Noise-to-signal transition of a Brownian particle in the cubic potential: I. general theory

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Abstract

The noise-to-signal transitions are very interesting processes in physics, as they might transform environmental noise to useful mechanical effects. We theoretically analyze stochastic noise-to-signal transition of overdamped Brownian motion of a particle in the cubic potential. The particle reaches thermal equilibrium with its environment in the quadratic potential which is suddenly swapped to the cubic potential. We predict a simultaneous increase of both the displacement and signal-to-noise ratio in the cubic potential for the position linearly powered by the temperature of the particle environment. The short-time analysis and numerical simulations fully confirm different dynamical regimes of this noise-to-signal transition.

Keywords: optically trapped particles, Brownian motion, optomechanics

1. Introduction

Recent progress in the optical control of mechanical systems opens new ways to investigate a wide variety of mechanical effects in thermodynamics and statistical physics [1–4], stochastic dynamics [5–9] and quantum mechanics [10, 11]. In the near future, the most interesting in these fields are likely to be nonlinear stochastic effects which autonomously transform environmental noise to useful mechanical effects. A nonlinearity of potential brings mechanical systems out of thermal equilibrium with their environment, changing their statistics far away from thermal statistics of the environment. Such mechanical processes can therefore be principally used as a primary source of coherent mechanical displacement and mechanical oscillations. They might allow us to do autonomously mechanical work at micrometer and nanometer distances. Thinking ahead, about a long-term vision, such nonlinear processes might be extremely interesting in the underdamped regime, which in some cases they have already reached [6, 7, 9]. We envision that as the environmental noise approaches a floor of minimal quantum noise [10, 11], we reach the area of novel and interesting phenomena of unexplored highly nonlinear quantum dynamics. These nonlinear quantum phenomena are also required for quantum information processing with complex systems. Importantly, the basic cubic nonlinearity is required to principally achieve any nonlinear operation in the quantum regime [12–14]. In future, the quantum-mechanical systems can substitute pioneering all-optical methods [15–17] exhibiting only limited strength of highly nonlinear quantum interactions.

Stimulated by this knowledge, we focus on basic stochastic dynamics provided by a static cubic nonlinearity. In general terminology of nonlinear dynamics, the damped oscillator with a cubic nonlinearity is known as the Helmholtz oscillator [18, 19]. The dynamical features have been discussed, for example, in references [20, 21], however, the stochastic properties of a damped Helmholtz oscillator have not been examined yet. It is attractive, since the static cubic nonlinearity is a candidate to generate mechanical displacement induced by the environmental fluctuations. Importantly, the cubic nonlinearity is also an element of many more
complex nonlinear stochastic effects in various fields of science, see references in [20, 21]. In the past, the main focus of attention has been to analyze and experimentally test an escape time from a metastable state in the potential (famous Kramers problem) [22–24]. Our direction is complementary, as it is focused on dynamical noise-to-signal transitions in the potential, positively exploiting instabilities. It seems to us that with all the progress in optical trapping the cubic potential for a single Brownian particle can be versatilely prepared by two pairs of counter-propagating laser beams [25].

In the paper, we principally analyze short-time nonlinear stochastic dynamics of an overdamped particle in the cubic potential and numerically confirm the predicted behavior. In the proof-of-principle theoretical analysis, we focus on the noise-induced position displacement. Importantly, we also predict that signal-to-noise ratio is simultaneously powered noise-induced position displacement. Importantly, we also predict the nontrivial character of cubic nonlinearity as an element of many nonlinear effects. We compare different dynamical regimes, when the displacement in the position is dominantly induced by initial thermal noise or environmental thermal noise during the cubic nonlinear dynamics.

