A LIEB TYPE RESULT AND APPLICATIONS INVOLVING A CLASS OF NON-REFLEXIVE ORLICZ-SOBOLEV SPACE

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Abstract. In this paper we prove a Lieb type result in an Orlicz-Sobolev space that can be non-reflexive and use this result to show the existence of solution for a large class of quasilinear problem on a non-reflexive Orlicz-Sobolev space.

1. Introduction

This paper concerns the existence of weak solutions for a class of quasilinear elliptic problem of the type

\[(P_\lambda) \begin{cases} -\Delta \Phi u + \phi(|u|)u = \lambda f(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,\Phi}(\mathbb{R}^N), \end{cases} \]

where \(N \geq 1\), \(\lambda \in J = [a, b]\) with \(0 < a < b < +\infty\), and \(f : \mathbb{R} \to \mathbb{R}\) is a continuous function verifying some conditions that will be mentioned later on. It is important to recall that

\[\Delta \Phi u = \text{div}(\phi(|\nabla u|)\nabla u),\]

where \(\Phi : \mathbb{R} \to \mathbb{R}\) is a N-function of the form

\[\Phi(t) = \int_0^{|t|} s\phi(s) \, ds\]

and \(\phi : (0, +\infty) \to (0, +\infty)\) is a continuous function verifying some assumptions. More specifically, we shall consider the following conditions:

1. \(t \mapsto t\phi(t)\) is increasing for \(t > 0\).
2. \(\lim_{t \to 0} t\phi(t) = 0\) and \(\lim_{t \to +\infty} t\phi(t) = +\infty\).
3. \(\frac{t^2\phi(t)}{\Phi(t)} \geq l > 1, \quad \forall t > 0\).
4. \(\int_1^\infty \frac{\Phi^{-1}(t)}{t^{1+1/N}} \, dt < +\infty\).

The assumption (\(\phi_4\)) implies that the embedding

\[(1.1) \quad W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)\]

is continuous, see [1, Theorem 8.35]. In what follows, let us denote by \(\Lambda > 0\) the best constant that satisfies

\[(1.2) \quad \|u\|_{L^\infty(\mathbb{R}^N)} \leq \Lambda \|u\|, \quad \forall u \in W^{1,\Phi}(\mathbb{R}^N),\]

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More recently, Bocea and Mihăilescu \cite{bocea2008some}, where \( \Omega \) is a bounded domain, \( \Phi : \mathbb{R} \to \mathbb{R} \) with \( p > N \) satisfy \((\phi_1) - (\phi_4)\). Moreover, we recall that \( u \in W^{1,\Phi}(\mathbb{R}^N) \) is a weak solution of \((P_\lambda)\) whenever
\[
\int_{\mathbb{R}^N} \phi(|\nabla u|)|\nabla u| v \, dx + \int_{\mathbb{R}^N} \phi(|u|)u v \, dx = \lambda \int_{\mathbb{R}^N} f(u) v \, dx, \quad \forall v \in W^{1,\Phi}(\mathbb{R}^N).
\]

Quasilinear elliptic problems have been considered using different assumptions on the \(N\)-function \( \Phi \). Here we refer the reader to \cite{alves2006existence}, \cite{figueiredo2003existence}, \cite{silva2009existence} and references therein. In all of these works the so called \( \Delta_2 \)-condition has been assumed on the functions \( \Phi \) and \( \Phi^* \) (Complementary function of \( \Phi \)), which ensures that the Orlicz-Sobolev space \( W^{1,\Phi}(\Omega) \) is a reflexive Banach space. This assertion is used several times in order to get a nontrivial solution for elliptic problems taking into account the weak topology. In this paper, the main goal is the use of techniques that allows one to deal with problem \((P_\lambda)\) without assuming the \( \Delta_2 \)-condition on the \( N \)-function \( \Phi \). This difficulty brings us many difficulties when we intend to apply variational methods directly in \( W^{1,\Phi}(\mathbb{R}^N) \). In order to overcome these difficulties, the weak* topology together with the space \( W^1E^\Phi(\mathbb{R}^N) \) apply an important role in our approach.

In the recent years many researchers have studied the non-reflexive case. For example, in \cite{garci2001eigenvalues}, García-Huidobro, Khoi, Manásevich and Schmitt have considered existence of solution for the following nonlinear eigenvalue problem
\[
\begin{aligned}
\begin{cases}
-\Delta_{\Phi} u = \lambda \Psi(u), & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\tag{1.3}
\]
where \( \Omega \) is a bounded domain, \( \Phi : \mathbb{R} \to \mathbb{R} \) is a \( N \)-function and \( \Psi : \mathbb{R} \to \mathbb{R} \) is a continuous function verifying some others technical conditions. In that paper, the authors have studied the situation where \( \Phi \) does not satisfy the well known \( \Delta_2 \)-condition. More precisely, in the first part of that paper the authors consider the function
\[
\Phi(t) = (e^{t^2} - 1)/2, \quad \forall t \in \mathbb{R}.
\tag{1.4}
\]
More recently, Bocea and Mihăilescu \cite{bocea2008some} made a careful study about the eigenvalues of the problem
\[
\begin{aligned}
\begin{cases}
div(e^{\lambda u}|\nabla u|^2) - \Delta u = \lambda u, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\tag{1.5}
\]
After that, Silva, Gonçalves and Silva \cite{silva2009existence} considered existence of multiple solutions for a class of problem like \((1.3)\). In that paper the \( \Delta_2 \)-condition is not also assumed and the main tool used was the truncation of the nonlinearity together with a minimization procedure for the energy functional associated to the quasilinear elliptic problem \((1.3)\).

In \cite{silva2009existence}, Silva, Carvalho, Silva and Gonçalves study a class of problem \((1.3)\) where the energy functional satisfies the mountain pass geometry and the \( N \)-function \( \Phi^* \) does not satisfies the \( \Delta_2 \)-condition and has a polynomial growth. Still related to the mountain geometry, in \cite{alves2006existence}, Alves, Silva and Pimenta also considered the problem \((1.3)\) for a large class of function \( \Psi \), but supposing that the \( N \)-function \( \Phi \) has an exponential growth like \((1.4)\).

