ON THE MULTILINEAR BOHNEBLUST–HILLE CONSTANTS: COMPLEX VERSUS REAL CASE

J.R. CAMPOS, D. NUÑEZ-ALARCÓN*, D. PELLEGRINO**, J.B. SEOANE-SEPÚLVEDA***, D. M. SERRANO-RODRÍGUEZ*

Abstract. The results of this note arise a rupture between the behavior of the real and complex best known constants for the multilinear Bohnenblust–Hille inequality; in one side, for real scalars, we show that new upper bounds for the real Bohnenblust–Hille inequality (the best up to now) can be obtained via a somewhat “chaotic” combinatorial approach, while in the complex case the combinatorial approach giving the best known constants seems to be fully controlled. We believe that the understanding of this fact is a challenging problem that may shed some new light to the subject. As a byproduct of our results we present new estimates for the constants of the Bohnenblust–Hille inequality as well as new closed formulas.

1. Introduction

Let \( K \) be the real or complex scalar field. The multilinear Bohnenblust–Hille inequality (\([2]\), see also \([1]\) for a more recent approach) asserts that there exists a sequence of positive scalars \((B_n)_{n=1}^\infty \) in \([1, \infty)\) such that

\[
\left( \sum_{i_1, \ldots, i_n = 1}^N |U(e_{i_1}, \ldots, e_{i_n})|^\frac{2n}{n+1} \right)^{\frac{n+1}{n}} \leq B_n \sup_{z_1, \ldots, z_n \in \mathbb{D}^N} |U(z_1, \ldots, z_n)|
\]

for all \( n \)-linear forms \( U : \mathbb{K}^N \times \cdots \times \mathbb{K}^N \rightarrow \mathbb{K} \) and every positive integer \( N \), where \((e_i)_{i=1}^N\) denotes the canonical basis of \( \mathbb{K}^N \) and \( \mathbb{D}^N \) represents the open unit polydisc in \( \mathbb{K}^N \). The exact values for the optimal constants \( B_n \) satisfying (1.1) remains a mystery and are being improved throughout the time. It is worth mentioning that the Bohnenblust–Hille inequality (and the growth of its constants) have been shown to have applications in Quantum Information Theory (see the recent work by Montanaro, \([10]\)).

From now on we denote the optimal constants of the Bohnenblust–Hille inequality by \( K_n \) (for the sake of simplicity we keep the same notation for complex and real scalars, although the values are quite likely not the same).

The first estimates (\([2, 3, 9, 16]\)) suggested an exponential growth and only very recently quite different results have arisen. The ultimate information related to the search of optimal values for constants satisfying (1.1) is:

- For real scalars,
  \[
  2^{1-\frac{1}{n+1}} \leq K_n \leq C_n,
  \]
  where \((C_n)_{n=1}^\infty\) have a subpolynomial growth and is given by a puzzling recursive formula (see \([2, 3]\)).

- For complex scalars,
  \[
  1 \leq K_n \leq \tilde{C}_n,
  \]
  where \((\tilde{C}_n)_{n=1}^\infty\) have a subpolynomial growth and is given by a similar recursive formula \([2, 4]\).

Up to now, it was an open problem, for real scalars, if \( K_n = 2^{1-\frac{1}{n+1}} \) or \( K_n = C_n \) or whether \( K_n \) lies strictly between these bounds. The only known precise value appears in the case \( n = 2 \), since \( 2^{1-\frac{1}{2}} = K_2 \). For the complex case, the similar question is unsolved for the estimates (1.2).

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One of the main goals of this note is to show a somewhat surprising rupture between the complex and real constants of the Bohnenblust–Hille inequality. We show that the constants $C_n$ are not the optimal ones; in fact improved constants can be obtained via a somewhat chaotic combinatorial induction. Moreover, this result seems to be in strong contrast to the case of complex scalars, in which the best known constants are obtained via a quite controlled (and by now standard) approach.

