HEAT-SMOOTHING FOR HOLOMORPHIC SUBALGEBRAS OF FREE GROUP VON NEUMANN ALGEBRAS

HAONAN ZHANG

Abstract. The heat semigroup on discrete hypercubes is well-known to be contractive over $L^p$-spaces for $1 < p < \infty$. A question of Mendel and Naor [MN14] concerns a stronger contraction property in the tail spaces, which is known as the heat-smoothing conjecture. Eskenazis and Ivanisvili [EI20] considered a Gaussian analog of this conjecture and resolved some special cases. In particular, they proved that heat-smoothing type conjecture holds for holomorphic functions in the Gaussian spaces with sharp constants. In this paper, we prove analogous sharp inequalities for holomorphic subalgebras of free group von Neumann algebras. Similar results also hold for $q$-Gaussian algebras and quantum tori. In the case of free group von Neumann algebras, the weaker formulation of heat-smoothing is proved with optimal order.

1. Introduction

For $n \geq 1$, let $\{\pm 1\}^n = \{-1, 1\}^n$ be the $n$-dimensional discrete hypercube. Any function $f : \{\pm 1\}^n \to \mathbb{R}$ admits the Fourier–Walsh expansion

$$f = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S,$$

where for each $S \subseteq [n] := \{1, \ldots, n\}$, $\hat{f}(S) \in \mathbb{R}$ and

$$\chi_S(x) := \prod_{j \in S} x_j, \quad x = (x_1, \ldots, x_n) \in \{\pm 1\}^n.$$

Then the heat semigroup $T_t = e^{-t L}$, $t \geq 0$ is defined as

$$T_t f := \sum_{S \subseteq [n]} e^{-t |S|} \hat{f}(S) \chi_S,$$

or equivalently, $L(\chi_S) = |S| \chi_S$. Here $|S|$ denotes the volume of $S$.

Let $1 \leq p \leq \infty$ and denote by $L_p(d\mu_n)$ the $L_p$-space with respect to the uniform probability measure $\mu_n$ on $\{\pm 1\}^n$. Each $T_t$ is unital positive and $\mu_n(T_t f) = \mu_n(f)$ for all $f$, thus it extends to a contraction over $L_p(d\mu_n)$ for all $1 \leq p \leq \infty$. The so-called heat-smoothing conjecture states that for any $1 < p < \infty$, there exists $c_p > 0$ depending only on $p$ such that for any $n \geq 1$, $1 \leq d \leq n$ and any $f : \{\pm 1\}^n \to \mathbb{R}$ with

$$(1) \quad \hat{f}(S) = 0 \quad \text{whenever} \quad |S| < d,$$

we have

$$\|T_t f\|_{L_p(d\mu_n)} \leq e^{-c_p d t} \|f\|_{L_p(d\mu_n)}, \quad t \geq 0.$$

A weaker formulation is that, for all $f$ satisfying (1) we have

$$\|L f\|_{L_p(d\mu_n)} \geq c_p d \|f\|_{L_p(d\mu_n)}.$$

This question was first asked by Mendel and Naor [MN14] Remark 5.5] in the vector-valued setting, but the scalar case still remains open. Some partial results were known, and we refer to [MN14] [HMO17] [EI20] for more information.
In [EI20] Eskenazis and Ivanisvili considered a Gaussian analog of this conjecture by replacing the heat semigroup $T_t = e^{-tL}$ on the discrete hypercube $\{\pm 1\}^n$, $d\mu_n$) with the Ornstein–Uhlenbeck semigroup $T_t = e^{-tN}$ on the Gaussian space $(\mathbb{R}^n, d\gamma_n)$, where $\gamma_n$ is the standard Gaussian on $\mathbb{R}^n$ and $N = -\Delta + x \cdot \nabla$. The Ornstein–Uhlenbeck semigroup $T_t = e^{-tN}$ is unital positive and preserves the standard Gaussian measure, thus it acts a contraction over polynomials. This allows to define for each $d \geq 1$ the $d$-th tail spaces $\mathcal{P}^{\geq d}$, which is spanned by Hermite polynomials of degree $\geq d$, satisfying a condition similar to (1). Then it is natural to ask the Gaussian analog of heat-smoothing conjecture. By central limit theorem [EI20], the discrete hypercube heat-smoothing implies its Gaussian counterpart. In the Gaussian setting, Eskenazis and Ivanisvili [EI20] resolved the conjecture in some special cases. In particular, they obtained sharp Gaussian heat-smoothing when restricted to holomorphic functions. More precisely, they showed that for any $n \geq 1$, $d \geq 1$ and $1 \leq p < \infty$, we have [EI20] Theorem 3 and Lemma 9

$$\left\| \sum_{|\alpha| \geq d} e^{-t|\alpha|} c_\alpha z^\alpha \right\|_{L_p(\mathbb{R}^n)} \leq e^{-td} \left\| \sum_{|\alpha| \geq d} c_\alpha z^\alpha \right\|_{L_p(\mathbb{R}^n)},$$

and

$$\left\| \sum_{|\alpha| \geq d} |\alpha| c_\alpha z^\alpha \right\|_{L_p(\mathbb{R}^n)} \geq d \left\| \sum_{|\alpha| \geq d} c_\alpha z^\alpha \right\|_{L_p(\mathbb{R}^n)},$$

Here we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $z_j = x_j + i y_j$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$, and we have used the multi-index convention $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. The polynomials $z^\alpha, \alpha \in \mathbb{N}^n$ are identified with the Hermite polynomials via the Segal–Bargmann transform. Then the condition $|\alpha| := \alpha_1 + \cdots + \alpha_n \geq d$ is equivalent to saying that the holomorphic polynomials (finite sum) $f(z) = \sum_{|\alpha| \geq d} c_\alpha z^\alpha$ belong to the $d$-th tail space $\mathcal{P}^{\geq d}$. A similar result [EI20] Lemma 10] holds for holomorphic $f \in \mathcal{P}^{\leq d}$ i.e. $f$ is of degree at most $d$. Eskenazis and Ivanisvili [EI20] Theorem 5] also obtained some moment comparison results, and all these inequalities are sharp.

The proofs of (2) and (3) are very simple, and they also work in the vector-valued setting for all Banach spaces without any modifications. This highlights a big difference between the holomorphic and non-holomorphic settings.

As already seen in the Gaussian setting, the heat-smoothing conjecture can be formulated in a much more general context. In this paper, we shall prove several analogous sharp holomorphic heat-smoothing results in the noncommutative setting, following [EI20]. Our main results hold for more general examples; see Section 3.3 at the end. Here we choose the free group von Neumann algebras, $q$-Gaussian algebras and quantum tori, three representative examples in noncommutative analysis, quantum probability and noncommutative geometry, to present the results and ideas. In the introduction and following, we will formulate and prove the main results for the free group von Neumann algebras in more detail. For the $q$-Gaussian algebras and the quantum tori, we will only briefly summarize the main results and their proof ingredients as they are very similar to the free group von Neumann algebras setting.

