AN ALGEBRA INVOLVING BRAIDS AND TIES

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Abstract. In this note we study a family of algebras \( \mathcal{E}_n(u) \) with one parameter defined by generators and relations. The set of generators contains the generators of the usual braids algebra, and another set of generators which is interpreted as ties between consecutive strings. We also study the representations theory of the algebra when the parameter is specialized to 1.

Foreword

This article was written in the 2000 year and published as ICTP preprint [1]. The authors submitted it to a journal, but it was rejected since according to the referee report the considered algebra was the Hecke algebra with another vesture; moreover, this algebra should be not related to knot theory.

However, in the last years the algebras \( \mathcal{E}_n(u) \) was revisited by Ryom-Hansen [5], who found explicit bases and classified the irreducible representations. Moreover, in 2013 Banjo [6] determined the complex generic representation theory, showing that a certain specialization of this algebra is isomorphic to the small ramified partition algebra. The authors of the article have successively proved that \( \mathcal{E}_n(u) \mathcal{E}_n(u) \) admits a trace [2] and introduced the tied links, defining different invariants for them [3]. More recently, they have considered Kauffman type invariants for tied links [4] and have introduced a new algebra (a generalization of the BMW algebra) which is related to \( \mathcal{E}_n(u) \). A Referee of [4] suggested to put in ArXiv the present article.

1. Introduction

In this note we continue to study an algebra \( \mathcal{E}_n(u) \) of type A with one parameter \( u \) defined over the fields of rational functions \( \mathbb{C}(u) \). This algebra was introduced in [10], and the definition arose from an abstracting procedure of a non-standard presentation for the Yokonuma-Hecke algebra (9). The definition make use of generators and braids relations between the generators.

Notice that the definition of the algebra \( \mathcal{E}_n(u) \) that we give in section 2 includes less relations than the original definition [10]. The definition of our algebra is now using a reduced system of relations.

In sections 3 and 4, we introduce and study a diagrammatic interpretation for the generators defining of \( \mathcal{E}_n(u) \), which become useful to obtain information on the linear part of our algebra.

Also, in this note we study the representation theory of the algebra, whenever the parameter is specialized to 1. In fact, in section 5, we construct families of representations arising from the hyperoctahedral group. Using our methods, we describe the theory of representation for low dimension. For instance, we show the representation theory for \( \mathcal{E}_3(1) \).
2. Generators

Definition 1. Let $n$ be a natural number. The algebra $\mathcal{E}_n(u)$ is defined as the associative algebra over the field of rational function $\mathbb{C}(u)$ with generators

$$1, T_1, \ldots, T_{n-1}, E_1, \ldots, E_{n-1}$$

subject to the following relations:

1. $$T_i T_j = T_j T_i \quad \text{if } |i - j| > 1$$
2. $$T_i T_j T_i = T_j T_i T_j \quad \text{if } |i - j| = 1$$
3. $$E_i^2 = E_i$$
4. $$E_i E_j = E_j E_i \quad \forall i, j$$
5. $$E_i T_i = T_i E_i$$
6. $$E_i T_j = T_j E_i \quad \text{if } |i - j| > 1$$
7. $$E_j T_i T_j = T_i T_j E_i \quad \text{if } |i - j| = 1$$
8. $$E_i E_j T_j = E_i T_j E_i = T_j E_i E_j \quad \text{if } |i - j| = 1$$
9. $$T_i^2 = 1 + (u^{-1} - 1) E_i (1 - T_i)$$

The original definition of the algebra $\mathcal{E}_n(u)$ given in [11] contained 2 superfluous relations with respect to the above definition. Throughout of the diagrammatical interpretation of the generators, in terms of braids and ties, it was easy to remove such relations.

As in Lemma 3.1 in [11] one can prove that the algebras $\mathcal{E}_n(u)$ are finite dimensional. In fact, firstly note that any word of $\mathcal{E}_2(u)$ is a linear combination of words in $1, T_1, E_1$ and $T_1 E_1$. Now, suppose that any words of $\mathcal{E}_n(u)$ in $1, T_1, \ldots, T_{n-1}, E_1, \ldots, E_{n-1}$ can be written as a linear combination of words having at most one $T_{n-1}, E_{n-1}$, or $T_{n-1} E_{n-1}$. Then, using the defining relations of $\mathcal{E}_n(u)$ it is not hard to verify that any word below is a linear combination of words having at most one $T_n, E_n$ or $E_n T_n$:

$$T_n R_{n-1} T_n \quad T_n R_{n-1} T_n E_n \quad T_n R_{n-1} E_n$$
$$T_n E_n R_{n-1} T_n \quad T_n E_n R_{n-1} T_n E_n \quad T_n E_n R_{n-1} E_n$$
$$E_n R_{n-1} T_n \quad E_n R_{n-1} T_n E_n \quad E_n R_{n-1} E_n,$$

where $R_{n-1} \in \{1, T_{n-1}, E_{n-1}, T_{n-1} E_{n-1}\}$. Then from an inductive argument one can deduce the following proposition.

