Stable capillary hypersurfaces in a half-space or a slab

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Abstract We study stable immersed capillary hypersurfaces in a domain $B$ which is either a half-space or a slab in the Euclidean space $\mathbb{R}^{n+1}$. We prove that such a hypersurface is rotationally symmetric in the following cases:

(1) $n = 2$, $B$ is a slab and the surface has genus zero,

(2) $n \geq 2$, $B$ is a slab, the angle of contact is $\pi/2$ and each component of the boundary of the hypersurface is embedded,

(3) $n \geq 2$, $B$ is a half-space, the angle of contact is $< \pi/2$ and each component of the boundary of the hypersurface is embedded.

Moreover, in case (2), if not a right circular cylinder, the hypersurface has to be graphical over a domain in $\partial B$. In case (3), the hypersurface is a spherical cap.

Keywords Capillary hypersurfaces, constant mean curvature hypersurfaces, stability.

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1. Introduction

Consider a closed domain $B$ with smooth boundary in the Euclidean space $\mathbb{R}^{n+1}$. A capillary hypersurface in $B$ is a compact CMC hypersurface (i.e with constant mean curvature) with non-empty boundary, which is contained in $B$ and meets the frontier $\partial B$ at a constant angle along its boundary. When the angle of contact is $\pi/2$, that is, when the hypersurface is orthogonal to $\partial B$, it is said to be a CMC hypersurface with free boundary.

Capillary surfaces in $\mathbb{R}^3$ model incompressible liquids inside containers in the absence of gravity. Indeed, the free surface of the liquid (locally) minimizes an energy functional under a volume constraint. More precisely, given an angle $\theta \in (0, \pi)$, for a compact surface $\Sigma$ inside $B$ such that $\partial \Sigma \subset \partial B$ and $\partial \Sigma$ bounds a compact domain $W$ in $\partial B$, the energy of
is by definition the quantity
\[ E(\Sigma) := \text{Area}(\Sigma) - \cos \theta \text{Area}(W). \]
The stationary surfaces of \( E \) for variations preserving the enclosed volume are precisely the CMC surfaces which make a constant angle \( \theta \) with \( \partial B \). In the physical interpretation, \( \Sigma \) represents the liquid-air interface and \( W \) the region of the container wetted by the liquid and \( c := \cos \theta \) is a physical constant. For more information on capillary surfaces we refer to [7] and [11].

From the geometrical viewpoint, capillary hypersurfaces raise several interesting questions. Given a domain \( B \subset \mathbb{R}^{n+1} \), one would like to know whether there exist immersed capillary hypersurfaces in \( B \) of a given topology and to characterize, in particular when \( B \) has symmetries, the embedded ones. These are quite difficult problems in general. For instance, even in the case of the unit ball in \( \mathbb{R}^{3} \), these issues are still not completely understood. They are related to problems in spectral geometry when the surfaces are minimal [8, 23]. In a pioneering work, Nitsche [15] has shown that an immersed capillary disk in the unit ball in \( \mathbb{R}^{3} \) has to be a spherical cap or a flat disk. Recently, Fraser and Scheon [8] proved the existence of embedded minimal surfaces of genus zero with free boundary in the unit ball of \( \mathbb{R}^{3} \) with arbitrarily many boundary components. In the case of a closed slab or a closed half-space of \( \mathbb{R}^{n+1} \), one can apply Alexandrov’s reflection technique to prove that in the embedded case, capillary hypersurfaces have rotational symmetry around an axis orthogonal to the boundary of the domain [26]. Wente [27] has also proven the existence of immersed, non embedded, capillary surfaces of annular type in a closed slab and in the unit ball in \( \mathbb{R}^{3} \) which are not pieces of Delaunay surfaces. In the case of a wedge in \( \mathbb{R}^{3} \), it has been shown that pieces of spheres are the only embedded capillary surfaces of annular type [17, 14]. Recently, in [1], it was shown there exists a large family of embedded capillary surfaces in convex polyhedral domains in \( \mathbb{R}^{3} \). The surfaces have genus zero and one boundary component lying on each face of the polyhedra.

Motivated by the physical interpretation, it is natural to restrict one’s attention to the capillary hypersurfaces which are stable, that is, those which minimize the energy \( E \) up to second order under the volume constraint. This has been addressed for the case of the unit ball [12, 21, 22] and for the case of a wedge or a half-space in \( \mathbb{R}^{n+1} \) [6, 13], and also for convex bodies in \( \mathbb{R}^{3} \) [19].

In this paper we will be concerned with stable immersed capillary hypersurfaces in a domain \( B \) which is either a closed slab or a closed half-space in \( \mathbb{R}^{n+1} \). For simplicity, we will refer to the latter as slabs and hyperplanes, meaning implicitly they are closed. We will prove that such a hypersurface is rotationally invariant around an axis orthogonal to \( \partial B \) in the following cases:

1. \( n = 2, B \) is a slab and the surface has genus zero (Theorem 3.1),
2. \( n \geq 3, B \) is a slab, the hypersurface has free boundary, that is, the angle of contact is \( \pi/2 \) and each boundary component of the hypersurface is embedded. Furthermore, if not a right circular cylinder, the hypersurface has to be a graph over a domain in \( B \) (Theorem 4.1),
3. \( n \geq 3, B \) is a half-space, the angle of contact either is \( \pi/2 \), or is \( < \pi/2 \) and each boundary component of the hypersurface is embedded. Moreover the hypersurface is then a spherical cap (Theorem 5.1).
2. Preliminaries

2.1. The variational problem and stability. Consider a domain $B$ with smooth boundary in an oriented Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ of dimension $n + 1$. We let $\bar{N}$ denote the unit normal to $\partial B$ pointing outwards $B$. The orientation of $M$ induces an orientation on $\partial B$ in the usual way: a local orthonormal frame $\{e_1, \ldots, e_n\}$ on $\partial B$ is positively oriented if the local frame $\{\bar{N}, e_1, \ldots, e_n\}$ has positive orientation in $M$.

In all what follows, unless otherwise stated, $\Sigma$ will denote a compact orientable smooth manifold of dimension $n$ with non empty boundary, $\partial \Sigma$, and $\psi : \Sigma \to B$ an immersion which smooth on the interior of $\Sigma$ and of class $C^2$ up to the boundary, and which is proper, that is, verifying $\psi(\text{int} \Sigma) \subset \text{int} \ B$ and $\psi(\partial \Sigma) \subset \partial B$.

Fix a global unit normal $N$ to $\Sigma$ along $\psi$. This determines an orientation on $\Sigma$ as above. Denote by $\nu$ the exterior unit normal to $\partial \Sigma$ in $\Sigma$. Again this induces an orientation on $\partial \Sigma$. Let now $\bar{\nu}$ the unit normal to $\partial \Sigma$ in $\partial B$ compatible with this orientation. Otherwise said, $\bar{\nu}$ is such that the the bases $\{N, \nu\}$ and $\{\bar{N}, \bar{\nu}\}$ have the same orientation in $(T \partial \Sigma)^\perp$.

By a variation of the immersion $\psi$ we mean a differentiable map $\Psi : (-\epsilon, \epsilon) \times \Sigma \to M$ such that $\psi_t : \Sigma \to M$, $t \in (-\epsilon, \epsilon)$, defined by $\psi_t(p) = \Psi(t,p), p \in \Sigma$, is an immersion for each $t \in (-\epsilon, \epsilon)$ and $\psi_0 = \psi$.

