NEAR-HORIZON MODES AND SELF-ADJOINT EXTENSIONS OF THE SCHRODINGER OPERATOR

A. P. BALACHANDRAN, A. R. DE QUEIROZ, AND ALBERTO SAA

Abstract. We investigate the dynamics of scalar fields in the near-horizon exterior region of a Schwarzschild black hole. We show that low-energy modes are typically long-living and might be considered as being confined near the black hole horizon. Such dynamics are effectively governed by a Schrödinger operator with infinitely many self-adjoint extensions parameterized by $U(1)$, a situation closely resembling the case of an ordinary free particle moving on a semiaxis. Even though these different self-adjoint extensions lead to equivalent scattering and thermal processes, a comparison with a simplified model suggests a physical prescription to choose the pertinent self-adjoint extensions. However, since all extensions are in principle physically equivalent, they might be considered in equal footing for statistical analyses of near-horizon modes around black holes. Analogous results hold for any non-extremal, spherically symmetric, asymptotically flat black hole.

1. Introduction

The dynamics of quantum and classical fields in the vicinity of black holes have received considerable attention recently. Several aspects of the so-called soft photons theorems and the asymptotic symmetries in black hole spacetimes depend ultimately upon the dynamics and the underlying algebraic structure of test fields in the near-horizon region of black holes. For a recent comprehensive review on these subjects, see, for instance, [1]. Here, we revisit the case corresponding to the simplest classical configuration of a field in the near-horizon region of a black hole: a massless Klein-Gordon field $\phi$ around a Schwarzschild black hole, which metric in standard coordinates reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\Omega^2.$$  (1)

As we will see, massive scalar fields can be easily accommodated in our discussion, without altering our main conclusions. By exploring the standard decomposition for the scalar field

$$\varphi_{\ell m} = \frac{e^{-i\omega t}}{r}u_{\ell m}(r)Y_{\ell m}^{m}(\theta, \phi)$$  (2)

and the usual tortoise coordinates

$$r_s = r + 2M \log \left(\frac{r}{2M} - 1\right),$$  (3)

* Dedicated to Alberto Ibort on the occasion of his sixtieth birthday.
one has the following effective Schrödinger equation for the radial function $u_{\ell m}$

$$\left( -\frac{d^2}{dr_{*}^2} + V_{\ell}(r) \right) u_{\ell m} = \omega^2 u_{\ell m},$$

(4)

where the effective potential $V_{\ell}(r)$ is given by

$$V_{\ell}(r) = \left( 1 - \frac{2M}{r} \right) \left( \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} \right),$$

(5)

which well known aspect is depicted in Fig. 1. The tortoise coordinate $r_{*}$ runs over $(-\infty, \infty)$, with the near-horizon region corresponding to $r \to 2M$ and $r_{*} \to -\infty$, where the effective potential can be well approximated as

$$V_{\ell}(r_{*}) \approx V_{\ell}^{nh}(r_{*}) = \frac{\ell(\ell+1) + 1}{4M^2e} \exp\left( \frac{r_{*}}{2M} \right).$$

(6)

For $r \to \infty$, which corresponds to $r_{*} \to \infty$, the effective potential decreases as a power law. For scalar fields with mass $m_{\phi} \neq 0$, there will be an extra term $m_{\phi}^2$ inside the parenthesis of the second term in (5). It will not alter the effective potential exponential decay in the near-horizon region, nor the power law decay at infinity, although in this case $V_{\ell} \to m_{\phi}^2$ for $r \to \infty$. Since the near-horizon potential (6) is not qualitatively altered by the mass term, our main conclusions will also hold for the massive case.

The effective Schrödinger equation (4) governs all dynamical processes involving scalar fields around Schwarzschild black holes. Scattering problems, in particular, involve certain boundary conditions at horizon and at infinity.

