Non self-adjoint correct restrictions and extensions with real spectrum

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Abstract. The work is devoted to the study of the similarity of a correct restriction to some self-adjoint operator in the case when the minimal operator is symmetric. The resulting theorem was applied to the Sturm-Liouville operator and the Laplace operator. It is shown that the spectrum of a non self-adjoint singularly perturbed operator is real and the corresponding system of eigenvectors forms a Riesz basis.

1 Introduction

Let a linear operator \( L \) be given in a Hilbert space \( H \). The linear equation

\[ Lu = f \]  

(1.1)

is said to be correctly solvable on \( R(L) \) if \( \|u\| \leq C\|Lu\| \) for all \( u \in D(L) \) (where \( C > 0 \) does not depend on \( u \)) and everywhere solvable if \( R(L) = H \). If (1.1) is simultaneously correct and solvable everywhere, then we say that \( \hat{L} \) is a correct operator. A correctly solvable operator \( L_0 \) is said to be minimal if \( R(L_0) \neq H \). A closed operator \( \hat{L} \) is called a maximal operator if \( R(\hat{L}) = H \) and \( \text{Ker} \hat{L} \neq \{0\} \). An operator \( A \) is called a restriction of an operator \( B \) and \( B \) is said to be an extension of \( A \) if \( D(A) \subset D(B) \) and \( Au = Bu \) for all \( u \in D(A) \).

Note that if a correct restriction \( L \) of a maximal operator \( \hat{L} \) is known, then the inverses of all correct restrictions of \( \hat{L} \) have in the form

\[ L^{-1}_K f = L^{-1} f + Kf, \]  

(1.2)

where \( K \) is an arbitrary bounded linear operator from \( H \) into \( \text{Ker} \hat{L} \).

Let \( L_0 \) be some minimal operator, and let \( M_0 \) be another minimal operator related to \( L_0 \) by the equation \( (L_0u,v) = (u,M_0v) \) for all \( u \in D(L_0) \) and \( v \in D(M_0) \). Then
\( \hat{L} = M_0^* \) and \( \hat{M} = L_0^* \) are maximal operators such that \( L_0 \subset \hat{L} \) and \( M_0 \subset \hat{M} \). A correct restriction \( L \) of a maximal operator \( \hat{L} \) such that \( L \) is simultaneously a correct extension of the minimal operator \( L_0 \) is called a boundary correct extension. The existence of at least one boundary correct extension \( L \) was proved by Vishik in [2], that is, \( L_0 \subset L \subset \hat{L} \).

The inverse operators to all possible correct restrictions \( L_K \) of the maximal operator \( \hat{L} \) have the form (1.2), then \( D(L_K) \) is dense in \( H \) if and only if \( \text{Ker}(I + K^*L^*) = \{0\} \).

All possible correct extensions \( M_K \) of \( M_0 \) have inverses of the form

\[
M_K^{-1} f = (L_K^*)^{-1} f = (L^*)^{-1} f + K^* f,
\]

where \( K \) is an arbitrary bounded linear operator in \( H \) with \( R(K) \subset \text{Ker} \hat{L} \) such that

\[
\text{Ker}(I + K^*L^*) = \{0\}.
\]

**Lemma 1.1** (Hamburger [3, p. 269]). Let \( A \) be a bounded linear transformation in \( H \) and \( N \) a linear manifold. If we write \( A(N) = M \) then

\[
A^*(M^\perp) = N^\perp \cap R(A^*).
\]

**Proposition 1.1** ([4, p. 1863]). A correct restrictions \( L_K \) of the maximal operator \( \hat{L} \) are correct extensions of the minimal operator \( L_0 \) if and only if \( R(K) \subset \text{Ker} \hat{L} \) and \( R(M_0) \subset \text{Ker} K^* \).

The main result of this work is the following.

**Theorem 1.2.** Let \( L_0 \) be symmetric minimal operator in a Hilbert space \( H \), \( L \) be self-adjoint correct extension of the \( L_0 \), and \( L_K \) be correct restriction of the maximal operator \( \hat{L}(\hat{L} = L_0^*) \). If

\[
R(K^*) \subset D(L), \quad I + KL \geq 0,
\]

and \( I + KL \) is invertible, where \( L \) and \( K \) are the operators in representation (1.2), then \( L_K \) similar to a self-adjoint operator.

