Unlensing the CMB in real space: a new approach to extract the weak lensing convergence

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Abstract. We present a new estimator defined in real space to extract the weak lensing convergence from temperature maps of the Cosmic microwave background radiation. This estimator is built upon the minimum variance estimator defined in harmonic space. Despite being by construction suboptimal, its implementation proves less sensitive to the experimental noise and to the bias of the mode coupling derived from the finite size of the map.

1. Introduction

The Cosmic microwave background (CMB) radiation propagates almost unperturbed after recombination, so that what it contains are mainly the fluctuations at the last-scattering surface $z \sim 10^3$ of order $10^{-5}$ about the black body profile of temperature $2.725 \text{ K}$ \cite{1}. It also contains a polarization signal of order $10^{-6}$ as the imprint of a quadrupole produced during recombination \cite{2}. Observations of the CMB temperature and polarization are thus a powerful probe of the physics of the early Universe and a robust discriminant of cosmological models.

At large angular scales ($\theta > 5'$), the observed fluctuations are mainly primary anisotropies described by small linear perturbations in the primordial plasma’s density and velocity, which are reflected in the characteristic series of acoustic peaks. At small angular scales, the primary anisotropies become negligible due to Silk damping and non-linear effects dominate, which developed at more recent epochs from the primordial potential–matter coupling, forming secondary sources of anisotropies. Among the non-linear effects is weak gravitational lensing, which consists of the deflection of CMB photons by the matter fluctuations along their trajectory (see Ref. \cite{3} for a review). Gravitational lensing generates small scale power at temperature gradients and distorts the temperature-temperature (TT) correlation power spectrum by smearing the acoustic peaks \cite{4}. It also distorts and mixes the EE and BB polarization power spectra \cite{5}. Finally, weak lensing also correlates with other secondary effects, such as the integrated Sachs-Wolfe effect and the Sunyaev-Zel’dovich effect, thus inducing non-Gaussianities which are manifest in higher-order point correlation functions \cite{6, 7}.

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\textsuperscript{4} This manuscript is based on a talk delivered by the author at the NEB 14, Ioannina, June 2010

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The relevance of the CMB lensing reconstruction for cosmology is thus threefold. First, by changing the power spectra, CMB lensing affects the estimation of cosmological parameters. Second, by converting a fraction of the dominant E-polarisation mode into a B mode, CMB lensing introduces a contaminant in the measurement of the primordial B-polarisation mode and consequently of the inflation energy scale. Third, CMB lensing is a powerful probe of the gravitational potential, integrated from the last-scattering surface to the presence, that governed the formation of the large scale structure. Thus it is also sensitive to the neutrino mass, the dark energy equation of state and the geometry of the University.

Current experiments, such as ACT\(^5\), SPT\(^6\) and PLANCK\(^7\) are making robust measurements of unprecedented arcminute precision, so that weak lensing should soon become apparent both in the temperature and in the polarization observations. This calls for new statistical and modelling methods to be explored to extract the newly observable signals.

In this manuscript we report on a study where we explored the properties of a novel estimator of the CMB weak lensing, introduced in Ref. \([8]\). In section 2 we describe the lensing of the CMB temperature. In section 3 we present the conventional minimum variance estimator of the lensing field, upon which in section 4 we derive the newly proposed estimator. In section 5 we reconstruct the lensing field, implementing both estimators, and discuss the results. In section 6 we conclude.

2. Weak lensing of the CMB

Weak lensing of the CMB consists of the deflection of CMB photons from the original propagation direction \(\theta\) on the last scattering surface (the source plane) to an observed direction \(\tilde{\theta}\) on the sky today (the image plane). This deflection amounts to remapping the unlensed temperature anisotropies \(T\) to the lensed ones \(\tilde{T}\) according to \(\tilde{T}(\theta) = T(\theta + \alpha)\), where \(\alpha = \alpha + \theta\). For small deflection angles \(\alpha\), we can expand

\[
\tilde{T}(\theta) = T(\theta + \alpha) = T(\theta) + \alpha \cdot \nabla T(\theta) + O[\alpha^2].
\]

