N=2 gauge theories on the hemisphere $HS^4$

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Abstract

Using localization techniques, we compute the path integral of $N = 2$ SUSY gauge theory coupled to matter on the hemisphere $HS^4$, with either Dirichlet or Neumann supersymmetric boundary conditions. The resulting quantities are wave-functions of the theory depending on the boundary data. The one-loop determinant are computed using $SO(4)$ harmonics basis. We solve kernel and co-kernel equations for the relevant differential operators arising from gauge and matter localizing actions. The second method utilizes full $SO(5)$ harmonics to reduce the computation to evaluating $Q_{SUSY}^2$ eigenvalues and its multiplicities. In the Dirichlet case, we show how to glue two wave-functions to get back the partition function of round $S^4$. We will also describe how to obtain the same results using $SO(5)$ harmonics basis.

Keywords: Supersymmetry, Localization, One-loop determinant, $SO(4)$ harmonics, Dirichlet and Neumann wave functions
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1. Introduction

Since the seminal work [14] on supersymmetric localization on round $S^4$ there has been an intense activity in the field. This computation was soon generalized to more general curved manifolds in various dimensions [7, 11, 8] and to various number of supersymmetries [13]. Manifolds with boundaries in two and three dimensions were also considered and various quantities computed [16, 10, 15, 9].

For manifolds without boundaries Atiyah-Singer Index Theory, together with a clever cohomological reorganization of fields introduced in [14], provides a shortcut to the computation of the one-loop determinants factor and avoids going through the diagonalization of the relevant differential operators arising from the quadratic part of the localizing action expanded around the saddle points. In particular, in [14], it has been shown that the one-loop factor is given by the ratio of determinants of the operator $Q^2$ on the kernel and cokernel of certain differential operator determined from the localizing action $QV$, $Q$ being the supercharge entering in the localization procedure and $V$ being some fermionic field. To compute the latter determinants one then uses Atiyah-Bott fixed point theorem which localizes the computation near the fixed points of $Q^2$ on the base manifold.

In this note we will work out the partition function of $N=2$ SUSY gauge theory coupled to matter in some $G$-representation $R$, on the hemisphere $HS^4$, both for Dirichlet and Neumann boundary conditions at the equator of the round $S^4$. Since applying the above mathematical machinery in the case of manifolds with boundary may be subtle, we will rather solve the partial differential equations which determine the kernel and cokernel of the relevant differential operators, together with the corresponding $Q^2$ quantum numbers and multiplicities. With this data we write down the one-loop determinant for both the vector multiplet and the hypermultiplets, for the corresponding boundary conditions. This direct approach turns out to be feasible because of the symmetry of the problem: we choose the $S^4$ metric with a manifest $SO(4)$ symmetry which is preserved by the localizing action. This allows to reduce the system of (coupled) partial differential equations depending on the four coordinates of $S^4$, to a system of ordinary differential equations depending on a radial coordinate $r$.

We will then write the expressions for the wave functions on $HS^4$ with Dirichlet and Neumann boundary conditions respectively. We finally show that the partition function of $N=2$ theory on the round $S^4$ can be obtained by gluing two Dirichlet wave functions corresponding to the two hemispheres $HS^4$s.

Next, for the purpose of completion, we briefly describe how one can obtain the same results using spherical harmonics of the full isometry group $SO(5)$. The logic is essentially the same as that for $SO(4)$.

It is interesting to note that the vector multiplet one loop determinant for Dirichlet BCs is not 'half' of that for the full round $S^4$ [14] as one would naively expect. As one solves the zero mode equations for kernel and co-kernel of vector multiplet, to find the multiplicity of the unpaired bosonic and fermionic modes, it turns out that for the Dirichlet BCs the net multiplicity is one unit less and for the Neumann BCs one unit more that that required for the one loop determinant to be exactly 'half'. This deficiency or access of one unit of multiplicity for bulk theory has interesting interpretation in terms of whether one chooses to freeze some degrees of freedom or to add some extra degrees of freedom at the boundary.

The contents in this note are arranged as follows: After a lightening review of $N=2$ SUSY gauge theory coupled to matter in section 2 we move on to compute one-loop determinants of vector multiplet and hypermultiplet in sections 3 and 4. Everything in the path integral computation is put
together in section 5 to get the sought after wave functions of \( N = 2 \) system on hemisphere with Dirichlet and Neumann boundary conditions. Factorization of round sphere \( S^4 \) \( N = 2 \) partition function is discussed in section 6. Discussion of \( SO(5) \) harmonics and subsequent computation of \( Z_{1\text{-loop}}^{\text{vec}} \) is given in sections 7 and 8. A brief set of conclusions are given in section 9.
2. $N = 2$ SUSY Field theory

We will consider 4d $N = 2$ extended supersymmetric field theories which are defined by eight supercharges in the flat space limit. These supercharges correspond to choosing a pair of chiral, anti-chiral Killing spinors denoted in four component notation by $\xi$, which satisfies following set of equations.

\[ D_\mu \xi + T_{\nu\rho} \gamma^{\nu\rho} \gamma_\mu \xi = \xi_\rho, \]

\[ \gamma^\mu \gamma^\nu D_\mu D_\nu \xi + 4 D_\mu T_{\nu\rho} \gamma^{\nu\rho} \gamma_\mu \xi = M \xi \]  

(2.1)

where $\xi_\rho \equiv \gamma^\mu D_\mu \xi$ and $T_{\nu\rho}$ are background non-dynamical fields belonging to parent supergravity theory $[7]$. Moreover the covariant derivative $D_\mu$ also contains a background $SU(2)_R$ R-symmetry gauge field $V^A_{\mu}$. The values of these auxiliary fields are found, if they exist, for which one is able to solve the Killing spinor equations. This Killing spinor $\xi$ defines the supercharge $Q$ with respect to which we localize the path integral.

2.1. Physical actions

The physical actions for vector multiplet and the matter multiplet of $N = 2$ SUSY on curved background are written in $[7]$. For SYM action we have:

\[ L_{YM} = \text{Tr} \left[ \frac{1}{2} F^{mn} F_{mn} + 16 F_{mn} (\phi T^{mn} + \phi T^{mn}) + 64 \phi^2 T_{mn} T^{mn} + 64 \phi^2 T^{mn} T_{mn} - 4 D_m \phi D^n \phi + 2 M \phi \bar{\phi} - 2 i \lambda^A \sigma^m D_m \lambda_A - 2 \lambda^A[\phi, \lambda_A] + 2 \bar{\lambda}^A[\phi, \bar{\lambda}] + 4[\phi, \bar{\phi}]^2 - \frac{1}{2} D^{AB} D_{AB} \right]. \]

When restricted to the round sphere $S^4$ (appendix [B]), the other supergravity background fields reduce to

\[ T_{mn} = 0, \quad T_{mn} = 0, \quad V_{m=0} = 0. \]

(2.2)

There are also reality conditions for the fields, chosen to ensure a well defined path integral. In particular, $\phi = \phi_2 + i \tilde{\phi}_1$ and $\phi = -\phi_2 + i \tilde{\phi}_1$. The gauginos $\lambda^A$ and $\lambda_A$ of opposite $SO(4)$ chirality carry and $SU(2)_R$ doublet index. $D_{AB}$ is an $SU(2)_R$ triplet of auxiliary fields. For non-trivial topological sectors, characterized by instanton number $k$, one has to add $\theta$-term to the full action:

\[ S_{YM} = \frac{1}{g_{YM}^2} \int d^4 x \sqrt{g} L_{YM} + i \frac{\Theta}{8\pi^2} \int \text{Tr}(F \wedge F). \]  

(2.3)

The action for for the matter hypermultiplets is:

\[ L_{\text{mat}} = \frac{1}{2} D_m q^A D^m q_A - q^A(\phi, \bar{\phi}) q_A + \frac{1}{2} q^A D_{AB} q^B + \frac{1}{8} (R + M) q^A q_A - i \frac{1}{2} \bar{\psi} \bar{\sigma}^m D_m \psi - \frac{1}{2} \bar{\psi} \psi \bar{\psi} + i \frac{1}{2} \psi \sigma^{kl} T_{kl} \psi - \frac{i}{2} \bar{\psi} \bar{\sigma}^{kl} T_{kl} \bar{\psi} - q^A \lambda_A \psi + \bar{\psi} \lambda q^A - \frac{1}{2} F^A F_A. \]  

(2.4)

where again the supergravity background satisfies eq.(2.2) and a with proper reality conditions for the fields is understood. The scalars $q^A$ carry an $SU(2)_R$ index $A$ and in addition an index $I = 1, \ldots, 2q$ of a symplectic representation of the gauge symmetry group $G \subset Sp(q)$, therefore it is possible to impose a reality condition on them in the usual way. The spinors $\psi$ carry index $I$ and $F^A$ are auxiliary fields.
2.2. Localizing actions

The localization technique proceeds first by identifying a supercharge $\hat{Q}$ and then a $\hat{Q}$-exact localizing action, with positive definite bosonic part, by which one perturbs the physical action in such a way that the path integral is independent of the perturbation. One then shows that the path integral localizes at the supersymmetric saddle points of the localizing action, in the sense that the one-loop approximation around them is exact.

The localizing supercharge $\hat{Q}$ depends on a choice of Killing spinors, $\xi^{A\alpha}$ and $\bar{\xi}^{\dot{A}\dot{\alpha}}$, which we arrange in a four-by-two matrix using four component $SO(4)$ spinors which are also $SU(2)_R$ doublets. Killing spinor $\xi$ is taken as Grassmann-even and $\hat{Q}$ is Grassmann-odd. With the background metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dr^2 + \frac{\sin(r)^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2).$$

(2.5)

which can be written in terms of $SU(2)$ left-invariant one-forms, a solution of the $N = 2$ Killing spinor equations is given by:

$$\xi = \begin{pmatrix} \cos(\bar{\zeta}) & 0 \\ 0 & \cos(\bar{\zeta}) \\ i \sin(\bar{\zeta}) & 0 \\ 0 & -i \sin(\bar{\zeta}) \end{pmatrix}.$$ 

(2.6)

with the index structure $\xi \equiv (\xi^{A\alpha}, \bar{\xi}^{\dot{A}\dot{\alpha}})$ where $\alpha, \dot{\alpha}$ are Lorentz indices and $A$ is R symmetry index. And it is normalized to $\xi^A \xi_A + \bar{\xi}^{\dot{A}} \bar{\xi}_{\dot{A}} = 1$. These Killing spinors give rise to a Killing vector $v = 2 \frac{\partial}{\partial \psi}$. The corresponding supercharge $\hat{Q}$ squares on all fields to bosonic symmetry generators:

$$\hat{Q}^2 = \mathcal{L}_v + R + \text{Gauge}_A.$$ 

(2.7)

where $\mathcal{L}_v$ is the Lie derivative along $v$, $R$ and $\text{Gauge}_A$ are, respectively, R-symmetry and field dependent gauge transformation parameter $\Lambda$:

$$\Lambda = -v^\mu A_\mu - 2i \phi_1 - 2 \cos(r) \phi_2.$$ 

(2.8)

The localizing action, $S_{\text{loc}} = \hat{Q}V$, is determined by the fermionic field $V$, for which, a convenient expression in terms of original fields is:

$$V = \text{Tr}[(\hat{Q} \lambda_{\alpha A})^\dagger \lambda_{\alpha A} + (\hat{Q} \bar{\lambda}^{\dot{A}})^\dagger \bar{\lambda}_{\dot{A}}],$$ 

(2.9)

\[1\] In four component notation we use the matrix \[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\] to raise and and lower the $\alpha, \dot{\alpha}$ indices.

\[2\] $\hat{Q}$ should be defined by including the BRST component and correspondingly $V$ should include the ghost part, as it will be shown in the following.
where $\dagger$ means complex conjugation. One can show that the localization locus is given by

$$F_{\mu\nu} = 0, \quad \phi = \bar{\phi} = a_0, \quad D_{AB} = -ia_0\omega_{AB},$$

(2.10)

where $\omega_{AB} = -8(\xi_A\xi_{pB} + \xi_B\xi_{pA})$ and $\xi_p = \gamma^\mu D_\mu \xi$. Therefore $\phi_2 = 0$ and $\phi_1 = a_0$, $a_0$ being a constant element of the Lie algebra of the gauge group $G$. Note that at the saddle point, choosing $A_\mu = 0$, $A$ reduces to a constant gauge parameter, $\Lambda = -2ia_0$, given by the v.e.v. of the scalar $\phi$. Here we stress that although $A_\mu$ is a pure gauge, due to non-trivial $\pi_3(SU(2)) \simeq \pi_3(S^3) \simeq \mathbb{Z}$ of the boundary there are equivalence classes of gauge connections characterized by integer $k$. And to compute perturbative part the quadratic fluctuations will be in general around vacuum labelled by $k$.

As mentioned in the introduction, it is convenient to introduce new fermionic fields which are linear combinations of the gauginos and carry integer spins, which are part of the cohomological fields,

$$\psi = \bar{Q}\phi_2, \quad \psi_\mu = \bar{Q}A_\mu, \quad \bar{\chi} = \xi^A\lambda_B\bar{\sigma}^B.$$  

(2.11)

The main point about introducing these fields is that, after gauge fixing and introducing the ghost-antighost system $c,\bar{c}$, and extending $\bar{Q}$ to include the BRST generator, the bosonic fields $X_0 = (A_\mu, \phi_2)$ and the fermionic ones, $X_1 = (\bar{\chi}, c, \bar{c})$, are lowest components of $\bar{Q}$ multiplets.

One can rewrite (2.9) in terms of cohomological variables $\bar{Q}$, however after some linear algebra, one can show that the relevant super determinant corresponding to the quadratic part of $\bar{Q}V$ can be expressed in terms of the super determinant of the $\bar{Q}$ operator on the spaces $X_0$ and $X_1$. Furthermore, since these spaces are related by a differential operator $D_{10}$ which commutes with $\bar{Q}V$ and can be read off from $V$, at the end the super determinant, up to an overall sign ambiguity, reduces to

$$Z_{1-loop} = \frac{\det_{\text{Coker}D_{10}} Q^2}{\det_{\text{Ker}D_{10}} Q^2}^\frac{1}{2}.$$  

(2.12)

that is, it is enough to compute the spectrum of $Q^2$ on the kernel and cokernel of $D_{10}$, where the latter equals the kernel $D^0_{10}$. The differential operator $D_{10}$ is identified from the terms in $V$ which are bilinear in $X_0$ and $X_1$, after expanding around the saddle point field configuration. The terms relevant to our analysis are the following

$$V_{vec} = e\text{Tr}\left(\chi_a(\bar{Q}\chi_a)^\dagger + \frac{1}{e}eD_\nu(e\bar{Q}\psi_\nu)^\dagger + \frac{1}{e}\bar{e}D_\nu(eA_\nu)^\dagger\right).$$  

(2.13)

Here $e = \sqrt{g}$.

Similarly, for the matter part the localizing action is obtained from:

$$V_{mat} = e\text{Tr}[\bar{Q}(\psi_\alpha I)^\dagger \psi_\alpha I + (\bar{Q}\bar{\psi}_\alpha I)^\dagger \bar{\psi}_\alpha I].$$  

(2.14)

and a trivial localization locus

$$q_{IA} = 0, \quad F_{IA} = 0.$$  

(2.15)

As for the vector case, it is convenient to change fermionic variables from $\psi_\alpha$ to $\Sigma_\alpha = \xi_\alpha\psi$, where $\xi$ is a spinor orthogonal to $\xi$. The cohomological fields are $X_0 = q_\Sigma$ and $X_1 = \Sigma_\alpha$ and the same linear algebra argument as before shows that the matter one-loop contribution is given by the superdeterminant of $\bar{Q}$ on the kernel and cokernel of the operator $D_{10}^0$ mapping $X_0$ to $X_1$, which can be read off from $V_{mat}$ by keeping the terms bilinear in $q$ and $\Sigma$.

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4Here we neglect the ghost-for-ghost system, which takes care of the $c,\bar{c}$ zero modes.
2.3. Main content of computation

The main content of this work is the computation of the partition function (actually, wave-function) of $N = 2$ supersymmetric gauge theory on a hemisphere $HS^4$, with supersymmetric boundary conditions. The relevant one-loop determinants for gauge and matter multiplets are computed by direct analysis of partial differential equations defining the kernel and cokernel of $D_{10}$ differential operators. In other words, we take the differential operator $D_{10}$ and its adjoint counterpart $(D_{10})^\dagger$ from the fermionic functional $V$ and then solve the zero mode partial differential equations for them. We explicitly find the solutions and their multiplicities by diagonalizing the differential equations by expanding fields in $SO(4) \sim SU(2)_L \times SU(2)_R$ harmonics. The kernel equations for $X_0$ fields are obtained by varying $V_{loc}$ with respect to $X_1$ fields and vice versa of the cokernel equations. We write them in the appendix. The differential equations can be expressed in terms of $SU(2)_L$ generators and $r$-derivatives, so that $SU(2)_R$ is spectator and provides the multiplicities, once we solve the ordinary differential equations in the $r$ coordinate.

We work with the round $S^4$ and then we adapt it to the case of hemisphere $HS^4$, where we impose appropriate boundary conditions to the zero modes of $D_{10}$ and $(D_{10})^\dagger$ at the equator of $S^4$, taken to be at $r = \pi/2$. As it turns out the solution set for the zero modes of $D_{10}$ is empty and those of $(D_{10})^\dagger$ is non-empty for the vector multiplet, whereas the converse is true for the matter multiplet.

The $Q^2_2$ eigenvalues and their multiplicities will then give us, by using eq. (2.12), the expression for the determinants. The analysis for kernel and cokernel equations respectively in sections 3 and 4 is done for $U(1)$ gauge group. The generalization for a non-abelian gauge group $G$ is straightforward. For vector multiplet we have to multiply the index by the character $\sum_{\alpha \in \text{Rots}} e^{i\alpha.a}$ of adjoint representation of $G$ and for matter multiplet by the character $\sum_{\rho \in R} e^{i\rho.a}$ in the representation $R$ of $G$.

Some comments on BRST analysis are in order. The standard covariant way to fix the gauge redundancy of the action is BRST formalism. In an abelian gauge theory like $U(1)$ the BRST charge $Q_B$ which parametrized the gauge freedom is nilpotent. To generalize it to non-abelian gauge group the corresponding BRST charge $Q_{B}$ squares to a constant gauge transformation $a$ that is ultimately identified with the zero mode of the scalar field as a solution to localization equations. This constant gauge parameter $a$ is integrated over to get the partition function on $S^4$.

