PROJECTIVE GEOMETRY FROM POISSON ALGEBRAS

FRANCESCA AICARDI

ABSTRACT. In analogy with the Poisson algebra of the quadratic forms on the symplectic plane, and the notion of duality in the projective plane introduced by Arnold in [1], where the concurrence of the triangle altitudes is deduced from the Jacobi identity, we consider the Poisson algebras of the first degree harmonics on the sphere, the pseudo-sphere and on the hyperboloid, to obtain analogous duality notions and similar results for the spherical, pseudo-spherical and hyperbolic geometry. Such algebras, including the algebra of quadratic forms, are isomorphic, as Lie algebras, either to the Lie algebra of the vectors in \( \mathbb{R}^3 \), with vector product, or to algebra \( \mathfrak{sl}_2(\mathbb{R}) \). The Tomihisa identity, introduced in [2] for the algebra of quadratic forms, holds for all these Poisson algebras and has a geometrical interpretation. The relation between the different definitions of duality in projective geometry inherited by these structures is shown.

INTRODUCTION

In [1], Arnold has shown that a hyperbolic version of the concurrence altitudes theorem for triangles holds in the projective plane, to which the space of binary quadratic forms projects. In fact, this theorem is a direct consequence of the Jacobi identity for the Poisson brackets, in the Poisson algebra of the binary quadratic forms \( ap^2 + 2bpq + cq^2 \) on the symplectic plane \( (p, q) \).

The space of the coefficients \((a, b, c)\) of the quadratic forms is also endowed in [1] with a scalar product (defining its metrics) as well as with a vector product (the Poisson bracket), so that the orthogonality of two forms is defined as the vanishing of their scalar product, and the Poisson bracket of two forms is a third form orthogonal to the two initial forms.

A geometrical notion of duality between points and lines in the projective plane is also introduced in [1], which allows to give a geometrical meaning to any expression involving Poisson brackets.

The starting point of the present work was the following consideration.

The real vector space \( \mathbb{R}^3 \) is a Lie algebra if endowed with the vector product

\[
(x, y, z) \wedge (x', y', z') = (yz' - zy', zx' - xz', xy' - yx').
\]

The vector product (1) is \( \text{SO}(3) \) invariant and has indeed a geometrical meaning in the oriented \( \mathbb{R}^3 \), provided with Euclidean metric.

However, the six vector products in \( \mathbb{R}^3 \), corresponding to choices of the signs \( \sigma_i \) for the components of (1) such that not all signs are coinciding

\[
(x, y, z) \wedge (x', y', z') = (\sigma_1(yz' - zy'), \sigma_2(zx' - xz'), \sigma_3(xy' - yx'))
\]
satisfy the Jacobi identity and the corresponding Lie algebras are all isomorphic to the algebra \( \mathfrak{sl}_2(\mathbb{R}) \).

Each of these vector products has a geometrical meaning in \( \mathbb{R}^3 \), endowed with an orientation and with the (pseudo)metric \( g_{ij} = \sigma_i \delta_{ij} \). The vector product of two vectors \( \mathbf{v} \) and \( \mathbf{v}' \) is orthogonal to both \( \mathbf{v} \) and \( \mathbf{v}' \), the orthogonality being the vanishing of the scalar product defined by the metric.
Now, the Poisson algebra of quadratic forms, together with the scalar product introduced by Arnold, is, up to a change of coordinates, the Lie algebra of $\mathbb{R}^3$ with vector product (2) and pseudo Euclidean metric given by $\sigma_1 = \sigma_2 = 1, \sigma_3 = -1$.

A natural question is: does it exist a Poisson algebra on a symplectic manifold corresponding to the Euclidean $\mathbb{R}^3$?

In Section 1 we show that such algebra is the 3-space of spherical harmonics of first degree, with $L^2$-metric.

Projecting this space to the unit 2-sphere, we associate with each harmonic a pair of antipodal points and its dual object, the great circle equidistant from these points. Like in [1], the Poisson bracket has a geometrical interpretation and the Jacobi identity assumes the meaning of the altitudes concurrence theorem for spherical triangles.

In Section 2 we show that the pseudo Euclidean geometry of the Poisson algebra of quadratic forms can be obtained as well as the Poisson algebras of the hyperbolic harmonics of first degree on the one sheet and on the two sheeted hyperboloids. In these cases the scalar product and the metrics are not defined intrinsically (as in the spherical case, by the $L^2$ norm), but the vanishing of the scalar product of two harmonics has a geometrical meaning in terms of the metrics of the hyperboloids similar to the meaning of the vanishing of the scalar product of two spherical harmonics in terms of the metric of the sphere.

In Section 3 we see the relation between the projective geometries obtained projectivizing the pseudo-Euclidean space (as in [1]) and the Euclidean one. Moreover, we observe that in the case of pseudo-Euclidean geometry, any non degenerate conic in the plane can play the role of the absolute of the Lobachevsky disc for a suitable projection: the orthogonality and duality can be therefore defined using this conic, obtaining the concurrence theorem for the corresponding triangle ‘altitudes’, defined in this projection.

Following the Arnold suggestion: “One might use the Jacobi identity and other theorems of the quadratic forms symplectic algebra to obtain new results of projective geometry”, Tomihisa [6] has recently shown one identity holding for the Poisson brackets in this algebra, from which all basic theorems of Projective Geometry follow.

In Section 4 we show that the Pappus theorem itself is contained in the Tomihisa identity. We observe, moreover, that the geometrical interpretation of the Tomihisa identity in projective geometry is independent from the metric structure and holds in the Lie-algebra of the Euclidean $\mathbb{R}^3$ as well.

**Notations and preliminary facts**

In this paper we use the following notations:

- $\mathbb{R}^3$ is the oriented three-dimensional vector space.
- $\mathbb{R}_E^3$ is the Lie algebra of the vectors in the Euclidean 3-space, with vector product (1). The scalar product of two vectors $\mathbf{v} = (x, y, z)$ and $\mathbf{v}' = (x', y', z')$ is:
  \begin{equation}
  \mathbf{v} \cdot \mathbf{v}' = xx' + yy' + zz',
  \end{equation}
  and the square norm of $\mathbf{v}$ is
  \begin{equation}
  ||\mathbf{v}|| := \mathbf{v} \cdot \mathbf{v} = x^2 + y^2 + z^2.
  \end{equation}
\( \mathbb{R}^3_E \) is the Lie algebra of the vectors in the pseudo-Euclidean 3-space, with metric \( g_{i,j} = \sigma_i \delta_{i,j}, \) \( \sigma_1 = \sigma_2 = 1, \sigma_3 = -1. \) I.e., the vector product of two vectors \( \mathbf{v} = (x, y, z) \) and \( \mathbf{v}' = (x', y', z') \) is:

\[
\mathbf{v} \wedge \mathbf{v}' = ((yz' - zy'), (zx' - xz'), -(xy' - yx')) ,
\]

their scalar product

\[
\mathbf{v} \cdot \mathbf{v}' = xx' + yy' - zz' ,
\]

and the square norm of \( \mathbf{v} \) is

\[
\lVert \mathbf{v} \rVert := \mathbf{v} \cdot \mathbf{v} = x^2 + y^2 - z^2 .
\]

