order-norm continuous operators and order weakly compact operators

Sajjad Ghanizadeh Zare, Kazem Haghnejad Azar, Mina Matin, Somayeh Hazrati

Abstract  Let $E$ be a sublattice of a vector lattice $F$. A continuous operator $T$ from the vector lattice $E$ into a normed vector space $X$ is said to be order-norm continuous whenever $x_\alpha \overset{F_0}{\to} 0$ implies $Tx_\alpha \overset{\|}{\to} 0$ for each $(x_\alpha)_\alpha \subseteq E$. Our mean from the convergence $x_\alpha \overset{F_0}{\to} x$ is that there exists another net $(y_\alpha)$ in $F$ with the same index set satisfying $y_\alpha \downarrow 0$ in $F$ and $|x_\alpha - x| \leq y_\alpha$ for all indexes $\alpha$. In this paper, we will study some properties of this new class of operators and its relationships with some known classifications of operators. We also define the new class of operators that named order weakly compact operators. A continuous operator $T : E \to X$ is said to be order weakly compact, if $T(A)$ in $X$ is a relatively weakly compact set for each $F_0$-bounded $A \subseteq E$. In this manuscript, we study some properties of this class of operators and its relationships with order-norm continuous operators.

Keywords  Vector lattice · property $(F)$ · $\hat{o}$-convergence · order-to-norm continuous operator · order-norm continuous operator · order weakly compact.

Mathematics Subject Classification (2010) 47B65 · 46B40 · 46B42

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1 Introduction and Preliminaries

Our motivation in writing this article is to communicate and expand the concepts which have been introduced in the articles [4] and [8]. In these papers some concepts such as $\sigma$-convergence, the property $(F)$ and order-to-norm continuous operators have been introduced, studied and authors have been investigated some of their properties and the relationships with other lattice properties. We introduce the new classes of operators as the set of order-norm continuous operators and we study some of their properties and their relationships with others knowns operators.

To state our results, we need to fix some notations and recall some definitions. A net $(x_\alpha)_{\alpha \in A}$ in a vector lattice $E$ is said to be order convergent to $x \in E$ if there is a net $(y_\beta)_{\beta \in B}$ in $E$ such that $y_\beta \downarrow 0$ and for every $\beta \in B$, there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq y_\beta$ whenever $\alpha \geq \alpha_0$. In short, we will denote this convergence by $x_\alpha \xrightarrow{\sigma} x$ and write that $x_\alpha$ is $\sigma$-convergent to $x$. A net $(x_\alpha)_{\alpha}$ in vector lattice $E$ is unbounded order convergent to $x \in E$ if $|x_\alpha - x| \leq y_\beta$ for all $\beta \in B$. We denote this convergence by $x_\alpha \xrightarrow{uo} x$ and write that $x_\alpha$ is $uo$-convergent to $x$. It is clear that for order bounded nets, $uo$-convergence is equivalent to $\sigma$-convergence. A net $(x_\alpha) \subseteq E$ is said to be $\sigma$-convergent (in short, $\sigma$-convergent) to $x$ if there is a net $(y_\beta) \subseteq F$, possibly over a different index set, such that $y_\beta \downarrow 0$ in $F$ and for every $\beta$, there exists $\alpha_0$ such that $|x_\alpha - x| \leq y_\beta$ whenever $\alpha \geq \alpha_0$. We denote this convergence by $x_\alpha \xrightarrow{\sigma} x$ and write that $(x_\alpha)$ is $\sigma$-convergent to $x$. It is clear that if $E$ is regular in $F$ and $x_\alpha \xrightarrow{\sigma} x$ in $E$, then $x_\alpha \xrightarrow{\sigma} x$ in $E$. The converse is not true in general. For example, $c_0$ is a sublattice of $\ell^\infty$ and $(e_n) \subseteq c_0, e_n \xrightarrow{\sigma} 0$ in $c_0$, but it is not order convergent to 0 in $c_0$. A subset $A$ of $E$ is said to be $\sigma$-order bounded (in short, $F_0$-bounded), if there exist $x, y \in E$ that $A \subseteq [x, y]$. A vector lattice $E$ is said to have the property $(F)$, if $A \subseteq E$ is order bounded whenever $A$ is $F_0$-bounded. (see [4]). A Banach lattice $E$ is said to be an $AM$-space if we have $\|x + y\| = \max\{\|x\|, \|y\|\}$ for each $x, y \in E$ such that $|x| \wedge |y| = 0$. A Banach lattice $E$ is said to be an $AL$-space if we have $\|x + y\| = \|x\| + \|y\|$ for each $x, y \in E$ such that $|x| \wedge |y| = 0$. A Banach lattice $E$ is said to be $KB$-space whenever each increasing norm bounded sequence of $E^+$ is norm convergent. Let $E$ and $G$ be vector spaces. We will denote $L(E, G)$ by the collections of operators from $E$ into $G$. $L_0(E, G)$ is the all of order bounded operators in this manuscript. An operator $T$ from a Banach space $X$ into a Banach space $Y$ is weakly compact if $T(B_X)$ is weakly compact where $B_X$ is the closed unit ball of $X$. A continuous operator $T$ from Banach lattice $E$ into Banach space $X$ is called $M$-weakly compact if $\lim \|Tx_n\| = 0$ holds for every norm bounded disjoint sequence $(x_n)_n$ of $E$. An operator $T : E \rightarrow F$ is said to preserve disjointness whenever $x \perp y$ in $E$ implies $Tx \perp Ty$. A subset $A$ of a vector lattice $E$ is called $b$-order bounded in $E$ if it is order bounded in $E^{\sim}$. If each $b$-order bounded subset of $E$ is order bounded in $E$, then $E$ is said to have the property $(b)$. Jalili, Haghnejad and Moghimi characterized $L_{\sigma}(E, G)$ and $L_0^{\sigma}(E, G)$ spaces in [8]. An operator $T$ from a vector lattice $E$ into topological vector space $G$ is said to be order-to-topology continuous whenever $x_\alpha \xrightarrow{\sigma} 0$ implies $Tx_\alpha \xrightarrow{\tau} 0$ for each $(x_\alpha) \subseteq E$. For each sequence $(x_n) \subseteq E$, if $x_n \xrightarrow{\sigma} 0$ implies $Tx_n \xrightarrow{\tau} 0$, then $T$ is called $\sigma$-order-to-topology continuous operator. The collection of all order-to-topology continuous operators.
operators will be denoted by $L_{o\tau}(E, G)$; the subscript $o\tau$ is justified by the fact that the order-to-topology continuous operators; that is,

