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A NON-HOMOGENEOUS ORBIT CLOSURE OF A DIAGONAL SUBGROUP

FRANÇOIS MAUCOURANT

Abstract. Let \( G = \text{SL}(n, \mathbb{R}) \) with \( n \geq 6 \). We construct examples of lattices \( \Gamma \subset G \), subgroups \( A \) of the diagonal group \( D \) and points \( x \in G/\Gamma \) such that the closure of the orbit \( Ax \) is not homogeneous and such that the action of \( A \) does not factor through the action of a one-parameter non-unipotent group. This contradicts a conjecture of Margulis.

1. Introduction

1.1. Topological rigidity and related questions. Let \( G \) be a real Lie group, \( \Gamma \) a lattice in \( G \), meaning a discrete subgroup of finite covolume, and \( A \) a closed connected subgroup. We are interested in the action of \( A \) on \( G/\Gamma \) by left multiplication; we will restrict ourselves to the topological properties of these actions, referring the reader to [4] and [5] for references and recent developments on related measure theoretical problems.

Two linked questions arise when one studies continuous actions of topological groups: what are the closed invariant sets, and what are the orbit closures?

In the homogeneous action setting we are considering, there is a class of closed sets that admit a simple description: a closed subset \( X \subset G \) is said to be homogeneous if there exists a closed connected subgroup \( H \subset G \) such that \( X = Hx \) for some (and hence every) \( x \in X \). Let us say that the action of \( A \) on \( G/\Gamma \) is topologically rigid if for any \( x \in G/\Gamma \), the closure \( \overline{Ax} \) of the orbit \( Ax \) is homogeneous.

The most basic example of a topologically rigid action is when \( G = \mathbb{R}^n \), \( \Gamma = \mathbb{Z}^n \), \( A \) any vector subspace of \( G \). It turns out that the behavior of elements of \( A \) for the adjoint action on the Lie algebra \( \mathfrak{g} \) of \( G \) plays a important role for our problem. Recall that an element \( g \in G \) is said to be \( \text{Ad} \)-unipotent if \( \text{Ad}(g) \) is unipotent, and \( \text{Ad} \) \(-\)split over \( \mathbb{R} \) if \( \text{Ad}(g) \) is diagonalizable over \( \mathbb{R} \). If the closed, connected subgroup \( A \) of \( G \) is generated by \( \text{Ad} \)-unipotent elements, a celebrated theorem of Ratner [13] asserts that the action of \( A \) is always topologically rigid, settling a conjecture due to Raghunathan.

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When $A$ is generated by elements which are $\text{Ad}$-split over $\mathbb{R}$, much less is known. Consider the model case of $G = \text{SL}(n, \mathbb{R})$ and $A$ the group of diagonal matrices with nonnegative entries. If $n = 2$, it is easy to produce non-homogeneous orbit closures (see e.g. [7]); more generally, a similar phenomenon can be observed when $A$ is a one-parameter subgroup of the diagonal group (see [8], 4.1). However, for $A$ the full diagonal group, if $n \geq 3$, to the best of our knowledge, the only nontrivial example of a nonhomogeneous $A$-orbit closure is due to Rees, later generalized in [7]. In an unpublished preprint, Rees exhibited a lattice $\Gamma$ of $G = \text{SL}(3, \mathbb{R})$ and a point $x \in G/\Gamma$ such that for the full diagonal group $A$, the orbit closure $Ax$ is not homogeneous. Her construction was based on the following property of the lattice: there exists a $\gamma \in \Gamma \cap A$ such that the centralizer $C_G(\gamma)$ of $\gamma$ is isomorphic to $\text{SL}(2, \mathbb{R}) \times \mathbb{R}^*$, and such that $C_G(\gamma) \cap \Gamma$ is, in this product decomposition and up to finite index, $\Gamma_0 \times \langle \gamma \rangle$, where $\Gamma_0$ is a lattice in $\text{SL}(2, \mathbb{R})$ (see [4], [7]). Thus in this case the action of $A$ on $C_G(\gamma)/C_G(\gamma) \cap \Gamma$ factors to the action of a 1-parameter non-unipotent subgroup on $\text{SL}(2, \mathbb{R})/\Gamma_0$, which, as we saw, has many non-homogeneous orbits.

Rees’ example shows that factor actions of 1-parameter non-$\text{Ad}$-unipotent groups are obstructions to the topological rigidity of the action of diagonal subgroups. The following conjecture of Margulis [8, conjecture 1.1] (see also [6, 4.4.11]) essentially states that these are the only ones:

**Conjecture 1.** Let $G$ be a connected Lie group, $\Gamma$ a lattice in $G$, and $A$ a closed, connected subgroup of $G$ generated by $\text{Ad}$-split over $\mathbb{R}$ elements. Then for any $x \in G/\Gamma$, one of the following holds:

(a) $Ax$ is homogeneous, or
(b) There exists a closed connected subgroup $F$ of $G$ and a continuous epimorphism $\phi$ of $F$ onto a Lie group $L$ such that

- $A \subset F$,
- $Fx$ is closed in $G/\Gamma$,
- $\phi(F_x)$ is closed in $L$, where $F_x$ denotes the stabilizer $\{g \in F | gx = x\}$,

- $\phi(A)$ is a one-parameter subgroup of $L$ containing no nontrivial $\text{Ad}_L$-unipotent elements.

