Auction design with ambiguity: 
Optimality of the first-price and all-pay auctions

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Abstract

We study the optimal auction design problem when bidders’ preferences follow the maxmin expected utility model. We suppose that each bidder’s set of priors consists of beliefs close to the seller’s belief, where “closeness” is defined by a divergence. For a given allocation rule, we identify a class of optimal transfer candidates, named the win-lose dependent transfers, with the following property: each type of bidder’s transfer conditional on winning or losing is independent of the competitor’s type report. Our result reduces the infinite-dimensional optimal transfer problem to a two-dimensional optimization problem. By solving the reduced problem, we find that: (i) among efficient mechanisms with no premiums for losers, the first-price auction is optimal; and, (ii) among efficient winner-favored mechanisms where each bidder pays smaller amounts when she wins than loses: the all-pay auction is optimal. Under a simplifying assumption, these two auctions remain optimal under the endogenous allocation rule.

Keywords: Auctions, mechanism design, ambiguity.

JEL Classification Numbers: D44, D81, D82.

*October 19, 2021. Corresponding author. The research of S.-H. H. was supported by the National Research Foundation of Korea.

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1. Introduction

We study the optimal auction design problem with ambiguity where bidders’ preferences follow the maxmin expected utility (MMEU) model (Gilboa and Schmeidler, 1989). In the MMEU model, the decision maker holds multiple priors on the state and maximizes the worst-case utility, the minimum expected utility over the set of priors. Despite a growing interest in auctions with ambiguity\(^1\), existing results in the popular setup of independent private values (IPV) auctions are somewhat negative. The optimal mechanism turns out to be a full insurance auction, where the seller provides premiums to losers so that each bidder’s payoff remains constant with respect to the competitor’s type report (Bose et al., 2006). However, the full insurance auction is rarely observed in reality. This gap between theory and practice poses a puzzle to researchers. In addition, the revenue ranking between the first-price and second-price auctions is sensitive to the choice of the bidders’ sets of priors (Bodoh-Creed, 2012).

In this paper, we ask a question rather different from the existing literature: under which circumstances does one of the standard auction formats, such as the first-price, second-price or all-pay auctions, become optimal? Our interest in these mechanisms is motivated by their being easily implementable since they do not require the designer’s specific knowledge, for example, the bidders’ beliefs about each other’s types (Wilson, 1987).\(^2\) We find that under plausible assumptions on the set of priors, (i) the first-price auction is optimal among efficient mechanisms in which the seller provides no premiums; and, (ii) the all-pay auction is optimal among efficient mechanisms in which each bidder pays smaller amounts when she wins than loses.

To derive our results, we first suppose that each bidder’s set of priors consists of beliefs close to some reference belief, interpreted as either the true proba-

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\(^1\)Lo (1998); Bose et al. (2006); Bose and Daripa (2009); Bodoh-Creed (2012); Chiesa et al. (2015); Laohakunakorn et al. (2019); Auster and Kellner (2020); Kocyigit et al. (2020); Ghosh and Liu (2021); see Section 6.1 for details.

\(^2\)Unlike these standard formats, the implementation of the full insurance auction requires the seller to know the bidders’ sets of priors, which depends on the specific environment.
bility law or the focal point distribution (Lo, 1998; Bose et al., 2006). We measure “closeness” between beliefs using a *divergence*—a coefficient of discrepancy between two probability distributions used in statistics and information theory (Kullback and Leibler, 1951; Kullback, 1959; Ali and Silvey, 1966). Our assumption on the set of priors is general enough to incorporate interesting examples in the existing literature (Example 1), yet specific enough to produce a definite answer to the optimal mechanism problem (Theorem 1 and Propositions 3-4).

Then, we study the problem of finding the revenue-maximizing transfer rule for a given allocation rule, called the *optimal transfer problem*. Besides the usual feasibility constraint (i.e., incentive compatibility and individual rationality), to identify environments where some standard format becomes optimal, we assume that the seller faces an additional constraint (Assumption T). Since this additional constraint can be chosen to be trivially satisfied, our setup also incorporates the standard setup with only the feasibility constraint.

Our first contribution lies in identifying a class of optimal transfer candidates (Theorem 1) with the following special properties: (i) each bidder’s transfer (and hence the payoff) conditional on winning or losing is independent of the competitor’s type report, and (ii) her winning payoff is higher than the losing payoff. We name the mechanism (or transfer) with this property as the *win-lose dependent* mechanism (or transfer) (Definition 3). Hence, the win-lose dependent mechanism generalizes the full insurance mechanism by allowing the winning and losing payoffs to differ. Moreover, for win-lose dependent mechanisms, the feasibility constraint simplifies to a formula for the bidder’s interim worst-case utility analogous to the standard envelope characterization in Bayesian mechanism design (Myerson, 1981). These results reduce the infinite-dimensional problem of choosing the transfer rule to a two-dimensional problem of determining each type of bidder’s winning and losing payoffs.

Theorem 1 can be regarded as a generalization of Bose et al.’s (2006) result on the optimality of the full insurance mechanisms. To prove Theorem 1, we establish the following principle: to an ambiguity averse decision maker, the con-
ditional expectation of a payoff schedule (with respect to coarser information) yields a higher worst-case utility than the original (Proposition 2). Thus, given any transfer, by offering its conditional expectation with respect to the bidders’ winning and losing events, the seller can create room to extract greater revenues. Exploiting this property along with characterizations of feasible mechanisms (Proposition 1), for any feasible transfer, we construct a win-lose dependent transfer which yields a greater revenue than the original while preserving incentive compatibility. Our result provides a novel insight that seemingly unrelated formats—the full insurance, first-price and all-pay auctions—share the common property of being win-lose dependent, which makes them superior over other formats such as the second-price auction and the war of attrition.

The second contribution of our paper is that using Theorem 1, we find two plausible classes of transfers within which the first-price and all-pay auctions become optimal, respectively: (i) the limited premium transfers (Section 4.1), and (ii) the winner-favored transfers (Section 4.2). Under limited premium transfers, the maximum premium provided to losers is limited by a certain amount. This class describes situations in which the seller provides only partial premiums to losers, as in real-world premium auctions such as Amsterdam auctions; see the related literature in Section 6.1.

Under winner-favored transfers, each bidder’s winning transfer is less than (or equal to) the losing transfer. This class interests us because it contains auctions whose revenues are frequently compared in the contest literature—the all-pay auction vs. the war of attrition (Krishna and Morgan, 1997; Hörisch, 2010; Bos, 2012), and the all-pay auction vs. the sad loser auction (Riley and Samuelson, 1981; Minchuk, 2018). In all three auctions, the loser pays exactly her bid (pos-

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3In other words, the MMEU preference exhibits monotonicity with respect to (a special kind of) the second-order stochastic dominance.

4Milgrom (2004); Goeree and Offerman (2004); Hu et al. (2011); Brunner et al. (2014); Hu et al. (2018).

5For instance, Krishna and Morgan (1997) show that the war of attrition revenue dominates the all-pay auction in the affiliated interdependent values environment (Milgrom and Weber, 1982). Also, Minchuk (2018) compares the all-pay and sad loser auctions with non-linear effort functions, and finds that the sad loser auction revenue dominates the all-pay auction when the
sibly because bids are sunk). On the other hand, the winner pays her bid in the all-pay auction, the second-highest bid in the war of attrition, and nothing in the sad loser auction. Thus, the relationship between the all-pay auction and the war of attrition is analogous to that between the first-price and second-price auctions. Also, the sad loser auction is obtained from the all-pay auction via the full reimbursement of the winner’s bid. In contests, the winner’s cost of efforts (bids) are often reimbursed to encourage contestants’ higher efforts; see the related literature in Section 6.1.

Within the first class of limited premium transfers, our results are as follows (Proposition 3 and Corollary 2; Figures 2 and 3). When the seller cannot provide premiums, the first-price auction is optimal among efficient mechanisms. This class includes many standard formats, for instance, the second-price and all-pay auctions. In contrast, when the seller can provide sufficiently large premiums, a full insurance auction is optimal, reproducing Bose et al.’s (2006) result. When the seller can provide some premiums but cannot fully insure all types, the optimal mechanism is a hybrid of the two mechanisms in which low types are fully insured, whereas high types are only partially insured.

Within the second class of winner-favored mechanisms, the all-pay auction is optimal among efficient mechanisms (Proposition 4 and Corollary 3; Figure 4). This result implies that within the class of auctions where the loser pays her own bid, the all pay auction revenue dominates any auctions (i) in which the winner pays any amount between the second-highest bid and her own bid, and (ii) in which the winner is partially or fully reimbursed for her bid.

The results explained thus far focus on finding the optimal transfer rule, taking the allocation rule as given. In Section 5, under a simplifying assumption on the divergence (Example 1 (b)), we endogenously determine the optimal allocation rule. With endogenous allocation, the first-price and all-pay auctions with suitable reserve prices remain optimal in the classes of no premium and

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6Kaplan et al. (2002); Cohen et al. (2008); Matros and Armanios (2009); Matros (2012); Minchuk and Sela (2020).
winner-favored mechanisms (Corollary 4). This suggests that the assumption of the exogenous allocation rule may be unessential for our results, and that the two formats have superior performances in a wide range of environments.

This paper is organized as follows. Section 2 introduces the setup. Section 3 presents and proves the optimality of the win-lose dependent transfers. In Section 4, we solve the optimal transfer problem within the classes of limited premium transfers and winner-favored transfers. Section 5 studies the endogenous optimal allocation rule. Section 6 discusses the related literature and concludes.

2. Model

2.1. Agents and preferences

We follow the standard setup in the literature on mechanism design with ambiguity averse agents (Bose et al., 2006; Bodoh-Creed, 2012; Wolitzky, 2016; Kocherlakota and Song, 2019). A seller wants to sell an indivisible object to two bidders. The assumption of two bidders is unessential; our results readily extend to the model with an arbitrary number of bidders. Each bidder has a privately known type \( \theta \in \Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R} \) representing her valuation for the object, where \( 0 < \underline{\theta} < \bar{\theta} \) and \( \Theta \) is equipped with the Borel \( \sigma \)-algebra \( B \). The bidders’ types are drawn independently.

Each bidder, unsure of the exact type distribution, holds a set of priors \( \Delta \subset \Delta(\Theta, \mathcal{B}) \) rather than a single prior about the competitor’s type, where \( \Delta(\Theta, \mathcal{A}) \) denotes the set of probability measures on a \( \sigma \)-algebra \( \mathcal{A} \). Following the MMEU model (Gilboa and Schmeidler, 1989), each bidder evaluates a payoff schedule\(^7\) (or often called an act in the literature) by its worst-case utility, defined as the minimum expected payoff over the set of priors. A belief that minimizes the expected payoff is called a worst-case belief.

The seller is ambiguity neutral, and hence has a single prior \( P \in \Delta(\Theta, \mathcal{B}) \) over each bidder’s type, assumed to be atomless. We call \( P \) the reference belief;

\(^7\)Throughout the paper, a payoff schedule is a bounded measurable function \( \pi : \Theta \to \mathbb{R} \); we interpret \( \pi(\theta) \) as the payoff when state \( \theta \) is realized.
this can be interpreted as either the true probability distribution of types (Lo, 1998) or a focal point around which bidders’ beliefs are perturbed (Bose et al., 2006). We discuss the ambiguity averse seller in the concluding section.

