Generating the Möbius group with involution conjugacy classes

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Abstract

A $k$-involution is an involution with a fixed point set of codimension $k$. The conjugacy class of such an involution, denoted $S_k$, generates $\text{Möb}(n)$-the the group of isometries of hyperbolic $n$-space-if $k$ is odd, and its orientation preserving subgroup if $k$ is even. In this paper, we supply effective lower and upper bounds for the $S_k$ word length of $\text{Möb}(n)$ if $k$ is odd, and the $S_k$ word length of $\text{Möb}^+(n)$, if $k$ is even. As a consequence, for a fixed codimension $k$ the length of $\text{Möb}^+(n)$ with respect to $S_k$, $k$ even, grows linearly with $n$ with the same statement holding in the odd case. Moreover, the percentage of involution conjugacy classes for which $\text{Möb}^+(n)$ has length two approaches zero, as $n$ approaches infinity.

1 Introduction and results.

Let $G$ be a group and $S$ a set of symmetric generators for a supergroup of $G$; $S$ is not necessarily a subset of $G$ but every element can be written as a product of elements from $S$. For $g \in G$, the length of $g$ with respect to $S$ (or $S$-length) is the minimal number of elements of $S$ needed to express $g$ as their product. The supremum over all group element lengths is called the length of $G$ with respect to $S$ (or simply the $S$-length of $G$), and is denoted $|G|$. We are interested in the set $S_k \subset \text{Möb}(n)$ of involutions with a codimension $k$ fixed point set acting on hyperbolic space, $\mathbb{H}^n$.

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Theorem 1.1. Let $n \geq 2$ and $k = 1, 2, \ldots, n - 1$.

- If $k$ is even, $S_k$ generates $\text{Möb}^+(n)$ and satisfies,
  \[
  \frac{n(n + 1)}{2k(n - k + 1)} \leq |\text{Möb}^+(n)|_k \leq 2n + 4.
  \]  
  \[\text{(1)}\]

- If $k$ is odd, $S_k$ generates $\text{Möb}(n)$ and satisfies,
  \[
  \frac{n(n + 1)}{2k(n - k + 1)} \leq |\text{Möb}(n)|_k \leq 2n + 2 + k.
  \]  
  \[\text{(2)}\]

where $| \cdot |_k$ denotes $S_k$-length.

In particular we have,

Corollary 1.2 (Linear growth). For a fixed codimension $k$,

- If $k$ is even, $|\text{Möb}^+(n)|_k \asymp n$.
- If $k$ is odd, $|\text{Möb}(n)|_k \asymp n$.

Our main tool is a dimension count which yields the fact that the set of elements of the form $\alpha_1 \cdots \alpha_m$, where $\alpha_i$ is a $k$-involution and $m < \frac{n(n + 1)}{2k(n - k + 1)}$, has measure zero in $\text{Möb}(n)$ (see theorem 4.1 and corollary 2.2).

We next consider the percentage of involution conjugacy classes for which $|\text{Möb}^+(n)| = 2$. More precisely, define

\[
\Phi(n) = \frac{|\{k : \text{the } S_k\text{-length of } \text{Möb}^+(n) \text{ is } 2\}|}{|\text{involution conjugacy classes in dimension } n|}.
\]  
  \[\text{(3)}\]

Recall that the conjugacy class of an involution is determined by the dimension of its (totally geodesic) fixed point set. Thus a $k$-involution determines a conjugacy class, and the denominator above is $n$. Setting $|\text{Möb}^+(n)| = 2$ and using the lower bound in inequality (13), after a straightforward computation to find bounds on $k$ in terms of $n$ we have,

Corollary 1.3. $\Phi(n) = O(n^{-\frac{1}{2}})$.

