ABSENCE OF EIGENVALUES OF ANALYTIC QUASI-PERIODIC
SCHRÖDINGER OPERATORS ON $\mathbb{R}^d$

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Abstract. In this paper we study on $L^2(\mathbb{R}^d)$ the quasi-periodic Schrödinger operator $H = -\Delta + \lambda V(x)$, where $V$ is a real analytic quasi-periodic function and $\lambda > 0$. We first show that $H$ has no eigenvalues in low energy region. We also provide in low energy region the new phase transition parameter, i.e. the competition between the strength of coupling and the length for frequencies.

1. Introduction and main results

Let $H = -\Delta + V(x)$ be the Schrödinger operator defined on $L^2(\mathbb{R}^d)$. The question of determining for which potential $V(x)$ such $H$ has no positive eigenvalues attracted a great deal of attention over years, see e.g., [Kat59, Sim69, Agm70, FH83, JK85, IJ03, Mar19]. In general, those works require $V(x)$ to obey certain decaying law at $\infty$. This may exclude a large class of potentials without decaying, such as the quasi-periodic one. The aim of the present paper tries to establish absence of positive eigenvalues for some quasi-periodic Schrödinger operators on $\mathbb{R}^d$ for arbitrary $d \geq 1$.

Indeed, the quasi-periodic Schrödinger operator has important applications in condensed matter physics, and has been extensively studied in recent years, in particular in its discrete case (see [MJ17, Dam17] and references therein).

Considerable research results have been obtained for quasi-periodic Schrödinger operators on $\mathbb{R}$. In 1D and periodic potentials case, the Floquet theory works well, and as a result the spectrum consists of intervals (bands) and is purely absolutely continuous ($ac$). When the periodic potential is replaced by a quasi-periodic one, the spectral properties change dramatically resulting from the so-called small-divisors effect. Dinaburg-Sinai [DS75] first proved the existence of $ac$ spectrum by a using a KAM reducibility argument. This result was further developed later by Rüssmann [Rüs80] and Moser-Pöschel [MP84]. Surace [Sur90] showed for potentials of the form

$$V(x) = \lambda (\cos x + \cos(\theta + \omega x)),$$

the whole spectrum contains no eigenvalues if $\lambda$ is small and the frequency $\omega$ is Diophantine. He extended the multi-scale analysis (MSA) of [FS83, FSW90] to work in momentum space. The breakthrough came from Eliasson [Eli92], in which he proved for any nonconstant analytic quasi-periodic potential and Diophantine frequency, the spectrum is purely $ac$ in high energy region. His result also applied to the whole spectrum for small quasi-periodic potentials. Moreover, he showed

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the spectrum is a Cantor set for a generic set of analytic quasi-periodic potentials, which exhibits completely different spectral features compared with periodic potentials. For quasi-periodic potentials $\lambda V(x)$ with large $\lambda > 0$, Anderson localization (AL, i.e., pure point spectrum with exponentially decaying eigenfunctions) and delocalization transition [YZ14] may be expected. Sorets-Spencer [SS91] improved Herman’s subharmonic trick to obtain the positivity of Lyapunov exponent at low energy for potentials (1.1) if $\lambda \gg 1$. Fröhlich-Scneider-Wittwer [FSW90] proved the first AL for potentials of the form (1.1) at low energy if $\lambda \gg 1$. They performed a MSA for Green’s functions in $x \in \mathbb{R}$ space directly. It has been proven by You-Zhou [YZ14] that there exists the phase transition from singular to purely ac spectrum for 1D and two frequencies quasi-periodic operators if the coupling is large. Very recently, Binder-Kinzebulatov-Voda [BKV17] proved a non-perturbative AL (i.e. AL under only positive Lyapunov exponents assumption) for general analytic potentials in finite energy intervals by applying methods of [BG00, GS08]. For more results in 1D quasi-periodic setting, we refer to [Bje06, DG14, Liu18].

If one increases the space dimension to $d \geq 2$, the situation becomes significantly more complicated and much less results were obtained in this setting. Unlike the 1D case, it was conjectured (i.e. the Bethe-Sommerfeld conjecture [Par08, PS10]) that the spectrum of any periodic Schrödinger operators in higher dimensions contains finitely many gaps. The proof of this elegant result is based on their new MSA in momentum space. Before this work, they [KS13] got a similar result for $(-\Delta)^l + V(x)$ with $l \geq 2$. We also mention the work of Karpeshina-Lee-Shterenberg-Stolz [KLSS17], in which the existence of ballistic transport for the Schrödinger operator with limit-periodic or quasi-periodic potential in dimension two was established. There are also some results [PS16, PS12] for quasi-periodic (even almost periodic) operators on $\mathbb{R}^d$ in dealing with asymptotic expansion of the spectral functions (such as the IDS).

If $d \geq 3$, as far as we know, there are no localization or delocalization results available for a quasi-periodic Schrödinger operator on $\mathbb{R}^d$. This is the other motivation of the present work.

In this paper we study on $\mathbb{R}^d$ ($d \geq 1$) the Schrödinger operators with analytic quasi-periodic potentials $\lambda V(x)$ and prove the absence of eigenvalues in low energy region. Combined our main theorem and results of [Bje06, Bje07, YZ14, BKV17], we also provide the new phase transition parameter in low energy region (see (4) of Remark 1.1 in the following).

Here is the set up of our main result:

Let $b_i \in \mathbb{N}$ $(1 \leq i \leq d)$ and $b = \sum_{i=1}^d b_i > d$. Define $\mathbb{T}^b = \prod_{i=1}^d (\mathbb{R}/2\pi\mathbb{Z})^{b_i}$. Write $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_d) \in \mathbb{T}^b$ for $\tilde{\theta}_i = (\theta_{i1}, \ldots, \theta_{ib_i}) \in \mathbb{T}^{b_i}$ $(1 \leq i \leq d)$. Similarly, let $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_d) \in [0, 2\pi]^b$ with $\tilde{\omega}_i \in [0, 2\pi]^{b_i}$ $(1 \leq i \leq d)$. Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and define $x\tilde{\omega} = (x_1\tilde{\omega}_1, \ldots, x_d\tilde{\omega}_d) \in \mathbb{R}^b$.

We consider on $L^2(\mathbb{R}^d)$ the following Schrödinger operators with quasi-periodic potentials

$$H(\tilde{\theta}) = -\Delta + \lambda V(\tilde{\theta} + Kx\tilde{\omega}),$$

(1.2)
where the real function $V$ is the potential, $\lambda \geq 0$ is the coupling, $\vec{\omega}$ is the frequency, $\vec{\theta}$ is the phase and $K > 0$.

We focus on the analytic function $V \in C^\omega (\mathbb{T}^b, \mathbb{R})$ satisfying $\int_{\mathbb{T}^b} V(\vec{\theta}) d\vec{\theta} = 0$. Without loss of generality, we may assume for some $\rho > 0$ and $\forall \mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{Z}^b$ ($k_i = (k_{i1}, \cdots, k_{ib_i}) \in \mathbb{Z}^{b_i}, 1 \leq i \leq d$)

$$|\hat{V}_k| \leq e^{-\rho |k|},$$

(1.3)

where $\hat{V}_k = \int_{\mathbb{T}^b} V(\vec{\theta}) e^{-i \mathbf{k} \cdot \vec{\theta}} d\vec{\theta}$ (with $\mathbf{k} \cdot \vec{\theta} = \sum_{i=1}^{d} k_{i} \cdot \vec{\theta}_i = \sum_{i=1}^{d} \sum_{j=1}^{b_i} k_{ij} \theta_{ij}$) is the Fourier coefficient of $V(\vec{\theta})$ and $|k| = \max_{1 \leq i \leq d, 1 \leq j \leq b_i} |k_{ij}|$.

Denote by $\text{mes}(\cdot)$ the Lebesgue measure. We have

**Theorem 1.1.** Let $H(\vec{\theta})$ be defined by (1.2) with $V$ satisfying (1.3). Then there is $c_* = c_*(b, d) > 0$ (depending on $b_1, \cdots, b_d$) such that, for $\delta > 0$ and $0 < c_1 < c_*$, there exists $\varepsilon_0 = \varepsilon_0(\delta, c_1, b, d) > 0$ such that the following holds: If $0 < \frac{1}{K^2} \leq \varepsilon_0$, then there is some $\Omega = \Omega(K, \lambda) \subset [0, 2\pi)^b$ satisfying $\text{mes}([0, 2\pi)^b \setminus \Omega) \leq \delta$ such that $H(\vec{\theta})$ has no eigenvalues in $\left[ -K^2(\log \frac{K^2}{\lambda})^{\frac{1}{2d}}, K^2(\log \frac{K^2}{\lambda})^{\frac{1}{2d}} \right]$ for every $\vec{\theta} \in \mathbb{T}^b$ and $\vec{\omega} \in \Omega$.

**Remark 1.1.**

1. **Theorem 1.1** is non-vacuous, i.e., there exists a portion of the spectrum in $\mathcal{I} = \left[ -K^2(\log \frac{K^2}{\lambda})^{\frac{1}{2d}}, K^2(\log \frac{K^2}{\lambda})^{\frac{1}{2d}} \right]$. Indeed (see Lemma A.1 in the Appendix), one can show for any $E \geq 0$ and $M \geq \lambda |V|_{\text{max}}$, $\sigma(H(\vec{\theta})) \cap [E - M, E + M] \neq \emptyset$, where $\sigma(\cdot)$ denotes the spectrum. In particular, $\sigma(H(\vec{\theta})) \cap [-M, M] \neq \emptyset$ for any $M > \lambda |V|_{\text{max}}$. Moreover, it is obvious that $\mathcal{I} \supset \mathbb{R}$ as $\lambda \to 0$ or $K \to \infty$.

