An Extended Goodwin Model with Horizontal Trade: A Sheaf Theoretic Approach

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Abstract

The Goodwin model of endogenous growth looks to study the dynamic interaction between employment rate and worker’s share of national income in an economy. The model is simplistically and elegantly described by a set of differential equations that predicts cyclic behavior between the two variables in an economy. While this model is simplistic, and most likely does not accurately represent reality, the mathematical modeling of cyclic behavior is attractive to economists. Cycles are at the heart of many macroeconomic theories and a mathematical model allows for future predictions to be made. Thus, over the years, it has been updated and extended. Ishiyama (2001) takes one such approach at updating the Goodwin model. He considers two countries engaged in horizontal trade. Through the lens of sheaf theory, this paper describes Ishiyama’s complex model through various dependency diagrams. Sheaves allow us to encode all information reflected in the equations of the model into a dependency diagram, yielding a visual representation of the variable relationship structure. These dependency diagrams are powerful, and much of the analysis typically done on equations can be done on the diagrams themselves. Further, these dependency diagrams allow for a new way to consider complex models, such as Ishiyama’s model. More specifically, it also allows us to analyze his system in a way not previously done. New questions regarding local sections of this sheaf and their possible extensions to global sections are considered. These questions lend themselves to unique analysis about the system. They also provide a practical way to check the accuracy of Ishiyama’s model, which has many obvious benefits. It is meaningful to conduct this approach to a system of equations given its novelty, its applicability, and its importance for modeling.
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1 Introduction

To model complex phenomena, mathematicians, physicists, engineers, chemists, and economists have long turned toward differential equations to provide insight and analysis. Because differential equations describe phenomena in a functional way, they can be used to make predictions about how systems evolve over time. Before the rise of computers and the ease to collect and analyze data, models built using differential equations were the first type of predictive machine; and were greatly successful, too. Such a simple concept was used to great effectiveness by Newton to model the trajectories of heavenly bodies. Many of his equations are still widely used today.

Economics, too, have come to understand the power of differential equations to model and predict phenomena. Economists, however, are not the only people who seek to understand and predict how a country’s economy grows over time. Such insight would be useful to politicians and businessmen alike. Provided the models are accurate, they can be quite informative. Therefore, there has been a great deal of interest in the rigor and accuracy with which these model economic phenomena which we observe.

The behavior of economic systems have been viewed in a few distinct ways over time. The first models claimed that markets obtain stable equilibrium over time. Random shocks, such as an oil crisis, are observable, but eventually the systems return to equilibrium. Later models assumed more complex growth behavior. These models assumed that growth paths are not steady, but rather cyclic, and equilibria are effected by past motions. This appears to be more in line with reality, too.

In 1967, Goodwin proposed a simple model which describes the dynamic relationship between the employment rate and worker’s share of national income. While Goodwin admitted that it was a “quite unrealistic model of cycles in growth rates,” the beauty of his model lies in its simplicity, while still yielding interesting cyclic dynamics. Over time, economists have attempted to update Goodwin’s model with two goals in mind. The first is to make it more applicable to reality and herein lies the second point of interest: how do the dynamics of the updated system change? This paper will ultimately focus on one of these updated models. Ishiyama proposed a model which extends the original Goodwin beyond its closed country assumption to an open one by building horizontal trade into the model. It is, at its core, a multi-model system. His new model demonstrates the emergence of chaotic behavior by allowing for horizontal trade. But how accurate is Ishiyama’s model? What else can be said about Ishiyama’s extended model? A branch of mathematics called sheaf theory may provide some answers.

The recent application of sheaf theory to multi-model systems looks to ease the burden of constructing and analyzing complex models. Sheaves provide a “tool box” for constructing predictive model’s described by a system of equations. Sheaves also provide a way to check the models accuracy. The power of sheaves comes about since smaller and easier to construct models can be systematically stitched together to form a larger, more complex one. Sheaves are
represented diagrammatically as dependency diagrams, where arrows relating variables to each other represent actual functions. Any analysis done to the sheaf is equivalent to analysis done on the system of equations. Ishiyama extends the Goodwin Model to horizontal trade, whereby he takes two separate countries and effectively “stitches” them together via a trading scheme. At its core, Ishiyama’s model is that of a multi-model system and sheaves have the power to shed light on his model. This paper will use Ishiyama’s proposed model for international horizontal trade and construct a number of dependency diagrams and sheaves. Each will be analyzed using the applicable tools. This analytical approach to a system of equations is meaningful given its novelty, its applicability, and its importance for modeling.

This paper is organized in the following way: Section 2 discusses nonlinear dynamical systems and introduces important concepts relating to the Goodwin model. Section 3 introduces the predator-prey model. Section 4 discusses the Goodwin model and various updates to it. Section 5 provides background information on sheaf theory and how to construct and analyze sheaves. These four sections provide a comprehensive and sufficient amount of background information to make the final section understandable, in which the extended Goodwin model is introduced, sheafified, and analyzed. The final section concludes and discusses areas of further research.

2 Nonlinear Dynamical Systems

2.1 Preliminary Concepts

We begin our background discussion with nonlinear dynamical systems. It is not a stretch to assume that the world is not linear in nature; thus to model phenomena that take place in it, it is natural to craft nonlinear models. While this process allows us to model phenomena in a better way, there are some tradeoffs; most notably, most nonlinear dynamical systems cannot be solved in closed form. What is available, however, is an analysis of the qualitative behavior of the system. It is possible to determine under certain conditions whether a dynamical system exhibits a closed orbit or displays other dynamic behavior. The rest of this section is devoted to providing some background on the concept of closed orbits in dynamical systems. We will introduce key definitions, concepts, and theorems.

Since we are dealing with dynamical systems of differential equations rather than difference equations, we will be exclusively dealing with continuous time. We begin by considering the $n$-dimensional ordinary differential equations system defining the motion of the state variables $x_i, i = 1,\ldots,n$

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, \ldots, x_n), \\
\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n).
\end{align*}
\]  

(2.1)
Alternatively, this can be written more succinctly in vector notation as
\[ \dot{x} = f(x) \] where \( x \in W \subseteq \mathbb{R}^n \).

(2.2)

Here, \( W \) is an open subset of \( \mathbb{R}^n \) and the dot over a variable denotes the derivative with respect to time.

**Definition 2.1.** Consider a subset \( S \) of the domain \( D \), under the mapping \( T : D \to D \). An **invariant set** is the subset \( S \) such that for all \( x \in S \), \( T(x) \in S \).

A solution curve, trajectory, or orbit is defined as \( \phi_t(x(0)) \). For a certain set of initial conditions \( x(0) \) given, \( \phi_t(x(0)) \) describes the values of \( x \) at \( t \). The flow of the system, \( \phi_t(x) : \mathbb{R}^n \to \mathbb{R}^n \), describes the future development of all \( x(0) \in W \).

An important concept in nonlinear dynamical systems is that of an attractor. An attractor is a set of numerical values that a system tends to evolve toward. This happens for a wide variety of starting conditions of the system. More formally:

**Definition 2.2.** A closed invariant set \( A \subset W \) is called an **attracting set** if there is some neighborhood \( U \) of \( A \) such that \( \phi_t(x(t)) \in U \) for all \( t > 0 \) and \( \lim_{t \to \infty} \phi_t(x(t)) = A \) for all \( x \in U \).

Additionally, a repelling set is simply defined by letting \( t \to -\infty \) in Definition 2.1. Thus, an attracting set is simply a set to which trajectories starting at an initial value in a neighborhood of the set will eventually converge. The basin of attraction \( A \) is therefore the set of all initial points which are attracted to \( A \).

**Definition 2.3.** Let \( U \) be a neighborhood of an attracting set \( A \). The **basin of attraction** \( B(A) \) is the region of the phase space \( U \) such that any initial condition in that region will eventually be iterated into the attractor.

Attracting sets can also be detected by a trapping region.

**Definition 2.4.** A closed and connected set \( D \) is a trapping region if \( \phi_t(D) \subset D \) for all \( t \geq 0 \).

There are two types of attractors which we will focus on in this section: fixed-point attractors and closed orbits.

**2.1.1 Fixed Point Attractors**

A fixed point of a function is a point that is mapped to itself by the function. In other words, if we observe the dynamical system evolving over time, the final state at which the system is in corresponds to an attracting fixed point of the function describing the system. Connecting fixed points to attractors, we can define a fixed point in the following way:

**Definition 2.5.** An invariant set \( A \) that consists of only a single element is called a **fixed point**.
A fixed point, or equilibrium point, is a type of attractor that has been featured heavily in most economic research. When discussing nonlinear systems, local and global stability properties are considered.

**Local Stability of Fixed Points**

Let \( \mathbf{x}^* = (x_1^*, \ldots, x_n^*) \) be a fixed point of (2.1) i.e. \( \dot{\mathbf{x}} = 0 = \mathbf{f}(\mathbf{x}^*) \). We introduce the following two local stability concepts, which are relevant in economic fixed point analysis:

**Definition 2.6.** A fixed point is **locally stable** (locally Lyapunov stable) if for every \( \epsilon > 0 \) there exits a \( \delta > 0 \) such that for all \( t, |\phi_t(\mathbf{x}(0)) - \mathbf{x}^*| \leq \epsilon \) whenever \( |\mathbf{x}(0) - \mathbf{x}| \leq \delta \).

We also consider asymptotically stable fixed points. These points are asymptotically stable if they are elements of the attracting set \( \mathcal{A} \).

**Definition 2.7.** A fixed point is **asymptotically stable** if for every \( \epsilon > 0 \) there exits a \( \delta > 0 \) such that \( \lim_{t \to \infty} |\phi_t(\mathbf{x}(0)) - \mathbf{x}^*| = 0 \) whenever \( |\mathbf{x}(0) - \mathbf{x}| \leq \delta \).

Figure 1 illustrates these two definitions. Note that in order to be locally stable, a trajectory starting in the neighborhood of \( \mathbf{x}^* \) is required to stay within an \( \epsilon \)-neighborhood. Alternatively, a trajectory is asymptotically stable if it starts in a \( \delta \)-neighborhood of \( \mathbf{x}^* \) and converges toward the fixed point. As time progresses forward, the distance, measured in Euclidian space, between the points \( \phi_t(\mathbf{x}(0)) \) and \( \mathbf{x}^* \) decreases.

**Global Stability of Fixed Points**

It is important to make the distinction between local and global stability of fixed points of dynamical systems. Unlike in linear systems, where local stability points imply global stability points, nonlinear systems can be characterized.
by multiple fixed points which are locally asymptotically stable or unstable. Below, we define global asymptotic stability in a similar way to local asymptotic stability, however with one modification:

**Definition 2.8.** A fixed point is **globally asymptotically stable** if it is stable and 
\[ \lim_{t \to \infty} |\phi_t(x(0)) - x^*| = 0 \]
for every \( x(0) \) in the domain of (2.1).

A useful tool to determine the global stability of a fixed point is the concept of a Lyapunov function:

**Theorem 2.1.** Let \( x^* \) be a fixed point of a differential equation system and let \( V: U \to \mathbb{R} \) be a differentiable function on some neighborhood \( U \subset \mathcal{W} \subset \mathbb{R}^n \) of \( x^* \) such that the following hold:

- (i) \( V(x^*) = 0 \) and \( V(x) > 0 \) if \( x \neq x^* \)
- (ii) \( \dot{V}(x) \leq 0 \) in \( U - \{x^*\} \)

Then \( x^* \) is stable. Further if

- (iii) \( \dot{V}(x) < 0 \) in \( U - \{x^*\} \)

then \( x^* \) is asymptotically stable.

It is important to observe that the neighborhood \( U \subset \mathcal{W} \) can be arbitrarily large. As a result, a fixed point is globally asymptotically stable if conditions (i-iii) are satisfied on the domain of (2.2).

### 2.1.2 Cyclic Attractors

The second kind of attractor is a cyclic one. The following discussion on cyclic attractors concentrates on attractors in the form of closed orbits. We say a point \( x \) is in a **closed orbit** if there exists a \( t \neq 0 \) such that \( \phi_t(x) = x \). A **limit cycle** occurs when a closed orbit is an attractor.

**Definition 2.9.** A closed orbit \( \Gamma \) is called a **limit cycle** if there is a tubular neighborhood \( U(\Gamma) \) such that for all \( x \in U(\Gamma) \), any flow \( \phi_t(x) \) approaches the closed orbit.

In most nonlinear economic applications, we wish to comment on the global behavior of the dynamical system. However, restrictions arise since it is only possible to completely categorize the global behavior of a dynamical system in two dimensions. Classifying behavior in higher dimensions is not as easy. To be able to classify the global behavior of a dynamical system in two dimensions, we turn to the Poincaré-Bendixson Theorem. We begin with a definition:

**Definition 2.10.** A \( \omega \)-**limit set** of a point \( x \in \mathcal{W} \) is the set of all points \( l \in \mathcal{W} \) with the property that there exists a sequence \( t_i \to \infty \) such that \( \lim_{i \to \infty} \phi_{t_i}(x) = l \). The \( \alpha \)-**limit set** is defined in the same way, however the sequence \( t_i \to -\infty \).