We also provide better closed formulas for the cases of real and complex scalars. More precisely, we show that, for all $n \geq 2$,

$$K_n \leq \sqrt{2} (n - 1)^{\log_2 \left( \frac{1 - \frac{1}{\sqrt{2}}}{\sqrt{2}} \right)} \leq \sqrt{2} (n - 1)^{0.526322}$$

for the case of real scalars, and

$$K_n \leq \frac{2}{\sqrt{\pi}} (n - 1)^{\log_2 \left( e^{\frac{1}{2} - \frac{1}{4}} \right)} \leq \frac{2}{\sqrt{\pi}} (n - 1)^{0.304975},$$

for the case of complex scalars. Above, $\gamma \approx 0.5772$ denotes the famous Euler–Mascheroni constant.

2. Background: the best known formulas up to today

Let

$$(2.1) \quad A_p := \sqrt{2} \left( \frac{\Gamma \left( \frac{p + 1}{2} \right)}{\sqrt{\pi}} \right)^{1/p},$$

for $p > p_0 \approx 1.847$ and

$$(2.2) \quad A_p := 2^{\frac{1}{2} - \frac{1}{p}}$$

for $p \leq p_0 \approx 1.847$. The exact definition of $p_0$ is given by the following equality: $p_0 \in (1, 2)$ is the unique real number with

$$\Gamma \left( \frac{p_0 + 1}{2} \right) = \frac{\sqrt{\pi}}{2}.$$ 

The constants $A_p$ are the best constants satisfying Khinchine’s inequality (due to Haagerup, [7]). Up to today, the best constants satisfying the multilinear Bohnenblust–Hille inequality for real scalars appeared in [14] and obey the following recursive formula:

$$(2.3) \quad C_m = \begin{cases} 1 & \text{if } m = 1, \\ \left( A_{\frac{m/2}{m+2}} \right)^{-1} C_{\frac{m}{2}} & \text{if } m \text{ is even, and} \\ \left( A_{\frac{1-m}{m+1}} C_{\frac{m-1}{2}} \right)^{\frac{m+1}{2m}} \left( A_{\frac{2m+2}{m+3}} C_{\frac{m+1}{2}} \right)^{\frac{m-1}{2m}} & \text{if } m \text{ is odd.} \end{cases}$$

For complex scalars the best known constants satisfying the multilinear Bohnenblust–Hille inequality appear in [12], and given by the formula

$$(2.4) \quad \tilde{C}_m = \begin{cases} 1 & \text{if } m = 1, \\ \left( \tilde{A}_{\frac{m}{m+2}} \right)^{-1} \tilde{C}_{\frac{m}{2}} & \text{if } m \text{ is even, and} \\ \left( A_{\frac{2m}{m+2}} C_{\frac{m-1}{2}} \right)^{\frac{m-1}{2m}} \left( A_{\frac{2m+2}{m+3}} C_{\frac{m+1}{2}} \right)^{\frac{m+1}{2m}} & \text{if } m \text{ is odd,} \end{cases}$$

where

$$\tilde{A}_p = \left( \Gamma \left( \frac{p+2}{3} \right) \right)^{\frac{1}{p}}.$$
3. New upper estimates: the exhaustive combinatorial approach

3.1. Real case. Let $f : [1, 2)^2 \to \mathbb{R}$ be given by

$$f(x, y) = \frac{4x - 2xy}{4x + 4y - 4xy},$$

$r : \mathbb{N} \to \mathbb{R}$ be defined by

$$r(x) = \frac{2x}{1 + x},$$

and $A : [1, 2) \to \mathbb{R}$ be given by

$$A(p) = \left\{ \begin{array}{ll} 
2^{\frac{1}{p} - \frac{1}{p_0}} & \text{if } p \leq p_0 \\
\sqrt[p]{\left(\frac{r(x + 1)}{p}\right)^{\frac{1}{p}}} & \text{if } p > p_0.
\end{array} \right.$$ 

From [4, Theorem 4.1] and using the best known constants for the Khinchine inequality from [7] we can see that the optimal constants $(K_m)_{m=1}^\infty$ satisfying the real multilinear Bohnenblust–Hille inequality are such that

$$K_m \leq J(k, m),$$

for all $k = 1, ..., \frac{m}{2}$ (when $m$ is even) and $k = 1, ..., \frac{m-1}{2}$ (when $m$ is odd), with

$$J(k, m) := \left(K_k \times (A(r(k)))^{k-m}\right)^{f(r(k), r(m-k))} \times \left(K_{m-k} \times (A(r(m-k)))^{-k}\right)^{f(r(m-k), r(k))}.$$ 

