Lifespan of Solutions to MHD Boundary Layer Equations with Analytic Perturbation of General Shear Flow

Feng XIE\textsuperscript{1,}, Tong YANG\textsuperscript{2,3}

\textsuperscript{1}School of Mathematical Sciences, and LSC-MOE, Shanghai Jiao Tong University, Shanghai 200240, China
(E-mail: txxief@sjtu.edu.cn)

\textsuperscript{2}Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong
(E-mail: matyang@cityu.edu.hk)

\textsuperscript{3}School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China

This paper is dedicated to Professor Philippe G. Ciarlet on the occasion of his 80th birthday.

Abstract  In this paper, we consider the lifespan of solution to the MHD boundary layer system as an analytic perturbation of general shear flow. By using the cancellation mechanism in the system observed in [12], the lifespan of solution is shown to have a lower bound in the order of \(\varepsilon^{-2}\) if the strength of the perturbation is of the order of \(\varepsilon\). Since there is no restriction on the strength of the shear flow and the lifespan estimate is larger than the one obtained for the classical Prandtl system in this setting, it reveals the stabilizing effect of the magnetic field on the electrically conducting fluid near the boundary.

Keywords  MHD boundary layer, analytic perturbation, lifespan estimate, shear flow

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1 Introduction

Consider the high Reynolds number limit to the MHD system near a no-slip boundary, the following MHD boundary layer system was derived in [12] when both of the Reynolds number and the magnetic Reynolds number have the same order in two space dimensions. Precisely, the MHD boundary layer system is in the domain \(\{ (x, Y) | x \in \mathbb{R}, Y \in \mathbb{R}_+ \}\) with \(Y = 0\) being the boundary,

\[
\begin{align*}
\partial_t u^\varepsilon + (u^\varepsilon \partial_x + v^\varepsilon \partial_Y)u^\varepsilon + \partial_x p^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_Y)h^\varepsilon &= \varepsilon (\partial_x^2 u^\varepsilon + \partial_Y^2 u^\varepsilon), \\
\partial_t v^\varepsilon + (u^\varepsilon \partial_x + v^\varepsilon \partial_Y)v^\varepsilon + \partial_Y p^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_Y)g^\varepsilon &= \varepsilon (\partial_x^2 v^\varepsilon + \partial_Y^2 v^\varepsilon), \\
\partial_t h^\varepsilon + (u^\varepsilon \partial_x + v^\varepsilon \partial_Y)h^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_Y)u^\varepsilon &= \kappa \varepsilon (\partial_x^2 h^\varepsilon + \partial_Y^2 h^\varepsilon), \\
\partial_t g^\varepsilon + (u^\varepsilon \partial_x + v^\varepsilon \partial_Y)g^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_Y)h^\varepsilon &= \kappa \varepsilon (\partial_x^2 g^\varepsilon + \partial_Y^2 g^\varepsilon), \\
\partial_x u^\varepsilon + \partial_Y v^\varepsilon &= 0,
\end{align*}
\tag{1.1}
\]

where both the viscosity and resistivity coefficients are denoted by a small positive parameter \(\varepsilon\), \((u^\varepsilon, v^\varepsilon)\) and \((h^\varepsilon, g^\varepsilon)\) represent the velocity and the magnetic field respectively. The no-slip boundary condition is imposed on the velocity field

\[
(u^\varepsilon, v^\varepsilon)|_{Y=0} = 0,
\tag{1.2}
\]

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and the perfectly conducting boundary condition is given for the magnetic field

\[ (\partial_Y h^e, g^e)|_{Y=0} = 0. \]  

(1.3)

Formally, when \( \varepsilon = 0 \), (1.1) is reduced into the following incompressible ideal MHD system

\[
\begin{aligned}
\partial_t u^0_c + (u^0_c \partial_x + v^0_e \partial_y) u^0_c + \partial_x p^0_c = \frac{1}{2} (h^0_e \partial_x + g^0_e \partial_y) h^0_c = 0, \\
\partial_t v^0_e + (u^0_c \partial_x + v^0_e \partial_y) v^0_e + \partial_y p^0_e = \frac{1}{2} (h^0_e \partial_x + g^0_e \partial_y) g^0_e = 0, \\
\partial_t h^0_e + (u^0_c \partial_x + v^0_e \partial_y) h^0_e - \frac{1}{2} (h^0_e \partial_x + g^0_e \partial_y) u^0_c = 0, \\
\partial_t g^0_e + (u^0_c \partial_x + v^0_e \partial_y) g^0_e - \frac{1}{2} (h^0_e \partial_x + g^0_e \partial_y) v^0_e = 0, \\
\partial_x u^0_c + \partial_y v^0_e = 0, \quad \partial_x h^0_e + \partial_y g^0_e = 0.
\end{aligned}
\]

(1.4)

Since the solvability of the system (1.4) requires only the normal components of the velocity and magnetic fields \((v^0_c, g^0_e)\) on the boundary

\[ (v^0_c, g^0_e)|_{Y=0} = 0, \]

(1.5)

in the limit from (1.1) to (1.4), a Prandtl-type boundary layer can be derived to resolve the mismatch of the tangential components between the viscous flow \((u^\varepsilon, h^\varepsilon)\) and inviscid flow \((u^0, h^0)\) on the boundary \(\{Y = 0\}\). And this system governing the fluid behavior in the leading order of approximation near the boundary is derived in [7,12,13]:

\[
\begin{aligned}
\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 &= b_1 \partial_x b_1 + b_2 \partial_y b_1 + \partial^2_{xx} u_1, \\
\partial_t b_1 + \partial_y (u_2 b_1 - u_1 b_2) &= \kappa \partial^2_{yy} b_1, \\
\partial_x u_1 + \partial_y u_2 &= 0, \quad \partial_x b_1 + \partial_y b_2 = 0
\end{aligned}
\]

(1.6)

in \( \mathbb{H} = \{(x, y) \in \mathbb{R}^2 | y \geq 0\} \) with the fast variable \( y = Y/\sqrt{\varepsilon} \). Here, the trace of the horizontal ideal MHD flow (1.4) on the boundary \( \{Y = 0\} \) is assumed to be a constant vector so that the pressure term \( \partial_x p^0_e(t, x, 0) \) vanishes by the Bernoulli’s law.

Consider the system (1.6) with initial data

\[ u_1(t, x, y)|_{t=0} = u_0(x, y), \quad b_1(t, x, y)|_{t=0} = b_0(x, y), \]

(1.7)

and the boundary conditions

\[
\begin{aligned}
|_{y=0} u_1 &= u_0(x, t, 0) \equiv \mathbf{u}, \\
|_{y=0} u_2 &= 0, \quad \partial_y|_{y=0} b_1 = 0, \\
|_{y=0} b_2 &= 0.
\end{aligned}
\]

(1.8)

And the far field state is denoted by \((\mathbf{u}, \mathbf{b})\):

\[ \lim_{y \to +\infty} u_1 = u_0(t, x, 0) \equiv \mathbf{u}, \quad \lim_{y \to +\infty} b_1 = h^0_c(t, x, 0) \equiv \mathbf{b}. \]

(1.9)

First of all, a shear flow \((u_s(t, y), 0, \mathbf{b}, 0)\) is a trivial solution to the system (1.6) with \( u_s(t, y) \) solving

\[
\begin{aligned}
\partial_t u_s(t, y) - \partial^2_{yy} u_s(t, y) &= 0, \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}_+, \\
|_{y=0} u_s(t, y) &= 0, \quad \lim_{y \to -\infty} u_s(t, y) = \mathbf{u}, \\
|_{y=0} u_s(t, 0, y) &= u_{s0}(y).
\end{aligned}
\]

(1.10)

In the following discussion, we assume the shear flow \( u_s(t, y) \) has the following properties:

\( (\mathbf{H}) \quad \| \partial_y u_s(t, \cdot) \|_{L^\infty_y} \leq \frac{C}{(t)^{1/4}}, \quad i = 1, 2, \quad \int_0^\infty |\partial_y u_s(t, y)| dy < C, \quad \| \partial_{y_i} \partial^2_{yy} u_s(t, \cdot) \|_{L^\infty_y} \leq \frac{C}{(t)^{1/4}}, \)
for some generic constant $C$. Here and after $(t) = 1 + t$.

**Remark 1.1.** The assumption (H) on the shear flow holds for a large class of initial data $u_{0}$. For example, it holds for the initial data $u_{0} = \chi(y)$ with $\chi(y) \in C^{\infty}(\mathbb{R})$, $\chi(y) = 0$ for $y \leq 1$ and $\chi(y) = \overline{\alpha}$ for $y \geq 2$ considered in [23] for the Prandtl system. Note that here we do not assume the smallness of the shear flow. In addition, it also holds when $u_{0}(y) = \frac{1}{\sqrt{4\pi}} \int_{y}^{y} \exp(-\frac{z^{2}}{4}) dz$ considered in [9] for the Prandtl system where the almost global solution is obtained. Note that for the classical Prandtl equations, such shear flow in the form of Guassian error function yields a time decay damping term in the time evolution equation of $u_{1}$, however, it does not lead to any damping effect in the MHD boundary layer system (1.6).