As an example note that in \( \mathbb{R}^2 \), there are three different types of limit sets that exist:
1. Fixed Point Attractors
2. Limit Cycles
3. Saddle loops

The first two types of limit sets have already been discussed. An example of a saddle loop is described in Figure 2. The dynamical system has two unstable fixed points (point A and point C) and a saddle as a third fixed point (point B). The $\omega$-limit set in this example is the union of the two loops, or the trajectory that leaves the saddle and returns to it, and the saddle point itself. Note that saddle loops can enclose closed orbits.

The Poincaré-Bendixson Theorem provides sufficient conditions for the existence of limit cycles in sub-areas of the plane. To begin, consider the two-dimensional differential equation system:

$$\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2), \\
\dot{x}_2 &= f_2(x_1, x_2),
\end{align*}\quad (2.3)$$

and let the initial point $x(0) = (x_1(0), x_2(0))$ be located in an invariant set $\mathcal{D} \subset \mathbb{R}^2$. Fixed point attractors, limit cycles, and saddle loops are all possible, when the set contains limit sets. The Poincaré-Bendixson Theorem discriminates between these different types:

**Theorem 2.2. (Poincaré-Bendixson)** A non-empty compact limit set of a $C^1$ dynamical system in $\mathbb{R}^2$ that contains no fixed points is a closed orbit.

While the fixed point has been excluded from the limit set in $\mathcal{D}$, a closed orbit in $\mathbb{R}^2$ will always enclose a fixed point.

**Theorem 2.3.** A closed trajectory of a continuously differentiable dynamical system in $\mathbb{R}^2$ must enclose a fixed point with $\dot{x}_1 = \dot{x}_2 = 0$. 
To summarize, the steps outlined are a procedure in applying the Poincaré-Bendixson Theorem to a dynamical system in $\mathbb{R}^2$:

- Locate a fixed point of the dynamical system and examine its stability properties.
- If the fixed point is unstable, proceed to search for an invariant set $D$ enclosing the fixed point. When a closed orbit does not coincide with the boundary of $D$, the vector field described by the function $f_1$ and $f_2$ must point into the interior of $D$.

The search for the set $D$ is the difficult aspect of applying the Poincaré-Bendixson Theorem to dynamical systems. However, it is easy to exclude the existence of closed orbits in a system like (2.3); consider $S$: a simply connected domain in $\mathcal{W} \subseteq \mathbb{R}^2$.

The Poincaré-Bendixson Theorem provides sufficient conditions for the existence of closed orbits in a set $D$. However the number of these orbits is not known. It is possible that more than a single orbit exists, and if several cycles exist it is impossible that all cycles are limit cycles. Given that a fixed point is unstable, the innermost cycle in $D$ is stable. Yet the most serious disadvantage of the Poincaré-Bendixson Theorem is that it is restricted to dynamical systems of dimension two only. Analogous theorems for higher dimensions don’t exist, severely limiting the analysis that can be done.

### 2.2 Conservative Dynamical Systems

If a system has a single limit cycle, then the trajectories starting at initial points in the basin of attraction are attracted by this cycle. In addition to these limit cycle systems, another type of dynamical system exists - conservative dynamical systems. The fundamental property of a conservative system is the existence of a function for the dependent variables which is constant in motion. This plays the equivalent role of “energy” in physical systems. This dynamical system is able to generate oscillations, however it is characterized by different dynamic behavior.

Consider a two-dimensional dynamical system:

\[
\begin{align*}
\dot{x} &= f_1(x, y) \\
\dot{y} &= f_2(x, y)
\end{align*}
\]

(2.4)

The Jacobian Matrix is as follows:

\[
J = \begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{pmatrix}.
\]

Let the determinant for $J$ be positive for all $(x, y)$. Note that the sign of the trace of the Jacobian - i.e. the sum of the elements in the main diagonal - plays an important role in determining the kind of oscillating behavior of a two dimensional dynamical system. We would like to be able to assign a qualitative
description to the meaning of the trace of $J$. There are three possibilities: the trace is positive, negative, or zero.

If the trace is positive, the fixed point is considered unstable, i.e. there exists a tendency away from the fixed point in all directions. We can think of this as a sort of negative friction - every point in the phase space would spiral outward, and no closed orbit would exist. Note that by Theorem 2.3, the trace of the Jacobian needs to change sign if limit cycles are to be generated. Thus a negative trace represents this idea of positive friction. Formerly exploding behavior will be dampened for all points that are sufficiently far away from the equilibrium. Therefore, a closed orbit emerges where the exploding and imploding forces both tend toward zero. Dynamical systems that exhibit this type of behavior are called dissipative systems.

Definition 2.11. A system of ordinary differential equations $\dot{x} = f(x)$ is called dissipative if there are numbers $R > 0$ and $t_1 > 0$ such that for all solutions $x^*$ of the system if $|x(0)| \leq R$ then $|x(t)| \leq R$ whenever $t > t_1$.

The term stems from the consideration of physical systems where there exists a permanent input of energy. That energy dissipates throughout the system. If the energy input is interrupted, the system collapses to its equilibrium state.

There is another class of systems that we will consider: a conservative dynamical system. A conservative system is one where no friction exists, i.e. there is no additional input or loss of energy. Keeping with the theme of classifying the value of the trace, the absence of friction seen in a conservative dynamical system corresponds to a zero trace for all points in the phase space.

Definition 2.12. Consider the Jacobian matrix for the dynamical system described in (2.4). Then, a conservative dynamical system is one where $\text{tr}(J) = 0$.

In geometric terms, conservative systems are characterized by the fact that throughout the evolutionary process, an element in the phase space changes only its shape, but the volume remains the same. Points rotate around an elliptic fixed point and volume is conserved. In dissipative systems, trajectories are attracted to a fixed point and volume shrinks. Figures 3 and 4 help further illustrate this point. Assume that a dynamical system has infinitely many closed orbits and that every initial point is located in such a closed orbit. If we consider the area A in the top diagram, initial points contained in this subset of the plane eventually move to the area B under the action of the flow. Note that the area of A is identical to the area of B. We call this dynamical system, thus, area preserving. A dynamical system that is area preserving is also conservative. This is in contrast to a dissipative system where areas get smaller, i.e. the trajectories converge to an attractor. The bottom diagram shows two trajectories which start at different initial points and approach an attractor. The area between the two trajectories is continually getting smaller and in the limit approaches zero as the trajectories approach the fixed point. This is formalized in the following definition:
Definition 2.13. Consider a differential equation in $\mathbb{R}^2$ that implies for some function $f(x)$ that $f'(x) = 0$, then

$$f(x) = a,$$

where $a$ is a constant along the trajectories of the solutions. The equation $f(x) = a$ is called the conservation law.

One such example of a conservative dynamical system in economics is the Goodwin Class struggle model, which is another form of the Lokta-Volterra Predator-Prey Model.
3 The Predator-Prey Model

We begin with a brief introduction to the predator-prey model. For further reading, see Lotka [12] and Volterra [23]. The predator-prey model is one of the first mathematical models for biological systems. Originally, it was based on interaction between predatory and non-predatory fish in the Adriatic sea. First, we derive it intuitively, and then through a more rigorous mathematical analysis, consider the system.

In the 1920’s, the observed amount of predatory fish in the Adriatic was higher than expected. Given the destruction of many of the fisheries during the first world war, one would anticipate that less prey in a given environment would also lead to less predators. However, the observations contradicted this notion.

Volterra, a mathematical biologist, wanted to model this dynamic based on his observations. To begin, Volterra assumed that under the absence of a predator, \( y = 0 \), the growth rate of prey is given by a constant, \( a \). Further, this growth rate would be dependent on the density of the predator population, \( y \), with a linear factor \( b \). This leads to the construction of our first equation:

\[
\frac{\dot{x}}{x} = a - by
\]

where \( a, b > 0 \). Simplifying this, we get

\[
\dot{x} = (a - by)x.
\]

When considering the growth of the predator, Volterra assumed if there is no prey available, \( x = 0 \), then the predator population dies, which is given by a constant decay rate \( -c \). Just as the prey population depends on the density of the predator, the predator depends on the population of the prey, leading to the following equation:

\[
\frac{\dot{y}}{y} = -c + dx
\]

where \( c, d > 0 \). This simplifies to

\[
\dot{y} = (-c + dx)y.
\]

Thus we are left with a system of equations, which forms the Lotka-Volterra predator-prey model:

\[
\begin{align*}
\dot{x} &= ax - bxy \\
\dot{y} &= -cy + dxy
\end{align*}
\] (3.1)

\( x \) represent the total prey population and \( y \) represents the total predator population. These equations describe the dynamical change in both populations at a given point in time. The dependency between the two variables is illustrated in Figure 5. This is an example of a variable dependency graph, which will be formally defined in Definition 5.8. Notice that there are two main reinforcing loops: the stock of prey is dependent on the net growth of prey and the
stock of predators changes with prey’s net growth. This balancing loop creates oscillations.

A more formal mathematical consideration of the system and its dynamically relevant attributes is now in order. Consider again the system outlined in (3.1). It is easy to see that system (3.1) has two fixed points when \( \dot{x} = \dot{y} = 0 \). These are the trivial fixed point \( (x^*, y^*) = (0, 0) \) - a saddle point - and \( (x^*, y^*) = \left( \frac{c}{b}, \frac{a}{b} \right) \), a stable center. The Jacobian matrix is:

\[
J = \begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{pmatrix}
= \begin{pmatrix}
a - by & -bx \\
y & -c + dx
\end{pmatrix}
\]

To see that \( (x^*, y^*) = (0, 0) \), we note that the Jacobian simplifies to:

\[
J = \begin{pmatrix}
a & 0 \\
0 & -c
\end{pmatrix}
\]

which has the corresponding eigenvalues \( a \) and \( -c \). Thus it is an unstable saddle point. Considering the other equilibrium, note that when we substitute \( (x^*, y^*) = \left( \frac{c}{b}, \frac{a}{b} \right) \) into the Jacobian matrix we get:

\[
J = \begin{pmatrix}
0 & \frac{-bc}{ab} \\
\frac{da}{c} & 0
\end{pmatrix}.
\]

Notice that \( \det(J) = ac \) and \( \text{tr}(J) = 0 \). Thus the eigenvalues associated with this matrix are purely imaginary, meaning the fixed point is neutrally stable. As a result, we cannot draw any conclusions on the dynamic behavior of system (3.1) by inspecting the Jacobian.

To study the global dynamic behavior of system (3.1), we introduce the concept of the first integral:
Definition 3.1. A continuously differentiable function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a first integral of a system $\dot{x} = f(x)$ where $x \in \mathbb{R}^2$, if $F$ is constant for any solution $x(t)$ of the system.

It is important to note that if a first integral exists, it is not unique. If $F(x)$ is a first integral, then so is $F(x) + C$, for some constant $C$. Further, the constancy of $F(x)$ is expressed as $\frac{dF(x)}{dt} = 0$; the constant expression $F(x) + C$ defines level curves for different values of the constant $C$. If the saddle point is the only fixed point, the level curves are given by stable and unstable manifolds. However, if the unique fixed point is a center, the level curves are closed orbits and any initial point is located in a closed orbit (except the fixed points). Figure 6 illustrates this concept. For different values of $C$, the curves $L_1$, $L_2$, and $L_3$ represent different level curves and each level curve has the property that $\frac{dF(x)}{dt} = 0$.

Proposition 3.1. When a system has a first integral, the dashed line in the diagram above does not exist.

Proof. Consider the point $x(0)$ located in $L_2$. Its trajectory for $t < 0$ and $t > 0$, which is passing through this point, is given by $\phi_t(x(0))$. Note that the point is a point in a level curve and can thus be described by the constant $F(x(0))$. The point $\phi_t(x(0))$ when $t > 0$ must also be located in the level curve, else $F(x)$ would not be constant for any solution. Then it follows that $F(x(0)) = F(\phi_t(x(0)))$ for all $t > 0$ and $t < 0$, i.e. the trajectory indicated above cannot exist when the system has a first integral.

All initial points are located on one of the infinitely many level curves determined by different values of $C$.