**Remark 1.** Observe that \( \mathbf{v} \wedge \mathbf{v}' \cdot \mathbf{v} = \mathbf{v} \wedge \mathbf{v}' \cdot \mathbf{v}' = 0 \) and \( \mathbf{v} \wedge \mathbf{v}' \cdot \mathbf{v} = \mathbf{v} \wedge \mathbf{v}' \cdot \mathbf{v}' = 0. \) Moreover, \( \mathbf{v} \wedge \mathbf{v}' = 0 (\mathbf{v} \wedge \mathbf{v}' = 0) \) iff \( \mathbf{v}' = \lambda \mathbf{v} \) for a real \( \lambda. \)

**Remark 2.** Given three independent vectors in the oriented Euclidean 3-space, the scalar product of any one of them (say, \( \mathbf{v} \)) with the vector product of the two others vectors (say, \( \mathbf{u}, \mathbf{w} \)) is equal to the volume of the parallelepiped defined by the three vectors, with positive (negative) sign if the ordered triple \( (\mathbf{v}, \mathbf{u}, \mathbf{w}) \) orients \( \mathbb{R}^3 \) positively (negatively). The same holds in \( \mathbb{R}^3_E. \) This allows to give a definition of the vector product in \( \mathbb{R}^3_E, \) independent of the coordinates and based only on the \( \text{SO}(2,1) \)-invariant metric: the vector product of two vectors \( (\mathbf{u}, \mathbf{w}) \) is the vector such that its scalar product with any third vector \( \mathbf{v} \) lying outside the plane of \( \mathbf{u} \) and \( \mathbf{w} \) is equal to the oriented volume defined by \( (\mathbf{u}, \mathbf{w}, \mathbf{v}). \)

**Remark 3.** The algebra \( R^3_E \) is isomorphic to the algebra of trace zero \( 2 \times 2 \) real matrices. The isomorphism \( \mu : R^3_E \rightarrow sl(2, \mathbb{R}) \) is given by

\[
\mu(X_1, X_2, X_3) = \frac{1}{2} \left( \begin{array}{cc} X_2 & -(X_1 + X_3) \\ X_3 - X_1 & -X_2 \end{array} \right) .
\]

We will consider also:

i. \( S^2 \subset \mathbb{R}^3_E, \) the sphere \( x^2 + y^2 + z^2 = 1, \) with coordinates \( \theta \) and \( \phi, \) related to \( x, y, z \) by:

\[
x := \sin \theta \cos \phi, \quad y := \sin \theta \sin \phi, \quad z := \cos \theta .
\]

The restriction of the metric of \( \mathbb{R}^3_E \) to the sphere yields to the metric

\[
ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 .
\]

ii. \( L^2 \subset \mathbb{R}^3_E, \) the pseudo-sphere \( x^2 + x^2 - z^2 = -1, \) \( z > 0, \) with coordinates \( \chi \) and \( \phi, \) related to \( x, y, z \) by:

\[
x := \sinh \chi \cos \phi, \quad y := \sinh \chi \sin \phi, \quad z := \cosh \chi .
\]

The restriction of the pseudometric of \( \mathbb{R}^3_E \) to the hyperboloid \( L^2 \) yields to the metric

\[
ds^2 = d\chi^2 + \sinh^2 \chi d\phi^2 .
\]

iii. \( D^2 \subset \mathbb{R}^3_E, \) the one-sheeted hyperboloid \( x^2 + y^2 - z^2 = 1, \) with coordinates \( \chi \) and \( \phi, \) related to \( x, y, z \) by:

\[
x := \cosh \chi \cos \phi, \quad y := \cosh \chi \sin \phi, \quad z := \sinh \chi .
\]

The restriction of the pseudometric of \( \mathbb{R}^3_E \) to the hyperboloid \( D^2 \) yields to the metric

\[
ds^2 = -d\chi^2 + \cosh^2 \chi d\phi^2 .
\]

Note that metric (10) is indefinite.
Remark 4. A coordinates-independent description of the metrics (9) and (10) is given in [2].

1. Spherical geometry and the algebra \( S_1 \)

In this section we obtain the spherical geometry from the Poisson algebra of the spherical harmonics of degree 1.

Consider the space \( S_1 \) of the linear combinations with real coefficients of the following spherical harmonics of first degree,

\[
\begin{align*}
  f_1 &:= \sin \theta \cos \phi; \\
  f_2 &:= \sin \theta \sin \phi; \\
  f_3 &:= \cos \theta,
\end{align*}
\]

\[ S_1 = \{ Xf_1 + Yf_2 + Zf_3, \quad (X, Y, Z) \in \mathbb{R}^3 \}. \]

The \( L^2 \)-metric with respect to the standard area form on the sphere defines the scalar product:

\[
\langle F, F' \rangle = \frac{3}{4\pi} \int_{S^2} FF' \sin \theta d\theta d\phi. \tag{11}
\]

We identify the function \( F = Xf_1 + Yf_2 + Zf_3 \) with the triple of its coefficients: \( F = (X, Y, Z) \in \mathbb{R}^3 \).

**Proposition 1.1.** The scalar product (11) coincides with the scalar product (3) in \( \mathbb{R}^3 \).

**Proof.** The three functions: \( f_1, f_2 \) and \( f_3 \) constitute a orthonormal basis:

\[
\langle f_i, f_j \rangle = \delta_{ij}.
\]

\( \square \)

**Definition.** Two functions in \( S_1 \) are orthogonal iff their scalar product vanishes.

Consider the Poisson structure associated with the standard symplectic form on the sphere:

\[
\{ F, F' \} = \frac{1}{\sin \theta} \left( \frac{\partial F}{\partial \theta} \frac{\partial F'}{\partial \phi} - \frac{\partial F}{\partial \phi} \frac{\partial F'}{\partial \theta} \right).
\]

**Proposition 1.2.** The Poisson bracket of two functions in \( S_1 \) satisfies:

\[
\{ F, F' \} = F \wedge F'.
\]

**Proof.** If \( F = (X, Y, Z) \) and \( F' = (X', Y', Z') \) are in \( S_1 \), then \( \{ F, F' \} \in S_1 \). For basic functions \( f_i, i = 1, 2, 3 \), we get

\[
\{ f_1, f_2 \} = -\{ f_2, f_1 \} = f_3, \quad \{ f_2, f_3 \} = -\{ f_3, f_2 \} = f_1, \quad \{ f_3, f_1 \} = -\{ f_1, f_3 \} = f_2.
\]

Since the Poisson bracket is bilinear and antisymmetric, Proposition 1.2 follows. \( \square \)

**Definition.** Two functions having non zero Poisson bracket are said independent.

We see now a geometrical implication of the orthogonality of two spherical functions of \( S_1 \).

