HERMITE TRACE POLYNOMIALS AND CHAOS DECOMPOSITIONS FOR THE HERMITIAN BROWNIAN MOTION

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ABSTRACT. For a non-zero parameter $q$, we define Hermite trace polynomials, which are multivariate polynomials indexed by permutations. We prove several combinatorial properties for them, such as expansions and product formulas. The linear functional determined by these trace polynomials is a state for $q = \frac{1}{N}$ for a non-zero integer. For such $q$, Hermite trace polynomials of different degrees are orthogonal. The product formulas extend to the closure with respect to the state. The state can be identified with the expectation induced by the $N \times N$ Hermitian Brownian motion. Hermite trace polynomials are martingales for this Brownian motion, while the elements in the closure can be interpreted as stochastic integrals with respect to it. Using the grading on the algebra, we prove several chaos decompositions for such integrals, as well as analyse corresponding creation and annihilation operators. In the univariate, pure trace polynomial case, trace Hermite polynomials can be identified with the Hermite polynomials of matrix argument.

1. INTRODUCTION

Let \( \{B(t) : t \geq 0\} \) be the Brownian motion. It is well known that there exist polynomials \( H_n(x, t) \) (the Hermite polynomials) such that for each \( n \), \( H_n(B(t), t) \) is a martingale with respect to the filtration induced by \( \{B(t)\} \). These polynomials have numerous other familiar properties. The list of properties relevant to this article includes:

- Orthogonality: \( \mathbb{E}[H_n(B(t), t) H_k(B(t), t)] = 0 \) for \( n \neq k \);
- Three-term recursion: \( xH_n(x, t) = H_{n+1}(x, t) + ntH_{n-1}(x, t) \);
- Expansion: \( H_n(x, t) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n!}{(n-2j)!2^j j!} t^j x^{n-2j} \), where the coefficients are related to the number of incomplete matchings;
- Product formulas: \( H_n(x, t) H_k(x, t) = \sum_{j=0}^{\lfloor n/k \rfloor} \binom{n}{j} \binom{k}{j} t^j H_{n+k-2j}(x, t) \), where the coefficients are related to the number of inhomogeneous incomplete matchings;
- Solution of the heat equation: \( \partial_t H_n(x, t) + \frac{1}{2} \partial^2_x H_n(x, t) = 0 \) with the initial condition \( H_n(x, 0) = x^n \);
- Stochastic integral representation: \( H_n(B(t), t) = \int_{[0,t]^n} dB(t_1) \ldots dB(t_n) \).

Moreover, some of these results, notably the product formulas, extend to more general stochastic integrals \( \int_{[0,t]^n} f(t_1, \ldots, t_n) dB(t_1) \ldots dB(t_n) \) for \( f \in L^2 \).

Now let \( \{X(t) : t \geq 0\} \) be the \( N \times N \) Hermitian Brownian motion: Hermitian random matrices with jointly complex Gaussian entries and the covariance function \( \mathbb{E}[X_{ij}(s) X_{kl}(t)] = \frac{1}{N} \delta_{i-l} \delta_{j-k} (s \wedge t) \).

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Then as is also well-known by now, there is no polynomial \( P(x, t) \) of degree greater than 2 such that \( P(X(t), t) \) is a martingale. For example, a simple calculation shows that for \( s < t \),

\[
\mathbb{E} \left[ X(t)^3 \right] \leq s = X(s)^3 + (t - s) \left( 2X(s) + \frac{1}{N} \text{Tr}[X(s)]I_N \right).
\]

It is therefore natural to replace the algebra of polynomials by a larger algebra of trace polynomials. Here a trace monomial is, roughly speaking, a product of a regular monomial and traces of regular monomials. See Section 6.1 for details. An incomplete but representative list of related work involving trace polynomials includes the study of random matrices \([\text{Céb}13, \text{DHK}13, \text{Kem}16, \text{Kem}17]\), noncommutative functions \([\text{KS}17, \text{KPV}21]\), and operator algebras \([\text{DGS}21, \text{Jek}20, \text{JLS}22]\). Other related work includes \([\text{GV}06, \text{Lév}08, \text{Hub}21]\).

In this article, we study the \(*\)-algebra of trace polynomials, the state on it induced by the Hermitian Brownian motion, the corresponding basis of Hermite trace polynomials, and the larger algebra obtained as its completion. However, most of the results in the article are stated in terms of permutations (rather than trace monomials), a general parameter \( q \) (rather than \( \frac{1}{N} \)), and a general Hilbert space \( \mathcal{H} \) (rather than \( L^2(\mathbb{R}_+, dx) \)). Thus, denote by \( S_0(n) \) the symmetric group on the set \( \{0, 1, \ldots, n\} \). For each \( n, \alpha \in S_0(n) \) and \( h_1, \ldots, h_n \in \mathcal{H}_\mathbb{R} \), we will consider symbols

\[
T(\alpha \otimes (h_1 \otimes \ldots \otimes h_n)) = T((\sigma \alpha \sigma^{-1}) \otimes_t U_\sigma(h_1 \otimes \ldots \otimes h_n))
\]

for any \( \sigma \in S(n) \), where

\[
U_\sigma(h_1 \otimes \ldots \otimes h_n) = h_{\sigma^{-1}(1)} \otimes \ldots \otimes h_{\sigma^{-1}(n)}.
\]

Extending linearly in both arguments, we obtain

\[
\mathcal{T}(\mathcal{H}_\mathbb{R}) = \{ T(\eta \otimes_t F) : n \geq 0, \eta \in \mathbb{C}[S_0(n)], F \in \mathcal{H}_\mathbb{R}^{\otimes n} \},
\]

where for now we are considering the algebraic tensor product of Hilbert spaces. We now define a star-algebra structure on \( \mathcal{T}(\mathcal{H}_\mathbb{R}) \) by

\[
T(\alpha \otimes_t F) T(\beta \otimes_t G) = T((\alpha \cup \beta) \otimes_t (F \otimes G))
\]

and

\[
T(\eta \otimes_t F)^* = T(\eta^* \otimes_t F).
\]

Here for \( \alpha \in S_0(n) \), \( \beta \in S_0(k) \), the permutation \( \alpha \cup \beta \in S_0(n + k) \) is obtained by shifting the cycles of \( \beta \) by \( n \), and merging the cycles of \( \alpha \) and \( \beta \) containing 0. See Notation 3.2.

Next, let \( q \) be a non-zero parameter. For a transposition \( \tau \), we define the contraction \( C_\tau \) as follows. For \( \alpha \in S_0(n) \), \( C_\tau(\alpha) \) is obtained by: multiplying \( \tau \alpha \); erasing the support of \( \tau \) and shifting to obtain a permutation in \( S_0(n - 2) \); and multiplying by a weight depending on \( q \) and the number of cycles in \( \tau \alpha \). See Definition 3.9. Then as usual, we define the Laplacian as the sum over transpositions \( \mathcal{L} = \sum_\tau C_\tau \), and the Hermite trace polynomial \( I(\eta \otimes_t F) = T(e^{-\mathcal{L}}(\eta \otimes_t F)). \)

Here the contraction on the tensor part of the argument is the usual tensor contraction. Hermite trace polynomials satisfy several properties which parallel those of ordinary Hermite polynomials.

We may now define a linear functional \( \varphi \) on \( \mathcal{T}(\mathcal{H}_\mathbb{R}) \) by requiring it to be unital and zero on each \( I(\eta \otimes_t F) \). If we use \( \{ T(\eta \otimes_t F) \} \) as a spanning set for \( \mathcal{T}(\mathcal{H}_\mathbb{R}) \), multiplication does not depend on \( q \), while the state \( \varphi \) does. On the other hand, if we use \( \{ I(\eta \otimes_t F) \} \) as a spanning set, the state does not depend on \( q \), but multiplication does. \( \varphi \) is positive semi-definite if and only if \( q \) is zero or of the form \( \pm \frac{1}{N} \) for \( N \in \mathbb{N} \). For such \( q \), we can define the GNS Hilbert space \( \mathcal{F}_q(\mathcal{H}) \) as the
Finally, we show that, as expected, for matrices is \([NK20]\). We have not elucidated the connection between their work and ours.

Another article exploring the connection between characters of the symmetric groups and GUE

Since this process. For various corollaries follow. In particular, we obtain several versions of the chaos decomposition for this space. For \(q = 0\), with a different scaling, one obtains objects related to free probability.

To the best of our knowledge, the Hermite trace polynomials \(I(\eta \otimes_s F)\) are new even in the univariate case \(\mathcal{H} = \mathbb{C}\). However if we further restrict to \(\eta \in \mathbb{C}[S(n)]\) (rather than \(S_0(n)\)), the corresponding objects have appeared in the literature. Indeed, denoting \(\chi^\lambda\) the character of the irreducible representation indexed by the partition \(\lambda\), \(I(\chi^\lambda)\) is closely related to the corresponding Hermite polynomial of matrix argument. In particular, these elements are all orthogonal with respect to \(\varphi\). Moreover, one can form a more general set of characters \(\chi^{\lambda,\lambda'}\) indexed by pairs of partitions which differ by one box, such that \(\{I(\chi^{\lambda,\lambda'})\}\) form an orthogonal basis for \(\mathcal{F}_{1/N}(\mathbb{C})\). This collection of trace polynomials is clearly deserving of additional study.

The paper is organized as follows. After the introduction and a background section, in Section 3, we discuss the kernel of the character \(\chi_q\), define the multiplication and contractions on the tensor algebra of symmetric groups, and study their properties. In Section 4, we define the Fock space \(\mathcal{F}_q(\mathcal{H})\), describe the kernel of the inner product, and list three chaos decompositions for this space for different choices of \(\mathcal{H}\). In Section 5, we upgrade the algebra structure from \(\mathcal{T}\mathcal{P}(\mathcal{H}_{\mathbb{R}})\) to \(\overline{\mathcal{T}\mathcal{P}(\mathcal{H}_{\mathbb{R}})}\), define the Hermite trace polynomials \(I(\eta \otimes_s F)\) and the functional \(\varphi\), and prove conditional expectation and product formulas. We also study creation and annihilation decompositions on the Fock space, and three subalgebras which arise: the Gaussian subalgebra, the commutative subalgebra corresponding to pure trace polynomials, and the subalgebra corresponding to pure polynomials, which is not closed under conditional expectations. We finish the section by describing the relation to a construction by Bożejko and Guţă. Finally, in Section 6, we give some background on trace polynomials and the Hermitian Brownian motion, prove the isomorphism with the random matrix.
picture for $q = \frac{1}{N}$, and list several corollaries. We show that in the $q = 0$ case, there is an isomorphism involving the free Fock space, and the case $q = -\frac{1}{N}$ is isomorphic to that for $q = \frac{1}{N}$. We also describe the relation with Hermite polynomials of matrix argument.

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2. Preliminaries

2.1. Permutations and partitions. Denote $[n] = \{1, \ldots, n\}$ and $[0, n] = \{0, 1, \ldots, n\}$.

A permutation in $S(n)$ induces a (cycle) set partition in $\mathcal{P}(n)$. Conversely, a partition $\pi \in \mathcal{P}(n)$ will be identified with a permutation the elements of whose cycles are ordered in increasing order. In particular, a partition whose blocks are pairs and singletons will be identified with the corresponding involutive permutation. For such a partition, we denote by $\text{Sing} (\pi)$ its single-element blocks, and $\text{Pair} (\pi)$ its two-element blocks.

Similarly, a set partition $\pi$ induces a (number) partition whose parts are the sizes of blocks of $\pi$. Conversely, a partition $\lambda \in \text{Par}(n)$ with parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ induces a set partition whose blocks are intervals of size $\lambda_1, \ldots, \lambda_k$.

Denote by $S_0(n)$ the permutations of $[0, n]$. $S(n)$ is a subgroup of $S_0(n)$, which acts on it by conjugation. The equivalence classes under this action are subsets of the standard conjugacy classes where the number of elements of the cycle containing 0 is preserved. So they are in a natural bijection with number partitions of $n+1$ with a marked element. Equivalently, they are indexed by pairs of partitions $(\lambda, \lambda')$, where $\lambda \in \text{Par}(n)$, $\lambda' \in \text{Par}(n+1)$, and the corresponding diagrams differ by one box.

For $m > n$, we will identify the element $\alpha \in S_0(n)$ with the corresponding element of $S_0(m)$ under the natural inclusion $[0, n] \subset [0, m]$.

For $\alpha \in S_0(n)$, denote $\text{cyc}_0(\alpha) = \text{cyc}(\alpha) - 1$, where $\text{cyc}(\alpha)$ is the number of cycles of $\alpha$. In other words, $\text{cyc}_0(\alpha)$ is the number of cycles of $\alpha$ not containing 0. Denote

$$|\alpha| = (n + 1) - \text{cyc}(\alpha) = n - \text{cyc}_0(\alpha),$$

which is the usual length function on the symmetric group on $[0, n]$.

To make the paper more readable, we will write the elements of the group algebra $\mathbb{C}[S_0(n)]$ as linear combinations $\sum_{\alpha \in S_0(n)} c_{\alpha} \alpha$ rather than the more standard $\sum_{\alpha \in S_0(n)} c_{\alpha} \delta_{\alpha}$.

A partial permutation $\alpha \in \mathcal{P}S_0(n)$ is a bijection from a subset of $[0, n]$ onto a subset of $[0, n]$; proper subsets and the empty subset are allowed. Orbits of a partial permutation fall into two types. Cyclic orbits are in the usual permutation sense. Linear orbits have the initial and the final element. Note that a linear orbit has at least two elements. It is convenient to abuse the terminology and consider elements of $[0, n]$ which do not belong to any orbit of $\alpha$ as single-element linear orbits of $\alpha$. Denote $\mathcal{P}S_0(n, N)$ the set of partial permutations of $[0, n]$ with $N$ linear orbits.
2.2. **Structure theory of the symmetric group.** For a partition \( \lambda \in \text{Par}(n + 1) \), denote by \( \chi^\lambda \) the character of the irreducible representation of \( S_0(n) \) corresponding to \( \lambda \). We will identify \( \chi^\lambda \) with the element
\[
\sum_{\sigma \in S_0(n)} \chi^\lambda(\sigma)\sigma \in \mathbb{C}[S_0(n)]
\]
(recall that the characters of the symmetric group are real-valued). These elements span the center \( Z(\mathbb{C}[S_0(n)]) \), and are orthogonal, in the sense that for \( \lambda \neq \mu \),
\[
\left( \sum_{\sigma \in S_0(n)} \chi^\lambda(\sigma)\sigma \right) \left( \sum_{\tau \in S_0(n)} \chi^\mu(\tau)\tau \right) = \sum_{\rho \in S_0(n)} (\chi^\lambda * \chi^\mu)(\rho)\rho = 0.
\]
In particular, for any character \( \chi \), \( \chi^\lambda \) and \( \chi^\mu \) are orthogonal with respect to the inner product induced by \( \chi \).

The centralizer of \( \mathbb{C}[S(n)] \) in \( \mathbb{C}[S_0(n)] \) is
\[
Z(\mathbb{C}[S_0(n)] : \mathbb{C}[S(n)]) = \{ \eta \in \mathbb{C}[S_0(n)] : \sigma^{-1}\eta\sigma = \eta \text{ for all } \sigma \in S(n) \}.
\]

The following are well-known [VOK04], [Gil05].

- \( Z(\mathbb{C}[S_0(n)] : \mathbb{C}[S(n)]) \) is generated (as an algebra) by \( Z(\mathbb{C}[S(n)]) \) and the Jucys–Murphy element \( (01) + \ldots + (0n) \).
- For \( \lambda \in \text{Par}(n) \), write \( \lambda' = \lambda + \square \) if the Young diagram for \( \lambda' \) is obtained by adding one box to the Young diagram for \( \lambda \). Denote \( \chi^{\lambda' : \lambda} \) the character of the compression of the \( \lambda' \)-irreducible representation of \( S_0(n) \) to the (unique) component giving a \( \lambda \)-irreducible representation of \( S(n) \). Then \( \{ \chi^{\lambda' : \lambda} : \lambda \in \text{Par}(n), \lambda' = \lambda + \square \} \) are orthogonal and span \( Z(\mathbb{C}[S_0(n)] : \mathbb{C}[S(n)]) \).

Let \( W \) be the isomorphism
\[
W : \sum_{\lambda \in \text{Par}(n+1)} M_{d_\lambda}(\mathbb{C}) \rightarrow \mathbb{C}[S_0(n)],
\]
where \( d_\lambda \) is the dimension of the irreducible representation of \( S_0(n) \) corresponding to \( \lambda \). Then \( W(M_{d_\lambda}(\mathbb{C})) \) is the ideal generated by (any one of) the Young symmetrizer(s) \( c_\lambda \), and is spanned by these symmetrizers (for different choices of the tableau corresponding to \( \lambda \)). In particular,
\[
W(I_{M_{d_\lambda}(\mathbb{C})}) = \frac{d_\lambda}{(n+1)!} \chi^\lambda,
\]
which can be characterized as minimal central projections.

Denote
\[
\mathbb{C}[S_0(n)]_{\leq N} = W \left( \sum_{\lambda \in \text{Par}(n+1; \leq N)} M_{d_\lambda}(\mathbb{C}) \right)
\]
and
\[
\mathbb{C}[S_0(n)]_{> N} = W \left( \sum_{\lambda \in \text{Par}(n+1; \geq N+1)} M_{d_\lambda}(\mathbb{C}) \right).
\]

Let \( \chi \) be a character of \( S_0(n) \). Then \( \chi \circ W \) is a trace on \( \sum_{\lambda \in \text{Par}(n+1)} M_{d_\lambda}(\mathbb{C}) \), and so has the form
\[
\chi \circ W = \sum_{\lambda \in \text{Par}(n+1)} n_\lambda \text{Tr}_{M_{d_\lambda}(\mathbb{C})}.
\]
Here

\[ n_\lambda = \frac{\sum_\alpha \chi[\alpha]|\chi^\lambda[\alpha]|}{\sum_\alpha \chi^\lambda[\alpha]|\chi[\alpha]|} = \frac{1}{(n+1)!} \sum_{\alpha \in S_0(n)} \chi[\alpha]|\chi^\lambda[\alpha]|. \]

Denote \( E_{ij}^\lambda \) the matrix units in \( M_{d_\lambda}(\mathbb{C}) \). Then for any \( \chi \),

\[ \{ W(E_{ij}^\lambda) : \lambda \in \text{Par}(n+1), 1 \leq i, j \leq d_\lambda \} \]

span \( \mathbb{C}[S_0(n)] \), and are orthogonal with respect to the (typically degenerate) inner product induced by \( \chi \),

\[ \chi[W(E_{ij}^\lambda)W(E_{ij}^{\mu})^*] = \delta_{\lambda=\mu}\chi[W(E_{ij}^\lambda)] = \delta_{\lambda=\mu}n_\lambda. \]

See for example [BMP16] for a detailed exposition and explicit expressions for \( W(E_{ij}^\lambda) \).

