A tiling proof of Euler’s Pentagonal Number Theorem and generalizations

Dennis Eichhorn · Hayan Nam · Jaebum Sohn

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Abstract
In two papers, Little and Sellers introduced an exciting new combinatorial method for proving partition identities which is not directly bijective. Instead, they consider various sets of weighted tilings of a $1 \times \infty$ board with squares and dominoes, and for each type of tiling they construct a generating function in two different ways, which in turn generates a $q$-series identity. Using this method, they recover quite a few classical $q$-series identities, but Euler’s Pentagonal Number Theorem is not among them. In this paper, we introduce a key parameter when constructing the generating functions of various sets of tilings which allows us to recover Euler’s Pentagonal Number Theorem along with an uncountably infinite family of generalizations.

Keywords Pentagonal Number Theorem · Rank · Generalized rank · Tiling

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1 Introduction
Proofs of $q$-series identities generally fall into one of the three categories: proofs by classical by $q$-series manipulations, proofs appealing to modular forms, or proofs using bijective methods showing that two sets of combinatorial objects generated by the two sides of an identity are equinumerous. In [5] and [6], Little and Sellers introduced a new combinatorial method for proving partition identities which is not directly bijective. Instead, they consider various sets of weighted tilings of a $1 \times \infty$ board with squares...
and dominoes, and for each type of tiling they construct a generating function for the set of all tilings in two different ways. Since this creates two different expressions for the same object, it generates a $q$-series identity. Using this method, they recover quite a few classical $q$-series identities, ranging from identities like

$$
\sum_{n \geq 0} \frac{(-z; q)_n}{(q^2; q^2)_n} q^{n^2+n} = \prod_{n \geq 1} (1 + q^{4n-2})(1 + zq^{4n-2})(1 + q^{4n})
$$

due to Göllnitz to more complicated identities such as

$$
\sum_{n=0}^{\infty} z^n q^{n^2+n} \frac{(1 - z^2 q^{2n+3})}{(q; q)_n} = (-zq; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{3n^2}}{(q^2; q^2)_n (-zq; q)_{2n+1}}.
$$

Interestingly, Euler’s Pentagonal Number Theorem is not among the results they recover using their method.

In this paper, we modify their method of generating $q$-series identities by introducing a new statistic on tilings, the rank. By considering tilings of every possible rank, we construct our generating functions in a new way which allows us to recover Euler’s Pentagonal Number Theorem along with a one-parameter generalization. Fortunately, the notion of the rank lends itself to significant generalization. For any two increasing sequences of nonnegative integers $X = \{x_m\}$ and $Y = \{y_m\}$, we can define another statistic on tilings called the $(X, Y)$-rank. Considering tilings of every possible $(X, Y)$-rank ultimately yields an uncountably infinite family of identities, each of which is a one-parameter generalization of Euler’s Pentagonal Number Theorem.

## 2 Tilings and the rank

As in [5,6], we consider tiling a $1 \times \infty$ board with tiles of various colors. For the bulk of the paper, we will only consider tilings with white squares and black squares where each tiling has a finite number of black squares. We define the position of a square to be its location on the board, a positive integer, and every tiling has exactly one square at each position. We define the weight of a tile $t$ to be

$$
w(t) = \begin{cases} 
zq^i & \text{if } t \text{ is a black square in position } i, \\
1 & \text{if } t \text{ is a white square in position } i, 
\end{cases}
$$

and the weight of a tiling $T$ is

$$
w(T) = \prod_{t \in T} w(t),
$$

the product of the weights of its tiles. Notice if a tiling $T$ has $m$ black tiles and $n$ is the sum of the positions of the black tiles in the tiling, then the weight of the tiling is $w(T) = z^m q^n$.
Fig. 1 A tiling with black squares at positions 3, 4, 6, 7, 8, 11, 12, 13, 14, 15, 16, and 18

Example 2.1 Consider the tiling with black squares at positions 3, 4, 6, 7, 8, 11, 12, 13, 14, 15, 16, and 18. Since the white squares at the other positions have weight 1, they do not contribute to the weight of the tiling. Thus the weight of the tiling is the product of the weights of the black squares, which is (Fig. 1): 

\[ zq^3 \times zq^4 \times zq^6 \times zq^7 \times zq^8 \times zq^{11} \times zq^{12} \times zq^{13} \times zq^{14} \times zq^{15} \times zq^{16} \times zq^{18} = z^{12} q^{127}. \]

