Multicriterial synthesis of the control law for a semi-active spatial model of the vehicle suspension

Julian Genov
Technical University – Sofia, 8 Kliment Ohridski Blvd., 1000 Sofia, Bulgaria
E-mail: j_genov@mail.bg

Abstract. This article explores the task for the multicriterial synthesis of the control-law from the type Linear Quadratic Regulator (LQR), for a semi-active vehicle suspension. It consists of obtaining the optimal compromise between the requirements for sufficient ride comfort and vehicle stability on the road pavement. The relevant object-functions are defined. It is proposed an algorithm for a multicriterial hierarchical approach for identifying in the Pareto’s set of solutions, this solution that realizes the optimal compromise in the sense of John Nash’s sustained strategy from the antagonistic games theory. Based on this approach, a new algorithm for the synthesis of LQR is proposed, combined with a compensator, in which the weight matrices being the subject of optimization. This synthesis was applied for a spatial model of a semi-active suspension of a vehicle.

1. Introduction
The quality criteria of the vehicle vibration insulation are contradictory. The main problem in the suspension synthesis is to be obtained such compromise that satisfied all of them in maximal extent. In general, ensuring enough ride comfort requires one softer suspension, while better vehicle stability and better traction of the tire with the pavement require a rigid one. Obviously, synthesis requires the solution of a multi-criterion task [1-12], allowing the reaching to the optimal compromise, satisfying to the maximum extent of both requirements. The using of active and semi-active suspensions gives significantly bigger possibilities in this direction. The semi-active suspension is more widely spread because of its reliability, low price and exploitation period and not at last position due to the absence of a necessity of a source of external energy.

The most commonly used approach to achieving the required compromise control is through the linear quadratic regulator. A synthesis of this type of control is described in [13, 14]. Its applications for control of the semi-active vehicle suspensions are discussed in [15-30]. In these researches are defined characteristics related to the comfort and stability which, together with the control (the force in the semi-active damper), by means of weight matrices, form the minimized function. These matrices are set by various means, most often intuitively or on the basis of normalizing of the characteristics with respect to their mean, maximum or acceptable values. However, in this way it is obtained different laws of the control. Depending on the choice of the weights they combine the quality criteria with varying degrees of importance. Thus in the best case, are obtained solutions that belong in the Pareto-set of non-dominated solutions [31], but for which ones no ground to be claimed, that they represent namely the optimal compromise. Another important moment is that due to the presence of saturation and hysteresis in the characteristic of the controllable semi-active damper [32], the generated solution, based on the theory of linear regulator, is quasi-optimal.
2. Synthesis of LQR

The state space representation of the spatial model of the vehicle suspension is:

\[ \dot{X} = AX + Bu + B_u U, \]
\[ Y = CX + D_1 A + D_u U, \]

(1)

where:

- the general coordinates \( q \) and state vector \( X \) are:
  \[ q = \begin{bmatrix} q_1 \n q_2 \n q_3 \n q_4 \end{bmatrix}^{T}, \quad X = \begin{bmatrix} q \n \dot{q} \end{bmatrix}; \]

- the vector of the input excitations \( A \), which are due to the roughness \( \xi(t) \) of the road pavement, and acting under the four vehicle wheels is:
  \[ \xi(t) = [\xi_r(t), \xi_c(t), \xi_a(t), \xi_b(t)]^{T} \]

- the vector of the control, consisting of the forces in the semi-active dampers, acting on the sprung mass, is:
  \[ U = [f_r(I, q, q), f_c(I, q, q), f_a(I, q, q), f_b(I, q, q)]^{T}, \quad I \text{ is the damper's controlling current} \]

- the vector of the outputs variables is:
  \[ Y = [q^{(w)}_1, q^{(w)}_2, -F^{(w)}_d]^{T}, \quad F^{(w)}_d \text{ – the filtered acceleration in the place of the driver seat} \]

- \( A, B_u, B_u \) – state matrices, \( C, D_1, D_u \) – matrices of the output variables.