2. We make no smallness assumptions on $\lambda > 0$, and prove the first purely continuous spectrum result in low energy region for quasi-periodic operators on $\mathbb{R}^d$ for arbitrary $d \geq 1$. This can be regarded as a supplement of results of [KS19], where the existence of ac spectrum for a 2D quasi-periodic Schrödinger operator was established in high energy region.

3. The theorem suggests that the length of frequencies is relevant for the spectral properties of continuous quasi-periodic Schrödinger operators.

4. If $d = 1, b_1 > 1$ and $V$ satisfies some conditions, it was proved in [BKV17] (combined with results of [Bje06]) that for $\lambda \geq \lambda_0(K\vec{\omega}, b_1) > 0$ the spectrum in $[0, \lambda^{2/3}]$ is pure point with exponential decay eigenfunctions (i.e. AL) for a.e. Diophantine $\vec{\omega}$. By contrast, our Theorem 1.1 implies that there are no eigenvalues in $[0, C\lambda]$ at all for any $\lambda > 0$ if $K \geq C\sqrt{\lambda}$. Those motivate us to provide the new phase transition parameter $\frac{\log K}{\log \lambda}$ from pure point to purely continuous spectrum in low energy region. Precisely, there may exist a competition between $\lambda$ and $K$: $\frac{\log K}{\log \lambda} > C > 0$ implies purely continuous spectrum; and $\frac{\log K}{\log \lambda} \leq C$ may show pure point spectrum (even AL). We refer to [YZ14] for the phase transition parameter involving the coupling and energy: $\frac{\log E}{\log \lambda}$.

5. Our results can be extended to the case $H = (-\Delta)^l + \lambda V(\vec{\theta} + Kx\vec{\omega})$, where $l \in \mathbb{Z}, l \geq 1$ and $V(\cdot)$ is Gevrey regular (see [Shi19b]).
It should be true that the spectrum in $I$ is purely absolutely continuous for $0 < \frac{1}{K^2} \ll 1$ and for most $\vec{\omega}$. However, we think Anderson localization occurs at low energy for $\frac{1}{K^2} \gg 1$ and for most $\vec{\omega}$.

We outline the proof of Theorem 1.1:

The main scheme is to perform MSA in momentum space $\Theta \in \mathbb{R}^d$. Let $E$ be an eigenvalue of $H(\hat{\theta})$, i.e. $H(\hat{\theta})\Psi = E\Psi$ for some $0 \neq \Psi \in L^2(\mathbb{R}^d)$. Let $k\vec{\omega} = (k_1 \cdot \vec{\omega}_1, \cdots, k_d \cdot \vec{\omega}_d)$.

Denote by $\hat{\Psi}$ the Fourier transformation of $\Psi$. Then by direct computations (see Lemma 4.1 for details), the vector

$$Z_{k} = \{Z_{k} = e^{-ik\cdot\hat{\omega}/(E)}\}_{k \in \mathbb{Z}^b}$$

will satisfy $h(\Theta)Z = \frac{E}{\varepsilon}Z$, where

$$(h(\Theta)Z)_k = \sum_{k' \in \mathbb{Z}^b} \varepsilon \hat{V}_{k-k'}Z_{k'} + \sum_{i=1}^d (\Theta_i + k_i \cdot \vec{\omega}_i)^2 Z_k, \varepsilon = \frac{K}{\lambda^2}, \Theta = \Theta' \frac{K}{\lambda}.$$

In addition, we can even show $Z_k$ grows at most polynomially in $k$ for a.e. $\Theta$. Thus the asymptotic property of $Z_k$ can be controlled by fast off-diagonal decaying properties of the Green’s functions (if exist)

$$G \Lambda(E; \Theta) = (R \Lambda(h(\Theta) - E)R \Lambda)^{-1}, \Lambda \subset \mathbb{Z}^b.$$

At this stage, it suffices to get certain off-diagonal decaying estimates on $G \Lambda(E; \Theta)$ as $|\Lambda| \to \infty$. For this purpose, it needs to make quantitative restrictions on $\vec{\omega}, \Theta$ and even on $E$ due to the small-divisors difficulty. Usually, the KAM and MSA methods are powerful to overcome the small-divisors. In continuous quasi-periodic operators case, Surace [Sur90] performed the first MSA scheme in momentum space $\Theta \in \mathbb{R}$ but restricted to diagonal part of the form

$$(\Theta + k_1 + k_2\omega)^2, (k_1, k_2) \in \mathbb{Z}^2.$$

Because of this special structure, Surace can even obtain uniform in $E \in \mathbb{R}$ estimate. However, the proof of [Sur90] depends heavily on eigenvalue variations and is hard to work for $\Theta$ in higher dimensions. For recent $\Theta \in \mathbb{R}$ type MSA, we also refer to [DG14] in dealing with the inverse spectral theory for quasi-periodic operators on $\mathbb{R}$.

For general $\Theta \in \mathbb{R}^d$ in momentum space the only known MSA is developed in recent papers [KS13, KS19] for $d = 2$. However, for arbitrary $d \geq 1$ similar type of questions has been extensively studied in case $\Theta \in \mathbb{T}^d$ [BGS02, Bou07, BK19, JLS20, Shi19b] via techniques of semi-algebraic geometry arguments and subharmonic estimates. As compared with compact momentum case (i.e. $\Theta \in \mathbb{T}^d$), there comes new difficulty for general $\Theta \in \mathbb{R}^d$. For any finite $\Lambda \subset \mathbb{Z}^b$,

$$\sup_{k \in \Lambda} \left| \sum_{i=1}^d (\Theta_i + k_i \cdot \vec{\omega}_i)^2 - E \right| \ll 1 \quad (1.4)$$

1The main scheme is similar that of [FSW90].
may frequently occur while $|\Theta| \gg 1$ and $|E| \gg 1$. To avoid this difficulty, one may make the assumption that for some $C > 0$ \footnote{This assumption is essentially satisfied in [BGS02, Bou07, BK19, JLS20, Shi19b].},

$$-C \leq E \leq C. \quad (1.5)$$

Once (1.5) is satisfied, $\Theta$ stays essentially in a compact set of size $\sim |\Lambda|$, and the small divisors presented in (1.4) could be handled via methods of [Bou07, JLS20]. The elimination frequencies argument of Bourgain [Bou07] based on semi-algebraic geometry theory is likely applicable. Consequently, we may expect that $H(\hat{\theta})$ has no eigenvalues in interval $[-CK^2, CK^2]$ if $0 < \varepsilon \leq \varepsilon_0(C)$.

It is reasonable that the absence of eigenvalues for $H(\hat{\theta})$ should hold in a much longer energy interval, which of course needs a proof with new ideas. In this paper we will provide such a proof based on the following observations:

- The first step of MSA is in general based on perturbative argument, in particular the Neumann series expansions. It is important that the large deviation estimate (LDE) for Green’s functions is valid for every $E \in \mathbb{R}$ in this step.
- Once the first step of MSA is finished, we will deal with the second iteration step. To propagate the LDE, some restrictions on $\omega$ are necessary, which is essentially necessary even in $\Theta \in \mathbb{T}^d$ case. What’s new here is that we can make quantitative restrictions on $E$ in this step. More precisely, the first step MSA implies that LDE holds in scales interval $N_0 \leq N \leq |\log \varepsilon|^C$. The second MSA step will start with $N = |\log \varepsilon|^C$. To avoid (1.4), we can actually set $|E| \leq |\log \varepsilon|^C$. In this energy interval, we could perform the iteration similar to that of [Bou07, JLS20] together with some technical improvements.
- Since $|E| \leq |\log \varepsilon|^C$ is negligible as compared with the later general MSA scales $N \gg |\log \varepsilon|^C$, the iterations become similar to that done in the second step.

In conclusion, our new aspect here is that we focus on the MSA (in the momentum space $\Theta \in \mathbb{R}^d$) in the first two steps and make effective restrictions on $E$ in the second step.

Of course, the final aim is to show $H(\hat{\theta})$ has no eigenvalues on $\mathbb{R}$, which seems difficult to handle via the present method.

The structure of the paper is as follows. Some preliminaries are introduced in §2. The MSA in momentum space is established in §3. In §4, we finish the proof of Theorem 1.1. Some useful estimates are included in the Appendix.
For example, \( X(\Theta_j) = \{ \Theta_j \in \mathbb{R} : (\Theta_j, \Theta_j) \in X \} \) if \( \emptyset \neq X \subset \mathbb{R}^d \).

Throughout this paper, we assume \( \rho \in (0, 1) \) for simplicity.

### 2.2. Aubry duality

It is easy to check that if

\[
\Psi(x) = e^{i\Theta \cdot x} \sum_{k \in \mathbb{Z}^b} \Psi_k e^{i k \cdot (\vec{\theta} + x \vec{\omega})}
\]

is a solution of \((-\Delta + \varepsilon V(\vec{\theta} + x \vec{\omega}))\Psi(x) = E \Psi(x)\) (i.e. the Floquet-Bloch solution), then \( \{ \Psi_k \}_{k \in \mathbb{Z}^b} \) satisfies the following equation

\[
\sum_{k' \in \mathbb{Z}^b} \varepsilon \hat{\Psi}_{k-k'} \Psi_{k'} + \sum_{i=1}^{d} (\Theta_i + k_i \cdot \vec{\omega}_i)^2 \Psi_k = E \Psi_k.
\]

This motivates us to study the following \textit{unbounded} quasi-periodic operators on \( \mathbb{Z}^b \)

\[
(h(\Theta) Z)_k = \sum_{k' \in \mathbb{Z}^b} \varepsilon \hat{Z}_{k-k'} Z_{k'} + \sum_{i=1}^{d} (\Theta_i + k_i \cdot \vec{\omega}_i)^2 Z_k.
\] (2.1)

We call \( h(\Theta) \) the Aubry duality of \(-\Delta + \varepsilon V(\vec{\theta} + x \vec{\omega})\). For Aubry duality results in discrete case, we refer to [BJ02, JK16, Shi19b].