2. Overdamped Brownian motion in a quadratic potential

Einstein–Ornstein–Uhlenbeck classical dynamics of Brownian motion of a particle trapped in potential $V(x)$ is described by the stochastic differential equation

$$m \frac{d^2 x(t)}{dt^2} = -\frac{\partial}{\partial x} V(x) - \gamma_0 \frac{dx(t)}{dt} + \sqrt{2 \gamma_0 k_B T} F(t)$$

for trajectory $x(t)$ of particle motion, where $V(x)$ is the optical potential, $\gamma_0 = 6 \pi \eta r$ is the friction coefficient for a spherical particle (from Stokes’s law) of radius $r$ and mass $m$, $\eta$ is the viscosity of liquid, $T$ the absolute temperature, $k_B$ the Boltzmann constant, and the Langevin force $F(t)$ satisfying $\langle F(t) \rangle = 0$, $\langle F(t) F(t') \rangle = \delta(t - t')$. In the overdamped regime the inertial term can be neglected and the Brownian motion can be approximated by

$$\frac{dx(t)}{dt} = -\frac{1}{\gamma_0} \frac{\partial}{\partial x} V(x) + \sqrt{2 D} F(t)$$

where

$$D = \frac{k_B T}{\gamma_0}$$

denotes the diffusion constant. For numerical simulations the corresponding stochastic differential equation with Wiener process can be rewritten as

$$dx(t) = -\frac{1}{\gamma_0} \frac{\partial}{\partial x} V(x) dt + \sqrt{2D} dW$$

where $\langle W(t) \rangle = 0$, $\langle W(t) W(t') \rangle = t' - t$, $\langle W^2(t) \rangle = t$ for any $t \geq 0$. The time derivative $dW(t)/dt = F(t)$ describes white noise.

To set an initial distribution of the particle for the investigation of cubic potential, let us first describe the overdamped motion of a Brownian particle moving in the quadratic potential forming a trap for the particle

$$V_2(x) = \frac{1}{2} \mu_2 x^2 = \frac{1}{2} \gamma_0 \omega_C x^2,$$

where $\mu_2$ denotes the quadratic nonlinear coefficient in the potential power series corresponding to the stiffness of the trap and $\omega_C = \mu_2/\gamma_0$ is the cut-off or corner frequency. This transition dynamics in the overdamped regime is described by

$$\frac{dx(t)}{dt} = -\omega_C x(t) + \sqrt{2D} F(t)$$

having a formal solution

$$x(t) = e^{-\omega_C (t-t_0)} x(t_0) + \sqrt{2D} \int_{t_0}^t e^{\omega_C s} F(s) ds.$$  

This solution gives the exponentially damped first moment

$$\langle x(t) \rangle = e^{-\omega_C (t-t_0)} \langle x(t_0) \rangle$$

and the saturating second moment

$$\langle x(t)^2 \rangle = e^{-2\omega_C (t-t_0)} \langle x(t_0)^2 \rangle + \frac{D}{\omega_C} (1 - e^{-2\omega_C (t-t_0)}).$$

As $\omega_C > 0$ increases (i.e. trap stiffness increases), the particle can faster approach a steady state represented by a limit $t \to \infty$. At the steady state, the variance of the position of zero-mean Gaussian distribution is equal to

$$\langle (\Delta x)^2 \rangle_{\infty} = \frac{D}{\omega_C} = \frac{k_B T}{\mu_2}.$$  

Clearly, this variance is inversely proportional to the trap stiffness $\mu_2$ and directly proportional to the temperature of the surroundings.

3. Overdamped Brownian motion in a cubic potential

Let us consider the cubic potential of the following form

$$V_3(x) = \frac{1}{3} \mu_3 x^3 = \frac{1}{3} \gamma_0 \kappa x^3$$

where the dynamics of an overdamped Brownian particle can be described as

$$\frac{dx(t)}{dt} = -\kappa x(t) + \sqrt{2D} F(t).$$

3.1. Deterministic motion

Let us first ignore the stochastic term in equation (12). The deterministic particle trajectory is done by the formal solution of

$$\frac{dx(t)}{dt} = -\kappa x^2(t)$$

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As $\omega_C > 0$ increases (i.e. trap stiffness increases), the particle can faster approach a steady state represented by a limit $t \to \infty$. At the steady state, the variance of the position of zero-mean Gaussian distribution is equal to

$$\langle (\Delta x)^2 \rangle_{\infty} = \frac{D}{\omega_C} = \frac{k_B T}{\mu_2}.$$  

