Ward Identities
and Integrable Differential Equations
in the Ising Field Theory

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We show that the celebrated Painlevé equations for the Ising correlation functions follow in a simple way from the Ward Identities associated with local Integrals of Motion of the doubled Ising field theory. We use these Ward Identities to derive the equations determining the matrix elements of the product $\sigma(x)\sigma(x')$ between any particle states. The result is then applied in evaluating the leading mass corrections in the Ising field theory perturbed by an external magnetic field.
1. Introduction

It is well known since the pioneering work of Barouch, Tracy, Wu and McCoy [1] that the spin-spin correlation function of the Ising model at zero external field is expressed in terms of suitable solution of certain ordinary differential equation, the so-called Painlevé III equation. Later this result was rederived and generalized via different approaches [2,3,4]. While Sato, Miwa and Jimbo [2] used the theory of monodromy preserving deformations of ordinary differential equations, the derivation of Babelon and Bernard [3] was based on exact form-factors [5]. One of the goals of this paper is to present yet another derivation of this classic result, which we believe is somewhat simpler and more straightforward from the field-theoretical point of view. It is based on the well known fact that the Ising field theory, being a theory of free fermions, possesses an infinite set of local integrals of motion. The differential equation for the spin-spin correlation function follows directly from the associated Ward identities. The simplest way of derivation along these lines utilizes the local integrals of motion of the “doubled” Ising field theory, which consists of two identical copies of the Ising field theory, with no interaction between the copies. Various advantages of such “doubling” were recognized before, see e.g. [6,7]. The doubled Ising field theory is equivalent to free Dirac fermion theory, and it is known to have $\hat{S}L(2)$ Kac-Moody symmetry generated by local charges [8]. The derivation based on the corresponding Ward identities is presented in Section 4 below.

The local integrals of motion act on particle states in a simple way, and therefore the approach based on the Ward identities applies as well to all matrix elements of the form

$$
\langle A(\beta_1') \ldots A(\beta_M') \mid \sigma(x)\sigma(x') \mid A(\beta_1) \ldots A(\beta_N) \rangle,
$$

where $A(\beta_i)$ stand for the Ising free fermions with rapidities $\beta_i$, and $\sigma(x)$ denotes the spin field. We derive the equations determining these matrix elements in Section 4.

Our particular interest in the matrix elements (1.1) was motivated by the applications to the Ising field theory with nonzero external field $h$. Adding the term $h \int \sigma(x) d^2x$ to the free-fermion action gives rise to an interesting interacting field theory [4] which describes the scaling domain of the Ising model in a magnetic field. Generally, this field theory is not integrable, hence no exact solution is available. However, at small field, $h << m^{15/8}$, perturbation theory in $h$ can produce useful results. As usual, such perturbative calculations involve matrix elements of the type (1.1). For instance, perturbative calculation of
the vacuum energy requires the correlation functions, and numerical solution of the corresponding exact differential equation yields the coefficients of the $h^2$ expansion with high precision [11]. The matrix elements (1.1) involving particles are used in the evaluation of the perturbative corrections to particle masses and scattering amplitudes. In particular, the $\sim h^2$ correction to the mass is obtained by evaluating the self-energy part, i.e.

$$\delta m^2 = -h^2 \int \langle A(\beta) | \sigma(x) \sigma(0) | A(\beta) \rangle_{\text{irred}} d^2 x ,$$

where $\langle \ldots \rangle_{\text{irred}}$ denotes the connected one-particle irreducible matrix element, and the integration (which makes the result independent of $\beta$) is taken over the Euclidean plane $\mathbb{R}^2$. The integral here can be computed numerically, using exact differential equation determining the matrix element involved. The results were previously quoted in [12] without derivation. Here, in Section 5 below, we fill this gap by presenting some details of this calculation.

2. The Ising Field Theory

In this section we describe the structure of the Ising field theory in some detail. This is done mostly in order to fix our notations and conventions; the reader familiar with the subject can safely skip this Section (except maybe for the important Eqs. (2.16), (2.17)).

As is well known (see e.g. [9]), the Ising Field Theory with zero magnetic field is a free Majorana fermion theory, described by the standard euclidean action

$$A_{\text{FF}} = \frac{1}{2\pi} \int \left[ \psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} + im \bar{\psi} \psi \right] d^2 x .$$

Here we have assumed that the theory is defined on an infinite plane $\mathbb{R}^2$, whose points $x$ are labelled by cartesian coordinates $(x, y) = (x(x), y(x))$, and $d^2 x \equiv dx dy$; In what follows we typically interpret $y$ as the euclidean time direction. Complex coordinates are defined as $z(x) = x + iy$, $\bar{z}(x) = x - iy$, and the derivatives $\partial$, $\bar{\partial}$ in (2.1) stand for $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, respectively. The chiral components $\psi(x), \bar{\psi}(x)$ of the Majorana fermi field obey the linear field equations

$$\bar{\partial} \psi = \frac{im}{2} \bar{\psi} , \quad \partial \bar{\psi} = -\frac{im}{2} \psi ,$$

and their normalization in the action (2.1) corresponds to the following short-distance limit of the two-point operator products

$$(z - z') \psi(x) \bar{\psi}(x') \to 1 , \quad (\bar{z} - \bar{z}') \bar{\psi}(x) \bar{\psi}(x') \to 1 , \quad \text{as} \quad |x - x'| \to 0 .$$
Typical observables of the Ising field theory are described in terms of the order and disorder fields, \( \sigma(x) \) and \( \mu(x) \) respectively \([13]\). These fields are non-local with respect to the fermi fields; the operator products

\[
\psi(x)\sigma(x_0), \quad \bar{\psi}(x)\sigma(x_0), \quad \psi(x)\mu(x_0), \quad \bar{\psi}(x)\mu(x_0),
\]

(2.4)

acquire a minus sign when the point \( x \) is taken around \( x_0 \). This property does not define the fields \( \sigma \) and \( \mu \) uniquely, but a precise definition can be given in terms of the “radial quantization” of the theory (2.1). Since the products (2.4) as functions of \( x \) obey the field equations (2.2), it is useful to introduce a complete set of solutions,

\[
\begin{pmatrix}
  u_n(x) \\
  \bar{u}_n(x)
\end{pmatrix} = \left( \frac{m}{2} \right)^{\frac{1}{2}-n} \Gamma \left( n + \frac{1}{2} \right) \begin{pmatrix}
  e^{i(n-\frac{1}{2})\theta} I_{n-\frac{1}{2}}(mr) \\
  -ie^{i(n+\frac{1}{2})\theta} I_{n+\frac{1}{2}}(mr)
\end{pmatrix},
\]

(2.5)

\[
\begin{pmatrix}
  v_n(x) \\
  \bar{v}_n(x)
\end{pmatrix} = \left( \frac{m}{2} \right)^{\frac{1}{2}-n} \Gamma \left( n + \frac{1}{2} \right) \begin{pmatrix}
  i e^{-i(n+\frac{1}{2})\theta} I_{n+\frac{1}{2}}(mr) \\
  e^{-i(n-\frac{1}{2})\theta} I_{n-\frac{1}{2}}(mr)
\end{pmatrix},
\]

(here \( n \in \mathbb{Z}; \ r, \ \theta \) are polar coordinates around \( x_0 \), i.e. \( z - z_0 = r e^{i\theta}, \bar{z} - \bar{z}_0 = r e^{-i\theta}; \ I_\nu \) are modified Bessel functions) and write the fields \( \psi(x) \) and \( \bar{\psi}(x) \) in the products (2.4) as

\[
\psi(x) = \sum_{n=-\infty}^{\infty} a_n u_{-n}(x) + \bar{a}_n v_{-n}(x),
\]

\[
\bar{\psi}(x) = \sum_{n=-\infty}^{\infty} a_n \bar{u}_{-n}(x) + \bar{a}_n \bar{v}_{-n}(x).
\]

(2.6)

The coefficients \( a_n \) and \( \bar{a}_n \) here are understood as operators acting on the space of fields; they can be easily shown to satisfy the canonical anticommutation relations

\[
\{a_n, a_{n'}\} = \delta_{n+n',0}, \quad \{\bar{a}_n, \bar{a}_{n'}\} = \delta_{n+n',0}, \quad \{a_n, \bar{a}_{n'}\} = 0.
\]

(2.7)

The fields \( \sigma \) and \( \mu \) are defined as the “primary” fields with respect to the above algebra (2.7). In particular, they satisfy

\[
a_n \sigma(x) = 0, \quad \bar{a}_n \sigma(x) = 0, \quad a_n \mu(x) = 0, \quad \bar{a}_n \mu(x) = 0
\]

(2.8)

for \( n > 0 \), as well as

\[
a_0 \sigma(x) = \frac{\omega}{\sqrt{2}} \mu(x), \quad a_0 \mu(x) = \frac{\bar{\omega}}{\sqrt{2}} \sigma(x),
\]

\[
\bar{a}_0 \sigma(x) = \frac{\bar{\omega}}{\sqrt{2}} \mu(x), \quad \bar{a}_0 \mu(x) = \frac{\omega}{\sqrt{2}} \sigma(x),
\]

(2.9)
where \( \omega = e^{i\pi/4} \) and \( \bar{\omega} = e^{-i\pi/4} \). These equations do not determine the normalization of the fields \( \sigma \) and \( \mu \); we fix it through the short-distance limit of their operator products

\[
|x - x'|^{1/4} \sigma(x)\sigma(x') \to 1, \quad |x - x'|^{1/4} \mu(x)\mu(x') \to 1, \quad \text{as} \quad |x - x'| \to 0. \tag{2.10}
\]