Figure 1. Aspect of the effective potential $V_{\ell}(r_{*})$ given by (5) for some values of $\ell$. The potential decreases exponentially in the near-horizon region $(r_{*} \to -\infty)$, see (6), and as a power law for $r_{*} \to \infty$ (the asymptotically flat region $r \to \infty$).
In these problems, typically, one starts with an incoming wave from infinity which is scattered by the effective potential barrier (Fig. 1), leading to a reflected wave towards infinity and a transmitted wave that plunges into the black hole horizon. Such a typical situation corresponds to the following boundary conditions for $u_{\ell m}$

$$u_{\ell m}(r_*) = \begin{cases} A_{\ell m}^{\text{in}}(\omega)e^{-i\omega r_*} + A_{\ell m}^{\text{out}}(\omega)e^{i\omega r_*}, & r_* \to \infty, \\ B_{\ell m}^{\text{in}}(\omega)e^{-i\omega r_*}, & r_* \to -\infty. \end{cases} \quad (7)$$

The (complex) values of $\omega$ such that $A_{\ell m}^{\text{in}}(\omega) = 0$ are known to correspond to the so-called quasinormal modes, which dominate the asymptotic evolution of non-stationary configurations of the scalar field, see [2, 3] for comprehensive reviews on the subject. Here, we are interested in a different field configuration. We will consider processes which originate in the near-horizon region of the black hole and eventually are transmitted to the infinity through the potential barrier. This situation corresponds to the following boundary conditions

$$u_{\ell m}(r_*) = \begin{cases} A_{\ell m}^{\text{out}}(\omega)e^{i\omega r_*}, & r_* \to \infty, \\ B_{\ell m}^{\text{in}}(\omega)e^{-i\omega r_*} + B_{\ell m}^{\text{out}}(\omega)e^{i\omega r_*}, & r_* \to -\infty. \end{cases} \quad (8)$$

We will focus in the lower energy limit, which of course corresponds to small $\omega$, which we assume to be positive. We will discuss the possibility of imaginary $\omega$, which would correspond to negative eigenvalues $\omega^2$ in the effective Schrödinger eigenproblem (4), in the last section. In the low-energy limit, we expect on physical grounds to have some oscillatory behavior in the near-horizon region and an exponential suppression, due to the effective potential barrier, as one departs from the horizon. It is rather natural to expect that $A_{\ell m}^{\text{out}} \to 0$ (or, to be more precise, $A_{\ell m}^{\text{out}}/B_{\ell m}^{\text{out}} \to 0$) for small $\omega$, and that the near-horizon modes $B_{\ell m}^{\text{in}}$ and $B_{\ell m}^{\text{out}}$ could be considered as long-living in the sense that the tunneling probability to infinity is extremely low, implying that near-horizon low-energy perturbations of the scalar fields tend to be confined near the black hole horizon. Moreover, since they are long-living and spatially confined, it is also natural to assume that such near-horizon modes could in principle attain thermal equilibrium, possibly with the black hole Hawking temperature $T_H = 1/8\pi M$.

Our analysis is based on the assumption that the dynamics of the near-horizon $B_{\ell m}^{\text{in}}$ and $B_{\ell m}^{\text{out}}$ modes, for small $\omega$, can be well approximated by employing the Schrödinger operator

$$\mathcal{H} = -\frac{d^2}{dr_*^2} + V_{\ell m}^{\text{nh}}(r_*) \quad (9)$$

on the domain $[-\infty, r_*^{\text{max}}]$, for some finite $r_*^{\text{max}}$ corresponding to a $r$ not far from the horizon $r = 2M$. This is, of course, equivalent to assume that, for small $\omega$, $A_{\ell m}^{\text{out}} = 0$, leading to a perfect reflection due to the effective potential barrier and, consequently, to a confinement of the near-horizon modes. This approach closely resembles the so-called “brick wall” proposal for the thermodynamical analysis of fields around black holes [4], even though we are concerned here with the dynamics in the interior region of the wall. As we will see, our approach may indeed be considered a generalization of the standard brick wall hypothesis.
It is a well known problem in standard Quantum Mechanics that the free-particle Schrödinger operator on the semi-axis has infinitely many self-adjoint extensions parameterized by a phase $\theta \in U(1)$, see [5, 6], for instance, for further references. We will show that similar results also hold for our problem, i.e., the Schrödinger operator (9) on the domain $(-\infty, r^\ast_{\max}]$ has infinitely many self-adjoint extensions determined by the boundary condition at $r^\ast_{\max}$. Moreover, all self-adjoint extensions in this case will give origin to physically acceptable descriptions for the near-horizon modes. Nevertheless, the comparison with a simplified model suggests a physical prescription to choose the pertinent extensions.