2.3. Algebraic conditional expectation.

**Proposition 2.1.** Let \( \mathcal{A} \) be a unital star-algebra, \( \mathcal{B} \) a unital star-subalgebra, and \( \varphi \) a faithful, tracial state on \( \mathcal{A} \), where positivity means that \( \varphi[a^*a] \geq 0 \) for any \( a \in \mathcal{A} \). Denote by \( L^2(\mathcal{A}, \varphi) \) and \( L^2(\mathcal{B}, \varphi) \) the corresponding GNS Hilbert spaces, with the common state vector \( \Omega \). Suppose \( F : \mathcal{A} \to \mathcal{B} \) is a function such that the map \( a\Omega \mapsto F(a)\Omega \) extends to the orthogonal projection \( P : L^2(\mathcal{A}, \varphi) \to L^2(\mathcal{B}, \varphi) \). Then

- \( \varphi[F(a)] = \varphi[a] \).
- \( F \) is a \( \mathcal{B} \)-bimodule map.
- For any \( a \in \mathcal{A} \), the operator on \( L^2(\mathcal{B}, \varphi) \) induced by \( F(a) \) is \( PaP \). In particular, the operator induced by \( F(a^*a) \) is positive.

We call such a map \( F \) an algebraic conditional expectation.

If \( \mathcal{A} \) is a \( C^* \) algebra, it follows that \( F \) is a genuine \( \varphi \)-preserving conditional expectation.

**Proof.** By assumption, for any \( a \in \mathcal{A} \) and \( b \in \mathcal{B} \),

\[ \varphi[b^*a] = \varphi[b^*F(a)], \]

and \( F(a) \) is uniquely determined by this condition. By taking \( b = 1 \), we get the first property. Next, for \( b' \in \mathcal{B} \), using the fact that \( \varphi \) is tracial,

\[ \varphi[b^*F(ab')] = \varphi[b^*ab'] = \varphi[b'b^*a] = \varphi[b'b^*F(a)] = \varphi[b^*F(a)b'], \]

so \( F \) is a right \( \mathcal{B} \)-module map. The proof for the left action is similar, and does not require the tracial property. Finally,

\[ \langle b\Omega, F(a)b'\Omega \rangle_\varphi = \langle b\Omega, ab'\Omega \rangle_\varphi = \langle b\Omega, PaPb'\Omega \rangle_\varphi \]

and

\[ \langle b\Omega, F(a^*a)b\Omega \rangle_\varphi = \langle ab\Omega, ab\Omega \rangle_\varphi \geq 0. \]
3. The Tensor Algebra of Symmetric Groups

3.1. A function on the symmetric group. On the symmetric group $S_0(n)$, consider the function $\chi^{n+1}_q : \alpha \mapsto q^{||\alpha||}$, and extend it to a function on the group algebra $\mathbb{C}[S_0(n)]$. As is well-known (see, for example, [GGK13, NK20]), this function is positive semi-definite for $q \in \mathbb{Z}_{n+1} = \{-1, 0, 1\} \cup \{\pm 1/N : 1 \leq N \leq n - 1\}$ and is not positive semi-definite for other values of $q$. It follows that these functions are positive for all $n$ if

$$q \in \mathcal{Z} = \bigcap_n \mathcal{Z}_n = \{0\} \cup \left\{\pm \frac{1}{N} : N \in \mathbb{N}\right\}.$$  

We will typically omit the superscript on $\chi^{n+1}_q$. For $q \in \mathbb{Z}$, the positivity of $\chi_q$ follows from the fact that $\chi^{1/N}$ is the normalized character of the standard representation

$$\pi_{n,q} : \mathbb{C}[S_0(n)] \to \text{End} \left((\mathbb{C}^N)^{\otimes(n+1)}\right),$$

while $\chi_0$ is the normalized character of its regular representation (and also the standard trace on $\mathbb{C}[S_0(n)]$).

It is well-known [GGK13] (see also [Bia01, Ker03]) that

$$\chi^{n+1}_1 = \frac{1}{N^{n+1}} \sum_{\lambda \in \text{Par}(n+1)} |SS_N(\lambda)| \chi^\lambda,$$

where $SS_N(\lambda)$ is the number of semistandard Young tableaux of shape $\lambda$ with entries belonging to the set $\{1, \ldots, N\}$. In particular, the coefficients are zero if $\lambda$ has more than $N$ parts, and non-zero if it has at most $N$ parts. Also

$$\chi^{n+1}_0 = \frac{1}{(n+1)!} \sum_{\lambda \in \text{Par}(n+1)} d_\lambda \chi^\lambda.$$

We now discuss the kernel of the normalized character $\chi_q$. See Section 4 of [Pro76] for a related discussion.

**Proposition 3.1.** Denote

$$\mathcal{N}_{gr,q,n} = \{\eta \in \mathbb{C}[S_0(n)] : \chi_q[\eta \eta^*] = 0\} = \{\eta \in \mathbb{C}[S_0(n)] : \pi_{n,q}(\eta) = 0\}$$

and

$$\mathcal{N}_{gr,q} = \bigoplus_{n=0}^\infty \mathcal{N}_{gr,q,n} \subset \bigoplus_{n=0}^\infty \mathbb{C}[S_0(n)].$$

(a) $\mathcal{N}_{gr,q,n} = \{0\}$ for $q = 0$ or for $q = \frac{1}{N}$, $n + 1 \leq N$.

(b) The following are equivalent descriptions of $\mathcal{N}_{gr,1/N,n}$ for $n + 1 > N$.

- $\mathcal{N}_{gr,1/N,n}$ is the ideal of the group algebra of $S_0(n)$ spanned by the Young symmetrizers corresponding to diagrams with at least $N + 1$ rows.
- $\mathcal{N}_{gr,1/N,n} = \mathbb{C}[S_0(n)]_{>N}$, so that on $\mathbb{C}[S_0(n)]_{\leq N}$, $\chi_q$ is faithful.
• More explicitly, $N_{gr,1/N,n}$ is the ideal generated by

$$\sigma^{(1/N)}_n = \sum_{\sigma \in S_0(N-1)} (-1)^{|\sigma|}\sigma$$

under the natural embedding of $S_0(N-1)$ into $S_0(n)$. Similarly, $N_{gr,-1/N,n}$ is the ideal generated by

$$\sigma^{(-1/N)}_n = \sum_{\sigma \in S_0(N-1)} \sigma.$$  

• Let $\pi \in \mathcal{PS}_0(n)$ be a partial permutation of $[0,n]$ with $N$ linear orbits. Denote $a_0, \ldots, a_{N-1}$ and $b_0, \ldots, b_{N-1}$ the initial, respectively, final elements of these orbits (recall that if an element does not belong to any actual orbit of $\pi$, we consider it as a single-element linear orbit, in which case $a_i = b_i$). Here $a_0, b_0$ belong to the orbit containing 0. A permutation $\sigma \in S_0(N-1)$ naturally acts on the linear orbits of $\pi$ by concatenating them. Denote the result of this action by $\sigma \circ \pi \in S_0(n)$. Thus, $(\sigma \circ \pi)(b_i) = a_{\sigma(i)}$, and $(\sigma \circ \pi)(x) = \pi(x)$ otherwise. Then

$$N_{gr,1/N} = \text{Span}\left( \sum_{\sigma \in S_0(N-1)} (-1)^{|\sigma|}\sigma \circ \pi : \pi \in \mathcal{PS}_0(n, N) \right)$$

and

$$N_{gr,-1/N} = \text{Span}\left( \sum_{\sigma \in S_0(N-1)} \sigma \circ \pi : \pi \in \mathcal{PS}_0(n, N) \right).$$

Proof: (a) and the equivalence between the first three entries in (b) are well-known. For the final entry, denote by $\bar{\pi}$ the permutation obtained by “closing” the orbits of $\pi$; that is, $\bar{\pi}(b_i) = a_i$ and $\bar{\pi}(x) = \pi(x)$ otherwise. Also let $\alpha \in S(n)$ be defined by $\alpha(a_i) = i$ and arbitrarily otherwise; thus, $\alpha$ maps $\{a_0, \ldots, a_{N-1}\}$ bijectively onto $[0, N-1]$. Then a calculation shows that $\alpha^{-1}\sigma\alpha\bar{\pi} = \sigma \circ \pi$. Therefore

$$\sum_{\sigma \in S(n-1)} (-1)^{|\sigma|}\sigma \circ \pi = \sum_{\sigma \in S(n-1)} (-1)^{|\sigma|}\alpha^{-1}\sigma\alpha\bar{\pi} \in N_{gr,q}.$$

The argument for the opposite inclusion is very close to the proof of Theorem 4.5 in [Pro76], in a somewhat different language. Start with $\alpha^{-1}\sigma^{(1/N)}_n\beta\alpha$ in the ideal. Possibly by replacing $\beta$ by its multiple by appropriate transpositions, we may assume that $0, 1, \ldots, N-1$ lie in different cycles of $\beta$. Denote $a_i = \alpha^{-1}(i)$ for $0 \leq i \leq N - 1$. Then $a_0, \ldots, a_{N-1}$ lie in different cycles of $\alpha^{-1}\beta\alpha$. So $\alpha^{-1}\beta\alpha = \bar{\pi}$, where linear orbits of $\pi$ are cycles of $\alpha^{-1}\beta\alpha$ with initial elements $a_0, a_1, \ldots, a_{N-1}$, and cyclic orbits of $\pi$ are the remaining cycles of $\alpha^{-1}\beta\alpha$. It follows that

$$(\alpha^{-1}\sigma^{(1/N)}_n\alpha)(\alpha^{-1}\beta\alpha) = \sum_{\sigma \in S(n-1)} (-1)^{|\sigma|}\sigma \circ \pi.$$  

The argument for $q = -1/N$ is similar. \qed
3.2. Algebra structure.

**Notation 3.2.** Let \( \alpha \in S(n) \), \( \beta \in S_0(k) \). Define \( \sigma_{n,k} \in S(n+k) \) by

\[
\sigma_{n,k}(i) = \begin{cases} 
  i + k, & 1 \leq i \leq n, \\
  i - n, & n + 1 \leq i \leq n + k.
\end{cases}
\]

Thus in word notation, \( \sigma_{n,k} = k + 1, k + 2, \ldots, k + n, 1, 2, \ldots, k \). Note for future reference that \( \sigma_{k,n} = \sigma_{n,k}^{-1} \).

Define \( \alpha \cup \beta \in S_0(n+k) \) by

\[
(1) \quad \alpha \cup \beta = \sigma_{n,k}^{-1} \beta \sigma_{n,k} \alpha.
\]

That is, \( \alpha \cup \beta \) is obtained by: combining the cycles of \( \alpha \) and \( \beta \) containing 0 into

\[
(0, \alpha(0), \ldots, \alpha^{-1}(0), \beta(0) + n, \ldots, \beta^{-1}(0) + n),
\]

keeping the remaining cycles of \( \alpha \), and letting the remaining cycles of \( \beta \) act on the shifted set \( \{n + 1, \ldots, n + k\} \).

We will now define a version of tensor multiplication on \( \bigoplus_{n=0}^{\infty} \mathbb{C}[S_0(n)] \) and its quotient by \( \mathcal{N}_{gr,q} \). To distinguish this algebra structure from the usual multiplication on the group algebra, we will denote \( \alpha \) by \( T(\alpha) \). We will use this identification to talk about \( \chi_q, \mathcal{N}_{gr,q}, \) etc., as applied to \( T(\eta) \).

**Definition 3.3.** Define the multiplication on \( \bigoplus_{n=0}^{\infty} \mathbb{C}[S_0(n)] \) by the linear extension of

\[
T(\alpha) T(\beta) = T(\alpha \cup \beta).
\]

We use the ordinary adjoint on the group algebra, defined by the anti-linear extension of the relation

\[
T(\alpha)^* = T(\alpha^{-1}).
\]

**Remark 3.4.** The subalgebra \( \bigoplus_{n=0}^{\infty} \mathbb{C}[S(n)] \) is called the generic tensor algebra in [Rai14].

**Proposition 3.5.**

(a) The multiplication in Definition 3.3 is associative.

(b) \( (0) \) is the identity.

(c) \( \mathcal{N}_{gr,q} \) is an ideal for this multiplication. Consequently the multiplication factors through to the quotient

\[
\mathcal{T P}_q = \bigoplus_{n=0}^{\infty} \mathbb{C}[S_0(n)]/\mathcal{N}_{gr,q}.
\]

For \( q = \frac{1}{N} \),

\[
\mathcal{T P}_q = \bigoplus_{n=0}^{N-1} \mathbb{C}[S_0(n)] \oplus \bigoplus_{n=N}^{\infty} \mathbb{C}[S_0(n)]_{\leq N}.
\]

**Proof.** (a) and (b) are immediate. (c) follows from (1) since \( \mathcal{N}_{gr,q} \) is an ideal for the usual group algebra multiplication.

**Notation 3.6.** For \( \alpha \in S_0(n) \) and \( S \subset [n] \), the restriction \( \alpha|_S \) of a permutation is the permutation on \( [0, n] \setminus S \) defined by \( \alpha|_S(x) = \alpha^m(x) \), where \( m = \min \{ k > 0 \mid \alpha^k(x) \in S^c \} \).
**Notation 3.7.** For $A, B \subseteq \mathbb{Z}$, $|A| = |B|$, denote $P_B^A$ the unique order-preserving bijection from $A$ to $B$, as well as (by abuse of notation) the corresponding bijection between the collections of permutations $S(A)$ and $S(B)$.

**Example 3.8.** For $\alpha = (13524)$ and $S = \{2, 5\}$, $\alpha|_{S^c} = (134)$ and $P_{\{5\}\setminus S}\alpha|_{S^c} = (123)$.

For $\pi \in \mathcal{P}_1(n)$, denote $\text{supp} (\pi) = [n] \setminus \text{Sing} (\pi)$.

**Definition 3.9.** Let $q \neq 0$. For a transposition $\tau = (ij) \in S(n)$ and $\alpha \in S_0(n)$, define the $\tau$-contraction by the linear extension of

$$C_\tau (\alpha) = q^{\text{cyc}_0(\tau \alpha)_{\text{supp}(\tau^c)} - \text{cyc}_0(\tau \alpha) + 1} P_{[0, n-2]}^{[0, n-1]\setminus \{ij\}} (\tau \alpha)_{\text{supp}(\tau^c)}$$

More generally, for $\pi \in S(n)$ with the cycle structure $\pi \in \mathcal{P}_{1, 2}(n)$, define the contraction

$$C_\pi (\alpha) = q^{\text{cyc}_0((\tau \alpha)_{\text{supp}(\tau^c)} - \text{cyc}_0(\tau \alpha) + \ell} P_{[0, n-2]}^{[0, n-1]\setminus \text{supp}(\tau)} (\tau \alpha)_{\text{supp}(\tau^c)}$$

Extend $C_\pi$ linearly to $\mathbb{C}[S_0(n)]$.

**Remark 3.10.** It is easy to check that for a transposition $\tau = (ij)$,

$$q^{\text{cyc}_0((\tau \alpha)_{\text{supp}(\tau^c)} - \text{cyc}_0(\tau \alpha) + 1} = \begin{cases} q^{-1}, & (ij) \text{ is a cycle in } \alpha, \\ 1, & (i), (j) \text{ are cycles in } \alpha, \\ 1, & i, j \text{ are consecutive elements} \\ q, & \text{in the same cycle of } \alpha \text{ of length at least } 3, \\ \end{cases}$$

In particular, $C_\tau$ is defined for $q = 0$ unless $(ij)$ is a cycle in $\alpha$. See Section 6.2.2.

**Lemma 3.11.** Let $q \in \mathbb{Z}\setminus \{0\}$. Each $C_\pi$ preserves $N_{gr, q}$. Therefore it factors through to the quotient $\mathcal{P}_q$.

**Proof.** We consider the case $q = \frac{1}{N}$; for $q = -\frac{1}{N}$, the argument similar. We will use the representation in Proposition 3.11. Let $\eta \in \mathcal{N}_{gr, q}$. Since $\mathcal{N}_{gr, q}$ is an ideal, $\tau \eta \in \mathcal{N}_{gr, q}$. So it suffices to show that if $\pi \in \mathcal{P}(S(n))$ and $\eta = \sum_{\sigma \in S(n)} (-1)^{|\sigma|} \sigma \circ \pi$, then for any $S \subseteq [0, n]$,

$$\sum_{\sigma \in S(N)} q^{\text{cyc}_0((\sigma \circ \pi)_{|S^c}} - \text{cyc}_0(\sigma \circ \pi) - (1)^{|\sigma|}(\sigma \circ \pi)|_{S^c} \in \mathcal{N}_{gr, q}.$$ 

Moreover, it suffices to take $S = \{s\}$. We consider two cases.

Suppose $s = a_i = b_j$. Then $(\sigma \circ \pi)|_{\{s\}} = (\sigma|_{\{i\}} \circ (\pi|_{\{s\}})^c)$ and $\text{cyc}_0((\sigma \circ \pi)|_{\{s\}}) = \text{cyc}_0(\sigma|_{\{i\}})$. Each $\sigma' \in S(N-1)$ appears as $\sigma|_{\{i\}} N$ times, once when $(i)$ is a cycle in $\sigma$ (so that $\sigma'$ has one less cycle than $\sigma$), and $N - 1$ times corresponding to $\sigma(i) = j$ for each $j \in [N]\setminus \{i\}$ (so that $\sigma'$ has the same number of cycles as $\sigma \circ \pi$). Thus

$$\sum_{\sigma \in S(N)} q^{\text{cyc}_0((\sigma \circ \pi)|_{S^c}} - \text{cyc}_0(\sigma \circ \pi) - (1)^{|\sigma|}(\sigma \circ \pi)|_{S^c} = \sum_{\sigma' \in S(N-1)} (N - 1 - q^{-1})(-1)^{|\sigma'|}(\sigma' \circ (\pi|_{\{s\}}^c)$$

$$= - \sum_{\sigma' \in S(N-1)} (-1)^{|\sigma'|}(\sigma' \circ (\pi|_{\{s\}}^c) \in \mathcal{N}_{gr, q}.$$ 

If $\{s\}$ is not a single-element linear orbit of $\pi$, then $\sigma \circ (\pi|_{\{s\}}^c)$, and has the same number of cycles as $\sigma \circ \pi$. 