The synthesis of the linear quadratic regulator is related to minimizing the functional:

\[ J(U) = X^{T}(T_r) FX(T_r) + \int_{0}^{T_r} \left[ Y^{T} Q Y + U^{T} R U \right] dt \]

(2)

where: \( F \in \mathbb{R}^{14 \times 14} \) and \( Q \in \mathbb{R}^{12 \times 12} \) are positive semi-definite, and \( R \in \mathbb{R}^{4 \times 4} \) is positive definite weighting matrices (in this consideration \( F = 0_{14 \times 14} \) and \( Q, R \) are diagonal);

\( T_r \) is time-interval in which the regulating process is realizes.

Taking into account (1) in (2) it is obtained:

\[ J = \int_{0}^{T_r} \left[ (X^{T} Q X + 2X^{T} N_{x1} U + U^{T} R U) + 2X^{T} N_{x1} A + 2U^{T} N_{u1} A + A^{T} R_{x1} A) \right] dt \]

(3)

where:

- \( Q = C^{T} QC, \quad N_{x1} = C^{T} Q D_A, \quad N_{u1} = D_u^{T} Q D_A \in \mathbb{R}^{4 \times 8} \)
- \( R_{1} = D_{u}^{T} Q D_{u} \in \mathbb{R}^{8 \times 8}, \quad R_u = D_{u}^{T} Q D_{u} + R \in \mathbb{R}^{4 \times 4} \)

Hamiltonian in normalized form \( \overline{H} \) and canonical equation about conjugate momentum \( P \) are:

\[ \overline{H} = (X^{T} Q X + 2X^{T} N_{x1} U + U^{T} R_{u} U) + 2X^{T} N_{x1} A + 2U^{T} N_{u1} A + A^{T} R_{x1} A) + P^{T} (A X + B_u U + B_{u1} A) \]

\[ \dot{P} = -\frac{\partial \overline{H}}{\partial X} = \left[ 2Q_{x1} X + 2N_{x1} U + A^{T} P + 2N_{u1} A \right], \quad P(T_r) = FX(T_r) = 0_{2n \times 1} \]

(4)

The kinematic excitation may be directly measured or identified in real time. In this case, control efficiency can be increased by including additional compensation that depends on the value of the input excitation. Based on this idea, the solution from equation (4) for conjugate momentum is proposed to be in the form:

\[ P(t) = 2 \left[ K(t) X(t) - \Phi (t, A(t)) \right], \]

(5)
where $\Psi(t, A(t))$ is this additional part represented the compensation by the input excitation in the controller.

Substituting the time-derivative of the conjugate momentum:

$$\dot{P} = 2\left( \dot{K}X + KAX + KB_0 U + KB_A A - \dot{\Psi} \right),$$

in equation (4) it is obtained:

$$\left[ (\dot{K} + KA) + A^T K + Q_x \right] X + (KB_0 + N_{xU}) U \right] + \left[ -\dot{\Psi} - A^T \dot{\Psi} + (KB_A + N_{xA}) A \right] = 0$$

(6)

According the Pontryagin’s minimum principle the optimal control is:

$$\left[ \frac{\partial H}{\partial U} \right]_{U = U^\text{opt}} = 0 \Rightarrow U^\text{opt} = -R_u^{-1} \left( N_{xU}^T + B_U^T K \right) X - R_{uA}^{-1} \left( N_{UA} A - B_U^T \dot{\Psi} \right)$$

(7)

After substitution $U^\text{opt}$ in (6) it is obtained a system of two matrix Riccati differential equations:

$$\dot{K} + KA + A^T K + Q_x - (KB_0 + N_{xU}) R_u^{-1} \left( N_{xU}^T + B_U^T K \right) = 0$$

$$\dot{\Psi} + \left[ A^T - (KB_0 + N_{xU}) R_u^{-1} B_U^T \right] \dot{\Psi} \left[ (KB_0 + N_{xU}) R_u^{-1} N_{UA} - (KB_A + N_{xA}) \right] A = 0$$

