Distribution of rooks on a chess-board representing a Latin square partitioned by a subsystem (Part 1.)*

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Abstract

A $d$-dimensional generalization of a Latin square of order $n$ can be considered as a chess-board of size $n \times n \times \ldots \times n$ ($d$ times), containing $n^d$ cells with $n^{d-1}$ non-attacking rooks. Each cell is identified by a $d$-tuple $(e_1, e_2, \ldots, e_d)$ where $e_i \in \{1, 2, \ldots, n\}$. For $d = 3$ we prove that such a chess-board represents precisely one main class.

A subsystem $T$ induced by a family of sets $< E_1, E_2, \ldots, E_d >$ over $\{1, 2, \ldots, n\}$ is real if $E_i \subset \{1, 2, \ldots, n\}$ for each $i \in \{1, 2, \ldots, d\}$. The density of $T$ is the ratio of contained rooks to the number of cells in $T$. The distance between two subsystems is the minimum Hamming distance between cell pairs. Replacing $k$ sets of $< E_1, E_2, \ldots, E_d >$ by their complements, a subsystem $U$ is obtained with distance $k$ between $T$ and $U$. All these subsystems, including $T$, form a partition of the chess-board. We prove that in such a partition, the number of rooks in a $U$ and the density of $U$ can be determined from the number of rooks in $T$ and the number of cells in $T$ and $U$ and the value of $(−1)^k$. We examine the subsystem couple $(T, U)$ in the 2- and 3-dimensional cases, where $U$ is the most distant unique subsystem from a real $T$. On the fly, a new identity of binomial coefficients is proved.

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Abbreviations

| Abbreviation | Description                      |
|--------------|----------------------------------|
| LS           | Latin square                     |
| LSC          | Latin Super Cube                 |
| $d$-LSC      | $d$-dimensional Latin Super Cube |
| RBC          | remote brick couple              |

*We intend to publish several consecutive papers on Latin and partial Latin squares and on bipartite graphs, so it is worth studying them in the order of publishing.
1 Introduction

The Hamming distance between two $d$-tuples is the number of positions at which the corresponding coordinates differ. Formally if $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ then $d(a, b) = |\{i: a_i \neq b_i\}|$.

A Latin square of order $n$ is an $n \times n$ array consisting of $n$ different symbols such that each symbol appears exactly once in each row and column. The property that no row or column contains any symbol more than once is known as the Latin property.

From now on, the set of symbols is always $Z_n = \{1, 2, \ldots, n\}$, so the Latin square can also be considered as a set of ordered triples of the form $(i, j, k)$, where $i, j, k \in Z_n$, and each Hamming distance between two distinct triples is at least 2.

Two Latin squares are isotopic if each can be obtained from the other by permuting the rows, columns, and symbols. This is an equivalence relation, with the equivalence classes are called isotopy classes.

Each Latin square $Q$ has six conjugate Latin squares obtained by uniformly permuting the coordinates in each of its triples. They are denoted by $Q(i, j, k)$, $Q(j, k, i)$, $Q(k, i, j)$, $Q(j, i, k)$, $Q(i, k, j)$, $Q(k, j, i)$, where $Q(i, j, k) = Q$.

Two Latin squares are called paratopic, if one of them is isotopic to a conjugate of the other. This is an equivalence relation, where the equivalence classes are called main classes.

2 d-LSC

A 3-dimensional chess-board of size $n \times n \times n$ is denoted by $H^3_n$ or simply $H^3$. Each cell is identified by a triple $(i, j, k)$, where $i, j, k \in Z_n$ and the distance between two cells is the Hamming distance. The chess-board $H^3_n$ can be regarded as a Rubik’s cube with $n^3$ cells (cubelets) and can be placed in a coordinate system with one vertex at the origin, as depicted in Figure 2.1.

![Figure 2.1](image-url)
There is an immediate generalization of this structure to dimension \(d\), where \(d > 3\). The chess-board \(H^d_n\) contains \(n^d\) cells and each cell is identified by a \(d\)-tuple \((e_1, e_2, \ldots, e_d)\) where \(e_i \in \mathbb{Z}_n\) for each \(i \in \mathbb{Z}_d\). In case of \(d = 2\) and \(d = 3\) we generally denote the coordinate axes \(x, y\) and \(x, y, z\), respectively. An illustration of the latter case is shown in Figure 2.1.

If \(d > 3\), then the axes of the chess-board \(H^d_n\) are denoted by \(t_1, t_2, \ldots, t_d\). If a cell is identified by a \(d\)-tuple \((e_1, e_2, \ldots, e_d)\), then we consider \(e_i\) as a coordinate of the cell on the axis \(t_i\) for each \(i \in \mathbb{Z}_d\).

**Definition 2.1.** Let \(H^d_n\) be a \(d\)-dimensional chess-board and \(j\) be a fixed coordinate on the axis \(t_i\), where \(j \in \mathbb{Z}_n\) and \(i \in \mathbb{Z}_d\). The set of cells of the chess-board whose coordinate is \(j\) on the axis \(t_i\), is called the \((d-1)\)-dimensional subspace of \(H^d_n\) and denoted by \(H^i_j\).

**Corollary 2.2.** For any axis \(t_i\), the subspaces \(H^i_j\) form a partition of \(H^d_n\) if \(j\) goes from 1 to \(n\).

**Definition 2.3.** A subspace of the chess-board of dimension 1 is called a file, of dimension 2 is called a layer.

**Definition 2.4.** If the chess-board \(H^d_n\) contains exactly \(n^d-1\) non-attacking rooks, then the structure is called a Latin Super Cube of dimension \(d\) or a \(d\)-LSC for short.

A brief description of the concept can be found in [1]. Instead of the term “Latin property” we often say that the rooks are non-attacking, or they do not see each other. Each subspace of the chess-board of dimension \(k\), where \(k \in \mathbb{Z}_d\), is itself considered as a \(k\)-LSC with \(n^k\) cells and \(n^{k-1}\) non-attacking rooks. Consequently, a file contains 1 rook, a layer contains \(n\) non-attacking rooks.

## 3 One Cube Represents One Main Class

The first element of a double identifying the cell \((i, j)\) of a Latin square is the row coordinate on the \(x\)-axis, the second is the column coordinate on the \(y\)-axis and the plane containing the both axes is denoted by \([xy]\). The cube, in which rooks are placed according to symbols, lies on the plane and one vertex, denoted by \(A\), is at the origin of \([xy]\) and the two neighboring vertices, denoted by \(B\) and \(D\) are on the \(x\)-axis and \(y\)-axis, respectively. The third neighboring vertex of \(A\), denoted by \(F\), is on the \(z\)-axis.

**Definition 3.1.** Building a 3-LSC from a Latin square by placing a rook into the cublet \((i, j, k)\) when the cell \((i, j)\) of the Latin square contains a symbol \(k\) is called composition.

This implies that a 3-LSC derived by composition from a Latin square is always created in a right-handed coordinate system and the symbols no longer play a special role in the LSC (a symbol value is just one of the three coordinates).

