CRITICAL SCALING FOR THE SIMPLE SIS STOCHASTIC EPIDEMIC

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Abstract. We exhibit a scaling law for the critical SIS stochastic epidemic: If at time 0 the population consists of $\sqrt{N}$ infected and $N - \sqrt{N}$ susceptible individuals, then when time and number currently infected are both scaled by $\sqrt{N}$, the resulting process converges, as $N \to \infty$, to a diffusion process related to the Feller diffusion by a change of drift. As a consequence, the rescaled size of the epidemic has a limit law that coincides with that of a first-passage time for the standard Ornstein-Uhlenbeck process. These results are the analogues for the SIS epidemic of results of Martin-Löf [9] for the simple SIR epidemic.

1. Introduction

Among the most thoroughly studied stochastic epidemic models are the simple SIR, and SIS epidemics (see Weiss and Dishon [13] for the origin of the SIS model); and among the many problems associated with these models perhaps the most basic and most interesting have to do with the duration and size of the epidemic. When the epidemic is either sub- or supercritical, the large-population behavior of the duration and size is reasonably well-understood: in the subcritical case, the epidemic is stochastically dominated by a subcritical Galton-Watson process (see below), and in the supercritical case the epidemic may become endemic (see Norden [12] and Kryscio and Lefèvre [7]). For critical epidemics, large-population asymptotics are more delicate. In 1998, Martin-Löf [9] (see also Aldous [11]) showed that the size $S = S_N$ has an interesting and nontrivial asymptotic behavior as the population size $N \to \infty$: If the number $X_0$ of individuals initially infected is of order $bN^{1/3}$, then the size $S_N$ (defined to be the sum of the total infection times over the whole population) has a limit distribution

$$S_N/N^{2/3} \overset{D}{\to} T_b^*$$

where $T_b^*$ is the first passage time of $W(t) + t^2/2$ to the level $b$, and $W(t)$ is a standard Wiener process. Furthermore, $1/3$ is the critical exponent, in that the quadratic drift is not felt if $X_0$ is much smaller than $N^{1/3}$. If
$X(0) \sim bN^\alpha$ for some $\alpha < 1/3$ then $S_N/N^{2\alpha}$ converges in law to the first passage time of $W(t)$ to the level $b$. (Martin-Löf neither proved nor stated this, but it can be deduced from his methods.)

The purpose of this note is to establish an analogous scaling law for the critical SIS epidemic: If the number $X_0$ of individuals initially infected is of order $b\sqrt{N}$, then as $N \to \infty$,

$$(2) \quad S_N/N \xrightarrow{D} \tau_b,$$

where $\tau_b$ is the time of first passage to 0 by a standard Ornstein-Uhlenbeck process started at $b$. Our approach has a rather different character than those of Aldous and Martin-Löf. We shall establish that the SIS epidemic process itself, suitably rescaled, converges in law to a diffusion process we dub the *attenuated Feller diffusion*. This has a law absolutely continuous with respect to, but not equal to, that of the standard Feller diffusion. (See equation (5) below for the stochastic differential equation governing the Feller diffusion, and (6) for the attenuated Feller diffusion.) Furthermore, we will show that 1/2 is the critical exponent, in the following sense: If $X(0) \sim bN^\alpha$ for some $\alpha < 1/2$ then the rescaled SIS process converges in law to a standard Feller diffusion with no drift. It will follow that the duration of the epidemic, rescaled by $\sqrt{X(0)}$, converges in law as $N \to \infty$ to the first passage time to 0 of the corresponding Feller diffusion.

Our analysis will show that the critical *scaling window* for the transmissivity parameter $\beta$ (see below) is on the order $1/\sqrt{N}$. This scaling window has also been observed — but in a different context — by Nasell [10], [11]. Nasell shows that the quasistationary distribution of the SIS epidemic undergoes a scaling transition when the transmissivity parameter varies from below $1 - O(N^{-1/2})$ to above $1 + O(N^{-1/2})$. This phenomenon does not seem to be directly linked to the critical scaling in our Theorem 1.

