Chaotic, regular and unbounded behaviour in the elastic impact oscillator

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Abstract

A discontinuous area-preserving mapping derived from a sinusoidally-forced impacting system is studied. This system, the elastic impact oscillator, is very closely related to the accelerator models of particle physics such as the Fermi map. The discontinuity in the mapping is due to grazing which can have a surprisingly large effect upon the phase space. In particular, at the boundary of the stochastic sea, the discontinuity set and its images can act as a partial barrier which allows trajectories to move between chaotic and regular regions. The system at higher energies is also analysed and Moser’s invariant curve theorem is used to find sufficient conditions for the existence of invariant curves that bound the energy of the motion. Finally the behaviour of the system under more general periodic forcing is briefly investigated.
1 Introduction

We study a forced oscillator, such as a mass on a spring, moving in one dimension and repeatedly impacting against a fixed wall (Figure 1). The motion between impacts is undamped and the impact is modelled as an instantaneous reversal of direction with a constant coefficient of restitution. For a linear spring with sinusoidal forcing this system, called the impact oscillator, is described after suitable rescaling by the following equations

\[
\begin{align*}
\ddot{x} + x &= \cos(\omega t), \quad x < \sigma \\
\dot{x} &\mapsto -r \dot{x}, \quad x = \sigma
\end{align*}
\]

The inelastic impact oscillator \((r < 1)\), was first studied \([1, 2, 3, 4]\) as a model for many important engineering phenomena including the rattling of engine components and the behaviour of structures in earthquakes. Further work has been done on such aspects as the bifurcation structure and the form of the chaotic attractors \([5, 6, 7, 8, 9]\). Here we consider the elastic case \(r = 1\) where no energy is lost at the impacts. For engineering applications \(r\) is often close to 1 and the elastic limit is an efficient way of exploring large areas of the phase space. It is also possible to exploit the time-reversal symmetry of the elastic case to locate the periodic orbits. However the elastic case has unusual behaviour of its own, and that is the subject of this paper.

The motion of the impact oscillator consists of a series of impacts separated by smooth motion between the impacts. It is natural to describe these systems in terms of the impact map, \(P_I\), relating the state of the system at one impact to the state at the next. Each impact is described by the (positive) velocity of the mass just before the impact and the phase (from 0 to \(2\pi\)) of the forcing cycle at the time of impact so that the phase space is a half-cylinder. Impacting systems that have been studied before include the accelerator models of particle physics \([10, 11, 12, 13, 14, 15, 16]\) and billiards, see \([17, 18, 19]\) and references therein.

Although a mechanical impacting system is clearly discontinuous when regarded as a continuous-time dynamical system, the resulting impact map may be analytic and indeed most of the literature concerns impacting systems with impact maps that are either smooth or have discontinuities deliberately introduced into a higher derivative. The map \(P_I\) is discontinuous because of grazes, or zero-velocity impacts, see trajectory B of Figure 2, where the mass approaches the wall, touches it, and is then pulled away again. Nearby trajectories (trajectory C) will either (just) hit the wall and be deflected by it or miss the wall and go on to impact at a later time (trajectory A). Surprisingly, low velocity impacts strongly distort the phase space and so have a large effect upon the overall dynamics. This process is described in \([3]\) and is an important source of chaotic behaviour. Previous work \([7, 8, 20, 21, 22]\) has shown that grazing can introduce additional bifurcations such as the
sudden appearance or disappearance of periodic orbits and the interruption of period-doubling cascades.

The phase space of $P_I$ resembles that of the Fermi map [12, 14, 15] in that the motion becomes more regular as the velocity of the impacts increases. Figure 3 shows a phase space plot of $P_I$ for parameter values $\sigma = 0.1$, $\omega = 2.5$ which has clearly separated regions of phase space giving rise to chaotic and regular behaviour. In particular, at low velocities the phase space is filled by a stochastic sea which, as the velocity increases, is interrupted by regular islands associated with elliptic fixed and periodic points. Above the stochastic sea the behaviour is more regular, with bounding invariant curves and elliptic islands. Note that the upper boundary of the stochastic sea is the lowest bounding invariant curve.

However for slightly different values of $\sigma$ and $\omega$ we observe a new phenomenon. Figure 4 shows a single trajectory for the parameter values $\sigma = 0$, $\omega = 2.8$. The trajectory starts in the stochastic sea, exhibiting a typical sensitivity to initial data, but then, at irregular intervals, leaves the sea and moves along a smooth curve before reentering the stochastic sea. There is, for this example, less of a distinction between chaotic and regular motion as both can occur on the same trajectory. This very unusual behaviour is due to the discontinuity introduced by the grazing. The grazing discontinuity set and its iterates form structures at the upper boundary of the stochastic sea which act as a partial barrier allowing trajectories to move between the chaotic and the regular regions. Partial barriers also occur for smooth systems although the mechanisms are quite different (see MacKay et al. [23]). The existence of trajectories that can behave both regularly and chaotically is of great importance to the understanding and modelling of this system. In an engineering context, it is also important that such trajectories involve a large number of high-velocity impacts (with associated high wear-rates) and is all the more surprising since they arise directly from low velocity impacts.

The outline of the paper is as follows. In §2 we define the impact oscillator and the impact map and in §3 we analyse the effect of the grazing discontinuity on the nearby dynamics. In §4 we examine the upper boundary of the stochastic sea and show how interactions between the grazing discontinuity set and invariant curves can allow trajectories to alternate between stochastic and regular regions. §5 describes the relationship between the elastic impact oscillator and other impacting systems such as the accelerator models and billiards. Then in §6, motivated by the links with the accelerator models, we turn our attention to the behaviour of the system at high velocities where the impact map is smooth. We show that the existence of bounding invariant curves has an interesting dependence on the parameter values and briefly look at the case of more general periodic forcing.
2 The impact oscillator and the impact map

The impact oscillator is a mass on a linear spring, moving in one dimension subject to a sinusoidal forcing, which impacts against a rigid obstacle referred to as the wall (Figure 1). The impact itself is modelled as an instantaneous process with coefficient of restitution \( r, 0 \leq r \leq 1 \). By rescaling the system so that the amplitude of the forcing and the natural frequency of the spring are equal to 1 we obtain equation (1) where \( \omega \) is the forcing frequency and \( \sigma \), the clearance, is the position of the wall relative to the equilibrium position of the mass in the unforced system. Thus the system has three parameters \( r, \sigma \) and \( \omega \). In this paper \( r = 1 \).