$$L_{o\tau}(E, G) = \{ T \in L(E, G) : T \text{ is order-to-topology continuous } \}.$$  

Similarly, $L_{o\sigma}^\sigma(E, G)$ represents the collection of all $\sigma$-order-to-topology continuous operators, that is,

$$L_{o\sigma}^\sigma(E, G) = \{ T \in L(E, G) : T \text{ is } \sigma-\text{order-to-topology continuous } \}.$$  

For a normed space $G$, $L_{o\tau}(E, G)$ is collection of order-to-norm topology continuous operators.

Let $E$, $G$ be two normed vector lattices. Recall that from [9], a continuous operator $T : E \to G$ is said to be $\sigma$-wun-continuous, if each norm bounded $w\sigma$-null sequence $(x_n) \subseteq E$ implies $Tx_n \xrightarrow{\|\|} 0$. Remember that an operator $T$ from Banach lattice $E$ into Banach space $X$ is a wun-Dunford-Pettis whenever $x_n \xrightarrow{\text{wun}} 0$ in $E$ implies $Tx_n \xrightarrow{\|\|} 0$ in $X$ for each sequence $(x_n) \subseteq E$ (See [10] for more information).

Recall that a Banach lattice $E$ is said to have the property $(P)$ if there exists a positive contractive projection $P : E^{**} \to E$ where $E$ is identified with a sublattice of its topological bidual $E^{**}$.

In a Banach lattice $E$, a subset $A$ is said to be almost order bounded if for any $\varepsilon > 0$ there exists $u \in E^+$ such that $A \subseteq [-u, u] + \varepsilon B_E$ ($B_E$ is the closed unit ball of $E$). One should observe the following useful fact, which can be easily verified using Riesz decomposition Theorem, that $A \subseteq [-u, u] + \varepsilon B_E$ if and only if $\sup_{x \in A} \|(x - u)^+\| = \sup_{x \in A} \|x - |x| \wedge u\| \leq \varepsilon$. By Theorems 4.9 and 3.44 of [1], each almost order bounded subset in order continuous Banach lattice is relatively weakly compact.

A vector subspace $G$ of an ordered vector space $E$ is majorizing, whenever there exists some $y \in G$ with $x \leq y$ for each $x \in E$. A sublattice $G$ of a vector lattice $E$ is said to be order dense in $E$ whenever there exists some $y \in G$ with $0 < y \leq x$ for each $0 < x \in E$. Recall that a Banach lattice $E$ is said to have the positive Schur property (the dual positive Schur property) if every positive $w$-null sequence in $E$ (positive $w^*$-null sequence in $E^*$) is norm null. A vector lattice is called laterally complete whenever every subset of pairwise disjoint positive vectors has a supremum.

Unless otherwise stated, throughout this paper, $F$ is a vector lattice, $E$ is a sublattice of $F$, and $X$ is a normed vector space.

2 The property $(F)$ in vector lattices

In this section, we investigate on the property $(F)$ which is defined in [4]. We will try to get new content from this property.