\qed
Notation 3.12. Denote
\[ \mathcal{L}_n = \sum_{\tau \text{ a transposition in } S(n)} C_\tau \]
and \( \mathcal{L} \) the direct sum of these operators. Note that for \( \pi \in \mathcal{P}_{1,2}(n) \), \( C_\pi \) is a product of several transposition-type contractions, and
\[ \mathcal{L}^\ell = \ell! \sum_{\substack{\pi \in \mathcal{P}_{1,2}(n) \\mid |\pi| = n-\ell}} C_\pi. \]
Denote also
\[ \mathcal{P}_{1,2}(n, k) = \{ \pi \in \mathcal{P}_{1,2}(n+k) : \text{if } (ij) \in \pi, i < j, \text{ then } i \leq n < j \} \]
the inhomogeneous partitions, and
\[ \mathcal{L}_{n,k} = \sum_{\substack{(ij) \in S(n+k) \\mid i \leq n < j}} C_{(ij)}. \]
Finally, for \( \ell \leq n \wedge k \), denote
\[ \mathcal{L}_{n,k}^{(\ell)} = \ell! \sum_{\pi \in \mathcal{P}_{1,2}(n,k) \mid |\pi| = n-\ell} C_\pi. \]

Definition 3.13. For \( \eta \in \bigoplus_{n=0}^{\infty} \mathbb{C}[S_0(n)] \), define
\[ I(\eta) = T(e^{\mathcal{L}}\eta) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{1}{\ell!} T(\mathcal{L}^{\ell}(\eta)) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} (-1)^{|\pi|} T(C_\pi(\eta)). \]
For \( q \in \mathbb{Z} \setminus \{0\} \), \( I(\eta) \) is also well-defined for \( \eta \in TP_q \).

Proposition 3.14.

(a) \[ T(\eta) = I(e^\mathcal{L} \eta) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} I(\mathcal{L}^{\ell}(\eta)) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} I(C_\pi(\eta)). \]

(b) For \( \alpha \in S_0(n) \), \( \beta \in S_0(k) \),
\[ I(\alpha) I(\beta) = I(\alpha \cup \beta) + \sum_{\ell=1}^{\min(n,k)} \frac{1}{\ell!} I(\mathcal{L}_{n,k}^{(\ell)}(\alpha \cup \beta)) \]
\[ = \sum_{\pi \in \mathcal{P}_{1,2}(n,k)} I(C_\pi(\alpha \cup \beta)). \]
Proof. (a) follows by composing (terminating) power series in \( L \). For (b), the argument is very similar to the standard one for Hermite polynomials \([dSCV85]\). We expand

\[
I(\alpha) I(\beta) = \sum_{\sigma_1 \in \mathcal{P}_{1,2}(n)} (-1)^{n-|\sigma_1|} T(C_{\sigma_1}(\alpha)) \sum_{\sigma_2 \in \mathcal{P}_{1,2}(k)} (-1)^{k-|\sigma_2|} T(C_{\sigma_2}(\beta))
\]

\[
= \sum_{\sigma_1 \in \mathcal{P}_{1,2}(n)} \sum_{\sigma_2 \in \mathcal{P}_{1,2}(k)} (-1)^{n+k-|\sigma_1|-|\sigma_2|} T(C_{\sigma_1}(\alpha) \cup C_{\sigma_2}(\beta))
\]

\[
= \sum_{\sigma_1 \in \mathcal{P}_{1,2}(n)} \sum_{\sigma_2 \in \mathcal{P}_{1,2}(k)} (-1)^{n+k-|\sigma_1|-|\sigma_2|} \sum_{\tau \in \mathcal{P}_{1,2}([\text{Sing}(\sigma_1)] + [\text{Sing}(\sigma_2)])} I(C_{\tau}(C_{\sigma_1}(\alpha) \cup C_{\sigma_2}(\beta)))
\]

\[
= \sum_{\pi \in \mathcal{P}_{1,2}(n+k)} I(C_{\pi}(\alpha \cup \beta)) \sum_{\tau \in \text{Pair}(\pi|_{[n]}) \cap \text{Pair}(\pi|_{[n+1,n+k]})} (-1)^{|\tau_1|+|\tau_2|}.
\]

The final sum is zero unless both \( \text{Pair}(\pi|_{[n]}) = \emptyset = \text{Pair}(\pi|_{[n+1,n+k]}) \), in which case \( \pi \in \mathcal{P}_{1,2}(n,k) \). \( \square \)

4. Fock space

Construction 4.1. Let \( \mathcal{H} \) be a real Hilbert space, and \( \mathcal{H} \) its complexification. We will denote by \( \mathcal{H}^{\otimes n} \) the algebraic tensor product, which is spanned by simple tensors, and by \( \overline{\mathcal{H}}^{\otimes n} \) the Hilbert space tensor product. For each \( n \), form the algebraic Fock space

\[
\mathbb{C}(0) \oplus \bigoplus_{n=1}^{\infty} \left( \mathbb{C}[S_0(n)] \otimes \overline{\mathcal{H}}^{\otimes n} \right).
\]

On each component of this space, we have a natural action of \( S(n) \): for \( \sigma \in S(n) \)

\[
\alpha \otimes F \mapsto \sigma \alpha \sigma^{-1} \otimes U_{\sigma} F,
\]

where

\[
U_{\sigma}(h_1 \otimes \ldots \otimes h_n) = h_{\sigma^{-1}(1)} \otimes \ldots \otimes h_{\sigma^{-1}(n)}
\]

extends to \( \overline{\mathcal{H}}^{\otimes n} \) as an isometry. We denote by

\[
\mathcal{T}\mathcal{F}(\mathcal{H}) = \mathbb{C}(0) \oplus \bigoplus_{n=1}^{\infty} \left( \mathbb{C}[S_0(n)] \otimes \overline{\mathcal{H}}^{\otimes n} \right)
\]

the vector space quotient under this action. \( \mathcal{T}\mathcal{F}(\mathcal{H}) \) may be identified with the fixed point subspace of this action, which is the image of the direct sum of the projections

\[
P_n : \alpha \otimes F \mapsto \frac{1}{n!} \sum_{\sigma \in S(n)} \sigma \alpha \sigma^{-1} \otimes U_{\sigma} F
\]

Thus

\[
\mathbb{C}[S_0(n)] \otimes \overline{\mathcal{H}}^{\otimes n} = \left\{ \sum_{\alpha \in S_0(n)} \alpha \otimes F_{\alpha} \in \mathbb{C}[S_0(n)] \otimes \overline{\mathcal{H}}^{\otimes n} : \frac{1}{n!} \sum_{\sigma \in S(n)} U_{\sigma} F_{\sigma^{-1}\alpha \sigma} = F_{\alpha} \right\}.
\]
To simplify notation, we will denote
\[ \alpha \otimes_s F = P_n(\alpha \otimes F) = \frac{1}{n!} \sum_{\sigma \in S(n)} \sigma \alpha \sigma^{-1} \otimes U_\sigma F. \]

On \( \mathbb{C}(0) \oplus \bigoplus_{n=1}^{\infty} (\mathbb{C}[S_0(n)] \otimes \mathcal{H}^{\otimes n}) \), we have the canonical inner product
\[ \langle \alpha \otimes F, \beta \otimes G \rangle_0 = \delta_{n=k} \chi_0[\alpha \beta^{-1}] \langle F, G \rangle_{\mathcal{H}^{\otimes n}} = \delta_{n=k} \delta_{\alpha=\beta} \langle F, G \rangle_{\mathcal{H}^{\otimes n}}, \]
where \( \chi_0 \) is the canonical trace on \( \mathbb{C}[S_0(n)] \). On this space, define the operator
\[ K_q(\alpha \otimes F) = \sum_{\beta \in S_0(n)} \chi_q[\alpha \beta^{-1}] \beta \otimes F. \]

Note that \( K_0 \) is the identity operator. It is easy to check that for \( q \in \mathbb{Z} \), \( K_q \) is a positive semi-definite operator. Moreover,
\[ K_q P_n(\alpha \otimes F) = \frac{1}{n!} \sum_{\beta \in S_0(n)} \sum_{\sigma \in S(n)} \chi_q[\sigma \alpha \sigma^{-1} \beta^{-1}] \beta \otimes U_\sigma F = \frac{1}{n!} \sum_{\beta \in S_0(n)} \sum_{\sigma \in S(n)} \chi_q[\alpha \beta^{-1}] \beta \sigma^{-1} \otimes U_\sigma F = P_n K_q(\alpha \otimes F), \]
so \( K_q \) restricts to the subspace \( \overline{TP}(\mathcal{H}) \). The resulting inner product on \( \overline{TP}(\mathcal{H}) \) is
\[ \left\langle \sum_{\alpha \in S_0(n)} \alpha \otimes_s F_\alpha, \sum_{\beta \in S_0(n)} \beta \otimes_s G_\beta \right\rangle = n! \sum_{\alpha, \beta \in S_0(n)} \chi_q(\alpha \beta^{-1}) \langle F_\alpha, G_\beta \rangle_q. \]

On the unsymmetrized Fock space, it is more natural to use the inner product coming from \( K_q P \), that is,
\[ \langle \alpha \otimes F, \beta \otimes G \rangle_q = \delta_{n=k} \sum_{\sigma \in S(n)} \chi_q(\alpha \sigma \beta^{-1} \sigma^{-1}) \langle F, U_\sigma G \rangle_{\mathcal{H}^{\otimes n}}. \]

For this inner product, \( P_n \) is an isometric projection.

The inner product (6) is positive semi-definite on \( \overline{TP}(\mathcal{H}) \) for \( q \in \mathbb{Z} \). It is not in general positive definite. However, for \( q = 0 \),
\[ \left\langle \sum_{\alpha \in S_0(n)} \alpha \otimes_s F_\alpha, \sum_{\beta \in S_0(n)} \beta \otimes_s G_\beta \right\rangle_0 = n! \sum_{\alpha, \beta \in S_0(n)} \langle F_\alpha, G_\beta \rangle_q, \]
and so the 0-inner product on \( \overline{TP}(\mathcal{H}) \) is positive definite.

**Remark 4.2.** In [GM02], Guță and Maassen considered a general Fock space construction. Let \( \mathcal{H} \) and \( \{ V_n : n \in \mathbb{N} \} \) be Hilbert spaces, such \( S(n) \) acts unitarily on \( V_n \). Then one can define \( V_n \otimes_s \mathcal{H}^{\otimes n} \) as the fixed point subspace of the action
\[ v \otimes F \mapsto (\sigma \cdot v) \otimes U_\sigma F, \]
and a symmetrized Fock space as the orthogonal sum of these subspaces. In our case, \( V_n = \mathbb{C}[S_0(n)] \), with the inner product induced by \( \chi_q \), on which \( S(n) \) acts by conjugation (which preserves \( \chi_q \)). One can then define the creation operator based on a sequence of maps \( j_n : V_n \to V_{n+1} \).
which commute with the action of $S(n)$; the annihilation operator as its adjoint; and study the algebra generated by field operators. In our setting, there are several natural equivariant choices of the map $j_n$. One possibility is the standard embedding $\mathbb{C}[S_0(n)] \hookrightarrow \mathbb{C}[S_0(n+1)]$. Another possibility is the linear extension of the map $\alpha \mapsto (0 \ n + 1)\alpha$. See Sections 5.1.1 and 5.1.3. The algebra considered throughout most of the article is much larger than the subalgebras generated by these field operators; in fact, the vacuum vector is cyclic for it.

**Proposition 4.3.** For $q \in \mathbb{Z}$, the kernel of the inner product is

$$N_{vs,q} = \left\{ \xi \in \mathcal{T}P(H) : \langle \xi, \xi \rangle_q = 0 \right\} = \text{Span} (\eta \otimes_s F : \eta \in N_{gr,q}, F \in \mathcal{F}_f(H)),$$

where $\mathcal{F}_f(H)$ is the full Fock space of $H$.

**Proof.** The kernel of $K_q$ as an operator on $\mathbb{C}[S_0(n)] \otimes H \otimes n$ with the (tensor) inner product (4) is clearly $N_{gr,q,n} \otimes \mathcal{H}^{\otimes n}$. The kernel of the inner product on $\mathcal{T}P(H)$ is the intersection of the kernel of $K_q$ and $\mathcal{T}P(H)$. □

**Definition 4.4.** For $q \in \mathbb{Z}$, denote

$$\mathcal{T}P_q(H) = \mathcal{T}P(H)/N_{vs,q}.$$

In particular,

$$\mathcal{T}P_{1/N}(H) = \bigoplus_{n=0}^{N-1} \left( \mathbb{C}[S_0(n)] \otimes_s \mathcal{H}^{\otimes n} \right) + \bigoplus_{n=N}^{\infty} \left( \mathbb{C}[S_0(n)] \otimes_s \mathcal{H}^{\otimes n} \right).$$

Denote by $\mathcal{F}_q(H)$ the completion of $\mathcal{T}P_q(H)$ with respect to the inner product $\langle \cdot, \cdot \rangle_q$.

Note that $\mathcal{T}P_q(\mathbb{C})$ is not $\mathcal{T}P_q$ from Proposition 3.5, but its symmetrized version,

$$\mathcal{T}P_q(\mathbb{C}) = \bigoplus_{n=0}^{\infty} Z(\mathbb{C}[S_0(n)] : \mathbb{C}[S(n)])/N_{gr,q}.$$

**Lemma 4.5.**

(a) If $F_i, F \in \mathcal{H}^{\otimes n}$ and $\|F_i - F\| \to 0$, then $\|\alpha \otimes F_i - (\alpha \otimes F)\|_q^2 \to 0$.

(b) If $H$ is infinite-dimensional, the linear span of the elements of the form $(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))$ for mutually orthogonal $h_1, \ldots, h_n$ is dense in $\mathcal{F}_q(H)$.

**Proof.** For part (a), note that

$$\|\alpha \otimes F\|_q^2 = \sum_{\sigma \in S(n)} \chi_q(\alpha \sigma \alpha^{-1} \sigma^{-1}) \langle F, U_\sigma F \rangle \leq n! \|F\|^2.$$

Part (b) follows from the fact that for infinite-dimensional $H$, the linear span of the elements of the form $h_1 \otimes \ldots \otimes h_n$ for mutually orthogonal $h_1, \ldots, h_n$ is dense in $\mathcal{H}^{\otimes n}$. □

**Remark 4.6.** Recall that we denote by $\mathcal{H}^{\otimes n}$ the algebraic tensor product. Then we could equally well consider the Fock space

$$\mathbb{C}(0) \oplus \bigoplus_{n=1}^{\infty} (\mathbb{C}[S_0(n)] \otimes \mathcal{H}^{\otimes n}),$$
its symmetrized subspace

\[ TP(H) = \mathbb{C}(0) \oplus \bigoplus_{n=1}^{\infty} \left( \mathbb{C}[S_0(n)] \otimes_s H^{\otimes n} \right), \]

and its quotient \( TP_q(H) \) by the kernel of \( K_q \) for \( q \in \mathcal{Z} \). By the preceding lemma, \( TP(H) \) is dense in \( TP(H) \) (with respect to the seminorm), and \( TP_q(H) \) is dense in \( TP_q(H) \). In particular, \( F_q(H) \) is the completion of either set.

**Theorem 4.7** (Chaos decomposition I). Let \( \{\xi_i : i \in \Xi\} \) be an orthonormal basis for \( H \), where \( \Xi = [d] \) or \( \Xi = \mathbb{N} \). Denote

\[ \Delta(\Xi^n) = \{ u \in \Xi^n : u(1) \leq u(2) \leq \ldots \leq u(n) \}, \]

and for \( u \in \Xi^n \), denote \( \ker(u) \) the interval partition \( \pi = (I_1, \ldots, I_k) \in \text{Int}(n) \) such that \( u(i) = u(j) \iff i \sim j \). Finally, denote the centralizer

\[ Z(\mathbb{C}[S_0(n)] : \pi) = Z(\mathbb{C}[S_0(n)] : S(I_1) \times \ldots \times S(I_k)). \]

(a) We have a decomposition

\[ TP(H) = \bigoplus_{n=0}^{\infty} \bigoplus_{u \in \Delta(\Xi^n)} Z(\mathbb{C}[S_0(n)] : \ker(u)) \otimes_s (\xi_{u(1)} \otimes \ldots \otimes \xi_{u(n)}) \]

which is orthogonal with respect to any \( q \)-inner product.

(b) Any \( A \in F_{1/N}(H) \) has a unique decomposition

\[ A = \sum_{n=0}^{\infty} \sum_{u \in \Xi^n} \eta_u \otimes_s \xi_u, \]

where \( \eta_u \in Z(\mathbb{C}[S_0(n)] : \ker(u)) \cap \mathbb{C}[S_0(n)]_{\leq N} \), and

\[ \langle A, A \rangle_q = \sum_{n=0}^{\infty} \sum_{u \in \Xi^n} \ker(u)! \chi_q[\eta_u^* \eta_u] < \infty, \]

where as usual \( \pi! = \prod_{V \in \pi} |V|! \).

(c) For each \( n \), for sufficiently large \( N \), \( \mathbb{C}[S_0(n)]_{\leq N} \otimes_s H^{\otimes n} \) is complete with respect to the \( \frac{1}{N} \)-norm.

**Proof.** The span of the vectors of the form \( \eta \otimes (\xi_{u(1)} \otimes \ldots \otimes \xi_{u(n)}) \) is dense in the left-hand side of equation (9). Using invariance (3), we first see that we may take \( u(1) \leq u(2) \leq \ldots \leq u(n) \). Choosing \( \pi = \ker(u) \), we further see that \( \xi_u \) is invariant under the action of \( S(I_1) \times \ldots \times S(I_k) \). So we may take \( \eta \) to be invariant under the corresponding action.

