A Convergent Star Product on the Poincaré Disc

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Abstract
On the Poincaré disc and its higher-dimensional analogs one has a canonical formal star product of Wick type. We define a locally convex topology on a certain class of real-analytic functions on the disc for which the star product is continuous and converges as a series. The resulting Fréchet algebra is characterized explicitly in terms of the set of all holomorphic functions on an extended and doubled disc of twice the dimension endowed with the natural topology of locally uniform convergence. We discuss the holomorphic dependence on the deformation parameter and the positive functionals and their GNS representations of the resulting Fréchet algebra.

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1 Introduction

Deformation quantization comes in two principle flavours: formal deformation quantization as introduced in [1] considers formal associative deformations of the algebra $\mathcal{O}^\infty(M)$ of smooth functions on a Poisson manifold $M$ in the sense of Gerstenhaber [17]. Here the existence as well as the classification of such formal star products is well-understood and established, see [19] for the final case of Poisson manifolds and e.g. [5, 11, 16, 25] for the symplectic case. While the formal deformations are well-understood, they still suffer from being not yet physically relevant: the deformation parameter plays the role of Planck’s constant $\hbar$ and thus should be treated not as formal. This leads to the second flavour of deformation quantization where one tries to find a more analytic framework in such a way that the deformation of the algebra of functions on the classical phase space depends continuously or even analytically on $\hbar$. Here the situation is less clear as there is not yet a general framework which would allow to prove general existence or classification results. Instead, one has several competing definitions of what a non-formal version of a formal star product should be, most notably a $C^*$-algebraic approach as taken e.g. in [7, 21–23, 27] where several classes of examples are studied. There the notion of deformation is based on continuous fields of $C^*$-algebras which satisfy the correct asymptotic for $\hbar \to 0$ on a sufficiently large domain. While the $C^*$-algebraic environment is of course best suited to purposes in quantum theory, the dependence on $\hbar$ is typically not much better than continuous. Moreover, the construction of the continuous fields is fairly complicated. As a final remark on these constructions one should note that on the technical level, all of them are based on certain integration procedures which are limited to finite-dimensional phase spaces. Thus the integral formulas for the non-formal products can not be generalized to field-theoretic situations.

The alternative approach consists now in finding suitable subalgebras of the formal star product algebra where the series defining the product actually converges. To make sense out of such an attempt one needs to specify the notion of convergence. Here a wide variety of possibilities exists, ranging from pointwise or even weaker notions of convergence to more uniform versions taking into account also convergence of derivatives etc. In previous works, such subalgebras with natural locally convex topologies have been identified and studied in several examples, allowing also for infinite-dimensional ones: based on the earlier works [3, 26] for finite dimensions, the case of a locally convex Poisson vector space with continuous constant Poisson structure was treated in detail in [30] and specified further in [29] for the case where the underlying topology is pro-Hilbert. Moreover, in [15] the case of linear Poisson structures, i.e. the Poisson structure on the dual of a Lie algebra, with the Gutt star product [18] was studied in detail. Also in this case infinite-dimensional Lie algebras can be included once the Lie bracket has a certain continuity property always satisfied in the finite-dimensional case. Finally, the first truly curved example, the Poincaré disc $D_n$, was studied in [2, 4] where the convergence of the Wick star product obtained by reduction as in [8, 9] is shown for a locally convex topology based on the growth of the coefficients with respect to a particular Schauder basis of functions on the disc.

It is this example which we investigate more closely: first, we improve the results of [4] in so far as we prove the continuity of the Wick product with respect to a slightly coarser topology on the same underlying linear span of the (to become a) Schauder basis. This gives a slightly larger completion to a Fréchet algebra as before. The impact of this construction is now that we are able to explicitly characterize this Fréchet algebra in terms of the Fréchet algebra $\mathcal{O}(\hat{D}_n)$ of all holomorphic functions on a complex manifold $\hat{D}_n$, which is obtained by doubling the Poincaré disc $D_n$ in a geometrically faithful way. The original functions on the disc $D_n$ are then obtained by restricting these holomorphic functions on $\hat{D}_n$ to the “diagonal” disc $D_n$.

The explicit characterization of the completion opens the doors to many further investigations of this example. In particular, we are able to determine the (classically) positive functionals and show that the evaluation functionals on the disc stay positive with respect to the Wick star product, too. In the corresponding GNS representation we are able to prove (essential) self-adjointness of many
elements of the algebra, among them the generators of the \( \mathfrak{su}(1,n) \) symmetry.

As an open question in the one-dimensional case \( n = 1 \) we mention that the convergent star product we construct will immediately pass to Riemann surfaces of higher genus provided we have enough invariant functions under the corresponding Fuchsian group on the unit disk \( D_1 \cong \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) in our algebra. At the moment, this is not completely clear and therefore we postpone this question to a future project. In any case, it is fairly easy to see that we have many bounded functions in the completion even though in the linear span of the Schauder basis there are only the constant functions bounded. In any case, this question becomes an entirely classical question independent of the deformed product: one needs to understand the holomorphic functions on the extended doubled disc. This will also allow a more detailed comparison with the very recent approach of Bieliavsky [6] to quantum Riemann surfaces where the \( C^* \)-algebraic approach is used.

Ultimately, we believe that this example contains many options to approach also more complicated situations of Wick type quantization of certain Kähler manifolds. The rich geometry of the doubled and extended disc will be present in further sufficiently symmetric cases as well.

The paper is organized as follows: in Section 2 we recall the construction of the Wick star product on the Poincaré disc by phase space reduction and describe the geometry of the extended disc. In Section 3 we define the topology based on the coefficient expansion with respect to the basis functions on the disc. The main result is then the characterization of the completion and the convergence of the Wick star product as a series for all elements in the completion. In the last section we investigate many properties of the Fréchet algebra we obtained: the dependence on the deformation parameter \( \hbar \) turns out to be holomorphic on \( \mathbb{C} \setminus \{ 0, -\frac{1}{2m} \mid m \in \mathbb{N} \} \) and we have an asymptotic in the Fréchet topology of the product for \( \hbar \to 0^+ \) establishing the correct semi-classical limit. Moreover, we determine the positive functionals and their GNS representations showing the essential self-adjointness of many important algebra elements in all continuous representations. Finally, in the case of the disc, i.e. in complex dimension \( n = 1 \), we have an additional discrete symmetry not present in higher dimensions.

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## 2 Preliminaries: The Construction of the Star Product

In this section we are going to present the essential steps of the construction of the star product on the Poincaré disc, introduce our notation, and also discuss some additional structures that will be helpful later on. Essentially everything is either standard concerning the Marsden-Weinstein reduction or can be found in the previous works [2, 4, 8, 9] concerning the construction of the star product, see also [10–13] for yet another approach to this star product on the Poincaré disc.

First we are going to introduce various manifolds and maps between them: In addition to the original manifold \( \mathbb{C}^{1+n} \), the level set \( Z \), and the reduced manifold \( D_n \) from the Marsden-Weinstein reduction this includes extensions of these to complex manifolds \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}, \hat{Z}, \) and \( \hat{D}_n \) that will allow us to define certain spaces of real-analytic functions on \( \mathbb{C}^{1+n} \) and \( D_n \).

Next we construct the classical Poisson \( * \)-algebra on the space \( \mathcal{C}^\infty(D_n) \) of all complex-valued smooth functions on \( D_n \), its subalgebras \( \mathcal{A}(D_n) \) and \( \mathcal{P}(D_n) \) of analytic and polynomial functions, as well as the classical reduction map \( \Psi_0 \) from functions on \( \mathbb{C}^{1+n} \) to such on \( D_n \).

The deformed quantum algebra on \( D_n \) can then be obtained by a similar reduction procedure starting from the space of polynomials on \( \mathbb{C}^{1+n} \) with a Wick-type star product.

### 2.1 The Poincaré disc \( D_n \)

The whole geometric construction can be summarized by the following commutative diagram (the detailed description follows). The upper horizontal row consists of complex manifolds, each of which
is equipped with an anti-holomorphic involution $\tau$ and a smooth action of the Lie group $U(1, n) = U(1) \times SU(1, n)$ by holomorphic automorphisms that commute with $\tau$. The arrows between them are holomorphic maps and equivariant with respect to the $U(1, n)$-action and the involution $\tau$. All other objects are (at least) smooth manifolds equipped with a smooth action of $U(1, n)$ and all other arrows are $U(1, n)$-equivariant smooth maps:

$$
\begin{align*}
\mathbb{C}^{1+n} \times \mathbb{C}^{1+n} & \overset{i \ i}{\longleftarrow} \ \hat{Z} \overset{\hat{pr}}{\longrightarrow} \ \hat{D}_n \overset{r \times p}{\longleftarrow} \ \mathbb{C}^n \times \mathbb{C}^n \\
\Delta & \overset{\Delta}{\longleftarrow} \ \Delta_Z \overset{\Delta_D}{\longrightarrow} \ D_{n, \text{ext}} \overset{r_{\text{ext}} \times p}{\longleftarrow} \ C_n
\end{align*}
$$

Bottom row:

The Poincaré disc $D_n$ can be constructed as a quotient of a subset $Z$ of $\mathbb{C}^{1+n}$. This procedure will be linked to Marsden-Weinstein reduction in Section 2.2.

On the smooth manifold $\mathbb{C}^{1+n}$ with standard (complex) coordinates $z^0, \ldots, z^n : \mathbb{C}^{1+n} \to \mathbb{C}$, the Lie group $U(1, n)$ acts from the left via $U \triangleright z := Ur$ for $U \in U(1, n)$ and $r \in \mathbb{C}^{1+n}$. Define

$$
g := h_{\mu\nu} z^\mu z^\nu \in C^\infty(\mathbb{C}^{1+n}),
$$

where $h_{00} := -1$, $h_{ii} := 1$ for $i \in \{1, \ldots, n\}$ and $h_{\mu\nu} := 0$ otherwise, then

$$
Z := g^{-1}(\{-1\}) = \left\{ r \in \mathbb{C}^{1+n} \mid |z^0(r)|^2 = 1 + \sum_{i=1}^{n} |z^i(r)|^2 \right\}
$$

is the orbit of $(1, 0, \ldots, 0)^T \in \mathbb{C}^{1+n}$ under the $U(1, n)$-action and a submanifold of $\mathbb{C}^{1+n}$. Consequently, the action of $U(1, n)$ can be restricted to $Z$ and the canonical inclusion $i$ of the subset $Z$ in $\mathbb{C}^{1+n}$ is of course $U(1, n)$-equivariant.

Next consider the (compact) Lie subgroup $U(1) \cong \{ e^{i\phi} 1_{1+n} \mid \phi \in \mathbb{R} \} \subseteq U(1, n)$ and construct the quotient manifold

$$
D_n := Z/U(1).
$$

As the $U(1)$-subgroup lies in the center of $U(1, n)$, the action of $U(1, n)$ remains well-defined on $D_n$ (of course, only the complementary $SU(1, n)$-subgroup acts non-trivially) and is still transitive. The canonical projection $pr : Z \to D_n$ then is an $U(1, n)$-equivariant smooth map.

Finally, $D_n$ can be embedded injectively in $\mathbb{C}^n$ via $i_p : D_n \to \mathbb{C}^n$, where we denote the equivalence classes with respect to the different group actions by $[\cdot]_{U(1)}$ and $[\cdot]_{\mathbb{C}^n}$, respectively, to indicate the different equivalence relations. This embedding is also $U(1, n)$-equivariant with respect to the $U(1, n)$-action on $\mathbb{C}^n$ inherited from $\mathbb{C}^{1+n} \setminus \{0\}$, i.e. $U \triangleright [r] := [Ur]$ for $U \in U(1, n)$ and $[r] \in \mathbb{C}^n$. Note that $\mathbb{C}^n$ is the disjoint union

$$
\mathbb{C}^n = \left\{ [r] \in \mathbb{C}^n \mid g(r) < 0 \right\} \cup \left\{ [r] \in \mathbb{C}^n \mid g(r) = 0 \right\} \cup \left\{ [r] \in \mathbb{C}^n \mid g(r) > 0 \right\},
$$

and that the image of $i_p$ is $\{ [r] \in \mathbb{C}^n \mid g(r) < 0 \}$. From $\mathbb{C}^n$, the disc $D_n$ inherits the structure of a complex $n$-dimensional manifold, and can even be covered by a single holomorphic chart $\phi^{\text{std}} = (w^1, \ldots, w^n)^T : D_n \to \mathbb{D}_n \subseteq \mathbb{C}^n$, where $\mathbb{D}_n$ is the $n$-dimensional polydisc $\mathbb{D} \times \cdots \times \mathbb{D}$ and

$$
w^i([r]) := \frac{z^i(r)}{z^0(r)}
$$

(2.6)
for all \( i \in \{1, \ldots, n\} \) and \([r] \in D_n\). This chart actually is a biholomorphic mapping from \( D_n \) to \( \mathbb{D}_n \) and the action of \( U(1, n) \) on \( D_n \) in this chart is described by Möbius transformations

\[
\phi_{\text{std}}^e(U \triangleright [r]) = \phi_{\text{std}}^e([Ur]) = \frac{c + A\phi_{\text{std}}([r])}{\alpha + b \cdot \phi_{\text{std}}([r])}
\]

for \([r] \in D_n\) and \( U = \left( \begin{array}{cc} \alpha & b \\ c & A \end{array} \right) \in \text{SU}(1, n), \quad (2.7)
\]

with \( \alpha \in \mathbb{C}, b, c \in \mathbb{C}^n \), and \( A \in \mathbb{C}^{n \times n} \) such that \( U \in \text{SU}(1, n) \).

Top row:

As we are interested in a non-formal star product on \( D_n \), which cannot be constructed on all smooth functions but only on certain analytic functions on \( D_n \), we have to extend the above construction in a certain way that allows us to describe these analytic functions appropriately, i.e. as pullback with a smooth map \( \Delta_D: D_n \to \hat{D}_n \) of holomorphic functions on some complex manifold \( \hat{D}_n \). This leads to the top row of the diagram above.

On the complex manifold \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) with standard holomorphic coordinate functions \( x^0, \ldots, x^n, y^0, \ldots, y^n \), the group \( U(1, n) \) acts from the left via \( U \triangleright (p, q) := (Up, Uq) \) for all \( U \in \text{SU}(1, n) \) and \( p, q \in \mathbb{C}^{1+n} \). Define the \( U(1, n) \)-equivariant anti-holomorphic involution \( \tau(p, q) := ([\overline{q}, \overline{p}] \) on \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) as well as the \( U(1, n) \)-equivariant diagonal inclusion \( \Delta: \mathbb{C}^{1+n} \to \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \)

\[
r \mapsto \Delta(r) := (r, \tau),
\]

then \( \Delta \) describes a diffeomorphism from \( \mathbb{C}^{1+n} \) to \( \{ (p, q) \in \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \mid \tau(p, q) = (p, q) \} \). Moreover, define

\[
\tilde{g} := h_{\mu\nu}x^\mu y^\nu \in O(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}),
\]

then \( \tilde{g} \circ \Delta = g \) holds. Let

\[
\tilde{Z} := \tilde{g}^{-1}(\{-1\}),
\]

then \( \tilde{Z} \) is a holomorphic submanifold on \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \), the \( U(1, n) \)-action as well as the anti-holomorphic involution \( \tau \) can be restricted to \( \tilde{Z} \) and so the canonical inclusion \( \tilde{i}: \tilde{Z} \to \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) is \( U(1, n) \)- and \( \tau \)-equivariant. Define \( \Delta_Z: Z \to \tilde{Z} \) as the restriction of \( \Delta \), then \( \Delta_Z \) is still \( U(1, n) \)-equivariant, the above rectangle of the above diagram commutes and \( \Delta_Z \) describes a diffeomorphism from \( Z \) to \( \{ (p, q) \in Z \mid \tau(p, q) = (p, q) \} \).

The action of the Lie subgroup \( U(1) \cong \{ e^{i\phi}1_{1+n} \mid \phi \in \mathbb{R} \} \subseteq U(1, n) \) on \( \tilde{Z} \) can be extended to a holomorphic action of \( \mathbb{C}_s \) via \( z \triangleright (p, q) := (zp, qz) \) for all \( (p, q) \in Z \) and \( z \in \mathbb{C}_s \). As this action is free and proper, we can construct the holomorphic quotient manifold

\[
\hat{D}_n := \tilde{Z}/\mathbb{C}_s.
\]

The \( \mathbb{C}_s \)-action on \( \tilde{Z} \) commutes with the \( U(1, n) \)-action, so the \( U(1, n) \)-action remains well-defined on \( \hat{D}_n \) (again, only the \( SU(1, n) \)-subgroup acts non-trivially) and the canonical projection \( \tilde{p}: \tilde{Z} \to \hat{D}_n \) is \( U(1, n) \)-equivariant. Moreover, there is a unique anti-holomorphic involution \( \tau \) on \( \hat{D}_n \) such that \( \tilde{p} \) becomes also \( \tau \)-equivariant, namely \( \tau([p, q]) := ([\overline{q}, \overline{p}] \), which is indeed well-defined and commutes with the action of \( U(1, n) \). The map \( \Delta_Z \) remains well-defined on the quotients, so \( \Delta_D: D_n \to \hat{D}_n \),

\[
[r] \mapsto \Delta_D([r]) := \Delta_Z(r)
\]

is well-defined and a smooth and \( U(1, n) \)-equivariant map. It is now easy to see that the central rectangle in the above diagram commutes as well.

