Discrete phase-space structures and Wigner functions for $N$ qubits

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Abstract We further elaborate on a phase-space picture for a system of $N$ qubits and explore the structures compatible with the notion of unbiasedness. These consist of bundles of discrete curves satisfying certain additional properties and different entanglement properties. We discuss the construction of discrete covariant Wigner functions for these bundles and provide several illuminating examples.

Keywords Wigner function · Mutually Unbiased Bases keyword · Entanglement

1 Introduction

Phase-space methods offer the remarkable advantage that quantum mechanics appear as similar as possible as a classical statistical theory, by avoiding the operator formalism [123].

The relevant role of discrete quantum systems, which live in a $d$-dimensional Hilbert space, was early anticipated by Weyl [4]. The related problem of generalizing the Wigner function to these finite systems has a long history. A plausible approach was taken by Hannay and Berry [5], considering a phase space constrained to admit only periodic probability distributions, which implies that the corresponding manifold is effectively a $2d \times 2d$-dimensional torus. Other surrogates using a $2d \times 2d$ grid were also investigated [67], and used to deal with different aspects of quantum information [89].

Another important line of research has focused on a phase space is pictured as a $d \times d$ lattice. It was started by Buot [10], who introduced a discrete Weyl transform that generates a Wigner function on the toroidal lattice $\mathbb{Z}_d$. This is in the same vein of the pioneer work of Schwinger [11], who clearly recognized that the expansion of arbitrary operators in terms of certain operator basis was the crucial concept in setting a proper phase-space description. Indeed, he identified the finite counterpart

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of the Weyl-Heisenberg group, which describes the canonical conjugacy position-momentum and that can be used to define a $d \times d$ phase space \[12\]. More recently, these ideas have been rediscovered and developed further by other authors \[13,14,15,16,17,18,19\].

Actually, when the dimension $d$ is a power of a prime, points in the $d \times d$ grid must be labeled with elements of the Galois field $\mathbb{F}_d$: only by doing this we can endow the phase space with geometric properties similar to those of the ordinary plane. Note also that though the restriction to powers of primes rules out many quantum systems, this formulation is ideally suited for the time-honored example of $N$ qubits we deal in this paper.

These satisfactory geometrical properties are in the realm of the most popular approach to deal with the discrete Wigner function, which is due to Wootters \[20,21\]. This leads to a non-unique procedure of relating states in the Hilbert space with lines in the grid. Such a map exhibits an important property inherited from the continuous case: the sum of the Wigner function along the line associated with a given state gives the probability distribution in this state. Furthermore, this construction also satisfies all the bona fide requirements: invertibility, Hermiticity, normalization and covariance under discrete displacements generated by the Pauli group \[22\].

On the other hand, these straight lines are intimately related with the concept of mutually unbiased bases (MUBs) \[23,24\]: eigenstates of sets of $N$ commuting operators labelled with points of mutually non-intersecting rays (lines passing through origin) determine MUBs.

A complete set of MUBs can be reduced to an arrangement of $d^2 - 1$ disjoint operators into $d + 1$ classes each containing $d - 1$ commuting operators. Eigenstates of lines in such a table with $(d - 1) \times (d + 1)$ entries form MUBs \[25\]. Interestingly, these operators can be organized in several nontrivial tables, leading to different factorization properties \[26\]. Here, we are interested only in unitary equivalent sets of MUBs. It has been noticed \[27\] that such arrangements are related with special types of geometric structures in the discrete phase space, the so-called commutative curves. A bundle of $d + 1$ non-intersecting curves determines the set. In principle, to each of these bundles one can link a Wigner function with all the required properties, in such a way that the traditional Wootters approach is recovered for the special case of rays. Obviously, to a given state correspond different Wigner functions based on different MUBs and the suitable choice of these MUBs depends on the entanglement structure of the state.

