Number of Measurements in Sparse Signal Recovery

Paul Tune, Sibi Raj Bhaskaran, Stephen Hanly
University of Melbourne, VIC-3010, Australia.
{1.tune, sibib, hanly}@unimelb.edu.au

Abstract—We analyze the asymptotic performance of sparse signal recovery from noisy measurements. In particular, we generalize some of the existing results for the Gaussian case to subgaussian and other ensembles. An achievable result is presented for the linear sparsity regime. A converse on the number of required measurements in the sub-linear regime is also presented, which cover many of the widely used measurement ensembles. Our converse idea makes use of a correspondence between compressed sensing ideas and compound channels in information theory.

I. INTRODUCTION

Sparse support recovery has been given much attention of late, due to the fact that many signals dealt with are sparse in some basis. We will consider the model,

\[ y = Ax + z \tag{1} \]

where \( y \in \mathbb{R}^m \), \( A \in \mathbb{R}^{m \times n} \), \( z \in \mathbb{R}^m \), distributed with \( \mathcal{N}(0, \sigma^2 I) \). The support of \( x \) is the index set \( \mathcal{I} \), \( \text{supp}(x) = |\mathcal{I}| = k \). The signal power, \( |x|^2_2 = P \). Each column of \( A \) is normalized to have unit \( \ell_2 \) norm.

Our main motivation in this paper is to study a wider class of measurement matrices. Previous studies have specifically focussed on the Gaussian measurement matrix [1], [13]. Two distinct sparsity regimes are often considered in literature:

- **Sublinear**: \( \frac{k}{n} \to 0 \) as both \( k, n \to \infty \), and
- **Linear**: \( k = \rho n \) for \( \rho \in (0, 1) \).

The following three performance estimates were studied in [1], [13].

- **Error metric 1**:
  \[ d_1(x, \hat{x}) = \mathbb{I}(\{ \hat{x}_i \neq 0 \ \forall i \in \mathcal{I} \} \cap \{ \hat{x}_j = 0 \ \forall j \notin \mathcal{I} \}) \]

- **Error metric 2**:
  \[ d_2(x, \hat{x}) = \mathbb{I} \left( \frac{|\{\hat{x}_i \neq 0\} |}{|\mathcal{I}|} > 1 - \alpha \right) \]

- **Error metric 3**:
  \[ d_3(x, \hat{x}) = \mathbb{I} \left( \sum_{k \in \{i: \hat{x}_i \neq 0\} \cap \mathcal{I}} |x_k|^2 > (1 - \delta)P \right) \]

where \( \mathbb{I}(\cdot) \) is the binary valued indicator function which is unity when the argument is true, and \( \alpha, \delta \) are in \((0, 1)\). In Section III we focus on subgaussian measurement matrices.

**Definition 1.1**: A random variable \( x \) is **subgaussian** if there is a constant \( B > 0 \) such that

\[ \Pr(|x| \geq t) \leq 2 \exp(-t^2/B^2) \]

for all \( t > 0 \). The smallest \( B \) is called the **subgaussian moment** of \( x \).

An example of a subgaussian measurement matrix is the matrix with i.i.d. entries of \( \pm 1/\sqrt{m} \) distributed according to Bernoulli(\(1/2\)).

We show that centered subgaussian measurement matrices achieve the same asymptotic results as Gaussian measurement matrices in the linear sparsity regime, i.e. \( m = O(k) \) measurements suffice for signal recovery. For the linear case, we are taking the **pessimistic** point of view that good measurement (sensing) schemes should have an exponentially decaying error probability in the number of measurements, which will also have a bearing on the practical constructions. On the other hand, if we take an **optimistic** (see [5]) viewpoint, that a sub-exponential decay in error is acceptable, our analysis remains valid for the sub-linear regime also.

In Section III we present some converse results, which lower bounds the required number of measurements for asymptotically exact support recovery. Our converse results give the required scaling of \( m \) with respect to \( n \) and \( k \) in both the regimes. Specifically, we invoke a correspondence between compressed sensing schemes and compound channels in information theory. Here we consider general measurement matrices and the underlying assumptions are mild.

II. ACHIEVABILITY

Our setup for achievability is similar to [1]. In particular, we extend Theorems 2.1, 2.5 and 2.9 from [1] which provide results for the number of measurements needed using Gaussian measurement matrices for the error metrics considered in this paper. For Gaussian measurement matrices, the number of measurements required for all three error metrics in the linear sparsity regime is \( m = O(k) \), where the hidden constant value differs for each error metric. For completeness, we state Theorem 2.1 from [1] here.

