Embedded minimal surfaces in $\mathbb{R}^n$

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Abstract In this paper we prove that every conformal minimal immersion of an open Riemann surface into $\mathbb{R}^n$ for $n \geq 5$ can be approximated uniformly on compacts by conformal minimal embeddings. Furthermore, we show that every open Riemann surface carries a proper conformal minimal embedding into $\mathbb{R}^5$. One of our main tools is a Mergelyan approximation theorem for conformal minimal immersions to $\mathbb{R}^n$ for any $n \geq 3$ which is also proved in the paper.

Keywords Riemann surfaces, minimal surfaces.

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1. Introduction and main results

One of the central questions in Geometric Analysis is to understand when is an abstract manifold of a certain kind embeddable as a submanifold of a Euclidean space. Among many others, some celebrated results were obtained by Whitney [33] in Differential Geometry, Nash [28] in Riemannian Geometry, Greene and Wu [19] in Harmonic mapping theory, and by Remmert [30], Bishop [9], Narasimhan [26, 27], Eliashberg and Gromov [12] and Schurmann [32] in Complex Geometry.

The same question is highly interesting in the context of minimal submanifolds; a fundamental subject in Differential Geometry. Two dimensional minimal submanifolds (i.e., surfaces) in $\mathbb{R}^n$ are especially interesting objects. They lie at the intersection of several branches of Mathematics and Physics and enjoy powerful tools coming from Differential Geometry, Topology, PDEs, and Complex Analysis; see [29] for a classical survey and [24, 25] for more recent ones, among others.

The aim of this paper is to obtain general embedding results for minimal surfaces in $\mathbb{R}^n$, $n \geq 5$. Our first main result is the following.

Theorem 1.1. Let $M$ be an open Riemann surface. If $n \geq 5$ then every conformal minimal immersion $u : M \to \mathbb{R}^n$ can be approximated uniformly on compacts in $M$ by conformal minimal embeddings. The same holds if $M$ is a compact bordered Riemann surface with nonempty boundary and $u$ is of class $C^r(M)$ for some $r \in \mathbb{N}$; in such case the approximation takes place in the $C^r(M)$ topology.

Recall that a conformal immersion $u : M \to \mathbb{R}^n$ $(n \geq 3)$ of an open Riemann surface $M$ is minimal if and only if it is harmonic: $\Delta u = 0$. An immersion $u : M \to \mathbb{R}^n$ is said to be an embedding if $u : M \to u(M)$ is a homeomorphism.

More precisely, our proof will show that for any $n \geq 5$ the set of all conformal minimal embeddings $M \hookrightarrow \mathbb{R}^n$ is of the second category in the Fréchet space of all conformal minimal immersions $M \to \mathbb{R}^n$, endowed with the compact-open topology.
Theorem 1.1 obviously fails in dimensions \( n \leq 4 \) since transverse self-intersections are stable in these dimensions. By using the tools of this paper it can be seen that a generic conformal minimal immersion \( M \to \mathbb{R}^4 \) has only simple double points (normal crossings).

We now come to the following second main result of the paper.

**Theorem 1.2.** Every open Riemann surface \( M \) carries a proper conformal minimal embedding into \( \mathbb{R}^5 \). Furthermore, if \( K \) is a compact holomorphically convex set in \( M \) and \( n \geq 5 \) then every conformal minimal embedding from a neighborhood of \( K \) into \( \mathbb{R}^n \) can be approximated uniformly on \( K \) by proper conformal minimal embeddings \( M \hookrightarrow \mathbb{R}^n \).

Our methods also provide some control of the flux of the conformal minimal embeddings that we construct (see Sec. 2 for a precise definition). In particular, the approximation in Theorem 1.1 can be done by embeddings with the same flux as the original immersion, whereas the proper conformal minimal embeddings in the first assertion of Theorem 1.2 can be found with any given flux; see the more precise Theorems 6.1 and 7.1.

One of our main tools is a Mergelyan approximation result for conformal minimal immersions to \( \mathbb{R}^n \) for any \( n \geq 3 \) that is also proved in the paper (cf. Theorem 5.3), extending the result of Alarcón and López [5] which applies to \( n = 3 \). The proof in [5] uses the Weierstrass representation of conformal minimal immersions \( M \to \mathbb{R}^3 \) and hence it does not generalize to the case \( n > 3 \).

Let us place Theorem 1.2 in the context of results in the literature. It has been known since the 1950’s that every open Riemann surface embeds properly holomorphically in \( \mathbb{C}^3 \) [9, 26, 27, 30]. Since a holomorphic embedding is also conformal and harmonic, it follows that every open Riemann surface carries a proper conformal minimal embedding into \( \mathbb{R}^6 \).

In a different direction, Greene and Wu showed in 1975 [19] that every open \( k \)-dimensional Riemannian manifold \( M^k \) admits a proper (not necessarily conformal) harmonic embedding into \( \mathbb{R}^{2k+1} \); hence surfaces \((k = 2)\) embed properly harmonically into \( \mathbb{R}^5 \). However, the image of a non-conformal harmonic map is not necessarily a minimal surface, hence Theorem 1.2 is a refinement of their result when \( M \) is an orientable surface.

The optimal result for immersions was obtained by Alarcón and López who proved that every open Riemann surface carries a proper conformal minimal immersion into \( \mathbb{R}^3 \) [5, 6].

It is well known that Theorem 1.2 fails in dimension \( n = 3 \). Indeed, the existence of a proper conformal minimal embedding \( M \hookrightarrow \mathbb{R}^3 \) is a very restrictive condition on \( M \) and there is a rich literature on this subject; see the recent surveys [24, 25] and references therein. However, every Riemann surface (open or closed) admits a smooth proper conformal (not necessarily minimal) embedding into \( \mathbb{R}^3 \) according to Rüedy [31] (see also Garsia [18]).

It remains an open problem whether Theorem 1.2 holds in dimension \( n = 4 \):

**Problem 1.3.** Does every open Riemann surface admit a proper conformal minimal embedding into \( \mathbb{R}^4 \)?

Motivated by the result of Greene and Wu [19], another interesting but less ambitious open question is whether every open (orientable) Riemannian surface admits a harmonic embedding into \( \mathbb{R}^4 \). These problems seem nontrivial also for nonproper embeddings.

Since every holomorphic embedding is conformal and harmonic (hence minimal), Problem 1.3 is related to the analogous long standing open problem whether every open Riemann surface admits a proper holomorphic embedding into \( \mathbb{C}^2 \). (See the survey of
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Bell and Narasimhan \[8\], Conjecture 3.7 on p. 20, and Sections 8.9–8.10 in \[14\]. For recent progress on this problem we refer to the articles of Forstnerič and Wold \[16, 17\] and the references therein. In particular, the following result is proved in \[16\]: Let $M$ be a compact bordered Riemann surface with nonempty boundary $bM$. If $M$ admits a (nonproper) holomorphic embedding into $\mathbb{C}^2$, then its interior $\bar{M} = M \setminus bM$ admits a proper holomorphic embedding into $\mathbb{C}^2$. The same question makes sense for conformal minimal embeddings and it naturally appears as a first approach to Problem 1.3.

**Problem 1.4.** Let $M$ be a compact bordered Riemann surface with nonempty boundary $bM$. Assume that $M$ admits a conformal minimal embedding $M \hookrightarrow \mathbb{R}^4$ of class $\mathcal{C}^1$ $(M)$. Does the interior $\bar{M}$ admit a proper conformal minimal embedding into $\mathbb{R}^4$?

In \[17\] it was proved that every (possibly infinitely connected) circular domain in the Riemann sphere $\mathbb{C}P^1$ admits a proper holomorphic embedding into $\mathbb{C}^2$, hence a proper conformal minimal embedding into $\mathbb{R}^4$.

The constructions in \[16, 17\] strongly depend on the theory of holomorphic automorphisms of complex Euclidean spaces: after exposing and sending to infinity a point in each boundary component of $M$, a proper holomorphic embedding $M \hookrightarrow \mathbb{C}^2$ is obtained by successively pushing the image of the boundary $bM$ to infinity by holomorphic automorphisms of $\mathbb{C}^2$. Such approach seems impossible for conformal minimal embeddings since this class of maps is preserved only under rigid transformations of $\mathbb{R}^n$. It is therefore a challenging problem to find suitable methods that could work in this situation.

On the other hand, there are no topological obstructions to these questions since Alarcón and López \[7\] proved that every open orientable surface admits a smooth proper embedding in $\mathbb{C}^2$ whose image is a complex curve. For bordered orientable surfaces of finite topology this was shown earlier by Cerne and Forstnerič \[10\].

The results in this paper, and the methods used in the proof, are influenced by the recent work \[3\] of the first two named authors who proved results analogous to Theorems 1.1 and 1.2 for a certain class of directed holomorphic immersions (including null curves) of Riemann surfaces into $\mathbb{C}^n$, $n \geq 3$. A null curve $M \rightarrow \mathbb{C}^n$ is a holomorphic immersion whose derivative lies in the punctured null quadric $\mathcal{A}^* = \mathcal{A} \setminus \{0\} \subset \mathbb{C}^n$; see (2.2) below. The real part of a null curve is a conformal minimal immersion $M \rightarrow \mathbb{R}^n$. (Obviously, the real part of an embedded null curve is not necessarily embedded.) These techniques exploit the close connection between minimal surfaces in $\mathbb{R}^n$ and Complex Analysis, in particular, modern Oka theory. The most relevant point for the purposes of this paper is that the punctured null quadric $\mathcal{A}^* = \mathcal{A} \setminus \{0\}$ is an Oka manifold. The simplest explanation of this notion can be found in Lárusson’s AMS Notices article \[22\]. Roughly speaking, maps $M \rightarrow X$ from any Stein manifold $M$ (in particular, from any open Riemann surface) to an Oka manifold $X$ satisfy all forms of Oka principle. For a more comprehensive expository article on this subject see \[15\], and a complete treatment is given in \[14\].

We include a few words concerning the organization of the paper. In Sec. 2 we collect some preliminary material and introduce the relevant definitions. In Sec. 3 we obtain some basic local properties of the space of conformal minimal immersions of bordered Riemann surfaces to $\mathbb{R}^n$; cf. Theorem 3.1. In Sec. 4 we prove the special case of Theorem 1.1 for bordered Riemann surfaces (cf. Theorem 4.1). In Sec. 5 we prove a Mergelyan type approximation theorem for conformal minimal immersions of open Riemann surfaces into $\mathbb{R}^n$ for any $n \geq 3$; see Theorem 5.3. By combining all these results we then prove Theorem 1.1 in Sec. 6 (cf. Theorem 6.1) and Theorem 1.2 in Sec. 7 (cf. Theorem 7.1).
2. Notation and preliminaries

Let \( n \geq 3 \) be a natural number that will be fixed throughout the paper.

Let \( M \) be a Riemann surface. An immersion \( u = (u_1, u_2, \ldots, u_n) : M \to \mathbb{R}^n \) is conformal (angle preserving) if and only if, in any local holomorphic coordinate \( z = x + iy \) on \( M \), the partial derivatives \( u_x = (u_{1,x}, \ldots, u_{n,x}) \) and \( u_y = (u_{1,y}, \ldots, u_{n,y}) \) at any point, considered as vectors in \( \mathbb{R}^n \), have the same Euclidean length and are orthogonal:

\[
|u_x| = |u_y| > 0, \quad u_x \cdot u_y = 0.
\]

Equivalently, \( u_x \pm iu_y \in \mathbb{C}^n \setminus \{0\} \) are null vectors, i.e., they belong to the null quadric

\[
\mathfrak{A} = \{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + \cdots + z_n^2 = 0 \}.
\]

Since \( \mathfrak{A} \) is defined by a homogeneous quadratic holomorphic equation and is smooth away from the origin, the punctured null quadric \( \mathfrak{A}^* = \mathfrak{A} \setminus \{0\} = \mathfrak{A}_{\text{reg}} \) is an Oka manifold (see Example 4.4 in [3, p. 743]). This means that:

**Remark 2.1.** Maps \( M \to \mathfrak{A}^* \) from any Stein manifold (in particular, from any open Riemann surface) satisfy all forms of the \( \tilde{\text{O}} \text{ka principle} \) [14, Theorem 5.4.4].

