Fundamental irreversibility and time’s arrow of the classical three-body problem. New approaches and ideas in the study of dynamical systems

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The three-body general problem is formulated as a problem of geodesic trajectories flows on the Riemannian manifold. It is proved that a curved space with local coordinate system allows to detect new hidden symmetries of the internal motion of a dynamical system and reduce the three-body problem to the system of 6th order. It is shown that the equivalence of the initial Newtonian three-body problem and the developed representation provides coordinate transformations in combination with the underdetermined system of algebraic equations. The latter makes a system of geodesic equations relative to the evolution parameter, i.e., to the arc length of the geodesic curve, irreversible. Equations of deviation of geodesic trajectories characterizing the behavior of the dynamical system as a function of the initial parameters of the problem are obtained. To describe the motion of a dynamical system influenced by the external regular and stochastic forces, a system of stochastic differential equations (SDE) is obtained. Using the system of SDE, a partial differential equation of the second order for the joint probability distribution of the momentum and coordinate of dynamical system in the phase space is obtained. A criterion for estimating the degree of deviation of probabilistic current tubes of geodesic trajectories in the phase and configuration spaces is formulated. The mathematical expectation of the transition probability between two asymptotic subspaces is determined taking into account the multichannel character of the scattering.

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I. INTRODUCTION

The three-body problem is one of the oldest and most complex problem in classical mechanics \[2–6\]. However, despite this, the study of this problem is still relevant in connection with its wide application in solving various applied problems, from celestial mechanics to the atom-molecular collision. As Bruns showed \[7\], the problem in the phase space is described by 18 degrees-of-freedom, while the integrals-of-motion are only by 10. The latter condition does not allow to solve the problem in the same way as it is done for two bodies, and therefore it is considered to belong to the class of non-integrable systems, which currently is also called Poincaré’s systems. It is important to note, that the three-body problem has certain symmetries and singularities. In particular, the reduction procedure is based on using the symmetries to reduce the number of degrees-of-freedom. In the light of the above, it is obvious that the Newtonian three-body problem can, generally speaking, be reduced to a system of 8th order or, what is the same, to the system of 8 ordinary differential equations of 1th order describing the evolution of a dynamical system in phase space. As it is known, the three-body problem served as the main source of development in many directions in mathematics and physics since the time of Newton, however just Poincaré opened a new era, developing geometric, topological and probabilistic methods for studying a nontrivial and highly complex behavior of this dynamical problem.

The three-body problem arising from celestial mechanics \[8–10\] continues to develop rapidly and it is widely used in particularly in the connection with the problems of the problems of atom-molecular physics \[11–16\]. It suffices to say that a significant number of elementary atomic-molecular processes, including chemical reactions that take into account external influences, are described within the framework of the classical three-particle scattering model. This makes new mathematical research extremely important for the creation of effective algorithms allowing to carry out the calculations of complex multi-channel processes. It should be noted that the area of atom-molecular collisions has its own specific features that can stimulate the development of fundamentally new ideas in the realm of the dynamical systems. Recall that one of the important and insufficiently studied problems of the atom-molecular collision is the multichannel character of flowing elementary atom-molecular processes. Another important unsolved problem, which is of great importance for modern chemistry, is the consideration of the regular and stochastic influences of the
environment on the dynamics of three bodies.

At substantiating statistical mechanics, Krylov investigated the dynamical problem of \( N \) classical particles (gas relaxation) on the energy hypersurface of a particle system \( 17 \), which in mathematical sense is equivalent to studying the properties of a geodesic flow on a Riemannian manifold. Later, this method was successfully used to study the statistical properties of the non-Abelian Yang-Mills gauge field \( 18 \) and the relaxation properties of stellar systems \( 19, 20 \).

In the present paper we also use this idea and formulate the general classical three-body problem on a Riemannian manifold (on the hypersurface of the energy of the bodies system). However, in contrast to the Krylov’s representation we prove a necessary and sufficient conditions under which, the formulated representation is equivalent to the original Newtonian three-body problem. As it is shown in our previous articles \( 21, 23 \), the representation based on curved geometry and using a local coordinate system, allows us to reveal new hidden internal symmetries of the dynamical system. The latter ensures the integration of a non-integrable problem more complete, but, very importantly, it leads to the irreversibility of representation regarding to the timing parameter of the dynamical problem.

Finally, the article deals with a more general case where the interaction potential between bodies depends on their relative distances and, in addition, it has a random component, which can be explained, for example, as the influence of the environment. For this case, second-order partial differential equations for geodesic flows are obtained both in phase and in configuration spaces. The criterion for the deviation of the tubes of geodetic trajectories is determined. Based on the probability flows and the use of the law of large numbers, the mathematical expectation of the transition probability between two different asymptotic subspaces of scattering is constructed. The latter, obviously, creates new opportunities and prospects for studying the three-body problem, taking into account its wide application in various applied problems of physics and chemistry.

II. THE CLASSICAL THREE-BODY SYSTEM

The classical three-body problem in its most general formulation, is the problem of multichannel scattering, with series of possible asymptotic outcomes. Schematically, the scat-
tering process can be represented as:

\[
1 + (23) \rightarrow \begin{cases} 
1 + (23), \\
1 + 2 + 3, \\
(12) + 3, \\
(13) + 2, \\
(123)^* \rightarrow \begin{cases} 
(12) + 3, \\
(13) + 2, \\
(123)^* \rightarrow \{ \ldots \}
\end{cases}
\end{cases}
\]

Scheme 1. Here 1, 2 and 3 indicate single bodies, the bracket (.) denotes the two-body bound state, while "*" and "**" denote, respectively, some transition states of the three-body system.

Definition 1. The dynamics of three bodies in the laboratory coordinate system is described by the Hamiltonian:

\[
H(\{r\}; \{p\}) = \sum_{i=1}^{3} \frac{|p_i|^2}{2m_i} + V(\{r\}),
\]

(1)

where \(\{r\} = (r_1, r_2, r_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3\) and \(\{p\} = (p_1, p_2, p_3) \in \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3\) are the sets of radius vectors and momenta of bodies with masses \(m_1, m_2\) and \(m_3\), respectively, "*" over a symbol denotes the transposed space, and finally, \(|.|\) denotes the Euclidean norm.

Below we shall consider the most general form of the total interaction potential of a system of bodies, which depending on the relative distances between the bodies:

\[
V(\{r\}) = \widetilde{V}(|r_{12}|, |r_{13}|, |r_{23}|),
\]

(2)

where \(r_{12} = r_1 - r_2\), \(r_{13} = r_1 - r_3\), and \(r_{23} = r_2 - r_3\) are relative displacements between the bodies, in addition, the set of radius-vectors \((r_{12}, r_{13}, r_{23}) \in \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \setminus \{0\}\), which means that there are no two bodies occupying the same position. Note that in the framework of the potential (2), besides of two-particle interactions, it is also possible to take into account the contribution of three-particle interactions and as well as the influence of external fields, which makes possible, to substantially expand the range of problems associated with the
classical three-body problem. It is obvious that the configuration space for three-body
dynamics without any restriction has to be \( \mathbb{R}^9 \). In this regard, it is important to note that \( V: \mathbb{R}^9 \to \mathbb{R}^1 \) and \( \bar{V}: \mathbb{R}^3 \to \mathbb{R}^1 \), in addition, \( H: \mathbb{R}^{18} \to \mathbb{R}^1 \), recall that the Hamiltonian is a
function on the 18-dimensional phase space \( \mathbb{R}^{18} \).

The three-body Hamiltonian (1) after the Jacobi coordinate transformations [24] acquires
the form:
\[
\tilde{H} = \sum_{i=1}^{3} \frac{P_i^2}{2\mu_i} + \bar{V}(||r - \lambda_- R||, ||R||, ||r + \lambda_+ R||),
\]
where the radius-vector \( R \) denotes the relative displacement between 2 and 3 bodies, the
\( r = r_1 - r_0 \) denotes relative displacement between the particle 1 and center-of-mass of the
pair (2, 3), while \( r_0 = (m_2 r_2 + m_3 r_3)/(m_1 + m_2) \) is the radius vector the center-of-mass of
the pair (2, 3) (see Fig. 1). In addition, the following notations are made in the equation
(3) (see also [21]):
\[
P_1 = p_1 + p_2 + p_3, \quad P_2 = \frac{m_3 p_2 - m_2 p_3}{m_2 + m_3}, \quad P_3 = \frac{(m_2 + m_3)p_1 - m_1(p_2 + p_3)}{\mu_1},
\]
\[
\mu_1 = m_1 + m_2 + m_3, \quad \mu_2 = \frac{m_2 m_3}{m_2 + m_3}, \quad \mu_3 = \frac{m_1(m_2 + m_3)}{\mu_1}, \quad \lambda_- = \frac{\mu_2}{\mu_2}, \quad \lambda_+ = \frac{\mu_3}{m_3}.
\]
Removing the motion of the center-of-mass of the three-body system, that is equivalent to
the condition \( P_1 = 0 \), leads the equation (3) to the form (see [23]):
\[
\tilde{H} = \frac{1}{2\mu_0} \left( \tilde{P}_2^2 + \tilde{P}_3^2 \right) + \bar{V}(||r - \lambda_- R||, ||R||, ||r + \lambda_+ R||),
\]
where
\[
\mu_0 = \left( \frac{m_1 m_2 m_3}{\mu_1} \right)^{1/2}, \quad \tilde{P}_2 = \sqrt{\mu_2 \mu_0} \tilde{R}, \quad \tilde{P}_3 = \sqrt{\mu_3 \mu_0} \tilde{r}, \quad \dot{x} = dx/dt.
\]
Since the potential \( \bar{V} \) in the (4) actually depends on three variables \( (r = ||r||, R = ||R||, \theta) \),
then it can be written in the form \( \bar{V}(r, R, \theta) = \bar{V}(||r - \lambda_- R||, ||R||, ||r + \lambda_+ R||) \), where \( \theta \)
denotes the angle between the radius-vectors \( R \) and \( r \). More clearly, the expression (4) can
be represented as an one-particle Hamiltonian with the effective mass \( \mu_0 \) in a 12-dimensional
phase space:
\[
\mathbb{H}(r, p) = \frac{1}{2\mu_0} p^2 + \bar{V}(r, R, \theta),
\]
where \( r = r \oplus R \in \mathbb{R}^6 \) and \( p = \tilde{P}_2 \oplus \tilde{P}_3 \in \mathbb{R}^{*6} \) are the radius-vector and momentum of the
effective mass \( \mu_0 \). It is obvious that \( \bar{V}: \mathbb{R}^3 \to \mathbb{R}^1 \) and \( \mathbb{H}: \mathbb{R}^{12} \to \mathbb{R}^1 \).
FIG. 1: The Cartesian coordinate system where the set of radius-vectors $r_1, r_2$ and $r_3$ denote positions of the 1, 2 and 3 bodies, respectively. The circle ” o ” denotes the center-of-mass of pair (23) which in the Cartesian system is denoted by $r_0$. Here we have the Jacobi coordinates system described by the radius-vectors $R$ and $r$, in addition, $\theta$ denotes the scattering angle.

Let us consider the following system of hyper-spherical coordinates:

$$
\rho_1 = r = ||r||, \quad \rho_2 = R = ||R||, \quad \rho_3 = R_0\theta, \quad \rho_4 = R_0\Theta, \quad \rho_5 = R_0\Phi, \quad \rho_6 = R_0\Psi, \quad (6)
$$

where the first set of three coordinates; $\{\hat{\rho}\} = (\rho_1, \rho_2, \rho_3)$ determines the position of the imaginary point (effective mass $\mu_0$) on the plane formed by three bodies (these coordinates hereinafter will be called the internal coordinates), while; $\Theta \in (-\pi, +\pi]$, $\Phi = (-\pi, +\pi]$ and $\Psi \in [0, \pi]$ are the Euler angles describing the rotation of a plane in 3D space (the external coordinates). The parameter $R_0$ denotes the equilibrium distance between the bodies of the coupled pair (23) in the absence of the third body.