Finally, we can again embed \( \hat{D}_n \) injectively in \( \mathbb{C}P^n \times \mathbb{C}P^n \) via \( \iota_{P^P}: \hat{D}_n \to \mathbb{C}P^n \times \mathbb{C}P^n \)

\[
[p, q]_{\mathbb{C}_s} \mapsto \iota_{P^P}([p, q]_{\mathbb{C}_s}) := ([p]_{\mathbb{C}_s}, [q]_{\mathbb{C}_s}),
\]

(2.13)
which is equivariant with respect to the $U(1)$-action on $\mathbb{CP}^n \times \mathbb{CP}^n$ defined as $U \cdot ([p],[q]) := ([Up],[Uq])$ for all $[p],[q] \in \mathbb{CP}^n$ and $U \in U(1)$. It is also $\tau$-equivariant if one defines the anti-holomorphic involution $\tau$ on $\mathbb{CP}^n \times \mathbb{CP}^n$ as $\tau([p],[q]) := ([\overline{q}],[\overline{p}])$. The image of $\iota_{\mathbb{P} \times \mathbb{P}}$ in $\mathbb{CP}^n \times \mathbb{CP}^n$ is then $\{([p],[q]) \in \mathbb{CP}^n \times \mathbb{CP}^n \mid \hat{g}(p,q) \neq 0\}$ and pulling back the usual charts from $\mathbb{CP}^n \times \mathbb{CP}^n$ to $\hat{D}_n$ yields suitable holomorphic charts on $\hat{D}_n$. We especially define the standard chart $\hat{\Delta}^{\text{std}} = (u^1, \ldots, u^n, v^1, \ldots, v^n)^T : \hat{D}^{\text{std}} \rightarrow C^{\text{std}} \subseteq \mathbb{C}^n \times \mathbb{C}^n$ by

$$u^i([p,q]) := \frac{x^i(p,q)}{x^0(p,q)} \quad \text{and} \quad v^i([p,q]) := \frac{y^i(p,q)}{y^0(p,q)}$$

(2.14)

for all $i \in \{1, \ldots, n\}$ and $[p,q] \in \hat{D}_n$, with domain

$$\hat{D}^{\text{std}} := \{[p,q] \in \hat{D}_n \mid x^0(p,q) \neq 0 \text{ and } y^0(p,q) \neq 0\}$$

(2.15)

and image $C^{\text{std}} := \{(p,q) \in \mathbb{C}^n \times \mathbb{C}^n \mid p \cdot q \neq 1\}$. Moreover, let $\Delta_p : \mathbb{CP}^n \rightarrow \mathbb{CP}^n \times \mathbb{CP}^n$ be the $U(1,n)$-equivariant diagonal inclusion

$$[r] \mapsto \Delta_p([r]) := ([r],[\overline{r}])$$

(2.16)

then the right rectangle in the above diagram commutes.

The extended disc $D_{n,\text{ext}}$:

Comparing the images of the embeddings of $D_n$ and $\hat{D}_n$ in $\mathbb{CP}^n$ and $\mathbb{CP}^n \times \mathbb{CP}^n$, respectively, shows that the image of $D_n$ under $\Delta_D$ in $\hat{D}_n$ is only contained in, but not the whole set,

$$D_{n,\text{ext}} := \{[p,q] \in \hat{D}_n \mid \tau([p,q]) = [p,q]\},$$

(2.17)

which is a smooth submanifold of $\hat{D}_n$ and stable under the $U(1,n)$-action. Let $\Delta_{\times} : D_{n,\text{ext}} \rightarrow \hat{D}_n$ be the canonical embedding, which is of course $U(1,n)$-equivariant. Then the diagonal inclusion $\Delta_D$ of $D_n$ in $\hat{D}_n$ factors through $\Delta_{\times}$, such that $\Delta_D = \Delta_{\times} \circ \iota_{\text{ext}}$ with a unique $U(1,n)$-equivariant smooth map $\iota_{\text{ext}} : D_n \rightarrow D_{n,\text{ext}}$. At last, the above commutative diagram is completed by the smooth $U(1,n)$-equivariant map $\iota_{\text{ext},\mathbb{P}} : D_{n,\text{ext}} \rightarrow \mathbb{CP}^n$,

$$[p,q]_{\mathbb{C}_*} \mapsto \iota_{\text{ext},\mathbb{P}} ([p,q]_{\mathbb{C}_*}) := [p]_{\mathbb{C}_*},$$

(2.18)

which is an injective embedding of $D_{n,\text{ext}}$ in $\mathbb{CP}^n$ with image $\{[p] \in \mathbb{CP}^n \mid g(p) \neq 0\}$. This is a consequence of the observation that, given $p \in \mathbb{C}^{1+n}$ with $g(p) \neq 0$, there is a unique $q \in \mathbb{C}^{1+n}$ such that $[p,q] \in D_{n,\text{ext}}$, namely $q = -\overline{p}/g(p)$, and $\iota_{\text{ext},\mathbb{P}}([p,q]) = [p]$.

2.2 The Classical Poisson Algebra

The Poisson structure on $D_n$ comes from a Kähler structure which can be obtained from a variant of Marsden-Weinstein reduction from $\mathbb{C}^{1+n}$: consider the complex 2-form $\hbar = h_{\mu\nu} \, d\overline{\tau}^\mu \otimes dz^\nu \in \Gamma(\Lambda^2 T^*\mathbb{C}^{1+n})$, then $h$ endows $\mathbb{C}^{1+n}$ with the structure of a pseudo Kähler manifold with pseudo Riemannian metric $g := \operatorname{Re}(h) = \frac{1}{2} h_{\mu\nu} \, d\overline{\tau}^\mu \wedge dz^\nu$ and Kähler symplectic form $\omega := \operatorname{Im}(h) = \frac{1}{2\hbar} h_{\mu\nu} \, d\overline{\tau}^\mu \wedge dz^\nu$. The inverse tensors are then $g^{-1} = 2h_{\mu\nu} \delta_{\tau^\nu} \wedge \delta_{\overline{\tau}}$ and $\pi := \omega^{-1} = 2h_{\mu\nu} \delta_{\overline{\tau}^\nu} \wedge \delta_{\tau}$ with $h_{\mu\nu} = h_{\overline{\nu}\mu}$ for all $\mu, \nu \in \{0, \ldots, n\}$, so that the Poisson bracket $\{\cdot, \cdot\}$ on $\mathbb{C}^{1+n}$ is given by

$$\{a, b\} := \pi(\, da \otimes db) = 2i\hbar^{\mu\nu} \left( \frac{\partial a}{\partial \overline{\tau}^\mu} \frac{\partial b}{\partial z^\nu} - \frac{\partial b}{\partial \tau} \frac{\partial a}{\partial \overline{\tau}^\nu} \right)$$

(2.19)
which cannot be constructed on the whole spaces $C\infty(C^{1+n})$. It fulfills $\{a, b\}^* = \{a^*, b^*\}$ because the Poisson tensor $\pi$ is real. Note that $h$, hence also $g$ and $\omega$, are $U(1,n)$-invariant, so the Poisson bracket is $U(1,n)$-equivariant, i.e. $\{a \triangleleft U, b \triangleleft U\} = \{a, b\} \triangleleft U$ for all $a, b \in C\infty(C^{1+n})$ and $U \in U(1,n)$.

The action of $U(1,n)$ is not only Kähler, i.e. preserves $\omega$ and $g$, but also Hamiltonian with an equivariant momentum map. Explicitly, the infinitesimal action of $u(1,n)$ on spaces of tensor fields on $C^{1+n}$ is given by $X \triangleleft u = L_{\xi_u} \circ \hat{\omega}$, where $u(1,n) \ni u \ni \xi_u = u^{-1}_{\alpha \nu} \partial_{\nu} \in \Gamma(TC^{1+n})$ is an anti-morphism of Lie algebras. An equivariant momentum map for this action is then given by $\mathcal{J}(u) : u(1,n) \to C\infty(C^{1+n})$,

$$\mathcal{J}(u) := \frac{1}{2i} h_{\mu \nu} u^\mu_{\nu} w^\nu,$$

because a straightforward calculation shows that $\{f, \mathcal{J}(u)\} = \xi_u(f) + \overline{\xi_u}(f) = f \triangleleft u$ as well as $\{\mathcal{J}(u), \mathcal{J}(v)\} = \mathcal{J}([u, v])$ holds for all $u, v \in u(1,n)$ and $f \in C\infty(C^{1+n})$.

Note that $g = \mathcal{J}(2i1_{1+n})$ with $2i1_{1+n} \in u(1,n)$ generating the $U(1)$-subgroup $\{e^{i\phi} 1_{1+n} | \phi \in \mathbb{R}\}$ of $U(1,n)$, so $Z$ is a $u(1)$-level set and $D_n = Z/\mathbb{U}(1)$ the resulting Marsden-Weinstein quotient. The Kähler structure on $D_n$ in the standard coordinates $\phi^{\text{std}} = (w^1, \ldots, w^n)^T$ is then given by the 2-form

$$h_d = (1 - w \cdot w)^{-2}(\delta_{ij}(1 - w \cdot w) + \overline{w}^i \overline{w}^j \delta_{kl} \delta_{ij}) \, d\overline{w}^i \wedge dw^j,$$

and thus the Kähler metric becomes $g_d = \text{Re}(h_d)$ while the Kähler symplectic form is $\omega_d = \text{Im}(h_d)$.

Finally, the Poisson tensor is

$$\pi_d = \omega_d^{-1} = 2i(1 - w \cdot \overline{w})(\delta^{ij} - \overline{w}^i w^j) \partial_{w^i} \wedge \partial_{\overline{w}^j}.$$

From the point of view of deformation quantization, the description of the geometric reduction in terms of function algebras becomes more important: we outline here how this can be accomplished. Let $C\infty(C^{1+n})_Z$ be the *-ideal in $C\infty(C^{1+n})$ consisting of all functions in $C\infty(C^{1+n})$ that vanish on $Z$, then $C\infty(C^{1+n})_Z$ is the ideal generated by $g + 1$ and $C\infty(Z) \cong C\infty(C^{1+n})/C\infty(C^{1+n})_Z$ as *-algebras. Moreover, restricted to $C\infty(C^{1+n})_Z$, the Poisson *-subalgebra of $C\infty(C^{1+n})$ consisting of all $U(1)$-invariant functions, the *-ideal $C\infty(C^{1+n})_Z$ is even a Poisson *-ideal, and consequently $C\infty(C^{1+n})_Z/(C\infty(C^{1+n})_Z \cap C\infty(C^{1+n})_Z)$ is even a Poisson *-algebra and isomorphic to the reduced Poisson *-algebra $C\infty(D_n)$. This isomorphism is described by the classical reduction map:

**Definition 2.1 (Classical reduction map)** Define the map $\Psi_0 : C\infty(C^{1+n})_Z \to C\infty(D_n)$,

$$a \mapsto \Psi_0(a) \text{ with } \Psi_0(a)([r]) := a(r) \text{ for all } [r] \in D_n.$$  

One can check that $\Psi_0$ is indeed a well-defined unital and $U(1,n)$-equivariant Poisson *-homomorphism. Moreover, its kernel is clearly $C\infty(C^{1+n})_Z \cap C\infty(C^{1+n})_Z$, and as every smooth $U(1)$-invariant function on $Z$ can be extended to a smooth $U(1)$-invariant function on $C^{1+n}$ (just take an arbitrary extension and make it $U(1)$-invariant by averaging over the action of $U(1)$), $\Psi_0$ is also surjective and thus descends to an isomorphism from the quotient $C\infty(C^{1+n})_Z/(C\infty(C^{1+n})_Z \cap C\infty(C^{1+n})_Z)$ to $C\infty(D_n)$.

However, as we are interested in non-formal deformation quantizations of these Poisson algebras, which cannot be constructed on the whole spaces $C\infty(C^{1+n})$ and $C\infty(D_n)$, we have to restrict our attention to suitable subalgebras:

**Definition 2.2 (Spaces of analytic functions on $C^{1+n}$ and $D_n$)** We define

$$\mathcal{A}(C^{1+n}) := \{a \circ \Delta | a \in \mathcal{C}(C^{1+n} \times C^{1+n})\} \text{ and } \mathcal{A}(D_n) := \{a \circ \Delta_D | a \in \mathcal{C}(D_n)\}.$$  

Note that these are unital Poisson *-subalgebras of $C\infty(C^{1+n})$ and $C\infty(D_n)$, respectively, because the coefficients of the Poisson tensors $\pi$ and $\pi_d$ with respect to the standard charts are polynomials.
in the coordinate functions and because \((\hat{a} \circ \Delta)^* = \tau \circ \hat{a} \circ \tau \circ \Delta\) as well as \((\hat{b} \circ \Delta_D)^* = \tau \circ \hat{b} \circ \tau \circ \Delta_D\) holds for all \(\hat{a} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})\) and \(\hat{b} \in \mathcal{O}(\hat{D}_n)\), where \(\tau \circ \hat{a} \circ \tau\) and \(\tau \circ \hat{b} \circ \tau\) are compositions of two anti-holomorphic and one holomorphic function, hence are holomorphic. Moreover, the pullback \(\Delta^*: \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \to \mathcal{A}\) is an isomorphism of vector spaces, because \(\frac{\partial}{\partial \nu} \circ \Delta = \frac{\partial}{\partial \nu} (\hat{a} \circ \Delta)\) holds for all \(\mu \in \{0, \ldots, n\}\) and \(\hat{a} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})\), so \(\hat{a} \circ \Delta = 0\) implies that all derivatives of \(\hat{a}\) vanish at all points in the image of \(\Delta\), and thus \(\hat{a} = 0\). Similarly, the pullback \(\Delta_D^*: \mathcal{O}(\hat{D}_n) \to \mathcal{A}\) is an isomorphism as well. The inverse of these isomorphisms will simply be denoted by \(\hat{\cdot}\), i.e. given \(a \in \mathcal{A}\), then \(\hat{a}\) is the unique element in \(\mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})\) that fulfills \(a = \hat{a} \circ \Delta\), similarly for \(\mathcal{A}\). Note that we have already used this notation for \(g = \hat{g} \circ \Delta \in \mathcal{A}\) in \((\ref{eq:1}).

As \(\mathcal{A}\) and \(\mathcal{A}\) are isomorphic to spaces of holomorphic functions, they are Fréchet spaces with respect to the topology of locally uniform convergence of the holomorphic extensions. We will only need the topology of \(\mathcal{A}\):

**Definition 2.3 (Norms on \(\mathcal{A}\))** Let \(K \subseteq \hat{D}_n\) be compact, then define the seminorm \(\| \cdot \|_{\mathcal{A}, K}\) on \(\mathcal{A}\) as

\[
\|a\|_{\mathcal{A}, K} := \sup_{[p, q] \in K} |\hat{a}(p, q)|
\]

for all \(a \in \mathcal{A}\).

It will be helpful to describe \(\mathcal{A}\) and \(\mathcal{A}\) as completions of algebras of polynomials:

**Definition 2.4 (Polynomials on \(\mathbb{C}^{1+n}\))** For all multiindices \(P, Q \in \mathbb{N}_0^{1+n}\) we define the monomial

\[
d_{P, Q} := z^P \bar{z}^Q := (z^0)^{P_0} \cdots (z^n)^{P_n} (\bar{z}^0)^{Q_0} \cdots (\bar{z}^n)^{Q_n} \in \mathcal{A} \quad \text{and write } \mathcal{P}(\mathbb{C}^{1+n}) \text{ for their span, i.e. for the space of polynomial functions.}
\]

It is then clear that the polynomial functions form a dense unital Poisson \(*\)-subalgebra of \(\mathcal{A}\).

Similarly like before, we denote the (closed) subspaces of \(U(1)\)-invariant analytic functions and polynomials on \(\mathbb{C}^{1+n}\) by \(\mathcal{A}(\mathbb{C}^{1+n})^{U(1)}\) and \(\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}\). Then \(\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}\) is spanned by the \(U(1)\)-invariant monomials \(d_{P, Q}\) with \(P, Q \in \mathbb{N}_0^{1+n}\) and \(|P| = |Q|\), and is dense in \(\mathcal{A}(\mathbb{C}^{1+n})^{U(1)}\). This both is a consequence of the observation that the Cauchy formula for reconstructing the Taylor coefficient in front of \(d_{P, Q}\) by means of circular integrals around the origin of \(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}\) is \(U(1)\)-invariant only if \(|P| = |Q|\). Of course, \(\mathcal{A}(\mathbb{C}^{1+n})^{U(1)}\) and \(\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}\) are again unital Poisson \(*\)-subalgebras of \(\mathcal{C}^{\infty}(\mathbb{C}^{1+n})\). Using the reduction map we can now also construct “polynomials” on \(\mathcal{D}_n\):

**Definition 2.5 (Polynomials on \(\mathcal{D}_n\))** We define \(f_{P, Q} := \Psi_0(d_{P, Q})\) for all \(P, Q \in \mathbb{N}_0^{1+n}\) with \(|P| = |Q|\) and write \(\mathcal{P}(\mathcal{D}_n)\) for the image of \(\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}\) under \(\Psi_0\), i.e. for the span of the functions \(f_{P, Q}\).