If the impacting condition is removed then the system is just the forced harmonic oscillator — a linear system whose behaviour is completely understood. But the impact oscillator is highly nonlinear and cannot be regarded as a small perturbation of a linear or integrable system. It is convenient to replace the time, \( t \), by the phase, \( \phi \), which is defined by \( \phi = (\omega t) \mod 2\pi \). So we have the state vector \( u = (\phi, x, v) \), where \( v \) is the velocity \( \dot{x} \), and the phase space \( \Omega = 2\pi S^1 \times (-\infty, \sigma] \times \mathbb{R} \). The vector \( u \) evolves under the discontinuous flow \( \Phi \).

A common approach to studying a system with periodic forcing is to define a Poincaré surface \( \Sigma_S = \{(\phi, x, v) : \phi = \phi_0\} \) (different values of \( \phi_0 \) give different maps but the choice is unimportant). This defines the stroboscopic map, \( P_S \), which maps the position and velocity at the phase \( \phi_0 \) of the forcing cycle onto the position and velocity one cycle later. Note that \( \Sigma_S \) is always crossed transversely since time is increasing. But it is the impacts themselves that are of interest and the map \( P_S \) tells us very little about these — for example, during a single forcing cycle there may be no impacts or there may be many and determining which is the case would appear to be a very hard problem.

Instead we study the mapping \( P_1 \) which maps one impact onto the next one (\( \Pi, \Xi, \Omega \)). \( P_1 \) is an extremely useful tool for the analysis of the impact oscillator. It contains all of the interesting dynamical information, since the impacts are the source of the nonlinearity and the motion between them is just that of a simple harmonic oscillator.

The map \( P_1 \) is discontinuous due to grazing trajectories. These are trajectories which pass through a point \( (\phi, \sigma, 0) \), known as a graze. Nearby orbits either have a low velocity impact close to \( (\phi, \sigma, 0) \) or miss the wall and impact at a later time and so \( P_1 \) is discontinuous in the neighbourhood of grazes.

We now examine the map \( P_1 \) and its discontinuities in more detail.
2.1 The impact map

We define the Poincaré surface \( \Sigma_I = \{ (\phi, x, v) : x = \sigma, v > 0 \} \). The resulting map takes the phase and velocity \( (\phi_n, v_n) \) at one impact to the phase and velocity \( (\phi_{n+1}, v_{n+1}) \) of the next impact. Conventionally, the velocity is recorded just before the impact and we find it convenient to not regard grazes as impacts. Thus the recorded velocity is always positive. More formally, the impact map, \( P_I \), is defined by

\[
P_I : \Sigma_I \rightarrow \Sigma_I \text{ where } \Sigma_I = 2\pi S^1 \times \mathbb{R}^+ \text{ and } P_I : (\phi_n, v_n) \mapsto (\phi_{n+1}, v_{n+1})
\]

The phase space of \( P_I \) is therefore an open half-cylinder. Technically, the impact map is not a Poincaré map since the surface \( \Sigma_I \) is not everywhere transverse to the flow because of grazing trajectories.

The precise definition of \( P_I \) is actually more cumbersome than we have described here. It is straightforward to show that since the motion between impacts is recurrent one impact must lead to another and so the map is defined everywhere. However, if \( |\sigma| < 1 \), it is possible for the mass to stick to the wall until the acceleration becomes negative and it moves away. This behaviour occurs with measure zero and if it occurs more than once on a single orbit, the motion must be periodic (in contrast, for the inelastic case \( r < 1 \), the mass can stick to the wall via an infinite sequence of bounces which brings the mass to rest in finite time and for some parameter ranges this behaviour can be extremely important).

The free motion (between impacts) is described by the first equation of (1) which is just the forced harmonic oscillator and can be solved exactly. If \( \omega \neq 1 \) and the time and velocity of an impact are given by \( t_0 \) and \( v_0 \) then the position and velocity of the mass at time \( t \) are given by

\[
x(t; t_0, v_0) = (\sigma - \gamma \cos \omega t_0) \cos(t - t_0)
+ (-v_0 + \omega \gamma \sin \omega t_0) \sin(t - t_0) + \gamma \cos \omega t
\]

\[
v(t; t_0, v_0) = (-v_0 + \omega \gamma \sin \omega t_0) \cos(t - t_0)
- (\sigma - \gamma \cos \omega t_0) \sin(t - t_0) - \omega \gamma \sin \omega t
\]

where \( \gamma = 1/(1 - \omega^2) \). This solution is only valid for \( t_0 \leq t < t_1 \) where \( t_1 \) is the time of the next impact given by the first solution of the transcendental equation

\[
x(t_1; t_0, v_0) = \sigma
\]

To solve the system numerically we only have to find the time of the next impact using a root-finding scheme. This is both quick and accurate since no numerical integration is involved although there is always the possibility of not detecting extremely low-velocity impacts.
Implicit differentiation of (2), (3) and (4) gives an expression for the Jacobian derivative of $P_I$ (see [1])

$$DP_I(\phi_0, v_0) = \begin{pmatrix}
(N_0S - v_0C)/v_1 & S/v_1 \\
(N_0N_1/v_1 - v_0)S - (N_0 + N_1v_0/v_1)C & N_1S/v_1 - C
\end{pmatrix}$$

(5)

where $S = \sin(t_1 - t_0)$, $C = \cos(t_1 - t_0)$ and $N_i = \cos(\omega t_i) - \sigma$, $i = 0, 1$. The $N_i$ are the accelerations of the mass just before the impacts. Note that $t$ appears instead of $\phi$ on the right-hand side of (5). This is purely for convenience.

It is immediate that $|DP_I| = \frac{v_0}{v_1}$ as stated in [1]. In fact this is equivalent to showing that $P_I$ preserves the measure $vd\phi dv$ and we can simplify the determinant by introducing the coordinate change $z = v^2$. Writing $P_I$ as a map from $(\phi, z)$ to $(\phi, z)$ gives us the Jacobian derivative

$$DP_I(\phi_0, z_0) = \begin{pmatrix}
(N_0S - \sqrt{z_0}C)/\sqrt{z_1} & S/2\sqrt{z_0z_1} \\
2(N_0N_1 - \sqrt{z_0z_1})S - 2(N_0\sqrt{z_1} + N_1\sqrt{z_0})C & (N_1S - \sqrt{z_1}C)/\sqrt{z_0}
\end{pmatrix}$$

(6)

which has determinant equal to 1 and so is an area-preserving mapping.

