THE MOTIVE OF SOME MODULI SPACES OF VECTOR BUNDLES OVER A CURVE.

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Abstract. We study the motive of the moduli spaces of semistable rank two vector bundles over an algebraic curve. When the degree is odd the moduli space is a smooth projective variety, we obtain the absolute Hodge motive of this, and in particular the Hodge-Poincaré polynomial. When the degree is even the moduli space is a singular projective variety, we compute pure Euler characteristics and show that only two weights can occur in each cohomology group, we also see that its cohomology is pure up to a certain degree. As a by-product we obtain the isogeny type of some intermediate jacobians of the moduli spaces.

§0. Introduction.

The moduli space of stable vector bundles over an algebraic curve is a relatively well-known object, it has received great attention for the last twenty years, in particular when the rank and degree are coprime its cohomology has been shown to be torsion free and its Betti numbers are known. However the methods used in studying its cohomology are topological ([Ne]), number theoretical ([HN]) or infinite-dimensional ([AB], see also [LPV]), and these, at least in principle, do not yield any information of the analytic/algebraic structure of the moduli space and the dependence of this on the curve, for instance its Hodge numbers remain unknown.

In this paper we use a recent construction by M. Thaddeus ([Th]) to give a description of the motivic Poincaré polynomial of the moduli space of rank two semistable vector bundles of fixed determinant on an algebraic curve. It is an idea of Grothendieck (see [S]) that one should work in the Grothendieck group $K_0$ of the category of motives, this is where the motivic Poincaré polynomial lives. We believe that the theory of motives is an effective language to express clearly and precisely how the algebro-geometric properties of $C$ influence those of the moduli space $N_0(r,d)$. However at the present moment we do not have at our disposal the true category of motives $\mathcal{M}_k$ of Grothendieck, since the Standard Conjectures remain unproven, so we use the definition by Deligne of absolute Hodge motives $\mathcal{M}_k^{AH}$.

We start by giving a quick review of the theory of absolute Hodge motives, the natural language in which motives are expressed is that of tannakian categories so we recall the basic facts of these, we also define the motivic Poincaré polynomial, this is done in §1.
In order to carry out the calculations we need a motivic version of MacDonald’s formula for the Betti numbers of a symmetric power, we do this in §2. Note that in fact we get an expression for the motive of $X^{(n)}$ and not only $P_1X^{(n)} \in K_0\mathcal{M}_k^{AH}$.

Then in §3 we give a short account of Thaddeus’ construction of the moduli spaces of pairs.

In §4 is where with the aid of Thaddeus’ construction we manage to calculate the motivic Poincaré polynomial of the moduli space $N_0(2,1)$ of stable rank 2 vector bundles with fixed odd determinant.

In §5 we study the singular moduli space $N_0(2,0)$ of rank 2 semistable vector bundles with fixed even determinant. Our results differ from those of Kirwan ([K]) in that we use pure Poincaré polynomials whereas she works with a canonical partial desingularization of the moduli space to get the intersection cohomology Poincaré polynomial.

Finally in §6 we extract information concerning the intermediate jacobians of the moduli spaces from the motivic Poincaré polynomial.

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§1. Absolute Hodge motives.

In this section we give a brief review of the theory of absolute Hodge motives and related topics. For proofs of theorems and more precise statements refer to [DM].

Let $k$ be a field of characteristic zero.

Tannakian Categories.

By a tannakian category we shall mean a $k$-linear abelian neutral rigid tensor category with $\text{End}(\mathbb{1}) = k$. Let $C$ be a tannakian category, the fact that $C$ is neutral means there is a faithful exact functor from $C$ to the category of finite dimensional vector spaces over $k$:

$$\omega : C \longrightarrow \text{Vec}_k$$

called a fibre functor.

In a rigid tensor category there is a concept of rank. However in the case of a tannakian category this definition of rank gets simplified by the use of the fibre functor, $\text{rank}M = \text{dim}_k\omega(M)$.

An example of tannakian category is the category of finite dimensional $k$-representations of an affine group scheme $G$ over $k$, the fibre functor being the obvious one. In fact a fundamental theorem states that all tannakian categories arise in this way, so that if $C$ is a tannakian category with fibre functor $\omega$ then there exists an equivalence of categories $C \longrightarrow \text{Rep}G$ compatible with the fibre functors, $G$ is then called the Galois group or fundamental group of the tannakian category $C$, $\text{Gal}(C)$.

Examples.

1. Consider the category of finite dimensional graded vector spaces over $k$, it is a tannakian category which is easily seen to be equivalent to $\text{Rep}\mathbb{G}_m$.

2. The category of local systems of finite dimensional complex vector spaces over a topological space $X$ is a tannakian category. A fibre functor is obtained by taking the stalks.
assigning to each local system $\mathcal{L}$ the complex vector space $\mathcal{L}_x$, where $x$ is a point of $X$.

3. The category of rational pure Hodge structures, $\mathcal{H}S_\mathbb{Q}$ is a tannakian category.

There is also the notion of a graded tannakian category. It is one where every object has a direct sum decomposition compatible with $\text{Km}$. This is better expressed by using the Galois group, a graded tannakian category is a tannakian category $\mathcal{C}$ together with a central morphism $\mathbb{G}_m \rightarrow \text{Gal}(\mathcal{C})$.

However a richer structure appears quite naturally in the theory of motives: $(\mathcal{C}, w, T)$ is called a Tate triple if $\mathcal{C}$ is a tannakian category graded by $w : \mathbb{G}_m \rightarrow \text{Gal}(\mathcal{C})$, and $T$ is a weight $-2$ invertible object called the Tate object. The result of tensoring an object by the Tate object is usually referred to as a Tate twist. A standard way to abbreviate $A \otimes T^\otimes i$ is $A(i)$.

The following definition will be useful

**Definition.** If $\mathcal{C}$ is a tensor category then $\mathcal{C}[[T]]$ is the tensor category whose objects are

$$\text{Ob}(\mathcal{C}[[T]]) = \{(A_i)_{i \in \mathbb{N}} | A_i \in \text{Ob}(\mathcal{C})\}$$

often written as $\sum A_i T^i$. The morphisms are defined by

$$\text{Hom}(\sum A_i T^i, \sum B_i T^i) = \prod \text{Hom}(A_i, B_i)$$

there is a natural functor

$$\text{Coef} : \mathcal{C}[[T]] \rightarrow \mathcal{C}$$

sending $\sum A_i T^i$ to $A_n$, it is not however a tensor functor.

Recall that in a tensor category there are commutation constraints, that is for every pair of objects $M, N \in \text{Ob}(\mathcal{C})$ isomorphisms

$$M \otimes N \xrightarrow{\varphi} N \otimes M$$

Let $\mathcal{C}$ be a graded tensor category. Consider the new commutation constraints, given on pure degree objects $M_i, N_j$ of weights $i$ and $j$ by

$$\psi : M_i \otimes N_j \rightarrow N_j \otimes M_i$$

$$\psi = (-1)^{ij} \varphi$$

where $\varphi$ are the old commutation constraints. Call $\hat{\mathcal{C}}$ the resulting tensor category.

In the case when $\mathcal{C}$ is a tannakian category $\hat{\mathcal{C}}$ need not be tannakian. For instance, if $\mathcal{C}$ is $\mathcal{M}_k^{AH}$ or $\mathcal{M}_k$ then $\hat{\mathcal{C}}$ is called the false category of motives $\hat{\mathcal{M}}_k^{AH}$ or $\hat{\mathcal{M}}_k$.

**Absolute Hodge Motives.**

Let $k$ be a field of characteristic zero embeddable in $\mathbb{C}$. We shall work with the category of smooth projective varieties over $k$, $\mathcal{V}_k$.

The main problem in the theory of motives is to find a tannakian category that factors all possible cohomology functors. Grothendieck gave a construction of such a category $\mathcal{M}_k$ (see [Ma]) but in order to prove it has the required properties one needs the Standard Conjectures which remain unproven.
Deligne ([DM]) has given a temporary working definition for motives, these are the absolute Hodge motives which we shall use in what follows. The category $\mathcal{M}^{AH}_k$ is constructed in exactly the same way as $\mathcal{M}_k$ but using absolute Hodge cycles instead of algebraic cycles. We recall that an absolute Hodge cycle of $X$ of codimension $p$ is an element of

$$F^0 H^{2p}_{DR}(X)(p) \times \prod_l H^{2p}_{et}(\overline{X}, \mathbb{Q}_l)(p) \times \prod_{\sigma: k \hookrightarrow \mathbb{C}} H^i_{sing}(X_{\sigma}, \mathbb{Q})(p)$$

such that it is compatible with the comparison isomorphisms. We denote the group of such cycles by $Z^{p}_{AH}(X)$

In the same manner as with Grothendieck motives we get a functor, $h$, from the category of smooth projective varieties over $k$ to the category of AH-motives.

