On Generalized Moment Maps for Symplectic Compact Group Actions

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Abstract

In this paper a moment map is defined for an arbitrary symplectic action of a compact connected Lie group on a closed symplectic manifold, in the spirit of the circle-valued map introduced by McDuff in the case of non-Hamiltonian circle actions. We show that, for torus actions, the Atiyah-Guillemin-Sternberg convexity theorem remains valid in our context. Also, we study the equivariance properties of generalised moment map and some facts about Marsden-Weinstein reduction procedure, allowing e.g. to reformulate a proof of Kim’s result that ”complexity one” symplectic torus actions are Hamiltonian. As illustration of the use McDuff moment maps, we give a symplectic proof of the finiteness of certain symmetry groups of genus $\geq 2$ compact oriented surfaces.

Generalized moment maps for non-Hamiltonian symplectic $S^1$-actions on closed symplectic manifolds were introduced by McDuff in [19], allowing her to prove that, for closed symplectic 4-manifolds, any $S^1$-action with a fixed point is Hamiltonian. A McDuff moment map is defined as follows:

**Definition 0.1** Let $(M, \omega)$ be a closed symplectic manifold endowed with a symplectic non-Hamiltonian $S^1$-action. Let $\eta \in s_1 \cong \text{Lie}(S^1)$ generate $\ker \exp \cong \mathbb{Z}$, $X_\eta$ be the corresponding fundamental field on $(M, \omega)$, $\alpha$ be the basis of $s^*_1$ dual to $\{\eta\}$ generating the invariant 1-form $\tilde{\alpha}$ on $S^1$. Then a McDuff moment map is a map

$$\mu : M \to S^1$$

with

$$\mu^*\tilde{\alpha} = i_{X_\eta}\omega'$$

where $\omega'$ is an integral symplectic form constructed from $\omega$.

(more details will be given in section 1 below)

Important properties of McDuff moment maps are (see [2] p.77):

- $\mu$ is $S^1$-invariant
- $\mu(M) = S^1$ (surjectivity)

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• each fiber of $\mu$ has the same number of connected components

• $\mu$ admits no local extremum

The motivation behind the present paper is to gain a basic understanding of the possible "moment map" obtained when McDuff’s construction is extended to compact group actions (extensions to non-compact groups belong to an other story). Here is an outline of the work:

We start with an application of the $S^1$ case: using a result of Ono [21] based on McDuff moment map, we will obtain a symplectic proof of the following well-known result:

**Theorem 0.1** Let $S$ be a smooth compact oriented genus $\geq 2$ surface. Then

1. for any Riemann metric on $S$, the isometry group is finite.

2. for any symplectic structure $\omega$ on $S$, the symplectomorphism group of $(S, \omega)$ has no connected compact Lie subgroup.

3. for any complex structure on $S$, the group of holomorphic transformations is finite.

As generalization of McDuff moments, we introduce for any symplectic action $\sigma$ of a connected compact Lie group $G$ on a compact symplectic manifold $(M, \omega)$ a "generalized moment". Its existence relies on the facts that

• (see section 2) up to a covering $G$ decomposes as a direct product $G = C \times T^r$ where $C$ is the maximal connected compact subgroup of $G$ on which $\sigma$ is Hamiltonian, and $T^r$ is a torus in $G$ acting in a totally non-Hamiltonian way

• (section 3) an invariant integral symplectic form $\omega'$ exists on $M$, relative to which $\sigma$ is exactly as Hamiltonian as it is relative to $\omega$.

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**Definition 0.2** A generalized moment map for $\sigma$ is a map

$$\mu : M \to C^* \times (S^1)^r$$

where the first factor $\mu_1 : M \to C^*$ is a genuine Hamiltonian moment map and the $r$ other factors are McDuff moments associated to each circle factor of a decomposition $T^r \cong (S^1)^r$, all this evaluated relative to a unique integral invariant symplectic form $\omega'$

There natural notion of equivariance for generalized moment maps: $\mu$ is said to be equivariant if $\mu_1$ is $Ad^*(C)$-equivariant and $T^r$-invariant, while $\mu_2 : M \to (S^1)^r$ is $G$-invariant. For an equivariant generalized moment, the Marsden-Weinstein reduction procedure can be xeroxed from the Hamiltonian case. But the basic example of the 2-torus acting on itself with moment $\mu : (\mathbb{R}/\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2 : [p, q] \to [-q, p]$ shows that, while equivariance can always be assumed in the Hamiltonian case ([7], prop.2), it fails here. However, we will show that things are nicer in the presence of fixed points:

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1It is worth to draw a parallel between the present work and the two papers [4] and [6].
Proposition 0.1 If any fixed point exists for the \( T^r \)-action on \( M \), \( \mu \) can be chosen equivariant.

This is not so surprising: in many (but not every!) situations that we recall in section 3, the presence of a fixed point even implies that the action is Hamiltonian.

When the moment map is not equivariant in the above "coadjoint" sense, we will show using arguments as in [17] and [3] that some "affine" action \( \text{Aff} \) exists on \( (S^1)^r \) regarding to which \( \mu_2 \) is equivariant. This also authorizes reduction procedure, in the less usual way described in [17].

Section 4 is devoted to a non-Hamiltonian version of the Atiyah-Guillemin-Sternberg convexity theorem [1, 9]:

Theorem 0.2 When \( G \) is a torus, \( M \) compact connected, let \( \Delta = \mu_1(M) \) be the convex polytope image of \( \mu_1 \). Then

\[
\mu(M) = \Delta \times (S^1)^r
\]

Moreover \( \mu : M \to \mu(M) \) is an open map.

