Confinement and focusing of geodesics in warped spacetimes

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Abstract. We have explored certain characteristic features of test particle trajectories in five dimensional, warped bulk geometries with a single thick brane. After a brief introduction on such spacetimes, we have first discussed timelike geodesic motion. The geodesic equations, which reduce to a first order autonomous dynamical system, have been solved using analytical and numerical methods. We have demonstrated how a growing (decaying) warp factor leads to oscillatory (runaway) trajectories, suggesting confinement (deconfinement). Further, we have pointed out differences that arise when we have a cosmological brane and/or a time-dependent extra dimension. Next, we have moved on to the kinematics of geodesic congruences. The evolution of the kinematical variables (expansion, rotation and shear) along geodesic flows have been obtained using analytical and numerical approaches, with particular emphasis on the required conditions and occurrence of geodesic focusing.

1. Introduction

Warped braneworld models have been a topic of active research interest for more than a decade now [1]. Such models involve extra dimensions (beyond the usual four). It is assumed that we live on an embedded timelike membrane. The higher (usually five) dimensional line element has a warp factor, which leads, in general, to a scaled Minkowski metric on the embedded four dimensional hypersurface. The models are useful in (i) solving the hierarchy problem without introducing new hierarchies, and (ii) providing an alternative to the usual notion of compactification, via a mechanism for localizing fields on the brane [2]. The zero modes of the bulk fields provide the usual classical physics on the brane, while the contributions of the higher modes to various physical processes lead to corrections reflecting the presence of extra dimensions. For extensive discussions on these models, the reader is referred to well-known review articles [2, 3].

The placing of branes in this class of models is done in various ways. Introducing a derivative discontinuity (say, a $|x|$ dependence) in the metric functions (specifically, the warp factor) yields a $\delta$-function on the right hand side of the bulk Einstein’s equations, which acts as a localized...
source (the brane). This is known as the thin brane model. The original Randall–Sundrum (RS) model has two such thin–branes and the bulk spacetime is orbifolded \((S_1/Z_2)\). Alternatively, the brane can be realised as a scalar field domain wall, wherein the metric functions are smooth, resulting in the so-called thick brane models [2].

In this article, we have considered thick branes only. Our primary aim is to show how the nature (growing/decaying) of the warp factor is reflected in the behaviour of geodesics and geodesic congruences. We have considered a wider class of models, wherein a cosmological on–brane metric and a time dependent extra dimension have been introduced and their effects on geodesics and geodesic flows have also been studied.

2. Geodesics and confinement
The line element of interest to us here is:

\[
ds^2 = e^{2f(\sigma)} \left( -dt^2 + a^2(t)|d\vec{x}|^2 \right) + b^2(t)d\sigma^2 ,
\]

where \(e^{2f(\sigma)}\), \(a(t)\) and \(b(t)\) are the warp factor, cosmological scale factor and the extra dimensional scale factor respectively. The warp factor is chosen with \(f(\sigma) = \pm \ln \cosh k\sigma\), where the plus corresponds to a growing one and the minus, to a decaying. The bulk timelike geodesic equations reduce to the following set of equations:

\[
\dot{\eta} = \frac{e^{-f(\sigma)}}{a(\eta)} \left[ \epsilon + \frac{\chi^2}{b^2(\eta)} + \sum_{i=1}^{2} \frac{D_i^2 e^{2f(\sigma)}a^2(\eta)}{e^{2f(\sigma)}a^2(\eta)} \right],
\]

\[
\dot{x}_i = \frac{D_i e^{-2f(\sigma)}}{a^2(\eta)},
\]

\[
\dot{\sigma} = \frac{\chi}{b^2(\eta)}, \text{ and}
\]

\[
\dot{\chi} = -f'(\sigma) \left( \epsilon + \frac{\chi^2}{b^2(\eta)} \right),
\]

where we have introduced a new variable \(\chi\) and also used the on–brane conformal time \(\eta\).

The above set of equations represent a first–order dynamical system, which can be analysed.
In order to understand the role of the warp factor, we have switched off \( b(t) \) (i.e., set it to 1). The \( \sigma-\chi \) set then reduces to an independent dynamical system, which can be solved analytically (for details see [4]). In Figure 1, we have shown the phase plots for this simpler dynamical system. It is evident that for a growing (decaying) warp factor, we have oscillatory (runaway) motion. This result is further substantiated by numerically solving the dynamical system and obtaining the trajectories for specific forms of \( a(t) \) and \( b(t) \). The oscillatory behavior of geodesics for the growing warp factor indicates confinement of trajectories. Varying \( a(t) \) and \( b(t) \) produces qualitative changes as shown in the numerous figures in [4].