One advantage of working with AH-motives is that the K"unneth components of the diagonal in $H^{2d}(X \times X)$ are again AH-cycles, so we get a decomposition $hX = h^0 X \oplus h^1 X \oplus \cdots \oplus h^{2d} X$. This makes $\mathcal{M}^{AH}_k$ into a graded tannakian category, it is customary to refer to this grading as the weight grading. A motive that is zero in all degrees except in one is called a pure weight motive. As $\mathcal{M}^{AH}_k$ is a graded tannakian category one has a graded fibre functor $\mathcal{M}^{AH}_k \rightarrow \text{Grad-Vect}_k$

$$M = \bigoplus M_i \rightarrow \bigoplus H^i_{DR}(M)$$

It is proven in [DM] that $\mathcal{M}^{AH}_k$ is in fact a polarized Tate triple, the Tate object is $1 \otimes 1$. Using the fact that $\mathcal{M}_k^{AH}$ is polarized, one can prove that the category $\mathcal{M}_k^{AH}$ is semisimple, that is, every motive is the direct sum of simple motives.

**Remarks.**

1. The cycle maps $Z^p(X) \rightarrow H^{2p}(X)(p)$ produce an absolute Hodge cycle for each $p$-codimensional algebraic cycle so one gets a morphism $Z^p(X) \rightarrow Z^p_{AH}(X)$. This way we get a functor $\mathcal{M}_k \rightarrow \mathcal{M}_k^{AH}$.

2. An important thing to know about the category of AH-motives is that it is a full subcategory of the category of realization systems defined in [J]. The motive $h^i(X)$ can thus be seen as a triple $(H^i_{DR}(X, k), H^i_{et}(X, \mathbb{Q}), H^i_{et}(\overline{X}, \mathbb{Q}_l))$, where $H^i_{DR}(X, k)$ is a finite dimensional $k$-vector space with a filtration (the Hodge filtration), for each embedding $k \hookrightarrow \mathbb{C}$ $H^i_\sigma(X, \mathbb{Q})$ is a rational pure Hodge structure of weight $i$ and for each prime $\ell$ $H^i_{et}(X, \mathbb{Q}_\ell)$ is a $\text{Gal}(\overline{k}, k)$-module, together with comparison isomorphisms.

$K_0\mathcal{M}_k^{AH}$ and the motivic Poincaré polynomial.

Recall that to every abelian category $\mathcal{C}$ one can attach the Grothendieck group $K_0\mathcal{C}$. Moreover if $\mathcal{C}$ is a (graded) tensor category then $K_0\mathcal{C}$ is a (graded) unitary commutative ring. Given an object $A$ we shall use the notation $[A]$ for its image in $K_0\mathcal{C}$.

In particular if we put $\mathcal{C} = \mathcal{M}_k^{AH}$ we get a graded ring $K_0\mathcal{M}_k^{AH}$. In the category $\mathcal{M}_k^{AH}$ one has the Tate twist $\mathcal{M}_k^{AH} \rightarrow \mathcal{M}_k^{AH}$.
and the dualising functor
\[ \mathcal{M}_k^{AH} \xrightarrow{\vee} \mathcal{M}_k^{AH} \]
\[ A \mapsto A^\vee \]
both of which are exact functors so they descend to additive morphisms of the graded ring \( K_0 \mathcal{M}_k^{AH} \).

**Definition.** Let \( M \) be an AH-motive and \( M = \oplus M_i \) its weight grading, then its motivic Poincaré polynomial is defined to be its class in the graded ring \( K_0 \mathcal{M}_k^{AH} \)
\[ P^\text{mot}_t(M) = [M] = \sum [M_i] \in K_0 \mathcal{M}_k^{AH} \]

Note that this is not really a polynomial, it is an element of a graded ring. We shall often drop the \( \text{mot} \) and just write \( P_t \). If \( X \) is a smooth projective variety over \( k \) then we shall write \( P_t(X) = P_t(hX) \).

This is a generalisation of the usual Poincaré polynomial as can be seen by following \( h(X) = \oplus h^i(X) \) through the commutative diagram

\[ \begin{array}{ccc}
\text{Ob}(\mathcal{M}_k^{AH}) & \xrightarrow{[\cdot]} & K_0 \mathcal{M}_k^{AH} \\
\omega \downarrow & & \downarrow K_0(\omega) \\
\text{Ob}(\text{Grad-Vect}_k) & \xrightarrow{[\cdot]} & K_0 \text{Grad-Vect}_k = \mathbb{Z}[t, t^{-1}] 
\end{array} \]

In general the map from isomorphism classes of objects of an abelian category \( C \) to \( K_0 C \) is not injective, but as an application of the fact that \( \mathcal{M}_k^{AH} \) is semisimple we show now that this is the case for \( \mathcal{M}_k^{AH} \).

The following proposition is easily proven.

**Proposition.** Let \( C \) be an artinian abelian semisimple category (for example \( \mathcal{M}_k^{AH} \)) and \( A, B, C \in \text{Ob}(C) \). If \( A \oplus C \simeq B \oplus C \) then \( A \simeq B \).

**Corollary.** Let \( M, N \) be AH-motives. If \( P_t M = P_t N \) then \( M \simeq N \).

**Proof.** \( P_t M = P_t N \) means that there exists \( P \) with \( M \oplus P \simeq N \oplus P \), now use the previous proposition. \( \square \)

Therefore whenever we need to prove an equality of motives it will be enough to prove it in \( K_0 \), and this is normally easier to write.

**Mixed absolute Hodge motives.**

The geometric methods in the definition of AH-motives do not extend at the present moment to the case of open or singular varieties. As already mentioned \( \mathcal{M}_k^{AH} \) is a full subcategory of the category of realization systems \( \mathcal{R}_k \), this is very useful to construct a category of mixed absolute Hodge motives as there is a reasonable candidate for category of mixed systems of realizations, \( \mathcal{MR}_k \) together with natural functors \( h^i : \mathcal{W}_k \longrightarrow \mathcal{MR}_k \). Where \( \mathcal{W}_k \) denotes the category of varieties over \( k \) (not necessarily smooth or proper).

Let \( \mathcal{W}_k^0 \) denote the category of smooth varieties over \( k \) (not necessarily proper). Jannsen ([J]) defines \( \mathcal{M}_{\mathcal{MR}}_{k}^{AH} \) to be the full tannakian subcategory of \( \mathcal{MR}_k \) generated by the image of the \( h^i : \mathcal{W}_k^0 \longrightarrow \mathcal{MR}_k \).
There is a functor
\[ h : \mathcal{W}_k \to \text{Grad-} \mathcal{M}_k^{AH} \]
which assigns \( \oplus h^i(X) \) to the variety \( X \).

There is a natural fully faithful functor \( \mathcal{M}_k^{AH} \to \mathcal{M}_k^{AH}, \mathcal{M}_k^{AH} \) can thus be seen as a full subcategory of \( \mathcal{M}_k^{AH} \). The fact that every object in \( \mathcal{M}_k^{AH} \) is an extension of objects in \( \mathcal{M}_k^{AH} \) implies that the previous functor induces an isomorphism of rings \( K_0\mathcal{M}_k^{AH} \isom K_0\mathcal{M}_k^{AH} \).

We now define a polynomial which via the mentioned isomorphism extends the motivic Poincaré polynomial.

