Research Article

On the Reformulated Second Zagreb Index of Graph Operations

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Topological indices (TIs) are expressed by constant real numbers that reveal the structure of the graphs in QSAR/QSPR investigation. The reformulated second Zagreb index (RSZI) is such a novel TI having good correlations with various physical attributes, chemical reactivities, or biological activities/properties. The RSZI is defined as the sum of products of edge degrees of the adjacent edges, where the edge degree of an edge is taken to be the sum of vertex degrees of two end vertices of that edge with minus 2. In this study, the behaviour of RSZI under graph operations containing Cartesian product, join, composition, and corona product of two graphs has been established. We have also applied these results to compute RSZI for some important classes of molecular graphs and nanostructures.

1. Introduction

In the whole study, we only consider the molecular graph [1, 2], a graphical representation of molecular structure, in which every vertex corresponds to the atoms and the edges to the bonds between them. Assume J as a simple (molecular) graph with \( |V_J| \) vertex set and \( |E_J| \) edge set. The notations \( |V_J| \) and \( |E_J| \) represent the number of elements of \( J \) in \( V_J \) and \( E_J \), respectively. Also, \( d_J(x) \) denotes the degree of a vertex \( x \) in \( J \) and is defined as the number of edges incident to \( x \).

TIs can be expressed by real numbers related to graphs. There exist many applications as tools for modelling chemical and other properties of molecules for TIs. They determine the correlation between the specific properties of molecules and the biological activity with their configuration in the study of quantitative structure-activity relationships (QSARs) and quantitative structure-property relationships (QSPRs) [3]. To develop the scientific knowledge in 20th century, the concept of molecular structure plays an important role in chemical graph theory, a branch of mathematical chemistry which is closely related to chemical graph.

The molecular structure descriptor, namely, topological index expresses the numerical value obtained from the molecular graph that represents its topology and is necessarily invariant under the automorphism of graphs.

The Zagreb indices, namely, first Zagreb index [4] and second Zagreb index [5] were introduced by Gutman et al. in 1972 and 1975, respectively. These two indices are, respectively, defined for molecular graph \( J \) as

\[
M_1(J) = \sum_{x \in V_J} d_J^2(x) = \sum_{x \in V_J} (d_J(x) + d_J(y)),
\]

\[
M_2(J) = \sum_{x \in V_J} d_J(x) d_J(y).
\]

Let \( x, y, \) and \( z \) be its three vertices of \( J \) forming a path of length two. If \( x \) and \( y \) are two adjacent vertices of \( J \), we express as \( x \sim y \) and similarly for \( y \) and \( z \) as \( y \sim z \). The edge connecting these vertices will be denoted by \( xy \sim yz \). Then, the three auxiliary Zagreb-type indices are introduced by Basavanagoud et al. [6] in 2015. They are denoted as follows:
\[ M_3(J) = \sum_{xy \sim y \in E_J} d^2_j(y), \quad (3) \]

\[ M_4(J) = \sum_{xy \sim y \in E_J} (d_j(x) + 2d_j(y) + d_j(z)), \quad (4) \]

\[ M_5(J) = \sum_{xy \sim y \in E_J} (d_j(x)d_j(y) + d_j(y)d_j(z) + d_j(z)d_j(x)). \quad (5) \]

In 2004, Milicevic et al. [7] reformulated the Zagreb indices by replacing vertex degree with edge degree, and the edge degree of an edge is defined as \( d(e) = d(x) + d(y) - 2 \). The first and second reformulated Zagreb indices [8] of a graph \( J \) are defined as

\[ EM_1(J) = \sum_{e \in E_J} d^2(e), \quad (6) \]

\[ EM_2(J) = \sum_{e \sim f} d(e)d(f), \quad (7) \]

where \( e \sim f \) means that the edges \( e \) and \( f \) share a common end vertex, and \( e \) and \( f \) are adjacent. In 2015, Furtula and Gutman [9] introduced forgotten index (F-index) and is defined as

\[ F(J) = \sum_{x \in V_J} d^2_j(x) = \sum_{xy \in E_J} (d_j^2(x) + d_j^2(y)). \quad (8) \]

In mathematical chemistry, graph operations perform a significant role in the formation of new classes of graphs. By different graph operations on some general or particular graphs, some chemically interesting graphs can be obtained. In [10], Khalifeh et al. computed the first and second Zagreb indices under some graph operations. Some explicit formulae of Zagreb coincides under some graph operations were presented by Ashrafi et al. [11]. In [12], Das et al. derived multiplicative Zagreb indices of different graph operations. In [13], De et al. computed the reformulated first Zagreb index under some graph operations. Recently, the analytical expressions for various topological indices under some binary graph operations have been discussed in [14–16]. We also refer to [17–23] in this regard for interested readers.

\section*{2. Main Results}

In the following, we study different binary graph operations such as join, Cartesian product, composition, and corona product of two molecular graphs and compute some exact formulae for RSZI with respect to those operations separately. Suppose \( J_1 \) and \( J_2 \) be two molecular graphs with the vertex sets \( V_{J_1}, V_{J_2} \), such that \(|V_{J_1}| = n_1, |V_{J_2}| = n_2\), and the edge sets \( E_{J_1}, E_{J_2} \), such that \(|E_{J_1}| = m_1, |E_{J_2}| = m_2\), respectively. We consider the notations \( P_n, C_n, K_n \) for path, cycle, and complete graph with \( n \) vertices. To establish the main results, we also follow equations (1)–(8).

\subsection*{2.1. Join}

The join [24] of two graphs \( J_1 \) and \( J_2 \), denoted by \( J_1 \cup J_2 \), contains the vertex set \( V_{J_1 \cup J_2} = V_{J_1} \cup V_{J_2} \) and the edge set \( E_{J_1 \cup J_2} = E_{J_1} \cup E_{J_2} \cup \{xy: x \in V_{J_1}, y \in V_{J_2}\} \).

In the following theorem, we compute RSZI for join of two graphs.

\textbf{Theorem 1.} Let \( J_1 \cup J_2 = J \) be the join of \( J_1 \) and \( J_2 \) graphs. Then, RSZI of \( J \) is given by

\[ EM_2(J) = EM_2(J_1) + EM_2(J_2) + 2|V_{J_1}|M_4(J_1) + 2|V_{J_2}|M_4(J_2) \]

\[ + 2|V_{J_1}||V_{J_2}||V_{J_1} - 2(M_1(J_1) - 2|E_{J_1}|) + 4|E_{J_1}|M_4(J_1) + 4|E_{J_2}|M_4(J_2) \]

\[ + 2|V_{J_1}||V_{J_2}||V_{J_1} - 2(M_1(J_2) - 2|E_{J_2}|) + |V_{J_1}|F(J_2) + |V_{J_2}|F(J_1) \]

\[ + 2|V_{J_1}||2|V_{J_1}| + |V_{J_2}|- 3)M_1(J_2) + 2|V_{J_1}|||V_{J_1}|| + 2|V_{J_1}|- 3)M_1(J_1) \]

\[ + \frac{|V_{J_1}|}{2} (4|M_{1}^{2} - M_1(J_1)) + \frac{|V_{J_2}|}{2} (4|M_{1}^{2} - M_1(J_2)) \]

\[ + (|V_{J_1}|C_2)M_1(J_1) + (|V_{J_2}|C_2)M_1(J_2) + \frac{1}{2}|V_{J_1}||V_{J_2}||V_{J_1}|| + |V_{J_1}|- 3)^3 \]

\[ + 12|E_{J_1}||E_{J_2}||V_{J_1}|| + |V_{J_1}|- 2)^2 + 2|V_{J_1}|M_2(J_2) + 2|V_{J_2}|M_2(J_1) \]

\[ + 2|V_{J_1}||E_{J_2}||V_{J_1}|| + |V_{J_1}|- 2)^3 + 3|V_{J_1}|| + |V_{J_1}|- 4) \]

\[ + 2|V_{J_1}||E_{J_2}||V_{J_1}|| + |V_{J_1}|- 2)^3 + 3|V_{J_1}|| + |V_{J_1}|- 4) \]
Proof. Consider \((d_f(x) + d_f(y) - 2)(d_f(y) + d_f(z) - 2) = X\). Let \(J_1\) and \(J_2\) be two graphs with \(|V_{J_1}|, |V_{J_2}|\) vertices and \(|E_{J_1}|, |E_{J_2}|\) edges, respectively. Then, by Table 1, we obtain

\[
EM_2(J) = \sum_{xy \sim yz \in E_1} X
\]