Applying the operators \( a_n \) and \( \bar{a}_n \) with negative \( n \) to the “primary” fields \( \sigma \) and \( \mu \) creates an infinite tower of “descendent” order and disorder fields. It will become important below that the lowest of these “descendents” are expressed in terms of the coordinate derivatives of the “primary” order and disorder fields themselves. Indeed, for any local field \( \mathcal{O}(x_0) \), its derivatives \( \partial_{z_0} \mathcal{O}(x_0) \) and \( \partial_{\bar{z}_0} \mathcal{O}(x_0) \) can be expressed as

\[
\partial_{z_0} \mathcal{O}(x_0) = \frac{1}{2\pi i} \oint_{x_0} \left[ T(x) \, dz(x) + \Theta(x) \, d\bar{z}(x) \right] \mathcal{O}(x_0),
\]

\[
\partial_{\bar{z}_0} \mathcal{O}(x_0) = -\frac{1}{2\pi i} \oint_{x_0} \left[ \bar{T}(x) \, d\bar{z}(x) + \Theta(x) \, dz(x) \right] \mathcal{O}(x_0), \tag{2.11}
\]

where the contour integrals in the \( x \)-plane are taken around the point \( x_0 \) in counterclockwise direction and

\[
T = -\frac{1}{2} \psi \partial \psi, \quad \Theta = -\frac{i m}{4} \bar{\psi} \psi, \quad \bar{T} = -\frac{1}{2} \bar{\psi} \partial \bar{\psi}, \tag{2.12}
\]

are related to the components of the energy-momentum tensor and normalized according to the usual conformal field theory convention, \( T = -2\pi T_{zz}, \Theta = 2\pi T_{z\bar{z}} \) and \( \bar{T} = -2\pi T_{\bar{z}\bar{z}} \).

Of course, the contour integrals (2.11) are nothing else but the commutators with the energy-momentum operators \( \mathbf{P} \) and \( \bar{\mathbf{P}} \),

\[
\partial_{z_0} \mathcal{O}(x_0) = -i \left[ \mathbf{P}, \mathcal{O}(x_0) \right], \quad \partial_{\bar{z}_0} \mathcal{O}(x_0) = i \left[ \bar{\mathbf{P}}, \mathcal{O}(x_0) \right], \tag{2.13}
\]

where

\[
\mathbf{P} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ T(x) + \Theta(x) \right] \, dx, \quad \bar{\mathbf{P}} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \bar{T}(x) + \Theta(x) \right] \, dx, \tag{2.14}
\]

with the integration taken along an “equal-time” slice \( y = \text{constant} \). If the local field \( \mathcal{O} \) is either \( \sigma \) or \( \mu \) or one of their descendents, it is straightforward to express these commutators in terms of the mode operators \( a_n, \bar{a}_n \) in (2.6),

\[
\partial_{z_0} \mathcal{O}(x_0) = \frac{1}{2} \sum_{n=0}^{\infty} \left[ (2n+1) a_{-n-1} a_n + \frac{m^2}{2n+1} a_{-n} a_{n+1} \right] \mathcal{O}(x_0),
\]

\[
\partial_{\bar{z}_0} \mathcal{O}(x_0) = \frac{1}{2} \sum_{n=0}^{\infty} \left[ (2n+1) \bar{a}_{-n-1} \bar{a}_n + \frac{m^2}{2n+1} \bar{a}_{-n} \bar{a}_{n+1} \right] \mathcal{O}(x_0). \tag{2.15}
\]
Using the defining equations for $\sigma$ and $\mu$, (2.8) and (2.9), the above equations (2.15) yield the lowest descendents

\[
a_{-1} \sigma(x) = \frac{\omega}{\sqrt{2}} 4\partial \mu(x), \quad a_{-1} \mu(x) = \frac{\bar{\omega}}{\sqrt{2}} 4\partial \sigma(x),
\]

\[
\bar{a}_{-1} \sigma(z, \bar{z}) = \frac{\bar{\omega}}{\sqrt{2}} 4\partial \mu(x), \quad \bar{a}_{-1} \mu(x) = \frac{\omega}{\sqrt{2}} 4\partial \sigma(x),
\]

and

\[
a_{-2} \sigma(x) = \frac{\omega}{\sqrt{2}} \frac{8}{3} \partial^2 \mu(x), \quad a_{-2} \mu(x) = \frac{\bar{\omega}}{\sqrt{2}} \frac{8}{3} \partial^2 \sigma(x),
\]

\[
\bar{a}_{-2} \sigma(x) = \frac{\bar{\omega}}{\sqrt{2}} \frac{8}{3} \partial^2 \mu(x), \quad \bar{a}_{-2} \mu(x) = \frac{\omega}{\sqrt{2}} \frac{8}{3} \partial^2 \sigma(x).
\]

(2.16)

(2.17)

Of course, the fields $\sigma(x)$ and $\mu(x)$ are not mutually local; again, the product $\sigma(x) \mu(x')$ picks up a minus sign when $x$ is brought around $x'$. In fact, this product admits operator product expansion which involves the Majorana fields $\psi, \bar{\psi}$ and local composite fields built from them. Thus, the leading short-distance terms are [13]

\[
\sigma(x) \mu(x') \sim \frac{1}{\sqrt{2|x - x'|^{1/4}}} \left[ \omega (z - z')^{1/2} \psi(x') + \bar{\omega} (\bar{z} - \bar{z}')^{1/2} \bar{\psi}(x') \right]
\]

(2.18)
as $|x - x'| \to 0$.

The Majorana theory (2.1) describes the critical behaviour of the Ising model in the scaling regime near its phase transition point $T_c$. The theory applies to both phases, just below and just above the $T_c$, depending on the choice of the sign of the mass parameter $m$ in (2.1). Our definition in (2.9) corresponds to the identification of the case $m > 0$ with the ordered phase $T < T_c$, while the case $m < 0$ is identified with the disordered phase $T > T_c$. Of course, the theories at the opposite signs of $m$ are related to each other through the well-known Kramers-Wannier duality $(\psi, \bar{\psi}) \leftrightarrow (\psi, -\bar{\psi})$, $\sigma \leftrightarrow \mu$, and there is no need to consider them separately. For this reason in what follows we assume that $m$ is positive, i.e. we interpret our results in terms of the ordered phase. In this case it is the order field $\sigma$ which develops nonzero vacuum expectation value [14],

\[
\langle \sigma(x) \rangle \equiv \bar{\sigma} = m^{1/8} \bar{s}, \quad \bar{s} = 2^{1/12} e^{-1/8} A^{3/2}
\]

(2.19)

($A = 1.28243 \ldots$ is Glaisher’s constant), while the disorder field $\mu$ does not, $\langle \mu(x) \rangle = 0$.

Also, (2.1) is a free field theory, and of course its Hilbert space of states $\mathcal{H}$ associated with an "equal-time" slice $y = \text{constant}$ is the Fock space of free fermions. It is generated
by canonical fermionic creation and annihilation operators, $a^\dagger(\beta)$ and $a(\beta)$, subject to the canonical anticommutation relations
\[
\{a^\dagger(\beta), a(\beta')\} = 2\pi \delta(\beta - \beta') , \quad \{a^\dagger(\beta), a^\dagger(\beta')\} = 0 , \quad \{a(\beta), a(\beta')\} = 0 . \quad (2.20)
\]

Here $\beta$ is the usual rapidity variable describing the energy-momentum state of the particle, $(p^0, p^1) = (mc^2, m\sinh \beta)$. The canonical operators are the Fourier modes of the Heisenberg field operators $\psi(x)$ and $\bar{\psi}(x)$, i.e.
\[
\psi(x) = \sqrt{\frac{m}{2}} \int_{-\infty}^{\infty} \frac{d\beta}{\sqrt{2\pi}} e^{\beta/2} \left[ a^\dagger(\beta) e^{ym \cosh \beta - ix m \sinh \beta} - i a(\beta) e^{-ym \cosh \beta + ix m \sinh \beta} \right] , \\
\bar{\psi}(x) = \sqrt{\frac{m}{2}} \int_{-\infty}^{\infty} \frac{d\beta}{\sqrt{2\pi}} e^{-\beta/2} \left[ a^\dagger(\beta) e^{ym \cosh \beta - ix m \sinh \beta} + i a(\beta) e^{-ym \cosh \beta + ix m \sinh \beta} \right] , \quad (2.21)
\]

Below we use the notation
\[
| A(\beta_1) \ldots A(\beta_N) \rangle = a^\dagger(\beta_1) \ldots a^\dagger(\beta_N) | 0 \rangle \quad (2.22)
\]
for an $N$-particle state.

3. Conserved Charges of the Doubled Ising Field Theory

It was observed in the past that many aspects of the Ising field theory simplify if one considers a system of two identical copies of the Ising model, with no interaction between the copies - the “doubled” Ising field theory \[3,4\]. Here we also use this trick, introducing two copies henceforth referred to as “copy $a$” and “copy $b$”. We will use the subscript $a$ or $b$ to distinguish between the fields belonging to the corresponding copy; thus, $\psi_a(x), \bar{\psi}_a(x), \sigma_a(x), \mu_a(x)$ will stand for the fermionic and the order-disorder fields from the copy $a$, and $\psi_b(x), \bar{\psi}_b(x), \sigma_b(x), \mu_b(x)$ will denote these fields from the copy $b$. The correlation functions of the doubled theory factorize in terms of the correlation functions of the individual copies, in an obvious manner.