**Corollary 1.1.** If \( K \) satisfies the assumptions of Theorem 1.2 then the spectrum of \( L_K \) is real, that is, \( \sigma(L_K) \subset \mathbb{R} \).

**Corollary 1.2.** If \( K \) satisfies the assumptions of Theorem 1.2 and \( L^{-1} \) is the compact operator, then the system of eigenvectors of \( L_K \) forms a Riesz basis in \( H \).

**Corollary 1.3.** The results of Theorem 1.2 are also valid if conditions “\( I + KL \geq 0 \) and \( I + KL \) is invertible” replase to condition “\( KL \geq 0 \)”.

**Corollary 1.4.** The results of Theorem 1.2, Corollary 1.1-1.3 are also valid for the \( L_{K}^* \).
2 Preliminaries

In this section, we present some results for correct restrictions and extensions which are used in Section 3.

If \( A \) is bounded linear transformation from a complex Hilbert space \( H \) into itself, then the numerical range of \( A \) is by definition the set

\[
W(A) = \{(Ax, x) : x \in H, \|x\| = 1\}.
\]

It is well known and easy to prove that if \( \sigma(A) \) denotes the spectrum of \( A \), then

\[
\sigma_p(A) \subset W(A), \quad \sigma(A) \subset \overline{W(A)},
\]

for the point spectrum \( \sigma_p(A) \) and the spectrum \( \sigma(A) \) of \( A \), where the bar indicates closure. The numerical range of an unbounded operator \( A \) in a Hilbert space \( H \) is defined as

\[
W(A) = \{(Ax, x) : x \in D(A), \|x\| = 1\},
\]

and similarly to the bounded case, \( W(A) \) is convex and satisfies \( \sigma_p(A) \subset W(A) \). In general, the conclusion \( \sigma(A) \subset \overline{W(A)} \) does not surely hold for unbounded operators \( A \) (see [5]).

**Theorem 2.1** (Theorem 2 in [6, p.181]). The following are equivalent conditions on an operator \( T \):

1. \( T \) is similar to a self-adjoint operator.
2. \( T = PA \), where \( P \) is positive and invertible and \( A \) is self-adjoint.
3. \( S^{-1}TS = T^* \) and \( 0 \not\in W(S) \).

**Theorem 2.2** (Theorem 1 in [7, p.215]). Let \( A \) and \( B \) operators on the complex Hilbert space \( H \). If \( 0 \not\in W(A) \) then

\[
\sigma(A^{-1}B) \subset \overline{W(B)/W(A)}.
\]

**Corollary 2.1** (Corollary in [7, p.218]). If \( A > 0 \), \( B \geq 0 \) and \( C = C^* \), then \( \sigma(AB) \) is positive and \( \sigma(AC) \) is real.

**Theorem 2.3** (Theorem A in [8, p.508]). The numerical range \( W(T) \) of \( T \) is convex and \( W(aT + b) = aW(T) + b \) for all complex numbers \( a \) and \( b \).

3 Proof of Theorem 1.2

We transform (1.2) to the form

\[
L_K^{-1} = L^{-1} + K = (I + KL)L^{-1}.
\]  

(3.1)

Then \( L_K \) is defined as the restriction of the maximal operator \( \hat{L} \) on the domain

\[
D(L_K) = \{u \in D(\hat{L}) : (I - K\hat{L})u \in D(L)\}.
\]
Now let us prove Theorem 1.2. It was proved in [9, p. 27] that $KL$ is bounded on $D(L)$ (that is, $KL \in B(H)$) if and only if

$$R(K^*) \subset D(L^*).$$

It follows from $D(L) = H$ that $KL$ is bounded on $H$. In the future, instead of $KL$, we will write $KL$. Then, by virtue of Theorem 2.1 and taking into account the conditions of Theorem 1.2 that $I + KL \geq 0$ and $I + KL$ is invertible, we obtain proof of Theorem 1.2.

The proof of Corollary 1.1 follows from Corollary 2.1. Corollary 1.2 is easy to obtain from the fact that the operator

$$C = (I + KL)^{1/2}L^{-1}(I + KL)^{1/2}$$

is self-adjoint and

$$L^{-1}_K = (I + KL)^{1/2}C(I + KL)^{-1/2} = (I + KL)L^{-1}. \quad (3.2)$$

Let us proof Corollary 1.3. By Theorem 2.3, we get that $0 \notin W(I + KL)$. Then $I + KL \geq 0$ and $I + KL$ is invertible.