Let $E$ be regular in $F$ and order continuous in its own right and closed, and let $(x_\alpha) \subseteq E$ be an order bounded net and $F$ be a Dedekind complete Banach lattice. By Lemma 4.5 of [1], it is clear that $x_\alpha \rightarrow^o x$ in $E$ if and only if $x_\alpha \rightarrow x$ in $E$.
Remark 1  i) Let $E^{**}$ be lattice isomorphic by one sublattice of $F$. If $E$ has the property (b), then $E$ has the property $(F)$. And if $E^{**}$ is lattice isomorphic with $F$, then the properties (b) and $(F)$ are equivalent.

ii) If $E$ is a majorizing sublattice of $F$, then $E$ has the property $(F)$. Let $A \subseteq E$ be a $F$-order bounded set. Therefore, there exists a $u \in F^+$ that $A \subseteq [-u,u]$. Since $E$ is majorizing sublattice of $F$, therefore, there exists $v \in E^+$ that $u \leq v$ and $-v \leq -u$. So $A \subseteq [-v,v]$. Hence $A$ is order bounded in $E$.

iii) Let $E$ be a sublattice of $F$ and $F$ be a sublattice of $G$. If $F$ has the property $(G)$, necessarily $E$ does not have the property $(G)$. For example, $c_0$ is a sublattice of $c$ and $c$ is a sublattice of $\ell^\infty$. $c$ has the property $(\ell^\infty)$ while $c_0$ has not property $(\ell^\infty)$.

iv) Let $E$ be a sublattice of $F$ and $F$ be a sublattice of $G$. It is clear that if $E$ has the property $(G)$, then it has the property $(F)$. If $A \subseteq E$ and it is order bounded in $F$, so it is order bounded in $G$. By assumption $A$ is order bounded in $E$.

Theorem 1 Let $E$ and $F$ be two Banach lattices, and let $E$ has the property $(F)$. Then, by one of the following assertions, if $A \subseteq E$ is an almost order bounded in $F$, $A$ is an almost order bounded in $E$.

i) $F$ has an order unit.

ii) $E$ is an ideal of $F$.

Proof Let $F$ has an order unit and $A \subseteq E$ be an almost order bounded set in $F$. By Theorem 4.21 of [1], $A$ is an order bounded set in $F$. By the assumption, $A$ is an order bounded set in $E$ and therefore, it is an almost order bounded set in $E$.

Suppose that $E$ is an ideal of $F$. Let $A \subseteq E$ be an almost order bounded set in $F$. It means that for each $\varepsilon > 0$, there exists a $u \in F^+$ that $A \subseteq [-u,u] + \varepsilon B_F$. For each $x \in A$, we have $x = x_1 + x_2$ that $x_1 \in [-u,u]$ and $x_2 \in \varepsilon B_F$. We assume that $x \neq 0$. It is obvious that $\|x_2\| \leq \varepsilon \frac{1}{1+10}$. Since $E$ is an ideal of $F$, therefore, $x_2 \in E$. Because $\|x_2\| \leq \varepsilon$, therefore, $x_2 \in \varepsilon B_E$. On the other hand, we have $x_1 = x - x_2$. So $x_1 \in E$. Since $E$ has the property $(F)$, there exists a $v \in E^+$ that $x_1 \in [-v,v]$. Therefore, $A \subseteq [-v,v] + \varepsilon B_E$. Hence $A$ is order bounded in $E$, as regard.

Corollary 1  i) Let $F$ has an order unit, $E$ has the property $(F)$ with order continuous norm and let $(x_n) \subseteq E$ be disjoint and almost order bounded sequence in $F$.

Then $x_n \overset{\|\cdot\|}{\longrightarrow} 0$ in $E$.

ii) Let $(x_n) \subseteq E$ be a disjoint and almost order bounded sequence in $F$, and $E$ be an ideal of $F$ and has the property $(F)$. If $E$ has order continuous norm, then $x_n \overset{\|\cdot\|}{\longrightarrow} 0$ in $E$.

Proof  i) By proof of Theorem 1, $(x_n)$ is order bounded in $E$ and so by Theorem 4.14 of [1], $x_n \overset{\|\cdot\|}{\longrightarrow} 0$ in $E$.

ii) Since $(x_n)$ is a disjoint sequence, therefore, by Corollary 3.6 of [6], we have $x_n \overset{\text{maz}}{\longrightarrow} 0$ in $E$. By Theorem 1, $(x_n)$ is almost order bounded in $E$. By Proposition 3.7 of [2], $x_n \overset{\|\cdot\|}{\longrightarrow} 0$ in $E$. 

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3 δ-order-norm continuous operators

A continuous operator $T : E \to X$ is said to be δ-order-norm continuous (or, δn-continuous for short), if $(x_n) \subseteq E$ is δ-null in $E$, then $(Tx_n)$ is convergent to 0 in norm. A continuous operator $T : E \to X$ is said to be σ-δ-order-norm continuous (or, σ-δn-continuous for short), if $(x_n) \subseteq E$ is δ-null in $E$, then $(Tx_n)$ in $X$ is convergent to 0 in norm.

The collection of all δn-continuous operators from a vector lattice $E$ into a Banach space $X$ (resp. σ-δn-continuous) will be denoted by $L_{\delta n}(E, X)$ (resp. $L_{\sigma \delta n}^{\ell}(E, X)$).

It is clear that, if $T : E \to X$ is a δn-continuous, then $T$ is an order-to-norm topology continuous. By the following example, the converse is not true in general.