We have the

Corollary 0.1 Let \( b_1(M) \) denote the first Betti number of \( M \). Then

\[
r \leq b_1(M)
\]

Also, we deduce from the convexity theorem some "heredity" property: when \( T_1 \times T_2 \) acts on \( (M, \omega) \), \( T_1, T_2 \) being torus groups, with a naturally equivariant moment, and the \( T_2 \)-action is non-Hamiltonian, the induced \( T_2 \)-action on a \( T_1 \)-reduced space \( M_{T_1}^{T_2} \) remains non-Hamiltonian. This allows us to reformulate in the Appendix a proof of a result of M.K. Kim generalizing the introducing McDuff result to "complexity one" symplectic action, i.e. the situation where a \( n \)-torus acts symplectically on a \((2n+2)\)-dimensional \((M, \omega)\).

**Notation conventions:** Throughout this paper, \((M, \omega)\) will denote a closed connected symplectic manifold, \(G, C\) compact connected Lie groups (as far as generalized moments are around), \(T\) a torus group, \(G, C, T\) their respective Lie algebras, \(G^*, C^*, T^*\) their duals. To each element \( \xi \) in the Lie algebra of a group acting on \( M \), \( X_\xi \) will be the associated fundamental field on \( M \): \( X_\xi(x) = \frac{d}{dt} |_{t=0} (\exp(-t\xi) \cdot x) \). Finally, by \( > S < \subset V \) we denote the vector space generated by a subset \( S \) in a vector space \( V \).

1 McDuff moments and genus \( \geq 2 \) surfaces

Let \( \sigma : S^1 \times M \to M \) be a symplectic non-Hamiltonian \( S^1 \)-action on \((M, \omega)\). The construction of a McDuff moment map goes as follows: an invariant symplectic form \( \omega'' \) with a rational cohomology class is chosen near \( \omega \), preserving the non-Hamiltonian property. Then \( \omega' = k \cdot \omega'' \) is in an integral class for some \( k \in \mathbb{Z} \).

Now, trying to define the moment map in the usual manner as

\[
\mu(x) = \int_{\gamma_x} i_{X_\xi} \omega' \quad x \in M, \gamma_x : [0, 1] \to M, \gamma_x(0) = \tilde{x}, \gamma_x(1) = x
\]
for some base point \( \tilde{x} \), we see that the obstructions lie in the non-vanishing of
the periods \( \int_C i_{X_\eta} \omega' \) for some 1-cycles in \( M \). But those periods are in \( \mathbb{Z} \), because
\( C \) can be pushed to a 2-cycle \( \sigma(C) : [0, 1]^2 \to M : (s, t) \to (\exp s \eta).C(t) \) and
\[
\mathbb{Z} \ni \int_{\sigma(C)} \omega' = \int_{[0,1]^2} \omega'_{(\exp s \eta).C(t)}(X_\eta((\exp s \eta).C(t)), (\exp s \eta)_*(\dot{C}(t))) = \int_C i_{X_\eta} \omega'
\]
where the invariance and the integrality of \( \omega' \) was used. The McDuff moment
\( \mu \) is then simply \( \mu_{\text{usu}} \) seen as a map to \( \mathbb{R}/\mathbb{Z} \). The properties of \( \mu \) recalled in
the introduction are either immediate or consequence of the fact that \( \mu \) is a
Morse-Bott function with critical points of even indices (see [2]).

**Remark 1.1 (Aside)** It is possible to render \( \mu \) fiber-connected.

Indeed, the non-existence of a local extremum for \( \mu \) implies that \( \mu \) is an open
map onto \( S^1 \), and defining as in [3, 11] the equivalence relation \( \sim \) on \( M \) by saying
\( x \sim y \) if \( \mu(x) = \mu(y) \) and \( x \) and \( y \) belong to the same connected component of
\( \mu^{-1}(\mu(x)) \), we see that \( \tilde{\mu} : (M/ \sim) = \tilde{M} \to S^1 \) is a finite covering. Such a
covering is always of the form \( \pi : S^1 \to S^1 : t \to t^k \) for some \( k \in \mathbb{Z} \), whence \( \mu \)
factorizes as
\[
M \overset{\mu_{fc}}{\to} \tilde{M} \cong S^1 \overset{\pi}{\to} S^1
\]
where \( \mu_{fc} \) is fiber-connected. We will discuss implications of this observation
below in section 4 and in the Appendix.

As example of the use of McDuff moment maps, we will now provide a proof of
Theorem 0.1. Compact connected group actions on surfaces are naturally
not very mysterious (see e.g. the full classification of \( S^1 \)-actions on compact
surfaces in [2]), the point here is to show that a symplectic argument can be
relevant. The starting point is the following theorem of Ono, obtained from the
properties of McDuff moments :

**Theorem 1.1 ([21])** Let \((M, \omega)\) be a closed symplectic manifold.

1. if the second homotopy group \( \pi_2(M) \) vanishes, then a circle group action
   on \( M \) preserving \( \omega \) has no fixed point.

2. Moreover if every abelian subgroup of \( \pi_1(M) \) is cyclic, there is no circle
   group action on \( M \) preserving \( \omega \).

It is also useful to collect here some standard facts about (symplectic) \( G \)-actions
on \((M, \omega)\):

- Isotropy groups \( G_x = \{ g \in G | g.x = x \} \) are divided into conjugation classes
  characterizing the "orbit type" of point in \( M \).

- There is only a finite number of orbit types ([10], prop.27.4 or [2] prop. 2.2.3)

- There exists a "principal" orbit type s.t. the corresponding points (the
  "principal stratum" \( M_{\text{princ}} \)) in \( M \) form a dense open subset.

- For \( H \) a closed subgroup of \( G \), define
  \[
  M_H = \{ x \in M | G_x = H \}.
  \]
Then each connected component of \( M_H \) is a symplectic submanifold of \( M \)
([10], p.203)
Proof of theorem 0.1: We remark that contradicting the conclusion of the theorem amounts to the existence of some non-trivial $S^1$-action $\sigma$ preserving a symplectic form $\omega$ on $S$: for 1. it follows from the fact that the isometry group of a Riemann structure on $S$ is a compact Lie transformation group ([15], p.39) preserving the induced (symplectic) volume form; for 2. it is obvious; for 3. we get an holomorphic transformation invariant metric $g$ by use of the uniformization theorem ([12], theorem 4.4.1) and the fact that holomorphic transformations of the hyperbolic half-plane are isometries ([12], pp.28-29). $\omega$ is again the Riemannian volume $v_g$.