First part:

\[
EM_2(J) = \sum_{xy \sim yz \in E_1} X + \sum_{xy \sim yz \in E_2} X + \sum_{xy \sim yz \in E_3} X + \sum_{xy \sim yz \in E_4} X
\]

\[
= D_1 + D_2 + D_3 + D_4 + D_5 + D_6 (\text{say}), \text{respectively.}
\]

(10)

Second part:

\[
D_1 = \sum_{L_1: \{xy \sim yz \in E_1\}} X
\]

(11)

\[
D_1 = \sum_{xy \sim yz \in E_1} \left( d_{J_1}(x) + d_{J_1}(y) + 2|V_{J_1}| - 2 \right) \left( d_{J_2}(y) + d_{J_2}(z) + 2|V_{J_2}| - 2 \right)
\]

\[
= \sum_{xy \sim yz \in E_1} \left( d_{J_1}(x) + d_{J_1}(y) - 2 \right) \left( d_{J_2}(y) + d_{J_2}(z) - 2 \right) + 4 \sum_{xy \sim yz \in E_1} |V_{J_1}|^2 + 2|V_{J_2}|
\]

\[
= M_2(L(J_1)) + 4|V_{J_1}| \left( \frac{1}{2} M_1(J_1) - |E_{J_1}| \right) + 2|V_{J_1}| |M_4(J_1) - 8|V_{J_1}| \left( \frac{1}{2} M_1(J_1) - |E_{J_1}| \right)
\]

\[
= EM_2(J_1) + 2|V_{J_1}| |M_4(J_1) - 2|V_{J_1}|| \left( M_1(J_1) - 2|E_{J_1}| \right).
\]
Third part:

\[ D_3 = \sum_{x \in V_{J_1}, y \in E_{J_2}, z \in E_{J_2}} X \]
\[ = \sum_{x \in V_{J_1}} \left( d_{J_1}(x) + d_{J_2}(y) + \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \right) \left( d_{J_1}(y) + d_{J_2}(z) + 2 \left( |V_{J_1}| - 1 \right) \right) \]
\[ = 2 \sum_{x \in V_{J_1}} d_{J_1}(x) \sum_{y \in E_{J_2}} \left( d_{J_2}(y) + d_{J_2}(z) \right) + 4 \left( |V_{J_1}| - 1 \right) \sum_{y \in E_{J_2}} \sum_{x \in V_{J_1}} d_{J_1}(x) \]
\[ + \sum_{x \in V_{J_1}} \sum_{y \in E_{J_2}} \left( d_{J_1}^2(y) + d_{J_2}^2(z) \right) \]
\[ + 2 \sum_{x \in V_{J_1}} \sum_{y \in E_{J_2}} d_{J_1}(y)d_{J_2}(z) + \left( 3|V_{J_1}| + |V_{J_2}| - 4 \right) \sum_{x \in V_{J_1}} \sum_{y \in E_{J_2}} \left( d_{J_1}(y) + d_{J_2}(z) \right) \]
\[ + \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \sum_{x \in V_{J_1}} \sum_{y \in E_{J_2}} \left( d_{J_1}(y) + d_{J_2}(z) \right) \]
\[ + 4 \left( |V_{J_1}| - 1 \right) \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \sum_{x \in V_{J_1}} \sum_{y \in E_{J_2}} 1 \]
\[ = 4|E_{J_1}|M_1(J_2) + 8|E_{J_1}||E_{J_2}||\left( |V_{J_1}| - 1 \right) + |V_{J_2}|F(J_2) \]
\[ + 2|V_{J_1}|M_2(J_2) + 2|V_{J_1}|\left( 2|V_{J_1}| + |V_{J_2}| - 3 \right) M_1(J_2) \]
\[ + 4|E_{J_2}||V_{J_1}||\left( |V_{J_1}| - 1 \right)\left( |V_{J_1}| + |V_{J_2}| - 2 \right). \]

Fourth part:

\[ D_4 = \sum_{x \in V_{J_2}, y \in E_{J_1}, z \in E_{J_1}} X \]
\[ = \sum_{x \in V_{J_2}} \left( d_{J_2}(x) + d_{J_1}(y) + \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \right) \left( d_{J_1}(y) + d_{J_2}(z) + 2 \left( |V_{J_1}| - 1 \right) \right) \]
\[ = 2 \sum_{x \in V_{J_2}} d_{J_2}(x) \sum_{y \in E_{J_1}} \left( d_{J_1}(y) + d_{J_2}(z) \right) + 4 \left( |V_{J_1}| - 1 \right) \sum_{y \in E_{J_1}} \sum_{x \in V_{J_2}} d_{J_2}(x) \]
\[ + \sum_{x \in V_{J_2}} \sum_{y \in E_{J_1}} \left( d_{J_2}^2(y) + d_{J_1}^2(z) \right) + \sum_{x \in V_{J_2}} \sum_{y \in E_{J_1}} d_{J_1}(y)d_{J_2}(z) \]
\[ + \left( 3|V_{J_1}| + |V_{J_2}| - 4 \right) \sum_{x \in V_{J_2}} \sum_{y \in E_{J_1}} \left( d_{J_1}(y) + d_{J_2}(z) \right) \]
\[ + 4 \sum_{x \in V_{J_2}} \sum_{y \in E_{J_1}} \left( |V_{J_1}| - 1 \right) \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \]
\[ = 4|E_{J_2}|M_1(J_1) + 8|E_{J_1}|\left( |V_{J_1}| - 1 \right) + |V_{J_2}|F(J_1) + 2|V_{J_1}|M_2(J_1) \]
\[ + 2|V_{J_2}|\left( |V_{J_1}| + 2|V_{J_2}| - 3 \right) M_1(J_1) + 4|E_{J_1}||V_{J_2}||\left( |V_{J_1}| - 1 \right)\left( |V_{J_1}| + |V_{J_2}| - 2 \right). \]
\[ D_5 = \sum_{x,z \in V_{J_1}, y \in V_{J_2}} X \]
\[ = \sum_{x,z \in V_{J_1}, y \in V_{J_2}} \left( d_{J_1}(x) + d_{J_2}(y) + \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \right) \left( d_{J_1}(y) + d_{J_2}(z) + \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \right) \]
\[ = \sum_{y \in V_{J_2}} \sum_{x,z \in \left( V_{J_1} \right)} d_{J_1}(x)d_{J_2}(z) + \sum_{x,z \in \left( V_{J_1} \right)} \sum_{y \in V_{J_2}} d_{J_2}(y) \]
\[ + \sum_{y \in V_{J_2}} \sum_{x,z \in \left( V_{J_1} \right)} \left( d_{J_1}(x) + d_{J_2}(z) \right) \sum_{x,z \in \left( V_{J_1} \right)} \sum_{y \in V_{J_2}} d_{J_2}(y) \]
\[ + \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \sum_{y \in V_{J_2}} \sum_{x,z \in \left( V_{J_1} \right)} \left( d_{J_1}(x) + d_{J_2}(z) \right) \]
\[ + 2 \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \sum_{x,z \in \left( V_{J_1} \right)} \sum_{y \in V_{J_2}} d_{J_2}(y) \]
\[ + \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \sum_{x,z \in \left( V_{J_1} \right)} \sum_{y \in V_{J_2}} 1 \]
\[ = \frac{|V_{J_1}|}{2} \left( 4|E_{J_1}|^2 - M_1(J_1) \right) + 4 \left( |V_{J_1}| - 1 \right) |E_{J_1}|^2 + \left( (v_{11}^2)C_2 \right)M_1(J_2) \]
\[ + 2|V_{J_1}| \left( |V_{J_1}| - 1 \right) \left( |V_{J_1}| + |V_{J_2}| - 2 \right) |E_{J_1}| \]
\[ + 4 \left( (v_{11}^2)C_2 \right)|E_{J_1}| \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \]
\[ + \left( (v_{11}^2)C_2 \right)|V_{J_1}| \left( |V_{J_1}| + |V_{J_2}| - 2 \right)^2 \]

