On the dependence of the affine coherent states quantization on the parametrization of the affine group

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Abstract

Affine coherent states quantization is a promising approach to quantize Hamiltonian systems when the phase space includes at least one conjugate pair of canonical variables which take values from a half plane. The construction of the quantization map includes the identification of the half plane with the affine group. Here we demonstrate that the details of this identification, how one parameterizes the affine group with the phase space variables, in general lead to unitarily inequivalent quantum theories.

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I. INTRODUCTION

Coherent state quantization is applicable to Hamiltonian systems where the physical phase space can be identified with a Lie group acting on itself. This group is expected to have a unitary irreducible representation on a Hilbert space. The latter allows us to construct a resolution of the identity in that Hilbert space, which can be used to map almost any observable of that system into a Hermitian operator acting in the Hilbert space.

This quantization method is especially useful in cases where the physical phase space includes at least one variable with a nontrivial topology, e.g., $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$. This is common, e.g., in quantum cosmological models with gravitational singularities.

One may expect that the result of quantization does not depend on how exactly one identifies the phase space with the Lie group, or in other words, how one parameterizes the group, but this is not the case. In fact, different parameterizations lead to quantum systems which are not unitarily equivalent, i.e. represent different quantum systems. The aim of this letter is showing explicitly this property of coherent states quantization.

In what follows we first present two parameterizations of the affine group used in the literature and derive the two corresponding quantizations which turn out to be unitarily inequivalent. Next, we extend this result to the general case. We also discuss reproduction of a classical algebra of observables as an example of criterion to choose between parameterizations. Finally, we conclude.

II. TWO KNOWN PARAMETRIZATIONS

To make the present paper self-contained we first recall general ideas underlying coherent states (see e.g. [1]):

For a Lie group $G$, let $U(g)$, where $g \in G$, be a unitary irreducible representation of it on some Hilbert space $\mathcal{H}$. One can take an (at this point) arbitrary $|\Phi\rangle \in \mathcal{H}$, called fiducial vector, and let the representation of $G$ act on it,

$$|g\rangle = U(g)|\Phi\rangle,$$

(1)

to construct a family of coherent states. Most importantly, this means that one can
construct a resolution of the identity from $|g\rangle$. Consider the operator

$$\mathcal{O} = \int_G d\mu(g) |g\rangle \langle g| = \int_G d\mu(g) \ U(g)|\Phi\rangle \langle \Phi| U^\dagger(g),$$

(2)

where $d\mu(g_0 \cdot g) = d\mu(g)$ is a left invariant measure on $G$. It is easy to see that $\mathcal{O}$ intertwines $U(g)$,

$$U(g) \cdot \mathcal{O} = \int_G d\mu(g') U(g \cdot g') |\Phi\rangle \langle \Phi| U(g'^{-1})$$

$$= \int_G d\mu(h) U(h)|\Phi\rangle \langle \Phi| U(h^{-1} \cdot g) = \mathcal{O} \cdot U(g).$$

(3)

As we know from Schur’s Lemma, any non-trivial intertwiner is a scalar multiple of the identity, i.e. $\mathcal{O} \propto I_H$. The factor of proportionality has to be decided on a case by case basis, and may come with a restriction on the fiducial vector $|\Phi\rangle$.

In what follows we discuss coherent states constructed as above from the affine group, affine coherent states (ACS), and show how they can be used in quantization. This procedure is called affine coherent state quantization (ACSQ), and we give here a short introduction to this method.

Suppose the phase space of some physical system is a half plane, $\Pi = \{(p, q) \in \mathbb{R} \times \mathbb{R}_+\}$. It can be identified with the affine group $G := \text{Aff}(\mathbb{R})$ by defining the multiplication law either by (see [2] for more details)

$$(p_1, q_1) \cdot (p_2, q_2) := (q_1 p_2 + p_1, q_1 q_2),$$

(4)

or by (see [3] for more details)

$$(\tilde{p}_1, \tilde{q}_1) \cdot (\tilde{p}_2, \tilde{q}_2) := (\tilde{p}_2/\tilde{q}_1 + \tilde{p}_1, \tilde{q}_1 \tilde{q}_2).$$

(5)

Eqs. (4)–(5) define two different parameterizations of $\text{Aff}(\mathbb{R})$. They correspond, respectively, to the two actions of this group on $\mathbb{R}_+$:

$$x' = (p_1, q_1) \cdot x := xq + p, \quad \text{and} \quad x' = (\tilde{p}_1, \tilde{q}_1) \cdot x := x/\tilde{q} + \tilde{p}.$$  

(6)

The affine group has two (nontrivial) inequivalent irreducible unitary representations, [4] and [5, 6], defined on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}_+, d\nu(x))$, where $d\nu(x) := dx/x$.

