MIXED INEQUALITIES FOR OPERATORS ASSOCIATED TO CRITICAL RADIUS FUNCTIONS WITH APPLICATIONS TO SCHRÖDINGER TYPE OPERATORS

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Abstract. We obtain weighted mixed inequalities for operators associated to a critical radius function. We consider Schrödinger Calderón-Zygmund operators of \((s, \delta)\) type, for \(1 < s \leq \infty\) and \(0 < \delta \leq 1\). We also give estimates of the same type for the associated maximal operators. As an application, we obtain a wide variety of mixed inequalities for Schrödinger type singular integrals.

As far as we know, these results are a first approach of mixed inequalities in the Schrödinger setting.

1. Introduction

One of the most classical results in Harmonic Analysis is the characterization of all the measurable and nonnegative functions \(w\) for which the Hardy-Littlewood maximal operator \(M\) maps \(L^p(w)\) into \(L^p(w)\), for \(1 < p < \infty\). B. Muckenhoupt solved this problem in [18], showing that \(M\) is bounded in \(L^p(w)\) if and only if \(w\) belongs to the \(A_p\) class. Later on, this result was extended to the general context of spaces of homogeneous type (see, for example, [13] and [17]).

In 1985, E. Sawyer proved in [21] that if \(u\) and \(v\) are \(A_1\) weights then the inequality

\[
   uv \left( \left\{ x \in \mathbb{R} : \frac{M(fw)(x)}{v(x)} > t \right\} \right) \leq C t \int_{\mathbb{R}} |f| uv
\]

holds for every positive \(t\). This result can be seen as the weak \((1, 1)\) type of the operator \(Sf = M(fv)/v\) with respect to the measure \(d\mu(x) = u(x)v(x)\, dx\). The proof of this inequality is highly non-trivial and involves a very subtle decomposition of dyadic intervals into a family with certain properties, called "principal intervals". The idea is an adaptation of a technique that appears in [19]. It is immediate that (1) generalizes the well-known fact that \(M : L^1(u) \to L^{1,\infty}(u)\) when \(u \in A_1\). On the other hand, one of the motivations to establish and prove (1) is that it allows to show, in an alternative way, the fact that \(M\) is bounded in \(L^p(w)\) when \(w \in A_p\), by combining Jones’ factorization theorem with Marcinkiewicz’s interpolation result. The main difficulty that appears in proving the inequality above is that the classical covering lemmas do not apply for the operator \(S\), which is a perturbation of \(M\) via an \(A_1\) weight. Moreover, the product \(uv\) may be very singular. Indeed, if we take \(u = v = |x|^{-1/2}\) then \(uv\) is not even locally integrable. These reasons do not allow to apply classical techniques to solve the problem.

Later, in [14], D. Cruz Uribe, J. M. Martell and C. Pérez proved some extensions of inequality (1) to higher dimensions and also for other operators. Particularly, they proved that if \(u\) and \(v\) are weights that satisfy \(u \in A_1\) and \(v \in A_\infty(u)\), then the inequality

\[
   uv \left( \left\{ x \in \mathbb{R}^d : \frac{|T(fw)(x)|}{v(x)} > t \right\} \right) \leq C t \int_{\mathbb{R}^d} |f| uv
\]

holds for every positive \(t\), where \(T\) is either the Hardy-Littlewood maximal function or a Calderón-Zygmund operator (CZO). The main idea in that paper is to obtain the desired estimate for \(M_D\),
the dyadic Hardy-Littlewood maximal operator and then, by using an extrapolation result, derive the corresponding estimate for $M$ and CZOs. The condition on the weights guarantees that the product $uv$ is an $A_\infty$ weight and therefore some classical techniques, such as Calderón-Zygmund decomposition, can be applied to achieve the estimates.

We will refer to these type of estimates as mixed inequalities. Similar results for more general operators were also studied in the literature (see for example [3] and [5] for commutators of CZO, [4] for fractional operators, [2] for generalized maximal functions and [6] for mixed inequalities involving Fefferman-Stein estimates).

In this paper we establish and prove mixed inequalities for classes of operators and weights associated to a critical radius function. More precisely, we will consider the space $\mathbb{R}^d$ equipped with a function $\rho : \mathbb{R}^d \to (0, \infty)$ whose variation is controlled by the existence of $C_0$ and $N_0 \geq 1$ such that for every $x, y \in \mathbb{R}^d$

$$C_0^{-1} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq C_0 \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{N_0+1}. \tag{3}$$

This functions and its associated operators appear naturally when dealing with a Schrödinger operator $L = -\Delta + V$ as we shall discuss in § 4. It is worth noting that if $\rho$ is a critical radius function, then for any $\gamma > 0$ the mapping $\gamma \rho$ is also a critical radius function. Moreover, if $0 < \gamma \leq 1$ then $\gamma \rho$ satisfies (3) with the same constants as $\rho$.

Given a locally integrable function $f$ and $\sigma \geq 0$, the Hardy-Littlewood maximal operator $M^{\rho, \sigma}f$ is defined by

$$M^{\rho, \sigma}f(x) = \sup_{Q(x_0, r_0) \ni x} \left(1 + \frac{r_0}{\rho(x_0)}\right)^{-\sigma} \left(\frac{1}{|Q|} \int_Q |f(y)| \, dy\right), \tag{4}$$

where $Q(x_0, r_0)$ stands for the cube with sides parallel to the coordinate axes centered at $x_0$ and radius $r_0$, that is, $r_0 = \sqrt{d} \ell(Q)/2$. Notice that $M^{\rho,0} = M$, the classical Hardy-Littlewood maximal function. This family of maximal operators is an adapted version of the Hardy-Littlewood maximal operator to the Schrödinger context and it is connected to the corresponding Muckenhoupt classes of weights $A^\rho$ associated to a critical radius function (see Proposition 3 in [12]).

Our first result involves a mixed type inequality for $M^{\rho, \sigma}$. The weights involved in the estimate are a generalization of the classical Muckenhoupt classes and are associated to a critical radius function $\rho$ (see § 2 for the definition).

**Theorem 1.** Let $u \in A^\rho_t$ and $v \in A^\rho_v(u)$. Then there exists $\sigma \geq 0$ such that the inequality

$$uv \left\{ x \in \mathbb{R}^d : \frac{M^{\rho, \sigma}(fv)(x)}{v(x)} > t \right\} \leq \frac{C}{t} \int f(x)u(x)v(x) \, dx$$

holds for every positive $t$ and every bounded function with compact support.

We shall also be dealing with singular integral operators related to a critical radius function $\rho$. For $0 < \delta \leq 1$ we shall say that a linear operator $T$ is a Schrödinger-Calderón-Zygmund operator (SCZO) of $(\infty, \delta)$ type if

(I) $T$ is bounded from $L^1$ into $L^{1,\infty}$;

(II) $T$ has an associated kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, in the sense that

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y) \, dy, \quad f \in L^\infty_c \text{ and a.e. } x \notin \text{supp}f.$$

(III) for each $N > 0$ there exists a constant $C_N$ such that
\begin{equation}
|K(x, y)| \leq \frac{C_N}{|x - y|^d} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N}, \quad x \neq y,
\end{equation}
and there exists $C$ such that
\begin{equation}
|K(x, y) - K(x, y_0)| \leq C \frac{|y - y_0|^\delta}{|x - y|^{d+s}}, \quad \text{when } |x - y| > 2|y - y_0|.
\end{equation}

Remark 1. It is known that if $T$ is a SCZO of $(\infty, \delta)$ type then it is bounded on $L^p(w)$ for $1 < p < \infty$ as long as $w \in A_\infty^p$ (see § 2 for the definition) and it is of weak type $(1, 1)$ with respect to $w$ as long as $w \in A_1^p$ (see [8, Theorem 1], [12, Theorem 5 and Proposition 5] and [7, Theorem 3.6]).

Our main result dealing with this type of operators is contained in the following theorem.

**Theorem 2.** Let $\rho$ be a critical radius function, $u \in A_1^p$ and $v \in A_\infty^p(u)$. If $0 < \delta \leq 1$ and $T$ is a SCZO of $(\infty, \delta)$ type, then the inequality
\[
uw \left( \left\{ x \in \mathbb{R}^d : \frac{|Tfv(x)|}{v(x)} > t \right\} \right) \leq C \frac{1}{t} \int_{\mathbb{R}^d} |f(x)|u(x)v(x)dx
\]
holds for every positive $t$ and every bounded function with compact support.

We shall also consider a wider class of operators with kernels satisfying another type of regularity. For $1 < s < \infty$ and $0 < \delta \leq 1$, we shall say that a linear operator $T$ is a Schrödinger-Calderón-Zygmund operator (SCZO) of $(s, \delta)$ type if
1. $T$ is bounded on $L^p$ for $1 < p < s$.
2. $T$ has an associated kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, in the sense that
\[
Tv(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy, \quad f \in L^s_c \quad \text{and} \quad x \notin \text{supp}f.
\]

Further, for each $N > 0$ there exists a constant $C_N$ such that
\begin{equation}
\left( \int_{R^d \setminus |x_0 - x| < 2R} |K(x, y)|^s \, dx \right)^{1/s} \leq C_N R^{-d/s} \left(1 + \frac{R}{\rho(x_0)}\right)^{-N},
\end{equation}
for $|y - x_0| < R/2$, and there exists $C$ such that
\begin{equation}
\left( \int_{R^d \setminus |x - y_0| < 2R} |K(x, y) - K(x, y_0)|^s \, dx \right)^{1/s} \leq CR^{-d/s} \left(\frac{r}{R}\right)^\delta,
\end{equation}
for $|y - y_0| < r \leq \rho(y_0)$, $r < R/2$.

Remark 2. It is known that if $T$ is a SCZO of $(s, \delta)$ type for some $s > 1$, then it is bounded on $L^p(w)$ for $1 < p < s$ as long as $w^{1-p'} \in A_{\rho/s'}^p$ and it is of weak type $(1, 1)$ with respect to $w$ as long as $w^{s'} \in A_1^p$ (see [8, Theorem 1], [12, Theorem 6 and Proposition 6] and [7, Theorem 3.6]).

Our result for these type of operators is the following.

**Theorem 3.** Let $\rho$ be a critical radius function, $1 < s < \infty$ and $0 < \delta \leq 1$. Let $T$ be a SCZO of $(s, \delta)$ type. If $u$ is a weight verifying $u^{s'} \in A_1^p$ and $v \in A_\infty^p(u^\delta)$ for some $\beta > s'$, then the inequality
\[
uw \left( \left\{ x \in \mathbb{R}^d : \frac{|Tfv(x)|}{v(x)} > t \right\} \right) \leq C \frac{1}{t} \int_{\mathbb{R}^d} |f(x)|u(x)v(x)dx
\]
holds for every positive $t$. 

Remark 3. Notice that we require a bit stronger assumption on \( v \) for this theorem to hold. It is easy to see that the hypothesis on \( u \) and \( v \) in this theorem imply the corresponding ones in Theorem 2 by virtue of Proposition 3 (see § 2).

Theorem 2 and Theorem 3 extend the known weak \((1,1)\) type of \( T \) with \( A^p_1 \) weights when we take \( v = 1 \) (see Theorem 3.6 in [7]).

The article is organized as follows. In § 2 we give the definitions for the reading and we state and prove some auxiliary results required for the main proofs, contained in § 3. Finally in § 4 we give mixed inequalities for operators in the Schrödinger setting, as an application.

2. Preliminaries and auxiliary results

We begin by introducing the basic definitions involved in our estimates. Throughout the article \( \rho : \mathbb{R}^d \to (0, \infty) \) will denote a fixed critical radius function. By a weight \( w \) we understand a function that is locally integrable and verifies \( 0 < w(x) < \infty \) almost everywhere.

We shall now introduce the classes of weights involved in our estimates. These are an extension of the classical Muckenhoupt \( A_p \) classes and were first defined by B. Bongioanni, E. Harboure and O. Salinas in [11].