If $\psi$ is a proper immersion into $B$, then a variation $\Psi$ is said to be admissible if $\psi_t$ is a proper immersion into $B$ for each for each $t \in (-\epsilon, \epsilon)$. In the sequel we will consider only admissible variations.

We let $\xi(p) = \frac{d\Psi}{dt}(0,p), p \in \Sigma$, be the variation vector field of $\Psi$. The volume function $V : (-\epsilon, \epsilon) \to \mathbb{R}$ is by definition

$$ V(t) = \int_{[0,t] \times \Sigma} \Psi^* \omega, $$

where $\omega$ is the volume form of $M$. The deformation is volume-preserving if $V(t) = 0$ for each $t \in (-\epsilon, \epsilon)$.

For $t \in (-\epsilon, \epsilon)$, we denote by $A(t)$ the $n$-dimensional volume of $\Sigma$ for the metric induced by $\psi_t$. We also consider the wetted area function $W(t) : (-\epsilon, \epsilon) \to \mathbb{R}$ defined by

$$ W(t) = \int_{[0,t] \times \partial \Sigma} \Psi^* \omega, $$

where $\nu$ is the volume form of $\partial B$.

Let $\theta \in (0, \pi)$ be a given number, the energy functional $E : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined as follows

$$ E(t) = A(t) - \cos \theta W(t). $$

The first variation formulae for the energy and volume functions are as follows (cf. [3])

\begin{align*}
(2.1) & \quad E'(0) = -n \int_\Sigma Hf \, d\Sigma + \int_{\partial \Sigma} \langle \xi, \nu - \cos \theta \bar{\nu} \rangle \, d(\partial \Sigma) \\
(2.2) & \quad V'(0) = \int_\Sigma f \, d\Sigma.
\end{align*}
Here $d\Sigma$ (resp. $d(\partial \Sigma)$) denotes the $n-$volume element on $\Sigma$ (resp. the $(n-1)-$volume element on $\partial \Sigma$), $H$ is the mean curvature of the immersion $\psi$ computed with respect to the unit normal $N$ and $f = (\xi, N)$.

Extending a result in [4], we have the following infinitesimal characterization of admissible volume preserving variations.

**Proposition 2.1.** Assume the proper immersion $\psi$ into $\mathcal{B}$ is transversal to $\partial \mathcal{B}$. Let $f$ be a smooth function on $\Sigma$ satisfying $\int_{\Sigma} f d\Sigma = 0$. Then there exists an admissible volume-preserving variation $\Psi$ of $\psi$ in $\mathcal{B}$ such that $f = (\frac{\partial \Psi}{\partial t}|_0, N)$.

**Proof.** Start with an admissible variation $F : (-\epsilon, \epsilon) \times \Sigma \to \mathcal{B}$, $\epsilon > 0$, of the immersion $\psi$ such that $F$ is a local diffeomorphism. An example of such a variation can be constructed as follows. Consider an open neighborhood $W$ of $\psi(\partial \Sigma)$ in $\mathcal{M}$ which is diffeomorphic to a product $(-\delta, \delta) \times U$, where $U \subset \partial \mathcal{B}$ is an open relatively compact neighborhood of $\psi(\partial \Sigma)$. Endow $W$ with the product metric and extend it to a Riemannian metric on $\mathcal{M}$. For this new metric, $U$ is totally geodesic. It suffices to the mapping $F : (-\epsilon, \epsilon) \times \Sigma \to M$ defined for $\epsilon > 0$ small enough and $(t, p) \in (-\epsilon, \epsilon) \times \Sigma$ by $F(t, p) = \exp_{\psi(p)}(N(p))$. Here $\exp$ is the exponential map and $N$ a unit normal field to $\Sigma$ for the new metric.

We now endow the manifold $\Sigma := (-\epsilon, \epsilon) \times \Sigma$ with the pull-back metric $F^*\langle \cdot, \cdot \rangle$, of the original metric on $\mathcal{M}$, which we denote also by $\langle \cdot, \cdot \rangle$. It is enough to prove the result for the hypersurface $\{0\} \times \Sigma$, in the domain $\Sigma$ for this metric. Indeed, if $\Phi$ is an admissible volume-preserving variation of $\{0\} \times \Sigma$ in $\Sigma$ then $F \circ \Phi$ is an admissible variation of $\psi$ in $\mathcal{B}$. Furthermore if $N$ is a unit field normal to $\{0\} \times \Sigma$ in $\Sigma$ then $dF(N)$ is a unit field normal to $\Sigma$ in $\mathcal{M}$ and $(\frac{\partial (F \circ \Phi)}{\partial t}|_0, dF(N)) = (\frac{\partial \Phi}{\partial t}|_0, dF(N)) = (\frac{\partial \Phi}{\partial t}|_0, N)$. Moreover,

$$\int_{[0, t] \times \Sigma} (F \circ \Phi)^* \Omega = \int_{[0, t] \times \Sigma} \Phi^*(F^* \Omega) = \int_{[0, t] \times \Sigma} \Phi^* \tilde{\Omega},$$

so that the volume functions of the deformations $F \circ \Phi$ and $\Phi$ coincide and hence $F \circ \Phi$ is volume-preserving too.

We now prove the result in $\Sigma$. For each point $p \in \partial \Sigma$, let $N_0(p) = N(p) - \langle N(p), \tilde{N}(p) \rangle \tilde{N}(p)$ be the projection of $N(p)$ on $T_p(\partial \Sigma)$; note that $\partial \Sigma = (-\epsilon, \epsilon) \times \Sigma$. We now consider the vector $w(p) = \frac{1}{\langle N_0(p), N_0(p) \rangle} N_0(p) - N(p)$ in $T_p(\{0\} \times \Sigma)$ which is well defined by the transversality assumption. We can extend $w$ to a vector field on $\{0\} \times \Sigma$, still denoted $w$. Set $z = w + N$, we can extend $z$, for instance in a trivial way using the product structure, to a vector field on $\Sigma$ which is tangent to $\partial \Sigma$ along $\partial \Sigma$. Call $Z$ such an extension. By construction $Z$ satisfies $\langle Z, N \rangle = 1$ on $\{0\} \times \Sigma$. Let $(\phi_t)|_{t < \delta}, \delta > 0$, denote the local flow of $Z$ and consider the map: $\Phi : (-\delta, \delta) \times \Sigma \to \Sigma$, defined by $\Phi(t, p) = \phi_t(p)$, for $(t, p) \in (-\delta, \delta) \times \Sigma$. Let now $u : (-\epsilon_0, \epsilon_0) \times \Sigma \to \Sigma$, $\epsilon_0 > 0$, be a differentiable function and define a variation $\mathcal{X} : (-\epsilon_0, \epsilon_0) \times \Sigma \to \Sigma$ as follows:

$$(t, p) \to \mathcal{X}(t, p) = \Phi(u(t, p), p), \quad t \in (-\epsilon_0, \epsilon_0), \; p \in \Sigma$$

