The electromagnetic fields and the radiation of a spatio-temporally varying electric current loop

Markus Lazar\textsuperscript{a,b,*}

\textsuperscript{a} Heisenberg Research Group, Department of Physics, Darmstadt University of Technology, Hochschulstr. 6, D-64289 Darmstadt, Germany
\textsuperscript{b} Department of Physics, Michigan Technological University, Houghton, MI 49931, USA

May 11, 2014

Abstract

The electric and magnetic fields of a spatio-temporally varying electric current loop are calculated using the Jefimenko equations. The radiation and the nonradiation parts of the electromagnetic fields are derived in the framework of Maxwell’s theory of electromagnetic fields. In this way, a new, exact, analytical solution of the Maxwell equation is found.

Keywords: electrodynamics; current loop; radiation; retardation; non-uniform motion.

1 Introduction

A current loop is an important item in the theory of electromagnetic induction and magnetostatics [1, 2, 3, 4]. Usually, such considerations are restricted to the static case or quasistatic case. No single work has been considered the general case of the non-uniform motion.

*E-mail address: lazar@fkp.tu-darmstadt.de (M. Lazar);
Tel.: +49(0)6151/163686; Fax.: +49(0)6151/163681.
motion of an arbitrary current loop or the electromagnetic radiation fields of such a loop in the general framework of Maxwell’s theory of electromagnetic fields, until now.

One important feature of the theory of electrodynamics is the retardation which is a consequence of the finite speed of the propagating electromagnetic fields. There is always a time delay since an effect observed by the receiver at the present position and present time was caused by the sender at some earlier time (retarded time) and at the retarded position.

The aim of this paper is the study of a spatio-temporally varying (or time-depending) closed current loop and the corresponding radiation fields. In particular we investigate the so-called electrokinetic field. The exact solutions of all the electromagnetic fields induced by a non-stationary electric current loop are determined. The Liénard-Wiechert fields produced by the electric current loop are calculated. The generalized Faraday law and the generalized Biot-Savart law for a spatio-temporally varying electric current loop are found. The electromagnetic fields are decomposed into radiation and nonradiation parts and finally, it is shown that the classical expression for an electric wire is recovered as static limit from our general expressions.

2 Basic framework

The basic electromagnetic field laws are represented by the inhomogeneous and homogeneous Maxwell equations [2, 3]

\[
\begin{align*}
\nabla \cdot D &= \rho , \quad \nabla \times H - \partial_t D = J , \\
\nabla \cdot B &= 0 , \quad \nabla \times E + \partial_t B = 0 ,
\end{align*}
\]

(1) (2)

where \( D \) is the electric displacement vector (electric excitation), \( H \) is the magnetic excitation vector, \( B \) is the magnetic field strength vector, \( E \) is the electric field strength vector, \( J \) is the electric current density vector, and \( \rho \) is the electric charge density. \( \partial_t \) denotes the differentiation with respect to the time \( t \) and \( \nabla \) is the Nabra operator. In addition, the electric current density vector and the electric charge density fulfill the continuity equation

\[
\nabla \cdot J + \partial_t \rho = 0 .
\]

(3)

The constitutive equations for the fields in a vacuum (Maxwell-Lorentz relations) read

\[
\begin{align*}
D &= \epsilon_0 E , \\
H &= \frac{1}{\mu_0} B ,
\end{align*}
\]

(4)

where \( \epsilon_0 \) is the vacuum permittivity and \( \mu_0 \) is the vacuum permeability. The speed of light in vacuum is given by

\[
\begin{align*}
e^2 &= \frac{1}{\epsilon_0 \mu_0} .
\end{align*}
\]

(5)

From the Maxwell equations (1) and (2), inhomogeneous wave equations for the electromagnetic field strengths follow

\[
\Box E = -\frac{1}{\epsilon_0} \left( \nabla \rho + \frac{1}{c^2} \partial_t J \right) \]

(6)
\[ \Box \mathbf{B} = \mu_0 \mathbf{\nabla} \times \mathbf{J}, \]  