After a bibliographic review, we have observed that there is no any paper involving problem \((P_\lambda)\), by supposing that the \( N \)-function \( \Phi \) does not satisfies the \( \Delta_2 \)-condition. When the \( N \)-function \( \Phi \) satisfies the \( \Delta_2 \)-condition, the problem \((P_\lambda)\) has been considered by Alves, Figueiredo and Santos \cite{alves2006existence} for \( \lambda = 1 \). One of the main contribution of \cite{alves2006existence} was the following version of Lions type Lemma for Orlicz-Sobolev.
Lemma 1.1. (A Lions-type result for Orlicz-Sobolev spaces)
Assume that φ satisfies the following conditions:
The function φ(t)t is increasing in (0, +∞), that is,
\[(φ(t)t)' > 0 \ \forall t > 0. \] \tag{i}

There exist l, m ∈ (1, N) such that
\[l ≤ \frac{φ(|t|)t^2}{Φ(t)} ≤ m \ \forall t ≠ 0, \] \tag{ii}

where \(l ≤ m < l^*\), \(l^* = \frac{2N}{N-1}\) and \(m^* = \frac{mN}{N-m}\). If \((u_n) ⊂ W^1Φ(\mathbb{R}^N)\) is a bounded sequence such that there exists \(R > 0\) satisfying
\[\lim_{n → +∞} \sup_{y ∈ \mathbb{R}^N} \int_{B_R(y)} Φ(|u_n|) = 0, \]
then for any N-function \(B\) verifying \(Δ_2\)-condition with
\[\lim_{t → 0} \frac{B(t)}{Φ(t)} = 0 \] \tag{B1}
and
\[\lim_{|t| → +∞} \frac{B(t)}{Φ_*(t)} = 0, \] \tag{B2}
we have
\[u_n → 0 \ \text{in} \ L_B(\mathbb{R}^N). \]

In the lemma above, \(Φ_*\) denotes the Sobolev conjugate function of \(Φ\) defined by
\[Φ_*^{-1}(t) = \int_0^t \frac{Φ^{-1}(s)}{s(N+1)/N} ds \ \text{for} \ t > 0, \]
when
\[\int_1^{+∞} \frac{Φ^{-1}(s)}{s(N+1)/N} ds = +∞. \]

This lemma combined with the fact that the energy functional associated with \((P_λ)\) is invariant by translation yields in the existence of a nontrivial critical point, which is a nontrivial solution for \((P_λ)\). Here, it is very important to say that \((i) - (ii)\) ensure that \(Φ\) and \(Φ\) satisfy the \(Δ_2\)-condition, and so, the space \(W^1Φ(\mathbb{R}^N)\) is reflexive. Since we intend to work with a situation that \(Φ\) does not satisfy the \(Δ_2\)-condition, the Lemma 1.1 does not work in our case, and so, we need to develop a new strategy. To overcome this difficulty, we have proved a Lieb type result for Orlicz-Sobolev space \(W^1Φ(\mathbb{R}^N)\), whose the N-function \(Φ\) does not need to satisfy the \(Δ_2\)-condition. The Lieb type result that we have proved is the following:

Lemma 1.2. (A Lieb type result) Let \(Φ ∈ C^1(\mathbb{R}, [0, +∞))\) be a N-function and \((u_n) ⊂ W^1Φ(\mathbb{R}^N)\) such that \(\int_{\mathbb{R}^N} Φ(|∇u_n|) dx ≤ M\). If there are \(ε, δ > 0\) such that
\[mes(||u_n|| > ε) ≥ δ, \ \forall n ∈ \mathbb{N}, \]
then there is \((y_n) ⊂ \mathbb{R}^N\) such that \(v_n(x) = u_n(x + y_n)\) has a subsequence whose limit in \(L^Φ_{loc}(\mathbb{R}^N)\) is non trivial.
The reader is invited to see that the Lieb type result works as a Lions type result, in the sense that it permits to find a \((PS)\) sequence whose the weak limit is not trivial.

Hereafter, the continuous function \(f : \mathbb{R} \to \mathbb{R}\) satisfies the following assumptions:

\[(f_1)\]
\[
\lim_{t \to 0} \frac{f(t)}{\Phi(t/2)} = 0.
\]

\[(f_2)\]
\[
f(t) \leq \frac{1}{2b} \Phi'(t), \quad \forall t \in [0,2\Lambda],
\]

where \(\Lambda\) was given in \((1.2)\).

There exists \(\theta > 1\) such that

\[(f_3)\]
\[
0 < \theta F(t) \leq h(t)f(t)t, \quad \text{for } t > 0
\]

holds true with \(h(t) = \frac{\Phi(t)}{t^2 \Phi(t)}\), where \(F(t) = \int_0^t f(s)ds, t \in \mathbb{R}\).

The condition \((f_3)\) suggests that \(F\) is \(\Phi\)-superlinear, that is,

\[(1.6)\]
\[
\lim_{|t| \to +\infty} \frac{F(t)}{\Phi(t)} = +\infty.
\]

In fact, by fixing \(R > 0\) and integrating the sentence

\[
\theta \frac{t^2 \phi(t)}{\Phi(t)} \leq \frac{f(t)}{F(t)}, \quad t > R,
\]

we find that

\[(1.7)\]
\[
\frac{F(t)}{\Phi(t)} \geq \frac{F(M)}{\Phi(M)^q} \Phi(t)^{q-1} \to +\infty \quad \text{as } t \to +\infty.
\]

Here, we would like to point out that \(f(t) = \mu q(\Phi(t))^{q-1} \phi(t)t, \quad \text{for } q > 1\), satisfies the conditions \((f_1) - (f_3)\) for a convenient constant \(\mu\), when \(\Phi(t) = (e^{t^2} - 1)/2\) or \(\Phi(t) = |t|^p/p\) with \(p > N\).

Under these conditions our main result involving the existence of nontrivial solution for \((P_\lambda)\) is the following:

**Theorem 1.3.** Suppose that \(f\) satisfies \((f_1) - (f_3)\). Assume that \(\Phi\) satisfies \((\phi_1) - (\phi_4)\). Then, for almost every \(\lambda \in J = [a,b]\), problem \((P_\lambda)\) has a nontrivial solution.

It is important to stress that, to the best of our knowledge, Theorem 1.3 is the first existence result where the Mountain Pass Theorem has been used to deal with a quasilinear elliptic problem driven by a \(N\)-function that can have an exponential growth in whole \(\mathbb{R}^N\). Since we were not able to show the boundedness of \((PS)\) sequence for any \(\lambda \in J = [a,b]\), it was necessary to use a seminal result due to Jeanjean [16], see Theorem 4.3, to show the existence of bounded Palais-smale sequence associated with the mountain level for almost very \(\lambda \in J\). Furthermore, since we do not assume the \(\Delta_2\)-condition to hold, the space \(W^{1,\Phi}(\mathbb{R}^N)\) can be non-reflexive, which brings much more difficulty to ensure some convergences. Have this in mind, we have decide to work in the space \(W^{1,E^{\Phi}}(\mathbb{R}^N)\), because it is topologically more rich than \(W^{1,\Phi}(\mathbb{R}^N)\), for example, it is possible to prove that the energy functional is \(C^1(W^{1,E^{\Phi}}(\mathbb{R}^N), \mathbb{R})\).
2. Basics on Orlicz-Sobolev spaces

In this section we recall some properties of Orlicz and Orlicz-Sobolev spaces, which can be found in [1, 24]. First of all, we recall that a continuous function $\Phi : \mathbb{R} \to [0, +\infty)$ is a N-function if:

(i) $\Phi$ is convex.
(ii) $\Phi(t) = 0 \iff t = 0$.
(iii) $\lim_{t \to 0} \frac{\Phi(t)}{t} = 0$ and $\lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty$.
(iv) $\Phi$ is even.