**Definition** ([Na]). Let \( M = \oplus M_i \in \text{Ob} (\text{Grad-} \mathcal{M}_k^{AH}) \) be a graded mixed motive, then the pure motivic Poincaré polynomial is
\[
P^\text{mot}(M) = \sum_m \left( \sum_i (-1)^{m+i} [\text{Gr}_m^W M_i] \right) \in K_0\mathcal{M}_k^{AH}
\]

**Remarks.**
1. If \( M = \oplus M_i \) with \( M_i \) a pure motive of weight \( i \), \( \text{Gr}_m^W M_i \) is equal to \( M_i \) if \( i = m \) and zero if \( i \neq m \) so that this polynomial coincides with the one already defined.
2. Note that in the mixed case \( P^t M \) does not coincide with the class of \( M \) in \( K_0\mathcal{M}_k^{AH} \).
3. Let \( X \) be a variety over \( k \), \( \oplus h^i X \) its mixed motive and \( P^\text{mot}_t \) its motivic Poincaré polynomial. Composition with the ring morphism
\[
K_0\mathcal{M}_k^{AH} \to K_0 \text{Grad-Vec}_k = \mathbb{Z}[t, t^{-1}]
\]
does not yield the classical Poincaré polynomial \( P^t X := \sum \dim H^i(X, \mathbb{Q}) t^i \) but rather the pure Poincaré polynomial defined by
\[
P^\text{pur}_t(X) = \sum_m \chi^\text{pur}(X) t^m, \text{ where } \chi^\text{pur}(X) = \sum_i (-1)^{i+m} \dim \text{Gr}_m^W H^i(X, \mathbb{Q})
\]
(c.f. [G,185-191], [S] and [Na]) this is better suited for computations than the ordinary Poincaré polynomial. For example if \( Y \) is a closed subvariety of \( X \) of codimension \( d \) and both \( X \) and \( Y \) are smooth one has the Gysin exact sequence,
\[
\cdots \to h^{i-2d} Y(-d) \to h^i X \to h^i(X-Y) \to \cdots
\]
as the functor \( \text{Gr}_m^W \) is exact one gets an equality in \( K_0\mathcal{M}_k^{AH} \)
\[
\sum_i (-1)^i [\text{Gr}_m^W h^i X] = \sum_i (-1)^i [\text{Gr}_m^W h^i(X-Y)] + \sum_i (-1)^i [\text{Gr}_{m-2d}^W h^{i-2d} Y](-d)
\]
so that \( P^\text{mot}_t X = P^\text{mot}_t (X-Y) + P^\text{mot}_t Y(-d) \).

§2. A motivic MacDonald formula.

Let \( X \) be a compact polyhedron and consider \( X^{(n)} \) the symmetric power of \( X \), this is the quotient of \( X^n \) by the natural action of the symmetric group \( \mathfrak{S}_n \). MacDonald gave a formula ([M]) that produces Betti numbers of \( X^{(n)} \) in terms of those of \( X \), explicitly
\[
P^t X^{(n)} = \text{Coef} \left( (1+t^2) h^1(X) \cdot (1+t^3) h^2(X) \cdots \right)
\]
In this paragraph we intend to give a motivic version of MacDonald’s formula valid in any neutral \( k \)-linear graded tannakian category, in particular that of Absolute Hodge Motives or conjecturally Grothendieck’s category of pure motives.

Throughout all this section \( k \) will denote a field of characteristic 0. Let \( C \) be a tannakian category, in [DM, pg.106] it is shown that the commutation constraints can be extended in a natural way to cover the case of more than two factors, for every \( \sigma \in S_n \), we get isomorphisms

\[
\varphi_\sigma : M_1 \otimes \cdots \otimes M_n \rightarrow M_{\sigma^{-1}(1)} \otimes \cdots \otimes M_{\sigma^{-1}(n)}
\]

in particular if \( M \in \text{Ob}(C) \) this defines an action of \( S_n \) on \( M \otimes^n S_n \phi \rightarrow \text{Aut}(M \otimes^n) \).

Let \( \epsilon : S_n \rightarrow \{+1,-1\} \) denote the signature

**Definition.** Recall that a tannakian category is abelian, in particular any morphism has an image. Given a \( M \in \text{Ob}(C) \) define \( S^i M \) (resp. \( \wedge^i M \)) to be the image of the morphism \( \frac{1}{i!} \sum_{\sigma \in S_n} \varphi_\sigma : M \otimes^i \rightarrow M \otimes^i \) (resp. \( \frac{1}{i!} \sum_{\sigma \in S_n} \epsilon(\sigma) \cdot \varphi_\sigma \)). Extend this definition to the case \( i = \text{rank}(M) \) by putting \( S^0 M = \wedge^0 M = 1 \).

**Remarks.**
1. As the fibre functor \( \omega \) is a tensor functor it sends \( \varphi_\sigma \) to the canonical commutation constraints in \( \text{Vec}_k \), this combined with the fact that \( \omega \) is exact gives immediately that \( \omega(\wedge^i M) = \wedge^i \omega(M) \). Using the faithfulness of \( \omega \) we see that \( \wedge^i M = 0 \) for \( i > \text{rank}(M) \).
2. If \( M \) is a rank one object then using again the fibre functor one sees that \( S^i M = M \otimes^i \).

**Definition.** If \( M \in \text{Ob}(C) \) define

\[
(1 + T)^M = \sum \wedge^i M \cdot T^i \in \text{Ob}(C[T])
\]
\[
(1 - T)^{-M} = \sum S^i M \cdot T^i \in \text{Ob}(C[[T]])
\]

If the rank of \( M \) is one then \( (1 - T)^{-M} = \sum M \otimes^i T^i \) so we shall also use the notation \( \frac{1}{1-MT} \) in this case. If \( A \) is an invertible object then \( \frac{1}{A-BT} \) will mean the reasonable thing

\[
\frac{1}{A-BT} = A^{-1} \otimes \left( \frac{1}{1-A^{-1}BT} \right) \in \text{Ob}(C[[T]])
\]

For the rest of the section \( C \) will denote a graded tannakian category over \( k \).

Recall that for a graded tensor category \( C \) we defined in §1 a tensor category \( \hat{C} \) by changing certain signs in the commutation constraints.

**Definition.** Define the symmetric power of \( M \), \( M^{(i)} \), the same way as \( S^i M \) but using the commutation constraints from \( \hat{C} \).
**Proposition.** Let $M$ be a pure degree object of weight $n$ then $M^{(i)}$ is $S^i M$ if $n$ is even and $\wedge^i M$ if $n$ is odd.

**Proof.** If $M$ is pure of even weight then the commutative constraints

$$M \otimes M \rightarrow M \otimes M$$

are the same in $\hat{\mathcal{C}}$ and in $\mathcal{C}$ so $M^{(n)} = S^n M$.

In the odd weight case the commutation constraints change sign and when more than one factor appears then the sign is given by the signature $\varepsilon$ so we get $M^{(n)} = \wedge^n M$. □

The next theorem gives an expression for the symmetric power of an object in terms of symmetric powers of its pure components, it is our motivic version of the MacDonal formula.

**Theorem.** Let $M = \oplus M_i$ be an object in a graded neutral $k$-linear tannakian category, then

$$M^{(n)} = \text{Coeff}_{T^n} \left( \cdots \otimes (1 + T)^{M_{-1}} \otimes (1 + T) M_i \otimes (1 + T) M_2 \otimes \cdots \right)$$

**Proof.** We need to see that $M^{(n)} = (M \otimes^n)^{S_n}$ is

$$\text{Coeff}_{T^n} \left( \cdots \otimes \sum_i \wedge^i M_{-1} T^i \otimes \sum_i \wedge^i M_1 T^i \otimes \sum_i \wedge^i M_3 T^i \otimes \cdots \right)$$

$$\otimes \sum_i S^i M_{-2} T^i \otimes \sum_i S^i M_0 T^i \otimes \sum_i S^i M_2 T^i \otimes \cdots \right)$$

$$= \sum_{\lambda_1 + \cdots + \lambda_k = n} M_{\lambda_1}^{(\lambda_1)} \otimes \cdots \otimes M_{\lambda_k}^{(\lambda_k)}$$

On the other hand, by Kneth

$$M \otimes^n = \bigoplus_{r \in \mathbb{Z}} \left( \bigoplus_{r_1 + \cdots + r_n = r} M_{r_1} \otimes \cdots \otimes M_{r_n} \right)$$

$$(M \otimes^n)_r = \bigoplus_{r_1 + \cdots + r_n = r} M_{r_1} \otimes \cdots \otimes M_{r_n}$$

using next lemma we get that the $S_n$-invariant subobject is isomorphic to

$$\bigoplus_{\sum \lambda_i s_i = r} \left( M_{s_1} \otimes \lambda_1 \otimes M_{s_1} \otimes \cdots \otimes M_{s_k} \otimes \lambda_k \otimes M_{s_k} \right)^{S_{\lambda_1} \times \cdots \times S_{\lambda_k}}$$

Now according to whether $s_i$ is even or odd, the action of $S_{\lambda_i} \hookrightarrow S_n$ on $M_{s_i} \otimes \cdots \otimes M_{s_i}$ is the canonical $(\varphi)$ or the anticanonical one $(\varphi \cdot \varepsilon)$ and thus $(M \otimes^n)^{S_{\lambda_i}}$ is $S_{\lambda_i} M_{s_i}$ or either $\wedge^{\lambda_i} M_{s_i}$. This proves the theorem. □
Lemma. There is a natural isomorphism

\[(M^\otimes n)^{S_n}_r \cong \bigoplus_{\sum \lambda_i = n} \bigoplus_{r_1 < \cdots < r_k} (M_{r_1} \otimes \lambda_1 \otimes \cdots \otimes M_{r_k} \otimes \lambda_k) \otimes \cdots \otimes (M_{r_k} \otimes \lambda_k) \bigotimes_{\lambda_1} \cdots \bigotimes_{\lambda_k}\]