Example 1 The identity operator $I : c_0 \to c_0$ is order-to-norm topology continuous. Let $(x_n) \subseteq c_0$ be order-null. Since $c_0$ has order continuous norm, therefore, $x_n \to 0$. Consider $(e_n) \subseteq c_0$, $e_n \ell \to 0$ in $c_0$. But $(I(e_n))$ is not convergent to zero in norm in $c_0$. Hence $I : c_0 \to c_0$ is not $\ell_\sigma$-on-continuous. Obviously $L_{\delta n}(E, X)$ is a subspace of $L_{\sigma \delta n}^{\ell}(E, X)$. In the following, we have some examples from δn-continuous operators.

Example 2 i) If $E$ has the property $(F)$, $E^*$ has order continuous norm and $X$ has the Schur property, then each continuous operator from $E$ to $X$ is a σ-δn-continuous operator. Let $(x_n) \subseteq E$ be δ-null sequence. Therefore, $(x_n)$ is order null in $F$ and so it is order bounded in $F$. Since $E$ has the property $(F)$, hence $(x_n)$ is order bounded in $E$. On the other hand $(x_n)$ is $u_0$-null in $F$ and so it is $u_0$-null in $E$. Because $E^*$ has order continuous norm, therefore, by Theorem 6.4 of [5], $x_n \ell \to 0$ in $E$. By continuity of $T$, we have $Tx_n \ell \to 0$ in $X$. Since $X$ has the Schur property, hence $Tx_n \ell \to 0$ in $X$.

The Banach lattice $c$ has the property $(\ell^\infty)$, $c^*$ has order continuous norm and $\ell^1$ has the Schur property, therefore, each continuous operator $T : c \to \ell^1$ is $\sigma$-$\ell_\sigma$-on-continuous.

ii) Let $F$ be a Dedekind complete Banach lattice. If $E$ has the property $(F)$ and order continuous norm, then each continuous operator from $E$ to $X$ is a δn-continuous operator. Let $(x_n) \subseteq E$ be δ-null net. Therefore, $(x_n)$ is order null in $F$ and so it is order bounded in $F$. Since $E$ has property $(F)$, hence $(x_n)$ is order bounded in $E$. On the other hand $(x_n)$ is $u_0$-null in $F$ and so by Lemma 4.5 of [7], it is $u_0$-null in $E$. Since $(x_n)$ is order bounded, hence $(x_n)$ is order-null in $E$. Because $E$ has order continuous norm, we have $x_n \ell \to 0$ in $E$. So $Tx_n \ell \to 0$ in $X$.

iii) If $T : F \to X$ is a $u_0$-continuous operators, then $T|_E : E \to X$ is a δn-continuous operator. Let $(x_n) \subseteq E$ be a δ-null net. It is clear that $x_n \ell \to 0$ in $F$. By assumption, $Tx_n \ell \to 0$ in $X$.

The class of δn-continuous operators differs from the class of order continuous operators. Since the identity operator $I : c_0 \to c_0$ is order continuous while it is not $\ell_\sigma$-on-continuous (see Example 1).
Proposition 1  i) Let \( T \in L_{\text{oin}}(E, X) \), \( S : E \to X \) be a continuous operator, and \( 0 \leq S \leq T \), then \( S \) is a \( \delta\)-continuous operator.

ii) Moreover, if \( X \) also is a sublattice of \( F \), \( T \in L_{\text{oin}}(E, X) \) and \( S \in L_{\text{oin}}(X, Y) \), then \( S \circ T \in L_{\text{oin}}(E, Y) \).

Proof i) Let \( (x_\alpha) \subseteq E \) be a \( \delta\)-null net. It is obvious that \( |x_\alpha| \xrightarrow{F_0} 0 \) in \( E \). We have
\[
|Sx_\alpha| \leq |S||x_\alpha| = S|x_\alpha| \leq T|x_\alpha|.
\]
By assumption, \( T|x_\alpha| \xrightarrow{||\|} 0 \). So \( |Sx_\alpha| \xrightarrow{||\|} 0 \). It means that \( S \) is a \( \delta\)-continuous operator.

ii) Let \( (x_\alpha) \subseteq E \) and \( x_\alpha \xrightarrow{F_0} 0 \). By assumption, we have \( Tx_\alpha \xrightarrow{||\|} 0 \). Because each \( \delta\)-continuous operator is continuous, therefore, \( STx_\alpha \xrightarrow{||\|} 0 \). Hence \( S \circ T \in L_{\text{oin}}(E, Y) \).

Remark 2  Let \( T : E \to G \) be an order continuous lattice homomorphism from a Dedekind complete vector lattice \( E \) to an Archimedean laterally complete normed vector lattice \( G \). If \( E \) is an order dense in the Archimedean vector lattice \( F \), then by Theorem 2.32 of [1], \( T \) extended from \( F \) to \( G \) that is an order continuous lattice homomorphism. Now if \( G \) has order continuous norm, then \( T \) is a \( \delta\)-continuous.

