On a Lie algebraic structure associated with a non-linear dynamical system

J.R. Guzmán *

February 13, 2019

Abstract

A family of Lie algebras of minimal dimension associated with vector fields that define a non-linear dynamical system is calculated. These Lie algebras contain the Heisenberg algebra.

An element that distinguishes these vector fields is called evapotranspiration function. This function can be calculated solving equations in partial derivatives that arise in determining the Heisenberg algebra.

Using Kozsul homology for this Lie algebras, Euler characteristic is calculated.

1 Introduction

The system that appears in [2] provided by the differential equation system below, is considered:

*Economía Aplicada. Instituto de Investigaciones Económicas. Universidad Nacional Autónoma de México. e-mail: jrg@unam.mx. This article was carried out with the support of a grant from from UNAM-DGAPA during the sabbatical year in the PUIMECI of the UACH.
\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix}
= \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
+ \begin{pmatrix}
\beta_{11} x + \beta_{12} E(x, y) \\
\beta_{22} y + \beta_{22}' E(x, y) + \beta_{13} z \\
\beta_{32} y + \beta_{33} z
\end{pmatrix}
+ \begin{pmatrix}
\gamma_{11} \\
\gamma_{21} \\
0
\end{pmatrix}
+ f \begin{pmatrix}
\gamma_{12} \\
\gamma_{22} \\
0
\end{pmatrix}
+ w \begin{pmatrix}
\gamma_{13} \\
\gamma_{23} \\
0
\end{pmatrix}.
\]

This dynamical system is a simplification of the original system; \( x, y \) and \( z \) denote \( x_i, T_i, T_m \); are the state variables that represent relative humidity, air temperature and thermal mass temperature. It is a dynamic system that has been formulated using thermodynamic principles.

In this system, all the thermodynamic functions the authors propose have been substituted. The letters \( \alpha, \beta, \gamma \), and their sub-indices depend on original parameters.

\( \alpha, f, w \) are control variables that represent the window opening angle, nebulization system intensity and heating system intensity, respectively. Although all of this work is based on what is itself a dynamic control system, these variables are not used in the subsequent.

\( \alpha, \beta, \gamma \) and sub-indices are dynamic system parameters; the signs of the parameters are \( \alpha_i > 0 \), for \( i = 1, 2, 3; \beta_{11}, \beta_{22}, \beta_{33}, \beta_{22}' < 0; \beta_{12}, \beta_{13}, \beta_{32} > 0; \gamma_{11}, \gamma_{12}, \gamma_{23} > 0; \gamma_{21}, \gamma_{22} < 0; \gamma_{13} = 0. \)

Although in the original article \([2]\) appears \( \gamma_{13} \) as null, is considered here \( \gamma_{13} > 0 \); for consistency with the units specified in the dynamic system should be \( \gamma_{13}[kg_{air}^{-1}] \) units. This change represents a heating on the humidity variable.

It is worth pointing out that in the original dynamic system, \( E(x, y) \) is the evapotranspiration function, only appears as a function of \( y \). In this article, evapotranspiration \( E \) is generalized and made to depend on \( x \).

Crops in a greenhouse are subject to an internal dynamic that creates evapotranspiration \( E[kgH_2Os^{-1}] \); this function provides a number of estimation proposals, depending on each author. Evapotranspiration research is an active research field, as testified to by \([4]\), for example. Function \( E \), is functioning as an integral part of the model, and dynamic richness is lent to the system in question. This definition can be stipulated since evapotranspiration estimation is an open problem.
In this article, a family of functions of evapotranspiration depending on the $\gamma_{ij}$ parameters, which naturally arises from the algebraic structure, can be calculated to define Lie algebra structure on the dynamic system into consideration.