3. Vector-multiplet contribution

Perturbative part of partition function corresponds to computing one loop determinant of the quadratic fluctuations around classical field configuration given by the saddle point solutions (2.10) (2.15). The differential operator $D_{10}$ from which we get kernel and cokernel equations, contains only the fluctuating part of the quantum fields. In the vector multiplet the fluctuating part of only $\phi_2 \in X_0$ is relevant, whereas $\phi_1$ and $D_{AB}$ contributes only classically. Hence in our discussion of kernel equations we will set $\phi_1 = 0$, $D_{AB} = 0$. As detailed in the appendix, it is convenient to work with tangent space basis for the gauge fields $A_\mu$: for the $S^3$ directions $\mu = 1, 2, 3$, the flat basis is $A_a = l^{\mu A}_a A_\mu$ where $a = 1, 2, 3$ or $a = \pm, -, 3$ where $A_+ \equiv A_1 + iA_2, A_- \equiv A_1 - iA_2$. Similarly for the fermionic fields in $X_1$ we choose a complex basis $\chi_+ \equiv \chi_1 + i\chi_2, \chi_- \equiv \chi_1 - i\chi_2$.

The first thing to observe is that, since $D_{10}$ commutes with $Q^2_2$, it will close in $X_0$ on fields of the same $Q^2_2$ eigenvalue and similarly for $D_{10}^\dagger$ on $X_1$. As we will see in the following sections that $D_{10}$ and $D_{10}^\dagger$ will only involve $SU(2)_L$ operators, so all the fields in $X_0$ will have the same $SU(2)_R$
weight denoted here by $q_R$ and all the fields in $X_1$ will carry weight $-q_R$. The first data to compute are the $Q^2$ eigenvalues on the cohomological fields. We start with the gauge fields in $X_0$:

\[
Q^2 A_3 = 2 \partial_\psi A_3, \quad Q^2 A_4 = 2 \partial_\psi A_4,
\]

\[
Q^2 A_+ = 2(\partial_\psi - i)A_+, \quad Q^2 A_- = 2(\partial_\psi + i)A_-. \tag{3.1}
\]

and $Q^2 \phi_2 = 2 \partial_\psi \phi_2$. Notice the shifts in $A_\pm$. As for the fields in $X_1$ we have:

\[
Q^2 \chi_3 = 2 \partial_\psi \chi_3, \quad Q^2 c = -i \partial_\psi c, \quad Q^2 \bar{c} = 2 \partial_\psi \bar{c},
\]

\[
Q^2 \chi_+ = 2(\partial_\psi - i)\chi_+, \quad Q^2 \chi_- = 2(\partial_\psi + i)\chi_. \tag{3.2}
\]

Again notice here the shifts in $\chi_\pm$.

It is then easy to see that, Fourier transforming every field $\Phi$ in $V_{loc}$ as $e^{i(q_L \psi + q_R \phi)}\Phi(\theta, r)$, the pairing of $X_0$ and $X_1$ involves only terms with net $q_R = 0$ and net $Q^2 = 0$ and the $\phi, \psi$ dependence drops out. In this way partial differential equations in four variables are converted into partial differential equations in terms of functions of two variables $(\theta, r)$. These functions are the Fourier coefficients depending on $SU(2)_L \times SU(2)_R$ charges $(q_L, q_R)$. Moreover in tangent space basis kernel and cokernel equations can be written in terms of $SU(2)_L$ generators with an inert $SU(2)_R$ charge $q_R$.

3.1. Kernel equations

The fact that kernel as well as the cokernel differential equations can be written in terms of $SU(2)_L$ generators $J_L$ only, implies in particular that $SU(2)_R$ commutes with the equations, and therefore the solutions will organize in $SU(2)_R$ multiplets of dimensions $2j_R + 1$, where the possible values of $j_R$ are $j \leq 1, j_L$, depending on the spherical harmonics involved, as it is detailed in the appendix \[. \] In more detail, if one introduces the expressions:

\[
J^- A_+ = e^{-iq_L \psi}e^{-iqr \phi}l^{-\mu}\partial_{\mu}(e^{i(q_L+1)\psi}e^{i(q_R)\phi} A_+(\theta, r)),
\]

\[
J^+ A_+ = e^{-iq_L \psi}e^{-iqr \phi}l^{+\mu}\partial_{\mu}(e^{i(q_L-1)\psi}e^{i(q_R)\phi} A_-(\theta, r)),
\]

\[
J^- A_3 = e^{-i(q_L-1)\psi}e^{-iqr \phi}l^{-\mu}\partial_{\mu}(e^{iql \psi}e^{i(q_R)\phi} A_3(\theta, r)),
\]

\[
J^+ A_3 = e^{-i(q_L+1)\psi}e^{-iqr \phi}l^{+\mu}\partial_{\mu}(e^{iql \psi}e^{i(q_R)\phi} A_3(\theta, r)),
\]

\[
J^- A_4 = e^{-i(q_L-1)\psi}e^{-iqr \phi}l^{-\mu}\partial_{\mu}(e^{iql \psi}e^{i(q_R)\phi} A_4(\theta, r)),
\]

\[
J^+ A_4 = e^{-i(q_L+1)\psi}e^{-iqr \phi}l^{+\mu}\partial_{\mu}(e^{iql \psi}e^{i(q_R)\phi} A_4(\theta, r)),
\]

\[
J^+ \Lambda = e^{-i(q_L+1)\psi}e^{-iqr \phi}l^{+\mu}\partial_{\mu}(e^{iql \psi}e^{i(q_R)\phi} \Lambda(\theta, r)). \tag{3.3}
\]
one can show that the kernel equations can be written in the following form:

\[ E_1 = \frac{i}{4} \tan(r) \sin(\theta) \left( (J^- A_3 + \frac{1}{4} \sin(2r) J^- A_4) + \frac{1}{2} \cos(r)(2q_L A_-(\theta, r) \cos(r), \right. \\
- \sin(r) \partial_r A_-(\theta, r) \right) \\
E_2 = \frac{i}{4} \tan(r) \sin(\theta) \left( (-J^+ A_3 + \frac{1}{4} \sin(2r) J^+ A_4) - \frac{1}{2} \cos(r)(2q_L A_+(\theta, r) \cos(r) + \\
x \sin(r) \partial_r A_+(\theta, r) \right), \\
E_3 = \frac{i}{8} \sec(r) \sin(\theta) \left( \sin(2r)(-J^- A_+ + J^+ A_-) - (2q_L A_4(\theta, r) \cos(r)^2 \sin(r)^2 \\
+ \tan(r)(A_3(\theta, r)(3 + \cos(2r)) + \sin(2r) \partial_r A_3(\theta, r))) \right), \\
E_4 = \frac{1}{8} \sin(r) \sin(\theta) \left( 2(J^- A_+ + J^+ A_-) - (4q_L A_3(\theta, r) + \sin(r)(3A_4(\theta, r) \cos(r) \\
+ \sin(r) \partial_r A_4(\theta, r)) \right), \\
E_5 = \frac{1}{2} \sin(\theta) \sin(r) \left( -4\partial_r^2 \Lambda(\theta, r) - \sin^2(r) \partial_r^2 \Lambda(\theta, r) - 4\cot(\theta) \partial_r \Lambda(\theta, r) \\
- (3\sin(r) \cos(r) \partial_r \Lambda(\theta, r) + 4\csc^2(\theta) \Lambda(\theta, r) \left( q_L^2 - 2q_L q_R \cos(\theta) + q_R^2 \right)) \right). \tag{3.4} \\
\]

Where, in the last equation we have traded \( \phi_2 \) with \( \Lambda \), for fluctuating part of \( \phi_1 = 0 \), in order to simplify the equation. Note that \( E_5 = 0 \) can be written as Laplacian acting of \( \Lambda \)

\[ \nabla^2 \Lambda = 0. \tag{3.5} \]

This equation has no smooth solution on \( S^4 \) apart from the constant and since we will be considering here in \( j_L > 0 \), we set it to 0. Whereas on hemisphere \( HS^4 \) with boundary there is constant function as solution for the special value \( j_L = 0 \). But if we impose supersymmetric boundary conditions this constant solution must be set to zero. This is discussed briefly in appendix. Therefore \( E_5 \) drops from our further discussions.

The coefficient \( \tan(r) \) or \( \sec(r) \) in kernel equations \( E_1, E_2 \) may appear problematic because it blows up at \( r = \frac{\pi}{2} \). However it is only an artifact of the redefinition of \( \phi_1(\theta, r) \) in favor of \( \Lambda(\theta, r) \) to simplify the form of differential equations. On the other hand if redefine \( A_3(\theta, r) \) instead of \( \phi_1(\theta, r) \)

\[ A_3(\theta, r) = -\frac{1}{2}(2i\phi_1(\theta, r) + 2\cos(r)\phi_2(\theta, r) + \Lambda(\theta, r)) \tag{3.6} \]

we will get kernel equations which will be well defined at \( r = \frac{\pi}{2} \). The only change in the kernel equations will be that \( A_3(\theta, r) \) is replaced by \( \phi_1(\theta, r) \) but the rest of the analysis will remain the same resulting in the same eigenvalue spectrum and degeneracies. A similar argument holds for the cokernel equations.

The following modes carry the same value of \( Q^2 \), taking into account the shifts in \( A_3 \):

\[ A_+(\theta, r) = Y^{(j_L, q_L + 1, q_R)}(\theta) A_+^{(j_L, q_L + 1, q_R)}(r), \quad A_-(\theta, r) = Y^{(j_L, q_L - 1, q_R)}(\theta) A_-^{(j_L, q_L - 1, q_R)}(r), \]

\[ A_3(\theta, r) = Y^{(j_L, q_L, q_R)}(\theta) A_3^{(j_L, q_L, q_R)}(r), \quad A_4(\theta, r) = Y^{(j_L, q_L, q_R)}(\theta) A_4^{(j_L, q_L, q_R)}(r). \tag{3.7} \]
where the $Y$ functions are the $\theta$ dependent part of the scalar spherical harmonics with the indicated quantum numbers. Now we analyze the kernel equations separately for all the possible values of $q_L$ for which we may get a non-trivial solution.

i) $q_L = j_L + 1, -j_L - 1$

The kernel equations evaluate to

$$
\begin{align*}
\mathcal{E}_1 &= 0, \quad \mathcal{E}_2 = 0, \quad \mathcal{E}_3 = 0, \\
\mathcal{E}_4 &= -i \frac{6}{8} \sin(r) \sin(\theta) Y_j^{(j_L, -j_L, -q_R)}(\theta)(2(1 + j_L) \cos(r) A_{+}^{(j_L, -j_L, -q_R)}(r) \\
&\quad +\sin(r) \partial_r A_{+}^{(j_L, -j_L, -q_R)}(r)).
\end{align*}
$$

Solving the differential equation $\mathcal{E}_4 = 0$ we get the solution

$$
A_{+}^{(j_L, -j_L, -q_R)}(r) = A_0^0 \sin(r)^{-2(1 + j_L)}. \tag{3.8}
$$

This is clearly a singular solution at the two poles of round $S^4$. So we have to set $A_+^0 = 0$. If there is a physical boundary at $r = \frac{\pi}{2}$ the result does not change.

For $q_L = -j_L - 1$:

$$
\begin{align*}
\mathcal{E}_1 &= 0, \quad \mathcal{E}_2 = 0, \quad \mathcal{E}_3 = 0, \\
\mathcal{E}_4 &= -i \frac{6}{8} \sin(r) \sin(\theta) Y_{j}^{(j_L, J_L, -q_R)}(\theta)(2(1 + j_L) \cos(r) A_{+}^{(j_L, J_L, -q_R)}(r) \\
&\quad +\sin(r) \partial_r A_{+}^{(j_L, J_L, -q_R)}(r)).
\end{align*}
$$

Here $\mathcal{E}_3$ can be solved to give

$$
A_{+}^{(j_L, J_L, -q_R)}(r) = A_0^0 \sin(r)^{-2(1 + j_L)}. \tag{3.11}
$$

which is again a singular solution and $A_+^0 = 0$.

ii) $q_L = j_L - j_L$

First for $q_L = j_L$ solving $\mathcal{E}_1 = 0$ gives the solution

$$
A_{+}^{(j_L, 1 - j_L, -q_R)}(r) = -\frac{1}{2} \cos(r)(\sec(r) A_{+}^{(j_L, 1 - j_L, -q_R)}(r) + 2(j_L \sin(r) A_{+}^{(j_L, 1 - j_L, -q_R)}(r) \\
&\quad + \sec(r) \tan(r) \partial_r A_{+}^{(j_L, 1 - j_L, -q_R)}(r)).
$$

for $\mathcal{E}_2 = 0$

$$
A_{+}^{(j_L, -j_L, -q_R)}(r) = A_0^0 \cos(r)^{\frac{3}{2} + j_L} \sin(r)^{-(\frac{3}{2} + j_L)} \sin(2r)^{-(j_L - \frac{1}{2})} - \frac{1}{2} \cos(r) \sin(r) A_{+}^{(j_L, j_L, -q_R)}(r). \tag{3.13}
$$

The last equation is singular at the North or South Poles $r = 0, \pi$ which implies that $A_0^0 = 0$. So

$$
A_{+}^{(j_L, -j_L, -q_R)}(r) = -\frac{1}{2} \cos(r) \sin(r) A_{+}^{(j_L, -j_L, -q_R)}(r). \tag{3.14}
$$

Next $\mathcal{E}_3 = 0$ and $\mathcal{E}_4$ can be solved to give

$$
A_{+}^{(j_L, -j_L, -q_R)}(r) = (a_0^0 \cos(r) - b_0^0 \text{F}_1 \left(-\frac{1}{2}, -2j_L; \frac{1}{2}; \cos^2(r)\right) ) \sin(r)^{-3 - 2j_L}. \tag{3.15}
$$
$a_4^0$ comes with $\cos(r)$ while $b_4^0$ is multiplied by a polynomial in $\cos(r)^2$, this means that the absence of singularity at $r = 0$ or $r = \pi$ implies that $a_4^0 = 0, b_4^0 = 0$.

Similarly for $q_L = -j_L$ we get a singular solutions. So far there is no Kernel, now we go to $|q_L| < j_L$ case.

For the isolated case of $q_L = 0$, solving the kernel equations we get the result that the solution set is empty. The details are given in appendix H. Hence the conclusion is that Kernel of $D_{10}^{vec}$ is empty.

3.2. Cokernel equations

The Fourier series expansion of fields contributing to the cokernel is given in eq. (3.19), except the following field redefinition

\[
\tilde{c}(\theta, r) = c_{a_1}(\theta, r) + 2i q_L c(\theta, r),
\]

with $c_{a_1}(\theta, r)$ an auxiliary variable. Now we show that all these equations can be expressed in terms of $SU(2)_L$ generators.

\[
\begin{align*}
J^+ \chi_+ &= e^{-i q_L \psi} \psi e^{-i q_R \phi} l^\mu \partial_\mu (e^{i(q_L+1)\psi} \psi), \\
J^- \chi_- &= e^{-i q_L \psi} \psi e^{-i q_R \phi} l^\mu \partial_\mu (e^{i(q_L-1)\psi} \psi), \\
J^- \chi_3 &= e^{-i(q_L-1)\psi} \psi e^{-i q_R \phi} l^\mu \partial_\mu (e^{i(q_L)\psi} \psi), \\
J^+ \chi_3 &= e^{-i(q_L+1)\psi} \psi e^{-i q_R \phi} l^\mu \partial_\mu (e^{i(q_L)\psi} \psi), \\
J^- c_{a_1} &= e^{-i(q_L-1)\psi} \psi e^{-i q_R \phi} l^\mu \partial_\mu (e^{i(q_L)\psi} \psi), \\
J^+ c_{a_1} &= e^{-i(q_L+1)\psi} \psi e^{-i q_R \phi} l^\mu \partial_\mu (e^{i(q_L)\psi} \psi), \\
J^+ J^- c &= e^{-i(q_L)\psi} \psi e^{-i q_R \phi} l^\mu D_\nu (l^{+\nu} D_\mu (e^{i(q_L)\psi} \psi)), \\
J^- J^+ c &= e^{-i(q_L)\psi} \psi e^{-i q_R \phi} l^\mu D_\nu (l^{-\nu} D_\mu (e^{i(q_L)\psi} \psi)), \\
J^3 J^3 c &= q_L^2 c(\theta, r).
\end{align*}
\]
Next the Cokernel equations can be written in terms of these generators as

\[ CE_1 = -\frac{1}{8} \sin(r) \sin(\theta) (-2J^- (c_{a_1}) + 2iJ^- (\chi_3) + i(-2(-1 + qL) \cos(r) \chi_-(\theta, r)) + \sin(r) \partial_r \chi_-(\theta, r)), \]

\[ CE_2 = -\frac{1}{8} \sin(r) \sin(\theta) (-2J^+ (c_{a_1}) - 2iJ^+ (\chi_3) + i(2(1 + qL) \cos(r) \chi_+(\theta, r)) + \sin(r) \partial_r \chi_+(\theta, r)), \]

\[ CE_3 = -\frac{1}{4} \tan(r) \sin(\theta) (-iJ^- (\chi_+) + iJ^+ (\chi_-) - (2qL \cos(r) c_{a_1} (\theta, r) - i \sin(r) \partial_r \chi_3 (\theta, r)), \]

\[ CE_4 = -\frac{1}{8} \sin(r)^2 \sin(\theta) (-iJ^- (\chi_+) - iJ^+ (\chi_-) + (-2iqL \cos(r) \chi_3 (\theta, r) + \sin(r) \partial_r c (\theta, r)), \]

\[ CE_5 = -\frac{1}{8} \sin(r)^3 \sin(\theta) \left( \frac{4}{\sin(r)^2} \left( \frac{1}{2} (J^+ J^- - J^- J^+) \right) + J^3 J^3 \right) + (2iqL c_{a_1} (\theta, r) + 3 \chi_3 (\theta, r) - 3 \cot(r) \partial_c (\theta, r) - \partial_r^2 c (\theta, r)). \]  

(3.18)

We can express the fields in terms of scalar harmonics \( i.e. j_R = j_L \) whose \( \psi \) and \( \phi \) coordinates dependence has already been extracted above. Similar to the case of kernel equations,

\[ \chi_+ (\theta, r) = Y^{(jL, qL-1, qR)}(\theta) \chi_+^{(jL, qL-1, qR)}(r), \quad \chi_- (\theta, r) = Y^{(jL, qL-1, qR)}(\theta) \chi_-^{(jL, qL-1, qR)}(r), \]

\[ \chi_3 (\theta, r) = Y^{(jL, qL, qR)}(\theta) \chi_3^{(jL, qL, qR)}(r), \quad c_{a_1} (\theta, r) = Y^{(jL, qL, qR)}(\theta) c_{a_1}^{(jL, qL, qR)}(r), \]

\[ c (\theta, r) = Y^{(jL, qL, qR)}(\theta) c^{(jL, qL, qR)}(r). \]  

(3.19)

using the inventory of various identities given in appendix G. Let’s begin the analysis.

\( \alpha \) \( qL = j_L + 1 \)

In this case \( CE_1 = 0, CE_2 = 0, CE_4 = 0, CE_5 = 0 \) give empty solution set and \( CE_3 = 0 \) can be solved to give the following solution for \( \chi_-^{(jL, qL, qR)}(r) \)

\[ \chi_-^{(jL, qL, qR)}(r) = \chi_-^0 \sin(r)^{2j_L}. \]  

(3.20)

Since the \( \hat{Q}^2 \) eigenvalue on \( \chi_-^{(jL, qL, qR)}(r) \) is \( n = 2(j_L + 1) \), the multiplicity of this solution is \( 2j_L + 1 = n - 1 \).