As it can be seen, the full interaction potential $V(r, R, \theta) \equiv V(\{\hat{\rho}\})$ is the function that depends on the internal coordinates. As shown in the works ($25-32$), it is convenient to represent the motion of a three-body system as translational and rotational motion of a triangle of bodies $\triangle(1, 2, 3)$, and also deformation of sides of the same triangle. In particular, the kinetic energy of a system of bodies in this case can be written in the form ($33$):

$$
T = \frac{\mu_0}{2} \left\{ \dot{R}^2 + \dot{r}^2 \right\} = \frac{\mu_0}{2} \left\{ R^2 + \left[ \omega \times k \right]^2 + \left( \dot{r} + [\omega \times r] \right)^2 \right\}, \quad (7)
$$

where the direction of the unit vector $k$ in the moving reference frame $\{\hat{\rho}\}$ is determined by the expression; $R||R||^{-1} = \pm k$. Below we will assume that the vector $k = (0, 0, 1)$ is
directed toward the positive direction of the axis \( OZ \) (below will be denoted as the \( z \) axis), and the angular velocity \( \omega \) describes the rotation of the frame \( \{ \vec{\rho} \} \) relative to the laboratory system.

Having carried out simple calculations in the expression (7) it is easy to find:

\[
T = \frac{\mu_0}{2} \left\{ \dot{R}^2 + \dot{r}^2 + r^2 \dot{\theta}^2 + AR^2 + Br^2 \right\},
\]

where the following notations are made:

\[
A = \omega_x^2 + \omega_y^2, \quad B = \omega_y^2 + (\omega_x \cos \theta - \omega_z \sin \theta)^2.
\]

Note that when deriving the expression (8) we used the definition of a moving system \( \{ \vec{\rho} \} \), suggesting that the unit vector \( \gamma = \frac{r}{||r||} - 1 \) lies on the plane \( OXZ \) at the angle \( \theta \) relative to the axis \( OZ \), i.e. \( \gamma = (\sin \theta, 0, \cos \theta) \). As for angular velocity projections, they satisfy the following equations:

\[
\begin{align*}
\omega_x &= \dot{\Phi} \sin \Theta \sin \Psi + \dot{\Theta} \cos \Psi, \\
\omega_y &= \dot{\Phi} \sin \Theta \cos \Psi - \dot{\Theta} \sin \Psi, \\
\omega_z &= \dot{\Phi} \cos \Theta - \dot{\Psi}.
\end{align*}
\]

(9)

Taking into account (8) and (9), the kinetic energy can be written in the tensor form:

\[
T = \frac{\mu_0}{2} \gamma^{\alpha \beta} \frac{d\rho_\alpha}{dt} \frac{d\rho_\beta}{dt}, \quad \alpha, \beta = 1, 6,
\]

where \( \gamma^{\alpha \beta} \) is the metric tensor, which has the form:

\[
\gamma^{\alpha \beta} = \begin{pmatrix}
\gamma^{11} & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma^{22} & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma^{33} & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma^{44} & \gamma^{45} & \gamma^{46} \\
0 & 0 & 0 & \gamma^{54} & \gamma^{55} & \gamma^{56} \\
0 & 0 & 0 & \gamma^{64} & \gamma^{65} & \gamma^{66}
\end{pmatrix},
\]

(10)

where the following notations are made (Appendix A):

\[
\begin{align*}
\gamma^{11} &= \gamma^{22} = 1, \quad \gamma^{33} = \left( \frac{r}{R_0} \right)^2, \quad \gamma^{44} = \left( \frac{R}{R_0} \right)^2 + \left( \frac{r}{R_0} \right)^2 (1 - \sin^2 \theta \cos^2 \Psi), \quad \gamma^{55} = \left( \frac{R}{R_0} \right)^2 \times \\
&\quad \sin^2 \Theta + \left( \frac{r}{R_0} \right)^2 (\sin^2 \Theta \cos^2 \Psi + \cos^2 \theta \sin^2 \Theta \sin^2 \Psi + \sin^2 \theta \cos^2 \Theta + \frac{1}{2} \sin 2\theta \sin 2\Theta \sin \Psi),
\end{align*}
\]
\[ \gamma^{66} = \left( \frac{r}{R_0} \right)^2 \sin^2 \theta, \quad \gamma^{45} = \gamma^{54} = -\frac{1}{2} \left( \frac{r}{R_0} \right)^2 (\sin^2 \theta \sin \Psi \sin 2\Theta \cos \Psi + \sin 2\theta \cos \Theta \cos \Psi), \]
\[ \gamma^{46} = \gamma^{64} = \frac{1}{2} \left( \frac{r}{R_0} \right)^2 \sin 2\theta \cos \Psi, \quad \gamma^{56} = \gamma^{65} = \frac{1}{2} \left( \frac{r}{R_0} \right)^2 \sin 2\theta \sin \Theta \sin \Psi - 2 \sin^2 \theta \cos \Theta. \]

Using the metric tensor \([10]\), one can write a linear infinitesimal element of Euclidean space in hyperspherical coordinates:

\[ (ds)^2 = \gamma^{\alpha\beta} \{\rho\} d\rho_\alpha d\rho_\beta, \quad \alpha, \beta = \overline{1,6}. \quad (11) \]

**Definition 2.** Let \((F, G) : \mathbb{R}^{12} \rightarrow \mathbb{R}^1\) be functions of 12 variables \((r_\alpha, p_\alpha)\), where \(\alpha = \overline{1,6}\). The Poisson bracket on the phase space \(\mathcal{P} \cong \mathbb{R}^{12}\) is defined by the following form:

\[ \{F, G\} = \sum_{\alpha=1}^{6} \left( \frac{\partial F}{\partial r_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial r_\alpha} \right). \quad (12) \]

Note that the variables \(r_\alpha\) and \(p_\alpha\) denote the projections of the 6-dimensional radius-vector \(r\) and the momentum \(p\), respectively.

**Definition 3.** Let \(\mathbb{H} : \mathbb{R}^{12} \rightarrow \mathbb{R}^1\) be the Hamiltonian of the imaginary point with the mass \(\mu_0\) in the 12-dimensional phase space. The Hamiltonian vector field \(X_{\mathbb{H}} : \mathbb{R}^{12} \rightarrow \mathbb{R}^{12}\) satisfies the equation:

\[ X_{\mathbb{H}}(z) = \{z, \mathbb{H}\}, \quad z \in \mathbb{R}^{12}. \quad (13) \]

**Definition 4.** The Hamiltonian equations on the phase space \(\mathcal{P} \cong \mathbb{R}^{12}\) will be defined as follow:

\[ \dot{z} = X_{\mathbb{H}}, \quad \dot{z} = dz/dt \in \mathbb{R}^{12}, \quad (14) \]

or, equivalently:

\[ \dot{r}_\alpha = \frac{\partial \mathbb{H}}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial \mathbb{H}}{\partial r_\alpha}, \quad \alpha = \overline{1,6}. \quad (15) \]

Without going into details, we note that the problem under consideration in the general case has 10 independent integrals of motion, with the help of which we can reduce the initial system of the 18th order to the system of the 8th order. Note that for a fixed total energy the reduction the problem leads to a system of the 7th order (see [1] also [2]).

**III. THE CLASSICAL THREE-BODY PROBLEM AS A PROBLEM OF GEODESIC FLOWS ON A HYPERSURFACE OF INTRINSIC ENERGY**

As it is easy to see, the classical system of three bodies at motion in the 3D Euclidean space continuously forms the triangle, and hence Newton’s equations describe the dynamical
system on the space of such triangles. The latter means that we can formally consider
the motion of a body-system, consisting of two parts. The first is the rotational motion of
the triangle body in the 3D Euclidean space and the second is the internal motion of the
bodies in the plane defined by the triangle. As well-known, the configuration manifold of
solid body $\mathbb{R}^6$ can be represented as a direct product of two subspaces:

$$\mathbb{R}^6 :\leftrightarrow \mathbb{R}^3 \times S^3,$$

where $\mathbb{R}^3$ is the manifold which is defined as an orthonormal space of relative distances
between bodies, while $S^3$ denotes the space of the rotation group $SO(3)$. However, in the
considered problem the connections between the bodies are not holonomic, and therefore
the representation (16) for the configuration space is incorrect.

**Definition 5.** Let $\mathcal{M}$ be a 6D Riemannian manifold on which the local coordinate system
is defined:

$$\overline{x^1, x^6} = \{x\} \in \mathcal{M}.$$  \hspace{1cm} (17)

It is assumed that the manifold $\mathcal{M}$ has a conformally Euclidean form, which is determined
by the metric tensor:

$$g_{\mu\nu}(\{\bar{x}\}) = g(\{\bar{x}\})\delta_{\mu\nu}, \quad g(\{\bar{x}\}) = [E - U(\{\bar{x}\})]U_0^{-1} > 0, \quad \mu, \nu = 1, 6,$$

where $\delta_{\mu\nu}$ is the Kronecker symbol, $E$ is the total energy of three-body system, $U(\{x\})$ is
the total interaction potential between bodies and $U_0 = \max|U(\{x\})|$.\hfill (18)

In the case when the total potential energy of the bodies system depends only on the
three coordinates $\{\bar{x}\} = (x^1, x^2, x^3)$, then the manifold $\mathcal{M}$ is representable as:

$$\mathcal{M} \cong \mathcal{M}^{(3)} \times S_t^3,$$  \hspace{1cm} (19)

where the set of internal coordinates $\{\bar{x}\} \in M_t$ and $M_t$ is the tangent bundle of a smooth
manifold $\mathcal{M}^{(3)}$, which has a map atlas. In the representation $\mathcal{M}^{(3)}$, $S_t^3$ denotes the space
of rotation group $SO(3)$ in a neighborhood of the internal point $M_i\{(x^1, x^2, x^3)_i\} \in M_t$,
which forms a layer on the base $\mathcal{M}^{(3)}$. Note that the second set of coordinates (external
coordinates) $\{\overline{x}\} = (x^4, x^5, x^6) \in S_t^3$.

**Definition 6.** Let the function $U(\{\bar{x}\})$ is in the one-to-one mapping with the potential
energy function $V(\{\bar{\rho}\})$:

$$f : U(\{\bar{x}\}) \mapsto V(\{\bar{\rho}\}), \quad f^{-1} : V(\{\bar{\rho}\}) \mapsto U(\{\bar{x}\}),$$
where \( f \) and \( f^{-1} \) denote the operation of direct and inverse mapping, respectively.