**Definition 3.2.** Place a 3-LSC in a right-handed coordinate system with one vertex at the origin. Creating a Latin square by placing a symbol \(k\) into the cell \((i, j)\) of the square when the cublet \((i, j, k)\) contains a rook is called projection.
Since the rooks are non-attacking, the result of the projection of a 3-LSC to any face of the cube is clearly a Latin square.

**Definition 3.3.** Two 3-LSCs are isotopic if each can be obtained from the other by permuting the rows layers, columns layers and symbols layers.

**Remark 3.4.** Let \( L \) be a 3-LSC derived from a Latin square \( Q \) by composition. There is a one-to-one correspondence between the permutations of the rows, columns and symbols of \( Q \) and the permutations of the row, column and symbol layers of \( L \). Therefore, there is a one-to-one correspondence between the isotopes of \( Q \) and isotopes of \( L \), that is if \( \mathcal{P} \) is a series of permutations that transforms \( Q \) to \( Q' \) then applying \( \mathcal{P} \) on \( L \), the resulting LSC \( L' \) can be derived from \( Q' \) by composition.

**Definition 3.5.** Since the permutations of the layers and the rotations of the cube leave the property unchanged that the rooks are non-attacking, they are called Latin transformations.

If we consider a 3-LSC \( L \) derived from a Latin square \( Q \) by composition the \( L \) does not “remember” which face of the cube was on the plane when the composition was applied, so applying Latin transformations, each face of the cube can be the face to apply a projection. To apply a projection to a specific face of the cube, without losing generality, we always rotate the cube so that the given face is on the plane \([xy]\) and a pre-selected vertex is at the origin.

**Definition 3.6.** A face of the cube can be identified by 3 adjacent vertices, so the face that contains vertices \( A, B \) and \( D \) is denoted by \([BAD]\) where the middle vertex \( A \) is at the origin, the first vertex \( B \) is on the \( x \)-axis and the third vertex \( D \) is on the \( y \)-axis, as shown in the Figure 2.1. The face \([BAD]\) is called bottom face, and the parallel face is called cover face and can be denoted \([EFG]\) or \([GHE]\).

**Remark 3.7.** Note that in a right-handed coordinate system with the face \([EFG]\) on the plane \([xy]\), the edge \( EF \) is on the \( x \)-axis and the edge \( FG \) is on the \( y \)-axis and similarly with the face \([GHE]\), the edge \( GH \) is on the \( x \)-axis and the edge \( HE \) is on the \( y \)-axis, since the cube is located under the given faces, so the coordinates on the \( z \)-axis increase downwards.

**Definition 3.8.** Let \( Q[BAD] \) denote the Latin square derived from the 3-LSC \( L \) by projection to the bottom face \([BAD]\).

**Theorem 3.9.** If \( L \) is a 3-LSC with \( n^2 \) non-attacking rooks and \( Q \) is the Latin square derived from \( L \) by projection to an arbitrary face of the cube then the LSCs obtained from \( L \) by Latin transformations are all and only those LSCs, that can be derived from a paratope of \( Q \) by composition.

**Proof.** Let \( L \) be a 3-LSC in a right-handed coordinate system with vertex \( A \) at the origin, \( B \) on the \( x \)-axis, \( D \) on the \( y \)-axis and \( F \) on the \( z \)-axis as indicated in Figure 2.1. Let \( Q \) be the Latin square on the face \( BAD \) derived from \( L \) by projection, hence \( Q[BAD] = Q = Q(i, j, k) \), according to our notations.

Derive \( Q[DAF] \) from \( L \) by projection onto the face \([DAF]\) in the right-handed coordinate system \((y, z, x)\), the rook in the cell, originally identified by 3-tuple, \((i, j, k)\) results the symbol \( i \)
in the cell \((j, k)\), so \(Q[DAF] = Q(j, k, i)\). In the same way, in the right-handed coordinate system \((z, x, y)\), the rook in the cell \((i, j, k)\) projected onto the face \([FAB]\) results the symbol \(j\) in the cell \((k, i)\), thus \(Q[FAB] = Q(k, i, j)\). So, if we take the Latin squares derived from \(L\) by projection onto the 3 faces that coincide at the origin of the cube, we get 3 conjugate Latin squares \(Q = Q(i, j, k), Q(j, k, i)\) and \(Q(k, i, j)\).

**Definition 3.10.** The Latin squares \(Q(i, j, k), Q(j, k, i)\) and \(Q(k, i, j)\), where \(Q(i, j, k) = Q\) are called primary conjugates of \(Q\).

Permute the layers on each axis of \(L\) in reverse order. The resulting LSC and \(L\) are isotopic and the cell \((i, j, k)\) is moved to cell \((n + 1 - i, n + 1 - j, n + 1 - k)\). Place a right-handed coordinate system with the origin at vertex \(H\) such that the \(z\)-axis is aligned with the edge \(HC\) of the cube. Then the edge \(HG\) is on the \(x\)-axis and the edge \(HE\) is on the \(y\)-axis. So, the new \(z\)-axis is parallel to the old one, however the new \(x\)-axis is parallel to the old \(y\)-axis and the new \(y\)-axis is parallel to the old \(x\)-axis, according to Remark 3.7. Consequently, the new coordinates of the cell \((n + 1 - i, n + 1 - j, n + 1 - k)\) are \((j, i, k)\), so, the projection to the face \(GHE\) gives the Latin square \(Q(j, i, k)\), the projection to the face \(EHC\) gives the Latin square \(Q(i, k, j)\), projection to the face \(CHG\) gives the Latin square \(Q(k, i, j)\).

**Definition 3.11.** The Latin squares \(Q(j, i, k), Q(i, k, j)\) and \(Q(k, j, i)\) are called secondary conjugates of \(Q\).

The primary conjugates of \(Q\) can be derived from \(L\) by projection, the secondary conjugates of \(Q\) can be derived from a specific isotope of \(L\) by projection. Consequently, based on Remark 3.4 each element of the main class of \(Q\) can be derived from the proper isotope of \(L\).

Now we prove that a Latin square \(Q^*\) derived from an LSC \(L^*\) produced from \(L\) by Latin transformations is an element of the main class of \(Q\).

All 6 faces of the cube can be moved to the bottom face and all four vertices of the bottom face can be rotated to the origin without changing the bottom face. Rotating the bottom face at -90 degrees is called face rotation. If the face \(ABCD\) in our example in Figure 2.1 is rotated by -90 degrees, vertex \(B\) is moved to the origin and the face \(ABCD\) remains the bottom face. It is easy to see, that from each position of the cube all 24 positions can be achieved with a sequence of face rotations. An equivalence class is transitive, so, it is enough to prove that a Latin square \(Q^*\) derived from an LSC \(L^*\) produced from \(L\) by a single face rotation is an element of the main class of \(Q\).

