All the three dimensional Lorentzian metrics admitting three Killing vectors

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Abstract

We obtain all the three-dimensional Lorentzian metrics which admit three Killing vectors. The classification has been made with the aid of the formalism which exploits the obstruction criteria for the Killing equations recently developed by the present authors. The current classification method does not rely on the transitivity property of the isometry group. It turns out that the Lorentzian manifold harbors a much richer spectrum of metrics with various Segre types, compared to the Riemannian case.

Keywords: Killing vectors, Isometry group, exact solutions to Einstein’s equations

1. Introduction and summary

A Killing vector (KV) on a manifold $M$ with a (pseudo-)Riemannian metric $g$ describes a vector field whose flow preserves the metric. The isometry group generated by the KVs is of fundamental touchstone in Lorentzian signature, since it has a direct link to the construction of exact solutions to Einstein’s equations. The importance of KVs in general relativity is even more highlighted by the relevance to the globally conserved quantities and constants of geodesic motion.

Despite the above-mentioned importance of KVs, the existence of any KV for a given a (pseudo-)Riemannian manifold $(M, g)$ is not instantly evident. A patient difficulty for this issue is how to find, if any, the explicit form and exhaustive list of KVs. More than a century ago, Darboux has put forward an inventive idea to determine the exact number of linearly independent KVs in two dimensions [1]. His criterion for the local existence of KVs has been formulated in terms of differential invariants and has provided a direct interplay between local geometric invariants and solvability of an overdetermined PDEs, namely the Killing
equations. Recently we have witnessed considerable developments in this direction [2, 3]. Notably, the complete set of local obstructions to the existence of KVs in three dimensions has been identified in a coordinate invariant fashion both for Riemannian [4] and for Lorentzian [5] signatures, which augmented a revival interest for Darboux’s methodology. The obstructions can be applied to count the exact number of KVs, resulting in severe restrictions on the curvature of $M$. This method has overcome some shortcomings inherent in other prescriptions proposed in the literature [6, 7] in that classification is indeed tractable.

Conversely, one might ask whether one can determine the local form of $g$ by prescribing the obstructions and/or the curvature of $M$. Indeed, the existence of KVs has traditionally been used to classify the manifolds, see e.g. [8] for a review. A three-dimensional (pseudo-)Riemannian manifold is said to be maximally symmetric if it has six KVs, or equivalently, if it has constant sectional curvature. Ricci [9] and Bianchi [10] have proved that the next maximum number of KVs is four, and have carried out the complete classification of all possible local forms of $g$ with four KVs. The pursuit for the Lorentzian counterpart has been implemented by Kručíkovič [11] and recently refined in [5] by making use of obstruction criteria for KVs (for the underlying geometric analysis of these geometries, see [12]). This is another intriguing opportunity offered by recent developments in [4, 5].

The difficulty for the complete classification of local metrics tends to increase as the total number of KVs becomes fewer. A primary study in this line was due to Bianchi [10], who elaborated on the three-dimensional Riemannian manifolds with an isometry group of transitive action. The geometry of their metrics has been intensively studied in the literature [13] since it has a prominent application to homogeneous cosmological models [14]. Consequently, local forms of three-dimensional Riemannian metrics with 3 KVs have been known. However, the Lorentzian counterpart is still lacking in spite of the intensive results on left-invariant Lorentzian metrics on three-dimensional Lie groups [15–17]. Our aim in this paper is to present a complete catalogue of local Lorentzian metrics that admit three KVs$^4$. We would like to emphasize that not all the Lorentzian metrics with three KVs are recovered by a mere Wick-rotation of Riemannian metrics with three KVs. A principal reason is that the Lorentzian metrics may admit a null KV, which cannot be captured by complexification of the coordinates.

Bona and Coll have obtained the necessary and sufficient conditions for the three-dimensional Lorentzian metrics to admit isometry group according to its isotropy subgroup, in terms of eigenvalues and eigenvectors of the Ricci tensor [19]. Nevertheless, the local form of the metrics has not been explored. To fill this gap is one of the main result in the present paper. It is also noteworthy to remark that our technique allows the complete classification without appealing to how the isometry group acts on the manifold.

Our strategy for the classification of metrics admitting KVs is based on the obstructions for the existence of KVs, which amount to the algebraic constraints obeyed by curvature tensors and connections. Conceptually, this blueprint is closely parallel with the classification of supersymmetric solutions in supergravity [20, 21]. The existence of Killing spinors puts constraints on the intrinsic torsion of $G$ structures, which tightly restricts the possible form of metrics and fluxes. In our case, the role of intrinsic torsion is played by the two obstruction matrices described below.

Before stating our result, let us first evoke the definition and some key properties of KVs. Let $(M, g)$ denote the three-dimensional Lorentzian manifold. A vector field $K^\alpha$ on $(M, g)$ is a KV iff it satisfies the Killing equation

$^4$The classification device employed by Petrov [18] based on the canonical real structures has been successfully able to construct canonical form of metrics admitting KVs in four dimensional (pseudo-)Riemannian manifold. However, this method seems to call for excessive and brute-force computations.
where $\mathcal{L}_K$ is the Lie derivative along $K^a$, $\nabla_a$ denotes the Levi-Civita connection and the round brackets denote symmetrization over the enclosed indices. We raise and lower the indices by $g_{ab}$ and its inverse $g^{ab}$. The compatibility of (1.1) leads to the defining equation of a curvature collineation [22]

$$\mathcal{L}_K R_{abc}^d = 0,$$

which gives rise to the following zero eigenvalue problem

$$(R^{1st})_a K^a = 0, \quad (R^{1st})_a \equiv \begin{pmatrix} \nabla_a R \\ \nabla_a S^{(2)} \\ \nabla_a S^{(3)} \end{pmatrix},$$

where $R_{abc}^d$ is the Riemann–Christoffel tensor defined by $[\nabla_a, \nabla_b]V_c = R_{abc}^d V_d$ for any covector $V_a$. A triple of curvature invariants $\{R,S^{(2)},S^{(3)}\}$ are constructed as $R \equiv R^{a}_{\ a}$, $S^{(2)} \equiv S^a_{\ b} S^b_{\ a}$ and $S^{(3)} \equiv S^a_{\ b} S^b_{\ c} S^c_{\ a}$ with the Ricci tensor $R_{ab} \equiv R_{abc}^c$ and its traceless part $S_{ab} \equiv R_{ab} - (1/3)R g_{ab}$. We call $R^{1st}$ the first obstruction matrix, since its minors put restrictions on the dimension of the solution space of (1.1). A general solution of (1.3) can be written as

$$K^a = \sum_{\alpha=1}^d \omega_\alpha e^a_{\alpha},$$

where $d \equiv \dim \ker R^{1st} \leq 3$, $\{e^a_{\alpha}\}$ are linearly independent vectors that span $\ker R^{1st}$ and $\{\omega_\alpha\}$ are arbitrary functions on $M$. Substituting equation (1.4) into (1.1), we end up with a PDE system of the form

$$\nabla_a \omega = \Omega^{\ a} \omega, \quad \omega \equiv \begin{pmatrix} \omega_\alpha \\ \omega_{\alpha\beta} \end{pmatrix},$$

where $\{\omega_{\alpha\beta} \equiv \mathcal{L}_{[\alpha} \omega_{\beta]}\}$ are the 1-jet variables and the abbreviation $\mathcal{L}_a$ denotes the Lie derivative along $e^a_{\alpha}$. The square brackets over indices is used for skew-symmetrization. The connection $\Omega_a$ is expressed in terms of the Ricci rotation coefficients, curvatures and their derivatives. Since the system (1.5) is first-order, its compatibility yields a set of algebraic equations

$$(R^{2nd})_a \omega = 0, \quad (R^{2nd})_a \equiv \nabla_a \Omega_b - \Omega^b_{\ [a} \Omega_{b]}.$$

We call $R^{2nd}$ the second obstruction matrix, whose entries are also obstructions to local existence of KVs. Note that the matrix depends upon maximal possible dimension of an orbit space under an isometric group action, and of its isotropy subgroup. At the outset, they are given by $d \equiv \dim \ker R^{1st}$ and $d(d-1)/2$, respectively. By deriving a complete list of the second obstruction matrices, all the conditions to ensure that the manifold $M$ attains three KVs have already been derived in [5]. Turning the logic around, these obstructions can be used to obtain the explicit local metrics with three KVs. Our principal result of this paper can be boiled down as follows:

Theorem 1. Let $(M, g)$ be a three-dimensional Lorentzian manifold. If it admits exactly three linearly independent Killing vectors, the manifold is locally isometric to any one of the metrics described in table 1.
An inspection of the explicit second obstruction matrices in [5] shows that the three KVs appear for ‘class 2’ and ‘class 3’ in the terminology therein. We shall moniker class 2 as ‘inhomogeneous’ and class 3 as ‘homogeneous’ in this paper, since this epithet represents how the isometry group acts on the manifold. The classification of class 3 in [5] relies on the property of Segre classification for the traceless Ricci tensor $S_{ab}^i$. Note also that our classification of metrics with three KVs does not represent the existence of the global isometry group, which may be broken by the discrete identification of spacetime points. We refer the reader to e.g. [24] for the discussion of related aspects.

We note that in the ‘homogeneous’ case the Lie algebra and the local metric do not have a one-to-one correspondence. This is in sharp contrast to the Riemmanian case. For instance, metrics (4.9), (7.28) and (8.12) admit $so(1, 2)$ algebra. The metric (4.9) deserves a Lorentzian counterpart of Bianchi IX metric, while the latter two metrics are not. This illustrates the richness of Lorentzian signature. Therefore, we avoided to refer to the Bianchi types throughout the text to circumvent the confusion. In addition, each Bianchi class counterpart is not realized as the three dimensional metrics with three KVs, since we are focusing on the three dimensional intrinsic metric, rather than the three dimensional subspace.

Our result obtained in this paper may be generalized into higher dimensions, if we properly find out the obstruction matrices. Research in this direction is an intriguing future study. On top of this, some of the metrics (8.12) and (9.13) derived in the present paper appear to be new, as far as we know. It seems interesting to further explore physical aspects of these spacetimes.

The remainder of this paper is devoted to the proof of this theorem and is organized as follows. Section 2 classifies all the inhomogeneous metrics with three KVs, which turn out to be all conformally flat. The following seven sections 3–9 aim to the classification of homogeneous metrics for each Segre type. Useful formulae for curvature tensors are summarized in appendix.

### 2. Inhomogeneous metrics

This section identifies all the 3D Lorentzian metrics endowed with an isometry group of dimension 3 with 1-dimensional isotropy subgroup, acting on 2D orbits. Hereon, any KV can be written as

$$K^a = \omega_2 e_2^a + \omega_3 e_3^a, \quad \text{(2.1)}$$

where $\{e_2, e_3\}$ are two annihilators of $R^{1st}$. Depending on the causal character of annihilators, we divide the following analysis into three cases: section 2.1 treats the case in which the annihilators are both spacelike, section 2.2 deals with the case one is spacelike while the other is timelike, and lastly section 2.3 handles the case one of them is null. As it turns out, all metrics falling into this category are conformally flat with a conformal factor depending only on a single variable. The three subclasses are characterized by the dependence on the timelike/spacelike/null coordinate. For the definition and useful formulae for these quantities, we refer the reader to appendix.

#### 2.1. Both annihilators are spacelike

We suppose in this subsection that an orthonormal frame $\{e_i\} (i = 1, 2, 3)$ is assigned to each point of $M$ as

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\(^{5}\) For a classification of $2 + 1$ dimensional asymptotically AdS spacetimes based on Segre types, see [23].
Table 1. Normal forms of homogeneous metrics with three linearly independent KVs. The metrics in referred equations in this table are \(-(e_1)^2 + (e_2)^2 + (e_3)^2\). \(d\Sigma^2\) describes the two-dimensional maximally symmetric space with a constant curvature \(k = 0, \pm 1\), and \(d\Sigma^2\) corresponds to its Lorentzian counterpart. The quantity \(\sigma\) is constant in this table and is defined by (6.3). \(\lambda, C_0, C_1, C_2\) are also constants. For the Ricci rotation coefficients \(\{\kappa_i, \eta_j, \tau_r\}\) (\(i = 1, 2, 3\) or \(u, v, \tau\)) which are all constants in this table, we refer the reader to (A.3a) and (A.13a).

| Class          | Segre type | Branch | Canonical metric |
|----------------|------------|--------|------------------|
| Inhomogeneous  | [1, (11)]  | —      | \(\tau_2 \neq 0\) |
|                | [(1, 1)1]  | —      | \(\tau_2 = 0\) |
|                | [(21)]     | —      | \(\tau_2 - \kappa^2 < 0\) |
| Homogeneous     | [1, (11)]  | \(\tau_2 + \tau_3 = 0\) | \(\tau_2 = 0\) |
|                | [(1, 1)1]  | \(\tau_2 + \tau_3 \neq 0\) | \(\tau_2^2 - \kappa^2 < 0\) |
|                | [(1, 1)1]  | \(\tau_2 + \tau_3 \neq 0\) | \(\tau_2^2 - \kappa^2 > 0\) |
|                | [(1, 1)1]  | \(\tau_2 + \tau_3 \neq 0\) | \(\tau_2 \tau_3 > 0\) |
|                | [1, 11]    | \(\eta_1 = \eta_2 = 0\) | \(\eta_1^2 \neq (\tau_1 - \tau_3)^2\) |
|                | [2, 1]     | \(\eta_1 = \eta_2 = 0\) | \(\eta_1 = 0\) |
|                | [2, 1]     | \(\eta_1 = \eta_2 = 0\) | \(\eta_1^2 \neq (\tau_1 - \tau_3)^2\) |
|                | [2, 1]     | \(\eta_1 = \eta_2 = 0\) | \(\eta_1^2 = (\tau_1 - \tau_3)^2\) |

| Class          | Segre type | Branch | Canonical metric |
|----------------|------------|--------|------------------|
| [2, 1]         | \(\sigma \neq 0\) | \(\tau_2 + \tau_3 \neq 0\) | \(2\eta_1 \eta_2^2\) |
| [2, 1]         | \(\sigma = 2\tau_2\) | \(\tau_2 + \tau_3 \neq 0\) | \(2\tau_2^2\) |
| [3]            | \(\sigma = 2\tau_2\) | \(\sigma_0 \equiv \sigma - 2\tau_2 \neq 0\) | \(2\tau_2\) |
| [zz1]          | \(\sigma = 2\tau_2\) | \(\sigma_0 \equiv \sigma - 2\tau_2 \neq 0\) | \(2\tau_2\) |

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\[ g^{ab} = -e_1^a e_1^b + e_2^a e_2^b + e_3^a e_3^b. \]  
\(2.2\)

By defining the Ricci rotation coefficients \(\{\kappa_i, \eta_i, \tau_i\}\) as (A.3a), the criterion for \((M, g)\) to have three linearly independent KV reads (this can be read off from (3.6), (3.9) and (3.12) in [5])

\[ 0 = \kappa_2 - \kappa_3, \quad (2.3a) \]

\[ 0 = \kappa_1 = \eta_1 = \tau_2 = \tau_3, \quad (2.3b) \]

\[ 0 = \mathcal{L}_2 \kappa_2 = \mathcal{L}_3 \kappa_2, \quad (2.3c) \]

\[ 0 = \mathcal{L}_2 R_{22} = \mathcal{L}_3 R_{22}, \quad (2.3d) \]

where \(\mathcal{L}_i\) is the Lie derivative along \(e^{ia}\) and \(R_{ij} \equiv R^{a}{}_{beiaejb}(i,j = 1, 2, 3)\). It immediately follows from (2.3a), (2.3b) and (A.4a) that

\[ \nabla_{[a} e_{b]} = 0, \quad W_{[a} \nabla_{b} W_{c]} = 0, \quad (2.4) \]

where \(W_a \equiv e_{2a} + i e_{3a}\) is a complex vector. Then, there exists a local coordinate chart \((t, x, y)\) such that

\[ e_1^a = -\nabla_a t, \quad W_a = e^{\theta_1(t, x, y) + \theta_2(t, x, y)} (\nabla_a x + i \nabla_a y), \quad (2.5) \]

where \(\theta_1\) and \(\theta_2\) are real functions. It is easy to see that the following gauge transformation with a parameter \(\lambda\)

\[ e_2^a \to (\cos \lambda) e_2^a + (\sin \lambda) e_3^a, \quad e_3^a \to -(\sin \lambda) e_2^a + (\cos \lambda) e_3^a, \quad (2.6) \]

leaves (2.3) invariant\(^6\). One can exploit this gauge freedom to set \(\theta_1 = 0\) without loss of generality. Equations (2.3c) and (2.3d) then boil down to

\[ \theta_2(t, x, y) = \log a(t) + \psi(x, y), \quad (2.7) \]

with

\[ (\partial_x^2 + \partial_y^2) \psi = -k e^{2 \psi}, \quad (2.8) \]

where \(a\) and \(\psi\) are arbitrary functions and \(k\) is a constant which can be normalized to be 0 or \(\pm 1\). The equation (2.8) stands for Liouville’s equation on a surface of constant Gaussian curvature \(k\), whose metric takes the form \(d\Sigma_k^2 \equiv e^{2\psi(x, y)} (dx^2 + dy^2)\). One finds that (2.2) becomes

\[ ds^2 = -dt^2 + a^2(t) d\Sigma_k^2. \quad (2.9) \]

This metric describes the 2 + 1 dimensional Friedmann–Lemaître–Robertson–Walker (FLRW) universe. Note that the set of all isometries of \(d\Sigma_k^2\) forms the isometry group of the whole spacetime (2.9).

