Construction of multi-instantons in eight dimensions

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Abstract

We consider an eight-dimensional local octonionic theory with the seven-sphere playing the role of the gauge group. Duality conditions for two- and four-forms in eight dimensions are related. Dual fields—octonionic instantons—solve an 8D generalization of the Yang-Mills equation. Modifying the ADHM construction of 4D instantons, we find general $k$-instanton 8D solutions which depend on $16k - 7$ effective parameters.

1 Introduction.

The discovery of instantons [1] was an important advance in our understanding of non-perturbative quantum field theory. These objects are (anti-)self-dual ($*F = \pm F$) Euclidean solutions to Yang-Mills field equations in 4D. They have lead to a deeper understanding of the QCD vacuum ($\theta$ vacuum [2]), and have been conjectured to play a part in the confinement of color charges [3]. Instantons also have a broad significance in mathematics, specifically in the theory of fake $\mathbb{R}^4$-manifolds [4]. The most general multi-instanton solutions have been constructed [5], and these again played a part in broadening our understanding of gauge theories.

A single instanton solution is spherically symmetric and, in mathematical language, corresponds to the third Hopf map, which is the principal fibre bundle $S^7 \xrightarrow{\Sigma^3} S^4$, where $S^4$ is the one-point compactification of $\mathbb{R}^4$, $S^3 \sim SU(2)$ is the fibre (gauge group) and $S^7$ is the total space.

As string theory and M-theory live in higher dimensions, it is of interest to consider higher dimensional analogs of 4D instantons; in particular, there exists a natural generalization of instantons to 8D, where the last Hopf map $S^{15} \xrightarrow{S^7} S^8$ resides. The original 4D instanton had gauge group $SU(2)$ embedded in $Spin(4) \sim SU(2) \times SU(2)$, so that the bundle became $Spin(5) \xrightarrow{Spin(4)} S^4$. 

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The analogous single instanton solution in 8D was found in [6], and has a generalized self-duality $\star F^2 = \pm F^2$ with the bundle $Spin(9) \xrightarrow{Spin(8)} S^8$. The higher dimensional instanton have conformal features similar to those of 4D instantons. The 8D case, and especially its multi-instanton generalization, appears more complicated than its 4D counterpart for the following reasons:

1. The fibre that is twisted with the 4D base space is a three-sphere, but this is a group, while the twisted part of the $Spin(8) \sim S_5^L \times S_5^R \times G_2$ fibre is a seven-sphere. $S^7$ is the only paralellizable manifold that is not a Lie group, but it does have a close resemblance to a gauge group.

2. As $S^7$ can be represented by unit octonions, and $G_2$ is the automorphism group of the octonions, there is a hidden nonassociativity that comes into play.

3. There is only one choice (via the Hodge star) for the form of duality in 4D, but in 8D other possibilities arise, e.g., a tensor form of duality $\lambda F_{\mu\nu} = \frac{1}{2} T_{\mu\rho\sigma} F_{\rho\sigma}$ has been studied [7, 8].

Attempts [9] to obtain multi-instantons in a $Spin(8)$ gauge theory meet with a number of difficulties. To circumvent these obstructions, we turn to a theory with only $S^7$ fibre, but to do this, we first need to review the properties of the octonions. Here we will construct multi-instanton solutions in 8D through a generalization of the ADHM procedure, and to do this we must deal with all of the above complications. We will introduce products and operators in a way that nonassociativity is tamed. Next, a new generalized duality is used to provide results that allow us to relate the topologically significant quadratic duality on $F^2$ to a specific form of tensor duality. We then consider the symmetries of our multi-instanton solutions and show that in 8D the $k$-instanton $S^7$ bundles contain $16k - 7$ parameters in analogy with the $8k - 3$ parameters of the most general 4D $k$-instanton $S^3$ bundles.

