An Exact Elliptic Superpotential for $\mathcal{N} = 1^*$
Deformations of Finite $\mathcal{N} = 2$ Gauge Theories

Nick Dorey, Timothy J. Hollowood and S. Prem Kumar

Department of Physics, University of Wales Swansea, Swansea, SA2 8PP, UK
E-mail: n.dorey@swan.ac.uk, t.hollowood@swan.ac.uk, s.p.kumar@swan.ac.uk

Abstract: We study relevant deformations of the $\mathcal{N} = 2$ superconformal theory on the world-volume of $N$ D3 branes at an $A_{k-1}$ singularity. In particular, we determine the vacuum structure of the mass-deformed theory with $\mathcal{N} = 1$ supersymmetry and show how the different vacua are permuted by an extended duality symmetry. We then obtain exact, modular covariant formulae (for all $k$, $N$ and arbitrary gauge couplings) for the holomorphic observables in the massive vacua in two different ways: by lifting to M-theory, and by compactification to three dimensions and subsequent use of mirror symmetry. In the latter case, we find an exact superpotential for the model which coincides with a certain combination of the quadratic Hamiltonians of the spin generalization of the elliptic Calogero-Moser integrable system.

Keywords: .
1. Introduction and summary

Conformal field theories (CFTs) play a key role in quantum field theory as fixed points of the renormalization group which classify possible universality classes. They can also give rise to massive theories of direct phenomenological interest after deformation by relevant operators. In general, little is known about the dynamics of interacting CFTs in four dimensions. Superconformal field theories (SCFTs) provide an important exception to this. In particular, strong evidence for several remarkable properties of these theories has emerged. Firstly these models often have marginal operators which give rise to continuous families of superconformal field theories. They also exhibit exact dualities which act on the marginal parameters relating regions of strong and weak coupling. Finally, these theories have a dual description in terms of superstring theory on a ten-dimensional space whose non-compact part is AdS$_5$ [1]. The example which is best understood is $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with gauge group U($N$) or SU($N$). This theory describes the world-volume dynamics of $N$ D3 branes in Type IIB string theory. The model has a single marginal dimensionless coupling, $\tau = 4\pi i/g^2 + \theta/2\pi$, formed from the gauge coupling $g$ and the vacuum angle $\theta$. In the D-brane realization, this coincides with the complexified IIB string coupling. The theory has an exact SL(2,$\mathbb{Z}$) duality which acts by modular transformations of $\tau$ and is inherited from the $S$-duality of the IIB theory. Finally, at large $N$, the SU($N$) theory has a dual description in terms of weakly-coupled IIB superstring theory on the near horizon geometry of the D3 branes which is AdS$_5 \times S^5$.

In this paper, we will consider a more general class of superconformal field theories with $\mathcal{N} = 2$ supersymmetry known as $A_{k-1}$ quiver theories. These theories arise in IIB string theory when $N$ D3 branes are placed at an elliptic $A_{k-1}$ singularity in spacetime [2]. The $\mathcal{N} = 4$ theory is naturally thought of as the $k = 1$ member of this family. We will focus on relevant mass-deformations which preserve either $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supersymmetry. Following the convention of the $\mathcal{N} = 4$ case, we will refer to these as the $\mathcal{N} = 2^*$ and $\mathcal{N} = 1^*$ quiver theories respectively. Our main results, which generalize previous work in the $k = 1$ case [3, 4], concern the vacuum structure and holomorphic observables of the $\mathcal{N} = 1^*$ quiver theories. We study the massive vacua of these theories using two different methods. Firstly, we analyse the maximal degenerations of the complex curve $\Sigma$ which governs the Coulomb branch of the corresponding $\mathcal{N} = 2^*$ theory [5]. Secondly, we propose an exact superpotential for the theory which coincides with the Hamiltonian of a certain classical integrable system known as the elliptic spin Calogero-Moser system [6-8]. The $\mathcal{N} = 1^*$ vacua can then be analysed directly by stationarizing this superpotential. The two approaches are shown to be in complete agreement. In the rest of this introduction we will outline our results. The detailed calculations are provided in subsequent sections.

The $A_{k-1}$ quiver theory has gauge group $G = U(1) \times SU(N)^k$ with matter in bi-fundamental
representations as determined by a quiver construction based on the Dynkin diagram of the Lie group $A_{k-1}$. The matter content ensures that the beta functions of each of the $k$ non-abelian gauge couplings $g_i, i = 1, \ldots, k$ vanish exactly and that the theory therefore has $k$ exactly marginal complex couplings

$$\tau_i = \frac{4\pi i}{g_i^2} + \frac{\theta_i}{2\pi}, \quad (1.1)$$

$i = 1, \ldots, k$, where $\theta_i$ are the vacuum angles of each SU($N$) factor in $G$. As mentioned above, these theories arise on the world-volume of $N$ D3 branes placed at an elliptic $A_{k-1}$ singularity in spacetime. In this construction, the IIB string coupling $\tau$ is identified with the coupling $\sum_{i=1}^{k} \tau_i$ of the diagonal SU($N$) subgroup of $G$. An alternative IIA brane configuration of $N$ D4-branes intersecting $k$ NS5 branes wrapped on a circle arises after T-duality in the compact direction of the elliptic singularity. As we review below, the strong coupling properties of the theory can then be analysed, as in [5], by lifting the IIA branes to M-theory five-branes in eleven dimensions. The M-theory spacetime includes a torus of complex structure $\tau$ with $k$ marked points whose relative positions on the torus encode the remaining $k-1$ independent gauge couplings. In the $\mathcal{N} = 4$ case ($k = 1$) modular transformations of the spacetime torus yield the familiar geometrical realization of IIB $S$-duality in M-theory. For $k > 1$, the theory has an enlarged $S$-duality group corresponding to the modular group of a torus with $k$ marked points which encode the positions of the NS5’s. In addition to modular transformations of $\tau = \sum_{i=1}^{k} \tau_i$, this includes shifts of the individual marked points by periods of the torus, physically realized as the movement of NS5-branes around non-trivial cycles, and other non-trivial dualities. The IIB set-up of branes at an orbifold singularity also yields a large-$N$ closed string dual of the quiver theories. In this case the relevant near-horizon geometry is the orbifold $AdS_5 \times S^5/\mathbb{Z}_k$ [9].

As usual, a useful way to study a conformal theory is via its relevant perturbations. The only such perturbations which preserve the full $\mathcal{N} = 2$ supersymmetry are mass-terms for the $k$ bi-fundamental hypermultiplets. After the deformation the resulting $\mathcal{N} = 2$ theory has a non-trivial Coulomb branch which can be studied by the methods of Seiberg and Witten [10]. As usual the Coulomb branch of the $A_{k-1}$ theory can be described as the moduli-space of a complex curve $\Sigma$ whose genus is equal to the rank of the gauge group, which is $r = k(N-1)+1$. The periods of a certain holomorphic one-form on the curve then yield the exact mass formula for BPS states of the theory. In fact, hypermultiplet masses are easily included in the IIA brane construction described above and lifting the IIA branes to M-theory yields a single M5-brane wrapped on $\Sigma \times \mathbb{R}^4$ [5]. This provides an explicit construction of $\Sigma$ as an $N$-fold cover of the torus in spacetime with prescribed singularities at the $k$ marked points. The action of the extended $S$-duality group described above is manifest in this approach.

As for any $\mathcal{N} = 2$ theory in four dimensions, the resulting Coulomb branch is a special Kähler manifold of complex dimension $r$, with singular submanifolds where BPS states become massless. For generic values of the masses, there are isolated points on the Coulomb branch at
which the maximal number of mutually-local BPS states become massless. These points are special because they survive soft-breaking of $\mathcal{N} = 2$ SUSY down to $\mathcal{N} = 1$ vacua where the theory is realized in different massive phases. Most famously, there are points on the Coulomb branch where magnetic monopoles become massless and, after soft-breaking, condense in the vacuum leading to the confinement of electric charges. When combined with the known holomorphy properties, this also provides a powerful approach for obtaining exact results for the vacuum properties of $\mathcal{N} = 1$ theories. In this paper we will present an exact analysis of the maximally singular points on the Coulomb branch of the $A_{k-1}$ quiver theories for all $k$ and $N$ and the resulting massive $\mathcal{N} = 1$ vacua. These results generalize the existing knowledge about massive deformations of the $\mathcal{N} = 4$ theory [3, 4] which we will now briefly review. In terms of $\mathcal{N} = 2$ supersymmetry, the $\mathcal{N} = 4$ theory contains a vector multiplet and a single hypermultiplet in the adjoint representation of the gauge group. The most general relevant deformation which preserves $\mathcal{N} = 2$ supersymmetry consists of introducing a mass term for the adjoint hypermultiplet. Following [11], the resulting theory will be referred to as $\mathcal{N} = 2^* SUSY \text{ Yang-Mills}$. The Coulomb branch of the $\mathcal{N} = 2^*$ theory with gauge group $SU(N)$ is described by a branched $N$-fold cover of the standard flat torus $E_\tau$ with complex structure parameter $\tau$ which is part of the spacetime of the M-theory brane construction. The points on the Coulomb branch which yield massive $\mathcal{N} = 1^*$ vacua correspond to the maximal degenerations of the curve. These are unbranched (unramified) $N$-fold covers of the torus $E_\tau$ [3], which are themselves tori with complex structure parameter $\tilde{\tau}$ of the form

$$\tilde{\tau} = \frac{q\tau + l}{p} \quad \text{with} \quad p q = N \quad \text{and} \quad l = 0, \ldots, p - 1. \quad (1.2)$$

The total number of massive $\mathcal{N} = 1^*$ vacua is therefore equal to $\sum_{p|N} p$. As each of these vacua is associated with the condensation of a BPS state with definite electric and magnetic charges, the theory in each vacuum is not invariant under S-duality. Instead, S-duality permutes the massive $\mathcal{N} = 1$ vacua, relating the physics of the theory in one ground-state at one value of the coupling to that of the theory in another ground-state at a different value of the coupling. Hence, one might say that the S-duality of the underlying $\mathcal{N} = 4$ theory is “spontaneously broken” in the $\mathcal{N} = 1^*$ theory. The action of $SL(2, \mathbb{Z})$ on the vacua is the same as the natural action of $SL(2, \mathbb{Z})$ on the set of $N$-fold covers of the torus $E_\tau$ [3].

Despite the “spontaneous breaking” of S-duality described above, the theory in each massive $\mathcal{N} = 1^*$ vacuum has a novel kind of duality named $\tilde{S}$-duality in [12]. This duality reflects the fact that the degenerate curve describing each massive vacuum is a torus with complex structure parameter $\tilde{\tau}$ given above. The low-energy physics of the $\mathcal{N} = 2^*$ theory (and its $\mathcal{N} = 1^*$ deformation) only depends on the complex structure of the curve and therefore is invariant under modular transformations acting on $\tilde{\tau}$. $\tilde{S}$-duality therefore has the same status as the IR electric-magnetic duality of Seiberg-Witten theory and is not expected to be valid at all length-scales. Nevertheless, this new duality leads to interesting predictions for the behaviour
of the theory in the limit of large ’t Hooft coupling which can be compared directly with the IIB dual background studied by Polchinski and Strassler [11].

One of the main aims of this paper is to extend our understanding of supersymmetry-preserving relevant deformations to the $A_{k-1}$ quiver models. In fact we find natural generalizations of the phenomena described above which are familiar from the $\mathcal{N} = 4$ case. Specifically, we study the maximally singular curves of the $\mathcal{N} = 2^*$ quiver theories which are again unbranched $N$-fold covers of the spacetime torus $E_\tau$. We find:

1. The extended $S$-duality group described above is “spontaneously broken” and has a non-trivial action on the set of massive $\mathcal{N} = 1^*$ vacua. In particular, shifts of the $k$ individual marked points by periods of $E_\tau$ now generate an additional degeneracy of $N^{k-1}$ massive vacua for each unbranched $N$-fold cover $E_\tilde{\tau}$. The total number of massive vacua is therefore $N^{k-1}\sum_{p|N} p$.

2. The $\tilde{S}$-duality group of each massive vacuum is extended in the obvious way to the modular group of a torus of complex structure parameter $\tilde{\tau}$ with $k$ marked points. In particular shifts of the individual marked points by periods of $E_\tilde{\tau}$ are exact dualities of the IR physics in each vacuum.

One of the most fascinating and mysterious aspects of $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions is their relation to finite-dimensional classical integrable systems [3, 13, 14]. In this paper, we will provide results which extend this correspondence in new directions and harness it to provide a practical method of computing physical quantities. We will now review the connection between $\mathcal{N} = 2$ SUSY and classical integrability in the context of the mass-deformed $\mathcal{N} = 4$ theory and briefly summarize our new results.

As discussed above, the $\mathcal{N} = 2^*$ theory with gauge group $U(N)$ has a Coulomb branch of complex dimension $N$ which is the moduli-space of a Riemann surface $\Sigma$ of genus $N$. In [3], Donagi and Witten gave a concrete recipe for constructing $\Sigma$ as the spectral curve of a certain classical integrable system. The phase space of the system in question is the moduli space of solutions of a set of two-dimensional field equations for a $U(N)$ gauge field and adjoint Higgs known as Hitchin’s equations. The equations are obtained from the dimensional reduction of the self-dual $U(N)$ Yang-Mills equation in four dimensions to the two dimensional torus $E_\tau$ by including certain $\delta$-function source terms. As usual for self-duality equations, the resulting moduli-space is a hyper-Kähler manifold. One of the three linearly-independent complex structures of this manifold will play a special role in the following discussion. In particular, the resulting complex symplectic manifold has a very concrete realization as the complexified phase space of the elliptic Calogero-Moser model [8].

The Calogero-Moser model is an integrable system of $k$ non-relativistic particles with po-
sitions $X_a$ and momenta $p_a$, $a = 1, \ldots, N$, interacting via the pairwise potential \[15\],

$$V = \sum_{a>b} \varphi(X_a - X_b), \quad (1.3)$$

where $\varphi(X)$ is the Weierstrass function. The system is integrable because of the existence of $N$ Poisson-commuting Hamiltonians $H_a$, $a = 1, \ldots, N$. The explicit integration of Hamilton’s equations (see the Appendix of \[15\]) proceeds by identifying the appropriate set of action-angle variables with respect to which the equations of motion become linear. While the action variables are just the $k$ commuting Hamiltonians themselves, the corresponding angle variables are the flat coordinates on a $k$-dimensional torus in phase space. It is a highly non-trivial fact that this torus is precisely the Jacobian variety of an appropriate Riemann surface of genus $N$, known as the spectral curve. Changing back to the original variables, an explicit solution for the $N$ positions $X_a(t)$, for any set of initial data, can then be given in terms of $\theta$-functions on the Jacobian. The connection to $\mathcal{N} = 2$ supersymmetry in four dimensions starts from the observation that the complex curve $\Sigma$ which governs the Coulomb branch of the $\mathcal{N} = 2^*$ theory with gauge group $U(N)$ is precisely the spectral curve of the Calogero-Moser system. In particular, the curves coincide if we promote the $N$ positions and momenta to complex numbers and identify the resulting holomorphic Hamiltonians $H_a$ with the $N$ order parameters $u_a = \langle \text{Tr} \Phi^a \rangle$ of the $\mathcal{N} = 2$ theory. (Here, $\Phi$ is the adjoint scalar in the $U(N)$ vector multiplet.)

In particular, the first non-trivial Hamiltonian

$$H = \sum_{a=1}^{N} \frac{p_a^2}{2} - m^2 \sum_{a \neq b} \varphi(X_a - X_b) \quad (1.4)$$

is identified with the quadratic order parameter $u_2$.

The correspondence as stated above is rather abstract as, so far we only have an $\mathcal{N} = 2$ interpretation for the conserved quantities and not the corresponding angle variables. Fortunately, a much clearer picture emerges after compactification of the four-dimensional theory to three dimensions on a circle [16]. Now the theory acquires new scalar degrees of freedom from the dimensional reduction of the $N$ massless abelian gauge fields on the four-dimensional Coulomb branch. The component of each four-dimensional gauge field in the reduced direction gives rise to a Wilson line while the remaining components can be dualized in three dimensions in favour of another scalar field. As first explained by Seiberg and Witten [17], these variables naturally lie on a torus whose periods are controlled by the effective abelian couplings of the low-energy theory in four dimensions. More precisely, they take values on the Jacobian variety of $\Sigma$ and can therefore be identified with the angle-variables of the complexified Calogero-Moser system.

After including both the $N$ complex moduli $u_a$ of the four-dimensional theory and the $2N$ new periodic scalars discussed above, the Coulomb branch, which we denote throughout as $\mathfrak{M},$
of the compactified theory is a manifold of real-dimension $4N$. The low-energy effective action is a three-dimensional non-linear $\sigma$-model with target $\mathcal{M}$. As the theory has eight supercharges this manifold must be hyper-Kähler. In fact the hyper-Kähler manifold in question is precisely the phase space of the (complexified) classical integrable system described above. In particular, Seiberg and Witten [17] have argued that $\mathcal{M}$ has a preferred complex structure which is independent of the radius of compactification. With respect to this complex structure, the manifold has a description as a toric fibration where the four-dimensional Coulomb branch parameterized by the moduli $u_a$ and the Jacobian of the curve $\Sigma$ is the fibre above each point. As above, the moduli $u_a$ are identified with the $N$ commuting Hamiltonians of the integrable system, while the holomorphic coordinates on the fibre are just the corresponding angle variables. Thus, with respect to the preferred complex structure, the Coulomb branch of the compactified theory coincides with the complexified phase space of the elliptic Calogero-Moser model. In fact, one may also demonstrate directly the equivalence of $\mathcal{M}$ and the moduli space of Hitchin’s equations as hyper-Kähler manifolds. This equivalence turns out to be a generalization of the mirror symmetry between three-dimensional gauge theories with eight supercharges discovered in [18].

It is also interesting to consider soft breaking to $\mathcal{N} = 1$ supersymmetry from the point of view of the compactified theory. As in four dimensions, this is accomplished by introducing a non-zero mass $\mu$ for the adjoint chiral multiplet in the $\mathcal{N} = 2$ vector multiplet, or in other words adding the perturbation $\mu u_2$ to the superpotential. This superpotential lifts the Coulomb branch leaving only isolated vacua as in the uncompactified theory. One of the main results of [4] was that the integrable systems viewpoint provides a simple and quantitative description of this effect. Essentially all we need is the identification of the Coulomb branch of the compactified theory as the phase-space of the Calogero-Moser model with the complex positions $X_a$ and momenta $p_a$ providing a convenient choice of holomorphic coordinates. In terms of these variables, the superpotential $u_2$ is simply given by the Calogero-Moser Hamiltonian given in (1.4) above. This leads to a new connection between $\mathcal{N} = 1$ supersymmetric gauge theories and integrable systems: the vacua of the former can be identified with the equilibrium configurations of the latter. Numerous checks of this identification were presented in [4]. It also provides a practical method of computing the condensates of chiral operators for the $\mathcal{N} = 1^*$ theory [12, 19].

In the preceding discussion, we outlined the correspondence between the massive deformations of $\mathcal{N} = 4$ SUSY Yang-Mills and a certain classical integrable system. One of the main goals of the present paper is to extend this to massive deformations of the $A_{k-1}$ quiver models for $k > 1$. The first step is to identify the appropriate integrable model. Fortunately, several relevant results already exist. In particular, Kapustin has used the mirror symmetry of [18] to show that the Coulomb branch of the $\mathcal{N} = 2^*$ quiver theories coincides with the moduli space of an appropriate system of self-duality equations. For the $A_{k-1}$ theory, the relevant equations
are the $U(N)$ Hitchin equations on the torus $E_r$ with specified behaviour at $k$ punctures. In the following we will confirm that the spectral curve of this system coincides with the complex curve for the $A_{k-1}$ elliptic model given by Witten. However our main interest will be to recast this system as a system of interacting particles with “spin” in one dimension, analogous to the elliptic Calogero-Moser system of the $k=1$ case. Again, the required result is available in the existing literature. Nekrasov has shown that the $U(N)$ Hitchin system on a torus with $k$ punctures can be recast as the spin generalization of the elliptic Calogero-Moser system [8]. This system consists of $N$ particles moving on a circle, each carrying $k$ “spin” variables. The Hamiltonian is given explicitly in (4.36) below. As in the $k=1$ case, we investigate the connection between the maximally degenerate curves, the vacua of the theory obtained by soft-breaking of $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$ and the equilibrium configurations of the integrable system.

In line with the arguments presented above, our main result is that the exact superpotential of the theory is precisely a certain linear combination of the quadratic Hamiltonians of the elliptic spin-Calogero-Moser system. In particular, we show that the equilibrium configurations of the elliptic spin system are in one-to-one correspondence with the genus-one degenerations of the complex curve $\Sigma$. Finally we perform an explicit computation of condensates in both approaches and demonstrate precise agreement.

The paper is organized as follows. In Section 2, we deduce the vacuum structure of the mass-deformed quiver theory by analysing its classical superpotential. The resulting structure has a very nice interpretation in terms of brane configurations in Type IIA string theory. In Section 3, we recall, following Witten [5], how these elliptic models can be solved by lifting the Type IIA brane configurations to M Theory. In this picture, the $\mathcal{N}=1^*$ deformation can be described in terms of a rotation whose description is relegated to Appendix B. The net result is that the massive vacua of the mass deformed theory have a very simple interpretation in the M Theory picture which allows us to solve for the holomorphic observables. Section 4 describes a rather different way to probe the vacuum structure via the compactification scenario described above. The net result is an expression for the exact superpotential of the low energy three-dimensional $\sigma$-model which encodes via its minima all the holomorphic structure of the vacua.