Each bidder’s set of priors $\Delta$ consists of probability measures close to the reference belief $P$, where the “closeness” between two beliefs is measured by a divergence. To explain the concept of divergence, we clarify some terminologies. By a $\sigma$-algebra, we always mean a sub-$\sigma$-algebra of the Borel $\sigma$-algebra $B$. Also, whenever we say “almost everywhere” (abbreviated as a.e.), we mean “almost everywhere with respect to $P$ (or the product measure $P^2$, depending on the context)”. Finally, given two $\sigma$-algebras $\mathcal{E} \subset \mathcal{A}$ and a probability measure $P \in \Delta(\Theta, \mathcal{A})$, the restriction of $P$ to $\mathcal{E}$ is denoted as $P_\mathcal{E} \in \Delta(\Theta, \mathcal{E})$.

A divergence $D(\cdot \mid \cdot)$ assigns a non-negative number (possibly infinite) to each pair of probability measures defined on the same $\sigma$-algebra. Given two probability measures $P$ and $Q$ on the same $\sigma$-algebra, we measure how far, or distinguishable, $Q$ is from $P$ by the divergence of $Q$ from $P$, $D(Q\mid\mid P)$. Following the literature on statistics and information theory (Kullback and Leibler, 1951; Kullback, 1959; Ali and Silvey, 1966), we require that the divergence satisfies the following basic properties:

**Assumption D** (Divergence). For every $\sigma$-algebra $\mathcal{A}$ and $P, Q \in \Delta(\Theta, \mathcal{A})$, we have:

**D1.** $D(Q\mid\mid P) = 0$ if $Q = P$.

**D2.** If $Q \ll P$ and $\frac{dQ}{dP}$ is bounded, $D(\epsilon Q + (1 - \epsilon)P\mid\mid P)$ is continuous in $\epsilon \in [0, 1]$.

**D3.** If $D(Q\mid\mid P) < \infty$, then $Q \ll P$.

**D4.** For a sub-$\sigma$-algebra $\mathcal{E}$ of $\mathcal{A}$, $D(Q_\mathcal{E}\mid\mid P_\mathcal{E}) \leq D(Q\mid\mid P)$.

**D5.** For a sub-$\sigma$-algebra $\mathcal{E}$ of $\mathcal{A}$, $D(Q_\mathcal{E}\mid\mid P_\mathcal{E}) = D(Q\mid\mid P)$ if $\frac{dQ_\mathcal{E}}{dP_\mathcal{E}} = \frac{dQ}{dP}$ a.e.

Properties **D1** (indistinguishability) and **D2** (continuity) are minimal requirements. **D3** says that if one can observe an event generated by $Q$ that never occurs under $P$ (i.e., $Q \nless P$), then $Q$ is perfectly distinguishable from $P$ (i.e.,

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8Formally, $D : \bigcup \Delta(\Theta, \mathcal{A}) \times \Delta(\Theta, \mathcal{A}) \to [0, \infty]$, with the union taken over all $\sigma$-algebras.

9Properties **D4** and **D5** are related to Kullback and Leibler’s (1951) Theorem 4.1, Kullback’s (1959) Corollary 3.2, and Ali and Silvey’s (1966) second property.
\(D(Q||P) = \infty\). **D4** means that under coarser information (a sub-\(\sigma\)-algebra), one is less able to distinguish between two probability measures. **D5** means that if the likelihood ratio \(\frac{dQ}{dP}\) is preserved under the coarser information, then the distinguishability between the two probability measures is also preserved. This is a natural requirement because the likelihood ratio \(\frac{dQ}{dP}\) is a sufficient statistic for distinguishing between \(P\) and \(Q\) (Cox and Hinkley, 1974, Chapter 2).

The bidder’s set of priors is given as the divergence neighborhood of the reference belief \(P\):

\[
\Delta = \{Q \in \Delta(\Theta, \mathcal{B}) : D(Q||P) \leq \eta\},
\]

where \(\eta > 0\) is the degree of ambiguity. We provide two examples of divergence; in Appendix A, we verify that they indeed satisfy Assumption D.

**Example 1.**

(a) The \(\phi\)-divergence and the relative entropy (Hansen and Sargent, 2001).

For a convex continuous function \(\phi : \mathbb{R}_+ \rightarrow \mathbb{R}\) with \(\phi(1) = 0\), define the \(\phi\)-divergence as follows (Ali and Silvey, 1966):

\[
D(Q||P) := \int_{\Theta} \phi \left(\frac{dQ}{dP}\right) dP \text{ for } Q \ll P
\]

and \(D(Q||P) = \infty\) otherwise. For example, when \(\phi(x) \equiv (x - 1)^2\), the \(\phi\)-divergence becomes the variance of the likelihood ratio \(\frac{dQ}{dP}\). More generally, the \(\phi\)-divergence measures the dispersion of the likelihood ratio \(\frac{dQ}{dP}\) by evaluating the expectation of a convex function of the likelihood ratio.

If \(\phi(\alpha) \equiv \alpha \log \alpha\),\(^{10}\) the \(\phi\)-divergence becomes the popular relative entropy, also known as the Kullback-Leibler divergence (Kullback and Leibler, 1951):

\[
D(Q||P) := \int_{\Theta} \frac{dQ}{dP} \log \frac{dQ}{dP} dP = \int_{\Theta} \log \frac{dQ}{dP} dQ \text{ for } Q \ll P.
\]

This divergence is used, among many others, in Hansen and Sargent (2001).

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\(^{10}\)When \(\alpha = 0\), we adopt the convention that \(0 \log 0 = \lim_{\alpha \rightarrow 0^+} \alpha \log \alpha = 0\).
(b) The contamination model (Bose et al., 2006).

Define the contamination divergence as follows:

\[ D(Q || P) := \text{ess sup}_P \left( 1 - \frac{dQ}{dP} \right) \quad \text{if } Q \ll P, \]

and \( D(Q || P) = \infty \) otherwise. Intuitively, this divergence measures the maximum downward deviation of the likelihood ratio \( \frac{dQ}{dP} \) from 1. In Appendix A, we show that this divergence generates the set of priors known as the contamination model (hence the name contamination divergence):

\[ \Delta = \{ Q \in \Delta(\Theta, \mathcal{B}) : Q = (1 - \eta)P + \eta R \text{ for some } R \in \Delta(\Theta, \mathcal{B}) \text{ with } R \ll P \}. \]

This model has been used extensively in the literature on mechanism design with ambiguity (e.g., Bose et al., 2006; Bose and Daripa, 2009; Auster, 2018). □

2.2. Mechanism design

The seller’s problem is to design a mechanism which maximizes the expected revenue under the reference belief \( P \). By the revelation principle,\(^{11}\) we can restrict our attention to direct mechanisms, defined as follows:

**Definition 1.** A direct mechanism \((x, t)\) consists of two bounded measurable functions, namely an allocation rule \( x = (x_1, x_2) : \Theta^2 \to \mathbb{R}^2 \) and a transfer rule \( t = (t_1, t_2) : \Theta^2 \to \mathbb{R}^2 \). The allocation rule \( x \) satisfies the following probability conditions: (i) \( x_1(\theta, \theta') \geq 0 \) and \( x_2(\theta', \theta) \geq 0 \), and (ii) \( x_1(\theta, \theta') + x_2(\theta', \theta) \leq 1 \) for \( \theta, \theta' \in \Theta \).

When bidder \( i \) reports \( \theta \) and her competitor reports \( \theta' \), bidder \( i \) wins the object with probability \( x_i(\theta, \theta') \) and pays \( t_i(\theta, \theta') \) to the seller.

\(^{11}\)Following most of the literature on mechanism design with ambiguity (Bose et al., 2006; Bodoh-Creed, 2012; Wolitzky, 2016; Kocherlakota and Song, 2019), we focus on static mechanisms; in dynamic mechanisms, due to dynamic inconsistency of the MMEU model, the equilibrium outcome may not be replicable by a static direct mechanism (Bose and Daripa, 2009). In this setup, assuming that bidders cannot hedge against ambiguity by randomization, the proof of the revelation principle in our setup is standard; see Wolitzky (2016) for details.
Given a mechanism \((x,t)\), we introduce the following notation:

\[
X_i(\theta) := \int_\Theta x_i(\theta, \theta') dP(\theta') \tag{1}
\]

\[
X_i^{\min}(\theta) := \inf_Q \{ \int_\Theta x_i(\theta, \theta') dQ(\theta') : D(Q||P) \leq \eta \} \tag{2}
\]

\[
U_i^{\min}(\theta) := \inf_Q \{ \int_\Theta [\theta x_i(\theta, \theta') - t_i(\theta, \theta')] dQ(\theta') : D(Q||P) \leq \eta \} \tag{3}
\]

\[
T_i(\theta) := \int_\Theta t_i(\theta, \theta') dP(\theta'). \tag{4}
\]

We call \(X_i(\theta)\) and \(X_i^{\min}(\theta)\) the reference and the minimum winning probability of bidder \(i\) with type \(\theta\), respectively. Also, we call \(U_i^{\min}(\theta)\) the bidder’s interim worst-case utility and \(T_i(\theta)\) the interim expected revenue. Likewise, the interim worst-case utility and revenue under \((x,\hat{t})\) are written as \(\hat{U}_i^{\min}(\theta)\) and \(\hat{T}_i(\theta)\).

Since the bidders’ preferences follow the MMEU model, the feasibility constraint (incentive compatibility and individual rationality) is as follows:

**Definition 2.**

(i) We say that \((x,t)\) is incentive compatible if, for every \(i\) and \(\theta\),

\[
\theta \in \arg\max_{\hat{\theta} \in \Theta} \inf_Q \{ \int_\Theta [\theta x_i(\hat{\theta}, \theta') - t_i(\hat{\theta}, \theta')] dQ(\theta') : D(Q||P) \leq \eta \}. \]

(ii) We say that \((x,t)\) is individually rational if, for every \(i\) and \(\theta\), \(U_i^{\min}(\theta) \geq 0\).

(iii) We say that \((x,t)\) is feasible if it is incentive compatible and individually rational.

Our study mainly focuses on the problem of finding the optimal transfer rule, taking an allocation rule \(x\) as exogenously given (Sections 3-4); the endogenous determination of the optimal allocation rule is studied in Section 5. We impose the following regularity conditions on the allocation rule:

**Assumption X.**

(i) For \(\theta' \neq \theta\), we have \(x_i(\theta, \theta') \in \{0,1\}\).

(ii) If \(X_i(\theta) = 1\), then \(\theta = \hat{\theta}\).

(iii) \(X_i^{\min}(\theta)\) is non-decreasing in \(\theta\).

Assumption X (i) means that the winner is chosen deterministically based on the reported type profile (except in the case of a tie). Assumption X (ii) means
that no type of bidder wins with certainty except for the highest possible type. Assumption X (iii), analogously to the usual monotonicity condition (Myerson, 1981), ensures that $x$ is implementable.$^{12}$

As mentioned in the introduction, our aim is to identify environments where a commonly used auction format—such as the first-price, second-price, or all-pay auction—becomes optimal. To do this, we consider two classes of transfer rules: (i) the limited premium transfers under which the premium provided to losers is limited by a certain amount (Section 4.1), and (ii) the winner-favored transfers under which each bidder pays smaller amounts when she wins than loses (Section 4.2). The following assumption on the class of transfer rules $\mathcal{T}$ incorporates these two classes as special cases:

**Assumption T.** Let $0 \leq \alpha \leq \beta \leq 1$ and $K \geq 0$ be constants. The class of transfer rules $\mathcal{T}$ is given as follows: $t \in \mathcal{T}$ if and only if for every $i$, $\theta < \bar{\theta}$, $\theta^w$ and $\theta^l$,

$$x_i(\theta, \theta^w) = 1 \text{ and } x_i(\theta, \theta^l) = 0 \implies \alpha t_i(\theta, \theta^w) - \beta t_i(\theta, \theta^l) \leq K.$$

The limited premium transfers correspond to $\alpha = 0$ and $\beta = 1$, and the winner-favored transfers correspond to $\alpha = \beta = 1$ and $K = 0$.