## 2 Octonions

We recall (for a review, see e.g. [10]) that the nonassociative octonionic algebra has the multiplication rule $e_ie_j = -\delta_{ij} + f_{ijk}e_k$, where the $f_{ijk}$’s are completely anti-symmetric structure constants. The seven-sphere is described by a unit octonion $g$ satisfying $g^*g = 1$. The octonions’ nonassociativity complicates construction of the analog of a gauge theory. For example, for imaginary octonionic $A$ and $F = dA + A^2$, the corresponding $S^7$-gauge transformed quantities are

$$A_g = g^*Ag + g^*dg$$

and

$$F_g = g^*Fg$$

$$+ dg^*dg - (dg^*g)(g^*dg) + dg^*(Ag) - (dg^*g)(g^*Ag)$$

$$- g^*(Adg) + (g^*Ag)(g^*dg) - g^*A^2Ag + (g^*Ag)(g^*Ag).$$

(1)
Nonassociativity prevents the terms in the last two lines of (1) from canceling. Using \( g(g^\ast h) = h \), which holds for any octonions \( g \) and \( h \), we note that the terms do cancel in \( LF_g \), where \( L \) is the operator of left octonionic multiplication,

\[
L(a_1 \ldots a_n) = a_1(a_2(\ldots a_n)) \ldots .
\]

(2)

Any arrangement of parentheses in the argument of \( L \) give the same results on the right-hand side of (2). Use of the operator \( L \) allows us to perform various operations on the octonions as if they were associative. For simplicity in notations, we omit \( L \) in the following. Instead of left octonionic multiplication we could use right multiplication with the same result. From (1) we now find the familiar result \( F_g = g^\ast F_g \).

For associative \( A \) and \( F \), the forms \( tr F^n \) are closed. To extend this to octonions, which do not admit a matrix representation, we need an octonionic operator with some of the properties of the matrix trace. Consider the operator \( tr_O \) defined by

\[
tr_O L(a_1 \ldots a_n) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{(r_k+\ldots+r_n)(r_1+\ldots+r_{k-1})} L(a_k \ldots a_n a_1 \ldots a_{k-1}),
\]

(3)

where differential forms \( a_k \) are of degrees \( r_k \). The operators \( tr_O \) and \( d \) commute and so the forms \( tr_O F^n \) are closed; thus we arrive at the familiar Lie algebra result [11]:

\[
tr_O F^n = dQ_{2n-1},
\]

(4)

where

\[
Q_{2n-1} = n tr_O \int_{0}^{1} dt A \left[ tF + (t^2 - t)A^2 \right]^{n-1}.
\]

(5)

3 Linear duality

Since any pair of imaginary octonions generate a quaternionic subalgebra, we expect to find an octonionic duality condition which is reducible to its quaternionic counterpart. For example, let us define dual octonionic 2-forms according to

\[
\diamond (dx_\mu dx_\nu) = \frac{1}{2} f_{\mu\nu\rho\sigma} dx_\rho dx_\sigma,
\]

(6)

and determine the tensor \( f_{\mu\nu\rho\sigma} \) from the following two requirements: (i) any 2-form can be written as a sum of its self-dual and anti-self-dual parts, or equivalently, \( \diamond^2 = 1 \); (ii) \( dx dx^* \) is self-dual and \( dx^* dx \) is anti-self-dual. Consequently, for octonionic forms we obtain

\[
f_{\mu \nu \rho \sigma} = f_{\nu \mu \rho \sigma},
\]

\[
f_{\mu \nu \rho \sigma} = \frac{1}{3} f_{\mu \rho \nu \sigma},
\]

\[
f_{\mu \nu \rho \sigma} = \pm \frac{1}{3} f_{\mu \rho \nu \sigma} f_{\kappa \lambda \mu \tau} \mp (\delta_{\mu \kappa} \delta_{\nu \lambda} - \delta_{\mu \lambda} \delta_{\nu \kappa}).
\]

(7)
From Eqs. (6) and (7), the components of the \( \diamond \)-dual field strength \( F = \frac{1}{2} F_{\mu \nu} dx_\mu dx_\nu \) are subject to the following 21 relations:

\[ F_{ij} = \pm f_{ijk} F_{0k}. \quad (8) \]

Applied to the quaternions, the above requirements lead to the familiar relations \( f_{0abc} = f_{n0bc} = \epsilon_{abc} \) and \( f_{abcd} = 0 \). In both the quaternionic and octonionic cases, the components \( f_{\mu \nu \rho \sigma} \) are the matrix elements of the corresponding groups and cosets in the products \( Spin(4) = S_3^L \times S_3^R \) and \( Spin(8) = S_7^L \times G_2 \times S_7^R \).