Let \(f\) be a face rotation. Let \(L^*\) be the resulting structure and \(Q^*\) the Latin square derived from \(L^*\) by projection to the bottom face. Then \(Q^*\) is in the main class of \(Q\). The transformation \(f\) moves the row layer \(i\) of \(L\) into the column layer \((n + 1 - i)\), the column layer \(j\) into the row layer \(j\), and the coordinates of the symbol layers remain unchanged. The order of the column layers does not change, but the layers are moved to the other side of the origin, so their coordinates change from \(1, 2, \ldots, n\) to \(n, n - 1, \ldots, 1\), therefore the new column coordinate is \((n + 1 - i)\).

So, the rows of the Latin square \(Q^*\) derived from the resulting structure \(L^*\) by projection to the bottom face contain the columns of \(Q\) in the same order, the columns contain the rows of \(Q\) in reverse order, the order of symbol layers unchanged. Denote the derived Latin square by
$Q(j, n+1-i, k)$. In $Q(j, n+1-i, k)$, permuting the the columns in reverse order yields the Latin square $\overline{Q}(j, i, k)$, so $Q(j, n+1-i, k)$ is a paratope of $Q$. \hfill \Box

**Remark 3.12.** Permuting the layers of $H^3$ perpendicular to a given axis in reverse order corresponds to a plane reflection in the Euclidean sense. Permuting the layers on each axis of $H^3$ in reverse order corresponds to central symmetry.

**Remark 3.13.** Secondary conjugates do not give us additional information, so from now on we only deal with primary conjugates.

## 4 Hamming Bricks

Let $X$ a subsystem induced by a family of sets $<E_1, E_2, \ldots, E_d>$ over $\mathbb{Z}_n$. Then $X$ contains all cells identified by the $d$-tuples $(e_1, e_2, \ldots, e_d)$ for which $e_i \in E_i$ for each $i \in \mathbb{Z}_d$.

**Definition 4.1.** The subsystem $X$ is visible if $\max\{E_i\} - \min\{E_i\} = |E_i| - 1$ for each $i \in \mathbb{Z}_d$. If $X$ is visible, it is called a *Hamming brick* or simply a *brick*.

For each set $E_i$, there exists a permutation $p_i$ which places the different elements of the set $E_i$ as coordinates on the axis $t_i$ into the set of coordinates $1, 2, \ldots, |E_i|$, leaving the coordinates on the other axes unchanged. This permutation only changes the order of the $(d-1)$-dimensional subspaces $H_i(j)$, where $j \in \mathbb{Z}_n$. After executing all the permutations $p_1, p_2, \ldots, p_d$, the subsystem $X$ is a brick permuted to the origin. We generally examine the subsystems in this form.

**Remark 4.2.** A brick permuted to the origin of the chess-board is a solid rectangular cuboid of size $|E_1| \times |E_2| \times \ldots \times |E_d|$ in the Euclidean geometry.

**Definition 4.3.** Let $X$ be a set of cells and $c$ be a cell of $H_n^d$. We define the Hamming distance between $X$ and $c$ as follows:

$$d(X, c) = \min\{d(x, c) \mid x \in X\},$$

where $d(x, c)$ is the Hamming distance between the cells $x$ and $c$.

Clearly $d(X, c) > 0$ if and only if $c \notin X$.

**Definition 4.4.** Let $T \subseteq H_n^d$ be a brick. The Hamming sphere of radius $r$, center $T$ is a set of cells for which

$$S_r(T) = \{c \in H_n^d \mid d(T, c) = r\}.$$

Obviously $S_0(T) = T$.

So, we generalized the cell-centered Hamming sphere to a brick-centered Hamming sphere. The Figure 4.1 shows a Hamming sphere around the cubelet $K$ with $r = 1$, a Hamming sphere around the brick $T$ with $r = 1$.

The Figure 4.2 shows a Hamming sphere around the brick $T$ with $r = 2$ and a Hamming sphere around the brick $T$ with $r = 3$. 

6
Definition 4.5. Let $T, U \subseteq H^d_n$ be two bricks. The Hamming distance between these two bricks is:
\[
d(T, U) = \min\{ d(x, y) \mid x \in T, y \in U \}
\]
It is evident that $d(T, U) = 0$ if and only if $T \cap U \neq \emptyset$.

For $d = 3$, \( \begin{pmatrix} 3 \\ 0 \end{pmatrix} \) brick has distance 0 from brick $T$ (itself), \( \begin{pmatrix} 3 \\ 1 \end{pmatrix} \) bricks have distance 1 from $T$ and \( \begin{pmatrix} 3 \\ 2 \end{pmatrix} \) bricks have distance 2 from $T$ and \( \begin{pmatrix} 3 \\ 3 \end{pmatrix} \) brick has distance 3 from $T$.

Definition 4.6. A brick of size $e_1 \times e_2 \times \ldots \times e_d$ is a real brick, if $1 \leq e_1, e_2, \ldots, e_d < n$.

Let $T_0$ be a brick of size $e_1 \times e_2 \times \ldots \times e_d$, where $1 \leq e_1, e_2, \ldots, e_k < n$, $e_{k+1} = \ldots = e_d = n$. In this case $k$ is the maximum distance between the brick $T_0$ and a cell, so the maximum distance between $T_0$ and another brick.
Definition 4.13. Let $I$ be a set of indices, where $I \subseteq \{1, 2, \ldots, k\}$ and $r = |I|$. Let us define the set $T_I$ as follows

$$T_I = \{ x = (x_1, x_2, \ldots, x_d) \in H_n^d \mid n \geq x_i > e_i \text{ for } i \in I \text{ and } 1 \leq x_i \leq e_i \text{ for } i \notin I \}. $$

Obviously, $T_I$ is a brick as well and $d(T_0, T_I) = r$.

Consequently, we have $\binom{k}{r}$ subsets of cardinality $r$, so we have $\binom{k}{r}$ distinct disjoint bricks that have distance $r$ from $T_0$. Using the Hamming distance from $T_0$, $T_0$ generates a partition of $H_n^d$ into $2^r$ disjoint bricks based on the index set $I$. If $T_0$ is a real brick, then $k = d$, and each Hamming sphere of radius $r$, with center $T_0$ consists of $\binom{d}{r}$ disjoint bricks for $r \in \mathbb{Z}_d$ and $r = 0$.

Remark 4.7. A brick $T_k$ at a distance $k$ from a real brick $T_0$ has exactly $k$ edges that are obtained by taking the edge $(n-e_i)$ instead of the edge $e_i$ of $T_0$.

Remark 4.8. Let $T$ be an arbitrary brick in the partition generated by $T_0$. Then $T$ generates the same bricks as $T_0$, so we get the the same partition of the space $H_n^d$.

Remark 4.9. Let $T_0$ be a real brick of size $e_1 \times e_2 \times \ldots \times e_d$ permuted to the origin. If we look at the partition generated by $T_0$ in Euclidean geometry, then each brick has a vertex that coincides with a vertex of the chess-board, different bricks fit on different chess-board vertices and the opposite (the only inner) vertex of each brick is the Euclidean point with coordinates $(e_1, e_2, \ldots, e_d)$.