Proposition 1. In the algebra $\mathcal{E}_n(u)$ any word in $1, T_1, \ldots, T_{n-1}, E_1, \ldots, E_{n-1}$ is a linear combination of words in $T_i$’s, $E_i$’s having at most one $R_{n-1}$, where $R_{n-1} \in \{T_{n-1}, E_{n-1}, T_{n-1} E_{n-1}\}$. Hence $\mathcal{E}_n(u)$ is finite dimensional.
Now, as in the Iwahori-Hecke algebra we can take a system of linear generates for $\mathcal{E}_n(u)$ (which in our case will be redundant) in the following way: we define $U_1 = \{1, T_1, E_1, T_1E_1\}$, and $U_i$ by

$$U_i := \{1\} \cup T_iU_{i-1} \cup E_iU_{i-1} \cup T_iE_iU_{i-1} \quad (2 \leq i \leq n).$$

Using induction and Proposition 1 we deduce that $\mathcal{E}_n(u)$ is generated linearly by all the products of the form $u_1 u_2 \cdots u_{n-1}$, where $u_i \in U_i$. From where we deduce

$$(10) \quad \mathcal{E}_{n+1}(u) = \sum_{1 \leq i \leq n} Y_iY_{i+1}\cdots Y_n\mathcal{E}_n(u) + \mathcal{E}_n(u),$$

where $Y_j \in \{T_j, E_j, T_jE_j\}$.

**Corollary 2.** A basis for $\mathcal{E}_2(u)$ is $\{1, T_1, E_1, T_1E_1\}$. And $\mathcal{E}_3(u)$ is spanned linearly by:

$$L, LE_1, LE_2, LE_1E_2, LE_2T_1,$$

where $L \in \{1, T_1, T_2, T_1T_2, T_2T_1, T_1T_2T_1\}$.

**Proof.** The proof follows directly from (10) and the lemma below. □

**Lemma 3.** For all $i, j$ such that $|i - j| = 1$, we have:

1. $E_jT_i = T_iE_j + (u-1)(T_iE_j + T_iE_iE_j)$
2. $T_jE_jT_i = T_iT_jT_iE_j + (u-1)(T_iT_jE_j + T_iT_jE_iE_j)$.

**Proof.** From (7) we get

$$(T_iT_jE_i)T_j = E_jT_iT_j^2$$

$$= E_jT_i(1 + (u-1)(E_j - E_jT_j)) \quad \text{(from (9))}$$

$$= E_jT_i + (u-1)(T_iE_jE_j - T_iT_jE_iE_j) \quad \text{(from (8))}.$$

Thus the assertion (3.1) follows.

Multiplying 3.1 on the left by $T_j$, and using (2) we get 3.2. □

### 3. A diagram representation

In this section we define an isomorphism $\phi$ from the algebra $\mathcal{E}_n(u)$ to the algebra $\mathcal{B}H_n(u)$ of special diagrams. The diagrams corresponding to the words of $\mathcal{E}_n(u)$ look like the standard diagrams of the elements of the braid group $B_n$ (braids with $n$ strings), with a new structure between adjacent strings.

#### 3.1. Word’s representation.** We first introduce a one-to-one correspondence $\phi$ between the free algebra $\mathcal{A}$ over $\mathbb{C}(u)$ generated by $1, T_i, E_i$ and the free algebra $\mathcal{B}$ over $\mathbb{C}(u)$ generated by diagrams-generators as follows.

**Definition 2.** The image by $\phi$ in $\mathcal{B}$ of a generator $(1, T_i, E_i)$ of $\mathcal{A}$ is called row

Let $\phi(1)$ be the row-diagram.
Let $\varphi(T_i)$ be the row-diagram

\[
\begin{array}{cccccccc}
1 & 2 & \cdots & i & i+1 & \cdots & n \\
\end{array}
\]

Let $\varphi(E_i)$ be the row-diagram

\[
\begin{array}{cccccccc}
1 & 2 & \cdots & i & i+1 & \cdots & n \\
\end{array}
\]

The application $\varphi$ sends the multiplication $AB$ of two words $A$ and $B$ (in particular of two generators) to the diagram obtained putting the diagram $\varphi(B)$ below $\varphi(A)$.

