Affine Super Yangians and Rectangular $W$-superalgebras

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Abstract

Motivated by the AGT conjecture, we construct a homomorphism from the affine super Yangian $Y_{\epsilon_1, \epsilon_2}(\hat{\mathfrak{sl}}(m|n))$ to the universal enveloping algebra of the rectangular $W$-superalgebra $W_k(\mathfrak{gl}(m|n), (l^{(m|n)})$ for all $m \neq n$, $m, n \geq 2$ or $m \geq 3$, $n = 0$. Furthermore, we show that the image of this homomorphism is dense provided that $k + (m - n)(l - 1) \neq 0$.

1 Introduction

A $W$-algebra $W^k(\mathfrak{g}, f)$ is a vertex algebra associated with a finite dimensional reductive Lie algebra $\mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$. It appeared in the study of two dimensional conformal field theories ([40]) and has been studied by both physicists and mathematicians since 1980’s. When $\mathfrak{g}$ is $\mathfrak{sl}_2$ and $f$ is a nonzero nilpotent element of $\mathfrak{g}$, $W^k(\mathfrak{g}, f)$ is nothing but the Virasoro algebra. However, when $\mathfrak{g}$ is a general finite dimensional reductive Lie algebra, it is no longer a Lie algebra and a presentation by generators and relations is not known in general. On the other hand, this problem has been solved ([7], [9]) in the case for the finite $W$-algebras ([32]). A finite $W$-algebra $W_{\text{fin}}(\mathfrak{g}, f)$ is an associative algebra associated with a finite dimensional reductive Lie algebra $\mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$ and it can be regarded as a finite analogue of $W$-algebra $W^k(\mathfrak{g}, f)$ ([8], [2]). In [7], Brundan and Kleshchev resolved the problem by using a relationship between Yangians of type $A$ and finite $W$-algebras of type $A$.

The Yangian is a quantum group which is a deformation of the current algebra $\mathfrak{g} \otimes \mathbb{C}[z]$. Drinfeld ([10], [11]) introduced a Yangian associated with a finite dimensional simple Lie algebra $\mathfrak{g}$ in order to solve the Yang-Baxter equation. The Yangian of type $A$ has several presentations; the RTT presentation, the parabolic presentation, the Drinfeld presentation and the Drinfeld $J$ presentation. It was shown in [33] that there exist surjective homomorphisms from Yangians of type $A$ to finite rectangular $W$-algebras of type $A$. The homomorphism is given by the Drinfeld $J$ presentation. More generally, Brundan and Kleshchev ([7]) constructed a surjective homomorphism from a shifted Yangian, a subalgebra of the Yangian of type $A$, to an arbitrary finite $W$-algebra of type $A$ by using the parabolic presentation. Moreover, the defining relations of finite $W$-algebras of type $A$ have been written down explicitly as a quotient of shifted Yangians in [7].

Similar results are known in the super setting. We can define the $W$-superalgebras and finite $W$-superalgebras, which are attached with finite dimensional reductive Lie superalgebras $\mathfrak{g}$ and nilpotent elements of $\mathfrak{g}$ in the even parity. In the case of the Lie superalgebra $\mathfrak{sl}(m|n)$, the corresponding Yangian in the Drinfeld presentation was first introduced by Stukopin ([50], see also [17]). It is called the super Yangian. A relationship between super Yangians and finite $W$-superalgebras was constructed by Briot and Ragoucy ([6]) for the rectangular case and by Peng ([31]) for more general case.

It is natural to ask whether there exists a similar result in the affine setting. The definition of Yangian naturally extends to the case that $\mathfrak{g}$ is a Kac-Moody Lie algebra in the Drinfeld
Definition 2.1. Suppose that $W$ devoted to the proof of the fact that the associative superalgebra over $C$ the construction of $\Phi$. In Section 5, we construct an algebra homomorphism from the affine super $W$ to the universal enveloping algebras of the principal $W$-algebras of type $A$ and have proved the celebrated AGT conjecture (16, 5). Gaberdiel, Li, Peng and Zhang (15) defined the Yangian for the affine Lie superalgebra $\mathfrak{gl}(1|1)$ and obtained a result similar to [35] in the super setting.

In this article, we give a result similar to the one of [33] in the affine super setting. The corresponding Yangian is the affine super Yangian, which is the deformation of the universal enveloping algebra of the current algebra of $\mathfrak{sl}_{2|1}$ subject to the following defining relations:

$$\alpha = \frac{\alpha}{m - n}, \quad \varepsilon_2 = 1 - \frac{\alpha}{m - n}.$$ 

Then, there exists an algebra homomorphism

$$\Phi: Y_{\varepsilon_1,\varepsilon_2}(\mathfrak{sl}(m|n)) \to \mathcal{U}(\mathcal{W}(\mathfrak{gl}(ml|nl), (l^{(m|n)})), $$

where $\mathcal{U}(\mathcal{W}(\mathfrak{gl}(ml|nl), (l^{(m|n)})))$ is the universal enveloping algebra of $\mathcal{W}(\mathfrak{gl}(ml|nl), (l^{(m|n)}))$. Moreover, the image of $\Phi$ is dense in $\mathcal{U}(\mathcal{W}(\mathfrak{gl}(ml|nl), (l^{(m|n)})))$ provided that $\alpha \neq 0$.

By Theorem 1.1 provided that $\alpha \neq 0$, any irreducible representation of $\mathcal{W}(\mathfrak{gl}(ml|nl), (l^{(m|n)}))$ can be seen as an irreducible representation of $Y_{\varepsilon_1,\varepsilon_2}(\mathfrak{sl}(m|n))$. In the case that $l = 1$, the corresponding result was previously shown in [19], [25], [26], [24], [37] and [39].

We expect that the above result will be useful for studying the AGT correspondence for parabolic sheaves. Precisely speaking, Feigin-Finkelberg-Negut-Rybnyikov (12) constructed an action of the Guay’s affine Yangian on the equivariant cohomology for the affine Laumon spaces. Showing that the kernel of $\Phi$ acts trivially on the equivariant cohomology for the affine Laumon parabolic sheaves. Precisely speaking, Feigin-Finkelberg-Negut-Rybnyikov (12) constructed an algebra homomorphism from the affine super Yangian to the universal enveloping algebra of the $W$-superalgebras of type $A$. The appendix is devoted to the proof of the fact that $W_{i,j}^{(1)}$ and $W_{i,j}^{(2)}$ generate the rectangular $W$-superalgebra.

## 2 Affine Super Yangians

First, we recall the definition of the affine super Yangian (see 37 Definition 3.1). In this paper, we set $\{x, y\}$ as $xy + yx$. We also fix $m, n \in \mathbb{Z}_{\geq 0}$ and set the following notation:

$$p(i) = \begin{cases} 0 & (1 \leq i \leq m), \\ 1 & (m + 1 \leq i \leq m + n). \end{cases}$$

**Definition 2.1.** Suppose that $m, n \geq 2$ and $m \neq n$. The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\mathfrak{sl}(m|n))$ is the associative superalgebra over $\mathbb{C}$ generated by $x_{i,r}^+, x_{i,r}^-, h_{i,r} (i \in \{0, 1, \ldots, m + n - 1\}, r \in \mathbb{Z}_{\geq 0})$ with two parameters $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ subject to the following defining relations:

$$[h_{i,r}, h_{j,s}] = 0,$$ 

(2.2)
is finite, as we explain below. Thus, using (2.11) and (2.12),
\[
[x^+_i,r, x^-_j,s] = \delta_{ij} h_{i,r+s},
\]
and
\[
[h_{i,0}, x^+_j,r] = \pm a_{ij} x^+_j,r,
\]
\[
[h_{i,1}, x^+_j,r] = \pm a_{ij} x^+_j,r,
\]
\[
[h_{i,r+1}, x^+_j,s] - [h_{i,r}, x^+_j,s] = \pm a_{ij} \frac{\varepsilon_1 + \varepsilon_2}{2} \{ h_{i,r}, x^+_j,s \} - m_{ij} \frac{\varepsilon_1 - \varepsilon_2}{2} [h_{i,r}, x^+_j,s],
\]
\[
x^+_i,r+1 = x^+_i,r - m_{ij} x^+_j,s,
\]
\[
\sum_{w \in \mathbb{Z}^+} x^+_j,r \{ [x^+_i,r, x^{-}_{i,r}] x^+_j,r \ldots \} = 0 \quad (i \neq j),
\]
\[
[x^+_i,r, x^-_i,r] = 0 \quad (i = 0, m),
\]
\[
[[x^+_i,r, x^-_i,0], x^+_i,0] = 0 \quad (i = 0, m),
\]
where
\[
a_{ij} = \begin{cases} (-1)^{p(i)} + (-1)^{p(j+1)} & \text{if } i = j, \\
(-1)^{p(i+1)} & \text{if } j = i+1, \\
(-1)^{p(i+1)} & \text{if } j = i-1, \\
1 & \text{if } (i, j) = (0, m + n - 1), (m + n - 1, 0), \\
0 & \text{otherwise}, \end{cases}
\]
and the generators $x^+_{m,r}$ and $x^-_{0,r}$ are odd and all other generators are even and we define $x^-_{-1,0}$ as $x^-_{m+n-1,0}$.

Note that in Definition 2.4 the number of generators of the affine super Yangian is infinite. It is possible to give a presentation of the affine super Yangian such that the number of generators is finite, as we explain below.

First, we show that $\hat{\mathfrak{y}}_{\varepsilon_1, \varepsilon_2}(\mathfrak{sl}(m|n))$ is generated by $x^+_i,r, x^-_i,r, h_{i,r} \quad (i \in \{0, 1, \ldots, m+n-1 \}, r = 0, 1)$. Let $\hat{h}_{i,1}$ be $h_{i,1} - \varepsilon_1 + \varepsilon_2 \mu^2 h^2_{i,0}$. When $r = 0$, the relation (2.10) is equivalent to
\[
[h_{i,1}, x^+_j,s] = \pm a_{ij} \left( x^+_j,s - m_{ij} \varepsilon_1 - \varepsilon_2 x^+_j,s \right).
\]
By (2.10) and (2.13), we have the following two relations for all $r \geq 1$;
\[
x^+_i,r+1 = \pm \frac{1}{\alpha_{i,i}} \hat{h}_{i,i}, x^+_i,r, \quad h_{i,r+1} = [x^+_i,r+1, x^-_i,0] \quad (i \neq m, 0),
\]
\[
x^+_i,r+1 = \pm \frac{1}{\alpha_{i,i}} \hat{h}_{i,i}, x^+_i,r, + m_{i+1,j} \varepsilon_1 - \varepsilon_2 x^+_i,r, \quad h_{i,r+1} = [x^+_i,r+1, x^-_i,0] \quad (i = m, 0). \quad (2.12)
\]
Thus, using (2.11) and (2.12), \{ $h_{i,r}, x^+_i,r \mid i \in \{0, 1, \ldots, m+n-1\}, r \geq 2$ \} are generated inductively by \{ $h_{i,r}, x^+_i,r \mid i \in \{0, 1, \ldots, m+n-1\}, r = 0, 1$ \}. The following theorem describes the presentation of the affine super Yangian $\hat{\mathfrak{y}}_{\varepsilon_1, \varepsilon_2}(\mathfrak{sl}(m|n))$ whose generators are $x^+_i,r, x^-_i,r, h_{i,r} \quad (i \in \{0, 1, \ldots, m+n-1\}, r = 0, 1)$.

**Theorem 2.13** (Ueda [37], Theorem 3.13). Suppose that $m, n \geq 2$ and $m \neq n$. The affine super Yangian $\hat{\mathfrak{y}}_{\varepsilon_1, \varepsilon_2}(\mathfrak{sl}(m|n))$ is isomorphic to the associative superalgebra generated by $x^+_i,r, x^-_i,r, h_{i,r} \quad (i \in \{0, 1, \ldots, m+n-1\}, r = 0, 1)$ subject to the following defining relations:
\[
[h_{i,r}, h_{j,s}] = 0,
\]
\[
(x^+_i,r)^{n+1} = 0, \quad (x^-_i,r)^{n+1} = 0.
\]
where the generators $x_{m,n}^\pm$, and $x_{0,r}^\pm$ are odd and all other generators are even and we define $x_{-1,0}^\pm$ as $x_{m+n-1,0}^\pm$.

There exists another presentation of the affine super Yangian.

**Proposition 2.23.** Suppose that $m, n \geq 2$ and $m \neq n$. The affine super Yangian $Y_{\varphi_1, \varphi_2} (\hat{\mathfrak{sl}}(m|n))$ is isomorphic to the associative superalgebra generated by $X_{i,r}^\pm, X_{i,0}^\pm, H_{i,r} (i \in \{0, 1, \ldots, m+n-1\}, r = 0, 1)$ subject to the following defining relations:

\[
[H_{i,r}, H_{j,s}] = 0, \quad (2.24)
\]

\[
[X_{0,0}^+, X_{j,0}^-] = \delta_{ij} H_{i,0}, \quad (2.25)
\]

\[
[X_{i,1}^+, X_{j,0}^-] = \delta_{ij} H_{i,1} = [X_{i,0}^+, X_{j,1}^-], \quad (2.26)
\]

\[
[H_{i,0}, X_{i,r}^\pm] = \pm a_{ij} X_{j,r}^\pm, \quad (2.27)
\]

\[
[H_{i,1}, X_{j,0}^\pm] = \pm a_{ij} (X_{j,1}^\pm), \quad \text{if } (i, j) \neq (0, m+n-1), (m+n-1, 0), \quad (2.28)
\]

\[
[H_{0,1}, X_{m+n-1,0}^\pm] = \mp (1)^{p(m+n)} (X_{m+n-1,1}^\pm - (\varepsilon + \frac{m-n}{2}h)X_{m+n-1,0}^\pm), \quad (2.29)
\]

\[
[H_{m+n-1,1}, X_{0,0}^\pm] = \mp (1)^{p(m+n)} (X_{0,1}^\pm + (\varepsilon + \frac{m-n}{2}h)X_{0,0}^\pm), \quad (2.30)
\]

\[
[X_{i,1}^\pm, X_{j,0}^\pm] - [X_{i,0}^\pm, X_{j,1}^\pm] = \pm \frac{h}{2} (X_{0,0}^\pm, X_{m+n-1,0}^\pm) \quad \text{if } (i, j) \neq (0, m+n-1), (m+n-1, 0), \quad (2.31)
\]

\[
[X_{0,1}^\pm, X_{m+n-1}^\pm] - [X_{0,0}^\pm, X_{m+n-1,1}^\pm] = \pm \left(1^{p(m+n)} \frac{h}{2} (X_{0,0}^\pm, X_{m+n-1,0}^\pm) - (\varepsilon + \frac{m-n}{2}h)[X_{0,0}^\pm, X_{m+n-1,0}^\pm], \quad (2.32)
\]

\[
(ad X_{i,0}^\pm)^{1+|a_{ij}|} (X_{j,0}^\pm) = 0 \quad (i \neq j), \quad (2.33)
\]

\[
[X_{i,0}^\pm, X_{i,0}^\pm] = 0 \quad (i = 0, m), \quad (2.34)
\]

\[
[X_{i,0}^\pm, X_{i,0}^\pm] = 0 \quad (i = 0, m), \quad (2.35)
\]

where $h = \varepsilon_1 + \varepsilon_2$, $\tilde{H}_{i,1} = H_{i,1} - \frac{h}{2} H_{i,0}$, $\varepsilon = -(m-n)\varepsilon_2$, the generators $X_{m,r}^\pm$ and $X_{0,r}^\pm$ are odd and all other generators are even and we define $X_{-1,0}^\pm$ as $X_{m+n-1,0}^\pm$.

**Proof.** The homomorphism $\Psi$ from $Y_{\varphi_1, \varphi_2} (\hat{\mathfrak{sl}}(m|n))$ to the superalgebra defined in Proposition 2.23 is given by

\[
\Psi(h_{i,0}) = H_{i,0}, \quad \Psi(x_{i,0}^\pm) = X_{i,0}^\pm.
\]

\[
\Psi(h_{i,1}) = \begin{cases} 
H_{0,1} & \text{if } i = 0, \\
H_{i,1} - \frac{i - 2\delta(i > m)(i - m)}{2} (\varepsilon_1 - \varepsilon_2) H_{i,0} & \text{if } i \neq 0,
\end{cases}
\]
It is clear that $\Psi$ is an isomorphism.

