceding and the same cycle) for the SIDC data set are in good agreement with the results of Vitinskij et al [36] and Hathaway et al [13]. A similar conclusion can be drawn if we compare our analysis of the SIDC data set for the relation between rise rate and amplitude of the cycles with the analysis given by Vitinskij et al [36].

3. Results

The typical time–latitude diagrams for the dynamo models are shown in figures 2 and 3. The shape of the simulated sunspot cycles in the 1D1 model can be seen in the right panel of figure 3. The simulated sunspot cycles for the 2D1 and 2D2 models are shown in figure 5. We can conclude that the shape of the simulated sunspot cycles (and perhaps the associated Waldmeier relations) is directly related to the spatial shape of the toroidal magnetic field evaluational patterns. For example, in the 1D1 model the maximum of the butterfly diagram is very close to the equator and the butterfly wing is elongated toward the pole. In such a pattern of the toroidal magnetic field evolution, the decay phase of the sunspot activity is shorter than the rise phase. The opposite situation is in the models 2D1 and 2D2. The physical mechanisms that produce the short rise and long decay of the toroidal magnetic field activity were discussed recently by Pipin and Kosovichev [30].

To proceed further, we would like to discuss the statistical properties of the cycle parameters involved in the Waldmeier relations. The 1D models have much lower cycle period than the diffusive time of the system. Therefore, we scale the periods of these models by a factor \( \sim 50 \). Table 2 shows the
results for the mean and variance (standard deviations) for the period, amplitude, rise rate and shape of the sunspot cycles in different data sets. From that table we see that the 1D1 model has smaller variance in the period, amplitude and rise rate of the cycles as compared with the other data sets. The shape asymmetry of the cycles in 1D1 is opposite to the other cases as well. Also, we can see that the mechanism of helicity loss in the dynamo model influences the mean and variance of the sunspot cycle parameters. In particular, the model 2D2 with increased diffusive loss of the magnetic helicity has lower variance of the period and amplitude of the sunspot cycles and has a more symmetric shape of the cycle as compared to the model 2D1. The difference in the synthetic data set of the sunspot cycles provided by NIMV as compared with SIDC is likely due to the fact that the SIDC data set does not cover the periods with low magnetic activity. This argument also applies if we compare NIMV and, e.g., the 2D1 model. The parameters of the 2D1 model do not allow us to have extended periods of time with very low sunspot cycles.

The difference between the statistical properties of the given data sets can be seen in more detail using the cumulative distribution probability functions. The cumulative distributions are constructed as follows. At the beginning, we sort each distribution for each parameter and each model in increasing order. After this, we compute the following:

$$\text{CDF}(P_i) = \frac{\sum_{k=1}^{i} k}{N},$$

where $P_i$ is the parameter under consideration (say, the cycle period) having the order number $i$ (after sorting the set in increasing order) and $N$ is the total number of instances of the given parameter in the set. Equation (9) approximates the probability for the parameter $P$ to have values in the interval between $P_{\text{min}}$ and $P_i$. The accuracy of the approximation improves as $N \to \infty$. We will use the log-normal cumulative distribution constructed on the basis of the SIDC data set as the reference distribution. The SIDC data set has only 23 instances of sunspot cycles. To construct the

