DEFORMATION QUANTIZATION OF POLYNOMIAL POISSON ALGEBRAS

MICHAEL PENKAVA AND POL VANHAECKE

ABSTRACT. This paper discusses the notion of a deformation quantization for an arbitrary polynomial Poisson algebra $A$. We examine the Hochschild cohomology group $H^3(A)$ and find that if a deformation of $A$ exists it can be given by bidifferential operators. We then compute an explicit third order deformation quantization of $A$ and show that it comes from a quantized enveloping algebra. We show that the deformation extends to a fourth order deformation if and only if the quantized enveloping algebra gives a fourth order deformation; moreover we give an example where the deformation does not extend. A correction term to the third order quantization given by the enveloping algebra is computed, which precisely cancels the obstruction.

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1. Introduction

Deformation theory for associative commutative algebras was first considered by Gerstenhaber in [10]. A formal deformation of an associative commutative algebra $A$ over a ground field $F$ is by definition an associative multiplication $\ast$ on $A^{\hbar} = A[[\hbar]]$,

$$ p \ast q = pq + \hbar \pi_1(p, q) + \hbar^2 \pi_2(p, q) + \cdots ,$$

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where $pq$ denotes the original product of elements $p, q \in A$. The main tools which are used by Gerstenhaber are the Hochschild cohomology groups $H^n(A)$ (introduced in [11]) and the Gerstenhaber bracket $[\cdot, \cdot]$ (introduced in [9]). In fact, the deformed product
\[
\pi_* : A^h \times A^h \to A^h
\]
defines an associative product if and only if $[\pi_*, \pi_*] = 0$, an equation which can be rewritten by using the Hochschild coboundary operator $\delta$ as
\[
\delta \pi_k = \frac{1}{2} \sum_{i+j=k} [\pi_i, \pi_j] \quad k = 1, 2, \ldots
\]
It is a fundamental fact that the right hand side of this equation is a Hochschild 3-cocycle: if a deformation is associative up to order $n$ then it extends to order $n + 1$ if and only if some given 3-cocycle is a coboundary, hence this question is cohomological in nature. One immediate consequence is that the vanishing of $H^3(A)$ implies that every $n$-th order deformation extends to a formal deformation.

The relevance of deformation theory to physics was first pointed out in [1]. The main idea is that the non-commutative (associative) operator product which appears in quantum mechanics is a deformation of the commutative product of classical observables, making deformation theory a tool for describing the transition from classical to quantum mechanics. In this context one often speaks of a deformation quantization (in the present paper we will reserve the term deformation quantizations for deformations that alternate in the sense that $\pi_i$ is even or odd according to the parity of $i$). A new object which appears in their treatment of deformation theory is a Poisson bracket. Indeed, it is easy to show that if (1) defines a deformation (of at least order 2) and $\pi_1$ is antisymmetric then $\pi_1$ is a Poisson bracket.

This observation leads to the following question. Let $A$ an associative commutative algebra, equipped with a Poisson bracket, i.e., an antisymmetric biderivation $\{\cdot, \cdot\} : A \times A \to A$ which satisfies the Jacobi identity
\[
\{p, \{q, r\}\} + \{q, \{r, p\}\} + \{r, \{p, q\}\} = 0, \quad \text{for all } p, q, r \in A.
\]
(2)

Does there exist a formal deformation (1) for which $\pi_1$ is given by the Poisson bracket, $\pi_1 = \frac{1}{2}\{\cdot, \cdot\}$? In two important cases an affirmative answer is given in [1]: when $A$ is the Poisson algebra of functions on the dual of a Lie algebra and when $A$ is the algebra of functions on a Poisson manifold which admits a flat connection. In the Lie algebra case, this result had already been observed by Berezin (see [3]), who also pointed out the relation with the enveloping algebra. The case of symplectic manifolds was settled later by De Wilde and Lecomte (see [5]; for a geometric proof of their result, see [8]).

The starting point of our research was to try to understand the case of a general Poisson algebra by first investigating the case of polynomial Poisson algebras (in any number of variables). For the latter we use a subcomplex of the Hochschild complex, consisting of differential operators. Indeed, for a polynomial ring, the third cohomology group of the subcomplex of differential operators maps injectively to the ordinary Hochschild cohomology group $H^3(A)$. Since the Poisson bracket is a differential operator and the Gerstenhaber bracket of differential operators is again a differential operator, the problem of extending a given deformation quantization can be described by the cohomology of differential operators, in which one can do
explicit computations more easily. The fact that one can replace the Hochschild cohomology groups with the smaller groups is not trivial and is based upon a careful investigation of $H^3(A)$. For a 3-cocycle $\phi$ we will show that its flip symmetric part $\phi_+$, which is defined by $\phi_+(p, q, r) = \frac{1}{2}(\phi(p, q, r) + \phi(r, q, p))$ is always a coboundary and that its flip antisymmetric part $\phi_- = \phi - \phi_+$ is a coboundary if and only if $\phi_-(\phi_-(p, q, r) + \phi_-(\phi_-(q, r), p) + \phi_-(\phi_-(r, p), q) = 0$. In either case we give a recursion formula for the cochain whose coboundary $\phi$ is given and find that this cochain is a bidifferential operator when $\phi$ is a tridifferential operator. A characterization of $H^3(A)$ can already be found in [18], but our proofs have the advantage of being purely cohomological and allow the latter conclusion.

Armed with the above explicit description of $H^3(A)$ and explicit formulas for the Gerstenhaber bracket and the Hochschild coboundary operator we easily find that a deformation quantization of order three always exists; moreover, using the Jacobi identity we can actually write down an explicit formula for that a deformation quantization of order three always exists; moreover, using the Jacobi identity we can actually write down an explicit formula for $\pi_3$, as a result of a lot of non-trivial computations which not only involve the Jacobi identity but also its derivative. Thus we find that every polynomial Poisson algebra admits a third order deformation quantization. Surprisingly enough this seemingly “natural” deformation quantization does not (in general) extend to a fourth order deformation.

This fact is even more striking once one realizes that the third order deformation which we construct comes from a quantized universal enveloping algebra, making this deformation most natural. We define this enveloping algebra for any polynomial Poisson algebra $(A, \{\cdot, \cdot\})$ as follows. First notice that $A$ can be seen as the symmetric algebra $S(V)$ over a vector space $V$; then the Poisson bracket is a linear map $S(V) \otimes S(V) \to S(V)$. We take the tensor algebra $T(V)$ of $V$ and we consider the two-sided ideal $J^h$ of $T(V)^h$ generated by all elements of the form $x \otimes y - y \otimes x - h\sigma(x, y)$, where $x, y \in V$ and $\sigma : S(V) \to T(V)$ is the symmetrization map, defined by

$$\sigma \left( \prod_{i=1}^{n} a_i \right) = \frac{1}{n!} \sum_{p \in S_n} a_{p(1)} \otimes a_{p(2)} \otimes \cdots \otimes a_{p(n)}.$$

The quantized universal enveloping algebra is defined as $U(V)^h = T(V)^h/J^h$. Notice that in the case of a linear Poisson bracket we recover the usual definition of the enveloping algebra of a Lie algebra. It is a well-known but non-trivial fact that for a linear Poisson bracket the enveloping algebra does give a deformation quantization in the following way: the natural map $S(V)^h \to U(V)^h$ is a linear isomorphism, so the product on $U(V)^h$ determines a product on $S(V)^h$, which is a deformation quantization. In general, i.e., for non-linear Poisson brackets the map $S(V)^h \to U(V)^h$ fails to be injective, but surprisingly enough, for a general Poisson bracket it is injective precisely up to order 3 (in $h$). In fact, one computes an obstruction to the injectivity of the map, which turns out to coincide with the obstruction which we found earlier when trying to extend the deformation to a fourth order deformation quantization. Thus the third order deformation which we construct using Hochschild cohomology extends to a fourth order deformation precisely when the quantized enveloping algebra gives a fourth order deformation. An explanation of this will be given in the text.

The recent result by Kontsevich, which states that every Poisson manifold has a deformation quantization (see [14]) was the motivation for us to look what was
“wrong” with our third deformation. It is easy to see that one can always add any antisymmetric biderivation to $\pi_3$ and obtain a new, non-equivalent (third order) deformation quantization. Even more, for some choices of biderivation the third order deformation extends while for others it doesn’t. We will give such a biderivation for which the extension to a fourth order deformation is always possible. The check that it does depends on a skillful use of the Jacobi identity, and the first and second derivatives of the Jacobi identity.

The structure of this paper is as follows. In Section 2 we explain the precise relation between deformation theory and Hochschild cohomology for associative commutative algebras and we study the second and third Hochschild cohomology groups for a polynomial algebra in Sections 3 and 4. An explicit third order deformation for any polynomial Poisson algebra is computed in Section 5 and in Section 6 we compute the obstruction for this deformation to extend to a fourth order deformation. In Section 7 we introduce the quantized enveloping algebra of a polynomial Poisson algebra and we show that our third order deformation comes from this algebra. We show in Section 8 how to modify the third order deformation such that it extends to a fourth order deformation. In the final section a few examples with very different characteristics are worked out, in particular we give an example which shows that the third order deformation which is given by the quantized universal enveloping algebra does not extend in general.

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2. Deformations of polynomial algebras

In this section we briefly discuss the notion of deformation of a commutative associative algebra $A$ over a field $F$. (We assume throughout this paper that the characteristic of $F$ is not 2.) We describe the obstruction to the existence of a deformation using the Hochschild cohomology group $H^3(A)$, which has an explicit description in the case of a polynomial algebra.

We will denote the product $pq$ of elements $p, q$ in $A$ by $\pi(p, q)$. Let $h$ be a formal parameter and let $A^h$ (resp. $F[[h]]$) denote the algebra of formal power series with coefficients in $A$ (resp. in $F$). For $n \in \mathbb{N}$ we will also use the algebra $A^h_n$ which is obtained from $A^h$ by dividing out by the ideal generated by $h^{n+1}$. For elements $p, q \in A^h$ we write $p = q \mod h^{n+1}$ when they project to the same elements in $A_n^h$.

Definition 2.1. An $F[[h]]$-bilinear map

$$\pi_* : A^h \times A^h \to A^h$$

is called a (formal) deformation of $A$ when it satisfies the associativity condition

$$\pi_* (\pi_* (p, q), r) = \pi_* (p, \pi_* (q, r))$$

for all $p, q$ and $r$ in $A^h$ and reduces to $\pi$ on $A \cong A^h_0$, i.e.,

$$\pi_* (p, q) = \pi(p, q) \mod h.$$
More generally, when associativity merely holds on $A^h$ we say that $\pi_*$ defines an $n$-th order deformation. A first-order deformation is also called an infinitesimal deformation.

When a (formal) deformation has the additional property that for any $p, q \in A$ the product $q \ast p$ is obtained from $p \ast q$ by applying the involution of $A^h$ determined by $h \mapsto -h$, then we say that it defines a (formal) deformation quantization of $A$.

Two ($n$-th order or formal) deformations $\pi_*$ and $\pi'_*$ are called equivalent if there exists an $F[[h]]$-linear map $F : A^h \to A^h$ such that

\[ F(p) = p \mod h \]

for any $p \in A$ and such that $F(\pi_*(p, q)) = \pi'_*(F(p), F(q))$ for any $p, q \in A$.

For $n \geq 0$ the space of $n$-cochains is given by

\[ C^n(A) = \text{Hom}(A^n, A), \]

and note that $\pi_*$ is given by a sequence of elements $\pi_i$ in $C^2(A)$,

\[ \pi_* = \pi + h\pi_1 + h^2\pi_2 + \cdots. \]

A deformation $\pi_*$ is a deformation quantization if $\pi_k$ is symmetric when $k$ is even and antisymmetric when $k$ is odd.

The condition that $\pi_*$ be associative is most conveniently expressed in cohomological language. A graded bracket on cochains, called the Gerstenhaber bracket (see [10]), is given by

\[
[\varphi, \psi](p_1, \cdots, p_{m+n-1}) = \\
\sum_{k=1}^{m} (-1)^{(k-1)(n-1)} \varphi(p_1, \cdots, p_{k-1}, \psi(p_k, \cdots, p_{k+n-1}), p_{k+n}, \cdots, p_{m+n-1}) \\
- (-1)^{(m-1)(n-1)} \times \\
\sum_{k=1}^{n} (-1)^{(k-1)(m-1)} \psi(p_1, \cdots, p_{k-1}, \varphi(p_k, \cdots, p_{k+m-1}), p_{k+m}, \cdots, p_{m+n-1}),
\]

for $\varphi \in C^m(A), \psi \in C^n(A)$. We will use the bracket in the case of elements $\varphi, \psi \in C^2(A)$ in which case $[\varphi, \psi] = [\psi, \varphi]$ and the formula specializes to

\[
[\varphi, \psi](p, q, r) = \varphi(\psi(p, q), r) - \varphi(p, \psi(q, r)) + \psi(\varphi(p, q), r) - \psi(p, \varphi(q, r)).
\]

In view of the above formula the proof of the following lemma is trivial.

**Lemma 2.2.** An element $\varphi \in C^2(A)$ defines an associative multiplication on $A$ if and only if $[\varphi, \varphi] = 0$.