### 3. Geodesic focusing

Having dealt with geodesics, we have now turned to geodesic congruences. It is well known that the kinematics of such congruences is described by the nature of evolution (along the geodesic flow) of the expansion \( \theta \), the shear \( \sigma_{ij} \) and the rotation \( \omega_{ij} \). The evolution equations are the Raychaudhuri equations, which constitute a set of first order, coupled ordinary differential equations with the affine parameter \( \lambda \) as the independent variable. Briefly, one defines:

\[
B_{ij} = \nabla_j v_i \equiv \sigma_{ij} + \omega_{ij} + \frac{1}{3} h_{ij} \theta \quad \text{and} \quad h_{ij} = g_{ij} + v_i v_j ,
\]

and the evolution of \( B_{ij} \) is given as:

\[
u^k \nabla_k B_{ij} = -B_{ik} B^k_j - R_{ikjl} u^k u^l .
\]

One may separate the trace, symmetric traceless and antisymmetric parts of the above equation, in order to obtain individual equations for \( \theta, \sigma_{ij} \) and \( \omega_{ij} \). The main aim is: Given initial values at some \( \lambda \), what is the nature of \( \theta, \sigma_{ij} \) and \( \omega_{ij} \) at another future \( \lambda \)? To this end, we have solved the equation for \( B_{ij} \) numerically, with appropriate initial conditions, and then obtained the expansion, shear and rotation evolution. This is done for the geodesics obtained numerically in [4]. We have discussed below the behaviour of the expansion \( \theta \) for various cases. It is known that if the timelike convergence condition \( R_{ij} u^i u^j \geq 0 \) holds, geodesics focus (meet) at a finite \( \lambda \). This is the focusing theorem.

**Figure 2.** \( \theta \) vs. \( \lambda \) (growing warp factor); zero (\( \theta_0 \)) and high (\( \theta_h \)) initial rotation.

**Figure 3.** \( \theta \) vs. \( \lambda \) (decaying warp factor); zero (\( \theta_0 \)) and high (\( \theta_h \)) initial rotation.

Figure 2 shows the numerically obtained behaviour for a growing warp factor, and FRW radiation dominated \( a(\eta) = 2\eta, \) and \( b(\eta) = 1 + \frac{1}{\eta} \). Congruences without initial rotation, expand slowly at first, but later become focused at a finite \( \lambda \). For high initial rotation, expansion inflates numerically.

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for a very short while, but eventually the geodesics gets focused again at another finite but larger value of $\lambda$. Rotation is always dominated by shear, which grows faster. An initial large rotation delays focusing.

Figure 3 is for a decaying warp factor with $a(\eta), b(\eta)$ as in Figure 2. Here, we have noted that congruences without any initial rotation, do come closer to each other monotonically, but focus only at $\sigma \to \infty$ (follows from the solution of the geodesic equation). When high initial rotation is introduced, the congruence expands initially, but eventually, geodesics tend to focus again, asymptotically.

Figure 4 is for a decaying warp factor, de Sitter $a(\eta) = \frac{1}{(1-\eta)}$, and $b(\eta) = 1 - \frac{3}{\eta}$. In the presence of a decaying warp factor, a de Sitter brane with an asymptotically static extra dimension, we have an example where geodesics are de-focused irrespective of initial rotation. Even though, it seems that at late times shear totally dominates rotation, it is the curvature term $(R_{AB}u^Au^B)$ in the Raychaudhuri equation for the expansion that becomes dominant and causes the de-focusing.

Figure 5 shows the behaviour of this term and justifies the above mentioned behaviour of the expansion.

4. Conclusions
In summary, we have shown how the nature of warping is reflected in the behaviour of geodesics. Confinement of trajectories is linked to a growing warp factor. We have also obtained the evolution of $\theta$ for varying initial conditions (rotation, in particular), and different $a(\eta)$ and $b(\eta)$. Some curious aspects of the evolution of $\theta$ have been pointed out. The interested reader is referred to [4, 5] for further details.

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