BEILINSON’S HODGE CONJECTURE FOR SMOOTH VARIETIES

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Abstract. Let $U/\mathbb{C}$ be a smooth quasi-projective variety of dimension $d$, $\text{CH}^r(U, m)$ Bloch’s higher Chow group, and $\text{cl}_{r,m} : \text{CH}^r(U, m) \otimes \mathbb{Q} \to \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{2r-m}(U, \mathbb{Q}(r)))$ the cycle class map. Beilinson once conjectured $\text{cl}_{r,m}$ to be surjective [Be]; however Jannsen was the first to find a counterexample in the case $m = 1$ [Ja1]. In this paper we study the image of $\text{cl}_{r,m}$ in more detail (as well as at the “generic point” of $U$) in terms of kernels of Abel-Jacobi mappings. When $r = m$, we deduce from the Bloch-Kato conjecture (now a theorem) various results, in particular that the cokernel of $\text{cl}_{m,m}$ at the generic point is the same for integral or rational coefficients.

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1. Introduction

Let $U/\mathbb{C}$ be a smooth and quasi-projective variety of dimension $d$, $\text{CH}^r(U, m)$ Bloch's higher Chow group, and

$$c_{r,m} : \text{CH}^r(U, m; \mathbb{Q}) \to \Gamma(H^{2r-m}(U, \mathbb{Q}(r)))$$

the Betti cycle class map, where

$$\Gamma(H^{2r-m}(U, \mathbb{Q}(r))) := \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{2r-m}(U, \mathbb{Q}(r))).$$

If $m = 0$, then by a standard weight argument (see [Ja1, pp.62-63]), the Hodge conjecture implies that $c_{r,0}$ is surjective. Beilinson once conjectured that $c_{r,m}$ is always surjective [Be]. However unless $U$ is given by base extension from a smooth quasi-projective variety over a number field, it is known that this conjecture is too optimistic [Ja1, Cor. 9.11].

Consider these three statements:

(S1) $c_{r,m} : \text{CH}^r(X, m; \mathbb{Q}) \to \Gamma(H^{2r-m}(X, \mathbb{Q}(r)))$ is surjective for all smooth complex projective varieties $X$;

(S2) $c_{r,m} : \text{CH}^r(U, m; \mathbb{Q}) \to \Gamma(H^{2r-m}(U, \mathbb{Q}(r)))$ is surjective for all smooth complex quasi-projective varieties $U$;

(S3) $\lim(c_{r,m}) : \text{CH}^r(\text{Spec}(\mathbb{C}(X)), m; \mathbb{Q}) \to \Gamma(H^{2r-m}(\mathbb{C}(X), \mathbb{Q}(r)))$ is surjective for all smooth complex projective varieties $X$.

Note that (S1) for $m = 0$ is equivalent with the Hodge conjecture (with rational coefficients as opposed to the original version with integral coefficients that was disproven by Atiyah-Hirzebruch by showing the existence of non-algebraic torsion classes), and that it is trivially true for $m > 0$ because then $\Gamma(H^{2r-m}(X, \mathbb{Q}(r))) = 0$. Also, $\text{CH}^r(\text{Spec}(\mathbb{C}(X)), m; \mathbb{Q}) = 0$ for $r > m$ in (S3) because of dimension reasons.

When $m = 0$, all three statements (for all $r \geq 0$) are equivalent (as one sees using a localization sequence argument, and Deligne's mixed Hodge theory, as on [Ja1, pp.62-63]). However as we shall see in this paper, the statements are independent of each other: (S1) is expected to be true, we conjecture (S3) to be true, and show (S2) to be false in general. There is some evidence ([A-S], [Sa], [A-K], [MSa]) that (S2) holds in the special case $r = m$, and the results in this paper are consistent with this. In [SJK-L] we provided some evidence that (S3) is always true, and in particular, (S3) can be viewed as an appropriate generalization of the Hodge conjecture.

In this paper we address a number of issues, namely:

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• Necessary and sufficient conditions for $c_{r,m}$ and $\lim(cl_{r,m})$ to be surjective, in terms of kernels of (reduced) higher Abel-Jacobi maps. This is worked out in Theorem 4.9 and subsequent examples, as well as in Corollary 6.9 below. Naturally this leads to a generalized notion of decomposable classes, which is discussed in Section 8. Also, in Theorem 5.1 we exhibit counterexamples to the surjectivity of $c_{r,m}$ in (S2) in all cases where it is not trivially true or where one might reasonably expect this surjectivity (namely $r = m$ or $m = 0$).

• The story can be worked out with integral coefficients in the case $r = m$, and in particular, we are interested in the nature of the map $d \log_m : CH^m(\text{Spec}(\mathbb{C}(X)), m) \to H^m(\mathbb{C}(X), \mathbb{Z}(m)) \cap F^mH^m(\mathbb{C}(X), \mathbb{C})$.

We prove in Section 7 that the torsion subgroup $H^m(\mathbb{C}(X), \mathbb{Z}(m))^\text{tor}$ of $H^m(\mathbb{C}(X), \mathbb{Z}(m))$ is trivial\(^1\) (hence this intersection makes sense!). The combination of Theorem 1.1 and Conjecture 1.2 below would imply that $d \log_m$ is surjective. We also relate $d \log_m$ to the map

$$\frac{CH^m(\text{Spec}(\mathbb{C}(X)), m)}{l} \to \text{H}_{\text{et}}^m(\text{Spec}(\mathbb{C}(X)), \mu_l^{\otimes m}),$$

for $l$ a non-zero integer, which is now known to be an isomorphism. (This is the former Bloch-Kato conjecture\(^2\), for the field $\mathbb{C}(X)$.) Thus the conjectured surjectivity of $d \log_m$ can be thought of as a Hodge theoretic version of the Bloch-Kato conjecture. Note that $\lim(cl_{m,m})$ equals

$$d \log_m \otimes \mathbb{Q} : CH^m(\text{Spec}(\mathbb{C}(X)), m; \mathbb{Q}) \to \Gamma(H^m(\mathbb{C}(X), \mathbb{Q}(m))).$$

As mentioned above, the classical Hodge conjecture, as originally formulated by Hodge with integral coefficients, is false. But we wish to remind the reader that it is false with integral coefficients even modulo torsion (see [Lew, p.67]), albeit expected (by optimists) to be true with rational coefficients. The following statement, again proven in Section 7, therefore seems rather remarkable.

**Theorem 1.1.** coker($d \log_m$) $\simeq$ coker($\lim(cl_{m,m})$). In particular, $d \log_m$ is surjective $\iff$ $\lim(cl_{m,m})$ is surjective.

What this theorem tells us is that the Hodge theoretic analog of the Bloch-Kato conjecture is the surjectivity of $\lim(cl_{m,m})$. Quite generally we expect that the following is true.

**Conjecture 1.2.** For all $r, m \geq 0$, statement (S3) holds.

\(^1\)Already known to some experts.

\(^2\)This is now a theorem ([We]).
By our earlier remarks this conjecture includes the Hodge conjecture and relates it to the (now proved) Bloch-Kato conjecture.

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2. Notation

(i) Unless otherwise specified, $X$ is a smooth complex projective variety of dimension $d$, and $U$ is a smooth complex quasi-projective variety.

Let $\mathbb{A} \subseteq \mathbb{R}$ be a subring.

(ii) $\mathbb{A}(r) := (2\pi i)^r \mathbb{A}$ (Tate twist).

(iii) If $H$ is an $\mathbb{A}$-mixed Hodge structure (MHS), then we write $\Gamma(H) := \text{hom}_{\mathbb{A}-\text{MHS}}(\mathbb{A}(0), H)$ and $J(H) := \text{Ext}_{\mathbb{A}-\text{MHS}}^1(\mathbb{A}(0), H)$.

(iv) For a quasi-projective variety $V$, $\text{CH}^r(V, m)$ is the higher Chow group defined in [Bl], and $\text{CH}^r(V) := \text{CH}^r(V, 0)$.

(v) $\text{CH}^r(V, m; \mathbb{Q}) := \text{CH}^r(V, m) \otimes \mathbb{Q}$.

(vi) We write $H^{2r-m}(\mathbb{C}(X), \mathbb{Q}(r)) = \lim_{\substack{\longrightarrow \ \ U \subset X}} H^{2r-m}(U, \mathbb{Q}(r))$, the limit taken over all Zariski open subsets of $X$.

(vii) We let $H^2_U(U, \mathbb{A}(r))$ denote the (algebraic) Deligne-Beilinson cohomology [EV] of $U$.

3. Weight filtered spectral sequence

We provide a breezy review of some of the ideas in [K-L, Section 3.1]. We first recall the definition of the higher Chow groups. Let $V/\mathbb{C}$ be a quasi-projective variety. Put $z^r(V) = \text{free abelian group generated by subvarieties of codimension } r \text{ in } V$, $\Delta^m$ the standard $m$-simplex, and $z^r(V, m) = \{\xi \in z^k(V \times \Delta^m) \mid \xi \text{ meets all faces properly}\}$. We let $\partial = \sum_i (-1)^i \partial_i$ where $\partial_i$ is the restriction to the $i$-th codimension 1 face.

**Definition 3.1.** ([Bl]) $\text{CH}^\bullet(V, \bullet) = \text{homology of } \{z^\bullet(V, \bullet), \partial\}$. We put $\text{CH}^k(V) := \text{CH}^k(V, 0)$.

We also need to recall the cubical version. Let $\square^m := (\mathbb{P}^1 \setminus \{1\})^m$ with coordinates $z_i$ and $2^m$ codimension one faces obtained by setting $z_i = 0, \infty$, and boundary maps $\partial = \sum (-1)^{i-1}(\partial^0_i - \partial^\infty_i)$, where $\partial^0_i, \partial^\infty_i$ denote the restriction maps to the faces $z_i = 0, z_i = \infty$ respectively.
The rest of the definition is completely analogous for \( z^r(V, m) \subset z^r(V \times \Delta^m) \), except that one has to quotient out by a subgroup of degenerate cycles. It is known that both complexes are quasi-isomorphic ([Bl]).

Now write \( U = X \setminus Y \), where \( X/\mathbb{C} \) is a smooth projective variety of dimension \( d \), \( Y = Y_1 \cup \cdots \cup Y_n \subset X \) a NCD with smooth components. For an integer \( t \geq 0 \), put \( Y^{[t]} = \text{disjoint union of } t\text{-fold intersections of the various components of } Y \), with corresponding simplicial scheme \( Y^{[t]} \to Y \implies Y^{[0]} = X \). There is a third quadrant double complex

\[
\begin{array}{c}
\mathbb{Z}_{i,j} \qquad \\
\uparrow \partial \\
\mathbb{Z}_{i,j} \\
\mathbb{Z}_{i,j+1} \\
\end{array}
\]

whose differentials are \( \partial \) vertically (\( \partial \) as coming from the definition of Bloch's higher Chow groups), and \( \text{Gy} (= \text{Gysin}) \) horizontally. To the corresponding total complex \( s^* \mathcal{Z}(r) \) with \( D = \partial \pm \text{Gy} \) are associated the two Grothendieck spectral sequences \( E_2^{i,j} \) and \( E_2'^{i,j} \) with

\[
E_2^{p,q} = H^p_{\text{Gy}}(H^q_0(\mathcal{Z}_0^*(r))); \quad E_2'^{p,q} = H^p_0(H^q_{\text{Gy}}(\mathcal{Z}^{**}_0(r))).
\]

The second spectral sequence, together with Bloch's quasi-isomorphism

\[
\begin{array}{c}
z^*(X,*) \qquad \\
\downarrow \text{Restriction} \\
z^*(U,*) \\
\end{array}
\]

shows that ([K-L](sect. 3.1))

\[
H^{-m}(s^* \mathcal{Z}(r)) = E_2^{'0,-m} = \text{CH}^r(U, m).
\]

The first spectral sequence has \( E_2^{i,j} = \text{CH}^{r+i}(Y^{-i}, -j) \) and

\[
E_2^{i,j} = \ker \left( \text{Gy} : \text{CH}^{r+i}(Y^{-i}, -j) \to \text{CH}^{r+i+1}(Y^{-i-1}, -j) \right) \bigg/ \text{Gy} \left( \text{CH}^{r+i+1}(Y^{-i+1}, -j) \right).
\]