Define

\[ F(z, q) = \sum_{\text{tilings } T} w(T). \]

We call \( F(z, q) \) the generating function of all tilings of the \( 1 \times \infty \) board. Notice that the contribution to the total weight of the tiling of the tile at position \( i \) must be either 1 or \( zq^i \), and thus

\[ F(z, q) = \prod_{i=1}^{\infty} (1 + zq^i) = (-zq; q)_{\infty}, \quad (1) \]

where we are using the usual \( q \)-series notation that defines

\[ (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i) \quad \text{and} \quad (a; q)_{\infty} = \lim_{n \to \infty} (a; q)_n. \]

Since tilings are determined completely by the positions of their black tiles, we will often only discuss the positions of black tiles. Some of our treatment of tilings will involve changing the positions of various tiles; it is to be assumed that whenever we move black tiles, any displaced white tiles are simply removed, and any vacancies left by moving black tiles are filled in with white tiles as needed to complete the tiling. Little and Sellers [6] defined one particular moving process which they called the projection of tiles, which we adopt here. Define projection to be moving some number of black tiles to the right in a way that does not change the relative positions of the black tiles. Observe that projecting one black tile \( k \) positions to the right increases the weight of a tiling by a factor of \( q^k \).

We are now ready to recompute \( F(z, q) \) in a new way. Notice that every tiling with \( m \) black tiles could be constructed by first placing black tiles in positions 1 through \( m \) (with white tiles at all other positions) and then projecting those \( m \) black tiles out to their proper positions. The weight of the initial placement is \( z^m q^{m(m+1)/2} \), and there
are several ways in which we could group tiles as we project them from positions 1 through \( m \) out to any \( m \) given positions \( p_1 < p_2 < \cdots < p_m \). One method is to first project all \( m \) tiles to the right until the leftmost tile is in position \( p_1 \), then project the \( m - 1 \) remaining tiles to the right until the second leftmost tile is in position \( p_2 \), and then iteratively continuing to project the remaining tiles to the right until the tiles are in positions \( p_1 \) through \( p_m \). Projecting \( m \) tiles \( k \) positions to the right will increase the weight by a factor of \( q^{mk} \). To generate any possible tiling, we must allow for projections every possible nonnegative distance, and so generating the first projection of all \( m \) tiles introduces a factor of
\[
\frac{1}{1 - q^m},
\]
and in general, the projection of the next \( m - i \) tiles introduces a factor of
\[
\frac{1}{1 - q^{m-i}}.
\]
Summing over every possible number of black tiles \( m \), we have
\[
F(z, q) = \sum_{m=0}^{\infty} z^m q^{m(m+1)/2} \frac{(q; q)_m}{(q; q)_m}.
\]
Equating our two expressions for \( F(z, q) \) in (1) and (2), we recover an identity of Euler [4], namely,
\[
(-zq; q)_\infty = \sum_{m=0}^{\infty} z^m q^{m(m+1)/2} \frac{(q; q)_m}{(q; q)_m}.
\]
This is perhaps the simplest example of a \( q \)-series identity that can be proven using tilings and projection. Little and Sellers actually start with more sophisticated identities by treating tilings with squares and dominoes.

In the next subsection, we introduce the rank of a tiling, which is the critical statistic that will allow us to use this same method to recover Euler’s Pentagonal Number Theorem and a one-parameter generalization due to Sylvester.

2.1 The rank

Indexing tilings by their ranks instead of their total number of black squares gives us a new way to prove \( q \)-series identities. For a given tiling, let \( b(m) \) be the number of black squares at positions greater than \( m \).

**Definition 2.2** The *rank* of a tiling is the least \( m \) such that \( b(m) \leq m \).

**Example 2.3** Consider the tiling with black squares at positions 3, 4, 6, 7, 8, 11, 12, 13, 14, 15, 16, and 18. For each position \( m \), we compute \( b(m) \) in Table 1. Since \( b(7) = 8 \) and \( b(8) = 7 \), we see that the rank of this tiling is 8.

