Robust Finite-Time Stabilization for Uncertain Discrete-Time Linear Singular Systems

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ABSTRACT This paper considers the robust finite-time stabilization problem for a class of discrete-time linear singular systems with norm-bounded parameter uncertainties. With the introduction of an additional matrix to describe the algebraic relationship between the slow subsystem and fast subsystem of a discrete-time linear singular system, a matrix inequality condition is first given for a discrete-time linear singular system to be regular, causal and finite-time stable. With this condition, the robust finite-time stability and robust finite-time stabilization problems are also resolved, and the explicit expression of the desired state feedback control law is also given in terms of a set of matrix inequalities. A numerical example is given to show the effectiveness of the proposed method.

INDEX TERMS Uncertain linear singular system, discrete-time singular system, finite-time stability, finite-time stabilization, matrix inequality.

I. INTRODUCTION

In many practical applications, the main concern of the system behavior is focused in a finite time interval, which is different from the traditional Lyapunov stability since its performance is defined over an infinite time interval. For example, the missile attitude control system is only defined in the time interval between its launch time and the time hitting the target, and the dynamic performance in a short time interval is more preferred for the automotive suspension control system. In those cases, the stability and performance in a given short time interval are more preferred. The concept of finite-time stability, which is a quantitative concept, is more suitable to describe such system performance [1]. This concept is different from the traditional Lyapunov stability dealing with the behavior of a system within an infinite time interval, which is a qualitative concept. The concept of finite-time stability was first introduced in the Russian literature in 1950s [2], and latter appeared in the western journals [3]. For a dynamic system, it is said to be finite-time stable if its state does not exceed a certain threshold during a specified time interval when the bound on the initial condition is given prior [4]. In the last fifteen years, the problems of finite-time stability and finite-time stabilization have been fully investigated in both the continuous-time and discrete-time contexts for state-space linear systems [5], [6]. Recently, the concept of $H_\infty$ finite-time boundedness was introduced for discrete-time switched delay systems and the problem of delay-dependent uniform finite-time $H_\infty$ stabilization problem was considered in [7], and further the problem of asynchronous finite-time filtering of networked switched systems was studied by using an event-driven method in [8], where some sufficient conditions were established to check the properties of the finite-time boundedness and the input-output finite-time stability of the event-driven asynchronous filtering error system by constructing a reasonable Lyapunov-Krasovskii functional.

Singular systems are also known as descriptor systems and generalized state-space systems. It is known that the singular system model can preserve the dynamic structure of practical systems and have extensive applications in neutral networks, robotic systems, power systems, etc., [9], [10]. Recently, the finite-time stability and finite-time stabilization problems were extended to continuous-time singular systems. For example, the finite-time stabilization problem of linear singular systems subject to parametric uncertainty and disturbances was studied in [11], where some fundamental definitions for state-space linear systems were extended to
singular linear systems, and the finite-time $H_{\infty}$ stabilization problem for linear singular systems with parameter uncertainty was studied in [12]. The finite-time dissipative control problem for singular Markovian jump time-delay systems was studied in [13], where the time-varying transition rate and actuator saturation were considered. The finite-time observer design problem was considered recently for a class of singular systems subject to unknown inputs in both the state and the output equations in [14], where some linear matrix inequality results were given. For discrete-time linear singular systems, the finite-time stability analysis problem was first studied in [15], where the matrix norm approach was used to derive sufficient conditions for finite-time stability. The matrix inequality approach was also used to give a similar finite-time stability criterion [16]. The finite-time control problem was considered in [17], where the positive semi-definiteness problem and the matrix equality constraint $PB = BL$ ($P$ and $L$ are matrices to be determined) were involved. It is well known that the equality constraint is fragile and thus this constraint is difficult to be verified numerically, and thus the analysis and synthesis criteria in [17] are difficult to be verified numerically. The finite-time $H_{\infty}$ control problem was considered in [18], but only the existence condition instead of explicit expression of the desired control law was given. The finite-time dissipative control problem was also considered for linear singular systems with Markovian jump parameters and actuator saturation in [19], where the positive semi-definiteness problem was also involved. Since the designed Lyapunov-like function for singular systems is not positive-definite, the positive semi-definiteness problem is often contained in the stability criteria [9]. Beside this, to keep this Lyapunov-like function to be symmetric, an additional equality constraint is often used [10]. Those make the obtained stability criteria for singular systems commonly difficult to be verified numerically.