By using Theorem 2.13 we can construct a non-trivial homomorphism from the affine super Yangian to the completion of the universal enveloping algebra of the affinization of $\mathfrak{gl}(m|n)$. The homomorphism is called as the evaluation map. In this paper, we deal with two different affinizations of $\mathfrak{gl}(m|n)$ corresponding to different cocycles. The first one is denoted by $\hat{\mathfrak{gl}}(m|n)^{\text{str}}$ and is defined as $\mathfrak{gl}(m|n) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}z$ whose commutator relations are given by

$$[x \otimes t^u, y \otimes t^v] = \begin{cases} [x, y] \otimes t^{u+v} + \delta_{u+v,0} \text{str}(xy)\hat{c} & \text{if } x, y \in \mathfrak{sl}(m|n), \\ [e_{a,b}, e_{i,i}] \otimes t^{u+v} + \delta_{u+v,0} \text{str}(E_{a,b}E_{i,i})\hat{c} + \delta_{u+v,0}\delta_{a,b}u(-1)^{p(a)+p(i)}z & \text{if } x = e_{a,b}, \ y = e_{i,i}, \end{cases}$$

where $E_{i,j} \in \mathfrak{gl}(m|n)$ is a matrix unit whose parity is $p(i) + p(j)$ and $\text{str}$ is a supertrace of $\mathfrak{gl}(m|n)$, that is, $\text{str}(E_{i,j}E_{k,l}) = \delta_{i,k}\delta_{j,l}(-1)^{p(i)}$. The other one is denoted by $\hat{\mathfrak{gl}}(m|n)^{\text{comp}}$ and is defined as $\mathfrak{gl}(m|n) \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}x$ whose commutator relations are

$$\hat{c} \text{ and } x \text{ are central elements of } \hat{\mathfrak{gl}}(m|n),$$

$$[u \otimes t^a, v \otimes t^b] = [u, v] \otimes t^{a+b} + \delta_{a+b,0} \text{str}(uv)\hat{c}, \text{ if } u \text{ or } v \in \mathfrak{sl}(m|n),$$

$$[E_{i,j} \otimes t^a, E_{j,i} \otimes t^b] = \delta_{a+b,0}(\text{str}(E_{i,j}E_{j,i})\hat{c} - \delta_{a+b,0}al(lc - 1)(-1)^{p(i)+p(j)}x.$$  

Next, we introduce a completion of $U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})/U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})(z-1)$ following [27] and [20]. For all $s \in \mathbb{Z}$, we denote $E_{i,j}^s t^s$ by $E_{i,j}(s)$. We also set the grading of $U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})/U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})(z-1)$ as $\deg(X(s)) = s$ and $\deg(c) = 0$. Then, $U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})/U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})(z-1)$ becomes a graded algebra and we denote the set of the degree $d$ elements of $U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})/U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})(z-1)$ by $U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})_d$. We obtain the completion

$$U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})^{\text{comp}} = \bigoplus_{d \in \mathbb{Z}} U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})_{d, \text{comp}}.$$

where

$$U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})_d = \lim_{N \to \infty} U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})_d/\sum_{r > N} U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})_{d-r, \text{comp}} U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})_r.$$

Now, we can define the evaluation map. Let us denote

$$\hat{h} = \varepsilon_1 + \varepsilon_2, \quad \delta(i \leq j) = \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{if } i > j, \end{cases}$$

$$h_i = \begin{cases} (-1)^{p(m+n)}E_{m+n,m+n} - E_{1,1} + \hat{c} & (i = 0), \\ (-1)^{p(i)}E_{ii} - (-1)^{p(i+1)}E_{i+1,i+1} & (1 \leq i \leq m + n - 1), \end{cases}$$

$$x_i^+ = \begin{cases} E_{m+n,1} \otimes t^i & (i = 0), \\ E_{i+1,i+1} & (\text{otherwise}), \end{cases} \quad x_i^- = \begin{cases} (-1)^{p(m+n)}E_{1,m+n} \otimes t^{-i} & (i = 0), \\ (-1)^{p(i)}E_{i+1,i} & (\text{otherwise}). \end{cases}$$

**Theorem 2.36** (Ueda [57], Proposition 5.2). Set $\hat{c} = \frac{(-m+n)\varepsilon_1}{\hat{h}}$. Then, there exists an algebra homomorphism

$$\text{ev}_{\varepsilon_1, \varepsilon_2} : \mathfrak{gl}(m|n) \to U(\hat{\mathfrak{gl}}(m|n)^{\text{str}})^{\text{comp}}$$
uniquely determined by

\[ \text{ev}_{\varepsilon_1, \varepsilon_2}(X^+_{i,0}) = x^+_i, \quad \text{ev}_{\varepsilon_1, \varepsilon_2}(X^-_{i,0}) = x^-_i, \quad \text{ev}_{\varepsilon_1, \varepsilon_2}(H_{i,0}) = h_i, \]

\[
\begin{cases}
\hbar\mathcal{C}h_0 - (-1)^{p(m+n)}\hbar E_{m+n, m+n}(E_{1,1} - \bar{c}) \\
-\hbar \sum_{s \geq 0} \sum_{k=1}^{m+n} (-1)^{p(k)} E_{m+n,k}(-s) E_{k,m+n}(s) \\
- h \sum_{s \geq 0} \sum_{k=1}^{m+n} (-1)^{p(k)} E_{1,k}(-s-1) E_{k,1}(s+1)
\end{cases}
\]

if \( i = 0, \)

\[
\text{ev}_{\varepsilon_1, \varepsilon_2}(H_{i,1}) = \left\{ \begin{array}{ll}
\hbar c x^+_0 + \hbar \sum_{s \geq 0} \sum_{k=1}^{m+n} (-1)^{p(k)} E_{m+n,k}(-s) E_{k,1}(s+1) & \\
- \frac{i - 2\delta(i \geq m+1)(i - m)}{2} \hbar x^+_i + \hbar \sum_{s \geq 0} \sum_{k=1}^{m+n} (-1)^{p(k)} E_{i,k}(-s) E_{k,i+1}(s) & \\
+ \hbar \sum_{s \geq 0} \sum_{k=1}^{m+n} (-1)^{p(k)} E_{i,k}(-s-1) E_{k,i+1}(s+1) & \\
\end{array} \right.
\]

if \( i \neq 0, \)

\[
\text{ev}_{\varepsilon_1, \varepsilon_2}(X^+_{i,1}) = \left\{ \begin{array}{ll}
\hbar c x^-_0 + (-1)^{p(m+n)}\hbar \sum_{s \geq 0} \sum_{k=1}^{m+n} (-1)^{p(k)} E_{1,k}(-s-1) E_{k,m+n}(s), & \\
- \frac{i - 2\delta(i \geq m+1)(i - m)}{2} \hbar x^-_i + (-1)^{p(i)}\hbar \sum_{s \geq 0} \sum_{k=1}^{m+n} (-1)^{p(k)} E_{i+1,k}(-s) E_{k,i}(s) & \\
+ (-1)^{p(i)}\hbar \sum_{s \geq 0} \sum_{k=1}^{m+n} (-1)^{p(k)} E_{i+1,k}(-s-1) E_{k,i}(s+1) & \\
\end{array} \right.
\]

if \( i \neq 0. \)

It was shown in [39] that the image of \( \text{ev}_{\varepsilon_1, \varepsilon_2} \) is dense in \( U(\tilde{\mathfrak{g}}(m|n)^{\text{str}})_{\text{comp}} \) in the case when \( \varepsilon_1 \neq 0. \)
Remark 2.37. In [19], the evaluation map was defined in terms of the generators $h_{i,r}$ and $x_{i,r}^\pm$ ($r = 0, 1$).

In the non-super case, the affine Yangian was defined in Definition 3.2 of [18] and Definition 2.3 of [19] as follows.

**Definition 2.38.** Suppose that $m \geq 3$ and set two $m \times m$-matrices $(a_{i,j})$ and $(m_{i,j})$ as

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ -1 & \text{if } (i,j) = (0,m-1),(m-1,0), \\ 0 & \text{otherwise}, \end{cases} \quad m_{i,j} = \begin{cases} 1 & \text{if } i = j - 1, \\ -1 & \text{if } i = j + 1, \\ 1 & \text{if } (i,j) = (0,m-1), \\ -1 & \text{if } (i,j) = (m-1,0), \\ 0 & \text{otherwise}. \end{cases}$$

The affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\hat{\mathfrak{g}}(m))$ is the associative algebra over $\mathbb{C}$ generated by $x_{i,r}^\pm, x_{i,r}^-, h_{i,r}$ ($i \in \{0, 1, \cdots, m-1\}, r \in \mathbb{Z}_{\geq 0}$) with parameters $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ subject to the defining relations (2.24), (2.27) and

$$[\tilde{H}_{i,1}, X_{j,0}^\pm] = \pm a_{ij} (X_{j,0}^\pm), \quad [\tilde{H}_{0,1}, X_{m-1,0}^\pm] = \mp \left( X_{m-1,1}^\pm - (\varepsilon + \frac{m}{2} \hbar) X_{m-1,0}^\pm \right),$$

$$[\tilde{H}_{m-1,1}, X_{0,0}^\pm] = \mp \left( X_{0,1}^\pm + (\varepsilon + \frac{m}{2} \hbar) X_{0,0}^\pm \right),$$

$$[X_{i,1}, X_{j,0}^\pm] - [X_{i,0}^\pm, X_{j,1}] = a_{ij} \frac{\hbar}{2} (X_{i,0}^\pm, X_{j,1}^\pm) \quad \text{if } (i,j) \neq (0,m-1), (m-1,0),$$

$$[X_{0,1}, X_{m-1,0}^\pm] - [X_{0,0}^\pm, X_{m-1,1}] = \pm \frac{\hbar}{2} (X_{0,0}^\pm, X_{m-1,1}^\pm) - (\varepsilon + \frac{m}{2} \hbar) [X_{0,0}^\pm, X_{m-1,1}^\pm].$$

where $\hbar = \varepsilon_1 + \varepsilon_2$, $\tilde{H}_{i,1} = H_{i,1} - \frac{\hbar}{2} H_{i,0}$, and $\varepsilon = -m \varepsilon_2$.

The evaluation map for the affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\hat{\mathfrak{g}}(m))$ was constructed in Section 6 of [19] and Theorem 3.8 of [25]. In fact, the evaluation map of [19] and [25] was defined in the same formula as that of Theorem 2.36 by setting $n = 0$ and assuming all of the parity is equal to zero. In the non-super case, the surjectivity of the evaluation map was shown in Theorem 4.18 of [24].

3 Generators of rectangular $W$-superalgebras of type $A$

We fix some notations for vertex algebras. For a vertex algebra $V$, we denote the generating field associated with $v \in V$ by $v(z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1}$. We also denote the OPE of $V$ by

$$u(z)v(w) \sim \sum_{s \geq 0} \frac{(u(s)v)(w)}{(z-w)^{s+1}},$$

for all $u, v \in V$. We denote the identity vector (resp. the translation operator) by $|0\rangle$ (resp. $\partial$).
First, we recall the definition of rectangular $W$-superalgebras of type $A$ (see [22], [23], and [1]). Let us set

$$\mathfrak{g} = \mathfrak{gl}(ml|nl) = \bigoplus_{1 \leq i,j \leq m+n, \ 0 \leq s,t \leq l} \mathbb{C} e_{(s-1)(m+n)+i,(t-1)(m+n)+j},$$

where $e_{(s-1)(m+n)+i,(t-1)(m+n)+j}$ is the unit matrix whose parity is $p(i) + p(j)$. Since $\mathfrak{gl}(ml|nl)$ is isomorphic to $\mathfrak{gl}(m|n) \otimes \mathfrak{gl}(l)$ as a graded vector space, we identify $e_{(s-1)(m+n)+i,(t-1)(m+n)+j} \in \mathfrak{gl}(ml|nl)$ with $e_{i,j} \otimes e_{s,t} \in \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(l)$. We set a parity of $e_{i,j} \in \mathfrak{gl}(m|n)$ as $p(i) + p(j)$. We take an even nilpotent element $f = \sum_{s=1}^{m+n} \sum_{i=1}^{l-1} e_{s(m+n)+i,(s-1)(m+n)+i} \in \mathfrak{gl}(ml|nl)$ and fix $k \in \mathbb{C}$. We also take $(\ | \ )$ as a supersymmetric invariant inner product of $\mathfrak{g}$ such that

$$(u|v) = \begin{cases} k \text{str}(uv) & \text{if } u \text{ or } v \text{ is an element of } \mathfrak{sl}(ml|nl), \\ k \text{str}(uv) + (-1)^{p(i) + p(j)}(1-c) & \text{if } u = e_{i,i} \otimes e_{r_1,r_1}, v = e_{j,j} \otimes e_{r_2,r_2}, \end{cases}$$

(3.1)

where $c$ is a complex number and str is a supertrace of $\mathfrak{gl}(ml|nl)$. We set

$$\mathfrak{g}_t = \bigoplus_{1 \leq i,j \leq m+n, \ 0 \leq s,t \leq l-1} \mathbb{C} e_{s(m+n)+i,(s+t)(m+n)+j},$$

and fix a $\mathfrak{sl}_2$-triple $(x, e, f)$ such that

$$\mathfrak{g}_t = \{ y \in \mathfrak{g} \mid [x, y] = ty \}.$$

Let us set

$$S = \{ (i,j,s,t) \mid 1 \leq i,j \leq m+n, \ 0 \leq s,s+t \leq l-1 \},$$

$$S_+ = \{ (i,j,s,t) \mid 1 \leq i,j \leq m+n, \ 0 \leq s,s+t \leq l-1, t \geq 1 \}.$$

For all $\beta = (i,j,s,t) \in S$, we also set $u_\beta$ as $e_{s(m+n)+i,(s+t)(m+n)+j}$ and $p(\beta)$ as the parity of $u_\beta$. Then, we have

$$\mathfrak{g} = \bigoplus_{\beta \in S} \mathbb{C} u_\beta, \quad \mathfrak{g}_{\geq 0} = \bigoplus_{t \geq 0} \mathfrak{g}_t = \bigoplus_{\beta \in S_+} \mathbb{C} u_\beta.$$

Moreover, let $\mathfrak{b}$ be $\bigoplus_{j \leq 0} \mathfrak{g}_j$, which is a subalgebra of $\mathfrak{g}$. We define $\kappa$ as an inner product of $\mathfrak{b}$ such that

$$\kappa(u, v) = (u|v) + \frac{1}{2}(\kappa_{\mathfrak{g}}(u, v) - \kappa_{\mathfrak{g}_0}(p_0(u), p_0(v)))$$

for all $u,v \in \mathfrak{b}$, where $p_0: \mathfrak{b} \to \mathfrak{g}_0$ is the projection map and $\kappa_{\mathfrak{g}}$ (resp. $\kappa_{\mathfrak{g}_0}$) is the Killing form on $\mathfrak{g}$ (resp. $\mathfrak{g}_0$). By the definition of $\kappa$, we have

$$\kappa(e_{x_1(m+n)+i_1,t_1(m+n)+j_1}, e_{x_2(m+n)+i_2,t_2(m+n)+j_2})$$

$$= \delta_{x_1,x_2} \delta_{t_1,t_2} \delta_{i_1,j_1} \delta_{i_2,j_2} (-1)^{p(i_1)}(k + (l - 1)(m - n))$$

$$- \delta_{x_1,x_2} \delta_{t_1,t_2} \delta_{i_1,j_1} \delta_{i_2,j_2} (-1)^{p(i_1) + p(i_2)}(c - \delta_{i_1,i_2}).$$

Let $\hat{\mathfrak{b}}$ be the Lie superalgebra $\mathfrak{b} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C} y$ whose commutator relations are

$$[at^u, bt^v] = [a, b]t^{u+v} + \delta_{u+v,0} \kappa(a, b)y,$$

$y$ is a central element.
We also set a left \( \mathfrak{b} \)-module \( V^\kappa(\mathfrak{b}) \) as \( U(\mathfrak{b})/U(\mathfrak{b}) (\mathfrak{b}[t] \oplus \mathbb{C}(y-1)) \cong U(\mathfrak{b}[t^{-1}] t^{-1}) \). Then, it has a vertex algebra structure whose identity vector is \( 1 \) and the gene rating field \( (ut^{-1}) \) is equal to \( \sum_{s \in \mathbb{Z}} (ut^s) z^{-s-1} \) for all \( u \in \mathfrak{b} \). We call \( V^\kappa(\mathfrak{b}) \) the universal affine vertex algebra associated with \( (\mathfrak{b}, \kappa) \).

In order to simplify the notation, we denote the generating field \( (ut^{-1})(z) \) as \( u(z) \). By the definition of \( V^\kappa(\mathfrak{b}) \), generating fields \( u(z) \) and \( v(z) \) satisfy the OPE

\[
u(z)v(w) \sim \frac{[u, v](w)}{z-w} + \frac{\kappa(u, v)}{(z-w)^2}
\]

for all \( u, v \in \mathfrak{b} \).

We set a Lie superalgebra \( a_{m,n} = \bigoplus_{u, v \in \mathbb{B}} \mathbb{C} J(u, v) \) with the following commutator relations:

\[
[j(u), j(v)] = J([u, v]), \quad [j(e_{ij}), \psi_{e_{ij}}] = \delta_{i,j,\kappa} - \delta_{i,t}(-1)^{p(e_{ij})} \psi_{e_{ij}}, \quad [\psi_u, \psi_v] = 0,
\]

where the parity of \( J(u) \) (resp. \( \psi_u \)) is equal to \( p(\beta) \) (resp. \( p(\beta) + 1 \)) and we denote \( \sum_{u, v \in \mathbb{B}} a_{\beta} J(u, v) \) (resp. \( \sum_{u, v \in \mathbb{B}} a_{\beta} \psi_u \)) by \( J(\sum_{u, v \in \mathbb{B}} a_{\beta} u)^{+} \) (resp. \( \psi(\sum_{u, v \in \mathbb{B}} a_{\beta} u) \) for all \( a_{\beta} \in \mathbb{C} \). We define an affinization of \( a_{m,n} \) by using the inner product on \( a_{m,n} \) such that

\[
k_{m,n}(J(u), J(v)) = \kappa(u, v), \quad k_{m,n}(J(u), \psi_v) = \kappa_{m,n}(\psi_u, \psi_v) = 0.
\]

By (3.2), \( V^{\kappa_{m,n}}(a_{m,n}) \) contains \( V^{\kappa}(\mathfrak{b}) \). We identify \( ut^{-1} \in V^{\kappa}(\mathfrak{b}) \) with \( J(ut^{-1})^{-1} \in V^{\kappa_{m,n}}(a_{m,n}) \).