(1)

The deflection angle is given by \(\alpha = \nabla \psi\), where the lensing potential \(\psi\) is the integral of the equation for a null geodesic from the source along the line of sight and \(\nabla\) is the covariant derivative on the image plane. Hence, for the line element

\[
ds^2 = a^2(\eta) \left[ - (1 - 2\Psi) d\eta^2 + (1 + 2\Psi) \left( dr^2 + f_k^2(r) \left( d\theta_x^2 + d\theta_y^2 \right) \right) \right],
\]

(2)

where the functional form of \(f_k\) depends on the spatial curvature \(k\) of the two-dimensional surface \((\theta_x, \theta_y)\), integrating along the unperturbed path, \(\theta_x = \text{const}\) and \(\theta_y = \text{const}\) (the Born approximation), we find for the lensing potential \(\psi\) generated by the gravitational potential \(\Psi\)

\[
\psi(\theta) = -2 \int_0^{r_{LS}} dr \frac{f_k(r_{LS} - r)}{f_k(r_{LS}) f_k(r)} \Psi(r, \eta_0 - r), \quad f_k(r) = \begin{cases} \sin[r] : k = +1, \\ \sinh[r] : k = 0, \end{cases} \quad \text{with} \quad \eta_0 = \text{the conformal time at recombination},
\]

(3)

Here, \(r\) is the comoving distance to the lens plane, \(r_{LS}\) is the comoving distance to the last-scattering surface and \(\theta\) is the two-dimensional position vector on the image plane.

It is useful to have an order of magnitude of the weak lensing effect on the CMB. On their way from the last-scattering surface, the CMB photons encounter gradients of matter densities, with

\[\text{http://www.physics.princeton.edu/act/about.html}\]
\[\text{http://pole.uchicago.edu/}\]
\[\text{http://www.rssd.esa.int/index.php?project=planck}\]
the depth of the gravitational potential of order $\Psi \sim 10^{-4}$. The matter is clustered predominantly at a slice of thickness 300 Mpc, which corresponds to the characteristic size of the potential wells given by the peak of the matter power spectrum. Given that the distance to the last-scattering surface is 14000 Mpc, the number of potentials passed through (hence of deflections suffered) is around 50, which for a random walk corresponds to a root mean square deflection of $\alpha(\theta) \approx \sqrt{50} \times 10^{-4} \approx 7 \times 10^{-4} \text{rad} = 2.4 \text{ arcmin}$. Hence weak lensing is expected to become an important effect for $\ell > 3000$. The deflections will be correlated over the sky by the angular size of a characteristic potential halfway to the last-scattering surface, $300/7000 \approx 2.5 \text{ deg}$. Hence although the deflections are much smaller than the scale of the first acoustic peak, they are correlated over comparable scales. This effect is called weak because the deflections are small ($\sim \text{arcmin}$) and can only be detected as a statistical effect from coherent distortions ($\sim \text{deg}$).

There are three ways to describe the lensing distortion of the CMB: a) the lensing potential $\psi$, b) the deflection vector $\alpha = \nabla \psi$, and c) the convergence tensor $\kappa = -\nabla \nabla \psi/2$ decomposed as

$$\kappa = \begin{pmatrix} \kappa_0 + \kappa_+ \\ \kappa_- \\ \kappa_0 - \kappa_+ \end{pmatrix}. \quad (4)$$

The descriptions of $\psi$ and $\alpha$ suffer from an ambiguity upon translation, since a patch of the sky and its translation have the same likelihood on account of isotropy. In contrast, the description of the convergence, which is a gradient of the deflection vector field, is locally well defined. Here we will develop the estimator to reconstruct the isotropic component of the tensor $\kappa$, the convergence $\kappa_0(\theta) = -(\partial_x^2 + \partial_y^2)\psi(\theta)/2$, which magnifies or demagnifies a feature on the last-scattering surface. We will treat the shear components, $\kappa_\pm(\theta) = -(\partial_x^2 - \partial_y^2)\psi(\theta)/2$ and $\kappa_x(\theta) = -\partial_x \partial_y \psi(\theta)$, in a forthcoming study.