Let \( u \) be a weight. For \( 1 < p < \infty \) and \( \theta \geq 0 \), we say that \( w \in A^{p,\theta}_1(u) \) if the inequality

\[
\left( \frac{1}{u(Q)} \int_Q w u \right)^{1/p} \left( \frac{1}{u(Q)} \int_Q w^{1-p'} u \right)^{1/p'} \leq C \left( 1 + \frac{r}{\rho(x)} \right)^\theta
\]

holds for every cube \( Q = Q(x, r) \) and \( C \) independent of \( Q \). We recall that we consider cubes with sides parallel to the coordinate axes, \( \ell(Q) \) denotes the length of the sides of \( Q \) and \( Q(x, r) \) denotes the cube centered in \( x \) with radius \( r \), that is, \( r = \sqrt{d} \ell(Q)/2 \). This notation will be used throughout the article.

Similarly, \( w \in A^{p,\theta}_1(u) \) if

\[
\frac{1}{u(Q)} \int_Q w u \leq C \left( 1 + \frac{r}{\rho(x)} \right)^\theta \inf_Q w,
\]

for every cube \( Q \). The smallest constants in the corresponding inequalities above will be denoted by \([w]_{A^{p,\theta}_1(u)}\). For \( 1 \leq p < \infty \), the \( A^p_0 \) class is defined as the collection of all the \( A^{p,\theta}_p \) classes for \( \theta \geq 0 \), that is

\[
A^p_0(u) = \bigcup_{\theta \geq 0} A^{p,\theta}_p(u).
\]

We also define

\[
A^\infty_p(u) = \bigcup_{p \geq 1} A^p_0(u).
\]

There are many ways to characterize the class above. A weight \( w \) belongs to \( A^\infty_\infty(u) \) if there exist \( \theta \geq 0 \) and \( \varepsilon > 0 \) such that the inequality

\[
\frac{wu(E)}{wu(Q)} \leq C \left( 1 + \frac{r}{\rho(x)} \right)^\theta \left( \frac{w(E)}{w(Q)} \right)^\varepsilon
\]

holds for every cube \( Q \) and every measurable subset \( E \) of \( Q \).

When \( u = 1 \) we shall simply denote \( A^p_0(u) = A^p_0 \) and \( A^{p,\theta}_p(u) = A^{p,\theta}_p, 1 \leq p \leq \infty \) and \( \theta \geq 0 \).

A very important property of \( A^p_0 \) weights is that they verify a reverse Hölder inequality. More precisely, given \( \theta \geq 0 \) and \( 1 < s < \infty \), we say that a weight \( w \) belongs to the reverse Hölder class \( RH^s_\theta \) if there
exists a positive constant $C$ such that for every cube $Q$ the inequality
\begin{equation}
\left( \frac{1}{|Q|} \int_Q w^s \right)^{1/s} \leq C \left( 1 + \frac{r}{\rho(x)} \right)^\theta \left( \frac{1}{|Q|} \int_Q w \right),
\end{equation}
holds. When $s = \infty$, we say $w \in \text{RH}^p_{\infty, \theta}$ if
\begin{equation}
\sup_Q w \leq C \left( 1 + \frac{r}{\rho(x)} \right)^\theta \left( \frac{1}{|Q|} \int_Q w \right)
\end{equation}
holds for every ball $Q$ and $C$ independent of $Q$. The smallest constant $C$ for which these estimates hold will be denoted by $[w]_{\text{RH}^p_s}$.

As in the case of $A_p^\infty$ classes, we define
\[ \text{RH}^p_s = \bigcup_{\theta \geq 0} \text{RH}^p_{s, \theta}, \quad 1 < s \leq \infty. \]

The next results give some useful properties of weights belonging to $A_p^\infty$ and $\text{RH}^p_s$ classes, generalizing some classical versions which deal with usual $A_p$ and $\text{RH}_s$ classes.

In order to do so, we shall introduce an auxiliary operator to establish some properties of weights that will be useful in our main results. For $\theta \geq 0$ we define the *Hardy-Littlewood minimal operator* to be
\begin{equation}
\mathcal{M}^{\rho, \theta} f(x) = \inf_{Q(x_0, r_0) \ni x} \left( 1 + \frac{r_0}{\rho(x_0)} \right)^\theta \left( \frac{1}{|Q|} \int_Q |f(y)| \, dy \right)
\end{equation}
The operators $M^{\rho, \theta}$ and $\mathcal{M}^{\rho, \theta}$ are closely related to weights belonging to $A_1^{\rho, \theta}$ and $\text{RH}^{\infty, \theta}$, respectively. This relation is established and proved in Lemma 1 (see §2).

The next lemma establishes a characterization for weights belonging to $A_1^{\rho, \theta}$ and $\text{RH}^{\infty, \theta}$. The proof is straightforward so we will omit it.

**Lemma 1.** Let $w$ be a weight and $\theta \geq 0$. Then:
(a) $w \in A_1^{\rho, \theta}$ if and only if there exists a positive constant $C$ such that $M^{\rho, \theta} w(x) \leq Cw(x)$, for almost every $x$;
(b) $w \in \text{RH}^{\infty, \theta}$ if and only if there exists $C > 0$ such that the inequality $w(x) \leq C \mathcal{M}^{\rho, \theta} w(x)$ holds for almost every $x$.

The lemma below states some properties of $A_p^\infty$ weights. These properties are well-known in the literature. A proof can be found in [11] and [9].

**Lemma 2.** Let $\theta \geq 0$ and $1 < p < \infty$. Then:
(a) if $u \in A_p^\rho$, then $u^{1-p'} \in A_p^\rho$;
(b) if $u$ and $v$ belong to $A_p^\rho$, then $w^{1-p} \in A_p^\rho$;
(c) if $w \in A_p^\rho$, there exist weights $u$ and $v$ in $A_1^\rho$ that verify $w = uv^{1-p}$.

The following proposition gives us another factorization property of $A_p^\infty$ weights. The corresponding result for classical Muckenhoupt and reverse Hölder classes was proved in [15].

**Proposition 1.** Let $s > 1$, $1 < p < \infty$ and $w \in A_p^\infty \cap \text{RH}_s^\rho$. Then there exist weights $w_1$ and $w_2$ such that $w = w_1 w_2$, with $w_1 \in A_1^\rho \cap \text{RH}^\rho_s$ and $w_2 \in A_p^\rho \cap \text{RH}^\rho_\infty$.

The proof of this result will follow from the next two lemmas, which are extensions of classical properties given also in [15].

**Lemma 3.** Let $s, p > 1$ and $q$ defined by $q = s(p-1) + 1$. Then $w \in A_p^\rho \cap \text{RH}^\rho_s$ if and only if $w^s \in A_q^\rho$. 
Proof. Let us first assume that \( w \in A^q_p \cap \text{RH}^p_q \). Then there exist two nonnegative numbers \( \theta_1 \) and \( \theta_2 \) such that \( w \in \text{RH}^p_{s,\theta_1} \cap A^p_{s,\theta_2} \). By using the relation between \( p, q \) and \( s \) we have, for every cube \( Q = Q(x, r) \)
\[
\left( \frac{1}{|Q|} \int_Q w^s \right)^{1/q} \left( \frac{1}{|Q|} \int_Q w^{s(1-q')} \right)^{1/q'} = \left( \frac{1}{|Q|} \int_Q w^s \right)^{1/q} \left( \frac{1}{|Q|} \int_Q w^{1-p} \right)^{(q-1)/(qs)}
\leq [w]^{s/q}_{\text{RH}^p_{s,\theta_1}} \left( \frac{1}{|Q|} \int_Q w^{s(q-1)/q} \right)^{(1/q)(p-1)/q}
\times \left( \frac{1}{|Q|} \int_Q w^{1-p} \right)^{(q-1)/q}
\leq [w]^{s/q}_{\text{RH}^p_{s,\theta_1}} [w]^{ps/q}_{A^p_{s,\theta_2}} \left( 1 + \frac{r}{\rho(x)} \right)^{ps\theta_2/q + s\theta_1/q},
\]
which implies that \( w^s \in A^q_{s,\theta_0} \subseteq A^q_{s,\theta} \), where \( \theta_0 = ps\theta_2/q + s\theta_1/q \).

Conversely, assume that \( w^s \in A^q_{s,\theta} \). Then there exists \( \theta_1 \geq 0 \) such that \( w^s \in A^q_{s,\theta_1} \). By Jensen’s inequality we obtain that
\[
\left( \frac{1}{|Q|} \int_Q w^s \right)^{1/p} \left( \frac{1}{|Q|} \int_Q w^{1-p} \right)^{1/p'} \leq \left( \frac{1}{|Q|} \int_Q w^s \right)^{1/(ps)} \left( \frac{1}{|Q|} \int_Q w^{(1-q')}^{s(1-q')} \right)^{(q-1)/(sp)}
\leq \left[ \left( \frac{1}{|Q|} \int_Q w^s \right)^{1/q} \left( \frac{1}{|Q|} \int_Q w^{(1-q')} \right)^{1/q} \right]^{q/(sp)}
\leq [w]^{q/(sp)}_{A^q_{s,\theta_1}} \left( 1 + \frac{r}{\rho(x)} \right)^{q\theta_1/(sp)}.
\]
that is, \( w \in A^p_{s,\theta_1/(sp)} \subseteq A^p_{s,\theta} \). In order to prove that \( w \in \text{RH}^p_{s,\theta} \), notice that
\[
\frac{1}{|Q|} \int_Q w^s = \left( \frac{1}{|Q|} \int_Q w^s \right) \left( \frac{1}{|Q|} \int_Q w^{s(1-q')} \right)^{q-1} \left( \frac{1}{|Q|} \int_Q w^{s(1-q')} \right)^{q-1}
\leq [w^s]^{q}_{A^q_{s,\theta_1}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 q} \left( \frac{1}{|Q|} \int_Q w^{(1-p)} \right)^{(1-p)q}
\leq [w^s]^{q}_{A^q_{s,\theta_1}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 q} \left( \frac{1}{|Q|} \int_Q w \right)^{q}.
\]
This implies that
\[
\left( \frac{1}{|Q|} \int_Q w \right)^{1/s} \leq [w^s]^{q/s}_{A^q_{s,\theta_1}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 q/s} \left( \frac{1}{|Q|} \int_Q w \right)
\]
and therefore \( w \in \text{RH}^p_{s,\theta_1 q/s} \subseteq \text{RH}^p_{s,\theta} \). \( \Box \)

Remark 4. The previous result remains true when \( p = 1 \). If \( u \) is a weight in \( A^1_{s,\theta} \), then there exists \( s > 1 \) such that \( u \in \text{RH}^p_{s,\theta} \). Therefore, this lemma establishes that \( u^s \in A^q_{s,\theta} \). In conclusion, if \( u \in A^1_{s,\theta} \) then there exists \( \varepsilon_0 > 0 \) such that \( u^{1+\varepsilon} \in A^1_{s,\theta} \), for every \( 0 < \varepsilon \leq \varepsilon_0 \).

Lemma 4. Let \( w \in A^p_{s,\theta} \). Then \( w^{1-p} \in A^p_{s,\theta} \cap \text{RH}^{p}_{\infty} \), for every \( 1 < p < \infty \).