**Theorem 2.1 (Achievability for error metric 1)**: Let a sequence of sparse vectors, \( \{x^{(n)} \in \mathbb{R}^n\}_n \) \( (x^{(n)}) \) denotes a dependence on \( n \) with \( \text{supp}(x^{(n)}) = k = \lfloor \rho n \rfloor \). Then asymptotic reliable recovery is possible for \( \{x^{(n)}\} \) with respect to error metric 1 if \( k\mu(x^{(n)}) \to \infty \) as \( k \to \infty \) and

\[ m > c_1 k \]

where \( \mu(x) = \min_{i \in \mathcal{I}} |x_i| \) and \( c_1 \) is a constant depending on \( \rho, \mu(x) \) and \( \sigma \).

Our result here shows that these results apply to subgaussian measurement matrices.
On the other hand, in the sublinear sparsity regime, measurements required are now in the order of \( m = O(k \log(n - k)) \) for all three error metrics for Gaussian measurement matrices. As mentioned earlier, if we take an optimistic viewpoint, then subgaussian measurement matrices also achieves the same performance as the Gaussian counterpart. The Lasso scheme was shown to perform optimally in the sublinear regime \([12]\) but the results show that there is a significant gap of the performance of Lasso in the linear regime.

Let \( D(y) \) be a decoder, which outputs a set of indices, depending on the problem objective. Our achievability results show the existence of asymptotically good measurement matrices. Similar to the random coding arguments in information theory, the average error probability attained by using random measurement matrices chosen from an ensemble can be made arbitrarily small asymptotically. However, good matrices are not explicitly identified.

The probability of decoding error for \( D \), averaged over all measurement matrices \( A \), is defined as

\[
p_{err}(D|x) = \mathbb{E}_A(p_{err}(A|x)) = \mathbb{E}_A(Pr(D(y) \neq I)).
\]

We focus on decoders using joint typicality. We define the projection matrix of \( B \) as \( \Pi_B = B(B^TB)^{-1}B^T \). The orthogonal projection is defined as \( \Pi_B^\perp = I - B(B^TB)^{-1}B^T \).

**Definition 2.2 (Joint Typicality):** [1] The noisy observation vector \( y \) and a set of indices \( J \subset \{1, 2, \ldots, n\} \), with \( |J| = k \), are \( \delta \)-jointly typical if \( \text{rank}(A_J) = k \) and

\[
\left| \frac{1}{m} \| \Pi_{\perp J} y \|_{\ell_2}^2 - \frac{m - k}{m} \sigma^2 \right| < \delta
\]

Denote the events,

\( \Omega_J = \{ y \text{ and } J \text{ are } \delta\text{-typical} \} \)

and

\( \Omega_0 = \{ \text{rank}(A_J) < k \} \).

The decoder has three sources of error:

- the decoder searches incorrect subspaces, event \( \Omega_0 \),
- the true support set \( I \) is not \( \delta \)-jointly typical, event \( \Omega_J^\perp \), and
- the decoder recovers another support set \( J \) such that \( J \neq I \), event \( \Omega_J \).

Hence, the upper bound to the decoder error is given by union bound of the three sources of error,

\[
p_{err}(D|x) \leq Pr(\Omega_0) + Pr(\Omega_J^\perp) + \sum_{J,J \neq I,|J| = k} Pr(\Omega_J).
\]

It suffices to find bounds on each error probability that vanishes asymptotically as \( n \to \infty \). We show this below.

### A. Proof of Achievability

We first find bounds on the probability that \( \Omega_0 \) occurs by using the following result \([11, \text{Theorem 1.1}]\).

**Lemma 2.3:** Let \( X \) be a subgaussian random variable with zero mean, variance one and subgaussian moment \( B \). Let \( X \in \mathbb{R}^{m \times k}, m \geq k \) be the random matrix whose entries are i.i.d. copies of \( X \). Then there are positive constants \( c_1, c_2 \) (depending polynomially on \( B \)) such that for any \( t > 0 \)

\[
Pr(\|X\|_{\ell_2} \leq t(\sqrt{m} - \sqrt{k - 1})) \leq (c_1 t)^{m-k+1} + e^{-c_2 t m}.
\]

where \( s_k(X) \) denotes the smallest singular value of \( X \).

In particular, the above lemma suggests that for subgaussian matrices, there is an exponentially small positive probability that \( s_n = 0 \). We use this in the following result.

**Theorem 2.4:** Assume \( m > k \). Given an index set \( I \subset \{1, 2, \ldots, n\} \) with \( |I| = k \),

\[
Pr(\text{rank}(A_I) < k) \leq e^{-c_0 m}
\]

for some constant \( c_0 > 0 \).

