ON A REGULARIZED FAMILY OF MODELS FOR HOMOGENEOUS INCOMPRESSIBLE TWO-PHASE FLOWS

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Abstract. We consider a general family of regularized models for incompressible two-phase flows based on the Allen-Cahn formulation in \( n \)-dimensional compact Riemannian manifolds for \( n = 2, 3 \). The system we consider consists of a regularized family of Navier-Stokes equations (including the Navier-Stokes-\( \alpha \)-like model, the Leray-\( \alpha \) model, the Modified Leray-\( \alpha \) model, the Simplified Bardina model, the Navier-Stokes-Voight model and the Navier-Stokes model) for the fluid velocity \( u \) suitably coupled with a convective Allen-Cahn equation for the order (phase) parameter \( \phi \). We give a unified analysis of the entire three-parameter family of two-phase models using only abstract mapping properties of the principal dissipation and smoothing operators, and then use assumptions about the specific form of the parameterizations, leading to specific models, only when necessary to obtain the sharpest results. We establish existence, stability and regularity results, and some results for singular perturbations, which as special cases include the inviscid limit of viscous models and the \( \alpha \to 0 \) limit in \( \alpha \)-models. Then, we also show the existence of a global attractor and exponential attractor for our general model, and then establish precise conditions under which each trajectory \((u, \phi)\) converges to a single equilibrium by means of a Lojasiewicz-Simon inequality. We also derive new results on the existence of global and exponential attractors for the regularized family of Navier–Stokes equations and magnetohydrodynamics models which improve and complement the results of [44]. Finally, our analysis is applied to certain regularized Ericksen-Leslie (RSEL) models for the hydrodynamics of liquid crystals in \( n \)-dimensional compact Riemannian manifolds.

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1. Introduction

Modelling and simulating the behavior of binary fluid mixtures in various turbulent regimes can be rather challenging [3]. A possible approach is based on the so-called diffuse-interface method (see [3] [14] [63] and their references). This method consists in introducing an order parameter,
accounting for the presence of two species, whose dynamics interacts with the fluid velocity. For incompressible fluids with matched densities, a well-known model consists of the classical Navier-Stokes equation suitably coupled with either a convective Cahn-Hilliard or Allen-Cahn equation (see \[8, 22, 23, 30, 38, 43, 68, 74, 78\] cf. also \[5, 16, 48, 56, 60, 67, 70\]). Denoting by \(u = (u_1, ..., u_n)\), \(n \geq 2\), the velocity field and by \(\phi\) the order parameter, where we suppose that \(\phi\) is normalized in such a way that the two pure phases of the fluid are \(-1\) and \(+1\), respectively, the Cahn-Hilliard-Navier-Stokes and the Allen-Cahn-Navier-Stokes systems can be written in a unified form. Indeed, if additionally we assume that the viscosity of fluid is constant, and temperature differences are negligible, we have

\[
\begin{align*}
(1.1) & \quad \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = -\varepsilon \text{div}(\nabla \phi \otimes \nabla \phi) + g, \\
(1.2) & \quad \text{div}(u) = 0, \\
(1.3) & \quad \partial_t \phi + u \cdot \nabla \phi + A_K \mu = 0, \\
(1.4) & \quad \mu = -\varepsilon \Delta \phi + \varepsilon^{-1} f(\phi),
\end{align*}
\]

in \(\Omega \times (0, +\infty)\), where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), \(n = 2, 3\), with a sufficiently smooth boundary \(\Gamma\), \(\varepsilon > 0\) is a parameter related to the thickness of the interface separating the two fluids, and \(g = g(t)\) is an external body force. Moreover, the operator \(A_K\) has a two-fold definition according to the case \(K = \text{CH}\) (Cahn-Hilliard fluid) or \(K = \text{AC}\) (Allen-Cahn fluid), namely,

\[
(1.5) \quad A_{\text{CH}} \mu = -m \Delta \mu, \quad A_{\text{AC}} \mu = \mu,
\]

where \(m > 0\) is the mobility of the mixture. The so-called chemical potential \(\mu\) is obtained, under an appropriate choice of boundary conditions, as a variational derivative of the following free energy functional

\[
(1.6) \quad \mathcal{F}(\phi) = \int_{\Omega} \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \varepsilon^{-1} F(\phi) \right) \, dx,
\]

where \(F(r) = \int_0^r f(y) \, dy\), \(r \in \mathbb{R}\). Here, the potential \(F\) is either a double-well logarithmic-type function

\[
(1.7) \quad F(r) = \gamma_1 ((1 + r) \log (1 + r) + (1 - r) \log (1 - r)) + \gamma_2 (1 - r^2), \quad r \in (-1, 1),
\]

for some \(\gamma_1, \gamma_2 > 0\), or a polynomial approximation of the type

\[
(1.8) \quad F(r) = \gamma_3 (r^2 - 1)^2,
\]

for some \(\gamma_3 > 0\).

Both the two systems \((1.1) - (1.4)\) for \(K = \text{AC}\) and \(K = \text{CH}\) capture basic features of binary fluid behavior. There are several key differences between the two formulations:

(I) With the Allen-Cahn formulation, both singular and regular potentials can be treated since a maximum principle holds under appropriate assumptions on \(F\) when \(K = \text{AC}\). In this case, we recall that the phase-field \(\phi\) takes values in a given bounded interval (i.e., the domain of \(F\) in the singular potential case \((1.7)\) or \(\phi \in [-1, 1]\) in the regular potential case, when \(F\) is of the form \((1.8)\), see \[33, Section 6\]). On the other hand, with the Cahn-Hilliard formulation the latter property is only true when \(F\) is a singular potential, like \((1.7)\), see, for instance, \[1, 2\]. Indeed, it is well-known that in the case of the regular potential \((1.8)\) the order parameter does not remain in the physically relevant interval \([-1, 1]\), see \[20\]. Numerical simulations show that the system \((1.1) - (1.4)\) for \(K = \text{AC}\) captures basic features of two-phase flow behavior, including vesicle dynamics or drop formation processes (cf. \[23, 71, 78\] and references therein). Moreover, from the numerical point of view it is easier to implement the system based on the Allen-Cahn equation than the system \((1.1) - (1.4)\) for \(K = \text{CH}\), see \[30, 74\].

(II) When \(K = \text{CH}\), the phase-field component \(\phi\) enjoys ”good” regularity properties and, therefore, the system \((1.1) - (1.4)\) is only weakly coupled through the Korteweg force in \((1.1)\) even in the inviscid case when \(\nu \equiv 0\), see \[15\]. Theoretical aspects (i.e., well-posedness, regularity and asymptotic behavior as time goes to infinity) for the system \((1.1) - (1.4)\) in the case \(K = \text{CH}\) have been investigated in a sufficiently large number of papers in both two and three dimensions. Well-posedness results for \(K = \text{CH}\) when \(F\) is a smooth polynomial potential can be found in \[12, 10, 11, 15, 73\], and in \[1, 2, 13\] when \(F\) is a singular potential of the form \((1.7)\). Regarding the longtime behavior when \(K = \text{CH}\), results about the stability of stationary solutions were given in \[2, 10, 31\], while theorems about the convergence to single equilibria and existence of global
and exponential attractors were proven in [1, 34, 32, 80]. Various numerical aspects of (1.1)-(1.4) when \( K = CH \) were investigated in [6, 5, 16, 51, 52, 59, 56]. However, for the system when \( K = AC \) one expects lower regularity for the \( \phi \)-component and, hence, in this case (1.1)-(1.4) is strongly coupled. This feature has already been present in [33, 62, 77, 76, 79]. Finally, some existence results in the compressible case for (1.1)-(1.4) in both formulations (1.5) are contained in [3, 29, 24, 50]. A comparison of these models, providing further insight on the behavior of the full problem (1.1)-(1.4), is given in [55]. The relationship between (1.1)-(1.4) and the standard incompressible models, the simplified 3D Bardina models, the 3D Navier-Stokes-Voight (NSV) equations, and their inviscid counterparts. For instance, it has been observed that computational simulations of the 3D Navier-Stokes-\( \alpha \) equations are statistically indistinguishable from the simulations of the Navier-Stokes equations. Furthermore, the 3D Navier-Stokes-\( \alpha \) model provides tremendous computational savings as shown in simulations of both forced and decaying turbulence. Finally, the 3D Navier-Stokes-\( \alpha \) model arises from a variational principle in the same fashion as the 3D Navier-Stokes equation (NSE) is strongly coupled. 

**III** For both these approaches, one expects a less or more incomplete theory for (1.1)-(1.4) in three dimensions because a full mathematical theory for the 3D Navier-Stokes equation (NSE) is still lacking at present. Moreover, as noted in [44] direct numerical simulation of the 3D NSE for many physical applications with high Reynolds number flows is "intractable even using state-of-the-art numerical methods on the most advanced supercomputers available nowadays". Recently, many applied mathematicians have developed regularized turbulence models for the 3D NSE as an attempt to overcome this simulation barrier. Their aim is to capture "the large, energetic eddies without having to compute the smallest dynamically relevant eddies, by instead modelling the effects of small eddies in terms of the large scales in the 3D NSE". Since 1998, many such regularized models have been proposed, tested and investigated from both the numerical and the mathematical point of views. Among these models, one can find the globally well-posed 3D Navier-Stokes-\( \alpha \) (NS-\( \alpha \)) equations (also known as the viscous Camassa-Holm equations and Lagrangian averaged Navier-Stokes-\( \alpha \) model), the 3D Leray-\( \alpha \) models, the modified 3D Leray-\( \alpha \) models, the simplified 3D Bardina models, the 3D Navier-Stokes-Voight (NSV) equations, and their inviscid counterparts. For instance, it has been observed that computational simulations of the 3D Navier-Stokes-\( \alpha \) (NS-\( \alpha \)) equations are statistically indistinguishable from the simulations of the Navier-Stokes equations. Furthermore, the 3D Navier-Stokes-\( \alpha \) model provides tremendous computational savings as shown in simulations of both forced and decaying turbulence. Finally, the 3D Navier-Stokes-\( \alpha \) model arises from a variational principle in the same fashion as the Navier-Stokes equations. We refrain from giving an exhaustive list of references but we refer the reader to [44] for a complete bibliography and detailed description of the results available for these regularized models.

In this paper, upon taking the point of view described in (III), we consider first the following prototype of initial value problem for two-phase incompressible flows on an \( n \)-dimensional compact Riemannian manifold \( \Omega \) with or without boundary, when \( n = 2, 3 \):

\[
\begin{align*}
&\partial_t u + A_0 u + (M u \cdot \nabla)(Nu) + \chi \nabla (Mu)^T \cdot (Nu) + \nabla p = -\varepsilon \text{div}(\nabla \phi \otimes \nabla \phi) + g, \\
&\partial_t \phi + Nu \cdot \nabla \phi + \varepsilon A_1 \phi + \varepsilon^{-1} f(\phi) = 0, \\
&\text{div}(u) = 0, \\
&u(0) = u_0, \\
&\phi(0) = \phi_0,
\end{align*}
\]

(1.9)

where \( A_0, A_1, M, \) and \( N \) are linear operators having certain mapping properties, and where \( \chi \) is either 1 or 0. All kinds of boundary conditions (i.e., periodic, no-slip, no-flux, Navier boundary conditions, etc) can be treated and are included in our analysis; they will be incorporated in the weak formulation for the problem (1.9), see Section 2. We introduce three parameters which control the degree of smoothing in the operators \( A_0, M \) and \( N \), namely \( \theta, \theta_1 \) and \( \theta_2 \), while \( A_1 \) is a differential operator of second order. Thus we will only focus on the case when \( K = AC \) in (1.5), which is actually the harder case (see (II) above). Some examples of operators \( A_0, A_1, M, \) and \( N \) which satisfy the mapping assumptions we will need in this paper are

\[
A_0 = \nu(-\Delta)^\theta, \quad A_1 = -\Delta, \quad M = (I - \alpha^2 \Delta)^{-\theta_1}, \quad N = (I - \alpha^2 \Delta)^{-\theta_2},
\]

(1.10)

for fixed positive real numbers \( \alpha, \nu \) and for specific choices of the real parameters \( \theta, \theta_1 \) and \( \theta_2 \). We note that the Korteweg force in (1.9) can be equivalently rewritten in the following form

\[
-\varepsilon \text{div}(\nabla \phi \otimes \nabla \phi) = \varepsilon \mu \nabla \phi - \nabla (\frac{\varepsilon^2}{2} |\nabla \phi|^2 + F(\phi))
\]

\[
= -\varepsilon^2 \Delta \phi \nabla \phi - \frac{\varepsilon^2}{2} \nabla |\nabla \phi|^2,
\]

where \( \mu \) is given by (1.4). As in [44], we emphasize that the abstract mapping assumptions we employ are more general, and as a result do not require any specific form of the parametrizations of
A₀, A₁, M, and N. This abstraction allows (1.9) to recover some of the existing models that have
been previously studied, as well as to represent a much larger three-parameter family of models
that have not been explicitly studied in detail. For clarity, some of the specific regularization
models recovered by (1.9) for particular choices of the operators A₀, M, N and χ are listed in
Table 1.

| Model | NSE-AC | Leray-AC-α | ML-AC-α | SBM-AC | NSV-AC | NS-AC-α | NS-AC-α-like |
|---|---|---|---|---|---|---|---|
| A₀ | −νΔ | −νΔ | −νΔ | −νΔS | −νΔ | ν(−Δ)| |
| M | I | S | I | S | S | S | Sθ₂ |
| N | I | I | S | S | S | I | I |
| χ | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

Recall that α-models of turbulence were intended as a basis for regularizing numerical schemes
for simulating turbulence in single-like fluids [22] (see the point (III) above). Thus, it is important
to verify whether the ad hoc smoothed systems from Table 1 inherit some of the original properties
of the Navier-Stokes-Allen-Cahn (NSE-AC) system. In particular, one would like to see if the
natural energy of the smoothed systems can be identified with the energy of the original NSE-AC
system under suitable boundary conditions. For the NSE-AC system there is one essential ideal
invariant (for instance, under rectangular periodic boundary conditions or in the whole space),

\[ E₀ = \frac{1}{2} \int_Ω \left( |u(x)|^2 + ν|∇\phi(x)|^2 \right) dx + \varepsilon^{-1} \int_Ω F(\phi) dx. \]

In the case of the α-models from Table 1, the corresponding ideal invariant is the energy

\[ E_\alpha = \frac{1}{2} \int_Ω \left( |u(x)|^2 + ν|∇\phi(x)|^2 \right) dx + \varepsilon^{-1} \int_Ω F(\phi) dx, \]

which reduces, as \( \varepsilon \to 0 \), to the dissipated energy \( E₀ \) of the NSE-AC equations.

Our main goal in this paper is to develop well-posedness and long-time dynamics results for
the entire three-parameter family of models, and then subsequently recover the existing results
of this type for the specific regularization models that have been previously studied. We first
aim to establish a number of results for the entire three-parameter family, including results on
existence, regularity, uniqueness, continuous dependence with respect to initial data, linear and
nonlinear perturbations (with the inviscid and \( \alpha \to 0 \) limits as special cases), existence and
finite dimensionality of global attractors, and existence of exponential attractors (also known
as inertial sets). Elaborating further on the latter issue, we recall that in the global attractor
theory, it is usually extremely difficult (if not impossible) to estimate and to express the rate of
convergence of trajectories to the global attractor in terms of the physical parameters of the system
considered. This constitutes the main drawback of the theory. Simple examples show that the
rate of convergence can be arbitrarily slow and non-uniform with respect to the parameters of the
system considered. As a consequence, the global attractor becomes sensitive to small perturbations
and, moreover, it may miss important transient behaviors because the global attractor consists
only of states in the final stage. Another suitable object which always contains the global attractor,
and thus is more structurally rich in content than the global attractor is the so-called exponential
attractor. The concept of exponential attractor overcomes the difficulties we mentioned earlier.
Indeed, in contrast to the global attractor theory, the relevant constants can be explicitly found
in terms of the physical parameters, and the exponential attractor theory can provide a direct
way to estimate the fractal dimension of the global attractor even when the classical machinery
fails. Furthermore, an exponential attractor attracts bounded subsets of the energy phase-space
at an exponential rate, which makes it a more useful object in numerical simulations than the
global attractor. We refer the reader for more details to the survey article [65]. Following [44],
our main goal is to analyze a generalized model based on abstract mapping properties of the principal
operators A₀, M, and N allowing for a simple analysis that helps bring out the core common
structure of the various regularized and unregularized Navier-Stokes-Allen-Cahn systems. In [35],
a direct relationship between the long-term dynamics of the 3D NSE-AC system and the three

Table 1. Some special cases of the model (1.9) with \( \alpha > 0 \), and with \( S = (I - αΔ)^{-1} \) and \( Sθ₂ = |I + (-αΔ)^θ₂|^{-1} \).
dimensional system based on the NS-α-model coupled with the Allen-Cahn equation (NS-AC-α) was established. Note that the NS-AC-α system corresponds to a subset of those problems studied here. For the other regularized models considered in Table 1, as far as we know mathematical and numerical results have not been previously established in the literature. The global existence, uniqueness and regularity of solutions for these models have been mostly known only for the two-dimensional NSE-AC system [33, 77, 79] and the three-dimensional NS-AC-α model [55]. Here, as a consequence of a more general result, we develop complete well-posedness and global regularity results for models of turbulence in two-phase flows described by (1.9). Furthermore, we establish various convergence results for the global weak solutions of (1.9) as either one of the parameters α, ν goes to zero. In addition, we prove results on the existence of finite-dimensional global and exponential attractors. Then, by the Lojasiewicz–Simon technique, we also establish the convergence of any bounded solution of (1.9) to single steady states, provided that $F$ is a real analytic function, and that the time-dependent body force $g$ is asymptotically decaying in a precise way, i.e.,

$$\int_0^\infty \|g(s)\|^2_{H^{-\theta-\varepsilon}} \, ds \lesssim (1 + t)^{-\varepsilon}$$

for all $t \geq 0$.

In particular, for any fixed initial datum $(u_0, \phi_0)$ the corresponding trajectory satisfies the estimate

$$\|u(t)\|_{H^{-\theta-\varepsilon}} + \|\phi(t)\|_{H^1} \lesssim (1 + t)^{-\varepsilon},$$

for some $\xi \in (0, 1)$, depending on $\phi_*$ and $\delta > 0$, where $\phi_*$ is a steady-state of $A_1 \phi_* + \varepsilon^{-2} f(\phi_*) = 0$, $\varepsilon > 0$.

We emphasize again that all these results are all new for the models mentioned in Table 1 and that the abstract mapping assumptions we employ for (1.9) are more general, and as a result do not require any specific form of the parametrizations of $A_0$, $M$, and $N$, as in (1.10). As a consequence, the framework we exploit allows us to derive new results for a much larger three-parameter family of models that have not been included in Table 1 and explicitly studied anywhere in detail. Finally, it is also worth emphasizing that any nondissipative (i.e., $\theta = 0$) regularized Navier-Stokes equation in (1.9) can be thought as an inviscid regularization of the usual (unregularized) Navier-Stokes equation (NSE). Thus, in contrast to the case $\theta > 0$ the regularized system (1.9) for $\theta = 0$ is even more strongly coupled than before. Consequently, this feature will make the analysis even more delicate, especially in the treatment of the long-term dynamic behavior as time goes to infinity. Indeed, in this case the Korteweg force (1.11) can be less regular than the convective term $(Mu \cdot \nabla)(Nu)$ from (1.9), especially in three space dimensions. Besides, our analysis can be applied verbatim to certain regularized simplified Ericksen-Leslie (RSEL) models for the hydrodynamics of liquid crystals in $n$-dimensional compact Riemannian manifolds with or without boundary. Recently, the simplified (unregularized) Ericksen-Leslie system which consists of the $n$-dimensional NSE coupled with the Allen-Cahn equation (in this context, also known as the Ginzburg-Landau equation) for the orientation parameter $\phi \in \mathbb{R}^n$ was considered in [9, 23, 57, 58, 69, 82] (and the references therein). Moreover, the same system where the 3D Navier-Stokes equation is replaced by the Lagrangian averaged 3D Navier-Stokes-α model was also considered in [69, Section 7]. We note that the problems studied in these references correspond to only a subset of those regularized (RSEL) systems contained here (see Section 7).

It is also important to note that the general framework of [44], also exploited and extended further here, allows for the development of new results for certain (regularized or un-regularized) Navier-Stokes equations and magnetohydrodynamics (MHD) models. For instance, our results on the existence of exponential attractors, and the existence of global attractors in non-dissipative systems (e.g., when $\theta = 0$) in (1.9), and when there is no coupling) are completely new and complementary to the results of [44, Section 5]. In fact, in this paper we will show how to close a gap in the proof of [44, Section 5, Corollary 5.4] whose assumptions can only be verified in the case $\theta > 0$. Indeed, one can easily observe that when $\theta = 0$, the assumptions of [44, Theorem 5.1, (b)] do not longer provide the existence of a compact absorbing set as claimed on [44, pg. 550]. For instance, the 3D Navier-Stokes-Voight model, or any other non-dissipative system when $\theta = 0$, is no longer covered by the result of [44, Corollary 5.4] (see also Section 6).

The remainder of the paper is structured as follows. In Section 2, we establish our notation and give some basic preliminary results for the operators appearing in the general regularized model. In Section 3, we build some well-posedness results for the general model; in particular, we establish existence results (Section 3.1), regularity results (Section 3.2), and uniqueness and
continuous dependence results (Section 3.2). In Section 4 we establish some results for singular perturbations, which as special cases include the inviscid limit of viscous models and the $\alpha \to 0$ limit in $\alpha$ models; this involves a separate analysis of the linear (Section 4.1) and nonlinear (Section 4.2) terms. In Section 5 we show existence of a global attractor for the general model by dissipation arguments (Sections 5.1 and 5.2), and then by employing the approach from [37, 65], to show the existence of exponential attractors (Sections 5.1 and 5.2). In Section 5.3 we establish asymptotic stability results as time goes to infinity of solutions to our regularized models, with the help from a Lojasiewicz–Simon technique. Section 6 contains several important new theorems and remarks for the systems considered by [44]. Section 7 contains some additional remarks on a regularized system for the simplified Ericksen-Leslie model for the hydrodynamics of liquid crystals. To make the paper sufficiently self-contained, our final Section 8 contains supporting material on Sobolev and Grönnwall-type inequalities, and several other abstract results which are needed to prove our main results.

2. Preliminary material

We follow the same framework and notation as in [44]. To this end, let $\Omega$ be an $n$-dimensional smooth compact manifold with or without boundary and equipped with a volume form, and let $E \to \Omega$ be a vector bundle over $\Omega$ equipped with a Riemannian metric $h = (h_{ij})_{n \times n}$. With $C^\infty(E)$ denoting the space of smooth sections of $E$, let $V \subseteq C^\infty(E)$ be a linear subspace, let $A_0 : V \to V$ be a linear operator, and let $A_1 : V \times V \to V$ be a bilinear map. At this point $V$ is conceived to be an arbitrary linear subspace of $C^\infty(E)$; however, later on we will impose restrictions on $V$ implicitly through various conditions on certain operators such as $A_0$. Furthermore, we let $W \subseteq C^\infty(\Omega)$ be a linear subspace and let $A_1 : W \to W$ be a linear operator satisfying various assumptions below. In order to define the variational setting for the phase-field component we also need to introduce the bilinear operators $R_0 : W \times W \to V$, $B_1 : V \times W \to W$, as follows:

$$B_1 (u(x), \phi(x)) := Nu(x) \cdot \nabla \phi(x), \quad R_0 (\psi(x), \phi(x)) := \psi(x) \nabla \phi(x).$$

Recalling (1.11) and assuming that $\varepsilon = 1$ (for the sake of simplicity), the initial data $u_0 \in V$, $\phi_0 \in W$ and forcing term $g \in C^\infty(0, T; V)$ with $T > 0$, consider the following system

$$\begin{cases}
\partial_t u + A_0 u + B_0 (u, u) = R_0 (A_1 \phi, \phi) + g, \\
\partial_t \phi + B_1 (u, \phi) + \mu = 0, \\
\mu = A_1 \phi + f(\phi), \\
u(0) = u_0, \phi(0) = \phi_0,
\end{cases}$$

on the time interval $[0, T]$. Bearing in mind the model (1.9), we are mainly interested in bilinear maps of the form

$$B_0(w, w) = \tilde{B}_0(Mv, Nw),$$

where $M$ and $N$ are linear operators in $V$ that are in some sense regularizing and are relatively flexible, and $\tilde{B}_0$ is a bilinear map fixing the underlying nonlinear structure of the fluid equation. In the following, let $P : C^\infty(E) \to V$ be the $L^2$-orthogonal projector onto $V$. Finally, concerning the derivative $f$ of the function $F$ in (1.6) we will focus mostly on the regular potential case when $f \in C^2(\mathbb{R})$ satisfies $f(1) \geq 0$, $f(-1) \leq 0$ and obeys the following condition

$$\liminf_{|r| \to \infty} f'(r) > 0.$$

However, when $f$ is a singular potential, see Remark 5.20.

We will study the regularized system (2.3) by extending it to function spaces that have weaker differentiability properties. To this end, we interpret (2.2) in distribution sense, and need to continuously extend $A_0$, $A_1$ and $B_0$, $B_1$ and $R_0$ to appropriate smoothness spaces. Namely, we employ the spaces $V^s = \text{clos}_H \mathcal{V}$, $W^s = \text{clos}_H \mathcal{W}$, which will informally be called Sobolev spaces in the following. The pair of spaces $V^s$ and $V^{-s}$ are equipped with the duality pairing $\langle \cdot , \cdot \rangle$, that is, the continuous extension of the $L^2$-inner product on $V^0$. Same applies to the triplet $W^s \subset W^0 = (W^0)^* \subset W^{-s}$. Moreover, we assume that there are self-adjoint positive operators $\Lambda$ and $A_1$, respectively, such that $\Lambda^s : V^s \to V^0$, $A_1^{s/2} : W^s \to W_0$ are isometries for any $s \in \mathbb{R}$, and $\Lambda^{-s}$, $(A_1)^{-1}$ are compact operators. For arbitrary real $s$, assume that $A_0$, $A_1$, $M$, and $N$ can be continuously extended so that

$$A_0 : V^s \to V^{s-2\theta}, \quad A_1 : W^s \to W^{s-2}, \quad M : V^s \to V^{s+2\theta}, \quad N : V^s \to V^{s+2\theta},$$

where $\theta > 0$. We will assume that for all $s \in \mathbb{R}$ there exist positive constants $C_\theta$ and $C'$ such that

$$\|A_0 u\|_{V^{s-2\theta}} \leq C_\theta \|u\|_{V^s},$$

$$\|A_1 u\|_{W^{s-2}} \leq C' \|u\|_{W^s}.$$
are bounded operators. Again, we emphasize that the assumptions we will need for \( A_0, M, \) and \( N \) are more general, and do not require this particular form of the parametrization (see (2.6) - (2.8) below). We will assume \( \theta \geq 0 \) and no \textit{a priori} sign restrictions on \( \theta_1, \theta_2 \). We remark that \( s \) in (2.6) is assumed to be arbitrary for the purpose of the discussion in this section; of course, it suffices to assume (2.6) for a limited range of \( s \) for most of the results in this paper. The canonical norm in the Hilbert spaces \( V^s \) and \( W^s \), respectively, will be denoted by the same quantity \( \| \cdot \|_s \) whenever no further confusion arises, while we will use the notation \( \| \cdot \|_{L^p} \) for the \( L^p \)-norm. Furthermore, we assume that \( A_0 \) and \( N \) are both self-adjoint, and coercive in the sense that for \( \beta \in \mathbb{R} \\
\begin{align}
\langle A_0v, v \rangle &\geq c_{A_0} \| v \|^2_{0,\beta} - C_{A_0} \| v \|^2_{\beta}, \\
\langle Nv, v \rangle &\geq c_N \| v \|^2_{-\theta_2}, \\
\end{align}
\] with \( c_{A_0} = c_{A_0}(\beta) > 0 \), and \( C_{A_0} = C_{A_0}(\beta) \geq 0 \), and that (2.7)
\[\langle Nv, v \rangle \geq c_N \| v \|^2_{-\theta_2}, \quad v \in V^{-\theta_2},\]
with \( c_N > 0 \). We also assume that (2.8)
\[\langle A_0v, Nv \rangle \geq c_{A_0} \| v \|^2_{\beta-\theta_2}, \quad v \in V^{\theta-\theta_2},\]
Note that if \( \theta = 0 \), (2.6) is strictly speaking not coercivity and follows from the boundedness of \( A \), and note also that (2.7) implies the invertibility of \( N \).

As examples, one may typically consider the following operators in various combinations in (2.2).

