Modeling of the effective universal constitutive relations for elastic laminated composites with finite strains

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Abstract. The article considers the modeling results of layered composites with finite strains deformation according to the individual layers characteristics. The article proposes an asymptotic averaging method version for layered composites with finite deformations and periodic structure. The method allows calculating the effective deformation diagrams connecting the averaged Piola-Kirchhoff stress tensors components and the strain gradient.

Introduction
Composite materials, consisting of rubber-like or elastomeric matrices reinforced with fibers, dispersed particles or fabric fillers [1-3], are actively used in various industries. They are of considerable interest because they have successful combinations of properties, in particular, relatively high strength and a sufficiently large limiting deformation of destruction, due to the ability of rubbers to deform without failure in the region of large deformations. The calculation of the exact effective elastic characteristics of such composites is a rather complicated task due to the strong physical nonlinearity of the mechanical behavior of the phases of the composite, as well as the geometric nonlinearity of the problem [2, 4, 5]. The method of asymptotic homogenization is quite well developed; the numerical simulation of the micromechanics of composites has been successfully implemented, but mainly for linear problems [6, 7].

Asymptotic expansions for laminated elastic composites with finite strains
Consider a heterogeneous elastic solid medium \( V \) with finite deformations, which in the reference configuration \( K \) has a periodic structure and for it can be selected a repeating element - a periodicity cell (PC) \( V^\varepsilon \), which consists of \( N \)-component, \( \alpha = 1, \ldots, N \). The Euler coordinates of each material point in the reference and actual configurations are denoted as \( x^k \) and \( x^\varepsilon \), respectively, and Lagrangian coordinates are denoted as \( X^\iota \). The latter are assumed to coincide with rectangular coordinates, i.e. \( X^\iota = x^\iota \).

For a given inhomogeneous medium, we consider the problem of the nonlinear theory of elasticity in the Lagrangian description using the universal models \( A_n \) proposed by Yu.I. Dimitrienko, for medium with finite deformations [8,9]:

\[
\nabla^i P^i + \rho f^i = 0, \quad x^\iota \in V,
\]

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\[ P^j = \mathcal{F}^{(0)}(F^{kl}, X^m), \quad X^i \in V \cup \Sigma, \] 

\[ F^{kl}_i = \delta^{ik} + \tilde{V}_i u^k, \quad X^i \in V \cup \Sigma, \]

\[ n_i [P^j] = 0, \quad [u^i] = 0, \quad X^i \in \Sigma_{\alpha}, \]

\[ n_i P^j = t^i, \quad X^i \in \Sigma_1, \quad u^i = u^i, \quad X^i \in \Sigma_2. \]

Introduce a small parameter \( \kappa = l/L \ll 1 \), as the ratio of the characteristic size \( l \) of the PC to the characteristic size \( L \) of the entire composite (the sizes are defined for \( K \)) and also introduce local \( \xi^i \) Lagrangian coordinates in \( K \) which are related to the following relations:

\[ \xi^i = X^i / \kappa, \quad \tilde{X}^i = X^i / L. \]

The solution of problem (1) with respect to the displacement vector is sought as a quasiperiodic function of the coordinates and is written as asymptotic expansion in the parameter \( \kappa \):

\[ u^i (X^i, \xi^i) = u^{(0)}(X^i) + \kappa u^{(1)}(X^i, \xi^i) + \kappa^2 \ldots \]

Find asymptotic expansions for the strain gradient, the defining relations and the Piola – Kirchhoff tensor, the equilibrium equations, and the boundary conditions:

\[ F^{k}_{ij} = F^{(0)}(X^i, \xi^i) + \kappa F^{(1)}(X^i, \xi^i) + \kappa^2 \ldots, \]

\[ F^{k(0)}(X^i, \xi^i) = \delta^{ik} + u^{(0)} + u^{(1)} + \kappa^{(2)}, \quad F^{k(1)} = u^{(1)} + u^{(2)} \]

\[ P^i = \mathcal{F}^{(0)}(X^i, \xi^i) + \kappa \mathcal{F}^{(1)}(X^i, \xi^i) + \kappa^2 \ldots, \]

\[ P^{(n)} = \mathcal{F}^{(n)}(F^{k(0)}, \xi^m), \quad P^{(n)} = \frac{\partial}{\partial F^{k(0)}} \mathcal{F}^{(n)}(F^{k(0)}, \xi^m) F^{k(1)} \]