2. The $\mathcal{N}=1^*$ deformations of the quiver theory

We begin by recalling the matter content of the $A_{k-1}$ quiver gauge theories. They have $\mathcal{N}=2$ supersymmetry and gauge group $U(N)^k$ with $k$ hypermultiplets $(Q_{i,i+1}, \tilde{Q}_{i+1,i})$, $i = 1, \ldots, k$ (the subscripts being defined modulo $k$) transforming in the bi-fundamental representation $((N,\overline{N}),(\overline{N},N))$ of the $i$-th and $i+1$-th $U(N)$ factors. In terms of $\mathcal{N}=1$ superfields we denote the $\mathcal{N}=2$ vector multiplets as $\{W_i^a, \Phi_i\}; i = 1, 2, \ldots, k$. As each $U(N)$ factor
sees $2N$ flavors in the fundamental representation, these $\mathcal{N} = 2$ theories have vanishing $\beta$-functions. In particular, in the absence of hypermultiplet masses they are superconformal theories. Consequently there are $k$ finite and independent gauge couplings

$$\tau_i = \frac{4\pi i}{g_i^2} + \frac{\theta_i}{2\pi},$$

$i = 1, \ldots, k$, each corresponding to an exactly marginal direction in the space of quiver theories [9]. Introducing an overall gauge coupling $\tau \equiv 4\pi i / g^2 + \theta / 2\pi = \sum_{i=1}^{k} \tau_i$ to be thought of as the coupling for the diagonal $U(N)$, the theory has a superpotential

$$W = \frac{1}{g^2} \sum_{i=1}^{k} \text{Tr} \Phi_i[\Phi_{i,i+1}, \Phi_{i+1,i}] .$$

We emphasize that the labels are defined modulo $k$. In what follows the dynamics of the $U(1)$ factors of the $U(N)$ groups is often irrelevant and can be ignored as they decouple from the interacting theory in the infra-red.

We will consider “$\mathcal{N} = 1^*$ deformations” of these theories, i.e. soft breaking to $\mathcal{N} = 1$ with generic hypermultiplet masses $m_i$ and generic masses $\mu_i$ for the adjoint chiral multiplets via the following $\mathcal{N} = 1$ tree-level superpotential,

$$W = \frac{1}{g^2} \sum_{i=1}^{k} \text{Tr} \left\{\Phi_i[\Phi_{i,i+1}, \Phi_{i+1,i}] - m_i \Phi_{i,i+1} \Phi_{i+1,i} - \mu_i \Phi_i^2\right\} .$$

We refer to this theory as the “$\mathcal{N} = 1^*$ quiver theory”. The theory with $\mu_i = 0$ and $m_i \neq 0$ which we dub the “$\mathcal{N} = 2^*$ quiver theory”, will play a central rôle in our analysis of the classical and quantum vacuum structure of the $\mathcal{N} = 1^*$ deformation.

The $A_{k-1}$ quiver theories with and without hypermultiplet masses (also known as the elliptic models) are obtained in Type IIA/M-theory [5] on the world-volume of $N$ D4-branes suspended between $k$ parallel NS5-branes arranged on a compact direction in spacetime as in Fig. 1. Following the usual conventions of [5] we choose our chain of $k$ NS5-branes to be located at $x^7 = x^8 = x^9 = 0$ and at specific values $\{x^6_i\}$ of the $x^6$-coordinate. The $x^6$ direction is compactified on a circle of radius $L$. There are $N$ (fractional) D4-branes between the $i-1$-th and $i$-th five-branes, with world-volumes parameterized by $x^0, x^1, x^2, x^3$ and $x^6$. The (classical) position of a D4-brane may be conveniently described by the complex variable $v = x^4 + ix^5$. At low energies this construction actually gives rise to a 4-dimensional $SU(N)^k \times U(1)$ theory with the desired matter content. The missing $U(1)$ factors corresponding to the center-of-mass

\[\text{As is well known, in the T-dual Type IIB picture, the quiver gauge theory may be obtained as the low-energy world volume dynamics of } N \text{ D3-branes at a } \mathbb{Z}_k \text{ orbifold singularity in an } A_{k-1} \text{ ALE space [2].}\]
Figure 1: The Type IIA setup of the elliptic model consisting of four-branes suspended between five-branes arranged on a compact direction.

Figure 2: $\mathcal{N} = 2^*$ SUSY Yang-Mills realized in Type IIA theory.

motion in the $x^4 - x^5$ plane of all the four-branes in a given segment between adjacent NS5-branes are IR-free and decouple. The distance between the $i$-th and $i+1$-th five-branes is directly related to the gauge coupling of the $i$-th SU($N$) factor, $g_i^{-2} = (x_{i+1}^6 - x_i^6)/(8\pi^2 g_s L)$ where $g_s$ is the Type IIA string coupling. Clearly then, the overall coupling $1/g^2 = \sum_i 1/g_i^2$ is proportional to the radius of the $x^6$-dimension and is also the gauge coupling for the U(1) factor. As we have already mentioned, the difference in positions of the centers-of-mass of two neighbouring stacks in Fig. 1 is frozen in the infra-red and is therefore a modulus which is identified with the the hypermultiplet mass-parameter $m_i$ in the low energy theory.

The periodicity of the spacetime implies that there are only $k-1$ independent masses with $\sum_{i=1}^k m_i = 0$. As explained in [5] this restriction may be avoided by modifying the M-theory spacetime so that upon going around the $x^6$ circle the value of $v$ is shifted at some point by an amount $m$ leading to

$$\sum_{i=1}^k m_i = m. \tag{2.4}$$

The parameter $m$ is often referred to as the “global mass”. The position of the global shift is completely unphysical and can be chosen anywhere. The simplest such setup (Fig. 2) with a single five-brane and $N$ D4-branes yields the SU($N$), $\mathcal{N} = 2$ theory with a massive adjoint hypermultiplet—the so-called $\mathcal{N} = 2^*$ theory, first analyzed by Donagi and Witten [3].

\footnote{Actually this is not quite true since the overall U(1) remains dynamical. But this decouples from the other fields and we ignore it in what follows.}
Soft breaking of the above set-up to the $\mathcal{N} = 1^*$ quiver theory is achieved by introducing generic masses $\mu_i$ for the adjoint chiral multiplets $\Phi_i$. This can be realized in the brane setup by rotating the NS5-branes relative to each other in the $(v, w)$-plane ($w = x^8 + ix^9$). This feature is described separately in Appendix B where we generalize the approach of [20] to the elliptic models.

2.1 Ground states of the $\mathcal{N} = 1^*$ quiver theory

In this section, we determine the vacuum states of the $\mathcal{N} = 1^*$ quiver theories generalizing the structure in the basic $\mathcal{N} = 1^*$ case [3, 21]. It is worth reviewing the latter in some detail because this will provide us with the necessary intuition for the quiver generalization.

In the mass deformed $\mathcal{N} = 4$ theory, the vacuum structure can be uncovered by first establishing a classical picture which can then be rather easily refined to give the full quantum description. The classical vacua are governed by the tree-level superpotential (2.3) for $k = 1$. As usual in an $\mathcal{N} = 1$ theory, we can avoid solving the $D$-flatness conditions by solving the $F$-flatness conditions modulo complex gauge transformations. In the $k = 1$ case, these are

\[ [\Phi, \Phi^\pm] = m\Phi^\pm, \]
\[ [\Phi^+, \Phi^-] = 2\mu\Phi. \]

modulo $\text{SL}(N, \mathbb{C})$. The independent solutions are associated to the partitions $N = q_1 + \cdots + q_p$ with

\[ \Phi_c = M_c \begin{pmatrix} J_c^{(q_1)} \\ J_c^{(q_2)} \\ \vdots \\ J_c^{(q_p)} \end{pmatrix}, \]

where $c = 1, 2, 3$, $\Phi \equiv \Phi_3$, $\Phi^\pm \equiv \Phi_1 \pm i\Phi_2$ and $J_c^{(q)}$ are the generators of the $q$-dimensional representation of $\text{SU}(2)$. The mass parameters are $M_1 = M_2 = \sqrt{2\mu m}$ and $M_3 = m$. The solution generically leaves abelian factors of the gauge group unbroken and the vacuum is massless. However, for equipartitions, $N = q \cdot p$, the unbroken gauge symmetry is $\text{SU}(p)$. In these cases, the non-abelian symmetry leads to confinement in the infra-red with a mass gap. The number of quantum vacua of an $\text{SU}(p)$ gauge theory with $\mathcal{N} = 1$ supersymmetry is $p$ and so the total number of massive vacua is

\[ \#(\text{massive vacua of } \mathcal{N} = 1^*) = \sum_{p|N} p, \]

a sum over the divisors $p$ of $N$. 
Figure 3: (a) The Coulomb branch singularity of $\mathcal{N} = 2^*$ SUSY Yang-Mills which yields the Higgs vacuum of $\mathcal{N} = 1^*$ theory. (b) The singular point with $\text{SU}(p)^q \times \text{U}(1)^{q-1}$ gauge symmetry which descends to the $\text{SU}(p)$ vacuum of $\mathcal{N} = 1^*$ theory.

The Type IIA brane picture

Before analyzing the classical vacuum structure of the $\mathcal{N} = 1^*$ quiver theory, it will be extremely instructive to see how the standard enumeration of massive vacua of $\mathcal{N} = 1^*$ theory (mass-deformed $\mathcal{N} = 4$ theory) described above arises from the viewpoint of the Type IIA brane picture. In particular, the massive vacua are associated to special (classical) brane configurations corresponding to maximally singular points on the Coulomb branch of the $\mathcal{N} = 2^*$ theory, where new massless (string) states appear. Importantly, the massless string states should give rise to charged hypermultiplets for all the classically visible U(1) factors at the aforementioned Coulomb branch singularities. This ensures that upon further soft breaking to $\mathcal{N} = 1^*$ all these U(1) factors get Higgsed and only a non-Abelian gauge symmetry survives and a mass gap is generated. It is easy to see that these requirements lead us to configurations where the branes are arranged in “helices” as in Figs. 3(a) and 3(b). The helices arise in the following way. With only an $\mathcal{N} = 2^*$ mass ($\mu = 0$) the $F$-flatness conditions (2.5a)-(2.5b) allow a more general solution for $\Phi$:

$$\Phi = m \begin{pmatrix}
c_1 \frac{1}{[q_1] \times [q_1]} + J_{3}^{(q_1)} \\
c_2 \frac{1}{[q_2] \times [q_2]} + J_{3}^{(q_2)} \\
\vdots \\
c_p \frac{1}{[q_p] \times [q_p]} + J_{3}^{(q_p)}
\end{pmatrix},$$

for arbitrary constants $c_r$. Each of the $p$ blocks corresponds to a helix and the parameters $c_r$ encode the centre-of-mass position of each helix. Notice that the pitch of a helix as one goes around $x^6$ is $m$, where $m$ is the $\mathcal{N} = 2^*$ global mass. In Fig. 3(a), there is a $\text{U}(1)^{N-1}$ gauge symmetry with massless hypermultiplets from strings stretching between two four-branes on either side of the five-brane, meeting each other at the same spacetime point. Each such hypermultiplet carries charges under two different U(1) factors. Thus in the low energy theory there are light states charged under each of the $N - 1$ U(1) factors. Soft breaking to $\mathcal{N} = 1^*$ will cause all these states to condense, as is apparent from the expressions for $\Phi^\pm$ in (2.6),
and therefore Higgses all the $U(1)$ factors. This is the Higgs vacuum. Fig. 3(b) illustrates a more general situation where the branes are arranged in $p$ coincident helices, where $p$ is a divisor of $N$. The gauge symmetry is $SU(p)^q \times U(1)^{q-1}$ with $pq = N$. The massless strings stretching across the five-brane, between two stacks of D4-branes meeting each other, are bi-fundamentals under the two $SU(p)$ factors and charged under the two $U(1)$’s associated with the motion the two stacks. Soft breaking to $\mathcal{N} = 1^*$ breaks the $U(1)$ factors and simultaneously Higgses the off-diagonal parts of the non-Abelian gauge symmetry leaving only an unbroken $SU(p)$ gauge group with $\mathcal{N} = 1$ supersymmetry. Since the low energy effective theory is $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $SU(p)$, each such configuration gives $p$ vacua, so that the total number of massive vacua of $\mathcal{N} = 1^*$ theory is given by the sum over divisors of $N$.

We can now extend the above picture fairly straightforwardly to the $\mathcal{N} = 1^*$ deformation of the $\mathcal{N} = 2$ elliptic models. As far as the massive vacua are concerned once again we expect the relevant Coulomb branch singularities to be those where a maximal number of massless hypermultiplets appear, Higgsing all the classically visible $U(1)$ factors upon soft breaking to $\mathcal{N} = 1$ supersymmetry. At these positions the branes arrange themselves in $k$ helices as shown in Fig. 4. As in the $\mathcal{N} = 1^*$ theory, the global mass sets the pitch of a helix as we go around the $x^6$ circle. We can choose the shift of $m$ to be anywhere, for instance between the $k^{th}$ and $1^{st}$ five-brane. The massive vacua correspond to precisely $k$ helices one of which begins, and one ends, on each of the $k$ five-branes. Any fewer than $k$ helices and one cannot accommodate the $k$ hypermultiplet masses. Any more, and there would be additional moduli in the configuration leading to a massless vacuum upon soft breaking to $\mathcal{N} = 1$. In general each of the $k$ helices can be composite consisting of a number of coincident $D4$-branes, yielding unbroken non-Abelian factors. For a massive vacuum it is necessary that the number in each helix be the same, namely $p$ a divisor of $N$: $pq = N$. The low energy theory at such a point (as in Fig. 4) on the moduli space of the $\mathcal{N} = 2^*$ quiver theory has $SU(p)^{kq} \times U(1)^{k(q-1)}$ gauge symmetry with $k(q-1)$ massless hypermultiplets. The latter originate, as before, from massless strings ending on two

**Figure 4:** A Coulomb branch singularity of the elliptic model with $SU(p)^{kq} \times U(1)^{k(q-1)}$ gauge symmetry. Upon soft breaking to $\mathcal{N} = 1$ SUSY it yields $p^k$ massive vacua.
coincident four-brane stacks, one on either side of an NS5. The $\mathcal{N} = 1^*$ deformation leads to the condensation of these massless hypermultiplets. This not only Higgses the classically visible U(1)’s, but also breaks the product gauge group associated with each composite helix to its diagonal SU($p$) subgroup. As there are $k$ composite helices, this leaves a classically unbroken SU($p$)$^k$ gauge theory with massive matter and $\mathcal{N} = 1$ supersymmetry which yields $p^k$ massive vacua in the quantum theory. There is an additional degeneracy that arises in the following way. There are $q = N/p$ stacks of D4-branes intersecting a five-brane from either side. Since there is one helix that starts and one that ends on each five-brane, $q - 1$ of the stacks are continuous across the five-brane but one has a jump. As the jump may be chosen to be at any of the $q$ stacks there is an additional degeneracy of $q$ per five-brane. Actually the total degeneracy turns out to be $q^{k-1}$ since the jump at one of the five-branes can be fixed without-loss-of-generality. Thus the total number of massive vacua of the $\mathcal{N} = 1^*$ quiver theory is

$$\#(\text{massive vacua of } \mathcal{N} = 1^* \text{ quiver}) = \sum_{p|N} p^k q^{k-1} = N^{k-1} \sum_{p|N} p . \quad (2.9)$$

In the following sections, we will show how the exact solution of the elliptic model can be used to reproduce this vacuum counting in the quantum theory. In addition the picture described above in terms of “four-brane helices” will have a natural origin in the quantum solution to be described from the M-theory point-of-view.

**Systematic semi-classical analysis**

After the intuitive discussion explored above, we now consider the problem of the vacuum structure of the quiver theories more systematically. As in the $\mathcal{N} = 1^*$ case, we first consider the classical vacua that arise from the tree-level superpotential (2.3). This can be re-cast as

$$W = \frac{1}{g^2} \text{Tr} \left\{ (\Phi + X)[\Phi^+, \Phi^-] - \frac{m}{k} \Phi^+ \Phi^- - \sum_i \mu_i \Phi_i^2 \right\} . \quad (2.10)$$

Here, $X$ is a diagonal matrix with $k$ independent parameters $x_i$ corresponding to the VEVs of the U(1)-components of the U($N$) gauge groups which are frozen in the infra-red. These parameters are determined in terms of the masses $m_i$ via

$$m_i - \frac{m}{k} = x_i - x_{i+1} . \quad (2.11)$$

Choosing $\sum_i x_i = 0$ then fixes the $x_i$. $\Phi$ and $\Phi^\pm$ are $kN \times kN$-dimensional matrices and if we choose an ordering for the elements so that the $i$th SU($N$) factor of the gauge group is associated to the $i$ mod $k$th row and columns, then

$$X = \text{diag}(x_1, \ldots, x_k, x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k) . \quad (2.12)$$

With this ordering $\Phi$ and $\Phi^\pm$ can only have non-zero elements in positions $(u, v)$, $1 \leq u, v \leq Nk$:

$$\Phi : \quad (u, u + nk) , \quad \Phi^\pm : \quad (u, u \pm 1 + nk) , \quad n \in \mathbb{Z} . \quad (2.13)$$
We also define $\Phi_i, i = 1, \ldots, k$ as the elements of $\Phi$ pertaining to the $i^{\text{th}}$ SU($N$) gauge group factor. The matrix $\Phi_i$ has non-zero elements in position $(i + mk, i + nk), m, n \in \mathbb{Z}$. The classical vacuum structure of the $\mathcal{N} = 1^*$ deformation is obtained from the $F$-flatness conditions

\begin{align}
[\Phi + X, \Phi^\pm] &= \frac{m}{k}\Phi^\pm, \\
\mathcal{P}\{[\Phi^+, \Phi^-]\} &= 2 \sum_i \mu_i \Phi_i,
\end{align}

modulo complex SL($N, \mathbb{C}$)$^k$ gauge transformation. In (2.14b) $\mathcal{P}$ is a projection onto the traceless part of each of the U($N$) factors.

Complex gauge transformations can be used to diagonalize $\Phi$. This uses up all of the complex gauge transformations apart from the abelian transformations that fix diagonal matrices and the Weyl group acting as permutations. The solutions of (2.14a) are associated to the partitions of the ordered set of $kN$ objects which are identified with the diagonal elements of $\Phi$:

\begin{equation}
\bigg\{ 1, \ldots, k, 1, \ldots, k, \ldots, 1, \ldots, k \bigg\} \longrightarrow \bigg\{ 1, \ldots, i_1, i_1 + 1, \ldots, i_2, \ldots, i_{n-1} + 1, \ldots, k \bigg\}. \tag{2.15}
\end{equation}

Here the labels $1, 2, \ldots, k$ refer to the $k$ SU($N$) factors. The subsets $\mathcal{A}_r$ are themselves ordered sets and we will define

$$\ell_r = \dim \mathcal{A}_r, \quad \sum_{r=1}^n \ell_r = Nk. \tag{2.16}$$

The labelling on the subsets $\mathcal{A}_r$ is irrelevant. In other words, a partition giving subsets $\mathcal{A}'_r$ is equivalent to the first if they are some re-labelling of the $\mathcal{A}_r$. This equivalence is due to the action of the Weyl group of SU($N$)$^k$. On the ordered set of $kN$ objects, the Weyl group of the $i^{\text{th}}$ SU($N$) gauge group factor acts by permuting the $i^{\text{th}}$, $i + k^{\text{th}}$, $\ldots$, $i + (kN - 1)^{\text{th}}$ elements:

$$\bigg\{ 1, \ldots, \hat{i}, \ldots, k, 1, \ldots, \hat{i}, \ldots, k, \ldots, 1, \ldots, \hat{i}, \ldots, k \bigg\}. \tag{2.16}$$

For example, take SU(2)$^2$. The partition $\{\{1, 2\}, \{1\}, \{2\}\}$ is equivalent to $\{\{1\}, \{2\}, \{1, 2\}\}$ but inequivalent from $\{\{1\}, \{2, 1\}, \{2\}\}$. In addition to these equivalences, there is a restriction on the allowed partitions, proved below, which requires that the set$^3$

$$\{i_1, i_2, \ldots, i_n\} \supset \{1, 2, \ldots, k\}. \tag{2.17}$$

$^3$Notice $i_n$ is fixed to be $k$. 

14
So in $SU(2)^2$, $\{1, 2, 1, 2\}$ and $\{1, 2\}$ are not allowed partitions, but $\{1, 2, 1\}$, $\{2\}$, for instance, is allowed.