**Main results for free group von Neumann algebras.** For $n \geq 1$, we denote $F_n$ the free group on $n$ generators and $\hat{F}_n$ the von Neumann subalgebra of $B(t_2(\hat{F}_n))$ generated by $\lambda(F_n)$. Here $\lambda$ is the left regular representation. Let $\tau$ be the canonical
Proposition 1.3. For any $d \geq 0$, $x \in \mathbb{F}_n$, $t \geq 0$, and holomorphic polynomial $x$ in $\mathbb{F}_n$, we have

\[
\|P_t(x)\|_p \leq e^{-td}\|x\|_p, \quad t \geq 0,
\]

and

\[
\|L(x)\|_p \geq d\|x\|_p;
\]

(ii) if $x$ belongs to the low-degree space $\mathcal{P}^{\leq d}$, then

\[
\|L(x)\|_p \leq d\|x\|_p,
\]

and

\[
\|P_t(x)\|_p \geq e^{-td}\|x\|_p, \quad t \geq 0.
\]

Remark 1.2. Clearly, the inequalities in Theorem 1.1 are sharp, as can be seen by choosing $d$-homogeneous holomorphic polynomials, e.g. $x = \lambda t^n$.

The heat-smoothing is related to the hypercontractivity. On one hand, the latter implies the former for large time $t$ when $p \geq 2$ and the reference measure is finite. This is a standard argument [MNT] Lemma 5.4, and we will repeat it in Section 2.2 for reader’s convenience.

Proposition 1.3. For any $2 \leq p < \infty$, there exists $c_p > 0$ such that for any $d \geq 0$, $x \in \mathcal{P}^{\geq d}$ and $t \geq 0$, we have

\[
\|P_t(x)\|_p \leq e^{-c_p d \min(t, t^2)}\|x\|_p.
\]

Hence for some $c_p' > 0$ we have

\[
\|L(x)\|_p \geq c_p' \sqrt{d}\|x\|_p.
\]

On the other hand, combining hypercontractivity and heat-smoothing, one obtains the following moment comparison estimates:

Proposition 1.4. For holomorphic $x \in \mathcal{P}^{\leq d} \subset \mathbb{F}_n$, we have

\[
\|x\|_q \leq c(p, q, d)\|x\|_p,
\]

for

(i) $1 < p \leq q < \infty$ and $c(p, q, d) = \left(\frac{p}{p-1}\right)^d$;

(ii) $p = 2$, $q \geq q_0 \approx 3.82$, and $c(p, q, d) = (q - 1)^{d/2}$.

In fact, (9) can be improved to $O(d)$ which is the right order, and it is valid for all $1 \leq p \leq \infty$. This is better than the estimates in the classical case and is special for free group von Neumann algebras (compared with the other examples that we are going to discuss). We are indebted to the referee for the following...
Theorem 1.5. For any $1 \leq p \leq \infty$, $d \geq 1$, $x \in \mathcal{D}^d$ and $t \geq 0$, we have
\[ \| P_t(x) \|_p \leq (d + 3) e^{-td} \| x \|_p, \]
and
\[ \| L(x) \|_p \geq \frac{d}{4} \| x \|_p. \]

See the proof of Theorem 1.5 in Sect. 2.2 for better constants in (10) and (11).

Main results for $q$-Gaussian algebras and quantum tori. Our main results for $q$-Gaussian algebras are similar sharp heat-smoothing type inequalities for holomorphic polynomials:

Theorem 1.6. For any $q \in (-1, 1)$, any real Hilbert space $H$ and any $1 \leq p \leq \infty$, we have the same sharp holomorphic inequalities as in Theorem 1.1 for the $q$-Ornstein–Uhlenbeck semigroup $P^q_t = e^{-tN_q}$ over the $q$-Gaussian algebra $\Gamma_q(H \otimes H)$ with adapted conventions introduced in Section 2.3.

For quantum tori, the sharp holomorphic heat-smoothing is still valid, but we do not have moment comparison due to the lack of hypercontractivity estimates. We refer to Section 2.4 for the definitions.

Theorem 1.8. For any $n \geq 1$, any real skewed symmetric $n$-by-$n$ matrix $\theta$ and any $1 \leq p \leq \infty$, we have the same sharp holomorphic inequalities as in Theorem 1.1 for the Poisson semigroup $P^\theta_t = e^{-tL_\theta}$ over the quantum tori $A_\theta$ with adapted conventions introduced in Section 2.4.

We conclude the introduction with the following two remarks. When $d = 1$, then (weak formulation of) heat-smoothing reduces to the $L_p$-spectral gap inequality. This was proved both in the commutative [HMO17] and noncommutative setting [CAPR18].

That many inequalities take stronger forms when restricting to holomorphic functions have already been studied in noncommutative setting. For example, Kemp [Ken08] proved a stronger version of hypercontractivity for $q$-Gaussian algebras (see also Section 2 below) as a noncommutative analog of Janson’s strong hypercontractivity theorem [Jan83]. Kemp and Speicher [KS07] observed that the celebrated Haagerup inequality, which plays a crucial role in many different areas, can be improved when restricting to the holomorphic ($\mathcal{F}$-diagonal) elements.

2. Preliminary

2.1. Noncommutative $L_p$-spaces. As it will be used in different examples, we briefly recall the noncommutative $L_p$-space here for convenience. Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a normal faithful tracial state $\tau$. For any $1 \leq p < \infty$, we define for any $x \in \mathcal{M}$
\[ \| x \|_p^p := \tau(x^* x)^{p/2}. \]
Then $\| \cdot \|_p$ is a norm and the noncommutative $L_p$-space associated with $(\mathcal{M}, \tau)$ is defined as the completion of $(\mathcal{M}, \| \cdot \|_p)$, denoted $L_p(\mathcal{M}, \tau)$ or $L_p(\mathcal{M})$ for short. For $p = \infty$, we define $L_\infty(\mathcal{M})$ as $\mathcal{M}$ equipped with the operator norm $\| \cdot \|_\infty := \| \cdot \|$. Noncommutative $L_p$-spaces share many properties with the classical ones, such as Hölder’s inequality: For $1 \leq p, q, r \leq \infty$ and $\theta \in [0, 1]$ such that $1/r = 1/p + 1/q$,

$$
\| xy \|_r \leq \| x \|_p \| y \|_q, \quad x \in L_p(\mathcal{M}), y \in L_q(\mathcal{M}).
$$

We refer to [PX03] for more detail. In the following, we will use $\| \cdot \|_p$ among different examples of $(\mathcal{M}, \tau)$ as there is no ambiguity.