To define the function space of the solution considered in this paper, the following Gaussian weighted function $\theta_{\alpha}$ will be used:

$$\theta_{\alpha}(t, y) = \exp\left(\frac{\alpha z(t, y)^{2}}{4}\right), \quad \text{with } z(t, y) = \frac{y}{\sqrt{t}} \quad \text{and } \alpha \in [1/4, 1/2].$$

With this and

$$M_{m} = \frac{\sqrt{m + 1}}{m!},$$

define the Sobolev weighted semi-norms by

$$X_{m} = X_{m}(f, \tau) = \|\theta_{\alpha} \partial_{x}^{m} f\|_{L^{2} \tau^{m} M_{m}}, \quad D_{m} = D_{m}(f, \tau) = \|\partial_{y} \partial_{x}^{m} f\|_{L^{2} \tau^{m} M_{m}},$$

$$Y_{m} = Y_{m}(f, \tau) = \|\theta_{\alpha} \partial_{y}^{m} f\|_{L^{2} \tau^{m-1} M_{m}}.$$  \hfill (1.11)

Then the following space of analytic functions in the tangential variable $x$ and weighted Sobolev in the normal variable $y$ is defined by

$$X_{\tau, \alpha} = \{ f(t, x, y) \in L^{2}(\mathbb{R}; \theta_{\alpha} \, dx \, dy) : \| f \|_{X_{\tau, \alpha}} < \infty \}$$

with $\tau > 0$ and the norm

$$\| f \|_{X_{\tau, \alpha}} = \sum_{m \geq 0} X_{m}(f, \tau).$$

In addition, the following two semi-norms will also be used:

$$\| f \|_{D_{\tau, \alpha}} = \sum_{m \geq 0} D_{m}(f, \tau) = \| \partial_{y} f \|_{X_{\tau, \alpha}}, \quad \| f \|_{Y_{\tau, \alpha}} = \sum_{m \geq 1} Y_{m}(f, \tau).$$

Here, the summation over $m$ is considered in the $l^{1}$ sense that is similar to the definition used in [9,23] rather than in the $l^{2}$ sense used in [10]. With the above notations, we are now ready to state the main Theorem as follows.

**Theorem 1.1.** For any $\lambda \in [3/2, 2]$, there exists a small positive constant $\varepsilon_{*}$ depending on $2 - \lambda$. Under the assumption (H) on the background shear flow $(u_{s}(t, y), 0, \overline{b}, 0)$ with $\overline{b} \neq 0$, assume the initial data $u_{0}$ and $b_{0}$ satisfy

$$\| u_{0} - u_{s}(0, y) \|_{X_{2\tau_{0}, 1/2}} \leq \varepsilon, \quad \| b_{0} - \overline{b} \|_{X_{2\tau_{0}, 1/2}} \leq \varepsilon,$$  \hfill (1.12)

for some given $\varepsilon \in (0, \varepsilon_{*}]$. Then there exists a unique solution $(u_{1}, u_{2}, b_{1}, b_{2})$ to the MHD boundary layer equations (1.6)–(1.9) such that

$$(u_{1} - u_{s}(t, y), b_{1} - \overline{b}) \in X_{\tau, \alpha}, \quad \alpha \in [1/4, 1/2],$$

with analyticity radius $\tau$ larger than $\tau_{0}/4$ in the time interval $[0, T_{\varepsilon}]$. And the lifespan $T_{\varepsilon}$ has the following low bound estimate

$$T_{\varepsilon} \geq C\varepsilon^{-\lambda},$$  \hfill (1.13)
where the constant $C$ is independent of $\varepsilon$.

As is well-known that the leading order characteristic boundary layer for the incompressible Navier-Stokes equations with no-slip boundary condition is described by the classical Prandtl equations derived by [19] in 1904. In the two space dimensions, under the monotonicity assumption on the tangential velocity in the normal direction, Oleinik firstly obtained the local existence of classical solutions by using the Crocco transformation, cf. [17] and Oleinik-Samokhin’s classical book [18]. Recently, this well-posedness result was re-proved by using an energy method in the framework of Sobolev spaces in [1] and [16] independently by observing the cancellation mechanism in the convection terms. And by imposing an additional favorable condition on the pressure, a global in time weak solution was obtained in [22].

When the monotonicity condition is violated, singularity formation or separation of the boundary layer is well expected and observed. For this, E-Engquist constructed a finite time blowup solution to the Prandtl equations in [3]. Recently, when the background shear flow has a non-degenerate critical point, some interesting ill-posedness (or instability) phenomena of solutions to both linear and nonlinear classical Prandtl equations around shear flows are studied, cf. [4–6]. All these results show that the monotonicity assumption on the tangential velocity plays a key role for well-posedness theory except in the frameworks of analytic functions and Gevrey regularity classes. Indeed, in the framework of analytic functions, Sammartino and Caflisch [2, 20] established the local well-posedness theory of the Prandtl system in three space dimensions and also justified the Prandtl ansatz in this setting by applying the abstract Cauchy-Kowalewskaya (CK) theorem initiated by Asano’s unpublished work. Later, the analyticity requirement in the normal variable was removed by Lombardo, Cannone and Sammartino in [15] because of the viscous effect in the normal direction.

Recently, Zhang and Zhang obtained the lifespan of small analytic solution to the classical Prandtl equations with small analytic initial data in [23]. Precisely, when the strength of background shear flow is of the order of $\varepsilon^{5/3}$ and the perturbation is of the order of $\varepsilon$, they showed that the classical Prandtl system has a unique solution with a lower bound estimate on the lifespan in the order of $\varepsilon^{-4/3}$. Furthermore, if the initial data is a small analytic perturbation of the Guassian error function (1.10), an almost global existence for the Prandtl boundary layer equations is proved in [9].

On the other hand, to study the high Reynolds number limits for the MHD Equations (1.1) with no-slip boundary condition on the velocity (1.2) and perfect conducting boundary condition (1.3) on the magnetic field, one can apply the Prandtl ansatz to derive the boundary layer System (1.1) as the leading order description on the flow near the boundary. For this, readers can refer to [7, 12–14, 21] about the formal derivation of (1.6), the well-posedness theory of the system and the justification of the Prandtl ansatz locally in time.

This paper is about long time existence of solutions to (1.6)–(1.9). Precisely, we will show that if the initial datum is a small perturbation of a shear flow analytically in the order of $\varepsilon$, then there exists a unique solution to (1.6)–(1.9) with the lifespan $T_\varepsilon$ of the order of $\varepsilon^{-2}$.

Compared with the estimate on the lifespan of solutions to the classical Prandtl system studied in [23], the lower bound estimate is larger and there is no requirement on the smallness of the background shear flow because the mechanism in the system is used due to the non-degeneracy of the tangential magnetic field. However, it is not known whether one can obtain a global or almost global in time solution like the work on the Prandtl system when the background shear velocity is taken to be a Guassian error function in [9]. We mention that even though Lin and Zhang showed the almost global existence of solution to MHD boundary layer equations with zero Dirichlet boundary condition on the magnetic field in [11] when the components of both the background velocity and magnetic fields are Guassian error functions, it is not clear weather the system (1.6) holds with zero Dirichlet boundary condition even in formal derivation.

The analysis on the lifespan of the perturbed system in this paper relies on the introduction
of some new unknown functions that capture the cancellation of some linear terms. Unlike the
work in [9] on the Prandtl system for which the cancellation yields a damping term in the time
evolution of the perturbation of the tangential velocity field, there is no such damping effect
observed for the MHD boundary layer system.

Finally, the rest of the paper is organized as follows. After giving some preliminary esti-
mates, a uniform estimate on the solution will be proved in the next section. Based on this
uniform estimate, a low bound of the lifespan of solution is derived in Section 3. The uniqueness
part is given in Section 4. Throughout the paper, constants denoted by $C, C'$, $C_0, C_1$ and $C_2$
are generic and are independent of the small parameter $\varepsilon$.

# 2 Uniform Estimate

We first list the following two preliminary estimates on the functions in the norms defined in
the previous section. The first estimate indeed is from Lemma 3.3 in [9] (also see [8]).

**Lemma 2.1.** (Poincaré type inequality with Gaussian weight). Let $f$ be a function such that
$f|_{y=0} = 0$ (or $\partial_y f|_{y=0}$) and $f|_{y=\infty} = 0$. Then, for $\alpha \in [1/4, 1/2]$, $m \geq 0$ and $t \geq 0$, it holds that

$$
\frac{\alpha}{\langle t \rangle} \|\theta_\alpha \partial_y \partial_y \theta_\alpha f\|_{L^2}^2 \leq \|\theta_\alpha \partial_y \theta_\alpha f\|_{L^2}^2.
$$

(2.1)

The second lemma is used in [9] and we include it here with a short proof for convenience
of readers.

**Lemma 2.2.** Let $f$ be a function such that $f|_{y=0} = 0$ (or $\partial_y f|_{y=0}$) and $f|_{y=\infty} = 0$. Then

$$
\sum_{m \geq 0} \|\theta_\alpha \partial_y \partial_y \theta_\alpha f\|_{L^2}^2 \tau^m M_m \geq \frac{\alpha^{1/2} \beta}{2 \langle t \rangle^{1/2}} \|f\|_{D^{1,0}} + \frac{\alpha (1 - \beta)}{\langle t \rangle} \|f\|_{X_{r,\alpha}},
$$

(2.2)

for $\beta \in (0, 1/2)$.

**Proof.** In fact, by Lemma 2.1, one has

$$
\frac{\|\theta_\alpha \partial_y \partial_y \theta_\alpha f\|_{L^2}^2}{\|\theta_\alpha \partial_y \theta_\alpha f\|_{L^2}^2} \geq \frac{\beta}{2} \frac{\alpha^{1/2}}{\langle t \rangle^{1/2}} \|\theta_\alpha \partial_y \theta_\alpha f\|_{L^2}^2 + \frac{2 - \beta \alpha^{1/2}}{2 \langle t \rangle^{1/2}} \|\theta_\alpha \partial_y \theta_\alpha f\|_{L^2}^2

$$

$$
\geq \beta \frac{\alpha^{1/2}}{2 \langle t \rangle^{1/2}} \|\theta_\alpha \partial_y \theta_\alpha f\|_{L^2}^2 + \frac{\alpha (1 - \beta)}{\langle t \rangle} \|\theta_\alpha \partial_y \theta_\alpha f\|_{L^2}^2
$$

Multiplying the above inequality by $\tau^m M_m$ and summing up in $m \geq 0$ give (2.2).

