Madsen-Weiss for geometrically minded topologists

Yakov Eliashberg ∗  Søren Galatius †
Stanford University     Stanford University
USA         USA

Nikolai Mishachev ‡
Lipetsk Technical University
Russia

To D.B. Fuchs on his 70th birthday

July 27, 2009

Abstract

We give an alternative proof of Madsen-Weiss’ generalized Mumford conjecture. Our proof is based on ideas similar to Madsen-Weiss’ original proof, but it is more geometrical and less homotopy theoretical in nature. At the heart of the argument is a geometric version of Harer stability, which we formulate as a theorem about folded maps.

Contents

1 Introduction and statement of results 3
  1.1 Diffeomorphism groups and mapping class groups  . . . . . . . . 4
  1.2 A Thom spectrum                          6
  1.3 Oriented bordism                          10

∗Partially supported by NSF grants DMS-0707103 and DMS 0244663
†Partially supported by NSF grant DMS-0805843 and Clay Research Fellowship
‡Partially supported by NSF grant DMS 0244663
1.4 Harer stability .......................................................... 11
1.5 Outline of proof ......................................................... 13
  1.5.1 From formal fibrations to folded maps ......................... 13
  1.5.2 Getting rid of elliptic folds .................................... 14
  1.5.3 Generalized Harer stability theorem ............................ 14
  1.5.4 Getting rid of hyperbolic folds ................................. 14
  1.5.5 Organization of the paper ....................................... 15

2 Folded maps ............................................................. 16
  2.1 Folds ...................................................................... 16
  2.2 Double folds and special folded mappings .......................... 17
  2.3 Enriched folded maps and their suspensions ....................... 21
  2.4 Fold surgery ........................................................... 26
    2.4.1 Surgery template ................................................ 26
    2.4.2 Surgery ........................................................... 30
    2.4.3 Bases for fold surgeries ........................................ 33
    2.4.4 The case $d = 2$ ............................................... 36
  2.5 Destabilization ....................................................... 37
  2.6 From formal epimorphisms to enriched folded maps .......... 43

3 Cobordisms of folded maps ............................................. 43

4 Generalized Harer stability theorem ................................ 45
  4.1 Harer stability for enriched folded maps .......................... 46
  4.2 Nodal surfaces and their unfolding ................................. 47
  4.3 Proof of Theorem 4.1 ............................................... 51

5 Proof of Theorem 1.8 ..................................................... 52
  5.1 Proof of Proposition 3.4 ............................................ 53
  5.2 Proof of Proposition 3.5 ............................................ 55

6 Miscellaneous ............................................................. 56
  6.1 Appendix A: From wrinkles to double folds ..................... 56
    6.1.1 Cusps .......................................................... 56
    6.1.2 Wrinkles and wrinkled mappings .............................. 56
    6.1.3 Cusp eliminating surgery ...................................... 56
    6.1.4 From wrinkles to double folds .............................. 61
  6.2 Appendix B: Hurewicz theorem for oriented bordism ........... 63
1 Introduction and statement of results

Our main theorem gives a relation between fibrations (or surface bundles) and a related notion of formal fibrations. By a fibration we shall mean a smooth map \( f : M \to X \), where \( M \) and \( X \) are smooth, oriented, compact manifolds and \( f \) is a submersion (i.e. \( df : TM \to f^*TX \) is surjective). A cobordism between two fibrations \( f_0 : M_0 \to X_0 \) and \( f_1 : M_1 \to X_1 \) is a triple \((W,Y,F)\) where \( W \) is a cobordism from \( M_0 \) to \( M_1 \), \( Y \) is a cobordism from \( X_0 \) to \( X_1 \), and \( F : W \to Y \) is a submersion which extends \( f_0 \amalg f_1 \).

Definition 1.1. (i) An unstable formal fibration is a pair \((f,\varphi)\), where \( f : M \to X \) is a smooth proper map, and \( \varphi : TM \to f^*TX \) is a bundle epimorphism.

(ii) A stable formal fibration (henceforth just a formal fibration) is a pair \((f,\varphi)\), where \( f \) is as before, but \( \varphi \) is defined only as a stable bundle map. Thus for large enough \( j \) there is given an epimorphism \( \varphi : TM \oplus \epsilon^j \to TX \oplus \epsilon^1 \), and we identify \( \varphi \) with its stabilization \( \varphi \oplus \epsilon^1 \). A cobordism between formal fibrations \((f_0,\varphi_0)\) and \((f_1,\varphi_1)\) is a quadruple \((W,Y,F,\Phi)\) which restricts to \((f_0,\varphi_0) \amalg (f_1,\varphi_1)\).

(iii) The formal fibration induced by a fibration \( f : M \to X \) is the pair \((f,df)\), and a formal fibration is integrable if it is of this form.

Our main theorem relates the set of cobordism classes of fibrations with the set of cobordism classes of formal fibrations. Let us first discuss the stabilization process (or more precisely “stabilization with respect to genus”. This should not be confused with the use of “stable” in “stable formal fibration”. In the former, “stabilization” refers to increasing genus; in the latter it refers to increasing the dimension of vector bundles.) Suppose \( f : M \to X \) is a formal fibration (we will often suppress the bundle epimorphism \( \varphi \) from the notation) and \( j : X \times D^2 \to M \) is an embedding over \( X \) (i.e. \( f \circ j = \text{Id} : X \to X \)), such that \( f \) is integrable on the image of \( j \). Then we can stabilize \( f \) by taking the fiberwise connected sum of \( M \) with \( X \times T \) (along \( j \)), where \( T = S^1 \times S^1 \) is the torus. If \( f \) happens to be integrable, this process increases the genus of each fiber by 1.

Our main theorem is the following.

Theorem 1.2. Let \( f : M \to X \) be a formal fibration which is integrable over the image of a fiberwise embedding \( j : X \times D^2 \to M \). Then, after possibly
stabilizing a finite number of times, $f$ is cobordant to an integrable fibration with connected fibers.

We will also prove a relative version of the theorem. Namely, if $X$ has boundary and $f$ is already integrable, with connected fibers, over a neighborhood of $\partial X$, then the cobordism in the theorem can be also be assumed integrable over a neighborhood of the boundary.

Theorem 1.2 is equivalent to Madsen-Weiss’ “generalized Mumford conjecture” [MW07] which states that a certain map

$$Z \times B\Gamma_\infty \to \Omega^\infty CP_{-1}^{-1}$$

induces an isomorphism in integral homology. In the rest of this introduction we will explain the equivalence and introduce the methods that go into our proof of Theorem 1.2. Our proof is somewhat similar in ideas to the original proof, but quite different in details and language. The general scheme of reduction of some algebro-topological problem to a problem of existence of bordisms between formal and genuine (integrable) fibrations was first proposed by D. B. Fuchs in [Fu74]. That one might prove Madsen-Weiss’ theorem in the form of Theorem 1.2 was also suggested by I. Madsen and U. Tillmann in [MT01].

1.1 Diffeomorphism groups and mapping class groups

Let $F$ be a compact oriented surface, possibly with boundary $\partial F = S$. Let $\text{Diff}(F)$ denote the topological group of diffeomorphisms of $F$ which restrict to the identity on the boundary. The classifying space $B\text{Diff}(F)$ can be defined as the orbit space

$$B\text{Diff}(F) = \text{Emb}(F, \mathbb{R}^\infty)/\text{Diff}(F),$$

and it is a classifying space for fibrations: for a manifold $X$, there is a natural bijection between isomorphism classes of smooth surface bundles $E \to X$ with fiber $F$ and trivialized boundary $\partial E = X \times S$, and homotopy classes of maps $X \to B\text{Diff}(F)$.

The mapping class group is defined as $\Gamma(F) = \pi_0\text{Diff}(F)$, i.e. the group of isotopy classes of diffeomorphisms of the surface. It is known that the identity component of $\text{Diff}(F)$ is contractible (as long as $g \geq 2$ or $\partial F \neq \emptyset$), so $B\text{Diff}(F)$ is also a classifying space for $\Gamma(F)$ (i.e. an Eilenberg-Mac Lane
space $K(\Gamma(F), 1))$. When $\partial F = \emptyset$, this is also related to the moduli space of Riemann surfaces (i.e. the space of isomorphism classes of Riemann surfaces of genus $g$) via a map

$$(2) \quad B\text{Diff}(F) \to \mathcal{M}_g$$

which induces an isomorphism in rational homology and cohomology. Mumford defined characteristic classes $\kappa_i \in H^{2i}(\mathcal{M}_g; \mathbb{Q})$ for $i \geq 1$ and conjectured that the resulting map

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \to H^*(\mathcal{M}_g; \mathbb{Q})$$

is an isomorphism in degrees less than $(g-1)/2$. This is the original Mumford Conjecture.

It is convenient to take the limit $g \to \infty$. Geometrically, that can be interpreted as follows. Pick a surface $F_{g,1}$ of genus $g$ and with one boundary component. Also pick an inclusion $F_{g,1} \to F_{g+1,1}$. Let $T_\infty$ be the union of the $F_{g,1}$ over all $g$, i.e. a countably infinite connected sum of tori. We will consider fibrations $E \to X$ with trivialized “$T_\infty$ ends”.

![Figure 1: Surface with $T_\infty$-end](image)

We are considering pairs $(f, j)$ where $f : E \to X$ is a smooth fiber bundle with fiber $T_\infty$, and

$$(3) \quad j : X \times T_\infty \leadsto E$$

is a germ at infinity of an embedding over $X$. This means, for $X$ compact, that representatives of $j$ are defined on the complement of some compact set.
in \( X \times T_\infty \), and their images contain the complement of some compact set in \( E \).

Let us describe a classifying space for fibrations with \( T_\infty \) ends. Let \( B\Gamma_\infty \) be the the mapping telescope (alias homotopy colimit) of the direct system

\[
BDiff(F_0,1) \to BDiff(F_{1,1}) \to BDiff(F_{2,1}) \to \ldots.
\]

**Lemma 1.3.** \( \mathbb{Z} \times B\Gamma_\infty \) is a classifying space for fibrations with \( T_\infty \) ends.

**Proof.** Any compact \( K \subset T_\infty \) is contained in \( F_{g,1} \subset T_\infty \) for some finite \( g \). Let \( T^g_\infty \subset T_\infty \) be the complement of \( F_{g,1} \). Let us consider for a moment fibrations with fiber \( T_\infty \) and an embedding as in (3), but which is actually defined on \( T^k_\infty \). Call such bundles \( k \)-trivialized. Specifying a \( k \)-trivialized bundle is the same thing as specifying a fibration \( E' \to X \) with connected, compact fibers, and trivialized boundary \( \partial E' = X \times S^1 \) (namely \( E' \) is the complement of the image of \( j \)). Thus, the disjoint union

\[
B = \bigsqcup_g BDiff(F_{g,1})
\]

is a classifying space for \( k \)-trivialized bundles (notice \( B \) is independent of \( k \)). A \( k \)-trivialized bundle is also a \( (k+1) \)-trivialized bundle, and increasing \( k \) is represented by a “stabilization” self-map \( s : B \to B \). In the representation (5), \( s \) maps \( BDiff(F_{g,1}) \) to \( BDiff(F_{g+1,1}) \) and this is induced by the same map as in (4).

Now the statement follows by taking the direct limit as \( k \to \infty \), and noticing that \( \mathbb{Z} \times B\Gamma_\infty \) is the homotopy colimit of the direct system

\[
B \xrightarrow{s} B \xrightarrow{s} \ldots.
\]

\[\blacksquare\]

### 1.2 A Thom spectrum

In this section we define a space \( \Omega^\infty CP_{\infty}^{-1} \) and interpret it as a classifying space for formal fibrations. The forgetful functor from fibrations to formal fibrations is represented by a map

\[
BDiff(F) \to \Omega^\infty CP_{-1}^{-1}.
\]
We then consider the same situation, but where fibrations and formal fibrations have \( T_\infty \) ends. This changes the source of the map (6) to \( \mathbb{Z} \times B\Gamma_\infty \), but turns out to not change the homotopy type of the target. We get a map

\[
\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbb{C} P_{-1}^\infty,
\]

representing the forgetful functor from fibrations with \( T_\infty \) ends to formal fibrations with \( T_\infty \) ends.

The space \( \Omega^\infty \mathbb{C} P_{-1}^\infty \) is defined as the Thom spectrum of the negative of the canonical complex line bundle over \( \mathbb{C} P^\infty \). In more detail, let \( \text{Gr}_2^+(\mathbb{R}^N) \) be the Grassmannian manifold of oriented 2-planes in \( \mathbb{R}^N \). It supports a canonical 2-dimensional vector bundle \( \gamma^N \) with an \((N-2)\)-dimensional complement \( \gamma_N^\perp \) such that \( \gamma_N^\perp \oplus \gamma_N^\perp = \epsilon_N \). There is a canonical identification

\[
\gamma_N^\perp \mid_{\text{Gr}_2^+(\mathbb{R}^N)} = \gamma_N^\perp \oplus \epsilon^1.
\]

The Thom space \( \text{Th}(\gamma_N^\perp) \) is defined as the one-point compactification of the total space of \( \gamma_N^\perp \), and the identification (7) induces a map \( S^1 \wedge \text{Th}(\gamma_N^\perp) \to \text{Th}(\gamma_{N+1}^\perp) \). The space \( \Omega^\infty \mathbb{C} P_{-1}^\infty \) is defined as the direct limit

\[
\Omega^\infty \mathbb{C} P_{-1}^\infty = \lim_{N \to \infty} \Omega^N \text{Th}(\gamma_N^\perp).
\]

Like we did for \( B\text{Diff}(F) \), we shall think of \( \Omega^\infty \mathbb{C} P_{-1}^\infty \) as a classifying space, i.e. interpret homotopy classes of maps \( X \to \Omega^\infty \mathbb{C} P_{-1}^\infty \) from a smooth manifold \( X \) in terms of certain geometric objects over \( X \). Recall the notion of formal fibration from Definition 1.1 above. A cobordism \((W, Y, F, \Phi)\) of formal fibrations is a concordance if the target cobordism is a cylinder: \( Y = X \times [0, 1] \).

**Lemma 1.4.** There is a natural bijection between set

\[
[X, \Omega^\infty \mathbb{C} P_{-1}^\infty]
\]

of homotopy classes of maps, and the set of concordance classes of formal fibrations over \( X \).

**Proof sketch.** This is the standard argument of Pontryagin-Thom theory. In one direction, given a map \( X \to \Omega^N \text{Th}(\gamma_N^\perp) \), one makes the adjoint map \( g : X \times \mathbb{R}^N \to \text{Th}(\gamma_N^\perp) \) transverse to the zero section of \( \gamma_N^\perp \) and sets \( M = g^{-1}(\text{zero-section}) \). Then \( M \) comes with a map \( c : M \to \text{Gr}_2^+(\mathbb{R}^N) \) and the normal bundle of \( M \subset X \times \mathbb{R}^N \) is \( c^*(\gamma_N^\perp) \). This gives a stable isomorphism \( TM \cong_{st} TX \oplus c^*(\gamma_N) \) and hence a stable epimorphism \( TM \to TX \).
In the other direction, given a formal fibration \((f, \varphi)\) with \(f : M \to X\), we pick an embedding \(M \subset X \times \mathbb{R}^N\). Letting \(\nu\) be the normal bundle of this embedding, we get a “collapse” map

\[
X_+ \wedge S^N \to \text{Th}(\nu).
\]

We also get an isomorphism of vector bundles over \(M\)

\[
TM \oplus \nu \cong f^*TX \oplus \epsilon^N.
\]

Let \(\xi : M \to \text{Gr}_2^+(\mathbb{R}^N)\) be a classifying map for the kernel of the stable epimorphism \(\varphi : TM \to TX\) (this is a two-dimensional vector bundle with orientation induced by the orientations of \(X\) and \(M\)), so we have a stable isomorphism \(TM \cong_{st} TX \oplus \xi^*\gamma_N\). Combining with \((9)\) we get a stable isomorphism \(\xi^*\gamma_N \oplus \nu \cong_{st} \epsilon^N\). By adding \(\xi^*(\gamma_N^\perp)\) we get a stable isomorphism \(\nu \cong_{st} \xi^*(\gamma_N^\perp) \oplus \epsilon^N\) which we can assume is induced by an unstable isomorphism (since we can assume \(N \gg \dim M\))

\[
\nu \cong \xi^*\gamma_N^\perp.
\]

This gives a proper map \(\nu \to \gamma_N^\perp\) and hence a map of Thom spaces \(\text{Th}(\nu) \to \text{Th}(\gamma_N^\perp)\). Compose with \((8)\) and take the adjoint to get a map \(X \to \Omega^N \text{Th}(\gamma_N^\perp)\). Finally let \(N \to \infty\) to get a map \(X \to \Omega^\infty \mathbb{C}P^\infty_{-1}\).

The homotopy class of the resulting map \(X \to \Omega^\infty \mathbb{C}P^\infty_{-1}\) is well defined and depends only on the concordance class of the formal fibration \(f : M \to X\).

A fibration naturally gives rise to a formal fibration, and this association gives rise to a map of classifying spaces which is the map \((2)\). We would like to make this process compatible with the stabilization procedure explained in section \([1,1]\) above. To this end we consider formal fibrations with \(k\)-trivialized \(T_\infty\) ends. This means that \(f : M \to X\) is equipped with an embedding over \(X\)

\[
j : X \times T_\infty^k \to M,
\]

and that \((f, \varphi)\) is integrable on the image of \(j\). Of course, we also replace the requirement that \(M\) be compact by the requirement that the complement of the image of \(j\) be compact.
Lemma 1.5. Formal fibrations with $k$-trivialized ends are represented by the space $\Omega^\infty \mathbb{CP}^{-1}_\infty$.

Proof sketch. This is similar to the proof of Lemma 1.4 above. Applying the Pontryagin-Thom construction from the proof of that lemma to the projection $X \times T^k_\infty \to X$ gives a path $\alpha_0 : [k, \infty) \to \Omega^{N-1} \text{Th}(\gamma_N^\perp)$. Applying the Pontryagin-Thom construction to an arbitrary $k$-trivialized formal fibration gives a path $\alpha : [0, \infty) \to \Omega^{N-1} \text{Th}(\gamma_N^\perp)$ whose restriction to $[k, \infty)$ is $\alpha_0$. The space of all such paths is homotopy equivalent to the loop space $\Omega^N \text{Th}(\gamma_N^\perp)$.

Increasing $k$ gives a diagram of stabilization maps

$$
\begin{array}{ccc}
\bigsqcup_g B\text{Diff}(F_{g,1}) & \longrightarrow & \Omega^\infty \mathbb{CP}^{-1}_\infty \\
\downarrow s & & \downarrow s \\
\bigsqcup_g B\text{Diff}(F_{g,1}) & \longrightarrow & \Omega^\infty \mathbb{CP}^{-1}_\infty.
\end{array}
$$

On the right hand “formal” side, the stabilization map is up to homotopy described as multiplication by a fixed element (multiplication in the loop space structure. See the proof of Lemma 1.5.) In particular it is a homotopy equivalence, so the direct limit has the same homotopy type $\Omega^\infty \mathbb{CP}^{-1}_\infty$. Taking the direct limit we get the desired map

$$
\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbb{CP}^{-1}_\infty.
$$

This is the map 1. The target of this map should be thought of as the homotopy direct limit of the system

$$
\Omega^\infty \mathbb{CP}^{-1}_\infty \xrightarrow{s} \Omega^\infty \mathbb{CP}^{-1}_\infty \xrightarrow{s} \ldots
$$

and as a classifying space for formal fibrations with $T_\infty$ ends. (In some sense it is a “coincidence” that the classifying space for formal fibrations and the classifying space for formal fibrations with $T_\infty$ ends have the same homotopy type).
1.3 Oriented bordism

For a pair \((X,A)\) of spaces, oriented bordism \(\Omega_n(X,A) = \Omega_n^{SO}(X,A)\) is defined as the set of bordism classes of continuous maps of pairs \(f : (M,\partial M) \to (X,A)\) for smooth oriented compact \(n\)-manifolds \(M\) with boundary \(\partial M\). To be precise, a bordism between two maps \(f_{\pm} : (M_\pm, \partial M_\pm) \to (X,A)\) is a map \(F : (W, \partial'W) \to (X,A)\), where \(W\) is a compact, oriented manifold with boundary with corners, so that \(\partial W = \partial_-W \cup \partial'W \cup \partial_+ W\), where \(\partial_-W = \partial M_\pm\) and \(\partial'W\) is a cobordism between closed manifolds \(\partial M_-\) and \(\partial M_+\), and the map \(F : (W, \partial'W) \to (X,A)\) such that \(F|_{\partial'W} = f_{\pm}\).