For a given allowed partition

$$\Phi + X = \frac{m}{k} \begin{pmatrix} c_1 1_{[\ell_1] \times [\ell_1]} + J_3^{(\ell_1)} & c_2 1_{[\ell_2] \times [\ell_2]} + J_3^{(\ell_2)} & \cdots & c_n 1_{[\ell_n] \times [\ell_n]} + J_3^{(\ell_n)} \end{pmatrix}, \tag{2.18}$$

where $J_3^{(\ell)}$ is the diagonal generator of the $\ell$-dimensional representation of $SU(2)$. The solution can be visualized in the Type IIA brane picture as $n$ helices, the $r$th having $\ell_r/k$ complete rotations around $x^6$ and a centre-of-mass in the $v$-plane at $c_r$. Notice that the pitch of a helix as one goes around $x^6$ is $m$, where $m$ is the $\mathcal{N} = 2^*$ global mass. The $\Phi^\pm$ share the same block structure:

$$\Phi^\pm = \begin{pmatrix} K_1^\pm & & & \\ & K_2^\pm & & \\ & & \ddots & \\ & & & K_n^\pm \end{pmatrix}, \tag{2.19}$$

where the blocks $K_r^\pm$ have the same pattern of non-vanishing elements as $J_1^{(\ell_r)} \pm i J_2^{(\ell_r)}$, the conventional raising and lowering matrices of the $\ell_r$-dimensional $SU(2)$ representation. However, unlike in the $\mathcal{N} = 1^*$ case, the numerical values are not the same. Nevertheless, we can use the remaining abelian symmetries to set

$$K_r^- = \begin{pmatrix} 0 \\ 1 & 0 \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 1 \end{pmatrix}, \tag{2.20}$$

leaving

$$K_r^+ = \begin{pmatrix} 0 & k_1^{(r)} & \cdots & \cdots \\ \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & k_{\ell_r - 1}^{(r)} \end{pmatrix} \tag{2.21}$$

for constants $k_l^{(r)}$, $l = 1, \ldots, \ell_r - 1$. Notice that the non-zero elements of $\Phi$ and $\Phi^\pm$ are consistent with the grading (2.13).

The restriction on the allowed partitions (2.17) means that that there must be at least one block of the partition that ends on each of the $k$ gauge group factors, \textit{i.e.} at least one of the $i$, \textit{\ldots}}
equals each of $1, 2, \ldots, k$. Suppose this were not true for the $i$th gauge group factor. By taking the difference of the trace of (2.18) for the $i$th and $i+1$th gauge group factors, one would find
\[ x_i - x_{i+1} = -\frac{m}{k}, \] (2.22)
in contradiction with $m_i$ being generic in (2.11). In particular, notice that this means the allowed partitions must have $n \geq k$; in other words there must be a least $k$ helices. Notice that in terms of the Type IIA diagrams that the condition means that at least one helix must end (and consequently at least one must start) on each NS5-brane.

Finally, we must solve (2.14b). This equation can be re-cast as
\[ [\Phi^+, \Phi^-] = 2 \sum_i \mu_i \Phi_i + A, \] (2.23)
where $A$ is accounts for the trace parts of the $U(N)$ factors:
\[ A = \text{diag}(a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots, a_1, \ldots, a_k), \] (2.24)
for, as yet, arbitrary constants $a_i$. All in all, we have $n + k + kN - n$ complex unknowns in $c_r$, $r = 1, \ldots, n$, $a_i$, $i = 1, \ldots, k$ and $k^{(r)}_l$, $l = 1, \ldots, \ell_r - 1$ and $r = 1, \ldots, n$. These are subject to the following conditions. Firstly, $k$ complex conditions arise from the tracelessness of $\Phi$, (2.18), in each of the $SU(N)$ factors. Secondly Eq. (2.23) gives $Nk$ complex conditions. So there are $(N+1)k$ complex conditions for $(N+1)k$ complex unknowns and so generically there exists a solution for each allowed partition. Generically, the solutions have no additional moduli and so the low energy theories will be pure $N = 1$ supersymmetric gauge theory with a gauge group that is the subgroup of $SU(N)^k$ fixing $\Phi$ and $\Phi^\pm$. A given allowed partition generically leads to a solution with unbroken $U(1)$ factors and so to a massless, or Coulomb, vacuum. However, for very special partitions, the unbroken gauge group is empty or non-abelian giving Higgs and confining vacua with a mass gap. We now identify these massive vacua. Firstly, there are partitions into $n = k$, the smallest number of possible, subsets:
\[ \text{Higgs Partition} = \{A_1, A_2, \ldots, A_k\}. \] (2.25)
For these vacua, the unbroken gauge group is empty, so they are Higgs vacua. The number of them can be determined as follows. Since $n = k$, there is a single $i_r$ in (2.15) associated to each of the $k$ gauge group factors. A given $i_r$ can therefore be situated in one of $N$ places. However, $i_k$ is fixed, so the total degeneracy of Higgs vacua is $N^{k-1}$. The partition with into $Nk$ sets $A_r$ with a single element each corresponds to $\Phi = \Phi^\pm = 0$. This vacuum leaves unbroken the whole $SU(N)^k$ gauge group, and since each $SU(N)$ factor yields $N$ confining quantum vacua, overall there are $N^k$ massive vacua of this type. More generally there are a number of massive vacua associated to each divisor of $N$, $N = pq$. The associated partition are of the form
\[ \text{Confining Partition} = \left\{ \underbrace{A_1, \ldots, A_1}_{p \text{ times}}, \underbrace{A_2, \ldots, A_2}_{p \text{ times}}, \ldots, \underbrace{A_k, \ldots, A_k}_{p \text{ times}} \right\}, \] (2.26)
where \( \sum_{i=1}^{k} \ell_i = qk \) and \( \{ A_1, \ldots, A_k \} \) is an allowed—Higgs vacuum—partition of an SU\((q)^k\) theory. The unbroken gauge group is SU\((p)^k\), giving a quantum degeneracy of \( p^k \), and since there are \( q^{k-1} \) inequivalent Higgs vacua for an SU\((q)^k\) theory, the total number of massive vacua associated to the divisor \( p \) of \( N \), is \( q^{k-1}p^k = N^{k-1}p \). This gives the total number of massive vacua as in Eq. (2.9). Hence, our more intuitive discussion in terms of helices produces the right combinatorics as this more detailed semi-classical analysis.

An example is in order. Consider the SU\((2)^2\) theory. The vacua are associated to the partitions of the ordered set \( \{1,2,1,2\} \). The partitions \( \{\{1,2,1,2\}\} \) and \( \{\{1,2\},\{1,2\}\} \) are ruled out by the restriction (2.17). This leaves: (i) \( \{\{1,2,1\},\{2\}\} \) and \( \{\{1\},\{2,1,2\}\} \) which correspond to two inequivalent Higgs vacua; (ii) \( \{\{1,2\},\{1\},\{2\}\}, \{\{1\},\{2\},\{2,1\}\} \) which have unbroken gauge group U\((1)\) and so are massless; and (iii) \( \{\{1\},\{2\},\{1\},\{2\}\} \) which leads to an unbroken SU\((2)^2\) gauge symmetry and four massive confining vacua in the quantum theory. Overall there are six massive vacua which is consistent with (2.9) and two additional massless vacua.

**Phase structure**

We end this section on the (semi-)classical analysis of the \( \mathcal{N} = 1^* \) quiver theory with some remarks on the phase structure of the massive vacua that we found above. Each SU\((N)\) factor in the quiver has matter fields in the fundamental representation which therefore transform under the center of that SU\((N)\) factor. However the matter fields are neutral under a simultaneous gauge rotation in all the SU\((N)\) factors by an element of the center \( U = e^{2\pi ia/N}1_{[N] \times [N]}, \ a = 0, \ldots, N-1 \). This can be interpreted as the center of the diagonal gauge group SU\((N)_D\) under which all fields appear to transform in the adjoint representation. This may then be used to classify the phases of the \( \mathcal{N} = 1^* \) quiver theory. However since there is only one kind of \( \mathbb{Z}_N \) symmetry—the center \( (\mathbb{Z}_N)_D \) of the diagonal SU\((N)\), the number of possible phases is much smaller than the number of massive vacua. As in the case of the basic \( \mathcal{N} = 1^* \) theory [3,11], the possible phases are given by the order \( N \) subgroups of \( (\mathbb{Z}_N)_D \times (\mathbb{Z}_N)_D \). The latter characterizes the \((\mathbb{Z}_N)_D\)-valued electric \((n_e,0)\) and magnetic \((0,n_m)\) charges respectively, of the sources used to probe the theory. This implies, for instance, that all the vacua with unbroken SU\((p)^k\) gauge symmetry may be classified into \( p \) distinct phases with \( N^{k-1} \) vacua in each phase. These \( p \) phases correspond to the subgroups of \( (\mathbb{Z}_N)_D \times (\mathbb{Z}_N)_D \), generated by the screened charges \((p,0)\) and \((l,q)\) with \( pq = N \) and \( l = 0,1,\ldots,p-1 \). The \( N^{k-1} \) Higgs vacua correspond to \( p = 1 \). Apart from computing the condensates, there is however no physical way to distinguish the \( N^{k-1} \) vacua in a given phase of the diagonal gauge group. Recall that there are two sources of the degeneracy of vacua in a given phase of the diagonal gauge group: (i) the multiplicity \( p^{k-1} \) associated with the Witten index for the remaining \( k-1 \) SU\((p)\) factors (excluding the diagonal one); and (ii) a factor of \( q^{k-1} \) associated to the position of the jumps in the four-brane helices. In the limit of infinitely massive matter fields, the former may be associated with the
emergence of an accidental $\mathbb{Z}_{2p}$ symmetry in each SU($p$) factor which is spontaneously broken to $\mathbb{Z}_2$ by gaugino condensation in each factor. The additional degeneracy of $q^{k-1}$ has a more unusual interpretation which we return to later.

3. Lifting to M Theory

3.1 Review of the solution to the elliptic model

We now consider the solution of the elliptic model, i.e. the $\mathcal{N} = 2^*$ quiver theory with the goal of extracting exact results for the holomorphic observables and vacuum structure of the $\mathcal{N} = 1^*$ quiver theory. To this end, we will follow the prescription outlined in [5] and begin with a brief review.

The exact low energy effective action for the elliptic model can be obtained from the Type IIA brane setup in the limit of large string coupling $g_s \gg 1$ accompanied by a simultaneous rescaling of the five-brane separations so that the $k$ gauge couplings $\{\tau_i\}$ are unchanged. In this limit the compact M-dimension $x^{10}$ opens up and the system of intersecting branes in Type IIA lifts to a single smooth M5-brane in M-theory. There are now two circles in spacetime: the compact $x^6$ dimension with radius $L$ and the M-theory dimension $x^{10}$ with radius $R$. The locations of the IIA NS5-branes are conveniently described by the holomorphic coordinate $z = x^6 + ix^{10}$. The $(v, z)$ space is locally $\mathbb{C} \times E_\tau$, with $v \in \mathbb{C}$, $z \in E_\tau$ and $E_\tau$ a genus one Riemann surface. Specifically, the complex structure $\tau$ of the spacetime torus $E_\tau$ is determined by obtaining it as a quotient of the $z$-plane by the two equivalences

\begin{align}
(i) & \quad x^6 \rightarrow x^6 + 2\pi L, \text{ combined with } x^{10} \rightarrow x^{10} + \theta R, \\
(ii) & \quad x^{10} \rightarrow x^{10} + 2\pi R.
\end{align}

Thus $E_\tau$ has complex structure $\tau = iL/R + \theta/2\pi$ which is the coupling associated with the overall U(1) factor in the IIA setup; $L/R = 4\pi/g^2 = 4\pi \sum_i 1/g_i^2$ and $\theta = \sum_i \theta_i$. The $x^{10}$-positions of the IIA NS5-branes can be directly related to the thet a-angles of each gauge group factor in the quiver,

$$\theta_i = \frac{x_{i+1}^{10} - x_i^{10}}{R}. \quad (3.2)$$

Thus the M-theory spacetime contains a torus $E_\tau$ with complex structure $\tau$ and $k$ distinct, unordered marked points $z_i, i = 1, \ldots, k$, to be identified with the positions of the NS5-branes. Different choices of $\tau$ and variations in the positions of the NS5-branes lead to different gauge couplings and theta angles for the SU($N$) factors in the gauge group; explicitly

$$\tau_i = \frac{i(z_{i+1} - z_i)}{2\pi R}, \quad (3.3)$$
where the $z_i$'s are labelled according to their order around $x^6$. In the following, for convenience, we re-scale the torus $E_{\tau}$ so that the M-theory radius $R$ is unity. This makes the $z_i$'s dimensionless.

In the presence of the overall mass shift $m$ described earlier, the $(v, z)$ space in which the M5-brane propagates is to be thought of as a $\mathbb{C}$ bundle over $E_{\tau}$ which is defined as a quotient $\mathbb{Q}_m$ obtained by dividing by the combined operations

$$x^6 \rightarrow x^6 + 2\pi L,$$
$$x^{10} \rightarrow x^{10} + \theta R,$$
$$v \rightarrow v + m. \quad (3.4)$$

This twisting of the spacetime can be undone almost everywhere at the expense of a special point on $E_{\tau}$, the location of which is arbitrary, at which certain discontinuities will occur. Although arbitrary, this point (chosen at $z = 0$ below) will appear in the solution of the model as we see below.

The M5-brane itself has world-volume $\Sigma \times \mathbb{R}^{1,3}$ where $\Sigma$ is a Riemann surface embedded in $\mathbb{Q}_m$. This is the Seiberg-Witten curve for the theory. The Type IIA setup of intersecting branes naturally lifts to a Riemann surface $\Sigma$ with genus $(N - 1)k + 1$ which is precisely the rank of the low-energy gauge group. More importantly, the curve $\Sigma$ is a branched $N$-fold cover of the torus $E_{\tau}$. This is clear from the Type IIA limit wherein the curve reproduces $N$ D4-branes, each wrapped around $E_{\tau}$. This curve is described by the equation

$$F(z, v) = v^N - f_1(z)v^{N-1} + f_2(z)v^{N-2} + \ldots + (-1)^N f_N(z) = \prod_{a=1}^{N} (v - v_a(z)) = 0, \quad (3.5)$$

with $z \in E_{\tau} \simeq \mathbb{C}/\Gamma$ where $\Gamma$ is the lattice $2\omega_1 \mathbb{Z} \oplus 2\omega_2 \mathbb{Z}$ with $\omega_2/\omega_1 = \tau$. Thus, for every point $z$ in $E_{\tau}$, the equation (3.5) has $N$ roots representing the $v$-coordinates of the $N$ D4-branes. The locations $z_i$ of simple poles of the $v_a(z)$ represent the positions of the NS5-branes. Let us now recall in detail following Witten [5] the properties of the functions $v_a(z)$ and $f_a(z)$.

The functions $f_a(z)$ are: (i) single-valued elliptic functions on $E_{\tau}$ with only simple poles at $z = z_i$ and a pole of order $a$ at $z = 0$; and (ii) the singularity at $z = 0$ may be removed by a linear re-definition of $v$. These imply that the $v_a(z)$ which need not be single-valued on $E_{\tau}$ must have the properties: (i) that near $z = z_i$, exactly one of the $v_a(z)$ has a simple pole; and (ii) all the $v_a$'s have a simple pole at $z = 0$ (the special point of discontinuity described above) with exactly the same residue.

Note that our parameterization of the curve differs from that presented in [5] which was written in terms of variables $x$ and $y$ specifying a point on $E_{\tau}$ via the Weierstrass equation $y^2 = 4x^3 - g_2(\tau; \omega_1, \omega_2)x - g_3(\tau; \omega_1, \omega_2)$. This is in fact solved by $x = \wp(z; \omega_1, \omega_2)$ and $y = \wp'(z; \omega_1, \omega_2)$ where $\wp(z; \omega_1, \omega_2)$ is the Weierstrass function.
The parameters of the theory, namely the hypermultiplet bare masses $m_i$ and the order parameters on the Coulomb branch of the $\mathcal{N} = 2$ elliptic model must be encoded in the solution $F(z, v) = 0$. The hypermultiplet bare masses are contained in the singular part of $f_1(z) = \sum_a v_a(z)$. In particular $m_i$ is equal to $1/N$ times the residue at $z = z_i$ (the location of an NS5-brane) \[3, 5\]. Since $f_1(z)$ is meromorphic on $E_\tau$, the sum of its residues must vanish and therefore it must have yet another simple pole which may be chosen at $z = 0$ with residue $N \sum_i m_i = N m$ incorporating the global mass. The constant part of $f_1(z)$ is the order parameter for the overall $U(1)$ factor in the elliptic model. Each of the remaining $N - 1$ functions $f_a(z)$ have $k - 1$ independent residues at the simple poles and a constant piece leading to a total of $(N - 1)k$ parameters. This is consistent with the Riemann-Roch Theorem which determines the space of meromorphic functions on $E_\tau$ with $k$ distinct simple poles to be $k$-dimensional. These $(N - 1)k$ parameters are to be identified with the order parameters on the Coulomb branch of the $SU(N)^k$ elliptic model. In the following sections our goal will be to compute these order parameters at the special singular points on the Coulomb branch which descend to $\mathcal{N} = 1$ vacua upon mass perturbation of the theory.

3.2 Duality symmetries

The $\mathcal{N} = 2$ quiver theories (without hypermultiplet masses) have a nontrivial duality symmetry that is a generalization of the $SL(2, \mathbb{Z})$ duality group of the $\mathcal{N} = 4$ theory. This duality symmetry is manifest in the Type IIA/M-theory construction of the quiver theory at its conformal point where all VEVs vanish. At this point the $N$ D4-branes lift to an M5-brane which is multiply wrapped around the M-theory spacetime torus $E_\tau$. The low-energy four dimensional field theory then inherits the duality symmetry of M-theory under the action of $SL(2, \mathbb{Z})$ on the complex structure of the spacetime torus $E_\tau$. This duality acts on the overall gauge coupling $\tau$ in the usual way and may be interpreted as modular invariance of the diagonal $SU(N)$ factor.

However, in addition to this the quiver theory has a much more unusual symmetry at the origin of the Coulomb branch which we now describe. The $k$ individual gauge-couplings are encoded in the complex structure of $E_\tau$ and the positions of the $k$ indistinguishable marked points $z_i$, representing the five-brane positions. Moving a given NS5-brane all the way around one of the cycles of the torus (with the remaining $k - 1$ NS5’s fixed), leaves the brane configuration unaltered and therefore is a symmetry of the low energy theory. But this operation changes the gauge couplings of the individual $SU(N)$ factors in the quiver as they are given by

\[ \text{In the parameterization of [5] the masses are } 1/N \text{ times the residues of the differential form } f_1(x, y)dx/y \text{ which is precisely the residue of } f_1(z)dz \text{ in our notation.} \]
separations of the five-branes. The positions of the latter change according to
\[ \begin{align*}
  z_i & \rightarrow z_i + 2m\omega_1 + 2n\omega_2; \quad m, n \in \mathbb{Z}, \\
  z_l & \rightarrow z_l, \quad \text{all } l \neq i.
\end{align*} \]
Clearly this transformation includes the familiar shift in the theta angles \( \theta_i \rightarrow \theta_i + 2m\pi \) and \( \theta_i+1 \rightarrow \theta_i+1 + 2m\pi \) which is a symmetry of the theory. However, in addition the transformation also implies a periodicity of the inverse gauge couplings \( 1/g_i^2 \) with period \( 1/g^2 \). In the T-dual Type IIB picture of the quiver theory, the periodicity of the inverse gauge couplings arises from the periodicity of the scalar fields obtained by integrating the NS-NS 2-form over the vanishing 2-cycles at the orbifold singularity [22, 23].

As in the case of the \( \mathcal{N} = 4 \) theory the duality group must be “spontaneously broken” at a generic point on the Coulomb branch of the \( A_{k-1} \) quiver theory. Specifically, the Coulomb branch of the quiver theory contains certain special singular points. These are points which become \( \mathcal{N} = 1 \) vacua upon perturbing the theory with non-zero hypermultiplet and adjoint masses, \( m_i \neq 0 \) and \( \mu_i \neq 0 \). At these singularities new massless BPS states appear in the low energy theory and this is signalled by the degeneration of the corresponding cycles of the Seiberg-Witten curve. As in the case of mass deformed \( \mathcal{N} = 4 \) SYM, one expects each massive vacuum of the \( \mathcal{N} = 1^* \) quiver theory to be associated with the condensation of a specific BPS state belonging to the spectrum of the parent conformal quiver theory. As the generalized duality symmetry of the quiver model described above must leave the spectrum of the conformal theory invariant, it therefore acts on the BPS states in the spectrum via permutation. This in turn implies that the action of the duality group takes the \( \mathcal{N} = 1^* \) quiver theory from one vacuum to another since each of these vacua is associated with the condensation of a specific BPS particle in the theory.

In addition to the above duality properties, we will argue below that the low-energy physics, in particular the chiral sector of the \( \mathcal{N} = 1^* \) quiver theory in each vacuum is invariant under a symmetry which we will refer to as extended \( \tilde{S} \)-duality.

### 3.3 The quantum vacuum structure of the \( \mathcal{N} = 1^* \) quiver theory

We now turn to the description of the quantum vacuum structure of the \( \mathcal{N} = 1^* \) quiver theory. As mentioned above, and described in Appendix B from the point of view of brane rotation, the massive vacua of the theory correspond to the singular points on the Coulomb branch of the \( \mathcal{N} = 2^* \) quiver theory where the Seiberg-Witten curve undergoes maximal degeneration to a surface of genus one.

The locations of these special singular points on the \( \mathcal{N} = 2 \) moduli space are completely specified by the \( k(N - 1) \) gauge-invariant order parameters \( u_a^{(i)} \equiv \langle \text{Tr } \Phi_i^a \rangle, \quad i = 1, 2, \ldots k, \) and
Figure 5: The N-fold cover of $E_\tau$ with $\tilde{\tau} = q\tau/p$. The $\times$’s represent the positions of the $k$ NS5-branes.

$a = 2, \ldots, N$. As we have seen above, these order parameters are encoded in the curves for the theory (modulo ambiguities to be discussed later). The values of these order parameters at the singular points (as a function of the $k$ couplings $\tau_i$ and mass parameters $m_i$) yield the condensates of the $\mathcal{N} = 1^*$ quiver theory and provide complete information on the chiral sector of that theory.