Proof. The fact that \( w^{1-p} \in A^p_{s,\theta} \) follows from item (b) in Lemma 2. To prove that \( w^{1-p} \in \text{RH}^{p}_{\infty} \) we shall see that there exist \( C > 0 \) and \( \theta \geq 0 \) such that \( w^{1-p}(x) \leq C M^{p,\theta} w^{1-p}(x) \), for almost every \( x \). Then, the conclusion follows immediately from item (b) of Lemma 1.
Since $w \in A_1^0$, there exists $\theta \geq 0$ such that $w \in A_1^{\rho, \theta}$. For $x \in \mathbb{R}^d$ and every cube $Q = Q(x_0, r_0)$ that contains $x$ we have

\[
1 \leq \left( \frac{1}{|Q|} \int_Q w \right)^{1/p'} \left( \frac{1}{|Q|} \int_Q w^{1-p} \right)^{1/p} \left( 1 + \frac{r_0}{\rho(x_0)} \right)^{-\theta/p'} \left( 1 + \frac{r_0}{\rho(x_0)} \right)^{\theta/p'} \leq \left( M^{\rho, \theta} w(x) \right)^{1/p'} \left( \frac{\rho}{\rho(x_0)} \right)^{(p'/p) - 1} \left( \frac{1}{|Q|} \int_Q w^{1-p} \right)^{1/p}.
\]

By taking the infimum over such cubes we obtain

\[
1 \leq \left( M^{\rho, \theta} w(x) \right)^{1/p'} \left( M^{\rho, \theta} w^{1-p}(x) \right)^{1/p},
\]

or equivalently

\[
1 \leq \left( M^{\rho, \theta} w(x) \right) \left( M^{\rho, \theta} w^{1-p}(x) \right)^{p'/p},
\]

where $\hat{\theta} = \theta/(p' - 1)$. This inequality combined with Lemma 1 allow us to conclude that there exists $C > 0$ such that

\[
w^{-1/(p' - 1)}(x) \leq C M^{\rho, \hat{\theta}} w^{1-p}(x),
\]

or

\[
w^{1-p}(x) \leq C M^{\rho, \hat{\theta}} w^{1-p}(x),
\]

which implies, again by item (b) in Lemma 1, that $w^{1-p} \in \text{RH}_{\rho, \hat{\theta}} \subseteq \text{RH}_{\rho}^\infty$. \qed

As an immediate consequence of this lemma we have the following result.

**Corollary 1.** If $w \in A_1^0$ then $w^{-r} \in \text{RH}_{\rho}^\infty$, for every $r > 0$.

We now proceed with the proof of Proposition 1.

**Proof of Proposition 1.** By virtue of Lemma 3, we have that $w^s \in A_1^0$. From item (c) in Lemma 2, there exist two weights $v_1$ and $v_2$ in $A_1^0$ such that $w^s = v_1 v_2^{1-q}$. Then $w = v_1^{1/s} v_2^{(1-q)/s}$. We define $w_1 = v_1^{1/s}$ and $w_2 = v_2^{(1-q)/s} = v_2^{1-p}$.

Observe that $w_1 \in A_1^0$, since $v_1 \in A_1^0$. Thus

\[
\left( \frac{1}{|Q|} \int_Q w_1^s \right)^{1/s} = \left( \frac{1}{|Q|} \int_Q v_1 \right)^{1/s} \leq \left[ v_1 \right]_{A_1^{\rho, \theta_1}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1/s} \inf_Q \left( v_1 \right)^{1/s} \leq \left[ v_1 \right]_{A_1^{\rho, \theta_1}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1/s} \inf_Q \left( v_1 \right)^{1/s} \leq \left[ v_1 \right]_{A_1^{\rho, \theta_1}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1/s} \left( \frac{1}{|Q|} \int_Q w_1 \right),
\]

for every cube $Q$. This establishes that $w_1 \in \text{RH}_{\rho}^{\theta_1/s}$ and consequently $w_1 \in A_1^0 \cap \text{RH}_{\rho}^\infty$.

On the other hand, Lemma 4 yields $w_2 \in A_1^0 \cap \text{RH}_{\rho}^\infty$. This completes the proof. \qed

The following properties will be useful in the sequel.
Lemma 5. Let $1 < p < \infty$. Let $u$ and $v$ two weights that verify $u \in A^p_\rho$ and $v \in A^p_\rho$. Then, there exist $C > 0$ and $\theta \geq 0$ such that

$$
\left( \frac{1}{|Q|} \int_Q u \right)^{1/p} \left( \frac{1}{|Q|} \int_Q v \right)^{1/p'} \leq C \left( \frac{1}{|Q|} \int_Q u^{1/p} v^{1/p'} \right)^{(1 + \frac{r}{\rho(x)})^\theta},
$$

for every cube $Q$.

Proof. By hypothesis, there exist two nonnegative numbers $\theta_1$ and $\theta_2$ such that

$$
\left( \frac{1}{|Q|} \int_Q u \right)^{1/p} \left( \frac{1}{|Q|} \int_Q u^{1-p} \right)^{1/p'} \leq [u]_{A^{p,\rho_1}_\rho} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1}
$$

and

$$
\left( \frac{1}{|Q|} \int_Q v \right)^{1/p'} \left( \frac{1}{|Q|} \int_Q v^{1-p} \right)^{1/p} \leq [v]_{A^{p,\rho_2}_\rho} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_2}.
$$

By multiplying these two inequalities, we get

(15) \quad \left( \frac{1}{|Q|} \int_Q u \right)^{1/p} \left( \frac{1}{|Q|} \int_Q v \right)^{1/p'} \leq C \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2} \left( \frac{1}{|Q|} \int_Q u^{1-p} \right)^{-1/p'} \left( \frac{1}{|Q|} \int_Q v^{1-p} \right)^{-1/p},

where $C = [u]_{A^{p,\rho_1}_\rho} [v]_{A^{p,\rho_2}_\rho}$. On the other hand, by Hölder inequality we have

(16) \quad \frac{1}{|Q|} \int_Q u^{-1/p} v^{-1/p'} \leq \left( \frac{1}{|Q|} \int_Q u^{1-p} \right)^{1/p'} \left( \frac{1}{|Q|} \int_Q v^{1-p} \right)^{1/p}.

By combining (15) and (16) with Jensen inequality we obtain

$$
\left( \frac{1}{|Q|} \int_Q u \right)^{1/p} \left( \frac{1}{|Q|} \int_Q v \right)^{1/p'} \leq C \left( \frac{1}{|Q|} \int_Q u^{1-p} \right)^{-1/p'} \left( \frac{1}{|Q|} \int_Q v^{1-p} \right)^{-1/p'} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2} \left( \frac{1}{|Q|} \int_Q u^{1-p} v^{-1/p'} \right)^{1/p}.
$$

Proposition 2. Let $w$ be a weight. Then $w \in \text{RH}^p_{s}$ if and only if $w^s \in A^p_{\infty}$.

Proof. Assume that $w \in \text{RH}^p_{s}$. Then there exists $\theta_1 \geq 0$ such that $w \in \text{RH}^{p,\theta_1}_{s}$. If we prove that $\text{RH}^p_{s} \subseteq A^p_{\infty}$, then there would exist $p_0 > 1$ such that $w \in A^{p_0,\theta_1}_{p_0} \cap \text{RH}^p_{s}$. Lemma 3 establishes that $w^s \in A^{p_0,\theta_1}_{p_0}$, where $q_0 = s(p_0 - 1) + 1$. This would imply that $w^s \in A^p_{\infty}$. Therefore, let us prove that $\text{RH}^p_{s} \subseteq A^p_{\infty}$. Fix a cube $Q$ and $E$ a measurable subset of $Q$. We have

$$
w(E) \leq \left( \int_E w^s \right)^{1/s} |E|^{1/s'} \leq \left( \frac{1}{|Q|} \int_Q w^s \right)^{1/s} |Q|^{1/s} |E|^{1/s'},
$$

$$
\leq [w]_{\text{RH}^s_{\theta_1}} w(Q) \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1} \left( \frac{|E|}{|Q|} \right)^{1/s'},
$$

and this yields

$$
\frac{w(E)}{w(Q)} \leq [w]_{\text{RH}^s_{\theta_1}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1} \left( \frac{|E|}{|Q|} \right)^{1/s'},
$$

that is, $w \in A^p_{\infty}$.

Conversely, assume that $w^s \in A^p_{\infty}$. This implies that $w^s \in A^p_{q_0}$, for some $q_0 > 1$. By virtue of Lemma 3, we have that $w \in \text{RH}^p_{s} \cap A^p_{q_0}$, where $p_0 = 1 + (q_0 - 1)/s$. \qed
Lemma 6. Let $p > 1$, $u \in \text{RH}_p^\rho$ and $v \in \text{RH}_p^{p'}$. Then there exist $\theta \geq 0$ and a positive constant $C$ such that
\[
\left( \int_Q u^p \right)^{1/p} \left( \int_Q v^{p'} \right)^{1/p'} \leq C \left( 1 + \frac{r}{\rho(x)} \right)^\theta \left( \int_Q uv \right),
\]
for every cube $Q$.

Proof. Let us prove, as a first step, that there exists $0 < \delta < 1$ such that $u^{\delta p} \in A_p^\rho \cap \text{RH}_1^{1/\delta}$ and $v^{\delta p} \in A_{p'}^\rho \cap \text{RH}_1^{1/\delta}$. Since $u \in \text{RH}_p^\rho$ and $v \in \text{RH}_{p'}^\rho$, Proposition 2 implies that there exist $1 < q, r < \infty$ verifying $u \in A_q^\rho$ and $v \in A_r^{p'}$. Define $p_0 = \min\{p, p'\}$, $q_0 = \max\{q, r\}$ and $\delta = (p_0 - 1)/(q_0 - 1)$. Observe that $0 < \delta < 1$ because $q_0$ can be chosen arbitrarily large. Also notice that $u^\rho$ and $v^{p'}$ belong to the $A_{q_0}^\rho$ class, with $q_0 = 1 + (p_0 - 1)/\delta$. Lemma 3 yields $u^p \in A_{p_0}^\rho \cap \text{RH}_1^{1/\delta}$ and $v^{p'} \in A_{p_0}^{p'} \cap \text{RH}_1^{1/\delta}$. Therefore
\[
u^{p} \in A_{p_0}^\rho \cap \text{RH}_1^{1/\delta}
\quad \text{and} \quad v^{p'} \in A_{p_0}^{p'} \cap \text{RH}_1^{1/\delta},
\]
By combining these facts with Lemma 5 we obtain that
\[
\left( \frac{1}{|Q|} \int_Q u^p \right)^{\delta/p} \left( \frac{1}{|Q|} \int_Q v^{p'} \right)^{\delta/p'} \leq \left[ u^{p_0} \right]_{\text{RH}_{1/\delta}^{1/\delta}}^{1/p} \left[ v^{p'_0} \right]_{\text{RH}_{1/\delta}^{1/\delta}}^{1/p'} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2}
\times \left( \frac{1}{|Q|} \int_Q u^{p_0} \right)^{1/p} \left( \frac{1}{|Q|} \int_Q v^{p'_0} \right)^{1/p'}
\leq \left[ u^{p_0} \right]_{\text{RH}_{1/\delta}^{1/\delta}}^{1/p} \left[ v^{p'_0} \right]_{\text{RH}_{1/\delta}^{1/\delta}}^{1/p'} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2 + \theta_3} \left( \frac{1}{|Q|} \int_Q uv^{\delta} \right)^{\theta_1 + \theta_2 + \theta_3}
\leq \left[ u^{p_0} \right]_{\text{RH}_{1/\delta}^{1/\delta}}^{1/p} \left[ v^{p'_0} \right]_{\text{RH}_{1/\delta}^{1/\delta}}^{1/p'} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2 + \theta_3} \left( \frac{1}{|Q|} \int_Q uv^{\delta} \right)^{\theta_1 + \theta_2 + \theta_3}.
\]
By raising both sides to the power $1/\delta$ we can conclude the thesis.

As a corollary, we have that the product of two weights in $\text{RH}_p^\rho$ is a weight in the same class.

Corollary 2. Let $u$ and $v$ be two weights belonging to the $\text{RH}_p^\rho$ class. Then $uv \in \text{RH}_p^\rho$.

Proof. The hypothesis implies that both $u$ and $v$ belong to the $\text{RH}_p^\rho$ class. By applying Lemma 6 with $p = p' = 2$ we get
\[
u(x)v(x) \leq \left[ u \right]_{\text{RH}_\infty^{\rho, \theta_1}} \left[ v \right]_{\text{RH}_\infty^{\rho, \theta_2}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2} \left( \frac{1}{|Q|} \int_Q u \right) \left( \frac{1}{|Q|} \int_Q v \right)
\leq \left[ u \right]_{\text{RH}_\infty^{\rho, \theta_1}} \left[ v \right]_{\text{RH}_\infty^{\rho, \theta_2}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2} \left( \frac{1}{|Q|} \int_Q u^2 \right)^{1/2} \left( \frac{1}{|Q|} \int_Q v^2 \right)^{1/2}
\leq C \left[ u \right]_{\text{RH}_\infty^{\rho, \theta_1}} \left[ v \right]_{\text{RH}_\infty^{\rho, \theta_2}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2 + \theta_3} \left( \frac{1}{|Q|} \int_Q uv \right). \square
\]

The following lemma establishes an important relation between the classes $A_p^\rho(u)$ and $A_p^\rho$. The corresponding classical version is given in [14].