A first step toward this conjecture has been done by Lindenstrauss and Weiss [8], who proved that in the case $G = \text{SL}(n, \mathbb{R})$ and $A$ the full diagonal group, if the closure of a $A$-orbit contains a compact $A$-orbit that satisfy some irrationality conditions, then this closure is homogeneous. See also [15]. Recently, using an approach based on measure theory, Einsiedler, Katok and Lindenstrauss proved that if moreover $\Gamma = \text{SL}(n, \mathbb{Z})$, then the set of bounded $A$-orbits has Hausdorff dimension $n - 1$ [3, Theorem 10.2].

1.2. Statement of the results. In this article we exhibit some counterexamples to the above conjecture when $G = \text{SL}(n, \mathbb{R})$ for $n \geq 6$ and $A$ is some strict subgroup of the diagonal group of matrices with nonnegative entries. Let $D$ be
the diagonal subgroup of $G$; note that $D$ has dimension $n - 1$. Our main result is:

**Theorem 1.** Assume $n \geq 6$.

1. There exists a $(n - 3)$ dimensional closed and connected subgroup $A$ of $D$, and a point $x \in \text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$ such that the closure of the $A$-orbit of $x$ satisfies neither condition (a) nor condition (b) of the conjecture.

2. There exists a lattice $\Gamma$ of $\text{SL}(n, \mathbb{R})$, a $(n - 2)$ dimensional closed and connected subgroup $A$ of $D$ and a point $x \in \text{SL}(n, \mathbb{R})/\Gamma$ such that the closure of the $A$-orbit of $x$ satisfies neither condition (a) nor condition (b) of the conjecture.

It will be clear from the proofs that these examples however satisfy a third condition:

3. There exists a closed connected subgroup $F$ of $G$ and two continuous epimorphisms $\phi_1, \phi_2$ of $F$ onto Lie groups $L_1, L_2$ such that
   - $A \subset F$,
   - $Fx$ is closed in $G/\Gamma$,
   - For $i = 1, 2$, $\phi_i(Fx)$ is closed in $L_i$,
   - $(\phi_1, \phi_2) : F \to L_1 \times L_2$ is surjective
   - $(\phi_1, \phi_2) : A \to \phi_1(A) \times \phi_2(A)$ is not surjective.

Construction of these examples is the subject of Section 2, whereas the proof that they satisfy the required properties is postponed to Section 3.

1.3. **Toral endomorphisms.** To conclude this introduction, we would like to mention that the idea behind this construction can be also used to yield examples of 'non-homogeneous' orbits for diagonal toral endomorphisms.

Let $1 < p_1 < \cdots < p_q$, with $q \geq 2$, be integers generating a multiplicative non-lacunary semigroup of $\mathbb{Z}$ (that is, the $\mathbb{Q}$-subspace $\oplus_{1 \leq i \leq q} \mathbb{Q}\log(p_i)$ has dimension at least 2). We consider the abelian semigroup $\Omega$ of endomorphisms of the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ generated by the maps $z \mapsto p_i z \mod \mathbb{Z}^n$, $1 \leq i \leq q$.

In the one-dimensional situation, described by Furstenberg [5], every $\Omega$-orbit is finite or dense. If $n \geq 2$, Berend [1] showed that minimal sets are the finite orbits of rational points, but there are others obvious closed $\Omega$-invariant sets, namely the orbits of rational affine subspaces. Meiri and Peres [10] showed that closed invariant sets have integer Hausdorff dimension.

Note that the study of the orbit of a point lying in a proper rational affine subspace reduces to the study of finitely many orbits in lower dimensional tori, although some care must be taken about the pre-periodic part of the rational affine subspace (for example, if $q = n = 2$, and if $\alpha \in T^1$ is irrational with non-dense $p_1$-orbit, the orbit closure of the point $(\alpha, 1/p_2) \in T^2$ is the union of a horizontal circle and a finite number of strict closed infinite subsets of some horizontal circles).
With this last example in mind, Question 5.2 of [10] can be re-formulated: is a proper closed invariant set necessarily a subset of a finite union of rational affine tori? Or, equivalently, if a point is outside any rational affine subspace, does it necessarily have a dense orbit? It turns out that this is not the case at least for \( n \geq 2q \), as the following example shows.

**Theorem 2.** Let \( N \) be an integer greater than \( q \log \frac{p_2}{\log p_1} \), and let \( z \) be the point in the \( 2q \)-dimensional torus \( T^{2q} \) defined by the coordinates modulo 1:

\[
z = (z_1, \ldots, z_{2q}) = \left( \sum_{k \geq 1} p_1^{-N^{2k}}, \ldots, \sum_{k \geq 1} p_q^{-N^{2k}}, \sum_{k \geq 1} p_1^{-N^{2k+1}}, \ldots, \sum_{k \geq 1} p_q^{-N^{2k+1}} \right).
\]

Then the point \( z \in T^{2q} \) is not contained in any rational affine subspace, but its orbit \( \Omega z \) is not dense.

The proof of Theorem 2 will be the subject of Section 4.

2. **Sketch of proof of Theorem 1**

2.1. **The direct product setup.** We now describe how these examples are built. Choose two integers \( n_1 \geq 3, n_2 \geq 3 \), such that \( n_1 + n_2 = n \). For \( i = 1, 2 \), let \( \Gamma_i \) be a lattice in \( G_i = \text{SL}(n_i, \mathbb{R}) \).

