Nonlinear Schrödinger problems: symmetries of some variational solutions

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Abstract. In this paper, we are interested in the nonlinear Schrödinger problem $-\Delta u + Vu = |u|^{p-2}u$ submitted to the Dirichlet boundary conditions. We consider $p > 2$ and we are working with an open bounded domain $\Omega \subseteq \mathbb{R}^N (N \geq 2)$. Potential $V$ satisfies $\max(V,0) \in L^{N/2}(\Omega)$ and $\min(V,0) \in L^+\infty(\Omega)$. Moreover, $-\Delta + V$ is positive definite and has one and only one principal eigenvalue. When $p \simeq 2$, we prove the uniqueness of the solution once we fix the projection on an eigenspace of $-\Delta + V$. It implies partial symmetries (or symmetry breaking) for ground state and least energy nodal solutions. In the literature, the case $V \equiv 0$ has already been studied. Here, we generalize the technique at our case by pointing out and explaining differences. To finish, as illustration, we implement the (modified) mountain pass algorithm to work with $V$ negative, piecewise constant or not bounded. It permits us to exhibit direct examples where the solutions break down the symmetries of $V$.

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1. Introduction

Let $N \geq 2, p > 2, \lambda > 0$ and an open bounded domain $\Omega \subseteq \mathbb{R}^N$. We study the nonlinear Schrödinger problem

$$-\Delta u(x) + V(x)u(x) = \lambda|u(x)|^{p-2}u(x)$$

submitted to the Dirichlet boundary conditions (DBC). We are interested in the symmetry of solutions.

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When $V$ belongs to $L^{N/2}(\Omega)$, the solutions can be defined as the critical points of the energy functional 
$$
E_p : H^1_0(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2} \int_{\Omega} |\nabla u|^2 + Vu^2 - \frac{\lambda}{p} \int_{\Omega} |u|^p.
$$

Clearly, $0$ is solution. Concerning other solutions, if we assume that $-\Delta + V$ is positive definite and $V^- := \min(V, 0) \in L^{+\infty}(\Omega)$, then the norm $\|u\|^2 = \int_{\Omega} |\nabla u|^2 + Vu^2$ defined on $H^1_0(\Omega)$ is equivalent to the traditional norm $\|u\|^2_{H^1_0} = \int_{\Omega} |\nabla u|^2$ (see Proposition 2.1). By working in the same way as in [8], it directly implies the existence of ground state solutions (g.s.) and least energy nodal solutions (l.e.n.s.); i.e. one-signed (resp. sign-changing) solutions with minimal energy. These solutions are characterized as minima of $E_p$ respectively on the (resp. nodal) Nehari set

$$
\mathcal{N}_p := \left\{ u \in H^1_0(\Omega) \setminus \{0\} \mid \int_{\Omega} |\nabla u|^2 + Vu^2 = \lambda \int_{\Omega} |u|^p \right\}
$$

(resp. $\mathcal{M}_p := \{ u : u^\pm \in \mathcal{N}_p \}$). The Morse index is 1 (resp. 2).

In this paper, we study the structure of these two types of solutions. We verify whether they are odd or even with respect to the hyperplanes leaving $V$ invariant (i.e. $V$ respects an orthogonal symmetry with respect to the hyperplane). When it is the case, we say that the solution respects the symmetries of $V$. When $V \equiv 0$, this type of questions has already been studied. First, on the square in dimension 2, we can mention a result of Arioli and Koch (see [1]). They proved the existence of a positive symmetric $C^\infty$-function $w$ such that $-\Delta u = wu^3$ possesses a non-symmetric positive solution (with 1 as Morse index). The same kind of result has also been obtained for a solution with 2 as Morse index. The proof is partially computer-assisted. Second, in collaboration with Bonheure, Bouchez, Troestler and Van Schaftingen (see [2,3,7]), we proved for $p$ close to 2 that the symmetries are related to the symmetries of eigenfunctions of $-\Delta$. We generalize here the technique at some non-zero potentials $V$ and we make numerical experiments to illustrate it.

