CHROMATIC CONVERGENCE FOR THE ALGEBRAIC 
K-THEORY OF THE SPHERE SPECTRUM

ANDREW J. BLUMBERG, MICHAEL A. MANDELL, AND ALLEN YUAN

Abstract. We show that the map from $K(S)$ to its chromatic completion is a connective cover and identify the fiber in $K$-theoretic terms. We combine this with recent work of Land-Mathew-Meier-Tamme to prove a form of “Waldhausen’s Chromatic Convergence Conjecture”: we show that the map $K(S(p))_{(p)} \rightarrow \text{holim} K(L_n S(p))$ is the inclusion of a wedge summand.

1. Introduction

The chromatic approach to homotopy theory is geared to study spectra that are\textit{chromatically convergent}, meaning that (for a fixed prime $p$) the natural map

$$X_{(p)} \rightarrow \text{holim}_n L_n X$$

is a weak equivalence, where $L_n$ denotes localization with respect to the Johnson-Wilson spectrum $E(n)$, or equivalently, the sum of the Morava $K$-theory spectra $K(0) \vee \cdots \vee K(n)$. A celebrated theorem of Hopkins-Ravenel, the\textit{chromatic convergence theorem}, is that finite spectra are chromatically convergent.

Waldhausen [37, §4] (following work of Dwyer-Friedlander [17] Thomason [33]) tied the chromatic approach to algebraic $K$-theory in his reformulation of the Quillen-Lichtenbaum conjecture: the Quillen-Lichtenbaum conjecture holds for a ring or scheme $Z$ at the prime $p$ if and only if the localization map $K(Z) \rightarrow L_1 K(Z)$ is an isomorphism on homotopy groups in high degrees. Mitchell [26] (Corollary) shows that the maps $L_n K(Z) \rightarrow L_1 K(Z)$ are always weak equivalences. We can then rephrase the Quillen-Lichtenbaum conjecture for $Z$ as the assertion that $K(Z)$ is chromatically semiconvergent in the following sense.

**Definition 1.1.** A spectrum $X$ is \textit{chromatically $N$-semiconvergent} for some $N \in \mathbb{Z}$ if the chromatic completion map

$$X_{(p)} \rightarrow \text{holim}_n L_n X$$

has $N$-truncated fiber (homotopy groups are zero in degree above $N$). It is \textit{chromatically semiconvergent} if it is chromatically $N$-semiconvergent for some $N$, or equivalently, when the chromatic completion map induces an isomorphism on homotopy groups in high degrees.

Mitchell’s result and elementary properties of chromatic semiconvergence discussed in Section 2 then imply the following theorem.
Theorem 1.2. For a ring $R$ (not necessarily commutative) or a scheme $Z$, the map $K(R) \rightarrow L_1K(R)$ or $K(Z) \rightarrow L_1K(Z)$ is an isomorphism on homotopy groups in high degrees if and only if $K(R)$ or $K(Z)$ is chromatically semiformal.

The chromatic semiformal perspective now allows a straightforward generalization from rings to ring spectra of the underlying question of the Quillen-Lichtenbaum conjecture: for which ring spectra $R$ is $K(R)$ chromatically semiformal? As an example of this perspective, we offer the following theorem (proved in Section 2) that in the cases it covers reduces chromatic semiformality of algebraic $K$-theory to the corresponding question for $TC$, which tends to be reasonably accessible to homotopy theoretic arguments; see for example [9, §4]. For a more general result closely related to Quillen-Lichtenbaum conjectures for rings, see Corollary 2.6 and the remarks that follow it.

Theorem 1.3. Let $R$ be a connective ring spectrum with $\pi_0R = \mathbb{Z}$, $\mathbb{Z}(p)$, or $\mathbb{Z}_p^\wedge$. Then $K(R)$ is chromatically semiformal if and only if $TC(R)$ is chromatically semiformal if and only if $TC(R; p)^{\wedge}_p$ is chromatically semiformal.

We showed in [9, 1.3] that for $S = \mathbb{S}, \mathbb{S}(p)$, or $\mathbb{S}_p^\wedge$, the spectrum $TC(S)^{\wedge}_p$ is chromatically convergent (see also Proposition 2.7 below). This resolves the question for $S, \mathbb{S}(p)$, and $\mathbb{S}_p^\wedge$.

Corollary 1.4. For $S = \mathbb{S}, \mathbb{S}(p)$, or $\mathbb{S}_p^\wedge$, $K(S)$ is chromatically semiformal.

The main purpose of this paper is to study the fiber of the chromatic completion map for $K(S)$, $K(\mathbb{S}(p))$, and $K(\mathbb{S}_p^\wedge)$. The case of $K(\mathbb{S}_p^\wedge)$ is easiest to describe.

Theorem 1.5. The homotopy fiber of $K(\mathbb{S}_p^\wedge)(p) \rightarrow \text{holim} L_nK(\mathbb{S}_p^\wedge)$ is an Eilenberg-Mac Lane spectrum $\Sigma^{-3}H(\mathbb{Z}/p^\infty)$. In particular, for $q > 0$,

$$\pi_{-q} \text{holim} L_nK(\mathbb{S}_p^\wedge) \simeq \begin{cases} \mathbb{Z}/p^\infty & q = 2 \\ 0 & q \neq 2 \end{cases}$$

The case of $L_nK(\mathbb{S})$ is more interesting.

Theorem 1.6. The homotopy fiber of $K(\mathbb{S})(p) \rightarrow \text{holim} L_nK(\mathbb{S})$ is weakly equivalent to $\Sigma^{-3}I_{\mathbb{Z}/p^\infty}K(\mathbb{Z})$, where $I_{\mathbb{Z}/p^\infty}$ denotes the $p$-local Brown-Comenetz dual. In particular, for $q > 0$

$$\pi_{-q}(\text{holim} L_nK(\mathbb{S})) \cong \pi_{-q-1}(\Sigma^{-3}I_{\mathbb{Z}/p^\infty}K(\mathbb{Z})) \cong \text{Hom}(\pi_{q-2}K(\mathbb{Z}), \mathbb{Z}/p^\infty)$$

In the case of $\mathbb{S}(p)$, we have a fiber sequence.

Theorem 1.7. The homotopy fiber $F(K(\mathbb{S}(p)))$ of $K(\mathbb{S}(p))(p) \rightarrow \text{holim} L_nK(\mathbb{S}(p))$ fits in a fiber sequence

$$F(K(\mathbb{S}(p))) \rightarrow \bigvee_{\ell \neq p} \Sigma^{-2}I_{\mathbb{Z}/p^\infty}K(\mathbb{F}_\ell) \rightarrow \Sigma^{-2}I_{\mathbb{Z}/p^\infty}K(\mathbb{Z}) \rightarrow \Sigma \cdots,$$

where the wedge is over primes $\ell \neq p$, and for $p > 2$ the map $I_{\mathbb{Z}/p^\infty}K(\mathbb{F}_\ell) \rightarrow I_{\mathbb{Z}/p^\infty}K(\mathbb{Z})$ is Brown-Comenetz dual to the map $K(\mathbb{Z}) \rightarrow K(\mathbb{F}_\ell)$ induced by $\mathbb{Z} \rightarrow \mathbb{F}_\ell$.

We conjecture that the description of the map also holds in the case $p = 2$; see Conjecture 3.3 and Theorem 6.9 for a more specific formulation.

As particular consequences of the theorems above, for $S = \mathbb{S}, \mathbb{S}(p)$, or $\mathbb{S}_p^\wedge$, the chromatic completion map $K(S)(p) \rightarrow \text{holim} L_nK(S)$ is a connective cover.
So far all our discussion has been for the chromatic localization functors $L_n$. However, all previous work on Quillen-Lichtenbaum conjectures for ring spectra have been based on periodic homotopy groups and the finite telescopic localization functors $L_f^n$ (see, for example, [25] for a definition). In particular, Rognes’ redshift conjectures [30, p. 8], [32, 4.1(a)], [20, 6.0.5] and results in this direction by Angelini-Knoll [3], Angelini-Knoll-Quigley [2], Ausoni-Rognes [4], Carmeli-Schlank-Yanovski [12], Clausen-Mathew-Naumann-Noel [13], Hahn-Wilson [20], Land-Mathew-Meier-Tamme [23], and Veen [36] ask or answer the following question about particular ring spectra $R$:

**Question (K-Theoretic Periodicity Question for $R$).** Does the map

$$K(R) \to L_f^n K(R)$$

induce an isomorphism on homotopy groups in all sufficiently high degrees?

There is an essentially unique natural transformation of localizations $L^n_f \to L_n$, and Ravenel’s telescope conjecture [28, 10.5] is the assertion that this is a weak equivalence. It is known to hold for $n = 1$, but is now widely expected by experts to fail for $n > 1$. If the telescope conjecture fails to hold for all sufficiently high $n$, then the relationship between $K$-theory chromatic semiconvergence questions and the cited redshift work above is not easily described.

Instead, the $K$-Theoretic Periodicity Question for ring spectra $R$ is intimately related to the concept of **telescopic** semiconvergence, which coincides with chromatic semiconvergence (in particular) if the telescope conjecture holds for infinitely many $n$. Say that a spectrum $X$ is telescopically convergent when the map

$$X(p) \to \text{holim}_n L_f^n X$$

is a weak equivalence and telescopically semiconvergent when it induces an isomorphism on homotopy groups in high degrees. Many of the formal properties of telescopic semiconvergence are analogous to the properties of chromatic semiconvergence; in particular, for a ring $R$ (as opposed to ring spectrum) or scheme $Z$, the map $K(R) \to L_f^n K(R)$ or $K(Z) \to L_f^n K(Z)$ induces an isomorphism in homotopy groups in high degrees if and only if $K(R)$ or $K(Z)$ is telescopically semiconvergent. More generally, for a spectrum $X$, the map $X \to L_f^n X$ induces an isomorphism on homotopy groups in high degrees if and only if $X$ is telescopically semiconvergent and its periodic homotopy groups $T(m)_* X$ vanish for all $m > n$ (where $T(m) = v_m^{-1} V(m)$ for some chosen finite type $m$ spectrum $V(m)$). Moreover, the main result of Land-Mathew-Meier-Tamme [23] implies that $T(m)_* K(R)$ vanishes for all $m > n$ when $T(m)_* R$ vanishes for all $m \geq n$ or equivalently (see [23, 2.3]) when $K(m)_* R$ vanishes for all $m \geq n$. Thus, in the context of redshift conjectures, the $K$-theoretic Periodicity Question for $R$ is equivalent to telescopic semiconvergence for $K(R)$. If we ask instead the question:

**Question.** Is $K(R)$ telescopically semiconvergent?

This then expands the scope of Quillen-Lichtenbaum conjectures for ring spectra beyond those that vanish above a given chromatic or telescopic height. In particular, we can ask this question for $S$ or $MU$.

In the setting of the current paper, the analogue of Theorem 1.2 (and the more general Corollary 2.6 in Section 4) hold with telescopic semiconvergence replacing chromatic semiconvergence by the analogous proof with $L_f^n$ replacing $L_n$. We do
not, however, have the telescopic analogue of Corollary 1.4 (telescopic semiconvergence of $K(S)$): it is not known whether $TC(S)$ or $TC(S; p)$ is telescopically semiconvergent. Indeed, it is not known whether the sphere spectrum is telescopically semiconvergent. Because $S$ is a wedge summand of $K(S)$, telescopic semiconvergence of $K(S)$ would imply telescopic semiconvergence of $S$ and hence telescopic semiconvergence of all finite spectra. Such a result is currently out of reach; the best we can say is that the map $S(p) \to \text{holim} L_n^p S$ is an inclusion of a wedge summand.

Returning to the ideas in Waldhausen [37], the original Ravenel conjectures [28, §10] motivated Waldhausen to make the following conjecture:

**Conjecture 1.8** (Waldhausen’s Chromatic Convergence Conjecture).

$$K(S(p)) \xrightarrow{\cong} \text{holim} K(L_n^p S).$$

In fact, Waldhausen’s original conjecture concerns the category of finite spectra with telescopic equivalences, which (by [38, 1.6.7]) gives the “free” algebraic $K$-theory of $L_n^p S$ (the $K$-theory of finite cell complexes rather than of perfect complexes). The previous conjecture is equivalent (by [35, 1.10.2]) to Waldhausen’s original conjecture plus an additional conjecture that $K_0(S) \to \lim K_0(L_n^p S)$ is an isomorphism and $\lim K_0(L_n^p S) = 0.$

If telescopic semiconvergence of $S$ fails, it still makes sense to ask for the map in Conjecture 1.8 to be the inclusion of a wedge summand. Combining our chromatic semiconvergence theorem for $K(S(p))$ with the results of Land-Mathew-Meier-Tamme [23], we prove the following theorem in Section 7.

**Theorem 1.9.** The natural map

$$K(S(p)) \xrightarrow{(p)} \text{holim} K(L_n^p S)$$

is the inclusion of a wedge summand.

Land-Mathew-Meier-Tamme [23, 2.5] formulate a finite localization functor $L_n^{p,f}$ that is a version of $L_n^p$ which does not $p$-localize (does not invert the primes $\ell \neq p$). Using this localization functor, we also have the following result for $K(S)$.

**Theorem 1.10.** The natural map

$$K(S) \xrightarrow{(p)} \text{holim} K(L_n^{p,f} S)$$

is the inclusion of a wedge summand.

At the time Waldhausen was writing, it was widely expected that the telescope conjecture would hold for all $n$ and [37] assumes this for some of its conclusions; for this reason, the conjecture is sometimes formulated with the chromatic localizations.

**Conjecture 1.11** (Waldhausen’s Chromatic Convergence Conjecture, 2nd Version).

$$K(S(p)) \xrightarrow{\cong} \text{holim} K(L_n S).$$

For the chromatic localizations $L_n$, the analogue of Theorem 1.9 is a slightly weaker version of Conjecture 1.11 that seems approachable and still retains much of the power of the original. We discuss the current obstacle to a complete proof at the end of Section 7.

**Conventions.** Throughout this paper the unmodified term “ring spectrum” and the term “commutative ring spectrum” mean $A_\infty$ and $E_\infty$ ring spectrum or (equivalently) $\mathcal{S}$-algebra and commutative $\mathcal{S}$-algebra.
Acknowledgments. The authors thank Ben Antieau, Prasit Bhattacharya, Jeremy Hahn, Mike Hopkins, and John Rognes for helpful conversations.

2. Basic properties of chromatic semiconvergence and proof of Theorem 1.3

In this section, we review some easy generalities about chromatic convergence and chromatic semiconvergence. Theorem 1.3 follows from these properties, the resolved Quillen-Lichtenbaum conjecture, and the work of Mitchell [24], Dundas-Goodwillie-McCarthy [18, 15, 24], and Hesselholt-Madsen [21] on $K$-theory of rings and ring spectra.

Because localizations and homotopy limits commute with fiber sequences, cofiber sequences, and suspensions, so does chromatic convergence. This leads to the following observation about chromatic semiconvergence.

Proposition 2.1. If $W \to X \to Y \to \Sigma \cdots$ is a fiber sequence of spectra then so is

$$\text{holim} L_n W \to \text{holim} L_n X \to \text{holim} L_n Y \to \Sigma \cdots.$$ 

In particular, if $W, Y$ are chromatically $N$-semiconvergent, then so is $X$.

We used the following proposition in the introduction in the example of algebraic $K$-theory spectra, and we record it here for convenience.

Proposition 2.2. Let $X$ be a spectrum and assume that there exists $m \geq 0$ such that $K(n)_*X = 0$ for all $n > m$. Then the natural map $\text{holim} L_n X \to L_m X$ is a weak equivalence.

Because $N$-truncated spectra are $K(n)$-acyclic for all $n > 0$, the previous propositions then imply the following one.

Proposition 2.3. Let $X$ be a spectrum and assume that there exists $m \geq 0$ and $N \in \mathbb{Z}$ such that the localization map $X \to L_m X$ is $N$-truncated. Then $X$ is chromatically $N$-semiconvergent. In particular, if $X$ is $L_m$-local, then $X$ is chromatically convergent.