Sixth part:

\[ D_6 = \sum_{x,z \in V_{J_1}, y \in V_{J_2}} X \]
\[ = \sum_{x,z \in V_{J_1}, y \in V_{J_2}} \left( d_{J_1}(u) + d_{J_2}(v) + \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \right) \left( d_{J_1}(v) + d_{J_2}(w) + \left( |V_{J_1}| + |V_{J_2}| - 2 \right) \right) \]
\[ = \sum_{y \in V_{J_2}} \sum_{x,z \in \left( V_{J_1} \right)} d_{J_1}(x)d_{J_2}(z) + \sum_{y \in V_{J_2}} \sum_{x,z \in \left( V_{J_1} \right)} d_{J_1}(y) \]
\[ + \sum_{y \in V_{J_2}} \sum_{x,z \in \left( V_{J_1} \right)} \left( d_{J_1}(x) + d_{J_2}(z) \right) + \sum_{x,z \in \left( V_{J_1} \right)} \sum_{y \in V_{J_2}} d_{J_2}(y) \]
Example 2. The RSZI of suspension of H that contains n vertices is the join or sum of H with a single vertex \( n + m \).

\[
|V_H| = \frac{1}{2} \left( |V_{H_1}|^2 - M_1(J_2) + 4\left( |V_{H_1}| - 1\right) |E_{H_2}| + \left( |V_{H_2}| \right) M_1(J_1) + 2 |E_{H_1}| M_1(J_1) \right).
\]

By adding \( \sum_{i=1}^{6} D_i \), we get the desired result.

2.2. Applications. The suspension of a graph H is the join or sum of H with a single vertex \( K_1 \).

**Corollary 1.** The RSZI of suspension of H that contain \( |V_H| = n \) and \( |E_H| = m \) can be expressed as \( EM_2(H + K_1) = EM_2(H) + 2M_2(H) + 2M_2(H) + (1/2)(4n - 9)M_1(H) + 2m^2 + 4m + 2m(n - 1)^2 + (n/2)(n - 1)^3 \).

**Example 1.** The cone graph \( C_{m,n} \) is defined as \( C_m + K_n \). Then, \( EM_2(C_{m,n}) = 2m(11n^2 + 5mn + 2) + (1/2)mn(m + n - 2)(m + n)^2 + 4(2n - 3) \).

**Example 2.** The RSZI of suspension of graph such as \( C_n, K_{(n-1)}, P_n, mK_2 \) are expressed in Table 2.

\[
EM_2(J) = |V_{H_1}| EM_2(J_1) + |V_{H_2}| EM_2(J_2) + 4|E_{H_1}| M_4(J_1) + 4|E_{H_2}| M_4(J_2) + 4M_1(J_1)M_1(J_2)
\]

\[
+ 4\left( M_1(J_1) - 2|E_{H_1}| \right) \left( M_1(J_2) - 3|E_{H_2}| \right) + 4\left( M_1(J_2) - 2|E_{H_1}| \right) \left( M_1(J_1) - 3|E_{H_2}| \right).
\]

**Example 3.** The complete bipartite graph \( K_{p,q} \) is the join of \( K_p + K_q \). Using Theorem 1, \( EM_2(K_{p,q}) = (1/2) pq(p + q - 2) \).

2.3. The Cartesian Product. The Cartesian product (CP) [25] of \( J_1 \) and \( J_2 \), denoted by \( J_1 \times J_2 \), is a graph with \( V_{J_1 \times J_2} = V_{J_1} \times V_{J_2} \) and any \( (f, p)(g, q) \in E_{J_1 \times J_2} \) if and only if \( \{ f \in E_{J_1}, p = q \} \) or \( \{ q \in E_{J_2}, f = g \} \). Also, \( d_{J_1 \times J_2}(f, p) = d_{J_1}(f) + d_{J_2}(p); i = 1, 2, \ldots, |V_{J_1}| \) and \( j = 1, 2, \ldots, |V_{J_2}| \).

Now, we obtain RSZI for Cartesian product of two graphs.

**Theorem 2.** If \( J_1 \times J_2 = J \) be the CP of \( J_1 \) and \( J_2 \) graphs, then RSZI of \( J \) is

**Proof.** By definition of RSZI, from equation (7) and the degree distribution for CP of two graphs, we have

\[
EM_2(J) = \sum_{(f, p)(g, q) \in E_J} (d_f(f, p) + d_f(g, q) - 2)(d_f(h, r) + d_f(h, r) - 2)
\]

\[
= \sum_{f \in V_{J_1}, pq \in d_f(f, q) \in E_{J_2}} (d_f(f, p) + d_f(f, q) - 2)(d_f(f, r) + d_f(f, r) - 2)
\]

\[
\frac{1}{2}(d_f(T_f, p) + d_f(T_f, q) - 2)(d_f(T_f, r) + d_f(T_f, r) - 2).
\]

\[
\frac{1}{2}(d_f(T_f, p) + d_f(T_f, q) - 2)(d_f(T_f, r) + d_f(T_f, r) - 2).
\]
Table 1: The degree distribution of join $J = J_1 + J_2$.

| Suspension of | Namely          | Join of          | $EM_2$ values          |
|--------------|----------------|-----------------|-----------------------|
| $C_n$        | Wheel graph ($W_n$) | $C_n + K_1$    | $(n/2)(n^2 + n^2 + 47n - 15)$ |
| $K_{(n-1)}$  | Star graph ($S_n$)     | $K_{(n-1)} + K_1$ | $(1/2)(n - 1)(n - 2)$ |
| $P_n$        | Fan graph ($F_n$)      | $P_n + K_1$     | $(1/2)(n + 4)(n - 1) + 26n^2 - 46n - 17$ |
| $mK_2$       | Flower graph          | $mK_2 + K_1$    | $4m^2(2m^2 - m + 2)$   |

$$
\begin{align*}
&= U_1 + U_2 + U_3 \text{ (say).} \\
&= \sum_{f \in V_{J_1}, pq \in E_{J_1}} (d_j(f, p) + d_j(f, q) - 2)(d_j(f, q) + d_j(f, r) - 2) \\
&= \sum_{pz \in E_{J_1}} \sum_{f \in V_{J_1}} \left(2d_{j_1}(f) + d_{j_2}(p) + d_{j_2}(q) - 2\right) \left(2d_{j_1}(f) + d_{j_2}(q) + d_{j_2}(r) - 2\right) \\
&= \sum_{f \in V_{J_1}} \sum_{pq \in E_{J_1}} \left(d_{j_1}(p) + d_{j_2}(q) - 2\right) \left(d_{j_2}(q) + d_{j_2}(r) - 2\right) + 4 \sum_{pq \in E_{J_1}} \sum_{f \in V_{J_1}} \left(d_{j_1}(f) - 2\right) \\
&= 2 \sum_{f \in V_{J_1}} d_{j_1}(f) \sum_{pq \in E_{J_1}} \left(d_{j_1}(p) + 2d_{j_2}(q) + d_{j_2}(z)\right) - 8 \sum_{pq \in E_{J_1}} \sum_{f \in V_{J_1}} d_{j_1}(f) \\
&= |V_{J_1}| M_2(L(J_2)) + 4|E_{J_1}| M_4(J_2) + 4\left(\frac{1}{2}M_1(J_2) - |E_{J_1}|\right)M_1(J_1) - 16\left(\frac{1}{2}M_1(J_2) - |E_{J_1}|\right)|E_{J_1}| \\
&= |V_{J_1}| EM_2(J_2) + 4|E_{J_1}| M_4(J_2) + 2\left(M_1(J_2) - 2|E_{J_1}|\right)\left(M_1(J_1) - 4|E_{J_1}|\right).
\end{align*}
$$

The notations $U_1, U_2,$ and $U_3$ represent the sum of above terms in order.

