We derive the special and general relativistic hydrodynamic equations of motion for ideal fluids from a variational principle. Our approach allows to find approximate solutions, whenever physically motivated trial functions can be used. Illustrating this, a Rayleigh-Plesset type equation for the relativistic motion of a spherical (QGP) droplet is obtained. Also the corresponding general relativistic effective Lagrangian for spherically symmetric systems is presented.

Dedicated to the memory of Peter A. Carruthers

1 Introduction

First applications of relativistic hydrodynamics to the process of multiparticle production in high-energy hadronic collisions were studied by Fermi and Landau in the early 1950's. Recently, the relativistic motion of fluids has been studied extensively with respect to the analysis of relativistic heavy-ion collisions. The hydrodynamic description of high-energy hadronic and nuclear collisions has been successful in reproducing many characteristic features of these processes, such as multiplicity and transverse energy distributions.

Theoretically, however, the foundation of the hydrodynamical picture for these processes is not fully understood. The application of hydrodynamics implicitly assumes local thermal equilibrium via an equation of state. Therefore, the relaxation time and the mean free path should be much smaller compared to, respectively, the hydrodynamical time scale and spatial size of the system, which may be satisfied best in collisions of heavy nuclei.
From a kinematical point of view, besides the equation of state, the equations of hydrodynamics simply express the conservation laws of energy, momentum, and possibly various charges. Thus, for processes where flow is an important factor, a hydrodynamic description seems natural to begin with. Effects of finite relaxation time and mean-free path may be implemented later on, viscosity and heat conductivity in particular, or some simplified transport equations might be applicable, for example, see Ref. 4.

Relativistic hydrodynamics is a local description of the conservation laws, i.e. in terms of the energy-momentum tensor,

\[ \partial_\mu T^{\mu\nu} = 0 , \]

where we momentarily assume cartesian coordinates. It is difficult to solve this set of coupled partial differential equations in any generality. Few analytical solutions are known, but even for simple geometries, like one-dimensional or spherically symmetric cases, one often has to resort to numerical solutions. Under these circumstances a physically motivated and more qualitative approach, such as via a variational formulation, should provide useful insight, before or without full scale computer simulations.

Another important arena for relativistic hydrodynamics is found in cosmology and high-energy astrophysics, such as the gravitational collapse of stellar cores forming a neutron star or a black hole, relativistic blast waves in models of gamma ray bursts, etc. The assumption of local thermodynamical equilibrium is considered to be well justified in most cases, cf. Ref. 7. However, often the fluid dynamics and the local changes of the gravitational field have to be considered simultaneously. Again, this is generally very complicated and, therefore, a variational scheme seems extremely helpful.

Frequently even the equation of state of matter is not known precisely. Rather, hydrodynamics is employed to infer its properties involved in the process. In such cases we need to describe the flow, which characterizes the dynamics, assuming a specific equation of state.

In conclusion, a variational method is highly desirable which allows to solve the dynamical equations of the system more effectively. Such an effective Lagrangian and variational principle has previously been employed to study the effect of local turbulent motion on the supernova mechanism. It has also been useful in deriving the Rayleigh-Plesset equation and various generalizations for the gas bubble dynamics in a liquid. Furthermore, the global features of high-energy hadronic and nuclear collisions are described well by the fireball model, see Ref. 11 for a review. We aim at a dynamical scheme which improves the simplest fireball model in the direction of a realistic hydrodynamical description, incorporating the QCD phase transition and important...
dissipative processes.

In the present work, we reformulate Special and General Relativistic (SR and GR, respectively) hydrodynamics as a variational principle. We derive a SR generalization of the Rayleigh-Plesset equation in this approach. Furthermore, we obtain the GR effective Lagrangian for spherically symmetric systems.