The exterior derivative on \( M \) splits into the sum \( d = \partial + \bar{\partial} \) of the \((1,0)\)-part \( \partial \)

and the \((0,1)\)-part \( \bar{\partial} \). In any local holomorphic coordinate \( z = x + iy \) on \( M \) we have

\[
2\partial u = (u_x - iu_y)dz, \quad 2\bar{\partial} u = (u_x + iu_y)d\bar{z}.
\]

Hence (2.1) shows that \( u \) is conformal if and only if the differential \( \partial u = (\partial u_1, \ldots, \partial u_n) \) satisfies the nullity condition

\[
\mathfrak{h} = (\partial u_1)^2 + (\partial u_2)^2 + \cdots + (\partial u_n)^2 = 0.
\]

The expression \( \mathfrak{h} \) is the Hopf differential of \( u \).

Assume now that \( M \) is an open Riemann surface and \( u: M \to \mathbb{R}^n \) is a conformal immersion. It is classical (cf. Osserman [29]) that \( \Delta u = 2\mu H \) where \( H: M \to \mathbb{R}^n \) is the mean curvature vector of \( u \) and \( \mu > 0 \) is a positive function. (In local isothermal coordinates \( x + iy \) on \( M \) we have \( \mu = ||u_x||^2 = ||u_y||^2 \). In \( \mathbb{R}^3 \) this is usually written in the form \( \Delta u = 2H\nu \) where \( H \) is the mean curvature function and \( \nu \) is the Gauss map.) Hence \( u \) is minimal (\( H = 0 \)) if and only if it is harmonic (\( \Delta u = 0 \)). If \( v \) is any local harmonic conjugate of \( u \) on \( M \) then the Cauchy-Riemann equations imply that

\[
\partial (u + iv) = 2\partial u = 2i \partial v.
\]

In particular, the differential \( \partial u \) of any conformal minimal immersion is a \( \mathbb{C}^n \)-valued holomorphic 1-form satisfying (2.4).

The conjugate differential of a smooth map \( u : M \to \mathbb{R}^n \) is defined by

\[
d^* u = i(\overline{\partial u} - \partial u) = 2\Im(\partial u).
\]

We have that

\[
2\partial u = du + id^* u, \quad dd^* u = 2i\partial \overline{\partial u} = \Delta u \cdot dx \wedge dy.
\]

Thus \( u \) is harmonic if and only if \( d^* u \) is a closed vector valued 1-form on \( M \), and \( d^* u = dv \) holds for any local harmonic conjugate \( v \) of \( u \).
The flux map of a harmonic map \( u : M \to \mathbb{R}^n \) is the group homomorphism \( \text{Flux}_u : H_1(M; \mathbb{Z}) \to \mathbb{R}^n \) given by
\[
\text{Flux}_u([C]) = \int_C \bar{\theta} u_i, \quad [C] \in H_1(M; \mathbb{Z}).
\]
The integral on the right hand side is independent of the choice of a path in a given homology class, and we shall write \( \text{Flux}_u(C) \) for \( \text{Flux}_u([C]) \) in the sequel.

Fix a nowhere vanishing holomorphic 1-form \( \theta \) on \( M \). (Such exists by the Oka-Grauert principle, cf. Theorem 5.3.1 in [14, p. 190].) It follows from (2.4) that
\[
2 \partial u = f \theta
\]
where \( f = (f_1, \ldots, f_n) : M \to \mathbb{A}^n \) is a holomorphic map satisfying
\[
\int_C \mathcal{R}(f \theta) = \int_C du = 0 \quad \text{for any closed curve} \ C \in M.
\]

Conversely, associated to any holomorphic map \( f : M \to \mathbb{C}^n \) is the period homomorphism \( \mathcal{P}(f) : H_1(M; \mathbb{Z}) \to \mathbb{C}^n \) defined on any closed curve \( C \subset M \) by
\[
\mathcal{P}(f)(C) = \int_C f \theta.
\]
The map \( f \) corresponds to a conformal minimal immersion \( u : M \to \mathbb{R}^n \) as in (2.6) if and only if \( f(M) \subset \mathbb{A}^n \) and \( \mathcal{R}(\mathcal{P}(f)) = 0 \), and in this case we have that \( u(x) = \int^x \mathcal{R}(f \theta) \) \( (x \in M) \) and
\[
\text{Flux}_u = \Im(\mathcal{P}(f)) : H_1(M; \mathbb{Z}) \to \mathbb{R}^n.
\]

The previous discussion connects the theory of minimal surfaces in \( \mathbb{R}^n \) to the one of Oka manifolds; see Remark 24. (See for instance [22, 15] for expository articles on Oka theory.)

Next we introduce the mapping spaces used in the paper. If \( M \) is an open Riemann surface then \( \mathcal{O}(M) \) is the algebra of holomorphic functions \( M \to \mathbb{C} \), \( \mathcal{O}(M, X) \) is the space of holomorphic mappings \( M \to X \) to a complex manifold \( X \), and \( \text{CMI}(M, \mathbb{R}^n) \) is the set of conformal minimal immersions \( M \to \mathbb{R}^n \). These spaces are endowed with the compact-open topology.

If \( K \) is a compact subset of \( M \) then \( \mathcal{O}(K) \) denotes the set of holomorphic functions on open neighborhoods of \( K \) in \( M \) (in the sense of germs along \( K \)). A compact set \( K \subset M \) is \( \mathcal{O}(M) \)-convex if \( K \) equals its holomorphically convex hull
\[
\hat{K} = \{ x \in M : |f(x)| \leq \sup_K |f| \ \forall f \in \mathcal{O}(M) \}.
\]

If \( M \) is an open Riemann surface then by Runge’s theorem \( K = \hat{K} \) if and only if \( M \setminus K \) does not contain any relatively compact connected components, and this holds precisely when every function \( f \in \mathcal{O}(K) \) is the uniform limit of functions in \( \mathcal{O}(M) \); for this reason such \( K \) is also called a Runge set in \( M \). (See e.g. [21] for these classical results.)

Assume now that \( M \) is a compact bordered Riemann surface, i.e. a compact connected Riemann surface with smooth boundary \( \emptyset \neq bM \subset M \) and interior \( M = M \setminus bM \). If \( g \geq 0 \) is the genus and \( m \geq 1 \) is the number of boundary components of \( M \) then the first homology group \( H_1(M; \mathbb{Z}) \) is a free abelian group on \( l = 2g + m - 1 \) generators whose basis is given by smoothly embedded loops \( \gamma_1, \ldots, \gamma_l : \mathbb{S}^1 \to M \) that only meet at a chosen base point \( p \in M \). Let \( C_j = \gamma_j(\mathbb{S}^1) \subset M \) denote the trace of \( \gamma_j \). Their union \( C = \bigcup_{j=1}^l C_j \) is a wedge of \( l \) circles with vertex \( p \).
Given $r \in \mathbb{Z}_+$, we denote by $\mathcal{A}^r(M)$ the space of all functions $M \to \mathbb{C}$ of class $\mathcal{C}^r(M)$ that are holomorphic in $M$. More generally, for any complex manifold $X$ we let $\mathcal{A}^r(M, X)$ denote the space of maps $M \to X$ of class $\mathcal{C}^r$ which are holomorphic in $M$. We write $\mathcal{A}^0(M) = \mathcal{A}(M)$ and $\mathcal{A}^0(M, X) = \mathcal{A}(M, X)$. Note that $\mathcal{A}^r(M, \mathbb{C}^n)$ is a complex Banach space, and for any complex manifold $X$ the space $\mathcal{A}^r(M, X)$ is a complex Banach manifold modeled on $\mathcal{A}^r(M, \mathbb{C}^n)$ with $n = \dim X$ (see [13, Theorem 1.1]). Likewise we denote by $\text{CM} \mathcal{I}^r(M, \mathbb{R}^n)$ the set of maps $M \to \mathbb{R}^n$ of class $\mathcal{C}^r(M)$ which are conformal minimal immersions on $M$. We write $\text{CM}^1(M, \mathbb{R}^n) = \text{CM}(M, \mathbb{R}^n)$.

A compact bordered Riemann surface $M$ can be considered as a smoothly bounded compact domain in an open Riemann surface $R$. It is classical that each function in $\mathcal{A}^r(M)$ can be approximated in the $\mathcal{C}^r(M)$ topology by functions in $\mathcal{C}(M)$. The same is true for maps to an arbitrary complex manifold or complex space (see [11, Theorem 5.1]).

The following notions will play an important role in our analysis.

**Definition 2.2.** Let $M$ be a connected open or bordered Riemann surface, let $\theta$ be a nowhere vanishing holomorphic 1-form on $M$, and let $\mathfrak{A}$ be the null quadric (2.2).

- A holomorphic map $f : M \to \mathfrak{A}^\ast$ is said to be nonflat if the image $f(M)$ is not contained in any complex ray $\mathbb{C} \nu \subset \mathfrak{A}$ of the null quadric. Likewise, a conformal minimal immersion $u : M \to \mathbb{R}^n$ is nonflat if the map $f = 2\partial u/\partial \theta : M \to \mathfrak{A}^\ast$ is nonflat, or equivalently, if the image $u(M)$ is not contained in a plane.
- A holomorphic map $f : M \to \mathfrak{A}^\ast$ is said to be nondegenerate if the image $f(M) \subset \mathfrak{A}^\ast$ is not contained in any linear complex hyperplane of $\mathbb{C}^n$. Likewise, a conformal minimal immersion $u : M \to \mathbb{R}^n$ is nondegenerate if the map $f = 2\partial u/\partial \theta : M \to \mathfrak{A}^\ast$ is nondegenerate.
- A conformal minimal immersion $u : M \to \mathbb{R}^n$ is full if the image $u(M)$ is not contained in a hyperplane.

For a conformal minimal immersion $M \to \mathbb{R}^3$, nonflat, full, and nondegenerate are equivalent notions. However, if one passes to higher dimensions then no two of these conditions are equivalent; we have

$$\text{Nondegenerate} \implies \text{Full} \implies \text{Nonflat},$$

but the converses are not true (see [29]).

Since the tangent space $T_z \mathfrak{A}$ is the kernel at $z$ of the $(1, 0)$-form $\sum_{j=1}^{n} z_j \, d\bar{z}_j$, we have $T_z \mathfrak{A} = T_w \mathfrak{A}$ for $z, w \in \mathbb{C}^n \setminus \{0\}$ if and only if $z$ and $w$ are colinear. This implies

**Lemma 2.3.** A holomorphic map $f : M \to \mathfrak{A}^\ast$ is nonflat if and only if the linear span of the tangent spaces $T_{f(x)} \mathfrak{A}$ over all $x \in M$ equals $\mathbb{C}^n$.

The condition in Lemma 2.3, which appears in [3] (see Definition 2.2 in [3, p. 736]), is used in the construction of period dominating sprays of holomorphic maps $M \to \mathfrak{A}^\ast$ (cf. Lemma 3.2 below).

If $M$ is an open Riemann surface, we denote by $\text{CM} \mathcal{I}_x(M, \mathbb{R}^n) \subset \text{CM} \mathcal{I}(M, \mathbb{R}^n)$ the subset consisting of all immersions which are nondegenerate on every connected component of $M$. The analogous notation is used for compact bordered Riemann surfaces. In particular, $\text{CM} \mathcal{I}^r_x(M, \mathbb{R}^n)$ is the space of all conformal minimal immersions $M \to \mathbb{R}^n$ of class $\mathcal{C}^r(M)$ which are nondegenerate on every component.
3. Conformal minimal immersions of bordered Riemann surfaces

The following result gives some basic local properties of the space of conformal minimal immersions of a bordered Riemann surface to \( \mathbb{R}^n \). It is analogous to Theorem 2.3 in [3] where similar properties were proved for certain classes of directed holomorphic immersions; in particular, for null holomorphic immersions \( M \to \mathbb{C}^n \) for any \( n \geq 3 \).

**Theorem 3.1.** Let \( M \) be a compact bordered Riemann surface with nonempty boundary \( \partial M \), and let \( n \geq 3 \) and \( r \geq 1 \) be integers.

(a) Every conformal minimal immersion \( u \in \text{CMI}^r(M, \mathbb{R}^n) \) can be approximated in the \( \mathcal{C}^r(M) \) topology by nondegenerate conformal minimal immersions \( \hat{u} \in \text{CMI}^r_c(M, \mathbb{R}^n) \) satisfying \( \text{Flux}_u = \text{Flux}_{\hat{u}} \).

(b) For any \( r \in \mathbb{N} \) the space \( \text{CMI}^r(M, \mathbb{R}^n) \) is a real analytic Banach manifold with the natural \( \mathcal{C}^r(M) \) norm.

(c) If \( M \) is a smoothly bounded compact domain in a Riemann surface \( R \) and \( r \in \mathbb{N} \) then every conformal minimal immersion \( u \in \text{CMI}^r(M, \mathbb{R}^n) \) can be approximated in the \( \mathcal{C}^r(M) \) topology by conformal minimal immersions defined on open neighborhoods of \( M \) in \( R \).