A linear combination of words with coefficients in $\mathbb{C}(u)$ is sent by $\varphi$ in the same formal combinations of diagrams corresponding to the words with the same coefficients.

**Proposition 4.** The above construction defines a one to one correspondence between $A$ and $B$.

**Proof.** It suffices to see that the inverse map associates to every diagram one and only one word. In fact, this word is obtained reading the diagram from top to bottom, and writing in the order the $\varphi^{-1}$ of the encountered rows. □

3.2. **Definition of the algebra $\mathcal{B}H_n(u)$.** We translate into the diagram language the set of relations 1 to 9. Adding these relations to the free algebra $\mathcal{B}$ we obtain a new algebra $\mathcal{B}H_n(u)$.

Note that the first eight relations involve only single words. Because of Proposition 4 we will use the same symbols for elements of $\mathcal{E}_n(u)$ and their images $\mathcal{B}H_n(u)$.
The following relations between $T_i$ are the standard relations of the braid groups.

(11) $T_j T_i = T_i T_j$, if $|i - j| > 1$.

(12) $T_i T_j T_i = T_j T_i T_j$, if $|i - j| = 1$.

The $E_i$ generators commute one another, and any (natural) power of $E_i$ coincides with $E_i$.

(13) $E_i^2 = E_i$.

(14) $E_j E_i = E_i E_j$, $\forall i, j$.

Relations involving $E_i$’s and $T_i$’s generators:

(15) $E_i T_i = T_i E_i$. 
\( E_j T_i = T_i E_j, \quad \text{if} \quad |i - j| > 1. \)

\( E_j T_i = T_i T_j E_i, \quad \text{if} \quad |i - j| = 1. \)

\( E_i E_j T_j = E_i T_j E_i = T_j E_i E_j, \quad \text{if} \quad |i - j| = 1. \)

The last relation, corresponding to (9), is of type “Skein rule”. By this relation a single diagram is equivalent to a linear combination of diagrams. By eq. (9) and relation \( T_i T_i^{-1} = 1 \) one obtains the following expression for by \( T_i^{-1} 

\( T_i^{-1} = T_i + (u - 1) E_i T_i + (1 - u) E_i. \)

Hence a diagram containing an inverse generator \( T_i^{-1} \) is equivalent to the following combination of diagrams containing \( T_i, E_i T_i \) and \( E_i \) in place of \( T_i^{-1} 

Proposition 5. The diagram relations 17 to 19 define the algebra $BH_n(u)$. This algebra is naturally isomorphic to $E_n(u)$ by construction.

4. Diagram’s reduction

The representation of the elements of an abstract algebra by some geometrical objects (which may be taken as models of some physical objects) is interesting if the equivalence relations of the algebra become, in terms of these objects, natural moves which leave unaltered some property of them. Thus the geometrical intuition can help to find immediately equivalence relations between objects.

In this section an interpretation of the element $E_i$ is given so that the allowed moves involving $E_i$’s relate equivalent objects with respect to the algebra.

4.1. Useful relations. We need some relations coming from 1 to 9.

Proposition 6. The relations 5 to 8 are true substituting $T_i$ for $T_i^{-1}$.

Proof. If we substitute $T_i^{-1}$ into relations 5 to 8 given by relation 19 we find identities by means of equations 3 to 8. □

Multiplying the terms of equations 7 to the left by $T_i^{-1}E_jT_i$ and to the right by $T_j^{-1}E_iT_j$ we obtain

$$T_i^{-1}E_jT_i = T_jE_iT_j^{-1}. \tag{20}$$

Now, we have

$$T_i^{-1}E_jT_i = T_i^{-1}E_jT_iT_i^{-1}E_iT_i$$

$$= T_i^{-1}E_jT_jE_iT_j^{-1} \quad \text{ (from 20)}$$

$$= T_i^{-1}T_iT_jT_j^{-1}E_iE_jT_j^{-1} \quad \text{ (from 17).}$$

Then

$$T_i^{-1}E_jT_i = T_j^{-1}E_iT_j, \quad \text{ (21)}$$

or equivalently

$$u(T_iE_jT_i - T_jE_iT_j) = (u - 1)(E_jT_iE_i - E_iT_jE_i). \tag{22}$$

Finally we consider relation 2. We multiply it by $T_i^{-1}E_jT_i$ and by the preceding relation we obtain

$$T_iT_jT_i^{-1}E_jT_i = T_jT_iT_j^{-1}E_iT_j,$$

i.e.

$$T_iT_jT_i^{-1}E_jT_i = T_jT_iE_iT_j. \tag{23}$$
4.2. \( E_i \)'s as ties. The element \( E_i \), represented in the diagrams by a dashed line between the strings \( i \) and \( i + 1 \), can be interpreted as a rigid bar free to move up and down between two adjacent strings as far as they remain at the same distance. This is the meaning of relations 14, 16, 17 and 15 if we interpret element \( T_i \) as a twist in the 3-space. The non commutativity of \( E_i \) with \( T_j \) when \( |i - j| = 1 \) is thus interpreted as the obstacle to movements of the bar.