Lemma 7. Let $1 \leq p \leq \infty$. If $u \in A_p^\rho$ and $v \in A_p^\rho(u)$ then $uv \in A_p^\rho$.

Proof. We shall first consider the case $p = 1$. Since $v \in A_p^\rho(u)$, we can find $\theta_1 \geq 0$ for which the inequality
\[
\frac{1}{u(Q)} \int_Q uv \leq \left[ v \right]_{A_p^{\rho, \theta_1}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1} \inf_Q v
\]
holds for every cube $Q$. Similarly, $u \in A^p_\rho$ implies that there exists $\theta_2 \geq 0$ such that

$$\frac{1}{|Q|} \int_Q u \leq [u]_{A^p_\rho} \left(1 + \frac{r}{\rho(x)}\right)^{\theta_2} \inf_Q u,$$

for every cube $Q$. Therefore,

$$\frac{1}{|Q|} \int_Q uv = \frac{u(Q)}{|Q|} \frac{1}{u(Q)} \int_Q uv \leq [u]_{A^p_\rho} \left(1 + \frac{r}{\rho(x)}\right)^{\theta_2} \inf_Q u [v]_{A^p_\rho} \left(1 + \frac{r}{\rho(x)}\right)^{\theta_1} \inf_Q v \leq [u]_{A^p_\rho} [v]_{A^p_\rho} \left(1 + \frac{r}{\rho(x)}\right)^{\theta_1 + \theta_2} \inf_Q uv,$$

which means that $uv \in A^p_\rho$.

We turn now to the case $1 < p < \infty$. If $v \in A^p_\rho$, then

$$\left(\frac{1}{|Q|} \int_Q uv\right)^{1/p'} \left(\frac{1}{|Q|} \int_Q (uv)^{1-p'}\right)^{1/p'} = \frac{u(Q)}{|Q|} \left(\frac{1}{u(Q)} \int_Q uv\right)^{1/p} \left(\frac{1}{u(Q)} \int_Q v^{1-p'} w^{p'-p}\right)^{1/p'} \leq [u]_{A^p_\rho} \left(1 + \frac{r}{\rho(x)}\right)^{\theta_2} \left(\frac{1}{u(Q)} \int_Q uv\right)^{1/p} \times \left(\frac{1}{u(Q)} \int_Q v^{1-p'} w^{p'-p}\right)^{1/p'} \leq [u]_{A^p_\rho} [v]_{A^p_\rho} \left(1 + \frac{r}{\rho(x)}\right)^{\theta_1 + \theta_2} 
n,$$

which gives us the desired estimate.

Finally, if $p = \infty$ there exists $1 < q < \infty$ such that $v \in A^q_\rho(u)$. By applying the previous case we get $uv \in A^q_\rho$ and so $uv \in A^\infty_\rho$.

We are now in a position to state and prove a result that will play a fundamental role in our estimates.

**Theorem 4.** Let $u \in A^p_1$ and $v$ a weight such that $uv \in A^p_\infty$. Then $u \in A^p_1(v)$, that is, there exist $C > 0$ and $\theta \geq 0$ such that

$$\frac{uv(Q)}{v(Q)} \leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta \inf_Q u,$$

for every cube $Q = Q(x,r)$.

**Proof.** Since $uv \in A^p_\infty$, there exist $1 < p, s < \infty$ verifying $uv \in A^p_\rho \cap RH^p_s$. By Proposition 1, $uv = w_1 w_2$, where $w_1 \in A^p_\rho \cap RH^p_s$ and $w_2 \in A^s_\rho \cap RH^s_\infty$. On the other hand, $u \in A^p_1$ implies that $u^{-1} \in RH^p_\infty$, by virtue of Lemma 4 with $p = 2$. Corollary 2 allows us to conclude that $w_2 u^{-1} \in RH^p_\infty$. Given a cube
Let us first assume that $	heta \geq 0$. By following the same argument we can obtain the inequality which yields

\[
\frac{uv(Q)}{v(Q)} = \frac{w_1 w_2(Q)}{w_1 w_2 u^{-1}(Q)}
\]

\[
\leq \frac{1}{\inf_Q w_1 w_2 u^{-1}(Q)} \left( \frac{1}{|Q|} \int_Q w_1^s \right)^{1/s} \left( \frac{1}{|Q|} \int_Q w_2^{s'} \right)^{1/s'}
\]

\[
\leq \frac{1}{\inf_Q w_1 \sup_Q w_2 u^{-1}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2 + \theta_3} [w_1]_{R_{\infty}^p, \rho} |w_1|_{R_{\infty}^p, \rho} [w_2]_{R_{\infty}^p, \rho} \inf_Q u,
\]

which leads us to the desired estimate.

An immediate consequence of this result is the following.

**Corollary 3.** Let $u \in A^p_1$ and $v \in A^p_{\infty}(u)$. Then, there exist $C > 0$ and $\theta \geq 0$ such that for every cube $Q$ the inequality

\[
\frac{uv(Q)}{v(Q)} \leq C \left( 1 + \frac{r}{\rho(x)} \right)^{\theta} \inf_Q u
\]

holds.

**Proof.** It follows straightforwardly by combining Lemma 7 with Theorem 4.

**Proposition 3.** Let $0 < \alpha < 1, 1 < p \leq \infty, u \in A^p_1$ and $v \in A^p_{\infty}(u)$. Then $v \in A^p_{\infty}(u^\alpha)$.

**Proof.** Let us first assume that $p < \infty$. By Corollary 1 we have that $u^{\alpha - 1} \in RH_{\infty}^p$, which implies that there exists $\theta_1 \geq 0$ such that $u^{\alpha - 1} \in RH_{\infty}^{p, \theta_1}$. On the other hand, there exist nonnegative constants $\theta_2$ and $\theta_3$ verifying $u \in A^p_1$ and $v \in A^p_{\infty}(u)$. Fixed a cube $Q$, we have

\[
\frac{1}{u^\alpha(Q)} \int_Q v u^\alpha \leq \left( \sup_Q u^{\alpha - 1} \right) \frac{u(Q)}{|Q|} \frac{|Q|}{u^\alpha(Q) u(Q)} \int_Q v u
\]

\[
\leq [u^{\alpha - 1}]_{R_{\infty}^{p, \theta_1}} [u]_{A^{p, \theta_2}} \frac{1}{|Q|} \int_Q u^{\alpha - 1} \left( \inf_Q u \right) \frac{|Q|}{u^\alpha(Q)} \left( \frac{1}{u(u(Q))} \int_Q v u \right) \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2}
\]

\[
= [u^{\alpha - 1}]_{R_{\infty}^{p, \theta_1}} [u]_{A^{p, \theta_2}} \left( \frac{1}{u(u(Q))} \int_Q v u \right) \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2}.
\]

Therefore,

\[
\left( \frac{1}{u^\alpha(Q)} \int_Q v u^\alpha \right)^{1/p} \leq [u^{\alpha - 1}]_{R_{\infty}^{p, \theta_1}} [u]_{A^{p, \theta_2}} \left( \frac{1}{u(u(Q))} \int_Q v u \right)^{1/p} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2}.
\]

By following the same argument we can obtain

\[
\frac{1}{u^\alpha(Q)} \int_Q v^{1-p'} u^\alpha \leq \left( \sup_Q u^{\alpha - 1} \right) \frac{u(Q)}{|Q|} \frac{1}{u^\alpha(Q) u(Q)} \int_Q v^{1-p'} u
\]

\[
\leq [u^{\alpha - 1}]_{R_{\infty}^{p, \theta_1}} [u]_{A^{p, \theta_2}} \left( \frac{1}{u(u(Q))} \int_Q v^{1-p'} u \right) \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2},
\]

which yields

\[
\left( \frac{1}{u^\alpha(Q)} \int_Q v^{1-p'} u^\alpha \right)^{1/p'} \leq [u^{\alpha - 1}]_{R_{\infty}^{p, \theta_1}} [u]_{A^{p, \theta_2}} \left( \frac{1}{u(u(Q))} \int_Q v^{1-p'} u \right)^{1/p'} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2}.
\]
By combining these two estimates we get
\[
\left( \frac{1}{u^\alpha(Q)} \int_Q v \right) u^\alpha \right)^{1/p} \left( \frac{1}{u^\alpha(Q)} \int_Q v^{1-p'} u^{\alpha} \right)^{1/p'} \leq [u^{\alpha-1}]_{RH^{\alpha,s_1}} [u]_{A^{\rho_{s_2}} v} [v]_{A^\alpha_{\beta,s_3}(u)} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2 + \theta_3},
\]
which implies that \( v \in A^\rho(u) \).

If \( p = \infty \), there exists \( q > 1 \) that verifies \( v \in A^\rho_q(u) \). By the previous case, \( v \in A^\rho_q(u) \subseteq A^\rho_{\infty}(u) \). □

The next result will be useful in the proof of Theorem 3.

**Proposition 4.** Let \( 1 < s < \infty \) be fixed, \( u^s \in A^\rho_1 \) and \( v \) a weight that verifies \( v \in A^\rho_{\infty}(u^\beta) \), for some \( \beta > s \). Then there exists a number \( q > 1 \), \( 1 < q < s \), such that \( u^{1-q'} v \in A^\rho_{q'/s'} \).

**Proof.** By virtue of Remark 4 we can find \( \varepsilon_0 \) such that \( u^{s(1+\varepsilon)} \in A^\rho_1 \), for every \( 0 < \varepsilon < \varepsilon_0 \). Since \( v \in A^\rho_\infty(u^\beta) \), there exists \( r > 1 \) such that \( v \in A^\rho_r(u^\beta) \). Let us fix \( \varepsilon \) such that
\[
0 < \varepsilon < \min \left\{ \frac{\beta}{s'} - 1, \frac{1}{s(r-1)}, \varepsilon_0 \right\},
\]
and define \( q \) such that
\[
q' = \frac{s'(1+\varepsilon) - 1}{\varepsilon}.
\]
Therefore we have that \( q' > s' > 1 \), or equivalently, \( 1 < q < s \). Let \( \alpha = s'(q' - 1)/(q' - 1) \). Then, by our choice of \( \varepsilon \) we can conclude that \( u^\alpha \in A^\rho_{1,\beta_1} \), for some \( \beta_1 \geq 0 \). Indeed, by the definition of \( q' \) we have
\[
q' - s' = \frac{s - 1}{\varepsilon}.
\]
So we can obtain
\[
\frac{s'(q' - 1)}{q' - s'} = \frac{s' \varepsilon}{s' - 1} \left( \frac{s'(1+\varepsilon) - 1}{\varepsilon} - 1 \right)
\]
\[
= \frac{s'}{s' - 1} (s' - 1 + \varepsilon(s' - 1))
\]
\[
= \frac{s'(s' - 1)(1+\varepsilon)}{s' - 1}
\]
\[
= s'(1+\varepsilon).
\]
Let us now check that \( u^{1-q'} v \in A^\rho_{q'/s'} \). Notice first that
\[
\frac{q'}{s'} = \frac{1 + \varepsilon}{\varepsilon} - \frac{1}{s' \varepsilon} = 1 + \frac{1}{s' \varepsilon},
\]
and again \( r < q'/s' \) by the choice of \( \varepsilon \). Combining this fact with Proposition 3 applied with \( \alpha/\beta < 1 \) we conclude that \( v \in A^\rho_{q'/s'}(u^\alpha) \), for some \( \beta_2 \geq 0 \). Given a cube \( Q \) of \( \mathbb{R}^d \), since \( (1 - q' - \alpha)s'/q' = -\alpha \) we obtain
\[
\left( \frac{1}{|Q|} \int_Q u^{1-q'} v \right)^{s'/q'} \left( \frac{1}{|Q|} \int_Q u^{1-q'} \right)^{1-(s'/q')} \left( \frac{1}{|Q|} \int_Q v^{1-q'} u^{\alpha} \right)^{s'/q'}
\]
\[
= \left( \frac{1}{|Q|} \int_Q u^{s' \rho_{s_1}}(q') \right)^{s'/q'} \left( \frac{1}{|Q|} \int_Q u^{1-q'} \right)^{1-s'/q'}
\]
\[
\leq [v]_{A^\rho_{q'/s'}(u^{s' \rho_{s_1}})} [u^\alpha]_{A^\rho_{\beta_2}} \left( \inf_{Q} u^{\alpha} \right) \left( \inf_{Q} u^{\alpha} \right)
\]
\[
\times \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2}
\]
\[
\leq [v]_{A^\rho_{q'/s'}(u^{s' \rho_{s_1}})} [u^\alpha]_{A^\rho_{\beta_2}} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_1 + \theta_2},
\]
and thus $u^{1-q'/q}v \in A_{q'/s'}^{\rho,\theta_1+\theta_2} \subseteq A_{q'/s'}^{\rho}$. 

\[ \Box \]

3. PROOF OF THE MAIN RESULTS

We devote this section to prove our main theorems established in § 1. Before proving them we give some auxiliary definitions and results which will be needed in the sequel.

