Torsion of anisotropic and inhomogeneous prismatic rods with a rectangular cross section

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Abstract. The limiting state of prismatic rods made of anisotropic material under torsion under the condition of translational plasticity is investigated. The rod of heterogeneous material is represented by a composite rod, when different plasticity conditions are true in different parts of the rod. Basic relations are determined by the method of characteristics. A field of characteristics of the basic relations for anisotropic and composite rods is constructed under the condition of translational plasticity, relations are obtained by characteristics, a tension break lines are found.

1. Introduction

Torsion is one of the types of deformation of the solid, characterized by mutual rotation of cross sections of the rod, the shaft under the influence of the moments acting in these sections. The cross sections of round rods remain flat at torsion. Torsion of rods is quite common in engineering practice. The theory of torsion of isotropic and anisotropic rods belongs to the number of developed sections of the theory of an ideal rigid-plastic solid. At the same time, researches on the theory of torsion of non-uniform rods are not enough. John T. Katsikadelis, George C. Tsiatas [1] investigated orthotropic and anisotropic bar with elliptical cross-section. In the works of Mironov B. G., Mironov Yu. B. [2, 3] is limit state of cylindrical and prismatic rods from anisotropic ideal rigid-plastic material is investigated under torsion for arbitrary condition of plasticity and the torsion of anisotropic and non-uniform rods with elliptic section under the condition of Mises-Hill plasticity is considered. Torsion of non-uniform and compound rods is considered in works [4]-[6]. L. S. Kozlova [7], [8] are devoted to the torsion of cylindrical and prismatic rods, a sector of circular ring, the rods of variable cross section under the action of external pressure.

Translational anisotropy was first introduced in the work [9]. New results that take into account the influence of translational ideally plastic anisotropy are important and relevant. They make it possible to take into account the effect of translational anisotropy during torsion of bodies. The results of the work can be used in studying the properties of anisotropy; using the values of ultimate resistance allows us to reduce energy costs.
Consider a rectangular anisotropic prismatic rod. The axis \( z \) to the right is the constant generatrix of the rod. Suppose the rod spins around equal and opposite forces. The lateral surface of the rod is considered to be free of load. The influence of mass forces can be neglected. We will build the tension break lines.

2. Problem statement and the solution of the problem

In this paper, we consider the torsion of rods of a perfectly plastic material. It is assumed that the plastic properties of the core material depend on the direction or coordinates of the point, i.e. the core material has anisotropy or heterogeneity. In the work, the anisotropy of the core material is determined by the condition of translational plasticity. A rod of dissimilar material is represented by a composite rod when different plasticity conditions are met in different parts of the rod. The Saint-Venant semi-inverse method was used in constructing the solution of the main relations. The basic relationships are determined by the method of characteristics.

3. Results and discussion

3.1. Torsion of anisotropic prismatic rods

Consider the torsion of anisotropic prismatic rods in the case when the lateral surface of rod is free from tangential forces.

Consider anisotropy of the type

\[ A \tau_{xz}^2 + B \tau_{yz}^2 = 1. \]  

Rewrite condition (1) as

\[ \frac{\tau_{xz}^2}{a^2} + \frac{\tau_{yz}^2}{b^2} = 1, \quad a = \frac{1}{\sqrt{A}}, \quad b = \frac{1}{\sqrt{B}}. \]  

If

\[ \tau_{xz} = k(\theta) \cos \theta, \quad \tau_{yz} = k(\theta) \sin \theta, \quad \tan \theta = \frac{\tau_{yz}}{\tau_{xz}}. \]  

Then from (1), (2), (3) we get

\[ k(\theta) = \frac{1}{\sqrt{A \cos^2 \theta + B \sin^2 \theta}} = \frac{1}{\sqrt{\frac{A + B}{2} + \frac{A - B}{2} \cos 2\theta}}. \]  

or

\[ k(\theta) = \frac{ab}{\sqrt{\frac{a^2 + b^2}{2} - \frac{a^2 - b^2}{2} \cos 2\theta}}, \quad \tan \theta = \frac{\tau_{yz}}{\tau_{xz}}. \]  

Consider a rod of rectangular cross section. Then, according to (3) and (5), we have (figure 1):

- along the border \( OC \) the value \( \theta = 0, \tau_{xz} = a, \tau_{yz} = 0, \tau = ai. \)
- along the border \( CD \) the value \( \theta = \frac{\pi}{2}, \tau_{xz} = 0, \tau_{yz} = b, \tau = bj. \)
- along the border \( DF \) the value \( \theta = \pi, \tau_{xz} = -a, \tau_{yz} = 0, \tau = -ai. \)
- along the border \( FO \) the value \( \theta = \frac{3\pi}{2}, \tau_{xz} = 0, \tau_{yz} = -b, \tau = -bj. \)
Figure 1. Cross section of the rod with the lines of tension rupture.