Clearly, this variance is inversely proportional to the trap stiffness $\mu_2$ and directly proportional to the temperature of the surroundings.
\[ x(t) = \frac{x(t_0)}{1 + \kappa x(t_0)(t - t_0)}. \] (14)

Obviously for initial \( x(t_0) > 0 \) one gets \( x(t) \to 0 \) for \( t \to \infty \).
However, for \( x(t_0) < 0 \) one gets diverging solution \( x(t) \to -\infty \) for \( t = t_0 + \tau_\infty^0 \), where \( \tau_\infty^0 = 1/(\kappa|x(t_0)|) \) is a finite divergence time for deterministic dynamics. Thus \( \tau_\infty^0 \) can be understood as the characteristic time-limiting deterministic particle motion from a fixed initial position \( x(t_0) < 0 \) in this type of potential. It is the maximal time of dynamical evolution, therefore the dynamics runs only for a time interval shorter than \( \tau_\infty^0 \) for \( x(t_0) < 0 \).

From \( \tau_\infty^0 = 1/(\kappa|x(t_0)|) \) it follows that more negative \( x(t_0) \) means that a divergency of \( x(t) \) can appear for a shorter time \( \tau_\infty^0 \). For random initial values \( x(t_0) \) reaching negative values, that divergency can appear although \( \langle x(t_0) \rangle = 0 \). If any trajectory approaches that divergency, then the statistics of the whole ensemble of the trajectories is very strongly influenced by the divergency. It cannot be statistically analyzed because the statistical moments calculated from the trajectories may start to diverge as well. Assuming the probability density function of initial positions \( x(t_0) \) in the form of normal distribution

\[ P(x, 0) = \frac{1}{\sqrt{2\pi(\langle \Delta x(t_0) \rangle)^2}} \exp \left[ -\frac{(x - \langle x(t_0) \rangle)^2}{2\langle (\Delta x(t_0))^2 \rangle} \right], \] (15)

with initial mean \( \langle x(t_0) \rangle \) and variance \( \langle (\Delta x(t_0))^2 \rangle \), one can derive the probability density function of the particle positions at time \( t \) as

\[ P(x, t) = \frac{\exp \left\{ -\frac{t}{1 - \kappa x(t_0)^2 - \langle x(t_0) \rangle} \right\}}{\sqrt{2\pi(\langle \Delta x(t_0)^2 \rangle)(1 - \kappa x(t_0)^2)}}. \] (16)

Note, \( P(x, t) \) satisfies the diffusion Fokker–Planck equation

\[ \frac{\partial}{\partial t} P(x, t) = \kappa \frac{\partial}{\partial x} [x^2 P(x, t)] \] (17)

for the overdamped Brownian motion. Note, the normalization of \( P(x, t) \) is over half plane \( t \geq t_0 \) and \(-\infty < x < \infty \). However from (14), \( x(t) \) diverges near \( \kappa x(t - t_0) = -1 \) and therefore, when any trajectory approaches the area \( x(t - t_0) < -\kappa^{-1} \), we can no longer use (16) to describe the statistics by the moments. Quantitatively, if the divergence probability \( P_\infty(t) = 1 - \int_{\infty}^{\infty} H(1 + \kappa x) P(x, t) \, dx \), where \( H(z) \) is a Heaviside step function, does not reach negligible values, the real statistics are greatly influenced by the divergency. To carefully keep very small \( P_\infty \leq 10^{-5} \), the numerically estimated maximal time interval can be approximatively bounded by

\[ t - t_0 < \frac{1}{2\pi \kappa \sqrt{\langle (\Delta x(t_0))^2 \rangle}} \equiv \tau_\infty. \] (18)

This boundary was numerically verified for large scale of \( \kappa(t - t_0) \) and \( \langle (\Delta x(t_0))^2 \rangle \), as can be seen later in the numerical simulations. Apparently, for \( \langle x(t_0) \rangle = 0, \tau_\infty \) becomes limiting for the time duration instead of the previous infinite \( \tau_\infty^0 \) for \( x(t_0) = 0 \) corresponding to deterministic dynamics.