For orthogonality, we observe that if \( u, v \in \Delta(\Xi^n) \),

\[ \langle \eta \otimes \xi_u, \xi \otimes \xi_v \rangle_q = \sum_{\sigma \in S(n)} \chi_q[\eta \sigma^* \sigma^{-1}] \langle \xi_{u(1)} \otimes \ldots \otimes \xi_{u(n)}, \xi_{v(\sigma^{-1}(1))} \otimes \ldots \otimes \xi_{v(\sigma^{-1}(n))} \rangle \]

\[ = \delta_{u=v} \sum_{\sigma \in S(I_1) \times \ldots \times S(I_k)} \chi_q[\eta \sigma^* \sigma^{-1}] \]

\[ = \ker(u)! \chi_q[\eta^* \xi^*]. \]
Part (b) follows from the fact that each subspace \( Z(\mathbb{C}[S_0(n)] : \ker(u)) \otimes (\xi_{u(1)} \otimes \ldots \otimes \xi_{u(n)}) \) is finite dimensional, and thus closed.

For (c), note that
\[
\left\| \sum_{\alpha \in S_0(n)} \alpha \otimes_s F_{\alpha} \right\|_q^2 = \sum_{\alpha, \beta \in S_0(n)} \chi_q[\alpha \beta^{-1}] \langle F_\alpha, F_\beta \rangle \\
\geq \sum_{\alpha \in S_0(n)} \| F_\alpha \|^2 - |q| \sum_{\alpha \neq \beta \in S_0(n)} \| F_\alpha \| \| F_\beta \| \\
\geq (1 - |q|) (n + 1)! \sum_{\alpha \in S_0(n)} \| F_\alpha \|^2 + \frac{1}{2} |q| \sum_{\alpha, \beta \in S_0(n)} (\| F_\alpha \| - \| F_\beta \|)^2,
\]
and so for \(|q| < \frac{1}{(n+1)!}\), the norm \( \left\| \sum_{\alpha \in S_0(n)} \alpha \otimes_s F_{\alpha} \right\|_q \) is equivalent to \( \sqrt{\sum_{\alpha \in S_0(n)} \| F_\alpha \|^2} \). \( \square \)

In two special cases, we have alternative chaos decompositions. The first one follows from the comments in Section 2.2.

**Proposition 4.8** (Chaos decomposition II). Let \( \mathcal{H} = \mathbb{C} \), so that \( \mathcal{T}\mathcal{P}(\mathbb{C}) = Z(\mathbb{C}[S_0(n)] : \mathbb{C}[S(n)]) \). Then \( \{ \chi^{x,\lambda} : \lambda \in \text{Par}(n), x = \lambda + \square \} \) are orthogonal and span \( \mathcal{T}\mathcal{P}(\mathbb{C}) \). Moreover,\n\[
\left\{ \chi^{x',\lambda} : x' = \lambda + \square, \lambda \in \text{Par}(n+1; \leq N) \right\}
\]
is an orthogonal basis for \( \mathcal{F}_{1/N}(\mathbb{C}) \).

**Notation 4.9.** In the case \( \mathcal{H}_\mathbb{R} = L^2(\mathbb{R}_+, dx) \), denote
\[
\Delta(\mathbb{R}^n_+) = \{ (t_1, \ldots, t_n) \in \mathbb{R}^n : t_1 \leq t_2 \leq \ldots \leq t_n \}.
\]

**Proposition 4.10** (Chaos decomposition III). Let \( \mathcal{H}_\mathbb{R} = L^2(\mathbb{R}_+, dx) \).

(a) We have a decomposition
\[
\mathcal{T}\mathcal{P}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \bigoplus_{\lambda \in \text{Par}(n+1)} \bigoplus_{i,j=1} d_\lambda W(E_{ij}^\lambda) \otimes_s L^2(\Delta(\mathbb{R}^n_+) \otimes \mathcal{H}_\mathbb{R}),
\]
which is orthogonal with respect to any \( q \)-inner product. Here we use the notation from Section 2.2.

(b) Any \( A \in \mathcal{F}_{1/N}(\mathcal{H}) \) has a unique decomposition
\[
A = \sum_{n=0}^{\infty} \sum_{\lambda \in \text{Par}(n+1; \leq N)} \sum_{i,j=1} d_\lambda W(E_{ij}^\lambda) \otimes_s F_{ij}^\lambda,
\]
where \( F_{ij}^\lambda \in L^2(\Delta(\mathbb{R}^n_+) \otimes \mathcal{H}_\mathbb{R}) \) and
\[
\sum_{n=0}^{\infty} \sum_{\lambda \in \text{Par}(n+1; \leq N)} \sum_{i,j=1} d_\lambda \| F_{ij}^\lambda \|^2 < \infty
\]
for \( n_\lambda = \frac{|SS(N)|}{N^{n+1}} \). For \( A \in \mathcal{F}_0(\mathcal{H}) \) the same decomposition holds with no restrictions on \( \lambda \) and \( n_\lambda = \frac{d_\lambda}{(n+1)^2} \). However, in that case we also have the simpler isometry (8).
Proof. Every element in the $n$’th component of $\mathcal{T}\mathcal{P}(\mathcal{H}_\mathbb{R})$ is equivalent to a unique element of the form

$$\sum_{\lambda \in \text{Par}(n+1)} \sum_{i,j=1}^{d_\lambda} W(E_{ij}^\lambda) \otimes_s F_{ij}^\lambda$$

for some $F_{ij}^\lambda \in L^2(\Delta(\mathbb{R}^n_+), dx^\otimes n)$. For $F, G \in L^2(\Delta(\mathbb{R}^n_+), dx^\otimes n)$,

$$\langle W(E_{ij}^\lambda) \otimes_s F, W(E_{k\ell}^\mu) \otimes_s G \rangle = \sum_{\sigma \in S(n)} \chi_q[W(E_{ij}^\lambda)\sigma W(E_{k\ell}^\mu)^*\sigma^{-1}] \langle F, U_\sigma G \rangle$$

$$= \chi_q[W(E_{ij}^\lambda)W(E_{k\ell}^\mu)^*] \langle F, G \rangle$$

$$= \delta_{i=k}\delta_{j=\ell}\delta_{\lambda=\mu}n_\lambda \langle F, G \rangle. \quad \square$$

5. The operator algebra

We now define a star-algebra structure on $\mathcal{T}\mathcal{P}(\mathcal{H}_\mathbb{R})$, and eventually on $\mathcal{T}\mathcal{P}(\mathcal{H}_\mathbb{R})$. To distinguish the elements of the algebra from the corresponding elements of the inner product space, we will denote $\alpha \otimes_s F$ by $1(\alpha \otimes_s F)$. Note that this identification differs from the one in Section 3.2.

Definition 5.1. For a transposition $\tau = (ij) \in S(n)$, define the $\tau$-contraction on $\mathcal{H}_\mathbb{R}^\otimes n$ by the linear extension of

$$C_\tau(h_1 \otimes \ldots \otimes h_n) = \langle h_i, h_j \rangle h_1 \otimes \ldots \otimes \hat{h}_i \otimes \ldots \otimes \hat{h}_j \otimes \ldots \otimes h_n,$$

More generally, for $\pi \in S(n)$ with the cycle structure $\pi \in \mathcal{P}_{1,2}(n)$,

$$\pi = \{(v_1, w_1), \ldots, (v_\ell, w_\ell), (u_1) \ldots (u_{n-2\ell})\},$$

with $u_1 < u_2 < \ldots < u_{n-2\ell}$, define the contraction

$$C_\pi(h_1 \otimes \ldots \otimes h_n) = \prod_{i=1}^{\ell} \langle h_{v(i)}, h_{w(i)} \rangle h_{u_1} \otimes \ldots \otimes h_{u_{n-2\ell}}$$

For $q \neq 0$, denote $C_\tau(\alpha \otimes F) = C_\tau(\alpha) \otimes C_\tau(F)$ following Definition 3.9.

Remark 5.2. The (tensor) contraction is not a bounded operator and so does not extend to the Hilbert space tensor product $\mathcal{H}_\mathbb{R}^\otimes n$. But if $\pi \in \mathcal{P}(n, k)$, it is easy to check that

$$C_\pi : \mathcal{H}_\mathbb{R}^\otimes n \times \mathcal{H}_\mathbb{R}^\otimes n \to \mathcal{H}_\mathbb{R}^\otimes n+k-2$$

is a contraction.

Lemma 5.3.

$$C_\pi(\sigma\alpha\sigma^{-1} \otimes U_\sigma F) = \tilde{\sigma}C_{\tilde{\pi}}(\alpha)\tilde{\sigma}^{-1} \otimes U_\tilde{\sigma}C_{\tilde{\pi}}(F),$$

where $\tilde{\pi} = \sigma^{-1}\pi\sigma$ and

$$\tilde{\sigma} = P_{[0,n]}^{\supp(\pi)}\sigma F_{[0,n-2\ell]}^{\supp(\tilde{\pi})} \in S(n-2\ell).$$

Proof. The relation

$$C_\pi(U_\sigma F) = U_\tilde{\sigma}C_{\sigma^{-1}\pi\sigma}F$$
is not hard to check. For the second relation, we compute
\[
C_\pi(\sigma\alpha\sigma^{-1}) = q^{\text{cyc}_0((\pi\sigma\alpha\sigma^{-1})_{\supp(\pi)^c}) - \text{cyc}_0((\pi\sigma\alpha\sigma^{-1})_{\supp(\pi)^c}) + \ell P_{[0,n] \setminus \supp(\pi)}([0,n-2\ell] \setminus \supp(\pi)^c)}
\]
\[
= q^{\text{cyc}_0((\pi\alpha)_{\supp(\pi)^c}) - \text{cyc}_0((\pi\alpha)_{\supp(\pi)^c}) + \ell P_{[0,n] \setminus \supp(\pi)}([0,n-2\ell] \setminus \supp(\pi)^c)}
\]
\[
= q^{\text{cyc}_0((\pi\alpha)_{\supp(\pi)^c}) - \text{cyc}_0((\pi\alpha)_{\supp(\pi)^c}) + \ell P_{[0,n] \setminus \supp(\pi)}([0,n-2\ell] \setminus \supp(\pi)^c)}
\]
\[
= \tilde{\sigma}C_{\tilde{\pi}}(\alpha)\tilde{\sigma}^{-1}.
\]

Lemma 5.4. We keep the notation \(\mathcal{L}_n, \mathcal{L}, \mathcal{L}_{n,k}, \mathcal{L}_{n,k}^{(\ell)}\) as in Notation\[\text{3.12}\] Then \(\mathcal{L}\) descends to a map on \(\mathcal{TP}(\mathcal{H}_R)\), and \(\mathcal{L}_{n,k}^{(\ell)}\) to a map \((\mathbb{C}[S_0(n)] \otimes_s \mathcal{H}_R^{\otimes n}) \times (\mathbb{C}[S_0(k)] \otimes_s \mathcal{H}_R^{\otimes k}) \rightarrow \mathbb{C}[S_0(n+k-2\ell)] \otimes_s \mathcal{H}_R^{\otimes n+k-2\ell}\).

Proof. \(\mathcal{L}\) is invariant under the action of \(S(n)\), and \(\mathcal{L}_{n,k}\) under the action of \(S(n) \times S(k)\).

Definition 5.5. For \(\eta \otimes F \in \mathbb{C}[S_0(n)] \otimes \mathcal{H}_R^{\otimes n}\), define
\[
T(\eta \otimes F) = I\left(e^\mathcal{L}(\eta \otimes F)\right) = \sum_{k=0}^{\infty} \frac{1}{k!} I\left(\mathcal{L}^k(\eta \otimes F)\right) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} I\left(C_\pi(\eta \otimes F)\right).
\]

Then \(T\) is also well-defined on \(\mathcal{TP}(\mathcal{H}_R)\). Note that we cannot in general extend it to \(\overline{\mathcal{TP}(\mathcal{H}_R)}\).

The following result follows immediately from Proposition\[\text{3.14}\]

Proposition 5.6. For \(\alpha \in S_0(n)\),
\[
I(\alpha \otimes F) = T\left(e^{-\mathcal{L}}(\alpha \otimes F)\right) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} T\left(\mathcal{L}^k(\alpha \otimes F)\right)
\]
\[
= \sum_{\pi \in \mathcal{P}_{1,2}(n)} (-1)^n \text{supp}(\tilde{\pi}) T\left(C_\pi(\alpha \otimes F)\right).
\]

Definition 5.7. Define the star-algebra structure on \(\mathcal{TP}(\mathcal{H}_R)\) by
\[
T(\alpha \otimes_s F) T(\beta \otimes_s G) = T((\alpha \cup \beta) \otimes_s (F \otimes G))
\]
and
\[
T(\alpha \otimes_s (h_1 \otimes \ldots \otimes h_n))^* = T(\alpha^{-1} \otimes_s (h_1 \otimes \ldots \otimes h_n)).
\]

Proposition 5.8. The multiplication on \(\mathcal{TP}(\mathcal{H})\) is well defined, and
\[
(T(\alpha \otimes_s F) T(\beta \otimes_s G))^* = T(\beta \otimes_s F)^* T(\alpha \otimes_s G)^*.
\]

Proof. For \(\sigma, \tau \in S(n)\), taking \(\rho = \sigma\sigma^{-1}T\sigma\),
\[
T\left(\sigma\alpha\sigma^{-1} \otimes U_\sigma F\right) T\left(\tau \beta^{-1} \otimes U_\tau G\right)
\]
\[
= T\left(\sigma^{-1} T\beta^{-1} \sigma_{n,k} \sigma\alpha\sigma^{-1} \otimes (U_\sigma F \otimes U_\tau G)\right)
\]
\[
= T\left(\sigma^{-1} T\beta^{-1} \sigma_{n,k} \sigma\alpha\sigma^{-1} \otimes (U_\sigma F \otimes U_\tau G)\right)
\]
\[
= T\left(\sigma^{-1} T\beta^{-1} \sigma_{n,k} \sigma\alpha\sigma^{-1} \otimes (U_\sigma F \otimes U_\tau G)\right)
\]
\[
= T\left(\rho(\alpha \cup \beta) \rho^{-1} \otimes U_\rho (F \otimes G)\right).
\]
Similarly,

\[ T \left( \beta^{-1} \otimes_s (g_1 \otimes \ldots \otimes g_k) \right) T \left( \alpha^{-1} \otimes_s (f_1 \otimes \ldots \otimes f_n) \right) \]

\[ = T \left( \sigma_{n,k} \alpha^{-1} \sigma_{n,k}^{-1} \otimes_s (g_1 \otimes \ldots \otimes g_k \otimes f_1 \otimes \ldots \otimes f_n) \right) \]

\[ = T \left( \sigma_{n,k} (\alpha \cup \beta)^{-1} \sigma_{n,k}^{-1} \otimes_s (g_1 \otimes \ldots \otimes g_k \otimes f_1 \otimes \ldots \otimes f_n) \right) \]

\[ = T \left( (\alpha \cup \beta)^{-1} \otimes_s (f_1 \otimes \ldots \otimes f_n \otimes g_1 \otimes \ldots \otimes g_k) \right). \]

\[ \Box \]

**Proposition 5.9.** For \( q \in \mathbb{Z} \setminus \{0\} \), \( N_{vs,q} \) is an ideal for multiplication in Definition\( \ref{def:ideal} \) and is preserved by each \( C_q \). Consequently, \( \mathcal{L} \) is defined as a map on \( TP_q(\mathcal{H}_\mathbb{R}) \), and \( \mathcal{L}^{(\ell)} \) on the appropriate quotient. Also, \( T(\eta) \) is well-defined for \( \eta \in TP_q(\mathcal{H}_\mathbb{R}) \).

**Proof.** Apply Lemma\( \ref{lem:ideal} \) and Propositions\( \ref{prop:ideal} \) and\( \ref{prop:ideal} \).

The following result also follows from Proposition\( \ref{prop:ideal} \).

**Proposition 5.10.** Let \( \alpha \in S_0(n) \), \( \beta \in S_0(k) \), \( F \in \mathcal{H}_{\mathbb{R}}^\otimes_n \), \( G \in \mathcal{H}_{\mathbb{R}}^\otimes_k \). Then

\[ I(\alpha \otimes F) I(\beta \otimes G) = I(\alpha \cup \beta \otimes (F \otimes G)) + \sum_{\ell=1}^{\min(n,k)} \frac{1}{\ell!} \left( \mathcal{L}^{(\ell)}_{n,k}((\alpha \cup \beta) \otimes (F \otimes G)) \right) \]

\[ = \sum_{\pi \in \mathcal{P}_{1,2}(n,k)} I(C_\pi(\alpha \cup \beta \otimes (F \otimes G))). \]

**Proposition 5.11.**

(a) By abuse of notation, define \( \mathcal{L} T(\eta \otimes F) = T(\mathcal{L}(\eta \otimes F)) \). Then also \( \mathcal{L} I(\eta \otimes F) = I(\mathcal{L}(\eta \otimes F)) \).

(b) Define the Euler operator on \( TP(\mathcal{H}_\mathbb{R}) \) by

\[ ET(\eta \otimes F) = nT(\eta \otimes F) \]

for \( \eta \in \mathbb{C} S_0(n) \). Then for such \( \eta \),

\[ (E - 2L)I(\eta \otimes F) = nI(\eta \otimes F), \]

and it is the unique eigenfunction of this operator with eigenvalue \( n \) and leading term \( T(\eta \otimes F) \). In particular, it follows that \( E \) maps \( TP_q(\mathcal{H}_\mathbb{R}) \) to itself.

**Proof.** Part (a) follows from the expansion in Proposition\( \ref{prop:ideal} \). For part (b), we first note that

\[ EI(\eta \otimes F) - 2I(\mathcal{L}(\eta \otimes F)) \]

\[ = E \sum_{k=0}^{n} \frac{(-1)^k}{k!} T\left( \mathcal{L}^k(\alpha \otimes F) \right) - 2 \sum_{k=0}^{n} \frac{(-1)^k}{k!} T\left( \mathcal{L}^{k+1}(\alpha \otimes F) \right) \]

\[ = \sum_{k=0}^{n} \frac{(-1)^k(n - 2k)}{k!} T\left( \mathcal{L}^k(\alpha \otimes F) \right) + 2 \sum_{k=1}^{n} \frac{(-1)^k}{k!} T\left( \mathcal{L}^k(\alpha \otimes F) \right) \]

\[ = \sum_{k=0}^{n} \frac{(-1)^k n}{k!} T\left( \mathcal{L}^k(\alpha \otimes F) \right) \]

\[ = nI(\alpha \otimes F). \]
In particular, anything of lower degree is in the sum of eigenspaces with eigenvalues 0, 1, \ldots, n - 1. It follows that specifying the leading term of an eigenfunction with a given eigenvalue determines it.

**Notation 5.12.** If $C_\pi(\eta \otimes F)$ is a scalar multiple of (0) (which is the identity for the algebra), we will identify it with a scalar.

