An inhomogeneous universe with thick shells and without a cosmological constant

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Abstract
We build an exact inhomogeneous universe composed of a central flat
Friedmann zone up to a small redshift \(z_1\), a thick shell made of anisotropic
matter and a hyperbolic Friedmann metric up to the scale where dimming
galaxies are observed (\(z \approx 1.7\)) that can be matched to a hyperbolic Lemaître–
Tolman–Bondi spacetime to best fit the WMAP data at early epochs. We
construct a general framework which permits us to consider a non-uniform
clock rate for the universe. As a result, for both a uniform time and a uniform
Hubble flow, the deceleration parameter extrapolated by the central observer
is always positive. Nevertheless, by taking a non-uniform Hubble flow, it is
possible to obtain a negative central deceleration parameter that, with certain
parameter choices, can be made the one observed currently. Finally, it is
conjectured as a possible physical mechanism to justify a non-uniform time
flow.

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1. Introduction

Supernovae-type Ia (SNIa) observations of the past decade seem to indicate an accelerating
universe [1–3]. In the standard approach with the Friedmann–Lemaître models (FRLW), an
accelerating universe invokes the presence of a large amount of the so-called dark energy. In
the FRLW picture, this dark energy is given by the cosmological constant. The dark energy
represents a puzzle and perhaps the biggest problem in modern cosmology. In fact, a direct
detection of a cosmological constant is still lacking. In the last decade, many attempts have
been made (see [4–24] and references therein) to obtain physically sensible inhomogeneous
models. Some authors (see for example [16, 17, 20–24]) showed that inhomogeneities
can generate an accelerating universe by using Lemaître–Tolman–Bondi (LTB) metrics (see
[25–27]), but several conditions must be imposed (see [4]) in order to build regular physically
viable models. In particular, in [21, 22] it is shown that LTB metrics can mimic the distance–redshift relation of the FLRW models at least at the third order in a series expansion with respect to the redshift near the center where the observer is located. More generally, in the LTB solutions, apparent acceleration in the redshift–distance relation seen by a central observer can be shown to coexist with a volume average deceleration on a spacelike hypersurface (see [28]). The assumption of spherical symmetry is (obviously) not in agreement with the Copernican principle. In any case, a spherical symmetry can be justified as the outcome of a smoothing out with respect to the angles: the metric so obtained becomes spherical.

An accelerating universe can also be built by averaging inhomogeneities (see [4–15]) by means of the techniques depicted in [8, 10, 11]. The approaches dealing by averaging on spatial domains are different with respect to the ones dealing with exact spherical solutions. First of all, averaged cosmologies retain the Copernican principle, although in a generalized statistical sense. Further, the exact spherically symmetric models try to describe apparent acceleration by means of a large amount of inhomogeneities. Conversely, the ‘Copernican’ cosmologist introduce inhomogeneities without special symmetries and then try to understand the modifications to the average evolution from backreaction. For a review of inhomogeneous cosmological models, see [29, 30]. In particular, we study the idea developed in Wiltshire’s papers [13–15]. There, the dimming of the distant galaxies is interpreted as a ‘mirage’ effect by means of a ‘Copernican’ statistical model. This effect is due to the different rate of clocks located in averaged not expanding galaxies, where the metric is spatially flat, with respect to clocks in voids where the spatial curvature is negative. With a negative spatial curvature can be associated a positive quasilocal energy. This gravitational energy is non-local, according to the strong equivalence principle. In this picture the universe is composed of a cosmic web of regions evolving asymptotically like an Einstein-de Sitter universe (our local universe) ‘matched’ with local voids evolving like a Milne universe. The matching conditions are imposed with a uniform Hubble flow by equating the radial null sections of the cosmic web. Although this matching is reasonable, it cannot be enough to define a geometry. In this context, inspired by Wiltshire’s idea, we introduce clock effects, but with an exact spherically symmetric model and without introducing backreaction. In this way we can build a geometry with a non-uniform time flow by imposing the usual matching conditions required by general relativity, which are missing in [13–15]. To mimic Wiltshire’s idea, our model is built with a central zone up to a small redshift \( z_1 \) (<100/\( h \) Mpc) made of a flat Friedmann metric, where the observer is located. In practice, the inhomogeneities can be taken into account by glueing together different homogeneous portions of the universe (see for example [24]). It should be noted that cosmological models with only two metrics can be exhaustively found in [31]. By means of a thick shell of anisotropic matter, the flat zone can be smoothly matched with a hyperbolic Friedmann metric up to the zone where dimming galaxies are observed [32]. Finally, we stress that the hyperbolic metric can be smoothly matched to a hyperbolic LTB solution on a comoving boundary surface, according to WMAP data [33]. Actually, we could choose the flat Friedmann one (without a cosmological constant!), but WMAP predicts a flat metric only at early times, while at later epochs a negatively curved metric is more appropriate. Our solution is exact and no approximations have been made, thus providing a general framework to analyze the effects of a possible non-uniform time flow. In this context, we show that, after a formal Taylor expansion of the distance–redshift relation near the center at \( z = 0 \), a negative deceleration parameter is only compatible with a non-uniform Hubble flow within a non-uniform time flow. The physical content of the thick shell is studied together with the energy conditions. Finally, it is conjectured that a non-vanishing heat flow term in the energy–momentum tensor of the thick shell can explain the possibly non-uniform time flow.
In section 2 we introduce the metrics of our model. In section 3 the junction conditions are discussed. In section 4 the matter content of the thick shell is studied. In section 5 the case of a uniform time flow is analyzed, while in section 6 the case of a uniform Hubble flow is presented. Section 7 is devoted to the study of the general case. Finally, section 8 collects some final remarks and conclusions.