The seller’s problem, studied in subsequent sections, is given as follows.

**Optimal Transfer Problem (P).** Given an allocation rule $x$, solve

$$\sup_t \left\{ \sum_i \int_\Theta T_i(\theta)dP(\theta) : (i) \ (x, t) \text{ is feasible} \right\}.$$

Our setup extends the standard setup where the seller faces only constraint (i) in problem (P), since constraint (ii) is trivially satisfied when $\alpha = \beta = K = 0.$

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$^{12}$However, unlike in Myerson (1981), Assumption X (iii) is only sufficient, not necessary, for implementability.
3. Main Result

3.1. Statement

This section presents our main result. We first define the class of win-lose dependent mechanisms:

Definition 3. We say that \((x, t)\) is a win-lose dependent mechanism if there exist \(t^w_i, t^l_i : \Theta \rightarrow \mathbb{R}\) such that, for every \(i, \theta\) and \(\theta'\),

\[
t_i(\theta, \theta') = t^w_i(\theta)x_i(\theta, \theta') + t^l_i(\theta)[1 - x_i(\theta, \theta')] \quad (5)
\]

\[
\theta - t^w_i(\theta) \geq -t^l_i(\theta). \quad (6)
\]

If \((x, t)\) is a win-lose dependent mechanism, we say \(t\) is a win-lose dependent transfer.

Equation (5) means that each type of bidder’s transfers (and hence her payoffs) conditional on winning and losing are independent of the competitor’s type report. Specifically, bidder \(i\) with type \(\theta\) pays \(t^w_i(\theta)\) to the seller and receives a payoff of \(\theta - t^w_i(\theta)\) if she wins, whereas she pays \(t^l_i(\theta)\) and receives \(-t^l_i(\theta)\) if she loses. Inequality (6) means that the winning payoff is higher than the losing payoff, a common property of most auctions. If inequality (6) holds with equality, then \((x, t)\) becomes a full insurance mechanism (Bose et al., 2006) where each bidder’s payoff is constant with respect to the competitor’s type report. Hence, the win-lose dependent mechanism generalizes the full insurance mechanism by allowing the winning and losing payoffs to differ.

Our main result, Theorem 1, identifies a class of optimal transfer candidates satisfying two properties: (i) \(t\) is win-lose dependent, and (ii) a type \(\theta\) bidder’s interim worst-case utility equals the sum of the minimum winning probabilities of all types below \(\theta\).

Theorem 1. Suppose \(x\) is given, and Assumptions D, X and T hold. Consider the following problem:
**Reduced Problem (R).** Given an allocation rule $x$, solve

$$\sup_{t} \left\{ \sum_i \int_{\Theta} T_i(\theta) dP(\theta) : (i) \ t \text{ is win-lose dependent} \right\}$$

(i) $U^\text{min}_i(\theta) = \int_{\theta} X^\text{min}_i(z) dz$ for every $i$ and $\theta$

(ii) $t \in T$  \hspace{1cm} \text{(R)}

where $X^\text{min}_i(\theta)$, $U^\text{min}_i(\theta)$ and $T_i(\theta)$ are defined in equations (2)-(4). Then,

$$\text{solution set of problem (R)} \subset \text{solution set of problem (P)}. \hspace{1cm} (7)$$

**Proof.** See Section 3.2. \hfill \square

It can be shown that the inclusion relation (7) becomes an equality under suitable regularity conditions on the divergence.\footnote{This result is omitted because of space limitation, but its formal statement and proof are available upon request.} In this case, whenever the reduced problem (R) has a unique solution (which is the case in Sections 4.1-4.2), so does the optimal transfer problem (P).

Theorem 1 essentially reduces an infinite-dimensional problem (P) to a two-dimensional problem, which can be solved graphically (Figure 1). Concretely, the ex-ante formulation of the reduced problem (R) is equivalent to the interim formulation of choosing an interim transfer $t_i(\theta, \cdot)$ for each fixed $i$ and $\theta$. We now express the interim formulation in terms of a two-dimensional vector $(w, l)$, where $w$ and $l$ denote the winning and losing payoffs of bidder $i$ with type $\theta$. First, the interim expected revenue can be written as

$$T_i(\theta) = \theta X_i(\theta) - \int_{\Theta} \left[ \theta X_i(\theta, \theta') - t_i(\theta, \theta') \right] dP(\theta')$$

$$= \theta X_i(\theta) - [w X_i(\theta) + l(1 - X_i(\theta))]. \hspace{1cm} (8)$$
Equation (8) also shows that maximizing the revenue $T_i(\theta)$ is equivalent to minimizing the bidder’s share of the surplus $[wX_i(\theta) + l(1 - X_i(\theta))]$. Next, since $t$ is win-lose dependent, we have $w \geq l$ (see inequality (6)). This implies that the bidder’s expected payoff is minimized under a belief which minimizes the winning probability; hence, her interim worst-case utility becomes

$$U_i^{\text{min}}(\theta) = \inf_{Q} \{ wQ\{\theta' : x_i(\theta, \theta') = 1\} + lQ\{\theta' : x_i(\theta, \theta') = 0\} : D(Q||P) \leq \eta \} = wX_i^{\text{min}}(\theta) + l(1 - X_i^{\text{min}}(\theta)). \quad (9)$$

Finally, by Assumption $T$, the condition $t \in \mathcal{T}$ becomes $\alpha(\theta - w) + \beta l \leq K$ because the bidder’s winning and losing transfers are $\theta - w$ and $-l$.

Thus, the reduced problem (R) simplifies as follows:

**Reduced Problem-Interim (R-Int).** Let $x$ be given. For given $i$ and $\theta$, solve

$$\inf_{(w,l)} \{ wX_i(\theta) + l(1 - X_i(\theta)) : (i) \ w \geq l$$

$$\quad (ii) \ wX_i^{\text{min}}(\theta) + l(1 - X_i^{\text{min}}(\theta)) = U_0$$

$$\quad (iii) \ \alpha(\theta - w) + \beta l \leq K \}, \quad (\text{R-Int})$$

where $U_0 := \int_{\theta}^{\theta} X_i^{\text{min}}(z)dz$ is a constant (since $x$, $i$, and $\theta$ are fixed). Figure 1
illustrates problem (R-Int) on the \((w, l)\)-plane. In Section 4, by solving problem (R-Int), we derive the optimal transfer rule in two classes \(\mathcal{T}\).

3.2. Proof

In this section, we prove Theorem 1. Bose et al. (2006) derive the optimality of full insurance mechanisms by showing that, for a given feasible transfer, the full insurance transfer providing the bidders with the same interim worst-case utility as the given satisfies the feasibility constraint and generates a higher revenue than the original. This argument does not extend to our setup: a win-lose dependent transfer providing the bidders with the same interim worst-case utility as the given is not necessarily feasible. We overcome this difficulty by using an alternative two-step construction. First, we take the conditional expectation of the given transfer with respect to the bidders’ winning and losing events; the resulting transfer yields the same revenue as the given for the seller and provides higher interim worst-case utilities than the given for the bidders (Proposition 2), but needs not be feasible. Second, we adjust the transfer obtained in the first step so that it satisfies the feasibility constraint (Proposition 1) and generates a higher revenue than the original.

We first explain Proposition 1, which provides a necessary condition and a sufficient condition for a mechanism to be feasible. When bidders are ambiguity averse, the so-called “payoff equivalence theorem” (Myerson, 1981) no longer holds; i.e., the bidder’s interim worst-case utilities under two feasible mechanisms with the same allocation rule need not be equal (up to a constant). Thus, the bidder’s worst-case utility under a feasible mechanism cannot be expressed in terms of the allocation rule alone, but depends also on the transfer rule.\(^{14}\) Despite these difficulties, the following characterizations hold:

**Proposition 1** (Characterizations of feasible mechanisms).

\(^{14}\)Bodoh-Creed (2012, 2014) shows that under suitable regularity conditions, a type \(\theta\) bidder’s worst-case utility under a feasible mechanism equals the sum of the winning probabilities under worst-case beliefs (i.e., beliefs minimizing the bidders’ expected payoffs) of all types below \(\theta\). Since worst-case beliefs depend on the transfer rule, so does the worst-case utility.
Let $x$ be given. If Assumptions D and X hold, the following statements hold:

(i) (Bose et al., 2006) Suppose that $(x,t)$ is feasible. Then, for every $i$ and $\theta$,

$$ U_i^{\min}(\theta) \geq \int_{\theta}^{\theta} X_i^{\min}(z)dz. $$

(ii) Suppose that $\bar{t}$ is a win-lose dependent transfer such that for every $i$ and $\theta$,

$$ \bar{U}_i^{\min}(\theta) = \int_{\theta}^{\theta} X_i^{\min}(z)dz, \quad (10) $$

where $\bar{U}_i^{\min}(\theta)$ denotes the bidder’s interim utility under $(x, \bar{t})$. Then, $(x, \bar{t})$ is feasible.

**Proof.** See Appendix B.

Proposition 1 (i) states that a necessary condition for feasibility is that a type $\theta$ bidder’s interim worst-case utility exceeds a certain lower bound—the sum of the minimum winning probabilities of all types below $\theta$. This result has been proved by Bose et al. (2006); we provide the proof for completeness. Proposition 1 (ii) states that a sufficient condition for a win-lose dependent mechanism to be feasible is that each bidder’s interim worst-case utility achieves the lower bound in Proposition 1 (i). Intuitively, under a win-lose dependent mechanism, the bidder’s interim worst-case utility is evaluated under the belief minimizing her winning probability (see equation (9)); hence, the interim worst-case utility equals the sum of minimum winning probabilities $X_i^{\min}(z)$ (equation (10)). Thus, our result partially generalizes the standard sufficiency result (Myerson, 1981) in the sense that equation (10) reduces to the standard envelope formula when ambiguity is absent ($\eta = 0$).