The cofiber of the map from the localization to the completion $X(p) \to X_p^\wedge$ is always rational (and hence $L_0$-local). Combining Propositions 2.1 and 2.2 we get the following result.

Proposition 2.4. For any spectrum $X$ the square

$$\begin{array}{ccc}
X(p) & \longrightarrow & \text{holim} L_n X \\
\downarrow & & \downarrow \\
X_p^\wedge & \longrightarrow & \text{holim} L_n (X_p^\wedge)
\end{array}$$

is homotopy cartesian. In particular, a spectrum $X$ is chromatically $N$-semiconvergent if and only if $X_p^\wedge$ is chromatically $N$-semiconvergent.

We now apply the previous observations to $K$-theory. For any connective ring spectrum $R$, Celebrated work of Dundas-Goodwillie-McCarthy [18, 24] (see
also \cite[VII.2.2.1)]{14} gives a homotopy cartesian square

\[
\begin{array}{ccc}
K(R) & \longrightarrow & K(R) \\
\downarrow & & \downarrow \\
TC(\pi_0 R) & \longrightarrow & TC(\pi_0 R)
\end{array}
\]

where \(TC\) denotes Goodwillie’s integral \(TC\) \cite[see also \cite{14} VI.3.3.1]}{19}, the horizontal maps are the linearization maps (induced by the map of ring spectra \(R \rightarrow H\pi_0 R\)) and the vertical maps are the cyclotomic trace maps. Applying chromatic completion \(\text{holim}\, L_n(-)\), we get a homotopy cartesian square

\[
\begin{array}{ccc}
\text{holim}\, L_n K(R) & \longrightarrow & \text{holim}\, L_n K(\pi_0 R) \\
\downarrow & & \downarrow \\
\text{holim}\, L_n TC(R) & \longrightarrow & \text{holim}\, L_n TC(\pi_0 R).
\end{array}
\]

Because \(K(\pi_0 R)\) and \(TC(\pi_0 R)\) are \(K(\mathbb{Z})\)-modules and \(K(n)_* K(\mathbb{Z}) = 0\) for \(n \geq 1\) by the work of Mitchell \cite[Thm A]{26}, we have that

\[
K(n)_* K(\pi_0 R) = 0 \quad \text{and} \quad K(n)_* TC(\pi_0 R) = 0
\]

for \(n > 1\). By Proposition \ref{prop:linearization}, we have that the maps

\[
\text{holim}\, L_n K(\pi_0 R) \longrightarrow L_1 K(\pi_0 R) \quad \text{and} \quad \text{holim}\, L_n TC(\pi_0 R) \longrightarrow L_1 TC(\pi_0 R)
\]

are weak equivalences. This proves the following theorem.

**Theorem 2.5.** Let \(R\) be a connective ring spectrum. Then the diagram induced by trace and linearization maps

\[
\begin{array}{ccc}
\text{holim}\, L_n K(R) & \longrightarrow & \text{holim}\, L_n K(\pi_0 R) \\
\downarrow & & \downarrow \\
\text{holim}\, L_n TC(R) & \longrightarrow & \text{holim}\, L_n TC(\pi_0 R).
\end{array}
\]

is homotopy cartesian.

Combining with the elementary properties of chromatic semiconvergence in the propositions above, we get the following corollary. In it \(TC(R; p)\) denotes \(p\)-typical \(TC\) and we are using the well-known result that the canonical map \(TC(R)^\wedge \rightarrow TC(R; p)^\wedge\) is a weak equivalence \cite[14.1.(ii)]{19}, \cite[2.2]{9}.

**Corollary 2.6.** Let \(R\) be a connective ring spectrum and assume that \(K(\pi_0 R)\) and \(TC(\pi_0 R)\) are chromatically semiconvergent. Then \(K(R)\) is chromatically semiconvergent if and only if \(TC(R)\) is chromatically semiconvergent if and only if \(TC(R; p)^\wedge\) is chromatically semiconvergent.

We note that the requirement that \(K(\pi_0 R)\) is chromatically semiconvergent is equivalent to requiring that the localization map

\[
K(\pi_0 R)^\wedge \longrightarrow (L_1 K(\pi_0 R))^\wedge \simeq L_{K(1)} K(\pi_0 R)
\]

be \(N\)-truncated for some \(N\); this is precisely the requirement that the Quillen-Lichtenbaum conjecture hold for \(\pi_0 R\) (as reformulated by Waldhausen). Work of Hesselholt-Madsen \cite[Thm D, Add 5.2]{21} shows that under finite generation
Proposition 2.7. Let \( S = \mathbb{S}, S_{(p)}, \) or \( S^\wedge_p \). Then \( TC(S; p)^\wedge \) is chromatically convergent.

Proof. In all three cases, we have by \([10]\) that \( TC(S; p)^\wedge \) is \( S^\wedge_p \vee X^\wedge_p \) where \( X \) is the fiber of the \( T \)-transfer \( \Sigma\Sigma^\infty_+ BT \rightarrow \mathbb{S} \). We have that \( S \) is chromatically convergent by the Hopkins-Ravenel chromatic convergence theorem, and \( \Sigma^\infty_+ BT \) is chromatically convergent by \([9, 4.2]\).

The proof of the Hopkins-Ravenel chromatic convergence theorem (and Barthel’s elaboration \([9, 3.8]\) used via \([9]\) in the argument above) proves more than a weak equivalence; it actually proves pro-isomorphisms on each homotopy group. Specifically, for fixed \( q \in \mathbb{Z} \), the map of towers in \( n \)
\[
\{\pi_q(TC(S; p^\wedge))\} \rightarrow \{\pi_q(L_n(TC(S; p^\wedge)))\}
\]
is a pro-isomorphism, where on the left we have the constant tower. In particular, when \( S = \mathbb{S}, S_{(p)}, \) or \( S^\wedge_p \), then for each fixed \( q \), the towers \( \{\pi_q(L_n(TC(S; p^\wedge)))\} \) and therefore \( \{\pi_q(TC(S; p^\wedge))\} \) are pro-constant. Since for \( Z = \pi_0 S \), the towers \( \{\pi_q(L_nK(Z))\} \) and \( \{\pi_q(L_nTC(Z))\} \) are actually constant (for \( n \geq 1 \)), we get the following proposition.

Proposition 2.8. Let \( S = \mathbb{S}, S_{(p)}, \) or \( S^\wedge_p \), and fix \( q \in \mathbb{Z} \). The tower in \( n \)
\[
\{\pi_q(L_nK(S))\}
\]
is pro-constant.

3. Tools for Studying Fiber of the Chromatic Completion Map on \( K(\mathbb{S}) \)

In this section, we start work on Theorem 1.6, 1.7, and 1.8 which identify the fiber of the chromatic completion map for \( K(\mathbb{S}), K(S_{(p)}), \) and \( K(S^\wedge_p) \), respectively. In the first part of this section, we argue that the statements reduce to assertions for the \( p \)-completions. In the case of \( K(S^\wedge_p) \), this is all that is needed to prove Theorem 1.5 and the proof is completed in this section. Theorems 1.6 and 1.7 for \( \mathbb{S} \) and \( S_{(p)} \) are stated in terms of the Brown-Comenetz duality, and for the \( p \)-complete versions of these statements, we work with Anderson duality, which is an equivalent theory but involves a suspension. The second part of the section introduces terminology for Anderson duality that makes it easier to state and prove precise duality statements for spectra and for maps.

In this section, we treat the three cases as uniformly as possible, writing \( S \) for \( \mathbb{S}, S_{(p)}, \) or \( S^\wedge_p \) and \( Z \) for \( \pi_0 S \). For convenience in this section and the remaining sections, we use the following notation.

Notation 3.1. We write \( F(X) \) for the homotopy fiber of the chromatic completion map \( X_{(p)} \rightarrow \text{holim} L_n X \) for any spectrum \( X \).
In this notation, Proposition 2.7 states that $F(\text{TC}(S;p)^\wedge_p)$ is trivial, and Proposition 2.4 then implies that $F(\text{TC}(S))$ is trivial. Using this, Theorem 2.6 gives a fiber sequence

$$F(K(S)) \to F(K(Z)) \to F(\text{TC}(Z)) \to \Sigma \cdots.$$ 

Writing $\text{trc}_Z : K(Z)_{(p)} \to TC(Z)_{(p)}$ for the $p$-localization of the cyclotomic trace and $Fib(\text{trc}_Z)$ for its fiber, swapping homotopy limits, we get a fiber sequence

$$(3.2) \quad F(K(S)) \to Fib(\text{trc}_Z) \to L_1 Fib(\text{trc}_Z) \to \Sigma \cdots.$$ 

In other words, we have a weak equivalence

$$F(K(S)) \simeq F(\text{Fib(\text{trc}_Z)}).$$

Since the $L_1$-localization map is always a rational equivalence, we get the following immediate consequence.

**Proposition 3.3.** For $S = S$, $S_{(p)}$, or $S^\wedge_p$, the fiber $F(K(S))$ of the chromatic completion map $K(S)_{(p)} \to \text{holim}_n K(S)$ is rationally trivial. In particular,

$$F(K(S)) \simeq (F(K(S))_p^\wedge) \wedge \Sigma^{-1} M_{Z/p^\infty}$$

(where $M_{Z/p^\infty}$ is the Moore spectrum for $Z/p^\infty$).

Theorem 1.5 (the case $S = S^\wedge_p)$ is now immediate: work of Hesselholt-Madsen [21, Thm. D] implies

$$\text{Fib(\text{trc}_Z^\wedge}_p) \simeq \Sigma^{-2} HZ^\wedge_p.$$ 

The $p$-completion of $L_1 \text{Fib(\text{trc}_Z^\wedge}_p)$ is therefore trivial, and we see that $F(K(S^\wedge_p))_p^\wedge \simeq \Sigma^{-2} HZ^\wedge_p$. By the proposition, we get

$$F(K(S^\wedge_p)) \simeq \Sigma^{-2} HZ^\wedge_p \wedge \Sigma^{-1} M_{Z/p^\infty} \simeq \Sigma^{-3} HZ/p^\infty.$$ 

This completes the proof of Theorem 1.5.

Both remaining cases, $S = S$ and $S = S_{(p)}$, require the identification of $p$-local Brown-Comenetz duals of certain spectra and of certain maps. For a spectrum $X$, the $p$-local Brown-Comenetz dual $I_{Z/p^\infty} X$ is characterized (up to isomorphism in the stable category) by its representable functor

$$[-, I_{Z/p^\infty} X] \cong \text{Hom}(\pi_0(\Sigma(-) \wedge X), Z/p^\infty).$$ 

For $p$-complete spectra, it works slightly better to work in terms of $Z^\wedge_p$-Anderson duals. We use the following terminology:

**Definition 3.4.** For $p$-complete spectra $X, Y$, an Anderson duality pairing on $X, Y$ is a homomorphism

$$\mu : \pi_0(X \wedge Y; Z/p^\infty) \to Z/p^\infty$$

such that the induced maps

$$\pi_q(X; Z/p^\infty) \otimes \pi_{-q}(Y) \to \pi_0(X \wedge Y; Z/p^\infty) \xrightarrow{\mu} Z/p^\infty$$

$$\pi_q(X) \otimes \pi_{-q}(Y; Z/p^\infty) \to \pi_0(X \wedge Y; Z/p^\infty) \xrightarrow{\mu} Z/p^\infty$$

are perfect pairings for all $q \in \mathbb{Z}$, i.e., if they adjoint to isomorphisms

$$\pi_q(X; Z/p^\infty) \to \text{Hom}(\pi_{-q}(Y), Z/p^\infty)$$

$$\pi_{-q}(Y; Z/p^\infty) \to \text{Hom}(\pi_q(X), Z/p^\infty).$$
The homomorphism $\mu$ induces a map of spectra $X \wedge Y \wedge M_{Z_p} \to I_{Z/p^\infty} S$, which is then adjoint to maps

$$X \wedge M_{Z/p^\infty} \to I_{Z/p^\infty} Y \quad \text{and} \quad Y \wedge M_{Z/p^\infty} \to I_{Z/p^\infty} X.$$  

The perfect pairing condition implies that these maps are weak equivalences. The following proposition then restates this in the form we use to study (3.2).

**Proposition 3.5.** For $p$-complete spectra $X,Y$, an Anderson duality pairing induces isomorphisms in the stable category

$$X \xrightarrow{\sim} \Sigma^{-1}(I_{Z/p^\infty} Y)^\wedge \quad \text{and} \quad Y \xrightarrow{\sim} \Sigma^{-1}(I_{Z/p^\infty} X)^\wedge,$$

and isomorphisms in the stable category

$$X \wedge \Sigma^{-1} M_{Z/p^\infty} \xrightarrow{\sim} \Sigma^{-1} I_{Z/p^\infty} Y \quad \text{and} \quad Y \wedge \Sigma^{-1} M_{Z/p^\infty} \xrightarrow{\sim} \Sigma^{-1} I_{Z/p^\infty} X.$$

As an example, we have an Anderson duality pairing for $KU^\wedge$ with itself given by the map

$$\pi_0(KU_p^\wedge \wedge KU_p^\wedge; \mathbb{Z}/p^\infty) \to \pi_0(KU_p^\wedge; \mathbb{Z}/p^\infty) \cong \mathbb{Z}/p^\infty$$

with the first map induced by multiplication and the second map the isomorphism

$$\pi_0(KU_p^\wedge; \mathbb{Z}/p^\infty) \cong \pi_0(KU_p^\wedge) \otimes \mathbb{Z}/p^\infty \cong \mathbb{Z}/p^\infty$$

deriving from the canonical isomorphism $\pi_0(KU) \cong \mathbb{Z}$. For $p > 2$, $KU_p^\wedge$ decomposes into a wedge of suspensions of the so-called Adams summand $L_p^\wedge$,

$$KU_p^\wedge \cong L_p^\wedge \vee \Sigma^2 L_p^\wedge \vee \cdots \vee \Sigma^{2p-4} L_p^\wedge.$$  

The Adams summand admits an Anderson duality pairing with itself via the multiplication $L_p^\wedge \wedge L_p^\wedge \to L_p^\wedge$ and canonical isomorphism $\pi_0(L_p^\wedge; \mathbb{Z}/p^\infty) \cong \mathbb{Z}/p^\infty$ as well. For $p = 2$, we have a Anderson duality pairing for $KO_2^\wedge$ and $\Sigma^4 KO_2^\wedge$ again coming from multiplication and canonical isomorphism $\pi_0(\Sigma^4 KO_2^\wedge; \mathbb{Z}/p^\infty) \cong \mathbb{Z}/p^\infty$; see [1] 4.16] for a proof.

For the theorems of the introduction, we also need to be able to identify when maps are Brown-Comenetz dual. We use the following terminology.

**Definition 3.6.** Given Anderson duality pairings $(X_1,Y_1,\mu_1)$ and $(X_2,Y_2,\mu_2)$, we say that maps $f: X_1 \to X_2$ and $g: Y_2 \to Y_1$ are Anderson dual when the diagram

$$\begin{array}{ccc}
\pi_0(X_1 \wedge Y_2; \mathbb{Z}/p^\infty) & \xrightarrow{g^*} & \pi_0(X_1 \wedge Y_1; \mathbb{Z}/p^\infty) \\
\downarrow f_* & & \downarrow \mu_1 \\
\pi_0(X_2 \wedge Y_2; \mathbb{Z}/p^\infty) & \xrightarrow{\mu_2} & \mathbb{Z}/p^\infty
\end{array}$$

commutes.

Unwinding the adjunctions, we get the following proposition.

**Proposition 3.7.** Let $(X_1,Y_1,\mu_1)$ and $(X_2,Y_2,\mu_2)$ be Anderson duality pairings. Given $f: X_1 \to X_2$ and $g: Y_2 \to Y_1$, the following are equivalent:

(i) The maps $f$ and $g$ are Anderson dual.