**Proof.** Fix a nowhere vanishing holomorphic 1-form \( \theta \) on \( M \). Choose a basis \( \{ \gamma_j \}_{j=1}^l \) of the homology group \( H_1(M;\mathbb{Z}) \) and denote by
\[
P = (P_1, \ldots, P_l) : \mathcal{A}(M, \mathbb{C}^n) \to (\mathbb{C}^n)^l
\]
the period map whose \( j \)-th component, applied to \( f \in \mathcal{A}(M, \mathbb{C}^n) \), equals
\[
P_j(f) = \int_{\gamma_j} f \theta = \int_0^1 f(\gamma_j(t)) \theta(\gamma_j(t), \gamma_j(t)) \, dt \in \mathbb{C}^n.
\]

**Proof of (a).** For simplicity of notation we assume that \( r = 1 \); the same proof applies for any \( r \in \mathbb{N} \).

Let \( u: M \to \mathbb{R}^n \) be a degenerate conformal minimal immersion. The map \( f = 2\partial u/\theta: M \to \mathbb{R}^n \) is continuous on \( M \) and holomorphic in \( M \), and the linear span of \( f(M) \) is a \( k \)-dimensional linear complex subspace \( \varPi \subset \mathbb{A} \), \( 1 \leq k < n \). Fix distinct points \( \{p_1, \ldots, p_k, q_1, \ldots, q_{n-k}\} \in M \) such that \( \{f(p_1), \ldots, f(p_k)\} \) is a basis of \( \varPi \). Choose a nonconstant function \( h \in \mathcal{A}(M) \) such that \( h(p_i) = 0 \) for all \( i = 1, \ldots, k \), and \( h(q_j) = 1 \) for all \( j = 1, \ldots, n-k \). Choose a holomorphic vector field \( V \) on \( \mathbb{C}^n \) tangential to \( \mathbb{A} \) such that \( \{f(p_1), \ldots, f(p_k), V(f(q_1)), \ldots, V(f(q_{n-k}))\} \) is a basis of \( \mathbb{C}^n \). Let \( t \to \phi(t, z) \) denote the flow of \( V \) for small complex values of time \( t \), with \( \phi(0, z) = z \). For any \( g \in \mathcal{A}(M) \) near the zero function we define the map \( \Phi(g) \in \mathcal{A}(M, \mathbb{A}^*) \) by
\[
\Phi(g)(x) = \phi(g(x)h(x), f(x)), \quad x \in M.
\]
Clearly \( \Phi(0) = f \). Consider the holomorphic map
\[
\mathcal{A}(M) \ni g \mapsto P(\Phi(g)) \in (\mathbb{C}^n)^l.
\]
Since the space \( \mathcal{A}(M) \) is infinite dimensional, there is a function \( g \in \mathcal{A}(M) \setminus \{0\} \) arbitrarily close to the zero function such that \( P(\Phi(g)) = P(\Phi(0)) = P(f) \); in particular, \( \Re P(\Phi(g)) = 0 \). For such \( g \) the map \( \tilde{f} = \Phi(g): M \to \mathbb{A}^* \) integrates to a conformal minimal immersion \( \tilde{u}(x) = u(p) + \int_p^x \Re(\tilde{f}(t)) \) that is close to \( u \) and satisfies \( \text{Flux}_{\tilde{u}} = \text{Flux}_u \).

For a generic choice of points \( q_j' \in M \) near \( q_j, \ j = 1, \ldots, n - k \), we have that
In the proof of part (b) we shall need the following lemma from [3]. (In [3] the result is stated in the case when \( P(f) = 0 \) and \( r = 0 \), but the proof given there applies verbatim in the more general situation stated here.)

**Lemma 3.2 (Lemma 5.1 in [3]).** Let \( r \in \mathbb{Z}_+ \), and let \( f \in \mathcal{A}^r(M, \mathfrak{A}^*) \) be a nonflat map. There exist an open neighborhood \( U \) of the origin in a Euclidean space \( \mathbb{C}^N \) and a map

\[
U \times M \ni (\zeta, x) \mapsto \Phi_f(\zeta, x) \in \mathfrak{A}^* \tag{3.2}
\]

of class \( \mathcal{A}^r(U \times M, \mathfrak{A}^*) \) such that \( \Phi_f(0, \cdot) = f \) and the period map \( \zeta \mapsto P(\Phi_f(\zeta, \cdot)) \in (\mathbb{C}^n)^l \) is submersive at \( \zeta = 0 \). Furthermore, there is a neighborhood \( V \) of \( f \) in \( \mathcal{A}^r(M, \mathfrak{A}^*) \) such that the map \( V \ni h \mapsto \Phi_h \) can be chosen holomorphic in \( h \).

A holomorphic family of maps \( \Phi_f(\zeta, \cdot) : M \to \mathfrak{A}^* (\zeta \in U \subseteq \mathbb{C}^N) \) as in Lemma 3.2 is called a period dominating spray with core \( \Phi_f(0, \cdot) = f \) and with values in \( \mathfrak{A}^* \) (cf. [3]).

A map \( \Phi_f \) as in Lemma 3.2 constructed in [3], is of the form

\[
\Phi_f(\zeta, x) = \Phi(\zeta, x, f(x)) \in \mathfrak{A}^*, \tag{3.2}
\]

for a map \( \Phi \) of the form

\[
\Phi(\zeta, x, z)\phi_{1,G1(x)}^1 \circ \cdots \circ \phi_{N,GN(x)}^N (z) \in \mathfrak{A}, \tag{3.3}
\]

where \( z \in \mathfrak{A}, x \in M, \zeta = (\zeta_1, \ldots, \zeta_N) \) belongs to a neighborhood \( U \subseteq \mathbb{C}^N \) of the origin, \( g_1, \ldots, g_N \) are suitably chosen holomorphic functions on \( M \), and every \( \phi_i^1 \) is the flow of some holomorphic vector field on \( \mathbb{C}^n \) that is tangential to \( \mathfrak{A} \).

**Proof of (b).** By [13, Theorem 1.1] the space \( \mathcal{A}^{r-1}(M, \mathfrak{A}^*) \) is a complex Banach manifold modeled on the complex Banach space \( \mathcal{A}^{r-1}(M, \mathbb{C}^{n-1}) \) (since \( \dim \mathfrak{A}^* = n - 1 \)). Set

\[
\mathcal{A}^{r-1}_{0,\star}(M, \mathfrak{A}^*) = \{ f \in \mathcal{A}^{r-1}(M, \mathfrak{A}^*) : \mathcal{R}(P(f)) = 0 \},
\]

where \( P : \mathcal{A}^{r-1}(M, \mathfrak{A}^*) \to (\mathbb{C}^n)^l \) is the (holomorphic) period map (3.1). Let \( \mathcal{A}^{r-1}_{0,\star}(M, \mathfrak{A}^*) \) denote the open subset of \( \mathcal{A}^{r-1}(M, \mathfrak{A}^*) \) consisting of all nondegenerate maps (cf. Def. 2.2). Since nondegenerate maps are nonflat, Lemma 5.2 implies that the differential \( dP_{f_0} \) of the restricted period map \( P : \mathcal{A}^{r-1}(M, \mathfrak{A}^*) \to (\mathbb{C}^n)^l \) at any point \( f_0 \in \mathcal{A}^{r-1}_{0,\star}(M, \mathfrak{A}^*) \) has maximal rank equal to \( ln \). By the implicit function theorem it follows that \( f_0 \) admits an open neighborhood \( \Omega \subseteq \mathcal{A}^{r-1}(M, \mathfrak{A}^*) \) such that \( \Omega \cap \mathcal{A}^{r-1}_{0,\star}(M, \mathfrak{A}^*) = \mathcal{A}^{r-1}_{0,\star}(M, \mathfrak{A}^*) \) is a real analytic Banach submanifold of \( \Omega \) which is parametrized by the kernel of the real part \( \mathcal{R}(dP_{f_0}) \) of the differential of \( P \) at \( f_0 \); this is a real codimension \( ln \) subspace of the complex Banach space \( \mathcal{A}^{r-1}(M, \mathbb{C}^{n-1}) \) (the tangent space of the complex Banach manifold \( \mathcal{A}^{r-1}(M, \mathfrak{A}^*) \)). This shows that \( \mathcal{A}^{r-1}_{0,\star}(M, \mathfrak{A}^*) \) is a real analytic Banach manifold.

The integration \( x \mapsto v + \int_x^p \mathcal{R}(f\theta) (x \in M) \), with an arbitrary choice of the initial value \( v \in \mathbb{R}^n \) at a chosen point \( p \in M \), provides an isomorphism between the Banach manifold \( \mathcal{A}^{r-1}_{0,\star}(M, \mathfrak{A}^*) \times \mathbb{R}^n \) and the space \( \text{CMF}^r(M, \mathbb{R}^n) \), so the latter is also a Banach manifold. This completes the proof of Theorem 3.1(b).

**Proof of (c).** Let \( u \in \text{CMF}^r(M, \mathbb{R}^n) \). By part (a) we may assume that \( u \) is nondegenerate, hence nonflat. Write \( 2\partial u = f\theta \) and let \( \Phi_f \) be the period dominating spray of conformal
minimal immersions furnished by Lemma 3.2. We can approximate $f$ uniformly on $M$ by holomorphic maps $\tilde{f}: V \to \mathbb{A}^*$ defined on an open neighborhood $V$ of $M$ in $\mathbb{R}^n$. The associated spray $\Phi_{\tilde{f}}$ is then defined and holomorphic on a neighborhood $\tilde{U} \times \tilde{V} \subset \mathbb{C}^N \times R$ of $\{0\} \times M$. If $\tilde{f}$ is sufficiently uniformly close to $f$ on $M$, then the domain and the range of the period map $P(\Phi_{\tilde{f}})$ are so close to those of $P(\Phi_f)$ that the range of $P(\Phi_{\tilde{f}})$ contains the point $P(f) \in \mathbb{C}^{\mathbb{A}^*}$. (Note that the components of $P(f)$ are purely imaginary since $f$ corresponds to a conformal minimal immersion.) Hence $\tilde{f}$ can be approximated in $\mathcal{C}^r(M)$ by a holomorphic map $h \in \mathcal{O}(W, \mathbb{A}^*)$ on a connected open neighborhood $W \subset R$ of $M$ satisfying $P(h) = P(f)$; in particular, we have that $\mathbb{R}(P(h)) = 0$. The integral $\tilde{u}(x) = u(p) + \int_p^x \Phi(h \theta)$ is then a conformal minimal immersion in a neighborhood of $M$ in $R$ which approximates $u$ in $\mathcal{C}^r(M, \mathbb{R}^n)$.

\[ \square \]

4. Desingularizing conformal minimal immersions

In this section we prove the following general position theorem (a special case of Theorem 1.1) for conformal minimal immersions of bordered Riemann surfaces.

**Theorem 4.1.** Let $M$ be a compact bordered Riemann surface and let $n \geq 5$ and $r \geq 1$ be integers. Every conformal minimal immersion $u \in \mathcal{CM}^r(M, \mathbb{R}^n)$ can be approximated arbitrarily closely in the $\mathcal{C}^r(M)$ topology by a conformal minimal embedding $\tilde{u} \in \mathcal{CM}^r(M, \mathbb{R}^n)$ satisfying $\mathrm{Flux}_\tilde{u} = \mathrm{Flux}_u$.

Since the set of embeddings $M \to \mathbb{R}^n$ is clearly open in the set of immersions of class $\mathcal{C}^r(M)$ for any $r \geq 1$ and $\mathcal{CM}^r(M, \mathbb{R}^n)$ is a closed subset of the Banach space $\mathcal{C}^r(M, \mathbb{R}^n)$ (hence a Baire space), we immediately get

**Corollary 4.2.** Let $M$ be a compact bordered Riemann surface. For every pair of integers $n \geq 5$ and $r \geq 1$ the set of conformal minimal embeddings $M \to \mathbb{R}^n$ of class $\mathcal{C}^r(M)$ is residual (of the second category) in $\mathcal{CM}^r(M, \mathbb{R}^n)$.

**Proof of Theorem 4.1.** In view of Theorem 3.1 (parts (a) and (c)) we may assume that $M$ is a smoothly bounded domain in an open Riemann surface $R$ and $u$ is a nondegenerate (in the sense of Def 2.2) conformal minimal immersion in an open neighborhood of $M$ in $R$.