It follows that $\pi_*$ is associative if and only if

\[ 0 = [\pi_*, \pi_*] = 2h[\pi, \pi_1] + h^2(2[\pi, \pi_2] + [\pi_1, \pi_1]) + \cdots \]

The cochains form a complex $C^\bullet(A)$ for the Hochschild coboundary operator

\[ \delta : C^n(A) \to C^{n+1}(A) \]
which is defined by
\[
\delta \varphi(p_1, \ldots, p_{n+1}) = p_1 \varphi(p_2, \ldots, p_{n+1})
\]
\[+
\sum_{k=1}^{n} (-1)^k \varphi(p_1, \ldots, p_{k-1}, p_k p_{k+1}, p_{k+2}, \ldots, p_{n+1}) + (-1)^{n+1} \varphi(p_1, \ldots, p_n) p_{n+1},
\]
for \( \varphi \in C^n(A) \) (see [11]). The \( n \)-th cohomology group of this complex will be denoted by \( H^n(A) \). It is easy to see that the Hochschild coboundary operator \( \delta \) can be written in terms of the Gerstenhaber bracket as
\[
\delta \varphi = -[\varphi, \pi],
\]
so that the associativity condition (3) can be expressed by an infinite list of relations
\[
\begin{align*}
\delta \pi_1 &= 0, \\
\delta \pi_2 &= \frac{1}{2} [\pi_1, \pi_1], \\
\delta \pi_3 &= [\pi_1, \pi_2], \\
\delta \pi_4 &= [\pi_1, \pi_3] + \frac{1}{2} [\pi_2, \pi_2], \\
&\vdots
\end{align*}
\]
More precisely, if the cochains \( \pi_1, \ldots, \pi_{n-1} \) define an \((n-1)\)-th order deformation of \( \pi \) then it extends to an \( n \)-th order deformation if and only if the equation
\[
\delta \pi_n = \frac{1}{2} \sum_{i+j=n} [\pi_i, \pi_j]
\]
has a solution \( \pi_n \). If such a solution exists it is clearly unique up to addition of any cocycle \( \delta \varphi, \varphi \in C^1(A) \). As for its existence it is important to note that the right hand side in (5) is always a cocycle:
\[
\sum_{i+j=n} \delta [\pi_i, \pi_j] = - \sum_{i+j=n} [[\pi_i, \pi_j], \pi],
\]
\[= 2 \sum_{i+j=n} [[\pi_j, \pi], \pi_i],
\]
\[= - \sum_{i+k+l=n} [[\pi_k, \pi_l], \pi_i],
\]
\[= 0.
\]
In this computation the Jacobi identity
\[
[[[\varphi, \chi], \psi]] + [[[\chi, \psi], \varphi]] + [[[\psi, \varphi], \chi]] = 0
\]
which is valid for any 2-cochains \( \varphi, \psi \) and \( \chi \), was used twice (when the characteristic of \( F \) is 3 then the equation \( [[[\varphi, \varphi], \varphi] = 0 \), which holds for any 2-cochain \( \varphi \), but not as a consequence of (5), is also used.) The upshot is that the extendibility of a deformation of order \( n - 1 \) to a deformation of order \( n \) depends on whether or not a certain Hochschild 3-cocycle is a coboundary. However, the particular \( \pi_n \) chosen for the extension of the deformation to order \( n \) will have a pronounced impact on the further extendibility of the deformation. If the deformation does not extend to order \( n + 1 \), it may be that a different choice of \( \pi_n \) would allow such an extension. Moreover, this effect is not limited to the next term in the extension, so that the
extendibility of an extension up to order \( n \) is influenced by all of the choices of the cochains \( \pi_k \) for \( k < n \).

One can describe the extendibility of the deformation in terms of Massey powers of \( \pi_1 \) (see [10]), so that there is an extension of order \( n \) when the \( n \)-th Massey power of \( \pi_1 \) vanishes, but this description does not yield any immediate advantage, since the problem of computation of the Massey powers may be more difficult to solve than the problem of finding an explicit sequence of cochains yielding a deformation.

Another important consideration is the uniqueness of deformations, which is partially governed by the second Hochschild cohomology group \( H^2(\mathbf{A}) \). The following lemma shows that if two deformations differ by a coboundary then they are equivalent.

**Proposition 2.3.** If the \( n \)-th cochain \( \pi_n \) in a formal (resp. \( m \)-th order with \( n \leq m \)) deformation \( \sum h^i \pi_i \) is altered by a coboundary then the new \( n \)-th order deformation extends to an equivalent formal (resp. \( m \)-th order) deformation.

**Proof.** In the case of a formal deformation, let us denote the coboundary which is added to \( \pi_n \) by \( \delta E \), where \( E \in C^1(\mathbf{A}) \). Define an \( \mathbb{F}[h] \)-linear map \( F : \mathbf{A}^h \rightarrow \mathbf{A}^h \) by

\[
F(p) = p + h^n E(p)
\]

with inverse

\[
F^{-1}(p) = p - \sum h^{kn} E^k(p)
\]

for any \( p \in \mathbf{A} \). Then \( \pi'_n(p, q) := F^{-1}(\pi_n(F(p), F(q))) \) is a new (equivalent) deformation whose first \( n \) cocycles \( \pi'_n \) coincide with the cocycles \( \pi_i \) and \( \pi'_n = \pi_n + \delta E \).

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### 3. Hochschild Cohomology

Our aim in this section is to analyze \( H^2(\mathbf{A}) \) and \( H^3(\mathbf{A}) \) more thoroughly. We will give an explicit characterization in the case of polynomial algebras, in Theorems 3.1, 3.2 and 3.3. These results should be regarded as classical, and complete proofs of these theorems appear in [18]; but our treatment here is more straightforward, relying on simple cohomological arguments. In the proofs below, for a polynomial algebra given by an ordered basis (free generating set) \( \{x_i\}_{i \in I} \), we will denote elements of the basis by the letters \( x \) and \( y \), while arbitrary polynomials will be denoted by the letters \( p, q, r \) and \( s \). For \( x \in I \) the statement \( x \leq p \) means that the basis elements appearing in the monomials in \( p \) have index greater than or equal to that of \( x \), so that in particular \( x \leq c \) for any constant \( c \).

Any 2-cochain \( \varphi \) can be uniquely decomposed as the sum of a symmetric cochain \( \varphi^+ \) and an antisymmetric cochain \( \varphi^- \). Then \( \varphi \) is a cocycle precisely when both its symmetric and antisymmetric parts are cocycles. To see this fact, suppose that \( \varphi \) is a 2-cocycle. Then 

\[
\delta \varphi(p, q, r) = p \varphi(q, r) - \varphi(pq, r) + \varphi(p, qr) - \varphi(p, q)r.
\]

Let \( \bar{\varphi}(p, q) = \varphi(q, p) \). Then

\[
\delta \bar{\varphi}(p, q, r) = p \varphi(r, q) - \varphi(r, pq) + \varphi(gr, p) - \varphi(q, p)r = -\delta \varphi(r, q, p) = 0.
\]

Since \( \varphi^+ \) and \( \varphi^- \) are linear combinations of \( \varphi \) and \( \bar{\varphi} \), this shows the desired result. Furthermore, the coboundary of any 1-cochain is symmetric, which is immediate from the fact that if \( \lambda \) is a 1-cochain, then

\[
\delta \lambda(p, q) = p\lambda(q) - \lambda(pq) + \lambda(p)q.
\]
Similarly, Using the symmetry of $\phi$ $\delta \phi$ factored out first. Consider the expansion $\psi$ a cocycle precisely when Jacobi map 2-cochain is flip symmetric. The symmetric 2-cochain is flip antisymmetric, and the coboundary of an antisymmetric $\delta \lambda$ has degree 1. The property $\lambda$ be chosen arbitrarily for basis elements. In particular, it can be chosen to satisfy $\lambda(x) = 0$ when $x$ is a basis element.

**Proof.** For a symmetric 2-cocycle $\varphi$, we construct recursively a 1-cochain $\lambda$ whose coboundary coincides with $\varphi$. Now

$$\delta \varphi(1, 1, q) = \varphi(1, q) - \varphi(1, q) + \varphi(1, q) - \varphi(1, 1)q = 0,$$

so that $\varphi(1, q) = \varphi(1, 1)q$. Let $\lambda(1) = \varphi(1, 1)$, and define $\lambda(x)$ arbitrarily when $x$ has degree 1. The property $\delta \lambda = \varphi$ holds precisely when

$$\lambda(pq) = p\lambda(q) + q\lambda(p) - \varphi(p, q).$$

When either $p$ or $q$ is constant, this equation holds by the preceding remarks; otherwise $\lambda$ is evaluated at terms of lower degree on the right hand side, so the left hand side is defined recursively by this formula. But we need to check that if $pq = p'q'$ then the right hand sides of the decomposition above agree. Equivalently, it is enough to check that expanding $\lambda(pqr)$ does not depend on whether $p$ or $q$ is factored out first. Consider the expansion

$$\lambda(pqr) = p\lambda(qr) + qr\lambda(p) - \varphi(p, qr)$$
$$= p(q\lambda(r) + r\lambda(q) - \varphi(q, r)) + qr\lambda(p) - \varphi(p, qr).$$

Similarly,

$$\lambda(qpr) = q(p\lambda(r) + r\lambda(p) - \varphi(p, r)) + pr\lambda(q) - \varphi(q, pr).$$

Using the symmetry of $\varphi$, we find that the difference of these two expressions is $\delta \varphi(p, r, q)$, and so vanishes.

Let us now turn our attention to the third Hochschild cohomology group, wherein lie the obstructions to extension of a deformation to higher order. A 3-cochain $\psi$ is called flip symmetric if it satisfies $\psi(p, q, r) = \psi(r, q, p)$, and flip antisymmetric if $\psi(p, q, r) = -\psi(r, q, p)$. Every 3-cochain $\psi$ can be uniquely decomposed as the sum of a flip symmetric cochain $\psi^+$ and a flip antisymmetric cochain $\psi^-$. Moreover, $\psi$ is a cocycle precisely when $\psi^+$ and $\psi^-$ are cocycles. Furthermore, the coboundary of a symmetric 2-cochain is flip antisymmetric, and the coboundary of an antisymmetric 2-cochain is flip symmetric. The Jacobi map $J : C^3(A) \rightarrow C^3(A)$ is given by

$$J\psi(p, q, r) = \psi(p, q, r) + \psi(q, r, p) + \psi(r, p, q).$$
Then, if $\psi$ is the coboundary of a symmetric cochain, it satisfies the Jacobi identity $J\psi = 0$. For any 3-cocycle $\psi$, $\psi(1,1,1) = \delta\psi(1,1,1) = 0$ and $\psi(1,p,1) = \delta\psi(1,1,p) = 0$ for all $p$. Suppose that $\psi(p,1,1) = \psi(1,1,p) = 0$ for all $p$. Then $\psi(1,p,q) = \delta\psi(1,1,p,q)$, $\psi(p,1,q) = \delta\psi(p,1,1,q)$, and $\psi(p,q,1) = \delta\psi(p,q,1,1)$, so these terms vanish for all $p$ and $q$. These remarks are easy to check, and apply to any commutative algebra $A$, not just a polynomial algebra.

**Theorem 3.2.** Let $A$ be a polynomial algebra and suppose that $\psi$ is a 3-cocycle. Then $\psi$ is flip symmetric if and only if $\psi$ is a Hochschild coboundary of an antisymmetric cochain $\varphi$.

**Proof.** By the above remarks we only need to verify that a flip symmetric cocycle $\psi$ is a coboundary of an antisymmetric cochain. If we define $\theta(p,1) = \psi(1,1,p) = -\theta(1,p)$, and extend $\theta$ in an arbitrary manner to an antisymmetric cochain, then $\delta\theta(1,1,p) = -\psi(1,1,p)$, so that by replacing $\psi$ by $\psi + \delta\theta$, we may assume that $\psi(1,1,p) = 0$, so that $\psi$ vanishes when any of its arguments is a constant. We define $\varphi$ recursively, by first setting $\varphi(1,1) = \varphi(1,p) = \varphi(p,1) = 0$. In addition, let us assume that $\varphi(x,y)$ is defined in an arbitrary manner for basis elements $x$ and $y$. Consider the following equalities which must be satisfied if $\psi = \delta\varphi$:

\[
\begin{align*}
\psi(p,q,r) &= p\varphi(q,r) - \varphi(pq,r) + \varphi(p,qr) - \varphi(p,q)r, \\
\psi(p,r,q) &= p\varphi(r,q) - \varphi(pr,q) + \varphi(p,qr) - \varphi(p,r)q, \\
\psi(r,p,q) &= r\varphi(p,q) - \varphi(rp,q) + \varphi(r,pq) - \varphi(r,p)q.
\end{align*}
\]

Adding the first and third and subtracting the second of these equations, and using the desired antisymmetry property for $\varphi$ yields the following equation:

\[2\varphi(pq,r) = 2p\varphi(q,r) + 2q\varphi(p,r) - \psi(p,q,r) + \psi(p,r,q) - \psi(r,p,q),\]

The expression above is evidently symmetric in $p$ and $q$, and holds when either $p$ or $q$ is constant, because $\psi$ vanishes when any of its arguments is a constant. Otherwise the right hand side involves terms of smaller degree than the left, so we obtain a recursive definition of $\varphi$. To check that this definition does not depend on the decomposition of the term $pq$, one should check that expanding a term of the form $\varphi(pqr,s)$ in two ways leads to the same result. Factoring $pqr$ as $p$ times $qr$ one obtains

\[2\varphi(pqr,s) = 2p\varphi(qr,s) + 2qr\varphi(p,s) - \psi(p,qr,s) + \psi(p,s,qr) - \psi(s,p,qr)\]

\[= p(2q\varphi(r,s) + 2r\varphi(q,s) - \psi(q,r,s) + \psi(q,s,r) - \psi(s,q,r)) + 2qr\varphi(p,s) - \psi(p,qr,s) + \psi(p,s,qr) - \psi(s,p,qr).\]

Subtracting the expression obtained by interchanging the roles of $p$ and $q$, the terms involving $\varphi$ drop out and we are left with

\[\begin{align*}
-p\psi(q,r,s) + p\psi(q,s,r) - p\psi(q,s,r) - p\psi(q,qr,s) + p\psi(p,s,qr) - \psi(s,p,qr) \\quad + q\psi(p,r,s) - q\psi(p,s,r) + q\psi(s,p,r) + q\psi(q,pr,s) - \psi(q,s,pr) + \psi(s,q,pr) \\
= \delta\psi(s,p,r,q) - \delta\psi(p,r,q,s) - \delta\psi(p,r,q,s) + \delta\psi(q,s,r,p),
\end{align*}\]
which is zero. To verify that \( \varphi \) is antisymmetric, we need to compute \( \varphi(pq, rs) + \varphi(rs, pq) \), using antisymmetry of lower degree terms. We have

\[
2\varphi(pq, rs) = 2p\varphi(q, rs) + 2q\varphi(p, rs) - \psi(p, q, rs) + \psi(p, rs, q) - \psi(rs, p, q)
\]

\[
= p(2r\varphi(q, s) + 2s\varphi(q, r) + \psi(r, q, s) - \psi(r, q, s) + \psi(q, r, s)) +
\]

\[
q(2r\varphi(p, s) + 2s\varphi(p, r) + \psi(r, s, p) - \psi(r, s, p) + \psi(p, r, s)) +
\]

\[
\psi(p, q, rs) + \psi(p, rs, q) - \psi(rs, p, q).
\]

From this we see that \( 2(\varphi(pq, rs) + \varphi(rs, pq)) \) equals

\[
\delta\psi(p, q, r, s) - \delta\psi(p, r, q, s) + \delta\psi(q, s, r, p) + \delta\psi(s, q, p, r) - \delta\psi(q, s, p, r),
\]

and thus vanishes. It is immediately checked that \( \delta\varphi = \psi \).

\[
\square
\]

**Theorem 3.3.** Let \( A \) be a polynomial algebra and suppose that \( \psi \) is a \( \delta \)-cocycle. Then \( \psi \) is a Hochschild coboundary of a symmetric cochain \( \varphi \) if and only if \( \psi \) is flip antisymmetric and satisfies the Jacobi identity \( J\psi = 0 \). Moreover, if we take an ordered basis of \( A \), then \( \varphi \) can be chosen to satisfy \( \varphi(x, p) = 0 \) whenever \( x \) is a basis element satisfying \( x \leq p \).

