Velocity Discontinuity of Particles Flowing in a Mass-Flow Hopper and the Analysis of Its Characteristics†

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Abstract

The velocity distribution of particles flowing in a two-dimensional mass-flow hopper under gravity was obtained for the entire region of a hopper by measuring the position of tracer particles against time. The characteristic regions of discontinuity or rapid change in velocity and flow direction were found from a detailed analysis of the velocity distribution obtained. On the other hand, an approximate calculus for the estimation of the "velocity characteristic curve" and "stress characteristic curve" (i.e., slip line) was proposed on the basis of the plasticity theory. Theoretically, discontinuity in velocity is possible across the velocity characteristic curve. It was confirmed that both the characteristic regions estimated and obtained experimentally were identical. Also, it was confirmed that the slip lines appear in the flow field that developed immediately after the yield of material occurred.

Introduction

One of the important requirements in designing moving beds and storage bins for granular materials is to estimate the velocity characteristics of the particles. Measuring the particle velocity requires difficult techniques. The flow characteristics of granular materials have gradually been uncovered by research2~8). Perry et al.6) have experimentally discovered that particles flow with velocity discontinuity in a mass-flow hopper. One of the present authors observed the flow patterns of particles flowing in a hopper and suggested that, in the incipient stage of particle flow immediately after the yield of material occurred, the flow was discontinuous and accompanied by slip lines (a stress characteristic curve and discontinuous curve of stress and velocity). Also, it was observed that another type of discontinuous flow, differing from that observed in the incipient stage, developed with the advance of the flowing stage. In this case, the discontinuous flow characteristics appearing in the advanced flow stage are estimated to be identical to the ones reported by Perry. Thus, the discontinuity of particle flow has two different forms, but the particle behavior has not yet been sufficiently investigated.

In this investigation, two types of two-dimensional hoppers having a different angle of hopper cone were used, and the moving velocity of the samples (coal particles) was measured through the observation window. The velocity in the entire hopper was analyzed in detail, experimentally confirming that a discontinuity in the velocity did in fact occur. The movement characteristics of particles flowing with a velocity discontinuity were investigated to obtain the velocity characteristic curve, i.e. the regions where rapid change in the velocity occurred, for the respective hopper. An approximate calculus for estimating this characteristic curve and the slip line characterizing the incipient stage of particle flow was proposed on the basis of the plasticity theory in order to attempt to estimate the discontinuity of particles flowing in a hopper.

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1. The equipment and method used in the experiment

The hoppers used were two-dimensional, and the inclination surfaces were made of steel plates. The parallel front and rear surfaces were made of transparent acrylic sheets. Two hoppers of the following dimensions were used:

1. Angle of the hopper cone $\theta = 20^\circ$
   - Diameter of the hopper outlet $D = 5 \text{ cm}$
   - Height $H = 1.1 \text{ m}$
   - Distance between the parallel sheets $l = 12.5 \text{ cm}$

2. Angle of the hopper cone $\theta = 40^\circ$
   - Diameter of the hopper outlet $D = 10 \text{ cm}$
   - Height $H = 1.5 \text{ m}$
   - Distance between the parallel sheets $l = 20 \text{ cm}$

Coal particles, the diameters of which were arranged within a range of 0.8 to 1.0 cm, were used as specimens. The specimens were rinsed using water, then packed into the hopper channel through a 2 x 2 cm screen provided in the upper portion of the hopper, so that the particles were packed in a uniform and horizontal manner in the channel cross section. At this time, a layer of tracer particles that were colored white and a layer of specimen particles were horizontally and alternately provided. Thus packed coal particles were made to discharge at a constant flow velocity by an electromagnetic feeder situated below the outlet at a right angle to the observation window of the hopper. Photographs of the motion of respective tracer particles in the entire area of the hopper were taken simultaneously and over a specific time interval using three or four cameras. The flow velocity was set to approximately 1.1 kg/min when a hopper of $\theta = 20^\circ$ was used, and to approximately 4.4 kg/min when a hopper of $\theta = 40^\circ$ was used. To facilitate the easy reading of the coordinates of tracer particles, squares at 4 cm intervals were provided in the front acrylic plate, as shown in Fig. 1. As described below, a pseudo-steady flow of particles, which was the feature of the mass-flow hopper, was obtained at these flow velocities. As a result, the obtained particle behavior could be explained using a theory which assumes steady flow.