In the paper [1] the authors show that in each dimension $n$ there exists an involution conjugacy class $S_k$ for which $|\text{Möb}^+(n)| = 2$. When $n$ is even $k$ may be taken to be $\frac{n}{2}$, and when $n$ is odd, $k$ can be taken to be $\frac{n+1}{2}$. The results of this paper show that away from the middle codimensions (relative to $n$) one cannot expect the length of $\text{Möb}^+(n)$ to be small. Factoring of isometries from a geometric viewpoint in hyperbolic 4-space is also studied in [3] where a slightly different formulation and generalization of two and three dimensional half-turns is given.
Hyperbolic $n$-space is denoted $\mathbb{H}^n$. The orientation preserving isometries of $\mathbb{H}^n$ (the Möbius group) is $\text{M" ob}^+(n)$, and the full group is $\text{M" ob}(n)$. An involution is an order two isometry of $\mathbb{H}^n$ and a $k$-involution is an isometry with a fixed point set of codimension $k$. A reflection is an involution with a codimension one fixed point set, and a half-turn is an involution with a codimension two fixed point set. The involution is orientation reversing if and only if the codimension of the fixed point set is odd. For $k = 1, 2, 3, \ldots, n$, let $S_k$ be the set (conjugacy class) of $k$- involutions. For the basics on hyperbolic space and its isometry group we refer to Maskit or Ratcliffe ([2], [4]). A good background for $n$-dimensional hyperbolic geometry and some lower dimensional factorization results are given in [6]. For standard material on differential topology and Lie groups the reader is referred to [5] and [7].

We will use the well-known facts that the dimension of $O(n)$, as well as $SO(n)$, is $\frac{n(n-1)}{2}$ and the dimension of $\text{M" ob}^+(n)$, as well as $\text{M" ob}(n)$, is $\frac{n(n+1)}{2}$.

The paper is organized as follows. Section 2 contains the proofs that conjugacy classes of involutions generate the Möbius group as well as upper bounds on word length. In section 3, we show that the space of $k$-involutions is a submanifold of $\text{M" ob}(n)$ having dimension $k(n-k+1)$. Finally, we prove theorem 1.1 in section 4.

2 $k$-involutions in the orthogonal and Möbius groups

Throughout this section, we fix an integer $n \geq 2$ and an integer $k = 1, 2, \ldots, n-1$. The case $k = n$ is excluded since $S_n \cap O(n)$ has only one element and hence does not generate the orthogonal group.

This section is devoted to proving,

**Theorem 2.1.** $S_k \cap O(n)$ generates $O(n)$ if $k$ is odd, and generates $SO(n)$ when $k$ is even. Furthermore for $g \in O(n)$,

$$|g| \leq \begin{cases} 2n, & \text{if } g \text{ is orientation preserving} \\ 2n - 2 + k, & \text{if } g \text{ is orientation reversing} \end{cases},$$

where $|g|$ is the $S_k \cap O(n)$-length of $g$.

**Corollary 2.2.** $S_k$ generates $\text{M" ob}(n)$ if $k$ is odd, and generates $\text{M" ob}^+(n)$ when $k$ is even. Furthermore for $g \in \text{M" ob}(n)$,

$$|g| \leq \begin{cases} 2n + 4, & \text{if } g \text{ is orientation preserving} \\ 2n + 2 + k, & \text{if } g \text{ is orientation reversing} \end{cases},$$

where $|g|$ is the $S_k$-length of $g$. 


Remark 2.3. In both theorem 2.1 and corollary 2.2, we note that $k$ is necessarily odd when $g$ is orientation reversing.

The stabilizer of any point in $\mathbb{H}^n$ has a natural identification with $O(n)$. We fix such a copy of $O(n) \subset \text{M" ob}(n)$.

Denote the $n \times n$ diagonal matrices with $k$ entries being $-1$ and $n-k$ being 1 by $\mathcal{D}(n,k)$. Since an involution in $O(n)$ is $O(n)$--conjugate to a diagonal matrix, it is immediate that a $k$-involution in $O(n)$ is conjugate to a diagonal matrix in $\mathcal{D}(n,k)$. There are $\binom{n}{k}$ such matrices.