**Example 2.1.** (a) When \( \Omega \) is a closed Riemannian manifold, and \( E = T\Omega \) the tangent bundle, an example of \( V \) is \( \mathcal{V}_\text{per} \subseteq \{ u \in C^\infty (T\Omega): \text{div} u = 0 \} \), a subspace of the divergence-free functions. The space of periodic functions with vanishing mean on a torus \( \mathbb{T}^n \) is a special case of this example. In this case, one typically has \( A_0 = (-\Delta)^\theta, M = (I - \alpha^2\Delta)^{-\theta_1}, N = (I - \alpha^2\Delta)^{-\theta_2} \) and \( A_1 = -\Delta \), as operators that satisfy (2.5), cf. also [44] Example 2.1, (a).

(b) When \( \Omega \) is a compact Riemannian manifold with boundary, and again \( E = T\Omega \) the tangent bundle, a typical example of \( V \) is \( \mathcal{V}_\text{hom} = \{ u \in C^\infty_0 (T\Omega): \text{div} u = 0 \} \) the space of compactly supported divergence-free functions. In this case, one may consider the choices \( A_0 = (-\Delta\Delta)^\theta, A_1 = -\Delta, M = (I - \alpha^2\Delta\Delta)^{-\theta_1}, N = (I - \alpha^2\Delta\Delta)^{-\theta_2} \), respectively, as operators satisfying (2.5), cf. also [44] Example 2.1, (b).

**Example 2.2.** (a) In Example 2.1 above, the bilinear map \( \theta \) can be taken to be (2.9)
\[\theta_{0x}(v, w) = P[(v \cdot \nabla)w + \chi(\nabla w^T)v],\]
which correspond to the models with \( \chi \in \{0, 1\} \) as discussed in the Introduction (see Table 1).

(b) Let \( \Omega \) be connected Riemannian \( n \)-dimensional manifold with non-empty (sufficiently smooth) boundary \( \partial \Omega \). Define \( A_1 = -\Delta \), as the Laplacian of the metric \( h \), acting on \( D(A_1) = \{ \phi \in W^2 : \partial_i \phi = 0 \text{ on } \partial \Omega \}, \)
where \( \zeta \) is an outward unit normal vector field of \( \partial \Omega \). Recall that in local coordinates \( \{x_i\}_{i=1}^n \), the Laplacian reads \(\Delta(\zeta) = \frac{1}{\sqrt{\det(h)}} \sum_{i,j=1}^n \partial_{x_j} \left( h^{ij} \sqrt{\det(h)} \partial_{x_i}(\cdot) \right),\)
where the matrix \( (h^{ij}) \) is the inverse matrix of \( h \). We have that \( A_1 \) is a nonnegative self-adjoint operator on \( W^0 \). Next, consider \( A_1 = A_1 + \gamma I \), for some \( \gamma > 0 \) and define \( f_1(r) = f(r) - \gamma r \). In this case, we can rewrite the second and third equations of (2.2) in the form
\[\partial_t \phi + B_1(u, \phi) + A_1 \phi + f_1(\phi) = 0,\]
with the function \( f_1 \) still obeying assumption (2.4). Clearly, \( A_1 \) is positive and it can now be continuously extended so that it satisfies the corresponding condition from (2.3). Hence, our original restriction that \( A_1 \) is \textit{positive} is indeed not necessary and, thus, the above framework also allows us to deal with a nonnegative selfadjoint operator \( A_1 \).

**Example 2.3.** (Navier boundary conditions). Let \( \Omega \subset \mathbb{R}^n \) be a bounded connected domain with sufficiently smooth boundary \( \partial \Omega \). We take \( V \) as \( \mathcal{V}_\text{Nbc} = \{ u \in C^\infty (T\Omega): \text{div} u = 0, u \cdot \zeta = 0 \text{ on } \partial \Omega \} \) and recall the classical decomposition \( V^0 = V^1 \oplus (V^1)\perp, \quad (V^1)\perp = \{ \nabla u : u \in V^1 \}. \)
When two vectors \( u \) and \( v \) are divergent free, \( u \) satisfies the Navier boundary condition
\[
2 \left[ (Du) n \right] \cdot \tau + \sigma u \cdot \tau = 0, \quad \text{on } \partial \Omega,
\]
(2.10) \( 2 \left[ (Du) n \right] \cdot \tau + \sigma u \cdot \tau = 0, \quad \text{on } \partial \Omega, \)

(2Du := \nabla u + (\nabla u)^T \) is the “usual” deformation tensor and \( \sigma > 0 \) is some friction coefficient), and \( u, v \in \mathcal{V}_\text{Nbc} \), the Green formula \( \text{[72]} \) yields
\[
\int_\Omega (-\Delta u) \cdot v dx = 2 \int_\Omega Du : Dv dx + \sigma \int_\Gamma (u \cdot \tau) (v \cdot \tau) dS.
\]
(2.11)

\[ \int_\Omega (-\Delta u) \cdot v dx = 2 \int_\Omega Du : Dv dx + \sigma \int_\Gamma (u \cdot \tau) (v \cdot \tau) dS, \]

On the basis of (2.11), one can define the bilinear form
\[
\rho_\sigma (u, v) := 2 \int_\Omega Du : Dv dx + \sigma \int_\Gamma (u \cdot \tau) (v \cdot \tau) dS, \quad \text{for } u, v \in V^1.
\]
(2.12)

\[ \rho_\sigma (u, v) := 2 \int_\Omega Du : Dv dx + \sigma \int_\Gamma (u \cdot \tau) (v \cdot \tau) dS, \quad \text{for } u, v \in V^1. \]

Note that \( \rho_\sigma (\cdot, \cdot) \) is bounded on \( V^1 \) and \( \rho_\sigma (u, u) > 0 \) for all \( u \in V^1 \). Then, by Korn’s inequality \( \text{[49]} \) we see that there exists a constant \( C = C(\sigma) > 0 \) (independent of \( u \)) such that
\[
(2.13) \quad C \| u \|^2_1 \leq \rho_\sigma (u, u), \quad \text{for all } u \in V^1.
\]

In view of (2.13), we see that the bilinear form \( \rho_\sigma \) is symmetric and coercive on \( V^1 \); for vector fields that satisfy (2.10) we set
\[
V^2_\sigma := \{ u \in V^2 : (2.10) \text{ holds on } \partial \Omega \}.
\]

The corresponding Stokes operator \( A_0 \) associated with the form \( \rho_\sigma \) can be constructed on the basis of first and second representations, as follows:
\[
A_0 u = \Theta_\sigma u, \quad u \in D(A_0) := \{ u \in V^1 : \Theta_\sigma u \in V^0 \},
\]
where \( \Theta_\sigma, \sigma \in \mathcal{L}(V^1, (V^1)^*) \) is a bounded one-to-one mapping such that
\[
\rho_\sigma (u, v) = \langle u, \Theta_\sigma v \rangle_{V^1, (V^1)^*},
\]
for all \( u, v \in V^1 \). As a byproduct, we also obtain the following compact embeddings \( D(A_0) \subset V^1 \subset V^0 \subset (V^1)^* \), the operator \( A_0 \) is a positive, selfadjoint operator on \( V^0 \), and \( (I + A_0)^{-1} \) is compact. Exploiting the formula (2.11) once more, we see that \( P v = v, \) for all \( v \in V^1 \) and \( \rho_\sigma (u, v) = \langle P(-\Delta u), v \rangle, \) for all \( u \in V^2_\sigma \) and \( v \in V^1 \). This implies, on the basis of standard regularity theory \( \text{[72]} \) for the Stokes operator, that \( D(A_0) = V^2_\sigma \) and \( A_0 u = P(-\Delta u), \) for all \( u \in D(A_0) \). Finally, fractional powers \( A_0^\theta \) are also well-defined for all \( \theta \geq 0 \).

To refer to the above examples, let us further introduce the shorthand notation:
\[
(2.14) \quad B_{0_{\chi}} (v, w) = \overline{B_{0_{\chi}} (M v, N w)}, \quad \chi \in \{0, 1\}.
\]

For clarity, we list in Table 2 the corresponding values of the parameters and bilinear maps discussed above for special cases listed in Table 1:

| Table 2. Values of the parameters \( \theta, \theta_1 \) and \( \theta_2 \), and the particular form of the bilinear map \( B_0 \) for some special cases of the model (2.2). (The bilinear maps \( B_{0_0} \) and \( B_{0_1} \) are as in (2.14)). |
|---|---|---|---|---|---|---|---|
| Model | NSE-AC | Leray-AC-\( \alpha \) | ML-\( \alpha \)-AC | SBM-AC | NS-AC | NS-AC-\( \alpha \) | NS-AC-\( \alpha \)-like |
| \( \theta \) | 1 | 1 | 1 | 1 | 0 | 1 | \( \theta \) |
| \( \theta_1 \) | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| \( \theta_2 \) | 0 | 1 | 0 | 1 | 1 | \( \theta_2 \) | \( \theta_2 \) |
| \( B_0 \) | \( B_{0_0} \) | \( B_{0_0} \) | \( B_{0_0} \) | \( B_{0_0} \) | \( B_{0_0} \) | \( B_{0_1} \) | \( B_{0_1} \) |

Next, we denote the trilinear forms
\[
(2.15) \quad b_0 (u, v, w) = \langle B_0 (u, v), w \rangle, \quad b_1 (u, \phi, \psi) = \langle B_1 (u, \phi), \psi \rangle,
\]
and similarly the forms \( \overline{b}_{0_{\chi}} \) and \( \overline{b}_{0_{\chi}} \), following (2.1), (2.9) and (2.14). Then our notion of weak solution for problem (2.2) can be formulated as follows.
Definition 2.4. Let \( g(t) \in L^2(0,T;V^*) \) for some \( s \in \mathbb{R} \), and \((u_0, \phi_0) \in \mathcal{V}_b := V^{-\theta_2} \times (W^1 \cap \{ \phi_0 \in L^\infty(\Omega) : |\phi_0| \leq 1 \}) \).

Find a pair of functions
\[
(\theta, \phi) \in L^\infty(0,T;\mathcal{V}_b) \cap L^2(0,T;V^{d-\theta_2} \times W^2)
\]

such that
\[
\partial_t u \in L^p(0,T;V^{-\gamma}), \quad \partial_t \phi \in L^2(0,T;W^{-2})
\]
for some \( p > 1 \) and \( \gamma \geq 0 \), such that \((u, \phi)\) fulfills \( u(0) = u_0, \phi(0) = \phi_0 \) and satisfies
\[
\int_0^T \left( -\langle u(t), w(t) \rangle + \langle A_0 u(t), w(t) \rangle + b_0(u(t), w(t)) \right) dt = 0,
\]
\[
\int_0^T \left( -\langle \phi(t), \psi(t) \rangle + \langle \mu(t), \psi(t) \rangle + b_1(u(t), \phi(t), \psi(t)) \right) dt = 0,
\]
for any \((w, \psi) \in C^\infty_0(0,T;V \times \mathcal{W})\), such that \( \mu(t) = A_1 \phi(t) + f(\phi(t)) \) a.e. on \( \Omega \times (0,T) \).

Remark 2.5. As far as the interpretation of the initial conditions \( u(0) = u_0, \phi(0) = \phi_0 \) is concerned, note that properties (2.16)-(2.17) imply that \( u_0, \phi_0 \) is concerned, note that properties (2.16)-(2.17) imply that \( u \in C(0,T;V^{-\gamma}) \) and \( \phi \in C(0,T;W^{\theta_2}) \). Thus, the initial conditions are satisfied in a weak sense.

3. Well-posedness results

Analogous to the theory for the Navier-Stokes-Allen-Cahn system \([34, 33]\), we begin to develop a solution theory for the general three-parameter family of regularized models. We begin by showing energy estimates that will be used to establish existence and regularity results, and under appropriate assumptions also uniqueness and stability. At the end of the proof of each theorem, we give the corresponding conditions for \((\theta, \theta_1, \theta_2)\) which allow us to not only recover old results, but also establish new results in the literature especially for the cases listed in Table 1. Throughout the paper, \( C \geq 0 \) will denote a generic constant whose further dependence on certain quantities will be specified on occurrence. The value of the constant can change even within the same line. Furthermore, we introduce the notation \( a \lesssim b \) to mean that there exists a constant \( C > 0 \) such that \( a \leq Cb \). This notation will be used when the constant \( C \) is irrelevant and becomes tedious.

3.1. Existence of weak solutions. In this subsection, we establish sufficient conditions for the existence of weak solutions to the problem (2.2) (cf. Definition 2.4). As noted in the Introduction, in the case \( K = AC \) a maximum principle holds for the phase-field component of any weak solution.

Proposition 3.1. Suppose that \( f \) satisfies (2.4), \( f(1) \geq 0, f(-1) \leq 0, \) and \( b_1(v,\psi,\psi) = 0 \), for any \( v \in V^{\theta-\theta_2}, \psi \in W^1 \). Let \( \phi_0 \in L^\infty(\Omega) \) such that \( |\phi_0| \leq 1 \) a.e. in \( \Omega \). Then for any weak solution \((u, \phi)\) to problem (2.2) in the sense of Definition 2.4 we have \( \phi \in L^\infty(0,T;L^\infty(\Omega)) \) and
\[
|\phi(t)|_{L^\infty(\Omega)} \leq 1, \quad \text{a.e. on } (0,T).
\]

Proof. For the reader’s convenience, a proof of the above statement is contained in [33, Theorem 6.1].

Theorem 3.2. Let the assumptions of Proposition 3.1 and the following conditions hold.

i) \((u_0, \phi_0) \in \mathcal{V}_b \) with any \( \theta_2 \geq 1, \) and \( g \in L^2(0,T;V^{-\theta_2}) \), \( T > 0 \).

ii) \( b_0(v,\nu) = 0, \) for any \( v \in V^{\theta-\theta_2} \).

iii) \( b_0 : V^{\sigma_1} \times V^{\sigma_2} \times \Gamma \rightarrow \mathbb{R} \) is bounded for some \( \sigma_i < \theta - \theta_2, i = 1,2, \) and \( \gamma \geq \gamma_i; \)

iv) \( b_0 : V^{\sigma_1} \times V^{\sigma_2} \times \Gamma \rightarrow \mathbb{R} \) is bounded for some \( \sigma_i \in [-\theta_2, \theta_2], i = 1,2, \) and \( \gamma \in [\theta + \theta_2, \infty) \cap (\theta_2, \infty) \cap (\frac{\theta}{2}, \infty); \)

Then, there exists at least one weak solution \((u, \phi)\) satisfying (2.10)-(2.17) such that
\[
p = \begin{cases} \min\{2 \frac{2\theta}{\sigma_1 + \sigma_2 + \theta_2}, 2\}, & \text{if } \theta > 0, \\ 2, & \text{if } \theta = 0. \end{cases}
\]
Proof. Let \( \{V_m : m \in \mathbb{N}\} \subset V^{\theta - \theta_2} \), \( \{W_m : m \in \mathbb{N}\} \subset D(A_1) \cap L^\infty(\Omega) \) be sequences of finite dimensional subspaces of \( V^{\theta - \theta_2} \) and \( D(A_1) \), respectively, such that

1. \( V_m \subset V_{m+1}, W_m \subset W_{m+1} \), for all \( m \in \mathbb{N} \);
2. \( \cup_{m \in \mathbb{N}} V_m \) is dense in \( V^{\theta - \theta_2} \), and \( \cup_{m \in \mathbb{N}} W_m \) is dense in \( D(A_1) \);
3. For \( m \in \mathbb{N}, \) with \( V_m = NV_m \subset V^{\theta + \theta_2} \), the projectors \( P_m : V^{\theta - \theta_2} \to V_m, Q_m : D(A_1) \to W_m \), defined by
   \[
   \langle P_m v, w_m \rangle = \langle v, w_m \rangle, \quad w_m \in \tilde{V}_m, \quad v \in V^{\theta - \theta_2},
   \]
   \[
   \langle Q_m \phi, \psi_m \rangle = \langle \phi, \psi_m \rangle, \quad \psi_m \in W_m, \quad \phi \in D(A_1),
   \]
   are uniformly bounded as maps from \( V^{-\gamma} \to V^{-\gamma} \) and \( W^{-2} \to W^{-2} \), respectively.

Such sequences can be constructed e.g., by using the eigenfunctions of the isometries \( A_1^{\theta} : V^{\theta} \to V^{\theta}, A_1 : D(A_1) \to W_0 \). Consider the problem of finding \( (u_m, \phi_m) \in C^1(0, T; V_m \times W_m) \) such that for all \( (w_m, \psi_m) \in \tilde{V}_m \times W_m \),

\[
\begin{aligned}
\langle \partial_t u_m, w_m \rangle + \langle A_0 u_m, w_m \rangle + b_0 (u_m, w_m) &= \langle g, w_m \rangle + \langle R_0 (\mu_m, \phi_m), w_m \rangle, \\
\langle \partial_t \phi_m, \psi_m \rangle + \langle \mu_m, \psi_m \rangle + b_1 (u_m, \phi_m, \psi_m) &= 0, \\
\langle u_m(0), w_m \rangle &= \langle u_0, w_m \rangle, \\
\langle \phi_m(0), \psi_m \rangle &= \langle \phi_0, \psi_m \rangle.
\end{aligned}
\]

(3.2)

Upon choosing a basis for \( V_m \times W_m \), the above becomes an initial value problem for a system of ODE’s, and moreover since \( N \) is invertible by (2.7), the standard ODE theory gives a unique local-in-time solution. Furthermore, this solution is global if its norm is finite at any finite time instance. The fourth equality in (3.2) gives

\[
c_N \|u_m(0)\|_{\theta_2}^2 \leq \langle u_m(0), Nu_m(0) \rangle = \langle u(0), Nu_m(0) \rangle \leq \|u(0)\|_{-\theta_2} \|Nu_m(0)\|_{\theta_2},
\]

so that

\[\|u_m(0)\|_{-\theta_2} \leq \frac{\|N\|_{-\theta_2, \theta_2}}{c_N} \|u(0)\|_{-\theta_2}.\]

Now in the first and second equalities of (3.2), taking \( w_m = Nu_m \) and \( \psi_m = \mu_m \), respectively, and using the condition ii) on \( b_0 \), we get after standard transformations}

\[
\frac{d}{dt} \left( \langle u_m, Nu_m \rangle + \|A_1^{1/2} \phi_m\|_{L^2}^2 + 2 \int_\Omega F(\phi_m) \, dx \right) + 2 \langle A_0 u_m, Nu_m \rangle + \|\mu_m\|_{L^2}^2 \\
= 2 \langle g, Nu_m \rangle \\
\leq \varepsilon^{-1} \|g\|_{\theta_2}^2 + \varepsilon \|N\|_{\theta_2, \theta_2}^2 \|u_m\|_{\theta_2}^2,
\]

for any \( \varepsilon > 0 \). Let us now set

\[\mathcal{E}(u, \phi) := \langle u, Nu \rangle + \|A_1^{1/2} \phi\|_{L^2}^2 + 2 \int_\Omega F(\phi) \, dx + C_F,\]

for some sufficiently large positive constant \( C_F \) such that \( \mathcal{E} \geq 0 \) (indeed, such a constant exists due to assumption (2.4)). Choosing \( \varepsilon > 0 \) small enough, we can ensure

\[2 \langle A_0 u_m, Nu_m \rangle + \varepsilon \|N\|_{\theta_2, \theta_2}^2 \|u_m\|_{\theta_2}^2 \leq -\|u_m\|_{\theta_2}^2,
\]

so that by Grönwall’s inequality we have

\[\mathcal{E}(u_m(t), \phi_m(t)) \leq \mathcal{E}(u_m(0), \phi_m(0)) + C \int_0^t \|g\|_{\theta_2}^2,
\]

for some \( C > 0 \). For any fixed \( T > 0 \), this gives

\[(u_m, \phi_m) \in L^\infty(0, T; V^{-\theta_2} \times W^{1})\]

with uniformly (in \( m \)) bounded norm. By Proposition 3.1 we also have the uniform bound \( \phi_m \in L^\infty(0, T; L^\infty(\Omega)) \) such that \( |\phi_m(t)| \leq 1 \) a.e. in \( \Omega \times (0, T) \). Moreover, integrating (3.4), and taking into account (3.3), we infer

\[\int_0^t \langle A_0 u_m(s), Nu_m(s) \rangle \, ds \leq C_T, \quad \int_0^t \|\mu_m(s)\|_{L^2}^2 \, ds \leq C_T, \quad t \in (0, T),
\]

(3.6)
If $\theta > 0$, by the coerciveness of $A_0$, the above bounds imply $u_m \in L^2(0, T; V^{\theta - \theta_2})$, $\mu_m \in L^2(0, T; L^2(\Omega))$, with uniformly bounded norms. The latter bound also yields from the third equation in (3.2), the uniform bound

$$\phi_m \in L^2(0, T; D(A_1)).$$

Summarizing, $(u_m, \phi_m)$ is uniformly bounded in $L^\infty(0, T; \mathcal{Y}_{\theta_2}) \cap L^2(0, T; V^{\theta - \theta_2} \times D(A_1))$, and passing to a subsequence, there exists $(u, \phi) \in L^\infty(0, T; \mathcal{Y}_{\theta_2}) \cap L^2(0, T; V^{\theta - \theta_2} \times D(A_1))$ such that

$$\begin{align*}
(u_m, \phi_m) \rightharpoonup (u, \phi) \quad &\text{weak-star in } L^\infty(0, T; \mathcal{Y}_{\theta_2}), \\
(u_m, \phi_m) &\to (u, \phi) \quad \text{weakly in } L^2(0, T; V^{\theta - \theta_2} \times D(A_1)).
\end{align*}$$

As usual, for passing to the limit as $m \to \infty$ in (3.2), we will need a strong convergence result, which is obtained by a standard compactness argument. We proceed by deriving bounds on the derivatives of $u_m$ and $\phi_m$, respectively. Note that (3.2) can also be written as

$$\begin{align*}
\begin{cases}
u_m^\prime + P_m A_0 u_m + P_m B_0(u_m, u_m) = P_m (g + R_0 (\mu_m, \phi_m)), \\
\phi_m + Q_m \mu_m + Q_m B_1 (u_m, \phi_m) = 0, \\
u_m(0) = P_m u(0), \phi_m(0) = Q_m \phi(0).
\end{cases}
\end{align*}$$

Therefore,

$$\begin{align*}
\|u_m\|_{-\gamma} \lesssim & \|u_m\|_{\theta - \theta_2} + \|P_m B_0(u_m, u_m)\|_{-\gamma} + \|P_m g\|_{-\gamma} + \|P_m R_0 (\mu_m, \phi_m)\|_{-\gamma} \\
= & : I_1 + I_2 + I_3 + I_4.
\end{align*}$$

First, we notice that one has $I_4 \lesssim \|\mu_m\|_{L^2} \|\phi_m\|_1$ provided that $\gamma > \frac{\theta}{2}$. Therefore, $R_0 (\mu_m, \phi_m) \in L^2(0, T; V^{-\gamma})$ with uniform bound on account of (3.6)-(3.8). By the boundedness of $B_0$ (see (4)), we have as in [43, Theorem 3.1],

$$\begin{align*}
I_2 \lesssim & \|u_m\|_{\sigma_1} \|u_m\|_{\sigma_2}.
\end{align*}$$

If $\theta = 0$, then the norms in the right-hand side are the $V^{-\theta_2}$-norm which is uniformly bounded. If $\theta > 0$, since

$$\|u_m\|_{\sigma_i} \lesssim \|u_m\|_{\theta - \theta_2}^{1 - \lambda_i} \|u_m\|_{\theta - \theta_2}^{\lambda_i}, \quad \lambda_i = \frac{\sigma_i + \theta_2}{\theta}, \quad i = 1, 2,$$

by the uniform boundedness of $u_m$ in $L^\infty(V^{-\theta_2})$ we have

$$\begin{align*}
I_2 \lesssim & \|u_m\|_{\theta - \theta_2}^{\lambda_1 + \lambda_2} \|u_m\|_{\theta - \theta_2}^{\lambda_1 - \lambda_2} \lesssim \|u_m\|_{\theta - \theta_2}^{\lambda_1 + \lambda_2}.
\end{align*}$$

Hence, with $\lambda := \lambda_1 + \lambda_2 = \frac{\sigma_1 + \sigma_2 + 2\theta_2}{\theta}$ if $\theta > 0$, and with $\lambda = 1$ if $\theta = 0$, we get

$$\begin{align*}
\|u_m\|_{L^p(V^{-\gamma})}^p \lesssim & \|u_m\|_{L^p(V^{\theta_2})}^p + \|u_m\|_{L^p(V^{-\theta_2})}^p + \|g\|_{L^p(V^{-\theta_2})}^p \\
+ & \|R_0 (\mu_m, \phi_m)\|_{L^p(V^{-\gamma})}.
\end{align*}$$

The first and last terms on the right-hand side are bounded uniformly when $p \leq 2$. The second term is bounded if $p \lambda \leq 2$, that is $p \leq 2/\lambda$. We conclude that $u_m^\prime$ is uniformly bounded in $L^p(V^{-\gamma})$, with $p = \min\{2, 2/\lambda\}$. Concerning the time derivative of $\phi_m$, we use the second equation of (3.9) and the previous uniform bounds. We have

$$\begin{align*}
\|\phi_m^\prime\|_{-2} \lesssim & \|\mu_m\|_{L^2} + \|Q_m B_1 (u_m, \phi_m)\|_{-2} \\
\lesssim & \|\mu_m\|_{L^2} + \|u_m\|_{\theta - \theta_2} \|\phi_m\|_2,
\end{align*}$$

provided that $\theta_2 \geq -1$. From this we can conclude that $\phi_m^\prime$ is uniformly bounded in $L^2(W^{-2})$. By the Aubin-Lions-Simon compactness criterion (see, e.g., [73]), we can now obtain strong convergence properties for our sequence $(u_m, \phi_m)$, as follows. There exists

$$(u, \phi) \in C(0, T; V^{-\gamma} \times W^0) \cap L^\infty(0, T; \mathcal{Y}_{\theta_2})$$

such that, in addition to (3.3), we also have

$$\begin{align*}
(u_m, \phi_m) &\to (u, \phi) \quad \text{strongly in } L^2(0, T; V^s \times W^{2s}) \\
\phi_m &\to \phi \quad \text{strongly in } C(0, T; W^{1})
\end{align*}$$

for any $s < \theta - \theta_2$, where $W^{s-\delta}$ denotes $W^{s-\delta}$, for some sufficiently small $\delta \in (0, s]$. 
Now we will show that this limit \((u, \phi)\) indeed satisfies the weak formulation \((2.18)-(2.19)\). To this end, let \((w, \psi) \in C^\infty(0; T; \mathcal{V} \times \mathcal{W})\) be an arbitrary vector-valued function with \((w, \psi) (T) = (0, 0)\), and let \((w_m, \psi_m) \in C^1(0; T; \mathcal{V}_m \times W_m)\) be such that \((w_m, \psi_m) (T) = (0, 0)\) and
\[
\begin{align*}
w_m &\to w \text{ strongly in } C^1(0; T; \mathcal{V}), \\
\psi_m &\to \psi \text{ strongly in } C^1(0; T; \mathcal{W}).
\end{align*}
\]
We have
\[
\begin{align*}
&- \int_0^T \langle u_m(t), w_m'(t) \rangle dt + \int_0^T \langle A_0 u_m(t), w_m(t) \rangle dt \\
&+ \int_0^T b_0(u_m(t), u_m(t), w_m(t)) dt - \int_0^T \langle R_0(\mu_m, \phi_m), w_m(t) \rangle dt \\
&= \langle u_m(0), w_m(0) \rangle + \int_0^T \langle g(t), w_m(t) \rangle dt,
\end{align*}
\]
and
\[
\begin{align*}
&- \int_0^T \langle \phi_m(t), \psi_m'(t) \rangle dt + \int_0^T \langle \mu_m(t), \psi_m(t) \rangle dt + \int_0^T b_1(u_m(t), \phi_m(t), \psi_m(t)) dt \\
&= \langle \phi_m(0), \psi_m(0) \rangle.
\end{align*}
\]
We would like to show that each term in the above equations converge to the corresponding terms in \((2.18)-(2.19)\). The convergence in the nonlinear term \(b_0\) is shown in \cite{44} Theorem 3.1. Here we only show it for the Korteweg term on the left-hand side of equation \((3.16)\). The convergence in the nonlinear term \(b_1\) is analogous. We have
\[
\begin{align*}
\int_0^T |\langle R_0(A_1 \phi_m, \phi_m), w_m(t) \rangle - \langle R_0(A_1 \phi, \phi), w(t) \rangle| dt &\leq J_m + J J_m + J J J_m,
\end{align*}
\]
where the terms \(J_m, J J_m,\) and \(J J J_m\) are defined below. Firstly, it holds that
\[
\begin{align*}
J_m &= \int_0^T |\langle R_0(A_1 \phi_m, \phi_m), w_m(t) - w(t) \rangle| dt \\
&\lesssim \int_0^T \|A_1 \phi_m(t)\|_{L^2} \|\phi_m(t)\| \|w_m(t) - w(t)\|_{\mathcal{L}} dt \\
&\leq \|\phi_m\|_{L^2(D(A_1))} \|\phi_m\|_{L^2(W^1)} \|w_m - w\|_{C(\mathcal{V})}.
\end{align*}
\]
thus, we get \(\lim_{m \to \infty} J_m = 0\). For \(J J_m\) we have
\[
\begin{align*}
J J_m &= \int_0^T |\langle R_0(A_1 (\phi_m - \phi), \phi_m, w(t) \rangle| dt \\
&\lesssim \int_0^T \|\phi_m(t) - \phi(t)\|_{2-s} \|\phi_m(t)\|_2 \|w(t)\|_{\mathcal{V}} dt \\
&\leq \|\phi_m - \phi\|_{L^2(W^1)} \|\phi_m\|_{L^2(W^2)} \|w\|_{C(\mathcal{V})},
\end{align*}
\]
so \(\lim_{m \to \infty} J_m = 0\) by \((3.15)\) for as long as \(s \in (0, 2)\). Similarly, we have
\[
\begin{align*}
J J J_m &= \int_0^T |\langle R_0(A_1 \phi(t), \phi_m(t) - \phi(t), w(t) \rangle| dt \\
&\lesssim \|A_1 \phi\|_{L^2(L^1)} \|\phi_m - \phi\|_{L^2(W^1)} \|w\|_{C(\mathcal{V})},
\end{align*}
\]
so \(\lim_{m \to \infty} J J J_m = 0\) by \((3.13)\). The proof of the theorem is now finished. \(\square\)

Keeping in mind the Examples \((2.1)-(2.4)\) from Section 2, our theorem covers the following special cases listed in Table 2.