Definition 4.10. Let $T, U \subseteq H_n^d$ be two bricks. The bricks $T$ and $U$ are remote if $d(T, U) = d$. The pair of $T$ and $U$ is called a remote couple, if $U$ contains all the cells that have a distance $d$ from $T$. If $T$ and $U$ form a remote couple, then $T$ is called the remote mate of $U$ and $U$ is called the remote mate of $T$.

If $T$ and $U$ are remote, then $U$ and $T$ are remote as well. If $T$ and $U$ are remote, then $T \cap U = \emptyset$.

Definition 4.11. Let $X$ be a set of cells. We define $V(X)$, the volume of $X$ as the number of cells in $X$.

If $T \subseteq H_n^d$ is a brick of size $e_1 \times e_2 \times \ldots \times e_d$, then the volume of $T$ is $V(T) = \prod_{i=1}^{d} e_i$.

Definition 4.12. Let $T_0 \subseteq H_n^d$ be a brick of size $e_1 \times e_2 \times \ldots \times e_d$. The area of $T_0$ orthogonal to $e_i$ or $t_i$ is

$$A(T_0, e_i) = A(T_0, t_i) = V(T_0)/e_i = e_1 e_2 \ldots e_{i-1} e_{i+1} \ldots e_{d-1} e_d.$$

Definition 4.13. Let $T_0 \subseteq H_n^d$ be a brick of size $e_1 \times e_2 \times \ldots \times e_d$. The brick $T_i$, in the partition generated by $T_0$, is the auxiliary brick of $T_0$ along $t_i$, if $T_i$ contains all of the cells $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$ for which $x_k \leq e_k$ if $k \neq i$ and $x_k > e_k$ if $k = i$. $T_0$ and $T_i$ together are called auxiliaries along $t_i$. 

8
Clearly, \( d(T_0, T_i) = 1 \) and \( T_0 \) has \( \left( \frac{d}{1} \right) = d \) auxiliary bricks. Figure 4.1 shows the case \( d = 3 \).

**Remark 4.14.** If the bricks \( U_1 \) and \( U_2 \) are auxiliaries along an axis \( t \), then \( U_1 \) and \( U_2 \) are disjoint, so \( V(U_1 \cup U_2) = V(U_1) + V(U_2) \) and \( U_1 \cup U_2 \) is a brick with an edge of length \( n \) on the axis \( t \).

**Definition 4.15.** A brick \( T \) is an \( n \)-brick, if \( T \) has at least one edge of length \( n \). A brick \( T \) is an \( n^2 \)-brick, if \( T \) has at least two edges of length \( n \).

**Remark 4.16.** A subspace is always an \( n \)-brick.

Let \( L \) be a \( d \)-LSC and \( T \subseteq H_n^d \) be an \( n \)-brick of size \( e_1 \times e_2 \times \ldots \times e_d \), where \( e_i = n \). It is clear from the construction, that each file has exactly one rook. Therefore, the number of rooks in \( T \) is the size of the area of \( T \) orthogonal to \( e_i \), i.e, \( V(T)/n \). Based on this observation the following results:

Let \( L \) be a \( d \)-LSC and \( T_0 \) be a brick of size \( e_1 \times e_2 \times \ldots \times e_k \times e_{k+1} \times \ldots \times e_d \), which has \( c_0 \) rooks. Let \( T_k \) be a brick with distance \( k \) from \( T_0 \) in the partition generated by \( T_0 \) and contain \( c_k \) rooks. Based on Remark 4.7 and without loss of generality, we assume that \( T_k \) is a brick of size

\[
(n-e_1) \times (n-e_2) \times \ldots \times (n-e_k) \times e_{k+1} \times \ldots \times e_d,
\]

otherwise, we change the order of the axes.

**Theorem 4.17** (Distribution Theorem).

\[
c_k = \frac{V(T_k) - (-1)^k V(T_0)}{n} + (-1)^k c_0 = \frac{V(T_k)}{n} - (-1)^k \left[ \frac{V(T_0)}{n} - c_0 \right] \tag{4.1}
\]

**Proof.** Let \( T_1 \) be the auxiliary brick of \( T_0 \) along \( t_1 \) and let \( T_1 \) contain \( c_1 \) rooks. Since \( T_0 \cup T_1 \) is an \( n \)-brick, so \( T_0 \cup T_1 \) has

\[
\frac{V(T_0 \cup T_1)}{n} = \frac{V(T_1)}{n} + \frac{V(T_0)}{n}
\]

rooks, so \( c_0 + c_1 = \frac{V(T_1)}{n} + \frac{V(T_0)}{n} \), and

\[
c_1 = \frac{V(T_1)}{n} + \frac{V(T_0)}{n} - c_0.
\]

Let the brick \( T_2 \) be the auxiliary brick of \( T_1 \) along \( t_2 \) and let \( T_2 \) contain \( c_2 \) rooks. So \( T_1 \cup T_2 \) is an \( n \)-brick, so \( T_1 \cup T_2 \) has

\[
\frac{V(T_2 \cup T_1)}{n} = \frac{V(T_2)}{n} + \frac{V(T_1)}{n}
\]

rooks, so \( c_1 + c_2 = \frac{V(T_2)}{n} + \frac{V(T_1)}{n} \), so

\[
c_2 = \frac{V(T_2)}{n} + \frac{V(T_1)}{n} - c_1
\]

\[
= \frac{V(T_2)}{n} + \frac{V(T_1)}{n} - \left( \frac{V(T_1)}{n} + \frac{V(T_0)}{n} - c_0 \right) = \frac{V(T_2)}{n} - V(T_0) + c_0.
\]

Let the brick \( T_3 \) be the auxiliary brick of \( T_2 \) along \( t_3 \) and let \( T_3 \) contain \( c_3 \) rooks. So \( T_2 \cup T_3 \) is an \( n \)-brick, so \( T_2 \cup T_3 \) has

\[
\frac{V(T_3 \cup T_2)}{n} = \frac{V(T_3)}{n} + \frac{V(T_2)}{n}
\]

rooks, so \( c_2 + c_3 = \frac{V(T_3)}{n} + \frac{V(T_2)}{n} \), so

\[
c_3 = \frac{V(T_3)}{n} + \frac{V(T_2)}{n} - c_2
\]

\[
= \frac{V(T_3)}{n} + \frac{V(T_2)}{n} - \left( \frac{V(T_2)}{n} - V(T_0) + c_0 \right) = \frac{V(T_3)}{n} + V(T_0) - c_0.
\]
When we take $T_{i+1}$, the auxiliary brick of $T_i$, and calculate $c_{i+1}$, then $V(T_i)$ falls out, the sign of $V(T_0)$ and $c_0$ change, in the next step $V(T_{i-1})$ falls out, the sign of $V(T_0)$ and $c_0$ change, etc.

\[ c_{i+1} = \frac{V(T_{i+1}) + V(T_i)}{n} - c_i = \frac{V(T_{i+1}) + V(T_i)}{n} - \left( \frac{V(T_i) + V(T_{i-1})}{n} - c_{i-1} \right) = \ldots \]

\[ = \frac{V(T_{i+1}) - (-1)^{i+1}V(T_0)}{n} + (-1)^{i+1}c_0. \]

So (4.1) holds, and we can write it in the following form:

\[ (-1)^k \left[ \frac{V(T_0)}{n} - c_0 \right] = \frac{V(T_k)}{n} - c_k. \]

(4.2)

So, for a given $k$, $c_k$ depends only on $c_0$ and the volume of $T_0$ and $T_k$ and the parity of $k$.