| Parameters  | 1D1      | 2D1      | 2D2      | SIDC     | NIMV (2004) |
|------------|----------|----------|----------|----------|-------------|
| Period     | 11.02 ± 0.66 | 11.07 ± 1.08 | 10.97 ± 0.92 | 11.01 ± 1.12 | 11.02 ± 1.49 |
| Amplitude  | 115.7 ± 33.6 | 103.3 ± 40.5 | 96.3 ± 25.7 | 108.2 ± 38.1 | 87.6 ± 43.9 |
| Rise rate  | 18.62 ± 6.14 | 25.39 ± 11.95 | 19.91 ± 5.95 | 25.81 ± 12.74 | 19.48 ± 13.38 |
| Rise time  | 6.11 ± .33 | 4.06 ± .77 | 4.73 ± .36 | 4.32 ± .107 | 4.82 ± 1.32 |
| Shape      | 1.27 ± 0.2 | 0.59 ± 0.15 | 0.77 ± 0.08 | 0.69 ± 0.31 | 0.83 ± 0.34 |
| Rise rate–amplitude | 5.4x ± 14.2 ± 3.0 | 3.3x ± 18.8 ± 7.6 | 4.2x ± 12.4 ± 5.6 | 2.9x ± 32.2 ± 8.9 | 3.1x ± 27.8 ± 15.7 |
| Period–amplitude(a) | -31.7x ± 463.9 ± 26.2 | -17.5x ± 298.0 ± 34.0 | -17.25x ± 2856 ± 20.3 | -23.6x ± 368.5 ± 28.0 | -8.4x ± 179.9 ± 42.0 |
| Period–amplitude(b) | -17.9x ± 312.3 ± 31.4 | -8.9x ± 202.9 ± 38.9 | -6.3x ± 165.4 ± 25.0 | -11.2x ± 231.7 ± 35.9 | -6.9x ± 163.4 ± 42.7 |
| Rise time–amplitude | -82.1x ± 617.4 ± 18.3 | -25.6x ± 207.5 ± 35.3 | -33.0x ± 252.8 ± 22.7 | -26.7x ± 234.2 ± 24.0 | -16.1x ± 165.4 ± 38.5 |
| Rise rate–decay rate | 1.0x ± 4.0 ± 3.1 | 0.45x ± 3.3 ± 2.2 | 0.68x ± 1.6 ± 1.6 | 0.34x ± 6.4 ± 2.6 | 0.42x ± 5.3 ± 4.1 |

**Figure 5.** The left panel shows cycle distributions for the model 2D1 and the right panel those for the model 2D2.
reference log-normal distribution we use the standard mean and variance of the cycle parameters (period, amplitude, rise rate and asymmetry) given in table 2. Then we take the natural logarithm of them and construct the log-normal distribution of the length 1000 using those means and variances. The results are shown in figure 6.

It is clearly seen that log-normal distribution is a good fit for the distributions of the sunspot cycle period in the SIDC data set and also for the model 2D2. We find that the probability distributions of the SIDC data set differ from the log-normal distribution for the rise rates and the shape of the cycles. It is, however, unclear if these differences are due to the limited data set of cycles covered by SIDC. The data set produced by the models and the NIMV data set can be equally well approximated by the log-normal distributions (with different means and variances). The difference between the distributions computed by equation (9) and the log-normal approximations for them is less clearly visible for the dynamo models, than for the SIDC and NIMV data sets.

Figure 7 shows the Waldmeier relations for the 1D1 and 2D1 models together with their linear fits and also fits for the SIDC and NIMV data sets. The parameters of the linear fits are summarized in table 2. It is seen that the model 2D1 well reproduces the SIDC data set, and the difference compared to the NIMV data is not very large. The correspondence of the 2D2 model to the SIDC and NIMV is not as good as for the 2D1 model. This can also be expected from the results presented in figure 6 and table 2. Finally, we can conclude that the 1D1 model shows only qualitative agreement with the relation between the rise rate and amplitude and the relation between the period and amplitude of the sunspot cycles.

4. Discussion and conclusions

In the paper, we have studied the possibility of reproducing the statistical relations of the sunspot activity cycle, such as the so-called Waldmeier relations, by means of the mean-field dynamo model with fluctuating \( \alpha \)-effect. The dynamo model includes the long-term fluctuations of the \( \alpha \)-effect. The dynamo models employ two types of nonlinear feedback of the mean field on the \( \alpha \)-effect, including algebraic quenching and dynamic quenching due to the magnetic helicity generation. This paper presents the results for three particular dynamo models.

The presented 1D model is similar to the model discussed by Moss et al [25]. It uses the constant shear and the algebraic quenching of the \( \alpha \)-effect. The results for this model disagree with observations (SIDC data set) about the shape of the simulated sunspot (decay rate is higher than rise rate) even though they qualitatively reproduce the basic Waldmeier relations for the rise rate–amplitude and the cycle period–amplitude (see the left column in figure 7). It was found that the variance of the cycle parameters in the long-term evolution is less than that in 2D models. It is interesting that, under the level of noise, 1D models involving magnetic helicity show smaller mean despite having stronger variances of the simulated sunspot parameters. Although we could scale the mean parameters of those models to the observational values, we have not presented the results for these models because they have Waldmeier relations that are quantitatively the same as those presented for the 1D1 model in table 2 and figure 7.