2. Self-adjoint extensions of the effective Schrödinger operator

Let us introduce the dimensionless variable $x = r^\ast / 2M$, in terms of which one has the following effective Schrödinger equation for near-horizon modes

$$H u_{\ell m} = \left( -\frac{d^2}{dx^2} + \frac{c_{\ell}^2}{4} e^x \right) u_{\ell m} = \lambda^2 u_{\ell m},$$

(10)

where

$$c_{\ell}^2 = \frac{4}{e} (\ell^2 + \ell + 1)$$

(11)

and $\lambda = 2M\omega$, which we assume initially to be positive. (The possibility of having imaginary $\lambda$ will be discussed in the last section.) The functions $u_{\ell m}$ are defined over the domain $(-\infty, x^\ast_{\max}]$. As we will see, our conclusions are independent of the precise value of $x^\ast_{\max}$, provided, of course, it is finite. We will drop the indices $\ell$ and $m$ for all functions and constants hereafter. It is natural to consider the initial domain $D(H)$ of the effective Schrödinger operator (9) as $C^\infty_0 (-\infty, x^\ast_{\max}]$, i.e., the smooth (complex) functions $u$ with compact support on the domain $(-\infty, x^\ast_{\max}]$. Notice that $H$ is a symmetric operator with respect to the inner product

$$\langle v, u \rangle = \int_{-\infty}^{x^\ast_{\max}} \bar{v} u \, dx$$

(12)

since

$$\langle v, Hu \rangle = \langle H v, u \rangle$$

(13)

for all $u, v \in D(H)$. However, it is clear too that $D(H) \subset D(H^\dagger)$ since (13) is valid also for functions $v \notin D(H)$, and this is the start point of the self-adjointness analysis of unbounded operators on Hilbert spaces [5, 6]. On physical grounds, we should expect $D(H^\dagger)$ to be the set of all smooth functions with finite norm $||v|| = \sqrt{\langle v, v \rangle}$, or at least finite norm per length unit in order to accommodate some possible plane wave solutions. Hence, we will consider $D(H^\dagger)$ as the set of smooth functions $v \in L^2 (-\infty, x^\ast_{\max}]$, with the norm induced by (12). The von Neumann theorem assures that $H$ will admit self-adjoint extensions provided the so-called deficiency index $n_+$ and $n_-$ be equal and greater than zero, where $n_\pm$ are the dimension of the deficiency subspaces $N_\pm \subset D(H^\dagger)$ defined by

$$N_\pm = \left\{ v \in D(H^\dagger), \ H v = \pm i v \right\}.$$
In order to determine the deficiency subspaces $N_{\pm}$, notice that the change of variable $z = e^{z\frac{i}{2}}$ reduces \cite{10} to a modified Bessel equation, allowing us to write down the general solution of $\mathcal{H}v = \pm iv$ in terms of standard modified Bessel functions
\begin{equation}
    v(x) = a I_{\mu_\pm} \left( ce^{z\frac{i}{2}} \right) + b K_{\mu_\pm} \left( ce^{z\frac{i}{2}} \right),
\end{equation}
where $a$ and $b$ are constants and
\begin{equation}
    \mu_\pm = \sqrt{2} (1 \mp i).
\end{equation}
From the standard asymptotic expressions for modified Bessel functions \cite{7}, one has for $x \to -\infty$
\begin{equation}
    I_{\mu_\pm} \left( ce^{z\frac{i}{2}} \right) \approx \frac{(\sqrt{2}(1+i))}{\sqrt{2} \Gamma(1+i)} e^{\mp \frac{\sqrt{2}z}{2}},
\end{equation}
and
\begin{equation}
    K_{\mu_\pm} \left( ce^{z\frac{i}{2}} \right) \approx \frac{1}{2} \left( \frac{c}{\sqrt{2}} \right)^{\sqrt{2}(1+i)} e^{-\frac{\sqrt{2}z}{2}}.
\end{equation}
It is clear that the modified Bessel function $K_{\mu_\pm}$ will give origin to solutions $v \notin D(\mathcal{H}^\dagger)$ since they will diverge exponentially for $x \to -\infty$. Hence, only the solutions involving $I_{\mu_\pm}$ are allowed, and we have $n_+ = n_\mp = 1$. The deficiency subspaces $N_{\pm}$ are then vector spaces with dimension 1 generated by $I_{\mu_\pm}$, and von Neumann theorem assures that $\mathcal{H}$ has a family of self-adjoint extensions parameterized by a phase $\theta \in U(1)$ \cite{5,6}.