Since $\pi_2(S) = 0$, part 1. of the Ono theorem applies and $\sigma$ has no fixed point. From the fact that the $M_H$ are symplectic and in particular even dimensional, we deduce that $S = S_{princ}$, i.e. all points have the same isotropy group $\cong \mathbb{Z}/k\mathbb{Z}$ for some $k$. $\sigma$ can then be replaced by a free action, so that $S$ becomes a principal $S^1$-bundle over a 1-dimensional compact base, forcing $S \cong S^1 \times S^1$ contradicting the genus hypothesis.

Remark 1.2 Hurwitz’s theorem ([12], p.88) states that the order of the group of holomorphic transformations of $S$ is at most $84(g-1)$, and it is even known that it reduces to identity when $g \geq 3$ ([8] p.276). Naturally, our technique doesn’t give access to those finer results.

2 The basic lemmas

The two lemmas below will legitimate the definition of generalized moments in the next section.

Lemma 2.1 Let $H$ be a dense connected Lie subgroup of $G$, and suppose given an Hamiltonian action of $H$ on $(M, \omega)$. If this action extends to $G$, the extension is Hamiltonian too.

Remark 2.1 No question of equivariance is considered until the next section.

The tools we need are:

1. Symplectic actions are always Hamiltonian when the group is semisimple (by the Whitehead lemmas and Theo. 26.1 in [10])

2. Up to a covering that is not relevant here, $G = K \times T$, where $K$ is compact semisimple and $T$ is a torus ([24] p.297)

3. If a torus $T$ acts symplectically on $(M, \omega)$ and $S^1 \subset T$ is a circle subgroup acting in a non-Hamiltonian way, then the modified symplectic form $\omega'$ used to define the associated McDuff moment can be chosen $T$-invariant. (this follows from the starting argument of the proof of the next lemma).

4. Near each fixed point $x$ of a symplectic $T$-action, there exists a local moment map which reads in an appropriate Darboux chart $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ as

$$\mu(x_i, y_j) = \mu(0) + \frac{1}{2} \sum_{i=1}^{n} \alpha_i (x_i^2 + y_i^2)$$
where the $\alpha_i \in T^*$ are the weights of the linearized $T$-action on the tangent space $T_x M$ ([10] p.250). Those weights are independant of the choice of $\omega$ or $\omega'$ above and belong to the weight lattice in $T^*$, meaning that they take integral values on the $\mathbb{Z}^n$ lattice ker exp in $T$.

**Proof of lemma 2.1:** By continuity considerations, the $G$-action is symplectic. Also, by tools 1) and 2) above, it is readily seen that the question reduces to the case of a dense 1-parameter subgroup $H$ in a torus $T$, the $H$-action being Hamiltonian. The associated Hamiltonian $f_H : M \to \mathbb{R}$ takes a minimum value ($M$ is compact) at some $H$-fixed point $p \in M$. Then $p$ is $T$-fixed, and in the notations of tool 4. above $f_H$ can be written near $p$ as

$$f_H = f_H(p) + \frac{1}{2} \sum_{i=1}^{n} \alpha_i(h)(x_i^2 + y_i^2)$$

with $\alpha_i(h) \geq 0 \forall i$, $h$ being some generator of the Lie algebra of $H$. Since the coordinates of $h$ are $\mathbb{Q}$-linearly independent in a natural basis $\{e_1, \ldots, e_r\}$ of $T$ on which the $\alpha_i$ take integral values, we see that $\alpha_i(h) = 0$ is only possible if $\alpha_i = 0$ in $T^*$, and that $\alpha_i(h) > 0$ otherwise. Now we can choose linearly independent vectors $r_1, \ldots, r_r$ in $T$ with rational coordinates in the basis $\{e_k\}$ and close enough to $h$ to have $\alpha_i(r_j) > 0 \ \forall j, \alpha_i \neq 0$. Each $r_j$ generates a circle subgroup $C_j$ of $T$. The lemma will be proved if we show that every $C_j$-action on $M$ is Hamiltonian. If not then the action of at least one circle, say $C_1$, admits a McDuff moment $\mu_1 : M \to S^1$. But, using the independence of the weights to a small change of symplectic structure in tool 4. above, we see that near $p$ and in well-chosen coordinates on $M$ and $S^1$, $\mu_1$ looks like

$$\mu_1(p) + \frac{1}{2} \sum_{i=1}^{n} \alpha_i(r_1)(x_i^2 + y_i^2)$$

furnishing a local extremum of $\mu_1$. This is impossible.

**Remark 2.2** This lemma is perhaps true even without compactness assumption on $G$.

In fact it is linked with the important Flux Conjecture (see [19]) that we now restate :

**Definition 2.1** an Hamiltonian isotopy on $(M, \omega)$ is a family $\Phi_t$ of symplectomorphisms obtained from a smooth family of Hamiltonian functions $H_t : M \to \mathbb{R}$ $t \in [0,1]$ as the flow of the time-dependent vector field $X_t$ s.t. $i_{X_t}\omega = dH_t$. The set $\text{Ham}(M, \omega) = \{\Phi_t|\Phi_t$ is a Hamiltonian isotopy } is a normal path-connected subgroup of the identity component $\text{Symp}_0(M, \omega)$ of the symplectomorphism group (see [20] p.311). The Flux Conjecture asserts :

" $\text{Ham}(M, \omega)$ is $C^1$-(or/and)$C^0$-closed in $\text{Symp}_0(M, \omega)$ "

Our lemma for more general $G$ would be a corollary of the conjecture as follows : since each element in $H$ belongs to $\text{Ham}(M, \omega)$ the Flux Conjecture gives that the entire $G$ belongs to $\text{Ham}(M, \omega)$. Now every smooth path
\[ \Psi_t \in \text{Ham}(M, \omega) \] is generated by Hamiltonian vector fields (Prop. 10.17 in [20]). This implies that \( \forall X \in G \), the 1-parameter subgroup \( \{ \exp tX \} \) in \( G \) corresponds to a Hamiltonian isotopy \( \Phi^X_t \) s.t. \( \Phi^X_{t+\mu} = \Phi^X_t \circ \Phi^X_\mu \). Then it is easy to show that the corresponding Hamiltonian \( H^X_t \) is time-independent, providing the desired Hamiltonian function \( H^X \).