Now, using the variational principle of Maupertuis on the manifold \( \mathcal{M} \), one can obtain geodesic equations \([34, 35]\):

\[
\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}(\{x\}) \dot{x}^\beta \dot{x}^\gamma = 0, \quad \alpha, \beta, \gamma = 1, 6,
\]  

(20)

where \( \dot{x}^\alpha = dx^\alpha/ds \) and \( \ddot{x}^\alpha = d^2x^\alpha/ds^2 \); in addition, ”s” denotes the length of a curve along the geodesic trajectory and is used as a proper time, \( \Gamma^\alpha_{\beta\gamma}(\{x\}) \) is the Christoffel symbol, which is defined by the formula:

\[
\Gamma^\alpha_{\beta\gamma}(\{x\}) = \frac{1}{2} g^{\alpha\mu} \left( \partial_\gamma g_{\mu\beta} + \partial_\beta g_{\gamma\mu} - \partial_\mu g_{\beta\gamma} \right), \quad \partial_\alpha \equiv \partial / \partial x^\alpha.
\]

Taking into account the definition for the metric tensor \([18]\) from \([20]\) we can find the following system of equations describing geodesic flows on the potential energy hypersurface:

\[
\begin{align*}
\ddot{x}^1 &= a_1 \left\{ (\dot{x}^1)^2 - \sum_{\mu\neq1, \mu=2}^6 (\dot{x}^\mu)^2 \right\} + 2 \dot{x}^1 \left\{ a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \\
\ddot{x}^2 &= a_2 \left\{ (\dot{x}^2)^2 - \sum_{\mu=1, \mu\neq2}^6 (\dot{x}^\mu)^2 \right\} + 2 \dot{x}^2 \left\{ a_3 \dot{x}^3 + a_1 \dot{x}^1 \right\}, \\
\ddot{x}^3 &= a_3 \left\{ (\dot{x}^3)^2 - \sum_{\mu=1, \mu\neq3}^6 (\dot{x}^\mu)^2 \right\} + 2 \dot{x}^3 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 \right\}, \\
\ddot{x}^4 &= 2 \dot{x}^4 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \\
\ddot{x}^5 &= 2 \dot{x}^5 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \\
\ddot{x}^6 &= 2 \dot{x}^6 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\},
\end{align*}
\]

(21)

where \( g(\{x\}) = g_{11}(\{x\}) = \ldots = g_{66}(\{x\}) \) since the metric is the conformally Euclidean, in addition:

\[
a_i(\{x\}) = -\partial_{x^i} \ln \sqrt{g(\{x\})}, \quad \partial_{x^i} \equiv \partial / \partial x^i.
\]

(22)

In the system \((21)\), the last three equations are integrated exactly:

\[
\dot{x}^\mu = J_{\mu-3}/g(\{x\}), \quad J_{\mu-3} = \text{const}_{\mu-3},
\]

(23)

where \( \mu = 4, 6 \).

Note that \( J_1, J_2 \) and \( J_3 \) are integrals of motion. They can be interpreted as projections of the total angular momentum of three-body system \( J = \sqrt{J_1^2 + J_2^2 + J_3^2} = \text{const} \) on corresponding axis.
Substituting (23) into the equations (21), we obtain the following system of second-order nonlinear ordinary differential equations:

\[
\begin{align*}
\dot{x}^1 &= a_1 \{ (\dot{x}^1)^2 - (\dot{x}^2)^2 - (\dot{x}^3)^2 - \Lambda^2 \} + 2 \dot{x}^1 \{ a_2 \dot{x}^2 + a_3 \dot{x}^3 \}, \\
\dot{x}^2 &= a_2 \{ (\dot{x}^2)^2 - (\dot{x}^3)^2 - (\dot{x}^1)^2 - \Lambda^2 \} + 2 \dot{x}^2 \{ a_3 \dot{x}^3 + a_1 \dot{x}^1 \}, \\
\dot{x}^3 &= a_3 \{ (\dot{x}^3)^2 - (\dot{x}^1)^2 - (\dot{x}^2)^2 - \Lambda^2 \} + 2 \dot{x}^3 \{ a_1 \dot{x}^1 + a_2 \dot{x}^2 \},
\end{align*}
\]

where \( a_i \equiv a_i(\{\bar{x}\}) \) and \( \Lambda^2 \equiv \Lambda^2(\{\bar{x}\}) = (J/g(\{\bar{x}\}))^2 \).

The system of equations (24) can be represented as a system of the sixth order, i.e., a system consisting of six equations of the first order:

\[
\begin{align*}
\dot{\xi}^1 &= a_1 \{ (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - \Lambda^2 \} + 2 \xi^1 \{ a_2 \xi^2 + a_3 \xi^3 \}, \quad \xi^1 = \dot{x}^1, \\
\dot{\xi}^2 &= a_2 \{ (\xi^2)^2 - (\xi^3)^2 - (\xi^1)^2 - \Lambda^2 \} + 2 \xi^2 \{ a_3 \xi^3 + a_1 \xi^1 \}, \quad \xi^2 = \dot{x}^2, \\
\dot{\xi}^3 &= a_3 \{ (\xi^3)^2 - (\xi^1)^2 - (\xi^2)^2 - \Lambda^2 \} + 2 \xi^3 \{ a_1 \xi^1 + a_2 \xi^2 \}, \quad \xi^3 = \dot{x}^3.
\end{align*}
\]

Thus, the system of equations (24) or the 6th order system (25) describes the dynamics of an imaginary point with an effective mass \( \mu_0 \) that performs a motion on the Riemannian manifold: \( \mathcal{M}^3 = \{ \bar{x} \equiv (x^1, x^2, x^3) \in \mathcal{M}_i; g_{ij}(\{\bar{x}\}) = g(\{\bar{x}\}) \delta_{ij}; g(\{\bar{x}\}) > 0 \} \). Note that the manifold \( \mathcal{M}^3 \) (internal space) is immersed in the 6D manifold \( \mathcal{M} \) and, in addition, is invariant under the local rotation group \( SO(3) \) (external space \( S^3 \)).

It is important to note, that with consideration (18) and (23) we can reduce the Hamiltonian and get the following representation for it:

\[
\mathcal{H}(\{\bar{x}\}; \{\dot{x}\}) = \frac{1}{2} g^{\mu\nu}(\{\bar{x}\}) p_\mu p_\nu = \frac{1}{2g(\{\bar{x}\})} \delta^{\mu\nu} p_\mu p_\nu \\
= \frac{1}{2g(\{\bar{x}\})} \left\{ \sum_{i=1}^{3} (\dot{x}^i)^2 + \left( \frac{J}{g(\{\bar{x}\})} \right)^2 \right\}, \quad \mu, \nu = 1, 2, 3.
\]

Note that in the representation there is no explicit dependence of the Hamiltonian on the masses of the bodies. It is hidden, present in coordinate transformations. The system of geodesic equations (21) can be obtained using the following Hamilton equations:

\[
\dot{x}^i = \frac{\partial \mathcal{H}}{\partial p_i} = g^{ik}(\{\bar{x}\}) p_k, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x^i} = -\frac{1}{2} \frac{\partial g^{kl}(\{\bar{x}\})}{\partial x^i} p_k p_l, \quad i, k, l = 1, 2, 3,
\]

describe geodesic flow on the manifold \( \mathcal{M}^3 \).

And finally, it is obvious that if the total energy of the three-body system is fixed:

\[
E = \mathcal{H}(\{\bar{x}\}; \{\dot{x}\}) = const,
\]

then the dynamical problem will be of the 5th order.
IV. THE MAPPINGS BETWEEN 6D EUCLIDEAN AND 6D CONFORMAL-EUCLIDEAN SPACES

Now the main problem is to prove that the system $6^{th}$ order (24) or (25) is equivalent to the original three-body Newtonian problem (15). Recall, that these representations will be equivalent, if we prove that there exists continuous one-to-one mappings between the following two manifolds $\mathbb{E}^6$ and $\mathcal{M}$. Note, that $\mathbb{E}^6 \subset \mathbb{R}^6$ is a subspace, which is stands out from the Euclidean space $\mathbb{R}^6$ taking into account the condition:

$$ E - \mathbb{V} (\{\bar{\rho}\}) > 0. $$

(28)

In other words, we must prove that between two sets of coordinates; $\bar{\rho}^1, \bar{\rho}^6 = \{\rho\} \in \mathbb{E}^6$ and $\bar{x}^1, \bar{x}^6 = \{x\} \in \mathcal{M}$, there exist continuous direct and inverse one-to-one mappings.

A. On a homeomorphism between the subspace $\mathbb{E}^6$ and the manifold $\mathcal{M}$

**Proposition 1.** If the interaction potential between the three bodies has the form (2) and, at least belongs to the class $C^1$, then the Euclidean subspace $\mathbb{E}^6 \subset \mathbb{R}^6$ is homeomorphic to the manifold $\mathcal{M}$.

Let us consider a linear infinitesimal element $ds$ in both coordinate systems $\{\rho\} \in \mathbb{E}^6$ and $\{x\} \in \mathcal{M}$. Equating them, we can write:

$$ (ds)^2 = \gamma^{\alpha\beta}(\{\bar{\rho}\})d\rho_\alpha d\rho_\beta = g_{\mu\nu}(\{\bar{x}\})dx^\mu dx^\nu, \quad \alpha, \beta, \mu, \nu = 1, 6, $$

(29)

from which one can find the following system of algebraic equations:

$$ \gamma^{\alpha\beta}(\{\bar{\rho}\})\rho_{\alpha,\mu}\rho_{\beta,\nu} = g_{\mu\nu}(\{\bar{x}\}) = g(\{\bar{x}\})\delta_{\mu\nu}, $$

(30)

where $\rho_{\alpha,\mu} = \partial \rho_\alpha / \partial x^\mu$.

Recall that the set of $\rho_{\alpha,\mu}$ derivatives allows us to perform coordinate transformations between the sets of coordinates $\{\bar{\rho}\}$ and $\{\bar{x}\}$, respectively, which we call direct transformations. Similarly, from (29) one can obtain a system of algebraic equations for the derivatives of the corresponding coordinates that determine the inverse transformations:

$$ \gamma_{\alpha\beta}(\{\rho\})g^{-1}(\{\bar{x}\}) = x^\mu_{\alpha}\bar{x}^\nu_{\beta} \delta_{\mu\nu}, $$

(31)
where \( x_{\mu}^{\alpha} = \partial x^{\mu} / \partial \rho^{\alpha} \) and \( \gamma_{\alpha \beta}(\{\rho\}) = \gamma_{\alpha \delta}(\{\rho\}) \gamma_{\beta \bar{\delta}}(\{\rho\}) \gamma_{\bar{\alpha} \bar{\beta}}(\{\rho\}) \), in addition, we will assume that \( f : \tilde{g}(\{\bar{\rho}\}) \mapsto g(\bar{x}) \).

At first we consider the system of equations (30), which is related to direct coordinate transformations. It is not difficult to see that the system of algebraic equations (30) is underdetermined with respect to the variables \( \rho_{\alpha, \mu} \), since it consists of 21 equations, while the number of unknown variables is 36. Obviously, when these equations are compatible, then the system (30) has an infinite number of real and complex solutions. Note that for the classical three-body problem, the real solutions of the system (30) are important, which form a 15-dimensional manifold. Since the system of equations (31) is still defined in a rather arbitrary way we can impose additional conditions on it in order to find the minimal dimension of the manifold allowing a separation of the base \( \mathcal{M}^{(3)} \) from the layer \( S^{3}_{t} \) (see the expression (19)).