For all \( u \in a_{m,n} \), let \( u[-s] \) be \( ut^{-s} \). In this section, we regard \( V^{\kappa_{m,n}}(a_{m,n}) \) (resp. \( V^{\kappa}(\mathfrak{b}) \)) as a non-associative superalgebra whose product \( \cdot \) is defined by

\[
u(-t) \cdot \nu[-s] = (u[-t])(-1)^{t-s}u[-s].
\]

We sometimes omit \( \cdot \) and denote \( \psi_{e_{i+j+i}}(m+n)_{+} \) by \( \psi_{e_{i+j+i}}(m+n)_{+} \) in order to simplify the notation. A rectangular \( W \)-superalgebra \( \mathcal{W}^k(\mathfrak{gl}(m|n), (l(m|n))) \) can be realized as the subalgebra of \( V^{\kappa_{m,n}}(a_{m,n}) \) \( (\mathfrak{22} \text{ and } \mathfrak{23}) \) as follows.

Let us set \( a = k + (l - 1)(m - n) \). We can define an odd differential \( d_0 \): \( V^{\kappa}(\mathfrak{b}) \to V^{\kappa_{m,n}}(a_{m,n}) \) determined by

\[
d_0 1 = 0, \quad [d_0, \partial] = 0, \quad [d_0, \partial] \]

\[
(3.3) \quad (3.4)
\]

\[
d_0 e_{s-1}(m+n)_{+} + \beta(t-1)(m+n)_+ i^{-1} = \sum_{t \leq 0 \leq s, \leq m+n} (-1)^{p(e_{i,j})} p(e_{i,j}) e_{s-1}(m+n)_{+} + \beta(t-1)(m+n)_+ i^{-1}
\]

\[
= \sum_{t \leq 0 \leq s, \leq m+n} (-1)^{p(e_{i,j})} p(e_{i,j}) e_{s-1}(m+n)_{+} + \beta(t-1)(m+n)_+ i^{-1} e_{s-1}(m+n)_{+} + \beta(t-1)(m+n)_+ i^{-1}
\]

\[
+ \delta(s < t)(-1)^{p(j)} \alpha \psi_{s-1}(m+n)_{+} + \beta(t-1)(m+n)_+ i^{-1} \psi_{s-1}(m+n)_{+} + \beta(t-2)(m+n)_+ i^{-1} \psi_{s-1}(m+n)_{+} + \beta(t-1)(m+n)_+ i^{-1}.
\]

\[
(3.5)
\]

\textbf{Definition 3.6} (Kac-Roan-Wakimoto [21], Theorem 2.4). The rectangular \( W \)-superalgebra associated with a Lie superalgebra \( \mathfrak{gl}(m|n) \) and a nilpotent element \( f = \sum_{s=1}^{l-1} \sum_{i=1}^{m+n} e_{s(m+n)_{+} + i, (s-1)(m+n)_{+} + i} \) is the vertex subalgebra defined by

\[
\mathcal{W}^k(\mathfrak{gl}(m|n), (l(m|n))) = \left\{ y \in V^{\kappa}(\mathfrak{b}) \subset V^{\kappa_{m,n}}(a_{m,n}) \mid d_0(y) = 0 \right\}.
\]
We denote the rectangular $W$-superalgebra associated with a Lie superalgebra $\mathfrak{gl}(m|n)$ and a nilpotent element $f$ by $W^k(\mathfrak{gl}(m|n), (l^{(m|n)})$. The rest of this section is devoted to the construction of two kinds of elements $W_{ij}^{(1)}$ and $W_{ij}^{(2)}$, which are generators of $W^k(\mathfrak{gl}(m|n), (l^{(m|n)}))$.

We regard $V^\kappa(\mathfrak{b}) \otimes \mathbb{C}[\tau]$ and $V^{\kappa,n}(\mathfrak{m}, \mathfrak{n}) \otimes \mathbb{C}[\tau]$ as non-associative superalgebras whose defining relations are given by

$$u[-t] \cdot v[-s] = (u[-t])(-1)v[-s], \quad [\tau, u[-s]] = su[-s],$$

where $\tau$ is an even element. Let $\overline{m,n} : V^{\kappa,n}(\mathfrak{m}, \mathfrak{n}) \otimes \mathbb{C}[\tau] \rightarrow V^{\kappa,n}(\mathfrak{m}, \mathfrak{n}) \otimes \mathbb{C}[\tau]$ be the odd differential determined by

$$\overline{m,n} u[-s] = [d_0, u[-s]], \quad \overline{m,n}, \tau = 0.$$

First, let us recall how to construct generators of the principal $W$-algebra $W^k(\mathfrak{gl}(l), (l^1))$ (\cite{3}, Section 2). We denote by $T(C)$ a non-associative free algebra associated with a vector space $C$ and by $\mathfrak{gl}(l) \leq \mathfrak{gl}(l) \leq_0$ the Lie algebra $\bigoplus_{1 \leq j \leq t} C_{e_{i,j}}$. In the principal case, $\mathfrak{b}$ is equal to $\mathfrak{gl}(l) \leq_0$. By Definition \cite{3}, the principal $W$-algebra can be defined as

$$W^k(\mathfrak{gl}(l), (l^1)) = \{ x \in V^\kappa(\mathfrak{gl}(l) \leq_0) \otimes \mathbb{C}[\tau] \mid d_0(x) = 0 \}.$$

Similarly, to $V^\kappa(\mathfrak{b}) \otimes \mathbb{C}[\tau]$, we define a non-associative algebra $T(\mathfrak{gl}(l) \leq_0[t^{1}][t^{-1}] \otimes \mathbb{C}[\tau]$. Let us set $\tau$ as $k + l - 1$ and an $l \times l$ matrix $B = (b_{i,j})_{1 \leq i,j \leq l}$ as

$$\begin{bmatrix}
\pi\tau + e_{1,1}[-1] & -1 & 0 & \cdots & 0 \\
e_{2,1}[-1] & \pi\tau + e_{2,2}[-1] & -1 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
e_{l-1,1}[-1] & e_{l-1,2}[-1] & \cdots & \pi\tau + e_{l-1,l-1}[-1] & -1 \\
e_{l,1}[-1] & e_{l,2}[-1] & \cdots & e_{l,l-1}[-1] & \pi\tau + e_{l,l}[-1]
\end{bmatrix}$$

(3.7)

whose entries are elements of $T(\mathfrak{gl}(l) \leq_0[t^{1}][t^{-1}] \otimes \mathbb{C}[\tau]$. For any matrix $A = (a_{i,j})_{1 \leq i,j \leq s}$, we define $cdet(A)$ as

$$\sum_{\sigma \in S_s} \text{sgn}(\sigma)a_{\sigma(1),1}(a_{\sigma(2),2}(a_{\sigma(3),3} \cdots a_{\sigma(s-1),s-1})a_{\sigma(s),s}) \in T(\mathfrak{gl}(l) \leq_0[t^{1}][t^{-1}] \otimes \mathbb{C}[\tau].$$

By the commutator relation of $T(\mathfrak{gl}(l) \leq_0[t^{1}][t^{-1}] \otimes \mathbb{C}[\tau]$, we can rewrite $cdet(B)$ as $\sum_{r=0}^{l} \tilde{W}^{(r)}(\pi\tau)^{l-r}$ such that $\tilde{W}^{(r)} \in T(\mathfrak{gl}(l) \leq_0[t^{1}][t^{-1}] \otimes \mathbb{C}[\tau]$. Let $p$ be the projection map from $T(\mathfrak{gl}(l) \leq_0[t^{1}][t^{-1}]$ to $V^\kappa(\mathfrak{gl}(l) \leq_0) = U(\mathfrak{gl}(l) \leq_0[t^{-1}] \otimes \mathbb{C}[\tau]$ and $W^{(r)}$ be $p(\tilde{W}^{(r)})$. Proving that $[\overline{m,n}, p(cdet(B))] = 0$, we obtain the following theorem (see Theorem 2.1 of \cite{3}).

**Theorem 3.8.** The $W$-superalgebra $W^k(\mathfrak{gl}(l), (l^1))$ is generated by $\{W^{(r)}\}_{1 \leq r \leq l}$. 

**Remark 3.9.** In \cite{3}, the tensor algebra $T(C)$ should have been defined as a non-associative superalgebra as above since $V(\mathfrak{g}) \leq_0$ is non-associative.

Let $A_{1,0}$ be a quotient algebra of $T(\mathfrak{gl}(l) \leq_0[t^{1}][t^{-1}] \otimes \mathbb{C}[\tau]$ subjected to the relation

$$(e_{a,1}[-1]|\psi_{i,a}[1])cdet(C^{l-i}) - e_{a,1}[-1]|\psi_{i,a}[1]cdet(C^{l-i}) = 0$$

for all $1 \leq a \leq l$, where $C^{l-i}$ is a submatrix of $B$ consisting of the last $(l-i)$ rows and columns. Constructing a homomorphism

$$D : T(\mathfrak{gl}(l) \leq_0[t^{1}][t^{-1}] \otimes \mathbb{C}[\tau] \rightarrow A_{1,0}$$
determined by
\[ D(e_{s,u}[-1]) = \sum_{u < a < s} e_{a,u}[-1] \psi_{s,a}[-1] - \sum_{u \leq a < s} \psi_{a,u}[-1] e_{s,a}[-1] + \delta(s < u) \pi \psi_{s,u}[-2] + \psi_{s,u+1}[-1] - \psi_{s-1,u}[-1], \]
we obtain the relation \( D(c\det(B)) = 0 \) in the way similar to the one of Theorem 2.1 of [3].

We regard \( \mathfrak{g}l(m|n) \) as an associative superalgebra whose product \( \cdot \) is determined by \( e_{i,j} \cdot e_{s,u} = \delta_{j,s} e_{i,u} \). Then, we obtain a non-associative superalgebra \( \mathfrak{g}l(m|n) \otimes V^\kappa(b) \otimes \mathbb{C}[\tau] \). We construct a homomorphism
\[ T : T(\mathfrak{g}l(l) \otimes [t^{-1}]) \otimes \mathbb{C}[\tau] \to \mathfrak{g}l(m|n) \otimes V^\kappa(b) \otimes \mathbb{C}[\tau] \]
determined by
\[ T_{i,j}(x) = (-1)^{p(i)} x \otimes e_{i,j} \in \mathfrak{g}l(l) \otimes [t^{-1}] \otimes \mathfrak{g}l(m|n) = b[t^{-1}] \otimes \mathfrak{g}l(m|n), \quad T(\tau) = \tau, \]
where \( T_{i,j}(x) \) is defined as \( e_{i,j} \otimes T_{i,j}(x) = T(x) \). Since \( T \) is a homomorphism, we obtain
\[ T_{i,j}(xy) = \sum_{r=1}^{m+n} (-1)^{p(e_{i,r})} p(e_{j,r}) T_{r,i}(x) T_{j,r}(y). \]

By the commutator relation of \( V^\kappa(b) \) and \( \mathbb{C}[\tau] \), \( W_{i,j}^{(r)} \in V^\kappa(b) \) is defined by
\[ T_{j,i}(c\det(B)) = \sum_{r=0}^{l} (-1)^{p(j)} W_{i,j}^{(r)} (\alpha \tau)^{l-r}, \quad (3.10) \]
where \( B \) is defined by replacing \( \pi \) in \([3,7]\) with \( \alpha \).

**Theorem 3.11.** For all \( m, n \geq 0 \) such that \( m \neq n \), the \( W \)-superalgebra \( W^k(\mathfrak{g}l(ml|nl), (l^{(m|n)})) \) is freely generated by \( \{W_{i,j}^{(r)} \mid 1 \leq r \leq l, 1 \leq i, j \leq m+n\} \).

**Remark 3.12.** In the case when \( n = 0 \), Theorem 3.11 is shown in Theorem 3.1 of [3].

**Proof.** Under the assumption that \( \pi \) is equal to \( \alpha \), we denote \( \tilde{A}_{1,0} \) (resp. \( \hat{D} \)) as \( \tilde{A}_{1,0} \) (resp. \( \hat{D} \)). We construct a homomorphism \( T^p : \tilde{A}_{1,0} \to \mathfrak{g}l(m|n) \otimes V^{\kappa,m-n}(a_{m,n}) \otimes \mathbb{C}[\tau] \) determined by
\[ T_{i,j}^p(e_{s,w}u) = (-1)^{p(j)} e_{s-1(m+n)+i, w-1(m+n)+j}[u], \quad T_{i,j}^p(\psi_{s,w}u) = \psi_{s-1(m+n)+i, w-1(m+n)+j}[u], \quad T^p(\tau) = \tau, \]
where \( T_{i,j}^p(x) \) is defined as \( e_{i,j} \otimes T_{i,j}^p(x) = T^p(x) \). Since \( T^p \) is a homomorphism, we obtain
\[ T_{i,j}^p(e_{s,w}[-1] \psi_{u,v}[-1]) = \sum_{r=1}^{m+n} (-1)^{p(e_{i,r})} p(e_{j,r}) T_{r,i}^p(e_{s,w}[-1]) T_{j,r}^p(\psi_{u,v}[-1]), \]
\[ T_{i,j}^p(\psi_{u,v}[-1] e_{s,w}[-1]) = \sum_{r=1}^{m+n} (-1)^{p(e_{i,r}) + p(e_{j,r})} T_{i,r}^p(\psi_{u,v}[-1]) T_{j,r}^p(e_{s,w}[-1]). \]

By the definition of \( T_{i,j} \) and \( d_0 \), we have
\[
\begin{align*}
& [a^m_{0,n}, T_{j,i}(e_{s,w})] \\
& = [a^m_{0,n}, (-1)^{p(j)} e_{s-1(m+n)+j, w-1(m+n)+i}[\cdot]] \\
& = \sum_{w < a \leq \cdot, s, 1 \leq r \leq m+n} (-1)^{p(i) + p(e_{i,r})} e_{a-1(m+n)+r, w-1(m+n)+i}[\cdot] \psi_{s-1(m+n)+j, a-1(m+n)+r}[\cdot].
\end{align*}
\]
\[-\sum_{w \leq a < s, 1 \leq r \leq m+n} (-1)^\gamma \psi(s-1)(m+n) + r, (w-1)(m+n) + i][-1] e(s-1)(m+n) + j, (a-1)(m+n) + r[-1]
+ \delta(s < w)\alpha \psi(s-1)(m+n) + j, (w-1)(m+n) + i[-2]
+ \psi(s(m+n) + j, (w-1)(m+n))[-1] - \psi(s-1)(m+n) + j, (w-2)(m+n) + i[-1]
= T_{p,j,i}^p([\bar{D}, e, s, w]), \] (3.13)

where \( \gamma = p(j) + p(e, i) p(e, r) \). Thus, the relation \( [\bar{d}^m, T_{j,i}(a)] = T_{p,j,i}^p([\bar{D}, a]) \) holds for all \( a \in T(\mathfrak{gl}(l) \leq 0) \). Then, we obtain
\[
[d_{a}, W_{i,j}^{(r)}] = 0. \] The rest of the proof is same as [3].

In particular, by (3.10), we have
\[
W_{i,j}^{(1)} = \sum_{1 \leq s \leq l} e(s-1)(m+n) + j, (s-1)(m+n) + i[-1], \] (3.15)
\[
W_{i,j}^{(2)} = \sum_{1 \leq s \leq l-1} e(s(m+n) + j, (s-1)(m+n))[-1] + \alpha \sum_{1 \leq s \leq l} (s-1) e(s-1)(m+n) + j, (s-1)(m+n) + i[-2]
+ \sum_{r_1 \leq r} (-1)^{p(r) + p(e, i) p(e, r)} e(r_1) e_{r_1} e_{r_1}[-1] e_{r_1}, \] (3.16)

where we set \( e_{r_j} \) as \( e(r_j) \).

**Theorem 3.17.** The rectangular \( W \)-superalgebra \( W^k(\mathfrak{gl}(m|n), (l|m|n)) \) is generated by \( W_{i,j}^{(1)} \) and \( W_{i,j}^{(2)} \) \( 1 \leq i, j \leq m + n \) provided that \( \alpha = k + (l-1)(m-n) \neq 0, m \neq n \) and \( m + n \geq 2 \).

Theorem 3.17 is proved in the appendix.

**Remark 3.18.** In the case when \( (m, n) = (1, 0) \) or \( (0, 1) \), the elements \( W_{i,i+1}^{(1)} \) or \( W_{i,i+1}^{(2)} \) do not exist. This is the reason why we need the condition that \( m + n \geq 2 \) in Theorem 3.17.

## 4 OPEs of rectangular \( W \)-superalgebras

First, let us recall the definition of the universal enveloping algebras of vertex algebras. For all vertex algebra \( V \), let \( L(V) \) be the Borchards Lie algebra, that is,
\[
L(V) = V \otimes \mathbb{C}[t, t^{-1}] / \text{Im}(\partial \otimes \text{id} + \text{id} \otimes \frac{d}{dt}), \] (4.1)

where the commutation relation is given by
\[
[u^a, v^b] = \sum_{r \geq 0} \frac{a}{r} (u_r v)t^{a+b-r},
\]
for all \( u, v \in V \) and \( a, b \in \mathbb{Z} \). Now, we define the universal enveloping algebra of \( V \).