In Sect. 2, by denoting $\lambda_1$ (resp. $E_1$) the distinct eigenvalues (resp. eigenspaces) of $-\Delta + V$ with DBC in $H^1_0(\Omega)$, we prove the following Theorem 1.1. For this, we assume that $\lambda_1$ is the unique principal eigenvalue, i.e. an eigenvalue with a related eigenspace of dimension 1 possessing an one-signed eigenfunction. We also require that eigenfunctions in $E_2$ have a nodal line of measure 0. By using a maximum principle, these assumptions are satisfied at least when $V \in L^{+\infty}(\Omega)$ (see [6]).

**Theorem 1.1.** When $V \in L^{N/2}(\Omega), V^- \in L^{+\infty}(\Omega)$ and $-\Delta + V$ is positive definite such that $\lambda_1$ is the unique principal eigenvalue, for $p$ close to 2, the ground state (resp. least energy nodal) solutions respect the symmetries of their orthogonal projections in $H^1_0(\Omega)$ on $E_1$ (resp. $E_2$).

In particular, when the eigenspace has a dimension 1, the solutions respect the symmetries of $V$. As we assumed that $\lambda_1$ is the unique principal eigenvalue, ground state solutions respect the symmetries of $V$. 
Depending on the structure of \( E_2 \), some symmetry breaking exist for l.e.n.s. (see Sect. 2.3). In fact, by a traditional bootstrap, a family of ground state (resp. least energy nodal) solutions \( (u_p)_{p>2} \) converges for \( C \)-norm to functions in \( E_1 \) (resp. \( E_2 \)). So, for l.e.n.s., \( u_p \) does not respect the symmetries of \( V \) for \( p \) small when the projection is not symmetric in \( E_2 \) (see Sect. 3 for an example). For larger \( p \), it is depending on the case. In Sect. 2.3, we exhibit rectangles and \( V \) (such that the eigenfunctions in \( E_2 \) are symmetric) where l.e.n.s. do not respect symmetries of \( V \) for \( p \) large enough. So, the result 1.1 cannot be extended to all \( p \).

In Sect. 3, as illustration, we implement the (modified) mountain pass algorithm (see [4,9,10]) to study the cases of \( V \) negative constant, piecewise constant or singular. We exhibit direct examples such that the solutions break down the symmetries of \( V \).

2. Main results

The proofs are related to the technique defined in [3]. This is why we just point out and explain the differences and we do not make all the details. The first result implies that the traditional Poincaré’s and Sobolev’s inequalities are available for \( \|\cdot\|^2 := \int_{\Omega} |\nabla \cdot|^2 + V(\cdot)^2 \).

**Proposition 2.1.** If \( -\Delta + V \) is positive definite, \( V^+ \in L^{N/2}(\Omega) \) and \( V^- \in L^{+\infty}(\Omega) \), the norm \( \|u\|^2 := \int_{\Omega} |\nabla u|^2 + Vu^2 \) and the traditional norm \( \|u\|_{H^1_0}^2 : = \int_{\Omega} |\nabla u|^2 \) are equivalent.

**Proof.** Using the Sobolev’s inequalities on \( \int_{\Omega} Vu^2 \) and as \( V^- \in L^{+\infty}(\Omega), \exists C > 0 \) such that

\[
\|u\|^2 \leq \int_{\Omega} |\nabla u|^2 + C \int_{\Omega} |\nabla u|^2.
\]

Using the Poincaré’s inequalities and as \( V^- \in L^{+\infty}(\Omega), \exists C > 0 \) and a real \( K \) such that

\[
\|u\|^2 = \varepsilon \int_{\Omega} |\nabla u|^2 + (1 - \varepsilon)\|u\|^2 + \varepsilon \int_{\Omega} Vu^2
\]

\[
\geq \varepsilon \int_{\Omega} |\nabla u|^2 + ((1 - \varepsilon)C + \varepsilon K) \int_{\Omega} u^2 \geq \varepsilon \int_{\Omega} |\nabla u|^2,
\]

where the last inequality is obtained for \( \varepsilon \) small enough. \( \square \)

Then, the proof of Theorem 1.1 is based on two main results. The first one shows that, for \( p \approx 2 \), a priori bounded solutions can be distinguished by their projections on \( E_i \).