The structure of the differential operator $\mathcal{H}$ is rather simple and will allow us to determine explicitly all of its self-adjoint extensions $\mathcal{H}_\alpha$. Notice that, for smooth $u, v \in L^2(-\infty, x^{\max}]$, one has
\begin{equation}
    \langle v, \mathcal{H}u \rangle - \langle \mathcal{H}v, u \rangle = \overline{v}'(x^{\max})u(x^{\max}) - \overline{v}(x^{\max})u'(x^{\max}),
\end{equation}
from where we see that $\mathcal{H}$ will be self-adjoint provided
\begin{equation}
    \frac{v(x^{\max})}{v'(x^{\max})} = \frac{u(x^{\max})}{u'(x^{\max})} = \alpha = \tan \frac{\theta}{2},
\end{equation}
with $\theta \in (-\pi, \pi)$, and we have finally established
\begin{equation}
    D(\mathcal{H}_\alpha) = D \left( \mathcal{H}_\alpha^\dagger \right) = \{ v \in L^2(-\infty, x^{\max}] \suchthat v(x^{\max}) = \alpha v'(x^{\max}) \},
\end{equation}
with $\mathcal{H}_\infty$ corresponding to the boundary condition $v'(x^{\max}) = 0$. It is worthy to notice that the case $\mathcal{H}_0$, which corresponds to $v(x^{\max}) = 0$, corresponds to the brick wall hypothesis \cite{4}. Our analysis, besides of involving more general boundary conditions, is restricted to the other side of the wall, \textit{i.e.} to the modes confined in the near-horizon region. Notice that the differential expression for the operator $\mathcal{H}_\alpha$ is independent of $\alpha$, it alters only $D(\mathcal{H})$.

In order to interpret the physical meaning of the self-adjoint extensions $\mathcal{H}_\alpha$, let us consider now the eigenproblem \cite{10} for positive $\lambda$. It has also solutions in terms of modified Bessel functions $I_\mu$ and $K_\mu$, but now with pure imaginary order $\mu = 2i\lambda$. However, it is more convenient for our purposes here to write down the solution as a linear combination of $I_{2i\lambda}$ and $I_{-2i\lambda}$. One has
\begin{equation}
    u(x) = a \lambda I_{2i\lambda} \left( ce^{z\frac{i}{2}} \right) + b \lambda I_{-2i\lambda} \left( ce^{z\frac{i}{2}} \right),
\end{equation}
with $a_\lambda$ and $b_\lambda$ constants. For $x \to -\infty$, we have [7]

$$u(x) \approx \frac{a_\lambda (\frac{c}{2})^{2i\lambda}}{2i\lambda \Gamma(2i\lambda)} e^{i\lambda x} - \frac{b_\lambda (\frac{c}{2})^{-2i\lambda}}{2i\lambda \Gamma(-2i\lambda)} e^{-i\lambda x},$$

(23)

from where one can read the scattering coefficients in the region very close to the horizon

$$B^\text{in}_\lambda = -\frac{b_\lambda (\frac{c}{2})^{-2i\lambda}}{2i\lambda \Gamma(-2i\lambda)}, \quad \text{and} \quad B^\text{out}_\lambda = \frac{a_\lambda (\frac{c}{2})^{2i\lambda}}{2i\lambda \Gamma(2i\lambda)}.$$ 

(24)

Defining the reflection coefficient as

$$R_\lambda = \frac{B^\text{in}_\lambda}{B^\text{out}_\lambda} = -\frac{b_\lambda}{a_\lambda} \frac{\Gamma(2i\lambda)}{\Gamma(-2i\lambda)} \left( \frac{c}{2} \right)^{-4i\lambda},$$