3.2. Torsion of transmission and anisotropic prismatic rods
Consider the case of transmission anisotropy [9] of the form
\[(\tau_{xz} - k_1)^2 + (\tau_{yz} - k_2)^2 = 1, \quad k_1, k_2 \text{ - const.} \]

From (3), (5) we obtain
\[k(\theta) = \rho \cos(\theta - \mu) + \sqrt{1 - \rho^2 \sin^2(\theta - \mu)}, \]
where
\[\rho = \sqrt{k_1^2 + k_2^2}, \quad \frac{k_1}{\rho} = \cos \mu, \quad \frac{k_2}{\rho} = \sin \mu, \quad \tan \mu = \frac{k_2}{k_1}.\]

Consider the various cases of torsion of rectangular rods at transmission anisotropy shown in figure 2.
When \(\mu = 0\):
along the border \(OC\) the value \(\theta = 0, \tau_{xz} = \rho + 1, \tau_{yz} = 0\);
along the border \(DC\) the value \(\theta = \frac{\pi}{2}, \tau_{xz} = 0, \tau_{yz} = \sqrt{1 - \rho^2}\);
along the border \(DF\) the value \(\theta = \pi, \tau_{xz} = -\rho + 1, \tau_{yz} = 0\);
along the border \(FO\) the value \(\theta = \frac{3\pi}{2}, \tau_{xz} = 0, \tau_{yz} = \sqrt{1 - \rho^2}\).

When \(\mu = \frac{\pi}{4}\):
along the border \(OC\) the value \(\theta = 0, \tau_{xz} = \frac{\sqrt{2}(\rho + \sqrt{2 - \rho^2})}{2}, \tau_{yz} = 0\);
along the border \(DC\) the value \(\theta = \frac{\pi}{4}, \tau_{xz} = 0, \tau_{yz} = \frac{\sqrt{2}(\rho + \sqrt{2 - \rho^2})}{2}\);
along the border \(DF\) the value \(\theta = \frac{\pi}{2}, \tau_{xz} = \frac{\sqrt{2}(-\rho + \sqrt{2 - \rho^2})}{2}, \tau_{yz} = 0\);
along the border \(FO\) the value \(\theta = \frac{3\pi}{4}, \tau_{xz} = 0, \tau_{yz} = \frac{\sqrt{2}(\rho + \sqrt{2 - \rho^2})}{2}\).
Figure 2. The lines of tension rupture in various cases of translational anisotropy.

When \( \mu = \frac{\pi}{2} \):

- along the border \( OC \) the value \( \theta = 0, \tau_{xz} = \sqrt{1-\rho^2}, \tau_{yz} = 0 \);
- along the border \( DC \) the value \( \theta = \frac{\pi}{2}, \tau_{xz} = 0, \tau_{yz} = \rho + 1 \);
- along the border \( DF \) the value \( \theta = \pi, \tau_{xz} = \sqrt{1-\rho^2}, \tau_{yz} = 0 \);
- along the border \( FO \) the value \( \theta = \frac{3\pi}{2}, \tau_{xz} = 0, \tau_{yz} = -\rho + 1 \).

3.3. Torsion of composite inhomogeneous transmission and anisotropic prismatic rods

Consider a composite rectangular anisotropic prismatic rod divided into two areas by a line of inhomogeneity \( DE \) coming from the vertex \( D \).

The axis \( \zeta \) is directed parallel to the generatrix of rod. Assume that the rod twists around the axis \( \zeta \) by equal and opposite pairs of forces with a moment \( M \). The lateral surface of rod is considered free from stress. The influence of mass forces can be neglected.