### 2.2. Free group von Neumann algebras.

Let $\hat{F}_n$ be the free group on $n$ generators $g_j, 1 \leq j \leq n$ with $e$ the unit element. Let $\hat{F}_n$ denote the group von Neumann algebra of $\mathbb{F}_n$, that is, the von Neumann algebra acting on $\ell_2(\mathbb{F}_n)$ generated by $\lambda_g, g \in \mathbb{F}_n$, where $\lambda$ is the left regular representation, i.e. $\lambda_g(f) := f(g^{-1}h), f \in \ell_2(\mathbb{F}_n)$. We use $\tau$ to denote the canonical tracial state on $\hat{F}_n$, that is, $\tau(x) = \langle x\delta_e, \delta_e \rangle$ with $\langle \cdot, \cdot \rangle$ being the usual inner product on $\ell_2(\mathbb{F}_n)$ and $\delta_e$ being the delta function at $e$. All the polynomials $\sum_g x_g \lambda_g, x_g \in \mathbb{C}$, where $\sum_g$ always denotes the finite sum, form a norm-dense *-subalgebra of $\hat{F}_n$ that is enough for our use. In particular,

$$
\tau \left( \sum_g x_g \lambda_g \right) = x_e.
$$

Any element $g \in \mathbb{F}_n$ has a word representation, i.e. it can be uniquely represented as a finite product of $g_i, g_i^{-1}, 1 \leq i, j \leq n$, and it does not contain $g_k g_k^{-1}$ or $g_k^{-1} g_k$. This gives the word length function $| \cdot | : \mathbb{F}_n \to \mathbb{N}$, that is, $| g |$ denotes the number of $g_i, g_i^{-1}, 1 \leq i, j \leq n$ in the word representation of $g$. For example, $| e | = 0$ and $| g_1 g_2^{-1} g_3 | = 4$. A classical result of Haagerup [Haa79, Lemma 1.2] states that for any $t > 0$, $e^{-t| \cdot |}$ is positive semi-definite on $\mathbb{F}_n$. So the semigroup of linear operators $P_t = e^{-tL}, t \geq 0$ over $\hat{F}_n$ given by

$$
P_t \left( \sum_g x_g \lambda_g \right) := \sum_g e^{-t|g|}x_g \lambda_g,
$$

is unital completely positive trace preserving. It extends to a contraction over $L_p(\hat{F}_n)$ for all $1 \leq p \leq \infty$. Note that the generator $L$ acts as

$$
L \left( \sum_g x_g \lambda_g \right) = \sum_g |g| x_g \lambda_g.
$$

For any $d \geq 0$, we denote $\mathcal{P}^{\leq d}$ (resp. $\mathcal{P}^{\geq d}$) the $\mathbb{C}$-linear span of $\{ \lambda_g, |g| \leq d \}$ (resp. $\{ \lambda_g, |g| \geq d \}$). Denote $\mathbb{F}_n^+$ the semigroup generated by the generators $\{ g_1, \ldots, g_n \}$:

$$
\mathbb{F}_n^+ := \{ g \in \mathbb{F}_n : g = g_{j_1}^{k_1} \cdots g_{j_m}^{k_m}, \quad k_i > 0, 1 \leq j_i \leq n, j_i \neq j_{i+1}, m \geq 0 \}.
$$

A polynomial $x \in \hat{F}_n$ is holomorphic if it is of the form $x = \sum_{g \in \mathbb{F}_n} x_g \lambda_g$. All the holomorphic polynomials form a subalgebra of $\hat{F}_n$.

It is well-known that for any $z \in \mathbb{T}$, there exists a *-automorphism $\pi_z$ of $\hat{F}_n$ such that $\pi_z(\lambda(g_j)) = z \lambda(g_j), 1 \leq j \leq n$. Clearly, $\pi_z$ is trace preserving. So it extends to an isometry on $L_p(\hat{F}_n)$: For any $1 \leq p \leq \infty$ and $x \in L_p(\hat{F}_n)$,

$$
\| \pi_z(x) \|_p = \| x \|_p.
$$
Note that for any holomorphic \( x = \sum_{g \in F_n} x_g \lambda_g \), we have
\[
\pi_c(x) = \sum_{g \in F_n} e^{g|\theta|} x_g \lambda_g.
\]

As already mentioned in the introduction, one may derive the heat-smoothing for large time and \( p \geq 2 \) using a standard argument \[MN14\] Lemma 5.4, provided that the reference measure is finite. Although we do not have the optimal hypercontractivity for free group von Neumann algebras, the known results already yield the heat-smoothing with constants \( e^{-c_p d_{\min}(t,t^2)} \) for \( p \geq 2 \) that is of the same asymptotic behavior as in \[MN14\]. The proof is essentially the same as in \[MN14\]. We provide it here for completeness.

**Proof of Proposition 2.3** Compared with the proof of \[MN14\] Lemma 5.4, the only difference is that we use the following hypercontractivity estimates for free group von Neumann algebras over twice the optimal time \[JPP+15\] Theorem A: If \( 1 < p \leq q < \infty \),
\[
\|P_t(x)\|_q \leq \|x\|_p \quad \text{for} \quad t \geq \log \frac{q - 1}{p - 1}.
\]
For \( p \geq 2 \), take \( q = 1 + e^t (p - 1) \geq p \). Since \( x \in P_{\geq d} \) and using Hölder’s inequality \[12\] (recall that \( \tau \) is a state):
\[
\|P_t(x)\|_2 \leq e^{-t d} \|x\|_2 \leq e^{-t d} \|x\|_p, \quad t \geq 0.
\]
This, together with Hölder’s inequality \[12\] and above hypercontractivity estimates, yields
\[
\|P_t(x)\|_p \leq \|P_t(x)\|_2^{\theta} \|P_t(x)\|_2^{1 - \theta} \leq e^{-t d \theta} \|x\|_p, \quad t \geq 0,
\]
where \( \theta \in [0, 1] \) is such that
\[
\frac{1}{p} = \frac{\theta}{2} + \frac{1 - \theta}{q}.
\]
So we have proved
\[
\|P_t(x)\|_p \leq \exp \left( \frac{2(p - 1) dt (e^t - 1)}{p (e^t (p - 1) - 1)} \right) \|x\|_p, \quad 2 \leq p < \infty.
\]
With this, we see that \[3\] holds with \( c_p = 2/p \). In fact, this follows from
\[
\min \{t, t^2\} \leq t \leq \frac{t (e^t - 1)}{e^t - (p - 1)} = \frac{(p - 1) t (e^t - 1)}{e^t (p - 1) - 1},
\]
where we have used the fact that \( p \geq 2 \). As argued in \[MN14\] Proof of Lemma 5.4, \[4\] follows immediately from \[3\]. \[ \square \]