We are now ready to study a uniform estimate on the solution. For this, we first rewrite
the solution to (1.6)–(1.9) as a perturbation $(u, v, b)$ of the $(u_0(t, y), 0, 0)$ by denoting

$$
\begin{align*}
\{ & u_1 = u_0(t, y) + u, \\
& u_2 = v, \\
& b_1 = b, \\
& b_2 = g.
\end{align*}
$$

(2.3)

Without loss of generality, take $\bar{b} = 1$ and $\kappa = 1$. Then (1.6) becomes

$$
\begin{align*}
\{ & \partial_t u + (u + u)\partial_x u + v\partial_y (u + u) - (1 + b)\partial_x b - g\partial_y b - \partial_y^2 u = 0, \\
& \partial_t b - (1 + b)\partial_x u - g\partial_y (u + u) + (u + u)\partial_x b + v\partial_y b - \partial_y^2 b = 0.
\end{align*}
$$

(2.4)
And the initial and boundary data of \((u, v)\) and \((b, g)\) are given by

\[
\begin{align*}
&u(t, x, y)|_{t=0} = u_0(x, y) - u_s(0, y), \quad b(t, x, y)|_{t=0} = b_0(x, y) - 1, \\
&\begin{cases}
  u|_{y=0} = 0, \\
  v|_{y=0} = 0,
\end{cases} \quad \text{and} \quad \begin{cases}
  \partial_y b|_{y=0} = 0, \\
  g|_{y=0} = 0,
\end{cases}
\end{align*}
\]

with the corresponding far field condition

\[
\lim_{y \to +\infty} u = 0, \quad \lim_{y \to +\infty} b = 0.
\]

It suffices to establish the long time existence of solutions to (2.4)–(2.7). In this section, we focus on the uniform a priori estimate on the solution to (2.4) in the analytical framework defined in Section 1.

Integrating equation (2.4) over \([0, y]\) gives that

\[
\partial_t \int_0^y b \, d\tilde{y} + v(1 + b) - (u_s + u)g = \partial_y^2 \int_0^y b \, d\tilde{y},
\]

where the boundary conditions that \(\partial_y b|_{y=0} = v|_{y=0} = g|_{y=0} = 0\) are used.

Define

\[
\psi(t, y) = \int_0^y b \, d\tilde{y},
\]

one has

\[
\partial_t \psi + v(1 + b) - (u_s + u)g = \partial_y^2 \psi.
\]

Now we introduce new unknown functions by taking care of the cancellation mechanism in the system as observed in [12] as follows

\[
\tilde{u} = u - \partial_y u_s \psi, \quad \tilde{b} = b.
\]

Then \((\tilde{u}, \tilde{b})\) satisfies the following equations.

\[
\begin{align*}
\begin{cases}
\partial_t \tilde{u} - \partial_y^2 \tilde{u} + (u_s + u)\partial_y \tilde{u} + v\partial_y \tilde{u} - (1 + b)\partial_y \tilde{b} - g\partial_y \tilde{b} - 2\partial_y^2 u_s \tilde{b} + v\partial_y^2 u_s \psi = 0,
\partial_t \tilde{b} - \partial_y^2 \tilde{b} - (1 + b)\partial_y \tilde{u} - g\partial_y \tilde{u} + (u_s + u)\partial_y \tilde{b} + v\partial_y \tilde{b} - g\partial_y^2 u_s \psi = 0.
\end{cases}
\end{align*}
\]

Here we have used the following fact that \(u_s\) is the solution to the heat equation. That is,

\[
\partial_t u_s - \partial_y^2 u_s = 0, \quad \partial_t \partial_y u_s - \partial_y^3 u_s = 0.
\]

By a direct calculation, the boundary conditions of \((\tilde{u}, \tilde{b})\) are given by

\[
\begin{align*}
&\tilde{u}|_{y=0} = 0, \quad \partial_y \tilde{b}|_{y=0} = 0, \\
&\tilde{u}|_{y=\infty} = 0, \quad \tilde{b}|_{y=\infty} = 0.
\end{align*}
\]

We then turn to show the existence of solution \((\tilde{u}, \tilde{b})\) to (2.11)–(2.13) with the corresponding initial data.

\[
\tilde{u}(0, x, y) = u(0, x, y) - \partial_y u_s(0, y) \int_0^y b(0, x, \tilde{y}) \, d\tilde{y}, \quad \tilde{b}(0, x, y) = b(0, x, y).
\]

Note that

\[
\|\tilde{u}(0, x, y)\|_{X_{2\tau_0, 0}} \leq \|u(0, x, y)\|_{X_{2\tau_0, 0}} + C\|b(0, x, y)\|_{X_{2\tau_0, 0}},
\]

where
for $\alpha \in [1/4, 1/2]$.

Moreover, once the existence of solution $(\tilde{u}, \tilde{b})$ to (2.11)–(2.14) is obtained, one can define $(u, b)$ by

$$u(t, x, y) = \tilde{u}(t, x, y) + \partial_y u_*(t, y) \int_0^y \tilde{b}(t, x, \tilde{y}) d\tilde{y}, \quad b(t, x, y) = \tilde{b}(t, x, y).$$

(2.16)

It is straightforward to check that $(u, b)$ is a solution to (2.4)–(2.7) with the following estimates

$$\|u\|_{X_{\tau, \alpha}} \leq \|\tilde{u}\|_{X_{\tau, \alpha}} + C\|\tilde{b}\|_{X_{\tau, \alpha}}, \quad \|b\|_{X_{\tau, \alpha}} = \|\tilde{b}\|_{X_{\tau, \alpha}}.$$

Therefore, we only need to estimate the solution $(\tilde{u}, \tilde{b})$ to (2.11)–(2.14) in the analytic norms as shown in the next two subsections.

### 2.1 A Priori Estimate on Velocity Field

For $m \geq 0$, by applying the tangential derivative operator $\partial_x^m$ to (2.11) and multiplying it by $\theta^2 \partial_x^m \tilde{u}$, the integration over $\mathbb{H}$ yields

$$\int_{\mathbb{H}} \partial_x^m (\partial_t \tilde{u} - \partial_y^2 \tilde{u} + (u_x + u) \partial_x \tilde{u} + v \partial_y \tilde{u}) - (1 + b) \partial_x \tilde{b} - g \partial_x \tilde{b} - 2 \partial_x^2 \tilde{u} + v \psi \partial_y^2 u_x) \theta^2 \partial_x^m \tilde{u} dx dy = 0. \quad (2.17)$$

We now estimate each term in (2.17) as follows. Firstly, note that

$$\frac{d}{dt} \int_{\mathbb{H}} \partial_x^m \tilde{u} \partial_x^2 \partial_x^m \tilde{u} dx dy - \frac{1}{2} \int_{\mathbb{H}} (\partial_x^m \tilde{u})^2 \theta^2 dx dy - \int_{\mathbb{H}} (\partial_x^m \tilde{u})^2 \theta^2 \frac{d}{dt} \|\theta\|_{\mathbb{H}} dx dy$$

$$= \frac{1}{2} \int_{\mathbb{H}} \|\partial_x^m \tilde{u}\|_{L^2}^2 + \frac{\alpha}{4(t)} \|\theta \partial_x^m \tilde{u}\|_{L^2}^2, \quad (2.18)$$

and

$$- \int_{\mathbb{H}} \partial_y^2 \partial_x^m \tilde{u} \partial_y^2 \partial_x^m \tilde{u} dx dy = \|\partial_x^m \tilde{u}\|_{L^2}^2 + \int_{\mathbb{H}} \partial_y^2 \partial_x^m \tilde{u} \partial_y (\theta^2) \partial_x^m \tilde{u} dx dy.$$

The boundary term vanishes because of the boundary condition $\partial_x^m \tilde{u} |_{y=0} = 0$. Furthermore,

$$\int_{\mathbb{H}} \partial_y \partial_x^m \tilde{u} \partial_y (\theta^2) \partial_x^m \tilde{u} dx dy = -\frac{1}{2} \int_{\mathbb{H}} (\partial_x^m \tilde{u})^2 \partial_y^2 (\theta^2) dx dy$$

$$= -\frac{\alpha}{2} \frac{1}{t} \|\partial_x^m \tilde{u}\|_{L^2}^2 - \frac{\alpha^2}{2} \frac{1}{t} \|\theta \partial_x^m \tilde{u}\|_{L^2}^2,$$

where we have used

$$\partial_y^2 (\theta^2) = \frac{\alpha}{t} \theta^2 + \frac{\alpha^2}{t} z^2 (t, y) \theta^2.$$

Consequently,

$$- \int_{\mathbb{H}} \partial_y^2 \partial_x^m \tilde{u} \partial_y^2 \partial_x^m \tilde{u} dx dy = \|\partial_x^m \tilde{u}\|_{L^2}^2 - \frac{\alpha}{2} \frac{1}{t} \|\theta \partial_x^m \tilde{u}\|_{L^2}^2 - \frac{\alpha^2}{2} \frac{1}{t} \|\theta \partial_x^m \tilde{u}\|_{L^2}^2. \quad (2.19)$$

For the nonlinear terms in (2.17), we have

$$\int_{\mathbb{H}} \partial_x^m ((u_x + u) \partial_x \tilde{u}) \theta^2 \partial_x^m \tilde{u} dx dy = \sum_{j=0}^m \binom{m}{j} \int_{\mathbb{H}} \partial_x^{m-j} u \partial_x^{j+1} \tilde{u} \partial_x^m \tilde{u} dx dy \triangleq R_1.$$
and

\[ |R_1| \leq \sum_{j=0}^{[m/2]} \left( \frac{m}{j} \right) \| \partial_{x}^{m-j} u \|_{L_x^2 L_y^\infty} \| \theta_\alpha \partial_{x}^{j+1} \tilde{u} \|_{L_x^2 L_y^\infty} \| \theta_\alpha \partial_{x}^{m-j} \tilde{u} \|_{L^2_x} + \sum_{j=[m/2]+1}^{m} \left( \frac{m}{j} \right) \| \partial_{x}^{m-j} u \|_{L_x^2} \| \theta_\alpha \partial_{x}^{j+1} \tilde{u} \|_{L^2_x} \| \theta_\alpha \partial_{x}^{m-j} \tilde{u} \|_{L^2_x}. \]