We associate to $u$ the difference map $\delta u: M \times M \to \mathbb{R}^n$ defined by

$$\delta u(x, y) = u(y) - u(x), \quad x, y \in M.$$ 

Clearly $u$ is injective if and only if $(\delta u)^{-1}(0) = D_M := \{(x, x) : x \in M\}$. Since $u$ is an immersion, it is locally injective, and hence there is an open neighborhood $U \subset M \times M$ of the diagonal $D_M$ such that $\delta u$ does not assume the value $0 \in \mathbb{R}^n$ on $U \setminus D_M$. To prove the theorem, it suffices to find arbitrarily close to $u$ another conformal minimal immersion $\tilde{u}: M \to \mathbb{R}^n$ whose difference map $\delta \tilde{u}$ is transverse to the origin $0 \in \mathbb{R}^n$ on $M \times M \setminus U$. Since $\dim_{\mathbb{R}} M \times M = 4 < n$, this will imply that $\delta \tilde{u}$ does not assume the value zero on $M \times M \setminus U$, so $\tilde{u}(x) \neq \tilde{u}(y)$ if $(x, y) \in M \times M \setminus U$. If $(x, y) \in U \setminus D_M$ then $\tilde{u}(x) \neq \tilde{u}(y)$ provided that $\tilde{u}$ is close enough to $u$, so $\tilde{u}$ is an embedding.

To find such $\tilde{u}$ we shall construct a neighborhood $\Omega \subset \mathbb{R}^N$ of the origin in a Euclidean space and a real analytic map $H: \Omega \times M \to \mathbb{R}^n$ satisfying the following properties:

(a) $H(0, \cdot) = u$, 

Clearly it is holomorphic in the variable \( \psi \) vanish at the endpoints the vectors \( V_i \) and points \( \xi, x \in \mathbb{C}^n \). Hence the map \( \delta H : \Omega \times M \times M \to \mathbb{R}^n \), defined by
\[
\delta H(\xi, x, y) = H(\xi, y) - H(\xi, x), \quad \xi \in \Omega, \quad x, y \in M,
\]
is a submersive family on \( M \times M \setminus U \), in the sense that the partial differential
\[
d_{\xi} \delta H(\xi, x, y) : \mathbb{R}^N \to \mathbb{R}^n
\]
is surjective for every \((x, y) \in M \times M \setminus U \).

Assume for the moment that such \( H \) exists. By compactness of \( M \times M \setminus U \) the partial differential \( d_{\xi} \delta H \) is surjective for all \( \xi \) in a neighborhood \( \Omega' \subset \Omega \) of the origin in \( \mathbb{R}^N \). Hence the map \( \delta H : M \times M \setminus U \to \mathbb{R}^n \) is transverse to any submanifold of \( \mathbb{R}^n \), in particular, to the origin \( 0 \in \mathbb{R}^n \). The standard transversality argument due to Abraham [11] (a reduction to Sard’s theorem; see also [14, Sec. 7.8]) shows that for a generic choice of \( \xi \in \Omega' \) the difference map \( \delta H(\xi, \cdot, \cdot) \) is then transverse to \( 0 \in \mathbb{R}^n \) on \( M \times M \setminus U \), and hence it omits the value \( 0 \) by dimension reasons. By choosing \( \xi \) sufficiently close to \( 0 \in \mathbb{R}^N \) we thus obtain a conformal minimal embedding \( \tilde{u} = H(\cdot, \cdot) : M \to \mathbb{R}^n \) close to \( u \), thereby proving the theorem.

We construct a spray \( H \) of conformal minimal immersions satisfying properties (a)--(c) above by suitably modifying the proof of the corresponding result for directed holomorphic immersions given in [3, Theorem 2.4].

Fix a nowhere vanishing holomorphic 1-form \( \theta \) on \( M \) and write \( 2\partial \theta = f \theta \), where \( f : M \to A^* \) is a nondegenerate holomorphic map (see [2.3] and Def. 2.2). The main step is furnished by the following lemma.

**Lemma 4.3.** (Notation and assumptions as above.) For every \((p, q) \in M \times M \setminus D_M \) there exists a spray \( H = H^{(p,q)}(\cdot, \cdot) : M \to \mathbb{R}^n \) of conformal minimal immersions of class \( \mathcal{C}^r(M) \), depending analytically on the parameter \( \xi \) in a neighborhood of the origin in \( \mathbb{R}^n \), satisfying properties (a) and (b) above, but with (c) replaced by the following property:

(c') the differential \( d_{\xi} \delta H(\xi, p, q) : \mathbb{R}^n \to \mathbb{R}^n \) is an isomorphism.

**Proof.** Let \( \Lambda \subset M \) be a smooth embedded arc connecting \( p \) to \( q \). Pick a point \( p_0 \in M \setminus \Lambda \) and closed loops \( C_1, \ldots, C_l \subset M \setminus \Lambda \) based at \( p_0 \) and forming a basis of \( H_1(M; \mathbb{Z}) \). Set \( C = \bigcup_{j=1}^l C_j \). Let \( \gamma_j : [0, 1] \to C_j \) \((j = 1, \ldots, l) \) and \( \lambda : [0, 1] \to \Lambda \) be smooth parametrizations of the respective curves.

Since \( u \) is nonflat, Lemma 2.3 shows that there exist tangential holomorphic vector fields \( V_1, \ldots, V_n \) on \( A \) and points \( x_1, \ldots, x_n \in \Lambda \setminus \{p, q\} \) such that, setting \( z_i = f(x_i) \in A^* \), the vectors \( V_i(z_i) \) for \( i = 1, \ldots, n \) span \( \mathbb{C}^n \). Let \( t_i \in (0, 1) \) be such that \( \lambda(t_i) = x_i \). Let \( \phi^{1}_{t} \) denote the flow of \( V_i \). Choose smooth functions \( h_i : C \cup \Lambda \to \mathbb{R}_+ \) \((i = 1, \ldots, n) \) that vanish at the endpoints \( p, q \) of \( \Lambda \) and on the curves \( C \); their values on \( \Lambda \) will be chosen later. Let \( \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n \). Consider the map
\[
\psi_f(\zeta, x) = \phi^{1}_{h_1(x)} \circ \cdots \circ \phi^{n}_{h_n(x)}(f(x)) \in A^*, \quad x \in C \cup \Lambda.
\]
Clearly it is holomorphic in the variable \( \zeta \in \mathbb{C}^n \) near the origin, \( \psi_f(0, \cdot) = f \), and \( \psi_f(\zeta, x) = f(x) \) if \( x \in C \cup \{p, q\} \) (since \( h_i = 0 \) on \( C \cup \{p, q\} \)). We have that
\[
\frac{\partial \psi_f(\zeta, x)}{\partial \zeta_i} \bigg|_{\zeta=0} = h_i(x) V_i(f(x)), \quad x \in C \cup \Lambda, \quad i = 1, \ldots, n.
\]
By choosing the function \( h_t \) to have support concentrated near the point \( x_i = \lambda(t_i) \in \Lambda \) we can arrange that for every \( i = 1, \ldots, n \) we have that

\[
\int_0^1 h_t(\lambda(t)) V_i(f(\lambda(t))) \theta(\lambda(t), \dot{\lambda}(t)) \, dt \approx V_i(z_i) \theta(\lambda(t_i), \dot{\lambda}(t_i)) \in \mathbb{C}^n.
\]

Assuming as we may that the above approximations are close enough, the vectors on the left hand side of the above display form a basis of \( \mathbb{C}^n \).

Fix a number \( \epsilon > 0 \); its precise value will be chosen later. We apply Mergelyan’s theorem to find holomorphic functions \( g_i \in \mathcal{O}(M) \) such that

\[
\sup_{C \cup \Lambda} |g_i - h_i| < \epsilon \quad \text{for} \quad i = 1, \ldots, n.
\]

Consider the holomorphic maps

\[
\begin{align*}
\Psi(\zeta, x, z) &= \phi^1_{c_1 g_1(x)} \circ \cdots \circ \phi^n_{c_n g_n(x)}(z) \in \mathfrak{A}, \\
\Psi_f(\zeta, x) &= \Psi(\zeta, x, f(x)) \in \mathfrak{A},
\end{align*}
\]

(4.3)

where \( x \in M, z \in \mathfrak{A} \), and \( \zeta \) is near the origin in \( \mathbb{C}^n \). Note that \( \Psi_f(0, \cdot) = f \). If the approximations of \( h_t \) by \( g_i \) are close enough then the vectors

\[
\frac{\partial}{\partial \zeta} \bigg|_{\zeta=0} \int_0^1 \Psi_f(\zeta, \lambda(t)) \theta(\lambda(t), \dot{\lambda}(t)) \, dt = \int_0^1 g_i(\lambda(t)) V_i(\lambda(t)) \theta(\lambda(t), \dot{\lambda}(t)) \, dt
\]

in \( \mathbb{C}^n \) are so close to the vectors \( V_i(z_i) \theta(\lambda(t_i), \dot{\lambda}(t_i)) (i = 1, \ldots, n) \) that they are \( \mathbb{C} \)-linearly independent.

The \( \mathbb{C}^n \)-valued 1-form \( \Psi_f(\zeta, \cdot) \theta \) need not have exact real part, so it may not correspond to the differential of a conformal minimal immersion. We shall now correct this.

From the Taylor expansion of the flow of a vector field we see that

\[
\Psi_f(\zeta, x) = f(x) + \sum_{i=1}^n \zeta_i g_i(x) V_i(f(x)) + o(|\zeta|).
\]

Since \( |g_i| < \epsilon \) on \( C \), the periods over the loops \( C_j \) can be estimated by

\[
\left| \int_{C_j} \left( \Psi_f(\zeta, \cdot) - f \right) \theta \right| \leq \eta_0 \epsilon |\zeta|
\]

(4.5)

for some constant \( \eta_0 > 0 \) and for sufficiently small \( |\zeta| \).

Lemma \([3,2]\) gives holomorphic maps \( \Phi(\tilde{\zeta}, x, z) \) and \( \Phi_f(\tilde{\zeta}, x) = \Phi(\tilde{\zeta}, x, f(x)) \) (see \([3,2]\) and \([3,3]\)), with the parameter \( \tilde{\zeta} \) near \( 0 \in \mathbb{C}^N \) for some \( N \in \mathbb{N} \) and \( x \in M \), such that \( \Phi(0, x, z) = z \) and the differential of the associated period map \( \tilde{\zeta} \mapsto P(\Phi_f(\tilde{\zeta}, \cdot)) \in \mathbb{C}^{1n} \) (see \([3,1]\)) at the point \( \tilde{\zeta} = 0 \) has maximal rank equal to \( 1n \). The same is true if we let the map \( f \in \mathcal{O}(M, \mathfrak{A}^*) \) vary locally near the given initial map. In particular, we can replace \( f \) by the spray \( \Psi_f(\zeta, \cdot) \) and consider the composed map

\[
\mathbb{C}^N \times \mathbb{C}^n \times M \ni (\tilde{\zeta}, \zeta, x) \mapsto \Phi(\tilde{\zeta}, x, \Psi_f(\zeta, x)) \in \mathfrak{A}^*
\]

which is defined and holomorphic for \((\tilde{\zeta}, \zeta) \) near the origin in \( \mathbb{C}^N \times \mathbb{C}^n \) and for \( x \in M \). The implicit function theorem furnishes a \( \mathbb{C}^N \)-valued holomorphic map \( \tilde{\zeta} = \rho(\zeta) \) near
ζ = 0 ∈ \mathbb{C}^n$, with ρ(0) = 0 ∈ \mathbb{C}^N, such that the \mathbb{C}^n-valued holomorphic 1-form on $M$ given by

$$\Theta_f(\zeta, x, v) = \Phi(\rho(\zeta), x, \Psi_f(\zeta, x)) \theta(x, v), \quad x \in M, \ v \in T_x M$$

satisfies

$$\int_{\gamma_\zeta} \Theta_f(\zeta, \cdot, \cdot) = \int_{\gamma_\zeta} f \theta, \quad j = 1, \ldots, l$$

for every ζ ∈ \mathbb{C}^n near the origin. In particular, the real parts of these periods vanish. (The map $\rho = (\rho_1, \ldots, \rho_n)$ also depends on $f$, but we shall suppress this dependence in our notation.) It follows that the integral

$$(4.6) \quad H_u(\zeta, x) = u(p_0) + \int_{p_0}^x \Re(\Theta_f(\zeta, \cdot, \cdot')) = u(p_0) + \int_0^1 \Re(\Theta_f(\zeta, \gamma(t), \dot{\gamma}(t)))$$

is independent of the choice of the path from the initial point $p_0$ to the variable point $x \in M$. Clearly $H_u$ is analytic, $H_u(0, \cdot) = u$, $H_u(\zeta, \cdot): M \to \mathbb{R}^n$ is a conformal minimal immersion for every ζ ∈ \mathbb{C}^n sufficiently close to 0, and the flux homomorphism of $H_u(\zeta, \cdot)$ equals that of $u$ for every fixed ζ. Furthermore, in view of (4.5) we have the estimate

$$(4.7) \quad |\rho(\zeta)| \leq \eta_1 |\zeta|$$

for some $\eta_1 > 0$ independent of $\epsilon$ and ζ.