We have $\frac{\partial \mathcal{X}}{\partial t}|_0 = \frac{\partial u}{\partial t}|_0 Z$ and so $\langle \frac{\partial \mathcal{X}}{\partial t}|_0, N \rangle = \frac{\partial u}{\partial t}|_0$. Clearly $\mathcal{X}$ is an admissible variation. We will see that we can choose the function $u$ to fulfill the required properties. To compute the volume function of $\mathcal{X}$ we write: $\mathcal{X} = \Phi \circ \Psi$ where $\Psi : (-\epsilon_0, \epsilon_0) \times \Sigma \to \Sigma$ is such that
The volume function associated to $\mathcal{X}$ is

$$V(t) = \int_{[0,t] \times \Sigma} \mathcal{X}^* \Omega = \int_{[0,t] \times \Sigma} \Psi^* (\Phi^* \Omega).$$

Set $\Phi^* \Omega = E \, dt \wedge d\Sigma$ where $E : (-\epsilon_0, \epsilon_0) \times \Sigma \to \tilde{\Sigma}$ is a smooth function. Then $\Psi^* (\Phi^* \Omega)(t,p) = E(u(t,p),p) \frac{\partial u}{\partial t}(t,p) \, dt \wedge d\Sigma$ and so

$$V(t) = \int_{\Sigma} \left( \int_0^t E(u(t,p),p) \frac{\partial u}{\partial t}(t,p) dt \right) d\Sigma.$$

Now, we choose $u$ to be the function which solves the following initial value problem:

$$\frac{\partial u}{\partial t}(t,p) = \frac{f(p)}{E(u(t,p),p)}, \quad u(0,p) = 0, \quad \text{for each } p \in \Sigma.$$

Then

$$V(t) = 0, \quad \text{for all } t \in (-\epsilon_0, \epsilon_0),$$

so that $\mathcal{X}$ is volume-preserving.

To finish, let us check that $\frac{\partial \Psi}{\partial t}(0,p) = f(p)$ for all $p \in \Sigma$. To see this, take a local orthonormal frame $\{e_1, \ldots, e_n\}$ in a neighborhood of $p \in \Sigma$, then

$$E(0,p) = \Omega \left( d\Phi_{(0,p)}(e_1), \ldots, d\Phi_{(0,p)}(e_n), \frac{\partial \Psi}{\partial t}(0,p) \right) = \Omega (e_1, \ldots, e_n, w(p) + N(p)) = \Omega (e_1, \ldots, e_n, N(p)) = 1,$$

where we used the fact that $\Phi(0,q) = q$ for all $q \in \Sigma$ and that $w(p)$ is tangent to $\Sigma$.

Thus $\frac{\partial \Psi}{\partial t}(0,p) = f(p)$ and so $\left( \frac{\partial \mathcal{X}}{\partial t}\right|_0, N \right) = f$ as required. \qed

The proper immersion $\psi : \Sigma \to \mathcal{B}$ is said to be a capillary immersion into $\mathcal{B}$ if $\psi$ is a critical point of the energy functional $\mathcal{E}$ for admissible volume-preserving variations. It follows, from formulae (2.1), (2.2) and Proposition 2.1, that $\psi$ is capillary if and only if it has constant mean curvature and the angle $\angle$ between $\nu$ and $\tilde{\nu}$, called the angle of contact, depends on the chosen unit normal $N$. With the opposite choice, $-N$, the angle would be $\pi - \theta$.

In the sequel we will always take the unit normal $N$ so that the (constant) mean curvature $H$ is $\geq 0$.

Given a function $f \in \mathcal{C}^\infty(\Sigma)$ satisfying $\int_\Sigma f \, d\Sigma = 0$ and an admissible volume-preserving variation $\Psi$ with $f = \left( \frac{\partial \Psi}{\partial t} \right|_0, N \right)$, we have (cf. [22])

$$\mathcal{E}''(0) = -\int_\Sigma \left( f \Delta f + (|\sigma|^2 + \text{Ric}(N)) f^2 \right) \, d\Sigma + \int_{\partial \Sigma} f \left( \frac{\partial f}{\partial \nu} - q \, f \right) \, d(\partial \Sigma),$$

where $\Delta$ is the Laplacian for the metric induced by $\psi$ on $\Sigma$ and $\sigma$ is the second fundamental form of $\psi$, Ric is the Ricci curvature of $M$ and

$$q = \frac{1}{\sin \theta} \Pi(\tilde{\nu}, \tilde{\nu}) + \cot \theta \, \sigma(\nu, \nu).$$

Here $\Pi$ denotes the second fundamental form of $\partial \mathcal{B}$ associated to the unit normal $-N$.

A capillary immersion $\psi$ is said stable if $\mathcal{E}''(0) \geq 0$ for all admissible volume-preserving variations. Let $\mathcal{F} = \left\{ f \in H^1(\Sigma), \int_\Sigma f \, d\Sigma = 0 \right\}$, where $H^1(\Sigma)$ is the first Sobolev space

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of $\Sigma$. The index form $\mathcal{I}$ of $\psi$ is the symmetric bilinear form defined on $H^1(\Sigma)$ by

$$\mathcal{I}(f, g) = \int_{\Sigma} \left( \langle \nabla f, \nabla g \rangle - (|\sigma|^2 + \text{Ric}(N))f g \right) d\Sigma - \int_{\partial \Sigma} q f g d(\partial \Sigma),$$

where $\nabla$ stands for the gradient for the metric induced by $\psi$. Consequently $\psi$ is capillarily stable if and only if $\mathcal{I}(f, f) \geq 0$ for all $f \in \mathcal{F}$.

A function $f \in \mathcal{F}$ is said to be a Jacobi function of $\psi$ if it lies in the kernel of $\mathcal{I}$, that is, if $\mathcal{I}(f, g) = 0$ for all $g \in \mathcal{F}$. By standard arguments, this is equivalent to saying that $f \in C^\infty(\Sigma)$ and satisfies

$$\Delta f + (|\sigma|^2 + \text{Ric}(N))f = \text{constant on } \Sigma$$

$$\frac{\partial f}{\partial \nu} = q f \quad \text{on } \partial \Sigma.$$

More generally, we may assume that the contact angle is constant along each component of $\partial \Sigma$. If $\Gamma_1, \ldots, \Gamma_k$ denote the connected components of $\partial \Sigma$ and $\theta_1, \ldots, \theta_k$ are given angles in $(0, \pi)$, the capillary hypersurfaces in $\mathcal{B}$ with contact angle $\theta_i$ along $\Gamma_i$ for each $i = 1, \ldots, k$ are the critical points for admissible volume-preserving variations of the energy functional

$$\mathcal{E}(t) = A(t) - \sum_{i=1}^k \cos \theta_i W_i(t)$$

where, $W_i(t)$ denotes the wetted area function corresponding to $\Gamma_i$, $i = 1, \ldots, k$. The first and second variations formulae as well as the previous discussion are valid, with obvious modifications, in this more general setting.

A fact that we will use repeatedly is the following:

**Lemma 2.2.** Suppose that $\partial \mathcal{B}$ is totally umbilical. Let $\psi$ be a capillary immersion into $\mathcal{B}$. Then, the unit outwards normal $\nu$ along $\partial \Sigma$ in $\Sigma$ is a principal direction of $\psi$.