Let \( g_i \) be an element of \( G_i \) such that \( g_i \Gamma_i g_i^{-1} \) intersects the diagonal subgroup \( D_i \) of \( \text{SL}(n_i, \mathbb{R}) \) in a lattice, in other words \( g_i \Gamma_i \) has a compact \( D_i \)-orbit; such elements exist, see [11]. In fact, we will need an additional assumption on \( g_i \), namely that the tori \( g_i^{-1}D_i g_i \) are irreducible over \( \mathbb{Q} \). The precise definition of this property and the proof of the existence of such a \( g_i \), a consequence of a theorem of Prasad and Rapinchuk [12, Theorem 1], will be the subject of Section 3.1.

Let \( \pi_i : G_i \to G_i/\Gamma_i \) be the canonical quotient map. Define for \( i = 1, 2 \):

\[
y_i = \pi_i \left( \begin{array}{cccc}
1 & 0 & \ldots & 0 & 1 \\
0 & 1 & \ddots & & 0 \\
& & \ddots & \ddots & \ddots \\
& & & 1 & 0 \\
0 & \ldots & \ldots & 0 & 1
\end{array} \right) g_i.
\]

The \( D_i \)-orbit of \( y_i \) is dense, by the following argument. It is easily seen that the closure of \( D_i y_i \) contains the compact \( D_i \)-orbit \( T_i = \pi_i(D_i g_i) \). The \( \mathbb{Q} \)-irreducibility of \( T_i \) is sufficient to show that the assumptions of the theorem of Lindenstrauss and Weiss [5, Theorem 1.1] are satisfied (Lemma 3.1); thus, by this theorem, we obtain that there exists a group \( H_i < G_i \) such that \( H_i y_i = D_i y_i \). Again because
of $\mathbb{Q}$-irreducibility, the group $H_i$ is necessarily the full group, i.e. $H_i = G_i$ (proof of Lemma 3.2) \(^1\).

Let $A_1$ be the $(n - 3)$ dimensional subgroup of $G_1 \times G_2$ given by:

\[
A_1 = \left\{ (\text{diag}(a_1, \ldots, a_{n_1}), \text{diag}(b_1, \ldots, b_{n_2})) : \prod_{i=1}^{n_1} a_i = \prod_{j=1}^{n_2} b_j = \frac{a_1 b_1}{a_{n_1} b_{n_2}} = 1, a_i > 0, b_j > 0 \right\}.
\]

Then the $A_1$-orbit of $(y_1, y_2)$ is not dense in $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ (Lemma 3.3), but $G_1 \times G_2$ is the smallest closed connected subgroup $F$ of $G_1 \times G_2$ such that $A_1(y_1, y_2) \subset F(y_1, y_2)$ (Lemma 3.7).

This yields a counterexample to Conjecture \(4\) which can be summarized as follows:

**Proposition 1.** For $i = 1, 2$, let $n_i \geq 3$ and $\Gamma_i$ be a lattice in $G_i = \text{SL}(n_i, \mathbb{R})$. For $A_1$, $y_1, y_2$ depicted as above, the $A_1$-orbit of $(y_1, y_2)$ in $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ satisfies neither condition (a) nor condition (b) of Conjecture \(4\).

2.2. **Theorem \(4\), part (1).** In order to obtain the first part of Theorem \(4\), choose $\Gamma_i = \text{SL}(n_i, \mathbb{Z})$, $\Gamma = \text{SL}(n, \mathbb{Z})$ and consider the embedding of $G_1 \times G_2$ in $G$, where matrices are written in blocks:

\[
\Psi : (M_{n_1, n_1}, N_{n_2, n_2}) \mapsto \begin{bmatrix} M_{n_1, n_1} & 0_{n_1, n_2} \\ 0_{n_2, n_1} & N_{n_2, n_2} \end{bmatrix}.
\]

This embedding gives rise to an embedding $\overline{\Psi}$ of $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ into $G/\Gamma$. Let $y_1, y_2$ be two points as above, let $x = \overline{\Psi}(y_1, y_2)$ and take $A = \Psi(A_1)$. We claim that this point $x$ and this group $A$ satisfy Theorem \(4\), part (1). In fact, since the image of $\overline{\Psi}$ is a closed connected $A$-invariant subset of $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$, everything takes place in this direct product.

2.3. **Theorem \(4\), part (2).** The second part of Theorem \(4\) is obtained as follows. Let $\sigma$ be the nontrivial field automorphism of the quadratic extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}(\sqrt{2})$. Consider for any $m \geq 1$:

\[
\text{SU}(m, \mathbb{Z}[\sqrt{2}], \sigma) = \left\{ M \in \text{SL}(m, \mathbb{Z}[\sqrt{2}]) : (\text{tr} M) M = I_m \right\}.
\]

Then $\text{SU}(m, \mathbb{Z}[\sqrt{2}], \sigma)$ is a lattice in $\text{SL}(m, \mathbb{R})$, as will be proved in Section 3.3 (see \(7\), Appendix) for $m = 3$). Define for $i = 1, 2$, $\Gamma_i = \text{SU}(n_i, \mathbb{Z}[\sqrt{2}], \sigma)$, and $\Gamma = \text{SU}(n, \mathbb{Z}[\sqrt{2}], \sigma)$. Now consider the map:

\[
\varphi : G_1 \times G_2 \times \mathbb{R} \to G,
\]

\(^1\)The reader only interested in the case $n = 6$ and $\Gamma = \text{SL}(6, \mathbb{Z})$ might note that when $\Gamma_1 = \Gamma_2 = \text{SL}(3, \mathbb{Z})$, \(6\), Corollary 1.4] can be used directly in the proof of Lemma 3.3; then the notion of $\mathbb{Q}$-irreducibility becomes unnecessary, and the entire Section 3.1 can be skipped.
\((X, Y, t) \mapsto \begin{bmatrix} e^{\alpha t} X & 0 \\ 0 & e^{-\alpha t} Y \end{bmatrix} \).  