(7)

where the d’Alembert operator is defined by

\[ \Box := \frac{1}{c^2} \partial_t - \Delta \quad \text{with} \quad \Delta = \mathbf{\nabla} \cdot \mathbf{\nabla}. \]  

(8)

For zero initial conditions, using the retarded Green function of the wave equation and some mathematical manipulations, the causal solutions of the inhomogeneous wave equations (6) and (7) are given by retarded electromagnetic field strength vectors:

\[ \mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \left( \frac{\rho(\mathbf{r}', t - R/c)}{R^3} \mathbf{R} + \frac{\partial_t \rho(\mathbf{r}', t - R/c)}{cR^2} \mathbf{R} - \frac{\partial_t \mathbf{J}(\mathbf{r}', t - R/c)}{c^2 R} \right) \, d\mathbf{r}', \]  

(9)

\[ \mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \left( \frac{\mathbf{J}(\mathbf{r}', t - R/c)}{R^3} + \frac{\partial_t \mathbf{J}(\mathbf{r}', t - R/c)}{cR^2} \right) \times \mathbf{R} \, d\mathbf{r}', \]  

(10)

where \( \mathbf{R} = \mathbf{r} - \mathbf{r}', \quad R = |\mathbf{r} - \mathbf{r}'|, \quad \mathbf{r} \in \mathbb{R}^3, \quad t \in \mathbb{R} \) and \( \mathcal{V} \) denotes the whole 3-dimensional space. Eq. (9) is the time-dependent generalized Coulomb-Faraday law and Eq. (10) is the time-dependent generalized Biot-Savart law (see also [4]). Eqs. (9) and (10) express the electromagnetic fields in terms of their retarded sources \( \rho, \mathbf{J}, \partial_t \rho \) and \( \partial_t \mathbf{J} \) with full generality. They were originally derived by Jefimenko [5] (see also [6, 7]). They appear also in the book of Clemmow and Dougherty [8] and in the third edition of Lorrain, Corson and Lorrain [9]. An equivalent representation was given by Panofsky and Phillips [1] (see also [7]). Both equations are nowadays called the Jefimenko equations in standard books on electrodynamics (e.g. [2, 3, 4]). They are fundamental, elegant and very useful equations.

3  A spatio-temporally varying electric current loop

We investigate a spatio-temporally varying electric current loop. We consider a closed loop of arbitrary shape (planar or non-planar) that can move arbitrary. The current density vector of a time-variable or spatio-temporally varying electric current loop at the time-dependent position \( \mathbf{s}(t) \) is given by a line integral of the form

\[ \mathbf{J}(\mathbf{r}, t) = \oint_{L(t)} I(t) \delta(\mathbf{r} - \mathbf{s}(t)) \, d\mathbf{L}(\mathbf{s}(t)), \]  

(11)

where \( L(t) \) is the loop curve at time \( t \) and \( d\mathbf{L} \) denotes the line element along the loop. \( I(t) \) is the time-dependent magnitude of the current. In addition, there is no charge density, \( \rho(\mathbf{r}, t) = 0 \), and the current vector fulfills \( \nabla \cdot \mathbf{J} = 0 \). For this particular situation, Eq. (9) simplifies to

\[ \mathbf{E}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\partial_t \mathbf{J}(\mathbf{r}', t - R/c)}{R} \, d\mathbf{r}', \]  

(12)
where $\frac{\partial}{\partial t} J$ is to be evaluated at the retarded time, which is less than the present time $t$. The electrostatic field (see also [6, 7]). The current density vector appearing in Eqs. (12) and the current is changing in time. Therefore, the electrokinetic field is different from the current distribution is non-stationary even when the velocity is constant or zero due to the time-dependent current magnitude $I(t)$. A spatiotemporal current field may be achieved without any motion, with sources turned on and off at different times in different locations. It can be seen in Eq. (12) that a time-variable electric current creates an electric field (anti)parallel to the current $\frac{\partial}{\partial t} J$ and the electrokinetic field exists only as long as the current is changing in time. Therefore, the electrokinetic field is different from the electrostatic field (see also [6, 7]). The current density vector appearing in Eqs. (12) and (10) is to be evaluated at the retarded time, which is less than the present time $t$. The retarded current density vector is given by