Now, let us consider the metric (2.9) together with a particular solution to (2.8) and its isometry algebra. A solution to Liouville’s equation (2.8) can be taken as

\[ \psi = -\log \left[ 1 + k \frac{4}{4} (c^2 + y^2) \right]. \quad (2.10) \]

\(^6\)We illustrate in appendix A.1 how the Ricci rotation coefficients (A.3a) are transformed under the transformation (2.6).
Performing the change of variables \((x, y) \rightarrow (\theta, \phi)\), given by the relation \(x + iy = \frac{1}{\sqrt{k}}e^{i\phi} \tan \left( \frac{\sqrt{k}}{2} r \right)\), the metric reads \(d\Sigma^2 = d\theta^2 + \left[ \frac{1}{\sqrt{k}} \sin(\sqrt{k}r) \right]^2 d\phi^2\). Then, the three linearly independent KVs can be found in the form
\[
K_1 = \sin \phi \partial_x + \sqrt{k} \cos \phi \cot(\sqrt{k}r) \partial_y, \quad K_2 = \cos \phi \partial_x - \sqrt{k} \cot(\sqrt{k}r) \sin \phi \partial_y, \quad K_3 = \partial_y. \tag{2.11}
\]
with the commutation relations
\[
[K_1, K_2] = kK_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = K_2. \tag{2.12}
\]
This algebra corresponds to \(so(3)\) for \(k = 1\), \(\epsilon^2\) for \(k = 0\) and \(so(2, 1)\) for \(k = -1\).

### 2.2. One annihilator is timelike

This subsection is essentially a duplication of section 2.1. As in the previous subsection, we assume that \(\{e_i\} (i = 1, 2, 3)\) forms an orthonormal frame
\[
g^{ab} = e^a_1 e^b_1 + e^a_2 e^b_2 - e^a_3 e^b_3. \tag{2.13}
\]
In terms of the Ricci rotation coefficients \(\{\kappa, \eta, \tau\}\) defined in (A.3a), the requirement that the manifold \((M, g)\) admits exactly three linearly independent KVs can be written as (2.3b)–(2.3d) and \(\kappa_2 + \kappa_3 = 0\). It follows that
\[
\nabla \varepsilon_{[ab]} = 0, \quad W^\pm_a \nabla_b W^\pm_c = 0, \tag{2.14}
\]
where \(W^\pm_a \equiv e^\pm_2 \pm e^3_a\) denote two real null vectors. Thereupon, a local coordinate chart \((t, x, y)\) can be chosen as
\[
e^a_1 = \nabla_t, \quad W^+_a = e^{\theta(t,x,y)} \nabla_x, \quad W^-_a = e^{\phi(t,x,y)} \nabla_y. \tag{2.15}
\]
Since there exists a gauge transformation with an arbitrary function \(\lambda\)
\[
e^a_2 \rightarrow (\cosh \lambda) e^a_2 + (\sinh \lambda) e^a_3, \quad e^a_3 \rightarrow (\sinh \lambda) e^a_2 + (\cosh \lambda) e^a_3, \tag{2.16}
\]
which enables us to choose \(\theta_1 = \theta_2 \equiv \theta\). The remaining equations yield
\[
\theta(t, x, y) = \log a(t) + \psi(x, y), \tag{2.17}
\]
with
\[
\partial_t \partial_x \psi = -\frac{k}{4} e^{2\psi}, \tag{2.18}
\]
where \(a\) and \(\psi\) are arbitrary functions. The PDE for \(\psi\) describes a Liouville surface equipped with the metric \(d\Sigma^2 = e^{2\psi} dx dy\) and corresponding Gaussian curvature \(k = 0, \pm 1\). The metric then reads
\[
ds^2 = dr^2 + a^2(t) d\Sigma^2. \tag{2.19}
\]
This is the double Wick-rotated metric of (2.9), as expected.

For further consideration, let us take a particular solution to (2.18) of the form
\[
\psi = - \log \left( 1 + \frac{k}{4} t^2 \right), \tag{2.20}
\]
and thereby the local metric of $d\Sigma^2_k$ is then represented as $d\Sigma^2_k = d\theta^2 - \frac{1}{\sqrt{k}} \sin(\sqrt{k}r^2) d\phi^2$, together with new variables $(r, \phi)$ defined by $x = \frac{2}{\sqrt{k}} \tan \left( \frac{\sqrt{k}}{2} r \right) e^\phi$ and $y = \frac{2}{\sqrt{k}} \tan \left( \frac{\sqrt{k}}{2} r \right) e^{-\phi}$.

Now, the three KVs are given by

$$K_1 = - \sinh \phi \partial_r + \sqrt{k} \cosh \phi \cot(\sqrt{k}r) \partial_\phi,$$
$$K_2 = \cosh \phi \partial_r - \sqrt{k} \sinh \phi \cot(\sqrt{k}r) \partial_\phi,$$
$$K_3 = \partial_\phi,$$

(2.21)

entailing the commutators

$$[K_1, K_2] = kK_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = - K_2.$$  

(2.22)

This defines $\mathfrak{so}(1, 2)$ for $k = \pm 1$ and Poincaré algebra $\mathfrak{e}^{1,1}$ for $k = 0$.

### 2.3. One annihilator is null

Let us consider the case in which one of the annihilators is null. One can write the KV as

$$K^a = \omega_a u^a + \omega_e e^a,$$  

(2.23)

where we have adopted a null frame $\{u, v, e\}$

$$g^{ab} = 2u^a v^b + e^a e^b,$$  

(2.24)

that satisfies

$$u^a v_a = e^a e_a = 1, \quad u^a u_a = v^a v_a = e^a e_a = 0.$$  

(2.25)

In this case, the 3 KV can arise when $u^a$ is a geodesic tangent and the one-jet variable $\varpi \equiv \mathcal{L}_u \omega_e$ is functionally independent of $\omega_u$ and $\omega_e$ [5]. In terms of the Ricci rotation coefficients $\{\kappa_i, \eta_i, \tau_i\}$ (i = u, v, e) defined in (A.13a) and frame components of the Ricci tensor (A.16a), this only occurs for (see equations (3.19), (3.22) and (3.25) in [5])

$$\eta_u = \tau_v = \kappa_e = 0,$$  

(2.26a)

and

$$R_{uv} = R_{ve} = 0, \quad \mathcal{L}_u R_{vv} + 2\tau_v R_{vy} = 0.$$  

(2.26b)

The KV in the form (2.23) allows two kinds of frame transformations (A.18a) and (A.18b). One can exploit this gauge freedom (A.18a) to obtain $\tau_u + \tau_e = 0$, while the other conditions (2.26a) and (2.26b) remain inert under (A.18a). Since $u^a$ and $e^a$ are now commutative (see, (A.15a)), we can work in the coordinate system $(x, y, z)$ such that

$$u^a = (\partial_i)^a, \quad v^a = V_1 (\partial_i)^a + V_2 (\partial_i)^a + V_3 (\partial_i)^a, \quad e^a = (\partial_i)^a,$$  

(2.27)

where $V_i$ are functions of $x, y, z$. Lowering indices, we have

$$u_a = \frac{1}{V_1} \nabla_a x, \quad v_a = \nabla_a y - \frac{V_2}{V_1} \nabla_a x, \quad e_a = \nabla_a z - \frac{V_3}{V_1} \nabla_a x.$$  

(2.28)

In this coordinate system, $\tau_i = 0$ is solved as

$$V_1 = \frac{1}{\partial_i F}, \quad V_3 = - \frac{\partial_i F}{\partial_j F}.$$  

(2.29)
where $F = F(x, y, z)$. The first two conditions in (2.26) are twice integrated to yield
\[
V_2 = -\frac{1}{2(\partial_x F)^2}[2Ff_1 + f_2 - (\partial_x F)^2 + 2\partial_x F],
\]
where $f_1 = f_1(x)$ and $f_2 = f_2(x, z)$. The last condition in (2.26) is satisfied, provided that $\partial_x^2 f_2 = 0$ holds. This gives
\[
f_2(x, z) = f_{20}(x) + f_{21}(x)z + f_{22}(x)z^2.
\]
Changing variables to $\hat{x} = h(x)$, $\hat{y} = F(x, y, z)/h'(x)$ and choosing $h''(x) = -f_1(x)h'(x)$, we obtain
\[
d\hat{x}^2 = 2d\hat{y}\hat{x} + d\hat{x}^2[f_{20}(\hat{x}) + f_{21}(\hat{x})z + f_{22}(\hat{x})z^2] + dz^2,
\]
where $f_{20}(\hat{x}) = h'(x)^{-2}f_{20}(x)$. The functions $f_{20}$ and $f_{21}$ can be made to vanish by the transformation $z = \hat{z} + h_1(\hat{x})$ and $y = \hat{y} + h_2(\hat{x}) - h_1'(\hat{x})\hat{z}$, thus giving
\[
d\hat{x}^2 = 2d\hat{y}\hat{x} + d\hat{x}^2 + \hat{z}^2f_{22}(\hat{x})d\hat{z}^2,
\]
where we have chosen $h_1, h_2$ to satisfy $f_{21} + 2f_{22}h_1 - 2h_1'' = 0$ and $f_{20} + f_{21}h_1 + f_{22}h_1^2 + h_1'' + 2h_2'' = 0$. One can verify that the Cotton tensor for this metric vanishes, meaning that the metric is conformally flat. To render the conformal flatness manifest, we perform further coordinate transformations $\hat{x} = \int e^{\hat{y}(x)}dx$, $\hat{z} = e^{\hat{y}(x)}z$, $\hat{y} = y - \frac{1}{2}\hat{z}^2f'(x)$ with $e^{\hat{y}}f_{22} + f' - f'' = 0$, which brings the metric into
\[
d\hat{x}^2 = e^{2\hat{y}(x)}(2d\hat{y}d\hat{x} + d\hat{z}^2),
\]
where $f(x)$ should meet the condition $f'' \neq f'^2$.

The three KVs read
\[
K_1 = -z\partial_y + x\partial_z, \quad K_2 = \partial_y, \quad K_3 = \partial_z,
\]
whose nonvanishing commutators are
\[
[K_1, K_3] = K_2.
\]
This defines the Heisenberg algebra. Since $K_2$ is covariantly constant null vector ($\nabla_a K_{2b} = 0$, $K_a K_a^* = 0$), the metric describes the pp-wave spacetime.

3. Homogeneous metrics: Segre type [1,(11)]

Let us next move on to the homogeneous case, for which the first obstruction matrix trivially vanishes. As discussed in our previous paper [5], one can address this case by resorting to the Jordan decomposition of the trace-free Ricci tensor $S_{ab}$, i.e. Segre type. When the metric is given, the transformation to the Jordan basis requires us to solve the eigenvalue problem, which is a prime impediment in practice. In contrast, the Jordan basis is of great help in reducing total amount of computations, as far as the classification of the metrics with KVs is concerned.

Let us begin with our discussion for the the Segre [1,(11)] type, for which the trace-free Ricci tensor takes the following form
\[
S_{ab} = -\lambda(2\epsilon_{1a}\epsilon_{1b} + \epsilon_{2a}\epsilon_{2b} + \epsilon_{3a}\epsilon_{3b}),
\]
where \( \{ e_i \} \) is the orthonormal frame (A.1) with \( \varepsilon = -1 \). Class 3 condition in [5] requires that any KV takes the form
\[
K^a = \omega_1 e_{1a} + \omega_2 e_{2a} + \omega_3 e_{3a},
\]
and the curvature components must be subjected to
\[
\{ \lambda, R \} = \text{constants}. \tag{3.3}
\]
From the Bianchi identity and the invariance condition \( L_K S_{ab} = 0 \), we have
\[
\kappa_1 = 0, \quad \eta_1 = 0, \quad \kappa_3 = -\kappa_2. \tag{3.4}
\]
For the case \( \kappa_2 = \tau_2 + \tau_3 = 0 \), we have 4 KVs which we shall not discuss. The 3 KVs may appear in the two cases below, which are distinguished according to \( \tau_2 + \tau_3 = 0 \) or not.

### 3.1 \( \tau_2 + \tau_3 = 0 \)

When \( \tau_2 + \tau_3 = 0, \quad \kappa_2 \neq 0, \tag{3.5} \)
the one-jet variable \( \bar{\omega}_3 = L_{2\omega_3} \) is linearly dependent on \( \{ \omega_i \} \) and the second obstruction matrix is given by (4.12c) in [5], which vanishes iff
\[
\{ \tau_2, \eta_2, \eta_3 \} = \text{constants}. \tag{3.6}
\]
Here \( L_1 \tau_2 = 0 \) follows from \( R_{23} = R_{32} \). Segre type [1,(11)] requires the constancy of \( R_{11} = 2(\tau_2^2 - \kappa_2^2) \), implying that \( \kappa_2 (\neq 0) \) is also a constant. Since \( S_{12} = S_{13} = S_{23} = 0 \) must be fulfilled in the Jordan basis of type [1,(11)], we obtain \( \eta_2 = \eta_3 = \tau_1 = 0 \). It follows that only the nonvanishing Ricci rotation coefficients are \( \kappa_2 \) and \( \tau_2 \), both of which are constants parameterizing the homogeneous solution. The curvature tensors now reduce to
\[
\lambda = \frac{2}{3}(\kappa_2^2 - \tau_2^2) \neq 0, \quad R = 2(\kappa_2^2 - \tau_2^2). \tag{3.7}
\]
Plugging these expressions into Einstein’s equations, these correspond to the cosmological constant and the energy density of the dust fluid.

Then, the orthonormal frame \( e_i \) obeys the following first-order relations
\[
\nabla_b e_{1a} = \kappa_2 (-e_{2b} e_{2a} + e_{3b} e_{3a}) + \tau_2 (e_{2b} e_{3a} - e_{3b} e_{2a}), \tag{3.8a}
\]
\[
\nabla_b e_{2a} = -\kappa_2 e_{2b} e_{1a} - \tau_2 e_{3b} e_{1a}, \tag{3.8b}
\]
\[
\nabla_b e_{3a} = \tau_2 e_{2b} e_{1a} + \kappa_2 e_{3b} e_{1a}. \tag{3.8c}
\]
It turns out that \( W_a = e_{2a} + \alpha e_{3a} \) satisfies
\[
\nabla_b W_a = (-\kappa_2 + \alpha \tau_2) e_{1a} W_b, \tag{3.9}
\]
where \( \alpha \) is the solution to the quadratic equation
\[
\tau_2 \alpha^2 - 2\kappa_2 \alpha + \tau_2 = 0. \tag{3.10}
\]
Depending on the nature of roots for this equation, we can obtain individual local metrics. We remark that the none of the above conditions imposed on Ricci rotation coefficients are invariant under further transformations (A.7a). Specifically, the triad frame has been already fixed completely.
3.1.1. \( \tau_2 = 0 \). Let us first consider the \( \tau_2 = 0 \) case. Since \( e_1 \) are all hypersurface-orthogonal, we can introduce local coordinates by

\[
e_{1a} = -e^{\phi_0(t,x,y)} dr, \quad e_{2a} = e^{\phi_1(t,x,y)} dx, \quad e_{3a} = e^{\phi_2(t,x,y)} dy.
\]

(3.11)

\( \kappa_1 = \eta_1 = 0 \) implies that \( \partial_t \phi_0 = \partial_t \phi_0 = 0 \), so that we can set \( \phi_0 = 0 \) by the redefinition of \( t \). Conditions \( \kappa_2 + \kappa_3 = \eta_2 = \eta_3 = 0 \) are now solved to give \( \phi_1(t,x,y) = \log a(t) + \Phi_1(x) \) and \( \phi_2(t,x,y) = \log a(t) + \Phi_2(y) \), hence we can set \( \Phi_1 = \Phi_2 = 0 \) by absorbing into the definition of \( x \) and \( y \). Lastly, \( \kappa_3 = \text{constant} \) gives \( a(t) = e^{-\kappa_3 t} \) and the metric belongs to the \((2+1)\)-dimensional Bianchi I class

\[
d s^2 = -dt^2 + e^{-2\kappa_3} dx^2 + e^{2\kappa_3} dy^2.
\]

(3.12)

The Killing vectors are given by

\[
K_1 = \partial_t, \quad K_2 = \partial_y, \quad K_3 = \frac{1}{\kappa_2} \partial_x + x \partial_x - y \partial_y.
\]

(3.13)

The nonvanishing commutation relations are

\[
[K_2, K_3] = -K_2, \quad [K_3, K_1] = -K_1.
\]

(3.14)

This corresponds to \( \mathfrak{e}^{1,1} \) algebra.