(ii) The map $g$ coincides with $\Sigma^{-1}(I_{Z/p^\infty} f)^\wedge_p$ under the induced weak equivalences of Proposition 3.3.

(iii) The map $f$ coincides with $\Sigma^{-1}(I_{Z/p^\infty} g)^\wedge_p$ under the induced weak equivalences of Proposition 3.6.
4. Proof of Theorem 1.6 for \( p > 2 \)

In this section, we study the fiber of the chromatic completion map for \( K(\mathbb{S}) \) in the case \( p > 2 \). As we explain, the results in this section depend on recent work of the first and second author on arithmetic duality in algebraic \( K \)-theory \([8]\), and the results for \( p = 2 \) in the next section depend on the older work of Rognes \([31]\) that partially inspired it.

We start from Proposition 3.3 and (3.2) which identify \( F(K(\mathbb{S})) \) in terms of the fiber of the cyclotomic trace \( \text{trc}_\mathbb{Z} \) as
\[
F(K(\mathbb{S})) \simeq F(\text{Fib}(\text{trc}_\mathbb{Z}^p))_p \wedge \Sigma^{-1}M_{\mathbb{Z}/p^\infty}.
\]

In the case \( p > 2 \), the first two authors identify \( \text{Fib}(\text{trc}_\mathbb{Z}^p) \) in terms of the Anderson dual of \( \Sigma L_{K(1)}K(\mathbb{Z}) \) by constructing an explicit Anderson duality pairing on \( L_{K(1)}K(\mathbb{Z}), \Sigma L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}) \). This pairing comes from the module multiplication
\[
L_{K(1)}K(\mathbb{Z}) \wedge \Sigma L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}) \to \Sigma L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z})
\]
and a canonical isomorphism
\[
u: \pi_-(L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}); \mathbb{Z}/p^\infty) \xrightarrow{\cong} \mathbb{Z}/p^\infty
\]
constructed in \([7] \S 1\) (compare \([\text{ibid.}, (1.6)]\) and the discussion following Conjecture \([8,4]\) below). The \( L_{K(1)}K(\mathbb{Z}) \)-module structure on \( L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}) \) comes from a \( K(\mathbb{Z})_p^\wedge \)-module structure on \( \text{Fib}(\text{trc}_\mathbb{Z})_p^\wedge \) and we have a \( K(\mathbb{Z})_p^\wedge \)-module structure on the fiber of the map
\[
\text{Fib}(\text{trc}_\mathbb{Z})_p^\wedge \to L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}),
\]
which is weakly equivalent to \( F(\text{Fib}(\text{trc}_\mathbb{Z}))_p^\wedge \). This gives a pairing
\[
K(\mathbb{Z})_p^\wedge \wedge \Sigma^2 F(\text{Fib}(\text{trc}_\mathbb{Z}^p))_p^\wedge \to \Sigma^2 F(\text{Fib}(\text{trc}_\mathbb{Z}))_p^\wedge.
\]
Because \( \pi_-(\text{Fib}(\text{trc}_\mathbb{Z}^p); \mathbb{Z}/p^\infty) = 0 \) and the map \( \text{Fib}(\text{trc}_\mathbb{Z}^p) \to L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}) \) is injective on \( \pi_*(-; \mathbb{Z}/p^\infty) \) (see specifics on \( \text{Fib}(\text{trc}_\mathbb{Z})_p^\wedge \) and \( L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}) \) below), the map
\[
\pi_-(L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}); \mathbb{Z}/p^\infty) \to \pi_{-2}(F(\text{Fib}(\text{trc}_\mathbb{Z}))_p^\wedge; \mathbb{Z}/p^\infty)
\]
in the long exact sequence of homotopy groups is an isomorphism. We prove the following theorem; together with the weak equivalence \( F(K(\mathbb{S})) \simeq F(\text{Fib}(\text{trc}_\mathbb{Z})) \) of (3.2), it implies Theorem 1.6 in the case \( p > 2 \).

**Theorem 4.2.** For \( p > 2 \), the homomorphism
\[
\pi_0(K(\mathbb{Z})_p^\wedge \wedge \Sigma^2 F(\text{Fib}(\text{trc}_\mathbb{Z}^p))_p^\wedge; \mathbb{Z}/p^\infty) \to \pi_0(\Sigma^2 F(\text{Fib}(\text{trc}_\mathbb{Z}))_p^\wedge; \mathbb{Z}/p^\infty)
\]
\[
\cong \pi_{-1}(L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}); \mathbb{Z}/p^\infty) \xrightarrow{\nu} \mathbb{Z}/p^\infty
\]
makes \( K(\mathbb{Z})_p^\wedge, \Sigma^2 F(\text{Fib}(\text{trc}_\mathbb{Z}))_p^\wedge \) an Anderson duality pair. In particular, it gives a weak equivalence
\[
F(\text{Fib}(\text{trc}_\mathbb{Z})) \xrightarrow{\cong} \Sigma^{-3}I_{\mathbb{Z}/p^\infty}(K(\mathbb{Z})_p^\wedge) \simeq \Sigma^{-3}I_{\mathbb{Z}/p^\infty}(K(\mathbb{Z})).
\]

To prove Theorem 4.2, we need to break \( K(\mathbb{Z})_p^\wedge \) and \( F(\text{Fib}(\text{trc}_\mathbb{Z}))_p^\wedge \) into “eigen-spectra” summands. The spectra \( L_{K(1)}K(\mathbb{Z}) \) and \( \Sigma L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}) \) split into the summands that would be expected from an action of \((p\text{-adically interpolated})\)
Adams operations, and [7] showed that the Anderson duality on this pair respects this splitting. Following the notation of [7, §2],

\[ L_{K(1)} K(\mathbb{Z}) \simeq \bigvee_{p} J_{p}^{\wedge} \vee \cdots \vee Y_{p-2} \]
\[ L_{K(1)} \text{Fib}(\text{trc}_{2}) \simeq \bigvee_{p} J_{p}^{\wedge} \vee \cdots \vee X_{p-2} \]

(numbered mod \( p - 1 \)) with \( Y_{i} \), \( \Sigma X_{1-i} \) Anderson dual and \( J_{p}^{\wedge}, \Sigma J_{p}^{\wedge} \) Anderson dual under the restriction of the pairing above. Here \( Y_{i} \) is the fiber of a map of the form

\[ \bigvee \Sigma^{2i-1} L_{p}^{\wedge} \longrightarrow \bigvee \Sigma^{2i-1} L_{p}^{\wedge} \]

for some wedges of copies of \( \Sigma^{2i-1} L_{p}^{\wedge} \), where \( L_{p}^{\wedge} \) denotes the Adams summand of \( KU_{p}^{\wedge} \); it follows that \( X_{k} \) is the fiber of a map of the form

\[ \bigvee \Sigma^{2k-1} L_{p}^{\wedge} \longrightarrow \bigvee \Sigma^{2k-1} L_{p}^{\wedge} \]

In terms of Brown-Comenetz duality, we have:

\[ J_{p}^{\wedge} \simeq \Sigma^{-2} (I_{\mathbb{Z}/p^{\infty}} (J_{p}^{\wedge}))^{\wedge} \]
\[ X_{k} \simeq \Sigma^{-2} (I_{\mathbb{Z}/p^{\infty}} Y_{1-k})^{\wedge} \]

Again following the notation of [7, §2], we let

\[ y_{i} = \tau_{\geq 2} Y_{i}, \quad x_{k} = \tau_{\geq 2} X_{k}, \quad k \neq 0, \quad x_{0} = \tau_{\geq -2} X_{0} \]

and then

\[ K(\mathbb{Z})^{\wedge}_{p} \simeq \bigvee_{p} J_{0}^{\wedge} \vee \cdots \vee y_{p-2} \]
\[ \text{Fib}(\text{trc}_{2})^{\wedge}_{p} \simeq \bigvee_{p} J_{0}^{\wedge} \vee \cdots \vee x_{p-2}. \]

We note that the homotopy groups of \( X_{i} \) and \( Y_{i} \) are concentrated in degrees congruent to \( 2i - 1 \) and \( 2i - 2 \) mod \( 2(p - 1) \), so there is a lot of leeway in most of the truncations.

The notation above gives an identification of \( F(\text{Fib}(\text{trc}_{2})) \) in terms of Whitehead truncations

\[ \Sigma F(\text{Fib}(\text{trc}_{2}))^{\wedge}_{p} \simeq \tau_{\leq -1} J_{p}^{\wedge} \vee \tau_{\leq -3} X_{0} \vee \tau_{\leq 1} X_{1} \vee \cdots \vee \tau_{\leq 1} X_{p-2}. \]

We can now prove Theorem 4.2.

**Proof of Theorem 4.2.** From the work above we see that \( K(\mathbb{Z})^{\wedge}_{p} \to L_{K(1)} K(\mathbb{Z}) \) induces a split injection on \( \pi_{*} \) and \( \Sigma L_{K(1)} \text{Fib}(\text{trc}_{2}) \to \Sigma^{2} F(\text{Fib}(\text{trc}_{2}))^{\wedge}_{p} \) induces a split surjection on \( \pi_{*} \). By construction of the pairings, the diagrams

\[ \pi_{q}(\Sigma L_{K(1)} \text{Fib}(\text{trc}_{2}); \mathbb{Z}/p^{\infty}) \xrightarrow{=} \text{Hom}(\pi_{-q}(L_{K(1)} K(\mathbb{Z})); \mathbb{Z}/p^{\infty}) \]
\[ \pi_{q}(\Sigma^{2} F(\text{Fib}(\text{trc}_{2}))^{\wedge}_{p}; \mathbb{Z}/p^{\infty}) \xrightarrow{=} \text{Hom}(\pi_{-q}(K(\mathbb{Z})^{\wedge}_{p}); \mathbb{Z}/p^{\infty}) \]
\[ \pi_{q}(K(\mathbb{Z})^{\wedge}_{p}; \mathbb{Z}/p^{\infty}) \longrightarrow \text{Hom}(\pi_{-q}(\Sigma^{2} F(\text{Fib}(\text{trc}_{2}))^{\wedge}_{p}); \mathbb{Z}/p^{\infty}) \]
\[ \pi_{q}(L_{K(1)} K(\mathbb{Z}); \mathbb{Z}/p^{\infty}) \xrightarrow{=} \text{Hom}(\pi_{-q}(\Sigma L_{K(1)} \text{Fib}(\text{trc}_{2})); \mathbb{Z}/p^{\infty}) \]
In terms of the fiber of the cyclotomic trace $\text{trc}_\mathbb{Z}$ characterized by the commutative diagram

$$\begin{array}{c}
\text{Fiber of \text{trc}_\mathbb{Z}} \\
\Downarrow \\
\text{Fiber of a map} \\
\Downarrow \\
\text{KU-ification map from \text{KU}} \\
\Downarrow \\
\text{Correspond. For the summands} \\
\Downarrow \\
\text{KU-stable map (even a map of commutative ring spectra) for the homotopy groups match up since $\pi_k(K(Z))$ is torsion free and $\pi_2(LK(1)K(Z))$ is torsion free and $\pi_2(LK(1)K(Z))$ is torsion.)}
\end{array}$$

we just need to check that the summands for $K(Z)_p^\wedge$ and $\Sigma^2F(\text{Fib}(\text{trc}_\mathbb{Z}))_p^\wedge$ correspond. For the summands

$$j_p^\wedge = \tau_{\geq 0}J_p^\wedge \text{ in } K(Z)_p^\wedge \quad \text{and} \quad \Sigma\tau_{\leq -1}J_p^\wedge \simeq \tau_{\leq 0}\Sigma J_p^\wedge \text{ in } \Sigma^2F(\text{Fib}(\text{trc}))_p^\wedge,$$

the homotopy groups match up since $\pi_{-1}(J_p^\wedge)$ is torsion free. The summands

$$y_0 = \tau_{\geq 2}Y_0^\wedge \text{ in } K(Z)_p^\wedge \quad \text{and} \quad \Sigma\tau_{\leq -1}X_1J \simeq \tau_{\leq 2}\Sigma X_1 \text{ in } \Sigma^2F(\text{Fib}(\text{trc}))_p^\wedge,$$

are both the trivial spectrum; see [16, 2.4, 2.9]. Because $X_k$ has homotopy groups only in degrees congruent to $2k - 1$ and $2k - 2 \mod 2(p - 1)$, we have $\tau_{\leq 1}X_k \xrightarrow{\simeq} \tau_{\leq -3}X_k$ for $k = 2, \ldots, p - 2$. Then for all $k$, $i = p - k \mod (p - 1)$, we see that the homotopy groups match up in the summands

$$y_i = \tau_{\geq 2}Y_i \text{ in } K(Z)_p^\wedge \quad \text{and} \quad \Sigma\tau_{\leq -3}X_k \simeq \tau_{\leq -2}\Sigma X_k \text{ in } \Sigma^2F(\text{Fib}(\text{trc}))_p^\wedge,$$

since $\pi_{-3}(X_k) = 0$ for all $k$. (We have $\pi_{-3}(LK(3)\text{Fib}(\text{trc}_\mathbb{Z})) = 0$ since $\pi_1(LK(1)K(Z))$ is torsion free and $\pi_2(LK(1)K(Z))$ is torsion.)

5. Proof of Theorem 1.6 for $p = 2$

As in the previous section, we use Proposition 3.3 and (5.2) to identify $F(K(S))$ in terms of the fiber of the cyclotomic trace $\text{trc}_\mathbb{Z}$ as

$$F(K(S)) \simeq F(\text{Fib}(\text{trc}_\mathbb{Z}))_2^\wedge \wedge \Sigma^{-1}M_{\mathbb{Z}/2\infty}.$$ 

Now in the case $p = 2$, work of Rognes [31, 3.13] identifies the 2-completion of the fiber of the cyclotomic trace as the fiber of a map

$$\xi: \Sigma^{-2}ku_2^\wedge \rightarrow \Sigma^4ko_2^\wedge$$

characterized by the commutative diagram

$$\begin{array}{ccc}
\Sigma^{-2}ku_2^\wedge & \xrightarrow{\xi} & \Sigma^4ko_2^\wedge \\
\downarrow & & \downarrow \\
\Sigma^{-2}KU_2^\wedge & \xrightarrow{\psi_{-1}} & \Sigma^4KO_2^\wedge
\end{array}$$

where the vertical maps are the $K(1)$-localization maps, the unlabeled weak equivalences are induced by complex Bott periodicity, and the map $r$ is the usual “realification” map from $KU$ to $KO$. The map $\psi^3$ is the Adams operation, which is a stable map (even a map of commutative ring spectra) for $KU_2^\wedge$. We write

$$\Xi: \Sigma^{-2}KU_2^\wedge \rightarrow \Sigma^4KO_2^\wedge$$

for the composite on the bottom.

On the other hand, work of Bökstedt [11, Theorem 2], Rognes-Weibel [29] and Weibel [39], collated in Rognes [31, 3.4] shows that $K(Z)_2^\wedge$ is the fiber of a map

$$\chi: ko_2^\wedge \rightarrow \Sigma^4ku_2^\wedge$$

characterized by the commutative diagram

$$\begin{array}{ccc}
ko_2^\wedge & \xrightarrow{\chi} & \Sigma^4ku_2^\wedge \\
\downarrow & & \downarrow \\
KO_2^\wedge & \xrightarrow{\psi_{-1}} & \Sigma^4KO_2^\wedge
\end{array}$$
Theorem 5.2. \[ \pi_4 \Sigma \] the unit under the Bott periodicity isomorphism \[ \Sigma \] of the map \[ \pi_4 \Sigma \] for the bottom composite.