At the special singular points which descend to massive $\mathcal{N} = 1$ vacua, the Seiberg-Witten curve degenerates maximally to an unbranched (unramified) $N$-fold cover of the spacetime torus $E_\tau$. This fact and the associated ellipticity properties will allow us to explicitly construct the functions $v_a(z)$ at the singular points in the moduli space of the curve Eq. (3.5) and extract the condensates for the $\mathcal{N} = 1$ theory.

An unbranched $N$-fold cover of $E_\tau$ is also a torus, but with a different complex structure $\tilde{\tau}$. As noted in [3] and [12] such covers are classified by 3 integers $p$, $q$ and $l$ where $pq = N$ and $l = 0, 1, \ldots, p - 1$. They are genus one Riemann surfaces $E_{\tilde{\tau}}$ with the complex structure $\tilde{\tau} = (q\tau + l)/p$. $E_{\tilde{\tau}}$ can be defined as usual as the quotient of the complex plane by the $\tilde{\tau}$-lattice, namely $E_{\tilde{\tau}} \simeq \mathbb{C}/\Gamma_{\tilde{\tau}}$. In what follows, we focus primarily on the cases with $l = 0$, since the more general covering can be obtained by suitable modular transformations. Fig. 5 illustrates the case $\Gamma_{\tilde{\tau}} \simeq \tilde{\omega}_1\mathbb{Z} \oplus \tilde{\omega}_2\mathbb{Z}$ with $\tilde{\omega}_1 = p\omega_1$ and $\tilde{\omega}_2 = q\omega_2$ and $\tilde{\tau} = \tilde{\omega}_2/\tilde{\omega}_1$. The fundamental parallelogram of the $\tilde{\tau}$ lattice shown in Fig. 5 consists of $N$-copies of the fundamental parallelogram of the $\tau$-lattice with $l = 0$. The NS5-branes which extend in the 012345 directions are represented by marked points (the $\times$’s) on the $\tilde{\tau}$-torus. Each $\tau$-parallelogram represents a single Type IIA four-brane that has grown an extra dimension in the $\omega_1$ direction (which is roughly speaking, the M-dimension).

The Seiberg-Witten curve controls the physics of the holomorphic sector of the theory and as discussed above, at the maximally singular points, the curve degenerates into a torus with $p = a, \ldots, N$. As we have seen above, these order parameters are encoded in the curves for the theory (modulo ambiguities to be discussed later). The values of these order parameters at the singular points (as a function of the $k$ couplings $\tau_i$ and mass parameters $m_i$) yield the condensates of the $\mathcal{N} = 1^*$ quiver theory and provide complete information on the chiral sector of that theory.

Note usually in $\mathcal{N} = 2$ theories as in [10] the curves maximally degenerate into a sphere. In the elliptic models the curves degenerate at most into a genus one Riemann surface. In the M-theory construction this reflects the presence of the unbroken and decoupled U(1) factor.

---

6Here, we use the same symbol to denote the chiral superfield and its lowest component.

7Note usually in $\mathcal{N} = 2$ theories as in [10] the curves maximally degenerate into a sphere. In the elliptic models the curves degenerate at most into a genus one Riemann surface. In the M-theory construction this reflects the presence of the unbroken and decoupled U(1) factor.
complex structure $\tilde{\tau}$. As usual, the low-energy physics depends only on the complex structure $\tilde{\tau}$ and must be invariant under modular transformations acting on $\tilde{\tau}$. Thus VEVs of chiral operators in each vacuum of the $\mathcal{N} = 1^*$ quiver theory are expected to be modular functions of $\tilde{\tau}$. This duality, referred to as $\tilde{S}$-duality [12] in the $\mathcal{N} = 1^*$ theory is further extended in the case of the mass-deformed quiver theories. In addition to modular transformations on $\tilde{\tau}$, we may also move each NS5-brane independently around non-trivial cycles of the $\tilde{\tau}$-torus, inducing corresponding shifts in the individual gauge-couplings. Such $\tilde{\tau}$-elliptic shifts in the positions of each NS5-brane are also symmetries of each vacuum of the $\mathcal{N} = 1^*$ quiver theory and form an extended version of $\tilde{S}$-duality.

It is worth noting that there is a simple and direct mapping between the $\tilde{\tau}$-lattice sketched in Fig. 5 and the Type IIA picture for a massive $\text{SU}(p)^k$ vacuum. Recall that $v_a(z)$ is simply the $v$-coordinate of the $a$-th Type IIA four-brane expressed as a function of the $(x^6, x^{10})$ coordinates of the compact dimensions. In the classical (Type IIA) limit the $x^{10}$ dimension is vanishingly small and the $\tilde{\tau}$-parallelogram describes $p$ coincident four-brane helices. In the classical limit all the NS5’s are lined along the $x^6$ direction leading to discontinuities in the four-brane helices. Quantum effects (or the growth of the M-dimension) not only cause the four-branes to grow an extra dimension but also lead to a separation between the helices in the $x^{10}$-direction.

**Vacuum counting in the quantum theory**

Let us now see how the quantum theory reproduces the degeneracy factor of $pN^{k-1}$ associated with each classical $\text{SU}(p)^k$ vacuum. Recall that the duality group of the $A_{k-1}$ quiver gauge theory acts on the massive vacua of the $\mathcal{N} = 1^*$ quiver model via permutations. Thus one may sweep out all the vacua of the theory by performing the duality operations on a given vacuum configuration in the IIA/M-theory setup.

Firstly, for a given $p$ that divides $N$ there are $p$ inequivalent $N$-fold covers of $E_{\tau}$ related to one another by $\text{SL}(2, \mathbb{Z})$ operations on $\tau$ that sweep out the $p$ values of $\tilde{\tau} = (q\tau + l)/p$ with $l = 0, 1, \ldots, p - 1$. Secondly, each NS5-brane may be shifted by integer multiples of $\omega_1$ and $\omega_2$ and placed in any one of the $N \tau$-parallelograms that make up the $N$-fold cover. While the first operation constitutes the action of $\text{SL}(2, \mathbb{Z})$ on $E_{\tau}$, the second represents the motion of an NS5-brane around the cycles of the spacetime torus $E_{\tau}$. These operations are precisely the duality symmetries of the parent conformal theory discussed in the previous section. Thus they act on the vacua of the $\mathcal{N} = 1^*$ quiver by permutation. Since only the relative separations (and not the actual positions) of the NS5’s are physically meaningful, and each five-brane may intersect any one of the $N$ four-branes, there are $N^{k-1}$ such distinct configurations. Thus the total number of vacua from configurations with a classically unbroken $\text{SU}(p)^k$ gauge symmetry is $pN^{k-1}$. We have already argued that the degeneracy factor of $p$ arises from $\text{SL}(2, \mathbb{Z})$ transformations on the coupling of the diagonal part of the gauge group. Displacing the NS5-branes by integer
multiples of $\omega_1$ leads to a shift in the theta angles $\theta_i$ of each gauge group factor in the quiver and yields $p^{k-1}$ distinct configurations. Moving the five-branes along the other period gives rise to $q^{k-1}$ ground states. The former is in line with our semiclassical arguments wherein a multiplicity of $p^k$ appears as a consequence of the Witten index for $\mathcal{N} = 1$ SUSY gauge theories with massive matter and SU$(p)^k$ gauge symmetry. The classically obscure degeneracy factor of $q^{k-1}$ associated with the position of the discontinuities in the four-brane helices in the Type IIA limit, can now be clearly understood to be a manifestation of the non-trivial duality symmetry of the underlying conformal quiver theory.

The singular curves and condensates

The calculation of the condensates of chiral operators in the vacua with $\mathcal{N} = 1$ SUSY relies on the key property that at the associated singular points in the $\mathcal{N} = 2$ moduli space the curve $\Sigma$ degenerates into the torus $E_\tau$. This translates directly into the requirement that the functions $v_a(z)$ describing the individual four-brane positions be invariant under a $\tilde{\tau}$-elliptic transformation i.e. under translations of $z$ by periods of $E_\tau$. At this point the functions $v_a(z)$ must be thought of as different branches of a complex function that is single-valued on $E_{\tilde{\tau}}$ but not on $E_\tau$. Translations by periods of the $\tau$-torus will smoothly take us from one branch to the next i.e. from one four-brane to the next. In particular the complete set of functions $v_a(z)$ will still be invariant under periodic shifts on $E_{\tilde{\tau}}$, but the individual $v_a$’s will get permuted by this action. We can readily write down the explicit form of these functions at the singular points. It will be sufficient to consider a generic vacuum with $N = pq$ and $l = 0$. All other massive vacua may be obtained by the operations generating the duality group of the theory.

In a vacuum with a classically unbroken SU$(p)^k$ gauge symmetry, it is convenient to denote the positions of the four-branes with two subscripts $v_{sr}(z) \equiv v_{sr}(z)$, $s = 0, 1, 2, \ldots p - 1$; $r = 0, 1, 2, \ldots q - 1$. It is well-known that every elliptic function has an expansion in terms of the Weierstrass Zeta function $\zeta(z)$ (see Appendix A for details) and its derivatives [24]. This function has only a simple pole at $z = 0$ with unit residue. Hence, knowledge of the singularities of an elliptic function and associated residues completely determines its expansion in terms of the zeta function and its derivatives. In addition, the Weierstrass Zeta function $\zeta(z|\tau)$ on a torus with complex structure $\tau$ has the following anomalous transformation property:

$$\zeta(z + 2\omega_\ell|\tau) = \zeta(z|\tau) + 2\zeta(\omega_\ell|\tau), \quad \ell = 1, 2. \quad (3.7)$$

We have argued that the four-brane positions $v_{sr}(z)$ must be elliptic on the $\tilde{\tau}$-torus. Thus they must have an expansion in terms of the zeta function on the $\tilde{\tau}$-torus, which we denote as $\tilde{\zeta}(z)$:

$$\tilde{\zeta}(z) \equiv \zeta(z|\tilde{\tau}); \quad \tilde{\zeta}(z + 2p\omega_1) = \tilde{\zeta}(z) + 2\tilde{\zeta}(p\omega_1); \quad \tilde{\zeta}(z + 2q\omega_2) = \tilde{\zeta}(z) + 2\tilde{\zeta}(q\omega_2). \quad (3.8)$$

Based on the properties of $v_{sr}(z)$ reviewed in the Section 4.1, we also know that these functions can only have simple poles located at the positions of the NS5-branes with residues
given by the hypermultiplet masses $Nm_i$. Finally, we must also have that at each NS5 location, one and only one $v_{sr}(z)$ has a simple pole. Following these requirements, we can deduce the form of the four-brane coordinates $v_{sr}(z)$. In particular, taking the vacuum with $l = 0$ where the $i$-th NS5-brane lies on the $(s_i, r_i)$ sheet of the covering, we have

$$v_{sr}(z) = N \sum_{i=1}^{k} m_i \tilde{\zeta}(z - z_i + 2(s - s_i)\omega_1 + 2(r - r_i)\omega_2)$$

$$- m \left\{ \sum_{t=0}^{p-1} \sum_{u=0}^{q-1} \tilde{\zeta}(z + 2t\omega_1 + 2u\omega_2) + 2qs \tilde{\zeta}(p\omega_1) + 2pr \tilde{\zeta}(q\omega_2) \right\}^{s = 0, 1, 2, \ldots p - 1; \ r = 0, 1, 2, \ldots q - 1.} \tag{3.9}$$

Note in particular that this satisfies the requirement that at each five-brane position, exactly one $v_{sr}(z)$, namely $v_{s_i r_i}(z)$, has a simple pole and the residue is given by the hypermultiplet mass: $Nm_i$. All the $v_{sr}(z)$ also have a simple pole at $z = 0$ with exactly the same residue $-m$. Our expressions Eq.(3.9) for the four-brane coordinates $v_{sr}(z)$ are uniquely determined up to additive constants, by the required singularity structure and the property of $\tau$-ellipticity. However, these additive constants do not affect the computation of condensates of gauge-invariant chiral operators. Due to the anomalous transformation properties of the zeta functions, the functions $v_{sr}(z)$ transform into one another upon going around one of the cycles of $E_\tau$. For example, $v_{sr}(z + 2\omega_1) = v_{s+1 r}(z)$ and $v_{sr}(z + 2\omega_2) = v_{s r+1}(z)$ where the subscripts are defined modulo $(p, q)$ as usual. Going around the periodic dimensions takes us smoothly from one four-brane to the next. This reflects the fact that the Type IIA four-branes lifted to M-theory form an unbranched (unramified) $N$-fold cover of the torus $E_\tau$. The other vacuum configurations with $l \neq 0$, may be obtained by using suitable SL$(2, \mathbb{Z})$ modular transformations on the overall coupling $\tau$.

We will now argue that the condensate $u_2^{(i)} = \langle \text{Tr} \Phi_i^2 \rangle$, namely the expectation value of the adjoint scalar in the $i$-th SU$(N)$ factor is encoded in the function

$$u_2(z) = \frac{1}{2N} \sum_{s,s' = 1}^{p} \sum_{r,r' = 1}^{q} (v_{sr}(z) - v_{sr'}(z))^2. \tag{3.10}$$

The simplest way to see that this is the relevant object is to first recall the classical (Type IIA) limit. In this limit the $v_{sr}(z)$ corresponding to the four-brane positions become independent of $z$ at any given point between two adjacent five-branes. The only position dependence in the four-brane coordinates arises from discontinuities or breaks as we move across an NS5-brane. In the classical limit, the positions of the D4-branes in a given gauge group factor correspond precisely to the eigenvalues of the adjoint scalar $\Phi_i$ in that gauge group factor. It is then straightforward to show that classically $\langle \text{Tr} \Phi_i^2 \rangle = \frac{1}{2N} \sum_{s,s' = 1}^{p} \sum_{r,r' = 1}^{q} (v_{sr} - v_{sr'})^2$ provided $\Phi_i$ is defined to be traceless.
When the same quantity is lifted to M-theory it naturally acquires non-trivial position-dependence ($z$-dependence) in that the classical discontinuities at the five-brane locations get smoothed out and the simple geometrical relationship between the classical condensates $u_2^{(i)}$ and Eq (3.10) is lost. Now $u_2$ is a smooth function of $z$ with $k$ double poles and $k$ simple poles. The strengths of the double poles are simply determined by the mass parameters $m_i$, while the values of the condensates $u_2^{(i)}$ are encoded in the residues at the simple poles and the constant part of $u_2(z)$. The simple pole residues and the constant part of $u_2(z)$ are moduli of the Seiberg-Witten curve and are related to the Coulomb branch moduli of the $\mathcal{N} = 2^*$ quiver theory.

It is easy to see that the function $u_2(z)$ is an elliptic function on the $\tau$-torus with modular weight 2. This follows from the fact that the “four-brane positions” $v_{sr}(z)$ essentially transform into one another under $z \rightarrow z + 2m\omega_1 + 2n\omega_2$; $m, n \in \mathbb{Z}$, and that the $\zeta$-functions have modular weight 1. The $\tau$-ellipticity of $u_2(z)$ provides a natural way to separate out the singular and constant pieces since every elliptic function has an expansion in terms of $\zeta$-functions (see Appendix A)

$$u_2(z) = N(N - 1) \sum_{i=1}^{k} m_i^2 \wp(z - z_i) + \sum_{i=1}^{k} \zeta(z - z_i)H_i + H_0. \quad (3.11)$$

There are only $k - 1$ independent simple pole residues $H_i$ since we must have $\sum_{i=1}^{k} H_i = 0$. Thus, including the constant part of $u_2(z)$ there are $k$ independent parameters which coincides with the number of condensates $u_2^{(i)} = \langle \text{Tr} \Phi_i^2 \rangle$. The connection between $u_2(z)$ and the condensates is also clearly seen via the intimate relationship between the Coulomb branch physics of the $\mathcal{N} = 2^*$ quiver theory and integrable models. This aspect will be explored in great detail in Section 4 and via a completely different physical picture, will lead to an extremely non-trivial check of the results for the condensates presented in this section.

At this stage it must be pointed out that the exact relationship between the parameters $H_0$ and $H_i$ on the one hand and the physical condensates $u_2^{(i)}$ on the other, is far from obvious. There is no unambiguous way to map the residues in the above expressions to the VEVs of the operators $\text{Tr} \Phi_i^2$ in each gauge group factor. In general, one expects the physical $u_2^{(i)}$ to be some linear combination of the residues $H_i$ and $H_0$ and the identity operator. Specifically, we may identify a set of $k$ condensates that are modular covariant with the correct modular weight, respecting the symmetries of a given vacuum. However, there is no unambiguous way to determine the exact relation between such a set of condensates and the VEVs of the operators $\text{Tr} \Phi_i^2$. This is a generalization of the ambiguity encountered in the basic $\mathcal{N} = 1^*$ case [12], [25], [19]. However although it appears difficult to obtain the condensates within a given gauge group factor, there is a natural unambiguous combination of the condensates $H_0$ and $H_i$ that can be identified with the average condensate $\frac{1}{k} \sum_{i=1}^{k} u_2^{(i)} = \frac{1}{k} \sum_{i=1}^{k} \langle \text{Tr} \Phi_i^2 \rangle$. As we will argue below, this combination will have the correct transformation properties under the
duality group and only suffer from a mild vacuum-independent additive ambiguity \textit{i.e.}, mixing with the identity operator.

Using Eqs. (3.9) and (3.10) we may readily calculate the residues of the function \(u_2(z)\) at the simple poles \(z = z_i\), corresponding to the location of the NS5-branes. The calculation of the \(z\)-independent part \(H_0\) is tedious and can be performed in two different ways. One method is to use \(\tilde{\tau}\)-ellipticity of each term \((v_{sr} - v_{sr'})^2\) in the summation in Eq.(3.10), expand in terms of \(\tilde{\zeta}\)-functions and subsequently obtain the constant pieces by evaluating the expressions at certain special points. The other technique is to make use of various identities for elliptic functions provided in Appendix A, particularly Eq.(A.20). Below, we simply present our final results for the condensates \(H_0\) and \(H_i\) for the vacuum with \(l = 0\) and \((s_i, r_i) = (0, 0)\). (As we have emphasized the values of the condensates in the other massive vacua can then be deduced by moving NS5-branes by periods of the torus and also by modular transformations in \(\tau\).)

It is convenient to introduce the following shorthand notation for certain combinations of variables that appear in the formulas for the condensates:

\[
z_{ij} \equiv z_i - z_j, \quad \Omega_{sr} = 2s\omega_1 + 2r\omega_2, \quad \tilde{\wp}(z) = \wp(z|\tilde{\tau}), \quad \tilde{\zeta}(z) = \zeta(z|\tilde{\tau}),
\]

(3.12)

We then find that the simple pole residues are given by

\[
H_i \bigg|_{l=0; s_i=r_i=0} = 2Nm_i \sum_{j \neq i} m_j \left\{ (N - 1)\tilde{\zeta}(z_{ij}) - \sum_{\substack{r,s \neq (0,0) \atop \neq (0,0)}} (\tilde{\zeta}(z_{ij} + \Omega_{sr}) - \tilde{\zeta}(\Omega_{sr})) \right\} .
\]

(3.13)

Note that \(\sum H_i = 0\) as required. Note also that they are modular functions with weight 1 under the action of SL(2, \(\mathbb{Z}\)) on \(\tau\). Furthermore, shifting one of the NS5-branes by a period of the \(\tilde{\tau}\)-torus leaves the \(H_i\) invariant. As argued earlier, shifting the five-branes by periods of the \(\tau\)-torus has the effect of permuting the vacua of the \(\mathcal{N} = 1^*\) quiver theory. But since the degenerate curve at a given vacuum is simply the \(\tilde{\tau}\)-torus, taking an NS5-brane around one of the cycles of this torus must be a symmetry of that vacuum and must leave the order parameters unchanged. Hence the \(H_i\) have the requisite properties to be identified as condensates characterizing a given massive vacuum of the \(\mathcal{N} = 1^*\) quiver theory. However, it is clear from Eq.(3.11), that \(H_0\) cannot share this property. Moving a five-brane around the \(\tilde{\tau}\)-torus will necessarily lead to shifts in our definition of \(H_0\) simply because of the anomalous transformation properties of the \(\zeta\)-functions characterizing the simple pole terms. This can be explicitly checked from our expression for
\[ H_0: \]
\[
H_0 \bigg|_{l=0,s_i=r_i=0} = - \frac{N^2}{12} \left( \sum_{i=1}^{k} m_i^2 + \frac{1}{2} \sum_{i \neq j} m_i m_j \right) \left[ E_2(\tau) - \frac{2}{p} E_2(\tilde{\tau}) \right] \\
+ N^2 \sum_{i \neq j} m_i m_j \left\{ \frac{1}{2} \left( \wp(z_{ij}) - N \wp(z_{ij}) \right) + \frac{N - 1}{2} \zeta^2(z_{ij}) \right. \\
+ \frac{1}{2N} \sum_{(r,s) \neq (r',s')} \left( - \zeta^2(z_{ij} + \Omega_{sr} - \Omega_{sr'}) + \left[ \frac{2(s-s')}{p} \zeta(p\omega_1) + \frac{2(r-r')}{q} \zeta(q\omega_2) \right]^2 \right) \\
- \frac{1}{N} q \zeta(p\omega_1) - \zeta(\omega_1) z_{ij} \left( (N - 1) \zeta(z_{ij}) - \sum_{(r,s) \neq (0,0)} \left( \zeta(z_{ij} + \Omega_{sr}) - \zeta(\Omega_{sr}) \right) \right) \right\}. \tag{3.14}
\]

In writing the above expression we have used the identities Eq. (A.16) and Eq. (A.21). Eqs. (3.13) and (3.14) are our results for \( H_i \) and \( H_0 \) respectively, in the vacuum with \( \tilde{\tau} = q\tau/p \) and with all NS5’s intersecting the same IIA four-brane. The expressions for the vacua with \( l \neq 0 \) may be obtained by the replacement \( \omega_2 \rightarrow \omega_2 + l\omega_1/q \) in the above equations. All other massive vacua can be obtained by displacing the five-branes by periods of the \( \tau \)-torus to yield inequivalent brane configurations. As discussed earlier, for every choice of \( p, q \) and \( l \), there are \( N^{k-1} \) such vacua.