More specifically, this dynamic system is of the form

$$\dot{X} = f_0(X) + \sum_{i=1} u_i f_i(X),$$

often investigated in control theory; $X$ is a curve in $\mathbb{R}^n$, $f_i$'s are sufficiently differentiable functions, $u_i$'s are the control variables. In the case that concerns,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad f_0(X) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \begin{pmatrix} \beta_{11}x + \beta_{12}E(x, y) \\ \beta_{22}y + \beta_{22}'E(x, y) + \beta_{13}z \\ \beta_{32}y + \beta_{33}z \end{pmatrix}$$

and $f_i(X) = \begin{pmatrix} \gamma_{1i} \\ \gamma_{2i} \\ 0 \end{pmatrix}$, $i = 1, 2, 3$.

In addition $u_1 = \alpha$, $u_2 = f$, $u_3 = w$. With these functions it is possible to define vector fields that are used to elucidate this algebraic structure of this dynamic system, (see below, in section of proof of proposition 1). The Lie bracket is the usual for vector fields.

From these calculations and contained in the infinite dimensional family of Lie algebras it is had a minimal dimension family of Lie algebras in which is contained the Heisenberg algebra.

As it is well known, this algebra plays an important role in investigations of energy of quantum systems. This algebra began to be considered by the physicist Werner Heisenberg in relation to the quantic oscillators. For current research on this topic see [5]. For a pleasant way of explanation about this, see [1]. Another example is in its application to gravitational physics of [8]. The Heisenberg algebras play a central role in the research of new ways of understanding the gravitational field.

By completeness is calculated the homology of the family of Lie algebras $\mathfrak{a}_E$. The homology is coincident with the results known to the case of the $\mathfrak{h}_3$ algebra. The homology of these algebras to been an object of a deep study. The pure Lie algebra homology has been considered in work starting with [7] up to [3], for example.
In general, there are investigations that are a central part of the homology and dynamical systems. For a review of the most important theorems, see [6].

The relevance of these results is that when conditions searching for the definition of the dynamic system underlying Lie algebra, a function of evapotranspiration is the solution of partial differential equations that arise in applying algebraic Lie tools.

2 Main results

The following are the main results.

**Proposition 2.1** The family of Lie algebras $\mathfrak{a}_E$, of minimal dimension 7, is specified by conditioning that $\Delta_{ij}E(x,y) = c_{ij}$; and $\Delta_{ijk}E(x,y) = 0$ and $\Lambda_{ijk}E(x,y) = 0$.

$\mathfrak{a}_E$ is the Heinsenberg algebra if $\Delta_{ij}E(x,y)$ does not equal 0 for $i = j$ and $\Delta_{ij}E(x,y) = 0$, for $i \neq j$; where

$$\Delta_{ij}E(x,y) = \gamma_{1i} \gamma_{1j} \frac{\partial^2 E}{\partial x^2} + 2 \left( \gamma_{1i} \gamma_{2j} + \gamma_{1j} \gamma_{2i} \right) \frac{\partial^2 E}{\partial x \partial y} + \gamma_{2i} \gamma_{2j} \frac{\partial^2 E}{\partial y^2};$$

and

$$\Delta_{ijk}E(x,y) = \gamma_{1i} \gamma_{1j} \gamma_{1k} \frac{\partial^3 E}{\partial x^3} + \gamma_{2i} \gamma_{2j} \gamma_{2k} \frac{\partial^3 E}{\partial y^3} + \frac{\partial^3 E}{\partial x^2 \partial y} \sum_{(\sigma_1, \sigma_2, \sigma_3) \in m_3(4)} \gamma_{\sigma_1} \gamma_{\sigma_2} \gamma_{\sigma_3} + \frac{\partial^3 E}{\partial x \partial y^2} \sum_{(\sigma_1, \sigma_2, \sigma_3) \in m_3(5)} \gamma_{\sigma_1} \gamma_{\sigma_2} \gamma_{\sigma_3};$$

where $m_k(n)$ is the magic square of $k \times k$ with magic sum $n$. The notation $\sigma \in m_k(n)$ means taking line $\sigma$ from $m_k(n)$.