The elements of $DP_I$ depend on both $(\phi_0, v_0)$ and $(\phi_1, v_1)$ and so the Jacobian is defined implicitly. Also the elements of (5) become unbounded as $v_1 \to 0$, that is, as the next impact tends to a graze. $P_I$ is therefore a map from the half-cylinder to itself which is everywhere smooth except on the 1-dimensional set $S$, the discontinuity set, where

$$S = \lim_{\epsilon \to 0} \left\{(\phi_0, v_0) : P_I(\phi_0, v_0) = (\phi, \epsilon) \text{ for some } \phi \in 2\pi S^1\right\}$$

The set $S$ can be thought of as the pre-image of the line $v = 0$. The sets $S$ are included in Figures 3 and 4 and a more complicated example is shown in Figure 5. $S$ is not, as one might expect, a smooth closed curve that spans the phase space but instead consists of one or more smooth curve segments. This is because for part of the line $v = 0$ the acceleration is positive and the mass sticks to the wall. Also there are trajectories that have consecutive grazes and these cause the smooth segments of $S$ to connect to one another in a complicated way as shown in Figure 5. For a detailed description of $S$ see [7]. An important result that we shall need in §4 and §6 is that $S$ is bounded above. This means that for sufficiently high velocities the impact map will be smooth.
The time-reversal symmetry of the system means that the phase space of $P_1$ is invariant under the transformation
\[ t \mapsto -t, \quad \phi \mapsto 2\pi - \phi \] (7)
and is symmetric about the midline $\phi = \pi$. The forward image of the line $v = 0$ is therefore the image of $S$ under the time reversal symmetry (7). We call this line $W$. We define further iterates and preiterates of these sets by
\[ S^n = P_1^{1-n}(S), \quad W^n = P_1^{n-1}(W) \]

The map $P_1$ behaves very differently on each side of $S$. On one side of the curve the trajectories have low velocity impacts and the phase space becomes very distorted as can be seen from the singular Jacobian for $v = 0$. On the other side, the trajectory just misses the wall and hits it at a later time and the Jacobian does not have large elements.

3 The dynamics of the impact map near $S$

To illustrate the nature of $P_1$ close to the discontinuity set $S$ we imagine a line segment $I$ of initial conditions in phase space which transversely crosses $S$ (Figure 6). Let $I$ and $S$ meet at point $B$ and call the resulting two sections of $I$, $I^+$ and $I^-$. The particle motions corresponding to the points $A, B$ and $C$ are those shown in Figure 6. As we move along $I^+$ from $A$ to $B$ the image curve $P_1(I^+)$ is traced out and moves towards the line $W$ and meets it transversely. The endpoint $P_1(B)$ lies on the line $W$ but is not (generically) the image of the point $B$ under the symmetry transformation. As we continue from $B$ to $C$ the next impact is now a low velocity impact. So the curve $P_1(I^-)$ grows out of the line $v = 0$ and the line $I$ would appear to be split in two. However, the second iterate of $I^-$ rejoins $P_1(I^+)$ meeting it at $P_1(B)$ and Whiston [7] showed that $P_1^2(I^-)$ meets $W$ tangentially. The side of $S$ that does not map directly to low velocity impacts is called the non-grazing side and the side that does is the grazing side. The line $P_1^2(I^-)$ is locally stretched by a factor of $O(\epsilon^{-\frac{1}{2}})$ where $\epsilon$ is the distance from $S$. This stretching is described by a square-root singularity. Together the cutting and stretching have important implications for the dynamics. Because grazes tend to collapse trajectories onto $W$, $W$ and its iterates strongly influence the overall dynamics. The stretching also means that periodic orbits which include low velocity impacts, $v \ll 1$, are highly likely to be unstable and so lie in the stochastic sea as can be easily seen by examining the Floquet multipliers derived from (5) or (6).

We end this section with a few final comments about the map $P_1$. The impact oscillator system is unusual in that the preferred Poincaré section
permits tangential intersections with the flow. This is not because there is no available everywhere-transverse section ($\Sigma_S$ for example) but because the impact map is a much more powerful tool for studying the impacts which are the source of chaos in the system. It is the tangential intersections with the Poincaré surface that define the sets $S$ and $W$, but it is the low velocity impacts that are responsible for the cutting and stretching that is observed near $W$. Because $x = \sigma$ is both the Poincaré surface and the impacting surface these effects are seen together.

4 Chaotic motion and partial barriers

We now study the main chaotic region, often known as the stochastic sea. The two phase space plots, Figures 3 and 4, show the stochastic sea lying between the line $v = 0$ and the regular region which exists at high velocities. It is our intention to explain the novel behaviour seen in Figure 4 where trajectories move between the stochastic sea and the regular region lying above it. To do this we must look carefully at the interaction between the discontinuity set and regular curves at the upper boundary of the stochastic sea.

4.1 The discontinuity set and the stochastic sea

As we have already shown, the set $S$ is a very strong source of chaos in the system due to the arbitrarily large stretching close to a graze. From numerical experiments it appears to be impossible for an invariant curve to cross $S$ without being destroyed. This agrees with previous studies [11, 19] which observed that invariant curves do not usually survive crossing lines of discontinuities in the first or second derivative and implies that $S$ can only lie within chaotic regions. It is easy to obtain an upper bound on $S$ from equations (2) and (3) and also straightforward to show that $S$ must touch the line $v = 0$ (for all $r, \sigma$ and rational $\omega$ there exists a smooth motion which never hits the wall but repeatedly grazes it). These facts strongly suggest (but do not prove) that the set $S$ will lie in a single bounded chaotic region extending upwards from $v = 0$, namely the stochastic sea. This is supported by the numerical studies.

4.2 The boundary of the stochastic sea

First let us consider the case where $S$ lies well away from the boundary of the sea. This is illustrated by Figure 3 where $S$ lies well inside the sea. The boundary is the lowest invariant curve that spans the cylindrical phase
space and since the map is smooth in the neighbourhood of this curve the boundary is the same as those found in smooth area-preserving maps, with regular curves above it and a sticky chaotic layer just below it.

We now examine Figure 4. This shows a single trajectory, with initial condition lying within the stochastic sea and followed for 20000 impacts. In that time it makes 3 excursions (each containing many hundreds of impacts) onto regular curves surrounding two different period-5 orbits. There is no smooth boundary to the stochastic sea, but a partial barrier which allows trajectories to move between the stochastic sea and the elliptic islands where the trajectory moves slowly around regular curves until it reenters the sea. This is because for these parameter values $S$ does not lie within the stochastic sea, but touches the boundary and intersects the islands associated with the two elliptic period-5 orbits.

