Convolution and deconvolution based estimates of galaxy scaling relations from photometric redshift surveys

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ABSTRACT
In addition to the maximum likelihood approach, there are two other methods which are commonly used to reconstruct the true redshift distribution from photometric redshift datasets: one uses a deconvolution method, and the other a convolution. We show how these two techniques are related, and how this relationship can be extended to include the study of galaxy scaling relations in photometric datasets. We then show what additional information photometric redshift algorithms must output so that they too can be used to study galaxy scaling relations, rather than just redshift distributions. We also argue that the convolution based approach may permit a more efficient selection of the objects for which calibration spectra are required.

Key words: methods: analytical, statistical – galaxies: formation — cosmology: observations.

1 INTRODUCTION
The next generation of sky surveys will provide reasonably accurate photometric redshift estimates, so there is considerable interest in the development of techniques which can use these noisy distance estimates to provide unbiased estimates of galaxy scaling relations. While there exist a number of methods for estimating photometric redshifts (Budavari 2009 and references therein), there are fewer for using these to estimate accurate redshift distributions (Padmanabhan et al. 2005; Sheth 2007; Lima et al. 2008; Cunha et al. 2009), the luminosity function (Sheth 2007), or the joint luminosity-size, color-magnitude, etc. relations (Rossi & Sheth 2008; Christlein et al. 2009; Rossi et al. 2010).

Ideally, the output from a photometric redshift estimator is a normalized likelihood function which gives the probability that the true redshift is \( z \) given the observed colors (i.e. Bolzonella et al. 2000; Collister & Lahav 2004; Cunha et al. 2009). Let \( \mathcal{L}(z|c) \) denote this quantity; it may be skewed, bimodal, or more generally it may assume any arbitrary shape.

Let \( \zeta \) denote the mean or the most probable value of this distribution (it does not matter which, although some of the logic which follows is more transparent if \( \zeta \) denotes the mean). Often, \( \zeta \) (sometimes with an estimate of the uncertainty on its value) is the only quantity which is available.

Therefore, in Section 2.1 we first consider how \( \zeta \) compares with the true redshift \( z \), and contrast the convolution and deconvolution methods for estimating \( dN/dz \) – while in Section 2.2 we describe how to reconstruct the redshift distribution directly from colors. Section 2.3 shows what this implies if one wishes to use the full distribution \( \mathcal{L}(z|c) \). Section 2.4 shows how to extend the logic to the luminosity function, and Section 2.5 to scaling relations, again by contrasting the convolution and deconvolution methods, and showing what generalization of \( \mathcal{L}(z|c) \) is required from the photometric redshift codes if one wishes to do this. A final section summarizes our results.

Where necessary, we write the Hubble constant as \( H_0 = 100h \, \text{km s}^{-1} \, \text{Mpc}^{-1} \), and we assume a spatially flat cosmological model with \( (\Omega_M, \Omega_\Lambda, h) = (0.3, 0.7, 0.7) \), where \( \Omega_M \) and \( \Omega_\Lambda \) are the present-day densities of matter and cosmological constant scaled to the critical density.

2 TO CONVOLVE OR DECONVOLVE?
In what follows, we will use spectroscopic and photometric redshifts from the SDSS to illustrate some of our arguments. Details of how the early-type galaxy sample was selected are in Rossi et al. (2010); the photo-zs for this sample are from Csabai et al. (2003).
Figure 1. Distribution of the difference between spectroscopic and photometric redshifts (\(z\) and \(\zeta\)), at fixed \(z\) (top) and \(\zeta\) (bottom), in the SDSS early-type galaxy sample. Note that \(p(\zeta|z)\) is rather well centered on \(z\), whereas \(p(z|\zeta)\) is not centered on \(\zeta\).

2.1 The redshift distribution

Suppose that the true redshifts \(z\) are available for a subset of the objects; for now, assume that the subset is a random subsample of the objects in a magnitude limited catalog. Ideally, this subset would have the same geometry as the full survey, as cross-correlating the objects with spectra and those without allows the use of other methods (e.g. Caler et al. 2009). In practice, this may be difficult to achieve – and this is not required for the analysis which follows, provided that the photometric redshift estimator does not have spatially dependent biases (e.g., as a result of photometric calibrations varying across the survey).