Let us make a new notations:

\[
\alpha_{\mu} = \rho_{1, \mu}, \quad \beta_{\mu} = \rho_{2, \mu}, \quad \gamma_{\mu} = \rho_{3, \mu}, \quad u_{\mu} = \rho_{4, \mu}, \quad v_{\mu} = \rho_{5, \mu}, \quad w_{\mu} = \rho_{6, \mu}. \tag{32}
\]

We demand that its elements obey to the following additional conditions:

\[
\alpha_{4} = \alpha_{5} = \alpha_{6} = 0, \quad \beta_{4} = \beta_{5} = \beta_{6} = 0, \quad \gamma_{4} = \gamma_{5} = \gamma_{6} = 0, \\
u_{1} = u_{2} = u_{3} = 0, \quad v_{1} = v_{2} = v_{3} = 0, \quad w_{1} = w_{2} = w_{3} = 0. \tag{33}
\]

Using (10), (32) and conditions (33) from the equation (30) we can obtain two independent systems of algebraic equations:

\[
\alpha_{1}^{2} + \beta_{1}^{2} + \gamma_{1}^{33} \gamma_{1}^{2} = \tilde{g}(\{\bar{\rho}\}), \quad \alpha_{1} \alpha_{2} + \beta_{1} \beta_{2} + \gamma_{33} \gamma_{1} \gamma_{2} = 0, \\
\alpha_{2}^{2} + \beta_{2}^{2} + \gamma_{2}^{33} \gamma_{2}^{2} = \tilde{g}(\{\bar{\rho}\}), \quad \alpha_{1} \alpha_{3} + \beta_{1} \beta_{3} + \gamma_{33} \gamma_{1} \gamma_{3} = 0, \\
\alpha_{3}^{2} + \beta_{3}^{2} + \gamma_{3}^{33} \gamma_{3}^{2} = \tilde{g}(\{\bar{\rho}\}), \quad \alpha_{2} \alpha_{3} + \beta_{2} \beta_{3} + \gamma_{33} \gamma_{2} \gamma_{3} = 0. \tag{34}
\]
and, correspondingly:

\[ \gamma^{44}u_4^2 + \gamma^{55}v_5^2 + \gamma^{66}w_6^2 + 2(\gamma^{45}u_4v_5 + \gamma^{46}u_4w_6 + \gamma^{56}v_5w_6) = \ddot{g}(\{\bar{\rho}\}), \]

\[ \gamma^{44}u_4^2 + \gamma^{55}v_5^2 + \gamma^{66}w_6^2 + 2(\gamma^{45}u_5v_5 + \gamma^{46}u_5w_6 + \gamma^{56}v_5w_5) = \ddot{g}(\{\bar{\rho}\}), \]

\[ \gamma^{44}u_6^2 + \gamma^{55}v_6^2 + \gamma^{66}w_6^2 + 2(\gamma^{46}u_6v_6 + \gamma^{46}u_6w_6 + \gamma^{56}v_6w_6) = \ddot{g}(\{\bar{\rho}\}), \]

\[ a_4u_4 + a_5v_5 + a_6w_6 = 0, \]

\[ b_4u_5 + b_5v_5 + b_6w_5 = 0, \]

\[ c_4u_6 + c_5v_6 + c_6w_6 = 0. \quad (35) \]

In equations (35) the following notations are made:

\[ a_i = \gamma^{i4}u_4 + \gamma^{i5}v_5 + \gamma^{i6}w_6, \quad b_j = \gamma^{j4}u_5 + \gamma^{j5}v_5 + \gamma^{j6}w_6, \quad c_k = \gamma^{k4}u_4 + \gamma^{k5}v_5 + \gamma^{k6}w_6, \]

where \( i, j, k = 4, 5, 6 \).

It should be noted that the solutions of algebraic systems (34) and (35) form two different 3D manifolds \( \mathcal{S}^{(3)} \) and \( \mathcal{R}^{(3)} \), respectively. The manifold \( \mathcal{S}^{(3)} \) is in a one-to-one mapping on the one hand with the subspace \( \mathbb{E}^3 \in \{\bar{\rho}\} \) (where \( \mathbb{E}^3 \subset \mathbb{E}^6 \) the internal space of three-body system in the hyper-spherical coordinates system), and on the other hand with the manifold \( \mathcal{M}^{(3)} \) (see Fig. 2). This approvement follows from the fact that all points of the manifold \( \mathcal{M}^{(3)} \) and the subspace \( \mathbb{E}^3 \), pairwise connected through the corresponding derivatives (see (30)), which, as unknown variables, enter the algebraic equations (34), and, in addition, as shown there exist also inverse coordinate transformations (see Appendix B). Now we prove continuity of these mappings.

Recall that the unknowns in the equations (34) are in fact functions of coordinates \( \{\bar{\rho}\} \). If to make a shift in the coordinates ie, \( \{\bar{\rho}\} \rightarrow \{\bar{\rho}\} + \{\delta \bar{\rho}\} \), then instead (34), we get the following system of equations:

\[ \alpha_1^2 + \beta_1^2 + \gamma^{33}\xi_1^2 = \ddot{g}(\{\bar{\rho}\}), \quad \alpha_1\bar{\alpha}_2 + \beta_1\bar{\beta}_2 + \gamma^{33}\bar{\gamma}_1\bar{\gamma}_2 = 0, \]

\[ \alpha_2^2 + \beta_2^2 + \gamma^{33}\xi_2^2 = \ddot{g}(\{\bar{\rho}\}), \quad \alpha_1\bar{\alpha}_3 + \beta_1\bar{\beta}_3 + \gamma^{33}\bar{\gamma}_1\bar{\gamma}_3 = 0, \]

\[ \alpha_3^2 + \beta_3^2 + \gamma^{33}\xi_3^2 = \ddot{g}(\{\bar{\rho}\}), \quad \alpha_2\bar{\alpha}_3 + \beta_2\bar{\beta}_3 + \gamma^{33}\bar{\gamma}_2\bar{\gamma}_3 = 0. \quad (36) \]

In (36) the following notation are made for the functions; \( \bar{\sigma}(\{\bar{\rho}\}) = \sigma(\{\bar{\rho}\} + \{\delta \bar{\rho}\}) \), where \( \{\delta \bar{\rho}\} = (\delta \bar{\rho}^1, \delta \bar{\rho}^2, \delta \bar{\rho}^3) \). Assuming that \( || \delta \{\bar{\rho}\} || \ll 1 \), in the equations (36), we can expand the functions in a Taylor series on these small parameters and taking into account the system
of equations (34), we get:

\[ \delta \rho^i \left\{ 2(\alpha_1 \alpha_{1,i} + \beta_1 \beta_{1,i} + \gamma^{33} \gamma_{1,i} \gamma_{1,i}) + \gamma^i_{,i} \gamma^2_1 - \tilde{g}_{,i} \{ \tilde{\rho} \} \right\} + O(||\delta \{ \tilde{\rho} \}||^2) = 0, \]

\[ \delta \rho^i \left\{ 2(\alpha_1 \alpha_{1,i} + \beta_1 \beta_{1,i} + \gamma^{33} \gamma_{1,i} \gamma_{1,i}) + \gamma^i_{,i} \gamma^2_1 - \tilde{g}_{,i} \{ \tilde{\rho} \} \right\} + O(||\delta \{ \tilde{\rho} \}||^2) = 0, \]

\[ \delta \rho^i \left\{ 2(\alpha_1 \alpha_{1,i} + \beta_1 \beta_{1,i} + \gamma^{33} \gamma_{1,i} \gamma_{1,i}) + \gamma^i_{,i} \gamma^2_1 - \tilde{g}_{,i} \{ \tilde{\rho} \} \right\} + O(||\delta \{ \tilde{\rho} \}||^2) = 0, \]

\[ \delta \rho^i \left\{ \alpha_1 \alpha_{2,i} + \alpha_2 \alpha_{1,i} + \beta_1 \beta_{2,i} + \beta_2 \beta_{1,i} + \gamma^{33} (\gamma_{1,2,i} + \gamma_{2,1,i}) + \gamma^i_{,i} \gamma_{1,2} \right\} + O(||\delta \{ \tilde{\rho} \}||^2) = 0, \]

\[ \delta \rho^i \left\{ \alpha_1 \alpha_{3,i} + \alpha_3 \alpha_{1,i} + \beta_1 \beta_{3,i} + \beta_3 \beta_{1,i} + \gamma^{33} (\gamma_{1,3,i} + \gamma_{3,1,i}) + \gamma^i_{,i} \gamma_{1,3} \right\} + O(||\delta \{ \tilde{\rho} \}||^2) = 0, \]

\[ \delta \rho^i \left\{ \alpha_2 \alpha_{3,i} + \alpha_3 \alpha_{2,i} + \beta_2 \beta_{3,i} + \beta_3 \beta_{2,i} + \gamma^{33} (\gamma_{2,3,i} + \gamma_{3,2,i}) + \gamma^i_{,i} \gamma_{2,3} \right\} + O(||\delta \{ \tilde{\rho} \}||^2) = 0, \]

(37)

where \( i = 1, 2, 3 \) and by dummy indices the summation is performed. The system of algebraic equations (37) at a fixed point \( \{ \tilde{\rho} \} \in \mathbb{E}^3 \) and for fixed increments \( \delta \rho^i \) with respect to the unknown quantities \( (\alpha_{1,i}, \beta_{1,i}, \gamma_{1,i}, \ldots, \gamma_{3,i}) \) is underdetermined, since only 6 equations are specified for finding 27 unknowns. In particular, we can require that the multipliers for the same increments \( \delta \rho^i \) be zero. This allows instead of one system (37) to obtain three systems of algebraic equations, of the form:

\[ 2(\alpha_1 \alpha_{1,i} + \beta_1 \beta_{1,i} + \gamma^{33} \gamma_{1,i} \gamma_{1,i}) + \gamma^i_{,i} \gamma^2_1 - \tilde{g}_{,i} \{ \tilde{\rho} \} = 0, \]

\[ 2(\alpha_1 \alpha_{1,i} + \beta_1 \beta_{1,i} + \gamma^{33} \gamma_{1,i} \gamma_{1,i}) + \gamma^i_{,i} \gamma^2_1 - \tilde{g}_{,i} \{ \tilde{\rho} \} = 0, \]

\[ 2(\alpha_1 \alpha_{1,i} + \beta_1 \beta_{1,i} + \gamma^{33} \gamma_{1,i} \gamma_{1,i}) + \gamma^i_{,i} \gamma^2_1 - \tilde{g}_{,i} \{ \tilde{\rho} \} = 0, \]

\[ \alpha_1 \alpha_{2,i} + \alpha_2 \alpha_{1,i} + \beta_1 \beta_{2,i} + \beta_2 \beta_{1,i} + \gamma^{33} (\gamma_{1,2,i} + \gamma_{2,1,i}) + \gamma^i_{,i} \gamma_{1,2} = 0, \]

\[ \alpha_1 \alpha_{3,i} + \alpha_3 \alpha_{1,i} + \beta_1 \beta_{3,i} + \beta_3 \beta_{1,i} + \gamma^{33} (\gamma_{1,3,i} + \gamma_{3,1,i}) + \gamma^i_{,i} \gamma_{1,3} = 0, \]

\[ \alpha_2 \alpha_{3,i} + \alpha_3 \alpha_{2,i} + \beta_2 \beta_{3,i} + \beta_3 \beta_{2,i} + \gamma^{33} (\gamma_{2,3,i} + \gamma_{3,2,i}) + \gamma^i_{,i} \gamma_{2,3} = 0. \]

(38)

The system of equations (38) has a continuum of solutions that form a 3D manifold if the first derivatives of the function \( \tilde{g}(\{ \tilde{\rho} \}) \) by coordinates \( \tilde{g}_{,i}(\{ \tilde{\rho} \}) = \partial_{\rho^i} \tilde{g}(\{ \tilde{\rho} \}) \) exist. In the case when the metric tensor \( \tilde{g}(\{ \tilde{\rho} \}) \) is sufficiently smooth, that is, \( n \gg 1 \) times differentiable, similar algebraic equations can be found that provide transformations for higher derivatives of the expansion. In other words, we have proved the continuity of direct mappings in the environment with radius \( ||\delta \{ \tilde{\rho} \}|| \) of any chosen point \( \forall \{ \tilde{\rho} \} \in \mathbb{E}^3 \).

The same is easily proved for inverse mappings (see Appendix B).