**Theorem 5.13.** Define a unital linear functional on $\bigoplus_{n=0}^{\infty} \mathbb{C}[S_0(n)] \otimes \mathcal{H}_\mathbb{R}^\otimes n$ by

$$\varphi[I(\alpha \otimes F)] = 0, \quad \varphi[I((0))] = 1.$$  

Then

\begin{itemize}
  \item[(a)] $\varphi[I(\beta \otimes G)^* I(\alpha \otimes F)] = \langle (\alpha \otimes F), (\beta \otimes G) \rangle_q$.
  
  In particular, $*$ is the adjoint with respect to this inner product, and $\varphi$ is well-defined on the quotient $TP_q(\mathcal{H}_\mathbb{R})$.
  \item[(b)] $\varphi$ is tracial.
  \item[(c)] $\varphi$ is positive for $q \in \mathbb{Z}$.
  \item[(d)] $\{A : \varphi[A^* A] = 0\} = \text{Span} \left( I(\zeta) : \zeta \in N_{vs,q} \right)$,
  
  and $\varphi$ is faithful $TP_q(\mathcal{H}_\mathbb{R})$.
  \item[(e)] For $\alpha \in S_0(2n)$,

$$\varphi[T(\alpha \otimes F)] = \sum_{\pi \in P_2(n,2n)} q^{n-\text{cyc}_0(\pi \alpha)} C_\pi(F) = \sum_{\pi \in P_2(n,2n)} q^{\text{cyc}_1 C_\pi(F)}.$$  

and it is zero if $\alpha \in S_0(2n + 1)$
\end{itemize}

**Proof.** For $\alpha \in S_0(n)$, $\beta \in S_0(k)$, $\varphi[I(\alpha \otimes F)I(\beta \otimes G)] = 0$ if $n \neq k$. If $n = k$, using Proposition 5.10 the definition of $\varphi$, and Definition 3.9

$$\varphi[I(\beta \otimes G)^* I(\alpha \otimes F)] = \sum_{\pi \in P_2(n,n)} I(C_\pi((\beta^{-1} \cup \alpha) \otimes (G \otimes F)))$$

$$= \sum_{\pi \in P_2(n,n)} q^{n-\text{cyc}_0(\pi(\beta^{-1} \cup \alpha))} C_\pi(G \otimes F).$$

The map $\sigma \mapsto \sigma^{-1} \sigma_{n,n} \sigma$ maps $S(n)$ bijectively onto $P_2(n, n)$. Clearly

$$C_{\sigma^{-1} \sigma_{n,n} \sigma}(G \otimes F) = \langle G, U_{\sigma^{-1}} F \rangle = \langle F, U_{\sigma} G \rangle.$$  

Moreover each cycle of

$$\sigma^{-1} \sigma_{n,n} \sigma(\beta^{-1} \cup \alpha) = \sigma^{-1} \sigma_{n,n} \sigma \sigma_{n,n} \sigma_{n,n} \alpha \sigma_{n,n} \sigma_{n,n} \beta^{-1}$$

intersects $[0, n]$. A computation shows that its restriction to $[0, n]$ is $\sigma^{-1} \alpha \sigma \beta^{-1}$. Therefore

$$q^{n-\text{cyc}_0(\sigma^{-1} \sigma_{n,n} (\beta^{-1} \cup \alpha))} = q^{n-\text{cyc}_0(\sigma^{-1} \alpha \sigma \beta^{-1})} = \chi_q(\sigma^{-1} \alpha \sigma \beta^{-1})$$

It follows that

$$\varphi[I(\beta \otimes F)^* I(\alpha \otimes G)] = \sum_{\sigma \in S(n)} \chi_q(\sigma^{-1} \alpha \sigma \beta^{-1}) \langle F, U_{\sigma} G \rangle_{\mathcal{H}_\mathbb{R}^\otimes n} = \langle (\alpha \otimes F), (\beta \otimes G) \rangle_q.$$
Since $\chi_q[\beta^{-1}\alpha] = \chi_q[\alpha^{-1}\beta]$, (b) follows from (a), as do (c) and (d). For (e), using the expansion in Proposition 5.6(a),

$$\varphi [T(\alpha \otimes F)] = \sum_{\pi \in \mathcal{P}_2(2n)} I(C_\pi(\alpha \otimes F)) = \sum_{\pi \in \mathcal{P}_2(2n)} q^{n-\text{cyc}}(\pi, \alpha) C_\pi(F). \quad \Box$$

**Proposition 5.14.** Let $\alpha \in S_0(n)$, $\beta \in S_0(k)$, $F \in \mathcal{H}_R^{\otimes n}$, $G \in \mathcal{H}_R^{\otimes k}$. Then

$$\|I(\alpha \otimes F)I(\beta \otimes G)\varphi \leq (n + k)! \|F\| \|G\|. \quad \Box$$

Consequently, the star-algebra structure extends to $\overline{\mathcal{T}\mathcal{P}}(\mathcal{H})$ and $\overline{\mathcal{T}\mathcal{P}}_q(\mathcal{H})$.

**Proof.** Combining Proposition 5.10 with the estimate in Lemma 4.5

$$\|I(\alpha \otimes F)I(\beta \otimes G)\varphi \leq (n + k)! |\mathcal{P}_{1,2}(n, k)| \|F\| \|G\|. \quad \Box$$

$|\mathcal{P}_{1,2}(n, k)|$ is sequence A086885 in OEIS, and has an easy estimate

$$|\mathcal{P}_{1,2}(n, k)| = \sum_{\ell=0}^{\min(n, k)} \ell! \binom{n}{\ell} \binom{k}{\ell} = \sum_{\ell=0}^{\min(n, k)} \frac{n!}{(n - \ell)!} \binom{k}{\ell} \leq n^k 2^k. \quad \Box$$

The proof of the following proposition is very similar to Proposition 5.10 and is omitted.

**Proposition 5.15.** For $\alpha_i \in S_0(n_i)$, and $F_i \in \overline{\mathcal{T}\mathcal{P}}(\mathcal{H})$,

$$\varphi [I(\alpha_1 \otimes_s F_1) \ldots I(\alpha_k \otimes_s F_k)] = \sum_{\pi \in \mathcal{P}_2(n_1, \ldots, n_k)} I(C_\pi((\alpha_1 \cup \ldots \cup \alpha_k) \otimes_s (F_1 \otimes \ldots \otimes F_k))).$$

Here $\mathcal{P}_2(n_1, \ldots, n_k)$ are the inhomogeneous pair partitions [dSCV85].

**Remark 5.16.** We have the (right) GNS representation of $\mathcal{T}\mathcal{P}(\mathcal{H}_R)$ on $L^2(\mathcal{T}\mathcal{P}(\mathcal{H}_R), \varphi) \simeq \mathcal{F}_q(\mathcal{H})$, for which the state vector $(0)$ is cyclic. For the corresponding representation of $\mathcal{T}\mathcal{P}_q(\mathcal{H}_R)$, it is cyclic and separating. It follows from Proposition 5.14 that this representation extends to $\overline{\mathcal{T}\mathcal{P}}_q(\mathcal{H}_R)$. Similarly, we have the left representation, which commutes with the right one, and for which $(0)$ is also cyclic.

**Proposition 5.17.** Let $\mathcal{H}' \subset \mathcal{H}_R$ be a closed subspace, and $P_{\mathcal{H}'} : \mathcal{H} \to \mathcal{H}'$ the orthogonal projection fixing $\mathcal{H}'_R$.

(a) The map defined by $\mathcal{F}(P_{\mathcal{H}'})(\alpha \otimes F) = (\alpha \otimes (P_{\mathcal{H}'_R} \otimes F))$ extends to the orthogonal projection $\mathcal{F}(P_{\mathcal{H}'_R}) : \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H}')$.

(b) The map $\Gamma(P_{\mathcal{H}'_R}) : \overline{\mathcal{T}\mathcal{P}}_q(\mathcal{H}_R) \to \overline{\mathcal{T}\mathcal{P}}_q(\mathcal{H}'_R)$ obtained by the linear extension of

$$\Gamma(P_{\mathcal{H}'_R})(I(\alpha \otimes F)) = I(\alpha \otimes (P_{\mathcal{H}'_R} \otimes F))$$

is an algebraic conditional expectation, which we will denote by $\varphi [\cdot | \mathcal{H}']$. In the GNS representation on $\mathcal{F}_q(\mathcal{H}')$, it is implemented by

$$\Gamma(P_{\mathcal{H}'_R})(I(\alpha \otimes F)) = \mathcal{F}(P_{\mathcal{H}'_R})I(\alpha \otimes F) \mathcal{F}(P_{\mathcal{H}'_R}).$$
Proof. We first verify that for $F \in \mathcal{H}^{\otimes n}$ and $G \in (\mathcal{H}')^{\otimes k}$,
\[
\langle \alpha \otimes F, \beta \otimes G \rangle_q = \delta_{n=k} \sum_{\sigma \in S(n)} \chi_q(\alpha \sigma \beta \sigma^{-1}) \langle F, U_\sigma G \rangle
\]
\[
= \delta_{n=k} \sum_{\sigma \in S(n)} \chi_q(\alpha \sigma \beta \sigma^{-1}) \langle P_{\mathcal{H}'}^{\otimes n} F, U_\sigma G \rangle
\]
\[
= \langle \alpha \otimes (P_{\mathcal{H}'}^{\otimes n} F), \beta \otimes G \rangle_q ,
\]
which implies part (a). Part (b) follows from Proposition 2.11. \hfill \square

Proposition 5.18. In the single-variable case, for $\alpha \in S_0(n)$, we have
\[
\varphi \left[ T \left( \alpha \otimes h^{\otimes n} \right) \right] = \sum_{\pi \in \mathcal{P}_{1,2}(n)} \| P_{(\mathcal{H}')}^{|\pi|} h \|^2 |\text{Pair}(\pi)| T \left( C_\pi(\alpha) \otimes (P_{\mathcal{H}'} h)^{\otimes |\text{Sing}(\pi)|} \right).
\]

Proof. We compute
\[
\varphi \left[ T \left( \alpha \otimes h^{\otimes n} \right) \right] = \sum_{\pi \in \mathcal{P}_{1,2}(n)} \| h \|^n |\text{Sing}(\pi)| I \left( C_\pi(\alpha) \otimes (P_{\mathcal{H}'} h)^{\otimes |\text{Sing}(\pi)|} \right)
\]
\[
= \sum_{\pi \in \mathcal{P}_{1,2}(n)} \| h \|^2 |\text{Pair}(\pi)|
\]
\[
\sum_{\sigma \in \mathcal{P}_{1,2}(\text{Sing}(\pi))} (-1)^{|\text{Pair}(\sigma)|} \| P_{\mathcal{H}'} h \|^2 |\text{Pair}(\sigma)| T \left( C_\sigma C_\pi(\alpha) \otimes (P_{\mathcal{H}'} h)^{\otimes |\text{Sing}(\sigma)|} \right)
\]
\[
= \sum_{\rho \in \mathcal{P}_{1,2}(n)} \sum_{S \subseteq \text{Pair}(\rho)} (-1)^{|S|} \| h \|^2 |\text{Pair}(\rho)| - |S| \| P_{\mathcal{H}'} h \|^2 |\text{Pair}(\rho)| T \left( C_\rho(\alpha) \otimes (P_{\mathcal{H}'} h)^{\otimes |\text{Sing}(\rho)|} \right)
\]
\[
= \sum_{\rho \in \mathcal{P}_{1,2}(n)} \left( \| h \|^2 - \| P_{\mathcal{H}'} h \|^2 \right)^{|\text{Pair}(\rho)|} T \left( C_\rho(\alpha) \otimes (P_{\mathcal{H}'} h)^{\otimes |\text{Sing}(\rho)|} \right).
\]

5.1. Three subalgebras. For $h \in \mathcal{H}_\mathbb{R}$, denote $\ell^+(h)$ the standard right creation operator on the full Fock space $\bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, and by $\ell^-_k(h)$ the annihilation operators
\[
\ell^-_k(h) F = C_{(k \ n+1)}(F \otimes h).
\]

5.1.1. Gaussian subalgebra. Denote
\[
\mathcal{G}(\mathcal{H}_\mathbb{R}) = \text{Alg} \left( T ((0)(1) \otimes h) : h \in \mathcal{H}_\mathbb{R} \right) = \text{Span} \left( T ((0)(1) \ldots (n) \otimes F) : h \in \mathcal{H}_\mathbb{R}^{\otimes n} \right).
\]

Theorem 5.19. In the right GNS representation from Remark 5.16, we may decompose
\[
I ((0)(1) \otimes h) = T ((0)(1) \otimes h) = a^+_0(1)(h) + a^-_0(1)(h),
\]
where
\[
a^+_0(1)(h)(\alpha \otimes F) = \alpha \otimes \ell^+(h) F.
\]
and
\[
a_{(0)(1)}^- (h)(\alpha \otimes F) = \sum_{k: (k) \in \alpha} P_{[0,n-1]}^\{0,n\}\{k\} \alpha |\{k\}\rangle \otimes \ell_k^- (h) F \\
+ q \sum_{k: (k) \not\in \alpha} P_{[0,n-1]}^\{0,n\}\{k\} \alpha |\{k\}\rangle \otimes \ell_k^- (h) F
\]

These operators are adjoints of each other.

The distribution of $I ((0)(1) \otimes h)$ is Gaussian with mean 0 and variance $\|h\|$.

**Proof.** According to Proposition 5.10
\[
I (\alpha \otimes F) I ((0)(1) \otimes h) \\
= I (\alpha \otimes \ell^+ (h) F) \\
+ \sum_{k=1}^n q^{cyc_\alpha (\alpha |\{k\}\rangle)} cyc_\alpha (\alpha) + 1 I \left( P_{[0,n-1]}^\{0,n\}\{k\} \alpha |\{k\}\rangle \otimes \ell_k^- (h) F \right) \\
= I (\alpha \otimes \ell^+ (h) F) \\
+ \sum_{k: (k) \in \alpha} I \left( P_{[0,n-1]}^\{0,n\}\{k\} \alpha |\{k\}\rangle \otimes \ell_k^- (h) F \right) + q \sum_{k: (k) \not\in \alpha} I \left( P_{[0,n-1]}^\{0,n\}\{k\} \alpha |\{k\}\rangle \otimes \ell_k^- (h) F \right)
\]

Also
\[
\varphi [I ((0)(1) \otimes h)^n] = \varphi [T ((0)(1) \ldots (n) \otimes h^\otimes n)] = \begin{cases} |P_2 (n)| \|h\|^n, & n \text{ even, } \\
0, & n \text{ odd. } \end{cases}
\]

Thus the distribution of $I ((0)(1) \otimes h)$ is Gaussian. Finally, since $T ((0)(1) \otimes h)$ is symmetric, $a_{(0)(1)}^+ (h)$ maps the $n$'th graded component into the $(n+1)$'st, and $a_{(0)(1)}^- (h)$ maps it into the $(n-1)$'st component, these operators have to be each other’s adjoints. $\square$

**Remark 5.20.** The subspace generated by this algebra’s action on $(0)$ is
\[
\text{Span } \{(0)(1) \ldots (n) \otimes s F : F \in \mathcal{H}_\mathbb{R}^\otimes n, n \in \mathbb{N} \}.
\]
Each element of this subspace has a unique permutation component, which can be dropped. Moreover, the action of $a_{(0)(1)}^\pm (h)$ on this subspace is independent of $q$, and the induced inner product is the usual symmetric inner product. Therefore this space is isomorphic to the symmetric Fock space $\mathcal{F}_s (\mathcal{H}_\mathbb{R})$, with the usual Bosonic creation and annihilation operators.

**Remark 5.21.** For $q = 1$, the inner product on the Fock space is
\[
\langle (\alpha \otimes F), (\beta \otimes G) \rangle_1 = \delta_{n=k} \sum_{\sigma \in S(n)} \langle F, U_\sigma G \rangle_{\mathcal{H}_\mathbb{R}^\otimes n}.
\]
So the non-degenerate quotient of the space is isomorphic to the symmetric Fock space $\mathcal{F}_s (\mathcal{H}_\mathbb{R})$, and $TP_1 (\mathcal{H}_\mathbb{R}) = \mathcal{G} (\mathcal{H}_\mathbb{R})$. 
5.1.2. Pure trace polynomial subalgebra.

**Theorem 5.22.**

Span \((I(\alpha \otimes_s F) : \alpha(0) = 0)\)

is the center of \(\overline{TP}(\mathcal{H}_\mathbb{R})\).

**Proof.** Note that

\[\sigma_{n,k}(\alpha \cup \beta)\sigma^{-1}_{n,k} = \beta \sigma_{n,k} \alpha \sigma^{-1}_{n,k}.\]

If \(\alpha \in S(n), \beta \in S_0(k)\), then \(\beta\) and \(\sigma_{n,k} \alpha \sigma^{-1}_{n,k}\) commute, and the expression above is \(\beta \cup \alpha\). Since also \(U_{\sigma_{n,k}}(F \otimes G) = G \otimes F\),

\[(\alpha \cup \beta) \otimes_s (F \otimes G) = (\beta \cup \alpha) \otimes_s (G \otimes F).\]

So using Lemma 5.3 for such \(\alpha\),

\[
I(\alpha \otimes_s F) I(\beta \otimes_s G) = \sum_{\pi \in \mathcal{P}_{1,2}(n,k)} I(C_{\pi}(\alpha \cup \beta) \otimes_s C_{\pi}(F \otimes G))
\]

\[
= \sum_{\pi \in \mathcal{P}_{1,2}(n,k)} I(C_{\pi}(\sigma_{n,k}((\beta \cup \alpha) \sigma^{-1}_{n,k} \otimes_s U_{\sigma_{n,k}}(G \otimes F))))
\]

\[
= \sum_{\pi \in \mathcal{P}_{1,2}(n,k)} I(C_{\sigma^{-1}_{n,k} \pi \sigma_{n,k}}((\beta \cup \alpha) \otimes_s (G \otimes F)))
\]

\[
= I(\beta \otimes_s G) I(\alpha \otimes_s F).
\]

For the converse, suppose that for some \(A \in \overline{TP}(\mathcal{H}_\mathbb{R}), A I((01) \otimes h) = I((01) \otimes h) A\) for all \(h \in \mathcal{H}\). \(A\) has the form

\[
A = \sum_{k=0}^{n} \sum_{\alpha \in S_0(k)} I(\alpha \otimes_s F_{\alpha}).
\]

It suffices to show that for each \(\alpha \in S_0(n), \alpha(0) \neq 0\) implies that \(F_{\alpha} = 0\) (since such a term can be subtracted from \(A\) with the difference still in the center). Comparing only the terms in the \((n+1)\)'th component, it follows that

\[
\sum_{\alpha \in S_0(n)} I(((\alpha \cup (01)) \otimes_s (F_{\alpha} \otimes h))) = \sum_{\beta \in S_0(n)} I(((01) \cup \beta) \otimes_s (h \otimes F_{\beta})).
\]

Recall that \(U_{\sigma_{1,n}}(h \otimes F_{\beta}) = F_{\beta} \otimes h\) and

\[\sigma_{1,n}((01) \cup \beta)\sigma^{-1}_{1,n} = \beta \sigma_{1,n}(01)\sigma^{-1}_{1,n} = \beta(0 n + 1).\]

Therefore

\[
\sum_{\alpha \in S_0(n)} I(((0 n + 1) \alpha) \otimes_s (F_{\alpha} \otimes h))) = \sum_{\beta \in S_0(n)} I(((\beta(0 n + 1)) \otimes_s (F_{\beta} \otimes h))).
\]

Suppose \(F_{\alpha} \neq 0\), and for some \(\sigma \in S(n + 1),\)

\[(0 n + 1) \alpha = \sigma \beta(0 n + 1)\sigma^{-1}\) and \(F_{\alpha} \otimes h = U_{\sigma}(F_{\beta} \otimes h).\)

Then from the second relation, in fact \(\sigma \in S(n)\). Applying the first relation to 0, it follows that \(\alpha(0) = 0.\) \(\square\)
5.1.3. Polynomial subalgebra.