Next, Proposition 2 states that if a payoff schedule $\bar{\pi}$ is less variable than another payoff schedule $\pi$—in the sense that $\bar{\pi}$ is the conditional expectation of $\pi$ with respect to coarser information—then $\bar{\pi}$ yields a higher worst-case utility than $\pi$. Since $\bar{\pi}$ second-order stochastically dominates $\pi$ in this case,\(^{15}\) Prope-

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\(^{15}\)To see this, suppose $\pi = \mathbb{E}_P[\pi|\mathcal{E}]$ for some $\sigma$-algebra $\mathcal{E}$. Let $\epsilon := \pi - \pi$. Then,

$$ \mathbb{E}_P[\epsilon|\pi] = \mathbb{E}_P[\mathbb{E}_P[\epsilon|\mathcal{E}]|\pi] = \mathbb{E}_P[\mathbb{E}_P[\pi|\mathcal{E}] - \pi|\pi] = 0. $$
sition 2 implies that ambiguity averse agents prefer payoff schedules that rank higher in a special kind of the second-order stochastic dominance (SOSD).\footnote{This shows that $\bar{\pi}$ is a mean-preserving contraction of $\pi$, or equivalently second-order stochastically dominates $\pi$.}

**Proposition 2** (Monotonicity with respect to SOSD). Suppose that Assumption D holds. Let $\pi : \Theta \to \mathbb{R}$ be a bounded measurable function. Given a $\sigma$-algebra $\mathcal{E}$, let

$$\bar{\pi} := \mathbb{E}_P[\pi|\mathcal{E}].$$

Then,

$$\inf_Q \{\mathbb{E}_Q[\bar{\pi}] : D(Q||P) \leq \eta\} \geq \inf_Q \{\mathbb{E}_Q[\pi] : D(Q||P) \leq \eta\}.$$

**Proof.** See Appendix B. \qed

The intuition of Proposition 2 is that the minimum expectation of $\bar{\pi}$ equals the minimum expectation of $\pi$ over the restricted set of “coarse” beliefs—whose likelihood ratios are measurable with respect to the coarser information $\mathcal{E}$—and therefore exceeds the unrestricted minimum. Specifically, given a belief $Q$ in the set of priors, define a coarse belief $\bar{Q}$ as $d\bar{Q} := \frac{dQ}{d\bar{P}} d\bar{P}$. Then, properties D4-D5 of Assumption D imply that the coarse belief $\bar{Q}$ is closer to $P$ than the original belief $Q$ is, and hence lies in the set of priors: $D(\bar{Q}||P) \leq D(Q||P) \leq \eta$. Hence

$$\mathbb{E}_Q[\bar{\pi}] = \mathbb{E}_Q[\pi] \geq \inf_Q \{\mathbb{E}_Q[\pi] : D(Q||P) \leq \eta\},$$

where the first equality follows from the basic properties of conditional expectation. Thus, Proposition 2 holds.

We now explore the mechanism design implication of Proposition 2. Corollary 1 states that holding each bidder’s interim worst-case utility above a lower bound $\int_{\theta}^{\theta} x_{i}^{\min}(z)dz$, the win-lose dependent mechanisms yield greater revenues than others.

\footnote{Under slightly stronger assumptions than Assumption D, Proposition 2 extends as follows: if $\bar{\pi}$ second-order stochastically dominates $\pi$, then $\bar{\pi}$ yields a higher worst-case utility than $\pi$. For our purpose, Proposition 2 suffices.}
Corollary 1. Given $x$, suppose Assumptions $D, X$ and $T$ hold. Consider $t$ such that

$$U_i^\text{min}(\theta) \geq \int_\theta^\theta X_i^\text{min}(z)dz \quad \text{and} \quad t \in T.$$ 

Then, there exists a win-lose dependent transfer rule $\hat{t}$ such that

$$\hat{T}_i(\theta) \geq T_i(\theta), \quad \hat{U}_i^\text{min}(\theta) = \int_\theta^\theta X_i^\text{min}(z)dz \quad \text{and} \quad \hat{t} \in T,$$

where $\hat{T}_i(\theta)$ and $\hat{U}_i^\text{min}(\theta)$ denote the interim variables under $(x, \hat{t})$.

Proof. See Appendix B. \qed

The intuition of Corollary 1 is as follows. Suppose that $\pi$ is the payoff schedule of bidder $i$ with type $\theta$ under $t$ (i.e., $\pi \equiv \theta x_i(\theta, \cdot) - t_i(\theta, \cdot)$). Let $\mathcal{E}$ be the $\sigma$-algebra generated by the bidder’s winning and losing events, and $\hat{\pi}$ be the conditional expectation of $\pi$. Note that $\mathbb{E}_P[\pi] = \mathbb{E}_P[\hat{\pi}]$ by the law of iterated expectations. Proposition 2 implies that if

$$\inf_Q \mathbb{E}_Q[\pi] \geq U_0 := \int_\theta^\theta X_i^\text{min}(z)dz,$$

then the following holds:

There exists $\mathcal{E}$-measurable $\pi$ s.t. \hspace{1cm} There exists $\mathcal{E}$-measurable $\pi$ s.t.

\begin{align*}
(i) \mathbb{E}_P[\pi] &= \mathbb{E}_P[\hat{\pi}] &\iff (i) \mathbb{E}_P[\pi] \leq \mathbb{E}_P[\hat{\pi}] \\
(ii) \inf_Q \mathbb{E}_Q[\pi] &\geq U_0 &\iff (ii) \inf_Q \mathbb{E}_Q[\hat{\pi}] = U_0,
\end{align*}

where the equivalence follows from the usual duality argument. Thus, we can construct a win-lose dependent transfer rule $\hat{t}$ which (i) extracts a higher revenue than $t$ for the seller, and (ii) guarantees the bidder with the lower bound for the interim worst-case utility:

\begin{align*}
(i) \quad &\hat{T}_i(\theta) \geq T_i(\theta) \iff \mathbb{E}_P[\pi] \leq \mathbb{E}_P[\hat{\pi}] \\
(ii) \quad &\hat{U}_i^\text{min}(\theta) = \int_\theta^\theta X_i^\text{min}(z)dz \iff \inf_Q \mathbb{E}_Q[\hat{\pi}] = U_0.
\end{align*}
Table 1: Comparison of transfers. FIA, FPA, SPA, APA, WoA, and SLA refer to the full insurance auction, first-price auction, second-price auction, all-pay auction, war of attrition and sad loser auction, respectively. Each type of bidder’s winning and losing transfers are denoted by \( t^w \) and \( t^l \), respectively.

**Proof of Theorem 1.** First, by Proposition 1 (ii), the constraint set of the reduced problem \((R)\) is contained in the constraint set of the optimal transfer problem \((P)\). Now, let \( t \) be a transfer rule in the constraint set of problem \((P)\). By Proposition 1 (i), \( t \) satisfies the hypothesis of Corollary 1. This implies that there exists a transfer rule \( \hat{t} \) in the constraint set of problem \((R)\) which yields a higher revenue than \( t \). Thus, Theorem 1 holds.

\[ \square \]

4. Optimal transfer rules: limited premium and winner-favored transfers

Theorem 1 simplifies the optimal transfer problem \((P)\) to the reduced problem \((R)\). Here, by solving the reduced problem \((R)\), we find the optimal transfer rule in two classes: (i) the limited premium transfers (Section 4.1) and (ii) the winner-favored transfers (Section 4.2).

4.1. Limited premium transfers

As mentioned in the introduction, in many standard formats such as the first-price and second-price auctions, bidders receive no premiums. Even when premiums are provided, their amounts are mostly limited and hence insufficient to fully insure bidders (Milgrom, 2004; Goeree and Offerman, 2004; Hu et al., 2011; Brunner et al., 2014; Hu et al., 2018; see Section 6.1). This motivates the following class of limited premium transfers:
**Definition 4.** Let $K \geq 0$. We say $(x, t)$ is a $K$-limited premium mechanism if the premium provided to the loser is limited by $K$: for every $i$, $\theta < \bar{\theta}, \theta^w$ and $\theta^l$,

\[
x_i(\theta, \theta^w) = 1 \text{ and } x_i(\theta, \theta^l) = 0 \implies t_i(\theta, \theta^l) \leq K.
\] (11)

If $(x, t)$ is a $K$-limited premium mechanism, we say $t$ is a $K$-limited premium transfer. Especially, if $K = 0$, we simply say a no premium mechanism (or transfer).

As mentioned, this class satisfies Assumption $T$ with $\alpha = 0$ and $\beta = 1$.

We solve the optimal transfer problem ($P$) in the class of $K$-limited premium transfers. The interim formulation of the reduced problem ($R$-Int) becomes:

**Reduced Problem, Limited Premium.** Let $x$ be given. For given $i$ and $\theta$, solve

\[
\begin{align*}
\inf_{(w, l)} \{ wX_i(\theta) + l(1 - X_i(\theta)) : & \ (i) \ w \geq l \\
& (ii) wX_i^{\min}(\theta) + l(1 - X_i^{\min}(\theta)) = U_0 \\
& (iii) l \leq K \},
\end{align*}
\] (12)

where $U_0 := \int_\theta^\bar{\theta} X_i^{\min}(z)dz$. Constraint (iii) corresponds to condition (11).

Figure 2 illustrates problem (12) and its solution $(w^*, l^*)$. We show that the solution is the point minimizing the difference between $w$ and $l$, i.e., the point closest to the 45° line. The ambiguity averse bidder underestimates the winning probability (i.e., $X_i^{\min}(\theta) < X_i(\theta)$) and overestimates the losing probability (i.e., $1 - X_i^{\min}(\theta) > 1 - X_i(\theta)$) relative to the ambiguity neutral seller. Hence, the ratio of the winning probability to the losing probability under the bidder’s worst-case belief—the relative effectiveness of $w$ in terms of $l$ in providing the bidder’s worst-case utility—is less than the corresponding ratio under the seller’s belief—the relative cost of $w$ in terms of $l$ to the seller:

\[
\frac{\text{relative effectiveness of } w \text{ in terms of } l}{\text{relative cost of } w \text{ in terms of } l} = \frac{X_i^{\min}(\theta)}{1 - X_i^{\min}(\theta)} < \frac{X_i(\theta)}{1 - X_i(\theta)} = \frac{\text{relative cost of } w \text{ in terms of } l}{\text{relative cost of } w \text{ in terms of } l}.
\] (13)

This means that holding the bidder’s worst-case utility fixed (constraint (ii)),

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the cost of substituting $w$ for $l$ exceeds the benefit; hence, the seller tries to reduce $w$ and raise $l$ as much as possible. Thus, the point with the minimum difference between $w$ and $l$ is optimal.

For instance, consider the case $K = 0$ (Panel A). Since the bidder receives no premiums, her losing payoff is at most 0; hence, $l^* = 0$ (point A). Next, suppose $K > 0$. For each $i$ and $\theta$ such that $U_0 := \int_0^{\theta} X_i^{\min}(z)dz \leq K$ (Panel B), the seller fully insures the bidder, i.e., $w^* = l^*$ (point B). In contrast, if $U_0 := \int_0^{\theta} X_i^{\min}(z)dz > K$ (Panel C), unable to provide full insurance, the seller provides the maximum available premium for the loser; hence, $l^* = K$ (point C). Thus, there exists a threshold type $\theta_i^K$ such that (i) bidder $i$ with type $\theta \leq \theta_i^K$ receives full insurance, whereas (ii) bidder $i$ with type $\theta > \theta_i^K$ receives only partial insurance, i.e., her winning payoff is strictly higher than her losing payoff.

As a result, the optimal transfer rule is given as follows:

**Proposition 3.** Suppose Assumption D holds. Let $x$ be an allocation rule satisfying
Assumptions \( X, \) and \( T \) be the class of \( K \)-limited premium transfers. Define \( t^* \) as

\[
t^*_{i} (\theta, \theta') := t^w_{i}(\theta)x_{i}(\theta, \theta') + t^l_{i}(\theta)[1 - x_{i}(\theta, \theta')],
\]

where \( t^w_{i}, t^l_{i} : \Theta \rightarrow \mathbb{R} \) are given as follows:\(^\text{17}\)

\[
(t^w_{i} (\theta), t^l_{i} (\theta)) := \begin{cases} 
(\theta - \int_{\theta}^{\theta'} X^\text{min}_i(z)dz, -\int_{\theta}^{\theta'} X^\text{min}_i(z)dz) & \text{if } \theta \leq \theta^K_i \\
(\theta - K - \frac{1}{X^\text{min}_i(\theta)}(\int_{\theta}^{\theta'} X^\text{min}_i(z)dz - K), -K) & \text{if } \theta > \theta^K_i,
\end{cases}
\]

and

\[
\theta^K_i := \sup \{ \theta : \int_{\theta}^{\theta'} X^\text{min}_i(z)dz \leq K \}. \tag{15}
\]

Then, \( t^* \) is the unique solution to the reduced problem \( (R) \), and hence a solution to the optimal transfer problem \( (P) \).