The proof of Corollary 1.4 follows from (3.2), since $C$ is a self-adjoint operator and in the case Corollary 1.2 the self-adjoint operator $C$ is compact.

4 Non self-adjoint perturbations for some differential operators

Example 1. We consider the Sturm-Liouville equation on the interval $(0, 1)$

$$\hat{L}y = -y'' + q(x)y = f, \quad (4.1)$$

where $q(x)$ is the real-valued function of $L^2(0, 1)$. We denote by $L_0$ the minimal operator and by $\hat{L}$ the maximal operator generated by the differential equation (4.1) in the space $L_2(0, 1)$. It’s clear that

$$D(L_0) = \hat{W}_2^2(0, 1)$$

and

$$D(\hat{L}) = \{y \in L^2(0, 1) : y, y' \in AC[0, 1], y'' - q(x)y \in L^2(0, 1)\}.$$ 

Then $\text{Ker} \hat{L} = \{a_{11}c(x) + a_{12}s(x)\}$, where $a_{11}$, $a_{12}$ are arbitrary constants, and the functions $c(x)$ and $s(x)$ are defined as follows

$$c(x) = 1 + \int_0^x \mathcal{K}(x, t; 0) \, dt, \quad s(x) = x + \int_0^x \mathcal{K}(x, t; \infty) \, t \, dt,$$

where

$$\mathcal{K}(x, t; 0) = \mathcal{K}(x, t) + \mathcal{K}(x, -t), \quad \mathcal{K}(x, t; \infty) = \mathcal{K}(x, t) - \mathcal{K}(x, -t),$$

and

$$\mathcal{K}(x, t; \infty) = \begin{cases} \mathcal{K}(x, t) & \text{if } x > t, \\ 0 & \text{if } x < t, \end{cases} \quad \mathcal{K}(x, -t; \infty) = \begin{cases} 0 & \text{if } x > -t, \\ \mathcal{K}(x, t) & \text{if } x < -t. \end{cases}$$
and \( \mathcal{K}(x, t) \) is the solution of the following Goursat problem

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial^2 \mathcal{K}(x, t)}{\partial x^2} - \frac{\partial^2 \mathcal{K}(x, t)}{\partial t^2} = q(x)\mathcal{K}(x, t), \\
\mathcal{K}(x, -x) = 0, \quad \mathcal{K}(x, x) = \frac{1}{2} \int_0^x q(t)dt,
\end{array} \right.
\end{align*}
\]

in the domain

\[
\Omega = \{(x, t) : 0 < x < 1, \ -x < t < x\}.
\]

Note that \( c(0) = s'(0) = 1, \ c'(0) = s(0) = 0 \) and Wronskian

\[
W(c, s) \equiv c(x)s'(x) - c'(x)s(x) = 1.
\]

As a fixed boundary correct extension \( L \) we take the operator corresponding to the Dirichlet problem for equation (4.1) on \((0, 1)\). Then

\[
D(L) = \{ y \in W^2_2(0, 1) : y(0) = 0, \ y(1) = 0 \}.
\]

Therefore the description of the inverse of all correct restrictions \( L_K \) of the maximal operator \( \hat{L} \) has the form

\[
y \equiv L_K^{-1} f = \int_0^x \left[ c(x)s(t) - s(x)c(t) \right] f(t) dt
\]

\[
- \frac{s(x)}{s(1)} \int_0^1 \left[ c(1)s(t) - s(1)c(t) \right] f(t) dt
\]

\[
+ c(x) \int_0^1 f(t)\sigma_1(t)dt + s(x) \int_0^1 f(t)\sigma_2(t) dt,
\]

where \( \sigma_1(x), \ \sigma_2(x) \in L_2(0, 1) \) which uniquely determine the operator \( K \) from (1.2) in the following form

\[
Kf = c(x) \int_0^1 f(t)\sigma_1(t)dt + s(x) \int_0^1 f(t)\sigma_2(t) dt, \quad \text{for all} \ f \in L_2(0, 1).
\]