**Proposition 1.3.** Let \( F_1 = X_1 f_1 + Y_1 f_2 + Z_1 f_3 \) and \( F_2 = X_2 f_1 + Y_2 f_2 + Z_2 f_3 \) be two non zero spherical functions. If \( F_1 \cdot F_2 = 0 \), then the great circles \( F_1 = 0 \) and \( F_2 = 0 \) on the sphere meet orthogonally.

**Proof.** Let \( s \) be the parameter along the curves on the sphere where \( F_1 \) and \( F_2 \) vanish:

\[
F_1(\theta_1(s), \phi_1(s)) = 0, \quad F_2(\theta_2(s), \phi_2(s)) = 0,
\]
so that at $s = s^*$, $\theta_1 = \theta_2 := \theta^*$ and $\phi_1 = \phi_2 := \phi^*$. Since (8) holds, we have to prove that at $s = s^*$ the following equation is fulfilled, whenever $X_1X_2 + Y_1Y_2 + Z_1Z_2 = 0$:

$$
\left. \frac{d\theta_1}{ds} \right|_{s=s^*} \frac{d\theta_2}{ds} \left|_{s=s^*} \right. + \sin^2 \theta^* \left. \frac{d\phi_1}{ds} \right|_{s=s^*} \left. \frac{d\phi_2}{ds} \right|_{s=s^*} = 0.
$$

Take as parameter $s$ the angle $\phi$. The functions $F_i$ vanish on the curves

$$
\theta_i = \arctan \left( \frac{Z_i}{X_i \cos \phi + Y_i \sin \phi} \right).
$$

From the equation $\theta_1(\phi^*) = \theta_2(\phi^*)$ we obtain

$$
\phi^* = \pm \arctan((X_1Z_2 - X_2Z_1)/(Y_1Z_2 - Y_2Z_1)),
$$

and

$$
\theta^* = \arctan \left( \frac{\sqrt{(X_1Z_2 - X_2Z_1)^2 + (Y_1Z_2 - Y_2Z_1)^2}}{(Y_1X_2 - X_1Y_2)} \right).
$$

The left member of Eq. (12) becomes:

$$
\left. \frac{d\theta_1}{d\phi} \right|_{\phi=\phi^*} \left. \frac{d\theta_2}{d\phi} \right|_{\phi=\phi^*} + \sin^2 \theta^* \frac{(X_1Z_2 - X_2Z_1)^2 + (Y_1Z_2 - Y_2Z_1)^2}{Z_1Z_2 \||\{F_1, F_2\}||} (F_1 \cdot F_2),
$$

which vanishes if $F_1 \cdot F_2 = 0$.

1.1. From Algebra to Geometry. We define the following duality on the unit 2-dimensional sphere.

**Definition.** A pair of antipodal points is dual of the great circle equidistant from these points. For example, the pair of North-South poles is dual of the equator.

We denote by Greek letters (e.g. $\alpha$) the great circles, by small letters (e.g. $a$) the pairs of antipodal points, and by $(a|\alpha)$ the pair of dual objects $a$ and $\alpha$.

With every vector $a = (X, Y, Z)$ in $\mathbb{R}^3$ we associate the pair of antipodal points

$$
a = (a, -a), \quad a = \frac{a}{\sqrt{||a||}}
$$

on the unit sphere.

Every function $F \in S_1$ is represented by a vector $a = (X, Y, Z)$, therefore we associate with every function $F \in S_1$ the corresponding pair $(a|\alpha)$.

**Proposition 1.4.** Let $F, F' \in S^1$ be two independent functions, and let $\{F, F'\} = F''$. Let $a, a', a''$ be the corresponding vectors, $a, a', a''$ the corresponding pairs of antipodal points on the sphere and $\alpha, \alpha', \alpha''$ their dual great circles. Then $(a''|\alpha'')$ has the following geometrical meaning: the antipodal points of the pair $a''$ are the intersection of the two great circles $\alpha$ and $\alpha'$. The circle $\alpha''$ is the great circle joining the two pairs of points in $a$ and $a'$.

**Proof.** The vector $a''$ is, by Proposition 1.2, the vector product of $a$ and $a'$, which is orthogonal to the plane containing $a$ and $a'$. The great circle containing $a$ and $a'$ is therefore the intersection of this plane with the sphere, which is the circle $\alpha''$ dual of $a''$. The circles $\alpha$ and $\alpha'$ lie in two planes orthogonal respectively to the vectors $a$ and $a'$. Since $a''$ is orthogonal to both $a$ and $a'$, $a''$ lies in the intersection of these planes, and therefore the pair $a''$ is the intersection of $\alpha$ with $\alpha'$.

$\square$
Definition. A spherical triangle is said to be proper, if no one of its vertices belong to the pair of points dual of the great circle containing the other vertices, or, equivalently, no one of its sides belongs to a great circle dual of a pair containing the opposite vertex.

Proposition 1.5. A spherical triangle defined by the normalized vectors $a_1$, $a_2$, $a_3$, corresponding to the functions $F_1, F_2, F_3 \in S_1$ is proper iff \{$\{F_i, F_j\}, F_k\} \neq 0$ for any choice of the indices $i, j, k$ among the permutations of $1, 2, 3$.

Proof. The identity $\{\{F_i, F_j\}, F_k\} = 0$ holds if and only if $\{F_i, F_j\} = \lambda F_k$, by Remark 2. If $\{F_i, F_j\} = \lambda F_k$ for some $i, j, k$, then the normalized vector $a_3$ is orthogonal to the plane of $a_1$ and $a_2$, i.e., $a_3$ is dual of the great circle containing the pairs $a_1$ and $a_2$. □

Definition. The altitude from a vertex $a_i$ of a proper spherical triangle is the great circle through $a_i$ intersecting orthogonally the great circle containing the opposite side to $a_i$.

Proposition 1.6. The altitude from a vertex $a_i$ of a proper spherical triangle is the great circle through the pair $a_i$ passing through the pair $b$, dual of the great circle $\beta$ containing the other two vertices of the triangle.

Proof. Any circle passing through $b$ is orthogonal to the great circle $\beta$, by the definition of duality. □

Remark 5. The altitude from a vertex $a_i$ of a proper spherical triangle is also the altitude of the antipodal triangle (with vertices the antipodal $-a_1, -a_2, -a_3$), and of the other six triangles cut off on the sphere by the great circles, sides of the triangle.

Proposition 1.7. Let $F_1, F_2, F_3 \in S_1$ define the vertices of a proper spherical triangle. Let $a_1, a_2, a_3$ be the normalized vectors corresponding to $F_1, F_2, F_3$, $a_1, a_2, a_3$ the corresponding pairs of antipodal points on the sphere and $\alpha_1, \alpha_2, \alpha_3$ their dual great circles. Then the Jacobi identity:

$$\{\{F_1, F_2\}, F_3\} + \{\{F_2, F_3\}, F_1\} + \{\{F_3, F_1\}, F_2\} = 0$$

has the following geometrical meaning: i) The altitudes of the eight spherical triangles with vertices $\pm a_1, \pm a_2, \pm a_3$ meet at the same pair $h := (h, h)$ of points; ii) These altitudes are also the altitudes of the eight spherical triangles defined by the three great circles $\alpha_1, \alpha_2, \alpha_3$.