### 3. Minimum variance estimator of the weak lensing convergence

To construct an estimator for the lensing potential from the observed lensed CMB, we start from the observation that the weak lensing effect is apparent in the temperature power spectrum. For small angular scales, we can expand the lensing potential in plane waves

$$\psi(\theta) = \int \frac{d^2 \ell}{(2\pi)^2} \exp[i \ell \cdot \theta] \psi(\ell) \quad (5)$$

so that $(\nabla \psi)(\ell) = i \ell \psi(\ell)$. We can then write the linear order correction of the lensed temperature anisotropy in harmonic space as the convolution of the lensing potential with the unlensed...
temperature anisotropy

$$\tilde{T}(\ell) = T(\ell) - \int d^2\ell' \cdot (\ell - \ell') \psi(\ell') T(\ell - \ell') + O[\psi^2(\ell)]. \tag{6}$$

Defining the temperature power spectrum by \( \langle T(\ell) T(\ell') \rangle = (2\pi)^2 \delta(\ell + \ell') C_\ell \), we find that the power spectrum of the lensed temperature anisotropy is related to that of the unlensed as follows

$$\langle \tilde{T}(\ell') \tilde{T}(\ell - \ell) \rangle = (2\pi)^2 \delta(\ell) C_{\ell'} + (2\pi)^2 \delta \cdot \left[ \ell' \cdot C_{\ell'} + (\ell - \ell') \cdot C_{|\ell - \ell'|} \right] \psi(\ell). \tag{7}$$

The first term in Eqn. (7) describes a statistically isotropic ensemble where modes with different \( \ell \) are uncorrelated. The anisotropies generated by the lensing potential introduce correlations among different \( \ell \) modes. Hence, an estimator for \( \psi \) is a weighted average of the \( \ell \neq 0 \) term \[9\]

$$\hat{\psi}(\ell, \ell') = \frac{\tilde{T}(\ell') \tilde{T}(\ell - \ell')}{\ell \cdot \ell' C_{\ell'} + \ell \cdot (\ell - \ell') C_{|\ell - \ell'|}}. \tag{8}$$

The optimal estimator is the weighted combination of \( \hat{\psi}(\ell, \ell') \) which makes the estimator unbiased and minimizes to leading order the variance. For the minimum variance estimator \( \hat{\psi}(\ell) \) we find the convolution of the square of \( \tilde{T}(\ell) \) by the weight function \( Q^\psi(\ell, \ell') \)

$$\hat{\psi}(\ell) = \int \frac{d^2\ell'}{(2\pi)^2} \tilde{T}(\ell') \tilde{T}(\ell - \ell') Q^\psi(\ell, \ell'), \tag{9}$$

where

$$Q^\psi(\ell, \ell') = N_\ell \frac{1}{2} \frac{\ell \cdot \ell' C_{\ell'} + \ell \cdot (\ell - \ell') C_{|\ell - \ell'|}}{C_{\ell'} C_{|\ell - \ell'|}} \tag{10}$$

and

$$N_\ell = \left[ \int \frac{d^2\ell'}{(2\pi)^2} \frac{1}{2} \frac{[\ell \cdot \ell' C_{\ell'} + \ell \cdot (\ell - \ell') C_{|\ell - \ell'|}]^2}{C_{\ell'} C_{|\ell - \ell'|}} \right]^{-1} \equiv \text{Var}[\hat{\psi}(\ell)] \tag{11}$$

(see Appendix A of Ref. [8] for a derivation). This reconstruction, based on the quadratic estimator in the lensed CMB map, will be noisy due to spurious correlations in the CMB map. The minimum variance estimator of the convergence \( \kappa_0(\theta) = -\nabla^2 \hat{\psi}(\theta)/2 \) is then \( \hat{\kappa}_0(\ell) = \ell^2 \hat{\psi}(\ell) \).