(25)

we have

$$|R_\lambda| = \left| \frac{b_\lambda}{a_\lambda} \right|. \quad \text{(26)}$$

On the other hand, one can determine $b_\lambda/a_\lambda$ from the boundary condition $u(x^\text{max}) = \alpha u'(x^\text{max})$. One has

$$\frac{b_\lambda}{a_\lambda} = -\frac{\chi}{\chi} \quad \text{(27)}$$

where

$$\chi = I_{2i\lambda} \left( ce^{\frac{1}{2}x^\text{max}} \right) - \frac{\alpha c}{2} e^{\frac{1}{2}x^\text{max}} I'_2(ce^{\frac{1}{2}x^\text{max}}), \quad \text{(28)}$$

which clearly implies that $|R_\lambda| = 1$, meaning that, irrespective of the value of $\alpha$, we have always full reflection of the near-horizon modes on the effective potential barrier, which is compatible with $A^\text{out}_\lambda = 0$ as expected. From the scattering point of view, it is possible to implement a brick wall which effectively confine the modes in the near-horizon region without imposing $u(x^\text{max}) = 0$. Moreover, any value of $\alpha$ is perfectly admissible in this context, all self-adjoint extensions give origin to physically acceptable descriptions for the near-horizon modes. We will have a complete set of (continuous) eigenvalues and eigenvectors for (10) for any value of $\alpha$. As we will see below, all self-adjoint extensions will lead also to consistent thermodynamics for the near-horizon modes.

2.1. Statistical mechanics and thermal equilibrium. All self-adjoint extensions $H_\alpha$ describe confined incoming and outgoing near-horizon modes characterized by the coefficients $B^\text{in}_\lambda$ and $B^\text{out}_\lambda$, see (24). The probability of having incoming and outgoing modes with energy $\lambda$ in the horizon are, respectively,

$$|B^\text{in}_\lambda|^2 = \frac{\sinh 2\pi \lambda}{2\pi \lambda}|b_\lambda|^2, \quad |B^\text{out}_\lambda|^2 = \frac{\sinh 2\pi \lambda}{2\pi \lambda}|a_\lambda|^2. \quad \text{(29)}$$

Notice that for small $\lambda$ we have essentially $|B^\text{in}_\lambda|^2 \approx |b_\lambda|^2$ and $|B^\text{out}_\lambda|^2 \approx |a_\lambda|^2$. Let us suppose now that the near-horizon modes are at thermal equilibrium with temperature $T = \tau/2M$ (the Hawking temperature corresponds to $\tau = 1/4\pi$). Assuming a grand canonical ensemble and the detailed balance principle [8], we expect that incoming and outgoing modes be separately at thermal equilibrium, meaning that we should expect that both $|B^\text{in}_\lambda|^2$ and
$|b_\lambda|^2$ obey Boltzmann distributions and, hence, both should be proportional to $e^{-\lambda/\tau}$. Interestingly, such detailed balance condition, which implies that incoming and outcoming modes are equally probable in a regime of thermal equilibrium, is compatible with any value of $\alpha$, i.e., all self-adjoint extensions $\mathcal{H}_\alpha$ are equivalent also from the thermal equilibrium point of view. The compatibility is assured by the fact that $|a_\lambda|^2 = |b_\lambda|^2$ for any value of $\alpha$, see (27) and (28). Hence, if one of the modes is assumed to be at thermal equilibrium, by (29) the other automatically be also at thermal equilibrium. It is fundamental for the detailed balance that the boundary condition implies

$$b_\lambda = e^{i\psi_\lambda} a_\lambda,$$

where the phase $\psi_\lambda$ depends on all parameters of the problem, see (27) and (28), and particularly on the energy $\lambda$. Nevertheless, irrespective of the value of $\alpha$, we have always $|a_\lambda|^2 = |b_\lambda|^2$.

2.2. A prescription for the extension selection. Rigorously, for each value of $\alpha$ we have a fixed domain on the Hilbert space and a complete, physically consistent, description for the low-energy modes. We should not mix modes with different $\alpha$ since they belong to different domains. The physical interpretation of the parameter $\alpha$ is still rather unclear, but a simplified model can help to shed some light here. Let us consider the well-known elementary problem of the scattering by a rectangular barrier

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & 0 \leq x \leq L, \\ 0, & x > L, \end{cases}$$

with both $V_0$ and $L$ positives. We are interested on scattering problems of the type (31), i.e., on solutions of the type

$$u(x) = \begin{cases} B_{\text{in}} e^{-i\lambda x} + B_{\text{out}} e^{i\lambda x}, & x < 0, \\ C_\lambda e^{\sqrt{V_0 - \lambda^2} x} + D_\lambda e^{-\sqrt{V_0 - \lambda^2} x}, & 0 \leq x \leq L, \\ A_{\text{out}} e^{i\lambda x}, & x > L, \end{cases}$$

with $\lambda^2 < V_0$. The standard matching conditions at $x = 0$ and $x = L$ read