We, however, do not focus on the Kramers problem in the cubic potential, as we are interested in the dynamical evolution as a promising candidate for the noise-to-signal transition. For the time duration substantially shorter than \( \tau_\infty \), the evolution of the mean and the variance of stochastic position \( x(t) \) can be determined directly by numerical integrations of the first and second moment \( \langle x(t) \rangle = \int_{-\infty}^{\infty} P(x, t) x \, dx, \langle x(t)^2 \rangle = \int_{-\infty}^{\infty} P(x, t) x^2 \, dx \). If the diffusion can be neglected, the short-term dynamics can be approximated by an expansion of (16) in time interval \( t - t_0 < \tau_\infty \), giving

\[ \langle x(t) \rangle \approx \langle x(t_0) \rangle - \kappa(t - t_0)(\langle (\Delta x(t_0))^2 \rangle + \langle x(t_0) \rangle^2), \]
\[ \langle (\Delta x(t))^2 \rangle \approx \langle (\Delta x(t_0))^2 \rangle - 4\kappa(t - t_0)\langle x(t_0) \rangle \langle (\Delta x(t_0))^2 \rangle. \] (19)

For \( \langle x(t_0) \rangle = 0 \), the mean \( \langle x(t) \rangle \) becomes more negative for large \( \langle (\Delta x(t_0))^2 \rangle \), whereas the variance \( \langle (\Delta x(t))^2 \rangle \) remains unchanged for the considered short time. Note here, that even without a diffusion term, the initial thermal noise induces pure drift \( \langle x(t) \rangle \) of the particle without any increase in its variance \( \langle (\Delta x(t))^2 \rangle \). This is the first evidence of a nontrivial noise-induced displacement effect in the cubic potential.

### 3.2. Short-time approximation of stochastic motion

Let us now consider the Brownian stochastic dynamics of a particle involving the diffusion \( D > 0 \) for a short interval \( (t_0, t) \) of time, where \( t - t_0 < \tau_\infty \) is satisfied. We assume that at \( t = t_0 \) we start from the thermal equilibrium state developed in the quadratic potential described in equation (5) by its first \( \langle (x(t_0))^2 \rangle \) and second \( \langle (x(t_0))^3 \rangle \) moment. Since the origins of these two potentials \( V_2(x) \) and \( V_3(x) \) do not necessarily coincide, we consider a displacement \( x(t_0) \) as an initial mean position relative to the origin of the cubic potential. After a formal integration of equation (12) in an interval of time \( (t_0, t_0, \ldots, t_{n-1}, t_n = t) \), where \( n \) is the order of our short-time approximation, we obtain

\[ x(t_n) = x(t_0) - \kappa \int_{t_0}^{t_n} x(t_{n-1})^2 \, dt_{n-1} + \sqrt{2D} \int_{t_0}^{t_n} F(t) \, dt. \] (20)

The relevant order \( n \) will be used to explain expected noise-induced effects without destructive influence of the above-described divergency.
3.2.1. The first-order approximation. In the first order of the short-time approximation, we simply obtain
\[ x(t) = x(t_0) - \kappa x(t_0)^3 (t - t_0) + \sqrt{2D} \int_{t_0}^{t} F(t) dt, \]
where we considered that \( x(t) \) does not change very much during time interval \((t_0, t)\). The mean displacement
\[ \langle x(t) \rangle = \langle x(t_0) \rangle - \kappa \langle x(t_0)^3 \rangle (t - t_0), \]
evolves linearly in time after we swap to the cubic potential. Noise-to-signal transition transforms the initial second moment \( \langle x(t_0)^2 \rangle \) into \( \langle (\Delta x(t))^2 \rangle \) of the Gaussian distribution at time \( t_0 \) to the mean displacement \( \langle x(t) \rangle \) at time \( t \). For the first-order approximation in the cubic potential, the effect of Langevin force \( F(t) \) does not contribute. At the beginning, a stochastic nature of the environment does not therefore affect the particle motion. The displacement \( \langle x(t) \rangle \) is only dictated by the initial position distribution \( \langle x(t_0)^3 \rangle \) as in equation \((19)\). For vanishing \( \langle x(t_0) \rangle = 0 \), the mean displacement simply approaches
\[ \langle x(t) \rangle = -\kappa \langle (\Delta x(t_0))^3 \rangle (t - t_0) \]
and thus the nonzero mean is purely induced by the initial variance \( \langle (\Delta x(t_0))^2 \rangle \) of the noise.