We say that a N-function $\Phi$ verifies the $\Delta_2$-condition, if

$$
\Phi(2t) \leq K \Phi(t), \quad \forall t \geq 0,
$$

for some constant $K > 0$. For instance, it can be shown that $\Phi(t) = |t|^p/p$ for $p > 1$ satisfies the $\Delta_2$-condition, while $\Phi(t) = (e^t - 1)/2$ does not satisfy it.

In what follows, fixed an open set $\Omega \subset \mathbb{R}^N$ and a N-function $\Phi$, we define the Orlicz space associated with $\Phi$ as

$$
L^\Phi(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_\Omega \Phi\left(\frac{|u|}{\alpha}\right) dx < +\infty \text{ for some } \alpha > 0 \right\}.
$$

The space $L^\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm given by

$$
\| u \|_\Phi = \inf \left\{ \alpha > 0 : \int_\Omega \Phi\left(\frac{|u|}{\alpha}\right) dx \leq 1 \right\}.
$$

The complementary function $\tilde{\Phi}$ associated with $\Phi$ is given by its Legendre’s transformation, that is,

$$
\tilde{\Phi}(s) = \max_{t \geq 0} \{ st - \Phi(t) \}, \quad \text{for } s \geq 0.
$$

The functions $\Phi$ and $\tilde{\Phi}$ are complementary each other. Moreover, we also have a Young type inequality given by

$$
st \leq \Phi(t) + \tilde{\Phi}(s), \quad \forall s, t \geq 0.
$$

Using the above inequality, it is possible to prove a Hölder type inequality, that is,

$$
\left| \int_\Omega uv dx \right| \leq 2\| u \|_\Phi \| v \|_{\tilde{\Phi}}, \quad \forall u \in L^\Phi(\Omega) \text{ and } \forall v \in L^{\tilde{\Phi}}(\Omega).
$$

The corresponding Orlicz-Sobolev space is defined by

$$
W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega), \quad i = 1, ..., N \right\},
$$

endowed with the norm

$$
\| u \| = \| \nabla u \|_\Phi + \| u \|_\Phi.
$$

The space $W^{1,\Phi}_0(\Omega)$ is defined as the weak$^*$ closure of $C^\infty_0(\Omega)$ in $W^{1,\Phi}(\Omega)$. Here we refer the readers to the important works [14, 15]. The spaces $L^\Phi(\Omega)$, $W^{1,\Phi}(\Omega)$ and $W^{1,\Phi}_0(\Omega)$ are separable and reflexive, when $\Phi$ and $\tilde{\Phi}$ satisfy the $\Delta_2$-condition.

If $|\Omega| < +\infty$, $E^\Phi(\Omega)$ denotes the closure of $L^\infty(\Omega)$ in $L^\Phi(\Omega)$ with respect to the norm $\| . \|_\Phi$. When $|\Omega| = +\infty$, $E^\Phi(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in $L^\Phi(\Omega)$ with respect to the norm $\| . \|_\Phi$. In any one of these cases, $L^\Phi(\Omega)$ is the dual space of $E^\Phi(\Omega)$, while $L^{\tilde{\Phi}}(\Omega)$ is the dual
Since \( \int_\Omega \Phi(w) \) still denoted by itself, and \( \Phi \) satisfies the \( \Delta_2 \)-condition, and so, \( W^1_0E^\Phi(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) in \( W^1_0E^\Phi(\Omega) \) with respect to the norm \( \| \cdot \| \).

Before concluding this section, we would like to state a lemma whose proof follows directly from a result by Donaldson [11, Proposition 1.1].

**Lemma 2.1.** Assume that \( \Phi \) is a N-function. If \( (u_n) \subset W^1,\Phi(\Omega) \) is a bounded sequence, then there are a subsequence of \( (u_n) \), still denoted by itself, and \( u \in W^1,\Phi(\Omega) \) such that

\[
 u_n \xrightarrow{w} u \quad \text{in} \quad W^1,\Phi(\Omega)
\]

and

\[
 \int_\Omega u_nv \, dx \rightarrow \int_\Omega vw \, dx, \quad \int_\Omega \frac{\partial u_n}{\partial x_i}w \, dx \rightarrow \int_\Omega \frac{\partial u}{\partial x_i}w \, dx, \quad \forall v, w \in E^\Phi(\Omega).
\]

As an immediate consequence of the last lemma is the following result that applies an important role in our work.

**Corollary 2.2.** Assume that \( \Phi \) is a N-function. If \( (u_n) \subset W^1,\Phi(\Omega) \) is a bounded sequence with \( u_n \rightarrow u \) in \( L^\Phi_{\text{loc}}(\Omega) \), then \( u \in W^1,\Phi(\Omega) \).

The lemma just above is crucial when the space \( W^1,\Phi(\Omega) \) is not reflexive, for example if \( \Phi(t) = (e^t - 1)/2 \). However, if \( \Phi(t) = |t|^p/p \) and \( p > 1 \), the above lemma is not necessary since \( \Phi \) satisfies the \( \Delta_2 \)-condition, and so, \( W^1,\Phi(\Omega) \) is reflexive. Here we would like to point out that the condition \( (\phi_3) \) ensures that \( \tilde{\Phi} \) verifies the \( \Delta_2 \)-condition, for more details see Fukagai and Narukawa [12].

The next lemma is a technical results that will be used later on. It will be important because we are only supposing that \( \Phi \) is a N-function.

**Lemma 2.3.** (Almost weak converge in \( L^\Phi(\mathbb{R}^N) \)) Let \( (w_n) \subset L^\Phi(\mathbb{R}^N) \) be a bounded sequence with \( w_n(x) \rightarrow w(x) \) a.e. in \( \mathbb{R}^N \). Then, \( w \in L^\Phi(\mathbb{R}^N) \) and

\[
 \int_{\mathbb{R}^N} w_nv \, dx \rightarrow \int_{\mathbb{R}^N} vw \, dx, \quad \forall v \in C_0^\infty(\mathbb{R}^N).
\]

**Proof.** To begin with, we will prove that \( w \in L^\Phi(\mathbb{R}^N) \). If \( \|w_n\|_{L^\Phi(\mathbb{R}^N)} \rightarrow 0 \) we have that \( w_n \rightarrow 0 \) in \( L^\Phi(\mathbb{R}^N) \), and so, \( w = 0 \), finishing the proof.