Theorem 3.1. The predator-prey system is a first integral.
Proof. To begin, it is helpful to eliminate the time component, which is done by dividing both equations:

\[ \frac{dy}{dx} = \frac{(c - dx)y}{(a - by)x} \]

Note we are left with two integrable equations. After rearrangement, dividing the equation by \( xy \) and integrating, we have:

\[ -aln(y) + by - cln(x) + dx = A \tag{3.3} \]

Where A is a constant. Further (3.3) can be rewritten as:

\[ y^{-a}e^{by}x^{-c}e^{dx} = B \tag{3.4} \]

if we exponentiate through. Here \( B = e^A \). If we set (3.4) equal to \( F(x, y) \), we can see that the predator-prey system is a first integral by differentiating with respect to time.

\[ \frac{d}{dt}F(x, y) = \frac{\partial F(x, y)}{\partial x} \dot{x} + \frac{\partial F(x, y)}{\partial y} \dot{y}. \]

The partial derivatives are

\[ \frac{\partial F(x, y)}{\partial x} \dot{x} = F(x, y)(-\frac{c}{x} + d) \]

and

\[ \frac{\partial F(x, y)}{\partial y} \dot{y} = F(x, y)(-\frac{a}{y} + b). \]

Substituting in, we have

\[ \frac{d}{dt}F(x, y) = F(x, y)(-\frac{c}{x} + d)(a - by)x + F(x, y)(-\frac{a}{y} + b)(-c + dx)y = 0 \]

Thus the predator prey system is a first integral. \( \Box \)

This result, combined with an earlier theorem, yields the following theorem.

**Theorem 3.2.** Every trajectory of the Lotka-Volterra equation is a closed orbit, except for the fixed point \((x^*, y^*)\) and the coordinate axes.

**Proof.** It follows that closed orbits cannot be limit cycles, otherwise the trajectories which approach limit cycles are not closed orbits. Since each point in the phase space is located in a closed orbit, the initial values of \((x(0), y(0))\) determine which of the infinitely many closed orbits describe the dynamical behavior of the system. \( \Box \)

With the help of the first integral, the predator-prey system is classified as a conservative dynamical system. A sample phase portrait for the predator-prey model is shown in Figure 7.
4 Goodwin’s Class Struggle Model

4.1 History of Macroeconomic Growth Models

The topic of modeling economic growth is one of the chief concerns of macroeconomics. Over its theoretical development, the behavior of economic systems organizes itself into a few main categories. The first and earliest type concluded that markets will always obtain a stable equilibrium; random shocks are possible and observable, but equilibrium will eventually be restored. This didn’t reflect reality very well, though.

It has long been known that growth happens in cycles. The first models were descriptive in nature. That changed when Lowe [13] called for a theoretical system to describe economic cycles. Hayek [5, 6] made the first attempt. He developed a theory of economic cycles based on an interdependent equilibrium system. This still failed to provide a full enough explanation. Keynes [10] was more successful, as he was able to develop a closed, interdependent, and consistent theoretical structure that was able to determine phenomena like aggregate output and unemployment. Yet, Keynes still failed to adequately explain the business cycle, a vital concept to understand and model in the growth of economies.

Keynes’ theoretical structure needed to be extended into a more dynamic long-run perspective. The Oxbridge Phase of the Keynesian Revolution provided the extension to Keynes’ General Theory. The Oxbridge Phase led to three developments in economics:

1. The development of multiplier-accelerator theories of cycles.
2. The development of non-linear endogenous mechanisms.
3. The development of Keynesian cycle theory.
The multiplier principle implies that investment will increase output, while the accelerator principle suggests that greater output will increase investment. Thus a feedback cycle is developed. This idea is attributed to Harrod [4] who adopted this theory to create cycles of growth. Harrod’s theory was later mathematized by Hicks [7]. Kaldor [9] and Goodwin [3] worked on the development of endogenous cycles. These different models, which are non-linear in structure, use mathematics to address the dynamics of income distribution. The structure and methods found in Kaldor and Goodwin’s work differ greatly in both the structure and the focus of their models. Finally, the development of Keynesian cycle theory reintroduced financial variables, moving beyond output growth and income distribution. The two most well known models within this realm are Keynes-Wicksell’s [21] model on monetary policy and Minsky’s [16] financial cycle theories.

The remainder of this paper focuses on Goodwin’s most famous model of endogenous growth cycles as well, as some of the modifications to it. In 1967 Goodwin [2] introduced a simplistic model about the dynamics of wages and employment. His model is analogous to the Lotka-Volterra predator-prey model, where wages correspond to predators and employment corresponds to prey. A brief explanation yields some intuition to why this model makes sense.

In this economy there are two groups: workers and employers; each has some bargaining power. At high levels of employment, the bargaining power of workers is high. They are able to drive up wages, thus reducing profits. As profit levels fall, fewer workers are able to be hired, yielding lower levels of employment. This low level of employment means lower wages and therefore higher profits. With higher profits, more workers are hired and the employment level rises again. A cyclic pattern emerges.

Goodwin’s model originates from an idea proposed by Karl Marx in Capital [15]. Marx argued “capitalism’s alternate ups and downs can be explained by the dynamic interaction of profits, wages, and employment.” The growth of profits fuels production and thereby labor demand; wages rise and shrink profits, which erode the basis for accelerated accumulation. Further, Marx argues that the employment rate and worker’s share of income triggers these cycles. Goodwin formalized this concept with mathematics, which is done through the framework of the predator-prey model.

These cycles are not to be confused with the business cycle which are also apparent in economic growth. However, the business cycle can affect wage-employment cycles, and fluctuations in wage-employment cycles can affect business cycles, leading to recessions. Figure 8 shows output growth, unemployment and workers’ share (net wages as a share of national income) in the United States from 1996 to 2015. A somewhat cyclic pattern is apparent. High levels of output growth, a high rate of employment (low rate of unemployment), and increases in real wages are tied with the economic expansion of the late 1990s. In the early 2000s, output growth slowed substantially, real wages fell, and unemployment slightly increased. From 2006-2008, there was strong economic growth, which was again mirrored by low unemployment and increases in worker’s share of national income. With the onset of the great recession in 2008, output growth
Figure 8: Real GDP Growth, Unemployment Rate, & Worker’s Share of Domestic Income (US 1996-2015)
again slowed substantially, unemployment rose drastically, and worker’s share of national income plummeted. It is important to observe that Goodwin’s cycle does not directly deal with business cycles, since changes in employment and wages are dynamically modeled. However, Goodwin cycles can coexist with business cycles.

Goodwin’s model found attention among political economists. It’s not entirely clear if it’s due to the oscillatory properties it predicts, the suggested analogy between predator-prey interdependence and worker’s class struggle, or some combination of the two. Nevertheless, Goodwin’s Model serves as a guide to show the power of mathematics in economic modeling.

4.2 Deriving the Model

We begin with the following assumptions. The theoretical economy being considered is a closed economy, simply meaning that there is no international trade.

(A1) Technical Progress grows at a constant rate.
(A2) The labor force grows at a constant rate.
(A3) There exist only two homogeneous and non-specific factors of production: capital and labor.
(A4) All quantities are real and net.
(A5) All wages are consumed; all profits are saved and invested.
(A6) There is a constant capital-output ratio.
(A7) A real wage that rises in the neighborhood of full employment, expressed by the Phillips Curve.

Further, the following list of abbreviations, definitions, and relations describes the framework of the economy.

- Output: $Y$
- National Income: $q$
- Labor: $L$
- Capital: $k$
- Wage Rate: $w$
- Labor Productivity: $a$
- Labor Income: $wL$
- Labor Income Share: $u$
- Profit Income Share: $\pi$
- Savings: $(1 - \frac{w}{a})Y$
- Capital Output Ratio: $\sigma$
Labor Supply: $N$

Employment Rate: $v$

We begin with the tautological equation

$$ q = \frac{Y}{L} \cdot \frac{L}{N} \cdot N $$

(4.1)

where $\frac{Y}{L}$ is defined to be labor productivity and $\frac{L}{N}$ is defined to be the employment rate. Thus the equation simplifies to:

$$ q = a \cdot v \cdot N. $$

(4.2)

We wish to take the logarithmic derivative with respect to time. Thus, taking the logarithmic derivative of (4.2) yields

$$ \frac{\dot{q}}{q} = \frac{\dot{a}}{a} + \frac{\dot{v}}{v} + \frac{\dot{N}}{N} $$

(4.3)

and simplifies to

$$ \frac{\dot{q}}{q} = \alpha + \frac{\dot{v}}{v} + \beta. $$

(4.4)

Here, $\alpha$ is the increase in productivity and $\beta$ is the growth rate in labor supply. Recall from our assumptions that both $\alpha$ and $\beta$ are constants.

The national income is distributed 100 percent between capitalist’s share of national income ($\pi$) and worker’s share of national income ($v$), which can be mathematically expressed by $1 = u + \pi$. Further, by assumption, since all profits are reinvested, it follows that

$$ \dot{k} = q \cdot \pi $$

$$ \dot{k} \quad q = 1 - u. $$

---

1With regard to the construction of our model, taking the logarithmic derivative allows us to understand how variables in our system change over time. Recall that if

$$ f = \frac{g}{h} $$

then by the quotient rule, the percent change over time is:

$$ \frac{\dot{f}}{f} = \frac{1}{f} \left[ \frac{d}{dt} \left( \frac{g}{h} \right) \right] $$

$$ = \frac{1}{f} \frac{gh - hg}{h^2} $$

$$ = \frac{h}{g} \left( \frac{\dot{g}}{h} - \frac{gh}{h^2} \right) $$

$$ = \frac{\dot{g}}{g} - \frac{h}{h}. $$

18
\( \sigma \) represents a constant capital output ratio. Mathematically, then, \( \sigma = \frac{k}{q} \).

Diving by \( \sigma \) yields:

\[
\frac{\dot{k}}{\sigma} = \frac{(1 - u)}{\sigma} \cdot q
\]
\[
q \cdot \frac{\dot{k}}{k} = \frac{(1 - u)}{k} \cdot q^2
\]
\[
\frac{\dot{k}}{k} = \frac{(1 - u)}{k} \cdot q
\]
\[
\frac{\dot{k}}{k} = \frac{(1 - u)}{\sigma}.
\]

We note some observations which will help us simplify our equation. Note first that the equation above states that the growth rate of capital is dependent on worker’s share of national income. As a result, we can replace the growth rate of capital with the growth rate of national income. Similarly, since we know that \( \sigma \) is constant, the growth rate of national income must be the same as the growth rate of the capital stock. Thus with these two assertions, a fluctuation in capital leads directly to a fluctuation in the national income:

\[
\frac{\dot{k}}{k} = \frac{\dot{q}}{q} = \frac{1 - u}{\sigma}.
\] (4.5)

Setting (4.4) equal to (4.5) and solving for \( \dot{v} \), we get the first differential equation describing employment rate:

\[
\dot{v} = v(t) \left( \frac{1}{\sigma} - (\alpha + \beta) - \frac{u(t)}{\sigma} \right).
\] (4.6)

Compared to the Lotka-Volterra model outlined above, (3.1) is equivalent to the function describing prey.

The second equation is established in a similar manner. To assist in defining it, we begin with the Phillips Curve. Phillips [17] attempted to estimate a correlation between changes in wages and unemployment rate. Goodwin took advantage of this relationship, however for simplicity, he proposed a linearized version. While the exact relationship is not known, this linearized version will suffice to model the relationship. To transform the unemployment rate into the employment rate, we begin by noting that the labor supply in our economy is composed fully of employed and unemployed workers, \( i.e. \; 1 = v + z \) where \( z \) is defined to be the unemployment rate. Figure 9 graphically notes the transformation of the Phillips curve from an unemployment rate to an employment rate.

With the transformed and linearized Phillips curve, we note

\[
\frac{\dot{w}}{w} = -\gamma + \rho v.
\] (4.7)
Here, $\gamma$ is defined to be the lines intersection with the $y$-axis and $\rho$ is the slope of the line. We define worker’s share of national income as

$$u = \frac{w}{a}. \quad (4.8)$$

A logarithmic differentiation with respect to time yields the growth rate of worker’s share of national income

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \frac{\dot{a}}{a}$$

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \alpha. \quad (4.9)$$

Finally, plugging in (4.7) into (4.9) and solving for $u$ yields the following equations for change in worker’s share of national income:

$$\dot{u} = u(t) \left(- (\alpha + \gamma) + (\rho v(t))\right) \quad (4.10)$$

The differential equation describing the dynamics of worker’s share of national income is analogous to the equation describing the predator population in the Lotka-Volterra model.

To summarize, we have a system of differential equations to describe the Goodwin Model:

$$\dot{v} = v(t) \left(\frac{1}{\sigma} - (\alpha + \beta) - \frac{u(t)}{\sigma}\right) \quad (4.11)$$

$$\dot{u} = u(t) \left(- (\alpha + \gamma) + (\rho v(t))\right).$$

This is simply a version of the predator-prey model, where $a = \frac{-(\alpha + \beta)}{\sigma}$, $b = \frac{1}{\sigma}$, $c = \alpha + \gamma$ and $d = \rho v$. Economically speaking the first summand in the
Figure 10: Behavior of Worker’s Share and Employment Rate in Different Sectors

parenthesis represents the natural growth rate of each variable, while the second summand gives the density. When there is no employment, worker’s share of national income tends to zero; when worker’s share of income tends to zero, the employment rate increases since no relative labor costs exist.