2.1. Abstract symmetry

**Lemma 2.2.** There exists \( \varepsilon > 0 \) such that if \( \|a(x) - \lambda_i\|_{L^{N/2}} < \varepsilon \) and \( u \) solves \(-\Delta u + Vu = a(x)u \) with DBC then \( u = 0 \) or \( P_{E_i}u \neq 0 \).

**Proof.** Similar as in Lemma 3.1 in [3], Poincaré’s and Sobolev’s inequalities are adapted using Proposition 2.1. \( \square \)
Then, we directly obtain our abstract symmetry result as in the proof of Proposition 3.2 in [3]. Let us remark that the result holds for any \( i \) and not just for \( i = 1 \) or 2 as stated in [3]. We denote by \( B(0, M) \) the ball in \( H^1_0(\Omega) \) centered at 0 and radius \( M \).

**Proposition 2.3.** Let \( M > 0 \). For \( i \in \mathbb{N}_0 \), \( \exists \tilde{p} > 2 \) such that, for \( p \in (2, \tilde{p}) \), if \( u_p, v_p \in \{ u \in B(0, M) : P_E, u \notin B(0, \frac{1}{M}) \} \) solve the boundary value problem with DBC \(-\Delta u + Vu = \lambda_i |u|^{p-2}u \) then \( P_{E_i} u_p = P_{E_i} v_p \) implies \( u_p = v_p \).

These two results permit us to conclude as in Theorem 3.6 in [3].

**Theorem 2.4.** Let \((G_\alpha)_{\alpha \in \mathcal{E}}\) with \( E = E_i \) be a group acting on \( H^1_0(\Omega) \) such that, for \( g \in G_\alpha \) and \( u \in H^1_0(\Omega) \),

\[
g(E) = E, \quad g(E^\perp) = E^\perp, \quad g\alpha = \alpha \quad \text{and} \quad \mathcal{E}_p(gu) = \mathcal{E}_p(u).
\]

For any \( M > 1, \exists \tilde{p} > 2 \) such that, for any family of solutions \((u_p)_{\tilde{p} > p > 2} \subseteq \{ u \in B(0, M) : P_E, u \notin B(0, \frac{1}{M}) \} \) of the boundary value problem with DBC \(-\Delta u + Vu = \lambda_i |u|^{p-2}u, u_p \) belongs to the invariant set of \( G_{\alpha_p} \) where \( \alpha_p \) is the orthogonal projection \( P_{E_p} u_p \).

Theorem 2.4 can be used for any bounded family of solutions staying away from 0. To apply Theorem 2.4 at a family \((u_p)_{p > 2} \) of ground state (resp. least energy nodal) solutions for the problem \((1)\), we study the asymptotic behavior when \( p \to 2 \). We prove that the expected upper and lower bounds are fine if and only if \( \lambda = \lambda_1 \) (resp. \( \lambda_2 \)). In some sense, \( \lambda_1 \) (resp. \( \lambda_2 \)) is the natural rescaling to work with ground state (resp. least energy nodal) solutions of problem \((1)\). Let us remark that this condition is not a restriction. Indeed, by homogeneity of \( |u|^{p-2}u \) in Eq. \((1)\), the symmetries of ground state (resp. least energy nodal) solutions are independent of \( \lambda \).

### 2.2. Asymptotic behavior

Let us denote \((u_p)_{p > 2} \) a family of ground state (resp. least energy nodal) solutions for the problem \((1)\). We consider \( \lambda = \lambda_1 \) for g.s. (resp. \( \lambda_n \) the first not principal eigenvalue for l.e.n.s.) and \( E = E_1 \) (resp. \( E_n \)).

**Lemma 2.5.** Concerning the upper bound, \( \limsup_{p \to 2} \|u_p\|^2 = \limsup_{p \to 2} \left( \mathcal{E}_p(u_p) \right) \left( \frac{1}{\frac{p}{2} - 1} \right)^2 \leq \|u_*\|^2 \) where \( u_* \in E \) minimizes the limit functional \( \mathcal{E}_* : E \to \mathbb{R} : u \mapsto \int_{\Omega} u^2 - \log u^2 \).