**2.3. q-Gaussian algebras.** Our references for this part are \[BS91, BS94, BKS97, LP99, KS07\]. Fix \( q \in [-1, 1] \). Let \( H \) be a real Hilbert space and \( H_\mathbb{C} \) the complexification with \( \langle \cdot, \cdot \rangle \) the inner product that is linear in the second argument. The algebraic Fock space \( \mathcal{F}(H) \) is
\[
\mathcal{F}(H) := \bigoplus_{k=0}^{\infty} H_\mathbb{C}^k,
\]
that is, \( \mathcal{F}(H) \) is the linear span of vectors of the form \( \xi_1 \otimes \cdots \otimes \xi_k \in H_\mathbb{C}^k, k \in \mathbb{N} \), where \( H_\mathbb{C}^0 := \mathbb{C} \Omega \). Here \( \Omega \) is some unit vector known as the vacuum vector. We define a Hermitian form \( \langle \cdot, \cdot \rangle_q \) on \( \mathcal{F}(H) \) given by
\[
\langle \Omega, \Omega \rangle_q := 1
\]
and
\[
\langle \xi_1 \otimes \cdots \otimes \xi_j, \eta_1 \otimes \cdots \otimes \eta_k \rangle_q := \delta_{j,k} \sum_{\sigma \in S_k} q^{|\sigma|} \langle \xi_1, \eta_{\sigma(1)} \rangle \cdots \langle \xi_k, \eta_{\sigma(k)} \rangle,
\]
When \( q \) each \( \langle \cdot, \cdot \rangle_q \) is positive semi-definite for all \( q \in [-1, 1] \), and strictly positive if \( q \in (-1, 1) \). In the latter case, it defines an inner product on \( \mathcal{F}(H) \) for \( q \in (-1, 1) \) and we denote by \( \mathcal{F}_q(H) \) the Hilbert space obtained by taking completion of \( \mathcal{F}(H) \) with respect to \( \langle \cdot, \cdot \rangle_q \). At the endpoint cases \( q = \pm 1 \), we need to take the quotient of \( \mathcal{F}(H) \) by the kernel of \( \langle \cdot, \cdot \rangle_q \) first. Then after completion, we obtain a Hilbert space that is also denoted by \( \mathcal{F}_q(H) \) for \( q = \pm 1 \). To avoid technical issues regarding \( q = \pm 1 \), we assume that \(-1 < q < 1 \) in the following.

Now we define the \( q \)-Gaussian algebras \( \Gamma_q(H) \) acting on the Fock space \( \mathcal{F}_q(H) \).

For any vector \( \xi \in H \), the creation operator \( c_q(\xi) \) is a linear operator over \( \mathcal{F}_q(H) \) defined via
\[
c_q(\xi)(\Omega) := \xi, \tag{1}
\]
and
\[
c_q(\xi) (\xi_1 \otimes \cdots \otimes \xi_k) := \xi \otimes \xi_1 \otimes \cdots \otimes \xi_k. \tag{2}
\]
The annihilation operator \( c_q^*(\xi) \) is the adjoint of the creation operator \( c_q(\xi) \) with respect to \( \langle \cdot, \cdot \rangle_q \), or equivalently,
\[
c_q^*(\xi)(\Omega) = 0,
\]
and
\[
c_q^*(\xi) (\xi_1 \otimes \cdots \otimes \xi_k) = \sum_{j=1}^k q^{-j-1} \langle \xi, \xi_j \rangle \xi_1 \otimes \cdots \otimes \xi_{j-1} \otimes \xi_{j+1} \otimes \cdots \otimes \xi_k. \tag{3}
\]
The creation and annihilation operators satisfy the \( q \)-commutation relations:
\[
c_q^*(\xi)c_q(\eta) - qc_q(\eta)c_q^*(\xi) = \langle \xi, \eta \rangle_q \text{id}. \tag{4}
\]
When \( q \in (-1, 1) \), the creation and the annihilation operators are bounded. For each \( \xi \in H \), denote \( X_q(\xi) := c_q(\xi) + c_q^*(\xi) \). Note that \( \xi \mapsto X_q(\xi) \) is \( \mathbb{R} \)-linear. The \( q \)-Gaussian algebra \( \Gamma_q(H) \) is the von Neumann subalgebra of \( B(\mathcal{F}_q(H)) \) generated by these self-adjoint operators \( X_q(\xi), \xi \in H \). We use \( \tau_q \) to denote the vacuum state given by \( \tau_q(A) := \langle A\Omega, \Omega \rangle_q \). It restricts to a faithful tracial state on \( \Gamma_q(H) \).

Any contraction \( T \) over \( H \) assigns to a bounded linear map \( \Gamma_q(T) \), the second quantization, over \( \Gamma_q(H) \) such that \( \Gamma_q(T)X_q(\xi) = X_q(T\xi) \) for all \( \xi \in H \). It is unital completely positive and trace preserving. If \( T \) is isometric, then \( \Gamma_q(T) \) is a faithful homomorphism. If \( S \) is another contraction over \( H \), then \( \Gamma_q(ST) = \Gamma_q(S)\Gamma_q(T) \). The \( q \)-Ornstein–Uhlenbeck semigroup \( P^q_t = e^{-tN_q} \) over \( \Gamma_q(H) \) is the second quantization of \( e^{-t\text{id}_H} \), i.e. \( P^q_1 = \Gamma_q(e^{-t\text{id}_H}) \), so it extends to a contraction over \( L_p(\Gamma_q(H), \tau_q) \) for \( 1 \leq p \leq \infty \). Moreover, it is hypercontractive:

**Theorem 2.1.** [Bia97] Theorem 3] For \( q \in (-1, 1) \) and \( 1 < p \leq r < \infty \), we have
\[
\| P^q_t(x) \|_r \leq \| x \|_p \quad \text{iff} \quad t \geq \frac{1}{2} \log \frac{r - 1}{p - 1}.
\]

Fix \( H \) a real Hilbert space as above. We now consider the \( q \)-Gaussian algebra \( \Gamma_q(H \oplus H) \), where \( H \oplus H \) denotes the \( \ell_2 \)-direct sum of \( H \). For any \( \xi \in H \), consider the vectors \( (\xi, 0) \), \((0, \xi) \) in \( H \oplus H \) and the associated \( X_q \)-operators \( X_q(\xi, 0), X_q(0, \xi) \in \Gamma_q(H \oplus H) \). Put
\[
Z_q(\xi) := \frac{1}{\sqrt{2}} (X_q(\xi, 0) + iX_q(0, \xi)).
\]
This way, we defined a holomorphic variable as \( z = (x + iy)/\sqrt{2} \) in the classical setting. Here the factor of \( \sqrt{2} \) is just to make \( Z_q(\xi) \) a unit vector in \( L_2(\Gamma_q(H \oplus H), \tau_q) \), as did in [Kem05]. The algebra generated by these \( Z_q(\xi) \), \( \xi \in H \) is the \( q \)-holomorphic algebra (actually Kemp considered the Banach algebra in [Kem05]).
Suppose that \((e_j)\) is an orthonormal basis of \(H\). Any element in the \(q\)-holomorphic algebra, which we shall call \textit{holomorphic polynomial}, can be written as the linear combination of

\[
Z_q(e_{j_1})^{n_1} \cdots Z_q(e_{j_k})^{n_k}, \quad n_\ell \geq 0, j_\ell \neq j_{\ell+1}, 1 \leq \ell \leq k-1, k \geq 1.
\]