For a single space \(X\) set \(\Omega_n(X) = \Omega_n(X,\emptyset)\). Oriented bordism is a generalized homology theory. This means that it satisfies the usual Eilenberg-Steenrod axioms for homology (long exact sequence etc) except for the dimension axiom. In particular a map \(A \to X\) induces an isomorphism \(\Omega_*(A) \to \Omega_*(X)\) if and only if the relative groups \(\Omega_*(X,A)\) all vanish. The following result is well known. It follows easily from the Atiyah-Hirzebruch spectral sequence (for completeness we give a geometric proof in Appendix B).

**Lemma 1.6.** Let \(f : X \to Y\) be a continuous map of topological spaces. Then the following statements are equivalent.

(i) \(f_* : H_k(X) \to H_k(Y)\) is an isomorphism for \(k < n\) and an epimorphism for \(k = n\).

(ii) \(f_* : \Omega_k(X) \to \Omega_k(Y)\) is an isomorphism for \(k < n\) and an epimorphism for \(k = n\).

In particular, \(f\) induces an isomorphism in homology in all degrees if and only if it does so in oriented bordism.

We apply this to the pair \((\Omega_\infty \mathbb{C} P_1,\mathbb{Z} \times B\Gamma_\infty)\). Interpreting \(\mathbb{Z} \times B\Gamma_\infty\) and \(\Omega_\infty \mathbb{C} P_1\) as classifying spaces for fibrations, resp. formal fibrations, with \(T_\infty\) ends, we get the following interpretation.

**Lemma 1.7.** There is a natural bijection between the relative oriented bordism groups \(\Omega_*(\Omega_\infty \mathbb{C} P_1,\mathbb{Z} \times B\Gamma_\infty)\) and cobordism classes of formal fibrations \(f : M \to X\) with \(T_\infty\) ends. The formal fibration is required to be integrable over a neighborhood of \(\partial X\), and cobordisms \(F : W \to Y\) are required to be integrable over a neighborhood of \(\partial Y\).
That induces an isomorphism in integral homology (Madsen-Weiss’ theorem) is now, by Lemma 1.6, equivalent to the statement that the relative groups

\[ \Omega_\ast(\Omega^\infty CP_{-1}^\infty, \mathbb{Z} \times B\Gamma_\infty) \]

all vanish. By Lemma 1.7, this is equivalent to

**Theorem 1.8.** Any formal fibration \( f : M \to X \) with \( T_\infty \) ends is cobordant to an integrable one. If \( (f, \varphi) \) is already integrable over \( \partial X \), then the cobordism can be assumed integrable over \( \partial' \).

Theorem 1.8 is our main result. It is a geometric version of Madsen-Weiss’ theorem. It is obviously equivalent to Theorem 1.2 above (with its relative form).

### 1.4 Harer stability

J. Harer proved a homological stability theorem in [Ha85] which implies precise bounds on the number of stabilizations needed in Theorem 1.2. At the same time, it will be an important part of the proof of the same theorem (as it does in [MW07]).

Roughly it says that the homology of the mapping class group of a surface \( F \) is independent of the topological type \( F \), as long as the genus is high enough. The result was later improved by Ivanov ([Iv89, Iv93]) and then by Boldsen [Bo09]. We state the precise result.

Consider an inclusion \( F \to F' \) of compact, connected, oriented surfaces. Let \( S = \partial F' \), and let \( \Sigma \subset F' \) denote the complement of \( F \). Thus \( F' = F \cup_{\partial F} \Sigma \).

There is an induced map of classifying spaces

\[ \text{BDiff}(F) \to \text{BDiff}(F'). \]

A map \( f : X \to \text{BDiff}(F') \) classifies a fibration \( E \to X \) with fiber \( F' \) and boundary \( \partial E = X \times S \), where \( S = \partial F' \). Lifting it to a map into \( \text{BDiff}(F) \) amounts to extending the embedding \( X \times S \to E \) to an embedding

\[ X \times \Sigma \to E \]

over \( X \).
The most general form of Harer stability states that the map \((11)\) induces an isomorphism in \(H_k(-;\mathbb{Z})\) for \(k < 2(g - 1)/3\), where \(g\) is the genus of \(F\). Consequently, by Lemma 1.6, it induces an isomorphism in oriented bordism \(\Omega_n(-)\) for \(n < 2(g - 1)/3\) or, equivalently, the relative bordism group

\[\Omega_n(\text{BDiff}(F'), \text{BDiff}(F))\]

vanishes for \(n < 2(g - 1)/3\). Thus, Harer stability has the following very geometric interpretation: For any fibration \(f : E \to X\) with fiber \(F'\) and boundary \(\partial E = X \times S\), \(f\) is cobordant to a fibration \(f' : E' \to X'\) via a cobordism \(F : W \to M\) (which is a fibration with trivialized boundary \(M \times S\), which restricts to \(f \amalg f'\)) where the embedding \(X' \times S \to E'\) extends to an embedding

\[X' \times \Sigma \to E'\]

Moreover, this can be assumed compatible with any given extension \((\partial X) \times \Sigma \to E\) over the boundary of \(X\). Here we assume \(F' = F \cup_{\partial F} \Sigma\) as above, that \(F\) and \(F'\) are connected, and that \(F\) has large genus. If the fibration has \(T_\infty\) ends, the genus assumption is automatically satisfied, and we get the following corollary.

**Theorem 1.9** (Geometric form of Harer stability). Let \(\Sigma_1 \subset \Sigma_2\) be compact surfaces with boundary (not necessarily connected). Let \(f : M \to X\) be a fibration with \(T_\infty\) ends, and let

\[j : (\partial X \times \Sigma_2) \cup (X \times \Sigma_1) \to M\]

be a fiberwise embedding over \(X\), such that in each fiber the complement of its image is connected. Then, after possibly changing \(f : M \to X\) by a bordism which is the trivial bordism over \(\partial X\), the embedding \(j\) extends to an embedding of \(X \times \Sigma_2\).

Explicitly, the bordism in the theorem is a fibration \(F : W \to Y\) with \(T_\infty\) ends, where the boundary \(\partial Y\) is partitioned as \(\partial Y = X \cup X'\) with \(\partial X = \partial X' = X \cap X'\) and \(F|_X = f\). The extension of \(j\) is a fiberwise embedding

\[J : (Y \times \Sigma_1) \cup (X' \times \Sigma_2) \to W\]

over \(Y\).
1.5 Outline of proof

1.5.1 From formal fibrations to folded maps

Given a formal fibration \((f : M \to X, \varphi)\) with \(T_\infty\) ends, the overall aim is to get rid of all singularities of \(f\) after changing it via bordisms. Our first task will be to simplify the singularities of \(f\) as much as possible using only homotopies. The simplest generic singularities of a map \(f : M \to X\) are folds. The fold locus \(\Sigma(f)\) consists of points where the rank of \(f\) is equal to \(\dim X - 1\), while the restriction \(f|_{\Sigma(f)} : \Sigma(f) \to M\) is an immersion. In the case when \(\dim M = \dim X + 2 = n + 2\), which is the case we consider in this paper, we have \(\dim \Sigma(f) = n - 1\). A certain additional structure on folded maps, called an enrichment, allows one to define a homotopically canonical suspension, i.e. a bundle epimorphism \(\varphi = \varphi_f : TM \oplus \epsilon^1 \to TX \oplus \epsilon^1\), such that \((f, \varphi)\) is a formal fibration. The enrichment of a folded map \(f\) consists of

- an \(n\)-dimensional submanifold \(V \subset M\) such that \(\partial V = \Sigma(f)\), and the restriction of \(f\) to each connected component of \(V_i \subset V\) is an embedding \(\text{Int } V_i \to X\);
- a trivialization of the bundle \(\text{Ker} df|_{\text{Int } V}\) with a certain additional condition on the behavior of this trivialization on \(\partial V = \Sigma(f)\).
Of course, existence of an enrichment is a strong additional condition on the fold map. In Section 2.3, we explain how to associate to an enriched folded map \((f, \varepsilon)\) a formal fibration \((f, \mathcal{L}(f, \varepsilon))\), where \(\mathcal{L}(f, \varepsilon) : TM \oplus \varepsilon^1 \to TX \oplus \varepsilon^1\) is a bundle epimorphism associated to the enrichment \(\varepsilon\). The main result of Section 2 is Theorem 2.22, which proves that any formal fibration can be represented in this way (plus a corresponding relative statement). This is proved using the \(h\)-principle type result proven in [EM97]. Note that Theorem 2.22 is a variation of the main result from [El72] and can also be proven by the methods of that paper. Also in Section 2 we recall some basic facts about folds and other simple singularities of smooth maps, and discuss certain surgery constructions needed for the rest of the proof of Theorem 1.8. This part works independently of the codimension \(d = \dim N - \dim M\), and hence the exposition in this section is done for arbitrary \(d > 0\).

1.5.2 Getting rid of elliptic folds

Each fold component has an index which is well defined provided that the projection of the fold is co-oriented. Assuming this is done, folds in the case \(d = 2\) can be of index 0, 1, 2 and 3. We call folds of index 1, 2 hyperbolic and folds of index 0, 3 elliptic. It is generally impossible to get rid of elliptic folds by a homotopy of the map \(f\). However, it is easy to do so if one allows to change \(f\) to a bordant map \(\tilde{f} : \tilde{M} \to X\). This bordism trades each elliptic fold component by a parallel copy of a hyperbolic fold, see Figure 11 and Section 5.1 below. A similar argument allows one to make all fibers \(\tilde{f}^{-1}(x), x \in X\), connected (comp. [MW07]).

1.5.3 Generalized Harer stability theorem

A generalization of Harer’s stability theorem to enriched folded maps, see Theorem 4.1 below, plays an important role in our proof. In Section 4.3 below we deduce Theorem 4.1 from Harer’s Theorem 1.9 by induction over strata in the stratification of the image \(f(\Sigma) \subset X\) of the fold, according to multiplicity of self-intersections.

1.5.4 Getting rid of hyperbolic folds

Let \(f\) be an enriched folded map with hyperbolic folds and with connected fibers. Let \(C\) be one of the fold components and \(\overline{C} = f(C) \subset X\) its image. For the purpose of this introduction we will consider only the following special
case. First, we will assume that $C$ is homologically trivial. As we will see, when $\dim X > 1$ this will always be possible to arrange. In particular, $\overline{C}$ bounds a domain $U_C \subset X$. Next, we will assume that the fold $C$ has index 1 with respect to the outward coorientation of the boundary of the domain $U_C$. In other words, when the point $x \in X$ travels across $\overline{C}$ inside $U_C$ then one of the circle in the fiber $f^{-1}(x)$ collapses to a point, so locally the fiber gets disconnected to two discs, see Figure 3. The inverse index 1 surgery makes a connected sum of two discs at their centers. Note that on

![Figure 3: Fibers of an index 2 fold](image)

an open collar $\Omega = \partial U_C \times (0,1) \subset \text{Int} U_C$ along $C$ in $U_C$ there exists two sections $S_{\pm}: \Omega \to M$ such that the 0-sphere $\{S_-(x), S_+(x)\}$ is the “vanishing cycle”, for the index 1 surgery when $x$ travels across $\overline{C}$. Moreover, the enrichment structure ensures that the vertical bundle along these sections is trivial. If one could extend the sections $S_{\pm}$ to all of $U_C$ preserving all these properties, then the fiberwise index 1 surgery, attaching 1-handle along small discs surrounding $S_{\pm}(x)$ and $S_{\mp}(x)$, $x \in U_C$, would eliminate the fold $\overline{C}$. This is one of the fold surgeries described in detail in Section 2.4. Though such extensions $S_{\pm}(x): \text{Int} U_C \to M$ need not exist for our original folded map $f$, Harer stability theorem in the form 2.4 states that there is an enriched folded map $\tilde{f}: \tilde{M} \to \tilde{X}$, bordant to $f$, for which such sections do exist, and hence the fold $\overline{C}$ could be eliminated.

### 1.5.5 Organization of the paper

As already mentioned, Section 2 recalls basic definition and necessary results and constructions involving fold singularities. In Section 2.1 we define folded
maps. The goal of Section 2.2 is Theorem 2.3 which is an h-principle for constructing so-called special folded maps, whose folds are organized in pairs of spheres. This theorem is a reformulation of the Wrinkling Theorem from [EM97]. We deduce Theorem 2.3 from the Wrinkling Theorem in Appendix A. In Section 2.3 we define the notion of enrichment for folded maps and prove that an enriched folded map admits a homotopically canonical suspension and hence gives rise to a formal fibration. The rest of Section 2 will prove that any formal fibration is cobordant to one induced by an enriched folded map. Section 2.4 is devoted to fold surgery constructions which we use later in the proof of the main theorem. These are just fiberwise Morse surgeries, in the spirit of surgery of singularities techniques developed in [El72]. For further applications we need a version of Theorem 2.3 applicable to a slightly stronger version of formal fibrations \((f, \phi)\) when \(\phi: TM \to TX\) is a surjective map between the non-stabilized tangent bundle. In Section 2.5 we explain the modifications which are necessary in the stable case, and in Section 2.6 we formulate and prove Theorem 2.22 reducing formal fibrations to enriched folded maps.

In Section 3 we introduce several special bordism categories and formulate the two remaining steps of the proof: Proposition 3.4 which allows us to get rid of elliptic folds, and Proposition 3.5 which eliminates the remaining hyperbolic folds.

Section 4 is devoted to the proof of the Harer stability theorem for folded maps (Theorem 4.1). We conclude the proof of Theorem 1.8 in Section 5 by proving Proposition 3.4 in 5.1 and Proposition 3.5 in 5.2. In Section 6 we collect two Appendices. In Appendix A we deduce Theorem 2.3 from the Wrinkling Theorem from [EM97]. Appendix B is devoted to a geometric proof of Lemma 1.6.

## 2 Folded maps

### 2.1 Folds

Let \(M\) and \(X\) be smooth manifolds of dimension \(m = n + d\) and \(n\), respectively.\(^1\) For a smooth map \(f: M \to X\) we will denote by \(\Sigma(f)\) the set of its

\(^1\)For applications in this paper we will need only the case \(d = 2\).
singular points, i.e. 

\[ \Sigma(f) = \{ p \in M, \ \text{rank} \ d_pf < n \} . \]

A point \( p \in \Sigma(f) \) is called a \textit{fold} type singularity or a \textit{fold} of index \( k \) if near the point \( p \) the map \( f \) is equivalent to the map

\[ \mathbb{R}^{n-1} \times \mathbb{R}^{d+1} \to \mathbb{R}^{n-1} \times \mathbb{R}^1 \]

given by the formula

\[
(y, x) \mapsto \left( y, Q(x) = -\sum_{1}^{k} x_i^2 + \sum_{k+1}^{d+1} x_j^2 \right)
\]

where \( x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \) and \( y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} \). We will also denote \( x_- = (x_1, \ldots, x_k) \), \( x_+ = (k_{k+1}, \ldots, x_{d+1}) \) and write \( Q(x) = -|x_-|^2 + |x_+|^2 \). For \( M = \mathbb{R}^1 \) this is just a non-degenerate index \( k \) critical point of the function \( f : V \to \mathbb{R}^1 \). By a \textit{folded map} we will mean a map with only fold type singularities. Given \( y \in \mathbb{R}^{n-1} \) and an \( \epsilon > 0 \) we will call a \((k-1)\)-dimensional sphere \( y \times \{ Q(x) = -\epsilon, x_+ = 0 \} \subset \mathbb{R}^{n-1} \times \mathbb{R}^{d+1} \) the \textit{vanishing cycle} of the fold over the point \((y, -\epsilon)\).

The normal bundle of the image \( \overline{C} = f(C) \) of the fold is a real line bundle over \( C \). A \textit{coorientation} of \( \overline{C} \) is a trivialization of this line bundle. A choice of coorientation allows one to provide each fold component \( C \) with a well-defined index \( s \), which changes from \( s \) to \( d + 1 - s \) with a switch of the coorientation. The normal bundle of \( C \) is \( \text{Ker} \ df \), and the second derivatives of \( f \) gives an invariantly defined non-degenerate quadratic form \( d^2f : \text{Ker} \ df|_C \to \text{Coker} \ df \). Denote \( \text{Cone}_{\pm}(C) := \{ z \in \text{Ker} \ df ; \pm d^2f(z) > 0 \} \). There is a splitting

\[ \text{Ker} \ df|_C = \text{Ker}_{-}(C) \oplus \text{Ker}_{+}(C), \]

which is defined uniquely up to homotopy by the condition \( \text{Ker}_{\pm}(C) \setminus 0 \subset \text{Cone}_{\pm}(C) \).

### 2.2 Double folds and special folded mappings

Given an orientable \((n-1)\)-dimensional manifold \( C \), let us consider the map

\[
w_C(n+d,n,k) : C \times \mathbb{R}^1 \times \mathbb{R}^d \to C \times \mathbb{R}^1
\]
given by the formula

\[(y, z, x) \mapsto \left( \frac{y}{z}, z^3 - 3z - \sum_{1}^{k} x_i^2 + \sum_{k+1}^{d} x_j^2 \right),\]

where \( y \in C, \ z \in \mathbb{R}^1 \) and \( x \in \mathbb{R}^d \).

The singularity \( \Sigma(w_C(n + d, n, k)) \) consists of two copies of the manifold \( C \):

\( C \times S^0 \times 0 \subset C \times \mathbb{R}^1 \times \mathbb{R}^d \).

The manifold \( C \times 1 \) is a fold of index \( k \), while \( C \times \{-1\} \) is a fold of index \( k + 1 \), with respect to the coorientation of the folds in the image given by the vector field \( \frac{\partial}{\partial u} \) where \( u \) is the coordinate \( C \times \mathbb{R} \to \mathbb{R} \). It is important to notice that the restriction of the map \( w_C(n + d, n, k) \) to the annulus

\( A = C \times \text{Int} \ D^1 = C \times \text{Int} \ D^1 \times 0 \subset C \times \mathbb{R}^1 \times \mathbb{R}^d \)

is an embedding.

![Diagram](image)

Figure 4: The radial projection to the cylinder has a double fold along \( C = S^1 \)

Although the differential

\( dw_C(n + d, n, k) : T(C \times \mathbb{R}^1 \times \mathbb{R}^d) \to T(C \times \mathbb{R}^1) \)

degenerates at points of singularity \( \Sigma(w_C) \), it can be canonically regularized over \( \mathcal{O}_p(C \times D^1) \), an open neighborhood of the annulus \( C \times D^1 \). Namely, we can change the element \( 3(z^2 - 1) \) in the Jacobi matrix of \( w_C(n + d, n, k) \) to a positive function \( \gamma \), which coincides with \( 3(z^2 - 1) \) on \( \mathbb{R}^1 \setminus [-1 - \delta, 1 + \delta] \) for sufficiently small \( \delta \). The new bundle map

\( R(dw_C) : T(C \times \mathbb{R}^1 \times \mathbb{R}^d) \to T(C \times \mathbb{R}^1) \)
provides a homotopically canonical extension of the map
\[ dw_C : T(C \times \mathbb{R} \times \mathbb{R}^d \setminus \mathcal{O}_p (C \times D^1)) \to T(C \times \mathbb{R}) \]
to an epimorphism (fiberwise surjective bundle map)
\[ T(C \times \mathbb{R} \times \mathbb{R}^d) \to T(C \times \mathbb{R}) \]
We call \( \mathcal{R}(dw_C) \) the regularized differential of the map \( w_C(n + d, n, k) \).