$$B_{\text{in}} + B_{\text{out}} = C_\lambda + D_\lambda,$$
$$-i\lambda (B_{\text{in}} - B_{\text{out}}) = \sqrt{V_0 - \lambda^2} (C_\lambda - D_\lambda),$$
$$A_{\text{out}} e^{i\lambda L} = C_\lambda e^{\sqrt{V_0 - \lambda^2} L} + D_\lambda e^{-\sqrt{V_0 - \lambda^2} L},$$
$$i\lambda A_{\text{out}} e^{i\lambda L} = \sqrt{V_0 - \lambda^2} \left( C_\lambda e^{\sqrt{V_0 - \lambda^2} L} - D_\lambda e^{-\sqrt{V_0 - \lambda^2} L} \right).$$

After some straightforward algebra, one can evaluate the usual reflection coefficient $R_\lambda$ leading to

$$|R_\lambda|^2 = \frac{V_0 \sinh^2 \sqrt{V_0 - \lambda^2} L}{4\lambda^2 (V_0 - \lambda^2) + V_0 \sinh^2 \sqrt{V_0 - \lambda^2} L}.$$ 

The problem of near-horizon modes is mimicked in this toy model by assuming $L \to \infty$, which implies $|R_\lambda| \to 1$, i.e., full reflection leading to a “confinement” of the solutions (32) in the negative semiaxis. Since $|R_\lambda| \to 1$, we
know that $A_{x}^{\text{cut}} \to 0$ and hence from (35) and (36) we have that $C_{\lambda} \to 0$, which implies the following condition for $u(x)$ on $x = 0$

$$u'(0) = -\sqrt{V_{0} - \lambda^{2}}u(0).$$

(38)

Thus, finally, in the low-energy limit, $\lambda^{2} \ll V_{0}$, we have that the dynamics of the totally reflect solutions for the barrier [31] may be viewed as an effective Schrödinger equation for a free particle on the negative semiaxis with the boundary condition corresponding to $\alpha^{-1} = -\sqrt{V_{0}}$. This simple results suggests that $\alpha^{-1} = -\sqrt{\max V_{\ell}}$ for the near-horizon modes. We would have different self-adjoint extensions for different angular momentum numbers $\ell$, but this is hardly a surprise since the effective potential [39], and consequently the Schrödinger operator [10], does depend explicitly on $\ell$. It is interesting to notice that the standard brick wall condition $\alpha = 0$ would require $\max V_{\ell} \to \infty$, which on the other hand demands $\ell \to \infty$. Nevertheless, all self-adjoint extensions act effectively as brick walls since we have full reflection for all values of $\alpha$. In fact, despite our prescription for the selection of $\alpha$, since all extensions are in principle physically equivalent, they might be considered in equal footing for statistical analyses of near-horizon modes around black holes.

3. Final remarks

We will revisit in this last section two previously noticed points. First, that our results do not depend on the details of the Schwarzschild black hole. They will also hold for any non-extremal, spherically symmetric, static, and asymptotically flat black hole. The metric of a generic spherically symmetric static black hole can be cast in the form

$$ds^{2} = -f(r)dt^{2} + \frac{dr^{2}}{h(r)} + r^{2}d\Omega^{2}.$$  

(39)

The event horizon corresponds to the outermost zero of $f(r)$, say at $r = r_{0}$. The black hole is said to be non-extremal if $f'(r_{0}) = k > 0$, and hence in the vicinity of the horizon we have $f(r) \approx k_{1}(r - r_{0})$. Regularity of the horizon area demands a smooth $\sqrt{-g}$, and from (39) we see also that $h(r) \approx k_{2}(r - r_{0})$, with $k_{2} > 0$. By using the standard decomposition (2) for the Klein-Gordon equation on the metric (39), we arrive to a Schrödinger-like equation as (41), but now with the effective potential

$$\tilde{V}_{\ell}(r) = \ell(\ell + 1)f/r^{2} + \frac{1}{2r}(f'h + fh')$$

(40)

and tortoise coordinates such that

$$\frac{dr_{*}}{dr} = \frac{1}{\sqrt{fh}}.$$  

(41)