**Remark 3.3.** Let \(\theta + \theta_1 > \frac{1}{2}\) and recall that \(\theta \geq 0\) and \(\theta_2 \geq -1\). By \cite{44} Proposition 2.5, the trilinear form \(b_{00}\), defined by \((2.14)-(2.15)\), fulfills the hypotheses (ii)-(iv) of Theorem 3.2 for \(-\gamma \leq \theta - \theta_2 - 1\) with \(-\gamma < \min\{2\theta + 2\theta_1 - \frac{\alpha + 2}{2}, \theta - \theta_2 + 2\theta_1, \theta + \theta_2 - 1\}\). Similarly, the trilinear form \(b_{01}\) satisfies (ii)-(iv) for \(-\gamma \leq \theta - \theta_2 - 1\) with \(-\gamma < \min\{2\theta + 2\theta_1 - \frac{\alpha + 2}{2}, \theta - \theta_2 + 2\theta_1 - 1, \theta + \theta_2\}\). In particular, our result yields the global existence of a weak solution for both the inviscid and viscous Leray-Allen-Cahn-\(\alpha\) models in two and three space dimensions, and for all the other regularized
models listed in Table 1. As far as we know, except for the 3D NS-AC-α-model [35], any of these results have not been reported previously.

3.2. Uniqueness and stability. Now we shall provide sufficient conditions for uniqueness and continuous dependence with respect to the initial data of weak solutions of the general three-parameter family of regularized models. Recall that $\theta_1 \in \mathbb{R}$ and $\theta \geq 0$.

**Theorem 3.4.** Let $(u_i, \phi_i) \in L^\infty(0, T; \mathcal{Y}_{\theta_i})$, $i = 1, 2$, be two solutions in the sense of Definition 2.4, corresponding to the initial conditions $(u_i(0), \phi_i(0)) \in \mathcal{Y}_{\theta_i}$, $i = 1, 2$. Assume the following.

(i) $b_0 : V^{\theta_1} \times V^{\theta_2} \times V^{\sigma_2} \to \mathbb{R}$ is bounded for some $\sigma_1 \leq \theta - \theta_2$ and $\sigma_2 \leq \theta + \theta_2$ with $\sigma_1 + \sigma_2 \leq \theta$,

(ii) $b_0(v, w, Nw) = 0$ for any $v \in V^{\sigma_1}$ and $w \in V^{\sigma_2}$.

We have the following cases:

(a) Case $n = 2$: $\theta + \theta_2 \geq 1$ and $\theta_2 \geq 0$.

(b) Case $n = 3$: $\theta_2 \geq 1$ and $\theta \geq 0$.

Then the following estimate holds

\[
\begin{align*}
\|u_1(t) - u_2(t)\|^2_{\theta_2} + \|\phi_1(t) - \phi_2(t)\|^2_{\theta_2} & = \int_0^t \left(\|u_1(s) - u_2(s)\|^2_{\theta_2} + \|\phi_1(s) - \phi_2(s)\|^2_{L^2}\right) ds \\
& \leq \rho(t) \left(\|u_1(0) - u_2(0)\|^2_{\theta_2} + \|\phi_1(0) - \phi_2(0)\|^2_{\theta_2}\right),
\end{align*}
\]

for $t \in [0, T]$, for some positive continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, $\rho(0) > 0$, which depends only on the initial data $(u_i(0), \phi_i(0))$ in $\mathcal{Y}_{\theta_2}$-norm.

**Proof.** Let $v = u_1 - u_2$ and $\psi = \phi_1 - \phi_2$. Then subtracting the equations for $(u_1, \phi_1)$ and $(u_2, \phi_2)$ we have

\[
\begin{align*}
\langle \partial_t v, w \rangle + \langle A_0(v, w) + (B_0(v, u_1), w) + \langle B_0(u_2, v), w \rangle \\
& = \langle R_0(A_1 \phi_2, \psi), w \rangle + \langle R_0(A_1 \psi, \phi_1), w \rangle,
\end{align*}
\]

and

\[
\begin{align*}
\langle \partial_t \psi, \eta \rangle + \langle A_1 \psi, \eta \rangle + \langle B_1(v, \phi_1), \eta \rangle + \langle B_1(u_2, v), \eta \rangle \\
& = \langle f(\phi_1) - f(\phi_2), \eta \rangle.
\end{align*}
\]

First, observe that by the assumptions on $\theta, \theta_2$, according to (3.20)-(3.32) below, the weak solution $(u, \phi)$ of (2.2) enjoys additional regularity. Indeed, by virtue of (2.16) it follows that $R_0(A_1 \phi_1, \phi_1) \in L^2(0, T; V^{\theta_2})$ and $B_1(u, \phi) \in L^2(0, T; L^2(\Omega))$. Thus, recalling the assumption on $b_0$ (see (i)-(ii) above and (3.20) below), it is easy to see that every pairing $\langle \cdot, \cdot \rangle$ in (3.23)-(3.24) is well-defined as a functional on the corresponding spaces for $w \in L^2(0, T; V^{\theta_2})$ and $\eta \in L^2(0, T; L^2(\Omega))$, respectively. Thus, by (2.10) we can take $w = Nv$ and $\eta = A_1 \psi$ into (3.23)-(3.24) to infer

\[
\begin{align*}
\frac{d}{dt} \left(\|v\|^2_{\theta_2} + \|A_1^{1/2} \psi\|^2_{L^2}\right) + 2c_{A_0} \|v\|_{\theta_2}^2 + 2\|A_1 \psi\|^2_{L^2} & \leq 2b_0(v, u_1, Nv) + 2b_1(v, \psi, A_1 \psi) + 2b_1(u_2, \psi, A_1 \psi) + 2\langle f(\phi_1) - f(\phi_2), A_1 \psi \rangle.
\end{align*}
\]

The first term on the right-hand side of (3.25) can be bounded as follows:

\[
\begin{align*}
b_0(v, u_1, Nv) & \lesssim \|v\|_{\theta_2} \|u_1\|_{\theta_2} \|\phi_1\|_{\sigma_2 - \theta_2} \|\phi_1\|_{\sigma_2 - \theta_2} \\
& \lesssim \|v\|_{\theta_2}^2 \|A_1\|_{\theta_2} \|\phi_1\|_{\sigma_2 - \theta_2} \|\phi_1\|_{\sigma_2 - \theta_2} \\
& \lesssim \delta^\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \|v\|_{\theta_2}^2 \|u_1\|_{\theta_2} \|\phi_1\|_{\sigma_2 - \theta_2}^2 + \delta \|v\|_{\theta_2}^2,
\end{align*}
\]

for any $\delta > 0$, where $\lambda_1 = \frac{\sigma_1 + \theta_2}{\theta}$ and $\lambda_2 = \frac{\sigma_2 - \theta_2}{\theta}$, and where in the last step we applied Young’s inequality. Exploiting the fact that $\phi_i \in L^\infty(0, \infty; L^\infty(\Omega))$ with $|\phi| \leq 1$ a.e. on $\Omega \times (0, \infty)$, the last term in (3.25) is easy:

\[
\langle f(\phi_1) - f(\phi_2), A_1 \psi \rangle \lesssim \delta \|A_1 \psi\|^2_{L^2} + \delta^{-1} \|\psi\|^2_{L^2}.
\]

In order to bound the second and third terms on the right-hand side of (3.25), we will treat each cases (a) and (b) separately.
Case (a): $n = 2$. Let $p \in (2, \infty)$. We have
\[
|b_1 (u_2, \psi, A_1 \psi)| = |\langle B_1 (Nu_2, \psi), A_1 \psi \rangle| \\
\leq \|Nu_2\|_{L^{2p/(p-2)}} \|A_1 \psi\|_{L^2}^{\frac{2}{p}} \|\nabla \psi\|_{L^2}^{\frac{2}{p}} \\
\leq \delta^{-1/(p-1)} \|Nu_2\|_{L^{2p/(p-2)}}^{2} \|\nabla \psi\|_{L^2}^{2} + \delta \|A_1 \psi\|_{L^2}^{2} \\
\lesssim \delta^{-1/(p-1)} \|Nu_2\|_{L^1}^{2} \|Nu_2\|_{L^{(p-2)}}^{2} \|\psi\|_{L^1}^{2} + \delta \|A_1 \psi\|_{L^2}^{2},
\]
where in the last step we have used the following inequality (see Appendix, Lemma 8.2).
\[
\|w\|_{L^{2p/(p-2)}} \lesssim \|w\|_{L^1}^{\frac{2}{p}} \|w\|_{L^p}^{\frac{2}{p} + 1 - \frac{2}{p}}, \quad n \geq 2.
\]
Exploiting now the fact that $N : V^s \to V^{s+2\theta}$ is bounded (see (2.5)), and since $V^{\theta-\theta_2} \subseteq V^{1-2\theta_2}$ and $V^{-\theta_2} \subseteq V^{-2\theta_2}$, one has
\[
|b_1 (u_2, \psi, A_1 \psi)| \leq \delta \|A_1 \psi\|_{L^2}^{2} + \delta^{-1/(p-1)} \|u_2\|_{L^{(p-2)}}^{2} \|\psi\|_{L^1}^{2}.
\]
For the last term, we have
\[
|b_1 (v, \psi, A_1 \phi_2)| = |\langle B_1 (Nv, \psi), A_1 \phi_2 \rangle| \\
\leq \|Nv\|_{L^4} \|\nabla \psi\|_{L^4} \|A_1 \phi_2\|_{L^2} \\
\lesssim \left( \|Nv\|_{L^1}^{1/2} \|Nv\|_{L^2}^{1/2} \right) \left( \|\nabla \psi\|_{L^2}^{1/2} \|A_1 \psi\|_{L^2}^{1/2} \right) \|A_1 \phi_2\|_{L^2} \\
\lesssim \|v\|_{L_{\theta_2}} \|v\|_{L_{\theta_2}} \|\psi\|_{L^1}^{1/2} \|A_1 \phi_2\|_{L^2} \\
\lesssim \delta \|v\|_{L_{\theta_2}} \|\psi\|_{L^1}^{2} + \delta \|A_1 \psi\|_{L^2}^{2} + \delta^{-1} \|v\|_{L_{\theta_2}} \|\psi\|_{L^1}^{1} \|A_1 \phi_2\|_{L^2}^{2}.
\]
\]
Case (b): $n = 3$. Let $p \in (2, 6)$ be fixed but otherwise arbitrary (a suitable value will be chosen below). Once again, we have
\[
|b_1 (u_2, \psi, A_1 \psi)| \\
\leq \|Nu_2\|_{L^{2p/(p-2)}} \|A_1 \psi\|_{L^2}^{\frac{2}{p}} \|\nabla \psi\|_{L^2}^{\frac{2}{p}} \\
\lesssim \delta^{-\frac{4(p-2)}{p}} \|Nu_2\|_{L^{2p/(p-2)}} \|\nabla \psi\|_{L^2}^{2} + \delta \|A_1 \psi\|_{L^2}^{2} \\
\lesssim \delta^{-\frac{2p-4}{p}} \|Nu_2\|_{L^1}^{\frac{2p}{p-2}} \|Nu_2\|_{L^{(p-2)}}^{\frac{4(p-3)}{p}} \|\psi\|_{L^1}^{2} + \delta \|A_1 \psi\|_{L^2}^{2},
\]
where we have once more used (6.28). By virtue of (2.5), and since $V^{\theta-\theta_2} \subseteq V^{-2\theta_2}$ and $V^{-\theta_2} \subseteq V^{1-2\theta_2}$, we can now conclude
\[
|b_1 (u_2, \psi, A_1 \psi)| \lesssim \delta^{-\frac{4(p-2)}{p}} \|u_2\|_{L_{\theta_2}} \|u_2\|_{L_{\theta_2}} \|\psi\|_{L^1}^{2} + \delta \|A_1 \psi\|_{L^2}^{2},
\]
for any $\delta > 0$, provided that we choose a suitable $p \in (2, 6)$ such that $\frac{4(p-3)}{6-p} \leq 2$ (this is easily the case for any fixed $p \in [3, 4]$). For the last term, since $\theta_2 \geq 1$ we have
\[
|b_1 (v, \psi, A_1 \phi_2)| \leq \|Nv\|_{L^2} \|\nabla \psi \phi \|_{L^2} \\
\lesssim \|v\|_{L_{\theta_2}} \|A_1 \phi_2\|_{L^2} \|\nabla \psi\|_{L^2} \\
\lesssim \|v\|_{L_{\theta_2}} \|A_1 \phi_2\|_{L^2} \|A_1 \psi\|_{L^2}^{1/2} \|\nabla \psi\|_{L^2}^{1/2} \\
\lesssim \delta \|A_1 \psi\|_{L^2}^{2} + \delta^{-1/3} \|v\|_{L_{\theta_2}}^{4/3} \|A_1 \phi_2\|_{L^2}^{1/3} \|\psi\|_{L^1}^{2/3} \\
\lesssim \delta \|A_1 \psi\|_{L^2}^{2} + \delta^{-1/3} \left( \|\psi\|_{L^1}^{2} + \|v\|_{L_{\theta_2}}^{2} \|A_1 \phi_2\|_{L^2}^{2} \right).
\]
Collecting all estimates from (3.29) to (3.32), and choosing a sufficiently small $\delta \sim \min (c_{A_0}, c_{A_1}) > 0$, we can now apply Grönwall’s inequality in (3.25) to deduce
\[
\|v(t)\|_{L^2_{\theta_2}} + \|\psi(t)\|_{L^2_{\theta_2}} \leq \left( \|v(0)\|_{L^2_{\theta_2}} + \|\psi(0)\|_{L^2_{\theta_2}} \right) \exp \int_0^t \Theta(s) \, ds,
\]
for a suitable function $\Theta \in L^1 (0, T)$. Integrating (3.25) over $(0, t)$ gives the desired inequality (3.22). The proof is finished. \qed
To clarify these results at least in the case of the specific models listed in Table 1, the corresponding conditions and stability results derived from Theorem 3.4 are given below. Recall that both conditions (a) and (b) of Theorem 3.4 are in force according to whether \( n = 2 \) or \( n = 3 \), respectively.

**Remark 3.5.** Exploiting [11, Proposition 2.5], the trilinear form \( b_{00} \) satisfies the hypotheses of Theorem 3.4 provided \( \theta + \theta_1 \geq \frac{k}{\kappa}, \theta + 2\theta_1 \geq k; \theta + \theta_2 \geq \frac{k}{2}, 2\theta + 2\theta_1 + \theta_2 > \frac{k}{2}, \) and \( 3\theta + 2\theta_1 + 2\theta_2 \geq 2 - k_1, k \in \{0, 1\} \). The trilinear form \( b_{01} \) satisfies the hypotheses of Theorem 3.4 for \( \theta + \theta_1 \geq 1, \theta + \theta_2 \geq \frac{1}{2}, \theta + \theta_2 \geq 0, 2\theta + 2\theta_1 + \theta_2 > \frac{k}{2}, \) and \( 3\theta + 2\theta_1 + 2\theta_2 \geq 1 \). For instance, these assumptions allow us to recover the stability and uniqueness of the weak solutions for the 3D Navier-Stokes-Allen-Cahn-\( \alpha \)-model included in Table 1 (see also Table 2). This result was reported previously in [35].

### 3.3. Regularity of weak solutions

In this subsection, we develop a regularity result on weak solutions for the general family of regularized models constructed in Section 3.1. Incidentally, the result below also allows us to obtain globally well-defined (unique) strong solutions for our regularized models, which will become important in Section 4 to the study of the asymptotic behavior as time goes to infinity. As in Section 3.2 recall that \( \theta \geq 0 \) and \( \theta_1 \in \mathbb{R} \). As before, due to the coupling of the regularized NSE with the Allen-Cahn equation, we will separately derive optimal estimates in dimensions \( n = 2, 3 \).

**Theorem 3.6.** Let

\[
(u, \phi) \in L^\infty(0, T; \mathcal{Y}_0) \cap L^2(0, T; V^{\theta-\theta_2} \times D(A_1))
\]

be a weak solution in the sense of Definition 2.4. Under the assumptions of Theorems 3.2 and 3.3, when \( n = 3 \), and some \( \theta \geq \beta \geq \max (1 - 2\theta_2, -\theta_2) \) with \( \beta \neq -\theta_2 \) when \( n = 2 \), let the following conditions hold.

(i) \( b_0 : V_\alpha \times V_\alpha \times V^{\theta-\beta} \to \mathbb{R} \) is bounded, where \( \alpha = \min \{\beta, \theta - \theta_2\} \);

(ii) \( b_0(u, v, w) = 0 \) for any \( u, v, w \in V \);

(iii) \( u_0 \in V^\beta, \phi_0 \in D(A_1), \) and \( g \in L^2(0, T; V^{\beta-\theta}) \).

Then we have

\[
(u, \phi) \in L^\infty(0, T; V^\beta \times D(A_1)) \cap L^2(0, T; V^{\beta+\theta} \times D(A_1^{3/2})).
\]

**Proof.** The following estimates will be also deduced by a formal argument. However, even in this case, they can be rigorously justified working with a sufficiently smooth approximating solution, see Theorem 3.2. Pairing the first equation of (2.2) with \( \Lambda^1 \), the second and third equations with \( A_1^2 \), respectively, we deduce

\[
(\partial_t u, A^2 u) + (A_0 u, A^2 u) + b_0(u, u, \Lambda^2 u) = (g + R_0(A_1 \phi, \phi), A^2 u), \quad \text{a.e. in } (0, T),
\]

and

\[
(\partial_t \phi, A_1^2 \phi) + (A_1 \phi, A_1^2 \phi) + b_1(u, \phi, A_1^2 \phi) = - (f(\phi), A_1^2 \phi), \quad \text{a.e. in } (0, T).
\]

By employing the boundedness of \( b_0 \) (see (i)) and the coercivity of \( A_0 \), we infer

\[
b_0(u, u, A^2 u) \lesssim \delta^{-1} u_{\beta-\theta} ||u||_{2, \theta}^2 + \delta ||u||_{2+\theta}^2, \quad \text{a.e. in } (0, T),
\]

for any \( \delta > 0 \), while for the first term on the right-hand side of (3.35), we have

\[
(g, A^2 u) \lesssim \delta ||u||_{2+\theta} + \delta^{-1} ||g||_{\beta-\theta}^2.
\]

Moreover, since \( \phi \in [-1, 1] \) a.e. on \( \Omega \times (0, T) \), one has

\[
||f(\phi), A_1^2 \phi) = ||A_1^{1/2} f(\phi), A_1^{1/2} \phi) ||_2^2 \lesssim \delta ||A_1^{3/2} \phi||_{L^2}^2 + \delta^{-1} ||\phi||_{1}^2.
\]

To bound the remaining terms in (3.35)-(3.38), we divide our proof according to the different assumptions we employed for \( n = 3 \) and \( n = 2 \).

**Case \( n = 3 \):** We begin to estimate the term involving the Korteweg force. Since \( V^{1/2} \subset L^3 \), we deduce

\[
|\langle R_0(A_1 \phi, \phi), A^3 u \rangle| \leq ||A^3 u||_{\theta-\theta} ||R_0(A_1 \phi, \phi)||_{\beta-\theta} \lesssim ||u||_{\theta+\beta} ||R_0(A_1 \phi, \phi)||_{-1/2}.
\]
In order to estimate the term $b_1$ in (3.36), we require the following basic inequality:

$$\|\nabla^s (f \cdot g)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{W^{s,p_2}} + \|f\|_{W^{s,p_2}} \|g\|_{L^{p_4}}$$

for any $p_i \in (1, \infty)$, $i = 1, \ldots, 4$, with $1/p_1 + 1/p_2 = 1/p_3 + 1/p_4 = 1/p$ and $s \in \mathbb{N}_0$. On account of (3.41), we have

$$b_1(u, \phi, A_1^2 \phi) = \left\langle A_1^{3/2} B_1(u, \phi), A_1^{3/2} \phi \right\rangle \leq \|B_1(u, \phi)\|_1 \left\| A_1^{3/2} \phi \right\|_{L^2}$$

$$\lesssim (\|Nu\|_{L^s} \|\nabla \phi\|_{W^{1,3}} + \|Nu\|_1 \|\nabla \phi\|_{L^\infty}) \left\| A_1^{3/2} \phi \right\|_{L^2}$$

$$= I_1 + I_2.$$ 

To estimate the $I_1$-term, we recall that $N : V^{1-2\theta_2} \to V^1$ is bounded, and $V^{-\theta_2} \subseteq V^{1-2\theta_2}$. Indeed, exploiting the following inequality $\|\cdot\|_{W^{1,3}} \lesssim \|\cdot\|_{1/4}^{1/4} \|\cdot\|_{L^4}^{7/4}$ (see Lemma 3.2), we infer

$$I_1 \lesssim \|u\|_{-\theta_2} \|\phi\|_1^{1/4} \left\| A_1^{3/2} \phi \right\|_{L^2}^{7/4}$$

$$\lesssim \delta \left\| A_1^{3/2} \phi \right\|_{L^2}^2 + \delta^{-3} \|u\|_{-\theta_2}^4 \|A_1 \phi\|_{L^2}^2.$$ 

Similarly, by the 3D Agmon’s inequality, we have

$$I_2 \lesssim \|u\|_{-\theta_2} \|A_1 \phi\|_{L^2}^{1/4} \left\| A_1^{3/2} \phi \right\|_{L^2}^{3/2}$$

$$\lesssim \delta \left\| A_1^{3/2} \phi \right\|_{L^2}^2 + \delta^{-3} \|u\|_{-\theta_2}^4 \|A_1 \phi\|_{L^2}^2.$$ 

Collecting all the above estimates from (3.37) to (3.39), and then adding together the relations (3.35) - (3.39), for a sufficiently small $\delta = \min (c_{A_1}, c_{A_0}) > 0$, we deduce the following inequality

$$\frac{d}{dt} \left( \|u\|_{\beta}^2 + \|A_1 \phi\|_{L^2}^2 \right) + \left\| A_1^{3/2} \phi \right\|_{L^2}^2 + c_{A_0} \|u\|_{\beta + \theta}^2$$

$$\leq \Xi(t) \left( \|u\|_{\beta}^2 + \|A_1 \phi\|_{L^2}^2 \right) + C \|g\|_{\beta - \theta}^2,$$

where we have set

$$\Xi := C_{\delta} \left( 1 + \|u\|_{\beta - \theta}^2 + \|\nabla \phi\|_{L^2}^2 + \|A_1 \phi\|_{L^2}^2 + \|u\|_{\beta - \theta}^2 \right),$$

for some $C_{\delta} > 0$. Notice that since $(u, \phi)$ is a weak solution in the sense of Definition 2.4, we have $\Xi \in L^1(0, T)$. Integrating (3.42) over $(0, t)$, and using Grönnwall’s inequality, we conclude

$$\|u(t)\|_{\beta}^2 + \|A_1 \phi(t)\|_{L^2}^2$$

$$\lesssim \left( \int_0^t \|g\|_{\beta - \theta}^2 + \|u(0)\|_{\beta}^2 + \|A_1 \phi(0)\|_{L^2}^2 \right) \exp \left( 2 \int_0^t \Xi(s) \, ds \right), \quad a.e. \quad (0, T).$$

Therefore, by assumption (iii) we have $(u, \phi) \in L^\infty(0, T; V^\beta \times D(A_1))$, which transfers to $(u, \phi) \in L^2(0, T; V^{\theta + \beta} \times D(A_1^{3/2}))$ on account of (3.41)-(3.46). 

**Case n = 2:** Concerning the Korteweg term, using the Agmon’s inequality in two dimensions, we have

$$\left\langle R_0 (A_1 \phi, \phi), A_2^{2\beta} u \right\rangle \leq \|A_2^{2\beta} u\|_{\theta - \beta} \|R_0 (A_1 \phi, \phi)\|_{\beta + \theta}$$

$$\lesssim \|u\|_{\theta + \beta} \|R_0 (A_1 \phi, \phi)\|_0$$

$$\lesssim \delta \|u\|_{\beta + \theta}^2 + \delta^{-1} \|\nabla \phi\|_{L^\infty}^2 \|A_1 \phi\|_{L^2}^2$$

$$\lesssim \delta \|u\|_{\beta + \theta}^2 + \delta \left( \|A_1^{3/2} \phi\|_{L^2}^2 + \|A_1 \phi\|_{L^2}^2 \right)$$

$$+ \delta^{-2} \|A_1 \phi\|_{L^2}^2 \left( \|A_1 \phi\|_{L^2}^2 \|\phi\|_{L^2}^2 \right),$$

for any $\delta > 0$. Next, using (3.41) we have

$$b_1(u, \phi, A_1^2 \phi) \leq \|B_1(u, \phi)\|_1 \left\| A_1^{3/2} \phi \right\|_{L^2}^2.$$
Now, since \( N \) is bounded from \( V^{-2\theta_2} \rightarrow V^0 \) and from \( V^{1-2\theta_2} \rightarrow V^1 \), the second term

\[
I_2 \lesssim \| Nu \|_{L^1}^{1/2} \| Nu \|_{L^1}^{1/2} \| A_1 \phi \|_{L^2}^{1/2} \| A_1^{3/2} \phi \|_{L^2}^{3/2}
\]

\[
\lesssim \| A_1^{3/2} \phi \|_{L^2}^2 + \delta^{-3} \| u \|_{L^2}^2 \| \nabla \phi \|_{L^2}^2 + \delta \| A_1^{3/2} \phi \|_{L^2}^2,
\]

provided that \( \beta \geq \max (1 - 2\theta_2, -\theta_2) \). Collecting the above estimates (3.47 - 3.49) and insert them into the right hand sides of (3.39 - 3.42), for a sufficiently small \( \delta \sim \min (c_{A_0}, c_{A_1}) \), we infer

\[
\frac{d}{dt} \left( \| u \|_{\beta}^2 + \| A_1 \phi \|_{L^2}^2 \right) + \| A_1^{3/2} \phi \|_{L^2}^2 + c_{A_0} \| u \|_{\beta + \theta}^2
\]

\[
\leq \Psi(t) \left( \| u \|_{\beta}^2 + \| A_1 \phi \|_{L^2}^2 \right) + C \| g \|_{\beta - \theta}^2,
\]

where we have set

\[
\Psi := C_\delta \left( 1 + \| u \|_{\beta - \theta_2}^2 \left( \| \nabla \phi \|_{L^2}^2 + \| u \|_{L^2}^2 \right) + \| A_1 \phi \|_{L^2}^2 \| \phi \|_{L^2}^2 \right),
\]

for some \( C_\delta > 0 \). We remark once again that \( \Psi \in L^1(0,T) \) for any weak solution \((u,\phi)\) to problem (2.2). Thus, the application of Gronwall’s inequality in (3.50) yields the desired conclusion. The proof of the theorem is finished. □

Remark 3.7. Let \( 4\theta + 4\theta_1 + 2\theta_2 > n + 2, 2\theta + 2\theta_1 \geq 1 - k, \theta + 2\theta_2 \geq 1, 3\theta + 4\theta_1 \geq 1, \theta + 2\theta_1 \geq \ell, \) and \( 3\theta + 2\theta_1 + 2\theta_2 \geq 2 - \ell, \) for some \( k, \ell \in \{0,1\} \). In addition to the assumptions of Theorem 3.6 let

\[
\beta \in \left( \frac{n + 2}{2} - 2(\theta + \theta_2) - \theta, 3\theta + 2\theta_1 - \frac{n + 2}{2} \right) \cap \left[ \frac{1}{2} - \theta_1 - \theta_2, \min \{2\theta + \theta_2 - 1, 2\theta - \theta_2 + 2\theta_1 - 1 \} \right].
\]

Then, by [44] Proposition 2.5, the trilinear form \( b_{00} \) satisfies the hypotheses of the above theorem.