We shall very often refer to critical cubes, meaning cubes of the type $Q(x_0, \rho(x_0))$, and we call subcritical cubes to those $Q(x_0, r)$ with $r \leq \rho(x_0)$. The family of all subcritical cubes will be denoted by $Q_{\rho}$. Observe that from (3), $\rho(y) \simeq \rho(x_0)$ whenever $y \in Q(x_0, \rho(x_0))$. The following result is a useful consequence of (3) and can be found in [16].

**Proposition 5.** There exists a sequence of points $\{x_j\}_{j \in \mathbb{N}}$ such that the family of critical cubes given by $Q_j = Q(x_j, \rho(x_j))$ satisfies

(a) $\bigcup_{j \in \mathbb{N}} Q_j = \mathbb{R}^d$.

(b) There exist positive constants $C$ and $N_1$ such that for any $\sigma \geq 1$, $\sum_{j \in \mathbb{N}} X_{\sigma Q_j} \leq C \sigma^{N_1}$.

We shall require the following Calderón-Zygmund decomposition on a fixed cube (see, for example, [1]).

**Lemma 8.** Let $R$ be a cube in $\mathbb{R}^d$, $v$ a doubling weight and a function $f \in L^1(R, v \, dx)$. Then for $t > \frac{1}{\|v\|_{\infty, R}} \int_R |f(x)| v(x) \, dx$ there exist functions $g$ and $h$ and a collection of dyadic subcubes $\{P_i\}_{i \in \mathbb{N}}$ of $R$ such that

(a) $f = g + h$;

(b) $|g(x)| \leq Ct$ for almost every $x$;

(c) $h = \sum_{i \in \mathbb{N}} h_i$ where each $h_i$ is supported on a dyadic cube $P_i$. Furthermore, these cubes are pairwise disjoint and

$$\int_{P_i} h_i(x) v(x) \, dx = 0.$$

We shall be dealing with local versions of the classical Hardy-Littlewood maximal operator. For a fixed cube $R \subseteq \mathbb{R}^d$ we define

$$M_R f(x) = \sup_{Q \ni x, Q \subseteq R} \frac{1}{|Q|} \int_Q |f(y)| \, dy. \tag{17}$$

We shall also consider a dyadic version of the operator above. In order to define it we shall introduce the following concept.

A dyadic grid $\mathcal{D}$ will be understood as a collection of cubes in $\mathbb{R}^d$ with the following properties:

1. every cube $Q$ in $\mathcal{D}$ verifies $\ell(Q) = 2^k$, for some $k \in \mathbb{Z}$;
2. if $P$ and $Q$ are in $\mathcal{D}$ and $P \cap Q \neq \emptyset$, then either $P \subseteq Q$ or $Q \subseteq P$;
3. $\mathcal{D}_k = \{Q \in \mathcal{D} : \ell(Q) = 2^k\}$ is a partition of $\mathbb{R}^d$, for every $k \in \mathbb{Z}$.

Given a dyadic grid $\mathcal{D}$ and a cube $R$, by $\mathcal{D}_R$ we shall understand the collection of cubes in the grid that are also contained in $R$, that is

$$\mathcal{D}_R = \{Q \in \mathcal{D} : Q \subseteq R\}.$$ 

We also denote by $\mathcal{D}(R)$ the collection of dyadic subcubes of $R$. Notice that if $R$ is a dyadic cube in $\mathcal{D}$, then $\mathcal{D}_R = \mathcal{D}(R)$. The weighted dyadic version of (17) is given by

$$M_{R,w} f(x) = \sup_{Q \ni x, Q \subseteq \mathcal{D}(R)} \frac{1}{w(Q)} \int_Q |f(y)| w(y) \, dy. \tag{18}$$
When \( w = 1 \) we simply write \( M^w_P \).

The following lemma establishes an important geometric relation between cubes in \( \mathbb{R}^d \) and dyadic grids (see [20]).

**Lemma 9.** There exists dyadic grids \( D^{(i)} \), \( 1 \leq i \leq 3^d \), such that for every cube \( Q \) in \( \mathbb{R}^d \) there exists an index \( 1 \leq i_0 \leq 3^d \) and a dyadic cube \( Q_0 \in D^{(i_0)} \) such that \( Q \subseteq Q_0 \) and \( \ell(Q_0) \leq 3\ell(Q) \).

The result above allows us to prove a useful relation between the localized maximal functions given above.

**Lemma 10.** There exists dyadic grids \( D^{(i)} \), \( 1 \leq i \leq 3^d \) with the following property: For every cube \( Q \) in \( \mathbb{R}^d \) there exists \( 3^d \) dyadic cubes \( Q_i \in D^{(i)} \), \( 1 \leq i \leq 3^d \) such that

\[
M_Qf(x) \leq 3^d \sum_{i=1}^{3^d} M_{Q_i}(fX_Q)(x),
\]

for every \( x \in Q \). Furthermore, each \( Q_i \) verifies \( Q_i \subseteq \lambda Q \), where \( \lambda \) depends only on \( d \).

**Proof.** Before proceeding with the proof, given a cube \( Q_0 \) and a dyadic grid \( D \) we shall denote by \( M_{Q_0,D}f \) the version of \( M_{Q_0} \) where the supremum is taken only for cubes in \( D \) contained in \( Q_0 \), that is

\[
M_{Q_0,D}f(x) = \sup_{Q \in D_{Q_0}} \frac{1}{|Q|} \int_Q |f|.
\]

Fix \( x \in Q \) and let \( P \subseteq Q \) be a subcube containing \( x \). By Lemma 9 there exists a dyadic grid \( D^{(i)} \) and \( P_i \in D^{(i)} \) such that \( P \subseteq P_i \) and \( \ell(P_i) \leq 3\ell(P) \). We claim that \( P_i \subseteq 8\sqrt{d}Q \). Indeed, if \( x_Q \) denotes the center of \( Q \), for \( y \in P_i \) we have that

\[
|y - x_Q| \leq |y - x| + |x - x_Q| \leq \sqrt{d}\ell(P_i) + \sqrt{d}\ell(Q) \leq 4\sqrt{d}\ell(Q),
\]

so \( P_i \subseteq B(x_Q, 4\sqrt{d}\ell(Q)) \subseteq 8\sqrt{d}Q \). Therefore we have that

\[
\frac{1}{|P|} \int_P |f| = \frac{1}{|P_i|} \int_{P_i} |f|X_Q \leq \frac{3^d}{|P_i|} \int_{P_i} |f|X_Q \leq 3^d M_{8\sqrt{d}Q,D^{(i)}}(fX_Q)(x) \leq 3^d \sum_{i=1}^{3^d} M_{8\sqrt{d}Q,D^{(i)}}(fX_Q)(x).
\]

By taking supremum over the cubes \( P \subseteq Q \) we arrive to

\[
M_Qf(x) \leq 3^d \sum_{i=1}^{3^d} M_{8\sqrt{d}Q,D^{(i)}}(fX_Q)(x),
\]

for every \( x \in Q \).

Fix \( 1 \leq i \leq 3^d \). There exists a unique \( k \in \mathbb{Z} \) such that

\[
2^k < 8\sqrt{d}\ell(Q) \leq 2^{k+1}.
\]

There also exist at most \( 2^d \) cubes in \( D^{(i)} \) with side length \( 2^k \) and that intersect \( 8\sqrt{d}Q \). Let \( Q_i \) be the cube in \( D^{(i)}_{k+1} \) that contains these cubes. We claim that

\[
8\sqrt{d}Q \subseteq Q_i \subseteq 48dQ.
\]
Indeed, the first inclusion is immediate. For the latter, if \( y \in Q_i \) and \( x \in 8\sqrt{d}Q \cap Q_i \) by (19) we have that

\[
| x_Q - y | \leq | x_Q - x | + | x - y | \\
\leq 8d \ell(Q) + \sqrt{d} \ell(Q_i) \\
\leq 8d \ell(Q) + 16d \ell(Q) \\
= 24d \ell(Q),
\]

so \( Q_i \subseteq B(x_Q, 24d \ell(Q)) \subseteq 48d Q. \)

By our choice of \( Q_i \), \( 1 \leq i \leq 3^d \), we must have

\[
M_{8 \sqrt{d}Q:D^i} (f \chi_Q) \leq M_{Q_i:D^i} (f \chi_Q) = M_{Q_i}(f \chi_Q).
\]

Finally,

\[
M_Q f(x) \leq 3^d \sum_{i=1}^{3^d} M_{Q_i}(f \chi_Q)(x),
\]

as desired. \( \square \)

In the sequel we shall consider local classes of weights associated to a critical radius function as it was done in [11]. Given a weight \( u \) and a critical radius function \( \rho \), we say that a weight \( w \in A^\rho_{p,\text{loc}}(u) \) if it satisfies (9) for each cube \( Q \in Q_\rho \), and \( w \in A^\rho_{p,\text{loc}}(u) \) if it satisfies (10) for each cube \( Q \in Q_\rho \). We also consider classes of weights associated to a cube \( R \). More precisely we say that a weight \( w \in A_p(R, u) \) if it satisfies (9) for each cube \( Q \subseteq R \), and \( w \in A_1(R, u) \) if it verifies (10) for each cube \( Q \subseteq R \). In all cases we will drop \( u \) from the notation when \( u = 1 \). It is easy to check that properties stated in Lemma 2 hold for these classes of weights.

It is well known that the classes of weights mentioned above are related to the continuity properties of the localized Hardy-Littlewood maximal operators \( M_R \) (see [17] and [13]).

The following proposition, that can be viewed as a localized version of Theorem 1.4 in [14], will play an important role in proving Theorem 1.

**Proposition 6.** Let \( R \) be a cube, \( u \in A_1(R) \) and \( v \in A_\infty(R, u) \). There exists a positive constant \( C \) depending only on the characteristic constant of the weights and the dimension, such that for all \( t > 0 \)

\[
uv \left( \left\{ x \in R : \frac{M_{R}^\rho(fv)(x)}{v(x)} > t \right\} \right) \leq \frac{C}{t} \int_R |f(x)|u(x)v(x)dx.
\]

**Proof.** It is enough to consider a nonnegative function \( f \) and \( t > f_R^{uv} = \frac{1}{uv(R)} \int_R fuv \), since in other case the estimate is trivial. We first apply Lemma 8 to decompose \( R \) with respect to the measure \( d\mu(x) = uv(x)dx \) at height \( t \), obtaining a sequence of pairwise disjoint dyadic cubes \( \{P_i\}_{i \in \mathbb{N}} \) such that

\[
t \leq f_{P_i}^{uv} \leq \gamma t,
\]

and \( f(x) \leq t \) for \( x \in R \setminus \Omega \), being \( \Omega = \bigcup_{i \in \mathbb{N}} P_i \), where \( \gamma = \gamma(d) > 1 \).