Each of the above basis is an eigenvector of \(N_q\) with eigenvalue \(n_1 + \cdots + n_k\):

\[
(17) \quad N_q(\sum_{\ell=1}^k Z_q(e_{j_\ell})^{n_\ell}) = (n_1 + \cdots + n_k) Z_q(e_{j_1})^{n_1} \cdots Z_q(e_{j_k})^{n_k},
\]

and thus

\[
(18) \quad P_t^q(\sum_{\ell=1}^k Z_q(e_{j_\ell})^{n_\ell}) = e^{-t(n_1 + \cdots + n_k)} Z_q(e_{j_1})^{n_1} \cdots Z_q(e_{j_k})^{n_k}.
\]

When restricting to the \(q\)-holomorphic algebra, Kemp \cite{Kemp2015} Theorem 1.4 proved some stronger hypercontractivity estimates as a noncommutative analog of Janson’s strong hypercontractivity theorem \cite[Theorem 11]{Janson1983}.

\textbf{Theorem 2.2.} \cite{Kemp2015} \textbf{Theorem 1.4} Let \(H\) be a real Hilbert space. For \(q \in [-1, 1)\) and \(r\) an positive even integer, we have for all holomorphic polynomial \(x \in \Gamma_q(H \oplus H)\):

\[
\|P_t^q(x)\|_r \leq \|x\|_2 \quad \text{iff} \quad t \geq \frac{1}{2} \log \frac{r}{2}.
\]

Since we have (optimal) hypercontractivity for \(q\)-Ornstein–Uhlenbeck semigroup \(P_t^q = e^{-t N_q}\) \cite[Theorem 2.1]{Kemp2015} we have a general heat-smoothing estimate analogous to Proposition \ref{prop:heat-smoothing} which will not be stated here.

We end this subsection on \(q\)-Gaussian algebras with the following crucial rotation-invariance property. Fix any \(z = e^{2\pi i \theta} \in \mathbb{T}\) with \(\theta \in [0, 2\pi]\). Denote by \(U_\theta\) the following unitary operator

\[
U_\theta := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \otimes \text{id}_H
\]

over \(H \oplus H\). The second quantization of these \(U_\theta, \theta \in [0, 2\pi]\) induces a semigroup \((i.e. \pi_\circ \pi_{\theta} = \pi_{\theta^2})\) \(\pi_z := \Gamma_q(U_\theta)\) of \(*\)-automorphisms of \(\Gamma_q(H \oplus H)\). They are trace preserving and extend to isometries over \(L_p(\Gamma_q(H \oplus H))\): For all \(x \in L_p(\Gamma_q(H \oplus H))\) and all \(1 \leq p \leq \infty\),

\[
(19) \quad \|\pi_z(x)\|_p = \|x\|_p.
\]

Since \(\pi_z(X_q(\eta)) = X_q(U_\theta \eta)\) for \(\eta \in H \oplus H\), we have for any \(\xi \in H\)

\[
\pi_z(X_q(\xi, 0)) = \cos \theta X_q(\xi, 0) - \sin \theta X_q(0, \xi),
\]

\[
\pi_z(X_q(0, \xi)) = \sin \theta X_q(\xi, 0) + \cos \theta X_q(0, \xi),
\]

and thus

\[
\pi_z(X_q(\xi, 0) + i X_q(0, \xi)) = (\cos \theta + i \sin \theta)(X_q(\xi, 0) + i X_q(0, \xi)).
\]

Hence for any \(z \in \mathbb{T}\) and any \(\xi \in H\) we have

\[
\pi_z(Z_q(\xi)) = z Z_q(\xi),
\]

and on the basis of \(q\)-holomorphic algebra

\[
(20) \quad \pi_z(\sum_{\ell=1}^k Z_q(e_{j_\ell})^{n_\ell}) = z^{n_1 + \cdots + n_k} Z_q(e_{j_1})^{n_1} \cdots Z_q(e_{j_k})^{n_k}.
\]
2.4. Quantum tori. Let \( n \geq 2 \) and \( \theta = (\theta_{jk})_{j,k=1}^n \) be a real skew-symmetric matrix. The quantum tori \( \mathcal{A}_\theta \) is the universal \( C^* \)-algebra generated by \( n \) unitary operators \( U_j, 1 \leq j \leq n \) satisfying
\[
U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j, \quad 1 \leq j, k \leq n.
\]
For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \), we denote
\[
U^\alpha := U_1^{\alpha_1} \cdots U_n^{\alpha_n}.
\]
Then the polynomials (finite sum here)
\[
\sum_{\alpha \in \mathbb{Z}^n} x_\alpha U^\alpha, \quad x_\alpha \in \mathbb{C},
\]
form a dense \(*\)-subalgebra of \( \mathcal{A}_\theta \). The functional
\[
\tau_\theta \left( \sum_{\alpha \in \mathbb{Z}^n} x_\alpha U^\alpha \right) := x_0,
\]
on polynomials extends to a faithful tracial state, which we still denote by \( \tau_\theta \), on \( \mathcal{A}_\theta \). Let \( \mathbb{T}_\theta \) be the von Neumann algebra arising from the GNS construction with respect to \( (\mathcal{A}_\theta, \tau_\theta) \). When \( \theta = 0 \), one recovers the classical \( n \)-dimensional tori, i.e. \( \mathcal{A}_0 \cong C(\mathbb{T}^n) \) and \( \mathbb{T}_0 \cong L_\infty(\mathbb{T}^n) \). We refer to [CXY13] for more discussion.

We shall work with the Poisson type semigroup \( P^\theta_t = e^{-tL_\theta} \) given by
\[
P^\theta_t \left( \sum_{\alpha \in \mathbb{Z}^n} x_\alpha U^\alpha \right) := \sum_{\alpha \in \mathbb{Z}^n} e^{-t|\alpha|} x_\alpha U^\alpha,
\]
where \( |\alpha| := \sum_{j=1}^n |\alpha_j| \). Its generator \( L_\theta \) acts as
\[
L_\theta \left( \sum_{\alpha \in \mathbb{Z}^n} x_\alpha U^\alpha \right) = \sum_{\alpha \in \mathbb{Z}^n} |\alpha| x_\alpha U^\alpha.
\]
Each \( P^\theta_t \) extends to a unital completely positive trace preserving map over \( \mathbb{T}_\theta \), as can be proven using the transference in [CXY13], then a contraction over \( L_p(\mathbb{T}_\theta), 1 \leq p \leq \infty \).