Let us consider the open set \( \forall \mathcal{G} = \cup_{\alpha} G_{\alpha} \), consisting of the union of cards \( G_{\alpha} \) arising at continuously mappings \( f : \{ \tilde{\rho} \} \mapsto \{ \tilde{x} \} \) using algebraic equations (34). Proceeding from the
foregoing, it is obvious that the maps can be chosen so that the immediate neighbors have intersections comprising at least one common point, that is a necessary condition for the continuity of the mappings. Proceeding from the above arguments, it we assert that the atlas \( G \) can be widen up to \( G \cong \mathcal{M}^{(3)} \).

Thus, all the conditions of the homeomorphism theorem between the metric spaces \( \mathbb{E}^3 \) and \( \mathcal{M}^{(3)} \) are carried out, and, therefore, we can say that these spaces are homeomorphic to \( f : \mathbb{E}^3 \mapsto \mathcal{M}^{(3)} \) or topologically equivalent.

As for the system of algebraic equations (35), then at each point of the internal space \( M_i(x^1, x^2, x^3), \in \mathcal{M}^{(3)} \), it generates a 3-dimensional manifold \( \mathfrak{R}^{(3)} \) that is a local analogue of the Euler angles and, consequently, \( S^3 \simeq \mathfrak{R}^{(3)} \). The layer, \( \mathfrak{R}^{(3)} \) continuously passing through all points of the basis \( \mathcal{M}^{(3)} \), fills the subspace \( \mathbb{E}^6 \).

Finally, taking into account the aforesaid, we can conclude that the spaces \( \mathbb{E}^6 \) and \( \mathcal{M} \), are homeomorphic too.

The proposition 1 is proved.

B. The classical three-body problem and the Poincaré conjecture

As it well known, Poincaré was the first to attempt to study of 3-dimensional manifolds in connection with problems of classical Hamiltonian systems, as a result of which, in 1904, he formulated his famous hypothesis (the Poincaré conjecture), which in the framework of modern mathematical conceptions could be formulated as follows:

*If a smooth compact 3-dimensional manifold \( \mathfrak{M}^3 \) has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that \( \mathfrak{M}^3 \) is homeomorphic to the sphere \( S^3 \) ?*

Recall that the 3-dimensional unit sphere \( S^3 \), that is, the locus of all points \( (x, y, z, w) \) in 4-dimensional Euclidean space which have distance exactly 1 from the origin \[36\]:

\[
x^2 + y^2 + z^2 + w^2 = 1.
\]

In 2002, Perelman proved Poincaré’s conjecture without any connection to dynamical systems \[37\]. In this sense it will be interesting to understand the relationship of this Poincaré conjecture to the classical three-body problem.

For this, in the equation (34) it is useful to make change of variables. In particular, the
new variables will be determined by the following formulas:

\[
\begin{align*}
\tilde{\alpha}_1 &= \frac{\alpha_1 + \alpha_2}{\sqrt{\sigma_1}}, & \tilde{\beta}_1 &= \frac{\beta_1 + \beta_2}{\sqrt{\sigma_1}}, & \tilde{\gamma}_1 &= \frac{\sqrt{\gamma_{33}^3} \gamma_1}{\sqrt{\sigma_1}}, & \tilde{\gamma}_{2(1)} &= \frac{\sqrt{\gamma_{33}^3} \gamma_2}{\sqrt{\sigma_1}}, \\
\tilde{\alpha}_2 &= \frac{\alpha_2 + \alpha_3}{\sqrt{\sigma_2}}, & \tilde{\beta}_2 &= \frac{\beta_2 + \beta_3}{\sqrt{\sigma_2}}, & \tilde{\gamma}_2 &= \frac{\sqrt{\gamma_{33}^3} \gamma_2}{\sqrt{\sigma_2}}, & \tilde{\gamma}_{3(2)} &= \frac{\sqrt{\gamma_{33}^3} \gamma_3}{\sqrt{\sigma_2}}, \\
\tilde{\alpha}_3 &= \frac{\alpha_3 + \alpha_1}{\sqrt{\sigma_3}}, & \tilde{\beta}_3 &= \frac{\beta_3 + \beta_1}{\sqrt{\sigma_3}}, & \tilde{\gamma}_3 &= \frac{\sqrt{\gamma_{33}^3} \gamma_3}{\sqrt{\sigma_3}}, & \tilde{\gamma}_{1(3)} &= \frac{\sqrt{\gamma_{33}^3} \gamma_1}{\sqrt{\sigma_3}},
\end{align*}
\]  

(39)

where

\[
\sigma_1(\bar{x}) = 2[g(\bar{x})] - \gamma_{33}^3(\{\bar{x}\}) \gamma_1 \gamma_2 > 0, \quad \sigma_2(\bar{x}) = 2[g(\bar{x})] - \gamma_{33}^3(\{\bar{x}\}) \gamma_2 \gamma_3 > 0, \\
\sigma_3(\bar{x}) = 2[g(\bar{x})] - \gamma_{33}^3(\{\bar{x}\}) \gamma_3 \gamma_1 > 0.
\]  

(40)

Now taking into account new notations (39), the system of algebraic equations (34) can be represented in the form:

\[
\begin{align*}
\tilde{\alpha}_1^2 + \tilde{\beta}_1^2 + \tilde{\gamma}_1^2 + \tilde{\gamma}_{2(1)}^2 &= 1, & \sigma_1 \rho_1^2 - \sigma_2 \rho_2^2 - \sigma_3 \rho_3^2 + \rho_{23} &= 0, \\
\tilde{\alpha}_2^2 + \tilde{\beta}_2^2 + \tilde{\gamma}_2^2 + \tilde{\gamma}_{3(2)}^2 &= 1, & \sigma_1 \rho_1^2 + \sigma_2 \rho_2^2 - \sigma_3 \rho_3^2 - \rho_{12} &= 0, \\
\tilde{\alpha}_3^2 + \tilde{\beta}_3^2 + \tilde{\gamma}_3^2 + \tilde{\gamma}_{1(3)}^2 &= 1, & \sigma_1 \rho_1^2 - \sigma_2 \rho_2^2 + \sigma_3 \rho_3^2 - \rho_{13} &= 0,
\end{align*}
\]  

(41)

where the following notations are made:

\[
\begin{align*}
\rho_i^2 &= \tilde{\alpha}_i^2 + \tilde{\beta}_i^2 + \tilde{\gamma}_i^2, \\
\rho_{ij} &= 2\sqrt{\sigma_i \sigma_j} (\tilde{\alpha}_i \tilde{\alpha}_j + \tilde{\beta}_i \tilde{\beta}_j + \tilde{\gamma}_i \tilde{\gamma}_j).
\end{align*}
\]

As we can be seen, the (41) is an underdetermined system of algebraic equations consisting of six equations and nine unknowns. Recall that in the set of six variables \(\{\gamma\} = (\gamma_1, \gamma_2, \gamma_3, \gamma_{1(2)}, \gamma_{3(2)}, \gamma_{1(3)})\) are linearly independent only three variables. Unlike the system of equations (34), whose domain of definition is limited by the condition (28), the domain of definition of the system (41) besides is limited by additional conditions (40). As a result, the algebraic system (41) generates a manifold in the form of a three-dimensional sphere with unit radius \(S^3 \subset \mathbb{S}^3\) for each group of variables \((\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1), (\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)\) and \((\tilde{\alpha}_3, \tilde{\beta}_3, \tilde{\gamma}_3)\).

In other words, the Poincaré conjecture for the Hamiltonian system, more precisely for the classical three-body problem, is a special case of proposition 1.
C. Transformations between global and local coordinate systems

To finally solve the question of the equivalence of representations (24) and (25), respectively, it remains to determine an explicit form of coordinate transformations. In particular, as analysis shows, the transformations between two sets of internal coordinates \( \{ \bar{\rho} \} \) and \( \{ \bar{x} \} \) can be represented only in a differential form:

\[
\begin{align*}
\frac{d\rho_1}{\rho_1} &= \alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dx_3, \\
\frac{d\rho_2}{\rho_2} &= \beta_1 dx_1 + \beta_2 dx_2 + \beta_3 dx_3, \\
\frac{d\rho_3}{\rho_3} &= \gamma_1 dx_1 + \gamma_2 dx_2 + \gamma_3 dx_3,
\end{align*}
\]

(42)

where the coefficients \((\alpha_1, \alpha_2, ..., \gamma_3)\), are defined by expressions (32).

Recall that in every subsequent step on a manifold there is an infinite number of possibilities for choosing a local system of coordinates \( \{ \bar{x} \} \in M_3^{(3)} \); however, it is obvious that this choice must be made taking into account the system of algebraic equations (34).

Thus, the system of equations (24) or the 6th order system (25) together with the system of algebraic equations (34) describes the classical three-body problem, which, in its turn, is equivalent to the original classical Newtonian problem of three bodies. It is important to note that the timing parameter \( s \in \mathbb{R}^1 \) with consideration of algebraic equations (34) is a non-trivial chronological parameter, a measure of the processes leaking in the system, which further will be called an internal time.

V. MOVEMENT WITHOUT ACCELERATION AND THE BOUND STATE OF A THREE-BODY SYSTEM

An important class of solutions of the classical three-body problem describes the bound state (123), when the motion of all bodies occurs in a confined space. Note that such a state cannot be formed as a result of classical scattering due to the absence of the mechanism of removing energy from the system (for example, the removal of radiation energy in an atomic-molecular collision), which is a necessary condition for the formation of a bound state. Nevertheless, it is not difficult to see that the character of the motions in the states (123) and (123)* in many of its manifestations should be similar. In the mathematical sense, the configuration space \( E^*_3 \) of the formation (123)*, by definition is noncompact (see
condition (18)). Regarding the stable coupled state (123), it is formed on a noncompact but a restricted subspace of the Euclidean space $E^3_{st} \subset E^3 \subset \mathbb{R}^3$.

In any case, all these solutions must satisfy the energy conservation law:

$$\mathcal{H}({\bar{x}}; {\bar{\dot{x}}}) \equiv \mathcal{H}({\bar{x}}; {\bar{\dot{\xi}}}) = E = \text{const},$$

(43)

that defines the 5D hypersurface in the 6D phase space, where the Hamiltonian $\mathcal{H}({\bar{x}}; {\bar{\dot{x}}})$ defines by the equation (26). Some important properties of this problem can be studied by algebraic methods without solving the equations of motion (24) or (25). In particular, it is very interesting to find a class of solutions for which the system of bodies (effective mass $\mu_0$) on the manifold $\mathcal{M}^3$ moves without acceleration and understand what configurations they form in the phase space $\mathcal{P} \cong E^6_{st}$. Under these circumstances, we can simplify the system of equations (25) presenting it as:

$$a_1 \left\{ (\xi^1)^2 - (\xi^3)^2 - (\xi^2)^2 - \Lambda^2 \right\} + 2\xi^1 \left\{ a_2\xi^2 + a_3\xi^3 \right\} = 0,$$

$$a_2 \left\{ (\xi^2)^2 - (\xi^3)^2 - (\xi^1)^2 - \Lambda^2 \right\} + 2\xi^2 \left\{ a_3\xi^3 + a_1\xi^1 \right\} = 0,$$

$$a_3 \left\{ (\xi^3)^2 - (\xi^1)^2 - (\xi^2)^2 - \Lambda^2 \right\} + 2\xi^3 \left\{ a_1\xi^1 + a_2\xi^2 \right\} = 0.$$  

(44)