For \( 0 \leq j \leq [m/2] \), by (2.16), one has

\[ \| \partial_{x}^{m-j} u \|_{L_x^2 L_y^\infty} = \| \partial_{x}^{m-j} (\tilde{u} + \partial_y u_0 \psi) \|_{L_x^2 L_y^\infty} \]

\[ \leq \| \partial_{x}^{m-j} \tilde{u} \|_{L_x^2 L_y^\infty} + \| \partial_y u_0 \partial_{x}^{m-j} \psi \|_{L_x^2 L_y^\infty} \]

\[ \leq C \| \partial_\alpha \partial_x^{m-j} \tilde{u} \|_{L_x^2} \| \theta_\alpha \partial_x^{m-j} \tilde{u} \|_{L^2} + C(t)^{-1/4} \| \theta_\alpha \partial_x^{m-j} \tilde{b} \|_{L^2}, \]

where in the last inequality, we have used

\[ \| \partial_\alpha u_0 \|_{L_x^\infty} \leq \frac{C}{\sqrt{\langle t \rangle)} \]

according to the assumption (H). Moreover,

\[ \| \partial_{x}^{m-j} \psi \|_{L_x^2 L_y^\infty} = \left\| \int_{0}^{y} \partial_{x}^{m-j} \tilde{b} dy \right\|_{L_x^2 L_y^\infty} \]

\[ = \left| \int_{0}^{y} \theta_\alpha \partial_x^{m-j} \tilde{b} \exp \left( -\frac{\alpha}{4} y^2 \right) dy \right|_{L_x^2 L_y^\infty} \leq C(t)^{1/4} \| \theta_\alpha \partial_x^{m-j} \tilde{b} \|_{L^2}. \]

And

\[ \| \theta_\alpha \partial_x^{j+1} \tilde{u} \|_{L_x^2 L_y^\infty} \leq C \| \theta_\alpha \partial_x^{j+1} \tilde{u} \|_{L^2} \| \theta_\alpha \partial_x^{j+2} \tilde{u} \|_{L^2}. \]

For \([m/2] + 1 \leq j \leq m\), we have

\[ \| \partial_{x}^{m-j} u \|_{L_x^2 L_y^\infty} \leq \| \partial_{x}^{m-j} \tilde{u} \|_{L_x^2 L_y^\infty} + \| \partial_y u_0 \partial_{x}^{m-j} \psi \|_{L_x^2 L_y^\infty} \]

\[ \leq C \| \theta_\alpha \partial_x^{m-j} \tilde{u} \|_{L_x^2} \| \theta_\alpha \partial_x^{m-j} \tilde{u} \|_{L^2} \| \partial_\alpha \partial_x^{m-j} \tilde{b} \|_{L^2} \]

\[ + C(t)^{-1/4} \| \theta_\alpha \partial_x^{m-j} \tilde{b} \|_{L^2} \| \theta_\alpha \partial_x^{m-j+1} \tilde{b} \|_{L^2}. \]

Hence,

\[ |R_1| \frac{\| \theta_\alpha \partial_x^m \tilde{u} \|_{L^2}}{\| \theta_\alpha \partial_x^m \tilde{u} \|_{L^2}} \]

\[ \leq \frac{C}{(\tau(t))^{1/2}} \left\{ \sum_{j=0}^{[m/2]} \left( X_{m-j}^{1/2} D_{m-j}^{1/2} + (t)^{-1/4} X_{m-j}^{1/2} Y_{j+1}^{1/2} \right) \right\} \]

\[ + \sum_{j=[m/2]+1}^{m} \left( X_{m-j}^{1/4} D_{m-j}^{1/4} + (t)^{-1/4} X_{m-j}^{1/2} Y_{j+1}^{1/2} \right) \]

\[ \geq \int_{\mathbb{R}} \partial_{x}^{m} (v \partial_y \tilde{u}) \theta_\alpha^2 \partial_x^m \tilde{u} \bar{d} x \bar{d} y = \sum_{j=0}^{[m/2]} \left( \frac{m}{j} \right) \int_{\mathbb{R}} \partial_{x}^{m-j} v \partial_y \tilde{u} \theta_\alpha^2 \partial_x^m \tilde{u} \bar{d} x \bar{d} y \equiv R_2. \]
Recall where we have used the assumption (H). Note that

\[ \| \partial_x^{m-j} \|_{L^2_0 L^\infty_y} \leq C \| \partial_x \partial_y \partial_y \|_{L^2_0} \leq C \| \partial_x \partial_y \partial_y \|_{L^2_0} \]
and

\[ |R_3| \leq \sum_{j=0}^{[m/2]} \left( \begin{array}{c} m \\ j \end{array} \right) \| \partial_x^{m-j} \tilde{b} \|_{L^2} \| \theta_\alpha \partial_x^{j+1} \tilde{b} \|_{L^2} \| \theta_\alpha \partial_x^m \tilde{u} \|_{L^2} \\
+ \sum_{j=[m/2]+1}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) \| \partial_x^{m-j} \tilde{b} \|_{L^2} \| \theta_\alpha \partial_x^{j+1} \tilde{b} \|_{L^2} \| \theta_\alpha \partial_x^m \tilde{u} \|_{L^2}. \]

For \(0 \leq j \leq [m/2]\),

\[ \| \partial_x^{m-j} \tilde{b} \|_{L^2} \leq C \| \theta_\alpha \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/2} \| \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/2}, \]

and

\[ \| \partial_x^{j+1} \tilde{b} \|_{L^2} \leq C \| \partial_x^{j+1} \tilde{b} \|_{L^2}^{1/2} \| \partial_x^{j+1} \tilde{b} \|_{L^2}^{1/2}. \]

For \([m/2]+1 \leq j \leq m\),

\[ \| \partial_x^{m-j} \tilde{b} \|_{L^2} \leq C \| \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/4} \| \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/4} \| \partial_x^{m-j+1} \tilde{b} \|_{L^2}^{1/4} \| \partial_x^{m-j+1} \tilde{b} \|_{L^2}^{1/4}. \]

Therefore,

\[
\frac{|R_3| \tau^m M_m}{\| \theta_\alpha \partial_x^m \tilde{u} \|_{L^2}} \leq C \frac{1}{(\tau(t))^{1/2}} \left\{ \sum_{j=0}^{[m/2]} \int_{\mathbb{R}} X_{m-j}^{1/4} D_{m-j}^{1/2} D_{j+1}^{1/2} + \sum_{j=[m/2]+1}^{m} \int_{\mathbb{R}} X_{m-j}^{1/4} D_{m-j}^{1/4} D_{m-j+1}^{1/4} D_{j+1}^{1/2} \right\}. \tag{2.22}
\]

Note that

\[ \int_{\mathbb{R}} \partial_x^m (g \partial_y \tilde{b}) \theta_\alpha^2 \partial_x^m \tilde{u} dx dy = \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) \int_{\mathbb{R}} \partial_x^{m-j} g \partial_x^j \partial_y \tilde{b} \theta_\alpha^2 \partial_x^m \tilde{u} dx dy \equiv R_4 \]

and

\[ |R_4| \leq \sum_{j=0}^{[m/2]} \left( \begin{array}{c} m \\ j \end{array} \right) \| \partial_x^{m-j} g \|_{L^2} \| \theta_\alpha \partial_x^m \tilde{b} \|_{L^2} \| \theta_\alpha \partial_x^m \tilde{u} \|_{L^2} \\
+ \sum_{j=[m/2]+1}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) \| \partial_x^{m-j} g \|_{L^2} \| \theta_\alpha \partial_x^m \tilde{b} \|_{L^2} \| \theta_\alpha \partial_x^m \tilde{u} \|_{L^2}. \]

For \(0 \leq j \leq [m/2]\),

\[ \| \partial_x^{m-j} g \|_{L^2} \leq C \| \partial_x^{m-j} \tilde{b} \|_{L^2}, \]

and

\[ \| \theta_\alpha \partial_x^m \tilde{b} \|_{L^2} \leq C \| \theta_\alpha \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/2} \| \theta_\alpha \partial_x^{m-j+1} \tilde{b} \|_{L^2}^{1/2}. \]

For \([m/2]+1 \leq j \leq m\),

\[ \| \partial_x^{m-j} g \|_{L^2} \leq C \| \partial_x^{m-j+1} \tilde{b} \|_{L^2} \leq C \| \partial_x^{m-j+2} \tilde{b} \|_{L^2}. \]
As a consequence, we have

\[
\frac{\|R_4\|^n M_m}{\|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2}} \leq \frac{C}{(\gamma(t))^{1/2}} \left\{ \sum_{j=0}^{[m/2]} \langle t \rangle^{1/4} \sum_{m-j+1}^{\gamma(t)} \sum_{j+\frac{1}{2}}^{[m/2]} \right\}.
\] (2.23)

And

\[
\left| \int_{\Omega} \partial_x^m u_x \partial_x^n \tilde{b} \partial_x^m \tilde{u} dxdy \right| \leq \|\partial_x^m u_x\|_{L^2\cdot L^\infty} \|\theta_\alpha \partial_x^m \tilde{b}\|_{L^2} \|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2}
\leq C(t)^{-1} \|\theta_\alpha \partial_x^m \tilde{b}\|_{L^2} \|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2},
\]

that is,

\[
\left| \int_{\Omega} \partial_x^m \tilde{b} \partial_x^n \tilde{u} dxdy \right| \leq C(t)^{-1} \|\theta_\alpha \partial_x^m \tilde{b}\|_{L^2},
\] (2.24)

where we have used \(\|\partial_x^m u_x\|_{L^2} \leq \frac{C(t)}{(\gamma(t))^{1/2}}\) by the assumption (H). We now consider

\[
R_5 = \sum_{j=0}^{[m/2]} \left( \sum_{j=0}^{[m/2]} \right) \int_{\Omega} \partial_x^m u_x \partial_x^n \psi \partial_x^m \tilde{u} dxdy.
\]