$$J(r', t_r) = \oint_{L(t_r)} I(t_r) \delta(r' - s(t_r)) dL(s(t_r)),$$

where $s(t_r)$ here denotes the position of the loop at the retarded time

$$t_r = t - |r - r'|/c = t - R/c.$$

Thus, substituting Eq. (13) in Eqs. (12) and (10), we obtain

$$E(r, t) = -\frac{\mu_0}{4\pi} \int \oint_{L(t_r)} \frac{I(t_r)}{R^3} \delta(r' - s(t_r)) \frac{dL(s(t_r))}{dL(r')} dr',$$

$$B(r, t) = -\frac{\mu_0}{4\pi} \int \oint_{L(t_r)} \frac{I(t_r)}{c R^2} \delta(r' - s(t_r)) \frac{dL(s(t_r))}{dL(r')} dr'$$

Here remain the integrals of the delta functions, which are done by changing the variable of integration from $r'$ to $z = r' - s(t_r)$ and using the Jacobian $J$ of this transformation (see, e.g., [10, 11])

$$J = \det \left( \frac{\partial z}{\partial r'} \right) = 1 - \frac{\mathbf{V}(t_r) \cdot (r - r')}{c |r - r'|},$$

$$\int F(r') \delta(r' - s(t_r)) dr' = \int F(r') \delta(z) \frac{1}{J} dz \quad \left| \begin{array}{c}
F(r') \\
F(r')
\end{array} \right|_{z=0} = \frac{F(r')}{1 - \frac{\mathbf{V}(t_r) \cdot (r - r')}{c |r - r'|}} |r' = s(t_r)|,$$

where $\mathbf{V} = \dot{s}$ is the velocity of every point of the current loop. From the argument of the $\delta$-function we get $r' = s(t_r)$. Therefore, now $R(t_r) = r - s(t_r)$ and $R(t) = |r - s(t_r)|$ are time-dependent and the retarded time is now given by

$$t_r = t - |r - s(t_r)|/c = t - R(t_r)/c,$$
where $s(t_r)$ is the retarded position of the time-dependent source point. Unfortunately, the retarded time $t_r(r, t)$ is not given directly, but must be determined by solving Eq. (19) what may be quite tedious. Only in some simple cases $t_r$ is easy to find (e.g. uniform motion). If the loop is moving with velocity less than the speed of light, the solution of Eq. (19) is unique. The retarded time is a result of the finite speed of the propagation for electrodynamic waves. In addition, we define

\[ P(t') = R(t') - V(t') \cdot R(t')/c, \tag{20} \]

and after the integration in $r'$, Eqs. (15) and (16) become

\[ E(r, t) = -\frac{\mu_0}{4\pi} \partial_t \left[ \oint_{L(t')} \frac{I(t')}{P(t')} dL(s(t')) \right]_{t' = t_r}, \tag{21} \]

\[ B(r, t) = -\frac{\mu_0}{4\pi} \left[ \oint_{L(t')} \frac{I(t')}{P(t') R^2(t')} R(t') \times dL(s(t')) \right]_{t' = t_r} - \frac{\mu_0}{4\pi c} \partial_t \left[ \oint_{L(t')} \frac{I(t')}{P(t') R(t')} R(t') \times dL(s(t')) \right]_{t' = t_r}. \tag{22} \]

Eqs. (21) and (22) have a remarkable structure. They are in some sense similar to the so-called Heaviside-Feynman formulas for a non-uniformly moving point charge (see, e.g., [2, 4]). Since we investigate an electric current loop, Eqs. (21) and (22) have the structure of time-dependent line integrals depending on the retarded time $t_r$. Therefore, Eqs. (21) and (22) are retarded line integrals. The fields (21) and (22) must be evaluated at some earlier times $t' = t_r$ (the retarded times) and for the corresponding point $s(t')$ of the current loop. Also the line element $dL$ depends at every point $s(t')$ on the retarded time.