It is easy to see that equations (13) and (14) at each point of $\{\bar{x}\}_i \in E^3_{st}$, form an algebraic system of four equations, whereas the unknown variables are six $\{\bar{\xi}\}_i = (\xi_1, \xi_2, \xi_3)_i$ and $\{\bar{a}\}_i = [a_1(\{\bar{x}\}_i), a_2(\{\bar{x}\}_i), a_3(\{\bar{x}\}_i)]$, respectively. Note that the set of coefficients $\{\bar{a}\}$ are defined by the formula (22), which can be represented as:

$$a_1(\{\bar{x}\}) = -\frac{1}{2} \frac{\partial \ln g(\{\bar{x}\})}{\partial x^1} = -\frac{1}{2} \frac{\partial \ln \hat{g}(\{\bar{\rho}\})}{\partial \rho_i} \frac{\partial \rho_i}{\partial x^1} = \pi_1 \alpha_1 + \pi_2 \beta_1 + \pi_3 \gamma_1,$$

$$a_2(\{\bar{x}\}) = -\frac{1}{2} \frac{\partial \ln g(\{\bar{x}\})}{\partial x^2} = -\frac{1}{2} \frac{\partial \ln \hat{g}(\{\bar{\rho}\})}{\partial \rho_i} \frac{\partial \rho_i}{\partial x^2} = \pi_1 \alpha_2 + \pi_2 \beta_2 + \pi_3 \gamma_2,$$

$$a_3(\{\bar{x}\}) = -\frac{1}{2} \frac{\partial \ln g(\{\bar{x}\})}{\partial x^3} = -\frac{1}{2} \frac{\partial \ln \hat{g}(\{\bar{\rho}\})}{\partial \rho_i} \frac{\partial \rho_i}{\partial x^3} = \pi_1 \alpha_3 + \pi_2 \beta_3 + \pi_3 \gamma_3,$$  

(45)

where $\pi_i(\{\bar{\rho}\}) = -(1/2) \partial_{\rho_i} \ln \hat{g}(\{\bar{\rho}\})$.

Recall that the set of nine variables $\{\alpha_1, ..., \beta_1, ..., \gamma_3\}$ satisfies an underdetermined system of six algebraic equations (34). It follows from the above that the sets of coefficients $\{\bar{a}\}$ form a certain continuous 3D manifold. Solving algebraic equations (14), (13) and (34), one can generate a two-dimensional manifold, which homeomorphic to it two-dimensional surface $S^{(2)} \subset E^3_{st}$ on which the effective mass $\mu_0$ moves without acceleration.
Note that the study of this system of algebraic equations will yield a lot of useful and important information about the dynamical system, in particular, on the geometric and topological properties of subspace on which a stable or quasistable three-body system can be formed.

**Proposition 2.** The three-body system forms a stable bound state, if:

on the closed restricted two-dimensional surface $\mathcal{S}_{0i}^{(2)} \subset \mathcal{S}^{(2)} \subset \mathbb{R}^3_{st}$, generated by the equation:

$$\ddot{g}(\{\bar{\rho}\}_0i) = h, \quad h \in \{h\} > 0, \quad \{\bar{\rho}\}_0i \in \mathcal{S}_{0i}^{(2)},$$

the equations (44), (46) and (34) form underdetermined algebraic system, which at each point $\{\bar{\rho}\}_0i \in \mathcal{S}_{0i}^{(2)}$ has an infinite set of solutions that continuously fill a two-dimensional manifold $\mathcal{S}^{(2)}$.

It is obvious, that from (46) implies also the equality:

$$g(\{\bar{x}\}_0i) = h, \quad \{\bar{x}\}_0i \in \mathcal{M}_{0i}^{(2)}.$$  

(47)

Note that $\mathcal{M}_{0i}^{(2)}$ is the two-dimensional manifold homeomorphic to $\mathcal{S}_{0i}^{(2)}$ due to the fact that $f^{-1} : \ddot{g}(\{\bar{\rho}\}_0i) \mapsto g(\{\bar{x}\}_0i) = h$.

Accordingly, the coefficients in the equations (44) on the manifold $\mathcal{M}_{0i}^{(2)}$ will be defined as functions:

$$a_j(\{\bar{x}\}_0i) = \lim_{\{\bar{x}\} \to \{\bar{x}\}_0i} \left[ -\frac{1}{2} \frac{\partial \ln g(\{\bar{x}\})}{\partial \bar{x}^j} \right] = \pi_1 \alpha_j + \pi_2 \beta_j + \pi_3 \gamma_j, \quad j = \overline{1, 3},$$

(48)

where $\{\bar{x}\} \in \mathcal{M}^{(2)} \subset \mathcal{M}^{(3)}$ and $\pi_j$ (see (45)).

Note that the manifold $\mathcal{M}^{(2)}$ we continuously deform and reduce to the manifold $\mathcal{M}_{0i}^{(2)}$.

Thus, we have ten algebraic equations (44), (46) and (34), while unknown variables twelve. Obviously, this underdetermined algebraic system at each point $\{\bar{\rho}\}_0i \in \mathcal{S}_{0i}^{(2)}$ and, respectively, at each point $\{\bar{x}\}_0i \in \mathcal{M}_{0i}^{(2)}$ (since $f : \mathcal{S}_{0i}^{(2)} \mapsto \mathcal{M}_{0i}^{(2)}$), has an infinite number of real solutions that form a 2-dimensional manifold $\mathcal{S}^{(2)}$.

★ The Proposition 2 is proved.

**VI. DEVIATION OF GEODESIC TRAJECTORIES OF ONE FAMILY**

Studying the linear deviations of the geodesic trajectories of one family, one can obtain valuable information about the properties of the dynamical system and, what is very
important, the relations between the behavior of the dynamical system and the geometric singularities of the Riemannian space.

**Definition 7.** Let \( x^i = x^i(s, \eta) \) be the equation of a one-parameter family of geodesics on the Riemannian manifold \( \mathcal{M}^{(3)} \), where "s" is an affine parameter along geodesic the trajectory, whereas the symbol \( \eta \) denotes the family parameter. The vector \( j(\{\zeta\}) \) in the direction of the normal of the geodesic \( l(\{\bar{x}\}) \) with components:

\[
\frac{\delta x^i(s, \eta)}{\delta \eta} = \zeta^i(s, \eta), \quad \{\zeta\} = (\zeta^1, \zeta^2, \zeta^3), \quad i = 1, 3,
\]

will be called the linear deviation of close geodesics.

The components of the deviation vector \( j(\{\zeta\}) \) satisfy the following equations [35]:

\[
\frac{D^2 \zeta^i}{Ds^2} = -R^i_{jkl}(\{\bar{x}\})x^j\zeta^kx^l, \quad i, j, k, l = 1, 3,
\]

where \( R^i_{jkl}(\{\bar{x}\}) \) is the Riemann tensor, which is represented as:

\[
R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{jk,l} + \Gamma^i_{k\lambda}\Gamma^\lambda_{lj} - \Gamma^i_{l\lambda}\Gamma^\lambda_{jk}, \quad \Gamma^i_{jk,l}(\{\bar{x}\}) = \partial \Gamma^i_{jk}(\{\bar{x}\})/\partial x^l.
\]

The equation (51) can be written in the form of an ordinary second-order differential equation:

\[
\ddot{\zeta}^i + 2\Gamma^i_{jkl}\dot{x}^j\dot{\zeta}^l + \left(\Gamma^i_{jk,l}\dot{x}^j\dot{x}^k + \Gamma^i_{jn}\Gamma^n_{kp}\dot{x}^j\delta^k_l\right)\zeta^l = -R^i_{jkl}x^j\zeta^kx^l,
\]

The explicit form of specific terms of the equation (52) can be found in the Appendix C. Solving equation (52) together with the equations systems (24) and (34), we can get a full view on deviation properties of close geodesic trajectories of a one-parameter family, which is a very important characteristic of a dynamical system.

**VII. CLASSICAL MOVEMENT WITH CONSIDERATION OF THE INFLUENCE OF THE ENVIRONMENT**

Let us suppose that a dynamical system is exposed to the action of the surroundings consisting of both regular and random terms. It is obvious that the reasons for such impact can be different. For example, there may be a situation when the three-body system is immersed in an environment in result of which it will be exposed to the external influences. As such environmental conditions can serve, for example a fundamental **physical vacuum** with its own stochastic fluctuations or an environment that can be interpreted as a **thermal bath**.
Impacts of the second type can be, for example, random collisions with other particles of the medium, accompanied by multichannel atomic-molecular processes. In this case, obviously, the total collision energy changes randomly, which is equivalent to random fluctuations of the metric of the internal space $g_{ij}(\{\bar{x}\})$.

In the first case, when the system of three-body is influenced by external random forces, using the system (25) we can write the following stochastic equations of motion:

$$
\dot{\chi}^\mu = A^\mu(\{\chi\}) + \eta^\mu(s), \quad \{\chi\} = (\{\xi\}, \{\bar{x}\}), \quad \mu = 1, 6,
$$

where for the set of independent variables $\{\chi\}$ the following notations are made:

$$
\chi^1 = \xi^1, \quad \chi^2 = \xi^2, \quad \chi^3 = \xi^3, \quad \chi^4 = x^1, \quad \chi^5 = x^2, \quad \chi^6 = x^3,
$$

which forming the Euclidean space, in addition:

$$
A^1(\{\chi\}) = a_1 (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - \Lambda^2 + 2\xi^1(a_2\xi^2 + a_3\xi^3), \quad A^4(\{\chi\}) = \xi^1, \\
A^2(\{\chi\}) = a_2 (\xi^2)^2 - (\xi^1)^2 - (\xi^3)^2 - \Lambda^2 + 2\xi^2(a_3\xi^3 + a_1\xi^1), \quad A^5(\{\chi\}) = \xi^2, \\
A^3(\{\chi\}) = a_3 (\xi^3)^2 - (\xi^2)^2 - (\xi^1)^2 - \Lambda^2 + 2\xi^3(a_1\xi^1 + a_2\xi^2), \quad A^6(\{\chi\}) = \xi^3.
$$

Recall that the set of functions $A^\mu(\{\chi\})$, where $\mu = 1, 6$, are regular functions.

As for the stochastic functions $\eta^\mu(s)$, it is assumed that they satisfy the correlation relations of white noise:

$$
\langle \eta^\mu(s) \rangle = 0, \quad \langle \eta^\mu(s)\eta^\mu(s') \rangle = 2\epsilon \delta(s - s'),
$$

where $\epsilon$ denotes the power of random fluctuations and $\delta(s - s')$ is the Dirac delta function.

Thus, all conditions for derivation of the equation of joint probability density for the independent variables $\{\chi\}$ are given.

Let us represent the joint probability density in the form [38]:

$$
P(\{\chi\}, s) = \prod_{\mu=1}^{6} \langle \delta[\chi^\mu(s) - \chi^\mu] \rangle.
$$

Using the standard technique (see [38, 39]), we can differentiate the expression (53) by the timing parameter $s$ and taking into account (53) and (54) we obtain the following equation:

$$
\frac{\partial P}{\partial s} = \sum_{\mu=1}^{6} \frac{\partial}{\partial \chi^\mu} \left[ A^\mu(\{\chi\}) + \epsilon \frac{\partial}{\partial \chi^\mu} \right] P.
$$
As it is easy to see, the function \( (56) \) defines the position and momentum of the *imaginary point* characterizing the three-body system in the phase space. In the case when \( \epsilon = \hbar \), this function will play the same role as the Wigner quasiprobability distribution \( (40) \). It is important to note, that unlike the Wigner function, the solution of the equation \( (56) \) is positive in the entire phase space, which is very important from a physical point of view. In other words the function \( (55) \) is a real probability distribution function.

We now consider the case when the metric of the internal space \( g_{\mu\nu}(\{\bar{x}\}) \) undergoes random fluctuations. The latter means that all functions in the equations \( (25) \), which depend on the metric tensor, will also be random. Mathematically, the above is equivalent to random mappings of the corresponding functions in the equations \( (25) \).