**Remark 5.23.** Denote $X(h) = T ((01) \otimes h)$. Then
\[
X(h_1) \ldots X(h_n) = T ((01 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n)) .
\]

Since any two long cycles are conjugate,
\[
\mathcal{P}(\mathcal{H}_\mathbb{R}) = \text{Alg} (T ((01) \otimes h)) = \text{Span} (T (\alpha \otimes_s F) : \text{cyc}_0 (\alpha) = 0)
\]
and is a unital star-subalgebra of $\mathcal{T} \mathcal{P}(\mathcal{H}_\mathbb{R})$. It is not closed under contractions or conditional expectations. In particular, the corresponding $I ((01 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n))$ operators do not in general belong to this subalgebra.

Clearly $\mathcal{P}(\mathcal{H}_\mathbb{R})$ and the center $\text{Span} (T (\alpha \otimes_s F) : \alpha(0) = 0)$ together generate $\mathcal{T} \mathcal{P}(\mathcal{H}_\mathbb{R})$.

**Theorem 5.24.** Suppose $\mathcal{H}_\mathbb{R}$ is infinite-dimensional. Then $\mathcal{T} \mathcal{P}(\mathcal{H}_\mathbb{R})$ is generated (as an algebra) by the conditional expectations
\[
\{ \varphi [ A | \mathcal{H}'] : A \in \mathcal{P}(\mathcal{H}_\mathbb{R}), \mathcal{H}'_\mathbb{R} \subset \mathcal{H}_\mathbb{R} \} .
\]

*Proof.* We will use induction on $n$. For $n = 0$, $\alpha = (0)$. Suppose that for each $\beta \in S_0(k), k < n$, $T (\beta \otimes_s F)$ is in the algebra generated by the conditional expectations of elements of $\mathcal{P}(\mathcal{H}_\mathbb{R})$. It suffices to show that $T ((0)(1 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n))$ is in it. Let $h \perp \mathcal{H}'_\mathbb{R} = \text{Span} (h_1, \ldots, h_n)$ be a non-zero vector. Then
\[
\varphi [ X(h) T ((01 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n)) X(h) | \mathcal{H}']
\]
\[
= \varphi [ C_{(1 \ldots n+2)} I ((01 \ldots n+2) \otimes (h \otimes h_1 \otimes \ldots \otimes h_n \otimes h)) | \mathcal{H}']
\]
\[
= \|h\|^2 I ((0)(1 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n)) .
\]

Therefore using Definition 5.5,
\[
\varphi [ X(h) T ((01 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n)) X(h) | \mathcal{H}']
\]
\[
= \varphi [ X(h) (I ((01 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n)) + \text{lower order terms}) X(h) | \mathcal{H}']
\]
\[
= \|h\|^2 I ((0)(1 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n)) + \text{lower order terms}.
\]

The result follows. \qed

**Corollary 5.25.** The $\varphi$-preserving conditional expectation from $\mathcal{T} \mathcal{P}(\mathcal{H}_\mathbb{R})$ onto its center is the map
\[
I (\alpha \otimes_s F) \mapsto I (\alpha |_{(0)} \otimes_s F) .
\]

On $\mathcal{T} \mathcal{P}(\mathcal{H}_\mathbb{R})$ it is implemented by
\[
\varphi [ X(h) I (\alpha \otimes_s (h_1 \otimes \ldots \otimes h_n)) X(h) | \mathcal{H}'] ,
\]
where $\mathcal{H}'_\mathbb{R} = \text{Span} (h_1, \ldots, h_n)$ and $h \perp \mathcal{H}'_\mathbb{R}$ is a unit vector.

A similar representation holds for general contractions.

**Lemma 5.26.** Let $\pi = \{(v_1, w_1), \ldots, (v_\ell, w_\ell), (u_1) \ldots (u_{n-2\ell})\} \in \mathcal{P}_{1,2}(n)$, and $\mathcal{H}'_\mathbb{R} \subset \mathcal{H}_\mathbb{R}$ a closed subspace. Let $h_1, \ldots, h_n \in \mathcal{H}_\mathbb{R}$ be vectors such that
\begin{itemize}
  \item $h_{v_i} = h_{w_i}, 1 \leq i \leq \ell$.
  \item The vectors $\{h_{v_1}, \ldots, h_{v_\ell}\}$ are an orthonormal subset of $(\mathcal{H}'_\mathbb{R})^\perp$.
  \item $\{h_{u_1}, \ldots, h_{u_{n-2\ell}}\} \subset \mathcal{H}'_\mathbb{R}$.
\end{itemize}
Then
\[ \varphi \left[ T(\alpha \otimes (h_1 \otimes \ldots \otimes h_n)) \mid \mathcal{H}' \right] = T(C_\pi(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))). \]

**Proof.** Using the assumptions on the vectors,
\[ \varphi \left[ T(\alpha \otimes (h_1 \otimes \ldots \otimes h_n)) \mid \mathcal{H}' \right] = \varphi \left[ \sum_{\sigma \in \mathcal{P}_{1,2}(n)} I(C_\sigma(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))) \mid \mathcal{H}' \right] = \sum_{\sigma \in \mathcal{P}_{1,2}(n)} I(C_\sigma(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))) = \sum_{\rho \in \mathcal{P}_{1,2}(\mathcal{S}(\pi))} I(C_\rho C_\pi(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))) = T(C_\pi(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))). \]

**Proposition 5.27.** In its representation on \( \mathcal{F}_q(\mathcal{H}) \), for \( h \in \mathcal{H}_\mathbb{R} \), \( X(h) \) is essentially self-adjoint.

**Proof.** Clearly \( X(h) \) is symmetric. So it suffices to show that for each \( \alpha \in S_0(k) \), \( T(\alpha \otimes F)(0) \) is an analytic vector for it. Indeed,
\[
\frac{1}{n} \left\| X(h)^n T(\alpha \otimes F)(0) \right\|^{1/n} = \frac{1}{n} \varphi \left[ T(\alpha^{-1} \cup (01 \ldots n) \cup \alpha \otimes (\tilde{F} \otimes h^{\otimes 2n} \otimes F)) \right]^{1/2n} = \frac{1}{n} \left( \sum_{\pi \in \mathcal{P}_2(n+2k)} q^{(n+k)-\text{cyc}(\pi(\alpha^{-1} \cup (01 \ldots n) \cup \alpha)))} C_\pi(\tilde{F} \otimes h^{\otimes 2n} \otimes F) \right)^{1/2n} \leq \frac{1}{n} \left( \frac{(2n + 2k)!}{(2n+k)(n+k)!} \left\| F \right\|^2 \left\| h \right\|^{2n} \right)^{1/2n} \sim \frac{1}{n} 2^{1/2} (n + k)^{1/2} \epsilon^{-1/2} \left\| h \right\| \to 0. \]

**Theorem 5.28.** In the right GNS representation, we may decompose \( X(h) = a^+_{(01)}(h) + a^-_{(01)}(h) \), where
\[ a^+_{(01)}(h)(\alpha \otimes F) = (0n + 1)\alpha \otimes \ell^+(h)F \]
and
\[ a^-_{(01)}(h)(\alpha \otimes F) = q \sum_{k \neq \alpha^{-1}(0)} \mathcal{P}_{0,0}^{(0,n+1)}((0k)\alpha) \alpha^{c} \otimes \ell^-(h)F \]
where if \( \alpha(0) = 0 \), the final term is absent.
The distribution of \( 1((01) \otimes h) \) is the unnormalized average empirical distribution of a GUE matrix with mean 0 and variance \( \left\| h \right\| \).
correspondence is more subtle. For a pair partition \( \pi \) with the same openers and closers as \( \sigma \), moreover (Lemma 5.1), the creation and annihilation operators satisfy a commutation relation

\[
\text{Here } c, \text{ and the corresponding expression is zero for an odd number of factors (compare with equation (10)).}
\]

Also,

\[
\varphi [I ((01) \otimes h)^n] = \varphi [T ((01 \ldots n) \otimes h^\otimes n)] = \begin{cases} 
\sum_{\pi \in \mathcal{P}_2(n)} q^{n/2 - cyco((0 \ldots n)\pi)} \|h\|^n, & n \text{ even}, \\
0, & n \text{ odd},
\end{cases}
\]

which should be compared with Theorem 22.12 in [NS06].

5.2. The relation to a construction by Bożejko and Guţă. We contrast the algebra \( \mathcal{P}(\mathcal{H}_\mathbb{R}) \) with a construction from [BG02]. In section 5 of that paper, Bożejko and Guţă considered the Fock space with the inner product

\[
\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_k \rangle_q = \delta_{n=k} \sum_{\sigma \in S(n)} \chi_q[\sigma] \prod_{i=1}^n \langle f_i, g_{\sigma(i)} \rangle
\]

for \( q = \pm \frac{1}{N} \). On this space, they defined the creation operator \( a^+(h) \) in the usual way, and the annihilation operator as its adjoint, which comes out to be

\[
a^-(h)(h_1 \otimes \ldots \otimes h_n) = \langle h_1, h \rangle + q \sum_{k=2}^n \langle h_k, h \rangle (h_2 \otimes \ldots \otimes h_{i-1} \otimes h_1 \otimes h_{i+1} \otimes \ldots \otimes h_n)
\]

(compare with Theorem 5.28). Then (Lemma 5.1) the operators \( \omega(h) = a^+(h) + a^-(h) \) satisfy (with our notation)

\[
\langle \Omega, \omega(h_1) \ldots \omega(h_{2n}) \rangle = \sum_{\pi \in \mathcal{P}_2(2n)} q^{n-c(\pi)} C_\pi(h_1 \otimes \ldots \otimes h_n)
\]

and the corresponding expression is zero for an odd number of factors (compare with equation (10)). Here \( c(\pi) \) is again the number of cycles of a permutation corresponding to a partition \( \pi \), but this correspondence is more subtle. For a pair partition \( \pi \), there is a unique non-crossing partition \( \bar{\pi} \) with the same openers and closers as \( \pi \). If \( i \sim j \) and \( i \bar{\sim} k \), then for the corresponding permutation \( \sigma, \sigma(i) = j \) if \( i < j \), and \( \sigma(i) = k \) if \( k < i \) (it is easy to check that this is an alternative). In other words, \( \sigma \) is a permutation with an upper partition \( \pi \) and the lower partition \( \bar{\pi} \) in the sense of Corteel [Cor07]. Then \( c(\pi) \) is the number of cycles of \( \sigma \).

Moreover (Lemma 5.1), the creation and annihilation operators satisfy a commutation relation

\[
a^-(f)a^+(g) = \langle f, g \rangle + q d\Gamma(|g\rangle\langle f|),
\]
where $d\Gamma(A)$ is the standard second quantization operator. We prove an analog of this relation in our context below. Note however that in other aspects, our construction behaves quite differently. For example, there is no simple commutation relation between $a^i_{(01)}$ and $d\Gamma(A)$ below. Also, for $q = -\frac{1}{N}$, the distribution of $\omega(h)$ only has finite support, in contrast to Theorem 5.13 and Propositions 6.17.

Remark 5.29. Combining our construction with [BG02], one could consider a Fock space with the inner product

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_k \rangle_q = \delta_{n=k} \sum_{\sigma \in S(n)} \chi_q[\sigma \alpha\sigma^{-1}\alpha^{-1}] \prod_{i=1}^n \langle f_i, g_{\sigma(i)} \rangle$$

for a fixed permutation $\alpha$, for example for $\alpha = (01 \ldots n)$. It is easy to see that for $q \in \mathbb{Z}$, this inner product is positive semi-definite for any $\alpha$. For particular choices of $\alpha$, it is positive semi-definite for a wider range of $q$.

Definition 5.30. Let $A$ be a (bounded for simplicity) linear operator on $H$. Define its differential second quantization

$$d\Gamma(A)^i = \sum_{i=1}^n (0^i)\alpha \otimes (h_1 \otimes \ldots \otimes h_n).$$

Note that

$$d\Gamma(I)(\alpha \otimes F) = \sum_{i=1}^n (0^i)\alpha \otimes F,$$

where $\sum_{i=1}^n (0^i)$ is the Jucys-Murphy element.

Lemma 5.31.

(a) For $P_n$ the symmetrizing projection from equation (2), $d\Gamma(A) P_n = P_n d\Gamma(A)$. Therefore $d\Gamma(A)$ restricts to an operator on $TP(H)$ and $TP_q(H)$.

(b) $(d\Gamma(A))^* = d\Gamma(A^*)$.

Proof. For part (a), we note that

$$d\Gamma(A) P_n(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))$$

$$\quad = \sum_{\sigma \in S(n)} \sum_{i=1}^n (0^i)\sigma\alpha\sigma^{-1} \otimes (h_{\sigma^{-1}(1)} \otimes \ldots \otimes Ah_{\sigma^{-1}(i)} \otimes \ldots \otimes h_{\sigma^{-1}(n)})$$

$$\quad = \sum_{\sigma \in S(n)} \sum_{i=1}^n \sigma(0^ {-1}(i))\alpha\sigma^{-1} \otimes (h_{\sigma^{-1}(1)} \otimes \ldots \otimes Ah_{\sigma^{-1}(i)} \otimes \ldots \otimes h_{\sigma^{-1}(n)})$$

$$\quad = P_n d\Gamma(A)(\alpha \otimes (h_1 \otimes \ldots \otimes h_n)).$$
The restriction to \( \mathcal{T}_{A}(\mathcal{H}) \) follows since \( N_{gr,q} \) is an ideal. Similarly, for part (b),
\[
\langle d\Gamma(A)(\alpha \otimes (f_1 \otimes \ldots \otimes f_n)), \beta \otimes (g_1 \otimes \ldots \otimes g_n) \rangle_q
\]
\[
= \sum_{\sigma \in S(n)} \sum_{i=1}^{n} \chi_{q}((0i)\alpha \sigma^{-1}\sigma^{-1}) \langle Af_{i}, g_{\sigma^{-1}(i)} \rangle \prod_{j \neq i} \langle f_{j}, g_{\sigma^{-1}(j)} \rangle
\]
\[
= \sum_{\sigma \in S(n)} \sum_{i=1}^{n} \chi_{q}(\alpha \sigma ((0\sigma^{-1}(i))\beta^{-1}\sigma^{-1}) \langle f_{i}, A^{*}g_{\sigma^{-1}(i)} \rangle \prod_{j \neq i} \langle f_{j}, g_{\sigma^{-1}(j)} \rangle
\]
\[
= \langle \alpha \otimes (f_1 \otimes \ldots \otimes f_n), d\Gamma(A^{*})\beta \otimes (g_1 \otimes \ldots \otimes g_n) \rangle_q.
\]
\( \square \)

**Proposition 5.32.** Splitting \( X(h) \) into the creation operator \( a^{+}(h) \) and annihilation operator \( a^{-}(h) \) as in Theorem 5.28 we have

\[
a^{-}(f)a^{+}(g) = \langle f, g \rangle + q d\Gamma(|g\rangle\langle f|).
\]

**Proof.** With the notation from Theorem 5.28,
\[
a^{-}(f)a^{+}(g)(\alpha \otimes_{s} (h_{1} \otimes \ldots \otimes h_{n}))(h_{1} \otimes \ldots \otimes h_{n} \otimes g)
\]
\[
= a^{-}(f)((0 n + 1)\alpha \otimes_{s} h_{1} \otimes \ldots \otimes h_{n} \otimes g)
\]
\[
= q \sum_{k=1}^{n} \langle h_{k}, f \rangle P_{[0,n+1]}^{(k)}((0k)(0 n + 1)\alpha) |_{\{k\}} \otimes_{s} (h_{1} \otimes \ldots \otimes h_{k} \otimes \ldots \otimes h_{n} \otimes g)
\]
\[
+ \langle f, g \rangle \alpha \otimes_{s} (h_{1} \otimes \ldots \otimes h_{n})
\]

Denoting \( \sigma_{k} = (k k + 1 \ldots n - 1 n) \), we have
\[
U_{\sigma_{k}}(h_{1} \otimes \ldots \otimes h_{k} \otimes \ldots \otimes h_{n} \otimes g) = h_{1} \otimes \ldots \otimes h_{k-1} \otimes g \otimes h_{k+1} \otimes \ldots \otimes h_{n}.
\]

On the other hand, the bijection
\[
\tau_{k} = P_{[0,n+1]}^{(k)} : i \mapsto \begin{cases} 
  i, & 1 \leq i \leq k - 1 \\
  i - 1, & k + 1 \leq i \leq n + 1.
\end{cases}
\]

and so \( \sigma_{k}\tau_{k}(i) = i \) for \( i \neq n + 1 \), \( \sigma_{k}\tau_{k}(n + 1) = k \). Therefore
\[
\sigma_{k}\tau_{k}((0k)(0 n + 1)\alpha) |_{\{k\}} \tau_{k}^{-1}\sigma_{k}^{-1} = (0k)\alpha.
\]
\( \square \)

**Remark 5.33.**

\[
\exp(d\Gamma(I)) = \sum_{k=0}^{\infty} \frac{1}{k!}(d\Gamma(I))^{k} = \sum_{\beta \in S_{0}(n)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{1}{k!} | \{ i \in [n]^{k} : \beta = (0i(0) \ldots (0i(1)) \} |
\]

Here the coefficient of \( \beta \) is the generating function of the number of primitive factorizations of \( \beta \).
See [MNI10].