The corresponding filtration on \( s^* \mathcal{Z}(r) \) also induces a "weight" filtration

\[
W_{-m} \text{CH}^r(U, m) \subseteq \cdots \subseteq W_0 \text{CH}^r(U, m) = \text{CH}^r(U, m),
\]

which is characterized by the injection

\[
\partial_R^{\ell,m} : G_{r,W}^\ell \text{CH}^r(U, m) = E_\infty^{-\ell-m,\ell} \hookrightarrow \left\{ \begin{array}{c} \text{A subquotient of} \\
\text{CH}^{r-\ell-m}(Y^{[\ell+m]}, -\ell) \end{array} \right\},
\]

for \( \ell = -m, \ldots, 0 \), where \( \partial_R^{\ell,m} \) is called a residue map in [K-L]. It is easy to check that

\[
E_\infty^{'0,-m} = W_{-m} \text{CH}^r(U, m) = \text{Image}(\text{CH}^r(X, m) \to \text{CH}^r(U, m)).
\]
4. The image of the cycle class map

Our main goal in this section is to prove Theorem 4.9, which provides necessary and sufficient conditions for the surjectivity of \( \text{cl}_{r,m} : \text{CH}^r(U, m; \mathbb{Q}) \to \Gamma \left( H^{2r-m}(U, \mathbb{Q}(r)) \right) \). The obstruction to surjectivity will be explained in terms of kernels of Abel-Jacobi maps for the higher Chow groups. We fix \( U, r \geq 0 \) and \( m \geq 0 \). Of particular interest is the top residue

\[
\partial_R(\xi) := \partial_{R}^{0,m}(\xi) \in E_{\infty}^{-m,0} \text{ for } \xi \in \text{CH}^r(U, m),
\]

where

\[
E_{\infty}^{-m,0} \subseteq E_2^{-m,0} = \frac{\ker \left( \text{Gy} : \text{CH}^{r-m}(Y^{[m]}) \to \text{CH}^{r-m+1}(Y^{[m-1]}) \right)}{\text{Gy} \left( \text{CH}^{r-m+1}(Y^{[m+1]}) \right)}.
\]

(In general, \( E_2^{-m,0} \neq E_{\infty}^{-m,0} \).) We use this to study the cycle class map \( \text{cl}_{r,m} \) via the commutative diagram

\[
\begin{array}{ccc}
\text{CH}^r(U, m; \mathbb{Q}) & \to & E_{\infty}^{-m,0} \otimes \mathbb{Q} \\
\text{cl}_{r,m} & \downarrow & \\
\Gamma \left( H^{2r-m}(U, \mathbb{Q}(r)) \right) & \to & \Gamma \left( \text{Gr}_0 \text{W} H^{2r-m}(U, \mathbb{Q}(r)) \right),
\end{array}
\]

where the injectivity of the map on the bottom row follows from the fact that \( \Gamma \left( \text{W}_{-1} H^{2r-m}(U, \mathbb{Q}(r)) \right) = 0 \). (In general we use \( \mathbb{Q} \)-coefficients because weight filtration in Hodge theory is defined for such coefficients. Exceptions to this situation are discussed in sections 6 and 7.) With regard to the morphism

\[
\text{Gy} : \text{CH}^{r-m}(Y^{[m]}) \to \text{CH}^{r-m+1}(Y^{[m-1]}),
\]

for \( m \geq 1 \), put

\[
\text{CH}^{r-m}(Y^{[m]})^\circ := \text{Gy}^{-1} \left( \text{CH}^{r-m+1}_{\text{hom}}(Y^{[m-1]}) \right),
\]

where \( \text{CH}^{r-m+1}_{\text{hom}}(Y^{[m-1]}) \subseteq \text{CH}^{r-m+1}(Y^{[m-1]}) \) is the subgroup of null-homologous cycles on \( Y^{[m-1]} \). Notice that for \( m \geq 1 \),

\[
E_{\infty}^{-m,0} = E_{m+1}^{-m,0} \subseteq E_2^{-m,0} \hookrightarrow \frac{\text{CH}^{r-m}(Y^{[m]})^\circ}{\text{Gy} \left( \text{CH}^{r-m+1}(Y^{[m+1]}) \right)},
\]

and for all \( m \geq 0 \), we have

\[
d_m : E_{m}^{-m,0} \to E_{m}^{0,-m+1}, \quad E_{\infty}^{-m,0} = \frac{E_{m}^{0,-m+1}}{d_m(E_{m}^{-m,0})}.
\]
For $k = 1, \ldots, m + 1$ put

\[
\tilde{E}_{k}^{m,0} = \begin{cases} \frac{\text{CH}_{r-m}(Y[m]; \mathbb{Q})}{\text{Gy}(\text{CH}_{r-m-1}(Y[m+1]; \mathbb{Q}))} & \text{if } k = 1 \\ E_{k}^{m,0} \otimes \mathbb{Q} & \text{for } k = 2, \ldots, m + 1 \end{cases}
\]

so that $\tilde{E}_{k}^{m,0} \subseteq \tilde{E}_{1}^{m,0}$ for $k \geq 1$. (Our main reason for introducing $\tilde{E}_{1}^{m,0}$ is so that the map $\beta$ in diagram (4.6) below has a chance of being surjective, as implied by the Hodge conjecture.) For $k = 1, \ldots, m$ we let $\tilde{E}_{k}^{m+k,-k+1}$ be

\[
\begin{cases} \ker \left( \text{CH}_{\text{hom}}^{r-m+1}(Y[m-1]; \mathbb{Q}) \xrightarrow{\text{Gy}} \text{CH}_{r-m+2}(Y[m-2]; \mathbb{Q}) \right) & \text{if } k = 1, \\ E_{k}^{m+k,-k+1} \otimes \mathbb{Q} & \text{for } k = 2, \ldots, m, 
\end{cases}
\]

hence $\tilde{E}_{k}^{m+k,-k+1} \subseteq E_{k}^{m+k,-k+1} \otimes \mathbb{Q}$ and $d_{k}(\tilde{E}_{k}^{m,0}) \subseteq \tilde{E}_{k}^{m+k,-k+1}$. Then (4.2) becomes

\[
\begin{array}{ccc}
\text{CH}_{r}(U, m; \mathbb{Q}) & \to & \tilde{E}_{\infty}^{m,0} \\
\begin{array}{c}
\text{cl}_{r,m} \\
\downarrow
\end{array} & & \downarrow \\
\Gamma(H^{2r-m}(U, \mathbb{Q}(r))) & \to & \Gamma(G_{r}^{W})
\end{array}
\]

(4.3)

where we abbreviate $W_{j}H^{2r-m}(U, \mathbb{Q}(r))$ to $W_{j}$ and similarly for $G_{r}^{W}$, and where $\tilde{E}_{\infty}^{m,0} \overset{\text{def}}{=} \tilde{E}_{m+1}^{m,0} = \bigcap_{k \geq 1} \ker(d_{k} : \tilde{E}_{k}^{m,0} \to \tilde{E}_{k}^{m+k,-k+1}) \subseteq \tilde{E}_{1}^{m,0}$, which equals $E_{\infty}^{m,0} \otimes \mathbb{Q}$ because $E_{\infty}^{m,0} = E_{m+1}^{m,0}$. Note that as in [K-L, 3.1],

\[
\Gamma(G_{r}^{W}) = \frac{\ker \left( \text{Gy} : H^{r-m,r-m}(Y[m], \mathbb{Q}(r-m)) \to H^{r-m+1,r-m+1}(Y[m+1], \mathbb{Q}(r-m+1)) \right)}{\text{Gy}(H^{r-m-r,m-1}(Y[m+1], \mathbb{Q}(r-m-1)))},
\]

and, for $0 \leq k \leq m$,

\[
Gr_{-k}^{W} = \frac{\ker \left( \text{Gy} : H^{2r-2m+k}(Y[m-k], \mathbb{Q}(r-m+k)) \to H^{2r-2m+k+2}(Y[m-k-1], \mathbb{Q}(r-m+k+1)) \right)}{\text{Gy}(H^{2r-2m+k-2}(Y[m-k+1], \mathbb{Q}(r-m+k-1)))}.
\]
Using the differentials $d_1 := \gamma_y, d_2, \ldots, d_m$ we claim that there is a commutative diagram of exact sequences for each $k = 1, \ldots, m$

\[
\begin{array}{cccc}
0 & \to & \tilde{E}_{k+1}^{-m,0} & \to & \tilde{E}_k^{-m,0} & \xrightarrow{d_k} & \tilde{E}_k^{-m+k,-k+1} \\
& & \alpha_{k+1} \downarrow & & \alpha_k \downarrow & & \lambda_k \downarrow \\
0 & \to & J(W_{-k-1}) & \to & J(W_{-k}) & \xrightarrow{h_k} & J(Gr^W_{r,m})
\end{array}
\]

(4.5)

where $h_k$ is the obvious map, $\lambda_k$ is the Abel-Jacobi map (defined explicitly in Section 10 below), and where the $\alpha_k$’s are characterized as follows. If we assume $\alpha_k$ is defined, then the definition of $\alpha_{k+1}$ is dictated by imposing commutativity in (4.5). Thus we need only define $\alpha_1$, and show that $h_k \circ \alpha_k = \lambda_k \circ d_k$. The latter will be proven in Section 10. Note that, as implicit in (4.3), $cl_{r,m}$ is the composition

\[\text{CH}^r(U, m; \mathbb{Q}) \to \tilde{E}_{\infty}^{-m,0} = \tilde{E}_{m+1}^{-m,0} \to \Gamma(H^{2r-m}(U, \mathbb{Q}(r))),\]

and that there is a map $\beta = \beta_{r,m} : \tilde{E}_1^{-m,0} \to \Gamma(Gr^W_{0,m})$, which is an isomorphism when $r = m$, and is surjective for all $r$ and $m$ under the assumption of the Hodge conjecture. We let $\alpha_1 = \kappa \circ \beta$ in the diagram (with $\kappa$ the obvious map)

(4.6)

Then (4.6) and (4.5) commute (see Theorem 10.1).

With regard to the diagram (4.5) above, it is obvious that

\[\ker(\alpha_{k+1}) = \ker (d_k|_{\ker(\alpha_k)}) \subseteq \ker(\alpha_k).\]

From the isomorphisms $\beta(\ker(\alpha_k)) = \ker(\alpha_k)/\ker(\beta) \cap \ker(\alpha_k)$ and $\beta(\ker(\alpha_{k+1})) = \ker (d_k|_{\ker(\alpha_k)})/\ker(\beta) \cap \ker (d_k|_{\ker(\alpha_k)})$, we arrive at the identification

\[\frac{\beta(\ker(\alpha_k))}{\beta(\ker(\alpha_{k+1}))} \simeq \frac{d_k(\ker(\alpha_k))}{d_k(\ker(\beta) \cap \ker(\alpha_k))}.\]

We have inclusions

\[\text{im}(cl_{r,m}) = \beta(\ker(\alpha_{m+1})) \subseteq \cdots \subseteq \beta(\ker(\alpha_1)) \subseteq \Gamma(H^{2r-m}(U, \mathbb{Q}(r)))\]

where on the left we have equality as $W_{-m-1} = 0$ and $\tilde{E}_{m+1}^{-m,0} = \tilde{E}_{-m,0}$, and the right-most inclusion is an equality if $\Gamma(H^{2r-m}(U, \mathbb{Q}(r))) \subseteq \cdots \subseteq \Gamma(H^{2r-m}(U, \mathbb{Q}(r)))$.
im(\beta) (e.g., if \beta is surjective). We mention in passing the following result:

**Proposition 4.7.** Suppose that \( \lambda_k \mid_{\text{im}(d_k)} \) is injective for \( k = 1, \ldots, m \). If \( \beta \) is surjective, then \( \text{cl}_{r,m} : \text{CH}^r(U, m; \mathbb{Q}) \to \Gamma(H^{2r-m}(U, \mathbb{Q}(r))) \) is surjective.