For each \( d \geq 1 \), we denote by \( \mathcal{P}^{\leq d} \) (resp. \( \mathcal{P}^{\geq d} \)) the \( \mathbb{C} \)-linear span of all \( U^\alpha \), \( |\alpha| \leq d \) (resp. \( |\alpha| \geq d \)). For any \( z \in \mathbb{T} \), there exists a \(*\)-automorphism \( \pi_z \) of \( \mathbb{T}_\theta \) such that
\[
\pi_z(U_j) = z U_j, \quad 1 \leq j \leq n.
\]
Clearly, it is also trace preserving. So it extends to an isometry over \( L_p(\mathbb{T}_\theta) \): For \( 1 \leq p \leq \infty \) and \( x \in L_p(\mathbb{T}_\theta) \),
\[
\|\pi_z(x)\|_p = \|x\|_p.
\]
A polynomial is holomorphic if it is of the form \( x = \sum_{\alpha \in \mathbb{N}^n} x_\alpha U^\alpha \), for which
\[
\pi_z(x) = \sum_{\alpha \in \mathbb{N}^n} z^{\alpha_1 + \cdots + \alpha_n} x_\alpha U^\alpha = \sum_{\alpha \in \mathbb{N}^n} z^{\alpha} x_\alpha U^\alpha.
\]

3. Proof of main results

This section is devoted to the proof of main results. We first give the proof of free group von Neumann algebra case in detail. Then we illustrate that similar arguments apply to the \( q \)-Gaussian algebras and quantum tori with minor modifications.
3.1. Proof of main results: free group von Neumann algebra case. We first prove Theorem 1.1. As in the proof in the classical Gaussian setting [EI20], there are two main ingredients. One is the rotation invariance property [13]. The other one is the following well-known lemma used in [EI20]. Let $C(\mathbb{T})$ denote the set of continuous functions on the torus $\mathbb{T}$. For any complex Borel measure $\mu$ on $\mathbb{T}$ we use $\|\mu\|$ to denote its total variation norm.

**Lemma 3.1.** [EI20] Fix $d \geq 1$. Then

(i) for any $m \geq d$ and $t \geq 0$, there exists a complex Borel measure $\mu = \mu_{d,m,t}$ on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} z^k d\mu(z) = e^{-tk}, \quad d \leq k \leq m, \quad \text{and} \quad \|\mu\| \leq e^{-td};$$

(ii) there exists a complex Borel measure $\nu = \nu_d$ on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} z^k d\nu(z) = k, \quad 0 \leq k \leq d, \quad \text{and} \quad \|\nu\| \leq d.$$

**Proof.** See the proof of [EI20] Lemma 9 & 10].

**Remark 3.2.** We are grateful to the referee who pointed out to us that (i) in the above lemma also holds for $m = \infty$. To see this, just consider the measure $d\mu(z) = f(z)dz$ with $f(z) = e^{-tdz-d\sum_{n \in \mathbb{Z}} e^{-tn}|z^n}$ a multiple of shifted Poisson kernel. Similarly, (ii) follows from by choosing $d\nu(z) = g(z)dz$ with $g(z) = z^{-d}\sum_{j=0}^{d-1} \sum_{i=-j}^{j} z^4$ a multiple of shifted Fejér kernel.

**Proof of Theorem 1.1.** Let us prove (i) first. Suppose that $x = \sum_{|g| \leq d} x_g \lambda_g$ is holomorphic for some $m \geq d$. Fix $t \geq 0$, and let $\mu = \mu_{d,m,t}$ be the measure in Lemma 3.1 (i). Then by (15) and (16)

$$P_t(x) = \sum_{|g| = d} \frac{e^{-t|g|}}{x_g \lambda_g} = \int_{\mathbb{T}} \sum_{|g| = d} \frac{z^{|g|} x_g \lambda_g}{\mu(\pi_g(x))} d\mu(z) = \int_{\mathbb{T}} \pi_z(x) d\mu(z).$$

This, together with the triangular inequality, (15) and the definition of $\mu$, yields

$$\|P_t(x)\|_p \leq \int_{\mathbb{T}} \|\pi_z(x)\|_p d\mu(\pi(z)) = \int_{\mathbb{T}} \|x\|_p d\mu(\pi(z)) \leq e^{-td}\|x\|_p.$$

This finishes the proof of (i). (ii) is an immediate consequence of (i) (note that $L(x)$ is also holomorphic and belongs to $\mathcal{P}^{\leq d}$):

$$\|x\|_p = \left\| \int_0^\infty P_t L(x) dt \right\|_p \leq \int_0^\infty \|P_t L(x)\|_p dt \leq \int_0^\infty e^{-td} dt \|L(x)\|_p = d^{-1}\|L(x)\|_p.$$

Now we prove (ii). Recall that $x = \sum_{|g| \leq d} x_g \lambda_g$ is holomorphic. Let $\nu = \nu_d$ be the measure in Lemma 3.1 (ii). Then by (15) and (16),

$$L(x) = \sum_{|g| \leq d} |g| x_g \lambda_g = \int_{\mathbb{T}} \sum_{|g| \leq d} z^{|g|} x_g \lambda_g d\nu(z) = \int_{\mathbb{T}} \pi_z(x) d\nu(z).$$

This, together with the triangular inequality, (15) and the definition of $\nu$, implies

$$\|L(x)\|_p \leq \int_{\mathbb{T}} \|\pi_z(x)\|_p d\nu(\pi(z)) = \int_{\mathbb{T}} \|x\|_p d\nu(\pi(z)) \leq d\|x\|_p.$$

This finishes the proof of (ii). By (ii), for any $m \geq 1$:

$$\|L^m(x)\|_p \leq d^m\|x\|_p.$$
and the proof of (7) is complete. \hfill \square

Remark 3.3. We would like to thank the referee who pointed out that (6) follows from the vector-valued Bernstein’s inequality. In fact, for holomorphic polynomial $x = \sum_{0 \leq k \leq d} x_k$ with $L(x_k) = kx_k$, consider

$$P(z) := \sum_{0 \leq k \leq d} z^k x_k = \pi_z(x)$$

with

$$P'(z) := \sum_{1 \leq k \leq d} z^{k-1} kx_k = \frac{1}{2} \pi_z(L(x)), \quad z \neq 0.$$ 

Note that (15) for $1 \leq p \leq \infty$ and $|z| = 1$:

$$\|P(z)\|_p \equiv \|x\|_p \quad \text{and} \quad \|P'(z)\|_p \equiv \|L(x)\|_p,$$

we have by vector-valued Bernstein’s inequality that

$$\|L(x)\|_p = \sup_{|z|=1} \|P'(z)\|_p \leq d \sup_{|z|=1} \|P(z)\|_p \leq d\|x\|_p.$$

Proposition 1.4 follows from the same argument in [EI20].