2. The model

All the astrophysical observations agree with the assumption that our ‘near’ local universe is rather inhomogeneous. Nevertheless, at sufficiently large scales \( \sim 100/h \) Mpc the inhomogeneities can be averaged to obtain a statistically homogeneous universe. In any case, from the perturbations of the primordial inflation, a model with an overdense density surrounded by voids seems to be the most probable. Hence, we can build an inhomogeneous universe by glueing together different homogeneous volumes. As a consequence, the central region, with the observer located at the center, is provided by a flat Friedmann metric

\[
\begin{align*}
\text{ds}^2_B &= -\text{d}t^2_B + a^2_B(t_B) \left( \text{d}\eta^2_B + \eta^2_B \text{d}\Omega^2 \right), \\
a_B(t_B) &= a_B(t_B) \left( \frac{t_B}{t_{Bi}} \right)^{\frac{2}{3}},
\end{align*}
\]

where in (1) we have assumed a dust model and \( a_{Bi}, t_{Bi} \) are initial values. With (1), the Hubble flow \( H_B \) is given by

\[
H_B = \frac{1}{a_B} \frac{\text{d}a_B}{\text{d}t_B} = \frac{2}{3t_B}.
\]

The zone where the dimming galaxies are observed \( z \leq 1.7 \) is modeled with a hyperbolic Friedmann metric with negative spatial curvature and a common center with (1). In appropriate coordinates, the metric can be put in the form

\[
\begin{align*}
\text{ds}^2_F &= -\text{d}t^2 + \bar{\pi}(t)^2 \left[ \text{d}\eta^2_F + \sinh^2 \eta_F \text{d}\Omega^2 \right], \\
\bar{H}_i &= \frac{\bar{\pi}_i}{2(1 - \bar{\Omega}_i)} (\sinh \xi - \xi), \\
\bar{\pi}(t) &= \frac{\bar{\pi}_i \bar{\Omega}_i}{2(1 - \bar{\Omega}_i)} (\cosh \xi - 1), \\
\bar{\pi}(t)^2 \bar{H}^2 (1 - \bar{\Omega}_i) &= 1,
\end{align*}
\]

where \( \bar{\Omega}_i \) is an initial density parameter, \( \bar{H}_i \) an initial Hubble constant and \( \bar{\pi}_i \) an initial expansion factor to be specified. An observer in the portion of the universe given by (3) measures the observables by means of the comoving time \( t \). With respect to this time, an observer in (3) measures a Hubble flow with a time-dependent Hubble constant \( \bar{H} \) given by

\[
\bar{H} = \frac{1}{\bar{\pi}} \frac{\text{d}\bar{\pi}}{\text{d}t} = \frac{2\bar{H}_i \sinh \xi}{\bar{\Omega}_i (\cosh \xi - 1)^2} (1 - \bar{\Omega}_i)^{\frac{2}{3}}.
\]

The only way to smoothly match the metrics (1) and (3) is by means of a thick shell living in the region \( z \in [z_1, z_2] \). Without loss of generality, we can choose the metric of the thick shell mimic the expression of a LTB metric

\[
\text{ds}^2_{\text{thick}} = -e^{G(\tau, \eta)} \text{d}\tau^2 + \frac{R^2(\tau, \eta)}{f^2(\eta)} \text{d}\eta^2 + R^2(\tau, \eta) \text{d}\Omega^2.
\]
where $e^{G(\tau, \eta)}$ denotes the lapse function. Furthermore, to achieve agreement with WMAP data [33], the universe beyond the dimming zone could be modeled with a hyperbolic LTB spacetime (see [25–27])

$$ds^2 = -d\tilde{\tau}^2 + \tilde{R}^2(\tilde{\tau}, \tilde{\eta}) \, d\tilde{\eta}^2 + \tilde{f}^2(\tilde{\eta}) \, d\Omega^2, \quad \tilde{f}^2(\tilde{\eta}) > 1.$$  

(6)

In fact, WMAP forces us to conclude that at early epochs the LTB metric must approach a flat one and, as a result, we must impose the condition that the density parameter $\Omega_m$ approaches unity at early times (recombination era), i.e. $\Omega_m(\tau_{rec}) \to 1$. According to [34], at later epochs the metric could as well be taken to have a negative spatial curvature.