In general, when the motion of particles is observed through a transparent plate, as described in this paper, the 1- to 2-particle layer that contacts the transparent plate suffers from the wall effect. This results in such a poor expression that the obtained velocity characteristics cannot sufficiently describe the motion of the interior particles. In this case, the tracer contacting the wall surface abruptly disappears into the interior particles, or it abruptly appears from the interior particles as a result of the shearing effects between particles, thereby making it impossible to estimate the velocity through the observation window. In our experiments, the wall effects described above were not observed, and so the particles were assumed to move at an equal velocity perpendicular to the observation window.

The angle of friction between the coal particles and the hopper inclination surface was measured as follows:
A container, devised so that the contained particles contact the steel plate but do not move inside the containers, was placed on the steel plate. The plate was gradually inclined to obtain the angle between the plate and the horizontal line when the container started sliding. The obtained angle was determined as the angle of wall friction, and $\phi_w = 26^\circ$ to $27^\circ$.

In addition, the angle of discharge ($\phi_d$) was measured using the method described below (the method is the same as that described in the previous report$^7$), obtaining $\phi_d = 38^\circ$.

Colored particles were packed into a two-dimensional box container that had a slit-shaped outlet provided in the center of the horizontal base. The packed colored particles were discharged from the outlet while continuously supplying non-colored coal particles from the upper portion of the container. When the particle flow reached a steady status, photographs of the stationary shape of the colored particles were taken. The angle between the horizontal base and the stationary layer of the colored particles at the outlet, was determined to be $\phi_d$. The angle of discharge approximately equals the angle of internal friction of the particles.

2. The results and discussion of the experiment

The moving distances $\Delta r$ and $\Delta z$ of the target particle in the horizontal and vertical directions during a time elapse $\Delta t$ starting from an arbitrary time were obtained. The horizontal and vertical velocities $u_r$ and $u_z$, and also, the angle between the direction of flow of the particles and horizontal plane were calculated from Eqs. (1) and (2), respectively.

$$u_r = -\Delta r/\Delta t, \quad u_z = \Delta z/\Delta t$$

$$B = \tan^{-1}(u_z/u_r)$$ (2)

Using these equations, the velocity distribution and the distribution in the direction of flow at any moment was calculated over the entire area of the hopper.

2.1 The flow pattern and velocity characteristics

Figure 1 shows one example of the flow patterns that was obtained in a hopper having a cone angle of $2\theta = 40^\circ$, and 180 seconds after the start of the flow operation. Each black dot in Fig. 1 indicates one tracer particle that was initially placed in the horizontal layer. The solid curves indicate the velocity characteristics calculated by the method described below. The figure represents two regions in the hopper: one is a region expanding upward from the discharge outlet where the deformation of the tracer layers is significant; the other is a region expanding outside the above-mentioned region toward the hopper wall surface where the deformation of the tracer layer is small. The boundary of these two regions is easily discernible as the points at which the deformation of the tracer layers suddenly changes. In a hopper having a cone angle of $2\theta = 20^\circ$, the moving velocity of the tracer layers in the region contacting the hopper wall was larger than that shown in Fig. 1. In this case, two regions having different velocity characteristics could be determined on the flow pattern, though the boundary was not as clear as that shown in Fig. 1. The relationship between the observed results obtained from the flow patterns and the velocity characteristics of the particles calculated using Eqs. (1) and (2) will be discussed next.