Because of the explicit dependence of the retarded times on $s(t')$, every point $s(t')$ on the loop $L(t')$ depends on its own retarded time. For a current loop, the time dependence of the electromagnetic fields is based on a retardation due to the retarded times which are functions of the variables $r$, $s$ and $t$. The electromagnetic fields (21) and (22) at the point $r$ and at time $t$ receive contribution from every point $s(t')$ on the electric current loop sending the electromagnetic waves at the retarded time $t_r$.

The electric and magnetic fields created by a non-stationary loop at the point of observation are the result of the signals sent out by all the individual points $s$ on the loop $L$ and simultaneously received at the point of observation at the instant time $t$. But different points on the loop are at different distances from the point of observation, and the times needed for the signals originating from different points of the loop to arrive at the point of observation are different. Thus, the signals that are received at the point of observation simultaneously at the time $t$ are sent out from different points of the loop at different retarded times (19). For a moving current loop these times are different not only due to different points on the loop, but also because the loop moves.

The evaluation of the electromagnetic fields (21) and (22) is not a trivial task. Eqs. (21) and (22) are complicated line integrals depending on the retarded times and time derivatives outside the line integrals. Since the loop is non-stationary and can move, the time derivative acts on the integrand as well as on the line element $dL$. The evaluation of
the time derivatives outside the integral can be carried out if we use a transport theorem for a time-dependent line integral. Using the transport theorem for a line integral (A.1), Eqs. (21) and (22) read

$$E(r, t) = -\frac{\mu_0}{4\pi} \left[ \oint_{L(t')} \left( \frac{\partial t}{P(t')} I(t') + V(t') \cdot \overrightarrow{n}_{s(t')} \frac{I(t')}{P(t')} \right) dL(s(t')) + \frac{I(t')}{P(t')} V(t') \overrightarrow{n}_{s(t')} \cdot dL(s(t')) \right]_{t'=t_r} \right)$$

(23)

and

$$B(r, t) = -\frac{\mu_0}{4\pi} \left[ \oint_{L(t')} \frac{I(t')}{P(t') R^2(t')} R(t') \times dL(s(t')) \right]_{t'=t_r}$$

$$- \frac{\mu_0}{4\pi c} \left[ \oint_{L(t')} \left( \frac{\partial t}{P(t') R(t')} R(t') + V(t') \cdot \overrightarrow{n}_{s(t')} \frac{I(t')}{P(t') R(t')} \right) \times dL(s(t')) + \frac{I(t')}{P(t') R(t')} V(t') \overrightarrow{n}_{s(t')} \cdot dL(s(t')) \right]_{t'=t_r} \right)$$

(24)

In Eqs. (23) and (24), the derivative with respect to the present time \( t \) and derivatives with respect to the time-dependent source point of the loop \( s(t') \), acting on the corresponding retarded time \( t' = t_r \), appear. The derivatives with respect to the time-dependent source points of the loop are a consequence that we investigate an electric current loop instead of an electric point charge.

To obtain the electromagnetic fields in a ‘Liénard-Wiechert’ type form, we must carry out the time derivatives and the derivatives with respect to the time-dependent source point \( s(t') \), which are not trivial because of the subtle relation between present and retarded times. Using the corresponding formulas for the derivatives (A.4)–(A.6) and (A.9)–(A.12), we find

$$E(r, t) = -\frac{\mu_0}{4\pi} \left[ \oint_{L(t')} \left( \frac{\dot{I}(t')}{P^2(t')} + \frac{V(t') \cdot R(t')}{c P(t') R(t')} \right) \right.$$

$$+ \frac{\dot{I}(t')}{P^3(t')} \left( \dot{V}(t') \cdot R(t') - V^2(t') \right) \frac{R(t')}{c} + \frac{V(t') \cdot R(t')}{c} \left. \right) \left[ \oint_{L(t')} \frac{\dot{I}(t')}{P(t') R(t')} V(t') \cdot R(t') \right. \frac{V(t') \cdot R(t')}{c^2} - \frac{R(t') V^2(t')}{c} + \frac{V(t') \cdot R(t')}{c} \right) \left. \right) \right]_{t'=t_r} \right)$$