3. Matching conditions

First of all, it should be noticed that the matching of the Friedmann metrics of this paper can be obtained only by taking a thick shell. We perform the matching along comoving surfaces (the boundary of the thick shell) given by

$$\eta_B = \eta_B(1), \quad \eta = \eta(1), \quad \eta = \eta(2), \quad \eta_F = \eta_F(2),$$  

(7)

where the subscripts (1) and (2) denote the boundaries of the shell. For the thick shell, the continuity of the first and the second fundamental form [35, 36] leads to

$$\left(\frac{d\tau_B}{d\tau}\right)_{(1)} = e^{G(\tau_B(1))}, \quad \left(\frac{d\tau}{d\tau}\right)_{(2)} = e^{G(\tau_B(2))},$$  

(8)

$$R(\tau, \eta(1)) = a_B(t_B),$$  

$$R(\tau, \eta(2)) = a(t) \sinh \eta_F(2),$$  

$$f(\eta(1)) = 1, \quad f(\eta(2)) = \cosh \eta_F(2),$$  

(9)

$$G,\eta(\tau, \eta(1)) = G,\eta(\tau, \eta(2)) = 0.$$  

(10)

Together with equations (8)–(11), we have the ‘gauge’ condition

$$\frac{d\tau}{d\tau_B} = J(\xi), \quad \overline{H}(\xi) = \alpha(\xi) H_B(t_B).$$  

(12)

The function $J(\alpha(\xi))$ depends upon the chosen function $\alpha$. With condition (11), the heat flow vanishes at the boundaries of the thick shell (see the next section). To integrate the system (8)–(11), we can fix an expression for $G$ satisfying equation (11) and the relations $t = t(\tau, \eta), \quad t_B = t_B(\tau, \eta)$ that satisfy equation (8). In this way, thanks to equations (9), the behavior of $R(\tau, \eta)$ is fixed at the boundaries (1)–(2), and, as a result, we have the freedom to choose $R(\tau, \eta)$ inside the shell and so also the function $f$ with conditions (10).

For a smooth matching between (3) and (6) on a comoving boundary surface $\eta_F = \eta_F(3), \quad \eta = \eta(3)$, we have

$$\tilde{\tau} = \tau, \quad \tilde{R}(\tilde{\tau}, \tilde{\eta}(3)) = \overline{a}(t(3)) \sinh \eta_F(3), \quad \tilde{f}(\tilde{\eta}(3)) = \cosh \eta_F(3).$$  

(13)

(14)

(15)

It should be noted that there exists a relationship between the metrics (5) and (6). First of all, at the recombination era (early epochs), we have

$$\tau_{rec} \simeq \tilde{\tau}_{rec}, \quad G(\tau_{rec}, \eta) \simeq 0, \quad R(\tau_{rec}, \eta) \simeq \tilde{R}(\tilde{\tau}_{rec}, \tilde{\eta}).$$  

(16)
Furthermore, thanks to the matching conditions (8) and (13), we have (remember that $\eta(2)$ is a constant)

$$e^{G(\tau, \eta(1))} d\tau^2 = g(\tau) \, d\tau^2 = d\tilde{\tau}^2.$$  

(17)

Finally, without loss of generality, we could also set $\eta = \tilde{\eta}$ in (5) and (6).

Since the matching conditions have been discussed, we can give the formal expressions for the angular distance $dA$ and the distance luminosity $dL$

where

$$dL = dA(1 + z)^2.$$  

(see [37–39]).

In fact, thanks to conditions (8)–(11) and (13)–(15), we obtain (see [24])

$$dA = a_B(t_B)\eta_B, \quad z \leq z_1,$n

$$dA = R(\tau, \eta), \quad z \in [z_1, z_2],$$

$$dA = \pi(t) \sinh \eta_F, \quad z \in [z_2, z_3],$$

$$dA = \tilde{R}(\tilde{\tau}, \tilde{\eta}), \quad z \geq z_3,$$

(18)

where $z_3 (>1.7)$ represents the ‘starting point’ of the LTB metric. For our purpose, we are interested in the patch of the universe where the dimming galaxies are observed, i.e. the third of equations (18).