\[ \frac{1}{\kappa} P^{(n)}_{ij} + (P^{(n)}_{ij} + \rho f^{(j)} + \kappa(P^{(n)}_{ij} + P^{(2)}_{ij}) + \kappa^2 \ldots = 0, \]

\[ n_i [P^{(n)}] + \kappa n_i [P^{(1)}] + \kappa^2 \ldots = 0, \quad [u^{(0)}] + \kappa [u^{(1)}] + \kappa^2 \ldots = 0, \quad X^i \in \Sigma_{\alpha}, \]

\[ n_i P^{(n)} + \kappa n_i P^{(1)} + \kappa^2 \ldots = t^i, \quad X^i \in \Sigma_1, \quad u^{(0)} + \kappa u^{(1)} + \kappa^2 \ldots = u^i, \quad X^i \in \Sigma_2. \]

Equating the terms in the expansions (2) - (3) with the same powers \( \kappa \) to zero a recurrent sequence \( L_x \) of local problems of nonlinear elasticity is derived.

**Local problem solution**

The local problem for a layered composite in which the periodicity cell is a system of parallel layers orthogonal to the direction \( \hat{O} \hat{Z} \) takes the form:

\[ P^{(0)}_{ij} = 0, \quad P^{(n)} = \mathcal{F}^{(n)}(F^{k(0)}, \xi), \]

\[ F^{k}_{ij} = \tilde{F}^{k}_{ij} = u^{(k(1))}, \quad \alpha = 1\ldots n - 1, \]

\[ \left[ u^{(k(1))} \right]_{ij} = 0. \]
The task is non-linear but one-dimensional - in it all functions depend only on $\xi$.

Integrating the equilibrium equations (4) can conclude that the stress $P^{(0)}_{(0)}$ are constant in the PC:

$$P^{(0)}_{(0)} = C' = \text{const},$$

(9)

From equation (6) it follows that of the 9 components of the deformation gradient $F_{\gamma}^{(0)}$ only 3 components depend on the coordinate $\xi$, and the remaining 6 coincide with the components of the averaged gradient:

$$F_{3}^{(0)} = \bar{F}_{3}, \quad F_{L}^{(0)} = \bar{F}_{L}, \quad L = 1, 2$$

(10)

Substituting (9) into (5) obtain a system of three nonlinear algebraic equations which can be considered with respect to three components $F_{3}^{(0)}$: $C' = \mathcal{F}^{(0)}_{j}(F_{3}^{(0)}, \bar{F}_{L}, \xi)$. Write the formal solution of this system in the form $F_{3}^{(0)} = \mathcal{G}^{(0)}(C', \bar{F}_{L}, \xi)$.

Using expression (10) obtain a system of three ordinary linear differential equations for displacements

$$u^{(1)}_{\gamma} = \bar{u}_{\gamma}^{(1)} = \mathcal{G}^{(0)}(C', \bar{F}_{L}, \xi)$$

which is easily integrated

$$u^{(1)}_{\gamma} = \int_{-0.5}^{\xi} \mathcal{G}^{(0)}(C', \bar{F}_{L}, \xi) d\xi - \bar{F}_{3} \xi + B^{x},$$

(11)

where $B^{x}$ - are the integration constants.

Conditions (7) of the ideal contact for functions (9) and (10) are automatically fulfilled.

After substituting the expression (11) into the periodicity condition (8), we obtain an equation

$$< \mathcal{G}^{(0)}(C', \bar{F}_{L}, \xi) > \bar{F}_{3}^{(0)}$$

that can be considered as a nonlinear algebraic equation for the constants $C'$. Write the formal solution of this equation: $C' = \mathcal{S}^{(0)}(\bar{F}_{3}^{(0)}, \bar{F}_{L}^{(0)})$ and find the ratio between the gradient $F_{\gamma}^{(0)}$ and the average gradient $\bar{F}_{\gamma}$:

$$F_{\gamma}^{(0)} = \bar{F}_{\gamma} + (\mathcal{G}^{(0)}(\mathcal{S}^{(0)}(\bar{F}_{3}^{(0)}, \bar{F}_{L}^{(0)}), \bar{F}_{\gamma}^{(0)} - \bar{F}_{3}^{(0)}) \delta_{3\gamma}$$

(12)