**Definition 4.2** (Frenkel-Zhu [14], Matsuo-Nagatomo-Tsuchiya [27]). We set \( \mathcal{U}(V) \) as the quotient algebra of the standard degreewise completion of the universal enveloping algebra of \( L(V) \) by the completion of the two-sided ideal generated by
\[
(u(a)v)^b - \sum_{i \geq 0} \frac{a}{i} (-1)^{i} (u^{a-i}v^{b+i} - (-1)^{p(u)} p(v) (-1)^{a+b-i} u^{a-i}v^b), \] (4.3)
\[|0\rangle t^{-1} - 1, \quad (4.4)\]

where \(|0\rangle\) is the identity vector of \(V\). We call \(\mathcal{U}(V)\) the universal enveloping algebra of \(V\).

**Lemma 4.5** (Kac-Roan-Wakimoto [21], Theorem 2.4). There exists a homomorphism from the universal enveloping algebra of \(\mathfrak{g}(m|n)^c\) to \(\mathcal{U}(\mathfrak{g}(m|n), (l^{(m|n)})^c)\) determined by

\[\xi(E_{i,j} t^s) = W_{i,j}^{(1)} t^s, \quad \xi(\delta) = \lambda t^{-1}, \quad \xi(x) = 1.\]

In order to construct a homomorphism from the affine super Yangian to the universal enveloping algebra of \(W\)-superalgebras in Section 6, we need to compute the following terms:

\[(W_{i,j}^{(1)}(u) W_{s,t}^{(2)} (u \geq 0), \quad (W_{i,j}^{(2)}(0) W_{s,t}^{(2)}), \quad (W_{i,j}^{(2)}(1) W_{s,t}^{(2)}).\]

First, we compute \((W_{i,j}^{(1)}(u) W_{s,t}^{(2)} (u \geq 0)).\) By direct computation, we obtain the below two terms.

**Lemma 4.6.** We obtain

\[(W_{w,u}^{(1)}(0) W_{i,j}^{(2)} = \delta_{j,u} W_{i,j}^{(2)} - \delta_{i,v} (-1)^{p|\alpha|} p(\alpha_{i,j,v}) W_{w,u}^{(2)}.\]

**Lemma 4.7.** The following equations hold:

\[(W_{v,w}^{(1)}(1) W_{i,j}^{(2)} = \delta_{j,v}(l - 1)\alpha W_{i,w}^{(1)} - \delta_{v,w} (-1)^{p|\alpha|} (l - 1)(l - 1) W_{i,j}^{(1)},\]
\[(W_{v,w}^{(1)}(2) W_{i,j}^{(2)} = l(l - 1)\alpha W_{i,j}^{(2)},\]
\[(W_{v,w}^{(1)}(s) W_{i,j}^{(2)} = 0 \quad (\text{for all } s \geq 3).\]

**proof of the first equation of Lemma 4.7** By the definition of \(W_{v,w}^{(1)}\) and \(W_{i,j}^{(2)}\), we obtain

\[
(W_{v,w}^{(1)}(1) W_{i,j}^{(2)} = \sum_{1 \leq r_1 < r_2 \leq l} (-1)^{p(i) + p(\alpha_{i,j,v})} \kappa(e_{w,u}^{(s)}, e_{i,j}^{(r_1)} e_{j,t}^{(r_2)} - 1)

+ \sum_{1 \leq r_1 < r_2 \leq l} (-1)^{p(i) + p(\alpha_{i,j,v})} p(\alpha_{i,j,v}) \kappa(e_{w,u}^{(s)}, e_{j,t}^{(r_1)} e_{i,j}^{(r_2)} - 1)

+ \alpha \sum_{1 \leq s \leq l} (t - 1) [e_{w,u}^{(s)}, e_{j,t}^{(t)} - 1]. \quad (4.8)\]

Let us compute each terms of the right hand side of (4.8). Since we obtain

\[\kappa(e_{w,u}^{(s)}, e_{i,j}^{(r_1)}) = \delta_{t,s} \delta_{i,j} \delta_{v,t} (-1)^{p(i)} \alpha + (c - \delta_{t,s}) \delta_{i,j} (-1)^{p(w) + p(i)}\]

the first term of (4.8) is equal to

\[
\sum_{1 \leq r_1 < r_2 \leq l} (-1)^{p(\alpha_{i,j,v})} \delta_{i,t} \alpha e_{j,v}^{(r_2)} - 1

- \sum_{1 \leq r_1 < r_2 \leq l} (-1)^{p(w) + p(\alpha_{i,j,v})} (c - \delta_{t,s}) \delta_{i,j} e_{j,v}^{(r_1)} - 1. \quad (4.9)\]

By a direct computation, we obtain

the first term of (4.9).
Lemma 4.14. We obtain

\[ W^{(2)}_{i,j}(0)W^{(2)}_{j,i} = (-1)^{p(i)}(W^{(1)}_{i,j})(-1)W^{(2)}_{j,i} - (-1)^{p(j)}(W^{(1)}_{j,i})(-1)W^{(2)}_{i,j} - (\delta_{i,j} - (1)^{p(i)}\partial W^{(2)}_{i,j} + (-1)^{p(j)}(l - 1)\alpha W^{(1)}_{i,j} - \{l - 1\}^{2}c - (l - 1)\}W^{(1)}_{j,i} - (l - 1)\partial W^{(2)}_{i,j} + \frac{l(l - 1)}{2}\alpha^{2}\partial^{2}W^{(1)}_{i,j} + (-1)^{p(j)}(l - 1)\alpha^{2}\partial^{2}W^{(1)}_{j,i} - \frac{l(l - 1)}{2}\alpha^{2}\partial^{2}W^{(1)}_{j,i} + \frac{1}{2}(-1)^{p(i)}(l - 1)\alpha^{2}\partial^{2}W^{(1)}_{i,j} - \left\{l - 1\right\}^{2}\alpha^{2}\partial^{2}W^{(1)}_{i,j}\]

and

\[ W^{(2)}_{i,i}(1)W^{(2)}_{j,j} = (l - 1)^{2}c - (l - 1)\}W^{(1)}_{j,i} - (l - 1)^{2}c - (l - 1)\}W^{(1)}_{i,j} - 2\delta_{i,j}\alpha W^{(2)}_{i,j} - (-1)^{p(i)}W^{(2)}_{j,j}\]
In this section, we prove the main result of this paper. Hereafter, we assume that

\[ 5 \text{ Affine Super Yangians and Rectangular } W\text{-superalgebras} \]

In [34], Rapčák defined two kinds of elements of rectangular \( W \)-superalgebras of type \( A \), which are called \( U_{1,i,j} \) and \( U_{2,i,j} \) (\( 1 \leq i, j \leq m+n \)) under the assumption that \( c = 0 \). The element \( U_{r,i,j} \) is corresponding to \( (-1)^{p(i)p(j)} W_{i,j}^{(r)} \) \( (r = 1, 2) \), where \( J_{a,b}^{(2)} \) in [34] is corresponding to \( (-1)^{p(a)p(b)} c_{b,a} \) in this paper.

**Theorem 5.1.** There exists an algebra homomorphism

\[ \Phi: Y_{\varepsilon_1,\varepsilon_2}(\mathfrak{gl}(m|n)) \rightarrow \mathcal{U}(\mathcal{W}(\mathfrak{gl}(m|n), (l^{(m|n)}))) \]

determined by

\[
\Phi(H_{i,0}) = \begin{cases} 
((-1)^{p(m+n)} W_{m,n,m+n}^{(1)} - W_{1,1}^{(1)} + l\alpha) & (i = 0), \\
((-1)^{p(i)} W_{i,i}^{(1)} - (-1)^{p(i+1)} W_{i+1,i+1}^{(1)}) & (i \neq 0),
\end{cases}
\]

\[
\Phi(X_{i,0}^+) = \begin{cases} 
W_{1,m+n}^{(1)} & (i = 0), \\
W_{1+1}^{(1)} & (i \neq 0),
\end{cases}
\]

\[
\Phi(X_{i,0}^-) = \begin{cases} 
(-1)^{p(m+n)} W_{m,n+1}^{(1)} + 1 & (i = 0), \\
(-1)^{p(i)} W_{i,i+1}^{(1)} & (i \neq 0),
\end{cases}
\]

\[
\Phi(H_{i,1}) = \begin{cases} 
\frac{2^i}{i} (1 - (-1)^{p(i)} W_{i,i}^{(2)} t - (-1)^{p(i+1)} W_{i+1,i+1}^{(2)} t) & (i = 0), \\
-(1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{m+n} (-1)^p W_{u,i}^{(1)} t^{-s} W_{i,u}^{(1)} t^s & (i \neq 0),
\end{cases}
\]
We note that following lemma.

Proof. It is enough to show that $\Phi$ is compatible with the defining relations (2.24)-(2.35). By Lemma 5.3, we find that $\Phi$ is compatible with $\Phi(X_{i,0})$, $\Phi(X_{i,0})$, and $\Phi(X_{i,0})$. Thus, it is enough to show that $\Phi$ is compatible with $\Phi(X_{i,0})$ and $\Phi(X_{i,0})$. We divide the proof into two pieces, that is, Claim 5.10 and Claim 5.11 below. In Claim 5.10, we show that $\Phi$ is compatible with $\Phi(X_{i,0})$ and $\Phi(X_{i,0})$. In Claim 5.11, we prove that $\Phi$ is compatible with $\Phi(X_{i,0})$.

In order to prove Claims 5.10 and 5.11, we relate $\Phi$ with the evaluation map of the affine super Yangian. We set $\tilde{ev}(H_{i,s})$ and $\tilde{ev}(X_{i,s}^{\pm})$ ($s = 0, 1$) as

$$
\tilde{ev}(H_{i,0}) = \Phi(H_{i,0}), \quad \tilde{ev}(X_{i,0}^{\pm}) = \Phi(X_{i,0}^{\pm}),
$$

$$
\tilde{ev}(H_{i,1}) = \begin{cases} 
\Phi(H_{0,1}) - \{(-1)^{p(m+n)} W_{m+n+1}^{(2)} - W_{1,1}^{(2)} t + (-1)^{p(m+n)} (l - 1) a W_{m+n+1}^{(1)} \} & \text{if } i = 0, \\
\Phi(H_{i,1}) - \{(-1)^{p(i)} W_{i+1}^{(2)} t - (-1)^{p(i+1)} W_{i+1}^{(2)} t \} & \text{if } i \neq 0,
\end{cases}
$$

$$
\tilde{ev}(X_{i+1,0}^{\pm}) = \begin{cases} 
\Phi(X_{i+1,0}^{\pm}) - \{W_{i+1}^{(2)} t^2 + (l - 1) a W_{1,1}^{(1)} t \} & \text{if } i = 0, \\
\Phi(X_{i+1,0}^{\pm}) - W_{i+1}^{(2)} t & \text{if } i \neq 0,
\end{cases}
$$

$$
\tilde{ev}(X_{i+1,1}^{\pm}) = \begin{cases} 
\Phi(X_{i+1,1}^{\pm}) - (-1)^{p(m+n)} W_{m+n,1}^{(2)} & \text{if } i = 0, \\
\Phi(X_{i+1,1}^{\pm}) - (-1)^{p(i)} W_{i+1,1}^{(2)} t & \text{if } i \neq 0.
\end{cases}
$$

We note that $\tilde{gl}(m|n)^{ad}$ is the same as $\tilde{gl}(m|n)^{ad}$ except for the inner product on the diagonal part. By Lemma 5.5, we can prove that $\tilde{ev}$ is compatible with $\tilde{ev}(H_{i,s})$ which are parts of the defining relations of the affine super Yangian $Y_{m+n,1-1} \in m-n$ (gl(m|n)) in a way similar to the proof of the existence of the evaluation map (see Theorem 5.2 in [37]). This is summarized as the following lemma.
Lemma 5.2. Let us set
\[ \tilde{\varepsilon}_1 = \frac{l\alpha}{m-n}, \quad \tilde{\varepsilon}_2 = -1 - \frac{l\alpha}{m-n}. \]

Then, \(\tilde{ev}\) is compatible with (2.25)-(2.32) which are parts of the defining relations of the affine super Yangian \(Y_{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2}(\mathfrak{s}(m|n))\).

We remark that \(\tilde{ev}\) is not an algebra homomorphism since \([\tilde{ev}(H_{i,1}), \tilde{ev}(H_{j,1})]\) is not equal to zero. See (5.32) below for the details.

Claim 5.3. For all \(i, j \in \{0, 1, \cdots, m+n-1\}\), \(\Phi\) is compatible with (2.26)-(2.32).

Proof. We only show that \(\Phi\) is compatible with (2.29). The other cases are proven in a similar way. It is enough to show that
\[ [\Phi(H_{0,1}), \Phi(X^+_{m+n-1,0})] = -(-1)^{p(m+n)} \{\Phi(X^+_{m+n-1,1}) - (m-n + \alpha - \frac{m-n}{2}) W^{(1)}_{m+n,m+n-1}\}, \]
\[ [\Phi(H_{0,1}), \Phi(X^-_{m+n-1,0})] = (-1)^{p(m+n)} \{\Phi(X^-_{m+n-1,1}) - (-1)^{p(m+n-1)} (m-n + \alpha - \frac{m-n}{2}) W^{(1)}_{m+n-1,m+n}\}. \]

By the definition of \(\Phi\), we can rewrite the left hand side of (5.4) as
\[ [\Phi(H_{0,1}), \Phi(X^+_{m+n-1,0})] = -[W^{(1)}_{m+n,m+n-1}, (-1)^{p(m+n)} W^{(2)}_{m+n,m+n} t] + [W^{(1)}_{m+n,m+n-1}, W^{(2)}_{1,1} t] \]
\[ -[W^{(1)}_{m+n,m+n-1}, (-1)^{p(m+n)} (l-1)\alpha W^{(1)}_{m+n,m+n}] + [\tilde{ev}(H_{0,1}), \tilde{ev}(X^+_{m+n-1,0})]. \]

By Corollary 4.13 we obtain
\[ -[W^{(1)}_{m+n,m+n-1}, (-1)^{p(m+n)} W^{(2)}_{m+n,m+n} t] = -(-1)^{p(m+n)} W^{(2)}_{m+n,m+n-1} t, \]
\[ [W^{(1)}_{m+n,m+n-1}, W^{(2)}_{1,1} t] = 0. \]

By Corollary 4.13 we have
\[ -[W^{(1)}_{m+n,m+n-1}, (-1)^{p(m+n)} (l-1)\alpha W^{(1)}_{m+n,m+n}] = -(-1)^{p(m+n)} (l-1)\alpha W^{(1)}_{m+n,m+n-1}. \]

By Lemma 5.2 we also obtain
\[ [\tilde{ev}(H_{0,1}), \tilde{ev}(X^+_{m+n-1,0})] = -(-1)^{p(m+n)} \tilde{ev}(X^+_{m+n-1,1}) + (-1)^{p(m+n)} (m-n + l\alpha - \frac{m-n}{2}) W^{(1)}_{m+n,m+n}. \]

The identity (5.3) follows by applying (5.7)-(5.10) to (5.9). We can prove that \(\Phi\) is compatible with (2.30) in a similar way.

Similarly, by the definition of \(\Phi\), we obtain
\[ [\Phi(H_{0,1}), (-1)^{p(m+n-1)} W^{(1)}_{m+n-1,m+n}] = -[(-1)^{p(m+n-1)} W^{(1)}_{m+n-1,m+n}, (-1)^{p(m+n)} W^{(2)}_{m+n,m+n} t] \]
\[ -[(-1)^{p(m+n-1)} W^{(1)}_{m+n-1,m+n}, -W^{(2)}_{1,1} t] \]
\[ -[-(-1)^{p(m+n-1)} W^{(1)}_{m+n-1,m+n}, (-1)^{p(m+n)} (l-1)\alpha W^{(1)}_{m+n,m+n}] \]
\[ + [\tilde{ev}(H_{0,1}), \tilde{ev}(X^-_{m+n-1,0})]. \]

(5.11)
By Corollary 4.14, we obtain

$$-\left[(-1)^{p(m+n-1)}W_{m+n-1,m+n}^{(1)}(-1)^{p(m+n)}W_{m+n,m+n}^{(2)} \right] = (-1)^{p(m+n-1)+p(m+n)}W_{m+n-1,m+n}^{(2)},$$

(5.12)

$$-\left[(-1)^{p(m+n-1)}W_{m+n-1,m+n}^{(1)}W_{1,1}^{(2)} \right] = 0.$$  

(5.13)

By Lemma 4.5 we have

$$- \left[(-1)^{p(m+n-1)}W_{m+n-1,m+n}^{(1)}(-1)^{p(m+n)}(l-1)\alpha W_{m+n,m+n}^{(1)} \right] = (-1)^{p(m+n-1)+p(m+n)}(l-1)\alpha W_{m+n-1,m+n}^{(1)}.$$  

(5.14)

By Lemma 5.2, we obtain

$$[\tilde{e}(\tilde{H}_{0,1}), \tilde{e}(X_{m+n-1,0})] = (-1)^{p(m+n)}\tilde{e}(X_{m+n-1,1})$$

$$- (-1)^{p(m+n)} \left( m - n \right) + l\alpha \frac{m - n}{2} \right) \left( (-1)^{p(m+n-1)}W_{m+n-1,m+n}^{(1)} \right).$$

(5.15)

The identity (5.15) follows by applying (5.12)-(5.14) to (5.11). Thus, we have shown that $\Phi$ is compatible with (2.20). 