This paper revisits the robust finite-time stabilization problem for a class of discrete-time linear singular systems with norm-bounded parameter uncertainty. With the introduction of an additional matrix to describe the algebraic relationship between the slow subsystem and fast subsystem of a discrete-time linear singular system, a sufficient condition is given for a nominal discrete-time linear singular system to be regular, causal and finite-time stable. By some matrix manipulation, this condition is further formulated in terms of strict matrix inequalities. Based on this condition, the robust finite-time stabilization problem is resolved, and the explicit expression of the desired state feedback control law is also given in terms of a set of matrix inequalities. As pointed out in [20], [21], the full state of a system is difficult or impossible to measure sometimes, and in this circumstance, the output feedback control law design is very important to synthesize this system. By following the static output feedback control law design algorithm for discrete-time singular systems in [22] and the dynamic output feedback control law design algorithm for linear singular parameter-varying systems [23], the finite-time output feedback control law design algorithm can be obtained, which is our further research.

The main contribution of this paper can be given as

1) A strict matrix inequality criterion for a nominal linear singular system is given. This matrix inequality can be further simplified to a linear matrix inequality condition by fixing a scalar to be constant.
2) A strict matrix inequality condition is given for the state feedback control law design such that the resultant closed-loop system is to be regular, causal and finite-time stable for all admissible uncertainties.

and we here highlight that the obtained criteria are formulated in terms of strict matrix inequalities, which has some mathematical elegance and comparatively easy to be verified numerically.

II. PROBLEM FORMULATION

Consider a class of uncertain discrete-time linear singular systems,

$$Ex(k + 1) = (A + \Delta A)x(k) + (B + \Delta B)u(k)$$

where $x(k) \in \mathbb{R}^n$ is the semi-state vector, $u(k) \in \mathbb{R}^m$ is the control input vector, $A, B, E$ are constant matrices with appropriate dimensions and rank $E \preceq n$, $\Delta A, \Delta B$ are unknown matrices describing the parameter uncertainties in system matrices with the form of

$$[\Delta A \, \Delta B] = MF(k) [N_a \, N_b]$$

where $M, N_a, N_b$ are constant matrices with appropriate dimensions and $F(k)$ is an unknown matrix bounded by $F^T(k)F(k) \leq I$.

The nominal counterpart of this system can be described as

$$Ex(k + 1) = Ax(k) + Bu(k)$$

For this system, some definitions and lemmas should be introduced first to facilitate the following discussion.

**Definition 1 (Dai [24]):**

1) The pair $(E, A)$ is said to be regular, if $\det(zE - A) \neq 0$.
2) The pair $(E, A)$ is said to be causal, if $\deg(\det(zE - A)) = \text{rank}E$.

**Lemma 1 (Dai [24]):** The solution to the discrete-time linear singular system (3) with $u(k) = 0$ exists and is unique and causal, if the pair $(E, A)$ is regular and causal.

**Definition 2:** The discrete-time linear singular system (3) is said to be finite-time stabilizable with respect to $(c_1, c_2, N, R)$ with $0 < c_1 < c_2$ and $R > 0$, if there exists a state feedback control law $u(k) = Kx(k)$ such that the resultant closed-loop system is regular, causal and such that $x^T(0)E^TREx(0) \leq c_1$ implies $x^T(k)E^TREx(k) < c_2$, $\forall k \in [0, 1, 2, \ldots, N]$.

**Definition 3:** The uncertain discrete-time linear singular system (1) with $u(k) = 0$ is said to be robustly finite-time stable with respect to $(c_1, c_2, N, R)$ with $0 < c_1 < c_2$ and
If $R > 0$, if this system is regular, causal for all admissible parameter uncertainties, and $x^T(0)E^T R E x(0) \leq c_1$ implies $x^T(k)E^T R E x(k) < c_2, \forall k \in [0, 1, 2, \ldots, N]$.