A map \( f : U \to X \) defined on an open \( U \subset M \) is called a double \( C \)-fold of index \( k + \frac{1}{2} \) if it is equivalent to the restriction of the map \( w_C(n + d, n, k) \) to \( \mathcal{O}_p (C \times D^1) \). For instance, when \( X = \mathbb{R} \) and \( C \) is a point, a double \( C \)-fold is a Morse function given in a neighborhood of a gradient trajectory connecting two critical points of neighboring indices. In the case of general \( n \), a double \( C \)-fold is called a spherical double fold if \( C = S^{n-1} \).

It is always easy to create a double \( C \)-fold as the following lemma shows. This lemma is a parametric version of the creation of two new critical points of a Morse function.

**Lemma 2.1.** Given a submersion \( f : U \to X \) of a manifold \( U \), a closed submanifold \( C \subset U \) of dimension \( n - 1 \) such that \( f|_C : C \to X \) is an embedding with trivialized normal bundle, and a splitting \( K_- \oplus K_+ \) of the vertical bundle \( \text{Vert} = \text{Ker} \, df \) over \( \mathcal{O}_p C \), one can construct a map \( \tilde{f} : U \to X \) such that

- \( \tilde{f} \) coincides with \( f \) near \( \partial U \);
- in a neighborhood of \( C \) the map \( \tilde{f} \) has a double \( C \)-fold, i.e. it is equivalent to the map (14) restricted to \( \mathcal{O}_p (A = C \times D^1) \), where the frames
  \[ \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \right) \quad \text{and} \quad \left( \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_{d+1}} \right) \]
  along \( A \) provide the given trivializations of the bundles \( K_- \) and \( K_+ \);
- \( df \) and \( \mathcal{R}(d\tilde{f}) \) are homotopic via a homotopy of epimorphisms fixed near \( \partial U \).

**Proof.** There exists splittings \( U_1 = C \times [-2, 2] \times D^d \), where \( D^d \) is the unit \( d \)-disc, and \( U_2 = \overline{C} \times [-2, 2] \) of neighborhoods \( U_1 \supset C = C \times 0 \times 0 \) in \( U \) and \( U_2 \supset \overline{C} = f(C) \) in \( X \), such that the map \( f \) has the form
\[ C \times [-2, 2] \times [-1, 1]^d \ni (v, z, x) \mapsto (\overline{v} = f(v), z) \in \overline{C} \times [-2, 2]. \]
Consider a $C^\infty$ function $\lambda : [-2, 2] \rightarrow [-2, 2]$ which coincides with $z^3 - 3z$ on $[-1, 1]$, with $z$ near $\pm 2$, and such that $\pm 1$ are its only critical points. Take two cut-off $C^\infty$-functions $\alpha, \beta : \mathbb{R}^+ \rightarrow [0, 1]$ such that $\alpha = 1$ on $[0, \frac{1}{4}]$, $\alpha = 0$ on $[\frac{1}{2}, 1]$, $\beta = 1$ on $[0, \frac{1}{2}]$ and $\beta = 0$ on $[\frac{3}{4}, 1]$. Set $Q(x) = -\sum_1^k x_i^2 + \sum_1^d x_j^2$ and define first a map $\tilde{f} : U_1 \rightarrow U_2$ by the formula

$$\tilde{f}(v, z, x) = (\bar{v} = f(v), \alpha(|x|)\lambda(z) + (1 - \alpha(|x|))z + \beta(|x|)Q(x)),$$

and then extend it to $M$, being equal to $f$ outside $U_1 \subset U$. The regularized differential of a linear deformation between $f$ and $\tilde{f}$ provides the required homotopy between $df$ and $\mathcal{R}(\tilde{df})$.

**Remark 2.2.** It can be difficult to eliminate a double $C$-fold. Even in the case $n = 1$ this is one of the central problems of Morse theory.

A map $f : M \rightarrow X$ is called *special folded*, if there exist disjoint open subsets $U_1, \ldots, U_l \subset M$ such that the restriction $f|_{M \setminus V}$, $V = \bigcup_1^l U_i$, is a submersion (i.e. has rank equal $n$) and for each $i = 1, \ldots, l$ the restriction $f|_{U_i}$ is a spherical double fold. In addition, we require that the images of all fold components bound balls in $X$.

The singular locus $\Sigma(f)$ of a special folded map $f$ is a union of $(n - 1)$-dimensional double spheres $S^{n-1} \times S^0_{(i)} = \Sigma(f|_{U_i}) \subset U_i$. It is convenient to fix for each double sphere $S^{n-1} \times S^0_{(i)}$ the corresponding annulus $S^{n-1} \times D^1_{(i)}$ which spans them. Notice that although the restriction of $f$ to each annulus $S^{n-1} \times \text{Int} D^1_{(i)}$ is an *embedding*, the restriction of $f$ to the union of all the annuli $S^{n-1} \times \text{Int} D^1_{(i)}$ is, in general, only an *immersion*, because the images of the annuli may intersect each other. Using an appropriate version of the transversality theorem we can arrange by a $C^\infty$-small perturbation of $f$ that all combinations of images of its fold components intersect transversally. The differential $df : TM \rightarrow TX$ can be regularized to obtain an epimorphism $\mathcal{R}(df) : TM \rightarrow TX$. To get $\mathcal{R}(df)$ we regularize $df|_{U_i}$ for each double fold $f|_{U_i}$.

In our proof of Theorem 1.18 we will use the following result about special folded maps.
Theorem 2.3. [Special folded mappings] Let \( F : TM \to TX \) be an epimorphism which covers a map \( f : M \to X \). Suppose that \( f \) is a submersion on a neighborhood of a closed subset \( K \subset M \), and that \( F \) coincides with \( df \) over that neighborhood. Then if \( d > 0 \) then there exists a special folded map \( g : M \to X \) which coincides with \( f \) near \( K \) and such that \( R(dg) \) and \( F \) are homotopic rel. \( TM|_K \). Moreover, the map \( g \) can be chosen arbitrarily \( C^0 \)-close to \( f \) and with arbitrarily small double folds.

Let us stress again the point that a special folded map \( f : M \to X \) by definition has only spherical double folds, each fold component \( C \subset M \) is a sphere whose image \( \bar{C} \subset X \) is embedded and bounds a ball in \( X \).

Remark 2.4. In the equidimensional case \( (d = 0) \) it is not possible, in general, to make images of fold components embedded. See Appendix A below.

Theorem 2.3 is a modification of the Wrinkling Theorem from [EM97]. We formulate the Wrinkling Theorem (see Theorem 6.1) and explain how to derive 2.3 from 6.1 in Appendix A below.

Special folded mappings give a nice representation of (unstable) formal fibrations. As a class of maps, it turns out to be too small for our purposes. Namely, we wish to perform certain constructions (e.g. surgery) which does not preserve the class of special folded maps. Hence we consider a larger class of maps which allow for these constructions to be performed, and still small enough to admit a homotopically canonical extension to a formal fibration. This is the class of enriched folded maps.

2.3 Enriched folded maps and their suspensions

Recall that a folded map is, by definition, a map \( f : M^{n+d} \to X^n \) which locally is of the form (12). In this section we study a certain extra structure on folded maps which we dub framed membranes.

Definition 2.5. Let \( M^{n+d}, X^n \) be closed manifolds and \( f : M \to X \) be a folded map. A framed membrane of index \( k \) if \( k = 0, \ldots, d \), for \( f \) is a compact, connected \( n \)-dimensional submanifold \( V \subset M \) with boundary \( \partial V = V \cap \Sigma(f) \), together with a framing \( K = (K_-, K_+) \) where \( K_- \), \( K_+ \) are trivialized subbundles of \( (\text{Ker} df)|_V \) of dimension \( k \) and \( d - k \), respectively, such that...
(i) the restriction \( f|_{\text{Int} V} : \text{Int} V \to X \) is an embedding;

(ii) \( K_\pm \) are transversal to each other and to \( TV \);

(iii) there exists a co-orientation of the image \( \overline{C} \) of each fold component \( C \subset \partial V \) such that \( K_\pm|_C \subset \text{Cone}_\pm(C) \);

Thus, over \( \text{Int} V \) we have \( \text{Ker} df = K_- \oplus K_+ \), while over \( \partial V \) \( \text{Ker} df \) splits as \( K_- \oplus K_+ \oplus \lambda \), where \( \lambda = \lambda(C) \) is a line bundle contained in \( \text{Cone}_+(C) \cup \text{Cone}_-(C) \).

A boundary component of a membrane \( V \) is called positive if \( \lambda(C) \subset \text{Cone}_+(C) \), and negative otherwise. We will denote by \( \partial_+(V, K) \) and \( \partial_-(V, K) \), respectively, the union of positive and negative boundary components of \( V \).

We will call a framed membrane \( (V, K) \) pure if either \( \partial_+(V, K) = \emptyset \), or \( \partial_-(V, K) = \emptyset \). Otherwise we call it mixed.

Switching the roles of the subbundles \( K_- \) and \( K_+ \) gives a dual framing \( \overline{K} = (\overline{K}_- = K_+, \overline{K}_+ = K_-) \). The index of the framed membrane \( (V, \overline{K}) \) equals \( d - k \), and we also have \( \partial_\pm(V, \overline{K}) = \partial_\pm(V, K) \).

**Definition 2.6.** An enriched folded map is a pair \( (f, \varepsilon) \) where \( f : M \to X \) is a folded map and \( \varepsilon \) is an enrichment of \( f \), consisting of finitely many disjoint framed membranes \( (V_1, K_1), \ldots, (V_N, K_N) \) in \( M \) such that \( \partial V = \Sigma(f) \), where \( V \) is the union of the \( V_i \).

We point out that while the definition implies \( f \) is injective on each \( V_i \), the images \( f(V_i) \) need not be disjoint.

**Example 2.7.** The double \( C \)-fold \( w_C(n + d, n, k) \) defined by (14) has the annulus \( A = C \times D^1 \times 0 \subset V \times \mathbb{R} \times \mathbb{R}^d \) as its membrane. Together with the frame \( K = (K_-, K_+) \), where the subbundles \( K_- \) and \( K_+ \) are generated, respectively by

\[
\left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \right) \quad \text{and} \quad \left( \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_d} \right)
\]

along \( A \) the membrane defines a canonical enrichment of the double \( C \)-fold \( w_C(n + d, n, k) \). In particular, any special folded map has a canonical enrichment. Note that we have \( \partial_+ A = A \times (-1) \) and \( \partial_- A = A \times 1 \).
From the tubular neighborhood theorem and a parametrized version of Morse’s lemma, we get

**Lemma 2.8.** Let \((V, K)\) be a connected framed membrane for an enriched folded map \((f, \epsilon)\). Let \(C\) be a connected component of \(V\), and \(\overline{C} = f(C)\) its image. Then there is a tubular neighborhood \(U_C \subset M\) of \(C\) with coordinate functions

\[
(y, u, x) : U_C \to C \times \mathbb{R} \times \mathbb{R}^d
\]

and a tubular neighborhood \(U_{\overline{C}} \subset X\) with coordinate functions

\[
(y, t) : U_{\overline{C}} \to C \times \mathbb{R},
\]

such that we have

- \(\frac{\partial}{\partial u} \in \lambda(C)\) along \(C\);

- the vector field \(\frac{\partial}{\partial t}\) defines the coorientation of \(\overline{C}\) implied by the framing of \(V\);

- the vector fields \(\frac{\partial}{\partial x_1}|_{V \cap O_p C}, \ldots, \frac{\partial}{\partial x_k}|_{V \cap O_p C}\) belong to \(K_-\) and provide its given trivialization, while \(\frac{\partial}{\partial x_{k+1}}|_{V \cap O_p C}, \ldots, \frac{\partial}{\partial x_d}|_{V \cap O_p C}\) provide the given trivialization of \(K_+\);

- in these local coordinates, \(f\) is given by

\[
f(y, u, x) = (y, t(x, u)),
\]

where

\[
t(x, u) = Q_k(x) \pm u^2 = -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{d} x_i^2 \pm u^2,
\]

and \(V \cap U_C\) coincides with \(\{x = 0, \pm u \geq 0\}\), where the signs in the above formulas coincide with the sign of the boundary component \(C\) of the framed membrane \((V, K)\).

**Remark 2.9.** While the use of normal form 2.8 is convenient but it is not necessary. Indeed, it is obvious that all the stated properties can be achieved by a \(C^1\)-small perturbation of our data near \(C\), and this will suffice for our purposes.
A suspension of a folded map \( f : M \to X \) is a surjective homomorphism \( \Phi : TM \oplus \epsilon^1 \to f^*TX \oplus \epsilon^1 \) such that \( \pi_X \circ \Phi|_{TM} \circ i_M = df \), where \( \pi_X \) is the projection \( TX \oplus \epsilon^1 \to TX \) and \( i_M : TX \to TX \oplus 0 \mapsto TX \oplus \epsilon^1 \) is the inclusion. The main reason for considering enrichments of folds is that an enriched folded map admits a suspension whose homotopy class depends only on the enrichment.

**Proposition 2.10.** To an enriched folded map \((f, \varepsilon)\) we can associate a homotopically well defined suspension \( \mathcal{L}(f, \varepsilon) \).

**Proof.** The suspension \( TM \oplus \epsilon^1 \to f^*TX \oplus \epsilon^1 \) will be of the form

\[
\begin{pmatrix}
    df & X \\
    \alpha & q
\end{pmatrix},
\]

where \( \alpha : TM \to \epsilon^1 \) is a 1-form, \( X \) is a section of \( f^*TX \), and \( q \) is a function. The 1-form \( \alpha \) is defined as \( \alpha = du \) near \( \partial V \), using the local coordinate \( u \) on \( U_C \). To extend it to a 1-form on all of \( M \), we will extend the function \( u \). We first construct convenient local coordinates near \( V \).

The map \( f : M \to X \) restricts to a local diffeomorphism on \( \text{Int}(V) \). The local coordinate functions \( x = (x_1, \ldots, x_d) \) near \( \partial V \) from the normal form \( 2.8 \) extend to \( \mathcal{O}_pV \) in such a way that the vector fields \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \) along \( V \) generate the bundle \( K_- \), while \( \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_d} \) along \( V \) generate the bundle \( K_+ \). The tubular neighborhood theorem then gives a neighborhood \( U_{\text{Int}V} \subset M \) with coordinate functions

\[
(\tilde{y}, x) : U_{\text{Int}V} \to \text{Int}V \times \mathbb{R}^d
\]

so that the fibers of \( \tilde{y} \) are the fibers of \( f \), or, more precisely, \( f(y, x) = f(y, 0) \). (The function \( \tilde{y} \) and the function \( y \) of the Lemma \( 2.8 \) are not directly related; in fact they have codomains of different dimension.) On this overlap, the function \( u \) of Lemma \( 2.8 \) can be expressed as a function of the local coordinates \( \{16\} \). Indeed, we have \( f(\tilde{y}, x) = Q(x) \pm u(\tilde{y}, x)^2 \), so

\[
\pm u(\tilde{y}, x)^2 = f(\tilde{y}, x) - Q(x) = f(\tilde{y}, 0) - Q(x) = \pm u(\tilde{y}, 0)^2 - Q(x)
\]

so

\[
u(\tilde{y}, x) = \sqrt{u(\tilde{y}, 0)^2 + Q(x)}.
\]
If we Taylor expand the square root, we get

\begin{equation}
\tag{17}
u(\tilde{y}, x) = \gamma(\tilde{y}) \pm \delta(\tilde{y}) Q(x) + o(|x|^2).
\end{equation}

for positive functions \(\gamma, \delta\). The function \(Q\) extends over \(U_{\text{Int} V}\) (use the same formula in the local coordinates of the tubular neighborhood of \(\text{Int} V\)), and hence we can also extend \(u\) to a neighborhood of \(V\) inside \(U_V = U_C \cup U_{\text{Int} V}\), such that on \(U_{\text{Int} V}\) is satisfies (17). Extend \(u\) to all of \(M\) in any way, and let \(\alpha = du\).

We have defined a bundle map \((df, \alpha) : TM \to f^*TX \oplus \varepsilon^1\) which is surjective whenever \(\alpha|_{\text{Ker} df} \neq 0\). Near \(V\), \(\alpha|_{\text{Ker} df} = 0\) precisely when \(x = 0 \in \mathbb{R}^d\). It remains to define the section \((X, q)|_{\text{Ker} df} \neq 0\). Pick a function \(\theta : U_V \to [-\pi, \pi]\) such that \(u = -\sin \theta\) near \(\partial V\), is negative on \(\text{Int} V\) and equal to \(-\pi\) on \(V - U_C\), and which is equal to \(\pi\) outside a small neighborhood of \(V\). Then set

\begin{align}
\tag{18}
X(u) &= (\cos \theta) \frac{\partial}{\partial u} \\
\tag{19}
q(u) &= \sin \theta
\end{align}

Remark 2.11. Changing the sign of \(X(u)\) in the formula (18) provides another suspension of the enriched folded map \((f, \varepsilon)\) which we will denote by \(L_-(f, \varepsilon)\). If \(n\) is even then the two suspensions \(L(f, \varepsilon)\) and \(L_-(f, \varepsilon)\) are homotopic.

Remark 2.12. Most (but not all) of the data of an enrichment \(\varepsilon\) of a folded map \(f : M \to X\) can be reconstructed from the suspension \(\Phi = L(f, \varepsilon)\). If we write \(\Phi\) in the matrix form (15), the manifold \(V\) is the set of points with \(q \leq 0\) and \((df, \alpha) : TM \to f^*TX \oplus \varepsilon^1\) not surjective. The partition of \(\partial V\) into \(\partial_{\pm}(V, K)\) is determined by the coorientation of images of the fold components. On the other hand, the splitting \(\text{Vert} = K_+ \oplus K_-\) cannot be reconstructed from the suspension.

Lemma 2.13. Let \(f : M \to X\) be a special folded map. Then the suspension \(L(f, \varepsilon)\) (as well as the suspension \(L_-(f, \varepsilon)\)) of the canonically enriched folded map \((f, \varepsilon)\) is homotopic to its stabilized regularized differential \(R df\).

Proof. This can be seen in the local models.

\[\blacksquare\]
2.4 Fold surgery

As in the previous section, we study a folded map \( f : M \to X \) with cooriented fold images. Let \( C \subset \Sigma(f) \) be a connected component of index \( k \), and let \( \overline{C} \subset X \) be its image. For \( p \in \overline{C} \), the fibers \( f^{-1}(p) \) has a singularity. There are two directions in which we can move \( p \) away from \( \overline{C} \) to resolve the singularity and get a manifold. The manifold we get by moving \( p \) to the positive side (with respect to the coorientation) differs from the manifold we get by moving \( p \) to the negative side by a surgery of index \( k \), i.e. it has an embedded \( D^k \times \partial D^{d-k} \) instead of a \( \partial D^k \times D^{d-k} \). If \( C \) bounds an embedded domain \( P \subset X \), then one can try to prevent the surgery from happening, or, which is the same, to perform an inverse Morse surgery fiberwise in each fiber \( f^{-1}(p), p \in \overline{P} \). In this section we describe this process, which we call fold eliminating surgery in more detail.