Step 1:

$$
U_1 = \sum_{f \in V_{J_1}, pq \in E_{J_1}} (d_j(f, p) + d_j(f, q) - 2)(d_j(f, q) + d_j(f, r) - 2) \\
= \sum_{f \in V_{J_1}} \sum_{pq \in E_{J_1}} \left(2d_{j_1}(f) + d_{j_2}(p) + d_{j_2}(q) - 2\right) \left(2d_{j_1}(f) + d_{j_2}(q) + d_{j_2}(r) - 2\right) \\
= \sum_{f \in V_{J_1}} \sum_{pq \in E_{J_1}} \left(d_{j_1}(p) + d_{j_2}(q) - 2\right) \left(d_{j_2}(q) + d_{j_2}(r) - 2\right) + 4 \sum_{pq \in E_{J_1}} \sum_{f \in V_{J_1}} \left(d_{j_1}(f) - 2\right) \\
= 2 \sum_{f \in V_{J_1}} d_{j_1}(f) \sum_{pq \in E_{J_1}} \left(d_{j_1}(p) + 2d_{j_2}(q) + d_{j_2}(z)\right) - 8 \sum_{pq \in E_{J_1}} \sum_{f \in V_{J_1}} d_{j_1}(f) \\
= \left|V_{J_1}\right| M_2(L(J_2)) + 4\left|E_{J_1}\right| M_4(J_2) + 4\left(\frac{1}{2}M_1(J_2) - |E_{J_1}|\right)M_1(J_1) - 16\left(\frac{1}{2}M_1(J_2) - |E_{J_1}|\right)|E_{J_1}|
$$

Step 2:

$$
U_2 = \sum_{f g \in E_{J_1}, pq \in E_{J_1}} (d_j(f, p) + d_j(f, q) - 2)(d_j(g, p) + d_j(h, p) - 2) \\
= \sum_{f \in V_{J_1}} \sum_{g \in V_{J_2}} \sum_{pq \in E_{J_1}} \left(2d_{j_1}(f) + d_{j_2}(f) + d_{j_2}(g) - 2\right) \left(2d_{j_2}(p) + d_{j_1}(g) + d_{j_1}(h) - 2\right) \\
= \sum_{f \in V_{J_1}} \sum_{g \in V_{J_2}} \sum_{pq \in E_{J_1}} \left(d_{j_1}(f) + d_{j_2}(g) - 2\right) \left(d_{j_1}(g) + d_{j_1}(h) - 2\right)
$$
\[ + 2 \sum_{p \in V_{J_1}} d_{f_1}(p) \sum_{f, g \in V_{E_{J_1}}} (d_{f_1}(f) + 2d_{f_1}(g) + d_{f_1}(h)) \]
\[ + 4 \sum_{f, g \in V_{E_{J_1}}} d_{f_2}(p) - 8 \sum_{f, g \in V_{E_{J_1}}} \sum_{p \in V_{J_1}} d_{f_1}(p) \]
\[ = |E_{J_1}|M_2(L(J_1)) + 4|E_{J_1}|M_4(J_1) + 4 \left( \frac{1}{2}M_1(J_1) - |E_{J_1}| \right) \]
\[ = |E_{J_1}|EM_2(J_1) + 4|E_{J_1}|M_4(J_1) + 2 \left( M_1(J_1) - 2|E_{J_1}| \right) \left( M_1(J_2) - 2|E_{J_1}| \right). \] (20)

Step 3:

\[
U_3 = \sum_{f, g \in E_{J_1}, p \in PJ_1} (d_{f_1}(f, p) + d_{f_1}(f, q) - 2)(d_{f_1}(f, q) + d_{f_1}(g, q) - 2)
\]
\[
= \sum_{f, g \in E_{J_1}, p \in PJ_1} (2d_{f_1}(f) + d_{f_1}(p) + d_{f_1}(q) - 2)(2d_{f_1}(q) + d_{f_1}(f) + d_{f_1}(g) - 2)
\]
\[
= 4 \sum_{f, g \in E_{J_1}, p \in PJ_1} d_{f_1}(f)d_{f_1}(q) + 2 \sum_{f, g \in E_{J_1}, p \in PJ_1} (d_{f_1}(p) + d_{f_1}(q) - 2)d_{f_1}(p)
\]
\[
+ 2 \sum_{f, g \in E_{J_1}, p \in PJ_1} (d_{f_1}(f) + d_{f_1}(g) - 2)(d_{f_1}(f) + d_{f_1}(g) - 2)
\]
\[
= 4 \sum_{f, g \in E_{J_1}, p \in PJ_1} (d_{f_1}(f) + d_{f_1}(g)) \sum_{p \in PJ_1} (d_{f_1}(p) + d_{f_1}(q))
\] (21)

By adding \(U_1, U_2, U_3\), we get the required result. □

2.4. Applications. Let \(P, Q, R, \) and \(S\) be the grids \((P, \times, P, m) = P,\) rook’s graph \((K, \times, K, m) = Q,\) \(C_n\)-nanotorus \(TC_n(n, m) = C_n \times C_m = R,\) and \(C_n\)-nanotube \(TUC_n(n, m) = (P, n \times C_m) = S.\) Then, by Theorem 2, we get the following results.

Example 4. The RSZI for \(P\) is given by \(EM_2(P) = 4 \left[ 54mn - 89 (n + m) + 132 \right],\) for \(m, n \geq 4.\)

Example 5. The RSZI for \(Q\) is given by \(EM_2(Q) = 2mn((m - 1)(m - 2) + 3m - 4n + (n - 1)(n - 2))\)

\[ (2mn + 3n - 2m - 4) + (n - 1)(m - 1)(6mn - 11m - 11n + 22),\] for \(m, n \geq 2.\)

Example 6. The RSZI for \(R\) is given by \(EM_2(R) = 216mn.\)

Example 7. The RSZI for \(S\) is given by \(EM_2(S) = 4m (54n - 89).\)

2.5. Lexicographic Product. The lexicographic product (LP) or composition [26] of two graphs \(J_1\) and \(J_2\) is denoted by \(J_1[J_2],\) and any two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are adjacent if and only if \(u_1, v_1 \in E_{J_1}\) or \(u_1 = v_1\) and \(u_2v_2 \in E_{J_2}\).
vertex set of \(J_1[J_2]\) is \(V_{J_1} \times V_{J_2}\), and the degree of a vertex \((a, b) \in J_1[J_2]\) is given by \(d_{J_1[J_2]}(a, b) = |V_{J_1}|d_{J_1}(a) + d_{J_2}(b)\).

In the following theorem, we compute RSZI for composition of two graphs \(J_1\) and \(J_2\).