3.1.2. \( \tau_2^2 - \kappa_2^2 < 0 \) and \( \tau_2 \neq 0 \). Since the equation (3.10) admits two real roots \( \alpha_\pm = \tau_2^{-1}(\kappa_2 \pm \sqrt{\kappa_2^2 - \tau_2^2}) \), it follows that \( W_a^\pm = e_{2a} + \alpha_\pm e_{3a} \) satisfy

\[
\nabla_b W_a^\pm = \pm \sqrt{\kappa_2^2 - \tau_2^2} e_{1a} W_b^\pm, \quad g^{ab} W_a^+ W_b^- = 2.
\]

(3.15)

One can then introduce coordinates by

\[
e_{1a} = -f(\sqrt{\kappa_2^2 - \tau_2^2} \nabla_a t + \chi_1 \nabla_a x + \chi_2 \nabla_a y), \quad W_a^+ = e^{\phi_+} \nabla_a x, \quad W_a^- = e^{\phi_-} \nabla_a y,
\]

(3.16)

where \( f = f(t,x,y) \), \( \chi_{1,2} = \chi_{1,2}(t,x,y) \) and \( \phi_\pm = \phi_\pm(t,x,y) \). By \( t \rightarrow \int f^{-1} dt \), one can achieve \( f \equiv 1 \) modulo the redefinition of \( \chi_{1,2} \). From \( \kappa_2 = -\kappa_3 \) const, one obtains \( \phi_- + \phi_+ = 2\psi(x,y) \) and \( \phi_+ = t\sqrt{\kappa_2^2 - \tau_2^2} + \Psi(x,y) \). The condition \( \kappa_1 = \eta_1 = 0 \) implies \( \partial_t \chi_{1,2} = 0 \), whereas \( \eta_3 = 0 \) gives

\[
\chi_2 = \frac{\partial_t \Psi}{\sqrt{\kappa_2^2 - \tau_2^2}}, \quad \chi_1 = \frac{\partial_t \Psi - 2\partial_x \psi}{\sqrt{\kappa_2^2 - \tau_2^2}}.
\]

(3.17)

The constancy of \( \tau_2 \) yields the Lorentzian Liouville equation (2.18) for \( \psi \) with \( k = -2\tau_2 \). Defining \( x = \frac{\sqrt{2}}{\tau_2} e^{\psi} \tanh(r/2) \) and \( y = \frac{\sqrt{2}}{\tau_2} e^{-\psi} \tanh(r/2) \), the orthonormal frame simplifies to

\[
e_1 = -(dt - t_0 \cosh \rho \phi), \quad e_3 = t_0 \sqrt{2} \left[ \sinh(t/t_0) dr + \cosh(t/t_0) \sinh \rho \phi \right],
\]

\[
e_2 = \frac{t_0 \cosh \beta [\cosh(t/t_0)(dr - \cosh \beta \sinh \rho \phi) - \sinh(t/t_0)(\cosh \beta dr - \sinh \rho \phi)]}{2}.
\]

(3.18)

where we have shifted the time coordinates by \( t \rightarrow t - t_0 [\psi + \phi - 2 \log(\cosh(r/2))] \) and defined new constants \((t_0, \beta)\) by \( \kappa_2 = t_0^{-1} \cosh \beta, \tau_2 = t_0^{-1} \sinh \beta \). The 3 KVs are given by
\[ K_1 = -t_0 \cosh \phi \operatorname{csch} r \partial_t + \sinh \phi \partial_r - \cosh \phi \coth r \partial_\phi, \]  
(3.19a)
\[ K_2 = t_0 \sinh \phi \operatorname{csch} r \partial_t - \cosh \phi \partial_r + \sinh \phi \coth r \partial_\phi, \]  
(3.19b)
\[ K_3 = \partial_\phi, \]  
(3.19c)

satisfying \( \mathfrak{so}(1,2) \) algebra
\[ [K_1, K_2] = -K_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = -K_2. \]  
(3.20)

3.1.3. \( \tau_2^2 - \kappa_2^2 > 0 \) and \( \tau_2 \neq 0 \). The equation (3.10) admits two distinct roots \( \alpha = \tau_2^{-1}(\kappa_2 + i\sqrt{\tau_2^2 - \kappa_2^2}) \) and its complex conjugation, so that \( \psi_a = e^{2\alpha} + \alpha e^{-2\alpha} \) satisfies
\[ \nabla^a W_a = i \frac{\tau_2^2}{\tau_2^2 - \kappa_2^2} e_{2a} W_b, \quad W_b W^a = 2. \]  
(3.21)

One can thus introduce coordinates
\[ e_{1a} = -(\nabla a t + \chi_1 \nabla a x + \chi_2 \nabla a y), \quad W_a = e^{i\psi} (\nabla a x + i \nabla a y), \]  
(3.22)

where \( \chi_1, \chi_2, \psi, \theta \) are functions of \( t, x, y \). From \( \kappa_2 + \kappa_3 = 0 \) and \( \kappa_2 = \text{constant} \), one finds \( \psi = \psi(x, y) \) and \( \theta = t \sqrt{\tau_2^2 - \kappa_2^2} + \Psi(x, y) \). The conditions \( \kappa_1 = t_1 \) give \( \partial_\tau \chi_1 = \partial_\tau \chi_2 = 0 \). Plugging this into \( \eta_3 = 0 \), one finds
\[ \chi_1 = \frac{\partial_\tau \psi + \partial_\phi \Psi}{\sqrt{\tau_2^2 - \kappa_2^2}}, \quad \chi_2 = \frac{-\partial_\tau \psi + \partial_\phi \Psi}{\sqrt{\tau_2^2 - \kappa_2^2}}. \]  
(3.23)

\( \tau_2 = \text{const.} \) requires that \( \psi = \psi(x, y) \) obeys Liouville’s equation (2.8) with \( \kappa \equiv -2\tau_2^2 < 0 \). The solution to Liouville’s equation can be chosen to be (2.10). By the change of the coordinates \( x + iy = \frac{\tau_2}{2} \tan \phi \) with \( t \to t - (\Psi + \phi)/\sqrt{\tau_2^2 - \kappa_2^2} \), one can bring the triad frame to
\[ e_1 = -(dt - t_0 \cosh \phi \text{rd} \phi), \quad e_3 = \frac{t_0}{\sqrt{2}} \left( \sin(t/t_0)dt + \cos(t/t_0) \sinh \phi \text{rd} \phi \right), \]
\[ e_2 = \frac{t_0}{\sqrt{2}} \text{sech} \beta \left[ \cos(t/t_0) \left( dt - \sinh \beta \sin \text{rd} \phi \right) - \sin(t/t_0) \left( \sinh \beta \text{dr} + \sin \text{rd} \phi \right) \right], \]  
(3.24)

where we have redefined constants as \( \tau_2 = t_0^{-1} \cosh \beta, \kappa_2 = t_0^{-1} \sinh \beta \). The KVs are
\[ K_1 = t_0 \cos \phi \csc \beta \partial_t + \sin \phi \partial_r + \cos \phi \coth \partial_\phi, \]
\[ K_2 = t_0 \sin \phi \csc \beta \partial_t - \cos \phi \partial_r + \sin \phi \coth \partial_\phi, \]
\[ K_3 = \partial_\phi. \]  
(3.25)

These KVs form the \( \mathfrak{so}(1,2) \) algebra
\[ [K_1, K_2] = K_3, \quad [K_2, K_3] = -K_1, \quad [K_3, K_1] = -K_2. \]  
(3.26)

3.2. \( \tau_2 + \tau_3 \neq 0 \)

In this branch, the second obstruction matrix is given by (4.13c) in [5]. This vanishes provided
\[ \tau_- = \text{constant,} \quad \Pi_{\mu}^{(\mu,11)} = \Xi_{\mu}^{(\mu,11)} = 0. \]  
(3.27)
\[ \tau_\pm \equiv \tau_1 \pm \tau_2, \]  
\[ \Pi_i^{[1,(1)]} \equiv L_i L_{(\lambda \pm \kappa)} \left( \ell_2 \left( \tau_- + 3 \tau_+ \right) - 4 \eta_2 \kappa_2 \right) \frac{L_i \kappa_2}{2 \tau_+^2}, \]  
\[ \Xi_i^{[1,(1)]} \equiv L_i \eta \left( \ell_i \kappa_2 \right) \left( \ell_3 \left( \tau_- + 3 \tau_+ \right) - 4 \eta_3 \kappa_2 \right) \frac{L_i \kappa_2}{2 \tau_+^2}. \]  

Plugging \( \tau_- = \text{const.} \) into the Ricci identity (A.6a) and using the Jordan normal form (3.1), we obtain the 1st-order system for \( \tau_2 \):

\[ L_1 \tau_2 = 2 \kappa_2 \tau_1, \quad L_2 \tau_2 = 2 \eta_2 \kappa_2 + \eta_3 (\tau_2 + \tau_3) - \ell_3 \kappa_2, \quad L_3 \tau_2 = -2 \eta_3 \kappa_2 + \eta_2 (\tau_2 + \tau_3) + \ell_2 \kappa_2. \]  

The compatibility (A.5a) of these equations gives rise to

\[ L_2 \kappa_2 = -\eta_2 (\tau_2 + \tau_3), \quad L_3 \kappa_2 = \eta_3 (\tau_2 + \tau_3). \]  

Further compatibility of these equations yields

\[ \lambda = \frac{2}{3} (\kappa_3^2 + \tau_2 \tau_3) \neq 0. \]  

One can verify that \( \Pi_i^{[1,(1)]} = \Xi_i^{[1,(1)]} = 0 \) are all fulfilled.

Thanks to the type II gauge transformations (A.7b), one can always achieve \( \kappa_2 = 0 \), while equations (3.27), (3.29), (3.30) and (3.31) remain unchanged. In this gauge, we have \( \eta_2 = \eta_3 = \tau_1 = 0 \) and \( \tau_2 = \text{const.} \). Then, the orthonormal frame obeys

\[ \nabla_b e_{1a} = \tau_2 e_{3a} e_{2b} + \tau_3 e_{2a} e_{3b}, \quad \nabla_b e_{2a} = \tau_3 e_{1a} e_{3b}, \quad \nabla_b e_{3a} = \tau_2 e_{1a} e_{2b}. \]  

The local metrics fall into two further subclasses depending on \( \tau_2 \tau_3 \geq 0 \). The \( \tau_2 \tau_3 = 0 \) possibility is excluded by (3.31). Since the solution possesses two constant Ricci rotation coefficients (\( \tau_2, \tau_3 \)), the metric is homogeneous.

### 3.2.1. \( \tau_2 \tau_3 > 0 \)

In this case, two real vectors \( W_a^\pm = e_{2a} \pm \sqrt{\tau_3 / \tau_2} e_{3a} \) are hypersurface-orthogonal \( W_a^\pm \nabla_b W_c^\pm = 0 \), which allows us to find local coordinates \((t, x, y)\) such that

\[ e_{1a} = \left( \nabla_a \chi_1 + \chi_1 \nabla_a x + \chi_2 \nabla_a y \right), \quad W_a^+ = \text{e}^{\phi^+} \, \text{d}x, \quad W_a^- = \text{e}^{\phi^-} \, \text{d}y. \]  

Substituting these into \( \kappa_2 = \kappa_3 = 0 \), we have \( \phi_- (t, x, y) = -\phi_+ (t, x, y) + 2 \psi (x, y) \). Conditions \( \kappa_4 = \eta_1 = 0 \) now yield \( \partial_\chi_1 = \partial_\chi_2 = 0 \), whereas the constancy of \( \tau_2, \tau_3 \) gives

\[ \partial_\chi_1 = \partial_\chi_2 = \frac{1}{2} (\tau_3 - \tau_2) \left( \frac{\tau_3}{\tau_2} \right)^{-1/2} \text{e}^{\phi}, \quad \phi_+ (t, x, y) = \tau_2 \sqrt{\frac{\tau_3}{\tau_2}} t + \Psi (x, y). \]  

Inserting these into \( \eta_2 = \eta_3 = 0 \), we find

\[ \partial_\chi \Psi = \tau_2 \sqrt{\frac{\tau_3}{\tau_2}} \chi_1 + 2 \partial_\chi \psi, \quad \partial_\chi \Psi = \tau_2 \sqrt{\frac{\tau_3}{\tau_2}} \chi_2. \]  

The integrability \( \partial_\chi \partial_\psi - \partial_\psi \partial_\chi \Psi = 0 \) yields Lorentzian Liouville’s equation (2.18) for \( \psi \) with \( k = \tau_2 (\tau_3 - \tau_2) \), whence we can choose \( \psi \) as (2.20). To simplify the form of the metric, let us define
and shift the time coordinate as \( t \rightarrow t - t_0 [\Psi + \phi - 2 \log(\cos(\sqrt{r}t))] \). Then the triad frame reduces to

\[
e_1 = - \left( dt - t_0 \cos(\sqrt{k}r) d\phi \right),
\]

\[
e_2 = \cosh(t/t_0) dt + \frac{\sin(\sqrt{k}r)}{\sqrt{k}} \sinh(t/t_0) d\phi,
\]

\[
e_3 = \left( \frac{\tau_1}{\tau_2} \right)^{-1/2} \left( \sinh(t/t_0) dt + \frac{\sin(\sqrt{k}r)}{\sqrt{k}} \cosh(t/t_0) d\phi \right).
\]

These expressions are well-defined irrespective of the sign of \( k \). The KVs are given by

\[
K_1 = -\sqrt{k}t_0 \cosh \phi \csc(\sqrt{k}r) \partial_t + \sinh \phi \partial_r - \sqrt{k} \cosh \phi \cot(\sqrt{k}r) \partial_\phi,
\]

\[
K_2 = \sqrt{k}t_0 \sinh \phi \csc(\sqrt{k}r) \partial_t - \cosh \phi \partial_r + \sqrt{k} \sinh \phi \cot(\sqrt{k}r) \partial_\phi,
\]

\[
K_3 = \partial_\phi,
\]

satisfying

\[
[K_1, K_2] = kK_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = -K_2.
\]

### 3.2.2. \( \tau_2 \tau_3 < 0 \)

Since the complex vector \( W_a = e_{2a} + i \sqrt{-\tau_3/\tau_2} e_{3a} \) is hypersurface-orthogonal, one can introduce the coordinates as

\[
e_1 = -(dt + \chi_1 dx + \chi_2 dy), \quad W = e^{\psi + i\theta} (dx + dy).
\]

\( \psi \) and \( \chi_{1,2} \) are \( t \)-independent on account of \( \kappa_1 = \eta_1 = \kappa_2 = 0 \). The constancy of \( \tau_{2,3} \) amounts to \( \dot{\theta} = t\tau_2 \sqrt{-\tau_3/\tau_2} + \Psi(x, y) \) and \( \partial_t \chi_1 = \partial_x \chi_2 + (\tau_2 - \tau_3)(-\tau_3/\tau_2)^{-1/2} e^{2\theta} \). One finds a set of equation for \( \psi \) from \( \eta_2 = \eta_3 = 0 \), whose integrability corresponds to the Liouville equation for \( \psi \) (2.8) with \( k = \tau_2(\tau_3 - \tau_2) \). Defining

\[
k \equiv \tau_2(\tau_3 - \tau_2), \quad t_0^{-1} \equiv \tau_2 \sqrt{-\tau_3/\tau_2}, \quad x + iy = \frac{2}{\sqrt{k}} \tan \left( \frac{\sqrt{k}r}{2} \right) e^{i\phi},
\]

and changing the time coordinate by \( t \rightarrow t - t_0(\Psi + \phi) \), the orthonormal frame is tantamount to

\[
e_1 = - \left( dt - t_0 \cos(\sqrt{k}r) d\phi \right),
\]

\[
e_2 = \cosh(t/t_0) dt - \frac{\sin(\sqrt{k}r)}{\sqrt{k}} \sin(t/t_0) d\phi.
\]
The 3 KVs are given by

\[ K_1 = \sqrt{k} t_0 \cos \phi \csc(\sqrt{k} r) \partial_t + \sin \phi \partial_r + \sqrt{k} \cos \phi \cot(\sqrt{k} r) \partial_\phi, \]

\[ K_2 = -\sqrt{k} t_0 \sin \phi \csc(\sqrt{k} r) \partial_t + \cos \phi \partial_r - \sqrt{k} \sin \phi \cot(\sqrt{k} r) \partial_\phi, \]

\[ K_3 = \partial_\phi, \]

satisfying

\[ [K_1, K_2] = kK_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = K_2. \] (3.44)

4. Homogeneous metrics: Segre \([(1,1)1]\)

This section considers the metrics belonging to the Segre \([(1,1)1]\). As deduced from the results in section 2.1, this case corresponds to the double Wick rotation of the \([1,(11)]\) case. Therefore, we shall not discuss the derivation in detail, but outline the argument and show quick results.