The map $\Phi(\zeta, x, z)$ furnished by Lemma 4.2 is obtained by composing flows of certain holomorphic vector fields $W_j$ on $\mathfrak{X}$ for the complex times $\tilde{\zeta}_j \tilde{g}_j(x)$, where $\tilde{g}_j \in \mathcal{O}(M)$ and $\tilde{\zeta}_j \in \mathbb{C}$. (See (3.3).) The Taylor expansion of the flow together with the estimate (4.7) gives

$$|\Phi(\rho(\zeta), x, \Psi_f(\zeta, x)) - \Psi_f(\zeta, x)| = \left| \sum \rho_j(\zeta) \tilde{g}_j(x) W_j(\Psi_f(\zeta, x)) + o(|\zeta|) \right| \leq \eta_2 |\zeta|$$

for some $\eta_2 > 0$ and for all $x \in M$ and all ζ near the origin in \mathbb{C}^n. Applying this estimate on the curve $\Lambda$ with the endpoints $p, q$ we get that

$$\left| \int_0^1 \Theta_f(\zeta, \lambda(t), \dot{\lambda}(t)) - \int_0^1 \Psi_f(\zeta, \lambda(t)) \theta(\lambda(t), \dot{\lambda}(t)) \, dt \right| \leq \eta_3 |\zeta|$$

for some constant $\eta_3 > 0$ independent of $\epsilon$ and ζ. If $\epsilon > 0$ is chosen small enough then it follows that the derivatives

$$\frac{\partial}{\partial \zeta_i} \Big|_{\zeta=0} \int_0^1 \Theta_f(\zeta, \lambda(t), \dot{\lambda}(t)) \in \mathbb{C}^n, \quad i = 1, \ldots, n,$$

are so close to the respective vectors in (4.4) that they are $\mathbb{C}$-linearly independent. This means that the holomorphic map $\zeta \mapsto \int_0^1 \Theta_f(\zeta, \lambda(t), \dot{\lambda}(t)) \in \mathbb{C}^n$ is locally biholomorphic near ζ = 0. By (4.6) its real part equals

$$\int_0^1 \Re(\Theta_f(\zeta, \lambda(t), \dot{\lambda}(t))) = H_u(\zeta, q) - H_u(\zeta, p) = \delta H_u(\zeta, p, q).$$

After a suitable $\mathbb{C}$-linear change of coordinates $\zeta = \xi + \eta$ on \mathbb{C}^n it follows that the partial differential $\frac{\partial}{\partial \zeta_i} \delta H_u(\xi, p, q): \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. The spray

$$(4.8) \quad H^{(p,q)}(\xi, \cdot) := H_u(\xi, \cdot)$$

satisfies the conclusion of Lemma 4.3. \qed
The spray \( H_u \) furnished by Lemma 4.3 depends real analytically on \( u \in \text{CM} \), \( M \) in a neighborhood of a given nondegenerate conformal minimal immersion \( u_0 \in \text{CM} \). In particular, if \( u(\eta,\cdot): M \to \mathbb{R}^n \) is a family of conformal minimal immersions depending analytically on a parameter \( \eta \), then \( H_u(\eta,\cdot) \) depends analytically on \( (\xi,\eta) \). This allows us to compose any finite number of such sprays just as was done in [3]. We recall this operation for two sprays. Suppose that \( S \) are sprays with \( H_u(\eta,\cdot) \) and \( G = G_u(\eta,\cdot) \), respectively. The operation \( \circ \) extends by induction to finitely many factors and is associative. (This is similar to the composition of sprays introduced by Gromov [20]; see also [14, p. 246].)

Pick an open neighborhood \( U \subset M \times M \) of the diagonal \( D_M \) such that \( U \cap (\delta u)^{-1}(0) = D_M \). Lemma 4.3 furnishes a finite open covering \( U = \{U_i\}_{i=1}^m \) of the compact set \( M \times M \setminus U \) and sprays of conformal minimal immersions \( H^i = H^i(\xi^i,\cdot): M \to \mathbb{R}^n \), with \( H^i(0,\cdot) = u \), such that the difference map \( \delta H^i(\eta,\cdot) \) is submersive at \( \xi^i = 0 \) for all \( \eta \in U_i \). By taking \( \xi = (\xi^1,\ldots,\xi^m) \in \mathbb{R}^N \), with \( N = \sum_{i=1}^m k_i \), and setting

\[
H(\xi,x) = (H^1\#H^2\#\cdots\#H^m)(\xi^1,\ldots,\xi^m,x)
\]

we obtain a spray satisfying properties (a) and (b) whose difference map \( \delta H \) is submersive on \( M \times M \setminus U \) for all \( \xi \in \mathbb{R}^N \) sufficiently near the origin. As explained earlier, a generic member \( H(\xi,\cdot) \) of this spray (for \( \xi \) sufficiently close to \( 0 \in \mathbb{R}^N \)) is a conformal minimal embedding \( M \hookrightarrow \mathbb{R}^n \).

This completes the proof of Theorem 4.1.

5. Mergelyan’s theorem for conformal minimal immersions to \( \mathbb{R}^n \)

In this section we prove a Mergelyan type approximation theorem for conformal minimal immersions of open Riemann surfaces into \( \mathbb{R}^n \) for any \( n \geq 3 \); see Theorem 5.3 below. The special case \( n = 3 \) has been already proved by Alarcón and López [3, Theorem 4.9] who used the López-Ros transformation for conformal minimal immersions \( M \to \mathbb{R}^3 \) (see [23]); a tool that is not available for \( n \geq 4 \). Here we use the more general approach developed in [3] for holomorphic null curves and more general directed holomorphic immersions of open Riemann surfaces to \( \mathbb{C}^n \).

We begin by introducing a suitable type of sets for the Mergelyan approximation. The same type of sets have been used in [3] (see Definition 7.1 there) and in several other papers.

Definition 5.1. A compact subset \( S \subset M \) of an open Riemann surface \( M \) is said to be admissible if \( S = K \cup \Gamma \), where \( K = \bigcup D_j \) is a union of finitely many pairwise disjoint, compact, smoothly bounded domains \( D_j \) in \( M \) and \( \Gamma = \bigcup \Gamma_j \) is a union of finitely many pairwise disjoint smooth arcs or closed curves that intersect \( K \) only in their endpoints (or not at all), and such that their intersections with the boundary \( bK \) are transverse.

Note that an admissible set \( S \subset M \) is Runge in \( M \) (i.e., \( \mathcal{C}^0(M) \)-convex) if and only if the inclusion map \( S \hookrightarrow M \) induces an injective homomorphism \( H_1(S;\mathbb{Z}) \to H_1(M;\mathbb{Z}) \) of the first homology groups. If this holds, then we have the classical Mergelyan approximation theorem: Every continuous function \( f: S = K \cup \Gamma \to \mathbb{C} \) that is holomorphic in the interior
\( K \) of the compact set \( K \) can be approximated, uniformly on \( S \), by functions holomorphic on \( M \). More generally, if \( f \) is of class \( \mathcal{C}^r \) on \( S \) for some \( r \geq 0 \), then the approximation can be made in the \( \mathcal{C}^r(S) \) topology.

Recall that \( \mathfrak{A} \) denotes the null quadric \((2.2)\) and \( \mathfrak{A}^* = \mathfrak{A} \setminus \{0\} \).

Given an admissible set \( S = K \cup \Gamma \subset M \), we denote by \( \mathcal{D}(S, \mathfrak{A}^*) \) the set of all smooth maps \( S \to \mathfrak{A}^* \) which are holomorphic on an unspecified open neighborhood of \( K \) (depending on the map). In accordance with Definition \( 2.2 \), we say that a map \( f \in \mathcal{D}(S, \mathfrak{A}^*) \) is nonflat if it maps no component of \( K \) and no component of \( \Gamma \) to a ray on \( \mathfrak{A}^* \). Likewise, we say that \( f \in \mathcal{D}(S, \mathfrak{A}^*) \) is nondegenerate if it maps no component of \( K \) and no component of \( \Gamma \) to a complex hyperplane of \( \mathbb{C}^n \). We denote by \( \mathcal{O}_*(S, \mathfrak{A}^*) \) the subset of \( \mathcal{D}(S, \mathfrak{A}^*) \) consisting of all nondegenerate maps.

Fix a nowhere vanishing holomorphic 1-form \( \theta \) on \( M \); the precise choice of \( \theta \) will be unimportant in the sequel.

The following definition of a conformal minimal immersion from an admissible subset emulates the spirit of the concept of marked immersion \([5]\) (cf. \([4]\) Def. 6.2). Such generalized conformal minimal immersions provide natural initial objects for the Mergelyan approximation.

**Definition 5.2.** Let \( M \) be an open Riemann surface and let \( S = K \cup \Gamma \subset M \) be an admissible subset (Def. 5.1). A generalized conformal minimal immersion on \( S \) is a pair \((u, f\theta)\), where \( f \in \mathcal{D}(S, \mathfrak{A}^*) \) and \( u: S \to \mathbb{R}^n \) is a smooth map which is a conformal minimal immersion on an open neighborhood of \( K \), such that the following properties hold:

\begin{itemize}
  \item \( f\theta = 2\partial u \) on an open neighborhood of \( K \), and
  \item \( \Re(f\theta)(x) = (u \circ \alpha)'(x)dx \) on every connected component \( \alpha \) of \( \Gamma \), where \( x \) is any smooth parameter along \( \alpha \).
\end{itemize}

A generalized conformal minimal immersion \((u, f\theta)\) is said to be nonflat (resp., nondegenerate) if \( f \in \mathcal{D}(S, \mathfrak{A}^*) \) is nonflat (resp., nondegenerate).

We denote by \( \text{GCMI}(S, \mathbb{R}^n) \) the set of all generalized conformal minimal immersions \( S \to \mathbb{R}^n \) and by \( \text{GCMI}_*(S, \mathbb{R}^n) \subset \text{GCMI}(S, \mathbb{R}^n) \) the subset consisting of all nondegenerate ones. We say that \((u, f\theta) \in \text{GCMI}(S)\) can be approximated in the \( \mathcal{C}^1(S) \) topology by conformal minimal immersions in \( \text{CMI}(M) \) if there exists a sequence \( v_i \in \text{CMI}(M) \) (\( i \in \mathbb{N} \)) such that \( v_i|_S \) converges to \( u|_S \) in the \( \mathcal{C}^0(S) \) topology and \( 2\partial v_i|_S \) converges to \( f\theta|_S \) in the \( \mathcal{C}^0(S) \) topology.

**Theorem 5.3** (Mergelyan’s theorem for conformal minimal immersions). Assume that \( M \) is an open Riemann surface and that \( S = K \cup \Gamma \) is a compact Runge admissible set in \( M \). Then every generalized conformal minimal immersion \((u, f\theta) \in \text{GCMI}(S, \mathbb{R}^n)\) for \( n \geq 3 \) can be approximated in the \( \mathcal{C}^1(S) \) topology by nondegenerate conformal minimal immersions \( \tilde{u} \in \text{CMI}_*(M, \mathbb{R}^n) \).

Furthermore, given a group homomorphism \( p : H_1(M; \mathbb{Z}) \to \mathbb{R}^n \) satisfying \( p(C) = \text{Flux}_\alpha(C) \) for every closed curve \( C \subset S \), we can choose \( \tilde{u} \) as above such that \( \text{Flux}_{\tilde{u}} = p \).

**Proof.** Let \( \rho : M \to \mathbb{R} \) be a smooth strongly subharmonic Morse exhaustion function and exhaust \( M \) by an increasing sequence \( M_1 \subset M_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} M_i = M \) of compact smoothly bounded domains of the form \( M_i = \{p \in M : \rho(p) \leq c_i\} \), where \( c_1 < c_2 < \cdots \) is an increasing sequence of regular values of \( \rho \) with \( \lim_{i \to \infty} c_i = +\infty \). Each domain \( M_i \)
is therefore a compact bordered Riemann surface, possibly disconnected. We may assume that \( \rho \) has at most one critical point \( p_i \) in each difference \( M_{i+1} \setminus M_i \). It then follows that \( M_i \) is Runge in \( M \) for every \( i \in \mathbb{N} \). Finally, since \( S \) is Runge, we may assume without loss of generality that \( S \subset M_1 \) and \( S \) is a strong deformation retract of \( M_1 \), hence the inclusion map \( S \hookrightarrow M_1 \) induces an isomorphism \( H_1(S; \mathbb{Z}) \cong H_1(M_1; \mathbb{Z}) \) of the homology groups.