If [39] is assumed to be asymptotically flat, we have $f(r) \to 1$ and $h(r) \to 1$ for $r \to \infty$, and hence [40] decays as a power law at infinity in the same way the Schwarzschild potential [39] does. On the other hand, in the near-horizon region one has

$$\tilde{V}_{\ell}(r) \approx k_{1}(r - r_{0}) \left( \frac{\ell(\ell + 1)}{r_{0}^{2}} + \frac{k_{2}}{r_{0}} \right).$$

(42)
The new tortoise coordinate (41) also obeys $r_\ast \to -\infty$ on the horizon and, moreover, we have

$$r - r_0 = r_0 e^{\sqrt{k_1 k_2} r_\ast}, \quad (43)$$

from where we conclude that the effective potential (40) also decays exponentially in the near-horizon region. Indeed, the aspect of the generic effective potential (40) of a non-extremal, spherically symmetric, static, and asymptotically flat black hole is qualitatively the same of the Schwarzschild case, Fig. 1. All the analyses we have done follow analogously for the generic black hole case.

The second point corresponds to the imaginary $\lambda$ case in (10). It is a well known and curious fact that the Schrödinger equation for the free particle on the semiaxis admits some bounded solutions, with negative energy, for certain self-adjoint extension choices, see [5]. We have the same interesting behavior here and they indeed correspond to the imaginary $\lambda$ solutions of the eigenproblem (10). For $\lambda = \sigma i$, the fundamental solutions of (10) will be linear combinations of the modified Bessel functions $I_{2\sigma}$ and $K_{2\sigma}$. From the asymptotic behavior near the origin, we can discharge the second solution. Using the standard series expansion [7] for $I_{2\sigma}$, we have the following solution for the eigenproblem (10) with eigenvalue $\lambda^2 = -\sigma^2$,

$$u(x) = a_{\sigma} I_{2\sigma} \left( c e^{\frac{x}{2}} \right) = \sum_{k=0}^{\infty} \frac{e^{(k+\sigma)x}}{k! \Gamma(k+2\sigma+1)} \left( \frac{c}{2} \right)^{2(k+\sigma)}, \quad (44)$$

where it is assumed $\sigma > 0$. It is clear from (44) that $u(x)$ and all of its derivative are monotonically increasing functions and, thus, in order to accommodate such bounded solution for (10), a self-adjoint extension with $\alpha > 0$ is required, which will never be selected by our prescription. In our case, such bounded solutions do not oscillate, see [2], but rather decrease exponentially. This kind of overdamped evolution for scalar fields is quite similar to some highly damped quasinormal modes that are known to exist for generic black holes, see [9]. This topic is now under investigation.

Acknowledgements

ARQ and AS thank the University of Zaragoza, where part of this work was carried on, for the warm hospitality. The authors acknowledge the financial support of CNPq and CAPES (ARQ and AS) and FAPESP (AS, Grant 2013/09357-9).

References

[1] A. Strominger, Lectures on the Infrared Structure of Gravity and Gauge Theory, [arXiv:1703.05448].
[2] E. Berti, V. Cardoso, and A.O. Starinets, Class. Quantum Grav. 26, 163001 (2009), [arXiv:0905.2975].
[3] R.A. Konoplya and A. Zhidenko, Rev. Mod. Phys. 83, 793 (2011), [arXiv:1102.4014].
[4] G. ’t Hooft, Nucl. Phys. B256, 727 (1985).
[5] G. Bonneau, J. Faraut, and G. Valent, Am. J. Phys. 69, 322 (2001), [arXiv:quant-ph/0103153].
[6] D.M. Gitman, I.V. Tyutin, and B.L. Voronov, Self-adjoint Extensions in Quantum Mechanics: General Theory and Applications to Schrödinger and Dirac Equations with Singular Potentials, Birkhuser (2012).
M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover (1974).

R.C. Tolman, *The Principles of Statistical Mechanics*, Dover (2010)

C. Chirenti, A. Saa, and J. Skakala, Phys. Rev. D 87, 044034 (2013), [arXiv:1211.1046]

Physics Department, Syracuse University, Syracuse, New York 13244-1130, USA

E-mail address: balachandran38@gmail.com

Instituto de Fisica, Universidade de Brasilia, C.P. 04455, 70919-970 Brasilia, DF, Brazil

E-mail address: amilcarq@unb.br

Department of Applied Mathematics, University of Campinas, 13083-859 Campinas, SP, Brazil

E-mail address: asaa@ime.unicamp.br