We need a variant of \( \chi \) that we denote \( \chi' \) and construct as follows. The Adams operation \( \psi^3 \) viewed as a map \( ku_2^\wedge \to ku_2^\wedge \) or \( KU_2^\wedge \to KU_2^\wedge \) is a weak equivalence typically denoted \( \psi^{1/3} \). The composite

\[
\Sigma^4 KU_2^\wedge \xrightarrow{\zeta} KU_2^\wedge \xrightarrow{\psi^{1/3}} KU_2^\wedge \xrightarrow{\zeta} \Sigma^4 KU_2^\wedge
\]

is \( \frac{1}{2} \Sigma^4 \psi^{1/3} \) and the self-map \( \frac{1}{2} \Sigma^4 \psi^{1/3} \) of \( ku_2^\wedge \) is the unique map making the evident diagram with the localization map \( ku_2^\wedge \to KU_2^\wedge \) commute. Let

\[
\chi' = -\frac{1}{9} \Sigma^4 \psi^{1/3} \circ \chi: ko_2^\wedge \to \Sigma^4 ku_2^\wedge
\]

\[
\chi' = -\frac{1}{9} \Sigma^4 \psi^{1/3} \circ \chi: KU_2^\wedge \to \Sigma^4 KU_2^\wedge.
\]

(Put another way, \( \chi' \) and \( X' \) are the maps defined analogously to \( \chi \) and \( X \) but using \( \psi^{1/3} - 1 \) in the diagram in place of \( \psi^3 - 1 \).) Because \( -\frac{1}{9} \psi^{1/3} \) is a weak equivalence, we also have that \( K(Z)_2^\wedge \) is the fiber of \( \chi' \).

The maps \( \Xi \) and \( X' \) are related by Anderson duality as follows. As discussed in Section 3.6, multiplication

\[
KO_2^\wedge \wedge \Sigma^4 KO_2^\wedge \to \Sigma^4 KO_2^\wedge
\]

and the canonical isomorphism \( \pi_0(\Sigma^4 KO_2^\wedge; \mathbb{Z}/2\infty) \cong \mathbb{Z}/2\infty \) gives an Anderson duality pairing on \( KO_2^\wedge, \Sigma^4 KO_2^\wedge \). We also have an Anderson duality pairing on \( \Sigma^4 KU_2^\wedge, \Sigma^2 KU_2^\wedge \) induced by the multiplication

\[
\Sigma^4 KU_2^\wedge \wedge \Sigma^2 KU_2^\wedge \to \Sigma^2 KU_2^\wedge
\]

and the isomorphism \( \pi_0(\Sigma^2 KU_2^\wedge; \mathbb{Z}/2\infty) \cong \mathbb{Z}/2\infty \) (using as generator the image of the unit under the Bott periodicity isomorphism \( \pi_0(KU_2^\wedge) \cong \pi_0(\Sigma^2 KU_2^\wedge) \)).

Theorem 5.2. Under the Anderson duality pairings on \( KO_2^\wedge, \Sigma^4 KO_2^\wedge \) and \( \Sigma^2 KU_2^\wedge, \Sigma^4 KU_2^\wedge \) above, the maps \( \Xi \) and \( X' \) are Anderson dual.

Proof. To check of Definition 3.6, we need to see that the following diagram commutes.

\[
\pi_0(KO_2^\wedge \wedge \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2\infty) \xrightarrow{\Xi} \pi_0(KO_2^\wedge \wedge \Sigma^4 KO_2^\wedge; \mathbb{Z}/2\infty) \to \pi_0(\Sigma^4 KO_2^\wedge; \mathbb{Z}/2\infty)\]

\[
\pi_0(\Sigma^4 KU_2^\wedge \wedge \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2\infty) \xrightarrow{\Xi} \pi_0(\Sigma^2 KU_2^\wedge; \mathbb{Z}/2\infty) \cong \mathbb{Z}/2\infty.
\]

Unwinding the definition of \( \Xi \), the top composite is the induced map on \( \pi_0(-; \mathbb{Z}/2\infty) \) of the map

\[
KO_2^\wedge \wedge \Sigma^{-2} KU_2^\wedge \xrightarrow{\zeta} KO_2^\wedge \wedge KU_2^\wedge \xrightarrow{id \wedge (\psi^3 - 1)} KO_2^\wedge \wedge KU_2^\wedge
\]

\[
\xrightarrow{\zeta} KO_2^\wedge \wedge \Sigma^4 KU_2^\wedge \xrightarrow{id \wedge \Sigma^1 r} KO_2^\wedge \wedge \Sigma^4 KO_2^\wedge \to \Sigma^4 KO.
\]
Because $\Sigma^4 r$ and the Bott map $KU_p^\wedge \simeq \Sigma^4 KU_p^\wedge$ are both $KO$-module maps, we can identify the previous composite as the composite

$$KO^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \xrightarrow{\cong} KO^\wedge_2 \wedge KU^\wedge_2 \xrightarrow{id \wedge (\psi^3 - 1)} KO^\wedge_2 \wedge KU^\wedge_2 \quad \rightarrow KU^\wedge_2 \xrightarrow{\cong} \Sigma^4 KU^\wedge_2 \xrightarrow{\Sigma^4 r} \Sigma^4 KO.$$ 

Since the $KO$-module structure on $\Sigma^4 KU^\wedge_2$ comes from the map of ring spectra $c: KO^\wedge_2 \to KU^\wedge_2$, we can further identify this map as

(*)  

$$KO^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \xrightarrow{c \wedge \text{id}} KU^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \xrightarrow{\cong} KU^\wedge_2 \wedge KU^\wedge_2 \xrightarrow{id \wedge (\psi^3 - 1)} KU^\wedge_2 \wedge KU^\wedge_2 \xrightarrow{\cong} \Sigma^4 KU^\wedge_2 \xrightarrow{\Sigma^4 r} \Sigma^4 KO.$$ 

Going around the diagram the other way, the map

$$\pi_0(KO^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2; \mathbb{Z}/2\infty) \xrightarrow{X^*} \pi_0(\Sigma^4 KU^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2; \mathbb{Z}/2\infty) \xrightarrow{p} \pi_0(\Sigma^2 KU^\wedge_2; \mathbb{Z}/2\infty)$$

is the induced map on $\pi_0(-; \mathbb{Z}/2\infty)$ of

$$KO^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \xrightarrow{c \wedge \text{id}} KU^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \xrightarrow{\psi^3 \wedge \text{id}} KU^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \xrightarrow{\cong} \Sigma^4 KU^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \rightarrow \Sigma^2 KU^\wedge_2$$

Since the complex Bott periodicity maps are $KU^\wedge_2$-module maps, we can identify this map as the composite

(**)  

$$KO^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \xrightarrow{c \wedge \text{id}} KU^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \xrightarrow{\psi^3 \wedge \text{id}} KU^\wedge_2 \wedge \Sigma^{-2}KU^\wedge_2 \xrightarrow{(\psi^3-1) \wedge \text{id}} KU^\wedge_2 \wedge KU^\wedge_2$$

Comparing (*) and (**), since the diagram

$$
\begin{array}{ccc}
\pi_0(KU^\wedge_2; \mathbb{Z}/2\infty) & \xrightarrow{\cong} & \pi_0(\Sigma^4 KU^\wedge_2; \mathbb{Z}/2\infty) \\
\downarrow & & \downarrow \\
\pi_0(\Sigma^2 KU^\wedge_2; \mathbb{Z}/2\infty) & \xrightarrow{\cong} & \mathbb{Z}/2\infty
\end{array}
$$

commutes, it suffices to see that the maps

$$KU^\wedge_2 \wedge KU^\wedge_2 \xrightarrow{id \wedge \psi^3 - \text{id}} KU^\wedge_2 \wedge KU^\wedge_2 \rightarrow KU^\wedge_2$$

$$KU^\wedge_2 \wedge KU^\wedge_2 \xrightarrow{\psi^3 \wedge \text{id} - \text{id}} KU^\wedge_2 \wedge KU^\wedge_2 \rightarrow KU^\wedge_2$$

induce the same map on $\pi_0(-; \mathbb{Z}/2\infty)$. This is clear because the diagram

\[\begin{array}{ccc}
KU^\wedge_2 \wedge KU^\wedge_2 & \xrightarrow{id \wedge \psi^3} & KU^\wedge_2 \wedge KU^\wedge_2 \\
\downarrow & & \downarrow \\
KU^\wedge_2 \wedge KU^\wedge_2 & \xrightarrow{\psi^3 \wedge \psi^{1/3} \wedge \text{id}} & KU^\wedge_2 \wedge KU^\wedge_2
\end{array}\]
commutes and $\psi^1/3: KU_2^\wedge \to KU_2^\wedge$ induces the identity map on $\pi_0(-; \mathbb{Z}/2^\infty)$. \qed

As a consequence, we have that

$$\text{Fib}(\mathbb{Z}) \simeq \Sigma^{-1}(I_{\mathbb{Z}/2^\infty}(\text{Cof}(X'))) \simeq \Sigma^{-2}I_{\mathbb{Z}/2^\infty}(K) \wedge \Sigma^{-1}M_{\mathbb{Z}/2^\infty}.$$  

We get the following “$K$-theoretic Tate-Poitou duality” style result at the prime 2 as in [8] at odd primes.

**Corollary 5.3.** At the prime 2,

$$L_{K(1)}\text{Fib}(\text{trc}_2) \simeq \Sigma^{-2}(I_{\mathbb{Z}/2^\infty}(K)) \simeq \Sigma^{-2}I_{\mathbb{Z}/2^\infty}(L_{K(1)}K) \wedge \Sigma^{-1}M_{\mathbb{Z}/2^\infty}.$$  

The odd case in [8] proves a little more, identifying the weak equivalence intrinsically as an Anderson duality pairing. For use in Section 6, we conjecture that the corresponding refinement also holds at the prime 2.

**Conjecture 5.4.** At the prime 2, for an isomorphism

$$u: \pi_1(L_{K(1)}\text{Fib}(\text{trc}_2); \mathbb{Z}/p^\infty) \cong \mathbb{Z}/2^\infty$$

the map

$$\pi_0(L_{K(1)}(K) \wedge \Sigma L_{K(1)}(\text{Fib}(\text{trc}_2)); \mathbb{Z}/2^\infty) \to \pi_0(\Sigma L_{K(1)}(\text{Fib}(\text{trc}_2)); \mathbb{Z}/2^\infty)$$

$$\cong \pi_1(L_{K(1)}(\text{Fib}(\text{trc}_2)); \mathbb{Z}/2^\infty) \xrightarrow{u} \mathbb{Z}/2^\infty$$

induced by the $K(1)$-module structure on the fiber makes $L_{K(1)}(K)$ and $\Sigma L_{K(1)}(\text{Fib}(\text{trc}_2))$ into an Anderson duality pair.

The conjecture does not depend on the choice of the isomorphism; the choice in [8] §1] was specified in terms of étale cohomology, but it admits a homotopy theoretic description that generalizes to the case $p = 2$ as follows. Let $U_1$ be the subgroup of 2-adic units that are congruent to 1 mod 4; then $U_1$ is non-canonically isomorphic to $\mathbb{Z}_2^\times$. The $K(1)$-localization of the $U_1$ Moore spectrum, $L_{K(1)}M_{U_1}$, is non-canonically isomorphic in the stable category to $J_2^\wedge = L_{K(1)}S$ and we have a canonical identification

$$\pi_1(L_{K(1)}M_{U_1}) \cong H^1_{\text{Gal}}((\mathbb{Z}_2^\wedge)^\times; U_1) \cong \text{Hom}_{c}((\mathbb{Z}_2^\wedge)^\times, U_1) \cong \mathbb{Z}_2^\wedge$$

where $\text{Hom}_{c}$ denotes continuous homomorphisms and the last isomorphism uses the usual map

$$(\mathbb{Z}_2^\wedge)^\times \cong \{\pm 1\} \times U_1 \to U_1$$

as the generator of the homomorphism group. Since $\pi_{-2} = 0$, we then get a canonical isomorphism

$$\pi_1(L_{K(1)}M_{U_1}; \mathbb{Z}/2^\infty) \cong \mathbb{Z}/2^\infty.$$  

Using the usual identification $\pi_1(K(Z^\wedge_2)) \cong (Z^\wedge_2)^\times$ and weak equivalences

$$L_{K(1)}TC(Z)_2^\wedge \sim L_{K(1)}TC(Z)_2^\wedge \xrightarrow{\sim} L_{K(1)}K(Z^\wedge_2),$$

the inclusion of $U_1$ in $(Z^\wedge_2)^\times$ induces a map $\Sigma L_{K(1)}M_{U_1} \to L_{K(1)}TC(Z)$. The composite map

$$\mathbb{Z}/2^\infty \cong \pi_0(\Sigma L_{K(1)}M_{U_1}; \mathbb{Z}/2^\infty) \to \pi_0(L_{K(1)}TC(Z); \mathbb{Z}/2^\infty) \to \pi_1(L_{K(1)}\text{Fib}(\text{trc}_2); \mathbb{Z}/2^\infty)$$

is an isomorphism and the inverse gives an isomorphism $u$ for Conjecture 5.3 analogous to the one used in the case $p > 2$ in the previous section (see [8] 1.5]).
To calculate $F(K(S))^\wedge_2$ by (6.1), we use the fiber sequences for $Fib(trc_2)^\wedge_2$ and $L_{K(1)}Fib(trc_2)$ in terms of topological $K$-theories. Specifically, we have the following calculation:

\[
F(K(S))^\wedge_2 \simeq F(Fib(trc_2))^\wedge_2 \simeq F(Fib(\xi: \Sigma^{-2}ku^\wedge_2 \to \Sigma^{4}ko^\wedge_2))_2^\wedge
\simeq Fib(F(\xi)_2^\wedge: F(\Sigma^{-2}ku^\wedge_2)_2^\wedge \to F(\Sigma^{4}ko^\wedge_2)_2^\wedge).
\]

(5.5)

We note that $\Sigma F(\Sigma^{-2}ku^\wedge_2)_2^\wedge$ is the cofiber of the map $\Sigma^{-2}ku^\wedge_2 \to \Sigma^{-2}KU^\wedge_2$, which is by definition the truncation $\tau_{\leq -4}\Sigma^{-2}KU^\wedge_2$. Working in the derived category of $ku^\wedge_2$-modules, we get a commutative diagram

$$
\begin{array}{c}
\Sigma^{-2}KU^\wedge_2 \land \Sigma^{4}ku^\wedge_2 \\
\downarrow \\
\tau_{\leq -4}\Sigma^{-2}KU^\wedge_2 \land \Sigma^{4}ku^\wedge_2 \\
\rightarrow \\
\tau_{\leq 0}\Sigma^{-2}KU^\wedge_2
\end{array}
$$

and the canonical isomorphism

$$
\pi_0(\tau_{\leq 0}\Sigma^{-2}KU^\wedge_2; \mathbb{Z}/2^\infty) \cong \pi_0(\Sigma^{-2}KU^\wedge_2; \mathbb{Z}/2^\infty) \cong \mathbb{Z}/2^\infty
$$

then gives us an Anderson duality pairing on $\tau_{\leq -4}\Sigma^{-2}KU^\wedge_2$, $\Sigma^{4}ku^\wedge_2$. Similarly, $\Sigma F(\Sigma^{4}ko^\wedge_2)$ is $\tau_{\leq 0}\Sigma^{4}KO^\wedge_2$, and using the commutative diagram

$$
\begin{array}{c}
\Sigma^{4}KO^\wedge_2 \land ko^\wedge_2 \\
\downarrow \\
\tau_{\leq 0}\Sigma^{4}KO^\wedge_2 \land ko^\wedge_2 \\
\rightarrow \\
\tau_{\leq 0}\Sigma^{4}KO
\end{array}
$$

and the canonical isomorphism

$$
\pi_0(\tau_{\leq 0}\Sigma^{4}KO^\wedge_2; \mathbb{Z}/2^\infty) \cong \pi_0(\Sigma^{4}KO^\wedge_2; \mathbb{Z}/2^\infty) \cong \mathbb{Z}/2^\infty
$$

we get a map

$$
\mu: \pi_0(\tau_{\leq 0}\Sigma^{4}KO^\wedge_2 \land ko^\wedge_2; \mathbb{Z}/2^\infty) \longrightarrow \mathbb{Z}/2^\infty.
$$

We claim that $\mu$ gives an Anderson duality pairing on $\tau_{\leq 0}\Sigma^{4}KO$, $ko^\wedge_2$. To see this, we use the commutative diagram

$$
\begin{array}{c}
\pi_q(\tau_{\leq 0}\Sigma^{4}KO^\wedge_2; A) \land \pi_q(\tau_{\leq 0}\Sigma^{4}KO^\wedge_2; B) \cong \pi_q(\tau_{\leq 0}\Sigma^{4}KO^\wedge_2; A) \land \pi_q(\tau_{\leq 0}\Sigma^{4}KO^\wedge_2; B) \cong \pi_q(\tau_{\leq 0}\Sigma^{4}KO^\wedge_2; A) \land \pi_q(\tau_{\leq 0}\Sigma^{4}KO^\wedge_2; B)
\end{array}
$$

for $q \geq 0$ and either $A = \mathbb{Z}$, $B = \mathbb{Z}/2^\infty$ or $A = \mathbb{Z}/2^\infty$, $B = \mathbb{Z}$.