2 The Variational Approach

2.1 Variables and Constraints

While the variational formulation of nonrelativistic hydrodynamics has been established for quite a while, we consider SR and GR equations of motion here. The most natural variable describing the flow is the velocity field, \( \vec{v}(\vec{r},t) \). Introducing the four-vector \( u^\mu(x) \), we define instead the variational variables,

\[
\begin{align*}
u^0 &\equiv \gamma , \\
\vec{u} &\equiv \gamma \vec{v} ,
\end{align*}
\]

i.e. functions of \( \vec{r} \) and \( t \), subject to the constraint,

\[
u_\mu u^\mu = 1 .
\]

The flow of matter provokes local changes in the occupied volume. Considering the case of a conserved chargelike quantity, say the baryon number, let its local density in the comoving frame be \( n \). Then,

\[
\partial_\mu (nu^\mu) = 0 .
\]

The energy of the matter is given by:

\[
E = \varepsilon V , \quad V = \frac{1}{n} ,
\]

where \( \varepsilon \) denotes the energy density and \( V \) the local specific volume. Assuming local equilibrium, we have the thermodynamical relations,

\[
\left( \frac{\partial E}{\partial V} \right)_S = -p , \quad \left( \frac{\partial \varepsilon}{\partial n} \right)_S = \frac{\varepsilon + p}{n} ,
\]

where \( S \) is the entropy and \( p \) the local pressure.
2.2 The SR Action and Hydrodynamic Equations of Motion

Let us first consider a single particle of rest mass $m_0$ with the action,

$$I_P = -m_0 \int ds = - \int dt \gamma^{-1} m_0 = - \int dt \varepsilon dV ,$$

(7)

where $\gamma^{-1} \equiv (1 - v^2)^{1/2}$; here we assumed an infinitesimal rest frame volume $dV_0$ of the particle, such that $m_0 \equiv \varepsilon dV_0 = \varepsilon \gamma dV$ in the laboratory frame, where the particle has velocity $v$.

This consideration leads to the action for a fluid, $I_M = - \int d^4x \varepsilon$, which is considered here as an aggregation of infinitesimal volume elements.

Varying the action with respect to the four-velocity field, $u^\mu \to u^\mu + \delta u^\mu$, and the density distribution, $n \to n + \delta n$, we have to respect the constraints, Eqs. (3), (4). Introducing Lagrange multipliers, we obtain the fluid action,

$$I_M = \int d^4x \left\{ -\varepsilon (n) + \xi(x) \partial_\mu [nu^\mu] + \frac{1}{2} \zeta(x) [u^\mu u_\mu - 1] \right\} ,$$

(8)

and the variational principle is stated as: $\delta I_M = 0$, for arbitrary variations of $u^\mu$, $n$, $\xi$, $\zeta$.

In practical applications of the variational principle it will be convenient to parametrize $u^\mu$, $n$ such that the constraints are automatically satisfied and the Lagrange multipliers $\xi$, $\zeta$ eliminated from the beginning.

We remark that the following derivation of the hydrodynamic equations of motion is equally valid in general coordinate systems. Then, the partial derivative $\partial^\mu$ should be replaced by the appropriate covariant derivative in the action and correspondingly in the following. Furthermore, the volume element $d^4x$ has to be replaced by the invariant volume element $\sqrt{-g} d^4x$.

Performing first a partial integration of the second term of the action, Eq. (8), and then the four variations, we obtain the set of equations:

$$u^2 = 1 ,$$

(9)

$$\partial_\mu (nu^\mu) = 0 ,$$

(10)

$$\frac{\partial \varepsilon}{\partial n} + u^\mu \partial_\mu \xi = 0 ,$$

(11)

$$n \partial_\mu \xi - \zeta u_\mu = 0 ,$$

(12)

from which the Lagrange multipliers can be eliminated.

In this calculation, it is useful to realize that $u^\mu \partial_\mu \equiv \partial_\tau$, i.e. the proper time derivative. Then, we calculate that $\partial_\mu \partial_\tau \xi = \partial_\tau \partial_\mu \xi$. Furthermore, for
adiabatic changes, cf. Eqs. (1), we make use of the relation,

$$d \left( \frac{\varepsilon + p}{n} \right) = \frac{dp}{n}.$$  

(13)

With these ingredients, we finally obtain:

$$(\varepsilon + p)u^\nu \partial_\nu u_\mu + u_\mu u^\nu \partial_\nu p = \partial_\mu p ,$$

(14)

which is the hydrodynamic four-vector equation of motion.