\( K \) is a bounded operator in \( L_2(0, 1) \) acting from \( L_2(0, 1) \) to \( \text{Ker} \ \hat{L} \). The operator \( L_K \) is the restriction of \( \hat{L} \) on the domain

\[
D(L_K) = \left\{ y \in W^2_2(0, 1) : y(0) = \int_0^1 \left[ -y''(t) + q(t)y(t) \right]\sigma_1(t)dt; \right. \]

\[
y(1) = c(1)y(0) + s(1) \int_0^1 \left[ -y''(t) + q(t)y(t) \right]\sigma_2(t) dt \left. \right\}.
\]

From the condition

\[
R(K^*) \subset D(L^*) = D(L)
\]

we have that

\[
KLy = c(x) \int_0^1 y(t)\left[ -\sigma_1''(t) + q(t)\sigma_1(t) \right] dt + s(x) \int_0^1 y(t)\left[ -\sigma_2''(t) + q(t)\sigma_2(t) \right] dt,
\]
where

\[ y \in D(L), \quad \sigma_1, \sigma_2 \in W_2^2(0, 1), \quad \sigma_1(0) = \sigma_1(1) = \sigma_2(0) = \sigma_2(1) = 0. \]

If \( I + KL \geq 0 \) and \( I + KL \) is invertible, then the spectrum of the operator \( L_K \) consists only of real eigenvalues \( \{\lambda_k\}_{k=1}^\infty \) and the corresponding eigenfunctions \( \{\varphi_k\}_{k=1}^\infty \) forms a Riesz basis in \( L^2(0,1) \), since \( L^{-1} \) is a compact self-adjoint positive operator. In particular, if

\[ \sigma_1(x) = \alpha(L^{-1}c)(x), \quad \sigma_2(x) = \beta(L^{-1}s)(x), \quad \alpha, \beta \geq 0, \]

then \( KL \geq 0 \). Therefore, by Corollary 1.3, the results of Theorem 1.2 are valid for \( L_K \). In this case, \( L_K^{-1} \) has the form

\[ y = L_K^{-1}f = L^{-1}f + c(x) \int_0^1 f(t)(L^{-1}c)(t)dt + s(x) \int_0^1 f(t)(L^{-1}s)(t)dt. \]

Then \( (L_K^{-1})^* = (L_K^*)^{-1} \) has form

\[ v(x) = (L^{-1}f)(x) + \alpha(L^{-1}c)(x) \int_0^1 f(t)c(t)dt + \beta(L^{-1}s)(x) \int_0^1 f(t)s(t)dt. \]

Thus, we have

\[ (L_K^*v)(x) = -v''(x) + q(x)v(x) + a(x)v'(0) + b(x)v'(1) = f(x), \]

\[ D(L_K^*) = \{ v \in W_2^2(0,1) : v(0) = v(1) = 0 \}; \]

where

\[ a(x) = \frac{\alpha \beta(c,s)s(x) - \alpha(1 + \beta\|s\|^2)c(x)}{(1 + \alpha \|c\|^2)(1 + \beta\|s\|^2) - \alpha \beta \|c,s\|^2}, \]

\[ b(x) = \frac{\alpha[1](1 + \beta\|s\|^2) - \beta s(1)(c,s)c(x) - \beta[\alpha c(1)(c,s) - s(1)(1 + \alpha\|c\|^2)]s(x)}{(1 + \alpha \|c\|^2)(1 + \beta\|s\|^2) - \alpha \beta \|c,s\|^2}, \]

\( a(x), b(x) \in \text{Ker} \hat{L} \) and \((\cdot,\cdot)\) is scalar product in \( L^2(0,1) \). The operator \( L_K^* \) acts as

\[ L_K^* = L^* + Q, \]

where

\[ L^* = -\frac{d^2}{dx^2} + q(x), \]

\[ (Qv)(x) = a(x) < \delta'(x), v(x) > + b(x) < \delta'(x - 1), v(x) > = a(x)v'(0) + b(x)v'(1), \]

which is, the function \( Q \in W_2^{-2}(0,1) \). Thus, we have constructed an example of a non self-adjoint singularly perturbed Sturm-Liouville operator with a real spectrum and the system of eigenvectors that forms a Riesz basis in \( L^2(0,1) \).
Example 2. In the Hilbert space $L^2(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^m$ with an infinitely smooth boundary $\partial \Omega$, let us consider the minimal $L_0$ and maximal $\hat{L}$ operators generated by the Laplace operator
\[ -\Delta u = -\left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2} \right). \] (4.2)