So, formally, the best estimate furnished by this method is

$$(3.1) \quad \begin{cases} 
K_m \leq P_m := \min \left\{ J(k, m) : k = 1, ..., \frac{m}{2} \right\} & \text{if } m \text{ is even} \\
K_m \leq P_m := \min \left\{ J(k, m) : k = 1, ..., \frac{m-1}{2} \right\} & \text{if } m \text{ is odd}.
\end{cases}$$

A first inspection shows that the choice

$$(3.2) \quad \begin{cases} 
k = \frac{m}{2} & \text{for } m \text{ even}, \\
k = \frac{m-1}{2} & \text{for } m \text{ odd}
\end{cases}$$

seems to be the best possible (i.e., the choice where the minimum of $J(k, m)$ is achieved). For this reason, in [14] this approach was selected and the formula (2.3) was presented.

As we mentioned before, at a first glance (or with the aid of some numerical tests) it seems clear that, in general, the choice (3.2) is better than other choices for $k$; for instance, the choice $k = 2$ was investigated in [11]. However, in some isolated cases we now identified that this choice of $k$ given by (3.2) was not the best one. Our main goal, rather than just a numerical approach, is to shed light to a curious rupture between the behavior of the best known constants for the real and complex Bohnenblust–Hille inequalities. For this reason, in this paper we look for the sharper constants by using the whole formula (3.1) which comes out with the chaotic way of generating the constants for the case of real scalars. In view of the amount of calculations involved and since a serious precision in the decimals is crucial, this new approach was done with a computer program. The program, which code is in the Appendix, calculates the constants by using the formula (3.1). The first improvement on the constants appear for $m = 26$ and since it is a recursive procedure, this improvement generates improvements in several other values of $m$. The following table is illustrative:

| $m$ | new constants $P_m$ | $C'_m$ (from [14]) |
|-----|---------------------|---------------------|
| 26  | < 5.22772           | > 5.22825           |
| 27  | < 5.31314           | > 5.31447           |
| 28  | < 5.39343           | > 5.39626           |
| 29  | < 5.47164           | > 5.47314           |
| 100 | < 10.509            | > 10.510            |

From $m = 27$ to 500 the only values of $m$ for which the new constants $P_m$ are not strictly smaller than $C_m$ are 31, 32, 33, 47, 48, 49, 63, 64, 65, 95, 96, 97, 127, 128, 129, 191, 192, 193, 255, 256, 257, 383, 384, 385. As $m$ goes to infinity, it is natural that the exhaustive combinatorial approach certainly achieves more
constants. Moreover, for certain higher values of \( m \), the difference \( C_m - P_m \) can be chosen arbitrarily large, as the following result illustrates. The proof is simple and we omit; we just mention that the nontrivial fact that the sequence \( \left( A_m^{-m/2} \right)_m \) is increasing (see [13 Lemma 6.1]) is crucial for the proof:

**Theorem 3.1.** Given any increasing sequence of positive real numbers \( (L_j)_{j=1}^{\infty} \) with \( \lim_{j \to \infty} L_j = \infty \), there is a strictly increasing sequence \( (m_k)_{k=1}^{\infty} \) of positive integers so that

\[
C_{m_j} - P_{m_j} > L_j \quad \text{for all} \quad j.
\]

The following table illustrates the previous result:

| \( m \) | \( C_m - P_m \) |
|-------|-----------------|
| 26 \cdot 2^{34} | > 3450 |
| 26 \cdot 2^{100} | > 1.19 \cdot 10^{11} |
| 26 \cdot 2^{150} | > 4.10 \cdot 10^{18} |

### 3.2. Complex case: open (and deep, we believe) questions.

The notation and terminology of this section are the same as those from [13], where it is proved that the optimal multilinear Bohnenblust–Hille constants \( (K_n)_{n=2}^{\infty} \) satisfy

\[
K_n < 1.65 (n-1)^{0.526322} + 0.13 \quad \text{(real scalars)}
\]

and

\[
K_n < 1.41 (n-1)^{0.304975} - 0.04 \quad \text{(complex scalars)}.
\]