As with the predator-prey model, (4.11) has two fixed points: the trivial fixed point at the origin and the non-trivial fixed point. Since the trivial fixed point makes little economic sense, it is ignored. The non-trivial fixed point is

\[ v^* = \frac{\alpha + \gamma}{\rho}, \]
\[ u^* = 1 - \sigma(\alpha + \beta). \]

The Jacobian evaluated at the non-trivial fixed points is

\[ J = \begin{pmatrix} 0 & -\frac{\alpha + \gamma}{\rho} \\ \rho(1 - \sigma(\alpha + \beta)) & 0 \end{pmatrix}. \]

(4.12)

As the system (4.11) and the Jacobian (4.12) are structurally identical to the Lotka-Volterra equations, the Goodwin model is a conservative dynamical system. Thus every initial point in the model is located in a closed orbit. The behavior is like that of the general predator-prey model, and is shown in Figure 10. There are four quadrants regarding the fixed point \((u^*, v^*)\). The small arrows indicate the behavior of worker’s share of national income and the employment rate. It can be described as follows: when labor share of national income \(u\) is greater than \(u^*\) and the employment ratio \(v\) is also greater than \(v^*\), the main economic variables considered in the Goodwin model all fall down because the pressure put on capitalist’s profits weakens the investment activities. The wage continues to rise until \(v\) becomes less than or equal to \(v^*\).
The period of the cycles is like that in the Lotka-Volterra model with respect to the variables:

\[ T = \frac{2\pi}{\sqrt{(\alpha + \gamma) \cdot \left(\frac{1}{\sigma} - (\alpha + \beta)\right)}} \]

Note the difference between the fixed points in the Goodwin model as compared to the Lotka-Volterra model. The increase in productivity, \( \alpha \), is implemented into both differential equations. To achieve a closed orbit around a defined coordinate \((u^*, v^*)\), the values of the variables must be in a specific ratio to each other.

The dynamics of this model lend theoretical credibility to Goodwin’s and thereby Marx’s belief that a capitalist economy is permanently oscillating. It is important to note, in an economic context, that the analogy is superficial, as it does not refer directly to the functional income shares of capitalists and their workers or even to their population size.

### 4.3 Updated Goodwin Models

As with the Lotka-Volterra model, the Goodwin model has been criticized as too simplistic; namely that the model is composed under an isolated set of assumptions which most likely do not reflect reality greatly. The model is simple and exhibits oscillations, as shown. However, the property of structural instability limits its applications. Many modifications have been made to the model to more accurately model changes in employment rate and worker’s share of national income. A few of these modifications, as well as their dynamical behavior, are discussed below.

The Lotka-Volterra model, and thus the original Goodwin model, are dynamical systems whose behavior is very sensitive to small alterations in their functional structure. We call dynamical systems of this sort \textit{structurally unstable}. A number of the updates to the Goodwin model focus on the structural instability that is found in all conservative dynamical systems. To demonstrate the effect of small perturbations, we consider an arbitrary but simple modification to the Phillips curve. Before we assumed that the changes in wages were dependent only on the employment rate. Now assume that they are dependent on employment rate and worker’s share of national income, a scenario that is a bit more realistic:

\[ \frac{\dot{w}}{w} = f(v) + g(u). \]

We assume that worker’s share increase if workers are at a disadvantage in the functional income distribution, which is reflected as \( g(u) > 0 \ \forall u \) and \( g'(u) < 0 \).

With this modified Phillips curve being considered, the updated Goodwin Model leads equations are as follows:

\[
\begin{align*}
\dot{v} &= v(t) \left( \frac{1}{\sigma} - (\alpha + \beta) - \frac{u(t)}{\sigma} \right) \\
\dot{u} &= u(t) \left( -\alpha - \gamma + (\rho v) + g(u) \right)
\end{align*}
\]
with the non new trivial fixed points being

\[ v^* = \frac{\alpha + \gamma - g(u)}{\rho} \]
\[ = \frac{\alpha + \gamma - \left( \frac{1}{\sigma} - (\alpha + \beta) \right)}{\rho} \]
\[ u^* = 1 - \sigma (\alpha + \beta). \]

The Jacobian evaluated at these fixed points is

\[ J = \begin{pmatrix} 0 & g(u^*) - (\alpha + \gamma) \\ \rho (1 - \sigma (\alpha + \beta)) & \rho g'(u^*) \left( \frac{1}{\sigma} - (\alpha + \beta) \right) \end{pmatrix}. \]

Unlike before, the determinant of \( J \) is not always positive. Suppose that \( g'(u^*) > 0 \) is sufficiently small such that \( J \) is in fact positive. The trace of \( J \) will be different from zero even for a small value of \( g(u^*) \), \( u^* \neq 0 \). By assumption of \( g'(u) < 0 \), the trace is also negative. Hence, the real parts of the complex conjugate eigenvalues are negative and the fixed point \( (v^*, u^*) \) is locally asymptotically stable. Thus system (4.13) possess an attractor and is no longer a conservative system but rather a dissipative system.

Similar modifications, with an extended Phillips curve \( f(u, v) \), done by Cugno and Montrucchio \[1\], were able to provide global stability results. Other modifications can also be constructed which yield similar results. Samuelson \[20, 19\] showed that when considering diminishing returns in the Lotka-Volterra framework, it can change the classical behavior of a conservative system. While Samuelson did not consider the Goodwin model directly - he commented directly on the general Lotka-Volterra model - diminishing or increasing returns to scale are considered by assuming the capital-output ratio changes with output, rather than being held constant. In fact, the addition of a term that influences the growth rate of some variable and which depends on on the value of this variable is equivalent to the introduction of a dampening effect. The conservative framework of the original Goodwin equations will change and result in a dissipative dynamical system.

Increasing the dimension of the original model by considering additional state variables is another way to modify Goodwin’s work. Additional state variables may include capital-output ratios, or non-constant growth rates of the labor supply and labor productivity. It is also possible to increase the dimension by introducing a lag-structure, which is also more reflective of economic reality. Vadasz \[22\] considers a number of these variations and implements them into one cohesive model.

Recall that in the absence of predators, prey grow exponentially. Goodwin’s original model assumed this. Economically, this means that in the absence of wages, employment grows at an exponential rate and quickly surpasses full employment, growing without bound. In reality, however, the average labor share - defined as the ratio of employed workers to the working age population (people between 15-65) - is steadily below 75 percent. A number of problems
arise when this type of growth is considered. First, employment cannot increase without limits and setbacks to productivity gains. In fact, diminishing returns to productivity are typically found. Further, workers are not homogenous and all do not have the same level of productivity. For example, during the periods where labor is being shed (employment rate is falling), the less trained and less productive workers will be the first to go. Similarly in the opposite case, when employment rates are increasing, the most productive will be hired first, though whatever is available in the labor market will also be hired, meaning less knowledgeable and skilled workers.

To make the model more representative of reality, Vadasz considers in the absence of wages, employment growing according to a logistic growth function, rather than an exponential growth function

\[
\dot{v} = \left( \frac{1}{\sigma} - (\alpha + \beta) \right) \frac{(1 - v)}{K} v
\]

where \( K \) represents the carrying capacity, or an upper bound on employment growth. In his model, Vadasz lets \( K = 1 \) which economically translates into labor not being able to surpass the total population in a closed economy.

Similarly, Vadasz considers the reaction of labor share of national income to employment and whether those two variables move in tandem. Changes in employment do not have an instantaneous effect on the labor’s share. Rather, labor’s share, reflected in wages, are sticky since they are determined in advance by contracts and rarely do those contracts take into effect future changes in demand for labor. In a recession, for example, wage rates react sluggishly to growing unemployment. The lowest wage rate usually is attained when the economy is already growing again. The delay can be modeled by substituting in a weight function to equation 2 of system (4.11). We replace \( v \) with \( z \)

\[
z = \int_0^t u(\tau)G(t - \tau)d\tau
\]

where \( G \) is a nonnegative integrable weight function with the following property:

\[
\int_{-\infty}^t G(t - \tau)d\tau = \int_0^\infty G(s)ds = 1.
\]

The growth rate of worker’s share of national income depends on the employment rate in the past. The way in which the growth rate depends on past values is determined by the choice of weight function. The choice of weight function further looks to remove the structural instability of original Goodwin model. Consider the weight function to be

\[
G(s) = G_1(s) = ae^{-as}, \quad a > 0.
\]

Here a \( \frac{1}{a} \) discount reaction of worker’s share of national income is considered. Employers take into consideration the changes in capital when setting wage contracts. Recall that, by assumption, the capital ratio is fixed. Since this is
the case, and output is directly related to employment, capital depreciation will
cause employers to discount past employment levels when setting wages by a
discount rate $a$. Further, past employment levels that happen further away will
have a smaller effect on labor’s share of national income, and will continue to
decrease exponentially as time goes on.

We then have an extended Goodwin model where

$$
\dot{v} = v(t)\left(\frac{1}{\sigma} - (\alpha + \beta)\right)\frac{(1 - v)}{K} - \frac{u}{\sigma}
$$

$$
\dot{u} = -u(t) \cdot (\alpha + \gamma) + \rho z
$$

$$
= -u(t) \cdot (\alpha + \gamma) + \rho \int_0^t u(\tau)ae^{-a(t-\tau)}d\tau.
$$

Note that differentiating $z$ with respect to time yields the ordinary differential
equation

$$
\dot{z} = a(v - z)
$$

and thus with the previously derived equations, the Goodwin model turns into
a three-dimensional system

$$
\dot{v} = v(t)\left(\frac{1}{\sigma} - (\alpha + \beta)\right)\frac{(1 - v)}{K} - \frac{u}{\sigma}
$$

$$
\dot{u} = -u(t) \cdot (\alpha + \gamma) + \rho \int_0^t u(\tau)ae^{-a(t-\tau)}d\tau
$$

$$
\dot{z} = a(v - z).
$$

The system has a fairly straightforward economic interpretation. In the labor
market, labor’s share of national income will not be set by actual employment
$v$ but according to expectations of future employment levels, which are based
on past employment levels. These expectations will continuously change and
correct themselves as expressed by $z$.

The system can be shown to have three fixed points. The trivial and unstable
fixed points $(0, 0, 0)$ and $(1, 0, 1)$, which represent the absence of wages. The
nontrivial fixed point is

$$
v^* = \frac{\alpha + \gamma}{\rho}
$$

$$
u^* = (\rho - (\alpha + \gamma)) \cdot \sigma(1 - \sigma(\alpha + \beta))
$$

$$
z^* = \frac{\alpha + \gamma}{\rho}.
$$

The conservative character of the original two-dimensional Goodwin model has
disappeared through the introduction of an exponential lag structure. As a
result, a dissipative system has emerged.

Since the Goodwin model is sensitive to small perturbations and suffers from
structural instabilities, as soon as a dissipative structure prevails, this modified
Goodwin system can exhibit converging or diverging oscillations and limit cycles depending on the assumed dampening or forcing terms present. These modifications still allow for oscillating behavior of the economically most relevant variables, like labor share of national income and employment rate. Further modifications to the model include relaxing the assumption that the economy being modeled is closed and considering an open economy with international trade, which will be explored further later in the paper.

5 Sheaf Methods for Analyzing Complex Systems

5.1 Motivation

Complex dynamical systems, including the ones discussed above, can be difficult to construct to accurately model specific phenomena. They are even harder to study and analyze. We seek to unlock a set of tools to help in this construction and analysis of these complex systems. Sheaf theory provides an avenue to do so, lending a hand in constructing and analyzing models that are described by a system of equations. Sheaves, which will be formally defined later, are a way to locally track data and synthesize these local bits into a consistent whole. Complex multi-model systems, then, are easily stitched together from diagrams of smaller models. This notion can be abstracted into a more general framework, which is done by Robinson [18].

In general, there is an interaction between the individual models and how they interact together. To abstract to its most general setting, the models consist of spaces and maps between them; thus it is natural to consider the system’s topology. By modeling the topology of the system first and then the spaces and maps of the individual models that are specified by the system, this construction and procedure naturally leads to sheaves. When dealing with sheaves, it is important to note which type of sheaf we are analyzing and a kind of topological space. The theory going forward is built over partial orders, which leads to a balance of expressivity and computational ability that other space types do not readily lend to. The multi-modal systems considered here are sheaves of smooth finite dimensional manifolds on posets.