**Proof.** As in the previous theorem, we reduce to the case where \( \psi(p, 1, 1) = 0 \). Take an ordered basis of \( A \). Define \( \varphi \) by \( \varphi(1, p) = 0 \) for all \( p \) and \( \varphi(x, p) = 0 \) when \( x \leq p \).

We extend the definition recursively by setting

\[
\varphi(xp, q) = x\varphi(p, q) + \psi(q, p, x) = \varphi(q, xp),
\]

when \( x \leq q \) and \( x \leq p \). To show \( \varphi \) is well defined and symmetric, we only need to show that if \( x \) is a basis element satisfying \( x \leq p \) and \( x \leq q \), then the expansion of \( \varphi(xp, xq) \) yields the same result as the expansion of \( \varphi(xq, xp) \). Now

\[
\varphi(xp, xq) = x\varphi(p, xq) + \psi(xq, p, x) = x(x\varphi(q, p) + \psi(p, q, x)) + \psi(xq, p, x),
\]

so that \( \varphi(xp, xq) - \varphi(xq, xp) = \delta\psi(x, p, q, x) = 0 \).

To show that \( \delta\varphi = \psi \), we note that if any of \( p, q \) or \( r \) is constant, then both \( \psi(p, q, r) \) and \( \delta\varphi(p, q, r) \) vanish (for the vanishing of the former, see the remarks preceding Theorem 3.2). We may proceed by induction on the sum of the degrees of \( p, q \) and \( r \). If \( p \) can be factored as \( xp' \), where \( x \) satisfies \( x \leq p' \), \( x \leq q \) and \( x \leq r \), then

\[
\delta\varphi(xp', q, r) = xp'\varphi(q, r) - \varphi(xp', q, r) + \varphi(xp', qr) - \varphi(xp', q)r
\]

\[
= xp'\varphi(q, r) - x\varphi(p'q, r) - \psi(r, p'q, x)
\]

\[
+ x\varphi(p', qr) + \psi(qr, p', x) - rx\varphi(p', q) - r\psi(q, p', x)
\]

\[
= x(p'\varphi(q, r) - \varphi(p'q, r) + \varphi(p', qr) - \varphi(p', q)r)
\]

\[
- \psi(r, p'q, x) + \psi(qr, p', x) - r\psi(q, p', x)
\]

\[
= x\varphi(p', q, r) - \psi(r, p'q, x) + \psi(qr, p', x) - r\psi(q, p', x)
\]

\[
= \psi(xp', q, r).
\]

On the other hand, if we can express \( r = xr' \), where \( x \leq p \), \( x \leq q \) and \( x \leq r \), then

\[
\psi(p, q, xr') = -\psi(xr', q, p) = -\delta\varphi(xr', q, p) = \delta\varphi(p, q, xr').
\]
since \( \psi \) is flip antisymmetric, and the coboundary of any symmetric cochain is also flip antisymmetric. The only other possibility is that \( q = xq' \), where \( x \leq q', x \leq p \) and \( x \leq r \). But then we have
\[
\psi(p, xq', r) = -\psi(xq', r, p) - \psi(r, p, xq') = -\delta\varphi(xq', r, p) - \delta\varphi(r, p, xq') = \delta\varphi(p, xq', r),
\]
using the Jacobi identity \( J\psi = 0 \) and the fact that the coboundary of any symmetric cochain satisfies the Jacobi identity. Note that it is only at this last step that the Jacobi identity is used.

For simplicity in the proof above, we constructed \( \varphi \) so that \( \varphi(x, p) = 0 \) for a basis element \( x \) satisfying \( x \leq p \). But for a polynomial algebra, one can always define \( \varphi(x, p) \) for \( x \leq p \) in an arbitrary manner and extend the definition to a cocycle, as we show below. Thus we could have assumed in Theorem 3.3 that \( \varphi(x, p) \) is defined arbitrarily for \( x \leq p \).

**Proposition 3.4.** Let \( A \) be a polynomial algebra with an ordered basis, and suppose that \( \varphi(x, p) \) is any cochain defined for \( x \leq p \), satisfying \( \varphi(x, 1) = 0 \). Then \( \varphi \) extends uniquely to a symmetric cocycle satisfying \( \varphi(1, 1) = 0 \).

**Proof.** From the condition \( \delta\varphi(x, p, q) = 0 \) one derives the property
\[
\varphi(xp, q) = x\varphi(p, q) + \varphi(x, pq) - \varphi(x, p)q.
\]
If either \( p \) or \( q \) is constant, then the formula above holds trivially. Otherwise, if \( x \leq p \) and \( x \leq q \), then the left hand side is defined recursively by the right hand side. The consistency and the symmetry condition \( \varphi(xp, xq) = \varphi(xq, xp) \) follow from
\[
\varphi(xp, xq) = xp\varphi(p, xq) + \varphi(x, xpq) - \varphi(x, p)xq = x^2\varphi(q, p) + x\varphi(x, pq) - \varphi(x, q)xp + \varphi(x, xpq) - \varphi(x, p)xq.
\]
If \( \varphi(q, p) = \varphi(p, q) \), then the above formula is already symmetric in \( p \) and \( q \), so the check of consistency and symmetry is trivial. To see that \( \delta\varphi(p, q, r) = 0 \), consider the case when \( p = xp' \), where \( x \leq p', x \leq q \) and \( x \leq r \). Then
\[
\delta\varphi(xp', q, r) = xp'\varphi(q, r) - \varphi(xp'q, r) + \varphi(xp', qr) - \varphi(xp', q)r = xp'\varphi(q, r) - x\varphi(p'q, r) - \varphi(x, p'qr) + \varphi(x, p'q)r + x\varphi(p', qr) + \varphi(x, p'qr) - \varphi(x, p'q)r - \varphi(x, p'q)r + \varphi(x, p'q)r = x(p'\varphi(q, r) - \varphi(p'q, r) + \varphi(p', qr) - \varphi(p', q)r) = x\delta\varphi(p, q, r),
\]
which is zero by the induction hypothesis. The other cases follow from the flip antisymmetry and the Jacobi identity \( J(\delta\varphi) = 0 \). \( \square \)

When applied to deformation theory Theorems 3.2 and 3.3 lead to the following result.
Theorem 3.5.

1. If $\pi + h\pi_1$ is an infinitesimal deformation of $A$ which extends to a second order deformation, then $\pi^-_1$ is an antisymmetric biderivation which satisfies the usual Jacobi identity:

$$\pi^-_1(p, q, r) + \pi^-_1(q, r, p) + \pi^-_1(r, p, q) = 0,$$

so that $\pi^-_1$ determines a Poisson algebra structure on $A$.

2. If $A$ is a polynomial Poisson algebra, then the converse is true. More precisely, if $\pi + h\pi_1$ is an infinitesimal deformation such that $\pi^-_1$ satisfies the usual Jacobi identity, then the infinitesimal deformation extends to a second order deformation $\pi + h\pi_1 + h^2\pi_2$. Furthermore, if $\pi_1$ is antisymmetric then $\pi_2$ can be chosen to be symmetric in which case the deformation can be extended to order 3. In particular, any Poisson algebra structure on a polynomial algebra determines a deformation quantization of order 3.

Proof. If $\varphi$ is antisymmetric, then $[\varphi, \varphi]$ satisfies the Jacobi identity $J([\varphi, \varphi]) = 0$ precisely when $\varphi$ satisfies the usual Jacobi identity (equation (7)). With this remark, the statements in the theorem follow from our previous results.

For a given Poisson algebra $(A, \{\cdot, \cdot\})$ we will say that a deformation $\pi_* = \pi + h\pi_1 + h^2\pi_2 + \cdots$ of $A$, in the sense of Definition 2.3, defines a deformation of $(A, \{\cdot, \cdot\})$ when $\pi_1 = \frac{1}{2}\{\cdot, \cdot\}$.

4. HOCHSCHILD COHOMOLOGY AND DIFFERENTIAL OPERATORS

In this section we will assume that $A$ is a polynomial algebra with a fixed basis $\{x_i\}_{i \in I}$ over a field $\mathbb{F}$ of characteristic 0. We will give a characterization of Hochschild cochains in terms of (possibly infinite order) differential operators. First, let us establish some conventions on our terminology. For a basis element $x_i$ of $A$ we will denote the derivation $\partial/\partial x_i$ by $\partial^i$. For a multi-index $I = (i_1, \cdots, i_m)$, $\partial^I$ will stand for the differential operator $\partial^{i_1}\cdots\partial^{i_m}$, $x_I$ will stand for $x_{i_1}\cdots x_{i_m}$, and $|I| = m$ is its order. For a polynomial $p$, we will denote $\partial^I(p)$ by $p^I$ and $\partial^I(x_I)$ by $I!$. The multi-index $I$ is said to be non-decreasing if $i_1 \leq \cdots \leq i_m$. Also, we shall write $I < I'$ to indicate that $I$ is obtained by removing some of the indices in $I'$. By $\partial^{I_1}\cdots\partial^{I_n}$ we shall denote the $n$-differential operator of order $|I_1| + \cdots + |I_n|$ given by

$$\partial^{I_1}\cdots\partial^{I_n}(p_1, \cdots, p_n) = \partial^{I_1}(p_1)\cdots\partial^{I_n}(p_n).$$

The differential operator is said to have type $(|I_1|, \cdots, |I_n|)$. An expression of the form

$$\varphi = \sum_{I_1, \cdots, I_n} \varphi_{I_1, \cdots, I_n} \partial^{I_1}\cdots\partial^{I_n},$$

where $\varphi_{I_1, \cdots, I_n}$ are polynomials, and we sum over all non-decreasing multi-indices, gives a well-defined $n$-cochain on the polynomial algebra. When only finitely many non-zero terms appear then we say that $\varphi$ is a (finite order) differential operator, otherwise such an expression is called a formal differential operator. The order of a differential operator $\varphi$ is the largest $m$ for which there is a nonzero term in $\varphi$ of
order \(m\). Every \(n\)-cochain can be expressed as a formal differential operator, since we can solve for the polynomials \(\varphi_{I_1, \ldots, I_n}\) above recursively by

\[
I_1! \ldots I_n! \varphi_{I_1, \ldots, I_n} = \varphi(x_{I_1}, \ldots, x_{I_n}) - \sum_{(J_1, \ldots, J_n) < (I_1, \ldots, I_n)} \varphi_{J_1, \ldots, J_n} \partial^{J_1} (x_{I_1}) \cdots \partial^{J_n} (x_{I_n}).
\]

In the following lemma, which characterizes when an \(n\)-cochain is a differential operator, we use the notation \((x - k)^I\) to stand for the product \((x_1 - k_1)^{i_1} \cdots (x_s - k_s)^{i_s}\) for \(k \in \mathbb{F}^I\), and \(I = (i_1, \ldots, i_s)\).

**Lemma 4.1.** An \(n\)-cochain is a (finite order) differential operator precisely when there is some \(N\) such that for any \(k \in \mathbb{F}^I\),

\[
\varphi((x - k)^{I_1}, \ldots, (x - k)^{I_n})(k) = 0,
\]

whenever \(|I_1| + \cdots + |I_n| \geq N\).

**Proof.** If \(\varphi\) is a (finite order) differential operator it suffices to take \(N = 1 + \text{ord} \varphi\). On the other hand, if the order of \(\varphi\) is infinite, we may find for any \(N\) a non-zero \(\varphi_{I_1, \ldots, I_n}\), with \(|I_1| + \cdots + |I_n| \geq N\), in particular this polynomial is non-zero at some point \((k_1, \ldots, k_n)\). Then

\[
\varphi((x - k)^{I_1}, \ldots, (x - k)^{I_n})(k) = I_1! \ldots I_n! \varphi_{I_1, \ldots, I_n}(k) \neq 0.
\]

\(\square\)

In the proof of the above lemma, it was necessary to evaluate a polynomial at a point. This is the only place where the arguments in this section cannot be extended to the ring of formal power series in the variables \(\{x_i\}_i \in \mathcal{I}\), because evaluation at a point is not well defined. By examining the recursion formulas in Theorems 3.2 and 3.3 and applying Lemma 4.1, one sees that if \(\psi\) is a differential operator, then the cochain \(\varphi\) satisfying \(\delta \varphi = \psi\) constructed in these theorems is also a differential operator. The cochain \(\lambda\) constructed in Theorem 3.1 will also be a differential operator when \(\varphi\) is a bidifferential operator.

The notation \(I' + I'' = I\) will be used to indicate a partitioning of the indices of the nondecreasing multi-index \(I\) into two nondecreasing multi-indices \(I'\) and \(I''\). Then we obtain a very simple description of the action of the Hochschild coboundary operator on cochains which are given by differential operators, namely if \(p \in \mathcal{A}\) and \(J\) is any multi-index then

\[
\delta (p \partial^J) = - \sum_{I'+I''=I} p \partial^{I'} \otimes \partial^{I''}.
\]

In general, if \(\alpha\) is an \(m\)-cochain, and \(\beta\) is an \(n\)-cochain, then the \((m+n)\)-cochain \(\alpha \otimes \beta\) is given by

\[
\alpha \otimes \beta(p_1, \ldots, p_{m+n}) = \alpha(p_1, \ldots, p_m)\beta(p_{m+1}, \ldots, p_{m+n}).
\]

The Hochschild coboundary operator acts as a graded derivation with respect to this product, i.e.,

\[
\delta (\alpha \otimes \beta) = \delta (\alpha) \otimes \beta + (-1)^m \alpha \otimes \delta (\beta).
\]

From this we obtain the following useful expression for the coboundary of a bidifferential operator,

\[
\delta (p \partial^J \otimes \partial^K) = p (\delta \partial^J) \otimes \partial^K - p \partial^J \otimes (\delta \partial^K).
\]
From the above, we see that the coboundary of an \( n \)-differential operator of order \( m \) is an \(( n + 1 )\)-differential operator of order \( m \). The following theorem is an easy consequence of the above remarks.