The above straightforward definition has an inconvenient feature that the fermi fields belonging to different copies commute. This can be fixed by introducing, in addition to the two Ising copies, two auxiliary variables, the so-called “Klein factors”, $\eta_a$ and $\eta_b$. These are assumed to commute with all the observables from the original Ising copies, and to satisfy the following defining relations
\[
\eta_a^2 = 1 , \quad \eta_b^2 = 1 , \quad \eta_a \eta_b = -\eta_b \eta_a , \quad (3.1)
\]
and
\[ \langle \eta_a \rangle = 0 , \quad \langle \eta_b \rangle = 0. \]  \hfill (3.2)

We then modify the definitions of the fields of the doubled Ising field theory as follows,
\[
\begin{aligned}
\psi_a(x) &\rightarrow \eta_a \psi_a(x) , & \psi_b(x) &\rightarrow \eta_b \psi_b(x) , \\
\bar{\psi}_a(x) &\rightarrow \eta_a \bar{\psi}_a(x) , & \bar{\psi}_b(x) &\rightarrow \eta_b \bar{\psi}_b(x) , \\
\sigma_a(x) &\rightarrow \sigma_a(x) , & \sigma_b(x) &\rightarrow \sigma_b(x) , \\
\mu_a(x) &\rightarrow \eta_a \mu_a(x) , & \mu_b(x) &\rightarrow \eta_b \mu_b(x) .
\end{aligned}
\]  \hfill (3.3)

This modification has no effect on any of the relations in Section 2, and hence it does not bring any change to the correlation functions involving only the fields from a single copy, either \(a\) or \(b\). On the other hand, the Klein factors give rise to additional sign factors when factorizing generic correlation functions with entries from both copies present. For example, \(\langle \sigma_a(x) \sigma_b(x) \sigma_a(x') \sigma_b(x') \rangle = \langle \sigma(x) \sigma(x') \rangle^2\), but \(\langle \mu_a(x) \mu_b(x) \mu_a(x') \mu_b(x') \rangle = -\langle \mu(x) \mu(x') \rangle^2\), where the expectation values in the right-hand sides are that of a single Ising field theory. The desirable effect of this modification is that fermi fields from different copies now anticommute inside the correlation functions, \(\psi_a(x) \psi_b(x') = -\psi_b(x') \psi_a(x)\), etc. Of course, with this definition the doubled Ising field theory is identical to the theory of a free Dirac fermi field
\[
\Psi(x) = (\psi_a(x) + i \psi_b(x), \bar{\psi}_a(x) + i \bar{\psi}_b(x)) .
\]  \hfill (3.5)

The space of states of the doubled Ising field theory involves two species of free fermi particles, \(A\) and \(B\), associated with the copies \(a\) and \(b\), respectively. The multiparticle states are generated by the corresponding creation and annihilation operators \(a^\dagger(\beta), a(\beta)\) and \(b^\dagger(\beta), b(\beta)\):
\[
|A(\beta_1) \ldots A(\beta_N)B(\beta_{N+1}) \ldots B(\beta_M)\rangle = a^\dagger(\beta_1) \ldots a^\dagger(\beta_N) b^\dagger(\beta_{N+1}) \ldots b^\dagger(\beta_M)|0\rangle .
\]  \hfill (3.6)

The operators \(a^\dagger(\beta), a(\beta)\) and \(b^\dagger(\beta), b(\beta)\) are related to the fields \(\psi_a(x), \bar{\psi}_a(x)\) and \(\psi_b(x), \bar{\psi}_b(x)\), respectively, through equations identical to Eqs. (2.21). Of course, each pair \(a^\dagger(\beta), a(\beta)\) and \(b^\dagger(\beta), b(\beta)\) obeys the canonical anticommutators (2.20), and \(a^\dagger(\beta), a(\beta)\) anticommute with \(b^\dagger(\beta), b(\beta)\).
Being a free fermion theory, the doubled Ising field theory exhibits an infinite set of local Integrals of Motion (IM). We have no need to describe the whole set here. Certain subset of these IM forms the commutator algebra \( \hat{SL}(2) \) of the level zero; its detailed description is given in Ref. [8]. Here we display only the fundamental generating elements of this subset, since these basic IM are directly used in the calculations below.

The simplest IM is the \( U(1) \) charge associated with phase rotations of the Dirac field (3.5),

\[
Z_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\psi_a \psi_b - \bar{\psi}_a \bar{\psi}_b] \, dx .
\]

(3.7)

Also, the energy-momenta \( (P, \bar{P}) \) associated with each copy are conserved separately, hence their linear combinations

\[
X_1 = P_a - P_b , \quad X_{-1} = \bar{P}_a - \bar{P}_b ,
\]

(3.8)

are conserved as well. Less trivially, the following integrals are conserved,

\[
Y_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \psi_a \partial \psi_b + i \frac{m}{2} \bar{\psi}_a \bar{\psi}_b \right] \, dx ,
\]

\[
Y_{-1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \bar{\psi}_a \partial \bar{\psi}_b - i \frac{m}{2} \psi_a \bar{\psi}_b \right] \, dx ,
\]

(3.9)

as one can check by a simple direct computation. Equally straightforward calculation yields the basic commutators

\[
[Z_0, X_1] = 2i Y_1 , \quad [Z_0, Y_1] = -2i X_1 ,
\]

\[
[Z_0, X_{-1}] = 2i Y_{-1} , \quad [Z_0, Y_{-1}] = -2i X_{-1} ,
\]

\[
[X_1, Y_{-1}] = 2i \left( \frac{m}{2} \right)^2 Z_0 , \quad [X_{-1}, Y_1] = 2i \left( \frac{m}{2} \right)^2 Z_0 ,
\]

\[
[X_1, X_{-1}] = 0 , \quad [Y_1, Y_{-1}] = 0 ,
\]

(3.10)

which show that the operators \( Z_0, X_{\pm 1}, Y_{\pm 1} \) generate the affine Lie algebra \( \hat{SL}(2) \) of the level zero \( \frac{1}{2} \). Further commutators of these basic generators give rise to other elements of

\[\text{\footnotesize 1}\]

The standard Chevalley generators \( \{ E_\pm, F_\pm, H_\pm \} \) for the \( \hat{SL}(2) \) algebra are the linear combinations

\[
E_\pm = (X_1 \pm i Y_1)/m , \quad F_\pm = (X_{-1} \mp i Y_{-1})/m , \quad H_\pm = \pm Z_0 .
\]

One can check validity of the Serre relations \([E_\pm, [E_\pm, [E_\pm, E_\mp]]]] = 0\) and \([F_\pm, [F_\pm, [F_\pm, F_\mp]]]] = 0\) by yet another straightforward computation.
\( \mathcal{SL}(2) \), which all have the form of integrals of local densities, quadratic in terms of the fermi fields and their derivatives \[^8\]. In our calculations below two such elements,

\[
2i \mathbf{Z}_2 = [\mathbf{X}_1, \mathbf{Y}_1], \quad -2i \mathbf{Z}_{-2} = [\mathbf{X}_{-1}, \mathbf{Y}_{-1}], \tag{3.11}
\]

will be particularly useful. Explicitly,

\[
\mathbf{Z}_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \partial \psi_a \partial \psi_b + \left( \frac{m}{2} \right)^2 \psi_a \psi_b \right] \, dx , \\
\mathbf{Z}_{-2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \bar{\partial} \bar{\psi}_a \bar{\partial} \bar{\psi}_b + \left( \frac{m}{2} \right)^2 \bar{\psi}_a \bar{\psi}_b \right] \, dx . \tag{3.12}
\]

In the next Section we study the Ward identities associated with this \( \mathcal{SL}(2) \) symmetry of the doubled Ising field theory. Derivations of the Ward identities involve commutators of the symmetry generators with local fields. Specifically, we will be interested in the commutators of the generators with the products of the order or the disorder fields from different copies, i.e. with the fields \( \sigma_a(x) \sigma_b(x) \), \( \sigma_a(x) \mu_b(x) \), \( \mu_a(x) \sigma_b(x) \) and \( \mu_a(x) \mu_b(x) \). Note that the densities of the IM (3.7)–(3.9), (3.12) are local with respect to such product fields, and therefore these commutators are again local fields. Of course, the commutators with \( \mathbf{X}_{\pm 1} \) can be written down immediately from the definitions (3.8),

\[
[X_1, \mathcal{O}_a(x) \mathcal{O}_b(x)] = i \partial \mathcal{O}_a(x) \mathcal{O}_b(x) - i \mathcal{O}_a(x) \partial \mathcal{O}_b(x) , \\
[X_{-1}, \mathcal{O}_a(x) \mathcal{O}_b(x)] = -i \bar{\partial} \mathcal{O}_a(x) \mathcal{O}_b(x) + i \mathcal{O}_a(x) \bar{\partial} \mathcal{O}_b(x) . \tag{3.13}
\]

Derivation of the other commutators are less elementary, but yet straightforward. One can use the decomposition (2.6) for each of the fields \( \psi_a \) and \( \psi_b \) to express any such commutator in terms of the corresponding radial mode operators \( a_n, \bar{a}_n \) and \( b_n, \bar{b}_n \). For instance,

\[
[Z_0, \mathcal{O}_a \mathcal{O}_b] = -i \left( a_0 b_0 + \bar{a}_0 \bar{b}_0 \right) \mathcal{O}_a \mathcal{O}_b - i \sum_{n=1}^{\infty} \left( a_{-n} b_n - b_{-n} a_n + \bar{a}_{-n} \bar{b}_n - \bar{b}_{-n} \bar{a}_n \right) \mathcal{O}_a \mathcal{O}_b . \tag{3.14}
\]