Back to the compact \( G \) case and a symplectic action \( \sigma \) on \((M, \omega)\) remember that every subtorus \( T^i \) in a torus \( T^n \) has a complementary subtorus \( T^n - i \) with \( T^n = T^i \times T^n - i \). Then we deduce from lemma 2.1 that (up to an irrelevant covering) \( G \) decomposes as a direct product \( C \times T^r \) where \( C \) is compact connected,

\[ \text{and} \quad \sigma \text{ being Hamiltonian on } C \text{ and totally non-Hamiltonian on } T^r \text{ (i.e. } i_X \omega \text{ is non-exact } \forall \eta \in T) \]

In order to define the generalized moment map, we first want to replace \( \omega \) with an integral symplectic 2-form s.t. the just mentioned decomposition remains valid.

**Lemma 2.2** The situation being as just described, there exists an integral \( G \)-invariant symplectic form \( \omega' \) on \( M \) with \( i_X \omega' \) exact \( \Leftrightarrow i_X \omega' \) exact, \( \forall \xi \in G \).

**proof**: borrowing an argument from [7], we deduce from the compacity of \( M \) that it is of "finite integral rank" i.e.

\[ H^i(M, \mathbb{R}) \cong H^i(M, \mathbb{Z}) \otimes \mathbb{R} \]

as a consequence of the universal coefficients formula in cohomology when the cochain complex is finitely generated (Ex.2 p.172 in [18]).

So we can write \( \omega \) as

\[ \omega = a_1 \omega_1 + \ldots + a_l \omega_l \quad a_i \in \mathbb{R}, \omega_i \in H^2(M, \mathbb{Z}) \]

and since averaging over \( G \) has no consequence on the cohomology class, we can furthermore assume that each \( \omega_i \) is \( G \)-invariant. Now let \( \{ \xi_1, \ldots, \xi_c \} \) be a basis of the Lie algebra \( \mathcal{C} \) of \( C \), consisting of circle generators satisfying \( \exp_C(t \xi_i) = e, \exp_C(t \xi) \neq e \forall t \in (0, 1) \). Let \( \{ \eta_1, \ldots, \eta_r \} \) be such a basis for \( T \), and denote \( \{ \mu_k \} \) the union of these two basis. Let \( \gamma_1, \ldots, \gamma_s \) be smooth 1-cycles generating the 1-homology of \( M \). Then the Hamiltonian character of the action is reflected in the properties

\[ \int_{\gamma_i} i_{X_j} \omega = 0 = \sum_{k=1}^l a_k \int_{\gamma_i} i_{X_k} \omega_k \]

\( \forall i = 1 \ldots s, \forall j = 1 \ldots c \) and also \( \forall j = 1 \ldots r, \exists k \in \{1 \ldots s\} \) s.t.

\[ \int_{\gamma_{k_j}} i_{X_{k_j}} \omega \neq 0 \]

Now we can as in section 1 push the 1-cycles with the various circle actions to obtain various 2-cycles \( \sigma_{\mu_j} \gamma_i \) and prove that

\[ \forall i, j, k, \quad \int_{\gamma_i} i_{X_{\mu_j}} \omega_k \in \mathbb{Z} \]
So \((a_1, \ldots, a_l)^t \in \mathbb{R}^l\) is a non-zero solution of a system of linear equations of the form \(A \cdot a = 0\), \(A \in \mathbb{Z}^{s \times l}\). The solution space of this system can thus be generated by vectors in \(\mathbb{Q}^l\). We may then replace \((a_1, \ldots, a_l)\) by an arbitrary close rational vector \((q_1, \ldots, q_l)\) s.t.

\[
\omega'' = q_1\omega_1 + \ldots + q_l\omega_l
\]

remains non-degenerate and still satisfies properties (A) and (B). Choosing \(\omega' = k.\omega''\) for a large enough integer \(k\) concludes this proof.

\[
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\]

3 Definition and equivariance of the generalized moment

Everything to guarantee the existence of generalized moment maps is now established. Keeping the notations of the preceding section, here is again the definition:

**Definition 3.1** A generalized moment map for \(\sigma\) is a map

\[
\mu : M \to \mathbb{C}^* \times (S^1)^r
\]

where the first factor \(\mu_1 : M \to \mathbb{C}^*\) is a genuine Hamiltonian moment map and the \(r\) other factors are McDuff moments associated to each circle factor of a decomposition \(T^r \cong (S^1)^r\), all this evaluated relative to a unique integral invariant symplectic form \(\omega'\).

This definition is the one that appears to us the closest in spirit to the original McDuff’s construction.

**Remark 3.1** Many choices are implicit in this definition, but we will study properties unaffected by those choices.

**Remark 3.2** The possibility of defining a moment map taking values in some cylinder obtained by quotienting \(\mathcal{G}^*\) adequately is already discussed in [3], in a completely general situation. The problem is to prevent this cylinder from being reduced to a point; this is the reason for our integrality obsession here, and for our seemingly limited framework.