Proof. The inclusion \((M^\otimes n)^{S_n}_r \hookrightarrow (M^\otimes n)_r\) followed by the obvious projection gives a morphism

\[(M^\otimes n)^{S_n}_r \rightarrow \bigoplus_{\sum \lambda_i = n} \bigoplus_{r_1 < \cdots < r_k} (M_{r_1} \otimes \cdots \otimes M_{r_k} \otimes \lambda_1 \otimes \cdots \otimes \lambda_k) \bigotimes_{\lambda_1} \cdots \bigotimes_{\lambda_k}\]

Applying the exactness and faithfulness of the fibre functor one sees that the image is contained in

\[\bigoplus_{\sum \lambda_i = n} \bigoplus_{r_1 < \cdots < r_k} (M_{r_1} \otimes \cdots \otimes M_{r_k} \otimes \lambda_1 \otimes \cdots \otimes \lambda_k) \bigotimes_{\lambda_1} \cdots \bigotimes_{\lambda_k}\]

and that in fact the induced morphism from \((M^\otimes n)^{S_n}_r\) to this is injective. So it is enough to see that they both have the same rank but this is just the statement of MacDonald’s main theorem ([M]). □

Let \(X\) be a smooth projective variety over \(k\). Write \(h(X) = \oplus h^i(X) \in Ob(M_{AH}^H)\), then by Proposition 6.8 in [DM] \(h(X^{(n)}) = (h(X)^{\otimes n})^{S_n}\), where the action of \(S_n\) is the one arising from the geometric commutations \(X \times \cdots \times X \rightarrow X \times \cdots \times X\), so that \((h(X)^{\otimes n})^{S_n} = h(X)^{(n)}\) and this is computed using the formula in the Theorem, so we have

Corollary.

\[h(X^{(n)}) = \text{Coef}_{T^n} \frac{(1 + T)^{h^1 X} \otimes (1 + T)^{h^3 X} \otimes \cdots}{(1 - T)^{h^0 X} \otimes (1 - T)^{h^2 X} \otimes \cdots}\]

and if \(C\) is a smooth projective curve

\[h(C^{(n)}) = \text{Coef}_{T^n} \frac{(1 + T)^{h^1 C}}{(1 - T)(1 - (-1)T)}\]

Remark. If we apply the graded fibre functor

\[M_{AH}^H \xrightarrow{H^*_{DR}} \text{Grad-Vec}_k\]

followed by \(Ob(\text{Grad-Vec}_k) \xrightarrow{[]} \mathbb{Z}[t, t^{-1}]\) we get the classical MacDonald formula, whereas if we do the same with

\[M_{AH}^H \xrightarrow{H^*_{DR}} \text{BiGrad-Vec}_k\]

and \(Ob(\text{BiGrad-Vec}_k) \xrightarrow{[]} \mathbb{Z}[x, y, x^{-1}, y^{-1}]\) we get the formula for the Hodge numbers in [Bu].
§3. Thaddeus’ construction.

In this section we review the basic construction of Thaddeus we shall use, for a more complete exposition see [Th].

Let $C$ be a fixed smooth projective algebraic curve of genus $g \geq 2$ over $k$ an algebraically closed field of zero characteristic and $L$ a line bundle over $C$ of large degree $d$. The moduli spaces we are primarily interested in are $N_0(2, d)(C)$ the moduli space of rank 2 semistable vector bundles with fixed determinant over $C$. They depend on the curve $C$ however we shall simply write $N_0(2, d)$.

Thaddeus considers the problem of giving a moduli space for pairs $(E, s)$, where $E$ is a rank 2 vector bundle over the curve $C$ with fixed determinant $L$ and $s$ is a non-zero section of $E$. It appears that there are many possible definitions for stability of a pair depending on a parameter $\sigma \in [0, \frac{d}{2}]$, for $\sigma$ varying in certain open disjoint intervals there are no strictly semistable pairs and one obtains a finite list of fine moduli spaces of pairs $M_0, \ldots, M_\omega$ ($\omega = \frac{d-1}{2}$).

These different moduli spaces are all birational and are related by a special kind of birational maps called flips. In this context a flip between two varieties $X$ and $Y$ means that $X$ and $Y$ have a common blow-up, $\tilde{X} \simeq \tilde{Y}$, with the same exceptional locus. Luckily the centers of these blow-ups are nonsingular so that one can use the standard formula for the Poincaré polynomial of a blow-up. Of course, to be able to work out Poincaré polynomials of the moduli spaces we need to know the centers of these blow-ups, these turn out to be a couple of subvarieties of $M_j$ called $PW_i^+$ and $PW_{i+1}^-$ isomorphic to certain projective bundles over symmetric products of the curve: $PW_i^+$ is a $\mathbb{P}^{d-2i+g-2}$-bundle over $C^{(i)}$ and $PW_{i+1}^-$ is a $\mathbb{P}^i$-bundle over $C^{(i+1)}$. To summarize, the blow-up of $M_i$ along $PW_i^-$ is isomorphic to the blow-up of $M_{i-1}$ along $PW_i^+$. We can picture this chain of flips:

\[
\begin{array}{cccccc}
M_1 & \searrow & \tilde{M}_2 & \searrow & \ldots & \searrow & \tilde{M}_\omega & \searrow & M_\omega \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
M_0 & & M_2 & & \cdots & & \cdots & & N_0(2, d)
\end{array}
\]

Moreover, it is easy to see that $M_0$ is a projective space of dimension $d + g - 2$. In the other extreme we have $M_\omega$; in the case when $\deg L$ is odd $M_\omega$ is a projective bundle of relative dimension $d - 2g + 1$ over $N_0(2, 1)$, the moduli space of rank two stable vector bundles over $C$ with fixed odd determinant, whereas if $\deg L$ is even we have a map from $M_\omega$ to the analogous moduli space which is only a projective fibration over the stable locus.

The above description of the centers of the flips enables Thaddeus to recover the formula by Harder-Narasimhan for the Poincaré polynomial of $N_0(2, 1)$. In the next sections we shall use the formulae for the motive of blow-ups and our motivic MacDonald formula in the same way to get an expression for the motive of $N_0(2, 1)$ and $N_0(2, 0)$.

§4. The motive of $N_0(2, 1)$.

The purpose of this section is to give an expression for $hN_0(2, 1)$ in terms of $h^1C$ and $\mathbb{L}(1)$. An immediate consequence is that $hN_0(2, 1)$ is in the tannakian subcategory of $\mathcal{MH}$, generated by $hC$ and $\mathbb{L}(1)$. 

The calculation of the Poincaré polynomial of the moduli space involves some infinite sums of motives thus falling outside of the ring $K_0\mathcal{M}_k^{AH}$, to formalise this we need to construct a greater ring $\hat{K}_0\mathcal{M}_k^{AH}$, the ring of Laurent series of motives, this is done as follows: first consider the subring $K_0\mathcal{M}_k^{AH} = \{x \in K_0\mathcal{M}_k^{AH} | \deg x \geq 0\} \subset K_0\mathcal{M}_k^{AH}$, complete it with respect to the ideal $I$ formed by the strictly positive degree elements, tensor the result by $K_0\mathcal{M}_k^{AH}$ over $K_0\mathcal{M}_k^{AH} + = \{x \in K_0\mathcal{M}_k^{AH} | \deg x \geq 0\}$ ⊂ $K_0\mathcal{M}_k^{AH}$, then the result is the ring we were looking for,

$$\hat{K}_0\mathcal{M}_k^{AH} = K_0\mathcal{M}_k^{AH} \otimes_{K_0\mathcal{M}_k^{AH} + I} \hat{K}_0\mathcal{M}_k^{AH}.$$  

If $A, B$ are invertible motives, with $\deg B > \deg A$, and $C$ is any motive then define $\frac{C}{A-B}$ to be the reasonable thing in $\hat{K}_0\mathcal{M}_k^{AH}$, that is

$$\frac{C}{A-B} := C \cdot (A^{-1} + A^{-2} \cdot B + A^{-3} \cdot B^2 + \cdots).$$