When the primary order or disorder fields are concerned, Eqs. (2.8), (2.9), and (2.16), (2.17) apply. It is important to keep track of the Klein factors \( \eta_a, \eta_b \) in these calculations to get the signs right. For the commutators involving \( Z_0 \) and \( Y_1 \), the results are

\[
[Z_0, \sigma_a(x) \sigma_b(x)] = 0 , \quad \quad \quad [Z_0, \mu_a(x) \mu_b(x)] = 0 , \\
[Z_0, \sigma_a(x) \mu_b(x)] = -i \mu_a(x) \sigma_b(x) , \quad \quad \quad [Z_0, \mu_a(x) \sigma_b(x)] = i \sigma_a(x) \mu_b(x) , \tag{3.15}
\]
and
\[
[Y_1, \sigma_a(x) \sigma_b(x)] = -\partial \mu_a(x) \mu_b(x) + \mu_a(x) \partial \mu_b(x) ,
\]
\[
[Y_1, \mu_a(x) \mu_b(x)] = -\partial \sigma_a(x) \sigma_b(x) + \sigma_a(x) \partial \sigma_b(x) ,
\]
\[
[Y_1, \sigma_a(x) \mu_b(x)] = i \partial \mu_a(x) \sigma_b(x) - i \mu_a(x) \partial \sigma_b(x) ,
\]
\[
[Y_1, \mu_a(x) \sigma_b(x)] = -i \partial \sigma_a(x) \mu_b(x) + i \sigma_a(x) \partial \mu_b(x) .
\]

Let us also present two useful commutators involving the operator \(Z_2\),
\[
[Z_2, \sigma_a(x) \sigma_b(x)] = 2 \partial \mu_a(x) \partial \mu_b(x) - \partial^2 \mu_a(x) \mu_b(x) - \mu_a(x) \partial^2 \mu_b(x) ,
\]
\[
[Z_2, \mu_a(x) \mu_b(x)] = 2 \partial \sigma_a(x) \partial \sigma_b(x) - \partial^2 \sigma_a(x) \sigma_b(x) - \sigma_a(x) \partial^2 \sigma_b(x) .
\]
The operators \(Y_{-1}\) and \(Z_{-2}\) satisfy relations similar to (3.16) and (3.17) with \(\partial\) replaced by \(\bar{\partial}\) and \(i\) replaced by \(-i\).

In addition to the above commutators, the calculations in Section 4 below will involve the action of the generators on the multiparticle states \(|A(\beta_1) \ldots A(\beta_N) B(\beta_{N+1}) \ldots B(\beta_M)\rangle\) of the doubled Ising model. The equations
\[
Z_0 = -i \int_{-\infty}^{\infty} \left[ a^\dagger(\beta) b(\beta) - b^\dagger(\beta) a(\beta) \right] \frac{d\beta}{2\pi} ,
\]
and
\[
Y_1 = \frac{m}{2} \int_{-\infty}^{\infty} e^{\beta} \left[ a^\dagger(\beta) b(\beta) + b^\dagger(\beta) a(\beta) \right] \frac{d\beta}{2\pi} ,
\]
\[
Y_{-1} = \frac{m}{2} \int_{-\infty}^{\infty} e^{-\beta} \left[ a^\dagger(\beta) b(\beta) + b^\dagger(\beta) a(\beta) \right] \frac{d\beta}{2\pi} ,
\]
follow directly from (3.18), (3.19) and the decompositions (2.21) of the fermi fields \(\psi_a\) and \(\psi_b\) in terms of the particle creation and annihilation operators \(a, a^\dagger\) and \(b, b^\dagger\).

4. Correlation Functions and Particle Matrix Elements

In this section we use the Ward identities associated with the \(\widehat{SL}(2)\) symmetry (3.10) of the doubled model to derive the differential equation determining the matrix elements of the type (1.1), involving order and/or disorder fields, between any particle states. As the warm-up exercise, let us first rederive the celebrated equations of Ref. [1] for the vacuum-vacuum matrix elements.
4.1. The Correlation Functions

Consider first the following simple identity

\[ \langle 0 | \left[ Z_2 , \sigma_a(x)\sigma_b(x) \, \mu_a(0)\mu_b(0) \right] | 0 \rangle = 0 . \tag{4.1} \]

Evaluating the commutator according to Eq. (3.17), one derives the equation

\[ \partial G \partial G - G \partial^2 G - \partial \tilde{G} \partial \tilde{G} + \tilde{G} \partial^2 \tilde{G} = 0 , \tag{4.2} \]

where

\[ G(x) = \langle \sigma(x)\sigma(0) \rangle , \quad \tilde{G}(x) = \langle \mu(x)\mu(0) \rangle , \tag{4.3} \]

and \( \partial = \partial_x \). Note the signs of the last two terms, which are due to the presence of the Klein factors in the definitions (3.4) of the fields \( \mu_a, \mu_b \). The equation similar to (4.2), with \( \partial \) replaced by \( \bar{\partial} = \partial_{\bar{z}} \), is derived when one takes \( Z_2 \) instead of \( Z_{-2} \) in (4.1).

Additional equations are derived in similar way. Again, since the vacuum state \( | 0 \rangle \) is annihilated by all the generators (3.8), (3.9), the following equations hold

\[ \langle 0 | \left[ X_{-1}, \sigma_a(x)\sigma_b(x) \right] \left[ Y_1, \mu_a(0)\mu_b(0) \right] | 0 \rangle = \frac{i}{2} m^2 \langle 0 | \left[ Z_0, \sigma_a(x)\sigma_b(x) \right] \mu_a(0)\mu_b(0) | 0 \rangle , \tag{4.4a} \]

\[ \langle 0 | \left[ X_{-1}, \sigma_a(x)\mu_b(x) \right] \left[ Y_1, \mu_a(0)\sigma_b(0) \right] - \left[ Y_1, \sigma_a(x)\mu_b(x) \right] \left[ X_{-1}, \mu_a(0)\sigma_b(0) \right] | 0 \rangle = \frac{i}{2} m^2 \langle 0 | \left[ Z_0, \sigma_a(x)\mu_b(x) \right] \mu_a(0)\sigma_b(0) | 0 \rangle , \tag{4.4b} \]

where \( Z_0 \) in the right hand sides comes from the commutator \( [X_{-1}, Y_1] \) (see Eq. (3.10)). Now using Eqs. (3.13) and (3.16) one derives

\[ G \partial \bar{\partial} G - G \bar{\partial} \partial G + \tilde{G} \bar{\partial} \partial \tilde{G} - \partial \tilde{G} \bar{\partial} \tilde{G} = 0 , \tag{4.5a} \]

\[ G \partial \bar{\partial} \tilde{G} - G \bar{\partial} \partial \tilde{G} + \tilde{G} \bar{\partial} \partial G - \partial \tilde{G} \bar{\partial} G = \left( \frac{m}{2} \right)^2 G \tilde{G} . \tag{4.5b} \]

The Eqs. (4.2) and (4.5a, b) are known as the quadratic form [15,16] of the famous differential equations of Wu, McCoy, Tracy and Barouch [1]. When \( G(x) \) and \( \tilde{G}(x) \) are written in terms of auxiliary functions \( \varphi(x) \) and \( \chi(x) \) as

\[ m^{-1/4} G = e^{\varphi/2} \cosh(\varphi/2) , \quad m^{-1/4} \tilde{G} = e^{\chi/2} \sinh(\varphi/2) , \tag{4.6} \]
the Eqs. (4.2), (4.5a, b) take the sinh-Gordon form,

\[ \partial \tilde{\varphi} = \frac{m^2}{8} \sinh(2\varphi), \quad \partial^2 \chi + (\partial \varphi)^2 = 0, \]

\[ \partial \tilde{\chi} = \frac{m^2}{8} \left[ 1 - \cosh(2\varphi) \right], \quad \partial^2 \chi + (\partial \varphi)^2 = 0. \] (4.7)

The correlation functions (4.3) are related to the rotationally-invariant solution \( \varphi = \varphi(r) \), \( \chi = \chi(r) \), where \( r = |x| \) is the distance between \( x \) and the origin; thus, Eqs. (4.7) reduce to ordinary differential equations with respect to \( r \). The solution relevant to the problem can be singled out by specifying its \( r \to 0 \) asymptotic behaviour,

\[ \varphi(r) \sim -\ln \frac{m r}{2} - \ln (-\Omega) + O \left( r^4 \Omega^2 \right), \quad \chi(r) \sim \frac{1}{2} \ln (4 m r) + \ln (-\Omega) + O \left( r^2 \right); \] (4.8)

here and below in the Appendix we use the notation

\[ \Omega = \ln \left( \frac{e^\gamma}{8 m r} \right), \] (4.9)

where \( \gamma \) is the Euler’s constant. This solution has the property that \( \varphi(r) \) decays at \( r \to \infty \), while \( \chi(r) \) approaches a constant \( 4 \ln \bar{s} \). All these properties are elaborated in Ref. [1], where more details can be found. We present some useful expansions in the Appendix.

4.2. One-Particle Matrix Elements

It is straightforward to extend this technique to the matrix elements of the form (1.1) involving the particle states. First, consider the one-particle matrix elements \( \langle 0 | \sigma(x)\mu(x') | A(\beta) \rangle \). It is useful to introduce the following notations

\[ \langle 0 | \sigma(x)\mu(x') | A(\beta) \rangle = E(x + x'; \beta) F(x - x'; \beta), \]

\[ \langle 0 | \mu(x)\sigma(x') | A(\beta) \rangle = E(x + x'; \beta) \tilde{F}(x - x'; \beta), \] (4.10)

where

\[ E(x + x'; \beta) = e^{- (y + y') \frac{m}{2} \cosh \beta + i (x + x') \frac{m}{2} \sinh \beta} \] (4.11)

are the “center of mass” plane waves. The fact that \( F \) and \( \tilde{F} \) depend on the separation \( x - x' \) only is a simple consequence of the kinematics. Likewise, from the rotational symmetry of the problem, both functions depend only on \( r \) and the combination \( \vartheta = \theta - i \beta \), where \( (r, \theta) \) are the polar coordinates associated with the separation \( x - x' \), i.e.

\[ z - z' = r e^{i\theta}, \quad \bar{z} - \bar{z}' = r e^{-i\theta}. \] (4.12)
And it follows from what was said in Section 2 about the local properties of the order versus the disorder field that $F(x - x'; \beta)$ and $\tilde{F}(x - x'; \beta)$ are double-valued functions of the coordinates, changing the sign when $x$ is brought around $x'$. With this understood, the above definition implies $F(x; \beta) = \pm \tilde{F}(-x; \beta)$, where the sign depends on the way of continuation $x \to -x$.