**Proof:** To ensure recovery of \( x \), it is essential that \( \text{rank}(A_I) = k \) or equivalently, the smallest singular value, \( s_k(A_I) \neq 0 \). Using Lemma 2.3 and choosing small \( t \), we have

\[
Pr(s_k(A_I) = 0) = \lim_{t \to 0} Pr(s_k(A_I) \leq t(\sqrt{m} - \sqrt{k - 1})) \leq e^{-c_0 m}.
\]

**Remark 2.5:** Reference \([1]\) uses the fact that if \( A \) has i.i.d. entries with \( N(0,1) \), then \( Pr(\text{rank}(A) < k) \) can never be singular. For subgaussian matrices, it is possible for such an error to occur. For example, with the random sign matrices distributed according to Bernoulli(\( \frac{1}{2} \)), it is easy to see that

\[
Pr(\text{rank}(A) < k) \geq \left( \frac{1}{2} \right)^k.
\]

Hence, Theorem 2.4 says that in the linear regime, the error decay for the event \( \Omega_0 \) is exponential with the number of measurements \( m > k \). However, a sub-exponential decay to zero can be achieved even for the sublinear case. The rest of our arguments are valid for both cases.

We first modify Lemma 3.3 from \([1]\) by introducing conditions under which the result is still valid. We then show that the subgaussian measurement matrices satisfy these conditions.

**Lemma 2.6:** 1) Let \( I = \text{supp}(x) \) and assume that \( \text{rank}(A_I) = k \). Then for \( \delta > 0 \),

\[
Pr \left( \left| \frac{1}{m} \| \Pi_{\perp J} y \|_{\ell_2}^2 - \frac{m - k}{m} \sigma^2 \right| > \delta \right) \leq 2 \exp \left( -\frac{\delta^2}{4\sigma^4 m^2} - \frac{m^2}{m - k + \frac{2\delta^2}{m}} \right).
\]

This result holds for any measurement matrix \( A \).

2) Let \( J \) be an index set such that \( |J| = k \) and \( |I \cap J| = p < k \), where \( I = \text{supp}(x) \) and assume that \( \text{rank}(A_J) = k \). Let

\[
V = \frac{\| \Pi_{\perp J} y \|_{\ell_2}^2}{\sigma_y^2} - (m - k)
\]
where \( \sigma_y^2 = \sum_{i \in \mathcal{I} \setminus \mathcal{J}} x_i^2 + \sigma^2 \). Then \( y \) and \( \mathcal{J} \) are \( \delta \)-joint typical with probability

\[
\Pr \left( \frac{1}{m} \| \Pi_{\mathcal{A}_\mathcal{J}} y \|_{\ell_2}^2 - \frac{m - k - \sigma^2}{m} < \delta \right) \\
\leq 2 \exp \left( - \frac{1}{2\gamma_1} \left( (m - k) \left( 1 - \frac{\sigma_y^2}{\sigma_y^2} \right) - \frac{\delta}{\sigma_y^2} m \right)^2 \right)
\]

if the moment condition

\[
\log \mathbb{E}[e^{TV}] \leq -\gamma_1 t - \frac{\gamma_1}{2} \log(1 - \gamma_2 t)
\]

is satisfied with constants \( \gamma_1, \gamma_2 > 0 \) for \( t < 1/\gamma_2 \).

**Proof:** The proof for the first item is the same as that of the proof of the first part given in [1, Lemma 3.3]. We have

\[
\Pi_{\mathcal{A}_\mathcal{J}} y = \Pi_{\mathcal{A}_\mathcal{J}} z,
\]

and

\[
\Pi_{\mathcal{A}_\mathcal{J}} y = \Pi_{\mathcal{A}_\mathcal{J}} \left( \sum_{i \in \mathcal{I} \setminus \mathcal{J}} x_i a_i + z \right).
\]

It can be seen that, by the property of symmetric projection matrices, \( \Pi_{\mathcal{A}^\top \mathcal{J}} \Pi_{\mathcal{A}_\mathcal{J}} = \Pi_{\mathcal{A}_\mathcal{J}} \). Furthermore, \( z \) is independent of the entries of \( \Pi_{\mathcal{A}_\mathcal{J}} \). Hence by [8, Chapter 18],

\[
\frac{\| \Pi_{\mathcal{A}_\mathcal{J}} y \|_{\ell_2}^2}{\sigma_y^2} = \left( \frac{z}{\sigma} \right)^T \Pi_{\mathcal{A}_\mathcal{J}} \left( \frac{z}{\sigma} \right) \sim \chi^2_{m-k}
\]

By using concentration inequalities of chi-squared random variables around their degrees of freedom \((m - k)\) here as in [1, Lemma 3.3], the same result is obtained.