**Proposition.** The motive of the moduli space of pairs $M_i$ is given by

$$h_{M_i} = \sum_{j=0}^{i} hC^{(j)} \otimes (\mathbb{1}(-j) \oplus \cdots \oplus \mathbb{1}(-d + 2j - g + 2))$$

and its motivic Poincaré polynomial is

$$P_{t}^{\text{mot}} M_i = \frac{\mathbb{1}}{\mathbb{1} - \mathbb{1}(-1)} \cdot \text{Coeff}_{T} \left( \frac{\mathbb{1}(-d + 2i - g + 1)}{\mathbb{1}(-2)T - \mathbb{1}} - \frac{\mathbb{1}(-i - 1)}{T - \mathbb{1}(-1)} \right) \cdot \frac{(1 + T)^{h^{1}C}}{(\mathbb{1} - T)(\mathbb{1} - \mathbb{1}(-1)T)}.$$ 

**Proof.** Notice that we suppress the $^{\text{mot}}$ in $P_{t}^{\text{mot}}$ and simply write $P_t$.

Recall that if $Y$ is a smooth subvariety of the smooth variety $X$ and $\tilde{X}$ denotes the blow-up of $X$ along $Y$ and $E$ is the exceptional divisor

$$P_t\tilde{X} = P_tX + P_tE - P_tY$$

In our case we get

$$P_t\tilde{M}_j = P_tM_{j-1} + P_tE_j - P_tP^W_j$$

$$P_t\tilde{M}_j = P_tM_{j} + P_tE_{j-1} - P_tP^W_j$$

combining both equalities

$$P_t M_j = P_tM_{j-1} + P^W_{j-1} + P^W_{j}$$

$$P_t M_j = P_tM_{j} + P_tE_{j-1} - P_tP^W_j$$
Projective bundles are rationally cohomologically trivial so
\[
P_t M_j = P_t M_{j-1} + P_t C^{(j)}(\mathbb{1} + \cdots + \mathbb{1}(-d + 2j - g + 2))
- P_t C^{(j)}(\mathbb{1} + \cdots + \mathbb{1}(-j + 1))
\]
this is
\[
P_t M_j = P_t M_{j-1} + P_t C^{(j)}(\mathbb{1}(-j) + \cdots + \mathbb{1}(-d + 2j - g + 2))
\]
when \(j = 0\) this is still valid taking \(M_{-1} = \emptyset\) since \(M_0\) is just \(\mathbb{P}^{d+g-2}\). Now add all these expressions for \(j = 0\) to \(j = i\)
\[
P_t M_i = \sum_{j=0}^{i} P_t C^{(j)}(\mathbb{1}(-j) + \cdots + \mathbb{1}(-d + 2j - g + 2))
\]
this proves the first part of the proposition. For the rest re-write the last expression
\[
P_t M_i = \sum_{j=0}^{i} P_t C^{(j)} \frac{\mathbb{1}(-j) - \mathbb{1}(-d + 2j - g + 1)}{\mathbb{1} - \mathbb{1}(-1)}
\]
and apply the corollary to our motivic MacDonald formula
\[
P_t M_i = \sum_{j=0}^{i} \text{Coef}_{T^j} \frac{(1+T)^{h^j C}}{(1 - \mathbb{1}T)(1 - \mathbb{1}(-1)T)} \frac{\mathbb{1}(-j) - \mathbb{1}(-d + 2j - g + 1)}{\mathbb{1} - \mathbb{1}(-1)}
\]
\[
= \text{Coef}_{T^i} \sum_{j=0}^{i} T^{i-j}(-j) - T^{i-j}(-d + 2j - g + 1)
\]
\[
= \text{Coef}_{T^i} \left( \frac{(1+T)^{h^j C}}{(1 - \mathbb{1}T)(1 - \mathbb{1}(-1)T)} \right)
\]
\[
\times \frac{T^{i+1} - (-i - 1)}{T - \mathbb{1}(-1)} + \frac{(-i + 1)(-2i + 2)\mathbb{1}(-d + g + 1 + 2i)}{(1+T)^{h^j C}} \frac{\mathbb{1}(-2i) - \mathbb{1}(-d + g + 1 + 2i)}{\mathbb{1}(-2i) - \mathbb{1}(-1)T}
\]
the Proposition is proved. \(\square\)

In the odd degree case, if \(d > 4g - 4\) \(M_\omega\) is a \(\mathbb{P}^{d-2g+1}\)-fibration over \(N_0(2,d)\). As \(N_0(2,d) \simeq N_0(2,1)\) we can choose any convenient value of \(d\), we use \(d = 4g - 3\), then \(\omega = 2g - 2\). Then as projective fibrations split in rational motives,
\[
P_t M_\omega = \frac{\mathbb{1} - \mathbb{1}(-2g + 1)}{\mathbb{1} - \mathbb{1}(-1)} P_t N_0.
\]
If we put the formula for \(P_t M_i\) into the previous expression we obtain \(P_t N_0(2,1)\) in the form.
Proof of theorem. We have seen that

\[ \frac{-\mathbb{I}(-g)}{\mathbb{I} - \mathbb{I}(-2g + 1)} \text{Coef}_f (1 + T)^{h^1 C} \frac{(1 + T)^{h^1 C}}{\mathbb{I} - \mathbb{I}(-2g + 1)} \]

Our problem is to simplify this in \( K_0 \mathcal{M}_k^{AH} \). We shall need a definition.

Definition. Let \( A, B \) and \( M \) be objects in \( \mathcal{M}_k^{AH} \) with \( r = \text{rank} M \), define \( (A+B)^M \) to be the Newton binomial

\[ (A + B)^M = \sum \wedge^i M \cdot A^{r-i} \cdot B^i \in K_0 \mathcal{M}_k^{AH} \]

Caution. It is not true that \( (A+B)^M = (B+A)^M \), maybe it would be a better idea to write \( \lambda_{A,B}(M) \) instead of \( (A+B)^M \). However in the case of interest \( M = h^1 C \) there is a certain relation.

Main Lemma. If \( M = h^1 C \), with \( C \) a curve of genus \( g \), we have

\[ (A + B)^M = (B(-1) + A)^M(g). \]

Proof. Poincaré duality on the Jacobian of \( C \) says \( \wedge^i M \simeq (\wedge^{2g-i} M(g))^\vee \), and by Poincaré duality on \( C \), \( M^\vee \simeq M(1) \) so that

\[ \wedge^i M \simeq (\wedge^{2g-i} M(g))^\vee \simeq (\wedge^{2g-i} M^\vee)(-g) \]

\[ \simeq \wedge^{2g-i}(M(1)(-g)) \simeq \wedge^{2g-i} M(g-i) \]

apply this to the definition of \( (A+B)^M \)

\[ (A + B)^M = \wedge^0 M A^{2g} + \wedge^1 M A^{2g-1} B + \wedge^2 M A^{2g-2} B^2 + \cdots + \wedge^{2g} M B^{2g} \]

\[ = \wedge^{2g} M(g) A^{2g} + \wedge^{2g-1} M(g-1) A^{2g-1} B + \cdots + \wedge^0 M(-g) B^{2g} \]

\[ = (\wedge^{2g} M A^{2g} + \wedge^{2g-1} M(-1) A^{2g-1} B + \cdots + \wedge^0 M(-2g) B^{2g})(g) \]

\[ = (B(-1) + A)^M(g). \]

\[ \square \]

Theorem. If \( N_0(2,1) \) denotes the moduli space of rank two vector bundles with fixed odd degree on a curve \( C \) then its motivic Poincaré polynomial in \( K_0(\mathcal{M}_k^{AH}) \) is

\[ P_t^{\text{mot}} N_0(2,1) = \frac{(\mathbb{I} + \mathbb{I}(-1)) h^1 C - (\mathbb{I} + \mathbb{I}) h^1 C(-g)}{(\mathbb{I} - \mathbb{I}(-1))(\mathbb{I} - \mathbb{I}(-2))}. \]

Proof of theorem. We have seen that

\[ P_t N_0 = \frac{-\mathbb{I}(-g)}{\mathbb{I} - \mathbb{I}(-2g + 1)} F(\mathbb{I}, \mathbb{I}(-1), \mathbb{I}(-2)) \]

\[ + \frac{\mathbb{I}(-2g + 2)}{\mathbb{I} - \mathbb{I}(-2g + 1)} F(\mathbb{I}, \mathbb{I}(-1), \mathbb{I}(1)) \]
where in analogy with [Th], \( F(a, b, c) \) means

\[
F(a, b, c) = \text{Coeff} \frac{(\mathbb{1} + T)^{h^1 C}}{(\mathbb{1} - aT)(\mathbb{1} - bT)(\mathbb{1} - cT)}
\]