**Proof.** From (4.5) we get \( \ker(\alpha_{m+1}) = \cdots = \ker(\alpha_1) \), and we apply the inclusions above. \( \square \)

A slight tweaking of Proposition 4.7 together with diagrams (4.5) and (4.6) leads to:

**Proposition 4.8.** [M. Saito [MSa], Also see [K-L].] Assume the Hodge conjecture, and that the Bloch-Beilinson conjecture holds, viz., for all smooth projective \( V/\mathbb{Q} \), the Abel-Jacobi map \( AJ : \text{CH}^r_{\text{hom}}(V/\mathbb{Q}, j; \mathbb{Q}) \to J(H^{2r-j-1}(V, \mathbb{Q}(r))) \), is injective for all \( r \) and \( j \). If \( U \) is obtained from a smooth quasi-projective variety over \( \mathbb{Q} \) by base change to \( \mathbb{C} \), then for all \( r \) and \( m \), \( \text{cl}_{r,m} : \text{CH}^r(U, m; \mathbb{Q}) \to \Gamma(H^{2r-m}(U, \mathbb{Q}(r))) \) is surjective.

**Proof.** The proof, which is similar to the one given in [K-L, Prop. 3.7], is omitted. \( \square \)

Next, \( \ker(\beta) \subseteq \ker(\alpha_1) \) by (4.6), hence \( \ker(\beta) \cap \tilde{E}_k^{-m,0} \subseteq \ker(\alpha_1) \cap \tilde{E}_k^{-m,0} = \ker(\alpha_k) \subseteq \tilde{E}_k^{-m,0} \) for \( k = 1, \ldots, m+1 \), with the last equality because of (4.5). Hence \( \ker(\beta) \cap \ker(\alpha_k) = \ker(\beta) \cap \tilde{E}_k^{-m,0} \). If \( h_k \) is an isomorphism then this gives

\[
\frac{d_k(\ker(\alpha_k))}{d_k(\ker(\beta) \cap \ker(\alpha_k))} = \frac{\ker(\lambda_k \mid_{\text{im}(d_k)})}{d_k(\ker(\beta) \cap \tilde{E}_k^{-m,0})}.
\]

This is the case when \( k = m \) again because \( W_{-m-1} = 0 \). Putting all these ideas together, we obtain the following.

**Theorem 4.9.** (i) if \( \text{cl}_{r,m} \) is surjective then

\[
\frac{d_k(\ker(\alpha_k))}{d_k(\ker(\beta) \cap \ker(\alpha_k))} = 0 \quad \text{for all } k = 1, \ldots, m;
\]

the converse is true if \( \beta \) is surjective; more precisely: the converse is true if \( \Gamma(H^{2r-m}(U, \mathbb{Q}(r))) \subseteq \text{im}(\beta) \) (e.g., if \( \beta \) is surjective);

(ii) \( \text{cl}_{r,m} \) is surjective implies that

\[
\frac{\ker(\lambda_m \mid_{\text{im}(\tilde{E}_m^{-m,0})})}{d_m(\ker(\beta) \cap \tilde{E}_m^{-m,0})} = 0
\]
(iii) if $\Gamma(H^{2r-m}(U, \mathbb{Q}(r))) \subseteq \text{im}(\beta)$ and

$$d_k(\ker(\alpha_k)) \quad \text{for all } k = 1, \ldots, m - 1,$$

then there is a short exact sequence

$$0 \to \text{im}(\text{cl}_{r,m}) \to \Gamma(H^{2r-m}(U, \mathbb{Q}(r))) \to \frac{\ker(\lambda_1|_{d_m(\tilde{E}^{-m,0})})}{d_m(\ker(\beta) \cap \tilde{E}^{-m,0})} \to 0.$$

Note that $\lambda_1$ is automatically injective when $r = m$ by the theory of the Picard variety, so that $d_1(\ker(\alpha_1)) = 0$ in this case. Since $\beta$ is an isomorphism here, we deduce

**Corollary 4.10.** Let us assume that $r = m$. Then $\text{cl}_{m,m}$ is surjective if and only if $d_k(\ker(\alpha_k)) = 0$ for all $k = 2, \ldots, m$. In particular, $\text{cl}_{1,1}$ is always surjective. (See also Example 6.3.)

**Example 4.11.** Assume $r \geq m = 1$. Note that $d_1: \tilde{E}^{-1,0} \rightarrow \tilde{E}^{0,0} = \text{CH}^r_{\text{hom}}(X; \mathbb{Q})$. Assuming $\beta$ is surjective, then Theorem 4.9(iii) we deduce the short exact sequence

$$0 \to \text{im}(\text{cl}_{r,1}) \to \Gamma(H^{2r-1}(U, \mathbb{Q}(r))) \to \frac{\ker(\lambda_1|_{d_1(\tilde{E}^{-m,0})})}{d_1(\ker(\beta))} \to 0.$$ 

Recalling $U = X \setminus Y$ of dimension $d$, we have that

$$\lambda_1: \text{im}(d_1) \rightarrow J\left(H^{2r-1}(X, \mathbb{Q}(r))/H^{2r-1}_Y(X, \mathbb{Q}(r))\right),$$

where the denominator term $H^{2r-1}_Y(X, \mathbb{Q}(r))$ in the jacobian is identified with its image in $H^{2r-1}(X, \mathbb{Q}(r))$, which apparently coincides with $G_Y(H^{2r-3}(Y^{[1]}, \mathbb{Q}(r-1)))$ by a standard mixed Hodge theory argument (Deligne). Taking limits, we arrive at the short exact sequence

$$0 \to \text{im}(\lim(\text{cl}_{r,1})) \to \Gamma(H^{2r-1}(X, \mathbb{Q}(r))) \to \frac{\ker(\text{CH}^r_{\text{hom}}(X; \mathbb{Q}) \rightarrow J(H^{2r-1}(X, \mathbb{Q}(r))/\mathbb{N}_1H^{2r-1}(X, \mathbb{Q}(r))))}{\lim_{U} d_1(\ker(\beta))} \to 0,$$

where $\mathbb{N}^p H^i(X, \mathbb{Q})$ is the $p$-th coniveau filtration. However, if for example $r = d$, then using the fact that a zero-cycle on a projective variety is homologous to zero if and only its degree is 0, we see $\beta$ is surjective in this case, and that

$$\frac{\text{CH}^d_{\text{hom}}(X; \mathbb{Q})}{\lim_{U} d_1(\ker(\beta))} = 0,$$

owing to the fact that any finite set of points on $X$ lies on a smooth divisor in $X$. Therefore we arrive at the following result.
Corollary 4.12 ([SJK-L]).
\[ \lim (\text{cl}_{d,1}) : \text{CH}^d(\text{Spec}(\mathbb{C}(X)), 1; \mathbb{Q}) \to \Gamma(H^{2d-1}(\mathbb{C}(X), \mathbb{Q}(d))) \]
is surjective.

Example 4.13. The case \( r = m = 2 \). We observe that \( \ker(\lambda_1) = 0 \), and that \( \beta \) is an isomorphism. Thus from Theorem 4.9(iii) and (4.5), we arrive at the short exact sequence
\[ 0 \to \text{im}(\text{cl}_{2,2}) \to \Gamma(H^2(U, \mathbb{Q}(2))) \to \ker(\lambda_2|_{\text{im}(d_2)}) \to 0. \]
We have
\[ d_2(\tilde{E}_2^{-2,0}) \subseteq E_2^{0,-1} = \frac{\text{CH}^2(X, 1; \mathbb{Q})}{\text{G} \text{y}(\text{CH}^1(Y^{[1]}, 1; \mathbb{Q}))} \xrightarrow{\lambda_2} J\left( \frac{H^2(X, \mathbb{Q}(2))}{H^2_1(X, \mathbb{Q}(2))} \right). \]
There is an exact sequence
\[ 0 \to d_2(\tilde{E}_2^{-2,0}) \to \frac{\text{CH}^2(X, 1; \mathbb{Q})}{\text{G} \text{y}(\text{CH}^1(Y^{[1]}, 1; \mathbb{Q}))} \to \text{CH}^2(U, 1; \mathbb{Q}), \]
hence
\[ d_2(\tilde{E}_2^{-2,0}) = \text{Image of CH}^1(Y, 1; \mathbb{Q}) \to \frac{\text{CH}^2(X, 1; \mathbb{Q})}{\text{G} \text{y}(\text{CH}^1(Y^{[1]}, 1; \mathbb{Q}))}. \]
Note that
\[ \text{CH}^1(Y^{[1]}, 1) = (\mathbb{C}^\times)^n, \]
and recall we have \( Y^{[1]} = \bigsqcup_{i=1}^n Y_i \). Thus \( \text{cl}_{2,2} \) is surjective if and only if \( \lambda_2 \) is injective on the subgroup of cycles in \( \text{CH}^2(X, 1; \mathbb{Q}) \) supported on \( Y \), modulo the image of the space of decomposables in \( \text{CH}(X, 1; \mathbb{Q}) \), supported on \( Y \). Now let \( \text{CH}^2_{\text{dec}}(X, 1; \mathbb{Q}) := \text{im}(\text{CH}^1(X, 0; \mathbb{Q}) \otimes \text{CH}^1(X, 1; \mathbb{Q}) \to \text{CH}^2(X, 1; \mathbb{Q})) \), under the product on the higher Chow groups, and
\[ \text{CH}^2_{\text{ind}}(X, 1; \mathbb{Q}) := \frac{\text{CH}^2(X, 1; \mathbb{Q})}{\text{CH}^2_{\text{dec}}(X, 1; \mathbb{Q})}. \]
Then
\[ \lim (\text{cl}_{2,2}) : \text{CH}^2(\text{Spec}(\mathbb{C}(X)), 2; \mathbb{Q}) \to \Gamma(H^2(\mathbb{C}(X), \mathbb{Q}(2))), \]
is surjective if and only if
\[ AJ : \text{CH}^2_{\text{ind}}(X, 1; \mathbb{Q}) \to J\left( \frac{H^2(X, \mathbb{Q}(2))}{N^1 H^2(X, \mathbb{Q}(2))} \right), \]
is injective\(^3\). In summary,
\(^3\)This is also pointed out in [MSa] (Remark 4.4).
Corollary 4.14.

\[
\lim (\text{cl}_{2,2}) (\Gamma(H^2(C(X), \mathbb{Q}(2))) \cong \ker AJ : \text{CH}^2_{\text{ind}}(X, 1; \mathbb{Q}) \to J \left( \frac{H^2(X, \mathbb{Q}(2))}{N^1 H^2(X, \mathbb{Q}(2))} \right).
\]

(See also Corollary 6.5.)

In particular, \(\lim (\text{cl}_{2,2})\) above is surjective if \(\text{CH}^2_{\text{ind}}(X, 1; \mathbb{Q}) = 0\). We recall Bloch’s conjecture which says in the case that \(X\) is a surface, \(p_g(X) = 0 \iff\) the Albanese map \(\kappa : \text{CH}^2_{\text{deg}, 0}(X) \to \text{Alb}(X)\) is an isomorphism. Equivalently, this amounts to saying that \(p_g(X) = 0 \iff\) the motive of \(X\) degenerates. The degeneration of the motive of \(X\) implies that \(\text{CH}^2_{\text{ind}}(X, 1; \mathbb{Q}) = 0\) ([CS]). So according to Bloch’s conjecture, if \(X\) is a surface with \(p_g(X) = 0\), then \(\lim (\text{cl}_{2,2})\) is surjective.

5. Amending the Beilinson-Hodge conjecture

For any smooth quasi-projective variety \(U/\mathbb{C}\) of dimension \(d\) we consider the following three regions for the pair \((r, m)\) (see Figure 5.1):

I: \(r > m > 0\) and \(r \leq d\);

II: \(r > m\) and \(r > d\);

III: \(r < m\).

![Figure 5.1. The three regions](image)

The corresponding cycle class map

\[
\text{cl}_{r, m} : \text{CH}^r(U, m; \mathbb{Q}) \to \Gamma(H^{2r-m}(U, \mathbb{Q}(r)))
\]

is surjective in regions II and III since there the right-hand side is trivial (see Corollary 6.6 on page 85 in [Ja1]). We shall show below that at every point in region I surjectivity fails in general. Thus the only open
cases are the diagonal \((r = m)\), where surjectivity is the Beilinson-Hodge conjecture as formulated in [A-S], and the \(r\)-axis, where surjectivity corresponds to the Hodge conjecture extended to such \(U\).

**Theorem 5.1.** For a fixed \(d\), assume \((r, m)\) lies in region I. Then there exists a smooth quasi-projective variety \(U/\mathbb{C}\) of dimension \(d\) such that \(\text{cl}_{r,m}\) fails to be surjective.