In this paper, we go one step further and provide an explicit form of phase-point operators for $N$ qubits corresponding to MUBs with different factorization structures. It results that these kernels are not equivalent under transformations connecting different sets of MUBs, but preserve the basic tomographic property, allowing to express the Wigner function of any state as a linear combination of measured probabilities. In addition, the Clifford inequivalence leads to the possibility of finding non-stabilizer states with non-negative Wigner functions, which clashes with previous results for the discrete case \[28,29,30\].
2 Curves in phase space

For a system of $N$ qubits, the Hilbert space is the tensor product $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2^N}$. Let $|k_1, \ldots, k_N\rangle$ ($k_i \in \mathbb{Z}_2$) an orthonormal basis in $\mathbb{C}^{2^N}$. We can label this basis by $\kappa \in \mathbb{F}_{2^N}$, so that

$$\kappa = \sum_{i=1}^N k_i \theta_i,$$

where $\{\theta_1, \ldots, \theta_N\}$ is a self-dual basis [i.e., $\text{tr}(\theta_i \theta_j) = \delta_{ij}$, with $\text{tr}(\alpha) = \alpha + \alpha^2 + \ldots + \alpha^{2^{N-1}}$, and $\alpha \in \mathbb{F}_{2^N}$].

The generators of the Pauli group $\mathcal{P}_N$ are

$$Z_\alpha = \sum_{\kappa} \chi(\kappa \alpha) |\kappa\rangle \langle \kappa|, \quad X_\beta = \sum_{\kappa} |\kappa + \beta\rangle \langle \kappa|$$

and satisfy the commutation relations $Z_\alpha X_\beta = \chi(\alpha \beta) X_\beta Z_\alpha$, with $\chi(\alpha) = \exp[i \pi \text{tr}(\alpha)]$ being an additive character. In addition, $Z_\alpha$ and $X_\beta$ are related through the finite Fourier transform [31].

The operators (2) can be factorized in the form

$$Z_\alpha = \sigma_1^{a_1} \otimes \cdots \otimes \sigma_N^{a_N}, \quad X_\beta = \sigma_1^{b_1} \otimes \cdots \otimes \sigma_N^{b_N},$$

where $a_i = \text{tr}(\alpha \theta_i)$ and $b_i = \text{tr}(\beta \theta_i)$ correspond to the expansion coefficients for $\alpha$ and $\beta$ in the self-dual basis.

The phase space can be appropriately labeled by the discrete points $(\alpha, \beta)$ [32], which are precisely the indices of the operators $Z_\alpha$ and $X_\beta$: $\alpha$ is the “horizontal” axis and $\beta$ the “vertical” one.

A stabilizer state is a simultaneous eigenvector of a maximal set of commuting observables in the Pauli group. A complete set of stabilizers is given by a set of $2^N$ disjoint commuting monomials $\{Z_{\alpha(\tau)}X_{\beta(\tau)}\}$, expressed as

$$\alpha(\tau) = \sum_{r=0}^{N-1} \alpha_r \tau^{2^r}, \quad \beta(\tau) = \sum_{r=0}^{N-1} \beta_r \tau^{2^r},$$

with $\alpha_r, \beta_r \in \mathbb{F}_{2^N}$ and such that

$$\sum_{r=0}^{N-1} \alpha_r^{2^r} \beta_q^{2^r} = \sum_{r=0}^{N-1} \alpha_q^{2^r} \beta_r^{2^r}.$$

We can look at these functions as curves $\Gamma = (\alpha(\tau), \beta(\tau))$ in phase space. We impose that they pass through the origin: $(\alpha(0), \beta(0)) = (0, 0)$; that is, $Z_{\alpha(0)}X_{\beta(0)} = 1$. We call them stabilizer curves. The disjointness is in agreement with the fact that they have no self-intersections: all the $2^N$ pairs $(\alpha(\tau), \beta(\tau))$ are different. Consequently, to each stabilizer curve $\Gamma$ corresponds a basis $\{|\psi_\kappa^\tau\rangle\}$, with $\kappa \in \mathbb{F}_{2^N}$.

It follows from [4] that summing the coordinates of any two points of a stabilizer curve we obtain another point on the curve

$$\alpha(\tau + \tau') = \alpha(\tau) + \alpha(\tau'), \quad \beta(\tau + \tau') = \beta(\tau) + \beta(\tau').$$
In other words, the stabilizers \( \{ Z_{\alpha(\tau)} X_{\beta(\tau)} \} \) form an Abelian group under multiplication, which is generated, e.g., by \( \{ Z_{\alpha(\theta)} X_{\beta(\theta)} \} \).

A stabilizer curve is called regular when it can be represented in the explicit form

\[ \beta = f(\alpha), \quad \text{or} \quad \alpha = g(\beta). \tag{7} \]

Otherwise, the curve are called degenerate \cite{33}. In that case, both \( \alpha \) and \( \beta \) do not take some values in \( \mathbb{F}_{2^N} \) and they are multivalued for some other values.