Figure 2 shows one example of the horizontal distribution of the calculated horizontal and vertical velocities $u_r$ and $u_z$ and flow direction defined by Eq. (2). Three different distances from the hopper apex ($y = 1.22, 1.10, 0.62$ m indicated on Fig. 1) were selected. The velocities in Fig. 2 were calculated from the moving distances during a period of from 60 to 120 seconds after the start of the flow operation. Therefore, the distribution was obtained at a time when the tracer deformation was smaller than that shown in Fig. 1. A maximum $u_z$ was obtained at the center line of the hopper at all levels. However, the shapes of the horizontal distribution of $u_z$ differed from each other depending on the distance from the hopper apex. At $y = 0.62$ m, the region in which the value of $u_z$ is approximately the same as that on the center line is spread out from the center line.
Fig. 2 Horizontal distributions of $u_r$, $u_z$ and $B$

to $r/R = 0.4$. However, when $y = 1.10$ m, the $u_z$ value sharply decreases with an increase in
the distance from the center line of the hopper. On the other hand, $u_r$ on the center line of the
hopper is theoretically zero because of the symmetry of the flow. The value of $u_r$ increases
with the increase in the distance from the center line of the hopper and reaches a maximum value at a specific point. This point differs depending on the $y$ level. Thus, it is revealed that the similarity rule cannot be applied to the variation with height in the velocity
distribution in the horizontal direction within the hopper. Jenike$^2$ assumed a radial velocity
field where each particle in a mass flow hopper linearly moves toward the imaginary apex. According to this assumption, the velocity distribution at each height should be similar. The results of our experiment revealed that this assumption is not correct. Perry et al.$^6$ also pointed out that the measured velocity field did not coincide with the radial velocity field. The deviation from the radial velocity field was most significant in the horizontal distribution of $B$ (flow direction of particle). For instance, in Fig. 2, when $y = 1.22$ m, the particle on the hopper center line moves toward the gravity direction (that is, $B = 90^\circ$). $B$ values gradually decrease as the distance from the center line increases, indicating that the particles in these areas move inwards toward the hopper center line. However, $B$ values increase in the region of $r/R \geq 0.68$. The graphs of the levels of $y = 1.1$ m and $y = 0.62$ m clearly indicate that two regions of different flow directions exist. In particular, in the graph of $y = 0.62$ m, a discontinuous change in $B$ values clearly appears. The change in the flow direction of the particle

Fig. 3 Distribution of the position of discontinuity in flowing direction of particles
passing through this point, where the inclination of the \( B \) distribution curve changes abruptly or discontinuously (indicated by the broken lines in Fig. 2), can be quantitatively illustrated in the circles in Fig. 2. In these circles, \( p_1 \) and \( p_2 \) are velocity characteristic curves that are defined according to the method of calculation described below. Clearly, these discontinuous points are not on a specific straight line leading to the virtual apex of the hopper. Experimentally, these points correspond to the points where a maximum value of \( U_r \) is obtained or \( U_z \) starts to increase. Furthermore, it has not been experimentally confirmed but it is mathematically possible that \( \partial u_r/\partial r \) need not necessarily be zero at the point where a maximum value of \( u_r \) is obtained. That is, it is possible that the differential coefficient may not be defined, and the inclination of the velocity distribution curve can discontinuously change. For the \( 2\theta = 20^\circ \) hopper, the same characteristics as the velocity field described above were obtained by analyzing the velocity characteristics.

The positions where \( B \) or the inclination of the \( B \) curve discontinuously changes, which were obtained at each height in Fig. 2, are indicated by the \( \otimes \) marks in Fig. 1. Figure 3 a plot of these points of discontinuity obtained for various different heights. The \( \Delta \) marks indicate the distribution of discontinuity at the same time as in Fig. 2. The \( \triangle \) marks indicate the distribution of discontinuity obtained using \( \Delta t = 15 \) seconds at the time when 180 seconds has elapsed from the start of flow. After the flow begins, the free surface of the sample in the upper portion of the hopper continuously changes in shape. However, the distribution of the position of discontinuity remains approximately the same shape independent of the time that has elapsed. This may be regarded as one characteristic of a mass flow. In a funnel flow, a flow region surrounded by a static region gradually expands from the outlet and propagates upwardly, in contrast to the steady behavior that appears in the mass flow. As can be seen by comparing Figs. 1 and 3, two regions of different velocity characteristics that appear in the flow pattern correspond to the two regions that are defined by the distribution of the position of discontinuity. To investigate the reproducibility of the results obtained, the same experiments were repeatedly carried out, thus obtaining the same flow patterns and the same distribution of the positions of discontinuity as those shown in Figs. 1 and 3.