**Proof.** Special instances of this are already established in [K-L]. Let 
\[ Z_0 \subset \mathbb{P}^{r+1} \]
be a hypersurface of sufficiently large degree, \(\mathbb{P}^r, \ldots, \mathbb{P}^r_{m-1} \subset \mathbb{P}^{r+1}\) general hyperplanes such that if we put
\[ W = \mathbb{P}^r \cap \cdots \cap \mathbb{P}^r_{m-1} \cap Z_0, \]
then \(W\) is smooth and \(H^0(W, \Omega^q_W) \neq 0\) where \(q = r - m + 1 = \dim W \geq 2\) since \(r > m\). Fix points \(P, Q\) in \(W\) such that the class of \(P - Q\) is non-trivial in \(\text{CH}_0(W; \mathbb{Q})\) (possible by Mumford/Roitman, see [Lew, Ch.15]), and consider the blow-up
\[ Z := B_{\{P,Q\}}(Z_0) \xrightarrow{\pi} Z_0. \]
Set \(E_1 = \pi^{-1}(P), E_2 = \pi^{-1}(Q), E_{2+i} = \pi^{-1}(\mathbb{P}^r_i \cap Z_0\setminus\{P, Q\})\) for \(i = 1, \ldots, m - 1\), and \(E = \bigcup_{j=1}^{m+1} E_j\). Observe that
\[ \bigcap_{i=1}^{m-1} E_{2+i} = B_{\{P,Q\}}(W). \]
Finally for \(k = d - r \geq 0\), put
\[ X = Z \times \mathbb{P}^k \]
\[ Y = \bigcup_{i=1}^{m+1} E_i \times \mathbb{P}^k \]
so that \(X\) has dimension \(d \geq 2\). Note that
\[ Y^{[m+1]} = \emptyset, \]
\[ Y^{[m]} = \left\{ \{E_1 \cap B_{\{P,Q\}}(W)\} \times \mathbb{P}^k \right\} \prod \left\{ \{E_2 \cap B_{\{P,Q\}}(W)\} \times \mathbb{P}^k \right\} \]
\[ \simeq \left\{ \mathbb{P}^{r-m} \times \mathbb{P}^k \right\} \prod \left\{ \mathbb{P}^{r-m} \times \mathbb{P}^k \right\}, \]
and that \(B_{\{P,Q\}}(W) \times \mathbb{P}^k\) is an irreducible component of \(Y^{[m-1]}\). Then with regard to diagram (4.6), \(\beta\) is an isomorphism, and yet for \(k = d - r\) there is a class
\[ \xi \in \text{CH}^{r-m}(Y^{[m]}; \mathbb{Q})^0 \]
of the form \(\{0\text{-cycle}\} \times \mathbb{P}^k\), for which
\[ G\text{y}(\xi) \neq 0 \in \text{CH}_{\text{hom}}^{r-m+1}(Y^{[m-1]}; \mathbb{Q}), \]
but \( \alpha_1(\xi) = 0 \in J(W_{-1}) \). To see why \( \alpha_1(\xi) = 0 \), observe that since \( W_{-1} \) has negative weight, it suffices to show that the values of \( \alpha_1(\xi) \) map to zero in \( J(Gr^W_{-k}) \) for \( k \geq 1 \). But the relevant part of \( Gr^W_{-k} \) involves the cohomology \( H^{2r-2m+k}(E[m-k]) \otimes H^0(P^k) \subset H^{2r-2m+k}(Y[m-k]) \), which in the end involves the mixed Hodge structure of \( \mathbb{Z} \setminus E = \mathbb{Z}_0 \setminus \{ \mathbb{P}_1 \cap Z_0 \cup \cdots \cup \mathbb{P}_{r-1} \cap Z_0 \} =: E_0 \).

But \( E_0^{[m-k]} \) is a union of smooth hypersurfaces of dimension \( r - m + k \). Since by Lefschetz, the cohomology of hypersurfaces is only “non-trivial” in the middle dimension, and in light of the description of \( Gr^W_{-k} \) in (4.4), it suffices to show that \( 2r - 2m + k \neq r - m + k \) (hence \( Gr^W_{-k} = 0 \), as it reduces to the same thing as the homology of a simplex). But \( 2r - 2m + k = r - m + k \iff r = m \) which is not the case for region I. Thus \( \alpha_1(\xi) = 0 \), hence \( \lambda_1 \circ d_1(\xi) = 0 \) as well. In particular, \( \xi \in \ker(\alpha_1) \), \( \ker(\beta) = 0 \), and \( d_1(\xi) = \text{Gy}(\xi) \neq 0 \). Thus by Theorem 4.9(i), \( c_{r,m} \) fails to be surjective. \( \square \)

6. Integral coefficients I

This section serves as a necessary forerunner to Section 7. Along the way we prove some results that are either new, or appear to be known only among experts. Let \( X/\mathbb{C} \) be a smooth projective variety and \( Y \subset X \) a proper subvariety. There is a short exact sequence

\[
0 \to \frac{H^{2r-m}(X, \mathbb{Z}(r))}{H^Y_{2r-m}(X, \mathbb{Z}(r))} \to H^{2r-m}(X \setminus Y, \mathbb{Z}(r)) \to H^Y_{2r-m+1}(X, \mathbb{Z}(r))^0 \to 0,
\]

where, for notational simplicity, we write \( H^Y_{2r-m}(X, \mathbb{Z}(r)) \) instead of \( \text{im}(H^{2r-m}(X, \mathbb{Z}(r))) \), and let

\[
H^Y_{2r-m+1}(X, \mathbb{Z}(r))^0 := \ker \left( H^Y_{2r-m+1}(X, \mathbb{Z}(r)) \to H^{2r-m+1}(X, \mathbb{Z}(r)) \right).
\]

Let us assume for the moment that

\[
\frac{H^{2r-m}(X, \mathbb{Z}(r))}{H^Y_{2r-m}(X, \mathbb{Z}(r))}
\]

is torsion-free. Except for the obvious case \( 2r - m = 0 \) this also holds in the following two cases:

(i) \( 2r - m = 1 \). Here

\[
\frac{H^1(X, \mathbb{Z}(r))}{H^Y_1(X, \mathbb{Z}(r))} = H^1(X, \mathbb{Z}(r)),
\]
is torsion-free, as can be seen from the long exact sequence of cohomology of $X$ associated to the short exact sequence

\[(6.1) \quad 0 \to \mathbb{Z}(1) \to \mathbb{C} \to \mathbb{C}^\times \to 0.\]

(ii) $2r - m = 2$. Let $Y$ be a divisor such that the image $H^2_r(X, \mathbb{Z}(2)) \to H^2(X, \mathbb{Z}(2))$ is precisely the algebraic part $H^2_{\text{alg}}(X, \mathbb{Z}(2))$. Then by the Lefschetz $(1, 1)$ theorem, $H^2(X, \mathbb{Z}(2))/H^2_{\text{alg}}(X, \mathbb{Z}(2))$ is torsion-free, and $H^2(X, \mathbb{Z}(2))/H^2_r(X, \mathbb{Z}(2))$ is isomorphic to this group.

Then by purity of negative weight and torsion-freeness,

\[\Gamma \left( \begin{array}{c} H^{2r-m}(X, \mathbb{Z}(r)) \\ H^r_Y(X, \mathbb{Z}(r)) \end{array} \right) = 0 \text{ for } m > 0.\]

Corresponding to this is a commutative diagram (use the fact that cycle class maps are compatible with localization sequences for the left-hand square, and the commutativity of the right-hand square can be deduced from an extension class interpretation of the Abel-Jacobi map (see [KLM]))

(6.2) \[
\begin{array}{ccc}
\text{CH}^r(X\setminus Y, m) & \to & \text{CH}^r_Y(X, m - 1)^0 \\
\text{cl}_{r,m} & \downarrow & \text{cl}_{r,m} \\
\Gamma H^{2r-m}(X\setminus Y, \mathbb{Z}(r)) & \to & \Gamma H^r_Y(X, \mathbb{Z}(r))^{0} \\
\beta & \downarrow & \text{AJ} \\
\end{array}
\]

where

\[\text{CH}^r_{\text{hom}}(X, m - 1) = \ker \left( \text{CH}^r(X, m - 1) \to H^{2r-m+1}(X, \mathbb{Z}(r)) \right),\]

and $\text{CH}^r_Y(X, m - 1)^0 := \ker \left( \text{CH}^r_Y(X, m - 1) \to H^{2r-m+1}(X, \mathbb{Z}(r)) \right) = \alpha^{-1} \text{CH}^r_{\text{hom}}(X, m - 1)$.

**Example 6.3.** Suppose that $r = m = 1$. Then $\beta$ and $\text{AJ}$ are isomorphisms, hence the same holds for $\text{cl}_{1,1}$ by (6.2). If we make the identifications $\text{CH}^1(X, 1) \simeq \mathbb{C}^\times$, $\text{CH}^1(X\setminus Y, 1) = \mathcal{O}^\times_{X\setminus Y}(X\setminus Y)$, then we arrive at the short exact sequence

\[0 \to \mathbb{C}^\times \to \mathcal{O}^\times_{X\setminus Y}(X\setminus Y) \xrightarrow{d\log} \Gamma H^1(X\setminus Y, \mathbb{Z}(1)) \to 0,\]

where $\text{cl}_{1,1} = d\log$ is well-known (see [KLM]). This is also a consequence of the identification

\[\mathcal{O}^\times_{X\setminus Y}(X\setminus Y) \simeq H^1_d(X\setminus Y, \mathbb{Z}(1)).\]
The surjectivity of $d\log$ in this case is also proven in [A-K].

**Example 6.4.** Suppose that $(r, m) = (2, 1)$, and $Y = Y_1 \cup \cdots \cup Y_n \subset X$ is a divisor. We observe that there is a short exact sequence

$$H^2_\gamma(X, \mathbb{C}) \cong F^2 H^2_\gamma(X, \mathbb{C}) + H^2_\gamma(X, \mathbb{Z}(2)) \rightarrow H^3_\gamma(X, \mathbb{Z}(2)) \\ \rightarrow \Gamma H^3_\gamma(X, \mathbb{Z}(2)),$$

where the first term may be identified with

$$\bigoplus_{j=1}^n H^0(Y_j, \mathbb{C}/\mathbb{Z}(1)) \simeq (\mathbb{C}^\times)^{\oplus n} =: \text{CH}_{Y, \text{dec}}^2(X, 1).$$

There is a canonical isomorphism $\text{CH}_{Y, \text{dec}}^2(X, 1) \simeq H^3_\gamma(X, \mathbb{Z}(2))$ by [Ja2, Lemma 3.1]. (In loc. cit. this is formulated in terms of $K$-theory, but because of the particular indices, it gives exactly this result.) We therefore obtain a short exact sequence (using the fact that $\beta$ factors through Deligne cohomology)

$$0 \rightarrow \text{CH}_{Y, \text{dec}}^2(X, 1) \rightarrow \text{CH}_{Y}^2(X, 1) \rightarrow \text{ker(}\beta\text{)} \rightarrow \text{cok(}\beta\text{)} \rightarrow 0,$$

so that the snake lemma applied to (6.2) yields an exact sequence

$$0 \rightarrow \ker(\text{cl}_{2, 2}) \rightarrow \text{CH}_{Y, \text{dec}}^2(X, 1) \rightarrow \ker(J_{\text{Im}(\alpha)}) \rightarrow \text{cok(}\text{cl}_{2, 2}\text{)} \rightarrow 0.$$

When taking limits over $Y$ in the above example $\alpha$ becomes surjective, and $\lim(\text{cl}_{2, 2}) = d\log$ in Section 1. So using the description of $\ker(\beta)$ for each $Y$ above we obtain the following result.