The simplest form of stabilizer curves are the straight lines

\[ \alpha(\tau) = \mu \tau, \quad \beta(\tau) = \nu \tau, \tag{8} \]

which can be represented in the regular form \( \beta = \lambda \alpha \) (or \( \alpha = 0 \) for the vertical axis). It is a well established result \cite{34} that the operators \( \{ Z_{\alpha} X_{\beta=\lambda \alpha} \} \) commute for any fixed value of \( \lambda \in \mathbb{F}_{2^\alpha} \), while the eigenstates of the set \( \{ Z_{\alpha} \} \) define the standard computational basis \( | \kappa \rangle \).

The regular curves can always be transformed into the horizontal [for curves \( \beta = f(\alpha) \)] or the vertical [for curves \( \alpha = g(\beta) \)] axes. This can be accomplished by a pair of symplectic operations (\( z \)- and \( x \)-rotations) such that

\[ P_f Z_{\alpha} P_f^{-1} \sim Z_{\alpha} X_{f}(\alpha), \quad Q_g X_{\beta} Q_g^{-1} \sim Z_{g(\beta)} X_{\beta}, \tag{9} \]

the symbol \( \sim \) indicating here equality except for a phase. Both \( P_f \) and \( Q_g \) are unitary operators, and can be written as

\[ P_f = \sum_{\kappa} c^{(f)}_{\kappa} | \kappa \rangle \langle \kappa |, \quad Q_g = \sum_{\kappa} c^{(g)}_{\kappa} | \kappa \rangle \langle \kappa |, \tag{10} \]

where \( | \kappa \rangle \) are the eigenstates of \( X_{\beta} \). The coefficients \( c^{(f)}_{\kappa} \) satisfy the recurrence relation

\[ c^{(f)}_{\kappa} c^{(f)}_{\kappa'} = \chi \left( \kappa' f(\kappa) \right) c^{(f)}_{\kappa + \kappa'}, \quad c^{(f)}_{\kappa 0} = 1, \tag{11} \]

and analogously for \( c^{(g)}_{\kappa} \). In general, Eq. (11) admits multiple solutions, as discussed in detail in Ref. \cite{35}. Thus, given a curve we can immediately obtain the eigenstates of the set of commuting monomials attached to this curve: for instance, \( | \Psi_{\kappa=f(\alpha)} \rangle = P_f | \kappa \rangle \).

One of the most fundamental characteristics of a stabilizer curve \( \Gamma = \{ Z_{\alpha(\tau)} X_{\beta(\tau)} \} \) is its factorization structure; that is, the possibility of parsing each monomial \( Z_{\alpha} X_{\beta} \) into smaller mutually commuting subsets containing \( 1 \leq k \leq N \) single-qubit operators:

\[ \text{fact}(\Gamma) = \{ m_1, m_2, \ldots, m_N \}, \tag{12} \]

where \( 0 < m_1 \leq m_2 \leq \ldots \leq m_N \) (\( m_j \in N \)) is the number of particles in the \( j \)-th block that cannot be factorized into commuting sub-blocks anymore. It is clear that \( \{ m_1, m_2, \ldots, m_N \} \) is just a partition of the integer \( N \), so the maximum number of terms is \( N \), which corresponds to a completely factorized curve, \( \text{fact}(\Gamma) = \{ 1, 1, \ldots, 1 \} \),

and the minimum number of terms is one, for a completely non-factorized curve \( \text{fact}(\Gamma) = \{ N \} \).
3 Mutually unbiased bases from curves

The bases related to nonintersecting curves $Γ$ and $Γ'$ are unbiased \[3\]; that is,

$$|⟨Ψ_κ^Γ|Ψ_κ'^Γ⟩|^2 = \frac{1}{2N},$$ (13)

so that a bundle of $2^N + 1$ mutually nonintersecting curves define a complete set of MUBs.

The simplest bundle is formed by the rays $\{ β = λα, α = 0 \}$. The corresponding (standard) set of MUBs will be denoted as $\{|Ψ_{λ;κ}\rangle, |Ψ_{κ}\rangle\}$, where $|Ψ_{κ}\rangle$ are the eigenstates of $X_κ$. The set $\{|Ψ_{λ;κ}\rangle\}$ is constructed as $|Ψ_{λ;κ}\rangle = P_j=λα|κ\rangle$. This allows one to establish a canonical association between basis elements and straight lines:

$$|Ψ_{λ;κ}\rangle \iff \{ β = λα + κ \}, \quad |Ψ_κ\rangle \iff \{ α = κ \},$$ (14)

where the ray $β = 0$ is associated with the state $|κ = 0\rangle$ (the only state with all positive eigenvalues), and the parallel lines $β = κ$ correspond to the shifted states $|κ = X_κ|0\rangle$.