Note that

\[
|R_5| \leq \sum_{j=0}^{[m/2]} \left( \sum_{j=0}^{[m/2]} \right) \|\partial_x^m u_x \|_{L^2\cdot L^\infty} \|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2} \|\partial_x^n \psi\|_{L^\infty} \|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2}
\]

+ \sum_{j=[m/2]+1}^{m} \left( \sum_{j=[m/2]+1}^{m} \right) \|\partial_x^m u_x \|_{L^2\cdot L^\infty} \|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2} \|\partial_x^n \psi\|_{L^\infty} \|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2}.
\]

For \(0 \leq j \leq [m/2]\), we have

\[
\|\partial_x^m u_x \|_{L^2\cdot L^\infty} \leq C(t)^{1/4} \|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2} + C(t)^{1/4} \|\theta_\alpha \partial_x^{m-j+1} \tilde{u}\|_{L^2},
\]

and

\[
\|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2} \leq \frac{C(t)^{3/2}}{(\gamma(t))^{1/2}},
\]

provided that \(\alpha < 1\) by the assumption (H). And

\[
\|\partial_x^n \psi\|_{L^\infty} \leq C(t)^{1/4} \|\theta_\alpha \partial_x^m \tilde{b}\|_{L^2} \|\theta_\alpha \partial_x^{m+j} \tilde{b}\|_{L^2}.
\]

For \([m/2] + 1 \leq j \leq m\), we have

\[
\|\partial_x^m u_x \|_{L^2\cdot L^\infty} \leq C(t)^{1/4} \|\theta_\alpha \partial_x^m \tilde{u}\|_{L^2} \|\theta_\alpha \partial_x^{m+j} \tilde{u}\|_{L^2} + C(t)^{1/4} \|\theta_\alpha \partial_x^{m+j+1} \tilde{u}\|_{L^2} \|\theta_\alpha \partial_x^{m+j} \tilde{b}\|_{L^2}.
\]

And

\[
\|\partial_x^n \psi\|_{L^2\cdot L^\infty} \leq C(t)^{1/4} \|\theta_\alpha \partial_x^m \tilde{b}\|_{L^2}.
\]
Hence,

\[
\frac{|R_5|^m M_m}{\|\theta_0 \partial_y^m \tilde{u}\|_{L^2}} \leq \frac{C}{(\tau(t))^{1/2}} \left\{ \sum_{j=0}^{[m/2]} (t)^{-1/4} Y_{m-j+1} + (t)^{-1/4} Y_{m-j+1} \right\} \\
+ \sum_{j=[m/2]+1}^{m} (t)^{-1/4} Y_{m-j+1}^{1/2} Y_{m-j+2}^{1/2} + (t)^{-1/4} Y_{m-j+1}^{1/2} Y_{m-j+2}^{1/2} \right\} \tag{2.25}
\]

Combining the estimates (2.18)–(2.25) and summing over \( m \geq 0 \) give

\[
\frac{d}{dt} \|\tilde{u}\|_{X_r,a} + \sum_{m \geq 0} \tau M M_m \frac{\|\theta_0 \partial_x^m \partial_y \tilde{u}\|_{L^2}}{\|\theta_0 \partial_y^m \tilde{u}\|_{L^2}} + \frac{\alpha(1-2\alpha)}{4(t)} \sum_{m \geq 0} \tau M M_m \frac{\|\theta_0 \partial_x^m \partial_y \tilde{u}\|_{L^2}}{\|\theta_0 \partial_y^m \tilde{u}\|_{L^2}} - \frac{\alpha}{2(t)} \|\tilde{u}\|_{X_r,a} - \frac{C}{(\tau(t))^{1/2}} (t)^{-1/4} \|\tilde{u}\|_{X_r,a} \tag{2.26}
\]

where we have used the fact that for any positive sequences \( \{a_j\}_{j \geq 0} \) and \( \{b_j\}_{j \geq 0} \),

\[
\sum_{j \geq 0} \sum_{j \geq 0} a_j b_{m-j} \leq \sum_{j \geq 0} a_j \sum_{j \geq 0} b_j.
\]

Choosing \( \alpha \leq 1/2 \) in (2.26) yields

\[
\frac{d}{dt} \|\tilde{u}\|_{X_r,a} + \sum_{m \geq 0} \tau M M_m \frac{\|\theta_0 \partial_x^m \partial_y \tilde{u}\|_{L^2}}{\|\theta_0 \partial_y^m \tilde{u}\|_{L^2}} - \frac{\alpha}{2(t)} \|\tilde{u}\|_{X_r,a} - \frac{C}{(\tau(t))^{1/2}} (t)^{-1/4} \|\tilde{u}\|_{X_r,a} + (t)^{1/4} \|\tilde{b}\|_{X_r,a} + \|\tilde{b}\|_{Y_r,a} \tag{2.27}
\]

\[
= \tau(t) \|\tilde{u}\|_{Y_{r,a}} + \left\{ (t)^{-1/4} (\|\tilde{u}\|_{X_r,a} + \|\tilde{b}\|_{X_r,a}) + (t)^{1/4} (\|\tilde{u}\|_{X_r,a} + \|\tilde{b}\|_{X_r,a}) \right\}
\]

\[
\times (\|\tilde{u}\|_{Y_{r,a}} + \|\tilde{b}\|_{Y_{r,a}}). \tag{2.27}
\]

\[
2.2 \quad A \text{ Priori Estimate on Magnetic Field}
\]

Similarly, for \( m \geq 0 \), by applying the tangential derivative operator \( \partial_x^m \) to (2.11)_2 and multiplying it by \( \tilde{b} \partial_y^m \), the integration over \( H \) gives

\[
\int_{\mathbb{H}} \partial_x^m (\partial_x \tilde{b} - \partial_y \tilde{b} - (1+b)\partial_x \tilde{u} + g\partial_x \tilde{u} + (u_n + u)\partial_x \tilde{b} + v\partial_y \tilde{b} - g\partial_y \tilde{u}) \tilde{b} \theta_0^2 \partial_y^m b \, dx \, dy = 0. \tag{2.28}
\]

We now estimate (2.28) term by term as follows. Firstly,

\[
\int_{\mathbb{H}} \partial_x \partial_x^m \tilde{b} \tilde{b} \theta_0^2 \partial_y^m b \, dx \, dy = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{H}} (\partial_x^m \tilde{b})^2 \theta_0^2 b \, dx \, dy - \int_{\mathbb{H}} (\partial_x^m \tilde{b})^2 \theta_0 \frac{d}{dt} \tilde{b} \, dx \, dy
\]

\[
= \frac{1}{2} \frac{d}{dt} \|\theta_0 \partial_x^m \tilde{b}\|_{L^2}^2 + \frac{\alpha}{4(t)} \|\theta_0 \partial_y^m \tilde{b}\|_{L^2}^2. \tag{2.29}
\]

And

\[
- \int_{\mathbb{H}} \partial_x^2 \partial_x \tilde{b} \partial_y \theta_0 \partial_y \tilde{b} \, dx \, dy = \|\theta_0 \partial_x^m \partial_y \tilde{b}\|_{L^2}^2 + \int_{\mathbb{H}} \partial_y \partial_x^m \tilde{b} \partial_y (\tilde{b} \partial_x) \partial_x^m \tilde{b} \, dx \, dy,
\]
where we have used the boundary condition $\partial_y \partial_x^{m} \tilde{b} |_{y=0} = 0$. Moreover,

$$\int_{\mathbb{H}} \partial_y \partial_x^{m} \tilde{b} \partial_y (\partial_x^{2}) \partial_x^{m} \tilde{b} dxdy = - \frac{1}{2} \int_{\mathbb{H}} (\partial_x^{m} \tilde{b})^2 \partial_x^{2} (\partial_x^{2}) dxdy$$

$$= - \frac{\alpha}{2} \frac{1}{\langle t \rangle} \| \partial_x \partial_x^{m} \tilde{b} \|_{L^2}^2 - \frac{\alpha^2}{2} \frac{1}{\langle t \rangle} \| \partial_x \partial_x^{m} \tilde{b} \|_{L^2}^2.$$

Hence,

$$- \int_{\mathbb{H}} \partial_y \partial_x^{m} \tilde{b} \partial_y \partial_x^{m} \tilde{b} dxdy = \| \partial_x \partial_x^{m} \tilde{b} \|_{L^2}^2 - \frac{\alpha}{2} \frac{1}{\langle t \rangle} \| \partial_x \partial_x^{m} \tilde{b} \|_{L^2}^2 - \frac{\alpha^2}{2} \frac{1}{\langle t \rangle} \| \partial_x \partial_x^{m} \tilde{b} \|_{L^2}^2.$$  \hspace{1cm} (2.30)

Similar to Subsection 2.1, the nonlinear terms can be estimated as follows. Firstly,

$$\int_{\mathbb{H}} \partial_x^{m} ((1 + b) \partial_x \tilde{u}) \partial_x^{2} \partial_x^{m} \tilde{b} dxdy = \sum_{j=0}^{m} \binom{m}{j} \int_{\mathbb{H}} \partial_x^{m-j} \tilde{b} \partial_x^{j+1} \tilde{u} \partial_x^{2} \partial_x^{m} \tilde{b} dxdy \triangleq R_6,$$

and

$$|R_6| \leq \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{j} \| \partial_x^{m-j} \tilde{b} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}$$

$$+ \sum_{j=\lfloor m/2 \rfloor + 1}^{m} \binom{m}{j} \| \partial_x^{m-j} \tilde{b} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}.$$