Separating the space and time components of Eq. (14), recalling Eqs. (2), we immediately obtain the results:

$$[\partial_t + \vec{v} \cdot \nabla] \gamma = \frac{1}{\gamma(\varepsilon + p)}(\partial_t p - \gamma^2 [\partial_t + \vec{v} \cdot \nabla]p) ,$$

(15)

$$[\partial_t + \vec{v} \cdot \nabla] \vec{v} = -\frac{1}{\gamma^2(\varepsilon + p)}(\nabla p + \vec{v} \partial_t p) ,$$

(16)

i.e. the relativistic energy and momentum flow equations, the latter being known as the SR generalization of the Euler equation.

2.3 Example: Relativistic Rayleigh-Plesset Equation for a Spherical Bubble

In all the applications of relativistic hydrodynamics to multiparticle production relatively little attention has been payed to the spherical geometry. The longitudinal expansion is usually considered predominant compared to the transverse one. However, in collisions of very heavy nuclei the transverse expansion might be equally important. Therefore, applying the variational principle, we derive an effective equation of motion for a spherically symmetric fluid bubble. We sketch the most simplified scenario, while more details can be found elsewhere.

Let $n_0(r, t)$ denote the baryon number density inside a bubble of radius $R(t)$ for a space fixed coordinate system (center-of-mass). In a comoving frame the density is $n = \gamma^{-1} n_0$, with $\gamma^{-1} = (1 - \vec{v}^2)^{1/2}$. We make the ansatz,

$$n_0(r, t) \equiv R^{-3} f(x; a(t)) , \quad x \equiv r/R ,$$

(17)

where $f$ denotes the shape function depending parametrically on $a$ (or several components $a_i$) and on the scaling variable $x$. Then, the continuity equation, $\partial_t n_0 + r^{-2} \partial_r (r^2 v n_0) = 0$, can be solved for the radial velocity,

$$v(x, t) = x \dot{R} - \frac{R \dot{a}}{x^2 f} \int_0^x dx' x'^2 \frac{\partial f}{\partial a} .$$

(18)
Since the constraints incorporated in the action $I_M$, Eq. (8), are automatically satisfied by now, the effective action becomes:

$$I_{RP} \equiv \int dt \ L = -4\pi \int dt \ R^3 \int_0^1 dx \ x^2 \varepsilon \left( f(x; a)/(\gamma R^3) \right) . \quad (19)$$

The corresponding Euler-Lagrange equations present coupled ordinary differential equations for $R(t)$ and $a(t)$, which demonstrates the essential simplification due to the variational ansatz.

In the limiting case of homologous motion, i.e. a time independent shape function $f(x)$ and consequently a linear velocity profile, $v(x; t) = x \dot{R}$, the resulting Rayleigh-Plesset type equation is:

$$\frac{d}{dt} \left\{ \dot{R} R^3 \int_0^1 dx \ x^4 \gamma^2 (p + \varepsilon) \right\} = 3R^2 \int_0^1 dx \ x^2 p , \quad (20)$$

which fully incorporates the necessary SR corrections.

Assuming an ultrarelativistic adiabatic equation of state for an idealized quark-gluon plasma fluid, we obtain the relations,

$$p = p_0 \left[ R_0 / R(t) \right]^{3\Gamma} \left( \frac{F(x)}{\gamma^\Gamma} \right) , \quad \varepsilon = \varepsilon_0 p / p_0 , \quad (21)$$

with the adiabatic index $\Gamma = 4/3$. The nonperturbative QCD vacuum pressure $B \approx (200\text{MeV})^4$ acting on such a plasma from the outside is incorporated here by replacing $p \rightarrow p - B$ on the right hand side of Eq. (20). Then, it is straightforward to solve Eq. (20) together with Eq. (21) numerically for suitable initial conditions, which illustrates our approach.

Along these lines, also an expanding hadronic phase outside the plasma bubble and dissipative processes due to the phase boundary can be incorporated and are presently under study.