For the objects with spectroscopic redshifts, one can study the joint distribution of \(\zeta\) and \(z\) (see Figure 1). Typically, most photometric redshift codes are constructed to return \(\langle \zeta|z \rangle \approx z\). The codes which do so are sometimes said to be unbiased, but they are not perfect: the scatter around the unbiased mean is of order \(\sigma_{\zeta|z} \approx 0.05 (1 + z)\). This scatter, combined with the fact that \(\langle z|\zeta \rangle \approx \zeta\) means that \(\langle z|\zeta \rangle \neq \zeta\); the fact that \(z|\zeta\) is guaranteed to be biased is not widely appreciated. However, we show below that it matters little whether \(\zeta|z\) or \(z|\zeta\) are unbiased – what matters is that the bias is accurately quantified.

In particular, if \(dN/d\zeta\) and \(dN/dz\) denote the distribution of \(\zeta\) and \(z\) values in the subset of the data where both \(z\) and \(\zeta\) are available, then what matters is that \(p(\zeta|z)\) and \(p(z|\zeta)\), where

\[
\frac{dN(z,\zeta)}{dz d\zeta} = \frac{dN(z)}{dz} p(\zeta|z) = \frac{dN(\zeta)}{d\zeta} p(z|\zeta),
\]

are known. Note that

\[
\frac{dN(\zeta)}{d\zeta} \equiv \int dz \frac{dN(z)}{dz} p(\zeta|z).
\]

The algorithm in Sheth (2007) assumes that \(p(\zeta|z)\), measured in the subset for which both \(z\) and \(\zeta\) are available, also applies to the full sample for which \(z\) is not available. Since \(dN/d\zeta\) is measured in the full dataset, and \(p(\zeta|z)\) is known, a deconvolution is then used to estimate the true \(dN/dz\).

Suppose, however, that one measured \(p(z|\zeta)\) instead. Then, because

\[
\frac{dN(z)}{dz} \equiv \int d\zeta \frac{dN(\zeta)}{d\zeta} p(z|\zeta),
\]

one could estimate the quantity on the left hand side by ‘convolving’ the two measurables on the right hand side. For the data-subset in which both \(z\) and \(\zeta\) are available, this is correct by definition. Clearly, to use this method on the larger dataset for which only \(\zeta\) is available, one must assume that \(p(z|\zeta)\) in the subset from which it was measured remains accurate in the larger dataset.

Rossi et al. (2010) have shown that the deconvolution method accurately reconstructs the true \(dN/dz\) distribution from \(dN/d\zeta\). Figure 2 shows that the convolution approach also works well, even when only a random 5% of the full dataset is used to calibrate \(p(z|\zeta)\) – as displayed in Figure...
Thus, for the dataset in which both $z$ and $\zeta$ are available, both the convolution and deconvolution approaches are valid, whether or not the means (or, for that matter, the most probable values) of $p(z|\zeta)$ and $p(\zeta|z)$ are unbiased, and however complicated (skewed, multimodal) the shape of these two distributions. This remains true in the larger dataset where only $\zeta$ is known. However, whereas the convolution approach assumes that $p(z|\zeta)$ is the same in the calibration subset as in the full one, the deconvolution approach assumes that $p(\zeta|z)$ is the same.

### 2.2 Convolution directly from colors

The integral in equation (3) is really a sum over all the objects in the photometric dataset, where each object with estimated $\zeta$ contributes to $dN/dz$ with weight $p(z|\zeta)$:

$$\frac{dN(z)}{dz} \equiv \int d\zeta \frac{dN(\zeta)}{d\zeta} p(z|\zeta) = \sum_i p(z|\zeta_i). \quad (4)$$

Now, recall that $\zeta$ was the mean (or most probable) value of a distribution returned by a photometric redshift code. In cases where the observed colours $c$ map to a unique value of $\zeta$, then this sum over $\zeta$ is really a sum over $c$, and the expression above is really

$$\frac{dN(z)}{dz} \equiv \int dc \frac{dN(c)}{dc} p(z|c) = \sum_i p(z|c_i). \quad (5)$$

Equation (5) is one of the key results of this paper.