Substituting expression (12) into (5), and integrating it over PC obtain the desired effective defining relations for the layered composite

$$\bar{p}_{ij} = \mathcal{F}^{(0)}_{ij}(\bar{F}_{3}^{(0)}),$$

(13)

where denotes the average stress and the averaged function of the defining relations

$$\bar{p}_{ij} = < P^{(0)}_{ij} >, \quad \mathcal{F}^{(0)}_{ij}(\bar{F}_{3}^{(0)}) = < \mathcal{F}^{(0)}_{ij}(\bar{F}_{j} + (\mathcal{G}^{(0)}(\mathcal{S}^{(0)}(\bar{F}_{3}^{(0)}, \bar{F}_{j}^{(0)}), \bar{F}_{j}^{(0)} - \bar{F}_{3}^{(0)}) \delta_{3j}, \xi) >$$

Universal models for effective constitutive relation of laminated composites

The tensor of defining relations for nonlinear-elastic components of a composite for models $A_{s}$ of elastic media with finite deformations has a complex implicitly defined form and depends on the components of the strain gradient and Lagrangian coordinates (discontinuous). The tensor function (13) has an analytical expression for models $A_{s}$, $A_{a}$; for models $A_{y}$, $A_{yv}$ was used the QUVF-system [11–13]

$$\mathcal{F}^{(0)}_{ij} = \sum_{\gamma=1}^{3} \mathcal{Q}_{ij}^{\gamma} \mathcal{F}_{\gamma}^{(0)} \mathcal{I}_{\gamma\gamma},$$

(14)

$$I_{1A} = E, \quad I_{2A} = EI_{C} - C, \quad I_{3A} = C^{2} - I_{1C} + EI_{C}^{2},$$
\[
\varphi_1 = (l_1 + 2l_2) I_{1}^{(n)} C, \quad \varphi_2 = -2l_1, \quad \varphi_3 = 0, \quad I_{1A}^{(n)} = C \cdot E, \quad I_{2A}^{(n)} = E \cdot C^2, \quad I_{3A}^{(n)} = \text{det} C.
\]

\[
C^{(n)} = \frac{1}{n-III} \sum_{a-1}^{3} \frac{(n-III)}{2} Q_a^{\alpha} Q_a^{\beta} - \frac{1}{n-III} g^{\alpha}, \quad E_A^{(n)} = \sum_{a-\beta=1}^{3} E_{\alpha\beta}^{a} Q_a^{\alpha} Q_a^{\beta} Q_a^{\gamma} Q_a^{\delta}.
\]

**Computational results**

The algorithm for the numerical finding of effective defining relations (13) [10] was implemented in the form of program code written in the C ++ programming language. The program produces a solution of a chain of systems of nonlinear algebraic equations by the method of multidimensional optimization.

**Figure 1.** Model \(A_1\) Deformation diagrams \(\bar{P}_{11}, \bar{P}_{33}\) with different relative thicknesses \(h_2: 1 - 0.0, 2 - 0.3, 3 - 0.5\).

**Figure 2.** Model \(A_2\) Deformation diagrams \(\bar{P}_{11}, \bar{P}_{33}\) with different relative thicknesses \(h_2: 1 - 0.0, 2 - 0.3, 3 - 0.5\).
The figures show the graphs of the dependence of the Piola-Kirchhoff stress tensor on the components of the averaged strain gradient for various input data. The constructed functions represent a numerical experiment for separate layers of a composite with two different sets of elastic constants on the first and third layers: $l_1 = 100$ MPa and $l_2 = 50$ MPa, on the second layer: $l_1 = 20$ MPa and $l_2 = 10$ MPa. Let us set the following relations between the layers $h_1 = h_3$ and $h_2 = 1 - 2 h_1$.

Cases $h_1 = 0.5$ and $h_1 = 0.0$ correspond to homogeneous materials with the characteristics of the first or second layer, respectively.

For models $A_\nu$ and $A_\nu$ graphs the calculations are based on analytical formulas (14) and using the QUVF-system (solid continuous lines).
Conclusions
For nonlinear elastic composites with large deformations and a periodic structure has been developed a method of asymptotic averaging over a small geometric parameter using a universal representation of the governing relations for a complex of various medium models.

The local and averaged problem of nonlinear elasticity is formulated and an algorithm is developed for finding effective defining relations for the composite material as a whole.

The resulting algorithm allows us to calculate the effective strain diagrams of layered composites with finite deformations, connecting the components of the averaged Piola-Kirchhoff stress tensors and the strain gradient.

The numerical calculations performed for models of nonlinear elastic medium with finite deformations using various elastic constants and layer thicknesses confirm the feasibility and effectiveness of the proposed method.

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