Theorem 2  For an order bounded operator \( T : E \to G \) between two Riesz spaces with \( G \) Dedekind complete and \( G \) has order continuous norm Banach lattice, the following statements are equivalent.

1. \( T \) is \( \delta\)-continuous.
2. If \( (x_\alpha) \subseteq E \) and \( x_\alpha \downarrow 0 \) holds in \( F \), then \( Tx_\alpha \) is norm convergent to 0 in \( G \).
3. If \( (x_\alpha) \subseteq E \) and \( x_\alpha \downarrow 0 \) holds in \( F \), then \( \inf \{||Tx_\alpha||\} = 0 \).
4. \( T^+ \), \( T^- \) are both \( \delta\)-continuous.
5. \( |T| \) is \( \delta\)-continuous.

Proof (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are obvious. Clearly, if \( T \) is a positive operator then (1), (2), and (3) are equivalent.

(3) \( \Rightarrow \) (4) It is enough to show that \( T^+ \) is \( \delta\)-continuous. To this end, let \( (x_\alpha) \subseteq E \) and \( x_\alpha \downarrow 0 \) in \( F \). Let \( ||T^+x_\alpha|| \downarrow z \geq 0 \). We have to show that \( z = 0 \). Fix some index \( \beta \) and put \( u = x_\beta \).

Now for each \( 0 \leq y \leq x \) and each \( \alpha \geq \beta \) we have
\[
0 \leq y-y \wedge x_\alpha = y \wedge x - y \wedge x_\alpha \leq x-x_\alpha,
\]
and consequently
\[
Ty - T(y \wedge x_\alpha) = T(y - y \wedge x_\alpha) \leq T^+(x-x_\alpha) = T^+x - T^+x_\alpha,
\]
from which it follows that
\[
0 \leq z \leq T^+x_\alpha \leq T^+x + |T(y \wedge x_\alpha)| - Ty
\]
holds for all \( \alpha \geq \beta \) and all \( 0 \leq y \leq x \). Now since for each fixed vector
\[
0 \leq y \leq x \text{ we have } y \wedge x_\alpha \downarrow 0 \text{ in } F, \text{ it then follows from our hypothesis that }
\]
\[
\inf \{||T(y \wedge x)||\} = 0, \text{ and hence from (3) we see that } 0 \leq ||z|| \leq ||T^+x - Ty|| \text{ holds}
\]
for all \( 0 \leq y \leq x \). In view of \( T^+x = \sup \{Ty : 0 \leq y \leq x \} \) and by order continuity of \( G \), the latter inequality yields \( z = 0 \), as desired.

(4) \( \Rightarrow \) (5) The implication follows from the identity \( |T| = T^+ + T^- \).

(5) \( \Rightarrow \) (1) The implication follows easily from the lattice inequality \( |Tx| \leq ||Tx|| \).
Theorem 3  If $E$ and $G$ two Riesz spaces with $G$ Dedekind complete and $G$ has order continuous norm Banach lattice, the following assertions are true.

i) The set of all order bounded $\bar{o}$n-continuous operators from $E$ into $G$ is a band of $L_0(E, G)$.

ii) The set of all order bounded $\sigma$-$\bar{o}$n-continuous operators from $E$ into $G$ is a band of $L_0(E, G)$.

Proof  i) Let $T : E \to G$ be an order bounded $\bar{o}$n-continuous operator. Note that if $|S| \leq |T|$ holds in $L_0(E, G)$, then by Theorem 2, $S$ is order bounded and $\bar{o}$n-continuous. That is, the set of all order bounded $\bar{o}$n-continuous operators from $E$ into $G$ is an ideal of $L_0(E, G)$. Let $0 \leq T \downarrow T$ in $L_0(E, G)$ that $T_{\lambda}$ is an $\bar{o}$n-continuous operator for all $\lambda$, and let $(x_{\alpha}) \subseteq E$ that $0 \leq x_{\alpha} \uparrow x$ in $F$. Then for each fixed index $\lambda$ we have

$$0 \leq T(x - x_{\alpha}) \leq (T - T_{\lambda})(x) + T_{\lambda}(x - x_{\alpha}),$$

and $x - x_{\alpha} \downarrow 0$, in conjunction with $T_{\lambda}$ is an $\bar{o}$n-continuous operator, implies

$$0 \leq \inf_{\alpha} \{ ||T(x - x_{\alpha})|| \} \leq ||(T - T_{\lambda})(x)||$$

for all $\lambda$. From $T - T_{\lambda} \downarrow 0$ and by order continuity of $G$, we see that $\inf_{\alpha} \{ ||T(x - x_{\alpha})|| \} = 0$. Thus, $T$ is $\bar{o}$n-continuous operator, and the proof follows.

ii) The proof is similar to (i).