In what follows, we will assume that \( \|w_n\|_{L^\Phi(\mathbb{R}^N)} \neq 0 \), consequently for some subsequence, still denoted by \( (w_n) \),

\[
 \|w_n\|_{L^\Phi(\mathbb{R}^N)} \geq \delta, \quad \forall n \in \mathbb{N},
\]

and

\[
 \|w_n\|_{L^\Phi(\mathbb{R}^N)} \rightarrow \alpha > 0.
\]

Since

\[
 \int_{\mathbb{R}^N} \Phi\left( \frac{|w_n|}{\|w_n\|_{L^\Phi(\mathbb{R}^N)}} \right) \, dx \leq 1, \quad \forall n \in \mathbb{N},
\]

the Fatou’s Lemma leads to

\[
 \int_{\mathbb{R}^N} \Phi\left( \frac{|w|}{\alpha} \right) \, dx \leq 1,
\]
from where it follows that \( w \in L^\Phi(\mathbb{R}^N) \).

Now, for a fixed \( v \in C_0^{\infty}(\mathbb{R}^N) \), we set \( \Omega = \text{supp}(v) \) and for \( k \in \mathbb{N} \)
\[
\Omega_k = \{ x \in \Omega : \forall n \geq k, \ |w_n(x) - w(x)| \leq 1 \}.
\]
Since \( w_n(x) \to w(x) \) a.e. in \( \mathbb{R}^N \), a simple computation gives
\[
\text{mes}(\Omega_k) \to \text{mes}(\Omega) \quad \text{and} \quad \text{mes}(\Omega \setminus \Omega_k) \to 0 \quad \text{as} \quad k \to +\infty.
\]
Given \( \epsilon > 0 \), let us fix \( k \) such that \( \|v\|_{L^\tilde{\Phi}(\Omega \setminus \Omega_k)} < \frac{\epsilon}{4M} \), where
\[
M = \max \left\{ \sup_{n \in \mathbb{N}} \|w_n\|_{L^\tilde{\Phi}(\Omega)}, \|w\|_{L^\tilde{\Phi}(\Omega)} \right\}.
\]
Using this information, we find
\[
\left| \int_\Omega w_n v \, dx - \int_\Omega w v \, dx \right| \leq \int_{\Omega_k} |w_n - u| v \, dx + \frac{\epsilon}{2}, \quad \forall n \in \mathbb{N}.
\]
By definition of \( \Omega_k \), for \( n \geq k \) we have
\[
|w_n(x) - w(x)| \leq 1, \quad \forall x \in \Omega_k.
\]
Hence by Lebesgue dominated convergence theorem
\[
\lim_{n \to +\infty} \int_{\Omega_k} |w_n - w| v \, dx = 0.
\]
Thus, there is \( n_0 = n_0(\epsilon, k) \in \mathbb{N} \) such that
\[
\left| \int_\Omega w_n v \, dx - \int_\Omega w v \, dx \right| < \epsilon, \quad \forall n \geq n_0,
\]
as asserted. \( \square \)

### 3. A Lieb type result

The main goal of this section is to show a Lieb type result for a large class of Orlicz-Sobolev spaces, without assuming the \((\Delta_2)\)-condition. A version of Lieb’s Lemma for Sobolev space can be found in Kavian [17, 6.2 Lemme].

The first lemma this section is a technical result that is a key point in the proof of the Lieb’s Lemma for Orlicz-Sobolev spaces.

**Lemma 3.1.** Let \( \Phi \in C^1(\mathbb{R}, [0, +\infty)) \) be a \( N \)-function and \( u \in W^{1, \Phi}(\mathbb{R}^N) \) such that \( \int_{\mathbb{R}^N} \Phi(|\nabla u|) \, dx \leq M \). Then, there is \( C_0 > 0 \) that does not depend on \( u \) and \( y_0 \in \mathbb{R}^N \) such that
\[
\left( 2 + M \left( \int_{\mathbb{R}^N} \Phi(|u/2|) \, dx \right)^{-1} \right)^N \text{mes}(B(y_0) \cap \text{supp}(u)) \geq C_0,
\]
where \( B(z) = \prod_{i=1}^N \left( z_i - \frac{1}{2}, z_i + \frac{1}{2} \right) \) for all \( z \in \mathbb{R}^N \).

**Proof.** First of all we claim that there is \( y_0 \in \mathbb{R}^N \) such that
\[
(3.1) \quad \int_{\mathbb{R}^N} \Phi(|\nabla u|) \chi_{B(y_0)} \, dx \leq \left( 1 + M \left( \int_{\mathbb{R}^N} \Phi(|u/2|) \, dx \right)^{-1} \right) \int_{\mathbb{R}^N} \Phi(|u/2|) \chi_{B(y_0)} \, dx,
\]
where \( \chi_{B(y_0)} \) is the characteristic function associated with the set \( B(y_0) \).
Otherwise, we must have

\[ M \geq \int_{\mathbb{R}^N} \Phi(|\nabla u|) \, dx \geq \left(1 + M \left(\int_{\mathbb{R}^N} \Phi(|u/2|) \, dx \right)^{-1}\right) \int_{\mathbb{R}^N} \Phi(|u/2|) \, dx > M, \]

which is impossible.

**Claim 3.2.** \( \Phi(|u/2|) \in W^{1,1}(B(y_0)) \).

Indeed, since \( \Phi \) is increasing

\[ \int_{B(y_0)} \Phi(|u/2|) \, dx \leq \int_{B(y_0)} \Phi(|u|) \, dx < +\infty. \]

On the other hand,

\[ \int_{B(y_0)} |\nabla \Phi(|u/2|)| \, dx = \frac{1}{2} \int_{B(y_0)} \Phi'(|u/2|) |\nabla u| \, dx. \]

By Young's inequality

\[ \int_{B(y_0)} |\nabla \Phi(|u/2|)| \, dx \leq \frac{1}{2} \int_{B(y_0)} \Phi(|\nabla u|) \, dx + \frac{1}{2} \int_{B(y_0)} \Phi'(|u/2|) \, dx. \]

Recalling that

\[ \bar{\Phi}(\Phi'(t)) \leq \Phi(2t), \quad \forall t > 0, \]

we get

\[ \int_{B(y_0)} |\nabla (\Phi(|u/2|))| \, dx \leq \frac{1}{2} \int_{B(y_0)} \Phi(|\nabla u|) \, dx + \frac{1}{2} \int_{B(y_0)} \Phi(|u|) \, dx. \]

The claim follows from (3.2) and (3.3).