The proof of the above estimates is achieved by following a series of technical steps. In the case of real scalars, using some previous lemmata, it is observed that the sequence

\[
M_n = \begin{cases} 
(\sqrt{2})^{n-1} & \text{if } n = 1, 2 \\
DM & \text{if } n \text{ is even, and} \\
D^M & \text{if } n \text{ is odd}
\end{cases}
\]

satisfies the multilinear Bohnenblust–Hille inequality, where \( D = e^{1+\frac{1}{\sqrt{2}}} \). Then, using a “uniform approximation” argument, the estimate (4.1) is achieved. In this section we remark that this final step of the proof, i.e., the uniform approximation argument, can be dropped and a quite simple argument provides even better constants. In fact, from [13] we know that, for all \( k \geq 1 \) and \( n \geq 2 \), we have

\[
M_n = \sqrt{2} D^{k-1} \text{ whenever } n \in B_k = \{2^{k-1} + 1, \ldots, 2^k\}.
\]

Thus, \( k-1 \leq \log_2 (n-1) \) and, hence,

\[
M_n \leq \sqrt{2} D^{\log_2 (n-1)} = \sqrt{2} (n-1)^{\log_2 \left( e^{1+\frac{1}{\sqrt{2}}} \right)} \leq \sqrt{2} (n-1)^{0.526322}.
\]

Using a similar argument (for complex scalars) it follows that

\[
M_n \leq \frac{2}{\sqrt{\pi}} (n-1)^{\log_3 \left( e^{1+\frac{1}{\sqrt{2}}} \right)} \leq \frac{2}{\sqrt{\pi}} (n-1)^{0.304975}
\]

for the complex scalar field. Summarizing, we have:
Theorem 4.1. The optimal constants satisfying the Bohnenblust–Hille multilinear inequality satisfy
\[ K_n \leq \sqrt{2}(n - 1)^{0.526322} \]
for \( n \geq 2 \) and real scalars, and
\[ K_n \leq \frac{2}{\sqrt{\pi}} (n - 1)^{0.304975} \]
for \( n \geq 2 \) and complex scalars.

Of course, the other estimates of [13] related to the above results can be straightforwardly improved by using these new estimates. Next, we shall cover both real and complex cases in order to improve our previous estimates for very large values of \( n \).

5. Closed formulas for “large” values of \( n \)

In this section we illustrate how the recursive essence of the best known constants of the Bohnenblust–Hille inequality affects the calculation of closed formulas. More precisely, we show that for big values of \( n \) the previous estimates can be pushed further. As before, the notation and terminology of this section are the same as those from [13].

5.1. Real case. If \((C_n)_{n=1}^\infty\) denotes the sequence in [13, (4.3)], if we fix any \( k_0 \), it is obvious that
\[
J_n = \begin{cases} 
C_n & \text{if } n \leq 2^{k_0}, \\
D (J_{n+1}) \frac{n+1}{n} & \text{if } n > 2^{k_0} \text{ is even, and} \\
D (J_{n+1}) \frac{n+1}{n} & \text{if } n > 2^{k_0} \text{ is odd,}
\end{cases}
\]
with \( D = \frac{e^{1 - \frac{\gamma}{2}}}{\sqrt{2}} \), satisfies the multilinear Bohnenblust–Hille inequality. For \( n > 2^{k_0} \), let \( k_1 > k_0 \) be such that
\[ 2^{k_1 - 1} + 1 \leq n \leq 2^{k_1}. \]
Then
\[ k_1 - k_0 \leq \log_2 \left( \frac{n - 1}{2^{k_0} - 1} \right) \]
and, since \((J_n)_{n=1}^\infty\) is increasing, the optimal constants \( K_n \) satisfying the multilinear Bohnenblust–Hille inequality are so that
\[
K_n \leq J_{2^{k_1}} = D^{k_1 - k_0} C_{2^{k_0}} \\
\leq C_{2^{k_0}} D^{\log_2 \left( \frac{n - 1}{2^{k_0} - 1} \right)} \\
= C_{2^{k_0}} \frac{D^{k_0 - 1}}{n - 1} (n - 1)^{\log_2 D}.
\]
We thus have
\[
K_n \leq \frac{C_{2^{k_0}}}{D^{k_0 - 1}} (n - 1)^{\log_2 \left( \frac{1}{2^{k_0}} \right)}.
\]
From [17, Theorem 3.1] we know that
\[ C_{2^{k_0}} \leq 4 D^{k_0 - 4} \]
whenever \( k_0 \geq 4 \). Thus,
\[
K_n \leq \frac{4}{(\frac{e^{1 - \frac{\gamma}{2}}}{\sqrt{2}})^{3}} (n - 1)^{\log_2 \left( \frac{1}{2^{k_0}} \right)} \approx 1.338887 (n - 1)^{0.526322}.
\]
Summarizing, we have:
Theorem 5.1. If \( n > 16 \), then
\[
K_n \leq \frac{4}{(\frac{1 - \frac{1}{2} \gamma}{\sqrt{2}})^3} (n - 1)^{\log_2 \left( \frac{1 - \frac{1}{2} \gamma}{\sqrt{2}} \right)}.
\]