Also, the components turn out to coincide with the elements of the torsion and curvature tensors of \( Spin(4)/Spin(3) \) and \( Spin(7)/G_2 \) respectively (for the latter see [12]). Note the two choices of sign for the curvature tensor \( f_{\mu \nu \rho \sigma} \) in (7) and the two choices of orientation, \( S_7^L,R = Spin(7)_{L,R}/G_2 \). Neither corresponds to the two choices of sign in Eq. (8).

Dual fields satisfy \( \diamond F = \pm F \) and, in view of the octonionic Bianchi identity \( D F = 0 \), they also solve an 8D generalization of the Yang-Mills equation \( D \diamond F = 0 \). Below we find multi-particle solutions to the duality equations.

4 Quadratic duality

In addition to the linear form of duality considered above, a quadratic form of duality is also possible in 8D. In the latter case, dual octonionic 4-forms are related via the Hodge star, \( "^\ast" \).

A conformally invariant action \( I = \text{tr}_O \int F^2 \ast F^2 \) yields the equation of motion \( \{ F, D \ast F^2 \} = 0 \). The \( \ast \)-dual fields, which are defined by

\[ \ast F^2 = \pm F^2, \quad (9) \]

solve the equation of motion by means of the Bianchi identity \( DF = 0 \).

In terms of (anti-)self-dual \( F^2 \pm = \frac{1}{4} (F^2 \pm \ast F^2) \), the action becomes

\[ I = \text{tr}_O \int \left( F^2 \ast F^2 + F^2 \ast F^2 \right). \quad (10) \]

On the other hand, the topological charge (the forth Chern number) is

\[ N = \frac{1}{384 \pi^4} \text{tr}_O \int F^4 = \frac{1}{384 \pi^4} \text{tr}_O \int (F^2 \ast F^2 - F^2 \ast F^2), \quad (11) \]

where we have used \( F^2 \ast F^2 = 0 \). It follows from (10) and (11) that the action is bounded from below,

\[ I \geq 384 \pi^4 |n|, \]

\[ \text{While our octonionic duality condition (8) is similar in form to one of the two duality conditions for } SO(8) \text{ considered in Ref. [7], the latter were not constructed to satisfy either of the two above-mentioned requirements. Consequently, our octonionic instantons are different from the } SO(8) \text{ solutions in Ref. [8].} \]
with minima achieved when \( F^2_{\pm} = 0 \), i.e. for the \(*\)-dual fields (9). There are one-particle solutions to the quadratic duality equations (9), and these solutions have a geometric interpretation in terms of the forth Hopf map [6].

It is remarkable but straightforward to verify that \( \diamond F = \pm F \) implies \( \ast F^2 = \mp F^2 \). To check this, we need the identity

\[
\delta^{(i[jf^j]_{lm]} = -\frac{1}{24} \epsilon^{klmnopr} f^{i}_{np} f^{j}_{qr},
\]

where indices included in braces (brackets) are to be symmetrized (antisymmetrized). We can also view \( \ast \) as a “square” of \( \diamond \). The relation between the linear and quadratic dualities allows us to proceed with the construction of octonionic multi-instantons.

5 Solution

The ADHM construction [5] gives the most general multi-instanton solutions to the duality equations in four dimensions. We construct octonionic dual fields by a suitable 8D generalization of the ADHM formula. Namely, consider a gauge potential [13]

\[
A = \frac{U^\dagger dU - dU^\dagger U}{2(1 + U^\dagger U)}, \quad U^\dagger = V(B - xI)^{-1},
\]

where the \( k \)-dimensional vector \( V \) and the \( k \times k \) matrix \( B \) have constant octonionic entries. The operator \( L \) is suppressed as usual, and the symbol “\( \dagger \)” means matrix transposition combined with octonionic conjugation. The corresponding field strength is

\[
F = (1 + U^\dagger U)^{-2} U^\dagger dxW dx^* U,
\]

where \( W^{-1} = V^\dagger V + (B^\dagger - x^* I)(B - xI) \).