Of special importance is the case where \( x \) is the efficient allocation rule, which grants the object to the highest-valuation bidder (ties are randomly broken). Since \( (x, t^*) \) is symmetric in this case, we drop the subscript \( i \). Corollary 2 states that depending on \( K \), the first-price auction, a full insurance mechanism, and a hybrid between these formats become optimal (see Figure 3 and Table 2).

**Corollary 2.** In Proposition 3, suppose that \( x \) is the efficient allocation rule. Let

\[
\bar{K} := \int_{\theta}^{\theta'} X^\text{min}(z)dz.
\]

Then, the following statements hold:

(i) If \( K = 0 \), the first-price auction implements \( (x, t^*) \).

(ii) If \( K \geq \bar{K} \), \( (x, t^*) \) is a full insurance auction.

(iii) If \( 0 < K < \bar{K} \), \( (x, t^*) \) is a hybrid of the first-price and the full insurance auctions.

In the no premium case \( (K = 0) \), the optimal mechanism \( (x, t^*) \) has two properties: first, it is win-lose dependent; second, each bidder’s losing transfer

\(^\text{17}\)The fraction in equation (14) is well-defined because, by the definition of \( \theta^K_i \) and Assumption \( X \) (iii), we have \( X^\text{min}_i(\theta) > 0 \) whenever \( \theta > \theta^K_i \).
Figure 3: The bidder’s winning and losing payoffs under the optimal transfer in Corollary 2. Horizontal axes represent the type $\theta$. Starred lines represent the winning payoff $\theta - t^w(\theta)$, and circled lines represent the losing payoff $-t^l(\theta)$.

| $K = 0$ | Optimal transfer |
| --- | --- |
| $t^w(\theta) = \begin{cases} \theta & \text{if } \theta \leq \theta^K \\ \theta - \int_{\theta}^{\theta^K} X_{\min}(z) \, dz & \text{if } \theta > \theta^K \end{cases}$ | $\theta^K = \sup \{ \theta : X_{\min}(\theta) = 0 \}$ |
| $t^l(\theta) = 0$ |

| $K \geq \bar{K}$ | Optimal transfer |
| --- | --- |
| $t^w(\theta) = \theta - \int_{\theta}^{\theta^K} X_{\min}(z) \, dz$ | $\theta^K = \bar{\theta}$ |
| $t^l(\theta) = -\int_{\theta}^{\theta^K} X_{\min}(z) \, dz$ |

| $0 < K < \bar{K}$ | Equation (14) | Equation (15) |

Table 2: The optimal transfer in Corollary 2. Since $x$ is the efficient allocation rule, we have $X_{\min}(\theta) = \inf_Q \{ Q(\theta, \theta) : D(Q||P) \leq \eta \}$.

is zero. A feasible and efficient mechanism with these properties is precisely the first-price auction. Corollary 2 (ii) reproduces Bose et al.’s (2006) result as explained earlier. In addition, this result identifies the exact range of the maximum available premium $K$ for which the full insurance mechanism is optimal.

When the seller can provide some premiums but cannot fully insure all types ($0 < K < \bar{K}$), low types are fully insured ($\theta \leq \theta^K$) whereas high types are only partially insured ($\theta > \theta^K$). For high types, the winning and losing transfers (the second line of equation (14)) are identical to the equilibrium transfers of a hybrid indirect mechanism where (i) the bidders first compete in the first-price auction.

18Alternatively, one can directly verify that the winning transfer $t^w(\theta)$ in Table 2 is an equilibrium bidding strategy of the first-price auction (Lo, 1998).
auction with reserve price $\theta^K$, and (ii) then receive a premium of $K$ regardless of winning or losing. Therefore, the optimal mechanism can be regarded as a hybrid of the first-price auction and a full insurance mechanism.

Corollary 2 extends naturally to the case where the allocation rule excludes bidders with valuations below $r \geq 0$. In this case, if $K = 0$, the first-price auction with reserve price $r$ implements the optimal mechanism.

4.2. Winner-favored transfers

Winner-favored mechanisms are commonly observed in the context of contests where contestants’ bids are sunk. Typically, the loser pays her own bid, while the winner pays her own bid (the all-pay auction), the second-highest bid (the war of attrition; Krishna and Morgan, 1997), or less than her own bid with some reimbursement from the designer—in the extreme case of the full reimbursement, the winner pays nothing (the sad loser auction; Riley and Samuelson, 1981). In contests, the designer often provides reimbursements (or any “rewards” other than the auctioned object) to encourage higher bids (Kaplan et al., 2002; Cohen et al., 2008; Matros and Armanios, 2009; Matros, 2012; Minchuk and Sela, 2020; see Section 6.1).

**Definition 5.** We say $(x, t)$ is a winner-favored mechanism if each bidder pays smaller amounts to the seller when she wins than loses: for every $i$, $\theta < \theta^w$ and $\theta^l$,

$$x_i(\theta, \theta^w) = 1 \text{ and } x_i(\theta, \theta^l) = 0 \implies t_i(\theta, \theta^w) \leq t_i(\theta, \theta^l).$$

(16)

If $(x, t)$ is a winner-favored mechanism, we say $t$ is a winner-favored transfer.

As mentioned, this class satisfies Assumption $T$ with $\alpha = \beta = 1$ and $K = 0$.

We solve the optimal transfer problem ($P$) in the class of winner-favored transfers. The interim formulation of the reduced problem ($R$-Int) becomes:
Reduced Problem, Winner-Favored. Let \( x \) be given. For given \( i \) and \( \theta \), solve

\[
\inf_{(w,l)} \{ wX_i(\theta) + l(1 - X_i(\theta)) : (i) \ w \geq l \\
(ii) \ wX_i^{\min}(\theta) + l(1 - X_i^{\min}(\theta)) = U_0 \\
(iii) \ l \leq w - \theta \},
\]  

where \( U_0 := \int_0^\theta X_i^{\min}(z)dz \). Constraint (iii) corresponds to condition (16).

Figure 4 illustrates problem (17) and its solution \((w^*,l^*)\) (point \(A\)). The intuition for the solution is the same as in Section 4.1: points with smaller differences between \(w\) and \(l\) yield greater revenues for the seller. Under the constraint that the winning transfer (= \(\theta - w\)) is less than or equal to the losing transfer (= \(-l\)), the difference between \(w\) and \(l\) is minimized when the winning and losing transfers are equal; hence, \(l^* = w^* - \theta\).

**Proposition 4.** Suppose Assumption \(D\) holds. Let \( x \) be an allocation rule satisfying Assumptions \(X\), and \(\mathcal{T}\) be the class of winner-favored transfers. Define \(t^*\) as

\[
t^*_i(\theta,\theta') = t^w_i(\theta)x_i(\theta,\theta') + t^l_i(\theta)(1 - x_i(\theta,\theta')),
\]
where

\[ t^*_i(\theta) = t^i_1(\theta) = \theta X^\text{min}_i(\theta) - \int_\theta^\theta X^\text{min}_i(z) dz. \] (18)

Then, \( t^* \) is the unique solution to the reduced problem \((R)\), and hence a solution to the optimal transfer problem \((P)\).

Again, consider the case of the efficient allocation rule. The optimal mechanism \((x, t^*)\) satisfies two properties: first, it is win-lose dependent; second, each bidder pays the same amount whether she wins or loses. A feasible and efficient mechanism with these properties is precisely the all-pay auction.\(^{19}\)

**Corollary 3.** In Proposition 4, suppose further that \( x \) is the efficient allocation rule. Then, the all-pay auction implements \((x, t^*)\).

As mentioned in the introduction, Corollary 3 implies that within the class of auctions where the loser pays her own bid, the all-pay auction revenue dominates any auctions (i) in which the winner pays any amounts between the second-highest bid and her own bid, and (ii) in which the winner receives a partial or full reimbursement of her bid.

Corollary 3 also extends to the case where the allocation rule excludes bidders with valuations below \( r \geq 0 \). In this case, the optimal mechanism can be implemented by the all-pay auction with reserve price

\[ r \cdot \inf_Q \{ Q([\theta, r]) : D(Q||P) \leq \eta \}. \]

5. **Optimal allocation rules**

Sections 3-4 focus on finding the optimal transfer rule, taking the allocation rule as exogenously given. In this section, under a simplifying assumption that the set of priors is defined by the contamination divergence (Example 1), we endogenously determine the optimal allocation rule. We show that the main results in Section 4 remain valid under endogenous allocation: (i) in the class of

\(^{19}\)Alternatively, one can directly verify that expression (18) is the equilibrium bidding strategy in the all-pay auction (Stong, 2018).
no premium mechanisms, the first-price auction with a suitable reserve price is optimal; and, (ii) in the class of winner-favored mechanisms, the all-pay auction with a suitable reserve price is optimal (Corollary 4).

In the case of the contamination divergence, the worst-case utility is simply the weighted average of the expected payoff under $P$ and the lowest possible payoff: for a payoff schedule $\pi$,

$$\inf_Q \{ \mathbb{E}_Q[\pi] : D(Q||P) \leq \eta \} = (1 - \eta) \cdot \mathbb{E}_P[\pi] + \eta \cdot \text{ess inf}_P \pi.$$ 

Accordingly, the minimum winning probability simplifies to

$$X_{i}^{\text{min}}(\theta) = (1 - \eta)X_{i}(\theta) + \eta \cdot \text{ess inf}_P x_i(\theta, \cdot) = (1 - \eta)X_{i}(\theta) \text{ for } \theta < \bar{\theta}. \quad (19)$$

Now, we find the optimal mechanism in the class of no premium mechanisms, where the allocation rule is determined endogenously. For a given $x$, by Proposition 3 and equation (19), the optimal winning and losing transfers are

$$t_{i}^{w}(\theta) = \begin{cases} \theta & \text{if } X_{i}(\theta) = 0 \\ \theta - \int_{\theta}^{X_{i}(\theta)} X_{i}^{\text{min}}(z) \, dz = \theta - \int_{\theta}^{X_{i}(\theta)} X_{i}(z) \, dz & \text{if } X_{i}(\theta) > 0 \end{cases}$$

and $t_{i}^{l}(\theta) = 0$.