3 General Relativistic Hydrodynamics

For most astrophysical or cosmological applications of the variational approach it will be necessary to incorporate GR gravity. According to our considerations in Sec. 2.2, we have to combine the usual action for gravity, $I_G$, with a suitable fluid/matter action, $I_M$. The following total action suggests itself:

$$I_G + I_M \equiv \int d^4x \sqrt{-g} \left\{ \frac{R}{16\pi G} - \varepsilon (n) + \xi (x) \nabla_{\mu} [nu^\mu] + \frac{1}{2} \zeta (x) [u^\mu u^\mu] - 1 \right\} , \quad (22)$$

where the only difference in the matter part, cf. Eq. (8), arises from co-/invariance requirements; we use standard notation. Here the variational
principle is: \( \delta S = 0 \), for all independent variations of the metric \( g_{\mu\nu} \) and of \( u^\mu, n, \xi, \zeta \), as before.

Performing the variations, using \( \partial_\mu \ln \sqrt{-g} = \Gamma^\alpha_{\mu\alpha} \), we obtain Eqs. (8), (11), and (12). Furthermore, varying \( \xi \) and \( g_{\mu\nu} \) leads to, respectively:

\[
\nabla_\mu [nu^\mu] \equiv (\partial_\mu + \Gamma^\alpha_{\mu\alpha})[nu^\mu] = 0 ,
\]

(23)

\[
(8\pi G)^{-1} G^{\mu\nu} - \frac{1}{2} \sqrt{-g}(\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu} = 0 ,
\]

(24)

where \( G^{\mu\nu} \) is the Einstein tensor, and we used \( \partial g_{\mu\nu} \ln \sqrt{-g} = -g_{\mu\nu}/2 \).

Identifying the energy-momentum tensor by \( \delta I_M / \delta g_{\mu\nu} \equiv \sqrt{-g}T^{\mu\nu}/2 \), we observe that Eq. (24) is nothing but Einstein’s equation for an ideal fluid, \( G^{\mu\nu} = 8\pi GT^{\mu\nu} \), with \( T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu} \). Then, the Bianchi identity, \( \nabla_\mu G^{\mu\nu} = 0 \), implies the GR hydrodynamic equations: \( \nabla_\mu T^{\mu\nu} = 0 \).

We conclude by briefly presenting the related effective Lagrangian for the most general spherically symmetric system, where the metric is:

\[
ds^2 = e^{2\phi} dt^2 - e^{2\lambda} dr^2 - R^2 d\Omega^2 ,
\]

(25)

with \( \phi, \lambda, R \) unknown (variational) functions of \( r, t \). Choosing the comoving frame with \( u^\mu = (u^0, 0, 0, 0) \), the normalization \( u^2 = 1 \) implying \( u^0 = \exp(-\phi) \), and with the (baryon) density \( n \) being proportional to \( R^{-2} \exp(-\lambda) \), the current conservation follows. Thus, the constraints being satisfied, the Lagrangian corresponding to Eq. (22) for spherical symmetry is obtained:

\[
\mathcal{L} = e^\phi e^\lambda R^2 [\varepsilon(n) + \frac{1}{16\pi GR^2} \{R^2[2R\phi' + R']e^{-2\lambda} - \dot{R}[2R\lambda + \dot{R}e^{-2\phi} + 1] \} ,
\]

(26)

where \( \dot{f} \equiv \partial f / \partial t \) and \( f' \equiv \partial f / \partial r \); we omitted a total derivative which does not influence the related Euler-Lagrange equations for \( \phi, \lambda, R \).

We have demonstrated that known cases like the Misner-Sharp equation relevant for stellar collapse or explosion, the Tolman-Oppenheimer-Volkov equation describing neutron star structure, and the Friedmann-Robertson-Walker cosmological model can be re-derived from this action.

4 Conclusions

All known results involving ideal fluids in SR and GR, respectively, can be obtained directly from the actions, Eqs. (8) and (22), respectively, which we derived. Some earlier attempts can be found in the literature. However, we motivated in Sec. 1 and stress here once more that the application of the variational principle together with physically motivated trial functions, as illustrated in Sec. 2.3, may provide a powerful tool to study new and interesting
fluids under extreme conditions, such as in ultrarelativistic nuclear collisions, or some more complex systems in GR. We will report the details of our calculations and such new applications elsewhere.

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