**Definition 4.18.** The *density* of a set of cells $X$ is $\rho(X) = c/V(X)$, where $c$ is the number of rooks in $X$.

In case of a $d$-LSC the density of the entire chess-board is: $\rho(H_d^n) = n^{d-1}/n^d = 1/n$.

**Definition 4.19.** The set of cells $X$ has **standard density** if $\rho(X) = 1/n$.

**Definition 4.20.** We define the *deflection* of the set of cells $X$ from the standard density as follows

\[ \text{df}(X) = V(X)/n - c \]

where $c$ is the number of rooks in $X$.

The set of cells $X$ has standard density if and only if $\text{df}(X) = 0$. Using the deflection, the equality in (4.1) can be written in the following two ways:

**Theorem 4.21** (Deflection Theorem).

\[ c_k = \frac{V(T_k)}{n} - (-1)^k \text{df}(T_0). \]

(4.3)

**Theorem 4.22** (Main Theorem).

\[ \text{df}(T_k) = (-1)^k \text{df}(T_0). \]

(4.4)

If $T_0$ has standard density, then $\text{df}(T_k) = 0$, ergo, each Hamming brick also has standard density. If $\text{df}(T_0) \neq 0$, then each brick in the Hamming sphere $S_k(T_0)$ has the same deflection, either $\text{df}(T_0)$ if $k$ is even or $-\text{df}(T_0)$ if $k$ is odd. Ergo, the sign of the deflection of the bricks alternates when we step to the bricks of the next Hamming sphere $S_{k+1}(T_0)$. Note, that the deflection is not necessarily integer.

If $T_0$ is not a real brick, then $T_0$ has at least one edge of length $n$, so $\text{df}(T_0) = 0$, thus, $T_0$ has a standard density. Consequently, in this case, each brick of the partition has a standard density.

**Corollary 4.23.** If $n$ is a prime number, then there is no real Hamming brick of standard density.
Case $d = 2$:

The brick $T_0$, colored yellow in the Figure 4.3, is a real brick of size $a \times b$. The bricks $T_{1a}$ and $T_{1b}$ are the two auxiliaries of $T_0$ and the brick $T_2$ is the remote mate of $T_0$. Each file has exactly one rook and let $T_0$ have $c_0$ rooks. Then $T_{1a}$ has

$$\frac{V(T_0) + V(T_{1a})}{n} - c_0 = b - c_0$$

rooks and so $T_2$ has

$$\frac{V(T_{1a}) + V(T_2)}{n} - \left(\frac{V(T_0) + V(T_{1a})}{n} - c_0\right) = \frac{V(T_2) - V(T_0)}{n} + c_0$$

rooks. So

$$c_2 = \frac{V(T_2) - V(T_0)}{n} + c_0 = (n - a - b) + c_0.$$  

Another form of this equality is

$$c_0 - c_2 = a + b - n.$$

![Figure 4.3](image)

**Definition 4.24.** If $T_0$ is a real brick of size $a \times b$, then the number $(a + b - n)$ is called the Ryser-number of $T_0$ and is denoted by $\text{Ry}(T_0)$.

The Ryser-number of $T_0$ is the difference between the numbers of rooks in $T_0$ and $T_2$. If $c_0$ is known, then we know exactly how many rooks are in $T_2$, that is $c_2 = c_0 - \text{Ry}(T_0)$.

It is clear, that $\text{Ry}(T_2) = -\text{Ry}(T_0) = (n - a - b)$ and $c_2 = c_0 + \text{Ry}(T_2)$.

**Corollary 4.25.** If $a + b = n$, then the white bricks $T_{1a}$ and $T_{1b}$ are squares and $\text{Ry}(T_0) = 0$, so $T_0$ and $T_2$ have the same number of rooks.
Case $d = 3$:
The brick $T_0$, colored yellow in the Figure 4.4, is a real brick of size $a \times b \times c$. The bricks $T_{1a}$, $T_{1b}$ and $T_{1c}$ are the auxiliaries of $T_0$, the brick $T_3$ is the remote mate of $T_0$ and the bricks $T_{2ab}$, $T_{2bc}$ and $T_{2ac}$ are the auxiliaries of $T_3$.

Each file has exactly 1 rook and let $T_0$ have $c_0$ rooks. Then $T_0 \cup T_{1a}$ has $\frac{V(T_0) + V(T_{1a})}{n} = bc$ rooks, so $T_{1a}$ has

$$
\frac{V(T_0) + V(T_{1a})}{n} - c_0 = bc - c_0
$$

rooks.

![Figure 4.4](image)

$T_{1a} \cup T_{2ab}$ has $\frac{V(T_{1a}) + V(T_{2ab})}{n} = (n-a)c$ rooks, hence $T_{2ab}$ has

$$
\frac{V(T_{1a}) + V(T_{2ab})}{n} - \left( \frac{V(T_0) + V(T_{1a})}{n} - c_0 \right) = \frac{V(T_{2ab}) - V(T_0)}{n} + c_0
$$

$T_{2ab} \cup T_3$ has $\frac{V(T_{2ab}) + V(T_3)}{n} = (n-a)(n-b)$ rooks, so $T_3$ has

$$
\frac{V(T_{2ab}) + V(T_3)}{n} - \left( \frac{V(T_{2ab}) - V(T_0)}{n} + c_0 \right) = \frac{V(T_3) + V(T_0)}{n} - c_0.
$$

As a result,

$$
c_0 + c_3 = \frac{V(T_0) + V(T_3)}{n} = \frac{abc + (n-a)(n-b)(n-c)}{n} = n^2 - (a+b+c)n + (ab+bc+ca).
$$
If $c = 0$ or $c = n$, then we also get the right numbers.

**Remark 4.26.** Let $M$ be a $d$-fold stochastic matrix of dimension $d$. Then the sum of the numbers in each file is exactly $1$. Let $c_0$ be the sum of the numbers in $T_0$ and $c_k$ be the sum of the numbers in $T_k$. Then the Distribution Theorem 4.17 holds for $d$-fold stochastic matrix, you only have to change the expression from “number of rooks” to “sum of the numbers” in the proof.

**Corollary 4.27.** The sum of the numbers in $T_k$ depends only on the sum of the numbers in $T_0$ and the volume of $T_0$ and $T_k$ and the parity of $k$.

**Remark 4.28.** The Distribution Theorem 4.17 holds for any types of numbers of $d$-fold stochastic matrices too, if the sum of the numbers is exactly $1$ in each file, but we are only dealing with the non-negative case.