Note that we will show in Theorem 3.10 that \(\mathcal{P}(\mathcal{D}_n)\) is dense in \(\mathcal{A}(\mathcal{D}_n)\) with respect to its Fréchet topology. As \(\Psi_0\) is not injective, we cannot expect the monomials \(f_{P, Q}\) on \(\mathcal{D}_n\) to be a basis of \(\mathcal{P}(\mathcal{D}_n)\). A suitable choice for a basis is the following (see [3, Lemma 4.20]):

**Definition 2.6 (Fundamental monomials on \(\mathcal{D}_n\))** For all \(P, Q \in \mathbb{N}_0^n\) we define the fundamental monomial

\[
f_{r, P, Q} := \begin{cases} f_{(|Q| - |P|, P_0, \ldots, P_n, 0, Q_1, \ldots, Q_n)} & \text{if } |Q| \geq |P| \\ f_{(0, P_0, \ldots, P_n, |P| - |Q|, Q_1, \ldots, Q_n)} & \text{if } |Q| \leq |P| \end{cases}
\]

Note that, with respect to the coordinates of the standard chart \(\phi^{std} = (w^1, \ldots, w^n)^T\), the monomials on \(\mathcal{D}_n\) are represented as

\[
f_{P, Q} = \frac{(w^1)^{P_1} \cdots (w^n)^{P_n} (\bar{w}^1)^{Q_1} \cdots (\bar{w}^n)^{Q_n}}{(1 - w \cdot \bar{w})^{|P|}}
\]
2.2 The Classical Poisson Algebra

for all $P, Q \in \mathbb{N}_{0}^{1+n}$ with $|P| = |Q|$. In particular,

$$f_{r, P, Q} = \frac{(w^{1})^{P_{1}} \cdots (w^{n})^{P_{n}}(\overline{w}^{1})^{Q_{1}} \cdots (\overline{w}^{n})^{Q_{n}}}{(1 - w \cdot \overline{w})^{\max\{|P|, |Q|\}}} = \frac{w^{r}P_{r}Q_{r}}{(1 - w \cdot \overline{w})^{\max\{|P|, |Q|\}}}$$

(2.29)

for all $P, Q \in \mathbb{N}_{0}^{n}$. Using this one can already show that the $f_{r, P, Q}$ are linearly independent, and they span $\mathcal{D}(D_{n})$ because every $f_{P, Q}$ with $P, Q \in \mathbb{N}_{0}^{1+n}$ and $|P| = |Q|$ can be rewritten as

$$f_{P, Q} = \sum_{T \in \mathbb{N}_{0}^{n}} \left( \frac{\min\{|P_{0}, Q_{0}|\}}{|T|} \right) \frac{|T|!}{T!} f_{r, P', T, Q', T},$$

(2.30)

where $P' = (P_{1}, \ldots, P_{n}) \in \mathbb{N}_{0}^{n}$ and analogously for $Q$. So we get (\cite{4} Lemma 4.20):

Proposition 2.7 The fundamental monomials on $D_{n}$ form a basis of $\mathcal{D}(D_{n})$.

Note that the product of two fundamental monomials on $D_{n}$ is more complicated than the product of monomials on $\mathbb{R}^{n}$: Given $f_{r, P, Q}, f_{r, R, S}$ with $P, Q, R, S \in \mathbb{N}_{0}^{n}$, then $f_{r, P, Q} f_{r, R, S} = f_{r, P+R, Q+S}$ holds only in the cases that $|P| \geq |Q|$ and $|R| \geq |S|$ or that $|P| \leq |Q|$ and $|R| \leq |S|$. If $|P| \geq |Q|$ and $|R| \leq |S|$ then

$$f_{r, P, Q} f_{r, R, S} = \sum_{T \in \mathbb{N}_{0}^{n}} \left( \min\{|S| - |R|, |P| - |Q|\} \right) \frac{|T|!}{T!} f_{r, P+R+T, Q+S+T},$$

(2.31)

and similarly if $|P| \leq |Q|$ and $|R| \geq |S|$. This especially shows that identifying $f_{r, P, Q}$ with a monomial $z^{P} \overline{z}^{Q}$ on $\mathbb{C}^{n}$ does not extend to an isomorphism of algebras.

The unital Poisson-*-algebras $\mathcal{P}(\mathbb{C}^{1+n})$ and $\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$ are of course graded by the degree of polynomials. However, the induced filtration will be even more important in the following, because it remains well-defined after reduction to $\mathcal{D}(D_{n})$ and will also be respected by the deformed product:

Definition 2.8 (Filtration on polynomials) For all $m \in \mathbb{N}_{0}$ we define $\mathcal{P}(\mathbb{C}^{1+n})^{U(1),(m)}$ as the space of $U(1)$-invariant polynomials of up to degree $2m$, i.e. as the span of $dP, Q$ for all $P, Q \in \mathbb{N}_{0}$ with $|P| = |Q| \leq m$. Similarly, $\mathcal{D}(D_{n})^{(m)}$ is defined as the image of $\mathcal{P}(\mathbb{C}^{1+n})^{U(1),(m)}$ under $\Psi_{0}$, i.e. as the span of $f_{P, Q}$ for all $P, Q \in \mathbb{N}_{0}$ with $|P| = |Q| \leq m$.

Similarly to \cite{4} Lemma 4.18, we get:

Proposition 2.9 For all $m \in \mathbb{N}_{0}$ the following holds:

i.) $\dim \mathcal{P}(\mathbb{C}^{1+n})^{U(1),(m)} = \sum_{k=0}^{m} \binom{n+k}{k}^{2}$.

ii.) $\dim \mathcal{D}(D_{n})^{(m)} = \binom{n+m}{m}^{2}$.

Moreover, the kernel of the restriction of $\Psi_{0}$ to $\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$ is the ideal in $\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$ generated by $g + 1$, i.e.

$$\ker \Psi_{0} \cap \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} = \left\{ (g + 1)a \mid a \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \right\}.$$  

(2.32)

Proof: Given $k \in \mathbb{N}_{0}$ and $\ell \in \mathbb{N}$ then the set $\left\{ P \in \mathbb{N}_{0}^{\ell} \mid |P| = k \right\}$ has $\binom{\ell - 1 + k}{k}$ elements. From this one can easily deduce the first dimension formula and also the second because

$$\dim \mathcal{D}(D_{n})^{(m)} = \left( \sum_{k=0}^{m} \binom{n-1+k}{k} \right)^{2} = \left( \frac{n+m}{m} \right)^{2}.$$
Moreover, it is easy to see that the *-ideal in \( \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \) that is generated by \( g+1 \) is in the kernel of \( \Psi_0 \) and in order to show that it is the whole of \( \ker \Psi_0 \cap \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \), it is sufficient to show for all \( m \in \mathbb{N}_0 \) that \( \ker \Psi_0 \cap \mathcal{P}(\mathbb{C}^{1+n})^{U(1),(m)} = \{ (g+1)a \mid a \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1),(m)} \} \), or

\[
\dim \mathcal{P}(\mathbb{C}^{1+n})^{U(1),(m)} - \dim \mathcal{P}(D_n)^{(m)} \leq \dim \left( \{ (g+1)a \mid a \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \} \cap \mathcal{P}(\mathbb{C}^{1+n})^{U(1),(m)} \right)
\]

Due to the above dimension formulas, the left hand side of this reduces to \( \sum_{k=0}^{m-1} \binom{n+k}{k}^2 \). In the case that \( m = 0 \), this inequality is certainly true. But if it holds for one \( m \in \mathbb{N}_0 \) then also for \( m + 1 \) because the span of all \( (g+1)d_{P,Q} \) with \( P, Q \in \mathbb{N}_0 \) and \( |P| = |Q| = m \) has dimension \( \binom{n+m}{m}^2 \) and is a subspace of \( \{ (g+1)a \mid a \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \} \cap \mathcal{P}(\mathbb{C}^{1+n})^{U(1),(m+1)} \) that has trivial intersection with \( \mathcal{P}(\mathbb{C}^{1+n})^{U(1),(m)} \).

So the algebraic description of the reduction stays the same for the polynomials, i.e. \( \mathcal{P}(D_n) \) is isomorphic as a Poisson-*-algebra to the quotient of \( \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \) over the ideal generated by \( g+1 \).

### 2.3 The Deformed Quantum Algebra

Analogously to the Marsden-Weinstein reduction in the classical case, the star product on the Poincaré disc can be constructed by quantum reduction: In order to obtain a formal deformation quantisation of \( D_n \), one can start with the Wick star product on \( \mathbb{C}^{1+n} \) given by

\[
a \star b := \sum_{l=0}^{\infty} \frac{(2\hbar)^l}{l!} \sum_{i_1,\ldots,i_l=0}^{n} h^{i_1j_1} \cdots h^{i_lj_l} \frac{\partial^l a}{\partial z^{i_1} \cdots \partial z^{i_l}} \frac{\partial^l b}{\partial \bar{z}^{j_1} \cdots \partial \bar{z}^{j_l}}
\]

for all \( a, b \in \mathcal{C}^\infty(\mathbb{C}^{1+n})[\hbar] \), which is a \( \mathcal{C}[\hbar] \)-bilinear associative multiplication and compatible with the *-involution of pointwise complex conjugation, i.e. \( (a \star b)^* = b^* \star a^* \) holds for all \( a, b \in \mathcal{C}^\infty(\mathbb{C}^{1+n})[\hbar] \). Its commutator yields the classical Poisson bracket up to terms of higher order, \( \frac{1}{\hbar}[a, b]_* = \{ a, b \} + \hbar \cdots \). Note that \( \mathcal{J} \) is not only a classical moment map but even a quantum moment map, i.e. \( \frac{1}{\hbar}[a, \mathcal{J}(u)]_* = \{ a, \mathcal{J}(u) \} = a \circ u \) for all \( u \in \mathfrak{u}(1, n) \), because in commutators with \( \mathcal{J}(\cdot) \), only the first order in \( \hbar \) contributes due to the linearity of the moment map in the \( z \)-coordinates. Analogously to the classical Poisson bracket, the Wick star product is also \( U(1,n) \)-equivariant, i.e. \( a \circ U \mathcal{J}(b \circ U) = (a \star U b) \circ U \) holds for all \( a, b \in \mathcal{C}^\infty(\mathbb{C}^{1+n})[\hbar] \) and \( U \in U(1,n) \). The reduced star product algebra on \( D_n \) can then be obtained from the one on \( \mathbb{C}^{1+n} \) by restriction to the subalgebra of \( U(1) \)-invariant elements in \( \mathcal{C}^\infty(\mathbb{C}^{1+n})[\hbar] \) and dividing out the ideal generated by \( g + 1 \).

On \( \mathcal{P}(\mathbb{C}^{1+n}) \), the Wick star product converges trivially for every \( h \in \mathbb{C} \) which yields an associative product \( \hat{\star} \) that fulfills \( (a \hat{\star} b)^* = b^* \hat{\star} a^* \), hence \( (\mathcal{P}(\mathbb{C}^{1+n}), \hat{\star}, \cdot) \) is a *-algebra for all \( h \in \mathbb{R} \). On the basis of \( \mathcal{P}(\mathbb{C}^{1+n}) \) given by the monomials \( d_{P,Q} \), the Wick star product can be expressed as

\[
d_{P,Q} \hat{\star} d_{R,S} = \sum_{T=0}^{\min\{P,S\}} (-1)^T h (2\hbar)^{|T|} T! \left( \begin{array}{c} P \\ T \end{array} \right) \left( \begin{array}{c} S \\ T \end{array} \right) \left( \begin{array}{c} 1 \\ 2\hbar \end{array} \right)^{|P-T|} \left( \begin{array}{c} 1 \\ 2\hbar \end{array} \right)^{|Q-T|} d_{P+R-T, Q+S-T} \tag{2.34}
\]

for all \( P, Q, R, S \in \mathbb{N}_0^{1+n} \). As the classical *-ideal generated by \( g+1 \) in \( \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \), i.e. the kernel of \( \Psi_0 \), is no longer an ideal with respect to the Wick star product, one has to perform an equivalence transformation first that assures that the star product with \( g \) is the classical product, see [4,8], and can then restrict to functions on \( D_n \). This procedure results in the following deformed reduction map:

#### Definition 2.10 (Deformed reduction map)

Let \( H := \mathbb{C}_\times \backslash \{-1/(2m) \mid m \in \mathbb{N} \} \) and define for all \( h \in H \) the deformed reduction map \( \Psi_h : \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \to \mathcal{P}(D_n) \) by linear extension of

\[\Psi_h(d_{P,Q}) := (2\hbar)^{|P|} \left( \begin{array}{c} 1 \\ 2\hbar \end{array} \right)^{|P|} \Psi_0(d_{P,Q}) = (2\hbar)^{|P|} \left( \begin{array}{c} 1 \\ 2\hbar \end{array} \right)^{|P|} f_{P,Q} \tag{2.35}\]
for all $P,Q \in \mathbb{N}_0$ with $|P| = |Q|$, where $(z)_m$ denotes the Pochhammer symbol, or rising factorial,

$$(z)_m := \prod_{k=0}^{m-1} (z + k)$$  \hspace{1cm} (2.36)

for all $z \in \mathbb{C}$ and $m \in \mathbb{N}_0$.

**Proposition 2.11** For all $h \in H$ the kernel of the deformed reduction map $\Psi_h$ is the $^*$-ideal generated by $g + 1$ with respect to the Wick product $\ast_h$ on $\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$.

**Proof:** Using the explicit formula (2.34) one can check that indeed

$$d_{P,Q} \ast_h (g + 1) = (g + 1) \ast_h d_{P,Q} = (g + 1 + 2h|P|)d_{P,Q}$$

is in the kernel of $\Psi_h$ for all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$, because

$$\Psi_h((g + 1 + 2h|P|)d_{P,Q}) = (2h)^{|P|+1} \left( \frac{1}{2h} \right)_{|P|+1}^{\Psi_0((g + 1) d_{P,Q})} = 0.$$ 

So the $^*$-ideal generated by $g + 1$ with respect to the Wick product on $\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$ is in the kernel of $\Psi_h$. Conversely, in order to show that this is indeed the whole kernel of $\Psi_h$ one can use the same argument as in the proof of Proposition 2.9 and count dimensions. □

As a consequence, the following product on $\mathcal{P}(D_n)$ is indeed well-defined and associative:

**Definition 2.12 (Reduced non-formal star product on $D_n$)** For all $h \in H$ we define the product $\ast_h : \mathcal{P}(D_n) \times \mathcal{P}(D_n) \to \mathcal{P}(D_n)$ as

$$(a \ast_h b) := a' \ast_h b' ,$$  \hspace{1cm} (2.37)

where $a', b' \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$ are arbitrary preimages of $a$ and $b$ under $\Psi_h$.

Note also that $(a \ast_h b) = a^* \ast_h b^*$ holds for all $h \in H$ and $a, b \in \mathcal{P}(D_n)$, so pointwise complex conjugation is a $^*$-involution for $\ast_h$ if $h \in H \cap \mathbb{R}$. An explicit formula for $\ast_h$ on the monomials on $D_n$ is

$$f_{P,Q} \ast_h f_{R,S} = \sum_{T=0}^{\min\{P,S\}} (-1)^T (\frac{1}{2h})_{|P+S-T|}^{T!} (\frac{1}{2h})_{|P|}^{P} (\frac{1}{2h})_{|S|}^{S} f_{P+S-T,Q+S-T}$$  \hspace{1cm} (2.38)

for all $P, Q, R, S \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and $|R| = |S|$. Moreover, as the Wick star product on $\mathbb{C}^{1+n}$ and the deformed reduction map are $U(1,n)$-equivariant, the reduced star product $\ast_h$ is also $U(1,n)$-equivariant.

### 3 The Construction of the Fréchet Algebra

In this section we now construct a topology for the algebra $\mathcal{P}(D_n)$ for which the star product becomes continuous. This will allow to complete the polynomial functions to a Fréchet $^*$-algebra. In a second step we show that this (abstract) completion is still a space of functions on the disc by noting that the evaluation functionals are continuous. Then the size of this completion is determined by establishing a bijection to the space $\mathcal{A}(D_n)$, i.e. those real-analytic functions on the disc $D_n$ which arise as “diagonal” restrictions of holomorphic functions on $D_n$.