Definition 4: The uncertain discrete-time linear singular system (1) is said to be robustly finite-time stabilizable with respect to $(c_1, c_2, N, R)$ with $0 < c_1 < c_2$ and $R > 0$, if there exists a state feedback control law $u(k) = K x(k)$ such that the resultant closed-loop system is regular, causal for all admissible parameter uncertainties, and such that $x^T(0)E^T R E x(0) \leq c_1$ implies $x^T(k)E^T R E x(k) < c_2, \forall k \in [0, 1, 2, \ldots, N]$.

The following theorem gives a sufficient condition for the discrete-time linear singular system (3) with $u(k) = 0$ to be regular, causal and finite-time stable.

**Theorem 1:** The discrete-time linear singular system (3) with $u(k) = 0$ is regular, causal and finite-time stable with respect to $(c_1, c_2, N, R)$, if there exist symmetric positive-definite matrices $P, Q, A, H$, a matrix $S$, and a scalar $\alpha > 1$ such that

$$
A^T P A - \alpha E^T P E + A^T H S^T + S H^T A < 0 \quad (4a)
$$

$$
Q = R^{-\frac{1}{2}} P R^{-\frac{1}{2}} \quad (4b)
$$

$$
\alpha^N \lambda_{\min}(Q) c_1 < \lambda_{\min}(Q) c_2 \quad (4c)
$$

where $H$ is a full-column-rank matrix satisfying $E^T H = 0$.

**Proof:** We first prove that the system $E x(k+1) = A x(k)$ is regular and causal. Since the matrix $E$ is singular, there must exist nonsingular matrices $T_L$ and $T_R$ such that

$$
\tilde{E} = T_L E T_R = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}
$$

and then, we can further define

$$
\tilde{A} = T_L A T_R = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},
$$

$$
\tilde{P} = T_L^{-T} P T_L^{-1} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix},
$$

$$
\tilde{S} = T_R^T S = \begin{bmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{bmatrix},
$$

$$
\tilde{H} = T_L^{-T} H = \begin{bmatrix} 0 \\ \Phi \end{bmatrix}
$$

where $\Phi$ is a nonsingular matrix. Then, we multiply $(4a)$ by $T_R^T$ and $T_R$, on the left and on the right respectively, which gives

$$
\begin{bmatrix}
* \\
* \tilde{S}_2 \Phi^T \tilde{A}_{22} + \tilde{A}_{22} \Phi \tilde{S}_2
\end{bmatrix} < 0
$$

where $*$ is the matrix element irrelevant to the following discussion and thus is omitted here. Then, we can show that $\tilde{A}_{22}$ is nonsingular and then the pair $(\tilde{E}, \tilde{A})$ is regular and causal [25].

Now, we prove the finite-time stability of this system. We construct a Lyapunov-like function

$$
V(x(k)) = x^T(k) E^T P E x(k) \quad (5)
$$

Then, we have $V(x(k+1)) = x^T(k+1) E^T P E x(k+1)$ and then

$$
V(x(k+1)) = \alpha V(x(k))
$$

$$
= x^T(k+1) E^T P E x(k+1)
$$

$$
= \alpha x^T(k) E^T P E x(k)
$$

$$
= x^T(k) A^T P A x(k) - \alpha x^T(k) E^T P E x(k)
$$

$$
+ 2 x^T(k+1) E^T H S^T x(k)
$$

$$
= x^T(k) (A^T PA - \alpha E^T P E + A^T H S^T + S H^T A) x(k)
$$

It can be concluded from (4a) that $V(x(k+1)) < \alpha V(x(k))$, and therefore we can obtain

$$
V(x(k)) < \alpha^k V(x(0)) \quad (6)
$$

If we define $Q = R^{-\frac{1}{2}} P R^{-\frac{1}{2}}$, we can obtain

$$
V(x(k)) = x^T(k) E^T P E x(k)
$$

$$
= x^T(k) E^T R^2 Q R^2 E x(k)
$$

$$
\geq \lambda_{\min}(Q) x^T(k) E^T R E x(k) \quad (7)
$$

and

$$
V(x(0)) = x^T(0) E^T P E x(0)
$$

$$
= x^T(0) E^T R^2 Q R^2 E x(0)
$$

$$
\leq \lambda_{\min}(Q) x^T(0) E^T R E x(0) \quad (8)
$$

Thus, from (4c) and (6), (7), (8), it follows that $x^T(k) E^T R E x(k) < c_2, \forall k \in [0, 1, 2, \ldots, N]$. This concludes the proof.