### 2.3. Green’s functions and elementary regions

If \( \Lambda \subset \mathbb{Z}^b \), denote \( h_{\Lambda}(\Theta) = R_{\Lambda} h(\Theta) R_{\Lambda} \), where \( R_{\Lambda} \) is the restriction operator and \( h(\Theta) \) is given by (2.1). Define the Green’s function as

\[
G_{\Lambda}(E; \Theta) = (h_{\Lambda}(\Theta) - E + i0)^{-1}.
\]

Denote by \( Q_N \) an elementary region of size \( N \) centered at 0 (as in [JLS20]):

\[
Q_N = [-N, N]^b \text{ or } Q_N = [-N, N]^b \setminus \{ k \in \mathbb{Z}^b : k_{ij} \in \varsigma_{ij}, 1 \leq i \leq d, 1 \leq j \leq b_i \},
\]

where \( \varsigma_{ij} \in \{ \{n < 0\}, \{n > 0\}, \emptyset \} \) and at least two \( \varsigma_{ij} \) are not \( \emptyset \). Denote by \( \mathcal{E}_N^0 \) the set of all elementary regions of size \( N \) centered at 0. Let \( \mathcal{E}_N \) be the set of all translates of elementary regions: \( \mathcal{E}_N = \bigcup_{k \in \mathbb{Z}^b, Q_N \in \mathcal{E}_N^0} \{ k + Q_N \} \).

### 2.4. Semi-algebraic sets

**Definition 2.1** (Chapter 9, [Bou05]). A set \( S \subset \mathbb{R}^n \) is called a semi-algebraic set if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More precisely, let \( \{ P_1, \cdots, P_s \} \subset \mathbb{R}[x_1, \cdots, x_n] \) be a family of real polynomials whose degrees are bounded by \( d \). A (closed) semi-algebraic set \( S \) is given by an expression

\[
S = \bigcup_{j} \bigcap_{\ell \in \mathcal{L}_j} \{ x \in \mathbb{R}^n : P_\ell(x)\zeta_{j\ell} = 0 \},
\] (2.2)

where \( \mathcal{L}_j \subset \{ 1, \cdots, s \} \) and \( \zeta_{j\ell} \in \{ \geq, \leq, = \} \). Then we say that \( S \) has degree at most \( sd \). In fact, the degree of \( S \) which is denoted by \( \deg(S) \), means the smallest \( sd \) over all representations as in (2.2).

**Lemma 2.2** (Tarski-Seidenberg Principle, [Bou05]). Denote by \( (x, y) \in \mathbb{R}^{d_1+d_2} \) the product variable. If \( S \subset \mathbb{R}^{d_1+d_2} \) is semi-algebraic of degree \( B \), then its projections \( \text{Proj}_x S \subset \mathbb{R}^{d_1} \) and \( \text{Proj}_y S \subset \mathbb{R}^{d_2} \) are semi-algebraic of degree at most \( BC^d \), where \( C = C(d_1, d_2) > 0 \).
Lemma 2.3 ([Bou07]). Let $S \subset [0,1]^{d_1+d_2}$ be a semi-algebraic set of degree $\deg(S) = B$ and $\mes_d(S) \leq \eta$, where $\log B \ll \log \frac{1}{\eta}$.

Denote by $(x_1,x_2) \in [0,1]^{d_1} \times [0,1]^{d_2}$ the product variable. Suppose $\eta^\frac{1}{d} \leq \epsilon$. Then there is a decomposition of $S$ as $S = S_1 \cup S_2$

with the following properties. The projection of $S_1$ on $[0,1]^{d_1}$ has small measure $\mes_{d_1}(\proj_{x_1}S_1) \leq B^{C(d)} \epsilon$, and $S_2$ has the transversality property $\mes_{d_2}(L \cap S_2) \leq B^{C(d)} \epsilon^{-1} \eta^\frac{1}{d}$, where $L$ is any $d_2$-dimensional hyperplane in $\mathbb{R}^d$ s.t., $\max_{1 \leq j \leq d_1} |\proj_L(\mathbf{e}_j)| < \epsilon$, where we denote by $\mathbf{e}_1, \cdots, \mathbf{e}_{d_1}$ the $x_1$-coordinate vectors.

3. LDT for Green’s functions

The main result of this section is the following LDT for Green’s functions.

Theorem 3.1 (LDT). There exists $c_*(b,d) > 0$ such that the following holds: For any $0 < c_1 \leq c_*$, there exists $N_0 = N_0(c_1, \rho, b, d) > 0$ such that if $\log \log \frac{1}{\epsilon} \geq N_0$,

then for all $N \geq N_0$ we have

(i) There is some semi-algebraic set $\Omega_N = \Omega_N(c_1, \epsilon, \rho, b, d) \subset [0,2\pi]^b$ with $\deg(\Omega_N) \leq N^{d_1}$, and as $\epsilon \to 0$,

$$\mes \left( [0,2\pi]^b \setminus \bigcap_{N \geq N_0} \Omega_N \right) \to 0.$$

(ii) If $\omega \in \Omega_N$ and $E \in \mathbb{R}$ satisfying $|E| \leq \frac{1}{\log \epsilon^{\frac{1}{d_1}}}$, then there exists some set $X_N = X_N(\omega, E) \subset \mathbb{R}^d$ such that

$$\sup_{1 \leq j \leq d} \mes(\Theta_j^N) \leq e^{-N^{c_1}},$$

and for $\Theta \notin X_N$, $Q \in \mathcal{E}_N^0$,

$$|G_Q(E; \Theta)(\mathbf{n}, \mathbf{n}')| \leq e^{-\frac{1}{\sqrt{d}}|\mathbf{n} - \mathbf{n}'|} \text{ for } |\mathbf{n} - \mathbf{n}'| \geq N/10.$$

Proof of Theorem 3.1. The proof is based on the multi-scale analysis scheme as in [Bou07, JLS20]. However, we formulate it into three steps.

**STEP 1: The First Step**

In this initial step we use a perturbative argument. It is important that in this step we obtain *uniform* measure estimates both on $E \in \mathbb{R}$ and $\omega$.

Lemma 3.2. Given $\delta > 0$, let

$$X_N = \bigcup_{|k| \leq N} \left\{ \Theta \in \mathbb{R}^d : \left| \sum_{i=1}^d (\Theta_i + k_i \cdot \omega_i)^2 - E \right| < \delta \right\}.$$
Then
\[
\sup_{1 \leq j \leq d, \Theta_j \in \mathbb{R}^{d-1}} \mes(X_N(\Theta_j^{-})) \leq C(2N + 1)^b \sqrt{d},
\]
where \( C > 0 \) is an absolute constant.

Moreover, if
\[
\varepsilon^{-1} \geq 2\delta^{-1}(2N + 1)^b,
\]
then for any \( \Theta \notin X_N \) and \( \Delta \subset [-N, N]^b \),
\[
\|G_{\Delta}(E; \Theta)\| \leq 2\delta^{-1},
\]
\[
|G_{\Delta}(E; \Theta)(n, n')| \leq 2\delta^{-1} e^{-\rho(n-n')}.
\]

Proof. The proof is perturbative and based on the Neumann series argument: The proof of measure bound is based on Lemma 4.7 in [Shi19a], and the estimate on Green’s functions can be found in the proof of Theorem 4.3 in [JLS20].

If we take \( C(2N + 1)^b \sqrt{d} = e^{-N^{c_1}} \) in the above lemma, we obtain

**Proposition 3.3.** Let \( 0 < c_1 < 1/4 \). Then there exists \( N_0 = N_0(c_1, \rho, b, d) > 0 \) such that the following: If \( N_0 \leq N \leq \lfloor \log \varepsilon \rfloor \), then for all \( E \in \mathbb{R} \) and \( \omega \in [0, 2\pi]^b \) there is some \( X_N = X_N(E, \omega) \subset \mathbb{R}^d \) such that
\[
\sup_{1 \leq j \leq d, \Theta_j \in \mathbb{R}^{d-1}} \mes(X_N(\Theta_j^{-})) \leq e^{-N^{c_1}},
\]
and, if \( \Theta \notin X_N \) and \( Q \in \mathcal{E}_N^0 \),
\[
\|G_Q(E; \Theta)\| \leq e^{-\sqrt{N}} \quad \text{(3.2)}
\]
\[
|G_Q(E; \Theta)(n, n')| \leq e^{-\frac{\delta}{\rho}|n-n'|} \quad \text{for } |n-n'| \geq N/10. \quad \text{(3.3)}
\]

Proof. Let \( X_N \) be given by Lemma 3.2. Then the measure bound (3.1) can be derived from choosing \( C(2N + 1)^b \sqrt{d} = e^{-N^{c_1}} \), i.e.,
\[
\delta = C^{-2}(2N + 1)^{-2b} e^{-2N^{c_1}}. \quad \text{(3.4)}
\]

We then turn to the Green’s function estimates. Let \( \Theta \notin X_N \). Since (3.4), the condition \( \varepsilon^{-1} \geq \delta^{-1}(2N + 1)^b \) is equivalent to
\[
\varepsilon^{-1} \geq 2C^2(2N + 1)^b e^{2N^{c_1}}
\]
It suffices to let \( N \geq N_0(c_1, b, d) > 0 \) and
\[
\varepsilon^{-1} \geq e^{N^{c_1}} \geq 2C^2(2N + 1)^b e^{2N^{c_1}},
\]
that is \( N_0(c_1, b, d) \leq N \leq \lfloor \log \varepsilon \rfloor \). From (3.5) and \( 0 < c_1 < 1/4 \), we have
\[
2\delta^{-1} \leq e^{N^{c_1}} \leq e^{\sqrt{N}},
\]
which implies (3.2). Finally, the exponential decay bound (3.3) follows from: If \( |n-n'| \geq N/10 \) and \( N \geq N_0(c_1, \rho, b, d) > 0 \), then
\[
|G_Q(E; \Theta)(n, n')| \leq e^{-\sqrt{N} \frac{\delta}{\rho} N} e^{-\frac{\delta}{\rho}|n-n'|} \leq e^{-\frac{\delta}{\rho}|n-n'|}.
\]