Similarly for \( qL = -j_L - 1 \) only \( \chi_+ \) survives and is given as

\[ \chi_+^{(jL, -jL, qR)}(r) = \chi_+^0 \sin(r)^{2j_L}, \]  

(3.21)

With \( \hat{Q}^2 \) eigenvalue on \( \chi_+^{(jL, -jL, qR)}(r) \) as \( n = -2j_L - 2 \), the multiplicity of this solution is \( 2j_L + 1 = |n| - 1 \).

\( \beta \) \( qL = j_L \)

For this value of \( qL, CE_1 = 0 \) can be solved to give

\[ \chi_-^{(jL, -jL, qR)}(r) = -i \cos(r) c_{a_1}^{(jL, jL, qR)}(r) - \frac{\sin(r) \partial_r \chi_3^{(jL, jL, qR)}(r)}{2j_L}. \]  

(3.22)
Using this value of $\chi_{-}^{(j_{L},-1+j_{L},q_{R})}(r)$, $CE_{2} = 0$ yields

$$
\chi_{3}^{(j_{L},j_{L},q_{R})(r)} = \frac{i}{2} \sin(r) + c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r).
$$

(3.23)

By considering the regularity of the solution at $r = 0$ and $r = \pi$, solving $CE_{3} = 0$ yields for $c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)$

$$
c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)} = \frac{i}{2} \sin(r) + c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r),
$$

(3.24)

where $c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)$ is a new function to be determined. Plugging this solution into $CE_{3}$, it is converted to a differential equation for $c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)$.

$$
\frac{1}{\sin(r)} \sin(r) \sin(\theta) Y_{r}^{(j_{L},-1+j_{L},q_{R})(\theta)}(jL(7 + 2jL + (-3 + 2jL) \cos(2r))
$$

$$
\times c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r) + \sin(r) \sin(r) \sin(\theta) R \partial_{r} c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r) + \sin(r) \partial_{r} c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)) = 0.
$$

(3.25)

We multiply this equation by $c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)$ and integrate over $r$ from 0 to $\pi$, if there is smooth solution the result must be zero. On the other hand by partial integrating the term containing $\partial_{r} c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)$ we get

$$
\sin(r)(2jL((2jL - 3) \cos^{2}(r) + 5)c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)^{2} + \sin(r) \partial_{r} c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)^{2}) = 0.
$$

(3.26)

Note that $2jL((2jL - 3) \cos^{2}(r) + 5)$ is always positive for all $j_{L}$ and all $r$, and $\sin(r)$ is positive, so for this integral to be zero, $c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)$ must be zero.

$$
c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r) = 0.
$$

(3.27)

$C\mathcal{E}_{4}$ is just the complex conjugate of $C\mathcal{E}_{3}$, so the same analysis goes through. The $C\mathcal{E}_{5}$ is the conjugate equation to the kernel equation for $\phi_{\theta}$ or $\phi_{p}$ and is given for $q_{L} = j_{L}$ by

$$
C\mathcal{E}_{5} = \frac{1}{10} \sin(r) \sin(\theta) Y_{r}^{(j_{L},j_{L},q_{R})(\theta)}(8jL(1 + jL)c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r) + \sin(r)
$$

$$
\times (-6 \cos(r) \partial_{r} c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r) + \sin(r)((3 + 2jL) \chi_{3}^{0} \sin(2jL) - 2\partial_{r} c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)}(r)).
$$

(3.28)

If the homogeneous piece is zero then this is just the Laplacian, which has not smooth solution on $S^{4}$. So it is sufficient to construct one smooth solution of the inhomogeneous equation and that will be the unique solution. It is given by

$$
c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)} = -\frac{1}{4jL} \chi_{3}^{0} \sin(r) 2jL.
$$

(3.29)

Summarizing the solution set for $q_{L} = j_{L}$ is

$$
\chi_{-}^{(j_{L},j_{L},q_{R})(r)} = 0, \quad \chi_{3}^{0} \sin(r) 2jL,
$$

$$
c_{a_{2}}^{(j_{L},j_{L},q_{R})(r)} = -\frac{1}{4jL} \chi_{3}^{0} \sin(r) 2jL.
$$

(3.30)
For $q_L = -j_L$

We get only one new solution from this

$$\chi_{(j_L, -j_L - 1, t)}(r) = 0. \quad (3.31)$$

In this case the $Q^2$ eigenvalue $= n = 2j_L$ and the multiplicity is $|n| + 1$.

Combining the results for $q_L = \pm (j_L + 1)$ and $q_L = \pm (j_L)$ the total multiplicity for $\hat{Q}^2$ is $(|n| - 1) + (|n| + 1) = 2|n|$.

For $q_L = 0$ there is no solution of the cokernel equations. Next we consider the case when $q_L \neq 0$ by following the same argument as given in \[11\] one can show that solution set is empty for this range of $q_L$. For round $S^4$ solution set of only cokernel differential equations is non-empty.

Eq.(3.30) shows that for $q_L = \pm j_L$ the solution set for $\chi_3, c, \bar{c}$ depends on a single constant parameter and we will count it only once. For our choice of normalization the eigenvalue of $Q^2 = n = \pm 2j_L$, with multiplicity $2j_L + 1 = |n| + 1$. Therefore it will contribute a factor $(n + i\alpha.a)^{|n|+1}$.

Similarly from eqs.(3.20),(3.21), for $q_L = \pm (j_L + 1)$ we have eigenvalue $Q^2 = n = \pm 2(j_L + 1)$, with multiplicity $2j_L - 1 = |n| - 1$ and the corresponding contribution $(n + i\alpha.a)^{|n|+1-1}$. Using this data the one loop determinant for round $S^4$ can be immediately written down

$$Z_{\text{vec-1-loop}}^{\text{round}S^4} = \prod_{\alpha \in \Delta} \prod_{n \neq 0} (n + i\alpha.a)^{|n|+1} (n + i\alpha.a)^{|n|-1}$$

$$= \prod_{\alpha \in \Delta} \prod_{n \neq 0} (n + i\alpha.a)^{2|n|}. \quad (3.32)$$

where $\Delta$ is the set of roots of $G$, which matches with Pestun’s result \[14\].

4. Hypermultiplet contribution

For matter multiplet the fields in the kernel and cokernel of $D_{10}^{\text{hyper}}$ in cohomological form are

$$q_{IA} = \begin{pmatrix} q_{11}(\psi, \theta, \phi, r) \\ q_{21}(\psi, \theta, \phi, r) \\ q_{12}(\psi, \theta, \phi, r) \\ q_{22}(\psi, \theta, \phi, r) \end{pmatrix}, \Sigma_{IA} = \begin{pmatrix} \Sigma_{11}(\psi, \theta, \phi, r) \\ \Sigma_{21}(\psi, \theta, \phi, r) \\ \Sigma_{12}(\psi, \theta, \phi, r) \\ \Sigma_{22}(\psi, \theta, \phi, r) \end{pmatrix}. \quad (4.1)$$

The $\hat{Q}^2$ action is given by:

$$\hat{Q}^2 q = e^{-i\alpha.a/2} \partial_{\psi}(e^{i\alpha.a/2} q),$$

$$\hat{Q}^2 \Sigma = e^{-i\alpha.a/2} \partial_{\psi}(e^{i\alpha.a/2} \Sigma). \quad (4.2)$$

Notice here too the shift due to the R-charge. The relevant kernel and cokernel equations are obtained by varying the localizing fermionic field of eq. (2.15) with respect to $\Sigma$'s and $q$'s respectively[^4]

[^4]: Explicit expression for $V^{\text{hyper}}$ is given in appendix \[16\]

16
4.1. Analysis of kernel and cokernel equations

Since the set of fields with flavor index \( I = 2 \) form a copy of that of \( I = 1 \) we will discuss the kernel and co-kernel equations for \( I = 1 \) only. After Fourier transforming in coordinates \( \phi \) and \( \psi \) as follows:

\[
q_{11}(\psi, \theta, \phi, r) = q_{11}(\theta, r) e^{-i(q_L + \frac{1}{2})\psi - iq_R\phi}, \quad q_{21}(\psi, \theta, \phi, r) = q_{21}(\theta, r) e^{-i(q_L - \frac{1}{2})\psi - iq_R\phi}.
\]

the kernel equations become:

\[
\begin{align*}
\frac{1}{32} \sin^2(r) (q_{11}(\theta, r) (\cos(\theta) - 2q_L \cos(\theta) + 2q_R) + i \sin(\theta) (-2i\partial_\theta q_{11}(\theta, r) - \sin(r) \partial_r q_{21}(\theta, r))) \\
\quad + \left(2(q_L - 1) \cos(r) q_{21}(\theta, r)\right) = 0,
\end{align*}
\]

and the cokernel are:

\[
\begin{align*}
\frac{1}{32} \sin^2(r) (q_{11}(\theta, r) (\cos(\theta) - 2q_L \cos(\theta) + 2q_R) + i \sin(\theta) (2\partial_\theta q_{11}(\theta, r) - i \sin(r) \partial_r q_{21}(\theta, r))) \\
\quad + 2i((q_L - 1) \cos(r) q_{21}(\theta, r)) = 0,
\end{align*}
\]

As in the vector multiplet case, the isometry group \( SO(4) \simeq SU(2)_L \times SU(2)_R \) of foliated \( S^3 \)s plays an important role in solving these equations. It turns out that kernel and cokernel equations can be written in terms of generators of \( SU(2)_L \), whereas the \( SU(2)_R \) remains a spectator. For this reason the degeneracy of the solutions to these equations is determined by \( q_R \) quantum number.

To convert these partial differential equations into ordinary ones in the variable \( r \) we further expand the fields in terms of spherical harmonics:

\[
\begin{align*}
q_{11}(\theta, r) &= q_{11}(r) (j_L \cdot q_L - \frac{1}{2} \cdot q_R) (r) Y(j_L \cdot q_L - \frac{1}{2} \cdot q_R)(\theta), \\
q_{21}(\theta, r) &= q_{21}(r) (j_L \cdot q_L - \frac{1}{2} \cdot q_R) (r) Y(j_L \cdot q_L + \frac{1}{2} \cdot q_R)(\theta),
\end{align*}
\]

\[
\begin{align*}
\Sigma_{11}(\theta, r) &= \Sigma_{11}(r) (j_L \cdot q_L - \frac{1}{2} \cdot q_R) (r) Y(j_L \cdot q_L - \frac{1}{2} \cdot q_R)(\theta), \\
\Sigma_{21}(\theta, r) &= \Sigma_{21}(r) (j_L \cdot q_L - \frac{1}{2} \cdot q_R) (r) Y(j_L \cdot q_L + \frac{1}{2} \cdot q_R)(\theta).
\end{align*}
\]

and get the solutions which we summarize below.

For kernel equations, solutions, which are regular at the North or South poles of \( S^4 \), exists only
for \( q_L = \pm (j_L + \frac{1}{2}) \).

For \( q_L = j_L + \frac{1}{2} \)

\[
q_{21}^{(j_L,j_L-q_R)}(r) = C_2 \sin^{2j_L}(r), \tag{4.8}
\]

with eigenvalue for the \( \hat{Q}^2 \) action equal to \((2j_L + 1)\).

For \( q_L = -(j_L + \frac{1}{2}) \)

\[
q_{21}^{(j_L,j_L-q_R)}(r) = C_3 \sin^{2j_L}(r), \tag{4.9}
\]

with eigenvalue for the \( \hat{Q}^2 \) action equal to \(-(2j_L + 1)\) for constants \( C_1, C_2, C_3, C_4 \).

Analysis of kernel and cokernel equations with physical boundary at \( r = \frac{\pi}{2} \), with regularity at one of the poles, and following the logic of appendix [H] again shows that only the solution set of kernel is non-empty.

For instance for \( q_L = (j_L + \frac{1}{2}) \):

\[
\Sigma^{(j_L,j_L,q_R)}_{11}(r) = s_{11} \sin^{-2j_L-3}(r), \tag{4.10}
\]

while for \( q_L = -(j_L + \frac{1}{2}) \):

\[
\Sigma^{(j_L,j_L,q_R)}_{21}(r) = s_{21} \sin(r)^{-2j_L-3}(r), \tag{4.11}
\]

which are not regular at \( r = 0 \) or \( \pi \).

For \( q_L = (j_L - \frac{1}{2}) \): we get identical set of ordinary differential equations for two fields

\[
\left((2j_L^2 - j_L - 3) \cos(2r) + 2j_L^2 - 3j_L + 4\right) \Sigma^{(j_L,j_L-q_R)}_{21}(r) - \sin(r) \left(5 \cos(r) \Sigma^{(j_L,j_L,q_R)}_{21}'(r) - \sin(r) \Sigma^{(j_L,j_L,q_R)}_{21}''(r)\right) = 0. \tag{4.12}
\]

with the solution

\[
\Sigma^{(j_L,j_L,q_R)}_{21}(r) = \frac{c_1 P_{2(j_L-1)}^{\sqrt{4j_L^2-4j_L+5}}(\cos(r))}{\cos^2(r) - 1} + \frac{c_2 Q_{2(j_L-1)}^{\sqrt{4j_L^2-4j_L+5}}(\cos(r))}{\cos^2(r) - 1}. \tag{4.13}
\]

For \( q_L = -(j_L - \frac{1}{2}) \):

\[
\left((2j_L^2 - 3j_L - 2) \cos(2r) + 2j_L^2 - j_L + 3\right) \Sigma^{(j_L,j_L-q_R)}_{21}(r) - \sin(r) \left(5 \cos(r) \Sigma^{(j_L,j_L,q_R)}_{21}'(r) - \sin(r) \Sigma^{(j_L,j_L,q_R)}_{21}''(r)\right) = 0. \tag{4.14}
\]

with the solution

\[
\Sigma^{(j_L,j_L-q_R)}_{21}(r) = \frac{c_1 P_{2(j_L-1)}^{\sqrt{4j_L^2-4j_L+5}}(\cos(r))}{\cos^2(r) - 1} + \frac{c_2 Q_{2(j_L-1)}^{\sqrt{4j_L^2-4j_L+5}}(\cos(r))}{\cos^2(r) - 1}. \tag{4.15}
\]

where \( P \) and \( Q \) are Legendre functions. These solutions are not regular at \( r = 0 \) or \( \pi \) for the case of round \( S^4 \) or at one pole and the equator \( r = \frac{\pi}{2} \) in case of half-\( S^4 \). Therefore the solution set is empty for cokernel equations.
5. Wave function on hemisphere $HS^4$

As is clear from the above analysis that for the round $S^4$ solution set of kernel equations is empty and that for the cokernel equations is nonempty. In order for the boundary to preserve supersymmetry, the component of the supercurrent normal to the boundary must vanish. Also the boundary conditions must be consistent with the localization locus given in eqs. (2.10), (2.13). If we consider hemisphere $HS^4$ with supersymmetric BCs at $r = \frac{\pi}{2}$, the analysis of kernel and cokernel equations remains identical except that one has to take account of possible boundary contributions. Also spectrum remains same with the result that kernel of $D^{vec}_{10}$ is empty and cokernel solutions set is non-trivial with the same eigenvalues and multiplicities if one imposes supersymmetric boundary conditions. Thus practically the only change is to take the range of coordinate $r$ to be $0 \leq r \leq \frac{\pi}{2}$.

5.1. Supersymmetric boundary conditions

Vector multiplet

First we recall that for manifolds with boundary e.g. $HS^4$ the supersymmetric variation of physical action $\hat{Q}S$ vanishes up to total derivative terms. These total derivative terms break supersymmetry at the boundary unless one adds extra terms to Lagrangian such that the $\hat{Q}$ variation of the modified action vanishes. The other way to get rid of the boundary terms is to impose supersymmetric boundary conditions on all the fields. For the vector multiplet the boundary contribution is

\[
\hat{Q}^{vec}_{\text{Boundary}} = -\frac{1}{8} \sin(\theta) \sin^2(r) \left( c(\theta, r) \left( -6 \cos(r) \Lambda(\theta, r) + \sin(r) \partial_r \Lambda(\theta, r) \right) \right) \\
- \left( \left( 2i q L \Lambda_r(\theta, r) \right) + \sin(r) \Lambda_r(\theta, r) c(\theta, r) + i \Lambda_{-}(\theta, r) \chi_{+}(\theta, r) + i \Lambda_{+}(\theta, r) \chi_{-}(\theta, r) \right) \\
- \left( \sin(r) \partial_r c(\theta, r) \Lambda(\theta, r) + 2 \chi_{3}(\theta, r) \phi_{2}(\theta, r) + \cos(r) \chi_{3}(\theta, r) \Lambda(\theta, r) \right) .
\]

(5.1)

where $\Lambda$ is the gauge parameter which takes the scalar zero mode as its value at the localization locus. For supersymmetry consistent BCs the boundary contribution vanishes. Dirichlet type boundary conditions correspond to choosing

\[
\chi_{3}(\theta, \frac{\pi}{2}) = 0, \quad c(\theta, \frac{\pi}{2}) = 0, \quad \bar{c}(\theta, \frac{\pi}{2}) = 0,
\]

(5.2)

while keeping $\chi_{+}, \chi_{-}$ arbitrary. Whereas for Neumann type boundary conditions:

\[
\chi_{+}(\theta, \frac{\pi}{2}) = 0, \quad \chi_{-}(\theta, \frac{\pi}{2}) = 0,
\]

(5.3)

keeping $\chi_{3}, c, \bar{c}$ arbitrary at the boundary. This can be understood in the following way: at the boundary $r = \frac{\pi}{2}$ the Killing spinor satisfies

\[
i \tau^{3} \xi_{A}|_{\frac{\pi}{2}} = \xi_{A}|_{\frac{\pi}{2}},
\]

(5.4)

which motivates choosing the following BC’s on the gaugino

\[
i \tau^{3} \lambda_{A}|_{\frac{\pi}{2}} = \pm \lambda_{A}|_{\frac{\pi}{2}},
\]

(5.5)

The above conditions on $\chi$ follow from its definition in terms of $\lambda$. Moreover for the consistency of supersymmetry

\[
i \tau^{3} \hat{Q} \lambda_{A}|_{\frac{\pi}{2}} = \pm \hat{Q} \lambda_{A}|_{\frac{\pi}{2}},
\]

(5.6)
it follows that
\[ F_{\mu\nu}(\theta, \pi/2) = 0 \quad \mu, \nu = \psi, \theta, \phi, \quad \Lambda(\theta, \pi/2) = a, \quad \partial_r \phi_2(\theta, \pi/2) = 0, \] (5.7)
for the lower sign. On the other hand, for the upper sign choice,
\[ i\tau^3 \lambda_A|_{\pi/2} = \bar{\lambda}_A|_{\pi/2}, \] (5.8)
we get:
\[ F_{\mu\nu}(\theta, \pi/2) = 0 \quad \mu, \nu = \psi, \theta, \phi, \quad \phi_2(\theta, \pi/2) = a, \quad \partial_r \Lambda(\theta, \pi/2) = 0. \] (5.9)
So, for lower sign choice we get Dirichlet and for the upper sign we get Neumann.