Consider now the following diagram relations whose validity we showed in the preceding subsection: The first one

\[
E_j T_i^{-1} T_j = T_i^{-1} T_j E_i, \quad \text{if} \quad |i - j| = 1.
\]

shows that the bar can slide passing through the strings. The following relations still indicate that in a rigid horizontal move of the bar, its bypassing of a string is allowed. Note that the corresponding moves generalize combinations of Reidemeister moves of second type.

\[
T_i^{-1} E_j T_i = T_i E_j T_i^{-1} = T_j^{-1} E_i T_j = T_j E_i T_j^{-1}, \quad (|i - j| = 1).
\]

Finally, equation 23

\[
T_i T_j E_j T_i = T_j T_i E_i T_j, \quad (|i - j| = 1).
\]

shows that also in the generalized third Reidemeister move the bar is allowed to bypass the string.
We remark that relation 18 is the only exception to the rule of sliding of the bar representing an $E_i$. We can interpret this relation as a sort of cooperation between two adjacent bars: $E_i$ do not commute with $T_{i+1}$, but commutes with $E_{i+1}T_{i+1}$.

4.3. Equivalent relations of Skein type. To conclude, we add some useful relations equivalent to 9.

Multiplying relation 9 by $E_i$ we get the following relations:

\begin{align}
(27) & \quad uE_iT_i - E_iT_i^{-1} = (u - 1)E_i. \\
(28) & \quad E_iT_i - E_iT_i^{-1} = T_i - T_i^{-1}.
\end{align}

5. Representation theory of $\mathcal{E}_n(1)$

In this section we study the representation theory of $\mathcal{E} := \mathcal{E}_n(1)$.

We will use the standard definition of partition, and as usual we will regard the partition as Young diagram. Also we will denote again by $\alpha$ the irreducible representation of $S_n$ associated to a partition $\alpha$ of $n$.

The symmetric group $S_n$ is regarded as a Coxeter group generated by the set of elemental transpositions $S$, that is, $S = \{s_1, \ldots, s_{n-1}\}$, $s_i = (i, i + 1)$.

We are going to show a series of irreducible representations of $\mathcal{E}$ that come from the hyper-octahedral group $W_n$. In other words $W_n$ is the wreath product $C_2 \wr S_n$, where $C_2$ is the group with two elements, say, $C_2 = \{1, t\}$. Thus, $W_n$ has a presentation defined by the following Dynkin diagram

\[ t \quad s_1 \quad \cdots \quad s_{n-2} \quad s_{n-1} \]

We have $W_n = C \rtimes S_n$, where $C = C_2 \times \cdots \times C_2$ ($n$-times).

We define the elements $t_i \in C$ inductively as $t_1 = t$, $t_{i+1} = s_it_is_i$.

We shall represent $W_n$ as the subgroup of the monomials matrices of $GL_n$, whose non-zero entries are 1 or $t$. Then $C$ is the diagonal subgroup of $GL$ with 1's and $t$'s on the diagonal; and $t_i$ is the diagonal matrix with $t$ in the position $(i, i)$, and 1 otherwise.

Let $a, b \geq 0$ such that $a + b = n$. Let $W_{(a,b)}$ be the subgroup of $W_n$ isomorphic to the group $W_a \times W_b$, according to the diagonal embedding $GL_a \times GL_b \subseteq GL_n$.

In order to describe the representation theory of $W_n$, we consider the following linear character, $\epsilon : W_n \rightarrow \{-1, 1\}$, defined by $\epsilon(t) = -1$, $\epsilon(s_i) = 1$. We consider, using the natural
surjection \( \pi \) of \( W_m \) onto \( S_m \) and pull-back mechanism, a representation of \( S_m \) as a representation of \( W_m \).

**Theorem 7.** (Specht, see [8]) The types of irreducible representations of \( W_n \) are parametrized by the bipartitions of \( n \). Moreover, all irreducible representations of \( W_n \) can be underlying like an induced representation of the form

\[
V_{(\alpha,\beta)} = \text{Ind}_{W_{(\alpha,\beta)}}^{W_n} \alpha \otimes \epsilon \beta,
\]

where \( \alpha \) is a partition of \( a \), and \( \beta \) is a partition of \( b, a + b = n \).

The key point for taking representations of \( E \) from the group \( W_n \) is the following proposition.