And
\( \Lambda_{ijk}(x, y) = \)
\[
\beta_{12}\gamma_{1i}\gamma_{1j}\gamma_{1k} \left( \frac{\partial^2 E}{\partial x^2} \right)^2 + \beta_{22}\gamma_{2i}\gamma_{2j}\gamma_{2k} \left( \frac{\partial^2 E}{\partial y^2} \right)^2 \\
+ (\beta_{12}\gamma_{2i}\gamma_{2j}\gamma_{1k} + \beta_{22}\gamma_{1i}\gamma_{1j}\gamma_{2k}) \frac{\partial^2 E}{\partial x^2} \frac{\partial^2 E}{\partial y^2} \\
+ (\beta_{12}\gamma_{1i}\gamma_{2j}\gamma_{2k} + \beta_{12}\gamma_{1i}\gamma_{1j}\gamma_{2k} + \beta_{22}\gamma_{1i}\gamma_{2j}\gamma_{1k} + \beta_{22}\gamma_{1i}\gamma_{1j}\gamma_{2k}) \frac{\partial^2 E}{\partial x^2} \\
+ \left( (\beta_{12}\gamma_{1i}\gamma_{1j}\gamma_{2k} + \beta_{12}\gamma_{1i}\gamma_{2j}\gamma_{1k} + \beta_{22}\gamma_{1i}\gamma_{1j}\gamma_{1k} + \beta_{22}\gamma_{1i}\gamma_{2j}\gamma_{1k}) \frac{\partial^2 E}{\partial x^2} \right) \frac{\partial^2 E}{\partial x \partial y} \\
- \left( \beta_{12}\gamma_{1i}\gamma_{1j}\gamma_{1k} + \beta_{12}\gamma_{1i}\gamma_{2j}\gamma_{2k} + \beta_{22}\gamma_{1i}\gamma_{1j}\gamma_{2k} + \beta_{22}\gamma_{1i}\gamma_{2j}\gamma_{1k} \right) \frac{\partial^2 E}{\partial x \partial y} \\
- \left( \beta_{12}\gamma_{1i}\gamma_{2j}\gamma_{2k} + \beta_{12}\gamma_{2i}\gamma_{1j}\gamma_{2k} \right) \frac{\partial^2 E}{\partial y^2} \\
- \left( \beta_{12}\gamma_{1i}\gamma_{2j}\gamma_{2k} + \beta_{22}\gamma_{2i}\gamma_{1j}\gamma_{2k} \right) \frac{\partial^2 E}{\partial x^2} \\
- \left( \beta_{12}\gamma_{2i}\gamma_{2j}\gamma_{1k} + \beta_{22}\gamma_{2i}\gamma_{1j}\gamma_{2k} \right) \frac{\partial^2 E}{\partial x^2} \\
- \left( \beta_{22}\gamma_{2i}\gamma_{2j}\gamma_{1k} + \beta_{22}\gamma_{2i}\gamma_{1j}\gamma_{2k} \right) \frac{\partial^2 E}{\partial x \partial y} \\
- \left( \beta_{22}\gamma_{2i}\gamma_{2j}\gamma_{2k} + \beta_{22}\gamma_{1i}\gamma_{2j}\gamma_{1k} \right) \frac{\partial^2 E}{\partial y^2}
\]

Proposition 2.2

\[ E_i(x, y) = \frac{1}{C_4 + C_5 \tanh \left( \frac{C_1\gamma_{1i} - C_2\gamma_{2i} x + C_3\gamma_{1i} y}{\gamma_{1i}} \right)}; \]
the \( C_i \)'s are arbitrary constants.

**Proposition 2.3** The Euler Characteristic of the generated subalgebras \( a_E \) for the dynamic systems is \( \mathcal{X}(a_E) = 0 \).

### 3 Proof of Proposition 2.1

The proof of proposition 1 is as follows.

To elucidate the structure of Lie algebras is must associated vector fields the dynamic system mentioned above.