Our next observation is that the “rotated” bases

$$|Ψ_{λ;κ}^{(f,g,h)}\rangle = P_βQ_{gβ}P_α|Ψ_{λ;κ}\rangle, \quad |Ψ_κ^{(f,g,h)}\rangle = P_βQ_{gβ}P_κ|Ψ_κ\rangle,$$ (15)

preserve the mutually unbiasedness inherited from the standard set $\{|Ψ_{λ;κ}\rangle, |Ψ_{κ}\rangle\}$, so that

$$|⟨Ψ_{λ;κ}^{(f,g,h)}|Ψ_{λ';κ'}^{(f,g,h)}⟩|^2 = δ_{λ,λ'}δ_{κ,κ'} + \frac{1}{2N}(1 - δ_{λ,κ'}), \quad |⟨Ψ_κ^{(f,g,h)}|Ψ_κ'^{(f,g,h)}⟩|^2 = \frac{1}{2N}.$$ (16)

Accordingly, $|Ψ_{λ;κ}^{(f,g,h)}\rangle$ are eigenstates of commuting sets $\{Z_{α_λ(τ)}X_{β_λ(τ)}\}$ with

$$α_λ(τ) = τ + g(λτ) + g(f(τ)),$$
$$β_λ(τ) = λτ + f(τ) + h(τ) + h(g(λτ)) + h(g(f(τ))),$$ (17)

and $|Ψ_κ^{(f,g,h)}\rangle$ are eigenstates of $Z_κX_κ + h(g(τ))$. Therefore, for any fixed $λ$, the eigenstates of the set $\{Z_{α_λ(τ)}X_{β_λ(τ)}\}$ can be associated with $2^N$ mutually nonintersecting curves parallel to (17), whereas the eigenstates of $Z_κX_κ + h(g(τ))$ are associated with curves parallel to

$$β = τ + h(g(τ)), \quad α = g(τ).$$ (18)

Such sets of parallel curves, known as striations, have the following structure:

a.– Curves parallel to $α_λ(τ), β_λ(τ)$ are of the form

$$α_λ(τ, κ) = τ + g(λτ) + g(f(τ)) + g(κ),$$
$$β_λ(τ, κ) = λτ + f(τ) + κ + h(τ) + h(g(λτ)) + h(g(f(τ))) + h(g(κ)).$$ (19)

b.– Curves parallel to $β = τ + h(g(τ)), α = g(τ)$ are of the form

$$α(τ, κ) = κ + g(τ) + g(f(κ)),$$
$$β(τ, κ) = κ + f(κ) + h(κ) + h(g(τ)) + h(g(f(κ))).$$ (20)
We conclude then that the bundle (17) and (18) is unitarily equivalent to the standard one formed by the rays \( \{ \beta = \lambda \alpha, \alpha = 0 \} \). The advantage of the parametrization in (19) and (20) is that it preserves the same association between states and curves as in (14); viz,

\[
| \psi_{\lambda;\kappa}^{(f,g,h)} \rangle \Longleftrightarrow \Gamma^{(f,g,h)}_{\lambda;\kappa} = \{ \alpha_\kappa (\tau, \kappa), \beta_\lambda (\tau, \kappa) \},
\]

\[
| \tilde{\psi}_{\kappa}^{(f,g,h)} \rangle \Longleftrightarrow \Gamma^{(f,g,h)}_{\tilde{\kappa}} = \{ \alpha (\tau, \kappa), \beta (\tau, \kappa) \}.
\] (21)

The resulting \( \Gamma^{(f,g,h)}_{\lambda;\kappa} \) and \( \Gamma^{(f,g,h)}_{\tilde{\kappa}} \) satisfy an important property: any pair of curves crosses at a single point, much in the same way as straight lines. This property is quite obvious for regular curves (7), but far from trivial for degenerate curves.