Define \(M\) to be the image of \(\varphi\). This time, \(\varphi\) factors into a finite covering \(\overline{\varphi}\) of homogeneous spaces:

\[
\overline{\varphi} : G_1 \times G_2 \times \mathbb{R} / \Gamma_1 \times \Gamma_2 \times (\log \alpha) \mathbb{Z} \to M / M \cap \Gamma \subset G / \Gamma,
\]

where \(\alpha = (3 + 2\sqrt{2}) + \sqrt{2}(2 + 2\sqrt{2})\) satisfies \(\alpha^{-1} = \sigma(\alpha)\). Consider the points \(y_i\) constructed above, and let \(x = \overline{\varphi}(y_1, y_2, 0)\). Choose:

\[
A = \left\{ \text{diag}(a_1, \ldots, a_n) \mid \prod_{i=1}^{n} a_i = \frac{a_1 a_{n+1}}{a_{n+1} a_n} = 1, a_i > 0 \right\} \subset \text{SL}(n, \mathbb{R}).
\]

We claim that this lattice \(\Gamma\), this point \(x\) and this group \(A\) satisfy Theorem 1 part (2). What happens here is that the \(A\)-orbit of \(x\) is a circle bundle over an \(A_1\)-orbit (up to the finite cover \(\overline{\varphi}\)), like in Rees’ example.

3. Proof of Theorem 1

3.1. \(\mathbb{Q}\)-irreducible tori. Fix \(i \in \{1, 2\}\). Recall that \(\Gamma_i\) is a lattice in \(G_i = \text{SL}(n_i, \mathbb{R})\). Since \(n_i \geq 3\), by Margulis’s arithmeticity Theorem [40, Theorem 6.1.2], there exists a semisimple algebraic \(\mathbb{Q}\)-group \(H_i\) and a surjective homomorphism \(\theta\) from the connected component of identity of the real points of this group \(H_i^0(\mathbb{R})\) to \(\text{SL}(n_i, \mathbb{R})\), with compact kernel, such that \(\theta(H_i(\mathbb{Z}) \cap H_i^0(\mathbb{R}))\) is commensurable with \(\Gamma_i\).

Following Prasad and Rapinchuk, we say that a \(\mathbb{Q}\)-torus \(T \subset H_i\) is \(\mathbb{Q}\)-irreducible if it does not contain any proper subtorus defined over \(\mathbb{Q}\). By [12, Theorem 1, (ii)], there exists a maximal \(\mathbb{Q}\)-anisotropic \(\mathbb{Q}\)-torus \(T_i \subset H_i\), which is \(\mathbb{Q}\)-irreducible. Because any two maximal \(\mathbb{R}\)-tori of \(\text{SL}(n_i, \mathbb{R})\) are \(\mathbb{R}\)-conjugate, there exists \(g_i \in G_i\) such that \(\theta(T_i(\mathbb{R})) = g_i^{-1} D_i g_i\). The subgroup \(T_i(\mathbb{Z})\) is a cocompact lattice in \(T_i(\mathbb{R})\) since \(T_i\) is \(\mathbb{Q}\)-anisotropic [2 Theorem 8.4 and Definition 10.5]. Because \(\theta(H_i(\mathbb{Z}) \cap H_i^0(\mathbb{R}))\) and \(\Gamma_i\) are commensurable and \(\theta\) has compact kernel, it follows that both \(\Gamma_i \cap g_i^{-1} D_i g_i\) and \(\theta(T_i(\mathbb{Z})) \cap \Gamma_i \cap g_i^{-1} D_i g_i\) are also cocompact lattices in \(g_i^{-1} D_i g_i\). The resulting topological torus \(\pi_i(D_i g_i) \subset G_i / \Gamma_i\) will be denoted \(T_i\). Write \(z_i = \pi_i(g_i)\), so that \(T_i = D_i z_i\).

For every \(1 \leq k < l \leq n_i\), define as in [6]:

\[
N_{k,l}^{(i)} = \left\{ \text{diag}(a_1, \ldots, a_{n_i}) : \prod_{s=1}^{n_i} a_s = 1, a_k = a_l, a_s > 0 \right\} \subset D_i.
\]

Of interest to us amongst the consequences of \(\mathbb{Q}\)-irreducibility is the fact that an element of \(\Gamma_i \cap g_i^{-1} D_i g_i\) lying in a wall of a Weyl chamber is necessarily trivial. This is expressed in the following form:

**Lemma 3.1.** For every \(1 \leq k < l \leq n_i\), and any closed connected subgroup \(L\) of positive dimension of \(N_{k,l}^{(i)}\), the \(L\)-orbit of \(z_i\) is not compact.
Proof. Assume the contrary, that is \( Lz_i \) is compact. This implies that \( g_i^{-1}Lg_i \cap \Gamma_i \) is a uniform lattice in \( g_i^{-1}Lg_i \), so \( g_i^{-1}Lg_i \cap \theta(H_i(Z)) \) is also a uniform lattice. Since \( L \) is nontrivial, there exists an element \( \gamma \in H_i(Z) \cap H_i^0(R) \) of infinite order, such that \( g_i\theta(\gamma)g_i^{-1} \) is in \( L \). Note that since \( \theta \) has compact kernel, \( T_i(Z) \) is a lattice in \( \theta^{-1}(\theta(T_i^0(R))) \) and is then a subgroup of finite index in \( H_i(Z) \cap H_i^0(R) \cap \theta^{-1}(\theta(T_i^0(R))) \), so there exists \( n > 0 \) such that \( \gamma^n \) belongs to \( T_i(Z) \). Consider the representation:

\[
\rho : H_i^0(R) \to GL(sl(n_i, R)),
\]

\[
x \mapsto Ad(g_i\theta(x)g_i^{-1}).
\]

Recall that \( \chi(diag(a_1, \ldots, a_{n_i})) = a_k/a_l \) is a weight of \( Ad \) with respect to \( D_i \), so \( \chi \) is a weight of \( \rho \) with respect to \( T_i \). By [12, Proposition 1, (iii)], the \( Q \)-irreducibility of \( T_i \) implies that \( \chi(\gamma^n) \neq 1 \), but this contradicts the fact that \( \theta(\gamma^n) \in g_i^{-1}N_1^{(i)}g_i \). \( \square \)

3.2. Contraction and expansion. For real \( s \), denote by \( a_i(s) \) the following \( n_i \times n_i \)-matrix:

\[
a_i(s) = diag(e^{s/2}, 1, \ldots, 1, e^{-s/2}),
\]

and write simply \( N_i \) for \( N_1^{(i)} \). Write also:

\[
h_i(t) = \begin{bmatrix}
1 & 0 & \ldots & 0 & t \\
0 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & 0 \\
0 & \ldots & \ldots & 0 & 1
\end{bmatrix}.
\]

Then the following commutation relation holds:

\[
a_i(s)h_i(t) = h_i(e^st)a_i(s),
\]

that is the direction \( h_i \) is expanded for positive \( s \); note that both \( h_i \) and \( a_i \) commute with elements of \( N_i \). It is easy to check from Equation (4) that

\[
A_1 = \{(a_1(s)d_1, a_2(-s)d_2) : s \in R, d_i \in N_i, i = 1, 2 \}.
\]

Recall that \( y_i = h_i(1)z_i \).

Lemma 3.2. (1) If \( s \leq 0 \), for any \( d \in N_i \) the point \( a_i(s)dy_i \) lies in the compact set \( K_i = h_i([[0, 1]])T_i \).

(2) The \( D_i \)-orbit of \( y_i \) is dense in \( G_i/\Gamma_i \).

(3) The set \( \{a_i(s)dy_i : s \geq 0, d \in N_i \} \) is dense in \( G_i/\Gamma_i \).

Proof. The first statement is clear from the commutation relation. It also implies that \( D_iy_i \) contains the compact torus \( T_i \) in its closure.
Proof. Consider the open set \( \text{Lemma 3.3} \). So the following holds: there exists a reductive subgroup \( H_i \), containing \( D_i \), such that \( \overline{D_iy_i} = H_iy_i \), and \( H_i \cap \Gamma_i \) is a lattice in \( H_i \). Write \( L = D_i \cap C_G(H_i) \).

Since \( D_iy_i \) is not closed, \( H_i \neq D_i \), so there exists a nontrivial root relatively to \( D_i \) for the Adjoint representation of \( H_i \) on its Lie algebra, which is a subalgebra of \( \mathfrak{s}(n_i, \mathbb{R}) \). Thus there exist \( k, l \) such that \( L \subset N_{k,l}^{(i)} \). By \cite{b}, step 4.1 of Lemma 4.2, \( Lz_i \) is compact, so by Lemma 3.4, \( L \) is trivial. By \cite{b}, Proposition 3.1, \( H_i \) is the connected component of the identity of \( C_G(L) \), so \( H_i = G_i \), as desired.

The third claim follows from the first and second claim together with the fact that \( K_i \) has empty interior. \( \square \)

3.3. Topological properties of the \( A_1 \)-orbit.

Lemma 3.3. The \( A_1 \)-orbit of \((y_1, y_2)\) is not dense in \( G_1 \times G_2 / \Gamma_1 \times \Gamma_2 \).

Proof. Consider the open set \( U = K_1^i \times K_2^i \). We claim that the \( A_1 \)-orbit of \((y_1, y_2)\) does not intersect \( U \). Indeed, if \((a_1(s)d_1, a_2(s)d_2) \in A_1 \) with \( s \in \mathbb{R} \) and \( d_1 \in N_i \), the previous Lemma implies that if \( s \geq 0 \), \( a_2(-s)d_2y_2 \in K_2 \), and if \( s \leq 0 \), \( a_1(s)d_1y_1 \in K_1 \).

The following elementary result will be useful:

Lemma 3.4. Let \( p_i : G_1 \times G_2 \to G_i \) be the first (resp. second) coordinate morphism. If \( F \subset G_1 \times G_2 \) is a subgroup such that \( p_i(F) = G_i \) for \( i = 1, 2 \), and \( A_1 \subset F \), then \( F = G_1 \times G_2 \).