In [4] Thm. 4.21, (viii) a topology was constructed for which the star product is also continuous. However, the topology we discuss here is slightly coarser which is ultimately the reason that we are able to determine the completion explicitly in geometric terms.
3.1 The Topology

By constructing a locally convex topology on $\mathcal{P}(D_n)$ under which $\star_h$ is continuous, we can extend the star product to the completion of $\mathcal{P}(D_n)$. A well-behaved topology on $\mathcal{P}(\mathbb{C}^{1+n})$ has already been examined in [29,30]. Transferring these results to the Poincaré disc is straightforward:

**Definition 3.1 (Norms on $\mathcal{P}(\mathbb{C}^{1+n})$)** For all $\rho > 0$ we define the norm

$$\left\| \sum_{P,Q \in \mathbb{N}^{1+n}_0} a_{P,Q} d_{P,Q} \right\|_{\mathcal{C}^{1+n},\rho} := \sum_{P,Q \in \mathbb{N}^{1+n}_0} |a_{P,Q}| \rho^{P+Q} \sqrt{|P+Q|!}$$

(3.1)

on $\mathcal{P}(\mathbb{C}^{1+n})$, where $a_{P,Q} \in \mathbb{C}$ for all $P,Q \in \mathbb{N}^{1+n}_0$.

Note that the locally convex topology defined by all these norms on $\mathcal{P}(\mathbb{C}^{1+n})$ is the same as the one in [29] (even though we are using a different fundamental system of continuous seminorms here), if one identifies $\mathcal{P}(\mathbb{C}^{1+n})$ with the symmetric tensor algebra over $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$. Then [29] Prop. 2.11 and Lemma 2.12] prove the continuity and absolute convergence of $\tilde{\star}_h$:

**Lemma 3.2** For every compact $K \subseteq \mathbb{C}$ and every $\rho > 0$ there exist $C,\rho' > 0$ such that the estimate

$$\|a \tilde{\star}_h b\|_{\mathcal{C}^{1+n},\rho} \leq \sum_{P,Q,R,S \in \mathbb{N}^{1+n}_0} |a_{P,Q}| \|b_{R,S}\| d_{P,Q,R,S} \|\tilde{\star}_h d_{R,S}\|_{\mathcal{C}^{1+n},\rho'} \leq C \|a\|_{\mathcal{C}^{1+n},\rho'} \|b\|_{\mathcal{C}^{1+n},\rho'}$$

(3.2)

holds for all $h \in K$ and all $a = \sum_{P,Q \in \mathbb{N}^{1+n}_0} a_{P,Q} d_{P,Q}, b = \sum_{R,S \subseteq \mathbb{N}^{1+n}_0} b_{R,S} d_{R,S} \in \mathcal{P}(\mathbb{C}^{1+n})$.

It is not very hard to show directly that the estimate in this lemma holds. However, we will later also need another technical result about the growth of $\tilde{\star}_h$-powers from [29]:

**Lemma 3.3** Let $h \in \mathbb{R}$ as well as a linear functional $\omega: \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \rightarrow \mathbb{C}$ be given, such that $\omega$ is continuous with respect to the locally convex topology on $\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$ defined by the norms $\|\|_{\mathcal{C}^{1+n},\rho}$ for all $\rho > 0$ and such that $\omega$ is positive with respect $\tilde{\star}_h$, i.e. such that $\omega(a \tilde{\star}_h b) \geq 0$ holds for all $a \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$. Then for all $k \in \mathbb{N}_0$ and all $a \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1),k}$ there exist $C,D > 0$ with the property that

$$\omega\left( (a^*)^{\tilde{\star}_h m} \tilde{\star}_h (a^*)^{\tilde{\star}_h m} \right)^{\frac{1}{2}} \leq CD^{m}(km)!$$

(3.3)

holds for all $m \in \mathbb{N}_0$, where $a^{\tilde{\star}_h m}$ denotes the $m$-th power of $a$ with respect to the product $\tilde{\star}_h$.

**Proof:** Let such $h,\omega, k$ and $a$ be given, then it follows from the previous Lemma 3.2 and the continuity of $\omega$ that there exists a $C',\rho > 0$ with the property that $\omega(b^* \tilde{\star}_h b)^{1/2} \leq C' \|b\|_{\mathcal{C}^{1+n},\rho}$ holds for all $b \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$, and especially

$$\omega\left( (a^*)^{\tilde{\star}_h m} \tilde{\star}_h (a^*)^{\tilde{\star}_h m} \right)^{\frac{1}{2}} \leq C' \|a^{\tilde{\star}_h m}\|_{\mathcal{C}^{1+n},\rho}$$

for all $m \in \mathbb{N}_0$. Then [29] Lemma 3.34] shows that there exist $C'',D' > 0$ such that $\|a^{\tilde{\star}_h m}\|_{\mathcal{C}^{1+n},\rho} \leq C'' \sqrt{(2km)!}D^{2m}$ holds for all $m \in \mathbb{N}_0$. As $\sqrt{(2km)!} \leq 2^{km}(km)!$, this proves the claim with $C = C'C''$ and $D = 2^{k}D'$.

Note that we explicitly have to require $\omega$ to be continuous in this statement: the polynomial functions in $\mathcal{P}(\mathbb{C}^{1+n})$ are not yet complete. Hence the usual argument that positive functionals on Fréchet $^*$-algebras are automatically continuous [28 Thm. 3.6.1], does not apply here.

Using the explicit basis for $\mathcal{P}(D_n)$ we define norms in the same spirit as before. Note that with our normalization conventions for the basis functions $f_p P_Q$ the following weighted $\ell^1$-like norms yield a slightly coarser topology than the original one in [4]:
3.1 The Topology

Definition 3.4 (Norms on $\mathcal{P}(D_n)$) For all $\rho > 0$ we define the norm

$$\left\| \sum_{P,Q \in \mathbb{N}_0^n} a_{P,Q} f_{P,Q} \right\|_{D_n,\rho} := \sum_{P,Q \in \mathbb{N}_0^n} |a_{P,Q}| \rho^{P+Q}$$

(3.4)

on $\mathcal{P}(D_n)$, where $a_{P,Q} \in \mathbb{C}$ for all $P,Q \in \mathbb{N}_0^n$.

These norms turn out to be convenient in two essential ways. First, the practitioners will perhaps anticipate (and we will make this precise later by showing that the completion of $\mathcal{P}(D_n)$ w.r.t. these norms is isomorphic to the Fréchet space of all entire functions on $\mathbb{C}^n \times \mathbb{C}^n$) that the induced topology is exactly the topology of locally uniform convergence on $D_n$. Second, we shall now identify the resulting topology as the quotient topology with respect to the reduction map $\Psi_h$! We need the following well-known estimate for the Pochhammer symbols:

Lemma 3.5 For every compact subset $K \subseteq \mathbb{C}\setminus(-\mathbb{N}_0)$ there exist two constants $\alpha, \omega > 0$ such that

$$\alpha^m m! \leq |(z)_m| \leq \omega^m m!$$

(3.5)

holds for all $z \in K$ and all $m \in \mathbb{N}_0$.

With the next lemma we are able to relate the two locally convex topologies before and after the (quantum) reduction procedure.

Lemma 3.6 Let $K \subseteq H$ be a compact subset and let $\rho > 0$. Then there exists a $\rho' > 0$ such that

$$\|\Psi_h(a)\|_{D_n,\rho} \leq \|a\|_{C^{1+n},\rho'}$$

(3.6)

holds for all $h \in K$ and all $a \in \mathcal{P}(\mathbb{C}^{1+n})$. Conversely, there exists a $\rho'' > 0$ such that

$$\|\Phi_h(a)\|_{C^{1+n},\rho} \leq \|a\|_{D_n,\rho''}$$

(3.7)

holds for all $h \in K$ and all $a \in \mathcal{P}(D_n)$, where $\Phi_h : \mathcal{P}(D_n) \rightarrow \mathcal{P}(\mathbb{C}^{1+n})$ is the right inverse of $\Psi_h$ that is defined as the linear extension of

$$\Phi_h(f_{P,Q}) := \left( (2\rho)^{\max\{|P|,|Q|\}} \left( \frac{1}{2\rho} \right)^{\max\{|P|,|Q|\}} \right)^{-1} d_{P,Q}$$

(3.8)

with $\hat{P} := (\max\{|Q|-|P|,0\}, P_1, \ldots, P_n) \in \mathbb{N}_0^{1+n}$ and $\hat{Q} := (\max\{|P|-|Q|,0\}, Q_1, \ldots, Q_n) \in \mathbb{N}_0^{1+n}$.

PROOF: Let $K \subseteq H$ and, without loss of generality, $\rho \geq 1$ be given. Then the previous Lemma 3.5 shows that there exist $\alpha, \omega > 0$ such that $\alpha^m m! \leq |(1/(2\rho))_m| \leq \omega^m m!$ holds for all $m \in \mathbb{N}_0$ and all $h \in K$. Define $r_{\text{max}} := \max_{h \in K} |2h|$ and $r_{\text{min}} := \min_{h \in K} |2h| > 0$. For all $h \in K$ and $a = \sum_{P,Q \in \mathbb{N}_0^{1+n}, |P|=|Q|} a_{P,Q} d_{P,Q} \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$ we get the following estimate with the help of identity (2.30) and the prime-notation for omission of the 0-component in tuples used there:

$$\|\Psi_h(a)\|_{D_n,\rho}$$

$$\leq \left| \sum_{P,Q \in \mathbb{N}_0^{1+n}, |P|=|Q|} a_{P,Q} |2h|^{|P|} \left( \frac{1}{2|P|} \right) f_{P,Q} \right|_{D_n,\rho}$$

$$\leq \sum_{P,Q \in \mathbb{N}_0^{1+n}, |P|=|Q|} |a_{P,Q}| |2h|^{|P|} \left( \frac{1}{2|P|} \right) \sum_{T \in \mathbb{N}_0^{|T| \leq \min\{P_0,Q_0\}}} (\min\{P_0, Q_0\})^{T} (T!^{P'+Q'+2T})$$

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\[ \sum_{P,Q \in \mathbb{N}^{1+n}_0 : |P| = |Q|} |a_{P,Q}| \left( \rho \sqrt{\omega_{\max}^{P+Q}} \right)^{|P+Q|} |P|! \sum_{T \in \mathbb{N}^n_0 : |T| \leq \min \{P_0, Q_0\}} \left( \min \{P_0, Q_0\} / |T| \right)^{|T|!} / T! \]

\[ = \sum_{P,Q \in \mathbb{N}^{1+n}_0 : |P| = |Q|} |a_{P,Q}| \left( \rho \sqrt{\omega_{\max}^{P+Q}} \right)^{|P+Q|} (1 + n)^{\min \{P_0, Q_0\}} |P|! \]

\[ \leq \sum_{P,Q \in \mathbb{N}^{1+n}_0 : |P| = |Q|} |a_{P,Q}| \left( \rho \sqrt{\omega_{\max}^{P+Q}} \sqrt{|P + Q|} \right) \]

\[ = \|a\|_{C^{1+n}, \rho \sqrt{\omega_{\max}^{P+Q}}}. \]

This shows the first estimate with \( \rho' = \rho \sqrt{\omega_{\max}^{P+Q}} \). Conversely, for all \( h \in K \) and all \( b = \sum_{P,Q \in \mathbb{N}^n_0} b_{P,Q} \), \( P,Q \in \mathcal{P}(D_n) \) we get

\[ \|\Psi_h(b)\|_{C^{1+n}, \rho} \]

\[ = \left\| \sum_{P,Q \in \mathbb{N}^n_0} b_{P,Q} \left( (2h)^{\max \{|P|, |Q|\}} \left( \frac{1}{2h} \right)^{\max \{|P|, |Q|\}} \right)^{-1} d_{P,Q} \right\|_{C^{1+n}, \rho} \]

\[ \leq \sum_{P,Q \in \mathbb{N}^n_0} b_{P,Q} \left( (2h)^{\max \{|P|, |Q|\}} \left( \max \{|P|, |Q|\} \right)! \right)^{-1} \rho^{2 \max \{|P|, |Q|\}} \left( 2 \max \{|P|, |Q|\} \right) \]

\[ = \sum_{P,Q \in \mathbb{N}^n_0} b_{P,Q} \left( \frac{\rho^2}{r_{\min}^\alpha} \right)^{\max \{|P|, |Q|\}} \left( 2 \max \{|P|, |Q|\} \right)^{\frac{1}{2}} \]

\[ \leq \sum_{P,Q \in \mathbb{N}^n_0} b_{P,Q} \left( \frac{\rho^2}{r_{\min}^\alpha} \right)^{\max \{|P|, |Q|\}} \]

\[ \leq \|b\|_{D_n, \rho''}. \]

With \( \rho'' = \max \{2/\rho, 1\} \) as \( \max \{|P|, |Q|\} \leq |P| + |Q| \).

The previous Lemma [5.6] shows that for all \( h \in H \) the norms \( \| \cdot \|_{D_n, \rho} \) with \( \rho > 0 \) induce the quotient topology of \( \mathcal{P}(C^{1+n}) / \ker \Psi_h \) with the locally convex topology of the norms \( \| \cdot \|_{C^{1+n}, \rho} \) with \( \rho > 0 \). Together with the continuity of \( \hat{\Psi}_h \) from Lemma [5.2] this yields:

**Theorem 3.7 (Continuity of the star product)** For every \( h \in H \) the product \( \star_h \) on \( \mathcal{P}(D_n) \) is continuous with respect to the locally convex topology defined by the norms \( \| \cdot \|_{D_n, \rho} \) for all \( \rho > 0 \).

### 3.2 Characterization of the Completion

Having constructed a suitable locally convex topology on \( \mathcal{P}(D_n) \), the next step is to characterize the topology as well as the completion of the space \( \mathcal{P}(D_n) \) under this topology. Understanding various charts on \( \hat{D}_n \) will be especially helpful. Recall that we have already defined the standard chart \( \hat{\phi}^{\text{std}} : \hat{D}_n^{\text{std}} \to C^{\text{std}} \) in \( (2.14) \) such that

\[ \hat{\phi}^{\text{std}} \circ \hat{\pi} := \left( \frac{x^1}{x^0}, \ldots, \frac{x^n}{x^0}, \frac{y^1}{y^0}, \ldots, \frac{y^n}{y^0} \right) \]

\[ \text{where } \hat{D}_n^{\text{std}} = \{ [p, q] \in \hat{D}_n \mid x^0(p, q) \neq 0 \text{ and } y^0(p, q) \neq 0 \} \text{ and } C^{\text{std}} := \{ (p, q) \in \mathbb{C}^n \times \mathbb{C}^n \mid p \cdot q \neq 1 \}. \]

We will also need the following two charts:
Definition 3.8 (P- and Q-chart on \( D_n \)) Let \( \hat{D}_n^P := \{ [p,q] \in D_n \mid y^0(p,q) \neq 0 \} \) and \( \hat{D}_n^Q := \{ [p,q] \in \hat{D}_n \mid x^0(p,q) \neq 0 \} \) and define the P-chart \( \hat{\phi}^P : \hat{D}_n^P \to \mathbb{C}^n \times \mathbb{C}^n \) as well as the Q-chart \( \hat{\phi}^Q : \hat{D}_n^Q \to \mathbb{C}^n \times \mathbb{C}^n \) by

\[
\hat{\phi}^P \circ \hat{p}_r := \left( x^1 y^0, \ldots, x^n y^0, \frac{y^1}{y^0}, \ldots, \frac{y^n}{y^0} \right) \bigg|_Z \quad (3.10)
\]

and

\[
\hat{\phi}^Q \circ \hat{p}_r := \left( \frac{x^1}{x^0}, \ldots, \frac{x^n}{x^0}, x^0 y^1, \ldots, x^0 y^n \right) \bigg|_Z , \quad (3.11)
\]

respectively.

Note that \( \hat{\phi}^{\text{std}}, \hat{\phi}^P \) and \( \hat{\phi}^Q \) are all well-defined biholomorphic maps. With respect to these charts, the monomials \( \hat{f}_{P,Q} \) with \( P,Q \in \mathbb{N}_0^{1+n}, |P| = |Q| \), are represented as

\[
\hat{f}_{P,Q} \circ (\hat{\phi}^{\text{std}})^{-1} = \frac{x^P y^Q}{(1 - x \cdot y)^{|P|}} \bigg|_{\mathbb{C}^{\text{std}}}, \quad (3.12)
\]

and

\[
\hat{f}_{P,Q} \circ (\hat{\phi}^P)^{-1} = (1 + x \cdot y)^{P_0} x^P y^Q, \quad (3.13)
\]

and

\[
\hat{f}_{P,Q} \circ (\hat{\phi}^Q)^{-1} = (1 + x \cdot y)^{Q_0} x^P y^Q, \quad (3.14)
\]

where \( P' = (P_1, \ldots, P_n) \in \mathbb{N}_0^n \) and \( Q' = (Q_1, \ldots, Q_n) \in \mathbb{N}_0^n \). In particular, this implies that \( \hat{f}_{r,P,Q} \circ (\hat{\phi}^P)^{-1} = x^P y^Q \) for all \( P,Q \in \mathbb{N}_0^n \) with \( |P| \geq |Q| \) and \( \hat{f}_{r,P,Q} \circ (\hat{\phi}^Q)^{-1} = x^P y^Q \) for all \( P,Q \in \mathbb{N}_0^n \) with \( |P| \leq |Q| \), which motivates the following definition:

Definition 3.9 (Coordinate functionals for fundamental monomials) For all \( P,Q \in \mathbb{N}_0^n \), define the linear functional \( \hat{f}_{r,P,Q} : \mathcal{A}(D_n) \to \mathbb{C} \) as the Cauchy integral

\[
\hat{f}_{r,P,Q}(a) := \frac{1}{(-4\pi)^n} \int_C \cdots \int_C \hat{a} \circ (\hat{\phi}^P)^{-1} \frac{d^n x \wedge d^n y}{x^{P+1} y^{Q+1}} \quad \text{if } |P| \geq |Q| \quad (3.15)
\]

and

\[
\hat{f}_{r,P,Q}(a) := \frac{1}{(-4\pi)^n} \int_C \cdots \int_C \hat{a} \circ (\hat{\phi}^Q)^{-1} \frac{d^n x \wedge d^n y}{x^{P+1} y^{Q+1}} \quad \text{if } |P| < |Q| \quad (3.16)
\]

for all \( a \in \mathcal{A}(D_n) \), where \( C \subset \mathbb{C} \) is a circle around 0 with arbitrary positive radius and where \( P + 1 := (P_1 + 1, \ldots, P_n + 1) \), analogous for \( Q \).