**Theorem 5.6.** Under the Anderson duality pairings above, the map

$$
\Sigma F(\xi)_2^\wedge: \Sigma F(\Sigma^{-2}ku^\wedge_2)_2^\wedge \longrightarrow \Sigma F(\Sigma^{4}ko^\wedge_2)_2^\wedge
$$

for the fiber in (5.5) is Anderson dual to the map $\chi'': ko^\wedge_2 \to \Sigma^{4}ku^\wedge_2$. 
Proof. We need to check that the diagram

\[(**)
\begin{array}{c}
\pi_0(ko_2^\wedge \& \tau_{\leq -4} \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty) \\ \xrightarrow{\chi_2^*} \\ \pi_0(\Sigma^4 ku_2^\wedge \& \tau_{\leq -4} \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty)
\end{array}
\ \xrightarrow{\Sigma F(\xi)} \pi_0(\tau_{\leq 0} \Sigma^4 KO_2^\wedge; \mathbb{Z}/2^\infty)
\begin{array}{c}
\pi_0(ko_2^\wedge \& \tau_{\leq 0} \Sigma^4 KO_2^\wedge; \mathbb{Z}/2^\infty) \\ \xrightarrow{\chi_2^*} \\ \pi_0(\tau_{\leq 0} \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty)
\end{array}
\xrightarrow{\sim} \mathbb{Z}/2^\infty
\]

commutes. Consider the not-necessarily commuting diagram

\[(***)
\begin{array}{c}
\pi_0(ko_2^\wedge \& \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty) \\ \xrightarrow{\chi_2^*} \\ \pi_0(\Sigma^4 ku_2^\wedge \& \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty)
\end{array}
\ \xrightarrow{\Sigma F(\xi)} \pi_0(\Sigma^4 KO_2^\wedge; \mathbb{Z}/2^\infty)
\begin{array}{c}
\pi_0(ko_2^\wedge \& \Sigma^4 KO_2^\wedge; \mathbb{Z}/2^\infty) \\ \xrightarrow{\chi_2^*} \\ \pi_0(\Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty)
\end{array}
\xrightarrow{\sim} \mathbb{Z}/2^\infty
\]

and the diagram

\[(***)
\begin{array}{c}
\pi_0(\Sigma^4 KU_2^\wedge \& \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty) \\ \xrightarrow{\chi_2^*} \\ \pi_0(\Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty)
\end{array}
\xrightarrow{\sim} \mathbb{Z}/2^\infty
\]

that we know commutes by Theorem 5.2. The two maps

\[
\pi_0(ko_2^\wedge \& \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty) \rightarrow \mathbb{Z}/2^\infty
\]

in \[(***)\] are the composite of

\[
\pi_0(ko_2^\wedge \& \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty) \rightarrow \pi_0(KO_2^\wedge \& \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty)
\]

with the two maps in \[(***)\], and therefore coincide, proving that \[(***)\] commutes. Likewise these maps are composite of the map

\[
\pi_0(ko_2^\wedge \& \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty) \rightarrow \pi_0(ko_2^\wedge \& \tau_{\leq -4} \Sigma^{-2} KU_2^\wedge; \mathbb{Z}/2^\infty)
\]

with the maps in \[(*)\]. This map is surjective because the fiber of

\[
ko_2^\wedge \& \Sigma^{-2} KU_2^\wedge \rightarrow ko_2^\wedge \& \tau_{\leq -4} \Sigma^{-2} KU_2^\wedge
\]
is \(ko_2^\wedge \& \Sigma^{-2} ku_2^\wedge\) and

\[
\pi_{-1}(ko_2^\wedge \& \Sigma^{-2} ku_2^\wedge; \mathbb{Z}/2^\infty) = 0.
\]

It follows that \[(*)\] commutes as well. \(\square\)

We have

\[
F(K(S)) \simeq F(Fib(\xi))_2^\wedge \& \Sigma^{-1} M_{\mathbb{Z}/2^\infty}
\]

from \[(**)\] and the previous theorem gives a weak equivalence

\[
\Sigma F(Fib(\xi)) \& \Sigma^{-1} M_{\mathbb{Z}/2^\infty} \simeq \Sigma^{-1} I_{\mathbb{Z}/2^\infty}(\text{Cof}(\chi')).
\]

Since \(K(\mathbb{Z})_2^\wedge \simeq \text{Fib}(\chi') \simeq \Sigma^{-1} \text{Cof}(\chi')\), we get

\[
F(K(S)) \simeq \Sigma^{-3} I_{\mathbb{Z}/2^\infty}(K(Z)_2^\wedge) \simeq \Sigma^{-3} I_{\mathbb{Z}/2^\infty}(K(Z)).
\]

This proves Theorem 1.6 in the case \(p = 2\).
6. The fiber of the chromatic completion map on $K(S_{(p)})$

The Quillen localization sequence gives a fiber sequence
\[ \bigvee_{\ell \neq p} K(F_\ell) \rightarrow K(\mathbb{Z}) \rightarrow K(\mathbb{Z}_{(p)}) \rightarrow \Sigma \cdots \]
where the wedge is indexed on the prime numbers $\ell \neq p$. Because the maps $TC(S) \rightarrow TC(S_{(p)})$ and $TC(\mathbb{Z}) \rightarrow TC(\mathbb{Z}_{(p)})$ becomes weak equivalences after $p$-completion, the Dundas-Goodwillie-McCarthy homotopy cartesian square implies an analogous fiber sequence for $K(S)$ and $K(S_{(p)})$ after $p$-completion.

**Proposition 6.1.** There exists a fiber sequence
\[ \bigvee_{\ell \neq p} K(F_\ell)^{\wedge}_p \rightarrow K(S)^{\wedge}_p \rightarrow K(S_{(p)})^{\wedge}_p \rightarrow \Sigma \cdots \]
where the map
\[ K(S_{(p)})^{\wedge}_p \rightarrow \Sigma \left( \bigvee_{\ell \neq p} K(F_\ell)^{\wedge}_p \right) \]
is the $p$-completion of the composite of the map $K(S_{(p)}) \rightarrow K(\mathbb{Z}_{(p)})$ and the map in the Quillen localization sequence.

Because we know from \[\text{(3.2)}\] that $F(K(S))$ and $F(K(S_{(p)}))$ are rationally trivial, the fiber of the map $F(K(S)) \rightarrow F(K(S_{(p)}))$ is rationally trivial. We will see below that $F(K(F_\ell))$ is rationally trivial. Putting this together with the previous proposition, we get the following one.

**Proposition 6.2.** There exists a fiber sequence
\[ \bigvee_{\ell \neq p} F(K(F_\ell)) \rightarrow F(K(S)) \rightarrow F(K(S_{(p)})) \rightarrow \Sigma \cdots \]
where the connecting map is induced by connecting the map in Proposition \[\text{(6.1)}\]

We identified $F(K(S))$ as $\Sigma^{-3}I_{\mathbb{Z}/p}\wedge K(\mathbb{Z})$ in the previous sections, so to prove Theorem \[\text{(1.7)}\] we need to identify $F(K(F_\ell))$ and a map $F(K(F_\ell)) \rightarrow F(K(S))$ that fits in a fiber sequence in the form in Proposition \[\text{(6.2)}\]

To identify $F(K(F_\ell))$, we use the usual identification of $L_{K(1)}K(F_\ell)$ (for $\ell \neq p$) as the homotopy fixed points of the action of $\psi^\ell$ on $KU^{\wedge}_p$, or equivalently, as the homotopy fiber of the map
\[ KU^{\wedge}_p \xrightarrow{\psi^\ell - 1} KU^{\wedge}_p. \]
Then $K(F_\ell)^{\wedge}_p$ is the connective cover and fits into a fiber sequence of spectra
\[ K(F_\ell)^{\wedge}_p \rightarrow ku^{\wedge}_p \xrightarrow{\psi^\ell - 1} \Sigma^2 ku^{\wedge}_p \rightarrow \Sigma \cdots. \]
The $L_1$-localization of $ku^{\wedge}_p$ is the fiber of the map
\[ KU^{\wedge}_p \rightarrow L_1(\tau_{<0} KU^{\wedge}_p) \simeq \Sigma^{-2} H\mathbb{Q}^{\wedge}_p \vee \Sigma^{-4} H\mathbb{Q}^{\wedge}_p \vee \cdots. \]
and therefore has $\pi_q = 0$ for $q = -1, -2$. We see from this that the map $K(F_\ell)^{\wedge}_p \rightarrow L_1(K(F_\ell)^{\wedge}_p)$ is $(-3)$-truncated and it follows that the map $K(F_\ell) \rightarrow L_1(K(F_\ell))$ is also $(-3)$-truncated. This implies that the map
\[ \text{holim} L_n(K(F_\ell)) \rightarrow L_1(K(F_\ell)) \]
is a weak equivalence. In particular, we get that $F(K(\mathbb{F}_\ell))$ is rationally trivial (which we needed above for Proposition 6.2). In addition, we get weak equivalences

$$K(F(\mathbb{F}_\ell))^\wedge \xrightarrow{\sim} \tau_{\geq 0}L_{K(1)}(K(\mathbb{F}_\ell))$$

$$\Sigma F(K(\mathbb{F}_\ell))^\wedge \xrightarrow{\sim} \tau_{\leq -1}(L_{K(1)}K(\mathbb{F}_\ell)).$$

It will be slightly more convenient below to work with the suspension of the latter weak equivalence,

$$\Sigma^2 F(K(\mathbb{F}_\ell))^\wedge \xrightarrow{\sim} \tau_{\leq 0}(\Sigma L_{K(1)}K(\mathbb{F}_\ell)).$$

As in the previous section, we reformulate the previous calculation in terms of an Anderson duality pairing. The identification of $L_{K(1)}K(\mathbb{F}_\ell)$ as the homotopy fixed points of the action of $\psi^\ell$ on $KU_p^\wedge$ gives a canonical isomorphism

$$v_\ell: \pi_{-1}(L_{K(1)}K(\mathbb{F}_\ell); \mathbb{Z}/p^\infty) \cong H^1(\mathbb{Z}; \pi_0(KU_p^\wedge; \mathbb{Z}/p^\infty)) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}/p^\infty) \cong \mathbb{Z}/p^\infty.$$  

(Although Quillen’s identification of $K(\mathbb{F}_\ell)$ depends on choices, $v_\ell$ does not, since $\pi_0(KU_p^\wedge; \mathbb{Z}/p^\infty) \cong \pi_0(KU_p^\wedge) \otimes \mathbb{Z}/p^\infty$ and the identification of $\pi_0(KU_p^\wedge) \cong \mathbb{Z}^\wedge_0$ is canonical, using the unit of $K(\mathbb{F}_\ell)$.) The isomorphism $v_\ell$ and the module multiplication

$$L_{K(1)}K(\mathbb{F}_\ell) \wedge \Sigma L_{K(1)}K(\mathbb{F}_\ell) \longrightarrow \Sigma L_{K(1)}K(\mathbb{F}_\ell)$$

define a homomorphism

$$\pi_0(L_{K(1)}K(\mathbb{F}_\ell) \wedge \Sigma L_{K(1)}K(\mathbb{F}_\ell); \mathbb{Z}/p^\infty) \longrightarrow \mathbb{Z}/p^\infty.$$  

**Proposition 6.5.** The homomorphism in (6.4) provides an Anderson duality pairing on $L_{K(1)}K(\mathbb{F}_\ell), \Sigma L_{K(1)}K(\mathbb{F}_\ell)).$

**Proof.** The homotopy fixed point spectral sequence gives an identification

$$\pi_{2q-\epsilon}(L_{K(1)}K(\mathbb{F}_\ell); \mathbb{Z}_p^\wedge(q)) \cong H^q(\mathbb{Z}; \mathbb{Z}_p^\wedge(q))$$

$$\pi_{2q-\epsilon}(L_{K(1)}K(\mathbb{F}_\ell); \mathbb{Z}/p^\infty) \cong H^q(\mathbb{Z}; \mathbb{Z}/p^\infty(q))$$

where the generator acts by multiplication by $\ell^q$ on $\mathbb{Z}_p^\wedge(q)$ and $\mathbb{Z}/p^\infty(q)$. The cup product on group cohomology with coefficients

$$H^i(\mathbb{Z}; \mathbb{Z}_p^\wedge(q)) \otimes H^j(\mathbb{Z}; \mathbb{Z}/p^\infty(r)) \longrightarrow H^{i+j}(\mathbb{Z}; \mathbb{Z}/p^\infty(q + r))$$

converges to the product on homotopy groups and the isomorphisms in Definition 3.6 follow from standard calculations in homological algebra.

Since the $K(\mathbb{F}_\ell)^\wedge_p$-module structure on $\Sigma L_{K(1)}K(\mathbb{F}_\ell)$ restricts to a $K(\mathbb{F}_\ell)^\wedge_p$-module structure on

$$\tau_{\geq 1}\Sigma L_{K(1)}K(\mathbb{F}_\ell) \simeq \Sigma K(\mathbb{F}_\ell)^\wedge,$$

we get a $K(\mathbb{F}_\ell)^\wedge_p$-module structure on the cofiber $\tau_{\leq 0}\Sigma L_{K(1)}K(\mathbb{F}_\ell)$,

$$K(\mathbb{F}_\ell)^\wedge_p \wedge \tau_{\leq 0}(\Sigma(L_{K(1)}K(\mathbb{F}_\ell))) \longrightarrow \tau_{\leq 0}(\Sigma(L_{K(1)}K(\mathbb{F}_\ell))).$$

Composing the induced map on $\mathbb{Z}/p^\infty$ homotopy groups with $v_\ell$ and inspecting the resulting pairing on homotopy groups, we get an Anderson duality pairing.