6. Trace polynomials in GUE matrices

**6.1. Background.** Let \( \{ x_{i} : i \in S \} \) be a collection of non-commuting variables. Informally, a trace polynomial in these variables is a polynomial in these variables as well as in a tracial functional \( \text{tr} \) applied to these variables. See [Ceb13] for a formal definition using a universal property.
For the purposes of this paper, we will use a more constructive definition. Let \( \alpha \in S_0(n) \). Denote
\[
\text{tr}_\alpha [x_1, \ldots, x_n] = \prod_{i \text{ in the cycle starting with } 0} x_i \prod_{\text{other cycles}} \text{tr} \left[ \prod_{i \text{ in the cycle}} x_i \right].
\]
For example, for \( \alpha = (024)(137)(56) \),
\[
\text{tr}_\alpha [x_1, \ldots, x_7] = x_2 x_4 \text{tr}[x_1 x_3 x_7] \text{tr}[x_5 x_6].
\]
Compare with Notation 22.29 in [NS06]. Then \( \text{tr}_\alpha [x_1, \ldots, x_n] \) is a trace monomial, and a trace polynomial is a linear combination of trace monomials.

Next, let \( \mathcal{A} \) be a unital algebra, \( \mathcal{C} \) its center, and \( F : \mathcal{A} \to \mathcal{C} \) a unital \( \mathcal{C} \)-bimodule linear map. For any \( \alpha \in S_0(n) \), we can similarly form \( F_\alpha (a_1, \ldots, a_n) \in \mathcal{A} \), and consider it as the application of the trace monomial \( \text{tr}_\alpha [x_1, \ldots, x_n] \) to the elements \( a_1, \ldots, a_n \). See \cite{Ceb13}.

We extend the notation to \( F_\eta \) for \( \eta \in \mathbb{C}[S_0(n)] \) by linearity.

6.1.1. Invariant theory of \( N \times N \) matrices. Let \( \left\{ x_{ij}^{(k)} : k \in S, 1 \leq i, j \leq N \right\} \) be formal commuting variables subject to the relation \( x_{ji}^{(k)} = (x_{ij}^{(k)})^* \). For each \( k \), form a matrix \( X^{(k)} = (x_{ij}^{(k)})_{i,j=1}^N \). Let \( \mathcal{A}_{N,S} \) be the collection of all matrices with polynomial entries
\[
\mathcal{A}_{N,S} = M_N(\mathbb{C}) \otimes \mathbb{C}[x_{ij}^{(k)} : k \in S, 1 \leq i, j \leq N].
\]
Let \( Y \in \mathcal{A}_{N,S} \), \( Y = P(X^{(k)} : k \in S) \), where each entry \( P_{ab} \) is a polynomial in the entries of its argument. We say that \( Y \) is equivariant if for any \( U \in U_N(\mathbb{C}) \),
\[
P(U X^{(k)} U^* : k \in S) = U Y U^*.
\]
Denote
\[
\mathcal{A}_{N,S}^{\text{equiv}} = \{ \text{equivariant } Y \in \mathcal{A}_{N,S} \}.
\]
Then
\[
\mathcal{A}_{N,S}^{\text{equiv}} = \text{Span} \left( \text{Tr}_\alpha (X^{(k(1))}, \ldots, X^{(k(n))}) : \alpha \in S_0(n), n \geq 0, k(1), \ldots, k(n) \in S \right),
\]
where \( \text{Tr} \) is the (un-normalized) trace on \( M_N(\mathbb{C}) \). Indeed, since for the purposes of this expansion, \( x_{ij}^{(k)} \) and \( (x_{ij}^{(k)})^* \) can be considered as independent variables, this follows directly from the first Procesi-Razmyslov theorem \cite{Pro76}. The result is usually formulated using \( GL(n) \)-invariance. However, the argument ultimately reduces to Schur-Weyl duality, for which (in the case of inner product spaces) unitary invariance is sufficient.

6.1.2. Hermitian Brownian motion. Let \( \{ b_{ij}(h) : h \in \mathcal{H}_\mathbb{R} \} \) be \( N^2 \) standard independent Gaussian processes indexed by the same real Hilbert space, represented on the same probability space. Define the \( N \times N \) Hermitian Gaussian process \( \{ X(h) : h \in \mathcal{H}_\mathbb{R} \} \) by
\[
X(h)_{ij} = \begin{cases} 
\frac{1}{\sqrt{2N}} (b_{ij}(h) + \sqrt{-1} b_{ji}(h)), & i < j, \\
\frac{1}{\sqrt{N}} b_{ij}(h), & i = j, \\
\frac{1}{\sqrt{2N}} (b_{ij}(h) - \sqrt{-1} b_{ji}(h)), & i > j.
\end{cases}
\]
Equivalently, each $X(h)$ is a Hermitian random matrix, whose entries are centered Gaussian variables with the joint covariance

$$
\mathbb{E}[X(f)_{ij}X(g)_{k\ell}] = \frac{1}{N} \delta_{i=\ell} \delta_{j=k} \langle f, g \rangle,
$$

so that

$$(I \otimes \mathbb{E})[X(f)X(g)] = \langle f, g \rangle I_N.$$ 

Note that $(I \otimes \mathbb{E})[\text{Tr}_\alpha(X(h_1), \ldots, X(h_n))]$ is always a scalar. Indeed, since the distribution of this random matrix is unitarily invariant, so is the distribution of its entry-wise expectation, which then has to be a multiple of identity. By a slight abuse of notation, we will denote this scalar-valued functional by $\mathbb{E}$ again.

**Proposition 6.1.** Let $\{D_i : i \in I\}$ be non-random $N \times N$ matrices. For even $n$,

$$
D^{(0)}(I \otimes \mathbb{E}) \left[ \text{Tr}_\alpha[X(h_1)D^{(1)}, X(h_2)D^{(2)}, \ldots, X(h_n)D^{(n)}] \right] = \frac{1}{N^{n/2}} \sum_{\pi \in \mathcal{P}_2(n)} C_\pi(h_1 \otimes \ldots \otimes h_n) D^{(0)} \text{Tr}_{\pi \alpha}[D^{(1)}, D^{(2)}, \ldots, D^{(n)}].
$$

In particular,

$$
\mathbb{E}[\text{Tr}_\alpha[X(h_1), X(h_2), \ldots, X(h_n)]] = \sum_{\pi \in \mathcal{P}_2(n)} \frac{1}{N^{n/2-c_{\pi \alpha}(\pi \alpha)}} C_\pi(h_1 \otimes \ldots \otimes h_n).
$$

**Proof.**

$$
\left( D^{(0)}(I \otimes \mathbb{E}) \left[ \text{Tr}_\alpha[X(h_1)D^{(1)}, X(h_2)D^{(2)}, \ldots, X(h_n)D^{(n)}] \right] \right)_{tr}
$$

$$= \sum_{\pi \in \mathcal{P}_2(n)} \sum_{\sum_{i}} \prod_{i=0}^{n} D_{v(i),u(\alpha(i))}^{(i)} \mathbb{E}\left[ \prod_{i=1}^{n} X_{u(i),v(i)}(h_i) \right]
$$

$$= \sum_{\sum_{i}} \prod_{i=0}^{n} D_{v(i),u(\alpha(i))}^{(i)} \mathbb{E}\left[ \prod_{(i,j) \in \pi} X_{v(\pi(i)),v(i)}(h_i) X_{v(i),v(\pi(i))}(h_{\pi(i)}) \right]
$$

$$= \frac{1}{N^{n/2}} \sum_{\pi \in \mathcal{P}_2(n)} C_\pi(h_1 \otimes \ldots \otimes h_n)
$$

$$= \frac{1}{N^{n/2}} \sum_{\pi \in \mathcal{P}_2(n)} C_\pi(h_1 \otimes \ldots \otimes h_n).$$
Notation 6.2. Denote
\[ A_N(\mathcal{H}) = M_{N}(\mathbb{C}) \otimes \mathbb{C}[b_{ij}(h) : h \in \mathcal{H}_R, 1 \leq i, j \leq N] \]
\[ = \{ P(b_{ij}(h) : h \in \mathcal{H}_R, 1 \leq i, j \leq N) : P \in A_{N,\mathcal{H}_R} \} \]
and
\[ A_{N}^{\text{equiv}}(\mathcal{H}) = \left\{ P(b_{ij}(h) : h \in \mathcal{H}_R, 1 \leq i, j \leq N) : P \in A_{N}^{\text{equiv}} \right\} \]
\[ = \text{Span} \left( \text{Tr}_\alpha(X(h_1), \ldots, X(h_n) : \alpha \in S_0(n), n \geq 0, h_1, \ldots, h_n \in \mathcal{H}_R) \right). \]
Denote \( A_{N}^{2,\text{equiv}}(\mathcal{H}) \) the \( L^2 \) completion of \( A_{N}^{\text{equiv}}(\mathcal{H}) \) with respect to the expectation functional \( \mathbb{E} \).

6.2. The isomorphisms.

Theorem 6.3. Let \( q = 1/N \). Define the evaluation map \( \mathcal{E} \) from \( \bigoplus_{n=0}^{\infty} \mathbb{C}[S_0(n)] \otimes \mathcal{H}_R^{\otimes n} \) to the algebra \( A_{N}^{\text{equiv}}(\mathcal{H}) \) by the linear extension of
\[ \mathcal{E}[\alpha \otimes (h_1 \otimes \ldots \otimes h_n)] = \text{Tr}_\alpha(X(h_1), \ldots, X(h_n)). \]

(a) This map factors through to \( TP(\mathcal{H}_R) \).
(b) \( \mathcal{E} \) is a star-homomorphism of algebras.
(c) \( \mathbb{E} \circ \mathcal{E} = \varphi \). It follows that \( \mathcal{E} \) extends to a homomorphism from \( \overline{TP(\mathcal{H}_R)} \) to \( A_{N}^{2,\text{equiv}}(\mathcal{H}) \), and to an isometry from \( F_{1/N}(\mathcal{H}) \) to \( A_{N}^{2,\text{equiv}}(\mathcal{H}) \).
(d) For \( A \in \overline{TP(\mathcal{H}_R)} \),
\[ \mathbb{E} \left[ \mathcal{E}[A] \mid X(h) : h \in \mathcal{H}_R' \right] = \mathcal{E}[\varphi \left[ A \mid \mathcal{H}' \right]], \]
so in particular,
\[ \mathbb{E} \left[ \text{Tr}_\alpha(X(h_1), \ldots, X(h_n)) \mid X(h) : h \in \mathcal{H}_R' \right] = \mathcal{E}[\varphi \left[ T(\alpha \otimes F) \mid \mathcal{H}' \right]]. \]

Proof. (a) follows from
\[ \text{Tr}_{\alpha \sigma^{-1}}(X(h_{\sigma^{-1}(1)}), \ldots, X(h_{\sigma^{-1}(n)})) = \text{Tr}_\alpha(X(h_1), \ldots, X(h_n)), \]
and (b) from
\[ \text{Tr}_\alpha(X(h_1), \ldots, X(h_n)) \text{Tr}_\beta(X(h_{n+1}), \ldots, X(h_{n+k})) = \text{Tr}_{\alpha \beta}(X(h_1), \ldots, X(h_{n+k})). \]

(c) follows by comparing Theorem 5.13(e) and Proposition 6.1.

For part (d), for \( h_1, \ldots, h_k \in \mathcal{H}_R' \), using earlier parts and properties of conditional expectations,
\[ \mathbb{E} \left[ \mathbb{E} \left[ \mathcal{E}[A] \mid X(h) : h \in \mathcal{H}_R' \right] \text{Tr}_\beta(X(h_1), \ldots, X(h_k)) \right] \]
\[ = \mathbb{E} \left[ \mathcal{E}[A] \text{Tr}_\beta(X(h_1), \ldots, X(h_k)) \right] \]
\[ = \mathbb{E} \left[ \mathcal{E}[A] \mathcal{E}[T (\beta \otimes (h_1 \otimes \ldots \otimes h_n)) \right] \]
\[ = \varphi \left[ \mathcal{E}[A] \mid \mathcal{H}' \right] T (\beta \otimes (h_1 \otimes \ldots \otimes h_n)) \]
\[ = \mathbb{E} \left[ \mathcal{E}[\varphi \left[ A \mid \mathcal{H}' \right] \text{Tr}_\beta(X(h_1), \ldots, X(h_k)) \right]. \]

Since \( \mathbb{E} \) is faithful on \( A_{N}^{\text{equiv}}(\mathcal{H}) \), the result follows. \( \square \)
Numerous corollaries follow by combining Theorem 6.3 with results from earlier in the article. We only list a few of them explicitly. Others include Proposition 4.8 (chaos decomposition in the univariate case), 5.6 (expansion of the Hermite polynomial and stochastic integral), 5.10 (product formula for Hermite polynomials and stochastic integrals), 5.15 (linearization coefficients).

**Corollary 6.4.** Define the \( \alpha \)-Hermite polynomial

\[
H_\alpha(X(h_1), \ldots, X(h_n)) = \mathcal{E}[I(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))].
\]

Let \( \mathcal{H} = L^2(\mathbb{R}_+, dx) \). Then for each \( \alpha \), \( H_\alpha(X(1_{[0,t]})) \) is a martingale.

For Hermite polynomials of matrix argument, the martingale property was proved in [Law08] by generating function methods. See Section 6.2.1 for the connection.

**Corollary 6.5.** (Compare with Proposition 5.18)

\[
\mathbb{E} \left[ \text{Tr}_\alpha(X(h)) \mid X(h) : h \in \mathcal{H}' \right] = \sum_{\rho \in \mathcal{P}_{1,2}(n)} \left\| P(h') h \right\|_{\text{Pair}(\rho)} \text{Tr}_{C_\rho(\alpha)}(X(P_h h)).
\]

In particular, for \( \mathcal{H} = L^2(\mathbb{R}_+, dx) \) and \( \mathcal{H}_s = L^2([0,s], dx) \), for \( s \leq t \),

\[
\mathbb{E} \left[ \text{Tr}_\alpha(X(1_{[0,t]})) \mid X(h) : h \in \mathcal{H}_s \right] = \sum_{\rho \in \mathcal{P}_{1,2}(n)} (t-s)^{\text{Pair}(\rho)} \text{Tr}_{C_\rho(\alpha)}(X(1_{[0,s]})).
\]

It follows that \( \mathcal{L} \) is the generator of the process \( \{ X(t) = X(1_{[0,t]}) : t \geq 0 \} \), in the sense that for formal univariate trace polynomial \( \text{tr}_\alpha(x) \),

\[
\left. \frac{d}{dt} \right|_{t=s} \mathbb{E} \left[ \text{tr}_\alpha(X(t)) \mid X(h) : h \in \mathcal{H}_s \right] = \text{tr}_\mathcal{L} \alpha(X(s)).
\]

**Remark 6.6.** In the case \( \mathcal{H}_\mathbb{R} = L^2(\mathbb{R}_+, dx) \) and \( F \in L^2(\mathbb{R}^n_+, dx^{\otimes n}) \), we may identify \( \mathcal{E}[I(\eta \otimes_s F)] \) with a stochastic integral

\[
\int F(t_1, \ldots, t_n) \, \text{Tr}_\eta[dX(t_1), \ldots, dX(t_n)].
\]

Indeed, consider first \( F = 1_{J_1} \otimes \ldots \otimes 1_{J_n} \), where all \( J_j \) are disjoint. Then from Proposition 5.6

\[
\text{I}(\eta \otimes (1_{J_1} \otimes \ldots \otimes 1_{J_n})) = T(\eta \otimes (1_{J_1} \otimes \ldots \otimes 1_{J_n}))
\]

and so

\[
\mathcal{E}[\text{I}(\eta \otimes (1_{J_1} \otimes \ldots \otimes 1_{J_n}))] = \text{Tr}_\eta[X(1_{J_1}), \ldots, X(1_{J_n})]
\]

which we define to be

\[
\int 1_{J_1}(t_1) \ldots 1_{J_n}(t_n) \, \text{Tr}_\eta[dX(t_1), \ldots, dX(t_n)].
\]

A general \( F \in L^2(\mathbb{R}^n_+, dx^{\otimes n}) \) can be approximated by linear combinations of such function in the \( L^2 \) norm, and by Lemma 4.5 we also get the approximation of \( \text{I}(\eta \otimes_s F) \).

**Corollary 6.7** (Chaos decomposition IV; compare with Proposition 4.10). Each element \( A \in \mathcal{A}_N^{2, \text{equiv}}(L^2(\mathbb{R}_+, dx)) \) has a unique decomposition

\[
A = \sum_{n=0}^{\infty} \sum_{\lambda \in \text{Par}(n+1; \leq N)} \sum_{i,j=1}^{d_N} \int F_{ij}^\lambda(t_1, \ldots, t_n) \, \text{Tr}_{W(E_{ij}^\lambda)}[dX(t_1), \ldots, dX(t_n)],
\]

where \( E_{ij}^\lambda \) are eigenvalues of \( F_{ij}^\lambda \), and \( W(E_{ij}^\lambda) \) is the corresponding eigenspace.
where $F_{ij}^\lambda \in L^2(\Delta(\mathbb{R}_+^n), dx\otimes^n)$ and

$$\sum_{n=0}^{\infty} \sum_{\lambda \in \text{Par}(n+1; \leq N)} n_\lambda \sum_{i,j=1}^{d_\lambda} \|F_{ij}^\lambda\|^2 < \infty$$

for $n_\lambda = \frac{|SS_N(\lambda)|}{N^{n+1}}$.

6.2.1. Hermite polynomials of matrix argument.

**Remark 6.8.** If $\alpha \in S(n)$ rather than $S_0(n)$, in the case of a single variable, $\text{Tr}_\alpha[X]$ depends only on the conjugacy class of $\alpha$, in other words on the number partition $\lambda \in \text{Par}(n)$. Moreover $\text{Tr}_\lambda[X] = p_\lambda(x_1, \ldots, x_N)$, where $\{x_1, \ldots, x_N\}$ are (random) eigenvalues of $X$ and $p_\lambda$ is the power sum symmetric polynomial.