**Corollary 6.5.**

$$\frac{\Gamma H^2(\mathbb{C}(X), \mathbb{Z}(2))}{\text{im}(d\log_2)} \cong \ker \left[ \frac{\text{CH}_{2, \text{hom}}^2(X, 1)}{\text{CH}_{2, \text{dec}}^2(X, 1)} \xrightarrow{\Delta J} J \left( \frac{H^2(X, \mathbb{Z}(2))}{H^2_{\text{alg}}(X, \mathbb{Z}(2))} \right) \right].$$

Let us now consider the general situation of $(r, m)$ with $m > 0$, so that we have the diagram (6.2) tensored with $\mathbb{Q}$. Then $\beta \otimes \mathbb{Q}$ need not be surjective; moreover a detailed description of this map when $Y$ is a NCD leads to the same kind of analysis as in Section 4. Recall that by a weight argument, $\Gamma H^{2r-m}(X \setminus Y, \mathbb{Q}(r)) = 0$ for $r \leq m - 1$. With this in mind let us assume that $r \geq m$. Then as $Y \subset X$ ranges over all pure codimension one subvarieties, the image of $\alpha$ in (6.2) is $\text{CH}_{r, \text{hom}}^i(X, m-1)$ because of dimensions. Referring to (6.2)$_{\mathbb{Q}}$, let us put

$$N^1 \text{CH}^r(X, m-1; \mathbb{Q}) := \lim_{\rightarrow Y} \alpha(\ker(\beta \otimes \mathbb{Q})),$$

where $Y \subset X$ ranges over all pure codimension one algebraic subsets of $X$. Note that from Example 6.4, $N^1 \text{CH}^2(X, 1; \mathbb{Q}) = \text{CH}_{2, \text{dec}}^2(X, 1; \mathbb{Q})$.

---

$^4$Working with subvarieties of higher pure codimension, this gives rise to a descending filtration $\{N^p \text{CH}^i(X, j; \mathbb{Q})\}_{p \geq 0}$ which is finer that the coniveau filtration on $\text{CH}^i(X, j; \mathbb{Q})$. 
Now fix a \( j : Y \hookrightarrow X \) of pure codimension one, with desingularization \( \lambda : \tilde{Y} \xrightarrow{\sim} Y \), and composite morphism \( \sigma = j \circ \lambda : \tilde{Y} \to X \). By a weight argument (Deligne) together with the purity of \( H^{2r-m}(X, \mathbb{Q}(r)) \), both Gysin images \( (\sigma_*, j_*) \) in the commutative diagram below are the same.

\[
\begin{array}{ccc}
H^{2r-m-2}(\tilde{Y}, \mathbb{Q}(r-1)) & \xrightarrow{\lambda_*} & H^{2r-m}(X, \mathbb{Q}(r)) \\
\downarrow{\sigma_*} & & \downarrow{j_*} \\
H^Y_{2r-m}(X, \mathbb{Q}(r)) & & 
\end{array}
\]

Assuming the Hodge conjecture, we can find \( w \) in \( \text{CH}^{d-1}(X \times \tilde{Y}; \mathbb{Q}) \) with \( \sigma_* [w] = \text{Id}_{\text{im}(\sigma_*)} \), where

\[
[w]_* : H^{2r-m}(X, \mathbb{Q}(r)) \to H^{2r-m-2}(\tilde{Y}, \mathbb{Q}(r-1))
\]

is induced by \( w \) (see [Lew, Prop. 7.4]). Note that (6.7)

\[
w_* \text{CH}^r_{\text{hom}}(X, m-1; \mathbb{Q}) \subset \text{CH}^{r-1}_{\text{hom}}(\tilde{Y}, m-1; \mathbb{Q}) \to \ker \beta_Q,
\]

where \( \beta_Q := \beta \otimes \mathbb{Q} \). Let us similarly write \((6.2)_Q\) for \((6.2)\) tensored with \( \mathbb{Q} \), and let

\[
AJ_Q : \text{CH}^r_{\text{hom}}(X, m-1; \mathbb{Q}) \to J(H^{2r-m}(X, \mathbb{Q}(r)))
\]

be the (full) Abel-Jacobi map. Referring to \((6.2)_Q\), we deduce from the Hodge conjecture that

\[
\ker (AJ_Q|_{\text{im}(\alpha)}) = \ker (AJ_Q|_{\text{im}(\alpha)}) + \alpha(\ker \beta_Q).
\]

Indeed, if \( \xi \in \ker (AJ_Q|_{\text{im}(\alpha)}) \), then \( AJ_Q(\xi) = AJ_Q(\sigma_* \circ w_*(\xi)) \) by functoriality of the Abel-Jacobi map. Thus \( \xi = (\xi - \sigma_* \circ w_*(\xi)) + \sigma_* \circ w_*(\xi) \in \ker (AJ_Q|_{\text{im}(\alpha)}) + \alpha(\ker \beta_Q) \) by \((6.2)_Q\). In particular,

**Proposition 6.8.** Under the assumption of the Hodge conjecture, for a fixed \( Y \) as above there are short exact sequences

\[
\frac{\ker AJ_Q|_{\text{im}(\alpha)} + \alpha(\ker \beta_Q)}{\alpha(\ker \beta_Q)} \hookrightarrow \frac{\Gamma H^{2r-m}(X \setminus Y, \mathbb{Q}(r))}{\text{im}(\text{ch}_{r,m})} \twoheadrightarrow \cok(\beta_Q)^0,
\]

where

\[
\cok(\beta_Q)^0 := \ker (\cok(\beta_Q) \to \cok(AJ_Q)),
\]

\[
AJ_Q : \text{im}(\alpha) \to J\left(\frac{H^{2r-m}(X, \mathbb{Q}(r))}{H_Y^{2r-m}(X, \mathbb{Q}(r))}\right).
\]
Taking the direct limit over $Y$ we obtain a short exact sequence

$$\frac{\ker AJ_Q + N^1\text{CH}^r(X, m - 1; \mathbb{Q})}{N^1\text{CH}^r(X, m - 1; \mathbb{Q})} \xrightarrow{\Gamma H^{2r-m}(\mathbb{C}(X), \mathbb{Q}(r))} \frac{\text{im}(\lim cl_{r,m})}{\lim \text{cok}(\beta_Q^0)},$$

where

$$\text{lim cok}(\beta_Q^0) = \ker \left( \lim \text{cok}(\beta_Q) \rightarrow \text{cok}(AJ_Q) \right).$$

**Corollary 6.9.** Assume the Hodge conjecture and let $r \geq m$. Then

$$\frac{\Gamma H^{2r-m}(\mathbb{C}(X), \mathbb{Q}(r))}{\text{im}(\lim cl_{r,m})} = 0$$

implies $\ker AJ_Q \subset N^1\text{CH}^r(X, m - 1; \mathbb{Q})$.

**Remark 6.10.** (i) Note that Corollary 6.9 for $m = 1$ is essentially a conjectural type question of Jannsen [Ja3, p.227].

(ii) Let us (again) take the direct limit over $Y$ of diagram (6.2)$_Q$. By applying the snake lemma to the limit diagram, we deduce (unconditionally) that for $r \geq m$,

$$\frac{\Gamma H^{2r-m}(\mathbb{C}(X), \mathbb{Q}(r))}{\text{im}(\lim cl_{r,m})} = 0$$

implies that

$$AJ_Q : \frac{\text{CH}^r(X, m - 1; \mathbb{Q})}{N^1\text{CH}^r(X, m - 1; \mathbb{Q})} \rightarrow J \left( \frac{H^{2r-m}(X, \mathbb{Q}(r))}{N^1H^{2r-m}(X, \mathbb{Q}(r))} \right),$$

is injective, which in turn implies by a generalization of Beilinson rigidity theorem given in [MS] that

$$\frac{\text{CH}^r(X, m - 1; \mathbb{Q})}{N^1\text{CH}^r(X, m - 1; \mathbb{Q})}$$

is countable for $m \geq 2$. Note that in the case $r = m = 2$, we have

$$\text{CH}^2_{\text{ind}}(X, 1; \mathbb{Q}) = \frac{\text{CH}^2(X, 1; \mathbb{Q})}{N^1\text{CH}^2(X, 1; \mathbb{Q})},$$

where $\text{CH}^2_{\text{ind}}(X, 1; \mathbb{Q})$ was defined in Example 4.13, and the statement of countability of $\text{CH}^2_{\text{ind}}(X, 1; \mathbb{Q})$ is a conjecture of Voisin.
7. Integral coefficients II

As always, \( X/\mathbb{C} \) is a smooth projective variety. This section concerns the following integrally defined map:

\[
d \log_m : \text{CH}^m(\text{Spec}(\mathbb{C}(X)), m) \to \Gamma(H^m(\mathbb{C}(X), \mathbb{Z}(m))),
\]

(7.1)

\[
\{f_1, \ldots, f_m\} \mapsto \bigwedge_1^m d \log(f_j)
\]

mentioned in Section 1. Of course, \( d \log_m \otimes \mathbb{Q} = \lim(cl_{m,m}) \). We shall prove in this section that \( H^m(\mathbb{C}(X), \mathbb{Z}(m)) \) is torsion-free, so that by a weight argument

\[
\Gamma(H^m(\mathbb{C}(X), \mathbb{Z}(m))) = H^m(\mathbb{C}(X), \mathbb{Z}(m)) \cap F^m H^m(\mathbb{C}(X), \mathbb{C}).
\]

Clearly, if \( d \log_m \) is surjective then so is \( \lim(cl_{m,m}) \), but we shall show that the converse also holds. In fact, we expect the following to be true:

**Conjecture 7.2.** The map in (7.1) is surjective.

For the moment, let us restrict to the case \( r = m = 1 \). It is then easy (and also follows from Example 6.3) that

\[
d \log : \text{CH}^1(\mathbb{C}(X), 1) = \mathbb{C}(X)^\times \to \Gamma(H^1(\mathbb{C}(X), \mathbb{Z}(1))),
\]

is surjective, with divisible kernel: \( \ker(d \log) = \mathbb{C}^\times \). Thus for any integer \( l \neq 0 \),

\[
\frac{\text{CH}^1(\mathbb{C}(X), 1)}{l \cdot \text{CH}^1(\mathbb{C}(X), 1)} \cong \frac{\Gamma(H^1(\mathbb{C}(X), \mathbb{Z}(1)))}{l \cdot \Gamma(H^1(\mathbb{C}(X), \mathbb{Z}(1)))}.
\]

If \( U \) is a Zariski open part of \( X \), then there is an exact sequence

\[
0 \to \frac{H^1(U, \mathbb{Z}(1))}{\Gamma(H^1(U, \mathbb{Z}(1)))} \to \frac{H^1(U, \mathbb{C})}{F^1 H^1(U, \mathbb{C})} \to H^2_D(U, \mathbb{Z}(1)),
\]

(7.3)

where

\[
H^2_D(U, \mathbb{Z}(1)) = \text{CH}^1(U).
\]

Indeed, from mixed Hodge theory (see [De](Cor. 3.2.13(ii)), the restriction map induces an isomorphism

\[
H^1(X, \mathcal{O}_X) \cong \frac{H^1(U, \mathbb{C})}{F^1 H^1(U, \mathbb{C})}.
\]

This together with the surjectivity of \( \text{CH}^1(X) \to \text{CH}^1(U) \), implies that the restriction map \( H^2_D(X, \mathbb{Z}(1)) \to H^2_D(U, \mathbb{Z}(1)) \) is surjective; moreover by a weak purity argument, \( H^2_{D,X \setminus U}(X, \mathbb{Z}(1)) \cong H^2_{X \setminus U}(X, \mathbb{Z}(1)) \).
Via the Deligne cycle class maps, we have identifications, $\text{CH}^1(X) = H^2_\mathbb{D}(X, \mathbb{Z}(1))$, $\text{CH}^1_{X\setminus U}(X) = H^2_{X\setminus U}(X, \mathbb{Z}(1))$, and hence by a localization sequence argument, the aforementioned identification $H^2_\mathbb{D}(U, \mathbb{Z}(1)) = \text{CH}^1(U)$. Next, by shrinking $U$, and using that $\text{CH}^1(\text{Spec}(\mathbb{C}(X))) = 0$, we obtain from (7.3) and (7.4) a short exact sequence

$$0 \to \Gamma H^1(\mathbb{C}(X), \mathbb{Z}(1)) \to H^1(\mathbb{C}(X), \mathbb{Z}(1)) \to H^1(X, \mathcal{O}_X) \to 0,$$

where the last term is uniquely divisible. Hence, for $l \neq 0$, we find an isomorphism

$$\frac{\Gamma H^1(\mathbb{C}(X), \mathbb{Z}(1))}{l \cdot \Gamma H^1(\mathbb{C}(X), \mathbb{Z}(1))} \simeq \frac{H^1(\mathbb{C}(X), \mathbb{Z}(1))}{l \cdot H^1(\mathbb{C}(X), \mathbb{Z}(1))}.$$