This dynamic system (1) is characterized by vector fields
\[ f_0 = (\alpha_1 + \beta_{11}x + \beta_{12}E(x,y)) \frac{\partial}{\partial x} + \\
(\alpha_2 + \beta_{22}y + \beta'_{22}E(x,y) + \beta_{13}z) \frac{\partial}{\partial y} + \\
(\alpha_3 + \beta_{32}y + \beta_{33}z) \frac{\partial}{\partial z}, \]
\[ f_i = \gamma_{1i} \frac{\partial}{\partial x} + \gamma_{2i} \frac{\partial}{\partial y}, \text{ for } i = 1, 2, 3. \]

The vector field \( B \) is defined as

\[ B = -\beta_{12} \frac{\partial}{\partial x} - \beta'_{22} \frac{\partial}{\partial y}, \]

called here the air vector; \( \beta_{12} \) features \( \text{kg}^{-1} \text{air} \), physical units, and is a dimensionless constant of the contained air volume. \( \beta'_{22} \) features \( \text{joule} \cdot \text{kg}^{-1} \text{air} \), units, i.e., energy contained in the air.

\( \mathfrak{a}_E \) sub-algebras can be defined using the multiplication table given by the following relationships

\[ [f_i, [f_0, f_j]] = \Delta_{ij}E(x,y)B, \text{ where } i, j = 1, 2, 3 \]
\[ [f_i, f_j] = 0 \]
\[ [[f_0, f_i], [f_0, f_j]] = 0, \]

where

\[ [f_0, f_j] = -\left( \gamma_{1j} \beta_{11} + \gamma_{1j} \beta_{12} \frac{\partial E}{\partial x} + \gamma_{2j} \beta_{12} \frac{\partial E}{\partial y} \right) \frac{\partial}{\partial x} - \left( \gamma_{2j} \beta_{22} + \gamma_{1j} \beta'_{22} \frac{\partial E}{\partial x} + \gamma_{2j} \beta'_{22} \frac{\partial E}{\partial y} \right) \frac{\partial}{\partial y} - \gamma_{2j} \beta_{32} \frac{\partial}{\partial z}. \]

The relationship \( \Delta_{ij}E(x,y) = \Delta_{ji}E(x,y) \) is satisfied.

In order that the non-trivial relationship \([f_i, [f_0, f_j]] = \Delta_{ij}E(x,y)B \) define a multiplication table, the first condition which must be met is that
\[[f_k, \Delta_{ij}E(x,y)B] = \Delta_{ijk}E(x,y)B = 0.\] And the second condition that must be met is that

\[[[f_0, f_k], \Delta_{ij}E(x,y)B] = \Lambda_{ijk}E(x,y)B = 0.\]

In sum, in order for Lie algebra \(a_E\) be defined it must have \(\Delta_{ij}E(x,y) = c_{ij}\); where \(c_{ij}\) are constants and as well, \(\Delta_{ijk}E(x,y) = 0\) and \(\Lambda_{ijk}E(x,y) = 0\).

It should be noted that the \(B\) continues to appear when other Lie parentheses are calculated, like \([f_{k1}, [f_{k2}, ..., [f_{kl}, \Delta_{ij}E(x,y)B]][] = (\cdot)B\), where \((\cdot)\) is an expression that contains partial derivatives of \(E\) of some determined order. And the sums contained in these expressions are defined in terms of superior order magic squares and greater magic sums.

The fact that the vector field \(B\) appears frequently in the calculations as a factor in the final result, was decisive for searching a Lie algebra structure.

## 4 Proof of Proposition 2.2

To solve the system of partial differential equations

\[
\begin{align*}
\Delta_{ij}E(x,y) &= \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \\
\Delta_{ijk}E(x,y) &= 0 \\
\Lambda_{ijk}E(x,y) &= 0,
\end{align*}
\]

can have the function of evapotranspiration \(E\).