Lemma 2.4. Assume $n \geq 2$ and $k = 1, \ldots, n-1$.

1. If $k$ is odd, then any element of $\mathcal{D}(n,1)$ can be written as the product of $k$ elements of $\mathcal{D}(n,k)$.

2. Any element of $\mathcal{D}(n,2)$ can be written as the product of two elements of $\mathcal{D}(n,k)$.

Proof. For ease of notation, we identify the diagonal matrices of size $n$ having $\pm 1$ entries with the group $\mathbb{Z}_2^n$. That is, $\bigcup_{k=0}^n \mathcal{D}(n,k) = \mathbb{Z}_2^n$. We write an element of $\mathbb{Z}_2^n$ as a vector with the obvious component-wise multiplication in $\mathbb{Z}_2$.

To prove item (1), Consider $A=[-1,1,1,\ldots,1] \in \mathcal{D}(n,1)$. It suffices to show that $A$ can be written as the desired product. For $i = 1, \ldots, k$, let $C_i \in \mathcal{D}(n,k)$ with $j$-th component being,

$$C_i^j = \begin{cases} 1, & \text{if } j = i+1 \text{ or } k+2 \leq j \leq n \\ -1, & \text{if } 1 \leq j \leq k+1 \text{ and } j \neq i+1 \end{cases} \quad \text{(6)}$$

Then $A = \prod_{i=1}^{k} C_i$ and we have the desired decomposition of $A$.

To prove item (2), consider $A = [-1,-1,1,\ldots,1] \in \mathcal{D}(n,2)$. It suffices to show that $A$ can be written as the desired product. Let $R \in \mathcal{D}(n,k)$ be such that its $j$-th component is,

$$R^j = \begin{cases} 1, & \text{if } j = 1 \text{ or } k+2 \leq j \leq n \\ -1, & \text{if } 2 \leq j \leq k+1 \end{cases} \quad \text{(7)}$$

and let $S \in \mathcal{D}(n,k)$ have $j$th entry

$$S^j = \begin{cases} 1, & \text{if } j = 2 \text{ or } k+2 \leq j \leq n \\ -1, & \text{otherwise.} \end{cases} \quad \text{(8)}$$

Then $RS = A$ and we are finished with the proof of item (2). 

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Lemma 2.5. Let $a$ and $b$ be reflections in hyperplanes $\alpha$ and $\beta$ in $\mathbb{H}^n (n \geq 3)$ and let $g = ab$. Then, there exist half-turns $h$ and $k$ such that $g = hk$.

**Proof.** Consider the upper half-space model of $\mathbb{H}^n$. Let $\alpha \cap \hat{\mathbb{R}}^{n-1} = \tilde{\alpha}$ and $\beta \cap \hat{\mathbb{R}}^{n-1} = \tilde{\beta}$. Then $\tilde{\alpha}$ and $\tilde{\beta}$ are $(n-2)$-spheres in $\hat{\mathbb{R}}^{n-1}$. We may assume that neither $\tilde{\alpha}$ nor $\tilde{\beta}$ contains the point at infinity ($\infty$). Consider the unique circle $\rho$, through $\infty$ and each of the centers of $\tilde{\alpha}$ and $\tilde{\beta}$. It is clear that any $(n-2)$-sphere containing $\rho$ is orthogonal to each of $\tilde{\alpha}$ and $\tilde{\beta}$. Let $\tilde{\gamma}$ be one such $(n-2)$-sphere. Then, $\tilde{\gamma} = \gamma \cap \hat{\mathbb{R}}^{n-1}$, where $\gamma$ is a hyperplane in $\mathbb{H}^n$ which is orthogonal to each of $\alpha$ and $\beta$. Let $c$ denote reflection in $\gamma$. Then, $h = ac$ and $k = cb$ are half-turns in $\mathbb{H}^n$ such that $hk = (ac)(cb) = accb = ab = g$.