In order to understand this mechanism we first look at how a trajectory moves from an elliptic curve into the stochastic sea. Figure 8 shows a schematic blowup of the last few iterates of the motion along the elliptic curve that is intersected by $S$ (so only every 5th iterate is shown). The trajectory moves slowly down the curve until it crosses $S$. The trajectory is now on the grazing side of $S$ and so the next impact (which is not shown) will be a low velocity impact which did not occur on the previous cycle. The trajectory is now cut, stretched and reconnected as described in §5 so that it returns to the neighbourhood of the elliptic curve having been stretched tangentially along the line $W^5$ and then very rapidly disappears into the stochastic sea.

The opposite mechanism, where the trajectory suddenly jumps out of the stochastic sea onto the regular curves is of course just the time reversal of the above mechanism. So the corresponding diagram is Figure 8 with the order of the points reversed, $S$ replaced by $W$ and $W^5$ replaced by $S^5$. It is not necessary for the system to be time-reversible in order for trajectories to be able to leave the sea, the existence of such trajectories can be deduced from the area-preserving property of the map — if trajectories can enter a region then trajectories must also leave that region.

We refer to the boundary in this case as a partial barrier since the discontinuity set and its iterates form structures which severely limit the rate of transport (of phase space area) across them but do not stop it entirely. Partial barriers for smooth systems (see [23]) include cantori and the turnstiles associated with periodic orbits. The mechanism described here is completely different and, unlike these other cases, the motion on one side of the barrier is regular. Some sections of the partial barrier allow trajectories to move upwards while others allow trajectories to move down. A section of the barrier is enlarged in Figure 9. For this piece of the barrier the upward and downward sections are separated by the invariant curve which just touches $S$ and surrounds the higher of the period-5 elliptic orbits.
The lower of the two period-5 elliptic orbits (the one that appears to lie in the stochastic sea) does not actually exist, even though part of the surrounding curves do. It corresponds to an unphysical motion — that is, one in which the mass moves through the wall instead of experiencing a low-velocity impact. The position of these unphysical orbits can easily be found by modifying the numerical code so that low-velocity impacts are ignored and the mass is allowed to move freely for a short time in the forbidden region $x > \sigma$.

The precise nature of the partial barrier depends upon the number, type and orientation of the regular curves and regions that are intersected by $S$ and it is possible to find parameter values for which the pictures can become extremely complex, for example when $S$ intersects higher order chains of Birkhoff elliptic points.

There is however another simple example that is very important, shown in Figure 10, and that is when $S$ intersects a region filled with bounding invariant curves rather than a elliptic region. At first glance Figure 10 looks just like Figure 3 with a stochastic sea bordered by a smooth invariant curve. However $S$ now touches the boundary and the difference lies in the position and the nature of this invariant curve. It is the lowest bounding curve not because it is on the point of breaking up, with regular curves above it and a sticky chaotic border below, but because it lies immediately above $S$ and all the invariant curves below it have been destroyed. Let us now consider the region just below this bounding curve. Here the phase space consists of invariant curves interrupted by $S$. Once a trajectory lands on one of these curves (having been stretched along $W$ by a low velocity impact) it will move regularly along that curve until it lands on that small part which lies on the grazing side of $S$. It then has a low velocity impact and moves back down into the sea. This process is essentially the same as that of Figure 4 — the only difference being that in Figure 10 the regular curves interrupted by $S$ are bounding curves rather than elliptic curves and it is only the latter which rise above $S$ and produce startling pictures such as Figure 4. So, while the stochastic sea in Figure 10 looks like a perfectly ordinary chaotic region from a smooth system it does in fact display the same alternating regular/chaotic behaviour as Figure 4.

The grazing mechanism provides an interesting new transition between the regular curves that exist at high velocities and the truly chaotic motion at very low velocities. If we replace the wall by a steep potential gradient then the resulting smooth system should also display similar behaviour.

We end by noting that exactly the same interactions between the discontinuity set and regular curves that occur at the upper boundary can occur at the boundaries between the chaotic region and elliptic islands that lie within it.
5 Related impacting systems

Impacting systems and discontinuous mappings have been studied before. To help put our work into perspective we briefly look at two closely related classes of impacting system which have been fundamental in the study of both smooth and discontinuous dynamical systems.

5.1 Billiards

Billiards are an important class of impacting system [17, 18, 19]. A billiard is a point mass moving in a bounded 2-dimensional region with boundary $\mathcal{D}$. When the mass hits $\mathcal{D}$ it bounces away elastically, following the usual ‘angle-of-incidence equals angle-of-reflection’ law. A boundary component is called concave or dispersing if it curves away from the bounded region and convex or focussing if it bends inwards. A billiard is called convex if the bounded region is convex. Natural, area-preserving, coordinates to use for the impact map are the curvilinear distance, $\eta$, along the boundary from some arbitrary point and $s = \sin \alpha$ where $\alpha$ is the oriented angle between the normal to the boundary and the incoming trajectory (see Figure 7). The mapping from one impact $(\eta_0, s_0)$ to the next $(\eta_1, s_1)$ has an implicitly defined Jacobian [18, 19]

$$J = \frac{\partial (\eta_1, s_1)}{\partial (\eta_0, s_0)} = \begin{pmatrix}
\frac{C_0 d - \cos \alpha_0}{\cos \alpha_1} & -\frac{d}{\cos \alpha_0 \cos \alpha_1} \\
C_0 \cos \alpha_1 + C_1 \cos \alpha_0 - C_0 C_1 d & \frac{C_1 d - \cos \alpha_1}{\cos \alpha_0}
\end{pmatrix}$$

where $C_0, C_1$ denote the curvature at the points of impact and $d$ is the Euclidean distance between the points of impact. The form of this Jacobian in the limit $\cos \alpha_1 \ll 1$, which corresponds to the mass hitting a non-convex piece of the boundary almost tangentially, is the same as that of the impact oscillator with area-preserving coordinates (6) in the grazing limit $z_1 \ll 1$. In other words, a low velocity impact in the impact oscillator corresponds to a nearly tangential impact in a non-convex billiard which will have a discontinuity set at the pre-images of such impacts. In fact the cutting and stretching close to grazing trajectories described in §3 and Figure 6 is exactly the same for these non-convex billiards.

The most frequently studied billiards are those with boundaries that are either everywhere-focussing or everywhere-dispersing. Billiards that have boundaries with focussing and dispersing components are likely to have both regular and chaotic regions and the mechanism described in §4 will also be of importance to the study of these systems.
5.2 Accelerator models

In 1949 Fermi [24] proposed a mechanism for the acceleration of cosmic rays that involved collisions with magnetic field structures. Much work followed in which this process was modelled by particles impacting repeatedly (and elastically) against oscillating heavy objects. The idea also found a natural application to particle accelerators where imperfections in the confining plasma ring could be modelled as periodic ‘kicks’ in a very similar manner. Because of these applications, and the simple form that the mappings take, they have become standard problems in Hamiltonian systems with two degrees of freedom. Some work has also been done for inelastic systems [25, 26].