2.4.1 Surgery template

We begin with a local model for the surgery. Let \( P \) be an \( n \)-dimensional oriented manifold with collared boundary \( \partial P \). The collar consists of an embedding \( \partial P \times [-1,0] \to P \), mapping \( (p,0) \mapsto p \). Extend \( P \) by gluing a bicollar \( U = \partial P \times [-1,1] \),

\[
\tilde{P} = P \cup_{\partial P \times [-1,0]} U.
\]

Let \( Q \) be a quadratic form of index \( k \) on \( \mathbb{R}^{d+1} \):

\[
Q(x) = -||x_-||^2 + ||x_+||^2,
\]

where \( x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1}, \ x_- = (x_1, \ldots, x_k), \ x_+ = (x_{k+1}, \ldots, x_{d+1}). \) Let \( H \subset \mathbb{R}^{d+1} \) be the domain

\[
H = \{|Q| \leq 1, \ ||x_+|| \leq 2\},
\]

We are going to use the map \( Q : H \to [-1,1] \) as a prototype of a fold. The boundary of the (possibly singular) fiber \( Q^{-1}(t) \) can be identified with \( S^{k-1} \times S^{d-k-1} \) via the diffeomorphism

\[
\{Q = t, \ ||x_+|| = 2\} \to S^{k-1} \times S^{d-k-1}
\]

\[
(x_-, x_+) \mapsto \left( \frac{x_-}{||x_-||}, \frac{x_+}{||x_+||} \right).
\]
The map $\text{Id} \times Q : \tilde{P} \times H \to \tilde{P} \times [-1,1]$ is a folded map which has $P \times 0 \subset P \times H$ as its fold of index $k$ with respect to the coorientation of the fold defined by the second coordinate of the product $P \times [-1,1]$. Given a smooth function

\[ \varphi : \tilde{P} \to [-1,1] \]

we define

\[ \tilde{P}^{\varphi} = \{(p, x) \in \tilde{P} \times H \mid Q(x) = \varphi(p)\} , \]

\[ Z^{\varphi} = \{(p, x) \in \partial \tilde{P} \times H \mid Q(x) = \varphi(p)\} , \]

\[ R^{\varphi} = P^{\varphi} \cap \{||x_+|| = 2\}. \]

We then have $\partial \tilde{P}^{\varphi} = Z^{\varphi} \cup R^{\varphi}$.

We have $\partial \tilde{P}^{\varphi t} = Z^{\varphi t} \cup R^{\varphi t} \cup \tilde{P}^{\varphi_0} \cup \tilde{P}^{\varphi_1}$. We will consider the projection

\[ \pi : \tilde{P} \times H \times [0,1] \to \tilde{P} \times [0,1] \]

and especially its restriction to the subsets \((24)\). Using \((21)\), the set $R^{\varphi t}$ can be identified with $(\tilde{P} \times [0,1]) \times S^{k-1} \times S^{d-k-1}$ via a diffeomorphism over $\tilde{P} \times [0,1]$. In particular we get a diffeomorphism

\[ R^{\varphi_0} \times [0,1] \to R^{\varphi_t} , \]
which scales the \( x_- \) coordinates. In fact it can be seen to be given by the formula

\[
(p, (x_-, x_+), t) \mapsto (p, (\sqrt{\frac{4 - \varphi_1(p)}{4 - \varphi_0(p)}} x_-, x_+), t),
\]

although we shall not need this explicit formula.

The restriction \( \pi : \tilde{P}^{\varphi_t} \rightarrow \tilde{P} \times [0, 1] \) is our “template cobordism”, and we record its properties in a lemma.

**Lemma 2.14.**  

a) Suppose that 0 is not a critical value of \( \varphi \). Then \( \tilde{P}^{\varphi} \) is a smooth manifold of dimension \( n + d \), and the projection \( \pi|_{\tilde{P}^{\varphi}} : \tilde{P}^{\varphi} \rightarrow \tilde{P} \) is a folded map with the fold \( C = \tilde{P}^{\varphi} \cap (\tilde{P} \times \{0\}) \), which projects to \( \overline{C} = \pi(C) = \varphi^{-1}(0) \subset \tilde{P} \). In particular, the map \( \pi|_{\tilde{P}^{\varphi}} \) is non-singular if \( \varphi \) does not take the value 0. The fold \( C \) has index \( k \) with respect to the coorientation of \( \overline{C} \) by an outward normal vector field to the domain \( \{ \varphi \leq 0 \} \).

b) Let \( \varphi_t : \tilde{P} \rightarrow [-1, 1] \), \( t \in [0, 1] \), be a one-parameter family of functions such that 0 is not a critical value of \( \varphi_0 \) and \( \varphi_1 \) and of the function \( \tilde{P} \times [0, 1] \rightarrow [-1, 1] \) defined by \( (p, t) \mapsto \varphi_t(p), p \in \tilde{P}, t \in [0, 1] \). We also assume \( \varphi_t(p) \) is independent of \( t \) for \( p \) near \( \partial P \). Then \( \pi|_{\tilde{P}^{\varphi_t}} : \tilde{P}^{\varphi_t} \rightarrow \tilde{P} \times [0, 1] \) is a folded cobordism between the folded maps \( \tilde{P}^{\varphi_0} \rightarrow \tilde{P} \) and \( \tilde{P}^{\varphi_1} \rightarrow \tilde{P} \). We have \( Z^{\varphi_t} = Z^{\varphi_0} \times [0, 1] \), so together with (25) we get a diffeomorphism

\[
(26) \quad (Z^{\varphi_0} \cup R^{\varphi_0}) \times [0, 1] \rightarrow Z^{\varphi_t} \cup R^{\varphi_t}.
\]

We will need to apply the above lemma to two particular functions on \( \tilde{P} \). Recall that \( U = \partial P \times [-1, 1] \subset \tilde{P} \) denotes the bicollar. Take \( \varphi_0 \equiv 1 \) and pick a function \( \varphi_1 \) with the following properties:

- \( \varphi_1 = 1 \) near \( \partial \tilde{P} \),
- \( \varphi_1 = -1 \) on \( \tilde{P} \setminus U \),
- For \( (p, v) \in U \), \( \varphi_1(p, v) \) is a non-decreasing function of \( v \),
- \( \varphi_1(p, v) = v \) for \( |v| < .5 \).
We will write $\tilde{P}^0$ and $\tilde{P}^1$ for $\tilde{P}^{\varphi_0}$ and $\tilde{P}^{\varphi_1}$, and denote by $\pi^0$ and $\pi^1$ the respective projections $\tilde{P}^0 \to \tilde{P}$ and $\tilde{P}^1 \to \tilde{P}$. Similarly, we will use the notation $Z^0, Z^1, R^0$ and $R^1$ instead of $Z^{\varphi_0}, Z^{\varphi_1}, R^{\varphi_0}$ and $R^{\varphi_1}$. The map $\tilde{P}^0 \to \tilde{P}$ is non-singular, while the map $\tilde{P}^1 \to \tilde{P}$ has a fold singularity with image $\partial P \subset \tilde{P}$. The index of this fold with respect of the outward coorientation to the boundary of $P$ is equal to $k$.

Taking linear interpolations between $\varphi_0$ and $\varphi_1$ in one order or the other, we get folded cobordisms in two directions between the map $\tilde{P}^0 \to \tilde{P}$ and $\tilde{P}^1 \to \tilde{P}$. We will denote the corresponding cobordisms by $\tilde{P}^{01}$ and $\tilde{P}^{10}$, respectively. The projections $\pi^{01} : \tilde{P}^{01} \to \tilde{P} \times [0, 1]$ and $\pi^{10} : \tilde{P}^{10} \to \tilde{P} \times [0, 1]$ are folded bordisms in two directions between $\pi^0 : \tilde{P}^0 \to \tilde{P}$ and $\pi^1 : \tilde{P}^1 \to \tilde{P}$. We think of $\tilde{P}^{\varphi_t}$ as a one-parameter family of (possibly singular) manifolds, interpolating between $\tilde{P}^0$ and $\tilde{P}^1$. Using the trivialization (26), these manifolds all have the same boundary, so $\pi^{01}$ and $\pi^{10}$ may be used as local models for cobordisms. They allow us to create, or annihilate a fold component, respectively. We describe the fold eliminating surgery more formally in the next section and leave the formal description of the inverse process of fold creating surgery to the reader. In fact fold creating surgery will not be needed for the proof of the main theorem.

In the context of enriched folded maps there are two versions of fold eliminating surgery. One will be referred to as membrane eliminating. In this case the membrane will be eliminated together with the fold. The second one will be referred to as membrane expanding. In that case the membrane after the surgery will be spread over $\overline{P}$, the image of $P$ in the target.

For the membrane eliminating case we choose the submanifold

$$ V_− = \{ (x_2, \ldots, x_{d+1}) = 0, x_1 \leq 0 \} \cap \tilde{P}^{10} \subset \tilde{P} \times H \times [0, 1] $$

as a template membrane for the folded bordism $\pi^{10} : \tilde{P}^{10} \to \tilde{P}$. Next we choose the subbundles $K_−$ and $K_+$ spanned by vector fields $\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_{d+1}}$, respectively, as a template framing. Note that with this choice we have $\partial V_− = \partial_− (V_−, K)$ and the index of the membrane $V_−$ is equal to $k − 1$.

For the membrane expanding surgery we choose as a template membrane the submanifold

$$ V_+ = \{ (x_1, \ldots, x_k, x_{k+2}, \ldots, x_{d+1}) = 0, x_k \geq 0 \} \cap \tilde{P}^{10} \subset \tilde{P} \times H \times [0, 1] $$
with boundary $\Sigma(\pi^{10})$ as the membrane for the folded bordism $\pi^{10} : \tilde{P}^{10} \to \tilde{P} \times [0, 1]$. We choose the subbundles $K_-$ and $K_+$ spanned by vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial x_{k+2}}, \ldots, \frac{\partial}{\partial x_{d+1}}$, respectively, as a template framing. Note that with this choice we have $\partial V_+ = \partial_+(V_+, K)$ and the index of the membrane $V_+$ is equal to $k$.

Note that in the membrane eliminating case the restriction of the membrane $V_-$ to $\tilde{P}^1$ projects diffeomorphically onto $P \subset \tilde{P}$, while in the membrane expanding case the restriction of the membrane $V_+$ to $\tilde{P}^1$ projects diffeomorphically onto $\tilde{P} \setminus \text{Int } P \subset \tilde{P}$.

2.4.2 Surgery

Membrane eliminating surgery. Let $(f : M \to X, \varepsilon)$ be an enriched folded map and $(V, K)$ one of its membranes. Suppose that the framed membrane $(V, K)$ is pure and assume first that $\partial_+(V, K) = \emptyset$, and that the index of the membrane is equal to $k - 1 \geq 0$. Note that in this case the boundary fold $\partial V$ has index $k$ with respect to the outward coorientation of $\partial V$. Consider the model enriched folded map $\pi^1 : \tilde{P}^1 \to \tilde{P}$ where $P$ is diffeomorphic to $V$. Fix a diffeomorphism $\psi : P \to \overline{V} = f(V) \subset X$. Let $U^1$ denote a neighborhood of $\partial P \subset \tilde{P}^1$. According to Lemma 2.8 there exist an extension $\tilde{\psi} : \tilde{P} \to X$ of the embedding $\psi$ and an embedding $\Psi : U^1 \to M$ such that

- the diagram

\[
\begin{array}{ccc}
U^1 & \xrightarrow{\Psi} & M \\
\downarrow & & \downarrow f \\
U & \xrightarrow{\tilde{\psi}} & X
\end{array}
\]

commutes;

- $\Psi^{-1}(V) = V_\cap U^1$;

- the canonical framing of the membrane $V_\cap U^1$ is sent by $\Psi$ to the given framing of the membrane $V$.

The data needed for eliminating the fold $\partial V$ by surgery consists of an extension of $\Psi$ to all of $\tilde{P}^1$ such that
• the diagram

\[
\begin{array}{ccc}
\tilde{P}^1 & \xrightarrow{\Psi} & M \\
\downarrow{\pi^1} & & \downarrow{f} \\
\tilde{P} & \xrightarrow{\tilde{\psi}} & X
\end{array}
\]

commutes;

• \(\Psi^{-1}(V) = V_+^1 := V_- \cap \tilde{P}^1\);

• the canonical framing of the membrane \(V_+^1\) is sent by \(\Psi\) to the given framing of the membrane \(V\).

**Construction 2.15.** Fold eliminating surgery consists of replacing \(\tilde{P}^1\) by \(\tilde{P}^0\) with the projection \(\pi^0\). More precisely, cut out \(\Psi(\tilde{P}^1) \times [0, 1]\) from \(M \times [0, 1]\) and glue in \(\tilde{P}^0\) along the identification (26). This gives an enriched folded map \(W \to X \times [0, 1]\) which is a cobordism starting at \(f\), and ending in an enriched folded map where the fold \(\partial V\), together with its membrane \(V\), has been removed.

![Figure 6: Fold eliminating surgery \((n = 2, d = 0)\)](image)

The case \(\partial_-(V, K) = \emptyset\) can be reduced to the previous one by the following procedure. Let \(K\) be the dual framing of \(V\), see Section 2.3 above. Then \(\partial V = \partial_-(V, K), \partial_+(V, K) = \emptyset, d > k\), and the membrane \((V, K)\) has index \(d - k - 1\). Hence, we can use for the membrane eliminating surgery the above template of index \(d - k\), and then switch the framing of the constructed membrane in the cobordism to the dual one.
Membrane expanding case. Let $P$ be a domain in $X$ with boundary bounded by image $\overline{C}$ of a fold $C$ of index $k$ with respect to the outward orientation of $\overline{C} = \partial P$. Suppose that the membrane $V$ adjacent to $C$ projects to the complement of $P$ in $X$, i.e. $\overline{V} = f(V) \subset X \setminus \text{Int} P$, and $C \subset \partial_+(V, K)$. The case $C \subset \partial_-(V, K)$ can be reduced to the positive by passing to the dual framing as it was explained above in the membrane eliminating case.

Consider the template enriched folded map $\pi^1 : \tilde{P}^1 \to \tilde{P}$ as in Section 2.4.1. Let $\psi$ denote the inclusion $P \hookrightarrow X$. According to Lemma 2.8 there exist an extension $\tilde{\psi} : \tilde{P} \to X$ of the embedding $\psi$ and an embedding $\Psi : U^1 \to M$ such that

- the diagram

\[ U^1 \xrightarrow{\Psi} M \]
\[ \downarrow \pi^1 \quad \downarrow f \]
\[ U \xrightarrow{\tilde{\psi}} X \]

commutes;

- $\Psi^{-1}(V) = V_+ \cap U^1$;
• the canonical framing of the membrane $V_+ \cap U^1$ is sent by $\Psi$ to the given framing of the membrane $V$.

In this case the data needed for eliminating the fold $\partial V$ by surgery consists of an extension of $\Psi$ to all of $\tilde{P}^1$ such that

• the diagram

\[
\begin{array}{ccc}
\tilde{P}^1 & \xrightarrow{\Psi} & M \\
\downarrow \pi^1 & & \downarrow f \\
\tilde{P} & \xrightarrow{\tilde{\psi}} & X
\end{array}
\]

commutes;

• the canonical framing of the membrane $V_+ \cap \tilde{P}^1$ is sent by $\Psi$ to the given framing of the membrane $V$.

Construction 2.16. Fold eliminating surgery consists of replacing $\tilde{P}^1$ by $\tilde{P}^0$ with the projection $\pi_0$. Exactly as in Construction 2.15 we get an enriched folded map $W \to X \times [0,1]$ which is a cobordism starting at $f$, and ending in an enriched folded map where the fold $\partial V$ has been removed.

Both constructions eliminate the fold $\partial V$. The difference between them is that in the membrane expanding case, the above surgery spreads the membrane $V$ over the domain $P$.

2.4.3 Bases for fold surgeries

The embedding $\Psi : \tilde{P}^1 \to M$ required for the surgeries in constructions 2.15 and 2.16 is determined up to isotopy by slightly simpler data. To any smooth manifold $P$ with collared boundary, let $\varphi_1 : P \to [-1,1]$ be the function defined in Section 2.4.1 and let

\[ S^{k-1}P = \{(p,x_-) \in \tilde{P} \times D^k | \varphi_1(p) = -\|x_-\|^2\}. \]

This is a closed manifold, which up to diffeomorphism depends only on $P$. In fact, it is diffeomorphic to the boundary of $P \times D^k$ (after smoothing the corners of $P \times D^k$). The projection $(p,x_-) \mapsto p$ restricts to a folded
map \( \pi : S^{k-1}P \to P \) with fold \( \partial P \) of index \( k \) with respect to the outward orientation of the boundary of \( P \). We have an embedding

\[
S^{k-1}P \to \tilde{P}^1 \subset \tilde{P} \times H
\]
given by \((p, x_-) \mapsto (p, x, 0)\). The normal bundle of this embedding has a canonical frame given by projections of the frame

\[
\frac{\partial}{\partial x_+} = \left( \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_{d+1}} \right)
\]
to \( TS^{k-1}P \).

We also have an embedding \( \partial P \to S^{k-1}P \) as \( p \mapsto (p, 0) \), which identifies \( \partial P \) with the folds of the projection \( S^{k-1}P \to P \), and the normal bundle of \( \partial P \subset S^{k-1}P \) is framed by

\[
\frac{\partial}{\partial x_-} = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \right).
\]

Let us denote \( P_\pm = V_- \cap S^{k-1}P \). Thus we have

\[
P_- = S^{k-1}P \cap \{(x_2, \ldots, x_k) = 0, x_1 \leq 0\} = \{(x_2, \ldots, x_k) = 0, x_1 = -\sqrt{-\varphi_1(p)}\}.
\]

**Definition 2.17.** Let \((f : M^{n+d} \to X^n, e)\) be an enriched folded map.

a) **Membrane eliminating case.** Let \((V, K) \subset M\) be a framed membrane with \( \partial_+(V, K) = \emptyset \). A basis for membrane-eliminating surgery consists of a pair \((h : S^{k-1}P \to M, \mu)\), where \( P \) is a compact \( n \)-manifold with boundary, \( h : S^{k-1}P \to M \) is an embedding, and \( \mu = (\mu_{k+1}, \ldots, \mu_{d+1}) \) is a framing of the normal bundle of \( h \), such that the following conditions are satisfied.

- \( h(P_-) = V; \)
- the map \( f \circ h \) factors through an embedding \( g : P \to X \), i.e. \( f \circ h = g \circ \pi \), and hence \( f(h(S^{k-1}P)) = \overline{V} = f(V) \);
- the vectors \( \mu_{k+1}, \ldots, \mu_{d+1} \) belong to \( \text{Ker} df|_{h(S^{k-1}P)} \) and along \( h(P_-) \) coincide with the given framing of the bundle \( \overline{\text{Ker}_+ df} \).
- the vectors \( dh(\frac{\partial}{\partial x_2}), \ldots, dh(\frac{\partial}{\partial x_k}) \) defines the prescribed framing of the bundle \( \overline{\text{Ker}_-|_V} \);
Figure 8: The oval is the image $h(S^{k-1}P)$, where $k = 1$, $P = I$

- $h(S^{k-1}P)$ is disjoint from membranes of $\varepsilon$, other than $V$.

b) **Membrane expanding case.** Let $P$ be a domain in $X$ bounded by folds of index $k$ with respect to the outward orientation of $C = \partial P$. Let $C$ be the union of the corresponding fold components, and $(V,K)$ the union of framed membranes adjacent to $C$. Suppose that $C \subset \partial_+(V,K)$ and $f(V) \subset X \setminus \text{Int} P$.

A basis for membrane-expanding surgery consists of a pair $(h : S^{k-1}P \to M, \mu)$, where $h : S^{k-1}P \to M$ is an embedding, and $\mu = (\mu_{k+1}, \ldots, \mu_{d+1})$ is a framing of the normal bundle of $h$, such that the following conditions are satisfied.

- The map $f \circ h$ factors through an embedding $g : P \hookrightarrow X$, i.e. $f \circ h = g \circ \pi$;
- $dh(\frac{\partial}{\partial x_k+1}|_{\partial P}) \subset \text{Ker} df$ and coincides with the given framing of the bundle $\text{Ker} df$ over the fold $C = h(\partial P)$;
- $h(S^{k-1}P) \cap V = C$ and $h(S^{k-1}P)$ is disjoint from any other membranes different from $V$.
- The vector field
  
  $$dh \left( (-1)^k \frac{\partial}{\partial x_{k+1}} \right) \bigg|_C$$

  is tangent to $V$ and inward transversal to $\partial V$.