For $X = X(h)$, we also get non-homogeneous symmetric polynomials $h_\lambda(x_1, \ldots, x_N) = \mathcal{E}[I (\lambda \otimes h \otimes^n)]$.

With respect to the inner product induced from $\mathcal{F}_{1/\mathbb{N}}(\mathbb{C})$, these polynomials are orthogonal for different $n$ but not necessarily for different $\lambda \in \text{Par}(n)$. We now recall a different and more familiar basis of polynomials which are fully orthogonal with respect to this inner product.

**Definition 6.9.** Fix $N \in \mathbb{N}$, and denote

$$D^* = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{1}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)$$

and

$$E^* = \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i}.$$ 

For $\lambda \in \text{Par}(n)$, the Hermite polynomial of matrix argument (for $\beta = 2$) is the symmetric polynomial in $\{x_1, \ldots, x_N\}$ with leading term $\frac{|\lambda|!}{n!} s_\lambda$ which is an eigenfunction of the operator $D^* - E^*$ with eigenvalue $-n$ (note a misprint in [DES07]). Here (see Corollary 7.1.7.4 in [Sta99])

$$s_\lambda = \frac{1}{n!} \sum_{\nu \in \text{Par}(n)} \frac{n!}{z_\nu} \chi^\lambda(\nu)p_\nu = \frac{1}{n!} \sum_{\alpha \in S(n)} \chi^\lambda(\alpha)p_\alpha$$

is the Schur polynomial and $c_\lambda = \prod((\lambda_i + \lambda_j - i - j + 1)$ is the hook length. See [BF97, DES07, For10] for more details.

**Proposition 6.10.** For $\eta \in \mathbb{C}[S_0(n)]$,

$$D^* \mathcal{E}[T (\eta \otimes h \otimes^n)] = 2N \mathcal{E}[T (L(\eta) \otimes h \otimes^{n-2})].$$

and

$$E^* \mathcal{E}[T (\eta \otimes h \otimes^n)] = \mathcal{E}[ET (\eta \otimes h \otimes^n)].$$
Comparing these two results, we see that $D$tion. The second relation follows from $E$

However the contractions, and so the operators $T$ $(\text{Definition } 6.12)$. we get well-defined contractions, but a more degenerate Fock space structure.

$\bullet$ Eliminates a cycle of length 2: in $k_2$ cases, with weight $q^{-1}$.
$\bullet$ Turns a cycle of length $l > 2$ into a cycle of length $l - 2$: in $k_3 l$ cases, with weight $1$.
$\bullet$ Splits a cycle of length $l$ into those of different lengths $a < l - 2 - a$: in $k_4 l$ cases, with weight $q$.
$\bullet$ Splits a cycle of length $l$ into those of equal lengths $(l - 2)/2$: in $k_5 l/2$ cases, with weight $q$.
$\bullet$ Eliminates two cycles of length 1: in $k_6 (k_1 - 1)/2$ cases, with weight 1.
$\bullet$ Eliminates a cycle of length 1 and turns a cycle of length $l$ into a cycle of length $l - 1$: in $k_7 k_2 l$ cases, with weight $q$.
$\bullet$ Glues together two cycles of different lengths $2 \leq l_1 < l_2$: in $k_8 l_1 l_2$ cases, with weight $q$.
$\bullet$ Glues together two cycles of equal length $l \geq 2$: in $k_9 (k_1 - 1) l^2/2$ cases, with weight $q$.

Comparing these two results, we see that $D^* p_\mu$ corresponds to $2 N \mathcal{E}[T (\mathcal{L} \mu)]$, giving the first relation. The second relation follows from $E^* p_\mu = n p_\mu$.

**Corollary 6.11.** Let $\|h\|^2 = \frac{1}{N}$. Then $\mathcal{E}[I (\chi^\lambda \otimes h^{\otimes n})]$ is a multiple of the Hermite polynomial of matrix argument.

**Proof.** Since their leading terms differ by a factor of $\frac{\lambda!}{\lambda^n}$, it suffices to verify that $\mathcal{E}[I (\chi^\lambda \otimes h^{\otimes n})]$ is an eigenfunction of $D^* - E^*$ with eigenvalue $-n$. Indeed, for $\|h\|^2 = \frac{1}{N}$,

$$(D^* - E^*) \mathcal{E}[T (\eta \otimes h^{\otimes n})] = \mathcal{E}[(2 \mathcal{L} - E) T (\eta \otimes h^{\otimes n})].$$

So using Proposition\[5.11\]

$$(D^* - E^*) \mathcal{E}[I (\eta \otimes h^{\otimes n})] = \mathcal{E}[(2 \mathcal{L} - E) I (\eta \otimes h^{\otimes n})] = -n \mathcal{E}[I (\eta \otimes h^{\otimes n})].$$

**6.2.2. $q = 0$.** The scaling used throughout most of the article (corresponding to the un-normalized trace) gives well-defined inner products and Fock space structure for $q = 0$, see equation \[8\]. However the contractions, and so the operators $T (\alpha \otimes F)$, may not be defined. In this section we consider a different scaling, corresponding to the normalized trace. Under this normalization, we get well-defined contractions, but a more degenerate Fock space structure.

**Definition 6.12.** We will now denote $\alpha \otimes_s F$ by $\tilde{I}(\alpha \otimes_s F)$. Define the normalized contractions

$$\tilde{C}_\mu(\alpha) = q^{\text{cyc}_0(\alpha) - \text{cyc}_0((\pi_\mu)_{\text{supp}(\pi)^c})} \text{C}_\mu(\alpha)$$

$$= q^{\text{cyc}_0(\alpha) - \text{cyc}_0(\alpha) + t P_{[0, n - 2]} \text{supp}(\pi) \text{supp}(\pi)^c},$$
and the operators

\begin{equation}
\tilde{T}(\alpha \otimes_s F) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} I(\tilde{C}_\pi(\alpha) \otimes_s C_\pi(F)),
\end{equation}

so that

\begin{equation}
I(\alpha \otimes_s F) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} (-1)^{n-|\pi|} \tilde{T}(\tilde{C}_\pi(\alpha) \otimes_s C_\pi(F))
\end{equation}

and

\begin{equation}
\tilde{T}(\alpha \otimes_s F) \tilde{T}(\beta \otimes_s G) = \tilde{T}((\alpha \cup \beta) \otimes_s (F \otimes G)).
\end{equation}

Finally, define

\begin{equation}
\varphi \left[ I(\alpha \otimes_s F) \right] = 0, \quad \varphi \left[ I((0)) \right] = 1.
\end{equation}

**Remark 6.13.** If we assume that $\tilde{T}(\alpha \otimes F) = q^{\varepsilon \kappa(\alpha)} T(\alpha \otimes F)$, then for $q = 0$, such a multiple is zero unless $\alpha$ is a single cycle (containing 0). As seen below, this need not be the case in general.

**Remark 6.14.** Let $\lambda$ be an interval partition of $[0, n]$. As discussed in Section 2.2, each permutation $\alpha \in S_0(n)$ is conjugate, under the action of $S(n)$, to a permutation with cycle structure $\lambda$ in which the elements in each cycle, as well as the cycles, appear in increasing order. Such permutation is not unique. In the results below, we will only consider permutations of this type; the results are easily extended to general $\alpha \in S_0(n)$, but the notation gets heavier.

**Remark 6.15.** Let $\mathcal{F}_f(\mathcal{H}) = \mathbb{C} \Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}_{\overline{\mathbb{R}}}$ be the full Fock space of $\mathcal{H}_{\overline{\mathbb{R}}}$. For $f \in \mathcal{H}_{\overline{\mathbb{R}}}$, denote by $S(f)$ the semicircular element corresponding to $f$ in its standard representation on $\mathcal{F}_f(\mathcal{H}_{\overline{\mathbb{R}}})$. Denote by $\Gamma_f(\mathcal{H}_{\overline{\mathbb{R}}})$ the algebra generated by $\{X(h) : h \in \mathcal{H}_{\overline{\mathbb{R}}}, \}$, and by $\Phi$ the vacuum state on it. Denote by $U(h_1 \otimes \ldots \otimes h_n)$ the multivariate Chebyshev polynomials, which are the elements of $\Gamma_f(\mathcal{H}_{\overline{\mathbb{R}}})$ which satisfy $U(h_1 \otimes \ldots \otimes h_n) \Omega = h_1 \otimes \ldots \otimes h_n$. They are also determined by the recursion

\begin{equation}
S(f)U(h_1 \otimes \ldots \otimes h_n) = U(f \otimes h_1 \otimes \ldots \otimes h_n) + \langle f, h_1 \rangle U(h_2 \otimes \ldots \otimes h_n).
\end{equation}

More generally, for $F \in \mathcal{H}^{\otimes n}_{\overline{\mathbb{R}}}$, $U(F)$ is the unique element of the weak closure of $\Gamma_f(\mathcal{H}_{\overline{\mathbb{R}}})$ satisfying $U(F) \Omega = F$.

**Proposition 6.16.** Let $q = 0$, and $\lambda, \mu \in S_0(n)$ as in Remark 6.14

(a) $\tilde{C}_\pi(\lambda) = 0$ unless $\pi$ is non-crossing and $\pi \leq \lambda$ as a cycle partition, in which case

$\tilde{C}_\pi(\lambda) = P_{[0, n]}[\sup(\pi)]\sup(\pi)^e$.

(b) Define the map $\mathcal{E}_0 : TP(\mathcal{H}) \rightarrow \Gamma_f(\mathcal{H}_{\overline{\mathbb{R}}})$ by

$\mathcal{E}_0[\tilde{T}(\alpha \otimes (h_1 \otimes \ldots \otimes h_n))] = \Phi_\alpha[S(h_1), \ldots, S(h_n)]$.

Then $\mathcal{E}_0$ extends to a homomorphism from $TP(\mathcal{H})$, and to an isometric isomorphism from $\mathcal{F}_0(\mathcal{H})$ to $L^2(\Gamma_f(\mathcal{H}_{\overline{\mathbb{R}}}), \Phi) \simeq \mathcal{F}_f(\mathcal{H})$.

(c) If $\lambda$ is not a single cycle (containing 0), then $\tilde{T}(\lambda \otimes F) = 0$. Otherwise,

$\mathcal{E}_0[\tilde{T}((01 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n))] = U(h_1 \otimes \ldots \otimes h_n)$

and more generally

$\mathcal{E}_0[\tilde{T}((01 \ldots n) \otimes F)] = U(F)$. 

Proof. Note first that
\[cyc_0(\alpha) - cyc_0(\pi\alpha) + \ell = -|\alpha| + |\pi^{-1}\alpha| + |\pi| \geq 0.\]
Moreover, for \(\alpha = \lambda\) as in Remark 6.14, this is equal to 0 if and only if \(\pi \leq \lambda\), and on each block of \(\lambda, \pi\) is non-crossing. Therefore
\[\tilde{T}(\lambda \otimes F) = \sum_{\pi \in NC_1, 2(n)} \tilde{I}(\hat{C}_\pi(\lambda) \otimes C_\pi(F)),\]
and so for \(F = h_1 \otimes \ldots \otimes h_n\),
\[\varphi\left[\tilde{T}(\lambda \otimes F)\right] = \sum_{\pi \in NC_2(n)} C_\pi(F) = \Phi[\Phi_\lambda[S(h_1), \ldots, S(h_n)]].\]
The homomorphism property follows as before, so \(\mathcal{E}_0\) is an isometric isomorphism. In particular, if (0) is a cycle in \(\lambda\), then
\[(13) \quad \tilde{T}(\lambda \otimes F) = \varphi\left[\tilde{T}(\lambda \otimes F)\right] = \sum_{\pi \in NC_2(n)} C_\pi(F)\]
is a scalar. Next,
\[\tilde{I}(\lambda \otimes F) = \sum_{\pi \in NC_1, 2(n)} (-1)^{n-|\pi|} \tilde{T}(\hat{C}_\pi(\lambda) \otimes C_\pi(F)).\]
Suppose \(\lambda\) contains a cycle \(V\) not containing 0. Then the sum above is
\[\tilde{I}(\alpha \otimes F) = \sum_{\pi \in NC_1, 2([n]\setminus V)} (-1)^{n-|V| - |\pi|} \tilde{T}(\hat{C}_\pi(\lambda|_V) \otimes C_\pi(\bigotimes_{i \in [n]\setminus V} h_i))\]
\[\cdot \sum_{\tau \in NC_1, 2(V)} (-1)^{|V| - |\tau|} \tilde{T}(\hat{C}_\tau(V) \otimes C_\tau(\bigotimes_{i \in V} h_i)).\]
Since \(V\) does not contain 0, denoting \(F_V = \bigotimes_{i \in V} h_i\) and using (13), the second of these sums is
\[\sum_{\tau \in NC_1, 2(V)} (-1)^{|V| - |\tau|} \sum_{\sigma \in NC_2(\text{Sing}(\tau))} C_\sigma C_\tau F_V = \sum_{\rho \in NC_2(V)} \sum_{S \in \text{Pair}(\rho)} (-1)^{|V| - |S|} C_\rho F_V = 0.\]
On the other hand, \(\mathcal{E}_0[\tilde{I}((01 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n))]\) is a polynomial in \(\{S(h_1), \ldots, S(h_n)\}\), with leading term \(S(h_1) \ldots S(h_n)\), and
\[\Phi[\mathcal{E}_0[\tilde{I}((01 \ldots k) \otimes (g_1 \otimes \ldots \otimes g_k))]*\mathcal{E}_0[\tilde{I}((01 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n))]]\]
\[= \varphi[\tilde{I}((01 \ldots k) \otimes (g_1 \otimes \ldots \otimes g_k))]*\tilde{I}((01 \ldots n) \otimes (h_1 \otimes \ldots \otimes h_n))]\]
\[= \delta_{n=k} \prod_{i=1}^n \langle h_i, g_i \rangle\]
\[= \Phi[U(g_1 \otimes \ldots \otimes g_k)*U(h_1 \otimes \ldots \otimes h_n)]\]
since there is only one non-crossing pair partition in \(\mathcal{P}(n, n)\). Since \(\mathcal{E}_0\) is an isometry, the final statement follows. \(\square\)
Finally we consider the case of \( q = -\frac{1}{N} \). We will use the notation \( C^T_\pi (\alpha) \), \( T^{(q)} [\alpha \otimes F] \), \( \varphi_q \), and the multiplication \( \cdot_q \) to indicate the dependence on \( q \).

**Proposition 6.17.** Define the map \( R : \mathcal{T} \mathcal{P} (\mathcal{H}_\mathbb{R}) \rightarrow \mathcal{T} \mathcal{P} (\mathcal{H}_\mathbb{R}) \) by

\[
R(\alpha \otimes F) = (-1)^{|\alpha|} \alpha \otimes F,
\]

so that \( R(1 (\alpha \otimes F)) = (-1)^{|\alpha|} 1 (\alpha \otimes F) \). Then

- \( T^{(-q)} (R(\alpha \otimes F)) = R(T^{(q)} (\alpha \otimes F)) \)
- \( R \) is a star-isomorphism from \( (\mathcal{T} \mathcal{P} (\mathcal{H}_\mathbb{R}), \cdot_q) \) to \( (\mathcal{T} \mathcal{P} (\mathcal{H}_\mathbb{R}), \cdot_{-q}) \)
- For \( \alpha \in S_0(2n) \), \( R \) satisfies
  \[
  \varphi_{-q} [R(T^{(q)} (\alpha \otimes F))] = (-1)^n \varphi_q [T^{(q)} (\alpha \otimes F)].
  \]
- For \( q = 1/N \), \( R \) is an isometry from \( \mathcal{F}_{1/N} (\mathcal{H}_\mathbb{R}) \) onto \( \mathcal{F}_{-1/N} (\mathcal{H}_\mathbb{R}) \).

**Proof.** It suffices to consider the case \( \mathcal{H} = \mathbb{C} \). \( R \) is clearly a bijection. We first compute

\[
T^{(-q)} (\alpha) = \sum_{\pi \in P_{1,2}(n)} I \left( C^{(-q)}_\pi (\alpha) \right)
= \sum_{\pi \in P_{1,2}(n)} (-1)^{cyc_0([\pi \alpha]_{supp(\pi)^c}) - cyc_0(\pi \alpha) + \ell} I \left( C^{(q)}_\pi (\alpha) \right)
= \sum_{\pi \in P_{1,2}(n)} (-1)^{-2\ell - |[\pi \alpha]_{supp(\pi)^c}| + |\pi \alpha| + \ell} I \left( C^{(q)}_\pi (\alpha) \right)
= (-1)^{|\alpha|} \sum_{\pi \in P_{1,2}(n)} (-1)^{-\ell - |\pi|} (-1)^{|\pi \alpha|_{supp(\pi)^c}} I \left( C^{(q)}_\pi (\alpha) \right)
= (-1)^{|\alpha|} \sum_{\pi \in P_{1,2}(n)} I (R(C^{(q)}_\pi (\alpha)))
= (-1)^{|\alpha|} R(T^{(q)} (\alpha))
\]

Since

\[
R(T^{(q)} (\alpha)) \cdot_{-q} R(T^{(q)} (\beta)) = (-1)^{|\alpha| + |\beta|} T^{(-q)} (\alpha) \cdot_{-q} T^{(-q)} (\beta)
= (-1)^{|\alpha| + |\beta|} T^{(-q)} (\alpha \cup \beta) = R(T^{(q)} (\alpha) T^{(q)} (\alpha)),
\]

\( R \) is a homomorphism. It clearly commutes with the adjoint operation. Since the action of the state \( \varphi \) on \( 1 (\alpha \otimes F) \) does not depend on \( q \), and \( \langle A, B \rangle_q = \varphi_q [B^* A] \), the isometry property follows from
the homomorphism property. Finally, for $\alpha \in S_0(2n)$,
\[
\varphi_q \left[ (-1)^{|\alpha|} T^{(q)}(\alpha \otimes F) \right] = \sum_{\pi \in P_2(2n)} (-1)^{|\alpha|} q^{|\pi\alpha|} C_\pi(F)
\]
\[
= \sum_{\pi \in P_2(2n)} (-1)^{|\pi|} (-q)^{\pi\alpha} C_\pi(F)
\]
\[
= (-1)^n \sum_{\pi \in P_2(2n)} (-q)^{\pi\alpha} C_\pi(F)
\]
\[
= (-1)^n \varphi_{-q} \left[ T^{(-q)}(\alpha \otimes F) \right]. \quad \square
\]

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