Therefore, we can write \( f = g + h \), where

\[
g(x) = \begin{cases} 
  f_{P_i}^{uv} & \text{if } x \in P_i \\
  f(x) & \text{if } x \in R \setminus \Omega,
\end{cases}
\]

and \( h(x) = \sum h_i(x) \) where

\[
h_i(x) = (f - f_{P_i}^{uv}) \chi_{P_i}.
\]
Let $Q \in \mathcal{P}(R)$, then for all $x \in Q$,
\[
\frac{1}{|Q|} \int_Q f v \leq \frac{|u|_{A_1(R)}}{u(Q)} \int_Q f uv \leq |u|_{A_1(R)} (M^p_{R,u}(gv)(x) + \tilde{M}_u(hv)(x)),
\]
since $u \in A_1(R)$, where
\[
\tilde{M}_u(x) = \sup_{Q \ni x \in \mathcal{P}(R)} \frac{1}{u(Q)} \int_Q \phi u.
\]
Therefore,
\[
M^p_R(fv)(x) \leq M^p_{R,u}(gv)(x) + \tilde{M}_u(hv)(x).
\]
Now,
\[
\begin{aligned}
uv \left( \left\{ x \in R : \frac{M^p_R(fv)(x)}{v(x)} > t \right\} \right) &\leq uv \left( \left\{ x \in R : \frac{M^p_{R,u}(gv)(x)}{v(x)} > \frac{t}{2} \right\} \right) \\
&\quad + uv \left( \left\{ x \in \Omega : \frac{\tilde{M}_u(hv)(x)}{v(x)} > \frac{t}{2} \right\} \right) \\
&\quad + uv \left( \left\{ x \in R \setminus \Omega : \frac{\tilde{M}_u(hv)(x)}{v(x)} > \frac{t}{2} \right\} \right) \\
&= I_1 + I_2 + I_3.
\end{aligned}
\]
Let us take care of $I_1$. By Chebyshev inequality, the continuity properties of the localized Hardy-Littlewood maximal operator with weights and the properties of $g$ we have
\[
I_1 \leq C \frac{1}{t^p} \int_R M^p_{R,u}(gv)(x)^p u(x)v(x)^{1-p} dx \\
\leq C \frac{1}{t^p} \int_R g(x)^p u(x)v(x) dx \\
\leq \frac{C \gamma^p - 1}{t} \int_R g(x)u(x)v(x) dx \\
= \frac{C}{t} \int_R f(x)u(x)v(x) dx + \frac{C}{t} \sum_{i \in \mathbb{N}} \int_{P_i} f(x)u(x)v(x) dx \\
= \frac{C}{t} \int_R f(x)u(x)v(x) dx.
\]
To deal with $I_2$,
\[
I_2 \leq uv(\Omega) = \sum_{i \in \mathbb{N}} uv(P_i) \leq \frac{1}{t} \sum_{i \in \mathbb{N}} \int_{P_i} f(x)u(x)v(x) dx \leq C \frac{1}{t} \int_R f(x)u(x)v(x) dx.
\]
Finally, by following a similar argument as in Theorem 1.4 in [14] we can show that $I_3 = 0$. 

We are now in a position to prove Theorem 1.

**Proof of Theorem 1.** Let us begin by observing that, for $\varphi \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\sigma > 0$,
\[
M^{p,\sigma} \varphi(x) \leq \sup_{Q \in \mathcal{P}(\mathbb{R}^d)} \frac{1}{|Q|} \int_Q |\varphi| + \sup_{Q \in \mathcal{P}(\mathbb{R}^d)} \left( \frac{\rho(x_Q)}{r_Q} \right)^\sigma \frac{1}{|Q|} \int_Q |\varphi| \\
= M^{p}_{\text{loc}} \varphi(x) + M^{p,\sigma}_{\text{glob}} \varphi(x).
\]
We can assume without losing generality that \( f \) is nonnegative. Let \( \{Q_j\}_{j \in \mathbb{N}} \) the family of critical cubes given in Proposition \( 5 \) and satisfying (\( a \)). For \( t > 0 \) we write

\[
uv \left( \left\{ x \in \mathbb{R}^d : \frac{M^{\rho,\sigma}(fv)(x)}{v(x)} > t \right\} \right) \leq \sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{M^{\rho,\sigma}(fv)(x)}{v(x)} > \frac{t}{2} \right\} \right)
\]

\[
\leq \sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{M^{\rho,\sigma}_{\text{glob}}(fv)(x)}{v(x)} > \frac{t}{2} \right\} \right)
\]

\[
+ \sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{M^\rho_{\text{loc}}(fv)(x)}{v(x)} > \frac{t}{2} \right\} \right).
\]

We will consider first the term corresponding to \( M^{\rho,\sigma}_{\text{glob}} \). Observe that for \( x \in Q_j \), there exist a constant \( c > 0 \) such that

\[
M^{\rho,\sigma}_{\text{glob}}(fv)(x) \leq C \sup_{k \geq 1} \frac{2^{-ck\sigma}}{|2^k Q_j|} \int_{2^k Q_j} fv
\]

\[
\leq C \sum_{k \geq 1} \frac{2^{-k(c\sigma - \theta)}}{u(2^k Q_j)} \int_{2^k Q_j} fvu
\]

\[
\leq \frac{C}{u(Q_j)} \sum_{k \geq 1} 2^{-k(c\sigma - \theta)} \int_{2^k Q_j} fvu = \frac{A_j}{u(Q_j)}.
\]

Now, applying part (\( b \)) of Proposition \( 5 \),

\[
\sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{M^{\rho,\sigma}_{\text{glob}}(fv)(x)}{v(x)} > \frac{t}{2} \right\} \right) \leq \sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{A_j}{u(Q_j)v(x)} > \frac{t}{2} \right\} \right)
\]

\[
\leq \frac{C}{t} \sum_{j \in \mathbb{N}} \frac{A_j}{u(Q_j)} \int_{Q_j} u(x) \, dx
\]

\[
\leq \frac{C}{t} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} 2^{-k(c\sigma - \theta)} \int_{2^k Q_j} fvu
\]

\[
\leq \frac{C}{t} \sum_{k \in \mathbb{N}} 2^{-k(c\sigma - \theta)} \int_{\mathbb{R}^d} \left( \sum_{j \in \mathbb{N}} \chi_{2^k Q_j}(y) \right) f(y)v(y)u(y) \, dy
\]

\[
\leq \frac{C}{t} \int_{\mathbb{R}^d} f(y)v(y)u(y),
\]

choosing \( \sigma > (N_1 + \theta + 1)/c. \)

To deal with the term corresponding to \( M^\rho_{\text{loc}} \) we will use the local theory developed in Lemma \( 10 \) and Proposition \( 6 \). We first claim that given \( Q_j \), there exists a cube \( R_j \supseteq Q_j \), \( R_j = Q(x_j, C_\rho \rho(x_j)) \) such that if \( x \in Q_j \cap Q \) and \( Q \in \mathcal{Q}_\rho \) then \( Q \subseteq R_j \). To see this, let \( x \in Q_j \cap Q \) with \( Q = Q(x_Q, \rho(x_Q)) \). If \( z \in Q \), using (\( 3 \)),

\[
|x_j - z| \leq |x_j - x| + |x - z|
\]

\[
\leq \rho(x_j) + 2\rho(x_Q)
\]

\[
\leq \rho(x_j) + 2C_\rho 2^{N_0} \rho(x)
\]

\[
\leq \rho(x_j) + C_\rho^2 2^{N_0+2} \rho(x_j).
\]
Therefore, we have that

\[
\sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{M_{p_j}(fv)(x)}{v(x)} > \frac{t}{2} \right\} \right) \leq \sum_{j \in \mathbb{N}} uv \left( \left\{ x \in R_j : \frac{M_{p_j}(fv)(x)}{v(x)} > \frac{t}{2} \right\} \right)
\]

\[
\leq C \sum_{j \in \mathbb{N}} \sum_{i=1}^{3^d} uv \left( \left\{ x \in P_{i,j} : \frac{M_{p_j}^v(fv)(x)}{v(x)} > \frac{t}{3^{3^d}} \right\} \right),
\]

where in the last inequality we used Lemma 10.

We have to make some observations in order to apply Proposition 6. First, notice that if \( u \in A_1^{s'} \) and \( v \in A_\infty^{s'}(u) \), then there exists \( \theta > 0 \) such that \( u \in A_1^{s,\theta} \) and \( v \in A_\infty^{s,\theta}(u) \) with constants \( [u]_{A_1^{s,\theta}} \), \( [v]_{A_\infty^{s,\theta}(u)} \). Therefore, \( u \in A_1^{s,\text{loc}} \) and \( v \in A_\infty^{s,\text{loc}}(u) \) with constants \( [u]_{A_1^{s,\text{loc}}} \), \( [v]_{A_\infty^{s,\text{loc}}(u)} \). Now, by means of Corollary 1 in [11] we have that \( u \in A_1^{s,\text{loc},\text{type}} \) and \( v \in A_\infty^{s,\text{loc},\text{type}}(u) \). Therefore, we have that \( u \in A_1(P_{i,j}) \) and \( v \in A_\infty(P_{i,j},u) \), with constants \( C(\rho,d)[u]_{A_1^{s,\text{loc}}} \) and \( C(\rho,d)[v]_{A_\infty^{s,\text{loc}}(u)} \) respectively, since \( R_j = C_\rho Q_j \) and \( P_{i,j} \subseteq \lambda R_j \), for each \( j \in \mathbb{N} \) and \( 1 \leq i \leq 3^d \). The thesis follows by combining Proposition 6 with part (b) of Proposition 5.

Before giving a proof for Theorem 2 and Theorem 3 we need to establish the following tools. The first one assures that the smoothness estimate for SCZO holds also with an extra decay if necessary. For a proof we refer to Lemma 4 in [8].

**Lemma 11.** Let \( \delta > 0 \) and \( T \) be a SCZO of \((s,\delta')\) type for some \( 1 < s \leq \infty \) and for every \( \delta' \in (0,\delta) \). Then, for each \( N \in \mathbb{N} \), there exists a constant \( C_N \) such that the associated kernel \( K \) satisfies

\[
\left( \int_{R < |x-y_0| < 2R} |K(x,y) - K(x,y_0)|^s dy \right)^{1/s} \leq C_N R^{-d/s'} \left( \frac{r}{R} \right)^{\delta'} \left( 1 + \frac{R}{\rho(x)} \right)^{-N},
\]

for \( |y-y_0| < r \leq \rho(y_0) \), \( r < R/2 \) if \( s < \infty \) and

\[
|K(x,y) - K(x,y_0)| \leq C_N \frac{|y-y_0|^{\delta}}{|x-y|^{d+\delta}} \left( 1 + \frac{|x-y|}{\rho(x)} \right)^{-N}, \text{ when } |x-y| > 2|y-y_0|,
\]

if \( s = \infty \). In particular, this applies to any SCZO of \((s,\delta)\) type.

The next geometrical lemma gives an equivalent size condition for SCZO of \((s,\delta)\) type when \( s < \infty \) that will be used to prove Theorem 3. For a proof we refer to Lemma 3 in [10].

**Lemma 12.** Let \( 1 < s < \infty \) and \( T \) be a SCZO of \((s,\delta)\) type. Then for each \( N > 0 \) there exists \( C_N \) such that

\[
\left( \int_{B(x_0,R/2)} |K(x,y)|^s dx \right)^{1/s} \leq C_N R^{-d/s'} \left( 1 + \frac{R}{\rho(x_0)} \right)^{-N},
\]

whenever \( R < |y - x_0| < 2R \).

We now proceed with the proof of Theorem 2.
Proof of Theorem 2. We shall first assume, without loss of generality, that $f$ is nonnegative. Let $\{Q_j\}_{j \in \mathbb{N}}$ be the family of critical cubes given by Proposition 5. We fix $t > 0$ and write

$$uv \left( \left\{ x \in \mathbb{R}^d : \frac{|T(fv)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \leq \sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{|T(fv)(x)|}{v(x)} > \frac{t}{2} \right\} \right)$$

$$\leq \sum_{j \in \mathbb{N}}uv \left( \left\{ x \in Q_j : \frac{|T(f_1^j v)(x)|}{v(x)} > \frac{t}{2} \right\} \right)$$

$$+ \sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{|T(f_2^j v)(x)|}{v(x)} > \frac{t}{2} \right\} \right),$$

where $f_1^j = f\chi_{2Q_j}$ and $f_2^j = f\chi_{(2Q_j)^c}$.