Note that in both cases it follows from the above definitions that $f \circ h : S^{k-1}P \to X$ is a folded map with the definite fold $\Sigma(f \circ h) = h(\partial P)$.

Given a basis $(h, \mu)$, one can extend, uniquely up to isotopy the embedding $h$ to an embedding $\Psi : \tilde{P}^1 \to M$ such that the diagram (28) or (30) commutes. This, in turn, enables us to perform a membrane eliminating or membrane expanding surgery.
Remark 2.18. Fold creating surgeries. Fold creating surgeries are inverse to fold eliminating surgeries. For our purposes we will need only one such surgery which creates a fold of index 1 with respect to the inward co-orientation of its membrane. A basis of such a surgery is given by a pair \((h, \mu)\), where \(h\) is an embedding \(h : P \times \{-1, 1\} \to M\) over a domain \(P \subset X\) disjoint from the folds of the map \(f\), and \(\mu\) is a framing of the vertical bundle \(\text{Ker} df|_{h(P \times \{-1, 1\})}\). See Figure 9.

![Figure 9: Fold creating surgery](image.png)

2.4.4 The case \(d = 2\)

Let us review fold eliminating surgeries in the case \(d = 2\). These surgeries can be of index 0, 1, 2 or 3. Let \(C\) be a union of fold components whose projections bound a domain \(P \subset X\). In the membrane eliminating case, \(P\) is the projection \(\overline{V} = f(V)\) of the membrane which spans \(C\). In the membrane expanding case the membranes adjacent to \(C\) projects to the complement of the domain \(P\). All fold indices below are with respect to the outward coorientation of the boundary of the domain \(P\).

Index 0. We have \(S^{-1}P = \partial P\), i.e. the basis of the surgery in this case is a framed embedding \(h : \partial P \to M\) which sends \(\partial P\) to the fold \(C\).
Only the membrane expanding surgery is possible in this case. When a point \( p \in X \) approaches \( \partial P \) from outside, a spherical components of the fiber \( f^{-1}(p) \) dies. The surgery prevents it from dying end prolongs its existence over all points of \( P \).

**Index 1.** The surgery basis in this case consists of 2 sections \( s_\pm : P \to M \), together with framings of the bundle \( \text{Ker} \, df \) over them. As \( p \to \bar{z} \in C \) the sections \( s_\pm(p) \) converge to the same point \( z \in C, f(z) = \bar{z} \). In the membrane eliminating case, one of these sections is the membrane \( V \).

The manifold \( M' \) is obtained by a fiberwise index 1 surgery (i.e. the connected sum) along the framed points \( s_+(p) \) and \( s_-(p), p \in P \). This eliminates the fold \( C \) together with the membrane \( V \) in the membrane eliminating case, and spreads the membrane over \( P \) in the membrane expanding case. In the latter case the newly created membrane is a section over \( P \) which takes values in the circle bundle over \( P \) formed by central circles of added cylinders \( S^1 \times [-1, 1] \).

**Index 2.** The surgery basis in this case is a circle-subbundle over the domain \( P \subset X \), i.e. a family of circles in fibers \( f^{-1}(p), \, p \in P \), which collapse to points in \( C \) when \( p \) converges to a boundary point of \( P \). In the fold eliminating case, the membrane \( V \) is a section over \( P \) of this circle bundle.

The surgery consists of fiberwise index 2 surgery of fibers along these circles, which eliminates the fold \( C \) together with its membrane in the eliminating case, and spreads it over \( P \) in the expanding one.

**Index 3.** The basis of the surgery in this case is a connected component of \( M \) which forms an \( S^2 \)-bundle over \( P \). The 2-spheres collapse to points of \( C \) when approaching the boundary of \( P \). The surgery eliminates this whole connected component, in particular removing the fold and its membrane. The membrane expanding case is not possible for \( k = 3 \).

### 2.5 Destabilization

So far (in Theorem 2.3 and Lemma 2.13) we have related unstable formal fibrations to (enriched) folded maps. In Theorem 1.8 we need to work with stable formal fibrations (because that is what \( \Omega^\infty C P^\infty_{-1} \) classifies). In this
section we study the question of whether an epimorphism $\Phi : TM \oplus e^1 \to TX \oplus e^1$ can be “destabilized”, i.e. homotoped to be of the form $\Phi \oplus \text{Id}$ for some unstable epimorphism $\overrightarrow{\Phi} : TM \to TX$. This is not possible in general of course (the obstruction is an Euler class). Instead we prove the following.

Proposition 2.19. Let $\Phi : TM \oplus e^1 \to TX \oplus e^1$ be a bundle epimorphism with underlying map $f : M \to X$. Assume $M$ and $X$ are connected. Then there is a compact codimension 0 submanifold $M_0 \subset M$ which is homotopy equivalent to a simplicial complex of dimension at most 1, such that the following hold, after changing $f$ and $\Phi$ by a homotopy (in the class of bundle epimorphisms).

(i) $f|_{M_0}$ is folded and has an enrichment $e$ such that
   - $\Phi|_{M_0} = \mathcal{L}(f|_{M_0}, e)$ if $n := \dim X > 1$;
   - $\Phi|_{M_0} = \mathcal{L}(f|_{M_0}, e)$ or $\Phi|_{M_0} = \mathcal{L}_-(f|_{M_0}, e)$ in the case $n = 1$;

(ii) $\Phi$ is integrable near $\partial M_0$, i.e. it equals $Df \oplus e^1$ there.

(iii) $\Phi$ destabilizes outside $M_0$, i.e. it equals $\Phi \oplus e^1$ there, for some bundle epimorphism $\overrightarrow{\Phi} : TM|_{M \setminus M_0} \to TX$.

The strategy of the proof of the proposition is as follows. First we forget about (iii) in the proposition, and only worry about how $\varphi$ and $\Phi$ looks like on $M_0$. We prove that this can be done for a large class of possible $M_0$’s. After that, we consider the obstruction to destabilizing $\Phi$ outside $M_0$ without changing it on $M_0$. This obstruction is essentially an integer, and we prove that $M_0$ can be chosen so that the obstruction vanishes.

We first give a local model for the enriched folded map $M_0 \to X$. Let $I = [-1, 1]$ be the interval, and let $K \subset \text{Int } (I^n)$ be a simplicial complex. We will consider $I^n \subset I^{n+d}$ as the subset $I^n \times \{0\}$. Let $U_0 \subset I^n$ be a regular neighborhood of $K \subset I^n$, and let $U \subset I^{n+d}$ be a regular neighborhood of $K \subset I^{n+d}$. Let $\pi : I^{n+d} \to I^n$ be the projection. In order to avoid confusion we will write $\overline{U}_0 = U_0 \times \{0\} \subset I^{n+d}$, and hence $U_0 = \pi(\overline{U}_0)$. We can assume that $\pi|_{\partial U} : \partial U \to I^n$ is a folded map, with fold $\partial \overline{U}_0 \subset \partial U$ which has index 0 with respect to the inward coorientation of $\partial U_0 \subset U_0$.

Let $N = \partial U \times [-1, 1]$ be a bicollar of $\partial U$, i.e. an embedding $N \to I^{n+d}$, which maps $(u, 0) \mapsto u \in \partial U$ and $\partial U \times [-1, 0]$ to $U$. Let $M_0 = U \cup N$. We construct a folded map $M_0 \to M_0$ in the following way. First pick a function $\varphi : [-1, 1] \to [-1, 1] \times \mathbb{R}$ with the following properties.
(i) \( \varphi(\pm s) = (\pm s, 0) \) for \( s > .5 \);
(ii) \( \varphi'_1(s) > 0 \) for \( s < 0 \), \( \varphi'_1(s) < 0 \) for \( s > 0 \), and \( \varphi''_1(0) < 0 \);
(iii) \( \varphi'_2(0) < 0 \).

In particular \( \varphi \) is an immersion of codimension 1, and \( \varphi_1 : [-1, 1] \) is a folded map with fold \( \{0\} \). Extend to an immersion \( \varphi : [-1, 1] \times \mathbb{R} \to [-1, 1] \times \mathbb{R} \) with the property that

\[
\varphi(\pm s, t) = (\pm s, \pm t)
\]

for \( s > .5 \). Recall that \( N = \partial U \times [-1, 1] \) and construct a codimension 0 immersion \( \gamma_0 : N \times \mathbb{R} \to N \times \mathbb{R} \) by

\[
\gamma_0(u, s, t) = (u, \varphi(s, t)),
\]

for \( u \in \partial U \). Extend to a codimension 0 immersion \( \gamma_1 : M_0 \times \mathbb{R} \to M_0 \times \mathbb{R} \) by

\[
\gamma_1(x, t) = (x, -t)
\]

for \( x \in M_0 - N \). For \( m \in M_0 \), let \( \gamma_1(m) \in M_0 \) be the first coordinate of \( \gamma_1(m, 0) \in M_0 \times \mathbb{R} \). Differentiating \( \gamma_1 \) then gives a bundle map

\[
\Gamma : TM_0 \oplus \mathfrak{e}^1 \to TM_0 \oplus \mathfrak{e}^1
\]

with underlying map \( \gamma : M_0 \to M_0 \). We record some of its properties in a lemma.

Lemma 2.20. (i) The map \( \gamma_1 \) is homotopic in the class of submersions (=immersions) to the map \( \mathrm{Id} \times (-1) : M_0 \times \mathbb{R} \to M_0 \times \mathbb{R} \).

(ii) Let \( M_0 \subset I^{n+d} \) and \( \gamma \) and \( \Gamma \) be as above. Then \( \gamma \) is a folded map. The fold is \( \partial U \subset N \subset M_0 \), and the image of the fold is also \( \partial U \). Near \( \partial M_0 \), the bundle map \( \Gamma \) is integrable, i.e. \( \Gamma = D\gamma \oplus \mathfrak{e}^1 \).

(iii) Let \( \pi : I^{n+d} \to I^n \) be the projection and define a bundle map

\[
H : TM_0 \oplus \mathfrak{e}^1 \to T(I^n) \oplus \mathfrak{e}^1
\]

by \( H = (D\pi \oplus \mathfrak{e}^1) \circ \Gamma \). It covers the folded map \( h = \pi \circ \gamma : M_0 \to I^n \), which has fold \( \partial U_0 \subset \partial U \subset M_0 \). The image of the fold is \( \partial U_0 \subset I^n \), and it has index 0 with respect to the inward coorientation of \( U_0 \).
(iv) A membrane for the underlying map \( h \) can be defined as \( V = U_0 \) with the framing \( K = (K_+ = \text{Span}(\partial/\partial x_{n+1}, \ldots, \partial/\partial x_{n+d}), \text{Ker}_-(V)), K_- = \{0\} \). This defines an enrichment \( \varepsilon \) for \( h \). Finally, \( H \) is integrable over \( \partial M_0 \) and \( H = L(h, \varepsilon) \).

**Proof.** We leave this as an easy exercise for the reader. Cf also Section 2.4.3 and the proof of Proposition 2.10.

Let us also point that we could equally well have based the construction on a map \( \varphi_- : [-1, 1] \to [-1, 1] \times \mathbb{R} \) defined as \( \varphi \) above, except that we replace the condition \( \varphi_2'(0) < 0 \) by \( \varphi_2'(0) > 0 \). Using this map as a basis for the construction gives a bundle epimorphism

\[
H_- : TM_0 \oplus \varepsilon^1 \to T(I^n) \oplus \varepsilon^1
\]

which also satisfies the conclusion of Lemma 2.20 except that (iv) gets replaced by \( H = L_-(h, \varepsilon) \).

**Proof of Proposition 2.19 (i) and (ii).** Let \( \Phi : TM \oplus \varepsilon^1 \to TX \oplus \varepsilon^1 \) be as in the proposition. By Phillips’ theorem we can assume that \( \Phi \) is induced by a submersion \( \Psi : M \times \mathbb{R} \to X \times \mathbb{R} \).

Pick cubes \( D = I^{n+d} \subset M \) and \( I^n \subset X \). We regard \( M_0 \subset D \subset M \) and let \( \pi : D \to I^n \) denote the projection to the first \( n \) coordinates. We can assume that \( \Psi(D \times \mathbb{R}) \subset I^n \times \mathbb{R} \). The space of submersions \( D \times \mathbb{R} \to I^n \times \mathbb{R} \) is, by Phillips’ theorem, homotopy equivalent to \( O(n + d + 1)/O(d) \) which is connected, and hence we may assume that \( \Psi|_D = \pi \times (-1) \). (Here we used \( d \geq 1 \). In the case \( d = 0 \), we get \( O(n + d + 1) \) which has two path components, but after possibly permuting coordinates on \( D = I^{n+d} \), we may assume that \( \Psi|_D \) is in the same path component as \( \pi \times (-1) \).) Hence, after a homotopy of \( \Psi \) in the class of submersions, we may assume that \( \Psi|_{M_0} = \pi \times (-1) \).

By Lemma 2.20(i), we can assume after a further deformation of \( \Psi \) in a neighborhood of \( M_0 \subset D \), that it agrees with the map \( (\pi \times \mathbb{R}) \circ \gamma_1 \). This proves (i) and (ii) in the proposition.

\[\square\]
Remember that the domain $M_0$ is a regular neighborhood of a simplicial complex $K \subset I^n$. The local model in Lemma 2.20 worked for any such $K$, but in the proof of (i) and (ii) in the proposition we used that $K$ had dimension at most 1. On the other hand, $K$ could still be arbitrary within that restriction. It remains to prove that $\Phi$ can be destabilized outside $M_0$, for a suitable choice of $K$. There is an obstruction to doing this, which we now describe.

Let $s$ denote a section of $TM \oplus \epsilon^1$ such that $\Phi \circ s$ is the constant section $(0,1) \in f^*(TX) \oplus \epsilon^1$. This defines $s$ uniquely up to homotopy (in fact $s$ is unique up to translation by vectors in the kernel of $\Phi$). Over $M_0$ the epimorphism $\Phi$ is induced by a composition

$$M_0 \times \mathbb{R} \xrightarrow{\gamma_1} M_0 \times \mathbb{R} \xrightarrow{\text{proj}} X \times \mathbb{R},$$

and on $M_0$ we may choose $s$ so that $D\gamma_1$ takes $s$ to a unit vector in the $\mathbb{R}$-direction. Another relevant section is the constant section $s_\infty(x) = (0,1) \in S(TM \oplus \epsilon^1)$. We have $s(x) = s_\infty(x)$ for $x \in \partial M_0$. Our aim is to change $\Phi$ by a homotopy and achieve $s(x) = s_\infty(x)$ for all $x$ outside $M_0$. In each fiber, $s(x) \in S(T_xM \oplus \mathbb{R}) = S^{n+d}$, so by induction of cells of $M - D$, we can assume that $s_\infty(x) = s(x)$ outside $D$, since $M - D$ can be built using cells of dimension at most $n + d - 1$. It remains to consider $D - M_0$. Let $s_K$ be the section which agrees with $s$ on $M_0$ and with $s_\infty$ outside $M_0$. Thus $s$ and $s_K$ are both sections of $S(TM \oplus \epsilon^1)$ which equal $s_\infty$ outside $D$. We study their homotopy classes in the space of such sections.

Using stereographic projection, the fiber of $S(TM \oplus \epsilon^1)$ at a point $x \in M$ can be identified with the one-point compactification of $T_xM$. Hence we can think of sections as continuous vector fields on $M$, which are allowed to be infinite. The section at infinity is $s_\infty(x) = (0,1) \in S(T_xM \oplus \mathbb{R})$. In this picture we have the following way of thinking of $s_K$: For $x \in \partial U$, $s_K(x)$ is a unit vector orthogonal to $\partial U$ pointing outwards. Moving $x$ away from $\partial U$ to the inside makes $s_K(x)$ smaller and it gets zero as we get far away from $\partial U$. Moving $x$ away from $\partial U$ to the outside makes $s_K(x)$ larger and it gets infinite as we get far away from $\partial U$. The section $s_K$ depends up to homotopy only on the simplicial complex $K \subset I^n$, hence the notation.

**Lemma 2.21.** There is a bijection between $\mathbb{Z}$ and sections of $S(TM \oplus \epsilon^1)$ which agree with $s_\infty$ outside $D$. The bijection takes $s_K$ to $\chi(K) \in \mathbb{Z}$.

**Proof.** The tangent bundle $TM$ is trivial over $D$, so the space of such sections
is just the space of pointed maps $S^n \to S^n$ and homotopy classes of such are classified by their degree, which is an integer.

We have assumed $U \subset D = I^{n+d}$ so using the standard embedding $D \subset \mathbb{R}^n$ we can work entirely inside $\mathbb{R}^n$. The geometric interpretation of $s_K$ given above can then be rephrased even more conveniently. Let $r : U \to K$ be the retraction in the tubular neighborhood, and let

$$\tilde{s}_K(x) = x - r(x)$$

for $x \in U$. Pick any continuous extension of $\tilde{s}_K$ to $D$ with the property that when $x \not\in U$, we have

$$\tilde{s}_K(x) \in (T_xM - \{0\}) \cup \{\infty\}.$$

Up to homotopy there is a unique such extension because we are picking a point $\tilde{s}_K(x)$ in a contractible space. Then $\tilde{s}_K \simeq s_K$.

To calculate the degree of the corresponding map we perturb even further. Remember that any simplicial complex $K$ has a standard vector field with the following property: The stationary points are the barycenters of simplices and the flowline starting at a point $x$ converges to the barycenter of the open cell containing $x$. Let $\psi_\epsilon : K \to K$ be the time $\epsilon$ flow of this vector field, and define a vector field $\tilde{s}_K$ just like $\tilde{s}_K$, except that we replace the right hand side of (32) by $x - \psi_\epsilon \circ r(x)$ for some small $\epsilon > 0$.

The resulting vector field vanishes precisely at the barycenters of $K$, and the index of the vector field at the barycenter of a simplex $\sigma$ is $(-1)^{\dim(\sigma)}$. The claim now follows from the Poincaré-Hopf theorem.

---

**Proof of Proposition 2.19 (iii).** Let us first consider the case $n > 1$. We have proved that all possible sections of $S(TM \oplus \epsilon^1)$ which agree with $s_\infty$ outside $D$, are homotopic to $s_K$ for some $K$. Then we can choose $K$ such that $s_K \simeq s$. Since $s_K$ and $s$ agree on $M_0$ there is a homotopy of $s$, fixed over $M_0$, so that $s(x) = s_\infty(x)$ for all $x \in M - M_0$. This homotopy lifts to a homotopy of bundle epimorphisms $\Phi : TM \oplus \epsilon^1 \to TX \oplus \epsilon^1$, and then (iii) is satisfied.

For $n = 1$ we may not be able to choose a $K \subset X$ with the required Euler characteristic, since subcomplexes of 1-manifolds always have non-negative Euler characteristic. However, vector fields of negative index can be achieved as $-s_K$, and that is the vector field that arises if we use the negative suspension $L_-(f, \epsilon)$. 

\[ \blacksquare \]
2.6 From formal epimorphisms to enriched folded maps

The following theorem summarizes the results of Section 2.

**Theorem 2.22.** Let \( \Phi : TM \oplus \epsilon^1 \rightarrow TX \oplus \epsilon^1 \) be a bundle epimorphism. Suppose that \( d > 0 \) and \( \Phi \) is integrable in a neighborhood of a closed set \( A \subset M \). Then there is a homotopy of epimorphisms \( \Phi_t : TM \oplus \epsilon^1 \rightarrow TX \oplus \epsilon^1 \), \( t \in [0,1] \), fixed near \( A \), which covers a homotopy \( \varphi_1 : M \rightarrow X \), such that \( \varphi_1 : M \rightarrow X \) is folded, and \( \Phi_1 = \mathcal{L}(\varphi_1, \epsilon) \) for some enrichment \( \epsilon \) of \( \varphi_1 \). If \( n > 1 \) then the image \( C \subset X \) of each fold component \( C \subset M \) of \( \varphi_1 \) bounds a domain in \( X \).

**Proof.** First use Proposition 2.19 to make \( \varphi \) enriched folded over a domain \( M_0 \), such that \( \Phi \) destabilizes outside \( M_0 \). Then use Lemma 2.13 to make \( (\varphi, \Phi) \) special enriched outside \( M_0 \).