**Remark 1:** It is easy to show that the equality $0 = 2 x^T(1) E^T H S^T x(k) = x^T(k) (A^T H S^T + S H^T A) x(k)$ describes the relationship between the fast subsystem and slow subsystem of a discrete-time linear singular system. This relationship is motivated by [26], where the robust $H_{\infty}$ stabilization problem for uncertain discrete-time linear singular systems was studied. It is noted that the explicit expressions of the fast subsystem and slow subsystem is not given, since this system decomposition is only used to prove the regularity and causality of the original linear singular system, which is different from [26]. In fact, the fast subsystem and slow subsystem couple each other in system (3). The criterion presented in Theorem 1 in [27] is the same as $(4a)$ with $\alpha = 1$. In this sense, the result in Theorem 1 can be regarded as an extension of Theorem 1 in [27] to finite-time stability analysis for discrete-time linear singular systems.

**Remark 2:** The matrix $H$ contained in matrix inequality (4a) should be chosen beforehand. A natural question one may raise then is whether the value of $H$ influence the feasibility of condition (4a). The answer is negative. To show this, we assume that condition (4) exists a feasible solution $P, Q, S, \alpha$ for an arbitrarily prescribed matrix $H \in \mathbb{R}^{n \times (n-r)}$. Now, we select a different $H$ as $H' \in \mathbb{R}^{n \times (n-r)}$. By noting the fact that $H$ and $H'$ are both with full column rank, it can be verified that there must exist a nonsingular matrix $U \in \mathbb{R}^{(n-r) \times (n-r)}$. 

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such that \( H' = HU \). Then, \( S(H')^T = SU^TH' \) and then condition (4a) also holds for \( P, Q, S' = SU^{-T} \). In a practical system, we only choose this matrix to be full column rank.

**Remark 3:** From inequality (4c), we can obtain

\[
N < \frac{\ln \frac{\lambda_{\text{min}}(Q)}{\lambda_{\text{max}}(Q)} + \ln \frac{\sigma_2}{\sigma_1}}{\ln \alpha}
\]

For given matrices \( P, Q, S \) and scalars \( \alpha, \sigma_1 \), the time interval \( N \) is affected by the shape of boundedness set \( R \) and the size of boundedness set \( c_2 \). Thus, the boundedness set has relationship with the time interval. When \( N \) trends to be larger, the boundedness of the state can be guaranteed if the inequality (4c) holds for some \( P, Q, S \) and \( \alpha \). When \( N \) trends to be larger and larger, it is more difficult to find a solution \( \alpha < \infty \) and thus (4c) cannot be guaranteed. In this case, Lyapunov stability can be applied to resolve this problem. This verifies the natural difference between the finite-time stability and Lyapunov stability: the former is valid in a finite time interval, while the latter is valid in an infinite time interval. For this reason, the former is a quantitative concept while the latter is a qualitative concept.

**Remark 4:** There are different types of parameter uncertainties in the system and control theory, in which the typical ones are rank-one parameter uncertainty, linear parameter uncertainty, norm-bounded parameter uncertainty and convex polytopic uncertainty. They can be transformed each other, but some conservativeness may be introduced. For example, the rank-one type uncertainty can be regarded as a special case of linear parameter uncertainty, and the linear parameter uncertainty can be transformed to the norm-bounded parameter uncertainty by noting \( \Delta A = a_1(t)A_1 + a_2(t)A_2 + \cdots + a_m(t)A_m = MF(t)N \), where \( M = [a_1 A_1 \ a_2 A_2 \cdots a_m A_m] \),

\( F(t) = \text{diag}([a_1(t) I, \ a_2(t) I, \cdots, a_m(t) I]), \ N = [I \ 1 \cdots 1]^T, \ a_i(t), (i = 1, 2, \ldots, m) \) are unknown time-varying scalars, \( a_i(t) = \text{max}(a_i(t)), (i = 1, 2, \ldots, m) \) are known scalars, and \( A_i, (i = 1, 2, \ldots, m) \) are known matrices with appropriate dimensions. It is obvious that the bounded norm condition \( F^T(t)F(t) \leq I \) is satisfied. It is also noted that the additional conservativeness is introduced in this transformation.