The second momentum of position simultaneously approaches
\[ \langle x(t)^2 \rangle = \langle x(t_0)^2 \rangle - 2\kappa \langle x(t_0)^3 \rangle (t - t_0) + 2D(t - t_0) \]
up to the terms being linear in time duration. For \( \langle x(t_0) \rangle = 0 \), we can use \( \langle x(t_0)^3 \rangle = \langle x(t_0)^3 \rangle^3 + 3 \langle x(t_0) \rangle \langle (\Delta x(t_0))^2 \rangle = 0 \) due to Gaussian statistics of initial thermal equilibrium distribution. The variance
\[ \langle (\Delta x(t))^2 \rangle = \langle (\Delta x(t_0))^2 \rangle + 2D(t - t_0) \]
does not depend on \( \kappa \) and increases linearly in time with a slope determined by the diffusion constant \( D \). It is analogous to pure diffusion of a free particle without any influence of the cubic nonlinearity. For very small diffusion, it converges to previous results in equation \((19)\). Moreover, for narrow initial distribution \( \langle (\Delta x(t_0))^2 \rangle \to 0 \), the deterministic dynamics approaches the short-time approximation of \((14)\) derived previously.

We can define the cubic characteristic time as
\[ \tau_3 \equiv \frac{1}{\kappa \sqrt{\langle (\Delta x(t_0))^2 \rangle}} \]
giving a time \( t = \tau_3 \) needed for the system to reach \( \langle x(t_3) \rangle = \sqrt{\langle (\Delta x(t_0))^2 \rangle} \), i.e. the mean displacement is equal to the root-mean-square of the initial variance of the noise. This quantity is important for an estimate of the speed of the process going towards high-quality mechanical displacement. Since the cubic characteristic time \( \tau_3 \) is comparable to \( \tau_{\text{sec}} \), we can reach in this simple setting only a fraction of the cubic characteristic time for very small \( P_c < 10^{-7} \). The displacement therefore cannot even reach the root-mean-square of the initial variance. However, \((22)\) and \((25)\) are sufficient to indicate the suitability of the cubic potential for an observation of the noise-to-signal transition.

3.2.2. The second-order approximation. From equation \((23)\) it follows that the second-order short-time approximation is necessary for negligible \( \langle (\Delta x(t_0))^2 \rangle \), otherwise the mean displacement in equation \((23)\) is vanishing. By iterating the formal solution from equation \((20)\) up to the second order, we get
\[ x(t) = x(t_0) - \kappa \int_{t_0}^{t} [x(t_0) - \kappa x(t_0)^3 (t_1 - t_0) + \sqrt{2D} \int_{t_0}^{t_1} F(t) dt \]
\[ \times [x(t_0) - \kappa x(t_0)^3 (t_1 - t_0) + \sqrt{2D} \int_{t_0}^{t_1} F(t) dt] dt_1 \]
\[ + \sqrt{2D} \int_{t_0}^{t} F(t) dt_1. \]
It gives the mean value
\[ \langle x(t) \rangle = \langle x(t_0) \rangle - \kappa \langle x(t_0)^3 \rangle (t - t_0) - \kappa D(t - t_0)^2 \]
\[ + \kappa^2 \langle x(t_0)^3 \rangle (t - t_0)^2 \]
expressed up to quadratic time duration \( (t - t_0)^2 \). The first and the last terms in equation \((28)\) vanish for \( \langle x(t_0) \rangle = 0 \), and the mean displacement from the origin
\[ \langle x(t) \rangle = -\kappa \langle (\Delta x(t_0))^3 \rangle (t - t_0) - \kappa D(t - t_0)^2 \]
increases due to two different stochastic contributions. The term linear in time duration has already been discussed and it comes from an uncertainty of the initial state. A new quadratic term in time duration arises from the diffusion in the cubic nonlinearity. The noise from the environment can therefore turn to a displacement, moreover, it increases quadratically in time. It dominates for the large diffusion when \( D(t - t_0) > \langle (\Delta x(t_0))^2 \rangle \). We can therefore define another characteristic time
\[ \tau_D = \frac{\langle (\Delta x(t_0))^2 \rangle}{D} \equiv \frac{1}{\omega_c}, \]
which has to be substantially exceeded to be in the regime of dominant diffusion. Of course, it is only possible if \( \tau_D \) is shorter than a time when trajectories approach the divergency.