(8)

For $T_i \Rightarrow \infty$ and if the matrixes $A, B, C, D, Q, R$ are constant, the boundary conditions are [34]:

$$\lim_{t \to \infty} K(t) = K = \text{const}$$

$$\lim_{t \to \infty} \Psi(t, A) = \Psi(A) \Rightarrow \dot{\Psi} = \dot{\Psi}(A)$$

by which the upper system is transformed into an algebraic:

$$\bar{K} \bar{A} + \bar{A}^T \bar{K} - \bar{K}B_0 R_u^{-1} B_U^T \bar{K} + \bar{Q}_x = 0$$

$$\bar{\Psi}(A) = \left[ A^T - (KB_0 + N_{xU}) R_u^{-1} B_U^T \right] \left[ \bar{K}B_A + N_{xA} - (KB_U + N_{xU}) R_u^{-1} N_{UA} \right] A$$

(9)

where: $\bar{A} = A - B_U R_u^{-1} N_{xU}^T$, $\bar{Q}_x = Q_x - N_{xU} R_u^{-1} N_{xU}^T$.

The solution $\bar{K}$ of the first equation is obtained by numerical determination (using Q-R algorithm [35]) of the 2n-of number stable eigenvalues of corresponding Hamiltonian matrix [36]:

$$H = \begin{bmatrix} \bar{A} & B_U R_u^{-1} B_U^T \\ -\bar{Q}_x & -\bar{A}^T \end{bmatrix}$$

(10)

The solution of the second equation is obtained directly by substitution in it the solution $\bar{K}$ of the first equation.

Finally the control formed from LQR is:

$$U^\text{LQR} = -G_x X - G_A A,$$

(11)

where:

$$G_x = R_u^{-1} \left( N_{xU}^T + B_U^T \bar{K} \right), \quad G_A = R_u^{-1} \left[ N_{UA} - B_U^T \left[ \bar{A}^T - G_x B_U^T \right] \right] \left[ \bar{K}B_A + N_{xA} - G_x N_{xU} A \right]$$

The structural scheme of a vehicle’s suspension implemented with magneto-rheological semi-active dampers and their control realized by the synthesized LQR is shown in figure 1. The inverse model of the MR damper is realized by artificial neural network.
3. Multi-objective approach

The control obtained in the synthesis of the linear quadratic regulator minimizes the integral weighted function, formed by the target characteristics and by the control itself. It depends on the concrete values of the weighting matrices and essentially represents some solution from the Pareto-set. In order to obtain the optimal compromise solution, it is proposed determination of the weighting matrices based on a two-level hierarchical optimization approach, whereby the optimal compromise is considered in the sense of John Nash's sustained strategy from the antagonistic games theory [37].

3.1. The definition of the optimization task

The target functions are the mean square values (RMS) of: the unsprung co-ordinates, the sprung coordinates, the acceleration at the driver’s seat attachment and the dynamic components of forces between the tires and road pavement. In general, the first and fourth criteria relate to sustainability and the second and third to ride comfort but do not overlap entirely with the requirements. In this way, the including of the first two criteria leads to a more effective reduction of the level of the vibration field. The vector criterion is:

\[
J(\Theta) = \left[ J_1(\Theta), J_2(\Theta), J_3(\Theta), J_4(\Theta) \right],
\]

\[
J_1(\Theta) = 0.25 \sum_{i=1}^{4} \left( \frac{1}{T_0} \int_{0}^{T} \left\| \ddot{x}_i(t, \Theta) \right\|^2 dt \right),
\]

\[
J_2(\Theta) = \frac{1}{T_0} \int_{0}^{T} \left\| \ddot{u}_d(t, \Theta) \right\|^2 dt,
\]

\[
J_3(\Theta) = 0.25 \sum_{i=1}^{4} \left( \frac{1}{T_0} \int_{0}^{T} \left\| \dot{u}_{i}(t, \Theta) \right\|^2 dt \right),
\]

\[
J_4(\Theta) = 0.33 \left( \mu_{r} \frac{1}{T_0} \int_{0}^{T} \left\| \ddot{z}_u(t, \Theta) \right\|^2 dt + \mu_{d} \frac{1}{T_0} \int_{0}^{T} \left\| \ddot{z}_d(t, \Theta) \right\|^2 dt + \mu_{c} \frac{1}{T_0} \int_{0}^{T} \left\| \ddot{z}_c(t, \Theta) \right\|^2 dt \right).
\]