Notice that changing the colors of any of the tiles in the first seven position of the tiling above does not change the rank. The tiling with black squares at 1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, and 18 also has rank 8, as does the tiling with black squares at just 8, 11, 12, 13, 14, 15, 16, and 18, for example (see Fig. 2).
Table 1 \(b(m)\) is the number of black squares at positions greater than \(m\)

| \(m\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|
| \(b(m)\) | 12 | 12 | 11 | 10 | 10 | 9 | 8 | 7 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 1 | 0 | 0 |}

Remark 2.4 Notice that for any tiling, \(b(m)\) is a nonincreasing sequence, and \(b(m - 1) - b(m) \leq 1\) for all \(m \geq 1\). That being the case, we see that the rank of a tiling will be the unique \(m\) such that either

1. \(b(m) = m\), or
2. \(b(m - 1) = m\) and \(b(m) = m - 1\),

and these two cases are mutually exclusive. This will allow us to take a sum over all tilings by summing these two cases over every \(m\) in our results below.

Conceptually, the reader may find it helpful to think of the graph of \(b(m)\) as a path along lattice points. Remark 2.4 reflects the fact that every step of the lattice path is either directly to the right or at a slope of \(-1\), and as this lattice path goes from the vertical axis down to the horizontal axis, there are two possible cases. Either (1), it will pass through a lattice point where \(b(m) = m\), or (2), it will pass through a pair of lattice points such that \(b(m - 1) = m\) and \(b(m) = m - 1\). In Fig. 3, we have indicated this by graphing the points \((m, m)\) in gray along with the line segments connecting \((m - 1, m)\) and \((m, m - 1)\) for each \(m\). For every tiling, the corresponding graph of \(b(m)\) will either intersect one of the points, or it will traverse one of the line segments (but not both).

We are now ready to give a third expression for \(F(z, q) = (-z q; q)_\infty\) originally due to Sylvester [7, p. 281].

Theorem 2.5 (Sylvester) For \(|q| < 1\),

\[
(-z q; q)_\infty = 1 + \sum_{m=1}^{\infty} \left( \frac{(-z q; q)_{m-1}}{(q; q)_{m-1}} z^m q^{m(3m-1)/2} \right.
\]

\[
+ \left. \frac{(-z q; q)_m}{(q; q)_m} z^m q^{m(3m+1)/2} \right) = \sum_{m=0}^{\infty} \frac{(-z q; q)_m}{(q; q)_m} z^m q^{m(3m+1)/2} (1 + z q^{2m+1}).
\]
Fig. 3 The gray dots are the points \((m, m)\) and the line segments connect pairs of points \((m - 1, m)\) and \((m, m - 1)\). The black squares graph the sequence \(b(m)\) for the tiling from Examples 2.1 and 2.3. Since \(b(7) = 8\) and \(b(8) = 7\), we know that rank of the tiling is 8.

**Proof** We proceed by writing down the generating function for all tilings of rank \(m\), and then summing over every possible rank of a tiling. We do this by first considering the possible weights of the first \(m\) positions of a tiling of rank \(m\), and then generating the tiles at positions greater than \(m\) by placing them in the first available positions and projecting them.

For tilings of rank \(m \geq 1\), notice that the weight of the tile at each position \(i\) for \(1 \leq i \leq m - 1\) must be either 1 or \(zq^i\), and thus the first \(m - 1\) positions are generated by \((-zq; q)_{m-1}\). The remainder of the generating function comes from two cases depending on whether \(b(m) = m\) or \(m - 1\).

If \(b(m) = m - 1\), as we have seen above we also have \(b(m - 1) = m\), and so position \(m\) must have a black square. Thus the contribution to the generating function of position \(m\) is \(zq^m\). For positions greater than \(m\), we can generate all possibilities by first placing \(m - 1\) black tiles at locations \(m + 1\) through \(2m - 1\), and then projecting them in all possible ways. The initial placement of those \(m - 1\) tiles has weight \(z^{m-1}q^{3m(m-1)/2}\), and all possible projections are generated by \(1/(q; q)_{m-1}\). Thus rank \(m\) tilings with \(b(m) = m - 1\) and \(b(m - 1) = m\) are generated by

\[
(-zq; q)_{m-1} \times zq^m \times \frac{z^{m-1}q^{3m(m-1)/2}}{(q; q)_{m-1}} = \frac{(-zq; q)_{m-1}}{(q; q)_{m-1}} z^m q^{m(3m-1)/2}. \tag{6}
\]