**Proposition 6.6.** The homomorphism

$$\pi_0(K(\mathbb{F}_\ell)^\wedge_p \wedge \tau_{\leq 0}(\Sigma(L_{K(1)}K(\mathbb{F}_\ell))); \mathbb{Z}/p^\infty) \longrightarrow \mathbb{Z}/p^\infty$$
induced by the pairing above and \( v_{\ell} \) gives an Anderson duality pairing on \( K(\mathbb{F}_{\ell})_p^\wedge \), \( \tau_{\leq 0}(\Sigma(L_{K(1)}K(\mathbb{F}_{\ell}))) \) The weak equivalence

\[
\Sigma^2 F(K(\mathbb{F}_{\ell})) \simeq \tau_{\leq 0}(\Sigma(L_{K(1)}K(\mathbb{F}_{\ell})))
\]

then induces an Anderson duality pairing on \( K(\mathbb{F}_{\ell})_p^\wedge \), \( \Sigma^2 F(K(\mathbb{F}_{\ell})) \). In particular, we have a weak equivalence

\[
F(K(\mathbb{F}_{\ell})) \simeq \Sigma^{-3}(I_{\mathbb{Z}/p^{\infty}}(K(\mathbb{F}_{\ell})_p^\wedge)) \simeq \Sigma^{-3}(I_{\mathbb{Z}/p^{\infty}}(K(\mathbb{F}_{\ell}))).
\]

To finish the proof of Theorem 1.7, we need to identify the maps

\[
F(K(\mathbb{F}_{\ell})) \longrightarrow F(K(\mathbb{S})) \simeq F(\text{Fib}(\text{trc}_{\mathbb{Z}})).
\]

After \( p \)-completion, each map is the induced map on \( F(-)_p^\wedge \) of a map \( K(\mathbb{F}_{\ell})_p^\wedge \rightarrow \text{Fib}(\text{trc}_{\mathbb{Z}})_p^\wedge \) arising as the horizontal fiber from the homotopy cartesian square

\[
\begin{array}{ccc}
\text{Fib}(\text{trc}_{\mathbb{Z}})_p^\wedge & \longrightarrow & \text{Fib}(\text{trc}_{\mathbb{Z}}_{\frac{1}{p}})_p^\wedge \\
\downarrow & & \downarrow \\
K(\mathbb{Z})_p^\wedge & \longrightarrow & K(\mathbb{Z}_{\frac{1}{p}})_p^\wedge
\end{array}
\]

(6.7)

As such, we can construct it in the derived category of \( K(\mathbb{Z})_p^\wedge \)-modules (where the \( K(\mathbb{Z})_p^\wedge \)-module structure on \( K(\mathbb{F}_{\ell})_p^\wedge \) comes from the map of commutative ring spectra \( K(\mathbb{Z}) \rightarrow K(\mathbb{F}_{\ell}) \)). As a consequence, the map

\[
F(K(\mathbb{F}_{\ell}))_p^\wedge \longrightarrow F(\text{Fib}(\text{trc}_{\mathbb{Z}}))_p^\wedge
\]

lifts to the derived category of \( K(\mathbb{Z})_p^\wedge \)-modules. Thus, the following diagram commutes.

\[
\begin{array}{ccc}
K(\mathbb{Z})_p^\wedge \wedge \Sigma^2 F(K(\mathbb{F}_{\ell}))_p^\wedge & \longrightarrow & K(\mathbb{F}_{\ell})_p^\wedge \wedge \Sigma^2 F(K(\mathbb{F}_{\ell}))_p^\wedge \\
\downarrow & & \downarrow \\
\Sigma^2 F(K(\mathbb{F}_{\ell}))_p^\wedge & \longrightarrow & \Sigma^2 F(\text{Fib}(\text{trc}_{\mathbb{Z}}))_p^\wedge
\end{array}
\]

This diagram almost asserts Anderson duality between the usual map of \( K \)-theory ring spectra \( K(\mathbb{Z})_p^\wedge \rightarrow K(\mathbb{F}_{\ell})_p^\wedge \) and \( \Sigma^2 \) of the map \( F(K(\mathbb{F}_{\ell})) \rightarrow F(\text{Fib}(\text{trc}_{\mathbb{Z}})) \) in question; such an assertion would follow from the diagram

\[
\begin{array}{ccc}
\pi_{-2}(F(\mathbb{F}_{\ell}); \mathbb{Z}/p^{\infty}) & \xrightarrow{\cong} & \pi_{-1}(L_{K(1)}K(\mathbb{F}_{\ell}); \mathbb{Z}/p^{\infty}) \\
\downarrow & & \downarrow \\
\pi_{-2}(F(\text{Fib}(\text{trc}_{\mathbb{Z}})); \mathbb{Z}/p^{\infty}) & \xrightarrow{\cong} & \pi_{-1}(L_{K(1)}\text{Fib}(\text{trc}_{\mathbb{Z}}); \mathbb{Z}/p^{\infty})
\end{array}
\]

commuting. The following lemma, proved at the end of the section, asserts that it commutes up to multiplying by a unit in \( \mathbb{Z}_p^\wedge \). (In fact, for the isomorphism \( u \) chosen in [5 §1] in the case \( p > 2 \) and in the discussion following Conjecture 5.4 in the case \( p = 2 \), a careful analysis of the proof shows that the unit is ±1, with the sign depending on the convention for the attaching map in the Quillen localization sequence, which was not specified above.)
Lemma 6.8. The map $L_{K(1)} K(\mathbb{F}_\ell) \to L_{K(1)} \text{Fib}(\text{trc}_2)$ induces an isomorphism on $\pi_{-1}(\mathbb{Z}/p^\infty)$.

Combining the lemma with the observations in the paragraph that proceeds it, we deduce the following theorem.

Theorem 6.9. Assume either $p = 2$ and Conjecture 5.4 holds or $p > 2$. Under the weak equivalences

$$F(K(\mathbb{F}_\ell)) \simeq \Sigma^{-3}I_{\mathbb{Z}/p^\infty} K(\mathbb{F}_\ell)$$

$$F(\text{Fib}(\text{trc}_2)) \simeq \Sigma^{-3}I_{\mathbb{Z}/p^\infty} K(\mathbb{Z})$$

of Theorem 1.7 and Proposition 6.6, the map $\text{Fib}(\text{trc}_2)$ is (up to a possible sign) the connecting map in the Mayer-Vietoris sequence associated the homotopy cartesian square (6.7), so coincides with $\Sigma^{-3}I_{\mathbb{Z}/p^\infty} K(\mathbb{Z})$.

We now finish with the proof of Lemma 6.8.

Proof of Lemma 6.8. Because $L_{K(1)} K(\mathbb{F}_\ell)$ and $L_{K(1)} \text{Fib}(\text{trc}_2)$ both have $\pi_{-1}$ isomorphic to $\mathbb{Z}_p^\times$ and $\pi_{-2}$ torsion free (0 for $L_{K(1)} K(\mathbb{F}_\ell)$ and $\mathbb{Z}_p^\times$ for $L_{K(1)} \text{Fib}(\text{trc}_2)$), it suffices to check that the map induces an isomorphism on $\pi_{-1}$ (without coefficients).

Before starting, we fix the following notation: write $\ell = \omega p^n$ where:

(i) For $p > 2$, $r \in (\mathbb{Z}_p^\times)^\times$ is a topological generator of the subgroup $U_1 \subset (\mathbb{Z}_p^\times)^\times$ congruent to 1 mod $p$, $n$ is a positive integer, and $\omega$ is a $(p-1)$st root of unity.

(ii) For $p = 2$, $r \in (\mathbb{Z}_2^\times)^\times$ is a topological generator of the subgroup $U_1 \subset (\mathbb{Z}_2^\times)^\times$ congruent to 1 mod 4, $n$ is a positive integer, and $\omega = \pm 1$.

Now consider the composite map obtained by precomposing the suspension of the map in question with the $L_{K(1)}$-localization of the map in Quillen’s localization sequence for $Z \to \mathbb{Z}[\frac{1}{\ell}]$,

$$L_{K(1)} K(\mathbb{Z}[\frac{1}{\ell}]) \to \Sigma L_{K(1)} K(\mathbb{F}_\ell) \to \Sigma L_{K(1)} \text{Fib}(\text{trc}_2).$$

This composite is (up to a possible sign) the connecting map in the Mayer-Vietoris (co)fiber sequence associated the homotopy cartesian square (6.7), so coincides with the composite

$$L_{K(1)} K(\mathbb{Z}[\frac{1}{\ell}]) \to L_{K(1)} TC(\mathbb{Z}[\frac{1}{\ell}]) \simeq L_{K(1)} TC(\mathbb{Z}) \to \Sigma L_{K(1)} \text{Fib}(\text{trc}_2).$$

Because

$$\pi_1((K(\mathbb{Z}[\frac{1}{\ell}])_p^\wedge) \to \pi_1(L_{K(1)} K(\mathbb{Z}[\frac{1}{\ell}])) \to \pi_1(L_{K(1)} K(\mathbb{Z}[\frac{1}{\ell}])_p^\wedge),$$

we have a canonical isomorphism

$$\pi_1(L_{K(1)} K(\mathbb{Z}[\frac{1}{\ell}])) \simeq (\mathbb{Z}[\frac{1}{\ell}]^\times)_p^\wedge.$$

The element $\ell \in (\mathbb{Z}[\frac{1}{\ell}])^\times_p^\wedge$ then specifies a map

$$\Sigma J^\wedge_p = \Sigma L_{K(1)} S \to L_{K(1)} K(\mathbb{Z}[\frac{1}{\ell}]).$$
Since \( \pi_0(\Sigma J_p^\wedge) \cong \mathbb{Z}_p^\wedge \), to complete the argument it suffices to show that the composite maps

\[
\pi_0(\Sigma J_p^\wedge) \rightarrow \pi_0(\Sigma L_{K(1)}K(\mathbb{F}_\ell)) \\
\pi_0(\Sigma J_p^\wedge) \rightarrow \pi_0(\Sigma L_{K(1)}(\text{Fib}(\text{trc}_\mathbb{Z})))
\]

both have cokernels cyclic of order \( p^n \) (for \( n \) the positive integer in the notation fixed above).

First consider the composite map \( \Sigma J_p^\wedge \rightarrow \Sigma L_{K(1)}K(\mathbb{F}_\ell) \). Under the canonical isomorphisms (induced by the inclusion of the unit)

\[
\pi_1(\Sigma J_p^\wedge) \cong \pi_0(J_p^\wedge) \cong \mathbb{Z}_p^\wedge \\
\pi_1(\Sigma L_{K(1)}K(\mathbb{F}_\ell)) \cong \pi_0(L_{K(1)}K(\mathbb{F}_\ell)) \cong \mathbb{Z}_p^\wedge,
\]

the map \( \Sigma J_p^\wedge \rightarrow \Sigma L_{K(1)}K(\mathbb{F}_\ell) \) in question induces on \( \pi_1 \) plus or minus the identity on \( \mathbb{Z}_p^\wedge \) (depending on the sign convention in the attaching map in the Quillen localization sequence that we have not specified). We have a canonical isomorphism

\[
\pi_{-1}(L_{K(1)}K(\mathbb{F}_\ell)) \cong H^1(\mathbb{Z}; \mathbb{Z}_p^\wedge) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}_p^\wedge) \cong \mathbb{Z}_p^\wedge
\]

specified by taking the usual inclusion \( \iota \colon \mathbb{Z} \rightarrow \mathbb{Z}_p^\wedge \) as the generator. We also have an isomorphism

\[
\pi_{-1}(J_p^\wedge) \cong H_{\text{Gal}}((\mathbb{Z}_p^\wedge)^\wedge; \mathbb{Z}_p^\wedge) \cong \text{Hom}_c((\mathbb{Z}_p^\wedge)^\wedge, \mathbb{Z}_p^\wedge) \cong \mathbb{Z}_p^\wedge
\]

(where \( \text{Hom}_c \) denotes continuous homomorphisms of profinite groups) specified by taking as the generator of \( \text{Hom}_c((\mathbb{Z}_p^\wedge)^\wedge, \mathbb{Z}_p^\wedge) \) the unique continuous homomorphism \( \phi_r \) sending \( r \) to 1. The map

\[
\pi_{-1}(J_p^\wedge) \cong \pi_0(\Sigma J_p^\wedge) \rightarrow \pi_0(\Sigma L_{K(1)}K(\mathbb{F}_\ell)) \cong \pi_{-1}(L_{K(1)}K(\mathbb{F}_\ell))
\]

is the map induced by the continuous homomorphism \( \mathbb{Z} \rightarrow (\mathbb{Z}_p^\wedge)^\wedge \) sending the generator to \( \ell \in (\mathbb{Z}_p^\wedge)^\wedge \); this is the map sending \( \phi_r \) to \( p^n \iota \), and we see that its cokernel is isomorphic to \( \mathbb{Z}/p^n \) as claimed.

For the map \( \Sigma J_p^\wedge \rightarrow \Sigma L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}) \), we look at the factorization \( \Sigma J_p^\wedge \rightarrow L_{K(1)}\text{TC}(\mathbb{Z}) \); write \( a_\ell \) for this map. The isomorphisms

\[
\pi_1(L_{K(1)}\text{TC}(\mathbb{Z})) \xrightarrow{\cong} \pi_1(L_{K(1)}\text{TC}(\mathbb{Z}_p^\wedge)) \leftarrow \pi_1(L_{K(1)}K(\mathbb{Z}_p^\wedge)) \xleftarrow{\cong} \pi_1(L_{K(1)}K(\mathbb{Q}_p^\wedge))
\]

identify \( \pi_1(L_{K(1)}\text{TC}(\mathbb{Z})) \) as \( (\mathbb{Q}_p^\wedge)^\wedge \) and by construction \( a_\ell \) sends the fundamental class of \( \pi_1(\Sigma J_p^\wedge) \) to \( \ell \) under this identification. Then for any \( \nu \in (\mathbb{Q}_p^\wedge)^\wedge \), we have a map \( a_\nu : \Sigma J_p^\wedge \rightarrow L_{K(1)}\text{TC}(\mathbb{Z}) \) that sends the fundamental class in \( \pi_1(\Sigma J_p^\wedge) \) to the image of \( \nu \) in \( \pi_1L_{K(1)}\text{TC}(\mathbb{Z}) \). In the case \( p > 2 \), \( \omega \) goes to the identity in \( (\mathbb{Q}_p^\wedge)^\wedge \), and we have that \( a_\ell = a_p = p^n a_r \). In the case \( p = 2 \), while that equality may not hold, \( \pi_1(\Sigma L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z})) \) is torsion free, so \( a_\ell \) and \( p^n a_r \) become the same after composing with the map \( \Sigma L_{K(1)}\text{TC}(\mathbb{Z}) \rightarrow \Sigma L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}) \) and in particular induce the same map \( \pi_0(\Sigma J_p^\wedge) \rightarrow L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}) \). Since the composite

\[
J_p^\wedge \xrightarrow{a_\ell} L_{K(1)}\text{TC}(\mathbb{Z}) \rightarrow \Sigma L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z})
\]

is an isomorphism on \( \pi_0 \), the map in question

\[
J_p^\wedge \xrightarrow{a_\ell} L_{K(1)}\text{TC}(\mathbb{Z}) \rightarrow \Sigma L_{K(1)}\text{Fib}(\text{trc}_\mathbb{Z}),
\]

has cokernel isomorphic to \( \mathbb{Z}/p^n \) as claimed. This completes the proof. \( \square \)
7. Proof of Theorems 1.9 and 1.10

In this section, we prove Theorems 1.9 and 1.10 which state that the maps

\[ K(S_{(p)})_{(p)} \longrightarrow \text{holim}_n K(L^n_{/S})_{(p)} \]
\[ K(S)_{(p)} \longrightarrow \text{holim}_n K(L^{p,f}_{/S})_{(p)} \]

are inclusions of wedge summands. The argument relies on recent work of Land-Mathew-Meier-Tamme [23]; we use the following result that we have stated in substantially weaker form than it appears there.

**Theorem 7.1** (Land-Mathew-Meier-Tamme [23], Theorem A). Let \( R \rightarrow R' \) be an \( L^{p,f}_{/}\)equivalence of ring spectra. Then

\[ (L_n K(R))^\wedge_p \longrightarrow (L_n K(R'))^\wedge_p. \]

is a weak equivalence.

The maps \( S_{(p)} \rightarrow L^n_{/S} \) and \( S \rightarrow L^{p,f}_{/S} \) are \( L^{p,f}_{/}\)localization maps and in particular \( L^{p,f}_{/}\)-equivalences. We therefore get the following corollary.

**Corollary 7.2.** The maps

\[ \text{holim}_n (L_n K(S_{(p)}))^\wedge_p \longrightarrow \text{holim}_n (L_n K(L^n_{/S}))^\wedge_p \]
\[ \text{holim}_n (L_n K(S))^\wedge_p \longrightarrow \text{holim}_n (L_n K(L^{p,f}_{/S}))^\wedge_p \]

are weak equivalences.