Remark 3.8. Let \( 4\theta + 4\theta_1 + 2\theta_2 > n + 2, \theta + 2\theta_2 \geq 0, \) and \( \theta + 2\theta_1 \geq 1 \). Let

\[
\beta \in \left( \frac{n + 2}{2} - 2(\theta + \theta_2) - \theta, 3\theta + 2\theta_1 - \frac{n + 2}{2} \right) \cap \left[ \frac{1}{2} - \theta_1 - \theta_2, \min \{2\theta + \theta_2 - 1, 2\theta - \theta_2 + 2\theta_1 - 1 \} \right].
\]

By [44] Proposition 2.5, the trilinear form \( b_{01} \) satisfies the hypotheses of the above theorem.

The corresponding conditions and results of Theorem 3.6 above are listed in the Table 3 below for the three-dimensional regularized models listed in Table 1. For the NS-AC-\( \alpha \)-like model in Table 1, the allowed values for \( \beta \) are \( \beta \leq 2\theta - \theta_2 - 1 \) with \( \beta < 3\theta - \frac{n}{2} \) and \( \beta \leq \theta - \frac{n}{2} \), provided that \( \theta \geq \frac{1}{2} \) and \( 4\theta + 2\theta_2 > 5 \).

| 3D Model | NSE-AC | Leray-AC-\( \alpha \) | ML-AC-\( \alpha \) | SBM-AC | NSV-AC | NS-AC-\( \alpha \) |
|----------|-------|----------------|----------------|--------|--------|----------------|
| Regularity | N/A | \( \beta \in (0, \frac{1}{2}) \) | \( \beta \in (-\frac{1}{2}, \frac{1}{2}) \) | \( \beta \in (-1, \frac{1}{2}) \) | \( \beta \in (-1, \frac{1}{2}) \) | \( \beta \in (-1, 0) \) |
4. Singular perturbations

In this section, following [14] we will consider the situation where the operators \(A_0\) and \(B_0\) in the general three-parameter family of regularized models represented by problem (2.2) have values from a convergent (in a certain sense) sequence, and study the limiting behavior of the corresponding sequence of solutions. As special cases we have inviscid limits \((\nu = 0)\) in the viscous equations and \(\alpha \to 0^+\) limits in the \(\alpha\)-models.

4.1. Perturbations to the linear part of the flow component. Consider the problem

\[
\begin{align*}
\partial_t u + A_0 u + B_0 (u, u) &= g + R_0 (A_1 \phi, \phi), \\
\partial_t \phi + A_1 \phi + B_1 (u, \phi) + f (\phi) &= 0
\end{align*}
\]

and its perturbation

\[
\begin{align*}
\partial_t u_i + A_0 u_i + B_0 (u_i, u_i) &= g + R_0 (A_1 \phi_i, \phi_i), \\
\partial_t \phi_i + A_1 \phi_i + B_1 (u_i, \phi_i) + f (\phi_i) &= 0,
\end{align*}
\]

for \(i \in \mathbb{N}\), where \(A_0, B_0, B_1, R_0\) and \(N\) satisfy the assumptions stated in Section 2 and for \(i \in \mathbb{N}\), \(A_{0i} : V^s \to V^{s-2e}\) is a bounded linear operator satisfying

\[
\|A_{0i} v\|_{2,\theta_2}^2 + \|v\|_{2,\theta_2}^2 \lesssim \langle A_{0i} v, N v \rangle + \|v\|_{2,\theta_2}^2, \quad v \in V^{\varepsilon-\theta_2}.
\]

Assuming that both problems (4.1) and (4.2) have the same initial condition \((u_0, \phi_0)\), and that \(A_{0i} \to A_0\) in some topology, we are concerned with the behavior of \((u_i, \phi_i)\) as \(i \to \infty\). We will also assume that \(\varepsilon \geq \theta\).

**Theorem 4.1.** Assume the above setting, and in addition let the following conditions hold.

1. \((u_0, \phi_0) \in \mathcal{Y}_{\theta_2}^\varepsilon\), with any \(\theta_2 \geq -1\), and \(g \in L^2(0, T; V^{\varepsilon-\theta_2})\), \(T > 0\);
2. \(b_0(v, \nu) = 0\) for any \(v \in V\);
3. \(b_1 : V^{\sigma_1} \times V^{\sigma_2} \times V^\gamma \to \mathbb{R}\) is bounded for some \(\sigma_j \in [-\theta_2, \theta - \theta_2]\), \(j = 1, 2\), and \(\gamma \in \varepsilon + \theta_2, \infty) \cap (\theta_2, \infty) \cap \left(\frac{1}{2}, \infty\right)\);
4. \(b_0 : V^{\sigma_1} \times V^{\sigma_2} \times V^\gamma \to \mathbb{R}\) is bounded for some \(\sigma_j < \theta - \theta_2\), \(j = 1, 2\), and \(\gamma \geq \gamma\);
5. \(A_{0i}\) converge weakly to \(A_0\) as \(i \to \infty\).

Then, there exists a solution \((u, \phi) \in L^\infty(0, T; \mathcal{Y}_{\theta_2}^\varepsilon) \cap L^2(0, T; V^{\varepsilon-\theta_2} \times D(A_1))\) to (4.1) such that, up to a subsequence,

\[
\begin{align*}
(u_i, \phi_i) &\to (u, \phi) \text{ weak-star in } L^\infty(0, T; \mathcal{Y}_{\theta_2}^\varepsilon), \\
(u_i, \phi_i) &\to (u, \phi) \text{ weakly in } L^2(0, T; V^{\varepsilon-\theta_2} \times D(A_1)), \\
(u_i, \phi_i) &\to (u, \phi) \text{ strongly in } L^2(0, T; V^{\varepsilon-\theta_2} \times W^\gamma) \\
\phi_i &\to \phi \text{ strongly in } C(0, T; W^\gamma) \text{ for any } \zeta < 1,
\end{align*}
\]

for any \(s \leq \theta - \theta_2, l < 2\), as \(i \to \infty\).

**Proof.** First, from Theorem 3.2 we know that for \(i \in \mathbb{N}\) there exists a solution

\[
(u_i, \phi_i) \in L^\infty(0, T; \mathcal{Y}_{\theta_2}^\varepsilon) \cap L^2(0, T; V^{\varepsilon-\theta_2} \times D(A_1))
\]

to (4.2). Moreover, \(\phi_i \in L^\infty(0, T; L^\infty(\Omega))\) with \(\phi_i \in [-1, 1]\) for all \(i \in \mathbb{N}\). Duality pairing of the first and second equations of (4.2) with \(Nu_i\) and \(A_1 \phi_i + f (\phi_i)\), respectively, arguing as in the proof of Theorem 3.2, we deduce

\[
\begin{align*}
\frac{d}{dt} \mathcal{E} (u_i (t), \phi_i (t)) + 2 \langle A_{0i} u_i, Nu_i \rangle + \|A_1 \phi_i + f (\phi_i)\|_{L^2}^2 \\
&\lesssim \delta^{-1} \|g\|_{2,\theta_2}^2 + \delta \|u_i\|_{2,\theta_2}^2,
\end{align*}
\]

Choosing \(\delta > 0\) small enough, then using (4.3) and integrating over \((0, t)\) we have

\[
\mathcal{E} (u_i (t), \phi_i (t)) \lesssim C_T (1 + \mathcal{E} (u(0), \phi(0))).
\]

Moreover, integrating (4.5), and taking into account (4.6) and (4.7), we once again infer

\[
\|A_{0i} u_i\|_{L^2(0,t;V^{\varepsilon-\theta_2})}^2 + \|A_1 \phi_i + f (\phi_i)\|_{L^2(0,t;V^{\varepsilon-\theta_2})}^2 + \|u_i\|_{L^2(0,t;V^{\varepsilon-\theta_2})}^2 \leq C_T, \quad t \in (0, T).
\]

For any fixed \(T > 0\), this gives \(u_i \in L^\infty(0, T; V^{\varepsilon-\theta_2}) \cap L^2(0, T; V^{\varepsilon-\theta_2})\) and \(\phi_i \in L^2(0, T; D(A_1))\), respectively, with uniformly (in \(i\)) bounded norms. On the other hand, we have

\[
\begin{align*}
\|u_i\|_{-\gamma} &\leq \|A_0 u_i\|_{-\gamma} + \|B_0 (u_i, u_i)\|_{-\gamma} + \|g + R_0 (A_1 \phi_i, \phi_i)\|_{-\gamma}, \\
\|\phi_i\|_{-\gamma} &\leq \|A_1 \phi_i + f (\phi_i)\|_{L^2} + \|B_1 (u_i, \phi_i)\|_{-2}.
\end{align*}
\]
By estimating these terms as in the proof of Theorem 4.2 and taking into account (4.6)-(4.7), we conclude that \( (u_i, \phi_i) \) is uniformly bounded in \( L^2(0,T;V^{-\gamma} \times W^{-2}) \). Passing now to a subsequence owing to compactness arguments, we infer the existence of \((u, \phi)\) satisfying (4.3). Now taking into account the weak convergence of \( A_{0i} \) to \( A_0 \), the rest of the proof proceeds similarly to that of Theorem 3.2.

For example, setting \( \varepsilon = 1 \), with \( \theta = 0 \) and \( \theta_2 = 1 \), and checking all the requirements (i)-(v) of Theorem 4.1, the viscous solutions to the 3D SBM-AC model converge to the inviscid solution as the viscosity \( \nu \) tends to zero. Recall that the global existence of a weak solution to the inviscid 3D SBM-AC model (see Table 1) is also contained in Theorem 3.2. Similarly, setting \( \varepsilon = 0 \), with \( \theta = 0 \) and \( \theta_2 = 1 \), the viscous solutions to the 3D Leray–AC-\( \alpha \) model converge to the inviscid solution as the viscosity tends to zero. This result gives another proof of the global existence of a weak solution for the inviscid \( (\nu = 0) \) 3D Leray–AC-\( \alpha \) model. Note that as in [44, Section 4, Theorem 4.1] where the inviscid problem for single-fluids was investigated, Theorem 4.1 establishes analogue global existence results for homogeneous two-phase Allen-Cahn flows without any viscosity terms. In particular, by Theorem 4.1 we recover the convergence results as \( \nu \to 0 \) for the global weak solutions of NSE-AC system in two space dimensions, which were previously reported in [77, 79].

### 4.2. Perturbations involving the nonlinear part of the flow component

We employ the same assumptions as in [44] Section 4. For \( i \in \mathbb{N} \), let \( A_{0i} : V^s \to V^{s-2\varepsilon} \) and \( N_i : V^s \to V^{s+2\varepsilon} \) be bounded linear operators, satisfying

\[
\|v\|^2_{\sigma + \theta_2} \lesssim \langle A_{0i} N_i^{-1} v, v \rangle + \|v\|^2_{\theta_2}, \quad v \in V^{\theta + \theta_2},
\]

and

\[
\|v\|^2_{\theta_2} \lesssim \langle N_i^{-1} v, v \rangle, \quad v \in V^{\theta_2},
\]

where we also assumed that \( N_i \) is invertible. In this subsection, we continue with perturbations of (4.1) of the form

\[
\begin{cases}
\partial_t u_i + A_{0i} u_i + B_{0i}(u_i, u_i) = g + R_0(A_1 \phi_i, \phi_i), \\
\partial_t \phi_i + A_1 \phi_i + B_1(u_i, \phi_i) + f(\phi_i) = 0, \quad (i \in \mathbb{N}),
\end{cases}
\]

where \( B_{0i} \) is some bilinear map. Again assuming that both problems 4.1 and 4.11 have the same initial condition \((u_0, \phi_0)\), and that \( A_{0i} \to A_0 \) and \( B_{0i} \to B_0 \) in some topology, we are concerned with the behavior of \((u_i, \phi_i)\) as \( i \to \infty \). For reference, define the trilinear form \( b_{0i}(u, v, w) = \langle B_{0i}(u, v), w \rangle \).

**Theorem 4.2.** Assume the above setting, and in addition let the following conditions hold.

i) \((u_0, \phi_0) \in Y_{\theta_2} \) with any \( \theta_2 \geq -1 \), and \( g \in L^2(0,T;V^{-\theta-\theta_2}) \), \( T > 0 \);

ii) \( b_{0i}(v, v, N_i v) = 0 \) for any \( v \in V \);

iii) \( b_{0i} : V^\sigma \times V^\sigma \times V^\sigma \to \mathbb{R} \) is uniformly bounded for some \( \sigma \in [-\theta_2, \theta + \theta_2 - 2\varepsilon_2] \), and \( \gamma \in [\theta + \varepsilon_2, \infty) \) \( \cap [\varepsilon_2, \infty) \) \( \cap (\theta, \infty) \);  

iv) \( A_{0i} : V^{\theta'-\theta_2} \to V^{-\gamma} \) is uniformly bounded and converges weakly to \( A_0 \);

v) \( N_i^{-1} : V^{s+2\varepsilon} \to V^{s+2\varepsilon-2\varepsilon_2} \) is uniformly bounded;

vi) \( N_i^{-1} N : V^{\theta-\theta_2} \to V^{\theta+\theta_2-2\varepsilon_2} \) converges strongly to the identity map;

vii) For any \( v \in V^{\theta-\theta_2} \), \( B_{0i}(v, v) \) converges weakly to \( B_0(v, v) \).

Then, there exists a solution

\[
(u, \phi) \in L^\infty(0,T;Y_{\theta_2}) \cap L^2(0,T;V^{\theta-\theta_2} \times D(A_1))
\]

to (4.1) such that, up to a subsequence, \( y_i = N_i^{-1} N_i u_i \) and \( \phi_i \) satisfy

\[
\begin{cases}
(y_i, \phi_i) \to (u, \phi) \text{ weak-star in } L^\infty(0,T;Y_{\theta_2}), \\
(y_i, \phi_i) \to (u, \phi) \text{ weakly in } L^2(0,T;V^{\theta-\theta_2} \times D(A_1)), \\
(y_i, \phi_i) \to (u, \phi) \text{ strongly in } L^2(0,T;V^s \times W^l), \\
\phi_i \to \phi \text{ strongly in } C(0,T;W^\zeta),
\end{cases}
\]

for any \( s < \theta - \theta_2 \), \( l < 2 \) and \( \zeta < 1 \), as \( i \to \infty \).

**Proof.** Recall that by Theorem 3.2 we know that for \( i \in \mathbb{N} \) there exists a solution to (4.11) with the following properties:

\[
(4.13)
\]

\[
(4.13)
\]

\[
(4.13)
\]
\( \phi_i \in L^\infty (0, T; L^\infty (\Omega)) \) with \( \phi_i \in [-1, 1] \),
\( \phi_i \in L^\infty (0, T; W^1) \cap L^2 (0, T; D (A_i)) \).

Pairing now the first and second equations of (4.11) with \( v_i := N_i u_i \) and \( \psi_i = A_1 \phi_i + f (\phi_i) \), respectively, after standard transformations, we have

\[
\frac{d}{dt} \left( \langle N_i^{-1} u_i, u_i \rangle + \left\| A_1^{1/2} \phi_i \right\|^2_{L^2} + 2 \int_\Omega F (\phi_i) \, dx \right) + 2 \langle A_{0i} N_i^{-1} u_i, u_i \rangle = \langle g, u_i \rangle \lesssim \delta^{-1} \| g \|_{H^{-\theta}}^2 + \delta \| v_i \|_{H^{\theta}}^2.
\]

Choosing \( \delta \in (0, 1) \) small enough, then using (4.10), by Grönwall’s inequality and (4.10) we have

\[
\| v_i (t) \|^2_{H^2} + \| \phi_i (t) \|_1 \lesssim C_\tau, \quad t \in (0, T).
\]

Moreover, integrating (4.14), and taking into account (4.12), (4.10) and (4.9), we infer that for any fixed \( T > 0 \),

\[
v_i = N_i u_i \in L^\infty (0, T; V^{\theta_2}) \cap L^2 (0, T; V^{\theta + \theta_2}),
\]
\( \phi_i \in L^\infty (0, T; W^1) \cap L^2 (0, T; D (A_i)) \),

with uniformly (in \( i \)) bounded norms. On the other hand, we again have

\[
\| u_i' \|_{-\gamma} \leq \| A_{0i} u_i \|_{-\gamma} + \| B_{0i} (u_i, u_i) \|_{-\gamma} + \| g + R_0 (A_i \phi_i, \phi_i) \|_{-\gamma}
\]

and

\[
\| \phi_i' \|_{-2} \leq \| A_1 \phi_i + f (\phi_i) \|^2_{L^2} + \| B_1 (u_i, \phi_i) \|_{-2}.
\]

By estimating the right hand sides as in the proof of Theorem 3.2 we conclude that \( (u_i', \phi_i') \) is uniformly bounded in \( L^2 (0, T; V^{-\gamma} \times W^{-2}) \); moreover, \( u_i' = N_i u_i \) is uniformly bounded in \( L^2 (0, T; V^{-\gamma}) \). Compactness arguments as in the proof of Theorem 3.2 allows us to pass to a subsequence, and, thus, infer the existence of

\[
v = N u \in L^\infty (0, T; V^{\theta_2}) \cap L^2 (0, T; V^{\theta + \theta_2}),
\]
\( \phi \in L^\infty (0, T; W^1 \cap L^\infty (\Omega)) \cap L^2 (0, T; D (A_1)) \)

satisfying

\[
v_i \rightarrow v \text{ weak-star in } L^\infty (0, T; V^{\theta_2}),
\]
\[
v_i \rightarrow v \text{ weakly in } L^2 (0, T; V^{\theta + \theta_2}),
\]
\[
v_i \rightarrow v \text{ strongly in } L^2 (0, T; V^s),
\]

for any \( s < \theta + \theta_2 \), and

\( \phi_i \rightarrow \phi \text{ weakly in } L^2 (0, T; D (A_1)), \phi_i \rightarrow \phi \text{ weak-star in } L^\infty (0, T; W^1 \cap L^\infty (\Omega)), \)

as \( i \rightarrow \infty \). Define \( u = N^{-1} v \) and \( y_i = N^{-1} v_i \), and note that these families satisfy (4.12).

We can now argue as in the proof of Theorem 3.2 to show that indeed the limit \((u, \phi)\) satisfies the problem (4.1). The procedure to pass to the limit is standard owing to the following identities

\[
\begin{cases}
    u_i - u = N_i^{-1} N y_i - u = N_i^{-1} N (y_i - u) + (N_i^{-1} N - I) u,
    \\
    A_{0i} u_i - A_0 u = A_{0i} (u_i - u) + (A_{0i} - A_0) u,
    \\
    B_{0i} (u_i, u_i) - B_0 (u, u) = B_{0i} (u_i, u_i - u) + B_{0i} (u_i - u, u) + B_{0i} (u, u) - B_0 (u, u),
\end{cases}
\]

the uniform boundedness and weak convergence of \( A_{0i} \rightarrow A_0, B_{0i} \rightarrow B_0 \), and assumptions (iv)-(vii), see [44, Theorem 4.2]. The proof is finished. \( \square \)

For example, setting \( \varepsilon = \varepsilon_2 = 1 \), with \( \theta = 1 \) and \( \theta_2 = 0 \), and checking all the requirements (i)-(vii) of Theorem 4.2 the weak solutions to the 3D NS–AC–\( \alpha \) model converge to a weak solution of the 3D NSE–AC model as the parameter \( \alpha \rightarrow 0 \). This result was previously reported in [35].
5. Longtime behavior

In this section we establish the existence of global and exponential attractors for the general three-parameter family of regularized models. Moreover, assuming the potential $F$ to be real analytic and under appropriate conditions on the external forces $g$ in (2.2), we also demonstrate that each trajectory converges to a single equilibrium, and find a convergence rate estimate. We recall that by Theorem 3.2 there exists a weak solution

$$(u, \phi) \in L^2_{loc}(0, \infty; \mathcal{Y}_{\theta_2}) \cap L^2_{loc}(0, \infty; V^{\theta - \theta_2} \times D(A_1))$$

to (2.1) with any given initial data $(u(0), \phi(0)) \in \mathcal{Y}_{\theta_2}$. By Theorem 3.4 the weak solution is unique and depends continuously on the initial data in a Lipschitz way. Therefore, we have a continuous (nonlinear) semigroup

$$(5.1)\quad S_{\theta_2}(t) : \mathcal{Y}_{\theta_2} \to \mathcal{Y}_{\theta_2}, \quad t \geq 0,$$

$$(u_0, \phi_0) \mapsto (u(t), \phi(t)).$$

Also for the sake of reference below, recall the following definition for the space of translation bounded functions

$$L^2_\alpha(\mathbb{R}^+; X) := \left\{ g \in L^2_{loc}(\mathbb{R}^+; X) : \|g\|_{L^2_\alpha(\mathbb{R}^+; X)}^2 := \sup_{t \geq 0} \int_0^{t+1} \|g(s)\|_X^2 \, ds < \infty \right\},$$

where $X$ is a given Banach space.

5.1. Global and exponential attractors in the case $\theta > 0$. The following proposition establishes the existence of an absorbing ball in $\mathcal{Y}_{\theta_2}$ in both cases $\theta > 0$ and $\theta = 0$. Moreover, with additional conditions in the case when $\theta > 0$, we show not only the existence of an absorbing ball in the space $V^{\theta} \times D(A_1)$, but also that any weak solution with initial condition in $\mathcal{Y}_{\theta_2}$ acquires additional smoothness in an infinitesimal time. For our first result it suffices to have $\theta \geq 0$.

**Proposition 5.1.** Let $(u, \phi) \in L^\infty_{loc}(0, \infty; \mathcal{Y}_{\theta_2}) \cap L^2_{loc}(0, \infty; V^{\theta - \theta_2} \times D(A_1))$ be a weak solution in the sense of Definition 2.4 with $(u(0), \phi(0)) \in \mathcal{Y}_{\theta_2}$. In addition, let the following conditions hold.

(i) $\langle A_0 v, N v \rangle \geq c \|v\|^2_{\theta - \theta_2}$ for any $v \in V^{\theta - \theta_2}$, with a constant $c > 0$;

(ii) $g \in L^2_\alpha(\mathbb{R}^+; V^{\theta - \theta_2})$.

Then for some constant $k > 0$ independent of time and initial conditions, we have

$$(5.2)\quad \|u(t)\|^2_{\theta_2} + \|\phi(t)\|^2 \leq \|u(0)\|^2_{\theta_2} + \|\phi(0)\|^2 + C_\ast,$$

for all $t \geq 0$, for some $C_\ast > 0$ independent of time and the initial data.

**Proof.** Let us now set

$$(5.3)\quad E(t) := \|u(t)\|^2_{\theta_2} + \|\phi(t)\|^2 \leq 2 (\|F(\phi(t))\|_{L^1} + \|\phi(t)\|_{L^2})^2 + c_E,$$

$$\Theta(t) := -2 \|u(t)\|^2_{\theta_2} + \|\mu(t)\|^2_{L^2} + \|\phi(t)\|^2_{L^2} + 2 \|\theta(t)\|^2_{\theta_2} - 2 \|\theta(t)\|^2_{\theta_2}$$

$$- 2 \|\phi(t)\|^2_{L^2} + 2 \kappa \|F(\phi(t)) - f(\phi(t))\|^2_{L^2} - (1 - \kappa) \|f(\phi(t))\|^2_{L^2},$$

where $\kappa \in (0, 1)$. In (5.3) we have set $c_E = 2C_F \text{vol}(\Omega)$, with $C_F$ taken large enough in order to ensure that $E(t)$ is nonnegative (note that $F$ is bounded from below). On account of this choice and recalling Proposition 3.1 we can find $C > 0$ such that

$$(5.4)\quad \|u(t)\|^2_{\theta_2} + \|\phi(t)\|^2 \leq E(t) \leq C \left( 1 + \|u(t)\|^2_{\theta_2} + \|\phi(t)\|^2 \right).$$

Following an argument from [33] Proposition 3.1, on account of (i) we have

$$(5.5)\quad \frac{d}{dt} E(t) + \kappa E(t) \leq \Theta(t) + \delta^{-1} \|g(t)\|^2_{\theta_2} + \delta \|N\|^2_{\theta_2} \|u(t)\|^2_{\theta_2},$$

for any $\delta > 0$. Observe preliminarily that, owing to the first assumption of (2.4), we have

$$(5.6)\quad |f(y)(1 + |y|)| \leq 2f(y) y + c_f,$$

$$(5.7)\quad F(y) - f(y) y \leq c'_f |y|^2 + c''_f,$$
for any \( y \in \mathbb{R} \). Here \( c_f, c'_f \) and \( c''_f \) are positive, sufficiently large constants that depend on \( f \) only. From (5.6)–(5.7) and elementary inequalities, it follows

\[
\Theta(t) \leq -(1 - \kappa c_\Omega |\Omega|) \|u(t)\|_{L^2}^2 - 2 \|\mu(t)\|_{L^2}^2 - (2 - \kappa) \|\nabla \phi(t)\|_{L^2}^2
\]

\[
- (2 - \kappa) (1 + 2c'_f) \|\phi(t)\|_{L^2}^2 - (1 - \kappa) \|f(\phi(t))\|_1 + 1 + |\phi(t)|_{L^2} + C,
\]

where \( c_\Omega \) depends on the shape of \( \Omega \), but not on its size, and \( C > 0 \) depends on \( \kappa, c_f \) and \( c''_f \) at most, but it is independent of time and the initial data. It is thus possible to adjust sufficiently small \( \kappa \in (0, 1) \) and \( \delta > 0 \), in order to have

\[
\frac{d}{dt} E(t) + \kappa E(t) + C \left( \|u(t)\|_{L^2}^2 + \|\nabla \phi(t)\|_{L^2}^2 + \|\phi(t)\|_{L^2}^2 \right) + 2 \|\mu(t)\|_{L^2}^2
\]

\[
+ C \|f(\phi(t))\|_1 + 1 + |\phi(t)|_{L^2} \leq C (1 + \|g(t)\|_{L^2}^2).
\]

Then, observing that assumption 2.2 implies that \( |F(y)| \leq |f(y)| (1 + |y|) + c_f \), for some positive constant \( c_f \) and all \( y \in \mathbb{R} \), and applying Gronwall’s inequality (see Appendix, Lemma 5.1), we deduce

\[
E(t) + \int_t^{t+1} \left( C \|u(s)\|_{L^2}^2 + 2 \|\mu(s)\|_{L^2}^2 + C \|F(y(s))\|_1 \right) ds
\]

\[
\leq E(0) e^{-\kappa t} + C(1 + \|g\|_{L^2(\mathbb{R}^N; L^2)}^2),
\]

for all \( t \geq 0 \). Taking (5.4) into account, by Proposition 5.1 we immediately obtain (5.2). This completes the proof. \( \square \)

Consequently, for any \( \theta \geq 0 \) we have the following proposition.

**Proposition 5.2.** For every \( R > 0 \), there exists \( C_* = C_* (R) > 0 \), independent of time, such that, for any \( \varphi_0 := (u_0, \phi_0) \in \mathcal{B}_y^{\theta_2} (R) \),

\[
\sup_{t \geq 0} \|S_{\theta_2} (t) \varphi_0\|_{\mathcal{Y}_\theta} + \int_t^{t+1} \left( \|u(s)\|_{L^2} + \|A_1 \phi(s)\|_{L^2} \right) ds \leq C_*,
\]

where \( \mathcal{B}_y^{\theta_2} (R) \) denotes the ball in \( \mathcal{Y}_\theta \) of radius \( R \), centered at 0.

The first main result of this subsection is the following.

**Theorem 5.3.** Let the assumptions of Theorems 3.3 and 3.4 be satisfied for some \( \theta > 0 \). In addition, for some

\[
\beta \in (-\theta_2, \min(\theta - \frac{1}{2}, \theta - \theta_2)],
\]

when \( n = 3 \), and

\[
\beta \in \left[ \max(1 - 2\theta_2, -\theta_2), \min(\theta, \theta - \theta_2) \right], \quad \beta \neq -\theta_2,
\]

when \( n = 2 \), provided that the above intervals are nonempty, let the following conditions hold.

(i) \( b_0 : V^\alpha \times V^\alpha \times V^{\theta - \beta} \to \mathbb{R} \) is bounded, where \( \alpha = \min\{\beta, \theta - \theta_2\} \);

(ii) \( g \in V^{\theta - \beta} \) is time independent.