**Proof.** It suffices to show that $\sigma(\nu, x) = 0$ for any $x \in T(\partial \Sigma)$. Along $\partial \Sigma$ we have $N = \cos \theta \nbar - \sin \theta \bar{\nu}$, where $\bar{\nu}$ is the unit normal to $\partial \Sigma$ in $\partial \mathcal{B}$ introduced above. So: $D_x N = \cos \theta D_x \nbar - \sin \theta D_x \bar{\nu}$. Let $\Pi$ denote, as above, the second fundamental form of $\partial \mathcal{B}$ associated to $-\nbar$. By hypothesis, $D_x \nbar = \Pi(x, x)$ $x$ is orthogonal to $\nu$. Furthermore, denoting by $\nabla$ the Levi-Civita connection on $\partial \mathcal{B}$, again since $\partial \mathcal{B}$ is totally umbilical: $D_x \bar{\nu} = -\Pi(x, \bar{\nu}) \nbar + \nabla_x \bar{\nu} = \nabla_x \bar{\nu}$ is tangent to $\partial \Sigma$. It follows that $\sigma(\nu, x) = -(D_x N, \nu) = 0$. $\square$

From now on, we will consider the case where the ambient manifold $M$ is the Euclidean space $\mathbb{R}^{n+1}$ and $\mathcal{B}$ is either a half-space or a slab. In this case, $\text{Ric} \equiv 0$ and $\Pi \equiv 0$.

### 2.2. Some formulae for hypersurfaces in Euclidean spaces

We gather here some general facts about hypersurfaces in Euclidean spaces that will be useful in the sequel. The first proposition contains well known formulae, see for instance [3].
Proposition 2.3. Let $\psi : \Sigma \to \mathbb{R}^{n+1}, n \geq 2$, be a $C^2$-immersion in the Euclidean space $\mathbb{R}^{n+1}$ of a smooth orientable $n$-dimensional manifold $\Sigma$, possibly with boundary. Let $N : \Sigma \to S^n \subset \mathbb{R}^{n+1}$ a global unit normal of $\psi$, $H$ its mean curvature and $\sigma$ its second fundamental form. Denote by $\Delta$ and $\text{div}$, respectively the Laplacian and divergence operators for the metric induced by $\psi$. Then the following equations hold on $\Sigma$

(i) $\Delta \psi = nH N,$
(ii) $\text{div} (\psi - \langle \psi, N \rangle N) = n + nH \langle \psi, N \rangle,$
(iii) for any constant vector field $\bar{a}$ on $\mathbb{R}^{n+1}$, $\text{div} (\bar{a} - \langle \bar{a}, N \rangle N) = nH \langle \bar{a}, N \rangle.$

Moreover, if the mean curvature $H$ is constant, then

(iv) $\Delta \langle \psi, N \rangle + |\sigma|^2 \langle \psi, N \rangle = -nH,$
(v) $\Delta N + |\sigma|^2 N = 0.$

A second result we will need is the following useful fact of independent interest.

Proposition 2.4. Let $\psi : \Sigma \to \mathbb{R}^{n+1}$ be a $C^1$-immersion in the Euclidean space $\mathbb{R}^{n+1}$ of a smooth compact orientable $n$-dimensional manifold $\Sigma$, possibly with boundary. Let $N : \Sigma \to S^n \subset \mathbb{R}^{n+1}$ be a global unit normal of $\psi$ and $\nu$ the unit outward conormal to $\partial \Sigma$ in $\Sigma$. Then

\begin{equation}
(2.3) \quad n \int_{\Sigma} N d\Sigma = \int_{\partial \Sigma} \{ \langle \psi, \nu \rangle N - \langle \psi, N \rangle \nu \} d(\partial \Sigma)
\end{equation}

where $d\Sigma$ and $d(\partial \Sigma)$ denote the volume elements of $\Sigma$ and $\partial \Sigma$, respectively.

In particular, if $\Sigma$ has no boundary, then

$$\int_{\Sigma} N d\Sigma = \bar{0}.$$ 

Proof. We prove the result for smooth immersions, the general case follows by approximation. Let $\bar{a}$ be a constant vector field on $\mathbb{R}^{n+1}$. Consider the following vector field on $\Sigma$

$$X = \langle \bar{a}, N \rangle \psi^T - \langle \psi, N \rangle \bar{a}^T,$$

where, $\psi^T = \psi - \langle \psi, N \rangle N$ (resp. $\bar{a}^T = \bar{a} - \langle \bar{a}, N \rangle N$ ) is the projection of $\psi$ (resp. of $\bar{a}$) on $T \Sigma$. Using (ii) and (iii) of Propositions 2.3 and denoting by $D$ the usual differentiation in $\mathbb{R}^{n+1}$, we compute the divergence of $X$:

$$\text{div} X = \langle \bar{a}, N \rangle \text{div} \psi^T + \langle \bar{a}, D_{\psi^T} N \rangle - \langle \psi, N \rangle \text{div} \bar{a}^T - \langle \bar{a}^T, N \rangle - \langle \psi, D_{\bar{a}^T} N \rangle$$

$$= n \langle \bar{a}, N \rangle + nH \langle \psi, N \rangle \langle \bar{a}, N \rangle + \langle \bar{a}^T, D_{\psi^T} N \rangle - nH \langle \bar{a}, N \rangle \langle \psi, N \rangle - \langle \psi^T, D_{\bar{a}^T} N \rangle$$

$$= n \langle \bar{a}, N \rangle,$$

where we used that $\langle \bar{a}^T, D_{\psi^T} N \rangle = \langle \psi^T, D_{\bar{a}^T} N \rangle = -\sigma(\bar{a}^T, \psi^T)$, $\sigma$ being, as above, the second fundamental form of the immersion. Integrating on $\Sigma$ and using the divergence theorem we get

$$n \int_{\Sigma} \langle \bar{a}, N \rangle d\Sigma = \int_{\partial \Sigma} \{ \langle \bar{a}, N \rangle \langle \psi, \nu \rangle - \langle \psi, N \rangle \langle \bar{a}, \nu \rangle \} d(\partial \Sigma).$$

Since this is true for any $\bar{a}$, (2.3) follows. \qed
3. Stable capillary surfaces of genus zero in a slab in $\mathbb{R}^3$

We consider here stable capillary surfaces in a slab in $\mathbb{R}^3$. In the free boundary case, that is, when the angle of contact is $\theta = \pi/2$, it was proved by Ros that the surface has to be a right circular cylinder. This follows from the results in [20]. For general values of $\theta$, we will show that in the genus zero case, a stable capillary surface in a slab has to be a surface of revolution. In particular, the capillary annuli constructed by Wente [27] are unstable. The result is true in the more general case where the contact angles $\theta_1$ and $\theta_2$ with the 2 planes $\Pi_1$ and $\Pi_2$ bounding the slab, are not necessarily equal. Although we do not explicitly state it below, we do not need to assume the surfaces are contained in the slab, only the assumption on the boundary is relevant. The stability of embedded CMC surfaces of revolution connecting 2 parallel planes was studied by Vogel [25] and Zhou [28].

**Theorem 3.1.** Let $\psi$ be a capillary immersion of a surface $\Sigma$ of genus zero in a slab of $\mathbb{R}^3$ bounded by 2 parallel planes $\Pi_1$ and $\Pi_2$ and having constant contact angles $\theta_1$ and $\theta_2$ with $\Pi_1$ and $\Pi_2$, respectively.

If $\psi$ is stable, then $\psi(\Sigma)$ is a surface of revolution around an axis orthogonal to $\Pi_1$.