Proof. Let \( F_1 = \text{Ker}(p_1) \cap F \). Since \( F_1 \) is normal in \( F \), \( p_2(F_1) \) is normal in \( p_2(F) = G_2 \). Note that \( N_2 \subset p_2(A_1 \cap \text{Ker}(p_1)) \subset p_2(F_1) \) is not finite, and that \( G_2 \) is almost simple, consequently the normal subgroup \( p_2(F_1) \) of \( G_2 \) is equal to \( G_2 \). Let \((a, b) \in G_1 \times G_2 \), by assumption there exists \( f \in F \) such that \( p_1(f) = a \). Let \( f_1 \in F_1 \) be such that \( p_2(f_1) = bp_2(f)^{-1} \), then \((a, b) = f_1f \in F \). \( \square \)

We will have to apply several times the two following well-known Lemmas:

Lemma 3.5. Let \( L \) be a Lie group, \( \Lambda \subset L \) a lattice, \( M, N \) two closed, connected subgroups of \( L \), such that for some \( w \in L / \Lambda \), \( Mw \) and \( Nw \) are closed. Then \((M \cap N)w \) is closed.

Proof. This is a weaker form of \cite{b}, Lemma 2.2. \( \square \)

Lemma 3.6. Let \( L \) be a connected Lie group, \( \Lambda \subset L \) a discrete subgroup, \( M, N \) two subgroups of \( L \), such that \( M \) is closed and connected, and \( N \) is a countable union of closed sets. For any \( w \in L / \Lambda \), if \( Mw \subset Nw \), then \( M \subset N \).

Proof. Up to changing \( \Lambda \) by one of its conjugate in \( L \), one can assume that \( w = \Lambda \in L / \Lambda \). By assumption, \( M\Lambda \subset N\Lambda \) so \( M \subset N \Lambda \subset L \). Recall that \( M \)
is closed, that $\Lambda$ is countable, and that $N$ is a countable union of closed sets, so Baire’s category Theorem applies, and there exists $\lambda \in \Lambda$ and an open set $U$ of $M$ such that $U \subset N\lambda$, so $UU^{-1} \subset N$. Since $M$ is a connected subgroup, $UU^{-1}$ generates $M$, so $M \subset N$.

The following lemma will be useful both for proving that the closure of $A_1(y_1, y_2)$ is not homogeneous, and for proving it does not fiber over a 1-parameter group orbit.

**Lemma 3.7.** Let $F$ be a closed connected subgroup of $G_1 \times G_2$ such that $F(y_1, y_2)$ contains the closure of $A_1(y_1, y_2)$. Then $F = G_1 \times G_2$.

**Proof.** By Lemma 3.2, the set of first coordinates of the set
\[ \{(a(s)d_1y_1, a(-s)d_2y_2) : s \geq 0, d_i \in N_i\}, \]
is dense in $G_1/\Gamma_1$ and the second coordinates lies in the compact set $K_2$, so the closure of $A_1(y_1, y_2)$ contains points of arbitrary first coordinate with their second coordinate in $K_2$. Consequently, the set of first coordinates of $F(y_1, y_2)$ is the whole $G_1/\Gamma_1$, and similarly for the set of second coordinates. For $i = 1, 2$, Lemma 3.6 now applies to $L = M = G_i$, $\Lambda = \Gamma_i$, $N = p_i(F)$, which is a countable union of closed sets because $G_1 \times G_2$ is $\sigma$-compact, and $w = y_i$, and so $p_i(F) = G_i$.

In order to apply Lemma 3.4 and finish the proof, we have to show that $A_1 \subset F$. Again, this follows from a direct application of Lemma 3.6 to $L = G_1 \times G_2$, $\Lambda = \Gamma_1 \times \Gamma_2$, $M = A_1$, $N = F$, $w = (y_1, y_2)$.

\[ \square \]

### 3.4. Proof of Theorem 1, part (1)

We now proceed to proving Theorem 1, part (1). The proof of Proposition 3 is similar and is omitted.

Recall that in this case, we fixed $A = \Psi(A_1)$ and $x = \Psi(y_1, y_2)$.

Assume $Ax$ is homogeneous, that is $Ax = Fx$ for a closed connected subgroup $F$ of $G$. Since $Ax \subset \Psi(G_1 \times G_2/\Gamma_1 \times \Gamma_2)$, which is closed in $G/\Gamma$, Lemma 3.6 imply that $F \subset \Psi(G_1 \times G_2)$. By Lemma 3.7, $F = \Psi(G_1 \times G_2)$, so $Fx = G/\Gamma$ and $Ax$ is dense in $\Psi(G_1 \times G_2)$, which is a contradiction.

Now assume $Ax$ fibers over the orbit of a one-parameter subgroup. Let $F$ be a closed connected subgroup, $L$ a Lie group and $\phi : F \to L$ a continuous epimorphism satisfying the (b) of the conjecture. Let $F' = F \cap \Psi(G_1 \times G_2)$, we have $A \subset F'$. By Lemma 3.3, $F'x$ is closed in $Fx \cap \Psi(G_1 \times G_2)$, so is closed in $G/\Gamma$. By Lemma 3.7, $F' = \Psi(G_1 \times G_2)$ necessarily. Let $H = \text{Ker}(\phi \circ \Psi) \subset G_1 \times G_2$, so $A_1/(A_1 \cap H)$ is a one-parameter group by assumption (b).