If we act once more with \( \hat{Q} \) they are closed and trivially satisfied. For example acting on eq. (5.6),
\[ i\tau^3 \hat{Q}^2 \lambda_A|_{\pi/2} = \pm \hat{Q}^2 \bar{\lambda}_A|_{\pi/2}, \]
\[ i\tau^3 \partial_\psi \lambda_A|_{\pi/2} = \pm \partial_\psi \bar{\lambda}_A|_{\pi/2}. \] (5.10)
The second line holds up to a constant gauge transformation \( a \). We see that it is trivially satisfied for Dirichlet BCs. Similar is the case for Neumann BCs. Therefore these boundary conditions are closed under the action of supersymmetry and hence consistent with it. However there is one subtle point about the SUSY closure of BCs. Since we are working in Euclidean signature and the fields entering the Lagrangian are analytically continued for Lorentzian signature, the BCs imposed on fields are closed under SUSY only if we take the fields as complex valued functions. If one tries to impose BCs on real and imaginary parts of various fields separately, it turns out that they are not closed under SUSY and will generate infinite number of differential constraints on gauge field and gaugino at the boundary. In other words the BC in eq. (5.6) is written in covariant form and due to this reason it is closed under SUSY trivially as shown in eq. (5.10). On the other hand if we do not work covariantly and instead consider the action of \( Q \) on \( \partial_r \phi_2(\theta, \pi/2) = 0 \) it is easy to see that
\[ \partial_r Q \phi_2(\theta, \pi/2) \neq 0, \] (5.11)
unless we impose extra BC on \( \lambda_A \)
\[ i\tau^3 \partial_\psi \lambda_A|_{\pi/2} = \mp \partial_\psi \bar{\lambda}_A|_{\pi/2}. \] (5.12)
Note the important inversion of sign from \( \pm \) to \( \mp \). Now this BC should itself be closed under SUSY. But it is easy to convince oneself that due to the inversion of sign \( \mp \) at each step if we act once more with \( Q \) it will generate BC other than already imposed and in fact one has to impose infinite number of boundary conditions.

However it turns out that our solution set of kernel and co-kernel equations satisfy BCs irrespective of whether we impose them covariantly or and separately in terms of real and imaginary parts of individual fields.

Second constraint that the BC conditions have to satisfy is the action principle. Taking arbitrary variations \( \delta \) of the fields in the the action, for round metric, to get the equations of motion, we obtain boundary contributions coming from integration by parts. Keeping in mind that the
boundary conditions have to be consistent with saddle point solutions and that the later break the
gauge symmetry $G$ to its maximal torus, some non abelian terms drop out and we get the following
\[
\delta \mathcal{L} = 2i\lambda^A \sigma^\tau \delta \lambda_A + 2F^\tau\mu \delta A_\mu - 4\delta \bar{\phi} \partial_r \phi - 4\delta \phi \partial_r \bar{\phi}.
\]
(5.13)
With the set of BCs (5.4),(5.5),(5.6), the boundary term from the action principle vanishes. $\delta$ is
an arbitrary variation in the sense that it may represent supersymmetry variation $Q$ too. SUSY
variation of the $\hat{Q}$ produces total derivative in the $\psi, \theta$, or $\phi$ direction and not in $r$ direction.
Hence there is no non-trivial contribution from here.

Hyper multiplet
In the case of hyper multiplet for the boundary conditions to preserve $N = 2$ SUSY in $3 - d$ at
$r = \frac{\pi}{2}$, one has to impose complementary BC’s on scalars with different $R$-charges .i.e. Dirichlet
BC’s on the scalars $q_{11}, q_{21}$ and Neumann BC’s on $q_{21}, q_{22}$ or vice versa. Only this choice of BC’s
satisfy the constraints coming from the vanishing of supercurrent normal to the boundary $\{ \underline{6}, \underline{\bar{6}} \}$.
In either case we get $\left( \frac{(2n+1)!(2n-1)!}{2} \right) = 2n$ multiplicity of the $\hat{Q}^2$ eigenstates with eigenvalue $n + i a, \rho$, with $\rho$ the weight vector of the complex conjugate representation $R$ of the Gauge group $G$. The one
loop factor for hypermultiplet on hemisphere with Dirichlet or Neumann boundary conditions at
the equator can immediately be written down
\[
Z_{HS}^{H^4} = \left( \prod_{\rho \in \text{weights}} \frac{1}{H(i a, \rho)} \right)^{\frac{1}{2}}.
\]
(5.14)
When one reads off the cokernel equations from $\hat{Q}_V$ matter, an integration by parts is done, which
in the case of a manifold with boundary produces following boundary terms
\[
\hat{Q}^\text{Boundary}_{\text{matter}} = \left( \frac{1}{32} i \sin(\theta) \sin^3(r)(-q_{11}(\theta, r)\Sigma_{22}(\theta, r) + q_{12}(\theta, r)\Sigma_{21}(\theta, r) - q_{21}(\theta, r)\Sigma_{12}(\theta, r)
+ q_{22}(\theta, r)\Sigma_{11}(\theta, r))) \right|_{r = \frac{\pi}{2}}.
\]
(5.15)
The discussion of $F_A$ auxiliary field is not important in boundary conditions and we will no more
discuss it. There are two choices for the boundary conditions for which
\[
\hat{Q}^\text{Boundary}_{\text{matter}} = 0.
\]
(5.16)
There are two choices:
1. $\psi_\alpha I|_{r = \frac{\pi}{2}} = +i \tilde{\psi}_\alpha J r^3 J|_{r = \frac{\pi}{2}}$
If we act with supersymmetry $Q$ on this BC,
\[
Q\psi_\alpha I|_{r = \frac{\pi}{2}} = -i Q\tilde{\psi}_\alpha J r^3 J|_{r = \frac{\pi}{2}},
\]
(5.17)
it will be closed if we choose Dirichlet BC’s on the following fields
\[
q_{11}(\theta, \frac{\pi}{2}) = 0, \quad q_{21}(\theta, \frac{\pi}{2}) = 0, \quad \Sigma_{11}(\theta, \frac{\pi}{2}) = 0 \quad \Sigma_{22}(\theta, \frac{\pi}{2}) = 0,
\]
\[
\partial_\theta q_{12}(\theta, \frac{\pi}{2}) = 0, \quad \partial_\theta q_{21}(\theta, \frac{\pi}{2}) = 0, \quad \partial_\theta \Sigma_{11}(\theta, \frac{\pi}{2}) = 0, \quad \partial_\theta \Sigma_{22}(\theta, \frac{\pi}{2}) = 0.
\]
(5.18)
and Neumann BCs on the following
\[
\partial_r q_{11}(\theta, \frac{\pi}{2}) = 0, \quad \partial_r q_{22}(\theta, \frac{\pi}{2}), \quad \partial_r \Sigma_{12}(\theta, \frac{\pi}{2}) = 0, \quad \partial_r \Sigma_{21}(\theta, \frac{\pi}{2}) = 0.
\] (5.19)

Acting once more with a supersymmetry operator \(Q\) we get
\[
Q^2 \psi_{\alpha I}|_{r=\pi/2} = -iQ^2 \bar{\psi}_{\alpha J}\tau^3_J|_{r=\pi/2}.
\] (5.20)

However note that
\[
Q^2 \psi_{IA} = 2 \partial_r \psi_{\alpha A} - i \psi_{IA}, \quad Q^2 \bar{\psi}_{\alpha A} = 2 \partial_r \bar{\psi}_{\alpha A} + i \bar{\psi}_{IA},
\] (5.21)

which means that eq. (5.20) is automatically satisfied. However there is a caveat here. Note that if we act with \(Q\) on say
\[
\partial_r q_{11}(\theta, \frac{\pi}{2}) |_{r=\pi/2} = \partial_r \Sigma_{12}(\theta, \frac{\pi}{2}) \neq 0,
\] (5.22)

using the BCs in case (1). It is clear that this will go on to produce infinite number of boundary conditions, not closed within themselves.

The resolution is that in working on \(HS^4\) with Euclidean signature, the real and imaginary parts of all fields on the Lorentzian signature get mixed when they are analytically continued to Euclidean signature. So when we check the SUSY closure of BCs on the individual fields thinking of them the same way as on Lorentzian space-time, the SUSY fails to close. On the other hand the full covariant expression for BCs \(\psi_{\alpha I}|_{r=\pi/2} = +i\bar{\psi}_{\alpha J}\tau^3_J|_{r=\pi/2}\) is closed under SUSY by construction. So the conclusion is that the BCs can closed under supersymmetry when written in covariant form in the Euclidean signature.

(2) \(\psi_{\alpha I}|_{r=\pi/2} = -i\bar{\psi}_{\alpha J}\tau^3_J|_{r=\pi/2}\)

If we act with supersymmetry \(Q\) on this BC, it will be closed if we choose Dirichlet BCs on the following fields
\[
q_{11}(\theta, \frac{\pi}{2}) = 0, \quad q_{22}(\theta, \frac{\pi}{2}) = 0, \quad \Sigma_{12}(\theta, \frac{\pi}{2}) = 0 \quad \Sigma_{21}(\theta, \frac{\pi}{2}) = 0,
\] (5.23)

and Neumann BCs on the following
\[
\partial_r q_{11}(\theta, \frac{\pi}{2}) = 0, \quad \partial_r q_{22}(\theta, \frac{\pi}{2}), \quad \partial_r \Sigma_{12}(\theta, \frac{\pi}{2}) = 0, \quad \partial_r \Sigma_{21}(\theta, \frac{\pi}{2}) = 0.
\] (5.24)

Like that previous case (1) this choice of BCs is closed under supersymmetry except for the properly writing the BCs in a covariant way with respect to Euclidean signature. However fortunately the solution set of kernel and co-kernel equations that we have found satisfy BCs irrespective of whether we impose them covariantly or and separately in terms of real and imaginary parts. Applying variational principle to \(S_{\text{hyper}}\), to get equations of motion ,we get boundary terms
\[
\delta L_{\text{hyper}} = \delta q^A D^r q_A - \frac{i}{2} \bar{\psi} \sigma^r \delta \psi.
\] (5.25)

Choosing one of the above BCs, the action principle will be satisfied as well as these BCs are consistent with supersymmetry.
Knowing that the eigenvalue of $\hat{Q}^2$ for $\chi_+(\psi,\theta,\phi,r)$ is $n+ia.\alpha$, with corresponding multiplicity $|n| - 1$ with $n \in \mathbb{Z}$, $a$ the zero mode of the imaginary part of the scalar of the vector multiplet and $\alpha$ the roots of the gauge group $G$. The expression for the one loop determinant can be written

$$Z^{1-\text{loop}}_{\text{vec.-Dir}} = \prod_{\alpha \in \Delta} \prod_{n \in \mathbb{Z}_+} (n + ia.\alpha) \frac{\Gamma(1 + ia.\alpha)}{a.\alpha}$$

$$= \prod_{\alpha \in \Delta_+} \prod_{n \in \mathbb{Z}_+} (n + ia.\alpha)^{n-1}(n-ia.\alpha)^{n-1}$$

$$= \prod_{\alpha \in \Delta_+} \prod_{n \in \mathbb{Z}_+} \frac{(n + ia.\alpha)^n(n-ia.\alpha)^n}{(n + ia.\alpha)(n-ia.\alpha)}$$

(5.26)

$\Delta$ representing the root system. The regularized form of this ill defined [14] product is

$$Z^{1-\text{loop}}_{\text{vec.-Dir}} = \prod_{\alpha \in \Delta_+} G(1 + ia.\alpha)G(1 - ia.\alpha)\Gamma(1 + ia.\alpha)\Gamma(1 - ia.\alpha).$$

(5.27)

Using the identity

$$\frac{1}{\Gamma(1 + ia.\alpha)\Gamma(1 - ia.\alpha)} = \frac{\sin(i\pi a.\alpha)}{a.\alpha}$$

(5.28)

$$Z^{1-\text{loop}}_{\text{vec.-Dir}} = \prod_{\alpha \in \Delta_+} G(1 + ia.\alpha)G(1 - ia.\alpha)\frac{a.\alpha}{\sin(i\pi a.\alpha)}$$

$$= \prod_{\alpha \in \Delta_+} H(ia.\alpha)\frac{a.\alpha}{\sinh(\pi a.\alpha)}$$

(5.29)

getting following hemisphere wave function

$$Z_{\text{Dir-hemi-}S^4}^{\text{Dir}} = Z^{1-\text{loop}}_{\text{vec.-Dir}} Z_{\text{inst}}^k$$

$$= \prod_{\alpha \in \Delta_+} e^{\frac{4\alpha^2 a}{g^2 YM}} H(ia.\alpha)\frac{a.\alpha}{\sinh(\pi a.\alpha)} Z_{\text{inst}}^k(a,\tau)$$

(5.30)

with $\tau = \frac{\theta}{2\pi} + \frac{4g_i}{g_{YM}}$ and $Z_{\text{inst}}^k$ is the contribution of $k-th$ sector of the Nekrasov instanton partition function.

Recall that the instanton configurations contributing to the path integral are point-like instantons localised at the pole of the hemisphere and in particular are pure (large) gauge at the boundary $S^3$. Given the large gauge transformation $T$, which maps the boundary $S^3$ to the $SU(N)$ Lie group, the corresponding winding number is given as:

$$k = \frac{1}{2\pi^2} \int_{S^3} \text{tr}(TdT)^3.$$

(5.31)
5.3. \( Z_{\mathrm{Neu}}^{S^4} \)

Neumann BCs by definition imply that the components of fields tangential to the boundary \( r = \frac{\pi}{2} \) are kept arbitrary and consequently in performing the path integral one has to integrate over all field configurations. However to be able to apply localization, the field configurations satisfying some BCs must be solution of the saddle point equations. As is evident from solution of saddle equations (2.10), (2.15), the infinite dimensional field space is reduced to a single scalar zero mode \( a \). For Dirichlet BCs this zero mode is fixed but for Neumann BCs \( a \) takes arbitrary values at the boundary and so one has to integrate over it to get the wave function. In general the Neumann wave function depends on the variables canonically conjugate to those fixed by the Dirichlet BCs. If at the boundary we see 4d vectormultiplet as composed of one 3d vectormultiplet plus a 3d chiral multiplet, then the Neumann BCs data corresponds to the fixed value of 3d chiral multiplet at the boundary. The dynamical fields of 3d chiral multiplet are given in terms of 4d fields as

\[
\{D^i \phi_2, D^r \phi_1, F_{ir}^r, \text{fermionic super - partners}\}
\]

with \( i = \psi, \theta \) and \( \phi \). Therefore

\[
Z^{1-\text{loop \ vec. - Neu}}_{\text{vec. - Neu}} = \prod_{\alpha \in \Delta} \prod_{n \in \mathbb{Z}_+} (n + ia.\alpha)^{\frac{n+1}{2}}(n - ia.\alpha)^{\frac{n+1}{2}}
\]

\[
= \prod_{\alpha \in \Delta_+} \prod_{n \in \mathbb{Z}_+} (n + ia.\alpha)^{n+1}(n - ia.\alpha)^{n+1}
\]

\[
= \prod_{\alpha \in \Delta_+} \prod_{n \in \mathbb{Z}_+} (n + ia.\alpha)^{n}(n - ia.\alpha)^{n}(n + ia.\alpha)(n - ia.\alpha)
\]

(5.33)

\( \Delta \) and \( g \) representing the root system and Lie algebra respectively, of \( SU(N) \). The regularized form of this infinite product is [14]

\[
Z^{1-\text{loop \ vec. - Neu}}_{\text{vec. - Neu}} = \prod_{\alpha \in \Delta_+} \frac{G(1 + ia.\alpha)G(1 - ia.\alpha)}{\Gamma(1 + ia.\alpha)\Gamma(1 - ia.\alpha)}
\]

(5.34)

Using the identity

\[
\frac{1}{\Gamma(1 + ia.\alpha)\Gamma(1 - ia.\alpha)} = \frac{\sin(i\pi a.\alpha)}{a.\alpha}
\]

(5.35)

\[
Z^{1-\text{loop \ vec. - Neu}}_{\text{vec. - Neu}} = \prod_{\alpha \in \Delta_+} \frac{G(1 + ia.\alpha)G(1 - ia.\alpha)\sinh(\pi a.\alpha)}{a.\alpha}
\]

(5.36)

For Neumann BC’s, instanton configurations contributing to the path integral are again localised at the pole and are pure (large gauge) at the equator \( S^3 \), at \( r = \frac{\pi}{2} \), with winding number \( k \) as before.

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Therefore the full partition function is:

\[ Z_{\text{Neu}}^{\text{hemi-}S^4} = \int_g da \prod_{\alpha \in \Delta_\pm} e^{-4\pi^2 a_\alpha^2 g_{YM}} H(i a_\alpha) \frac{\sinh(\pi a_\alpha)}{a_\alpha} Z_{\text{inst}}(a, \tau) \]  

(5.37)

and where \( Z_{\text{inst}} \) is the full holomorphic part of Nekrasov partition function. Note that here we have written the Neumann wave function after summing over all instanton sectors and thus getting \( Z_{\text{inst}}(a, \tau) \). However in principle Neumann wave function is computed for each instanton sector labelled by an integer \( k \) and hence depends on the discrete parameter \( k \). After summing over all values of \( k \) one gets the last expression.

5.4. Large Radius limit \( R \to \infty \)

In our computation we have set the radius \( R_{S^4} \equiv R = \frac{1}{\epsilon} = 1 \). For illustration let’s take \( G = SU(2) \) in this subsection. Then restoring it one gets the following expressions for the bulk one-loop part of \( HS^4 \) with Dirichlet and Neumann BCs.

\[
Z_{1-\text{loop}}^{\text{Dir}} = H(i 2R a) \frac{2\pi R}{\sinh(2\pi R a)}
\]

\[
Z_{1-\text{loop}}^{\text{Neu}} = H(i 2R a) \frac{\sinh(2\pi R a)}{2\pi R a}.
\]  

(5.38)

Now using the following identities

\[
\ln H(x) = -x^2 \ln(|x|) e^{\gamma - \frac{1}{2}} + \mathcal{O}(\ln(|x|)) \quad x \to \infty,
\]

\[
\ln\left(\frac{2\pi x}{\sinh(2\pi x)}\right)^\pm = \pm \ln(2\pi x) \mp 2\pi x \quad x \to \infty.
\]  

(5.39)

it is easy to see that to leading order in \( R \to \infty \) limit we get the following simplified expressions

\[
Z_{1-\text{loop}}^{\text{Dir}} = e^{2a^2 R^2 \ln(2)|a|R} e^{\gamma - \frac{1}{2}},
\]

\[
Z_{1-\text{loop}}^{\text{Neu}} = e^{2a^2 R^2 \ln(2)|a|R} e^{\gamma - \frac{1}{2}}.
\]  

(5.40)

where the positive sign in the exponential is accounted by taking \( a \) to be anti-hermitian.

So we reach an interesting conclusion that at leading order in \( R \to \infty \) limit Dirichlet and Neumann BCs lead to same perturbative result. This exponential contribution of one loop in the large radius limit can be interpreted as producing an RG flow that renormalizes the classical gauge coupling constant \( g_{YM} \).