If \(b(m) = m\), position \(m\) may have a white square or a black square, and thus the contribution to the generating function of position \(m\) is \(1 + zq^m\). For positions greater than \(m\), we can generate all possibilities by first placing \(m\) black tiles at locations \(m + 1\) through \(2m\), and then projecting them in all possible ways. The initial placement of those \(m\) tiles has weight \(z^m q^{m(3m+1)/2}\), and all possible projections are generated by \(1/(q; q)_m\). Thus rank \(m\) tilings with \(b(m) = m\) are generated by
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\[
(-zq; q)_{m-1} \times (1 + zq^m) \times \frac{z^m q^{m(3m+1)/2}}{(q; q)_m} = \frac{(-zq; q)_m z^m q^{m(3m+1)/2}}{(q; q)_m}.
\]  

(7)

Summing (6) and (7) over all \(m \geq 1\) and adding 1 for the weight of the tiling of rank 0 (the tiling of all white tiles), we have our new expression for the generating function of all tilings.

In addition to Sylvester’s original proof of Theorem 2.5 in [7, p. 281], there is also a very interesting combinatorial treatment of Theorem 2.5 by Alladi in [1]. Setting \(z = -1\) in (4) creates substantial cancelation and recovers Euler’s Pentagonal Number Theorem.

**Corollary 2.6** (Euler’s Pentagonal Number Theorem)

\[
(q; q)_\infty = 1 + \sum_{m=1}^{\infty} (-1)^m q^{m(3m-1)/2} (1 + q^m).
\]

3 The generalized rank

The search for a tiling proof of Euler’s Pentagonal Number Theorem is what led the authors to discover the original rank statistic. To recover Euler’s Pentagonal Number Theorem, the rank strikes the perfect balance between the number of projectiles and how far to the right we go on the \(1 \times \infty\) board to place those projectiles in their initial configuration. However, even when we do not strike this perfect balance, there are other perfectly legitimate ways to generate all possible tilings of the \(1 \times \infty\) board using projectiles, each of which generates a new identity for \(F(z, q) = (-zq; q)_\infty\). To explore this idea fully, we now generalize the notion of the rank.

The rank in Definition 2.2 is well defined because there is always a least \(m\) such that the threshold condition \(b(m) \leq m\) is met. Fortunately, this same idea can be applied to a much more general class of threshold conditions.

**Definition 3.1** For any two increasing sequences of nonnegative integers \(X = \{x_m\}_{m=0}^{\infty}\) and \(Y = \{y_m\}_{m=0}^{\infty}\) with \(x_0 = y_0 = 0\), we define the \((X, Y)\)-rank to be least \(m\) such that \(b(x_m) \leq y_m\).

Notice that if we set \(X\) and \(Y\) to each be the simplest possible increasing sequence, \(X = Y = \{m\}_{m=0}^{\infty}\), the \((X, Y)\) rank of a tiling is our original rank given in Definition 2.2. Also notice that in all cases, the only tiling with rank zero is the tiling with no black squares.

**Example 3.2** Let \(X = \{m\}_{m=0}^{\infty}\) and \(Y = \{2m\}_{m=0}^{\infty}\). Consider the tiling with black squares at positions 3, 4, 6, 7, 8, 11, 12, 13, 14, 15, 16, and 18. For each position \(m\), we compute \(b(m)\) in Table 2. Since \(b(4) = 10 \not\leq 2 \cdot 4\), but \(b(5) = 10 \leq 2 \cdot 5\), we see that the \((X, Y)\)-rank of this tiling is 5.

As mentioned earlier, the reader may find it helpful to think of the graph of \(b(m)\) as a path along lattice points. After Remark 2.4, we observed that a tiling has rank

\[\begin{align*}
(-zq; q)_{m-1} \times (1 + zq^m) \times \frac{z^m q^{m(3m+1)/2}}{(q; q)_m} &= \frac{(-zq; q)_m z^m q^{m(3m+1)/2}}{(q; q)_m},
\end{align*}\]
Table 2 The rank of this tiling is 8, and for \( X = \{ m \}_{m=0}^{\infty} \) and \( Y = \{ 2m \}_{m=0}^{\infty} \), the \((X, Y)\)-rank of this tiling is 5

| \( m \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|
| \( b(m) \) | 12 | 12 | 11 | 10 | 10 | 9 | 8 | 7 | 7 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 1 | 0 | 0 |

![Diagram](image)