Numerically,
\[
(5.2) \quad K_n < 1.338887 (n - 1)^{0.526322}.
\]

If we use the exact value of \( C_{2^k_0} \) instead of estimate (5.1) we can improve (5.2) as \( n \) grows. For example,
\[
\begin{align*}
& n > 2^6 \Rightarrow K_n < 1.310883 (n - 1)^{0.526322}, \\
& n > 2^7 \Rightarrow K_n < 1.306156 (n - 1)^{0.526322}, \\
& n > 2^8 \Rightarrow K_n < 1.303787 (n - 1)^{0.526322}.
\end{align*}
\]

5.2. Complex case. Let \((C_n)_{n=1}^\infty\) denote the sequence in [12, Theorem 2.3]. If we fix any \( k_0 \), and as the authors did in [13], we can show that
\[
J_n = \begin{cases} 
C_n & \text{if } n \leq 2^{k_0}, \\
D J_n^{\frac{n}{2}} & \text{if } n > 2^{k_0}\text{ is even, and} \\
D \left(J_{n-1}^{\frac{n}{2}} - J_{n+1}^{\frac{n}{2}}\right) & \text{if } n > 2^{k_0}\text{ is odd,}
\end{cases}
\]

with \( D = e^{\frac{1}{2} - \frac{1}{2} \gamma} \), satisfies the multilinear Bohnenblust–Hille inequality. For \( n > 2^{k_0} \), by mimicking the real case we obtain
\[
K_n \leq \frac{C_{2^{k_0}}}{D^{2^{k_0}-1}} (n - 1)^{\log_2 \left( e^{\frac{1}{2} - \frac{1}{2} \gamma} \right)}.
\]

Thus, using the values of \( C_{2^k} \) from [12] we have
\[
\begin{align*}
& n > 2^3 \Rightarrow K_n < 1.029610 (n - 1)^{0.304975}, \\
& n > 2^6 \Rightarrow K_n < 0.996322 (n - 1)^{0.304975}, \\
& n > 2^{15} \Rightarrow K_n < 0.991365 (n - 1)^{0.304975}.
\end{align*}
\]

6. Open Questions

Although the results of this note are simple to an expert, we believe that some new issues are bring into light, as the following open problems illustrate:

(1.-) Is there any explanation for the apparent “chaos” in the real case in contrast with the “perfect” behavior in the complex case?

(2.-) Is it a fault of the Rademacher system for the purposes we need? More precisely, is there any sequence of random variables (for the real case) which behaves better (with better constants) as it happens for Steinhaus variables in the case of complex scalars?

(3.-) Are the constants obtained here the optimal ones for the Bohnenblust–Hille inequality?

7. Appendix: the codes

In the code below, for real scalars, note that we replaced \( p_0 \) by 1.846999. We remark that this procedure does not cause any problem (no lack of precision in the estimates). The reason is simple. In fact, since the function \( r \) is increasing and
\[
\begin{align*}
r(12) &= \frac{24}{13} < 1.8463 < p_0 \\
r(13) &= \frac{26}{14} > 1.857 > p_0,
\end{align*}
\]