The construction of sheaves is the most fundamental way to express the topological relationships between the variables and equations. It also yields a diagrammatic intuition for how variables and equations are all simultaneously related. Further, with the sheaf model, any analysis done on the sheaf is equivalent to analysis done on the system of equations, allowing us to study and analyze locally and globally stable states of the system.

In summary, the sheaf construction makes apparent and possible two capabilities:

1. It allows one to combine seemingly different dynamical models into a multi-model system in a fundamental and consistent manner.

2. The sheaf diagram allows for analysis of local and global sections to be
done in an easier manner than would be on the system of equations.

5.2 Goodwin Model as a Sheaf

Recall the Lotka-Volterra model:

\[
\frac{dv}{dt} = \dot{v} = v(t) \left( \frac{1}{\sigma} - (\alpha + \beta) - \frac{u(t)}{\sigma} \right)
\]
\[
\frac{du}{dt} = \dot{u} = u(t) \left( -\gamma + (\rho v(t)) \right).
\]

The dynamical model presented here involves a collection of two state variables \((u, v)\) and two equations \((\dot{u}, \dot{v})\), which are both functions of time. For both equations, the values of \(u\) and \(v\) determine all future values for the equation. As discussed above, the solution exhibits interesting oscillatory behavior. One way to gain an understanding in the behavior of the solutions is to build out some visual representations of the system. These diagrams will highlight on the nuances of the causal relationships between the different state variables.

Figure 11 describes the most basic dependency relationship between the state variables. Here an arrow from one variable to the next - i.e. \(u \rightarrow v\) - states that the future value of \(v\) is partially dependent on the current value of \(u\). This dependency diagram, however, isn’t entirely representative of what the equations are describing, since there is no notion of a derivative present. Now, if we include the derivatives of the state variables as state variables, we construct a larger diagram which provides a bit more information about how the derivatives are determined by the values of the state variables. Yet, it is not complete. It fails to fully describe the relationship between the derivative and the state variable, since, for example, \(\frac{du}{dt}\) is determined both by the values of \(u\) (alone) and by \(u\) and \(v\) through the first equation in our system. Our next step is for the encoding of this model to contain all of this information into the diagram.

This encoding can be done simply by reinterpreting what the arrows mean in our dependency diagram. Previously we interpreted the arrow to mean that the variable at the head of the arrow is dependent on the variable at the tail of the arrow. If we interpret the arrow as a functional relationship rather than a simple dependency, this added information is encoded. This stronger requirement is not present in either Figure 11 or 12. It is fairly clear why it is not present in the Figure 11; a functional dependence is not present in Figure 12 since, from how our system is defined, the formula \(\frac{du}{dt}\) is dependent on both \(u\) and \(v\). To define the functional dependence between \(u, v\), and \(\frac{du}{dt}\), it needs to come from the pair
Figure 12: Expanded Dependency Graph

Figure 13: Functional Dependencies - Variable Names

(u, v). Performing this transformation to the dependency diagram, we obtain the following figures. The next two figures are composed of two different types of functional dependency graphs; Figure 13 shows the functional dependencies between the state variables and their derivatives according to variable names, while Figure 14 shows the same relationship according to the spaces of values involved.

There are a number of advantages to the functional dependency diagram described above. A useful property of their structure is that sheaves are path independent, i.e. the diagram commutes. Second, the arrows from the variables or spaces are actual functions, and if one so desires, they can be labeled as such. The arrows which correspond to the pairs of variables to an individual variable (i.e. (u, v) → u) are projections, while the others are determined the definition of derivative, and by the equation of the system. A third advantage to this diagram is that all information from our system is captured in the dependency diagram and the equations can be recovered from the diagram. Analysis that is applied to the diagram, such as finding and studying global and local sections, is equivalent to finding those in our system of equations.

It is useful to look at the number of functional equations that constrain a variable value. This is defined as the in-degree of a variable. Our functional dependence diagram easily allows us to identify the in-degree values of variables
in our system. Independent variables are those which have no arrows pointing into them, i.e. the pair \((u, v)\). However, simply because they are independent variables doesn’t mean that they have no constraints, just that there are no functional dependencies. Constraints, on the other hand, arise by requiring each value to take on only one value. For example, if a variable is determined by two functional equations, as is the case in the Goodwin model, the independent variables in those two equations must be chosen compatibly. There are two other possibilities for the variables: they are completely dependent or intermediate variables. Both \(\frac{du}{dt}\) and \(\frac{dv}{dt}\) are examples of dependent variables since there exist no arrows going out of them, while the variables \(u\) and \(v\) are intermediate variables since arrows go in and out of them.

Figure 13 is that of a partially ordered set. The advantage of this method is the partial order ranks the variables in our system according to their independence of each other. The arrows in the diagram point from lower variables to higher variables in our partial order. Thus we can think of the most independent variables as the minimal elements of the partial order and the completely dependent variables as the maximal elements of the partial order.

It is easy to see Figure 14 has the same diagrammatic structure as the partial order. However, the labeling is different. This mathematical representation presented here is the sheaf of the Goodwin model. Recall that the sheaf is a way to represent local consistency relationships. As will be outlined in the next subsection, sheaves are equipped with a number of useful properties which yield descriptive power for systems of equations. Further, simply observing the structure of the dependency diagram is generally illuminating in that it helps us understand the sometimes complex relationships among variables.

5.3 Mathematical Construction: Sheaves on Posets

According to the analysis outlined by Robinson, topological spaces in their full generality admit some properties that are not reflected in practical models and
thus needs constraints. Other topological spaces aside from partially ordered sets vary in expressivity. Cell complexes, locally finite topological spaces, abstract simplicial complexes, and partial orders are most useful for modeling systems. Partial orders, however, appear to be the most useful as each computational example can be expressed with them [18]. What follows, therefore, is a description of sheaves on partial orders, since they are the primary mathematical tool which will be used going forward. For computational ease, we will be dealing with locally finite posets.

**Definition 5.1.** A partial order on a set $P$ is a relation $\leq$ on the set that is
1. Reflexive: $x \leq x \forall x \in P$
2. Antisymmetric: If $x \leq y$ and $y \leq x$, then $x = y$
3. Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$.

A pair $(P, \leq)$ is called a partially ordered set (or poset for short). Typically when context is clear, we will denote $P = (P, \leq)$. Further, a poset is called locally finite if the set $\{z \in P : x \leq z \leq y\}$ is finite, given every pair $x, y \in P$.

Figure 15 shows a number of diagrams. The diagram to the left is that of a poset $P$ with four elements. The poset is to be interpreted as follows: $d \leq a \leq c$ in $P$. The figure also includes a diagram for a sheaf. The sheaf can be explained in terms of the diagram of the poset. In the poset, each vertex represents an element and each arrow points from lesser elements to greater ones. By replacing each vertex and arrow by a set (or space) and a function respectively, such that the composition is path independent (it does not matter which path is taken to get from $\mathcal{S}(d)$ to $\mathcal{S}(c)$), we have created a sheaf. If all of the functions’ inputs are at the tail of the arrow, then the diagram is that of a sheaf on the Alexandroff topology (denoted by $\mathcal{S}(\_)$).

**Definition 5.2.** [18] In a poset $(P, \leq)$, the collection of sets of the form
$$U_x = \{y \in P : x \leq y\}$$
for each $x \in P$ forms a base for a topology, called the Alexandroff topology, shown in Figure 15.

This topology is particularly important as it can be built from a partial order.
Definition 5.3. \[18\] Suppose that \( P = (P, \leq) \) is a poset. Then, a sheaf \( \mathcal{S} \) of sets on \( P \) with the Alexandroff topology consists of the following:

1. For each \( p \in P \), a set \( \mathcal{S}(p) \), is called the stalk at \( p \),
2. For each pair \( p \leq q \in P \), there exists a function \( \mathcal{S}(p \leq q) : \mathcal{S}(p) \to \mathcal{S}(q) \), which is called a restriction function (restriction), such that
3. For each triple \( p \leq q \leq r \in P \), \( \mathcal{S}(p \leq r) = \mathcal{S}(q \leq r) \circ \mathcal{S}(p \leq q) \), i.e. the diagram commutes.

If any of the conditions above are not satisfied, we do not have a sheaf. These constructions are called diagrams rather than sheaves.

Robinson observes that the stalks can have structure: they can be vector spaces or topological spaces, for example. Suppose that the given stalks have structure, and the restriction or extension functions preserve that structure. We are then left with a sheaf of that type of structure. For example, a sheaf of vector spaces that has linear functions for each restriction map.

The encoding of systems of equations as sheaves illustrates a number of consistencies and inconsistencies between the component models \[18\]. Elements of the stalks that are mutually consistent across the entire sheaf diagram - and thereby the entire system - are called sections. In other words, the output of the combined system that corresponds to satisfying the system of equations are the sections.

Definition 5.4. \[18\] A global section of a sheaf \( \mathcal{S} \) on a poset \( P \) is an element \( s \) of the direct product \( \prod_{x \in P} \mathcal{S}(x) \) such that for all \( x \leq y \in P \), \( \mathcal{S}(x \leq y)(s(x)) = s(y) \). A local section is defined similarly, however, it is only defined on a some subset \( Q \subseteq P \).

5.4 Systems of Equations

To begin, consider a multi-model system described by a system of equations. Here, the set of variables \( V \) values lie in the set \( W \) for all \( v \in V \). Each variable is further interrelated though a set of equations \( E \) such that each equation \( e \in E \) specifies a list of variables \( V_e \subset V \) and a subset of solutions \( S_e \subseteq \prod_{v \in V_e} W_v \). Notice that there exist a natural projection functions for each \( x \in V_e \) i.e. \( pr_x : \prod_{v \in V_e} W_v \to W_x \). This allows us to define a poset structure, since these projection functions restrict to functions on \( S_e \) such as \( pr_x : S_e \to W_x \). The poset structure is defined in the following way. Let \( P = V \cup E \), i.e. variables of \( P \) are either variables are equations and define \( e \leq v \) if \( v \in V_e \). This then defines a partial order on \( P \) if we assume that \( \leq \) is reflexive.

Definition 5.5. \[18\] A sheaf \( \mathcal{E} \) on \( (P, \leq) \) can be defined by specifying the following:

1. \( \mathcal{E}(v) = W_v \) for every variable \( v \),
2. \( \mathcal{E}(e) = \prod_{v \in V_e} W_v \) for every equation \( e \),
3. \( \mathcal{E}(e \leq v) = p_v \) whenever \( e \leq v \).

A number of important claims follow about the sections of the system of equations and how they correspond to the sections of the sheaf.
Proposition 5.1. Assume that each variable $v$ appears in at least one equation. Then the set of sections of $E'$ is in one-to-one correspondence with $\prod_{v \in V} W_v$.

Proof. Notice that each section of $E'$ specifies all values of all variables. This follows since each $v \in P$ and its stalk corresponds to its respective space of values. Further, specifying the value of each variable clearly specifies a section of $E'$.

The aggregation of sheaf $E'$ does not account for the actual equations; rather it simply specifies the variables which are involved. To account for this loss of information, we construct a sub-sheaf $E$ of $E'$.

Definition 5.6. The solution sheaf $E$ of a system of equations is given by the following:

1. $E(v) = W_v$ for every variable $v$,
2. $E(e) = S_e$ for every equation $e$,
3. $E(e \leq v) = p_v$ whenever $e \leq v$.

Recall that $S_e \subseteq \prod_{v \in V} W_v$ is the set of solutions to $e$. Further, the next proposition outlines a rather important fact about the relationship between the sections of $E$ and the solutions to the system of equations.

Proposition 5.2. The sections of $E$ consist of solutions to the simultaneous system of equations.

Proof. First observe that a section $s$ of $E$ specifies an element $s(e) \in S_e$ for every equation $e \in E$ that satisfies that condition. On the other hand, let the solution to a simultaneous system of equations be given. Then this is a specification of some element $x \in \prod_{v \in V} W_v$ where the projection of $x$ onto $\prod_{e \in V} W_v$ lies in $S_e$. In other words, an assignment onto each variable $v \in V$ is given by the following:

$$s(v) = pr_v x, \quad s(e) = pr_{E'(e)} x.$$ 

Observe that by construction $s(e) \in S_e = F(e)$.

Up to this point we have been dealing with systems of arbitrary equations. However, there is generally more structure available to us. When available, the stalks of the sheaf $E$ over the variables can be reduced in size. This results in computational savings. Further, equations usually take on the form

$$v_{n+1} = f_e(v_1, \ldots, v_n).$$

In the following cases, it is most helpful to employ a dependency graph to visualize the relationships among the variables.

Definition 5.7. A system of equations $E$ on variables $V$ is called explicit if there is an injective function $\gamma : E \to V$ that selects a specific variable from each equation such that each equation $e \in E$ has the form

$$\gamma(e) = f_e(v_1 \ldots v_n)$$
such that $\gamma(e) \in V(e)$ and $\gamma \notin \{v_1 \ldots v_n\}$. If any variable is outside the image of $\gamma$, then that variable is said to be free or independent. Conversely, if a variable is inside the image of $\gamma$ then that variable is said to be dependent.