Fig. 4 The circular dots are the points in the rectangular regions of lattice points of the form \( \{(i, j) \mid 4m - 4 \leq i \leq 4m, 3m - 3 < j \leq 3m\} \). The gray dots are the points \( \{(4m, j) \mid 3m - 3 < j \leq 3m\} \), and the line segments connect pairs of points \( (i - 1, 3m - 2) \) and \( (i, 3m - 3) \) for \( 4m - 4 < i \leq 4m \). The black squares graph the sequence \( b(m) \) for the tiling from Examples 2.1, 2.3, and 3.2. It is easy to see that the sequence \((m, b(m))\) passes through the third rectangular region, and thus for \( X = \{4m\}_{m=0}^{\infty} \) and \( Y = \{3m\}_{m=0}^{\infty} \), the \((X, Y)\)-rank is 3. Algebraically, this is witnessed by the fact that \( b(10) = 7 = 3 \cdot 3 - 2 \) and \( b(11) = 6 = 3 \cdot 3 - 3 \) with \( 4 \cdot 3 - 4 < 11 \leq 4 \cdot 3 \).

Example 3.3 Let \( X = \{4m\}_{m=0}^{\infty} \) and \( Y = \{3m\}_{m=0}^{\infty} \). To determine the \((X, Y)\)-rank of a tiling, we can graph all rectangular regions of lattice points of the form \( \{(i, j) \mid 4m - 4 \leq i \leq 4m, 3m - 3 < j \leq 3m\} \) and then plot the sequence \((m, b(m))\) to see which of the rectangular regions it intersects. In Fig. 4, we have done this for the tiling from Examples 2.1, 2.3, and 3.2. Since the sequence \((m, b(m))\) intersects the third rectangular region, the \((X, Y)\)-rank of the tiling is 3.

Remark 3.4 As the path leaves the unique rectangular region it intersects, either the path will pass through a lattice point where \( b(x_m) = j \), for some \( y_m - 1 + 1 \leq j \leq y_m \) (along the right edge of the rectangle), or it will pass through a pair of lattice points...
such that \( b(i - 1) = y_{m-1} + 1 \) and \( b(i) = y_{m-1} \) for some \( x_{m-1} + 1 \leq i \leq x_m \) (exiting the bottom of the rectangle). This allows us to algebraically characterize the \((X, Y)\)-rank as the unique \( m \) such that either

1. \( b(x_m) = j \), for some \( y_{m-1} + 1 \leq j \leq y_m \), or
2. \( b(i - 1) = y_{m-1} + 1 \) and \( b(i) = y_{m-1} \) for some \( x_{m-1} + 1 \leq i \leq x_m \).

In Fig. 4, we graph all of the points in each rectangular region, coloring the points \((x_m, j)\) in gray, and graphing the line segments connecting \((i - 1, y_{m-1} + 1)\) and \((i, y_{m-1})\) for each \( m \). For every tiling, the corresponding graph of \( b(m) \) will either intersect exactly one of the gray points, or it will traverse exactly one of the line segments (but not both).

Now, the algebraic characterization of the \((X, Y)\)-rank allows us to give our main theorem, an uncountably infinite family of identities for \( F(z, q) = (-zq; q)_\infty \).

**Theorem 3.5** Let \(|q| < 1\). For any two increasing sequences of nonnegative integers \( X = \{x_m\}_{m=0}^\infty \) and \( Y = \{y_m\}_{m=0}^\infty \) with \( x_0 = y_0 = 0 \), we have

\[
(-zq; q)_\infty = 1 + \sum_{m=1}^\infty \sum_{i=x_{m-1}+1}^{x_m} \frac{(-zq; q)_{i-1} z^{y_{m-1}+1} q^{(y_{m-1}+1)(y_m+2i)/2}}{(q; q)^{y_{m-1}}} + \sum_{m=1}^\infty \sum_{j=y_{m-1}+1}^{y_m} \frac{(-zq; q)_{x_m} z^j q^{(2x_m+j+1)/2}}{(q; q)^j}.
\]

**Proof** We proceed by writing down the generating function for all tilings of \((X, Y)\)-rank \( m \), and then summing over every possible \((X, Y)\)-rank of a tiling.