The expression of \( Q^\psi(\ell, \ell') \) can be straightforwardly modified to include the detector noise and the finite beam width by replacing \( C_{\ell'} \rightarrow C_{\ell'} + N_{\ell'} \). The noise power spectrum \( N_\ell \) is the inverse-sum over the number of channels of the detector noise \( n_i(\ell) \) of each channel \( i \)

$$N_\ell = \left( \sum_{i=0}^{\text{num.chann}} \frac{1}{n_i(\ell)} \right)^{-1}. \tag{12}$$

The detector noise is modelled by a Gaussian signal on the beam size and includes both the white noise amplitude and the beam profile attenuation factor

$$n_i(\ell) = (\theta_{fwhm_i} \sigma_{pix_i})^2 \exp[(\theta_{fwhm_i})^2 \ell(\ell + 1)/(8 \ln 2)], \tag{13}$$

where \( \theta_{fwhm_i} \) is the beam full-width at half-maximum of the beam and \( \sigma_{pix_i} \) is the white noise amplitude per beam width.
4. Real space estimator of the weak lensing convergence

The optimal reconstruction in harmonic space discussed above implicitly assumes a full-sky coverage [10] and moreover without: galactic cuts, bad pixels due to excision of point sources, or a nonuniform weighting for an uneven sky coverage. To surpass the difficulties of the analysis of realistic maps, we consider a slightly less optimal estimator modified to have a kernel with compact support which acts in real space.

An estimator for the convergence in real space would be

\[ \hat{\kappa}_0(\theta) = \int \frac{d^2 \ell}{(2\pi)^2} \exp[i \ell \cdot \theta] \int \frac{d^2 \ell'}{(2\pi)^2} \hat{T}(\ell') \hat{T}(\ell - \ell') Q(\ell, \ell') \]

where we define the corresponding weight function in real space by

\[ Q(\theta, \theta', \theta'') = \int \frac{d^2 \ell}{(2\pi)^2} \exp[i \ell \cdot \theta] \int \frac{d^2 \ell'}{(2\pi)^2} \exp[-i \ell' \cdot \theta'] \exp[-i(\ell - \ell') \cdot \theta''] Q(\ell, \ell'). \]

and \( Q(\ell, \ell') = \ell^2 Q^\ell(\ell, \ell') \). The kernel \( Q(\ell, \ell') \) is a function of the lengths of \( \ell \) and \( \ell' \) as well as of the angle \( \xi_\ell \) between \( \ell \) and \( \ell' \), and can be expanded in eigenfunctions which factorize the radial and the angular dependence as follows

\[ Q(\ell, \ell') = \sum_{m=-\infty}^{+\infty} \exp[im\xi_\ell] Q_m(\ell, \ell') \]

so that

\[ Q_m(\ell, \ell') = \frac{1}{2\pi} \int d\xi_\ell \exp[-im\xi_\ell] Q(\ell, \ell'). \]

In order to obtain the analogous factorization of the kernel in real space, which would make the implementation computationally less costly, we change to the variables \( \ell_+, \ell_- \) such that \( \ell = \ell_+, \ell = (\ell_+ + \ell_-)/2 \). Upon this change of variables, Eqn. (9) yields

\[ \hat{\kappa}_0(\ell_+) = \int \frac{d^2 \ell_-}{(2\pi)^2} \hat{T'}(\ell_+ - \ell_-) W(\ell_+, \ell_-), \]

where \( W(\ell_+, \ell_-) = Q(\ell, \ell') \). The corresponding estimator in real space becomes

\[ \hat{\kappa}_0(\theta) = \int d^2 \theta_+ d^2 \theta_- \hat{T}(\theta_+ + \theta_-) \hat{T}(\theta_+ - \theta_-) W(\theta, \theta_+, \theta_-), \]

where we define the corresponding weight function in real space by

\[ W(\theta, \theta_+, \theta_-) = \int \frac{d^2 \ell_+}{(2\pi)^2} \exp[i \ell_+ \cdot \theta] \int \frac{d^2 \ell_-}{(2\pi)^2} \exp[-i(\ell_+ \cdot \theta_+ + \ell_- \cdot \theta_-)] W(\ell_+, \ell_-). \]