\( a, b, c \) are now motives. By direct calculation one can prove the same identity as in [Th],

\[
F(a, b, c) = \frac{(a + 1)(a-b)(a-c)}{(a-b)(b-c)} + \frac{(b + 1)(b-c)(b-a)}{(b-c)(c-a)} + \frac{(c + 1)(c-a)(c-b)}{(c-a)(a-b)}
\]

then \( P_t N_0 \) equals

\[
\frac{\mathbb{1}(-2g+2)}{\mathbb{1} - \mathbb{1}(-2g+1)} \left( \frac{(\mathbb{1} + 1)^{h^1 C}}{(\mathbb{1} - 1)(\mathbb{1} - 2)} + \frac{(\mathbb{1}(-1) + 1)^{h^1 C}}{(\mathbb{1}(-1) - 1)(\mathbb{1}(-1) - 2)} + \frac{(\mathbb{1}(1) + 1)^{h^1 C}}{(\mathbb{1}(1) - 1)(\mathbb{1}(1) - 2)} \right)
\]

\[
+ \frac{-\mathbb{1}(g)(\mathbb{1}(-2) + 1)^{h^1 C}}{(\mathbb{1}(-2) - 1)(\mathbb{1}(-2) - 2)}
\]

Call \( S_1 \) the result of adding the third summand in both sums and \( S_2 \) the rest, we shall first calculate \( S_1 \),

\[
\frac{\mathbb{1}(-2g+2)(\mathbb{1}(1) + 1)^{h^1 C}}{(\mathbb{1}(1) - 1)(\mathbb{1}(1) - 2)} = \frac{\mathbb{1}(-2g+2)(\mathbb{1} + 1)^{h^1 C}}{(\mathbb{1} - 1)(\mathbb{1} - 2)}
\]

and

\[
\frac{-\mathbb{1}(g)(\mathbb{1}(-2) + 1)^{h^1 C}}{(\mathbb{1}(-2) - 1)(\mathbb{1}(-2) - 2)}
\]

Adding and dividing by \( (\mathbb{1} - \mathbb{1}(-2g+1)) \)

\[
S_1 = \frac{(\mathbb{1} + 1(-1))^{h^1 C}}{(\mathbb{1} - 1)(\mathbb{1} - 2)}
\]

similarly we calculate \( S_2 \)

\[
S_2 = -\frac{(\mathbb{1} + 1)^{h^1 C} \mathbb{1}(-g)}{(\mathbb{1} - 1)(\mathbb{1} - 2)}
\]

sum \( S_1 \) and \( S_2 \) to get the desired expression for \( P_t N_0 \). This proves the theorem. \( \square \)

The formula in the theorem contains the one found by Harder and Narasimhan in [HN],

\[
P_t N_0(2, 1) = \frac{(1 + t^3)^{2g} - t^{2g}(1 + t)^{2g}}{(1 - t^2)(1 - t^4)}
\]

but now we can also get the Hodge numbers and the level of the Hodge structure. Recall that the level of a Hodge structure is \( \text{Max} |p - q| \).
Corollary. The Poincaré-Hodge polynomial of $N_0(2, 1)$ is

$$P_{xy}N_0(2, 1) = \frac{(1 + x^2y)^g(1 + xy^2)^g - x^gy^g(1 + x)^g(1 + y)^g}{(1 - xy)(1 - x^2y^2)}$$

and the level of the Hodge structure $H^iN_0(2, 1)$ is less than or equal to $\left[\frac{i}{3}\right]$.

Proof. Let $\text{BiGrad-Vec}_C$ or $\text{Vec}_C[x, y]$ denote the category of vector spaces with a double graduation

$$V = \bigoplus_n \bigoplus_{i+j=n} V^{i,j}$$

Sometimes we shall write $V^{i,j}x^iy^j$ instead of $V^{i,j}$ to remind us of the graduation.

Note that $K_0\text{BiGrad-Vec}_C = \mathbb{Z}[x, y, x^{-1}, y^{-1}]$ and $[V] = \sum \dim V^{i,j}x^iy^j$.

We have to study the image of

$$(1 + 1(−1))^{h^iC} - 1(−g)(1 + 1)^{h^iC}$$

by the morphism

$$K_0M^A_{k} \to K_0\text{BiGrad-Vec}_C = \mathbb{Z}[x, y, x^{-1}, y^{-1}]$$

as this is a morphism of rings it is enough to calculate the image of $\frac{1}{1-xy}$, $\frac{1}{1-x^2y^2}$ and $x^gy^g$. Consider the functor

$$M^A_{k} \to \text{BiGrad-Vec}_C$$

It takes $1(-i)$ to $\mathbb{C}x^iy^j$ so the image of $\frac{1}{1-xy}$, $\frac{1}{1-x^2y^2}$ and $x^gy^g$ by the morphism of $K_0$ rings is just $\frac{1}{1-xy}$, $\frac{1}{1-x^2y^2}$ and $x^gy^g$.

This functor sends $(1 + 1)^{h^iC} = \bigoplus_n \wedge^n h^iC$ to $\bigoplus_n \wedge^n (\mathbb{C}^g x \oplus \mathbb{C}^g y) = \bigoplus_n \bigoplus_{i+j=n} \wedge^i(\mathbb{C}^g x) \otimes \wedge^j(\mathbb{C}^g y)$ and going down to $K_0$ we get $(1 + x)^g(1 + y)^g$.

Similarly we get the Poincaré-Hodge polynomial of the motive $(1 + 1(-1))^{h^iC}$ and putting it all together we obtain the Poincaré-Hodge polynomial of the moduli space.

Note that if $A = (1 + xy^2)(1 + x^2y)$ and $B = xy(1 + x)(1 + y)$ then

$$P_{xy}N_0(2, 1) = \frac{A^g - B^g}{A - B} = A^{g-1} + A^{g-2}B + \cdots + B^{g-1}$$

and as the only monomials in $A$ and $B$ are $x^i y^j$ with $i = j$, $i = 2j$ or $2i = j$ one can now see that the level of $H^i$ is less than or equal to $\left[\frac{i}{3}\right]$ □
§5. The mixed motive of \( N_0(2,0) \)

In the even determinant case the moduli space \( N_0(2,d) \) is not smooth and so the motive of \( N_0(2,d) \) is a mixed motive, in this section we shall not find this motive but rather the pure motivic Poincaré polynomial.

Recall that we have \( M_\omega \to N_0(2,d) \) and the motive of \( M_\omega \) is known. If \( d \) is odd this is a \( \mathbb{P} \)-fibration whereas if \( d \) is even it is only a \( \mathbb{P} \)-fibration over the stable locus (=nonsingular locus if \( g > 2 \)), \( N_0(2,d)^s \). Call \( M_\omega^s \) its preimage

\[
\begin{array}{ccc}
M_\omega^s & \longrightarrow & M_\omega \\
\downarrow & & \downarrow \\
N_0(2,d)^s & \longrightarrow & N_0(2,d)
\end{array}
\]

The strictly semistable locus of \( N_0(2,d) \) is known to be isomorphic to the Kummer variety of \( C \), \( \text{Kum}(C) = \text{Jac}(C) \), this is again a singular variety with \( 2^g \) ordinary double points.

So to calculate the pure motivic Poincaré polynomials the strategy will be:

**Step 1.** Calculate the motive of \( M_\omega - M_\omega^s \).

**Step 2.** To get then the motive of \( M_\omega^s \)

**Step 3.** Use \( M_\omega^s \to N_0(2,d)^s \) is a \( \mathbb{P} \)-bundle to obtain motive of \( N_0(2,d)^s \)

**Step 4.** Combine Step 3. with \( P_1 \text{Kum}(C) \) to calculate \( P_1N_0(2,d) \).

**Step 1. Motive of \( M_\omega - M_\omega^s \).**

We are working with vector bundles with fixed even determinant, fix this to be \( \mathcal{O}(dP) \), \( d \) is the degree and \( P \) is a point of \( C \). We are concerned with

\[
M_\omega - M_\omega^s = \{ (E,s) | \sigma \text{-stable pair with } E \text{ strictly semistable} \}
\]

\[
= \left\{ (E,s) \big| E \text{ is an extension } 0 \to \mathcal{L}(\frac{d}{2}P) \to E \to \mathcal{L}^{-1}(\frac{d}{2}P) \to 0 \right. \right.
\]

\[
\text{with } s \notin H^0(C, \mathcal{L}(\frac{d}{2}P)), \quad \text{deg} \mathcal{L} = 0
\]

Let us now code this information:

\[
\begin{array}{ccc}
H^0E & \to & H^0\mathcal{L}^{-1}(\frac{d}{2}P) \\
s & \to & \gamma \neq 0
\end{array}
\]

\( \gamma \) determines a divisor \( D \), linearly equivalent to \( \mathcal{L}^{-1}(\frac{d}{2}P) \), conversely any degree \( d \) divisor \( D \), gives a \( \mathcal{L} = \mathcal{O}(\frac{d}{2}P - D) \), this doesn’t suffice to recover \( (E,s) \). Consider

\[
\begin{array}{ccccccc}
0 & \to & \mathcal{L}(\frac{d}{2}P)|_D & \longrightarrow & E|_D & \longrightarrow & \mathcal{L}^{-1}(\frac{d}{2}P)|_D & \to 0 \\
0 & \to & H^0\mathcal{L}(\frac{d}{2}P)|_D & \longrightarrow & H^0E|_D & \longrightarrow & H^0\mathcal{L}^{-1}(\frac{d}{2}P)|_D & \to 0 \\
\gamma|_D & \to & 0
\end{array}
\]

so we get a \( p \in H^0\mathcal{L}(\frac{d}{2}P) = H^0\mathcal{O}(dP - D) \).