Now, by using the continuous Sobolev embedding \( W^{1,1}(B(y_0)) \hookrightarrow L^{1*}(B(y_0)) \) where \( 1^* = \frac{N}{N-1} \), there is \( C_1 > 0 \) such that

\[ C_1 \|w\|_{L^{1*}(B(y_0))} \, dx \leq \int_{B(y_0)} (|\nabla w| + |w|) \, dx, \quad \forall w \in W^{1,1}(B(y_0)). \]

Hence

\[ (3.4) \]

\[ C_1 \left( \int_{B(y_0)} |\Phi(|u/2|)|^{1^*} \, dx \right)^{\frac{1}{1^*}} \leq \int_{B(y_0)} (|\nabla (\Phi(|u/2|))| + |\Phi(|u/2|)|) \, dx, \quad \forall u \in W^{1,1}(\mathbb{R}^N). \]

From (3.1)-(3.4),

\[ C_1 \left( \int_{B(y_0)} |\Phi(u/2)|^{1^*} \, dx \right)^{\frac{1}{1^*}} \leq \left(2 + M \left(\int_{\mathbb{R}^N} \Phi(|u/2|) \, dx \right)^{-1}\right) \int_{B(y_0)} \Phi(|u/2|) \, dx, \]

leading to

\[ C_1 \leq \left(2 + M \left(\int_{\mathbb{R}^N} \Phi(|u/2|) \, dx \right)^{-1}\right) mes[B(y_0) \cap supp(u)]^N, \]

that is,

\[ C_0 \leq \left(2 + M \left(\int_{\mathbb{R}^N} \Phi(|u/2|) \, dx \right)^{-1}\right) \frac{N}{mes[B(y_0) \cap supp(u)]}. \]

\[ \square \]
Now, we are ready to prove our Lieb type result, see Lemma 1.2.

**Proof of Lemma 1.2.** To begin with, we will apply Lemma 3.1 for the function \((|u_n| - \frac{\epsilon}{2})^+\).

Note that
\[
\int_{\mathbb{R}^N} \Phi\left(\frac{1}{2} \left(|u_n| - \frac{\epsilon}{2}\right)^+ \right) dx \geq \int_{|u_n| > \epsilon} \Phi\left(\frac{1}{2} \left(|u_n| - \frac{\epsilon}{2}\right)^+ \right) dx \geq \Phi\left(\frac{\epsilon}{4}\right) \text{mes}[|u_n| > \epsilon] \geq \Phi\left(\frac{\epsilon}{4}\right) \delta,
\]
from where it follows that
\[
\left(\int_{\mathbb{R}^N} \Phi\left(\frac{1}{2} \left(|u_n| - \frac{\epsilon}{2}\right)^+ \right) dx \right)^{-1} \leq \frac{1}{\Phi\left(\frac{\epsilon}{4}\right) \delta}.
\]

Since
\[
C_0 \leq \left(2 + M \left(\int_{\mathbb{R}^N} \Phi\left(\frac{1}{2} \left(|u_n| - \frac{\epsilon}{2}\right)^+ \right) dx\right)^{-1}\right)^N \text{mes} \left[B(y_n) \cap \text{supp} \left(|u_n| - \frac{\epsilon}{2}\right)^+\right],
\]
we get
\[
C_0 \leq \left(2 + M \frac{1}{\Phi\left(\frac{\epsilon}{4}\right) \delta}\right)^N \text{mes} \left[B(y_n) \cap \text{supp} \left(|u_n| - \frac{\epsilon}{2}\right)^+\right].
\]

On the other hand, as \(\text{supp} \left(|u_n| - \frac{\epsilon}{2}\right)^+ = [|u_n| \geq \frac{\epsilon}{2}]\), we derive
\[
\text{mes}[B(y_n) \cap [|u_n| \geq \frac{\epsilon}{2}]] \geq C_2, \quad \forall n \in \mathbb{N},
\]
for some \(C_2 > 0\). Now, note that
\[
\int_{B(0)} \Phi(|v_n|) dx \geq \int_{B(y_n) \cap [|u_n| \geq \frac{\epsilon}{2}]} \Phi(|u_n|) dx \geq \Phi\left(\frac{\epsilon}{4}\right) \text{mes}[B(y_n) \cap [|u_n| \geq \frac{\epsilon}{2}]]
\]
that is,
\[
\int_{B(0)} \Phi(|v_n|) dx \geq \Phi\left(\frac{\epsilon}{4}\right) C_2 = C_3 > 0, \quad \forall n \in \mathbb{N}.
\]
As \((v_n)\) is bounded, the compact embedding \(W^{1,\Phi}(\mathbb{R}^N) \to L^\Phi(B_R(0))\) for all \(R > 0\) ensures that \(v_n \to v\) in \(L^\Phi_{loc}(\mathbb{R}^N)\) for some subsequence. Thus,
\[
\int_{B(0)} \Phi(|v|) dx \geq C_3 > 0,
\]
showing that \(v \neq 0\), as asserted. \(\square\)

4. **Technical results**

Note that under hypotheses \((\phi_1) - (\phi_4)\), it is well known that \(\Phi\) might not satisfy the \(\Delta_2\)-condition, and as a consequence, \(W^{1,\Phi}(\mathbb{R}^N)\) might be non-reflexive anymore. Under these conditions, it is also well known that there exists \(u \in W^{1,\Phi}(\mathbb{R}^N)\) such that
\[
\int_{\mathbb{R}^N} \Phi(|\nabla u|) dx = +\infty.
\]
In order to avoid this problem, we will work with the space \(X = W^{1,E^\Phi}(\mathbb{R}^N)\), because in this space the functional \(Q : X \to \mathbb{R}\) given by
\[
(4.1) \quad Q(u) = \int_{\mathbb{R}^N} (\Phi(|\nabla u|) + \Phi(|u|)) dx
\]
belongs to $C^1(X, \mathbb{R})$. The proof this claim follows as in [13, Lemma 3.4]. Moreover, it is easy to see that $Q$ is strictly convex and l.s.c. with respect to the weak* topology.