Using the explicit formulas (3.13) and (3.14) we immediately get:

**Proposition 3.10** For all \( P,Q,R,S \in \mathbb{N}_0^n \), the identity

\[
\hat{f}_{r,P,Q}(\hat{f}_{r,P,Q}) = \delta_{P,R} \delta_{Q,S} \quad (3.18)
\]

holds.

**Proposition 3.11** The two formulas for \( \hat{f}'_{r,P,Q} \) in the P- and Q-chart can be combined into one single formula in the standard-chart, namely

\[
\hat{f}'_{r,P,Q}(a) = \frac{1}{(-4\pi)^n} \int_C \cdots \int_C \hat{a} \circ (\hat{\phi}^{\text{std}})^{-1} \frac{d^n x \wedge d^n y}{x^{P+1} y^{Q+1}} \left(1 - x \cdot y\right)^{\max(|P|,|Q|)-1} \quad (3.19)
\]

for all \( a \in \mathcal{A}(D_n) \), where \( C \subset \mathbb{C} \) is a circle around 0 with radius in \( |0,1/\sqrt{11}| \) and where again \( P + 1 := (P_1 + 1, \ldots, P_n + 1) \), analogous for \( Q \).
3.2 Characterization of the Completion

**Proof:** The change of coordinates from the standard- to the \( P \)-chart is given by
\[
\Psi^P := \hat{\phi}^P \circ (\hat{\phi}^{\text{std}})^{-1} : C^{\text{std}} \to \{ (\xi, \eta) \in \mathbb{C}^n \times \mathbb{C}^n \mid \xi \cdot \eta = -1 \}
\]
\[
(\xi, \eta) \mapsto \Psi^P(\xi, \eta) = \left( \frac{\xi}{1 - \xi \cdot \eta} \right).
\]
Then
\[
t'_{r,P,Q}(a) := \frac{1}{(4\pi)^n} \int_{\Psi^P(C^2)} \frac{\hat{a} \circ (\hat{\phi}^P)^{-1}}{x^{P+1} y^{Q+1}} \, dx \wedge dy
\]
\[
= \frac{1}{(4\pi)^n} \int_{(C')^2} \frac{\hat{a} \circ (\hat{\phi}^{\text{std}})^{-1} \circ \Psi^P}{(x^{P+1} y^{Q+1}) \circ \Psi^P} \, d^n(x \circ \Psi^P) \wedge d^n(y \circ \Psi^P)
\]
which yields \[3.19\] if \(|P| \geq |Q|\). Note that the calculation of \( d^n(x \circ \Psi^P)|_{\xi,\eta} \) is easy for \( \eta = (1,0,\ldots,0)^T \in \mathbb{C}^n \), which is already sufficient by symmetry. If \(|P| < |Q|\), the argument is analogous using the \( Q \)-chart. \( \square \)

Recall that \( \mathcal{A}(D_n) \) is endowed with a Fréchet topology given by the seminorms \( \| \cdot \|_{D_n,K} \) defined for all compact \( K \subseteq \hat{D}_n \) in Definition 2.3. We would of course like to understand the relation between this topology and the topology defined by the norms \( \| \cdot \|_{D_n,\rho} \) for all \( \rho > 0 \).

**Proposition 3.12** For every \( P, Q \in \mathbb{N}_0^n \) the linear functional \( t'_{r,P,Q} : \mathcal{A}(D_n) \to \mathbb{C} \) is continuous. Moreover, for every \( \rho > 0 \) there exists a compact \( K \subseteq \hat{D}_n \) such that the estimate
\[
\sum_{P,Q \in \mathbb{N}_0^n} |t'_{r,P,Q}(a)| \rho^{P+Q} \leq 2^{2n+2} \|a\|_{D_n,K}
\]
holds for all \( a \in \mathcal{A}(D_n) \).

**Proof:** It is sufficient to show that the estimate \[3.20\] holds, which is much stronger than mere continuity of \( t'_{r,P,Q} \). So let \( \rho > 0 \) be given and define \( K^P \) and \( K^Q \) as the images of the polydiscs with radius \( 2\rho \) in \( \mathbb{C}^n \times \mathbb{C}^n \) under the holomorphic maps \((\hat{\phi}^P)^{-1}\) and \((\hat{\phi}^Q)^{-1}\), respectively, and \( K := K^P \cup K^Q \subseteq D_n \). Then \( K \) is compact and from the usual estimate for the Cauchy integral over the boundary of a polydisc with radius \( 2\rho \) it follows for all \( a \in \mathcal{A}(D_n) \) that
\[
\sum_{P,Q \in \mathbb{N}_0^n} |t'_{r,P,Q}(a)| \rho^{P+Q} \leq \sum_{P,Q \in \mathbb{N}_0^n} \|a\|_{D_n,K} \rho^{P+Q} |(2\rho)|^{P+Q} = 2^{2n+2} \|a\|_{D_n,K}.
\] \( \square \)

**Lemma 3.13** For all compact \( K \subseteq \hat{D}_n \) there exists a \( \rho > 0 \) such that the estimate \( \|t'_{r,P,Q}\|_{D_n,K} \leq \rho^{P+Q} \) holds for all \( P, Q \in \mathbb{N}_0^n \).

**Proof:** Given such a \( K \subseteq \hat{D}_n \), then one possible choice for \( \rho \) is the maximum of \( \|f_{E^\mu,E^\nu}\|_{D_n,K}^2 \) over all \( \mu, \nu \in \{1,\ldots,n\} \). Submultiplicativity of \( \| \cdot \|_{D_n,K} \) with respect to the pointwise product yields
\[
\|f_{P,Q}\|_{D_n,K} \leq \sqrt{\rho^{P+Q}} \quad \text{for all} \ P, Q \in \mathbb{N}_0^n
\]
and thus \( \|t'_{r,P,Q}\|_{D_n,K} \leq \sqrt{\rho^{P+Q}} = \rho^{P+Q} \) for all \( P, Q \in \mathbb{N}_0^n \). \( \square \)

**Proposition 3.14** On \( \mathcal{P}(D_n) \) the locally convex topology defined by the seminorms \( \| \cdot \|_{D_n,\rho} \) for all \( \rho > 0 \) coincides with the subspace topology inherited from \( \mathcal{A}(D_n) \).
3.2 Characterization of the Completion

Proof: Let a compact $K \subseteq \hat{D}_n$ be given, then by the previous Lemma 3.13 there exists a $\rho > 0$ such that $\|f_r, P, Q\|_{D_n, K} \leq \rho |P + Q|$ holds for all $P, Q \in \mathbb{N}_0^n$, so

$$\|a\|_{D_n, K} \leq \sum_{P, Q \in \mathbb{N}_0^n} |a_{P, Q}| \|f_r, P, Q\|_{D_n, K} \leq \|a\|_{D_n, \rho}$$

holds for all $a = \sum_{P, Q \in \mathbb{N}_0^n} a_{P, Q} f_r, P, Q \in \mathcal{P}(D_n)$ with complex coefficients $a_{P, Q}$. The converse estimate follows directly from Proposition 3.12 which shows that for every $\rho > 0$ there exists a compact $K \subseteq \hat{D}_n$ such that

$$\|a\|_{D_n, \rho} = \sum_{P, Q \in \mathbb{N}_0^n} |f_r, P, Q(a)| \rho |P + Q| \leq 2^{2n+2} \|a\|_{D_n, K}$$

holds for all $a \in \mathcal{P}(D_n)$. □

Lemma 3.15 If $a \in \mathcal{A}(D_n)$ fulfils $f_r, P, Q(a) = 0$ for all $P, Q \in \mathbb{N}_0^n$, then $a = 0$.

Proof: Given $a \in \mathcal{A}(D_n)$, then $\hat{a} \circ (\hat{\phi}^P)^{-1} \in \mathcal{O}(\mathbb{C}^n \times \mathbb{C}^n)$, so there exist unique complex coefficients $\hat{a}_{P, Q}$ such that $\hat{a} \circ (\hat{\phi}^P)^{-1} = \sum_{P, Q \in \mathbb{N}_0^n} \hat{a}_{P, Q} x^P y^Q$ (and the series converges absolutely and locally uniformly). It is sufficient to show that all these coefficients vanish, because the domain of the P-chart is dense in $\hat{D}_n$. From the definition of $f_r, P, Q$ it is immediately clear that $\hat{a}_{P, Q} = 0$ for all $P, Q \in \mathbb{N}_0^n$ with $|P| \geq |Q|$. Now assume that there is a non-vanishing coefficient $\hat{a}_{P, Q}$, then there is a minimal $N \in \mathbb{N}_0$ such that $\hat{a}_{P, Q} \neq 0$ for some $P, Q \in \mathbb{N}_0^n$ with $|P| < |Q|$ and $|P + Q| = N$, so

$$\hat{a} \circ (\hat{\phi}^P)^{-1} = \sum_{P, Q \in \mathbb{N}_0^n} \hat{a}_{P, Q} x^P y^Q.$$ 

Consider $\Psi := \hat{\phi}^P \circ (\hat{\phi}^Q)^{-1}|_{C^\Psi} : C^\Psi \to C^\Psi$ with $C^\Psi := \{ (\xi, \eta) \in \mathbb{C}^n \times \mathbb{C}^n \mid \xi \cdot \eta \neq 1 \}$, which is explicitly given by $\Psi(\xi, \eta) = (\xi(1 - \xi \cdot \eta), \frac{\eta}{1 - \xi \cdot \eta})$ and describes the change of coordinates between $P$- and $Q$-chart. Then $\hat{a} \circ (\hat{\phi}^P)^{-1} \circ \Psi = \hat{a} \circ (\hat{\phi}^Q)^{-1}|_{C^\Psi}$ can be represented as the absolutely and locally uniformly convergent series

$$\hat{a} \circ (\hat{\phi}^Q)^{-1}|_{C^\Psi} = \sum_{P, Q \in \mathbb{N}_0^n} \hat{a}_{P, Q} x^P y^Q\frac{1 - x \cdot y}{|P - |Q|}.$$ 

It follows that $\hat{a}_{P, Q} = f_r, P, Q(a)$ for all $P, Q \in \mathbb{N}_0^n$ with $|P| < |Q|$ and $|P + Q| = N$ by evaluating the Cauchy-integral for $f_r, P, Q(a)$ on sufficiently small circles in the $Q$-chart. So $\hat{a}_{P, Q} = 0$ and we have a contradiction. □

Theorem 3.16 (Completion of $\mathcal{P}(D_n)$) The Fréchet $^*$-algebra $\mathcal{A}(D_n)$ with the pointwise operations is the completion of the $^*$-algebra $\mathcal{P}(D_n)$ with the pointwise operations and the locally convex topology defined by the seminorms $\|\cdot\|_{P, \rho}$ for all $\rho > 0$. Moreover, the functions $f_r, P, Q$ with $P, Q \in \mathbb{N}_0^n$ form an absolute Schauder basis of $\mathcal{A}(D_n)$ and the coefficients of the expansion in this basis can be calculated explicitly by means of the integral formulas for $f_r, P, Q$ from Definition 3.2 or Proposition 3.17:

$$a = \sum_{P, Q \in \mathbb{N}_0^n} f_r, P, Q(a) f_r, P, Q = \sum_{P, Q \in \mathbb{N}_0^n} \frac{f_r, P, Q}{(-4\pi)^n} \oint_C \ldots \oint_C (1 - u \cdot v)^{\max\{|P|, |Q|\}-1} u^{P+1} v^{Q+1} d^n u \wedge d^n v,$$ 

(3.21)

for all $a \in \mathcal{A}(D_n)$, where $u^1, \ldots, u^n, v^1, \ldots, v^n$ are the coordinate functions of the standard chart (2.14).
The product can explicitly be calculated as the series
\[ D = A \] which converges absolutely and locally uniformly in \( H \). This makes the discussion of the limit
\[ \hbar \] converges absolutely in the topology inherited from \( P \). Element of the closure of \( A \), and due to the identity from Proposition 3.10, it follows from Lemma 3.15 that \( a = \tilde{a} \). So \( a \) is an element of the closure of \( D \) in \( A \). As the functions \( f_{r,P,Q} \) with \( P, Q \in \mathbb{N}_0^n \) are linearly independent, this also shows that they form an absolute Schauder basis of \( A \) and that the coefficients of \( a \) with respect to this basis are the \( f_{r,P,Q}(a) \).

Proof: Proposition 3.11 shows that the \( \| \cdot \|_{D_n,\rho} \)-_topology on \( A(D_n) \) coincides with the relative topology inherited from \( A(D_n) \). Moreover, given \( a \in A(D_n) \), then \( \tilde{a} := \sum_{P,Q \in \mathbb{N}_0^n} f_{r,P,Q}(a) f_{r,P,Q} \) converges absolutely in \( A(D_n) \) due to the estimates in Proposition 3.12 and Lemma 3.13. As \( f_{r,P,RS}(a) = f_{r,P,RS}(\tilde{a}) \) for all \( R, S \in \mathbb{N}_0^n \) due to the continuity of \( f_{r,P,RS} \) shown in Proposition 3.12 and due to the identity from Proposition 3.10 it follows from Lemma 3.15 that \( a = \tilde{a} \). So \( a \) is an element of the closure of \( D \) in \( A(D_n) \). As the functions \( f_{r,P,Q} \) with \( P, Q \in \mathbb{N}_0^n \) are linearly independent, this also shows that they form an absolute Schauder basis of \( A(D_n) \) and that the coefficients of \( a \) with respect to this basis are the \( f_{r,P,Q}(a) \).

Note that we have also shown that \( \mathcal{O}(\hat{D}_n) \) is isomorphic to \( \mathcal{O}(\mathbb{C}^n \times \mathbb{C}^n) \) as a Fréchet space via the isomorphism
\[
\mathcal{O}(\hat{D}_n) \ni \sum_{P,Q \in \mathbb{N}_0^n} a_{P,Q} \hat{f}_{r,P,Q} \mapsto \sum_{P,Q \in \mathbb{N}_0^n} a_{P,Q} x^P y^Q \in \mathcal{O}(\mathbb{C}^n \times \mathbb{C}^n).
\]

However, this is not an isomorphism of Fréchet algebras due to the more complicated formula (2.31) for the (commutative) product on \( A(D_n) \).

**Theorem 3.17 (The completed star product)** For all \( h \in H \) the product \( \ast_h \) on \( A(D_n) \) extends continuously to the completion \( A(D_n) \), such that \( A(D_n) \) with the product \( \ast_h \) becomes a Fréchet algebra. The product can explicitly be calculated as the series
\[
a \ast_h b = \sum_{P,Q,RS \in \mathbb{N}_0^n} a_{P,Q} b_{R,RS} f_{r,P,Q} \ast_h f_{r,RS},
\]

which converges absolutely and locally uniformly in \( h \in H \) for all \( a, b \in A(D_n) \) with coefficients \( a_{P,Q} := f_{r,P,Q}(a) \) and \( b_{R,RS} := f_{r,RS}(b) \). If \( h \) is also real, then this Fréchet algebra is even a Fréchet \( * \)-algebra with pointwise complex conjugation as \( * \)-involution.

Proof: Continuity of \( \ast_h \) on \( A(D_n) \) has already been shown in Theorem 3.7 so \( \ast_h \) extends continuously to the completion of \( A(D_n) \), which is \( A(D_n) \) by the previous Theorem 3.16.

From the construction of \( \ast_h \) out of the star product \( \hat{\ast}_h \) on \( \mathbb{C}^{1+n} \) in Definition 2.12, the locally uniform estimate for \( \hat{\ast}_h \) in Lemma 3.2 and the locally uniform estimates for the reduction map \( \Psi_h \) in Lemma 3.6 it follows that the explicit formula for \( \ast_h \) converges absolutely and locally uniformly in \( h \in H \).