Hence, the optimal ex-ante expected revenue for a given $x$ is

$$\sum_{i} \int_{\Theta}[t_{i}^{w}(\theta)X_{i}(\theta) + t_{i}^{l}(\theta)(1 - X_{i}(\theta))]dP(\theta) = \sum_{i} \int_{\Theta}[\theta X_{i}(\theta) - \int_{\theta}^{X_{i}(\theta)} X_{i}(z)dz]dP(\theta).$$

To find the optimal allocation, we maximize this expression with respect to $x$:

**Optimal Allocation Problem (A), No Premium.**

$$\sup_{x} \{ \sum_{i} \int_{\Theta}[\theta X_{i}(\theta) - \int_{\theta}^{X_{i}(\theta)} X_{i}(z)dz]dP(\theta) : x \text{ satisfies Assumption X} \}. \quad (A)$$

We solve problem (A) as in Myerson (1981). Integrating by parts, we rewrite
the objective function as

$$\sum_i \int_\Theta [\theta X_i(\theta) - \int_\Theta X_i(z) dz] dP(\theta) = \sum_i \int_\Theta \left( \theta - \frac{1 - F(\theta)}{f(\theta)} \right) X_i(\theta) dP(\theta)$$

$$= \int_{\Theta^2} \left[ \left( \theta_1 - \frac{1 - F(\theta_1)}{f(\theta_1)} \right) x_1(\theta_1, \theta_2) + \left( \theta_2 - \frac{1 - F(\theta_2)}{f(\theta_2)} \right) x_2(\theta_2, \theta_1) \right] dP^2(\theta_1, \theta_2),$$

where $F$ and $f$ denote the cumulative distribution and probability density functions of $P$. The standard argument yields the following optimal allocation rule:

**Proposition 5.** Let $D$ be the contamination divergence. Suppose that $F$ is absolutely continuous, $f > 0$ on $\Theta$ and the function $\theta \mapsto \theta - \frac{1 - F(\theta)}{f(\theta)}$ is strictly increasing in $\theta$. Define an allocation rule $x^*$ as

$$x^*_i(\theta, \theta') := \begin{cases} 1 & \text{if } \theta > \theta' \text{ and } \theta \geq r^* \\ \frac{1}{2} & \text{if } \theta = \theta' \text{ and } \theta \geq r^* \\ 0 & \text{otherwise}, \end{cases}$$

where $r^* := \sup\{\theta : \theta - \frac{1 - F(\theta)}{f(\theta)} \leq 0\}$. Then, $x^*$ is the solution to the optimal allocation problem $(A)$. The optimal transfer rule corresponding to the optimal allocation rule $x^*$ in Proposition 5 can be implemented by the first-price auction with reserve price $r^*$ (see Section 4.1). A similar argument shows that the optimal mechanism in the class of winner-favored mechanisms can be implemented by the all-pay auction with a reserve price. Corollary 4 summarizes our results.

**Corollary 4.** Under the assumptions of Proposition 5, the following statements hold:

(i) The first-price auction with reserve price $r^*$ is optimal in the class of no premium mechanisms satisfying Assumption X.

(ii) The all-pay auction with reserve price $(1 - \eta)r^*F(r^*)$ is optimal in the class of winner-favored mechanisms satisfying Assumption X.
6. Discussion

6.1. Related literature

Auctions with ambiguity. Our paper primarily contributes to the literature on auctions with ambiguity. Early works study sealed-bid IPV auctions where bidders’ preferences follow the MMEU model (Lo, 1998; Bose et al., 2006; Bodoh-Creed, 2012). Recent papers examine alternative setups: dynamic mechanisms (Bose and Daripa, 2009; Auster and Kellner, 2020; Ghosh and Liu, 2021), ambiguity over the joint distribution of valuations and signals (Laohakunakorn et al., 2019), or alternative models of preferences under ambiguity (Chiesa et al., 2015; Kocyigit et al., 2020). Our paper studies the standard setup of sealed-bid IPV auctions with MMEU bidders which has been analyzed most extensively, but differs from existing works in that we focus on empirically plausible classes of transfers: the limited premium transfers (Section 4.1) and the winner-favored transfers (Section 4.2).

Information theory. Recently, concepts in information theory are being used in various economic contexts, such as robust control under model uncertainty in macroeconomics (Hansen and Sargent, 2001, 2008), rational inattention (Sims, 2003; Matejka and McKay, 2015) and Bayesian persuasion (Gentzkow and Kamenica, 2014; Treust and Tomala, 2019). Among them, the most closely related to our paper is the literature on robust control under model uncertainty; this literature studies decision processes maximizing the worst-case performance over a set of priors, given by the relative entropy neighborhood around the “approxi-

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20 Bose and Daripa (2009) show that if dynamic mechanisms are allowed, the seller can extract almost all surplus using a variant of the Dutch auction. Auster and Kellner (2020) find that under ambiguity, the strategic equivalence between the first-price auction and the Dutch auction breaks down, because bidders gradually learn about other bidders’ types in the latter. Ghosh and Liu (2021) studies sequential auctions and show that bidders’ ambiguity aversion can provide an explanation of the “declining price anomaly”. Laohakunakorn et al. (2019) study the two-bidder, binary-value and binary-signal setup where the joint distribution of valuations and signals is ambiguous. They find that with common values, the optimal mechanism provides only partial insurance to bidders, in contrast to Bose et al.’s (2006) finding. Some studies assume that bidders’ preferences follow Bewley’s (2002) model instead of the MMEU model, and show that the second-price auction is desirable in terms of both revenue and efficiency (Chiesa et al., 2015; Kocyigit et al., 2020).
mating model”.21 Our paper generalizes this set of priors by assuming the basic properties of the relative entropy (Assumption D) motivated from the statistics and information theory literature (Kullback and Leibler, 1951; Kullback, 1959; Ali and Silvey, 1966). We use these properties to show that an ambiguity averse agent prefers payoff schedules with less variation (Proposition 2).

**Limited premium mechanisms.** The class of limited premium mechanisms is related to the literature on premium auctions (Milgrom, 2004; Goeree and Offerman, 2004; Hu et al., 2011; Brunner et al., 2014; Hu et al., 2018). In practice, the premiums are typically limited and insufficient to provide the full insurance. For example, in Amsterdam auctions, bidders first compete in an ascending auction until all but two bidders drop out; then, the two finalists submit bids, and the loser receives a premium equal to 10-30% of the difference between her bid and the price at which the first stage was terminated (Goeree and Offerman, 2004). Our approach is in line with this literature in that we allow various amounts of maximum available premiums that the seller can provide.

**Winner favored mechanisms.** The class of winner-favored mechanisms is related to the literature on contests with reimbursements (Cohen et al., 2008; Matros and Armanios, 2009; Matros, 2012; Minchuk and Sela, 2020), in which contestants’ bids (or efforts) are sunk and the seller reimburses contestants’ bids to encourage higher bids. In many cases, only the winner receives the reimbursement, or more generally, the winner receives greater amounts of reimbursements than the loser. For instance, the Bush administration’s R&D tax policy provides the winner with tax credits—a form of reimbursement (Matros, 2012; Minchuk and Sela, 2020). As another example, in Canadian elections, candidates may be reimbursed for up to 25% of their expenses if the candidate receives 5% of valid votes in the electoral division. In these cases, each contestant’s bid net of the reimbursement—which corresponds to the transfer in our

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21 This set of priors is also popular in the literature on robust optimization in operations research (Ben-Tal et al., 2013; Duchi et al., 2021) and uncertainty quantification in applied mathematics (Gourgoulias et al., 2020; Dupuis et al., 2020).
setup—is smaller when she wins than loses.

6.2. Conclusion

This paper studies the optimal auction design problem where bidders’ preferences follow the MMEU model. Assuming that the set of priors consists of beliefs that are close—in terms of a divergence—to the reference belief, we identify a class of optimal transfer candidates satisfying two properties: first, the transfer rule is win-lose dependent; second, a type $\theta$ bidder’s interim worst-case utility equals the sum of minimum winning probabilities of all types below $\theta$. Using this result, we find the optimal transfer rule in two empirically plausible classes of transfers: (i) in the class of efficient no premium mechanisms, the first-price auction is optimal; and, (ii) in the class of efficient winner-favored mechanisms, the all-pay auction is optimal.

Our paper focuses on the ambiguity neutral seller. The difficulty in studying the ambiguity averse seller lies in extending Corollary 1. Our proof of Corollary 1 relies on the fact that given a transfer, its conditional expectation with respect to winning and losing events provides the bidders with higher interim worst-case utilities than the original, while leaving the seller’s revenue unchanged. When the seller is ambiguity averse, it is unclear whether the new transfer yields a higher or lower worst-case revenue for the seller than the original. We leave a closer investigation of this case to future research.

Appendix

A. Proofs for Section 2

We use the following well-known fact: if $\mathcal{E} \subseteq \mathcal{A}$ are $\sigma$-algebras and $P, Q \in \Delta(\Theta, \mathcal{A})$ satisfy $Q \ll P$, then

$$\frac{dQ_{\mathcal{E}}}{dP_{\mathcal{E}}} = \mathbb{E}_P \left[ \frac{dQ}{dP} \mid \mathcal{E} \right].$$

(A.1)

**Lemma A.1.** The $\phi$-divergence (Example 1) satisfies Assumption D.
Proof. As properties D1-D3 and D5 are clear, we verify D4. Let \( \mathcal{E} \subset \mathcal{A} \) be \( \sigma \)-algebras and \( P, Q \in \Delta(\Theta, \mathcal{A}) \). By equation (A.1),

\[
D(Q_{\mathcal{E}} || P_{\mathcal{E}}) = \int_\Theta \phi \left( \mathbb{E}_P \left[ \frac{dQ}{dP} \bigg| \mathcal{E} \right] \right) dP
\leq \int_\Theta \mathbb{E}_P \left[ \phi \left( \frac{dQ}{dP} \right) \bigg| \mathcal{E} \right] dP = \int_\Theta \phi \left( \frac{dQ}{dP} \right) dP = D(Q || P),
\]

where the second inequality holds by Jensen’s inequality for conditional expectations.

\( \square \)

**Lemma A.2.** The contamination divergence (Example 1) satisfies Assumption D. In addition, it generates the contamination model: for \( P \in \Delta(\Theta, \mathcal{B}) \),

\[ \{ Q : D(Q || P) \leq \eta \} = \{ Q : Q = (1 - \eta)P + \eta R \text{ for some } R \in \Delta(\Theta, \mathcal{B}), R \ll P \}. \]  

(A.2)

**Proof.** We first verify Assumption D. As properties D1-D3 and D5 are clear, we show D4. Let \( \mathcal{E} \subset \mathcal{A} \) be \( \sigma \)-algebras and \( P, Q \in \Delta(\Theta, \mathcal{A}) \). By equation (A.1),

\[
D(Q_{\mathcal{E}} || P_{\mathcal{E}}) = \text{ess sup}_P \left( 1 - \mathbb{E}_P \left[ \frac{dQ}{dP} \bigg| \mathcal{E} \right] \right) \leq \text{ess sup}_P \left( 1 - \frac{dQ}{dP} \right) = D(Q || P),
\]

where the second inequality holds because the conditional expectation of a function has a smaller maximum than the original.

Next, we prove equation (A.2). We show only that the left-hand side is contained in the right-hand side, as the reverse direction is straightforward. Suppose \( Q \) satisfies \( D(Q || P) \leq \eta \). Define \( R := (1/\eta)[Q - (1 - \eta)P] \). Rearranging yields \( Q = (1 - \eta)P + \eta R \). Also, \( R \in \Delta(\Theta, \mathcal{B}) \) because (i) \( R(\Theta) = (1/\eta)[1 - (1 - \eta)] = 1 \), and (ii) \( D(Q || P) \leq \eta \) implies that \( \frac{dR}{dP} = (1/\eta)[\frac{dQ}{dP} - (1 - \eta)] \geq 0 \). Hence, \( Q \) belongs to the right-hand side of equation (A.2).