**Proposition.** \( D \in \text{Div}^d(C) = C(\frac{d}{2}) \) and \( p \in \mathbb{P}H^0\mathcal{O}(dP - D) \) determine \( (E,s) \) uniquely.

**Proof.** See [Th, (3.3)]

**Remark.** These data are parametrized by \( \mathbb{P}W^{-}_{\frac{d}{2}} \), a certain \( \mathbb{P}^{rac{d}{2}-1} \)-bundle over \( C(\frac{d}{2}) \).
Corollary. The inverse image of the singular locus by $M_\omega \to N_0(2, \mathcal{O}(dP))$ is isomorphic to a $\mathbb{P}^{d-1}$-bundle over $C(\frac{d}{2})$.

Remarks. 1. Recall from §3 that

$$\dim N_0 = 3g - 3 \quad \dim M_\omega - M_\omega^s = \frac{d}{2} - 1 + \frac{d}{2} = d - 1$$

$$\dim M_\omega = d + g - 2 \quad \text{codim}_{M_\omega}(M_\omega - M_\omega^s) = g - 1$$

2. $M_\omega \to N_0(2, d)$ is surjective for $d > 2g - 2$ for then every semistable bundle has nonzero sections. Also $H^1E = 0$ if $E$ stable for $d > 4g - 4$. So for $d$ odd one uses $d = 4g - 3$ and then $\omega = [\frac{d-1}{2}] = 2g - 2$, for $d$ even we use $d = 4g - 2$ and $\omega = \lceil \frac{4g-3}{2} \rceil = 2g - 2$.

We saw that $M_\omega - M_\omega^s$ is isomorphic to a $\mathbb{P}^{d-1}$-bundle over $C(\frac{d}{2})$ so

$$P_t(M_\omega - M_\omega^s) = P_t\mathbb{P}^{d-1}P_tC(\frac{d}{2}) = P_t\mathbb{P}^{2g-2}P_tC(2g-1)$$

now we don’t need any MacDonald formula as $C(2g-1)$ is a $\mathbb{P}^{g-1}$-bundle over $Jac(C)$ so

$$P_t(M_\omega - M_\omega^s) = \frac{1}{1 - \frac{1}{T}} \frac{1}{1 - \frac{1}{T}} \frac{(1 + T)^{g-1}C}{(1 - T)(1 - \frac{1}{T})}$$

Step 2. Motive of $M_\omega^s$.

Recall that $P_tM_t$ equals

$$P_tM_\omega = \frac{1}{1 - \frac{1}{T}} \frac{1}{1 - \frac{1}{T}} \frac{\text{Coeff} \left( \frac{(-d + 2i - g + 1)}{(-2)T - \frac{1}{T}} - \frac{(-i - 1)}{T - \frac{1}{T}} \right)}{(1 - T)(1 - \frac{1}{T})}$$

In our case $i = \omega = 2g - 2$ and $d = 4g - 2$ so that $-d + 2i + 1 - g = -g - 1$ and $-i - 1 = -2g + 1$, so

$$P_tM_\omega = \frac{1}{1 - \frac{1}{T}} \frac{1}{1 - \frac{1}{T}} \frac{\left( -\frac{1}{T} \frac{(-g - 1)}{(-2)T - \frac{1}{T}} - \frac{(-2g + 1)}{T - \frac{1}{T}} \right)}{(1 - T)(1 - \frac{1}{T})}$$

where $M_\omega(2, 1)$ indicates the moduli space for odd degree considered in the previous section, we know its Poincaré polynomial

$$P_tM_\omega(2, 1) = \frac{1}{1 - \frac{1}{T}} \frac{1}{1 - \frac{1}{T}} \frac{\left( -\frac{1}{T} \frac{(-g - 1)}{(-2)T - \frac{1}{T}} - \frac{(-2g + 1)}{T - \frac{1}{T}} \right)}{(1 - T)(1 - \frac{1}{T})}$$

Now $P_tM_\omega$ is

$$\frac{1}{1 - \frac{1}{T}} \frac{1}{1 - \frac{1}{T}} \frac{\left( -\frac{1}{T} \frac{(-g - 1)}{(-2)T - \frac{1}{T}} - \frac{(-2g + 1)}{T - \frac{1}{T}} \right)}{(1 - T)(1 - \frac{1}{T})}$$

and we use the formula for $F(a, b, c)$ to obtain

$$\left( \frac{1}{1 - \frac{1}{T}} \frac{1}{1 - \frac{1}{T}} \frac{\left( -\frac{1}{T} \frac{(-g - 1)}{(-2)T - \frac{1}{T}} - \frac{(-2g + 1)}{T - \frac{1}{T}} \right)}{(1 - T)(1 - \frac{1}{T})} \right)$$

Doing a bit of calculation we get...
Now is when pure Poincaré polynomials of mixed motives enter the scene,

\[
P_t M^s = P_t M_\omega - \mathbb{1}(1 - g) \cdot P_t (M_\omega - M^s)
\]

\[
\frac{(1 + \mathbb{1}(-1))^{h^1C} (1 - \mathbb{1}(-2g)) - \mathbb{1}(1 + \mathbb{1}(-g)) (1 + \mathbb{1}(2g))}{(1 - \mathbb{1}(-1))^2 (1 - \mathbb{1}(-2))}
\]

Step 3. Motive of \(N_0(2, 0)^s\).

As \(M^s \to N_0(2, 0)^s\) is a \(\mathbb{P}^{2g-1}\)-bundle, the cohomology of the \(N_0(2, 0)^s\) will be that of \(M^s\) divided by \(\mathbb{1} - \mathbb{1}(-2g)/\mathbb{1} - \mathbb{1}(-1)^s\), that is \(P_t N_0(2, d)^s\) equals

\[
\frac{(1 + \mathbb{1}(-1))^{h^1C}}{(1 - \mathbb{1}(-1))(1 - \mathbb{1}(-2))} - \frac{\mathbb{1}(-g)(1 + \mathbb{1})^{h^1C} (1 - \mathbb{1}(1)(1 + \mathbb{1}(2g)) - \mathbb{1}(1)(1 + \mathbb{1}(1)(1 + \mathbb{1}(2g)) (1 - \mathbb{1}(-2)))}{(1 - \mathbb{1}(-1))(1 - \mathbb{1}(-2)) (1 - \mathbb{1}(-2))}.
\]

Step 4. Motive of \(N_0(2, 0)\).

Use the previous expression together with

\[
P_t N_0(2, 0) = P_t N_0(2, 0)^s + P_t Kum(C)(-2g + 3)
\]

Note that the motivic Poincaré polynomial of the Kummer variety of \(C\) is given by

\[
P_t Kum(C) = \frac{1}{2} \left( (\mathbb{1} + \mathbb{1})^{h^1C} + (\mathbb{1} - \mathbb{1})^{h^1C} \right) = \sum_{i=0}^{g} \wedge_{i=0}^{2i} h^1C.
\]

Corollary. The mixed AH motive \(h^1N_0(2, 0)\) has only weights \(i\) and \(i + 1\). Furthermore, if \(i < 2g - 2\), \(h^1N_0(2, 0) \simeq h^1N_0(2, 1)\), in particular it is pure of weight \(i\).