Proof. Equation (14) says that the three vectors $\{\{F_1, F_2\}, F_3\}, \{\{F_2, F_3\}, F_1\}$ and $\{\{F_3, F_1\}, F_2\}$ have zero sum. In particular, they lie in the same plane. i) By Proposition 1.4 $\{\{F_1, F_2\}, F_3\}$ represents the great circle $\gamma_3$ containing $a_3$ and the pair $b_3$ of points, dual of the opposite side $\beta_3$, which is the great circle containing $a_1$ and $a_2$. But $\{\{F_1, F_2\}, F_3\}$ represents also the pair of points $c_3$ (dual of $\gamma_3$), which is the intersection of $\alpha_3$ with $\beta_3$. The equation implies that the pair $c_3$ and the analogue $c_1$ and $c_2$ lie in the same great circle. Therefore, the dual great circles $\gamma_3, \gamma_1$ and $\gamma_2$ meet on a unique pair of points $h$. The three altitudes $\gamma_1, \gamma_2$ and $\gamma_3$ are common to all eight spherical triangles, cut off by the 3 sides of the initial triangle.

ii) The spherical triangles defined by the three dual great circles $\alpha_1, \alpha_2, \alpha_3$ have the vertices at the points of the pairs $b_1, b_2, b_3$, dual of the sides of the initial triangle. The altitudes are the great circles containing each one a pair $b_i$ and the corresponding pair $a_i$, dual of $\alpha_i$. Therefore these altitudes coincide with the great circle $\gamma_i$, meeting at the pair of points $h$ (See Figure 1). □

Remark 6. Observe that, if the spherical triangle $a_1, a_2, a_3$ with sides $\beta_1, \beta_2, \beta_3$ is not proper, that is a vertex, say $a_1$, belongs to the pair of points dual to the great circle $\beta_1$ through $a_2$ and $a_3$ (i.e., $\beta_1 = \alpha_1$) then all great circles through $a_1$ are altitudes of the triangle, and both the altitude $\gamma_2$ from $a_2$ to $\beta_2$ and the altitude $\gamma_3$ from $a_3$ to $\beta_3$ coincide with the side $\alpha_1$. The common point of the altitudes of the triangle degenerates in this case to the entire circle $\alpha_1$. This is the meaning
Figure 1. Stereographic projection of the spherical triangles defined by the great circles $\mu, \nu, \rho$ through $a, b, c,$ and the dual triangles with great circles $\alpha, \beta, \gamma$ through $m, n, r.$ They have the same altitudes (dotted lines) meeting at $\pm h$.

of the Jacobi identity that in this case reads $\{\{F_1, F_2\}, F_3\} + \{\{F_3, F_1\}, F_2\} = 0$. Indeed, the points $\{\{F_1, F_2\}, F_3\}$ and $\{\{F_3, F_1\}, F_2\}$, being antipodal, define two coinciding pairs of points, $c_2$ and $c_3$, dual to the altitudes $\gamma_2$ and $\gamma_3$, that therefore coincide.

2. Hyperbolic geometry

2.1. Lobachevsky geometry. In analogy with the Poisson algebra $S_1$, we consider the Poisson algebra $H^{-1}$ of the linear combinations with real coefficients of the following hyperbolic harmonics of first degree, defined on the upper sheet of the two-sheeted hyperboloid (the pseudo-sphere):

$$ f_1 := \sinh \chi \cos \phi ; \quad f_2 := \sinh \chi \sin \phi ; \quad f_3 := \cosh \chi , $$

$$ H^{-1} = \{X f_1 + Y f_2 + Z f_3, \quad (X, Y, Z) \in \mathbb{R}^3 \}. $$

The Poisson bracket in $H^{-1}$ is:

$$ \{F, F'\} = \frac{1}{\sinh \chi} \left( \frac{\partial F}{\partial \chi} \frac{\partial F'}{\partial \phi} - \frac{\partial F}{\partial \phi} \frac{\partial F'}{\partial \chi} \right). $$

**Proposition 2.1.** The Poisson algebra $H^{-1}$ coincides with the Lie algebra $\mathbb{R}_H^3$.

**Proof.** If $F = (X, Y, Z)$ and $F' = (X', Y', Z')$ are in $H^{-1}$, then $\{F, F'\} \in H^{-1}$, and coincides with the vector product (5) in $\mathbb{R}_H^3$:

$$ \{F, F'\} = F \wedge F'. $$

Indeed, consider the basic functions $f_i, i = 1, 2, 3$, and their Poisson brackets. They satisfy:

$$ \{f_1, f_2\} = -\{f_2, f_1\} = -f_3, \quad \{f_2, f_3\} = -\{f_3, f_2\} = f_1, \quad \{f_3, f_1\} = -\{f_1, f_3\} = f_2. $$

Since the Poisson bracket is bilinear and antisymmetric, Eq. (16) is fulfilled. Proposition 2.1 follows.

Also in $H^{-1}$ two functions having non zero Poisson bracket will be said *independent.*
The Poisson algebra $\mathcal{H}^-_1$ can be therefore endowed with the same metric as $\mathbb{R}^3_H$, with scalar product (3), square norm (7), and the same definition of orthogonality. In this way the Poisson bracket of two independent functions is orthogonal to these functions.

**Remark 7.** The three functions $f_1$, $f_2$ and $f_3$ are orthogonal and satisfy $f_1^2 + f_2^2 - f_3^2 = -1$. They are the coordinates in $\mathbb{R}^3$ of the point with coordinates $(\chi, \phi)$ on the pseudo-sphere.

Also in this case we see the geometrical implication of the orthogonality of two functions in $\mathcal{H}^-_1$.

**Proposition 2.2.** Let $F_1 = X_1 f_1 + Y_1 f_2 + Z_1 f_3$ and $F_2 = X_2 f_1 + Y_2 f_2 + Z_2 f_3$ be two non zero functions of $\mathcal{H}^-_1$. If $F_1 \cdot F_2 = 0$, then the lines $F_1 = 0$ and $F_2 = 0$ on the pseudo-sphere meet orthogonally.