Finally, we prove that $\Phi$ is compatible with (2.24).

Claim 5.16. The following equation holds for all $i, j \in \{0, 1, \cdots, m + n - 1\}$, $r, s \in \{0, 1\}$:

$$[\Phi(H_{i,r}), \Phi(H_{j,s})] = 0.$$  

Proof. By Lemma 4.3, we obtain $[\Phi(H_{i,0}), \Phi(H_{j,0})] = 0$. In the similar way as that of Claim 5.3 we have $[\Phi(H_{i,0}), \Phi(H_{j,1})] = 0$. Thus, it is enough to show that $[\Phi(H_{i,1}), \Phi(H_{j,1})] = 0$. We only show the case when $i, j \neq 0$ and $i > j$. The other case is proven in a similar way. In order to simplify the notation, we set

$$X_{i} = -(-1)^{p(i)}\sum_{s \geq 0}^i \sum_{u=1}^i (-1)^{p(u)}W_{u,i}^{(1)}t^{-s}W_{i,u}^{(1)}t^s$$

$$- (-1)^{p(i)}\sum_{s \geq 0}^m \sum_{u=i+1}^n (-1)^{p(u)}W_{u,i}^{(1)}t^{-s-1}W_{i,u}^{(1)}t^{s+1}.$$  

By the definition of $\tilde{e}v$, we obtain

$$\tilde{e}v(H_{i,1}) = \frac{i - 2\delta(i \geq m + 1)(i - m)}{2}((-1)^{p(i)}W_{i,i}^{(1)} - (-1)^{p(i+1)}W_{i+1,i+1}^{(1)})$$

$$+ (-1)^{p(E_{i,i+1})}W_{i,i}^{(1)}W_{i+1,i+1}^{(1)} + X_{i} - X_{i+1} - (W_{i+1,i+1}^{(1)})^2$$

$$= X_{i} - X_{i+1} + \text{the term generated by} \{W_{i,i}^{(1)}t^l | 1 \leq i \leq m + n\}.$$  

(5.17)

By Lemma 4.3, Lemma 4.6 and (5.17), we obtain

$$[\tilde{e}v(H_{i,1}), \tilde{e}v(H_{j,1})] = [X_{i} - X_{i+1}, X_{j} - X_{j+1}],$$

(5.18)

$$\tilde{e}v(H_{i,1}), ((-1)^{p(j)}W_{j,j}^{(2)} - (-1)^{p(j+1)}W_{j+1,j+1}^{(2)}t]$$

$$= [X_{i} - X_{i+1}, ((-1)^{p(j)}W_{j,j}^{(2)} - (-1)^{p(j+1)}W_{j+1,j+1}^{(2)}t].$$

(5.19)

We remark that $[\tilde{e}v(H_{i,1}), \tilde{e}v(H_{j,1})]$ is not equal to zero since the inner products on the diagonal parts of $\mathfrak{gl}(m|n)^{\text{str}}$ and $\mathfrak{gl}(m|n)^{\text{str}}$ are different.
By (5.18), (5.19), and the definition of $\Phi$, we obtain

$$[\Phi(H_{i,1}), \Phi(H_{j,1})] = \left[\left((-1)^i W_{i,i}^{(2)} - (-1)^j W_{j,j}^{(2)}\right)_t, \left((-1)^i W_{i,j}^{(2)} - (-1)^j W_{j,i}^{(2)}\right)_t\right]$$

$$+ [X_i - X_{i+1}, \left((-1)^j W_{j,j}^{(2)} - (-1)^j W_{j,j}^{(2)}\right)_t]$$

$$+ [\left((-1)^i W_{i,i}^{(2)} - (-1)^i W_{i,i}^{(2)}\right)_t, X_j - X_{j-1}] + [X_i - X_{i+1}, X_j - X_{j-1}].$$

Thus, it is enough to show the relation

$$\left[(-1)^i W_{i,i}^{(2)} t, (-1)^j W_{j,j}^{(2)} t\right] + [X_i, (-1)^j W_{j,j}^{(2)} t] + [(-1)^i W_{i,i}^{(2)} t, X_j] + [X_i, X_j] = 0 \quad (5.20)$$

holds for all $i, j \in \{1, \cdots, m+n\}$, $i \geq j$. Let us compute each terms of the left hand side of (5.20). First, we compute the first term of the left hand side of (5.20). By Lemma 4.1.4 we obtain

$$(-1)^{i+j} [W_{i,i}^{(2)} t, W_{j,j}^{(2)} t]$$

$$= (-1)^{i+j} \left[\left(W_{i,i}^{(2)}\right)_t, \left(W_{j,j}^{(2)}\right)_t\right] + (-1)^{i+j} \left[\left(W_{i,i}^{(2)}\right)_t, \left(W_{j,j}^{(2)}\right)_t\right]$$

$$= (-1)^{i+j} \left[\left(-1\right) W_{i,i}^{(2)} t^2 - (-1)^i \left(W_{i,i}^{(2)}\right)_t^2 - \delta_{i,j} \alpha \partial W_{i,j}^{(2)} t^2\right]$$

$$- (-1)^{i+j} \left\{\left(l-1\right) c - \left(l-1\right) \left(W_{i,i}^{(2)}\right)_t^2\right\}$$

$$+ \delta_{i,j} \left(l\left(l-1\right) \alpha^2 \partial^2 W_{i,i}^{(1)} t^2 + (-1)^{i+j} \left[l\left(l-1\right) \alpha \partial W_{i,i}^{(1)} t\right]\right)$$

We can rewrite it as

$$(-1)^{i+j} W_{i,i}^{(2)} t + (-1)^{i+j} W_{j,j}^{(2)} t + (-1)^{i+j} \left(W_{i,i}^{(1)}\right)_t (-1)^{i+j} W_{i,j}^{(2)} t^2$$

$$+ (-1)^{i+j} \left(l-1\right) \alpha \left(W_{i,i}^{(1)}\right)_t + (-1)^{i+j} \left(l-1\right) \alpha \partial W_{i,i}^{(1)} t - (-1)^{i+j} \left(l-1\right)^2 \alpha^2 \partial W_{i,i}^{(1)}$$

since six relations

$$-\delta_{i,j} \alpha \partial W_{i,j}^{(2)} t^2 - 2\delta_{i,j} \alpha W_{j,j}^{(2)} t = 0,$$

$$(-1)^{i+j} W_{i,i}^{(2)} t - (-1)^{i+j} W_{j,j}^{(2)} t + (-1)^{i+j} \partial W_{i,j}^{(2)} t^2$$

$$= -((-1)^{i+j} W_{i,i}^{(2)} t - (-1)^{i+j} W_{j,j}^{(2)} t),$$

$$\delta_{i,j} \frac{l(l-1)}{2} \alpha^2 \partial^2 W_{i,i}^{(1)} t^2 + \delta_{i,j} l(l-1) \alpha^2 \partial W_{i,i}^{(1)} t = 0,$$

$$(-1)^{i+j} l\left(l-1\right) \alpha^2 \partial^2 W_{i,i}^{(1)} t^2 + (-1)^{i+j} l(l-1) \alpha \partial W_{i,i}^{(1)} t = 0,$$

$$-(-1)^{i+j} \frac{l(l-1)^2}{2} \alpha \partial W_{i,i}^{(1)} t^2 - (-1)^{i+j} l(l-1)^2 \alpha \partial W_{i,i}^{(1)} t = 0,$$

$$\frac{1}{2} (-1)^{i+j} (l-1) \alpha^2 \partial^2 W_{i,j}^{(1)} t^2 - \frac{1}{2} (-1)^{i+j} (l-1) \alpha^2 \partial^2 W_{i,i}^{(1)} t^2.$$
hold by the definition of the translation operator \( \partial \).

In order to rewrite (5.21), we remark that the following two relations

\[
(x(-1)y)_{t} = \sum_{s \geq 0} xt^{-s} yt^{s+1} + (-1)^{p(y)} yt^{-s} xt^{s},
\]

\[
(x(-1)\partial y)_{t}^{2} = \sum_{s \geq 0} xt^{-s} (\partial y)_{t} t^{s+2} + (-1)^{p(y)} \partial y_{t} t^{-s} xt^{s}
\]

\[
= \sum_{s \geq 0} (-s+2) xt^{-s} yt^{s+1} - (-1)^{p(y)} (1-s) yt^{-s} xt^{s}
\]

hold by (5.28) for all \( x, y \in W^{k} (\mathfrak{gl}(m|n), (t^{(m|n)}) \). By (5.28) and (5.29), we also obtain

\[
(x(-1)\partial y)_{t}^{2} + (x(-1)y)_{t} = \sum_{s \geq 0} (-1+s) xt^{-s} yt^{s+1} + (-1)^{p(y)} syt^{-s} xt^{s}.
\]

By (5.28) - (5.30), we can rewrite (5.21) as

\[
- (-1)^{p(i)} W_{i,j}^{(2)} t + (-1)^{p(j)} W_{j,i}^{(2)} t
+ (-1)^{p(j)} \sum_{s \geq 0} W_{i,j}^{(1)} t^{s-1} W_{j,i}^{(2)} t^{s+2} + (-1)^{p(i)} \sum_{s \geq 0} W_{j,i}^{(2)} t^{s-1} W_{i,j}^{(1)} t^{s}
- (-1)^{p(i)} \sum_{s \geq 0} W_{i,j}^{(1)} t^{s-1} W_{j,i}^{(2)} t^{s+2} - (-1)^{p(j)} \sum_{s \geq 0} W_{j,i}^{(2)} t^{s-1} W_{i,j}^{(1)} t^{s}
+ (-1)^{p(j)} (l-1) \alpha \sum_{s \geq 0} s W_{i,j}^{(1)} t^{-s} W_{j,i}^{(1)} t^{s} - (-1)^{p(i)} (l-1) \alpha \sum_{s \geq 0} s W_{j,i}^{(1)} t^{-s} W_{i,j}^{(1)} t^{s}
- (-1)^{p(i)+p(j)} (l-1)^{2} c - (l-1) \sum_{s \geq 0} (-s) W_{j,i}^{(1)} t^{-s} W_{i,j}^{(1)} t^{s} + s W_{j,i}^{(1)} t^{-s} W_{i,j}^{(1)} t^{s}.
\]

Next, let us compute the last term of (5.21). By a computation similar to the proof of the existence of the evaluation map (see Theorem 5.2 of [37]), it is equal to

\[
[X_{1}, X_{j}] = -(-1)^{p(i)+p(j)} (l-1)^{2} c - (l-1) \sum_{s \geq 0} s \{ W_{i,j}^{(1)} t^{-s} W_{j,i}^{(1)} t^{s} - W_{j,i}^{(1)} t^{-s} W_{i,j}^{(1)} t^{s} \}
- (-1)^{p(i)+p(j)} \sum_{s \geq 0} s \{ W_{i,j}^{(1)} t^{-s} W_{j,i}^{(1)} t^{s} - W_{j,i}^{(1)} t^{-s} W_{i,j}^{(1)} t^{s} \}.
\]

Finally, let us compute the second term and the third term of (5.21). By the definition of \( X_{1} \), we obtain

\[
[X_{1}, W_{j,j}^{(2)} t]
= -(-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{i} (-1)^{p(u)} W_{u,i}^{(1)} t^{-s} W_{i,u}^{(1)} t^{s} W_{j,j}^{(2)} t]
- (-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{i} (-1)^{p(u)} W_{j,j}^{(1)} t^{-s} W_{i,u}^{(2)} t W_{i,u}^{(1)} t^{s}
- (-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{i} (-1)^{p(u)} W_{u,i}^{(1)} t^{-s} W_{j,j}^{(1)} t^{s+1} W_{i,u}^{(2)} t]
- (-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{i} (-1)^{p(u)} W_{u,i}^{(1)} t^{-s} W_{i,u}^{(2)} t W_{i,u}^{(1)} t^{s+1}.
\]
By Corollary 4.13, the first term of the right hand side of (5.33) is equal to

\[ -(-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{i} (-1)^{p(u)} W_{u,i}^{(1)} t^{-s} [W_{i,u}^{(1)}, W_{j,j}^{(2)}] \]

\[ = -\delta(i,j)(-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{i} (-1)^{p(u)} W_{u,i}^{(1)} t^{-s} W_{i,j}^{(2)} W_{j,j}^{(1)} t^{s+1} + \delta(i \geq j)(-1)^{p(i) + p(j)} \sum_{s \geq 0} W_{j,i}^{(1)} t^{-s} W_{i,j}^{(2)} W_{j,j}^{(1)} t^{s+1} \]

\[ - \delta(i,j)(-1)^{p(i)} (l-1) \alpha \sum_{s \geq 0} (1)^{p(u)} sW_{u,i}^{(1)} t^{-s} W_{i,j}^{(1)} t^{s} \]

\[ + (-1)^{p(i)} (l-1) \alpha \sum_{s \geq 0} sW_{i,i}^{(1)} t^{-s} W_{i,j}^{(1)} t^{s} \]

(5.34)

since by (4.1) \( \kappa(e_{i,u}, e_{j,j}) t^{s-1} \) is equal to zero unless \( s = 0 \). Similarly to (5.34), we rewrite the second, third, and 4-th terms of the right hand side of (5.33). By Corollary 4.13, the second term of the right hand side of (5.33) is equal to

\[ -(-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{i} (-1)^{p(u)} [W_{u,i}^{(1)} t^{-s}, W_{j,j}^{(2)}] W_{i,u}^{(1)} t^{s} \]

\[ = -\delta(i \geq j)(-1)^{p(i) + p(j)} \sum_{s \geq 0} W_{j,i}^{(1)} t^{-s} W_{i,j}^{(2)} t^{s} + \delta(i,j)(-1)^{p(i)} \sum_{s \geq 0} (-1)^{p(u)} W_{u,i}^{(1)} t^{-1} sW_{i,j}^{(1)} t^{s} \]

\[ + \delta(i < j)(-1)^{p(i) + p(j)} (l-1) \alpha \sum_{s \geq 0} sW_{j,i}^{(1)} t^{-s} W_{i,j}^{(1)} t^{s} \]

\[ - (-1)^{p(i)} (l-1) (l-1) \alpha \sum_{s \geq 0} sW_{i,j}^{(1)} t^{-s} W_{i,j}^{(1)} t^{s} \]  

(5.35)

By Corollary 4.13, the third term of the right hand side of (5.33) is equal to

\[ -(-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{m+n} (-1)^{p(u)} W_{u,i}^{(1)} t^{-s-1} W_{i,u}^{(1)} t^{s+1}, W_{j,j}^{(2)} t] \]

\[ = -\delta_{i,j}(-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{m+n} (-1)^{p(u)} W_{u,i}^{(1)} t^{-s-1} W_{i,j}^{(2)} t^{s+2} \]

\[ + \delta(i < j)(-1)^{p(i) + p(j)} \sum_{s \geq 0} W_{j,i}^{(1)} t^{-s-1} W_{i,j}^{(2)} t^{s+2} \]

\[ - \delta_{i,j}(-1)^{p(i)} (l-1) \alpha \sum_{s \geq 0} \sum_{u=1}^{m+n} (s+1)(-1)^{p(u)} W_{u,i}^{(1)} t^{-s-1} W_{i,j}^{(1)} t^{s+1} \]

(5.36)

By Corollary 4.13, the 4-th term of the right hand side of (5.33) is equal to

\[ -(-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{m+n} (-1)^{p(u)} [W_{u,i}^{(1)} t^{-s-1}, W_{j,j}^{(2)}] W_{i,j}^{(1)} t^{s+1} \]

\[ = -\delta(i < j)(-1)^{p(i) + p(j)} \sum_{s \geq 0} W_{j,i}^{(2)} t^{-s} W_{i,j}^{(1)} t^{s+1} \]

\[ + \delta_{i,j}(-1)^{p(i)} \sum_{s \geq 0} \sum_{u=1}^{m+n} (-1)^{p(u)} W_{u,i}^{(2)} t^{-s} W_{i,j}^{(1)} t^{s+1} \]

\[ + \delta(i < j)(-1)^{p(i) + p(j)} (l-1) \alpha \sum_{s \geq 0} (s+1) W_{j,i}^{(1)} t^{-s-1} W_{i,j}^{(1)} t^{s+1} \]