We proceed by induction. The basis is given by the following lemma.

**Lemma 5.4.** Every \((u, f \theta) \in \text{GCMI}(S, \mathbb{R}^n)\) for \( n \geq 3 \) can be approximated in the \( C^1(S) \) topology by nondegenerate conformal minimal immersions in \( \text{CMI}_1(M_1, \mathbb{R}^n) \).

**Proof.** Since \( S = K \cup \Gamma \) is a strong deformation retract of \( M_1 \), we may assume that \( S \) is connected; the same argument works on any connected component.

By Theorem 5.1(a), and slightly deforming \((u, f \theta)\) on \( \Gamma \) if needed, we may assume that \((u, f \theta) \in \text{GCMI}_\ast(S, \mathbb{R}^n)\), i.e., it is nondegenerate in the sense of Def. 5.2.

We claim that it is possible to approximate \( f \in \mathcal{D}_\ast(S, \mathbb{A}^\ast) \) as closely as desired uniformly on \( S \) by a holomorphic map \( f_1 : M_1 \to \mathbb{A}^\ast \) such that

\[
\int_C (f_1 - f) \theta = 0 \quad \text{for every closed curve } C \subset S.
\]

(5.1)

Assume for a moment that this holds. Since \( H_1(S; \mathbb{Z}) \cong H_1(M_1; \mathbb{Z}) \) and \( f \theta \) has no real periods on \( S \), the same is true for \( f_1 \) on \( M_1 \) in view of (5.1). Hence \( f_1 \) provides a conformal minimal immersion \( u_1 \in \text{CMI}(M_1, \mathbb{R}^n) \) by the expression

\[
u_1(p) = u(p_0) + \int_{p_0}^p \Re(f_1 \theta), \quad p \in M_1,
\]

where \( p_0 \in K \) is any base point. Furthermore, since \( S \) is connected, \( u_1 \) can be assumed to be as close as desired to \( u \) in the \( C^1(S) \) topology provided that the approximation of \( f \) by \( f_1 \) is close enough. In particular, since \( u \) is nondegenerate, \( u_1 \) can be taken in \( \text{CMI}_\ast(M_1, \mathbb{R}^n) \).

This proves Lemma 5.4 provided that the above claim holds.

The construction of a holomorphic map \( f_1 : M_1 \to \mathbb{A}^\ast \) satisfying (5.1) is similar to the proof of the Mergelyan approximation theorem for null holomorphic curves (and other classes of directed holomorphic immersions) in [3 Theorem 7.2]. The main difference is that the period vanishing condition in the latter result is now replaced by the condition of matching the periods of a given map. Here is the outline.

By the assumption the map \( f \) is holomorphic on an open neighborhood \( U \subset M \) of \( K \) and is smooth on \( \Gamma \). By Theorem 5.1(a) we may assume that \( f \) is nondegenerate. Up to a shrinking of \( U \) around \( K \) we can apply [3 Lemma 5.1] to find a period dominating spray of smooth maps \( f_w : U \cup \Gamma \to \mathbb{A}^\ast \) which are holomorphic on \( U \) and depend holomorphically on a parameter \( w \) in a ball \( B \subset \mathbb{C}^N \), with \( f_0 = f \). (One deforms \( f \) by flows of holomorphic vector fields on \( \mathbb{A} \) which generate the tangent space at every point of \( \mathbb{A}^\ast \); see (3.2) and (3.3) above. The time variables of these flows are holomorphic functions on a neighborhood of \( S \) in \( M \), chosen so as to ensure the period domination property.)

By Mergelyan approximation we can approximate the map \( f = f_0 \) uniformly on \( S \) by a map \( \tilde{f}_0 \) which is holomorphic on a small open neighborhood \( V \subset M \) of the set \( S \). Here we can use Theorem 3.7.2 in [14, p. 81], noticing that our set \( S \) is a special case of the sets \( S = K_0 \cup E \) in the cited theorem. By applying the same flows to \( \tilde{f}_0 \) we get a new holomorphic spray of maps \( f_w : V \to \mathbb{A}^\ast \) which approximates the initial spray \( f_w \) uniformly.
on $S$, and uniformly with respect to the parameter $w$. (This part of the construction can be
done with an arbitrary complex manifold $X$ in place of $\mathfrak{X}^*$.)

Since $\mathfrak{X}^*$ is an Oka manifold (see Example 4.4 in \cite{3} and Remark 2.1 above) and $S$ is
Runge in $M$ and a deformation retract of $M_1$, we can apply \cite{14}, Theorem 5.4.4, p. 193
(the main result of Oka theory) to approximate the spray $\tilde{f}_w$ uniformly on $S$ (and uniformly
with respect to the parameter $w$) by a holomorphic spray of maps $g_w : M_1 \to X$. (The
parameter set $B \subset \mathbb{C}^N$ is allowed to shrink a little. The topological condition on the
inclusion $S \hookrightarrow M_1$ is used to get the existence of a continuous extension of the spray $\tilde{f}_w$
from $S$ to $M_1$, a necessary condition to apply the Oka principle.)

If both approximations made above are close enough then there exists a point $w_0 \in B$
close to the origin such that the map $g_{w_0} : M_1 \to \mathfrak{X}^*$ satisfies the period condition \textcircled{5.1}.
(The last argument is as in the proof of Theorem \textcircled{3.1}(c).) Taking this map as our $f_1$
completes the proof of Lemma \textcircled{5.4}.\hfill $\square$

The following result provides the inductive step in the recursive process.

**Lemma 5.5.** Assume that $p : H_1(M; \mathbb{Z}) \to \mathbb{R}^n$ is a group homomorphism. Let $i \in \mathbb{N}$
and let $u_i \in \text{CMI}_i(M_i, \mathbb{R}^n)$ be a nondegenerate conformal minimal immersion such
that $\text{Flux}_{u_i}(C) = p(C)$ for all closed curve $C \subset M_i$. Then $u_i$ can be approximated
in the $\mathcal{C}^1(M_i)$ topology by nondegenerate conformal minimal immersions $u_{i+1} \in
\text{CMI}_i(M_{i+1}, \mathbb{R}^n)$ satisfying $\text{Flux}_{u_{i+1}}(C) = p(C)$ for all closed curve $C \subset M_{i+1}$.

**Proof.** We consider two essentially different cases.

The noncritical case: Assume that $\rho$ has no critical value in $[c_i, c_{i+1}]$. In this case there
is no change of topology when passing from $M_i$ to $M_{i+1}$ and $M_i$ is a strong deformation
retract of $M_{i+1}$. The immersion $u_{i+1}$ can then be constructed as in the proof of Lemma \textcircled{5.4}.

The critical case: Assume that $\rho$ has a critical point $p_{i+1} \in M_{i+1} \setminus M_i$. By the assumptions
on $\rho$, $p_{i+1}$ is the only critical point of $\rho$ on $M_{i+1} \setminus M_i$ and is a Morse point. Since $\rho$
is strongly subharmonic, the Morse index of $p_{i+1}$ is either 0 or 1.

If the Morse index of $p_{i+1}$ is 0 then a new (simply connected) component of the sublevel
set $\{ \rho \leq r \}$ appears at $p_{i+1}$ when $r$ passes the value $\rho(p_{i+1})$. In this case we reduce the
proof to the noncritical case by defining $u_{i+1}$ on this new component as any nondegenerate
conformal minimal immersion.

Assume now that the Morse index of $p_{i+1}$ is 1. In this case the change of topology of the
sublevel set $\{ \rho \leq r \}$ at $p_{i+1}$ is described by attaching to $M_i$ a smooth arc $\gamma \subset M_{i+1} \setminus M_i$,
and it follows that $M_i \cup \gamma$ is a Runge strong deformation retract of $M_{i+1}$. We assume
without loss of generality that $M_i \cup \gamma$ is admissible (Def. \textcircled{5.1}). Since $u_i$ is nondegenerate
and $\text{Flux}_{u_i}(C) = p(C)$ for all closed curve $C \subset M_i$, we may extend $u_i$ to $M_i \cup \gamma$ as a
nondegenerate generalized conformal minimal immersion $(\hat{u}_i, \hat{f}_i) \in \text{GCMI}(M_i \cup \gamma, \mathbb{R}^n)$
such that $\hat{u}_i = u_i$ on $M_i$ and $\hat{f}_i \exists (f_i, \theta) = p(C)$ for all closed curve $C \subset M_i \cup \gamma$. This can
be done as in \cite{4}, Lemma 3.4] where the details are given for the case $n = 3$, but the same
proof works in general. To finish we reason again as in the proof of Lemma \textcircled{5.4}.\hfill $\square$

Combining Lemmas \textcircled{5.4} and \textcircled{5.5} we may construct a sequence of nondegenerate
conformal minimal immersions $\{ u_i \in \text{CMI}_i(M_i) \}_{i \in \mathbb{N}}$ such that:

- $u_i$ is as close to $(u, f\theta)$ as desired in the $\mathcal{C}^1(S)$ topology for all $i \in \mathbb{N}$.
Let \( u_i \) is as close to \( u_{i-1} \) as desired in the \( C^1(M, \mathbb{R}^n) \) topology for all \( i \geq 2 \).
- Flux\(_{u_i}(C) = p(C)\) for all closed curve \( C \subset M_i \) and all \( i \in \mathbb{N} \).

If these approximations are close enough, then the limit \( \tilde{u} := \lim_{i \to \infty} u_i : M \to \mathbb{R}^n \) is a nondegenerate conformal minimal immersion as close to \((u, f \theta)\) in the \( C^1(S) \) topology as desired and satisfying Flux\(_{\tilde{u}} = p\). This concludes the proof of Theorem 5.3. \( \square \)

The following Mergelyan type result for conformal minimal immersions into \( \mathbb{R}^n \) with \( n - 2 \) fixed components was essentially proved in [2]. It will play an important role in the proof of Theorem 1.2 in Sec. 7.

**Lemma 5.6.** Assume that \( M \) is a compact bordered Riemann surface and \( K \) is a union of finitely many pairwise disjoint, smoothly bounded compact Runge domains in \( M \). Assume that \( K \) contains a basis of \( H_1(M; \mathbb{Z}) \) and naturally identify \( H_1(K; \mathbb{Z}) = H_1(M; \mathbb{Z}) \). Let \( u = (u^1, \ldots, u^n) \in \text{CMI}_p(K, \mathbb{R}^n) \) be a nondegenerate conformal minimal immersion and assume that \( u^j \) extends harmonically to \( M \) for all \( j \geq 3 \). Then \( u \) can be approximated in the \( C^1(K) \) topology by nondegenerate conformal minimal immersions \( \tilde{u} = (\tilde{u}^1, \ldots, \tilde{u}^n) \in \text{CMI}_p(M, \mathbb{R}^n) \) such that Flux\(_{\tilde{u}} = \text{Flux}_u\) and \( \tilde{u}^j = u^j \) for all \( j \geq 3 \).

**Proof.** Let \( \theta \) be a nowhere vanishing holomorphic 1-form on \( M \). As usual we write \( f^j = 20u^j/\theta \in \mathcal{O}(K) \) for all \( j \) and notice that \( f^j \in \mathcal{O}(M) \) for all \( j \geq 3 \). Denote by \( \Theta \) the quadratic holomorphic form \( -(\sum_{j=3}^n (f^j)^2)\theta^2 \) on \( M \).

Let \( S = K \cup \Gamma \subset M \) be a Runge connected admissible set (Def. 5.1) such that \( \Theta \) does not vanish anywhere on \( \Gamma \). Choose a nondegenerate generalized conformal minimal immersion \((v, g\theta) \in \text{GCMI}_p(S, \mathbb{R}^n)\) such that

- \( v = u \) and \( g = f \) on \( K \),
- \( v^j = u^j \) and \( g^j = f^j \) on \( S \) for all \( j \geq 3 \), where \( v = (v^1, \ldots, v^n) \) and \( g = (g^1, \ldots, g^n) \).