The trace-free part of the Ricci tensor takes the form

\[ S_{ab} = -\lambda (e_1 e_1 e_1 + 2e_2 e_2 e_2 - e_3 e_3 e_3), \] (4.1)

where the metric is given by (A.1) with \( \varepsilon = -1 \). The Bianchi identity implies

\[ \kappa_2 = 0, \quad \eta_2 = 0, \quad \eta_3 = \kappa_1. \] (4.2)

4.1. \( \tau_1 = \tau_3 \)

When

\[ \tau_1 = \tau_3, \quad \kappa_1 \neq 0, \] (4.3)

the second obstruction matrix is given by (4.19c) of [5], which vanishes iff

\[ \{\tau_1, \eta_1, \kappa_3\} = \text{constants}. \] (4.4)

From the constancy of \( R_{22}, \kappa_1 \) turns out to be a nonvanishing constant. The Jordan form of \( S_{ab} \) then requests \( \kappa_3 = \eta_1 = \tau_2 = 0 \). The curvature tensors reduce to

\[ \lambda = \frac{2}{3} (\kappa_1^3 + \tau_1^3), \quad R = -2 (\kappa_1^2 + \tau_1^2). \] (4.5)

4.1.1. \( \tau_1 = 0 \). When \( \tau_1 = 0, e_1 \) and \( e_3 \) are hypersurface-orthogonal and \( e_2 \) is closed, for which the final metric reads

\[ ds^2 = -e^{2\kappa_1} dr^2 + dx^2 + e^{-2\kappa_1} dy^2. \] (4.6)

The KVs are given by

\[ K_1 = \partial_t, \quad K_2 = \partial_y, \quad K_3 = t \partial_t - \kappa_1^{-1} \partial_x - y \partial_y. \] (4.7)
with the commutators
\[ [K_2, K_3] = -K_2, \quad [K_3, K_1] = -K_1. \] (4.8)

4.1.2. \( \tau_3 \neq 0 \). In this case, the vectors \( W_\alpha^\beta = e_1 + \tau_3^{-1}(\kappa_1 \pm \sqrt{\kappa_1^2 + \tau_3^2})\eta_3 + a \) are hypersurface-orthogonal, and the final orthonormal frame boils down to

\[ e_1 = \frac{x_0}{\sqrt{2}} \csc \beta \left[ (dt - \cos \beta \sin t \phi) \cosh(x/x_0) + (\cos \beta dt - \sin t \phi) \sinh(x/x_0) \right], \]

\[ e_2 = dx + x_0 \cos t \phi, \quad e_3 = \frac{x_0}{\sqrt{2}} (-\sinh(x/x_0) dt + \cosh(x/x_0) \sin t \phi), \]

where we have defined \( \kappa_1 = x_0^{-1} \cos \beta, \tau_1 = x_0^{-1} \sin \beta \). The KVs are given by

\[ K_1 = \sinh \phi \partial_t + x_0 \cosh \phi \csc t \partial_x - \cosh \phi \cot t \partial_\phi, \]

\[ K_2 = -\cosh \phi \partial_t - x_0 \sinh \phi \csc t \partial_x + \sinh \phi \cot t \partial_\phi, \]

\[ K_3 = \partial_\phi, \]

satisfying the \( so(1,2) \) algebra

\[ [K_1, K_2] = K_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = -K_2. \] (4.11)

4.2. \( \tau_1 \neq \tau_3 \)

The second obstruction matrix in this case reads (4.20c) in [5], whose vanishing requires

\[ \tau_+ = \text{constant}, \quad \Pi^{[[1,1,1]]}_2 = \Xi^{[[1,1,1]]}_2 = 0, \] (4.12)

where \( \tau_\pm = \tau_3 \pm \tau_1 \) and

\[ \Pi^{[[1,1,1]]}_2 \equiv \mathcal{L}_{\eta_1} - \frac{\mathcal{L}_{\eta_1 \eta_3}}{\tau_-} + \left( \mathcal{L}_1 (\tau_+ + 3 \tau_-) - 4 \eta_1 \kappa_1 \right) \frac{\mathcal{L}_1 \kappa_1}{2 \tau_-}. \] (4.13)

\[ \Xi^{[[1,1,1]]}_2 \equiv \mathcal{L}_1 \kappa_3 + \frac{\mathcal{L}_1 \kappa_1 \kappa_3}{\tau_-} + \left( \mathcal{L}_3 (\tau_+ - 3 \tau_-) - 4 \kappa_1 \kappa_3 \right) \frac{\mathcal{L}_3 \kappa_1 \kappa_3}{2 \tau_-}. \] (4.14)

Inserting \( \tau_+ = \text{constant} \) into the Ricci identity, one gets

\[ \mathcal{L}_1 \tau_1 = -2 \eta_1 \kappa_1 + \kappa_3 (\tau_1 - \tau_3) - \mathcal{L}_1 \kappa_1, \quad \mathcal{L}_2 \tau_1 = -2 \kappa_1 \tau_2, \quad \mathcal{L}_3 \tau_1 = 2 \kappa_1 \kappa_3 + \eta_3 (\tau_3 - \tau_1) - \mathcal{L}_1 \kappa_3. \] (4.15)

The compatibility conditions (A.5a) for these equations yield

\[ (\tau_1 - \tau_3) [\mathcal{L}_1 \kappa_1 - 2 \kappa_1 \kappa_3 + \eta_1 (\tau_1 - \tau_3)] + 2 \kappa_1 \mathcal{L}_3 \kappa_1 = 0, \] (4.16a)

\[ 2 \kappa_1 \mathcal{L}_1 \kappa_1 + (\tau_1 - \tau_3) [\mathcal{L}_1 \kappa_1 + 2 \eta_1 \kappa_1 - \kappa_3 (\tau_1 - \tau_3)] = 0. \] (4.16b)

The following analysis proceeds according to \( 4 \kappa_1^2 = (\tau_1 - \tau_3)^2 \) or not.

4.2.1. \( 4 \kappa_1^2 \neq (\tau_1 - \tau_3)^2 \). If \( 4 \kappa_1^2 \neq (\tau_1 - \tau_3)^2 \), equations (4.16) can be solved to give \( \mathcal{L}_1 \kappa_1 = \eta_1 (\tau_1 - \tau_3) \) and \( \mathcal{L}_1 \kappa_3 = \kappa_3 (\tau_1 - \tau_3) \), whose additional compatibility condition yields \( \lambda = \frac{2}{3} (\kappa_1^2 + \tau_1 \tau_3) \). This implies that \( \Pi^{[[1,1,1]]}_2 = \Xi^{[[1,1,1]]}_2 = 0 \) are satisfied trivially.

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If $|2\kappa_1/(\tau_1 - \tau_3)| > 1$, one can use the gauge freedom (A.7c) to bring about $\tau_1 = \tau_3$. Since this criterion has been already discussed in the previous section 4.1, we do not discuss this case here. Supposed $|2\kappa_1/(\tau_1 - \tau_3)| < 1$, one can always achieve $\kappa_1 = 0$ using the gauge freedom (A.7c). This leads to $\eta_1 = \kappa_3 = \tau_2 = 0$ and constancy of $\tau_1$. The nonvanishing Ricci rotation coefficients are $(\tau_1, \tau_3)$, which are constants satisfying $\tau_1(\tau_3 - \tau_1) \neq 0$. Repeating the argument in section 3.2, the final orthonormal frame reduces to

$$e_1 = \cosh(\sqrt{\kappa}x/x_0)dt - \frac{\sqrt{\kappa}}{\sqrt{k}} \sin(\sqrt{\kappa}t) \sinh(\sqrt{\kappa}x/x_0)d\phi, \quad e_2 = dx + x_0 \cos(\sqrt{\kappa}t)d\phi,$$

$$e_3 = \sqrt{\kappa} \left(\frac{-1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}x/x_0)dt + \frac{1}{\sqrt{k}} \sin(\sqrt{\kappa}t) \cosh(\sqrt{\kappa}x/x_0)d\phi\right), \quad (4.17)$$

where we have denoted

$$k \equiv \tau_1(\tau_1 + \tau_3), \quad x_0 = \sqrt{\kappa} \left(\frac{\tau_1}{\tau_3}\right)^{-1/2} \quad \epsilon = \text{sgn}(\tau_1/\tau_3) = \pm 1. \quad (4.18)$$

The KVs are given by

$$K_1 = \sqrt{\kappa} \sinh(\sqrt{\kappa}x/x_0)dt + x_0 \sqrt{k} \cosh(\sqrt{\kappa}t) \csc(\sqrt{\kappa}t) \partial_t - \sqrt{k} \cosh(\sqrt{\kappa}t) \cot(\sqrt{\kappa}t) \partial_\phi, \quad (4.19a)$$

$$K_2 = -\cosh(\sqrt{\kappa}t) \partial_t - x_0 \sqrt{k} \csc(\sqrt{\kappa}t) \sinh(\sqrt{\kappa}t) \partial_t + \frac{\sqrt{k}}{\sqrt{\kappa}} \cot(\sqrt{\kappa}t) \sinh(\sqrt{\kappa}t) \partial_\phi, \quad (4.19b)$$

$$K_3 = \partial_\phi, \quad (4.19c)$$

satisfying

$$[K_1, K_2] = kK_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = -\epsilon K_2. \quad (4.20)$$

4.2.2. $4\kappa_1^2 = (\tau_1 - \tau_3)^2$. When $4\kappa_1^2 = (\tau_1 - \tau_3)^2$, equations in (4.16) are not independent. The gauge transformation (A.7c) enables us to employ the frame for which $\kappa_1$ is a nonvanishing constant. This leads to the constancy of $\tau_1$ and $\tau_3 = \tau_1 \pm 2\kappa_1$. From the Ricci identity in the Jordan form, we have $\tau_2 = 0$ and $\kappa_3 = \mp \eta_1$. Combining the Ricci identity with $\Pi^{(1,1)}_{11} = 0$, we get $\eta_1 = 0$. Thus, the solution is characterized by 2 parameters $(\kappa_1, \tau_1)$. The curvature tensors are reduced to

$$\lambda = \frac{2}{3}(\kappa_1 \pm \tau_1)^2, \quad R = -\frac{4}{3}(\kappa_1 \pm \tau_1)^2. \quad (4.21)$$

The metric is written as [5]

$$e_1 = \cosh(x/x_0)dt - \frac{\sin(\sqrt{\kappa}t)}{\sqrt{k}} \sinh(x/x_0)d\phi + x_0 \kappa_1 \left(\sinh(x/x_0)dt - \frac{\sin(\sqrt{\kappa}t)}{\sqrt{k}} \cosh(x/x_0)d\phi\right), \quad (4.22a)$$

$$e_2 = dx + x_0 \cos(\sqrt{\kappa}t)d\phi, \quad (4.22b)$$

$$e_3 = x_0 \left(-\tau_1 \sinh(x/x_0)dt \pm \frac{1}{2}x_0 \sqrt{k} \sin(\sqrt{\kappa}t) \cosh(x/x_0)d\phi\right). \quad (4.22c)$$
where we have denoted
\[ k \equiv 2\tau_1 (\tau_1 \pm \kappa_1), \quad \kappa_0^{-1} \equiv \kappa_1 \pm \tau_1. \] (4.23)

The KVs are given by
\[ K_1 = \sinh \phi \partial_t + \frac{2\tau_1}{\sqrt{\kappa}} \csc(\sqrt{\kappa} t) \cosh \phi \partial_s - \sqrt{\kappa} \cot(\sqrt{\kappa} t) \cosh \phi \partial_o, \] (4.24a)
\[ K_2 = -\cosh \phi \partial_t + \frac{2\tau_1}{\sqrt{\kappa}} \csc(\sqrt{\kappa} t) \sinh \phi \partial_s + \sqrt{\kappa} \cot(\sqrt{\kappa} t) \sinh \phi \partial_o, \] (4.24b)
\[ K_3 = \partial_o. \] (4.24c)

with the commutators
\[ [K_1, K_2] = kK_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = -K_2. \] (4.25)

5. Homogeneous metrics: Segre [1,11]

In this class, the trace-free Ricci tensor takes the form
\[ S_{ab} = -\lambda_1 e_{1a}e_{1b} + \lambda_2 e_{2a}e_{2b} + \lambda_3 e_{3a}e_{3b}, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0, \] (5.1)
where all eigenvalues \( \lambda_i \) are constant and take distinct values. From the above form of \( S_{ab} \), the only remaining degrees of freedom for the frame choice is the exchange of \( e_2 \leftrightarrow e_3 \) with \( \lambda_2 \leftrightarrow \lambda_3 \) corresponding to (A.7b) with \( \alpha_2 = \pi/2 \). The second obstruction matrix is given by (4.24) of [5], which vanishes if
\[ \{ \kappa_1, \kappa_2, \eta_1, \tau_1, \tau_2, \tau_3 \} = \text{constants}. \] (5.2)

The Bianchi identity puts further restrictions on Ricci rotation coefficients
\[ \kappa_3 = \frac{\lambda_1 - \lambda_2}{\lambda_3 - \lambda_1}, \quad \kappa_1 = -\frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_2} \eta_1, \quad \eta_2 = -\frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_3} \eta_1. \] (5.3)

It then follows that all the Ricci rotation coefficients are constants, leading to the homogeneity of the solution.

The condition \( R_{ab} = R_{ba} \) culminates in a zero eigenvalue equation
\[ N_{ab} X_b = 0 \] (5.4)
where the symmetric matrix \( N_{ab} = N_{ba} \) and the vector \( X_b \) are given by
\[ N_{ab} \equiv \begin{pmatrix} 2(\tau_2 - \tau_3) & -\eta_1 - \eta_2 & \kappa_1 + \kappa_3 \\ -\eta_1 - \eta_2 & 2(\tau_1 + \tau_3) & \kappa_2 - \kappa_3 \\ \kappa_1 + \kappa_3 & \kappa_2 - \kappa_3 & 2(\tau_1 - \tau_2) \end{pmatrix}, \quad X_a \equiv \begin{pmatrix} \kappa_1 + \kappa_3 \\ \eta_1 - \kappa_3 \\ \eta_2 - \eta_1 \end{pmatrix}. \] (5.5)

Let \( C^e_i = C^e_{i[j]} \) be a collection of Ricci rotation coefficients such that \([e_i, e_j] = C^e_{i[j]}e_k, \text{ see, (A.5a)} \). Since \( C^e_{i[j]} \) are all constants, one can view \( C^e_{i[j]} \) as the structure constants of the algebra and \( e_i \) as invariant dual forms. Then \( N_{ab} X_b = 0 \) is nothing but the condition that the matrices \( [C_i]^{e}_{[j]} = C^e_{i[j]} \) constitute the adjoint representation of the algebra \([C_i, C_j] = -C^e_{i[j]}C_k \). This is the Lorentzian version of the classification of Bianchi cosmology [8, 10, 14].

The standard method for the classification of Bianchi type is the diagonalization of \( N_{a[b}X^{b]} = 0 \). However, we do not attempt to follow this route by the 2 reasons. The first
rationale is that the diagonalization of $N^a_b$ is a cumbersome task since $N^a_b$ is no longer a symmetric matrix in the Lorentzian signature. Secondly, we wish to keep the Jordan form (5.1) of the traceless Ricci tensor. Further diagonalization of $N^a_b$ does not preserve this property. In lieu of diagonalization, we shall investigate all possible cases of zero eigenvalue problem (5.5) by utilizing (5.3), and check whether $S_{ab}$ is of diagonal form.

Let us first separate our inquiry into two categories (A) $\kappa_2 = 0$ or (B) $\kappa_2 \neq 0$. In the case (A), the first component of (5.5) reduces to $\eta_1 \eta_3 \lambda_1 = 0$, which falls into (i) $\eta_1 = 0$, (ii) $\eta_3 = 0$ with $\eta_1 \neq 0$, (iii) $\lambda_1 = 0$ with $\eta_1 \eta_3 \neq 0$. In the case (A-i), the second component of (5.5) requires $\eta_1 (\tau_1 + \tau_3) = 0$, which is further distinguished into (1) $\eta_1 = 0$ and (2) $\tau_1 + \tau_3 = 0$ with $\eta_1 \neq 0$. $S_{ab}$ becomes diagonal for (A-i-1), whereas for (A-i-2) $S_{13} = 0$ demands $\tau_1 = -3\lambda_2 \tau_2 / (\lambda_1 - \lambda_3)$. For (A-ii), we need $\tau_2 = \tau_1$ from (5.5) and $S_{12} = 0$ asks for $\tau_1 = 3\lambda_3 \tau_3 / (\lambda_1 - \lambda_2)$. For (A-iii), we have $\tau_1 = \tau_2 = \tau_3 = 0$ from (5.5) and $S_{13} = 0$ demands $\eta_1 \eta_3 = 0$, which is a contradiction to the assumption $\eta_1 \eta_3 \neq 0$. Hence the class (A-iii) does not admit solutions with 3 KVs.

For case (B) $\kappa_2 \neq 0$, the first component of (5.5) can be solved with respect to $\tau_2$. We split the analysis according to (i) $\eta_3 = 0$ and (ii) $\eta_3 \neq 0$. For (i) $\eta_3 = 0$, we have $\tau_1 = \tau_2$ and $\eta_1 (\tau_1 - \tau_2) = 0$ by the third component of (5.5), leading to the subclasses (1) $\eta_1 = 0$ and (2) $\tau_2 = \tau_1$ with $\eta_1 \neq 0$. For (B-i-1) $\tau_2 = -3\lambda_1 \tau_1 / (\lambda_2 - \lambda_3)$ follows from $S_{13} = 0$. For (B-i-2), we obtain $\lambda_2 = 0$ from the second component of (5.5). Then we have $S_{13} = -3\eta_1 \kappa_2 \neq 0$, inconsistent with the Jordan form. So the class (B-i-2) has no solutions. Consider next (B-ii), for which $\kappa_2 \eta_3 \neq 0$. Then, the second component of (5.5) is solved to give $\tau_2$. Supposed $\eta_1 = 0$, $S_{12} = 0$ is never satisfied, so that we have $\eta_1 \neq 0$. Then the third component of (5.5) is solved to give $\tau_1$. After some computations, one finds that the rest of the condition for the Jordan form (5.1) fails to be fulfilled. Hence, we have no solutions in the class (B-ii).

We thus obtained four classes to consider (A-i-1), (A-i-2), (A-ii) and (B-i-1). However, we have not yet made use of the freedom for the discrete frame change $e_2 \leftrightarrow e_3$. As it turns out, the (A-2) case is equivalent to (A-ii) under this frame change. We shall therefore consider separately (A-i-1), (A-ii) and (B-i-1) in the following.