Proof of Proposition 1.4. Fix holomorphic $x \in \mathcal{P}^d$. Note that following the same argument in the proof of Theorem 1.1 (ii), one has

$$\|P_{-t}(x)\|_q \leq e^{ctd}\|x\|_q, \quad t > 0,$$

where the definition of $P_{-t}$ is clear over $\mathcal{P}^d$. Suppose that for such $x$, we have

(27) $$\|P_t(x)\|_q \leq \|P_{-t}(x)\|_q \leq e^{ctd}\|P_t(x)\|_q \leq e^{ctd}\|x\|_p.$$ 

According to [JPP15 Theorem A ii]), the hypercontractivity (27) holds whenever $1 < p \leq q < \infty$ and $e^t \geq \frac{q}{p-1}$, which proves (i). By [RX16 Theorem 10] and its proof, the hypercontractivity (27) is valid for $p = 2$, $q \geq 4 - \epsilon_0 \approx 3.82$ and $e^{ct} \geq q-1$. This shows (ii) and finishes the proof. \hfill \square

Now we turn to the proof of Theorem 1.5 which comes from the referee.

Proof of Theorem 1.5. Let $m = (m_k)_{k \geq 0} \subset \mathbb{C}$ be a sequence that vanishes at $\infty$. By a result of Haagerup, Streenstrup and Szwarc [HSS10 Theorem 4.2, Lemma 1.10], the radial Fourier multiplier $T_m$ on $\mathbb{F}_n$ 

$$T_m : \lambda \mapsto m|\lambda| \lambda,$$

is completely bounded if and only if the matrix $H_m = [m_{i+j} - m_{i+j+2}]_{i,j \geq 0}$ is of trace class. Moreover (actually the left hand side can be replaced by the cb norm),

$$\|T_m : \mathbb{F}_n \rightarrow \mathbb{F}_n\| \leq \|H_m\|_{1}.$$ 

Now for each $d \geq 1$ and $r \in (0, 1)$ we take the sequence $m = m(d, r)$ as follows. When $k \geq d$, we choose $m_k = r^k$, so that 

$$m_k - m_{k+2} = r^k(1 - r^2), \quad k \geq d.$$

When $k < d$, we choose $m_k$ such that 

$$m_k - m_{k+2} = r^{2d-k}(1 - r^2), \quad 0 \leq k < d.$$ 

Clearly, $\lim_{k \to \infty} m_k = 0$. To estimate $\|H_m\|_1$, we divide $H_m$ into three parts:

$$H_m = p_d H_m + p_d H_m(1-p_d) + (1-p_d)H_m = A + B + C,$$
where \( p_d := \sum_{k=0}^{d} c_{i,k} \) is the projection over the first \( d+1 \) entries.

Note that for \( \alpha = (\alpha_i)_{i \geq 0} \) and \( \beta = (\beta_j)_{j \geq 0} \), the trace norm of \( [\alpha \beta]_{i,j \geq 0} \) is \( \|\alpha\|_2 \|\beta\|_2 \). So for \( B = [r^{i+j} (1-r^2)]_{0 \leq i,j \leq d} \) we have \( \|B\|_1 = r^{d+1} \sqrt{1-r^{2(d+1)}} \leq r^{d+1} \), and for \( C = [r^{i+j} (1-r^2)]_{i \geq d, j \geq 0} \), we have \( \|C\|_1 = r^{d+1} \).

By definition, \( A = [a_{ij}]_{0 \leq i,j \leq d} \) where
\[
a_{ij} = \begin{cases} r^{2d-(i+j)} (1-r^2) & i + j \leq d \\ r^{i+j} (1-r^2) & d < i + j \leq 2d \end{cases}
\]
or equivalently, \( a_{ij} = (1-r^2)^{r_d} \cdot r^{i+j-d} \). Now for each \( 0 \leq j \leq \lfloor d/2 \rfloor \), we permute column \( j \) and column \( d-j \) to obtain a matrix \( \hat{A} = AU = [\hat{a}_{ij}]_{0 \leq i,j \leq d} \) for some unitary \( U \), with \( \hat{a}_{ij} = (1-r^2)^{r_d} \cdot r^{i+j} \). Since the kernel \( (i,j) \mapsto |i-j| \) is conditionally negative definite (see e.g. [WW12 Example 3.5] or [Han79]), \( \hat{A} = [\hat{a}_{ij}] \) is positive semi-definite by Schoenberg’s theorem [Sch38]. So \( \|A\|_1 = \|\hat{A}\|_1 = \text{Tr}[\hat{A}] = (d+1)(1-r^2)^{r_d} \).

All combined, we get
\[
\|T_m : \mathcal{F}_n \to \mathcal{F}_\infty \| \leq \|H_m\|_1 \leq \|A\|_1 + \|B\|_1 + \|C\|_1 \leq (d+1)(1-r^2)^{r_d} + 2r^{d+1}.
\]
As a consequence, for any \( t > 0 \) and for any \( x \in P^{\geq d} \), we have (choosing \( r = e^{-t} \))
\[
\|P_t(x)\| \leq \|T_m (x)\| \leq [(d+1)(1-e^{-2t})e^{-td} + 2e^{-t(d+1)}] \|x\|.
\]
A rough estimate
\[
(d+1)(1-e^{-2t})e^{-td} + 2e^{-t(d+1)} \leq (d+1)e^{-td} + 2e^{-td} = (d+3)e^{-td},
\]
gives (10) for \( p = \infty \). By (30),
\[
\|x\| \leq \int_0^\infty \|P_t L(x)\| dt \leq \int_0^\infty [(d+1)(1-e^{-2t})e^{-td} + 2e^{-t(d+1)}] dt \|L(x)\| = \left[ (d+1) \left( \frac{1}{d} - \frac{1}{d+2} \right) + \frac{2}{d+1} \right] \|L(x)\| \leq \frac{4}{d} \|L(x)\|,
\]
which proves (11) for \( p = \infty \). Since (29) holds on the whole von Neumann algebra \( \mathcal{F}_n \) and \( T_m \) is self-adjoint, one obtains by duality and complex interpolation that for all \( 1 \leq p \leq \infty \):
\[
\|T_m : L_p(\mathcal{F}_n) \to L_p(\mathcal{F}_\infty)\| \leq (d+1)(1-r^2)^{r_d} + 2r^{d+1}.
\]
Then following a similar argument we get (10) and (11) for general \( 1 \leq p \leq \infty \).