(25)
and

\[ B(r, t) = -\frac{\mu_0}{4\pi c} \left[ \oint_{L(t')} \left( \dot{I}(t') \left( \frac{1}{P^2(t')} + \frac{V(t') \cdot R(t')}{c P(t') R^2(t')} \right) + \frac{\dot{V}(t') \cdot R(t')}{P^3(t')} \left( \frac{R(t')}{R(t')} - \frac{V(t') \cdot R(t')}{c^2} - \frac{V(t')}{c} + V(t') \cdot R(t') \right) \right. \right. \]
\[ + \frac{I(t')}{P^2(t') R^2(t')} \left( \dot{V}(t') \cdot R(t') \frac{V(t') \cdot R(t')}{c^2} - \frac{R(t') V^2(t')}{c} + V(t') \cdot R(t') \right) \]
\[ + I(t') V(t') \cdot R(t') \left( \frac{1}{P^2(t') R^2(t')} + \frac{1}{P(t') R^3(t')} \right) \]
\[ + \frac{c^2 I(t')}{P(t') R^2(t')} R(t') \times dL(s(t')) \]
\[ \left. - \oint_{L(t')} I(t') \left( \frac{1}{P^2(t')} + \frac{1}{P(t') R(t')} \right) V(t') \times dL(s(t')) \right. \]
\[ + \oint_{L(t')} \frac{I(t')}{P(t')} \frac{R(t') \times \dot{V}(t')}{c R^2(t')} R(t') \cdot dL(s(t')) \] \[ t' = t_r. \] \[ (26) \]

These are the electromagnetic fields induced by the spatio-temporally varying electric current loop. These expressions completely describe the electromagnetic fields of the spatio-temporally varying electric current loop. The expressions for the electromagnetic fields of the electric current loop are given in terms of the retarded positions, retarded velocities, and the retarded line element. For a general non-uniform motion the relation of the retarded position to the present position is not known. Eq. (25) is the generalized Faraday law or electrokinetic field of a spatio-temporally varying electric current loop and Eq. (26) is the generalized Biot-Savart law for a spatio-temporally varying electric current loop. There are two sources for the induction of the electromagnetic fields: the time-dependence of the current magnitude and the fact that positions of the electric current loop are time-dependent.

If the current loop is moving uniformly, which means with constant velocity \( V \) and zero acceleration, the induced electromagnetic fields are obtained from Eqs. (25) and (26) putting \( \dot{V} = 0 \). The solution of the uniform motion of a current loop is more complicated than the usual steady-state solution of a point charge (see, e.g., [1, 3]), due to the uniformly moving line element of the current loop. One non-trivial task for the uniform motion is to express the electromagnetic fields in terms of the so-called present position: \( R_0 = [R(t_r) - R(t_r) V / c] \) where \( t_r \) denotes the retarded time for the uniform motion\(^1\).

\(^1\)For the uniform motion, the retarded time reads (see, e.g., [3])

\[ t_r = \frac{(c^2 t - V \cdot r) - \sqrt{(c^2 t - V \cdot r)^2 + (c^2 - V^2)(r^2 - c^2 t^2)}}{c^2 - V^2}, \]

which is the solution of Eq. (19) for \( s(t) = V t \) and \( V = \text{constant} \).
Since the time derivative of the line integrals in Eqs. (21) and (22) is involved, it is clear that the electromagnetic fields (21) and (22) will be functions not only of the velocity $\mathbf{V}$, but also of the acceleration $\dot{\mathbf{V}}$. We may therefore separate $\mathbf{E}$ and $\mathbf{B}$ into two parts each, one which involves the acceleration and the time-derivative of the magnitude of the current and goes to zero for $\mathbf{V} = 0$ and $\dot{I} = 0$, and one which involves only the velocity $\mathbf{V}$:

\[
\begin{align*}
\mathbf{E}(\mathbf{r}, t) &= \mathbf{E}_{\text{nonrad}}(\mathbf{r}, t) + \mathbf{E}_{\text{rad}}(\mathbf{r}, t), \\
\mathbf{B}(\mathbf{r}, t) &= \mathbf{B}_{\text{nonrad}}(\mathbf{r}, t) + \mathbf{B}_{\text{rad}}(\mathbf{r}, t).
\end{align*}
\]