In general the conditions that define \(a_E\) sub-algebras and \(h_3\) sub-algebra are non linear differential equation systems in partial derivatives with dependent variable \(E\). Nevertheless within the complexity that searching for \(E\) solutions in these systems can represent, it is possible to find general evapotranspiration functions with adequate boundary conditions.

For the system of partial differential equations that define \(h_3\), the system of equations can be resolved separately, in three subsystems.

The first class of equations \(\Delta_{ij}E(x,y) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}\) has the general solution.

For \(i \neq j\)
$$E_{ij}(x, y) = F_1(-\gamma_{2j}x + \gamma_{1j}y) + F_2(-\gamma_{2i}x + \gamma_{1i}y).$$

For $i = j$

$$E_{ij}(x, y) = F_1(-\gamma_{2j}x + \gamma_{1j}y) + F_2(-\gamma_{2i}x + \gamma_{1i}y) + \frac{x^2}{2\gamma_{1i}\gamma_{1j}},$$

with $F_1, F_2$ arbitrary, sufficiently differentiable functions.

The second class of functions $\Delta_{ijk}E(x, y) = 0$, has the general solution.

$$E_{ijk}(x, y) = F_1(-\gamma_{2i}x + \gamma_{1i}y) + F_2(-\gamma_{2j}x + \gamma_{1j}y) + F_3(-\gamma_{2k}x + \gamma_{1k}y),$$

with $F_1, F_2, F_3$ arbitrary, sufficiently differentiable functions.

Because of the voluminous calculations, to resolve the third system of equations $\Lambda_{ijk}E(x, y) = 0$, for all $i, j, k$ is possible to use the algebraic package ??? program. The result of the search for the solution of the system is simple; it is the formula given in the proposition 2, above enunciated.

Thus, finding functions $F_i$'s in the first and second class of solutions it is possible to calculate the function $E(x, y)$.

## 5 Proof of Proposition 2.3

An important question is the general nature of the dynamic system that the greenhouse models, along with the conditions that define the Lie subalgebras. Is well known that the best invariant to see if the same sub-algebras are exhibited—given another dynamic system with three state variables and three control variables by example—is to calculate the homology of the sub-algebras that are found.

The results of the calculation of the homology is as follows.

The $a_E$ algebra is generated as a 7 dimensional vector space by

$$\{X_i \equiv f_i, Y_i \equiv [f_0, f_i], Z \equiv B\}_{i=1,2,3}.$$

Koszul’s homology is used on field-$\mathbb{R}$, of real numbers, and the trivial representation with the usual defined, $p$-chain to $p-1$-chains border operator in the alternate $\wedge a_E$, algebra.

$$\partial_p : \wedge a_E \rightarrow \wedge a_E,$$
\[ \partial_p (X_1 \wedge X_2 \wedge \ldots \wedge X_P) = \sum_{1 \leq i < j \leq P} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge \widehat{X_j} \wedge \ldots \wedge X_P. \]

The complex to calculate the homology is

\[ \begin{array}{cccccccc}
0 & \stackrel{\partial_8}{\rightarrow} & \wedge^7 a_E & \stackrel{\partial_7}{\downarrow} & \wedge^6 a_E & \stackrel{\partial_6}{\rightarrow} & \wedge^5 a_E & \stackrel{\partial_5}{\downarrow} & \wedge^4 a_E & \stackrel{\partial_4}{\rightarrow} & \wedge^3 a_E & \stackrel{\partial_3}{\downarrow} & \wedge^2 a_E & \stackrel{\partial_2}{\rightarrow} & a_E & \stackrel{\partial_1}{\downarrow} & a_E & \rightarrow & R & \rightarrow & 0 \\
0 & \rightarrow & R^7 & \rightarrow & R^{21} & \rightarrow & R^{25} & \rightarrow & R^{35} & \rightarrow & R^{21} & \rightarrow & R^{27} & \rightarrow & R & \rightarrow & R & \rightarrow & 0.
\end{array} \]