Although we arrived at equation (5) by requiring the mapping $c \rightarrow \zeta$ be one-to-one (as may be the case for, e.g., LRGs), it is actually more general. This is because one can simply measure $p(z|c)$ in the sample for which spectra are in hand, for the same reason that one could measure $p(z|\zeta)$. In fact, $p(z|c)$ is an easier measurement, since it does not depend on the output of a photo-z code! The constraint on the mapping between $c$ and $\zeta$ in the discussion above was simply to motivate the connection between photo-z codes and the convolution method. Once the connection has been made, however, there is no real reason to go through the intermediate step of estimating $\zeta$, since all photo-z codes use the observed colors $c$ anyway. In this respect, equation (5) is the more direct and natural expression to work with than is equation (4). In particular, because $p(z|c)$ is an observable, the convolution approach of equation (5) is independent of...
any photo-z algorithm. Of course, if this method is to work, then the subsample with spectral information must be able to provide an accurate estimate of $p(z|c)$.

2.3 Relation to photo-z algorithms

The convolution method of the previous subsection provides a simple way of illustrating how one should use the output from photo-z codes that actually provide a properly calibrated probability distribution $L(z|c)$ for each set of colors $c$, to estimate $dN/dz$. It also shows in what sense the codes should be ‘unbiased’.

In particular, equation (5) suggests that one can estimate $dN(z)/dz$ by summing over all the objects in the dataset, weighting each by its $L(z|c)$. This is because

$$
\sum_i L(z|c_i) = \frac{dN(z)}{dz} \quad \text{if} \quad L(z|c) = p(z|c).
$$

Equation (6) shows that if $L(z|c)$ does not have the same shape as $p(z|c)$, then use of $L(z|c)$ will lead to a bias; this is the pernicious bias which must be reduced – whether or not $\langle z|c \rangle$ equals the spectroscopic redshift is, in some sense, irrelevant. (In the case of a one-to-one mapping between $c$ and $\zeta$, $\langle z|c \rangle$ is the same as the quantity $\langle z|\zeta \rangle$ which we discussed in the previous subsections.)

Satisfying $L(z|c) = p(z|c)$ is nontrivial. This is perhaps most easily seen by supposing that the template or training set consists of two galaxy types (early- and late-types, say), for which the same observed colors are associated with two different redshifts. In this case, if the photo-z algorithms are working well, then $L(z|c)$ will be bimodal for at least some $c$. However, if the sample of interest only contains LRGs, then $p(z|c)$ may actually be unimodal. As a result, $L(z|c) \neq p(z|c)$ unless proper priors on the templates are used, or care has been taken to insure that the training set is representative of the sample of interest.

2.4 The luminosity function

We can perform a similar analysis of the luminosity function. In this case, the key is to recognize that, in a magnitude limited survey, the quantity which is most directly affected by the photometric redshift error is not the luminosity function $\phi(M)$ itself, but the luminosity distribution $N(M) \equiv V_{\max}(M) \phi(M)$ (Sheth 2007). In a spectroscopic survey, $N(M)$ differs from $\phi(M)$ because one sees the bright-
Scaling relations from photo-zs

Figure 4. Same as Figure 2, but for the absolute magnitudes. Crosses show the distribution one obtains by convolving the dotted histogram with the distributions shown in the bottom panel of Figure 3; solid histogram shows the true distribution of $M$.

![SDSS early types](image)

Sheth (2007) describes a deconvolution algorithm for estimating $N(M)$ given measurements of $N(M)$ and the assumption that $p(M|M)$, measured in a subset for which both $z$ and $\zeta$ (hence both $M$ and $M$) are available, also applies to the full photometric survey.

Following the discussion in the previous section, we could instead have measured $p(M|M)$, and then used the fact that

$$N(M) = \int dM N(M) p(M|M).$$

(7)

Sheth (2007) describes a deconvolution algorithm for estimating $N(M)$ given measurements of $N(M)$ and the assumption that $p(M|M)$, measured in a subset for which both $z$ and $\zeta$ (hence both $M$ and $M$) are available, also applies to the full photometric survey.