Several different accelerator models have been studied. Two of the most important are the Fermi model and the Pustyl’nikov model.

5.2.1 The Fermi models

Ulam et al. [10] performed the first study of the Fermi mechanism. The model they used was a particle bouncing elastically with constant speed between two walls — one fixed and one moving periodically. The impact maps arising from such systems are known as Fermi maps. They were primarily interested in the long-term behaviour of the particle and the maximum velocity that could be attained. Further studies can be found in [11, 12, 14, 15].

Typically the grazing discontinuity set lies far below the lowest bounding invariant curve which is why the chaotic/regular trajectories described in § 4 have not been observed for this model. A simplification that has been frequently made is to treat the position of the oscillating wall as being fixed while allowing its velocity to oscillate. In this way, the time of the next impact can be calculated explicitly for many wall motions and so the mapping itself can be written explicitly. This is a good approximation for high-velocity motions although at low velocities it precludes the possibility of grazing. For a sinusoidal wall motion the simplified mapping is

\[
\begin{align*}
    u_{n+1} & = |u_n + \sin(\psi_n)| \\
    \psi_{n+1} & = (\psi_n + A/u_{n+1}) \mod 2\pi
\end{align*}
\]

where \( A \) is the system parameter. Note that the time between impacts is inversely proportional to the velocity.

At low velocities there is a stochastic sea interspersed with elliptic islands. At higher velocities it has been proved (see [13]) that for wall velocities that are \( C^{3+\epsilon} \) there are invariant curves which cross the phase space and act as upper bounds to the velocity of the particle. If the wall motion is given by, for example, a saw-tooth function then these bounding curves disappear and the motion can become unbounded. Indeed much of the subsequent work on this
system was concerned with the existence of such curves and the smoothness conditions of KAM theory.

5.2.2 The Pustyl’nikov map

Pustyl’nikov [13] examined a different system consisting of a ball returning to an oscillating wall under the influence of gravity. Again the position of the wall can be regarded as fixed for high velocities and this gives rise to the following simplified mapping

\[ u_{n+1} = |u_n + \sin(\psi_n)|, \]
\[ \psi_{n+1} = (\psi_n + Au_{n+1}) \mod 2\pi \] (9)

for sinusoidal forcing. Pustyl’nikov proved that even for analytic wall velocities parameters can be chosen such that there is no bound to the velocity of the particle. The difference between this case and the Fermi maps is due to the time between impacts at high energies which now is proportional to the velocity.

5.2.3 The impact oscillator

In (1) the wall is stationary and the forcing is on the spring. However it is an easy exercise to show that, upon making the substitution \( y = x - \gamma \cos(\omega t) \), equation (1) becomes

\[ \ddot{y} + y = 0, \quad y < \sigma - \gamma \cos(\omega t) \]
\[ \dot{y} \rightarrow -r \dot{y} + (1 + r)\gamma \omega \sin(\omega t), \quad y = \sigma - \gamma \cos(\omega t) \] (10)

This is the equation of motion for an unforced mass moving on a linear spring and impacting against a wall whose position at time \( t \) is given by \( y = \sigma - \gamma \cos(\omega t) \). Therefore, except for \( \omega = 1 \), the forced oscillator problem defined by (1) is equivalent to a moving wall problem. This equivalence also holds for more general periodic forcing as long as there is no power at the frequency \( \omega = 1 \). This relationship between the two systems is highly relevant since in physical situations the driving oscillations may act on the wall or on the mass/spring component.

In the Fermi model the particle moves with constant speed between impacts with the vibrating wall. In the Pustyl’nikov model the particle moves freely under a constant acceleration (linear potential). The next ‘natural’ case to consider is a particle moving in a quadratic potential which, by the above result, is the impact oscillator (for \( \omega \neq 1 \)). So the impact oscillator fits very naturally into the family of accelerator models.

The impact oscillator is similar to the Fermi model with chaotic behaviour at low energies and regular behaviour at higher energies. But there are
important differences, especially at high velocities. These are also due to the
time between impacts. For the Fermi model the time between impacts is
$O(1/v)$ for large velocities while for the Pustyl’nikov model it is $O(v)$. For
the impact oscillator the time between impacts tends to $\pi$ as $v$ increases as
can be seen from equations (2) and (3). So at high velocities the period
of the oscillation is almost independent of the amplitude. This property is
reminiscent of linear oscillators and for this reason we expect resonance effects
to be important. This is indeed the case and the behaviour at high velocities
is much more complicated than for the Fermi or Pustyl’nikov models, as we
show in §6.

5.3 Discontinuities in impacting systems

We briefly discuss the different kinds of discontinuities that can appear, and
have been studied, in impact maps.

A billiard with a smooth boundary has a smooth impact map and in-
troducing a discontinuity into the $n^{th}$ derivative of the curvature at some
point of the boundary leads to an impact map with discontinuous $(n - 1)^{th}$
derivative. This situation has been much studied, especially with respect to
proving ergodic and mixing properties and testing the smoothness conditions
of KAM theory. Similarly, if for the simplified Fermi map, the wall velocity is
smooth, then the impact map is also smooth and introducing discontinuities
into the wall velocity or a higher derivative of the wall velocity results in a
corresponding discontinuity in the impact map.

When the map itself is discontinuous, nearby trajectories that straddle the
discontinuity are separated. When the discontinuity is in the first derivative
or higher, nearby trajectories remain close together. For examples of such
discontinuous systems, see [11, 19, 27, 28, 29].

The grazing discontinuity is of a different form. Nearby trajectories that
straddle $S$ are not separated. Instead, one trajectory has an extra (low
velocity) impact and the distance between the trajectories is stretched as
was described in the previous section. It is perhaps helpful to think of a
graze as being less severe than a discontinuity in the mapping but more
severe than a discontinuity in the first derivative (if the extra low velocity
impact is ignored then the impact map is Hölder continuous with exponent
$\frac{1}{2}$). Grazing discontinuities are a natural feature of impacting systems, much
more natural than a discontinuity in the $n^{th}$ derivative, but have been less
thoroughly studied.
6 Behaviour of the impact oscillator at high velocities

In this section we examine the impact oscillator at high velocities in the region above $S$ where $P_1$ is smooth.

We showed in §5 that the elastic impact oscillator fits very naturally into the family of accelerator models that includes the Fermi maps and the Pustyl’nikov maps. This link motivates us to ask the following question — under what conditions is it possible for the velocity of the mass to become unbounded? To answer this it is necessary to find conditions for the existence or non-existence of bounding invariant curves.