For the second part of the lemma, we have

\[
\Pr \left( \frac{1}{m} \| \Pi_{\mathcal{A}_\mathcal{J}} y \|_{\ell_2}^2 - \frac{m - k - \sigma^2}{m} < \delta \right) \\
= \Pr \left( \frac{1}{m} \| \Pi_{\mathcal{A}_\mathcal{J}} y \|_{\ell_2}^2 - \frac{m - k - \sigma^2}{m} < \delta \right) + \Pr \left( \frac{1}{m} \| \Pi_{\mathcal{A}_\mathcal{J}} y \|_{\ell_2}^2 - \frac{m - k - \sigma^2}{m} > -\delta \right) \\
= \Pr \left( V < -(m - k) \left( 1 - \frac{\sigma_y^2}{\sigma_y^2} + \frac{\delta}{\sigma_y^2} m \right) \right) + \Pr \left( V > -(m - k) \left( 1 - \frac{\sigma_y^2}{\sigma_y^2} - \frac{\delta}{\sigma_y^2} m \right) \right).
\]

Using Chernoff’s bound and the moment condition, it can be shown that for any \( \lambda > 0 \) (see Appendix),

\[
\Pr(V \geq \gamma_2 \lambda + \sqrt{2\gamma_1 \lambda}) \leq \Pr(V \geq \sqrt{2\gamma_1 \lambda}) \leq e^{-\lambda}
\]

and

\[
\Pr(V \leq -\sqrt{2\gamma_1 \lambda}) \leq e^{-\lambda}.
\]

We bound the first probability by choosing in equation (4),

\[
\lambda_1 = \frac{1}{2\gamma_1} \left( (m - k) \left( 1 - \frac{\sigma_y^2}{\sigma_y^2} \right) - \frac{\delta}{\sigma_y^2} m \right)^2
\]

and for the second,

\[
\lambda_2 = \frac{1}{2\gamma_1} \left( (m - k) \left( 1 - \frac{\sigma_y^2}{\sigma_y^2} \right) + \frac{\delta}{\sigma_y^2} m \right)^2
\]

in equation (4), we have

\[
\Pr \left( \frac{1}{m} \| \Pi_{\mathcal{A}_\mathcal{J}} y \|_{\ell_2}^2 - \frac{m - k - \sigma^2}{m} < \delta \right) \leq 2 \exp(-\lambda_1).
\]

since \( \lambda_1 \leq \lambda_2 \).

**Theorem 2.7:** Subgaussian measurement matrices satisfy Lemma 2.6 with \( \gamma_1 = m - k \) and \( \gamma_2 = 2 \).

**Proof:** We only need to show how subgaussian measurement matrices satisfy Lemma 2.6.2. We first note that subgaussian r.v.s have a closure property under addition. Hence, the vector

\[
y = \sum_{i \in \mathcal{I} \setminus \mathcal{J}} x_i a_i + z
\]

is still subgaussian since for some constant \( \alpha [9] \),

\[
\mathbb{E}[e^{TV}] \leq \exp \left( \frac{t^2}{2} \left( \sum_{i \in \mathcal{I} \setminus \mathcal{J}} x_i^2 \alpha^2 + \sigma^2 \right) \right) \leq \exp \left( \frac{t^2}{2} \left( \alpha' \sigma_y^2 \right) \right)
\]

where \( \lambda' > 0 \) is a constant and

\[
\sigma_y^2 = \sum_{i \in \mathcal{I} \setminus \mathcal{J}} x_i^2 + \sigma^2.
\]

Note that the vector is independent of the entries of \( \Pi_{\mathcal{A}_\mathcal{J}} \).

Since \( \Pi_{\mathcal{A}_\mathcal{J}} \) is a symmetric and idempotent, we rewrite

\[
\frac{\| \Pi_{\mathcal{A}_\mathcal{J}} y \|_{\ell_2}^2}{\sigma_y^2} = \left( \frac{y}{\sigma_y} \right)^T \Pi_{\mathcal{A}_\mathcal{J}} \left( \frac{y}{\sigma_y} \right).
\]

To bound the moment, we require an estimate using [9, Lemma 1.2,] for \(0 \leq t \leq \frac{1}{2}\alpha'\),

\[
\mathbb{E}[\exp(tV)] \leq e^{-t(m-k) \cdot (1 - 2t)^{-m-k/2}}.
\]

Note that the upper bound is the moment generating function of distribution \( \chi^2_{m-k} \).

The function \( \log \mathbb{E}[\exp(tV)] \) is monotonically decreasing in \( t < 0 \) and at \( t = 0 \), we have \( \log \mathbb{E}[\exp(tV)] \leq 0 \). On the other hand, the function \( (m-k)^2 \) is monotonically increasing for \( t < 0 \). As such, we have \( \log \mathbb{E}[\exp(tV)] \leq (m-k)^2 \) for \( t < 0 \). Hence, it can be easily seen that \( \gamma_1 = m - k \) and \( \gamma_2 = 2 \).