One of the important constructions allowed in the Hamiltonian case is Marsden-Weinstein reduction, working as follows (our reference here is [17]):

let \(\mu : M \to \mathcal{G}^*\) be the moment map associated with an Hamiltonian action of a Lie group \(G\) on \((M, \omega)\). Then there exists a (unique) affine action \(a_\theta\) of \(G\) on \(\mathcal{G}^*\) defined by a 1-cocycle \(\theta : G \to \mathcal{G}^*\) for the coadjoint action, giving the \(\mu\)-equivariance \((\mu(g.x) = a_\theta(g)\mu(x))\).

Given a weakly regular value \(v \in \mu(M) \subset \mathcal{G}^*\), \(G_v\) its \(a_\theta\) isotropy group, \(G_v^0\) its neutral component, the reduced space associated to \(v\) is

\[
M_{\text{red}}^v = \mu^{-1}(v)/G_v^0 \quad \text{or} \quad M_{\text{red}}^v = \mu^{-1}(v)/G_v
\]

In many circumstances, \(M_{\text{red}}\) is still a manifold (or an orbifold) and has a naturally induced symplectic structure. This reduction scheme appears as a very useful tool in symplectic geometry.
Now a fundamental observation is that the only global property of the moment used to construct the reduced space and its symplectic structure is its equivariance. This leads us to consider equivariance properties of our generalized moments. The starting point is that, when $G$ is compact and the action is Hamiltonian, $\mu$ can always be chosen $Ad^*$-equivariant, i.e. $\theta = 0$ ([7]). This brings our first equivariance property:

**Lemma 3.1** Let $\mu : M \to C^* \times (S^1)^r$ be a generalized moment map for a symplectic $G$-action. Then the first factor $\mu_1 : M \to C^*$ can be chosen $Ad^*(C)$-equivariant and $T^r$-invariant, the second factor $\mu_2 : M \to (S^1)^r$ being $C$-invariant.

**proof:** Starting with an $Ad^*(C)$-equivariant $\mu'_1$, we observe as in [7] that averaging $\mu'_1$ over the $T^r$-action still delivers an $Ad^*(C)$-equivariant moment (now $T^r$-invariant!) $\mu_1$ for the $C$-action. The $G$-invariance of $\mu_2$ is then natural since by the defining property of a moment map, infinitesimal $T$-invariance of $\mu_1$ means

$$\omega'(X_\xi, X_\eta) = 0 \quad \forall \xi \in C, \forall \eta \in T$$

But this is equivalent to the infinitesimal $C$-invariance of $\mu_2$. $C$ being connected, this implies the full $C$-invariance of $\mu_2$.

The equivariance property expected from the Hamiltonian situation is contained in the following definition

**Definition 3.2** $\mu$ is naturally equivariant if $\mu_1$ is $Ad^*(C)$-equivariant, $T^r$-invariant, and $\mu_2$ is $G$-invariant.

Natural equivariance is not always obtainable:

**Example 3.1 (The 2-torus)**

Consider the standard action of $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ on itself:

$$[p, q], [r, s] = [p+r, q+s].$$

The invariant and integral symplectic form

$$\omega = \omega dx \wedge dy/\mathbb{Z}^{2r}$$

gives a generalized moment map

$$\mu : T^2 \to T^2 : [p, q] \to [q, -p]$$

which can of course not be chosen as $T^2$-invariant.

This simple example shows that natural moment equivariance is not the rule for non-Hamiltonian actions, but there is a situation where things remain nice:

**Proposition 3.1** If any fixed point exists for the $T^r$-action on $M$, $\mu$ can be chosen naturally equivariant.

**Remark 3.3** The existence of a fixed point for a symplectic torus action simplifies in many cases the analysis radically by forcing the action to be Hamiltonian.

This is so e.g. when

- $M$ is Kähler or of Lefschetz type (see [20] p.150)
• $M$ is 4-dimensional (see [19])
• $M$ is monotone (see [20] p.152)
• The action is semifree with isolated fixed points (see [24])
• The action is effective and $\dim T = \dim M/2$ (see [4])

Nevertheless the subject of non-Hamiltonian symplectic actions with fixed points on a compact $(M, \omega)$ is not empty, as shown by a 6-dimensional example in [19].

The proof of prop. 3.1 will be based on the following two facts about symplectic actions:

1. For a torus $T$, any generalized moment $\mu$ for the $T$-action is $T$-invariant if and only if the $T$-orbits are isotropic in $(M, \omega')(\text{see }[2\text{ prop. 3.5.6)})$

2. Near any isotropic $G$-orbit $O$ in $(M, \omega')$ the equivariant isotropic embedding theorem provides a model for a $G$-invariant neighbourhood of $O$ in $(M, \omega')$, as described e.g. in [22], prop. 2.5:

**Proposition 3.2 (local normal form for the moment map)** Let $H$ be the stabilizer of $p \in O$ and $V$ be the symplectic slice to the orbit $O$. Then a neighbourhood of $O$ is equivariantly symplectomorphic to a neighbourhood of the zero section of $Y = G \times_H (m^* \times V)$ with the equivariant $G$-moment map $J$ given by the formula

$$J([g, \mu, v]) = Ad^*(g)(\mu + \Phi_V(v))$$

We will not explain the notations here, the important fact is that any isotropic orbits possesses an invariant neighbourhood on which the action is Hamiltonian with an $Ad^*$-equivariant moment.

**Proof of prop. 3.1:** By Lemma 3.1, we see that it is enough to prove $T$-invariance for generalized moment maps of symplectic $T$-actions with fixed points, $T$ being a torus. By fact 1) above it is equivalent to show that every $T$-orbit is isotropic. Clearly, since the isotropy of the orbit through $x \in M$ is equivalent to

$$\omega'(X_\xi, X_\eta) = 0 \quad \forall \xi, \eta \in T,$$

the set of points with isotropic orbits in $M$ is closed. But the Sjamaar-Lerman proposition in fact 2), applied to the torus case, imply the existence of a (genuine) equivariant moment map on an invariant neighbourhood of any isotropic orbit, hence again by fact 1) the isotropy of every orbit in this neighbourhood. So the set of points with isotropic orbit is open and closed in $(M, \omega')$, and nonempty by the fixed point hypothesis. $M$ being connected, the proposition follows.