4. Energy–momentum tensor for the thick shell

In this section, we study the metric (5). The most general energy–momentum tensor $T_{ab}$ compatible with it is

$$T_{ab} = E V_a V_b + P_\perp [W_a W_b + L_a L_b] + P_\eta S_a S_b + K [V_a S_b + S_a V_b],$$  

(19)

with

$$V_a = [-e^{G/2}, 0, 0, 0],$$

$$W_a = [0, 0, R, 0], \quad L_a = [0, 0, 0, R \sin \theta], \quad S_a = \begin{bmatrix} 0, & \frac{R}{f}, & 0, & 0 \end{bmatrix},$$

(20)

(21)

where $E$ is the energy-density, $P_\eta$ the radial pressure, $P_\perp$ the tangential pressure and $K$ the ‘heat flow term’ or radial energy flux. In particular, Einstein’s equations for $K$ give

$$K = -\frac{f R \, G_{\eta}}{RR_\eta \, e^{G/2}}.$$  

(22)

The regularity conditions require that $(R, \eta, R, f) \neq 0$. Hence, from equation (22), it follows that $K = 0$ if and only if $G_{\eta} = 0$. Therefore, a non-trivial lapse function is only compatible with a non-vanishing energy flux. If we take the most general expression for a spherically symmetric metric that is

$$ds^2 = -e^{G(\tau, \eta)} d\tau^2 + \frac{A^2(\tau, \eta)}{f^2(\eta)} \, d\eta^2 + B^2(\tau, \eta) \, d\Omega^2,$$

the energy flux for (23) vanishes if and only if

$$G_{\eta} = 2(\ln B, \eta) - 2(\ln A, \tau) \frac{R_\eta}{B, \tau}.$$  

(24)

To the best of our knowledge, the only non-static metric with a non-barotropic equation of state satisfying equation (24) is the Stephani metric [40]. Remember that a time-dependent spherical matter cannot have a barotropic equation of state ($E = E(P)$) within a finite radius [41]. It is a simple matter to see that the spherically symmetric Stephani spacetime cannot
satisfy the matching conditions (8)–(11). As a result, a link between the heat flow and a non-trivial lapse function can be conjectured, at least for spherically symmetric spacetimes. Should this argument be correct, we would have a possible physical mechanism to generate clock effects.

In what follows, starting from conditions (8)–(11), we analyze the possibility of building a physically reasonable anisotropic thick shell. For simplicity, we study the case $G = 0 (t = t_B)$. Hence, the matching conditions (9)–(10) can be fulfilled by taking, for example,

$$R = \eta \, i^2 \bar{Y}^2 + \alpha(\eta) \bar{Y}^2,$$

$$f(\eta) = Y^2 + \sqrt{1 + \eta^2 \bar{Y}^2}, \quad Y = 1 - \frac{(\eta - \eta_1)^2}{(\eta_2 - \eta_1)^2}, \quad \bar{Y} = Y = 1 - \frac{(\eta_2 - \eta)^2}{(\eta_2 - \eta_1)^2},$$

where, without lost of generality, we have taken (only for this section!)

$$a_{B_i}(t_{rec}) = 1, \quad t_{B_i} = t_{rec} = 1, \quad \eta = \sinh \eta_f, \quad \eta_B = \eta,$$

and $\eta_1, \eta_2$ denote the location of the comoving shell. Furthermore, we have set in (27) the scale of times to be unity at the recombination era.

Regularity conditions impose that $R > 0, R, \eta > 0$. By performing, after fixing the time, a Taylor expansion near the boundaries of the thick shell, we have

$$E(t, \eta \simeq \eta_1) = \frac{4}{3t^2} + o(1),$$

$$E(t, \eta \simeq \eta_2) = \frac{3\bar{\eta}^2 - 1}{\bar{\eta}^2} + o(1) \simeq \frac{2}{t^2} + o(1),$$

$$P_\eta(t, \eta \simeq \eta_1) = s(\eta - \eta_1)^2 + o(1),$$

$$P_\eta(t, \eta \simeq \eta_2) = \frac{2}{\bar{\eta}^2} \frac{(\eta_2 - \eta)}{\bar{\eta}^2} + o(1),$$

$$P_{\perp}(t, \eta \simeq \eta_1) = s\eta_1(\eta - \eta_1) + o(1),$$

$$P_{\perp}(t, \eta \simeq \eta_2) = \frac{2\bar{\eta}^2}{\bar{\eta}^2} \bar{\eta}^2 + \bar{\eta}^2 \sim -\frac{1}{t^2} + o(1),$$

$$s = \frac{4t^{-1}}{9\eta_1^2(\eta_2 - \eta_1)^2} [t^2 - (18\sqrt{1 + \eta_2^2 - 9} + 4\bar{\eta}^2 - 18\eta_1^2 t^2 \bar{\eta}^2 - 12\eta_1^2 \bar{\eta}^2)],$$

where the symbol $\simeq$ on the right-hand side means that the expression is evaluated for $t \gg 1$. From equations (28), although the tangential pressure can have a negative value near $\eta = \eta_2$, the energy conditions follow near the boundaries of the thick shell. This means that physically reasonable thick shells can be built with our matching conditions. It should be stressed again that the junction conditions (8)–(11) only fix the behavior on the boundaries (1)–(2).

In the next three sections we apply the technology developed above to physically interesting situations.