**Proof.** Up to isometries and a homothety of $\mathbb{R}^3$, we may suppose the slab is bounded by the horizontal planes $\Pi_1 = \{x_3 = 0\}$ and $\Pi_2 = \{x_3 = 1\}$. Let $\gamma$ be a connected component of $\partial \Sigma$ such that $\psi(\gamma)$ lies on the plane $\{x_3 = 0\}$. Consider in the plane $\{x_3 = 0\}$ the circumscribed circle $C$ about $\psi(\gamma)$. We will show that $\psi(\Sigma)$ is a surface of revolution around the vertical axis passing through the center of $C$.

We may assume that the center of $C$ coincides with the origin. Let us consider the Jacobi function $u$ on $\Sigma$ induced by the rotations around the $x_3$–axis. More precisely, for $p \in \Sigma$, $u(p) = \langle \psi(p) \wedge e_3, N(p) \rangle$, where $\wedge$ denotes the cross product on $\mathbb{R}^3$. The function $u$ verifies

$$\begin{cases}
\Delta u + |\sigma|^2 u = 0 & \text{on } \Sigma \\
\frac{\partial u}{\partial \nu} = q u & \text{on } \partial \Sigma
\end{cases}$$

(3.1)

We will prove that $u \equiv 0$ on $\Sigma$.

Suppose, by contradiction, $u$ is not identically zero. Then its nodal set $u^{-1}(0)$ in the interior of $\Sigma$ has the structure of a graph (cf. [5]). We will analyze the structure of $u^{-1}(0) \cap \partial \Sigma$.

Let $p \in \partial \Sigma$, then: $\frac{\partial u}{\partial \nu}(p) = \langle \nu(p) \wedge N(p), e_3 \rangle - \sigma(\nu, \nu)(p) \langle \psi(p) \wedge \nu(p), e_3 \rangle$. Since $\psi(\gamma)$ is a horizontal curve, the vectors $\nu(p)$ and $N(p)$ are contained in a vertical plane, so

$$\frac{\partial u}{\partial \nu}(p) = -\sigma(\nu, \nu)(p) \langle \psi(p) \wedge \nu(p), e_3 \rangle.$$

(3.2)

A point $p \in \partial \Sigma \cap u^{-1}(0)$ will be called a boundary vertex of $u^{-1}(0)$ if $\frac{\partial u}{\partial \nu}(p) = 0$. We observe that $u$ has to change sign in any neighborhood of a boundary vertex. Indeed, otherwise, as $\Delta u = -|\sigma|^2 u$, by the strong maximum principle (see [9], Theorem 3.5 and Lemma 3.4), at such a point $p$, we would have $\frac{\partial u}{\partial \nu}(p) \neq 0$, unless $u$ is identically zero in a neighborhood of $p$, but then, by the unique continuation principle of Aronszajn, $u$ would vanish everywhere on $\Sigma$, contradicting our assumption. It follows that each such point lies on the boundary of at least 2 components of the set $\{u \neq 0\}$.
Stable capillary hypersurfaces

We note that if \( p \in \partial \Sigma \) is a critical point of the function \(|\psi|^2\) restricted to \( \gamma \), then one can check that \( u(p) = 0 \) and \( \langle \psi(p) \wedge \nu(p), e_3 \rangle = 0 \), and so, according to (3.2), one also has \( \frac{\partial u}{\partial \nu}(p) = 0 \). Otherwise said, critical points of the function \(|\psi|^2\) on \( \gamma \) are boundary vertices of \( u^{-1}(0) \).

Now, we observe that by the choice of \( C \), there are at least 3 boundary vertices on \( \gamma \). Indeed, it is a known fact that \( C \) contains at least two points of \( \psi(\gamma) \), see [16]. This gives rise to 2 boundary vertices of \( u^{-1}(0) \). A third one is a point of \( \gamma \) whose image by \( \psi \) is a closest one to the origin.

Since by hypothesis \( \Sigma \) is topologically a planar domain, using the above information and the Jordan curve theorem, it is easy to see this implies that the set \( \{ u \neq 0 \} \) has at least 3 components.

Denote by \( \Sigma_1 \) and \( \Sigma_2 \) two of these components and consider the following function in the Sobolev space \( H^1(\Sigma) \):

\[
\tilde{u} = \begin{dcases}
  u & \text{on } \Sigma_1 \\
  \alpha u & \text{on } \Sigma_2 \\
  0 & \text{on } \Sigma \setminus (\Sigma_1 \cup \Sigma_2)
\end{dcases}
\]

where \( \alpha \in \mathbb{R} \) is chosen so that \( \int_{\Sigma} \tilde{u} \, dA = 0 \). Using (3.1) we compute

\[
\int_{\Sigma_1} \{ \langle \nabla \tilde{u}, \nabla \tilde{u} \rangle - |\sigma|^2 \tilde{u}^2 \} \, dA = \int_{\Sigma_1} \{ \langle \nabla u, \tilde{\nabla} u \rangle - |\sigma|^2 u \tilde{u} \} \, dA
\]

\[
= -\int_{\Sigma_1} (\Delta u + |\sigma|^2 u) \tilde{u} \, dA + \int_{\partial \Sigma_1} \tilde{u} \frac{\partial u}{\partial \nu} \, ds
\]

\[
= \int_{\partial \Sigma_1 \cap \partial \Sigma} \tilde{u}^2 \, ds
\]

Using a similar computation on \( \Sigma_2 \), we deduce that \( I(\tilde{u}, \tilde{u}) = 0 \). As \( \Sigma \) is stable, we conclude that \( \tilde{u} \) is a Jacobi function. Indeed, the quadratic form on \( F \) associated to \( I \) has a minimum at \( \tilde{u} \) and so \( \tilde{u} \) lies in the kernel of \( I \). However, \( \tilde{u} \) vanishes on a non empty open set. By the unique continuation principle of Aronszjan, \( \tilde{u} \) has to vanish everywhere, which is a contradiction.

Therefore \( u \equiv 0 \). This means that \( \psi(\Sigma) \) is a surface of revolution around the \( x_3 \)-axis. \( \square \)

4. Stable CMC hypersurfaces with free boundary in a slab in \( \mathbb{R}^{n+1} \)

Stability of embedded rotationally invariant CMC hypersurfaces connecting 2 parallel hyperplanes in \( \mathbb{R}^{n+1} \) and orthogonal to them was studied by Athanassenas [2] and Vogel [24] for \( n = 2 \), and by Pedrosa and Ritoré for any \( n \geq 2 \) [18]. It turns out that for \( 2 \leq n \leq 7 \) only circular cylinders can be stable. However for \( n \geq 8 \) there are certain unduloids which are stable [18].

We study here stability of general immersed hypersurfaces. Under a mild condition on the immersion along the boundary of the hypersurfaces, we show they have to be embedded and rotationally invariant around an axis orthogonal to the hyperplanes bounding the slab. Our proof is inspired by some ideas used in [21]. When \( n = 2 \), the conclusion is valid without any assumption. This follows from the results in [20]. We believe the assumption on the boundary behaviour of the immersion can also be removed for all \( n \geq 3 \). We note also
that our proof does not use the fact that the hypersurfaces are contained in the slab. Clearly, without loss of generality, the slab can be assumed to be horizontal. More precisely, our result is as follows:

**Theorem 4.1.** Let $\psi : \Sigma \to \mathbb{R}^{n+1}, n \geq 2$, be an immersed capillary hypersurface connecting two horizontal hyperplanes in $\mathbb{R}^{n+1}$ with contact angle $\theta = \pi/2$. Suppose that the restriction of $\psi$ to each component of $\partial \Sigma$ is an embedding.