The fields $\mathbf{E}_{\text{nonrad}}$ and $\mathbf{B}_{\text{nonrad}}$ are called nonradiation parts and the fields $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$ are called the radiation parts. The electric parts are

\[
\mathbf{E}_{\text{nonrad}}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \left[ \oint_{L(t')} \left( \frac{I(t')}{P^3(t')} \left( \frac{V^2(t') R(t')}{c} - \mathbf{V}(t') \cdot \mathbf{R}(t') \right) \right. \\
+ \frac{I(t')}{P^2(t') R(t')} \left( \frac{V^2(t') R(t')}{c} - \mathbf{V}(t') \cdot \mathbf{R}(t') \right) \bigg) d\mathbf{L}(\mathbf{s}(t')) \right]_{t'=t},
\]

\[
\mathbf{E}_{\text{rad}}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi} \left[ \oint_{L(t')} \left( \dot{I}(t') \left( \frac{R(t')}{P^2(t')} + \frac{\mathbf{V}(t') \cdot \mathbf{R}(t')}{c} \right) \right. \\
+ \frac{\dot{I}(t')}{c P(t') R(t')} \left( \frac{R(t')}{c P^3(t')} + \frac{\mathbf{V}(t') \cdot \mathbf{R}(t')}{c^2 P^2(t') R(t')} \right) \bigg) d\mathbf{L}(\mathbf{s}(t')) \right]_{t'=t},
\]

and the magnetic parts read

\[
\begin{align*}
\mathbf{B}_{\text{nonrad}}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi c} \left[ \oint_{L(t')} \left( \frac{I(t')}{P^3(t') R(t')} \left( \frac{V^2(t') R(t')}{c} - \mathbf{V}(t') \cdot \mathbf{R}(t') \right) \right. \\
+ \frac{I(t')}{P^2(t') R^2(t')} \left( \frac{V^2(t') R(t')}{c} - \mathbf{V}(t') \cdot \mathbf{R}(t') \right) \bigg) \mathbf{R}(t') \times d\mathbf{L}(\mathbf{s}(t')) \right]_{t'=t}, \\
\mathbf{B}_{\text{rad}}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi c} \left[ \oint_{L(t')} \left( \frac{\dot{I}(t')}{P^3(t')} + \frac{1}{P(t') R^3(t')} \right) \mathbf{V}(t') \times d\mathbf{L}(\mathbf{s}(t')) \right]_{t'=t}.
\end{align*}
\]
The nonradiation parts (29) and (31) are formally the ‘convective’ fields of a uniformly moving current loop at the retarded position. In addition, it can be seen in Eqs. (30) and (32) that the electromagnetic radiation fields produced by the spatio-temporally varying electric current loop are more complicated than the radiation fields of a point charge (see, e.g., [1, 2, 3, 4]). If the loop is accelerated, the electromagnetic fields are neither static nor convective, and there is a net change in the field energy which causes radiation. The electromagnetic radiation possesses two sources and is caused by the time-change of the magnitude of the current loop \( \dot{I} \) and the acceleration \( \ddot{V} \).

From these equations, it is apparent that the static limit (i.e., \( V = 0, \dot{V} = 0, I = \text{constant} \)) is satisfied, since

\[
\begin{align*}
E^{\text{(static)}} &= E^{\text{nonrad (static)}} + E^{\text{rad (static)}} \\
&= 0 + 0 \\
B^{\text{(static)}} &= B^{\text{nonrad (static)}} + B^{\text{rad (static)}} \\
&= B^{\text{nonrad (static)}} + 0 \\
&= -\frac{\mu_0 I}{4\pi} \oint_L \frac{1}{R^3} R \times dL',
\end{align*}
\]

which is the appropriate expression for the Biot-Savart law (see, e.g., [3]).