We then expand the kernel \( W(\ell_+, \ell_-) \) in terms of eigenfunctions that factorize the radial and the angular dependence so that

\[ W_m(\ell_+, \ell_-) = \frac{1}{2\pi} \int d\chi_\ell \exp[-im\chi_\ell] W(\ell_+, \ell_-). \]
and $\chi_\ell$ is the angle between $\ell_+$ and $\ell_-$. For the case that $\theta = 0$, Eqn. (20) becomes

$$ W(\theta_+, \theta_-, \chi_\theta) \equiv \sum_{m=-\infty}^{\infty} \exp[im\chi_\theta] W_m(\theta_+, \theta_-), \quad (22) $$

where

$$ W_m(\theta_+, \theta_-) = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty d\ell_+ d\ell_- J_m(\ell_+ \theta_+) J_m(\ell_- \theta_-) W_m(\ell_+, \ell_-) \quad (23) $$

and $\chi_\theta$ is the angle between $\theta_+$ and $\theta_-$. Having achieved the factorization of the angular and the radial dependence of the kernel in real space, Eqn. (19) can be written as

$$ \hat{\kappa}_0(\theta = 0) = \sum_{m=-\infty}^{\infty} \int d^2 \theta_+ \int d^2 \theta_- \exp[im\chi_\theta] \hat{T}(\theta_+ + \theta_-) \hat{T}(\theta_+ - \theta_-) W_m(\theta_+, \theta_-). \quad (24) $$

The convergence map thus estimated contains two contributions: the convergence of the lensing potential $\kappa_0|_\psi$ and the convergence of the CMB (Gaussian noise) $\kappa_0|_{\psi=0}$

$$ \hat{\kappa}_0 = \kappa_0|_\psi + \kappa_0|_{\psi=0}. \quad (25) $$

Hence, in the absence of lensing there is a non-vanishing contribution to the estimated convergence map and to the corresponding power spectrum. To remove the bias of the estimated map, i.e. $\langle \kappa_0|_{\psi=0} \rangle$, we take the pixel average of the convergence map obtained from a single realization of the lensed CMB and subtract from the convergence map. (This is equivalent to taking the average convergence map obtained from different realizations of the CMB lensed by the same realization of the lensing potential.) This causes the Gaussian noise to average out, yielding an unbiased estimate of the convergence map $\langle \hat{\kappa}_0 \rangle = \langle \kappa_0|_\psi \rangle$. To remove the bias of the estimated power spectrum, i.e. $\langle \kappa_0|_{\psi=0} \kappa_0^*|_{\psi=0} \rangle$, we compute the power spectrum of the convergence map obtained from different realizations of the unlensed CMB, take the average power spectrum and subtract from the power spectrum of the convergence map obtained from the individual realization of the lensed CMB map, thus yielding an unbiased estimate of the convergence power spectrum $\langle \hat{\kappa}_0 \hat{\kappa}_0^* \rangle = \langle \kappa_0|_\psi \kappa_0^*|_\psi \rangle$.

In this formulation, the kernel is the Bessel transform of the minimal variance weight function in harmonic space. Whereas in the harmonic space formulation the temperature map is transformed and then is weighted by the kernel in harmonic space, in the real space formulation the kernel is transformed and then weights the temperature map to yield the lensing field.

5. Implementation of the convergence estimator

We implement the real space estimator to reconstruct the convergence field. We will use two experiments based on the specification of the PLANCK experiment for the $\nu = 143$ GHz channel [11] which differ only in the detector noise. The experiments, denoted by Designer and Planck, both have $\theta_{fwhm} = 7.8$ arcmin for the beam full-width at half maximum, and $\sigma_{pix} = 0$ and $\sigma_{pix} = 6.8 f_{sky}^{1/2} \mu K/\text{rad}$ for the white noise amplitude per beam width respectively, where $f_{sky}$ is the fraction area of the sky covered by the patch [12, 13]. We chose the experiments to have the same beam size in order not to entangle the effect of the noise with that of the beam.