Then, there exists a compact attractor \( \mathcal{A} \subseteq \mathcal{Y}_\theta \) for the system (2.18)–(2.19) which attracts the bounded sets of \( \mathcal{Y}_\theta \). Moreover, \( \mathcal{A} \) is connected and it is the maximal bounded attractor in \( V^\beta \times D (A_1) \).

**Proof.** By Propositions 5.1 and 3.1 there is a ball \( \mathcal{B} \) in \( \mathcal{Y}_\theta \) which is absorbing in \( \mathcal{Y}_\theta \), meaning that for any bounded set \( U \subset \mathcal{Y}_\theta \), there exists \( t_0 = t_0 (\|U\|_{\mathcal{Y}_\theta}) > 0 \) such that \( S_{\theta_2} (t) U \subset \mathcal{B} \) for all \( t \geq t_0 \). Moreover, by Theorem 3.9 and application of the uniform Gronwall’s lemma [75] Lemma III.1.1] in (3.14) and (3.50), by virtue of (5.2) (cf. also (5.10)), we infer the existence of a new time \( t_1 = t_0 + 1 \) such that

\[
\sup_{t \geq t_1} \left( \|u(t)\|_{L^2}^2 + \|A_1 (\phi(t))\|_{L^2}^2 \right) \leq C,
\]

for some positive constant \( C \) independent of time and the initial data. Moreover, integration over \((t, t+1)\) of the previous inequalities (3.14), (3.50) yields

\[
\sup_{t \geq t_1} \int_t^{t+1} \left( \|u(s)\|_{L^2}^2 + \|A_1^{3/2} (\phi(s))\|_{L^2}^2 \right) \leq C,
\]
owing once again to (5.14). Thus, for any bounded set $U \subset \mathcal{Y}_2$, we have that $\bigcup_{t \geq t_0} S_{\theta_2}(t)U$ is relatively compact in $\mathcal{Y}_2$, when $\mathcal{Y}_2$ is endowed with the metric topology of $V^{-\theta_2} \times W^1$. Finally, applying [25, Theorem I.1.1] we have that the set $\mathcal{A} = \cap_{s \geq 0} \bigcup_{t \geq s} S_{\theta_2}(t)\mathcal{B}$ is a compact attractor for $S_{\theta_2}$, and the rest of the result is immediate. \hfill \Box

Remark 5.4. (i) All the special cases listed in Table 1 (except for the 3D NSE-AC and 3D NSV-AC) satisfy the conditions of Theorem 5.3 when the space dimension is $n = 3$, cf. also Table 3. For instance, the uniform Gronwall’s lemma cannot be applied to (5.14) until $n = 3$, and so we cannot infer that (5.13) is satisfied in the case $\theta = 0$. The NSV-AC system ($\theta = 0$, $\theta_1 = \theta_2 = 1$) will be handled in the next subsection. Note that an absorbing set $\mathcal{B}$ in $\mathcal{Y}_2$ for problem (2.18) - (2.19) can be constructed for all $\theta \geq 0$ on account of Proposition 5.1. Indeed, there exists $R_0 > 0$ independent of time and initial data such that the ball $\mathcal{B} := \mathcal{B}_{\mathcal{Y}_2}(R_0)$ is absorbing for $S_{\theta_2}(t)$ on $\mathcal{Y}_2$.

(ii) We emphasize that estimate (5.13) is also satisfied provided that the external force $g$ is time dependent and $g \in L^2_b(\mathbb{R}^+; V^{(\theta)} \cap D(A_1))$. On account of this observation, one can generalize the notion of global attractor and replace it by the notion of pullback attractor, for example. One can still study the set of all complete bounded trajectories, that is, trajectories which are bounded for all $t \in \mathbb{R}$. All the results that we have presented in this section are still true in that case.

(iii) Note that (5.13) also implies that the dynamical system $(S_{\theta_2}(t), \mathcal{Y}_2)$ possesses a compact absorbing set $\mathcal{B}_\beta$ which is contained in $V^{\beta} \times D(A_1)$.

Next we show the existence of exponential attractors for our regularized family of models (2.2), when $\theta > 0$. It turns out that in order to successfully construct an exponential attractor for problem (2.2), we need to derive a compact absorbing set with a higher degree of smoothness than the set $\mathcal{B}_\beta \subset V^\beta \times D(A_1)$. As in [33], this feature is due to the strong coupling of the regularized NSE with the Allen-Cahn equation. Indeed, as we will see below it does not seem possible to get the exponential attractor directly on the smooth set $\mathcal{B}_\beta$. The next lemma is concerned with this issue.

Lemma 5.5. Let the assumptions of Theorem 5.3 be satisfied. In addition, for the same $\beta$ as in (5.11) - (5.12), and some value $\gamma \in \mathbb{R}$ satisfying

$$1 - \theta - 2\theta_2 \leq \gamma \leq \min \left(\theta - \frac{n}{6}, \theta - 1, \beta - \theta, 0\right),$$

provided that the above interval is nonempty, let the following conditions hold.

(i) $\partial_t g \in L^2_b(\mathbb{R}^+; V^{\gamma - \theta})$, $g \in L^\infty(\mathbb{R}^+; V^\gamma)$;

(ii) $b_0 : V^\alpha \times V^\alpha \times V^{\gamma - \theta} \to \mathbb{R}$ is bounded, where $\alpha = \min (\gamma, \beta + \theta)$.

Then, there exists a time $t_2 \geq t_1$ such that

$$\sup_{t \geq t_2} \left(\|u(t)\|_{2\theta + \gamma}^2 + ||A_1^{3/2}(\phi(t))||_{L^2_2}^2\right) \leq C,$$

for some positive constant $C$ independent of time and the initial data.

Proof. The following arguments are formal for the sake of simplicity, but they can be rigorously justified within the Galerkin scheme used in the proof of Theorem 5.2. We first observe that, using the apriori bounds (5.13), (5.14), and arguing exactly as in the proof of Theorem 3.6, (3.41) - (3.49), from the second equation of (2.2) by comparison, we have

$$\sup_{t \geq t_1} \left(\|\partial_t \phi(t)\|_{L^2_2}^2 + \int_t^{t+1} \|\partial_t \phi(s)\|_{L^2_2}^2 \, ds\right) \leq C. \tag{5.16}$$

Similarly, by comparison in the first equation of (2.2), we also have

$$\sup_{t \geq t_1} \left(\|\partial_t u(t)\|_{L^2_2}^2 + \int_t^{t+1} \|\partial_t u(s)\|_{L^2_2}^2 \, ds\right) \leq C, \tag{5.17}$$

owing to the assumptions on $A_0, g \in L^2_b(\mathbb{R}^+; V^{\gamma - \theta})$ and boundedness of the form $b_0$, see assumptions (i)-(ii) of Theorem 5.3.

Set $v := \partial_t u, \psi := \partial_t \phi$, and observe that $(v, \psi)$ solves the following system

$$\begin{cases}
\partial_t v + A_0 v + B_0 (v, u) + B_1 (u, v, \psi) = R_0 (A_1 \psi, \phi) + R_0 (A_1 \phi, \psi) + \partial_t g, \\
\partial_t \psi + B_1 (v, \phi) + B_1 (u, \psi) + A_1 \psi = -f'(\phi) \psi.
\end{cases} \tag{5.18}$$
We repeat the arguments from the proof of Theorem 3.10 and whenever necessary show the new estimates. Pairing the first and second equations of (5.18) with $\Lambda^{2\gamma}v$ and $A_1\psi$, respectively, then adding the identities together, we deduce

\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|v\|_{\gamma}^2 + \|A_1^{1/2} \psi\|_{L^2}^2 \right) + \langle A_0 v, \Lambda^{2\gamma} v \rangle + \|A_1 \psi\|_{L^2}^2 \\
= \langle \partial_t g, \Lambda^{2\gamma} v \rangle - b_0 (v, u, \Lambda^{2\gamma} v) - b_0 (u, v, \Lambda^{2\gamma} v) \\
+ \langle R_0 (A_1 \psi, \phi), \Lambda^{2\gamma} v \rangle + \langle R_0 (A_1 \phi, \psi), \Lambda^{2\gamma} v \rangle - \langle f' (\phi) \psi, A_1 \psi \rangle \\
- b_1 (v, \phi, A_1 \psi) - b_1 (u, \psi, A_1 \psi) .
\end{equation}

Some of the estimates on the right-hand side of (5.19) are easy. We have

\begin{equation}
\langle \partial_t g, \Lambda^{2\gamma} v \rangle \lesssim \delta^{-1} \|\partial_t g\|_{\gamma-q}^2 + \delta \|v\|_{\gamma+q}^2, \quad \text{for any } \delta > 0
\end{equation}

and

\begin{equation}
\langle f' (\phi) \psi, A_1 \psi \rangle \leq \delta \|A_1 \psi\|_{L^2}^2 + C\delta^{-1} \|\psi\|_1^2.
\end{equation}

To bound the last term we argue verbatim as in the proof of Theorem 3.10. We have

\begin{equation}
b_1 (u, \psi, A_1 \psi) \leq \delta \|A_1 \psi\|_{L^2}^2 + C\delta^{-1/(p-1)} \|u\|_{\theta-\theta_2}^2 \|u\|_{\psi}^{(p-2)} \|\psi\|_1^2,
\end{equation}
in two space dimensions, and

\begin{equation}
b_1 (u, \psi, A_1 \psi) \leq C\delta^{-7} \|u\|_{\theta-\theta_2}^6 \|u\|_{\psi}^2 \|\psi\|_1^2 + \delta \|A_1 \psi\|_{L^2}^2,
\end{equation}
when $n = 3$. In order to bound the last remaining terms in (5.19), we argue as follows. Using (3.11), the fact that $V^* \subset L^3$ and $\gamma - \theta \leq -n/6$, we have

\begin{equation}
\langle R_0 (A_1 \phi, \psi), \Lambda^{2\gamma} v \rangle \lesssim \|\Lambda^{2\gamma} v\|_{\theta-\gamma} \|R_0 (A_1 \phi, \psi)\|_{\gamma-\theta} \\
\lesssim \|v\|_{\theta-\gamma} \|A_1 \phi\|_{L^2} \|\nabla \psi\|_{L^2} \\
\lesssim \delta \|\nabla \psi\|_{\theta+\gamma}^2 + \delta^{-1} \|A_1^{3/2} \phi\|_{L^2}^2 \|\psi\|_1^2 .
\end{equation}

Next, from standard interpolation inequalities and (3.11), we infer

\begin{equation}
\langle R_0 (A_1 \psi, \phi), \Lambda^{2\gamma} v \rangle \lesssim \|\Lambda^{2\gamma} v\|_{\theta-\gamma} \|R_0 (A_1 \psi, \phi)\|_{\gamma-\theta} \\
\lesssim \|v\|_{\theta+\gamma} \|A_1^{3/2} \phi\|_{L^2} \\
\leq \delta \|\nabla \psi\|_{\theta+\gamma}^2 + C\delta^{-1} \|\psi\|_1^2 \|A_1^{3/2} \phi\|_{L^2}^2 ,
\end{equation}
thanks to the inequality

\begin{equation}
\|R_0 (A_1 \psi, \phi)\|_{\gamma-\theta} \leq \|R_0 (A_1 \psi, \phi)\|_{-1} \lesssim \|\psi\|_1 \|A_1^{3/2} \phi\|_{L^2}.
\end{equation}

Finally, we can argue as above in (5.24) - (5.26) to obtain, owing to the boundedness of the map $N : V^s \to V^{s+2\theta_2}$,

\begin{equation}
b_1 (v, \phi, A_1 \psi) = \langle R_0 (A_1 \psi, \phi), N v \rangle \\
\lesssim \|N v\|_{\gamma+\theta+2\theta_2} \|R_0 (A_1 \psi, \phi)\|_{-(\gamma+\theta+2\theta_2)} \\
\lesssim \|v\|_{\theta+\gamma} \|R_0 (A_1 \psi, \phi)\|_{-1} \\
\leq \delta \|v\|_{\gamma+\theta}^2 + C\delta^{-1} \|A_1^{3/2} \phi\|_{L^2}^2 .
\end{equation}

The assumption (ii) on $b_0$ is enough to bound the following terms:

\begin{equation}
b_0 (v, u, \Lambda^{2\gamma} v) + b_0 (u, v, \Lambda^{2\gamma} v) \\
\lesssim \|v\|_{\gamma} \|u\|_{\theta+\gamma} \|\Lambda^{2\gamma} v\|_{\theta-\gamma} \\
\leq \delta \|v\|_{\gamma+\theta}^2 + C\delta^{-1} \|A_1^{3/2} \phi\|_{L^2}^2 .
\end{equation}

Setting

\begin{equation}
Z (t) := \|v (t)\|_{\gamma}^2 + \|\psi (t)\|_1^2 ,
\end{equation}

and inserting all the previous estimates \((5.20)-\(5.27\)) into \((5.14)\), then using the coercivity assumption on \(A_0\) and choosing a sufficiently small \(\delta \sim \min(c_{A_0}, c_{A_1}) > 0\), we obtain
\[
\frac{d}{dt} Z(t) + \|v(t)\|_{\gamma+\theta}^2 + \|A_1 \psi(t)\|_{L_2}^2 \leq \Delta(t) Z(t) + C \|\partial_t g\|_{\gamma-\theta}^2,
\]
where the function \(\Delta \in L^1(t, t+1)\) is
\[
\Delta := C_\delta \left(1 + \|\varphi_1^{3/2}\|_{L^2} + \|u\|_{\theta-\theta}^2 + \|u\|_{\beta+\theta}^2\),
\]
for some \(C_\delta > 0\). Observe now that, on account of \((5.13)-(5.14)\) and \((5.16)-(5.17)\), we have
\[
\sup_{t \geq t_1} \int_{t}^{t+1} \Delta(s) \, ds \leq C, \quad \sup_{t \geq t_1} \int_{t}^{t+1} Z(s) \, ds \leq C.
\]
Hence, we can exploit the uniform Gronwall’s lemma [75, Lemma III.1.1] in \((5.29)\) to infer the existence of a new time \(t_2 \geq t_1\) such that
\[
\sup_{t \geq t_2} \|\partial_t u(t)\|_{\gamma} + \|\partial_t \phi(t)\|_{1} \leq C,
\]
for some positive constant \(C\) independent of time and the initial data. Comparison in the second equation of \((2.2)\) yields the desired bound for \(\phi\) in \((5.14)\) owing once more to \((5.13)\) and \((5.30)\).

Finally, from the first equation of \((2.2)\), we have
\[
\|u\|_{\theta+\gamma} \approx \|A_0 u\|_{\gamma} \lesssim \|g\|_{\gamma} + \|B_0(u,v)\|_{\beta-\theta} + ||R_0(A_1,\phi,\phi)\|_{\beta-\theta},
\]
since \(\gamma \leq \beta - \theta\). Owing to the assumptions on \(\beta\), and on \(b_0\) in the statement of Theorem \(5.3\), each term on the right-hand side is essentially bounded for times greater than \(t_2\). Estimate \((5.31)\) entails the desired estimate for \(u\) from \((5.13)\), and the proof is finished.

The next lemma is concerned with the Lipschitz-in-time regularity of the semigroup \(S_{\theta_2}(t)\).

**Lemma 5.6.** Let the assumptions of Lemma \(5.5\) be satisfied. For any \(R > 0\), there exists a time \(t_1 = t_1(R) > 0\), such that
\[
\|S_{\theta_2}(t) \varphi_0 - S_{\theta_2}(\tilde{t}) \varphi_0\|_{W_{\gamma+1}} \leq C|t - \tilde{t}|,
\]
for all \(t, \tilde{t} \in [t_1, \infty)\) and any \(\varphi_0 = (u_0, \varphi_0) \in B_{\theta,\gamma}(R) \subset V^{2\theta+\gamma} \times D(A_1^{3/2})\). Here \(B_{\theta,\gamma}(R)\) denotes any ball in \(V^{2\theta+\gamma} \times D(A_1^{3/2})\) of radius \(R > 0\), centered at \(0\).

**Proof.** The claim \((5.32)\) follows from the basic equality
\[
S_{\theta_2}(t) \varphi_0 - S_{\theta_2}(\tilde{t}) \varphi_0 = \int_{t}^{\tilde{t}} \partial_t (S_{\theta_2}(y) \varphi_0) \, dy.
\]
and estimate \((5.30)\).

With the essential Lemma \(5.5\), the next result states the validity of the smoothing property for the semigroup \(S_{\theta_2}(t)\) in the case \(\theta > 0\).

**Lemma 5.7.** Let the assumptions of Lemma \(5.5\) be satisfied. Indicate by \((u_i, \phi_i)\) the solution to problem \((2.2)\) which corresponds to the initial data \((u_i(0), \phi_i(0)) \in \mathcal{B}\) where \(i = 1, 2\). Then the following estimate holds:
\[
\|u_1(t) - u_2(t)\|_{\beta}^2 + \|A_1(\phi_1(t) - \phi_2(t))\|_{L_2}^2 \leq C \frac{T + 1}{T} e^{Ct} \left(\|u_1(0) - u_2(0)\|_{\theta-\beta}^2 + \|\phi_1(0) - \phi_2(0)\|_{1}^2\right),
\]
for all \(T := t - t_2 > 0\), and some positive constant \(C\) which only depends on \(\mathcal{B}\).

**Proof.** First, recall that by \((5.14)\), there is a compact absorbing set for \(S_{\theta_2}(t)\) contained in \(V^{2\theta+\gamma} \times D(A_1^{3/2})\).

As before, let \(v = u_1 - u_2\) and \(\psi = \phi_1 - \phi_2\) and recall that \((v, \psi)\) solves equations \((5.23)\) and \((5.24)\). Taking \(w = A^{2\beta} v\) and \(\eta = A_1^{3/2} \psi\) into \((5.23)\), \((5.24)\), respectively, we infer
\[
\frac{1}{2} \frac{d}{dt} \left(\|v\|_{\beta}^2 + \|A_1 v\|_{L_2}^2\right) + c_{A_0} \|v\|_{\theta+\beta}^2 + \|A_1^{3/2} v\|_{L_2}^2 \leq b_0(v, u_1, A^{2\beta} v) + b_0(u_2, v, 1) + \langle R_0(A_1, \phi_2, \psi), A^{2\beta} v\rangle.
\]
In order to bound the last two terms on the right-hand side of (5.34), we repeat the same estimates
\begin{equation}
(5.39)
\end{equation}
\begin{equation}
(5.36)
\end{equation}
\begin{equation}
(5.35)
\end{equation}
when \( n = 3 \), and in two space dimensions,
\begin{equation}
(5.37)
\end{equation}
\begin{equation}
(5.38)
\end{equation}
in both two and three space dimensions, respectively. Moreover, it easy to see that
\begin{equation}
(5.39)
\end{equation}
\begin{equation}
(5.40)
\end{equation}
\begin{equation}
(5.41)
\end{equation}
when \( n = 3 \), and
\begin{equation}
(5.42)
\end{equation}
in two space dimensions \( (n = 2) \). Let us now set
\begin{equation}
(5.43)
\end{equation}
with the obvious definition for \( \Xi \). We emphasize that in both dimensions \( n = 2, 3 \), the function
\( \Xi \in L^\infty (t_2, \infty) \), thanks now to the uniform estimates (5.12), (5.13) and (5.15). Multiplying now
both sides of this inequality by \( T := t - t_2 \) and integrating the resulting relation over \((t_2, t)\), we get

\[
\bar{t} X (t) \leq C \int_{t_2}^{t} (s - t_2 + 1) X (s) \, ds, \quad \text{for all } t > t_2,
\]

which entails \( (5.33) \), owing to Theorem 3.4, \( (3.22) \). The proof is complete.

The second main result of this subsection is concerned with the existence of exponential attractors for problem \( (2.2) \) in the case \( \theta > 0 \).

**Theorem 5.8.** Let the assumptions of Lemma 5.5 be satisfied, and assume \( g \) is time independent. Then \((S_{\theta_2}, \mathcal{Y}_{\theta_2})\) possesses an exponential attractor \( \mathcal{M}_{\theta_2} \subset \mathcal{Y}_{\theta_2} \) which is bounded in \( V^{2\theta + \gamma} \times D(A_1^{3/2}) \). Thus, by definition, we have

(a) \( \mathcal{M}_{\theta_2} \) is compact and semi-invariant with respect \( S_{\theta_2} (t) \), that is,

\[
S_{\theta_2} (t) (\mathcal{M}_{\theta_2}) \subseteq \mathcal{M}_{\theta_2}, \quad \forall t \geq 0.
\]

(b) The fractal dimension \( \dim_F (\mathcal{M}_{\theta_2}, \mathcal{Y}_{\theta_2}) \) of \( \mathcal{M}_{\theta_2} \) is finite and an upper bound can be computed explicitly.

(c) \( \mathcal{M}_{\theta_2} \) attracts exponentially fast any bounded subset \( B \) of \( \mathcal{Y}_{\theta_2} \), that is, there exist a positive nondecreasing function \( Q \) and a constant \( \rho > 0 \) such that

\[
dist_{\mathcal{Y}_{\theta_2}} (S_{\theta_2} (t) B, \mathcal{M}_{\theta_2}) \leq Q(\|B\|_{\mathcal{Y}_{\theta_2}}) e^{-\rho t}, \quad \forall t \geq 0.
\]

Here \( \text{dist}_{\mathcal{Y}_{\theta_2}} \) denotes the Hausdorff semi-distance between sets in \( \mathcal{Y}_{\theta_2} \) and \( \|B\|_{\mathcal{Y}_{\theta_2}} \) stands for the size of \( B \) in \( \mathcal{Y}_{\theta_2} \). Both \( Q \) and \( \rho \) can be explicitly calculated.

**Proof.** Using Theorem \( 5.2 \), Theorem \( 5.3 \) and Lemma \( 5.5 \) we can find a bounded subset \( X_0 \) of \( V^{2\theta + \gamma} \times D(A_1^{3/2}) \) and a time \( t' > 0 \) such that, setting \( \Sigma = S_{\theta_2} (t') \), the mapping \( \Sigma : X_0 \to X_0 \) enjoys the smoothing property \( (5.33) \). Therefore Theorem \( 5.8 \) applies to \( \Sigma \) and there exists a compact set \( \mathcal{M}_{\theta_2} \subset X_0 \) with finite fractal dimension (with respect to the metric topology of \( V^{\gamma} \times W^1 \)) that satisfies \( (5.32) \) and \( (5.33) \). Hence, setting

\[
\mathcal{M}_{\theta_2} = \bigcup_{t \in [t', 2t']} S_{\theta_2} (t) \mathcal{M}_{\theta_2}^*,
\]

we deduce that (a) and (c) are fulfilled, while (b) is a consequence of Theorem \( 5.3 \) and \( (5.32) \). The attraction property (c) in the metric topology of \( V^{-\theta_2} \times W^1 \) is a standard corollary of the aforementioned properties, standard interpolation inequalities and the fact that \( \mathcal{M}_{\theta_2} \) is bounded in \( V^{2\theta + \gamma} \times D(A_1^{3/2}) \).

As a consequence of the above theorem, we have the following.

**Corollary 5.9.** Under the assumptions of Theorem 5.8 the global attractor \( \mathcal{A} \) is bounded in \( V^{2\theta + \gamma} \times D(A_1^{3/2}) \), and \( \mathcal{A} \) has finite fractal dimension.

Note that Theorem \( 5.3 \) provides many examples where the conditions of the Lemma 5.5 are satisfied (with \( \beta = \theta - \theta_2 \) and \( \gamma = \beta = \theta = -\theta_2 \)). For example, setting \( \theta_1 = \theta_2 = 0 \), with \( \theta = 1 \), and checking all the requirements (i)-(ii) of Theorem \( 5.3 \) and conditions (i)-(ii) of Lemma \( 5.5 \) the 2D NSE-AC system possesses an exponential attractor bounded in \( V^2 \times W^3 \). This result was previously reported in \( [33] \). Another example covered by the assumptions of Theorem 5.8 is the 3D NS-AC-\( \alpha \) system \( (\theta_1 = \theta_2 = \theta = 1) \) treated previously in \( [63] \).

**Remark 5.10.** The above results can be used to recover estimates on the dimension of the global attractor \( \mathcal{A} \) for the generalized model \( (2.2) \), through the application of the classical machinery previously used for the special case of the 2D NSE-AC system, see \( [33] \) Section 4. This is a somewhat long calculation that we do not include here.

### 5.2. Global and exponential attractors in the case \( \theta = 0 \)

In this subsection, we consider non-dissipative systems, which are represented in our generalized model \( (2.2) \) when \( \theta = 0 \). This is the case of the Navier-Stokes-Voight equation which can be seen as an inviscid regularization of the Navier-Stokes equation. The parabolic character of the NS equation is lost; indeed, the semigroup generated by the NSV equation is only asymptotically compact \( [53] [54] \). In this sense, NSV behaves more like a damped hyperbolic system, and the same is true for the NSV-AC system and all other systems when \( \theta = 0 \) (cf. also Remark 5.4).
Our first task is to prove that the evolution system under consideration possesses a global attractor, bounded in the energy phase space $\mathcal{Y}_{\theta_2}$, under rather general conditions on $\theta_1 \in \mathbb{R}$, $\theta_2 \geq 1$. The analogue of Theorem 5.3 in the case $\theta = 0$ is as follows.

**Theorem 5.11.** Let the assumptions of Theorems 3.2 and 3.4 be satisfied for $\theta = 0$. For $s \in (\frac{1}{2}, 1]$, define

$$I_s := (-\infty, 0] \cap (-\infty, 2s - \frac{n}{2}) \cap [2s - 1 - 2\theta_2, \infty) \cap (\frac{n}{2} + 2s - 2 - 2\theta_2, \infty)$$

such that $I_s \neq \emptyset$. In addition, for some $\beta$

$$\beta \in I_s := [\max (1 - 2\theta_2, -\theta_2), 0] \cap I_s$$

when $n = 2$, and for some $\beta$

$$\beta \in I_s := (-\theta_2, -\frac{1}{2}) \cap I_s$$

when $n = 3$, provided that $I_s \neq \emptyset$, let the following conditions hold.

(i) $\langle A_0 v, (I - \Delta)^\beta v \rangle \geq c_{A_0} \langle v \rangle^2_{\theta_2}$ for any $v \in V^{\theta_2}$, with a constant $c_{A_0} > 0$;

(ii) $\langle A_0 v, (I - \Delta)^\beta v \rangle \geq c_{A_0} \langle v \rangle^2_{\theta_2}$, for any $v \in V^{\beta}$, for some $c_{A_0} > 0$;

(iii) $b_0(v, w, Nw) = b_1(v, \phi, \phi) = 0$, for any $v, w \in V$, $\phi \in W$;

(iv) $b_0 : V^{\theta_2} \times V^{\beta} \rightarrow \mathbb{R}$ is bounded;

(v) $g \in V^{\beta}$ is time independent.

Then, there exists a compact attractor $A_0 \subset \mathcal{Y}_{\theta_2}$, for the system (2.13)-(2.19) which attracts the bounded sets of $\mathcal{Y}_{\theta_2}$. Moreover, $A_0$ is connected and it is the maximal bounded attractor in $\mathcal{Y}_{\theta_2}$.