**Proposition 8.** The map below defines a morphism \( \psi \) of algebras, from \( E \) to \( CW_n \).

\[
T_i \mapsto s_i, \quad E_i \mapsto e_i := \frac{1}{2}(1 + t_it_{i+1}).
\]

**Proof.** One verifies that \( s_i \)'s and \( e_i \)'s satisfy the relations 1 to 8 when we put \( s_i \) in place of \( T_i \), and \( e_i \) to the place of \( E_i \). \( \square \)

Thus, from the above proposition the representations \( V_{(\alpha,\beta)} \) are \( E \)-modules. We are going to prove that \( V_{(\alpha,\beta)} \) and \( V_{(\beta,\alpha)} \) are equivalent as \( E \)-modules, and that \( V_{(\alpha,\alpha)} \) is reducible as \( E \)-module.

To do this we need some facts. Firstly, we will use the Mackey model for describing the induced representations. More precisely, let us denote by \( V_\rho \) an underlying for the irreducible representation \( \rho \) of a subgroup \( H \) of \( G \). Then \( \text{Ind}_H^G \rho \) can be realized as the following vector space

\[
V_\rho := \{ f : G \to V : f(hg) = \rho(h)f(g), h \in H, g \in G \},
\]

with action given by \((gf)(x) = f(xg), g, x \in W_n \).

Let \( \{v_i\} \) be a basis of \( V \), and \( X \) a set of right coset representatives of \( H \) in \( G \). The Dirac basis for \( V_\rho \) with respect to \( X \) and \( \{v_i\} \) is by definition the basis \( \{\delta_{u,i} : u \in X, 1 \leq i \leq \dim V\} \), where \( \delta_{u,i} \in V_\pi \) is defined as

\[
\delta_{u,i}(g) = \begin{cases} 
\rho(h)v_i & \text{if } g = hu, \\
0 & \text{otherwise}.
\end{cases}
\]

(We have \( g\delta_{u,i} = \delta_{ug^{-1},i} \).)

Also we use some notations and facts from [7]. Let \( a, b \geq 0 \) such that \( a + b = n \). Let us consider the element \( w_{a,b} \) in \( S_n \) defined by \( w_{a,b} = 1 \) if either \( a \) or \( b \) is zero. And

\[
w_{a,b} := (s_{a+b,1})^b \quad \text{if } a, b > 0,
\]

where \( s_{i,i} = 1 \), and

\[
s_{i,j} = \begin{cases} 
s_i s_{i+1,j} & \text{if } i < j, \\
s_{i,j+1}s_j & \text{if } i > j.
\end{cases}
\]

We will use the following properties of the elements \( w \)'s:

\[
w_{a,b}^{-1} = w_{b,a}
\]

\[
w_{a,b}sw_{k} = \begin{cases} 
s_{a+k}w_{a,b} & \text{if } 1 \leq k < b, \\
s_{b-k}w_{a,b} & \text{if } b + 1 \leq k < n.
\end{cases}
\]
where $\mathcal{S}_{(a,b)}$ is the homomorphic image by $\pi$ of $W_{(a,b)}$. Thus $\mathcal{S}_{(a,b)}$ is the subgroup of $S_n$ isomorphic to the group $S_a \times S_b$.

Let us denote by $S_{(a,b)}$ the subset of $S = \{s_1, \ldots, s_{n-1}\}$ generating the subgroup $\mathcal{S}_{(a,b)}$ of $S_n$. We have $S_{(a,b)}w_{a,b} = w_{a,b}S_{(b,a)}$.

Let $X_{(a,b)}$ be the set of distinguished right coset representatives of $S_{(a,b)}$ in $S_n$. Thus $w_{a,b} \in X_{(a,b)}$, and

$$X_{(a,b)} = w_{a,b}X_{(b,a)}. \tag{32}$$

**Proposition 9.** We have $V_{(\alpha, \beta)} \cong V_{(\beta, \alpha)}$ as $\mathcal{E}$-module.

**Proof.** Let $\{v_i\}$ (respectively $\{v'_j\}$) be a basis for an underlying $V$ (respectively $V'$) of the representation $\alpha$ (respectively $\beta$) of $S_a$ (respectively $S_b$). Let $\{\delta_{u,(i,j)}\}$ be a Dirac basis for $V_{(\alpha, \beta)}$, relative to $X_{(a,b)}$ and $\{v_i \otimes v_j\}$. The maps $\delta_{u,(i,j)} \mapsto \delta_{w^{-1}u,(j,i)} (w = w_{a,b})$ define a linear isomorphism $\Phi$ from $V_{(\alpha, \beta)}$ to $V_{(\beta, \alpha)}$. Thus, for the proof of the proposition we need only to prove:

1. $T_r \circ \Phi = \Phi \circ T_r$, \tag{31.1}
2. $e_r \circ \Phi = \Phi \circ e_r$, \tag{31.2}

For any $1 \leq r \leq n - 1$.