Similar questions have also been asked of smooth systems. The boundedness or otherwise of a particle moving in a one-dimensional smooth time-periodic potential has been investigated by several authors, see Norris [30] and references therein. Norris gave a sufficient condition for the existence of bounding curves for a system of the form

$$\ddot{x} + g(x) = p(t)$$

where $p(t)$ is a sufficiently smooth periodic function, $g(x) \to \infty$ as $x \to \pm \infty$ and either

$$g(x)/x \to \infty \text{ as } x \to \pm \infty,$$  

or

$$g(x)/x \to 0 \text{ as } x \to \pm \infty$$

The method of proof relies on the construction of a twist map and breaks down for functions $g(x)$ which are ‘too close’ to the linear oscillator. The obvious example is the linear oscillator itself $g(x) = x$ for which (11) has unbounded solutions whenever $p(t)$ has a Fourier component with the resonant frequency 1.

The impact oscillator retains some of the characteristics of the linear oscillator which describes its motion between impacts, and for high velocity motions the time between impacts tends to a constant. Therefore, the frequency is approximately independent of the amplitude of the motion and we can expect there to be resonant forcing frequencies.

We first consider the response of the impact oscillator to sinusoidal forcing and show that it depends not just on the forcing frequency but also on the position of the wall. We use the impact map $P_1$ to examine (11) at high energies and give sufficient conditions on the parameter values for the velocities to be bounded for all time. We also find parameter values for which there exist trajectories whose velocity can become unbounded. We then briefly look at the case of more general periodic forcing.

We start by considering the approximate behaviour of the system at large velocities.
Lemma 6.1 For $\omega \neq 1$ and $v \gg \max(1, \omega \gamma)$ the impact map has the following form

\[ v_1 = v_0 + f(\phi_0) + O\left(\frac{1}{v_0}\right) \]
\[ \phi_1 = (\phi_0 + \alpha + \frac{g(\phi_0)}{v_0} + O\left(\frac{1}{v_0^2}\right)) \mod 2\pi \tag{12} \]

where $\alpha$ is a constant independent of $(\phi_0, v_0)$.

Proof We assume that $v_0$ is large and, in particular, that the point $(\phi_0, v_0)$ lies above the set $S$ which is bounded. From equation (4), $t_1$ satisfies

\[ \sigma = (\sigma - \gamma \cos \omega t_0) \cos(t_1 - t_0) - (v_0 - \omega \gamma \sin \omega t_0) \sin(t_1 - t_0) + \gamma \cos \omega t_1 \tag{13} \]

and for large $v_0$ the time between impacts is close to $\pi$. So we let $t_1 - t_0 = \pi + \delta$ where $\delta$ is small and expanding (13) in powers of $1/v_0$ gives

\[ \delta = \frac{2\sigma - \gamma(\cos \omega t_0 - \cos(\phi_0 + \alpha))}{v_0} + O\left(\frac{1}{v_0^2}\right) \]

where $g(t)$ as defined in the statement of the Lemma is

\[ g(t) = 2\sigma - \gamma(\cos \omega t + \cos(\phi_0 + \alpha)) \tag{14} \]

Substituting this value for $t_1$ into equation (3) we obtain the following expression for $v_1$

\[ v_1 = v_0 - \omega \gamma(\sin \omega t_0 + \sin(\phi_0 + \alpha)) \]
\[ + \frac{g(t_0)}{v_0} [\sigma - \gamma \cos \omega t_0 + (1 - \gamma) \cos(\phi_0 + \alpha)] + O\left(\frac{1}{v_0^2}\right) \tag{15} \]

We let

\[ f(t) = -\omega \gamma(\sin \omega t + \sin(\phi_0 + \alpha)) \tag{16} \]

Replacing $t$ by the phase $\phi$ in (14) and (16) we obtain

\[ v_1 = v_0 - \omega \gamma(\sin(\phi_0) + \sin(\phi_0 + \alpha)) + O\left(\frac{1}{v_0}\right) \]
\[ \phi_1 = [\phi_0 + \alpha + \frac{2\omega \sigma - \omega \gamma(\cos(\phi_0) + \cos(\phi_0 + \alpha))}{v_0} + O\left(\frac{1}{v_0^2}\right)] \mod 2\pi \tag{17} \]

where $\alpha = \omega \pi \mod 2\pi$.

We now prove the following result.

Theorem 6.2 If $\omega \neq 2n, n \in \mathbb{N}^+$ and $\sigma \neq 0$ then the velocity of the impact oscillator is bounded for all time for all initial conditions.
**Proof** We prove this by showing that at high velocities the impact map is a perturbation of an integrable twist map of the form

\[
\begin{align*}
    r_1 &= r_0 \\
    \psi_1 &= \psi_0 + \beta(r)
\end{align*}
\]

where \(\beta' \neq 0\). Then, if certain other conditions are satisfied, Moser’s small twist theorem [31] guarantees the existence of infinitely many bounding invariant curves. First we consider the case \(\omega \neq 1\).

We shall need the following Lemma.

**Lemma 6.3** Let \(n\alpha \neq 0 \mod 2\pi\) where \(n \in \mathbb{N}^+\). Then the functional equations

\[
\begin{align*}
    F(\phi + \alpha) - F(\phi) &= \sin(n\phi + n\alpha) + \sin(n\phi) \\
    G(\phi + \alpha) - G(\phi) &= \cos(n\phi + n\alpha) + \cos(n\phi)
\end{align*}
\]

are respectively solved by

\[
\begin{align*}
    F(\phi) &= -\cot\left(\frac{n\alpha}{2}\right) \cos(n\phi) \\
    G(\phi) &= \cot\left(\frac{n\alpha}{2}\right) \sin(n\phi)
\end{align*}
\]

**Proof** The result is easily established by using the Fourier transform. \(\square\)

We now assume \(v \gg \omega \gamma \cot(\alpha/2)\) and make the following coordinate change

\[
\begin{align*}
    w &= v + \omega \gamma \cot\left(\frac{\alpha}{2}\right) \cos(\phi) \\
    \psi &= \phi - \frac{\omega \gamma}{v} \cot\left(\frac{\alpha}{2}\right) \sin(\phi)
\end{align*}
\]

which, using the results of Lemma 6.3 for \(n = 1\), transforms (17) into

\[
\begin{align*}
    w_1 &= w_0 + \mathcal{O}\left(\frac{1}{w_0^3}\right) \\
    \psi_1 &= \psi_0 + \alpha + \frac{2\sigma \omega}{w_0} + \mathcal{O}\left(\frac{1}{w_0^2}\right)
\end{align*}
\]

Physically, this coordinate transformation corresponds to finding the approximate form of the bounding invariant curves, the existence of which we now prove.