Under the duality transformation \( z_1 \rightarrow z_1 + 2p\omega_1 \) the quantity \( H_0 \) is not invariant, rather \( H_0 \rightarrow H_0 + 2p\zeta(\omega_1)H_1 \). This implies that \( H_0 \) does not respect \( \tilde{S} \)-duality and by itself cannot be identified with the order parameter in a given vacuum. However, there is a linear combination of the quantities \( H_0 \) and \( H_i \) that respects the duality symmetries of the \( \mathcal{N} = 1^* \) vacuum; namely

\[
H^* = H_0 - \frac{1}{k} \sum_{i \neq l} \zeta(z_{il}) H_i. \tag{3.15}
\]

The redefined constant piece \( H^* \) appears to have the properties required of the average condensate \( \frac{1}{k} \sum_i \langle \text{Tr} \Phi_i^2 \rangle \). It has dimension 2, modular weight 2, and is invariant under permutations of the gauge group factors and importantly is invariant under \( \tilde{\tau} \)-elliptic shifts of each five-brane, i.e \( \tilde{S} \)-duality, which was argued to be a symmetry of each massive vacuum. As we will see below, up to an additive vacuum-independent ambiguity, in the classical limit it indeed corresponds to the average condensate \( \frac{1}{k} \sum_i \langle \text{Tr} \Phi_i^2 \rangle \).

**A Superpotential for the \( \mathcal{N} = 1^* \) quiver theory**

For generic \( \mathcal{N} = 1^* \) deformations of the quiver theory, the superpotential in each vacuum
of the low energy theory takes the value

\[ W = -\frac{1}{g^2} \sum_{i=1}^{k} \mu_i \langle \text{Tr} \Phi_i^2 \rangle. \]  

(3.16)

Our inability to map the coordinates \( H_0 \) and \( H_i \) to the physical condensates in each gauge group factor, implies that we cannot determine this superpotential for generic deformations.

However, for a subclass of deformations of the quiver theory where all the adjoint masses are equal \( \mu_i = \mu \neq 0 \), the effective superpotential in each vacuum is simply given by the expectation value of the average condensate:

\[ W = -\frac{1}{g^2 \mu} \sum_{i=1}^{k} \langle \text{Tr} \Phi_i^2 \rangle = -\frac{1}{g^2 \mu} k \ H^*. \]  

(3.17)

This result for the superpotential in the massive vacua of the \( \mathcal{N} = 1^* \) quiver theory along with Eqs.(3.14) and (3.15) are among the central results of this paper. When \( k = 1 \), Eq.(3.14) reduces to the expression for the elliptic superpotential for \( \mathcal{N} = 1^* \) gauge theory obtained in [4, 12].

A few remarks about the condensates \( H^*, H_i \) and the superpotential in Eq.(3.17) are in order. Under SL(2, \( \mathbb{Z} \)) transformations on \( \tau \), these order parameters in a vacuum with given \( p, q \) and \( l \) transform with a definite modular weight into the corresponding condensates in a vacuum with a different value of \( p, q \) and \( l \) and with different values of the gauge couplings. For example under S-duality, \( H^*(p, q, l = 0) \) transforms into \( \tau^2 H^*(q, p, l = 0) \) and with \( z_{ij} \rightarrow \tau z_{ij} \). It is also worth noting that in the classical theory (Type IIA picture), the relative positions of adjacent fivebranes appear to play a special role in that they correspond to gauge couplings of the individual SU(\( N \)) factors in the quiver. However, the exact expressions for the condensates in a given vacuum exhibit complete democracy in the relative positions of any two fivebranes in the setup.

Finally, the condensates contain a wealth of information on instanton and “fractional instanton” contributions in these theories. This can be clearly seen by studying the expressions in a semiclassical expansion i.e. in the \( g^2, g_i^2 \ll 1 \) limit. For example, in this limit, the condensates in the confining vacuum with \( p = N, q = 1 \) and \( l = 0 \) may be expanded in powers of \( e^{2\pi i \tau/N} \) and \( e^{-z_{ij}/N} \). There is no obvious semiclassical origin for such terms in the 4D theory as they appear to be contributions from objects with fractional topological charge in both the diagonal and the individual gauge group factors. However, as seen in [4], in the theory on \( \mathbb{R}^3 \times S^1 \), such terms can arise from semi-classical configurations corresponding to 3D monopoles carrying fractional topological charge.

*The classical limit*
We now demonstrate that $H^*$ indeed reduces to the average condensate in the classical theory in the appropriate limit. The classical limit of the theory is obtained by taking $\tau \to i\infty$ which corresponds to blowing up the radius of the $x^6$-circle, simultaneously scaling the five-brane positions so that $\tau_i \to i\infty$ in each gauge group factor. In this limit we should expect our general expression for $u_2(z)$ to yield the condensate in the $i$-th $\text{SU}(N)$ factor provided $z_i < z < z_{i+1}$, $z - z_i \to \infty$ and $z_{i+1} - z \to \infty$. For simplicity, we have taken all the $\{z_i\}$ to be real. The $\zeta$-functions then reduce to (see Appendix A):

$$\zeta(z - z_j) \to -\frac{(z - z_j)}{12} + \frac{1}{2}\text{sign}(z - z_j). \quad (3.18)$$

while the Weierstrass function simply tends to a constant $\wp(z - z_j) \to \frac{1}{12}$. Note the presence of discontinuities at the positions of the NS5-branes in the classical limit. The condensate in the $i$-th $\text{SU}(N)$ factor in the classical limit is then

$$u_2^{(i)} = \frac{N(N - 1)}{12} \sum_{j=1}^{k} m_j^2 + \sum_{j=1}^{k} \frac{z_j}{12} H_j + \sum_{j=1}^{i} H_j + H_0, \quad (3.19)$$

so that the average condensate is

$$\frac{1}{k} \sum_{i=1}^{k} u_2^{(i)} = \frac{N(N - 1)}{12} \sum_{j=1}^{k} m_j^2 + \sum_{j=1}^{k} \frac{z_j}{12} H_j + \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{i} H_j + H_0. \quad (3.20)$$

It is straightforward to see that in the classical limit our definition for $H^*$ in Eq.(3.15) approaches precisely this value up to a vacuum-independent additive constant:

$$H^* \to \sum_{j=1}^{k} \frac{z_j}{12} H_j + \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{i} H_j + H_0. \quad (3.21)$$

The discrepancy is vacuum-independent because we can trace its origin to the double pole terms in $u_2(z)$ involving the Weierstrass functions. The order parameters of the theory and therefore the vacuum-dependent information is completely encoded in the simple poles and constant pieces, $H_i$ and $H_0$. The quantity $H^*$ contains precisely the right combination of these parameters that reproduces not only the correct semiclassical limit of the average condensate but also has the right transformation properties under the duality symmetries.

4. Compactification to Three Dimensions

In the previous section, we “solved” for the vacuum structure of the mass deformed theory by lifting the Type IIA brane configuration to M Theory. As we saw, the massive vacua have
a very simple interpretation as unbranched multiple covers of the basic underlying torus $E_\tau$.
There is another way to “solve” the theory involving compactification on a circle to three
dimensions [4], [17]. On compactification, the $2n = 2(kN - k + 1)$ real dimensional Coulomb
branch of the quiver theory is enlarged. This is because the $U(1)^n$ gauge field can have a
non-trivial Wilson line around the compact dimension, which we choose as $x^3$. In addition,
the three-dimensional gauge field transforming in the unbroken $U(1)^n$, on the Coulomb branch,
can be exchanged, under $S$-duality, for $n$ real scalar fields. In four-dimensions, the effective
couplings of the $U(1)^n$ theory are encoded in the $n \times n$-dimensional matrix $\tau_{uv}$ which is precisely
the period matrix of the Seiberg-Witten Riemann surface $\Sigma$. The Wilson line and dual photons
naturally pair up to form $n$ complex scalars $X_u$ which are periodic variables

$$X_u \sim X_u + 2\pi im_u + 2\pi i\tau_{uv}n_v, \quad m_u, n_u \in \mathbb{Z}.$$  (4.1)

This means that the Wilson line and dual photon are valued in the Jacobian variety $J(\Sigma)$. So
the dimension of the Coulomb branch, denoted $\mathfrak{M}$, of the three-dimensional theory is $4n$ and
can be thought of as a fibration of the Jacobian variety $J(\Sigma)$ over the four-dimensional Coulomb
branch $\Sigma$. It turns out that $\mathfrak{M}$ is rather special: it is hyper-Kähler space.

The low-energy effective three-dimensional theory is described by a supersymmetric sigma
model with $\mathfrak{M}$ as target. When an $\mathcal{N} = 1^*$ mass deformation is added to the four-dimensional
theory, the sigma model is perturbed by a superpotential. The phase structure can then be
determined from the superpotential. The key point that makes the compactification approach
attractive even when one is interested in the four-dimensional theory is that the superpotential
is completely independent of the compactification radius. Therefore the vacuum structure and
the values of condensates apply also in the de-compactification limit. By “solving” the model,
we mean finding the exact form for the superpotential.

For the $\mathcal{N} = 1^*$ deformation of the $\mathcal{N} = 4$ theory, the form of the superpotential was
arrived at from purely physical reasoning: by imposing all the symmetries of the low-energy
three-dimensional theory [4]. In addition, in the limit of small radius, semi-classical reasoning
applies and provides a strong constraint on the functional form of the superpotential. However,
there is a more direct way to construct the superpotential which relies on the fact that the
Seiberg-Witten fibration, and hence $\mathfrak{M}$, is the phase space of a complexification of a classically
integrable dynamical system\(^8\) (for a recent thorough review of this connection see [26] and
references therein). In this description, the Coulomb branch of the four-dimensional theory
is parameterized by the set of $n$ (complex) Hamiltonians and the fibre is parameterized by $n$
(complex) angle variables. The $\mathcal{N} = 1^*$ deforming operator is then associated with a particular
combination of the Hamiltonians and this gives the superpotential. The problem of finding the
explicit superpotential is then the problem of finding a convenient and unconstrained parame-

---

\(^8\)This means that the dynamical variables, Hamiltonians, etc., are taken to be complex.
terization of the phase space $\mathcal{M}$ and then finding the appropriate Hamiltonian describing the $\mathcal{N} = 1^*$ deformation.

4.1 The three-dimensional Coulomb branch

The first problem is to find a convenient description of the hyper-Kähler manifold $\mathcal{M}$. Remarkably, following the approach adopted by Kapustin [16], this can be achieved by using a version of mirror symmetry. What results is a description of $\mathcal{M}$ in terms of a hyper-Kähler quotient [27] which can be made very explicit.

The most convenient way to describe the mirror transform is to return to the Type IIA brane set-up. For now let us ignore the NS5-branes and concentrate on the $N$ D4-branes alone. We also suppose that the D4-branes are all coincident. We now compactify $x^3$ on a circle of radius $R'$, to distinguish it from the radius of $x^{10}$ defined earlier. The theory on the $N$ D4-branes is effectively three-dimensional in the infra-red. The low energy degrees-of-freedom include the Wilson line around $x^3$ which is an adjoint-valued scalar field that generically breaks the $U(N)$ gauge symmetry of the three-dimensional gauge theory to $U(1)^N$ via the Higgs mechanism. We define the abelian components of the Wilson line as $\phi^a = \int_0^{2\pi R'} A_3^a dx^3$, $a = 1, \ldots, N$. Large gauge transformations on the circle imply that the Wilson line is actually a periodic variable

$$\phi^a \approx \phi^a + 2\pi n^a, \quad n^a \in \mathbb{Z}.$$ (4.2)

In additional to the Wilson line, is the three-dimensional $U(1)^N$ gauge field. The bosonic part of the classical effective three-dimensional action is

$$S_{\text{cl}} = \frac{2\pi R'}{g^2} \int d^3x \left\{ \frac{1}{4\pi^2 R'^2} (\partial_i \phi^a)^2 - \frac{1}{2} (F^a_{ij})^2 - \frac{i\theta}{8\pi^2} \epsilon_{ijk} \partial_i \phi^a F^a_{jk} \right\}. \quad (4.3)$$

Here, $i, j, k = 0, 1, 2$ and we assume that the adjoint scalar in the four-dimensional theory has no VEV. The gauge fields may be exchanged under $S$-duality for $N$ real scalars $\sigma^a$, the dual photons. To do this one adds a new term to the action

$$-\frac{i}{2\pi} \int d^3x \epsilon_{ijk} \partial_i \sigma^a F^a_{jk}, \quad (4.4)$$

involving $\sigma^a$ which acts as Lagrange multipliers for the Bianchi identity. The abelian field strength $F^a_{ij}$ can be now integrated out as a Gaussian field to give

$$S_{\text{cl}} = \frac{1}{2\pi R'} \int d^3x \left\{ \frac{1}{g^2} (\partial_i \phi^a)^2 + \frac{g^2}{16\pi^2} \left( \partial_i \sigma^a + \frac{\theta}{2\pi} \partial_i \phi^a \right)^2 \right\}. \quad (4.5)$$

Due to the quantization of magnetic charge, the dual photons are also periodic variables:

$$\sigma^a \sim \sigma^a + 2\pi m^a, \quad m^a \in \mathbb{Z}.$$ (4.6)
The classical effective theory has the form of a sigma model in the $N$ complex scalar fields

\[ X_a = i(\sigma^a + \tau \phi^a) , \quad (4.7) \]

which are valued on the torus that we defined earlier $E_\tau$ with periods $2\omega_1 \equiv 2\pi i$ and $2\omega_2 \equiv 2\pi i \tau$. The pairing of the Wilson line and dual photons is natural when one includes supersymmetry since they form the scalar component of a superfield. The effective three-dimensional theory, including quantum corrections, is a sigma model involving $N$ chiral superfields whose scalar components are the fields $X_a$.

This story can be given a brane interpretation in the following way. We start with the D4-branes in Type IIA wrapped around $x^3$ with radius $R'$ and $x^6$ with radius $L$. For small radius $R'$, we can now perform a T-duality in $x^3$ to yield the Type IIB configuration of D3-branes spanning $x^0, x^1, x^2, x^6$. Under this duality, the string coupling is transformed to $g'_s = g_s \sqrt{\alpha'}/R'$. We follow this with an S-duality on the four-dimensional theory on the D3-branes. Under this duality, the three-dimensional effective abelian gauge field is exchanged with the Wilson line of the dual gauge field around $x^6$. This means that the dual photons are

\[ \sigma^a = \int_0^{2\pi L} \tilde{A}_6^a \, dx^6 , \quad (4.8) \]

where $\tilde{A}$ is the dual gauge field of the D3-branes. Finally, we perform, once again, a T-duality in $x^3$ to return to a Type IIA configuration with D4-branes spanning $x^0, x^1, x^2, x^3, x^6$. However, due to the intervening S-duality, the radius of the $x^3$ is not returned to its original value. The new radius is $R' g'_s = g_s \sqrt{\alpha'}$. In other words, it is independent of the radius $R'$.\(^9\) The Wilson line $\phi^a$ in the original theory is now identified with the Wilson line of the dual gauge field component $\tilde{A}_3$ around the dual $x^3$ circle:

\[ \phi^a = \int_0^{2\pi R'} \tilde{A}_3^a \, dx^3 \equiv \int_0^{2\pi g_s \sqrt{\alpha'}} \tilde{A}_3^a \, dx^3 . \quad (4.9) \]

The theory describing the collective dynamics of these D4-branes is the mirror dual, or “magnetic”, theory. It is a five-dimensional theory compactified on $\mathbb{R}^3 \times T^2$. The most significant fact is that the torus $T^2$ has complex structure $\tau$ and, up to an overall rescaling, is therefore identified with $E_\tau$. At low energies the effective three-dimensional theory is described by the Wilson lines around the two cycles of the torus which are identified with the Wilson line and dual photons, (4.9) and (4.8), respectively.

The discussion so far has been simplified because we have ignored the fact that there are NS5-branes in the original Type IIA set-up on which the D4-branes can fractionate. The gauge group of the effective low theory should be $U(1)^n$ (recall $n = kN - k + 1$) rather than

\[^9\]All memory of $R'$ is not lost because the string coupling in the dual theory is $R'^2/(g_s \alpha')$.\]
U(1)^N. So there should be \( n \) complex fields associated to \( n \) Wilson lines and \( n \) dual photons. Furthermore, these complex fields will be valued on tori which are not simply copies of \( E_\tau \) but have “renormalized” complex structures. This is clear when one looks at the theory for large \( R' \). In that limit, one can think of the problem in two stages. First, there is an effective four-dimensional theory à la Seiberg and Witten. This is a U(1)^n gauge theory with coupling constants encoded in the period matrix \( \tau_{uv} \) of \( \Sigma \). The way that all these extra Wilson lines and dual photons arise in the dual theory will emerge in due course. For now we must put back the NS5-branes. Under the first T-duality the NS5-branes become Type IIB NS5-branes. Then under S-duality they becomes D5'-branes (to distinguish them from the other D-branes in the problem). Finally the T-duality around \( x^3 \) changes them into D4'-branes spanning \( x^0, x^1, x^2, x^4, x^5 \), but localized at points on the \( x^3, x^6 \) torus. In the original theory, the NS5-branes allow the D4-branes to fractionate and the gauge group is enhanced from U(\( N \)) to SU(\( N \))^k \times U(1). In the dual magnetic theory, the freedom for the branes to fractionate is encoded by presence of VEVs for the impurities. The impurities somehow must also encode all the extra Wilson lines and dual photons that we expect, as we shall see later. As usual in a mirror transform we have mapped the Coulomb branch of the original theory, where the D4-branes were prevented from moving off the NS5-branes, to the Higgs branch of the magnetic theory, where the impurities gain a VEV Higgsing the dual U(\( N \)) gauge group.

The configuration that we are considering preserves eight real supersymmetries. So we have a realization of \( \mathfrak{M} \) as the Higgs branch of an U(\( N \)) impurity gauge theory with eight real supercharges. This is why the mirror map is a useful devise. The Higgs branch will not be subject to quantum corrections and in this way we are able to “solve” the theory. It is naturally described by a set of \( D \)-flatness equations which involve the, suitably normalized, components of the dual U(\( N \)) gauge field \( \tilde{A}_{z,\bar{z}} = \frac{1}{2}(\tilde{A}_3 \pm i\tilde{A}_6) \) of the D4-branes along the torus\(^{10} \) and the adjoint-valued complex scalar field \( \phi,^{11} \) describing the fluctuations of the D4-branes in the \( x^4, x^5 \) direction. This field is directly related to the M-theory picture described in the previous section: the \( N \) eigenvalues of \( \phi \) are precisely the \( v_a \). In addition, the D4'-brane impurities give rise to \( k \) hypermultiplets \( (Q_i, \tilde{Q}_i) \) transforming in the \( (\mathbf{N}, \overline{\mathbf{N}}) \)-representation of U(\( N \)), which are localized on the torus at points \( z_i, i = 1, \ldots, k \). The \( D \)-flatness conditions, with some convenient choice of normalization of the hypermultiplets, read

\[
\tilde{F}_{z,\bar{z}} - [\phi, \phi^\dagger] = -2\pi i \sum_{i=1}^{k} \delta^2(z - z_i)(Q_i \tilde{Q}_i^\dagger - \tilde{Q}_i^\dagger Q_i) , \quad (4.10a)
\]

\[
\tilde{D}_{z,\bar{z}} \phi = 2\pi i \sum_{i=1}^{k} \delta^2(z - z_i)Q_i \tilde{Q}_i . \quad (4.10b)
\]

\(^{10}\)As previously we perform an overall re-scaling of the torus \( T^2 \) so that it becomes parameterized by the holomorphic coordinate \( z \) periodic on \( \Gamma = 2\omega_1 \mathbb{Z} \oplus 2\omega_2 \mathbb{Z} \).

\(^{11}\)Not to be confused with the Wilson line \( \phi^a \).
Here, (4.10a) is a real equation and (4.10b) is a complex equation and $\tilde{D}_z \phi = \partial_z \phi + [\tilde{A}_z, \phi]$. These equations are a generalization of Hitchin’s self-duality equations reduced to two dimensions [28].

The space $\mathcal{M}$ described as the solution to (4.10a)-(4.10b) modulo $U(N)$ gauge transformations, has another interesting interpretation. Consider a configuration of $N$ D2-branes lying inside $k$ coincident D6-branes. The D2-branes are embedded as a charge $N$ Yang-Mills instanton solution in the $U(k)$ gauge theory on the D6-branes in the four directions orthogonal to the D2-branes inside the D6-branes. In the brane picture, the moduli space of the instantons is described by the Higgs branch of an $U(N)$ gauge theory, with eight real supercharges and $k$ hypermultiplets transforming in the $(N, \overline{N})$. This yields nothing but the ADHM construction of instantons in $\mathbb{R}^4 U(k)$ gauge theory. Now imagine compactifying two spatial dimensions in the D6-branes orthogonal to the D2-branes. The D2-branes now correspond to Yang-Mills instantons in a $U(k)$ gauge theory on $\mathbb{R}^2 \times T^2$ rather than $\mathbb{R}^4$. Performing a $T$-duality along each compactified direction, we now have the configuration of $N$ D4-branes and $k$ D4'-branes described above. So the Coulomb branch of the compactified quiver theory is identified with the moduli space of $N$ instantons in $U(k)$ gauge theory on $\mathbb{R}^2 \times T^2$.