The subgroup $H$ is a normal subgroup of the semisimple group $G_1 \times G_2$, which has only four kind of normal subgroups: finite, $G_1 \times G_2$, $G_1 \times \text{finite}$ and finite $\times G_2$. None of these possible normal subgroups have the property that they intersect $A_1$ in a codimension 1 subgroup, so this is a contradiction.
3.5. The arithmetic lattice. Here we prove that $\text{SU}(n, \mathbb{Z}[\sqrt{2}], \sigma)$ is a lattice in $\text{SL}(n, \mathbb{R})$. Let $P, Q$ be the polynomials with coefficients in $\mathbb{Q}(\sqrt{2})$ such that for any $X, Y \in M_n(\mathbb{C})$
\[
\det(X + \sqrt{2}Y) = P(X, Y) + \sqrt{2}Q(X, Y).
\]
For an integral domain $A \subseteq \mathbb{C}$, consider the set of pairs of matrices:
\[
G(A) = \{(X, Y) \in M_n(A)^2 : tXX - \sqrt{2}YY = I_n, tXY - tYX = 0, P(X, Y) = 1, Q(X, Y) = 0\},
\]
which implies that $(tX - \sqrt{2}Y)(X + \sqrt{2}Y) = I_n$ and $\det(X + \sqrt{2}) = 1$ for all $(X, Y) \in G(A)$. Endow $G(A)$ with the multiplication given by
\[
(X, Y)(X', Y') = (XX' + \sqrt{2}YY', XY' + YX'),
\]
which is such that the map $\phi : G(A) \to \text{SL}(n, \mathbb{C})$, $(X, Y) \mapsto X + \sqrt{2}Y$ is a morphism. With this structure, $G$ is an algebraic group, which is clearly defined over $\mathbb{Q}(\sqrt{2})$. Let $\tau$ be the nontrivial field automorphism of $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, it can be checked that the map $\phi$ is an isomorphism between $G(\mathbb{R})$ and $\text{SL}(n, \mathbb{R})$, and that moreover $\phi' : G^+(\mathbb{R}) \to \text{SL}(n, \mathbb{C})$, $(X, Y) \mapsto X + i\sqrt{2}Y$ is an isomorphism onto $\text{SU}(n)$. Let $H = \text{Res}_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}} G = G \times G^+$. Then $H$ is defined over $\mathbb{Q}$ (see for example [10], 6.1.3, for definition and properties of the restriction of scalars functor). It follows from a Theorem of Borel and Harish-Chandra [10, Theorem 3.1.7] that $H(\mathbb{Z})$ is a lattice in $H(\mathbb{R})$. Since $\text{SU}(n)$ is compact, it follows that the projection of $H(\mathbb{Z})$ onto the first factor of $G(\mathbb{R}) \times G^+(\mathbb{R})$ is again a lattice. Using the isomorphism between $G(\mathbb{R})$ and $\text{SL}(n, \mathbb{R})$, this projection can be identified with
\[
G(\mathbb{Z}[\sqrt{2}]) = \text{SU}(n, \mathbb{Z}[\sqrt{2}] + \sqrt{2}\mathbb{Z}[\sqrt{2}], \sigma) = \text{SU}(n, \mathbb{Z}[\sqrt{2}]), \sigma).
\]
3.6. Proof of Theorem 1, part (2). Note that, as stated implicitely in Section 2.3,
\[
\varphi(\Gamma_1 \times \Gamma_2 \times (\log \alpha)Z) \subseteq \Gamma \cap M,
\]
so $\Gamma \cap M$ is a lattice in $M$, and $M/(M \cap \Gamma)$ is a closed, $A$-invariant subset of $G/T$. Notice also that the map $\Psi$ defined by Equation (2) defines an embedding $\overline{\Psi} : G_1 \times G_2/\Gamma_1 \times \Gamma_2 \to G/T$.

Assume $Ax$ is homogeneous, that is $Ax = Fx$ for a closed connected subgroup $F$ of $G$. Since $Ax \subseteq M/(M \cap \Gamma)$, which is closed in $G/T$, Lemma 3.3 applied twice gives that $A \subseteq F \subseteq M$. Let $F' = F \cap \Psi(G_1 \times G_2)$, again by Lemma 3.3, $F'x$ is a closed subset of $\text{Im}(\overline{\Psi})$. Since $A_1 \subseteq F'$, $\Psi(A_1)x \subseteq F'x$ and Lemma 3.7 implies that $F' = \Psi(G_1 \times G_2)$. Since $A$ contains $\varphi(e, e, t)$ for all $t \in \mathbb{R}$, we have $M = AF' \subset F$ so $F = M$ necessarily.

By Lemma 3.3, the $A_1$-orbit of $(y_1, y_2)$ is not dense; the topological transitivity of the action of $A_1$ on $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ implies that moreover the closure of this orbit has empty interior. Thus, the $A_1 \times \mathbb{R}$-orbit of $(y_1, y_2, 0)$ is also nowhere
dense in $G_1 \times G_2 \times \mathbb{R}/\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbb{Z}$. The map $\varphi$ being a finite covering, the $A$-orbit of $x$ is nowhere dense. This is a contradiction with $F = M$.