6. Gluing back two hemisphere \( HS^4 \) wave functions

Wave functions with Dirichlet BCs

If we see \( N = 2, d = 4 \) vector multiplet as a combination of an \( N = 2, d = 3 \) vector and a chiral multiplet, then imposing Dirichlet BC’s at the equator \( r = \frac{\pi}{2} \) of \( S^4 \) amounts to freezing the 3d vector multiplet to fixed value and consequently decoupling the gauge theory dynamics from two sides of the equator. Gluing the two wave functions then naturally implies that one has to put
back $N = 2, d = 3$ gauge multiplet at the equator. See also the discussion in [3]. It has being argued that the two wave functions for Dirichlet BC’s are glued along the equator $r = \frac{\pi}{2}$ by gauging the global symmetry under which the boundary values of the dynamical fields transform [4, 2]. In other words one has to put an $N = 2, 3d$ vector multiplet at the equator and include the corresponding partition function. In fact, our results confirm this general argument. As for the matter multiplet, the wave functions from two hemispheres are joined together by turning on a super potential coupling at the equator [5]. However being a Q-exact term, the superpotential does not contribute to the localization computation.

\[
Z = \int_g da \left[ \prod_{\alpha \in \Delta^+} \left( H(ia,\alpha) \frac{a,\alpha}{\sinh(\pi a,\alpha)} e^{-\frac{4\pi^2 a,\alpha^2}{g^2 s_M}} \frac{\sinh(\pi a,\alpha)^2}{(a,\alpha)^2} e^{-\frac{4\pi^2 a,\alpha^2}{g^2 s_M}} \frac{a,\alpha}{\sinh(\pi a,\alpha)} H(ia,\alpha) \right) \right] \times \left[ Z_{HS^4} Z_{HS^4} \right] |Z_{inst}|^2
\]

\[
= \int_g da \left[ \prod_{\alpha \in \Delta^+} e^{-\frac{8\pi^2 a,\alpha^2}{g^2 s_M} H(ia,\alpha)^2} \left[ \prod_{\rho \in \text{weights}} \frac{1}{H(ia,\rho)} \right] |Z_{inst}(a, q)|^2 \right]
\]

\[= Z_{pestun}^S \] (6.1)

where $q = e^{2\pi i \tau}$. The last identity can also be interpreted as factorization of round sphere $S^4$ partition function, though more precisely it is a convolution of two Dirichlet wave functions of two hemispheres with non-trivial integral kernel, the latter being due to a 3D vector multiplet at the equator. Perhaps the instanton contribution requires a comment: one would have naively thought that in glueing two Dirichlet wave functions one should have matched the $k$-th instanton sector on one side with the the $-k$-th anti-instanton sector on the other side. This would have produced a function of $|q|$, i.e. no $\Theta$ dependence. This is not the $S^4$ answer however, which is not diagonal in the instanton number: to get the $S^4$ result one has actually to sum over all (anti-)instanton sectors on each hemisphere before glueing. Put it differently, the identification of fields at the boundary is up to large gauge transformations.

**Wave functions with Neumann BCs**

We will only be sketchy here to describe the gluing of two Neumann wave functions. Roughly speaking Neumann BCs are canonically conjugate to Dirichlet BCs. In the semiclassical approximation Neumann wave function is related to Dirichlet wave function through Legendre transformation. From section 5.3 we know that the 4d vector multiplet when restricted to 3d boundary, can be decomposed into a 3d vector plus a 3d chiral multiplet. In the same vein Dirichlet wave function depends on the value of scalar of the 3d gauge multiplet at the boundary, whereas Neumann depends on the value of 3d chiral multiplet at the boundary. These boundary conditions are constrained by localizing equations and for this reason 3d chiral vev at the boundary is taken to be zero in our case.

Compared to the Dirichlet case, where we needed to include a 3d vector multiplet partition function in the glueing procedure, in the Neumann case we are facing an over counting problem, i.e. we count twice the contribution of the boundary 3D vector multiplet, since in this case the corresponding boundary degrees of freedom from each side are not frozen, as it is clear from eq.(5.36). Therefore in the glueing procedure we insert a factor:

\[
\prod_{\alpha} \frac{a(a)}{\sinh(\pi a(\alpha))} \] (6.2)
to remove the redundant degrees of freedom. This gluing procedure gives rise to the round $S^4$ partition function. However it would be very nice to have a better understanding of how this measure arises from the full path integral. See e.g. [12].

Wave functions with Dirichlet-Neumann BCs

If we impose Dirichlet BCs on the vector multiplet fields on one hemisphere with the resulting one-loop part

$$Z^{1-\text{loop}}_{\text{vec.-Dir}} = \prod_{\alpha \in \Delta_+} H(i a.\alpha) \frac{a.\alpha}{\sinh(\pi a.\alpha)}$$

and on the complementary hemisphere we impose Neumann BCs with the following one-loop part

$$Z^{1-\text{loop}}_{\text{vec.-Neu}} = \prod_{\alpha \in \Delta_+} H(-i a.\alpha) \frac{\sinh(\pi a.\alpha)}{a.\alpha}$$

it is obvious that if we naively glue these two hemispheres we get

$$Z^{1-\text{loop}}_{\text{vec.-Dir}} Z^{1-\text{loop}}_{\text{vec.-Neu}} \approx \prod_{\alpha \in \Delta_+} H(i a.\alpha) H(-i a.\alpha) \approx Z_{\text{vec.}}^{\text{vec.}}$$

This shows that no extra measure is needed to glue $Z_{HS^4}^{\text{vec.}}$ Dir with $Z_{HS^4}^{\text{vec.}}$ Neu to get $Z_{S^4}^{\text{vec.}}$. Intuitively the over counting of modes from the hemisphere with Neumann BCs is compensated by the removal of boundary modes by imposing Dirichlet BCs on the other hemisphere. However a more satisfactory explanation in terms of path integral will be illuminating.

7. One loop determinant and $SO(5)$ harmonics

One loop determinants can also be computed more directly using the full $SO(5)$ harmonics. Like the case of $SO(4)$ harmonics as given in the first part of work, we are only interested in the spectrum of $Q^2$ on kernel($D_{10}$) and cokernel($D_{10}$). The purpose of this and the next section is to compute the net multiplicity of $Q^2$ on kernel and cokernel of $D_{10}^{\text{vec}}$ for round $S^4$, hemisphere $HS^4$ with Dirichlet BCs , Neumann BCs at the equator. We show that it matches with the results obtained using $SO(4)$. The task can be simplified by observing how vector and scalar harmonics of $SO(4)$ irreps. are embedded in $SO(5)$ irreps. Here it is helpful to recall some useful results from Lie group Representation theory [1].

Irreducible representations of $SO(2k+1)$ determined by their highest weights $(n_1, n_2, ..., n_k)$ with integer or half-integer entries, when restricted to the subgroup $SO(2k)$, contains all irreps. of the later with highest weights $(p_1, p_2, ..., p_k)$ with integer or half-integer entries, satisfying the following constraints

$$n_1 \geq p_1 \geq n_2 \geq p_2 \geq ..., \geq n_{k-1} \geq p_{k-1} \geq |n_k|.$$  \hspace{1cm} (7.1)$$

If $n_i$ are integers ( half integers) so are $p_i$. Quadratic Casimir is an important operator for a Lie algebra whose eigenvalues for different irreps. are used to regularize infinite sums using heat kernel technique. For irreps. of orthogonal group it is given by

$$C_2(n_1, n_2, ..., n_{k+1}) = n.n + 2w.m$$  \hspace{1cm} (7.2)$$
with Euclidean dot product assumed and where the Weyl vector \( w \) given by

\[
    w_i = \begin{cases} 
        k - i + 1 & \text{for } SO(2k + 2) \\
        k + rac{1}{2} - i & \text{for } SO(2k + 1) 
    \end{cases}
\]

(7.3)

Assume that the weights are given in the basis of Cartan generators \((j^3_L, j^3_R)\) for \(SU(2)_L \times SU(2)_R \sim SO(4) \subset SO(5)\), with \(SO(4)\) being the isometry group of \(S^3\) at constant value of coordinate \(r\). Then the irreps. of scalar and vector \(SO(5)\) harmonics can be constructed by starting with the simple roots

\[
    (1, 0), \quad (-\frac{1}{2}, \frac{1}{2})
\]

(7.4)

in the above basis. Here we describe only the final results of the construction. First of all it is easy to check that \(SO(5)\) Lie algebra is generated by the following generators

\[
    j^+_R = (-e^{i\phi} \csc(\theta), -ie^{i\phi}, e^{i\phi} \cot(\theta), 0), \quad j^-_R = (-e^{-i\phi} \csc(\theta), ie^{-i\phi}, e^{-i\phi} \cot(\theta), 0),
\]

\[
    j^+_L = (e^{i\phi} \cot(\theta), -ie^{i\phi}, -e^{i\phi} \csc(\theta), 0), \quad j^-_L = (e^{-i\phi} \cot(\theta), ie^{-i\phi}, -e^{-i\phi} \csc(\theta), 0),
\]

\[
    j^3_R = (0, 0, 1, 0), \quad j^3_L = (1, 0, 0, 0),
\]

\[
    j^+_5 = e^{\frac{1}{2}i(\phi-\psi)}(\csc(\frac{\theta}{2})(-\cot(r)), -2i \cos(\frac{\theta}{2}) \cot(r), \csc(\frac{\theta}{2}) \cot(r), -i \sin(\frac{\theta}{2})),
\]

\[
    j^-_5 = e^{-\frac{1}{2}i(\phi-\psi)}(\csc(\frac{\theta}{2})(-\cot(r)), 2i \cos(\frac{\theta}{2}) \cot(r), \csc(\frac{\theta}{2}) \cot(r), i \sin(\frac{\theta}{2})),
\]

\[
    j^+_6 = e^{\frac{1}{2}i(\psi+\phi)}(\sec(\frac{\theta}{2})(-\cot(r)), -2i \sin(\frac{\theta}{2}) \cot(r), \sec(\frac{\theta}{2}) (-\cot(r)), i \cos(\frac{\theta}{2})),
\]

\[
    j^-_6 = e^{-\frac{1}{2}i(\psi+\phi)}(\sec(\frac{\theta}{2})(-\cot(r)), 2i \sin(\frac{\theta}{2}) \cot(r), \sec(\frac{\theta}{2}) (-\cot(r)), -i \cos(\frac{\theta}{2}))
\]

(7.5)

\(\mathbb{Z}_2\) action

Keeping in mind the fact that the above generators act on the fields as a differential and hence the fourth entry corresponds to derivative w.r.t. \(r\), we conclude that first six generators are even under \(\mathbb{Z}_2\) action \(r \to \pi - r\), whereas the last four generators are odd.

Harmonics

The logic for constructing \(SO(5)\) harmonics is simple. One repeatedly applies negative roots \((-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\) to the highest weight state of an \(SO(5)\) irrep. to get a state which is a linear combination \(SO(4)\) highest weight and \(SO(4)\) descendants. One then removes the descendants part to get the irreps. of \(SO(4)\) given by its highest weight. This construction has following properties

- The highest weights of \(SO(5)\) both for scalars and vectors are \(\mathbb{Z}_2\) even.
- the two lowering operator represented by negative roots \((-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\), which do not belong to \(SO(4)\), project out even modes w.r.t. \(\mathbb{Z}_2\) action.
- Since we already know from the branching rule given in 7.1 which \(SO(4)\) irreps. appear in a given \(SO(5)\) irrep. one can easily see that by counting how many times one needs to apply these two negative roots to reach an allowed \(SO(4)\) highest weight state starting from a given \(SO(5)\) highest weight state
- If the count is even (odd) the corresponding \(SO(4)\) irrep. is even(odd).
Scalar harmonics

For even irreps. of SU(2) it is clearly even under \( Z_2 \) action. So if we apply \( j_6^- \) or \( j_5^- \) on it the result will be odd. As a result decomposing this \( SU(2) \) representation in terms of irreps. of \( SO(4) \) we get the following

\[
(j_L - \frac{n}{2}, j_L - \frac{n}{2}) \quad \text{for} \quad n = 0, 1, ..., 2j_L.
\]

(7.7)

For \( n \) an even integer this irrep. is even under \( SO(2) \). Second \( SO(1) \). First \( SO(1) \). Vector harmonics

\[
\cos^2 \left( \frac{\theta}{2} \right) \sin^2 \left( \frac{\theta}{2} \right) e^{i \epsilon L (\psi + \phi)}
\]

(7.6)

and is clearly even under \( Z_2 \) action. Similarly for vector harmonics of \( SO(2) \) result decomposing this \( SO(4) \) representation in terms of irreps. of \( SO(4) \) we get the following

\[
(j_L - \frac{n}{2}, j_L - \frac{n}{2}) \quad \text{for} \quad n = 0, 1, ..., 2j_L.
\]

(7.7)

For \( n \) an even integer this irrep. is even under \( Z_2 \) and for \( n \) odd the irrep. is \( Z_2 \) odd.

Vector harmonics

Vector \( SO(5) \) harmonics come in two classes of \( SO(5) \) irreps. labeled by highest weights and they decompose in \( SU(2)_L \times SU(2)_R \) irreps. represented by the highest weights \( (j_L, j_R) \), as

1. First \( SO(5) \) irrep. is \( (j_L + 1, j_L) \), which decomposes into three \( SO(4) \) irreps.

   a. \( (j_L - \frac{n}{2}, j_L + 1 - \frac{n}{2}) \) for \( n = 0, ..., 2j_L \). Irreps. with even \( n \) are invariant under \( Z_2 \) while odd \( n \) modes are odd.

   b. \( (j_L + \frac{1}{2} - \frac{n}{2}, j_L + 1 - \frac{n}{2}) \) for \( n = 0, ..., 2j_L \). Importantly in this case irreps. with even \( n \) are odd and irreps. with odd \( n \) are even under \( Z_2 \) action.

   c. \( (j_L + 1 - \frac{n}{2}, j_L - \frac{n}{2}) \) for \( n = 0, ..., 2j_L \). These are invariant for even \( n \) and odd for odd \( n \).

2. Second \( SO(5) \) irrep. for vector harmonics is \( (j_L, j_L) \) and it decomposes into \( SO(4) \) irreps. as \( (j_L - \frac{n}{2}, j_L - \frac{n}{2}) \) for \( n = 0, ..., 2j_L \), which are even for even \( n \) and odd for odd \( n \).

8. \( Z_1 \)-loop via \( SO(5) \) harmonics

For \( SO(4) \) irreps. \( (j_L - \frac{n}{2}, j_L - \frac{n}{2}) \) for \( n = 0, 1, ..., 2j_L \) contained in \( (j_L, j_L) \) irrep. of \( SO(5) \), various dynamical scalar fields will contribute the following to the net multiplicity of the one-loop determinant. Scalar contribution for \( Z_2 \) even irreps. is denoted as \( S_e \) and for odd irreps. as \( S_o \).

\[
S_e = \sum_{j=\frac{m}{2}}^{\infty} [(j + 1)^2 - \frac{m^2}{4}], \quad S_o = \sum_{j=\frac{m}{2}}^{\infty} [(j + 1)^2 - \frac{m^2}{4}].
\]

(8.1)

Similarly for vector harmonics of \( SO(5) \) one gets the following individual contribution to the net multiplicity of one loop determinant.

For even irreps. of \( SO(4) \)

\[
V^0_e = \sum_{j=\frac{m}{2}}^{\infty} [(j + 1)^2 - \frac{m^2}{4}], \quad V^+ e = \sum_{j=\frac{m}{2}}^{\infty} [(j + 2)^2 - \frac{(m + 2)^2}{4}], \quad V^- e = \sum_{j=\frac{m}{2}}^{\infty} [(j + 1)^2 - \frac{(m - 2)^2}{4}],
\]

(8.2)

and for odd irreps. of \( SO(4) \)

\[
V^0_o = \sum_{j=\frac{m}{2}}^{\infty} [(j + \frac{3}{2})^2 - \frac{m^2}{4}], \quad V^+ o = \sum_{j=\frac{m}{2}}^{\infty} [(j + \frac{3}{2})^2 - \frac{(m + 2)^2}{4}], \quad V^- o = \sum_{j=\frac{m}{2}}^{\infty} [(j + \frac{1}{2})^2 - \frac{(m - 2)^2}{4}].
\]

(8.3)
**Regularizing the Infinite sums**

As an example we will describe in detail the regularization of \( S_e \). For the other contributions we will only give the final results. Consider the following expression in the heat kernel regularization

\[
S_e(t, m) = \sum_{j=m}^{\infty} \left[ (j + \frac{3}{4})^2 + \frac{1}{2}(j + \frac{3}{4}) + \left( \frac{1}{16} - \frac{m^2}{4} \right) \right] e^{-2t(j + \frac{3}{4})^2 + \frac{3}{4}t}
\]  

(8.4)

where \( 2j^2 + 3j \) is the regularization factor for \( j_L = j_R \equiv j \) representation of \( SO(5) \). Taking the Mellin transform of \( S_e \) w.r.t. \( t \)

\[
\tilde{S}_e(s, m) = \int_0^\infty t^{s-1} S_e(t, m) dt
\]  

(8.5)

we get

\[
\tilde{S}_e(s, m) = \sum_{j=m}^{\infty} \left[ (j + \frac{3}{4})^2 - 2s + \frac{1}{2}(j + \frac{3}{4})^{1-2s} + \left( \frac{1}{16} - \frac{m^2}{4} \right) (j + \frac{3}{4})^{-2s} \right] \Gamma(s)^2 \left[ 1 - \frac{9}{16} (j + \frac{3}{4})^{-2s} \right]^{-s}
\]  

(8.6)

Now applying the Binomial expansion

\[
[1 - \frac{9}{16}(j + \frac{3}{4})^{-2s}]^{-s} = \sum_k (-1)^k \frac{(-s)!}{k!(s+k)!} \left( \frac{9}{16} \right)^k (j + \frac{3}{4})^{-2k}
\]  

(8.7)

and using \( \Gamma \) function analytic continuation

\[
[1 - \frac{9}{16}(j + \frac{3}{4})^{-2s}]^{-s} = \sum_k (-1)^k \frac{\Gamma(1-s)}{\Gamma(k+1)\Gamma(1-s-k)!} \left( \frac{9}{16} \right)^k (j + \frac{3}{4})^{-2k}
\]  

(8.8)

Substituting this in the expression for \( \tilde{S} \)

\[
\tilde{S}_e(s, m) = \sum_{j=m}^{\infty} \sum_k \left[ (j + \frac{3}{4})^{2-2s-2k} + \frac{1}{2}(j + \frac{3}{4})^{1-2s-2k} + \left( \frac{1}{16} - \frac{m^2}{4} \right) (j + \frac{3}{4})^{-2s-2k} \right] \
\times \Gamma(s)^2 \left( -1 \right)^k \frac{\Gamma(1-s)}{\Gamma(k+1)\Gamma(1-s-k)!} \left( \frac{9}{16} \right)^k
\]  