A hyper-Kähler quotient

Lurking within the generalized Hitchin equations (4.10a)-(4.10b) is a complexified classical integrable dynamical system and uncovering it will be key to our story of the superpotential of the three-dimensional theory. But this is jumping ahead. The first thing to say about these Hitchin’s equations is they are an example of a hyper-Kähler quotient, although of a rather novel infinite-dimensional kind. The quotient construction [27] provides a way of constructing a new hyper-Kähler manifold $\mathcal{M}$ in terms of an old one $\tilde{\mathcal{M}}$ with some group of isometries $G$. These isometries must preserve both the metric and the three complex structures and so are described by tri-holomorphic Killing vectors $X_r, r = 1, \ldots, \dim G$. Each Killing vector defines a triplet of moment maps $\vec{\mu}^{X_r}$ defined by $d\vec{\mu}^{X_r} = i(X_r)\vec{\omega}$, where $\vec{\omega}$ are the triplet of Kähler forms. The quotient space is defined as the coset manifold $\mathcal{M} = \tilde{\mathcal{M}}/G$, where the level set $\mathfrak{N} \subset \tilde{\mathcal{M}}$ is defined by $(\vec{\mu}^{X_r})^{-1}(0)$, for all $r$, i.e. the subspace of $\tilde{\mathcal{M}}$ on which the moment maps vanish. It can be shown that $\mathcal{M}$ inherits the hyper-Kähler structure of $\tilde{\mathcal{M}}$. If $\tilde{\mathcal{M}}$ has real dimension $4n$ then the above construction leads to $\mathcal{M}$ with dimension $4(n - \dim G)$.

In the present context, the novelty arises from the fact that the space $\tilde{\mathcal{M}}$, parameterized by $\phi, \tilde{A}, Q$ and $\tilde{Q}$, is infinite dimensional because $\phi$ and the gauge field $\tilde{A}$ are functions on $E_r$. However, the quotient construction—at least formally—proceeds in the usual way. In this case the quotient group $G$ is the group of $U(N)$ local gauge transformations on $E_r$. The idea is that quotienting the infinite-dimensional space $\tilde{\mathcal{M}}$ by the infinite-dimensional gauge group leads to a finite-dimensional space $\mathcal{M}$. The equations (4.10a) and (4.10b) are really a triplet of equations since (4.10b) is a complex equation while (4.10a) is real, and they are nothing but
the moment maps for the local $U(N)$ action.

But, as pointed by Kapustin [16], there is a problem. The fields $\phi$ and $\tilde{A}$ must have simple poles at the punctures $z_i$ with residues that are variable, given as they are by the $Q$ and $\tilde{Q}$. This means that the variations of $\phi$ and $\tilde{A}$ will have simple poles as well and so their norm will be logarithmically divergent. As a consequence, the corresponding tangent vectors to $\mathfrak{M}$ will have infinite norm. Physically this is a different manifestation of the fact that in the Type IIA brane set-up the relative centres-of-mass of the stacks of D4-branes are not genuine moduli. These parameters are, of course, the masses $m_i$ of the bi-fundamental hypermultiplets. All this means is that we have not quite described the correct hyper-Kähler quotient. What we need to do is freeze the impurities $Q$ and $\tilde{Q}$ in some way to ensure that the non-normalizable modes are discarded. But we must do this in a way which still leads to a hyper-Kähler manifold. The way to do this was described by Kapustin [16], however, we will present a re-interpretation of his procedure which is more in keeping with the quotient construction.

We now hypothesize that the correct construction involves a hyper-Kähler quotient by a larger group which involves not only local $U(N)$ gauge transformations on the torus $E_\tau$, but also $U(1)$ transformations at each puncture:

$$Q_{ai} \to Q_{ai} e^{\psi_i}, \quad \tilde{Q}_{ia} \to e^{-\psi_i} \tilde{Q}_{ia} .$$

(4.11)

In fact the additional part of the quotient group is $U(1)^{k-1}$ because the transformation that rotates all the $Q$ and $\tilde{Q}$ by the same phase is part of the $U(N)$ gauge group. The associated moment maps are simply

$$\sum_a (Q_{ai} Q^\dagger_{ia} - \tilde{Q}^\dagger_{ai} \tilde{Q}_{ia}) = 0 ,$$

(4.12a)

$$\sum_a Q_{ai} \tilde{Q}_{ia} = 0 .$$

(4.12b)

The question is where are the mass parameters? In the hyper-Kähler quotient construction, the definition of the moment maps for any abelian components of $G$ is ambiguous. Any constant can be added. When the hyper-Kähler quotient construction is viewed as the Higgs branch of a supersymmetric gauge theory with eight supercharges, these parameters are the Fayet-Illiopolos (FI) couplings of the abelian part of the gauge group. In the present context, the masses $m_i$ are the complex FI couplings associated to the abelian subgroup $U(1)^k$ which acts as in (4.11). The complex components of the moment maps (4.10b) and (4.12b) are modified to

$$\tilde{D}_z \phi = 2\pi i \sum_{i=1}^k \delta^2(z - z_i) \left( Q_i \tilde{Q}_i - \frac{m}{k^{1/2} [N]^2} \right) ,$$

(4.13a)

$$\sum_a Q_{ai} \tilde{Q}_{ia} = Nm_i .$$

(4.13b)
As before \( m = \sum_i m_i \). The form of (4.13a) is consistent with (4.13b) in the following way. The quantity \( \varphi = \text{Tr} \phi \) satisfies

\[
\partial\bar{z}\varphi = 2\pi i N \sum_{i=1}^{k} \delta^2(z - z_i) \left(m_i - \frac{m}{k}\right) ;
\]

and hence, is a meromorphic function on \( E_\tau \) with simple poles at \( z = z_i \) and residues \( N(m_i - m/k) \). For consistency the sum of the residues must vanish and this is guaranteed by (2.4).

We can now go on and find a very concrete realization of the quotient. Firstly, as we have already mentioned, the real and complex moment maps, (4.10a) and (4.12a) verses (4.13a) and (4.13b), can be viewed as the \( D \)- and \( F \)-flatness conditions of a Higgs branch of supersymmetric gauge theory with four real supercharges. As usual, we can relax the \( D \)-flatness condition and simply impose the \( F \)-flatness conditions (4.13a)-(4.13b) at the expense of modding out by the complexified gauge group. Picking out the complex moment map amounts to picking out a preferred complex structure from the three independent complex structures of the hyper-Kähler manifold. The associated preferred Kähler form will turn out to be the symplectic form of the complexified dynamical system.

It is convenient to introduce the "spins" \( S^i \), \( N \times N \) matrices at each puncture, with elements

\[
S_{ab}^i \overset{\text{def}}{=} Q_{ai} \tilde{Q}_{ib} - \frac{m}{k} \delta_{ab} .
\]

The complex moment map equations (4.13a)-(4.13b) are then

\[
\tilde{D}\bar{z}\phi = 2\pi i \sum_{i=1}^{k} S^i \delta^2(z - z_i) ,
\]

\[
\sum_a S^i_{aa} = N \left(m_i - \frac{m}{k}\right) .
\]

The constraint (4.16b) is equivalent to Kapustin’s residue condition, generalizing the residue condition of Donagi and Witten [3], because it implies that the spins lie in a particular conjugacy class of the complexified quotient group:

\[
S^i = U_i \begin{pmatrix}
Nm_i - \frac{m}{k} & -\frac{m}{k} & & \\
-\frac{m}{k} & & & \\
& & \ddots & \\
& & & -\frac{m}{k}
\end{pmatrix} U_i^{-1}
\]

for elements \( U_i \) in the complexified quotient group. ¹²

¹²To see this identify \((U_i)_{a1} = Q_{ai}/\sqrt{Nm_i}\) and \((U_i^{-1})_{1a} = \tilde{Q}_{ia}/\sqrt{Nm_i}\). The fact that \( \sum_a (U_i^{-1})_{1a}(U_i)_{a1} = 1 \) follows from (4.13b).
To proceed, it is very convenient to use up (most of) the local part of the quotient group, \GL(N, \mathbb{C})
, to transform the the anti-holomorphic component \(A_{\bar{z}}\) into a constant diagonal matrix:

\[
\tilde{A}_{\bar{z}} = \frac{\pi i}{2(\bar{\omega}_2 \omega_1 - \bar{\omega}_1 \omega_2)} \text{diag}(X_1, \ldots, X_N) = \frac{1}{16g^2} \text{diag}(X_1, \ldots, X_N)
\]  

(4.18)

The only local transformations that remain act by shifting the \(X_a\) by periods of \(E_\tau\),

\[
X_a \rightarrow X_a + 2n\omega_1 + 2m\omega_2, \quad m, n \in \mathbb{Z}.
\]  

(4.19)

The \(X_a\) are nothing but the complex combination of the abelian Wilson lines and dual photons defined in (4.7). The global part of the gauge group is also fixed, up to permutations of the \(X_a\) and the \GL(1, \mathbb{C})^N\ diagonal subgroup, as well as the complexified \GL(1, \mathbb{C})^{k-1}\ “flavour” symmetries (4.11). Putting these symmetries together, we must mod out by the action

\[
Q_{ai} \rightarrow \zeta_a Q_{ai} \xi_i, \quad \tilde{Q}_{ia} \rightarrow \xi_i^{-1} \tilde{Q}_{ia} \zeta_a^{-1}
\]  

(4.20)

We can now solve explicitly for \(\phi\) to get a very concrete parameterization of the quotient space \(\mathcal{M}\) and the associated dynamical system. Roughly speaking, the elements of \(\phi\) must have simple poles at \(z = z_i\) to account for the \(\delta\)-functions. Candidate functions are \(\zeta(z - z_i)\) and \(\sigma(z - z_i)^{-1}\),\(^{13}\) however, these must be put together in the right way to ensure periodicity on \(E_\tau\). After some trial and error, one is led to the solution

\[
\phi_{aa}(z) = p_a + \sum_{i=1}^{k} S^i_{aa} \zeta(z - z_i)
\]  

(4.21)

for the diagonal components, where the \(p_a\) are new parameters. There are extra constraints on the spins, arising from the fact that in the gauge we have chosen, the diagonal elements \(\phi_{aa}(z)\) are meromorphic functions on \(E_\tau\) and so the sum of the residues must vanish:

\[
\sum_i S^i_{aa} = 0.
\]  

(4.22)

The off-diagonal elements are

\[
\phi_{ab}(z, \bar{z}) = e^{\psi(z, \bar{z})X_{ab}} \sum_{i=1}^{k} S^i_{ab} \frac{\sigma(X_{ab} + z - z_i)}{\sigma(X_{ab})\sigma(z - z_i)} e^{-\psi(z_i, \bar{z}_i)X_{ab}} \quad (a \neq b).
\]  

(4.23)

Here \(X_{ab} \equiv X_a - X_b\), and we have defined

\[
\psi(z, \bar{z}) \overset{\text{def}}{=} \frac{1}{\omega_2 \bar{\omega}_1 - \bar{\omega}_1 \omega_2} \left[\zeta(\omega_2)(\bar{\omega}_1 z - \omega_1 \bar{z}) - \zeta(\omega_1)(\bar{\omega}_2 z - \omega_2 \bar{z})\right].
\]  

(4.24)

\(^{13}\)A short review of elliptic functions and their properties is provided in Appendix A.
One can readily verify that $\phi_{ab}(z, \bar{z})$ is periodic on $E_\tau$. Furthermore, a shift of $X_a$ by a lattice vector $2\omega_\ell$, can be undone by a large gauge transformation on the torus as anticipated earlier.

We now have an explicit parameterization of $\mathcal{M}$ furnished by $p_a$, $S^i_{ab}$ and $X_a$. In order to determine the dimension of $\mathcal{M}$ we now count the number parameters. First of all the spins. There are $4Nk$ real quantities $Q$ and $\tilde{Q}$ subject to $2k$ and $2N$ real constraints, (4.22) and (4.16b), respectively. The group action (4.20) further reduces the number of variables by $2(N + k - 1)$ (the “1” arising from the fact that not all the parameters $\zeta_a$ and $\xi_i$ in (4.20) are independent). Hence the number of real independent spin variables is $4(Nk - N - k + 1)$. The remaining variables are $p_a$ and $X_a$, giving $4N$ real parameters. Hence, the overall dimension of the quotient space $\mathcal{M}$ is $4n \equiv 4(kN - k + 1)$. From the point of view of the dynamical system, the variables $p_a$ are naturally the momenta conjugate to the $X_a$. Notice that $\phi(z, \bar{z})$ is only dependent on the differences $X_{ab} \equiv X_a - X_b$; hence $\sum_a p_a$ is not dynamical. In fact $\sum_a p_a$, and its conjugate position $\sum_a X_a$, correspond to the decoupled overall U(1) factor of the gauge group in the quiver theory. We now choose to set

$$\sum_a p_a = \sum_a X_a = 0 .$$

(4.25)

Notice that although we have a concrete parameterization of $\mathcal{M}$, the relation with the physical parameters of the three-dimensional Coulomb branch is not obvious. In the basic $\mathcal{N} = 1^*$ case, recovered by taking $k = 1$, the spins are completely absent. The coordinates in this case are $p_a$ and $X_a$. Writing $X_a = -i(\sigma^a + \tau \phi^a)$, the real components $\sigma^a$ and $\phi^a$ are precisely the dual photons and Wilson lines of the effective U(1)$^N$ theory. In the general case, the relation with the dual photons and Wilson lines and the coordinates is less obvious. Intuitively, the $X_a$ are dual photons and Wilson lines of the diagonally embedded SU($N$).

4.2 A dynamical system and the exact superpotential

As we have already alluded to above, there is also a completely integrable dynamical system underlying the construction of $\mathcal{M}$, for which $\mathcal{M}$ is the phase space with symplectic form given by the Kähler form singled out by the complex moment map. It is the rather esoteric spin generalization of the elliptic Calogero-Moser system which was first described in Ref. [6] and further studied in Refs. [7, 8]. The integrable system has an associated spectral curve which is defined by [7]

$$F(z, v) = \det\left(v1_{N \times N} - \phi\right) = 0 .$$

(4.26)

This is precisely the Seiberg-Witten curve $\Sigma$, the branched $N$-fold cover of $E_\tau$ which appeared in the M-theory interpretation, Eq. (3.5). Since the dynamical system is completely integrable, there are $n$ (complex) Hamiltonians. These are identified with coordinates on the Coulomb
branch of the four-dimensional theory. The conjugate angle variables (also complex), $X_u$, $u = 1 \ldots, n$, take values in the Jacobian variety $J(\Sigma)$. Finally we have identified all the Wilson lines and dual photons of the three-dimensional effective theory. The explicit maps between the variables \{p_a, X_a, S^i\} and the angle variables was found in [7,8], but we shall not need them here.

Before proceeding, the resulting equations are cleaner if the spin variables are re-defined by

$$S_{ab}^i \rightarrow S_{ab}^i e^{\psi(z_i, \bar{z}_i)} X_{ab}.$$  \hspace{1cm} (4.27)

The Hamiltonians of the dynamical system arise as the residues of the gauge invariant quantities $\text{Tr}\phi^l(z)$, $l = 1, 2, \ldots, N$. Since the system is integrable, there will be $2(kN - k + 1)$ independent Hamiltonians which will be identified with coordinates on the Coulomb branch of the four-dimensional $\mathcal{N} = 2$ theory that we started with. Of particular importance will be the $k$ combinations of Hamiltonians that correspond to the condensates $u_2^{(i)} = \langle \text{Tr} \Phi_i^2 \rangle$ for each of the SU($N$) factors of the gauge group. They must come, on dimensional grounds, from expressions quadratic in $\phi(z)$. There are two such terms $(\text{Tr} \phi(z))^2$ and $\text{Tr} \phi(z)^2$. We have already argued that the quantity that relates to the $\mathcal{N} = 1^*$ deformation is $u_2(z) = \frac{1}{2N} \sum_{a \neq b} (v_a(z) - v_b(z))^2$, and since $v_a(z)$ are the eigenvalues of $\phi(z)$, this identifies

$$u_2(z) = \text{Tr} \phi^2 - \frac{1}{N} (\text{Tr} \phi)^2 .$$  \hspace{1cm} (4.28)

We now compute this quantity given our solution for $\phi(z)$ in (4.21) and (4.23).

Since $\phi(z)$ has simple poles, the quadratic invariant $u_2(z)$ has double poles at $z_i$. Since it is manifestly elliptic, the expansion must be of the form (compare (3.11))

$$u_2(z) = \sum_{i=1}^{k} \lambda_i \phi(z - z_i) + \sum_{i=1}^{k} \zeta(z - z_i) H_i + H_0 ,$$  \hspace{1cm} (4.29)

where $\sum_{i=1}^{k} H_i = 0$. It is tedious but a straightforward exercise to extract the residues and constant part. Firstly, the residues of the double poles are constants

$$\lambda_i = \sum_{ab} S_{ab}^i S_{ba}^i - \frac{1}{N} (\sum_{a} S_{aa}^i)^2 = N(N - 1)m_i^2 .$$  \hspace{1cm} (4.30)

The residues of the simple poles are non-trivial functions on $\mathfrak{m}$, to wit

$$H_i = 2 \sum_{a} p_a S_{aa}^i - 2N (m_i - \frac{m}{N}) \sum_{j(\neq i)} (m_j - \frac{m}{N}) \zeta(z_{ij})$$

$$+ 2 \sum_{a \neq b} \sum_{j(\neq i)} S_{aa}^i S_{ba}^j \zeta(z_{ij}) - 2 \sum_{a \neq b} \sum_{j(\neq i)} S_{ab}^i S_{ba}^j \frac{\sigma(X_{ab} + z_{ji})}{\sigma(X_{ab}) \sigma(z_{ji})} ,$$  \hspace{1cm} (4.31)
where \( z_{ij} \equiv z_i - z_j \). One can verify that \( \sum_i H_i = 0 \). To complete the expansion, the constant piece in the expansion is

\[
H_0 = \sum_a p_a^2 - \sum_a \sum_{i \neq b} S_{ab}^i S_{ba}^i \varphi(X_{ab}) + \sum_a \sum_i S_{ab}^i S_{ba}^i \sigma(X_{ab} + z_{ji}) \left( \zeta(X_{ab} + z_{ji}) \right) - \zeta(X_{ab}) \right) - \zeta(z_{ij})^2 \right) .
\]

The \( k \) independent functions on \( M \), \( H_0 \) and \( H_i \), are a subset of the Hamiltonians of the integrable system.

The Hamiltonians are not simply invariant under these shifts (3.6). Rather the shifts can be undone by an appropriate transformation on \( M \). To find this transformation, under a shift of \( z_l \to z_l + 2\omega_l \), we have

\[
H_i(p_a, X_a, S_{ab}^j | z_j + 2\delta_{jl}\omega_l) = H_i(p_a', X_a, S_{ab}^j | z_j) ,
\]

where

\[
p_a' = p_a - 2\zeta(\omega_l)(S_{aa}^i - 2(m_i - \frac{m_i}{k})) , \quad S_{ab}^{ij} = S_{ab}^j e^{2\delta_{jl}X_{ab}\zeta(\omega_l)} .
\]

The remaining Hamiltonian transforms similarly, but with an additive anomaly

\[
H_0(p_a, X_a, S_{ab}^j | z_j + 2\delta_{jl}\omega_l) = H_0(p_a', X_a, S_{ab}^j | z_j) + 2\zeta(\omega_l)H_i(p_a', X_a, S_{ab}^j | z_j) .
\]

We have already described, based on the structure of the massive vacua in the M Theory picture, how to relate the quantities \( H_i \) and \( H_0 \), now interpreted as Hamiltonians, to the condensates. The conclusion was that a basis of functions with the right modular properties is provided by the \( H_i \) and \( H^* \) defined in (3.15). But, using the parameterization of the Coulomb branch provided by the integrable system, we can now see that the basis \( \{ H_i, H^* \} \) is precisely the one with good modular properties, not just at the massive vacua, but also at generic points on the Coulomb branch. To see this, notice that under shifts of \( z_i \) on \( E_r \), \( H^* \) transforms covariantly as the \( H_i \) (4.33) since the anomaly piece in (4.35) is compensated. Based on the semi-classical
limit, the quantity $H^*$ was then identified with the diagonal combination. Explicitly

$$H^* = \sum_a p_a^2 - \sum_{a \neq b} \sum_{i} S_{ab}^i S_{ba}^i \varphi(X_{ab}) - \frac{2}{k} \sum_{i \neq l} \left\{ \sum_a p_a S_{aa}^i \right\}$$

$$- N (m_i - \frac{m}{k}) \sum_{j(\neq i)} (m_j - \frac{m}{k}) \zeta(z_{ij}) + \sum_a \sum_{j(\neq i)} S_{aa}^i S_{aa}^j \zeta(z_{ij}) \right\} \zeta(z_{il})$$

$$+ \sum_{a \neq b} \sum_{i \neq j} S_{ab}^i S_{ba}^j \frac{\sigma(X_{ab} + z_{ji})}{\sigma(X_{ab}) \sigma(z_{ji})} \left( \zeta(X_{ab} + z_{ji}) - \zeta(X_{ab}) + \frac{2}{k} \sum_{l(\neq i)} \zeta(z_{il}) \right)$$

$$- \frac{1}{2} \sum_{i \neq j} \left\{ \sum_a S_{aa}^i S_{aa}^j - N (m_i - \frac{m}{k}) (m_j - \frac{m}{k}) \right\} \left( \varphi(z_{ij}) - \zeta(z_{ij})^2 \right).$$

The exact superpotential of the three-dimensional theory corresponding to the diagonal $\mathcal{N} = 1^*$ deformation is then simply $W = -k\mu H^*/g^2$.