**Proof.** For instance, we follow our argument devised in [36] for the Navier-Stokes-Voight equation with memory. On account of (5.1) and Remark 5.4.1 let us take a fixed $\varphi_0 := (u_0, \phi_0) \in B (B$ is the absorbing set derived from Proposition 5.1) and consider the corresponding trajectory $(u(t), \phi(t)) = S_{\theta_2}(t) \varphi_0$. Recall that, by Proposition 5.2, we have

$$\sup_{\varphi_0 \in B} \| S_{\theta_2}(t) \varphi_0 \|_{\mathcal{Y}_{\theta_2}}^2 \leq C,$$

$$\sup_{\varphi_0 \in B} \int_0^t \| A_1 \phi(s) \|_{L^2}^2 ds \leq C (1 + t - s),$$

for all $t > s \geq 0$, with a constant $C > 0$ independent of time and initial data. Moreover, in this proof the generic constant $C > 0$ depends only on $B$, $g$ (see Remark 5.4.1 (i)) and other physical parameters of the problem. We split a given trajectory $(u(t), \phi(t))$ as follows:

$$(u(t), \phi(t)) = (u_d(t), \phi_d(t)) + (u_c(t), \phi_c(t)),$$

where

$$\begin{align*}
\partial_t u_d + A_0 u_d + B_0(u, u_d) &= R_0 (A_1 \phi_d, \phi_d), \\
\partial_t \phi_d + B_1 (u_d, \phi_d) + A_1 \phi_d &= 0, \\
u_d(0) &= u_0, \phi_d(0) = \phi_0,
\end{align*}$$

and

$$\begin{align*}
\partial_t u_c + A_0 u_c + B_0(u, u_c) &= R_0 (A_1 \phi_c, \phi_c) + R_0 (A_1 \phi_c, \phi_d) + g, \\
\partial_t \phi_c + B_1 (u_c, \phi_c) + B_1 (u_c, \phi_d) + A_1 \phi_c &= -f(\phi), \\
u_c(0) &= 0, \phi_c(0) = 0.
\end{align*}$$

Let us first show that $(u_d(t), \phi_d(t))$ decays exponentially to zero with respect to the norm of $V^{\theta_2} \times W^1$. We begin by noting that, since $(u, \phi) \in L^\infty(0, \infty; \mathcal{Y}_{\theta_2})$ we can easily adapt the proof of Proposition 5.1 (see also (5.46)), to find

$$\sup_{\varphi_0 \in B} \| (u_d(t), \phi_d(t)) \|_{\mathcal{Y}_{\theta_2}}^2 + \int_0^\infty \left( \| A_1 \phi_d(s) \|_{L^2}^2 + \| u_d(s) \|_{\mathcal{Y}_{\theta_2}}^2 \right) ds \leq C,$$

for all $t \geq 0$, which implies on account of (5.28) and (5.47), that

$$\sup_{\varphi_0 \in B} \| (u_c(t), \phi_c(t)) \|_{\mathcal{Y}_{\theta_2}}^2 \leq C,$$

$$\sup_{\varphi_0 \in B} \int_s^t \| A_1 \phi_c(s) \|_{L^2}^2 ds \leq C (1 + t - s),$$

for all $t > s \geq 0$, where $C > 0$ is obviously independent of time and initial data (cf. also Proposition 5.2 and (5.46)).
Let us now consider the functional
\[ E_d(t) := \| u_d(t) \|_{\theta_2}^2 + \| A_1^{1/2} \phi_d(t) \|_{L^2}^2. \]

We now pair the first and second equations of (5.48) with \( Nu_d \) and \( A_1 \phi_d \), respectively. Adding together the resulting relations, on account of assumptions (i)-(ii), we easily derive
\[
\frac{d}{dt} E_d(t) + 2 \left( c_{A_0} \| u_d(t) \|_{\theta_2}^2 + \| A_1 \phi_d(t) \|_{L^2}^2 \right) \leq 0.
\]

Thus, applying a suitable Gronwall’s inequality (see Appendix, Lemma 8.1) to (5.52), we obtain
\[
E_d(t) \leq E_d(0) e^{-\min\{c_{A_1}, c_{A_0}\} t}, \quad \text{for all } t \geq 0.
\]

This estimate gives the desired exponential decay of \( (u_d, \phi_d) \) in the norm of \( V^{-\theta_2} \times W^1 \).

Let us now obtain a bound for \( \| (u_c, \phi_c) \|_{Y_{\beta, s}} \). To this end, we argue as in the proof of Theorem 5.6, namely, we pair the first and second equations of (5.49) with \( \Lambda^{2\beta} u_c \) and \( A_1^{2\beta} \phi_c \), respectively, for \( s \in (\frac{1}{2}, 1] \), and then we add the resulting relationships. The analogue of (5.44)-(5.50) for (5.49) is the following identity:
\[
\frac{d}{dt} \left( \| u_c \|_{L^2}^2 + \| A_1^{1/2} \phi_c \|_{L^2}^2 \right) + 2 \| A_1^{1/2+1/2} \phi_c \|_{L^2}^2 + 2 \langle A_0 u_c, \Lambda^{2\beta} u_c \rangle \\
= -2 b_0 (u, u_c, \Lambda^{2\beta} u_c) - 2 \langle A_1^{1/2-1/2} f(\phi), A_1^{1/2+1/2} \phi_c \rangle + 2 \langle g, \Lambda^{2\beta} u_c \rangle \\
+ 2 \langle R_0 (A_1 \phi, \phi_c), \Lambda^{2\beta} u_c \rangle + 2 \langle R_0 (A_1 \phi, \phi_c), \Lambda^{2\beta} \phi_c \rangle \\
- 2 \langle A_1^{1/2-1/2} B_1 (u, \phi_c), A_1^{1/2+1/2} \phi_c \rangle - 2 \langle A_1^{1/2-1/2} B_1 (u_c, \phi_d), A_1^{1/2+1/2} \phi_c \rangle.
\]

Note that, using the fact that \( (u, \phi) \in L^\infty (0, \infty; Y_{\theta_2}) \), we can once again find a positive constant \( C_\delta \sim 1/\delta \) such that
\[
\langle A_1^{1/2-1/2} f(\phi), A_1^{1/2+1/2} \phi_c \rangle \leq \delta \| A_1^{1/2+1/2} \phi_c \|_{L^2}^2 + C_\delta^{-1} \| \phi \|_{L^2}^2, \quad \text{for any } \delta > 0.
\]

Moreover, by assumptions (iv)-(v) we have
\[
2 \langle B_0 (u, u_c), \Lambda^{2\beta} u_c \rangle \leq \| u \|_{\theta_2} \| u_c \|_{\beta} \| A^{2\beta} u_c \|_{-\beta} \leq \delta^{-1} \| u \|_{\theta_2}^2 \| u_c \|_{\beta}^2 + \| u_c \|_{\beta}^2,
\]

for any \( \delta > 0 \). In order to bound the last four terms on the right-hand side of (5.54), we argue as follows. By Lemma 8.3 the bilinear mapping \( B_1 : V^{\theta_2} \times V^{2s-\varepsilon} \rightarrow W^{2s-1} \) is continuous provided that \( \theta_2 \geq 1 \), for any \( s \in (\frac{1}{2}, 1], n = 2, 3 \), and some \( \varepsilon \in (0, \frac{1}{2}) \). Moreover, by the Sobolev inequality \( \| \cdot \|_{2s+1-\varepsilon} \leq \| \cdot \|_{2s+1} \| \cdot \|_{2s} \), we have
\[
2 \langle A_1^{1/2-1/2} B_1 (u, \phi_c), A_1^{1/2+1/2} \phi_c \rangle \leq \| A_1^{1/2+1/2} \phi_c \|_{L^2}^2 \| B_1 (u, \phi_c) \|_{2s-1} \leq \| N u \|_{\theta_2} \| \nabla \phi_c \|_{2s-\varepsilon} \| A_1^{1/2+1/2} \phi_c \|_{L^2} \leq \| u \|_{\theta_2} \| \phi_c \|_{2s+1}^2 \| \phi_c \|_{2s}^\varepsilon \| A_1^{1/2} \phi_c \|_{L^2}^2 \leq \delta \| A_1^{1/2+1/2} \phi_c \|_{L^2}^2 + C_\delta^{-1} \| u_c \|_{\beta}^2 \| A_1 \phi \|_{L^2}^2.
\]

Similarly, the mapping \( R_0 \) is continuous from \( W^{2s-1} \times V^1 \rightarrow V^\beta \), and from \( W^0 \times V^{2s} \rightarrow V^\beta \), for as long as \( \beta \in I_s \neq \emptyset \) and \( s \in (\frac{1}{2}, 1] \). Henceforth, we have the following bounds
\[
2 \langle R_0 (A_1 \phi, \phi_c), \Lambda^{2\beta} u_c \rangle \leq \| A^{2\beta} u_c \|_{-\beta} \| R_0 (A_1 \phi, \phi_c) \|_{\beta} \leq \delta \| A_1^{1/2+1/2} \phi_c \|_{L^2}^2 + C_\delta^{-1} \| u_c \|_{\beta}^2 \| A_1 \phi \|_{L^2}^2
\]

and
\[
2 \langle R_0 (A_1 \phi_c, \phi_d), \Lambda^{2\beta} u_c \rangle \leq \| u_c \|_{\beta} \| R_0 (A_1 \phi_c, \phi_d) \|_{\beta} \leq C_\delta^{-1} \| u_c \|_{\beta}^2 \| A_1 \phi_d \|_{L^2}^2 + \delta \| A_1^{1/2+1/2} \phi_c \|_{L^2}^2.
\]
Finally, for the last term we exploit Lemma \[5.33\] once more, and observe that for \(\beta \in \mathbb{I}_s\), the continuity of the mapping \(B_1 : V^{\beta+2\theta_2} \times V^1 \rightarrow W^{2s-1}\) and the boundedness of the map \(N : V^{\beta} \rightarrow V^{\beta+2\theta_2}\), yields

\[
(5.60) \quad \left\langle A_1^{(2s-1)/2}B_1(u_c, \phi_d), A_1^{(2s+1)/2}\phi_c \right\rangle \lesssim \|B_1(u_c, \phi_d)\|_{2s-1} \|A_1^{(2s+1)/2}\phi_c\|_{L^2} \\
\lesssim \|Nu_c\|_{\beta+2\theta_2} \|\nabla \phi_d\|_1 \|A_1^{(2s+1)/2}\phi_c\|_{L^2} \\
\leq \delta \|A_1^{(2s+1)/2}\phi_c\|_{L^2}^2 + C\delta^{-1} \|u_c\|_{\beta}^2 \|A_1\phi_d\|_{L^2}^2.
\]

Therefore, by setting

\[
E_c(t) := \|u_c(t)\|_{\beta}^2 + \|A_1^{1}\phi_c(t)\|_{L^2}^2,
\]

\[
\Xi_c(t) := C\delta \left(1 + \|u(t)\|_{\theta_2}^2 + \|A_1\phi(t)\|_{L^2}^2 + \|A_1\phi_d(t)\|_{L^2}^2\right),
\]
on account of \((5.54)\), we can choose a sufficiently small \(\delta = (1/8) \min (c_{A_1}, c_{A_2}) > 0\) to deduce

\[
(5.61) \quad \frac{d}{dt} E_c(t) + C \left(\|A_1^{(2s+1)/2}\phi_c(t)\|_{L^2}^2 + \|u_c(t)\|_{\beta}^2\right) \\
\leq \Xi_c(t) E_c(t) + C\delta \left(\|u\|_{\beta}^2 + \|\phi(t)\|_{L^2}^2\right),
\]

for all \(t \geq 0\). Next, integrate this relation over \((0, t)\) and note that \(E_c(0) = 0\). Hence, exploiting \((5.2), (5.10), (5.46), (5.50), (5.51)\), from the application of Gronwall’s lemma we obtain

\[
(5.62) \quad \|u_c(t)\|_{\beta}^2 + \|A_1^{1}\phi_c(t)\|_{L^2}^2 \lesssim e^{C(1+t)}, \quad \text{for all } t \geq 0.
\]

Finally, integrating \((5.61)\) once more between \(0\) and \(t\), owing to \((5.62)\) we also find the estimate

\[
(5.63) \quad \sup_{\varphi_0 \in B} \int_0^t \left\|A_1^{(2s+1)/2}\phi_c(s)\right\|_{L^2}^2 ds \lesssim e^{C(1+t)}, \quad \text{for all } t \geq 0.
\]

In particular, for every fixed \(T > 0\), we have found a compact subset \(V^{\beta} \times D(A_1) \subset V^{-\theta_2} \times W^1\) such that the mapping \(S_{\theta_2}^c(t) \varphi_0 := (u_c(t), \phi_c(t))\) satisfies

\[
(5.64) \quad \bigcup_{\varphi_0 \in B} S_{\theta_2}^c(t) \varphi_0 \subset V^{\beta} \times D(A_1), \quad \text{for all } t \in [0, T].
\]

This fact, together with the exponential decay \((5.60)\), implies that \(S_{\theta_2}(t) : \mathcal{Y}_{\theta_2} \rightarrow \mathcal{Y}_{\theta_2}\) is asymptotically smooth for \(t \geq 0\). The existence of the global attractor follows by means of standard methods in the theory of dynamical systems (see, for instance, \([10]\) Theorem 3.4.6). The proof is finished.

Although Theorem \[5.11\] yields the global attractor, no conclusion can be drawn at this stage about its optimal regularity. Ideally, one would like to directly check that \(A_0\) is bounded in \(V^{\beta} \times D(A_1)\) as in the statement of Theorem \[5.3\]. On the contrary, the proof of the above theorem seems to suggest that this is generally a much harder task, if not out of reach in just one step. In order to show that \(S_{\theta_2}(t)\) enjoys a stronger dissipativity property, we shall employ another semigroup decomposition, which is much more complicated than the one in \((5.38) - (5.40)\). This step is also crucial in order to demonstrate the existence of an exponential attractor below.

Our second result establishes the existence of a bounded exponentially attracting \(C_{\beta,s}\) set in \(\mathcal{V}_{\beta,s} := V^{\beta} \times D(A_1)\).

**Theorem 5.12.** Let the assumptions of Theorems \[3.2\] and \[3.4\] be satisfied for \(\theta = 0\), and assume that the conditions (i), (iii) of Theorem \[5.11\] are also satisfied. Fix \(s \in \left(\frac{1}{2}, 1\right) \cap \left(0, \frac{\theta_2}{2} - \frac{n}{4} + 1\right)\) and consider the nonempty interval

\[
\mathcal{K}_s := \mathcal{J}_s \cap \left(-\infty, \frac{1}{2} - \frac{\theta_2}{2} - \frac{n}{4}\right) \cap \left(-\infty, s - \frac{1}{2} - \frac{\theta_2}{2}\right).
\]

Suppose now that the conditions (ii), (v) of Theorem \[5.11\] hold for some \(\beta \in \mathcal{K}_s\), and the slightly stronger condition:

\(\text{(vi)} b_0 : V^{-\theta_2} \times V^{-\theta_2} \times V^{\beta} \rightarrow \mathbb{R}\) is bounded;
There exists $R_1 > 0$, and a closed ball $B_{\beta,s} (R_1) \subset V_{\beta,s} \cap \mathcal{Y}_{\theta}$ which attracts $\mathcal{B}$ exponentially fast, that is,

\begin{equation}
\text{dist}_{V^{-\theta_2} \times W^1} \left( S_{\theta_2} (t) \mathcal{B}, B_{\beta,s} (R_1) \right) \leq C e^{-\rho t}, \quad \text{for all } t \geq 0,
\end{equation}

for some positive constants $C$ and $\rho$ independent of time. Here $\text{dist}_{V^{-\theta_2} \times W^1}$ denotes the non-symmetric Hausdorff distance in $V^{-\theta_2} \times W^1$.

**Proof.** The main steps require nothing more than what is already contained in the proof of Theorem 5.11. Once again, we will rely on the semigroup decomposition developed there, and the corresponding estimates. First, we employ another semigroup decomposition and adopt a strategy devised in [21]. For $\varphi_0 \in \mathcal{B}$, we write $\varphi_0 = \varphi_0 + \varphi_1$ with $\varphi_0 \in \mathcal{B}$ and $\varphi_1 \in V^{\beta} \times D (A_1^s)$, and consider

\begin{equation}
S_{\theta_2} (t) \varphi_0 = U_{\varphi_0} (t) \varphi_0 + V_{\varphi_0} (t) \varphi_1.
\end{equation}

We define $U_{\varphi_0} (t) \varphi_0 = \varphi_d (t)$ and $V_{\varphi_0} (t) \varphi_1 = \varphi_c (t)$, where $\varphi_d (t) = (\varphi_d (t), \varphi_d (t))$ and $\varphi_c (t) = (\varphi_c (t), \varphi_c (t))$ solve the Cauchy problems:

\begin{equation}
\begin{aligned}
\partial_t \varphi_d &+ A_0 \varphi_d + B_0 (u, \varphi_d) = R_0 \left( A_1 \varphi_d, \phi_c \right), \\
\partial_t \varphi_c &+ B_1 \left( \varphi_d, \phi_c \right) + A_1 \varphi_c = -f (\phi), \\
\varphi_d (0) &= \varphi_0, \quad \varphi_c (0) = \varphi_0.
\end{aligned}
\end{equation}

Concerning the mapping $U_{\varphi_0}$, for every $\varphi_0 \in \mathcal{B}$ one argues exactly as in (5.52), (5.53) to deduce

\begin{equation}
\left\| U_{\varphi_0} (t) \varphi_0 \right\|_{\mathcal{Y}_{\theta_2}} \leq e^{-\rho t} \left\| \varphi_0 \right\|_{\mathcal{Y}_{\theta_2}}^2 \leq R_0 e^{-\rho t},
\end{equation}

for all $t \geq 0$, for some $\rho > 0$. The final step is to deduce an energy inequality for $\varphi_c (t)$. To this end, we set

\begin{align}
\Lambda_1 (t) &:= \left\| \phi_c (t) \right\|_{L^2}^2 + \left\| A_1^s \phi_c (t) \right\|_{L^2}^2, \\
\Lambda_2 (t) &:= C \left\| u \right\|_{L^2}^2 \left[ \left\| \phi_c (t) \right\|_{L^2}^2 + \left\| \phi_d (t) \right\|_{L^2}^2 \right] + C \left\| g \right\|_{L^2}^2 + \left\| \phi \right\|_{L^2}^2 \\
&+ C \left\| \phi_c (t) \right\|_{L^2}^2 \left\| \phi_c (t) \right\|_{L_{2s+1}}^2 + C \left\| \phi_c (t) \right\|_{L^2}^2 \left\| \phi_c (t) \right\|_{L_{2s+1}}^2.
\end{align}

and then observe, on account of (5.68), (5.69) and (5.46), (5.50), (5.63), that

\begin{equation}
\sup_{\varphi_0 \in \mathcal{B}} \int_0^t \Lambda_2 (s) \, ds \lesssim e^{C (1 + t)}, \quad \text{for all } t \geq 0.
\end{equation}

Once again we pair the first and second equations of (5.68) with $\Lambda^{2} \phi_c$ and $A_1^s \phi_c$, respectively. Adding the resulting relationships, we get the following identity:

\begin{equation}
\frac{d \Lambda_1}{dt} + 2 \left| \left| A_1^{(2s+1)/2} \phi_c \right| \right|_{L^2}^2 + 2 \left\langle A_0 \phi_c, \Lambda^{2} \phi_c \right\rangle,
\end{equation}

\begin{align}
&= -2 b_0 \left( u, \phi_c, \Lambda^{2s+1} \phi_c \right) - 2 \left\langle A_1^{(2s-1)/2} f (\phi), A_1^{(2s+1)/2} \phi_c \right\rangle + 2 \left\langle g, \Lambda^{2} \phi_c \right\rangle \\
&+ 2 \left\langle R_0 \Lambda_1 \phi_c, \Lambda^{2} \phi_c \right\rangle + 2 \left\langle R_0 A_1 \phi_c, \Lambda^{2} \phi_c \right\rangle - 2 \left\langle A_1^{(2s-1)/2} B_1 u, \phi_c \right\rangle - 2 \left\langle A_1^{(2s-1)/2} B_1 \phi_c \right\rangle.
\end{align}

As in the proof of Theorem 5.11, for every $\delta > 0$ we have

\begin{equation}
\left\langle A_1^{(2s-1)/2} f (\phi), A_1^{(2s+1)/2} \phi_c \right\rangle \lesssim \delta \left\| \phi_c \right\|_{L_{2s+1}}^2 + \delta^{-1} \left\| \phi \right\|_{L_1}^2
\end{equation}

and

\begin{equation}
2 \left\langle B_0, \phi_c \right\rangle - g, \Lambda^{2} \phi_c \right\rangle \lesssim \delta^{-1} \left( \left\| u \right\|_{L_2}^2 \left\| \phi_c \right\|_{L_{2s+1}}^2 + \left\| g \right\|_{L_2}^2 \right) + \delta \left\| \phi_c \right\|_{L_2}^2,
\end{equation}

owing to assumption (vi). Exactly as in (5.51), it follows

\begin{equation}
2 \left\langle A_1^{(2s-1)/2} B_1 \phi_c \right\rangle \lesssim \left\| \nabla \phi_c \right\|_{L_{2s+1}} \left\| \phi_c \right\|_{L_{2s+1}}.
\end{equation}
Remark 5.12. There exists a time $t$ such that
\[ \gamma \frac{d}{dt} \Delta_1 (t) + 4 \delta \Delta_1 (t) \leq \Lambda_2 (t), \quad \text{for } t \geq 0. \]
For the remaining two terms, we use the boundedness of the mapping $R_0$ from $V^{2s-1} \times V^{2s}$ to $V^{2s-1}$, and $W^0 \times V^1$ to $V^{2s-1}$, for every $\beta \in \mathbb{K}_s$ to derive
\[ 2 \langle R_0 (A_1 \phi, \phi_x), \Lambda^{2s} \Pi_c \rangle + 2 \langle R_0 (A_1 \Pi_c, \phi_x), \Lambda^{2s} \Pi_c \rangle \leq \delta \| \Pi_c \|^2_{2s+1} + \delta^{-1} \| \Pi_c \|^2_{2s+1} + \| \Pi_c \|_{-\theta_2}^2 \| A_1 \phi \|_{L^2} \| A_1 \phi_x \|_{L^2}. \]
Finally, taking $\delta = \frac{1}{4} \min (c_{A_1}, c_{A_0}) > 0$ sufficiently small, the basic energy identity (5.71) takes the form of an inequality
\[ \frac{d}{dt} \Delta_1 (t) + 4 \delta \Delta_1 (t) \leq \Lambda_2 (t), \quad \text{for } t \geq 0. \]
In view of (5.70), the classical Gronwall lemma gives
\[ \| V_{\varphi_0} (t) \|^2_{V_{\beta,s}} \leq \Delta_1 (t) \leq e^{-4 \delta t} \| \Pi_c \|^2_{V_{\beta,s}} + J (t), \]
for some positive continuous function $J : \mathbb{R}_+ \to \mathbb{R}_+$, $J (t) \sim e^{Ct}$ as $t \to \infty$. The thesis (5.65) then follows from (5.69), (5.77) and the application of [21 Theorem 3.1], which finishes the proof. \(\square\)

As a byproduct of the previous result, we also obtain a regularity result for the global attractor $A_0$ in the case $\theta = 0$.

Corollary 5.13. The global attractor $A_0$ is bounded in $V^3 \times D (A_1^2)$, for some $\beta \in \mathbb{K}_s$ and $s \in (\frac{1}{2}, 1) \cap (0, \frac{n}{2} - \frac{q}{2} + 1)$.

Remark 5.14. For example, setting $\theta = 0$ and $\theta_2 = \theta_1 = 1$, and checking all the requirements of Theorem 5.12, the 3D NSV-AC admits a global attractor in the sense of Corollary 5.13, provided that $s \in (0.5, 0.75)$. As far as we know this result was not reported anywhere else. On the other hand, once the regularity in $C_{\beta,s} := B_{V_{\beta,s}} (R_1)$ is established, it may be also possible to exploit the semigroup decomposition (5.45), (5.49) and a bootstrap argument to show that, for $(u, \phi) \in A_0$, it also holds $\phi \in D (A_1)$. We omit the details in order to avoid further technicalities.

Remark 5.15. Note that estimate (5.69) is also satisfied provided that the external force $g$ is time dependent and $g \in L^2_t (\mathbb{R}^+; V^2)$, even in the case $\theta = 0$, one can generalize the notion of global attractor and replace it by the notion of pullback attractor, see Remark 5.3 (ii).

In the second part, we show the existence of exponential attractors for our regularized family of models (2.2) when $\theta = 0$. First, we note the following straightforward proposition.

Proposition 5.16. Let the assumptions of Theorem 5.12 be satisfied. There exists a time $t_3 > 0$ such that $S_{\theta_2} (t) C_{\beta,s} \subset C_{\beta,s}$, for all $t \geq t_3$. Moreover, the following estimate holds:
\[ \sup_{\varphi_0 \in C_{\beta,s}} \| (u (t), \phi (t)) \|^2_{V_{\beta,s}} + \int_t^{t+1} \| \phi (s) \|^2_{2s+1} ds \leq R_1, \]
for all $t \geq 0$, for some $\beta \in \mathbb{K}_s$ and $s \in (\frac{1}{2}, 1) \cap (0, \frac{n}{2} - \frac{q}{2} + 1)$.

The next lemma is concerned with the Hölder-in-time regularity of the semigroup $S_{\theta_2} (t)$.

Lemma 5.17. Let the assumptions of Proposition 5.16 be satisfied. There exists a constant $C > 0$ such that
\[ \| S_{\theta_2} (t) \varphi_0 \|_{V_{\theta_2} \times W^{2s-1}} \leq C \| \varphi_0 \|_{V_{\theta_2} \times W^{2s-1}} \]
for all $t \in [0, \infty)$ and any $\varphi_0 = (u_0, \varphi_0) \in C_{\beta,s} \subset V^{-\theta_2} \times W^{2s-1}$.\]
Proof. The following basic equality plays an essential role:

\[
(5.80) \quad S_{\theta_2}(t)\varphi_0 - S_{\theta_1}(t)\varphi_0 = (u(t) - u(\tilde{t}), \phi(t) - \phi(\tilde{t})) = \int_{\tilde{t}}^{t} \partial_{\theta_2}(S_{\theta_2}(y)\varphi_0) \, dy.
\]

We claim that the bounds below hold for any strong solution \((u, \phi)\) of problem (2.2). More precisely, we have

\[
(5.81) \quad \int_{0}^{\infty} \|\partial_{t}u(s)\|^2_{\theta_2} \, ds \leq C \left( R(1), \int_{0}^{\infty} \|\partial_{t}\phi(s)\|^2_{2s+1} \, ds \leq C \left( R(1) \right) .
\]

As usual in order to rigorously justify (5.81) we can appeal once more to the approximation scheme used in the proof of Theorem 3.2 and Theorem 5.11. Indeed, exactly as in (3.10)-(3.14), and applying the same estimates derived in the proof of Theorem 3.2 and Theorem 5.11, we can now bound the second inequality of (5.81) and exploiting the fact that \(R_0 : W^{2s-2} \times V^{2s} \to V^{-1}\) is bounded for any \(s \in \left(\frac{1}{2}, 1\right)\), we have

\[
(5.82) \quad \|B_0(u, u)\|_{\theta_2} \lesssim \|u\|_{\theta_2} + \|R_0(A_1 \phi, \phi)\|_{\theta_2} + \|g\|_{\theta_2},
\]

where we recall that \(\lambda = \lambda_1 + \lambda_2 = 1\) if \(\theta = 0\). Thus, we get thanks to (5.78) that \(\partial_{t}u \in L^2(0, \infty; W^{2s-\theta})\) and the first estimate of (5.81) holds. On the other hand, recalling that for \(\beta \in J_\nu\), the map \(B_1 : V^{0+2\theta} \times V^1 \to W^{2s-1}\) is continuous, we can also bound the \(W^{2s-1}\)-norm of \(B_1(u, \phi)\) in the last inequality of (5.81). Hence, owing to (5.78) we obtain that \(\partial_{t}\psi \in L^2(0, \infty; W^{2s-1})\), such that the second estimate of (5.81) is satisfied. Thanks to (5.81), from (5.80) and by application of Hölder’s inequality, we infer

\[
(5.83) \quad \|u(t) - u(\tilde{t})\|_{\theta_2} \leq C|t - \tilde{t}|^{1/2}, \quad \|\phi(t) - \phi(\tilde{t})\|_{2s-1} \leq C|t - \tilde{t}|^{1/2}.
\]

Thus, we immediately arrive at the inequality (5.79) owing to (5.83). The proof is finished. \(\square\)

The second main result of this subsection is the following.

**Theorem 5.18.** Let the assumptions of Theorem 5.12 be satisfied. Then \((S_{\theta_2}, \mathcal{Y}_{\theta_2})\) possesses an exponential attractor \(\mathcal{M}_0 \subset \mathcal{Y}_{\theta_2}\) which is bounded in \(\mathcal{Y}_{\beta,s}\). Thus, by definition, we have

(a) \(\mathcal{M}_0\) is compact and semi-invariant with respect to \(S_{\theta_2}(t)\), that is,

\[
S_{\theta_2}(t)\left(\mathcal{M}_0\right) \subseteq \mathcal{M}_0, \quad \forall t \geq 0.
\]

(b) The fractal dimension \(\dim_F(\mathcal{M}_0, \mathcal{Y}_{\theta_2})\) of \(\mathcal{M}_0\) is finite and an upper bound can be computed explicitly.