Now, following [31] we introduce a small parameter \(\rho\) via the transformation \(w = \rho^{-1}r\) where \(0 < \rho \leq 1\) and \(1 \leq r \leq 2\). Equation (24) becomes

\[
\begin{align*}
    r_1 &= r_0 + \mathcal{O}(\rho^3) \\
    \psi_1 &= \psi_0 + \alpha + \frac{2\sigma \omega \rho}{r_0} + \mathcal{O}(\rho^2)
\end{align*}
\]
This is a small perturbation of a mapping of the form (18) and we are now almost in a position to use Moser’s small twist theorem. First we note that the map (25) is analytic and so easily satisfies the smoothness assumption of Moser’s theorem. Secondly, if \( \sigma \neq 0 \), then the twist condition \( \beta' \neq 0 \) in equation (18) is also satisfied. It only remains to demonstrate the curve intersection property for (25), namely that any bounding curve \( C \) intersects its image \( C' \). This follows from the area-preserving property of \( P_1 \) via the coordinate transformations we have made since. Even though at low velocities the phase space is cut and stretched by the set \( S \), the area under a bounding curve \( C \) which does not intersect \( S \) must be preserved.

The result for \( \omega \neq 1 \) now follows by a straightforward application of Moser’s small twist theorem which guarantees the existence of invariant bounding curves for (24) for arbitrarily small \( \rho \), which corresponds to arbitrarily high velocities of the original system. Therefore the velocity of the mass is bounded for all time.

Finally we consider the case \( \omega = 1 \). This is a special case because the equations of motion (3) and (4) do not apply. Instead the motion after an impact at \( (t_0, v_0) \) is given by

\[
x(t) = \sigma \cos(t - t_0) - (v_0 + \frac{1}{2} \sin(t_0)) \sin(t - t_0) \\
+ \frac{1}{2} (t - t_0) \sin(t)
\]

\[
v(t) = -\sigma \sin(t - t_0) - (v_0 + \frac{1}{2} \sin(t_0)) \cos(t - t_0) \\
+ \frac{1}{2} (t - t_0) \cos(t) + \frac{1}{2} \sin(t)
\]

Proceeding as before we find that for \( v \gg 1 \) the impact map has the form

\[
v_1 = v_0 - \frac{\pi}{2} \cos(\phi_0) + O\left(\frac{1}{v_0}\right)
\]

\[
\phi_1 = \phi_0 + \pi + \frac{1}{v_0} (2\sigma + \frac{\pi}{2} \sin(\phi_0)) + O\left(\frac{1}{v_0^2}\right) \mod 2\pi
\]

(26)

If we now consider \( P_1^2 \) then (23) becomes

\[
v_2 = v_0 + O\left(\frac{1}{v_0}\right)
\]

\[
\phi_2 = \phi_0 + \frac{4\sigma}{v_0} + O\left(\frac{1}{v_0^2}\right)
\]

(27)

and, for \( \sigma \neq 0 \), we can apply Moser’s invariant curve theorem as before. This completes the proof of the theorem. \( \square \)
6.1 Unbounded motion

The resonant cases $\omega = 2n$ do in fact lead to unbounded motion for certain values of $\sigma$. Since $\alpha = 0$, (17) becomes

$$v_1 = v_0 - 2\omega \gamma \sin(\phi_0) + O\left(\frac{1}{v_0}\right)$$

$$\phi_1 = \phi_0 + \frac{2\omega}{v_0}(\sigma - \gamma \cos(\phi_0)) + O\left(\frac{1}{v_0^2}\right) \mod 2\pi$$

(28)

We prove the following result

**Theorem 6.4** For $\omega = 2n, n \in \mathbb{N}^+$, there are unbounded trajectories if $|\sigma| < |\gamma|$.

**Proof** Making the substitution $s = 1/v$ into (28) we get

$$s_1 = s_0 + 2s_0^2\omega \gamma \sin(\phi_0) + O(s_0^3)$$

$$\phi_1 = \phi_0 + 2s_0\omega(\sigma - \gamma \cos(\phi_0)) + O(s_0^2)$$

(29)

We now define $\phi^*$ by $\phi^* = \cos^{-1}(\sigma/\gamma)$ and let $t_i = \phi_i - \phi^*$.

Assuming that $s, t \ll 1$ and calculating the first neglected term in (29) we obtain the following

$$t_1 = t_0(1 + 2\omega \gamma s_0 \sin(\phi^*)) + O(s_0^2t_0, s_0^2t_0^2)$$

$$s_1 = s_0(1 + 2\omega \gamma s_0 \sin(\phi^*)) + O(s_0^3, s_0^2t_0)$$

(30)

By the definition of $\phi^*, 2\omega \gamma \sin(\phi^*) < 0$ and so for sufficiently small $s, t$,

$$|s_n| < |s_{n-1}| \text{ and } |t_n| < |t_{n-1}|$$

and it is clear that $\lim_{n \to \infty} s_n = 0$ which corresponds to a motion with unbounded velocity.

We can summarise the above results as follows. For the generic case $\sigma \neq 0$ and $\omega \neq 2n$ the motion is bounded at sufficiently high velocities. When $\sigma = 0$ the boundedness of the motion is marginal and depends on the higher order terms that were neglected in the above analysis. If the energy is in fact unbounded, then the rate of energy gain is extremely slow. The resonant frequencies are at $\omega = 2n$ where unbounded motion occurs for $|\sigma| < |\gamma|$ and numerical experiments indicate that this inequality is tight and for $|\sigma| > |\gamma|$ the motion is bounded (in this case, the map (29) cannot be regarded as a small perturbation of an integrable map of the form (18) and Moser’s theorem cannot be applied).

Thus the existence of bounding invariant curves depends crucially on the position of the wall, $\sigma$. For $|\sigma|$ sufficiently large (which means $\sigma \neq 0$ for
non-resonant $\omega$) the motion is bounded. In the mechanical system described by (1) there are two ways of changing the value of $\sigma$. One way is of course simply to move the wall. The other is to add a constant term, $a$, to the forcing $\cos(\omega t)$. Then a simple coordinate change $x \rightarrow x - a$ recovers the initial impact oscillator system with the new value of $\sigma$ given by $\sigma \rightarrow \sigma - a$. This leads us to the physically interesting, and rather counterintuitive, result that to suppress any unbounded behaviour it is enough just to add a constant force of sufficient magnitude — pointing either towards or away from the wall.