**Proof.** The proof is analogous of that of [1,3] If $\chi_1(\phi)$ and $\chi_2(\phi)$ parametrize the curves $F_1 = 0$ and $F_2 = 0$ which meet at the point $(\chi^*, \phi^*)$, we obtain finally, using metric (5):

$$
\frac{d\chi_1}{d\phi} \bigg|_{\phi=\phi^*} \frac{d\chi_2}{d\phi} \bigg|_{\phi=\phi^*} + \sinh^2 \chi^* \frac{(X_1 Z_2 - X_2 Z_1)^2 + (Y_1 Z_2 - Y_2 Z_1)^2}{Z_1 Z_2 \|\{F_1, F_2\}\|} (F_1 \cdot F_2),
$$

which vanishes if $F_1 \cdot F_2 = 0$. \qed

2.2. De Sitter geometry. In complete analogy with the algebra $\mathcal{H}^-_1$, we consider the Poisson algebra $\mathcal{H}^+_1$ of the linear combinations with real coefficients of the following hyperbolic harmonics of first degree, defined on the one sheeted hyperboloid:

$$
f_1 := \cosh \chi \cos \phi; \quad f_2 := \cosh \chi \sin \phi; \quad f_3 := \sinh \chi,
$$

with Poisson bracket

$$
\{F, F'\} = \frac{1}{\cosh \chi} \left( \frac{\partial F}{\partial \chi} \frac{\partial F'}{\partial \phi} - \frac{\partial F}{\partial \phi} \frac{\partial F'}{\partial \chi} \right).
$$

**Proposition 2.3.** The Poisson algebra $\mathcal{H}^+_1$ is isomorphic to the Lie algebra $\mathbb{R}^3_H$; the isomorphism is given by an inversion of sign: $F \to -F$.

**Proof.** We prove that if $F = (X, Y, Z)$ and $F' = (X', Y', Z')$ are in $\mathcal{H}^+_1$, then $\{F, F'\} \in \mathcal{H}^+_1$, and is opposite to the vector product [5] in $\mathbb{R}^3_H$.

$$
\{F, F'\} = -F \wedge F'.
$$

Consider the basic functions $f_i$, $i = 1, 2, 3$, and their Poisson brackets. We obtain

$$
\{f_1, f_2\} = -\{f_2, f_1\} = f_3, \quad \{f_2, f_3\} = -\{f_3, f_2\} = -f_1, \quad \{f_3, f_1\} = -\{f_1, f_3\} = -f_2.
$$

Since the Poisson bracket is bilinear and antisymmetric, Eq. (19) is fulfilled. Therefore $\{-F, -F'\} = -F \wedge F'$ and the Proposition follows. \qed

Also here the same metric structure as in $\mathbb{R}^3_H$ can be introduced, so that the Poisson bracket of two functions in $\mathcal{H}^+_1$ is orthogonal to both functions.

**Remark 8.** The three functions $f_1$, $f_2$ and $f_3$ are orthogonal and satisfy $f_1^2 + f_2^2 - f_3^2 = 1$. They are the coordinates in $\mathbb{R}^3$ of the point with coordinates $(\chi, \phi)$ on the one sheeted hyperboloid.

The geometrical meaning of the orthogonality of two functions in $\mathcal{H}^+_1$ is the following.

**Proposition 2.4.** Let $F_1 = X_1 f_1 + Y_1 f_2 + Z_1 f_3$ and $F_2 = X_2 f_1 + Y_2 f_2 + Z_2 f_3$ be two non zero functions in $\mathcal{H}^+_1$. If $F_1 \cdot F_2 = 0$, then the lines $F_1 = 0$ and $F_2 = 0$ on the hyperboloid meet orthogonally.
Proof. The proof is analogous of that of Proposition 1.3. If \( \chi_1(\phi) \) and \( \chi_2(\phi) \) parametrize the curves \( F_1 = 0 \) and \( F_2 = 0 \) which meet at the point \((\chi^*, \phi^*)\), we obtain finally, using metric (10):

\[
-\frac{d\chi_1}{d\phi}_{\phi=\phi^*} \frac{d\chi_2}{d\phi}_{\phi=\phi^*} + \cosh^2 \chi^* = -\frac{(X_1Z_2 - X_2Z_1)^2 + (Y_1Z_2 - Y_2Z_1)^2}{Z_1Z_2 \|\{F_1, F_2\}\|}(F_1 \cdot F_2),
\]

which vanishes if \( F_1 \cdot F_2 = 0 \).

2.3. Quadratic forms on the symplectic plane. Consider the Poisson algebra \( \mathcal{Q} \) of the real binary quadratic forms

\[
F = ap^2 + 2bpq + cq^2,
\]

provided with the Poisson bracket in the symplectic plane \((p, q)\). We will rewrite the form \( F \) as linear combination of

\[
(21) \quad f_1 := 2pq; \quad f_2 := p^2 - q^2; \quad f_3 := p^2 + q^2.
\]

From \( F = Xf_1 + Yf_2 + Zf_3 \) we obtain:

\[
(22) \quad X = b, \quad Y = (a - c)/2, \quad Z = (a + c)/2.
\]

Arnold defined in [1] a scalar product \( \tilde{\Delta} \), as the symmetric bilinear form coinciding on the diagonal with the discriminant \( \Delta = ac - b^2 \) of the form; if \( F = ap^2 + 2bpq + cq^2 \), and \( F' = a'p^2 + 2b'pq + c'q^2 \), then

\[
\tilde{\Delta} = (ac' + ca')/2 - bb'.
\]

Definition. We define the scalar product of two forms as \(-\tilde{\Delta}\).

Proposition 2.5. In the new coordinates the scalar product of \( F = (X, Y, Z) \) with \( F' = (X', Y', Z') \) coincides with the scalar product (20) in \( \mathbb{R}^3_H \) and the square norm with (7).

Proof. Using (22) Eq. (6) becomes

\[
F \cdot F' = bb' - (ac' + ca')/2,
\]

and, on the diagonal:

\[
\|F\| = b^2 - ac.
\]

Definition. Two forms are said orthogonal iff their scalar product vanishes.

The choice of the base (21) for the quadratic forms allows us to write the Poisson bracket of two forms \( F \) and \( F' \) in the form analog to the vector product.

Proposition 2.6. The algebra \( \mathcal{Q} \) is isomorphic to the algebra \( \mathbb{R}^3_H \).

Proof. In the new coordinates the Poisson bracket of \( F \) and \( F' \) is four times the vector product (5) of \( F \) and \( F' \) in \( \mathbb{R}^3_H \):

\[
(23) \quad \{F, F'\} = 4F \wedge F'.
\]

Indeed, the Poisson brackets between the basic functions \( f_i, i = 1, 2, 3 \), are equal to:

\[
\{f_1, f_2\} = -\{f_2, f_1\} = -4f_3, \quad \{f_2, f_3\} = -\{f_3, f_2\} = 4f_1, \quad \{f_3, f_1\} = -\{f_1, f_3\} = 4f_2.
\]

Eq. (23) follows, being the Poisson bracket bilinear and antisymmetric. The isomorphism \( \mu : \mathbb{R}^3_H \rightarrow \mathcal{Q} \) associates with the vector \( F \) the form \( \mu(F) = F/4 \).