Proof of theorem 2.1. Fix $k = 1, \ldots, n - 1$. Using the block diagonal form an element in $g \in SO(n)$, it is easy to see that an element $g \in SO(n)$ can be written as a product $\rho_1 \ldots \rho_m$, where $\rho_i \in S_1 \cap O(n)$, $m$ is even, and $m$ is at most $n$. Now, using lemma 2.5 we write $g$ as a product of $m$ half-turns. Of course, the half-turns are $O(n)$-conjugate to a diagonal matrix in $D(n, 2)$ and hence using item (2) of lemma 2.4 we can write $g$ as the product of at most $2n$ elements in $S_k \cap O(n)$.

If $g \in O(n) - SO(n)$, then $g = \rho_1 \ldots \rho_m$, where $m$ is odd and at most $n$. Note that it must be that $k$ is odd. As above we write $\rho_1 \ldots \rho_{m-1}$ as the product of at most $2n - 2$ elements in $S_k \cap O(n)$. The reflection $\rho_m$, using item (1) of lemma 2.4 can be written as the product of $k$ elements in $S_k \cap O(n)$. Thus for such an element $g$, $|g| \leq 2n - 2 + k$.

Proof of corollary 2.2. For $g \in \text{M"{o}b}(n)$, it is well-known that $g = \Phi \tau \sigma$, where $\sigma$ and $\tau$ are reflections, and $\Phi$ is an element of $O(n)$. Moreover $g$ is orientation preserving if and only if $\Phi \in SO(n)$. Using lemma 2.5 we can replace $\tau \sigma$ by the product of two half-turns which by lemma 2.4 can be written as the product of 4 elements in $S_k$. The corollary now follows from theorem 2.1.

3 Involutions and the space of totally geodesic subspaces of $\mathbb{H}^n$.

Throughout this section, we fix an integer $n \geq 2$ and an integer $k = 1, 2, \ldots, n - 1$.

**Lemma 3.1.** $S_k \subset \text{M"{o}b}(n)$ is a (connected) differentiable submanifold of dimension $k(n-k+1)$.

**Proof.** Set $G = \text{M"{o}b}(n)$. Fix $\alpha \in S_k \subset \text{M"{o}b}(n)$ and denote its fixed point set by $\pi$, an $(n - k)$ dimensional plane. Consider the smooth conjugation
action of the Lie group $G$ on itself. Namely, $g \cdot f = gfg^{-1}$. Since an orbit of
a Lie group action is a submanifold, we have that the $G$-orbit of $\alpha$, that is
$S_k$, is a submanifold of $\text{M"{o}b}(n)$. Furthermore, the map from $G$ to $G$ given
by $g \mapsto g\alpha g^{-1}$, induces a one-to-one smooth map from $G/K$ onto $S_k$, where
$K = \text{Stab}_G(\alpha)$. (Note that $K$ is a closed subgroup of $G$). Next observe that
$\text{Stab}_G(\alpha) = \text{Stab}_G(\pi)$ and consider the map,

$$\Phi : \text{Stab}_G(\pi) \to \text{M"{o}b}(n - k)$$

given by $g \mapsto g|_{\pi}$. This is a surjective map with kernel being isomorphic to
$O(k) \leq \text{Stab}_G(\pi)$. Hence, $\text{Stab}_G(\pi)/O(k)$ is isomorphic to $\text{M"{o}b}(n - k)$ and thus,

$$\dim(K) = \dim(\text{Stab}_G(\alpha)) = \dim(\text{M"{o}b}(n - k)) + \dim(O(k)) \quad (10)$$

Thus we have

$$\dim(S_k) = \dim(G) - \dim(K) = \dim(G) - \dim(\text{M"{o}b}(n - k)) - \dim(O(k)) \quad (11)$$

Now plugging in the various quantities and simplifying yields the dimension of $S_k$ to be $k(n - k + 1)$.