A bundle may contain curves with different factorizations (12). We characterize different bundles with a set of numbers that indicate the number of completely factorized curves \( \{ 1, 1, ..., 1 \} \) structure, completely factorized except a single two-particle block (curves of the type \( \{ 1, 1, ..., 1, 2 \} \), etc., until completely nonfactorized curves \( \{ N \} \). We thus assign to the bundle the set of numbers \( (\ell_1, \ell_2, ..., \ell_{p(N)}) \),

\[
\sum_j \ell_j = 2^N + 1,
\] (22)

which indicate the number of curves factorized in \( N \) one-dimensional blocks, \( \ell_1 \); the number of curves factorized in \( N - 2 \) one-dimensional blocks and one two-dimensional block, \( \ell_2 \); etc, and \( p(N) \) is the number of partitions of an integer \( n \). For instance, the bundle of curves \( \{ \beta = \lambda \alpha, \alpha = 0 \} \) has the structure (3,0,6) in the three-qubit case. Examples of another bundles of curves corresponding to different type of factorizations of complete set of MUBs can be found in Ref. [33].

It is worth noting here that application of a set of three transformations in (15), which is an analog to the Euler decomposition in the discrete case [36], is the most general transformation that allows to obtain any curve bundle starting from the rays \( \{ \beta = \lambda \alpha, \alpha = 0 \} \). If two transformations \( Q_f P_f \) already produce an arbitrary curve from any of the ray \( \beta = \lambda \alpha, \alpha = 0 \), the last \( P_h \)-transformation is required to obtain a generic (in particular, a degenerate) curve from the \( x \)-axis, \( \alpha = 0 \). However, not always all three transformations are needed to generate a bundle with a given factorization structure as it will be exemplified below.

4 Wigner function on the curves

According to Wootters original proposal [20] the kernel of the discrete Wigner function (also called phase point operator) is constructed as

\[
\hat{\psi}(\alpha, \beta) = \sum_\lambda \hat{Q}(\lambda) - 1,
\] (23)
where $Q(\lambda)$ is a projector linked with a line $\beta = \lambda \alpha + \gamma$ passing through the point $(\alpha, \beta)$. The Wigner function for a state with density operator $\rho$ is then

$$W_{\rho}(\alpha, \beta) = \text{Tr}[\rho \hat{\omega}(\alpha, \beta)],$$

(24)
and it has the desired properties [37].

More explicitly, the kernel can be written down in terms of projectors on the standard MUBs, associated with rays, as follows

$$\hat{\omega}(\alpha, \beta) = |\Psi_{\alpha}\rangle \langle \Psi_{\alpha}| + \sum_{\lambda, \gamma} \delta_{\beta, \alpha \lambda + \gamma} |\Psi_{\lambda, \gamma}\rangle \langle \Psi_{\lambda, \gamma}| - \mathbb{1}. $$

(25)

In this way, the Wigner function of the state $|\Psi_{\lambda, \gamma}\rangle$ is just a straight line

$$W_{|\Psi_{\lambda, \gamma}\rangle}(\alpha, \beta) = \delta_{\beta, \lambda \alpha + \gamma}. $$

(26)

The Wigner kernel for a complete bundle $\{\Gamma^l = (\alpha(\tau) = f^l(\tau), \beta(\tau) = g^l(\tau))\}$ (with $l = 1, ..., 2^N$) can be constructed in the same way as in (23). Indeed, let us denote by

$$\{\Gamma^l_k\} = \{\alpha_k(\tau) = f^l_k(\tau), \beta_k(\tau) = g^l_k(\tau)\}$$

(27)

sets of parallel curves in the corresponding striations (i.e. the curves $\Gamma^l_k$, with $k \in \mathbb{F}_{2^N}$, do not intersect for a fixed value of $l$) and $\{|\Psi^l_k\rangle \equiv |\Psi^l_k\rangle$ are the associated states. Then, the Wigner kernel $\hat{\omega}(\alpha, \beta)$ can be jotted down exactly as in (23):

$$\hat{\omega}(\alpha, \beta) = \sum_{l=1}^{2^N-1} \sum_{\kappa \in \mathbb{F}_{2^N}} \delta_{\alpha, f^l_k(\tau)} \delta_{\beta, g^l_k(\tau)} |\Psi^l_k\rangle \langle \Psi^l_k| - \mathbb{1}. $$

(28)