2. 2 Discontinuity in velocity

In this section, the velocity characteristics will be discussed using the distribution of \( u_z \) in the vertical direction. Figure 4 shows the axial distributions of \( u_z \) at distances of \( r = 0.02, 0.06, \) and \( 0.1 \) m, from the central axis. These distributions were obtained in a hopper of \( 2\theta = 20^\circ \). The \( u_z \) was calculated based on the movement of the particles 180 to 210 seconds after the start of the flow operation. As can be seen in the figure, the axial distribution of \( u_z \) is complicated in shape. Especially at the positions of \( r = 0.02 \) and \( 0.06 \) m, significant velocity discontinuities appear at \( y = 0.6 \) m. The same behavior was observed in the hopper of \( 2\theta = 40^\circ \), so significant velocity discontinuity.

![Fig. 4 Axial distribution of \( u_z, \theta = 10^\circ \)](image)
in $u_z$ appeared at $y = 0.4$ to $0.5$ m. The broken lines in Fig. 4 indicate the positions where $u_z$ or $\partial u_z / \partial y$ changes discontinuously. The same velocity analysis was carried out for many positions, excluding the positions shown in Fig. 4. Thus, the distribution of the position of discontinuity in $u_z$ was obtained over the entire hopper as shown by the o marks in Fig. 5. The solid lines in the figure indicate the theoretical velocity characteristics described below. Each graph was obtained based on a velocity analysis executed at the times shown in the figure. These graphs show the experimental data from 0 to 210 seconds after flow started. The shape of each distribution curve is substantially the same independent of the time when the data was obtained thus indicating that the flow characteristics are pseudo-steady. The velocity distribution when the amount of particles was reduced to a low level was not obtained. However, the following assumption may be made: the particle flow in a hopper cannot be steady because the amount of particles in the hopper gradually decreases. However, it may be assumed that the velocity characteristics do not change as long as the stress conditions in the particle layer remaining in the hopper are the same as in the initial state (defined by Eq. (10) described below). As described in the preceding section, the distribution of the points of discontinuity of the $B$ value was regarded to be pseudo-steady. Therefore, it is necessary to discuss whether the discontinuity observed in $B$, $u_z$ or the inclination of these curves is caused by the same particle behavior.

2. 3 Analysis of the characteristic curves of particle behavior

The flow behavior of the particles will be discussed from the viewpoint of plasticity theory. The basic items theoretically supporting this research, such as the equations for the slip lines and the velocity characteristic curve, introduced from plasticity theory, are listed in the Appendix.

1) Discontinuity in the tangential velocity along the velocity characteristic curve

Assume that at the point $Q$ on the characteristic curve ($p_1$-line) in the hopper shown in Fig. 3, both the velocity and direction of movement change discontinuously, as shown in the figure, when particles pass from region $i$ to region $j$. (It will be indicated by the result of the calculation described below that the characteristic curve $ST$ equals $p_1$.) At this time, the tangential and normal velocities along the $p_1$-line in regions $i$ and $j$ are defined as shown in the figure. By applying the equations along the $p_1$-line (that are included in Geiringer’s equations [Eq. (A-4)] to the regions $i$ and $j$, and assuming $a^* = a^*_Q$ at point $Q$:

Region $i$ : $du_{1,i} + u_{2,i}d\alpha_Q^* = 0$  
Region $j$ : $du_{1,j} + u_{2,j}d\alpha_Q^* = 0$  

(3)

Even when a discontinuity in the velocity is permitted, at least the normal velocity that is at a right angle to the discontinuous line $p_1$ must be continuous, that is $u_{2,i} = u_{2,j}$, (10). Therefore,

$$d(u_{1,i} - u_{1,j}) = 0$$

and $u_{1,i} - u_{1,j} = C$ (constant)  

(4)
C represents the amount of discontinuity in the tangential velocity along the $p_1$-line. Assuming the normal axis of the $p_1$-line to be $n$, the equation indicates that a discontinuity is permitted in the shear strain velocity $\partial u/\partial n$ when exceeding the $p_1$-line. Thus, the velocity and the direction of movement of particles can change discontinuously when passing through the velocity characteristic curve (Eq. (A-3)).