5. Case with $G = 0$

First of all, we must integrate along the past null cone inward. Generally, we have there an equation given by

$$dT = -\frac{A(T, \eta)}{f(\eta)} \, d\eta,$$

where $t_B = \tau = t = T$. 

6
Following Célérier (see [21, 22]), we obtain for the redshift $z$
\[\frac{dT}{dz} = -\frac{A(T(\eta), \eta)}{(1 + z)A(T(\eta), \eta)}.\] (30)

We are interested in the determination of the distance–redshift formula for the dimming galaxies. Therefore, from the third of equations (18), we get
\[d_L = (1 + z)^2 \frac{\bar{a}(t) \sinh \eta_F}{(1 - \Omega_0)} F(1 - \Omega_1) \sinh(\xi_0 - \xi).\] (31)

Obviously, we must solve equation (30) in the three regions (hyperbolic Friedmann, thick shell and flat Friedmann) up to an observer located at $z = 0$.

For $z \in [z_2, z]$, $(A(T(\eta), \eta) = \bar{a}(T), f(\eta) = 1)$, we have
\[\frac{1 + z}{1 + z_2} = a(T_2) \frac{\bar{a}(T_2)}{\bar{a}(T)}.\] (32)

For $z \in [z_1, z_2]$, we have $(A(T(\eta), \eta) = R, \eta)$:
\[\ln \left(\frac{1 + z_2}{1 + z_1}\right) = -\int_{T_1}^{T_2} \frac{R_{R,T}}{\bar{a}(T)} dT = \gamma(T_2, T_1).\] (33)

From (33) we get
\[\frac{1 + z_2}{1 + z_1} = e^\gamma,\] (34)
where $\gamma > 0$ and physical plausibility requires that $e^\gamma$ is of the order of unity ($\geq 1$).

For the time flow, we read
\[t = t_B = T = \frac{\Omega_0}{2H_0(1 - \Omega_0)^2} (\sinh \xi - \xi).\] (35)

Finally, thanks to (35), for $z \leq z_1$ we obtain $(A(T(\eta), \eta) = a_B(T), f(\eta) = 1)$:
\[1 + z_1 = a_B(T_0) a_B(T_1) \left(\frac{\sinh \xi_0 - \xi_0}{\sinh \xi_1 - \xi_1}\right).\] (36)

with the subscript ‘0’ denoting the actual time at $z = 0$. By multiplying equations (32)–(36), we get
\[1 + z = e^\gamma \frac{\bar{a}(T_2)}{\bar{a}(T)} a_B(T_0) a_B(T_1).\] (37)

For $\eta_F$ along the past null cone, we have
\[\eta_F = \xi_0 - \xi, \quad \xi \leq \xi_0.\] (38)

As a result, since $\Omega(\xi) = \frac{2}{1 + \cosh \xi}$, equation (37) becomes
\[1 + z = 2F(1 + z_1) \frac{(1 - \Omega_1)}{\Omega_1(\cosh \xi - 1)},\] (39)
where ‘1’ refers to the time $T_1$ (or $\xi_1$) and
\[F = e^{\gamma_0} \frac{(\cosh \xi_2 - 1)}{(\cosh \xi_1 - 1)}.\] (40)

Physical plausibility requires that $F$ be of the order of unity. Thanks to (3) and (39), formula (31) becomes
\[d_L = \frac{\Omega_0(1 + z)(1 + z_1) F(1 - \Omega_1)}{H_0(1 - \Omega_0)^2} \sinh(\xi_0 - \xi).\] (41)
Equations (39) and (41) hold for $z \geq z_2$ and obviously the junction conditions only imply the continuity of $dL$ and not its analyticity when crossing the boundaries of the model. This obviously applies to any inhomogeneous model built by matching two or more metrics (as for example in [13, 15, 24]). Nevertheless, to make contact with the astrophysical data at intermediate redshifts, a central observer can formally expand expression (41) in a Taylor series in the range $z_1 \ll 1, z_2 - z_1 \ll 1$ ($\Omega_0 \simeq \Omega_1, F \simeq 1$). To the first order in $z$ we formally obtain

$$dL = \frac{z}{H_{\text{obs}}} + o(z),$$

$$H_{\text{obs}} = \frac{\epsilon H_0 (1 - \Omega_0^{1/3}) \Omega_1}{F (1 - \Omega_1) (1 + z_1) \Omega_0} P,$$

$$P = \sqrt{F \epsilon (1 - \Omega_1) (1 + z_1)},$$

$$\epsilon = -\frac{\Omega_1 [F (1 + z_1) - 1] + F (1 + z_1)}{\Omega_0^{2/3}}.$$ 

(42)