We shall first bound the term corresponding to $f_2^j$. By Chebyshev inequality we write

$$\sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{|T(f_2^j v)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \leq \sum_{j \in \mathbb{N}} \frac{2}{t} \int_{Q_j} |T(f_2^j v)(x)|u(x) \, dx$$

$$\leq \sum_{j \in \mathbb{N}} \frac{2}{t} \int_{Q_j} \left( \sum_{k \in \mathbb{N}} \int_{2^{k+1}Q_j \setminus 2^kQ_j} |K(x,y)||f(y)v(y)\, dy \right) u(x) \, dx.$$ 

Now, by applying (5) and $A_t^\varphi$ condition for $u$, we have that

$$\sum_{j \in \mathbb{N}} uv \left( \left\{ x \in Q_j : \frac{|T(f_2^j v)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \leq \frac{CN}{t} \sum_{j \in \mathbb{N}} \int_{2^{k+1}Q_j} \sum_{k \in \mathbb{N}} 2^{-kN} \left| \frac{1}{2^k\rho(x_j)^d} \int_{2^{k+1}Q_j} f(y)v(y)\, dy \right| u(x) \, dx$$

$$\leq \frac{CN}{t} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} 2^{-kN} \frac{u(2^{k+1}Q_j)}{|2^{k+1}Q_j|} \int_{2^{k+1}Q_j} f(y)v(y)\, dy$$

$$\leq \frac{CN}{t} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} 2^{-k(N-\theta)} \int_{2^{k+1}Q_j} f(y)v(y)u(y)\, dy$$

$$\leq \frac{CN}{t} \sum_{k \in \mathbb{N}} 2^{-k(N-\theta)} \int_{\mathbb{R}^d} \left( \sum_{j \in \mathbb{N}} \chi_{2^{k+1}Q_j}(y) \right) f(y)v(y)u(y)\, dy$$

$$\leq \frac{C}{t} \int_{\mathbb{R}^d} f(y)v(y)u(y),$$

where we have used part (b) of Proposition 5 and chosen $N > N_1 + \theta + 1$.

We now turn our attention to the term corresponding to $f_1^j$. We shall prove that

$$uv \left( \left\{ x \in Q_j : \frac{|T(f_1^j v)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \leq \frac{C}{t} \int_{2Q_j} f(x)v(x)u(x) \, dx,$$

for each $j \in \mathbb{N}$. Then the result directly follows by applying part (b) of Proposition 5. Fix $j \in \mathbb{N}$ and let us first suppose that

$$(f_1^j)_{2Q_j}^\varphi = \frac{1}{v(2Q_j)} \int_{2Q_j} f_1^j(x)v(x) \, dx \geq \frac{t}{2}.$$
Then, by Corollary 3

\[ uv \left( \left\{ x \in Q_j : \frac{|T(f^1_j v)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \leq \frac{uv(2Q_j)v(2Q_j)}{v(2Q_j)} \]

\[ \leq \frac{2uv(2Q_j)}{v(2Q_j)^2} \int_{2Q_j} f^1_j (x)v(x) \, dx \]

\[ \leq C \left( \inf \left\{ \frac{u}{2Q_j} \right\} \right) \int_{2Q_j} f(x)v(x) \, dx \]

\[ \leq C \int_{2Q_j} f(x)v(x)u(x) \, dx. \]

On the other hand, if \( (f^1)^q_{2Q_j} < t/2 \), we apply the decomposition given by Lemma 8 on \( 2Q_j \) at level \( t/2 \), obtaining a collection of pairwise disjoint dyadic cubes \( \{ P_{i,j} \}_{i \in \mathbb{N}} \) in \( \mathcal{Q}(2Q_j) \) and a pair of functions \( g_j \) and \( h_j \) such that \( f^1_j = g_j + h_j \). By setting \( \tilde{P}_{i,j} = 3P_{i,j} \) and \( \tilde{\Omega}_j = \bigcup_{i \in \mathbb{N}} \tilde{P}_{i,j} \), we write

\[ uv \left( \left\{ x \in Q_j : \frac{|T(f^1_j v)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \leq uv \left( \left\{ x \in Q_j : \frac{|T(g_j v)(x)|}{v(x)} > \frac{t}{4} \right\} \right) \]

\[ + uv \left( \left\{ x \in Q_j : \frac{|T(h_j v)(x)|}{v(x)} > \frac{t}{4} \right\} \right) \]

\[ \leq uv \left( \left\{ x \in Q_j : \frac{|T(g_j v)(x)|}{v(x)} > \frac{t}{4} \right\} \right) + uv(\tilde{\Omega}_j) \]

\[ + uv \left( \left\{ x \in Q_j \setminus \tilde{\Omega}_j : \frac{|T(h_j v)(x)|}{v(x)} > \frac{t}{4} \right\} \right) \]

\[ = I + II + III. \]

Let us first estimate \( I \). Since \( v \in A^\infty_q(u) \), there exists \( 1 < q' < \infty \) such that \( v \in A^p_{q'}(u) \). Therefore, \( v^{1-q} \in A^p_q(u) \) by item (a) of Lemma 2. Consequently, we have that \( uv^{1-q} \in A^p_q \) by virtue of Lemma 7. Then, by applying Chebyshev inequality with exponent \( q \) and using the strong \((q, q)\) type of \( T \) for \( A^p_q \) weights (see Remark 1) we obtain

\[ I = uv \left( \left\{ x \in Q_j : \frac{|T(g_j v)(x)|}{v(x)} > \frac{t}{4} \right\} \right) \leq \frac{C}{t^q} \int_{2Q_j} |T(g_j v)(x)|^q u(x)v^{1-q}(x) \, dx \]

\[ \leq \frac{C}{t^q} \int_{2Q_j} g_j(x)v(x)|^q u(x)v^{1-q}(x) \, dx \]

\[ \leq \frac{C}{t} \int_{2Q_j} g_j(x)u(x)v(x) \, dx, \]

where we have also used that \( g_j(x) \leq Ct \) in almost every \( x \in 2Q_j \). Finally, from the definition of \( g_j \) and Corollary 3 we arrive to

\[ I \leq \frac{C}{t} \int_{2Q_j \setminus \Omega_j} f(x)u(x)v(x) \, dx + \frac{C}{t} \sum_{i \in \mathbb{N}} \frac{uv(P_{i,j})}{v(P_{i,j})} \int_{P_{i,j}} f(x)v(x) \, dx \]

\[ \leq \frac{C}{t} \int_{2Q_j \setminus \Omega_j} f(x)u(x)v(x) \, dx + \frac{C}{t} \sum_{i \in \mathbb{N}} \int_{P_{i,j}} f(x)u(x)v(x) \, dx \]

\[ \leq \frac{C}{t} \int_{2Q_j} f(x)u(x)v(x) \, dx. \]
To deal with $II$, observe that $uv \in A^p_\infty$ by Lemma 7 and therefore it is a doubling weight over sub-critical cubes. We proceed as follows

$$II = uv(\tilde{Q}_j) \leq \sum_{i \in \mathbb{N}} uv(P_{i,j}) \leq C \sum_{i \in \mathbb{N}} v(P_{i,j}) \frac{uv(P_{i,j})}{v(P_{i,j})} \leq C t \sum_{i \in \mathbb{N}} \left( \inf_{P_{i,j}} u \right) \int_{P_{i,j}} f(x)v(x) \, dx \leq C t \int_{2Q_j} f(x)u(x)v(x) \, dx.$$ 

In order to estimate $III$, by using item (c) of Lemma 8 we can write

$$III = uv \left( \left\{ x \in Q_j \setminus \tilde{Q}_j : \frac{\left| T(h_{i,j})v(x) \right|}{v(x)} > \frac{t}{\pi} \right\} \right) \leq C t \sum_{i \in \mathbb{N}} \int_{2Q_j \setminus P_{i,j}} |T(h_{i,j})v(x)|u(x) \, dx \leq C t \sum_{i \in \mathbb{N}} \int_{2Q_j \setminus P_{i,j}} \left| \int_{P_{i,j}} K(x,y)h_{i,j}(y)v(y) \, dy \right| u(x) \, dx \leq C t \sum_{i \in \mathbb{N}} \int_{2Q_j \setminus P_{i,j}} \left| K(x,y) - K(x,x_{i,j}) \right| h_{i,j}(y)v(y) \, dy \, u(x) \, dx \leq C t \sum_{i \in \mathbb{N}} \int_{2Q_j \setminus P_{i,j}} |K(x,y) - K(x,x_{i,j})|u(x) \, dx \, dy.$$ 

Let us denote $r_{i,j}$ the radius of the ball inscribed in $P_{i,j}$ and let $\theta \geq 0$ such that $u \in A^{\theta}_1$. For $y \in P_{i,j}$, by applying condition (6) we have that

$$\int_{2Q_j \setminus P_{i,j}} |K(x,y) - K(x,x_{i,j})|u(x) \, dx \leq C_N \sum_{k \in \mathbb{N}} \int_{2k^{-1}r_{i,j} \leq |x-x_{i,j}| < 2k^{1/2}r_{i,j}} \frac{|y-x_{i,j}|^\delta}{|x-x_{i,j}|^{d+\delta}} \left( 1 + \frac{|x-x_{i,j}|}{\rho(x_{i,j})} \right)^{-N} u(x) \, dx \leq C_N \sum_{k \in \mathbb{N}} \frac{r_{i,j}^\delta}{(2k^{1/2}r_{i,j})^{d+\delta}} \left( 1 + \frac{2k^{1/2}r_{i,j}}{\rho(x_{i,j})} \right)^{-N} u(x) \, dx \leq C_N u(y) \sum_{k \in \mathbb{N}} 2^{-k\delta} \left( 1 + \frac{2k^{1/2}r_{i,j}}{\rho(x_{i,j})} \right)^{-N+\theta} \leq C u(y),$$

where we also used that $u \in A^{\theta}_1$ and chose $N > \theta$. Therefore, by Corollary 3 we get

$$III \leq C t \sum_{i \in \mathbb{N}} \int_{P_{i,j}} |h_{i,j}(y)|u(y)v(y) \, dy \leq C t \sum_{i \in \mathbb{N}} \left( \int_{P_{i,j}} f_1^\delta(y)u(y)v(y) \, dy + \int_{P_{i,j}} (f_1^{1/2}P_{i,j})u(y)v(y) \, dy \right) \leq C t \sum_{i \in \mathbb{N}} \left( \int_{P_{i,j}} f_1^\delta(y)u(y)v(y) \, dy + \frac{uv(P_{i,j})}{v(P_{i,j})} \int_{P_{i,j}} f(y)v(y) \, dy \right) \leq C t \int_{2Q_j} f(y)u(y)v(y) \, dy.$$
This completes the proof of (20). By summing over \( j \) we can conclude the thesis. \( \square \)

**Proof of Theorem 3.** The proof will follow similar lines as the previous one, so we will just focus on the parts where suitable changes are needed. We consider again \( \{Q_j\}_{j \in \mathbb{N}} \) the family of critical cubes given by Proposition 5. Fix \( t > 0 \) and write

\[
\begin{align*}
uv\left( \left\{ x \in \mathbb{R}^d : \left| \frac{T(fv)(x)}{v(x)} \right| > t \right\} \right) & \leq \sum_{j \in \mathbb{N}} uv\left( \left\{ x \in Q_j : \left| \frac{T(fv)(x)}{v(x)} \right| > t \right\} \right) \\
& \leq \sum_{j \in \mathbb{N}} uv\left( \left\{ x \in Q_j : \left| \frac{T(f_j^1v)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right) \\
& \quad + \sum_{j \in \mathbb{N}} uv\left( \left\{ x \in Q_j : \left| \frac{T(f_j^2v)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right),
\end{align*}
\]

where \( f_j^1 = f\mathcal{X}_{2Q_j} \) and \( f_j^2 = f\mathcal{X}_{(2Q_j)^c} \).