We note the following radial dependencies of the electromagnetic fields:

\[
\begin{align*}
E^{\text{nonrad}}, B^{\text{nonrad}} &\approx \left[ \oint_L \frac{1}{R^2} dL \right]_{ts} \\
E^{\text{rad}}, B^{\text{rad}} &\approx \left[ \oint_L \frac{1}{R} dL \right]_{ts}
\end{align*}
\]

The nonradiation fields \( E^{\text{nonrad}} \) and \( B^{\text{nonrad}} \) are the near fields since they fall off as \( 1/R^2 \) in the integrand. The radiation or acceleration fields \( E^{\text{rad}} \) and \( B^{\text{rad}} \) are the far fields falling off as \( 1/R \) in the integrand. The radiation fields then dominate at large distances and they must be used to calculate energy loss by radiation. As an application, the obtained results can be used for the radiation of non-uniformly moving loop antennas.

Another important limit of our general results, is the case of a current loop with time-variable magnitude \( I(t) \) and localized at the position \( r' \) with fixed loop shape \( L \). The corresponding current density vector reads

\[
J(r, t) = \oint_L I(t) \delta(r - r') dL'.
\]
Since the position of the loop is fixed at the point \( r' \), the velocity and the acceleration of the positions of the loop are zero \( V = 0 \) and \( \dot{V} = 0 \). The electric and magnetic fields can be obtained directly from Eqs. (25) and (26), or by substituting Eq. (37) into the Jefimenko equations (12) and (10). The electric field (electrokinetic field) is given by

\[
E(r, t) = -\frac{\mu_0}{4\pi} \oint_L \frac{i(t - R/c)}{R} \, dL',
\]
and the magnetic field is

\[
B(r, t) = -\frac{\mu_0}{4\pi} \oint_L \left( \frac{I(t - R/c)}{R^3} + \frac{i(t - R/c)}{c R^2} \right) R \times dL',
\]
which are in agreement with the result given earlier by Jefimenko [5].

**Acknowledgement**

The author gratefully acknowledges the grants from the Deutsche Forschungsgemeinschaft (Grant Nos. La1974/2-1, La1974/3-1). The author wishes to express his gratitude to Prof. Wolfgang Ellermeier for stimulating discussions and useful remarks.

**A Appendix**

For the convenience of the reader, we give all the formulas in the appendix in the index notation.

**A.1 Generalized transport theorem for a line integral**

The generalized transport theorem for a line integral reads (see, e.g., [12, 13])

\[
\partial_t \oint_{L(t)} f(r, r', t) \, dL' = \oint_{L(t)} \left[ \left( \partial_t f + V'_j \partial_j f \right) \, dL' + f \partial_j V'_i \, dL'_j \right],
\]
where \( V' = \dot{r}' \) is the velocity of every point of the moving line (drift velocity). The derivative \( \partial_t + V'_j \partial_j \) on the right hand side is often called the co-moving time derivative. \( dL' \) is a time-dependent line element. The last term in Eq. (A.1) is due to the time-derivative of the line element and contains the velocity gradient. In general, \( f \) may be a tensor-valued function.

**A.2 Derivatives at the retarded time**

Here we give some useful relations of derivatives of quantities depending on the retarded time, which is the unique solution of the relation

\[
t - t_r - |r - s(t_r)|/c = 0.
\]
First, we carry out the time derivatives, which are not trivial because of the subtle relation between present and retarded time (see also [14]):

\[
\left[ \frac{\partial t'}{\partial t} \right]_{t'=t_r} = \left[ \frac{R(t')}{P(t')} \right]_{t'=t_r} \tag{A.3}
\]

\[
\partial_t [I(t')]_{t'=t_r} = \left[ \frac{\partial t' \partial I(t')}{\partial t' \partial t} \right]_{t'=t_r} = \left[ \frac{R(t')}{P(t')} \right] \dot{I}(t') \tag{A.4}
\]

\[
\partial_t \left[ \frac{1}{P(t')} \right]_{t'=t_r} = \left[ \frac{1}{P^3(t')} \left( (\dot{V}_m(t')R_m(t') - V^2(t')) \frac{R(t')}{c} + V_m(t')R_m(t') \right) \right]_{t'=t_r} \tag{A.5}
\]

\[
\partial_t \left[ \frac{R_k(t')}{P(t')R(t')} \right]_{t'=t_r} = \left[ \frac{R_k(t')}{P^3(t')R(t')} \left( (\dot{V}_m(t')R_m(t') - V^2(t')) \frac{R(t')}{c} + V_m(t')R_m(t') \right) \right.
\]

\[+ \frac{1}{P^2(t')} \left( V_m(t')R_m(t') \frac{R_k(t')}{R^2(t')} - V_k(t') \right) \right]_{t'=t_r}. \tag{A.6}
\]