The matrices associated with \( \partial_k \) linear transformations that we will denote as \( \partial^{ij}_k \) are

\[ \partial^{ij}_7 = 0, \ i = 1, \ldots, 7; \ j = 1. \]

\[ \partial^{ij}_6 = -c_{33}, \ i = 8; \ j = 1 \]
\[ = c_{23}, \ i = 9, 12; \ j = 1 \]
\[ = -c_{13}, \ i = 10, 17; \ j = 1 \]
\[ = -c_{22}, \ i = 13; \ j = 1 \]
\[ = c_{12}, \ i = 14, 18; \ j = 1 \]
\[ = -c_{11}, \ i = 19; \ j = 1 \]
\[ = 0 \text{ in the other cases of } i, j. \]

Matrices are defined thus

\[ a_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ a_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ a_{-2} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \]

Note that the elements \( a_2, a_0, a_{-2} \), form non-trivial commutators relations \([a_i, a_j] = -a_{i+j} \), forming a base of some Lie algebra.

Some \( \partial^{ij}_k \) that follow are expressed by means of \( a_2, a_0, a_{-2} \).
\[ \partial_{ij}^5 = c_{k1}a_2 + c_{k2}a_0 + c_{k3}a_{-2}, \]

\[ k = 3, \; i = 7, 9, 10; \; j = 1, 2, 3 \]
\[ = 2, \; i = 13, 15, 16; \; j = 1, 2, 3 \]
\[ = 1, \; i = 23, 25, 26; \; j = 1, 2, 3 \]

\[ \partial_{ij}^5 = \begin{pmatrix} c_{\sigma 3} & c_{\tau 3} \\ c_{\sigma 2} & c_{\tau 2} \\ c_{\sigma 1} & c_{\tau 1} \end{pmatrix} \]

\[ (\sigma, \tau) = (2, 3), \; i = 18, 19, 20; \; j = 4, 5; \; \text{signs} \; +--\]
\[ = (1, 3), \; i = 28, 29, 30; \; j = 4, 6; \; \text{signs} \; --+ \]
\[ = (1, 2), \; i = 32, 33, 34; \; j = 5, 6; \; \text{signs} \; --+ \]

\[ \partial_{ij}^5 = 0 \text{ in other cases of } i, j. \]

\[ \partial_{ij}^4 = (c_{\sigma 1} + c_{\tau 1})a_2 + (c_{\sigma 2} + c_{\tau 2})a_0 + (c_{\sigma 3} + c_{\tau 3})a_{-2} \]

\[ (\sigma, \tau) = (2, 3), \; i = 12, 14, 15; \; j = 4, 5, 6 \]
\[ = (1, 3), \; i = 22, 24, 25; \; j = 4, 5, 6 \]
\[ = (1, 2), \; i = 28, 30, 31; \; j = 7, 8, 9 \]

\[ \partial_{ij}^4 = (c_{kl})_{k,l=1,2,3}; \; i = 5, 9, 19; \; j = 1, 2, 3; \; \text{signs} \; --++ -+ + - \]
\[ = (c_{kl})^T_{k,l=1,2,3}; \; i = 33, 34, 35; \; j = 10, 14, 15; \; \text{signs} \; --++ -+ + - \]

\[ \partial_{ij}^4 = 0 \text{ in another case of } i, j. \]
\[ \partial_{ij}^{3} = (c_{\sigma 1} + c_{\tau 1} + c_{\nu 1}) a_{2} + (c_{\sigma 2} + c_{\tau 2} + c_{\nu 2}) a_{0} + (c_{\sigma 3} + c_{\tau 3} + c_{\nu 3}) a_{-2} \]

\[ (\sigma, \tau, \nu) = (1, 2, 3), \ i = 18, 20, 21; j = 7, 8, 9; j = 13, 14, 15; j = 16, 17, 18 \]

\[ \partial_{ij}^{3} = \begin{pmatrix} c_{\tau 1} & c_{\tau 2} & c_{\tau 3} \\ -c_{\sigma 1} & -c_{\sigma 2} & -c_{\sigma 3} \end{pmatrix} \]