Next, we observe that $H^2(\mathbb{C}(X), \mathbb{Z}(1))$ is torsion-free, since by the Lefschetz (1, 1) theorem the torsion in $H^2(X, \mathbb{Z}(1))$ is algebraic. This implies that

$$\frac{H^1(\mathbb{C}(X), \mathbb{Z}(1))}{l \cdot H^1(\mathbb{C}(X), \mathbb{Z}(1))} \simeq H^1(\mathbb{C}(X), \mathbb{Z}(1)),$$

To see this in another context, let us work in the étale topology on a variety $V/\mathbb{C}$, and consider the sheaf $\mu_l$ on $V$, where for $U \to V$ étale, $\mu_l(U) = \{ \xi \in \Gamma(U, \mathcal{O}_U) \mid \xi^l = 1 \}$. Now let $V = \text{Spec}(\mathbb{C}(X))$. Then by Hilbert 90,

$$H^1_{\text{ét}}(\mathbb{C}(X), \mu_l) \simeq \mathbb{C}(X)^{\times}/[\mathbb{C}(X)^{\times}]^l,$$

where $[\mathbb{C}(X)^{\times}]^l = \{ x^l \mid x \in \mathbb{C}(X)^{\times} \}$. There are exact sequences

$$0 \to \mathbb{C}^{\times} \to \mathbb{C}(X)^{\times} \xrightarrow{d \log} \Omega^1_{\mathbb{C}(X)/\mathbb{C}},$$

$$0 \to \mathbb{C}^{\times} \to [\mathbb{C}(X)^{\times}]^l \xrightarrow{d \log} \Omega^1_{\mathbb{C}(X)/\mathbb{C}},$$

where $\Omega^1_{\mathbb{C}(X)/\mathbb{C}}$ are the Kähler differentials, which induces the short exact sequences

$$0 \to \mathbb{C}^{\times} \to \mathbb{C}(X)^{\times} \xrightarrow{d \log} \Gamma(H^1(\mathbb{C}(X), \mathbb{Z}(1))) \to 0$$

and

$$0 \to \mathbb{C}^{\times} \to [\mathbb{C}(X)^{\times}]^l \xrightarrow{d \log} l \cdot \Gamma(H^1(\mathbb{C}(X), \mathbb{Z}(1))) \to 0.$$
Note that $d \log : \text{CH}^1(\text{Spec}(\mathbb{C}(X), 1)) = \mathbb{C}(X)^\times \to H^1(\mathbb{C}(X), \mathbb{Z}(1))$, and $\text{CH}^1(\text{Spec}(\mathbb{C}(X), 1)) = \mathbb{C}(X)^\times \to H^1_{\text{et}}(\mathbb{C}(X), \mu_l)$ from (7.5) are the respective cycle class maps to Betti cohomology (analytic topology) and étale cohomology. In summary, we have a commutative diagram corresponding to a morphism of sites from the étale to the analytic topologies.

\[
\begin{array}{ccc}
\mathbb{C}(X)^\times & \xrightarrow{d \log} & H^1(\mathbb{C}(X), \mathbb{Z}(1)) \\
\downarrow & & \downarrow \\
H^1_{\text{et}}(\mathbb{C}(X), \mu_l) & \xrightarrow{\sim} & H^1(\mathbb{C}(X), \frac{\mathbb{Z}(1)}{l \cdot \mathbb{Z}(1)})
\end{array}
\]

where the isomorphism in the bottom row is from (7.5). Taking cup products, we have a similar diagram

\[
\begin{array}{ccc}
\left(\mathbb{C}(X)^\times\right)^\otimes m & \xrightarrow{(d \log)^\otimes m} & H^1(\mathbb{C}(X), \mathbb{Z}(1))^\otimes m \\
\downarrow & & \downarrow \\
H^1_{\text{et}}(\mathbb{C}(X), \mu_l)^\otimes m & \xrightarrow{\sim} & H^1(\mathbb{C}(X), \frac{\mathbb{Z}(1)}{l \cdot \mathbb{Z}(1)})^\otimes m \\
\cup & & \cup \\
H^m_{\text{et}}(\mathbb{C}(X), \mu_l^\otimes m) & \xrightarrow{\sim} & H^m(\mathbb{C}(X), \frac{\mathbb{Z}(m)}{l \cdot \mathbb{Z}(m)})^\otimes m
\end{array}
\]

where the isomorphism

\[
H^m_{\text{et}}(\mathbb{C}(X), \mu_l^\otimes m) \xrightarrow{\sim} H^m\left(\mathbb{C}(X), \frac{\mathbb{Z}(m)}{l \cdot \mathbb{Z}(m)}\right)
\]

here (and in the previous diagram for $m = 1$), which really arises from the Leray spectral sequence associated to a morphism of sites, can be deduced from [Mi, Thm 3.12 on p.117]. The Bloch-Kato/Milnor conjectures (now theorems [We]) tell us that for $l$ a non-zero integer,
the induced map
\[
\frac{\text{CH}^m(\text{Spec}(\mathbb{C}(X)), m)}{l \cdot \text{CH}^m(\text{Spec}(\mathbb{C}(X)), m)} \to H^m_{\text{et}}(\text{Spec}(\mathbb{C}(X)), \mu_l \otimes \mathbb{Z}^m)
\]
is an isomorphism. In our situation, this translates to saying

**Theorem 7.6.** For \(m \geq 0\), the map
\[
\frac{\text{CH}^m(\text{Spec}(\mathbb{C}(X)), m)}{l \cdot \text{CH}^m(\text{Spec}(\mathbb{C}(X)), m)} \to H^m(\mathbb{C}(X), \frac{\mathbb{Z}(m)}{l \cdot \mathbb{Z}(m)})
\]
is an isomorphism for any integer \(l \neq 0\).

We can now prove the following result. Note that part (i) for \(i = 1\) is immediate from the short exact sequence (6.1), and for \(i = 2\) follows from the Lefschetz \((1, 1)\) theorem.

**Theorem 7.7.** (i) \(H^i(\mathbb{C}(X), \mathbb{Z})\) is torsion-free for all \(i\). In particular, the torsion subgroup of \(H^i(X, \mathbb{Z})\) is supported in codimension 1.

(ii) \(\ker(d \log_m)\) in (7.1) is divisible.

(iii) The groups
\[
\frac{H^m(\mathbb{C}(X), \mathbb{Z}(m))}{\Gamma(H^m(\mathbb{C}(X), \mathbb{Z}(m)))} \quad \text{and} \quad \frac{\Gamma(H^m(\mathbb{C}(X), \mathbb{Z}(m)))}{\text{im}(d \log_m)}
\]
are uniquely divisible.

**Proof.** First observe that the map in Theorem 7.6 is the composition
\[
\text{CH}^m(\text{Spec}(\mathbb{C}(X)), m) \to H^m(\mathbb{C}(X), \mathbb{Z}(m)) \to H^m(\mathbb{C}(X), \frac{\mathbb{Z}(m)}{l \cdot \mathbb{Z}(m)})
\]
Notice that the short exact sequence \(0 \to \mathbb{Z} \overset{x \cdot l}{\to} \mathbb{Z} \to \mathbb{Z}/l\mathbb{Z} \to 0\) induces the short exact sequence
\[
\frac{H^m(\mathbb{C}(X), \mathbb{Z}(m))}{l} \to H^m(\mathbb{C}(X), \frac{\mathbb{Z}(m)}{l \cdot \mathbb{Z}(m)}) \to H^{m+1}(\mathbb{C}(X), \mathbb{Z}(m))_{l-\text{tor}}.
\]
By Theorem 7.6, it follows that \(H^{m+1}(\mathbb{C}(X), \mathbb{Z}(m))_{l-\text{tor}} = 0\), thus proving part (i). Next observe that
\[
\Gamma(H^m(\mathbb{C}(X), \mathbb{Z}(m))) = F^m \cdot H^m(\mathbb{C}(X), \mathbb{C}) \cap H^m(\mathbb{C}(X), \mathbb{Z}(m)),
\]
and hence
\[
\Gamma(H^m(\mathbb{C}(X), \mathbb{Z}(m))) \cap l \cdot H^m(\mathbb{C}(X), \mathbb{Z}(m)) = l \cdot \Gamma(H^m(\mathbb{C}(X), \mathbb{Z}(m))).
\]

---

5This generalizes the Merkurjev-Suslin theorem, where \(m = 2\).
Using also Theorem 7.6, we have the commutative diagram
\[
\begin{array}{c}
\text{CH}^m(\text{Spec}(\mathbb{C}(X)), m) \\
\downarrow \downarrow \downarrow \downarrow \\
\text{im}(d \log_m) \\
\downarrow \downarrow \\
\Gamma(\text{H}^m(\mathbb{C}(X), \mathbb{Z}(m))) \\
\end{array}
\begin{array}{c}
\text{( Spec)(} \mathbb{C}(X) \text{)), m) \\
\text{H}^m(\mathbb{C}(X), \mathbb{Z}(m)) \\
\text{lim(cl}_{r,m}) \\
\end{array}
\]
where all maps must be isomorphisms. Part (ii) follows by applying the snake lemma to multiplication by \( l \neq 0 \) on the short exact sequence
\[
0 \to \ker(d \log_m) \to \text{CH}^m(\text{Spec}(\mathbb{C}(X)), m) \to \text{im}(d \log_m) \to 0
\]
as \( \text{im}(d \log_m) \) is torsion-free. Using the obvious abbreviations, part (iii) follows similarly from \( 0 \to \Gamma \to \text{H}^m \to \text{H}^m/\Gamma \to 0 \) and \( 0 \to \text{im} \to \Gamma \to \Gamma/\text{im} \to 0. \)

Corollary 7.8.
\[
\frac{\Gamma(\text{H}^m(\mathbb{C}(X), \mathbb{Z}(m)))}{\text{im}(d \log_m)} = \frac{\Gamma(\text{H}^m(\mathbb{C}(X), \mathbb{Q}(m)))}{\text{im}(d \log_m \otimes \mathbb{Q})}.
\]

Corollary 7.9.
\[
\ker\left[AJ: \frac{\text{CH}^2_{\text{hom}}(X, 1)}{\text{CH}^2_{\text{dec}}(X, 1)} \to J\left(\frac{H^2(X, \mathbb{Z}(2))}{H^2_{\text{alg}}(X, \mathbb{Z}(2))}\right)\right]
\]
is uniquely divisible.

Proof. Apply Theorem 7.7(iii) with \( m = 2 \) to Corollary 6.5.

Note that one can define \( N^1 \text{CH}^r(X, m - 1) \) with integral coefficients analogous to (6.6). We assume \( r \geq m \). Since \( H^{2r-m}(\mathbb{C}(X), \mathbb{Z}(r)) \) is torsion-free by Theorem 7.7(i), diagram (6.2) becomes valid after passing to the generic point of \( X \). After applying the snake lemma, we arrive at the fact that
\[
\ker\left[AJ: \frac{\text{CH}^r(X, m - 1)}{N^1 \text{CH}^r(X, m - 1)} \to J\left(\frac{H^{2r-m}(X, \mathbb{Z}(m-1))}{N^1 H^{2r-m}(X, \mathbb{Z}(m-1))}\right)\right]
\]
jinjects into
\[
\frac{\Gamma(\text{H}^{2r-m}(\mathbb{C}(X), \mathbb{Z}(r)))}{\text{im}(\lim(\text{cl}_{r,m}))}.
\]
In the case \( r = m \) we deduce

**Corollary 7.10.**

\[
\ker \left( AJ : \frac{\text{CH}^m(X, m - 1)}{N^1 \text{CH}^m X, m - 1} \to J \left( \frac{H^m(X, \mathbb{Z}(m - 1))}{N^1 H^m(X, \mathbb{Z}(m - 1))} \right) \right)
\]

is torsion-free. Hence we have an injection of torsion subgroups

\[
AJ : \left\{ \frac{\text{CH}^m(X, m - 1)}{N^1 \text{CH}^m X, m - 1} \right\}_{\text{tor}} \hookrightarrow \left\{ J \left( \frac{H^m(X, \mathbb{Z}(m - 1))}{N^1 H^m(X, \mathbb{Z}(m - 1))} \right) \right\}_{\text{tor}}.
\]

**Remark 7.11.** Conjecture 7.2 would imply that the maps \( AJ \) in Corollaries 7.9 and 7.10 are injective.

### 8. Decomposables

Fix \( r, m \geq 1 \). If \( r = m \), then according to Theorem 4.9, surjectivity of \( \text{cl}_{m, m} \) implies that \( \lambda_m \) is injective on \( d_m(E_{m,0}^{-m,0}) \subseteq \tilde{E}_{m}^{0,-m+1} \). Thus it makes sense to calculate \( \text{im}(d_m) \) in general.