Our next goal is to see that some equivariance in the Libermann-Marle sense above remains without any fixed point assumption. Namely we will prove:
Proposition 3.3 There exists a matrix $Z \in \mathbb{Z}^{r \times r}$ with zero diagonal, defining an affine action $\text{Aff}^{Z}$ of $T^{r}$ on itself, s.t.

$$\mu_{2}(t.x) = \text{Aff}^{Z}(t)\mu_{2}(x) \quad \forall t \in T^{r}, x \in M$$

Remark 3.4 (Definition of $\text{Aff}^{Z}$)

The affine action $\text{Aff}^{Z}$ is defined by :

$$\text{Aff}^{Z}(s_{1}, \ldots, s_{r})(t_{1}, \ldots, t_{r}) = (\prod_{j}(s_{j})^{Z_{ij}.t_{1}}, \ldots, \prod_{j}(s_{j})^{Z_{ij}.t_{r}})$$

In the two-torus example, $Z = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$.

proof of prop. 3.3 : we see $\mu_{2}$ as taking values in $\mathbb{R}^{r}/\mathbb{Z}^{r}$, and then providing an ordinary moment $\mu_{O} : M \to T^{*} \cong \mathbb{R}^{r}$ on each open set $O \subset M$ around a point $o$, s.t. $\mu_{2}(O) \subset \mu_{2}(o) + \left[-\frac{1}{2}, \frac{1}{2}\right]^{r}$.

Let us choose the following data : two nested finite covers $O = \{O_{i}\} \subset V = \{V_{i}\}$ of $M$ (compact!), with $\mu_{2}|V_{i}$, an ordinary moment and an inversion-invariant identity neighbourhood $U \subset T^{r}$ small enough to have $t_{1}, t_{2}, O_{i} \subset V_{i} \forall t_{1}, t_{2} \in U$. Then a localization of the argument of Théo. 3.2 in [17] gives :

$$\forall i, \forall x \in O_{i}, \forall t \in U, \mu_{2}|V_{i}.(t.x) = a_{i}(t, \mu_{2}|V_{i}(x)),$$

where $a_{i}(t, v) = \text{Ad}^{r}(t)v + \theta_{i}(t)$, $\theta_{i} : U \subset T^{r} \to T^{*}$ being a 1-cocycle for the coadjoint action :

$$\theta_{i}(t_{1}, t_{2}) = \theta_{i}(t_{1}) + \text{Ad}^{r}_{t_{1}}\theta_{i}(t_{2}).$$

The coadjoint action being trivial in the torus case, we have

$$a_{i}(t, v) = v + \theta_{i}(t) \quad \text{and} \quad \theta_{i}(t_{1}, t_{2}) = \theta_{i}(t_{1}) + \theta_{i}(t_{2})$$

Now, using the connectedness of $M$, we deduce that the $\theta_{i}$ in fact are not dependent on $i$, so there is a unique (local) cocycle $\theta : U \subset T^{r} \to T^{*}$ associated to $\mu_{2}$. We have to show that $\theta$ can be extended from $U$ to $T$, keeping its relation to $\mu_{2}$.Remark that we are interested in the image of $\theta$ up to the $\mathbb{Z}^{r}$-lattice. The integrality assumption on $\omega'$ is crucial at this point : let $\mu^{2}_{1}, \ldots, \mu^{2}_{r}$ be the $r$ McDuff components of $\mu_{2}$, $S_{1}^{1}, \ldots, S_{r}^{1}$ the corresponding circles in $T^{r}$, $\eta_{1}, \ldots, \eta_{r} \in T$ the corresponding "unit" vectors, $\tilde{x} \in M$ a base point, $C_{1} = S_{1}^{1}.\tilde{x}, \ldots, C_{r} = S_{r}^{1}.\tilde{x}$ r 1-cycles generated by the action. Define

$$Z_{ij} = \int_{C_{j}} i_{x_{\eta_{i}}}^{\omega'} \in \mathbb{Z}^{r \times r}.$$  

Then it is not hard to see that $\theta$ induces the homomorphism

$$T^{r} \to T^{r} : s \to \text{Aff}^{Z}(s)(e)$$

and that $\mu_{2}$ is equivariant with respect to $\text{Aff}^{Z}$.

Remark 3.5 In the Hamiltonian case, $\theta$ is global and induces an homomorphism $T^{r} \to \mathbb{R}^{r}$, forcing $\theta$ to be zero.

Remark 3.6 This result was deduced from a corresponding discussion about affine Poisson structures in [3].
Corollary 3.1 If $\text{Aff} Z$ is free on some subtorus $T^s \subset T^r$, then $M$ is a (trivial) $T^s$-principal bundle.

Corollary 3.2 If $Z$ is of rank $r$, then the $T^r$-action is locally free, i.e. every point in $M$ has a finite stabilizer.

Those equivariance properties in principle give access to the machinery of symplectic reduction along the Hamiltonian lines. It may seem strange physically to look at objects constructed from a different symplectic structure (unless $\omega$ is integral); for us one motivation is to understand results as in [13], where the philosophy is to show that certain symplectic actions have to be Hamiltonian by using properties of an associated ‘generalized moment’.

4 Convexity

In this section the acting group is supposed to be a torus $S$, and we look at the image of the generalized moment

$$\mu : M \to C^* \times (S^1)^r \cong C^* \times \mathbb{R}^r / \mathbb{Z}^r$$

where $C$ is the maximal subtorus in $S$ with Hamiltonian action, $r$ its codimension in $S$. Then, by the Atiyah-Guillemin-Sternberg convexity theorem, we know that $\mu_1(M) = \Delta$, a convex polytope in $C^*$.