We will first bound the term corresponding to \( f_j^2 \). By applying Chebyshev inequality we have

\[
\begin{align*}
\sum_{j \in \mathbb{N}} uv\left( \left\{ x \in Q_j : \frac{T(f_j^2v)(x)}{v(x)} > \frac{t}{2} \right\} \right) & \leq \sum_{j \in \mathbb{N}} \frac{2}{t} \int_{Q_j} |T(f_j^2v)(x)|u(x) \, dx \\
& \leq \sum_{j \in \mathbb{N}} \frac{2}{t} \int_{Q_j} \left( \int_{(2Q_j)^c} |K(x,y)|f(y)v(y) \, dy \right) u(x) \, dx \\
& \leq \sum_{j \in \mathbb{N}} \frac{2}{t} \int_{(2Q_j)^c} f(y)v(y) \left( \int_{Q_j} |K(x,y)|u(x) \, dx \right) dy \\
& \leq \sum_{j \in \mathbb{N}} \frac{2}{t} \sum_{k \in \mathbb{N}} \int_{2^{k+1}Q_j \setminus 2^kQ_j} f(y)v(y) \left( \int_{Q_j} |K(x,y)|^s \, dx \right)^{\frac{1}{s'}} \left( \int_{Q_j} u^{s'}(x) \, dx \right)^{\frac{1}{s'}} dy.
\end{align*}
\]

By applying condition (7) on \( K \), for each \( N \) we have that

\[
\begin{align*}
\sum_{j \in \mathbb{N}} uv\left( \left\{ x \in Q_j : \frac{T(f_j^2v)(x)}{v(x)} > \frac{t}{2} \right\} \right) & \leq \frac{C_N}{t} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} |2^kQ_j|^{-1/s'} 2^{-kN} \int_{2^{k+1}Q_j} f(y)v(y) \left( \int_{Q_j} u^{s'} \right)^{1/s'} dy \\
& \leq \frac{C_N}{t} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} 2^{-kN} \int_{2^{k+1}Q_j} f(y)v(y) \left( \frac{1}{|2^{k+1}Q_j|} \int_{2^{k+1}Q_j} u^{s'} \right)^{1/s'} dy \\
& \leq \frac{C_N}{t} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} 2^{-k(N-\theta/s')} \int_{2^{k+1}Q_j} f(y)v(y)u(y) \, dy \\
& \leq \frac{C_N}{t} \sum_{k \in \mathbb{N}} 2^{-k(N-\theta/s')} \int_{\mathbb{R}^d} \left( \sum_{j \in \mathbb{N}} \mathcal{X}_{2^{k+1}Q_j}(y) \right) f(y)v(y)u(y) \, dy \\
& \leq \frac{C}{t} \int_{\mathbb{R}^d} f(y)v(y)u(y),
\end{align*}
\]

where we have used part (b) of Proposition 5 and chosen \( N > N_1 + \theta/s' + 1 \).
For $f_1^j$ we shall prove that, for every $j \in \mathbb{N}$, the inequality

$$ uv \left( \left\{ x \in Q_j : \frac{|T(f_1^j v)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \leq \frac{C}{t} \int_{2Q_j} f(x)v(x)u(x) \, dx $$

holds. Fixed $j \in \mathbb{N}$, we can assume that $(f_1^j)_{2Q_j} < t/2$, since the estimate follows exactly as in page 19 otherwise. We apply the decomposition given by Lemma 8 on $2Q_j$ at level $t/2$ obtaining the family $\{P_{i,j}\}_{i \in \mathbb{N}}$, $g_j$ and $h_j$ such that $f_1^j = g_j + h_j$. By adopting the same notation as in the proof of Theorem 2 we get

$$ uv \left( \left\{ x \in Q_j : \frac{|T(g_j v)(x)|}{v(x)} > \frac{t}{4} \right\} \right) + uv(\tilde{\Omega}_j) + uv \left( \left\{ x \in Q_j \setminus \tilde{\Omega}_j : \frac{|T(h_j v)(x)|}{v(x)} > \frac{t}{4} \right\} \right) = I + II + III. $$

We shall first estimate $I$. By Proposition 4, our hypotheses on $u$ and $v$ imply that there exists a number $1 < q < s$ such that $w^{1-q'} = u^{1-q'} v \in A_{q/s}^\rho$. By applying Chebyshev inequality with exponent $q$ we obtain that

$$ I \leq \frac{C}{t^q} \int_{Q_j} |T(g_j v)(x)|^q u(x)^{1-q}(x) \, dx $$

$$ \leq \frac{C}{t^q} \int_{Q_j} |T(g_j v)(x)|^q w(x) \, dx $$

$$ \leq \frac{C}{t^q} \int_{2Q_j} g_j(x)^q u(x)v(x) \, dx $$

$$ \leq \frac{C}{t} \int_{2Q_j} g_j(x)u(x)v(x) \, dx, $$

where we have applied the boundedness of $T$ on $L^q(w)$ stated in Remark 2 and used that $g_j(x) \leq Ct$ for almost every $x \in 2Q_j$. From this point we can obtain the desired estimate for $I$ by repeating the argument given in page 20.

The estimate of $II$ can be performed exactly as we have done in page 21. It only remains to estimate $III$. By proceeding as in page 21 we have that

$$ III \leq \frac{C}{t} \sum_{i \in \mathbb{N}} \int_{P_{i,j}} |h_{i,j}(y)| v(y) \int_{2Q_j \setminus P_{i,j}} |K(x,y) - K(x,x_{i,j})| u(x) \, dx \, dy. $$

Let $r_{i,j}$ be as in the proof of Theorem 2 and $\theta \geq 0$ such that $u^{s'} \in A_{q}^{\rho,\theta}$. By applying condition (8) on the kernel, for $y \in P_{i,j}$ we obtain that
\[
\int_{2Q_i \setminus \bar{P}_{i,j}} |K(x, y) - K(x, x_{i,j})| u(x) \, dx
\]
\[
\leq \sum_{k \in \mathbb{N}} \int_{2^{k-1} r_{i,j} \leq |x-x_{i,j}| < 2^k r_{i,j}} |K(x, y) - K(x, x_{i,j})| u(x) \, dx
\]
\[
\leq \sum_{k \in \mathbb{N}} \left( \int_{2^{k-1} r_{i,j} \leq |x-x_{i,j}| < 2^k r_{i,j}} |K(x, y) - K(x, x_{i,j})|^\sigma \, dx \right)^{1/s} \left( \int_{|x-x_{i,j}| < 2^k r_{i,j}} u^{s'}(x) \, dx \right)^{1/s'}
\]
\[
\leq C_N \sum_{k \in \mathbb{N}} (2^k r_{i,j})^{-d/s'} 2^{-k\delta} \left( 1 + \frac{2^k r_{i,j}}{\rho(x_{i,j})} \right)^{N} (2^k r_{i,j})^{d/s'} \left( 1 + \frac{2^k r_{i,j}}{\rho(x_{i,j})} \right)^{\theta/s'} u(y)
\]
\[
\leq C_N u(y) \sum_{k \in \mathbb{N}} 2^{-k\delta} \left( 1 + \frac{2^k r_{i,j}}{\rho(x_{i,j})} \right)^{N+\theta/s'}
\]
\[
\leq C u(y),
\]
where we have chosen \(N > \theta/s'\). Therefore
\[
III \leq \frac{C}{t} \sum_{i \in \mathbb{N}} \int_{P_{i,j}} |h_{i,j}(y)| u(y) v(y) \, dy
\]
and we can conclude the estimate as in page 21. This completes the proof of (21) and we are done. \(\square\)

4. Application: Schrödinger type singular integrals

In this section we are going to apply the results obtained along this work to some operators associated to the Schrödinger semigroup generated by \(L = -\Delta + V\) on \(\mathbb{R}^d\), \(d \geq 3\). We will assume that the potential \(V\) is a non-negative function, not identically zero and satisfying a reverse-Hölder condition of order \(q > d/2\), this is
\[
\left( \frac{1}{|B|} \int_B V^q \right)^{1/q} \leq C \frac{1}{|B|} \int_B V,
\]
for all ball \(B\). As usual, we denote this by \(V \in RH_q\).

In [22], Shen defined the function
\[
\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(x) \, dx \leq 1 \right\},
\]
and proved that is a critical radius function, under the above assumptions. There, he also proved the boundedness on Lebesgue spaces of some singular integral operators associated to \(L\). Later, Shen’s results were extended in several directions, considering different function spaces or different type of inequalities concerning these singular integral operators.

Our purpose in this section is to obtain mixed weak type inequalities for the Schrödinger Riesz transforms of first and second order \(R_1 = \nabla L^{-1/2}\) and \(R_2 = \nabla^2 L^{-1}\) as well as for the operators \(T_\gamma = V^{\gamma} L^{-\gamma}\) for \(0 < \gamma < d/2\) and \(S_\gamma = V^{\gamma-1/2} \nabla L^{-\gamma}\) for \(1/2 \leq \gamma < 1\).

As it is implied by Shen’s work, if \(V \in RH_q\), with \(q < \infty\) then the only candidate to be a SCZO of \((\infty, \delta)\) type is \(R_1\) provided \(q > d\). The rest of the operators above are not even bounded in the whole range \(1 < p < \infty\).

We summarize in the next proposition some known properties for the above operators. A proof can be found in [8] where SCZO classes were widely discussed, however most of the required estimates may be traced back to [22].
Proposition 7. Let $V \geq 0$ satisfying a reverse Hölder inequality of order $q > d/2$ with $d \geq 3$. Then, for some $\delta > 0$, which may be different at each occurrence, we have:

(a) $R_1$ is a SCZO of $(\infty, \delta)$ type if we further ask $q \geq d$.
(b) $R_1$ is a SCZO of $(p_0, \delta)$ type, with $p_0$ such that $1/p_0 = 1/q - 1/d$.
(c) $R_2$ is a SCZO of $(q, \delta)$ type.
(d) $T_\gamma$ is a SCZO of $(q/\gamma, \delta)$ type for $0 < \gamma < d/2$.
(e) $S_\gamma$ is a SCZO of $(q_\gamma, \delta)$ type where $q_\gamma$ is such that

$$1/q_\gamma = (1/q - 1/d)^+ + (2\gamma - 1)/2q$$

with $1/2 < \gamma \leq 1$.

The proposition above, together with Theorem 2 and Theorem 3 allow us to obtain mixed inequalities for the singular integrals operators associated to $L$. We state the results in the following theorem. When considering $R_1$, the openness of the reverse Hölder condition allow us to consider the cases $d/2 < q < d$ and $q > d$.

Theorem 5. Let $V \in \text{RH}_q$ for $q > d/2$. Then the following inequalities hold for every positive $t$.

(a) If $d < q < \infty$, $u \in A_1^p$ and $v \in A_\infty^p(u)$, then

$$uv \left( \left\{ x \in \mathbb{R}^d : \frac{|R_1(fu)(x)|}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^d} |f(x)|u(x)v(x)dx.$$

(b) If $d/2 < q < d$, $u^{\beta_0} \in A_1^p$ and $v \in A_\infty^p(u^{\beta})$ for some $\beta > p_0^0$ then

$$uv \left( \left\{ x \in \mathbb{R}^d : \frac{|R_1(fu)(x)|}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^d} |f(x)|u(x)v(x)dx.$$

(c) If $u^{\beta} \in A_1^p$ and $v \in A_\infty^p(u^{\beta})$ for some $\beta > q'$ then

$$uv \left( \left\{ x \in \mathbb{R}^d : \frac{|R_2(fu)(x)|}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^d} |f(x)|u(x)v(x)dx.$$

(d) If $u^{(q/\gamma)'} \in A_1^p$ and $v \in A_\infty^p(u^{\beta})$ for some $\beta > (q/\gamma)'$ then

$$uv \left( \left\{ x \in \mathbb{R}^d : \frac{|T_\gamma(fu)(x)|}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^d} |f(x)|u(x)v(x)dx.$$

(e) If $u^{\beta} \in A_1^p$ and $v \in A_\infty^p(u^{\beta})$ for some $\beta > q'_\gamma$ then

$$uv \left( \left\{ x \in \mathbb{R}^d : \frac{|S_\gamma(fu)(x)|}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^d} |f(x)|u(x)v(x)dx.$$

As expected, in all of the above theorems we generalize the weighted weak $(1, 1)$ type proved in [7] for $R_1$, and in [8] for $R_2$, $T_\gamma$ and $S_\gamma$.

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