The condition in Theorem 1 contains (4b) and (4c), which are difficult to verify numerically. To resolve this problem, we set

\[
\theta I < R^{-\frac{1}{2}}PR^{-\frac{1}{2}} < I
\]

where \( 0 < \theta < 1 \), and have \( \theta < \lambda_{\text{min}}(Q) < \lambda_{\text{max}}(Q) < 1 \) and further have \( \alpha^N \lambda_{\text{max}}(Q)c_1 < \alpha^N c_1, \theta c_2 < \lambda_{\text{min}}(Q) c_2 \).

If

\[
\alpha^N c_1 - \theta c_2 < 0
\]

holds, we have \( \alpha^N \lambda_{\text{max}}(Q)c_1 < \alpha^N c_1 < \theta c_2 < \lambda_{\text{min}}(Q) c_2 \) and then (4c) can be guaranteed.

It should be noted here that constraint (9) is without loss of generality. To show this, we assume that there exists a solution \( P, Q, S, \alpha \) such that condition (4) is satisfied and

\[
Q = R^{-1/2}PR^{-1/2} > I
\]

holds, then we can define \( P' = P/\lambda_{\text{max}}(Q), Q' = Q/\lambda_{\text{max}}(Q) \), and in this case the conditions

\[
Q' = R^{-1/2}P'R^{-1/2}, \ \alpha^N \lambda_{\text{max}}(Q)'c_1 < \lambda_{\text{min}}(Q)'c_2 \]

\[
A^T P'A - \alpha E^T P'E + A^T H S^T \frac{1}{\lambda_{\text{max}}(Q)} + S H^T A \frac{1}{\lambda_{\text{max}}(Q)} < 0
\]

hold. That is to say, condition (4) also holds for \( P', Q', S' = S/\lambda_{\text{max}}(Q) \) and \( \alpha \). Here, \( \theta I < R^{-1/2}P'R^{-1/2} < I \) holds with

\[
\theta = \lambda_{\text{min}}(Q) \frac{1}{\lambda_{\text{max}}(Q)} < 1
\]

With this observation, the following theorem can be obtained straightforwardly.

**Theorem 2:** Consider the discrete-time linear singular system (3) with \( u(k) = 0 \). It is regular, causal and finite-time stable with respect to \((c_1, c_2, N, R)\), if there exists a symmetric positive-definite matrix \( P \), a matrix \( S \), and positive scalars \( \alpha > 1, \theta < 1 \) such that matrix inequalities (4a), (9), and (10) hold, where \( H \in \mathbb{R}^{n \times (n-r)} \) is given in Theorem 1.

We are now in the position to give a robust finite-time stability condition for the uncertain discrete-time linear singular system (1) with \( u(k) = 0 \).

**Theorem 3:** Consider the uncertain discrete-time linear singular system (1) with \( u(k) = 0 \). It is robustly finite-time stable with respect to \((c_1, c_2, N, R)\), if there exists a symmetric positive-definite matrix \( P \), a matrix \( S \), and positive scalars \( \alpha > 1, \theta < 1, \varepsilon \) such that matrix inequalities (9), (10) and (11) hold.

\[
\Omega = \begin{bmatrix} \Omega_{11} & A^T P SH^T M \ 
M^T H S^T & M^T P - \varepsilon I \end{bmatrix} < 0
\]

where \( H \in \mathbb{R}^{n \times (n-r)} \) is given in Theorem 1, and

\[
\Omega = -\alpha E^T P'E + A^T H S^T + S H^T A + \varepsilon N^T N_a
\]