Let us consider the following random mappings:

\[
R_f : a_i(\{\bar{x}\}) \mapsto \bar{a}_i(\{\bar{x}(s)\}) = a_i(\{\bar{x}(s)\}) + \eta_i(s),
\]

and

\[
R_f : \Lambda^2(\{x\}) \mapsto \bar{\Lambda}^2(\{\bar{x}(s)\}) = \Lambda^2(\{\bar{x}(s)\}) + \eta_0(s),
\]

where \( a_i(\{\bar{x}(s)\}) \) and \( \Lambda^2(\{\bar{x}(s)\}) \) are regular functions from the geodesic trajectory \( \{\bar{x}(s)\} \), \( R_f \) denotes the operator of random mappings, the set of functions \( \{\eta_0(s), ..., \eta_3(s)\} \) denote random generators which will be refined below. Obviously, the random components \( \bar{a}_i \) are much larger in value than the random term in \( \bar{\Lambda} \), since \( \bar{a}_i \) is the first derivative of the stochastic function \( \bar{g}(\{\bar{x}(s)\}) \), where \( R_f : g(\{\bar{x}\}) \mapsto \bar{g}(\{\bar{x}(s)\}) \). Taking all this into account, the system of equations \( (25) \) can be decomposed and represented in the form of stochastic equations of the Langevin type:

\[
\dot{\xi}^i = A^i(\{\bar{\xi}\}|\{\bar{x}(s)\}) + \sum_{j=1}^{3} B^{ij}(\{\bar{\xi}\}|\{\bar{x}(s)\})\eta_j(s) + O(\eta^2), \quad i = 1, 3,
\]

(57)

where \( \{\bar{\xi}\} = (\xi^1, \xi^2, \xi^3) \), in addition:

\[
A^1(\{\bar{\xi}\}|\{\bar{x}(s)\}) = a_1 \{(\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - \Lambda^2\} + 2\xi^1(a_2 \xi^2 + a_3 \xi^3),
\]

\[
A^2(\{\bar{\xi}\}|\{\bar{x}(s)\}) = a_2 \{(\xi^2)^2 - (\xi^1)^2 - (\xi^3)^2 - \Lambda^2\} + 2\xi^2(a_3 \xi^3 + a_1 \xi^1),
\]

\[
A^3(\{\bar{\xi}\}|\{\bar{x}(s)\}) = a_3 \{(\xi^3)^2 - (\xi^2)^2 - (\xi^1)^2 - \Lambda^2\} + 2\xi^3(a_1 \xi^1 + a_2 \xi^2),
\]

and, respectively:

\[
B^{11} = (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - \Lambda^2, \quad B^{12} = 2\xi^1 \xi^2, \quad B^{13} = 2\xi^1 \xi^3,
\]

\[23\]
\[ B^{21} = 2\xi^2\xi^1, \quad B^{22} = (\xi^2)^2 - (\xi^1)^2 - (\xi^3)^2 - \Lambda^2, \quad B^{23} = 2\xi^2\xi^3, \]
\[ B^{31} = 2\xi^3\xi^1, \quad B^{32} = 2\xi^3\xi^2, \quad B^{33} = (\xi^3)^2 - (\xi^2)^2 - (\xi^1)^2 - \Lambda^2. \]

The joint probability density for the independent variables \{\bar{\xi}\} can be represented as:

\[ P(\{\bar{\xi}\}, s|\{\bar{x}(s)\}) = \prod_{i=1}^{3} \langle \delta \left[ \xi^i(s|\{\bar{x}(s)\}) - \xi^i \right] \rangle. \quad (58) \]

Assuming that random generators have to satisfy the correlation properties of the white noise:

\[ \langle \eta_i(s) \rangle = 0, \quad \langle \eta_i(s)\eta_j(s') \rangle = 2\epsilon_{ij}\delta(s - s'), \quad (59) \]

and, performing the calculations similar to (55)-(56), we obtain the following second order partial differential equation for the joint probability density:

\[ \frac{\partial P}{\partial s} = \sum_{i=1}^{3} \frac{\partial}{\partial \xi^i} \left( A^i P \right) + \sum_{i,j,l,k=1}^{3} \epsilon_{ij} \frac{\partial}{\partial \xi^l} \left[ B^{il} \frac{\partial}{\partial \xi^k} (B^{kj} P) \right]. \quad (60) \]

Note that in (59) and (60) constants \( \epsilon_{ij} \) determine power of fluctuations by different directions of 6D phase space.

Thus, we obtained equations describing the geodesic flows in the phase space (56) and the momentum space (60), respectively, which must be solved taking into account the system of algebraic equations (34).

**VIII. NEW CRITERIA FOR ESTIMATING CHAOS IN THE STATISTICAL SYSTEM**

When the three-body system is in an environment that has both regular and random influences on it, then it makes sense to talk about a statistical system. In this case, the main task is to construct the mathematical expectations of different elementary atomic-molecular processes flowing at the multichannel scattering (see Sch. 1). Note that the evolution equations (56) and (60), describing of geodesic flows depending on internal times have an important feature. Namely, the internal time, depending on the properties of the manifold geometry can branching. The latter circumstance forces us follow up the development of all solutions. In other words, it is necessary to establish criteria indicating the measure of deviation of probabilistic current tubes.
Following the definition of Kullbeck-Leibler on the distance between two continuous distributions, we can determine the criterion characterizing the deviation between the tubes of probabilistic currents of elementary processes [41].

**Definition 8.** The deviation between two different tubes of probabilistic currents in the phase space will be defined by the expression:

\[
d(s_a, s_b) = \int_{\mathcal{P}} P(\{\chi\}, s_a) \ln \left| \frac{P(\{\chi\}, s_a)}{P(\{\chi\}, s_b)} \right| \sqrt{g(\{\bar{x}\})} \prod_{\nu=1}^{6} d\chi^\nu,
\]

where \( P_a \equiv P(\{\chi\}, s_a) \) and \( P_b \equiv P(\{\chi\}, s_b) \) are two different tubes of probabilistic currents, which at the beginning of development of elementary processes are closely located or have an intersection.

In the case when the distance between two flows with time \( s = |s_a - s_b| \) grows linearly, that is:

\[
d(s_a, s_b) \sim k|s_a - s_b|, \quad k = \text{const} > 0,
\]

there is a reason to believe that the dynamical system, which is under the influence of the environment, is chaotic.

**Definition 9.** If \( P_{if}(s_n) \) be the transition probability between the asymptotic channels \( i \) and \( f \) with the internal time \( s_n \), then the total mathematical expectation \( P_{tot}^{ab} \) will be the sum of partial:

\[
P_{tot}^{if} = \lim_{N \to \infty} \left[ \frac{1}{N} \sum_{n=1}^{N} \left( \lim_{s_n \to \infty} P_{if}(s_n) \right) \right],
\]

where \( N \) denotes the number of solutions of the Cauchy problem.

Using similar reasoning, it is possible to calculate the total transition probabilities between different asymptotic channels using probabilistic currents in the internal space obtaining by the equation (60).

**IX. CONCLUSION**

The study of the classical three-body problem, with the aim of revealing new regularities of both celestial mechanics and elementary atomic-molecular processes, is still of great interest. In addition, it is very important to answer the question: is irreversibility fundamental for describing the classical world [42]? In this sense, the study of the three-body problem,
which is a typical example of a dynamical system with all its complexities, has not lost its fundamental significance both for the foundations of physics and for mathematics.

Note that if for the problems of celestial mechanics first of all it is important to find a stable trajectory solutions, then for an atomic-molecular collision, studies of multichannel scattering processes flowing through the formation of a transition state are of paramount importance (see Sch. 1, $ABC^*$). However, the study of these systems is complicated due to the fact that the dynamics of the three-body system is often influenced by the external factors, such as the environment, external fields, etc., which must be taken into account.

Following the idea of Krylov, we considered the general classical three-body problem on a Riemann surface, ie, on the hypersurface of the energy of a system of bodies. The new formulation of the well-known problem makes it possible to reveal a number of important fundamental features of the dynamical system. Here we list only three of them:

a. The Riemannian geometry with its local coordinate system in most general case makes it possible to reveal additional hidden symmetries of the dynamical system and thereby achieve a more complete reduction of the problem, up to the system of 6th order (see Eq.s (25)), instead of the generally accepted 8th order. In the case when the energy of the body-system is fixed, the dynamical problem is reduced to a 5th-order system, which is very important for creating effective algorithms for numerical simulation.

b. The equivalence between the Newtonian three-body problem (15) and the problem of geodesic flows on the Riemannian manifold (25) provides the coordinate transformations (42) together with the system of algebraic equations (34). Note that due to algebraic system, which is absent in the Krylov’s representation, the evolutionary or timing parameter $s$ of the dynamical system, conditionally called internal time, branches out, which essentially distinguishes it from ordinary time. Furthermore, as shows the analysis, the arrow of time in this microscopic problem, like the time arrow of more complex systems can have non-trivial [43], and sometimes unpredictable behavior, that obviously makes the system of equations (25) irreversible. The latter radically changes the content of the time of classical mechanics, as a trivial parameter, binding the past with the future through the present. And in spite of the pessimistic statements of Bergson and Prigogine [44–46], this, in our view will allow classical mechanics to describe the whole spectrum of diverse phenomena, including the irreversibility inherent in elementary atomic-molecular processes.

c. The developed approach allows to take into account external regular and random forces
on the evolution of the dynamical system without using perturbation theory methods. In particular, equations describing the propagation of the geodesic flow both in the phase space (56) and in the configuration space (60) are obtained.

It is obvious, that this approach will be useful and promising for studying the problem of few bodies, as well as statistical and relaxation properties of many-body systems. Lastly, it is easy to see that the quantization based on reduced Hamiltonian (26) with consideration the system of algebraic equations (34) and coordinate transformations (42) makes quantum mechanics irreversible, which is a necessary condition for the generation of chaos in the wave function. The latter, without violating Arnold’s theorem [47, 48], in the limit $\hbar \to 0$ allows us to make the transition from the quantum region to the region of classical chaotic motion, that solves an important open problem of the quantum-classical correspondence.

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XI. APPENDIX

A.

Let us consider vector product of vectors encountered in the expression of the kinetic energy (17). Taking into account the fact that the direction $k = R||R||^{-1}$ coincide with the axis $z$ we get:

$$[\omega \times k] = (\dot{x}\omega_x + \dot{y}\omega_y + \dot{z}\omega_z) \times (\dot{x} \cdot 0 + \dot{y} \cdot 0 + \dot{z} \cdot k_z) = \dot{x}\omega_y - \dot{y}\omega_x, \quad k = \dot{z} \cdot k_z, \quad (63)$$

and respectively,

$$[\omega \times k]^2 = \omega_x^2 + \omega_y^2, \quad ||\dot{x}|| = ||\dot{y}|| = ||\dot{z}|| = 1. \quad (64)$$

Similarly, we can calculate the second term:

$$[\omega \times r] = \dot{x}\omega_y r \cos \theta + \dot{y}r (\omega_z \sin \theta - \omega_x \cos \theta) - \dot{z}r \omega_y \sin \theta, \quad r = ||r||\gamma = r\gamma, \quad (65)$$
using which we can get:

\[
[\omega \times \mathbf{r}]^2 = r^2 \left\{ \omega_y^2 + (\omega_z \sin \theta - \omega_x \cos \theta)^2 \right\}, \quad \gamma = (\sin \theta, 0, \cos \theta),
\]

\[
\dot{\mathbf{r}}^2 = (||\mathbf{r}||\dot{\gamma} + ||\dot{\mathbf{r}}||\gamma)^2 = (r^2 \dot{\gamma}^2 + 2r \dot{\gamma} \dot{\gamma} + \dot{r}^2 \gamma^2) = r^2 \dot{\theta}^2 + \dot{r}^2, \quad \gamma \dot{\gamma} = 0,
\]

\[
\dot{\mathbf{r}} \cdot [\omega \times \mathbf{r}] = (r \dot{\gamma} + \dot{r} \gamma) \cdot [\omega \times \mathbf{r}] = r \dot{\mathbf{r}} \omega_y \sin \theta \cos \theta - r \dot{\mathbf{r}} \omega_y \sin \theta \cos \theta = 0. \quad (66)
\]

Taking into account (63)-(66), the expression of the kinetic energy (7) can be written in the form (8).