2) Estimation of the velocity characteristic curve and the slip line

It was implied\(^8\) that the direction of the failure (slip) was given by the stress characteristic curve (Eq. (A-1)) in the incipient flow status immediately after the particles reached yielding (failing) conditions. From the characteristics of the slip phenomena between the particle layers, a discontinuity in the tangential velocity components along the stress characteristic curve may be permitted in both sides of the curve. Thus, two types of discontinuities in particle flow exist: one related to the stress characteristic curve and the other related to the velocity characteristic curve. To estimate these discontinuities, it is first necessary to determine the stress parameter $\alpha$ (angle between the direction of the major principal stress and the horizontal plane) as a function of the position in the hopper.

Assuming two slip lines ($a$ and $b$) passing through an arbitrary point $P$ in the hopper as shown in Fig. 6, the change in $\alpha$ along the respective slip line can be approximately estimated using Eqs. (A-5) and (A-6) and Eqs. (A-7) and (A-8). The boundary conditions are given as follows:

\[
\begin{align*}
\alpha &= \alpha_c \quad \text{at } r = 0 \quad \text{(on the central axis)} \\
\alpha &= \alpha_R \quad \text{at } r = R, \quad y = y_R \quad \text{(on the wall surface)}
\end{align*}
\]

(5)

After applying Eq. (5) to the $a$-slip line and Eqs. (A-5) and (A-6) to eliminate the integration constant, introduce $r/R$ and $y/R$ as functions of $\alpha$. By taking the ratio between the above functions, the relationship between the coordinates $(r, y)$ of the point $P$ on the $a$-slip line is given as follows:

\[
y/r = \cot \psi = \frac{\cot \theta \cdot f(\alpha_R, \alpha_c, \eta) + g(\alpha_R, \alpha, \eta)}{f(\alpha, \alpha_c, \eta)}
\]

(6)

where functions $f$ and $g$ are defined as follows:

\[
\begin{align*}
\alpha &= (X - Y) \cos 2\eta - \sin 2\eta \\
\times \ln | \cos (Y - \eta)/\cos (X - \eta) | \end{align*}
\]

\[
\alpha = (X - Y) \sin 2\eta + \cos 2\eta \\
\times \ln | \cos (Y - \eta)/\cos (X - \eta) | \]

(7)

Since $\alpha$ value can be calculated at any point $P$ in the hopper by assuming an $a$-slip line passing through the said point $P$, the distribution of $\alpha$ over the entire hopper can be estimated using the above equations. Likewise for the point $P$ on the $b$-slip line in Fig. 6, by applying the boundary conditions (Eq. (5)) to Eqs. (A-7) and (A-8), the distribution of $\alpha$ can also be obtained. The result is expressed by the following equation that is obtained by replacing $\eta$ with $-\eta$ in Eq. (6).

\[
y/r = \cot \psi = \frac{\cot \theta \cdot f(\alpha_R, \alpha_c, \eta) + g(\alpha_R, \alpha, \eta)}{f(\alpha, \alpha_c, \eta)}
\]

(8)

If $\alpha_a$ is the value of $\alpha$ at point $P(x, y)$ obtained from Eq. (6) and $\alpha_b$ is the value obtained from
Eq. (8), their values do not necessarily coincide with each other. Rather, they are normally slightly different because Eqs. (A-5) through (A-9) are approximate equations. In the present research, the average values of these were used to estimate the distribution of $\alpha$.

$$\alpha = (\alpha_a + \alpha_b)/2$$

(9)

Jenike\(^2\) called the stress field where $\alpha$ is a function of only $\psi$ shown in Fig. 6 a "radial stress field". From Eqs. (6) and (8), the approximate solution of $\alpha$ obtained in the present research is also the solution for a radial stress field. To execute the calculation, the boundary conditions are given by the following equations\(^8\) based on the assumption that the horizontal stress within the particles in a mass-flow hopper is at a maximum on the center axis, that is, the particles are in the so-called passive state or arched field.