□
STEP 2: The Second Step

In the second step we will propagate LDT from (scales) \( [N_0, |\log \varepsilon|^c] \) to
\[
[|\log \varepsilon|^{c/4}, e^{|\log \varepsilon|^{c/4}}].
\]

For this purpose, further removal of \( \vec{\omega} \) is necessary and the semi-algebraic geometry arguments established by Bourgain [Bou07] play a key role. In what follows we always assume
\[
E \in \mathcal{I}_\varepsilon = \left[ -|\log \varepsilon|^{c/4}, |\log \varepsilon|^{c/4} \right].
\]  

(3.6)

Denote \( B(N) = \{ \Theta \in \mathbb{R}^d : |\Theta| \leq N \} \). Then we have

**Proposition 3.4.** Let \( |\log \varepsilon|^{c/4} \leq N \leq e^{|\log \varepsilon|^{c/4}} \), \( N_1 \sim (\log N)^{2/c_1} \). Then there exist constants
\[
0 < c_2(b, d) \leq c_3(b, d) \leq c_4(b, d) \leq 1,
\]
and semi-algebraic set \( \Omega_N \subset [0, 2\pi]^b \) satisfying
\[
\deg(\Omega_N) \leq N^{4d}, \quad \text{mes}(\mathbb{R} \setminus \Omega_N) \leq N^{-c_2}
\]
such that the following holds: If \( \log \frac{1}{\varepsilon} \geq N_0(c_1, b, d) > 0 \) and \( \vec{\omega} \in \Omega_N \), then for all \( (E, \Theta) \in \mathcal{I}_\varepsilon \times B(100bN^2) \) there is some \( \frac{1}{\varepsilon}N^{c_3} < M < 10N^{c_4} \) such that for all \( k \in [-M, M]^b \setminus [-\frac{1}{\varepsilon}N^{c_3}, \frac{1}{\varepsilon}N^{c_3}]^b \), one has \( \Theta + k\vec{\omega} \notin X_{N_1} \), here \( X_{N_1} \) is given by Proposition 3.3.

**Remark 3.1.** The conditions \( \log \frac{1}{\varepsilon} \geq N_0(c_1, b, d) > 0 \) and \( N \leq e^{|\log \varepsilon|^{c/4}} \) can ensure that \( N_0(c_1, b, d) \leq N_1 \leq |\log \varepsilon|^{c/4} \).

**Proof.** We will eliminate the variables \((\Theta, E) \in B(100bN^2) \times \mathcal{I}_\varepsilon \). This needs make quantitative restrictions on \( \vec{\omega} \).

Let
\[
N^{c_3} < L < N^{c_4},
\]
where \( 0 < c_3 < c_4 < 1 \) will be specified later. Let \( S_L \) be the set of all \((\vec{\omega}, \Theta, E) \in [0, 2\pi]^b \times B(100bN^2) \times \mathcal{I}_\varepsilon \) such that, for any \( n \in [-L, L]^b \) and \( \vec{\omega} \in [0, 2\pi]^b \), \( \Theta + n\vec{\omega} \notin X_{N_1}(E, \vec{\omega}) \). Applying Proposition 3.3 gives
\[
\deg(S_L) \leq L^{C(b, d)},
\]  

(3.7)

\[
\sup_{1 \leq j \leq d, \Theta_j \in \mathbb{R}^d} \text{mes}(\mathbb{R} \setminus S_L(\Theta_j^-)) \leq e^{-N^{c_4}/2}.
\]  

(3.8)

Fix \( I \subset \{1, \cdots, d\} \) and define \( A \) to be all \((\vec{\omega}, \Theta, y, E) \in [0, 2\pi]^b \times B(100bN^2) \times \mathbb{R}^l \times \mathcal{I}_\varepsilon \) satisfying
\[
(\vec{\omega}, (\Theta_j + y_j)_{j \in I}, (\Theta_j)_{j \notin I}, E) \notin S_L.
\]  

(3.9)

Obviously, by (3.7) and (3.9),
\[
\deg(A) \leq L^{C(b, d)}.
\]  

(3.10)

Fix \( \vec{\omega} \in [0, 2\pi]^b \) and consider
\[
A_1 := A(\vec{\omega}) \subset B(100bN^2) \times \mathbb{R}^l \times \mathcal{I}_\varepsilon.
\]
Assume for \( y_t = (y_t)_{t \in T}, y_t \notin B(x_t(200bN^2) = \{ y \in \mathbb{R}^n : |y| \leq 200bN^2 \}. \) Then for all \( n \in [-L, L]^b + c^0_{N,N}, \) one has since \(|\Theta| \leq 100bN^2\) and \(|E| \leq |\log \varepsilon|^\frac{1}{M} \leq N,\)

\[
\sum_{i=1}^{d}(\Theta_i + y_t + n_i \cdot \omega_i)^2 - E \geq \sum_{i \in I}(\Theta_i + y_t + n_i \cdot \omega_i)^2 - E \\
\geq N^4,
\]

(3.11) which shows that there is no resonances in this case. In other words, we must have \( A_1 \subset B(100bN^2) \times B(200bN^2) \times \mathcal{I}_e. \)

By (3.10) and Tarski-Seidenberg principle (i.e., Lemma 2.2), we obtain

\[
\deg(A_1) \leq L^{C(b,d)}.
\]

(3.13)

From (3.8), for all \( (\Theta, E) \in B(100bN^2) \times I_e, \) we have

\[
\text{mes}(A_1(\Theta, E)) \leq \eta := e^{-N^{11}/2}.
\]

(3.14)

At this stage, we need a lemma for eliminating multi-variable:

**Lemma 3.5.** Let \( S \subset [0, 100bN^2]^{s+d} \) be a semi-algebraic set of degree \( B \) and such that

\[
\text{mes}(S(y)) < \eta \text{ for } \forall y \in [0, 100bN^2]^{s+d} \text{ and } \log(BN) \ll \log \frac{1}{\eta}.
\]

Then the set

\[
\left\{ (x_1, \ldots, x_{2^r}) \in [0, 100bN^2]^{s+2^r} : \bigcap_{1 \leq i \leq 2^r} S(x_i) \neq \emptyset \right\}
\]

is semi-algebraic of degree at most \( B^C \) and measure at most

\[
N^C B^C \eta^{-2^{-r(r-1)/2}},
\]

where \( C = C(s, r, b) > 0. \)

**Proof of Lemma 3.5.** The degree bound follows from Tarski-Seidenberg principle Lemma 2.2. The measure bound is derived as follows. We first divide \([0, 100bN^2]^{s+d}\) into about \( N^{C(s,r,b)} \) many unit cubes. Then applying Lemma 1.18 of [Bou07] gives measure bound \( B^C \eta^{-2^{-r(r-1)/2}} \) on each such unit cube. Finally, it suffices to take account of all those cubes.

From (3.13) and (3.14), we have by using Lemma 3.5 (with \( s = |I|, r = d+1, B = L^C \) and \( \eta = e^{-N^{11}/2} \)) that

\[
A_2 = \left\{ (y^{(i)})_{1 \leq i \leq 2^{d+1}} : \bigcap_{1 \leq i \leq 2^{d+1}} A_1(y^{(i)}) \neq \emptyset \right\}
\]

is a semi-algebraic set with

\[
\deg(A_2) \leq L^{C(b,d,|I|)}, \text{ mes}(A_2) \leq \eta_1 := N^{C(b,d,|I|)} \eta^{|I|-d-1-2^{-d(d+1)/2}}.
\]

(3.15)

Define

\[
B = \left\{ (\omega, y^{(i)})_{1 \leq i \leq 2^{d+1}} : \omega \in [0, 2\pi]^b, \bigcap_{1 \leq i \leq 2^{d+1}} A(\omega, y^{(i)}) \neq \emptyset \right\}
\]

\[
\subset [0, 2\pi]^b \times [0, 200bN^2]^{2^{d+1}}.
\]
By Lemma 2.2, $\deg(B) \leq L^{C(d, |I|)}$. Write $\overline{\omega} = (\overline{\omega}_i)_{i \in I}$ and $\tilde{\omega} := (\tilde{\omega}_i)_{i \in I}$. Then by Fubini Theorem and (3.15),
\[
\mes(B(\overline{\omega})) \leq \eta_1, \quad \deg(B(\overline{\omega})) \leq L^C.
\tag{3.16}
\]
Notice that for any $C_1 > 1$, one has $\frac{1}{C_1} \gg N^C$ if $N \geq N_0(C_1, c_1, b, d, |I|) > 0$. One considers the set $B_1$ containing $\tilde{\omega}$, which is defined by the following: there is some sequence $n^{(1)}, \ldots, n^{(2^d-1)} \in \mathbb{Z}^{d}$ $(b_I = \sum_{i \in I} b_i)$ satisfying
\[
L^{C_1} \leq \min_{i \in I, 1 \leq j \leq b_i} |n^{(\alpha)}_i| \leq |n^{(\alpha)}_i| \leq L^{C_1} (1 \leq \alpha \leq 2^{d+1}),
\tag{3.17}
\]
such that
\[
(\tilde{\omega}, n^{(1)} \tilde{\omega}, \ldots, n^{(2^d-1)} \tilde{\omega}) \in B(\overline{\omega}),
\]
where
\[
1 \ll C_1 \ll C_1 \ll \cdots \ll C_2^{d+1} \ll C_{2^{d+1}}.
\]

At this stage, we will introduce a key lemma for eliminating frequencies:

Lemma 3.6. Let $S \subset \mathbb{R}^r$ be a semi-algebraic set of degree $B$ and $\mes(S) < \eta$ with $\eta > 0$.