5.1. Case (A-i-1): $\kappa_2 = \eta_1 = \eta_3 = 0$

The only nonvanishing Ricci rotation coefficients are $\tau_1, \tau_2, \tau_3$. We can suppose $\tau_1 \neq \tau_2$, since the $\tau_1 = \tau_2$ case is obtained by the $\eta_1 \rightarrow 0$ limit of (A-ii). In this case, $W^\pm_a = e_2 \pm \sqrt{2} (\tau_1 + \tau_3) / (\tau_1 - \tau_2) e_3$ is hypersurface-orthogonal, implying the existence of adapted coordinates. $W^\pm$ are real or complex conjugates. In either case, the derivation is similar to those described in sections 3.2.1 and 3.2.2. The final metric is written in a unified fashion as

$$e_1 = - (dt - t_0 \cos(\sqrt{k} r) d\phi), \quad e_2 = \cos(\sqrt{k} t / t_0) dr - \frac{\sqrt{2}}{\sqrt{k}} \sin(\sqrt{k} r) \sin(\sqrt{k} t / t_0) d\phi,$$

$$e_3 = \sqrt{\frac{\tau_1 + \tau_3}{\tau_1 - \tau_2}} \left( \frac{\sin(\sqrt{k} r / t_0)}{\sqrt{k}} dr \right) - \frac{1}{\sqrt{k}} \sin(\sqrt{k} r) \cos(\sqrt{k} t / t_0) d\phi),$$

where we have defined

$$k \equiv (\tau_1 - \tau_2)(\tau_2 - \tau_3), \quad t_0 \equiv - \frac{\epsilon}{\tau_1 + \tau_3} \sqrt{\frac{\tau_1 + \tau_3}{\tau_1 - \tau_2}}, \quad \epsilon = \text{sgn} \left( \frac{\tau_1 + \tau_3}{\tau_1 - \tau_2} \right).$$

(5.6, 5.7)
The KVs are given by
\[
K_1 = \sqrt{k_0} \cos(\sqrt{\tau}) \csc(\sqrt{\tau}) \partial_t + \sqrt{\tau} \sin(\sqrt{\tau}) \partial_\tau + \sqrt{k} \cos(\sqrt{\tau}) \cot(\sqrt{\tau}) \partial_\phi, \quad (5.8)
\]
\[
K_2 = -\frac{k}{\sqrt{\tau}} k_0 \sin(\sqrt{\tau}) \csc(\sqrt{\tau}) \partial_t + \cos(\sqrt{\tau}) \partial_\tau - \sqrt{k} \cosh(\sqrt{\tau}) \tan(\sqrt{\tau}) \cot(\sqrt{\tau}) \partial_\phi, \quad (5.9)
\]
\[
K_3 = \partial_\phi. \quad (5.10)
\]

The commutation relations are
\[
[K_1, K_2] = kK_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = \epsilon K_2. \quad (5.11)
\]

5.2. Case (A-ii): \(\kappa_2 = \eta_2 = 1/2 - \tau_1 = 0\) with \(\eta_1 \neq 0\)

The nonvanishing Ricci rotation coefficients are \(\tau_1 = \tau_2, \eta_1, \eta_2, \) which are constrained to be
\[
\tau_1 = \frac{2\lambda_3}{\lambda_1 - \lambda_2} \tau_3, \quad \eta_2 = -\frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} \eta_1, \quad 3\lambda_3 \left( \frac{4\tau_1^2}{(\lambda_1 - \lambda_2)^2} - \frac{\eta_1}{(\lambda_2 - \lambda_3)^2} \right) = 1. \quad (5.12)
\]

Since \([\epsilon_1, \epsilon_2]^a = (\partial_\tau)^a\) and \([\epsilon_2]^a = (\partial_\eta)^a\), Equation (5.12) can be integrated to give \(\eta_3 = \eta_1 t + (\tau_1 - \tau_3) x + \omega_1(y)[\partial_t + \partial_\tau + \omega_1(y) \partial_\eta + \omega_2(y) \partial_\phi]\), where \(\omega_1\) and \(\omega_2\) are arbitrary functions of \(y\). By \(\int dy/\omega_3 \rightarrow y\), one can set \(\omega_3 = 1\), whereas \(\omega_1\) and \(\omega_2\) can be made to vanish by coordinate transformations \(t \rightarrow t + g_1(y)\) and \(x \rightarrow x + g_2(y)\) where \(\eta_1 g_1 + (\tau_1 - \tau_3) g_2 + \omega_1 - g_1' = 0\) and \((\tau_1 + \tau_3) g_1 + \eta_2 g_2 - \omega_2 + g_2' = 0\). We thus arrive at
\[
e_1 = -[\partial_t + (\tau_1 - \tau_3) x] dy, \quad e_2 = dx + [\eta_2 x + (\tau_1 + \tau_3) y] dy, \quad e_3 = dy. \quad (5.13)
\]

The KVs are given by
\[
K_1 = \partial_\tau, \quad (5.14a)
\]
\[
K_2 = e^{\beta y}\left[ (\tau_1 - \tau_3) \cosh(\sqrt{\tau_2}y) \partial_t + \left( -\frac{1}{2} (\eta_1 + \eta_2) \cosh(\sqrt{\tau_2}y) + \sqrt{k} \sinh(\sqrt{\tau_2}y) \right) \partial_\tau \right], \quad (5.14b)
\]
\[
K_3 = e^{\beta y}\left[ \frac{(\tau_1 - \tau_3)}{\sqrt{k}} \sinh(\sqrt{\tau_2}y) \partial_t + \left( \cosh(\sqrt{\tau_2}y) - \frac{\eta_1 + \eta_2}{2\sqrt{k}} \sinh(\sqrt{\tau_2}y) \right) \partial_\tau \right], \quad (5.14c)
\]

where
\[
\beta \equiv \frac{\eta_1 - \eta_2}{2}, \quad k \equiv \frac{1}{4} \left( \eta_1 + \eta_2 \right)^2 - \left( \tau_1^2 - \tau_3^2 \right). \quad (5.15)
\]

The nonvanishing commutation relations are given by
\[
[K_1, K_2] = \beta K_2 + k K_3, \quad [K_2, K_3] = -k K_2 - \beta K_3. \quad (5.16)
\]

5.3. Case (B-1): \(\eta_3 = \tau_2 = \eta_1 = 0\) with \(\kappa_2 \neq 0\)

The nonvanishing Ricci rotation coefficients are \(\kappa_2, \kappa_3, \tau_1, \tau_2 = \tau_3\). The constraints to be imposed on these parameters are
\[
\kappa_3 = \frac{\lambda_1 - \lambda_2}{\lambda_3 - \lambda_1} \kappa_2, \quad \tau_2 = -\frac{3\lambda_1 \tau_1}{\lambda_3 - \lambda_3}, \quad 3\lambda_1 \left( \frac{\kappa_3^2}{(\lambda_1 - \lambda_2)^2} + \frac{4\tau_1^2}{(\lambda_2 - \lambda_3)^2} \right) = 1. \quad (5.17)
\]
Now \([e_2, e_3]^\mu = 0\), implying \(e_2 = \partial_x\) and \(e_3 = \partial_y\), \(e_1\) can be inferred from (A.5a) as

\[
e_1 = \omega_1(t) \partial_t + [-\kappa_2 x + (\tau_1 + \tau_2)y + \omega_2(t)] \partial_x + [(\tau_2 - \tau_1)x - \kappa_3 y + \omega_3(t)] \partial_y.
\]

Without loss of generality, one can set \(\omega_1 = 1\) and \(\omega_2 = \omega_3 = 0\), yielding

\[
e_1 = -d\tau, \quad e_2 = dx + [\kappa_2 x - (\tau_1 + \tau_2)y]d\tau, \quad e_3 = dy + [\kappa_3 y + (\tau_1 - \tau_2)x]d\tau.
\]

The KV's are given by

\[
K_1 = \partial_t, \quad K_2 = e^{\beta t} \left[ -\left(\frac{1}{2}(\kappa_3 - \kappa_2) \cosh(\sqrt{k} t) + \sqrt{k} \sinh(\sqrt{k} t)\right) \partial_x + (\tau_1 - \tau_2) \cosh(\sqrt{k} t) \partial_y \right],
\]

\[
K_3 = e^{\beta t} \left[ -\left(\cosh(\sqrt{k} t) + \frac{\kappa_3 - \kappa_2}{2\sqrt{k}} \sinh(\sqrt{k} t)\right) \partial_x + \frac{\tau_1 - \tau_2}{\sqrt{k}} \sinh(\sqrt{k} t) \partial_y \right],
\]

satisfying

\[
[K_1, K_2] = \beta K_2 + k K_3, \quad [K_3, K_1] = -K_2 - \beta K_3, \quad [K_2, K_3] = 0,
\]

where

\[
\beta \equiv -\frac{\kappa_2 + \kappa_3}{2}, \quad k \equiv \frac{1}{4}(\kappa_2 - \kappa_3)^2 - (\tau_1^2 - \tau_2^2).
\]

6. Homogeneous metrics: Segre [(21)]

This section discusses the Segre type [(21)], in which the trace-free part of the Ricci tensor takes the form

\[
S_{ab} = S_{vu} \eta_u \eta_b, \quad S_{vu} \neq 0,
\]

where we have worked in a null frame (A.11). The Bianchi identity implies

\[
\eta_u = 0, \quad \mathcal{L}_u S_{vu} + (2\kappa_u - \kappa_v)S_{vu} = 0.
\]

The following analysis is classified according to \(\sigma = 0\) or not, where

\[
\sigma \equiv \mathcal{L}_\nu \varphi + \tau_\nu + \tau_\nu.
\]

Here and in what follows, we set \(S_{vu} = e^{2\nu}\). The above form (6.1) of \(S_{ab}\) is preserved under (A.18a) and (A.18b), which will be used in the following analysis.

6.1. \(\sigma = 0, \kappa_\nu \neq 0\)

The second obstruction matrix reads (4.36c) in [5], which vanishes iff

\[
\{\tau_\nu, \Sigma, \kappa_\nu e^\varphi, \eta_\nu e^{-\varphi}, \eta_\nu e^{-2\varphi}\} = \text{constants},
\]

where

\[
\Sigma \equiv \mathcal{L}_\nu (e^{-\varphi}) + \kappa_\nu e^{-\varphi}.
\]

By virtue of \(\kappa_\nu \neq 0\), the component \(R_{ab} e^\mu \eta_b = 0\) gives rise to \(\tau_\nu = \tau_\nu\). From \(\sigma = 0\), \(\Sigma = \text{constant}\) and Bianchi identity, one can derive the first-order equations for \(\varphi\).
\[
\mathcal{L}_u \phi = \frac{1}{2} \kappa_e - \kappa_u, \quad \mathcal{L}_v \phi = \kappa_e - \Sigma \phi, \quad \mathcal{L}_v \phi = -(\tau_e + \tau_v). \tag{6.6}
\]

From the Ricci identity \(R_{ab}u^au^b = 0\) we have \(\mathcal{L}_u \kappa_e = \kappa_e (\kappa_e + \kappa_u)\). Inserting this into \(\mathcal{L}_u (\kappa_e \phi) = 0\) and using (6.6), we obtain \(\kappa_e = 0\), leading to the contradiction to the assumption \(\kappa_e \neq 0\). This means that there exist no solutions in this class, which admit precisely 3 KVs.

Let us emphasize that this does not mean that this class fails to admit KVs more than 3. As illustrated in our previous paper [5], the class 3 Segre [(21)] allows solutions with 4 KVs.

### 6.2. \(\sigma \neq 0\)

The second obstruction matrix is given by (4.37c) in [5], which vanishes if

\[
\mathcal{L}_i \left( \sigma - \frac{5}{2} \tau_i \right) = 0, \quad \mathcal{L}_i \tau_i - \sigma^{-1} \mathcal{L}_i \Sigma = 0, \quad \mathcal{L}_i(\kappa_v \phi) = 0, \quad \Phi_i^{(21)} = 0, \tag{6.7}
\]

where

\[
\Phi_i^{(21)} \equiv \mathcal{L}_i(\eta_v - 2 \eta_v \mathcal{L}_i \phi + \frac{e^\phi \mathcal{L}_i \Sigma}{\sigma} - \frac{e^\phi (\mathcal{L}_v \phi)(\mathcal{L}_v \Sigma)}{\sigma}, \quad + \left( e^\phi \mathcal{L}_v \Sigma - \mathcal{L}_v \sigma + (\mathcal{L}_v \phi - \kappa_v - \eta_v) \sigma \right) \frac{e^\phi \mathcal{L}_v \Sigma}{\sigma^2}. \tag{6.8}
\]

\[
\Theta_i^{(21)} \equiv \mathcal{L}_i(\eta_v - \eta_v \mathcal{L}_i \phi - \frac{e^\phi \mathcal{L}_i \Sigma}{\sigma} + \left( e^\phi \mathcal{L}_v \Sigma + \mathcal{L}_v \sigma + (\tau_v + \tau_v - \sigma) \sigma \right) \frac{e^\phi \mathcal{L}_v \Sigma}{\sigma^2}. \tag{6.9}
\]

From the Ricci identity \(R_{ab}u^au^b = 0\) and \(\mathcal{L}_u(\kappa_v \phi) = 0\), one must have \(\kappa_e = 0\). Since \(\sigma\) and \(\tau_v\) are now constant due to the first two conditions of (6.7), the Ricci identity \(S_{ab}u^au^b = S_{ab}e^ae^b = 0\) gives rise to \(R = -6 \tau_v^2 \Phi_e^{(21)} = 0\) is now reduced to

\[
e^\phi \phi = 2(\sigma - 2 \tau_v) [\eta_v, \eta_v] + \kappa_v (-\sigma + \tau_v) + \sigma \mathcal{L}_v \kappa_v + \kappa_v (\sigma - 2 \tau_v) \mathcal{L}_v \phi + (\tau_v \phi)^2 - \sigma \mathcal{L}_v \mathcal{L}_v \phi]. \tag{6.10}
\]

Since \(\tau_v = \frac{1}{2} \sigma\) leads to the contradiction to \(S_{vv} = e^{2\phi} \neq 0\), we assume \(\sigma \neq 2 \tau_v\) in the hereafter. Then \(\Phi_e^{(21)} = 0\) and \(\Theta_e^{(21)} = 0\) are automatically fulfilled. We have now three first-order system for \(\Sigma \equiv e^\phi \Sigma\) as

\[
\mathcal{L}_u \hat{\Sigma} = - \kappa_u \hat{\Sigma} - (\tau_u - \tau_u) \sigma, \quad \mathcal{L}_v \hat{\Sigma} = \kappa_v \hat{\Sigma} - \eta_v \sigma + \frac{\sigma}{2(\sigma - 2 \tau_v)} e^\phi \hat{\Sigma}^2 - \frac{\tau_v \hat{\Sigma}}{\sigma}, \quad \mathcal{L}_v \hat{\Sigma} = \eta_v \sigma - (\tau_v - \tau_v) \hat{\Sigma}. \tag{6.11}
\]

Since the compatibility of these equations are automatically satisfied, equation (6.10) is the unique constraint to be imposed.

Let us note that all the conditions obtained above are invariant under (A.18a). Making use of this freedom, one can choose \(\tau_v + \tau_u = 0\), permitting us to choose the triad (2.27). The constancy of \(\tau_v\) requires

\[
\partial_i V_3 = - 2 \tau_v + \frac{\partial_i V_1 + V_3 \partial_i V_1}{V_1}. \tag{6.12}
\]
Inserting this into $S_{ab} e^b = 0$, one finds a function $k_3(x)$ satisfying

$$\partial_1 V_2 = - k'_1(x)V_1 + V_3 \left( -2\tau_0 + \frac{\partial_1 V_1}{V_1} \right) + 2V_3 \partial_2 V_1 + \partial_1 V_1. \quad (6.13)$$

From (6.2) and (6.3), $\varphi$ is found to be

$$\varphi = (\sigma - 2\tau_0)z + \log(k_2(x)V_1), \quad (6.14)$$

where $k_2 = k_2(x)$ is an arbitrary function of $x$. Comparison of $S_{\varphi} = e^{2\varphi}$ in a coordinate basis with the one in (6.14), we find a function $k_3 = k_3(x, y)$ such that

$$k_3(x, y) = - \frac{\partial_1 V_3}{V_1} + \frac{2V_2 + V_1^2}{V_1^2} \partial_2 V_1 - \frac{V_3}{V_1^2} \partial_2 V_3 + \frac{V_3}{V_1^2} \partial_1 V_1 + \frac{\partial_2 V_2}{V_1^2} + \frac{V_2}{V_1^2} (-k'_1(x)V_1 - \tau_0 V_3) + \frac{2\tau_0 V_2}{V_1^2} - \frac{e^{2(\sigma - 2\tau_0)}}{2(\sigma - 2\tau_0)} k_3^2, \quad (6.15)$$

which can be rewritten into

$$\partial_1 \left[ \frac{V_3 e^{2\tau_0 + k_1}}{V_1} \right] = \partial_1 \left[ e^{2\tau_0 + k_1} \left( \frac{2V_2 - V_1^2}{2V_1^2} - \frac{k_3}{2\tau_0} - \frac{e^{2(\sigma - 2\tau_0)}}{4(\sigma - 2\tau_0)(\sigma - \tau_0)} k_3^2 \right) \right]. \quad (6.16)$$

This implies the existence of a function $F = F(x, y, z)$ such that the terms in the square bracket on the left-hand side is $\partial_1 F$ and the terms in the square bracket on the left-hand side is $\partial_2 F$, which amounts to

$$V_2 = \frac{1}{4} V_1^2 \left[ \frac{e^{2(\sigma - 2\tau_0)} k_3^2}{(\sigma - 2\tau_0)(\sigma - \tau_0)} + \frac{2k_3}{\tau_0} + 2e^{-2(2\tau_0 + k_1)} (\partial_1 F)^2 + 2e^{2\tau_0 + k_1} \partial_2 F \right], \quad (6.17a)$$

$$V_3 = e^{-2\tau_0 + k_1} V_1 \partial_1 F. \quad (6.17b)$$

From the compatibility of (6.12), (6.13) and (6.17), we have

$$V_1 = \frac{e^{2\tau_0 + k_1(x)}}{k_4(y) - \partial_y F}, \quad k_3 = k_3(x) \quad (6.18)$$

where $k_4$ is a function of $y$. $\bar{\Sigma}$ is now computed to yield

$$\bar{\Sigma} = - \frac{e^{2\tau_0 + k_1(x)}}{k_3(x, k_4(y) - \partial_y F)} \left( k_2(x) k'_1(x) + k_1(x) + \sigma k_2(x) \partial_2 F \right). \quad (6.19)$$

The obstruction (6.11) reduces to

$$(\sigma + 2\tau_0)k'_2(x)^2 + k_2(x)^2 [\sigma k'_3(x) + (\sigma + 2\tau_0)k'_1(x)^2 - \sigma k''_1(x)] - k_2(x) [(\sigma - 2\tau_0)k'_2(x) + \sigma k''_2(x)] = 0. \quad (6.20)$$

Defining $\hat{y} = \int k_4(y) dy = F$, $\hat{x} = \int e^{-k_1(x)} dx$, the metric now reads

$$ds^2 = 2e^{-2\tau_0} d\hat{y}^2 + dz^2 - \frac{e^{2(\sigma - 2\tau_0)}}{2(\sigma - 2\tau_0)(\sigma - \tau_0)} \left( \frac{\hat{k}_1(x)}{\tau_0} \right)^2 + \frac{\hat{k}_3(x)}{\tau_0}, \quad (6.21)$$

where $\hat{k}_2(x) = k_2(x) e^{k_1(x)}$ and $\hat{k}_3(x) = k_3(x) e^{2k_1(x)}$. A further coordinate change $\bar{y} = \hat{y} - \frac{e^{2\tau_0} \hat{k}_1(\hat{x})}{\tau_0}$, $\bar{z} = z + h_1(\hat{x})$, $\bar{x} = \int e^{2\tau_0 h_1(\hat{x})} d\hat{x}$ is used to simplify the metric as
\[ ds^2 = 2e^{-2\tau_2^2}d\tilde{x}d\tilde{y} + dz^2 - \frac{\tilde{k}_2(\tilde{x})^2}{2(\sigma - 2\tau_2)(\sigma - \tau_2)}e^{2(\sigma-2\tau_2)^2}d\tilde{z}^2, \] 

where \( \tilde{k}_2(\tilde{x}) = e^{-2\tau_2 h(\tilde{x})}\tilde{k}_2(\tilde{x}) \) and \( h_1 \) has been chosen to satisfy \( h''_1(\tilde{x}) = \tilde{k}_3(\tilde{x}) + \tau_1 h'_1(\tilde{x})^2 \). In this coordinate system, the last obstruction condition \( (6.20) \) becomes

\[ (\sigma + \tau_0)\tilde{k}_2'(\tilde{x})^2 - \sigma \tilde{k}_2(\tilde{x})\tilde{k}'_2(\tilde{x}) = 0. \] 

The solution to this differential equation branches into two cases, depending on \( \tau_0 = 0 \) or not.