To prove (31.1), we need only to check

$$(s \circ \Phi)(\delta_{u,(i,j)}) = (\Phi \circ s)(\delta_{u,(i,j)}),$$

for all $s \in S$. Now, from a result of V. Deodhar (Lemma 2.1.2 in [8]), we have either the cases: $us \in X_{(a,b)}$ or $us = s'u$ for some $s' \in S_{(a,b)}$. If we are in the first case the situation is trivial. So, suppose we are in the second case. Put $(\alpha \otimes \epsilon\beta)(s')(v_i \otimes v'_j) = \sum_{k,l} \lambda_{k,l} v_k \otimes v'_l$, then, it is easy to check that $(\beta \otimes \epsilon\alpha)(w^{-1}s'w)(v'_j \otimes v_i) = \sum_{k,l} \lambda_{k,l} v'_l \otimes v_k$. (notice that $\epsilon(s') = 1$).

Now, we have

$$s \delta_{u,(i,j)} = \sum_{k,l} \lambda_{k,l} \delta_{u,(k,l)}. \tag{31.1}$$

In fact, let $g = hu \in W_n, \ h \in W_{(a,b)}$, we have $(s \delta_{u,(i,j)})(g) = \delta_{u,(i,j)}(h(ut)) = \delta_{u,(i,j)}(hs'u)$. Thus,

$$(s \delta_{u,(i,j)})(g) = (\alpha \otimes \epsilon\beta)(h) \delta_{u,(i,j)}(s'v_i \otimes v'_j) = (\alpha \otimes \epsilon\beta)(h) \delta_{u,(i,j)}(s'v_i \otimes v'_j) = (\alpha \otimes \epsilon\beta)(h) \sum_{k,l} \lambda_{k,l} v_k \otimes v'_l = \sum_{k,l} \lambda_{k,l} (\alpha \otimes \epsilon\beta)(h) v_k \otimes v'_l = \sum_{k,l} \lambda_{k,l} \delta_{u,(k,l)}(g). \tag{31.2}$$

In a similar way we get

$$s \delta_{w^{-1}u,(j,i)} = \sum_{k,l} \lambda_{k,l} \delta_{w^{-1}u,(k,l)}$$
Thus,
\[ \delta_{u,(i,j)} \overset{s}{\rightarrow} \sum_{l,k} \lambda_{k,l} \delta_{u,(k,l)} \overset{\Phi}{\rightarrow} \sum_{k,l} \lambda_{k,l} \delta_{w^{-1}u,(l,k)}. \]

On the other hand, we have
\[ \delta_{u,(i,j)} \overset{\Phi}{\rightarrow} \delta_{w^{-1}u,(j,i)} \overset{s}{\rightarrow} s\delta_{w^{-1}u,(j,i)} = \sum_{k,l} \lambda_{k,l} \delta_{w^{-1}u,(l,k)}. \]

Hence $\Box$ follows.

To prove $\Box$, we see that
\[ t_r t_{r+1} \delta_{u,(i,j)} = (\alpha \otimes \epsilon \beta)(ut_r t_{r+1} u^{-1}) \delta_{u,(i,j)}. \]
(Notice that $\alpha \otimes \epsilon \beta$ is a linear character on $C$.) Then,
\[ (\Phi \circ t_r t_{r+1})(\delta_{u,(i,j)}) = (\alpha \otimes \epsilon \beta)(ut_r t_{r+1} u^{-1}) \delta_{w^{-1}u,(i,j)}, \]
\[ (t_r t_{r+1} \circ \Phi)(\delta_{u,(i,j)}) = (\beta \otimes \epsilon \alpha)(w^{-1}ut_r t_{r+1} u^{-1} w) \delta_{w^{-1}u,(i,j)}. \]

Therefore we deduce $\Box$, because $t_i t_{i+1}$ is a diagonal matrix, and
\[ w^{-1} \text{diag}(d_1, d_2, \ldots, d_a, d_{a+1}, \ldots, d_n) w = \text{diag}(d_{a+1}, \ldots, d_n, d_1, d_2, \ldots d_a). \]
\[ \square \]

From eq. (33) we get:

**Corollary 10.** For any $\delta$ in a Dirac basis, we have
\[ e_r \delta = \begin{cases} 1 & \text{if } r \neq a, \\ 0 & \text{if } r = a. \end{cases} \]

Now, we decompose the $E$-module $V_{(a,a)}$ into two modules. Note that this situation appears only when $n$ is an even number. Set $m = n/2$, and put $S' = S - \{s_m\}$. Thus $S_{(m,m)}$ is generated by $S'$.