However, independent of $\Delta_1$-condition, the condition $(f_1)$ guarantees that

$$\left| \int_{\mathbb{R}^N} F(u) \, dx \right| \leq C \int_{\mathbb{R}^N} \Phi(|u|) \, dx + \left( \max_{t \in [0, \Lambda]} |F(t)| \right) \text{mes}([|u| > \delta]), \quad \forall u \in X.$$ 

Having this in mind, the energy functional $I_\lambda : X \to \mathbb{R}$ associated with $(P_\lambda)$ given by

$$(4.2) \quad I_\lambda(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) \, dx + \int_{\mathbb{R}^N} \Phi(|u|) \, dx - \lambda \int_{\mathbb{R}^N} F(u) \, dx,$$

is well defined and $I_\lambda \in C^1(X, \mathbb{R})$ with

$$I_\lambda'(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u|)|\nabla u| \nabla v \, dx + \int_{\mathbb{R}^N} \phi(|u|)uv \, dx - \lambda \int_{\mathbb{R}^N} f(u)v \, dx, \quad \forall v \in X.$$

The next lemma is very important in our approach, because it shows that critical points of $I_\lambda$ in $X$ are in fact critical points in whole $W^{1, \Phi}(\mathbb{R}^N)$.

**Lemma 4.1.** If $u \in X$ is a critical point of $I_\lambda$ in $X$, then $u$ is critical point of $I_\lambda$ in $W^{1, \Phi}(\mathbb{R}^N)$, and so, $u$ is a weak solution of $(P_\lambda)$.

**Proof.** By hypothesis,

$$\int_{\mathbb{R}^N} \phi(|\nabla u|) \nabla u \nabla v \, dx + \int_{\mathbb{R}^N} \phi(|u|)uv \, dx = \lambda \int_{\mathbb{R}^N} f(u)v \, dx, \quad \forall v \in X.$$

This equality yields $\phi(|\nabla u|)|\nabla u|^2, \phi(|u|)|u|^2 \in L^1(\mathbb{R}^N)$. Since $\Phi(u), \Phi(|\nabla u|) \in L^1(\mathbb{R}^N)$, the below identity

$$(4.3) \quad \phi(s)s^2 = \Phi(s) + \Phi(\phi(s)), \quad s \geq 0,$$

ensures that $\phi(|\nabla u|)|\nabla u|, \phi(|u|)|u| \in L^\Phi(\mathbb{R}^N)$. On the other hand, from $(f_1)$, we claim that $f(u) \in L^\Phi(\mathbb{R}^N)$. In fact, by $(f_1)$, given $\tau > 0$, there is $\varepsilon > 0$ such that

$$|f(t)| \leq \tau \Phi'(|t|/2), \quad \forall t \in [-\varepsilon, \varepsilon].$$

Hence,

$$\int_{\mathbb{R}^N} \Phi(f(u)) \, dx \leq \tau \int_{|u| \leq \varepsilon} \Phi(|u|/2) \, dx + \max_{t \in [0, \Lambda]} |\Phi(f(t))| \text{mes}([|u| > \varepsilon]),$$

where $\Lambda$ was given in $(1.2)$. Hence, since $\Phi(\Phi'(s)) \leq \Phi(2s)$ for all $s \geq 0$, it follows that

$$\int_{\mathbb{R}^N} \Phi(f(u)) \, dx \leq \tau \int_{|u| \leq \varepsilon} \Phi(|u|) \, dx + \left( \max_{t \in [0, \Lambda]} |\Phi(f(t))| \right) \text{mes}([|u| > \varepsilon]) < +\infty.$$ 

This proves the claim.

These facts combined with the weak* density of $C_0^\infty(\mathbb{R}^N)$ in $W^{1, \Phi}(\mathbb{R}^N)$ yields

$$\int_{\mathbb{R}^N} \phi(|\nabla u|) \nabla u \nabla v \, dx + \int_{\mathbb{R}^N} \phi(|u|)uv \, dx = \lambda \int_{\mathbb{R}^N} f(u)v \, dx, \quad \forall v \in W^{1, \Phi}(\mathbb{R}^N),$$

and the proof is finished. \(\square\)

The next result shows that $I_\lambda$ possesses the mountain pass geometry for all $\lambda \in J = [a, b]$. 

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Theorem 4.2. Suppose $(\phi_1) - (\phi_2)$ and $(\phi_4)$. Assume also that $f$ verifies $(f_1) - (f_3)$. Then the functional $I_\lambda$ satisfies the mountain pass geometry for all $\lambda \in J = [a, b]$, that is, $(a)$

$$I_\lambda(u) \geq 1/2 \text{ for } \|u\| = 2.$$ 

$(b)$ There is $e \in X$ with $\|e\| > 2$ and $I_\lambda(e) < 0$ for all $\lambda \in J = [a, b]$. 

Proof. From (1.2),

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq \Lambda\|u\|, \quad \forall u \in X.$$ 

Then, by (f2),

$$I_\lambda(u) \geq \frac{1}{2}Q(u), \quad \text{ for } \|u\| = 2.$$ 

If $\|u\| = 2$, we must $\|\nabla u\|_\Phi \geq 1$ or $\|u\|_\Phi \geq 1$. Hence,

$$\int_{\mathbb{R}^N} \Phi(|\nabla u|) \, dx \geq \|\nabla u\|_\Phi \geq 1 \quad \text{or} \quad \int_{\mathbb{R}^N} \Phi(|u|) \, dx \geq \|u\|_\Phi \geq 1,$$

and so

$$Q(u) \geq 1 \quad \text{for } \|u\|_\Phi = 2,$$

implying that

$$\inf_{\{u \in X: \|u\| = 2\}} Q(u) = \rho \geq 1 > 0.$$ 

This proves $(a)$. 

From (1.6), there exist $A_0, B_0 > 0$ in such way that

$$(4.4) \quad F(t) \geq A_0 \Phi(t) - B_0, \quad \forall t \in \mathbb{R}.$$ 

Fixed $R > 1$, there is $\Psi \in C_0(\mathbb{R}^N) \setminus \{0\}$ verifying

$$\Psi(x) \geq 0 \quad \forall x \in B_R(0),$$

and

$$A_1 = 2R|\nabla \Phi|, \quad \Omega < B_1 = \inf_{x \in B_{R_0}(0)} \Psi(x) \quad 0 < R_0 < R.$$ 

Since $\Psi \in C_0(\mathbb{R}^N) \subset W_0^1(\mathbb{R}^N)$, we can use $2R > 1$, Poincaré’s Modular Inequality (see [13, Lemma 2.1]), $A_1 < B_1$ and $\Phi$ increasing to conclude

$$(4.5) \quad \int_{B_R(0)} \Phi(t|\nabla \Psi|) \, dx + \int_{B_R(0)} \Phi(t|\Psi|) \, dx \leq 2 \int_{B_R(0)} \Phi(tR|\nabla \Psi|) \, dx = 2\text{meas}(B_R(0))\Phi(B_1t).$$ 