For the purpose of reducing the number on the optimized parameters, the weighted matrices are presented in the form:

\[
Q = \mu_Q \bar{Q}, \quad R = \mu_R \bar{R},
\]

where the normalized matrices \( \bar{Q} \) and \( \bar{R} \) are constructed as:

\[
\bar{Q} = \text{diag} \left( [Q_{xx}, Q_{ux}, Q_{ux}, Q_{uu}, Q_{uu}, Q_{uu}, Q_{uu}, Q_{uu}] \right), \quad \bar{R} = \text{diag} \left( [R, R, R, R] \right),
\]

\[
\bar{Q}_{z_1} = \bar{Q}_{z_2} = \bar{Q}_{z_3} = \bar{Q}_{z_4} = Q_{xx}, \quad \bar{Q}_{u_1} = \bar{Q}_{u_2} = Q_{uu}, \quad \bar{Q}_{d_1} = \bar{Q}_{d_2} = Q_{uu}, \quad \bar{Q}_{c_1} = \bar{Q}_{c_2} = Q_{uu}, \quad \bar{Q}_{r_1} = \bar{Q}_{r_2} = Q_r, \quad \bar{Q}_{r_3} = Q_r, \quad \bar{Q}_{r_4} = Q_r,
\]

\[
\bar{R}_{z_1} = \bar{R}_{z_2} = \bar{R}_{z_3} = \bar{R}_{z_4} = \bar{R}_{u_1} = \bar{R}_{u_2} = \bar{R}_{d_1} = \bar{R}_{d_2} = \bar{R}_{c_1} = \bar{R}_{c_2} = \bar{R}_{r_1} = \bar{R}_{r_2} = \bar{R}_{r_3} = \bar{R}_{r_4} = \bar{R} = R.
\]
and the scaling coefficients for the parameters’ values reconciling $\mu_Q$ and $\mu_R$ are:

$\mu_Q = \text{diag} \left( \left[ \mu_{\alpha}, \mu_{\alpha}, \mu_{\alpha}, \mu_{\alpha}, 50 \mu_{\alpha}, 8.13 \mu_{\alpha}, \mu_{\beta}, \mu_{\beta}, \mu_{\beta}, \mu_{\beta} \right] \right)$

$\mu_R = \text{diag} \left( \left[ \mu_u, \mu_u, \mu_u, \mu_u \right] \right)$

$\mu_{\alpha} = 0.25 / \text{RMS}^2 \left( z_i(t) \right) = \mu_{\alpha}, \quad i = 1..4$

$\mu_{\beta} = 0.33 / \text{RMS}^2 \left( z_j(t) \right) = \mu_{\beta}, \quad i = 1..4$

$\mu_{\alpha} = 0.33 / \text{RMS}^2 \left( \theta_{\alpha_j}(t) \right) \approx 50 \mu_{\alpha}, \quad \mu_{\beta} = 0.33 / \text{RMS}^2 \left( \theta_{\beta_j}(t) \right) \approx 8.13 \mu_{\beta}$

in which the root mean square (RMS) values are obtained from implementation with passive suspension.