**Proof.** The proof is inspired by Lemma 4.1 in [3]. First, we define \( v_p := u_* + (p-2)w \) where \( w \in H^1_0(\Omega) \) solves the problem \(-\Delta w + Vw - \lambda_2 w = 2u_* \log |u_*| \) with \( P_{E_p} w = 0 \). Then, we prove that the projection of \( v_p \) on \( \mathcal{N}_p \) (resp. \( \mathcal{M}_p \)) converges when \( p \to 2 \).

Concerning l.e.n.s., in [3], the result has been stated for \( n = 2 \). Here, let us remark that it works with \( E_n \) which is not specially \( E_2 \). We just need to ensure that \( v_p \) is sign-changing for \( p \) close to 2.

Nevertheless, we need to assume \( n = 2 \) to obtain the lower bound. It is explained in the next Lemma.
Lemma 2.6. Concerning the lower bound, if \( n = 2 \) then \( \lim_{p \to 2} \| u_p \| > 0 \).

Proof. The proof is inspired by Lemma 4.4 in [3]. Concerning l.e.n.s. (the argument is easier for g.s.), let \( c_1 \) be a first eigenfunction in \( E_1 \). By considering

\[
\lambda^{-} := \frac{\int_{\Omega} u_p^+ c_1}{\| u_p^+ \|_{1,\Omega}}, \quad s_p^+ := 1 - s_p^{-},
\]

we show the existence of \( t_p > 0 \) such that \( v_p = t_p (s_p u_p^+ + s_p^{-} u_p^-) \) belongs to \( \mathcal{M}_p \cap E_1^+ \).

Then, we prove that \( v_p \) stays away from zero using Poincaré’s and Sobolev’s embeddings, which concludes the proof. For this part, we need to require that \( \lambda_1 \) is the unique principal eigenvalue, i.e. \( n = 2 \). Otherwise, we should prove that \( v_p \in (E_1 \oplus \cdots \oplus E_{n-1})^\perp \), which cannot be assumed. \( \square \)

The two previous results imply Theorem 2.7.

Theorem 2.7. Assume that \( -\Delta + V \) is positive definite and possesses one and only one principal eigenvalue (\( n = 2 \)), \( V^+ \in L^{N/2}(\Omega) \) and \( V^- \in L^{1+}\infty(\Omega) \). If \((u_p)_{p \geq 2}\) is a family of ground state (resp. least energy nodal) solutions for

\[
\text{Eq. (1)}
\]

then \( \exists C > 0 \) such that \( \| u_p \|_{H_0^1} \leq C (\frac{\lambda_1}{\lambda})^{\frac{1}{p-2}} \) for \( i = 1 \) (resp. 2). If \( p_n \to 2 \) and \( (\frac{\lambda_1}{\lambda})^{\frac{1}{p-2}} u_{p_n} \to u_* \) in \( H_0^1(\Omega) \), then \( (\frac{\lambda_1}{\lambda})^{\frac{1}{p-2}} u_{p_n} \to u_* \) in \( H_0^1(\Omega) \), \( u_* \) satisfies \( -\Delta u_* + V u_* = \lambda_1 u_* \) and \( \mathcal{E}_*(u_*) = \inf\{ \mathcal{E}_*(u) : u \in E_i \setminus \{0\}, \langle d\mathcal{E}_*(u), u \rangle = 0 \} \), where \( \mathcal{E}_*: E_i \to \mathbb{R} : u \mapsto \frac{\lambda_1}{2} \int_{\Omega} u^2 - u^2 \log u^2 \).