For $k = 1, \ldots, n - 1$, let $G_k$ denote the space of $k$-planes (that is, $k$-dimensional totally geodesic subspaces) in $\mathbb{H}^n$. The boundary (at infinity) of a $k$-plane is a round $(k - 1)$ sphere. The space of $(k - 1)$ spheres with the Gromov-Hausdorff topology induces a natural topology on $G_k$.

**Corollary 3.2.** $G_k$ is a differentiable manifold of dimension $(n - k)(k + 1)$.

**Proof.** Consider the map, $G_k \to S_{n-k}$ given by taking the $k$-plane $\pi$ to the
$(n - k)$-involution with fixed point set $\pi$. This map is a homeomorphism
(needs to be checked). Pulling back the differentiable structure from $S_{n-k}$,
$G_k$ becomes a differentiable manifold whose dimension by lemma (3.1) is
$(n - k)(k + 1)$.

**4 Bounds for the $S_k$-length of the Möbius group**

Given a subset $J \subseteq \{1, \ldots, n - 1\}$, let $S$ be the generating set $S = \cup_{k \in J} S_k$ and set $M = M(S) = \max_{c \in J} \{\dim(S_c)\} = \max_{k \in J} \{k(n - k + 1)\}$.

**Theorem 4.1.** Except for a set of measure zero no element of $\text{M"{o}b}(n)$ can be written as a product $\alpha_1 \ldots \alpha_m$, where $\alpha_i \in S$, and $m < \frac{n(n+1)}{2M}$. 


Proof. Consider the manifold which is the $m$-fold product of the M"obius group. Given a sequence $\{k_1, ..., k_m\}$ of $m$-elements from $J$ (repetition is allowed), consider the mapping $\Psi: S_{k_1} \times ... \times S_{k_m} \to \text{M"ob}(n)$ which assigns the $m$-tuple of ordered $k_i$-involutions $(\alpha_1, ..., \alpha_m)$ to the product $\alpha_1...\alpha_m$. This is a smooth mapping between manifolds. The dimension of $S_{k_1} \times ... \times S_{k_m}$ is bounded from above by $mM$ which is by assumption less than the dimension of $\text{M"ob}(n) = \frac{n(n+1)}{2}$. Hence $\Psi$ is a smooth mapping from a manifold of lower dimension to one of higher dimension. It is a standard fact that the image of a smooth map from a manifold of lower dimension to one of higher dimension has measure zero.

Finally, there are a finite number of (namely, $\sum_{i=1}^{\lfloor \frac{n(n+1)}{2M} \rfloor} |J|^i$) sequences from $J$ of length less than or equal to $\frac{n(n+1)}{2M}$, and hence a finite number of maps $\Psi$ above. Thus the set of elements in $\text{M"ob}(n)$ that are in the images of such maps $\Psi$ is the finite union of sets of measure zero, hence has measure zero. This precisely says that the elements of $\text{M"ob}(n)$ that can be written as a product of at most $m$ elements from $S$ has measure zero. 

\[\blacksquare\]

**Theorem 1.1.** Let $n \ge 2$ and $k = 1, 2, ..., n - 1$.

- If $k$ is even, $S_k$ generates $\text{M"ob}^+(n)$ and satisfies,
  \[ \frac{n(n+1)}{2k(n-k+1)} \le |\text{M"ob}^+(n)|_k \le 2n + 4. \]
  \[(12)\]

- If $k$ is odd, $S_k$ generates $\text{M"ob}(n)$ and satisfies,
  \[ \frac{n(n+1)}{2k(n-k+1)} \le |\text{M"ob}^+(n)|_k \le 2n + 2 + k. \]
  \[(13)\]

**Proof.** The upper bound in either the even or odd case follows from corollary 2.2. For the lower bound, if all $k_i = k$ then $M = k(n-k+1)$. Now if $|g| < \frac{n(n+1)}{2k(n-k+1)}$, for all $g \in \text{M"ob}(n)$ then theorem 4.1 is contradicted. \[\blacksquare\]

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