When dealing with an explicit system a dependency graph can be defined in the following way:

**Definition 5.8.** A **variable dependency graph** for an explicit system is a directed graph $G$ whose vertices are are given by the union $E \bigcup (V \setminus \gamma(E))$. This consists of the set of equations and free variables such that the following hold:

1. Free variables have in-degree zero,
2. If $e$ is a vertex of $G$ corresponding to an equation whose incoming edges are given by $(e_1 \rightarrow e), \ldots, (e_n \rightarrow e)$, then the equation $e \in E$ is of the form
   \[ \gamma(e) = f_e(\gamma(e_1), \ldots \gamma(e_n)). \]

**Example 5.1.** Recall the Goodwin model defined by (4.11). Then note that the Goodwin model is an explicit system and its dependency graph is shown in Figures 11 and 12 in subsection 5.2.

If $E$ is an explicit system of equations with variable $V$, then it is possible to construct the explicit solution sheaf $\mathcal{G}$. The underlying poset for $\mathcal{G}$ is still given by the union of the variables and the equations, however the stalks and restriction maps are slightly different.

**Definition 5.9.** The **explicit solution sheaf** $\mathcal{G}$ whose sections are the simultaneous solutions of $E$ is defined by the following:

1. $\mathcal{G}(v) = W_v$ for each variable $v \in V$,
2. $\mathcal{G}(v) = \prod_{x \in V \setminus \gamma(e)} W_x$,
3. $\mathcal{G}(e \leq \gamma(e)) = f_e$, and
4. $\mathcal{G}(e \leq v) : \prod_{x \in V \setminus \gamma(e)} W_x \rightarrow W_v$ is given by an appropriate projection if $v \neq \gamma(e).

**Example 5.2.** The explicit solution sheaf for the Goodwin system defined in (4.11) is shown in Figure 13 in subsection 5.2.

**Proposition 5.3.** The sections of an explicit solution sheaf $\mathcal{G}$ are in one-to-one correspondence with the simultaneous solutions of its system of equations.

**Proof.** Assume that $e$ is an equation in the explicit system of the form
\[ \gamma(e) = f_e(\gamma(e_1), \ldots \gamma(e_n)). \]
Then notice that $S_e = \{v_1, \ldots v_n, f_e(\gamma(e_1), \ldots \gamma(e_n)) : v_i \in W_{\gamma(e_i)}\}$. Thus, the Proposition follows directly from Proposition 5.2. \qed
5.4.1 Ordinary Differential Equations

The framework developed in the previous section extends nicely to ordinary differential equations, which is the main object being considered throughout this paper. Differential equations give rise to sheaves of solutions, upon which various types of analyses can be conducted. Consider an ordinary differential equation, given by

\[ u' = f(u), \]

where \( u \in C^1(\mathbb{R}, \mathbb{R}^d) \) is a continuously differentiable function. There are two ways to consider \( u \) and \( u' \): either as one variable or as two separate variables. If we first consider them as one variable, we note that this is essentially writing (5.1) as:

\[ 0 = F(u) = f(u) - \frac{d}{dt}u. \]

The solutions of (5.1) are the sections of the sheaf diagram:

\[ C^1(\mathbb{R}, \mathbb{R}^d) \]

\[ \{u : F(u) = 0\} \subseteq C^1(\mathbb{R}, \mathbb{R}^d) \]

When looking to conduct our analysis, this sheaf lacks some important information; too much of the structure of (5.1) is hidden in the function \( F \).

We now turn to considering \( u \) and \( u' \) as separate variables. The initial construction of the sheaf yields a similar structure,

\[ C^0(\mathbb{R}, \mathbb{R}^d) \]

\[ C^1(\mathbb{R}, \mathbb{R}^d) \]

\[ \{u : F(u) = 0\} \subseteq C^1(\mathbb{R}, \mathbb{R}^d) \]

where \( u \) is on the top right and \( u' \) is on the top left. However, there is still some information buried. Notice that the solutions of (5.1) correspond to sections of this sheaf. However, the converse is not true! Our diagram is missing another equation that links \( u \) and \( u' \); they are related through differentiation. Including this relationship yields the following sheaf diagram:

\[ C^0(\mathbb{R}, \mathbb{R}^d) \]

\[ C^1(\mathbb{R}, \mathbb{R}^d) \]

\[ \{u : F(u) = 0\} \subseteq C^1(\mathbb{R}, \mathbb{R}^d) \]

This sheaf’s sections are now in one-to-one correspondence with the differential equation.
From here, several stalks of the sheaf can be collapsed together, without disrupting the space of global sections. A cleaner sheaf arises, yet it still reflects all of the relevant information:

\[
\begin{array}{ccc}
C^0(\mathbb{R}, \mathbb{R}^d) & \xleftarrow{\frac{d}{dt}} & C^1(\mathbb{R}, \mathbb{R}^d) \\
\downarrow f & & \downarrow \text{id} \\
S \subseteq C^1(\mathbb{R}, \mathbb{R}^d)
\end{array}
\]

where \( S \) is the space of solutions.

6 Goodwin Models and International Trade

6.1 Two Countries with Horizontal Trade

It is at this point that we return to the Goodwin growth model to apply some of the sheaf theoretic applications just introduced. Recall the basic framework of the Goodwin model. In the original model, and all the updates that were discussed, one assumption reminded constant: the economy that was being analyzed was a closed economy. One possible way to extend the Goodwin model, thereby making it more representative of phenomena, is to consider an open economy that is trading with other countries. The framework employed here mimics that of Ishiyama [8], though with some divergences. For example, Ishiyama builds his two country Goodwin model through difference equations, while this paper will continue to do so with differential equations. These differences are subtle, and while they don’t affect phenomena such as equilibrium, dynamic results can vary.

The motivation for expanding the Goodwin model is three-fold. First, aside for the work done by Ishiyama, it seems this idea of extending the Goodwin framework to an international setting has not been done. It is not entirely clear why this is the case. The Goodwin framework elegantly models fluctuations and cycles within an economy, thus making it an attractive candidate to expand. It appears that this question was not tackled in the past not because it is uninteresting or of little value, but may be because the mathematics that surround extending the model to many countries is daunting. Second, extending the Goodwin model requires a systematic approach. As the number of countries increases, it becomes ever more useful to have a consistent framework which allows us to conduct our analysis. Sheaf theory provides that framework. Third, and most relevant toward the fields of economics and policy, the types of questions that can be asked and answered using a sheaf theoretic approach allows us to provide new insight into topics that were once difficult to answer.

For simplicity, we begin by considering a two-country two-good model in which we assume horizontal trade occurs, \textit{i.e.} the goods traded have no relation to each other in terms of factors of production. Considering two goods is suitable from a theoretical standpoint. However, if we want to test how good
this model is in reality, we would consider a basket of goods, rather than a single good. Consider, for example, guns and butter. We formally employ the following assumptions:

(A1) There are two countries that produce different goods.
(A2) Each country has a steady growth of technical progress.
(A3) Each country has a steady growth of the labor force.
(A4) In each country there exist only two homogeneous and non-specific factors of production: capital and labor.
(A5) There is no mobility of either capital or labor between the two countries.
(A6) All quantities are real and net.
(A7) All wages are consumed; all profits are saved and invested.
(A8) The utility of the consumer is maximized at a positive combination of two goods. Thus horizontal trade continues permanently.
(A9) The change in the money wage rate is determined by a Phillips Curve.
(A10) The capital-output ratio is constant.
(A11) The two countries use the same currency.

From these assumptions, and the derivation of the Goodwin model above, we can represent our two countries as four unique differential equations:

\[
\dot{v}_i = v_i \left( \frac{1}{\sigma_i} - (\alpha_i + \beta_i) - \frac{u_i(t)}{\sigma_i} \right) \\
\dot{u}_i = \frac{u_i(t)}{p_i(t)} \left( -(\alpha_i + \gamma_i) + (\rho_i u_i(t)) \right)
\]

for \(i = 1, 2\). Note that the equations for each country are in the exactly the same form, and defined by exactly the same variables and constants, but they differ in value depending on the country in question.

Concerning assumption (A8), a utility function is defined, which shows the preferences of the representative workers in each country as indifference curves. Figure 16 graphically represents this for country one, though country two is exactly the same. Consumers in each country can buy two different goods \(x_1\) and \(x_2\). As stated by (A7), all wages earned by workers are then spent on some bundle of these two goods. Mathematically this is described in the equation

\[ I = p_1 x_1 + p_2 x_2. \]

For a bundle of goods to be purchased, it must lie on or inside of the budget constraint. Further, the indifference curves represent some combination of \(x_1\) and \(x_2\) such that the utility achieved from all of those different bundles is the same; i.e. they are indifferent about which bundle they choose. For consumers in country one, this is represented mathematically as

\[ U_1 = x_1^{\theta_1 + \rho} \cdot x_2^{(1-\theta_1)^{-2}} \]

where \(\theta_1\) is a parameter strictly greater than zero and strictly less than one. The symbol \(\rho\) is defined as the price proportion, i.e. \(\rho = \frac{p_2}{p_1}\). This utility function
corresponds to a special Cobb-Douglas function which reflects changes in the price proportion. In other words, built into the utility function is a bias with respect to the goods produced in different countries.

Utility is maximized when the budget line and indifference curve are tangent to each other. Given the utility function, Ishiyama showed that the optimal combination for country one on a budget line for a given price vector is:

$$(x_1, x_2) = \left( \frac{(\theta_1 p_1 + p_2) a_1 u_1(t)v_1(t)N_1}{p_1(p_1 + p_2)}, \frac{(1 - \theta_1) p_1 a_1 u_1(t)v_1(t)N_1}{p_2(p_1 + p_2)} \right).$$

This is the demand function for a given price vector. Note that country one consumes domestic goods by $\theta_1$ of its income.

Similarly, country two’s utility curve can be defined as follows:

$$U_2 = x_1^{(1-\theta_2)^2} \cdot x_2^{\theta_2 + \rho^2},$$

and the optimal combination of goods for a given price vector is:

$$(x_1, x_2) = \left( \frac{(1 - \theta_1) p_1 a_1 u_1(t)v_1(t)N_1}{p_1(p_1 + p_2)}, \frac{\theta_2 p_2 a_2 u_2(t)v_2(t)N_2}{p_2(p_1 + p_2)} \right).$$

These show the preference and consumption behavior of country two.

Given these two demand curves, we can write down equations to describe price adjustments in each country, where the excess demand functions are on the right-hand side of the equation:

$$\dot{p}_1 = (1 - \theta_2)p_2 a_2 u_2(t)v_2(t)N_2 - (1 - \theta_1)p_1 a_1 u_1(t)v_1(t)N_1$$

$$\dot{p}_2 = (1 - \theta_1)p_1 a_1 u_1(t)v_1(t)N_1 - (1 - \theta_2)p_2 a_2 u_2(t)v_2(t)N_2.$$ (6.2)

**Proposition 6.1.** The Short Run Equilibrium (i.e. balance of trade) price proportion $\rho^* = \frac{(1-\theta_1)a_1 u_1(t)v_1(t)N_1}{(1-\theta_2)a_2 u_2(t)v_2(t)N_2}$ is stable.
Proof. To show the stability of the price proportion, consider system (6.2) rewritten in the form \( \dot{p} = A \cdot p \):

\[
\begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2
\end{bmatrix} = \begin{bmatrix}
-(1 - \theta_1) a_1 u_1(t) v_1(t) N_1 & (1 - \theta_2) a_2 u_2(t) v_2(t) N_2 \\
(1 - \theta_1) a_1 u_1(t) v_1(t) N_1 & -(1 - \theta_2) a_2 u_2(t) v_2(t) N_2
\end{bmatrix} \begin{bmatrix}
p_1 \\
p_2
\end{bmatrix}
\]

and \( A \)'s eigenvalues, which are found by evaluating:

\[
\det(\lambda I - A) = \det \left[ -\lambda - (1 - \theta_1) a_1 u_1(t) v_1(t) N_1 & (1 - \theta_2) a_2 u_2(t) v_2(t) N_2 \\
(1 - \theta_1) a_1 u_1(t) v_1(t) N_1 & -\lambda - (1 - \theta_2) a_2 u_2(t) v_2(t) N_2
\right]
\]

\[
= [-\lambda - (1 - \theta_1) a_1 u_1(t) v_1(t) N_1] \cdot [-\lambda - (1 - \theta_2) a_2 u_2(t) v_2(t) N_2] \\
- [(1 - \theta_2) a_2 u_2(t) v_2(t) N_2] \cdot [(1 - \theta_1) a_1 u_1(t) v_1(t) N_1]
\]

\[
= \lambda^2 + \lambda [(1 - \theta_1) a_1 u_1(t) v_1(t) N_1 + (1 - \theta_2) a_2 u_2(t) v_2(t) N_2] \\
+ \lambda [(1 - \theta_1) a_1 u_1(t) v_1(t) N_1 + (1 - \theta_2) a_2 u_2(t) v_2(t) N_2]
\]

The eigenvalues are 0 and \(-[(1 - \theta_1) a_1 u_1(t) v_1(t) N_1 + (1 - \theta_2) a_2 u_2(t) v_2(t) N_2]\).