Following the discussion in the previous section, we could instead have measured $p(M|M)$, and then used the fact that

$$N(M) = \int dM N(M) p(M|M)$$

(8)

to estimate the quantity on the left hand side by summing over the photometric catalog on the right hand side, weighting each object in it by $p(M|M)$; note that this weight depends on $M$. Figure 4 shows $p(M|M)$ and $p(M|M)$; notice how broad they are, and how much more skewed and biased $p(M|M)$ is than $p(M|M)$. Nevertheless, Rossi et al. (2010) have shown that the deconvolution algorithm produces good results. Figure 4 shows that the convolution algorithm does as well.

One estimates $\phi(M)$ by dividing $N(M)$ by $V_{\text{max}}(M)$. Since this weight is the same for all objects with the same $M$, one could have added an additional weighting term to the sum above to get

$$\phi(M) = \int dM N(M) \frac{p(M|M)}{V_{\text{max}}(M)}$$

$$\neq \int dM \frac{N(M)}{V_{\text{max}}(M)} p(M|M).$$

(9)

One might have written $\phi(M) = N(M)/V_{\text{max}}(M)$, so the expression above shows explicitly why the photometric errors should be thought of as affecting $N(M)$ and not $\phi(M)$.

To make the connection to $p(z|c)$ and then $L(z|c)$ it is worth considering how one computes $M$ from $z$ given the observed colors $c$. If there were no $k$-correction, then the luminosity in a given band would be determined from the
observed apparent brightness by the square of the (cosmology dependent) luminosity distance – the colors are not necessary. In practice however, one must apply a $k$-correction; this depends on the spectral type of the galaxy, and hence on its color. As a result, the mapping between $m$ and $M$ depends on $z$ and $c$. But it is still true that both $M$ and $z$ are determined by $c$. Therefore, the spectroscopic subsample which was previously used to estimate $p(z|c)$ also allows one to estimate $p(M, z|c)$. The quantity of interest in the previous section, $p(z|c)$, is simply the integral of $p(M, z|c)$ over all $M$. The quantity of interest here, $p(M|c)$, is the integral of $p(M, z|c)$ over all $z$. Thus, equation (3) becomes

$$N(M) = \int dc \frac{dN(c)}{dc} \int dz \, p(M, z|c)$$

where the second to last expression writes the integral of $p(M, z|c)$ over all $z$ as $p(M|c)$, and the final one writes the integral explicitly as a sum over the objects in the catalog.

The expression above is the convolution-type estimate of $N(M)$; it does not require a photometric redshift code. However, in principle, a photometric redshift code could output $L(M, z|c)$: the quantity codes currently output, $L(z|c)$, is the integral of $L(M, z|c)$ over all $M$. The relevant weighted sum becomes

$$N(M) = \sum_i L(M|c_i),$$

where $L(M|c)$ is the integral of $L(M, z|c)$ over all $z$, the sum is over all the objects in the catalog, and the method only works if $L(M|c) = p(M|c)$. Note that the luminosity density (in solar units) can, therefore, be written as

$$j = \int dM \phi(M) \left( \frac{10^{-0.4(M-M_\odot)}}{V_{\text{max}}(M)} \right)$$

$$= \int dM \, N(M) \left( \frac{10^{-0.4(M-M_\odot)}}{V_{\text{max}}(M)} \right)$$

$$= \int dM \, N(M) \int dM \, p(M|M) \left( \frac{10^{-0.4(M-M_\odot)}}{V_{\text{max}}(M)} \right)$$

$$= \int dM \, N(M) \left( \frac{10^{-0.4(M-M_\odot)}}{V_{\text{max}}(M)} \right) |_{M}$$

$$= \sum_i \left( \frac{10^{-0.4(M-M_\odot)}}{V_{\text{max}}(M)} \right) c_i.$$  (12)

The second to last line shows that one requires the average of $\langle L/V_{\text{max}}(L) \rangle$ summed over the distribution $p(M|M)$; this is easily computed from distributions like those shown in the bottom panel of Figure 3. The final expression writes this as a sum over the observed distribution of colors.