The real space implementation is described as follows:

- First we synthesise the lensed CMB map by pixel remapping of the unlensed CMB map by the lensing map [14]. Then we convolve the lensed map with the beam profile $b_\ell = \sum_i \exp[-(\theta_{fwhm})^2\ell(\ell + 1)/(8\ln 2)/2]$ and finally add the map constructed from the detector noise power spectrum $1/\sum_i[1/(\theta_{fwhm}\sigma_{pix})^2]$ (see Appendix B of Ref. [8]). Instead of deconvolving the lensed map with the beam profile, we deconvolve the kernel.
Figure 2. Contour plots of $Q_{m=0}(\ell, \ell')$ (left panel) and $Q_{m=0}^{\text{intrinsic}}(\ell, \ell')$ (right panel). The levels denote the orders of magnitude as fractions of the maximum value, delimited by $\{1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$ from the lightest to the darkest shade for PLANCK noise at $\nu = 143$ GHz.

- Second we generate the kernel in harmonic space (Eqn. (10)), deconvolve with the beam profile (which is equivalent to deconvolving the lensed map with the beam profile) and finally transform to real space (Eqn. (23)).
- Third we convolve the square of the lensed map with the kernel (Eqn. (24)) and compute the unbiased convergence map. We repeat this step for different realizations of the unlensed CMB map and use the average power spectrum to compute the unbiased power spectrum of the convergence.

For comparison, we also implement the harmonic space estimator by applying the procedure described in Appendix C of Ref. [8]. In order to reproduce periodic boundary conditions, before convolving with the beam profile we apodize the CMB maps by smoothing to zero over a strip around the boundary. The requirement of periodic boundary conditions makes the implementation of the harmonic space estimator easier but is not necessary for the implementation of the real space estimator.

5.1. The kernel

For the real space implementation, we compute $W_m(\ell_+, \ell_-)$ using Eqn. (21) and the corresponding inverse Fourier transform $W_m(\theta_+, \theta_-)$ using Eqn. (23).

For the number of eigenfunctions $m_{\text{max}}$ kept in the series in Eqn. (24), we took the order of the highest-order eigenfunction whose maximum amplitude was of order 1% the maximum amplitude of the dominant $m = 0$ eigenfunction, which corresponds to $m_{\text{max}} = 4$. For the kernel extent in harmonic space $\ell_{\text{max}}$, we used the highest $\ell$ moment used in the synthesis of the lensed CMB, i.e. $\ell_{\text{max}} = 4000$. This is also consistent with the value read off of $Q_{m=0}(\ell, \ell')$ which we plot in Fig. 2, left panel. Defining

$$Q(\ell, \ell') = N_\ell \ell^2 \frac{1}{2} \frac{\ell \cdot \ell' C_\ell + \ell \cdot (\ell - \ell') C_{|\ell - \ell'|}}{C_\ell C_{|\ell - \ell'|}} = N_\ell Q_{m=0}^{\text{intrinsic}}(\ell, \ell')$$

we also plot $Q_{m=0}^{\text{intrinsic}}$ in Fig. 2, right panel. Using higher values for $\ell_{\text{max}}$, we observe that whereas $Q_{m=0}^{\text{intrinsic}}$ is compact, $Q_m$ is largely $\ell_{\text{max}}$ invariant as a result of the presence of the estimator variance $N_\ell$. We thus conclude that $N_\ell$ weights the different $\ell$ modes in such a way as
to enhance the contribution of the highest $\ell$ modes accessible. In the absence of detector noise, the kernel is displaced to higher values of $\ell$ and $\ell'$ up to the chosen $\ell_{\text{max}}$.

In Fig. 3 we show the contour plot of the $m = 0$ eigenfunction of the kernel in real space for the Designer and the Planck experiment respectively. From these plots we read off the value for the spatial extent of the kernel $\theta_{\text{max}}$ used in the convolution to compute the convergence map. In harmonic space the kernel peaks at the scale of the beam but has support at smaller $\ell$ values. The spatial extent of the kernel is thus larger than the beam scale and scales roughly linearly with the beam size. We observe that the kernel in real space is compact and extends only up to $0.70^\circ$ and $0.60^\circ$ at the 1% level in the absence and in the presence of detector noise respectively. Since increasing the value of $\theta_{\text{max}}$ beyond $0.35^\circ$ did not improve the reconstruction of the power spectrum, whereas reducing the value of $\theta_{\text{max}}$ below $0.35^\circ$ degraded the reconstruction, we set $\theta_{\text{max}} = 0.35^\circ$.