On the other hand,

$$(4.6)\int_{B_R(0)} \Phi^\theta(t|\Psi|) \, dx \geq \int_{B_{R_0}(0)} \Phi^\theta(t|\Psi|) \, dx \geq \int_{B_{R_0}(0)} \Phi^\theta(B_1t) \, dx = \text{meas}(B_{R_0}(0))\Phi^\theta(B_1t).$$ 

Combining (4.5) and (4.6), we get

$$I_\lambda(t\Psi) \leq C_1\Phi(B_1t) - C_2(\Phi(B_1t))^\theta + C_3 \to -\infty \quad \text{as } \quad t \to +\infty,$$

where $C_i > 0$, for each $i = 1, 2, 3$ do not depend on $\lambda \in J = [a, b]$. The last limit ensures the existence of $t > 0$ large enough, which is independent of $\lambda \in J = [a, b]$, in such way that $(b)$ is verified with $e = t\Psi$. This ends the proof. 

In the sequel, we denote by $c_\lambda$ the mountain pass level associated with $I_\lambda$. In this moment, the theorem below due to Jeanjean is crucial in our approach.
We assume that there are two points \( v_1, v_2 \in Y \), such that setting
\[
\Gamma = \{ \gamma \in C([0,1], Y) : \gamma(0) = v_1, \gamma(1) = v_2 \},
\]
there hold, \( \forall \lambda \in J \),
\[
c\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_\lambda(\gamma(t)) > \max\{E_\lambda(v_1), E_\lambda(v_2)\}.
\]

Then, for almost every \( \lambda \in J \), there is a sequence \((u_n) \subset Y\) such that
(i) \((u_n)\) is bounded
(ii) \(E_\lambda(u_n) \to c\lambda\) and
(iii) \(E'\lambda(u_n) \to 0\) in the dual \(Y^*\) of \(Y\).

By Theorem 4.3, there is \(J_0 \subset J\) with \(|J_0| = 0\) such that for each \(\lambda \in J_0\) there is a sequence \((v_n) \subset X\) such that (i),(ii) and (iii) hold.

5. Proof of Theorem 1.3

By the previous section we can apply Theorem 4.3 to the functional \(I_\lambda\) given (4.2). In what follows, we fix \(\lambda \in J_0\) and \((u_n) \subset X\) is the \((PS)_{c\lambda}\) sequence associated with \(I_\lambda\) given by Theorem 4.3. Since \((u_n)\) is bounded, we can assume that for some subsequence, there is \(u \in L^p_{loc}(\mathbb{R}^N)\) such that \(u_n \rightharpoonup u\) in \(L^p_{loc}(\mathbb{R}^N)\). By Corollary 2.2 we derive that \(u \in W^{1,p}(\mathbb{R}^N)\).

Claim 5.1. For some subsequence, still denoted by itself,
\[
u_n(x) \to u(x) \quad \text{and} \quad \nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in} \quad \mathbb{R}^N.
\]

Indeed, for a fixed \(R > 0\), we consider \(\psi \in C_c^\infty(B_{2R})\) such that
\[
\inf_{x \in B_{2R}(0)} \psi(x) = A > 0 \quad \text{and} \quad \sup_{x \in \mathbb{R}^N} |\psi(x)|, \sup_{x \in \mathbb{R}^N} |\nabla \psi(x)| \leq 1/2.
\]

Using these information, a direct computation gives the sequence \((\psi u_n)\) is bounded in \(X\), more precisely, it is possible to prove that
\[
\|\psi u_n\| \leq 3\|u_n\|, \quad \forall n \in \mathbb{N}.
\]

Hence,
\[
\int_{\mathbb{R}^N} \phi(|\nabla u_n|)\nabla u_n \nabla (\psi u_n) \, dx + \int_{\mathbb{R}^N} \phi(|u_n|)u_n (\psi u_n) \, dx - \int_{\mathbb{R}^N} f(u_n)(\psi u_n) \, dx = o_n(1)
\]
and
\[
\int_{\mathbb{R}^N} \phi(|\nabla u_n|)\nabla u_n \nabla (\psi u) \, dx + \int_{\mathbb{R}^N} \phi(|u_n|)u_n (\psi u) \, dx - \int_{\mathbb{R}^N} f(u_n)(\psi u) \, dx = o_n(1).
\]

These limits ensure that
\[
\int_{\mathbb{R}^N} \phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u, \nabla u_n - \nabla u) \psi \, dx + \int_{\mathbb{R}^N} \phi(|u_n|)u_n - \phi(|u|)u)(u_n - u) \psi \, dx = o_n(1)
\]
and so,
\[
\int_{B_{2R}(0)} \phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u, \nabla u_n - \nabla u) \psi \, dx + \int_{\mathbb{R}^N} \phi(|u_n|)u_n - \phi(|u|)u)(u_n - u) \psi \, dx = o_n(1).
\]
The last limit together with the fact that $\Phi$ is convex permits to apply Dal Maso and Murat [10, Lemma 6] to get

$$u_n(x) \to u(x) \text{ and } \nabla u_n(x) \to \nabla u(x) \text{ a.e. in } B_R(0).$$

As $R > 0$ is arbitrary, the Claim 5.1 is proved.

Now, recalling that $\left(\int_{\mathbb{R}^N} \phi(|\nabla u_n|)|\nabla u_n|^2 \, dx\right)$, $\left(\int_{\mathbb{R}^N} \phi(|u_n|)|u_n|^2 \, dx\right)$, $\left(\int_{\mathbb{R}^N} \Phi(|\nabla u_n|) \, dx\right)$ and $\left(\int_{\mathbb{R}^N} \Phi(|u_n|) \, dx\right)$ are bounded, the identity (4.3) ensures that $(\phi(|\nabla u_n|)|\nabla u_n|)$ and $(\phi(|u_n|)|u_n|)$ are bounded sequences in $L^\Phi(\mathbb{R}^N)$. Gathering these information, we can apply the Lemma 2.3 with $\Phi$ replaced by $\tilde{\Phi}$ to obtain

$$\int_{\mathbb{R}^N} (\phi(|\nabla u_n|)\nabla u_n \nabla v + \phi(|u_n|)u_nv) \, dx \to \int_{\mathbb{R}^N} (\phi(|\nabla u|)\nabla u \nabla v + \phi(|u|)uv) \, dx, \quad \forall v \in C^\infty_0(\mathbb{R}^N).$$