3 Cobordisms of folded maps

Let us rephrase the results of the previous section more systematically, and put them in the context of the overall goal of the paper. So far we have mainly studied the relation between formal fibrations and enriched folded maps. Let us formalize the result. We consider various bordism categories of maps \( f : M \rightarrow X \) such that \( M \) and \( X \) are both oriented and \( X \) is a compact manifold, possibly with boundary, and which satisfy the following two conditions:

C1. \( f \) has \( T_\infty \) ends, i.e. there is (as part of the structure) a germ at infinity of a diffeomorphism \( j : T_\infty \times X \sim M \) such that \( f \circ j = \pi \), where \( \pi \) is a germ at infinity of the projection \( X \times T_\infty \rightarrow X \). This trivialized end will be called the **standard end of** \( M \).

C2. There is a neighborhood \( U \) of \( \partial X \) such that \( f^{-1}(U) \rightarrow U \) is a fibration (i.e. smooth fiber bundle) with fiber \( T_\infty \).

When \( \text{dim}(X) > 1 \) we will always assume that all fold components are homologically trivial, and in particular, the image \( \overline{C} \subset X \) of any fold component \( C \subset M \) bounds a domain in \( X \). Note that this condition is preserved by all fold surgeries which we discussed above in Section 2.4. We have been studying the following bordism categories of maps.
Definition 3.1.  

(i) **Fib** is the category of fibrations (smooth fiber bundles) with fiber $T_{\infty}$, which satisfy C1 and C2.

(ii) **Fold** is the category of enriched folded maps, satisfying C1 and C2.

(iii) **FFib** is the category of formal fibrations, i.e. bundle epimorphisms $\Phi : TM \oplus e^1 \to TX \oplus e^1$ with underlying map $f : M \to X$, such that $f$ satisfies C1 and C2, and such that $\Phi = df \oplus e^1$ near $f^{-1}(\partial X)$.

The objects in bordism categories, Bordisms in the categories **Fib**, **Fold**, and **FFib** are required to be trivial (as bordisms, and not as fibrations!) over a neighborhood of $\partial X$.

We have functors

$$\text{Fib} \to \text{Fold} \xrightarrow{\mathcal{L}} \text{FFib}.$$ 

The functor $\text{Fib} \to \text{Fold}$ is the obvious inclusion. Everything we said in Chapter 2 works just as well with the conditions C1 and C2 imposed, and hence Proposition 2.10 gives the functor $\mathcal{L} : \text{Fold} \to \text{FFib}$.

In this setup, the main goal of the paper is to prove that any object in **FFib** is cobordant to one in **Fib**. Theorem 2.22, which also works for manifolds with boundary, says that $\mathcal{L}$ is essentially surjective: any object in **FFib** is cobordant to an element in the image of $\mathcal{L}$. It remains to see that any object of **Fold** is cobordant to one in **Fib**. In fact Theorem 2.22 is a little stronger: any object of **FFib** is cobordant to one in the image of $\mathcal{L}$ using only homotopies of the underlying maps, i.e. no cobordisms of $M$ and $X$.

In contrast, comparing **Fib** to **Fold** involves changing $M$ and $X$ by surgery. The surgery uses the membranes in the enrichment, and also makes crucial use of a form of Harer’s stability theorem.

In fact, it is convenient to work with a slight modification of the category **Fold**.

Definition 3.2. Let $\text{Fold}^{\#}$ be the category with the same objects as **Fold**, but where we formally add morphisms which create double folds along a submanifold $C$, i.e. singularities of the form (14). We also formally add inverses of these morphisms. When $n > 1$ we additionally require the manifold $C$ to be homologically trivial. By Example 2.7 this has a canonical enrichment, and we formally add morphism in **Fold** in both directions between the map $f : M \to X$ and the same map $f' : M \to X$ with a double $C$-fold singularity created.
According to 2.13, adding a double $C$-fold together with its canonical enrichment changes $\mathcal{L}(f, \epsilon)$ only by a homotopy. Hence the functor $\mathcal{L} : \text{Fold}^\$ \to \text{FFib}$ extends to a functor $\tilde{\text{Fold}}^\$ \to \text{FFib}$. In the proof of our main theorem, $\text{Fold}^\$ is just a middle step, and it turns out to be more convenient to work with the modified category $\tilde{\text{Fold}}^\$.

We leave it as an exercise to the reader to show that when $n > 1$ there is, in fact, no real difference: if two objects in $\tilde{\text{Fold}}^\$ are cobordant, then they are already cobordant in $\text{Fold}^\$. We shall not need this fact.

We prove that any object of $\tilde{\text{Fold}}^\$ is cobordant to one in $\text{Fib}$ in two steps.

**Definition 3.3.** Let $\text{Fold}_h \subset \text{Fold}^\$ be the subcategory where objects and morphisms are required to satisfy the following conditions.

(i) Folds are hyperbolic, i.e. have no folds of index 0 and 3.

(ii) For all $x \in X$, the manifold $f^{-1}(x) - \Sigma(f)$, i.e. the fiber minus its singularities, is connected.

Let $\tilde{\text{Fold}}_h$ be the category with the same objects, but where we formally add morphisms (in both directions) which create double folds.

Our main result, Theorem 1.8, follows from Theorem 2.22 and the following two propositions.

**Proposition 3.4.** Let $d > 0$. Any enriched folded map $(f : M \to X, \epsilon) \in \text{Fold}^\$ is bordant in the category $\tilde{\text{Fold}}^\$ to an element in $\text{Fold}_h^\$.

**Proposition 3.5.** Let $d = 2$. Any enriched folded map $(f : M \to X, \epsilon) \in \text{Fold}_h^\$ is bordant in the category $\tilde{\text{Fold}}_h^\$ to a fibration from $\text{Fib} \subset \text{Fold}_h^\$.

The proof of the latter uses Harer stability. This is the only part of the whole story in which $d = 2$ is used in an essential way. The reason for the condition $d > 0$ in the first proposition is explained in Appendix A.

## 4 Generalized Harer stability theorem

The main ingredient in the proof of 3.5 is Harer’s stability Theorem 1.9. In this section we will deduce a version of 1.9 for enriched folded maps.
4.1 Harer stability for enriched folded maps

The proof of the following main theorem of this section will be given in Section 4.3. Section 4.2 contains necessary preliminary constructions.

**Theorem 4.1.** Let \((f : M \to X, \epsilon) \in \text{Fold}_h^k\) be an enriched folded map. Let \(U \subset X\) be a closed domain with smooth boundary transversal to the images of the folds and \(\Sigma_1 \subset \Sigma_2\) be compact surfaces with boundary. Let

\[
j : (\partial U \times \Sigma_2) \cup (U \times \Sigma_1) \to M
\]

be a fiberwise embedding over \(\partial U\) whose image does not intersect any fold or membrane and such that the complement of its image in each fiber is connected, even after removing the folds of \(f\).

Then, after possibly changing \((f, \epsilon)\) by a bordism which is constant outside \(\text{Int } U\), the embedding \(j\) extends to an embedding of \(U \times \Sigma_2\).

The following corollary of Theorem 4.1 will be the key ingredient in the proof of Proposition 3.5.

**Corollary 4.2.** Let \((f : M \to X, \epsilon) \in \text{Fold}_h^k\) be an enriched folded map. Let \(C_1, \ldots, C_K \subset M\) be fold components of \(f\) and \(\overline{C}_1, \ldots, \overline{C}_K \subset X\) their image. Assume that the \(\overline{C}_i, i = 1, \ldots, K\), are disjoint and their union bounds together a domain \(P \subset X\), all folds \(C_i\) have the same index with respect to the outward co-orientation of \(\partial P\) and that one of the following conditions holds:

- **M1.** there exists a pure framed membrane \(V\) which spans \(C = \bigcup_{1}^{K} C_j\) and projects to \(P\);
- **M2.** all framed membranes adjacent to \(C\) project to the complement of \(P\) and the folds \(C_j\) are either all positive or all negative (recall that the boundary of a membrane is split into a positive and negative part).

Then \((f, \epsilon)\) is bordant in the category \(\text{Fold}_h^k\) to an element \((\tilde{f}, \tilde{\epsilon})\) such that

- the bordism is constant over the complement of \(\text{Int } P\);
- \((\tilde{f}, \tilde{\epsilon})\) has no more membranes than \((f, \epsilon)\);
- \(\tilde{f}\) admits a basis for a surgery eliminating the fold \(C\).
The surgery eliminates the membrane of $C$ in Case $M1$ and spreads it over $P$ in Case $M2$.

Proof. Consider a slightly smaller domain $U \subset \text{Int } P$, so that $P \setminus \text{Int } U$ is an interior collar of $\partial P$ in $P$. If the index of $C$ with respect to the outward coorientation of $C$ is 1 then the 0-dimensional vanishing cycles over points of $\partial U$ form two sections $s_{\pm} : \partial U \to M$ of the map $f$. In case $M1$ we can assume that one of these sections, say $s_-$, consists of the points of the membrane $V$. The local structure near the membrane allows us to construct fiberwise embeddings $S_- : U \times D^2 \to M$ and $S_+ : \partial U \times D^2 \to M$ such that $S_-|_{U \times 0}$ extends the section $s_-$, $S_+|_{\partial U \times 0} = s_+$ and $S_-(U \times 0) \subset V$. Applying Theorem 4.1 with $\Sigma_1 = D^2$, $\Sigma_2 = \Sigma_1 \amalg D^2$, and $j = S_+ \amalg S_-$, we construct a basis for a membrane eliminating surgery which removes the fold $C$. In the case $M2$ the enrichment structure for the membranes adjacent to $C$ provides us with an extension of sections $s_{\pm}$ and $s_+$ to disjoint fiberwise embeddings $S_{\pm} : \partial U \times D^2 \to M$ such that $S_{\pm}|_{\partial U \times 0} = s_{\pm}$. To conclude the proof in this case we apply 4.1 with $\Sigma_1 = \emptyset$, $\Sigma_2 = D^2 \amalg D^2$, and $j = S_+ \amalg S_-$. Suppose now that the index of $C$ is 2. Consider first the case $M2$. Then the vanishing cycles over $\partial U$ define a fiberwise embedding $\partial U \times S^1 \to M$ over $\partial U$ which extends to a fiberwise embedding $j : \partial U \times A \to M$ disjoint from all folds and their membranes, where $A$ is the annulus $S^1 \times [-1, 1]$. It follows from the definition of the category $\text{Fold}_h$ that the complement of the image of $j$ is fiberwise connected even after all singularities being removed. Hence we can apply Theorem 4.1 with $\Sigma_1 = \emptyset$ and $\Sigma_2 = A$ to construct a basis for a membrane expanding surgery eliminating the fold $C$. Finally, in the case $M1$ each vanishing cycle $j(x \times (S^1 \times 0)), x \in \partial U$, has a unique point $p_x$ which is also in the membrane $V$ of the fold $C$. This point is the center of an embedded disk $D^2 \to A$, and the framed membrane gives a fiberwise embedding $j_1 : U \times D^2 \to M$ over $U$, which over the boundary extends to a fiberwise embedding $j_2 : \partial U \times A \to M$ over $\partial U$. Hence, we are in a position to apply Theorem 4.1 with $\Sigma_1 = D^2$ and $\Sigma_2 = A$.

4.2 Nodal surfaces and their unfolding

Consider a quadratic form

\begin{equation}
Q(x) = x_1^2 + x_2^2 - x_3^2,
\end{equation}

47
Take the handle $H = \{|Q| \leq 1; |x_3| \leq 2\}$ and denote by $K_t$ the level set $\{Q = t\} \cap H$, $t \in [-1, 1]$. When passing through the critical value 0, the level set $K_t$ experiences a surgery of index 1, i.e. changes from a 2-sheeted to a 1-sheeted hyperboloid. The critical level set $K_0$ is the cone $\{x_1^2 + x_2^2 - x_3^2 = 0, |x_3| \leq 2\}$.

Let us fix diffeomorphisms $\beta_{\pm} : \partial_{\pm} H = H \cap \{x_3 = \pm 2\} \to S^1 \times [-1, 1]$ which send boundary circles of $K_t$ to $S^1 \times t$, $t \in I = [-1, 1]$. We will write $\beta_{\pm}(x) = (\beta^S_{\pm}(x), \beta^I_{\pm}(x)) \in S^1 \times [-1, 1]$ for $x \in \partial_{\pm} H$.

We will call the singularity of $K_0$ a node and call surfaces with such singularities nodal. A singular surface $S$ is called $k$-nodal if it is a smooth surface in the complement of $k$ points $p_1, \ldots, p_k \in S$, while each of these points has a neighborhood diffeomorphic to $K_0$.

![Figure 10: Nodal surface and its unfolding](image)

The function $Q$ is a folded map $H \to \mathbb{R}$ with the origin $0 \in H$ as its fold $\Sigma$. If we want to add to it the structure of an enriched folded map, there should be considered four local possibilities for the choice of an enriched membrane depending on the membrane index and the sign of $\Sigma$ as the membrane boundary component:

(i) **Index 0 membrane with negative boundary**, $V_0^- = \{x_1, x_2 = 0, x_3 \leq 0\}$, $K_- = \{0\}, K_+ = \text{Span}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$;

(ii) **Index 1 membrane with positive boundary**, $V_1^+ = \{x_2, x_3 = 0, x_1 \geq 0\}$, $K_- = \text{Span}(\frac{\partial}{\partial x_3}), K_+ = \text{Span}(\frac{\partial}{\partial x_2})$;

(iii) **Index 1 membrane with negative boundary**, $V_1^- = \{x_2, x_3 = 0, x_1 \leq 0\}$, $K_- = \text{Span}(\frac{\partial}{\partial x_3}), K_+ = \text{Span}(\frac{\partial}{\partial x_2})$

48
(iv) Index 2 membrane with positive boundary, $V_2^+ = \{x_1, x_2 = 0, x_3 \geq 0\}$, $K_+ = \{0\}, K_- = \text{Span}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$.

In the cases (i)-(ii) the fold has index 1 and in the cases (iii)-(iv) index 2 with respect to the outward orientation of the boundary of the projection $\overline{V}$ of $V$.

A $k$-nodal fibration $f : Y \to Z$ is a fiber bundle whose fibers are $k$-nodal surfaces, equipped with $k$ disjoint fiberwise embeddings $\psi_i : Z \times K_0 \to Y$ over $Y$, such that the complement of the images of the $\psi_i$ forms a smooth fiber bundle over $Z$.

Let $f : Y \to Z$ be a $k$-nodal fibration. Denote $\hat{Z} := Z \times I^k$ and construct a manifold $\hat{Y}$ together with a map $\hat{f} : \hat{Y} \to \hat{Z}$ as follows. Set

$$\hat{Y} = \left(Y \setminus \bigcup_{i=1}^{k} \psi_i(Z \times K_0)\right) \times I^k \cup (Z \times H \times I^{k-1}) \cup \ldots \cup (Z \times H \times I^{k-1}),$$

where $\sigma_i : Z \times (\partial_+ H \cup \partial_- H) \times I^{k-1} \to \psi_i(Z \times \partial K_0) \times I^k, i = 1, \ldots, k$, are gluing diffeomorphisms defined by the formula

$$\sigma_i(z, x, t_1, \ldots, t_{k-1}) = \psi_i(z, \beta^S_+(x), t_1, \ldots, t_{i-1}, \beta^I_-(x), t_i, \ldots, t_{k-1})$$

for $x \in \partial_\pm H$, $z \in Z$ and $t_j \in I$ for $j = 1, \ldots, k - 1$. The map $f : Y \to X$ extends to a map $\hat{f} : \hat{Y} \to \hat{Z}$ as equal to the projection $(y, t) \mapsto (f(y), t) \in \hat{Z} = Z \times I^k$ for $(y, t) \in \left(Y \setminus \bigcup_{i=1}^{k} \psi_i(K_0 \times Z)\right) \times I^k$ and equal to the map

$$(z, x, t_1, \ldots, t_{k-1}) \mapsto (z, t_1, \ldots, t_{i-1}, Q(x), t_i, \ldots, t_k)$$

on the $i$-th copy of $Z \times H \times I^{k-1}$ glued with the attaching map $\sigma_i$. Note that the map $\hat{f}$ has $k$ fold components which are mapped to the hypersurfaces $C_i = \{t_i = 0\} \subset \hat{Z} = Z \times I^k$. Thus $Z = Z \times 0 = \bigcap_{i=1}^{k} C_i$ is the locus of $k$-multiple intersection of images of fold components of $\hat{f}$. We will call the folded map $\hat{f} : \hat{Y} \to \hat{Z}$ the universal unfolding of the $k$-nodal fibration $f : Y \to Z$.

The following lemma, which follows from the local description of an enriched folded map in a neighborhood of its fold, see Lemma 2.8, shows that the universal unfolding describes the structure of a folded map over a neighborhood of the locus of maximal multiplicity of fold intersection.
Lemma 4.3. Let $f : M \rightarrow X$ be a folded map with cooriented hyperbolic folds. Suppose that all combinations of (projections of) fold components intersect transversally among themselves and with $\partial X$. Let $k$ be the maximal multiplicity of the fold intersection and $Z$ be one of the components of the $k$-multiple intersection. Then $Z \subset X$ is a submanifold with boundary $\partial Z \subset \partial X$, and the restriction $f|_{Y = f^{-1}(Z)} : Y \rightarrow Z$ is a $k$-nodal fibration. Let $\hat{f} : \hat{Y} \rightarrow \hat{Z} = Z \times I^k$ be the universal unfolding of the $k$-nodal fibration $f|_Z$. Then there exist embeddings $\phi : \hat{Z} \rightarrow X$ and $\Phi : \hat{Y} \rightarrow M$ which extend the inclusions $Z \hookrightarrow X$ and $Y \hookrightarrow M$ such that the diagram

\[
\begin{array}{ccc}
\hat{Y} & \xrightarrow{\phi} & M \\
\downarrow{\hat{f}} & & \downarrow{f} \\
\hat{Z} & \xrightarrow{\varphi} & X
\end{array}
\]

commutes. If the folded map is enriched then, depending on indices of the membranes adjacent to the intersecting folds, and signs of the folds as boundary components of the membranes, one can arrange that the pre-images of the membranes and their framings under the embedding $\Phi : \hat{Y} \rightarrow M$ coincide with the submanifolds $Z \times V^+_j \times I^{k-1}$, $j = 0, 1, 2$, and their defined above model framings in the corresponding copies of $Z \times H \times I^{k-1}$ in $\hat{Y}$.

Notice that Theorem 1.9 is easily generalized to $k$-nodal fibrations as follows.

**Theorem 4.4** (Geometric form of Harer stability for nodal fibrations). Let $\Sigma_1 \subset \Sigma_2$ be compact surfaces with boundary (not necessarily connected). Let $f : M \rightarrow X$ be a $k$-nodal fibration with $T_\infty$ ends, and let

\[ j : (\partial X \times \Sigma_2) \cup (X \times \Sigma_1) \rightarrow M \]

be a fiberwise embedding over $X$, such that its image in each fiber is disjoint from the nodes, and that in each fiber the complement of its image is connected, even after removing all nodes.

Then, after possibly changing $f : M \rightarrow X$ by a bordism which is the trivial bordism over $\partial X$, the embedding $j$ extends to an embedding of $X \times \Sigma_2$, still disjoint from nodes and with connected complement.
4.3 Proof of Theorem 4.1

Let $(f : M \to X, e)$ be an enriched folded map from $\text{Fold}_k^X$. Let $U \subset X$ be a compact domain with smooth boundary. We assume that all combinations of (projections of) fold components intersect transversally among themselves and with $\partial U$. Let $k$ be the maximal multiplicity of the fold intersection. We denote by $U_j$, $j = 1, \ldots, k$, the set of intersection points of multiplicity $\geq j$ in $U$, and set $U_0 := U$. Thus we get a stratification $U = \bigcup_{0 \leq j \leq k} U_j$. Set $M_j = f^{-1}(U_j \setminus U_{j+1})$, and we have $U_j = \bigcup_{i \geq j} U_i$. Note that $U_k$ is a closed submanifold of $U$ with boundary $\partial U_k \subset \partial U$. The map $f_k = f|_{M_k} : M_k \to U_k$ is a $k$-nodal fibration. The membranes which are not adjacent to the fold components intersecting along $U_k$, together with their framings define fiberwise embeddings $s_1, \ldots, s_l : U_k \times D^2 \to M_k$ over $U_k$, disjoint from the image of $j$ and from each other.