Theorem 4  Let $T : E \to G$ be an order bounded operator. Then the following assertions are true.

i) If $G$ is Archimedean vector lattice and $T$ is a preserves disjointness and $\bar{o}$n-continuous, then $|T|$ exists and $|T| \in L_{\bar{o}n}(E, G)$.

ii) If $E$ is a band in $F$, $G$ is an atomic with order continuous norm Banach lattice and $T : E \to G$ is $\sigma$-$\bar{o}$n-continuous, then $|T|$ exists and $|T| \in L_{\bar{o}n}^\sigma(E, G)$.

Proof  i) Let $(x_{\alpha}) \subseteq E$ be a $\bar{o}$-null net. By assumption we have $Tx_{\alpha} \xrightarrow{\|\|} 0$. By Theorem 2.40 of [1], $|T|$ exists and $|T||x| = |T||x| = |Tx|$ for all $x \in E$. Since, $||T||x_{\alpha}| \leq |T||x_{\alpha}| \xrightarrow{\|\|} 0$, therefore, $|T||x_{\alpha}| \xrightarrow{\|\|} 0$. Hence $|T| \in L_{\bar{o}n}(E, G)$.

ii) Let $(x_{\alpha}) \subseteq E$ and $x_{\alpha} \xrightarrow{\sigma} 0$ in $E$. It is clear that $x_{\alpha} \xrightarrow{\sigma_0} 0$ in $E$. By assumption, $Tx_{\alpha} \xrightarrow{\|\|} 0$ in $G$. It is obvious that, $(x_{\alpha})$ is order bounded and therefore, $(Tx_{\alpha})$ is order bounded. Hence by Lemma 5.1 of [2], $Tx_{\alpha} \xrightarrow{\sigma} 0$ in $G$. Hence $T$ is an $\sigma$-order continuous operator. Note that since $G$ has order continuous norm, therefore, it is a Dedekind complete. So by Theorem 1.56 of [1], $|T|$ exists and it is an $\sigma$-order continuous. Let $(x_{\alpha}) \subseteq E$ be a $\bar{o}$-null net. We have $|x_{\alpha}| = P_E|x_{\alpha}| \leq P_E(y_m)$. We have $x_{\alpha} \xrightarrow{\sigma} 0$ in $E$. By assumption, $|T||x_{\alpha}| \xrightarrow{\sigma} 0$ in $G$. Because $G$ has order continuous norm, therefore, $|T||x_{\alpha}| \xrightarrow{\|\|} 0$ in $G$. Hence $|T| \in L_{\bar{o}n}^\sigma(E, G)$

Corollary 2  By part 2 of Theorem 4, if $E$ is a band in $F$, $G$ is an atomic with order continuous norm Banach lattice. It follows that $T : E \to G$ is a $\sigma$-$\bar{o}$n-continuous operator if and only if it is a $\sigma$-order continuous operator. Therefore, by Theorem 1.57 of [1], $L_{\bar{o}n}^\sigma(E, G)$ is a band of $L_0(E, G)$. 
4 \(\tilde{o}\)-order weakly compact operator

A continuous operator \(T : E \rightarrow X\) is said to be \(\tilde{o}\)-order weakly compact (or, \(\tilde{o}\)-weakly compact for short), if \(A \subseteq E\) is \(F_o\)-bounded in \(E\), then \(T(A)\) in \(X\) is a relatively weakly compact set. The collection of all \(\tilde{o}\)-weakly compact operators from vector lattice \(E\) into Banach space \(X\) will be denoted by \(W_0(E, X)\).

A subset \(A\) in a Banach lattice \(E\) is \(F\)-almost order bounded if for any \(\varepsilon > 0\) there exists \(u \in F^+\) such that \(A \subseteq [-u, u] + \varepsilon B_E\).

As following remark, very weakly compact operator \(T : E \rightarrow X\) is a \(\tilde{o}\)-weakly compact operator, the converse holds whenever \(E\) has an order unit.

**Remark 3** i) Let \(T : E \rightarrow X\) be an weakly compact operator. If \(A\) is a \(F\)-order bounded set in \(E\), so it is norm bounded set. Since \(T\) is a weakly compact operator, \(T(A)\) is relatively weakly compact set in \(X\). It means that \(T\) is a \(\tilde{o}\)-weakly compact operator.

ii) Let \(E\) has an order unit and \(T : E \rightarrow X\) is a \(\tilde{o}\)-weakly compact operator. If \(A\) is a norm bounded set in \(E\), then it is order bounded set in \(E\) and therefore, it is \(F\)-order bounded. By assumption, \(T(A)\) is relatively weakly compact set in \(X\). It means that \(T\) is a weakly compact operator.

**Proposition 2** If \(E\) has order continuous norm with property \((F)\), then the identity operator \(I : E \rightarrow E\) is \(\tilde{o}\)-weakly compact

**Proof** Let \(A \subseteq E\) be a \(F_o\)-bounded set. Since \(E\) has the property \((F)\), therefore, \(E\) is an order bounded set in \(E\). It is obvious that \(A\) is almost order bounded in \(E\). Since \(E\) has order continuous norm, \(A\) is a relatively weakly compact set in \(E\) by Theorem 4.9(5) and Theorem 3.44 of [1]. Hence \(I(A)\) is a relatively weakly compact set in \(E\). It means that \(I\) is a \(\tilde{o}\)-weakly compact operator.