The massive vacua

The superpotential $W = -k\mu H^*/g^2$ determines the vacuum structure of the $\mathcal{N} = 1^*$ deformation of the theory. A full analysis of the vacuum structure is beyond the scope of the present work. Actually, even in the $\mathcal{N} = 4$ case, where the superpotential is a good deal simpler, there is only a systematic treatment of the massive vacua for $N \geq 3$ \cite{4}. We shall achieve as much for the finite $\mathcal{N} = 2$ theories.

The massive vacua have a very beautiful interpretation from the point-of-view of the dynamical system: they are precisely equilibrium configurations with respect to the space of flows defined by the $n$ Hamiltonians.\textsuperscript{14} To see this, recall that the massive vacua correspond to points of the four-dimensional Coulomb branch for which $\Sigma$ degenerates to a torus: cycles pinch off and one is left with an $N$-fold un-branched cover of $E_{\tau}$. This means that the Jacobian Variety $\mathcal{J}(\Sigma)$ itself degenerates: at these points the period matrix only has rank 1, with non-zero eigenvalue $\tau$. The remaining torus is associated with the overall $\mathbf{U}(1)$ factor of the gauge group which we have removed from the integrable system (4.25). So at a massive vacuum, the remaining angle variables must stay fixed under any time evolution. Since the Hamiltonians are by definition constants of the motion, this means that the entire dynamical system must be static at a massive vacuum and the system is at an equilibrium point. Consequently, a massive vacuum is not only a critical point of the Hamiltonian describing the $\mathcal{N} = 1^*$ deformation, but simultaneously of all the other $n-1$ Hamiltonians. On the other hand, for the massless vacua, this is no longer true.

First of all, it is instructive to recall some details of the $\mathcal{N} = 4$ case described in \cite{4} which

\textsuperscript{14}Here, “time” is an auxiliary concept referring to evolution in the dynamical system and not a spacetime concept in the field theories under consideration.
is recovered in our formalism by setting $k = 1$. This will provide an important clue for quiver theories. When $k = 1$, the constraints on the single spin mean that it is not dynamical:

$$
k = 1 : \quad S_{ab} = m(1 - \delta_{ab}) . \tag{4.37}$$

There is a single quadratic Hamiltonian,

$$
k = 1 : \quad H_0 = \sum_a p_a^2 - m^2 \sum_{a \neq b} \varphi(X_{ab}) . \tag{4.38}$$

At the critical points of $H_0$ the momenta $p_a$ conjugate to $X_a$ vanish. The massive vacua are associated to the finer lattices $\Gamma'$ which contain $\Gamma$ as a sublattice of index $N$. This means that there are $N$ points of $\Gamma'$ contained in a period parallelogram of $\Gamma$. The simplest kind, labelled by two integers $p$ and $q$ with $pq = N$, are generated by $2\omega_1/q$ and $2\omega_2/p$. All the other cases can be generated from these by modular transformations, as we shall see later. Each $a \in \{1, \ldots, N\}$ is uniquely associated to the pair $(r_a, s_a)$, with $0 \leq r_a < q$ and $0 \leq s_a < p$. The critical point of $H_0$ associated to $(q, p)$ is then

$$
X_a = \frac{2r_a}{q} \omega_1 + \frac{2s_a}{p} \omega_2 , \quad 0 \leq r < q , \quad 0 \leq s < p . \tag{4.39}
$$

The proof that this is a critical point of $H_0$ is delightfully simple. One only needs to use the fact that $\varphi'(z)$ is an odd elliptic function. Terms in the sum $\sum_{b \neq a} \varphi'(X_{ab})$ either cancel in pairs or vanish because $X_{ab}$ is a half-lattice point. As we mentioned, the set (4.39) does not exhaust the set of massive vacua. For a given pair $(q, p)$ we can generate $p - 1$ additional vacua by performing the suitable modular transformation on $\tau$ to give

$$
X_a = \frac{2r_a}{q} \omega_1 + \frac{2s_a}{p} \left(2 \omega_2 + \frac{l}{q} \omega_1\right) , \quad 0 \leq l < p . \tag{4.40}
$$

So the total number of massive vacua is equal to $\sum_{p|N} p$, as expected on the basis of the semi-classical analysis.

Returning to the finite $N = 2$ theories, we hypothesize that the massive vacua are also associated to the finer lattices $\Gamma'$ with $X_a$ given in (4.40). Once again we can start with the configurations (4.39) and the additional massive vacua will be obtained by modular transformations. We will now find a set of critical points common to each of the Hamiltonians $H_i$ and $H_0$, and so, by implication, of $H^*$. Firstly, by varying $H_0$ with respect to the $p_a$, we find, as in the $\mathcal{N} = 4$ case described above, that $p_a = 0$. Varying each $H_i$ with respect to $p_a$, one finds

$$
S_{aa}^i = m_i - \frac{m}{k} . \tag{4.41}
$$

What remains is to impose the $X_a$ and $S_{ab}^i$ (more properly the $Q_{ai}$ and $\tilde{Q}_{ia}$) equations and hence find the spins at the critical points. Rather than write down these equations and find
their solutions, which is necessarily complicated and uninspiring, we can motivate the form of the solution and then verify that the ansatz is a critical point. The critical point equations are significantly simplified if, for $a \neq b$,

$$S^i_{ab} S^j_{ba} \frac{\sigma(X_{ab} + z_{ji})}{\sigma(X_{ab}) \sigma(z_{ji})} \sigma(X_{ab}) \sigma(z_{ji})$$

with $X_a$ as in (4.39), are periodic in the indices $r_a \mod q$ and $s_a \mod p$. This can be achieved if

$$S^i_{ab} \sim \rho_i^{r_a-r_b} \lambda_i^{s_a-s_b} e^{2\pi i \left[\frac{r_a-r_b}{q} \zeta(\omega_1) + \frac{s_a-s_b}{p} \zeta(\omega_2)\right]}.$$  

(4.43)

Here, $\rho_i$ and $\lambda_i$ are arbitrary $q^{th}$ and $p^{th}$ roots of unity, respectively. In fact, this periodicity requirement determines the spins completely, up to an overall factor which is easily fixed. With a little more work one is lead to the ansatz

$$S^i_{ab} = m_i \rho_i^{r_a-r_b} \lambda_i^{s_a-s_b} e^{2\pi i \left[\frac{r_a-r_b}{q} \zeta(\omega_1) + \frac{s_a-s_b}{p} \zeta(\omega_2)\right]} - \frac{m}{k} \delta_{ab}.$$  

(4.44)

Notice that this is consistent with (4.15) where

$$Q_{ai} = \sqrt{m_i} \rho_i^{r_a-r_b} \lambda_i^{s_a} e^{2\pi i \left[\frac{r_a-r_b}{q} \zeta(\omega_1) + \frac{s_a-s_b}{p} \zeta(\omega_2)\right]}, \quad \tilde{Q}_{ia} = \sqrt{m_i} \rho_i^{r_a-r_b} \lambda_i^{s_a} e^{-2\pi i \left[\frac{r_a-r_b}{q} \zeta(\omega_1) + \frac{s_a-s_b}{p} \zeta(\omega_2)\right]}.$$

(4.45)

In particular, the constraints (4.22) and (4.16b) are satisfied. In addition, the solution is consistent with (4.41). The undetermined roots of unity $\rho_i$ and $\lambda_i$ label inequivalent critical points. Hence the number of critical points appears to be $p^k q^k = N^k$. But this over counts. The residual $U(1)^k$ transformations described previously can be used to set $\rho_i = \lambda_i = 1$ for one particular $1 \leq i \leq k$, and so there is an $N^{k-1}$ additional degeneracy of vacua for each $N = pq$: precisely the same counting that we found in (2.9) for the massive vacua. This gives an important clue that the critical points we have found are to be identified with the massive vacua. But there is more. We saw that the massive vacua were related by $(2\omega_1, 2\omega_2)$ translations of the punctures $z_i$. This should be reflected by the critical points of the superpotential. Consider the quantity (4.42) with the spins as in (4.44). Under the shift $z_i \rightarrow z_i + 2m_1 + 2n_2$, for $a \neq b$,

$$S^i_{ab} S^j_{ba} \frac{\sigma(X_{ab} + z_{ji})}{\sigma(X_{ab}) \sigma(z_{ji})} \sigma(X_{ab}) \sigma(z_{ji}) \sigma(X_{ab}) \sigma(z_{ji}).$$

(4.46)

To show this one has to employ the identity $\omega_2 \zeta(\omega_1) - \omega_1 \zeta(\omega_2) = i\pi/2$. Consequently, these transformations change the roots of unity labelling the vacua as

$$z_i \rightarrow z_i + 2\omega_1: \quad \lambda_i \rightarrow \lambda_i e^{\frac{2\pi i}{p}}, \quad \rho_i \rightarrow \rho_i,$$

$$z_i \rightarrow z_i + 2\omega_2: \quad \lambda_i \rightarrow \lambda_i, \quad \rho_i \rightarrow \rho_i e^{\frac{2\pi i}{q}}.$$  

(4.47)

Hence, as found earlier, shifts of the punctures by the lattice $(2\omega_1, 2\omega_2)$ does indeed permute the $N^{k-1}$ massive vacua for a given $pq = N$. Shifts on the larger lattice, generated by $(2\tilde{\omega}_1, 2\tilde{\omega}_2) \equiv \cdots \equiv$
(2pω₁, 2qω₂), leave the vacua invariant. (Notice that the relation of p and q to ω₁ and ω₂ swaps over relative to (4.39).)

PROOF: We now prove that our ansatz (4.39) and (4.44) is a critical point of \( H_t \) and \( H_0 \). First of all, consider the \( X_a \) variations. The \( X_a \)-derivative of the second term of \( H_0 \) in Eq. (4.32) is

\[
-2 \sum_i m_i^2 \sum_{b(\neq a)} \varphi'(X_{ab}) .
\]  

(4.48)

This vanishes for the same reason as in the \( k = 1 \) case. For each \( a \) and \( b \) there exists a unique \( b' \) (possibly \( b' = b \)) such that

\[
X_{ab} = X_{b'a} \mod \Gamma .
\]  

(4.49)

Then we can see that terms cancel in pairs when \( b \neq b' \), since \( \varphi'(X_{ab}) + \varphi'(X_{ab'}) = 0 \), or \( \varphi'(X_{ab}) = 0 \) when \( b = b' \), since \( X_{ab} \) is then a half-lattice point and \( \varphi'(z) \) is an odd elliptic function. Now consider the \( X_a \)-derivatives of the third term of \( H_0 \), in (4.32), and the fourth term of \( H_t \), in (4.31). In both cases, the resulting expression involves terms like

\[
\sum_{b(\neq a)} \left\{ S_{ab}^i S_{ba}^j \frac{\sigma(X_{ab} + z_{ji})}{\sigma(X_{ab})\sigma(z_{ji})} f(X_{ab}) - S_{ba}^i S_{ab}^j \frac{\sigma(X_{ba} + z_{ji})}{\sigma(X_{ba})\sigma(z_{ji})} f(X_{ba}) \right\} ,
\]  

(4.50)

for some function \( f(X_{ab}) \) elliptic in \( X_{ab} \). Using the special periodicity properties of the quantity (4.42), one can show

\[
S_{ba}^i S_{ab}^j \frac{\sigma(X_{ba} + z_{ji})}{\sigma(X_{ba})\sigma(z_{ji})} = S_{ba}^i S_{ba}^j \frac{\sigma(X_{ab} + z_{ji})}{\sigma(X_{ab})\sigma(z_{ji})} .
\]  

(4.51)

Furthermore, \( f(X_{ba}, z_{ji}) = f(X_{ab'}, z_{ji}) \). After re-labelling the second term in Eq. (4.50) with \( b' \to b \), we see that the two terms in (4.50) cancel.

Now we turn to the spins. It is convenient to parameterize

\[ Q_{ai} = t_{ai} , \quad \tilde{Q}_{ia} = y_{ai}/t_{ai} . \]  

(4.52)

The constraints (4.22) and (4.16b) are then linear in \( y_{ai} \):

\[ \sum_a y_{ai} = N m_i , \quad \sum_i y_{ai} = m . \]  

(4.53)

The symmetries (4.20) can be used to set \( t_{a1} = t_{1i} = 1 \). Now consider

\[
\frac{\partial H_0}{\partial y_{ai}} = -2 \sum_{b(\neq a)} y_{ba} \varphi'(X_{ab}) - \sum_{j(\neq i)} \left( y_{aj} - \frac{m}{k} \right) \left( \varphi'(z_{ij}) - \zeta(z_{ij})^2 \right) + \frac{2}{y_{ai}} \sum_{b(\neq a)} \sum_{j(\neq i)} S_{ba}^i S_{ab}^j \frac{\sigma(X_{ba} + z_{ji})}{\sigma(X_{ba})\sigma(z_{ji})} \left( \zeta(X_{ba} + z_{ji}) - \zeta(X_{ba}) \right) + B_i + C_a .
\]  

(4.54)
Here, $B_t$ and $C_a$ are the Lagrange multipliers for the constraints (4.53). In order to show that the derivative vanishes, it is sufficient to show that the three terms in (4.54), besides the Lagrange multipliers, are independent of $a$, since then one can solve for the Lagrange multipliers. Recall that our ansatz for the solution has $y_{ai} = m_i/N$, independent of $a$. The second term in (4.54) is then manifestly independent of $a$. Contrary to appearances, the first term is also independent of $a$, since $\sum_{b(\neq a)} \phi(X_{ab})$ is, itself, independent of $a$ due to the form of $X_a$ (4.39) and the elliptic periodicity of $\phi(z)$. The second term involves

$$
\sum_{b(\neq a)} S_{ba}^i S_{ab}^j \frac{\sigma(X_{ba} + z_{ji})}{\sigma(X_{ab})\sigma(z_{ji})} \left( \zeta(X_{ba} + z_{ji}) - \zeta(X_{ba}) \right).
$$

(4.55)

This is also independent of $a$. To show this, one uses the special periodicity property that we established for the quantity (4.42). Hence, there exists Lagrange multipliers such that $\partial H_0 / \partial y_{ai} = 0$. The same kind of reasoning shows that $\partial H_j / \partial y_{ai} = 0$.

Finally, the $t_{ai}$ derivatives of $H_i$ and $H_0$ involve quantities like

$$
\frac{1}{t_{ai}} \sum_{b(\neq a)} \left\{ S_{ab}^i S_{ba}^j \frac{\sigma(X_{ab} + z_{ji})}{\sigma(X_{ab})\sigma(z_{ji})} f(X_{ab}, z_{ji}) - S_{ba}^i S_{ab}^j \frac{\sigma(X_{ba} + z_{ji})}{\sigma(X_{ba})\sigma(z_{ji})} f(X_{ba}, z_{ji}) \right\},
$$

(4.56)

where $f(X_{ab}, z_{ji})$ is elliptic in $X_{ab}$. But this is of the form (4.50) which we have already shown vanishes.

As in the softly broken $\mathcal{N} = 4$ case, there are additional massive vacua that are obtained by modular transformations in $\tau$ associated to the more general configurations (4.40).

Finally, we can evaluate the Hamiltonians on the vacua. It suffices to pick the vacua with $\rho_i = \lambda_i = 1$ and $l = 0$, since all the others are related either by $\tau$ modular transformations or shifts of the $\{z_i\}$. One finds

$$
H^*|_{l=0; \rho_i = \lambda_i = 1} = -N \sum_i m_i^2 \sum_{(r,s) \neq (0,0)} \phi \left( \frac{2r}{q} \omega_1 + \frac{2s}{p} \omega_2 \right) + \sum_{i \neq j} m_i m_j \left\{ -N^2 \tilde{\phi}(z_{ij}) + N \phi(z_{ij}) 
$$

$$
+ N^2 \sum_{(r,s) \neq (0,0)} \left[ \tilde{\zeta}(\Omega_{sr}) - \frac{2r}{q} \tilde{\zeta}(q \omega_2) - \frac{2s}{p} \tilde{\zeta}(p \omega_1) \right] \left[ \tilde{\zeta}(z_{ij} + \Omega_{sr}) - \tilde{\zeta}(\Omega_{sr}) \right]
$$

$$
+ N \left[ (N - 1) \tilde{\zeta}(z_{ij}) - \sum_{(r,s) \neq (0,0)} \left( \tilde{\zeta}(z_{ij} + \Omega_{sr}) - \tilde{\zeta}(\Omega_{sr}) \right) \right] \tilde{\zeta}(z_{ij}) - \frac{2}{k} \sum_{l(\neq i)} \zeta(z_{il})
$$

$$
- (N - 1) \sum_{(r,s) \neq (0,0)} \phi \left( \frac{2r}{q} \omega_1 + \frac{2s}{p} \omega_2 \right) \right\},
$$

(4.57)
along with

\[ H_i \bigg|_{l=0, \rho_i=\lambda_i=1} = 2N \sum_{j(\neq i)} m_i m_j \left[ (N - 1)\tilde{\zeta}(z_{ij}) - \sum_{(r,s) \neq (0,0)} \left( \tilde{\zeta}(z_{ij} + \Omega_{sr}) - \tilde{\zeta}(\Omega_{sr}) \right) \right]. \]  \hspace{1cm} (4.58)

The expression for the residues, \( H_i \), above matches (3.13) precisely. It now remains to show that the expression for \( H^* \) in (4.57) matches with the M-theory result (3.15) using the expression for \( H_0 \) in (3.14). This can be achieved by noting that both the expressions for \( H^* \) are in fact \( \tilde{\tau} \)-elliptic in the variables \( z_{ij} \). It is then a straightforward but tedious exercise (making use of various identities provided in Appendix A) to show that the residues of the double poles and simple poles and the constant pieces of these expressions treated as functions of \( z_{ij} \), are indeed identical. As the expressions are \( \tilde{\tau} \)-elliptic in the \( z_{ij} \), this is sufficient to demonstrate that they are in fact the same.

Recall that in the M theory picture one has a freedom to choose which of the \( N \) branches of the covering each NS5-brane lies on. This freedom, represented by the parameters \( (s_i, r_i) \) in (3.9), is encoded in the integrable system as the \( N \) choices for the roots of unity \( \rho_i \) and \( \lambda_i \). In particular, the covering where all the NS5-branes lie on the same branch, \( (s_i, r_i) = (0, 0) \) corresponds to \( \rho_i = \lambda_i = 1 \).

5. Conclusions and Future Directions

In this paper we have provided a classification of the vacuum structure and duality properties of the \( \mathcal{N} = 1^* \) deformations (mass-deformations) of the \( \mathcal{N} = 2 \) quiver theories. We have also obtained exact results for certain chiral condensates in the massive vacua of these theories, following two completely different approaches, namely: (i) by lifting the corresponding Type IIA brane set-ups to M-theory and (ii) by studying the theory on \( \mathbb{R}^3 \times S^1 \). For a certain class of mass-deformations, both approaches were used to independently evaluate the exact superpotential (Eqs.(3.13),(3.14) and (3.17)) in each massive vacuum and extremely non-trivial agreement was found. In particular, one of the main results of this paper is an exact superpotential Eq. (4.36) for the theory on \( \mathbb{R}^3 \times S^1 \), which coincides with a certain linear combination of the quadratic Hamiltonians of the spin-generalization of the elliptic Calogero-Moser system. This is a generalization of the corresponding results for the mass-deformed \( \mathcal{N} = 4 \) theory obtained in [4]. Although we have only concentrated on the massive vacua, the superpotential (4.36) also determines all the massless vacua. However, even in the basic \( k = 1 \) case the structure of the massless vacua is not known beyond \( N = 3 \). It would be interesting to understand the structure of these vacua as well.
Some immediate applications of our results include the calculation of physical quantities such as the gluino condensate and tensions of domain walls interpolating between the massive vacua [19, 30]. In particular, these quantities may be evaluated in the large $g^2N$, large-$N$ limit for a direct comparison with the appropriate deformation of the Type IIB backgrounds on $AdS_5 \times S^5/Z_k$. In the $k = 1$ case i.e. in the $N = 1^*$ theory, the corresponding string backgrounds [11] were asymptotically $AdS_5 \times S^5$ containing D3-branes polarized to 5-branes in the interior, wrapping 2-cycles of the $S^5$. From the point of view of the string dual, it would be interesting to understand the characterisation of the extra vacua which arise in the $k > 1$ theories. As remarked in [11], they are presumably associated with the values of twisted sector fields.