there is absolutely no difference in working with 1.846999 instead of \( p_0 \). As a matter of fact, we could have even used 1.847 instead of 1.846999. The new constants, up to 500, can be easily checked by means of, for instance, the code given below (that was made using the Mathematica package and provides the first 500 values of the constants).
Remark 7.1. (Added in October, 7, 2013) The present paper is no longer submitted to any journal, since the main question raised by this paper (i.e., the optimality of the constants presented here) was recently settled (in the negative) in consequence of a new interpolative approach to the Bohnenblust–Hille inequality, introduced in [1]. In fact, by combining ideas of [1, 15], Frédéric Bayart (in collaboration with D. Pellegrino and J. Seoane-Sepúlveda) proved that the optimal Bohnenblust–Hille constants are so that

\begin{equation}
K_n \leq C_n^{1 - \frac{1}{2^n}}
\end{equation}

for complex scalars and

\begin{equation}
K_n \leq C_n^{2 - \frac{1}{2^n - 2}}
\end{equation}

for real scalars and these estimates are quite better than the ones of the present note. The proof of the new estimates is done by an adequate use of the new interpolative procedure from [1] (it will appear in a joint paper of Frédéric Bayart, D. Pellegrino and J. Seoane-Sepúlveda). The argument is the following: from the multiple Khinchin inequality one can easily prove that the constant associated to the Bohnenblust–Hille exponent \((\frac{2n-2}{n}, \ldots, \frac{2n-2}{n}, 2)\) is dominated by \(A_{\frac{1}{2^n - 2}}^{-1} K_{n-1}\). From a simple variation of Proposition 3.1 of [1], varying the position of the power 2 in \((\frac{2n-2}{n}, \ldots, \frac{2n-2}{n}, 2)\) we still have the upper bound \(A_{\frac{1}{2^n - 2}}^{-1} K_{n-1}\) for the Bohnenblust–Hille exponents \((\frac{2n-2}{n}, \ldots, \frac{2n-2}{n}, 2, \frac{2n-2}{n}, \ldots, \frac{2n-2}{n})\) regardless of the position of the power 2.
Interpolating the $n$ Bohnenblust–Hille exponents

$$
\left( \frac{2n-2}{n}, \ldots, \frac{2n-2}{n}, \text{position } j, \frac{2n-2}{n}, \ldots, \frac{2n-2}{n} \right),
$$

with $j = 1, \ldots, n$, with $\theta_1 = \cdots = \theta_n$, one obtains the Bohnenblust–Hille exponent $\left( \frac{2n}{n+1}, \ldots, \frac{2n}{n+1} \right)$ with the constant $A^{-1}_{2n-2} K_{n-1}$, i.e.,

$$
K_n \leq A^{-1}_{2n-2} K_{n-1}.
$$

By using the optimal estimates for $A_{2n-2}$ and properties of the gamma function one obtains (7.1) and (7.2).

Alternatively, using the optimal constants of the Khinchin inequality, we have the formulas:

$$
K_m \leq \prod_{j=2}^{m} \left( 2 - \frac{1}{j} \right)^{-\frac{1}{j-2}}
$$

for complex scalars and

$$
(7.3)
K_m \leq 2^{4.588847 - \frac{1}{2} m} \prod_{j=14}^{m} \frac{\Gamma\left( \frac{2}{3} - \frac{1}{j} \right)}{\sqrt{\pi}^j}
$$

for real scalars (and $m \geq 14$). For $2 \leq m \leq 13$ we have

$$
K_m \leq \prod_{j=2}^{m} 2^{-\frac{1}{j-2}}.
$$

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DEPARTAMENTO DE MATEMÁTICA,
Universidade Federal da Paraíba,
Rio Tinto, Brazil.
E-mail address: jamilsonrc@gmail.com

DEPARTAMENTO DE MATEMÁTICA,
Universidade Federal da Paraíba,
58.051-900 - João Pessoa, Brazil.
E-mail address: danielnunezal@gmail.com

DEPARTAMENTO DE MATEMÁTICA,
Universidade Federal da Paraíba,
58.051-900 - João Pessoa, Brazil.
E-mail address: dmpellegrino@gmail.com and pellegrino@pq.cnpq.br

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO,
Facultad de Ciencias Matemáticas,
Plaza de Ciencias 3,
Universidad Complutense de Madrid,
Madrid, 28040, Spain.
E-mail address: jseoane@mat.ucm.es

DEPARTAMENTO DE MATEMÁTICA,
Universidade Federal da Paraíba,
58.051-900 - João Pessoa, Brazil.
E-mail address: dmserrano0@gmail.com