With \( \gamma_1 = m - k, \gamma_2 = 2, \sigma = 1 - \frac{\sigma_y^2}{\sigma_y^2} \) and \( \delta' = \frac{\delta m}{\sigma_y^2} \),

\[
\Pr(\Omega_{\mathcal{J}}) \leq 2 \exp \left( - \frac{1}{2(m-k)} \left( (m-k)\sigma - \frac{\delta m}{\sigma_y^2} \right)^2 \right) \\
\leq 2 \exp \left( - \frac{m-k}{4} \left( \frac{\sigma_y^2 - \sigma^2 - \delta'}{\sigma_y^2} \right)^2 \right) \\
= 2 \exp \left( - \frac{m-k}{4} \left( \sum_{k \in \mathcal{J}} x_k^2 - \delta' \frac{m}{\sigma_y^2} \right)^2 \right) \\
= 2 \exp \left( - \frac{m-k}{4} \left( \sum_{k \in \mathcal{J}} x_k^2 + \sigma^2 \right)^2 \right).
\]

Assuming \( \text{rank}(\mathcal{A}_\mathcal{J}) = k \), the number of subsets \( \mathcal{J} \) that overlaps \( \mathcal{I} \) in \( p \) indices is upper-bounded by \( \left( \frac{p}{k} \right) \left( \frac{n-k}{p-k} \right) \), implying
that by (2) and Theorem 2.6
\[ p_{err}(D|x) \leq \exp(-c_0 m) + 2\exp(-\frac{\delta^2}{4\sigma^2} m - k + \frac{2k}{\sigma^2} m) + 2 \sum_{p=1}^k \binom{k}{p} \binom{m-k}{p} \exp(-\frac{m-k}{4} \left( \sum_{k \in I \setminus J} x_k^2 - \delta^2 \right)^2 ). \]

We sketch an outline of the rest of the proof here. Only \( \Pr(\Omega_J) \) changes depending on the error metric. Let \( |I \cap J| = p \) for some particular set \( J \). For error metric 1, we note that \( \sum_{k \in I \setminus J} x_k^2 \geq (k-p)\mu^2(x) \). For error metric 2, since we only need \( \Pr(\Omega_J) \rightarrow 0 \) for \( p \leq (1-\alpha)k \) for \( \alpha \in (0,1) \), then we have \( \sum_{k \in I \setminus J} x_k^2 \geq \alpha k \mu^2(x) \) for error to occur. Finally, for error metric 3, we have \( \sum_{k \in I \setminus J} x_k^2 \geq \gamma P \) for error to occur. The rest of the arguments on bounding the error probability follows that of the analysis on Gaussian measurement ensembles in [1], both in the linear and sublinear sparsity regimes.

III. CONVERSE ON THE NUMBER OF MEASUREMENTS

Our starting point is again the signal recovery model in (1). For simplicity, assume that \( x \) has \( k \) non-zero entries. Further more, the entries of \( A \) are taken from some alphabet \( \mathcal{A} \), and normalized, i.e., for each column \( a_i \),
\[ \frac{1}{m} \|a_i\|_2^2 = 1, \quad a_{ij} \in \mathcal{A} \]
(6)
Note that the measurement matrix \( A \) is specified in advance without the knowledge of the instantaneous realization of \( x \). So \( A \) depends only on the global properties of \( x \) and the noise statistics. For simplicity (also for practical reasons), we make the mild assumption that there is no prior knowledge about the input values favoring any particular locations. This implies that the support of \( x \) is uniformly chosen from the \( \binom{k}{m} \) possible choices.

Our discussion in this section is for the error metric 1, but can be tailored for other purposes too. Recall that for the first metric, we are interested in recovering the support of \( x \) based on \( m \) measurements from (1). The error probability in recovering the support lower-bounds that of exact signal recovery. This can be easily seen by imagining a genie which tells the receiver about the non-zero components in the order of their appearance.

We need some notation to proceed. Let us define the following:
- \( \alpha \) - the vector of non-zero values of \( x \), in descending order of magnitude, the \( i^{th} \) entry being \( \alpha_i \).
- \( \beta \) - non-zero values of \( x \) in the order of appearance.
- \( I_0 \) - set of indices of \( x \) with zero magnitude.
- \( \rho(\alpha) \) - index in \( x \) corresponding to the \( i^{th} \) entry of \( \alpha \).
- \( \bar{R}(k,\alpha,\sigma^2) \) - capacity region of a \( k \)-user single antenna Gaussian MAC with channel gains \( \bar{\alpha} \), and input constraints as in (5).