Thus the price proportion is stable. \(\square\)

Figure 17 shows graphically that the short run equilibrium is stable. When the parameter \(\theta_1\) is equal to \(\theta_2\), the slope of a local change in the price vector \(p\) is expressed as a tangent to the hyperbolic curve given by

\[
p_2 = \frac{p_1(0)p_2(0)}{p_1}
\]

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where \((p_1(0), p_2(0))\) is some initial price vector. The tangent lines become smaller as the point comes near the equilibrium ray. This is the line where prices are no longer changing and is given by:

\[
p_2 = \frac{a_1 u_1(t) v_1(t) N_1}{a_2 u_2(t) v_2(t) N_2} \cdot p_1.
\]

Thus if the adjustment is sufficiently flexible, the trade between two countries at period \(t\) is conducted at an equilibrium, or the positive intersection point of the hyperbolic curve and equilibrium ray. \(E\) is defined to be the point:

\[
E = \left( \sqrt{\frac{p_1(0)p_2(0)a_2 u_2(t) v_2(t) N_2}{a_1 u_1(t) v_1(t) N_1}}, \sqrt{\frac{p_1(0)p_2(0)a_1 u_1(t) v_1(t) N_1}{a_2 u_2(t) v_2(t) N_2}} \right).
\]

Thus we summarize our two country horizontal trade model with the following equations:

\[
\begin{align*}
\dot{v}_i &= v_i(t) \left( \frac{1}{\sigma_i} - (\alpha_i + \beta_i) - u_i(t) \frac{1}{\sigma_i} \right) \\
\dot{u}_i &= u_i(t) \left( - (\alpha_i + \gamma_i) + (\rho_i v_i(t)) \right) \\
\dot{p}_1 &= \sqrt{\frac{p_1(0)p_2(0)a_2 u_2(t) v_2(t) N_2}{a_1 u_1(t) v_1(t) N_1}} \\
\dot{p}_2 &= \sqrt{\frac{p_1(0)p_2(0)a_1 u_1(t) v_1(t) N_1}{a_2 u_2(t) v_2(t) N_2}} \\
\end{align*}
\]

for \(i = 1, 2\). This system includes six sets of state variables.

We now explore the existence on long-run equilibria of the system. Recall the equilibrium of the Goodwin model when considering just one country. The system has two fixed points: the trivial fixed point at the origin and the non-trivial fixed point. The non-trivial fixed points are

\[
\begin{align*}
v^* &= \frac{\alpha + \gamma}{\rho} \\
u^* &= 1 - \sigma(\alpha + \beta).
\end{align*}
\]

The Jacobian evaluated at the non-trivial fixed points is

\[
J = \begin{pmatrix}
0 & -\frac{\alpha + \gamma}{\sigma \rho} \\
\rho(1 - \sigma(\alpha + \beta)) & 0
\end{pmatrix}.
\]

Notice that the equations describing a country’s dynamics in this section are slightly modified from the original Goodwin model. The change is small, but it is present in the equation:

\[
\dot{u}_i = \frac{u_i(t)}{p_i(t)} \left( -(\alpha_i + \gamma_i) + (\rho_i v_i(t)) \right).
\]
Figure 18: Country one (left) with trade and Country two (right) with trade

The function is updated by dividing $u_i$ by $p_i$. However, this doesn’t change the equilibrium point $v^*$. For the two country model, since each county exhibits its own unique Goodwin cycle, the non-trivial fixed points are

$$v_i^* = \frac{\alpha_i + \gamma_i}{\rho_i}$$
$$u_i^* = 1 - \sigma_i(\alpha_i + \beta_i)$$

for $i = 1, 2$, and it has a similar Jacobian matrix. Thus, each Goodwin model is a conservative dynamical system where every point is in a closed orbit around the non-trivial fixed points which are asymptotically stable. Notice that $p_1^*$ and $p_2^*$ occurs when the following are true: first, prices are in equilibrium i.e. we are somewhere along the equilibrium ray and when both $u_i$ and $v_i$ are in equilibrium. Thus, we get

$$p_1^* = \sqrt{\frac{p_1(0)p_2(0)a_2u_2^*v_2^*N_2}{a_1u_1^*v_1^*N_1}}$$
$$p_2^* = \sqrt{\frac{p_1(0)p_2(0)a_1u_1^*v_1^*N_1}{a_2u_2^*v_2^*N_2}}.$$

From the analysis above, it is shown that the two country Goodwin model describing horizontal international trade has a meaningful equilibrium point for real variables. Next, we will consider the mechanism of dynamics for the system. Recall the behavior of worker’s share and employment rate when considering a single Goodwin model, described in Figure 10. When labor share of national income ($u$) of the country $i$ is greater than $u_i^*$ and the employment ratio ($v$) is also greater than $v_i^*$, the main economic variables considered in the Goodwin model all fall down because the pressure put on capitalist’s profits weakens investment activities. The wage continues to rise until $v_i$ becomes less than or equal to $v_i^*$. However, when considering horizontal trade, as our two country model does, the behavior of labor share in country one, $u_1$, is unclear. In the case of our system, $u_1$, depends on many more variables than just $v_1$, as it did in the original Goodwin. This fact will be made more clear in the next section where we sheafify
the two country Goodwin model and these dependencies can be seen graphically. Further, the nominal capital stock also changes by the influence of one country’s behavior. For example, if foreign workers demand domestic products such that the rise of its price can be offset by the growth of money, the domestic labor share falls. Thus we expect the trajectories generated by our dynamic system to exhibit much more complex behavior than that of the Goodwin model without horizontal trade.

Ishiyama looks to confirm these conjectures through computer simulations. There are two different cases that are considered. First, the capital-output ratios between the two countries are equal, i.e. \( \sigma_1 = \sigma_2 \); second, the more general case of capital-output ratios not equal to each other is considered. To examine the relationship between the lack and presence of horizontal trade, note that when horizontal trade is not present, we are presented with the original Goodwin model, which generates closed orbit periodic solutions, which is expected of a conservative dynamical system. Next, the dynamics between the two countries
### System Dynamics

| System Dynamics | $\sigma_2$ |
|-----------------|------------|
| Chaotic         | 2.0        |
| Limit Cycle     | 2.2        |
| Chaotic         | 2.3-2.5    |
| Limit Cycle     | 2.6        |
| Chaotic         | 2.7-3.1    |
| Chaotic         | 3.2        |
| Limit Cycle     | 3.3-3.6    |
| Chaotic         | 3.7, 3.8   |
| Limit Cycle     | 3.9-4.1    |
| Chaotic         | 4.2-4.7    |
| Limit Cycle     | 4.8        |

**Table 1: Dynamics of System with varying $\sigma_2$’s**

is considered when horizontal trade is introduced and capital-output ratios are equal, and is represented in Figure 18. Country one is on the left side and country two is on the right. Note, the horizontal axis represents labor’s share of national income, while the vertical axis represents the employment ratio.

Next, the more general case is considered, where capital output ratios are not equal. Different values are assigned to each country, and the fluctuations within country one are considered. First, $\sigma_2 = 2.6$ is considered. Figure 19 shows that over time, country one’s economy reaches a limit cycle, in spite of some initial chaotic behavior. On the other hand, when $\sigma_2 = 4.7$ is considered, chaotic behavior appears to persist throughout, which is represented in Figure 20. Moreover, Ishiyama conducted many different simulations using various values for $\sigma_2$ while keeping $\sigma_1$ constant. The properties of fluctuations are classified in Table 1.

#### 6.2 Sheafifying The Two Country Goodwin Model

The sheaf theoretic concepts that were introduced in the last section are interesting from a purely mathematical standpoint. But, their applications to complex modeling make them a great asset to build and expand models in a systematic way. They also present a graphical understanding of how variables relate to each other with varying degrees of dependency graphs. Further, the notion of being able to “stitch together” a number of simple sub-models to form a more complex multi-model via sheaf theory has far reaching applications, such as the extended Goodwin model just introduced.

Recall the sheaf diagram made for the Goodwin model in Figure 13. Using that construction process as a model, we wish to build out a sheaf diagram for the extended Goodwin model using the same process. Our expectation, given the construction of the two country system, is that our sheaf diagram will in some way be made up of two original Goodwin models and they will be interlinked in some way. The two separate original Goodwin models each
Given our system of equations (6.3) for our extended Goodwin model, we can systematically construct the related dependency diagrams and sheaves. The easiest way to gain an understanding for how the solutions of (6.3) behave is to build out the first dependency diagram, or the visual representation of the causal relationships between the variables, seen in Figure 21. Recall that an arrow from one variable to the next - i.e. $u_1 \rightarrow v_1$ - states that the future value of $v_1$ is partially dependent on the current value of $u_1$. From this first dependency diagram, we begin to see the general structure of the system. It is easy to isolate the two individual countries and the variables that interlink them - i.e. $p_1$ and $p_2$. Taking this dependency diagram one step further to encode more information - the fact that derivatives of these functions are state variables - we end up with Figure 22. While more information has been encoded, it still fails to fully describe the relationship between the derivative and its state variable, since, for example, both $u_1$ and $v_1$ determine the value of $\frac{dv_1}{dt}$. This is fully remedied by reinterpreting the arrows as functional relationships between variables and considering the pair of variables to determine the functional dependence between variables and their derivatives. After this reinterpretation, we are left with Figures 23 and 27.

Through this process, we have successfully sheafified the Goodwin Model of two countries engaged in horizontal trade. There are a few things to note about the diagram. First, each country does not exactly represent the original Goodwin model, since the new country specific model introduced by Ishiyama contains a price variable. That being said, as expected, the diagram can be broken down into three separate sub-diagrams. On the left, we notice the sub-
Figure 23: Functional Dependency Graph of 2 Countries - Variable Names

Figure 24: Country one Sub-Diagram Embedded Within 2 Country Sheaf Diagram
Figure 25: Price Sub-Diagram Embedded Within 2 Country Sheaf Diagram

Figure 26: Country two Sub-Diagram Embedded Within 2 Country Sheaf Diagram

diagram representing country one’s Goodwin model. Similarly on the right, we notice the sub-diagram corresponding to county two’s Goodwin model. The sub-diagram in the middle corresponds to the interaction between the two countries via horizontal trade. These are illustrated in Figures 24, 25, and 26: The boxes outline the sub-sheaf diagrams present within the overall sheaf. This lends a more intuitive understanding of how the two country system is constructed; each of these sub-diagrams are “stitched” together yielding the larger diagram.

It is clear from its construction that the sheaf is quite symmetric. Actually, it is important to point out that Figure 23 and 27 aren’t quite sheaves yet since the diagram most likely does not commute - one of the requirement for a diagram to be a sheaf. Nevertheless, it is simple enough to consider the the largest sub-diagram that is a sheaf. This is done by considering the solution spaces on the state spaces, i.e. $S_1 \subset C^1(\mathbb{R}, \mathbb{R}^3)$, $S_2 \subset C^1(\mathbb{R}, \mathbb{R}^6)$ and $S_3 \subset C^1(\mathbb{R}, \mathbb{R}^3)$. Here, $S_1$ and $S_3$ represent the set of solutions satisfying the system of equations describing country one and two respectively. $S_2$ is the set of solutions satisfying the set of functions which describes the price-trading scheme within our model.

We can formally define $S_1$, $S_2$, $S_3$ in the following way:

$S_1 = \left\{ (u_1, v_1, p_1) : \begin{array}{l}
\dot{v}_1 = v_1(t) \left( \frac{1}{\sigma_1} - (\alpha_1 + \beta_1) - \frac{u_1(t)}{\sigma_1} \right) \\
\dot{u}_1 = \frac{u_1(t)}{p_1(t)} \left( -(\alpha_1 + \gamma_1) + (\rho_1 v_1(t)) \right) \end{array} \right\}$
\[ S_2 = \left\{ (u_1, v_1, p_1, u_2, v_2, p_2) : \begin{align*}
\dot{p}_1 &= \sqrt{\frac{p_1(0) p_2(0) a_2 u_2(t) v_2(t) N_2}{a_1 u_1(t) v_1(t) N_1}} \\
\dot{p}_2 &= \sqrt{\frac{p_1(0) p_2(0) a_1 u_1(t) v_1(t) N_1}{a_2 u_2(t) v_2(t) N_2}} \end{align*} \right\} \]

\[ S_3 = \left\{ (u_2, v_2, p_2) : \begin{align*}
\dot{v}_2 &= v_2(t) \left( \frac{1}{\sigma_2} - (\alpha_2 + \beta_2) - \frac{u_2(t)}{\sigma_2} \right) \\
\dot{u}_2 &= \frac{u_2(t)}{\rho_2(t)} \left( - (\alpha_2 + \gamma_2) + (p_2 v_2(t)) \right) \right\} \]

With this consideration, the diagram now is path independent and thus we are left with a sheaf.