If $\psi$ is stable then $\psi(\Sigma)$ is either a circular vertical cylinder or a vertical graph which is rotationally invariant around a vertical axis.

**Proof.** Call $\Pi_1$ and $\Pi_2$ the hyperplanes bounding the slab with $\Pi_1$ below $\Pi_2$. We denote, as usual, by $e_1, \ldots, e_{n+1}$ the vectors of the canonical basis of $\mathbb{R}^{n+1}$. We consider the function $v = \langle N, e_{n+1} \rangle$, that is, the $(n+1)$-coordinate function of $N$. If $v$ is identically zero then $\Sigma$ is a vertical cylinder whose base is an embedded CMC hypersurface in $\Pi_1$. The base is, by Alexandrov’s theorem, a round sphere in $\Pi_1$. Consequently, $\Sigma$ is a circular cylinder.

Assume $v$ is not identically zero. We will show it has a sign in the interior of $\Sigma$. Suppose by contradiction that $v$ changes sign and consider the functions $v_+ = \max\{v, 0\}$ and $v_- = \min\{v, 0\}$ which lie in the Sobolev space $H^1(\Sigma)$. We compute, using the fact that $v = 0$ on $\partial \Sigma$

$$
I(v_+, v_+) = \int_{\Sigma} \left\{ \langle \nabla v_+, \nabla v_+ \rangle - |\sigma|^2(v_+)^2 \right\} d\Sigma \\
= \int_{\Sigma} \left\{ \langle \nabla v, \nabla v_+ \rangle - |\sigma|^2 v v_+ \right\} d\Sigma \\
= -\int_{\Sigma} \left\{ \Delta v + |\sigma|^2 v \right\} d\Sigma \\
= 0
$$

and similarly $I(v_-, v_-) = 0$. As we supposed that $v$ changes sign, there exists $a \in \mathbb{R}$ such that $\int_{\Sigma}(v_+ + av_-) d\Sigma = 0$. So we can use $\tilde{v} := v_+ + av_-$ as a test function in the second variation. We have

$$
I(\tilde{v}, \tilde{v}) = I(v_+, v_+) + 2aI(v_+, v_-) + a^2I(v_-, v_-) = 0.
$$

Since $\Sigma$ is stable, we conclude that $\tilde{v}$ is a Jacobi function and so satisfies $\frac{\partial I}{\partial v} = 0$ on $\partial \Sigma$. Note that $\psi(\Sigma)$ extends analytically by reflection through the hyperplanes $\Pi_1$ and $\Pi_2$. Since we also have $\tilde{v} = 0$ on $\partial \Sigma$, by the uniqueness part in the Cauchy-Kowalevski theorem the function $\tilde{v}$ vanishes in a neighborhood of $\partial \Sigma$, i.e $v$ vanishes in a neighborhood of $\partial \Sigma$. This means $\psi(\Sigma)$ is a cylinder in a neighborhood of $\psi(\partial \Sigma)$ and, by analyticity of CMC hypersurfaces, $\psi(\Sigma)$ is a vertical cylinder and so $v$ is identically zero, a contradiction.

Therefore the function $v$ does not change sign in $\Sigma$. We will assume $v \geq 0$, the case $v \leq 0$ being similar. The function $v$ satisfies

$$
v \geq 0, \\
\Delta v = -|\sigma|^2 v \leq 0, \\
v = 0 \text{ on } \partial \Sigma.
$$

As we are assuming $v$ is not identically zero, by the maximum principle for superharmonic functions we know that $v > 0$ on the interior of $\Sigma$. So the interior of $\psi(\Sigma)$ is a local vertical graph. We will show $\psi(\Sigma)$ is globally a vertical graph by analyzing its behavior near its boundary. Let $\Gamma_1, \ldots, \Gamma_k$ denote the connected components of $\partial \Sigma$. By hypothesis,
ψ restricted to Π_i is an embedding and so ψ(Γ_i) separates the hyperplane containing it, among Π_1 and Π_2, into two connected components, for each i = 1, . . . , k.

Denote by P : \mathbb{R}^{n+1} \to \Pi_1 the orthogonal projection and set F = P \circ \psi. Fix i = 1, . . . , k and consider a point p ∈ Γ_i, and a curve \gamma : (-\epsilon, 0) \to \Sigma parametrized by arc-length with \gamma(0) = p and \gamma'(0) = \nu(p). It is easy to check that

\[
\frac{d}{dt}(F(\gamma(t)) - F(p), N(p))|_0 = 0
\]

\[
\frac{d^2}{dt^2}(F(\gamma(t) - F(p), N(p))|_0 = \langle D\psi(\gamma')|_0, N(p)\rangle
\]

\[
= \sigma(\nu, \nu)
\]

Note that

\[
\frac{\partial u}{\partial \nu} = -\sigma(\nu, \nu)(\nu, e_{n+1}) = \begin{cases}
+\sigma(\nu, \nu) & \text{if } \psi(\Gamma_i) \subset \Pi_1 \\
-\sigma(\nu, \nu) & \text{if } \psi(\Gamma_i) \subset \Pi_2
\end{cases}
\]

By the strong maximum principle, \frac{\partial u}{\partial \nu} < 0 on \partial \Sigma. So for t small, \(F(\gamma(t))\) lies in the component of \(\Pi_1 \setminus F(\Gamma_i)\) which has \(N(p)\) as outwards (resp. inwards) pointing normal at \(F(p)\) if \(\psi(\Gamma_i) \subset \Pi_1\) (resp. if \(\psi(\Gamma_i) \subset \Pi_2\)). It follows that there is a thin strip in the interior of \(\Sigma\) around \(\Gamma_i\) which projects on this component. We call \(D_i\) the component of \(\Pi_1 \setminus F(\Gamma_i)\) which does not intersect this projection. We define \(\tilde{\Sigma}\) to be the union of \(\Sigma\) with the disjoint union of all the domains \(D_i\), and \(\tilde{F} : \tilde{\Sigma} \to \Pi_1\) by

\[
\tilde{F} = \begin{cases}
F & \text{on } \Sigma \\
\text{projection on } \Pi_1 & \text{on } D_i, i = 1, \ldots, k.
\end{cases}
\]

It is clear that \(\tilde{F}\) is a local homeomorphism and a proper map, thus it is a covering map. Therefore \(\tilde{F}\) is a global homeomorphism onto \(\Pi_1\). So \(\psi(\Sigma)\) is a graph over a domain in \(\Pi_1\) and it is, in particular, embedded. Alexandrov’s reflection technique shows that \(\psi(\Sigma)\) is a hypersurface of revolution around a vertical axis, see [26].

5. Stable capillary hypersurfaces in a half-space in \(\mathbb{R}^{n+1}\)

We focus in this section on capillary immersions into a half-space in \(\mathbb{R}^{n+1}\). Spherical caps are examples of such immersions and are actually the only embedded ones [26]. It is known they are stable and even minimize the energy functional, see [10]. Marinov [13] characterized the spherical caps as the only stable immersed capillary surfaces in a half-space in \(\mathbb{R}^3\). He utilized an infinitesimal admissible variation which is the version in the capillarity setting of the one used, in the closed case, by Barbosa and do Carmo [3]. Recently, Choe and Koiso [6] proved the same result in \(\mathbb{R}^{n+1}\), for any \(n \geq 2\), assuming the contact angle is \(\geq \pi/2\) and the boundary of the hypersurface is convex. To achieve this, they computed the second variation of an admissible volume-preserving variation which is the integrated variation of the infinitesimal one used by Marinov.