Remark 9. The functions \( f_1, f_2 \) and \( f_3 \) satisfy \( f_1^2 + f_2^2 - f_3^2 = 0 \). Therefore they can be considered as the coordinates in \( \mathbb{R}^3 \) of a point on the upper part \((Z \geq 0)\) of the cone \( X^2 + Y^2 - Z^2 = 0 \).
2.4. **From Algebra to Geometry.** Let us denote by $L^2_\pm$ the union of the two sheets of the two-sheeted hyperboloid. The content of Section 1.1 can be translated for the hyperbolic geometry of $\mathbb{R}^3_H$. In fact, the definition of the duality does not depend on the sign of the vector product. A pair of antipodal points on the sphere is the intersection of a straight line through the origin with the sphere. Here it becomes a pair of antipodal points either in $L^2_\pm$ or in $D^2$, whenever the straight line does not belong to the cone. The great circle dual of a pair of points is the intersection of the plane orthogonal (in $\mathbb{R}^3_H$) to the line connecting the two points. Here the great circle dual of a pair of points is replaced with the intersection of the plane orthogonal (in $\mathbb{R}^3_H$) to $L^2_\pm$ or $D^2$. The great circle dual of a pair of points is the intersection of the plane orthogonal (in $\mathbb{R}^3_E$) to the line connecting these points of the hyperboloids. Observe that such a plane intersects either only $D^2$ or both $L^2_\pm$ and $D^2$ along a pair of hyperbolae. The conics obtained as intersection of $L^2_\pm$ and $D^2$ with the planes through the origin are in fact the geodesics of the hyperboloids, in the metrics (9) and (10) on $L^2$ and $D^2$ respectively.

The geodesic line $\alpha$ connecting two points belonging to $L^2_\pm \cup D^2$ is either an ellipse or a pair of hyperbolae (see Figure 2). A triangle is a subset of $L^2_\pm \cup D^2$ bounded by three geodesics.

Also here we say that a triangle is *proper* if no one of its vertices belong to the pair of points dual of the geodesics containing the other vertices, or, equivalently, no one of its sides belongs to a geodesic dual of a pair containing the opposite vertex.

The theorem of the altitudes holds here for proper triangles exactly as Proposition 1.6 and Remark 6 in the spherical case, substituting great circles with geodesics.

![Figure 2](image)

**Figure 2.** Left: section $X = 0$ of $L^2_\pm \cup D^2$. The pair of points $(a, -a)$ and $(b, -b)$ lie on the hyperboloids and $\alpha$ and $\beta$ are the geodesic dual of them (Right).

3. **Projective geometry**

We consider now the projective plane $\mathbb{R}P^2$, with coordinates $(x := \frac{X}{Z}, y := \frac{Y}{Z})$, to which the space $\mathbb{R}^3_H$ projects.

The unit circle $x^2 + y^2 = 1$ represents the cone $Z^2 = X^2 + Y^2$, its interior the Lobachevsky disc and the exterior the De Sitter world.

It is easy to verify that the duality between a point and a line (see [1]) on this projective plane is the duality between a vector based on the origin and the plane through the origin orthogonal in $\mathbb{R}^3_H$ to this vector.
Remark 10. From Propositions 2.1 and 2.3 it follows that the altitudes theorem obtained from the Jacobi identity of the Poisson bracket for the quadratic forms algebra $Q$ is obtained as well from the Jacobi identity of the Poisson bracket for the algebra $\mathcal{H}_1^+$ and for the algebra $\mathcal{H}_1^-$. Recall that in $\mathbb{R}P^2$ we associate with $F = (X,Y,Z)$, the pair $(\alpha, \alpha)$, where $\alpha = (X/Z,Y/Z)$ is the projection of $F$, and $\alpha$ is the line, projection of the plane orthogonal to $F$.

3.1. Relation between $\mathbb{R}P^2_H$ and $\mathbb{R}P^2_E$. Observe that the orthogonality between two lines, as well as the duality between point and line in the projective plane, has not an intrinsic meaning but is inherited by the orthogonality in $\mathbb{R}^3_H$ of their preimages by the projection.

In an analog way we can in fact define the orthogonality between two lines, as well as the duality between point and line in the projective plane as the orthogonality in $\mathbb{R}^3_E$ of their preimages by the projection.

If we do this, we obtain the following theorem, illustrated in Figure 3.

Let us denote by $\mathbb{R}P^2_H$ and $\mathbb{R}P^2_E$ the projective plane with the orthogonality inherited by projectivization from $\mathbb{R}^3_H$ and $\mathbb{R}^3_E$ respectively. Provide both planes with coordinates $(x := \frac{X}{Z}, y := \frac{Y}{Z})$.

**Theorem 3.1.** The point dual of a line in $\mathbb{R}P^2_E$ is symmetric with respect to the point $(0,0)$ of the point dual of the same line in $\mathbb{R}P^2_H$. The line dual of a point in $\mathbb{R}P^2_E$ is symmetric with respect to the point $(0,0)$ to the line dual of the same point in $\mathbb{R}P^2_H$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Three different mutual positions of a point $a$ and its dual line $\alpha$. Upper images: duality in $\mathbb{R}P^2_H$. Lower images: duality in $\mathbb{R}P^2_E$.}
\end{figure}

**Proof.** Let $a = (a,0)$ and let $b = (b,0)$ lie on the plane orthogonal in $\mathbb{R}^3_E$ to $(a,0,1)$ (see Fig. 3). The vectors $(a,0,1)$ and $(b,0,1)$ satisfy $ab + 1 = 0$. Hence $b = -1/a$. Let $c = (c,0)$ lie on the plane orthogonal in $\mathbb{R}^3_H$ to $(a,0,1)$. The vectors $(a,0,1)$ and $(c,0,1)$ satisfy $ac - 1 = 0$. Hence $c = 1/a$. Therefore the points $b$ and $c$ are symmetric with respect to the origin of $\mathbb{R}P^2$. 

The dual line $\alpha$ to $a$ is the line through $b$ (in $\mathbb{RP}^2_E$) and through $c$ (in $\mathbb{RP}^2_H$) which is orthogonal (in the plane $Z = 1$, with Euclidean metrics) to $o - a$ (see Figure 4).

**Corollary 3.2.** The Jacobi identity for the Poisson bracket in the algebra $S_1$ implies the concurrence of the altitudes of the triangle in $\mathbb{RP}^2_E$.

Figure 5 shows the comparison of the altitudes concurrence theorem for a triangle in $\mathbb{RP}^2_H$ and in $\mathbb{RP}^2_E$.

### 3.2. Other models for the projective plane $\mathbb{RP}^2_H$ and $\mathbb{RP}^2_E$

The projective plane $\mathbb{RP}^2_H$ can be obtained projecting the space $\mathbb{R}^3_H$ on any affine plane (not only the plane $Z = 1$). This implies that the image of the cone is any conic section.

**Definition.** Given a non degenerate conic $C$ on the projective plane, define the Lobachevsky disc ($\mathcal{L}$) as the region of the complement containing the foci of the conic, and de Sitter world ($\mathcal{D}$) the other region. The line dual of a point $a \in \mathcal{D}$ is the line joining the tangent points to $C$ from $a$. The point dual of a line $\alpha$ intersecting $C$ is the point of $\mathcal{D}$ where the tangents to $C$ at the points where $C$ intersects $\alpha$ meet. The line dual of a point $a \in \mathcal{L}$ is the line containing all points of $\mathcal{D}$ dual of all lines through $a$ (see Figure 6). The conic $C$ is called the **absolute**.

**Definition.** The altitude issued from a point to a line not containing it is the line joining the point with the point, dual of the line.