Arguing as in the proof of Lemma 4.1, we derive that $(f(u_n))$ is bounded in $L^\tilde{\Phi}(\mathbb{R}^N)$. Thus, again by Lemma 2.3,

$$\int_{\mathbb{R}^N} f(u_n)v \, dx \to \int_{\mathbb{R}^N} f(u)v \, dx, \quad \forall v \in C^\infty_0(\mathbb{R}^N).$$

The last two limits yield

$$\int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v + \int_{\mathbb{R}^N} \phi(|u|)uv \, dx = \int_{\mathbb{R}^N} f(u)v \, dx, \quad \forall v \in C^\infty_0(\mathbb{R}^N).$$

Now, the fact that $\phi(|\nabla u|)|\nabla u|, \phi(|u|)|u|, f(u) \in L^\tilde{\Phi}(\mathbb{R}^N)$ together with the density of $C^\infty_0(\mathbb{R}^N)$ in $X$ give

$$\int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v + \int_{\mathbb{R}^N} \phi(|u|)uv \, dx = \lambda \int_{\mathbb{R}^N} f(u)v \, dx, \quad \forall v \in X,$

that is, $u$ is a critical point of $I_\lambda$ in $X$. Now, we use Lemma 4.1 to conclude that $u$ is a weak solution of $(P_\lambda)$.

In this point we have the following question: Is $u$ nontrivial? If the answer is yes, we have finished the proof of Theorem 1.3. Otherwise, we must work more a little, and in this case, the next lemma is crucial in our approach

**Lemma 5.2.** Assume $(\phi_1) - (\phi_3)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $(f_1)$. If $(w_n) \subset W^{1,\phi}(\mathbb{R}^N)$ is a sequence such that $\int_{\mathbb{R}^N} (\Phi(|\nabla w_n|) + \Phi(|w_n|)) \, dx \leq M$ for all $n \in \mathbb{N}$ and for each $\epsilon > 0$

(*)

$$\text{mes}(|w_n| > \epsilon) \to 0, \quad \text{as } n \to +\infty,$$

then

$$\int_{\mathbb{R}^N} f(w_n)w_n \, dx \to 0 \quad \text{as } n \to +\infty.$$

**Proof.** By hypothesis, given $\tau > 0$, there is $\epsilon > 0$ such that

$$|f(t)| \leq \tau \Phi'(|t|/2), \quad \forall t \in [-\epsilon, \epsilon].$$

Fixing $M = \sup_{n \in \mathbb{N}} \|w_n\|_\Phi$, we get

$$\int_{\mathbb{R}^N} |f(w_n)w_n| \, dx \leq \tau \int_{|w_n| \leq \epsilon} |f(w_n)w_n| \, dx + \left(\max_{t \in [0,\Lambda M]} |f(t)t|\right) \text{mes}(|w_n| > \epsilon)$$
that is,
\[
\int_{\mathbb{R}^N} |f(w_n)w_n| \, dx \leq \tau \int_{||w_n|| \leq \epsilon} \Phi'(||w_n||/2)||w_n|| \, dx + \left( \max_{t \in [0,\Lambda \delta]} |f(t)| \right) \text{mes}(||w_n| > \epsilon),
\]
where \(\Lambda\) was given in (1.2). By Young inequality
\[
\int_{\mathbb{R}^N} |f(w_n)w_n| \, dx \leq \tau \int_{||w_n|| \leq \epsilon} (\Phi'(||w_n||/2) + \Phi(||w_n||)) \, dx + \left( \max_{t \in [0,\Lambda \delta]} |f(t)| \right) \text{mes}(||w_n| > \epsilon).
\]
and so
\[
\int_{\mathbb{R}^N} |f(w_n)w_n| \, dx \leq 2\tau \int_{||w_n|| \leq \epsilon} \Phi(||w_n||) \, dx + \left( \max_{t \in [0,\Lambda \delta]} |f(t)| \right) \text{mes}(||w_n| > \epsilon),
\]
finishing the proof.

We claim that the sequence \((u_n)\) does not satisfy the condition (*) in Lemma 5.2, otherwise we must have
\[
\int_{\mathbb{R}^N} f(u_n)u_n \, dx \to 0 \quad \text{as} \quad n \to +\infty.
\]
Since
\[
\int_{\mathbb{R}^N} \phi(||\nabla u_n||)|\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} \phi(||u_n||)u_n^2 \, dx = \lambda \int_{\mathbb{R}^N} f(u_n)u_n \, dx + o_n(1),
\]
it follows that
\[
\int_{\mathbb{R}^N} \phi(||\nabla u_n||)|\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} \phi(||u_n||)u_n^2 \, dx \to 0.
\]
From \((\phi_3)\) and \((f_3)\),
\[
\int_{\mathbb{R}^N} (\Phi(||\nabla u_n||) + \Phi(||u_n||)) \, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} F(u_n) \, dx \to 0
\]
These limits imply that
\[
c_\lambda + o_n(1) = I_\lambda(u_n) = \int_{\mathbb{R}^N} (\Phi(||\nabla u_n||) + \Phi(||u_n||)) \, dx - \lambda \int_{\mathbb{R}^N} F(u_n) \, dx \to 0,
\]
which is absurd, because we have that \(c_\lambda \geq 1\) for all \(\lambda \in J_0\), see Lemma 4.2.
From this, there are \(\epsilon, \delta > 0\) such that
\[
\text{mes}(||u_n|| > \epsilon) \geq \delta, \quad \forall n \in \mathbb{N}.
\]
By generalized Lieb’s Lemma 1.2, there is \((y_n) \subset \mathbb{R}^N\) such that \(w_n(x) = u_n(x + y_n)\) has a nontrivial limit \(w \in L_{\text{loc}}^{\Phi}(\mathbb{R}^N)\), that is, \(w_n \to w\) in \(L_{\text{loc}}^{\Phi}(\mathbb{R}^N)\) and \(w \neq 0\). Therefore, by Corollary 2.2, \(w \in W^{1,\Phi}(\mathbb{R}^N)\). Moreover, fixed \(v \in X\), we have
\[
\int_{\mathbb{R}^N} \phi(||\nabla w_n||)|\nabla w_n|v \, dx + \int_{\mathbb{R}^N} \phi(||w_n||)w_n v \, dx = \lambda \int_{\mathbb{R}^N} f(w_n)v \, dx + o_n(1).
\]
Arguing as above, we conclude that
\[
\int_{\mathbb{R}^N} \phi(||w||)|\nabla w|v \, dx + \int_{\mathbb{R}^N} \phi(||w||)w v \, dx = \lambda \int_{\mathbb{R}^N} f(w)v \, dx, \quad \forall v \in X.
\]
Now, it is enough to apply Lemma 4.1 to conclude that \(w\) is a nontrivial weak solution of \((P_\lambda)\).
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