When \( \tau_0 \neq 0, \) the solution to \( (6.23) \) is \( \tilde{k}_2(\tilde{x}) = c_0(\tau_0 \tilde{x})^{-\sigma/\tau}. \) One can perform the transformation \( \tilde{x} = -1/\sigma^2 x, \) \( \tilde{y} = y + \frac{\tilde{x}^{\sigma^2}}{\tilde{x}^2} \) and \( \tilde{z} = z - \frac{1}{\tau_0} \log(\sigma x) \) to bring the metric into the homogeneous plane wave

\[ ds^2 = 2e^{-2\tau_2^2}dxdy + dz^2 + C_0e^{2(\sigma-2\tau_2)^2}d\tilde{x}^2, \] 

where \( C_0 \) is a constant. On the other hand, the solution to \( (6.23) \) is \( \tilde{k}_2(\tilde{x}) = c_1e^{2\tilde{x}} \) for \( \tau_0 = 0. \) By \( x \rightarrow \tilde{x}, \) \( y = \frac{\tilde{x}^2}{2\sigma^2} + \frac{\tilde{x}}{\sigma^2}, \) \( z = z - \frac{\tilde{x}^2}{\sigma^2}, \) the metric can be written again into the form \( (6.24) \) with \( \tau_0 = 0. \) In either case, three KVs are

\[ K_1 = \partial_x, \quad K_2 = \partial_y, \quad K_3 = -\frac{\sigma - 2\tau_0}{\sigma}x\partial_x + y\partial_y + \sigma^{-1}\partial_z, \] 

with

\[ [K_1, K_3] = -\frac{\sigma - 2\tau_0}{\sigma}K_1, \quad [K_2, K_3] = K_2. \] 

For \( \sigma = \frac{3}{2}\tau_0, \) the above \([21]\) metric \( (6.24) \) is conformally flat.

### 7. Homogeneous metrics: Segre [21]

For the Segre [21] type, the trace-free Ricci tensor reads

\[ S_{ab} = 2\lambda u_{a} v_{b} + S_{uv} u_{a} h_{b} - 2\lambda e_{a} e_{b}, \] 

where the class 3 condition amounts to \( \lambda = \text{const} \neq 0. \) From the Bianchi identity, we have

\[ \kappa_e = 0, \quad \eta_e = -3\lambda e^{-2\varphi}(\tau_u + \tau_v), \quad \mathcal{L}_u \varphi = -\kappa_u + \frac{3}{2}e^{-2\varphi} \eta_u; \lambda, \] 

where \( S_{uv} = e^{2\varphi}. \) The second obstruction matrix is given by \((4.41a)\) in [5], which vanishes iff

\[ \{ \tau_u, \tau_v, \Sigma, \eta_e e^{-2\varphi}, \eta_u e^{-\varphi} \} = \text{constants}. \] 

In the Jordan basis \((7.1), \) the Ricci identity \((A.16a)\) requires

\[ (2\sigma + \tau_u - \tau_v)(\tau_u + \tau_v) = 0, \quad \Sigma(\tau_u + \tau_v) = 0. \] 

From the Bianchi identity, definition of \( \Sigma \) and \( \sigma \) give rise to the 1st-order system

\[ \mathcal{L}_u \varphi = -\kappa_u + \frac{3}{2}e^{-2\varphi} \eta_u; \lambda, \quad \mathcal{L}_v \varphi = \kappa_v - \Sigma e^\varphi, \quad \mathcal{L}_e \varphi = \sigma - \tau_e - \tau_v. \] 

The compatibility conditions \((A.15a)\) for these equations give

\[ 0 = (\tau_v - \tau_u)\sigma - 3\lambda[1 + e^{-\varphi} \eta_u \Sigma + 2\eta_e(\tau_u + \tau_v)e^{-2\varphi}], \]
\[ 0 = 2(\tau_u + \tau_v)e^\varphi \Sigma + \eta_e(\sigma + 3\tau_u + \tau_v), \]  
\[ 0 = -\frac{3}{2}e^{-2\varphi}\eta_e\eta_\lambda + \eta_e\sigma - (\sigma - 2\tau_v)e^\varphi \Sigma. \]

Inspecting (7.4), the following analysis falls into two cases, according to the vanishing of \( \tau_u + \tau_v \).

### 7.1. \( \tau_u + \tau_v \neq 0 \)

From (7.4), we obtain

\[ \Sigma = 0, \quad \sigma = \frac{1}{2}(\tau_v - \tau_u). \]  
(7.7)

From (7.6), we must have \( \eta_e = 0 \). Together with \( \mathcal{L}_\eta(e^{-2\varphi}) = 0 \), the rest of the Ricci identity demands

\[ \tau_u = \tau_v, \quad e^{3\varphi} = -4\eta_e \tau_v, \]  
(7.8)

leading to \( \sigma = 0 \).

Since we have extracted all the conditions to be imposed, we move on to introduce local coordinates. To this aim, let us consider the rescaled frame

\[ \hat{u}^a = e^{\varphi}u^a, \quad \hat{v}^a = e^{\varphi}v^a. \]  
(7.9)

Substituting this into (A.15a), one finds

\[ [\hat{u}, \hat{v}]^a = 0, \quad [e, \hat{u}]^a = -6\lambda\tau_v \hat{v}^a, \quad [e, \hat{v}]^a = -\frac{1}{4\tau_v} \hat{u}^a + 2\tau_v \hat{v}^a. \]  
(7.10)

The first equation implies that we can employ the coordinate system in such a way that \( \hat{u}^a \) and \( \hat{v}^a \) are coordinate vectors \( \hat{u}^a = (\partial_x)^a, \hat{v}^a = (\partial_y)^a \). The second and third equations in (7.10) are integrated to yield

\[ e^a = [6\lambda\tau_y, y - 2\tau_x + \omega_1(z)](\partial_x)^a + \left(\frac{x}{4\tau_v} + \omega_2(z)\right)(\partial_y)^a + \omega_3(z)(\partial_z)^a, \]  
(7.11)

where \( \omega_{1,2,3}(z) \) are arbitrary functions of \( z \). Since (7.9) corresponds just to the rescaling of the frame (A.18a), one can set \( \varphi = 0 \) without losing any generality. By \( z \to \int \omega_3^{-1}dz \), we can achieve \( \omega_3(z) = 1 \). A change of variables \( x \to x + g_1(z) \) and \( y \to y + g_2(z)/(4\tau_v) \) allows one to set \( \omega_{1,2} = 0 \) by choosing \( g_{1,2} \) to satisfy \( g_1 + 4\tau_v\omega_2 - g_2 = 0 \) and \( 2\tau_v g_1 - \frac{3}{2}\lambda g_2 - \omega_1 + g_1' = 0 \). Then, the triad frame simplifies to

\[ u_\mu dx^\mu = dx + 2\tau_x(x - 3\lambda y)dz, \quad v_\mu dx^\mu = dy - \frac{x}{4\tau_v}dz, \quad e_\mu dx^\mu = dz. \]  
(7.12)

We therefore arrive at

\[ ds^2 = 2[dx + 2(x - 3\lambda y)\tau_v dz] \left(dy - \frac{x}{4\tau_v} dz\right) + dz^2. \]  
(7.13)

The KVs are given by

\[ K_1 = \partial_x. \]  
(7.14a)
\[ K_2 = e^{-\tau_v} \left[ -4\tau_v \left( \tau_v \cosh(\sqrt{k}z) - \sqrt{k} \sinh(\sqrt{k}z) \right) \partial_z + \cosh(\sqrt{k}z)\partial_z \right], \]
\[ K_3 = e^{-\tau_v} \left[ 4\tau_v \left( \cosh(\sqrt{k}z) - \frac{\tau_v}{\sqrt{k}} \sinh(\sqrt{k}z) \right) \partial_z + \frac{1}{\sqrt{k}} \sinh(\sqrt{k}z)\partial_z \right], \]

with the commutators

\[ [K_1, K_2] = kK_3 - \tau_v K_2, \quad [K_1, K_3] = K_2 - \tau_v K_3. \]

(7.15)

Here we have defined \( k = \frac{1}{2} \lambda + \tau_v^2 \). The expressions in (7.14) are valid in either sign of \( k \).

7.2. \( \tau_u + \tau_v = 0 \)

In this case, we have \( \eta_u = 0 \) from the Bianchi identity (7.2). The Ricci identity in the Jordan basis (7.1) demands

\[ \eta_v = 0, \quad e^{2\varphi} = 2\eta_v(\sigma - \tau_v). \]

(7.16)

The compatibility conditions (7.6) give

\[ \lambda = \frac{2}{3} \sigma \tau_v, \quad (\sigma - 2\tau_v)\Sigma = 0. \]

(7.17)

Let us now introduce the local coordinates. We have the gauge freedom (A.18a), which preserves all the above conditions on connections and curvatures. This allows us to work in a gauge \( \tau_v + \tau_u = 0 \), whence we can employ the triad (2.27). Requiring \( \eta_v = 0 \), we have \( \partial_z(\log V_1) = \partial_z(\log V_1) \). \( \tau_u + \tau_v = 0 \) is solved as \( V_1 = (\partial_z F(x, y))^{-1} \). Inserting into the definition of \( \tau_v(= \text{const.}) \), we get \( V_1 = (-2\tau_vF + \omega_1(x))/(\partial_z F) \). From \( S_{ab\mu}e^\mu = 0 \), one can derive \( \partial_z V_2 = f_1(x, z)/(\partial_z F)^2 \). The constancy of \( \sigma \) and the second equation of (7.16) asks for

\[ f_1(x, z) = e^{2(\sigma - 2\tau_v)}f_{11}(x) \].

From \( \Sigma = \text{const.} \), condition, we obtain

\[ V_2 = \frac{1}{(\partial_z F)^2} \left[ \frac{1}{2} F\partial_z(\log f_{11}) + \tau_v(2\tau_v - \sigma)F^2 + V_2 f_{11}(x) z - \partial_z F \right. \]
\[ + \left. F \left( \sqrt{2(\sigma - \tau_v)e^{2(\sigma - 2\tau_v)}f_{11}(x) \Sigma} + (\sigma - 2\tau_v)\omega_1 \right) \right], \]

(7.18)

where \( V_2 f_{11}(x) \) is a function independent of \( y \). Comparing this expression with \( \partial_z V_2 = f_1(x, z)/(\partial_z F)^2 \), we get

\[ \partial_z V_2 f_{11}(x) = e^{2(\sigma - 2\tau_v)}f_{11}(x). \]

(7.19)

Redefining \( F(x, y) \to y \), the resulting metric is cast into

\[ ds^2 = 2dy^2 + [dz + (2\tau_v y - \omega_1(x)) dy] \]
\[- 2 \left[ \tau_v(2\tau_v - \sigma)y^2 + V_2 f_{11}(x) + y \left( \sqrt{2(\sigma - \tau_v)e^{2(\sigma - 2\tau_v)}f_{11}(x) \Sigma} + f_{11}(x) \right) \right] dy^2, \]

(7.20)

Our remaining task is to impose the second condition in (7.17) and (7.19). In view of (7.17), this will be implemented separately depending on \( \sigma = 2\tau_v \) or not.

7.2.1. \( \sigma = 2\tau_v \). Solving (7.19), we have \( V_2 f_{11}(x) = f_{11}(x)z + \omega_2(x) \). Let us consider the following coordinate transformations
\[ x \to h(x), \quad y \to \frac{y - g_1(x)}{h'(x)}, \quad z \to z + g_2(x), \]  
\tag{7.21}

and choose functions \( h \) and \( g_{1,2} \) to satisfy
\[ f_{11}(h(x))h'(x)^2 = C_1, \quad \omega_1(h(x))h'(x) = g_2'(x) - 2\tau_1 g_1(x), \quad C_1 \omega_2(x) + f_{11}(x)(-\sqrt{2C_1\tau_1}g_1(x) + C_1 g_2(x) + g_1'(x)) = 0, \]  
\tag{7.22}

where \( C_1 \) is a nonvanishing constant with \( \text{sgn}(C_1 \tau_1) \geq 0 \). This procedure amounts to set \( f_{11}(x) = C_1 \) and \( \omega_1(x) = \omega_2(x) = 0 \). The metric (7.20) is then brought into
\[ ds^2 = 2dx \left[ dy - \left( C_1 z + \sqrt{2\tau_1 C_1} \Sigma y \right) dx \right] + (dz + 2\tau_1 ydx)^2. \]  
\tag{7.23}

KV's are given by
\[ K_1 = \partial_x, \]  
\tag{7.24a}
\[ K_2 = e^{\Sigma_* x} \left[ \frac{\Sigma_*}{\sqrt{k}} \frac{\sqrt{C_1 \tau_1} \Sigma}{2} \right] \partial_x + 2\tau_1 \cosh(\sqrt{k} x) \partial_x, \]  
\tag{7.24b}
\[ K_3 = e^{\Sigma_* x} \left[ \frac{\coth(\sqrt{k} x)}{\sqrt{k}} \right] \partial_x + 2\tau_1 \sinh(\sqrt{k} x) \partial_x, \]  
\tag{7.24c}

where we have denoted
\[ \Sigma_* = \sqrt{\frac{C_1 \tau_1 \Sigma}{2}}, \quad k = \frac{1}{2} \tau_1 C_1 (-4 + \Sigma^2). \]  
\tag{7.25}

The KV's in (7.24a) are well-defined in either sign of \( k \) and obey the following commutation relations
\[ [K_1, K_2] = kK_3 + \Sigma_* K_2, \quad [K_1, K_3] = \Sigma_* K_3 + K_2. \]  
\tag{7.26}

This algebra is isomorphic to (7.15).

72.2. \( \sigma \neq 2\tau_1 \). Solving (7.19), we get \( V_{21}(x, z) = \frac{1}{2} \sigma_*^{-1} e^{2\sigma_* z} f_{11}(x) + \omega_3(x) \), where \( \sigma_* \equiv \sigma - 2\tau_1 \) is a constant. Imposing \( \Sigma = 0 \), we perform the coordinate transformations (7.21) and choose functions \( h, g_{1,2} \) to satisfy
\[ \omega_1(h(x))h'(x) = g_2'(x) - 2\tau_1 g_2(x), \quad 2\sigma_* g_2(x) + \log \left(\frac{f_{11}(h(x))h'(x)^2}{2C_2 \sigma_*}\right) = 0, \]  
\[ \sigma_* \tau_1 g_1(x)^2 + \frac{2C_2}{f_{11}(h(x))} e^{-2\sigma_* c(x)} \sigma_* \omega_3(x) + g_1'(x) = 0, \]  
\tag{7.27}

where \( C_2 \) is a nonvanishing constant. Then, the metric is transformed into
\[ ds^2 = 2dx \left[ dy + \left( -C_2 e^{2\sigma_* z} + \sigma_* \tau_1 y^2 \right) dx \right] + (dz + 2\tau_1 ydx)^2. \]  
\tag{7.28}

The KV's are given by
\[ K_1 = \partial_x, \quad K_2 = x \partial_y - y \partial_x - \sigma_*^{-1} \partial_x, \quad K_3 = \tau_x \sigma_0^2 \partial_x + \left( -2\tau_1 \tau y + \frac{1}{\sigma_*} \right) \partial_y - \frac{2\tau_1 \tau y}{\sigma_*} \partial_z. \]  
\tag{7.29}
satisfying $so(1,2)$ algebra

\[ [K_1, K_2] = K_1, \quad [K_2, K_3] = K_3, \quad [K_3, K_1] = -2\tau_1 K_2. \]  

(7.30)

When $\tau_v = 0$, the results in section 6.2 is recovered.

