A Simple Proof of Jung’ Theorem on Polynomial Automorphisms of $\mathbb{C}^2$

Nguyen Van Chau

Abstract. The Automorphism Theorem, discovered first by Jung in 1942, asserts that if $k$ is a field, then every polynomial automorphism of $k^2$ is a finite product of linear automorphisms and automorphisms of the form $(x, y) \mapsto (x + p(y), y)$ for $p \in k[y]$. We present here a simple proof for the case $k = \mathbb{C}$ by using Newton-Puiseux expansions.

1. In this note we present a simple proof of the following theorem on the structure of the group $GA(\mathbb{C}^2)$ of polynomial automorphisms of $\mathbb{C}^2$

Automorphism Theorem. Every polynomial automorphism of $\mathbb{C}^2$ is tame, i.e. it is a finite product of linear automorphisms and automorphisms of the form $(x, y) \mapsto (x + p(y), y)$ for one-variable polynomials $p \in \mathbb{C}[y]$.

This theorem was first discovered by Jung [J] in 1942. In 1953, Van der Kulk [Ku] extended it to a field of arbitrary characteristic. In an attempt to understand the structure of $GA(\mathbb{C}^n)$ for large $n$, several proofs of Jung’s Theorem have presented by Gurwith [G], Shafarevich [Sh], Rentchler [R], Nagata [N], Abhyankar and Moh [AM], Dicks [D], Chadzy’nski and Krasi’nski [CK] and McKay and Wang [MW] in different approaches. They are related to the mysterious Jacobian conjecture, which asserts that a polynomial map of $\mathbb{C}^n$ with non-zero constant Jacobian is an automorphism. This conjecture dated back to 1939 [K], but it is still open even for $n = 2$. We refer to [BCW] and [E] for nice surveys on this conjecture.

2. The following essential observation due to van der Kulk [Ku] is the crucial step in some proofs of Jung’ theorem.

Division Lemma: $F = (P, Q) \in GA(\mathbb{C}^2) \Rightarrow \deg P | \deg Q$ or $\deg Q | \deg P$.

Abhyankar and Moh in [AM] deduced it as a consequence of the theorem on the embedding of a line to the complex plane. McKay and Wang [MW] proved it by

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using formal Laurent series and the inversion formula. Chadzyński and Krasinski in [CK] obtained the Division Lemma from a formula of geometric degree of polynomial maps \((f, g)\) that the curves \(f = 0\) and \(g = 0\) have only one branch at infinity.

Here, we will prove this lemma by examining the intersection of irreducible branches at infinity of the curves \(P = 0\) and \(Q = 0\) in term of Newton-Puiseux expansions.

Our proof presented here is quite elementary and simpler than any proof mentioned above. It uses the following two elementary facts on Newton-Puiseux expansions (see, for example, [BK]).

Fact 1. Suppose the curve \(h = 0\) has only one irreducible branch at infinity and \(u\) is a Newton-Puiseux expansion at infinity of \(\gamma\) and the natural number \(\text{mult}(u) = \deg h\), where \(\epsilon\) is a primitive \(\deg h\)-th root of 1.

Let \(\phi(x, \xi)\) be a finite fractional power series of the form

\[
\phi(x, \xi) = \sum_{k=0}^{n-1} c_k x^{1-\frac{k}{m\varphi}} + \xi x^{1-\frac{n_{\varphi}}{m\varphi}},
\]

where \(\xi\) is a parameter and \(\gcd\{k = 0, \ldots, n_{\varphi} - 1 : c_k \neq 0\} \cup \{n_{\varphi}\} = 1\). Let us represent

\[
h(x, \phi(x, \xi)) = x^{\frac{a_{\varphi}}{m\varphi}}(h_0(\xi) + \text{lower terms in } x^{\frac{1}{m\varphi}}), \ h_0(\xi) \neq 0.
\]
The second fact is deduced from the Implicit Function Theorem.

**Fact 2.** Let $\varphi$ and $h_0$ be as in (1) and (2). If $c$ is a simple zero of $h_0(\xi)$, then there is a Newton-Puiseux expansion at infinity

$$u(x^{\frac{1}{m_{\varphi}}}) = \varphi(x, c + \text{lower terms in } x^{\frac{1}{m_{\varphi}}})$$

for which $h(x, u(x^{\frac{1}{m_{\varphi}}})) \equiv 0$. Furthermore, $\text{mult}(u)$ divides $m_{\varphi}$ and $\text{mult}(u) = m_{\varphi}$ if $c \neq 0$.