**Proposition 3.3.** Given any non degenerate conic in a plane, the theorems of projective geometry hold in that plane taking the above definitions of duality.

**Example.** An illustration of the hyperbolic altitudes concurrence theorem is shown in Figure 7.

In section 3.1 we have used the constructions of dual objects in $\mathbb{RP}^2_H$ to obtain the dual objects in $\mathbb{RP}^2_E$. In fact, to do this we have reflected points and lines about the origin of the projective plane.
Figure 5. The triangle $abc$ with its dual triangle $mnr$ in $\mathbb{R}P^2_H$ (upper image) and in $\mathbb{R}P^2_E$ (lower image). $h$ denotes in both images the common point of the altitudes of $abc$ and of $mnr$. The points $m, n, r$ in $\mathbb{R}P^2_E$ are symmetric with respect to the centre $o$ of the points $m, n, r$ in $\mathbb{R}P^2_H$.

If we consider, as a model of the projective plane, a projection of $\mathbb{R}^3$ to an affine plane, different from the plane $Z = 1$, the point $O$ about which we have to “reflect” points and lines, to obtain the $\mathbb{R}P^2_E$ duality, becomes the intersection point of that plane with the axis of the cone. By reflection of a point $c$, lying on $C$, we mean the second meeting point of $C$ with $Oc$. By reflection of a line through two points $c_1$ and $c_2$ of the conic $C$ with respect to $O$ we mean the line connecting the points, different from $c_1$ and $c_2$, intersection of the conic $C$ with the lines $Oc_1$ and $Oc_2$. By reflection of a point $p$, not lying on $C$, with respect to $O$, we mean the meeting point of two lines, obtained by reflection with respect to $O$ of two lines meeting at $p$. Finally, by reflection of a line $m$ nonintersecting $C$ we mean the line connecting two points, which are obtained by reflection of two points of $m$.

The point $O$ is well defined by the conic itself representing the absolute. This is however an indirect and complicated way to define duality in $\mathbb{R}P^2_E$.

Indeed, if we are interested in the geometry of $\mathbb{R}P^2_E$, obtained as projection of $\mathbb{R}^3_E$ on an arbitrary affine plane, for the SO(3) invariance we can take that plane as the plane $Z = 1$, the intersection
Figure 6. The lines $\alpha$, $\beta$ and $\gamma$ are dual of the points $a$, $b$ and $c$.

Figure 7. The altitudes of the triangle $abc$ meet at the same points. They are as well the altitudes of the dual triangle $mnr$.

of this plane with the so defined $Z$ axis as the origin, and the unit circle centered at the origin as the absolute, so that Theorem 3.1 holds.

Remark 11. The altitudes concurrence theorem is mainly stated for triangles in the Euclidean plane. In fact, in this case the orthogonality between an altitude and the relative side of a triangle is a metric property of the Euclidean plane, property that is not invariant under projective transformations. In the projective plane, the altitude from a vertex relative to the opposite
side of a triangle is the straight line connecting that vertex with the point, dual to the line containing the opposite side. The 'orthogonality' between altitude and side stands for the orthogonality (defined as the vanishing of a scalar product in a 3-dimensional metric space) of the corresponding planes through the origin containing respectively the altitude and the side, and is therefore invariant under projective transformations.

4. Tomihisa's identity

The following identity was found by Tomihisa \cite{6} for the algebra $\mathcal{Q}$.

Given 5 quadratic forms $F_1, F_2, F_3, F_4, F_5$,

(24) \[ \{F_1, \{\{F_2, F_3\}, \{F_4, F_5\}\}\} + \{F_3, \{\{F_2, F_5\}, \{F_4, F_1\}\}\} + \{F_5, \{\{F_2, F_1\}, \{F_4, F_3\}\}\} = 0. \]

The Tomihisa identity has been written here (24) in a little different form with respect to the expression in \cite{6} to point out the following symmetry: the indices (1,3,5) are cyclically permuted in the three terms (like the three indices of the Jacobi identity). The indices (2,4) are at the same place in all terms. The indices playing this role will be called \textit{fixed indices} of the identity.

We have seen that the algebra $\mathcal{Q}$ is isomorphic to the algebra $\mathbb{R}^3_L$. We will see in the last subsection that the Tomihisa identity holds in all real Lie-algebras of dimension 3, and hence in all Poisson algebras we have here considered. It has indeed a unique geometrical meaning using the geometrical interpretation of the Poisson bracket given in both sections 1.1 and 2.4.

4.1. Geometrical remarks about Tomihisa identity. An element of the Poisson algebras we have considered is associated with a pair of dual geometric objects. In this way, any expression concerning Poisson brackets has two geometrical meanings.

Moreover, we underline the following fact. Call $p(F)$ and $\lambda(F)$ the point and its dual line associated with $F$. The expression:

\[ \{F, F'\} = F'' \]

has the following meaning:

1. $\lambda(F'')$ is the line connecting $p(F)$ and $p(F')$,
2. $p(F'')$ is the intersection of $\lambda(F)$ with $\lambda(F')$,
3. $\lambda(F'')$ is the line from $p(F)$ orthogonal to $\lambda(F')$,
4. $\lambda(F'')$ is the line from $p(F')$ orthogonal to $\lambda(F)$.

We say that the interpretations (1) and (2) are \textit{homogeneous}, since the objects inside the brackets are both points or both lines.

\textit{Definition.} An expression concerning $n$ objects and many levels of Poisson brackets is said \textit{homogeneous} if there exists a homogeneous interpretation of each bracket, whenever the $n$ objects are interpreted either all as points, either all as the dual lines.

\textit{Example.} The Jacobi identity is not homogeneous, since if $F_1, F_2$ and $F_3$ are interpreted as points, $\{\{F_1, F_2\}, F_3\}$ is not homogeneous.

\textit{Remark 12.} The straight line connecting two points and the intersection point of two lines are not depending on the orthogonality definition. Hence a homogeneous expression of the considered Lie algebras has a geometrical meaning independent of the definition of duality (i.e., of the definition of orthogonality).
This leads us to the conclusion that the geometric implications of the Tomihisa identity must hold for the projectivizations of all spaces we have considered, independently of their metric structure.

Figures 8 and 9 illustrate the Tomihisa identity in $\mathbb{R}P^2$.

**Figure 8.** Tomihisa identity for five elements $F_i$. The numbers 1-5 indicates the indices $i$. Left: the $F_i$ are interpreted as points: the three bold lines intersect at the same point. Right: the $F_i$ are interpreted as lines: the three points marked by white discs lie on the same line.

We conclude this subsection with a comment to Remark 12 due to V.Timorin.