Therefore, in our inhomogeneous universe, after writing the correct matching conditions, a central observer extrapolates an effective Hubble flow given by (42). It is worth noticing that, in the limit $F = 1, \Omega_1 \rightarrow \Omega_0, z_1 = 0, \epsilon = 1$, in which the full spacetime is composed only with the hyperbolic Friedmann metric, we have $H_{\text{obs}} \rightarrow H_0$, a correct result. Furthermore, the inequality $H_{\text{obs}} \neq H_0$ in a general inhomogeneous universe is compatible with the fact that in such spacetimes we do not have a unique definition of an observed Hubble flow (see [42]): a direct way is to infer its value by a formal Taylor expansion (if this is possible) near the observer. The extrapolated central deceleration parameter $q_0$ is given by

$$q_0 = -H_{\text{obs}} \frac{d^2}{dz^2} (dL(z = 0)) + 1.$$ 

(43)

Note that, from equation (36), we could express $\Omega_1$ as a function of $z_1$, $\xi_0$, although this is not necessary for our purposes. Equation (43), thanks to (41), gives

$$q_0 = \frac{\Omega_1}{2 \epsilon}.$$ 

(44)

From equation (42), we must have $\epsilon > 0$ ($H_{\text{obs}} > 0$) and therefore $q_0 \geq 0$, for all times $T$, and no ‘formal’ acceleration is perceived by the central observer by considering the distance-redshift function. At early times ($\xi \simeq 0$) we have $q \rightarrow \frac{1}{2}$ and $q \rightarrow 0^+$ asymptotically (for $\xi_1 \rightarrow \infty$).

6. Uniform Hubble flow

The case of a uniform Hubble flow has been studied in [13, 15] in the context of the Buchert equations with backreaction [8]. In [13, 15] the overdensity evolves asymptotically as an Einstein–de Sitter spacetime, while the underdensity as a Milne universe. As a result, our model can be considered as representing the far future limit ($t \rightarrow \infty$) of [13, 15], where backreaction is asymptotically vanishing. With the uniform Hubble gauge, we have

$$\Omega_1 = H_B, \quad \alpha = 1, \quad J(\xi) = \frac{dt}{d\mu} = \frac{3}{2} \frac{1 + \cosh \xi}{2 (1 + \cosh \xi)}.$$ 

(45)

The calculations are similar to the ones of the last section. However, the central observer measures the redshift with respect to its proper time $t_B$. As a result, along the past null cone
we have

\[ dt_B = -\frac{\alpha}{J(\xi)} \frac{d\eta_F}{\bar{a}(\xi)} \rightarrow \eta_F = \xi_0 - \xi \]  

(46)

Instead of equation (33), we read

\[ \frac{1 + z_2}{1 + z_1} = e^{\gamma} \simeq \frac{J(\xi_2)}{J(\xi_1)} \left( 1 + \frac{\tau_2}{\tau_1} \right) \]  

(47)

\[ \frac{(1 + \tau_2)}{(1 + \tau_1)} = - \int_{\tau_1}^{\tau_2} \frac{e^{\gamma}}{R_{\eta}} \left( \frac{R_{\eta}}{e^{\gamma}} \right) d\tau, \]  

(48)

\( \bar{z}_1, \bar{z}_2 \) being the redshifts measured by a comoving observer with time \( \tau \). The approximation (\( \simeq \)) in (47) has been given as an example and is valid when \( \xi_2 \simeq \xi_1 \). It does not enter in the calculations of this section. Expression (31) becomes

\[ d_L = (1 + z)^2 \frac{\alpha(t)}{J(\xi)} \sinh(\xi_0 - \xi). \]  

(49)

Concerning the relation between \( t_B \) and \( t \), we get

\[ t_B = \frac{\overline{\Omega}_0}{3H_0 (1 - \overline{\Omega}_0)^{\frac{1}{2}}} \left( \frac{\cosh \xi - 1}{2 + \cosh \xi} \right). \]  

(50)

Instead of equation (36), we have

\[ 1 + z_1 = \left( \frac{t_B}{t_B^0} \right)^{\frac{1}{2}}. \]  

(51)

After defining

\[ e^\gamma = F \frac{(\cosh \xi_1 - 1)(2 + \cosh \xi_1)(1 + \cosh \xi_2) - (\cosh \xi_2 - 1)(2 + \cosh \xi_2)(1 + \cosh \xi_1)}{\cosh \xi - 1}, \]  

(52)

and with the same technique of the last section, we obtain

\[ \cosh \xi = -\frac{1}{2} + \frac{Q}{2(1 + z)} + \sqrt{\frac{9(1 + z)^2 + 2Q(1 + z) + Q^2}{2(1 + z)}}, \]  

(53)

\[ Q = F \frac{(1 + z_1)(1 - \overline{\Omega}_1)(2 + \overline{\Omega}_1)}{\overline{\Omega}_1^2}. \]  

(54)

\[ d_L = \frac{(2 + \cosh \xi)(\cosh \xi - 1) \sinh(\xi_0 - \xi)}{3H_0 (1 - \overline{\Omega}_0)^{\frac{1}{2}}(1 + \cosh \xi)} (1 + z)^2. \]  