It is clearly a noise-to-signal transition powered by thermal fluctuations of the environment, however, the evolution of noise has to be analyzed as well to get the complete picture. The variance calculated up to the quadratic terms in time duration has the following form
\[ \langle (\Delta x(t))^2 \rangle = \langle (\Delta x(t_0))^2 \rangle + 2D(t - t_0) \]
\[ + 8\kappa^2 (t - t_0)^2 \langle (\Delta x(t_0))^2 \rangle. \]
The first two terms have already been discussed. The last term depends on the initial variance, but not on the diffusion coefficient. If the initial variance is negligible, only the second term contributes and the variance increases only linearly in time. Now, it is important to compare the dynamics
of the mean \( \langle x(t) \rangle \) and \( \langle (\Delta x(t_0))^2 \rangle \) to further justify the noise-to-signal transition.

### 3.3. Quantification of signal-to-noise transition

Let us consider a possible complete transfer of the particle stochastic position \( x(t) \to \eta \eta(t) \) to the output momentum \( p_{out}(t) \) of free motion of another ideal thought particle representing the remaining output of our procedure, where \( \eta \) is an arbitrary multiplicative constant. The free motion of that thought particle has only average kinetic energy \( \langle E_k \rangle \propto \langle p_{out}^2 \rangle \). By the mapping, the average of the kinetic energy swaps to \( \langle E_k(t) \rangle \propto \langle x(t)^2 \rangle = \langle x(t)^2 \rangle + \langle (\Delta x(t))^2 \rangle \). It can be naturally decomposed to a sum of the coherent part of energy \( \langle E_C(t) \rangle \propto \langle x(t)^2 \rangle \), representing a useful signal in the motion, and the incoherent part of the energy \( \langle E_{NC}(t) \rangle \propto \langle (\Delta x(t))^2 \rangle \), representing a noise disturbing the motion. Thus, following this consideration, we can quantify the stochastic dynamics of the position during the overdamped dynamics as the following ratio

\[
\text{SNR} = \frac{\langle E_k(t) \rangle}{\langle E_{NC}(t) \rangle} = \frac{\langle x(t)^2 \rangle}{\langle (\Delta x(t))^2 \rangle}
\]  

(32)

simply describing the signal-to-noise ratio between the coherent and incoherent part of the principally extractable kinetic energy. The signal-to-noise ratio has been used for many purposes in science, as it is a simple quantifier of the quality of the operations under disturbing influence of the noise. To indicate a starting noise-to-signal transition during short-time dynamics, it is necessary that both \( x(t) \) and SNR\( (t) \) are monotonously increasing in time and are powered by initial thermal noise or thermal noise of the environment during the transition. It is a pre-requisite to later reach SNR > 1, when generated coherent motion dominates over the accompanied noise. If SNR\( (t) \) starts to decrease, then a tendency to the noise-to-signal transition stops, irrespective of increasing mean value \( \langle x(t) \rangle \). The maximum of signal-to-noise ratio therefore determines another upper-bound for the signal-to-noise transition, which can be much stricter than previous \( \tau_N \) and \( \tau_D \).

Now, we will check two limits of weak and strong diffusion, which are interesting to understand regarding the different aspects of the noise-to-signal transitions.

For \( t - t_0 \ll \tau_D \), when the effect of diffusion is negligible, by combining equations (23) and (25) the signal-to-noise ratio

\[
\text{SNR}(t) \approx \kappa^2 \langle (\Delta x(t_0))^2 \rangle (t - t_0)^2
\]  

(33)

increases quadratically in time, whereas the displacement (29) grows linearly. In this case, SNR\( (t) \) depends linearly on the initial variance \( \langle (\Delta x(t_0))^2 \rangle \), which quantifies the noise supplying the noise-to-signal transition. The signal-to-noise ratio is therefore powered by initial fluctuations of the particle.