Using the localization maps

\[ \text{holim}_n K(L^n_{/S})^\wedge_p \longrightarrow \text{holim}_n (L_n K(L^n_{/S}))^\wedge_p \]
\[ \text{holim}_n K(L^{p,f}_{/S})^\wedge_p \longrightarrow \text{holim}_n (L_n K(L^{p,f}_{/S}))^\wedge_p \]

and the inverse equivalence from Corollary 7.2, we get maps

\[ \text{holim}_n K(L^n_{/S})^\wedge_p \longrightarrow \text{holim}_n (L_n K(S_{(p)}))^\wedge_p \]
\[ \text{holim}_n K(L^{p,f}_{/S})^\wedge_p \longrightarrow \text{holim}_n (L_n K(S))^\wedge_p \]  

(7.3)

with the property that the composite maps

\[ K(S_{(p)})^\wedge_p \longrightarrow \text{holim}_n (L_n K(S_{(p)}))^\wedge_p \]
\[ K(S)^\wedge_p \longrightarrow \text{holim}_n (L_n K(S))^\wedge_p \]

are the \( p \)-completions of the chromatic completion maps.

We also have maps on rationalizations, constructed as follows. Since the ring spectrum \( Q \) is \( L^{p,f}_{/}\)-local, the maps of ring spectra \( S_{(p)} \rightarrow Q \) and \( S \rightarrow Q \) factor through maps \( L^n_{/S} \rightarrow Q \) and \( L^{p,f}_{/S} \rightarrow Q \). The maps

\[ K(S_{(p)}) \longrightarrow K(Q) \]
\[ K(S) \longrightarrow K(Q) \]

(7.4)

induce an isomorphism on homotopy groups except in degree 1; in this degree, \( \pi_1(K(S); Q) = 0 \) and we have a canonical splitting

\[ \pi_1(K(Q); Q) \cong (Q^\times) \otimes Q \cong \bigoplus_{\ell \text{ prime}} Q \cong \bigoplus_{\ell \neq p} Q \cong (Z^\times_{(p)}) \otimes Q \cong \pi_1(K(S_{(p)}); Q). \]
The splittings on rational homotopy groups specify splittings of rational spectra and we get maps

\[
\begin{align*}
(7.5) \quad & \quad \text{holim}_n K(L_n^f S)_Q \rightarrow K(Q)_Q \rightarrow K(S(p))_Q \xrightarrow{\sim} (\text{holim}_n L_n K(S(p)))_Q \\
& \quad \text{holim}_n K(L_n^{p,f} S)_Q \rightarrow K(Q)_Q \rightarrow K(S)_Q \xrightarrow{\sim} (\text{holim}_n L_n K(S))_Q
\end{align*}
\]

with the property that the composite maps

\[
K(S(p))_Q \rightarrow (\text{holim}_n L_n K(S(p)))_Q \\
K(S)_Q \rightarrow (\text{holim}_n L_n K(S))_Q
\]

are the rationalizations of the chromatic completion maps.

We have constructed a \(p\)-complete map and a rational map. The following theorem, proved in the next section, asserts that they are compatible with the arithmetic square.

**Theorem 7.6.** For \(S = S\) or \(S(p)\), the diagram in the stable category

\[
\begin{CD}
(\text{holim} K(L_n^{p,f} S))_{(p)} @>>> \text{holim} K(L_n^{p,f} S)^\wedge_p @>>> \text{holim} L_n K(S)^\wedge_p \\
\downarrow @VVV @VVV \\
(\text{holim} K(L_n^{p,f} S))_Q \\
\end{CD}
\]

commutes.

As we now explain, the previous theorem completes the proof of Theorems 1.9 and 1.10. The previous theorem implies that there exists a map

\[
(7.7) \quad (\text{holim} K(L_n^{p,f} S))_{(p)} \rightarrow \text{holim} L_n K(S)
\]

such that the composite maps

\[
K(S)_{(p)} \rightarrow (\text{holim} K(L_n^{p,f} S))_{(p)} \rightarrow (\text{holim} L_n K(S))^\wedge_p \\
K(S)_{(p)} \rightarrow (\text{holim} K(L_n^{p,f} S))_{(p)} \rightarrow (\text{holim} L_n K(S))_Q
\]

are the canonical ones, i.e., the chromatic completion map followed by the \(p\)-completion map and rationalization maps, respectively. We first need to observe that the map \((7.7)\) factors through \(K(S)_{(p)}\).

In the case \(S = S\), this is clear because as the limit of connective spectra, \(\text{holim} K(L_n^{p,f} S)\) is \((-2)\)-connected and the \((-2)\)-connected cover of \(\text{holim} L_n K(S)\) is \(K(S)_{(p)}\) by Theorem 1.6. In the case \(S = S(p)\), the \((-2)\)-connective cover of \(\text{holim} L_n K(S(p))\) has non-trivial \(\pi_{-1}\), and it suffices to see that the map on \(\pi_{-1}\) is trivial. In the notation of 3.1 the map

\[
\pi_{-1} (\text{holim} L_n K(S(p))) \rightarrow \pi_{-2} (F(K(S(p))))
\]

is injective; by Theorem 1.6, \(\pi_{-2} F(K(S)) = 0\), and that implies that the map

\[
F(K(S(p))) \rightarrow \bigvee_{\ell \neq p} F(K(F_\ell)) \simeq \bigvee_{\ell \neq p} F(\Sigma K(F_\ell))
\]

in the fibration sequence of Proposition 6.2 is injective on \(\pi_{-2}\). Since the maps

\[
L_1 \Sigma K(F_\ell) \rightarrow \Sigma F(\Sigma K(F_\ell))
\]
are injective on $\pi_{-1}$ and the diagram
\[
\begin{array}{c}
\text{holim } L_n K(\mathbb{S}(p)) \rightarrow \Sigma F(K(\mathbb{S}(p))) \\
\downarrow \\
L_1 \Sigma K(\mathbb{F}_l) \rightarrow \Sigma F(\Sigma K(\mathbb{F}_l))
\end{array}
\]
commutes by naturality, it suffices to observe that the maps
\[
\text{holim } K(L_n^{f} \mathbb{S}) \rightarrow L_1 \Sigma K(\mathbb{F}_l)
\]
are all zero on $\pi_{-1}$. But since by construction these maps factor through $L_1 K(Z(\mathbb{S}))$ (see Proposition 6.1), they factor through $K(L_n^{f} \mathbb{S})$, which is connective.

The factorization constructed above is a map
\[(7.8) \quad (\text{holim } K(L_n^{p,f} \mathbb{S}))(p) \rightarrow K(\mathbb{S})(p)\]
such that the composite maps
\[
K(\mathbb{S})(p) \rightarrow (\text{holim } K(L_n^{p,f} \mathbb{S}))(p) \rightarrow K(\mathbb{S})^\wedge
\]
\[
K(\mathbb{S})(p) \rightarrow (\text{holim } K(L_n^{p,f} \mathbb{S}))(p) \rightarrow K(\mathbb{S})_p
\]
are the $p$-completion and rationalization maps, respectively. In the fiber sequence for $K(\mathbb{S})(p)$, the connecting map
\[
\Sigma^{-1} K(\mathbb{S})_p \rightarrow K(\mathbb{S})(p)
\]
is the zero map on homotopy groups, because each homotopy group of $K(\mathbb{S})$ is finitely generated and each homotopy group of $K(\mathbb{S}(p))$ is the extension by a finitely generated group of a subgroup of a direct sum of finitely generated groups. It follows that the composite map
\[(7.9) \quad K(\mathbb{S})(p) \rightarrow (\text{holim } K(L_n^{p,f} \mathbb{S}))(p) \rightarrow K(\mathbb{S})(p)\]
induces the identity map on homotopy groups and in particular is a weak equivalence. Composing the map (7.8) with the inverse of this weak equivalence, we get a map
\[
(\text{holim } K(L_n^{p,f} \mathbb{S}))(p) \rightarrow K(\mathbb{S})(p)
\]
so that the composite map
\[
K(\mathbb{S})(p) \rightarrow (\text{holim } K(L_n^{p,f} \mathbb{S}))(p) \rightarrow K(\mathbb{S})(p)
\]
is the identity, completing the proof.

We close the section with a few remarks on the chromatic localization analogue of Theorem 1.9 specifically what is missing to prove that the map
\[
K(\mathbb{S}(p))(p) \rightarrow \text{holim } K(L_n \mathbb{S})(p)
\]
is the inclusion of a wedge summand. Most of the argument (in this section and the next) works for $L_n$ as it does for $L_1^{f}$ except the Land-Mathew-Meier-Tamme result (Theorem 7.1) that we used to construct the map (7.3). For the chromatic localization version, we need to construct a map
\[(7.10) \quad \text{holim } K(L_n \mathbb{S})^\wedge_p \rightarrow \text{holim}(L_n K(\mathbb{S}(p)))^\wedge_p\]
by some other means. We envision such a construction would go as follows. Previous work of the authors [9, 1.1; 2.6] shows that the map
\[
TC(\mathbb{S}(p))^\wedge_p \rightarrow \text{holim } TC(L_n \mathbb{S}(p))^\wedge_p \xrightarrow{\simeq} \text{holim}(L_n TC(L_n \mathbb{S}(p)))^\wedge_p
\]
is a weak equivalence (the latter weak equivalence holding because \( TC(L_n S(p)) \) is \( L_n \)-local). We also have by the Land-Mathew-Meier-Tamme result in Theorem 7.1 (and the telescope conjecture in the case \( n = 1 \)) that the map

\[
(L_1 K(S((p))))_p^\wedge \to (L_1 K(S((p))))_p^\wedge
\]

is a weak equivalence, which then gives us a map

\[
(L_1 K(L_1 S((p))))_p^\wedge \to (L_1 TC(S((p))))_p^\wedge.
\]

We would then expect (but have not been able to verify) that the resulting diagram

\[
\begin{array}{ccc}
\text{holim } K(S)^\wedge & \to & TC(S)^\wedge \\
& & \\
(L_1 K(L_1 S((p))))_p^\wedge & \to & (L_1 TC(S((p))))_p^\wedge
\end{array}
\]

should commute where the horizontal are maps as discussed and the vertical maps are induced by \( L_1 \)-localization. If this is the case, then using the commutative diagram

\[
\begin{array}{ccc}
(L_1 K(S((p))))_p^\wedge & \simeq & (L_1 K(S(p)))_p^\wedge \\
& & \\
(L_1 K(S((p))))_p^\wedge & \to & L_1 TC(S((p)))
\end{array}
\]

\[
\begin{array}{ccc}
(L_1 K(Q))_p^\wedge & \simeq & (L_1 K(Z((p))))_p^\wedge \\
& & \\
(L_1 K(Q))_p^\wedge & \to & L_1 TC(Z((p)))
\end{array}
\]

we get a commutative diagram

\[
\begin{array}{ccc}
\text{holim } K(L_n S)^\wedge & \to & TC(S)^\wedge \\
& & \\
(L_1 K(Z((p))))_p^\wedge & \to & (L_1 TC(Z((p))))_p^\wedge
\end{array}
\]

which then entitles us to a map of the form (7.10). The composite maps

\[
K(S((p)))_p^\wedge \to \text{holim } K(L_n S) \to TC(S)^\wedge
\]

\[
K(S((p)))_p^\wedge \to \text{holim } K(L_n S) \to L_1 K(Z((p)))_p^\wedge
\]

are the usual ones; we can then show in the case \( p > 2 \) (applying [6, 1.1] and an argument like the analysis of the map (7.9) above) that the map (7.10) can be chosen so that the composite map

\[
K(S((p)))_p^\wedge \to \text{holim } K(L_n S) \to \text{holim}(L_n K(S((p))))_p^\wedge
\]

is the \( p \)-completion of the chromatic completion map. In other words, the argument for the chromatic localizations reduces in the case \( p > 2 \) to showing that (7.11) commutes.

8. Proof of Theorem 7.6

The entirety of this section is devoted to the proof of Theorem 7.6. As in the statement, let \( S = S \) or \( S((p)) \) and then \( L_n^{p,f} S \) is \( L_n^{p,f} S \) or \( L_n^f S \) (respectively). By
construction, the diagram

\[
\begin{array}{c}
\text{holim } K(L_n^{p,f} S) \longrightarrow \text{(holim } L_n K(S))_p \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
K(S)_Q \longrightarrow (L_{K(1)} K(Q))_Q \leftarrow ((\text{holim } L_n K(S))_p)_Q \\
\end{array}
\]

\((8.1)\)

commutes: note the direction of the lower left diagonal arrow; the left vertical arrow is essentially defined in \((7.5)\) by the lefthand trapezoid. The proof of Theorem \((7.6)\) is based on the following two lemmas proved below. As in the statement of the theorem, \(S = \mathbb{S}\) or \(\mathbb{S}(p)\).

**Lemma 8.2.** The map \(\pi_* (\text{holim } K(L_n^{p,f} S); \mathbb{Q}) \rightarrow \pi_* (K(Q); \mathbb{Q})\) factors through the image of \(\pi_* (K(S); \mathbb{Q})\).

**Lemma 8.3.** The map \(\pi_* ((\text{holim } L_n K(S))_p^\wedge; \mathbb{Q}) \rightarrow \pi_* (L_{K(1)} K(Q); \mathbb{Q})\) is an injection for \(* \geq 0\).

**Proof of Theorem 7.6.** Lemma 8.2 together with diagram (8.1) give the commuting diagram

\[
\begin{array}{c}
\text{holim } K(L_n^{p,f} S) \longrightarrow \text{(holim } L_n K(S))_p^\wedge \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
K(S)_Q \longrightarrow (L_{K(1)} K(Q))_Q \leftarrow ((\text{holim } L_n K(S))_p)_Q \\
\end{array}
\]

By naturality, the diagram

\[
\begin{array}{c}
K(S)_Q \longrightarrow (\text{holim } L_n K(S))_Q \longrightarrow ((\text{holim } L_n K(S))_p)_Q \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
(L_{K(1)} K(Q))_Q \\
\end{array}
\]

also commutes. By Theorems 1.6 and 1.7 we have that \(\pi_{-1} (\text{holim } L_n K(S)_p^\wedge; \mathbb{Q}) = 0\). Lemma 8.3 and the fact that holim \(K(L_n^{p,f} S)\) is \((-2)\)-connected then implies that the diagram in Theorem 7.6 commutes.

We now prove the lemmas.

**Proof of Lemma 8.2.** By the remarks surrounding (7.4), it suffices to analyze the case \(* = 1\). As reviewed above, we have

\[\pi_1 (K(Q); \mathbb{Q}) \cong (\mathbb{Q}^\times) \otimes \mathbb{Q} \cong \bigoplus_{\ell \text{ prime}} \mathbb{Q},\]

and we write \(\nu_\ell : \pi_1 (K(Q); \mathbb{Q}) \rightarrow \mathbb{Q}\) for the projection to the \(\ell\) factor; this homomorphism is induced by \(\ell\)-adic valuation on \(\mathbb{Q}^\times\). The image of \(\pi_1 K(S(p); \mathbb{Q})\) is the kernel of \(\nu_\ell\) and the image of \(\pi_1 K(S; \mathbb{Q})\) is zero, or equivalently, the intersection of the kernel of \(\nu_\ell\) for all \(\ell\).
In the case $S = \mathbb{S}$, the map $\text{holim } K(L^n_p; \mathbb{S}) \to K(\mathbb{Q})$ factors through $K(L^n_0; \mathbb{Z}) = K(\mathbb{Z}[1/p])$. We see from the commuting diagram

$$\begin{array}{ccc}
\mathbb{Z}[1/p]^\times & \xrightarrow{\cong} & \pi_1(K(\mathbb{Z}[1/p])) \\
\downarrow & & \downarrow \\
\mathbb{Q}^\times & \xrightarrow{\cong} & \pi_1(K(\mathbb{Q}))
\end{array}$$

that the image of $\pi_1(K(\mathbb{Z}[1/p]); \mathbb{Q})$ in $\pi_1(K(\mathbb{Q}); \mathbb{Q})$ lands in the kernel of $\nu_\ell$ for all $\ell \neq p$. To see that the image of $\pi_1(\text{holim } K(L^n_p; \mathbb{S}); \mathbb{Q})$ also lands in the kernel of $\nu_p$, we note that it lands in the image of $\pi_1(\text{holim } K(L^n_0; \mathbb{S}); \mathbb{Q})$, and this reduces the case $S = \mathbb{S}$ to the case $S = \mathbb{S}(p)$.