5.2. The convergence map

We compute the estimator by convolving the kernel with the lensed CMB map according to Eqn. (24). We implement the convolution using the discrete version of this equation as follows

$$\kappa_0(\theta = 0) = \sum_{m=-m_{\text{max}}}^{m_{\text{max}}} \sum_{\theta_{+x}} \sum_{\theta_{+y}} \sum_{\theta_{-x}} \sum_{\theta_{-y}} \exp \left[ \text{im} \arctan \left( \frac{\theta_{+y} - \theta_{-y}}{\theta_{+x} - \theta_{-x}} \right) \right]$$

$$\times \tilde{T}(\theta_{+x} + \theta_{-x}, \theta_{+y} + \theta_{-y}) \tilde{T}(\theta_{+x} - \theta_{-x}, \theta_{+y} - \theta_{-y})$$

$$\times W_m \left( \sqrt{\theta_{+x}^2 + \theta_{+y}^2}, \sqrt{\theta_{-x}^2 + \theta_{-y}^2} \right).$$

(27)

The points in the temperature map which contribute to the convergence at a given point $\theta$ span the region from $(\theta_{+} - \theta_{-})$ to $(\theta_{+} + \theta_{-})$, where $\theta_{+}$ and $\theta_{-}$ are in the kernel. Hence we need a region of the temperature map that is four times the length of the kernel to compute the convolution, while setting the summation upper limits equal to the spatial extent of the kernel $\theta_{\text{max}}$.

We then average the reconstructed convergence map over all pixels to extract the Gaussian noise which is then subtracted from the map. Finally we compute the convergence power spectrum of a Monte Carlo realization of one hundred unlensed CMB maps and subtract the
average variance of the Gaussian CMB noise from the convergence power spectrum of the lensed map, thus removing the bias from the estimated power spectrum.

5.3. The convergence power spectrum

In Figs. 4 and 5, for the Designer and the Planck experiment respectively, we plot the output power spectrum $C^\kappa_{\text{out}}$, both before (light grey solid line) and after (dark grey solid line) the removal of the bias, against the input power spectrum $C^\kappa_{\text{in}}$ (black solid line). We also
plot the variance of the estimator $\text{Var} \left[ \hat{\kappa}_0(\ell) \right]$ (black dot-dashed line) as well as $\text{Var} \left[ \hat{\kappa}_0(\ell) \right]/f_{\text{sky}}$ (grey dot-dashed line) as an auxiliary tool for visualization. The error bars on the power spectra are computed from the standard deviation of the reconstructed power spectrum, binned over logarithmically spaced intervals in $\ell$ space and added in quadrature. The variance of the reconstructed power spectrum encompasses both the variance of the optimal estimator and the average value of the convergence fluctuation in the map, weighted by the fraction of the area of the sky covered by the patch, $f_{\text{sky}}$, as follows [13]

$$\text{Var}[C_{\ell}^{\kappa_0\kappa_0}] \approx \frac{1}{\ell f_{\text{sky}}} \left( \text{Var}[\hat{\kappa}_0(\ell)] + C_{\ell}^{\kappa_0\kappa_0} \right)^2.$$  

(28)

We observe that at small $\ell$ the cosmic variance dominates the error of the reconstruction, whereas at large $\ell$ the variance of the estimator takes over. The region where $C_{\ell}^{\kappa_0\kappa_0} > \text{Var}[\hat{\kappa}_0(\ell)]$ indicates the interval in $\ell$ where the lensing potential can in theory be mapped. Accordingly, this is also the region where the reconstructed convergence power spectra track more closely the input power spectrum. Also visible are the fluctuations in the recovered power spectrum which arise from the subtraction of the noise bias term estimated from a finite number of realizations of the unlensed CMB map.