### 2.5 Galaxy scaling relations

Although the previous section considered the luminosity function in a single band, it is clear that the photometric redshift codes could output $L(M, z|c)$, where $M$ is a set of absolute luminosities (typically, these will be those associated with the various band passes from which the colors $c$ were determined). Hence, the color magnitude relation, which is really a statement about the joint distribution in two bands, can be estimated by

$$N(M) = \int dc \frac{dN(c)}{dc} \int dz \, p(M, z|c)$$

$$= \int dc \frac{dN(c)}{dc} \, p(M|c) = \sum_i p(M|c_i).$$  (13)

Galaxy scaling relations can be estimated similarly, if we simply interpret $M$ as being the vector of observables which can include sizes, etc. (not just luminosities). In principle, quantities other than colors (e.g., apparent magnitudes, surface brightness, axis ratios) can play a role in the photometric redshift determination; this can be incorporated into the formalism simply by using $c$ to now denote the full set of observables from which the redshift and other intrinsic quantities $M$ were estimated.

If one wishes to use the output from a photo-$z$ code, rather than from the spectroscopic subset, one would use

$$N(M) = \sum_i \mathcal{L}(M|c_i),$$  (14)

having checked that, in the spectroscopic subset, $\mathcal{L}(M|c) = p(M|c)$. 

### 3 DISCUSSION

We showed how previous work on deconvolution algorithms for making unbiased reconstructions of galaxy distributions and scaling relations (Sheth 2007; Rossi & Sheth 2008; Rossi et al. 2010) could be related to convolution-based methods. Whereas deconvolution based methods require accurate knowledge of $p(\zeta|z)$, the distribution of the photometric redshift $\zeta$ given the true redshift $z$, convolution based methods require accurate knowledge of $p(z|\zeta)$. Since $\zeta$ is derived from photometry, this may more generally be written as $p(z|c)$, where $c$ is the vector of observed photometric parameters which were used to estimate the redshift. In both cases, $p(z|c)$ and $p(\zeta|z)$ are calibrated from a sample in which $z$ is known, and are then used in a larger sample where $z$ is not available. If the smaller training set has the same selection limits as the larger dataset (e.g., both have the same magnitude limit) then both approaches are valid. We illustrated our arguments with measurements in the SDSS (Figures [1] and [4]).

We also showed what additional information must be output from photometric redshift codes if their results are to be used in a convolution-like approach to provide unbiased estimates of galaxy scaling relations. In particular, we argued that only if the redshift distribution output by a photo-$z$ algorithm, $L(z|c)$, has the same shape as $p(z|c)$, can the algorithm be said to be unbiased. Only in this case its output (available for the full sample) can be used in place of $p(z|c)$ (which is typically available for a small subset). The safest way to accomplish this is for the training set to be a random subsample of the full dataset – and to then tune the algorithm so that $L(z|c) = p(z|c)$. If the training set is not representative, then care must be taken to ensure that $L(z|c)$ does not yield biased results.

Obtaining spectra is expensive, so the question arises
as to whether or not there is a more efficient alternative to the random sample approach. For the convolution method, which requires \(p(z|c)\), the answer is clearly ‘yes’. This is because some color combinations (e.g. the red sequence) might give rise to a narrow \(p(z|c)\) distribution, whereas others may result in broader distributions. Since it will take fewer objects to accurately estimate the shape of a narrow \(p(z|c)\) distribution than a broad one, observational effort would be better placed in obtaining spectra for those objects which produce broad \(p(z|c)\) distributions. For the deconvolution approach, one would like to preferentially target those redshifts \(z\) which produce broader \(p(\zeta|z)\) distributions — for similar reasons. But, since \(z\) is not known until the spectra are taken, this cannot be done, so taking a random sample of the full dataset is the safest way to proceed.

Our methods permit accurate measurement of many scaling relations for which spectra were previously thought to be necessary (e.g. the color-magnitude relation, the size-surface brightness relation, the Photometric Fundamental Plane), so we hope that our work will permit photometric redshift surveys to provide more stringent constraints on galaxy formation models at a fraction of the cost of spectroscopic surveys.

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