(5.37)
We prepare some notations. We denote the $i$-th term of the right hand side of (5.34) (resp. (5.35), (5.36), (5.37)) by (5.34)$_i$ (resp. (5.35)$_i$, (5.36)$_i$, (5.37)$_i$). Let us set

$$A_{i,j} = (1)^{(p)}(5.34)_1 + (1)^{(p)}(5.34)_2 + (1)^{(p)}(5.35)_1 + (1)^{(p)}(5.37)_2$$

$$= -\delta_{ij} \sum_{s\geq 0} \sum_{u=1}^{m+n} (-1)^{p(u)} W_{u,i} W_{i,u} t^{s-1} W_{i,j} t^{s+1} + \delta_{ij} \sum_{s\geq 0} \sum_{u=1}^{m+n} (-1)^{p(u)} W_{u,i} t^{s-1} W_{i,j} t^{s+1},$$

$$B_{i,j} = (1)^{(p)}(5.34)_2 + (1)^{(p)}(5.35)_1 + (1)^{(p)}(5.36)_3 + (1)^{(p)}(5.37)_1$$

$$= \delta(i \geq j)(-1)^{(p)} \sum_{s\geq 0} W_{j,i} W_{i,j} t^{s-1} W_{i,j} t^{s+1} - \delta(i < j)(-1)^{(p)} \sum_{s\geq 0} W_{j,i} W_{i,j} t^{s-1} W_{i,j} t^{s+1},$$

$$C_{i,j} = (1)^{(p)}(5.35)_3 + (1)^{(p)}(5.37)_3$$

$$= \delta(i \geq j)(-1)^{(p)} (l-1) \alpha \sum_{s\geq 0} W_{j,i} W_{i,j} t^{s},$$

$$D_{i,j} = (1)^{(p)}(5.36)_3 + (1)^{(p)}(5.38)_3$$

$$= -\delta_{ij} (l-1) \alpha \sum_{s\geq 0} \sum_{u=1}^{m+n} s(-1)^{p(u)} W_{u,i} W_{i,j} t^{s},$$

Then, we can rewrite $[X_i, (1)^{(p)} W_{j,i} t]$ as $A_{i,j} + B_{i,j} + C_{i,j} + D_{i,j} + \tilde{E}_{i,j}$. By exchanging $i$ and $j$, we find that $[X_j, (1)^{(p)} W_{i,j} t]$ is equal to $A_{j,i} + B_{j,i} + C_{j,i} + D_{j,i} + \tilde{E}_{j,i}$. We find that the left hand side of (5.20) is equal to

$$(5.31) + A_{i,j} + B_{i,j} + C_{i,j} + D_{i,j} + \tilde{E}_{i,j} - (A_{j,i} + B_{j,i} + C_{j,i} + D_{j,i} + \tilde{E}_{j,i}) + (5.32).$$

By the definition of $A_{i,j}$ and $D_{i,j}$, we have

$$A_{i,j} - A_{j,i} = 0, \quad D_{i,j} - D_{j,i} = 0. \quad (5.38)$$

By direct computation, we obtain

$$C_{i,j} = C_{j,i} + (5.31)_7 + (5.31)_8 = 0, \quad (5.39)$$

where we denote the $i$-th term of (5.31) by (5.31)$_i$. Hence, by (5.38), it is enough to obtain the following two relations

$$B_{i,j} - B_{j,i} = (5.31)_1 + (5.31)_2 + (5.31)_3 + (5.31)_4 + (5.31)_5 + (5.31)_6 = 0, \quad (5.40)$$
\[
\hat{E}_{i,j} - \hat{E}_{j,i} + b_{13} b_{13} + b_{23} = 0. 
\] 

(5.41)

First, we show that (5.40) holds. Let us compute \( B_{i,j} - B_{j,i} \). When \( i = j \), it is equal to zero and (5.40) holds. Suppose that \( i > j \). Then, we can rewrite \( B_{i,j} - B_{j,i} \) as

\[
(-1)^{p(i)} \sum_{s \geq 0} W_{j,i}^{(1)} t^{-s} W_{j,i}^{(2)} t_s + 1 - (-1)^{p(i)} \sum_{s \geq 0} W_{i,j}^{(2)} t^{1-s} W_{i,j}^{(1)} t_s \\
- (-1)^{p(j)} \sum_{s \geq 0} W_{i,j}^{(1)} t^{-s} W_{i,j}^{(2)} t_{s+2} + (-1)^{p(j)} \sum_{s \geq 0} W_{i,j}^{(2)} t^{-s} W_{i,j}^{(1)} t_{s+1}. 
\] 

(5.42)

By Corollary 4.13, we obtain

\[
(-1)^{p(i)} \sum_{s \geq 0} W_{j,i}^{(1)} t^{-s} W_{j,i}^{(2)} t_{s+1} + (-1)^{p(j)} \sum_{s \geq 0} W_{i,j}^{(2)} t^{1-s} W_{i,j}^{(1)} t_{s+1} \\
= (-1)^{p(i)} \sum_{s \geq 0} W_{j,i}^{(1)} t^{-s-1} W_{j,i}^{(2)} t_{s+2} + (-1)^{p(j)} \sum_{s \geq 0} W_{i,j}^{(2)} t^{1-s} W_{i,j}^{(1)} t_s \\
+ (-1)^{p(i)} W_{j,i}^{(2)} t - (-1)^{p(j)} W_{j,j}^{(2)} t.
\] 

(5.43)

Applying (5.43) to (5.42), we obtain

\[
B_{i,j} - B_{j,i} \\
= (-1)^{p(i)} \sum_{s \geq 0} W_{j,i}^{(1)} t^{-s-1} W_{j,i}^{(2)} t_{s+2} - (-1)^{p(i)} \sum_{s \geq 0} W_{i,j}^{(2)} t^{1-s} W_{i,j}^{(1)} t_s \\
- (-1)^{p(j)} \sum_{s \geq 0} W_{i,j}^{(1)} t^{-s-1} W_{i,j}^{(2)} t_{s+2} + (-1)^{p(j)} \sum_{s \geq 0} W_{i,j}^{(2)} t^{1-s} W_{i,j}^{(1)} t_s \\
+ (-1)^{p(i)} W_{j,i}^{(2)} t - (-1)^{p(j)} W_{j,j}^{(2)} t.
\]

We have shown that (5.40) holds.

Finally, let us compute the left hand side of (5.41). By direct computation, we obtain

\[
\hat{E}_{i,j} - \hat{E}_{j,i} \\
= 2(-1)^{p(i)+p(j)} (l - 1)(lc - 1) \sum_{s \geq 0} s W_{i,j}^{(1)} t^{-s} W_{i,j}^{(1)} t_s \\
- 2(-1)^{p(i)+p(j)} (l - 1)(lc - 1) \sum_{s \geq 0} s W_{j,j}^{(1)} t^{-s} W_{i,j}^{(1)} t_s.
\]

It follows that the left hand side of (5.41) is equal to

\[
2(-1)^{p(i)+p(j)} (l - 1)(lc - 1) \sum_{s \geq 0} s W_{i,j}^{(1)} t^{-s} W_{i,j}^{(1)} t_s \\
- 2(-1)^{p(i)+p(j)} (l - 1)(lc - 1) \sum_{s \geq 0} s W_{j,j}^{(1)} t^{-s} W_{i,j}^{(1)} t_s \\
- (-1)^{p(i)+p(j)} ((l - 1)^2 c - (l - 1)) \sum_{s \geq 0} \{ -W_{j,j}^{(1)} t^{-s} W_{i,i}^{(1)} t_s + s W_{i,i}^{(1)} t^{-s} W_{i,j}^{(1)} t_s \} \\
- (-1)^{p(i)+p(j)} l(lc - 1) \sum_{s \geq 0} \{ W_{i,i}^{(1)} t^{-s} W_{j,j}^{(1)} t_s - W_{i,j}^{(1)} t^{-s} W_{i,i}^{(1)} t_s \} \\
- (-1)^{p(i)+p(j)} \sum_{s \geq 0} \{ W_{j,j}^{(1)} t^{-s} W_{i,i}^{(1)} t_s - W_{j,j}^{(1)} t^{-s} W_{i,j}^{(1)} t_s \} \\
= -(-1)^{p(i)+p(j)} c \sum_{s \geq 0} \{ -W_{j,j}^{(1)} t^{-s} W_{i,i}^{(1)} t_s + s W_{i,i}^{(1)} t^{-s} W_{j,j}^{(1)} t_s \}.
\]

Since \( c = 0 \), this is equal to zero. Thus, (5.41) holds. We have shown that \([H,H \cdot 1,1,1,1]) = 0. \]
Since we have proved Claim 5.3 and Claim 5.16, we have proven that $\Phi$ is compatible with the defining relations of the affine super Yangian.

Next, let us show that $\Phi$ is essentially surjective when $\alpha \neq 0$.

**Theorem 5.44.** The image of $\Phi$ is dense in $U(W^\prime(\mathfrak{gl}(m|n)), (t^{(m)}))$ provided that $\alpha$ is nonzero.

**Proof.** Suppose that $\alpha \neq 0$. By Theorem 3.17, it is enough to show that the completion of the image of $\Phi$ contains $W_{i,j}^{(1)}t^s$ and $W_{i,j}^{(2)}t^s$ for all $1 \leq i, j \leq m + n$ and $s \in \mathbb{Z}$.

First, we show that $W_{j,j}^{(1)}t^s$ is contained in the completion of the image of $\Phi$. By the definition of $\Phi(H_{i,0})$ and $\Phi(X_{i,0}^\pm)$, the image of $\Phi$ contains $((-1)^{p(i)}W_{i,i}^{(1)} - (-1)^{p(j)}W_{j,j}^{(1)})t$ and $W_{i,j}^{(1)}t^s$ for all $i \neq j$ and $s \in \mathbb{Z}$. Then, by the definition of $\Phi(H_{i,1})$, the completion of the image of $\Phi$ contains

\[
((-1)^{p(j)}W_{j,j}^{(2)} - (-1)^{p(j+1)}W_{j+1,j+1}^{(2)})t
- \sum_{a \geq 0} W_{i,j}^{(1)}t^{-a}W_{i,j}^{(1)a} + \sum_{a \geq 0} W_{i,j}^{(1)}W_{j,j}^{(1)+1,a}t^{-a-1}W_{j,j}^{(1)+1,j+1}t^{a+1} - (-1)^{p(e_{j,j}+1)}W_{j,j}^{(1)}W_{j,j}^{(1)+1,j+1}.
\]

We take $1 \leq r, q \leq m + n$ such that $q \neq r, r + 1$. Then, by Corollary 4.13, we have

\[
[W_{q,r}^{(1)}t^{r-1},((-1)^{p(r)}W_{r,r}^{(2)} - (-1)^{p(r+1)}W_{r+1,r+1}^{(2)})t - \sum_{a \geq 0} W_{r,r}^{(1)}t^{-a}W_{r,r}^{(1)a}
+ \sum_{a \geq 0} W_{r,r}^{(1)}W_{r+1,r+1}^{(1)}t^{-a-1}W_{r+1,r+1}^{(1)}W_{r,r}^{(1)}t^{a+1} - (-1)^{p(e_{r,r}+1)}W_{r,r}^{(1)}W_{r,r}^{(1)}W_{r+1,r+1}^{(1)}]
= (-1)^{p(r)}W_{q,r}^{(2)}t^{r-1} + \sum_{a \geq 0} W_{q,r}^{(1)}t^{-a+s-1}W_{r,r}^{(1)a} + \sum_{a \geq 0} W_{r,r}^{(1)}t^{-a-1}W_{q,r}^{(1)a+s} - (-1)^{p(e_{r,r}+1)}W_{q,r}^{(1)}t^{r-1}W_{r+1,r+1}^{(1)}.
\]

Let us set $\sum_{a \geq 0} W_{q,r}^{(1)}t^{-a+s-1}W_{r,r}^{(1)a} + \sum_{a \geq 0} W_{r,r}^{(1)}t^{-a-1}W_{q,r}^{(1)a+s} - (-1)^{p(e_{r,r}+1)}W_{q,r}^{(1)}t^{r-1}W_{r+1,r+1}^{(1)}$ as $P_{q,r}^s$. By Lemma 4.3, we obtain

\[
[W_{q,r}^{(1)}, W_{q,r}^{s}] - [W_{r,q}^{(1)}, W_{q,r}^{s-1}]
= (-1)^{p(q)}\alpha W_{q,r}^{(1)}t^{q-1} + \delta_{s,1}((-1)^{p(e_{r,r}+1)} + p(q))\alpha W_{r+1,r+1}^{(1)}
+ (-1)^{p(e_{r,r})}W_{q,r}^{(1)}t^{r-1}W_{r,q}^{(1)} - W_{r,q}^{(1)}W_{q,r}^{(1)}t^{r-1}.
\]

Then, by Lemma 4.5 and Corollary 4.13, we have

\[
[W_{r,q}^{(1)}, W_{q,r}^{(2)}t^s] - (-1)^{p(r)}P_{q,r}^s - [W_{r,q}^{(1)}t, W_{q,r}^{(2)}t^{s-1} - (-1)^{p(r)}P_{q,r}^{s-1}]
= -l - 1\alpha W_{q,q}^{(1)}t^{q-1} + (-1)^{p(e_{r,r})}l\alpha W_{r,r}^{(1)}t^{q-1} - \delta_{s,1}((-1)^{p(e_{r,r}+1)}l\alpha W_{r+1,r+1}^{(1)}
- (-1)^{p(q)}W_{q,r}^{(1)}t^{s-1}W_{r,q}^{(1)} + (-1)^{p(r)}W_{r,q}^{(1)}W_{q,r}^{(1)}t^{s-1}
= \alpha W_{q,q}^{(1)}t^{q-1} + \delta_{s,1}((-1)^{p(e_{r,r}+1)}l\alpha W_{r+1,r+1}^{(1)}
- (-1)^{p(q)}W_{q,r}^{(1)}t^{s-1}W_{r,q}^{(1)} + (-1)^{p(r)}W_{r,q}^{(1)}W_{q,r}^{(1)}t^{s-1}.
\]

We find that $\alpha W_{q,q}^{(1)}t^{q-1} - \delta_{s,1}((-1)^{p(e_{r,r}+1)}l\alpha W_{r+1,r+1}^{(1)}$ is contained in the completion of the image of $\Phi$ by (5.45) we have shown that the completion of the image of $\Phi$ contains $W_{q,q}^{(1)}t^{s}$. Since we have already shown that the completion of the image of $\Phi$ contains $W_{i,j}^{(1)}t^{s}$ for all $1 \leq i, j \leq m + n$, we find that the completion of the image of $\Phi$ contains $((-1)^{p(i)}W_{i,i}^{(1)} - (-1)^{p(j)}W_{j,j}^{(2)})t$ and $W_{i,j}^{(2)}t^s$ for all $1 \leq i, j \leq m + n$ by the definition of $\Phi(H_{i,1})$ and $\Phi(X_{i,1}^\pm)$. Since there exists
Thus, the completion of the image of $\Phi$ contains $W_{i,j}^{(2)}t^s$ ($i \neq j$). By Corollary 4.13, we have

$$[W_{i,j}^{(2)}t^{s-1}, ((-1)^{p(i)}W_{i,i}^{(2)} - (-1)^{p(j)}W_{j,j}^{(2)}t^s, W_{j,j}^{(2)}t^s, \text{ and } W_{i,i}^{(2)}t^s + W_{j,j}^{(2)}t^s]$$

Next, we show that the completion of the image of $\Phi$ contains $W_{i,j}^{(2)}t^s$ ($i \neq j$). By Corollary 4.13, we obtain

$$[(W_{i,j}^{(2)}t^{s-1}, ((-1)^{p(i)}W_{i,i}^{(2)} - (-1)^{p(j)}W_{j,j}^{(2)}t^s, W_{j,j}^{(2)}t^s, \text{ and } W_{i,i}^{(2)}t^s + W_{j,j}^{(2)}t^s]$$

Thus, $W_{i,j}^{(2)}t^s$ ($i \neq j$) is contained in the completion of the image of $\Phi$. By Corollary 4.13, we obtain

$$[(W_{i,j}^{(2)}t^{s-1}, ((-1)^{p(i)}W_{i,i}^{(2)} - (-1)^{p(j)}W_{j,j}^{(2)}t^s, W_{j,j}^{(2)}t^s, \text{ and } W_{i,i}^{(2)}t^s + W_{j,j}^{(2)}t^s]$$

By Lemma 1.14 provided that $i \neq j$, it is equal to

$$-2\alpha(W_{i,i}^{(2)} + W_{j,j}^{(2)})t^s + 2(-1)^{p(i)}W_{i,i}^{(2)} - 2(-1)^{p(j)}W_{j,j}^{(2)} + 2((-1)^{p(i)}W_{i,i}^{(2)} + (-1)^{p(j)}W_{j,j}^{(2)})t^s$$

+ (the terms consisting of $\{W_{i,j}^{(1)} (1 \leq i, j \leq m + n), W_{i,j}^{(2)} (i \neq j)\})

$$= -2\alpha(W_{i,i}^{(2)} + W_{j,j}^{(2)})t^s$$

+ (the terms consisting of $\{W_{i,j}^{(1)} (1 \leq i, j \leq m + n), W_{i,j}^{(2)} (i \neq j)\}).$$

Thus, the completion of the image of $\Phi$ contains $W_{i,i}^{(2)}t^s + W_{j,j}^{(2)}t^s$.

Theorem 5.46. We assume that $m \geq 3$ and $l \geq 2$. Let us set

$$\varepsilon_1 = \frac{k + (l - 1)m}{m}, \varepsilon_2 = -1 - \frac{k + (l - 1)m}{m}.$$ 

Then, there exists an algebra homomorphism

$$\Phi: Y_{\varepsilon_1, \varepsilon_2}(\mathfrak{g}(m)) \rightarrow \mathcal{U}(\mathcal{W}^{(2)}(\mathfrak{gl}(ml), (l^m)))$$

determined by the same formula as that of Theorem 5.4 under the assumption that $n = 0$. Moreover, the image of $\Phi$ is dense in $\mathcal{U}(\mathcal{W}^{(2)}(\mathfrak{gl}(ml), (l^m)))$ provided that $k + (l - 1)m \neq 0$.