The differential equations for the functions $F$ and $\tilde{F}$ can be derived from the identities

$$
\langle 0 \mid Y_1 \sigma_a(x) \sigma_b(x') \sigma_a(x') \mu_b(x') \mid A(\beta) \rangle = 0,
$$

$$
\langle 0 \mid Y_1 \sigma_a(x) \mu_b(x) \sigma_a(x') \sigma_b(x') \mid A(\beta) \rangle = 0.
$$

(4.13)

One uses the commutation relations (3.16) to move the operator $Y_1$ to the right, and then applies (3.19) to evaluate the action of this operator on the particle state. The resulting equations are linear in $F$ and $\tilde{F}$, and involve the correlation functions (4.3) as the coefficients. They are brought to a nice form by introducing the functions $\Psi_+$ and $\Psi_-$ related to $F$ and $\tilde{F}$ as

$$
F + i \tilde{F} = \omega m^{1/4} e^{\chi/2} \Psi_+ ,
$$

$$
F - i \tilde{F} = \omega m^{1/4} e^{\chi/2} \Psi_- ,
$$

(4.14)

where again $\omega = e^{i\pi/4}$, $\bar{\omega} = e^{-i\pi/4}$, and $\chi$ is related to the two-point correlation functions as in (4.6). With these notations, the Ward identities (4.13) yield

$$
\partial \Psi_+ = -\frac{1}{2} \partial \varphi \Psi_+ + \frac{m}{4} e^\beta e^{\varphi} \Psi_- ,
$$

$$
\partial \Psi_- = \frac{1}{2} \partial \varphi \Psi_- - \frac{m}{4} e^\beta e^{-\varphi} \Psi_+ ,
$$

(4.15)

where $\partial = \partial_z$ and the function $\varphi$ is the same as in (4.6). Replacing $Y_1$ in (4.13) with $Y_{-1}$ one derives similar equations involving the derivative $\bar{\partial} = \partial_\bar{z}$,

$$
\bar{\partial} \Psi_+ = \frac{1}{2} \bar{\partial} \varphi \Psi_+ - \frac{m}{4} e^{-\beta} e^{-\varphi} \Psi_- ,
$$

$$
\bar{\partial} \Psi_- = -\frac{1}{2} \bar{\partial} \varphi \Psi_- + \frac{m}{4} e^{-\beta} e^{\varphi} \Psi_+ .
$$

(4.16)

One recognizes in (4.13), (4.14) the Lax representation of the sinh-Gordon equation: the first of the Eqs. (4.7) guarantees compatibility of (4.15) and (4.16). Thus, the one-particle matrix elements (4.10) are related to the solution of the linear problem (4.13), (4.16) associated with the sinh-Gordon equation (4.7).
Let us briefly describe here some elementary properties of the functions $\Psi_{\pm}$. As was mentioned above, they depend on two variables, $r$ and $\vartheta = \theta - i\beta$, where $(r, \theta)$ are the polar coordinates defined in (4.12). The monodromy properties

$$
\Psi_+(r, \vartheta + \pi) = i \Psi_+(r, \vartheta), \quad \Psi_-(r, \vartheta + \pi) = -i \Psi_-(r, \vartheta),
$$

(4.17)

follow directly from the definitions (4.10), (4.14). Also, it is possible to show that

$$
\Psi_+(r, \vartheta) = \Psi_-(r, -\vartheta),
$$

(4.18)

and that when both $r$ and $\vartheta$ are real (i.e. the rapidity $\beta$ is continued to pure imaginary values) $\Psi_+$ and $\Psi_-$ take complex conjugate values. The $r \to 0$ behaviour of $\Psi_+$ and $\Psi_-$ follows from the short-distance operator product expansion (2.18),

$$
e^{\chi(r)/2} \Psi_{\pm}(r, \vartheta) \sim \sqrt{2\pi} (mr)^{1/4} e^{\pm i\vartheta/2}.
$$

(4.19)

More details on the asymptotic behaviour is presented in the Appendix. Note that in view of (4.17), (4.18), $\Psi_+(r, \vartheta)$ and $\Psi_-(r, \vartheta)$ admit the Fourier decompositions of the form

$$
\Psi_{\pm}(r, \vartheta) = \sum_{n=-\infty}^{\infty} \Psi_{2n}(r) e^{\pm i(\frac{1}{2} + 2n)\vartheta},
$$

(4.20)

where $\Psi_{2n}(r)$ are real at real $r$. The coefficients $\Psi_{2n}(r)$ can be interpreted in terms of the finite-distance version of the operator product expansion (2.18), i.e.

$$
\sigma(x) \mu(x') = \sum_{n=0}^{\infty} \left[ C_n(x - x') \partial^n \psi \left( \frac{x + x'}{2} \right) + \bar{C}_n(x - x') \bar{\partial}^n \bar{\psi} \left( \frac{x + x'}{2} \right) \right] + \cdots,
$$

(4.21)

where the dots stand for the contributions of the multi-fermion operators (i.e. the composite fields built from three or more fermi fields, defined in such a way that their matrix elements between $\langle 0 \mid$ and $\mid A(\beta) \rangle$ vanish). Using the polar coordinates (4.12) to represent the separation $x - x'$, we have

$$
C_n(r, \theta) = \omega e^{i(\frac{1}{2} + n)\theta} c_n(r), \quad \bar{C}_n(r, \theta) = \bar{\omega} e^{-i(\frac{1}{2} + n)\theta} \bar{c}_n(r),
$$

(4.22)

where

$$
c_{2n}(r) = \frac{(-)^n 2^{2n-1}}{\sqrt{\pi} m^{2n+1/4}} e^{\chi(r)/2} \Psi_{2n}(r),
$$

$$
c_{2n-1}(r) = -\frac{(-)^n 2^{2n-2}}{\sqrt{\pi} m^{2n-3/4}} e^{\chi(r)/2} \Psi_{-2n}(r).
$$

(4.23)
4.3. Multi-Particle Matrix Elements

The matrix elements involving more then one particle can be determined in a similar way. Here we explicitly elaborate the matrix elements

\[ G(x, x'; \beta_1, \ldots, \beta_{2N}) = \langle 0 | \sigma(x)\sigma(x') | A(\beta_1) \ldots A(\beta_{2N}) \rangle, \]
\[ \tilde{G}(x, x'; \beta_1, \ldots, \beta_{2N}) = \langle 0 | \mu(x)\mu(x') | A(\beta_1) \ldots A(\beta_{2N}) \rangle. \]  

In order to avoid lengthy expressions, we shall drop the explicit indication of position dependence from now on and use instead the notations \( G(\beta_1, \ldots, \beta_{2N}) \) and \( \tilde{G}(\beta_1, \ldots, \beta_{2N}) \) for the matrix elements (4.24). Similarly, \( \Psi_\pm(\beta) \) will stand for the functions defined in (4.14) in relation to the one-particle matrix elements (4.10).

Let us start with the derivation of the two-particle matrix elements \( G(\beta_1, \beta_2) \) and \( \tilde{G}(\beta_1, \beta_2) \). It suffices to consider the identities

\[ \langle 0 \mid Z \sigma_a(x)\mu_b(x) \sigma_a(x')\mu_b(x') \mid A(\beta_1)B(\beta_2) \rangle = 0, \]
\[ \langle 0 \mid Y_1 \sigma_a(x)\sigma_b(x) \sigma_a(x')\sigma_b(x') \mid A(\beta_1)B(\beta_2) \rangle = 0, \]
\[ \langle 0 \mid Y_1 \mu_a(x)\mu_b(x) \mu_a(x')\mu_b(x') \mid A(\beta_1)B(\beta_2) \rangle = 0. \]  

Again, one uses the commutators (3.13), (3.16), and then Eqs. (3.18), (3.19) to evaluate the action of the operators \( Z \) and \( Y_1 \) on the two-particle states. The relations that come out allow one to find explicit expression for the desired two-particle matrix elements in terms of the one-particle ones. The result is

\[ i \frac{G_+(\beta_1, \beta_2)}{G + \tilde{G}} = \frac{E(\beta_1)E(\beta_2)}{e^{\beta_1} + e^{\beta_2}} \left[ e^{\beta_1}\Psi_-(\beta_1)\Psi_+(\beta_2) - e^{\beta_2}\Psi_+(\beta_1)\Psi_-(\beta_2) \right], \]
\[ i \frac{G_-(\beta_1, \beta_2)}{G - \tilde{G}} = \frac{E(\beta_1)E(\beta_2)}{e^{\beta_1} + e^{\beta_2}} \left[ e^{\beta_1}\Psi_+(\beta_1)\Psi_-(\beta_2) - e^{\beta_2}\Psi_-\beta_1(\beta_1)\Psi_+(\beta_2) \right], \]  

where

\[ G_\pm(\beta_1, \beta_2) = G(\beta_1, \beta_2) \pm \tilde{G}(\beta_1, \beta_2), \]

and \( E(\beta) \equiv E(x + x'; \beta) \) are the plane-wave factors (4.11).