5.4. Discussion

Using the real space estimator, we achieve a reasonable reconstruction of the input power spectrum in the interval where the input signal is larger than the variance of the estimator, although on smaller scales, $\ell > 600$, there is a decrease in the recovered power. Comparing the reconstructed power spectra for the Designer and Planck experiment, we observe that the real space implementation is fairly insensitive to the detector noise and returns a reconstruction consistent with the input power spectrum. We can understand these results as follows.

The loss of power at small scales, beginning around the angular scale corresponding to the size of the kernel, is a consequence of averaging modes smaller than the finite extent of the kernel and hence is an intrinsic constraint of the real space implementation. This intrinsic constraint can be overcome by shrinking the extent of the kernel in real space, which can be achieved with a smaller beam and detector noise, thereby moving the support of the kernel to larger $\ell$. The $\ell$ interval over which the power spectrum can be recovered by the real space estimator is determined as follows. The lower $\ell$ limit is determined by the size of the map, since it measures the longest wavelength mode that can be enclosed within the map. The upper $\ell$ limit is determined by the size of the kernel, since it measures the smallest wavelength mode that the kernel can probe, below which the averaging of modes smaller than the size of the kernel causes loss of power. In contrast, the $\ell$ range for a reasonable reconstruction with the harmonic space estimator is wider, being limited at higher $\ell$ by the angular scale corresponding to the beam size, since this is the scale which constrains the action of the kernel in harmonic space.

Within this $\ell$ range, the real space estimator seems to perform fairly well both in the absence and in the presence of detector noise. We explain the insensitivity of the real space estimator to the experimental noise as follows. Recalling that in the real space implementation the convergence in each pixel is given by the sum of the product of pairs of neighbouring pixels weighted by the kernel, the noise, being independent in each pixel, is averaged down. To test this result, we reconstructed the convergence map from a pure white noise input map using the real space estimator and the harmonic space estimator (Fig. 6). We observe that the power spectrum recovered by the real space estimator is about seven orders of magnitude smaller than the input power spectrum, whereas the power spectrum recovered by the harmonic space estimator is comparable to the input power spectrum.

Finally comparing the two implementations (Figs. 4 and 5), we observe that the removal of the bias was not effective in the harmonic space implementation. This remaining bias presumably
arises from a coupling between unlensed temperature modes due to the finite size of the map, providing an additional source of lensing which we have not yet treated. This mode coupling is absent in the reconstruction over the full sky [15]. The real space implementation appears to be insensitive to this bias.

6. Concluding remarks
We report on the introduction of a new estimator of the weak lensing convergence of the CMB. The new estimator acts locally in real space and is thus able to treat the excision of pixels and nonuniform sky coverage in a flexible manner. The finite extent of the kernel is a desirable property of the proposed estimator, since in theory it allows the reconstruction of the lensing convergence, manifested mainly at very small scales, from a small map of the sky as long as the kernel probes sufficiently small angular scales.

We implemented the estimator on two experimental setups, one without and one with detector noise, based on the specifications of PLANCK for the $\nu = 143$ GHz channel. From the comparative implementation of the new estimator and the conventional estimator defined in harmonic space, we found that the real space implementation is less sensitive to the experimental noise, as well as to the bias of the mode coupling derived from the finite size of the map. Even though the new estimator shows a loss of power on scales smaller than the finite extent of the kernel, this effect could be studied using simulations to determine a form factor to be applied to correct the loss of power [16].

A preliminary study of the effect of the excision of points (e.g. resulting from the application of a mask for the subtraction of point sources) on the reconstruction of the lensing convergence seems to strengthen the advantages of a real space implementation of the estimator [17]. Follow-up studies include the formulation of the analogous estimators for the shear components of the convergence tensor, as well as for the CMB polarisation.

Acknowledgments
The author was supported by the South African National Research Foundation. The author is currently supported by the Fundação para a Ciência e a Tecnologia under the fellowship BPD/65993/2009.
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