Let $b(r) = \sum_{i=1}^r b_i, b_i \in \mathbb{N}$. For $\tilde{\omega}_i \in [0, 2\pi]^{b_i}$, $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_r) \in [0, 2\pi]^{b(r)}$ and $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^{b(r)}(n_i \in \mathbb{Z}^{b_i})$, define
\[
\mathbf{n} \tilde{\omega} = (n_1 \tilde{\omega}_1, \ldots, n_r \tilde{\omega}_r).
\]

For any $C > 1$, define $\mathcal{N}_1, \ldots, \mathcal{N}_{s-1} \subset \mathbb{Z}^{b(r)}$ to be finite sets with the following property:
\[
\min_{1 \leq i \leq r, 1 \leq j \leq b_i} |n_{ij}| > (B \max_{1 \leq i \leq r} |m_i|)^C,
\]
where $\mathbf{n} \in \mathcal{N}_1, \mathbf{m} \in \mathcal{N}_{s-1}$ ($2 \leq l \leq s - 1$).

Then there is some $C = C(r, s, b) > 0$ such that for $\max_{\mathbf{n} \in \mathcal{N}_{s-1}} |\mathbf{n}|^C < \frac{1}{b(r)}$, one has
\[
\mes(\{\tilde{\omega} \in [0, 2\pi]^{b(r)} : \exists \mathbf{n} \in \mathcal{N}_s s.t., (\tilde{\omega}, \mathbf{n}^{(1)} \tilde{\omega}, \ldots, \mathbf{n}^{(s-1)} \tilde{\omega}) \in S\}) \leq B^C \left(\min_{\mathbf{n} \in \mathcal{N}_1, 1 \leq i \leq r} |n_{ij}|\right)^{-1}.
\]

Proof of Lemma 3.6. The proof is similar to that of Lemma 1.20 in [Bou07] by iterating Lemma 2.3. The main difference is that we allow $S$ to vary in $\mathbb{R}^r$ rather than in the unit cube. Bourgain’s proof remains applicable since Lemma 2.3 permits re-scaling. Moreover, at $i$-th ($i \leq s - 1$) iteration step, the valid sets are essentially contained in $[0, \max_{\mathbf{n} \in \mathcal{N}_{s-1}} |\mathbf{n}|^{r(s-i+1)}]$. We omit the details here.

Then from (3.16) and Lemma 3.6 (set $r = |I|, s = 2^{d+1} + 1$), we have
\[
\mes(B_1) \leq L^{-C_1} L^C \leq L^{-3}.
\tag{3.18}
\]

Define
\[
\mathcal{G}_L := \bigcap_{\emptyset \neq I \subset \{1, \ldots, d\}} \{\tilde{\omega} \in [0, 2\pi]^b : \tilde{\omega} = (\overline{\omega}, \tilde{\omega}), \tilde{\omega} \notin B_1\}.
\]

Thus $\tilde{\omega} \notin \mathcal{G}_L$ if and only if, there are $\emptyset \neq I \subset \{1, \ldots, d\}$ and some sequence $\mathbf{n}^{(1)}, \ldots, \mathbf{n}^{(2^d-1)} \in \mathbb{Z}^b$ satisfying (3.17) such that
\[
(\tilde{\omega}, \mathbf{n}^{(1)} \tilde{\omega}, \ldots, \mathbf{n}^{(2^d-1)} \tilde{\omega}) \in B.
\]
This implies that \( \mathcal{G}_L \) is a semi-algebraic set. Furthermore, by (3.17) and (3.18),
\[
\text{mes}([0, 2\pi]^b \setminus \mathcal{G}_L) \leq L^{-2}, \quad \text{deg}(\mathcal{G}_L) \leq L^{C_2d+1}.
\]
We should remark that \( \mathcal{G}_L \) also depends on \( C_1, \cdots, C_{2d+1} \).

Finally, for \( 0 < c_3(b, d) \ll c_4(b, d) \ll 1 \), we can choose appropriate \( L_t, C_{11}, \cdots, C_{2d+1} \) and then iterate along every axis direction (i.e., induction on \( I \)) as done by Bourgain [Bou07] (see the proof of the CLAIM) if \( \vec{\omega} \in \mathcal{G}_L \). The number of all possible \( L_t, C_{11}, \cdots, C_{2d+1} \) is finite and depends only on \( b, d \). In particular, we have
\[
\text{deg}(\Omega_N) \leq CN^{c_4C_2d+1} \leq N^{4d},
\]
\[
\text{mes}([0, 2\pi]^b \setminus \Omega_N) \leq CN^{-2c_3} \leq N^{-c_3},
\]
where \( \Omega_N = \bigcup_{L_t, C_{11}, \cdots, C_{2d+1}} \mathcal{G}_L \) and \( 0 < c_2(b, d) \ll c_3(b, d) \). In addition, if \( \vec{\omega} \in \Omega_N \), then for all \( (E, \Theta) \in \mathcal{E}_c \times \mathcal{B}(100bN^2) \) there is some \( 1 \leq N^c_3 < M < 10N^c_4 \) such that for all \( k \in [-M, M]^b \setminus [-M \frac{1}{10}, M \frac{1}{10}]^b \), one has \( \Theta + k\vec{\omega} \notin X_N \).

\( \square \)

Combining the above proposition and Cartan’s estimate on subharmonic functions, we can finish the proof of LDT in \([N, N^2] \).

Since (3.6) and (3.11), it suffices to consider in case
\[
\Theta \in \mathcal{B}(100bN^2).
\]

**Proposition 3.7.** Assume the assumptions of Proposition 3.4 are satisfied. Let \( \Theta \) satisfy (3.19). Then for \( \vec{\omega} \in \Omega_N \), we have

1. **Fix** \( 1 \leq j \leq d \) and \( \Theta_j^{-} \in \mathcal{B}_{d-1}(100bN^2) \). Write \( \Theta = (\Theta_j, \Theta_j^{-}) \in \mathbb{R}^d \). Assume there exist \( \bar{N} \in [N^{c_3}/4, N^{c_4}] \) and \( \Lambda \in \mathcal{E}_\bar{N} \) with \( \text{diam}(\Lambda) \leq 4\bar{N} \frac{1}{10} \) and \( \Lambda \subset \mathcal{B}(100bN^2) \) such that, for any \( k \in \Lambda \), \( \bar{N} \), there exists some \( \mathcal{E}_N, \exists W \subset \Lambda \setminus \bar{N} \) such that \( \text{dist}(k, \Lambda \setminus \bar{N}) \geq N_1/2 \), and \( \Theta + k\vec{\omega} \notin X_N \).
Let
\[
Y_\Theta = \left\{ y \in \mathbb{R} : |y - \Theta_j| \leq e^{-10\rho N_1}, |\Theta_j| \leq 100bN^2, ||G_A(E; (y, \Theta_j^{-}))|| \geq e^\sqrt{N} \right\}.
\]
Then
\[
\text{mes}(Y_\Theta) \leq e^{-\bar{N}^2/4}.
\]

2. **Fix** \( N_* \in [N, N^2] \). If \( E \in \mathcal{E}_c \) and \( 0 < c_1 < c_3/10 \), then there exists some set \( X_{N_*} = X_{N_*}(E, \vec{\omega}) \subset \mathbb{R}^d \) such that
\[
\sup_{1 \leq j \leq d, \Theta_j^{-} \in \mathbb{R}^{d-1}} \text{mes}(X_{N_*}(\Theta_j^{-})) \leq e^{-N_*^{c_1}},
\]
and for \( \Theta \notin X_{N_*}, Q \in \mathcal{E}_0^{N_*},
\]
\[
|G_Q(E; \Theta)(n, n')| \leq e^{-\left(\frac{c}{\rho^c_1} \frac{C_{b, a, d}(n, n')}{\sqrt{N_1}}\right) |n - n'|} \text{ for } |n - n'| \geq N_*/10.
\]

**Proof.** (1) Let \( D \) be the \( e^{-10\rho N_1} \) neighbourhood of \( \Theta_j \) in the complex plane, i.e.,
\[
D = \{ y \in \mathbb{C} : |\Re y| \leq e^{-10\rho N_1}, |\Im y - \Theta_j| \leq e^{-10\rho N_1} \}.
\]
From assumptions, we have for all $k \in \Lambda \setminus \tilde{\Lambda}$ and $Q \in \mathcal{E}^0_{N_1}$,
\[
\|G_Q(E; \Theta + k\omega)\| \leq e^{\sqrt{N_1}},
\]
\[
|G_Q(E; \Theta + k\omega)(n, n')| \leq e^{-\frac{3}{4}|n-n'|} \text{ for } |n-n'| \geq N_1/10.
\]
(3.20)
(3.21)
Note that for all $n, n' \in [-N_1, N_1]^b,
\[
e^{-10pN_1} < e^{-3pN_1 - p|n-n'|}.
\]
Then by Lemma B.1, (3.20) and (3.21), we have for any $y \in D$, $Q \in \mathcal{E}^0_{N_1}$ and $k \in \Lambda \setminus \tilde{\Lambda},$
\[
\|G_Q(E; (\Theta_j + y, \Theta_j^{-1}) + k\omega)\| \leq 2e^{\sqrt{N_1}},
\]
\[
|G_Q(E; (\Theta_j + y, \Theta_j^{-1}) + k\omega)(n, n')| \leq e^{-\frac{3}{4}|n-n'|} \text{ for } |n-n'| \geq N_1/10.
\]
(3.22)
(3.23)
Applying Lemma B.2 with $M_1 = M_0 = N_1$ implies for any $y \in D,$
\[
\|G_{\Lambda \setminus \tilde{\Lambda}}(E; (\Theta_j + y, \Theta_j^{-1}))\| \leq 4(2N_1 + 1)b e^{\sqrt{N_1}} \leq e^{2\sqrt{N_1}}.
\]
(3.24)
We can then use the following matrix-valued Cartan's estimate to propagate the “smallness of measure”:

**Lemma 3.8** (Cartan’s estimate, [Bou05]). Let $T(\theta)$ be a self-adjoint $N \times N$ matrix function of a parameter $\theta \in [-\delta, \delta]$ satisfying the following conditions:

(i) $T(\theta)$ is real analytic in $\theta \in [-\delta, \delta]$ and has a holomorphic extension to
\[
\mathcal{D}_{\delta, \delta} = \{\theta \in \mathbb{C} : |\Re \theta| \leq \delta, |\Im \theta| \leq \delta_1\}
\]
satisfying $\sup_{\theta \in \mathcal{D}_{\delta, \delta}} \|T(\theta)\| \leq K_1, K_1 \geq 1.$

(ii) For all $\theta \in [-\delta, \delta],$ there is subset $V \subset [1, N]$ with $|V| \leq M$ such that
\[
\|(R_{[1,N]} \setminus V)T(\theta)R_{[1,N] \setminus V})^{-1}\| \leq K_2, K_2 \geq 1.
\]

(iii) Assume
\[
\text{mes}\{\theta \in [-\delta, \delta] : \|T^{-1}(\theta)\| \geq K_3\} \leq 10^{-3}\delta(1 + K_1)^{-1}(1 + K_2)^{-1}.
\]

Let $0 < \epsilon \leq 1 + (K_1 + K_2)^{-10M}.$ Then
\[
\text{mes}\{\theta \in [-\delta/2, \delta/2] : \|T^{-1}(\theta)\| \geq \epsilon^{-1}\} \leq C\delta e^{-\frac{1}{M \log(\deltaK_1K_2K_3)}},
\]
(3.25)
where $C, c > 0$ are some absolute constants.

We will apply Lemma 3.8 with
\[
T(y) = h_\Lambda((\Theta_j + y, \Theta_j^{-1})) - E, \delta = \delta_1 = 2e^{-10pN_1}.
\]
(3.26)
It suffices to verify the assumptions of Lemma 3.8. By assumptions (3.19), $\Lambda \subset \mathbf{B}(1000bN^2)$ and (3.24), one has
\[
K_1 = O(N^2), M = |\tilde{\Lambda}| \leq C(b, d)N^{1/10}, K_2 = e^{2\sqrt{N_1}}.
\]
(3.27)
Since LDT holds at scale $N_2$ for $y$ being outside a set of measure at most $e^{-N_2^{1/2}},$ Applying Lemma B.2 yields
\[
\|T^{-1}(y)\| \leq 4(2N_2 + 1)b e^{\sqrt{N_2}} \leq e^{2\sqrt{N_2}} = K_3
\]
for $y$ being outside a set of measure at most
\[
(2N + 1)b e^{-N_2^{1/2}} \leq e^{-N_2^{1/2}}.
\]
It follows from $100N_1 < N_1^{3/2} < N_2^c$ that

$$10^{-3} \delta_1 (1 + K_1)^{-1} (1 + K_2)^{-1} \geq e^{-N_2^c/2}.$$ 

This verifies (iii) of Lemma 3.8. For $\epsilon = e^{-\sqrt{N}}$, one has by (3.26) and (3.27), $\epsilon < (1 + K_1 + K_2)^{-10M}$. By (3.25) of Lemma 3.8,

$$\operatorname{mes}(Y) \leq e^{-\sqrt{N}N^{1/3}e^{c_1N}} \leq e^{-N^{1/3}}.$$ 

(2) Fix $N_\ast \in [N, N^2]$ and $E \in \mathcal{E}_\ast$. As done in (3.11), to prove LDT at scale $N_\ast$ it suffices to restrict $\Theta \in B(10bN_\ast) \subset B(10bN^2)$. Fix $1 \leq j \leq d$, $\Theta_j \in \mathbb{R}^{d-1}$ and $\Theta = (\Theta_j, \Theta_j^\ast) \in B(10bN_\ast)$. Then $\Theta + n\bar{\omega} \in B(10bN^2)$ for all $n \leq N_\ast$. By using Proposition 3.4 (with $\Theta$ replaced by $\Theta + n\bar{\omega}$), for such $\Theta$ and any $n \in Q \in \mathcal{E}_{N_\ast}$, there exist $\bar{\epsilon} N \leq \bar{N} \leq N^{c_1}, \Lambda \in \mathcal{E}_{\bar{N}}$ and $\bar{\Lambda}$, such that

$$n \in \bar{\Lambda} \subset \Lambda \subset Q, \operatorname{dist}(n, Q) \Lambda \geq \bar{N}/2, \operatorname{diam}(\bar{\Lambda}) \leq 4 \bar{N}^{1/3}.$$ 

Moreover, for any $\mathbf{k} \in \Lambda \setminus \bar{\Lambda}$, $\Theta + \mathbf{k}\bar{\omega} \notin X_{N_1}$ and there exists some $E_{N_1} \ni W \subset \Lambda \setminus \bar{\Lambda}$ such that $\mathbf{k} \in W$, $\operatorname{dist}(\mathbf{k}, \Lambda \setminus \bar{\Lambda}/W) \geq N_1/2$. We should remark that in the above argument we had applied essentially methods of [JLS20] in dealing with elementary regions (of size $\Lambda$) near the boundary of $Q$.

We now fix above $N_\ast, \Lambda, \bar{\Lambda}$ throughout the set $\{(y, \Theta_j) \in \mathbb{R}^d : |y - \Theta_j| \leq e^{-10bN_1}\}$. Recalling Lemma B.1 and the above constructions, we have by (1) of Proposition 3.7 that there exists a $Y \subset \{y \in \mathbb{R} : |y - \Theta_j| \leq e^{-10bN_1}\}$ such that

$$\operatorname{mes}(Y) \leq e^{-N^{1/3}},$$ 

(3.28)

and for $\Theta_j \notin Y$, $\|G_A(E; \Theta)\| \leq e^{\sqrt{N}}$. Applying Lemma B.3 yields

$$|G_A(E; \Theta)(n, n')| \leq e^{-\left(\frac{\bar{N}}{2} - \sqrt{N}N^{1/3}\right) |n - n'|} \text{ for } |n - n'| \geq \bar{N}/10.$$ 

Cover $[0, 10bN_\ast]$ by pairwise disjoint $e^{-10bN_1}$-size intervals and let

$$X_{N_1}(\Theta_j) = \bigcup_{Q \in \mathcal{E}_{N_1}, n \in Q, \Theta = (\Theta_j, \Theta_j^\ast)} Y.$$ 

(3.29)

We remark that while $\Theta = (\Theta_j, \Theta_j^\ast)$ varies on a line for fixed $\Theta_j^\ast$, the total number of $Y$ is bounded by $10N_1e^{10bN_1}$. Thus by (3.28), (3.29) and $c_1 < c_3/10$, one has

$$\operatorname{mes}(X_{N_1}(\Theta_j^\ast)) \leq C(2N_\ast + 1)^b e^{10bN_1} e^{-N^{1/3}} \leq e^{-N_\ast^{c_3/7}} \leq e^{-N_\ast^{c_4}}.$$ 

Suppose now $\Theta \notin X_{N_1}$. Applying Lemma B.2 yields

$$\|G_Q(E; \Theta)\| \leq 4(2N^{c_4} + 1)^b e^{\sqrt{N_\ast}} \leq e^{\sqrt{N_\ast}}.$$ 

Recalling Lemma B.3, we obtain

$$|G_Q(E; \Theta)(n, n')| \leq e^{-\left(\frac{\bar{N}}{2} - \sqrt{N}N^{1/3}\right) |n - n'|} \text{ for } |n - n'| \geq \bar{N}/10.$$ 

\[\square\]

**STEP 3: The General Step**

The proof is based on similar arguments as in **STEP 2**.
We define for $N \geq e^{\log \varepsilon^{1/4}}$ the following scales

$$N_1 \sim (\log N)^{2/c_1}, \quad N_2 \sim N_1^{2/c_1}.$$  

Then we have

**Proposition 3.9.** Let $\Omega_{N_i} (i = 1, 2)$ be semi-algebraic set satisfying $\deg(\Omega_{N_i}) \leq N_i^{4d}$ and let $\bar{\rho}_i \in [\rho/2, \rho)$. Assume further the following holds: If $\bar{\omega} \in \Omega_{N_i}$ and $E \in I_{\varepsilon}$, then there exists some semi-algebraic set $X_{N_i} \subset \mathbb{R}^d$ satisfying $\deg(X_{N_i}) \leq N_i^{2d}$ such that

$$\sup_{1 \leq j \leq d, \Theta_j} \text{mes}(X_{N_i}(\Theta_j)) \leq e^{-N_i^{c_1}},$$

and for $\Theta \notin X_{N_i}$, $Q \in \mathcal{E}_{N_i}^0$,

$$\|G_Q(E; \Theta)\| \leq e^{\sqrt{N_i}},$$

$$|G_Q(E; \Theta)(n, n^i)| \leq e^{-\bar{\rho}_i|n-n^i|} \text{ for } |n-n^i| \geq N_i/10,$$

(i = 1, 2).

Then exists some semi-algebraic set $\Omega_N \subset \Omega_{N_1} \cap \Omega_{N_2}$ with $\deg(\Omega_N) \leq N^{4d}$ and $\text{mes}((\Omega_{N_1} \cap \Omega_{N_2}) \setminus \Omega_N) \leq N^{-c_2}$ such that, if $\bar{\omega} \in \Omega_N$, then for $N \geq N_0(c_1, \rho, b, d) > 0$

(i) For all $E \in I_{\varepsilon}$ and $\Theta \in B(100bN^2)$, there is $\frac{N_{M_{10}}^{2d}}{10} < M < 10N^{c_4} \text{ such that }$

for all $k \in [-M, M] \setminus [-M^{10b}, M^{10b}]$, $\Theta + k\bar{\omega} \notin X_{N_1}$.