**Remark 4.29.** For $d = 3$ we get $c_0 + c_3 = n^2 - (a + b + c)n + (ab + bc + ca)$, which is a result of Cruse [2] for triply stochastic matrices including permutation cubes, which are characteristic matrices of Latin squares. Consequently, for triply stochastic matrices $c_0 + c_3$ always is an integer.

**Remark 4.30.** The $d = 2$ we get $c_2 = (n - a - b) + c_0$, thus $c_0 - c_2 = a + b - n$, consequently, for doubly stochastic matrices $c_2 - c_0$ always is an integer.

**Definition 4.31.** A file of brick $T$ means the cells of a file, which are in $T$.

**Definition 4.32.** A layer of brick $T$ means the cells of a layer, which are in $T$.

**Definition 4.33.** The brick $T$ is large, if $T$ has a layer of size $a \times b$, for which $a + b > n$.

**Corollary 4.34.** Let $T$ be a large brick in a $d$-fold stochastic matrix or in a $d$-LSC. Then $T$ has no layer, for which the sum of numbers or the number of rooks is 0, respectively.

**Corollary 4.35.** If $T$ is a brick of size $a \times b \times c$ in a $3$-LSC and $a + b > n$, $b + c > n$, $a + c > n$, then each layer of the brick $T$ has at least one rook.

## 5 Taxicab Geometry

In the taxicab geometry the distance between two points is the sum of the absolute differences of their Cartesian coordinates or simply $d(O, P) = r + c$. $d(O, P)$ is the shortest path from $O$ to $P$. There can be more than one shortest path. If you start from $O$ and always drive right or down (never left or up), you get a shortest path from $O$ to $P$ when you arrive at $P$ as indicated in the Figure 5.1.

**Definition 5.1.** The diameter of the rectangular $T$, $diam(T) = r + c$.

The diameter is the distance between the two most remote points of $T$.

Let $T \subseteq H^d$ be a brick of size $e_1 \times e_2 \times \ldots \times e_d$. Based on the case of dimension 2 we generally define the diameter of a Hamming brick as follows:

**Definition 5.2.** The diameter of the brick $T$ is $diam(T) = e_1 + e_2 + \ldots + e_d$. 

13
6 Sum of Hamming Distances

First, we prove an identity for binomial coefficients.

**Theorem 6.1.**

\[
\sum_{k=0}^{d} k (-1)^k \binom{d}{k} = 0
\]

for any \(d \geq 2\).

**Proof.** We use the two well-known identities

\[
\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}
\]

(6.1a)

\[
\sum_{k=0}^{d} (-1)^k \binom{d}{k} = 0.
\]

(6.1b)

We prove the theorem by induction on \(d\). The assertion of the theorem is true if \(d = 2, 3, 4\).

\[
0 \cdot 1 - 1 \cdot 2 + 2 \cdot 1 = 0
\]

\[
0 \cdot 1 - 1 \cdot 3 + 2 \cdot 3 - 3 \cdot 1 = 0
\]

\[
0 \cdot 1 - 1 \cdot 4 + 2 \cdot 6 - 3 \cdot 4 + 4 \cdot 1 = 0
\]

We suppose that \(d \geq 2\) and that the induction hypothesis holds for \(d\), i.e,

\[
A_d = \sum_{k=0}^{d} k (-1)^k \binom{d}{k} = 0,
\]

then we prove that

\[
A_{d+1} = \sum_{k=0}^{d+1} k (-1)^k \binom{d+1}{k}.
\]
Using (6.1a)

\[ A_{d+1} = \sum_{k=0}^{d+1} (-1)^k C^d_k + \sum_{k=0}^{d+1} k(-1)^k C^d_{k-1} \]

\[ A_{d+1} = \sum_{k=0}^{d} (-1)^k \binom{d}{k} + (d+1)(-1)^{d+1} \binom{d}{d+1} + 0 \cdot (-1)^0 \binom{d}{1} + \sum_{k=1}^{d+1} k(-1)^k \binom{d}{k-1}. \]

Since \( \binom{d}{d+1} = \binom{d}{-1} = 0 \) we get

\[ A_{d+1} = \sum_{k=0}^{d} (-1)^k \binom{d}{k} + \sum_{k=1}^{d+1} k(-1)^k \binom{d}{k-1}. \]

\[ A_d = 0 \] by the assertion, so

\[ A_{d+1} = \sum_{k=1}^{d+1} k(-1)^k \binom{d}{k-1}. \]

Replacing \( k \) with \( (j+1) \):

\[ A_{d+1} = \sum_{j=0}^{d} (j+1)(-1)^{j+1} \binom{d}{j} = (-1) \sum_{j=0}^{d} (j+1)(-1)^j \binom{d}{j} \]

\[ = (-1) \left[ \sum_{j=0}^{d} j(-1)^j \binom{d}{j} + \sum_{j=0}^{d} 1 \cdot (-1)^j \binom{d}{j} \right] \]

\[ = (-1) \left[ A_d + \sum_{j=0}^{d} 1 \cdot (-1)^j \binom{d}{j} \right] = (-1) \left[ \sum_{j=0}^{d} (-1)^j \binom{d}{j} \right] = 0 \]

because of (6.1b).

So, the statement holds for \( (d+1) \), that means it holds for every \( d \geq 2 \).

Let \( L \) be a \( d \)-LSC and \( T_0 \) be a real brick. Each rook has a Hamming distance from \( T_0 \). Let us denote \( s_k \) the number of rooks in the Hamming sphere \( S_k(T_0) \), and \( V(S_k(T_0)) \) is the sum of the volumes of all the bricks of the Hamming sphere \( S_k(T_0) \).

We sum up all Hamming distances from \( T_0 \) for all rooks. The sum is denoted by \( h_d^d(T_0) \).
Theorem 6.2
We prove, that
Proof. Because of Theorem 6.1
rooks with distance 2. The sum is:
by induction on \( d \)
so
Let us assume that \( d \)

\[ T \]

Hamming brick in the partition generated by \( S \)

\[ k \]

\[ h^d_n(T_0) = \sum_{k=0}^{d} nhs_k = \sum_{k=0}^{d} kV(S_k(T_0)) \]

(6.2)
Theorem 6.2 (Distance Theorem).

\[ h^d_n(T_0) = n^{d-2} [(n-e_1) + (n-e_2) + \ldots + (n-e_d)] \]

(6.3)
Proof. We prove, that

\[ nh^d_n(T_0) = \sum_{k=0}^{d} kV(S_k(T_0)) = n^{d-1} [(n-e_1) + (n-e_2) + \ldots + (n-e_d)] \]

(6.4)
by induction on \( d \).