(8.9)

Since the above summation is absolutely convergent, one can exchange the order of summation and at the same time shift \( j \) to \( j + \frac{m}{2} \) and perform the \( j \) summation in terms of Hurwitz Zeta function to get

\[
\tilde{S}_e(s, m) = \sum_k \left[ \zeta \left( \frac{2k + 2s - 2}{2}, \frac{m}{2} + \frac{3}{4} \right) + \frac{1}{2} \zeta \left( \frac{2k + 2s - 1}{2}, \frac{m}{2} + \frac{3}{4} \right) + \left( \frac{1}{16} - \frac{m^2}{4} \right) \zeta \left( \frac{2k + 2s}{2}, \frac{m}{2} + \frac{3}{4} \right) \right] \
\times \Gamma(s)^2 \left( -1 \right)^k \frac{\Gamma(1-s)}{\Gamma(k+1)\Gamma(1-s-k)!} \left( \frac{9}{16} \right)^k
\]  

(8.10)
In the last step we take the inverse Mellin transform
\[ S_e(t, m) = \frac{1}{2\pi i} \int_C t^{-s} \tilde{S}_e(s, m) ds \] (8.11)
and perform complex integration along a contour \( C \) which encloses all the poles of integrand. Interestingly when we evaluate the last integral for various poles of \( s \), we find that the series terminates for finite values of \( k \). We thus obtain
\[ S_e(t, m) = \frac{s - 1}{t} - \frac{9 (16m^2 - 13)}{2048 \sqrt{t}} + \left( \frac{m^3}{12} - \frac{m^2}{16} - \frac{m}{12} + \frac{7}{48} \right)t^0 + \frac{1}{64} \left( 11 - 8m^2 \right) \sqrt{t} + \ldots \] (8.12)
Following the same procedure we find
\[ S_o(t, m) = \ldots + \left( \frac{m^3}{12} + \frac{m^2}{16} - \frac{m}{12} - \frac{7}{48} \right)t^0 + \ldots \] (8.13)
and similarly for vector harmonics
\[ V^0_e = \ldots + \left( \frac{m^3}{12} + \frac{m^2}{16} - \frac{m}{12} - \frac{1}{48} \right)t^0 + \ldots, \quad V^+_e = \ldots + \left( \frac{m^3}{12} + \frac{5m^2}{16} + \frac{m}{6} - \frac{1}{16} \right)t^0 + \ldots, \]
\[ V^-_e = \ldots + \left( \frac{m^3}{12} - \frac{7m^2}{16} + \frac{2m}{3} - \frac{13}{48} \right)t^0 + \ldots \] (8.14)
and
\[ V^0_o = \ldots + \left( \frac{m^3}{12} - \frac{m^2}{16} - \frac{m}{12} + \frac{1}{48} \right)t^0 + \ldots, \quad V^+_o = \ldots + \left( \frac{m^3}{12} + \frac{7m^2}{16} + \frac{2m}{3} + \frac{13}{48} \right)t^0 + \ldots, \]
\[ V^-_o = \ldots + \left( \frac{m^3}{12} - \frac{5m^2}{16} + \frac{m}{6} + \frac{1}{16} \right)t^0 + \ldots \] (8.15)
In the application of localization to supersymmetric theory, fermions are written in cohomological form. Different components of fermion transform as scalars and vector of SO(4). Therefore the above results will suffice in determining their contribution.

**Net multiplicity \( N \)**

It is easy to see that for round \( S^4 \) the net multiplicity can be found as
\[ N_{S^4} = [S_e(m) + S_e(m + 2) + S_e(m - 2)] + [S_o(m) + S_o(m + 2) + S_o(m - 2)] - [V^0_e(m) + V^+_e(m) + V^-_e(m)] - [V^0_o(m) + V^+_o(m) + V^-_o(m)] = 2m \] (8.16)
In the next step keeping in mind the Dirichlet and Neumann BCs on the hemisphere given in section 5.1
\[ N_{HS^4}^{Dir.} = [2S_e(m) + S_e(m + 2) + S_e(m - 2)] - [V^0_e(m) + V^+_e(m) + V^-_e(m)] - S_o(m) = m + 1 \] (8.17)
for Dirichlet BCs and
\[ N_{HS^4}^{Neum.} = [2S_o(m) + S_e(m + 2) + S_e(m - 2)] - [V^0_o(m) + V^+_o(m) + V^-_o(m)] - S_e(m) = m - 1 \] (8.18)
for Neumann BCs. Next since we know the eigenvalues of $\hat{Q}^2$, it is trivial to write down the expressions for one-loop determinant. However it is important to note that there is some arbitrariness in the regularization scheme used here. For instance if one multiplies a factor of $e^{iC}$ for constant $C \in \mathbb{R}$, the net multiplicity $N_{HS^4}^{\text{Dir}}$ and $N_{HS^4}^{\text{Neum.}}$ is modified to $m + p$ and $m - p$ respectively, for some positive integer $p$, in such a way that $N_{HS^4}^{\text{Neum.}} + N_{HS^4}^{\text{Dir}} = 2m$. 
9. Conclusions

Despite extensive activity in Supersymmetric Localization computations on curved manifolds in various dimensions, there were no first principle computations available on hemisphere in four dimensions, although some educated guesses were given in [2]. We have done detailed computation of wave functions on hemisphere $H S^4$ with supersymmetric BCs of Dirichlet type, and also discussed briefly the Neumann BCs. In the first part of this work, various one-loop determinants are computed using $SO(4)$ harmonics as the complete set of basis functions. The results obtained in the first part are re-checked in the second part where we do the same analysis in the framework of full $SO(5)$ harmonics. We have also briefly discussed how the $N = 2$ SUSY partition function on round $S^4$ à la Pestun [14], can be seen as composed of two Dirichlet type wave functions on southern and northern hemispheres properly glued together. The last observation can also be interpreted as kind of factorization of $Z_{S^4}$ wave function in terms of two hemisphere wave functions. Though this factorization should be seen as a convolution of two wave functions with non-trivial kernel, the later being the one-loop determinant of $N = 2, 3d$ gauge multiplet.

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APPENDICES

A. Notation

We use the same notation as in [7]. In the flat basis of the tangent space on $S^4$ we use the following set of Dirac matrices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\gamma^4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad \text{(A.1)}$$

Pauli matrices are defined as usual

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(A.2)}$$

In Weyl basis, with the decomposition $SO(4) \approx SU(2)_R \times SU(2)_L$ in chiral and anti-chiral basis, the sigma matrices $(\sigma^a)_{\alpha\dot{\beta}}, (\bar{\sigma}^a)_{\dot{\alpha}\beta}$ are related to Pauli matrices as follows

$$\sigma^1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^4 = -i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad \text{(A.3)}$$

$$\bar{\sigma}^1 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{\sigma}^2 = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \bar{\sigma}^3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{\sigma}^4 = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad \text{(A.4)}$$

The R-symmetry indices $A, B, ..$ and chiral and anti-chiral indices $\alpha, \dot{\alpha}$ are raised and lowered with antisymmetric matrices $\epsilon^{a\dot{b}}, \epsilon^{a\beta}, \epsilon^{a\dot{b}}, \epsilon^{AB}, \epsilon^{AB}$ with the following matrix elements

$$\epsilon^{12} = 1, \epsilon_{12} = -1, \epsilon^{1\dot{2}} = 1, \epsilon_{1\dot{2}} = -1. \quad \text{(A.5)}$$

B. $SO(4) \approx SU(2)_R \times SU(2)_L$ harmonics on $S^3$

Background geometry

The isometry group of round $S^4$ is $SO(5)$ and the most general way to compute the one-loop determinant is to use $SO(5)$ harmonics. However since we are interested in applying localization on a hemisphere, the boundary at $r = \frac{\pi}{2}$ breaks translational symmetry in $r$ coordinate, only $SO(4) \subset SO(5)$ is left intact and the best we can do is to use $SO(4)$ spherical harmonics. We take the following metric on $S^4$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dr^2 + \frac{\sin(r)^2}{4} (d\theta^2 + \sin \theta^2 d\phi^2 + (d\psi + \cos \theta d\phi)^2) \quad \text{(B.1)}$$
with coordinates \( \mu = (\psi, \theta, \phi, r) \), such that the coordinates for Hopf fibration of unit \( S^3 \) part are:

\[
z_1 = \sin \left( \frac{\theta}{2} \right) e^{i \frac{(\phi+\psi)}{2}}, \quad z_2 = \cos \left( \frac{\theta}{2} \right) e^{i \frac{(\phi+\psi)}{2}},
\]

for \( 0 \leq \theta \leq \pi \), \( 0 \leq \phi \leq 2\pi \) and \( 0 \leq \psi \leq 4\pi \). As for the radial coordinate \( r \), \( 0 \leq r \leq \pi \). The metric above describes \( S^4 \) as an \( S^3 \) fibration over the \( r \) interval \((0, \pi)\). \( S^3 \) has radius \( \sin(r) \) which vanishes at \( 0 \) and \( \pi \). The vielbeins for \( SU(2)_L \)-frame and \( SU(2)_R \)-frame are

\[
e^L_{\alpha \mu} = \begin{pmatrix}
0 & -\frac{i}{2} \cos(\psi) \sin(r) & -\frac{i}{2} \sin(r) \sin(\theta) \sin(\psi) & 0 \\
0 & 0 & \frac{1}{2} \cos(\psi) \sin(r) \sin(\theta) & 0 \\
-\sin(r) & 0 & 0 & \frac{1}{2} \cos(\theta) \sin(r) \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (B.3)
\]

and

\[
e^R_{\alpha \mu} = \begin{pmatrix}
-\frac{1}{2} \sin(r) \sin(\theta) \sin(\phi) & -\frac{1}{2} \cos(\phi) \sin(r) & 0 & 0 \\
\frac{i}{2} \cos(\phi) \sin(r) \sin(\theta) & -\frac{i}{2} \sin(r) \sin(\phi) & 0 & 0 \\
\frac{1}{2} \cos(\theta) \sin(r) & 0 & -\sin(r) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (B.4)
\]

**SU(2)_L \times SU(2)_R** Lie algebra

The \( SO(4) = SU(2)_L \times SU(2)_R \) Killing vectors of \( SU(2)_L \), \( J^a_L \) and \( SU(2)_R \), \( J^a_R \), \( a = 1, 2, 3 \) are given by \( J^a_L = t^a \mu \partial_\mu \) and \( J^a_R = r^a \mu \partial_\mu \), where:

\[
tl^a \mu \equiv \sin(r) \frac{i}{2} (1 - 2 \delta^a_2) g^{\mu \nu} e^L_{\alpha \nu}, \quad r^a \mu \equiv \sin(r) \frac{i}{2} g^{\mu \nu} e^R_{\alpha \nu}.
\]

They obey the algebra:

\[
\left[ J^a_L, J^b_L \right] = i \epsilon_{abc} J^c_L, \quad \left[ J^a_R, J^b_R \right] = i \epsilon_{abc} J^c_R, \quad \left[ J^a_L, J^b_R \right] = 0.
\]

Notice that

\[
l^3 \mu \equiv (-i, 0, 0, 0), \quad r^3 \mu \equiv (0, 0, -i, 0)
\]

and

\[
J^3_L e^{i(\eta L \psi + \eta R \phi)} = l^3 \mu \partial_\mu e^{i(\eta L \psi + \eta R \phi)} = q_L e^{i(\eta L \psi + \eta R \phi)}
\]

\[
J^3_R e^{i(\eta L \psi + \eta R \phi)} = r^3 \mu \partial_\mu e^{i(\eta L \psi + \eta R \phi)} = q_R e^{i(\eta L \psi + \eta R \phi)}
\]

showing that \( e^{i(\eta L \psi + \eta R \phi)} \) is an eigenfunction of \( J^3_L \) and \( J^3_R \).

**Scalar harmonics**

Highest weight states with respect to \( SU(2)_L \) and \( SU(2)_R \) for the scalar functions \( e^{i(\eta L \psi + \eta R \phi)} f(\theta) \) are constructed by forming raising (lowering) operators \( J^z_L = J^z_L \pm i J^3_L \) and \( J^z_R = J^z_R \pm i J^3_R \). Highest weight states are annihilated by \( J^z_{L,R} \); one can easily prove that this implies that \( j_L = j_R \) and

\[
f(\theta) = \left( \cos \left( \frac{\theta}{2} \right) \right)^{2j_L}, \quad \text{up to a constant:}
\]

\[
\Phi = e^{i(\eta L \psi + \eta R \phi)} \cos \left( \frac{\theta}{2} \right)^{2j_L}
\]

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Applying lowering operators on $\Phi$ we can get all the other harmonics. In particular by applying $J_R$ $s$ times we get states

$$\Phi_s = e^{i(j_L \psi + (j_L - s) \phi)} \cos\left(\frac{\theta}{2}\right)^{(2j_L - s)} \sin\left(\frac{\theta}{2}\right)^s$$ \hspace{1cm} (B.10)

and one can check that when $s = 2j_L$ this is annihilated by $J_R$, i.e. it is a lowest weight state.

Vector harmonics

Let us move on to the vector harmonics: now we have to consider Lie derivatives along the Killing vectors of $SO(4)$ acting on one-forms. The Lie derivative with respect to a vector field $K$ on a 1-form $\omega$ is defined as

$$L_X \omega \equiv (X, d\omega) + d(X, \omega)$$ \hspace{1cm} (B.11)

In component form for $SU(2)_L \subset SO(4)$

$$L^a_L \omega_{\mu\nu} = l^{a\mu}(D_\mu \omega_{\nu} - D_\nu \omega_{\mu}) + D_\mu(l^{a\mu} \omega_{\mu})$$ \hspace{1cm} (B.12)

and for $SU(2)_R \subset SO(4)$ subgroup

$$L^a_R \omega_{\mu\nu} = r^{a\mu}(D_\mu \omega_{\nu} - D_\nu \omega_{\mu}) + D_\mu(r^{a\mu} \omega_{\mu})$$ \hspace{1cm} (B.13)

One can verify that Lie derivatives satisfy the Lie algebra relations: $[L^a, L^b] = L^{[a,b]}$. A basis of eigenstates of of the Cartan generators of both $SU(2)_L,R$ is given by:

$$\omega_{\nu}(\psi, \theta, \phi) = e^{i(q_L \psi + q_R \phi)} \omega^1_{\nu}(\theta)$$ \hspace{1cm} (B.14)

Now for $q_L = j_L$ $q_R = j_R$, this will be a highest weight state if it is annihilated by the raising operators of $SU(2)_R \times SU(2)_L$ Lie algebra

$$L^1_L \omega_{\nu} + iL^2_L \omega_{\nu} = 0$$
$$L^1_R \omega_{\nu} + iL^2_R \omega_{\nu} = 0$$ \hspace{1cm} (B.15)

Solving these differential equations gives the following general solution:

$$\omega^\psi_1(\theta) = \alpha_1 \sin\left(\frac{\theta}{2}\right)^{-j_L} \cos\left(\frac{\theta}{2}\right)^{j_L} \sin(\theta)^{j_R}$$
$$\omega^\phi_1(\theta) = \alpha_2 \sin(\theta)^{j_L} \cos\left(\frac{\theta}{2}\right)^{j_R}$$

$$\omega^\theta_1(\theta) = \alpha_1 \cos\left(\frac{\theta}{2}\right)^{j_L} \sin\left(\frac{\theta}{2}\right)^{j_R} \cos(\theta)^{j_L - j_R - 1}$$
$$- \alpha_2 \sin\left(\frac{\theta}{2}\right)^{j_L} \sin(\theta)^{j_R} \cos(\theta)^{j_L - j_R - 1}$$ \hspace{1cm} (B.16)

along with the following three choices of constant parameters

a) $j_R = j_L$, $\alpha_1 \neq 0, \alpha_2 = \alpha_1$

b) $j_R = j_L + 1$, $\alpha_1 \neq 0, \alpha_2 = 0$. 

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To write the kernel equations in a more suggestive form, we redefine the $z$ of $SU_O$ of gauge fields to tangent ones by contracting with $l$ has to expand its components as polynomials in $z$:

\[ \text{Fields in the cokernel of } D \quad \text{and to check the regularity of } A_\mu \text{ one has to expand its components as polynomials in } z_1, z_2, \bar{z}_1, \bar{z}_2 \text{ and find the leading behavior in the limit of } z_1 \to 0, z_2 \to 0, \bar{z}_1 \to 0, \bar{z}_2 \to 0 \text{ or in terms of polar variable } r \to 0. \text{ It is easy to find that} \]

Projection to $SU(2)_L$ basis

The analysis simplifies if we work with the $SO(3)$ tangent space basis, i.e. convert world indices of gauge fields to tangent ones by contracting with $l^\mu$:

\[ \omega_j^\alpha = l^{\alpha \mu} \omega_{\mu} \]  

(B.17)

In this basis the gauge fields behave like scalars and will always be in the $j_L = j_R$ representation of $SU(2)_L \times SU(2)_R$, as can be explicitly verified.

Now we specialize gauge field $A_\mu$, $\mu = 1, 2, 3$, to the flat basis $A_\alpha = l^{\mu \alpha} A_\mu$ where $a = 1, 2, 3$ or $a = +, -, 3$ where $A_+ \equiv A_1 + i A_2, A_- \equiv A_1 - i A_2$ and similarly $\chi_+ \equiv \chi_1 + i \chi_2, \chi_- \equiv \chi_1 - i \chi_2$.

Taking into account the relation between $\hat{Q}_2$ eigenvalues and the $SU(2)_L$ weights $q_L$, including the shifts in the weights, we are led to the following Fourier expansions.

Fields in the kernel of $D_{10}^{vec}$:

\[ A_+ (\psi, \theta, \phi, r) = e^{-i(q_L+1)\psi} e^{-i q r \phi} A_+ (\theta, r), \quad A_- (\psi, \theta, \phi, r) = e^{-i(q_L+1)\psi} e^{-i q r \phi} A_- (\theta, r) \]
\[ A_3 (\psi, \theta, \phi, r) = e^{-i(q_L)\psi} e^{-i q r \phi} A_3 (\theta, r) \]
\[ \phi_2 (\psi, \theta, \phi, r) = e^{-i(q_L)\psi} e^{-i q r \phi} \phi_1 (\theta, r). \]  

(B.18)

Fields in the cokernel of $D_{10}^{vec}$:

\[ \chi_+ (\psi, \theta, \phi, r) = e^{i(q_L+1)\psi} e^{i q r \phi} \chi_+ (\theta, r), \quad \chi_- (\psi, \theta, \phi, r) = e^{i(q_L+1)\psi} e^{i q r \phi} \chi_- (\theta, r) \]
\[ \chi_3 (\psi, \theta, \phi, r) = e^{i(q_L)\psi} e^{i q r \phi} \chi_3 (\theta, r), \quad \chi_3 (\psi, \theta, \phi, r) = e^{i(q_L)\psi} e^{i q r \phi} \chi_3 (\theta, r) \]
\[ \bar{c} (\psi, \theta, \phi, r) = e^{i(q_L)\psi} e^{i q r \phi} \bar{c} (\theta, r). \]  

(B.19)

To write the kernel equations in a more suggestive form, we redefine the $\phi_1 (\theta, r)$ field as

\[ \phi_1 (\theta, r) = \frac{1}{2} (-2 i A_3 (\theta, r) - \Lambda (\theta, r)) \sec (r). \]  

(B.20)

where $\Lambda (\theta, r)$ is field dependent gauge transformation.