2. The SIS Epidemic and its Branching Envelope

2.1. SIS Model. The SIS epidemic is a continuous-time birth-death Markov chain $X_t = X(t)$ on the state space $[N] := \{0, 1, 2, \ldots, N\}$ whose infinitesimal transition probabilities are as follows:

$$(3) \quad P\{X(t+\delta t) = x+1 \mid X(t) = x\} = \beta x(1 - x/N)\delta t + o(x\delta t),$$

$$(4) \quad P\{X(t+\delta t) = x-1 \mid X(t) = x\} = x\delta t + o(x\delta t),$$

$$(5) \quad P\{X(t+\delta t) = x \mid X(t) = x\} = 1 - \beta x(1 - x/N)\delta t - x\delta t + o(x\delta t)).$$

These describe a population of $N$ individuals in which $X_t$ are infected and the remainder $N - X_t$ are susceptible to infection at time $t$. Infected individuals recover at rate 1, after which they once again become susceptible to (re-)infection, and susceptible individuals become infected at rate $\beta X_t/N$ proportional to the number of infected individuals in the population. The epidemic ends at the first time $T = T_N = t$ when $X_t = 0$ (note that state
0 is absorbing). The epidemic is said to be critical when $\beta = 1$, and nearly critical when $\beta = 1 + \lambda/\sqrt{N}$.

2.2. Branching Envelope. When the number of individuals infected is small compared to the population size, the epidemic evolves approximately as a continuous-time branching process $Z(t) = Z_t$ with infinitesimal transition probabilities

\begin{align}
P\{Z(t + \delta t) = z + 1 \mid Z(t) = z\} &= \beta z \delta t + o(z \delta t), \\
P\{Z(t + \delta t) = z - 1 \mid Z(t) = z\} &= z \delta t + o(z \delta t), \\
P\{Z(t + \delta t) = z \mid Z(t) = z\} &= 1 - (\beta + 1)z \delta t + o(z \delta t).
\end{align}

We shall refer to this process as the branching envelope of the SIS process. Observe that the death rate $x$ is the same as for the SIS epidemic, but the birth rate $\beta x$ dominates the birth rate $\beta x(1 - x/N)$ of the SIS process; the difference $\beta x^2/N$ will be called the attenuation or attrition rate. It is possible, by a standard construction, to build the SIS process $X(t)$ and its branching envelope $Z(t)$ on the same probability space in such a way that $X(0) = Z(0)$ and $X(t) \leq Z(t)$ for all $t \geq 0$. Thus, the size and duration of the SIS epidemic are stochastically dominated by the total progeny and extinction time of the branching envelope.

2.3. Critical Scaling for the Branching Envelope. It was proved by Feller [4] that a critical branching process, when properly renormalized, behaves approximately as a Feller diffusion with drift $\lambda Y_t$, that is, a solution of the stochastic differential equation

\begin{align}
dY_t = \lambda Y_t \, dt + \sqrt{Y_t} \, dW_t
\end{align}

where $W_t$ is a standard Wiener process. (Equivalently, the Feller diffusion with drift parameter $\lambda$ may be described as the diffusion process on $[0, \infty)$ with infinitesimal generator $G^\lambda = \lambda x \partial_x + x \partial_x^2/2$.) Feller’s theorem (see Jirina [8] and Lindvall [9] for the proof) asserts that if $Z^m(t)$ is a sequence of branching processes satisfying (4) with $\beta = \beta_m = 1 + \lambda/m$ and with $Z^m(0) \sim bm$ for some $b > 0$ and $\lambda \in \mathbb{R}$ then

\begin{align}
Z^m(mt)/m \xrightarrow{D} Y_t
\end{align}

where $Y_t$ is the Feller diffusion with drift parameter $\lambda$ and initial value $Y_0 = b$.

2.4. Critical Scaling for the SIS Process. Because the branching envelope stochastically dominates the SIS process, the scaling law (6) limits the duration and growth of the critical and near-critical SIS process. Since time is scaled by the factor $m$, where $Z^m(0) \sim bm$, it follows that the corresponding SIS started with $X(0) \sim bN^{\alpha}$ infected individuals cannot have duration longer than $O_P(N^{\alpha})$ time units. Consequently, we should expect that if the attenuation rate, divided by the scale factor $N^{\alpha}$ and integrated to time $N^{\alpha}$, is $o_P(1)$ then the limiting behavior of the rescaled SIS process $X(N^{\alpha}t)/N^{\alpha}$
should be no different from that of the branching envelope \(Z(N^\alpha t)/N^\alpha\). An easy calculation shows that this will be the case when \(\alpha < 1/2\). When \(\alpha = 1/2\), the accumulated attrition over the duration of the branching envelope will be on the same order of magnitude as the fluctuations, and so the rescaled SIS process should have a genuinely different asymptotic behavior from the branching envelope. Our main result makes this precise:

**Theorem 1.** Assume that process \(X(t) = X^N(t)\) has infinitesimal transition probabilities \(\beta N\). If for some constants \(\alpha \leq 1/2\) and \(b > 0\) the number of individuals initially infected satisfies \(X^N(0) \sim bN^\alpha\), and if the birth rate \(\beta\) satisfies \(\beta = \beta_N = 1 + \lambda/N^\alpha\), then as \(N \to \infty\),

\[
X^N(N^\alpha t)/N^\alpha \xrightarrow{D} Y_t
\]

where

(a) if \(\alpha < 1/2\) then \(Y_t\) is a Feller diffusion with drift \(\lambda\) and initial state \(Y_0 = b\);

(b) if \(\alpha = 1/2\) then \(Y_t\) is an “attenuated” Feller diffusion with drift \(\lambda\) and initial state \(Y_0 = b\), that is, \(Y_t\) is a solution to the stochastic differential equation

\[
dY_t = (\lambda Y_t - Y_t^2) \, dt + \sqrt{Y_t} \, dW_t
\]

with \(W_t\) a standard Wiener process.

Note that the “attenuation” term \(-Y_t^2\) in the drift of the limiting process \(Y_t\) can be guessed from the form of the attrition \(\beta x^2/N\). The proof of Theorem 1 will be given in section 4 below.

3. Size of the Epidemic

The size of an epidemic can be defined as the total number \(\xi\) of new infections during its entire course. Alternatively, it can be defined as the total infection time summed over the whole population:

\[
S = S_N = \int_0^T X_t \, dt.
\]

Although the two definitions are not the same, it can be shown that the two quantities have the same asymptotic behavior for large \(N\), that is, \(\xi_N \sim S_N\). (This follows from the fact that the length of the infection periods for infected individuals are i.i.d. unit exponential r.v.s.) Because the integral \(\int_0^T X_t \, dt\) is a continuous functional of the path \(X_t\) (relative to the Skorohod topology), Theorem 1 implies that if \(X(0) \sim b\sqrt{N}\) and \(\beta = 1 + \lambda/\sqrt{N}\) then

\[
S_N/N \xrightarrow{D} \int_0^{\tau(0)} Y_t \, dt
\]

where \(Y_t\) is the attenuated Feller diffusion \(Y_t\) with initial state \(Y_0 = b\) and \(\tau_0\) is the first passage time to 0 by \(Y_t\).
By an odd bit of luck, the instantaneous rate $Y_t\,dt$ at which infection time accrues coincides with the rate of change in accumulated quadratic variation of the semimartingale $Y_t$. (Note: In fact this is really no accident, but rather an artifact of the fundamental connection between Galton-Watson processes and random walks via the “depth-first search” algorithm. See [1] for more on this.) This suggests making the natural time change to the diffusion $Y_t$ so as to make the instantaneous quadratic variation constant. The new time scale $s = s(t)$ and the old $t$ are related by

\begin{equation}
 ds = Y_t\,dt,
\end{equation}

and so $\int Y_t\,dt = \int ds$ is the limit of the rescaled epidemic sizes $S_N/N$. The time-changed process $V_s = Y_{t(s)}$ satisfies the stochastic differential equation

\begin{equation}
 dV_s = (\lambda - V_s)\,ds + d\tilde{W}_s
\end{equation}

where $\tilde{W}_s$ is again a standard Wiener process. Setting $U_s = V_s - \lambda$, one obtains the stochastic differential equation for the standard Ornstein-Uhlenbeck process:

\begin{equation}
 dU_s = -U_s\,ds + d\tilde{W}_s
\end{equation}

This proves the following.

**Corollary 1.** If $X(0) \sim b\sqrt{N}$ and $\beta = 1 + \lambda/\sqrt{N}$ then

\begin{equation}
 S_N/N \xrightarrow{D} \tau(b - \lambda; -\lambda)
\end{equation}

where $\tau(x; y)$ is the time of first passage to $y$ by a standard Ornstein-Uhlenbeck process started at $x$.