### 3.2. Proofs for \( \varphi \)-Gaussian algebras and quantum tori

The proofs of Theorems 1.6 and 1.8 are essentially the same as that of Theorem 1.1.

**Proof of Theorems 1.6 and 1.8** To prove (i) in Theorems 1.6 and 1.8, note first that we still have an average identity similar to (25) by replacing (13) & (15) with (18) & (20), and (21) & (24) respectively. Then one can finish the proof using the rotation invariance properties (19) and (23), respectively, instead of (15).

The proof of (ii) can be adapted easily as we did for (i), by replacing (14) & (15) & (16) with (17) & (19) & (20), and (22) & (23) & (24), respectively. □

The proof of Proposition 1.4 is similar to that of Proposition 1.4.

**Proof.** The proof follows the same line as in the proof of Proposition 1.4. The only difference is to use the holomorphic heat-smoothing estimates in Theorem 1.6 and hypercontractivity results Theorems 2.1 and 2.2. □
3.3. **Remarks.** The main results of sharp holomorphic heat-smoothing hold as long as one has the rotation invariance property [15] and nice holomorphic structure that gives the average identities [25] and [26]. Here we treated $q$-Gaussian algebras for $q \in (-1, 1)$, but similar results hold for $q = \pm 1$ as well. When $q = 1$, it corresponds to the classical Gaussian setting, so the work in [EI20].

When $q = -1$, it gives the CAR algebra. By embedding $L_\infty(\{\pm 1\}^n, d\mu_n)$ into $2^n$-by-$2^n$ matrix algebra, Ben Efraim and Lust-Piquard [ELP08] proved many inequalities, such as $L_\infty$-Poincaré, over the discrete hypercubes using rotation on matrix algebras. The rotation technique is similar to what we used here, so they also obtained inequalities for CAR algebras, in parallel to discrete hypercubes. Having holomorphic heat-smoothing for CAR algebras, one may ask if we can obtain heat-smoothing for certain ("holomorphic") functions on discrete hypercubes using the idea of Ben Efraim and Lust-Piquard. Unfortunately, although we have nice rotation operators in [ELP08] Section 2] and we may derive inequalities for the holomorphic variable $Z_j := (Q_j + iP_j)/\sqrt{2}$, the operators $Z_j$’s, unlike $Q_j$’s, do not correspond to functions on the discrete hypercubes.

**Acknowledgement.** I am grateful to Alexandros Eskenazis and Paata Ivanisvili for helpful comments on an earlier version of the paper. I would like to thank the referee for valuable comments and remarks that improve the paper substantially, especially for Theorem [15]. Part of the work was finished during visits to LMB, Besançon, and IMPAN, Warsaw. I want to thank both places, and Professor Quanhua Xu and Adam Skalski for their warm hospitality. The research is supported by the Lise Meitner fellowship, Austrian Science Fund (FWF) M3337.

**References**

[Bia97] Philippe Biane. Free hypercontractivity. *Comm. Math. Phys.*, 184(2):457–474, 1997.

[BKS97] Marek Bożejko, Burkhard Kümmerer, and Roland Speicher. $q$-Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.*, 185(1):129–154, 1997.

[BS91] Marek Bożejko and Roland Speicher. An example of a generalized Brownian motion. *Comm. Math. Phys.*, 137(3):519–531, 1991.

[BS94] Marek Bożejko and Roland Speicher. Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. *Math. Ann.*, 300(1):97–120, 1994.

[CAPR18] José M. Conde-Alonso, Javier Parect, and Éric Ricard. On spectral gaps of Markov maps. *Israel J. Math.*, 226(1):189–203, 2018.

[CXY13] Zepian Chen, Quanhua Xu, and Zhi Yin. Harmonic analysis on quantum tori. *Comm. Math. Phys.*, 322(3):755–805, 2013.

[EL20] Alexandros Eskenazis and Paata Ivanisvili. Sharp growth of the Ornstein-Uhlenbeck operator on Gaussian tail spaces. to appear in *Isr. J. Math.*, page arXiv:2011.01359, November 2020.

[ELP08] Limor Ben Efraim and Françoise Lust-Piquard. Poincaré type inequalities on the discrete cube and in the CAR algebra. *Probab. Theory Related Fields*, 141(3-4):569–602, 2008.

[Haa79] Uffe Haagerup. An example of a nonnuclear $C^*$-algebra, which has the metric approximation property. *Invent. Math.*, 50(3):279–293, 1978/79.

[HMO17] Steven Heilman, Elchanan Mossel, and Krzysztof Oleszkiewicz. Strong contraction and influences in tail spaces. *Trans. Amer. Math. Soc.*, 369(7):4843–4863, 2017.

[HSS10] Uffe Haagerup, Troels Steenstrup, and Ryszard Szwarc. Schur multipliers and spherical functions on homogeneous trees. *International Journal of Mathematics*, 21(10):1337–1382, 2010.

[Jan83] Svante Janson. On hypercontractivity for multipliers on orthogonal polynomials. *Ark. Mat.*, 21(1):97–110, 1983.

[JPP+15] Marius Junge, Carlos Palazuelos, Javier Parect, Mathilde Perrin, and Éric Ricard. Hypercontractivity for free products. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):861–889, 2015.

[Ken05] Todd Kemp. Hypercontractivity in non-commutative holomorphic spaces. *Comm. Math. Phys.*, 259(3):615–637, 2005.
Todd Kemp and Roland Speicher. Strong Haagerup inequalities for free $\mathcal{B}$-diagonal elements. *J. Funct. Anal.*, 251(1):141–173, 2007.

Françoise Lust-Piquard. Riesz transforms on deformed Fock spaces. *Comm. Math. Phys.*, 205(3):519–549, 1999.

Manor Mendel and Assaf Naor. Nonlinear spectral calculus and super-expanders. *Publ. Math. Inst. Hautes Études Sci.*, 119:1–95, 2014.

Gilles Pisier and Quanhua Xu. Non-commutative $L^p$-spaces. In *Handbook of the geometry of Banach spaces*, Vol. 2, pages 1459–1517. North-Holland, Amsterdam, 2003.

Éric Ricard and Quanhua Xu. A noncommutative martingale convexity inequality. *Ann. Probab.*, 44(2):867–882, 2016.

Isaac J Schoenberg. Metric spaces and positive definite functions. *Transactions of the American Mathematical Society*, 44(3):522–536, 1938.

James Howard Wells and Lynn R Williams. *Embeddings and extensions in analysis*, volume 84. Springer Science & Business Media, 2012.

Institute of Science and Technology Austria (IST Austria), Am Campus 1, 3400 Klosterneuburg, Austria

*Email address*: haonan.zhang@ist.ac.at