Finally, if \( h \in \mathbb{R} \), then pointwise complex conjugation is a \( * \)-involution for this product by construction, and this also extends to the completion by continuity.

\[ \square \]

**4 Properties of the Construction**

In this section we investigate now some first properties of the algebra we obtained by completion of the polynomial functions. In particular, we examine the dependence on \( h \) which is now much more delicate due to the presence of the poles on the negative axis: unlike for the Weyl algebra on \( \mathbb{C}^{1+n} \) as discussed in [29, 30] we do not have an entire deformation anymore. Even worse, the classical limit \( h = 0 \) is only a boundary point of the domain where the deformation is holomorphic. This makes the discussion of the limit \( h \to 0 \) more involved. In a next step we investigate the classically positive linear functionals on \( A(D_n) \) and show that the characters of \( A(D_n) \), i.e. its (maximal) spectrum, are given by \( D_{n,ext} \), finally showing that the completion is indeed still a space of functions. Then the deformation \( \ast_h \) for \( h > 0 \) is shown to have a faithful representation as unbounded operators on a Hilbert space and we discuss the question whether and how the infinitesimal action of \( su(1, n) \) exponentiates to the global symmetry under the group SU(1, n). Finally, we shortly discuss an additional symmetry that occurs in the special case \( n = 1 \).
4.1 Holomorphic Dependence on $\hbar$ and Classical Limit

By now we have seen that for $\hbar = 0$, the completion of the $^*$-algebra $\mathcal{A}(D_n)$ is the function algebra $\mathcal{A}(D_n)$, and that the deformed product $\ast_\hbar$ extends continuously to $\mathcal{A}(D_n)$ for all $\hbar \in H$. We would like to understand how $a \ast_\hbar b$ with $a, b \in \mathcal{A}(D_n)$ depends on $\hbar$, especially in the limit $\hbar \to 0$.

**Theorem 4.1 (Holomorphic dependence on $\hbar$)** For all $a, b \in \mathcal{A}(D_n)$ the function

$$H \ni \hbar \mapsto a \ast_\hbar b \in \mathcal{A}(D_n)$$

is holomorphic. The singularities at $\hbar = -1/(2m)$ with $m \in \mathbb{N}$ are at most poles of order 1.

**Proof:** If $a, b \in \mathcal{A}(D_n)$, then this all is clear because the explicit formula (2.38) for $\ast_\hbar$ shows that $a \ast_\hbar b$ is even a rational function of $\hbar$ with finitely many poles of at most order 1 only at the points $-1/(2m)$ with $m \in \mathbb{N}$. From the explicit formula for $\ast_\hbar$ in Theorem 3.17 and its absolute and locally uniform convergence it follows that this result extends to the completion. \(\square\)

Note that the above theorem does not give any information about the classical limit $\hbar \to 0$. In fact, this limit is (contrary to the case of the ordinary Wick star product on $\mathbb{C}^{1+n}$) non-trivial because the following example shows that there can indeed occur a pole at every $\hbar = -1/(2m)$ with $m \in \mathbb{N}$:

**Example 4.2** Let $j, k \in \mathbb{N}$ be given and write $E_1 := (0, 1, 0, \ldots, 0) \in \mathbb{N}_0^{1+n}$, then

$$f_j E_{1,j} \ast_\hbar f_k E_{1,k} E_1 = \sum_{t=0}^{\min\{j,k\}} \left( \frac{1}{m} \right)^{j+k-t} \left( \begin{array}{c} j \\end{array} \right) \left( \begin{array}{c} k \\end{array} \right) \left( \begin{array}{c} t \\end{array} \right) f_{(j+k-t)E_1,(j+k-t)E_1}.$$  

Moreover,

$$\frac{(z)^{j+k-t}}{(z)^{j} (z)^{k}} = \prod_{i=\max\{j,k\}}^{j+k-t-1} \frac{(z+i)}{i!} \frac{1}{\prod_{i=0}^{\min\{j,k\}-1} (z+i)}$$

has first order poles at all $z = -m$ with $m \in \{0, \ldots, \min\{j,k\} - 1\}$ and residue

$$( -1)^m \frac{(j+k-t-m-1)!}{m!(j-m-1)!(k-m-1)!},$$

whose sign only depends on $m$, but not on $j, k$ or $t$. This implies that if $a, b \in \mathcal{A}(D_n)$ are of the form

$$a = \sum_{j=0}^{\infty} a_j f_{j} E_{1,j} E_1 \quad \text{and} \quad b = \sum_{k=0}^{\infty} b_k f_{k} E_{1,k} E_1,$$

with positive coefficients $a_j, b_k \in [0, \infty[$, e.g. $a_j = b_j = 1/j!$, then

$$a \ast_\hbar b = \sum_{j,k=0}^{\infty} a_j b_k (f_{j} E_{1,j} E_1 \ast_\hbar f_{k} E_{1,k} E_1)$$

has simple poles at each of the points $\hbar = -1/(2m)$, $m \in \mathbb{N}$.

**Lemma 4.3** For all $p, s \in \mathbb{N}_0$ and $x \in [0, 1]$, the estimate

$$1 \leq \frac{\prod_{i=0}^{p+s-1} (1 + xi)}{(\prod_{j=0}^{p-1} (1 + xj))(\prod_{k=0}^{s-1} (1 + xk))} \leq 1 + x2^{p+s}$$

holds.
4.1 Holomorphic Dependence on $\hbar$ and Classical Limit

Proof: Without loss of generality we can assume that $p \geq s$. Note that

$$\prod_{t=0}^{p+s-1}(1+xi) \over (\prod_{j=0}^{s-1}(1+xj))(\prod_{k=0}^{s-1}(1+xk)) = \prod_{k=0}^{s-1} {1+x(p+k) \over 1+xk}$$

holds. So the first estimate $1 \leq \ldots$ is trivial, for the second one we will show by induction over $s$ that

$$\prod_{k=0}^{s-1} {1+x(p+k) \over 1+xk} \leq 1+x({p+s} \over s) - x$$

holds. If $s = 0$ or $s = 1$, then this is certainly true, and if it holds for one $s \in \mathbb{N}$ with $s < p$, then also for $s + 1$, because then

$$\prod_{k=0}^{s} {1+x(p+k) \over 1+xk} \leq \left(1+x \left({p+s \over s} - x \right) \right) {1+x(p+s) \over 1+xs}$$

$$= 1 + x \left({p+s \over s} \right) {1+xp+xs \over 1+xs} - x + (1-x)xp$$

$$= 1 + x \left({p+s \over s} \right) {1+p+s \over 1+s} - x \left({p+s \over s} \right) \frac{(1-x)p}{1+xs(1+s)} - x + \frac{(1-x)xp}{1+xs}$$

$$\leq 1 + x \left({p+s+1 \over s+1} \right) - x.$$  \[\square\]

Lemma 4.4 For all $t, k_0 \in \mathbb{N}_0$ and all $x \in [0, 1]$, the estimate

$$x^t! \prod_{k=k_0}^{t-1} (1+sk) \leq x^m 2^m m!$$

holds for all $m \in \{0, \ldots, t\}$.

Proof: As $\frac{1}{1+xk} \leq \frac{1}{1+x(k-k_0)}$ it is sufficient to prove the estimate for the special case $k_0 = 0$. If $t = m = 0$ then this is certainly true, and otherwise $t \geq 1$ and we have

$$x^t! \prod_{k=0}^{t-1} (1+sk) = x^m \prod_{k=0}^{m-1} \frac{1}{1+xk} \left(\prod_{k=m}^{t-1} \frac{sk}{1+xk}\right) t(m-1)! \leq x^m t(m-1)! \leq x^m 2^m m!.$$  \[\square\]

Theorem 4.5 (Classical limit) For all $a, b \in \mathfrak{A}(D_n)$ the functions

$$]0, \infty[ \ni \hbar \mapsto a \star_{\hbar} b \in \mathfrak{A}(D_n) \quad \text{and} \quad ]0, \infty[ \ni \hbar \mapsto \frac{i}{\hbar} [a, b]_{\star_{\hbar}} \in \mathfrak{A}(D_n) \quad (4.3)$$

are continuous and can be extended continuously to $]0, \infty[$ by

$$\lim_{\hbar \to 0^+} a \star_{\hbar} b = ab \quad \text{and} \quad \lim_{\hbar \to 0^+} \frac{i}{\hbar} [a, b]_{\star_{\hbar}} = \{a, b\}. \quad (4.4)$$

Proof: The continuity of these functions on $]0, \infty[$ is a direct consequence of the holomorphic dependence of $\star_{\hbar}$ on $\hbar$ from Theorem 4.1. For the limit $\hbar \to 0^+$ we first consider only products of $f_{P,Q}$ and $f_{R,S}$ with $P, Q, R, S \in \mathbb{N}_0^{1+n}$ as well as $|P| = |Q|$ and $|R| = |S|$. It will be helpful to use both the fundamental system of continuous seminorms $\|\cdot\|_{D_n,K}$ of $\mathfrak{A}(D_n)$ with $K$ running over all compact subsets of $\mathcal{D}_n$ and the fundamental system of continuous seminorms $\|\cdot\|_{D_n,\rho}$ for all $\rho > 0$, extended continuously from $\mathcal{P}(D_n)$ to $\mathfrak{A}(D_n)$. Recall that these two systems are equivalent by Theorem 3.16. Let $\hbar \in ]0, 1/2]$ and a compact $K \subseteq \mathcal{D}_n$ be given, then the estimate

$$\|f_{P,Q} \star_{\hbar} f_{R,S} - f_{P,Q} f_{R,S}\|_{D_n,K} \leq \ldots$$

...
holds by the formula (2.38) for $*_{\hbar}$. Using the results of the previous two lemmas, we get

$$
\left| \left| \left( \frac{2\pi}{\hbar} \right)^{P+S} \right| \left( \frac{2\pi}{\hbar} \right)^{P+S-T} - 1 \right| = \left| \left| \prod_{i=0}^{P+S-1} (1 + 2\hbar i) \right| \left( \prod_{j=0}^{P+S-T-1} (1 + 2\hbar j) \right) \left( \prod_{k=0}^{S-1} (1 + 2\hbar k) \right) \right| \left| \left| \prod_{\ell=|S-T|}^{S-1} (1 + 2\hbar \ell) \right| \right| - 1 \leq 2h^2 |P+S|
$$

by Lemma 4.3 and

$$
\left( \frac{2\pi}{\hbar} \right)^{P+S-T} \left( \frac{2\pi}{\hbar} \right)^{P+S-T} = \left( \prod_{i=0}^{P+S-T-1} (1 + 2\hbar i) \right) \left( \prod_{j=0}^{S-1} (1 + 2\hbar j) \right) \left( \prod_{k=0}^{S-1} (1 + 2\hbar k) \right) \left( \prod_{\ell=|S-T|}^{S-1} (1 + 2\hbar \ell) \right) \leq \left( \prod_{P}^{P+S-T} 2h^2 |T| \right) \leq 2h^2 |P+S|
$$

by Lemma 4.4 with $m = 1$ and by using that $\frac{1+2\hbar}{1-2\hbar k} \leq \frac{1+2\hbar}{1+2h}$ as long as $i \geq k$. Now define

$$
\rho := 2 + \max_{\mu,\nu \in \{1, \ldots, n\}} \|f_{E_\mu, E_\nu}\|_{D_{\mu, K}},
$$

then $\|f_{P+R-T, Q+S-T}\|_{D_{\mu, K}} \leq \rho |P+R+Q+S-2T|/2 = \rho |P+S-T|$ by submultiplicativity, and this yields

$$
\|f_{P, Q} *_{\hbar} f_{R, S} - f_{P, Q} f_{R, S}\|_{D_{\mu, K}} \leq 2h^2 |P+S| \min_{T=0}^{\min\{P, S\}} \left( \frac{P}{T} \right) \left( \frac{S}{T} \right) \rho |P+S-T|
$$

$$
\leq 2h (4\rho)^{P+S} \sum_{T=0}^{\min\{P, S\}} \rho^{-|T|} \leq 2h^2 (4\rho)^{|P+S|}
$$

as $\rho \geq 2$. From the definition of the fundamental monomials it now follows that especially

$$
\|f_{P, Q} *_{\hbar} f_{R, S} - f_{P, Q} f_{R, S}\|_{D_{\mu, K}} \leq 2h^2 (4\rho)^{\max\{P, |Q|\} + \min\{R, |S|\}}
$$

holds for all $P, Q, R, S \in \mathbb{N}^0_0$. Thus, for all $a = \sum_{P, Q, R \in \mathbb{N}^0_0} a_{P, Q} f_{R, Q}$ and $b = \sum_{R, S \in \mathbb{N}^0_0} b_{R, S} f_{R, S}$ with complex coefficients $a_{P, Q}$ and $b_{R, S}$ we get

$$
\|a *_{\hbar} b - ab\|_{D_{\mu, K}} \leq \sum_{P, Q, R, S \in \mathbb{N}^0_0} |a_{P, Q} b_{R, S}| \|f_{P, Q} *_{\hbar} f_{R, S} - f_{P, Q} f_{R, S}\|_{D_{\mu, K}}
$$

$$
\leq 2h (4\rho)^{|P+S|} \sum_{P, Q, R, S \in \mathbb{N}^0_0} |a_{P, Q} b_{R, S}| (4\rho)^{\max\{P, |Q|\} + \min\{R, |S|\}}
$$

$$
\leq 2h (4\rho)^{|P+Q+R+S|} \sum_{P, Q, R, S \in \mathbb{N}^0_0} |a_{P, Q} b_{R, S}| (4\rho)^{|P+Q+R+S|}
$$

which proves that $\lim_{h \to 0^+} a *_{\hbar} b = ab$. In order to prove the result for the limit of the commutator, we proceed analogously and start with commutators of $f_{P, Q}$ and $f_{R, S}$ with $P, Q, R, S \in \mathbb{N}^0_0$. Let $h \in [0, 1/2]$ and a compact $K \subseteq \mathbb{D}_n$ be given, then the estimate

$$
\left\| \frac{1}{h} \left[ f_{P, Q}, f_{R, S} \right] \ast_{\hbar} \left\{ f_{P, Q}, f_{R, S} \right\} \right\|_{D_{\mu, K}} \leq \ldots
$$
4.2 Gel’fand Transformation and Classically Positive Linear Functionals

\[
\frac{1}{h} \sum_{m=0}^{n} \left| \frac{\left( \frac{1}{2\pi} \right)^{|P+S|-1}}{P(\frac{1}{2\pi})} \right| - 2h \left| \frac{\left( \frac{1}{2\pi} \right)^{|P+S|}}{P(\frac{1}{2\pi})} \right| \left( \frac{1}{1 + 2h(|P + S| - 1)} - 1 \right) \\
+ \frac{1}{h} \sum_{T \in N_0^{1+n}} \frac{\left( \frac{1}{2\pi} \right)^{|P+S-T|T^T}}{P(\frac{1}{2\pi})} \left( \left( \frac{P}{T} \right) \left( \frac{S}{T} \right) + \left( \frac{Q}{T} \right) \left( \frac{R}{T} \right) \right) \rho^{P+S-T} \leq 16h (4\rho)^{|P+S|} \sum_{T \in N_0^{1+n}, \ [T] > 0 \text{ and } T \leq \min\{P,S\}} (4\rho)^{|P+S|}
\]

holds by the formula (2.35) for \( \ast_h \) and because

\[
\{ f_{P,Q}, f_{R,S} \} = \lim_{h \to 0^+} \frac{1}{h} \{ f_{P,Q}, f_{R,S} \}_{\ast_h} = 2i (P_m S_m - Q_m R_m) f_{P+R-E_m, Q+S-E_m}
\]

by construction of \( \ast_h \). For the first term we can use

\[
\left| \frac{\left( \frac{1}{2\pi} \right)^{|P+S|-1}}{P(\frac{1}{2\pi})} \right| - 2h \left| \frac{\left( \frac{1}{2\pi} \right)^{|P+S|}}{P(\frac{1}{2\pi})} \right| \left( \frac{1}{1 + 2h(|P + S| - 1)} - 1 \right) \\
\leq 2h \left| \frac{\left( \frac{1}{2\pi} \right)^{|P+S|}}{P(\frac{1}{2\pi})} \right| - 1 \left( \frac{1}{1 + 2h(|P + S| - 1)} + \frac{(2h)^2(|P + S| - 1)}{1 + 2h(|P + S| - 1)} \right) \\
\leq (2h)^2 \left| \frac{2^{P+S} + |P + S|}{P+} \right|
\]

by Lemma 4.3 as long as \(|P + S| \geq 1\), which is of course the only case of interest. For the second term, an analogous argument as before using Lemma 4.4 with \( m = 2 \) yields

\[
\left| \frac{\left( \frac{1}{2\pi} \right)^{|P+S-T|T^T}}{P(\frac{1}{2\pi})} \right| \leq 2 (2h)^2 2^{P+S}\]

and by putting all of this together we see that

\[
\left\| \frac{1}{h} \{ f_{P,Q}, f_{R,S} \}_{\ast_h} - \{ f_{P,Q}, f_{R,S} \} \right\|_{D_n,K} \leq 8h 2^{P+S} \sum_{T \in N_0^{1+n}, \ [T] > 0 \text{ and } T \leq \min\{P,S\}} (4\rho)^{|P+S-T|} \rho^{-|T|} \\
\leq 16h (4\rho)^{|P+S|} \sum_{T \in N_0^{1+n}, \ [T] > 0 \text{ and } T \leq \min\{P,S\}} (4\rho)^{|P+S|}
\]

hence \( \lim_{h \to 0^+} \frac{1}{h} [a, b]_{\ast_h} = \{ a, b \} \).