**Theorem 4.2.** Suppose that \( A \) is a polynomial algebra, and \( \psi \) is an \( n \)-differential operator of order \( m \). If \( \psi \) is a Hochschild coboundary, then we can find an \(( n - 1 )\)-differential operator \( \varphi \) of order \( m \) such that \( \delta \varphi = \psi \).

Let us denote
\[
(p \partial I_1 \otimes \cdots \otimes \partial I_n)^J = \sum_{J_0 + \cdots + J_n = J} p^J_{J_0} \partial I_{J_1} \otimes \cdots \otimes \partial I_{J_n}.
\]

Then the bracket of differential operators is given by
\[
[p \partial I_1 \otimes \cdots \otimes \partial I_m, q \partial I_1 \otimes \cdots \otimes \partial I_n] = \sum_{k=1}^m (-1)^{(k-1)(n-1)} p \partial I_1 \otimes \cdots \otimes \partial I_{k-1} \otimes (q \partial I_k \otimes \cdots \otimes \partial I_n) \partial I_k \otimes \partial I_{k+1} \otimes \cdots \otimes \partial I_m \quad - (-1)^{(m-1)(n-1)} \times \sum_{k=1}^n (-1)^{(k-1)(m-1)} q \partial I_1 \otimes \cdots \otimes \partial I_{k-1} \otimes (p \partial I_k \otimes \cdots \otimes \partial I_m) \partial I_k \otimes \partial I_{k+1} \otimes \cdots \otimes \partial I_n.
\]

In particular, we obtain the following formula for the bracket of 2-cochains.
\[
(9) \quad [p \partial I_1 \otimes \partial I_2, q \partial I_1 \otimes \partial I_2] = p(q \partial I_1 \otimes \partial I_2)^I \otimes \partial I_2 - p \partial I_1 \otimes (q \partial I_1 \otimes \partial I_2)^J \partial I_2 + q(p \partial I_1 \otimes \partial I_2)^J \otimes \partial I_2 - q \partial I_1 \otimes (p \partial I_1 \otimes \partial I_2)^J \partial I_2.
\]

The formula above will be used in the calculations of the second and third order deformations of a polynomial Poisson algebra, which will be carried out in the next section.

Finally, we apply the results of this section to show that any deformation of a polynomial algebra is equivalent to one which is given by differential operators.

**Theorem 4.3.** Any deformation (deformation quantization) of a polynomial algebra is equivalent to a deformation (deformation quantization) whose cochains are differential operators.

**Proof.** Suppose that \( \pi_* = \pi + h \pi_1 + \ldots \) is the given deformation, and that \( \pi_1, \ldots, \pi_n \) are given by differential operators. Then we show that \( \pi_{n+1} \) can be replaced by a differential operator yielding an equivalent deformation. By Theorem 4.2, we can express \( \delta(\pi_{n+1}) = \delta(C) \) for some differential operator \( C \), so that \( \delta(\pi_{n+1} - C) = 0 \).

Therefore we can express \( \pi_{n+1} - C = A + S \) where \( A \) is an antisymmetric cocycle and \( S \) is a symmetric cocycle. Let \( \pi'_{n+1} = C + A \). \( A \) is a differential operator because it is a biderivation, hence \( \pi'_{n+1} \) is a differential operator. \( \pi_{n+1} \) and \( \pi'_n \) differ by \( S \), which is a coboundary, since it is a symmetric cocycle, so we can replace \( \pi_* \) by an equivalent deformation whose first \( n + 1 \) terms are given by differential operators. If \( \pi_* \) is a deformation quantization then (i.e., if it is alternating) then the new deformation will also be a deformation quantization because the differential operator \( C \) for which \( \delta \pi_{n+1} = \delta C \) has the same parity as \( \pi_{n+1} \) in view of Theorems 4.2 and 4.3. \( \square \)
5. Construction of the cochains $\pi_2$ and $\pi_3$

In this section we start from a polynomial Poisson algebra $(A, \{\cdot, \cdot\})$ over a field $\mathbb{F}$ of characteristic zero and construct an explicit third order deformation $\pi + h\pi_1 + h^2\pi_2 + h^3\pi_3$ of the multiplication $\pi$ in $A$. In principle the theorems of sections 3 and 4 allow one to construct a third order deformation. However, even in the case in which we are given a concrete example of $\pi_1$ it is difficult to determine $\pi_2$ and $\pi_3$ explicitly from these theorems. Therefore we will use a different method for constructing $\pi_2$ and $\pi_3$. It should be remarked that the two constructions do not give the same $\pi_2$ and $\pi_3$ terms.

Throughout this section a basis $\{x_i\}_{i \in I}$ will be fixed and we use $X_{ij}$ as a convenient notation for the Poisson bracket $\{x_i, x_j\}$ and we use superscripts to denote partial derivatives, as in the previous section. Without loss of generality we pick $\pi_1 = \frac{1}{2}\{\cdot, \cdot\}$. If we use the summation convention then $\pi_1$ can be written as $\frac{1}{2}X_{ij}\partial^i \otimes \partial^j$, the Jacobi identity for $\pi_1$ reads

\begin{equation}
X_{ij}^l X_{kl} + X_{jk}^l X_{kl} + X_{kl}^l X_{ij} = 0
\end{equation}

(for any $i, j, k \in I$), the derivative of the Jacobi identity is written as

\begin{equation}
X_{ij}^{lm} X_{kl} + X_{jk}^{lm} X_{kl} + X_{kl}^{lm} X_{ij} = 0,
\end{equation}

(for any $i, j, k, m \in I$ and there are similar expressions for higher derivatives. We first give a formula for $\pi_2$ and show that it solves the second equation in (12).

**Proposition 5.1.** Given an infinitesimal deformation $\pi + h\pi_1$ of $A$ where $\pi_1 = \frac{1}{2}X_{ij}\partial^i \otimes \partial^j$ is antisymmetric and satisfies the Jacobi identity, let $\pi_2$ be the following symmetric cochain

\begin{equation}
\pi_2 = \frac{1}{12}X_{ij}^l X_{kl} (\partial^i \otimes \partial^j \partial^k + \partial^j \otimes \partial^k \partial^i) + \frac{1}{8}X_{ij}^l X_{kl} \partial^i \otimes \partial^j \otimes \partial^k.
\end{equation}

Then $\pi + h\pi_1 + h^2\pi_2$ is a second order deformation of $A$.

**Proof.** Use equation (9) to compute the right hand side of

\begin{equation}
\delta \pi_2 = \frac{1}{2}[\pi_1, \pi_1]
\end{equation}

and use the Jacobi identity (10) to find

\begin{equation}
\frac{1}{2}[\pi_1, \pi_1] = \frac{1}{4}X_{ik}^j X_{ij} \partial^i \otimes \partial^j \otimes \partial^k + \frac{1}{4}X_{ij}^k X_{kl} (\partial^i \otimes \partial^j \partial^k - \partial^j \otimes \partial^k \partial^i - \partial^k \otimes \partial^i \partial^j).
\end{equation}

The third order part of $\delta \pi_2$ (with $\pi_2$ given by (12)) is computed using (8) to be given by

\begin{equation}
\frac{1}{12}X_{ij}^l X_{lk} (\partial^i \otimes \partial^j \otimes \partial^k + \partial^j \otimes \partial^k \otimes \partial^i - \partial^i \otimes \partial^k \partial^j - \partial^k \otimes \partial^i \partial^j).
\end{equation}

Since $i, j$ and $k$ are just summation indices this can be rewritten as

\begin{equation}
\frac{1}{12} (X_{ij}^l X_{lk} + 2X_{lk}^j X_{ij} - X_{ik}^j X_{lj}) \partial^i \otimes \partial^j \otimes \partial^k.
\end{equation}

Using the Jacobi identity (10) this reduces to a single term

\begin{equation}
\frac{1}{4}X_{ik}^l X_{lj} \partial^i \otimes \partial^j \otimes \partial^k,
\end{equation}

which is the third order term of $\frac{1}{2}[\pi_1, \pi_1]$. For the fourth order term one makes a similar computation (but the Jacobi identity is not used).
One concludes from these computations that it is not obvious how to guess a cochain whose coboundary is given; compare carefully (14) and (12).

Our next task is to find an explicit solution for the third equation in (4), namely the equation \( \delta \pi_3 = [\pi_1, \pi_2] \). The computation of the right hand side is long but straightforward. Writing it as a coboundary is non-trivial and we will concentrate on this aspect. Clearly every term of the right hand side is a differential operator of order 3, 4, 5 or 6. We will denote the \( i \)-th order part of a bidifferential operator by a subscript \((i)\). We start with the highest order, which is the easiest.

**Lemma 5.2.** The sixth order part of \([\pi_1, \pi_2]\) is the coboundary of an antisymmetric 2-cochain,

\[
[\pi_1, \pi_2]_{(6)} = \delta \left( \frac{1}{48} X_{ij} X_{kl} X_{mn} \partial^{ikm} \otimes \partial^{jln} \right).
\]

**Proof.** The sixth order terms in \([\pi_1, \pi_2]\) are the ones for which none of the coefficients in \(\pi_1\) or \(\pi_2\) are differentiated. There are twelve terms, they come from the bracket of \(\pi_1\) and the fourth order term of \(\pi_2\) only, and eight of them cancel in pairs, leaving the following expression for \([\pi_1, \pi_2]_{(6)}\).

\[
\frac{1}{16} X_{mn} X_{ij} X_{kl} (\partial^{ikm} \otimes \partial^{jln} + \partial^{imn} \otimes \partial^{jkl} + \partial^{jkl} \otimes \partial^{imn}).
\]

To compute \(\delta(X_{ij} X_{kl} X_{mn} \partial^{ikm} \otimes \partial^{jln})\), use (8) and find twelve terms which come in equal triples due to the order three symmetry \((i,j) \rightarrow (k,l) \rightarrow (m,n)\). Formula (15) follows.

Note that the computation did not involve the Jacobi identity. In the symplectic case this is the only term which survives. Next, we consider the terms of order 5.

**Lemma 5.3.** The fifth order term \([\pi_1, \pi_2]_{(5)}\) is also the coboundary of an antisymmetric 2-cochain, given by

\[
[\pi_1, \pi_2]_{(5)} = \frac{1}{24} \delta \left( X_{ij} X_{kl} X_{mn} (\partial^{jkl} \otimes \partial^{imn} - \partial^{jim} \otimes \partial^{klm}) \right).
\]

**Proof.** The bracket \([\pi_1, \pi_2]_{(5)}\) has a lot of terms, they are of types \((1,1,3)\), \((1,3,1)\), \((3,1,1)\), \((1,2,2)\), \((2,1,2)\) and \((2,2,1)\). The terms of type \((1,3,1)\) cancel and in the other ones there is some simplification. Since \([\pi_1, \pi_2]\) is flip symmetric and the coboundary of any antisymmetric 2-cochain is flip symmetric as well, we only need to consider the terms of types \((3,1,1)\), \((1,2,2)\) and \((2,1,2)\). We give the result below, omitting a global factor 1/24. Note the non-triviality of the coefficients.

\[
(3,1,1) : (X_{mn} X_{ij} X_{kl} + X_{im} X_{nj} X_{kl}) \partial^{ikm} \otimes \partial^{jln},
\]

\[
(1,2,2) : (X_{ij} X_{kl} X_{mn} + 2X_{kn} X_{ij} X_{kl} + X_{km} X_{ij} X_{kl} + 3X_{ml} X_{kn} X_{ij}) \partial^{imn} \otimes \partial^{jln},
\]

\[
(2,1,2) : (X_{ij} X_{kl} X_{mn} + 2X_{ij} X_{km} X_{in} + 3X_{ij} X_{kn} X_{ml}) \partial^{jim} \otimes \partial^{imn}.
\]

It is surprising that all these terms integrate to a single term, \(i.e.,\) as a whole they can be written as

\[
\delta \left( X_{ij} X_{kl} X_{mn} (\partial^{jim} \otimes \partial^{klm} - \partial^{jim} \otimes \partial^{klm}) \right).
\]

Before checking this, note that (16) produces indeed precisely terms of the appropriate types. Clearly the \((3,1,1)\) part of (16) is given by

\[
X_{ij} X_{kl} X_{mn} (\partial^{jim} \otimes \partial^{klm} + \partial^{jim} \otimes \partial^{ln}).
\]
and is easily rewritten in the form of type (3,1,1). Type (2,1,2) involves the Jacobi identity. The (2,1,2) part of (14) is given by
\[-X^k_{ij}X_{kl}X_{mn}(\partial^i \otimes \partial^m \otimes \partial^n + \partial^i \otimes \partial^n \otimes \partial^m + \partial^m \otimes \partial^n \otimes \partial^i + \partial^i \otimes \partial^m \otimes \partial^n)\]
which is easily rewritten as
\[(X^k_{ij}X_{km}X_{nl} + X^k_{ij}X_{km}X_{mn} + 2X^k_{ij}X_{km}X_{ml})\partial^m \otimes \partial^n \otimes \partial^i\]
Now use the Jacobi identity (11) on the last two terms to obtain the term of type (2,1,2).
\[-X^k_{ij}X_{kl}X_{mn} \left( \partial^m \otimes \partial^ij \otimes \partial^n + \partial^i \otimes \partial^jm \otimes \partial^n + \partial^j \otimes \partial^im \otimes \partial^n \right).\]
When this is rewritten as
\[(X^k_{ij}X_{kl}X_{mn} + X^k_{mn}X_{kl}X_{ij} + X^k_{ml}X_{kn}X_{ij}) \partial^n \otimes \partial^i \otimes \partial^m\]
then the first term matches with the first term of type (1,2,2) and the other two match up with the three remaining terms of type (1,2,2).