(c) \(\mathcal{M}_0\) attracts exponentially fast any bounded subset \(B\) of \(\mathcal{Y}_{\theta_2}\), that is, there exists a positive nondecreasing function \(Q\) and a constant \(\rho > 0\) such that

\[
dist_{\mathcal{Y}_{\theta_2}}(S_{\theta_2}(t)B, \mathcal{M}_0) \leq Q(\|B\|_{\mathcal{Y}_{\theta_2}}) e^{-\rho t}, \quad \forall t \geq 0.
\]

**Proof.** Step 1 (The smoothing property). For \(\varphi_{0i} = (u_{0i}, \phi_{0i}) \in C_{\beta,s}\), let \(\varphi_i = (u_i, \phi_i) = S_{\theta_2}(t)\varphi_{0i}, i = 1, 2,\) be the corresponding solutions. We decompose \(\varphi(t) := (u(t), \phi(t)) = \varphi_1(t) - \varphi_2(t)\), such that

\[
(5.84) \quad \varphi(t) = (v(t), \rho(t)) + (\omega(t), \psi(t)) \quad (v, \rho) \quad (\omega, \psi)
\]

where \((v, \rho)\) solves

\[
(5.85) \quad \begin{cases}
\partial_{t}v + A_0 v + B_0(u_2, v) = R_0(A_1 \rho, \phi_1), \\
\partial_{t}\rho + A_1 \rho + B_1(v, \phi_1) = 0, \\
v(0) = u_{01} - u_{02}, \rho(0) = \phi_{01} - \phi_{02},
\end{cases}
\]

and \((\omega, \psi)\) solves

\[
(5.86) \quad \begin{cases}
\partial_{t}\omega + A_0 \omega + B_0(u_1, \phi_1) + B_0(u_2, \omega) = R_0(A_1 \psi, \phi_1) + R_0(A_1 \phi_2, \phi), \\
\partial_{t}\psi + B_1(\omega, \phi_1) + B_1(u_2, \phi) + A_1 \psi = -(f(\phi_1) - f(\phi_2)),
\end{cases}
\]
supplemented with null initial data. It is apparent that upon pairing the first and second equations of (5.84) with \(N\nu\) and \(A_1\rho\), respectively, the norm \(\|(v, \rho)\|_{V_{-\nu_2} W^1}\) is exponentially decaying to zero. More precisely, we easily get

\[
(5.87) \quad \|(v(t), \rho(t))\|_{V_{-\nu_2} W^1}^2 \leq e^{-\kappa t} \|\varphi_{01} - \varphi_{02}\|_{V_{-\nu_2} W^1}^2,
\]

for all \(t \geq 0\), for some positive constant \(\kappa\) independent of time. Concerning the other component in (5.84), we pair the first and second equations of (5.85) with \(A^2\beta\omega\) and \(A^2\beta\psi\), respectively, for \(s \in (\frac{1}{2}, 1]\), and then we add the resulting relationships. We have

\[
(5.88) \quad \frac{d}{dt} \left( \|\omega\|^2 + \|A^1\psi\|^2 \right) + 2 \|A^{2s+1}/2\psi\|^2 \leq 2 \langle A_0\omega, A^2\beta\omega \rangle
\]

\[
= -2b_0 (u, u_1, A^2\beta\omega) - 2b_0 (u_2, \omega, A^2\beta\omega)
\]

\[
- 2 \left( A^{2s-1}/2 \langle f (\phi_1) - f (\phi_2) \rangle, A^{2s+1}/2\psi \right)
\]

\[
+ 2 \left( R_0 (A_1\psi, \phi_1), A^2\beta\omega \right) + 2 \left( R_0 (A_1\phi_2, \phi), A^2\beta\omega \right)
\]

\[
- 2 \left( A^{2s-1}/2 B_1 (\omega, \phi_1), A^{2s+1}/2\psi \right) - 2 \left( A^{2s-1}/2 B_1 (u_2, \phi), A^{2s+1}/2\psi \right).
\]

At this point, we basically repeat the proof of Theorem 5.11 up to the estimate (5.60). Note that, using the fact that any trajectory \(\varphi_i\), \(i = 1, 2\), satisfies 5.78, we can once again find a constant \(C > 0\) such that

\[
(5.89) \quad \left\langle A^{2s-1}/2 (f (\phi_1) - f (\phi_2)), A^{2s+1}/2\psi \right\rangle \leq \delta \|A^{2s+1}/2\psi\|_{L^2}^2 + C\delta^{-1} \|\phi\|^2, \text{ for any } \delta > 0.
\]

Moreover, exactly as in (5.58) it follows that

\[
2b_0 (u, u_1, A^2\beta\omega) \lesssim \delta^{-1} \|\omega\|^2 + \delta \|\omega\|^2 \|
\]

\[
2b_0 (u_2, \omega, A^2\beta\omega) \lesssim \delta^{-1} \|u_2\|^2 + \delta \|\omega\|^2 \|
\]

Owing to (5.58)-(5.59), we infer

\[
2\left\langle R_0 (A_1\psi, \phi_1), A^2\beta\omega \right\rangle \lesssim \delta \|\omega\|^2 \|A_1\phi_1\|^2_{L^2} + \delta^{-1} \|A^{2s+1}/2\psi\|_{L^2}^2,
\]

and, respectively,

\[
2\left\langle R_0 (A_1\phi_2, \phi), A^2\beta\omega \right\rangle \lesssim \|\omega\|^2 \|A^{2s+1}/2\phi_2\|^2_{L^2} + \|A_1\phi_1\|^2_{L^2}.
\]

Finally, by virtue of (5.61)-(5.60) the following inequalities also hold:

\[
(5.93) \quad 2 \left\langle A^{2s-1}/2 B_1 (\omega, \phi_1), A^{2s+1}/2\psi \right\rangle \lesssim \delta \left\|A^{2s+1}/2\psi\right\|^2_{L^2} + \delta^{-1} \|\omega\|^2 \|A_1\phi_1\|^2_{L^2},
\]

\[
(5.94) \quad 2 \left\langle A^{2s-1}/2 B_1 (u_2, \phi), A^{2s+1}/2\psi \right\rangle \lesssim \delta \left\|A^{2s+1}/2\psi\right\|^2_{L^2} + \delta^{-1} \|u_2\|^2 \|A_1\phi_1\|^2_{L^2}.
\]

Let us now set

\[
P (t) := \|\omega (t)\|^2 + \|A^1\psi (t)\|^2_{L^2},
\]

\[
N_1 (t) := C_5 \left( \|u_2 (t)\|^2 \|A_1\phi_1 (t)\|^2_{2s+1} + \|\phi_2\|^2_{2s+1} \right),
\]

\[
N_2 (t) := C_5 \left( 1 + \|u_2 (t)\|^2 \right) \|A_1\phi_1 (t) - A_1\phi_2 (t)\|^2_{L^2} + C_5 \|u (t)\|^2 \|u_2 (t)\|^2_{\delta_2} \|u_2 (t)\|^2_{\delta_2}.
\]

On account of 5.88, 5.91, we can choose a sufficiently small \(\delta \approx \min \{c_{A_0}, c_{A_1}\} > 0\) to deduce

\[
(5.95) \quad \frac{d}{dt} P (t) \leq N_1 (t) P (t) + N_2 (t),
\]

for all \(t \geq 0\). Hence, integrating 5.61 with respect to time on the interval \((0, t)\), noting that \(P (0) = 0\), and exploiting 3.22, 5.78, we can find a positive continuous function \(\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \lambda (0) > 0\), such that

\[
(5.96) \quad \|(\omega (t), \psi (t))\|_{\delta, \beta, s}^2 = P (t) \leq \lambda (t) \|\varphi_{01} - \varphi_{02}\|_{V_{-\nu_2} W^1}^2,
\]

for all \(t \geq 0\).

**Step 2** (The final argument). Thanks to Theorem 5.12 Propositions 5.1, 5.16 and the transitivity property of the exponential attraction [28, Theorem 5.1], the set \(C_{\beta, s}\) is positively invariant.
for $S_{\theta_2}(t)$ and attracts any bounded set of $\mathcal{Y}_{\theta_2}$ exponentially fast. Moreover, by Theorem 5.16 and Lemma 5.17 the map $(t, \varphi_0) \mapsto S_{\theta_2}(t) \varphi_0$ is H"older continuous on $[0, T] \times C_{\beta,s}$, provided that $C_{\beta,s}$ is endowed with the metric topology of $V^{-\theta_2} \times W^{2s-1}$. Also, by virtue of estimates (5.87), (5.96), $S_{\theta_2}(t)$ enjoys the smoothing property. Thus, using also the bound (5.78) and exploiting a well-known abstract result (see Appendix, Theorem 8.6), the existence of an exponential attractor with the properties stated in Theorem 5.18 follows. The proof is finished.

We finish this subsection with a simple corollary.

**Corollary 5.19.** Let the assumptions of Theorem 5.18 be satisfied. The global attractor $A_0$ has finite fractal dimension.

**Remark 5.20.** Note that Theorem 5.18 applies to the 3D NSV-AC system provided that $s \in (0.5, 0.75)$.

### 5.3. Convergence to steady states.

In this subsection, we show that any global-in-time bounded solution to the model (2.2) converges to a single equilibrium as time tends to infinity. The proof of the main result is based on a suitable version of the Lojasiewicz–Simon theorem and the regularity results provided in Sections 5.1, 5.2. The question of such convergence is usually a delicate matter since it is well known that the topology of the set of stationary solutions of (2.2) can be non-trivial even when $u \equiv 0$. In particular, there may be a continuum of stationary solutions for (2.2) even in the simplest cases, for instance when $\Omega$ is a disk, see e.g., [33] [11] (cf. also [1] [2] [80]). Since weak solutions $(u(t), \phi(t))$ for (2.2) become strong for times $t \geq t_1$ (for some $t_1 = t_1(\theta) > 0$), we can confine ourselves to considering only strong solutions.

We summarize the regularity results of the previous subsections in the following proposition.

**Proposition 5.21.** Let the assumptions of Theorem 5.13 when $\theta > 0$, and Theorem 5.18 when $\theta = 0$, be satisfied. Moreover, assume that $g \in L^2_{\alpha_0}(\mathbb{R}_+; V^{\beta-\theta})$. For every $R > 0$, there exists $C_* = C_*(R) > 0$, independent of time, such that,

\begin{align}
\sup_{\varphi_0 \in B_{\mathcal{Y}_\beta}(R)} \|S_{\theta_2}(t) \varphi_0\|^2_{V_\beta} + \int_t^{t+1} \left( \|u(s)\|^2_{\beta+\theta} + \left\| A_1^{3/2} \phi(s) \right\|^2_{L^2} \right) ds \leq C_*, \text{ for all } t \geq 0,
\end{align}

when $\theta > 0$, and

\begin{align}
\sup_{\varphi_0 \in B_{\mathcal{Y}_{\beta,s}}(R)} \|S_{\theta_2}(t) \varphi_0\|^2_{V_{\beta,s}} + \int_t^{t+1} \left( \|u(s)\|^2_{\beta} + \left\| A_1^{(2s+1)/2} \phi(s) \right\|^2_{L^2} \right) ds \leq C_*, \text{ for all } t \geq 0,
\end{align}

when $\theta = 0$. Here $B_X(R)$ denotes the ball in $X$ of radius $R$, centered at 0.

Note that $V_{\beta,1} = V_\beta$. For the sake of simplicity and notation, below we will make the following convention: $V_{\beta,s} = V_\beta$ when $\theta > 0$ (and, thus, always assume $s = 1$), while for $\theta = 0$, we recall that $s \in \left( \frac{1}{2}, 1 \right]$ is possibly sufficiently small. Next, we characterize the structure of the $\omega$-limit set for problem (2.2) corresponding to any initial datum $\varphi_0 = (u_0, \phi_0) \in V_{\beta,s}$. Recall that $\omega$-limit set is defined as follows:

\[ \omega(\varphi_0) = \left\{ (u_*, \phi_*) \in V_{\beta,s} : \exists t_n \to \infty \text{ such that } \lim_{n \to \infty} \|(u(t_n), \phi(t_n)) - (u_*, \phi_*)\|_{V_{\beta,s}} = 0 \right\}. \]

Clearly, $\omega(\varphi_0)$ is nonempty by virtue of Proposition 5.21.

**Lemma 5.22.** Let the assumptions of Proposition 5.21 be satisfied, and suppose that $g$ also obeys the following condition:

\begin{align}
\int_t^{t+1} \left( g(s) \right)^2_{-\theta_2} ds \lesssim (1 + t)^{-(1+\delta)}, \text{ for all } t \geq 0,
\end{align}

for some constant $\delta > 0$. Then, the $\omega$-limit set $\omega(\varphi_0)$ is a subset of

\[ \mathcal{L} = \{ (0, \phi_*) : \phi_* \in D(A^*_1), \phi_* \in [-1, 1] \text{ and (5.100) holds} \}, \]

for some $s \in \left( \frac{1}{2}, 1 \right]$, where (5.100) is the following problem:

\begin{align}
A_1 \phi_* + f(\phi_*) = 0.
\end{align}

Moreover, we have

\begin{align}
\lim_{t \to \infty} \|u(t)\|_{-\theta_2} = 0, \quad \lim_{t \to \infty} \left\| A_1 \phi(t) + f(\phi(t)) \right\|_{L^2} = 0.
\end{align}
Proof. First, we have by Proposition 5.21 the corresponding energy inequality (3.4), and assumption (5.99) that

$$\int_0^\infty g(s) \, ds < \infty, \quad y := \|u\|_{-\theta_2}^2 + \|\mu\|_{-\theta_2}^2,$$

where $\mu = A_1\phi + f(\phi)$. It follows from (5.97), (5.98), (5.99) and a higher-order differential inequality for the function $y(t)$:

$$\frac{d}{dt} y(t) \leq C \left( 1 + \|g(t)\|_{-\theta_2}^2 \right),$$

that

$$\lim_{t \to \infty} \|u(t)\|_{-\theta_2} + \|\mu(t)\|_{L^2} = 0.$$ 

Hence, for any $(u_*, \phi_*) \in \omega (\varphi_0)$ we have $u_*=0$. The assertion (5.101) is also immediate. Moreover, by (5.97), (5.98) and (5.102) it is easy to see that $\phi_* \in D(A_1^s)$, $\phi_* \in [-1, 1]$, and that the following inequality holds:

$$\lim_{n \to \infty} \|A_1^s (\phi(t_n) - \phi_*)\|_{L^2} = 0.$$ 

Let $\psi \in D(A_1^{1-s})$. Then, $\phi_*$ satisfies (5.100) on account of (5.101), (5.104), and the basic inequality

$$\|\langle A_1 \phi_0 + f(\phi_*), \psi \rangle\| \leq (\|A_1^s (\phi(t_n) - \phi_*)\|_{L^2} + \|f(\phi(t_n) - \phi_*)\|_{L^2} + \|A_1^s (\phi(t_n)) + f(\phi(t_n))\|_{L^2} \|A_1^{1-s} \psi\|_{L^2},$$

Passing now to the limit as $t_n \to \infty$, the proof of Lemma 5.22 is concluded. Finally, we only briefly sketch the details for getting (5.103) in the case $\theta > 0$ (the case $\theta = 0$ is similar). We observe that for smooth solutions the function $y(t)$ satisfies the identity

$$\frac{dy}{dt} + 2 \langle A_0 u, Nu \rangle + \left\| A_1^{1/2} u \right\|_{L^2}^2 = 2 \langle g, Nu \rangle + 2 \langle R_0 (\mu, \phi), Nu \rangle - \left\langle A_1^{1/2} B_1 (u, \phi), A_1^{1/2} \mu \right\rangle - \left\langle f'(\phi) B_1 (u, \phi), \mu \right\rangle.$$ 

We can bound the nonlinear terms on the right-hand side of (5.105), by using the following facts:

- $R_0 (\mu, \phi) \in L^2 (0, \infty; V^{0-\theta_2})$, as a mapping from $L^2 \times W^1 \to V^{-\theta-\theta_2}$ due to (5.102) and (5.97), since $\theta + \theta_2 \geq 1$ (see Lemma 8.3). It follows that

$$\|R_0 (\mu, \phi), Nu\| \leq \delta \|u\|_{-\theta_2}^2 + C_\delta \|\mu\|_{L^2} \|\phi\|_{L^2}^2.$$

- $B_1 (u, \phi) \in L^2 (0, \infty; L^2 (\Omega))$, as a mapping from $V^{0+\theta_2} \times W^1 \to L^2 (\Omega)$, due to (5.102) and (5.97). Since $\phi$ is also bounded, we have

$$\left\| f'(\phi) B_1 (u, \phi), \mu \right\| \leq \delta \|u\|_{-\theta_2}^2 + C_\delta \left\| f'(\phi) \right\|_{L^\infty} \|\phi\|_{L^2}^2 \|\mu\|_{L^2}^2.$$

- For the third term on the right-hand side of (5.105), we employ the same strategy of proof used in Theorem 3.6, 3.42-3.43, in the case $n = 3$, and 3.48-3.49 for $n = 2$. For instance, when $n = 3$ we derive

$$\left\| A_1^{1/2} B_1 (u, \phi), A_1^{1/2} \mu \right\| \leq (\|Nu\|_{L^6} \|\nabla \phi\|_{W^{1,3}} + \|Nu\|_1 \|\nabla \phi\|_{L^\infty}) \|A_1^{1/2} \mu\|_{L^2} = I_1 + I_2,$$

where

$$I_1 \leq 2\delta \left\| A_1^{1/2} \mu \right\|_{L^2}^2 + C_\delta \left( \|u\|_{-\theta_2}^2 \|\phi\|_{L^2}^2 + \|u\|_{-\theta_2}^2 \|\phi\|_{L^2}^2 \right),$$

$$I_2 \leq 2\delta \left\| A_1^{1/2} \mu \right\|_{L^2}^2 + C_\delta \left( \|u\|_{-\theta_2}^2 \|\phi\|_{L^2}^2 + \|u\|_{-\theta_2}^2 \|\phi\|_{L^2}^2 \right).$$

- The bound on the final term is basic

$$\left\| f'(\phi) \mu \right\| \leq \left\| f'(\phi) \right\|_{L^\infty} \|\mu\|_{L^2}^2.$$
We can now choose \( \delta \sim \min (\varepsilon, A_0, 1) > 0 \) sufficiently small in all these estimates, and use the coercivity of the operator \( A_0 \) together with (3.4) to handle the term \( (g, N u) \). Then recalling that for a strong solution, \( y(t) \leq C \) uniformly, for all \( t \geq 0 \) and reporting all the preceding estimates in (5.105) we can easily obtain the claim (5.101) on account of the application of [81, Lemma 6.2.1].

Consequently, for the energy functional
\[
E(u, \phi) := \frac{1}{2} (u, N u) + \tilde{E}(\phi),
\]
where
\[
\tilde{E}(\phi) := \frac{1}{2} ||A_{1/2}^1 \phi||^2_{L^2} + \int_{\Omega} F(\phi) \, dx,
\]
the following statement holds. Note that, by the above Lemma 5.22, \( \phi_* \) is a critical point of \( \tilde{E} \) over \( D(A_{1/2}^1) \cap L^\infty(\Omega) \).

**Proposition 5.23.** There exists a constant \( c_\infty \in \mathbb{R} \) such that \( \tilde{E}(\phi_*) = c_\infty \), for all \( (0, \phi_*) \in \mathcal{L} \), and we have
\[
\lim_{t \to \infty} E(u(t), \phi(t)) = c_\infty.
\]
Moreover, the functional \( \Phi(t) \) is decreasing along all strong trajectories \( (u(t), \phi(t)) \) and, for all \( t \geq 0 \),
\[
\frac{d}{dt} \Phi(t) \leq -\left(\frac{cA_0}{2} \|u(t)\|_{L^\infty}^2 + \|A_1 \phi(t) + f(\phi(t))\|_{L^2}^2\right),
\]
where
\[
\Phi(t) := E(u(t), \phi(t)) + \frac{\|N\|^2_{-\theta_2; \theta_2}}{2cA_0} \int_0^\infty \|g(s)\|^2_{-\theta_2; ds}.
\]

Even though we are dealing with an asymptotically decaying force due to (5.99), we cannot conclude that each strong solution of (2.2) converges to a single equilibrium, for \( \mathcal{L} \) can be a continuum (see, e.g., [41]). However, we can establish this fact when the nonlinear function \( f \) is real analytic. The version of the Łojasiewicz-Simon inequality we need is given by the following lemma (see [17, 47]).

**Lemma 5.24.** For the above setting, let \( f \) be real analytic. There exist constants \( \zeta \in (0, 1/2) \) and \( C_L > 0, \eta > 0 \) depending on \( (0, \phi_*) \) such that, for any \( \phi \in D(A_1^1) \), if \( \|\phi - \phi_*\|_1 \leq \eta \), denoting by \( \tilde{E} \) the Fréchet derivative of \( E \), we have
\[
C_L^\zeta \|\tilde{E}(\phi(t))\|_{-1} \geq \|\tilde{E}(\phi(t))\|_1^1 - \zeta.
\]

The result below is concerned with the convergence of trajectories of problem (2.2) to single equilibria, which shows, in a strong form, their (global) asymptotic stability.

**Theorem 5.25.** Let the assumptions of Lemma 5.22 hold. In addition, assume that \( f \) is real analytic. For any given initial datum \( \phi_0 = (u_0, \phi_0) \in \mathcal{V}_{\beta,s} \), the corresponding solution \( \varphi(t) = (u(t), \phi(t)) \) of problem (2.2) converges to a single equilibrium \( (0, \phi_*) \) in the strong topology of \( \mathcal{V}_{\beta,s} \), as time goes to infinity. Moreover, there exists \( \xi \in (0, 1) \), depending on \( \phi_* \) and the other physical parameters of the problem, such that
\[
\|u(t)\|_{L^\infty} + \|A_1^{1/2} (\phi(t) - \phi_*)\|_{L^2} \lesssim (1 + t)^{-\xi},
\]
for all \( t \geq 0 \).

**Proof.** We prove the case \( \theta > 0 \) (the case \( \theta = 0 \) is analogous and follows with minor modifications). We adapt the ideas of [17, 47, 53] (cf. also [11, 32, 80]) to prove the claim. On account of the first statement of (5.101), it suffices to prove the claim only for the phase-field component \( \phi \). First, by Lemma 5.22 the omega limit set \( \omega(\varphi_0) \) is a non-empty and compact subset of \( \mathcal{Y}_{\beta,s} \). Secondly, we can choose a sufficiently large time \( t_1 > 0 \) such that for all \( t \geq t_1 \), we have \( \|A_1^{1/2} (\phi(t) - \phi_*)\|_{L^2} < \eta \), such that the conclusion of Lemma 5.24 holds with \( \tilde{E}(\phi) = A_1 \phi + f(\phi) \) provided that we choose even a smaller constant \( \zeta \in (0, 1/2) \cap (0, \delta (1 + \delta)^{-1}) \). Without loss of generality, in what follows we let \( t_1 = 1 \) and assume \( \delta < 1 \). Next, define
\[
\Sigma := \{ t \geq 1 : ||A_1^{1/2} (\phi(t) - \phi_*)||_{L^2} \leq \eta/3 \}
and observe that $\Sigma$ is unbounded by Lemma $5.22$. For every $t \in \Sigma$, we define
\[
\tau (t) = \sup \{ t' \geq t : \sup_{s \in [t, t']} \| A_{1/2}^t (\phi (t) - \phi_s) \|_{L^2} < \eta \}.
\] By continuity, $\tau (t) > t$ for every $t \in \Sigma$. Let now $t_0 \in \Sigma$ and divide the interval $J := [t_0, \tau (t_0))$ into two subsets
\[
\Sigma_1 := \left\{ t \in J : \Upsilon (t) > \left( \int_t^{\tau (t_0)} \| g (s) \|^2_{-\theta - \theta_2} ds \right)^{1-\zeta} \right\}, \quad \Sigma_2 := J \setminus \Sigma_1,
\] where
\[
\Upsilon (t) := \| u (t) \|_{\theta - \theta_2} + \| A_1 \phi (t) + f (\phi (t)) \|_{L^2}.
\] Define further
\[
\hat{\Phi} (t) := E (u (t), \phi (t)) - \tilde{E} (\phi_s) + \frac{\| N \|^2_{-\theta_2 : \theta_2}}{2cA_0} \int_t^{\tau (t_0)} \| g (s) \|^2_{-\theta - \theta_2} ds
\] and notice that $\hat{\Phi} (t)$ differs from $\Phi (t)$ in (5.107) only by a constant. Hence, for every $t \in J$ we have
\[
\frac{d}{dt} \hat{\Phi} (t) \leq -\Upsilon^2 (t) \leq 0
\] so that $\hat{\Phi}$ is a decreasing function. Moreover, for every $t \in J$ we have
\[
\frac{d}{dt} \left( \hat{\Phi} (t) \right|_t sgn(\hat{\Phi} (t)) \right) = \zeta |\hat{\Phi} (t)|^{\zeta-1} \frac{d}{dt} \hat{\Phi} (t)
\] which implies that the functional $sgn(\hat{\Phi} (t)) |\hat{\Phi} (t)|^{\zeta}$ is decreasing on $J$. By $5.23$ (indeed, $\| u \|_{-\theta_2}^{2(1-\zeta)} \leq \| u \|_{\theta - \theta_2}$ since $\zeta < \frac{1}{2}$), for every $t \in \Sigma_1$ we have
\[
|\hat{\Phi} (t)|^{1-\zeta} \leq E (u (t), \phi (t)) - \tilde{E} (\phi_s) + \frac{\| N \|^2_{-\theta_2 : \theta_2}}{2cA_0} \int_t^{\tau (t_0)} \| g (s) \|^2_{-\theta - \theta_2} ds
\] which together with equation (5.110) yields
\[
-\frac{d}{dt} \left( |\hat{\Phi} (t)|^{\zeta} sgn(\hat{\Phi} (t)) \right) \geq \Upsilon (t).
\] Moreover, exploiting (5.112) we have
\[
\int_{\Sigma_1} \Upsilon (s) \, ds \leq -\int_{\Sigma_1} \frac{d}{ds} \left( |\hat{\Phi} (s)|^{\zeta} sgn(\hat{\Phi} (s)) \right) \, ds
\] which we interpret the term involving $\tau (t_0)$ on the right hand side of (5.113) as 0 if $\tau (t_0) = \infty$ (recall (5.106)). On the other hand, if $t \in \Sigma_2$, using assumption (5.99) we obtain
\[
\Upsilon (t) \leq \left( \int_t^{\tau (t_0)} \| g (s) \|^2_{-\theta - \theta_2} ds \right)^{1-\zeta} \leq (1 + t)^{-(1-\zeta)(1+\delta)},
\] so once again the function $\Upsilon$ is dominated by an integrable function on $\Sigma_2$ since $\zeta (1+\delta) < \delta$. Combining the inequalities (5.113), (5.114), we deduce that $\Upsilon$ is absolutely integrable on $J$ and
\[
\lim_{t_0 \to \infty, t_0 \in \Sigma} \int_{t_0}^{\tau (t_0)} \Upsilon (s) \, ds = 0.
\] On the other hand, recalling estimates (5.97), (5.102), from the second equation of (2.2) it follows that
\[
\| \partial_t \phi (t) \|_{L^2} \leq \| B_1 (u (t), \phi (t)) \|_{L^2} + \| A_1 \phi (t) + f (\phi (t)) \|_{L^2}
\leq \| u (t) \|_{\theta - \theta_2} \| A \phi (t) \|_{L^2} + \| A_1 \phi (t) + f (\phi (t)) \|_{L^2}
\leq \Upsilon (s),
\]
since \( N : V^{-2\theta_2} \to V^0 \) is bounded, and \( V^{-\theta_2} \subseteq V^{-2\theta_2}, \theta_2, \theta \geq 0. \) Consequently, we also have
\[
\lim_{t_0 \to \infty, \delta_0 \in \Sigma} \int_{t_0}^{\tau(t_0)} \| \partial_t \phi(s) \|_{L^2} \, ds = 0.
\]
For \( t \in \Sigma \), note that the inequality
\[
\| \phi(t) - \phi_\ast \|_{L^2} \leq \int_{t_0}^{t} \| \partial_t \phi(s) \|_{L^2} \, ds + \| \phi(t_0) - \phi_\ast \|_{L^2}
\]
implies that \( \tau(t_0) = \infty \), for some \( t_0 \in \Sigma \). Indeed, let us assume for a second that the latter statement is not true. Then, by definition of \( \tau(t_0) \), \( \| A_1^{1/2} (\phi(t) - \phi_\ast) \|_{L^2} = \eta \) for every \( t_0 \in \Sigma \). Let now \( \{ t_n \} \subset \Sigma \) be an unbounded sequence such that
\[
\lim_{t_n \to \infty} \| A_1^{1/2} (\phi(t_n) - \phi_\ast) \|_{L^2} = 0.
\]
By compactness and Lemma 5.22, we can now pass to a subsequence of \( \{ t_n \} \) if necessary to conclude that one can find \( \tilde{\phi} \in \omega(\varphi_0) \) such that \( \| \tilde{\phi} - \phi \|_1 = \eta > 0 \) and \( \lim_{t_n \to \infty} \| \tilde{\phi} - \phi(\tau(t_n)) \|_1 = 0 \). Then, the above inequality gives
\[
0 < \| \tilde{\phi} - \phi \|_{L^2} \leq \lim_{t_n \to \infty} \left\{ \int_{t_n}^{\tau(t_n)} \| \partial_t \phi(s) \|_{L^2} \, ds + \| \phi(t_n) - \phi_\ast \|_{L^2} \right\} = 0,
\]
which is a contradiction. Hence, we conclude that we must have \( \tau(t_0) = \infty \), for some \( t_0 \) sufficiently large. Thus, the above arguments (see, in particular, (5.117)) imply that \( \| \partial_t \phi \|_{L^2} \) is absolutely integrable on \([t_0, \infty)\), which implies that the limit of \( \phi(t) \) exists, as time goes to infinity. By compactness and a basic interpolation inequality, we have \( \phi(t) \to \phi_\ast \) in the strong topology of \( D(A_1) \). Hence, \( \omega(\varphi_0) = \{(0, \phi_\ast)\} \), as claimed. The estimate of the rate of convergence in (5.110) is a straight-forward consequence of (5.110), (5.111) and the definition of \( \Phi \) (see, e.g., [33, Theorem 5.7]). We leave the details to the interested reader. The proof of Theorem 5.25 is complete. □

Remark 5.26. All the results of the previous sections and subsections can be extended for singular potentials of the form (1.7). Let us suppose that \( F \in C^2(-1, 1) \), set \( f = F' \) and assume
\[
\lim_{r \to \pm 1} f(r) = \pm \infty \quad \text{and} \quad \lim_{r \to \pm 1} f'(r) = \mp \infty.
\]
It is easy to see that the derivative of \( F \) defined by (1.7) satisfies (5.118). Although the potential \( f \) is singular, we can still use the results derived in the previous sections, since the solutions to our problem are smooth and strictly separated thanks to [33, Theorem 6.1], if additionally we assume \( (u_0, \phi_0) \in \mathcal{Y}_{\theta_2}^\delta \), for some \( \delta \in (0, 1) \). Here, we have defined
\[
\mathcal{Y}_{\theta_2}^\delta := \left\{ (u_0, \phi_0) \in V^{-\theta_2} \times (D(A_1^{1/2}) \cap L^\infty(\Omega)) : \| \phi_0 \|_{L^\infty} \leq \delta < 1 \right\}.
\]
Consequently, for \( \phi \in \mathcal{Y}_{\theta_2}^\delta \), \( f(\phi) \) and any of its higher-order derivatives are bounded provided that we replace \( \mathcal{Y}_{\theta_2} \) by \( \mathcal{Y}_{\theta_2}^\delta \) everywhere in the paper. Thus, the arguments used in the previous (sub)sections are still valid in the present case.