The vector of the optimized parameters is formed from the elements forming the weighted matrices and from the coefficients of elasticity of the front and rear suspensions $\Theta = \left[ k_{j_f}, k_{j_r} \right]$:

$$\Theta = \left[ Q_{\alpha}, \varphi, Q_{\beta_j}, Q_{i_f}, R, \Theta \right] \in D,$$

where the definition area is of the form “lower-upper boundaries”

$$D = \left[ \Theta^\prime, \Theta^\prime \right].$$

### 3.2. Algorithm for multiobjective synthesis

The algorithm consists of the next steps:

- It is performed optimization for each of the criteria (12) in regard to $\Theta$ and are defined:

$$\Theta_i^{opt} \Rightarrow J_i \left( \Theta \right) = \min_{\Theta \in D} \left( \Theta \right) = \left\{ \Theta \mid n_j = \{1..4 \} \right\} ;$$

- A functional space is formed for a defining the strategy in the context of the Games theory:

$$J \left( \Theta \right) \in D_{\text{game}} = \left\{ J \left( \Theta \right) \in R_{\Theta} \mid \left[ J^\min \leq J \left( \Theta \right) \leq J^\max \right] \right\},$$

where:

$$J^\min = \left[ J_{1}^{\min} \ldots J_{4}^{\min} \right]^T; \quad J^\max = \max_{\Theta^{opt}} \left( J \left( \Theta \right) \right), \quad k = 1..4$$

and all object functions are transforming in sense of the cooperative game’s strategy:

$$J_k^{*} \left( \Theta \right) = \left( J_k \left( \Theta \right) - J_k^{\min} \right) \left( J_k^{\max} - J_k^{\min} \right)^{-1} \in \left[ 0,1 \right], \quad k = 1..4,$$

through which it goes to the task of minimizing the vector criterion:

$$J^{*} \left( \Theta \right) = \left[ J_{1}^{*} \left( \Theta \right), \ldots, J_{k}^{*} \left( \Theta \right), \ldots, J_{4}^{*} \left( \Theta \right) \right]^T;$$

- A generalized criterion is formed as:

$$S \left( \Theta, \Psi \right) = \Psi \cdot J^{*} \left( \Theta \right),$$

with a weighted vector $\Psi = \left[ \lambda_1 \ldots \lambda_4 \right] \in \Psi = \left\{ \Psi \in R_{\Psi} \mid \sum_{j=1}^{4} \lambda_j = 1 \right\}$,

and whose minima $\Theta_{\Psi} \left( \Psi \right)$ form the Pareto-set for the multi-criteria task:

$$D_{\Psi} = \left\{ \Theta_{\Psi} \left( \Psi \right) \in D \mid S \left( \Theta_{\Psi} \left( \Psi \right) \right) = \min_{\Theta \in D_{\text{game}}} \left( J \left( \Theta \right) \right) \right\}.$$
The stationary solution according to the Nash theory is sought as the solution from the Pareto set, which minimizes $L^p$-norm in the space of the transformed object-functions:

$$
\min_{\Psi \in \mathcal{P}} L^p \left( J^* \left( \Theta_p (\Psi) \right) \right) = \min_{\Psi \in \mathcal{P}} \left[ \sum_{k=1}^{n_k} \left( J^*_k \left( \Theta_p (\Psi) \right) \right)^p \right]^{1/p};
$$

(22)

Thus the optimization process takes place on two levels. For each search-attempt for minimization of (22), generates a value of the weighted vector $\Psi$ and is performed the minimization of (20), whereby is defines the Pareto-solution corresponding to $\Psi$. The search direction, setting the value of $\Psi$ for the next step, is successful if the corresponding Pareto-solution realizes a minimization of $L^p$-norm (22) in comparison with the norm’s value at the current search-step.

In figure 2 are shown the forms of the $L^p$-norm for different values of the power $p$. The norm, with the growth of the power $p$, tends towards the function of the maximum and respectively the solution of the optimization task, to the mini-max von Neumann’s solution [38], i.e. it will have a signification of a John Nash’s sustained strategy and respectively of the optimal compromise solution [37]. On figure 3 is shown the Pareto’s set for two object-functions (the part of the line between the points corresponded to $J_1^{\text{min}}$ and $J_2^{\text{min}}$) and the different optimal compromise’s solutions obtained for the values of the norm’s power $p$ respectively 1, 2 and $\infty$.