**Theorem 5** An operator \(T : E \rightarrow X\) is \(\tilde{o}\)-weakly compact if and only if for each disjoint and \(F_o\)-bounded sequence \((x_n) \subseteq E\) implies \(Tx_n \xrightarrow{\|\cdot\|_X} 0\).

**Proof** Let \(T : E \rightarrow X\) be a \(\tilde{o}\)-weakly compact operator. Therefore, for each \(u \in F^+\), \(T([-u, u])\) is relatively weakly compact. Let \(I_u\) be the ideal generated by \(u\) in \(E\). The operator \(T|_{I_u} : I_u \rightarrow X\) is weakly compact operator. Since \(I_u\) is an AM-space with order unit, therefore, \(T|_{I_u} : I_u \rightarrow F\) is \(M\)-weakly. Hence for each disjoint norm bounded sequence \((x_n) \subseteq I_u\), we have \(Tx_n \xrightarrow{\|\cdot\|_X} 0\).

Conversely, let \(A \subseteq E\) be a \(F\)-bounded set. Then there exists \(u \in F^+\) such that \(A \subseteq [-u, u]\). Let \(I_u\) be the ideal generated by \(u\) in \(E\) and \((x_n) \subseteq A\) be a disjoint sequence. It is clear that \((x_n)\) is norm bounded. By assumption, we have \(Tx_n \xrightarrow{\|\cdot\|_X} 0\) in \(F\). Therefore, \(T : I_u \rightarrow X\) is \(M\)-weakly compact and by Theorem 5.61 of [1], \(T : I_u \rightarrow X\) is a weakly compact operator. Let \(A \subseteq E\) be a \(F\)-bounded set. Then there exists \(u \in F^+\) such that \(A\) is norm bounded in \(I_u\) and \(T : I_u \rightarrow X\) is weakly compact. Therefore, \(T(A)\) is a relatively weakly compact in \(X\). So \(T : E \rightarrow X\) is a \(\tilde{o}\)-weakly compact operator.

**Corollary 3** i) Let \(T, S\) be two operators that \(0 \leq T \leq S\) and \(S\) is a \(\tilde{o}\)-weakly compact operator. If \((x_n) \subseteq E\) is disjoint and \(F_o\)-bounded, then by Theorem 5 \(Sx_n \xrightarrow{\|\cdot\|_X} 0\). It follows that \(Tx_n \xrightarrow{\|\cdot\|_X} 0\). So \(T\) is a \(\tilde{o}\)-weakly compact operator.
ii) Let $T, S$ be two $\tilde{o}$-weakly compact operators. By Theorem 5 it is clear that $S \circ T$ is a $\tilde{o}$-weakly compact operator.

It is obvious that, if $T : E \to X$ is a $\tilde{o}$-weakly compact operator, then is order weakly compact. By following example the converse is not true in general.

Example 3 The operator $T : \ell^1 \to \ell^\infty$ defined by

$$T(x_1, x_2, \ldots) = \left(\sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \ldots\right)$$

is an order weakly compact operator. Let $(x_n) \subseteq \ell^1$ be disjoint order bounded sequence. We have $x_n \overset{\text{o}}{\to} 0$ and order bounded. Therefore, $x_n \overset{\text{o}}{\to} 0$. Since $\ell^1$ has order continuous norm, therefore, $(x_n)$ is norm-null. Because $T$ is a continuous operator, hence $Tx_n \overset{\| \cdot \|}{\to} 0$ in $\ell^\infty$. So by Theorem 5.57 of [1], $T$ is an order weakly compact operator.

If we consider $(e_n) \subseteq \ell^1$ we have $e_n \overset{\text{o}}{\to} 0$ in $\ell^1$. On the other hand, $Te_n = (1, 1, 1, \ldots)$, therefore, $(Te_n)$ is not convergent to zero in norm topology. Thus, $T$ is not $\tilde{o}$-weakly compact.

Theorem 6 Let $G$ be a normed vector lattice that is a sublattice of normed vector lattice $H$ and $T : E \to G$ be a $\tilde{o}$-weakly compact operator. By one of the following conditions, the modulus of $T$ exists and it is a $\tilde{o}$-weakly compact operator.

i) $E$ is an AL-space and $X$ has the property $(P)$ and the $H$-property.

ii) $E$ and $G$ have an order unit.

iii) $G$ is Archimedean Dedekind complete and $T$ is an order bounded preserves disjointness.