Let \( \hat{x} \) be the recovered vector using some decoding method. In this section, we assume that \( m \) is large enough, with respect to \( k \) and in relation to \( n \), to ensure that the probability of decoding error tends to zero as \( m,n \) and \( k \) tend to infinity.

The error event can be written in terms of a random variable \( \Phi \), which is defined as,
\[ \Phi = \left( \prod_{i: x_i = 0} 1_{\{\hat{x}_i = 0\}} \right) \left( \prod_{i: x_i \neq 0} 1_{\{\hat{x}_i \neq 0\}} \right). \]
(7)

Given the \( k \) non-zero symbols \( \beta \), \( \Phi \) is induced by a uniform distribution on the \( \binom{k}{m} \) possible supporting indices of the vector \( x \). In many practical cases, \( \beta \) is drawn from some distribution. Our results can be extended to handle this, but presently we stick to fixed \( \beta \), and we assume all the components of \( \beta \) are distinct. The later assumption is just for saving some notation, and has no bearing on the technical details. The average error probability now becomes,
\[ P_{error} = \Pr(\Phi = 0). \]
(8)

The following lemma yields a lower bound on \( m \), the number of measurements required for asymptotically exact support recovery.

**Lemma 3.1:** For a given \( \beta \) with \( k \) non-zero elements, if \( P_{error} \) goes to zero with \( m \), then
\[ m \geq \frac{k \log(n/k)}{R_{CMAC}(k,\alpha,\sigma^2)} \]
(9)
where
\[ R_{CMAC}(k,\alpha,\sigma^2) = \min_{\bar{\alpha}} \max_{\alpha^*} \|R\|_{\ell_1} \mathbb{1}_{\{R \in \bar{R}(k,\bar{\alpha},\sigma^2)\}} \]
(10)
and \( \alpha^* \) is any permutation of the channel coefficients \( \bar{\alpha} \).

The proof of this lemma proceeds in number of stages. In the next few paragraphs, we will explain the essential ideas behind it. The arguments that we present shed light on some of the underlying bottlenecks in the detection problem.

To obtain a bound as above, we map the support recovery (SR) problem to a communication problem and then establish the connection between the number of measurements \( m \) and the required number of channel uses in the communication model, or alternatively to the maximal rate at which error-free transmissions are possible.

In principle, the communication setup that we describe can simulate any strategy for the support recovery problem. We briefly describe how any SR problem comes under our communication setup, see Figure below.

Recall the notations introduced in paragraph 3 of this section. Consider \( k \) encoders trying to communicate information
to a decoder. Each encoder corresponds to a non-zero value of the input vector $x$ in the support recovery problem. Perform a random permutation of the set $I_o$ and partition it into $k$ subsets $\{U_1, \ldots, U_k\}$, provide this to each encoder as side-information. The decoder is given the index set $Q_i$ of each encoder’s inputs, as well as the channel coefficient $\alpha_i$ from that encoder. Clearly this system can emulate the SR problem. We now take an alternate view to the keep the discussion as simple as possible. We describe a setup where the sparse vector $x$ for the SR problem, and the messages for the corresponding communication problem are generated together. There is no loss of generality in coupling the two systems like this.

To this end, randomly permute the indices of $x$ and partition them into $k$ sets $S_1, S_2, \ldots, S_k$. To partially emulate the SR problem, the support of $x$ is chosen by selecting one element from each of these sets, which correspond to the indices of the support of $x$. This selection will correspond to message selection in a $k$-user communication channel, in which $S_k$ is the message set of user $k$. A simple method of communication is for user $k$ to encode the chosen message by sending the corresponding column of $A$ directly (rather like a CDMA scheme, with no additional coding) and the decoder then receives

$$y = \sum_{i=1}^{k} \beta_{\pi(i)} a_i + z$$

where $a_i$ is the column corresponding to the message chosen by user $i$, and $\pi$ is a random permutation of $\{1, 2, \ldots, k\}$ that assigns a component of $x$ to user $k$. The decoder is given the vector $(\beta_{\pi(i)})_{i=1}^k$ as side information. This coherent $k$-user faded AWGN communication channel is a partial emulation of the CS decoding problem in [1], except that here the decoder has more information: it knows that each $S_i$ contains exactly one index from the support of the vector $x$, and it knows the corresponding value of $x$ in that component, namely $\beta_{\pi(i)}$. Note that user $i$ is conveying $\log(|S_i|)$ bits to the decoder, and the total number of bits being conveyed is $\sum_{i=1}^{k} \log(|S_i|)$ bits. The decoder in this communication set-up must do at least as well as the CS decoder in the original problem, so these bits are being conveyed reliably.