6. REMARKS ON A REGULARIZED FAMILY FOR THE NSE AND MHD MODELS

As in Section 2, consider the following system
\[
\begin{cases}
\partial_t u + A_0 u + B_0(u, u) = g, \\
u(0) = u_0,
\end{cases}
\]
on the time interval \([0, T]\). Bearing in mind the model (2.3), we are mainly interested in bilinear maps \( B_0 \) of the form (2.3). We recall that the formulation (6.1) here includes not only various regularized models for the Navier-Stokes (NSE) equation but also certain (regularized or un-regularized) magnetohydrodynamics (MHD) models (see [44]).

In this section, we show how to close a gap in the proof of [44, Section 5, Corollary 5.4] whose assumptions can only be verified in the case \( \theta > 0 \). Note that when \( \theta = 0 \), the assumptions of [44, Theorem 5.1, (b)] do not longer provide the existence of a compact absorbing set as claimed on [44, pg. 550] (see also Remark 5.42). Besides, it is well-known that in the non-dissipative case when \( \theta = 0 \), the regularized model for the Navier-Stokes equation looses its parabolic character, see Section 5.2 for related discussions.

Our analogue result of [44, Corollary 5.4] in the case \( \theta = 0 \) is as follows.
Theorem 6.1. Let the following conditions hold and recall that $\theta_1, \theta_2 \in \mathbb{R}$.

(i) $b_0(w, v, Nv) = 0$, for any $v, w \in V^{-\theta_2}$;
(ii) $b_0 : V^{\bar{\sigma}_1} \times V^{\sigma_2} \times V^{\bar{\gamma}} \to \mathbb{R}$ is bounded for some $\bar{\sigma}_i < -\theta_2$, $i = 1, 2$, and $\bar{\gamma} \in \mathbb{R}$;
(iii) $b_0 : V^{\sigma_1} \times V^{-\theta_2} \times V^{\sigma_2} \to \mathbb{R}$ is bounded for some $\sigma_1 < -\theta_2$ and $\sigma_2 \leq \theta_2$ with $\sigma_1 + \sigma_2 \leq 0$;
(iv) $\langle A_0 v, N v \rangle \geq c_{A_0} \| v \|^2_{\theta_2}$, for any $v \in V^{-\theta_2}$, with a constant $c_{A_0} > 0$.

In addition, for some $\beta > -\theta_2$, let the following conditions hold:
(v) $\langle A_0 v, (I - \Delta)^{\beta} v \rangle \geq c_{A_0} \| v \|^2_{\beta}$, for any $v \in V^{\beta}$, for some $c_{A_0} > 0$;
(vi) $g \in V^{\beta}$ is time independent.

Then, there exists a compact attractor $A_{reg} \subset V^{-\theta_2}$ for the system (6.1) which attracts the bounded sets of $V^{-\theta_2}$. Moreover, $A_{reg}$ is connected and it is the maximal bounded attractor in $V^{-\theta_2}$.

Remark 6.2. (a) For instance, the 3D Navier-Stokes-Voight (NSV) equation satisfies all the requirements (i)-(vii) of Theorem 6.1. Note that the existence of the global attractor and several properties for the 3D NSV were derived in [53, 54]. The assumptions (i)-(iv) of the above theorem insure that there is a continuous (nonlinear) semigroup for problem (6.1).

\begin{equation}
S_{NS}(t) : V^{-\theta_2} \rightarrow V^{-\theta_2}, \quad t \geq 0,
\end{equation}

$u_0 \rightarrow u(t)$.

(b) The final assumptions (v)-(vii) are used to derive that $S_{NS}$ is asymptotically smooth, as in the proof of Theorem 5.1. Indeed, actual modifications of the proof of Theorem 5.1 are not necessary (i.e., we can let $\phi \equiv 0$ in (2.2) in order to completely decouple the system); we use a suitable decomposition of the velocity $u = u_d + u_c$, such that $u_d(t) = S^d_{NS}(t) u_0$ solves

\begin{equation}
\begin{aligned}
\partial_t u_d + A_0 u_d + B_0(u, u_d) &= 0, \\
 u_d(0) &= u_0,
\end{aligned}
\end{equation}

and $u_c(t) = S^c_{NS}(t) u_0$ solves

\begin{equation}
\begin{aligned}
\partial_t u_c + A_0 u_c + B_0(u, u_c) &= g, \\
 u_c(0) &= 0.
\end{aligned}
\end{equation}

We can now obtain the following result which is also new in the literature. For instance, it allows us to derive the existence of an exponential attractor for the 3D NSV model, which was not reported anywhere else. Recall again that $\theta = 0$.

Theorem 6.3. Let the assumptions of Theorem 6.1 (i)-(v), (vii) be satisfied, and instead of (vi), assume that $b_0 : V^{-\theta_2} \times V^{-\theta_2} \times V^{\beta} \rightarrow \mathbb{R}$ is bounded. Then $(S_{NS}, V^{-\theta_2})$ possesses an exponential attractor $M_{reg} \subset V^\beta$ which is bounded in $V^\beta$. Thus, by definition, we have

(a) $M_{reg}$ is compact and semi-invariant with respect $S_{NS}(t)$, that is,

$S_{NS}(t)(M_{reg}) \subseteq M_{reg}, \quad \forall t \geq 0$.

(b) The fractal dimension $\dim_F (M_{reg}, V^{-\theta_2})$ of $M_{reg}$ is finite and an upper bound can be computed explicitly.

(c) $M_{reg}$ attracts exponentially fast any bounded subset $B$ of $V^{-\theta_2}$, that is, there exists a positive nondecreasing function $Q$ and a constant $\rho > 0$ such that

$\text{dist}_{V^{-\theta_2}}(S_{NS}(t) B, M_{reg}) \leq Q(\| B \|_{V^{-\theta_2}}) e^{-\rho t}, \quad \forall t \geq 0$.

Both $Q$ and $\rho$ can be explicitly calculated.

Proof. The proof requires only minor modifications from that of Theorem 5.1. We briefly sketch the main two (crucial) steps, the existence of a compact exponentially attracting set in $V^{-\theta_2}$, and the smoothing property for $(S_{NS}, V^{-\theta_2})$.

Step (i). We claim that there exists a bounded subset of $V^\beta$ which is positively invariant for $S_{NS}(t)$ and attracts any bounded set $B$ of $V^{-\theta_2}$ exponentially fast. We work with the semigroup decomposition $S(t) = S^d(t) S^c(t)$. First it is clear that $\| u_d(t) \|^2_{\theta_2} \leq e^{-c_{\theta_2} t} \| u_0 \|^2_{\theta_2}$, for all $t \geq 0$. In order to show the bound for the $u_c$-component, we pair the first equation of (6.1) with $A^\beta u_c$, and then
use the fact that \( b_0 : V^{-\theta_2} \times V^{-\theta_2} \times V^{-\beta} \to \mathbb{R} \) is bounded. After standard transformations, we finally have the following inequality:

\[
\frac{d}{dt} \| u_c \|_{\beta}^2 + c_{A_0} \| u_c \|_{\beta}^2 \lesssim \| u \|_{-\theta_2}^2 (\| u_d \|_{-\theta_2}^2 + \| u \|_{-\theta_2}^2) + \| g \|_{\beta}^2 \leq C,
\]

for some positive constant \( C \) which depends only on \( B_{\theta_2}, c_{A_0} \) and \( g \). Hence, noting once again that \( u_c(0) = 0 \), application of Lemma 5.1 (Appendix) yields

\[
\| u_c(t) \|_{\beta}^2 \leq R_\beta, \quad \text{for all } t \geq 0,
\]

for some constant \( R_\beta > 0 \) which is independent of time and initial data. Let now

\[
\mathcal{D}_\beta := \{ u \in V^\beta : \| u \|_{\beta} \leq R_\beta \}.
\]

On the other hand, recalling that \( \| S^d_{NS}(t) u_0 \|_{-\theta_2} \lesssim e^{-\rho t} \), for all \( t \geq 0 \), \( 6.6 \) implies that \( \mathcal{D}_\beta \) attracts \( B_{\theta_2} \), exponentially fast, that is,

\[
dist_{V^{-\theta_2}} (S_{NS}(t) B_{\theta_2}, \mathcal{D}_\beta) \leq \| S_{NS}(t) u_0 \|_{-\theta_2} \lesssim e^{-\rho t}, \quad \text{for all } t \geq 0.
\]

Since by 44 Theorem 5.1 we already know that for every nonempty bounded subset \( B \) of \( V^{-\theta_2} \),

\[
dist_{V^{-\theta_2}} (S_{NS}(t) B, \mathcal{D}_\beta) \lesssim e^{-\zeta t}, \quad \text{for all } t \geq 0,
\]

as usual we can appeal to (6.7) and the transitivity property of the exponential attraction 28 Theorem 5.1 to infer

\[
dist_{V^{-\theta_2}} (S_{NS}(t) B, \mathcal{D}_\beta) \lesssim e^{-\kappa t}, \quad \text{for all } t \geq 0,
\]

for some \( \kappa > 0 \) depending only on \( \rho, \zeta \). Note that (6.8) entails that \( \mathcal{D}_\beta \) is a compact (exponentially) attracting set in \( V^{-\theta_2} \) for \( S_{NS}(t) \). By enlarging \( R_\beta > 0 \) if necessary, the claim follows easily.

\textbf{Step (ii).} One argues as follows: for \( u_i \in \mathcal{D}_\beta \), let \( u_i = S_{NS}(t) u_{0i}, i = 1, 2 \), be the corresponding solutions of (6.1), and decompose \( u(t) = u_1(t) - u_2(t) \) such that \( u(t) = \nu(t) + \omega(t) \), where \( \nu(t) \) solves

\[
\begin{aligned}
\frac{d}{dt} \nu + A_0 \nu + B_0 (u_2, \nu) &= 0, \\
\nu(0) &= u_{01} - u_{02},
\end{aligned}
\]

and \( \omega(t) \) solves

\[
\begin{aligned}
\frac{d}{dt} \omega + A_0 \omega + B_0 (u, u_1) + B_0 (u_2, \omega) &= 0, \\
\omega(0) &= 0,
\end{aligned}
\]

As in the proof of Theorem 5.1.8 one can show that

\[
\| \nu(t) \|_{-\theta_2}^2 \leq \exp^{-c_{A_0} t} \| u_{01} - u_{02} \|_{-\theta_2}^2, \quad \text{for all } t \geq 0
\]

and

\[
\| \omega(t) \|_{\beta}^2 \leq \lambda(t) \| u_{01} - u_{02} \|_{-\theta_2}^2, \quad \text{for all } t \geq 0,
\]

for some positive continuous function \( \lambda : \mathbb{R}_+ \to \mathbb{R}_+, \lambda(0) > 0 \). These final estimates entail that the mapping \( S_{NS}(t) \) enjoys the smoothing property in the sense of Theorem 5.6 (Appendix), assumption (H4). We leave the other (minor) details for the interested reader to check.

As a consequence of this result, we also have the following.

\textbf{Corollary 6.4.} With the assumptions of Theorem 6.3 the global attractor \( \mathcal{A}_{\text{reg}} \) is bounded in \( V^\beta \) and has finite fractal dimension.

Finally, we state the analogue of Theorem 6.3 for problem (6.1) in the case \( \theta > 0 \). Note that in contrast to the case studied in Section 5.1 (see the proof of Theorem 5.8) we can construct the exponential attractor directly on the set \( V^\beta \), for some \( \beta > -\theta_2 \).

\textbf{Theorem 6.5.} Let the assumptions of Theorems 5.2 and 5.4 be satisfied for some \( \theta > 0 \). In addition, for some \( \beta \in (-\theta_2, \theta - \theta_2) \), let the following conditions hold.

(i) \( b_0 : V^\beta \times V^\beta \times V^{\theta - \beta} \to \mathbb{R} \) is bounded;

(ii) \( g \in V^\beta \) is time independent.

Then \( (S_{NS}, V^{-\theta_2}) \) possesses an exponential attractor \( \hat{\mathcal{M}} \subset V^\beta \) which is bounded in \( V^\beta \). Thus, by definition, we have

(a) \( \mathcal{M} \) is compact and semi-invariant with respect \( S_{NS}(t) \), that is,

\[
S_{NS}(t)(\mathcal{M}) \subset \mathcal{M}, \quad \forall t \geq 0.
\]
(b) The fractal dimension $\dim_P(\mathcal{M}, V^{-\theta_2})$ of $\mathcal{M}$ is finite and an upper bound can be computed explicitly.

(c) $\mathcal{M}$ attracts exponentially fast any bounded subset $B$ of $V^{-\theta_2}$, that is, there exist a positive nondecreasing function $Q$ and a constant $\rho > 0$ such that

$$\text{dist}_{V^{-\theta_2}}(S_{NS}(t)B, \mathcal{M}) \leq Q(\|B\|_{V^{-\theta_2}})e^{-\rho t}, \ \forall t \geq 0.$$ 

Both $Q$ and $\rho$ can be explicitly calculated.

Proof. We sketch the proof by only showing that the semigroup $S_{NS}(t)$ enjoys the smoothing property. By [44] Theorem 5.1, we recall that there exists a time $t_4 > 0$ such that

$$\sup_{t \geq t_4} \left( \|u(t)\|_2^2 + \int_{t_4}^{t_1} \|u(s)\|_{\theta + \theta_2}^2 ds \right) \leq C,$$

for some positive constant $C$ independent of $t$ and the initial data. Setting $v = u_1 - u_2$, and recalling that each $u_i$ is a solution of (6.1), we observe that $v$ solves the following problem

$$\partial_t v + A_0 v + B_0 (v, u_1) + B_0 (u_2, v) = 0.$$

First, owing to [44] Theorem 3.5, for all $t \geq 0$ we have the following estimate

$$\|v(t)\|_{-\theta_2}^2 + \int_0^t \|v(s)\|_{\theta - \theta_2}^2 ds \leq \lambda(t) \|v(0)\|_{\theta_2}^2,$$

for some positive continuous function $\lambda \in C([\mathbb{R}_+, \mathbb{R}_+),$ with $\lambda(0) > 0$. Pairing (6.12) with $\Lambda^{2\beta} v,$ we infer

$$\frac{1}{2} \frac{d}{dt} \|v\|_2^2 + c A_0 \|v\|_{\theta + \beta}^2 \leq b_0(v, u_1, \Lambda^{2\beta} v) + b_0(u_2, v, \Lambda^{2\beta} v).$$

The first two terms are bounded using the assumption (i) above. We have

$$b_0(v, u_1, \Lambda^{2\beta} v) \leq C \delta^{-1} \|u_1\|_3 \|v\|_3^3 + \delta \|v\|_{\beta + \theta}^2,$$

$$b_0(u_2, v, \Lambda^{2\beta} v) \leq C \delta^{-1} \|u_2\|_3 \|v\|_3^3 + \delta \|v\|_{\beta + \theta}^2,$$

for any $\delta > 0$. Inserting the above estimates from (6.15) into (6.14), and choosing $\delta = c A_0 / 2 > 0$ sufficiently small, we deduce the following inequality

$$\frac{d}{dt} \|v(t)\|_2^2 \leq C \delta^{-1} \left( \|u_1\|_3^3 + \|u_2\|_3^3 \right) \|v(t)\|_3^3 \leq C \|v(t)\|_2^3,$$

owing once more to the uniform estimate (6.11). Multiplying now both sides of this inequality by $\mathcal{I} := t - t_4$ and integrating the resulting relation over $(t_4, t),$ thanks to (6.13) we get

$$\|v(t)\|_2^3 \leq C (\mathcal{I} + 1) (\mathcal{I})^{-1} e^{C \mathcal{I}} \|v(0)\|_{\theta_2}^2,$$

for all $\mathcal{I} > 0$, which entails the required smoothing property for $S_{NS}(t)$. From this point on, the rest of the proof goes essentially as in the proof of Theorem 5.8. We leave the details to the interested reader. \hfill $\square$

Remark 6.6. We can also argue as in the proof of Lemma 5.5 to deduce further regularity properties for $\mathcal{M}$. Finally, note that the assumptions of the last two theorems apply to 3D Navier-Stokes-$\alpha$ (NS-$\alpha$) equations, the 3D Leray-$\alpha$ models, the modified 3D Leray-$\alpha$ models, the simplified 3D Bardina models, the 3D Navier-Stokes-Voigt (NSV) equations, and many other models not explicitly stated anywhere in the literature.

7. REMARKS ON A SIMPLIFIED ERICKSEN–LESLIE MODEL FOR LIQUID CRYSTALS

If we take $\phi$ as a vector, say $d \in \mathbb{R}^n$, then our regularized system (1.9) can be used to model the motion of liquid crystal flows in an $n$-dimensional compact Riemannian manifold $\Omega$, with $n = 2, 3$:

$$\begin{align*}
\partial_t u + A_0 u + (M u \cdot \nabla)(Nu) + \chi \nabla(Mu) \cdot (Nu) + \nabla p &= -\varepsilon \text{div} (\nabla d \otimes \nabla d) + g, \\
\text{div} (u) &= 0, \\
\partial_t d + N u \cdot \nabla d + \gamma (A_1 d + f (d)) &= 0, \\
u (0) &= u_0, \\
d (0) &= d_0,
\end{align*}$$

where, for instance, $A_0, A_1, M,$ and $N$ are the same bounded linear operators as (1.10). In this context, the positive constants $\varepsilon, \gamma$ stand for the competition between kinetic energy and
potential energy, and the macroscopic elastic relaxation time (Deborah number) for the molecular orientation field, respectively. Generally speaking, the system \((7.1)\) may be viewed as a macroscopic continuum description of the time evolutions of liquid crystal materials influenced by both the flow field \(u(x,t)\), and the microscopic orientational configuration \(d(x,t)\). The first and second equations combine the laws describing the incompressible (regularized) flow of fluid and an extra nonlinear coupling term, which is the induced elastic stress from the elastic energy through the transport, represented by the third equation which is a second-order parabolic equation with \(f(d) = \nabla dF(d)\). Here \(F(d) = (|d|^2 - 1)^2\) is a potential function used to approximate the constraint \(|d| = 1\), see [23, 57].

Problem \((7.1)\) with the following choice of operators
\[
A_0 = P (-\nu \Delta), \quad M = N = I \quad \text{and} \quad \chi = 0
\]
has been investigated on a two-dimensional compact Riemannian manifold in [69], where the existence of a global attractor was also proved. Global existence and regularity results in bounded domains \(\Omega \subset \mathbb{R}^n\) were also derived in [53] for the first time (see also [26, 27, 45] for related results in three dimensions). The longtime behavior of the system \((7.1)\), under the hypothesis \((7.2)\) and various boundary conditions, was also investigated recently in [9, 39, 82] in the case of bounded domains \(\Omega \subset \mathbb{R}^n, \ n = 2, 3\). In all these cases, a maximum principle as stated in Proposition 5.1 holds for the \(d\)-component of any weak solution \((u,d)\) of \((7.1)\). On account of this fact, all the results on well-posedness, regularity and singular perturbations, the existence of global and exponential attractors, and convergence to single equilibria, as stated in Sections 3, 4 and 5 remain valid without any essential modifications for the family of regularized problems \((7.1)\).

To avoid redundancy, we refrain from explicitly stating these results and their obvious proofs. In particular, we recover the result on existence of global attractors for the Lagrangian averaged liquid crystal equations, which consists of the Navier-Stokes-\(\alpha\)-model coupled with the equation for the orientation parameter \(d\) from \((7.1)\). This case was reported in [69, Theorem 7.1].

8. Appendix

In this section, we include some supporting material on Grönwall-type inequalities, Sobolev inequalities, some definitions and abstract results. The first lemma is a slight generalization of the usual Grönwall-type inequality [75]: its proof is quite elementary and thus omitted.

**Lemma 8.1.** Let \(E : \mathbb{R}_+ \to \mathbb{R}_+\) be an absolutely continuous function satisfying
\[
\frac{d}{dt}E(t) + 2\eta E(t) \leq h(t)E(t) + l(t) + k,
\]
where \(\eta > 0\), \(k \geq 0\) and \(\int_{t}^{t+1} h(\tau) \, d\tau \leq \eta(t-s) + m\), for all \(t \geq s \geq 0\) and some \(m \in \mathbb{R}\), and \(\int_{t}^{t+1} l(\tau) \, d\tau \leq \gamma < \infty\). Then, for all \(t \geq 0\),
\[
E(t) \leq E(0)e^{m}e^{-\eta t} + \frac{2\gamma e^{m+\eta}}{e^{\eta} - 1} + \frac{ke^{m}}{\eta}.
\]

With \(s, \rho \in \mathbb{R}_+\), let \(W^{s,p}\) be the standard Sobolev space on an \(n\)-dimensional compact Riemannian manifold with \(n \geq 2\). The following result states the classical Gagliardo-Nirenberg-Sobolev inequality (cf. [7, 42, 66] and [18, 19]).

**Lemma 8.2.** Let \(0 \leq k < m\) with \(k, m \in \mathbb{N}\) and numbers \(p, q, q \in [1, \infty]\) satisfy
\[
k - \frac{n}{p} = \tau \left(m - \frac{n}{q}\right) - (1 - \tau) \frac{n}{\tau}.
\]
Then there exists a positive constant \(C\) independent of \(u\) such that
\[
\|u\|_{W^{k,p}} \leq C \|u\|_{W^{m,q}}^{1 - \tau} \|u\|_{L^{p/q}}^{\tau},
\]
with \(\tau \in \left[\frac{k}{m}, 1\right]\) provided that \(m - k - \frac{n}{\tau} \notin \mathbb{N}_0\), and \(\tau = \frac{k}{m}\) provided that \(m - k - \frac{n}{\tau} \in \mathbb{N}_0\).
**Lemma 8.3.** Let $s$, $s_1$, and $s_2$ be real numbers satisfying

$$s_1 + s_2 \geq 0, \quad \min(s_1, s_2) \geq s, \quad \text{and} \quad s_1 + s_2 - s > \frac{n}{2},$$

where the strictness of the last two inequalities can be interchanged if $s \in \mathbb{N}_0$. Then, the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$H^{s_1} \otimes H^{s_2} \to H^s.$$

We have the following definition of exponential attractor (also known as inertial set).

**Definition 8.4.** Let $(S(t), K)$ be a dynamical system on a given Banach space $K$. A set $M \subset K$ is said to be an exponential attractor (also known as inertial set) for the semigroup $S(t)$ provided that the following statements hold:

(i) The sets $M$ are positively invariant with respect to the semigroup $S(t)$, that is, $S(t)M \subset M$, for all $t \geq 0$.

(ii) The fractal dimension of the sets $M$ is finite, that is, $\dim_F (M, K) \leq C < \infty$, where $C > 0$ can be computed explicitly.

(iii) Each $M$ attracts exponentially any bounded subset of $K$, that is, there exist a positive constant $\rho$ and a monotone nonnegative function $Q$, such that, for every bounded subset $B$ of $K$, we have

$$\text{dist}_K (S(t)B, M) \leq Q(\|B\|_K)e^{-\rho t},$$

where $\text{dist}_K (X, Y) := \sup_{x \in X} \inf_{y \in Y} \|x - y\|_K$ is the Hausdorff semidistance.

We report the following basic abstract results (see [37, Theorem 4.4], [33, 32]; cf. also [65]) which are needed in order to prove Theorem 5.8 when $\theta > 0$ and Theorem 5.18 in the case $\theta = 0$.

**Theorem 8.5.** Let $X_1$ and $X_2$ be two Banach spaces such that $X_2$ is compactly embedded in $X_1$. Let $X_0$ be a bounded subset of $X_2$ and consider a nonlinear map $\Sigma : X_0 \to X_0$ satisfying the smoothing property

$$(8.1) \quad \|\Sigma (x_1) - \Sigma (x_2)\|_{X_2} \leq d\|x_1 - x_2\|_{X_1},$$

for all $x_1, x_2 \in X_0$, where $d > 0$ depends on $X_0$. Then the discrete dynamical system $(X_0, \Sigma^n)$ possesses a discrete exponential attractor $\mathcal{E}_M \subset X_2$, that is, a compact set in $X_1$ with finite fractal dimension such that

$$(8.2) \quad \Sigma (\mathcal{E}_M) \subset \mathcal{E}_M,$$

$$(8.3) \quad \text{dist}_{X_1} (\Sigma^n (X_0), \mathcal{E}_M) \leq d_X e^{-\rho_* n}, \quad n \in \mathbb{N},$$

where $d_X$ and $\rho_*$ are positive constants independent of $n$, with the former depending on $X_0$.

**Theorem 8.6.** Let $K$, $K_c$ be two Banach spaces such that $K_c$ is compactly embedded in $K$, and let $(S(t), K)$ be a dynamical system. Assume the following hypotheses hold:

(H1) There exists a bounded subset $B \subset K$ which is positively invariant for $S(t)$ and attracts any bounded set of $K$ exponentially fast.

(H2) There exists a positive constant $C$ independent of time such that

$$\|S(t)\varphi_1 - S(t)\varphi_1\|_K \leq \rho(t) \|\varphi_1 - \varphi_2\|_K,$$

for every $t \geq 0$, and every $\varphi_1, \varphi_2 \in B$, where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is some continuous function with $\rho(0) > 0$.

(H3) There exist a positive constant $C$, $\kappa \in (0, 1)$ and a time $t^* > 0$ such that

$$\|S(t)\varphi_0 - S(\tilde{t})\varphi_0\|_K \leq C|t - \tilde{t}|^\kappa,$$

for all $t, \tilde{t} \in [t^*, 2t^*]$ and any $\varphi_0 \in B$.

(H4) For every $\varphi_{01}, \varphi_{02} \in B$, $S(t)$ can be decomposed as follows:

$$S(t)\varphi_{01} - S(t)\varphi_{02} = D(t) (\varphi_{01}, \varphi_{02}) + N(t) (\varphi_{01}, \varphi_{02})$$

where, for all $t \geq 0$, we have

$$\|D(t) (\varphi_{01}, \varphi_{02})\|_K \leq K e^{-\alpha t} \|\varphi_{01} - \varphi_{02}\|_K,$$

$$\|N(t) (\varphi_{01}, \varphi_{02})\|_K \leq \rho(t) \|\varphi_{01} - \varphi_{02}\|_K,$$

for some positive constants $\alpha, K$ independent of time, and some positive continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, $\rho(0) > 0$. 

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Then, if \((H1)-(H4)\) are satisfied, there exists an exponential attractors \(\mathcal{M}\) for \((S(t),\mathcal{K})\) in the sense of Definition \((i)-(iii)\).

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