\( \square \)
B. Proofs for Section 3.2

Proof of Proposition 1. (i) By incentive compatibility,

\[ U_i^{\min}(\theta) \geq \inf_{Q \in \Delta_1} \int_\Theta [\theta x_i(\hat{\theta}, \theta') - t_i(\hat{\theta}, \theta')]dQ(\theta') \]

\[ = \inf_{Q \in \Delta} \left[ \int_\Theta [\hat{\theta} x_i(\hat{\theta}, \theta') - t_i(\hat{\theta}, \theta')]dQ(\theta') + (\theta - \hat{\theta}) \int_\Theta x_i(\hat{\theta}, \theta')dQ(\theta') \right] \]

\[ \geq U_i^{\min}(\hat{\theta}) + \inf_{Q \in \Delta} (\theta - \hat{\theta}) \int_\Theta x_i(\hat{\theta}, \theta')dQ(\theta'), \quad (B.1) \]

where the last inequality holds because the minimum of the sum of two functions is greater than the sum of the minimums of two functions.

Now, suppose \( \theta > \hat{\theta} \). Then, inequality (B.1) becomes

\[ U_i^{\min}(\theta) \geq U_i^{\min}(\hat{\theta}) + (\theta - \hat{\theta}) X_{i}^{\min}(\hat{\theta}). \quad (B.2) \]

Interchanging the roles of \( \theta \) and \( \hat{\theta} \) in inequality (B.1) yields

\[ U_i^{\min}(\hat{\theta}) \geq U_i^{\min}(\theta) + \inf_{Q \in \Delta} (\theta - \hat{\theta}) \int_\Theta x_i(\hat{\theta}, \theta')dQ(\theta') = U_i^{\min}(\theta) + (\hat{\theta} - \theta) X_{i}^{\max}(\theta), \quad (B.3) \]

where the second equality holds because \( \theta > \hat{\theta} \). By inequalities (B.2)-(B.3),

\[ X_i^{\min}(\hat{\theta}) \leq \frac{U_i^{\min}(\theta) - U_i^{\min}(\hat{\theta})}{\theta - \hat{\theta}} \leq X_i^{\max}(\theta). \]

Thus, \( U_i^{\min} \) is Lipschitz and hence absolutely continuous. At a.e. \( \theta \), \( U_i^{\min} \) is differentiable and \( X_i^{\min} \) is continuous (by Assumption X (iii)). Taking \( \hat{\theta} \to \theta \),

\[ X_i^{\min}(\theta) \leq (U_i^{\min})'(\theta) \implies U_i^{\min}(\theta) \geq \int_\theta^\theta X_i^{\min}(z)dz. \]

(ii) As individual rationality is clear, we show only incentive compatibility. Let \( Q_{i,\theta}^* := \arg \min_{Q \in \Delta} \int_\Theta x_i(\theta, \theta')dQ(\theta') \) (Lemma B.2 shows its existence). Then,

\[ X_i^{\min}(\theta) = \int_\Theta x_i(\theta, \theta')dQ_{i,\theta}^*(\theta'). \quad (B.4) \]
Since \((x, t)\) is win-lose dependent, each type of bidder’s worst-case utility is evaluated under the belief minimizing the winning probability (equation (9)):

\[
U_i^{\text{min}}(\theta) = \int_{\Theta} [\theta x_i(\theta, \theta') - t_i(\theta, \theta')] dQ_i^*(\theta').
\]  \hspace{1cm} (B.5)

We establish incentive compatibility as follows:

\[
U_i^{\text{min}}(\theta) = U_i^{\text{min}}(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} X_i^{\text{min}}(z) dz \geq U_i^{\text{min}}(\hat{\theta}) + (\theta - \hat{\theta}) \int_{\hat{\theta}}^{\theta} x_i(\hat{\theta}, \theta') dQ_i^*(\theta')
\]

\[
= \int_{\Theta} [\theta x_i(\hat{\theta}, \theta') - t_i(\hat{\theta}, \theta')] dQ_i^*(\theta') + (\theta - \hat{\theta}) \int_{\hat{\theta}}^{\theta} x_i(\theta', \theta) dQ_i^*(\theta')
\]

\[
= \int_{\Theta} [\theta x_i(\hat{\theta}, \theta') - t_i(\hat{\theta}, \theta')] dQ_i^*(\theta') \geq \inf_{Q \in \Delta} \int_{\Theta} [\theta x_i(\hat{\theta}, \theta') - t_i(\hat{\theta}, \theta')] dQ(\theta')
\]

where the first equality follows by condition (10), the second inequality by Assumption X (iii) (this holds whether \(\hat{\theta} \leq \theta\) or \(\hat{\theta} \geq \theta\)), and the third equality by equations (B.4)-(B.5).

\[\Box\]

**Proof of Proposition 2.** We first prove the following: if \(g, h : \Theta \to \mathbb{R}\) are integrable functions such that \(gh\) is integrable, \(\bar{g} := \mathbb{E}_P[g|\mathcal{E}]\) and \(\bar{h} := \mathbb{E}_P[h|\mathcal{E}]\), then

\[
\mathbb{E}_P[\bar{g} \bar{h}] = \mathbb{E}_P[gh].
\]  \hspace{1cm} (B.6)

To see this, note that by the law of iterated expectations,

\[
\mathbb{E}_P[\bar{g} \bar{h}] = \mathbb{E}_P[\mathbb{E}_P[\bar{g} h|\mathcal{E}]] = \mathbb{E}_P[\bar{g} \cdot \mathbb{E}_P[h|\mathcal{E}]] = \mathbb{E}_P[\bar{g} \bar{h}].
\]

A similar argument yields \(\mathbb{E}_P[\bar{g} \bar{h}] = \mathbb{E}_P[\bar{g} \bar{h}]\). Thus, equation (B.6) holds.

To show Proposition 2, it suffices to prove the following equation:

\[
\inf_{Q} \{ \mathbb{E}_Q[\pi] : D(Q||P) \leq \eta \} = \inf_{Q} \{ \mathbb{E}_Q[\pi] : D(Q||P) \leq \eta, \frac{dQ}{dP} \text{ is } \mathcal{E}-\text{measurable} \}.
\]  \hspace{1cm} (B.7)

We first show that the left-hand side exceeds the right-hand side in equation
(B.7). Let \( Q \in \Delta(\Theta, \mathcal{B}) \) with \( D(Q\|P) \leq \eta \) be given. Define \( \bar{Q} \in \Delta(\Theta, \mathcal{B}) \) as
\[
d\bar{Q} := \frac{dQ_E}{dP} dP = \mathbb{E}_P[dQ_E|\mathcal{E}]dP,
\]
where the second equality holds by equation (A.1). By construction, the restrictions of \( \bar{Q} \) and \( Q \) to \( \mathcal{E} \) coincide, i.e., \( \bar{Q}_\mathcal{E} = Q_\mathcal{E} \). By D4-D5 of Assumption D, \( D(\bar{Q}\|P) = D(\bar{Q}_\mathcal{E}\|P_\mathcal{E}) = D(Q_\mathcal{E}\|P_\mathcal{E}) \leq D(Q\|P) \leq \eta \).

Thus, by equation (B.6), we obtain the desired inequality:
\[
\mathbb{E}_Q[\pi] = \mathbb{E}_P[\pi dQ] = \mathbb{E}_P[\pi d\bar{Q}] = \mathbb{E}_\bar{Q}[\pi] \geq \text{right-hand side of equation (B.7)}.
\]

It remains to prove the reverse inequality. Consider \( Q \in \Delta(\Theta, \mathcal{B}) \) such that \( D(Q\|P) \leq \eta \) and \( \frac{dQ}{dP} \) is \( \mathcal{E} \)-measurable. Since \( \mathbb{E}_P[\frac{dQ}{dP}|\mathcal{E}] = \frac{dQ}{dP} \), by equation (B.6),
\[
\mathbb{E}_Q[\pi] = \mathbb{E}_P[\pi \frac{dQ}{dP}] = \mathbb{E}_P[\pi dQ] = \mathbb{E}_Q[\pi] \geq \text{left-hand side of equation (B.7)}.
\]

\( \square \)

**Proof of Corollary 1.** To find \( \hat{t} \), for each \( i \) and \( \theta \), we construct an interim transfer
\[
\hat{t}_i(\theta, \cdot) = \hat{t}_w^i(\theta) x_i(\theta, \cdot) + \hat{t}_l^i(\theta)(1 - x_i(\theta, \cdot))
\]
satisfying the following properties:
\[
(i) \; \hat{T}_i(\theta) \geq T_i(\theta) \tag{B.8}
(ii) \; \hat{U}_i^{\min}(\theta) = \int_0^\theta X_i^{\min}(z)dz \tag{B.9}
(iii) \; \alpha \hat{t}_w^i(\theta) - \beta \hat{t}_l^i(\theta) \leq K \text{ if } \theta < \bar{\theta} \tag{B.10}
(iv) \; \theta - \hat{t}_w^i(\theta) \geq -\hat{t}_l^i(\theta). \tag{B.11}
\]

We divide into three cases: (A) \( 0 < X_i(\theta) < 1 \), (B) \( X_i(\theta) = 0 \) and (C) \( X_i(\theta) = 1 \).
Case A presents our main idea, while Cases B and C deal with boundary cases.

**Case A:** if \( 0 < X_i(\theta) < 1 \). We proceed in two steps \((t \to \bar{t} \to \hat{t})\).

**Case A-Step 1.** We find \( \bar{t}_i(\theta, \cdot) \) satisfying

\[
(i) \quad \bar{T}_i(\theta) \geq T_i(\theta) \tag{B.12}
\]

\[
(ii) \quad \bar{U}^\text{min}_i(\theta) \geq \int_\Theta x_i^\text{min}(z) dz \tag{B.13}
\]

\[
(iii) \quad \alpha \bar{t}_i(\theta) - \beta \tilde{t}_i(\theta) \leq K \tag{B.14}
\]

\[
(iv) \quad \theta - \bar{t}_i(\theta) \leq -\tilde{t}_i(\theta), \tag{B.15}
\]

where \( \bar{U}^\text{min}_i(\theta) \) and \( \bar{T}_i(\theta) \) denote the interim variables under \((x, \bar{t})\).

Define \( \bar{t}_w^\text{w}(\theta) \) and \( \bar{t}_l^\text{l}(\theta) \) as

\[
\bar{t}_w^\text{w}(\theta) = \int_{\{\theta^\text{w}: x_i(\theta, \theta^\text{w}) = 1\}} t_i(\theta, \theta^\text{w}) dP(\theta^\text{w}) X_i(\theta), \quad \bar{t}_l^\text{l}(\theta) = \int_{\{\theta^\text{l}: x_i(\theta, \theta^\text{l}) = 0\}} t_i(\theta, \theta^\text{l}) dP(\theta^\text{l}) 1 - X_i(\theta).
\]

Let \( E_{i,\theta} \) be the \( \sigma \)-algebra generated by the winning and losing events:

\[
E_{i,\theta} = \sigma(\{\theta^\text{w}: x_i(\theta, \theta^\text{w}) = 1\}, \{\theta^\text{l}: x_i(\theta, \theta^\text{l}) = 0\}).
\]

Then, \( \bar{t}_i(\theta, \cdot) \) is the conditional expectation of \( t_i(\theta, \cdot) \) with respect to \( E_{i,\theta} \):

\[
\bar{t}_i(\theta, \theta') = \mathbb{E}_P[t_i(\theta, \cdot)|E_{i,\theta}](\theta') \quad \text{a.e. } \theta'. \tag{B.16}
\]

Now, we verify that \( \bar{t}_i(\theta, \cdot) \) satisfies properties (B.12)-(B.15). The first property (B.12) follows directly from equation (B.16):

\[
\bar{T}_i(\theta) := \int_\Theta \bar{t}_i(\theta, \theta') dP(\theta') = \int_\Theta \mathbb{E}_P[t_i(\theta, \cdot)|E_{i,\theta}] dP(\theta') = \mathbb{E}_P[t_i(\theta, \cdot)] = T_i(\theta).
\]
The second property (B.13) follows by equation (B.16) and Proposition 2:

\[
\bar{U}_i^{\text{min}}(\theta) := \inf_Q \{ \int_\Theta [\theta x_i(\theta, \theta') - \bar{t}_i(\theta, \theta')] dQ(\theta') : D(Q||P) \leq \eta \}
\]

\[
\geq U_i^{\text{min}}(\theta) \geq \int_\theta^{\theta'} X_i^{\text{min}}(z) dz.
\]

We verify the third property (B.14) as follows. By Assumption T, for \(\theta^w\) and \(\theta^l\) with \(x_i(\theta, \theta^w) = 1\) and \(x_i(\theta, \theta^l) = 0\),

\[
\alpha t_i(\theta, \theta^w) - \beta t_i(\theta, \theta^l) \leq K \implies \alpha \sup_{\theta^w} t_i(\theta, \theta^w) - \beta \inf_{\theta^l} t_i(\theta, \theta^l) \leq K.
\]

Since \(\bar{t}_i^w(\theta) \leq \sup_{\theta^w} t_i(\theta, \theta^w)\) and \(\bar{t}_i^l(\theta) \geq \inf_{\theta^l} t_i(\theta, \theta^l)\), property (B.14) follows.