Remark 2.8. [(i)]

1. By a traditional bootstrap, \( (\frac{\lambda_1}{\lambda})^{\frac{1}{p-2}} u_{p_n} \to u_* \) for \( \mathcal{E}_- \)-norm (see [7]).
2. If \( \lambda < \lambda_1 \) (resp. \( \lambda_2 \)), a family of g.s. (resp. l.e.n.s.) blows up in \( H_0^1(\Omega) \). If \( \lambda > \lambda_1 \) (resp. \( \lambda_2 \)), it goes to 0. So, a family of g.s. (resp. l.e.n.s.) is bounded and stays away from \( 0 \) if and only if \( \lambda = \lambda_1 \) (resp. \( \lambda_2 \)).
3. By homogeneity of \( \lambda |u|^{p-2} u \), the study of symmetries for only one value of \( \lambda \) is enough to conclude symmetries for any \( \lambda > 0 \).
4. By combining Theorems 2.4 and 2.7, we obtain that ground state solutions for \( p \) close to 2 respect the symmetries of their projection on \( E_1 \). As first eigenfunctions are unique up to a constant, they keep symmetries of \( V \) for \( p \) close to 2.
5. By combining Theorems 2.4 and 2.7, we obtain that l.e.n.s. for \( p \) close to 2 respect the symmetries of their projection on \( E_2 \).

2.3. Symmetry breaking for least energy nodal solutions

For \( p \simeq 2 \), previous results showed that the structure of l.e.n.s. are related to the symmetries of \( u_* \) verifying \( \mathcal{E}_*(u_*) = \inf\{ \mathcal{E}_*(u) : u \in E_i \setminus \{0\}, \langle d\mathcal{E}_*(u), u \rangle = 0 \} \).

In [3] (see Section 6), on the square and for \( V \equiv 0 \), it is proved that if \( u_* \) does not respect the symmetries of the rectangle, i.e. \( u_* \) is not odd or even with respect to the medians (which is numerically observed), then there exists a symmetry breaking on rectangles sufficiently close to the square and \( p \) sufficiently large. In our case, this property can be stated as follows.

Theorem 2.9. Let us work on a square. If \( V \) is odd or even with respect to a median but \( u_* \) does not respect this symmetry, then there exist some rectangles and \( p \) such that l.e.n.s. \( u_p \) does not respect the symmetries of \( V \).
Moreover, as $u_p$ converges for $\mathcal{C}$-norm, we are able to directly construct $V$ such that the l.e.n.s. break down the symmetry of $V$. It will happen once $u_*$ is not symmetric. In the next section numerical experiments illustrate this interesting case.

3. Numerical illustrations: non-zero potentials $V$

In this section, we compute the (resp. modified) mountain pass algorithm to approach one-signed (resp. sign-changing) solutions (see [4,5,9,10]). While it is not sure that approximate solutions have least energy, all the other solutions that we have found numerically have a larger energy. So, we will assume that the approximations are ground state (resp. least energy nodal) solutions. We also give some level curves: 1 and 2 for g.s., $\pm 1$ and $\pm 2$ for l.e.n.s.

Numerically, we study $p = 4$. Let us remark that we always obtain the same kind of symmetry for smaller values of $p$. We work with $p = 4$ to illustrate that the result of Theorem 1.1 seems to hold at least for a non-negligible interval.

3.1. Negative constant potential on a square

As first example, we consider a constant $V$ such that $\lambda_1 > 0$, i.e. $V > -\tilde{\lambda}_1$ where $\tilde{\lambda}_1$ is the first eigenvalue of $-\Delta$. So, the required assumptions on $V$ are clearly satisfied. Theorem 1.1 holds. In particular, concerning symmetries, we obtain

(1) for $p$ close to 2, on convex domains, ground state solutions are even with respect to each hyperplane leaving $\Omega$ invariant;
(2) for $p$ close to 2, l.e.n.s. on a rectangle are even and odd with respect to a median;
(3) for $p$ close to 2, l.e.n.s. on radial domains are even with respect to $N - 1$ orthogonal directions and odd with respect to the orthogonal one;
(4) for $p$ close to 2, l.e.n.s. on a square are odd with respect to the barycenter.