Let us apply Theorem 4.4, the nodal version of Harer’s stability theorem, to the nodal fibration $f_k$ and the fiberwise embedding

\[ \widetilde{j} = j \sqcup \bigcup_i s_i : (\partial U_k \times \widetilde{\Sigma}_2) \cup (U_k \times \widetilde{\Sigma}_1) \to M_k, \]

where $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ are disjoint unions of $\Sigma_1$ and $\Sigma_2$, respectively, with $l$ copies of the disc $D^2$. As a result, we find a bordism $F_k : W_k \to Y_k$ (in the class of $k$-nodal fibrations) between the $k$-nodal fibrations $f_k : M_k \to U_k$ and $f'_k : M'_k \to U'_k$ which is constant over $\mathcal{O}p \partial U_k$ and such that the fiberwise embedding $\widetilde{j}$, extends to a fiberwise embedding

\[ J : (\partial Y_k = \partial U_k \times [0, 1]) \cup (\partial_+ Y_k = U'_k) \times \widetilde{\Sigma}_2 \to W_k. \]

Let $\widehat{F}_k : \widehat{W}_k \to \widehat{Y}_k = Y_k \times I^k$ be the universal unfolding of the $k$-nodal fibration $F_k : W_k \to Y_k$. We view $\widehat{F}_k$ as a bordism between the universal unfoldings $\widehat{f}_k : \widehat{M}_k \to \widehat{U}_k = U_k \times I^k$ and $\widehat{f}_k' : \widehat{M}_k \to \widehat{U}_k' = U'_k \times I^k$ of the $k$-nodal fibrations $f_k$ and $f_k'$. The embeddings $J$ extends to a fiberwise embedding

\[ \widehat{J} : (\partial Y_k = \partial Y_k \times I^k) \cup \widehat{U}_k' \times \widetilde{\Sigma}_2 \to \widehat{W}_k. \]

over $\widehat{Y}_k$. According to Lemma 4.3 the restriction of $f$ to a tubular neighborhood of $U_k$ is isomorphic to the universal unfolding $\widehat{f}_k : \widehat{M}_k \to \widehat{U}_k$ of the
$k$-nodal fibration $f_k$. In other words, there exist embeddings $\varphi_k : \hat{U}_k \to X$ and $\Phi_k : \hat{M}_k \to M$ which extend the inclusions $U_k \hookrightarrow X$ and $M_k \hookrightarrow M$ such that the diagram

$\begin{array}{ccc}
\hat{M}_k & \xrightarrow{\Phi_k} & M \\
\downarrow & & \downarrow \quad f \\
\hat{X}_k & \xrightarrow{\varphi_k} & X
\end{array}$

commutes. Moreover, $(\varphi_k, \Phi_k)$ can be chosen in such a way that the framed membranes adjacent to intersecting folds correspond to model framed membranes of Lemma 4.3.

Let us glue the bordism $\tilde{F}_k : \hat{W}_k \to \hat{Y}_k$ to the trivial bordism $F = f \times \text{Id} : W = M \times I \to Y = X \times I$ using the attaching maps $(\varphi_k, \Phi_k)$:

$\tilde{F} : \hat{W}_k \cup M \times I \to \hat{Y}_k \cup X \times I$

and then smooth the corners. The folded map $\tilde{F} : \hat{W} \to \hat{Y}$ resulting from this construction is a bordism between $f : M \to X$ and $f' : M' \to X'$, which is trivial over the complement of $\varphi_k(\hat{U}_k) \subset X$. Model framed membranes of Lemma 4.3 provide us with a canonical extension to $\hat{W}_k$ of framed membranes adjacent to $Y_k$. On the other hand, the restriction of $\hat{J}$ to $\hat{Y}_k \times \coprod_i D^2 \subset \hat{Y}_k \times \Sigma_2$ allows us to extend all the other framed membranes. Thus the constructed map $\tilde{F} : \hat{W} \to \hat{Y}$ together with the extended enrichment is a bordism in the category $\text{Fold}^\delta_k$. The fiberwise embedding $\tilde{J} : \hat{U}_k' \times \Sigma_2 \to M_k'$ extends to a closed neighborhood $\Omega \supset U_k'$ in $M'$. Consider the domain $U'' = U' \setminus \text{Int } \Omega$. The maximal multiplicity of fold intersection over $U''$ is equal to $k - 1$. Hence, we can repeat the previous argument to extend $\tilde{J}$ over the stratum $U_k'' \cup \partial U$, possibly after changing it by another bordism in the category $\text{Fold}^\delta_k$. Continuing inductively we find the required extension to the whole domain bounded by $\partial U$.

\[ \square \]

5 Proof of Theorem 1.8

As it was already mentioned above, Theorem 1.8 follows from Theorem 2.22 and Propositions 3.4 and 3.5. We prove these propositions in the next two sections.
5.1 Proof of Proposition 3.4

Let \((f, e)\) be an enriched folded map with \(f : M \to X\). First, we get rid of non-hyperbolic folds. Let \(Z\) be a non-hyperbolic fold component. The following procedure, which is illustrated in Fig. 11, replaces \(Z\) by a parallel hyperbolic fold.

Let \(V\) be the membrane of \(Z\), and \(N = \overline{Z} \times [-2, 0] \subset X\) be an interior collar of \(\overline{Z} = \overline{Z} \times 0\) in \(X \setminus \text{Int} \overline{V}\). Let us recall that the map \(f\) has a standard end, where it is equivalent to the trivial fibration \(T_\infty \times X \to X\).

Let \(A = \overline{Z} \times [-2, -1] \subset N\) and \(B = \overline{Z} \times [-2, -1] \subset N\), so that \(N = A \cup B\). Let us lift \(A\) to an annulus \(\overline{Z} \times (-1, 0) \subset M\), \(z \in T_\infty\). Write \(\overline{Z}_1 := \overline{Z} \times (-1), \overline{Z}_2 := \overline{Z} \times (-2), Z_1 := z \times \overline{Z}_1, Z_2 := z \times \overline{Z}_2\), so that \(\partial A = \overline{Z}_1 \cup \overline{Z}_2\) and \(\partial A = Z_1 \cup Z_2\). Using Lemma 2.1 we can create a double fold with the membrane \(A\) and folds \(Z_2\) of index 1 and \(Z_1\) of index 0 with respect to the coorientation of \(\overline{Z}_1, \overline{Z}_2\) by the second coordinate of the splitting \(N = \overline{Z} \times [-2, 0]\). The folds \(Z_1\) and \(Z\) have index 0 with respect to the outward coorientation of \(\partial B\), and hence we can use a membrane expanding surgery to kill both folds, \(Z\) and \(Z_1\), and spread their membranes over \(B\). As a result of this procedure we have replaced \(Z\) by a hyperbolic fold \(Z_2\).

![Figure 11: Replacing an elliptic fold by a hyperbolic one](image-url)
It remains to make the fibers of $f|_{M \setminus \Sigma(f)}$ connected. We begin with the following lemma from [MW07].

**Lemma 5.1.** Let $M \rightarrow X$ be an enriched folded map without folds of index 0. Then there exist disjoint $n$-discs $D_i$, $i = 1, \ldots, K$, embedded into $M$, such that

- $f|_{D_i}$ is embedding $D_i \rightarrow \text{Int} X$ for each $i = 1, \ldots K$;
- $V \cap \bigcup_{i=1}^{K} D_i = \emptyset$, where $V \subset M$ is the union of all membranes;
- for each $x \in X$ each irreducible component of $\pi^{-1}(x)$ intersects at least one of the discs $D_i$ at an interior point.

**Proof.** Note first, that the statement is evident for any fixed $x \in X$. Hence, without controlling the disjointness of the disks $D_i$ the statement just follows from the compactness of $X$. One can choose the required disjoint disks $D_i$ using the following trick: fix a function $h : M \rightarrow \mathbb{R}$ and take the disks $D_i$ such that each disk belongs to its own level hypersurface of $h$. When we choose such disks for $x \in X$ one needs to avoid the points $z \in F_x = f^{-1}(x)$ where the level hypersurface is tangent to the fiber $F_x$. It can be done by a small perturbation of disks, if the complement of all “bad” points (for all $x \in X$) is open and dense in $F_x$ for all $x \in X$. But Thom’s jet transversality theorem asserts that this is a generic situation for functions $h : M \rightarrow \mathbb{R}$.

Let the disks $D_i \subset M$ be as in Lemma 5.1. Let us also consider discs $\Delta_i = \overline{D_i} \times y_i$, $i = 1, \ldots, K$, where $y_1, \ldots, y_K$ are disjoint points at the end $T_{\infty}$ of the fiber $F$. Next, using each pair $(D_i, \Delta_i)$, $i = 1, \ldots, K$, as a basis for an index 1 fold creating surgery we create new folds $\Sigma_i = \partial \overline{D_i}$ of index 1 with the discs $\overline{D_i}$ serving as membranes, while making all the fibers connected, see Remark 2.18.

As a result of this step we arrange all fibers $F_x = f^{-1}(x) \setminus \Sigma(f), x \in X$, to be connected. This completes the proof of Proposition 3.4.

\[\blacksquare\]
5.2 Proof of Proposition 3.5

Let $V_1, \ldots, V_N$ be the collection of (connected) framed membranes of $f = (f, \epsilon)$. We are going to inductively remove all of them. If the membrane $V_1$ is pure then, in view of Corollary 4.2 we can assume, after a possible change of $f$ by a bordism in the category $\text{Fold}^1_\epsilon$, that there is a basis for the fold surgery which removes $\partial V_1$ together with the membrane.

Suppose now that the membrane $V_1$ is mixed. Consider first the case $n > 1$. By our assumption each boundary component $\Sigma$ of $\overline{V}_1$ bounds in this case a domain $U_C \subset X$. If $\partial X \neq \emptyset$ then the domain $U_C$ is uniquely defined. If $X$ is closed then we fix a point $p \in X \setminus \overline{V}_1$ and denote by $U_C$ the domain which does not contain $p$. There exists exactly one boundary component $C$ of $V_1$ such that $\overline{V}_1 \subset U_C$. For any other boundary component $C'$ of $V_1$ we have $U_{C'} \subset X \setminus \text{Int} \overline{V}_1$. We will refer to $C$ as the exterior fold of $V_1$, and to all other boundary folds of $V_1$ as its interior folds. First, we apply Corollary 4.2 in order to create a basis for a surgery which eliminates each interior fold $C'$ and spreads the membrane $V_1$ over $U_{C'}$. After applying this procedure to all interior folds of $V_1$ we will come to the situation when $\partial V_1 = C$ is connected, and hence the membrane $V_1$ is pure, which is the case we already considered above. Applying the same procedures subsequently to the membranes $V_2, \ldots, V_N$ we complete the proof of Proposition 3.5 when $n > 1$.

Finally consider the case $n = 1$, i.e. $X = I$ or $X = S^1$. We assume for determinacy that $X = I$, i.e. $f$ is a Morse function. A mixed framed membrane of $f$ connects two critical points $p_1, p_2$ of $f$ of index 1 and 2, and with critical values $c_1, c_2, c_1 < c_2$, respectively. For a small $\epsilon > 0$ let us consider vanishing circles $S_1 \subset F_{c_1 + \epsilon}, S_2 \subset F_{c_2 - \epsilon}$ of critical points $p_1$ and $p_2$, where we denote $F_t := f^{-1}(t)$, $t \in \mathbb{R}$. Choose an oriented embedded circle $S_1' \subset F_{c_1 + \epsilon}$ which intersects $S_1$ transversally in one point. Let $A$ denote the annulus $S^1 \times [-1, 1]$, and $D \subset \text{Int} A$ be a small 2-disc centered at a point $q \in S^1 \times 0$. Choose a fiberwise embedding $k : I_\epsilon \times D \to M_\epsilon = f^{-1}(I_\epsilon)$ over $I_\epsilon = [1 + \epsilon, 2 - \epsilon]$, such that $k(I_\epsilon \times 0) \subset V$ and the linearization of $k$ along $k(I_\epsilon \times 0)$ provides the given framing of the membrane $V$. Let us also choose embeddings $j_1 : A \to F_{1+\epsilon}$ and $j_2 : A \to F_{2-\epsilon}$ such that $j_1|_{S_1' \times 0} = S_1'$, $j_2|_{S_1' \times 0} = S_2$, and for $x \in D$ we have $j_1(x) = k(1 + \epsilon, x), j_2(x) = k(2 - \epsilon, x)$. Let us apply Theorem 4.1 to the Morse function (= the folded map) $f' = f_{M_\epsilon} : M_\epsilon \to I_\epsilon$ with $\Sigma_2 = A$, $\Sigma_1 = D \subset \Sigma_1$ and the embedding $j : (\partial I_\epsilon \times \Sigma_2) \cup (I_\epsilon \times \Sigma_1) \to M$, which is equal to $j_1 \cup j_2$ on $\partial I_\epsilon \times A$ and
to $k$ on $I_\epsilon \times D$. Theorem 4.1 then allows us, after possibly changing $f$ in its bordism class in $\text{Fold}_h$, to extend $j_1 \sqcup j_2$ to a fiberwise over $I_\epsilon \times I_\epsilon$ map $j : I_\epsilon \times A \to M$ which coincides with $k$ on $I_\epsilon \times D$. Choosing a metric for which the cylinder $f(I_\epsilon \times (S^1 \times 0))$ is foliated by gradient trajectories, one of which is $V$, we come to the situation when we can apply the standard Morse theory cancellation lemma, see for instance [Mi65], to kill both critical points $p_0$ and $p_1$ together with their membrane $V$. The cancellation deformation is inverse to the double fold creation, and hence it can be realized by a bordism in the category $\text{Fold}_h$.

This completes the proof of Theorem 1.8.

6 Miscellaneous

6.1 Appendix A: From wrinkles to double folds

6.1.1 Cusps

Let $n > 1$. Given a map $f : M \to X$, a point $p \in \Sigma(f)$ is called a cusp type singularity or a cusp of index $s + \frac{1}{2}$ if near the point $p$ the map $f$ is equivalent to the map

$$\mathbb{R}^{n-1} \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^{n-1} \times \mathbb{R}^1$$

given by the formula

$$y, z, x \mapsto \left( y, z^3 + 3y_1z - \sum_{i=1}^{s} x_i^2 + \sum_{j=s+1}^{d} x_j^2 \right) \tag{36}$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $z \in \mathbb{R}^1$, $y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$.

The set of cusp points is denoted by $\Sigma^{11}(f)$. It is a codimension 1 submanifold of $\Sigma(f)$ which is in the above canonical coordinates is given by $x = (x_1, \ldots, x_d) = 0, y_1 = z = 0$. The vector field $\frac{\partial}{\partial y_1}$ along $\Sigma^{11}(f)$ is called the characteristic vector field of the cusp locus. It can be invariantly defined as follows. Note that for any point $p \in \Sigma^{11}(f)$ there exists a neighborhood $\Omega \ni f(p)$ in $X$ such that $\Omega \cap f(\Sigma(f))$ can be presented as a union of two
manifolds $\Sigma_\pm$ with the common boundary $\partial \Sigma_\pm = \Omega \cap f(\Sigma^{11}(f))$, the common tangent space $T = T_{f(p)}\Sigma_\pm = df(T_pM)$ at the point $f(p)$, and the common outward coorientation $\nu$ of $T' = T_p\partial \Sigma_\pm \subset T$. On the other hand, the differential $df$ defines an isomorphism

$$T_pM/(\ker df + T_p\Sigma(f)) \rightarrow T/T'.$$

Hence, there exists a vector field $Y$ transversal to $\ker df + T\Sigma(f)$ in $TM$ along $\Sigma^{11}(f)$, whose projection defines the coorientation $\nu$ of $T'$ in $T$ for all points $p \in \Sigma^{11}(f)$. One can show that any vector field $Y$ defined that way coincides with the vector field $\frac{\partial}{\partial y_1}$ for some local coordinate system in which the map $f$ has the canonical form \([36]\).

Note that the line bundle $\lambda$ over $\Sigma^{11}(f)$ is always trivial. Indeed, $\lambda$ can be equivalently defined as the kernel of the quadratic form $d^2f : \ker df \rightarrow \text{Coker } df$, and thus one has an invariantly defined cubic form $d^3f : \lambda \rightarrow \text{Coker } df$ which does not vanish. The bundle $\ker df|_{\Sigma^{11}(f)}$ can be split as $\ker_+ \oplus \ker_- \oplus \lambda$, so that the quadratic form $d^2f$ is positive definite on $\ker_+$ and negative definite on $\ker_-$.  

### 6.1.2 Wrinkles and wrinkled mappings

Consider the map

$$w(n + d, n, s) : \mathbb{R}^{n-1} \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}^1$$

given by the formula

$$(y, z, x) \mapsto \left(y, z^3 + 3(|y|^2 - 1)z - \sum_{i=1}^{s} x_i^2 + \sum_{j=s+1}^{d} x_j^2\right),$$

where $y \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}^1$, $x \in \mathbb{R}^d$ and $|y|^2 = \sum_{i=1}^{n-1} y_i^2$.

The singularity $\Sigma(w(n + d, n, s))$ is the $(n - 1)$-dimensional sphere

$$S^{n-1} = \mathbb{S}^{n-1} \times 0 \subset \mathbb{R}^n \times \mathbb{R}^d,$$

whose equator $\Sigma^{11}(f) = \{|y| = 1, z = 0, x = 0\} \subset \Sigma(w(n + d, n, s))$ consists of cusp points of index $s + \frac{1}{2}$. The upper hemisphere $\Sigma(w) \cap \{z > 0\}$ consists of folds of index $s$, while the lower one $\Sigma(w) \cap \{z < 0\}$ consists of folds of
index $s + 1$. The radial vector field $Y = \sum_{j=1}^{n-1} y_j \frac{\partial}{\partial y_j}$ serves as a characteristic vector field of the cusp locus.

Although the differential $dw(n + d, n, s) : T(\mathbb{R}^{n+d}) \to T(\mathbb{R}^n)$ degenerates at points of $\Sigma(w)$, it can be canonically regularized over $\mathcal{O}p_{\mathbb{R}^{n+d}D^n}$, an open neighborhood of the disk $D^n = D^n \times 0 \subset \mathbb{R}^n \times \mathbb{R}^d$. Namely, we can substitute the element $3(z^2 + |y|^2 - 1)$ in the Jacobi matrix of $w(n + d, n, s)$ by a function $\gamma$ which coincides with $3(z^2 + |y|^2 - 1)$ on $\mathbb{R}^{n+d} \setminus \mathcal{O}p_{\mathbb{R}^{n+d}D^n}$ and does not vanish along the $n$-dimensional subspace $\{x = 0\} = \mathbb{R}^n \times 0 \subset \mathbb{R}^{n+d}$. The new bundle map $\mathcal{R}(dw) : T(\mathbb{R}^{n+d}) \to T(\mathbb{R}^n)$ provides a homotopically canonical extension of the map $dw : T(\mathbb{R}^{n+d} \setminus \mathcal{O}p_{\mathbb{R}^{n+d}D^n}) \to T(\mathbb{R}^n)$ to an epimorphism (fiberwise surjective bundle map) $T(\mathbb{R}^{n+d}) \to T(\mathbb{R}^n)$. We call $\mathcal{R}(dw)$ the regularized differential of the map $w(n + d, n, s)$.