6 Stukopin’s Yangians and rectangular finite $W$-superalgebras of type A

In this section, we explain how the homomorphism $\Phi$ deduce the map from the Stukopin’s Yangian to rectangular finite $W$-superalgebras of type A.

First, let us recall the definition of the Zhu algebra (11). Let $V$ be a vertex algebra and $H \in \text{End}(V)$ be the Hamiltonian of $V$. We can define a degree on $\mathcal{U}(V)$ by

$$\deg(v \otimes t^s) = r + s + 1 \text{ if } H(v) = -rv.$$
We denote the set of degree $r$ elements of $\mathcal{U}(V)$ by $\mathcal{U}(V)_r$. In the case that $V$ is a rectangular $W$-superalgebra $W^k(\mathfrak{gl}(ml|nl), (l^{m|n}))$, we obtain $\deg(W^r_{i,j} t^s) = s - r + 1$.

By Theorem A.2.11 in [28], the Zhu algebra of $V$ can be defined as an associative algebra

$$Z\mathcal{H}(V) = \mathcal{U}(V)_0/\sum_{r > 0} \mathcal{U}(V)_{-r} \mathcal{U}(V)_r.$$  

The finite $W$-superalgebra is defined by [32]. A finite $W$-superalgebra $W^{\text{fin}}(\mathfrak{g}, f)$ is an associative algebra associated with a finite dimensional reductive Lie superalgebra $\mathfrak{g}$ and its even nilpotent element $f$. By [2], [13], and [8] a finite $W$-superalgebra $W^{\text{fin}}(\mathfrak{g}, f)$ is the Zhu algebra of the $W^k(\mathfrak{g}, f)$. In the case when $\mathfrak{g} = \mathfrak{gl}(ml|nl)$ and $f$ is a nilpotent element whose Jordan block is of type $(l^{m|n})$, we call $W^{\text{fin}}(\mathfrak{g}, f)$ the rectangular finite $W$-superalgebra and denote it by $W^{\text{fin}}(\mathfrak{gl}(ml|nl), (l^{m|n}))$.

Peng [34] constructed a surjective homomorphism from shifted super Yangians to finite $W$-superalgebras of type $A$. Especially, in the rectangular case, he gave a surjective homomorphism from the Nazarov’s Yangian (29) to the rectangular finite $W$-superalgebras. The definition of the Nazarov’s Yangian is as follows.

**Definition 6.1.** The Nazarov’s Yangian $Y(\mathfrak{gl}(m|n))$ is an associative superalgebra whose generators are $\{t_{i,j}^{(r)} \mid r \geq 0, 1 \leq i, j \leq m + n\}$ and defining relations are

$$t_{i,j}^{(0)} = \delta_{i,j},$$

$$[t_{i,j}^{(r+1)}, t_{u,v}^{(s)}] - [t_{i,j}^{(r)}, t_{u,v}^{(s+1)}] = (-1)^{p(i)p(j) + p(u)p(v)} \delta_{i,j}^{(r)} t_{u,v}^{(s)} - t_{u,j}^{(s)} t_{i,v}^{(r)},$$

where $i_{i,j}$ is

- even if $p(i) + p(j) = 0$,
- odd if $p(i) + p(j) = 1$.

Let $\tilde{p}$ be a natural projection from $\mathcal{U}(W^k(\mathfrak{gl}(ml|nl), (l^{m|n})))$ to $\mathcal{U}(W^k(\mathfrak{gl}(ml|nl), (l^{m|n})))$. Then, $\{\tilde{p}(W_{i,j}^{(r+1)} t^{-1}) \mid 1 \leq i, j \leq m + n, 1 \leq r \leq l\}$ becomes the stronger generators of the finite $W$-superalgebra $W^{\text{fin}}(\mathfrak{gl}(ml|nl), (l^{m|n}))$. The homomorphism given by Peng is written down as follows;

$$\tilde{\Phi}: Y(\mathfrak{gl}(m|n)) \to W^{\text{fin}}(\mathfrak{gl}(ml|nl), (l^{m|n}));$$

determined by $t_{i,j}^{(r)} \to \begin{cases} (-1)^{p(i)p(j)} \tilde{p}(W_{i,j}^{(r)} t^{-1}) & \text{if } r \leq l, \\ 0 & \text{if } r > l. \end{cases}$

The Nazarov’s Yangian has a subalgebra which is a deformation of the universal enveloping algebra of the current algebra associated with $\mathfrak{sl}(m|n)$. The subalgebra is called the Stukopin’s Yangian (33). We recall the definition of the Stukopin’s Yangian (see [17]).

**Definition 6.2.** The Sukopin’s Yangian $Y_{h}(\mathfrak{sl}(m|n))$ is the associative superalgebra over $\mathbb{C}$ generated by $x_{i,r}, \bar{x}_{i,r}, h_{i,r} (i \in \{1, \cdots, m+n-1\}, r \in \mathbb{Z}_{\geq 0})$ with two parameters $h \in \mathbb{C}$ subject to the following defining relations:

$$[h_{i,r}, h_{j,s}] = 0, (6.3)$$

$$[x_{i,r}^+, \bar{x}_{j,s}^-] = \delta_{ij} h_{i,r+s}, (6.4)$$

$$[h_{i,0}, x_{i,r}^+] = \pm a_{ij} x_{j,r}^+, (6.5)$$

$$[h_{i,r+1}, x_{j,s}^+] - [h_{i,r}, x_{j,s}^+] = \pm a_{ij} \frac{h}{2} (h_{i,r}, x_{j,s}^+), (6.6)$$

$$[x_{i,r+1}, x_{j,s}^+] - [x_{i,r}, x_{j,s+1}^+] = \pm a_{ij} \frac{h}{2} (x_{i,r}, x_{j,s}^+), (6.7)$$

$$\sum_{u \in \mathbb{Z}_{\geq 0}[i,j]} [x_{i,r_u^+(1)}, [x_{i,r_u^+(2)}, \cdots, [x_{i,r_u(1+|u_{ij}|)}^+, x_{j,s}^+] \cdots] = 0 \text{ } (i \neq j), (6.8)$$

26
\[ [x^\pm_{m,r}, x^\pm_{m,s}] = 0, \quad (6.9) \]
\[ [[x^\pm_{m-1,r}, x^\pm_{m,0}], [x^\pm_{m,0}, x^\pm_{m+1,s}]] = 0., \quad (6.10) \]

where \( x^\pm_{m,r} \) is

Similarly to the affine super Yangian, we note that \( Y_\hbar(\mathfrak{sl}(m|n)) \) is generated by \( \{ h_{i,0}, x^\pm_{i,0}, h_{i,1} \} \).

For \( \hbar \neq 0 \), the embedding \( \iota_\hbar : Y_\hbar(\mathfrak{sl}(m|n)) \to Y(\mathfrak{gl}(m|n)) \) is given by

\[
\iota_\hbar(h_{i,0}) = t^{(1)}_{i,i} - t^{(1)}_{i+1,i+1}, \quad \iota_\hbar(x^+_{i,0}) = -(1)^p(i)t^{(1)}_{i,i+1}, \quad \iota_\hbar(x^-_{i,0}) = -(1)^p(E_{i,i+1})t^{(1)}_{i+1,i}, \\
\iota_\hbar(h_{i,1}) = -\hbar t^{(2)}_{i,i} + \hbar t^{(2)}_{i+1,i+1} \\
\quad \quad - \frac{(i - 2\delta(i \geq m + 1)(i - m))}{2} \hbar (t^{(1)}_{i,i} - t^{(1)}_{i+1,i+1}) - h^{(1)}_{i,i}t^{(1)}_{i,i+1} \\
\quad \quad + \hbar \sum u = 1 t^{(1)}_{i,u}u_{u,i} - \hbar \sum u = 1 t^{(1)}_{i+1,u}u_{u,i+1}.
\]

Setting \( \varepsilon_1 = \varepsilon_2 = -\frac{1}{2} \), by the relation \( \deg(W_{i,j}^{(r)}t^s) = s - r + 1 \), we find that the image of \( \Phi \circ \omega \) is contained in \( \mathcal{U}(W^k(\mathfrak{gl}(m|n)), (t^{(m|n)})_0) \). Then, we obtain a homomorphism

\[ \tilde{\Phi} = p \circ \Phi \circ \omega : Y_{-1}(\mathfrak{sl}(m|n)) \to Z(W^k(\mathfrak{gl}(m|n)), (t^{(m|n)})). \]

By \( \deg(W_{i,j}^{(r)}t^s) = s - r + 1 \), we can explicitly write down \( \tilde{\Phi} \) as follows;

\[
\tilde{\Phi}(H_{i,0}) = (-1)^pW^{(1)}_{i,i} - (-1)^{p+1}\bar{W}^{(1)}_{i+1,i+1}, \\
\tilde{\Phi}(X^+_{i,0}) = \bar{W}^{(1)}_{i,i+1}, \\
\tilde{\Phi}(X^-_{i,0}) = (-1)^p\bar{W}^{(1)}_{i,i},
\]

\[
\Phi(H_{i,1}) = (-1)^p\bar{W}^{(2)}_{i,i}t - (-1)^{p+1}\bar{W}^{(2)}_{i+1,i+1}t \\
\quad \quad + \frac{i - 2\delta(i \geq m + 1)(i - m)}{2} ((-1)^p\bar{W}^{(1)}_{i,i} - (-1)^{p+1}\bar{W}^{(1)}_{i+1,i+1}) \\
\quad \quad + (-1)^pE_{i,i+1}\bar{W}^{(1)}_{i,i} + (-1)^p\sum u = 1 (-1)^pW^{(1)}_{u,i+1}
\]

\[ \bar{\Phi}(X^+_{i,0}) = \bar{W}^{(1)}_{i,i+1}, \quad \bar{\Phi}(X^-_{i,0}) = (-1)^p\bar{W}^{(1)}_{i,i},
\]

\[ \tilde{\Phi}(t^{(2)}_{i,i}) - \Phi(t^{(2)}_{i+1,i+1}) \\
\quad \quad + \frac{(i - 2\delta(i \geq m + 1)(i - m))}{2} (\tilde{\Phi}(t^{(1)}_{i,i}) - \Phi(t^{(1)}_{i+1,i+1})) + \tilde{\Phi}(t^{(1)}_{i,i})\Phi(t^{(1)}_{i+1,i+1})
\]

**Theorem 6.11.** We obtain the commutativity \( \Phi = \Phi \circ \iota_{-1} \).

**Proof.** It is enough to show that

\[ \tilde{\Phi}(x) = \Phi \circ \iota_{-1}(x) \text{ for } x = h_{i,1}, x^\pm_{i,0}, h_{i,0}. \]

We only show the case when \( x = h_{i,1} \) since other cases are trivial. By a direct computation, we obtain

\[ \Phi \circ \iota_{-1}(h_{i,1}) \]
\[ = \Phi(t^{(2)}_{i,i}) - \Phi(t^{(2)}_{i+1,i+1}) \\
\quad \quad + \frac{(i - 2\delta(i \geq m + 1)(i - m))}{2} (\Phi(t^{(1)}_{i,i}) - \Phi(t^{(1)}_{i+1,i+1})) + \Phi(t^{(1)}_{i,i})\Phi(t^{(1)}_{i+1,i+1}) \]
Remark homomorphism \( \Phi \) deduce the map from A to \( \mathfrak{g}(m|n) \). We show that \( W \) generate these terms by two claims, that is, Claim A1.4 and Claim A1.5. In Claim A1.4 below, for all \( 0 \leq r \leq l - 1 \), \( 1 \leq i, j \leq m + n \) and \( i, j \) if \( i \neq j \), we prove that \( W_{i,j}^{(1)} \) and \( W_{i,j}^{(2)} \) generate the term whose form is

\[
(-1)^{p(i)} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i} \bigl[ -1 \bigr] + \text{higher terms}
\]

for all \( 0 \leq r \leq l - 1 \). In Claim A1.5 below, we prove that \( W_{i,j}^{(1)} \) and \( W_{i,j}^{(2)} \) generate the term whose form is

\[
\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i+1} \bigl[ -1 \bigr] + \text{higher terms}
\]

for all \( 1 \leq r \leq l - 1 \). Since \( \sum_{s=1}^{l-0} e_{(0+s-1)(m+n)+i,(s-1)(m+n)+i} \bigl[ -1 \bigr] \) is nothing but \( W_{i,i}^{(1)} \), Theorem 3.17 is derived from Claim A1.4 and Claim A1.5.

In order to prove Claims A1.4 and A1.5, we prepare the following claim.

A1 The proof of Theorem 3.17

In this section, we prove Theorem 3.17. We define a grading on \( \mathfrak{g} \) by setting \( \deg(x) = j \) if \( x \in \mathfrak{g} \). Since

\[
\{ \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+j,(s-1)(m+n)+i} \bigm| 0 \leq r \leq l - 1, \ 1 \leq i, j \leq m + n \}
\]

forms a basis of \( \mathfrak{gl}(m|n) \). It is enough to show that \( W_{i,j}^{(1)} \) and \( W_{i,j}^{(2)} \) generate the terms whose form is

\[
(-1)^{p(i)} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i} \bigl[ -1 \bigr] + \text{higher terms}
\]

for all \( 0 \leq r \leq l - 1 \), \( 1 \leq i, j \leq m + n \) and \( i, j \) if \( i \neq j \), by Theorem 6.11. We show that \( W_{i,j}^{(1)} \) and \( W_{i,j}^{(2)} \) generate these terms by two claims, that is, Claim A1.4 and Claim A1.5. In Claim A1.4 below, for all \( 0 \leq r \leq l - 1 \), \( 1 \leq i, j \leq m + n \) and \( i, j \) if \( i \neq j \), we prove that \( W_{i,j}^{(1)} \) and \( W_{i,j}^{(2)} \) generate the term whose form is

\[
(-1)^{p(i)} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i} \bigl[ -1 \bigr] + \text{higher terms}
\]

or

\[
\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i+1} \bigl[ -1 \bigr] + \text{higher terms}
\]

for all \( 0 \leq r \leq l - 1 \). In Claim A1.5 below, we prove that \( W_{i,j}^{(1)} \) and \( W_{i,j}^{(2)} \) generate the term whose form is

\[
\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i} \bigl[ -1 \bigr] + \text{higher terms}
\]

for all \( 1 \leq r \leq l - 1 \). Since \( \sum_{s=1}^{l-0} e_{(0+s-1)(m+n)+i,(s-1)(m+n)+i} \bigl[ -1 \bigr] \) is nothing but \( W_{i,i}^{(1)} \), Theorem 3.17 is derived from Claim A1.4 and Claim A1.5.

In order to prove Claims A1.4 and A1.5, we prepare the following claim.
Claim A1.1. (1) The following equation holds for all $0 \leq w \leq l - 1$, $1 \leq i, j, u, v \leq m + n$:

$$
\sum_{s=1}^{l-1} e_s(m+n)+j,(s-1)(m+n)+i[-1] j \sum_{t=1}^{l-w} e_{(w+t-1)(m+n)+u,(t-1)(m+n)+v}[−1] \\
= \delta_{i,u} \sum_{t=1}^{l-w-1} e_{(w+t)(m+n)+j,(t-1)(m+n)+v} \sum_{t=1}^{l-w-1} e_{(w+t)(m+n)+u,(t-1)(m+n)+v} [−1] \\
- \delta_{j,v} (−1)^{p(e_{i,j})p(e_{u,v})} \sum_{t=1}^{l-w-1} e_{(w+t)(m+n)+u,(t-1)(m+n)+v} [−1].
$$

(A1.2)

(2) We obtain

$$(W_{i,j}^{(1)})_{(0)} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+x,(s-1)(m+n)+y} [−1]$$

$$
= \delta_{i,x} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+j,(s-1)(m+n)+y} [−1] \\
- \delta_{j,y} (−1)^{p(e_{i,j})p(e_{x,y})} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+x,(s-1)(m+n)+y} [−1]
$$

(A1.3)

for all $0 \leq r \leq l - 1$, $1 \leq i, j, x, y \leq m + n$.

Claim A1.1 is proven by direct computation. We omit the proof. By (A1.2) and (A1.3), it is easy to obtain the following claim.

Claim A1.4. (1) For all $0 \leq r \leq l - 1$, the elements $W_{i,j}^{(1)}$ and $W_{i,j}^{(2)}$ generate the term whose form is

$$
\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+j} [−1] + \text{higher terms (} i \neq j \text{)}.
$$

(2) For all $0 \leq r \leq l - 1$, the elements $W_{i,j}^{(1)}$ and $W_{i,j}^{(2)}$ generate the term whose form is

$$
(-1)^{p(i)} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i+j} [−1] \\
- (-1)^{p(i+1)} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i+1,(s-1)(m+n)+i+1+j} [−1] + \text{higher terms}.
$$

Proof. First, let us show (1). Since $W_{i,j}^{(2)}$ has the form such that

$$
\sum_{s=1}^{l-1} e_{s(m+n)+j,(s-1)(m+n)+i} [−1] + \text{degree 0 terms},
$$

we obtain

$$
((W_{i,j}^{(2)})_{(0)})^r W_{j,i}^{(1)} = \left( \sum_{s=1}^{l-1} e_{s(m+n)+i,(s-1)(m+n)+i} [−1] \right)^r W_{j,i}^{(1)} + \text{higher terms}
$$

for all $i \neq j$, $0 \leq r \leq l - 1$. By (A1.2), we have

$$
((W_{i,j}^{(2)})_{(0)})^r W_{j,i}^{(1)} = \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+j} [−1] + \text{higher terms}.
$$
Thus, we have proved (1).