**Theorem 3.** Let \(J_1[J_2] = J\) be the composition of \(J_1\) and \(J_2\). The RSZI of \(J\) is given by

\[
EM_z[J] = |V_{J_1}|EM_z(J_1) + |V_{J_2}|EM_z(J_2) + |V_{J_1}|^2M_z(J_1) + 4|E_{J_1}||V_{J_1}|M_z(J_2)
\]

\[
+ |V_{J_1}|^3\left(|V_{J_1}|^2 - 4|E_{J_1}|\right)(F(J_1) + 2M_2(J_1)) + 2|E_{J_1}||V_{J_1}|(F(J_2) + 2M_2(J_2))
\]

\[
+ \frac{17}{2}|V_{J_1}|^2M_z(J_1)M_z(J_2) + 2|V_{J_1}|^3\left(2|E_{J_1}| - |V_{J_1}|\right)M_z(J_1)
\]

\[
+ 2|V_{J_1}|\left(4|E_{J_1}||V_{J_1}|^2 - 2|V_{J_1}|^3 + 2|V_{J_1}|^2 - 18|E_{J_1}||V_{J_1}| + 7|E_{J_1}|^2\right)M_z(J_1)
\]

\[
- 2|E_{J_1}||\left(9|V_{J_1}| - 4|E_{J_1}|\right)M_z(J_2) - 4|E_{J_1}|\left(|V_{J_1}|^2 - 12|E_{J_1}||V_{J_1}| + 6|E_{J_1}|^2\right).
\]

**Proof.** By using the definition of RSZI and from the equation (7), we have

\[
EM_z(J) = \sum_{(p, x) (q, y) \in (V_{J_1} \times V_{J_1}) \times (V_{J_2} \times V_{J_2})} \left(d_{J_1}(p, x) + d_{J_2}(q, y) - 2\right)\left(d_{J_1}(q, y) + d_{J_2}(r, z) - 2\right)
\]

\[
= \sum_{p \in V_{J_1}} \sum_{xy \in E_{J_1}} \sum_{yz \in E_{J_2}} \left(d_{J_1}(p, x) + d_{J_2}(q, y) - 2\right)\left(d_{J_1}(q, y) + d_{J_2}(r, z) - 2\right)
\]

\[
+ \sum_{p \in V_{J_1}} \sum_{x \in V_{J_1}} \sum_{y \in V_{J_1}} \sum_{z \in V_{J_2}} \left(d_{J_1}(p, x) + d_{J_2}(q, y) - 2\right)\left(d_{J_1}(q, y) + d_{J_2}(r, z) - 2\right)
\]

\[
+ \sum_{p \in V_{J_1}} \sum_{x \in V_{J_1}} \sum_{y \in V_{J_1}} \sum_{z \in V_{J_2}} \left(d_{J_1}(p, x) + d_{J_2}(q, y) - 2\right)\left(d_{J_1}(q, y) + d_{J_2}(p, y) - 2\right)
\]

\[
+ \sum_{xyz} \sum_{p \in V_{J_1}} \sum_{x \in V_{J_1}} \sum_{y \in V_{J_1}} \sum_{z \in V_{J_2}} \left(d_{J_1}(p, x) + d_{J_2}(q, y) - 2\right)\left(d_{J_1}(q, y) + d_{J_2}(p, z) - 2\right)
\]

\[
= T_1 + T_2 + T_3 + T_4 + T_5 (\text{say}), \text{ respectively.}
\]

Now,

\[
T_1 = \sum_{p \in V_{J_1}} \sum_{xy \in E_{J_1}} \left(d_{J_1}(p, x) + d_{J_1}(p, y) - 2\right)\left(d_{J_1}(p, y) + d_{J_1}(p, z) - 2\right)
\]

\[
= \sum_{p \in V_{J_1}} \sum_{xy \in E_{J_1}} \left(d_{J_1}(p, x) + d_{J_1}(p, y) - 2\right)\left(d_{J_1}(p, y) + d_{J_1}(p, z) - 2\right)
\]

\[
= \sum_{p \in V_{J_1}} \sum_{xy \in E_{J_1}} \left(2|V_{J_1}|d_{J_1}(p) + d_{J_1}(x) + d_{J_1}(y) - 2\right)\left(2|V_{J_1}|d_{J_1}(p) + d_{J_1}(y) + d_{J_1}(z) - 2\right)
\]

\[
= \sum_{p \in V_{J_1}} \sum_{xy \in E_{J_1}} \left(d_{J_2}(x) + d_{J_2}(y) - 2\right)\left(d_{J_2}(y) + d_{J_2}(z) - 2\right)
\]

\[
+ 2|V_{J_1}| \sum_{p \in V_{J_1}} d_{J_1}(p) \sum_{xy \in E_{J_1}} \left(d_{J_2}(x) + 2d_{J_2}(y) + d_{J_2}(z)\right)
\]
\[ T_2 = \sum_{p \in E_{j_1}} \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} (d_f(p, x) + d_f(p, y) - 2)(d_f(p, y) + d_f(q, z) - 2) \]
\[ = \sum_{p \in E_{j_1}} \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} \left( 2|V_{j_1}|d_{j_1}(p) + d_{j_1}(x) + d_{j_1}(y) - 2 \right) \left( |V_{j_1}|d_{j_1}(p) + d_{j_1}(q) \right) \]
\[ + d_{j_1}(y) + d_{j_1}(z) - 2 \]
\[ = 2|V_{j_1}| \sum_{p \in E_{j_1}} \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} d_{j_1}(p)(d_{j_1}(y) + d_{j_1}(z)) \]
\[ + \sum_{p \in E_{j_1}} \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} (d_{j_1}(x) + d_{j_1}(y) - 2)(d_{j_1}(y) + d_{j_1}(z)) \]
\[ + 2|V_{j_1}| \sum_{p \in E_{j_1}} \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} d_{j_1}(p) \left( |V_{j_1}|d_{j_1}(p) + |V_{j_1}|d_{j_1}(q) - 2 \right) \]
\[ + \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} (d_{j_1}(x) + d_{j_1}(y) - 2) \sum_{p \in E_{j_1}} \left( |V_{j_1}|d_{j_1}(p) + |V_{j_1}|d_{j_1}(q) - 2 \right) \]
\[ = 2|V_{j_1}| \sum_{p \in E_{j_1}} \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} \left( |V_{j_1}|d_{j_1}(x) + d_{j_1}(y) \right) + 4|E_{j_1}| \]
\[ + 2 \sum_{p \in E_{j_1}} \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} (d_{j_1}(x) + d_{j_1}(y) - 2) \left( |V_{j_1}|d_{j_1}(x) + d_{j_1}(y) \right) + 4|E_{j_1}| \]
\[ + 4|V_{j_1}| \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} \sum_{p \in E_{j_1}} (d_{j_1}(p) + d_{j_1}(q)) \left( |V_{j_1}|d_{j_1}(p) + d_{j_1}(q) - 2 \right) \]
\[ + \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} (d_{j_1}(x) + d_{j_1}(y) - 2) \sum_{p \in E_{j_1}} \left( |V_{j_1}|d_{j_1}(p) + |V_{j_1}|d_{j_1}(q) - 2 \right) \]
\[ = 2|V_{j_1}|M_1(J_1) \left( |V_{j_1}|M_1(J_2) + 4|E_{j_1}|^2 \right) + 2|E_{j_1}| |V_{j_1}| F(J_2) \]
\[ + 4|E_{j_1}| \left( |V_{j_1}| - |V_{j_1}| \right) M_1(J_2) + 16|E_{j_1}|^2|E_{j_1}| + 4|E_{j_1}| |V_{j_1}| M_2(J_2) \]
\[ + 4|E_{j_1}| \left( |V_{j_1}|^3 F(J_1) + 8|E_{j_1}| |V_{j_1}|^3 M_2(J_1) - 8|E_{j_1}| |V_{j_1}|^3 M_1(J_1) \right) \]
\[ + 4|V_{j_1}| \left( M_1(J_2) - 2|E_{j_1}| \right) \left( |V_{j_1}|M_1(J_1) - 2|E_{j_1}| \right) . \]