Once the two-particle matrix elements are found, all the multiparticle ones are obtained through the recursive equation

\[ GG(\beta_1, \ldots, \beta_{2N}) = \sum_{i=1}^{2N-1} (-)^i G(\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_{2N-1}) G(\beta_i, \beta_{2N}) \]  

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which follows directly from the Ward identity

\[ \langle 0 \mid Z_0 \sigma_a(x) \sigma_b(x) \sigma_a(x') \sigma_b(x') \mid A(\beta_1) \ldots A(\beta_{2N-1}) B(\beta_{2N}) \rangle = 0 . \]  

(4.29)

Of course the recursive equation (4.28) is nothing but the statement that the multiparticle elements (4.24) are expressed in terms of the two-particle ones, through all possible Wick pairings,

\[ G^{N-1} G(\beta_1, \ldots, \beta_{2N}) = \frac{1}{(2N)!!} \epsilon^{a_1 \ldots a_{2N}} G(\beta_{a_1}, \beta_{a_2}) \ldots G(\beta_{a_{2N-1}}, \beta_{a_{2N}}) , \]  

(4.30)

where \( \epsilon^{a_1 \ldots a_{2N}} \) denotes usual antisymmetric tensor and summation over \( a_i = 1, \ldots, 2N \) is implicit. This form reflects the free-fermion structure of the Ising field theory. The matrix elements \( \tilde{G}(\beta_1, \ldots, \beta_{2N}) \) have the same form in terms of the functions \( \tilde{G}(\beta_1, \beta_2) \).

5. Corrections to the Masses

As was already mentioned in the Introduction, the primary motivation of this study was the development of more efficient perturbation theory for the Ising field theory in the presence of external field. The last theory corresponds to adding the spin term to the free action (2.1),

\[ A = A_{FF} + h \int d^2x \sigma(x) , \]  

(5.1)

and it appears to be highly nontrivial interacting field theory. Its physics depends on a single scaling parameter \( \eta = m/|h|^{8/15} \) and shows rich behavior in terms of its particle spectrum [9]. Exact solution exists only in the limiting cases \( \eta = \pm \infty \), where of course the theory reduces back to the free fermions (2.1), and at \( \eta = 0 \), where it is integrable, but at generic values of this parameter the mass spectrum is understood mostly on a qualitative level, as described in [8,7,12]. In this situation, perturbative expansions around \( \eta = \pm \infty \), say, are useful. Here we will display the calculations of the leading perturbative correction, \( \sim h^2 \), to the particle mass in the disordered regime \( \eta \to -\infty \), and of the same order correction to the “quark mass” in the ordered regime \( \eta \to +\infty \).

Both these calculations are based on the formula (1.2), but of course their interpretations in terms of the physical mass spectrum are rather different. In the disordered regime \( \eta \to -\infty \) the spectrum contains a single particle, and the calculation of the mass correction according to (1.2) yields just that - the correction to its mass \( M_1 \), i.e. the coefficient \( a \) in

\[ M_1(\eta) = |m| + a \frac{|m|}{(-\eta)^{15/4}} + O((-\eta)^{-15/2}) , \quad \eta \to -\infty , \]  

(5.2)
where \( m \) is the mass parameter of the unperturbed theory (2.1). Note that we put \(|m|\) here because in our convention (2.9) the disordered regime corresponds to negative \( m \) in (2.1). Recall that all equations in the previous Sections are written down under the opposite assumption \( m > 0 \). But thanks to the duality \( m \leftrightarrow -m, \sigma \leftrightarrow \mu \) we can still use them in this computation, just taking in (1.2) the matrix elements of the disorder fields \( \mu \) instead of \( \sigma \). Therefore we have for the coefficient \( a \) in (5.2),

\[
a = -\frac{1}{2} |m|^{15/4} \int \langle \beta | \mu(x) \mu(0) | \beta \rangle_{\text{irred}} d^2 x .
\]

On the contrary, in the ordered regime \( \eta \to +\infty \) the mass spectrum is rather complex. The last term in (5.1) creates a long-range confining interaction between the original particles of (2.1), which become “quarks”. The particle spectrum consists of a tower of their meson-like bound states. The meson masses \( M_i, i = 1, 2, \ldots \), do not admit straightforward perturbative expansions in \( h^2 \), instead they expand in fractional powers of \( h \) [9,12],

\[
M_i(\eta) = 2 m_q(\eta) \left\{ 1 + \frac{(2 \bar{s})^{2/3}}{2 \eta^{5/4}} z_i - \frac{(2 \bar{s})^{4/3}}{40 \eta^{5/2}} z_i^2 + \frac{11 z_i^3}{2800} - \frac{57}{560} \frac{(2 \bar{s})^2}{\eta^{15/4}} + \ldots \right\} , \quad (5.4)
\]

where \( \bar{s} \) relates to the spontaneous magnetization at zero \( h \) as in Eq. (2.19), and \( -z_i \) are consecutive zeros of the Airy function, \( \text{Ai}(-z_i) = 0 \). Here \( m_q \) denotes the “quark mass” which admits usual perturbative expansion in powers of \( h^2 \),

\[
m_q(\eta) = m + a_q \frac{m}{\eta^{15/4}} + O(\eta^{15/2}) , \quad \eta \to +\infty , \quad (5.5)
\]

where the coefficient \( a_q \) is computed according to the Eq.(1.2) ,

\[
a_q = -\frac{1}{2} m^{15/4} \int \langle \beta | \sigma(x) \sigma(0) | \beta \rangle_{\text{irred}} d^2 x .
\]

Note that the leading radiative correction in (5.5) competes with the last term in (5.4). We also would like to stress here that, since the “quarks” do not appear in the asymptotic states of the theory, there is no definition of the “quark mass” independent of the perturbation theory. Therefore, while (5.5) is derivable as usual from the LSZ theory, the perturbative expansion (5.5) is rather a definition of the quark mass. Nonetheless, the leading correction in (5.5) shows up in many places besides the expansion (5.4) of the meson masses, notably in the \( \eta \to \infty \) expansion of the “false vacuum” resonance decay rate (see [12]).
With all this said, we just proceed with evaluating the integrals in (5.3) and (5.6). The matrix elements involved are obtained from the matrix elements in Section 4 through usual crossing equations. One writes the full matrix elements as

\[
\langle A(\beta_1) | \sigma(x)\sigma(x') | A(\beta_2) \rangle = 2\pi \delta(\beta_1 - \beta_2) G + G(\beta_1|\beta_2), \tag{5.7a}
\]

\[
\langle A(\beta_1) | \mu(x)\mu(x') | A(\beta_2) \rangle = 2\pi \delta(\beta_1 - \beta_2) \tilde{G} + \tilde{G}(\beta_1|\beta_2), \tag{5.7b}
\]

where the delta-function terms are the disconnected “direct propagation” parts shown schematically in Fig. 1, with \( G = G(x, x') \) and \( \tilde{G} = \tilde{G}(x, x') \) being the two-point functions (4.3), and the functions \( G(\beta_1|\beta_2) \) and \( \tilde{G}(\beta_1|\beta_2) \) (whose arguments \( x, x' \) are also suppressed) are related to the two-particle matrix elements \( G(\beta_1, \beta_2) \) and \( \tilde{G}(\beta_1, \beta_2) \), Eq. (4.24), as

\[
G(\beta_1|\beta_2) = G(\beta_1 - i\pi, \beta_2), \quad \tilde{G}(\beta_1|\beta_2) = \tilde{G}(\beta_1 - i\pi, \beta_2). \tag{5.8}
\]

![Fig. 1: Disconnected “direct propagation” part in (5.7a); the double line represents two-point correlation function \( G(x, x') \) and the plain line indicates a particle of rapidity \( \beta_1 \).](image)

The Eqs. (5.3) and (5.6) involve only the diagonal elements \( \tilde{G}(\beta|\beta) \) and \( G(\beta|\beta) \). From (4.26) we find

\[
G(\beta|\beta) = \Psi_+(\beta)\Psi_-(\beta) \left[ \hat{G} - G \frac{d}{d\beta} \ln \left( \frac{\Psi_+(\beta)}{\Psi_-(\beta)} \right) \right],
\]

\[
\tilde{G}(\beta|\beta) = \Psi_+(\beta)\Psi_-(\beta) \left[ G - \hat{G} \frac{d}{d\beta} \ln \left( \frac{\Psi_+(\beta)}{\Psi_-(\beta)} \right) \right]. \tag{5.9}
\]

Remaining disconnected and one-particle reducible parts still have to be subtracted before one plugs these functions in (5.3), (5.6),

\[
\langle A(\beta) | \mu(x)\mu(x') | A(\beta) \rangle_{\text{irred}} = \tilde{G}(\beta|\beta) - \tilde{S}(\beta|\beta),
\]

\[
\langle A(\beta) | \sigma(x)\sigma(x') | A(\beta) \rangle_{\text{irred}} = G(\beta|\beta) - S(\beta|\beta). \tag{5.10}
\]
Since the matrix elements \( \langle A(\beta) \mid \mu(x) \mid A(\beta') \rangle \) vanish, but
\[
\langle 0 \mid \mu(x) \mid A(\beta) \rangle = \bar{\sigma} e^{-y m \cosh \beta + i x m \sinh \beta},
\]
the matrix element of \( \mu(x) \mu(x') \) has no one-particle reducible parts, and one just has to subtract the disconnected pieces depicted in Fig. 2,
\[
\tilde{S}(\beta|\beta) = \langle A(\beta) \mid \mu(x) \mid 0 \rangle \langle 0 \mid \mu(x') \mid A(\beta) \rangle + \langle 0 \mid \mu(x) \mid A(\beta) \rangle \langle A(\beta) \mid \mu(x') \mid 0 \rangle
\]
\[
= 2 \sigma^2 \cosh (m r \sin \vartheta),
\]
where \( \vartheta = \theta - i \beta \) and \((r, \theta)\) are polar coordinates (4.12) associated with the separation \( x - x' \). On the other hand, \( \langle 0 \mid \sigma(x) \mid A(\beta) \rangle = 0 \), so that the matrix element of \( \sigma(x) \sigma(x') \) does not have such disconnected parts, but it has one-particle reducible components shown in Fig. 3.