Next, let us prove (2). By (1), the element whose form is

$$\sum_{s=1}^{l-r} e^{(r+s-1)(m+n)+i,(s-1)(m+n)+i+1}[-1] + \text{higher terms}$$

is generated by $W_{i,j}^{(1)}$ and $W_{i,j}^{(2)}$. By (A1.3), we have

$$(W_{i,i+1}^{(1)})(0)\left(\sum_{s=1}^{l-r} e^{(r+s-1)(m+n)+i,(s-1)(m+n)+i+1}[-1] + \text{higher terms}\right)$$

$$= \sum_{s=1}^{l-r} e^{(r+s-1)(m+n)+i+1,(s-1)(m+n)+i+1}[-1]$$

$$- (-1)^{p(e_{i,i+1})} \sum_{s=1}^{l-r} e^{(r+s-1)(m+n)+i,(s-1)(m+n)+i}[-1] + \text{higher terms.}$$

Thus, we have proved (2).

\[\square\]

Claim A1.5. The elements $W_{i,j}^{(1)}$ and $W_{i,j}^{(2)}$ generate the term whose form is

$$\sum_{1 \leq t \leq l-r} e^{(t+r-1)(m+n)+i,(t-1)(m+n)+i}[-1] + \text{higher terms}$$

for all $1 \leq r \leq l-1$.

Proof. It is enough to show that

$$(W_{i,i+1}^{(2)})(W_{i,i+1}^{(1)})(0)\left(\sum_{s=1}^{l-r} e^{(r+s-1)(m+n)+i,(s-1)(m+n)+i+1}[-1] + \text{higher terms}\right)$$

$$= (-1)^{p(e_{i,i+1})} \sum_{1 \leq t \leq l} e^{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]$$

$$+ (-1)^{p(i) r} \sum_{1 \leq t \leq l} e^{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]$$

$$- (-1)^{p(i) r} \sum_{1 \leq t \leq l} e^{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1] + \text{higher terms} \quad (A1.6)$$

since we have already shown that

$$\sum_{1 \leq t \leq l} (e^{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1] - (-1)^{p(e_{i,i+1})} e^{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1])$$

is generated by $W_{i,j}^{(1)}$ and $W_{i,j}^{(2)}$. Let us set

$$Z = \sum_{1 \leq s \leq l-1} e^{s(m+n)+i,(s-1)(m+n)+i}[-1], \quad W = W_{i,i}^{(2)} - Z.$$

The element $W_{i,j}^{(2)}$ is the sum of degree $-1$ element $Z$ and degree 0 element $W$. We can rewrite the left hand side of (A1.6) as

$$Z_{(1)}(W_{i,i+1}^{(1)})(0)(Z_{(0)})^{r}W_{i+1,i}^{(1)} + W_{(1)}(W_{i,i+1}^{(1)})(0)(Z_{(0)})^{r}W_{i+1,i}^{(1)}$$

$$+ \sum_{1 \leq d \leq r} Z_{(1)}(W_{i,i+1}^{(1)})(0)(Z_{(0)})^{r-d}W_{(0)}(Z_{(0)})^{d-1}W_{i+1,i}^{(1)} + \text{higher terms.} \quad (A1.7)$$
In order to simplify the notation, hereafter, we denote \[
\sum_{a \leq s \leq l} e_{(b+s-1)(m+n)+i,(s-a)(m+n)+j}[-u]
\] by \[
\sum_{1 \leq l \leq l} e_{(b+s-1)(m+n)+i,(s-a)(m+n)+j}[-u].
\] Let us compute the each terms of (A1.7). First, we compute the first term of (A1.7). By (A1.2) and (A1.3), we have

\[
\kappa \bigg( \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[1] \bigg).
\]

By the similar computation, the first term of (A1.9) is equal to

\[
\sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[1].
\]

Applying (A1.8) to the first term of (A1.7), we obtain

\[= \left( \sum_{1 \leq t \leq l} e_{s(m+n)+i,(s-1)(m+n)+i}[1] \right) (\text{the right hand side of (A1.8)}) = 0.\]

Next, let us compute the second term of (A1.7). By (A1.8), it is the sum of

\[
-(-1)^p \sum_{r_1 < r_2} e_{r_1, r_2} e_{s, 2} - 1 \sum_{1 \leq l \leq l} e_{(t-1)(m+n)+i, (t-r-1)(m+n)+i}[1]
\]

and

\[
-(-1)^p \alpha \sum_{2 \leq s \leq l} (s-1) e_{s, 2} \sum_{1 \leq l \leq l} e_{(t-1)(m+n)+i, (t-r-1)(m+n)+i}[1].
\]

Let us compute (A1.9) and (A1.10). By direct computation, the second term of (A1.9) is equal to

\[
-(-1)^p \sum_{r_1 < r_2} e_{r_1, r_2} e_{(t-1)(m+n)+i, (t-r-1)(m+n)+i}[1]
\]

By the similar computation, the first term of (A1.9) is equal to

\[
(-1)^p \sum_{1 \leq l \leq l} e_{(t-1)(m+n)+i, (t-r-1)(m+n)+i}[1].
\]
By direct computation, we rewrite the second term of (A1.10) as

\[ (-1)^{p(e_{rd})} \sum_{1 \leq s \leq l, 1 \leq t \leq l} (s-1)[e_{s,t}^{(e)}; e_{(t-1)(m+n)+i}(t-r-1)(m+n)+1][-1] \]

\[ = (-1)^{p(e_{rd})} \sum_{1 \leq s \leq l} e_{(t-1)(m+n)+i}(t-r-1)(m+n)+1[-1]. \]

By the similar computation, we find that the first term of (A1.10) is zero. Thus, we obtain

the sum of first two terms of (A1.7) is 

\[ (-1)^{p(e_{rd})} \sum_{1 \leq s \leq l} e_{(t-1)(m+n)+i}(t-r-1)(m+n)+1[-1]. \]

(A1.11)

Finally, we compute the third term of (A1.7). Since the relation \((\sum_{1 \leq s \leq l} (s-1)e_{s,t}^{(e)}[-2])_0 = 0\) holds, we can rewrite the third term of (A1.7) as

\[ \sum_{1 \leq d \leq r} Z(1)(W_{i,1+i}(0)(Z(0))^{r-d} \cdot (1-p(i)) \sum_{1 \leq r_1 \leq r_2} e_{i,1}(r_1) [-1]e_{i,1}(r_2)[-1])_0(Z(0))^{d-1}W_{i,1+i}^{(1)} \]

Let us set

\[ T_d = Z(1)(W_{i,1+i}(0)(Z(0))^{r-d}, \quad B_d = ((-1)^{p(i)} \sum_{1 \leq s \leq l} e_{s,t}^{(e)}[-1]e_{s,t}^{(e)}[-1])_0(Z(0))^{d-1}W_{i,1+i}^{(1)} \]

Then, the third term of (A1.7) is equal to \(\sum_{1 \leq d \leq r} T_d(B_d)\).

We rewrite \(B_d\) and \(T_d\). By (A1.2) and (A1.3), \(T_d\) is the sum of \(T_d^1\) and \(T_d^2\) such that

\[ T_d^1 = -\sum_{g=0}^{r-d} \left( \begin{array}{c} r-d \\ g \end{array} \right) (Z(0))^{r-d-g} \sum_{1 \leq s \leq l} e_{s+g(m+n)+i}(s-1)(m+n)+1[-1])_0 \]

(A1.12)

\[ T_d^2 = (W_{i,1+i}(0)(Z(0))^{r-d}Z(1). \]

(A1.13)

Since

\[ (Z(0))^{d-1}W_{i,1+i}^{(1)} = \sum_{1 \leq d \leq l} e_{(t-1)(m+n)+i}(t-d)(m+n)+1[-1]. \]

(A1.14)

by (A1.2) and (A1.3), \(B_d\) is equal to

\[ ((-1)^{p(i)} \sum_{1 \leq r_1 < r_2 \leq l} e_{u,i}^{(r_1)}[-1]e_{u,i}^{(r_2)}[-1])_0 \sum_{1 \leq d \leq l} e_{d}(t-1)(m+n)+i(t-d)(m+n)+1[-1] \]

\[ = (-1)^{p(i)} \sum_{1 \leq r_1 < r_2 \leq l} e_{u,i}^{(r_1)}[-1] e_{u,i}^{(r_2)}[-1] e_{(t-1)(m+n)+i}(t-d)(m+n)+1][-1] \]

\[ + \sum_{1 \leq u \leq m+n} \sum_{1 \leq r_1 < r_2 \leq l} e_{u,i}^{(r_1)}[-1] e_{u,i}^{(r_2)}[-1] e_{(t-1)(m+n)+i}(t-d)(m+n)+1][-1] \]

(A1.15)
By direct computation, we find that the first term of the right hand side of (A1.15) is equal to
\[ (-1)^{p(i)} \sum_{d \leq r_1 < t \leq l} e_{t,i}^{(r_1)} [-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1} [-1] \] 
and the second term of the right hand side of (A1.15) is equal to
\[ \sum_{d \leq l < r_2 \leq l} (-1)^{p(u)} e_{t,u}^{(r_2)} [-1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1} [-1] \]
\[ - (-1)^{p(i)} \sum_{t-d+1 < r_2 \leq l} e_{i,i+1}^{(r_2)} [-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i} [-1]. \] 
(A1.17)

By the definition of $\kappa$, the third term of the right hand side of (A1.15) is equal to
\[ \delta_{d,1} \alpha \sum_{1 \leq r_2 \leq l} (r_2 - 1) e_{i,i+1}^{(r_2)} [-2]. \] 
(A1.18)

Adding (A1.16), (A1.17), and (A1.18), we obtain
\[ B_d = (-1)^{p(i)} \sum_{d \leq r_1 < t \leq l} e_{t,i}^{(r_1)} [-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1} [-1] \]
\[ + \sum_{d \leq l < r_2 \leq l} (-1)^{p(u)} e_{t,u}^{(r_2)} [-1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1} [-1] \]
\[ - (-1)^{p(i)} \sum_{t-d+1 < r_2 \leq l} e_{i,i+1}^{(r_2)} [-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i} [-1] \]
\[ + \delta_{d,1} \alpha \sum_{1 \leq r_2 \leq l} (r_2 - 1) e_{i,i+1}^{(r_2)} [-2] \]
\[ = (-1)^{p(i)} \sum_{r_1 \neq t} e_{t,i}^{(r_1)} [-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1} [-1] \]
\[ + \sum_{r_2 > l} (-1)^{p(u)} e_{t,u}^{(r_2)} [-1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1} [-1] \]
\[ - (-1)^{p(i)} \sum_{t-d+1 < r_2 \leq l} e_{i,i+1}^{(r_2)} [-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i} [-1] \]
\[ + \delta_{d,1} \alpha \sum_{1 \leq r_2 \leq l} (r_2 - 1) e_{i,i+1}^{(r_2)} [-2]. \] 
(A1.19)

Now, we compute $T_d(B_d)$. We divide $B_d$ into two parts such that
\[ B^1_d = (-1)^{p(i)} \sum_{r_1 \neq t} e_{t,i}^{(r_1)} [-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1} [-1] \]
\[ + \sum_{r_2 > l} (-1)^{p(u)} e_{t,u}^{(r_2)} [-1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1} [-1] \]
\[ - (-1)^{p(i)} \sum_{t-d+1 < r_2 \leq l} e_{i,i+1}^{(r_2)} [-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i} [-1] \]
\[ B^2_d = \delta_{d,1} \alpha \sum_{1 \leq r_2 \leq l} (r_2 - 1) e_{i,i+1}^{(r_2)} [-2]. \]

First, let us compute $T_d(B^1_d)$. By (A1.2) and (A1.3), we obtain
\[ T_d(B^1_d) = -\delta_{d,1} (-1)^{p(e_i,i+1)} (r - 1) \alpha \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i} [-1]. \] 
(A1.20)
Next, let us compute \( T_d(B^1_d) = T^1_d(B^1_d) + T^2_d(B^1_d) \). We compute \( T^1_d(B^1_d) \) and \( T^2_d(B^1_d) \) respectively. In order to compute \( T^1_d(B^1_d) \), we prepare the following three relations;

\[
\sum_{1 \leq s \leq l} (e_{(s+g)(m+n)+i+1,(s-1)(m+n)+i}[1]) (1)
\cdot \left( (-1)^{p(i)} \sum_{r_1 \neq t} e_{i,i}^{(r_1)} [1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}[1] \right)
= -(-1)^{p(i+1)} \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-g-d-1)(m+n)+i}[1], \quad (A1.21)
\]

\[
\sum_{1 \leq s \leq l} (e_{(s+g)(m+n)+i+1,(s-1)(m+n)+i}[1]) (1)
\cdot \left( \sum_{r_2 > \ell \atop u \neq i} (-1)^{p(u)} e_{i,u}^{(r_2)} [1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1}[1] \right) = 0, \quad (A1.22)
\]

\[
\sum_{1 \leq s \leq l} (e_{(s+g)(m+n)+i+1,(s-1)(m+n)+i}[1]) (1)
\cdot \left( (-1)^{p(i)} \sum_{t-d+1 \leq r_2 \leq \ell} e_{i,i+1}^{(r_2)} [1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[1] \right)
= -(-1)^{p(i+1)} \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-g-d-1)(m+n)+i}[1]. \quad (A1.23)
\]

We only show the relation (A1.23) holds. The other relations are proven similarly. By direct computation, (A1.23) is equal to

\[
(-1)^{p(i)} \sum_{1 \leq s \leq l} \sum_{t-d+1 \leq r_2 \leq \ell} \left[ e_{(s+g)(m+n)+i+1,(s-1)(m+n)+i}^{(r_2)}, e_{i,i+1}^{(r_2)}, e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[1] \right][1] = 0.
\]

Thus, we have obtained (A1.23). By (A1.21), (A1.23) and (A1.12), we find the relation

\[
T^1_d(B^1_d) = 0. \quad (A1.24)
\]

Similarly to (A1.21)-(A1.23), we obtain the following three equations;

\[
\sum_{1 \leq s \leq l} (e_{s(m+n)+i,(s-1)(m+n)+i}[1]) (1)
\cdot \left( (-1)^{p(i)} \sum_{r_1 \neq t} e_{i,i}^{(r_1)} [1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[1] \right)
= -(-1)^{p(i)} \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-g-d-1)(m+n)+i}[1], \quad (A1.25)
\]

\[
\sum_{1 \leq s \leq l} (e_{s(m+n)+i,(s-1)(m+n)+i}[1]) (1)
\cdot \left( \sum_{r_2 > \ell \atop u \neq i} (-1)^{p(u)} e_{i,u}^{(r_2)} [1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i}[1] \right) = 0, \quad (A1.26)
\]

\[
\sum_{1 \leq s \leq l} (e_{s(m+n)+i,(s-1)(m+n)+i}[1]) (1)
\cdot \left( (-1)^{p(i)} \sum_{t-d+1 \leq r_2 \leq \ell} e_{i,i+1}^{(r_2)} [1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[1] \right) = 0. \quad (A1.27)
\]
By (A1.24)-(A1.27) and (A1.13), we obtain
\[
T_d^2(B_d) = -(-1)^{p(i)}(W_{i+1})_0(Z_0)^{r-d} \sum_{1 \leq i \leq l} e_{(t-1)(m+n)+i,(t-d-1)(m+n)+i+1}[1]
\]
\[
= -(-1)^{p(i)} \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[1] + (-1)^{p(i+1)} \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i+1}[1],
\]
(A1.28)

where the second equality is due to (A1.2) and (A1.3). By (A1.20), (A1.24) and (A1.28), we have
\[
\sum_{1 \leq d \leq r} T_d(B_d) = -(-1)^{p(e_i,i+1)}(r - 1) \alpha \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i+1}[1]
\]
\[
- (-1)^{p(i)} r \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[1] + (-1)^{p(i+1)} r \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i+1}[1].
\]
(A1.29)

Adding (A1.11) and (A1.29), (A1.7) is equal to
\[
(-1)^{p(e_i,i+1)} \alpha \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i+1}[1]
\]
\[
- (-1)^{p(i+1)} r \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[1] + (-1)^{p(i)} r \sum_{1 \leq t \leq l} e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[1] + \text{higher terms.}
\]

We have obtained (A1.6).

Since we complete the proof of Claims (A1.4) and (A1.5) we have proved Theorem (4.17).

Acknowledgement

The author wishes to express his gratitude to his supervisor Tomoyuki Arakawa for suggesting lots of advice to improve this paper. We express our sincere thanks to Ryosuke Kodera for carefully reading the manuscript and giving me lots of advice and comments. The author is particularly grateful for the assistance given by Naoki Genra and Shigenori Nakatsuka. This work was supported by Iwadare Scholarship and and JSPS KAKENHI, Grant-in-Aid for JSPS Fellows, Grant Number JP20J12072.

Data Availability

The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

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