For the fifth order term we used the Jacobi identity. For the fourth order term we will also use the derivative of the Jacobi identity (11).

Lemma 5.4. The fourth order term $[\pi_1, \pi_2]_{(4)}$ is the coboundary of an antisymmetric 2-cochain,
\[[\pi_1, \pi_2]_{(4)} = \frac{1}{48} \delta(X^k_{lm}X^l_{jn}X_{ki}(\partial^{mn} \otimes \partial^{ij} - \partial^{ij} \otimes \partial^{mn} + X^k_{ij}(\partial^m \otimes \partial^n \otimes \partial^{ij} - \partial^{ij} \otimes \partial^m \otimes \partial^n)).\]

Proof. As in the previous case we give the terms in $[\pi_1, \pi_2]_{(4)}$ by type. There are just three types, to wit, (1,1,2), (1,2,1) and (2,1,1). By flip symmetry we only need to consider the terms of type (1,1,2) and (1,2,1). They have the following form (we omit the global constant 1/48),
\[(1,1,2) : X_{kl}(4X^k_{ml}X^l_{jn} + 2X^k_{ij}X^l_{mn} + 2X^k_{ij}X_{lm} + 3X^k_{mn}X_{ij}) \partial^m \otimes \partial^n \otimes \partial^{ij},\]
\[(1,2,1) : 2(X^k_{ml}X_{ij}X_{ki} - X^k_{ij}X_{km}X_{ln}) \partial^m \otimes \partial^n \otimes \partial^j).\]
We already simplified these formulas by using the Jacobi identity (for (1,2,1) we used it twice). The verification for type (1,2,1) is straightforward: the six terms of type (1,2,1) in
\[\delta(X^k_{mn}X_{kl}X_{ij} \partial^m \otimes \partial^{nij} + X^k_{ij}X_{km}X_{ln} \partial^{mn} \otimes \partial^j)\]
come in pairs and reduce to (1,2,1) above. The terms of type (1,1,2) in
\[\delta(2X^k_{lm}X^l_{jn}X_{kl} \partial^m \otimes \partial^{nij} + X^k_{mn}X_{kl}X_{ij} \partial^m \otimes \partial^{nij})\]
are given by
\[X_{kl}(-2X^k_{mn}X^l_{jn} - 2X^k_{lm}X^l_{jn} + X^k_{mn}X_{ij} + 2X^k_{ij}X_{lm}) \partial^m \otimes \partial^n \otimes \partial^{ij}\]
which reduces to
\[X_{kl}(4X^k_{mn}X^l_{jn} + 2X^k_{ij}X^l_{mn} + 2X^k_{ij}X_{lm} + 3X^k_{mn}X_{ij}) \partial^m \otimes \partial^n \otimes \partial^{ij}\]
by using the derivative of the Jacobi identity. $\square$
Finally we consider the term of order 3. The proof does not involve the Jacobi identity and is left to the reader.

**Lemma 5.5.** The third order term $[\pi_1, \pi_2]_{(3)}$ is also the coboundary of an antisymmetric 2-cochain,

$$[\pi_1, \pi_2]_{(3)} = \frac{1}{24}\delta(X_{ij}X_{kl}X_{mn}(\partial^n \otimes \partial^{lm} - \partial^{lm} \otimes \partial^n)).$$

Our previous results lead to the following theorem.

**Theorem 5.6.** Let $(A, \{\cdot, \cdot\})$ be a polynomial Poisson algebra with basis $\{x_i\}_{i \in \mathcal{I}}$ and denote $\pi_1 = X_{ij}\partial^i \otimes \partial^j$, where $X_{ij} = \{x_i, x_j\}$. Then the following formula gives a third order deformation $\pi + h\pi_1 + h^2\pi_2 + h^3\pi_3$ of $A$,

$$\pi_* = \pi + \frac{h}{2}X_{ij}\partial^i \otimes \partial^j + \frac{h^2}{24}[2X_{ij}X_{kl}X_{mn}(\partial^n \otimes \partial^{lm} - \partial^{lm} \otimes \partial^n) + X_{ij}X_{kl}X_{mn}\partial^{km} \otimes \partial^{ln}$$

$$+ X^{k}_{lm}X^{l}_{jm}X_{ki}(\partial^{mn} \otimes \partial^{ij} - \partial^{ij} \otimes \partial^{mn}) + X^{k}_{mn}X_{ij}X_{kl}(\partial^{im} \otimes \partial^{nj} - \partial^{nj} \otimes \partial^{im}) + 2X^{k}_{ij}X_{kl}X_{mn}(\partial^{jm} \otimes \partial^{in} - \partial^{in} \otimes \partial^{jm})].$$

Up to equivalence every extension of $\pi + h\pi_1$ is of the form

$$\pi + h\pi_1 + h^2(\pi_2 + \varphi_2) + h^3(\pi_3 + \varphi_3 + \psi_3)$$

with $\varphi_2$ and $\varphi_3$ antisymmetric biderivations and $\psi_3$ a symmetric 2-cochain satisfying $\partial\psi_3 = [\pi_1, \varphi_2]$. Conversely, for such $\varphi_2$, $\varphi_3$ and $\psi_3$, $\pi_*$ is always a third order deformation.

**Proof.** We proved already that $\pi_*$ is a third order deformation. Suppose now that $\pi + h\pi_1 + h^2\pi_2' + h^3\pi_3$ is another deformation which extends the same infinitesimal deformation. Then $\varphi_2' = \pi_2 - \pi_2'$ is a cocycle which can be assumed to be an antisymmetric biderivation (Proposition 2.3). Since $\delta\pi_3' = [\pi_1, \pi_2 + \varphi_2]$ is a coboundary its flip antisymmetric part $[\pi_1, \varphi_2]$ satisfies the Jacobi identity (Theorem 3); we let $\psi_3$ be any symmetric cochain whose coboundary is $[\pi_1, \varphi_2]$. Then $\pi_3' - \pi_3$ must differ from $\pi_3$ by a cocycle $\varphi_3$ which we may assume, again without loss of generality, to be an antisymmetric biderivation. \hfill $\square$

6. THE OBSTRUCTION TO A FOURTH ORDER DEFORMATION

In this section we want to investigate the fourth order term of the explicit deformation which is given by (17). For a given polynomial Poisson algebra $(A, \{\cdot, \cdot\})$ over a field $F$ of characteristic zero we will denote the latter deformation by $\pi_* = \pi + h\pi_1 + h^2\pi_2 + h^3\pi_3$; as before $\pi_1 = \frac{1}{2}\{\cdot, \cdot\} = \frac{1}{2}X_{ij}\partial^i \otimes \partial^j$.

**Theorem 6.1.** The deformation (17) extends to a fourth, hence fifth, order deformation if and only if the following, non-trivial, condition is satisfied for any $a < b < c \in \mathcal{I}$:

$$2X_{ij}X_{kl}(X_{ab}X_{cm}^{jm} + X_{bc}^{km}X_{am}^{jl} + X_{ca}^{km}X_{bm}^{jl})$$

$$+ X_{ij}X_{kl}(X_{ab}X_{cm}^{jm} + X_{bc}^{km}X_{am}^{jl} + X_{ca}^{km}X_{bm}^{jl}) = 0.$$
Proof. The deformation (17) extends to a fourth order deformation if and only if $[\pi_1, \pi_3] + \frac{1}{2}[\pi_2, \pi_2]$ is a coboundary. Since this cocycle is flip antisymmetric this is equivalent to $J([\pi_1, \pi_3] + \frac{1}{2}[\pi_2, \pi_2]) = 0$. The fact that $\pi_2$ is symmetric implies at once that $J([\pi_2, \pi_3]) = 0$. As for the terms in $J([\pi_1, \pi_3])$, they have orders ranging from 3 to 8 only. We claim that the terms of order at least four all vanish, sketching the computation in the least trivial case when the order equals four. A direct application of (1) gives the following expression for the coefficient of $\partial^a \otimes \partial^b \otimes \partial^c$ in $J([\pi_1, \pi_3])$ (some indices have been relabelled for later convenience and a global constant $1/48$ has been omitted; note also that $a$ and $\bar{a}$ can be freely interchanged):

$$2X_{ji}X_{\bar{a}}^j(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$+ 2X_{ji}X_{\bar{a}}^j(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$+ 2X_{ji}X_{\bar{a}}^j(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$+ X_{ji}X_{\bar{a}}^j(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$- 2X_{ji}X_{\bar{a}}^j(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl}).$$

We now use the second derivative of the Jacobi identity, i.e., we use the formula (valid for any indices $a, b, c, j$ and $l$),

$$(X_{ak}X_{\bar{b}}^k + X_{bk}X_{\bar{a}}^k + X_{ck}X_{\bar{a}}^k)_{ji} = 0$$

to rewrite the first two lines (giving the first line below) and we use twice a derivative of the Jacobi identity to rewrite the third line (giving lines two and three below); the fourth line is simplified by a direct application of the Jacobi identity,

$$2(X_{ji}X_{\bar{a}}^j + X_{ji}X_{\bar{a}}^j)(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$+ 2X_{ji}X_{\bar{a}}^j(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$+ 2X_{ji}X_{\bar{a}}^j(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$+ X_{ji}X_{\bar{a}}^j(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$- 2X_{ji}X_{\bar{a}}^j(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl}).$$

Most terms in this expression cancel out in pairs, leaving

$$2(X_{ji}X_{\bar{a}}^j + X_{ji}X_{\bar{a}}^j)(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$+ 2(X_{ji}X_{\bar{a}}^j + X_{ji}X_{\bar{a}}^j)(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

$$+ 2(X_{ji}X_{\bar{a}}^j + X_{ji}X_{\bar{a}}^j)(X_{ak}X_{\bar{b}}^{kl} + X_{bk}X_{\bar{a}}^{kl} + X_{ck}X_{\bar{a}}^{kl})$$

which is zero, by a single application of the Jacobi identity on every line. It follows that the only non-zero terms in $J([\pi_1, \pi_3])$ are terms of type $(1, 1, 1)$. Using (2) we find that the coefficient of $\partial^a \otimes \partial^b \otimes \partial^c$ in $J([\pi_1, \pi_3])$ is given, (up to a global constant $-1/48$) by the left hand side of (19); since this expression is antisymmetric in $a, b, c$ it will hold in general when it holds for $a < b < c \in \mathbb{Z}$. We will see later an example for which (19) is non-zero, showing that our deformation (17) in general does not extend to a fourth order deformation. However, if (19) vanishes then $\pi_4$ can be chosen to be symmetric, which implies the existence of $\pi_5$ upon using Theorem 3.2.

We will show in Section 8 how to overcome this obstruction. Before doing this we will examine the quantized enveloping algebra as a natural candidate for a
deformation quantization. As it turns out the same obstruction found in Theorem 6.1 will arise. This surprising fact is a consequence of the non-trivial fact that the third order deformation quantization given by (17) coincides with the third order deformation quantization given by the quantized enveloping algebra.

7. THE QUANTIZED UNIVERSAL ENVELOPING ALGEBRA

In this section we will show that the third order deformation which we constructed for any polynomial Poisson algebra comes from a “quantized” enveloping algebra. The fact that an enveloping algebra appears here is not surprising. The symmetric algebra of a Lie algebra is a polynomial Poisson algebra in a natural way and it is well known that the quantized universal enveloping algebra of a Lie algebra is a deformation quantization of this Poisson algebra (see [1], [2]).

In order to describe the enveloping algebra of a polynomial Poisson algebra we will view polynomial algebras as symmetric algebras over a vector space. Let $V$ be a (possibly infinite-dimensional) vector space over a field $F$ of characteristic zero. For simplicity of notation we will denote elements in $V$ by lowercase roman letters.

For any positive integer $n$ we let $V^n = V \otimes V \otimes \ldots \otimes V$ ($n$ copies) and $V^0 = F$. The tensor algebra over $V$ is the $\mathbb{Z}$-graded associative algebra (with unit) defined by $T(V) = \bigoplus_{n=0}^{\infty} V^n$.

The symmetric algebra $S(V)$ is the quotient $S(V) = T(V)/I$, where $I$ is the homogeneous ideal in $T(V)$ generated by elements of the form $x \otimes y - y \otimes x$. The symmetric algebra is isomorphic to the polynomial algebra $F[x_j]_{j \in \mathcal{I}}$ where $\{x_j\}_{j \in \mathcal{I}}$ is any basis for $V$. (Of course, any polynomial algebra can be represented in this form.) In particular, we will use juxtaposition to denote the product in $S(V)$, just as we did for a polynomial algebra.

Any antisymmetric map $V \otimes V \rightarrow S(V)$ extends to a unique antisymmetric biderivation on $S(V)$. When this biderivation satisfies the Jacobi identity then $(S(V), \{\cdot, \cdot\})$ becomes a polynomial Poisson algebra, and every polynomial Poisson algebra arises in this fashion. The quotient map $\mu : T(V) \rightarrow S(V)$ has a $F$-linear right inverse $\sigma : S(V) \rightarrow T(V)$ which is defined by

$$\sigma \left( \prod_{i=1}^{n} a_i \right) = \frac{1}{n!} \sum_{p \in S_n} a_{p(1)} \otimes a_{p(2)} \otimes \cdots \otimes a_{p(n)},$$

where $S_n$ is the symmetric group on $n$ elements. We call $\sigma$ the symmetrization map. Note that $\mu$ is an algebra homomorphism but the symmetrization map $\sigma$ is not. Let $T(V)^h = (S(V))^h$ be the formal power series with coefficients in $T(V)$ $(S(V))$. Then $T(V)^h$ and $S(V)^h$ are naturally $F[[h]]$-algebras, $\mu$ extends to an $F[[h]]$-algebra homomorphism $\mu : T(V)^h \rightarrow S(V)^h$, and $\sigma$ extends to a $F[[h]]$-linear map $\sigma : S(V)^h \rightarrow T(V)^h$. Now we introduce a natural candidate for a deformation quantization of a polynomial Poisson algebra $(S(V), \{\cdot, \cdot\})$.