For $0 \leq j \leq \lfloor m/2 \rfloor$,

$$\| \partial_x^{m-j} \tilde{b} \|_{L^2} \leq C \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/2} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/2},$$

and

$$\| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2} \leq C \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/2} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/2}.$$

For $\lfloor m/2 \rfloor + 1 \leq j \leq m$,

$$\| \partial_x^{m-j} \tilde{b} \|_{L^2} \leq C \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/4} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/4} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/4} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/4}.$$

Hence,

$$\frac{|R_6|}{\tau^n M_m} \leq \frac{C}{\langle t \rangle^{1/2}} \left\{ \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{j} \| \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/2} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/2} \right\} + \sum_{j=\lfloor m/2 \rfloor + 1}^{m} \binom{m}{j} \| \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/4} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/4} \| \partial_x \partial_x^{m-j} \tilde{b} \|_{L^2}^{1/4}.$$

Moreover,

$$\int_{\mathbb{H}} \partial_x^{m} (g \partial_y \tilde{u}) \partial_x^{2} \partial_x^{m} \tilde{b} dxdy = \sum_{j=0}^{m} \binom{m}{j} \int_{\mathbb{H}} \partial_x^{m-j} (g \partial_x \tilde{u}) \partial_x^{j+1} \partial_x^{2} \partial_x^{m} \tilde{b} dxdy \triangleq R_7,$$

and

$$|R_7| \leq \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{j} \| \partial_x^{m-j} \tilde{g} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{u} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{u} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{u} \|_{L^2}$$

$$+ \sum_{j=\lfloor m/2 \rfloor + 1}^{m} \binom{m}{j} \| \partial_x^{m-j} \tilde{g} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{u} \|_{L^2} \| \partial_x \partial_x^{m-j} \tilde{u} \|_{L^2}.$$
For $0 \leq j \leq \lfloor m/2 \rfloor,$
$$\| \partial_x^{m-j} g \|_{L_x^\infty L_y^\infty} \leq C(t)^{1/4} \| \theta_\alpha \partial_x^{m-j+1} \tilde{b} \|_{L^2},$$
and
$$\| \theta_\alpha \partial_x^j \theta_\beta \tilde{u} \|_{L_x^\infty L_y^\infty} \leq C \| \theta_\alpha \partial_x^j \theta_\beta \tilde{u} \|_{L_x^2}^{1/2} \| \theta_\alpha \partial_x^{m+1} \theta_\beta \tilde{u} \|_{L_y^2}^{1/2}.$$  

For $\lfloor m/2 \rfloor + 1 \leq j \leq m,$
$$\| \partial_x^{m-j} g \|_{L_x^\infty L_y^\infty} \leq C(t)^{1/4} \| \theta_\alpha \partial_x^{m-j+1} \tilde{b} \|_{L_x^2} \| \theta_\alpha \partial_x^{m-j+2} \tilde{b} \|_{L_y^2}^{1/2}.$$  

Therefore,
$$\frac{|R_\gamma|^m M_m}{\| \theta_\alpha \partial_x^n b \|_{L^2}} \leq \frac{C}{(r(t))^{1/2}} \left\{ \sum_{j=0}^{\lfloor m/2 \rfloor} (t)^{1/4} \mathcal{M}_{m-j+1} \mathcal{D}^{1/2} \mathcal{D}^{1/2} + \sum_{j=\lfloor m/2 \rfloor+1}^m (t)^{1/4} \mathcal{M}_{m-j+1} \mathcal{M}^{1/2} \mathcal{M}_{m-j+1} \mathcal{D} \right\}. \quad (2.32)$$

Denote
$$R_\delta \triangleq \sum_{j=0}^m \binom{m}{j} \int_\mathbb{R} \partial_x^{m-j} u \partial_x^j \theta_\alpha \partial_x^n \tilde{b} \, dx \, dy,$$
then
$$|R_\delta| \leq \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{j} \| \partial_x^{m-j} u \|_{L_x^\infty L_y^\infty} \| \theta_\alpha \partial_x^j \tilde{b} \|_{L_x^2} \| \theta_\alpha \partial_x^n \tilde{b} \|_{L^2}$$
$$+ \sum_{j=\lfloor m/2 \rfloor+1}^m \binom{m}{j} \| \partial_x^{m-j} u \|_{L_x^\infty L_y^\infty} \| \theta_\alpha \partial_x^j \tilde{b} \|_{L_x^2} \| \theta_\alpha \partial_x^n \tilde{b} \|_{L^2}.$$  

Similar to the estimation on $R_1,$ we can obtain
$$\frac{|R_\delta|^m M_m}{\| \theta_\alpha \partial_x^n b \|_{L^2}} \leq \frac{C}{(r(t))^{1/2}} \left\{ \sum_{j=0}^{\lfloor m/2 \rfloor} \mathcal{X}_{m-j}^{1/2} \mathcal{D}^{1/2} + (t)^{-1/4} \mathcal{X}_{m-j} \mathcal{D}^{1/2} \mathcal{D}^{1/2} + \sum_{j=\lfloor m/2 \rfloor+1}^m \mathcal{X}_{m-j+1}^{1/2} \mathcal{X}^{1/2} \mathcal{X}_{m-j+1} \mathcal{D} \right\}. \quad (2.33)$$

And
$$\int_\mathbb{R} \partial_x^n (v \partial_x \tilde{b}) \theta_\alpha \partial_x^n \tilde{b} \, dx \, dy = \sum_{j=0}^m \binom{m}{j} \int_\mathbb{R} \partial_x^{m-j} v \partial_x^j \theta_\alpha \partial_x^n \tilde{b} \, dx \, dy \triangleq R_0.$$  

Thus
$$|R_0| \leq \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{j} \| \partial_x^{m-j} v \|_{L_x^\infty L_y^\infty} \| \theta_\alpha \partial_x^j \theta_\beta \tilde{u} \|_{L_x^2} \| \theta_\alpha \partial_x^n \tilde{b} \|_{L^2}$$
$$+ \sum_{j=\lfloor m/2 \rfloor+1}^m \binom{m}{j} \| \partial_x^{m-j} v \|_{L_x^\infty L_y^\infty} \| \theta_\alpha \partial_x^j \theta_\beta \tilde{u} \|_{L_x^2} \| \theta_\alpha \partial_x^n \tilde{b} \|_{L^2}.$$
Then for \(0 \leq j \leq [m/2]\),
\[
\|\partial_t^{m-j} g\|_{L^2_t L^\infty_x} \leq C(t)^{1/4} \|\theta_\alpha \partial_x^{m-j+1} \tilde{b}\|_{L^2_t}
\]
and
\[
\|\theta_\alpha \partial_x^2 u_s\|_{L^2_x} \leq \frac{C}{(t)^{1/4}},
\]
provided that \(\alpha < 1\) by the assumption (H). Moreover,
\[
\|\partial_t^j \psi\|_{L^2_t L^\infty_x} \leq C(t)^{1/4} \|\theta_\alpha \partial_x^{j+1} \tilde{b}\|_{L^2_t}^{1/2}.
\]

For \([m/2] + 1 \leq j \leq m\),
\[
\|\partial_x^{m-j} g\|_{L^2_t L^\infty_x} \leq C(t)^{1/4} \|\theta_\alpha \partial_x^{m-j+1} \tilde{b}\|_{L^2_t}^{1/2} \|\theta_\alpha \partial_x^{m-j+2} \tilde{b}\|_{L^2_t}^{1/2},
\]
and
\[
\|\partial_t^j \psi\|_{L^2_t L^\infty_x} \leq C(t)^{1/4} \|\theta_\alpha \partial_x^j \tilde{b}\|_{L^2_t}.\]

Consequently,
\[
\frac{|R_{10}| \tau^m M_m}{\|\theta_\alpha \partial_x^2 b\|_{L^2}} \leq \frac{C}{(\tau(t))^{1/2}} \left\{ \sum_{j=0}^{[m/2]} \langle t \rangle^{-1/4} \tilde{X}_{m-j+1} \tilde{X}_{j+1}^{1/2} \right\} + \sum_{j=[m/2]+1}^{m} \langle t \rangle^{-1/4} \tilde{X}_{m-j+2}^{1/2} \tilde{X}_{j}\right\} \right\}.
\]

From the estimates (2.29)–(2.35), summing over \(m \geq 0\) yields
\[
\frac{d}{dt} \|\tilde{b}\|_{X_{r,\alpha}} + \sum_{m \geq 0} \tau^m M_m \frac{\|\theta_\alpha \partial_x^m \tilde{b}\|_{L^2}^2}{\|\theta_\alpha \partial_x^m b\|_{L^2}} + \alpha(1-2\alpha) \sum_{m \geq 0} \tau^m M_m \frac{\|\theta_\alpha \partial_x^m \tilde{b}\|_{L^2}^2}{\|\theta_\alpha \partial_x^m b\|_{L^2}} - \alpha \frac{2\langle t \rangle}{(\tau(t))^{1/2}} (\|\tilde{b}\|_{X_{r,\alpha}})
\leq \hat{\tau}(t) \|\tilde{b}\|_{Y_{r,\alpha}} + \frac{C_0}{(\tau(t))^{1/2}} (\|\tilde{b}\|_{X_{r,\alpha}} + \|\tilde{b}\|_{X_{r,\alpha}} + \langle t \rangle^{1/4} (\|\tilde{u}\|_{D_{r,\alpha}} + \|\tilde{b}\|_{D_{r,\alpha}}) + \langle t \rangle^{1/4} (\|\tilde{u}\|_{D_{r,\alpha}} + \|\tilde{b}\|_{D_{r,\alpha}}) )
\times (\|\tilde{u}\|_{Y_{r,\alpha}} + \|\tilde{b}\|_{Y_{r,\alpha}}).
\]
Similarly, by choosing $\alpha \leq 1/2$, we have
\[
\frac{d}{dt} \| \tilde{b} \|_{X,\alpha} + \sum_{m \geq 0} \tau^m M_m \left( \frac{\| \theta_0 \partial_x^m \partial_y \tilde{b} \|_{L^2}^2}{\| \theta_0 \partial_x^m \tilde{u} \|_{L^2}} - \frac{\alpha}{2(t)} \| \tilde{b} \|_{X,\alpha} \right) \\
\leq \hat{\tau}(t) \| \tilde{b} \|_{Y,\alpha} + \frac{C_0}{(\tau(t))^{1/2}} \left( (t)^{-1/4} \left( \| \tilde{u} \|_{X,\alpha} + \| \tilde{b} \|_{X,\alpha} \right) + (t)^{1/4} \left( \| \tilde{u} \|_{Y,\alpha} + \| \tilde{b} \|_{Y,\alpha} \right) \right) \\
\times (\| \tilde{u} \|_{Y,\alpha} + \| \tilde{b} \|_{Y,\alpha}). \tag{2.37}
\]