Now assume $Ax$ fibers over the orbit of a one-parameter non-$A$-unipotent subgroup. Let $F$ be a closed connected subgroup, $L$ a Lie group and $\phi : F \to L$ a continuous epimorphism satisfying the (b) of the conjecture. Let $F' = F \cap \Psi(G_1 \times G_2)$ and $F'' = F \cap M$, we have $A_1 \subset F'$ and $A \subset F''$. Similarly, $F'x$ and $F''x$ are closed in $G/\Gamma$. Again, by Lemma 3.7, $F' = \Psi(G_1 \times G_2)$ necessarily, and like before, $AF \subset F'' \subset M$ so $F'' = M$. Let $H = \text{Ker}(\phi \circ \varphi) \subset G_1 \times G_2 \times \mathbb{R}$, so $A_1 \times \mathbb{R}/(A_1 \times \mathbb{R} \cap H)$ is a one-parameter group. This time, possibilities for the closed normal subgroup $H$ are: finite $\times \Lambda$, $G_1 \times G_2 \times \Lambda$, $G_1 \times \text{finite} \times \Lambda$ and finite $\times G_2 \times \Lambda$, where $\Lambda$ is a closed subgroup of $\mathbb{R}$. Of all these possibilities, only $G_1 \times G_2 \times \Lambda$, where $\Lambda$ is discrete, has the required property that $A_1 \times \mathbb{R}/(A_1 \times \mathbb{R} \cap H)$ is a one-parameter group. This proves that $\Psi(G_1 \times G_2) \subset \text{Ker}(\phi)$, so $F \subset N_G(\Psi(G_1 \times G_2))$. However, the normalizer of $\Psi(G_1 \times G_2)$ in $G$ is the group of block matrices having for connected component of the identity the group $M$. So by connectedness of $F$, $F \subset M$, and since $M = F'' \subset F$, we have $F = M$. Thus $L = F/\text{Ker}(\phi) = \mathbb{R}/\Lambda$ is abelian, and a fortiori every element of $L$ is unipotent; this contradicts (b).

4. Proof of Theorem 2

The proof of Theorem 2 is divided in two independent lemmas.

Lemma 4.1. The family $(z_1, \ldots, z_{2q}, 1)$ is linearly independent over $\mathbb{Q}$.

Proof. Consider a linear combination:

$$\sum_{i=1}^{q} a_i z_i + b_i z_{i+q} = c.$$ 

We can assume that $a_i, b_i$ and $c$ are integers. Let $k_0 \geq 1$, write

$$(3) \quad \left( \prod_{i=1}^{q} p_i \right)^{N^{2k_0+1}} \left( \sum_{i=1}^{q} \sum_{k=1}^{k_0} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} - c \right) =$$

$$- \left( \prod_{i=1}^{q} p_i \right)^{N^{2k_0+1}} \left( \sum_{i=1}^{q} \sum_{k \geq k_0+1} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} \right).$$
It is clear the left hand side is an integer. Since $1 < p_1 < \cdots < p_q$, the right hand side is less in absolute value than
\[
p_q^{N^{2k_0+1}} 2q \sup_i (|a_i|, |b_i|) \sum_{k \geq 0} \left( p_1^{-N^{2k_0+2}} \right)^{N^{2k}} \leq 4q \sup_i (|a_i|, |b_i|) p_q^{N^{2k_0+1}} p_1^{-N^{2k_0+2}} \leq 4q \sup_i (|a_i|, |b_i|) \exp(N^{2k_0+1}(q \log p_q - N \log p_1)).
\]
Since $N > q^{\log(p_q)}$, the last expression tends to zero. This proves the right-hand side of (3) is zero for large enough $k_0$, so for all large $k$,
\[
\sum_{i=1}^q a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} = 0.
\]
The $p_i$ being distincts, this implies that for $i \in \{1, \ldots, q\}$, $a_i = b_i = 0$. \hfill \Box

The following Lemma implies easily that the orbit of $z$ under $\Omega$ cannot be dense.

**Lemma 4.2.** For all $\epsilon > 0$, there exists $L > 0$, such that for all $n_1, \ldots, n_q \geq 0$ with $\sum_{i=1}^q n_i \geq L$, there exists $j \in \{1, \ldots, 2q\}$ such that $p_1^{n_1} \cdots p_q^{n_q} z_j$ lies in the interval $[0, \epsilon]$ modulo 1.

**Proof.** Let $s \in \{1, \ldots, q\}$ such that for all $r \in \{1, \ldots, q\}$, $p_r^{n_r} \geq p_s^{n_s}$. Let $k_0$ be the integer part of $\log(n_s)/2\log(N)$, then either $N^{2k_0} \leq n_s \leq N^{2k_0+1}$, or $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$. In the first case, take $j = s$, then:
\[
p_1^{n_1} \cdots p_q^{n_q} z_j = p_1^{n_1} \cdots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} = p_1^{n_1} \cdots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} \mod 1.
\]
We have
\[
\sum_{k \geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{-N^{2k_0+2}},
\]
so, using the fact that for all $r \in \{1, \ldots, q\}$, $p_r^{n_r} \leq p_s^{n_s} \leq p_s^{N^{2k_0+1}}$, we obtain:
\[
p_1^{n_1} \cdots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{N^{2k_0+1}-N^{2k_0+2}} \leq 2p_s^{N^{2k_0+1}(q-N)},
\]
but by hypothesis we have $N > q^{\log(p_q)} > q$, so the preceding bound is small whenever $k_0$ is large. Because of the definition of $k_0$, we have
\[
k_0 \geq \frac{\log \sum_{i=1}^q n_i \log p_i}{2\log N} \geq \frac{\log L \log p_1}{2\log N},
\]
so $k_0$ is arbitrary large when $L$ is large.

In the second case $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$, one can proceed similarly with $j = s + q$.
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