(We refer the reader to the proof of Lemma 3.3 in [2] for details on how to find such \((v, g\theta)\).) By the latter condition, \((g^1)^2 + (g^2)^2 = -\sum_{j=3}^n (g^j)^2 = \Theta/\theta^2 \) on \( S \). Further, since \( u \) is nondegenerate, the functions \( g^1 \) and \( g^2 \) are linearly independent in \( \mathcal{O}(K) \). By Lemma 3.3 in [2], \((g^1, g^2)\) can be uniformly approximated on \( S \) by a pair \((h^1, h^2) \subset \mathcal{O}(M)\) satisfying

- \((h^1)^2 + (h^2)^2 = \Theta/\theta^2 \),
- the 1-form \( ((h^1, h^2) - (g^1, g^2))\theta \) is exact on \( S \), and
- the zeros of \((h^1, h^2)\) on \( M \) are those of \((f^1, f^2)\) on \( K \) (in particular, \((h^1, h^2)\) does not vanish anywhere on \( M \setminus K \)).

Fix a point \( p_0 \in K \) and set \( \tilde{u}^j(p) := u^j(p) + \int_{p_0}^p h^j \theta, \ p \in M, \ j = 1, 2, \) and \( \tilde{u}^j := u^j \) for all \( j \geq 3 \). If the approximation of \((g^1, g^2)\) by \((h^1, h^2)\) is close enough on \( S \), then it is clear that \( \tilde{u} = (\tilde{u}^1, \ldots, \tilde{u}^n) \in \text{CMI}_p(M, \mathbb{R}^n) \) satisfies the conclusion of the lemma. \( \square \)

### 6. Approximation by conformal minimal embeddings

In this section we prove the following more precise version of Theorem 1.1.

**Theorem 6.1.** Let \( M \) be an open Riemann surface and let \( n \geq 5 \) be an integer. Given a conformal minimal immersion \( u : M \to \mathbb{R}^n \), a compact Runge set \( K \subset M \) and...
a number \( \epsilon > 0 \), there exists a conformal minimal embedding \( \tilde{u} \): \( M \to \mathbb{R}^n \) such that
\[
\sup_{x \in K} |\tilde{u}(x) - u(x)| < \epsilon \quad \text{and} \quad \text{Flux}(\tilde{u}) = \text{Flux}(u).
\]

**Proof.** Exhaust \( M \) by an increasing sequence \( M_1 \subset M_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} M_i = M \) of compact Runge smoothly bounded domains such that \( K \subset M_1 \) and each domain \( M_i \) is a compact bordered Riemann surface.

We proceed by induction.

Theorem 4.1 furnishes a conformal minimal embedding \( u_1 \in \text{CMI}(M_1, \mathbb{R}^n) \) which is as close as desired to \( u \) in the \( C^1(M_1) \) topology and satisfies \( \text{Flux}_{u_1}(C) = \text{Flux}_{u}(C) \) for all closed curve \( C \subset M_1 \). Further, we may assume by Theorem 5.3 (a) that \( u_1 \) is nondegenerate, i.e. \( u_1 \in \text{CMI}(M_1, \mathbb{R}^n) \).

Let \( i \in \mathbb{N} \) and assume the existence of a nondegenerate conformal minimal embedding \( u_i \in \text{CMI}(M_i, \mathbb{R}^n) \) with \( \text{Flux}_{u_i}(C) = \text{Flux}_{u}(C) \) for all closed curve \( C \subset M_i \). This process gives a sequence of nondegenerate conformal minimal embeddings satisfying \( \text{Flux}_{u_i+1}(C) = \text{Flux}_{u}(C) \) for all closed curve \( C \subset M_{i+1} \). Moreover, in view of Theorem 4.1 this approximation can be done by embeddings.

This process gives a sequence of nondegenerate conformal minimal embeddings \( \{u_i \in \text{CMI}_*(M_i, \mathbb{R}^n) \}_{i \in \mathbb{N}} \) such that
\begin{itemize}
  \item \( u_i \) is as close to \( u \) as desired in the \( C^1(K) \) topology for all \( i \in \mathbb{N} \).
  \item \( u_i \) is as close to \( u_{i-1} \) as desired in the \( C^1(M_{i-1}) \) topology for all \( i \geq 2 \).
  \item \( \text{Flux}_{u_i}(C) = \text{Flux}_{u}(C) \) for all closed curve \( C \subset M_i \) and all \( i \in \mathbb{N} \).
\end{itemize}

If these approximations are close enough, then the limit \( \tilde{u} := \lim_{i \to \infty} u_i : M \to \mathbb{R}^n \) is a nondegenerate conformal minimal embedding satisfying the conclusion of the theorem. (See for instance the proof of Theorem 4.5 in [7] for a similar argument.)

### 7. Construction of proper conformal minimal embeddings

In this section we prove Theorem 1.2 in the following more precise form.

**Theorem 7.1.** Let \( M \) be an open Riemann surface and let \( K \subset M \) be a compact smoothly bounded Runge domain in \( M \). Let \( u = (u^1, \ldots, u^n) : K \to \mathbb{R}^n \) be a conformal minimal immersion on a neighborhood of \( K \) and let \( p : H_1(M; \mathbb{Z}) \to \mathbb{R}^n \) be a group homomorphism satisfying \( p(C) = \text{Flux}_{u}(C) \) for every closed curve \( C \subset K \). Then, for any \( \epsilon > 0 \), there exists a nondegenerate conformal minimal immersion \( \tilde{u} = (\tilde{u}^1, \ldots, \tilde{u}^n) : M \to \mathbb{R}^n \) such that \( \sup_{x \in K} |\tilde{u}(x) - u(x)| < \epsilon \), \( \text{Flux}_{\tilde{u}} = p \). Furthermore, if \( n \geq 5 \), the approximating immersions \( \tilde{u} : M \to \mathbb{R}^n \) can be taken to be embeddings.

Theorem 7.1 was already proved for \( n = 3 \) by Alarcón and López in [5].

**Proof.** Let \( \rho : M \to \mathbb{R} \) be a smooth strongly subharmonic Morse exhaustion function and exhaust \( M \) by an increasing sequence \( M_1 \subset M_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} M_i = M \) of compact smoothly bounded Runge domains of the form \( M_i = \{ p \in M : \rho(p) \leq c_i \} \), where \( c_1 < c_2 < \cdots \) is an increasing sequence of regular values of \( \rho \) with \( \lim_{i \to \infty} c_i = +\infty \). Each domain \( M_i \) is a compact bordered Riemann surface, possibly disconnected. Assume that \( \rho \) has at most one critical point \( p_i \) in each difference \( M_{i+1} \setminus M_i \). Since \( K \) is Runge in \( M \), we may also assume that \( M_1 = K \).
Since $K$ is compact, we may assume up to a translation that $\max\{u_1, u_2\} > 1$ on $bK$. Further, by Theorem 3.1(a) we may assume that $u$ is nondegenerate (Def. 2.2).

We proceed by induction. The initial immersion is $u_1 = u \in \text{CMI}_s(M_1, \mathbb{R}^n)$. The inductive step is furnished by the following lemma.

**Lemma 7.2.** Let $i \in \mathbb{N}$ and let $u_i = (u_1^i, \ldots, u_n^i) \in \text{CMI}_s(M_i, \mathbb{R}^n)$ be a nondegenerate conformal minimal immersion such that

(I) $\text{Flux}_{u_i}(C) = p(C)$ for all closed curve $C \subset M_i$ and

(II) $\max\{u_1^i, u_2^i\} > i$ on $bM_i$.

Then $u_i$ can be approximated in the $C^1(M_i)$ topology by nondegenerate conformal minimal immersions $u_{i+1} = (u_1^{i+1}, \ldots, u_n^{i+1}) \in \text{CMI}_s(M_{i+1}, \mathbb{R}^n)$ such that

(i) $\text{Flux}_{u_{i+1}}(C) = p(C)$ for all closed curve $C \subset M_{i+1}$,

(ii) $\max\{u_1^{i+1}, u_2^{i+1}\} > i$ on $M_{i+1} \setminus M_i$, and

(iii) $\max\{u_1^{i+1}, u_2^{i+1}\} > i + 1$ on $bM_{i+1}$.

**Proof.** By Theorem 3.1(c) we may assume that $u_i$ extends as a conformal minimal immersion to an unspecified open neighborhood of $M_i$.

We consider two essentially different cases.

The noncritical case: Assume that $\rho$ has no critical value in $[e_i, e_{i+1}]$.

In this case there is no change of topology when passing from $M_i$ to $M_{i+1}$ and $M_i$ is a strong deformation retract of $M_{i+1}$. Denote by $m \in \mathbb{N}$ the number of boundary components of $bM_i$. It follows that $M_{i+1} \setminus M_i = \bigcup_{j=1}^m A_j$ where the sets $A_j$, $j = 1, \ldots, m$, are pairwise disjoint smoothly bounded compact annuli. For each $j \in \{1, \ldots, m\}$ write $bA_j = \alpha_j \cup \beta_j$ with $\alpha_j \subset bM_i$ and $\beta_j \subset bM_{i+1}$. In view of condition (II) in the statement of the lemma, there exists an integer $l \geq 3$ such that each $\alpha_j$ splits into $l$ compact subarcs $\alpha_{j,k}, k \in \mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$, satisfying the following conditions:

(a1) $\alpha_{j,k}$ and $\alpha_{j,k+1}$ intersect at a common endpoint $p_{j,k}$ and $\alpha_{j,k} \cap \alpha_{j,a} = \emptyset$ for all $a \in \mathbb{Z}_l \setminus \{k - 1, k, k + 1\}$, for all $(j, k) \in I := \{1, \ldots, m\} \times \mathbb{Z}_l$.

(a2) $\bigcup_{k \in \mathbb{Z}_l} \alpha_{j,k} = \alpha_j$ for all $j \in \{1, \ldots, m\}$.

(a3) There exist subsets $I_1, I_2$ of $I$ such that $I = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$, and $u_i^\sigma > i$ on $\alpha_{j,k}$ for all $(j, k) \in I_2$, $\sigma = 1, 2$.

For every $j = 1, \ldots, m$ let $\{\gamma_{j,k} \subset A_j : (j, k) \in I\}$ be a family of pairwise disjoint smooth Jordan arcs such that $\gamma_{j,k}$ connects $p_{j,k} \in \alpha_j$ with a point $q_{j,k} \in \beta_j$ and is otherwise disjoint with $bA_j$. We may assume in addition that the set

$$S = M_i \cup \bigcup_{(j,k)\in I} \gamma_{j,k}$$

is admissible in the sense of Def. 5.1. Recall that $M_i$ is Runge in $M$, and hence $S \subset M$ is Runge as well. Let $\theta$ be a nowhere vanishing holomorphic 1-form on $M$. Extend $(u_i, 2\partial u_i)$ to $S$ as a nondegenerate generalized conformal minimal immersion $(u_i, f \theta) \in \text{GCMI}_s(S, \mathbb{R}^n)$ satisfying that:

- $u_i^\sigma > i$ on $\gamma_{j,k} \cup \alpha_{j,k} \cup \gamma_{j,k-1}$ for all $(j, k) \in I_\sigma$, $\sigma = 1, 2$.
- $u_i^\sigma > i + 1$ on $\{q_{j,k}, q_{j,k-1}\}$ for all $(j, k) \in I_\sigma$, $\sigma = 1, 2$. 

The existence of such extension is trivially ensured by property (a3). Theorem 5.3 then provides $v = (v^1, \ldots, v^n) \in \text{CMI}_s(M_{i+1}, \mathbb{R}^n)$ enjoying the following properties:

(b1) $v$ is as close as desired to $u_i$ in the $\mathcal{C}^1(M_i)$ topology.
(b2) $v^\sigma > i$ on $\gamma_{j,k} \cup \alpha_{j,k} \cup \gamma_{j,k-1}$ for all $(j,k) \in I_\sigma, \sigma = 1, 2$.
(b3) $v^\sigma > i + 1$ on $\{q_{j,k}, q_{j,k-1}\}$ for all $(j,k) \in I_\sigma, \sigma = 1, 2$.
(b4) $\text{Flux}_{\alpha_i}(C) = \text{Flux}_{u_i}(C)$ for any closed curve $C \subset M_i$.