Let \( j : Y \hookrightarrow X \) be the inclusion of a NCD \( Y \) (with smooth components). Then

\[
\tilde{E}_{m}^{0,-m+1} \cong \text{im}(d_m) = \frac{\text{CH}^r(Y, m - 1; \mathbb{Q})}{j_* \text{CH}^{r-1}(Y, m - 1; \mathbb{Q})} \oplus \text{CH}^r(U, m - 1; \mathbb{Q}).
\]

Note that \( \text{CH}^{r-1}(Y, m - 1; \mathbb{Q}) \) can be calculated from the simplicial complex \( Y^{[m]} \to Y \). So there are residue maps

\[
\partial^Y_R : \text{CH}^{r-1}(Y, m - 1; \mathbb{Q}) \to \frac{\text{CH}^{r-m}(Y^{[m]}; \mathbb{Q})}{\text{Gy}(\text{CH}^{r-m-1}(Y^{[m+1]}; \mathbb{Q}))},
\]

\[
\partial^{X\backslash Y}_R : \text{CH}^r(X\backslash Y, m; \mathbb{Q}) \to \frac{\text{CH}^{r-m}(Y^{[m]}; \mathbb{Q})}{\text{Gy}(\text{CH}^{r-m-1}(Y^{[m+1]}; \mathbb{Q}))}.
\]

**Proposition 8.1.** For \( m \geq 1 \),

\[
\text{im}(d_m) = \frac{j_* \text{CH}^{r-1}(Y, m - 1; \mathbb{Q})}{j_* (\ker \partial^Y_R)}.
\]

**Corollary 8.2.** If \( m \geq 1 \) and \( \text{cl}_{m, m} \) is surjective, then the Abel-Jacobi map \( \lambda_m \) in (4.5) induces an injection

\[
\frac{j_* \text{CH}^{m-1}(Y, m - 1; \mathbb{Q})}{j_* (\ker \partial^X_R)} \hookrightarrow J(G_{r-W}^m).
\]
**Proof of Proposition 8.1.** Consider the exact sequence

\[ \text{CH}^r(X, m) \rightarrow \text{CH}^r(X \setminus Y, m) \rightarrow \text{CH}^{r-1}(Y, m-1) \rightarrow \text{CH}^r(X, m-1). \]

In the weight filtered spectral sequence obtained from (3.2) involved in computing \( \text{CH}^r(X \setminus Y, m) \) we can restrict our attention to those \( Z_{i,j}^r = z^{r+i}(Y^{|-i|}, -j) \) with \( i \leq -1 \), which converges to \( \text{CH}^{r-1}(Y, m-1) \). It also exists for \( \text{CH}^r(X, m-1) \) (using the column where \( i = 0 \)). We use indices to distinguish between the various spectral sequences.

Note that \( E_{1, X \setminus Y}^{0, -m+1} = E_{\infty, X}^{0, -m+1} = \text{CH}^r(X, m-1) \) and that for \( \ell \geq 1 \),

\begin{equation}
E_{\ell, Y}^{\ell, \ell - m} \rightarrow E_{\ell, X \setminus Y}^{\ell, \ell - m} \rightarrow E_{\infty, Y}^{\ell, \ell - m},
\end{equation}

since \( d_{\ell}^Y = 0 \) on \( E_{\ell, Y}^{\ell, \ell - m} \) as its target is trivial. Non-canonically, we have

\[ \text{CH}^{r-1}(Y, m-1; \mathbb{Q}) \cong \widetilde{E}_{\infty, Y}^{m, 0} \oplus \ker(\partial_Y^R) \]

and

\[ \text{CH}^r(X \setminus Y, m; \mathbb{Q}) \cong \widetilde{E}_{\infty, X \setminus Y}^{m, 0} \oplus \ker(\partial_Y^X \setminus Y). \]

Thus

\[ j_* \text{CH}^{r-1}(Y, m-1; \mathbb{Q}) \cong \frac{\text{CH}^{r-1}(Y, m-1; \mathbb{Q})}{\text{CH}^r(X \setminus Y, m; \mathbb{Q})} \cong \frac{\widetilde{E}_{\infty, Y}^{m, 0}}{E_{\infty, X \setminus Y}^{m, 0}} \oplus \frac{\ker(\partial_Y^R)}{\ker(\partial_Y^X \setminus Y)}, \]

with the last term isomorphic with \( j_* (\ker(\partial_Y^R)) \). One has a commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & \widetilde{E}_{\infty, X \setminus Y}^{m, 0} & \rightarrow \widetilde{E}_{\infty, Y}^{m, 0} & \rightarrow \widetilde{E}_{m, X \setminus Y}^{0, -m+1} \\
\downarrow & & \downarrow & & \\
E_{m, X \setminus Y}^{m, 0} & \rightarrow & E_{m, X \setminus Y}^{m, 0} & \rightarrow & E_{m, X \setminus Y}^{m, 0}
\end{array}
\]

with surjective vertical map by (8.3). One sees the bottom row is exact by comparing the spectral sequences \( E_{m, X \setminus Y}^{p,q} \) and \( E_{m, Y}^{p,q} \) at \( (p, q) = (-m, 0) \). Hence \( j_* \text{CH}^{r-1}(Y, m-1; \mathbb{Q}) \cong \text{im}(d_m) \oplus j_* (\ker(\partial_Y^R)) \). \( \square \)

We now restrict to the case \( r \geq m \) because \( \text{im}(d_m) = 0 \) for \( r < m \) as \( \text{CH}^{r-m}(Y^{[m]}) = 0 \) (see (4.1)). Let \( \xi \in \text{CH}^r(X, m-1) \). Then \( |\xi| \subset X \times \Delta^{m-1} \) is of codimension \( r \), and hence \( W := \text{Pr}_1(|\xi|) \) is of
codimension \((r-m)+1 \geq 1\) in \(X\). By Hironaka, there is a proper modification diagram

\[
\sigma^{-1}(W) =: \begin{array}{c}
Y' \\
\downarrow
\end{array} 
\begin{array}{c}
X
\end{array}
\]

(8.4)

\[
W \begin{array}{c}
\sigma
\end{array} \rightarrow X
\]

where \(Y'\) is a NCD. Let \(U = X \setminus W\). We have a corresponding diagram with \(\text{CH}^r_Y(X, m-1; \mathbb{Q}) = \text{CH}^{r-1}(Y', m-1; \mathbb{Q})\),

(8.5)

\[
\begin{array}{c}
\sigma_{1, *} \text{CH}_Y^r(X, m-1; \mathbb{Q}) \\
\downarrow
\end{array} \rightarrow \begin{array}{c}
\text{CH}^r(X, m-1; \mathbb{Q}) \\
\downarrow \sigma_*
\end{array}
\]

\[
\begin{array}{c}
\text{CH}^r(U, m-1; \mathbb{Q})
\end{array}
\]

where \(\xi\) is in \(\text{CH}^r_W(X, m-1; \mathbb{Q})\). Obviously \(\sigma_* \circ \sigma^* = \text{Identity}\) implies that \(\sigma_*\) is onto. A diagram chase shows that \(\sigma_{1, *}\) is a surjection as well.

As \(\overline{X}\) is obtained from \(X\) by a sequence of blow-ups with non-singular centres, it is clear from the well-known motivic decomposition of \(\overline{X}\) with respect \(X\), that the (higher) Chow group of \(\overline{X}\) involves that of \(X\) and of smooth irreducible divisors. Hence it follows that \(\ker(\sigma_*) \subset j_1(* \ker \partial^*_R)\).

We deduce the isomorphisms

\[
\begin{array}{c}
\text{CH}^r(Y', m-1; \mathbb{Q}) \\
\downarrow
\end{array} \sim \begin{array}{c}
\text{CH}^r(X, m-1; \mathbb{Q})
\end{array}
\]

\[
\begin{array}{c}
\text{CH}^r(U, m-1; \mathbb{Q})
\end{array}
\]

\[
\begin{array}{c}
\text{CH}^r_W(X, m-1; \mathbb{Q}) \\
\downarrow
\end{array} \sim \begin{array}{c}
\text{CH}^r(W, m-1; \mathbb{Q})
\end{array}
\]

\[
\begin{array}{c}
\sigma_1 \circ j_1(* \ker \partial^*_R)
\end{array}
\]

Let

\[
\Xi^r,m_X := \sum_{(X, Y)} \sigma_* \circ j_1(* \ker \partial^*_R),
\]

where \((X, Y)\) runs over all proper modifications (8.4) for all \(W \subset X\) of codimension at least 1. By construction we have

\[
\text{CH}^r(X, m-1; \mathbb{Q}) = \sum_{(X, Y)} \sigma_* \circ j_1(* \text{CH}_Y(X, m-1; \mathbb{Q})).
\]

Now recall the subspace \(N^1 \text{CH}^r(X, m-1; \mathbb{Q}) \subset \text{CH}^r_{\text{hom}}(X, m-1; \mathbb{Q})\) introduced in equation (6.6). It follows from the definitions that

\[
\Xi^r,m_X \subseteq N^1 \text{CH}^r(X, m-1; \mathbb{Q}),
\]
since, by (4.6), $\partial H^r_\partial(\xi) = 0$ in $\tilde{E}^{-m,0}$ implies $\beta(\partial H^r_\partial(\xi)) = 0$ in $\Gamma(Gr^W_0)$. The above inclusion is an equality if $r = m$, since in this case $\beta$ is an isomorphism. Let $\text{CH}^i_{\text{dec}}(X, m - 1; \mathbb{Q}) \subseteq \text{CH}^r(X, m - 1; \mathbb{Q})$ be the subspace generated by images of the form

$$\text{CH}^p(X, b; \mathbb{Q}) \otimes \text{CH}^i(X, s; \mathbb{Q}) \to \text{CH}^r(X, m - 1; \mathbb{Q})$$

under the product where

$$a + s = r, \quad b + t = m - 1, \quad (a, b) \neq (0, 0), \quad (s, t) \neq (0, 0).$$

**Proposition 8.6.** For $r \geq m \geq 1$ we have $\text{CH}^r_{\text{dec}}(X, m - 1; \mathbb{Q}) \subseteq \Xi^r_X$.

**Remark 8.7.** It is not clear if equality holds in the above proposition.

**Proof of Proposition 8.6.** Consider $\xi = \xi_1 \otimes \xi_2 \in \text{CH}^a(X, b; \mathbb{Q}) \otimes \text{CH}^s(X, t; \mathbb{Q})$, where $b, t \leq m - 1$. Note that $\text{CH}^r(X, j)$ is supported in codimension $i - j$, namely

$$\text{CH}^i(X, j) = \sum_{\text{cd}X, Y \geq i-j} \text{im}(\text{CH}^i(X, j) \to \text{CH}^i(X, j)).$$

Also $a \leq b$ and $s \leq t \Rightarrow r = a + s \leq b + t = m - 1$, which is not the case as we are assuming $r \geq m$. Therefore we can assume say $s > t$.