For tilings of \((X, Y)\)-rank \( m \geq 1 \), if we are in Case (1) from Remark 3.4, our only condition on the coloring of our tiles is that \( b(x_m) = j \), for some \( y_{m-1} + 1 \leq j \leq y_m \). Notice that in this case, the weight of the tile at each position \( i \) for \( 1 \leq k \leq x_m \) can be either 1 or \( zq^k \), and thus the first \( x_m \) positions are generated by \((-zq; q)_{x_m}\). For positions greater than \( m \), we can generate all possibilities by first placing \( j \) black tiles at locations \( x_m + 1 \) through \( x_m + j \), and then projecting them in all possible ways. The initial placement of those \( j \) tiles has weight \( z^j q^{(2x_m+j+1)/2} \), and all possible projections are generated by \( 1/(q; q)_j \). Thus \((X, Y)\)-rank \( m \) tilings with \( b(x_m) = j \) are generated by

\[
\frac{(-zq; q)_{x_m} z^j q^{(2x_m+j+1)/2}}{(q; q)^j}.
\]

If we are in Case (2) from Remark 3.4, our only condition on the coloring of our tiles is that \( b(i - 1) = y_{m-1} + 1 \) and \( b(i) = y_{m-1} \) for some \( x_{m-1} + 1 \leq i \leq x_m \). Notice that in this case the weight of the tile at each position \( i \) for \( 1 \leq k \leq i - 1 \) must be either 1 or \( zq^k \), and thus the first \( i - 1 \) positions are generated by \((-zq; q)_{i-1}\). Notice also that position \( i \) must have a black square, and thus the contribution to the generating function of position \( i \) is \( zq^i \). For positions greater than \( i \), we can generate all possibilities by first placing \( y_{m-1} \) black tiles at locations \( i + 1 \) through \( i + y_{m-1} \),

\[\text{Springer}\]
and then projecting them in all possible ways. The initial placement of those \( y_{m-1} \) tiles has weight \( z^{y_{m-1}}q^{y_{m-1}(y_{m-1}+2i+1)/2} \), and all possible projections are generated by \( 1/(q; q)_{y_{m-1}} \). Thus \((X, Y)\)-rank \( m \) tilings with \( b(i-1) = y_{m-1} + 1 \) and \( b(i) = y_{m-1} \) are generated by

\[
\frac{(-zq; q)_{i-1}}{q(q; q)_{y_{m-1}}} z^{y_{m-1}+1} q^{(y_{m-1}+1)(y_{m-1}+2i)/2}.
\] (9)

Summing (8) and (9) over all \( m \geq 1 \) and adding 1 for the weight of the tiling of rank 0 (the tiling of all white tiles), we have our new expression for the generating function of all tilings.

Theorem 3.5 has many interesting corollaries. Of course, it is quite natural to consider the case when the sequences \( X \) and \( Y \) are linear.

**Theorem 3.6** Let \(|q| < 1\), and for any positive integers \( k, \ell, r, \) and \( s \), let \( x_0 = y_0 = 0, x_m = k(m-1) + r, \) and \( y_m = \ell(m-1) + s \) for \( m > 0 \). Then

\[
(-zq; q)_{\infty} = 1 + \sum_{i=1}^{r} (-zq; q)_{i-1} zq^i + \sum_{j=1}^{s} \frac{(-zq; q)_{x_1} zq^j}{(q; q)_{j}} \left( -zq; q \right)_{i-1} z^{\ell(m-2)+s+1} q^{(\ell(m-2)+s+1)(\ell(m-1)+s+2i)/2} \]

\[
+ \sum_{m=2}^{\infty} \sum_{i=k(m-2)+r+1}^{k(m-1)+r} \frac{(-zq; q)_{i-1}}{(q; q)_{j}} z^{\ell(m-2)+s+1} q^{(\ell(m-2)+s+1)(\ell(m-1)+s+2i)/2} \]

\[
+ \sum_{m=2}^{\infty} \sum_{j=\ell(m-2)+s+1}^{\ell(m-1)+s+s+1} \frac{(-zq; q)_{k(m-1)+r}}{(q; q)_{j}} z^{\ell(m-1)+s} q^{(k(m-1)+r)(\ell(m-1)+s+2i)/2}.
\]

An astute referee of a previous version of this paper made the observation that when we set \( X = \{km\}_{m=0}^{\infty} \) and \( Y = \{\ell m\}_{m=0}^{\infty} \), we recover a family of identities given by Andrews [2]. Andrews obtained this same family of identities by analyzing Durfee rectangles whose side lengths have prescribed ratios inside partitions into distinct parts.