### 6.3 Sheaf theoretic Analysis

The dependency graphs presented above are a good introduction to the power that sheaves have when studying systems of differential equations. These dependency graphs lend an intuition to how the system is structured and how variables interact with each other. Sheaf theory goes beyond simply creating these dependency graphs, though. We turn to asking questions about properties of the sheaf, such as global and local sections, and how far can local sections be extended throughout the system.

One of the global sections for this sheaf is quite easy to compute. Recall Proposition 5.2 which states the sections of a sheaf are in one-to-one correspondence with the simultaneous solution of its system of equations. Thus, then, one global section of our sheaf is simply the equilibrium of our system

\[ v_i^* = \frac{\alpha_i + \gamma_i}{\rho_i} \]
\[ u_i^* = \frac{1}{\sigma_i} - (\alpha_i + \beta_i) \]
\[ p_1^* = \sqrt{\frac{p_1(0) p_2(0) a_2 u_2^* v_2^* N_2}{a_1 u_1^* v_1^* N_1}} \]
\[ p_2^* = \sqrt{\frac{p_1(0) p_2(0) a_1 u_1^* v_1^* N_1}{a_2 u_2^* v_2^* N_2}} \]

This is effectively equivalent to setting each of the leaves (of the sheaf) corresponding to a derivative equal to zero, as done in Figure 28. There are many other global sections within our sheaf that are not the equilibrium points; however, the only global sections that correspond setting the leaves equal to zero are the equilibrium points.

Recall that there is exactly one equilibrium for a single country Goodwin model and that the equation for a country’s Goodwin model are slightly different from the original Goodwin model; however, the equilibrium points and equilibrium analysis remain unchanged. Since the model for two countries engaged in horizontal trade includes two single country models as slightly modified Goodwin models that still poses the same equilibrium point, this means that each country’s equilibrium \((u_i, v_i)\) is determined from the outset. Further,
Figure 27: Functional Dependency Graph of 2 Countries - Variable Spaces
the pricing structure introduced by Ishiyama has these two countries \((u_1, v_1), (u_2, v_2)\) always uniquely determine commodity prices. Thus there is exactly one equilibrium. Always. Recall also that the Goodwin model for a single country is a conservative system, where level curves form around the equilibrium point \((u_i^*, v_i^*)\). While the countries engaged in trade at the equilibria do not change, Ishiyama shows via computer simulation the dynamics are not quite as straightforward. In fact, they are regularly chaotic.

The questions of equilibria for the extended Goodwin model, while vitally important from a dynamical systems perspective, are not the interesting questions that can be asked from a sheaf theoretic perspective. Since there is only one equilibrium point, it is unlikely that the system will ever be in equilibrium, nor will it converge to an equilibrium point, not much more can be said for this global section. More interesting questions, both mathematically and economically - from a theoretical and policy standpoint - are that of local sections, and their extensions. For example, what happens if we decide to fix \((u_1, v_1)\)? That is, assert that these values in country one are known. Given this, can we say anything that is happening in country two? Are there any constraints that country one forces upon country two via the equations in our system? If we have a local section at \((u_1, v_1)\), how far can we extend this section? To the price-trading scheme space? To \((u_2, v_2)\) or to the entire system? Does this allow us to inferences about whether the Goodwin model has broken down in country two?

We now turn to a local section analysis on our sheaf. The main questions we will be asking are:

- How far can a local section be extended?
- What constraints does it have on the other variables?
- For some given choices, will all variables remain free, or will some variables be required to be determined - i.e. not free - based on how these local sections are chosen?
We will consider fixing \((u_1, v_1)\) and observing its effects on \((u_2, v_2)\). The process of extending this local section amounts to something of a “diagram-chase,” whereby we “chase” the local section across the sheaf and thus see how far it can be extended.

We will first start by noting the number of degrees of freedom in each sub-diagram. In the sub-diagrams describing country one and country two respectively, there are three degrees of freedom present. Each sub-system has two equations and five variables, which can be seen from the equations describing each countries individual dynamics:

\[
\dot{v}_i = v_i(t) \left( \frac{1}{\sigma_i} - (\alpha_i + \beta_i) - \frac{u_i(t)}{\sigma_i} \right);
\]

\[
\dot{u}_i = \frac{u_i(t)}{p_i(t)} \left( - (\alpha_i + \gamma_i) + (\rho_i v_i(t)) \right).
\]

Hence, we can freely pick three out of the five variables of the set \(\{u_i, \dot{u}_i, v_i, \dot{v}_i, p_i\}\); after this selection, the final two will be fully dependent on the three picked.

In our sub-diagram describing the trading scheme, six degrees of freedom are present. Again, there are two equations and eight variables present; after picking six variables freely, the final two variables not picked are determined.

With this in mind, we will begin by placing a local section on country one and assert that three of the five variables present within the system are known. Without loss of generality, we assert that the variables \(u_1, v_1, \) and \(\dot{u}_1\) are known; \(\dot{v}_1\) and \(p_1\) are now fixed and determined based on the functional relationship described above. As Figure 29 shows, we have gone from extending the local section from a few variables to the entire system within country one.

All of country one’s variables are completely known, since we either assert they are known, or they can be determined based on the functional relationships presented with country one’s system. We will now try to extend to the price-trading system, found in the middle of the sheaf. Notice that three out of the five variables in country one are also present in the price-trading system: \(u_1, v_1,\)
and $p_1$. Since these three variables are now known in country one, they carry over to the price-trading system. The local section has partially been extended to the price-trading scheme. Recall that there are six degrees of freedom within this portion of the diagram. Since we have eliminated three of those degrees of freedom with the three known variables from country one, we are left with three degrees of freedom. Thus from our set of free variables $\{\dot{p}_1, p_2, \dot{p}_2, u_2, v_2\}$, choosing three freely will then determine the remaining two. Without loss of generality, we assert that $\dot{p}_1, p_2, \dot{p}_2; u_2$ and $v_2$ are now completely determined by their functional relationships with the other variables. The local section has thus been fully extended to the price trading scheme and partially extended to country two. All eight variables are “locked down,” as shown in Figure 30.

We turn now to country two to see if the local section can be extended further. Recall that from the outset, there are three degrees of freedom present within the country two, for the exact same reason as country one. Similarly, there is an overlap of three of the variables found within country two as there are in the price-trading system. As we move across to country two, these variables which are now determined also move across. This is the process of extending the local section to part of country two. The three degrees of freedom that were present within country two are now used up, leaving $\dot{u}_2$ and $\dot{v}_2$ completely determined. As Figure 31 shows, from this theoretical framework, the local section has been extended across the entire system.

This analysis by employing sheaves is powerful, to say the least. First, it is possible for all this analysis to be done without the aid of a dependency diagram simply using the equations. However, the sheaf diagram helps represent the system graphically, making the diagram chase an easier task. Similarly, because each arrow is a function - a product of the sheaf - and not just a relation, this local section diagram chasing using the sheaf is equivalent to conducting the analysis on the equations themselves. Further, this analysis is powerful since, by successfully extending the local section across the entire system, we can
say the following: country one, having a good understanding of their Goodwin cycle, and where they are within it can infer quite a bit about what is happening in the rest of the system. Further, with a little bit of information about the price-trading structure, country one can fully determine country two’s Goodwin cycle. Through “locking down” $u_2$ and $v_2$, country one knows where country two is within its Goodwin cycle and because $\dot{u}_2$ and $\dot{v}_2$ are completely determined, they know how country two is moving through their cycle. Thus, this local section can be extended to a global section in our sheaf and system.

6.4 Applications and Policy Implications

The discussion and analysis conducted above were quite theoretical. However, sheaves and their applications aren’t restricted to this theoretical level. This analysis extends quite nicely to a more practical discussion. We saw at the end of the last subsection the power of successfully extending local sections across a sheaf. Since we can theoretically always extend the local section, we should be able to in practice. In theory, theory and practice are the same. In practice, this is not always the case. Consider two countries each with accurate data about the different variables in the extended Goodwin model. Based on our analysis above, we should be able to fully extend a local section $(u_1, v_1)$ to the entire system, making it a global section, thereby accurately determining values of variables in country two. If we attempt to extend this local section and practice, there are two outcomes: there is disagreement based on the data or there isn’t. If there is disagreement on the data, this is valuable insight; it tells us that the model we have constructed is not as good as we thought! It forces us to reconsider the variable relationship structures we’ve built out using our differential equations. If the data agree, great! Our model appears to be a good representation of the phenomena we are trying to model. Still more can be said.

For the remainder of this subsection, suppose that our model is good. The following represents the power of extending local sections in practice. Consider
two countries are engaged in horizontal trade. Each has their own separate Goodwin model “running” within their economy. If country one has a strong understanding of its own Goodwin cycle, and understands a little bit of the trading structure, country one can completely determine the Goodwin cycle running in country two. This is what extending a local section of \((u_1, v_1)\) to the entire system effectively amounts to. Country one, could for example, control its own Goodwin cycle such that through trade, its dynamic interaction with country two could lead to positive or negative results. For example, it could manipulate its own Goodwin cycle in such a way that country two experiences negative effects, hindering growth; alternatively, it could help country two as well.

This application of the power of local sections is unlikely at best from an economic perspective. If a country is looking to help or hurt a fellow country, other programs or policies exist which have that specific goal. More likely than not, the power of local sections comes from a data collection and synthesis standpoint. For example, imagine two countries who are again engaged in horizontal trade. For the sake of this example, let us say that country one is a developed nation and its agencies that collect and report macroeconomic related statistics are trustworthy and reliable, while country two, a developing nation, is not trustworthy with its statistics reporting. Perhaps, country one has cause to be doubtful of their reports, or country two simply does not report this data. Given a strong understanding of its internal Goodwin Model and some reliable data about prices of goods exchanged, country one can effectively “peer inside” country two. Country one can accurately gain an understanding of where country two is in their Goodwin cycle as well as where they’re going and where they’ve been. Understanding the Goodwin cycle of country one’s trading parter can help them make more informed dynamic policy decisions.

7 Conclusion and Further Research

In this paper we have explored the extended Goodwin model from a sheaf theoretic approach. This paper has also introduced the concepts of conservative dynamical systems, the original Goodwin model, and the foundations of sheaf theory applied to systems of equations and differential equations. Finally, we considered the extended Goodwin model, developed by Ishiyama, which introduces horizontal international trade between two countries. Sheaf theory provides a new avenue to conduct analysis on differential equations by going beyond typical equilibrium and dynamic systems analysis. Through considering local sections of sheaves, and their extensions, theoretical inferences can be made from different “vantage points” within the model. Further, it allows for a suitable “check” on the theoretical model. Since local sections of \((u_1, v_1)\), in the case of the extended Goodwin model can always be extended, theoretically, they ought to be able to extend in practice; if not, the model is not as good as we originally thought.

Given the novelty of this type of research approach, there are multiple av-
venues for further research to build upon what has been started here. The Goodwin model beautifully and simplistically models cycles in an economy, making it a good candidate for updating and expanding.

One avenue for future research concerns expanding Ishiyama’s theoretical framework to more than two countries. From looking at the sheaf diagram, an extension to three countries engaged in horizontal pairwise trade simply involves a “cut and paste” approach to build out the sheaf. Extending to \( n \) countries generalizes in a similar way, generating a network of differential equations, described diagrammatically by a sheaf. Given this new network, do the dynamics still appear to be chaotic? Further, does the shape of the network result in different dynamics? For example, if country one trades with country two and three, the equilibrium prices and economic state in country one appears to be independent of whether countries two and three trade with each other; however the dynamics may change. The power of local sections can be similarly applied to this extended network, and similar questions arise. Given that two of country two’s variables were completely determined through country one’s choice of \((u_1, v_1)\), how does this generalize to a larger network?

A second approach to further research involves moving away from the theoretical framework and into the application of data. With regard to domestic countries, there is an ample amount of data available with which the hypotheses presented in the two country framework could be tested.

Finally, the sheaf theoretic approach need not be confined to updated Goodwin models. This approach is sufficiently general and malleable that it can be applied to other theories of economic growth and trade, supply chain evolution, or theories of financial markets. These problems remain to be attacked in future areas of research.
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