Regularity at North and South poles

At the two poles of $S^4$ the space locally looks like $R^4$ and to check the regularity of $A_\mu$ one has to expand its components as polynomials in $z_1, z_2, \bar{z}_1, \bar{z}_2$ and find the leading behavior in the limit of $z_1 \to 0, z_2 \to 0, \bar{z}_1 \to 0, \bar{z}_2 \to 0$ or in terms of polar variable $r \to 0$. It is easy to find that
the highest weight state with $q_L = j_L$ and $q_R = j_R$ and because of expansion in scalar harmonics $j_L = j_R$

\begin{align}
A_r &\sim r^{2j_L - 1}, \quad \text{for} \quad j_L = j_R \neq 0 , \\
A_r &\sim r \quad \text{for} \quad j_L = j_R = 0
\end{align}

(B.21)

whereas for $A_i$ with $i = \psi, \theta, \bar{\psi}$ one gets the following leading order behavior

\begin{align}
A_i &\sim r^{2j_L}, \quad \text{for} \quad j_L = j_R \neq 0 , \\
A_i &\sim r^2 \quad \text{for} \quad j_L = j_R = 0
\end{align}

(B.22)

However in tangent space basis all the components of gauge field $A_a = e_a^\mu A_\mu$ with $a = 1, 2, 3, 4$ have identical leading behavior

\begin{align}
A_a &\sim r^{2j_L - 1}, \quad \text{for} \quad j_L = j_R \neq 0 , \\
A_a &\sim r \quad \text{for} \quad j_L = j_R = 0
\end{align}

(B.23)

The consequences of these regularity properties are analyzed in detail in section 3 and in appendix A. Since for computational purposes fermions are written in terms of cohomological variables, their regularity behavior can be easily deduced from that of the gauge field $A_\mu$ given above.

C. $N = 2$ off-shell SUSY transformations

For completeness we reproduce here the supersymmetric transformation rules of vector and matter multiplet for general background auxiliary fields.

C.1. vector multiplet

\begin{align}
QA_\mu &= i\tilde{\xi}^A \sigma^\mu \tilde{\lambda}_A - i\tilde{\xi}^A \sigma^\mu \lambda_A , \\
Q\phi &= -i\tilde{\xi}^A \lambda_A , \quad \bar{Q}\bar{\phi} = i\tilde{\xi}^A \bar{\lambda}_A , \\
Q\lambda_A &= \frac{1}{2} \sigma^{\mu\nu} \xi_A (F_{\mu\nu} + 8\phi T_{\mu\nu}) + 2\sigma^\mu \xi_A D_\mu \phi + \sigma^\mu D_\mu \xi_A \phi + 2i\xi_A [\phi, \bar{\phi}] + D_A B , \\
Q\bar{\lambda}_A &= \frac{1}{2} \bar{\sigma}^{\mu\nu} \bar{\xi}_A (F_{\mu\nu} + 8\bar{\phi} \bar{T}_{\mu\nu}) + 2\bar{\sigma}^\mu \bar{\xi}_A D_\mu \bar{\phi} + \bar{\sigma}^\mu D_\mu \bar{\xi}_A \bar{\phi} - 2i\bar{\xi}_A [\bar{\phi}, \phi] + D_A B , \\
QD_{AB} &= -i\tilde{\xi}_A \bar{\sigma}^\mu D_\mu \lambda_B - i\bar{\xi}_B \sigma^\mu D_\mu \lambda_A + i\xi_A \sigma^\mu D_\mu \bar{\lambda}_B + i\xi_B \bar{\sigma}^\mu D_\mu \bar{\lambda}_A \\
&\quad - 2[\phi, \xi_A \bar{\lambda}_B + \bar{\xi}_B \lambda_A] + 2[\bar{\phi}, \xi_A \lambda_B + \bar{\xi}_B \bar{\lambda}_A] .
\end{align}

(C.1)

C.2. matter multiplet

\begin{align}
Qq_A &= -i\tilde{\xi}_A \psi + i\bar{\xi}_A \bar{\psi} , \\
Q\psi &= 2\sigma^\mu \xi_A D_\mu q^A + \sigma^\mu D_\mu \xi_A q^A - 4i\xi_A \bar{\phi} q^A + 2\xi A F^A , \\
Q\bar{\psi} &= 2\bar{\sigma}^\mu \bar{\xi}_A D_\mu \bar{q}^A + \bar{\sigma}^\mu D_\mu \bar{\xi}_A \bar{q}^A - 4i\bar{\xi}_A \bar{\phi} q^A + 2\bar{\xi} A \bar{F}^A , \\
QF_A &= i\tilde{\xi}_A \sigma^\mu D_\mu \bar{\psi} - 2\xi A \bar{\phi} \psi - 2\xi A \lambda B q^B + 2i\bar{\xi}_A (\sigma^\mu T_{\mu\nu}) \bar{\psi} \\
&\quad - i\bar{\xi}_A \bar{\sigma}^\mu D_\mu \psi + 2\bar{\xi}_A \bar{\phi} \bar{\psi} + 2\bar{\xi}_A \bar{\lambda} B \bar{q}^B - 2i\bar{\xi}_A (\bar{\sigma}^\mu \bar{T}_{\mu\nu}) \bar{\psi} .
\end{align}

(C.2)
D. Differential equations

Kernel

Varying $V_{vee}$ with respect to $\chi_3, \chi_+, \chi_-, c$ and $\bar{c}$ respectively generates following differential equations for the fields belonging to kernel of $D_{10}$:

\[
\mathcal{E}_1 = \frac{1}{2} (4 \sin(\theta) \tan(r) \partial_\theta A_3(\theta, r) - 4q_L \tan(r) \Lambda(\theta, r) + 4q_R \cos(\theta) \tan(r) A_3(\theta, r))
\]

\[
- \left( \sin(\theta) \sin^2(r) \partial_\theta A_\tau(\theta, r) + \sin^2(r) A_\tau(\theta, r)(q_L - q_R \cos(\theta)) + 2i \sin(\theta) \sin^2(r) \partial_r A_+ (\theta, r) \right)
\]

\[
+ \left( 2i q_R \sin(\theta) \sin(2r) A_+(\theta, r) - 2i \sin(\theta) \sin^2(r) \tan(r) \partial_\theta \Lambda(\theta, r) + 2i q_L \sin^2(r) \tan(r) \Lambda(\theta, r) \right)
\]

\[
- \left( 2i q_R \cos(\theta) \sin^2(r) \tan(r) \Lambda(\theta, r) \right),
\]

\[
\mathcal{E}_2 = \frac{1}{2} \tan(r) (i (2 \sin^2(r) \Lambda(\theta, r)(q_L - q_R \cos(\theta)) + \sin(\theta) (4i \partial_\theta A_3(\theta, r)) + (i \sin(r) \cos(\theta) \partial_\theta A_\tau(\theta, r))))
\]

\[
+ \left( \left( \left( \sin(2r) \partial_r A_- (\theta, r) - 4q_R \cos^2(r) A_- (\theta, r) \right) + \left( 2 \sin^2(r) \partial_\theta A_\Lambda(\theta, r) \right) \right) - 4 \Lambda(\theta, r)(q_L - q_R \cos(\theta))) \right)
\]

\[
+ \left( \sin(r) \cos(r) A_\tau(\theta, r)(q_R \cos(\theta) - q_L)) \right),
\]

\[
\mathcal{E}_3 = \frac{1}{4} \sin(r) \cos(r) (\sin(\theta) (\sec(r) (8i \tan(r) \partial_\theta A_3(\theta, r) - 8 \partial_\theta A_-(\theta, r) + 8 \partial_\theta A_+(\theta, r))))
\]

\[
+ \left( \left( \left( \sin(2r) \partial_r A_-(\theta, r) - 4q_R \cos^2(r) A_-(\theta, r) + 4 \tan^2(r) \Lambda(\theta, r) \right) + 4i(\cos(2r) + 3) \sec^3(r) A_3(\theta, r) \right) \right)
\]

\[
+ \left( 4i q_R \sin(r) A_\tau(\theta, r) - 8 \sec(r) A_-(\theta, r) ((q_R + 1) \cos(\theta) - q_L) + 8 \sec(r) A_+(\theta, r) \cos(\theta) \right)
\]

\[
+ \left( q_L - q_R \cos(\theta) \right),
\]

\[
\mathcal{E}_4 = \frac{1}{4} \sin(r) (\sin(\theta) (16q_R A_3(\theta, r) + \partial_\theta A_\tau(\theta, r) - \cos(2r) \partial_\theta A_\tau(\theta, r) + 3 \sin(2r) A_\tau(\theta, r))
\]

\[
+ \left( 8i \partial_\theta A_- (\theta, r) + 8i \partial_\theta A_+(\theta, r) \right) + 8i A_-(\theta, r) ((q_R + 1) \cos(\theta) - q_L) + 8i A_+(\theta, r) \cos(\theta)
\]

\[
+ q_L - q_R \cos(\theta) \right),
\]

\[
\mathcal{E}_5 = \frac{1}{2} \left( \left( \left( \sin(\theta) \sin^2(r) \partial_\theta A_\tau(\theta, r) + \sin^2(r) A_\tau(\theta, r)(q_L - q_R \cos(\theta)) + 2i \sin(\theta) \sin^2(r) \partial_r A_+ (\theta, r) \right) \right)
\]

\[
+ \left( 2i q_R \sin(\theta) \sin(2r) A_+(\theta, r) - 2i \sin(\theta) \sin^2(r) \tan(r) \partial_\theta A_\Lambda(\theta, r) + 2i q_L \sin^2(r) \tan(r) \Lambda(\theta, r) \right)
\]

\[
- \left( 2i q_R \cos(\theta) \sin^2(r) \tan(r) \Lambda(\theta, r) \right) \right). \quad (D.1)
\]
Cokernel

Similar to kernel equations, to get the zero mode differential equations for co-kernel fields one has to vary $V_{vec}$ with respect to $A_+, A_-, A_3, A_4$ and $\Lambda$ respectively to generate the following:

\[
\begin{align*}
\mathcal{CE}_1 &= \frac{1}{8} \sin(r) \left( 4q_L \partial_\theta c(\theta, r) + 2i \partial_\theta \bar{c}(\theta, r) - 2i (q_L - 1) \cos(r) \chi_- (\theta, r) \right) \\
&+ \left( (2 \partial_\theta \chi_3 (\theta, r) + i \sin(r) \partial_r \chi_- (\theta, r)) - 2q_R (2q_L c(\theta, r) + i \bar{c}(\theta, r) + \chi_3 (\theta, r)) \right) \\
&+ \left( 2q_L \cos(\theta) (2q_L c(\theta, r) + i \bar{c}(\theta, r) + \chi_3 (\theta, r)) \right),
\end{align*}
\]

\[
\begin{align*}
\mathcal{CE}_2 &= \frac{1}{8} \sin(r) \left( i \sin(\theta) \left( -4i q_L \partial_\theta c(\theta, r) + 2 \partial_\theta \bar{c}(\theta, r) + 2(q_L + 1) \cos(r) \chi_+ (\theta, r) \right) \\
&+ \left( (2i \partial_\theta \chi_3 (\theta, r) + \sin(r) \partial_r \chi_+ (\theta, r)) + 4q_L c(\theta, r) (q_R - q_L \cos(\theta)) \right) \\
&+ \left( 2i \bar{c}(\theta, r) (q_R q_L \cos(\theta)) + 2q_L \cos(\theta) \chi_3 (\theta, r) - 2q_R \chi_3 (\theta, r) \right),
\end{align*}
\]

\[
\begin{align*}
\mathcal{CE}_3 &= \frac{1}{4} \tan(\theta) \left( (2q_L \cos(r) (-\bar{c}(\theta, r) + 2i q_L c(\theta, r)) + i \sin(r) \partial_r \chi_3 (\theta, r)) \\
&+ \left( (\partial_\theta \chi_- (\theta, r) - \partial_\theta \chi_+ (\theta, r)) - \cos(\theta) (q_L \chi_- (\theta, r) + (q_L + 1) \chi_+ (\theta, r)) \right) \\
&+ \left( \cos(\theta) + q_R \right) \chi_- (\theta, r) + q_R \chi_+(\theta, r),
\end{align*}
\]

\[
\begin{align*}
\mathcal{CE}_4 &= \frac{1}{8} \sin^2(r) \left( (2q_L \cos(r) \chi_3 (\theta, r)) + \sin(r) \partial_r \bar{c}(\theta, r) \right) \\
&- \left( (\partial_\theta \chi_- (\theta, r) - \partial_\theta \chi_+ (\theta, r)) + \cos(\theta) ((q_L - 1) \chi_- (\theta, r) - (q_L + 1) \chi_+ (\theta, r)) \right) \\
&+ \left( q_R (\chi_+ (\theta, r) - \chi_- (\theta, r)) \right),
\end{align*}
\]

\[
\begin{align*}
\mathcal{CE}_5 &= -\frac{1}{16} \sin(r) \left( \sin(\theta) \left( 2 \sin^2(r) \left( (2q_L \cos(r) \chi_3 (\theta, r)) + \sin(r) \partial_r \bar{c}(\theta, r) - 3 \chi_3 (\theta, r) \right) + 8q_R^2 \cos(\theta) \right) \right) \\
&+ \left( (3 \sin(2r) \partial_r c(\theta, r)) + 8 \cos(\theta) \partial_\theta c(\theta, r) - 8 \cos(\theta) (\csc(\theta) \left( q_R^2 \right)) \right) \\
&+ \left( \left( q_R^2 - q_L^2 \sin(\theta) \sin^2(r) - 2q_L q_R \cos(\theta) \right) \right).
\end{align*}
\]

E. Localizing fermionic functional $V_{hyper}$

Instead of writing down the kernel and co-kernel differential equations for the hyper multiplet, we only provide the fermionic functional in cohomological variables is given

\[
\begin{align*}
V_{hyper} &= \frac{1}{64} \sin^2(r) \left[ \left( 2 (\sin(\theta) \partial_\theta q_{11}(\theta, r) \Sigma_{12}(\theta, r) - i \sin(\theta) \sin(r) \partial_r q_{11}(\theta, r) \Sigma_{22}(\theta, r)) \right) \\
&+ \left( (q_{11}(\theta, r) \Sigma_{12}(\theta, r) \cos(\theta) + 2q_L \cos(\theta) - 2q_R) - i (2q_L + 1) \sin(\theta) \cos(r) \Sigma_{22}(\theta, r)) \right) \\
&- \left( (2 \sin(\theta) \partial_\theta q_{12}(\theta, r) \Sigma_{11}(\theta, r) + i \sin(\theta) \sin(r) \partial_r q_{12}(\theta, r) \Sigma_{21}(\theta, r)) \right) \\
&- \left( (i \sin(\theta) \sin(r) \partial_r q_{12}(\theta, r) \Sigma_{11}(\theta, r) + 2 \sin(\theta) \partial_\theta q_{12}(\theta, r) \Sigma_{21}(\theta, r)) \right) \\
&+ 2 \left( (i q_L \sin(\theta) \cos(r) q_{12}(\theta, r) \Sigma_{12}(\theta, r) - 2q_L \cos(\theta) q_{12}(\theta, r) \Sigma_{22}(\theta, r)) \right) \\
&+ 2 \left( (q_L q_{12}(\theta, r) \Sigma_{22}(\theta, r) - i \sin(\theta) \sin(r) \partial_\theta q_{12}(\theta, r) \Sigma_{22}(\theta, r) + \cos(\theta) q_{12}(\theta, r) \Sigma_{22}(\theta, r)) \right) \\
&+ \left( \left( q_{12}(\theta, r) \Sigma_{12}(\theta, r) \right) \right) \\
&+ \left( \left( q_{12}(\theta, r) \Sigma_{11}(\theta, r) \right) \right) \\
&+ \left( \left( q_{12}(\theta, r) \Sigma_{21}(\theta, r) \right) \right)
\right].
\]
F. Analysis of kernel and cokernel equations for $j_L = 0$

**Hemisphere**

With a boundary at $r = \frac{\pi}{2}$, the regular solutions of kernel equations, with the unique choice $q_L = 0, q_R = 0$, are

$$
A_r^{(0,0,0)}(r) = -\frac{1}{12} b_0 \left( 3 \sin \left( \frac{r}{2} \right) + \sin \left( \frac{3r}{2} \right) \right) \sec^3 \left( \frac{r}{2} \right),
$$

$$
A_{3}^{(0,0,0)}(r) = \frac{1}{288} \left( \csc^2(r) \left( \cos(2r) \left( -24a_0 \log \left( \sin \left( \frac{r}{2} \right) \right) + 24a_0 \log(\sin(r)) \right) \right) \right)
+ \left( 24a_0 \log \left( \cos \left( \frac{r}{2} \right) \right) - 61a_0 + 72d_0 \right) + 2a_0 \cos(4r) - 72a_0 \log \left( \sin \left( \frac{r}{2} \right) \right)
+ 72 \left( \bar{a}_0 \log(\sin(r)) + 72a_0 \log \left( \cos \left( \frac{r}{2} \right) \right) - 96a_0 \cos(r) \log \left( \sec^2 \left( \frac{r}{2} \right) \right) - 57a_0 + 216d_0 \right)
- \frac{4 \cos(r)a_0(24 \log(2) - 29) + 72d_0 \sqrt{\sin(r) \cos(r)}}{\sin^2(r)},
$$

$$
A^{(0,0,0)}(r) = \frac{1}{24} \left( -9a_0 \cot^2(r) + 2a_0 \csc^2 \left( \frac{r}{2} \right) + a_0 \csc^2(r) - 2a_0 \sec^2 \left( \frac{r}{2} \right) \right)
- 8 \left( \bar{a}_0 \log \left( \sin \left( \frac{r}{2} \right) \right) + 8a_0 \log(\sin(r)) + 8a_0 \log \left( \cos \left( \frac{r}{2} \right) \right) + 24d_0 \right). \quad (F.1)
$$

Similarly cokernel equations can be solved easily for $j_L = 0$ to give following regular solutions

$$
\chi_3^{(0,0,0)}(r) = C_1 \frac{1}{2}, \quad \tilde{c}^{(0,0,0)}(r) = C_2,
$$

$$
\tilde{c}^{(0,0,0)}(r) = \frac{1}{16} \left( 2C_1 \cot^2(r) - 2C_1 \csc^2 \left( \frac{r}{2} \right) - C_1 \csc^2(r) + 2C_1 \sec^2 \left( \frac{r}{2} \right) + 8C_1 \log \left( \sin \left( \frac{r}{2} \right) \right) \right)
- 8 \left( C_1 \log(\sin(r)) - 8C_1 \log \left( \cos \left( \frac{r}{2} \right) \right) + 16C_0 \right). \quad (F.2)
$$

where $b_0, \bar{a}_0, d_0, C_0, C_1, C_2$ are constant functions. However with Dirichlet BCs $A_1|_{r=\frac{\pi}{2}} = 0, A_3|_{r=\frac{\pi}{2}} = 0$ and $\phi_2|_{r=\frac{\pi}{2}} = 0$ imposed at the boundary for the fluctuation fields and by the requirement that these BCs should be closed under supersymmetry, we get that all of the constants $b_0, \bar{a}_0, d_0, C_0, C_1, C_2$ must vanish.