(55)

Performing a formal Taylor expansion of (55) at \( z = 0 \), we again get an effective observed \( H_{\text{obs}} \), and by means of equation (43) we can obtain the central deceleration parameter. In any case, we always have \( q = \frac{1}{2} \) at early times. Furthermore, the extrapolated parameter \( q_0 \) remains positive and approaches zero as follows:

\[ q(\xi_1 \rightarrow \infty) \rightarrow \frac{1}{Q_0}. \]  

As a result, in the presence of a uniform Hubble flow, \( q_0 \) goes to zero more rapidly than in the case \( G = 0 \) (see equation (44)), albeit always from positive values. In practice, we recover the results of [14], but imposing the correct matching conditions.
7. The general case

In the general case, $\mathcal{H} = \alpha H_B$. Hence, the relation between $t_B$ and $t$ becomes

$$t_B = \alpha(\xi) \frac{\Omega_0}{3\mathcal{H}_0(1 - \Omega_0)} \frac{(\cosh \xi - 1)^{\frac{1}{2}}}{(1 + \cosh \xi)}.$$  

(56)

For the lapse factor $J(\xi)$, we obtain

$$J(\xi) = \frac{dt}{dt_B} = \frac{3}{2} \beta(\xi),$$

$$\alpha(\xi) \frac{(\cosh \xi - 1)}{\sinh \xi} + \alpha(2 + \cosh \xi) \frac{1}{(1 + \cosh \xi)} = \beta^{-1}(\xi).$$  

(57)

To explore the case with a non-uniform Hubble flow with $G \neq 0$, we can take

$$\beta(\xi) = \frac{(1 + \cosh \xi)}{(A \cosh \xi + 3 - A)}, \quad A > 0.$$  

(58)

The case with $A = 1$ has been studied in the section above. The calculations are similar to the ones of section 6 and after posing

$$e^\nu = \frac{1 + z_2}{1 + z_1} = \frac{F(\Omega_2)(1 - \Omega_1)(2A - 2A\Omega_2 + 3\Omega_1)}{\Omega_1(1 - \Omega_2)(2A - 2A\Omega_2 + 3\Omega_2)},$$  

(59)

we get

$$d_L = \frac{B(1 + z_1)\Omega_0}{3\mathcal{H}_0(1 - \Omega_0)^{\frac{1}{2}}} \sinh(\xi_0 - \xi),$$  

(60)

$$1 + z = \frac{B(1 + \cosh \xi)}{(A \cosh \xi + 3 - A)(\cosh \xi - 1)}$$

(61)

$$B = \frac{F(1 + z_1)(1 - \Omega_1)(2A - 2A\Omega_2 + 3\Omega_1)}{\Omega_1}. $$  

(62)

The expressions for $H_{obs}$ and $q_0$ are rather cumbersome. However, some general remarks can be made on the behavior of the extrapolated central deceleration parameter $q_0$ at different values of the parameter $A$. First of all, $\forall A \in (0, \infty)$, $q_0(\xi = 0) = \frac{1}{2}$. Furthermore, $\forall A \in [1, \infty)$, the parameter $q_0$ is always positive and asymptotically $q_0(\xi \rightarrow \infty) \rightarrow 0^+$. Conversely, $\forall A \in (0, 1)$, the parameter $q_0$ becomes negative at some value of $B$, and after an absolute minimum reaches $0^-$ asymptotically but from negative values. As an example, for $A = \frac{1}{2}, q_0 = 0$ for $B_0 \simeq 5.3$ and for $B \simeq 10$ we have $q_{0\text{min}} \simeq -0.075$. For $A = \frac{1}{2}, q_0 = 0$ for $B_0 \simeq 4.2$ with $q_{0\text{min}} \simeq -0.16$ at $B \simeq 8$. For $A = \frac{1}{4}, q_0 = 0$ for $B_0 \simeq 3.6$ with $q_{0\text{min}} \simeq -0.31$ at $B \simeq 6$. For $A = \frac{1}{5}, q_0 = 0$ for $B_0 \simeq 3.2$ and $q_{0\text{min}} \simeq -0.51$ at $B \simeq 5$. In any case, $\forall A \in (0, 1)$, it is always possible to have, from equation (62), reasonable values for $F$ ($\simeq 1$, with $z_2 - z_1 \ll 1$) and $\Omega_0$ compatible with an apparent acceleration at some later time calculated by means of equation (36), provided that $B > B_0$. The inequality $B > B_0$ imposes a constraint on $\Omega_1 \simeq \Omega_0$. As an example, for $F = 1.1, z_1 = 0.01$ and $A = \frac{1}{2}$ we have an accelerated universe if and only if $\Omega_1 \simeq \Omega_0 < 0.24$, while for $A = \frac{1}{4}$ and $F(1 + z_1) = 1, z_1 \ll 1$, we have $\Omega_1 \simeq \Omega_0 < 0.22$. Further, for $A = \frac{1}{5}, F = 1.1, z_1 = 0.01$, we have formal acceleration when $\Omega_1 \simeq \Omega_0 < 0.25$. It is also possible to mimic for $A \leq \frac{1}{5}$ a value for $q_0$ compatible with the actual observations. Furthermore, note that the models with $A \in (0, 1)$ represent the case with $\alpha(\xi) < 1$, i.e. $H_B > \mathcal{H}$ (this can be seen by noting that
equation (57) has, in the limit $\xi \gg 1$, the tracker solution $\alpha = A$. As a result, the clock effects depicted in this paper can mimic a model with a large underdensity surrounded by an overdensity (see [24]). In conclusion, if the clock effects depicted in this paper (and in [13, 15]) exist in the real universe, the actual data at intermediate redshifts are in agreement with a non-uniform Hubble flow.