4.2 Gel’fand Transformation and Classically Positive Linear Functionals

From the construction of the commutative \(*\)-algebra \( \mathcal{A}(D_n) \) with the pointwise product it is clear that \( \mathcal{A}(D_n) \) can be interpreted as a \(*\)-algebra of functions on \( D_n \). Nevertheless, this is to some extend artificial. The most natural representation of \( \mathcal{A}(D_n) \) as a \(*\)-algebra of functions is on the space of its characters (the unital \(*\)-homomorphisms to \( \mathbb{C} \)) via Gel’fand transformation. In the context of topological \(*\)-algebras, it seems reasonable to focus on continuous characters. Note, however, that by a theorem of Xia, see [28, Thm. 3.6.1], every positive linear functional on a unital Fréchet-*algebra is actually continuous, so in our case this is not a restriction.
**Proposition 4.6** The map $M : \hat{D}_n \to \mathbb{C}^{(1+n)\times(1+n)}$ with components $M^{\mu\nu} := \hat{\iota}_{E_\mu,E_\nu}$ is a holomorphic embedding that realizes $\hat{D}_n$ as the submanifold

\[
\mathcal{S} := \{ A \in \mathbb{C}^{(1+n)\times(1+n)} \mid h_{\mu\nu} A^{\mu\nu} = -1 \text{ and } A^{\mu\nu} A^{\sigma\nu} = A^{\mu\sigma} A^{\nu\nu} \text{ for } \mu, \nu, \rho, \sigma \in \{0, \ldots, n\} \}.
\] (4.5)

**Proof:** First of all we note that $h_{\mu\nu} M^{\mu\nu} = -\hat{\iota}_{E_\mu,E_\nu} + \sum_{i=1}^n \hat{\iota}_{E_i,E_\nu} = -1$ by (2.39), where $E_i$ is the unit vector with 1 at the $i$-th position and 0 elsewhere. Note also that $M^{\mu\nu} \circ \hat{\pi} = (x^{\mu} y^{\nu}) \circ \hat{\iota}$ by construction of the $\hat{\iota}_{E_\mu,E_\nu}$, so

\[
(M^{\mu\nu} M^{\sigma\nu})([p,q]) = x^{\mu}(p,q) y^{\nu}(p,q) x^{\rho}(p,q) y^{\sigma}(p,q) = (M^{\mu\nu} M^{\rho\nu})([p,q])
\]

for all $[p,q] \in \hat{D}_n$. As these polynomials are holomorphic, $M$ is a holomorphic mapping to $\mathcal{S}$.

Given $A \in \mathcal{S}$, then $h_{\mu\nu} A^{\mu\nu} = -1$ implies that there exists a $\rho \in \{0, \ldots, n\}$ such that $A^{\mu\rho} \neq 0$. Define $p,q \in \mathbb{C}^{1+n}$ as $p^{\mu} := A^{\mu\rho}$ and $q^{\nu} := A^{\rho\nu} / A^{\rho\rho}$, then $(p,q) \in \hat{Z}$ and $M([p,q]) = A$. Let $\| \cdot \|$ be any norm on $\mathbb{C}^{(1+n)\times(1+n)}$ and consider $B \in \mathcal{S}$ such that $\| A - B \| \leq \delta$ for some $\delta \in [0,\infty]$. If $\delta$ is sufficiently small, then also $B^{\mu\rho} \neq 0$ and we can construct $r, s \in \mathbb{C}^{1+n}$ as $r^{\mu} := B^{\mu\rho}$ and $s^{\nu} := B^{\rho\nu} / B^{\rho\rho}$, and again $(r,s) \in \hat{Z}$ with $M([r,s]) = B$. Now $|p^{\mu} - r^{\mu}| = |A^{\mu\rho} - B^{\mu\rho}| \xrightarrow{\delta \to 0} 0$ and $|q^{\nu} - s^{\nu}| = |A^{\rho\nu} / A^{\rho\rho} - B^{\rho\nu} / B^{\rho\rho}| \xrightarrow{\delta \to 0} 0$ show that $M$ is injective and a homeomorphism onto its image $\mathcal{S}$.

Moreover, the $2n + 1$ differentials $d(x^{\mu} y^{\rho}), d(x^{\nu} y^{\rho})$ and $d(x^{\mu} y^{\nu})$ for $\mu, \nu \in \{0, \ldots, n\}\setminus\{\rho\}$ are linearly independent in the point $(p,q) \in \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$. Restricted to the submanifold $\hat{Z}$ with (complex) codimension 1, the $x^{\mu} y^{\nu}$ are the components of $M \circ \hat{\pi}$, which shows that the tangent map of $M \circ \hat{\pi}$, hence of $M$, has (at least) rank $2n = \text{dim}_\mathbb{C} \hat{D}_n$. \hfill \square

Note that this especially implies that the holomorphic functions on $\hat{D}_n$ separate points, because the holomorphic functions on $\mathbb{C}^{(1+n)\times(1+n)}$ do.

**Definition 4.7** For all $[p,q] \in \hat{D}_n$ we define the evaluation functional $\delta_{[p,q]} : \mathcal{A}(D_n) \to \mathbb{C}$, $a \mapsto \delta_{[p,q]}(a) := \hat{a}([p,q])$.

**Proposition 4.8** Given $[p,q] \in \hat{D}_n$, then $\delta_{[p,q]}$ is a continuous unital homomorphism from $\mathcal{A}(D_n)$ with the pointwise product to $\mathbb{C}$ and it is a continuous character of $\mathcal{A}(D_n)$ if and only if $[p,q] \in D_{n,\text{ext}}$.

**Proof:** It is immediately clear from its definition that $\delta_{[p,q]}$ is a continuous unital homomorphism. Moreover, $\delta_{[p,q]}(a^*) = \langle \hat{a} \circ \tau \rangle([p,q])$ and $\delta_{[p,q]}(a) = \hat{a}([p,q])$, so $\delta_{[p,q]}$ is a character if and only if $(\hat{a} \circ \tau)([p,q]) = \hat{a}([p,q])$ holds for all $a \in \mathcal{A}(D_n)$. As the holomorphic functions on $\hat{D}_n$ separate points by Proposition 4.6 this is equivalent to $\tau([p,q]) = [p,q]$, i.e. to $[p,q] \in D_{n,\text{ext}}$. \hfill \square

**Theorem 4.9 (Gel’fand transformation)** Let $\text{Spec}(\mathcal{A}(D_n))$ be the set of continuous unital homomorphisms from $\mathcal{A}(D_n)$ to $\mathbb{C}$ with the weak-*-topology. Then $\delta : \hat{D}_n \to \text{Spec}(\mathcal{A}(D_n)), [p,q] \mapsto \delta_{[p,q]}$ is a well-defined homeomorphism. Moreover, let $\text{Spec}^*(\mathcal{A}(D_n)) \subseteq \text{Spec}(\mathcal{A}(D_n))$ be the set of characters of $\mathcal{A}(D_n)$ again with the weak-*-topology, then $\delta$ restricts to a homeomorphism from $D_{n,\text{ext}}$ to $\text{Spec}^*(\mathcal{A}(D_n))$.

**Proof:** Proposition 4.6 already shows that $\delta$ maps to the continuous unital homomorphisms, and $\delta$ is injective because the holomorphic functions on $\hat{D}_n$ separate points due to Proposition 4.6.

Now let a continuous unital homomorphism $\omega : \mathcal{A}(D_n) \to \mathbb{C}$ be given. Construct the matrix $A^{\mu\nu} := \omega(\hat{I}_{E_\mu,E_\nu})$. Then $h_{\mu\nu} A^{\mu\nu} = \omega(\hat{I}_{E_\mu,E_\nu} - \sum_{i=1}^n \hat{I}_{E_i,E_\nu}) = -\omega(\hat{I}_{0,0}) = -1$ by (2.39) and $A^{\mu\nu} A^{\rho\nu} = \omega(\hat{I}_{E_\mu,E_\nu} \hat{I}_{E_\nu,E_\rho}) = A^{\mu\sigma} A^{\rho\rho}$, so $A$ is in the image of the holomorphic embedding $M$ from Proposition 4.6 and there exists a unique $[p,q] \in \hat{D}_n$ with $\omega(\hat{I}_{E_\mu,E_\nu}) = A^{\mu\nu} = M^{\mu\nu}([p,q]) = \cdots$
\[ \delta_{[\mu, \nu]} f_{E_\mu, E_\nu} \text{ for all } \mu, \nu \in \{0, \ldots, n\}. \] As these monomials generate \( \mathcal{P}(D_n) \) as a unital \(*\)-algebra, \( \delta_{[\mu, \nu]} \) and \( \omega \) coincide on \( \mathcal{P}(D_n) \), and as \( \mathcal{P}(D_n) \) is dense in \( \mathcal{A}(D_n) \) we can conclude that \( \delta_{[\mu, \nu]} = \omega \).

By now we have seen that \( \delta \) is a bijection, and it is even a homeomorphism, because the embedding \( M \) of \( D_n \) in \( \mathbb{C}^{(1+n) \times (1+n)} \) shows that \( D_n \) carries the weak topology of its homomorphic functions (because \( \mathbb{C}^{(1+n) \times (1+n)} \) does), which under \( \delta \) corresponds to the weak-*-topology.

The analogous statements about the space of characters of \( \mathcal{A}(D_n) \) are now an immediate consequence of the above and of Proposition \[ \text{[4.R]} \]

Note that this result is to some extent unfortunate, as it shows that the interpretation of \( \mathcal{A}(D_n) \) as a *-algebra of functions on \( D_n \) is not really natural. At the center of the problem lies the fact that the function \( \frac{1}{\omega(A)} = f_{E_0, E_0} \in \mathcal{P}(D_n) \) is not algebraically positive even though \( f_{E_0, E_0} = \Psi_0((d_{E_0, 0})^* d_{E_0, 0}) \).

As the algebra \( \mathcal{P}(D_n) \) arises from \( \mathcal{P}(\mathbb{C}^{1+n}) \) by a reduction procedure (in the classical case as well as in the quantum case), one might consider only those positive linear functionals on \( \mathcal{P}(D_n) \) to be “relevant” that come from a \( U(1) \)-invariant functional on \( \mathcal{P}(\mathbb{C}^{1+n}) \). This would especially eliminate all characters \( \phi \in \text{Spec}^* (\mathcal{A}(D_n)) \) for which \( \phi(f_{E_0, E_0}) < 0 \) and leave only the evaluation functionals at points in \( D_n \).

**Corollary 4.10** Let \( \phi: \mathcal{A}(D_n) \to \mathbb{C} \) be a continuous positive linear functional (with respect to the pointwise product), then there exists a compact \( K \subseteq D_{n,\text{ext}} \) and a Radon measure \( \mu \) on \( K \) such that

\[
\phi(a) = \int_K \hat{a} \, d\mu
\]

holds for all \( a \in \mathcal{A}(D_n) \).

**Proof:** It is sufficient to treat the case that \( \phi \) is normalized to \( \phi(1) = 1 \), as \( \phi(1) = 0 \) implies \( \phi = 0 \) by the Cauchy-Schwarz inequality, and to show that there exists a compact \( K \subseteq D_{n,\text{ext}} \) such that \( |\phi(a)| \leq \|a\|_{D_n,K} \) holds for all \( a \in \mathcal{A}(D_n) \), in which case \( \phi \) extends continuously to \( \mathcal{A}(K) \), the completion of \( \mathcal{A}(D_n) \) under \( \| \cdot \|_{D_n,K} \) by the Stone-Weierstrass theorem, and can be represented by integration over a Radon measure \( \mu \) on \( K \) by the Riesz-Markov theorem.

As \( \phi \) is continuous, there exists a compact \( K' \subseteq D_n \), stable under \( \tau \), and a constant \( C \in \mathbb{R} \) such that \( |\phi(a)| \leq C \|a\|_{D_n,K'} \) holds for all \( a \in \mathcal{A}(D_n) \). From the submultiplicativity of \( \| \cdot \|_{D_n,K'} \) and the Cauchy-Schwarz inequality it then follows that \( \|a\|_{D_n,\phi,\infty} := \sup_{b \in \mathcal{A}(D_n)} \phi(b^*ab) = 1 \sqrt{\phi(b^*a^*ab)} \) is a continuous seminorm on \( \mathcal{A}(D_n) \), because

\[
\sqrt{\phi(b^*a^*ab)} \leq \sqrt{\phi(b^*(a^*a)b)} \leq \cdots \leq a^n = \sqrt{\phi(b^*(a^*a)2^{n-1}b)} \leq 2^n \sqrt{C \|b^*\|_{D_n,K'} \|b\|_{D_n,K'} \sqrt{\|a^*\|_{D_n,K'} \|a\|_{D_n,K'}}}
\]

holds for all \( n \in \mathbb{N} \) and all \( b \in \mathcal{A}(D_n) \) with \( \phi(b^*b) = 1 \), hence \( \sqrt{\phi(b^*a^*ab)} \leq \|a\|_{D_n,K} \) and \( \| \cdot \|_{D_n,\phi,\infty} \leq \| \cdot \|_{D_n,K} \). One can also check that \( \| \cdot \|_{D_n,\phi,\infty} \) is a C*-seminorm (it is the operator norm in the usual GNS representation associated to \( \phi \)). By dividing out the zeros of \( \| \cdot \|_{D_n,\phi,\infty} \) and completing with respect to \( \| \cdot \|_{D_n,\phi,\infty} \), we construct a commutative C*-algebra \( \mathcal{B} \) and the continuous \( \iota: \mathcal{A}(D_n) \to \mathcal{B} \) as the composition of the projection on the quotient and the inclusion in the completion. Let \( \text{Spec}^* (\mathcal{B}) \) be the (compact) set of characters of \( \mathcal{B} \), then the C*-norm \( \| \cdot \|_{D_n,\phi,\infty} \) on \( \mathcal{B} \) is the uniform norm on the Gel’fand transformation of \( \mathcal{B} \), hence especially \( \|a\|_{D_n,\phi,\infty} = \sup_{\tau \in \text{Spec}^* (\mathcal{B})} |\hat{\psi}(\iota(a))| \) for all \( a \in \mathcal{A}(D_n) \). The pullback \( \iota^*: \text{Spec}^* (\mathcal{B}) \to \text{Spec}^* (\mathcal{A}(D_n)) \) is weak-*-continuous by construction of \( \iota \) and by the previous Theorem \[ \text{[4.9]} \] the compact \( \iota^*(\text{Spec}^* (\mathcal{B})) \subseteq \mathcal{A}(D_n) \).
4.3 Positive Linear Functionals and Representations of the Deformed Algebra

\[ \text{Spec}^\ast(D_n)) \text{ is the image of a compact } K \subseteq D_{n,\text{ext}} \text{ under } \delta, \text{ so } \]
\[ |\phi(a)| \leq \phi(a^*a) \leq \|a\|_{D_n,\phi,\infty} = \sup_{\psi \in \text{Spec}^\ast(\mathcal{A})} |\psi(i(a))| = \sup_{[p,q] \in K} |\delta_{[p,q]}a| = \sup_{[p,q] \in K} |a(p, q)| = \|a\|_{D_n,K} \]
for all \( a \in \mathcal{A}(D_n) \).

4.3 Positive Linear Functionals and Representations of the Deformed Algebra

From the point of view of physics, the most important problem after having constructed a \(^\ast\)-algebra of observables is, whether there exist many positive linear functionals and thus faithful representations. Again, we focus on continuous positive linear functionals as we are dealing with a locally convex \(*\)-algebras, which is no restriction as mentioned before, because every positive linear functional on a Fréchet-\(^\ast\)-algebra is continuous.

**Proposition 4.11** For every \([r] \in D_n\), the evaluation functional \( \delta_{[r]} := \delta_{\Delta_D([r])}: \mathcal{A}(D_n) \to \mathbb{C}, a \mapsto \delta_{[r]}(a) = a([r]) \) is continuous and positive with respect to every product \( \ast_h \) for all \( h \geq 0 \), i.e. \( \delta_{[r]}(a^* \ast_h a) \geq 0 \) holds for all \( a \in \mathcal{A}(D_n) \).

**Proof:** Like in Proposition 4.8, continuity of all evaluation functionals \( \delta_{[r]} \) with \([r] \in D_n\) is clear. It is thus also sufficient to prove positivity of \( \delta_{[r]} \) only on the dense unital \(*\)-subalgebra \( \mathcal{P}(D_n) \) of \( \mathcal{A}(D_n) \), which has already been done in [4, Lemma 5.21].

**Corollary 4.12** Let \( h \geq 0 \). Then every positive linear functional \( \mathcal{C}^\ast(D_n) \) restricts to a positive linear functional on \( \mathcal{A}(D_n) \) with respect to \( \ast_h \).

**Proof:** Indeed, every such positive linear functional is an integration with respect to a Radon measure on a compact subset \( K \subseteq D_n \). Note that now the support \( K \) is contained in \( D_n \) instead of \( D_{n,\text{ext}} \), see Corollary 4.11. Since all evaluation functionals at points of \( D_n \) are positive with respect to \( \ast_h \), this also holds for the convex combinations needed for general Radon measures.