Fig. 2: Disconnected terms in the matrix element \( \langle A(\beta) \mid \mu(x) \mu(x') \mid A(\beta) \rangle \). Sum of these diagrams, Eq. (5.12), is the subtraction term \( \tilde{S}(\beta|\beta) \) in (5.10).

Fig. 3: One-particle reducible components in \( \langle A(\beta) \mid \sigma(x) \sigma(x') \mid A(\beta) \rangle \). These add up to the subtraction term \( S(\beta|\beta) \) in (5.10) - see Eq. (5.13).
This reducible part is written as the principal-value integral

\[
S(\beta|\beta) = \int_{-\infty}^{\infty} \frac{d\beta'}{2\pi} \left[ \langle A(\beta) | \sigma(x) | A(\beta') \rangle \langle A(\beta') | \sigma(x') | A(\beta) \rangle + \langle 0 | \sigma(x) | A(\beta)A(\beta') \rangle \langle A(\beta)A(\beta') | \sigma(x') | 0 \rangle \right]
\]

(5.13)

over the rapidity \(\beta'\) of the intermediate particle in Fig. 3; the matrix elements involved here are the \(\sigma\)-field form-factors [4],

\[
\langle 0 | \sigma(0) | A(\beta_1)A(\beta_2) \rangle = i \bar{\sigma} \tanh \left( \frac{\beta_1 - \beta_2}{2} \right),
\]

\[
\langle A(\beta_1) | \sigma(0) | A(\beta_2) \rangle = i \bar{\sigma} \coth \left( \frac{\beta_1 - \beta_2}{2} \right),
\]

(5.14)

and in writing (5.13) we have assumed that \(y \geq y'\). For our purposes the following equivalent form of this term will be more convenient,

\[
S(\beta|\beta) = 2\bar{\sigma}^2 mr \cos \vartheta + \bar{\sigma}^2 e^{-mr \sin \vartheta} \left[ \frac{1}{\pi} K_0(mr) + 2 \partial_\vartheta A(r, \vartheta - \pi/2) \right] + \bar{\sigma}^2 e^{mr \sin \vartheta} \left[ \frac{1}{\pi} K_0(mr) + 2 \partial_\vartheta B(r, \vartheta - \pi/2) \right],
\]

(5.15)

where

\[
A(r, \vartheta) = \int_{-\infty}^{\infty} \frac{d\beta'}{2\pi i} \tanh \left( \frac{\beta' - i\vartheta}{2} \right) e^{-mr \cosh \beta'}, \quad -\pi < \Re \vartheta < \pi,
\]

\[
B(r, \vartheta) = \int_{-\infty}^{\infty} \frac{d\beta'}{2\pi i} \coth \left( \frac{\beta' - i\vartheta}{2} \right) e^{-mr \cosh \beta'}, \quad 0 < \Re \vartheta < 2\pi.
\]

(5.16)

Here again \(\vartheta = \theta - i\beta\), and \((r, \theta)\) are the polar coordinates (4.12). The above integrals define the functions \(A(r, \vartheta)\), \(B(r, \vartheta)\) in the specified domains of \(\vartheta\) only; outside them, they are defined by analytic continuation.

Using (5.10), the integrals in (5.3) and (5.6) were evaluated numerically, after finding numerical solution of the differential equations (4.15), (4.16) for the functions \(\Psi_{\pm}\) in (5.9). Since the integrals do not depend on \(\beta\) we set

\[
\beta = 0,
\]

(5.17)

so that \(\vartheta = \theta\), just the euclidean angle between the separation \(x - x'\) and the x-axis. In this case (and indeed at any pure imaginary \(\beta\)) the functions \(\Psi_+\) and \(\Psi_-\) are complex-conjugate.
to each other. Few words about integration of the differential equations (4.15), (4.16) are worth saying. We found it most convenient to use the representation of the solution in terms of the Backlund transformation of the angular-symmetric solution $\varphi(r), \chi(r)$ of the sinh-Gordon system (4.7) appearing in the representation (4.6) of the correlation functions. The solution of the linear system (4.15), (4.16) can be written in terms of $\chi(r)$ and two auxiliary functions $\phi(r, \vartheta)$ and $\rho(r, \vartheta)$,

$$e^{\chi/2} \Psi_{\pm} = e^{(\rho \pm i\phi)/2},$$  \hspace{1cm} (5.18)

where $\phi$ and $\rho$ must satisfy the equations

$$\partial(\varphi + i\phi) = \frac{m}{2} e^{\beta} \cosh (\varphi - i\phi), \quad \bar\partial(\varphi - i\phi) = \frac{m}{2} e^{-\beta} \cosh (\varphi + i\phi),$$  \hspace{1cm} (5.19)

and

$$\partial(\rho - \chi) = \frac{m}{2} e^{\beta} \sinh (\varphi - i\phi), \quad \bar\partial(\rho + \chi) = \frac{m}{2} e^{-\beta} \sinh (\varphi + i\phi),$$  \hspace{1cm} (5.20)

and hence also the sine-Gordon system

$$\partial\bar\partial \phi = -\frac{m^2}{8} \sin(2\phi), \quad \partial^2 \rho - (\partial \phi)^2 = 0,$$

$$\bar\partial\bar\partial \rho = \frac{m^2}{8} [1 + \cos(2\phi)], \quad \bar\partial^2 \rho - (\bar\partial \phi)^2 = 0.$$  \hspace{1cm} (5.21)

The solution of (5.21) which appears through (5.19) is rather interesting. The phase $\phi$ is not a single-valued function of the coordinates. Instead, when written in terms of the polar coordinates $(r, \theta)$, it is quasiperiodic function of the angle, $\phi(r, \theta + 2\pi) = \phi(r, \theta) + 2\pi$, as demanded by the monodromy properties of the matrix elements (4.10) stated in Section 4. Qualitatively, it can be described as the juxtaposition of two sine-Gordon domain walls (i.e. the sine-Gordon soliton solutions, in the euclidean nomenclature) of opposite sign, extending along the x-axis in opposite directions; the solution is singular at $r = 0$, and its shape in the “junction” region, $r \sim m^{-1}$, is shown in Fig. 4.
Fig. 4: Plot of the phase $\phi(r, \vartheta)$ for fixed $r$ as function of $\vartheta = \theta$. It satisfies the symmetry properties $\phi(r, \vartheta) + \phi(r, \pi - \vartheta) = \pi$ and $\phi(r, \vartheta) = -\phi(r, -\vartheta)$. The “domain walls” at $\theta \approx 0$ and $\theta \approx \pi$ are clearly visible in the plots for $r \geq 3m^{-1}$.

The numerical integration was performed in three steps. First, the ordinary differential equations, the radial form of \( (4.7) \), were integrated numerically, the initial conditions at small $r$ being provided by the short-distance expansions \( (A.2) \), and the large-distance asymptotics \( (A.5) \) were used for precision control. Next, the functions $\phi(r, \vartheta)$ and $\rho(r, \vartheta)$ were computed by integrating the first-order differential equations \( (5.19), (5.20) \), with the initial conditions fixed using the short-distance expansions \( (A.6) \). This step provides the numerics for $G(0|0)$ and $\tilde{G}(0|0)$ as functions of $(r, \vartheta)$; according to \( (5.9) \),

\[
G(\beta|\beta) = e^{\rho - \chi/2} \left[ \sinh(\varphi/2) - \cosh(\varphi/2) \partial_{\vartheta} \phi \right],
\]

\[
\tilde{G}(\beta|\beta) = e^{\rho - \chi/2} \left[ \cosh(\varphi/2) - \sinh(\varphi/2) \partial_{\vartheta} \phi \right].
\]

(5.22)

After the subtractions in \( (5.10) \) are made to eliminate the reducible parts, the integrands in \( (5.3) \) and \( (5.6) \) decay exponentially at large $r$, rendering the integrals convergent. So, as the last step, the integrations over the euclidean coordinates $r, \theta$ were performed numerically, yielding

\[
a = 10.7619899(1),
\]

(5.23)

and

\[
a_{q} = \bar{s}^{2} 0.142021619(1).
\]

(5.24)

These numbers provide more accurate data for the mass corrections previously estimated in \[12\].
6. Discussion

The purpose of this paper was two-fold. Firstly, we have presented a new derivation of well-known result - the nonlinear differential equations of Ref. [1] for the correlation functions in the Ising field theory with zero magnetic field. This part has a methodical value at the best. Nonetheless, we decided to include it here because we believe our derivation is somewhat simpler than the traditional ones, and also it has certain potential for generalizations. It is based on the Ward identities associated with the special integrals of motion of the doubled Ising field theory. Incidentally, the “doubling”, albeit convenient, is not the most essential part of our approach. The single Ising field theory has a system of local integrals of motion powerful enough to render similar derivation possible; the “doubling” trick just makes it shorter. Anyhow, our approach can be adopted to yield simple derivations of other results in the Ising field theory. Thus, it was already used in [12] in deriving the finite-size form-factors of the spin operator, and recently in [18] in a simple derivation of the differential equations of [19] in the Ising field theory on a Poincaré disk. Among interesting potential applications let us mention the Ising field theory at finite temperature $T$ (equivalently, the theory defined on euclidean cylinder, with the points $(x, y)$ and $(x, y + T^{-1})$ identified). One can notice that all arguments of Section 4.1 remain valid when the vacuum expectation values $\langle 0 | \ldots | 0 \rangle$ are replaced by the thermal averages $\text{tr}(e^{-H/T} \ldots)$, because the total Hamiltonian $H$ of the doubled theory commutes with all the generators $Z_{\pm 2}$, $X_{\pm 1}$ and $Y_{\pm 1}$ in (4.1) and (4.4a,b). It follows that the Eqs. (4.6) and (4.7) remain valid in this situation as well - the result previously obtained by different methods in [20] and [21]. In this case the functions $\varphi$ and $\chi$ depend on two variables $(x, y)$, but the relevant solution of the Eqs. (4.7) still can be fixed uniquely by imposing the Matsubara boundary conditions $\varphi(x, y + T^{-1}) = \varphi(x, y)$, as well as appropriate asymptotic conditions at $|x| \to \infty$ and $(x, y) \to (0, 0)$. We plan to exploit advantages of our technique in relation to this problem elsewhere. Yet another potential application is in the Ising field theory with boundaries. On the other hand, extension of this approach to other integrable field theories is more problematic. The fact that all descendants of the spin fields up to sufficiently high level are expressed through the derivatives, as in the Eqs. (2.16), (2.17), was very essential for the whole scheme to work; this fact seems to be rather specific feature of the Ising field theory.