The Laplace transforms of the distributions of $\tau(x; y)$ can be expressed in terms of parabolic cylinder (Weber) functions: see Darling and Siegert [2]. These do not invert easily. However, in the special case $\lambda = 0$ (the case corresponding to the critical SIS epidemic!), the distribution of $\tau(b; 0)$ has a simple closed form:

\begin{equation}
 P\{\tau(b; 0) > s\} = P^b\{U_s > 0\} - P^b\{U_s < 0\}.
\end{equation}

This can be obtained from a reflection principle, using the symmetry of the Ornstein-Uhlenbeck process about the origin.

**4. Proof of Theorem 1**

We prove Theorem 1 using the weak machinery developed in Ethier and Kurtz [3], which reduces the problem to checking convergence, in an appropriate sense, of infinitesimal generators. Write

\begin{equation}
 Y_t^N = X^N(N^\alpha t)/N^\alpha
\end{equation}
for the rescaled epidemic process, and denote by $E_y^N$ the corresponding expectation operator under the initial condition $Y_0^N = y$. For $f \in \hat{C}(0, \infty)$, define

\begin{equation}
G^N f(y) = \lim_{h \to 0} \frac{E_y^N [f(Y_h) - f(y)]}{h} \quad \text{and}
\end{equation}

\begin{equation}
G f(y) = (\lambda y) \cdot \frac{\partial f}{\partial y}(y) + y \cdot \frac{\partial^2 f}{\partial y^2}(y) \quad \text{if } \alpha < 1/2,
\end{equation}

\begin{equation}
G f(y) = (\lambda y - y^2) \cdot \frac{\partial f}{\partial y}(y) + y \cdot \frac{\partial^2 f}{\partial y^2}(y) \quad \text{if } \alpha = 1/2.
\end{equation}

By Corollary 1.2, sec. 11.2 of [3], the operator $G$ restricted to $\hat{C}(0, \infty) \cap C^2(0, \infty)$ generates a Feller semigroup on $\hat{C}(0, \infty)$, and Proposition 3.3, sec. 1.5 implies that $C_C(0, \infty)$ is a core for the generator. (An easy calculation, which we omit, shows that 0 is an exit boundary and $\infty$ is a natural boundary in both cases.) Moreover, the Markov processes determined by these Feller semigroups can be constructed so as to satisfy the stochastic differential equations (5) and (6), respectively. By Theorem 2.5 on page 167 and Theorem 6.1 on page 28, to prove convergence (17) it is enough to show that for each $f$ in the core of $G$ the generators converge in the sense of the following lemma.

**Lemma 1.** Let $f \in C^{\infty}_c(0, \infty)$ then

\begin{equation}
\lim_{N \to \infty} \sup_{y \in [N] / N^\alpha} \left| G^N f(y) - G f(y) \right| = 0.
\end{equation}

**Proof.** Consider first the case $\alpha = 1/2$. The first step is to calculate $G^N f$ for $f \in C^{\infty}_c(0, \infty)$. Using the infinitesimal transition probabilities (21), we have (with $x = yN^{1/2}$ and $h = tN^{1/2}$)

\[
E_y^N [f(Y_h^N) - f(y)] = \left[ f(1 + N^{1/2}) - f(y) \right] \times \left[ (1 + \lambda N^{-1/2})yN^{1/2}(1 - yN^{-1/2})hN^{1/2} \right] \\
+ \left[ f(1 - N^{-1/2}) - f(y) \right] \times \left[ yN^{1/2} \cdot hN^{1/2} \right] + o(Nhy)
\]

\[
= \left[ f(1 + N^{-1/2}) - f(y) \right] \times \left[ (yN^{1/2} + \lambda y - y^2 - \lambda y^2 N^{-1/2})hN^{1/2} \right] \\
+ \left[ f(1 - N^{-1/2}) - f(y) \right] \times \left[ yN^{1/2} \cdot h\sqrt{N} \right] + o(Nhy).
\]

The error term $o(Nhy)$ is uniform in $y$, because $f$ is assumed to have compact support. Taking the limit of this expression as $h \to 0$ yields

\[
G^N f(y) = (f(1 + N^{-1/2}) - f(y)) \times N^{1/2}(\lambda y - y^2 - \lambda y^2 N^{-1/2}) \\
+ \left[ (f(1 - N^{-1/2}) - f(y)) - (f(y) - f(y - N^{-1/2})) \right] \times Ny.
\]
Since \( f \in C^\infty_c([0,\infty)) \), there exists a constant \( C > 0 \) so that \( f \) and all its partial derivatives vanish for \( y > C \). Therefore, uniformly in all \( y \in [N]/\sqrt{N} \),

\[
\lim_{N \to \infty} G_N^N f(y) = \frac{\partial f}{\partial y}(y) \times (\lambda y - y^2) + \frac{\partial^2 f}{\partial y^2}(y) \times y = G f(y).
\]