For real \( W \), i.e. when

\[
V^\dagger V + B^\dagger B \text{ is real}
\]

and \( B \) is symmetric,

\( F \) involves the expression \( L(\ldots dxdx^* \ldots) \). The \( \diamond \)-dual of this 2-form is \( L(\ldots \diamond (dxdx^*) \ldots) \) and, owing to the self-duality of \( dxdx^* \), \( F \) is \( \diamond \)-self-dual itself, but \( F^2 \) is \( \ast \)-anti-self-dual. Interchanging \( x \) and \( x^* \), interchanges self-dual and anti-self-dual objects for both dualities.

6 Instanton number

For the solution obtained above, the gauge potential vanishes at infinity faster than a pure gauge, and has singularities at the instanton locations. A physically acceptable solution results from a suitable gauge transformation.
The singularities are located at eigenvalues \( \{ b_i \} \) of the \( k \times k \) matrix \( B \). Expanding around each singularity, we have approximately

\[
A \approx \frac{y_i^* dy_i - dy_i^* y_i}{2|y_i|^2(1 + |y_i|^2)} \quad \text{for} \quad y_i \to 0,
\]

where \( y_i = (x - b_i) V_i^* \). A gauge transformation with the gauge function \( g_i = y_i^*/|y_i| \) removes the singularity at \( y_i = 0 \) in the potential (15), and leads to

\[
A_{g_i} \approx \frac{y_i^* dy_i - dy_i^* y_i^*}{2(1 + |y_i|^2)} \quad \text{for} \quad y_i \to 0.
\]

Similar to the quaternionic case [14], all singularities inside a finite \( S^7 \) can be removed. Inside this \( S^7 \), after using (4), the instanton number becomes

\[
N = \int_{R^4} \text{tr} O F^4_g = \int_{S^7} (Q_7)_g,
\]

where asymptotically \((Q_7)_g \sim -\frac{1}{4} \text{tr} (g^* dg)^7 \). Since the field strength corresponding to the gauge potential \( g^* dg \) is zero, we use Stokes’s theorem again to replace the integral over the large \( S^7 \) by the sum of the integrals over \( k \) small spheres \( S^7_i \) enclosing singularities \( b_i \). Around each singularity, \( F_g \) looks like the field of a single anti-instanton at the origin,

\[
F_g = \frac{dx dx^*}{(1 + |x|^2)^2}.
\]

Therefore, the topological charge \( N \) and minus the instanton number \(-k\) are one and the same.

### 7 Parameters

We now count the number of parameters needed to describe a \( k \)-instanton. The octonions \( V \) and \( B \) have, respectively, \( 8k \) and \( 8\frac{1}{2} k(k + 1) \) real parameters. There are \( 7\frac{1}{2} k(k - 1) \) real equations in (14) constraining \( V \) and \( B \). When \( V \) is replaced by \( g^* V \), where \( g \in S^7 \) is constant, the potential (12) is gauge transformed, \( A \to g^* Ag \), eliminating 7 more parameters. Also, a transformation \( V \to VT \), \( B \to T^{-1} BT \) with real and constant \( T \in O(k) \), which has \( \frac{1}{2} k(k - 1) \) parameters, does not change \( A \). Therefore, the number of effective degrees of freedom describing a \( k \)-instanton is

\[
8k + 8\frac{1}{2} k(k + 1) - 7\frac{1}{2} k(k - 1) - 7 - \frac{1}{2} k(k - 1) = 16k - 7.
\]

We do not have a proof that the above construction gives all dual fields, although we suspect it does. At least it does so for the case of a one-instanton [6], which is described by 9 parameters—instanton’s scale and location. Perhaps completeness of the construction can be proved by using octonionic projective
spaces [15] and generalized twistors in analogy with the 4D case ([5, 13]). Other multi-instanton solutions are subsets of our solutions. For example, one can generalize Witten’s and ‘t Hooft’s [16] 4D multi-instanton solutions to 8D.

The single 8D instanton has entered string theory and produced a solitonic member of the brane scan (for a review, see [17]). We hope our general construction will facilitate further applications to string and M-theory, and perhaps in pure mathematics.

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