Let $w$ be the element $w_{m,m}$, and let $X$ be the set of distinguished elements $X_{(m,m)}$. From 32 we have $X = wX$. Then we can choose a subset $Y$ of $X$ such that $X = Y \cup wY$ (disjoint union). Let $\{\delta_{u,i}\}$ be the Dirac basis of $V_{(a,a)}$ with respect to $X$ and the basis $\{v_i\}$ of $V \otimes V$, where $V$ is an underlying for the representation $\alpha$ of $S_m$. We have

**Proposition 11.** $V_{(a,a)} = V_{\alpha}^+ \oplus V_{\alpha}^-$ as $E$-module, where
\[ V_{\alpha}^+ := \langle \delta_{u,i} + \delta_{w_{u,i}}; u \in Y, 1 \leq i \leq (\dim V)^2 \rangle, \]
\[ V_{\alpha}^- := \langle \delta_{u,i} - \delta_{w_{u,i}}; u \in Y, 1 \leq i \leq (\dim V)^2 \rangle. \]

**Proof.** The decomposition of vector space is obvious. So, we need only to prove that $V_{\alpha}^+$ (respectively $V_{\alpha}^-$) is a $E$-module. For this, it is sufficient to prove:

\[ s(\delta_{u,i} + \delta_{w_{u,i}}) \in V_{\alpha}^+ \quad (s \in S), \]
\[ e_r (\delta_{u,i} + \delta_{w_{u,i}}) \in V_{\alpha}^+. \]

Let us see $\Box$. For any $s \in S$, we have $s \delta_{u,i} = \delta_{us,i}$. Again from V. Deodhar’s result, we have either the cases: $us \in X$ or $us = s' u$, for some $s' \in S'$. Suppose we are in the
first case. If \( us = u' \in Y \), the situation is trivial, if \( us = u' \in wY \), hence \( wu' \in Y \), then \( s(\delta_{u,i} + \delta_{wu,i}) = \delta_{wu',i} + \delta_{u',i} \in V_{\alpha}^+ \).

If we are in the second case, put \( us = s'u \), with \( s' \in S' \). Set

\[
(\alpha \otimes \epsilon\alpha)(s')v_i = \sum_j \lambda_{i,j}v_j.
\]

We claim that

\[
(a) \ s\delta_{u,i} = \sum_j \lambda_{i,j}\delta_{u,j}, \quad \text{and} \quad (b) \ s\delta_{wu,i} = \sum_j \lambda_{i,j}\delta_{wu,j}.
\]

Then we have obtained what we were looking for, since

\[
s(\delta_{u,i} + \delta_{wu,i}) = \sum_j \lambda_{i,j}(\delta_{u,i} + \delta_{wu,i}).
\]

Now, claim (a) is easy to check. Let us prove claim (b). Let \( x = h(wu) \in W_n \), with \( h \in S_{(m,m)} \), we have \( xs = h(wu)s = hws'u \). Then

\[
(s\delta_{wu,i})(x) = \delta_{wu,i}(hws'u) = (\alpha \otimes \epsilon\alpha)(hws'u)(v_i) = (\alpha \otimes \epsilon\alpha)(h)(\alpha \otimes \epsilon\alpha)(ws'w)(v_i) = (\alpha \otimes \epsilon\alpha)(h)(\alpha \otimes \epsilon\alpha)s'(v_i) \quad \text{(from \( \mathfrak{B1} \))}
\]

\[
= (\alpha \otimes \epsilon\alpha)(h)\sum_j \lambda_{i,j}v_j
\]

\[
= \sum_j \lambda_{i,j}(\alpha \otimes \epsilon\alpha)(h)v_j
\]

\[
= \sum_j \lambda_{i,j}\delta_{wu,j}v_j.
\]

Then \( s\delta_{wu,i} = \sum_j \lambda_{i,j}\delta_{wu,j} \).

Now \( \mathfrak{11} \) follows from \( \mathfrak{10} \).

Similarly one can prove that \( V_{\alpha}^- \) is an \( E \)-module.

Now, we have two homomorphisms \( \varphi_0 \) and \( \varphi_1 \) from \( E \) to the algebra \( \mathbb{C}S_n \): \( \varphi_0 \) is defined by sending \( T_i \) to \( s_i \), and \( E_i \) to 0, and \( \varphi_1 \) is defined by sending \( T_i \) to \( s_i \), and \( E_i \) to 1. Then, these morphisms yield two families of irreducible representations non-equivalent of \( E \). Set \{ \( (\alpha, 0) \) \} (respectively \{ \( (\alpha, 1) \) \}) the family of irreducible representations yielded by \( \varphi_0 \) (respectively \( \varphi_1 \)).