We checked the 1D models with the spatially variable shear just like that suggested by Kitchatinov et al [17].
In agreement with the helioseismology results, the given 1D models have realistic latitudinal profile of shear (see equation (6)). Although these models qualitatively reproduce the relation between the rise rate and the amplitude of the cycle, they fail with the other kinds of relations, having positive correlation between the period and amplitude of the cycle and equal rate for the rise and decay phases of the simulated sunspot cycles.

Similar to the 1D cases, the magnetic helicity contribution to the \( \alpha \)-effect results in a decrease of the toroidal magnetic field strength and to the growth of the variance of the cycle parameters in the long-term evolution of the magnetic activity. The strong variance of the cycle parameters is expected from the SIDC data set and from NIMV as well. For this reason, in the paper we discuss the 2D model which involves the effect of magnetic helicity. The 2D models employ two different descriptions of the magnetic helicity loss, to overcome the problem of the \( \alpha \)-effect catastrophic quenching. The term \(-\frac{\chi}{R_c \tau_c}\) in equation (3) describes the magnetic helicity loss with the dissipation rate \((\tau_c R_c)^{-1}\) without specifying the nature of the loss. Note that \(\tau_c\) is varied from about 2 months near the bottom of the convection zone to a few hours at the top of the integration domain (which is 0.99 \(R_\odot\)). Thus, for the \(R_c = 200\) used in the model 2D1, the typical decay time for the magnetic helicity is varied from about four solar cycles at the bottom of the convection zone to a time that is less than 1 month at the top of the convection zone. It is not clear if this simple description is a satisfactory approximation for the magnetic helicity loss. Therefore we checked the alternative possibility using the diffusive helicity flux. Although the model that employs the diffusive helicity flux is in satisfactory agreement with SIDC data, the correspondence to observation in this model is not as good as for the model 2D1. We find the variance of the cycle parameters in the model 2D2 is less than in the model 2D1, while the SIDC and NIMV data sets show higher variances than the model 2D1.

A detailed comparison of the results of our models with those given by Karak and Choudhuri [9] is not possible, because we have used a different definition for the amplitude of the cycle and the rise time. They did not give the results for the linear fit coefficients and only provided the correlation coefficients in the Waldmeier relations involving the rise rate–amplitude and the rise time–amplitude of

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**Figure 7.** The Waldmeier relations for the 1D1 (left) and 2D1 (right) models. The linear fits are shown by the solid lines, the dashed lines show the fits for the SIDC data and the dash-dot lines show the fit for the NIMV data.
the cycle. Bearing in mind the differences in definition, their ‘high-diffusivity model’ with fluctuating meridional circulation is comparable with our 2D1 and 2D2 models. It is unclear, however, what is the typical shape of the cycle in their model. This is an important issue as we have seen in the example given by the model 1D1. It has qualitative agreement with SIDC data about the period–amplitude and the rise rate–amplitude relations despite having the rise time of the cycle greater than the decay time.

In the models under consideration, the asymmetry between the ascent and descent phases of the sunspot cycle is inherent from the pattern of the toroidal magnetic field activity. In particular, the 1D1 model has the toroidal magnetic field butterfly diagram with a maximum located very close to the equator. Therefore, applying the definition (8) for this type of toroidal magnetic field evolutional pattern, we obtain the descent phase of the sunspot activity shorter than the ascent phase. The opposite situation is in 2D models. There, we relate the sunspot activity to the toroidal field in the ascent phase. The opposite situation is in 2D models. In particular, 1D models fail to reproduce the asymmetric shape of the sunspot cycle with short rise and long decay phases. The statistical distributions of the cycle parameters show the log-normal probability distributions for all the data sets analysed in this paper. The parameters of these distributions are different for all data sets. Again the 1D model is significantly different from the others in this sense. The 2D model that employs the simplest form of the helicity loss via the term \(-\chi/R\tau\) agrees well with the SIDC, even though the long-term variation in this model is not intermittent enough, and this seems to be the reason for its differing from the NIMV data set in some aspects. The use of diffusive loss in the magnetic helicity evolution equation results in a decrease in the variation of the cycle parameters. Further studies on the magnetic helicity transport mechanisms should help clarify the likely candidates that are responsible for the magnetic helicity loss from the dynamo region. We have seen that analysis of the statistical relations of the sunspot cycle may offer a valuable diagnostic tool for this study.