We consider here the case where the angle of contact is \(\leq \pi/2\). We recall that the hypersurfaces are oriented by the unit normal \(N\) for which the mean curvature \(H\) is \(\geq 0\). We will use two infinitesimal variations, the first one being the one used by Marinov, Choe and Koiso. However, contrarily to the previous authors, thanks to Proposition 2.4, to establish the admissibility of this variation we do not need to assume embeddedness of the boundary.
The second infinitesimal variation we use is a suitable combination of the negative and positive parts of the last coordinate of the Gauss map. Our result is the following:

**Theorem 5.1.** Let $\psi: \Sigma \to \mathbb{R}^{n+1}$, $n \geq 2$, be a stable immersed capillary hypersurface in a half-space in $\mathbb{R}^{n+1}$ with contact angle $0 < \theta \leq \pi/2$.

(i) If $\theta = \pi/2$, then $\psi(\Sigma)$ is a hemisphere.

(ii) If $\theta < \pi/2$ and the restriction of $\psi$ to each component of $\partial \Sigma$ is an embedding, then $\psi(\Sigma)$ is a spherical cap.

**Proof.** Without loss of generality, we may suppose the half-space is the upper half-space $x_{n+1} \geq 0$.

Integrating the equation in (ii) of Proposition 2.3, we get

$$\int_{\partial \Sigma} \langle \psi, \nu \rangle d(\partial \Sigma) = n \int_{\Sigma} \{1 + H \langle \psi, N \rangle \} d\Sigma.$$  

On $\partial \Sigma$, we have:

$$\cos \theta N + \sin \theta \nu = -e_{n+1},$$

where $e_{n+1}$ is the $(n+1)$-th vector of the canonical basis of $\mathbb{R}^{n+1}$. Therefore, Proposition 2.4 gives

$$n \int_{\Sigma} N d\Sigma = -\frac{1}{\cos \theta} \left( \int_{\partial \Sigma} \langle \psi, \nu \rangle d(\partial \Sigma) \right) e_{n+1}.$$  

From (5.1) and (5.2), we conclude that:

$$\int_{\Sigma} \{1 + H \langle \psi, N \rangle + \cos \theta \langle N, e_{n+1} \rangle \} d\Sigma = 0.$$  

So we may use $\phi := 1 + H \langle \psi, N \rangle + \cos \theta \langle N, e_{n+1} \rangle$ as a test function in the stability inequality. Set $u = \langle \psi, N \rangle$ and $v = \langle N, e_{n+1} \rangle$. From Proposition 2.3 we know that:

$$\Delta u + |\sigma|^2 u = -nH,$$

and

$$\Delta v + |\sigma|^2 v = 0.$$  

Using these equations, we compute:

$$\Delta \phi = H(-nH - |\sigma|^2 u) + \cos \theta |\sigma|^2 v$$

$$= -nH^2 - |\sigma|^2 (Hu - \cos \theta v)$$

$$= -nH^2 - |\sigma|^2 (\phi - 1).$$  

Therefore

$$\phi \Delta \phi + |\sigma|^2 \phi^2 = (|\sigma|^2 - nH^2) \phi.$$  

On the other hand:

$$\frac{\partial u}{\partial \nu} = \langle \nu, N \rangle + \langle \psi, D \nu N \rangle = -\sigma(\nu, \nu) \langle \psi, \nu \rangle$$

and

$$\frac{\partial v}{\partial \nu} = \langle D \nu N, e_{n+1} \rangle = -\sigma(\nu, \nu) \langle \psi, e_{n+1} \rangle = \sigma(\nu, \nu) \sin \theta.$$  

Using the relation: $\nu = -\cot \theta \ N - \frac{1}{\sin \theta} e_{n+1}$, one can check after direct computations that:

$$\frac{\partial \phi}{\partial \nu} = \cot \theta \sigma(\nu, \nu) \phi.$$
It follows that
\begin{equation}
I(\phi, \phi) = -\int_\Sigma (|\sigma|^2 - nH^2) \phi \, d\Sigma.
\end{equation}

Integrating (5.3) we have
\[ \int_{\partial \Sigma} \frac{\partial u}{\partial \nu} \, d(\partial \Sigma) + \int_\Sigma |\sigma|^2 u \, d\Sigma = -nH \int_\Sigma 1 \, d\Sigma. \]

Using this, we can write
\[ \int_\Sigma (|\sigma|^2 - nH^2) \phi \, d\Sigma = \int_\Sigma (|\sigma|^2 - nH^2) \, d\Sigma + H \int_\Sigma |\sigma|^2 u \, d\Sigma - nH^2 \int_\Sigma H u \, d\Sigma \]
\[ + \cos \theta \int_\Sigma (|\sigma|^2 - nH^2) \, d\Sigma \]
\[ = \int_\Sigma (|\sigma|^2 - nH^2) \, d\Sigma - nH^2 \int_\Sigma (1 + Hu) \, d\Sigma - H \int_{\partial \Sigma} \frac{\partial u}{\partial \nu} \, d(\partial \Sigma) \]
\[ + \cos \theta \int_\Sigma (|\sigma|^2 - nH^2) \, d\Sigma \]
\[ = \int_\Sigma (|\sigma|^2 - nH^2) \, d\Sigma + H n^2 \cos \theta \int_\Sigma v \, d\Sigma - H \int_{\partial \Sigma} \frac{\partial u}{\partial \nu} \, d(\partial \Sigma) \]
\[ + \cos \theta \int_\Sigma (|\sigma|^2 - nH^2) \, d\Sigma. \]

Integrating (5.4) we obtain
\[ \int_\Sigma |\sigma|^2 v \, d\Sigma = -\int_{\partial \Sigma} \frac{\partial v}{\partial \nu} \, d(\partial \Sigma). \]

So,
\[ \int_\Sigma (|\sigma|^2 - nH^2) \phi \, d\Sigma = \int_\Sigma (|\sigma|^2 - nH^2) \, d\Sigma - \int_{\partial \Sigma} \frac{\partial}{\partial \nu} (Hu + \cos \theta v) \, d\Sigma, \]
that is,
\begin{equation}
\int_\Sigma (|\sigma|^2 - nH^2) \phi \, d\Sigma = \int_\Sigma (|\sigma|^2 - nH^2) \, d\Sigma - \int_{\partial \Sigma} \frac{\partial \phi}{\partial \nu} \, d(\partial \Sigma).
\end{equation}

Let \( i \in \{1, \ldots, k\} \) and \( \{v_1, \ldots, v_{n-1}\} \) be a local orthonormal frame on \( \partial \Sigma \). Then:
\[ \sigma(\nu, \nu) = nH - \sum_{j=1}^{n-1} \sigma(v_j, v_j). \]

Now, considering the unit normal \( \bar{\nu} \) in \( \mathbb{R}^n \times \{0\} \) to \( \partial \Sigma \) along \( \psi \), as chosen in Sec. 2.1, we have \( N = -\sin \theta \bar{\nu} - \cos \theta e_{n+1} \). We can thus write
\[ \sigma(v_j, v_j) = -\langle \nabla_{\nu_j} N, v_j \rangle = \sin \theta \langle \nabla_{\nu_j} \bar{\nu}, v_j \rangle. \]