**Proposition 12.** We have the following equivalence of \( E \)-modules:

\[
(\alpha, 1) \simeq V_{(\alpha, \varphi_1)} \quad (\alpha \text{ partition of } n, \forall n).
\]

**Proof.** The equivalence follows from the fact that \( e_i \) acts trivially on the Delta basis. See Corollary \( \mathfrak{10} \). \( \square \)

We shall denote the found representation by “Young diagram”, which has encoded information of the dimension of the representation. The representations of Proposition \( \mathfrak{9} \) are denoted by \( (\alpha, \beta) \), and the representations \( V_{\alpha}^+ \) (respectively \( V_{\alpha}^- \)) of the Proposition \( \mathfrak{11} \) are denoted by \( (\alpha, +) \) (respectively \( (\alpha, -) \)).
In the case $n = 2$ the algebra $E$ is of dimension 4, and we have four non-equivalent representations of dimension 1 for $E$: $(\begin{array}{c} \phi \\ \phi \end{array})$, $(\begin{array}{c} \phi \\ 0 \end{array})$, $(\begin{array}{c} 0 \\ \phi \end{array})$, $(\begin{array}{c} 0 \\ 0 \end{array})$.

In the case $n = 3$ the irreducible representations of $E$ are:

| dimension | $(\begin{array}{c} \phi \\ \phi \end{array})$ | $(\begin{array}{c} \phi \\ 0 \end{array})$ | $(\begin{array}{c} 0 \\ \phi \end{array})$ | $(\begin{array}{c} 0 \\ 0 \end{array})$ |
|-----------|----------------------------------|-----------------|-----------------|-----------------|
|           | 1                               | 1               | 2               | 1               |
|           |                                  | 1               | 2               | 3               |
|           |                                  | 3               | 3               |                 |

**Theorem 13.** The algebra $E_3(1)$ is semisimple, and we have

$$E_3(1) = 4M_1(\mathbb{C}) \oplus 2M_2(\mathbb{C}) \oplus 2M_3(\mathbb{C}).$$

**Proof.** We shall prove that the dimension of $E = E_3(1)$ is 30. Then, the theorem follows.

Let us consider the linear generators for $E$ of Corollary 2:

$$\{T_w, T_wE_1, T_wE_2, T_wE_1E_2, T_wE_2T_1 ; w \in S_n\}.$$

We must prove that if:

$$\sum_{w \in S_3} a_w T_w + \sum_{w \in S_3} A_w T_w E_1 + \sum_{w \in S_3} B_w T_w E_2 + \sum_{w \in S_3} C_w T_w E_1 E_2 + \sum_{w \in S_3} D_w T_w E_2 T_1 = 0,$$

then $a_w = A_w = B_w = C_w = D_w = 0$, $\forall w$.

Applying $\varphi_0$ to equation (13.1), we deduce that $a_w = 0$, for all $w$.

Now, applying the morphism $\psi$ of Proposition 8 to (13.1), we get the equation

$$\sum_{w \in S_3} A_w we_1 + \sum_{w \in S_3} B_w we_2 + \sum_{w \in S_3} C_w we_1 e_2 + \sum_{w \in S_3} D_w we_3 = 0,$$

where $e_3 : = s_2 e_1 = \frac{1}{3}(1 + t_1 t_3)$.

The left part of (13.2) can be written in terms of elements of $D := \{w, wt_1 t_2, wt_2 t_3, wt_1 t_3\}$, which is a subset of the canonical basis $\{wt ; w \in S_n, t \in C\}$ of $\mathbb{C}W_n$. Hence $D$ is an independent linear set. Thus, one can deduce that $A_w = B_w = C_w = D_w = 0$, for all $w$, studying the coefficient of the elements of $D$. More precisely,

- The coefficient of $1$ in (13.2) is $A_1 + B_1 + C_1/2 + D_1$,
- The coefficient of $t_1 t_2$ in (13.2) is $A_1 + C_1/2$,
- The coefficient of $t_2 t_3$ in (13.2) is $B_1 + C_1/2$,
- The coefficient of $t_1 t_3$ in (13.2) is $C_1/2 + D_1$.

As all these coefficients are 0, we get $A_1 = B_1 = C_1 = D_1 = 0$.

Finally, multiplying (13.2) for convenient $w$ one can deduce $A_w = B_w = C_w = D_w = 0$, $\forall w$.

**Remark 1.** From the proof of Theorem 13 we deduce that $\psi$ is one-to-one.
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