This kernel satisfies the crucial tomographic property: summing the Wigner function (24) along any curve $\Gamma^l_k$ from the set (27) we obtain the probability of finding the system in the state $|\Psi^l_k\rangle$ associated with this curve:

$$\sum_{\alpha, \beta} \sum_{\tau} \delta_{\alpha, f^l_k(\tau)} \delta_{\beta, g^l_k(\tau)} \text{Tr}[\rho \hat{\omega}(\alpha, \beta)] = 2^N \langle \Psi^l_k| \rho |\Psi^l_k\rangle. $$

(29)

As a direct consequence of (25), we obtain that the Wigner function of a state $|\Psi^l_k\rangle$ has the form of the corresponding curve in the discrete phase-space:

$$W_{|\Psi^l_k\rangle} = \sum_{l} \delta_{\alpha, f^l_k(\tau)} \delta_{\beta, g^l_k(\tau)}. $$

(30)

As we have seen in previous Section, any extended bundle (that includes all the striations) can be obtained form the standard Wootters set of straight lines by unitary transformations. Then, for the bases related with the curves (17) one has

$$\hat{\omega}(\alpha, \beta) = |\Psi^f_{\alpha}\rangle \langle \Psi^f_{\alpha}| + \sum_{\lambda, \tau, \kappa} \delta_{\beta, \lambda + f(\tau) + g(\lambda) + g(\tau)} |\Psi^f_{\lambda, \kappa}\rangle \langle \Psi^f_{\lambda, \kappa}| - \mathbb{1}. $$

(31)
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Fig. 1 Wigner function of an eigestate of a commuting set element of the set (0,9,0) labeled by the curve \( \alpha = \sigma^3 + \sigma^2 \tau + \tau^4, \beta = \sigma^3 \tau^4 + \sigma^3 \tau^2 + \sigma^4 \).

which reduces to (25) when \( f(x) = 0, g(x) = 0, h(x) = 0 \), and consequently,

\[
\sum_{\tau} W_{\rho}(\alpha_{\lambda}(\tau, \kappa), \beta_{\lambda}(\tau, \kappa)) = 2^N \langle \Psi_{\kappa}^{(f,g,h)} | \rho | \Psi_{\kappa}^{(f,g,h)} \rangle,
\]

\[
\sum_{\kappa} W_{\rho}(\alpha(\tau, \kappa), \beta(\tau, \kappa)) = 2^N \langle \tilde{\Psi}_{\kappa}^{(f,g,h)} | \rho | \tilde{\Psi}_{\kappa}^{(f,g,h)} \rangle,
\]

where \( \{ \alpha_{\lambda}(\tau, \kappa) \}, \{ \alpha(\tau, \kappa) \} \) are defined in (19) and (20).

Observe, that by construction the kernel (28) satisfies the standard covariance (under discrete shifts) condition. Moreover, it cannot be obtained from (25) by a simple application of unitaries \( P \) and \( Q \) that transform one curve bundle into another. In particular, a transformed kernel

\[
\tilde{W}_{gfh}(\alpha, \beta) = P_{\beta} Q_{\beta} P_{\alpha} \tilde{W}(\alpha, \beta) P^\dag_{\alpha} Q^\dag_{\beta},
\]

(33)

does not satisfy the marginality (32). Instead, to get the probabilities associated to \( | \Psi_{\kappa}^{(f,g,h)} \rangle \), one should sum the Wigner function (33) over the straight lines \( \beta = \lambda \alpha + \kappa \).

To exemplify this approach we consider the case of three qubits, for which we know that the are four different sets of MUBs, with factorizations (3,0,6), (2,3,4), (1,6,2), and (0,9,0). The standard set, as discussed before in relation with rays, is the (3,0,6). The MUBs with factorization (1,6,2) can be obtained from the standard one with the transformation \( P_f \), with the curve \( f(\alpha) = \alpha + \alpha^2 + \alpha^4 \). The set with the factorization (2,3,4) requires application of two transformations \( P_f \) and \( Q_f \), generated by the single curve \( f(\alpha) = \mu \alpha + \alpha^2 + \alpha^4 \) with any \( \mu \neq 0 \). Finally, to generate the
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Fig. 2 Wigner function for a GHZ state $\frac{\langle 000 \rangle + \langle 111 \rangle}{\sqrt{2}}$ expressed in the four inequivalent sets of MUBs, with factorization $(3,0,6)$ (top left), $(2,3,4)$ (top right), $(1,6,2)$ (bottom left), and $(0,9,0)$ (bottom right).

set with the factorization $(0,9,0)$ a set of three transformations $P_f Q_f P_f$ is required, although still one curve $f(\alpha) = \alpha + \sigma^2 \alpha^2 + \sigma \alpha^4$ is sufficient. Here, $\sigma$ is a primitive element of $\mathbb{F}_{2^3}$, a root of $\sigma^3 + \sigma + 1 = 0$.