### 8. Homogeneous metrics: Segre [3]

In the Segre [3] case, the trace-free Ricci tensor can be cast into

\[ S_{ab} = S_{vv}u_av_b + 2u_a\epsilon_b, \]  

(8.1)

where we have exploited the rescaling freedom (A.18a) to fix $S_{vv} = 1$. The Bianchi identity puts the following restrictions

\[ \eta_a = 0, \quad \kappa_a = 2\kappa_v, \quad \mathcal{L}_a S_{vv} + 3\kappa_a S_{vv} = 2\tau_a + \tau_v - \tau_e. \]  

(8.2)

The second obstruction matrix boils down to (4.50a) in [5], which vanishes iff

\[ \{\kappa_v, \tau_e - 3\tau_v, \kappa_v S_{vv} + 2\tau_v\} = \text{constants}, \quad \Phi_i^{[3]} = \Xi_i^{[3]} = \Theta_i^{[3]} = 0. \]  

(8.3)

Here

\[ \Phi_i^{[3]} \equiv \mathcal{L}_i \kappa_v + \frac{\tau_v + \tau_e}{2} \mathcal{L}_v S_{vv}, \]  

(8.4a)

\[ \Xi_i^{[3]} \equiv \mathcal{L}_i \eta_v + \frac{1}{2} \mathcal{L}_v \mathcal{L}_i S_{vv} + \frac{1}{4} (\tau_v + \tau_e - 3\kappa_v S_{vv}) \mathcal{L}_i S_{vv}, \]  

(8.4b)

\[ \Theta_i^{[3]} \equiv \mathcal{L}_i \eta_v - \frac{1}{2} \mathcal{L}_v \mathcal{L}_i S_{vv} + \frac{1}{4} (\mathcal{L}_v S_{vv} + 2\kappa_v + 2\eta_e) \mathcal{L}_i S_{vv}. \]  

(8.4c)

From $S_{uu} = 0$, we have $\kappa_e = 0$, leading to the conclusion that $\tau_e$ and $\tau_v$ are constants as a result of (8.3). A simple computation yields

\[ \Phi_v^{[3]} = -\frac{1}{2} (\tau_e + \tau_v)(\tau_e - 3\tau_v), \quad \Theta_v^{[3]} = -1 - \frac{1}{4} (\tau_e - 3\tau_v)(2\eta_e + \mathcal{L}_e S_{vv}). \]  

(8.5)

It follows that $\Phi_v^{[3]} = \Theta_v^{[3]} = 0$ are simultaneously satisfied only for $\tau_e = -\tau_v$. $\Phi_v^{[3]} = 0$ gives rise to $2\kappa_v \tau_v = 1$. Other components of $\Xi_v^{[3]} = \Theta_v^{[3]} = 0$ are reduced to

\[ \mathcal{L}_v S_{vv} = 2(\tau_v + \tau_e), \quad \mathcal{L}_v S_{vv} = \frac{S_{vv} + 2\kappa_v^2 + 2\eta_e \tau_v}{\tau_v}, \quad \mathcal{L}_e S_{vv} = 2(\kappa_v - \eta_e). \]  

(8.6)

whose integrability is automatically assured.

Using the type II gauge transformations (A.18b), one can set $\tau_u + \tau_e = 0$, while all the above conditions are kept invariant. In this gauge, one sees $[u, e]^{\mu} = 0$, allowing us to employ the frame (2.27). $\kappa_u = 0$ implies $V_1 = V_1(x, z)$, whereas $\tau_e + \tau_v = 0$ is solved to give $V_3 = \partial_z V_{31}(x, z)$. The constancy of $\tau_v$ and $\kappa_v = (2\tau_v)^{-1}$ gives rise to $V_1 = e^{2\tau_v} H(x)$ and $V_2 = \frac{1}{2\tau_v} + V_{21}(x, z)$. Inserting these expressions into the first and the third equations in (8.6), one can conclude $S_{vv} = 4\tau_v + \frac{2}{\tau_v} - 4\tau_v V_{31}(x, z) + S_0(x) + 2\partial_z V_{31}(x, z)$. Substituting next into the second equation in (8.6), one can integrate in $z$ to derive
Lastly, we shall investigate the Segre $[z\bar{z}]$ class, for which the trace-free Ricci tensor $S^a_b$ obeys the following eigensystem

$$S^a_b \mid_{\pm} = \lambda_{\pm} j^a_{\pm} \mid_{\pm}, \quad S^a_b \mid_{\pm} = \lambda^a_j,$$

where $\lambda_{\pm}$ are complex eigenvalues corresponding to the complex eigenvectors $j^a_{\pm} = (j^a_{\bar{\pm}})^*$. $j^a$ is spacelike and $\lambda_+ + \lambda_- = \lambda = 0$. Denoting $j^a_{\pm} = \alpha^a \pm i \beta^a$ and $\lambda_{\pm} = \alpha \pm i \beta$ ($\beta \neq 0$), $S_{ab}$ then takes the form
Here \(\{e_i\}\) denotes the orthonormal frame with \(\varepsilon = -1\). From the class 3 condition, the eigenvalues \((\alpha, \beta)\) are constants.

The second obstruction matrix reads (4.59) in [5], which vanishes provided
\[
\{\kappa_3, \eta_1, \eta_2, \eta_3, \tau_2, \tau_3\} = \text{constants.} \tag{9.3}
\]
Through the Bianchi identity, the Ricci rotation coefficients are related by
\[
\eta_3 = 2\kappa_1 = \frac{3\alpha}{\beta}\kappa_3, \quad \kappa_3 + 2\kappa_2 + \frac{3\alpha}{\beta}\eta_3 = 0, \quad \tau_1 + \tau_2 = \frac{3\alpha}{\beta}(\eta_1 - \eta_2). \tag{9.4}
\]
Since \(\{\kappa_1, \kappa_2, \tau_1\}\) are also constants, we deduce the homogeneity of the solution.

Let us first suppose \(\alpha = 0\). Solving (9.4) with respect to \((\kappa_3, \eta_1, \tau_2)\) and inserting into \(2S_{11} = S_{33}\), we get \(\eta_1^2 - \eta_2^2 = 6(\kappa_1^2 + \kappa_2^2)\), allowing us to parameterize
\[
\eta_1 = \sqrt{6\kappa}\cos \phi_1, \quad \eta_2 = \sqrt{6\kappa}\sin \phi_1, \quad \kappa_1 = \kappa \cos \phi_2, \quad \kappa_2 = \kappa \sin \phi_2, \tag{9.5}
\]
where \(\kappa = \sqrt{\kappa_1^2 + \kappa_2^2}\). With this parameterization, one can derive \(\kappa = 0\) from \(S_{31} = S_{32} = 0\), which gives rise to \(S_{12} = 0\). This is a contradiction to the Segre \([\pm 1]\) condition \((\beta \neq 0)\). In the hereafter, we therefore assume \(\alpha \neq 0\).

Solving (9.4) with respect to \((\kappa_3, \eta_2, \eta_3)\) and plugging into the Ricci identity \(S_{12} = S_{21}\), one ends up with \((9\alpha^2 + \beta^2)(\tau_1^2 - \tau_2^2) = 18\alpha^2(\kappa_1^2 + \kappa_2^2)\). We can thus parametrize
\[
\tau_3 = \tau \cos \phi_1, \quad \tau_1 = \tau \sin \phi_1, \quad \kappa_1 = \kappa \cos \phi_2, \quad \kappa_2 = \kappa \sin \phi_2, \tag{9.6}
\]
where
\[
\tau \equiv \sqrt{\frac{18\alpha^2}{9\alpha^2 + \beta^2}} \kappa, \quad \kappa \equiv \sqrt{\kappa_1^2 + \kappa_2^2}. \tag{9.7}
\]
Substitution of this into \(S_{11} = -S_{22}\) leads to
\[
\kappa_1 = \frac{\beta \tau}{6\alpha}(3e^{-\phi_1} + e^{\phi_1}) + \frac{3\alpha}{\beta} \tau_3. \tag{9.8}
\]
From \(\beta(S_{11} + \alpha) - 9\alpha^2(S_{12} + \beta) = 0\), we get
\[
(\alpha^2 + \beta^2)\tau^2 + 8\alpha e^{\phi_1} \eta_1 \tau = 0. \tag{9.9}
\]
Supposed \(\tau \neq 0\), this equation gives \(\tau_3 = -e^{-\phi_1}(\alpha^2 + \beta^2)\tau/(8\alpha^2)\). Then the rest of the Ricci identity reduces to
\[
0 = 4\alpha\beta[2\beta^2 + e^{2\phi_1}(9\alpha^2 + \beta^2)] \cos \phi_2 - (27\alpha^4 + 18\alpha^2\beta^2 - \beta^4) \sin \phi_2, \tag{9.10a}
\]
\[
0 = (27\alpha^4 + 18\alpha^2\beta^2 - \beta^4) \cos \phi_2 - 4\alpha\beta[-2\beta^2 + e^{2\phi_1}(9\alpha^2 + \beta^2)] \sin \phi_2, \tag{9.10b}
\]
\[
0 = 3e^{2\phi_1}[\alpha(9\alpha^2 + 5\beta^2) \cos(2\phi_2) + \beta(\beta^2 - 3\alpha^2) \sin(2\phi_2)]
+ (9\alpha^2 + \beta^2)[3\alpha e^{4\phi_1} + \frac{9\alpha^2 + \beta^2}{\kappa^2} e^{2\phi_1} - \frac{9(3\alpha^2 - \beta^2)(\alpha^2 + \beta^2)}{16\alpha^2\beta^2}]. \tag{9.10c}
\]
One can verify that these equations are not satisfied simultaneously, leading to \( \tau = \kappa = 0 \). The Jordan form (9.2) then requires \( \alpha = \beta^2 / (6\tau^2) \) and the nonvanishing Ricci rotation coefficients are given by

\[
\eta_1 = \eta_2 = \frac{\beta}{2\tau_3}, \quad \tau_3 \neq 0.
\]

(9.11)

The adapted coordinate system is found by inspecting (A.5a), implying the existence of \((t, x, y)\) such that

\[
e_1^a = (\partial_t)^a, \quad e_2^a = (\partial_y)^a, \quad e_3^a = (-\tau_3 t + \eta_1 t + \omega_1(\eta)) (\partial_y)^a + (-\tau_3 t - \eta_1 x + \omega_2(\eta)) (\partial_x)^a + \omega_3(\eta) (\partial_y)^a,
\]

(9.12)

where \( \omega_{1-3}(y) \) are arbitrary functions of \( y \). By \( \int dy / \omega_3(y) \rightarrow y \), one can set \( \omega_3(y) = 1 \).

Apart from (5.12), this metric is obtained by \( \tau_1 \rightarrow 0, \eta_2 \rightarrow \eta_1 \) of (5.13). The 3 KVs are given by

\[
K_1 = \frac{1}{\sqrt{\eta_1^2 + \tau_3^2}} \partial_y, \quad K_\pm = e^{\pm y \sqrt{\eta_1^2 + \tau_3^2}} \left[ \tau_3 \partial_x - \left( \eta_1 \pm \sqrt{\eta_1^2 + \tau_3^2} \right) \partial_t \right],
\]

(9.14)

with the \( e^{1,1} \) algebra \((k = 0 \text{ in } (2.22)) \)

\[
[K_1, K_\pm] = \pm K_\pm.
\]

(9.15)

It has been pointed out recently that this class of stress–energy tensor which is a source of Einstein’s equations for the metric (9.13) violates the null energy condition [26]. It would be interesting to explore the quantum origin of this stress–energy tensor.

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**Appendix. Curvature and connections**

**A.1. Orthonormal frame**

The metric in the orthonormal frame is given by

\[
g_{ab} = \varepsilon e_{1a} e_{1b} + e_{2a} e_{2b} - \varepsilon e_{3a} e_{3b},
\]

(A.1)

where

\[
\varepsilon = e_1^a e_{1a} = -e_3^a e_{3a} = \pm 1, \quad e_2^a e_{2a} = 1, \quad e_1^a e_{2a} = e_1^a e_{3a} = e_2^a e_{3a} = 0.
\]

(A.2)

The Ricci rotation coefficients \( \{ \kappa_i, \eta_i, \tau_i \} \) are defined by
\[ \kappa_1 \equiv e_1^a e_2^b \nabla_a e_{1b}, \quad \eta_1 \equiv e_1^a e_3^b \nabla_a e_{1b}, \quad \tau_1 \equiv e_1^a e_3^b \nabla_a e_{2b}, \quad (A.3a) \]
\[ \kappa_2 \equiv e_2^a e_1^b \nabla_a e_{2b}, \quad \eta_2 \equiv e_2^a e_3^b \nabla_a e_{1b}, \quad \tau_2 \equiv e_2^a e_3^b \nabla_a e_{1b}, \quad (A.3b) \]
\[ \kappa_3 \equiv e_3^a e_1^b \nabla_a e_{3b}, \quad \eta_3 \equiv e_3^a e_2^b \nabla_a e_{3b}, \quad \tau_3 \equiv e_3^a e_2^b \nabla_a e_{1b}, \quad (A.3c) \]

implying the relation
\[ \nabla_b e_{1a} = e_{1b} (\varepsilon \kappa_1 e_{2a} - \eta_1 e_{3a}) + e_{2b} (-\varepsilon \kappa_2 e_{1a} + \eta_2 e_{3a}) + e_{3b} (-\varepsilon \kappa_3 e_{1a} - \eta_3 e_{2a}), \quad (A.4a) \]
\[ \nabla_b e_{2a} = e_{1b} (-(\varepsilon \kappa_1 e_{1a} - \eta_1 e_{3a}) + \varepsilon e_{3b} (\varepsilon \kappa_2 e_{1a} - \eta_2 e_{3a}) + e_{3b} (\varepsilon \kappa_3 e_{1a} - \eta_3 e_{2a}), \quad (A.4b) \]
\[ \nabla_b e_{3a} = e_{1b} (-(\varepsilon \kappa_1 e_{1a} - \eta_1 e_{3a}) + \varepsilon e_{2b} (\varepsilon \kappa_2 e_{1a} - \eta_2 e_{3a}) + e_{3b} (\varepsilon \kappa_3 e_{1a} - \eta_3 e_{2a}), \quad (A.4c) \]

These basis obeys the following commutation relations
\[ [e_1, e_2]^a = -\varepsilon \kappa_1 e_1^a + \kappa_2 e_2^a - \varepsilon (\tau_1 - \tau_2)e^a, \quad (A.5a) \]
\[ [e_2, e_3]^a = -\varepsilon (\tau_1 - \tau_2)e_1^a - \eta_2 e_2^a - \varepsilon \eta_3 e_3^a, \quad (A.5b) \]
\[ [e_3, e_1]^a = \varepsilon \eta_1 e_2^a + (\tau_1 + \tau_2)e_3^a + \kappa_3 e_3^a. \quad (A.5c) \]

The frame components \( R_{ij} = R_{ab} e^a_i e^b_j \) of the Ricci tensor are given by
\[ R_{11} = \eta_1^2 + \varepsilon \eta_1 \eta_2 - \kappa_1^2 + \varepsilon \eta_1 \kappa_1 - \varepsilon \kappa_1^2 - \kappa_2^2 + 2 \varepsilon \tau_2 \tau_1 - \varepsilon \kappa_2 \tau_2 - \varepsilon L_1 \kappa_3 - \varepsilon L_3 \eta_1 + L_1 \kappa_2 + L_2 \kappa_1, \quad (A.6a) \]
\[ R_{22} = -\eta_2^2 + \eta_1 \eta_2 + \varepsilon \eta_1 \kappa_1 - \varepsilon \kappa_1^2 - \kappa_2^2 - 2 \varepsilon \tau_1 \tau_1 - \varepsilon L_2 \kappa_2 - \varepsilon L_1 \kappa_2 + \varepsilon L_2 \kappa_3, \quad (A.6b) \]
\[ R_{33} = \eta_3^2 + \eta_1^2 - \kappa_3^2 + \varepsilon \eta_1 \kappa_1 - \varepsilon \kappa_1 \kappa_2 - 2 \varepsilon \tau_1 \tau_2 + \varepsilon L_3 \eta_1 + L_2 \tau_2 + \varepsilon L_1 \kappa_3 + L_2 \eta_3, \quad (A.6c) \]
\[ R_{12} = -\eta_3 \kappa_3 - \eta_1 \kappa_2 - \eta_1 \tau_1 - \eta_2 \tau_1 + \eta_2 \tau_2 - \varepsilon L_3 \tau_2 + \varepsilon L_3 \kappa_3, \quad (A.6d) \]
\[ R_{21} = -\eta_3 \kappa_3 - \eta_1 \kappa_2 - \eta_1 \tau_1 + \eta_2 \tau_1 - \eta_1 \tau_2 - \varepsilon L_1 \tau_2, \quad \varepsilon L_1 \eta_3 - \varepsilon L_1 \eta_3, \quad (A.6e) \]
\[ R_{23} = -\eta_3 \eta_1 - \eta_1 \kappa_1 - \kappa_3 \tau_1 + \varepsilon \kappa_2 \tau_3 + \kappa_3 \tau_1 + \varepsilon L_3 \kappa_1 + L_3 \tau_1, \quad (A.6f) \]
\[ = -\eta_3 \kappa_1 + \varepsilon \eta_2 \kappa_1 + \kappa_3 \tau_1 + \varepsilon \kappa_2 \tau_1 - \kappa_3 \tau_2 + \varepsilon \kappa_2 \tau_2 + \varepsilon L_2 \eta_1, \quad (A.6g) \]
\[ R_{31} = -\eta_3 \kappa_1 - \eta_3 \kappa_2 + \eta_3 \tau_2 - \eta_1 \kappa_1 - \eta_3 \tau_1 + \eta_3 \tau_2 + \kappa_3 \tau_1 + L_3 \kappa_2 + L_2 \tau_3, \quad (A.6h) \]
\[ = \varepsilon \eta_1 \kappa_2 - \eta_3 \kappa_2 - \varepsilon \eta_2 \tau_1 + \varepsilon \eta_1 \tau_1 + \varepsilon \eta_3 \tau_2 + \varepsilon \eta_3 \tau_1 + L_1 \eta_2 - L_2 \tau_1, \quad (A.6i) \]

Here and throughout the body of text, we denote \( L_i = L_{e_i} \) for the Lie derivative along \( e_i \).