Theorem 4.1 (Atiyah-Guillemin-Sternberg for $\mu$) The image of $\mu$ is $\mu(M) = \Delta \times (S^1)^r$ (convexity property), and $\mu$ is an open map from $M$ to $\mu(M)$.

Tool: we will use the local form of the image of a (non-necessarily equivariant) moment map for the $S$-action given by Theorem 32.3 of [10]: the image of a (non-necessarily invariant) neighbourhood $U$ of a point $x \in M$ for a local moment map $\Phi$ is given by

$$\Phi(U) = U' \cap (\Phi(x) + s^+_1 \oplus Co^+(\alpha_1, \ldots, \alpha_n)),$$

where $S_1$ is the identity component of the the isotropy group at $x$, $s_1$ its Lie algebra and $s^+_1$ its annihilator in $s^*$, $\alpha_1, \ldots, \alpha_n \in s^+_1 \subset s^*$ are the weights of the isotropy representation of $S_1$ on $T_x M$, and

$$Co^+(\alpha_1, \ldots, \alpha_n) \overset{\text{def}}{=} \{ \sum \lambda_i \alpha_i | \lambda_i \in \mathbb{R}^+ \}.$$

Moreover, $\Phi$ is open as a map $U \to \Phi(U)$ (see [11]).

Proof of Theo.4.1: The inclusion $\mu(M) \subset \Delta \times (S^1)^r$ is trivial. We will show that a set $\mu(U)$ described as $\Phi(U)$ above is always open in $\Delta \times (S^1)^r$, using again that McDuff moments have no local extrema. Combined with the fact that $\mu(M)$ is closed in $\Delta \times (S^1)^r$, this will give the result.

Now if $\mu(U)$ is not open, this means that $Co^+(\alpha_1, \ldots, \alpha_n)$ is not $s^+_1$. Let $V \subset s^*$ be the subspace generated by $s^+_1 \oplus Co^+(\alpha_1, \ldots, \alpha_n)$. Then $W = V^\perp \subset s_1$ is rationally generated (see a similar argument in the proof of lemma 2.2) so it is the algebra of a subtorus $Z \subset S$. We claim that $Z \subset C$; otherwise there would exist some circle in $Z$, not contained in $C$. This circle would have a non-Hamiltonian action on $M$, whose McDuff moment would be constant on
an open set, a contradiction. Now by looking at $\mu_1$ and by usual Hamiltonian arguments, one sees that $U$ is the identity component of the generic stabilizer (corresponding to the principal orbit type) of the points of $M$ for the $C$-action. This shows that in fact $V = W^\perp$ is independent of the point $x \in M$ and the neighbourhood $U$ chosen, and is given by $> \Delta < \oplus T^*$. Let us write $S = W \oplus K$, with $K$ the algebra of a complementary torus to $Z$ in $S$.

We aim to show $\mu(U) = U' \cap (\Delta \oplus T^*) = U' \cap V \cap (\Delta \oplus T^*)$. If $V = s_1^+ \oplus \text{Co}^+ (\alpha_1, \ldots, \alpha_n)$, we are done. Otherwise, by the elementary properties of convex cones, $s_1^+ \oplus \text{Co}^+ (\alpha_1, \ldots, \alpha_n)$ is a finite intersection of half-spaces in $V$, of the form

$$H_y^+ = \{ v \in V | v(y) \geq 0 \} \quad 0 \neq y \in K \cap S_1$$

Again by rationality considerations, $> y <$ is the algebra of a circle in $K$, and if $y \notin C$ we are led to a contradiction by creating an extremum for a McDuff moment. So $y \in C$, and the conclusion comes from the well-known openness of $\mu_1$.

\[\Box\]

**Corollary 4.1** If a $n$-torus $T^n$ acts in a totally non-Hamiltonian way on a compact connected symplectic manifold $(M, \omega)$, then $n \leq b_1(M)$.

**proof:** Let $\mu : M \to (S^1)^r$ be a generalized moment map. Then by the surjectivity and the openness of $\mu$, it is possible to build $n$ 1-cycles $\gamma_1, \ldots, \gamma_n$ in $M$ whose $\mu$-image are (perhaps up to an integral multiple due to the non-connectedness of the fibers of McDuff moments, which can force to make several revolutions before being back in the right component) the 1-cycles in $T^n$ corresponding to the $n$ $S^1$-factors. It is then easy to see that the subgroup generated by $\gamma_1, \ldots, \gamma_n$ in $H_1(M, \mathbb{Z})$ is a free subgroup of rank $n$, giving $b_1(M) \geq n$.

\[\Box\]

**Remark 4.1** This result has to be related to the "cohomologically free" actions in [6].

From the cycle construction in Corollary 4.1, we also obtain the other

**Corollary 4.2** Let $\mu : M \to C^* \times (S^1)^r$ be a generalized moment for a symplectic torus action. Let $(c, s) = (c, s_1, \ldots, s_{r-1}) \in C^* \times (S^1)^{r-1}$. Then there exists a 1-cycle $\gamma$ in $M$ with $\mu(\gamma) = (c, s_1, \ldots, s_{r-1}) \times (S^1)$.

This has an implication on the reduction side:

**Corollary 4.3** Suppose that $\mu$ in Corollary 4.2 is naturally equivariant. Decompose $\mu$ as $\mu = (\mu_1, \mu_2)$ where $\mu_1 : M \to C^* \times (S^1)^{r-1}$ and $\mu_2 : M \to S^1$. If $(c, s)$ is a regular value of $\mu_1$ and $M_{red} = \mu_1^{-1}(c, s)/C \times (S^1)^{r-1}$ is a smooth reduced space, then the last $S^1$ factor action induces a symplectic non-Hamiltonian action on $M_{red}$.
proof: For clarity, we write $K$ for the last $S^1$ factor.
Let $i$ denote the inclusion of $M_{cs} = \mu_1^{-1}(c, s)$ in $M$. The quotienting map $\pi : M_{cs} \to M_{red}$. Then we see by standard arguments that $i^*(\mu_2)$ is a $S$-invariant function, furnishing a function $\tilde{\mu}_2 : M_{red} \to S^1$ which is a generalized moment for the $K$-action on $M_{red}$. The cycle $\gamma$ in Corollary 4.2 sits in $M_{cs}$, and projects in $M_{red}$ to a cycle $\tilde{\gamma}$ with $\tilde{\mu}_2(\tilde{\gamma}) = S^1$, showing that really $\tilde{\mu}_2$ is a McDuff moment corresponding to a non-Hamiltonian action.