3 The Proof of Estimate of Lifespan

By the uniform a priori estimates obtained in Section 2, we now estimate the low bound on lifespan of the solution. Consider (2.27) + $K \times (2.37)$ with $K > 1$ to be determined later,
\[
\frac{d}{dt} (\| \tilde{u} \|_{X,\alpha} + K \| \tilde{b} \|_{X,\alpha}) + \sum_{m \geq 0} \tau^m M_m \left( \frac{\| \theta_0 \partial_x^m \partial_y \tilde{b} \|_{L^2}^2}{\| \theta_0 \partial_x^m \tilde{u} \|_{L^2}} + K \frac{\| \theta_0 \partial_x^m \partial_y \tilde{b} \|_{L^2}^2}{\| \theta_0 \partial_x^m \tilde{b} \|_{L^2}} \right) \\
- \frac{\alpha}{2(t)} \| \tilde{u} \|_{X,\alpha} - \left( C + \frac{K \alpha}{2} \right) \frac{1}{(t)} \| \tilde{b} \|_{X,\alpha} \\
\leq \left( \hat{\tau}(t) + \frac{C_0(K + 1)}{(\tau(t))^{1/2}} \right) \left( (t)^{-1/4} \left( \| \tilde{u} \|_{X,\alpha} + \| \tilde{b} \|_{X,\alpha} \right) + (t)^{1/4} \left( \| \tilde{u} \|_{Y,\alpha} + \| \tilde{b} \|_{Y,\alpha} \right) \right) \\
\times (\| \tilde{u} \|_{Y,\alpha} + K \| \tilde{b} \|_{Y,\alpha}). \tag{3.1}
\]

Choose the function $\tau(t)$ satisfies the following ODE.
\[
\frac{d}{dt} (\tau(t))^{1/2} + \frac{3C_0(K + 1)}{2} \left( (t)^{-1/4} \left( \| \tilde{u} \|_{X,\alpha} + \| \tilde{b} \|_{X,\alpha} \right) + (t)^{1/4} \left( \| \tilde{u} \|_{Y,\alpha} + \| \tilde{b} \|_{Y,\alpha} \right) \right) = 0. \tag{3.2}
\]

From (3.1) and (3.2), one has
\[
\frac{d}{dt} (\| \tilde{u} \|_{X,\alpha} + K \| \tilde{b} \|_{X,\alpha}) + \sum_{m \geq 0} \tau^m M_m \left( \frac{\| \theta_0 \partial_x^m \partial_y \tilde{b} \|_{L^2}^2}{\| \theta_0 \partial_x^m \tilde{u} \|_{L^2}} + K \frac{\| \theta_0 \partial_x^m \partial_y \tilde{b} \|_{L^2}^2}{\| \theta_0 \partial_x^m \tilde{b} \|_{L^2}} \right) \\
- \frac{\alpha}{2(t)} \| \tilde{u} \|_{X,\alpha} - \left( C + \frac{K \alpha}{2} \right) \frac{1}{(t)} \| \tilde{b} \|_{X,\alpha} \leq 0. \tag{3.3}
\]

By Lemma 2.2, we have
\[
\sum_{m \geq 0} \tau^m M_m \frac{\| \theta_0 \partial_x^m \partial_y \tilde{b} \|_{L^2}^2}{\| \theta_0 \partial_x^m \tilde{u} \|_{L^2}} \geq \frac{\alpha(1 - \beta_1)}{2(t)^{1/2}} \| \tilde{u} \|_{D,\alpha} + \frac{\alpha(1 - \beta_1)}{\langle t \rangle} \| \tilde{u} \|_{X,\alpha}, \tag{3.4}
\]
and
\[
\sum_{m \geq 0} \tau^m M_m \frac{\| \theta_0 \partial_x^m \partial_y \tilde{b} \|_{L^2}^2}{\| \theta_0 \partial_x^m \tilde{b} \|_{L^2}} \geq \frac{\alpha(1 - \beta_2)}{2(t)^{1/2}} \| \tilde{b} \|_{D,\alpha} + \frac{\alpha(1 - \beta_2)}{\langle t \rangle} \| \tilde{u} \|_{X,\alpha}, \tag{3.5}
\]
for $\beta_1, \beta_2 \in (0, 1/2)$.

From (3.3)–(3.5), it follows that
\[
\frac{d}{dt} (\| \tilde{u} \|_{X,\alpha} + K \| \tilde{b} \|_{X,\alpha}) + \frac{1}{2}(\alpha(1 - 2\beta_1)) \frac{1}{(t)} \| \tilde{u} \|_{X,\alpha} + \frac{1}{2}(\alpha(1 - 2\beta_2)) - \frac{2C_0}{K} \frac{1}{(t)} K \| \tilde{b} \|_{X,\alpha} \\
+ \frac{\alpha^{1/2} \beta_1}{2(t)^{1/2}} \| \tilde{u} \|_{D,\alpha} + \frac{K \alpha^{1/2} \beta_2}{2(t)^{1/2}} \| \tilde{u} \|_{D,\alpha} \leq 0. \tag{3.6}
\]
Choose
\[ \alpha = \frac{1}{2} - \delta, \quad \beta_1 = \frac{\delta}{2}, \quad \beta_2 = \frac{\delta}{2}, \quad K = \frac{4C}{\delta}, \]
where \(0 < \delta < 1/4\) is sufficiently small to be determined later, then
\[ \alpha(1 - 2\beta_1) = \frac{1}{2} - \frac{3}{2}\delta + \delta^2, \]
and
\[ \alpha(1 - 2\beta_2) - \frac{2C}{K} = \frac{1}{2} - 2\delta + \delta^2. \]

Then, there exist small positive constants \(\eta_1 = \delta\) and \(\eta_2 = \frac{\delta}{4}\) such that
\[
\frac{d}{dt}(\|\tilde{u}\|_{X,\alpha} + K\|\tilde{b}\|_{X,\alpha}) + \frac{1/4 - \eta_1}{\langle t \rangle} (\|\tilde{u}\|_{X,\alpha} + K\|\tilde{b}\|_{X,\alpha}) \\
+ \frac{\eta_2}{\langle t \rangle^{1/4}} (\|\tilde{u}\|_{D,\alpha} + K\|\tilde{b}\|_{D,\alpha}) \leq 0. \tag{3.7}
\]
It implies that
\[
\frac{d}{dt}(\|\tilde{u}\|_{X,\alpha} + K\|\tilde{b}\|_{X,\alpha}) \langle t \rangle^{1/4 - \eta_1} + \int_0^t \frac{\eta_2}{\langle s \rangle^{1/4 + \eta_1}} (\|\tilde{u}(s)\|_{D,\alpha} + K\|\tilde{b}(s)\|_{D,\alpha}) ds \\
\leq \|\tilde{u}(0)\|_{X,\alpha} + K\|\tilde{b}(0)\|_{X,\alpha} \leq C(1 + K)\varepsilon, \tag{3.9}
\]
where we have used (2.15). Then, by noting that \(K = \frac{4C}{\delta}\), one has
\[
\frac{3C_0}{2}(K + 1) \int_0^t \langle s \rangle^{-1/4} (\|\tilde{u}(s)\|_{X,\alpha} + \|\tilde{b}(s)\|_{X,\alpha}) ds \\
= \frac{3C_0}{2} \left( \frac{4C}{\delta} + 1 \right) \int_0^t \langle s \rangle^{-1/4} (\|\tilde{u}(s)\|_{X,\alpha} + \|\tilde{b}(s)\|_{X,\alpha}) ds \\
\leq \frac{3CC_0\varepsilon}{2} \left( \frac{4C}{\delta} + 1 \right)^2 \left( \frac{s}{t} \right)^{1/2 + \eta_1} ds \leq 3CC_0\varepsilon \left( \frac{4C}{\delta} + 1 \right)^2 (t)^{1/2 + \eta_1}, \tag{3.10}
\]
and
\[
\frac{3C_0}{2}(K + 1) \int_0^t \langle s \rangle^{1/4} (\|\tilde{u}(s)\|_{D,\alpha} + \|\tilde{b}(s)\|_{D,\alpha}) ds \\
= \frac{3C_0}{2} \left( \frac{4C}{\delta} + 1 \right) \int_0^t \langle s \rangle^{1/4} (\|\tilde{u}(s)\|_{D,\alpha} + \|\tilde{b}(s)\|_{D,\alpha}) ds \\
= \frac{3C_0}{2} \left( \frac{4C}{\delta} + 1 \right) \frac{8}{\delta} \int_0^t \langle s \rangle^{1/2 + \eta_1} \frac{\eta_2}{\langle s \rangle^{1/4 + \eta_1}} (\|\tilde{u}(s)\|_{D,\alpha} + K\|\tilde{b}(s)\|_{D,\alpha}) ds \\
\leq \left( \frac{4C}{\delta} + 1 \right)^2 \frac{12CC_0\varepsilon}{\delta} (t)^{1/2 + \eta_1}. \tag{3.11}
\]
On the other hand, (3.2) implies that
\[
\tau(t)^{3/2} = \tau(0)^{3/2} - \frac{3C_0(K + 1)}{2} \int_0^t (s)^{-1/4}(\|\tilde{u}\|_{X_{r,\alpha}} + \|\tilde{b}\|_{X_{r,\alpha}})
+ (s)^{1/4}(\|\tilde{u}\|_{D_{r,\alpha}} + \|\tilde{b}\|_{D_{r,\alpha}})ds.
\]

(3.12)

From (3.10)–(3.12), one has
\[
\tau(t)^{3/2} \geq \tau_0^{3/2} - \max \left\{3CC_0\left(\frac{4C}{\delta} + 1\right)^2 (t)^{1/2 + \eta} \epsilon, \left(\frac{4C}{\delta} + 1\right)^2 \frac{12C_0}{\delta} (t)^{1/2 + \eta} \epsilon\right\},
\]
for all \(t \geq 0\).