**Definition 7.1.** Let $J^h$ denote the two-sided ideal of $T(V)^h$, generated by all elements

$$(20) \quad x \otimes y - y \otimes x - h\sigma\{x, y\} \quad (x, y \in V).$$
The quantized universal enveloping algebra of \( (S(V), \{\cdot,\cdot\}) \) is given by
\[
U(V)^h = T(V)^h / J^h.
\]
The induced product on \( U(V)^h \) is denoted by \( \odot \) and the quotient map by
\[
\rho : T(V)^h \to U(V)^h.
\]
Thus, we have associated to a polynomial Poisson algebra \( (S(V), \{\cdot,\cdot\}) \) a new (non-commutative) associative algebra \( (U(V)^h, \odot) \) and they are linked by the \( F[[h]] \)-linear map (not a homomorphism!)
\[
\tau : S(V)^h \to U(V)^h
\]
given by \( \tau = \rho \circ \sigma \). The maps \( \tau, \rho \) and \( \sigma \) induce maps \( \tau_n, \rho_n \) and \( \sigma_n \) on the quotient spaces \( T(V)_{n}^h, S(V)_{n}^h \) and \( U(V)_{n}^h \) obtained by dividing out by the ideal \( (h^{n+1}) \). We also use the notation \( J_n^h \) for \( J^h / (h^{n+1}) \), so that \( U(V)_{n}^h = T(V)_{n}^h / J_n^h \). We will see that in some important cases the map \( \tau \) is a bijection, but that in general \( \tau_n \) is only injective for \( n \leq 3 \). If \( \tau \) is injective up to some order, the enveloping algebra provides a deformation quantization of \( (S(V), \{\cdot,\cdot\}) \) of the same order, as given by the following theorem.

**Theorem 7.2.** If \( \tau : S(V)^h \to U(V)^h \) (resp. \( \tau_n \)) is injective then the unique product \(* \) on \( S(V)^h \) which makes \( \tau \) (resp. \( \tau_n \)) into a homomorphism is a deformation quantization (resp. of order \( n \)) of the Poisson algebra \( (S(V), \{\cdot,\cdot\}) \).

*Proof.* \( \tau \) is always surjective: simply note that \( U(V)^h_1 \) is canonically isomorphic to \( S(V) \), so that for any \( q \in U(V)^h \) there exists a \( p \in S(V) \) such that \( \tau(p) = q \mod h \). Then \( \tau(p) - q = hq_1 \), for some \( q_1 \in U(V)^h \). Continuing this process, we obtain a sequence of polynomials \( p_k \) such that \( \tau(p + hp_1 + \cdots + h^k p_k) = q = h^k q_k \) for some \( q_k \in U(V)^h \). Then \( \tau(p + hp_1 + \ldots) = q \). It follows that \( \tau_n \) is also surjective.

If \( \tau_n \) is injective then the associative product which is induced by \( \tau_n \) is given for \( p, q \in S(V) \) by
\[
p \star q = \tau_n^{-1}(\tau_n(p) \odot \tau_n(q)).
\]
We show that it defines a deformation of \( (S(V), \{\cdot,\cdot\}) \) and that it is alternating. It is easy to see that
\[
\tau_n(p) \odot \tau_n(q) = \tau_n(pq) \mod h
\]
so that \( p \star q = pq \mod h \): the associativity of \(* \) on \( S(V)^h_n \) implies that \( p \star q = pq + h\pi_1(p,q) \mod h^2 \) for some cocycle \( \pi_1 \). If we can show that \( \pi_1 \) is antisymmetric then it is a biderivation and the fact that \( \pi_1 = \frac{1}{2}\{\cdot,\cdot\} \) follows from the following check for elements \( x, y \in V \),
\[
h\pi_1(x,y) = \frac{1}{2}(x \star y - y \star x) = \frac{h}{2}\{x,y\} \mod h^2.
\]
Now we show that \(* \) is alternating (up to order \( n \), which proves in particular that \( \pi_1 \) is antisymmetric. Let \( T \) be the anti-involution on \( T(V)^h \) induced by the map which reverses the order of elements in a tensor product, and let \( t \) be the involution of \( F[[h]] \) which is given by the map \( h \mapsto -h \). Then \( t \) determines involutions of \( S(V)^h \) and \( T(V)^h \), which we will also denote by \( t \). Let \( \iota = T \circ t = t \circ T \), so \( \iota \) is an anti-involution of \( T(V)^h \). Note that \( T \circ \sigma = \sigma \). Thus \( \iota(x \otimes y - y \otimes x - h(x,y)) = y \otimes x - x \otimes y - h(y,x) \), so \( \iota \) maps the ideal \( J^h \) to itself inducing an anti-involution \( \iota \). We also have the
Theorem 7.3. For $n \geq 1$ the following four statements are equivalent.

1. $\tau_n$ is injective;
2. For any $\alpha \in \mathcal{U}(V)^h_n$, $h\alpha = 0$ implies $\alpha = 0 \mod h^n$;
3. $\ast$ satisfies the $n$-th diamond relation $\Delta_n = 0$;
4. The restriction of $\rho_n$ to $\mathcal{O}(V)^h_n$ is injective.

Moreover, each of these statements is true for $n = 0$.

Proof. Let us first treat the case of $n = 0$ because this is used later in the proof. The fact that $\tau_0$ is injective follows immediately from the fact that the image of $J^h$ in $\mathcal{T}(V)$ is the ideal $I$, so that $\tau_0$ is essentially the identity map, from which it also follows that the restriction of $\rho_0$ to $\mathcal{O}(V)_0$ is injective. Statements 2) and 3) hold vacuously for $n = 0$, so all statements are true for $n = 0$.

Let us suppose that $\tau_n$ is injective and let $\alpha \in \mathcal{U}(V)^h_n$ be an element such that $h\alpha = 0$. Since $\tau_n$ is surjective there exists $\beta \in \mathcal{S}(V)^h_n$ such that $\tau_n(\beta) = \alpha$. Then $\tau_n(h\beta) = h\tau_n(\beta) = 0$, so that $h\beta = 0$ and $\beta \in (h^n)$. Then $\alpha = \tau_n(\beta) = 0 \mod h^n$, which shows that 1) implies 2).

That 2) implies 3) follows from the fact that $h\Delta_n = 0$.

We now show that 4) implies 1), so we assume that the restriction of $\rho_n$ to $\mathcal{O}(V)^h_n$ is injective. We show that $\tau_n$ is injective. By induction, we can assume that this theorem is true for $n - 1$, so that $\tau_{n-1}$ is injective, since $\Delta_{n-1} = 0$ if $\Delta_n = 0$. 

Therefore, if $\tau_n(\gamma) = 0$ for some $\gamma \in S(V)_n^h$, then since $\tau_{n-1}(\gamma) = 0$, we must have $\gamma = 0 \mod h^n$. Thus $\gamma = h^np$ for some $p \in S(V)$. But if $x_{i_1} \cdots x_{i_k}$ satisfies $i_1 \leq \cdots \leq i_k$, then $\tau_n(h^{n}x_{i_1} \cdots x_{i_k}) = h^n\rho_n(x_{i_1} \cdots x_{i_k})$, because we can always reorder the terms appearing in a tensor at the price of adding $h$ times something. If we express $p = \sum a^l_i x_{i_1} \cdots x_{i_k}$, where we sum over all increasing multi-indices $I = (i_1, \cdots, i_k)$, and $\beta = h^n \sum a^l_i x_{i_1} \cdots x_{i_k}$, then $\beta \in \mathcal{O}(V)_n^h$ and satisfies $\rho_n(\beta) = \tau_n(\gamma) = 0$, so that $\beta = 0$, by injectivity of $\rho_n$ on $\mathcal{O}(V)_n$. It follows that $p$ must also vanish, and thus $\gamma = 0$. This shows that $4)$ implies $1)$.

The rest of the proof is devoted to showing that $3)$ implies $4)$. We fix any $n \geq 1$ and assume that $\Delta_n = 0$. Since the kernel of $\rho_n$ restricted to $\mathcal{O}(V)_n^h$ is $\mathcal{O}(V)_n^h \cap J_n^h$, it suffices to show that $\mathcal{O}(V)_n^h \cap J_n^h \subseteq hJ_n^h$. An arbitrary element $\gamma$ of ker $\rho_n$ is of the form $\gamma = \gamma' + h\gamma''$ where $\gamma', \gamma'' \in J_n^h$ and

$$\gamma' = \sum_{1 \leq i \leq N} a_i \otimes (x_{i_1} \otimes x_{j_i} - x_{j_i} \otimes x_{i_1} - h\sigma(x_{i_1}, x_{j_i})) \otimes \beta_i$$

for some monomials $a_i$, $\beta_i$ in $\mathcal{T}(V)$, basis elements $x_{i_1}$, and $x_{j_i}$ and some positive integer $N$. We need to show that if $\gamma$ is ordered then $\gamma' \in hJ_n^h$. We first show that $\gamma'(0) = 0$. Since $\rho_n(\gamma) = 0$ also $\rho_n(\gamma(0)) = 0$ which implies that $\gamma(0) = 0$ because $\gamma$ and hence also $\gamma(0)$ is ordered. Then $\gamma'(0)$ also vanishes because $\gamma(0) = \gamma'(0)$. Now consider a fixed multi-index $I$ and define $\gamma'_I$ by (22) but summing only over those $l$ for which the indices in $a_l \otimes x_{i_1} \otimes x_{j_l} \otimes \beta_l$ coincide with the ones in $I$ (including multiplicities). Then evidently $\gamma'_I(0) = 0$. We will show that this implies that $\gamma'_I \in hJ_n^h$, from which it follows that $\gamma' \in hJ_n^h$ because $\gamma' = \sum_I \gamma'_I$.

First we consider the case when $I$ is a strictly ordered monomial, in which case we may assume that $I = (1, \cdots, m)$ for some $m$. We denote by $S_m$ the symmetric group and we consider its standard presentation with generators $\theta_k$, $k = 1, \cdots, m-1$, $(\theta_k$ corresponds to the transposition $(k, k+1)$) and relations $\theta_k^2$, $(\theta_k\theta_{k+1})^3$ and $(\theta_k\theta_j)^2$ for $|i-j| \geq 2$. For $\lambda \in S_m$, let $x_\lambda = x_{\lambda(1)} \otimes \cdots \otimes x_{\lambda(m)}$. Then we may express $\gamma'_I$ as

$$\gamma'_I = \sum_{\lambda \in S_m} \sum_{k=0}^{m-1} \alpha_{\lambda,k} (x_\lambda - x_{\theta_k\lambda} - h\chi_\lambda)$$

where $\alpha_{\lambda,k} \in \mathbb{F}$ and $\chi_\lambda,k = x_{\lambda(1)} \otimes \cdots \otimes \sigma(x_{\lambda(k)}, x_{\lambda(k+1)}) \otimes \cdots \otimes x_{\lambda(m)}$. Now consider the Cayley graph $\Gamma_m$ of the above presentation for $S_m$. The vertices of $\Gamma_m$ are given by the elements in $S_m$, with an edge connecting two vertices precisely when the permutations defining them differ by a transposition. The oriented edge connecting $\lambda$ and $\theta_k\lambda$ is denoted by $e_{\lambda,k}$, so that $\partial(e_{\lambda,k}) = \lambda - \theta_k\lambda$. We define a linear map $\Psi$ from the group $C^1(\Gamma_m, \mathbb{F})$ of (oriented) 1-chains on $\Gamma_m$ to $\mathcal{T}(V)^h$ by letting

$$\Psi(e_{\lambda,k}) = x_\lambda - x_{\theta_k\lambda} - h\chi_\lambda,k.$$ 

Notice that $\Psi$ is well-defined because although $e_{\theta_k\lambda,k}$ is the same edge as $e_{\lambda,k}$ but with the opposite orientation, it gets mapped to $-\Psi(e_{\lambda,k})$. Then obviously

$$\gamma'_I = \Psi \left( \sum_{\lambda \in S_m} \sum_{k=0}^{m-1} \alpha_{\lambda,k} e_{\lambda,k} \right)$$

and the fact that $\gamma'_I(0)$ vanishes means that $\sum_{\lambda \in S_m} \sum_{k=0}^{m-1} \alpha_{\lambda,k} e_{\lambda,k}$ is a cycle in the homology of the Cayley graph. By the universal coefficient theorem, every
cycle (with coefficients in an arbitrary group) on a graph can expressed as a sum of multiples of closed edge paths in the graph; moreover, any cycle on the Cayley graph of a presentation is a sum of cycles (with integral coefficients) which correspond to the basic relations which appear in the presentation. It follows that
\[ \sum_{n \in S_m} \sum_{k=1}^{n-1} a_{k,k} \epsilon_{k,k} = \sum_{i=1}^{t} b_i r_i \]
where each \( r_i \) corresponds to one of the basic relations appearing in the presentation and \( \beta_i \in \mathbb{F} \). Therefore we have that
\[ \gamma_j' = \sum_{i=1}^{t} b_i \Psi(r_i), \]
and it suffices to show that \( \Psi(f) \in hJ_n^h \) for any cycle \( f \) which corresponds to a basic relation. First, notice that the cycle \( f \) which corresponds to \( \theta^2 \) is zero because it consists of the sum of two copies of an edge with opposite orientation. Second, let \( i \) and \( j \) be such that \( |i - j| > 1 \) and let \( f_{ij} \) be the corresponding cycle, \( f_{ij} = e_{i,j} + e_{j,i} \). Then
\[ \Psi(f_{ij}) = -h(\chi_{i,j} + \chi_{j,i} + \chi_{i,j} + \chi_{j,i}). \]
Now both \( \chi_{i,j} + \chi_{j,i} \) and \( -\chi_{i,j} - \chi_{j,i} \) are given, up to an element of \( J_n^h \), by
\[ x_{(1)} \otimes \cdots \otimes \{ x_{(i,j)} , x_{(i+1)} \} \otimes \cdots \otimes \{ x_{(j)} , x_{(j+1)} \} \otimes \cdots \otimes x_{(m)}, \]
showing that \( \Psi(f_{ij}) \in hJ_n^h \). Finally, let us assume that \( f_l \) corresponds to the relation \( (\theta^2)^3 \). Then
\[ f_l = e_{i,l} + e_{j,l} + e_{i,l+1} + e_{j,l+1} + e_{i,l+1,l+2} + e_{j,l+1,l+2} + e_{i,l+1,l+2} + e_{j,l+1,l+2} \]
so that
\[ \Psi(f_l) = h x_{(1)} \otimes \cdots \otimes (x_{(l)} \otimes \sigma(x_{(l+1)} , x_{(l+2)})) \]
\[ - (x_{(l+1)} , x_{(l+2)}) \otimes \cdots \otimes x_{(m)} ] \]
Since \( \Delta_n = 0 \) the term between parentheses lies in \( J_{n-1}^h \). But now note that if \( \alpha \in J_{n-1}^h \), then \( \alpha = \beta + h^{n-1} \gamma \) for some \( \beta \in J_n^h \), so that \( h^{-1} \in hJ_n^h \). Thus we can conclude that \( \Psi(f_1) \in hJ_n^h \).