Proof

i) Let $(x_n) \subseteq E$ be a disjoint order bounded sequence. It is obvious that $(x_n)$ is $\text{Fo}$-bounded. By assumption and Theorem 5 $Tx_n \overset{\| \cdot \|}{\to} 0$. Hence by Theorem 5.57 of [1], $T$ is an order weakly compact operator. Since $E$ is an $\text{AL}$-space and $G$ has the property $(b)$, by Theorem 2.2 of [3], $|T|$ exists and is an order weakly compact operator. Since $G$ has the $H$-property, $|T|$ is a $\tilde{o}$-weakly compact operator.

ii) Let $A$ be a norm bounded set in $E$. Since $E$ has an order unit, therefore, $A$ is order bounded and so is $\text{Fo}$-bounded. By assumption, $T(A)$ is a relatively weakly compact set in $G$. Hence $T$ is a weakly compact operator. Since $G$ has an order unit, therefore, by Theorem 2.3 of [11], the modulus of $T$ exists and it is a weakly compact operator. It is obvious that $|T|$ is a $\tilde{o}$-weakly compact operator.

iii) By Theorem 2.40 of [11], $|T|$ exists and we have $|T||x| = |T|x| = |Tx|$ for all $x$. If $(x_n) \subseteq E$ is a $\text{Fo}$-bounded disjoint sequence, then by assumption $Tx_n \overset{\| \cdot \|}{\to} 0$. We have $|T||x_n| = |T|x_n| = |Tx_n| \overset{\| \cdot \|}{\to} 0$ in $G$ for each $n$. Now by inequality $|T|x_n| \leq |T||x_n|$, we have $|T|x_n \overset{\| \cdot \|}{\to} 0$. Hence $|T|$ is a $\tilde{o}$-weakly compact operator.

The following examples shows that $\tilde{o}$-weakly compact operators do not have the duality property.
Example 4  i) Consider the operator $T : C[0,1] \to c_0$, given by

$$T(f) = (\int_0^1 f(x) \sin x dx, \int_0^1 f(x) \sin 2x dx, \ldots).$$

By Example 3.15 of [10], $T$ is a weak-Dunford-Pettis and by Theorem 3.11 of [10], $T$ is a weakly compact operator. Therefore, $T$ is a \(\tilde{o}\)-weakly compact operator. We have $T^* : \ell^1 \to (C[0,1])^*$ that

$$T^* x_n(f) = \sum_{n=1}^{\infty} x_n \left( \int_0^1 f(t) \sin t dt \right).$$

Note that \((e_n) \subseteq \ell^1\) is \(\ell^\infty\)-order bounded and disjoint. Put $f_n(t) = \sin nt$ for all $n$. We have

$$\|T^* e_n\| \geq \|T^* e_n(f_n)\| = \int_0^1 (\sin nt)^2 dt \to 0.$$ 

Thus by Theorem 5, $T^*$ is not a \(\tilde{o}\)-weakly compact operator.

ii) Consider the functional $f : \ell^1 \to \mathbb{R}$ defined by

$$f(x_1, x_2, \ldots) = \sum_{i=1}^{\infty} x_i.$$

\((e_n) \subseteq \ell^1\) is \(\ell^\infty\)-order bounded and disjoint while $f(e_n) \not\to 0$. Therefore, by Theorem 5, $f$ is not a \(\tilde{o}\)-weakly compact operator, but it is obvious that $f^* : \mathbb{R} \to \ell^\infty$ is a \(\tilde{o}\)-weakly compact operator.

In the following, under some conditions, we show that if an operator $T$ is \(\tilde{o}\)-weakly compact, then its adjoint $T^*$ is also \(\tilde{o}\)-weakly compact and vice versa.

**Proposition 3** Let $G$ be a vector lattice such that $G^* \subseteq F$. Then the following assertions are true.

i) If $E$ has an order unit and $T : E \to G$ is \(\tilde{o}\)-weakly compact, then $T^*$ is \(\tilde{o}\)-weakly compact.

ii) If $G^*$ has an order unit and $T^* : G^* \to E^*$ is \(\tilde{o}\)-weakly compact, then $T$ is \(\tilde{o}\)-weakly compact.

**Proof**  i) Let $E$ has an order unit and $T : E \to G$ is a \(\tilde{o}\)-weakly compact operator. It is obvious that $T$ is a weakly compact operator. By Theorem 5.23 of [1], $T^*$ is a weakly compact operator and therefore, $T^*$ is a \(\tilde{o}\)-weakly compact operator.

ii) The proof is similar to (i).

**Theorem 7** Let $T : F \to X$ be an operator. $T|_E : E \to X$ is \(\tilde{o}\)-weakly compact if and only if $T(A)$ is relatively weakly compact for each $F$-almost order bounded set $A \subseteq E$.

**Proof** If $T(A)$ is relatively weakly compact for each $F$-almost order bounded subset $A$ of $E$, it is obvious that $T|_E$ is \(\tilde{o}\)-weakly compact.

Conversely, let $A \subseteq E$ be a $F$-almost order bounded. Therefore, there exists a $u \in F^+$ that $A \subseteq [-u, u] + \varepsilon B_F$. It is obvious that $T(A) \subseteq T[-u, u] + \varepsilon T(B_F)$. Since $T$ is \(\tilde{o}\)-weakly compact, hence $T[-u, u]$ is a relatively weakly compact. By Theorem 3.44 of [11], $T(A)$ is a relatively weakly compact in $X$. 
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