There are three kinds of transformations for the basis corresponding to local SO(2, 1):
\[ (I) \; e_{1a} \rightarrow \cos(\sqrt{z} a) e_{1a} + \frac{1}{\sqrt{z}} \sin(\sqrt{z} a) e_{2a}, \quad e_{2a} \rightarrow -\sqrt{z} \sin(\sqrt{z} a) e_{1a} + \cos(\sqrt{z} a) e_{2a}, \quad e_{3a} \rightarrow e_{3a}, \quad (A.7a) \]
\[ (II) \; e_{1a} \rightarrow e_{1a}, \quad e_{2a} \rightarrow \cosh(\sqrt{z} a) e_{2a} - \sqrt{z} \sinh(\sqrt{z} a) e_{3a}, \quad e_{3a} \rightarrow -\frac{\sinh(\sqrt{z} a)}{\sqrt{z}} e_{2a} + \cosh(\sqrt{z} a) e_{3a}. \quad (A.7b) \]
\( (\text{III}) \) \( e_{1a} \rightarrow \cosh(a_1)e_{1a} + \sinh(a_1)e_{3a}, \quad e_{2a} \rightarrow e_{2a}, \quad e_{3a} \rightarrow \sinh(a_1)e_{1a} + \cosh(a_1)e_{2a} \),

\( (A.7c) \)

where \( a_{1,2,3} \) are arbitrary functions. The Ricci rotation coefficients transform respectively as

\[
\begin{align*}
\kappa_1 & \rightarrow \cos(\sqrt{\varepsilon}a_1)(\kappa_1 + \mathcal{L}_1a_1) + \frac{1}{\sqrt{\varepsilon}} \sin(\sqrt{\varepsilon}a_1)(-\kappa_2 + \mathcal{L}_2a_1), \\
\eta_1 & \rightarrow \cos^2(\sqrt{\varepsilon}a_1)\eta_1 + \varepsilon \sin^2(\sqrt{\varepsilon}a_1)\eta_2 + \frac{1}{\sqrt{\varepsilon}} \sin(\sqrt{\varepsilon}a_1)\cos(\sqrt{\varepsilon}a_1)(\tau_1 + \tau_2), \\
\tau_1 & \rightarrow \cos^2(\sqrt{\varepsilon}a_1)\tau_1 - \sin^2(\sqrt{\varepsilon}a_1)\tau_2 + \frac{1}{\sqrt{\varepsilon}} \sin(\sqrt{\varepsilon}a_1)\cos(\sqrt{\varepsilon}a_1)(\eta_2 - \varepsilon\eta_1), \\
\kappa_2 & \rightarrow \cos(\sqrt{\varepsilon}a_1)(\kappa_2 - \mathcal{L}_2a_1) + \sqrt{\varepsilon} \sin(\sqrt{\varepsilon}a_1)(\kappa_1 + \mathcal{L}_1a_1), \\
\eta_2 & \rightarrow \cos^2(\sqrt{\varepsilon}a_1)\eta_2 + \varepsilon \sin^2(\sqrt{\varepsilon}a_1)\eta_1 - \sqrt{\varepsilon} \sin(\sqrt{\varepsilon}a_1)\cos(\sqrt{\varepsilon}a_1)(\tau_1 + \tau_2), \\
\tau_2 & \rightarrow \cos^2(\sqrt{\varepsilon}a_1)\tau_2 - \sin^2(\sqrt{\varepsilon}a_1)\tau_1 + \frac{1}{\sqrt{\varepsilon}} \sin(\sqrt{\varepsilon}a_1)\cos(\sqrt{\varepsilon}a_1)(\eta_2 - \varepsilon\eta_1), \\
\kappa_3 & \rightarrow \cos(\sqrt{\varepsilon}a_1)\kappa_3 + \frac{1}{\sqrt{\varepsilon}} \sin(\sqrt{\varepsilon}a_1)\eta_3, \\
\eta_3 & \rightarrow \cos(\sqrt{\varepsilon}a_1)\eta_3 - \sqrt{\varepsilon} \sin(\sqrt{\varepsilon}a_1)\kappa_3, \\
\tau_3 & \rightarrow \tau_3 + \mathcal{L}_3a_1, \\
\kappa_1 & \rightarrow \cosh(\sqrt{\varepsilon}a_2)\kappa_1 - \sqrt{\varepsilon} \sinh(\sqrt{\varepsilon}a_2)\eta_1, \\
\eta_1 & \rightarrow \cosh(\sqrt{\varepsilon}a_2)\eta_1 - \frac{1}{\sqrt{\varepsilon}} \sinh(\sqrt{\varepsilon}a_2)\kappa_1, \\
\tau_1 & \rightarrow \tau_1 + \mathcal{L}_1a_2, \\
\kappa_2 & \rightarrow \cosh^2(\sqrt{\varepsilon}a_2)\kappa_2 + \varepsilon \sinh^2(\sqrt{\varepsilon}a_2)\kappa_3 + \sqrt{\varepsilon} \sinh(\sqrt{\varepsilon}a_2)\cosh(\sqrt{\varepsilon}a_2)(\tau_2 + \tau_3), \\
\eta_2 & \rightarrow \cosh(\sqrt{\varepsilon}a_2)(\eta_2 + \mathcal{L}_2a_2) + \sqrt{\varepsilon} \sinh(\sqrt{\varepsilon}a_2)(\eta_3 - \mathcal{L}_3a_2), \\
\tau_2 & \rightarrow \cosh^2(\sqrt{\varepsilon}a_2)\tau_2 + \sinh^2(\sqrt{\varepsilon}a_2)\tau_3 + \frac{1}{\sqrt{\varepsilon}} \sinh(\sqrt{\varepsilon}a_2)\cosh(\sqrt{\varepsilon}a_2)(\kappa_2 + \varepsilon\kappa_3), \\
\kappa_3 & \rightarrow \cosh^2(\sqrt{\varepsilon}a_2)\kappa_3 + \varepsilon \sinh^2(\sqrt{\varepsilon}a_2)\kappa_2 + \frac{1}{\sqrt{\varepsilon}} \sinh(\sqrt{\varepsilon}a_2)\cosh(\sqrt{\varepsilon}a_2)(\tau_2 + \tau_3), \\
\eta_3 & \rightarrow \cosh(\sqrt{\varepsilon}a_2)(\eta_3 - \mathcal{L}_3a_2) + \frac{1}{\sqrt{\varepsilon}} \sinh(\sqrt{\varepsilon}a_2)(\eta_2 + \mathcal{L}_2a_2), \\
\end{align*} \]
\[ \tau_3 \to \cosh^2(\sqrt{a_2})\tau_3 + \sinh^2(\sqrt{a_2})\tau_2 + \frac{1}{\sqrt{\varepsilon}} \sinh(\sqrt{a_2}) \cosh(\sqrt{a_2})(\kappa_2 + \varepsilon \kappa_3). \] (A.9i)

\[ \kappa_1 \to \cosh^2(a_3)\kappa_1 + \sinh^2(a_3)\eta_3 + \cosh(a_3) \sinh(a_3)(\tau_3 - \tau_1), \] (A.10a)

\[ \eta_1 \to \cosh(a_3)(\eta - \varepsilon L_1 a_3) - \sinh(a_3)(\kappa_3 + \varepsilon L_3 a_3), \] (A.10b)

\[ \tau_1 \to \cosh^2(a_3)\tau_1 - \sinh^2(a_3)\tau_3 - \sinh(a_3) \cosh(a_3)(\kappa_1 + \eta_3), \] (A.10c)

\[ \kappa_2 \to \cosh(a_3)\kappa_2 + \sinh(a_3)\eta_2, \] (A.10d)

\[ \eta_2 \to \cosh(a_3)\eta_2 + \sinh(a_3)\kappa_2, \] (A.10e)

\[ \tau_2 \to \tau_2 - \varepsilon L_2 a_3, \] (A.10f)

\[ \kappa_3 \to \cosh(a_3)(\kappa_3 + \varepsilon L_3 a_3) + \sinh(a_3)(-\eta_1 + \varepsilon L_1 a_3), \] (A.10g)

\[ \eta_3 \to \cosh^2(a_3)\eta_3 + \sinh^2(a_3)\kappa_1 + \sinh(a_3) \cosh(a_3)(\tau_3 - \tau_1), \] (A.10h)

\[ \tau_3 \to \cosh^2(a_3)\tau_3 - \sinh^2(a_3)\tau_1 + \sinh(a_3) \cosh(a_3)(\kappa_1 + \eta_3). \] (A.10i)

### A.2. Null frame

In the null frame, the metric is given by

\[ g_{ab} = 2u_a v_b + e_a e_b, \] (A.11)

where

\[ g_{ab} u^a u^b = g_{ab} v^a v^b = g_{ab} e^a e^b = 0, \quad g_{ab} u^a v^b = g_{ab} v^a e^b = 1. \] (A.12)

The 9 Ricci rotation coefficients are defined by

\[ \kappa_u \equiv \nu^a u^b \nabla_b u_a, \quad \eta_u \equiv \nu^a u^b \nabla_b v_a, \quad \tau_u \equiv \nu^a u^b \nabla_b e_a, \] (A.13a)

\[ \kappa_v \equiv \nu^a v^b \nabla_b u_a, \quad \eta_v \equiv \nu^a v^b \nabla_b v_a, \quad \tau_v \equiv \nu^a v^b \nabla_b e_a, \] (A.13b)

\[ \kappa_e \equiv \nu^a e^b \nabla_b u_a, \quad \eta_e \equiv \nu^a e^b \nabla_b v_a, \quad \tau_e \equiv \nu^a e^b \nabla_b e_a, \] (A.13c)

leading to

\[ \nabla_u u_a = \kappa_u u_a v_b + \eta_u e_a v_b - \kappa_v u_a u_b + \tau_u e_a u_b - \kappa_e u_a e_b + \tau_v u_a e_b, \] (A.14a)

\[ \nabla_v v_a = -\kappa_u v_a v_b + \tau_u e_a v_b + \kappa_v v_a u_b + \eta_v e_a u_b - \kappa_e v_a e_b - \tau_v v_a e_b, \] (A.14b)

\[ \nabla_e e_a = -\eta_u v_a v_b - \tau_u u_a v_b + \eta_v v_a u_b - \tau_v v_a u_b + \kappa_v v_a e_b + \eta_e v_a e_b. \] (A.14c)
The commutation relations are
\[ [u, v]^a = \kappa_u u^a - \kappa_u v^a + (\tau_u - \tau_v) e^a, \]  
\[ [v, e]^a = -\eta_v u^a - (\tau_v - \tau_e) v^a + \eta_e e^a, \]  
\[ [e, u]^a = (\tau_u + \tau_e) u^a + \eta_u v^a - \kappa_e e^a. \]  
\[ (A.15) \]

The Ricci tensor is projected to
\[ R_{uu} = -\kappa_u - \kappa_v - 2\eta_u \tau_v - 2\eta_v \tau_u + \mathcal{L}_e \eta_u + \mathcal{L}_a \kappa_v, \]  
\[ R_{uv} = -\eta_v - \kappa_v + 2\eta_u \tau_v - \eta_v \tau_u + \mathcal{L}_e \eta_v + \mathcal{L}_v \kappa_v, \]  
\[ R_{ee} = -2\eta_v \eta_u - 2\eta_u \kappa_v + \kappa_v \kappa_v - \tau_v^2 - \tau_u^2 + \mathcal{L}_e \tau_u + \mathcal{L}_v \tau_v + \mathcal{L}_u \eta_u + \mathcal{L}_v \kappa_v, \]  
\[ R_{ev} = -\eta_v \kappa_v + \kappa_v \kappa_v - 2\eta_u \tau_v - 2\eta_v \tau_u + \mathcal{L}_e \tau_u + \mathcal{L}_v \tau_v + \mathcal{L}_u \eta_u - \mathcal{L}_u \kappa_v - \mathcal{L}_v \kappa_v, \]  
\[ R_{ve} = -\eta_v \kappa_v - \kappa_v \kappa_v + \eta_u \tau_v - \kappa_v \tau_u + \mathcal{L}_e \tau_v - \mathcal{L}_v \tau_u, \]  
\[ R_{ee} = -2\eta_v \kappa_v - \eta_u \tau_v + \mathcal{L}_a \eta_v + \mathcal{L}_v \tau_u, \]  
\[ (A.16) \]

where
\[ R_{uu} = R_{ah} u^a v^b, \quad R_{uv} = R_{ah} u^a v^b, \quad R_{ee} = R_{ah} v^a e^b. \]  
\[ (A.17) \]

Throughout the paper, we adopt the same notation for the trace-free Ricci tensor, e.g., \( S_{uu} = S_{ah} v^a u^b \).

The three-kinds of local Lorentz transformations are expressed as
\[ (I) \quad u_a \rightarrow b_1 u_a, \quad v_a \rightarrow b_1^{-1} v_a, \quad e_a \rightarrow e_a, \]
\[ (II) \quad u_a \rightarrow u_a, \quad v_a \rightarrow v_a - \frac{1}{2} b_2^2 u_a + b_2 e_a, \quad e_a \rightarrow e_a - b_2 u_a, \]
\[ (III) \quad u_a \rightarrow u_a - \frac{1}{2} b_3^2 v_a + b_3 e_a, \quad v_a \rightarrow v_a, \quad e_a \rightarrow e_a - b_3 v_a, \]  
\[ (A.18) \]

where \( b_1, b_2, b_3 \) are arbitrary functions. Under these frame change, the transformation rules for the Ricci rotation coefficients are summarized as
\[ \kappa_a \rightarrow b_1 \kappa_a + \mathcal{L}_a b_1, \quad \eta_a \rightarrow b_1 \eta_a, \quad \tau_a \rightarrow \tau_a, \quad \kappa_v \rightarrow b_1^{-1} \kappa_v - b_1^2 \mathcal{L}_a b_1, \]
\[ \tau_v \rightarrow \tau_v, \quad \eta_v \rightarrow b_1^{-1} \eta_v, \quad \tau_e \rightarrow \tau_e + \mathcal{L}_e \log b_1, \quad \kappa_e \rightarrow b_1 \kappa_e, \quad \eta_e \rightarrow b_1^{-1} \eta_e. \]  
\[ (A.19) \]
\[\begin{align*}
\kappa_a & \rightarrow \kappa_a + b_3 \eta_a, \quad \eta_a \rightarrow \eta_a, \quad \tau_a & \rightarrow \tau_a + b_2 \kappa_a + \frac{1}{2} b_2^3 \eta_a + \mathcal{L}_a b_2, \\
\kappa_v & \rightarrow \kappa_v - b_2 (\tau_v + \tau_e) + b_2^3 \left( \kappa_v + \frac{1}{2} \kappa_u \right) + \frac{1}{2} b_2^3 \eta_v, \quad \tau_v & \rightarrow \tau_v - b_2 \kappa_v - \frac{1}{2} b_2^3 \eta_v, \\
\eta_v & \rightarrow \eta_v + \mathcal{L}_v b_2 - b_2 (\eta_v - \kappa_v + \mathcal{L}_v b_2) + \frac{1}{2} b_2^3 (2 \tau_v - \tau_u + \tau_v - \mathcal{L}_v b_2) - \frac{1}{2} b_2^3 (\kappa_u + \kappa_v) - \frac{1}{4} b_2^3 \eta_v, \\
\tau_v & \rightarrow \tau_v - b_2 (\kappa_v + \kappa_u) - b_2^3 \eta_v, \quad \kappa_v \rightarrow \kappa_v + b_2 \eta_v, \\
\eta_v & \rightarrow \eta_v + b_2 (\tau_v + \tau_u + \mathcal{L}_u b_2) - \mathcal{L}_v b_2 + b_2^3 \left( \frac{1}{2} \kappa_v + \kappa_u \right) + \frac{1}{2} b_2^3 \eta_v. 
\end{align*}\]

(A.20)

\[\begin{align*}
\kappa_v & \rightarrow \kappa_v + b_3 (\tau_u - \tau_e) + b_3^3 \left( \eta_v + \frac{1}{2} \kappa_v \right) - \frac{1}{2} b_3^3 \eta_v, \quad \tau_u & \rightarrow \tau_u + b_3 \eta_u - \frac{1}{2} b_3^3 \eta_v, \\
\eta_u & \rightarrow \eta_u + b_3 (\kappa_u + \kappa_v + \mathcal{L}_v b_3) - \mathcal{L}_u b_3 + b_3^3 (\tau_u - \tau_e + \mathcal{L}_u b_3) + \frac{1}{2} b_3^3 (\eta_v + \kappa_v) - \frac{1}{4} b_3^3 \eta_v, \\
\kappa_v & \rightarrow \kappa_v + b_3 \eta_v, \quad \eta_v \rightarrow \eta_v, \quad \tau_v \rightarrow \tau_v - b_3 (\eta_v + \kappa_v) + b_3^3 \eta_v, \\
\kappa_e & \rightarrow \kappa_e + b_3 (\tau_v - \tau_e + \mathcal{L}_e b_3) + \mathcal{L}_e b_3 + b_3^3 \left( \frac{1}{2} \eta_v + \kappa_v \right) - \frac{1}{2} b_3^3 \eta_v, \quad \eta_v \rightarrow \eta_v - b_3 \eta_v. 
\end{align*}\]

(A.21)

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