8. Conclusions

We built a model for the universe without dark energy by means of an exact spherically symmetric solution taking into account the observed inhomogeneous universe. The main purpose of this paper is to show how the non-uniform time flow depicted in [13, 15] can be obtained within an exact solution of Einstein’s equations by imposing the correct matching conditions required by general relativity. In this sense, since the backreaction is absent in our model, our approach is different from Wiltshire’s, where the backreaction is analyzed in a statistical (‘Copernican’) model within the Buchert formalism.

The model is composed of three regions, a central flat Friedmann metric, a hyperbolic Friedmann zone and eventually a bulk LTB hyperbolic metric, according to WMAP. Within our exact solution, it is shown that, after a ‘formal’ Taylor expansion of the distance–redshift relation near the observer and by imposing the correct matching conditions, a uniform Hubble ‘gauge’ (present in [13, 15]) does not lead to an ‘apparent’ acceleration as extrapolated from the redshift–distance relation. In a purely spherically symmetric universe, such an acceleration can only be obtained with a non-uniform Hubble flow. Furthermore, in our model we have a parameter ($A$ in the paper) at our disposal that permits us to obtain a large amount of ‘apparent’ acceleration which is consistent with the other parameters of the model (for example the thickness $z_2 - z_1 \ll 1$). Furthermore, in the presentation given in [13, 15], the physical mechanism that can generate a non-uniform time flow is not yet clear. In fact it is always possible to build an inhomogeneous universe with a global cosmic time by means, for example, of LTB metrics. An exact formulation requires that the junction conditions are fulfilled only by means of a thick shell. We have shown a possible link between a non-uniform time flow and a radial energy flux present in the energy–momentum tensor of the thick shell. Hence, this ‘heat flow’ term can give a possible physical explanation for the clock effects depicted in [13, 15] (if they exist!). The introduction of thick shells can be useful to explore exact models obtained by glueing different Friedmann metrics. An anisotropic thick shell is certainly an unusual choice, but this alleviates the drawbacks of a whole universe filled with an exotic dark energy. It should be noted that the radial flux energy vanishes on the boundaries of the thick shell. In fact the radial symmetry inhibits a ‘heat flow’ between the flat central Friedmann metric and the ‘dimming’ hyperbolic Friedmann one. A more realistic model could be obtained by relaxing the spherical symmetry and substituting the Friedmann metrics with more general ones, admitting a non-vanishing energy flux. Unfortunately, nowadays such metrics are not at our disposal. In any case, our calculations can suggest an improvement over Wiltshire’s model. In fact, Wiltshire’s paper neglects the shear, but this encodes fundamental information related to the variation of non-local gravitational energy, which is a fundamental ingredient in [13, 15]. As is well known, it is not a simple task to relate the shear to physical observable quantities (see [34]). To this purpose, Wiltshire’s model could be amended by taking

\begin{align}
(\langle \theta \rangle)_{f1} &= 0, & (\langle \sigma^2 \rangle)_{f1} &= 0, \\
(\langle \theta \rangle)_s &= 3H_s, & (\langle \sigma^2 \rangle)_s &\neq 0,
\end{align}

(63) (64)
\[(\langle \theta \rangle)_v > 0, \quad (\langle \sigma^2 \rangle)_v = 0, \quad (65)\]

where, following the notation of [13], ‘fi’ stands for ‘finite-infinity’ and ‘v’ for ‘voids’ and \(H_s\) is the averaged Hubble parameter for the thick shell. Hence, as suggested by the matching conditions, a third scale between ‘fi’ and ‘v’ with a non-vanishing shear is introduced: this is the scale at which variations in the flux energy are appreciable.

Finally, note that the dark energy appears in two ‘phase transitions’ for the universe: the formation of the big structures and the end of it. Hence, since cosmic strings and superstrings in the context of the M-theory are supposed to have acted during the inflation epoch to give (in principle) observable effects a later time, a link between them and the thick structures depicted in this paper can be suggested (see [43]).

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