The above simple CDMA communication scheme is valid for the $k$-user, faded AWGN channel in which, in general, the user is allowed to encode his messages using symbols each taken from the same alphabet as the symbols in $A$, and each codeword satisfies the power constraint $E_0(\alpha, \sigma^2)$. Since the permutation $\pi$ is selected randomly, this is a compound MAC, and the rate region can in principle be calculated. In a compound MAC, the transmitter knows only a set of possible MACs from which one realization will be picked [5]. We do not go into the details of the coding theorems, rather we merely use the results on the achievable maximal sum-rate. Compound MAC capacity regions are contained in the intersection of MAC capacity regions of the individual components; in our case, the sum-rate is at best that in (10). The lemma is proved by noting that the communication scheme requires successful communication of $k \log(n/k)/m$ bits per channel-use, when we choose each set $S_i$ to have $n/k$ indices. This rate must be upper bounded by the sum-rate of the compound MAC.

**Corollary 3.2:** By using a Gaussian measurement ensemble,

$$m \geq \max \left\{ \frac{2 \log \frac{\alpha_k}{\sigma^2}}{\log(1 + \alpha_k^2/\sigma^2)}, \frac{2k \log \frac{\alpha_k}{\sigma^2}}{\log(1 + \|\alpha\|^2/\sigma^2)} \right\} . \quad (11)$$

and when $\alpha_k/\sigma << 1.0$,

$$m \geq \frac{\sigma^2 \log \frac{\alpha_k}{\sigma}}{\alpha_k^2}. \quad (12)$$

The corollary follows from Lemma 3.1 by noting that the maximal sum-rate in the compound MAC setting is less than $k \log(1 + \alpha_k^2/\sigma^2)$, since this is the sum of the single user constraints. The expression in (12) is identical to that obtained in [13], which can be further tightened by an alternative approach. Consider the above compound MAC, when we take $S_1$ to have size $n - k + 1$ and $|S_i| = 1, \forall i > 1$. In this case, user $1$ is conveying $\log(n - k + 1)$ bits to the decoder, and the other users are conveying zero bits, since the decoder knows apriori that these users have only one index (corresponding to telling the CS decoder $k - 1$ elements of the support set as side information). The single user rate constraint then tells us that

$$m \geq \frac{\log(n - k + 1)}{\log(1 + \alpha_k^2/\sigma^2)}. \quad (13)$$

**Corollary 3.3:** If the measurement matrix is chosen by Bernoulli($\frac{1}{2}$) on $\{+1, -1\}$,

$$m \geq \frac{2k \log_2 \frac{\alpha_k}{\sigma}}{\log \pi k / 2}. \quad (14)$$

With $\{+1, -1\}$ as the input alphabet, we can see that this channel has sum-rate strictly less than that available in a $k$ user binary-input adder channel [3]. The achievable sum-rate there is half that of the denominator in (14). This bound can be made tighter by considering an adder channel with noise, but we do not pursue it here.

Bounding the number of measurement as above also allows us to get insights about the speed at which exponential decay of recovery-error happens, this is given in the following lemma.

**Lemma 3.4:** The error probability in support recovery obeys,

$$P_{\text{error}} \geq \exp\left(-E_0(\alpha_k, \sigma^2) m)\right), \quad (15)$$

where $E_0(\alpha, \sigma^2)$ is the cut-off rate of a standard scalar AWGN channel with power constraint $\alpha^2/\sigma^2$. Notice that in the compound MAC we consider, the error probability in the scalar channel with gain $\alpha_k$ lowerbounds the total error probability. The best exponent of error-decay for this channel is given by the above $E_0(\cdot)$, which is also the maximal error exponent, happening at zero rate. We can extend this result to include the sphere-packing and straightline bounds, this is part of some ongoing work.

**IV. RELATED WORK**

A direct comparison can be made between our work and that of [1]. In that paper, it was shown that Gaussian measurement matrices are asymptotically optimal for joint typical
decoders with $O(k)$ measurements, with fixed SNR, for each error metric defined here. We extend this result to show that these sufficient conditions also hold for centered subgaussian measurement matrices in the linear sparsity regime. Necessary conditions are also established in [1] using arguments based on MACs, however, their bounds are not as refined as ours.

In [13], necessary and sufficient conditions are given for error metric 1. Sufficient conditions were established using an ML decoder while the necessary conditions exploited a corollary of Fano’s inequality. By comparing results in [13] and [12], it was shown that, in the sublinear sparsity regime, Lasso error metric defined here. We extend this result to show that Lasso has achieved the Fano’s inequality. By comparing results in [13] and [12], it was shown that, in the sublinear sparsity regime, Lasso is essentially information theoretically optimal. However, in the linear regime, there has been no practical algorithm that achieved the bound established in our paper and Fletcher et al. [6, Theorem 1].