The superpotential Eq. (4.36) and the expressions for the condensates Eqs. (3.13), (3.14), (3.15) and (4.57) contain a wealth of information regarding instanton and “fractional instanton” contributions in the massive vacua. In the three-dimensional picture, the superpotential receives contributions from semiclassical monopoles carrying fractional topological charge. The nature of these contributions is visible in the condensates in the semiclassical limit. For example in the $N^{k-1}$ confining vacua with $p = N, q = 1$ and $l = 0$, the semi-classical expansion reveals contributions from instantons as well as fractional instantons in each gauge group factor. In the $k = 1$ theory, i.e. mass-deformed $\mathcal{N} = 4$ SUSY Yang-Mills, in the large $g^2N$ limit, the fractional instanton series can be ‘resummed’ using $\tilde{S}$-duality to obtain a dual expansion. Terms in this dual expansion can be elegantly described in the IIB string dual of Polchinski and Strassler as arising from world-sheet instantons wrapping the 2-cycles of the 5-branes polarized from the D3-branes. While one expects a similar interpretation to arise in the $k > 1$ theories, the appearance of different types of fractional instantons (from each gauge group factor) and their associated actions needs to be understood better from the point-of-view of the string dual.

Finally, we point out that in the mass-deformed $\mathcal{N} = 4$ theory on $\mathbb{R}^3 \times S^1$, the appearance of the elliptic superpotential [4] encoding pairwise interactions between $N$ particles on a torus can be given a very nice, physical interpretation. Realizing the compactified $\mathcal{N} = 4$ theory on $N$ D3-branes wrapped on a circle, one may perform T-duality and lift the resulting setup of $N$ D2-branes to M-theory. We thus obtain $N$ M2-branes with two transverse compact directions (the second compact direction being the M-dimension), which may now be thought of as the $N$ particles on a torus. Upon introducing the $\mathcal{N} = 1^*$ mass-deformation the M2-branes exert forces on each other which is described by the Weierstrass superpotential obtained in [4]. It would be extremely interesting to generalize this picture to the superpotential Eq.(4.36) for the $\mathcal{N} = 1^*$ quiver theory and obtain an interpretation for the “spin-spin” interactions in the Hamiltonian in terms of appropriate interactions between M2-branes.

Appendix A: Some Properties of Elliptic Functions
In this appendix we provide some useful—but far from complete—details of elliptic functions and their near cousins. For a more complete treatment we refer the reader to one of the textbooks, for example [24]. An elliptic function \( f(z) \) is a function on the complex plane, periodic in two periods \( 2\omega_1 \) and \( 2\omega_2 \). We will define the lattice \( \Gamma = 2\omega_1 \mathbb{Z} \oplus 2\omega_2 \mathbb{Z} \) and define the basic period parallelogram as
\[
D = \left\{ z = 2\mu \omega_1 + 2\nu \omega_2, \ 0 \leq \mu < 1, \ 0 \leq \nu < 1 \right\} .
\] (A.1)

The archetypal elliptic function is the Weierstrass \( \wp(z) \) function. It is an even function which can be defined via
\[
\wp(z) = \frac{1}{z^2} + \sum_{m,n \neq (0,0)} \left( \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right) ,
\] (A.2)
where the sums is over all integer pairs except \( m = n = 0 \). The function \( \wp(z) \) is analytic throughout \( D \), except at \( z = 0 \) where it has a double pole:
\[
\wp(z) = \frac{1}{z^2} + \mathcal{O}(z^2) .
\] (A.3)

The other two important functions for our purposes, are \( \sigma(z) \) and \( \zeta(z) \). They are both odd functions but are not elliptic, since they have anomalous transformations under shifts by periods:
\[
\zeta(z + 2\omega \ell) = \zeta(z) + 2\zeta(\omega \ell) , \quad \sigma(z + 2\omega \ell) = -\sigma(z)e^{2(z+\omega \ell)\zeta(\omega \ell)} .
\] (A.4)

The three functions are related via
\[
\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} , \quad \wp(z) = -\zeta'(z) .
\] (A.5)

In \( D \), \( \zeta(z) \) has a simple pole and \( \sigma(z) \) a simple zero at \( z = 0 \):
\[
\zeta(z) = \frac{1}{z} + \mathcal{O}(z^3) , \quad \sigma(z) = z + \mathcal{O}(z^5) .
\] (A.6)

Some useful identities for \( \zeta(z) \) and \( \wp(z) \) evaluated on half-periods are
\[
\omega_2 \zeta(\omega_1) - \omega_1 \zeta(\omega_2) = \frac{\pi i}{2} ,
\] (A.7a)
\[
\zeta(\omega_1 + \omega_2) = \zeta(\omega_1) + \zeta(\omega_2) ,
\] (A.7b)
\[
\wp(\omega_1 + \omega_2) + \wp(\omega_1) + \wp(\omega_2) = 0 .
\] (A.7c)

An elliptic function \( f(z) \) can always be expressed as
\[
f(z) = c_0 + \sum_{k=1}^{n} \left\{ c_{k,1}\zeta(z - a_k) + \cdots + c_{k,r_k}\zeta^{(r_k-1)}(z - a_k) \right\}
\] (A.8)
for constants $c_0$ and $c_{k,i}$ with the restriction that the sum of the simple pole residues vanishes, $\sum_{k=1}^n c_{1,k} = 0$. In particular, an elliptic function which has no poles is a constant.

Of particular importance to us is the behaviour of our basic functions under modular transformations of the complex structure of the torus defined by $\Gamma$. These $\text{SL}(2,\mathbb{Z})$ transformations act as

$$
\begin{pmatrix}
\omega_2 \\
\omega_1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\omega_2 \\
\omega_1
\end{pmatrix},
$$

(A.9)

for $a, b, c, d \in \mathbb{Z}$ subject to $ad - bc = 1$. Since we choose $\omega_1 = i\pi$ and $\omega_2 = i\pi\tau$, after the transformation (A.9), one has to perform a re-scaling by $(c\tau + d)^{-1}$ so that the transformation on $\tau$ has the usual form:

$$
\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d}.
$$

(A.10)

A function $f(z)$ has modular weight $w$ if

$$
f(z|\tau') = (c\tau + d)^w f(z(c\tau + d)|\tau).
$$

(A.11)

The functions $\wp(z)$, $\zeta(z)$ and $\sigma(z)$ have modular weights 2, 1 and $-1$, respectively.

We will need to explore the semi-classical limit of our functions. This is the limit $g^2 \rightarrow 0$, $\tau \rightarrow i\infty$ or $\omega_2 \rightarrow -\infty$ with fixed $\omega_1 = i\pi$. In this limit, for $z \in \mathcal{D}$,

$$
\begin{align*}
\wp(z) &\rightarrow \frac{1}{12} + \frac{1}{4 \sinh^2 \frac{z}{2}}, \\
\zeta(z) &\rightarrow -\frac{z}{12} + \frac{1}{2} \coth \frac{z}{2}, \\
\sigma(z) &\rightarrow 2e^{-z^2/24} \sinh \frac{z}{2}.
\end{align*}
$$

(A.12)

In particular, for fixed $-2 < \alpha_i < 2$, $\sum_i \alpha_i = 0$,

$$
\lim_{\tau \rightarrow i\infty} \sum_i \zeta(\alpha_i \omega_2) = -\frac{1}{2} \sum_i \text{sign}(\alpha_i).
$$

(A.13)

Below we collect, without proof, various useful identities. The un-tilded functions are defined with respect to the lattice $\Gamma = 2\omega_1 \mathbb{Z} \oplus 2\omega_2 \mathbb{Z}$ while the tilded functions are defined with respect to the lattice $\tilde{\Gamma} = 2p\omega_1 \mathbb{Z} \oplus 2q\omega_2 \mathbb{Z}$.

$$
\zeta(z) = \sum_{(r,s)} \tilde{\zeta}(z + 2s\omega_1 + 2r\omega_2) - \sum_{(r,s), (r,s) \neq (0,0)} \tilde{\zeta}(2s\omega_1 + 2r\omega_2) - \frac{q\tilde{\zeta}(p\omega_1) - \zeta(\omega_1)}{\omega_1} z.
$$

(A.14)
\[ \psi(z) = \sum_{(r,s)} \bar{\psi}(z + 2s \omega_1 + 2r \omega_2) - \sum_{(r,s) \neq (0,0)} \bar{\psi}(2s \omega_1 + 2r \omega_2) \, . \] (A.15)

\[ \sum_{(r,s) \neq (0,0)} \psi(\frac{2r}{q} \omega_1 + \frac{2s}{p} \omega_2) = -pq \sum_{(r,s) \neq (0,0)} \bar{\psi}(2s \omega_1 + 2r \omega_2) \, . \] (A.16)

\[ \sum_{(r,s) \neq (0,0)} \left\{ \zeta(\frac{2r}{q} \omega_1 + \frac{2s}{p} \omega_2) - \frac{2r}{q} \zeta(\omega_1) - \frac{2s}{p} \zeta(\omega_2) \right\}^2 = \frac{pq - 2}{pq} \sum_{(r,s) \neq (0,0)} \psi(\frac{2r}{q} \omega_1 + \frac{2s}{p} \omega_2) \, . \] (A.17)

\[ \sum_{(r,s) \neq (0,0)} \left\{ \zeta(\frac{2r}{q} \omega_1 + \frac{2s}{p} \omega_2) - \frac{2r}{q} \zeta(\omega_1) - \frac{2s}{p} \zeta(\omega_2) \right\} e^{-2\pi i \frac{r' q + 2\pi i ss'}{p}} 
\[ = pq \left\{ \bar{\zeta}(2s' \omega_1 + 2r' \omega_2) - \frac{2r'}{q} \bar{\zeta}(q \omega_2) - \frac{2s'}{p} \bar{\zeta}(p \omega_1) \right\} \, . \] (A.18)

\[ \left[ \zeta(x + y) - \zeta(x) - \zeta(y) \right]^2 = \psi(x + y) + \psi(x) + \psi(y) \, . \] (A.19)

For \[ \sum_i \lambda_i = 0 \]
\[ \left( \sum_i \lambda_i \zeta(z_i) \right)^2 = \sum_i \lambda_i^2 \psi(z_i) + \sum_{i \neq j} \lambda_i \lambda_j \left\{ 2 \zeta(z_i) \zeta(z_j - z_i) - \frac{1}{2} \psi(z_i - z_j) + \frac{1}{2} \zeta(z_i - z_j)^2 \right\} \, . \] (A.20)

Finally
\[ \sum_{(r,s) \neq (0,0)} \psi(\frac{2r}{q} \omega_1 + \frac{2s}{p} \omega_2) = \frac{pq}{12} \left[ E_2(\tau) - \frac{q}{p} E_2(\bar{\tau}) \right] \, . \] (A.21)

where \( E_2(\tau) \) is the second Eisenstein series [29] which has the modular transformation properties \( E_2(\tau + 1) = E_2(\tau) \) and \( E_2(\tau) = E_2(-1/\tau)/\tau^2 - 6/i\pi \tau \).

**Appendix B: Rotating the brane configuration**

For the non-elliptic models it is well established that the breaking supersymmetry by adding mass terms for the adjoint chiral multiplets can be realized in the Type IIA brane configurations by relative rotations of the NS5-branes into the \( w = x^8 + ix^9 \) direction [20]. This way of realizing soft breaking to \( \mathcal{N} = 1 \) in the non-elliptic models gives a very simple picture of why the Riemann
surface $\Sigma$ has to degenerate at a vacuum. The point is that only very particular surfaces $\Sigma$ are “rotatable”. Let us quickly review the argument in the context of a non-elliptic model with two NS5-branes describing a model with $U(N)$ gauge symmetry. Breaking to $\mathcal{N} = 1$ is achieved by rotating the first NS5-brane into the $(v, w)$ plane, along the line $w = \mu v$, while leaving the second NS5-brane intact. Since $v$ diverges at the positions of the five-branes, $w$ must diverge at the first five-brane like $\mu v$, but vanish at the position of the second five-brane. Since $\mu$ is a free parameter we can construct the rotated surface $\tilde{\Sigma}$ order-by-order in $\mu$. Working to first order in $\mu$ allows us to find the constraints on the original $\Sigma$ in order that it be “rotatable”. To first order, the problem is to find a meromorphic function $w$ on the initial surface $\Sigma$, which has a simple pole at the position of the first five-brane. But a function with a single simple pole cannot exist on surfaces of genus $>0$, and so in order that $\Sigma$ be rotatable it must have completely degenerated to a surface of genus 0.

The goal of the present section will be to generalize the brane rotation story to the elliptic models. We shall see that the situation is rather more involved but the essence of the problem is the same. The condition that a surface $\Sigma$ be rotatable will boil down to the existence, or otherwise, a certain meromorphic function on $\Sigma$ with a prescribed pole structure. To start with we consider the original $\mathcal{N} = 1^*$ models where there is only a single NS5-brane. The obvious problem is that there is no immediate sense rotating a single NS5-brane. Thinking about the periodicity in terms of an infinite string of images, what we need to do is rotate each image relative to the last. The situation is rather similar to the introduction of the global $\mathcal{N} = 2$ mass $m$. Recall that in order to introduce this mass we had to modify the spacetime itself introducing a non-trivial bundle over $E_\tau$. We need to do an analogous thing in order to incorporate the $\mathcal{N} = 1^*$ rotation. The twist acts as a complex rotation on the complex combination $g = v + i w$:

$$g \to e^{i\xi} g, \quad e^{i\xi} \equiv \frac{1 + i\mu}{1 - i\mu}, \quad (B.1)$$

where $\mu$ is the $\mathcal{N} = 1^*$ mass. Comparing with the discussion of how the global mass $m$ was introduced, we can trivialize the bundle at the expense introducing a suitable singularity at an arbitrary point of $E_\tau$. Recalling that the shift $v \to v + m$ required that $v(z)$ to a simple pole at an arbitrary point (chosen to be $z = 0$) on each sheet with residue $-m/N$, we see that $g(z)$ must have an essential singularity of the form

$$g(z) \sim \exp \left( -\frac{i\xi/N}{z - z_0} \right) \quad (B.2)$$

at a new arbitrary point $z_0$ on each sheet. The function $g(z)$ must also have simple poles at $z = 0$, as before to incorporate the global mass $m$, on each sheet and also a simple pole of residue $m$ at $z = z_1$ on the single sheet where the NS5-brane is located.

---

15Technically $\Sigma$ is non-compact at the positions of the five-brane, so we must compactify it by adding these points to get a compact surface.
We now follow the logic of the non-elliptic case and work to first order in $\mu$ in order to derive the condition that $\Sigma$ is rotatable. To leading order we think of $g(z) = v^{(0)}(z)$, where $v^{(0)}(z)$ describes the un-rotated surface $\Sigma$. Consider the function $h(z) = g(z) - v^{(0)}(z)$. The simple pole at the NS5-brane is now cancelled and so $h(z)$ has simple poles at $z = 0$ on each sheet, of the same residue, and essential singularities at $z = z_0$ of the form (B.2) on each sheet. It is instructive to consider the function $\sum_a h_a(z)$, a sum over the images of $h(z)$ on each sheet. This is a bona fide meromorphic function on the torus $E_\tau$ itself with a simple pole at $z = 0$ and an essential singularity of the form (B.2) at $z = z_0$. The unique function with these properties is

$$\frac{\sigma(z + i\xi/N)}{\sigma(z)} e^{-i\xi(z-z_0)/N}. \tag{B.3}$$

We can find a necessary condition that $\Sigma$ be rotatable by working to first order in $\mu$ (or $\xi$). To this order, we expand (B.2) to find simple poles at $z = z_0$, with residue $-i\mu/N$ on each sheet. So the surface $\Sigma$ will be rotatable provided there exists a meromorphic function $h(z)$ on $\Sigma$, more properly its compactification, with simple poles on each sheet at $z = 0$ and $z_0$ of equal and opposite residue $\pm i\mu/N$, respectively.\(^\text{16}\)

We can translate the condition into something more convenient by the following chain of arguments. Since $z_0$ is arbitrary, we can take $z_0 \to 0$. In this limit the simple poles merge together in pairs on each sheet yielding double poles. So $\Sigma$ will be rotatable provided there exists a meromorphic function on $\Sigma$ with double poles at $z = 0$ on each sheet with the same coefficient and vanishing residue. Finally, by taking a linear combination of this function, $v^{(0)}(z)^2$ and $v^{(0)}(z)$ we can find a function which is now regular at $z = 0$, but now has a double pole at $z = z_1$ on the single sheet where the NS5-brane is located. So a necessary condition that $\Sigma$ be rotatable is that there exists a meromorphic function $f(z)$ on its compactification which has a single double pole at the position of the NS5-brane.

The Riemann-Roch Theorem implies that, generically, $f(z)$ can only exist on surfaces with genus 0 or 1. So generically in our example, in order for $\Sigma$ to be rotatable it must completely degenerate to an unbranched (unramified) $N$-fold cover of the torus. The function with a single double pole is then the Weierstrass function. In this case, as we shall see, the theory has a mass gap (ignoring the overall $U(1)$ factor). Exceptionally, however, a suitable function $f(z)$ can exist on a higher genus surface, in which case the deformed theory is massless since there is an unbroken $U(1)^{g-1}$ symmetry (in addition to the overall $U(1)$ factor) where $g$ is the genus. In fact, we can say more about these exceptional surfaces. In order that there exists a meromorphic function with a single double pole at the position of the NS5-brane $\Sigma$ must

\(^{16}\)This last requirement follows from the fact that $\sum_a h_a(z)$ is a meromorphic function on the torus $E_\tau$ with two simple poles the sum of whose residues must vanish, as can be verified by expanding (B.3) to linear order in $\xi$.\]
necessarily be hyper-elliptic and moreover the NS5-brane must necessarily be positioned at one of the $2g + 2$ Weierstrass Points of the surface.

The generalization to the case with more NS5-branes is now clear. As well as an overall rotation, there are $k - 1$ relative rotations of the NS5-branes parameterized by $\hat{\mu}_i = \mu_i - \mu/k$, $i = 1, \ldots, k$ and $\sum_i \mu_i = \mu$. The surface will be rotatable if $h(z)$ has the pole structure established in the $k = 1$ case above but, in addition, has a simple pole at the position of the $i$th NS5-brane with residue $\hat{\mu}_i$ for $i = 1, \ldots, k$. For example, if $\mu = 0$, then $h(z)$ only has simple poles at the NS5-branes with residues $\hat{\mu}_i$. As before the Riemann-Roch Theorem dictates that such a function will generically only exist when $\Sigma$ undergoes complete degeneration to the torus. However, exceptionally there will exist higher genus surfaces which admit such a function. Notice that these exceptional $g > 1$ surfaces will depend on the $\mu_i$: in other words, as we vary the $\mathcal{N} = 1$ deformation the rotatable surface $\Sigma$ will change accordingly.

References

[1] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323 (2000) 183 [hep-th/9905111].

[2] M. R. Douglas, G. Moore, hep-th/9603167.

[3] R. Donagi and E. Witten, Nucl. Phys. B460 (1996) 299 [hep-th/951010].

[4] N. Dorey, JHEP 9907 (1999) 021 [hep-th/9906011].

[5] E. Witten, Nucl. Phys. B 500 (1997) 3 [hep-th/9703166].

[6] J. Gibbons and T. Hermsen, Physica 11D (1984) 337.

[7] I. Krichever, O. Babelon, E. Billey and M. Talon, equation,” hep-th/9411160.

[8] N. Nekrasov, Commun. Math. Phys. 180 (1996) 587 [hep-th/9503157].

[9] S. Kachru and E. Silverstein, Phys. Rev. Lett. 80 (1998) 4855 [hep-th/9802183].

[10] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19 [hep-th/9407087].

[11] J. Polchinski and M. Strassler, hep-th/0003136.

[12] N. Dorey, O. Aharony and S.P. Kumar, JHEP 0006 (2000) 026 [hep-th/0006008].

[13] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B355 (1995) 466 [hep-th/9505035]. E. Martinec and N. Warner, Nucl. Phys. B459 (1996) 97 [hep-th/9509161].

[14] E. J. Martinec, Phys. Lett. B 367 (1996) 91 [hep-th/9510204].
[15] M. A. Olshanetsky and A. M. Perelomov, Phys. Rept. 71 (1981) 313.

[16] A. Kapustin, Nucl. Phys. B 534 (1998) 531 [hep-th/9804069].

[17] N. Seiberg and E. Witten, hep-th/9607163.

[18] K. Intriligator and N. Seiberg, Phys. Lett. B 387 (1996) 513 [hep-th/9607207].

[19] N. Dorey and S. P. Kumar, JHEP 0002 (2000) 006 [hep-th/0001103].

[20] K. Hori, H. Ooguri and Y. Oz, “theory fivebrane,” Adv. Theor. Math. Phys. 1 (1998) 1 [hep-th/9706082].

[21] C. Vafa and E. Witten, Nucl. Phys. B431 (1994) 3 [hep-th/9408074].

[22] J. Polchinski, Int. J. Mod. Phys. A 16 (2001) 707 [hep-th/0011193].

[23] S. Katz, P. Mayr and C. Vafa, Adv. Theor. Math. Phys. 1 (1998) 53 [hep-th/9706110].

[24] E.T. Whittaker and G.N. Watson, “A Course of Modern Analysis”, Cambridge University Press, 4th Edition 1927.

[25] N. Dorey, V. V. Khoze and M. P. Mattis, Phys. Lett. B396 (1997) 141 [hep-th/9612231].

[26] A. Gorsky and A. Mironov, hep-th/0011197.

[27] N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, Commun. Math. Phys. 108 (1987) 535.

[28] N. Hitchin, Duke Math. J. 54 (1987) 91.

[29] N. Koblitz, ‘Introduction to Elliptic Curves and Modular Forms’ (Springer-Verlag,1984).

[30] N. Dorey, T. J. Hollowood and S. P. Kumar, work in progress.