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2 Additional equation proposed in [21], the Eq. (4.42d), apparently does not hold.
Secondly, the results of Section 4 were used in Sect. 5 in computing the leading perturbative mass corrections in the Ising field theory with magnetic field (5.1), and from this point of view the numbers (5.23) and (5.24) constitute the main result of this work. This computation is a part of our ongoing project of systematic study of the field theory (5.1). We are planning to further apply the technique developed here to perturbative calculations of other relevant quantities, such as scattering amplitudes and various structure functions.

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Appendix A.

Here we present the short- and large-distance expansions of the functions $\varphi(r)$, $\chi(r)$ in (4.6) (mostly borrowed from Ref. [1]), as well as the analogous expansions of the functions $\Psi_{\pm}(r, \vartheta)$ in (4.14). Here again $\vartheta = \theta + i\beta$, where $(r, \theta)$ are the polar coordinates (4.12). These expansions are used in the numerical evaluation of the self-energy parts (5.3) and (5.6), as explained in Section 5.

A.1

The functions $\phi(r)$, $\chi(r)$ in (4.6) obey the radial form of the Eqs. (4.7), i.e.

$$\partial_r^2 \varphi + \frac{1}{r} \partial_r \varphi = \frac{m^2}{2} \sinh(2\varphi) , \quad \frac{2}{r} \partial_r \chi = (\partial_r \varphi)^2 + \frac{m^2}{2} (1 - \cosh(2\varphi)) . \quad (A.1)$$

As is explained in [1], the relevant solution is characterized uniquely by its short-distance asymptotic behavior (4.8). Corrections to this leading asymptotics can be obtained by iterations of (A.1),

$$\varphi(r) = - \ln \left( \frac{m r}{2} \right) - \ln (-\Omega) + (m r)^4 f_4 + (m r)^8 f_8 + O \left( (m r)^{12} \Omega^6 \right) , \quad \chi(r) = \frac{1}{2} \ln(4m r) + \ln(-\Omega) + \frac{(m r)^2}{8} + (m r)^4 h_4 + (m r)^8 h_8 + O \left( (m r)^{12} \Omega^6 \right) , \quad (A.2)$$
where Ω was defined in (4.9), and the coefficients $f_n$ and $h_n$ are rational functions of Ω,

\begin{align}
  f_4 &= -\frac{1}{2^{11}} (2 \Omega - 1) \left( 4 \Omega^2 - 2 \Omega + 1 \right), \\
  f_8 &= -\frac{1}{2^{28}} (2048 \Omega^6 - 4096 \Omega^5 + 3648 \Omega^4 - 1568 \Omega^3 + 136 \Omega^2 + 111 \Omega - 32),
\end{align}

(A.3)

and

\begin{align}
  h_4 &= -\frac{1}{2^{11}} (8 \Omega^3 - 8 \Omega^2 + 2 \Omega + 1), \\
  h_8 &= -\frac{1}{2^{28}} (2048 \Omega^6 - 4096 \Omega^5 + 3776 \Omega^4 - 1888 \Omega^3 + 496 \Omega^2 - 111 \Omega + 32).
\end{align}

(A.4)

These equations extend the corresponding results presented in [1].

Alternatively, the same solution can be characterized by its large-distance behavior [1],

\begin{align}
  \varphi(r) &= 2 \pi K_0(m r) + O \left( e^{-3m r} \right), \\
  \chi(r) &= 4 \ln \bar{s} - \frac{2m r}{\pi^2} \left[ m r \left[ K_0^2(m r) - K_1^2(m r) \right] + K_0(m r) K_1(m r) \right] + O \left( e^{-4m r} \right),
\end{align}

(A.5)

where $K_\nu$ are usual modified Bessel functions and $\bar{s}$ is defined in (2.19).

A.2

The short-distance expansions of $\Psi_{\pm}$ are best written in terms of the representation (5.18). The functions $\phi(r, \vartheta)$, $\rho(r, \vartheta)$ in (5.18) solve the first-order differential equations (5.19), (5.20), with the leading $r \to 0$ asymptotic $\phi(r, \vartheta) \to \vartheta$, $\rho(r, \vartheta) \to \frac{1}{2} \ln \left( 4\pi^2 m r \right)$ (see Eq. (4.19)). Corrections to this asymptotic are obtained directly, by iterating (5.19), (5.20), using the expansions (A.2) for $\varphi(r)$ and $\chi(r)$. We present just few first terms:

\begin{align}
  \phi(r, \vartheta) &= \vartheta + (m r)^2 p_2 + (m r)^4 p_4 + (m r)^6 p_6 + O \left( (m r)^8 \Omega^4 \right), \\
  \rho(r, \vartheta) &= \frac{1}{2} \ln(4\pi^2 m r) + (m r)^2 q_2 + (m r)^4 q_4 + (m r)^6 q_6 + O \left( (m r)^8 \Omega^4 \right),
\end{align}

(A.6)

where again Ω is the logarithm (4.9),

\begin{align}
  p_2 &= -\frac{1}{2^{23}} \Omega \sin 2\vartheta, \\
  p_4 &= \frac{1}{2^{10}} \Omega (4 \Omega - 1) \sin 4\vartheta, \\
  p_6 &= \frac{1}{2^{15} 3} \left[ 3 \left( 10 \Omega^2 - 7 \Omega + 2 \right) \sin 2\vartheta - \Omega \left( 16 \Omega^2 - 6 \Omega + 1 \right) \sin 6\vartheta \right].
\end{align}

(A.7)
and
\[
q_2 = \frac{1}{8} + \frac{1}{24} (2 \Omega - 1) \cos 2\vartheta,
\]
\[
q_4 = \frac{1}{212} \left[ (16 \Omega - 8) - (16 \Omega^2 - 4 \Omega + 1) \cos 4\vartheta \right],
\]
\[
q_6 = \frac{1}{21632} \left[ -9 (12 \Omega^2 - 10 \Omega + 3) \cos 2\vartheta + (96 \Omega^3 - 36 \Omega^2 + 6 \Omega - 1) \cos 6\vartheta \right].
\]

(A.8)

Large-distance behavior of these functions follows from exact expansions
\[
F + i \tilde{F} = \sigma^2 \sum_{n=0}^{\infty} \frac{e^{-i \pi n/2}}{n!} \left[ e^{\frac{m r \sin \vartheta}{2}} B_n(r, \vartheta - \pi/2) + i e^{-\frac{m r \sin \vartheta}{2}} A_n(r, \vartheta - \pi/2) \right],
\]
\[
F - i \tilde{F} = \sigma^2 \sum_{n=0}^{\infty} \frac{e^{i \pi n/2}}{n!} \left[ e^{\frac{m r \sin \vartheta}{2}} B_n(r, \vartheta - \pi/2) - i e^{-\frac{m r \sin \vartheta}{2}} A_n(r, \vartheta - \pi/2) \right],
\]

(A.9)

of the matrix elements (4.10). When \( \vartheta \) lays inside the strip \( 0 < \Re \vartheta < \pi \), the functions \( A_n(r, \vartheta) \) and \( B_n(r, \vartheta) \) here are defined as the \( n \)-fold integrals
\[
A_n(r, \vartheta) = \prod_{1 \leq j \leq n} \left[ \int_{-\infty}^{\infty} \frac{d\beta_j}{2\pi i} \tanh \left( \frac{\beta_j - i \vartheta}{2} \right) e^{-m r \cosh \beta_j} \right] \prod_{1 \leq j < k \leq n} \tanh^2 \left( \frac{\beta_j - \beta_k}{2} \right),
\]
\[
B_n(r, \vartheta) = \prod_{1 \leq j \leq n} \left[ \int_{-\infty}^{\infty} \frac{d\beta_j}{2\pi i} \coth \left( \frac{\beta_j - i \vartheta}{2} \right) e^{-m r \cosh \beta_j} \right] \prod_{1 \leq j < k \leq n} \tanh^2 \left( \frac{\beta_j - \beta_k}{2} \right),
\]

(A.10)

\( (A_0(r, \vartheta) = B_0(r, \vartheta) = 1) \) and analytic continuation of these functions to full complex \( \vartheta \)-plane is achieved using the following relations
\[
A_n(r, -\vartheta) = (-1)^n A_n(r, \vartheta),
\]
\[
B_n(r, -\vartheta) = (-1)^n B_n(r, \vartheta) - 2n(-1)^n e^{-r \cos \vartheta} A_{n-1}(r, \vartheta),
\]

(A.11)

and
\[
A_n(r, \vartheta + \pi) = B_n(r, \vartheta) - 2n e^{-r \cos \vartheta} A_{n-1}(r, \vartheta),
\]
\[
B_n(r, \vartheta + \pi) = A_n(r, \vartheta).
\]

(A.12)

The last relations ensure right monodromy properties of the matrix elements (4.10). The expansions (A.9) are easily obtained in a standard way, by writing down the intermediate-state decompositions of the matrix elements (4.10), and using the exact form-factors of [3]. In fact, the series representations (A.9) can be taken as the starting point of alternative derivation of the equations (4.15), (4.16) [3].

3 This was done independently by F. Smirnov (private communication).
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