**Remark 13.** (V. Timorin) For a real oriented vector space $V$ of dimension 3, we can define a bilinear skew-symmetric map $V \times V \to V^*$ (the dual space) in the following way. Let $u, v, z$ be three vectors of $V$. Denote by $P(u, v, z)$ the oriented volume spanned by the ordered triple of vectors $u, v, z$. The volume has a sign which is invariant under cyclic permutations of the three vectors and changes sign under odd permutations of them. We define a linear functional $P_{u, v}$ on $V$ (i.e. an element of the dual space $V^*$) sending $z$ to $P(u, v, z)$. The map: $V \times V \to V^*$, sending $(u, v)$ to $P_{u, v}$ is therefore bilinear and skew-symmetric. Since the volume form is well-defined up to a constant factor, $P$ is also well-defined up to a constant factor. It is completely independent of any kind of additional structure, say, orthogonality, duality between points and lines in the same projective plane etc. Similarly, if we take two vectors $x^*$ and $y^*$ in the dual space, then the linear functional $P_{x^*, y^*}$ on $V^*$, which sends $z^*$ to $P(x^*, y^*, z^*)$, is a vector in $V$ (since $V^{**}$ is canonically isomorphic to $V$). Thus the Tomihisa identity holds interpreting the Poisson bracket as the functional $P$, without any reference to any Poisson structure.

### 4.2. Tomihisa’s identity as Pappus’ theorem.

In [6], the Pappus theorem is deduced from the identity (24) and other equations [1] holding for the scalar product in the space of quadratic forms.

We show here that the identity (24) contains in fact in itself the Pappus theorem.

**Proposition 4.1.** The Tomihisa identity for the points $F_1, \ldots, F_5$ is equivalent to the Pappus theorem.

1In particular, in [6], the Pappus Theorem is deduced by an identity (Theorem 2) already shown in [3] as equivalent to the Pappus Theorem.
Proof. Pappus’ theorem states that given two triples \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\) of points, each one lying on a straight line, and defining the point \(c_i (i = 1, 2, 3)\) as the meeting of the two straight lines through \(a_j, b_k\) and \(a_k, b_j\) (\(i, j, k\) all different) then the points \(c_1, c_2, c_3\) are collinear.

In Figure 9 the points \(a_i\) are named \(A, B, C\), the points \(b_i\) are named \(D, E, H\) and the points \(c_i\) are named \(M, N, P\).

An equivalent formulation that the points \(M, N, P\) are collinear is that the straight line joining \(M\) and \(P\) passes through \(N\), the meeting point of the lines connecting \(A\) with \(H\) and \(D\) with \(C\).

Suppose the points \(A, B, C, D, E, H\) be assigned. \(P\) is the meeting point of \(BH\) with \(CE\). We set

\[
F_1 \equiv A, \quad F_2 \equiv B, \quad F_3 \equiv P, \quad F_4 \equiv E, \quad F_5 \equiv D.
\]

The points \(C\) and \(H\) are therefore

\[
C = \{\{F_1, F_2\}, \{F_3, F_4\}\}, \quad H = \{\{F_3, F_2\}, \{F_5, F_4\}\}.
\]

Moreover

\[
M = \{\{F_5, F_2\}, \{F_1, F_4\}\}.
\]

The Tomihisa identity is thus

\[
\{F_1, H\} + \{F_3, M\} + \{F_5, C\} = 0,
\]

i.e., the straight line connecting \(P\) and \(M\) passes through the meeting point of the lines connecting \(A\) with \(H\) and \(D\) with \(C\). □

![Figure 9](image-url)

4.3. An algebraic remark about Tomihisa’s identity. According to [5], there are 2 identities of degree 5 that constitute a base for the identities in the simple real Lie algebras of dimension 3 (The vector space \(\mathbb{R}^3\) and \(sl_2(\mathbb{R})\)): Given 5 elements \(X_0, X_1, X_2, X_3\) and \(X_4\), the following identities hold:

\[
\sum_{s \in S_4} \sigma(s)[X_{s(1)}, [X_{s(2)}, [X_{s(3)}, [X_{s(4)}, X_0]]]] = 0,
\]

where \(\sigma(s)\) is the sign of the permutation \(s\), positive or negative respectively for even or odd permutations; and

\[
[X_0, [X_0, [X_0, [X_1, X_2]]]] - [X_1, [X_0, [X_0, [X_2, X_0]]]] + [X_2, [X_0, [X_0, [X_0, X_1]]]] = 0.
\]
**Proposition 4.2.** Identity (26) follows from the Tomihisa identity.

**Proof.** Write Identity (24) renaming \(F_2\) and \(F_1\), respectively \(X_1\) and \(X_0\), and renaming \(F_1, F_3, F_5\), respectively, \(X_2, X_3, X_4\). The identity itself in the Lie algebra will be denoted by

\[ T(X_0, X_1; X_2, X_3, X_4) = 0. \]

Each term of (24) is of type \([X_i, [[X_j, [X_k, X_0]], X_0]]\), where \((i, j, k)\) is a cyclic permutation of \((2, 3, 4)\). We apply to each term the Jacobi identity in this way:

\[ [X_i, [[X_j, [X_k, X_0]], X_0]] = [X_i, [[X_j, X_k], X_0]] + [X_i, [[X_k, X_0], X_j]]. \]

We obtain:

\[
T(X_0, X_1; X_2, X_3, X_4) = [X_2, [X_3, [X_1, [X_4, X_0]]]] - [X_2, [X_1, [X_3, [X_4, X_0]]]] +

[X_3, [X_4, [X_1, [X_2, X_0]]]] - [X_3, [X_1, [X_4, [X_2, X_0]]]] +

+ [X_4, [X_2, [X_1, [X_3, X_0]]]] - [X_4, [X_1, [X_2, [X_3, X_0]]]].
\]

Observe that in all such terms \(X_0\) occupies the right position of the fourth Lie bracket, in all positive terms the element with fixed index 1 occupies the left position in the third Lie bracket, and in the negative terms it occupies the left position of the second Lie bracket. The other elements \(X_2, X_3, X_4\) are cyclically permuted in the remaining places, so that the sign of the term \([X_i, [[X_j, [X_k, X_0]], X_0]]\) coincides with the sign of the permutation \((i, j, k, l)\) of \((4, 3, 2, 1)\). The analogue terms obtained from the Tomihisa identities \(T(X_0, X_i; X_j, X_k, X_l)\), with fixed indices \(0, i\) for \(i = 2, 3, 4\), and such that \((i, j, k, l)\) is an even permutation of \((4, 3, 2, 1)\), are therefore all different. We obtain:

\[
\sum_{s \in S_4} \sigma(s)[X_{s(1)}, [X_{s(2)}, [X_{s(3)}, X_{s(4)}]]] = T(X_0, X_1; X_2, X_3, X_4) +

+ T(X_0, X_2; X_3, X_1, X_4) + T(X_0, X_3; X_1, X_2, X_4) + T(X_0, X_4; X_2, X_1, X_3).
\]

\[ \square \]

**Remark 14.** In fact, A. Dzhumadil’daev has recently proved [3] that also identity (26) follows from Tomihisa identity, that therefore constitutes itself a base. He provides also in [4] the expression of the Tomihisa identity using the base \(\{25, 26\}\).

**Remark 15.** In the vector algebras we have considered, identity (26) reads as the Jacobi identity multiplied by the square norm of \(X_0\). Its geometrical meaning coincides therefore with that of the Jacobi identity.

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