G. Identities used

**G.1. For kernel**

For simplicity we take the basis for the harmonics as $e^{i(q_L \psi + q_R \phi)} Y^{(j_L, q_L, q_R)}(\theta)$ satisfying

$$
J^- (e^{i(q_L \psi + q_R \phi)} Y^{(j_L, q_L, q_R)}(\theta)) = e^{i((q_L - 1) \psi + q_R \phi)} Y^{(j_L, q_L - 1, q_R)}(\theta)
$$

and

$$
J^+ (e^{i(q_L \psi + q_R \phi)} Y^{(j_L, q_L, q_R)}(\theta)) = (j_L - q_L)(j_L + q_L + 1)e^{i((q_L + 1) \psi + q_R \phi)} Y^{(j_L, q_L + 1, q_R)}(\theta)
$$

(G.1)
This basis is not normalized but it is irrelevant for the present analysis. It is trivial to see the following identities hold

\[ Y^{(j_L, j_L+1, q_R)}(\theta) = 0, \quad Y^{(j_L, -j_L-1, q_R)}(\theta) = 0, \]
\[ A_+^{(j_L, j_L+1, q_R)}(r) = 0, \quad A_+^{(j_L, -j_L-1, q_R)}(r) = 0, \]
\[ A_-^{(j_L, j_L+1, q_R)}(r) = 0, \quad A_-^{(j_L, -j_L-1, q_R)}(r) = 0, \]
\[ A_+^{(j_L, j_L+2, q_R)}(r) = 0, \quad A_+^{(j_L, -j_L-2, q_R)}(r) = 0, \]
\[ A_-^{(j_L, j_L+2, q_R)}(r) = 0, \quad A_-^{(j_L, -j_L-2, q_R)}(r) = 0, \]
\[ A_3^{(j_L, j_L+1, q_R)}(r) = 0, \quad A_3^{(j_L, -j_L-1, q_R)}(r) = 0, \]
\[ A_4^{(j_L, j_L+1, q_R)}(r) = 0, \quad A_4^{(j_L, -j_L-1, q_R)}(r) = 0. \] (G.3)

To evaluate the Kernel equations for different SU(2)_R charges we need the following identities. For \(E_1, E_2\)

\[ \partial_\theta Y^{(j_L, q_L+1, q_R)}(\theta) = i Y^{(j_L, q_L, q_R)}(\theta) - (q_R + (1 + q_L) \cos(\theta)) \csc(\theta) \times Y^{(j_L, q_L+1, q_R)}(\theta), \]
\[ \partial_\theta Y^{(j_L, q_L-1, q_R)}(\theta) = -(q_R + \cos(\theta)(1 - q_L)) \csc(\theta) Y^{(j_L, q_L-1, q_R)}(\theta) + i(1 + j_L - q_L)(j_L + q_L)s Y^{(j_L, q_L, q_R)}(\theta). \] (G.4)

For \(E_3\)

\[ \partial_\theta Y^{(j_L, q_L+q_R)}(\theta) = i Y^{(j_L, q_L-1, q_R)}(\theta) + (-q_L \cot(\theta) + q_R \csc(\theta)) \times Y^{(j_L, q_L, q_R)}(\theta). \] (G.5)

and for \(E_4\)

\[ \partial_\theta Y^{(j_L, q_L, q_R)}(\theta) = (-q_R + q_L \cos(\theta)) \csc(\theta) Y^{(j_L, q_L+q_R)}(\theta) + i(j_L - q_L)(1 + j_L + q_L)Y^{(j_L, 1+q_L,q_R)}(\theta). \] (G.6)

G.2. For Cokernel

We need the following relations

\[ Y^{(j_L, j_L+1, q_R)}(\theta) = 0, \quad Y^{(j_L, -j_L-1, q_R)}(\theta) = 0, \]
\[ \chi_+^{(j_L, j_L+1, q_R)}(r) = 0, \quad \chi_+^{(j_L, -j_L-1, q_R)}(r) = 0, \]
\[ \chi_-^{(j_L, j_L+1, q_R)}(r) = 0, \quad \chi_-^{(j_L, -j_L-1, q_R)}(r) = 0, \]
\[ \chi_+^{(j_L, j_L+2, q_R)}(r) = 0, \quad \chi_+^{(j_L, -j_L-2, q_R)}(r) = 0, \]
\[ \chi_-^{(j_L, j_L+2, q_R)}(r) = 0, \quad \chi_-^{(j_L, -j_L-2, q_R)}(r) = 0, \]
\[ \chi_3^{(j_L, j_L+1, q_R)}(r) = 0, \quad \chi_3^{(j_L, -j_L-1, q_R)}(r) = 0, \]
\[ c_{a_1}^{(j_L, j_L+1, q_R)}(r) = 0, \quad c_{a_1}^{(j_L, -j_L-1, q_R)}(r) = 0, \]
\[ c^{(j_L, j_L+1, q_R)}(r) = 0, \quad c^{(j_L, -j_L-1, q_R)}(r) = 0. \] (G.7)
Different useful identities involving derivative of harmonics are, for $C\mathcal{E}_1, C\mathcal{E}_2$
\[
\partial_\theta Y^{(jL,qL+1,qR)}(\theta) = iY^{(jL,qL,qR)}(\theta) - (-q_R + (1 + q_L \cos(\theta)) \csc(\theta)) \times Y^{(jL,qL+1,qR)}(\theta),
\]
\[
\partial_\theta Y^{(jL,qL-1,qR)}(\theta) = i(j_L + q_L)(1 + j_L - q_L)Y^{(jL,qL,qR)}(\theta) - (q_R + \cos(\theta)(1 - q_L)) \csc(\theta)Y^{(jL,qL-1,qR)}(\theta).
\]
for $C\mathcal{E}_3$
\[
\partial_\theta Y^{(jL,qL,qR)}(\theta) = iY^{(jL,qL-1,qR)}(\theta) + (-q_L \cot(\theta) + q_R \csc(\theta)) Y^{(jL,qL,qR)}(\theta),
\]
for $C\mathcal{E}_4$
\[
\partial_\theta Y^{(jL,qL,qR)}(\theta) = i(j_L - q_L)(1 + j_L + q_L)Y^{(jL,qL+1,qR)}(\theta) + (q_L \cos(\theta) - q_R) \csc(\theta)Y^{(jL,qL,qR)}(\theta),
\]
and for $C\mathcal{E}_5$
\[
\partial_\theta^2 Y^{(jL,qL,qR)} = -(j_L(1 + j_L) + 2q_Lq_R \cot(\theta) \csc(\theta) - (q_L^2 + q_R^2) \csc(\theta)^2 Y^{(jL,qL,qR)} - \cot(\theta)\partial_\theta Y^{(jL,qL,qR)}.
\]

**H. Analysis for $|q_L| < j_L$**

For $q_L = 0$
\[
A_4^{(jL,0,-qR)}(r) = \csc^2(r)(a_4^0((\cos(r) + 1) \csc(r))^{-2j_L-1} + b_4^0((\cos(r) + 1) \csc(r))^{2j_L+1})
\]
(H.1)

Again the only non-trivial solution here is $a_4^0 = 0, b_4^0 = 0$.

For the values of $q_L \neq 0$ in the range $|q_L| < j_L$ the analysis is a bit involved. We start from
\[
A_3(j_L, -q_L, -q_R)(r) = \frac{1}{4q_L}((4j_L + 4j_L^2 - 2q_L^2 + (-3 + 2q_R^2) \cos(2r)) \cot(r)A_4(j_L, -q_L, -q_R)(r) - \cos(r)(5 \cos(r)\partial_r A_4(j_L, -q_L, -q_R)(r) + \sin(r)\partial_\theta^2 A_4(j_L, -q_L, -q_R)(r))
\]
(H.2)

To get the solution for $A_4(j_L, -q_L, -q_R)(r)$ at $r = 0$ we plug the following ansatz into the kernel equation $\mathcal{E}_3$
\[
A_4^{(jL,-qL,-qR)}(r) = r^\alpha
\]
(H.3)

and after trivial rescaling get the indicial equation
\[
8(32j_L^3 + 16j_L^4 - 8j_L(1 + \alpha)^2 + (1 + \alpha)^2(-3 + 2\alpha + \alpha^2) - 8j_L^2(-1 + 2\alpha + \alpha^2)) = 0
\]
(H.4)

which is solved to yield
\[
\alpha = -3 - 2j_L, -1 - 2j_L, 1 + 2j_L, 1 + 2j_L
\]
(H.5)
and ansatz at \( r = \pi - r \)
\[
A_4^{(j_L, q_L, \nu)}(r) = (\pi - r)^{\alpha}
\]
(H.6)
gives the same indicial equation as above and hence the same solution for \( \alpha \). Smooth solutions correspond to \( \alpha = (2j_L + 1), (2j_L - 1) \).

Now in general it is very difficult to solve a fourth order ordinary differential equations. But fortunately in this case it is easy to see that no non-trivial solution exists by using a simple trick: if the smooth solution at \( r = 0 \) interpolates to smooth solution at \( r = \pi \) then we can try to construct out of the \( \mathcal{E}_3 \) a positive definite integral which will indicate a contradiction.

Let’s suppose a general solution exists for \( A_4^{(j_L, q_L, \nu)}(r) \) and define a function \( S(r) \)
\[
A_4^{(j_L, q_L, \nu)}(r) = F(r)
S(r) = F(r) \left( 4(3 + 8j_L + 4j_L^2) + 64j_L^3 + 32j_L^4 - 15q_L^2 - 32j_Lq_L^2 - 32j_Lq_L^2 + 12q_L^4 + 2(-9 + 14q_L^2 + 4j_L(-3 + 4q_L^2) \right)
+ 10(-3 + 2q_L^2) \cos(2r) \partial_r F(r) + 2 \sin(r)((-5 + 16j_L + 16j_L^2 + 8q_L^2 + (-37 + 8q_L^2) \cos(2r)) \partial^2 F(r) - 2 \sin(r)(10 \cos(r) \partial^3 F(r) + \sin(r) \partial^4 F(r))) \right). \tag{H.7}
\]
We perform integration by parts until the integrand is converted into a sum of positive terms plus the total derivative terms. The total derivative terms becomes the following boundary contributions
\[
S_{\text{boundary}} = -2(7 + 8j_L + 8j_L^2 - 4q_L^2 + 4(-1 + q_L^2) \cos(2r)) F(r)^2 \sin(2r)
- 8 \sin(r)^3 \partial_r F(r)(\cos(r) \partial_r F(r) + \sin(r) \partial^2 F(r))
- 4F(r) \sin(r)^2((7 + 16j_L + 16j_L^2 - 8q_L^2 + (-13 + 8q_L^2) \cos(2r) \partial_r F(r)
- 2 \sin(r)(6 \cos(r) \partial^3 F(r) + \sin(r) \partial^4 F(r))). \tag{H.8}
\]
We now first check that the boundary term vanishes. By assumption of the smoothness of \( F(r) \) it goes at least as \( r^{2j_L - 1} \) at \( r = 0 \). This means that \( \partial^3 F(r) F(r) \) and \( \partial^2 F(r) \partial_r F(r) \) goes like \( r^{4j_L - 5} \). The coefficient of these terms is \( \sin(r)^4 \) which goes as \( r^4 \). Combining this we get \( r^{2j_L - 1} \) which vanishes if \( j_L \geq 1 \). Similarly all the other terms can be easily seen to vanish at \( r = 0 \) and \( r = \pi \).

Next we define another function \( S_{\text{bulk}} \) related to \( S_{\text{boundary}} \) as
\[
S_{\text{bulk}}(r) = S(r) - \partial_r S_{\text{boundary}}
= 8 \sin(r)^4(\partial^2 F(r))^2 + 4(9 + 16j_L + 16j_L^2 - 8q_L^2
+ (9 + 8q_L^2) \cos(2r)) \sin(r)^2(\partial_r F(r))^2
+ 4(3 + 8j_L + 4j_L^2 + 64j_L^3 + 32j_L^4 - 15q_L^2 - 32j_Lq_L^2
- 32j_L^2q_L^2 + 16q_L^4 + (-11 + 24q_L^2 + 16j_L(-1 + 2q_L^2))
+ 16j_L^2(-1 + 2q_L^2)) \cos(2r) + (5 - 9q_L^2 + 4q_L^4) \cos(4r)). \tag{H.9}
\]

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Notice that the coefficient of $(\partial_r^2 F(r))^2$ is positive definite. Similarly in the coefficient of $(\partial_r F(r))^2$, $(1 - \cos(2r))$ is non-negative and bounded by 2 and since $j_L^2 > q_L^2$, the coefficient of $(\partial_r F(r))^2$ is non-negative.

Now looking at the coefficient of $F(r)^2$ term, since only even powers of $q_L$ appear we can take $q_L$ to be positive. Since $q_L < j_L$ in units of 1, we can set $j_L = q_L + n$ for $n = 0, 1, 2, 3,...$. This coefficient is a function of $r$, so we will find its minimum as a function of $r$ for fixed $q_L$ and $n$.

If the value of the coefficient at the minimum is non-negative for the allowed values of $q_L$ and $n$ then this means that $S_{bulk}$ is a sum of non-negative terms and hence positive definite, which is a contradiction to $E_3$.

$$
\partial_r \text{Coefficient}(F(r)) = \sin(2r)(4(-2(-11 + 24q_L^2) - 16q_L^2 + 16(1 + n + q_L) + 16(1 + n + q_L)^2(-1 - 2q_L^2)) - 8(5 - 9q_L^2 + 4q_L^2)\cos(2r)).
$$

(H.10)

For $q_L = 1$ it evaluates to $-8(93 + 80n + 16n^2)$, so one solution is $r = 0, \pi, \pi$ for $\sin(2r) = 0$.

For $q_L \neq 1$ the solution is

$$
\cos(2r) = \frac{8(-2q_L^4 - 52q_L^3 - 169q_L^2 + 20q_L + 80) + 48q_L + 43}{4(q_L^2 - q_L^2 + 5)}.
$$

(H.11)

It can be easily shown that the absolute value of the right hand side is greater than 1 and hence the solution does not exist. As an example for $q_L = \frac{1}{2}$

$$
\cos(2r) = \frac{8 + 8n + 2n^2}{3} > 1
$$

(H.12)

so this stationary point does not exist.

For general argument for $q_L > 1$ take the numerator and denominator of the general expression for $\cos(2r)$ separately as

$$
X_N = 8(-2q_L^4 - 52q_L^3 - 169q_L^2 + 20q_L + 80) + 48q_L + 43,
$$

$$
X_D = 4(q_L^2 - q_L^2 + 5).
$$

(H.13)

and observe that for $q_L = \frac{3}{4}, X_N < 0$ and $X_D > 0$. Now for all $q_L > 1$ i.e. for $q_L = p + 1$ for $p$ increasing in increments of $\frac{1}{2}$, let’s perform a Taylor series expansion of $X_N$ around $n = 0$

$$
X_N = -16n^2(2p^2 + 4p + 1) - 16n(4p^3 + 18p^2 + 22p + 5) - 16p^4 - 160p^3 - 456p^2 - 448p - 93 + O(n)^5.
$$

(H.14)

For $q_L > 1$ this is negative. Furthermore the denominator $X_D = 4(1 - q_L^2)(5 - 4q_L^2)$ is positive for $q_L > 1$ as then $q_L \geq \frac{3}{2}$.

Next we are going to show that $|X_N| = X_N > X_D$, or in other words $|X_N| - X_D$ is positive. To show this again perform Taylor series expansion of $|X_N| - X_D$ around $n = 0$ for $q_L = p + 1$

$$
|X_N| - X_D = 16n^2(2p^2 + 4p + 1) + 16n(4p^3 + 18p^2 + 22p + 5) + 96p^3 + 396p^2 + 456p + 93 + O(n)^5.
$$

(H.15)
It is clear that for $q_L > 1$ each of these terms is positive so $\frac{|X_N|}{X_D} > 1$ for all $q_L > 1$. Therefore $\cos(2r) = \frac{|X_N|}{X_D}$ has no solution for real $r$. This means that the only stationary point is $\sin(2r) = 0$ i.e. for $r = 0, \frac{\pi}{2}, \pi$. Next taking the second derivative of the coefficient of $F(r)^2$ and evaluating at $r = 0, \pi$, for $q_L = \frac{1}{2}$ we get $128(1 + n)(3 + n) > 0$, so stationary point at $r = 0, \pi$ are minimum for $q_L = \frac{1}{2}$. To go for $q_L = p + 1 \geq 1$, now perform Taylor series expansion of the second derivative around $p = 0$ we get

$$-16(93 + 16n(5 + n)) - 128(55 + 44n + 8n^2)p - 64(129 + 8n(9 + n))p^2 - 512(7 + 2n)p^3 - 512p^4 + O(p)^5$$

(H.16)

Note that each term is negative, so at $r = 0, \pi$ this is maximum for $q_L \geq 1$. Now for $r = \frac{\pi}{2}$ and $q_L = \frac{1}{2}$, evaluating the second derivative we get $-128(6 + 4n + n^2) < 0$. Next for $q_L = p + 1 \geq 1$, expand the second derivative in the Taylor series around $r = \frac{\pi}{2}, p = 0$ to get

$$16(93 + 16n(5 + n)) + 128(57 + 44n + 8n^2)p + 64(99 + 8n(9 + n))p^2 + 512(3 + 2n)p^3 + O(p)^5$$

(H.17)

Each term is positive, so stationary point at $r = \frac{\pi}{2}$ is minimum for $q_L \geq 1$. In the final step we evaluate the coefficient of $F(r)^2$ at the minimum and show that it is non-negative in all the cases. First consider $q_L = \frac{1}{2}$ and evaluating the coefficient at the minimum value $r = 0$

$$4(417 + 928n + 744n^2 + 256n^3 + 32n^4) > 0$$

(H.18)

Next consider $q_L \geq 1$, in this case the minimum is at $r = \frac{\pi}{2}$ and the series expansion of the coefficient around $n = 0$ and $q_L = p + 1$ evaluates to

$$12(305 + 328p + 88p^2) + 32(215 + 206p + 48p^2)n + 32(143 + 104p + 16p^2)n^2$$

$$+ 256(5 + 2p)n^3 + 128n^4 + O(n)^7$$

(H.19)

showing that each coefficient in this series expansion is positive, so the coefficient of $F(r)^2$ is already positive at the minimum value as a function of $r$.

This proves that $S_{bulk}$ is a sum of non-negative terms, therefore each term must vanish, which implies that there is no non-singular solution for $F(r)$. Hence the conclusion is that Kernel of $D_{10}$ is empty.
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