(ii) Fix $N_* \in [N, N^2]$. Then there exists some set $X_{N_*} = X_{N_*}(E, \omega) \subset \mathbb{R}^d$ such that

$$\sup_{1 \leq j \leq d, \Theta_j} \text{mes}(X_{N_*}(\Theta_j)) \leq e^{-N_*^{c_1}},$$

and for $\Theta \notin X_{N_*}$, $Q \in \mathcal{E}_{N_*}^0$,

$$\|G_Q(E; \Theta)\| \leq e^{\sqrt{N_*}},$$

$$|G_Q(E; \Theta)(n, n^i)| \leq e^{-\left(\frac{(\gamma + C_{\rho, b, d})}{\sqrt{N_*}}\right)|n-n^i|} \text{ for } |n-n^i| \geq N_*/10.$$

**Proof.** The proof is similar to that in **STEP 2**, and we omit the details here. \[ \square \]

From the above arguments, we can propagate LDT from (scales)

$$\left[N, N^{2/c_1}\right]$$

to

$$\left[e^{N^{c_1/2}}, e^{2N^{c_1/2}}\right].$$

Thus a standard MSA induction (on scales) can establish LDT on the whole interval $[N_0, \infty]$ (see [BGS02] or [JLS20] for details). This finishes the proof of Theorem 3.1. \[ \square \]
4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1 by using LDT and Aubry duality. We begin with a useful lemma:

**Lemma 4.1.** Let $\Psi \in H^2(\mathbb{R}^d)$ satisfy

$$(-\Delta + \varepsilon V(\vec{\theta} + x\vec{\omega}))\Psi(x) = E\Psi(x). \quad (4.1)$$

Let $\hat{\Psi}$ be the Fourier transformation of $\Psi$. Then for a.e. $\Theta \in \mathbb{R}^d$ and any $\vec{\theta} \in T^b$ the following holds: Let $Z = \{Z_k\}_{k \in \mathbb{Z}^b}$ satisfy $Z_k = e^{-ik \cdot \vec{\theta}}\hat{\Psi}(\Theta + k\vec{\omega})$. Then

$$h(\Theta)Z = EZ, \quad (4.2)$$

where $h(\Theta)$ is given by (2.1). Moreover, There is $C = C(\Theta, \Psi, b, d) \in (0, \infty)$ such that

$$|Z_k| \leq C(1 + |k|)^{5b} \text{ for } \forall k \in \mathbb{Z}^b. \quad (4.3)$$

**Proof.** It suffices to consider $\Psi \in C^\infty_0(\mathbb{R}^d)$. We note first (4.1) is equivalent to

$$\|\xi\|^2\hat{\Psi}(\xi) + \varepsilon \hat{V}(\xi) = E\hat{\Psi}(\xi), \quad (4.4)$$

where $||\xi||^2 = \sum_{i=1}^d |\xi_i|^2$. Since $V$ is quasi-periodic (and analytic), we have

$$V(\vec{\theta} + x\vec{\omega}) = \sum_{k \in \mathbb{Z}^b} \hat{V}_k e^{ik \cdot (\vec{\theta} + x\vec{\omega})},$$

and as a result,

$$\hat{V}\hat{\Psi}(\xi) = \int_{\mathbb{R}^d} V(\vec{\theta} + x\vec{\omega})\Psi(x)e^{-ix \cdot \xi}dx$$

$$= \sum_{m \in \mathbb{Z}^b} \int_{\mathbb{R}^d} \hat{V}_m e^{im \cdot (\vec{\theta} + x\vec{\omega})}\Psi(x)e^{-ix \cdot \xi}dx$$

$$= \sum_{m \in \mathbb{Z}^b} \hat{V}_m e^{im \cdot \vec{\theta}} \int_{\mathbb{R}^d} \Psi(x)e^{-i(\xi - m\vec{\omega}) \cdot x}dx$$

$$= \sum_{m \in \mathbb{Z}^b} \hat{V}_m e^{im \cdot \vec{\theta}} \hat{\Psi}(\xi - m\vec{\omega}). \quad (4.5)$$

Combining (4.4) and (4.5) yields

$$||\xi||^2\hat{\Psi}(\xi) + \varepsilon \sum_{m \in \mathbb{Z}^b} \hat{V}_m e^{im \cdot \vec{\theta}} \hat{\Psi}(\xi - m\vec{\omega}) = E\hat{\Psi}(\xi). \quad (4.6)$$

Given $k \in \mathbb{Z}^b$, we set $\xi = \Theta + k\vec{\omega}$ in (4.6). Then

$$e^{-ik \cdot \vec{\theta}}||\Theta + k\vec{\omega}||^2\hat{\Psi}(\Theta + k\vec{\omega}) + \varepsilon \sum_{m \in \mathbb{Z}^b} \hat{V}_{k-m} e^{-im \cdot \vec{\theta}} \hat{\Psi}(\Theta + m\vec{\omega})$$

$$= Ee^{-ik \cdot \vec{\theta}}\hat{\Psi}(\Theta + k\vec{\omega}),$$

which implies (4.2).
We then deal with the polynomial bound (4.3). Since $\hat{\Psi} \in L^2(\mathbb{R}^d)$, we have
\[
\int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^b} \frac{|Z_k|^2(\Theta)}{(1 + |k|)^{10b}} d\Theta = \sum_{k \in \mathbb{Z}^b} \frac{1}{(1 + |k|)^{10b}} \int_{\mathbb{R}^d} |\hat{\Psi}(\Theta + k\omega)|^2 d\Theta
\]
\[
= \|\hat{\Psi}\|_{L^2}^2 \sum_{k \in \mathbb{Z}^b} \frac{1}{(1 + |k|)^{10b}} < \infty,
\]
which implies for a.e. $\Theta \in \mathbb{R}^d$,
\[
\sum_{k \in \mathbb{Z}^b} \frac{|Z_k|^2(\Theta)}{(1 + |k|)^{10b}} := C(\Theta, \Psi, b, d) < \infty.
\]
This means that for a.e. $\Theta \in \mathbb{R}^d$,
\[
|Z_k| \leq C(\Theta, \Psi, b, d)(1 + |k|)^{5b} \text{ for } \forall k \in \mathbb{Z}^b.
\]

We then prove our main result Theorem 1.1

**Proof of Theorem 1.1.** Let
\[
\tilde{\Theta} = \frac{\Theta}{K}, \tilde{E} = \frac{E}{K^2}, \tilde{\varepsilon} = \frac{\lambda}{K^2}.
\]
Suppose now $H(\tilde{\theta})$ admits some eigenvalue $E \in \left[-K^2(\log \frac{K^2}{\lambda})^{\frac{1}{41}}, K^2(\log \frac{K^2}{\lambda})^{\frac{1}{41}}\right]$, i.e., there is some $\Psi \in H^2(\mathbb{R}^d)$ such that $H(\tilde{\theta})\Psi = E\Psi, \Psi \neq 0$. Then we have $\tilde{E} \in I_\varepsilon$ (see (3.6)) and
\[
(-\Delta + \lambda V(\tilde{\theta} + Kx\omega))\Psi(x) = E\Psi(x).
\]
Using Lemma 4.1, we know $Z(\Theta) = \{Z_k = e^{-k\tilde{\theta}\hat{\Psi}(\Theta + Kk\omega)}\}_{k \in \mathbb{Z}^b}$ satisfies
\[
\sum_{k \in \mathbb{Z}^b} \lambda \tilde{V}_{k-k'} Z_{k'}(\Theta) + \sum_{i=1}^d (\Theta_i + Kk_i \cdot \omega_i)^2 Z_k(\Theta) = E Z_k(\Theta).
\]
Obviously, (4.7) is equivalent to
\[
\sum_{k \in \mathbb{Z}^b} \frac{\lambda}{K^2} \tilde{V}_{k-k'} Z_{k'}(\Theta) + \sum_{i=1}^d (\Theta_i/K + k_i \cdot \omega_i)^2 Z_k(\Theta) = \tilde{E} Z_k(\Theta),
\]
or in the new coordinate $(\tilde{\Theta}, \tilde{E}, \tilde{\varepsilon})$,
\[
\sum_{k' \in \mathbb{Z}^b} \varepsilon \tilde{V}_{k-k'} \tilde{Z}_{k'}(\tilde{\Theta}) + \sum_{i=1}^d (\tilde{\Theta}_i + k_i \cdot \omega_i)^2 \tilde{Z}_k(\tilde{\Theta}) = \tilde{E} \tilde{Z}_k(\tilde{\Theta}),
\]
where
\[
\tilde{Z}_k(\tilde{\Theta}) := Z_k(\Theta) \text{ for } \forall k \in \mathbb{Z}^b.
\]

Note that (4.8) has the form of $h(\tilde{\Theta})\tilde{Z} = \tilde{E}\tilde{Z}$ with $h(\tilde{\Theta})$ being given by (2.1). Furthermore, recalling (4.3) and (4.9), we have for a.e. $\tilde{\Theta}$,
\[
|\tilde{Z}_k(\tilde{\Theta})| \leq C(\tilde{\Theta}, \Psi, b, d)(1 + |k|)^{5b}.
\]
Let $\delta > 0$ be given. We obtain by Theorem 3.1 that for $0 < \varepsilon = \frac{1}{N} \leq \varepsilon_0(\delta, c_1, \rho, b, d)$,

$$\text{mes} \left( [0, 2\pi]^b \setminus \bigcap_{N \geq N_0} \Omega_N \right) \leq \delta.$$  