Proof: From Definition 2, the uncertain discrete-time linear singular system (1) with \( u(k) = 0 \) is robustly finite-time stable, if there exists a symmetric positive-definite matrix \( P \), a matrix \( S \), and positive scalars \( \alpha > 1, \theta < 1 \) such that (4b), (4c) and

\[
(A + \Delta A)^T P(A + \Delta A) - \alpha E^T P E
\]

\[
+ (A + \Delta A)^T H S^T + S H^T A + \varepsilon N^T N_a < 0
\]

hold. It is easy to see that (4b), (4c) can be guaranteed by (9), (10). The inequality (12) can be equivalently rewritten as the following form by using the Schur complement lemma,

\[
\begin{bmatrix} -\alpha E^T P'E + A^T H S^T + S H^T A A^T P \\
PA & -P \end{bmatrix}
\]

\[
+ \begin{bmatrix} SH^T M \\
PM \end{bmatrix} F(k) \begin{bmatrix} N_a^T \\
0 \end{bmatrix}^T < 0
\]

\[
+ \begin{bmatrix} N_a^T \\
0 \end{bmatrix} F(k) \begin{bmatrix} SH^T M \\
PM \end{bmatrix}^T < 0
\]
This inequality holds for any $F(k)$ satisfying $F^T(k)F(k) \leq I$, if and only there exists a scalar $\varepsilon > 0$ such that
\[
\begin{bmatrix}
-\alpha E^T PE + A^T HS^T + SH^T A^T P \\
+\varepsilon^{-1} SH^T M \begin{bmatrix} SH^T M \\ PM \end{bmatrix}^T \\
+\varepsilon \begin{bmatrix} N_a^T \\ 0 \end{bmatrix} \begin{bmatrix} N_a^T \\ 0 \end{bmatrix}^T
\end{bmatrix} < 0
\]
which is equivalent to inequality (11) in the sense of Schur complement.

We are now in the position to give a state feedback gain design algorithm for the uncertain discrete-time linear singular system (1), which is summarized in the following theorem.

**Theorem 4:** Consider the uncertain discrete-time linear singular system (1). It is robustly finite-time stabilizable with respect to $(c_1, c_2, N, R)$, if there exists a symmetric positive-definite matrix $P$, a matrix $S$, and positive scalars $\alpha > 1$, $\theta < 1$, $\varepsilon$, $\delta$ such that matrix inequalities (9), (10), and (15) hold,
\[
\begin{align*}
\Gamma &= P^{-1} - \varepsilon^{-1} MM^T > 0 \quad (15a) \\
SH^T A + A^T HS^T - \alpha E^T PE + \varepsilon N_a^T N_a + \Pi \\
&\quad +\varepsilon^{-1} SH^T MM^T HS^T - \Psi \Phi^{-1} \Psi^T < 0 \quad (15b)
\end{align*}
\]
where $H \in \mathbb{R}^{n\times(n-r)}$ is given in Theorem 1, and
\[
\Phi = B^T \Gamma^{-1} B + \varepsilon N_a^T N_b + \delta I, \\
\Psi = SH^T B + (A^T + \varepsilon^{-1} SH^T MM^T) \Gamma^{-1} B + \varepsilon N_a^T N_b, \\
\Pi = (A^T + \varepsilon^{-1} SH^T MM^T) \Gamma^{-1} (A^T + \varepsilon^{-1} SH^T MM^T)^T.
\]
Furthermore, a suitable state feedback gain matrix is given as
\[
K = -\Phi^{-1} \Psi^T. \quad (16)
\]

**Proof:**
Applying the state feedback control law (16) to the system (1) results in the following closed-loop system,
\[
Ex(k+1) = (A_c + MF(k)N_c)x(k)
\]
where $A_c = A - B\Phi^{-1} \Psi^T, N_c = N_a - N_b \Phi^{-1} \Psi^T$. Noting the fact of $B^T \Gamma^{-1} B + \varepsilon N_a^T N_b < 0$. By tedious and straightforward computation, we can verify that
\[
\begin{align*}
SH^T A_c + A_c^T HS^T + \varepsilon N_a^T N_c \\
&\quad + (A_c^T + \varepsilon^{-1} SH^T MM^T) \Gamma^{-1} (A_c^T + \varepsilon^{-1} SH^T MM^T)^T
\leq SH^T A + A^T HS^T + \varepsilon N_a^T N_a - \varepsilon N_a^T N_b \Phi^{-1} \Psi^T \\
&\quad - \varepsilon \Psi \Phi^{-1} N_a^T N_a
\end{align*}
\]
\[
\begin{align*}
&\quad + (A^T + \varepsilon^{-1} SH^T MM^T) \Gamma^{-1} (A^T + \varepsilon^{-1} SH^T MM^T)^T \\
&\quad - (A^T + \varepsilon^{-1} SH^T MM^T) \Gamma^{-1} B \Phi^{-1} \Psi^T \\
&\quad - \Psi \Phi^{-1} B^T \Gamma^{-1} (A^T + \varepsilon^{-1} SH^T MM^T)^T + \Psi \Phi^{-1} \Psi^T \\
&\quad = SH^T A + A^T HS^T + \varepsilon N_a^T N_a - \Psi \Phi^{-1} \Psi^T \\
&\quad + (A^T + \varepsilon^{-1} SH^T MM^T) \Gamma^{-1} (A^T + \varepsilon^{-1} SH^T MM^T)^T < 0
\end{align*}
\]