Now important to calculate the terms \(A\) and \(B\) that enter in the expression (8). Taking into account the equations system (9), it is easy to calculate:

\[
A = \omega_x^2 + \omega_y^2 = (\dot{\Phi} \sin \Theta \sin \Psi + \dot{\Theta} \cos \Psi)^2 + (\dot{\Phi} \sin \Theta \cos \Psi - \dot{\Theta} \sin \Psi)^2 =
\]

\[
\dot{\Phi}^2 \sin^2 \Theta \sin^2 \Psi + 2\dot{\Phi} \dot{\Theta} \sin \Theta \sin \Psi \cos \Psi + \dot{\Theta}^2 \cos^2 \Psi + \dot{\Phi}^2 \sin^2 \Theta \cos^2 \Psi
\]

\[
-2\dot{\Phi} \dot{\Theta} \sin \Theta \cos \Psi \sin \Psi + \dot{\Theta}^2 \sin^2 \Psi = \dot{\Phi}^2 \sin^2 \Theta + \dot{\Theta}^2. \quad (67)
\]

\[
B = \omega_y^2 + (\omega_x \cos \theta - \omega_z \sin \theta)^2 = (\dot{\Phi} \sin \Theta \cos \Psi - \dot{\Theta} \sin \Psi)^2 + (\dot{\Phi} \sin \Theta \sin \Psi + \dot{\Theta} \cos \Psi)^2 \cos^2 \theta - 2(\dot{\Phi} \sin \Theta \sin \Psi + \dot{\Theta} \cos \Psi)(\dot{\Phi} \cos \Theta - \dot{\Psi}) \sin \theta \cos \theta +
\]

\[
(\dot{\Phi} \cos \Theta - \dot{\Psi})^2 \sin^2 \theta = \dot{\Phi}^2 \sin^2 \Theta \cos^2 \Psi - \dot{\Phi} \dot{\Theta} \sin \Theta \sin 2\Psi + \dot{\Theta}^2 \sin^2 \Psi
\]

\[
+ \dot{\Phi}^2 \sin^2 \Theta \sin^2 \Psi \cos^2 \theta + \dot{\Phi} \dot{\Theta} \sin \Theta \sin 2\Psi \cos^2 \Psi + \dot{\Theta}^2 \cos^2 \Psi \cos^2 \theta - \frac{1}{2} \dot{\Phi}^2 \sin 2\Theta \sin \Psi \sin 2\theta + \dot{\Phi} \dot{\Theta} \sin \Theta \sin \Psi \sin 2\theta - \dot{\Phi} \dot{\Theta} \cos \Theta \cos \Psi \sin 2\theta
\]

\[
+ \dot{\Theta} \dot{\Psi} \cos \Psi \sin 2\theta + \dot{\Phi}^2 \cos^2 \Theta \sin^2 \theta - 2\dot{\Phi} \dot{\Psi} \cos \Theta \sin^2 \theta + \dot{\Psi}^2 \sin^2 \theta. \quad (68)
\]

Finally, taking into account the calculations (67) and (68), it is easy to calculate the components of the tensor \(\gamma^{\alpha \beta}\) (see expression (10)).

**B.**

Since the existence of inverse coordinate transformations is very important for the proof of the proposition, we now consider the system of algebraic equations (51).

Let us make the following notations:

\[
\bar{\alpha}_\mu = x^{1, \mu}, \quad \bar{\beta}_\mu = x^{2, \mu}, \quad \bar{\gamma}_\mu = x^{3, \mu}, \quad \bar{\alpha}_\mu = x^{4, \mu}, \quad \bar{\epsilon}_\mu = x^{5, \mu}, \quad \bar{\omega}_\mu = x^{6, \mu}. \quad (69)
\]
In addition, we require the following conditions to be fulfilled:

\[
\begin{align*}
\dot{\alpha}_4 &= \dot{\alpha}_5 = \dot{\alpha}_6 = 0, \\
\dot{\beta}_4 &= \dot{\beta}_5 = \dot{\beta}_6 = 0, \\
\dot{\gamma}_4 &= \dot{\gamma}_5 = \dot{\gamma}_6 = 0, \\
\ddot{u}_1 &= \ddot{u}_2 = \ddot{u}_3 = 0, \\
\ddot{v}_1 &= \ddot{v}_2 = \ddot{v}_3 = 0, \\
\dddot{w}_1 &= \dddot{w}_2 = \dddot{w}_3 = 0.
\end{align*}
\]

Now, performing similar arguments and calculations, as in the case of direct coordinate transformations, from (3.1) it is easy to get the following two systems of algebraic equations:

\[
\begin{align*}
\ddot{\alpha}_1^2 + \ddot{\beta}_1^2 + \ddot{\gamma}_1^2 &= \frac{1}{g(\{x\})}, \\
\ddot{\alpha}_1 \ddot{\alpha}_2 + \ddot{\beta}_1 \ddot{\beta}_2 + \ddot{\gamma}_1 \ddot{\gamma}_2 &= 0, \\
\ddot{\alpha}_2^2 + \ddot{\beta}_2^2 + \ddot{\gamma}_2^2 &= \frac{1}{g(\{x\})}, \\
\ddot{\alpha}_1 \ddot{\alpha}_3 + \ddot{\beta}_1 \ddot{\beta}_3 + \ddot{\gamma}_1 \ddot{\gamma}_3 &= 0, \\
\ddot{\alpha}_3^2 + \ddot{\beta}_3^2 + \ddot{\gamma}_3^2 &= \frac{\gamma_{33}}{g(\{x\})}, \\
\ddot{\alpha}_2 \ddot{\alpha}_3 + \ddot{\beta}_2 \ddot{\beta}_3 + \ddot{\gamma}_2 \ddot{\gamma}_3 &= 0,
\end{align*}
\]

and, correspondingly:

\[
\begin{align*}
\dddot{u}_4^2 + \dddot{v}_4^2 + \dddot{w}_4^2 &= \frac{\gamma_{44}}{g(\{x\})}, \\
\dddot{u}_4 \dddot{u}_5 + \dddot{v}_4 \dddot{v}_5 + \dddot{w}_4 \dddot{w}_5 &= \frac{\gamma_{45}}{g(\{x\})}, \\
\dddot{u}_5^2 + \dddot{v}_5^2 + \dddot{w}_5^2 &= \frac{\gamma_{55}}{g(\{x\})}, \\
\dddot{u}_4 \dddot{u}_6 + \dddot{v}_4 \dddot{v}_6 + \dddot{w}_4 \dddot{w}_6 &= \frac{\gamma_{46}}{g(\{x\})}, \\
\dddot{u}_6^2 + \dddot{v}_6^2 + \dddot{w}_6^2 &= \frac{\gamma_{66}}{g(\{x\})}, \\
\dddot{u}_5 \dddot{u}_6 + \dddot{v}_5 \dddot{v}_6 + \dddot{w}_5 \dddot{w}_6 &= \frac{\gamma_{56}}{g(\{x\})},
\end{align*}
\]

where \( f^{-1} : g(\{x\}) \mapsto \tilde{g}(\{\tilde{\rho}\}) \).

Thus, we have proved that there are also inverse coordinate transformations.

C.

The equation for the covariant derivative \([52]\) can be written as:

\[
\frac{\mathcal{D}F^i}{\mathcal{D}s} = \dot{F}^i + Y^i, \quad Y^i = \Gamma^i_{jl}(\{\tilde{x}\}) \tilde{x}^j F^l, \quad \dot{q} = dq/ds, \quad i, j, l = 1, 3
\]

where \( Y^i \in \mathcal{M}^{(3)} \) is a component of the 3D vector.

Using [73], we can calculate the covariant derivative of the second order:

\[
\begin{align*}
\frac{\mathcal{D}^2 \zeta^i}{\mathcal{D}s^2} &= \ddot{\zeta}^i + \Gamma^i_{jl} \dot{x}^j \dot{\zeta}^l + Y^i + \Gamma^i_{jl} \dot{x}^j Y^l = \dddot{\zeta}^i + \Gamma^i_{jl} \ddot{x}^j \dot{\zeta}^l + \frac{d}{ds}(\Gamma^i_{jl} \ddot{x}^j \zeta^l) + \\
&\quad \Gamma^i_{jl} \ddot{x}^j (\dot{\Gamma}^l_{kp} \dot{x}^k \zeta^p) = \dddot{\zeta}^i + 2\Gamma^i_{jl} \dddot{x}^j \dddot{\zeta}^l + \dddot{\Gamma}^i_{jl} \dddot{x}^j \zeta^l + \Gamma^i_{jl} \dddot{x}^j \dddot{\zeta}^l + \Gamma^i_{jl} \Gamma^l_{kp} \ddot{x}^k \zeta^p \\
&= \dddot{\zeta}^i + 2\Gamma^i_{jl} \dddot{x}^j \dddot{\zeta}^l + (\dddot{\Gamma}^i_{jl} \dddot{x}^j \zeta^l - \Gamma^i_{jl} \Gamma^l_{kp} \dddot{x}^k \zeta^p) + \Gamma^i_{jl} \Gamma^l_{kp} \ddot{x}^k \zeta^p \zeta^l, \quad (74)
\end{align*}
\]

29
where \( k, n, p = 1, 3 \). In addition:

\[
\Gamma_{jl}^i = \frac{1}{2} g^{ip} (\partial_i g_{pj} + \partial_j g_{ip} - \partial_p g_{ij}) = -\delta^i_j a_j - \delta^i_p a_p + \delta^i_p \delta_{jl} a_p, \quad a_k = -\frac{1}{2} \partial_s \ln g, \tag{75}
\]

\[
\dot{\Gamma}_{jl}^i = \frac{d \Gamma_{jl}^i}{ds} = \frac{1}{2} g^{ip} (\partial_i \dot{g}_{pj} + \partial_j \dot{g}_{ip} - \partial_p \dot{g}_{ij}) + \frac{1}{2} g^{ip} (\partial_i \dot{g}_{pj} + \partial_j \dot{g}_{ip} - \partial_p \dot{g}_{ij})
= \frac{1}{g} \left( \sum_{k=1}^{3} a_k \dot{x}^k \right) \left[ (\delta^i_j a_j + \delta^i_p a_p - \delta^i_p \delta_{jl} a_p) - (\delta^i_j b_l + \delta^i_p b_j - \delta^i_p \delta_{jl} b_p) \right]
= \frac{1}{g} \left( \sum_{k=1}^{3} a_k \dot{x}^k \right) \left[ (\delta^i_j (a_j - b_l) + \delta^i_p (a_j - b_j) - \delta^i_p \delta_{jl} (a_p - b_p) \right], \tag{76}
\]

where \( b_k = -(1/2) \partial_s \ln | \sum_{i=1}^{3} g_{i} \dot{x}^i | \) and \( g_{i,k} = \partial g / \partial x^k \).

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