We fix $\tilde{\omega} \in \bigcap_{N \geq N_0} \Omega_N$. Then since $\tilde{E} \in \mathcal{I}_\varepsilon$ and $N \geq N_0$, there exists some set

$$X_N = X_N(\tilde{\omega}, \tilde{E}) \subset \mathbb{R}^d$$

such that

$$\sup_{1 \leq j \leq d, \tilde{\theta}_j \in \mathbb{R}^{d-1}} \text{mes}(X_N(\tilde{\theta}_j)) \leq e^{-N^{c_1}},$$  

and for $\tilde{\Theta} \notin X_N$, $Q \in \mathcal{E}_N^0$,

$$|G_Q(\tilde{E}; \tilde{\Theta})| \leq e^{\sqrt{N}},$$  

$$|G_Q(\tilde{E}; \tilde{\Theta})(n, n')| \leq e^{-\frac{1}{50}|n-n'|} \text{ for } |n-n'| \geq N/10.$$  

We should remark that (4.12) and (4.13) obviously hold true if $|\tilde{\Theta}| \geq 10bN$. Without loss of generality, we may assume $X_N \subset \mathcal{B}(10bN)$. As a result, we have by Fubini Theorem and (4.11) that

$$\text{mes}(X_N) \leq CN^{d-1}e^{-N^{c_1}} \leq e^{-N^{c_1}/2}$$

and

$$\sum_{N \geq N_0} \text{mes}(X_N) < \infty.$$  

Then we obtain by using Borel-Cantelli Theorem that

$$\text{mes}(X_\infty) = 0,$$

where

$$X_\infty = \bigcap_{N \geq N_0} \bigcup_{M \geq N} X_M.$$  

Assuming $\tilde{\Theta} \notin X_\infty$, then there exists $N_1(\tilde{\Theta}) \geq N_0$ such that $\tilde{\Theta} \notin X_N$ for all $N \geq N_1$. Recall that the Poisson’s identity: For $h(\tilde{\Theta})\tilde{Z} = \tilde{E}\tilde{Z}$ and $n \in \Lambda \subset \mathbb{Z}^b$,

$$\tilde{Z}_n(\tilde{\Theta}) = -\varepsilon \sum_{n' \in \Lambda, n'' \notin \Lambda} G_A(\tilde{E}; \tilde{\Theta})(n, n')\tilde{V}_{n-n'}\tilde{Z}_{n''}(\tilde{\Theta}).$$

Thus for a.e. $\tilde{\Theta}$ and $N \geq N_1$,

$$|\tilde{Z}_0(\tilde{\Theta})| = \left| \sum_{|n| \leq N, |n'| > N} G_{[-N, N]^b}(\tilde{E}; \tilde{\Theta})(0, n)\tilde{V}_{n-n'}\tilde{Z}_{n''}(\tilde{\Theta}) \right|$$

$$\leq C(\tilde{\Theta}, \Psi, b, d) \sum_{|n| \leq N, |n'| > N} e^{-\frac{1}{20}|n| + \frac{1}{100}N + \sqrt{N}} e^{-\rho|n-n'||n'|^{5b}}$$

$$\leq C(\tilde{\Theta}, \Psi, b, d)N^b \sum_{|n'| > N} e^{-\frac{1}{20}|n'| + \frac{1}{100}N + \sqrt{N}|n'|^{5b}}$$

$$\leq e^{-\frac{bN}{50}}.$$  

(4.14)
Letting $N \to \infty$ in (4.14), we have $\tilde{Z}_0(\tilde{\Theta}) = 0$ for a.e. $\tilde{\Theta}$. This implies

$$0 = \int_{\mathbb{R}^d} |\tilde{Z}_0(\tilde{\Theta})|^2 d\tilde{\Theta} = K^{-d} \int_{\mathbb{R}^d} |\tilde{\Psi}(\tilde{\Theta})|^2 d\tilde{\Theta},$$

and $\Psi = 0$, a contradiction.

We have shown $H(\tilde{\Theta})$ has no eigenvalues in $\left[-2^2(\log \frac{K^2}{\lambda})^\frac{1}{d^2}, 2^2(\log \frac{K^2}{\lambda})^\frac{1}{d^2}\right]$.

\[\square\]

**Appendix A.**

**Lemma A.1.** Let $H = -\Delta + V(x)$ be a Schrödinger operator with $M = \sup_{x \in \mathbb{R}^d} |V(x)| < \infty$. Then for any $E \geq 0$, we have

$$\sigma(H) \cap [E - M, E + M] \neq \emptyset,$$

where $\sigma(H)$ denotes the spectrum of $H$.

**Proof.** Let $\mathbb{E}(\cdot)$ be the (projection-valued) spectral measure of $H = -\Delta + V(x)$. Then the Spectral Theorem reads as

$$H = \int_{\sigma(H)} \lambda d\mathbb{E}(\lambda).$$

Note that $\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$. From Weyl criterion, for each $E \geq 0$ there is a sequence $\{F_n(x)\}_{n \in \mathbb{N}} \subset H^2(\mathbb{R}^d)$ such that

$$(-\Delta - E)F_n \to 0 \text{ as } n \to \infty,$$

$$\|F_n\|_{L^2} = 1 \text{ for } \forall n.$$

For $n \in \mathbb{N}$, we know $\mu_n(\cdot) = \langle \mathbb{E}(\cdot) F_n, F_n \rangle$ is a positive Borel measure and $\mu_n(\sigma(H)) = \|F_n\|_{L^2}^2 = 1$, where $\langle F, G \rangle = \int_{\mathbb{R}^d} F(x) \overline{G(x)} dx$. Obviously, $\|VF_n\|_{L^2} \leq M$. Hence

$$\inf_{\lambda \in \sigma(H)} |\lambda - E|^2 \|F_n\|_{L^2}^2 \leq \int_{\sigma(H)} |\lambda - E|^2 d\mu_n(\lambda) = \|(-\Delta + V - E)F_n\|_{L^2}^2 \leq \|(\Delta - E)F_n\|_{L^2}^2 + M^2. \quad (A.1)$$

Letting $n \to \infty$ in (A.1), we get

$$\text{dist}(E, \sigma(H)) \leq M, \quad (A.2)$$

which shows $[E - M, E + M] \cap \sigma(H) \neq \emptyset$. For otherwise, there must be $\text{dist}(E, \sigma(H)) \geq 2M$, which contradicts with (A.2).

\[\square\]

**Appendix B.**

We write $G(\cdot) = G(\cdot)(E; \Theta)$ for simplicity.

We have the following lemmas:

**Lemma B.1** (Lemma A.1, [Shi19a]). Fix $\tilde{\rho} > 0$. Let $\Lambda \subset \mathbb{Z}^b$ satisfy $\Lambda \in \mathcal{E}_N$ and let $A, B$ be two linear operators on $\mathbb{C}^\Lambda$. We assume further

$$\|A^{-1}\| \leq e^{\sqrt{N}},$$

$$|A^{-1}(n, n')| \leq e^{-\tilde{\rho}|n - n'|} \text{ for } |n - n'| \geq N/10.$$
Suppose that for all \( n, n' \in \Lambda \),
\[
| (B - A)(n, n') | \leq e^{-3\bar{M}N - \bar{\rho}n - n'}.
\]
Then
\[
\| B^{-1} \| \leq 2 \| A^{-1} \|, \\
| B^{-1}(n, n') | \leq | A^{-1}(n, n') | + e^{-\bar{\rho}n - n'}.
\]

**Lemma B.2** (Lemma 3.2, [JLS20]). Let \( \bar{\rho} \in (\rho, \rho], M_1 \leq N \) and \( \text{diam}(\Lambda) \leq 2N + 1 \). Suppose that for any \( n \in \Lambda \), there exists some \( W = W(n) \in E_M \) with \( M_0 \leq M \leq M_1 \) such that \( n \in W \subset \Lambda \), dist\((n, \Lambda \setminus W) \geq M/2 \) and
\[
\| G_W \| \leq 2e^{\sqrt{M}}, \\
| G_W(n, n') | \leq 2e^{-\bar{\rho}n - n'} \quad \text{for} \quad |n - n'| \geq M/10.
\]
We assume further that \( M_0 \geq M_0(\rho, \rho, b, d) > 0 \). Then
\[
\| G_A \| \leq 4(2M_1 + 1)^b e^{\sqrt{M_1}}.
\]

**Lemma B.3** (Theorem 3.3, [JLS20]). Let \( \Lambda_1 \subset \Lambda \subset \mathbb{Z}^b \) satisfy \( \text{diam}(\Lambda) \leq 2N + 1 \), \( \text{diam}(\Lambda_1) \leq N^{3/2} \). Let \( M_0 \geq (\log N)^2 \) and \( \bar{\rho} \in [\frac{\rho}{2}, \rho] \). Suppose that for any \( n \in \Lambda \setminus \Lambda_1 \), there exists some \( W = W(n) \in E_M \) with \( M_0 \leq M \leq N^{1/3} \) such that \( n \in W \subset \Lambda \setminus \Lambda_1 \), dist\((n, \Lambda_1 \setminus \Lambda) \geq M/2 \) and
\[
\| G_W \| \leq e^{\sqrt{M}}, \\
| G_W(n, n') | \leq e^{-\bar{\rho}n - n'} \quad \text{for} \quad |n - n'| \geq M/10.
\]
Suppose further that
\[
\| G_A \| \leq e^{\sqrt{N}}.
\]
Then
\[
| G_A(n, n') | \leq e^{-(\bar{\rho} - \frac{C}{\sqrt{M_0}})|n - n'|} \quad \text{for} \quad |n - n'| \geq N/10,
\]
where \( C = C(\rho, b, d) > 0 \) and \( N \geq N_0(\rho, b, d) > 0 \).

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