Numerically, we consider $-\Delta u - \frac{\pi^2}{4} u = u^3$ defined on the square $\Omega = (-1,1)^2$ in $\mathbb{R}^2$. First and second eigenvalues of $-\Delta$ are given by $\frac{\pi^2}{2}$ and $\frac{5\pi^2}{4}$. On the following graph, one-signed (resp. nodal) numerical solutions have the expected symmetries. Ground state solutions respect the symmetries of the square and l.e.n.s. are odd with respect to the center 0. Moreover, the nodal line of the sign-changing solutions seems to be a diagonal, as for $V \equiv 0$ (see [3]).
• For g.s.: max($u$) = 2.18, $\mathcal{E}_4(u) = 2.54$
• For l.e.n.s.: min($u$) = −4.61, max($u$) = 4.61, $\mathcal{E}_4(u) = 33.21$
• Starting function for g.s.: $(x - 1)(y - 1)(x + 1)(y + 1)$
• Starting function for l.e.n.s.: $\sin(\pi(x + 1)) \sin(2\pi(y + 1))$

3.2. Piecewise constant potential on a rectangle
As second example, $V$ is piecewise constant on $(0, 2) \times (0, 1)$. In [6], it is proved that there exists just one principal eigenvalue. So, our assumptions are satisfied and Theorem 1.1 is available. For $\lambda = 1, p = 4$ and $V(x, y) = V_- := 0$ when $x < 1$ (resp. $V_+ := 10$ otherwise), following graphs indicate that approximations are just even with respect to a direction. Ground state solutions are more
or less “located” in $x < 1$ (the side minimizing energy) and respect symmetries
of $V$. l.e.n.s. seem to be formed by g.s. on each nodal domains and are even
with respect to a direction.

- For g.s.: $\max(u) = 5.98, \mathcal{E}_4(u) = 30.98$
- For l.e.n.s.: $\min(u) = -8.67, \max(u) = 6.53, \mathcal{E}_4(u) = 76.23$
- Starting function for g.s.: $(x - 2)(y - 1)xy$
- Starting function for l.e.n.s.: $\sin(\pi(x + 1))\sin(2\pi(y + 1))$

If $V_+ = 0$ and $V_- = 35$, we get the same symmetry for g.s. but l.e.n.s.
are not symmetric. The mass is more or less “located” in the square defined
by $x < 1$. So, we obtain a direct symmetry breaking. To minimize the energy,
the difference in the potential is so large that it is better to locate the mass
in one side of the rectangle. On a square and for $V = 0$, it is conjectured
that l.e.n.s. is odd with respect to a diagonal. It explains the structure of the
approximation.

3.3. A singular potential on a ball

As last example, we study singular potentials. First, $\lambda = 1, p = 4$ and
$V(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$ on the ball $B(0, 1)$. Approximations show as expected that
the ground state solutions are radial and the l.e.n.s. are odd and even with
respect to a diagonal. We obtain the same symmetry as for the potential $V = 0$
(see [3]).
For g.s.: \( \max(u) = 4.15, \mathcal{E}_4(u) = 29.9 \)

For l.e.n.s.: \( \min(u) = -6.36, \max(u) = 6.36, \mathcal{E}_4(u) = 76.04 \)

Starting function for g.s.: \( \cos(\pi(x^2 + y^2)^{0.5}/2) \)

Starting function for l.e.n.s.: \( \cos(\pi(x^2 + y^2)^{0.5}/2) \cos(2\pi(x^2 + y^2)^{0.5}) \cos(\pi(x^2 + y^2)^{0.5}) \)

Second, \( V(x, y) = \frac{1}{\sqrt{(x-0.5)^2+y^2}} \) on the ball \( B(0, 1) \). Ground state solutions seem to be even with respect to a direction but are not radial. One can remark the work of the singularity on the level curve. l.e.n.s. are just odd with respect to a direction. The mass is a little bit attracted by the side \( x < 0 \) (the side minimizing the energy).
• For g.s.: \( \max(u) = 4.41, \varepsilon_4(u) = 18.74 \)
• For l.e.n.s.: \( \min(u) = -6.25, \max(u) = 6.25, \varepsilon_4(u) = 76.23 \)
• Starting function for g.s.: \( \cos(\pi (x^2 + y^2)^{0.5}/2) \)
• Starting function for l.e.n.s.: \( \cos(\pi (x^2 + y^2)^{0.5}/2) \cos(2\pi (x^2 + y^2)^{0.5} \cos(\pi (x^2 + y^2)^{0.5}) \)

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