Finally, if the fourth property (B.15) holds, the construction is complete. Otherwise, we adjust \(\bar{t}_i(\theta, \cdot)\) using Lemma B.1 to obtain \(\tilde{t}_i(\theta, \cdot)\) which satisfies properties (B.12)-(B.15), and then proceed to Step 2 with \(\tilde{t}_i(\theta, \cdot)\) instead of \(\bar{t}_i(\theta, \cdot)\).

**Case A-Step 2.** Define \(\hat{t}_i^w(\theta)\) and \(\hat{t}_i^l(\theta)\) as

\[
\hat{t}_i^w(\theta) = \bar{t}_i^w(\theta) + [\bar{U}_i^{\text{min}}(\theta) - \int_\theta^{\theta'} X_i^{\text{min}}(z) dz]
\]

\[
\hat{t}_i^l(\theta) = \bar{t}_i^l(\theta) + [\bar{U}_i^{\text{min}}(\theta) - \int_\theta^{\theta'} X_i^{\text{min}}(z) dz].
\]

We verify that \(\hat{t}_i(\theta, \cdot)\) satisfies the desired properties (B.8)-(B.11).

The first property (B.8) holds because properties (B.12) and (B.13) imply that

\[
\hat{T}_i(\theta) = \bar{T}_i(\theta) + [\bar{U}_i^{\text{min}}(\theta) - \int_\theta^{\theta'} X_i^{\text{min}}(z) dz] \geq T_i(\theta).
\]

To prove the second property (B.9), note that

\[
\theta x_i(\theta, \theta') - \hat{t}_i(\theta, \theta') = [\theta x_i(\theta, \theta') - \bar{t}_i(\theta, \theta')] - [\bar{U}_i^{\text{min}}(\theta) - \int_\theta^{\theta'} X_i^{\text{min}}(z) dz].
\]

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It follows that
\[ \hat{U}^\text{min}_i(\theta) = \bar{U}^\text{min}_i(\theta) - \int_\theta^\theta X^\text{min}_i(z)dz = \int_\theta^\theta X^\text{min}_i(z)dz. \]

We verify the third property (B.10) as follows:
\[ a\hat{I}^w_i(\theta) - b\hat{I}^l_i(\theta) = a\bar{I}^w_i(\theta) - b\bar{I}^l_i(\theta) - (b - a)[\bar{U}^\text{min}_i(\theta) - \int_\theta^\theta X^\text{min}_i(z)dz] \leq K, \]
where the last inequality holds because \( \alpha \leq \beta \) (see Assumption T).

The final property (B.11) follows immediately from property (B.15).

Case B: if \( X_i(\theta) = 0 \). Let \( \hat{I}^w_i(\theta) = \hat{I}^l_i(\theta) := 0 \). First, property (B.8) holds because
\[ \hat{T}_i(\theta) = 0 = \theta X_i(\theta) = E_p[\theta x_i(\theta, \cdot) - t_i(\theta, \cdot)] + T_i(\theta) \geq U^\text{min}_i(\theta) + T_i(\theta) \geq T_i(\theta). \]

Next, we verify property (B.9). Since \( X_i(\theta) = 0 \) implies \( X^\text{min}_i(\theta) = 0 \),
\[ \hat{U}^\text{min}_i(\theta) = (\theta - \hat{I}^w_i(\theta)) \cdot X^\text{min}_i(\theta) + (-\hat{I}^l_i(\theta)) \cdot (1 - X^\text{min}_i(\theta)) = \theta X^\text{min}_i(\theta) = 0. \]

By Assumption X (iii), \( X^\text{min}_i(z) = 0 \) for \( z \leq \theta \). Hence, \( \hat{U}^\text{min}_i(\theta) = 0 = \int_\theta^\theta X^\text{min}_i(z)dz. \)

The remaining properties (B.10) and (B.11) hold by construction.

Case C: if \( X_i(\theta) = 1 \). Let \( \hat{I}^w_i(\theta) := \theta - \int_\theta^\theta X^\text{min}_i(z)dz \) and \( \hat{I}^l_i(\theta) := -\int_\theta^\theta X^\text{min}_i(z)dz. \)

Then
\[ \theta x_i(\theta, \theta') - \hat{t}_i(\theta, \theta') = \int_\theta^\theta X^\text{min}_i(z)dz \quad \text{for every } \theta'. \quad \text{(B.17)} \]

To verify the first property (B.8), observe the following:
\[ \theta X_i(\theta) - \hat{T}_i(\theta) = \int_\theta^\theta X^\text{min}_i(z)dz \quad \text{(B.18)} \]
\[ \theta X_i(\theta) \geq U^\text{min}_i(\theta) + T_i(\theta) \geq \int_\theta^\theta X^\text{min}_i(z)dz + T_i(\theta), \quad \text{(B.19)} \]
where equation (B.18) follows from equation (B.17). Comparing equation (B.18) with inequality (B.19) yields \( \hat{T}_i(\theta) \geq T_i(\theta). \)

The second property (B.9) follows by equation (B.17). The third property
(B.10) holds automatically, since $\theta = \bar{\theta}$ by Assumption X (ii). The final property (B.11) holds by construction.

\[\square\]

**Lemma B.1.** Suppose that the assumptions of Corollary 1 hold. Fix $i$ and $\theta$ such that $0 < X_i(\theta) < 1$. Consider an interim transfer

\[ \bar{I}_i(\theta, \cdot) := I_i^u(\theta) x_i(\theta, \cdot) + I_i^l(\theta) (1 - x_i(\theta, \cdot)) \]

satisfying $\alpha I_i^u(\theta) - \beta I_i^l(\theta) \leq K$ and $\theta - I_i^u(\theta) < -I_i^l(\theta)$. Then, there exists

\[ \widetilde{I}_i(\theta, \cdot) := \tilde{I}_i^u(\theta) x_i(\theta, \cdot) + \tilde{I}_i^l(\theta) (1 - x_i(\theta, \cdot)) \]

satisfying the following properties:

\begin{align*}
(i) \quad & \bar{T}_i(\theta) \geq \tilde{T}_i(\theta) \quad \text{(B.20)} \\
(ii) \quad & \bar{U}_i^{\text{min}}(\theta) = \bar{U}_i^{\text{min}}(\theta) \quad \text{(B.21)} \\
(iii) \quad & \alpha \tilde{I}_i^u(\theta) - \beta \tilde{I}_i^l(\theta) \leq K \quad \text{(B.22)} \\
(iv) \quad & \theta - \tilde{I}_i^u(\theta) = -\tilde{I}_i^l(\theta), \quad \text{(B.23)}
\end{align*}

where $\bar{U}_i^{\text{min}}(\theta)$ and $\bar{T}_i(\theta)$ denote the interim variables under $(x, \bar{t})$.

**Proof.** Let $\tilde{I}_i^u(\theta) := \theta - \bar{U}_i^{\text{min}}(\theta)$ and $\tilde{I}_i^l(\theta) := -\bar{U}_i^{\text{min}}(\theta)$. A similar argument as in Case C in the proof of Corollary 1 yields properties (B.20), (B.21) and (B.23). To verify the remaining property (B.22), note that

\[ \theta - \tilde{I}_i^u(\theta) = \bar{U}_i^{\text{min}}(\theta) = \bar{U}_i^{\text{min}}(\theta) \geq \theta - \bar{I}_i^u(\theta), \]

where the first equality follows from $\theta - \tilde{I}_i^u(\theta) = -\tilde{I}_i^l(\theta)$, the second equality from property (B.21), and the third inequality because $\theta - \tilde{I}_i^u(\theta) < -\tilde{I}_i^l(\theta)$. It follows that $\tilde{I}_i^u(\theta) \leq \tilde{I}_i^u(\theta)$. A similar argument yields $\tilde{I}_i^l(\theta) \geq \tilde{I}_i^l(\theta)$. Hence,

\[ \alpha \tilde{I}_i^u(\theta) - \beta \tilde{I}_i^l(\theta) \leq \alpha \bar{I}_i^u(\theta) - \beta \bar{I}_i^l(\theta) \leq K. \]
Lemma B.2. Suppose that Assumption D holds. Given a Borel set $E$, let

$$p_{\text{min}} := \inf_{Q} \{ Q(E) : D(Q \| P) \leq \eta \}. \quad (B.24)$$

Define $Q^*$ as $\frac{dQ^*}{dP} = \frac{p_{\text{min}}}{p} \cdot 1_E + \frac{1-p_{\text{min}}}{1-p} \cdot 1_{\Theta \setminus E}$, where $p := P(E)$. Then, $Q^*$ is a minimizer of problem (B.24).

Proof. Plugging $\pi \equiv 1_E$ into equation (B.7) in the proof of Proposition 2 yields

$$\inf_{Q} \{ Q(E) : D(Q \| P) \leq \eta \} = \inf_{Q} \{ Q(E) : D(Q \| P) \leq \eta, \frac{dQ}{dP} \text{ is } \mathcal{E} \text{-measurable} \} \quad (B.25)$$

where $\mathcal{E}$ denotes the $\sigma$-algebra generated by $E$. For $q \in [0,1]$, there exists a unique $Q_q \in \Delta(\Theta, \mathcal{B})$ such that $\frac{dQ_q}{dP}$ is $\mathcal{E}$-measurable and $Q_q(E) = q$, defined as $
abla \frac{dQ_q}{dP} := \frac{q}{p} \cdot 1_E + \frac{1-q}{1-p} \cdot 1_{\Theta \setminus E}$. Hence, equation (B.25) can be rewritten as

$$\inf_{Q} \{ Q(E) : D(Q \| P) \leq \eta \} = \inf_{q \in [0,1]} \{ q : D(Q_q \| P) \leq \eta \}. \quad (B.26)$$

By D2 of Assumption D, the right-hand side of equation (B.26) has a minimizer, which must be $p_{\text{min}}$. Thus, $Q^* = Q_{p_{\text{min}}}$ is a minimizer to problem (B.24).

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