4. Illustration of the proposed algorithm

4.1. Parameters of the Full car model:

- front and rear unsprung masses, sprung mass, moments of inertia about principal transverse and longitudinal axes:
  $$ m_{w_f} = 40 \text{ kg}, \ m_{w_r} = 35 \text{ kg}, \ m_s = 580 \text{ kg}, \ I_{Cx} = 1580 \text{ kgm}^2, \ I_{Cz} = 430 \text{ kgm}^2; $$

- coefficients of elasticity and damping in the tires:
  $$ k_f = k_r = 175.5 \text{ kN/m}, \ c_f = c_r = 80 \text{ Ns/m}; $$

- distances from the centre of gravity to the front and rear axes, half-track, driver seat’s coordinates about longitudinal and transverse axes:
\[ \ell_1 = 1.2 \text{ m}, \ell_2 = 1.4 \text{ m}, \ell_3 = 0.75 \text{ m}, \ell_4 = 0.32 \text{ m}, \ell_5 = -0.4 \text{ m}; \]

- boundaries of variation of the optimized parameters:

\[
\begin{bmatrix}
x^e \\
x^u
\end{bmatrix} = 
\begin{bmatrix}
Q_{a_0} & Q_{a_0} & Q_{a_2} & Q_F & R & k_{s_f} & k_{r_s}
0 & 0 & 0 & 0.05 & 12000 & 12000
1.5 & 1.5 & 1.5 & 1.5 & 28000 & 28000
\end{bmatrix}
\]

- RMS and scaling coefficients for the parameters’ values reconciling are given at a table 1 (they are obtained from the optimal compromise solution for a passive vehicle suspension)

| Table 1. RMS and scaling coefficients for the parameters’ values reconciling. |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| RMS            | \(z_{a_0}, \text{m} \) | \(z_s, \text{m} \) | \(\dot{z}_d, \text{m/s}^2 \) | \(F^d, \text{N} \) | \(U, \text{N} \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
|                | 0.0074         | 0.0074         | 0.45           | 390            | 160            |
| \(\mu \)      | \(\mu_{a_0}, \text{m}^2 \) | \(\mu_s, \text{m}^2 \) | \(\mu_{\dot{z}_d} \) | \(\mu_F \) | \(\mu_U \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
|                | 4565           | 6026           | 4.94           | 1.64\times10^{-6} | 39.06\times10^{-6} |

4.2. Results

On table 2 are given the results from optimization by each of the fourth object-functions. On the table 3 is given the game strategy defined by the main object function \(J_3 \) and \(J_4 \). On table 4 is given the obtained optimal compromise solution for this strategy for the power of \(L^p\)-norm \(p = 6 \), which converges to this for the infinity. It is seen that both functions are equally far from their optimal values and their normalized values are approximately equal: \(J^*_3 = 0.418 \) and \(J^*_4 = 0.425 \), as what is the goal of the optimal compromise. This result is shown and in figure 4 for \(p = 6 \) (\(\approx\) such as for infinity), and also is shown and the result for \(p = 2 \). The values of the matrixes of the LQR - controller for the realization of this optimal compromise are given in table 5.

| Table 2. Optimization by the each of the object functions. |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(Q_{a_0} \) | \(Q_s \) | \(Q_{\dot{z}_d} \) | \(Q_F \) | \(R_U \) | \(k_g \text{ N/m} \) | \(k_r \text{ N/m} \) | \(J_1 \text{ m} \) | \(J_2 \text{ m} \) | \(J_3 \text{ m/s}^2 \) | \(J_4 \text{ N} \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(J_1 \)     | 0.811          | 0.202          | 0.788          | 1.192          | 0.053          | 21531          | 23906          | 0.00830         | 0.0089         | 0.321          | 352            |
| \(J_2 \)     | 1.392          | 1.286          | 0.161          | 0.015          | 0.364          | 15844          | 13218          | 0.00850         | 0.0067         | 0.264          | 439            |
| \(J_3 \)     | 0.010          | 0.022          | 0.005          | 0.013          | 0.050          | 12000          | 12000          | 0.00847         | 0.0077         | 0.246          | 450            |
| \(J_4 \)     | 0.812          | 0.202          | 0.788          | 1.182          | 0.053          | 21531          | 23906          | 0.00850         | 0.0089         | 0.321          | 352            |