Next,

\[ T_3 = \sum_{\Omega} \sum_{p \in E_{j_1}} \sum_{x \in \{V_{j_1}\}} \sum_{y \in \{V_{j_1}\}} \sum_{z \in \{V_{j_1}\}} (d_f(p, x) + d_f(q, y) - 2)(d_f(q, y) + d_f(r, z) - 2) \]
\[
\begin{align*}
&= \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \left( [V_{j_1}] (d_{j_1}(p) + d_{j_1}(q)) + d_{j_2}(x) + d_{j_1}(y) - 2 \right) \left( [V_{j_2}] (d_{j_1}(p) + d_{j_1}(r)) + d_{j_2}(y) + d_{j_2}(z) - 2 \right) \\
&= |V_{j_1}|^2 \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \left( d_{j_1}(p) + d_{j_1}(q) \right) \left( d_{j_1}(q) + d_{j_1}(r) \right) \\
&+ |V_{j_2}| \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \left( d_{j_2}(x) + d_{j_2}(y) + d_{j_2}(z) - 2 \right) \\
&+ \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \left( d_{j_2}(x) + d_{j_2}(y) - 2 \right) \left( d_{j_2}(y) + d_{j_2}(z) - 2 \right) \\
&= |V_{j_1}|^4 \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \left( \left( d_{j_1}(p) \right) d_{j_1}(q) + \left( d_{j_1}(q) \right) d_{j_1}(r) + \left( d_{j_1}(p) \right) d_{j_1}(r) + d_{j_1}(q) \right) \\
&+ |V_{j_2}| \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \left( d_{j_2}(x) + d_{j_2}(y) + d_{j_2}(z) - 2 \right) \\
&+ \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \left( d_{j_2}(x) + d_{j_2}(y) - 2 \right) \left( d_{j_2}(y) + d_{j_2}(z) - 2 \right) \\
&= |V_{j_1}|^4 \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \left( \left( d_{j_1}(p) \right) d_{j_1}(q) + \left( d_{j_1}(q) \right) d_{j_1}(r) + \left( d_{j_1}(p) \right) d_{j_1}(r) + d_{j_1}(q) \right) \\
&+ |V_{j_2}| \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \left( d_{j_2}(x) + d_{j_2}(y) + d_{j_2}(z) - 2 \right) \\
&+ \sum_{\Omega} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \left( d_{j_2}(x) + d_{j_2}(y) - 2 \right) \left( d_{j_2}(y) + d_{j_2}(z) - 2 \right) \\
&= |V_{j_1}|^5 \left( M_5(J_1) + 5 \right) + \left( 2 |E_{j_2}| - |V_{j_2}| \right) \left( M_4(J_1) \right) \\
&+ \left( \frac{1}{2} M_4(J_1) - |E_{j_1}| \right) \left( |V_{j_2}|^3 \left( M_4(J_1) - 8 |E_{j_2}| \right) + 4 |V_{j_2}|^3 \right) \\
&+ \left( 2 |E_{j_2}| - |V_{j_2}| \right) \left( 4 |E_{j_2}| \left( |V_{j_2}|^3 + 4 |V_{j_2}|^3 \right) \right) \\
&= |V_{j_1}|^5 \left( M_5(J_1) + 5 \right) + \left( 2 |E_{j_2}| - |V_{j_2}| \right) \left( M_4(J_1) + 4 |V_{j_2}|^3 \right) \left( \frac{1}{2} M_4(J_1) - |E_{j_1}| \right). 
\end{align*}
\]
For

\[
T_4 = \sum_{pq \in E_{j_1}} \sum_{x \in V_{j_2}} \sum_{y \in V_{j_2}} \sum_{z \in V_{j_3}} (d_f(p, x) + d_f(q, y) - 2)(d_f(q, z) + d_f(p, y) - 2)
\]

\[
= \sum_{pq \in E_{j_1}} \sum_{x \in V_{j_2}} \sum_{y \in V_{j_2}} \left( |V_{j_3}|(d_f(p) + d_f(q)) + 2(d_f(x) - 1)(d_f(q) + d_f(r)) + (d_f(x) + d_f(y) - 2) \right),
\]

\[
= \text{Part (A)} - \text{Part (B)} \quad \text{(say), where}
\]

**Part (A) =** \[
2 |V_{j_3}|^2 \sum_{x \in V_{j_2}} \sum_{y \in V_{j_2}} \left( d_f(p) + d_f(q) \right)^2
\]

\[
+ 2 |V_{j_3}| \sum_{x \in V_{j_2}} \sum_{y \in V_{j_2}} \left( 3d_f(x) + d_f(y) - 4 \right) \sum_{pq \in E_{j_1}} (d_f(p) + d_f(q))
\]

\[
+ 4 \sum_{pq \in E_{j_1}} \sum_{x \in V_{j_2}} \sum_{y \in V_{j_2}} \left( d_f(x) - 1 \right) \left( d_f(x) + d_f(y) - 2 \right)
\]

\[
= 2 |V_{j_3}|^2 \left( F(J_1) + 2M_2(J_1) \right) + 8 |V_{j_3}| \left( 2 |E_{j_3}| - |V_{j_3}| \right) M_1(J_1)
\]

\[
+ 4 |E_{j_3}| |V_{j_3}| M_1(J_2) + 16 |E_{j_3}| |E_{j_3}|^2 + 8 |E_{j_3}| |V_{j_3}|^2 - 32 |E_{j_3}| |E_{j_3}| |V_{j_3}|,
\]

**Part (B) =** \[
\sum_{pq \in E_{j_1}} \sum_{x \in V_{j_2}} \left( |V_{j_3}|(d_f(p) + d_f(q) + 2(d_f(x)) - 1) \right)^2
\]

\[
= 2 |V_{j_3}|^2 \sum_{x \in V_{j_2}} \sum_{y \in V_{j_2}} \left( d_f(p) + d_f(q) \right)^2
\]

\[
+ 8 |V_{j_3}| \sum_{x \in V_{j_2}} \left( d_f(x) - 1 \right) \sum_{pq \in E_{j_1}} (d_f(p) + d_f(q)) + 8 \sum_{pq \in E_{j_1}} \sum_{x \in V_{j_2}} \left( d_f(x) - 1 \right)^2
\]

\[
= 2 |V_{j_3}|^2 \left( F(J_1) + 2M_2(J_1) \right) + 8 |V_{j_3}| \left( 2 |E_{j_3}| - |V_{j_3}| \right) M_1(J_1) + 8 |E_{j_3}| |M_1(J_2)
\]

\[
+ 8 |E_{j_3}| |V_{j_3}| - 32 |E_{j_3}| |E_{j_3}| |V_{j_3}|.
\]

Therefore, we get

\[
T_4 = (\text{part (A)} - \text{part (B)})
\]

\[
= 2 |V_{j_3}|^2 \left( |V_{j_3}| - 1 \right) \left( F(J_1) + 2M_2(J_1) \right) + 8 |V_{j_3}| \left( |V_{j_3}| - 1 \right) \left( 2 |E_{j_3}| - |V_{j_3}| \right) M_1(J_1)
\]

\[
+ 4 |E_{j_3}| \left( |V_{j_3}| - 2 \right) M_1(J_2) + 16 |E_{j_3}| |E_{j_3}|^2 + 8 |E_{j_3}| \left( |V_{j_3}| - 1 \right) \left( |V_{j_3}| - 4 |E_{j_3}| \right).
\]
Lastly,

\[ T_5 = \sum_{x \in (V_{J_1} \setminus V_{J_2})} \sum_{p \in V_{J_1}} \sum_{q \in V_{J_2}} \left( d_f(p, x) + d_f(q, y) - 2 \right) \left( d_f(q, y) + d_f(p, z) - 2 \right) \]

\[ = \sum_{x \in (V_{J_1} \setminus V_{J_2})} \sum_{p \in V_{J_1}} \sum_{q \in V_{J_2}} \left[ \left| V_{J_1} \right| (d_f(p) + d_f(q)) + d_f(x) + d_f(y) - 2 \right] \left[ \left| V_{J_2} \right| (d_f(p) + d_f(q)) + d_f(z) + d_f(y) - 2 \right] \]

\[ = \left| V_{J_1} \right|^3 \left( \left| V_{J_1} \right| - 1 \right) \left( \left| V_{J_1} \right| - 2 \right) (F(J_1) + 2M_{J_1}(J_1)) \]
\[ + 4 \left| V_{J_1} \right| \left( \left| V_{J_1} \right| - 1 \right) \left( \left| V_{J_1} \right| - 2 \right) \left( 2 |E_{J_1}| - |V_{J_1}| \right) M_{J_1}(J_1) \]
\[ + 3 |E_{J_1}| \left( |V_{J_1}| - 2 \right) \left( 4 |E_{J_1}|^2 - M_{J_1}(J_2) \right) \]
\[ + |E_{J_1}| \left( |V_{J_1}| - 2 \right) \left( 2 |E_{J_1}| - M_{J_1}(J_2) \right) \]
\[ + 4 |E_{J_1}| \left( |V_{J_1}| - 1 \right) \left( |V_{J_1}| - 2 \right) \left( |V_{J_1}| - 4 |E_{J_1}| \right) \]