In the case \( d = 2 \), there are \( c_0 \) rooks with distance 0, \( c_{1a} + c_{1b} \) rooks with distance 1 and \( c_2 \) rooks with distance 2. The sum is:

\[ h^2_n(T_0) = 0 \cdot c_0 + 1 \cdot (c_{1a} + c_{1b}) + 2 \cdot c_2 \]

\[ = (b - c_0) + (a - c_0) + 2(c_0 - a - b + n) = 2n - a - b = (n-a) + (n-b), \]

so

\[ nh^2_n(T_0) = n[(n-a) + (n-b)] \]

Let us assume that \( d > 2 \) and (6.4) holds.

\[ \sum_{k=0}^{d+1} kV(S_k^{d+1}(T_0)) = n^{d+1} [(n-e_1) + (n-e_2) + \ldots + (n-e_d) + (n-e_{d+1})] \]

Because of (6.2)

\[ nh^{d+1}_n(T_0) = \sum_{k=0}^{d+1} kV(S_k^{d+1}(T_0)) \]

\[ T_0^{d+1} \] be a real brick in \( H^{d+1}_n \). The volume of \( T_0^{d+1} \) is \( V(T_0^{d+1}) = \prod_{i=1}^{d+1} e_i \). Let \( T_k^{d+1} \) be a Hamming brick in the partition generated by \( T_0^{d+1} \), and \( d(T_0^{d+1}, T_k^{d+1}) = k \), then there exist an
index set \( I = i_1, i_2, \ldots, i_k \) such that we changed \( e_i \) of \( T_0^{d+1} \) to \( n - e_i \) in \( T_0^{d+1} \) for \( i \in I \) and we kept \( e_j \) of \( T_0^{d+1} \) in \( T_k^{d+1} \) for \( j \notin I \). There are exactly \( \binom{d+1}{k} \) index set with this property and each of them defines a \( T_k^{d+1} \) brick.

The volume of \( T_0^{d+1} \) is \( V(T_0^{d+1}) = \prod_{i=1}^{d+1} e_i \) and the volume of \( T_k^{d+1} \) is

\[
V(T_k^{d+1}) = \prod_{i \in I} (n - e_i) \cdot \prod_{j \notin I} e_j
\]

There are two types of bricks that have the distance \( k \) from \( T_0 \). The first type has an edge \( e_{d+1} \) on the axis \( t_{d+1} \), the other has an edge \( (n - e_{d+1}) \) on the axis \( t_{d+1} \). Take all bricks from the first type.

In this case we changed \( k \) edges from the edges of \( T_0^d \). The sum of the volume of these bricks is \( e_{d+1} V(S_k(T_0^d)) \). In the other case we changed \( k - 1 \) edges from the edges of \( T_0^d \) and the \( e_{d+1} \) to \( (n - e_{d+1}) \).

The sum of the volume of these bricks is \( (n - e_{d+1}) V(S_{k-1}(T_0^d)) \). So

\[
V(S_k^d(T_0^{d+1})) = e_{d+1} V(S_k^d(T_0^d)) + (n - e_{d+1}) V(S_{k-1}(T_0^d))
\]

\[
= e_{d+1} \sum_{k=0}^{d+1} k V(S_k^d(T_0^d)) + (n - e_{d+1}) \sum_{k=0}^{d+1} k V(S_{k-1}(T_0^d)).
\]

Because of \((d + 1) V(S_{d+1}^d(T_0^d)) = 0 \text{ és } 0 \cdot V(S_{d-1}^d(T_0^d)) = 0\) we get

\[
\begin{align*}
nh_n^{d+1}(T_0) &= \sum_{k=0}^{d+1} k V(S_k^d(T_0^{d+1})) = e_{d+1} \sum_{k=1}^{d+1} k V(S_k^d(T_0^d)) + (n - e_{d+1}) \sum_{k=1}^{d+1} k V(S_{k-1}^d(T_0^d)) \\
&= e_{d+1} nh_n^d(T_0) + (n - e_{d+1}) \left[ \sum_{k=0}^{d} k V(S_k^d(T_0^d)) + \sum_{k=0}^{d} k V(S_{k}^d(T_0^d)) \right] \\
&= e_{d+1} nh_n^d(T_0) + (n - e_{d+1}) \left[ n^d + nh_n^d(T_0) \right] \\
&= e_{d+1} nh_n^d(T_0) + n \left[ n^d + nh_n^d(T_0) \right] - e_{d+1} \left[ n^d + nh_n^d(T_0) \right] \\
&= e_{d+1} nh_n^d(T_0) + n n^d + nh_n^d(T_0) - e_{d+1} n^d - e_{d+1} nh_n^d(T_0) \\
&= (n - e_{d+1}) n^d + n^2 nh_n^d(T_0).
\end{align*}
\]

Based on the assumption of the induction we change \( h_n^d(T_0) \) for the right part of the equality (6.3)

\[
\begin{align*}
nh_n^{d+1}(T_0) &= (n - e_{d+1}) n^d + n^2 n^d - 2 [(n - e_1) + (n - e_2) + \ldots + (n - e_d)] \\
&= n^d [(n - e_1) + (n - e_2) + \ldots + (n - e_d) + (n - e_{d+1})]
\end{align*}
\]

17
\[ h_n^{d+1}(T_0) = n^{d-1} [(n-e_1) + (n-e_2) + \ldots + (n-e_d)] = n^{d-1} \text{diam}(T_0) \quad (6.5) \]

So (6.3) holds for any \( d > 1 \).

If \( T_0 \) is not a real brick, then \( T_0 \) is an \( n \)-brick. Since \( \text{df}(T_0) = 0 \), so (6.3) holds. Let’s assume, that \( T_0 \) has \( m \) edges, \( e_1, e_2, \ldots, e_m < n \). Every \( T_k \) brick in the partition generated by \( T_0 \) has \( d - m \) edges of length \( n \). So for \( T_k \) of dimension \( d \) the \( V(T_k^d) = n^{d-m}V(T_k^m) \), where \( T_k^m \) means a brick of dimension \( m \) and edges \( e_1, e_2, \ldots, e_m \). In this case (6.2) holds for \( T_0^m \), so

\[ h_n^m(T_0^m) = n^{m-2} [(n-e_1) + (n-e_2) + \ldots + (n-e_m)] \]

The left-hand side is \( h_n^m(T_0) \), therefore, (6.2) holds for not real bricks too. \( \square \)

**Corollary 6.3.** If we take the sum of the Hamming distances of all rooks of a \( d \)-LSC from a brick \( T_0 \), then the result does not depend on the number of rooks in \( T_0 \).