Proof. \(M_\omega\) and \(M_\omega - M^s\) are smooth and projective so their cohomology is pure, now by Gysin

\[
\cdots \to h^{i-2g+2}(M_\omega - M^s)(-g+1) \to h^{i}M_\omega \to h^{i}M^s \to h^{i+1-2g+2}(M_\omega - M^s)(-g+1) \to \cdots
\]

so \(h^i(M^s)\) has weights \(i\) and \(i + 1\). As this is a projective bundle over \(N_0(2, 0)^s\), the same happens with \(N_0(2, 0)^s\) and the Gysin exact sequence associated to \(Kum(C) \to N_0(2, 0)\)

\[
\cdots \to h^{i-2g+6}Kum(C)(-2g+3) \to h^{i}N_0(2, 0) \to h^{i}N_0(2, 0)^s \to h^{i+1-2g+6}Kum(C)(-2g+3) \to \cdots
\]

gives the result.

The ring \(K_0 \mathcal{M}^{AH}_k\) is a graded ring so it makes sense to take the \(n\)-truncation of an element of \(K_0 \mathcal{M}^{AH}_k\), going down to Poincaré polynomials this is just taking the residue modulo \(t^n\).

The \(2g - 2\) truncation of \(P_t N(2, 0)\) coincides with the \(2g - 2\) truncation of

\[
\frac{(1 + \mathbb{1}(-1))^{h^1C}}{(1 - \mathbb{1}(-1))(1 - \mathbb{1}(-2))}
\]

and this coincides with that of \(P_t N(2, 1)\). \(\square\)

Remark. Of course if the genus of the curve is 2 then the moduli space \(N_0(2, 0)\) is smooth so one gets the pure motive of this.
§6. Jacobian considerations.

Let $X$ be a smooth projective complex variety. Griffiths defines the $i$-th intermediate jacobian of $X$ to be:

$$J^i(X) = \frac{H^{2i-1}(X, \mathbb{C})}{F^iH^{2i-1}(X, \mathbb{C}) + H^{2i-1}(X, \mathbb{Z})}$$

This is just a complex torus with a naturally defined 2-form, it is not an abelian variety in general.

Now let $k$ be a field as before. If $X$ is a smooth projective variety over $k$ for each embedding $\sigma : k \hookrightarrow \mathbb{C}$ we get a complex variety $X_\sigma$ and intermediate jacobians for each $X_\sigma$.

Note that if we know $h^{2i-1}(X)$ and are interested in $J^i(X_\sigma)$ there is only one piece of data missing: the entire structure on the singular cohomology group $H^{2i-1}(X_\sigma, \mathbb{Q})$, so we can recover $J^i(X_\sigma)$ up to isogeny from $h^{2i-1}(X)$.

As we have just seen one can recover the intermediate jacobians up to isogeny from the motive of the variety. We shall now exploit the expression found for the motivic Poincaré polynomial of the moduli spaces $N_0(2,1)$ and $N_0(2,0)$ to get information on the intermediate jacobians.

**Corollary.** Let $\delta \in \{0, 1\}$. For $i \leq g - 1 + \delta$ the $i$-th intermediate jacobian of the moduli space $N_0(2, \delta)$ is isogenous to

$$\prod_{\alpha=1}^{\lceil \frac{i+1}{2} \rceil} J^\alpha Jac(C)^{\lceil \frac{i+3-2\alpha}{2} \rceil}$$

**Proof.** We have proved that $h^i N_0(2,0) \simeq h^i N_0(2,0)$ if $i < 2g - 2$ so that it suffices to prove the theorem for the odd degree case.

In the smooth case, expand the formula in the theorem in power series to get

$$((1 + 1(-1))^{h^1C} - 1(-g)(1 + 1)^{h^1C})(\sum 1(-i))(\sum 1(-2i))$$

taking the $2g$-truncation

$$\sum_{n \geq 0} \wedge^i h^1C(-i)(1 + 1(-1) + 1(-2) + \cdots)(1 + 1(-2) + 1(-4) + \cdots)$$

The last expression is seen to be equal to

$$\sum_{n \geq 0} \wedge^i h^1C(-i) \sum_{n \geq 0} \left[ \frac{n}{2} + 1 \right] 1(-n)$$

Multiplying the series we get a weight decomposition

$$\sum \left( \sum \wedge^a h^1C(-a) \otimes \left[ \frac{b}{2} + 1 \right] 1(-b) \right)$$
As we are interested in odd cohomology $n = 2i - 1$, and if $3a + 2b = 2i - 1$ then $a$ has to be odd, $a = 2\alpha - 1 \alpha \geq 1$. And the condition $3a + 2b = n$ turns into $3\alpha + b = i + 1$. So that for $i \leq g$ we get

$$h^{2i-1}N_0 = \sum_{\alpha=1}^{i+1} \wedge^{2\alpha-1} h^1 C(1 - 2\alpha) \otimes \left[ \frac{i + 1 - 3\alpha}{2} + 1 \right] \mathbb{H}(3\alpha - i - 1)$$

$$= \sum_{\alpha=1}^{i+1} \wedge^{2\alpha-1} h^1 C(\alpha - i) \otimes \left[ \frac{i + 3\alpha}{2} \right]$$

now the result follows. □

Remarks.

- Newstead and Mumford proved in [MN] that $J^2N_0(2,1)$ is in fact isomorphic to $Jac(C)$ as polarized abelian varieties, as a corollary they obtain a Torelli-type theorem: if $N_0(2,1)$ is isomorphic for two curves then the curves are isomorphic.

- Balaji has calculated the second intermediate jacobian of the canonical desingularization of $N_0(2,0)$, $M([Ba])$, there is a canonical isogeny of degree $2^{2g} Jac(C) \rightarrow J^2M$. As in the case treated by Mumford and Newstead he obtains a Torelli-type theorem.

- Our corollary applies as well to the $\ell$-adic intermediate jacobians. For the case $i = 2$ treated by Mumford and Newstead one can re-do their proof in the $\ell$-adic setup to obtain an isomorphism and not only an isogeny.

References

[AB] M.F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1982), 523-615.

[Ba] V. Balaji, Intermediate jacobian of some moduli spaces of vector bundles on curves, Am. J. Math. 112 (1990), 611-630.

[BGL] E.Bifet, F.Ghione and M.Letizia, On the Abel-Jacobi map for divisors of higher rank on a curve, Math. Ann. 299 (1994), 641-672.

[BS1] V. Balaji and C.S. Seshadri, Cohomology of a moduli space of vector bundles, The Grothendieck Festschrift, vol. I, Birkhuser, 1990, pp. 87-120.

[BS2] V. Balaji and C.S. Seshadri, Poincaré polynomials of some moduli varieties, Algebraic Geometry and Analytic Geometry, ICM-90 Satellite Conference Proceedings, Springer, 1990, pp. 1-25.

[Bu] J. Burillo, El polinomio de Poincaré-Hodge de un producto simétrico de variedades kähleriannas compactas, Collect. Math. 41 (1990), 59-69.

[DM] P. Deligne and J. Milne, Tannakian Categories, Hodge Cycles, Motives and Shimura Varieties, Lecture Notes in Math. 900, 1982.

[G] A. Grothendieck, Recoltes et Semailles: réflexions et témoignage sur un passé de mathématicien, Montpellier, 1985.

[HN] G. Harder and M.S. Narasimhan, On the cohomology groups of moduli spaces of vector bundles over curves, Math. Ann. 212 (1975), 215-248.

[J] U. Jannsen, Mixed Motives and Algebraic K-Theory., Lecture Notes in Math. 1400, 1989.

[K] F. Kirwan, On the homology of compactifications of moduli spaces of vector bundles over a Riemann surface 53 (1986), Proc. London Math. Soc. (3), 237-266.

[LPV] J. Le Potier, J.L. Verdier, Module des fibrés stables sur les courbes algébriques, Progress in Math.54 Birkhuser, 1985.

[M] I.G. MacDonald, The Poincaré polynomial of a symmetric product, Proc. Camb. Phil. Soc. 58 (1962), 563-568.

[Ma] Y. Manin, Correspondences, motives and monoidal transformations, Math. USSR Sb. 6 (1968), 421-470.
[MN] D. Mumford and P. Newstead, *Periods of a moduli space of bundles on curves*, Am. J. Math. 90, 1201 (1968), 1200-1208.

[Na] V. Navarro - Aznar, *Stratifications parfaites et thorie des poids*, Pub. Mat. U.A.B. 36 (1992), 807-825.

[Ne] P. Newstead, *Topological properties of some spaces of stable bundles*, Topology 6 (1967), 241-262.

[S] J.P. Serre, *Motifs*, Journées Arithmetiques Luminy, Asterisque 198-200, 1991, pp. 333-349.

[Th] M. Thaddeus, *Stable Pairs, linear systems and the Verlinde formula*, Invent. Math. 117 (1994), 317-353.

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