By taking the summation of the five cases \( T_1, T_2, T_3, T_4, \) and \( T_5 \) and after simplification, we get the desired result. \( \square \)

2.6. Applications. The fence graph is the composition of \( P_n \) and \( P_2. \)

Example 8. From Theorem 3, RSZI of \((P_n[P_2])\) is calculated as \( EM_2(P_n[P_2]) = 160(8n - 17), \) where \( n > 3. \)

The closed fence graph is the composition of \( C_n \) and \( P_2. \)

\[ EM_2(J_1 \circ J_2) = EM_2(J_1) + |V_{J_1}| EM_1(J_1) + 2 |V_{J_1}| M_4(J_1) + |V_{J_1}| (M_2(J_2) + M_2(J_2)) \]
\[ + 2 \left( 3 |V_{J_1}|^2 - 2 |V_{J_1}| + 2 |E_{J_1}| \right) (M_1(J_1) - 2 |E_{J_1}|) + |V_{J_1}| F(J_2) \]
\[ + 2 |V_{J_1}| M_2(J_2) + 2 \left( |V_{J_1}| \left( |V_{J_1}| - 1 \right) + 2 |E_{J_1}| \right) \left( |E_{J_1}| \left( |V_{J_1}| - 1 \right) + M_1(J_2) \right) \]
\[ + \frac{|V_{J_1}|}{2} \left( 4 |E_{J_1}|^2 - M_1(J_2) \right) + \left( \binom{|V_{J_1}|}{2} C_2 \right) M_1(J_1) \]
\[ + 2 |E_{J_1}| |V_{J_1}| \left( 3 |V_{J_1}|^2 - 2 |V_{J_1}| + 4 |E_{J_1}| + 1 \right) + \frac{1}{2} |V_{J_1}| |V_{J_1}| \left( |V_{J_1}| - 1 \right)^3 \]

Example 9. The RSZI of \( C_n[P_2] \) is given by \( EM_2 C_n[P_2] = 1280n. \)

2.7. Corona Product. For the corona product (COP) [27] of \( J_1 \) and \( J_2, \) denoted by \( J_1 \circ J_2, \) the degree of a vertex \( r \in J_1 \circ J_2 \) is given in Table 3.

Now, we obtain the explicit expression of RSZI for corona product of two graphs.

Theorem 4. The RSZI of \( J_1 \circ J_2 \) is given by
Table 3: The degree distribution of COP for $J$ and $I_{2,j}$ is the $i^{th}$ copy of the graph $J_2$.

| $r \in V_{I_j}$ | $d_{I_j}(r) + |V_{I_j}|$ | $r \in V_{I_{2,j}}$, $i = 1, 2, \ldots, n_1$ | $d_{I_{2,j}}(r) + 1$ |
|----------------|--------------------------|--------------------------|--------------------------|

Proof. From definition of RSZI and from equation (7) and Table 3, we have

\[
\text{EM}_2(J_1 \circ I_2) = \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + 2d_{I_{1,j}}(s) - 2)(d_{I_{1,j}}(s) + d_{I_{1,j}}(t) - 2)
\]

\[
= \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + d_{I_{1,j}}(s) - 2)(d_{I_{1,j}}(s) + d_{I_{1,j}}(t) - 2)
\]

\[
+ |V_{I_j}| \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + d_{I_{1,j}}(s) - 2)(d_{I_{1,j}}(s) + d_{I_{1,j}}(t) - 2)
\]

\[
+ \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + d_{I_{1,j}}(s) - 2)(d_{I_{1,j}}(s) + d_{I_{1,j}}(t) - 2)
\]

\[
+ \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + d_{I_{1,j}}(s) - 2)(d_{I_{1,j}}(s) + d_{I_{1,j}}(t) - 2)
\]

\[
+ I_1 + I_2 + I_3 + I_4 + I_5 \text{(say) in order.}
\]

First,

\[
I_1 = \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + d_{I_{1,j}}(s) - 2)(d_{I_{1,j}}(s) + d_{I_{1,j}}(t) - 2)
\]

\[
= \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + d_{I_{1,j}}(s) - 2)(d_{I_{1,j}}(s) + d_{I_{1,j}}(t) - 2) + 2n_2
\]

\[
= \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + d_{I_{1,j}}(s) - 2)(d_{I_{1,j}}(s) + d_{I_{1,j}}(t) - 2) + 2n_2 \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + 2d_{I_{1,j}}(s) + d_{I_{1,j}}(t))
\]

\[
+ \sum_{rs \in E_{I_{1,j}}} 4n_2(n_2 - 2)
\]

\[
= M_2(L(J_1)) + 2n_2M_4(J_1) + 4n_2(n_2 - 2)\left(\frac{1}{2}M_1(J_1) - |E_{I_{1,j}}|\right)
\]

\[
= EM_2(J_1) + 2|V_{I_j}|M_4(J_1) + 2|V_{I_j}|(|V_{I_j}| - 2)\left(M_1(J_1) - 2|E_{I_{1,j}}|\right).
\]

Second,

\[
I_2 = |V_{I_j}| \sum_{rs \in E_{I_{1,j}}} (d_{I_{1,j}}(r) + d_{I_{1,j}}(s) - 2)(d_{I_{1,j}}(s) + d_{I_{1,j}}(t) - 2)
\]
Third,

\[ I_3 = \sum_{rs \in \mathcal{E}_{12}} (d_{r} + d_{s})(d_{r} + d_{s}) - 2 \left( d_{r} + d_{s} + d_{s} \right) + 4 \left( m_{2} + n_{2} \right) \]

\[ = n_{2} EM_{1} \left( J_{1} \right) + 4 \left( m_{2} + n_{2} \right) \left( M_{1} \left( J_{1} \right) - 2M_{1} \right) + 4n_{2} \left( 2m_{2} + n_{2} \right) m_{1} \]

\[ = |V_{1}| EM_{1} \left( J_{1} \right) + 4 \left( |M_{1}| + |M_{1}| \right) \left( M_{1} \left( J_{1} \right) - 2|E_{1}| \right) + 4|E_{1}| \left( 2|E_{1}| + |E_{1}| \right). \]

Fourth,

\[ I_4 = \sum_{rs \in \mathcal{E}_{12}} (d_{r} + d_{s})(d_{r} + d_{s}) - 2 \left( d_{r} + d_{s} + d_{s} \right) + 2 \left( m_{2} + n_{2} \right) \]

\[ = n_{2} EM_{1} \left( J_{1} \right) + 4 \left( m_{2} + n_{2} \right) \left( M_{1} \left( J_{1} \right) - 2M_{1} \right) + 2 \left( 2m_{2} + n_{2} \right) m_{1} \]

\[ = |V_{1}| F \left( J_{2} \right) + 2|V_{1}| M_{2} \left( J_{2} \right) + 4|E_{1}| M_{1} \left( J_{2} \right) + 2|V_{1}| \left( |V_{1}| - 1 \right) M_{1} \left( J_{2} \right) \]

\[ = |V_{1}| F \left( J_{2} \right) + 2|V_{1}| M_{2} \left( J_{2} \right) + 2 \left( 2|E_{1}| + |V_{1}| \right) M_{1} \left( J_{2} \right). \]
By simplifying the sum $\sum_{i=1}^{5} I_i$, we get the required result.