For both parameterizations we choose the one defined, respectively, by

$$U_1(p, q) \Psi(x) := e^{ipx} \Psi(qx), \quad \text{and} \quad U_2(\tilde{p}, \tilde{q}) \Psi(x) := e^{i\tilde{p}x} \Psi(x/\tilde{q}),$$

(7)

\[1\] The representation defined in [3] is $U(p, q) \psi(x) = e^{ipx/\sqrt{q}} \psi(x/q)$ with the carrier space $L^2(\mathbb{R}_+, dx)$, but takes the form $U(p, q) \psi(x) = e^{ipx}\psi(x/q)$ when acting on $L^2(\mathbb{R}_+, dx/x)$.  

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where \( \Psi \in L^2(\mathbb{R}_+, d\nu(x)) \).

Integration over the affine group is defined, respectively, as

\[
\int_G d\mu_1(p, q) := \int_{-\infty}^\infty dp \int_0^\infty dq/q^2, \quad \text{and} \quad \int_G d\mu_2(\tilde{p}, \tilde{q}) := \int_{-\infty}^\infty d\tilde{p} \int_0^\infty d\tilde{q}, \tag{8}
\]

where both measures in \( (8) \) are left invariant.

Any coherent state can be obtained as

\[
\langle x|p,q\rangle_1 = U_1(p,q)\Phi(x), \quad \text{or} \quad \langle x|\tilde{p},\tilde{q}\rangle_2 = U_2(\tilde{p}, \tilde{q})\Phi(x), \tag{9}
\]

where \( L^2(\mathbb{R}_+, d\nu(x)) \ni \Phi(x) = \langle x|\Phi \rangle \), with \( \langle \Phi|\Phi \rangle = 1 \), is the fiducial vector.

The resolutions of the identity in the Hilbert space \( \mathcal{H} \) read

\[
\int_G d\mu_1(p, q)|p,q\rangle_1 \langle p,q| = 2\pi A_\Phi \mathbb{I}, \quad \text{and} \quad \int_G d\mu_2(\tilde{p}, \tilde{q})|\tilde{p},\tilde{q}\rangle_2 \langle \tilde{p},\tilde{q}| = 2\pi A_\Phi \mathbb{I}, \tag{10}
\]

where

\[
A_\Phi = \int_0^\infty dx \frac{dx}{x^2} |\Phi(x)|^2. \tag{11}
\]

We now have to impose on \( \Phi(x) \) as an additional condition that \( A_\Phi \) be finite.

Making use of \( (10) \) one can (formally) map any observable \( f \) defined on the phase space \( \Pi \) into a Hermitian operator acting on \( \mathcal{H} \) as follows:

\[
\hat{f}_1 := \frac{1}{2\pi A_\Phi} \int_G d\mu_1(p, q)|p,q\rangle_1 f(p,q) \langle p,q|_1, \tag{12}
\]

or

\[
\hat{f}_2 := \frac{1}{2\pi A_\Phi} \int_G d\mu_2(\tilde{p}, \tilde{q})|\tilde{p},\tilde{q}\rangle_2 f(\tilde{p},\tilde{q}) \langle \tilde{p},\tilde{q}|_2. \tag{13}
\]

Due to the above we have

\[
\hat{f}_1 \Psi(x) = \frac{1}{2\pi A_\Phi} \int_G d\mu_1(p, q)e^{ipx}\Phi(qx)f(p,q) \langle p,q|\Psi \rangle, \tag{14}
\]

where

\[
\langle p,q|\Psi \rangle = \int_0^\infty d\nu(x')e^{-ipx'}\Phi(qx')^*\Psi(x'), \tag{15}
\]

and

\[
\hat{f}_2 \Psi(x) = \frac{1}{2\pi A_\Phi} \int_G d\mu_2(\tilde{p}, \tilde{q})e^{i\tilde{p}x}\Phi(x/\tilde{q})f(\tilde{p},\tilde{q}) \langle \tilde{p},\tilde{q}|\Psi \rangle, \tag{16}
\]
where
\[ 2\langle \tilde{p}, \tilde{q}|\Psi \rangle = \int_0^\infty d\nu(x')e^{-i\tilde{p}x'}\Phi(x'/\tilde{q})^*\Psi(x'). \tag{17} \]

Since \( \int_0^\infty d\frac{q}{\eta}F(q) = \int_0^\infty d\tilde{q}F(\frac{1}{\eta}) \), we get
\[ 1\langle p, 1/q|\Psi \rangle = 2\langle p, q|\Psi \rangle. \tag{18} \]

Therefore
\[ \hat{f}_1\Psi(x) = \frac{1}{2\pi A_\Phi} \int_G d\mu_2(\tilde{p}, \tilde{q})e^{i\tilde{p}x}\Phi(x/\tilde{q})f(\tilde{p}, 1/\tilde{q}) \cdot 2\langle \tilde{p}, \tilde{q}|\Psi \rangle. \tag{19} \]

Comparing (16) with (19) we can see that \( \hat{f}_2|\Psi \rangle \neq \hat{f}_1|\Psi \rangle \) for a generic \( |\Psi \rangle \in \mathcal{H} \), which means that these operators are quite different.