Denote by $\beta_{j,k}$ the subarc of $\beta_j$ connecting $q_{j,k-1}$ and $q_{j,k}$ and containing $q_{j,a}$ for no $a \in \mathbb{Z}_l \setminus \{k-1, k\}$, for all $(j,k) \in I$. Denote by $\Omega_{j,k} \subset \alpha_j$ the closed disc bounded by $\gamma_{j,k-1}, \alpha_{j,k}, \gamma_{j,k}$, and $\beta_{j,k}$, $(j,k) \in I$. By (b2), (b3) and the continuity of $v$ there exist compact, smoothly bounded discs $D_{j,k} \subset \Omega_{j,k} \setminus (\gamma_{j,k-1} \cup \alpha_{j,k} \cup \gamma_{j,k})$, $(j,k) \in I$, such that $D_{j,k} \cap \beta_{j,k} \neq \emptyset$ is a subarc of $\beta_{j,k}$ and the following conditions hold:

(b2’) $v^\sigma > i$ on $\overline{\Omega_{j,k} \setminus D_{j,k}}$ for all $(j,k) \in I_\sigma, \sigma = 1, 2$.
(b3’) $v^\sigma > i + 1$ on $\beta_{j,k} \setminus D_{j,k}$ for all $(j,k) \in I_\sigma, \sigma = 1, 2$.

Assume that $I_1 \neq \emptyset$, otherwise $I_2 \neq \emptyset$ and we would reason in a symmetric way.

Consider the compact smoothly bounded Runge domain

$$S_1 = M_1 \cup \left( \bigcup_{(j,k) \in I_2} \Omega_{j,k} \right) \cup \left( \bigcup_{(j,k) \in I_1} D_{j,k} \right).$$

Observe that $S_1$ is not connected; its components are $M_1 \cup \bigcup_{(j,k) \in I_2} \Omega_{j,k}$ and $D_{j,k}$, $(j,k) \in I_1$. Since $D_{j,k}$ is compact, there exists a constant $\tau_1 > 0$ such that

$$\tau_1 + v^2 > i + 1 \quad \text{on} \quad \bigcup_{(j,k) \in I_1} D_{j,k},$$

recall that $v = (v^1, v^2, \ldots, v^n)$.

Denote by $\hat{v}_1 = (\hat{v}_1^1, \hat{v}_1^2, \ldots, \hat{v}_1^n) \in \text{CMI}_s(S_1, \mathbb{R}^n)$ the conformal minimal immersion given by

$$\hat{v}_1 = v \quad \text{on} \quad M_1 \cup \bigcup_{(j,k) \in I_2} \Omega_{j,k},$$

$$\hat{v}_1 = (v^1, \tau_1 + v^2, \ldots, v^n) \quad \text{on} \quad \bigcup_{(j,k) \in I_1} D_{j,k}.$$

Observe that every component of $\hat{v}_1$ equals the restriction to $S_1$ of the corresponding component of $v$, except for $\hat{v}_1^2$. By Lemma 5.6 we may approximate $\hat{v}_1$ in the $\mathcal{C}^1(S_1)$ topology by a nondegenerate conformal minimal immersion $v_1 = (v_1^1, v_1^2, \ldots, v_1^n) \in \text{CMI}_s(M_{i+1}, \mathbb{R}^n)$ satisfying the following properties:

(c1) $v_1$ is as close as desired to $u_i$ in the $\mathcal{C}^1(M_i)$ topology.
(c2) $v_1^1 = v^1$ on $M_{i+1}$.
(c3) $v_1^a > i$ on $\bigcup_{(j,k) \in I_a} \Omega_{j,k} \setminus D_{j,k}, a = 1, 2$. Take into account (b2’), (7.2), and (c2).
(c4) $v_1^a > i + 1$ on $\bigcup_{(j,k) \in I_a} \beta_{j,k} \setminus D_{j,k}, a = 1, 2$. See (b3’), (7.2), and (c2).
(c5) $v_1^2 > i + 1$ on $\bigcup_{(j,k) \in I_1} D_{j,k}$. Take into account (7.2) and (7.1).
(c6) $\text{Flux}_{v_1}(C) = \text{Flux}_{u_i}(C)$ for any closed curve $C \subset M_i$. See (b4).
Assume that \( I_2 \neq \emptyset \); otherwise the immersion \( v_{i+1} = v_1 \) satisfies the conclusion of the lemma and we are done. Indeed, if \( I_2 = \emptyset \), then \( I_1 = I \) and we have

\[
M_{i+1} \setminus M_I = \bigcup_{(j,k) \in I_1} \Omega_{j,k} \setminus D_{j,k} \bigcup_{(j,k) \in I_2} D_{j,k}
\]

and

\[
bM_{i+1} \subset \bigcup_{(j,k) \in I_1} \beta_{j,k} \setminus D_{j,k} \bigcup_{(j,k) \in I_2} D_{j,k}.
\]

Therefore, properties (c3) and (c5) above imply Lemma 7.2 (ii), whereas (c4) and (c5) ensure (iii). Finally (c6), Lemma 7.2 (I), and the fact that \( M_i \) is a strong deformation retract of \( M_{i+1} \) give (iii). This and (c1) would conclude the proof.

Consider the compact Runge, smoothly bounded domain \( S_2 = M_2 \bigcup \bigcup_{(j,k) \in I_1} \Omega_{j,k} \bigcup_{(j,k) \in I_2} D_{j,k} \).

Since \( I_2 \neq \emptyset \), \( S_2 \) is not connected. Pick a constant \( \tau_2 > 0 \) such that

\[
\tau_2 + v_1 > i + 1 \quad \text{on} \bigcup_{(j,k) \in I_2} D_{j,k}.
\]

Define \( \hat{v}_2 = (\hat{v}_2^1, \hat{v}_2^2, \ldots, \hat{v}_2^n) \in \text{CMI}_n(S_1, \mathbb{R}^n) \) by

\[
\hat{v}_2 = v_1 \quad \text{on} \quad M_I \bigcup_{(j,k) \in I_1} \Omega_{j,k},
\]

\[
\hat{v}_2 = (\tau_2 + v_1, v_1^2, \ldots, v_1^n) \quad \text{on} \quad \bigcup_{(j,k) \in I_2} D_{j,k}.
\]

Now every component of \( \hat{v}_2 \) equals the restriction to \( S_2 \) of the corresponding component of \( v_1 \), except for \( \hat{v}_2^1 \). By Lemma 7.6 we may approximate \( \hat{v}_2 \) in the \( \mathcal{C}^1(S_2) \) topology by an immersion \( v_2 = (v_2^1, v_2^2, \ldots, v_2^n) \in \text{CMI}_n(M_{i+1}, \mathbb{R}^n) \) such that:

(d1) \( v_2 \) is as close as desired to \( u_i \) in the \( \mathcal{C}^1(M_i) \) topology.

(d2) \( v_2^1 = v_1^1 \) on \( M_{i+1} \).

(d3) \( v_2^a > i + 1 \) on \( \bigcup_{(j,k) \in I_a} \Omega_{j,k} \setminus D_{j,k} \), \( a = 1, 2 \). Take into account (c3), (7.5), and (d2).

(d4) \( v_2^a > i + 1 \) on \( \bigcup_{(j,k) \in I_a} \beta_{j,k} \setminus D_{j,k} \), \( a = 1, 2 \). See (c4), (7.5), and (d2).

(d5) \( v_2^a > i + 1 \) on \( \bigcup_{(j,k) \in I_a \setminus I_2} D_{j,k} \). Take into account (c5), (7.5), (7.6), and (7.4).

(d6) \( \text{Flux}_{u_1}(C) = \text{Flux}_{u_2}(C) \) for any closed curve \( C \subset M_i \). See (c6).

Set \( u_{i+1} = v_2 \in \text{CMI}_n(M_{i+1}, \mathbb{R}^n) \). Since obviously

\[
M_{i+1} \setminus M_I = \bigcup_{(j,k) \in I_1} \Omega_{j,k} \setminus D_{j,k} \bigcup_{(j,k) \in I_2} D_{j,k}
\]

and

\[
bM_{i+1} \subset \bigcup_{(j,k) \in I_1} \beta_{j,k} \setminus D_{j,k} \bigcup_{(j,k) \in I_2} D_{j,k},
\]

(d3) and (d5) ensure condition (ii) in the lemma, (d4) and (d5) give (iii), and (d6), Lemma 7.2 (I), and the fact that \( M_i \) is a strong deformation retract of \( M_{i+1} \) imply (i). Taking into account (d1), this concludes the proof of the lemma in the noncritical case.

The critical case: Assume that \( \rho \) has a critical point \( p_{i+1} \in M_{i+1} \setminus M_i \).
By the assumptions on \( \rho \), \( p_{i+1} \) is the only critical point of \( \rho \) on \( M_{i+1} \setminus M_i \) and it is a Morse point of Morse index either 0 or 1.

Assume first that the Morse index of \( p_{i+1} \) is 0. In this case a new (simply connected) component of the sublevel set \( \{ \rho \leq r \} \) appears at \( p_{i+1} \) when \( r \) passes the value \( \rho(p_{i+1}) \). We then reduce the proof of the lemma to the noncritical case by defining \( u_{i+1} = (u_{i+1}^1, \ldots, u_{i+1}^n) \) on this new component, \( D \), as any nondegenerate conformal minimal immersion with \( \max\{u_{i+1}^1, u_{i+1}^2\} > i + 1 \) on \( D \).

Assume now that the Morse index of \( p_{i+1} \) is 1. In this case the change of topology of the sublevel set \( \{ \rho \leq r \} \) at \( p_{i+1} \) is described by attaching to \( M_i \) a smooth arc \( \gamma \subset M_{i+1} \setminus M_i \), and hence \( M_i \cup \gamma \) is a Runge strong deformation retract of \( M_{i+1} \). We assume without loss of generality that \( M_i \cup \gamma \) is admissible in the sense of Def. 5.1. In view of Lemma 7.2 (I) and (II), we may assume that \( u_i \to u_{i+1} \) in \( \text{CMI}_n(M_i) \) and \( f_\theta \) is a nondegenerate generalized conformal minimal immersion \( (u_i, f_\theta) \in \text{GCM}_n(M_i \cup \gamma, \mathbb{R}^n) \) such that \( u_i = u_i' \) on \( M_i \), \( f_\theta \) is a Runge strong deformation retract of \( M_{i+1} \), and \( \max\{u_i^1, u_i^2\} > i \) on \( \gamma \). By Theorem 5.3 we find a Runge compact, smoothly bounded domain \( \tilde{M}_i \) and a nondegenerate conformal minimal immersion \( v = (v^1, \ldots, v^n) \in \text{CMI}_n(\tilde{M}_i, \mathbb{R}^n) \) such that

- \( M_i \cup \gamma \subset \tilde{M}_i \) and \( \tilde{M}_i \) is a strong deformation retract of \( M_{i+1} \),
- \( \text{Flux}_{v_i}(C) = p(C) \) for every closed curve \( C \subset \tilde{M}_i \), and
- \( \max\{v^1, v^2\} > i \) on \( \tilde{M}_i \setminus M_i \).

This reduces the proof to the noncritical case and proves the lemma.

By recursively applying Lemma 7.2 we may construct a sequence of nondegenerate conformal minimal immersions \( \{u_i \in \text{CMI}_n(M_i, \mathbb{R}^n)\}_{i \in \mathbb{N}} \) such that:

(a) \( u_i \) is as close to \( u \) as desired in the \( C^1(K) \) topology for all \( i \in \mathbb{N} \).
(b) \( u_i \) is as close to \( u_{i-1} \) as desired in the \( C^1(M_{i-1}) \) topology for all \( i \geq 2 \).
(c) \( \text{Flux}_{u_i}(C) = p(C) \) for every closed curve \( C \subset M_i \) for all \( i \in \mathbb{N} \).
(d) \( \max\{u_{i+1}^1, u_{i+1}^2\} > i \) on \( M_{i+1} \setminus M_i \) for all \( i \in \mathbb{N} \).
(e) \( \max\{u_{i+1}^1, u_{i+1}^2\} > i + 1 \) on \( \partial M_{i+1} \) for all \( i \in \mathbb{N} \).

Furthermore, if \( n \geq 5 \), applying Theorem 4.1 at each step in the recursive construction we may assume that

(f) \( u_i \) is an embedding for every \( i \in \mathbb{N} \).

If the approximations in (a) and (b) are close enough, then the limit \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n) := \lim_{i \to \infty} u_i : M \to \mathbb{R}^n \) is a nondegenerate conformal minimal immersion satisfying the conclusion of the theorem. Indeed, (c) trivially implies that \( \text{Flux}_{\tilde{u}} = p \), whereas properties (d) and (e) ensure that \( \max\{\tilde{u}_1, \tilde{u}_2\} : M \to \mathbb{R} \) and hence \( (\tilde{u}_1, \tilde{u}_2) : M \to \mathbb{R}^2 \) are proper maps. Finally, if \( n \geq 5 \), \( \tilde{u} \) can be taken an embedding; take into account (f) and see for instance the proof of Theorem 4.5 in [7] for a similar argument.

This concludes the proof of Theorem 7.1.

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Embedded minimal surfaces in $\mathbb{R}^n$

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