Assume that

\[
\begin{align*}
\sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0, \\
\tau_{xz} &= \tau_{xz}(x,y), \tau_{yz} &= \tau_{yz}(x,y),
\end{align*}
\]

where \( \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz} \) are normal and tangential stresses on the platforms, perpendicular to the coordinate axes \( x, y, \zeta \).

The plasticity condition in each area has the form

\[
(\tau_{xz} - k_{i1})^2 + (\tau_{yz} - k_{i2})^2 = k_{i0}^2, \quad k_{i0}, k_{i1}, k_{i2} - \text{const},
\]

(9)
where $k_{20} > k_{10}$, $i = 1, 2$.

We take all the values having the dimension of stresses to the value $k_{i0}$ ($i = 1, 2$), and proceed to the dimensionless values.

Condition (10) takes the form

$$\left(\tau_{xz} - k_{i1}\right)^2 + \left(\tau_{yz} - k_{i2}\right)^2 = 1,$$

where $i = 1, 2$.

Assume that in each area

$$k_{i1} = k_i(\theta)\cos\theta,$$

$$k_{i2} = k_i(\theta)\sin\theta,$$

$$\tan\theta = \frac{\tau_{yz}}{\tau_{xz}},$$

where $i = 1, 2$.

According to (12), (13), we find from (11)

$$\cos\theta \sin\theta = \frac{k_{i1}}{k_{i2}} = \cos\mu_i, \quad \frac{k_{i2}}{k_{i1}} = \sin\mu_i, \quad \tan\mu_i = \frac{k_{i2}}{k_{i1}}, \quad i = 1, 2.$$

The differential equilibrium equation at torsion has the form:

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0.$$

Substituting the expressions (12), (13) into the equilibrium equation (17), we obtain

$$(k'_i \cos\theta - k_i \sin\theta) \frac{\partial \theta}{\partial x} + (k'_i \sin\theta + k_i \cos\theta) \frac{\partial \theta}{\partial y} = 0,$$

where

$$k'_i = \frac{dk_i}{d\theta}, \quad i = 1, 2.$$

Corresponding equations for determining the characteristics have the form

$$\frac{dx}{k'_i \cos\theta - k_i \sin\theta} = \frac{dy}{k'_i \sin\theta + k_i \cos\theta} = \frac{d\theta}{0},$$

where $i = 1, 2$.

From equation (20), it follows that the characteristics of equations (18) take the form

$$y_i = \frac{k'_i \sin\theta + k_i \cos\theta}{k'_i \cos\theta - k_i \sin\theta} \cdot x + \Phi_i(\theta), \quad \theta = \text{const},$$

where $i = 1, 2$.

From equation (21), it follows that the characteristics are straight lines along which the tangential stresses $\tau_{xz}, \tau_{yz}$ are constant.

Figure 3 presents the case of torsion of rectangular anisotropic rods divided into two areas by a line of inhomogeneity $DE$ coming from the vertex $D$.

We define the line of tension rupture in each area.

The condition of conjugation of the vectors of tangential stresses should be satisfied on the line of inhomogeneity $DE$.
The direction vector of rupture line $AN$ of the first area is equal to the difference of vectors of tangential stresses $\overline{r}_{13}$ and $\overline{r}_{14}$. The direction vector of rupture line $BN$ is equal to the difference of vectors $\overline{r}_{14}$ and $\overline{r}_{11}$. Rupture lines $AN$ and $BN$ intersect at point $N$. By virtue of the conjugation of vectors of the tangential stresses $\overline{r}_{11}$ and $\overline{r}_{13}$, we have the rupture line $NM$ of the first area, coming from a point $N$ parallel to a straight line $AD$. The rupture line $NM$ crosses the rupture line $DE$ at a point $M$.

4. Conclusion
In the case of an inhomogeneous anisotropic rod, the vectors of tangential stresses, the characteristics of the relations determining the stress-strain state of the body are found in each region, and the lines of stress discontinuity are constructed. In all considered cases of torsion of rectangular rods, the characteristics are straight lines along which the tangential stresses $\tau_{xz}$, $\tau_{yz}$ are constant. The results obtained make it possible to reveal the influence of the properties of ideally plastic translational anisotropy during torsion of rods. In particular, the results can be applied in calculating the bearing capacity of structures and allow to take into account the influence of anisotropy in determining the limit forces at torsion of solids.

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