This completes the proof that 3) implies 4) in case 1 is strictly ordered. If I = \( (i_1, \ldots , i_m) \) is merely ordered then the proof can repeated verbatim after replacing \( S_m \) with a quotient group, whose presentation is obtained from the above standard presentation of \( S_m \) by adding the relations \( \theta_k \) for any \( k \) for which \( i_k = i_{k+1} \). The corresponding Cayley graph is obtained from the one for \( S_m \) by collapsing the edges which correspond to those \( \theta_k \).

The above theorem gives us an analytic criterion to check injectivity at some order. When we assume that injectivity at order \( n - 1 \) has been checked then we may think of the \( n \)-th diamond relation as being a relation in \( S(V)^h \). Since this is the way in which we will use the diamond relation below, we formulate this fact in a separate theorem.

**Theorem 7.4.** If \( \tau_n : S(V)^h \rightarrow U(V)^h \) is injective (hence bijective) then \( \tau_{n+1} \) is also injective if and only if the diamond relation
\[ x_a \ast \{ x_{b,c} \} - \{ x_{b,c} \} \ast x_a + \text{cyc}(a, b, c) = 0 \]
holds for any \( a, b, c \in I \). In this formula \( \ast \) is the product on \( S(V)^h \) which is induced using \( \tau_n \).
In this formulation the theorem will turn out to be very useful. For example we note that \( p \ast q = q \ast p \mod h \) and conclude from it that \( \tau_1 \) is injective. In order to use the theorem to prove injectivity of the higher \( \tau_i \) we need an explicit formula for the \( \ast \)-bracket which comes from the enveloping algebra. We will show now that such a formula is given exactly by (17) and derive injectivity of \( \tau_2 \) and \( \tau_3 \) from it.

Given a deformation \( (S(V)^h, \ast) \) of \( S(V) \) there is, besides the enveloping algebra \( U(V)^h \) another (in general) enveloping algebra which is associated to it.

**Definition 7.5.** Let \( (S(V)^h, \ast) \) be a deformation (of finite order or formal) of \( S(V) \) and denote the commutator in \( (S(V)^h, \ast) \) by \( [\cdot, \cdot]_\ast \). Define \( J_h \ast \) to be the two-sided ideal of \( T(V)^h \), generated by all elements of the form \( a \otimes b - b \otimes a - \sigma[a, b]_\ast \), \( (a, b \in V) \) and define the \( \ast \)-enveloping algebra \( U(V)^h_\ast \) of \( (S(V)^h, \ast) \) by

\[
U(V)^h_\ast = T(V)^h / J_h \ast.
\]

For a given deformation \( (S(V)^h, \ast) \) the enveloping algebras \( U(V)^h \) and \( U(V)^h_\ast \) coincide if and only if

\[
[x, y]_\ast = h\{x, y\} \quad (x, y \in V).
\]

We call a deformation which satisfies (24) **bracket-exact**. In terms of the cocycles \( \pi_i \) this means that

\[
\pi_i(x, y) = 0 \quad (x, y \in V, i > 1).
\]

For example, our general formula (17) defines a bracket-exact deformation quantization; adding any non-zero antisymmetric biderivation to \( \pi_3 \) defines a deformation quantization which is not bracket-exact.

We now give a property which characterizes \( \ast \)-enveloping algebras; in the case of bracket-exact deformations it characterizes enveloping algebras, showing that the \( \ast \)-product which comes from the enveloping algebra is given by (17).

**Definition 7.6.** Let \( (S(V)^h, \ast) \) be a deformation of \( S(V) \). The \( \mathbb{F}[[h]] \)-linear map,

\[
\sigma_* : S(V)^h \to S(V)^h
\]

which is defined by

\[
\sigma_* \left( \prod_{i=1}^n a_i \right) = \frac{1}{n!} \sum_{p \in S(n)} \frac{a_{p(1)} \ast a_{p(2)} \ast \cdots \ast a_{p(n)}}{a_{p}}.
\]

is called \( \ast \)-symmetrization. We will say that \( \ast \) is **\( s \)-balanced** if \( \sigma_* \) is the identity when restricted to elements of \( S(V) \) of degree \( \leq s \). If \( (S(V)^h, \ast) \) is a deformation (of order \( n \)) of \( S(V) \) then we call it a **balanced deformation** if \( \ast \) is \( s \)-balanced, where \( s \) is the degree of \( [\cdot, \cdot]_\ast \), i.e., the supremum of the degrees of all coefficients of \( [x, y]_\ast \), where \( x, y \) run over \( V \) (this degree may be infinite).

Note that when a deformation is bracket-exact then the degree of \( [\cdot, \cdot]_\ast \) is the degree of the corresponding Poisson bracket \( \{\cdot, \cdot\} \).

**Example 1.** Any deformation is equivalent to a 2-balanced deformation. Indeed, such an equivalence is given precisely by \( \sigma_* \), i.e., define an equivalent product \( \circ \) by

\[
p \circ q = \sigma_*^{-1}((\sigma_* p) \ast \sigma_* (q)).
\]
Then
\[ \sigma_\circ(xy) = \frac{1}{2}(x \circ y + y \circ x) = \frac{1}{2}\sigma_\circ^{-1}(x \star y + y \star x) = xy, \]
for any \( x, y \in V \), so that \( \circ \) is 2-balanced.

**Lemma 7.7.** Formula (17) gives, for any polynomial Poisson algebra, a bracket-exact balanced deformation of order 3.

**Proof.** The proof of balancing is by induction. Obviously any deformation is 1-balanced, so we assume that the deformation, given by Formula (17), is \( n \)-balanced and prove that it is \((n + 1)\)-balanced. To do this, take a monomial \( a \) of degree \( n + 1 \) and write \( a = a_1a_2\cdots a_{n+1} \). We denote the associative product (17) on \( \mathcal{S}(V)^h \) by \( \star \) and the corresponding cochains by \( \pi_i \). Using the associativity of \( \star \) one has
\[
\sum_{\tau \in S_{n+1}} a_{\tau(1)} \star a_{\tau(2)} \star \cdots \star a_{\tau(n+1)} = \sum_{i=1}^{n+1} a_i \star \left( \prod_{j \neq i}^{n+1} a_j \right)
\]
so \( \star \) is \((n + 1)\)-balanced when
\[
\sum_{i=1}^{n+1} \pi_k \left( a_i, \prod_{j \neq i} a_j \right) = 0,
\]
for \( k = 1, 2, 3 \). The verification is immediate.

The following theorem gives a precise relation between balanced deformations and the \( \star \)-enveloping algebra.

**Theorem 7.8.** If \((\mathcal{S}(V)^h, \star)\) is a balanced deformation of \( \mathcal{S}(V) \) then the \( \mathbb{F}[h]\)-algebra homomorphism
\[ F : (\mathcal{T}(V)^h, \otimes) \rightarrow (\mathcal{S}(V)^h, \star) \]
which is induced by the natural inclusion \( V \rightarrow \mathcal{S}(V) \) induces an \( \mathbb{F}[h]\)-algebra isomorphism
\[ f : (\mathcal{U}(V)^h, \otimes) \rightarrow (\mathcal{S}(V)^h, \star). \]
When \((\mathcal{S}(V)^h, \star)\) is moreover bracket-exact then \( \mathcal{U}(V)^h \star = \mathcal{U}(V)^h \) and we have an isomorphism
\[ f : (\mathcal{U}(V)^h, \otimes) \rightarrow (\mathcal{S}(V)^h, \star). \]

The corresponding statements for \( n \)-th order deformations also hold.

**Proof.** We will only prove the first statement. If we denote the canonical map \( \mathcal{T}(V)^h \rightarrow \mathcal{U}(V)^h \) by \( \rho^h \), then it suffices to prove that \( \text{ker } F = \text{ker } \rho_\star \) and that \( F \) is surjective. Let us first show that \( F \) is surjective. If \( p \in \mathcal{S}(V) \) then there exists an element \( \alpha \in \mathcal{T}(V) \) such that \( p = F(\alpha) \mod h \). Indeed, since \( \star \) is a deformation we have for any monomial \( \prod_{i=1}^n a_i \) that
\[
\prod_{i=1}^n a_i = a_1 \star a_2 \star \cdots \star a_n = F(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \mod h.
\]
More generally, for any \( k \in \mathbb{N} \), since \( F \) is \( \mathbb{F}[h]\)-linear we can find \( \alpha_0, \ldots, \alpha_k \in \mathcal{T}(V) \) such that \( p = F(\alpha_0 + \alpha_1h + \cdots + \alpha_kh^k) \mod h^{k+1} \). It follows that \( \mathcal{S}(V) \subset \mathcal{S}F \), which is sufficient to prove that \( F \) is surjective.
Let us show that $\ker \rho_* = \ker F$. Take $a, b \in V$ and compute

$$F(a \otimes b - b \otimes a - \sigma[a,b]_*) = F(a) * F(b) - F(b) * F(a) - F\sigma[a,b]_*$$

which is zero; we used in the computation that $\sigma_* = F\sigma$ and that $\sigma_*[a,b]_* = [a,b]_*$ (because the deformation is balanced). This shows that $\ker \rho_* \subset \ker F$.

To show that $\ker F \subset \ker \rho_*$ we pick any $X \in T(V)^h$ for which $F(X) = 0$ and show the existence of $Y \in T(V)^h$ such that $\rho_*(X) = \rho_*(Y)$ and whose degree (in $h$) is larger than the degree of $X$. This will imply that for any $j \in \mathbb{N}$ the composition

$$T(V)^h \xrightarrow{\rho_*} U(V)^h \xrightarrow{\rho^*} \mathbb{C}$$

maps $X$ to 0, hence $\rho_*(X) = 0$. To prove it, let $d$ denote the degree of $X$, i.e., $X = X_0 h^d \text{ mod } h^{d+1}$. Let $X_0$ denote the unique element in $3\sigma$ for which

$$\rho_*(X_0) = \rho_*(X_0) \mod h$$

If we write

$$X_0 = \frac{c}{n!} \sum_{p \in S_n} a_{p(1)} \otimes a_{p(2)} \otimes \cdots \otimes a_{p(n)}$$

then

$$F(X_0) = \frac{c}{n!} \sum_{p \in S_n} F(a_{p(1)}) \ast F(a_{p(2)}) \ast \cdots \ast F(a_{p(n)})$$

Thus $F(X) = 0$ implies that $c = 0$ so that $X_0 = 0$. So there exists a $Y_0$ such that $\rho_*(X_0) = \rho_*(h Y_0) \mod h^2$ and hence there exists an element $Y \in T(V)^h$ of the form $Y = Y_0 h^{d+1} \mod h^{d+2}$ such that $\rho_*(X) = \rho_*(Y)$. \hfill $\Box$

We have seen that Formula (17) defines a bracket-exact balanced deformation (of order three). Theorem 7.8 implies that this deformation comes from the enveloping algebra, via the symmetrization map. This fact has the important consequence that we can use (13) to check injectivity of the maps $\tau_n$. We used already the first term of our formula; i.e., we have used $p \ast q = pq \mod h$ to show that $\tau_1$ is injective. Further,

$$\pi_1(x_a, \{b,c\}) - \pi_1(\{b,c\}, x_a) + \text{cycl}(a,b,c)$$

which is zero in view of the Jacobi identity. This proves injectivity of $\tau_2$. Also

$$\pi_2(x_a, \{b,c\}) - \pi_2(\{b,c\}, x_a) + \text{cycl}(a,b,c) = 0$$

since $\tau_3$ is symmetric, hence $\tau_3$ is also injective. The fact that this step is easy is similar to the fact that the existence of $\pi_1$ is automatic (given the fact that $\pi_1$ is antisymmetric and that $\pi_2$ is symmetric). Finally, let us examine the injectivity
of \( \tau_4 \).

\[
\pi_3(x_a,\{b,c\}) - \pi_3(\{b,c\}, x_a) + \text{cycl}(a,b,c) \\
= \frac{1}{24} (2X_{ij}X_{kl}X_{ma}X_{bc} + X_{an}X_{ij}X_{kl}X_{bc}^n) + \text{cycl}(a,b,c) \\
= \frac{1}{12} X_{ij}X_{kl}(X_{ab}X_{cm} + X_{bc}X_{am} + X_{ca}X_{bm}) \\
= \frac{1}{24} X_{ij}X_{kl}(X_{ab}X_{cm} + X_{bc}X_{am} + X_{ca}X_{bm}).
\]

which is identical to the obstruction \((13)\) which we found when trying to extend the deformation given by \((17)\). We will see in the examples that in general the obstruction is non-zero, hence \( \tau_4 \) is not injective and the enveloping algebra leads in general only to a deformation of order three.

8. The Extension to a Fourth Order Deformation

We now come to the existence question of a fourth order deformation for a polynomial Poisson algebra \((A,\{\cdot,\cdot\})\) over a field \( \mathbb{F} \) of characteristic zero. We denote the third order deformation quantization that we obtained in \((17)\) by \( \pi_* = \pi + h\pi_1 + h^2\pi_2 + h^3\pi_3 \) where \( \pi_1 = \frac{1}{2} \{\cdot,\cdot\} \). We have shown in Theorem 5.6 that we get up to equivalence all possible third order deformations of \((A,\{\cdot,\cdot\})\) by adding any biderivations \( \varphi_2 \) and \( \varphi_3 \) to \( \pi_2 \) and \( \pi_3 \) and adding any symmetric cochain \( \psi_3 \) satisfying \( \delta\psi_3 = [\pi_1,\varphi_2] \) to \( \pi_3 \). Let us denote such an alternative deformation by \( \pi'_4 = \pi + h\pi_1 + h^2\pi'_2 + h^3\pi'_3 \). If \( \pi'_4 \) extends to a fourth order deformation by adding a term \( h^4\pi_4 \) then \( \pi_4 \) is a solution to

\[
\delta\pi_4 = [\pi_1,\pi'_3] + \frac{1}{2}[\pi'_2,\pi'_2],
\]

and the antisymmetric part of the right hand side must be in the kernel of \( J \), leading to

\[
J ([\pi_1,\pi_3] + [\pi_1,\varphi_3] + \frac{1}{2}[\varphi_2,\varphi_2]) = 0.
\]

In view of the following lemma, all terms in the left-hand side of \((27)\) are of type \((1,1,1)\).