The closure $L_0$, in the space $L^2(\Omega)$ of Laplace operator (4.2) with the domain $C^\infty_0(\Omega)$, is the minimal operator corresponding to the Laplace operator. The operator $\hat{L}$, adjoint to the minimal operator $L_0$ corresponding to Laplace operator, is the maximal operator corresponding to the Laplace operator. Then
\[ D(\hat{L}) = \{ u \in L^2(\Omega) : \hat{L}u = -\Delta u \in L^2(\Omega) \}. \]

Denote by $L$ the operator, corresponding to the Dirichlet problem with the domain
\[ D(L) = \{ u \in W^2_2(\Omega) : u|_{\partial \Omega} = 0 \}. \]

We have (1.2), where $K$ is an arbitrary linear operator bounded in $L^2(\Omega)$ with $R(K) \subset \ker \hat{L} = \{ u \in L^2(\Omega) : -\Delta u = 0 \}$.

Then the operator $L_K$ is defined by
\[ \hat{L}u = -\Delta u, \]
on
\[ D(L_K) = \{ u \in D(\hat{L}) : [(I-K\hat{L})u]|_{\partial \Omega} = 0 \}, \]
where $I$ is the identity operator in $L^2(\Omega)$. Note that $L^{-1}$ is a self-adjoint compact operator. If $K$ satisfies the conditions of Theorem 1.2 then $L_K$ is non self-adjoint operator with a real positive spectrum (i.e., $\sigma(L_K) \subset \mathbb{R}^+$), and the system of eigenvectors $L_K$ forms a Riesz basis in $L^2(\Omega)$. In particular, if
\[ Kf = \varphi(x) \int_{\Omega} f(t) \psi(t) dt, \]
where $\varphi \in W^2_{2,loc}(\Omega) \cap L^2(\Omega)$ is a harmonic function and $\psi \in L^2(\Omega)$, then $K \in B(L^2(\Omega))$ and $R(K) \subset \ker \hat{L}$. From $R(K^*) \subset D(L)$ it follows that $\psi \in W^2_2(\Omega)$ and $\psi|_{\partial \Omega} = 0$. From the condition $KL \geq 0$ we have that $\psi(x) = \alpha(L^{-1}\varphi)(x), \ \alpha \in \mathbb{R}^+$. Hence the operator $L_K$ is the restriction of $\hat{L}$ to the domain
\[ D(L_K) = \{ u \in D(\hat{L}) : \left( u - \frac{\varphi}{1 + \|\varphi\|^2} \int_{\Omega} u(y)\varphi(y)dy \right)|_{\partial \Omega} = 0 \}. \]

The inverse of $L^{-1}_K$ has the form
\[ u = L^{-1}_K f = L^{-1} f + \varphi \int_{\Omega} f(y)(L^{-1}\varphi)(y) dy. \] (4.3)
We find the adjoint operator \( L^*_K \). From (4.3) we have
\[
v = (L^{-1}_K)^* g = L^{-1} g + L^{-1} \varphi \int_{\Omega} g(y) \varphi(y) dy, \quad \text{for all } g \in L^2(\Omega).
\]
Then
\[
L^*_K v = -\Delta v + \frac{\varphi}{1 + \|\varphi\|^2} \int_{\Omega} (\Delta v)(y) \varphi(y) dy = g,
\]
\[
D(L^*_K) = D(L) = \{ v \in W^2_2(\Omega) : v |_{\partial \Omega} = 0 \}.
\]
By virtue of Corollary 1.4, the spectrum of the operator \( L^*_K \) consists only of real positive eigenvalues and the corresponding eigenfunctions forms a Riesz basis in \( L^2(\Omega) \). Note that
\[
(L^*_K v)(x) = -(\Delta v)(x) + \frac{\varphi(x)}{1 + \|\varphi\|^2} F(u) = g(x),
\]
where \( F \in W^{-2}_2(\Omega) \), since
\[
F(u) = \int_{\Omega} (\Delta v)(y) \varphi(y) dy.
\]
This is understood in the sense of the definition of the space \( H^{-s}(\Omega), \ s > 0 \) as in Theorem 12.1 (see [10, p. 71]).

Thus, we have shown the examples of a non self-adjoint singularly perturbed operator with a real spectrum. Moreover, the corresponding eigenvectors forms a Riesz basis in \( L^2(\Omega) \).

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