In Fig. 2 we plot the Wigner function, for the MUB with factorization $(0,9,0)$, of the eigenstate with all positive eigenvalues of the stabilizer (degenerate) curve $\alpha = \sigma^5 + \sigma^4 \tau + \tau^4, \beta = \sigma^4 \tau^4 + \sigma^3 \tau^2 + \sigma^4$ (actually this curve is obtained by the transformation $P_f Q_f P_f, f(\alpha) = \alpha + \sigma^2 \alpha^2 + \sigma \alpha^4$ from the straight line $\beta = \sigma^2 \alpha + \sigma^2$).

For an explicit construction of the operators (9) we take here the particular solution of the recurrence (11) where the first $N$ coefficients are chosen positive $c_{f_i}^{(f)} = +\sqrt{\chi(\theta_i f(\theta_i))}$. One can check the degeneracy of the corresponding curve.

The appearance of a quantum state may be very different under Wigner maps linked to MUBs with different factorizations. In Fig. 2 we plot the Wigner functions of a three-qubit Greenberger-Horne-Zeilinger (GHZ) state $\frac{\langle 000 \rangle + \langle 111 \rangle}{\sqrt{2}}$ for all possible factorizations. For the factorization $(0,9,0)$, the Wigner function contains only 8 points [although they do not form a curve, since (6) is violated], while for the other factorizations it has a form of a “real” distribution, spread over all the phase space. This suggests that some factorizations could be more appropriate for representation of states, with particular correlation properties, than others.
The Clifford inequivalence of the kernels (25) and (28) brings about an unforeseen consequence: the possibility of finding non-stabilizer states with positive Wigner functions. In this respect, we recall that, in the standard Wootters construction, corresponding to the set of MUBs (3,0,6), the only states with positive Wigner functions are stabilizer states [28,29,30]. Indeed, these states can be seen as the discrete counterparts of Gaussian for continuous variable systems [38,39] and the negativity of the Wigner function as a measure of quantum correlations [40,41].

As an example, one can show that the Wigner function of eigentstate $|\Psi_{\lambda,0}\rangle$ (with all positive eigenvalues) of the commuting set labelled with points of the ray $\beta = \lambda \alpha$ in the set $(0,9,0)$ has a form of a line

$$W_{|\Psi_{\sigma,0}\rangle}(\alpha,\beta) = \delta_{\beta,\lambda \alpha + \lambda^5}, \quad f(\alpha) = \alpha + \sigma_2^2 \alpha^2 + \sigma_4^4. \quad (34)$$

This can be clearly observed in Fig. 3. Actually, eigenstates of all the non-factorized rays $\beta = \lambda \alpha$ ($\lambda \neq 0,1$) are represented by positive Wigner distributions in this set. Observe, that such states, being completely non-factorized, are not eigenstates of any stabilizer set in the set (0,9,0), which contains only bi-factorized bases.

This property strongly depends on the set of MUBs. For instance the state $|\Psi_{\lambda,0}\rangle$ is represented as a positive distribution (actually as a non-degenerate curve) in the set (1,6,2), whereas in the set (2,3,4) the same state has a complicated distribution.

5 Concluding remarks

In summary, what we have shown is that for each complete set of MUBs, one can construct a discrete Wigner map following the original Wootters idea: the transformation kernel at a given point is obtained as a sum of projectors on the basis states
corresponding to the curves (associated with such states) passing through this point. This construction generalizes the standard one based on rays and cannot be obtained by a unitary transformation of the former map. As an immediate consequence, we obtain that the images of the basis states are not straight lines anymore, but some specific curves in the phase space.

In addition, it appears that Wigner functions based on certain set of MUBs may possess properties drastically different to the standard Wootters construction. In particular positive distributions not necessarily correspond to the stabilizer states.

In principle, it would be interesting to extend these notions to the continuous case. However, this would require know the limit of $d \to \infty$ of the MUBs. Although, this limit passing through prime dimensions suggests the existence of an unlimited number of MUBs, the question involve some subtle open questions [42].

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