**Lemma 8.1.** If \( \varphi \) and \( \psi \) are two biderivations then \( J([\varphi,\psi]) \) has type \((1,1,1)\).

**Proof.** Let \( \varphi = Y_{ij}\partial^i \otimes \partial^j \) and \( \psi = Z_{kl}\partial^k \otimes \partial^l \). Then the piece of \([\varphi,\psi]\) that does not contain terms of type \((1,1,1)\) is given by

\[
(Y_{ij}Z_{kl} + Y_{kl}Z_{ij})(\partial^k \otimes \partial^l \otimes \partial^j - \partial^i \otimes \partial^k \otimes \partial^j).
\]

Applying the Jacobi map every term appears twice with opposite signs hence they all cancel out.

By computing the terms of type type \((1,1,1)\) in \((25)\) we find that the existence of a fourth order deformation for a given \((A,\{\cdot,\cdot\})\) is equivalent to the existence of two antisymmetric biderivations \( \varphi_2 = \frac{1}{2} Y_{ij}\partial^i \otimes \partial^j \) and \( \varphi_3 = \frac{1}{12} Z_{ij}\partial^i \otimes \partial^j \) such that for any \( a < b < c \in \mathcal{I} \)

\[
X_{mc}Z_{ab} + Z_{mc}X_{ab} + 6Y_{mc}Y_{ab}m - X_{ij}X_{kl}X_{cm}X_{il} - 2X_{ij}X_{kl}X_{ab}X_{cm}X_{il} \\
+ \text{cycl} (a,b,c) = 0.
\]
Lemma 8.2. The 2-cocycles $Y_{ab} = 0$ and 

\begin{equation}
Z_{ab} = \frac{1}{2} X_{ab}^{ik} x_{ij}^l x_{kl}^j - X_{ab}^{ikl} x_{ij}^l x_{kl}^j, \quad (a, b \in I)
\end{equation}

solve equation (26) hence yield the correction term

\begin{equation}
\varphi_3 = \frac{1}{96} (X_{mn}^{ik} x_{ij}^l x_{kl}^j - 2 X_{mn}^{jkl} x_{ij}^l x_{kl}^j) \partial^m \otimes \partial^n
\end{equation}

to $\pi_3$ in (17) in order for the deformation quantization to extend to a fourth order deformation quantization.

Proof. Consider the following four equations, which are all a consequence of the Jacobi identity.

\begin{align*}
&\frac{1}{2} (X_{ab}^{ik} x_{cj}^l x_{jm}^k + \text{cycl} (a, b, c)) = 0, \\
&(X_{ab}^{ij} x_{jk}^l + X_{bc}^{jk} x_{ja}^l + X_{ca}^{jk} x_{ab}^l) x_{im}^k x_{km}^j + \text{cycl} (a, b, c) = 0, \\
&(X_{ab}^{ij} x_{jk}^l + X_{bc}^{jk} x_{ja}^l + X_{ca}^{jk} x_{ab}^l) x_{im}^k x_{km}^j + \text{cycl} (a, b, c) = 0, \\
&(X_{ab}^{ij} x_{jk}^l + X_{bc}^{jk} x_{ja}^l + X_{ca}^{jk} x_{ab}^l) x_{im}^k x_{km}^j + \text{cycl} (a, b, c) = 0.
\end{align*}

Expand now $X_{ij} Z_{ab}^{mn} + Z_{mc} X_{ab}^{nm} + \text{cycl} (a, b, c) = 0$, (where $Z_{ab}$ is given by (27)) and add the above four equations. After the smoke clears up you will find

\begin{equation}
X_{ij} x_{kl} x_{ab} X_{cm}^{i} x_{jm}^{k} + 2 X_{ij} x_{kl} x_{ab} x_{cm}^{i} x_{jm}^{k}
\end{equation}
as needed to solve (26). \hfill \square

9. Examples

In this section we will investigate some general and some more specific examples. We use the diamond relations to show that for constant and linear brackets the quantized enveloping algebra always gives a formal deformation quantization. For the quadratic case we give a few examples in which the quantized enveloping algebra gives a fifth order deformation (at least) and we give an example in which the quantized enveloping algebra gives a formal deformation quantization. We give in the cubic case a few examples for which the quantized enveloping algebra gives a deformation of order three but not of higher order thereby showing the non-injectivity of $\tau_4$ in general. All these examples are in $F^4$ (with coordinates $x_1, \ldots, x_4$; $F$ is a field of characteristic zero) but they have higher-dimensional counterparts. We will describe the Poisson structure by a $4 \times 4$ matrix whose $(i, j)$-th entry is the Poisson bracket $\{x_i, x_j\}$. We refer to this matrix as the Poisson matrix.

The simplest case is the one in which all $X_{ij}$ are constant (i.e., they belong to $F$). It is well-known that in this case a deformation quantization always exists. This follows also immediately from the diamond relations: since in this case

\begin{equation}
x \circ \tau \{y, z\} - \tau \{y, z\} \circ x = 0
\end{equation}

for any $x, y, z \in V$ we conclude that $\Delta = 0$ hence that $\tau$ is injective. Alternatively it is immediate to check that the following explicit formula defines a deformation quantization in this case,

\begin{equation}
\pi_* = \pi + \sum_{n=1}^{\infty} \sum_{k_1, \ldots, k_n} \frac{h^n}{2\pi i^n} X_{k_1 t_1} \cdots X_{k_n t_n} \partial^{k_1} \cdots \partial^{k_n} \otimes \partial^{l_1} \cdots \partial^{l_n}.
\end{equation}

If a linear map $V \otimes V \to V$ satisfies the Jacobi identity then its extension to $S(V)$ also satisfies the Jacobi identity, hence a Lie algebra leads in a natural way to a
polynomial Poisson algebra. We call it \textit{linear} because the bracket of any two basis elements is a linear combination of the basis elements. In this case it is known that the quantized enveloping algebra defines a formal deformation quantization. This is checked immediately using the diamond relations: in this case the fact that \( \{ y, z \} \in V \) for any \( y, z \in V \) implies that 
\begin{equation}
\tau \{ y, z \} - \tau \{ y, z \} = h \{ x, \{ y, z \} \} 
\end{equation}
so that the diamond relation holds in view of the Jacobi identity. Note also that, as a corollary of Theorem 7.3 all bracket-exact deformations of a linear bracket are isomorphic (to the one given by the enveloping algebra).

We can also consider brackets which have both linear and constant terms. Since in these particular examples the Poisson matrix is always of the form \( U \), we will only give the matrix \( U \) and the polynomial it derives from. Let us explain shortly how to compute \( U \) from \( \{ u, \varphi \} \) for a given bracket \( \varphi \) on \( \mathbb{C}^2 \). The coordinates are \( u_1, u_2, v_1 \) and \( v_2 \); also \( u(\lambda) = \lambda^2 + u_1 \lambda + u_2 \) and \( v(\lambda) = v_1 \lambda + v_2 \). Then the first row of \( U \) consists of the coefficients of \( \varphi(\lambda, v(\lambda)) \) mod \( u(\lambda) \) (just do Euclidean division) and the second row is given by the coefficients of \( \varphi(\lambda, u(\lambda))(\lambda + u_1) \) mod \( u(\lambda) \). For example, take \( \varphi = x^3 \). Then
\[ U = \begin{pmatrix} u_1^2 - u_2 & u_1 u_2 \\ \frac{1}{2} u_1 u_2 & u_2^2 \end{pmatrix}. \]

In this case direct substitution in the left hand side of \( \{ u, \varphi \} \) gives zero so that the deformation, as given by \( \{ L \} \), extends to a fifth order deformation. Another quadratic bracket is found by taking \( \varphi = y \). Then \( U \) is given by
\[ U = \begin{pmatrix} v_1 & v_2 \\ \frac{1}{2} u_1 u_2 - u_2 v_1 \\ \frac{1}{2} u_1 u_2 & u_2^2 \end{pmatrix}. \]

Again \( \{ L \} \) is satisfied. The same is also true for the sum, \( \varphi = x^3 + y \), which corresponds to taking the sum of the above \( U \) matrices. Another quadratic example
of interest is the quadratic bracket on $\mathfrak{gl}(2)$ (see (15)). It has a Poisson matrix
\[
U = \begin{pmatrix}
0 & x_1 x_2 & 0 & x_2 x_3 \\
-x_1 x_2 & 0 & 0 & x_2 x_4 \\
0 & 0 & 0 & 0 \\
-x_2 x_3 & -x_2 x_4 & 0 & 0
\end{pmatrix}.
\]

(19) is satisfied and the deformation extends to order five. In the following example of a quadratic bracket the quantized universal enveloping algebra gives a formal deformation quantization. If \((a_{ij})\) is a skew-symmetric matrix of size 4 then \(\{x_i, x_j\} = a_{ij} x_i x_j\) defines a quadratic Poisson bracket on \(\mathbb{C}^4\). In this case the relation

\[
x_i \circ x_j - x_j \circ x_i = h\tau \{x_i, x_j\} = ha_{ij}(x_i \circ x_j + x_j \circ x_i)
\]

can be rewritten as \(x_j \circ x_i = A_{ij} x_i \circ x_j\) where \(A_{ij} = (1 - ha_{ij})/(1 + ha_{ij})\). The verification of diamond relation then reduces to the following computation.

\[
\begin{align*}
&x_i \circ \{x_j, x_k\} - \{x_j, x_k\} \circ x_i + \text{cycl}(i, j, k) \\
&= x_i \circ x_j \circ x_k(a_{jk} - a_{ij}) + x_i \circ x_k \circ x_j(a_{ij} - a_{ki}) + x_j \circ x_i \circ x_k(a_{ki} - a_{ij}) \\
&+ x_j \circ x_k \circ x_i(a_{ij} - a_{jk}) + x_k \circ x_i \circ x_j(a_{ij} - a_{ki}) + x_k \circ x_j \circ x_i(a_{ij} - a_{jk}) \\
&= x_i \circ x_j \circ x_k((a_{jk} - a_{ij}) + (a_{ij} - a_{ki})A_{jk} + (a_{ki} - a_{ij})A_{ij} \\
&+ (a_{ki} - a_{jk})A_{ik}A_{ij} + (a_{ij} - a_{ki})A_{ik}A_{jk} + (a_{ij} - a_{jk})A_{ik}A_{jk}) \\
&= 0.
\end{align*}
\]

Therefore the quantized enveloping algebra of this quadratic Poisson bracket gives a formal deformation quantization.

Next we consider a few higher order brackets. As in the quadratic case, if you take \(\varphi = x^4\) then

\[
U = \begin{pmatrix}
-u_3^2 + 2u_1 u_2 & u_2^2 - u_1^2 u_2 \\
u_2^2 - u_1^2 u_2 & -u_1 u_2^2
\end{pmatrix}.
\]

In this case we find again that (19) is satisfied so that the enveloping algebra leads to a fifth order deformation. However, if you take \(\varphi = y^2\) then \(U\) is given by

\[
U = \begin{pmatrix}
2v_1 v_2 - u_1 v_1^2 & v_2^2 - u_2 v_1^2 \\
v_1 v_2^2 - u_1 v_2^2 & u_2 v_1^2 - 2u_2 v_1 v_2
\end{pmatrix},
\]

and (13) is not satisfied: if we denote \(x_1 = u_1, x_2 = u_2, x_3 = v_1\) and \(x_4 = v_2\) then the left hand side of (19) is given by

\[-96 x_3 (x_1^2 - 2x_1 x_2 x_3 + 2x_2 x_3^2 - 2x_1 x_2 x_3^2 x_4 + x_1^2 x_3^2 x_4 + x_2^2 x_3^2 x_4) \partial^1 \wedge \partial^2 \wedge \partial^4
\]

where the triple wedge is defined by

\[
\partial^i \wedge \partial^j \wedge \partial^k = \frac{1}{6} \sum_{\lambda} \text{sgn}(\lambda) \partial^{\lambda(i)} \otimes \partial^{\lambda(j)} \otimes \partial^{\lambda(k)}.
\]

It follows that in this case the quantized enveloping algebra only defines a third order deformation quantization. The choice \(\varphi = y^2 + xy\) gives another non-zero term; basically any higher order polynomial leads to an obstruction. Also the cubic
bracket on $\mathfrak{gl}(2)$ (see [13]), which is given by
\[
U = \begin{pmatrix}
0 & x_1^2 x_2 & x_2 x_3 (x_1 + x_4) \\
-x_1^2 x_2 & 0 & x_2 x_3 (x_4 - x_1) \\
-x_2 x_3 (x_1 + x_4) & -x_2 x_2^3 & 0
\end{pmatrix}
\]
leads to a non-zero obstruction, upon evaluating (19). Explicitly it is given by
\[
96 x_2^2 x_3 (2 x_1 x_4 + x_2 x_3) (x_4 - x_1)
\]
\[
(x_3 \partial^1 \wedge \partial^2 \wedge \partial^3 + (x_4 - x_1) \partial^1 \wedge \partial^2 \wedge \partial^4 - x_3 \partial^2 \wedge \partial^3 \wedge \partial^4).
\]
It follows that for most brackets the enveloping algebra only leads to a third order deformation.

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**University of Wisconsin, Department of Mathematics, Eau Claire, WI 54702-4004**  
*E-mail address:* penkovmr@uwec.edu

**University of California, 1015 Department of Mathematics, Davis, CA 95616-8633**  
*E-mail address:* vanhaeck@math.ucdavis.edu

**Université des Sciences et Technologies de Lille, U.F.R. de Mathématiques, 59655 Villeneuve D’Ascq, France**  
*E-mail address:* Pol.Vanhaecke@Univ-Lille1.fr