\[ (\sigma, \tau) = (2, 3), \ i = 11, 15; j = 10, 11, 12 \]
\[ = (1, 3), \ i = 6, 15; j = 4, 5, 6 \]
\[ = (1, 2), \ i = 6, 11; j = 1, 2, 3 \]

\[ \partial_{ij}^{3} = 0 \] in other cases of \( i, j \)

and finally

\[ \partial_{ij}^{2} = (-c_{\sigma 1}, -c_{\sigma 2}, -c_{\sigma 3}) \]

\[ \sigma = 1, \ i = 7, \ j = 1, 2, 3 \]

\[ \sigma = 2, \ i = 7, \ j = 4, 5, 6 \]

\[ \sigma = 3, \ i = 7, \ j = 7, 8, 9. \]

The homology groups are in general \( H_{i} (a_{E}) = R^{\dim \ker \partial_{i} - \dim \text{im} \partial_{i+1}} \). In this case

\[ H_{7} (a_{E}) = R; \ H_{6} (a_{E}) = R^{6} \]
\[ H_{5} (a_{E}) = R^{20 - \text{rank} \partial_{5}}, \ \text{rank} \partial_{5} = 0, 1, 2, ..., 6 \]
\[ H_{4} (a_{E}) = R^{35 - \text{rank} \partial_{4} - \text{rank} \partial_{5}}, \ \text{rank} \partial_{4} = 0, 1, 2, ..., 15; \ \text{rank} \partial_{5} = 0, 1, 2, ..., 6 \]
\[ H_{3} (a_{E}) = R^{35 - \text{rank} \partial_{3} - \text{rank} \partial_{4}} - \text{rank} \partial_{5}, \ \text{rank} \partial_{4} = 0, 1, 2, ..., 21; \ \text{rank} \partial_{5} = 0, 1, 2, ..., 15 \]
\[ H_{2} (a_{E}) = R^{20 - \text{rank} \partial_{3}}, \ \text{rank} \partial_{3} = 0, 1, 2, ..., 21 \]
\[ H_{1} (a_{E}) = R^{6}, \ H_{0} (a_{E}) = R. \]

The Euler characteristic is in every case \( \chi (a_{E}) = 0 \).

\[ \square \]

References

[1] Baez,J., Morton and Vicary On the Categorified Heisenberg Algebra. http://golem.ph.utexas.edu/category/2012/07/morton_and_vicary_on_the_categ.html
[2] Blasco, X., Martínez, M., Herrero, J.M., Ramos, C., Sanchis, J. Model-based predictive control of greenhouse climate for reducing energy and water consumption. Computers and Electronics in Agriculture 55. (2007) 49-70.

[3] Cairns, Grant; Jambor, Sebastian The cohomology of the Heinsenberg Lie algebras over fields of finite characteristic. Proc. Amer. Math. Soc. 136 (2008), no. 11, 3803–3807.

[4] Merlin Olivier, Ahmad Al Bitar, Vincent Rivalland, Pierre Béziat, Eric Ceschia, Gérard Dedieu, “An Analytical Model of Evaporation Efficiency for Unsaturated Soil Surfaces with an Arbitrary Thickness.” J. Appl. Meteor. Climatol. 50, (2011) 457–471.

[5] Morton, J. C. and Vicary, J. The Categorified Heisenberg Algebra I: A Combinatorial Representation. arXiv:1207.2054v2 [math.QA].

[6] Sánchez-Gabites, J. J. Dynamical systems and shapes. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 102 (2008), no. 1, 127–159.

[7] Santaroubane, L. J. Cohomology of Heisenberg Lie algebras. Proceedings of the American Mathematical Society 87 (1983), no. 1, 23–28.

[8] Tkachuk, V. M. Deformed Heisenberg algebra with minimal length and equivalence principle. arXiv:1301.1891v1 [gr-qc].