Choose \(\delta = \frac{1}{\ln(1/\epsilon)}\). It is straightforward to show that
\[
\tau(t) \geq \frac{\tau_0}{4},
\]
in the time interval \([0, T_\epsilon]\), where \(T_\epsilon\) satisfies
\[
T_\epsilon = \mathcal{C}\left(\frac{1}{\epsilon(\ln(1/\epsilon))^{2-4/(\ln(1/\epsilon)+2)}} - 1\right).
\]
(3.13)

This gives the estimate on the lifespan of solution stated in (1.13).

4 The Proof of Uniqueness

Assume there are two solutions \((\tilde{u}_1, \tilde{b}_1)\) and \((\tilde{u}_2, \tilde{b}_2)\) to (2.11) with the same initial data \((\tilde{u}_0, \tilde{b}_0)\), which satisfies \(\|\tilde{u}_0, \tilde{b}_0\|_{X_{2\tau_0,\alpha}} \leq \epsilon\). And the tangential radii of analyticity of \((\tilde{u}_1, \tilde{b}_1)\) and \((\tilde{u}_2, \tilde{b}_2)\) are \(\tau_1(t)\) and \(\tau_2(t)\), respectively.

Define \(\tau(t)\) by
\[
\frac{d(\tau(t))^{3/2}}{dt} + \frac{3C_0(K + 1)}{2} (t)^{-1/4}(\|\tilde{u}_1\|_{X_{r,\alpha}} + \|\tilde{b}_1\|_{X_{r,\alpha}})
+ (t)^{1/4}(\|\tilde{u}_1\|_{D_{r,\alpha}} + \|\tilde{b}_1\|_{D_{r,\alpha}}) = 0, \tag{4.1}
\]
with initial data
\[
\tau(0) = \frac{\tau_0}{8}. \tag{4.2}
\]
By the estimates obtained in Section 2, there exists a time interval \([0, T_0]\) with \(T_0 \leq T_\epsilon\) such that
\[
\frac{\tau_0}{32} \leq \tau(t) \leq \frac{\tau_0}{8} \leq \min\{\tau_1, \tau_2\} \tag{4.3}
\]
for all \(t \in [0, T_0]\).

Set \(U = \tilde{u}_1 - \tilde{u}_2\) and \(B = \tilde{b}_1 - \tilde{b}_2\). Then
\[
\partial_t U - \partial_y^2 U + (u + u_1)\partial_y U + (v_1 - v_2)\partial_y \tilde{u}_1 - (1 + b_1)\partial_y \tilde{b}_1 - (g_1 - g_2)\partial_y \tilde{b}_1
- 2\partial_y^2 u s B + (v_1 - v_2)\partial_y^2 u s \psi_1 + R_{s1} = 0, \tag{4.4}
\]
and
\[
\partial_t B - \partial_y^2 B - (1 + b_1)\partial_y U - (g_1 - g_2)\partial_y \tilde{u}_1 + (u_1 + u_1)\partial_y B
+ (v_1 - v_2)\partial_y \tilde{b}_1 - (g_1 - g_2)\partial_y^2 u s \psi_1 + R_{s2} = 0, \tag{4.5}
\]
with the source terms \( R_{s1} \) and \( R_{s2} \) given by

\[
R_{s1} = (u_1 - u_2) \partial_x \tilde{u}_2 + v_2 \partial_y U - (b_1 - b_2) \partial_x \tilde{b}_2 - g_2 \partial_y B + v_2 \partial_y^2 u_s(\psi_1 - \psi_2),
\]

and

\[
R_{s2} = -(b_1 - b_2) \partial_x \tilde{u}_2 - g_2 \partial_y U + (u_1 - u_2) \partial_x \tilde{b}_2 + v_2 \partial_y B - g_2 \partial_y^2 u_s(\psi_1 - \psi_2).
\]

Note that the initial data and the boundary conditions are

\[
U(t, x, y)|_{t=0} = 0, \quad B(t, x, y)|_{t=0} = 0,
\]

and

\[
\begin{cases}
U|_{y=0} = 0, \\
U|_{y=\infty} = 0, \\
\partial_y B|_{y=0} = 0, \\
B|_{y=\infty} = 0.
\end{cases}
\]
From (4.1), one has
\[ \dot{\tau}(t) + \frac{C_0(K + 1)}{(\tau(t))^{1/2}} \left( (t)^{-1/4}(\| \tilde{u}_1 \|_{X_{\tau,\alpha}} + \| \tilde{b}_1 \|_{X_{\tau,\alpha}}) ight. \\
+ \left. (t)^{1/4}(\| \tilde{u}_1 \|_{D_{\tau,\alpha}} + \| \tilde{b}_1 \|_{D_{\tau,\alpha}}) \right) (\| U \|_{Y_{\tau,\alpha}} + \| B \|_{Y_{\tau,\alpha}}) \leq 0, \] 
(4.13)
because \( \tau(t) \leq \tau_1(t) \) and the norms \( X_{\tau,\alpha} \) and \( D_{\tau,\alpha} \) are increasing in \( \tau \).

By the inequalities (3.4), (3.5) and (4.13), one has
\[ \frac{d}{dt}(\| U \|_{X_{\tau,\alpha}} + K \| B \|_{X_{\tau,\alpha}}) + \frac{\alpha(1 - 2\beta_1)}{2(t)} \| U \|_{X_{\tau,\alpha}} + \frac{\alpha(1 - 2\beta_2)}{2(t)} \| B \|_{X_{\tau,\alpha}} \]
\[ + \frac{\alpha^{1/2}\beta_1}{2(t)^{1/2}} \| U \|_{D_{\tau,\alpha}} + \frac{\alpha^{1/2}\beta_2}{2(t)^{1/2}} K \| B \|_{D_{\tau,\alpha}} \]
\[ \leq \frac{C_0(1 + K)}{(\tau(t))^{1/2}} (\| \tilde{u}_2 \|_{Y_{\tau,\alpha}} + \| \tilde{b}_2 \|_{Y_{\tau,\alpha}})(t)^{-1/4}(\| U \|_{X_{\tau,\alpha}} + \| B \|_{X_{\tau,\alpha}}) \]
\[ + (t)^{1/4}(\| U \|_{D_{\tau,\alpha}} + \| B \|_{D_{\tau,\alpha}}), \]
(4.14)
for \( \beta_1, \beta_2 \in (0, 1/2) \). Since
\[ \| \tilde{u}_2 \|_{Y_{\tau,\alpha}} \leq \frac{1}{\tau} \| \tilde{u}_2 \|_{X_{\tau,\alpha}} \leq \frac{1}{\tau} \| \tilde{u}_2 \|_{X_{\tau,\alpha}} \leq \frac{C(1 + K)}{\tau} \varepsilon(t)^{-1/4+\eta_1}, \]
(4.15)
and
\[ \| \tilde{b}_2 \|_{Y_{\tau,\alpha}} \leq \frac{1}{\tau} \| \tilde{b}_2 \|_{X_{\tau,\alpha}} \leq \frac{1}{\tau} \| \tilde{b}_2 \|_{X_{\tau,\alpha}} \leq \frac{C(1 + K)}{\tau} \varepsilon(t)^{-1/4+\eta_1}, \]
(4.16)
we have
\[ \frac{C_0(1 + K)}{(\tau(t))^{1/2}} (\| \tilde{u}_2 \|_{Y_{\tau,\alpha}} + \| \tilde{b}_2 \|_{Y_{\tau,\alpha}}) \leq \frac{2(1 + K)^2CC_0\varepsilon(t)^{1/4+\eta_1}}{(\tau(t))^{3/2}(t)^{1/4+\eta_1}}. \]
(4.17)
Notice that \( t \in [0, T_2] \) with \( T_2 = \varepsilon^{-2\delta_0} \), where \( \delta_0 \) is a fixed small positive constant. As in Section 3, we can choose \( \alpha = 1/2 - \delta, \beta_1 = \beta_2 = \frac{\delta}{2} \), \( K = \frac{4C_0}{\varepsilon} \) and \( \delta = 1/\ln(1/\varepsilon) \), then \( \eta_1 \) can be chosen to be \( \delta \). Let \( \varepsilon \) suitably small to have
\[ \frac{\alpha(1 - 2\beta_1)}{2} > \frac{2(1 + K)^2CC_0\varepsilon(t)^{1/2+\eta_1}}{(\tau(t))^{3/2}}, \quad \frac{\alpha(1 - 2\beta_2)}{2} > \frac{2(1 + K)^2CC_0\varepsilon(t)^{1/2+\eta_1}}{(\tau(t))^{3/2}}. \]
(4.14) and (4.17) imply that
\[ \frac{d}{dt}(\| U \|_{X_{\tau,\alpha}} + K \| B \|_{X_{\tau,\alpha}}) + \eta_3(\| U \|_{X_{\tau,\alpha}} + K \| B \|_{X_{\tau,\alpha}}) \leq 0 \]
(4.18)
for suitably small \( \eta_3 > 0 \) and any \( t \in [0, T_2] \). It implies uniqueness of solution to (2.11) in the time interval \([0, T_2]\).

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