6.2 General periodic forcing

We can use the fact that the motion between impacts is linear to extend the method to the case of more general periodic forcing functions. We consider a forcing function $p(t)$ with period $2\pi/\omega$ and Fourier components given by

$$p(t) = \sum_{n=1}^{\infty} p_n \cos(n\omega t) + q_n \sin(n\omega t)$$

where, as described above, the constant term in the Fourier series has been taken as zero and absorbed into the position of the wall. We now prove the following result.

**Theorem 6.5** Let $\sigma \neq 0$ and $p(t) \in C^{2+\epsilon}$. Then for a set of irrational $\omega$ having full measure the velocity of the impact oscillator is bounded for all time for all initial conditions.

**Proof** In order to simplify the algebra we shall only consider an even periodic forcing function

$$p(t) = \sum_{n=1}^{\infty} p_n \cos(n\omega t)$$

since the terms of the sine expansion are dealt with in an identical fashion.

The time of the next impact, $t_1$, is given by the first solution of the equation

$$\sigma = (\sigma - \sum_{n=1}^{\infty} \gamma_n p_n \cos(n\omega t_0)) \cos(t_1 - t_0) -$$

$$\left(v_0 - \sum_{n=1}^{\infty} n\omega \gamma_n p_n \sin(n\omega t_0)\right) \sin(t_1 - t_0) + \sum_{n=1}^{\infty} \gamma_n p_n \cos(n\omega t_1)$$

where $\gamma_n = 1/(1 - n^2 \omega^2)$. If $p(t) \in C^{2+\epsilon}$ then the LHS of (33) is $C^{3+\epsilon}$ since $n\gamma_n \approx 1/n$ as $n \rightarrow \infty$. So by the implicit function theorem $t_1(t_0, v_0) \in C^{3+\epsilon}$. 


Taking \( v \gg 1 \) and writing \( t_1 = t_0 + \pi + \delta \) we obtain the following equation for \( \delta \)

\[
\sigma = (-\sigma + \sum_{n=1}^{\infty} \gamma_n p_n \cos(n \omega t_0)) \cos(\delta) + (v_0 - \sum_{n=1}^{\infty} n \omega \gamma_n p_n \sin(n \omega t_0)) \sin(\delta) \\
+ \sum_{n=1}^{\infty} \gamma_n p_n \cos(n \omega (t_0 + \pi + \delta))
\]  

(34)

Proceeding as before, we get the following approximate mapping at sufficiently high velocities

\[
v_1 = v_0 - \sum_{n=1}^{\infty} n \omega \gamma_n p_n (\sin n \phi_0 + \sin(n \phi_0 + n \alpha)) + O\left( \frac{1}{v_0} \right) \\
\phi_1 = \phi_0 + \alpha + \frac{2 \sigma \omega - \sum_{n=1}^{\infty} \omega \gamma_n p_n (\cos n \phi_0 + \cos(n \phi_0 + n \alpha))}{v_0} + O\left( \frac{1}{v_0^2} \right)
\]  

(35)

where \( \alpha = \omega \pi \mod 2\pi \) and the mapping is \( C^{3+\epsilon} \). Once again, for \( \alpha \neq 0 \), Lemma 6.3 suggests the correct coordinate change to reduce the mapping to a sufficiently small perturbation of an integrable twist map. This change is

\[
w = v + \sum_{n=1}^{\infty} n \omega \gamma_n p_n \cot\left( \frac{n \alpha}{2} \right) \cos(n \phi) \\
\psi = \phi - \frac{1}{v_0} \sum_{n=1}^{\infty} \omega \gamma_n p_n \cot\left( \frac{n \alpha}{2} \right) \sin(n \phi)
\]  

(36)

and is only well-defined if both of the Fourier series in (34) converge. Manipulating the coefficients of the first series slightly (if this series converges then so does the second one), we get

\[
\sum_{n=1}^{\infty} \left| n \omega \gamma_n p_n \cot\left( \frac{n \alpha}{2} \right) \cos(n \phi) \right| \leq \sum_{n=1}^{\infty} \left| n \omega \gamma_n p_n \sin\left( \frac{n \alpha}{2} \right) \right| \leq \sum_{n=1}^{\infty} \left| 2n \omega \gamma_n p_n \right| \left| \frac{1}{1 - e^{in \alpha}} \right|
\]

This last sum is the classic ‘small-divisor’ problem and for \( p \in C^{2+\epsilon} \), convergence of the series is guaranteed if \( \omega \) is irrational and has a continued fraction expansion that satisfies certain conditions. The set of such \( \omega \) has full measure. Our smoothness condition on \( p(t) \) was chosen so that the perturbation is \( C^{3+\epsilon} \) and we can now apply Moser’s twist theorem exactly as for the sinusoidal case.

So for a set of forcing frequencies that has full measure, the motion will be bounded for all \( C^{2+\epsilon} \) forcing functions (for \( \sigma \neq 0 \)). It is not clear whether, for general periodic forcing, unbounded behaviour can still be removed by increasing the magnitude of the clearance.

\[\Box\]
From § 5.2.3 the impact oscillator with forcing

\[ p(t) = \sum_{n=1}^{\infty} p_n \cos(n\omega t) \]

is equivalent to the moving wall problem with wall motion given by

\[ y(t) = \sum_{n=1}^{\infty} \gamma_n p_n \cos(n\omega t) \]

So we have the immediate corollary that for the moving wall system the $C^{2+\epsilon}$ smoothness condition must be replaced by a $C^{4+\epsilon}$ condition on $y(t)$. This corresponds to a wall velocity which is $C^{3+\epsilon}$.

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Figure 1: The mechanical impact oscillator.

Figure 2: Trajectories in the neighbourhood of a graze.

Figure 3: Several trajectories of the impact oscillator showing the phase space of $P_1$ for parameter values $\sigma = 0.1, \omega = 2.5$. The thick line is the discontinuity set.

Figure 4: A single trajectory showing both regular and chaotic behaviour for $\sigma = 0, \omega = 2.8$. The thick line is the discontinuity set.

Figure 5: The discontinuity set $S$ for $\sigma = 0, \omega = 5.3$.

Figure 6: The dynamics close to $S$.

Figure 7: Billiard coordinates

Figure 8: Close-up of the partial barrier together with the last few iterates of a trajectory on an elliptic curve before it reenters the stochastic sea.

Figure 9: An invariant curve, denoted by the solid line, separating pieces of the partial barrier that allow trajectories up and down.
Figure 10: The phase space for $\sigma = 0, \omega = 2.85$. The discontinuity set is also shown.