We are left to establish the case $S = \mathbb{S}(p)$. (As above, we write $L^n_0; \mathbb{S}$ for $L^n_p; S$ in the case $S = \mathbb{S}(p)$.) Because $\pi_0 K(\mathbb{S}(p)) \cong \pi_0 K(\mathbb{Q}) \cong \mathbb{Z}$ is torsion-free, $\pi_1(K(\mathbb{S}(p))^\wedge_p)$ and $\pi_1(K(\mathbb{Q})^\wedge_p)$ are the left 0-derived $p$-completions of $\pi_1(K(\mathbb{S}(p))) \cong \mathbb{Z}^\wedge_p$, and $\pi_1(K(\mathbb{Q})) \cong \mathbb{Q}^\wedge$. The homomorphism $\nu_p$ then extends to a homomorphism $\nu_p : \pi_1(K(\mathbb{Q}^\wedge_p); \mathbb{Q}) \to \mathbb{Q}^\wedge_p$ with kernel $\pi_1(K(\mathbb{S}(p))^\wedge_p; \mathbb{Q})$. We have an analogous homomorphism $\nu_p : \pi_1(K(\mathbb{Q}^\wedge_p); \mathbb{Q}) \to \mathbb{Q}^\wedge_p$ and a commuting diagram of short exact sequences

$$\begin{array}{cccccc}
0 & \longrightarrow & \pi_1(K(\mathbb{S}(p)); \mathbb{Q}) & \longrightarrow & \pi_1(K(\mathbb{Q}); \mathbb{Q}) & \xrightarrow{\nu_p} & \mathbb{Q} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \ll & & \\
0 & \longrightarrow & \pi_1(K(\mathbb{S}(p))^\wedge_p; \mathbb{Q}) & \longrightarrow & \pi_1(K(\mathbb{Q})^\wedge_p; \mathbb{Q}) & \xrightarrow{\nu_p} & \mathbb{Q}^\wedge_p & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1(K(\mathbb{Z})^\wedge_p; \mathbb{Q}) & \longrightarrow & \pi_1(K(\mathbb{Q})^\wedge_p; \mathbb{Q}) & \xrightarrow{\nu_p} & \mathbb{Q}^\wedge_p & \longrightarrow & 0.
\end{array}$$

To complete the proof, we need to see that the composite of $\pi_1(\text{holim } K(L^n_0; \mathbb{S}); \mathbb{Q}) \to \pi_1(K(\mathbb{Q}^\wedge_p); \mathbb{Q})$ with $\nu_p$ is zero.

By [24, Thm D], we have a weak equivalence

$$K(\mathbb{Q})^\wedge_p \cong \tau_{\geq 0}(TC(\mathbb{Q}_p)^\wedge).$$

From this, the Quillen localization sequence for $\mathbb{Q}^\wedge_p \to \mathbb{Q}^\wedge_p$ (using $K(\mathbb{Q})^\wedge_p \cong H\mathbb{Q}_p$), and the resolved Quillen-Lichtenbaum conjecture [24, Thm. D] for $\mathbb{Q}^\wedge_p$, we get weak equivalences

$$L_{K(1)}TC(\mathbb{Q}_p)^\wedge \cong L_{K(1)}K(\mathbb{Q}^\wedge_p), \quad \tau_{\geq 1}L_{K(1)}K(\mathbb{Q})^\wedge \cong L_{K(1)}K(\mathbb{Q}^\wedge_p),$$

and therefore a commuting diagram of short exact sequences

$$\begin{array}{cccccc}
0 & \longrightarrow & \pi_1(K(\mathbb{Q})^\wedge_p; \mathbb{Q}) & \longrightarrow & \pi_1(L_{K(1)}K(\mathbb{Q}^\wedge_p); \mathbb{Q}) & \xrightarrow{\nu_p} & \mathbb{Q}^\wedge_p & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1(K(\mathbb{Q})^\wedge_p; \mathbb{Q}) & \longrightarrow & \pi_1(L_{K(1)}K(\mathbb{Q})^\wedge_p; \mathbb{Q}) & \xrightarrow{\nu_p} & \mathbb{Q}^\wedge_p & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1(\mathbb{Q}(\mathbb{Q})^\wedge_p; \mathbb{Q}) & \longrightarrow & \pi_1(L_{K(1)}TC(\mathbb{Q})^\wedge_p; \mathbb{Q}) & \xrightarrow{\nu_p} & \mathbb{Q}^\wedge_p & \longrightarrow & 0.
\end{array}$$
By (8.1) and naturality, the diagram:

\[
\begin{array}{ccc}
\pi_1(\text{holim} K(L_n S; \mathbb{Q})) & \to & \pi_1(\text{holim} L_n K(S; \mathbb{Q})), \\
\pi_1((\text{holim} L_n K(S; \mathbb{Q}))_p^\wedge; \mathbb{Q}) & \to & \pi_1(L_1 K_1(\mathbb{Q}_p); \mathbb{Q}), \\
\pi_1(\text{holim} L_n K(S; \mathbb{Q})) & \to & \pi_1(L_1 K_1(\mathbb{Q}_p); \mathbb{Q})
\end{array}
\]

commutes, and we see that the composite \(\pi_1(\text{holim} L_n K(S; \mathbb{Q})) \to \mathbb{Q}_p^\wedge\) factors through \(\pi_1(L_1 K_1(\mathbb{Q}_p); \mathbb{Q})\), which then establishes the claim for the right vertical map. This establishes the claim for the right vertical map, which then establishes the claim for the left hand bottom horizontal map. For the last map, we look at the \(K(1)\)-localization of the Quillen localization sequence for \(Q\),

\[
L_{K(1)} \left( \bigvee_{\ell \text{ prime}} K(\mathbb{F}_\ell) \right) \to L_{K(1)} K(\mathbb{Q}) \to \Sigma \cdots.
\]

Except in degrees 0 and \(-1\), the homotopy groups of each \(L_{K(1)} K(\mathbb{F}_\ell)\) are finite and in odd degrees. For \(q > 0\), we have \(\pi_q(L_{K(1)} K(\mathbb{Q}); \mathbb{Q}) = 0\) for \(q \equiv 0, 2, 3 \pmod{4}\), while \(\pi_q(L_{K(1)} K(\mathbb{Q})) \cong \mathbb{Z}_p\) or \(\mathbb{Z}_2^\times \oplus \mathbb{Z}/2\) for \(q \equiv 1 \pmod{4}\). Since the \(K(1)\)-localization of the wedge is the \(p\)-completion of the wedge of the \(K(1)\)-localizations, the map

\[
L_{K(1)} \left( \bigvee_{\ell \text{ prime}} K(\mathbb{F}_\ell) \right) \to L_{K(1)} K(\mathbb{Q})
\]

must be zero on \(\pi_q(\mathbb{Q}; \mathbb{Q})\) for \(q \geq 1\), and by inspection, it is also zero for \(q = 0\). This proves the claim for the map \(L_{K(1)} K(\mathbb{Q}) \to L_{K(1)} K(\mathbb{Q})\) and completes the proof of the lemma in the case \(S = \mathbb{Q}\).

For the case \(S = \mathbb{Q}(p)\), we have that the map \(L_{K(1)} K(\mathbb{Q}(p)) \to L_{K(1)} K(\mathbb{Q})\) is a weak equivalence (since the fiber is \(L_{K(1)} K(\mathbb{Q}(p)) \simeq \ast\)). Thus, it suffices to see that the map

\[
(8.4) \quad \text{holim} L_n K(S(\mathbb{Q}(p)))_p^\wedge \to L_{K(1)} K(\mathbb{Q}(p))
\]
is injective on rational homotopy groups in non-negative degrees. For the purposes of this argument, denote the cofiber of the map (8.4) as Cof. By Theorem 2.5 and Proposition 2.7, Cof is weakly equivalent to the cofiber of the map

$$TC(S_p)_p^\wedge \to L_{K(1)} TC(Z(p)).$$

In the diagram

$$\begin{array}{ccc}
TC(S)^\wedge_p & \to & L_{K(1)} TC(Z) \\
\downarrow & & \downarrow \\
TC(S_p)_p^\wedge & \to & L_{K(1)} TC(Z(p))
\end{array}$$

the vertical maps are weak equivalences (for example, by [9, 2.5]), and so the horizontal maps have equivalent cofibers. Applying Theorem 2.5 and Proposition 2.7 again, we see that the cofiber on top is weakly equivalent to the cofiber of the map (8.5)

$$\text{holim}_n L_{K(S)}^\wedge_p \to L_{K(1)} K(Z).$$

Work of the previous paragraph now shows that this cofiber (and therefore also Cof) satisfies $\pi_q(\blank; \mathbb{Q}) = 0$ for $q > 1$. Thus, the map (8.4) is injective on $\pi_q(\blank; \mathbb{Q})$ for $q \geq 1$.

It remains to check that the map (8.4) is injective on $\pi_0(\blank; \mathbb{Q})$. In the work on the case $S = S$, we showed that the map (8.5) is injective on $\pi_0(\blank; \mathbb{Q})$ and it follows from the work in the previous paragraph that the map $L_{K(1)} K(Z) \to \text{Cof}$ is surjective on $\pi_1(\blank; \mathbb{Q})$. Since this map factors through the map $L_{K(1)} K(Z(p)) \to \text{Cof}$, that map is also surjective on $\pi_1(\blank; \mathbb{Q})$. We conclude that the map (8.4) is injective on $\pi_0(\blank; \mathbb{Q})$, which completes the proof of the lemma. □

REFERENCES

[1] D. W. Anderson. There are no phantom cohomology operations in $K$-theory. Pacific J. Math., 107(2):279–306, 1983.
[2] Gabriel Angelini-Knoll and J. D. Quigley. The Segal conjecture for topological Hochschild homology of Ravenel spectra. J. Homotopy Relat. Struct., 16(1):41–60, 2021.
[3] Gabriel J. Angelini-Knoll. Periodicity in Iterated Algebraic $K$-Theory of Finite Fields. ProQuest LLC, Ann Arbor, MI, 2017. Thesis (Ph.D.)–Wayne State University.
[4] Christian Ausoni and John Rognes. Algebraic $K$-theory of topological $K$-theory. Acta Math., 188(1):1–39, 2002.
[5] Tobias Barthel. Chromatic completion. Proc. Amer. Math. Soc., 144(5):2263–2274, 2016.
[6] Andrew J. Blumberg and Michael A. Mandell. The homotopy groups of the algebraic $K$-theory of the sphere spectrum. Geom. Topol., 23(1):101–134, 2019.
[7] Andrew J. Blumberg and Michael A. Mandell. The eigensplitting of the fiber of the cyclotomic trace for the sphere spectrum. Preprint, arXiv:2012.07951, 2020.
[8] Andrew J. Blumberg and Michael A. Mandell. $K$-theoretic Tate-Poitou duality and the fiber of the cyclotomic trace. Invent. Math., 221(2):397–419, 2020.
[9] Andrew J. Blumberg, Michael A. Mandell, and Allen Yuan. A version of Waldhausen’s chromatic convergence for $TC$. Preprint, arXiv:2106.00849, 2021.
[10] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic $K$-theory of spaces. Invent. Math., 111(3):465–539, 1993.
[11] Marcel Bökstedt. The rational homotopy type of $\Omega \text{Wh}^{\text{Diff}}(\ast)$. In Algebraic topology, Aarhus 1982 (Aarhus, 1982), volume 1051 of Lecture Notes in Math., pages 25–37. Springer, Berlin, 1984.
[12] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski. Ambidexterity and height. Adv. Math., 385:107763, 90, 2021.
[13] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. Descent and vanishing in chromatic algebraic $K$-theory via group actions. Preprint, arXiv:2011.08233, 2020.
[14] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. The local structure of algebraic K-theory, volume 18 of Algebra and Applications. Springer-Verlag London, Ltd., London, 2013.
[15] Bjørn Ian Dundas. Relative K-theory and topological cyclic homology. Acta Math., 179(2):223–242, 1997.
[16] W. G. Dwyer. Twisted homological stability for general linear groups. Ann. of Math. (2), 111(2):239–251, 1980.
[17] William G. Dwyer and Eric M. Friedlander. Algebraic and etale K-theory. Trans. Amer. Math. Soc., 292(1):247–280, 1985.
[18] Thomas G. Goodwillie. Relative algebraic K-theory and cyclic homology. Ann. of Math. (2), 124(2):347–402, 1986.
[19] Thomas G. Goodwillie. Notes on the cyclotomic trace. MSRI Lecture Notes, 1991.
[20] Jeremy Hahn and Dylan Wilson. Redshift and multiplication for truncated brown-peterson spectra. Preprint, arXiv:2012.00864, 2020.
[21] Lars Hesselholt and Ib Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. Topology, 36(1):29–101, 1997.
[22] Lars Hesselholt and Ib Madsen. On the K-theory of local fields. Ann. of Math. (2), 158(1):1–113, 2003.
[23] Markus Land, Akhil Mathew, Lennart Meier, and Georg Tamme. Purity in chromatically localized algebraic K-theory. Preprint, arXiv:2001.10425, 2020.
[24] Randy McCarthy. Relative algebraic K-theory and topological cyclic homology. Acta Math., 179(2):197–222, 1997.
[25] Haynes Miller. Finite localizations. Bol. Soc. Mat. Mexicana (2), 37(1-2):383–389, 1992. Papers in honor of José Adem (Spanish).
[26] S. A. Mitchell. The Morava K-theory of algebraic K-theory spectra. K-Theory, 3(6):607–626, 1990.
[27] Daniel Quillen. Finite generation of the groups $K^i$ of rings of integers in number fields. J. Amer. Math. Soc., 13(1):1–54, 2000. Appendix A by Manfred Kolster.
[28] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. Amer. J. Math., 106(2):351–414, 1984.
[29] John Rognes. Two-primary algebraic K-theory of rings of integers in number fields. J. Amer. Math. Soc., 13(1):1–54, 2000. Appendix A by Manfred Kolster.
[30] John Rognes. Algebraic K-theory of finitely presented spectra. Topology, 41(5):873–926, 2002.
[31] John Rognes. Chromatic redshift. Lecture notes. Available at https://www.mn.uio.no/math/personer/vit/rognes/papers/chresh.pdf, 2014.
[32] R. W. Thomason. The Lichtenbaum-Quillen conjecture for $K/L[3^{-1}]$. In Current trends in algebraic topology, Part 1 (London, Ont., 1981), volume 2 of CMS Conf. Proc., pages 117–139. Amer. Math. Soc., Providence, R.I., 1982.
[33] R. W. Thomason. Algebraic K-theory and étale cohomology. Ann. Sci. École Norm. Sup. (4), 18(3):437–552, 1985.
[34] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In The Grothendieck Festschrift, Vol. III (New Brunswick, N.J., 1983), volume 1126 of Lecture Notes in Math., pages 247–435. Birkhäuser Boston, Boston, MA, 1990.
[35] Torleif Veen. Detecting periodic elements in higher topological Hochschild homology. Geom. Topol., 22(2):693–756, 2018.
Department of Mathematics, Columbia University, New York, NY 10027
Email address: blumberg@math.columbia.edu

Department of Mathematics, Indiana University, Bloomington, IN 47405
Email address: mmandell@indiana.edu

Department of Mathematics, Columbia University, New York, NY 10027
Email address: yuan@math.columbia.edu