Remark 4.2 Starting with a generalized moment $\mu : M \to C^* \times (S^1)^r$, we can apply to each McDuff factor $\mu_2, \ldots, \mu_{r+1} : M \to S^1$ the procedure of Remark 1.1, to obtain a "moment" $\mu^{fc} : M \to C^* \times (S^1)^r$ where each $S^1$ factor is fiber-connected. It is not difficult to check that every result in this section remains true for $\mu^{fc}$.

Remark 4.3 As reasonable extension of the result given here, we expect the Kirwan convexity theorem [14] to be true in our context (see also [4]). Generalizations to symplectic orbifolds are also imaginable.

A Relation with a result of M.K. Kim

As application of the "non-Hamiltonian heredity" in Corollary 4.3, we propose a reformulation of the main theorem in [13]. We have no familiarity with symplectic orbifolds, but we cannot avoid their appearance in some arguments below, so we will make the following

Assumption: the results used here and that we have proved in the manifold context are still valid in the symplectic orbifold context.

The interesting main result stated by Kim is the

Proposition A.1 Let $\sigma : T^{(n)} \times M^{(2n+2)} \to M^{(2n+2)}$ be an effective symplectic torus action action on a compact connected symplectic manifold $(M, \omega)$, the superscripts indicating the respective dimensions. Then $\sigma$ is Hamiltonian whenever it possesses a fixed point.

Suppose from now on that the proposition is false. Then a generalized moment $\mu : M \to C^* \times (S^1)^r$ exists, and can be chosen naturally equivariant because of the fixed point hypothesis. We will first prove a little lemma:

Lemma A.1 In the hypotheses of the proposition, there exists a decomposition $T = C \times (S^1)^{r-1} \times K$, $K \cong S^1$, and a regular value $(c, s) \in C^* \times (S^1)^{r-1}$ of $\mu_1$ (notation of Corollary 4.3) such that $M_{cs}$ contains a $C \times (S^1)^{r-1}$-orbit of $K$-fixed points. Equivalently, the $K$-action on $M_{red}$ has a fixed point.

Proof: We use the already invoked local form of any local moment map near a fixed point $p \in M$, in a coordinate system $(x_i, y_i)$ centered at $p$:

$$\mu(x_i, y_j) = \mu(0) + \frac{1}{2} \sum_{i=1}^{n+1} \alpha_i (x_i^2 + y_i^2).$$
By usual arguments, the effectiveness of the action implies $\{\alpha_i\} \leq \mathcal{T}^*$. Looking at the set $\{V_\alpha\}$ of (n-1)-dimensional subspaces in $\mathcal{T}^*$ of the form $\alpha_i, \ldots, \hat{\alpha}_i, \ldots, \alpha_{n+1} <$, and their annihilators $\xi_\alpha < \mathcal{T}$, we see that the $\xi_\alpha$'s are circle generators and at least one of those circles acts on $M$ in a non-Hamiltonian way by dimensionality reason. Pick one “non-Hamiltonian” $\xi_\alpha$, call $K$ the corresponding circle. For notation ease, reorder the coordinates so that the corresponding $V_\alpha$ is $\alpha_3, \ldots, \alpha_{n+1} <$. Remark that every point in $M$ having coordinates $(x_1, y_1)$ with $x_1 = x_2 = y_1 = y_2 = 0$ is $K$-fixed. Let $F$ be the relatively open cone $F = \{\sum_{i=3}^{n+1} \lambda_i^+ \alpha_i | \lambda^+ \in \mathbb{R}_0^+\}$ Choose a subgroup $C \times (S^1)^{r-1} \subset \mathcal{T}$ such that $C \times (S^1)^{r-1} \times K$ covers $\mathcal{T}$, and denote $i^*_1 : \mathcal{T}^* \to C^* \times (S^1)^{r-1}$ the dual of the inclusion $C \times (S^1)^{r-1} \subset \mathcal{T}$. Let also $\mu = (\mu_1, \mu_2) : M \to \mathcal{C}^* \times (S^1)^{r-1} \times K$ be a corresponding naturally equivariant generalized moment. Then, using for example Sard’s theorem to avoid singularities coming from outside our coordinate system, we see that there exists a regular value $(c, s)$ of $\mu_1$ in the regio near $\mu_1(p)$ corresponding to $i^*_1(F)$, satisfying our requirements.

Remark A.1 Again, this result remains valid if we replace $\mu$ by a $\mu^{fc}$ as in Remark 4.2.

Proof of Kim’s result: Still assuming that the result is false, consider a naturally equivariant ”fiber-connected” moment $\mu^{fc}$. Thanks to the results proved before, we see that a classical ”reduction in stages” procedure allows to construct from $M$ and $\mu^{fc}$ a connected compact 4-dimensional orbifold endowed with a non-Hamiltonian $K$-action possessing some fixed point. Our assumption that orbifolds behave like manifolds, applied to the McDuff result that symplectic $S^1$-actions on compact connected 4-manifolds are Hamiltonian in the presence of fixed points ([19]), furnishes our contradiction.

Remark A.2 Without a way to produce connected reduced spaces, the arguments above fail because there is a priori no reason for a “reduced” fixed point to be in a ”non-Hamiltonian” reduced component. This is why we used this somewhat inelegant trick with $\mu^{fc}$.

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