For \( t - t_0 \gg \tau_D \), when the diffusion plays a significant role, we can use equations (29) and (31) to get

\[
\text{SNR}(t) \approx \frac{1}{2} \kappa^2 D (t - t_0)^3
\]  

(34)

exhibiting a cubic temporal dependency. The signal-to-noise ratio therefore increases faster in time than in the previous case, due to the Brownian noise continuously supplying energy to the cubic potential. Different to the previous limit, the signal-to-noise ratio is therefore continuously powered by the environmental Brownian noise, not only the initial noise. However, without detailed numerical simulations, presented in the next Section, we cannot specify the maximal time for large-diffusion limit, limited by the divergency of trajectories.

In a real situation, the short-time noise-to-signal transition can be powered by a mixture of the noise of the initial state and the Brownian noise from the environment. The temporal dependency of SNR can therefore be between linear and cubic, however, it is monotonous and powered by the noise in any case.

### 4. Stochastic simulations of noise-to-signal transition

We obtain short-time analytical evidence of the noise-to-signal transitions in the cubic potential and its different aspects in the limits of small and large diffusion. To demonstrate a validity of short-time approximation and limitations by the divergency, we now use an example of stochastic simulations based on (4). To simplify the numerical simulations, we consider all quantities dimensionless and select suitable numerical values to be able to illustrate the main effects. The stochastic trajectories first evolve in a quadratic potential with cut-off frequency \( \omega_c = 1 \) and diffusion constant \( D = 0.1 \), until they approximately reach a steady state at time \( t_0 = 4 \). At the steady state, the Gaussian distribution of zero-mean position reaches the variance \( \langle (\Delta x(t_0))^2 \rangle = \frac{D}{\omega_c^2} = 0.1 \). The short-time dynamics in cubic potential with \( \kappa = 3 \) starts from \( t_0 = 4 \) and runs for a short time until \( t = 4.1 \).

In figure 1, the regime with small diffusion \( D = 0.1 \), \( t - t_0 < 0.1 < t_D = 0.168 \) and \( \langle (\Delta x(t_0))^2 \rangle = 0.1 \) is simulated. We observe very good agreement between the numerical simulations and the analytical short-time approximation. It verifies the linear increase of the displacement in equation (23) and the quadratic tendency of signal-to-noise ratio in equation (33) predicted by the short-time approximation. Apparently, the stability of numerical simulations in the cubic potential is limited. For time \( t > 4.2 \), the numerical simulations diverge in both displacement \( \langle x(t) \rangle \) and variance \( \langle (\Delta x(t))^2 \rangle \). The divergency in the variance is much faster, therefore signal-to-noise ratio vanishes quickly. The signal-to-noise ratio appears to be very good for the justification of the validity of the stochastic simulations. In figure 2, the linear dependency of (23) and (33) on the variance \( \langle (\Delta x(t_0))^2 \rangle \) of the initial noise for both the mean \( \langle x(t) \rangle \) and SNR\( (t) \) are also well certified.
In figure 3, the numerical simulations are performed for the identical parameters, only with larger diffusion constant $D = 10$. It is the range of validity of the large-diffusion approximation since for $t - t_0 < 10^{-3}$, it becomes larger than $(\Delta x(t_0)^2) = 0.1$. From time $t = 4.1$ we have already started to observe strong instability in the numerical simulations. It appears faster than in the case of small diffusion. In figure 4, the mean and signal-to-noise ratio is depicted as a function of the diffusion coefficient $D$ during the nonlinear evolution. The numerical simulations are still in good qualitative agreement with the theoretical predictions of the quadratic temporal dependency in (29) and the cubic temporal dependency in (34) from the short-time approximation already for 5000 trajectories, although the numerical stability is weaker compared to the previous case. For more precise evolution, a larger number of trajectories is required. In figure 4, the linear dependency of (29) and (34) on the diffusion coefficient $D$ for both the mean $\langle x(t) \rangle$ and $\text{SNR}(t)$ are also very well approved.