**3. Proof of the Division Lemma.** Given $F = (P, Q) \in GA(C^2)$. We may assume that $\deg P > \deg Q$ and we will prove that $\deg Q$ divides $\deg P$. By choosing a suitable linear coordinate, we can express

$$P(x, y) = y^{\deg P} + \text{lower terms in } y$$

$$Q(x, y) = y^{\deg Q} + \text{lower terms in } y.$$ 

Observe that $F$ is a polynomial diffeomorphism of $C^2$ and

$$J(P, Q) := P_x Q_y - P_y Q_x \equiv \text{const.} \neq 0.$$ 

Then, $P$ and $Q$ are reducible and each of the curves $P = 0$ and $Q = 0$ is diffeomorphic to $C$ which has only one irreducible branch at infinity. Let $\alpha$ and $\beta$ be the unique irreducible branches at infinity of $P = 0$ and $Q = 0$, respectively. Then, by Fact 1 we can find Newton-Puiseux expansion $u(x^{\frac{1}{\deg P}})$ and $v(x^{\frac{1}{\deg Q}})$ with $\text{mult}(u) = \deg P$ and $\text{mult}(v) = \deg Q$ such that

$$P(x, y) = \prod_{i=1}^{\deg P} (y - u(\sigma^i x^{\frac{1}{m_{\varphi}}}))$$

$$Q(x, y) = \prod_{j=1}^{\deg Q} (y - v(\delta^j x^{\frac{1}{m_{\varphi}}})).$$

where $\sigma$ and $\delta$ are primitive $\deg P$-th and $\deg Q$-th roots of 1, respectively.

Put $\theta := \min_{ij} \text{ord}(u(\sigma^i x^{\frac{1}{m_{\varphi}}}) - v(\delta^j x^{\frac{1}{m_{\varphi}}}))$. Without loss of generality, we can assume $\text{ord}(u(x^{\frac{1}{m_{\varphi}}}) - v(x^{\frac{1}{m_{\varphi}}})) = \theta$. We define a fractional power series $\varphi(x, \xi)$ with parameter $\xi$ by deleting in $u$ all terms of order no larger than $\theta$ and adding to it the term $\xi x^{\theta}$,$$

\varphi(x, \xi) = \sum_{k=0}^{n_{\varphi} - 1} c_k x^{1 - \frac{k}{m_{\varphi}}} + \xi x^{1 - \frac{n_{\varphi}}{m_{\varphi}}}

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with \( \gcd\{k = 0, \ldots, K - 1 : c_k \neq 0\} \cup \{n_{\varphi}\} = 1 \), where \( 1 - \frac{n_{\varphi}}{m_{\varphi}} = \theta \). Then, by definition

\[
u(x^{\frac{1}{m_{\varphi}}}) = \varphi(x, \xi_u(x)) \quad \text{with} \quad \xi_u(x) = \alpha_u + \text{lower terms in } x,
\]

\[
u(x^{\frac{1}{m_{\varphi}}}) = \varphi(x, \xi_v(x)) \quad \text{with} \quad \xi_v(x) = \beta_v + \text{lower terms in } x
\]

and \( \alpha_u - \beta_v \neq 0 \). Let us represent

\[
P(x, \varphi(x, \xi)) = x^{\frac{a_{\varphi}}{m_{\varphi}}} (P_{\varphi}(\xi) + \text{lower terms in } x^{\frac{1}{m_{\varphi}}})
\]

\[
Q(x, \varphi(x, \xi)) = x^{\frac{b_{\varphi}}{m_{\varphi}}} (Q_{\varphi}(\xi) + \text{lower terms in } x^{\frac{1}{m_{\varphi}}})
\]

where \( a_{\varphi} \) and \( b_{\varphi} \) are integers and \( 0 \neq P_{\varphi}, Q_{\varphi} \in C[\xi] \).

**Claim 1.**

(a) \( P_{\varphi}(\alpha_u) = 0 \) and \( Q_{\varphi}(\beta_v) = 0 \).

(b) The polynomials \( P_{\varphi}(\xi) \) and \( Q_{\varphi}(\xi) \) have no common zero.

**Proof.** (a) is implied from the equalities \( P(x, \varphi(x, \xi_u(x))) = 0 \) and \( Q(x, \varphi(x, \xi_v(x))) = 0 \). For (b), if \( P_{\varphi}(\xi) \) and \( Q_{\varphi}(\xi) \) have a common zero \( c \), then by Fact 2 there exists series

\[
\bar{\xi}_u(x) = c + \text{lower terms in } x,
\]

\[
\bar{\xi}_v(x) = c + \text{lower terms in } x
\]

such that \( \varphi(x, \bar{\xi}_u(x)) \) and \( \varphi(x, \bar{\xi}_v(x)) \) are Newton-Puiseux expansions at infinity of \( \alpha \) and \( \beta \), respectively. For these expansions \( \text{ord}(\varphi(x, \bar{\xi}_u(x)) - \varphi(x, \bar{\xi}_v(x)) < \theta. \) This contradicts to the definition of \( u \) and \( v \). \( \blacksquare \)

**Claim 2.** \( P_{\varphi} \) and \( Q_{\varphi} \) have only simple zeros.

**Proof.** First, observe that

\[
a_{\varphi} > 0, \quad b_{\varphi} > 0. \tag{3}
\]

Indeed, for instance, if \( a_{\varphi} \leq 0 \), then \( F(t^{-m_{\varphi}}, \varphi(t^{-m_{\varphi}}, \xi_v(t^{-m_{\varphi}}))) \) tends to a point \((a, 0) \in C^2 \) as \( t \to 0 \). This is impossible since \( F \) is a diffeomorphism.