| Table 3. Game strategy by \(J_3 \) and \(J_4 \). |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(J_{opt} \text{ m} \) | \(J_{opt} \text{ m/s}^2 \) | \(J_{opt} \text{ N} \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.00847         | 0.0077         | 0.246          | 352            |
| 0.00850         | 0.0089         | 0.321          | 450            |

| Table 4. Optimal compromise solution by \(J_3 \) and \(J_4 \) (\(p = 6 \) approximating infinity). |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(Q_{a_0} \) | \(Q_s \) | \(Q_{\dot{z}_d} \) | \(Q_F \) | \(R_U \) | \(k_g \text{ N/m} \) | \(k_r \text{ N/m} \) | \(J_1 \text{ m} \) | \(J_2 \text{ m} \) | \(J_3 \text{ m/s}^2 \) | \(J_4 \text{ N} \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.054          | 0.68           | 0.633          | 0.445          | 0.2086         | 13750          | 25750          | 8.51\times10^{-3} | 8.02\times10^{-3} | 0.278          | 388            |

normalized values: 0.418 0.425
Table 5. Values of the LQR matrices.

| $G_{x} \times 10^{-3}$ | -13.96 | -1.513 | 2.779 | 7.775 | 2.563 | 47.55 | -29.054 | -632.6 | 0.012 | 0.066 | -0.02 | 2.211 | 5.89 | -3.3 |
|-------------------------|--------|--------|------|------|------|------|--------|------|------|------|------|------|------|------|
| -1.201                  | -13.74 | 4.019  | -1.916 | -5.531 | -62.1 | -23.258 | 8.856 | -0.567 | -0.041 | 0.025 | 1.724 | -7.025 | -2.94 |
| 2.433                   | 2.678  | -2.78  | 9.780 | -99.61 | 40.9 | 6.122 | 72.277 | -0.044 | -503 | 0.095 | 2.294 | 5.408 | 2.07 |
| 6.040                   | -2.318 | 4.486  | -8.504 | -7.748 | -68.2 | 10.07 | -5.178 | 0.025 | 0.112 | -0.517 | 1.843 | -7.446 | 2.22 |

| $G_{A}$ | x10[^3] | x10[^3] | x10[^3] | x10[^3] | x10[^3] | x10[^3] | x10[^3] |
|---------|---------|---------|---------|---------|---------|---------|---------|
| 20.043  | 1.258   | 0.208   | -14.568 | 9.134   | 0.573   | 0.094   | -6.640  |
| 2.613   | 22.120  | -6.733  | 3.979   | 1.191   | 10.080  | -3.068  | 1.813   |
| 0.007   | -10.685 | 10.077  | -7.530  | 0.003   | -4.870  | 4.592   | -3.430  |
| -9.000  | 3.860   | -2.268  | 18.456  | -4.100  | 1.762   | -1.033  | 8.410   |

Figure 4. Pareto’s set and the optimal compromise solution for synthesis of LQR for $p=2$ and $p=6$.

5. Conclusion
A multicriterial approach for a vehicle suspension synthesis has been proposed, combining the theoretical formulations for V. Pareto’s non-dominated solutions and the Theory of antagonistic games by J. Nash. The solved task is related to the determination of the optimal compromise between ride comfort and on-road stability. Appropriate target functions being proposed for quantitative evaluations. The approach considered is applied for the multicriterial synthesis of a semi-active suspension controlled by a linear quadratic regulator. The LQR is combined with a compensator by the input excitations. Another news in the approach is including the elements of the weighted matrixes in the optimized parameters. It is given a concrete illustration of the proposed algorithm for a multicriterial synthesis of the control law for a semi-active spatial model of the vehicle suspension.

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