This further gives that inequality (11) holds, where $A$ and $N_a$ are replaced by $A_c$ and $N_c$, respectively. Then, the closed-loop system is robustly finite-time stable with respect to $(c_1, c_2, N, R)$, which completes the proof.

**Corollary 1:** The discrete linear singular system (3) is finite-time stabilizable with respect to $(c_1, c_2, N, R)$, if there exists a symmetric positive-definite matrix $P$, a matrix $S$, and positive scalars $\alpha > 1, \theta < 1, \varepsilon, \delta$ such that matrix inequalities (9), (10), and (18) hold,
\[
\begin{align*}
A^T PA + SH^T A + A^T HS^T - \alpha E^T PE \\
&\quad + (A^T PB + SH^T B)(B^T PB + \delta I)^{-1} \\
&\quad \cdot (A^T PB + SH^T B)^T < 0
\end{align*}
\]
where $H \in \mathbb{R}^{n\times(n-r)}$ is given in Theorem 1, and a suitable state feedback gain matrix is given as
\[
K = -(B^T PB + \delta I)^{-1} (A^T PB + SH^T B)^T
\]

It is noted here that the matrix $B^T PB + \delta I$ is reversible, since $B^T PB$ is positive semi-definite and $\delta I$ is positive-definite. To further show this, for a given nonzero vector $\zeta$, we have
\[
\zeta^T (B^T PB + \delta I)\zeta = \zeta^T B^T PB\zeta + \delta \|\zeta\|^2 > 0,
\]
and thus the gain matrix $K$ is feasible.

**III. NUMERICAL EXAMPLES**

We consider system (1) with the following parameters,
\[
E = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad N_a = \begin{bmatrix} 0.2 & 0.2 & 0.1 \end{bmatrix}, \quad N_b = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}.
\]
We set \( c_1 = 1, c_2 = 5, N = 5 \) and \( R = \text{diag}[1, 1, 1] \). By using the result of Theorem 4, we set \( H = [1 \ 1 \ -1]^T \), \( \alpha = 1.1 \) and solve the feasibility problem of inequalities (9), (10) and (15), and have that this system is robustly finite-time stabilizable with respect to \( (c_1, c_2, N, R) \). A feasible solution is given as follows,

\[
P = \begin{bmatrix}
0.7299 & -0.0329 & -0.0229 \\
-0.0329 & 0.7478 & -0.0050 \\
-0.0229 & -0.0050 & 0.6920
\end{bmatrix}, \quad S = \begin{bmatrix}
0.8556 \\
-0.7004 \\
-1.4498
\end{bmatrix},
\]

\[
\theta = 0.4183, \quad \varepsilon = 0.5951, \quad \delta = 0.6584
\]

and a suitable state feedback gain matrix is given as

\[
K = \begin{bmatrix}
-2.1669 & -2.3968 & -3.5391 \\
2.6453 & 5.6147 & 1.6342
\end{bmatrix}
\]

With \( F(k) = \sin(k) \), the state trajectory of is shown in Figure 1.

IV. CONCLUSION

In this paper, the problems of robust finite-time stability and robust finite-time stabilization for uncertain discrete-time linear singular systems are considered. A sufficient condition is first given for a discrete-time linear singular system to be regular, causal and finite-time stable, based on which, the robust finite-time stability problem is resolved and a sufficient criterion is given in terms of matrix inequalities. The robust finite-time stabilization problem is also resolved and the corresponding state feedback gain design algorithm is given in terms of a set of matrix inequalities.

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