Secondly, we give the relations for the derivatives of the quantities depending on the retarded time with respect to the retarded position of moving source point (see also [7]):

\[
\left[ \frac{\partial R_i(t')}{\partial s_j(t')} \right]_{t'=t_r} = -\delta_{ij} \tag{A.7}
\]

\[
\left[ \frac{\partial t'}{\partial s_j(t')} \right]_{t'=t_r} = \left[ \frac{R_j(t')}{c R(t')} \right]_{t'=t_r} \tag{A.8}
\]

\[
\left[ \frac{\partial}{\partial s_j(t')} I(t') \right]_{t'=t_r} = \left[ \frac{\partial t' \partial I(t')}{\partial t' \partial t} \right]_{t'=t_r} = \left[ \frac{R_j(t')}{c R(t')} \right] \dot{I}(t') \tag{A.9}
\]

\[
\left[ \frac{\partial}{\partial s_j(t')} V_i(t') \right]_{t'=t_r} = \left[ \frac{R_j(t')}{c R(t')} \right] \dot{V}_i(t') \tag{A.10}
\]

\[
\left[ \frac{\partial}{\partial s_j(t')} \frac{1}{P(t')} \right]_{t'=t_r} = \left[ \frac{1}{P^2(t')R(t')} \left( R_j(t') - \frac{R(t')V_j(t')}{c} + \dot{V}_m(t')R_m(t') \frac{R_j(t')}{c^2} \right) \right]_{t'=t_r} \tag{A.11}
\]

\[
\left[ \frac{\partial}{\partial s_j(t')} \frac{R_k(t')}{P(t')R(t')} \right]_{t'=t_r} = \left[ \frac{R_k(t')}{P^2(t')R^2(t')} \left( R_j(t') - \frac{R(t')V_j(t')}{c} + \dot{V}_m(t')R_m(t') \frac{R_j(t')}{c^2} \right) \right.
\]

\[+ \frac{\delta_{jk}}{P(t')R(t')} + \frac{R_j(t')R_k(t')}{P(t')R^3(t')} \right]_{t'=t_r}. \tag{A.12}
\]
References

[1] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed., Addison Wesley, Reading, MA (1962); Dover, New York (2005).

[2] J.D. Jackson, *Classical Electrodynamics*, 3rd ed., Wiley, New York (1999).

[3] D.J. Griffiths, *Introduction to Electrodynamics*, 3rd ed., Pearson, Addison Wesley, Prentice Hall, New Jersey (1999).

[4] M.A. Heald and J.B. Marion, *Classical Electromagnetic Radiation*, 3rd ed., Brooks/Cole Thomson Learning (1995); Dover, New York (2012).

[5] O.D. Jefimenko, *Electricity and Magnetism*, Appleton-Century-Crofts, New York (1966).

[6] O.D. Jefimenko, *Causality, Electromagnetic Induction and Gravitation*, 2nd ed., Electret Scientific, Star City (2000).

[7] O.D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity*, 2nd ed., Electret Scientific, Star City (2004).

[8] P.C. Clemmow and J.P. Dougherty, *Electrodynamics of Particles and Plasmas*, Addison Wesley, Reading, MA (1969).

[9] P. Lorrain, D.P. Corson and F. Lorrain, *Electromagnetic Fields and Waves*, 3rd ed., Freeman, New York (1988).

[10] D.S. Jones, *The Theory of Electromagnetism*, Pergamon, New York (1964).

[11] L. Eyges, *The Classical Electromagnetic Field*, Addison-Wesley, Reading, MA (1972).

[12] A.C. Eringen, *Mechanics of Continua*, Wiley, New York (1967).

[13] P. Chadwick, *Continuum Mechanics*, George Allen and Unwin Ltd., London (1976); Dover, New York (1998).

[14] G. Barton, *Elements of Green’s Functions and Propagation*, Oxford University Press, Oxford (1989).