Now, let us examine the traces of both operators in some orthonormal basis \( \{|e_k\} \) of \( \mathcal{H} \) to see whether
\[ \text{Tr}(\hat{f}_1) = \text{Tr}(\hat{f}_2). \tag{20} \]

Eq. (20) is satisfied if the operators \( \hat{f}_1 \) and \( \hat{f}_2 \) are unitarily equivalent, since all unitarily equivalent operators have the same trace:
\[ \text{Tr}(\bar{U}\hat{f}\bar{U}^{-1}) = \text{Tr}(\hat{f}) = \text{Tr}(\hat{f}_1) = \text{Tr}(\hat{f}_2), \tag{21} \]

where \( \bar{U} \) is unitary operator.

First we consider \( \hat{f}_1 \):
\[ \text{Tr}(\hat{f}_1) = \sum_n \langle e_n|\hat{f}_1|e_n \rangle = \frac{1}{2\pi A_\Phi} \int_G d\tilde{p}d\tilde{q} \sum_n \langle e_n|\tilde{p}\tilde{q}\rangle_2 f(\tilde{p}, 1/\tilde{q}) \cdot 2\langle \tilde{p}, \tilde{q}|e_n \rangle 
= \frac{1}{2\pi A_\Phi} \int_G d\tilde{p}d\tilde{q} \cdot 2\langle \tilde{p}, \tilde{q}|e_n \rangle \sum_n \langle e_n||\tilde{p}\tilde{q}\rangle_2 f(\tilde{p}, 1/\tilde{q}) 
= \frac{1}{2\pi A_\Phi} \int_G d\tilde{p}d\tilde{q} \cdot 2\langle \tilde{p}, \tilde{q}|\tilde{p}\tilde{q}\rangle_2 f(\tilde{p}, 1/\tilde{q}) 
= \frac{1}{2\pi A_\Phi} \int_G d\tilde{p}d\tilde{q} f(\tilde{p}, 1/\tilde{q}). \tag{22} \]

In the derivation of (22) we used
\[ \sum_n |e_n\rangle\langle e_n| = I \quad \text{and} \quad 2\langle \tilde{p}, \tilde{q}|\tilde{p}\tilde{q}\rangle_2 = \langle \Phi|U_2^{-1}U_2|\Phi \rangle = \langle \Phi|\Phi \rangle = 1. \tag{23} \]
Similarly, we get
\[ \text{Tr}(\hat{f}_2) = \frac{1}{2\pi A_\phi} \int_G d\tilde{p} d\tilde{q} f(\tilde{p}, \tilde{q}). \]  

Therefore, Eq. (20) cannot be satisfied so that the considered operators are unitarily inequivalent, which means that the two considered affine group parameterizations, defined by (4)–(5), lead to different quantum systems.

### III. GENERAL CASE

In this section we consider any affine group parameterization. Let \((\xi, \eta)\) be a fixed parametrization of the affine group, e.g. defined by (4) (it can be taken the one defined by (5) as well). Different parameterizations of the affine group can now be implemented by a family of one-to-one transformations \(\chi : \Pi \rightarrow \text{Aff}(R)\). Every function \(\chi\) provides a parametrization of the affine group by elements of the phase space \(\Pi\):

\[ \chi(p, q) = (\xi(p, q), \eta(p, q)). \]  

The composition law (4) has to be fulfilled so that we have

\[ (\xi(p_1, q_1), \eta(p_1, q_1)) \cdot (\xi(p_2, q_2), \eta(p_2, q_2)) = ((\eta(p_1, q_1), \xi(p_2, q_2) + \xi(p_1, q_1), \eta(p_1, q_1)\eta(p_2, q_2)). \]  

This determines uniquely the composition law for a new parametrization of the affine group.

The corresponding invariant measure can be obtained by the change of variables in the first measure in (8):

\[ d\xi \frac{d\eta}{\eta^2} = \left[ \frac{1}{\eta(p, q)} \right]^2 \left| \frac{\partial(\xi, \eta)}{\partial(p, q)} \right| dp dq = \sigma(p, q) dp dq \]  

Therefore, ACS quantization of the phase space function \(f\) yields

\[ \hat{f} = \frac{1}{2\pi A_\phi} \int_G dp dq \sigma(p, q) \| \xi(p, q), \eta(p, q) \| f(p, q) \langle \xi(p, q), \eta(p, q) \| . \]  

Following the idea of the preceding section, we calculate the trace of \(\hat{f}\). If the trace of \(\hat{f}\) is independent of the affine group parametrization, then the ACS method of quantization is universal. But we have

\[ \text{Tr}(\hat{f}) = \frac{1}{2\pi A_\phi} \int_G dp dq \sigma(p, q) f(p, q) \text{Tr}(\| \xi(p, q), \eta(p, q) \| f(p, q) \langle \xi(p, q), \eta(p, q) \| ) \]  

\[ = \frac{1}{2\pi A_\phi} \int_G dp dq \sigma(p, q) f(p, q). \]
Our result explicitly shows the dependence of the ACS quantization scheme on the parameterization of the group manifold by the phase space variables.