2.8. Applications. The t-thorny graph of a graph $J$, denoted as $J^t$, is obtained by joining t-number of thorns (pendent edges) to each vertex of $J$. It is defined as the corona product of complete graph $K_t$ and $J$. To know more about the thorn graphs, it may be followed in [28]. By using Theorem 4, we have the following results.

Example 10. The RSZI of $J^t$ is given by $EM_2(J^t) = EM_2(J) + t\cdot EM_1(J) + 2t\cdot M_1(J) + \frac{t}{2}(2t(3t - 2) + (C_2))M_1(J) + \frac{2mt}{t-1}(3t - 5) + (n/2)t(t - 1)^3$.

Example 11. The RSZI of t-thorny of $P_n$ is given by $EM_2(P_n^t) = (1/2)t(t - 1)(2(2n - 3) + (n - 1)(t - 1)(t + 3)) + 4(t + 1)(2t + 1) + t(4n - 10) + 4t^2(4n - 6) + 4(n - 1)t^2(t - 2)$.

Example 12. The RSZI of $C_n$ is given by $EM_2(C_n^t) = 2n(3t^3 + 5t^2 - 2t + 1) + (1/2)nt(t - 1)^3$.

The bottleneck graph of a graph $J$ is defined as the corona product of $K_2$ and $J$.

Example 13. The RSZI is given by $EM_2(K_2 \circ J) = 2M_2(J) + 2M_2(J^2) + 2F(J) + (4n - 1)M_1(J) + 4M_2(J) + (n^2 + 3n + 2m)(n^2 + 2m) - 2mn$.

3. Conclusion

In this study, we have executed the explicit expressions for RSZI under several graph operations such as join, Cartesian product, lexicographic product, and corona product. By applying these results, RSZI is also computed for some classes of graphs by specializing the components of graph operations. As a future work, we want to generalize the above theorems for $n$ graphs. These results will also be helpful for further development using remaining graph operations.
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that there are no conflicts of interest.

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References
[1] N. Trinajstic, Chemical Graph Theory, 2nd edition, 1992.
[2] X. Zhang, H. M. Awais, M. Javaid, and M. K. Siddiqui, “Multiplicative Zagreb indices of molecular graphs,” Journal of Chemistry, vol. 2019, Article ID 5294198, 19 pages, 2019.
[3] M. Dehmer, K. Varmuza, and D. Bonchev, Statistical Modeling of Molecular Descriptors in QSAR/QSPR, Wiley VCH, Weinheim, Germany, 2012.
[4] I. Gutman and N. Trinajstić, “Graph theory and molecular orbitals. Total $\varphi$-electron energy of alternant hydrocarbons,” Chemical Physics Letters, vol. 17, no. 4, pp. 535–538, 1972.
[5] I. Gutman, B. Rušcic, N. Trinajstić, and C. F. Wilcox, “Graph theory and molecular orbitals. XII. Acyclic polyenes,” The Journal of Chemical Physics, vol. 62, no. 9, pp. 3399–3405, 1975.
[6] B. Basavanagoud, I. Gutman, and C. S. Gail, “On second Zagreb index and coindex of some derived graphs,” Kraviceanski Journal of Science, vol. 37, pp. 113–121, 2015.
[7] A. Milčević, S. Nikolić, and N. Trinajstić, “On reformulated Zagreb indices,” Molecular Diversity, vol. 8, pp. 393–399, 2004.
[8] J. B. Liu, B. Ali, M. Aslam Malik, H. M. A. Siddiqui, and M. Imran, “Reformulated Zagreb indices of some derived graphs,” Mathematics, vol. 7, p. 283, 2019.
[9] B. Furtula and I. Gutman, “A forgotten topological index,” Journal of Mathematical Chemistry, vol. 53, no. 4, pp. 1184–1190, 2015.
[10] M. H. Khalifeh, H. Yousefi-Azari, and A. R. Ashrafi, “The first and second Zagreb indices of some graph operations,” Discrete Applied Mathematics, vol. 157, no. 4, pp. 804–811, 2009.
[11] A. R. Ashrafi, T. Došlić, and A. Hamzeh, “The Zagreb coindices of graph operations,” Discrete Applied Mathematics, vol. 158, no. 15, pp. 1571–1578, 2010.
[12] K. C. Das, A. Yurttaş, M. Togan, A. S. Cevik, and I. N. Cangul, “The multiplicative Zagreb indices of graph operations,” Journal of Inequalities and Applications, vol. 2013, no. 1, 2013.
[13] N. De, S. Nayeem, and A. Pal, “Reformulated first Zagreb index of some graph operations,” Mathematics, vol. 3, no. 4, pp. 945–960, 2015.
[14] M. S. Y. Al-Sharafi and M. M. Shubatah, “On the hyper-zagreb index of some graph binary operations,” Asian Research Journal of Mathematics, vol. 16, pp. 12–24, 2020.
[15] A. Modabish, A. Almeri, M. S. Gumaan, and M. Alsharafi, “The second Hyper-Zagreb index of graph operations,” Journal of Mathematical and Computational Science, vol. 11, pp. 1455–1469, 2021.
[16] S. Mondal, N. De, and A. Pal, “On neighborhood Zagreb index of product graphs,” Journal of Molecular Structure, vol. 1223, Article ID 129210, 2021.
[17] J. Cao, U. Ali, M. Javaid, and C. Huang, “Zagreb connection indices of molecular graphs based on operations,” Complexity, vol. 2020, Article ID 7385682, 15 pages, 2020.
[18] J.-B. Liu, S. Javaid, M. Javaid, and K. Shabbir, “Computing first general Zagreb index of operations on graphs,” IEEE Access, vol. 7, pp. 47494–47502, 2019.
[19] D. Maji and G. Ghorai, “A novel graph invariant: the third leap Zagreb index under several graph operations,” Discrete Mathematics, Algorithms and Applications, vol. 11, no. 5, pp. 1–16, 2019.
[20] D. Maji and G. Ghorai, “Computing F-index, coindex and Zagreb polynomials of the kth generalized transformation graphs,” Heliyon, vol. 6, no. 12, Article ID e05781, 2020.
[21] D. Maji and G. Ghorai, “The first entire Zagreb index of various corona products and its bounds,” Journal of Mathematical and Computational Science, vol. 11, no. 5, pp. 6018–6044, 2021.
[22] K. G. Mirajkar and A. Morajkar, “KCD indices and coindexes of graphs,” Ratio Mathematica, vol. 39, pp. 165–186, 2020.
[23] M. Veylaki, M. J. Nikmehr, and H. A. Tavalaee, “The third and hyper-Zagreb coindices of some graph operations,” Journal of Applied Mathematics and Computing, vol. 50, no. 1-2, pp. 315–325, 2015.
[24] M. S. Alsharafi, M. M. Shubatah, and A. Q. Alameri, “The first and second Zagreb index of complement graph and its applications of molecular graph,” Asian Journal of Probability and Statistics, vol. 8, pp. 15–30, 2020.
[25] A. Alameri, M. Al-Rumaima, and M. Almazah, “Y-coindex of graph operations and its applications of molecular descriptors,” Journal of Molecular Structure, vol. 1221, Article ID 128754, 2020.
[26] H. M. A. Siddiqui, S. Baby, M. K. Shafiq, and M. Faisal Nadeem, “Bounds of some degree based indices of lexicographic product of some connected graphs,” Polycyclic Aromatic Compounds, pp. 1–13, 2020.
[27] R. Khoeilar and A. Jahanbani, “General reduced second Zagreb index of graph operations,” Asian-European Journal of Chemistry, vol. 14, Article ID 2150082, 2021.
[28] Y. Alizadeh, A. Iranmanesh, T. Došlić, and M. Azari, “The edge wiener index of suspensions, bottlenecks, and thorny graphs,” Glasnik Matematički, vol. 49, no. 1, pp. 1–12, 2014.