Given a pre-Hilbert space \( \mathcal{H} \) with inner product \( (\cdot | \cdot)_{\mathcal{H}} \), then we write \( \mathcal{L}^\ast(\mathcal{H}) \) for the unital \(*\)-algebra of all adjointable endomorphisms of \( \mathcal{H} \), where being adjointable is to be understood in the purely algebraic sense that for an endomorphism \( a \in \mathcal{L}^\ast(\mathcal{H}) \) there exists a (necessarily unique) \( a^* \in \mathcal{L}^\ast(\mathcal{H}) \) such that \( \langle a^* \phi, \psi \rangle_{\mathcal{H}} = \langle a \phi, \psi \rangle_{\mathcal{H}} \) holds for all \( \phi, \psi \in \mathcal{H} \).

**Definition 4.13 (Continuous representation)** Let \( h \geq 0 \). A continuous representation of the Fréchet-\(*\)-algebra \( \mathcal{A}(D_n), \ast_h, \cdot^\ast \) is defined as a tuple \((\mathcal{H}, \pi)\) consisting of a pre-Hilbert space \( \mathcal{H} \) and a unital \(*\)-homomorphism \( \pi: \mathcal{A}(D_n) \to \mathcal{L}^\ast(\mathcal{H}) \) that is continuous with respect to the weak topology on \( \mathcal{L}^\ast(\mathcal{H}) \), i.e. the topology defined by the seminorms \( \mathcal{L}^\ast(\mathcal{H}) \ni a \mapsto |\langle a^* \phi, \psi \rangle_{\mathcal{H}}| \) for all \( \phi, \psi \in \mathcal{H} \).

The following is well-known: As the product \( \ast_h \) on \( \mathcal{A}(D_n) \) is continuous, one could equivalently demand that \( \pi \) be continuous with respect to the strong topology on \( \mathcal{L}^\ast(\mathcal{H}) \), i.e. the topology defined by the seminorms \( \mathcal{L}^\ast(\mathcal{H}) \ni a \mapsto \sqrt{\langle a^\dagger a \phi, \phi \rangle_{\mathcal{H}}} \) for all \( \phi \in \mathcal{H} \), because \( \mathcal{A}(D_n) \ni a \mapsto \sqrt{\langle \pi(a) \phi, \pi(a) \phi \rangle_{\mathcal{H}}} = \sqrt{\langle \phi, \pi(a^\dagger a) \phi \rangle_{\mathcal{H}}} \in \mathbb{R} \) is a continuous seminorm on \( \mathcal{A}(D_n) \) for all \( \phi \in \mathcal{H} \) if \( \pi \) is weakly continuous. Moreover, there exists a faithful continuous representation of \( \mathcal{A}(D_n), \ast_h, \cdot^\ast \) if and only if the continuous positive linear functionals on \( \mathcal{A}(D_n), \ast_h, \cdot^\ast \) separate points, i.e. if and only if \( \rho(a) = 0 \) for one \( a \in \mathcal{A}(D_n) \) and all continuous linear functionals \( \rho: \mathcal{A}(D_n) \to \mathbb{C} \) that are positive with respect to \( \ast_h \) implies that \( a = 0 \). This is because, on the one hand, given such a continuous representation, then every vector \( \phi \in \mathcal{H} \) yields a weakly continuous positive linear functional \( \mathcal{L}^\ast(\mathcal{H}) \ni a \mapsto \langle \phi, a^\dagger a \phi \rangle_{\mathcal{H}} \in \mathbb{C} \), that can be pulled back to a continuous positive linear functional on \( \mathcal{A}(D_n) \) by \( \pi \), and on the other, the GNS-construction allows to construct continuous representations out of continuous positive linear functionals. In this case, a \(*\)-algebra is also called \(*\)-semisimple [28, Def. 6.4.1]. In our case, this allows for the following conclusion:
Theorem 4.14 (Existence of faithful continuous representations) Let $\hbar \geq 0$ be given, then there exists a faithful continuous representation of the Fréchet *-algebra $(\mathcal{A}(D_n), \star_h, \cdot^*)$.

Proof: The evaluation functionals $\delta[r]$ for all $[r] \in D_n$ from the previous Proposition 4.11 are continuous and positive and clearly separate points. 

Having established the existence of interesting representations by (unbounded) operators, the question arises which algebra elements are actually essentially self-adjoint in representations. Therefore, recall Definition 2.8 of the filtration of $\mathcal{P}(D_n)$ by degree and Lemma 3.3 for the estimate on the growth of $\star_h$-powers. Note that the formula (2.38) also shows immediately that the deformed products $\star_h$ are also filtered with respect to the above filtration, i.e. $a \star_h b \in \mathcal{P}(D_n)^{(k+\ell)}$ holds for all $a \in \mathcal{P}(D_n)^{(k)}, b \in \mathcal{P}(D_n)^{(\ell)}$ and all $h \in H$.

Theorem 4.15 (Essential self-adjointness of observables) Fix $\hbar \geq 0$ and let $(\mathcal{H}, \pi)$ be a continuous *-representation of $(\mathcal{A}(D_n), \star_h, \cdot^*)$. Then $\pi(a)$ is essentially self-adjoint for every Hermitian $a \in \mathcal{P}(D_n)^{(1)}$ and for every Hermitian $a \in \mathcal{P}(D_n)^{(2)}$ that is semi-bounded, i.e. for which the set of all $\phi(a)/\phi(1)$ with $\phi$ running over all non-zero continuous positive linear functionals is bounded from above or below.

Proof: If $a$ is Hermitian then $\pi(a)$ is symmetric. Moreover, every vector $\phi \in \mathcal{H}$ yields a continuous positive linear functional $\mathcal{A}(D_n) \ni a \mapsto \langle \phi | \pi(a) \phi \rangle_{\mathcal{H}} \in \mathbb{C}$ on $\mathcal{A}(D_n)$, which can be pulled back to a continuous positive linear functional on $\mathcal{P}(\mathbb{C}^{1+n})_{\mathcal{U}(1)} \subseteq \mathcal{P}(\mathbb{C}^{1+n})$ with the Wick star product $\star_h$. The estimate from Lemma 3.3 then shows that every such $\phi \in \mathcal{H}$ is an analytic vector of every $\pi(a)$ if $a \in \mathcal{P}(D_n)^{(1)}$, and a semi-analytic vector of every $\pi(a)$ if $a \in \mathcal{P}(D_n)^{(2)}$. In both cases it follows from Nelson’s criterion for self-adjointness that $\pi(a)$ is essentially self-adjoint, see e.g. [20] for a direct proof.

4.4 Exponentiation of the $su(1,n)$-Action

By reduction to $D_n$, the $U(1,n)$-symmetry of $\mathbb{C}^{1+n}$ is reduced to a SU(1,1)-symmetry. Recall that $u(1,n) \ni u \mapsto \mathcal{J}(u) := \frac{1}{\hbar}h_{\mu\nu} u^\mu_\rho E_\rho E_\nu \in \mathcal{P}(\mathbb{C}^{1+n}) \subseteq \mathcal{C}^{\infty}(\mathbb{C}^{1+n})$ is a (classical) equivariant moment map for this action. As $\mathcal{J}(u)$ is linear in the $z$ and $\bar{z}$-coordinates, only terms up to first order in $\hbar$ will contribute to the Wick star product with $\mathcal{J}(u)$, so $f \star u = \{ f, \mathcal{J}(u) \} = \{ f, \frac{i}{\hbar} \mathcal{J}(u) \}$ and $\frac{i}{\hbar} \mathcal{J}([u,v]) = \frac{i}{\hbar} \{ \mathcal{J}(u), \mathcal{J}(v) \} = \left[ \frac{i}{\hbar} \mathcal{J}(u), \frac{i}{\hbar} \mathcal{J}(v) \right]$ hold for all $u, v \in u(1,n)$ and $f \in \mathcal{P}(\mathbb{C}^{1+n})$, i.e. $\frac{i}{\hbar} \mathcal{J}$ is an equivariant quantum moment map. Reduction to $D_n$ then yields the following well-known result, see e.g. [2], Lemma 5] for the case of reduction to $\mathbb{C}P^n$:

Proposition 4.16 The map $\mathcal{J}_{D_n} : su(1,n) \rightarrow \mathcal{P}(D_n), u \mapsto \mathcal{J}_{D_n}(u) := (\Psi_0 \circ \mathcal{J})(u) = \frac{1}{\hbar}h_{\mu\nu} u^\mu_\rho E_\rho E_\nu$ is a classical equivariant moment map (with respect to the Poisson tensor $\pi_d$ on $D_n$) and $\Psi_h \circ \frac{i}{\hbar} \mathcal{J} = \frac{1}{\hbar} \mathcal{J}_{D_n}$ an equivariant quantum moment map (with respect to $\star_h$) for all $h \in H$.

Proof: This follows directly from $\Psi_0$ and $\Psi_h$ being $U(1,n)$-equivariant and the algebraic version of the reduction procedure, i.e. that $\Psi_0$ is a morphism of Poisson-*-algebras, or the construction of $\star_h$ such that $\Psi_h$ becomes a morphism of *-algebras, respectively.
Example 4.17 Let \( n = 1, h = 1/2 \) and \( a = f_{E_0,E_1} = f_{r,0,1} \), then the \( m \)-th \( \ast_h \)-power of \( a \) is
\[
a^{*hm} = m! f_{mE_0,mE_1} = m! f_{r,0,m},
\]
(4.6) and so \( \lim_{M \to \infty} \sum_{m=0}^{M} a^{*hm}/m! = \lim_{M \to \infty} \sum_{m=0}^{M} f_{r,0,m} \) is not a Cauchy-sequence in any topology on \( \mathcal{P}(D_n) \) that makes the evaluation functionals \( \delta_{[r]} \) at all \( [r] \in D_n \) continuous, as this series does not converge in the point \( [r] \in D_n \) with \( \phi^{\text{std}}([r]) = u_1([r]) = 1/\sqrt{2} \), where \( f_{r,0,m}([r]) = 2^m/2 \).

Note that this also rules out the existence of any locally multiplicatively convex topology on \( \mathcal{P}(D_n) \) that makes all these evaluation functionals continuous: the example shows that there is no entire calculus which for a locally multiplicatively convex algebra would exist.

Nevertheless, by Theorem 4.15 all elements in the image of \( \mathcal{J}_{D_n} \) are essentially self-adjoint in every continuous \( \ast \)-representation for all \( h > 0 \). Moreover, Nelson’s theorem even allows to exponentiate this inner Lie algebra action to an inner Lie group action in such representations:

Theorem 4.18 (Exponentiation of \( \mathfrak{su}(1,n) \)-action) Fix \( h > 0 \) and let \( (\mathcal{H},\pi) \) be a continuous \( \ast \)-representation of \( (\mathfrak{A}(D_n),\ast_h,\cdot^\ast) \) and \( \mathcal{H}^\text{cpl} \) the completion of \( \mathcal{H} \). Then there exists a unique unitary representation \( U: \mathfrak{su}(1,n) \to \mathcal{U}(\mathcal{H}^\text{cpl}) \) such that \( \pi(\mathcal{J}_{D_n}(u))^\dagger = d\mathcal{U}(u)\dagger \) holds for all \( u \in \mathfrak{su}(1,n) \), where \( ,\dagger \) denotes the closure of an operator on \( \mathcal{H}^\text{cpl} \) and \( d\mathcal{U}(u) \) the derivation of the representation \( U \) at the neutral element in direction \( u \), i.e. \( d\mathcal{U}(u) \) is the operator in \( \mathcal{H}^\text{cpl} \) whose domain is \( C^\infty(\mathcal{U}) \), the set of all vectors \( \phi \in \mathcal{H}^\text{cpl} \) for which the map \( \mathfrak{su}(1,n) \ni g \mapsto \langle \psi | U(g)\phi \rangle_{\mathcal{H}^\text{cpl}} \in \mathbb{C} \) is smooth for all \( \psi \in \mathcal{H}^\text{cpl} \), and is defined as
\[
d\mathcal{U}(u)\phi := \frac{d}{dt} \big|_{t=0} \mathcal{U}(\exp(tu))\phi
\]
(4.7) for all \( \phi \in C^\infty(\mathcal{U}) \).

PROOF: By [24] Thm. 5 (see also [28] Thm. 10.5.6) we only have to show that the Nelson Laplacian \( \Delta := \sum \frac{1}{h^2} \mathcal{J}_{D_n}(u_i)^2 \), with \( u_i \) running over a basis of \( \mathfrak{su}(1,n) \), is represented by an essentially self-adjoint operator. As the image of \( \mathcal{J} \) is in \( \mathcal{P}(D_n)(1) \) and Hermitian, it follows that \( \Delta \in \mathcal{P}(D_n)(2) \), and as \( \Delta \) is clearly bounded from below, we can apply Theorem 4.15. \( \square \)

4.5 An Additional \( \mathbb{Z}_2 \)-Symmetry for \( n = 1 \)

The case \( n = 1 \) seems to be of special importance due to some interesting discrete symmetries. Here we can identify \( \mathbb{CP}^1 \) with \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \), the one point compactification of \( \mathbb{C} \), and thus \( D_{1,\text{ext}} \subseteq \mathbb{CP}^1 \) (via \( t\mathcal{P} \times \mathcal{P} \)) with \( \{v \in \mathbb{C} | |v| = 1\} \cup \{\infty\} \), which makes the relation between \( D_1 \cong \mathbb{D} \) and \( D_{1,\text{ext}} \cong \mathbb{D} \cup \mathbb{D} \) more apparent. Note also that analogously, \( \hat{D}_1 \subseteq \mathbb{CP}^1 \times \mathbb{CP}^1 \) (via \( t\mathcal{P} \times \mathcal{P} \)) is identified with \( (\overline{\mathbb{C}} \times \overline{\mathbb{C}}) \setminus \{(u,v) \in \mathbb{C}^2 \mid uv = 1\} \cup \{(0,\infty), (\infty,0)\}) \).

From the identification of \( D_{1,\text{ext}} \) with a subset of \( \overline{\mathbb{C}} \) one can already guess that there is another \( \mathbb{Z}_2 \)-symmetry that we did not discuss yet, which corresponds to the involution \( \overline{\mathbb{C}} \ni w \mapsto -1/w \in \overline{\mathbb{C}} \).

Definition 4.19 Define the holomorphic involution \( \Sigma \) of \( \hat{D}_1 \) as
\[
\Sigma: [p^0, p^1, q^0, q^1] \mapsto \Sigma([p^0, p^1, q^0, q^1]) := [-ip^1, ip^0, iq^1, iq^0].
\]
(4.8)

It is not hard to check that this is indeed a well-defined holomorphic involution of \( \hat{D}_1 \). Moreover, it acts on functions in \( \mathcal{O}(\hat{D}_1) \) via pullback as \( \Sigma^\ast(\hat{a}) := \hat{a} \circ \Sigma \) and especially
\[
\Sigma^\ast(\hat{f}(P_0,P_1),(Q_0,Q_1)) = (-1)^{P_0+Q_0}P_1+Q_1\hat{f}(P_1,P_0),(Q_1,Q_0) = (-1)^{P_1+Q_1}\hat{f}(P_1,P_0),(Q_1,Q_0)
\]
(4.9)
for all $P, Q \in \mathbb{N}_0^3$ with $|P| = |Q|$. As $\mathcal{O}(\hat{D}_1)$ is isomorphic to $\mathcal{A}(D_1)$, this of course yields an involution of $\mathcal{A}(D_1)$ as well. With respect to the standard chart this reads for the function $\hat{f}((P_0,P_0),(Q_0,Q_0)) = \frac{(-1)^{P_0+Q_0}P_0Q_0}{(1-|w|^2)^{P_0}}$ as
\[
\Sigma^*(\hat{f}(P_0,P_0),(Q_0,Q_0)) = (-1)^{P_1+Q_0^0}P_0Q_0 = \frac{(-1)^{P_1+Q_0^0}P_0Q_0}{(1-|w|^2)^{P_1}} = \frac{(-1/w)^P(-1/w^2)^Q}{(1-1/w|w|^2)^P} \tag{4.10}
\]
i.e. this involution, transferred to $\mathcal{A}(D_1)$, indeed describes the above mentioned involution $w \mapsto -1/w$. It is also worth noting that $\Sigma$ commutes with $\tau$, so its pullback commutes with the $\ast$-involution on $\mathcal{A}(D_1)$. With respect to the star product, we have:

**Proposition 4.20** Every $\ast$ on $\mathcal{A}(D_1)$ is $\Sigma$-equivariant for all $h \in H$, i.e. $\Sigma^* (a \ast_h b) = \Sigma^* (a) \ast_h \Sigma^* (b)$ holds for all $a, b \in \mathcal{A}(D_1)$.

**Proof:** As $\Sigma$ is a holomorphic involution of $\hat{D}_1$, its pullback acts by a continuous linear function on $\mathcal{A}(D_1)$ and so it is sufficient to check that $\Sigma^* (\hat{f}(P,Q) \ast_h \hat{f}(R,S)) = \Sigma^* (\hat{f}(P,Q)) \ast_h \Sigma^* (\hat{f}(R,S))$ holds for all $P, Q, R, S \in \mathbb{N}_0^3$ with $|P| = |Q|$ and $|R| = |S|$, which is easily done using equations (4.9) and (2.38). $\square$

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