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Appendix

We describe some aspects of the mean electromotive force. The basic formulation is given in P08. For this paper, we reformulate the tensor \(a_{i,j}^{\text{Mf}}\), which represents the hydrodynamical part of the \(\alpha\)-effect, by using equation (23) from P08 in the following form:

\[
\begin{align*}
\alpha_{i,j}^{\text{Mf}} &= \delta_{ij} \left[ 3\eta_{i} (f_{i}^{(a)} \cdot \Lambda^{(a)}) + f_{i}^{(a)} (e \cdot \Lambda^{(a)}) \right] \\
&+ e_{i} e_{j} \left[ 3\eta_{i} (f_{i}^{(a)} \cdot \Lambda^{(a)}) + f_{i}^{(a)} (e \cdot \Lambda^{(a)}) \right] \\
&+ 3\eta_{i} (e_{i} \Lambda_{j}^{(a)} + e_{j} \Lambda_{i}^{(a)}) f_{6}^{(a)} + (e_{i} \Lambda_{j}^{(a)} + e_{j} \Lambda_{i}^{(a)}) f_{8}^{(a)}. 
\end{align*}
\]  
(A.1)

The contribution of magnetic helicity \(\chi = a \cdot b\) (a is a fluctuating vector magnetic field potential) to the \(\alpha\)-effect is defined as \(a_{i,j}^{\text{Mf}} = C_{ij}^{(\alpha)} \chi\), where

\[
C_{ij}^{(\alpha)} = 2 f_{i}^{(a)} \delta_{i,j} \frac{\tau_{c}}{\mu_{0} \rho^{2}} e_{i} e_{j} \frac{\tau_{c}}{\mu_{0} \rho^{2}}.
\]  
(A.2)

The turbulent pumping, \(\gamma_{ij}\), is also part of the mean electromotive force in equation (1) and it is given by

\[
\gamma_{ij} = 3\eta_{i} (f_{3}^{(a)} \Lambda_{j}^{(a)} + f_{1}^{(a)} (e \cdot \Lambda^{(a)}) e_{a} \xi_{naj} - 3\eta_{i} f_{1}^{(a)} e_{j} \xi_{naj} \Lambda_{m}^{(a)}).
\]  
(A.3)

The effect of turbulent diffusivity, which is anisotropic due to the Coriolis force, is given by

\[
\eta_{ijk} = 3\eta_{i} (2 f_{1}^{(a)} - f_{1}^{(d)}) e_{i} e_{j} e_{k} - 2 f_{1}^{(a)} e_{i} e_{a} e_{nj} \{. \}
\]  
(A.4)

Functions \(f_{1}^{(a,d)}\) depend on the Coriolis number \(\mathcal{Q}^{2} = 2 \tau_{c} \Omega_{0}\) and the typical convective turnover time in the mixing-length
approximation: $\tau_c = \ell / u'$. They can be found in P08. The turbulent diffusivity is parameterized in the form $\eta_T = C_\eta \eta_T^{(0)}$, where $\eta_T^{(0)} = u'/T$ is the characteristic mixing-length turbulent diffusivity, $u'$ is the rms convective velocity, $\ell$ is the mixing length and $C_\eta$ is a constant to control the intensity of turbulent mixing. The other quantities in equations (A.1), (A.3) and (A.4) are as follows: $\Lambda^{(\alpha)} = \nabla \log \bar{T}$ is the density stratification scale, $\Lambda^{(\beta)} = \nabla \log (\eta_T^{(0)})$ is the scale of turbulent diffusivity and $\epsilon = \Omega / |\Omega|$ is a unit vector along the axis of rotation. Equations (A.1), (A.3) and (A.4) take into account the influence of the fluctuating small-scale magnetic fields, which can be present in the background turbulence and stem from the small-scale dynamo (see discussions in [6, 19]). In the present paper, the parameter $\epsilon = \frac{\Omega_T}{|\Omega_T|}$, which measures the ratio between the magnetic and kinetic energies of fluctuations in the background turbulence, is assumed to be equal to 1. This corresponds to the energy equipartition. The quenching function of the hydrodynamical part of the $\alpha$-effect is defined by

$$\psi_a = \frac{5}{128 \beta^2} \left( 16 \beta^2 - 3 - 3(4 \beta^2 - 1) \frac{\arctan (2 \beta)}{2 \beta} \right).$$

(A.5)

Note that, in the notation of P08, $\psi_a = -3/4 \psi_0^{(\alpha)}$ and $\beta = \frac{\nabla \log \bar{T}}{u' \sqrt{\mu n}}$.

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