IV. EXAMPLE OF CHOOSING SUITABLE PARAMETRIZATION

The question presents itself which parameterization one should choose when applying ACS quantization. This can be tailored to each application, but in this section we want to focus on the reproduction of the classical algebra of a subset of observables of the theory. In particular we want to focus on the classical relation \( \{q, d\} = q \), where \( d := pq \). For simplicity we only consider real \( \Phi(x) \), and restrict ourselves to the two specific parameterizations used in section II.

A quick calculation shows that the respective position operators are given as

\[
\hat{q}_1\psi(x) = \frac{1}{A_\Phi} \psi(x), \quad \text{and} \quad \hat{q}_2\psi(x) = \frac{B_\Phi}{A_\Phi} x \psi(x),
\]

where

\[
B_\Phi = \int_0^\infty dx \frac{x^3}{x^3+\Phi(x)^2},
\]

and the dilation operators as

\[
\hat{d}_1\psi(x) = -i \frac{d}{A_\Phi} \frac{\psi(x)}{x}, \quad \text{and} \quad \hat{d}_2\psi(x) = -i \frac{B_\Phi}{A_\Phi} x \frac{d}{dx} \psi(x).
\]

This leads to

\[
[\hat{q}_1, \hat{d}_1] = iA_\Phi \hat{q}_1^3, \quad \text{and} \quad [\hat{q}_2, \hat{d}_2] = \frac{i}{A_\Phi} B_\Phi \hat{q}_2.
\]

Apart from numerical prefactors depending on \( \Phi(x) \), which could be absorbed through a rescaling of \( p \), the parametrization of \([3]\) fulfills the affine commutation relation, and the one from \([2]\) does not. Based on this criterion, one should hence choose the former one. A simple calculation shows that the same is true for the canonical commutation relation. Note that the canonical commutation relation is problematic on the half line as \( \hat{p} \) is not strictly speaking an observable in the quantum theory, since it cannot be made self-adjoint. Of course, other choices of observables are possible, where the second choice of parametrization might be preferable.

V. CONCLUSIONS

The result we have obtained is quite general and can be applied to any type of coherent state quantization where the phase space is identified with a Lie group.
The dependence we have found could be seen as the advantage of this method over other quantization schemes as it allows us to construct quantum theories fulfilling certain requirements, depending on the application. However, high flexibility may lead to low predictability. Therefore, an additional constraint on the choice of the parametrization, well motivated physically, should be an essential element of the constructed quantization scheme.

We have here discussed the reproduction of the classical algebra of a subset of observables of the theory. Of the two parametrizations considered here explicitly, the one of \[3\] fulfills the affine commutation relation while the one of \[2\] does not. If one prefers to focus on a different set of observables, other parameterizations may be preferable.

Especially in the context of quantum gravity, a different important criterion would be the regularization of the classically singular dynamics of the theory. Our next paper will concern the ACS quantization of the Lemaître-Tolman-Bondi model, having singular dynamics at the classical level, to be compared with the results of canonical quantization \[7\].

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[1] A. M. Perelomolv, Coherent states for arbitrary Lie group, Commun. Math. Phys. 26, 222 (1972).
[2] A. Góźdź, W. Piechocki, and G. Plewa, “Quantum Belinski-Khalatnikov-Lifshitz scenario”, Eur. Phys. J. C (2019) 79:45.
[3] C. R. Almeida, H. Bergeron, J.-P. Gazeau, and A. C. Scardua, “Three examples of quantum dynamics on the half-line with smooth bouncing”, Annals of Physics 392, 206 (2018).
[4] I. M. Gel’fand and M. A. Naïmark, “Unitary representations of the group of linear transformations of the straight line”, Dokl. Akad. Nauk. SSSR 55, 567 (1947).
[5] E. W. Aslaksen and J. R. Klauder, “Unitary Representations of the Affine Group”, J. Math. Phys. 9, 206 (1968).
[6] E. W. Aslaksen and J. R. Klauder, “Continuous Representation Theory Using Unitary Affine Group”, J. Math. Phys. 10, 2267 (1969).
[7] C. Kiefer and T. Schmitz, “Singularity avoidance for collapsing quantum dust in the Lemaître-Tolman-Bondi model”, Phys. Rev. D 99, 126010 (2019).