A similar calculation establishes convergence of generators when \( \alpha < 1/2 \). \( \square \)

5. The SIR Epidemic Revisited

The continuous-time SIR epidemic differs from the SIS epidemic in that individuals may only be infected once: upon recovery, individuals are effectively removed from the population. Thus, the state at any time \( t \) is determined by two variables, the number currently infected (\( I(t) = I^N(t) \)) and the number removed (\( R(t) = R^N(t) \)). These take values in the set of nonnegative integer pairs \((i, r)\) such that \( 0 \leq i + r \leq N \), where \( N \) is the (original) population size. The instantaneous transition rates are as follows:

\[
(i, r) \mapsto (i - 1, r + 1) \quad \text{at rate } \ i dt; \\
(i, r) \mapsto (i + 1, r) \quad \text{at rate } \beta i(N - i - r) dt/N.
\]

All states \((i, r)\) with \( i = 0 \) are absorbing: the epidemic ends at the first time one of these states is visited.

As for the SIS epidemic, if the numbers of infected and removed individuals are small compared to the total population size \( N \), then the second transition rate in (21) reduces to \( \beta i dt \), and so the process \( I(t) \) evolves approximately as the branching process \( 4 \). Therefore, by the same logic as in sec. 2.4, the limiting behavior of the epidemic can be deduced by examination of the accumulated attrition over the duration of the branching process. The result is as follows.

**Theorem 2.** Assume that \((I^N(t), R^N(t))\) has instantaneous transition rates \( 21 \), and assume that \( R^N(0) = 0 \). If for some \( \alpha \leq 1/3 \) and \( b > 0 \) the number \( I^N(0) \) of individuals initially infected satisfies \( I^N(0) \sim bN^\alpha \), and if the birth rate \( \beta \) satisfies \( \beta = 1 + \lambda/N^\alpha \), then as \( N \to \infty \),

\[
\left( \begin{array}{c}
I^N(t) \\
R^N(t)
\end{array} \right) \xrightarrow{D} \left( \begin{array}{c}
I(t) \\
R(t)
\end{array} \right)
\]

where (i) if \( \alpha < 1/3 \), the limit process \((I(t), R(t))\) satisfies

\[
dI(t) = \lambda I(t) dt + \sqrt{I(t)} dW_t \\
dR(t) = I(t) dt;
\]

and (ii) if \( \alpha = 1/3 \),

\[
dI(t) = (\lambda I(t) - I(t)R(t)) dt + \sqrt{I(t)} dW_t \\
dR(t) = I(t) dt.
\]
Martin-Löf’s result [11] can be easily recovered from Theorem 2 by the same device as used in sec. 3 above. Define the new time scale \( s \) by \( ds = \int dt \) and corresponding time-changed process \( dJ(s) = dI(t) \). Then the total size of the epidemic is just the integral \( \int ds \) up to the time of first passage to 0 by \( J(s) \). But \( J(s) \) is just the Wiener process with a quadratic drift, and so (1) follows.

Theorem 2 can be proved either by martingale methods or by use of the Ethier-Kurtz machinery. The latter approach is mildly complicated by the fact that the generator

\[
G = (\lambda i - ir) \partial_i + \sqrt{i} \partial_{ii} + i \partial_r
\]

is not elliptic, but rather parabolic, and singular along the \( i = 0 \) axis. The singularity at \( i = 0 \) can be handled by truncating the state space: To prove (22), it suffices to prove weak convergence for the processes \( (I^N(t \land \tau_\epsilon), R^N(t \land \tau_\epsilon)) \), where \( \tau_\epsilon \) is the time of first passage to the level \( i = \epsilon \). Nonellipticity of the generator may be handled by using standard existence results from the theory of parabolic PDE (see [5]) to verify the hypotheses of the Hille-Yosida theorem (Th. 2.2 of [3]). Weak convergence of the truncated processes may then be proved by checking convergence of generators; this is another routine calculation similar to that carried out for the SIS epidemic in sec. 4 above.

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