$$\alpha_c = 0$$
$$\alpha_R = \left\{ \phi_w + 2\theta + \sin^{-1}(\sin \phi_w / \sin \phi) \right\}/2$$

(10)

The network of velocity characteristic curves obtained using Eqs. (6) through (10) and Eq. (A-3) is represented by the solid lines in Figs. 1, 3, and 5. The calculation was executed starting from the hopper wall on the discharge outlet. The friction characteristics of $\phi = 37^\circ$ and $\phi_w = 25^\circ$ were used in the calculation. These values are considered to be reasonable compared with the experimentally obtained values: the angle of discharge $\phi_d = 38^\circ$, and the angle of static friction = 26 to 27$. The theoretical velocity characteristics coincide approximately with the position where the deformation quantity of the tracer layers abruptly changes (shown in Fig. 1) and the position where the direction of particle flow changes discontinuously (shown in Fig. 3). Slight disagreement between the calculated and experimental results are observed in the upper portion of the hopper. This disagreement is considered to be caused by applying the theory (that is only effective when it is applied to regions which are not affected by the free surface) to the upper portion of the hopper without any modifications. From Fig. 5, it is seen that the calculated velocity characteristics coincide with the distribution of the position of discontinuity in $u_z$ determined by the experiments. According to Yamada\(^10\), the velocity discontinuity curve is equivalent to the limiting case, where the width of the narrow transitional region in which the velocity changes suddenly is infinitesimally small. This implies that it is reasonable for the velocity characteristics to be measured as two phenomena: discontinuity in $u_z$ and $B$; and abrupt change in the slope of these curves. In the flow pattern of sawdust given in the preceding report\(^8\), the existence of discontinuities in both the direction of flow and velocity, which did not appear in the incipient flow immediately after the start of flow, was implied in the velocity field of relatively advanced flow. The above described velocity characteristics may be considered to appear in the flow pattern. Thus, the theory described in Section 2.3.1), which predicts the existence of discontinuous change in the velocity vector at the boundary of the velocity characteristic curve, was proved to be effective when applied to the analysis of the flow behavior of the particles.

It was described above that the characteristics of the initial and subsequent flows were observed to be different. The breaks found in an incipient flow of sawdust and in the theo-
retical slip lines obtained using Eqs. (A-1), (6) through (10) are compared in Fig. 7. The experimental data in Fig. 7 were obtained by reading the positions of the breaks from several sheets of photos similar to those shown in the previous report\(^8\). In the previous report, these break surfaces were assumed to correspond to the slip lines. In the previous report\(^8\), \(\phi = 40^\circ\) to \(45^\circ\), and the angle of static friction \(\phi_w = 30^\circ\) were estimated. By assuming the angle of friction shown in the figure, the calculated results coincided approximately with the measured values. Although indefinite factors concerning the angle of friction remain, it may be concluded that the development of breaking surfaces in the incipient flow can be expressed by the slip lines.

In the above analysis, it was assumed that the passive stress conditions (Eq. (10)) developed with the start of flow. The results described above indicate that the particle flow under the conditions used in this experiment satisfies the passive stress field. However, according to a recent report by Moriyama and Jimbo\(^4\), the state of consolidation of packed particles depended on the method by which they had been fed into the hopper, and this affected the dynamic behavior of the wall pressure during discharge. Assuming that the stress conditions differ according to the method of supplying the particles into the hopper, the velocity characteristic curve and slip line generation will differ based on the stress conditions. Investigating this will be our objective in the future research.

### Conclusion

By measuring the velocity of coal particles flowing under gravity in a two-dimensional hopper and by analyzing the obtained velocity in detail, the following results were obtained:

1) The existence of the positions where the direction of particles flow and the velocity changed discontinuously or abruptly was confirmed experimentally, and the distribution of these positions in the hopper was determined. Judging from this distribution, the radial velocity field proposed by Jenike cannot be applied.

2) The particle flow characteristics obtained in the above 1) remained without any changes as time elapsed. That is, the flow was pseudo-steady.

3) The experimentally obtained distribution of velocity characteristics coincided with the velocity characteristic obtained using the plasticity theory. An approximate estimation method was described.