Therefore, if we denote by \( H_{\partial \Sigma} \) the mean curvature of \( \partial \Sigma \) in \( \mathbb{R}^n \times \{0\} \) computed with respect to the unit normal \( \bar{\nu} \), the following relation holds on \( \partial \Sigma \)
\begin{equation}
\sigma(\nu, \nu) = nH + (n-1) \sin \theta H_{\partial \Sigma}.
\end{equation}
Also, taking into account that $\nu = \cos \theta \bar{\nu} - \sin \theta e_{n+1}$, one has on $\partial \Sigma$
\[
\phi = 1 - \sin \theta H \langle \psi, \bar{\nu} \rangle - \cos^2 \theta = \sin^2 \theta - \sin \theta H \langle \psi, \bar{\nu} \rangle.
\]
Replacing in (5.5), we obtain:
\[
\frac{\partial \phi}{\partial \nu} = \cos \theta \{ nH \sin \theta - nH^2 \langle \psi, \bar{\nu} \rangle + (n-1) \sin^2 \theta H \partial \Sigma - (n-1) \sin \theta \partial \Sigma H \langle \psi, \bar{\nu} \rangle \}
\]
and so, using (ii) of Proposition 2.3 applied to the immersion $\psi|_{\partial \Sigma} : \partial \Sigma \to \mathbb{R}^n \times \{0\}$, we get
\[
\int_{\partial \Sigma} \frac{\partial \phi}{\partial \nu} d(\partial \Sigma) = nH \cos \theta \left[ \sin \theta \text{vol}_{n-1}(\partial \Sigma) - H \int_{\partial \Sigma} \langle \psi, \bar{\nu} \rangle d(\partial \Sigma) \right]
+ (n-1) \cos \theta \sin \theta \left[ H \text{vol}_{n-1}(\partial \Sigma) + \sin \theta \int_{\partial \Sigma} \partial \Sigma H d(\partial \Sigma) \right].
\]
Integrating equation (i) of Proposition 2.3, we get:
\[
\int_{\partial \Sigma} \nu d(\partial \Sigma) = nH \int_{\Sigma} \langle N, e_{n+1} \rangle d\Sigma.
\]
Therefore
\[
\int_{\partial \Sigma} \langle \nu, e_{n+1} \rangle d(\partial \Sigma) = nH \int_{\Sigma} \langle N, e_{n+1} \rangle d\Sigma,
\]
that is,
\[
\int_{\partial \Sigma} \langle \nu, e_{n+1} \rangle d(\partial \Sigma) = nH \int_{\Sigma} \langle N, e_{n+1} \rangle d\Sigma.
\]
Combining (5.10), (5.2) and taking into account the relation $\langle \psi, \nu \rangle = \cos \theta \langle \psi, \bar{\nu} \rangle$, we conclude that
\[
\sin \theta \text{vol}_{n-1}(\partial \Sigma) = H \int_{\partial \Sigma} \langle \psi, \bar{\nu} \rangle d(\partial \Sigma).
\]
Therefore (5.9) becomes
\[
\int_{\partial \Sigma} \frac{\partial \phi}{\partial \nu} d(\partial \Sigma) = (n-1) \sin \theta \cos \theta \left[ H \text{vol}_{n-1}(\partial \Sigma) + \sin \theta \int_{\partial \Sigma} \partial \Sigma H d(\partial \Sigma) \right].
\]
Combining this with (5.6) and (5.7), we finally obtain
\[
\mathcal{I}(\phi, \phi) = - \int_{\Sigma} \{ |\sigma|^2 - nH^2 \} d\Sigma
+ (n-1) \sin \theta \cos \theta \left[ H \text{vol}_{n-1}(\partial \Sigma) + \sin \theta \int_{\partial \Sigma} \partial \Sigma H d(\partial \Sigma) \right].
\]
By stability $\mathcal{I}(\phi, \phi) \geq 0$. In particular, if $\theta = \pi/2$, since $|\sigma|^2 \geq nH^2$, we conclude that necessarily $|\sigma|^2 = nH^2$ everywhere on $\Sigma$. So the immersion is totally umbilical, that is, $\psi(\Sigma)$ is a hemisphere. This proves (i).

To prove (ii), we will first show that the function $\nu = \langle N, e_{n+1} \rangle$, does not change sign on $\Sigma$. Suppose, by contradiction, this is not the case. Then, we can find a number $\alpha \in \mathbb{R}$ so that the function $w := v_+ + \alpha v_-$ satisfies $\int_{\Sigma} w d\Sigma = 0$. Under the hypothesis $\theta < \frac{\pi}{2}$, we
have \( w = v_- = -\cos \theta \) on \( \partial \Sigma \). Proceeding as in the proof of Theorem 4.1 and using that
\[
\frac{\partial v}{\partial \nu} = \sin \theta \sigma(\nu, \nu),
\]
we find that
\[
\mathcal{I}(v_+, v_+ + v_-, v_+) = \mathcal{I}(v_-, v_+ + v_-) = 0.
\]
Taking into account (5.8), we get
\[
\mathcal{I}(w, w) = \mathcal{I}(v_-, v_-) + 2\alpha \mathcal{I}(v_-, v_+ + v_-) + \alpha^2 \mathcal{I}(v_+, v_+) = -n H \cot \theta \sigma n_{-1}(\partial \Sigma) - (n - 1) \cos \theta \int_{\partial \Sigma} H n_{-1}(\partial \Sigma).
\]
By stability
\[
\mathcal{I}(\phi, \phi) + \sin^2 \theta \mathcal{I}(w, w) \geq 0.
\]
Taking into account (5.11), we conclude that
\[
-\int_{\Sigma} \left[ |\sigma|^2 - n H^2 \right] d\Sigma - \sin \theta \cos \theta H n_{-1}(\partial \Sigma) \geq 0.
\]
As \( |\sigma|^2 \geq n H^2 \), \( \theta < \pi/2 \) and \( H > 0 \) by our choice of orientation, this is a contradiction (note that \( H \neq 0 \) otherwise by the maximum principle \( \psi(\Sigma) \) would be planar). We conclude that the function \( v \) does not change sign on \( \Sigma \). Since \( v = -\cos \theta < 0 \) on \( \partial \Sigma \), we must have \( v \leq 0 \) everywhere on \( \Sigma \). As \( \Delta v = -|\sigma|^2 v \geq 0 \), by the Hopf maximum principle we have \( v < 0 \) everywhere. So \( \psi(\Sigma) \) is a local vertical graph around each of its interior points. Since the angle of contact \( \theta \) is not \( \pi/2 \), this is also the case near \( \partial \Sigma \). We can now conclude, as in the proof of Theorem 4.1, that \( \psi(\Sigma) \) is globally a graph over a domain in \( \mathbb{R}^n \times \{0\} \), hence it is embedded and a spherical cap by Alexandrov’s reflection argument [26].

**Remark 5.2.** The case \( \theta = \pi/2 \) can be treated more directly. Indeed, by reflection through the boundary hyperplane, the hypersurface gives rise to a closed CMC hypersurface which can be shown to be stable for volume-preserving variations and which is therefore a round sphere by the result of Barbosa and do Carmo [3].

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