Results from Fletcher et al. [6] is the closest to ours in terms of the scaling bounds they achieved. After submitting a first version here, we noticed that [6] describes some good bounds for the Gaussian case, along with a detailed comparison with existing bounds. Our converse bound generalizes their result, and we believe it is comparable for specific instances. A detailed study along this direction will be included in the final manuscript.

Partial support recovery was also addressed in [10] and necessary conditions are given. There a general bound was derived for deterministic and stochastic signals. A bound strictly focussed on Fourier measurement matrices is found in [7], which uses Fano’s inequality to establish the bound. In terms of the necessary condition in [10, Theorem 3.2], Theorem 3.1 is tighter and is also general as it applies to a variety of measurement ensembles. Theorem 3.1 is general enough to apply to structured codewords, such as Fourier measurement matrices, although the codewords now have a dependence. However, one needs to compute the capacity region of the a compound MAC channel using these structure codewords.

V. Conclusion

We have analyzed schemes for sparse signal recovery using subgaussian measurement matrices. Our achievability scheme used an impractical decoder. Future work intends to tackle the performance of subgaussian matrices and practical decoders.

where

$$g(\epsilon) = \sup_{t>0} \left( t\epsilon - \frac{\gamma_1 t^2}{2(1 - \gamma_2 t)} \right).$$

It can be shown that the supremum is achieved for

$$t = \gamma_2^{-1} \left[ 1 - \sqrt{\frac{1}{2}(2\gamma_2 + \gamma_1)^{-1}} \right]$$

and that

$$g(\epsilon) \geq \frac{\epsilon^2}{2\gamma_2 + 2\gamma_1},$$

and that $g(\epsilon) = \lambda$. To prove (5), we note that $\log \mathbb{E}[e^{tV}] \leq \gamma_1 t^2$ for $-1/\gamma_2 < t < 0$. The result then follows.

\section*{References}

[1] M. Akcakaya and V. Tarokh. Shannon theoretic limits on noisy compressive sensing. \textit{IEEE Trans. Info. Theory}, 2007. preprint, http://arxiv.org/PS_cache/arxiv/pdf/0711/0711.0366v1.pdf.

[2] L. Birgé and P. Massart. Minimum contrast estimators on sieves: exponential bounds and rates of convergence. \textit{Bernoulli}, 4(3):329–375, 1998.

[3] S.-C. Chang and J. K. Wolf. On the T-user M-frequency noiseless multiple-access channel with and without intensity information. \textit{IEEE Trans. Info. Theory}, 27(1):41–48, January 1981.

[4] T. M. Cover and J. A. Thomas. \textit{Elements of Information Theory}. John Wiley and Sons, Inc., 2nd edition, 2006.

[5] I. Csiszár and J. Körner. \textit{Information Theory}. Akadémiai Kiadó, 3rd edition, 1981.

[6] A. K. Fletcher, S. Rangan, and V. K. Goyal. Necessary and sufficient conditions on sparsity pattern recovery. 2008. preprint, http://arxiv.org/abs/0804.1839.

[7] M. Gastpar and Y. Bresler. On the necessary density for spectrum-blind nonuniform sampling subject to quantization. In \textit{Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing}, pages 1633–1636, Istanbul, Turkey, June 2000.

[8] N. L. Johnson, S. Kotz, and N. Balakrishnan. \textit{Continuous Univariate Distributions}, volume 1. John Wiley and Sons, 2nd edition, 1995.

[9] T. Mikosch. Estimates for tail probabilities of quadratic and bilinear forms in subgaussian random variables. \textit{Probability and Mathematical Statistics}, 11(2):169–178, 1991.

[10] G. Reeves. Sparse signal sampling using noisy linear projections. Technical Report UCB/EECS-2008-3, Univ. of California, Berkeley, Dept. of Elec. Eng. and Comp. Sci., January 2008.

[11] M. Rudelson and R. Vershynin. The smallest singular value of a random rectangular matrix. 2008. preprint, http://www-personal.umich.edu/~roman/papers/ry-rectangular-matrices.pdf.

[12] M. Wainwright. Sharp thresholds for noisy and high-dimensional recovery of sparsity using $l_1$-constrained quadratic programming. \textit{IEEE Tran. Info. Theory}, 2006. preprint, http://www.eecs.berkeley.edu/~wainwrig/Papers/Wai_SharpThres.pdf.

[13] M. Wainwright. Information-theoretic bounds on sparsity recovery in the high-dimensional and noisy setting. In \textit{ISIT ’07}, pages 961–965, June 2007.