**Theorem 3.7** (Andrews) Let \(|q| < 1\). For any positive integers \( k \) and \( \ell \), we have

\[
(-zq; q)_{\infty} = 1 + \sum_{m=1}^{\infty} \sum_{i=k(m-1)+1}^{km} \frac{(-zq; q)_{i-1}}{(q; q)_{\ell(m-1)}} z^{\ell(m-1)+1} q^{(\ell(m-1)+1)(\ell m+2i)/2} \]

\[
+ \sum_{m=1}^{\infty} \sum_{j=\ell(m-1)+1}^{\ell m} \frac{(-zq; q)_{km}}{(q; q)_{j}} z^{\ell(m-1)+1} q^{(2km+j+1)/2}.
\]

Several simpler interesting corollaries come from setting \( z = -1 \) because of the cancelation that then occurs between the numerators and denominators of the summands in Theorem 3.5. Of course, if we set \( X = Y = \{m\}_{m=0}^{\infty} \) to take us back to
the original rank and then set \( z = -1 \), we again recover Euler’s Pentagonal Number Theorem as mentioned in Sect. 2.

The next simplest corollary of Theorem 3.5 with \( z = -1 \) comes from setting
\[
X = \{ 2m \}_{m=0}^\infty \quad \text{and} \quad Y = \{ m \}_{m=0}^\infty.
\]

**Corollary 3.8** We have
\[
(q; q)_\infty = 1 + \sum_{m=1}^{\infty} (-1)^m \left\{ (q^{m+1}; q)_m q^{m(5m+1)/2} + (q^m; q)_m q^{m(5m-1)/2} \right\}
\]
\[
= 1 + \sum_{m=1}^{\infty} (-1)^m \left\{ (q^{m+1}; q)_{m-2} \left( 1 - q^{3m-1} - q^{4m} + q^{6m-1} \right) q^{m(5m-3)/2} \right\}.
\]

**Remark 3.9** Notice that the exponents \( m(5m - 3)/2 \) are the heptagonal numbers, and thus Corollary 3.8 is a “heptagonal number theorem” of sorts, where we have expressed \( (q; q)_\infty \) as a sum of heptagonal powers of \( q \) multiplied by simple polynomials. Notice also that if we apply Euler’s Pentagonal Number Theorem to the left-hand side of Corollary 3.8, we see that this is a way to construct the pentagonal number series from simple polynomials multiplied heptagonal powers of \( q \). If we set \( z = -1 \), \( X = \{ 3m \}_{m=0}^\infty \), and \( Y = \{ m \}_{m=0}^\infty \) in Theorem 3.5, we get a similar “nonagonal number theorem,” and more generally, any instance of setting \( z = -1 \) with \( X = \{ km \}_{m=0}^\infty \) and \( Y = \{ m \}_{m=0}^\infty \) in Theorem 3.5, leads to a “(2k + 3)-agonal number theorem.”

Setting \( z = 1 \) in Theorem 3.5 also gives some interesting corollaries, although they are of a very different flavor than Corollaries 2.6 and 3.8. Notice that with \( z = 1 \), if we expand the denominators in the sums on the right-hand side of Theorem 3.5 as geometric series, everything on both sides of the equation becomes positive. In contrast to corollaries where \( z = -1 \) and cancelation plays a huge role, when we let \( z = 1 \), there is no cancelation whatsoever, and our results are purely additive.

The astute reader will also notice that in the case where \( z = 1 \), while the left-hand side of Theorem 3.5 is the generating function for partitions into distinct parts, the summands on the right-hand side are related to a variation of overpartitions. We leave the exploration of this connection between partitions into distinct parts and a variation of overpartitions to the reader.

### 4 Conclusion

By introducing the rank, we were able to extend the scope of the method of tiling to include Euler’s Pentagonal Number Theorem and an uncountable family of generalizations. These generalizations have many interesting special cases, including a result of Sylvester, a “(2k + 3)-agonal number theorem,” identities relating partitions into distinct part to variations of overpartitions, and an already very general result of Andrews. In fact, there are several more possible generalizations that we have not listed above. In [5,6], Little and Sellers treat tilings with dominoes instead of squares quite
thoroughly. Introducing the concept of the rank into domino tilings reveals many more identities, all involving the $q$-Fibonacci polynomials of Carlitz [3]. These identities are of a somewhat different and more complicated form than most classical identities.

An open problem that remains is whether or not Jacobi’s Triple Product Identity can be proven using the method of tiling through the introduction of a rank-like statistic. In addition, it will be interesting to see if the scope of the method of tiling can be extended even further to include other families of identities by using the concept of the rank.

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