**Corollary 6.4.** If \( T_0 \) is a real brick, then \( T_d \) is the only brick that has a distance \( d \) from \( T_0 \), so because of (6.3)

\[ h_n^d(T_0) = n^{d-2} [(n-e_1) + (n-e_2) + \ldots + (n-e_d)] = n^{d-2} \text{diam}(T_d) \]
\[ h_n^d(T_d) = n^{d-2} [e_1 + e_2 + \ldots + e_d] = n^{d-2} \text{diam}(T_0) \quad (6.6) \]

### 7 Remote Brick Couples (RBCs)

From now on we are dealing with the cases \( d = 2 \) and \( d = 3 \). Based on the Distribution Theorem 4.17

\[ c_0 - c_2 = \frac{V(T_2) - V(T_0)}{n} = \frac{ab - (n-a)(n-b)}{n} = a + b - n \quad \text{and} \]
\[ c_0 + c_3 = \frac{V(T_3) + V(T_0)}{n} = \frac{abc + (n-a)(n-b)(n-c)}{n} = n^2 - (a + b + c)n + (ab + bc + ca) \]

**Definition 7.1.** For \( d = 2 \) the RBC \((T_0, T_2)\) is called balanced, if \( c_0 - c_2 = a + b - n \) holds.

**Definition 7.2.** For \( d = 3 \) let the capacity of RBC \((T_0, T_3)\) be

\[ \text{cap}(T_0, T_3) = n^2 - (a + b + c)n + (ab + bc + ca) \]

**Definition 7.3.** For \( d = 3 \) the RBC \((T_0, T_3)\) is called stuffed, if \( c_0 + c_3 = \text{cap}(T_0, T_3) \).

**Corollary 7.4.** For a 2-LSC \( L \) holds, that each RBC \((T_0, T_2)\) of \( L \) is balanced. For a 3-LSC \( M \) holds, that each RBC \((T_0, T_2)\) of any layer of \( M \) is balanced and each RBC \((T_0, T_3)\) of \( M \) is stuffed.

**Remark 7.5.** \( T_0 \) has different meanings for \( d = 2 \) and \( d = 3 \).
Remark 7.6. The expression \( \text{cap}(T_0, T_3) \) is referred sometimes by \( \text{cap}(n, a, b, c) \) to emphasize that \( \text{cap}(T_0, T_3) \) is a function with integer variables \( a, b, c \), where \( a, b, c \in \{1, 2, \ldots, n\} \) and the integer \( n \) is fixed. Cruse [2] proved some properties of this function.

Definition 7.7. The brick \( T \) is degenerated, if at least one edge of \( T \) is 0.

If \( T \) is degenerated, then it contains no rooks.

We consider the remote bricks \( (T_0, T_3) \) as degenerated RBC, if either \( T_0 \) or \( T_3 \) has one edge of size 0, for example \( c = 0 \) or \( n - c = 0 \). If \( c = 0 \) then the size of \( T_0 \) is \( a \times b \times 0 \), if \( n - c = 0 \) then the size of \( T_3 \) is \( (n - a) \times (n - b) \times 0 \) as shown in Figure 7.1. The capacity function provides the correct results for both cases, the number of rooks that the \( n \)-brick can contain.

![Figure 7.1](image)

The capacity function can be written as follows:

\[
\text{cap}(n, a, b, c) = n^2 - (a + b + c)n + (ab + bc + ca) = (n - a)(n - b) + c(a + b - n).
\]

If the integer variable \( c \) goes from 0 to \( n \), then the value of the function goes from \((n - a)(n - b)\), which is the area of \( T_3 \) orthogonal to \( z \), to \( ab \), which is the area of \( T_0 \) orthogonal to \( z \), as depicted on the left-hand side of Figure 7.1 and on the right-hand side of Figure 7.1, respectively. The change is \((a + b - n)\) in each step.

The layer \( c \) is balanced, so the yellow brick of the layer has \( c_0 \) rooks and the remote brick in this the layer has \( c_2 \) rooks and \( c_0 - c_2 = (a + b - n) \).

The step from \((c - 1)\) to \( c \) can be seen in the Figure 7.2.

After this step, the number of rooks in brick \( T_3 \) changes by \(-c_2\), the number of rooks in brick \( T_0 \) changes by \( c_0 \), so by \( c_0 - c_2 = (a + b - n) \) combined.

If \( a + b - n > 0 \), then \((n - a)(n - b) < ab\), so the capacity increases from \((n - a)(n - b)\) to \( ab \), if \( a + b - n < 0 \), then \((n - a)(n - b) > ab\), so the capacity decreases from \((n - a)(n - b)\) to \( ab \) and if \( a + b - n = 0 \), then \((n - a)(n - b) = ab\), so the capacity does not change, i.e, the RBCs have the same capacity for all \( c \).

The capacity of an RBC can be regarded as a generalization of the capacity of an \( n \)-brick (degenerated RBC).
Remark 7.8. In a 3-LSC $c_0 + c_3 = \frac{V(T_3) + V(T_0)}{n}$ for each RBC $(T_0, T_3)$.

$V(T_3) + V(T_0) = V(T_0 \cup T_3)$ because of $T_0 \cap T_3 = \emptyset$, hence

$$p(T_0 \cup T_3) = \frac{c_0 + c_3}{V(T_0 \cup T_3)} = \frac{c_0 + c_3}{V(T_0) + V(T_3)} = \frac{1}{n},$$

ero, each RBC $(T_0, T_3)$ has a standard density.

Definition 7.9. An $n$-brick is called an axis if it has exactly one edge of size $n$.

If we consider the bricks $T_{1b}$ and $T_{2bc}$ together, like the brown $n$-brick in the Figure 7.3, then the structure of $T_0, T_{1b}, T_{2bc}, T_3$ combined looks like a hinge (door hinge). The brown $n$-brick $T_{1b} \cup T_{2bc}$ is the axis of the hinge, the bricks $T_0$ and $T_3$ are the leafs of the hinge.

On one hand, a hinge can be considered as an axis with two leafs, on the other hand, a hinge is the union of two disjoint $n$-bricks $(T_0 \cup T_{1b})$ and $(T_3 \cup T_{2bc})$. That gives

**Corollary 7.10** (Hinge Volume).

$$V(T_0 \cup T_3) = V(T_0 \cup T_{1b}) + (T_3 \cup T_{2bc}) - V(T_{1b} \cup T_{2bc}) \quad (7.1)$$

and dividing both sides of (7.1) by $n$ we get

**Corollary 7.11** (Hinge Capacity),

$$\text{cap}(T_0, T_3) = \frac{V(T_0) + V(T_{1b})}{n} + \frac{V(T_3) + V(T_{2bc})}{n} - \frac{V(T_{1b}) + V(T_{2bc})}{n} \quad (7.2)$$
With the help of equality \((7.2)\) we can prove, without using the Distribution Theorem 4.17, that any RBC has as many rooks as its capacity.

**Theorem 7.12.** For any RBC \((T_0, T_3)\) in an LSC

\[
c_0 + c_3 = n^2 - (a + b + c)n + (ab + bc + ca)
\]

**Proof.** The \(n\)-brick \(T_0 \cup T_{1b}\) contains \(ac\) rooks, the \(n\)-brick \(T_{2bc} \cup T_3\) contains \((n - b)(n - c)\) rooks and the \(n\)-brick \(T_{1b} \cup T_{2bc}\) contains \(a(n - b)\) rooks. So the RBC \((T_0, T_3)\) contains

\[
ac + (n - b)(n - c) - a(n - b)
\]

rooks, i.e,

\[
c_0 + c_3 = ac + (n - b)(n - c) - a(n - b),
\]

but the right-hand side is equal to \(n^2 - (a + b + c)n + (ab + bc + ca)\). \(\square\)

**Remark 7.13.** The complement of a hinge is also a hinge, so the chess-board \(H^3\) consists of two disjoint hinges.

**References**

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