The numerical simulations have verified the theoretically predicted quantitative aspects of the noise-to-signal transition in both regimes of small and large diffusion. They also simultaneously checked the stability of the short-time predictions. It is visible that in the regime of the large diffusion, the time when divergency starts to limit the dynamics is shorter.

5. Noise-to-signal transition in an environment with constant temperature

In many experiments, the environment of the particle can be considered with a constant temperature $T$. For the selected value of $\gamma_0$, the experiment can be then controlled by the
potential parameters $\mu_2$ and $\mu_3$. Using equations (3) and (10) for the diffusion coefficient and the initial variance $\langle (\Delta x(t_0))^2 \rangle$ at the same thermal equilibrium, respectively, we obtain

$$\langle x(t) \rangle = -\mu_3\left[\frac{1}{\mu_2}(t - t_0) + \frac{1}{\gamma_0}(t - t_0)^2\right] \frac{k_B T}{\gamma_0}$$  \hspace{1cm} (35)

from equation (29) for the mean displacement. Both the terms in (35) are equally driven by the temperature of the environment. The particle therefore obtains the non-vanishing coherent part of its energy driven by thermal fluctuations from the environment. Using equations (3) and (10), the variance (31) can be rewritten to

$$\langle (\Delta x(t))^2 \rangle = \frac{2\gamma_0}{\mu_2} + 2(t - t_0)$$

$$+ 8\left[\frac{\mu_2^{3/2}}{2} \frac{k_B T}{\gamma_0}\right] \frac{k_B T}{\gamma_0}.$$  \hspace{1cm} (36)

The variance can depend linearly or quadratically on the temperature $T$ of the Brownian environment. A dominant quadratic dependency is detrimental because SNR is not then powered by the temperature $T$. The equations (35) and (36) allow us to discuss a control in the previous limits of weak and strong diffusion using the potential parameters $\mu_2$ and $\mu_3$ easily controllable for example by the methods of optical manipulation with particles [25].

For a short time $t - t_0 \ll \gamma_0/\mu_2$, when the diffusion is weak, the first term of (36) dominates and the variance does not substantially change. In this limit, the mean and the signal-to-noise ratio approaches

$$\langle x(t) \rangle \approx -\frac{\mu_3}{\mu_2} \frac{k_B T}{\gamma_0}(t - t_0),$$

$$\text{SNR}(t) \approx \frac{\mu_3}{\mu_2} \frac{k_B T}{\gamma_0}(t - t_0)^2.$$  \hspace{1cm} (37)

The mean can be linearly controlled by the ratio $\mu_3/\mu_2$, whereas the signal-to-noise ratio is additionally multiplied by $\mu_3/\gamma_0$. For $\gamma_0$ specifying the particle and liquid, the mean and the SNR can therefore be independently controlled by tailoring $\mu_2$ and $\mu_3$.

For a longer time $\gamma_0 \ll t - t_0 \ll \left(\frac{\mu_2^2 k_B T}{\gamma_0}\right)^{-1}$, the mean and the signal-to-noise ratio approaches

$$\langle x(t) \rangle \approx -\frac{\mu_3}{\gamma_0} \frac{k_B T}{\gamma_0}(t - t_0)^2,$$

$$\text{SNR}(t) \approx \frac{1}{2} \left(\frac{\mu_3}{\gamma_0}\right)^2 \frac{k_B T}{\gamma_0}(t - t_0)^3.$$  \hspace{1cm} (38)
6. Conclusion

In this part of the paper, we theoretically predict elementary and interesting noise-to-signal transition which can be, in principle, tested by a particle optically trapped in an intensity-shaped laser beam. The key results of the short-time approximation describing the stochastic dynamics of a Brownian particle in the cubic potential are two generic features of the mean displacement and signal-to-noise ratio. Both are linearly powered by the temperature of the environment and both monotonously increase in time. These are necessary conditions to later reach a large displacement simultaneously with the signal-to-noise ratio overcoming unity. The exact time dependency of the mean displacement and the signal-to-noise ratio can depend on the dominant regime of the transition. The transition is limited in time by the divergent character of the cubic potential. In the following part of the paper [25], the idea will be discussed from the experimental point of view. The main aim of the experimental part is to predict whether two generic features can be observed.

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