Now, let

\[
J_{\varphi} := a_{\varphi} P_{\varphi} \frac{d}{d\xi} Q_{\varphi} - b_{\varphi} Q_{\varphi} \frac{d}{d\xi} P_{\varphi}.
\]

Taking differentiation of \( DF(t^{-m_{\varphi}}, \varphi(t^{-m_{\varphi}}, \xi)) \), by (3) one can get that

\[
m_{\varphi} J(P, Q)^{m_{\varphi} - 2m_{\varphi} - 1} = -J_{\varphi} t^{-a_{\varphi} - b_{\varphi} - 1} + \text{higher terms in } t.
\]
Since $J(P, Q) \equiv \text{const.} \neq 0$,

$$J_\varphi \equiv \begin{cases} -m_\varphi J(P, Q), & \text{if } a_\varphi + b_\varphi + n_\varphi = 2m_\varphi \\ 0, & \text{if } a_\varphi + b_\varphi + n_\varphi > 2m_\varphi. \end{cases}$$

If $J_\varphi \equiv 0$, it must be that $P^{-b_\varphi} = C Q^{-a_\varphi}$ for $C \in \mathbb{C}^*$. This is impossible by Claim 1(b). Thus, $J_\varphi = -m_\varphi J(P, Q)$. In particular, $P_\varphi$ and $Q_\varphi$ have only simple zeros. ■

Now, we can complete the proof of the lemma. By Claim 2 the numbers $\alpha_u$ and $\beta_v$ are simple zero of $P_\varphi$ and $Q_\varphi$, respectively. Then, by Fact 2 there exists Newton-Puiseux expansions at infinity

$$\bar{u}(x^{\frac{1}{m_\varphi}}) = \varphi(x, \alpha_u + \text{lower terms in } x^{\frac{1}{m_\varphi}}),$$

$$\bar{v}(x^{\frac{1}{m_\varphi}}) = \varphi(x, \beta_v + \text{lower terms in } x^{\frac{1}{m_\varphi}}),$$

for which $P(x, \bar{u}(x^{\frac{1}{m_\varphi}})) \equiv 0$, $Q(x, \bar{v}(x^{\frac{1}{m_\varphi}})) \equiv 0$ and $\text{mult}(\bar{u})$ and $\text{mult}(\bar{v})$ divide $m_\varphi$. Since $\text{mult}(\bar{u}) = \deg P > \deg Q = \text{mult}(\bar{v})$ and $\alpha_u \neq \beta_v$, we get $\alpha_u \neq 0$, $\beta_v = 0$ and $\deg P = m_\varphi$. Hence, $\deg Q \mid \deg P$. ■

4. Proof of Automorphism Theorem. The proof uses Division Lemma and the following fact which is only an easy elementary exercise on homogeneous polynomial.

(*) Let $f, g \in \mathbb{C}[x, y]$ be homogeneous. If $f_x g_y - f_y g_x \equiv 0$, then there is a homogeneous polynomial $h \in \mathbb{C}[x, y]$ with $\deg h = \gcd(\deg f, \deg g)$ such that

$$f = ah^{\deg f \deg h} \text{ and } g = bh^{\deg g \deg h}, a, b \in \mathbb{C}^*.$$  

(See, for example [E, Lemma 10.2.4, p 253]).

Given $F = (P, Q) \in GA(\mathbb{C}^2)$. Assume, for instance, $\deg P \geq \deg Q$ and $\deg P > 1$. Then, by the Division Lemma $\deg P = m \deg Q$, and hence, by (*) $\deg(P - cQ^m) < \deg P$ for a suitable number $c \in \mathbb{C}$. By induction one can find a finite sequence of automorphisms $\phi_i(x, y), i = 1, \ldots, k$ of the form $(x, y) \mapsto (x + cy^i, y)$ and $(x, y) \mapsto (x, y + cx^n)$ such that the components of the map of $\phi_k \circ \phi_{k-1} \circ \ldots \circ \phi_1 \circ F$ are of degree 1. Note that $\phi_i^{-1}$ has the form as those of $\phi_i$. Then, we get the automorphism Theorem. ■

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Hanoi Institute of Mathematics, P.O. Box 631, Boho 10000, Hanoi, Vietnam.
E-mail: nvchau@thevinh.ntsc.ac.vn