4) It was confirmed that the flow immediately after the particles yielded was characterized by the slip line.

### Appendix

In the \(rz\) plane stress problems, the direction in which the yielded material initiates failure is given by the following equation for the stress characteristic line (slip line) if Coulomb's yield condition for non-adhesive particles is applied\(^1,3,7,8\).

\[
dz/dr = \tan(\alpha \pm \eta)
\]

\(A-1\)

\(\alpha\) is a variable indicating the angle between the plane perpendicular to the \(r\)-direction and the major principal plane. Breaking (slip) is possible in two directions, and these are called the \(a\)- and \(b\)-slip lines. The following equations can be applied to the particles that have already begun to deform only when it is assumed that the axes of major strain and major principal stress coincide\(^10\).

\[
\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0
\]

(Equation expressing continuity)

\[
(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z})(\frac{\partial u_r}{\partial r} - \frac{\partial u_z}{\partial z}) = \tan 2\alpha
\]

\(A-2\)

The velocity characteristic line can be given by the following two curves as the solution of the above simultaneous equations\(^10\). Designating each of them as \(p_1\) and \(p_2\):

\[
p_1 : dz/dr = -\cot \alpha*, \quad p_2 : dz/dr = \tan \alpha*
\]

\[
2\alpha* = 2\alpha - \pi/2
\]

\(A-3\)

Between the components of the strain velocity along the velocity characteristic line, the following equation is satisfied.

\[
p_1 : du_1 + u_2 d\alpha* = 0
\]

\[
p_2 : du_2 - u_1 d\alpha* = 0
\]

\(A-4\)

Where \(u_1\) and \(u_2\) are tangential velocities along lines \(p_1\) and \(p_2\), respectively.

In the previous reports\(^7,8,9\), the distribution of \(\alpha\) was estimated from the following approxi-
mations for the changes in $\alpha$ along the slip line, Eq. (A-1).

$a$-slip line

\[\begin{align*}
\text{r vs. } \alpha & : \ A \left\{ (\alpha - \eta) \cos 2\eta + \sin 2\eta \right\} \ln | \cos (\alpha - \eta) | = r + c_1 \\
y vs. \alpha & : \ A \left\{ (\alpha - \eta) \sin 2\eta - \cos 2\eta \right\} \ln | \cos (\alpha - \eta) | = -y + c_2
\end{align*}\] (A-5) (A-6)

$b$-slip line

\[\begin{align*}
\text{r vs. } \alpha & : \ A \left\{ (\alpha + \eta) \cos 2\eta - \sin 2\eta \right\} \ln | \cos (\alpha + \eta) | = r + c_3 \\
y vs. \alpha & : \ A \left\{ (\alpha + \eta) \sin 2\eta + \cos 2\eta \right\} \ln | \cos (\alpha + \eta) | = y + c_4
\end{align*}\] (A-7) (A-8)

where $c_1$ through $c_4$ are integration constants and

\[A \equiv 2 \sigma_m \sin \phi / (-\rho_bg)\] (A-9)

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Nomenclature

- $B$ : flow direction of particle, defined in Eq. (2) [°]
- $R$ : distance from center axis to hopper wall [m]
- $r, z$ : coordinates defined by Fig. 6 [m]
- $t$ : time [s]
- $u_r, u_z$ : horizontal and vertical velocities [m/s]
- $u_1, u_2$ : velocities tangential and normal to $p_1$-line
- $y$ : distance from hopper [m]
- $\alpha$ : angle between the plane perpendicular to the $r$-direction and major principal plane [rad]
- $\alpha_c, \alpha_R$ : values of $\alpha$ at $r = 0$ and $r = R$ [rad]
- $\eta$ : $\pi/4 - \phi/2$ [rad]
- $\theta$ : hopper half-angle [°]
- $\rho_b$ : density [kg/m³]
- $\sigma_m$ : mean principal stress [Pa]
- $\phi$ : angle of internal friction [°]
- $\phi_d$ : angle of discharge [°]
- $\phi_w$ : angle of wall friction [°]
- $\psi$ : radial angle defined by Fig. 6 [rad]

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