A map $f : U \to X$ defined on an open ball $U \subset M$ is called a wrinkle of index $s + \frac{1}{2}$ if it is equivalent to the restriction $w(n + d, n, s)|_{\mathcal{O}p_{\mathbb{R}^{n+d}D^n}}$. We will use the term “wrinkle” also for the singularity $\Sigma(f)$ of a wrinkle $f$.

Notice that for $n = 1$ the wrinkle is a function with two nondegenerate critical points of indices $s$ and $s + 1$ given in a neighborhood of a gradient trajectory which connects the two points.

A map $f : M \to X$ is called wrinkled if there exist disjoint open subsets $U_1, \ldots, U_l \subset M$ such that the restriction $f|_{M \setminus U} = \bigcup U_i$ is a submersion (i.e. has rank equal $n$) and for each $i = 1, \ldots, l$ the restriction $f|_{U_i}$ is a wrinkle.

The singular locus $\Sigma(f)$ of a wrinkled map $f$ is a union of $(n-1)$-dimensional spheres (wrinkles) $S_i = \Sigma(f|_{U_i}) \subset U_i$. Each $S_i$ has a $(n-2)$-dimensional
equator $S'_i \subset S_i$ of cusps which divides $S_i$ into two hemispheres of folds of two neighboring indices. The differential $df : T(M) \to T(X)$ can be regularized to obtain an epimorphism $R(df) : T(M) \to T(X)$. To get $R(df)$ we regularize $df|_{U_i}$ for each wrinkle $f|_{U_i}$.

The following Theorem 6.1 is the main result of the paper [EM1]:

**Theorem 6.1** (Wrinkled mappings). Let $F : T(M) \to T(X)$ be an epimorphism which covers a map $f : M \to X$. Suppose that $f$ is a submersion on a neighborhood of a closed subset $K \subset M$, and $F$ coincides with $df$ over that neighborhood. Then there exists a wrinkled map $g : M \to X$ which coincides with $f$ near $K$ and such that $R(dg)$ and $F$ are homotopic rel. $T(M)|_K$. Moreover, the map $g$ can be chosen arbitrarily $C^0$-close to $f$ and with arbitrarily small wrinkles.

### 6.1.3 Cusp eliminating surgery

We are going to modify each wrinkle to a spherical double fold using *cusp elimination surgery*, which is one of the surgery operations studied in [El72]. Unlike fold elimination surgeries described above in Section 2.4 cusp elimination surgery does not affect the underlying manifold and changes a map by a homotopic one. For maps $\mathbb{R}^2 \to \mathbb{R}^2$ the operation is shown on Fig. 13.

![Figure 13: Cusp eliminating surgery in the case $n = 2, d = 0$](image-url)
Definition 6.2. Let $C \subset \Sigma(f)$ be a connected component of the cusp locus. Let $Y$ be the characteristic vector field of $C$. Suppose that the bundles $\text{Ker}_-, \text{Ker}_+$ and $\lambda$ over $C$ are trivialized, respectively, by the frames
\[
\left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s} \right), \left( \frac{\partial}{\partial x_{s+1}}, \ldots, \frac{\partial}{\partial x_d} \right) \quad \text{and} \quad \frac{\partial}{\partial z}.
\]
A basis for a cusp eliminating surgery consists of an $(n-1)$-dimensional submanifold $A \subset M$ bounded by $C$, together with an extension of the above framing as a trivialization of the normal bundle $\nu$ of $A$ in $M$, such that
\begin{itemize}
  \item $f|_{\text{Int} A} : \text{Int} A \to X$ is an immersion;
  \item the characteristic vector field $Y$ is tangent to $A$ along $C$, and inward transversal to $C = \partial A$;
  \item $\frac{\partial}{\partial x_j} \in \text{Ker} df$ for all $j = 1, \ldots, d$.
\end{itemize}
Let us extend $A$ to a slightly bigger manifold $\tilde{A}$ such that $\text{Int} \tilde{A} \supset A$, and extend the framing over $\tilde{A}$. One can show (see [El72] and [Ar76]) that there exists a splitting $U \to \tilde{A} \times \mathbb{R} \times \mathbb{R}^d$ of a tubular neighborhood of $\tilde{A}$ in $M$, such that in the corresponding local coordinates $y \in \tilde{A}, z \in \mathbb{R}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ the map $f$ can be presented as a composition
\[
U \xrightarrow{F} \tilde{A} \times \mathbb{R} \xrightarrow{h} X,
\]
where $h$ is an immersion and $F$ has the form
\[
F(y, z, x) = \left( y, z^3 + \varphi_0(\tilde{y}) \sigma \left( \frac{1}{\epsilon} (z^2 + \sum_{i=1}^{d} x_i^2) \right) z - \sum_{i=1}^{s} x_i^2 + \sum_{s+1}^{d} x_j^2 \right),
\]
where the function $\varphi_0 : \mathbb{R} \to \mathbb{R}$ satisfies $\varphi_0 < 0$ on $\text{Int} A \subset \tilde{A}$ and $\varphi_0 > 0$ on $\tilde{A} \setminus A$, $\sigma : [0, 1] \to [0, 1]$ is a cut-off function equal to 1 near 0 and to 0 near 1, and $\epsilon > 0$ is small enough.

Consider another function $\psi_1 : \tilde{A} \to (-\infty, 0)$ which coincides with $\varphi_0$ outside $\text{Op} A \subset \tilde{A}$ and such that $|\psi_1| \leq |\varphi_0|$. Denote $\varphi_t := (1-t)\varphi_0 + t\varphi_1$, $t \in [0, 1]$, and consider homotopies
\[
F_t(y, z, x) = \left( y, z^3 + \varphi_t(\tilde{y}) \sigma \left( \frac{1}{\epsilon} (z^2 + \sum_{i=1}^{d} x_i^2) \right) z - \sum_{i=1}^{s} x_i^2 + \sum_{s+1}^{d} x_j^2 \right),
\]
where $\text{Ker}_-$, $\text{Ker}_+$ and $\lambda$ over $C$ are trivialized, respectively, by the frames
\[
\left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s} \right), \left( \frac{\partial}{\partial x_{s+1}}, \ldots, \frac{\partial}{\partial x_d} \right) \quad \text{and} \quad \frac{\partial}{\partial z}.
\]
\((y, z, x) \in U\), and \(f_t = h \circ F_t : U \to X\). The homotopy \(f_t\) is supported in \(U\) and hence can be extended to the whole manifold \(M\) as equal to \(f\) on \(M \setminus U\). The next proposition is straightforward.

**Proposition 6.3.** 1. The homotopy \(f_t\) removes the cusp component \(C\).

The map \(f_1\) coincides with \(f_0\) outside \(U\), has only fold type singularities in \(U\), and

\[\Sigma(f_1|_U) = \{x = 0, z^2 = -\tilde{\varphi}_1(\tilde{y})\}\.\]

2. Suppose that \(\Sigma(f) \setminus C\) consists of only fold points and that the restriction of the map \(f\) to \(\Sigma \cup A\) is an embedding. Then the restriction \(f_1|_{\Sigma(f_1)} : \Sigma(f_1) \to X\) is an embedding provided that the neighborhood \(U \supset A\) in the surgery construction is chosen small enough.

**6.1.4 From wrinkles to double folds**

**Proposition 6.4.** Let

\[w(n + d, n, s) : \mathbb{R}^{n-1} \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^{n-1} \times \mathbb{R}^1\]

be the standard wrinkled map with the wrinkle \(S^{n-1} \subset \mathbb{R}^n \times \mathbb{R}^d\). Suppose that \(n > 1\). Then

a) there exists an embedding

\[h : D^{n-1} \to \mathcal{O}_{\mathbb{R}^{n+d}}D^n\]

and a framing \(\mu\) of the normal bundle to \(A = h(D^{n-1}) \subset \mathbb{R}^n \times \mathbb{R}^d\) such that the pair \((A, \mu)\) forms a basis for a surgery eliminating the cusp \(\Sigma^11(w) = S^{n-2} \subset S^{n-1}\) of the wrinkle;

b) if \(d > 0\) then one can arrange that the map \(w(n + d, n, s)\) restricted to \(\Sigma(w(n + d, n, s)) \cup A\) is an embedding.

**Proof.** It is easy to construct an embedding \(h_0\) and a framing \(\mu\) to satisfy a). The construction is clear from Fig.14. The manifold \(A\) in this case is obtained from the boundary of the upper semi-ball \(|y|^2 + z^2 \leq 1 + \delta, z \geq 0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}\) by removing the open disc \(D^{n-1} = \{z = 0, |y| < 1\}\, and then smoothing the corner. Here \(\delta > 0\) should be chosen small enough so that \(A\) lie in the prescribed neighborhood of the wrinkle.
The framing $\mu$ is given by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}$ and the normal vector field to $A$ in $\mathbb{R}^{n-1} \times \mathbb{R}$ which coincides with $\frac{\partial}{\partial z}$ near $\partial A$.

Unfortunately the embedding $h_0$ does not satisfies the property b). However, if $d > 0$ this can be corrected as follows. We suppose that the index $s > 0$ (if $s = 0$ then one should start with an embedding $h$ obtained by smoothing the boundary of the lower semi-ball). Let us denote by $g_0$ the composition $w \circ h_0 : D^{n-1} \to \mathbb{R}^{n-1} \times \mathbb{R}$, and by $g_0^{n-1}$ and $g_1^{n-1}$ the projections of $g_0$ to the first and second factors, respectively.

For any $\epsilon > 0$ one can choose $\delta$ in the construction of $h_0$ so small that there exists a function $\alpha : D^{n-1} \to [0, \epsilon)$ such that

- $\alpha$ vanishes along $\partial D^{n-1}$ together with all its derivatives;
- $\alpha|_{\text{Int } D^{n-1}} > 0$;
- the function $g^1 = g_1^{n-1} - \alpha$ has a unique interior critical point, the minimum, at $0 \in D^{n-1}$;
- the map $g = (g_0^{n-1}, g^1) : D^{n-1} \to \mathbb{R}^{n-1} \times \mathbb{R}$ is an embedding, and the image $g(\text{Int } D^{n-1})$ does not intersect the image of the wrinkle.
Next, take an embedding \( h : D^{n-1} \to \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^d \) given by

\[
(y, z) = h_0(u), x_1 = \sqrt{\alpha(u)}, x_j = 0, j = 2, \ldots, d,
\]

\( y \in \mathbb{R}^{n-1}, z \in \mathbb{R}, x = (x_1, \ldots, x_d) \in \mathbb{R}^d \). Then we have \( g = w \circ h \), and hence the embedding \( h \) satisfies the property b) of Proposition 6.4.

Combining Propositions 6.4 and 6.3 we get

**Proposition 6.5.** There exists a \( C^0 \)-small perturbation of the map \( w(n + d, n, s) \mid_{\mathcal{CP}_{n+d} D^n} \) in an arbitrarily small neighborhood of the embedded disk \( h(D^n) \) constructed in 6.4 such that the resulting map \( \tilde{w}(n + d, d, s) \) is a special folded map with only one double fold (of index \( s + \frac{1}{2} \)). Moreover, the regularized differentials of \( w(n + d, n, s) \) and \( \tilde{w}(n + d, d, s) \) are homotopic.

Theorem 6.1 and Proposition 6.5 yield Theorem 2.3.

### 6.2 Appendix B: Hurewicz theorem for oriented bordism

As before, let \( \Omega_* = \Omega_*^{SO}(-) \) denote oriented bordism. In this appendix we give a proof of the following well known lemma.

**Lemma 6.6.** Let \( f : X \to Y \) be a continuous map of topological spaces. Then the following statements are equivalent.

(i) \( f_* : H_k(X) \to H_k(Y) \) is an isomorphism for \( k < n \) and an epimorphism for \( k = n \).

(ii) \( f_* : \Omega_k(X) \to \Omega_k(Y) \) is an isomorphism for \( k < n \) and an epimorphism for \( k = n \).

In particular, \( f \) induces an isomorphism in homology in all degrees if and only if it does so in oriented bordism.

For a pair \((X, A)\) of spaces, let \( \Omega_n(X, A) = \Omega_n^{SO}(X, A) \) denote the set of bordism classes of continuous maps of pairs

\[
f : (M^n, \partial M^n) \to (X, A)
\]
for smooth oriented compact manifolds $M^n$ with boundary $\partial M$. There is a “cycle map” $\Omega_*(X, A) \to H_*(X, A)$ that maps the class of $f$ to $f_*([M]) \in H_*(X, A)$.

Recall that $\Omega_*(X, A)$ is a “generalized homology theory”, i.e. it satisfies the same formal properties as singular homology (the Eilenberg-Steenrod axioms except the dimension axiom). In particular we have a long exact sequence for pairs of spaces. If $f : X \to Y$ is an arbitrary map, then we can replace $Y$ by the mapping cylinder of $f$. The axioms implies there is a natural long exact sequence

$$\cdots \to \Omega_n(X) \to \Omega_n(Y) \to \Omega_n(C(f), x) \to \Omega_{n-1}(X) \to \cdots$$

where $C(f)$ is the mapping cone and $x \in C(f)$ is the cone point.

It seems well known that a map $X \to Y$ induces an isomorphism in oriented bordism if and only if it induces an isomorphism in singular homology (with $\mathbb{Z}$-coefficients). This can be seen e.g. from the Atiyah-Hirzebruch spectral sequence. We offer a more geometrical argument, based on Hurewicz’ theorem. Recall that this says that a map $f : X \to Y$ of simply connected spaces is a weak equivalence if and only if it is a homology equivalence.

We will use the **unreduced suspension** $\Sigma X$ of a space $X$. This is the union $CX \cup_X CX$ of two cones. We will regard $\Sigma X$ as a pointed space, using one of the cone points (which we denote $N$) as basepoints. It is easily seen that $\Sigma X$ is simply connected if and only if $X$ is arcwise connected. Also it follows from the Mayer-Vietoris exact sequence that $h_*(\Sigma X, N) \cong \tilde{H}_{*-1}(X)$ for any homology theory $h_*$ (e.g. singular homology or oriented bordism).

**Lemma 6.7.** A map $f : X \to Y$ of topological spaces induces a homology isomorphism if and only if the unreduced suspension $\Sigma f$ is a weak equivalence.

**Proof.** If $\Sigma f$ is a weak equivalence, then it induces a homology isomorphism $(\Sigma f)_*$, but $\tilde{H}_n(X) = H_{n+1}(\Sigma X)$, so $f_*$ is an isomorphism as well.

For the converse assume $f_*$ is an isomorphism. Then $\pi_0(f)$ is an isomorphism so we can assume that $X$ and $Y$ are arc connected. Then $\Sigma X$ and $\Sigma Y$ are simply-connected. Then Hurewicz’ theorem implies that $(\Sigma f)$ is a weak equivalence.

Below we will sketch a proof of the following Hurewicz theorem for oriented bordism.
Proposition 6.8. A map \( f : X \rightarrow Y \) of simply connected spaces is a weak homotopy equivalence if and only if it induces an isomorphism in oriented bordism.

Using this we can easily prove the main theorem

**Theorem 6.9.** A map \( f : X \rightarrow Y \) of topological spaces is a homology isomorphism if and only if it induces an isomorphism in oriented bordism.

**Proof.** We proved above that \( f \) is a homology equivalence if and only if \( \Sigma f \) is a weak equivalence. Using the bordism Hurewicz theorem we can prove in the same way that \( f \) is a bordism isomorphism if and only if \( \Sigma f \) is a weak equivalence.

We sketch a proof of a bordism Hurewicz theorem. Let \( h_n : \pi_n(X, x_0) \rightarrow \Omega_n(X, x_0) \) be the map that maps the homotopy class of a map \( (D^n, \partial D^n) \rightarrow (X, x_0) \) to the bordism class of the same map. The composition of \( h_n \) with the cycle map is the classical Hurewicz homomorphism.

**Lemma 6.10.** Let \( n \geq 2 \). Let \( (X, x_0) \) be a pointed topological space with \( \pi_k(X, x_0) = 0 \) for all \( k < n \). Then

\[
h_n : \pi_n(X, x_0) \rightarrow \Omega_n(X, x_0)
\]

is an isomorphism.

**Proof.** We construct an inverse. Let \( f : (M, \partial M) \rightarrow (X, x_0) \) represent an element of \( \Omega_n(X, x_0) \). We can assume \( M \) is connected. Choose a CW-structure on \( (M, \partial M) \) with only one top cell \( e : D^n \rightarrow M^n \). Let \( M^{[n-1]} \) and \( M^{[n-2]} \) denote the skeleta in the chosen structure. These fit into a cofibration sequence

\[
M^{[n-1]}/M^{[n-2]} \rightarrow M/M^{[n-2]} \rightarrow M/M^{[n-1]} \rightarrow \Sigma(M^{[n-1]}/M^{[n-2]}).
\]

The last of these maps is a pointed map \( S^n \rightarrow \vee^k S^n \), where \( k \) is the number of \( n-1 \) cells in \( (M, \partial M) \). It must induce the zero map in homology, because otherwise we would have \( H_n(M, \partial M) = 0 \). Hence it is null-homotopic because \( n \geq 2 \). Applying the functor \([-,(X, x_0)]\), pointed homotopy classes of maps to \( (X, x_0) \), then gives a isomorphisms

\[
\pi_n(X, x_0) \rightarrow [M/M^{[n-2]}, (X, x_0)] \rightarrow [(M, \partial M), (X, x_0)]
\]
Thus $f$ defines an element of $\pi_n(X,x_0)$ which is easily seen to depend only on the cobordism class of $f$. This defines an inverse.

Notice how surjectivity of the Hurewicz homomorphism is easier than injectivity. Surjectivity uses only connectivity estimates, but injectivity uses a property of the attaching map of the top dimensional cell in an oriented manifold.

**Theorem 6.11.** Let $n \geq 2$. Let $(X,x_0)$ be a pointed topological space with $\pi_1(X,x_0) = 0$ and $\Omega_k(X,x_0) = 0$ for all $k < n$. Then $\pi_k(X,x_0) = 0$ for all $k < n$.

**Proof.** This follows from the previous lemma by induction.

**Proof of bordism Hurewicz theorem.** Assume $f : X \to Y$ is a map of simply connected spaces that is a bordism isomorphism. Let $C(f)$ be the mapping cone and $x \in C(f)$ the cone point. It follows from the long exact sequence in oriented bordism that $\Omega_*(C(f),x) = 0$. Hence by the previous theorem (using that $C(f)$ is simply connected) we get that $C(f)$ is weakly contractible. Therefore $H_*(C(f),x) = 0$, so $f$ is a homology isomorphism and hence a weak equivalence by the classical Hurewicz theorem.

**References**

[Ar76] V.I. Arnold, *Wave front evolution and equivariant Morse lemma*, Comm. in Pure and Appl. Math. 29(1976), 557-582.

[Bo09] S. Boldsen, *Improved homological stability for the mapping class group with integral or twisted coefficients*, arXiv:0904.3269v1.

[El72] Y. Eliashberg, *Surgery of singularities of smooth maps*, Izv. Akad. Nauk SSSR Ser. Mat., 36(1972), 1321–1347.

[EM97] Y. Eliashberg and N. Mishachev, *Wrinkling of smooth mappings and its applications - I*, Invent. Math., 130(1997), 345–369.
[Fu74] D.B. Fuchs, *Quillenization and bordism*. Funkcional. Anal. i Priložen. 8 (1974), no. 1, 36–42.

[Iv89] N.V. Ivanov, *Stabilization of the homology of Teichmüller modular groups*, Algebra i Analiz 1 (1989), no. 3, 110–126.

[Iv93] N.V. Ivanov, *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), 149–194, Contemp. Math., 150, Amer. Math. Soc., Providence, RI, 1993.

[Ha85] J. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2) 121 (1985), no. 2, 215–249.

[MT01] I. Madsen and U. Tillmann, *The stable mapping class group and $Q(\mathbb{C} P^\infty)$*, Invent. Math. 145 (2001), no. 3, 509–544.

[MW07] I. Madsen, and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford’s conjecture*, Ann. of Math. (2) 165 (2007), no. 3, 843–941.

[Mi65] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, Princeton, NJ, 1965