Next, if $t = m - 1$, then $b = 0$, and thus $a > 0 = b$ implies $\xi_1$ is supported on a divisor in $X$. This scenario can be handled in the same way as in the case where we assume that $s > t$ and $t < m - 1$. Namely, since $s > t$, we can assume that $\xi_2$ is supported on some $Y \subset X$ of codimension $1 \leq s - t$, and by the surjectivity of $\sigma_{Y, *}$ in diagram (8.5), we can assume without loss of generality that $Y \subset X$ is a NCD. Thus for some $\xi_2^Y \in \text{CH}^{s-1}(Y, t; \mathbb{Q})$, $\xi_2^Y \Rightarrow \xi_2$. Since $X$ is smooth, we have the product ([Bl]):

$$\text{CH}^a(X, b) \times \text{CH}^{s-1}(Y, t; \mathbb{Q}) \to \text{CH}^{r-1}(Y, m - 1; \mathbb{Q}),$$

which defines $\xi_1 \cap \xi_2^Y \in \text{CH}^{r-1}(Y, m - 1; \mathbb{Q})$. But since $\xi_1$ can be assumed in general position with respect to $Y$, and together with $t < m - 1$, we have $\partial H^r_\partial(\xi_1 \cap \xi_2^Y) = 0$. \qed

9. **Noether-Lefschetz for a family of surfaces**

Let $X/\mathbb{C}$ be a smooth projective surface, $Y = \bigcup_{j=1}^n Y_j \subset X$ a NCD (with smooth $Y_i$'s) with open complement $U := X\setminus Y$. The work of Deligne ([De], Cor. 3.2.13 and 3.2.14) implies that $F^2H^2(U, \mathbb{C}) = H^0(X, \Omega^2_X(Y))$, where $\Omega^2_X(Y)$ is the sheaf of rational 2-forms on $X$, regular on $U$ with logarithmic poles along $Y$. In particular if we denote by
$H(U)$, the space of regular algebraic 2-forms on $U$ with $\mathbb{Q}(2)$-periods, then

$$\Gamma H^2(U, \mathbb{Q}(2)) := H^2(U, \mathbb{Q}(2)) \cap F^2H^2(U, \mathbb{C}) \subset H(U),$$

and where we allow for the possibility that the left hand side of (9.1) is non-zero. Let $Z(Y)$ be the singular set of $Y$. For each $j$ we choose distinct points $\{P_j, Q_j\} \subset Y_j \setminus \{Y_j \cap Z(Y)\}$. Now let’s modify $X$ by blowing it up along $\{P_1, Q_1, ..., P_n, Q_n\}$ and call this $X'$. So in particular the strict transform $Y'$ of $Y$ is a copy of $Y$ itself. On $U' := X' \setminus Y'$ we have for each $j = 1, ..., n$ an interesting real 2-cycle $\gamma_j$ obtained as follows: Take the complement of a small disc in the blowup of $P_j$, the complement of a small disc in the blowup of $Q_j$, and a small tube in $X \setminus Y$ along a path in $Y_j$ from $P_j$ to $Q_j$ that so that the end circles meet the circles of the two previous parts. Then by a standard residue argument, integrating an element $\omega \in H(U')$ against $\gamma_j$ gives us essentially $2\pi i$ times the integral along the part in $Y_j$ from $P_j$ to $Q_j$ of the residue of $\omega$ along $Y$, where we observe that $\omega$ restricts to zero on the other two parts of $\gamma_j$, viz., there are no non-zero holomorphic 2-forms living on any subset of $\mathbb{P}^1$.

Note that this integral is determined by the finite dimensional $\mathbb{Q}$-vector space of residues on $Y$. So pick $P_j$ and $Q_j$ sufficiently general in $Y_j$ so that we can never end up in $\mathbb{Q}(2)$ with this integral except with the trivial residue. Then doing this for all $j = 1, ..., n$, we arrive at $U'$ for which $\Gamma H^2(U', \mathbb{Q}(2)) = 0$, using the fact that $H^0(X', \Omega^2_{X'}) \cap H^2(X', \mathbb{Q}(2)) = 0$. We deduce:

**Theorem 9.2.** Let

$$t \in B := \left\{ (P_1, Q_1, ..., P_n, Q_n) \in \prod_{j=1}^{n} Y_j^2 \left| \begin{array}{l} P_j \neq Q_j \\
_{j=1, ..., n} \{P_j, Q_j\} \in Y_j \setminus \{Y_j \cap Z(Y)\} \end{array} \right. \right\},$$

with corresponding family $\{U'_t\}_{t \in B}$. Then $\Gamma H^2(U'_t, \mathbb{Q}(2)) = 0$ for a very general point $t$ in $B$. 

\[\text{Figure 9.1. The cycle } \gamma_j\]
10. Appendix: The Abel-Jacobi map revisited

The purpose of this appendix is to establish the commutativity of diagram (4.5) above. Due to its technical nature, the reader with pressing obligations can easily skip this without losing sight of the main results of this paper. We first digress by describing the Abel-Jacobi map

\[ \lambda_k : \widetilde{E}_{m+k,-k+1} \to J(Gr^W_{-k}), \]

where \( Gr^W_{-k} \) is described in (4.4). In the case \( k = 1 \), viz.,

\[ \lambda_1 : \widetilde{E}_{m+1,0} \to J(Gr^W_{-1}), \]

this is induced by the classical Abel-Jacobi map

\[ \text{CH}^r_{m+1}(Y^{m-1}; \mathbb{Q}) \xrightarrow{\xi \mapsto \int_{\partial_{-1}\xi^{(-)}}} J(H^{2r-2m+1}(Y^{m-1}, \mathbb{Q}(r-m+1))). \]

The Abel-Jacobi map

\[ \Phi_k : \widetilde{E}_{m+k,-k+1} \to J(Gr^W_{-k}), \]

is defined using the formula in [KLM]. By degeneration of the mixed Hodge complex spectral sequence at \( E_2 \) (Deligne), and a map of spectral sequences from Chow groups to Hodge cohomology, together with functoriality of the Abel-Jacobi map, the map \( \Phi_k \) induces \( \lambda_k \) for all \( k \geq 2 \). So we need only describe the map \( \Phi_k \) explicitly. Let \( n^m := (\mathbb{P}^1 \setminus \{1\})^m \) with coordinates \( z_i \) and \( 2^m \) codimension one faces obtained by setting \( z_i = 0, \infty \), and boundary maps \( \partial = \sum (-1)^{i-1}(\partial^0_i - \partial^\infty_i) \), where \( \partial^0_i, \partial^\infty_i \) denote the restriction maps to the faces \( z_i = 0, z_i = \infty \) respectively. Here we adopt the notation in [KLM] adapted to the cubical description of \( \text{CH}^r(X, m; \mathbb{Q}) \), with cycles lying in \( z^r(X \times n^m; \mathbb{Q}) \), in general position with respect to the \( 2^m \) faces of \( n^m \) as well as the real part \( [-\infty, 0]^m \subset n^m \).

Recall the Tate twist \( \mathbb{Q}(r) \), let \( X/\mathbb{C} \) be smooth projective with \( d = \dim X \), and put \( D^p_X := \text{sheaf of currents that act on compactly supported } \mathbb{C} \)-valued \( C^\infty \) forms of degree \( 2d-k \). Note that

\[ D^p_X = \bigoplus_{p+q=\bullet} D^{p,q}_X, \]

where \( D^{p,q}_X \) acts on corresponding \( (d-p, d-q) \) forms. Let \( \mathcal{C}^k_X(\mathbb{Q}(r)) \) be the sheaf of Borel-Moore chains of real codimension \( k \) in \( X \) with \( \mathbb{Q}(r) \) coefficients. One has an inclusion \( \mathcal{C}^k_X(\mathbb{Q}(r)) \subset D^k_X \). Now put

\[ \mathcal{M}^\bullet_D = \text{Cone}\{\mathcal{C}^\bullet_X(X, \mathbb{Q}(r)) \bigoplus F^\bullet D^\bullet_X(X) \to D^\bullet_X(X)\}[-1]. \]

The cohomology of this complex at \( \bullet = k \) is precisely the Deligne cohomology \( H^k_D(X, \mathbb{Q}(r)) \) (see [KLM]).
We recall the cycle class map
\[ \Psi_{r,m} : \text{CH}^r(X, m; \mathbb{Q}) \to H^{2r-m}_D(X, \mathbb{Q}(r)), \]
in terms of the cubical description of \( \text{CH}^r(X, m) \). Note that for \( m \geq 1 \),
\[ H^{2r-m}_D(X, \mathbb{Q}(r)) \simeq J(H^{2r-m-1}(X, \mathbb{Q}(r))), \]
and under this identification, \( \Psi_{r,m} \) is the Abel-Jacobi map. On \( \square^m \) we introduce the currents
\[ \Omega_{\square^m} := \bigwedge^m d \log z_j \]
\[ T_{\square^m} := T_{z_1} \cap \cdots \cap T_{z_m} = \int_{[-\infty,0]^m} (-) =: \delta_{[-\infty,0]^m}, \]
where \( T_{z_j} \) is integration on \( \{(z_1, \ldots, z_m) \in \square^m \mid z_j \in [-\infty,0]\} \).
\[ R_{\square^m} := \log z_1 d \log z_2 \wedge \cdots \wedge d \log z_m - (2\pi i) \log z_2 d \log z_3 \wedge \cdots \wedge d \log z_m T_{z_1} \]
\[ + \cdots + (-1)^{m-1}(2\pi i)^{m-1} \log z_m \cdot \{ T_{z_1} \cap \cdots \cap T_{z_{m-1}} \}, \]
where \( \log \) has the principle branch. One has
\[ dR_{\square^m} = \Omega_{\square^m} - (2\pi i)^m T_{\square^m} = 2\pi i \partial \Omega_{\square^m}. \]
We consider a cycle \( \xi \in \zeta^r(X \times \square^m) \) in general position. One considers the projections \( \pi_1 : [\xi] \to X, \pi_2 : [\xi] \to \square^m \). We put
\[ R_\xi = \pi_1 \ast \circ \pi_1^{\ast} R_{\square^m}, \quad \Omega_\xi = \pi_1 \ast \circ \pi_1^{\ast} \Omega_{\square^m}, \quad T_\xi = \pi_1 \ast \circ \pi_2^{\ast} T_{\square^m}. \]
Note that when \( m = 0, R_\xi = 0 \) (classical case!). Correspondingly
\[ dR_\xi = \Omega_\xi - (2\pi i)^m T_\xi - 2\pi i \partial \xi. \]
Up to a normalizing constant, the cycle class map \( \Psi_{r,m} \) is induced by
\[ \xi \mapsto ((2\pi i)^m T_\xi, \Omega_\xi, R_\xi). \]
Recall \( d = \dim X \). The Abel-Jacobi map
\[ \Phi_{r,m} : \text{CH}^r_{\text{hom}}(X, m; \mathbb{Q}) \to \frac{F^{d-r+1}H^{2d-2r+m+1}(X, \mathbb{C})^\vee}{H^{2d-2r+m+1}(X, \mathbb{Q}(d-r))}, \]
where we described \( J(H^{2r-m-1}(X, \mathbb{Q}(r))) \) using the Carlson isomorphism, is defined as follows. If \( \xi \in \text{CH}^r(X, m; \mathbb{Q}) \), then by a moving lemma [K-L, Lemma 8.14], we can assume that \( \xi \) is in general position with respect to the real cube \( [-\infty,0]^m \subset \square^m \). Furthermore, \( m > 0 \) implies \( T_\xi = dT_\xi \), i.e., \( \pm \partial \xi = \xi \cap \{ X \times [-\infty,0]^m \} \), \( \Omega_\xi = -dS \), for some \( S \in F^rD^{2r-m-1}_X \), so up to a normalizing constant and for \( \omega \in F^{d-r+1}H^{2d-2r+m+1}(X, \mathbb{C}) \), we have
\[ \Phi_{r,m}(\xi)(\omega) = S(\omega) + (-2\pi i)^m \int_\xi \omega + R_\xi(\omega) = (-2\pi i)^m \int_\xi \omega + R_\xi(\omega), \]
where the latter equality stems from Hodge type considerations.

Now referring to (4.5), we fulfill a promise made earlier, viz.,

**Theorem 10.1.** The diagram (4.5) is commutative. Specifically,

\[ h_k \circ \alpha_k = \lambda_k \circ d_k. \]

**Proof.** We have to unravel the definitions. We use the simplicial complex \( Y^{[\bullet]} \to Y \hookrightarrow X \) as a way of describing \( H^2r-m(U, \mathbb{Q}(r)) \). Let

\[ K^r_{\mathbb{Q}}(Y^{[m-i]}) = C^{2r-2m+i}(Y^{[m-i]}, \mathbb{Q}(r - m + i)) \]

\[ K^r_{\mathbb{C}}(Y^{[m-i]}) = D^{2r-2m+i}(Y^{[m-i]}) \]

\[ \mathbb{D} = d \pm \text{Gy} \]

A class \( \xi \in W_0H^2r-m(U, \mathbb{Q}(r)) \) is represented by a \( \mathbb{D} \)-closed \( (m + 1) \)-tuple

\[ \xi = (\xi_0, \xi_1, \ldots, \xi_m) \in \bigoplus_{i=0}^m K^r_{\mathbb{Q}}(Y^{[m-i]}). \]

With \( W_j := W_jH^2r-m(U, \mathbb{Q}(r)) \), consider the short exact sequence

\[ 0 \to W_{-1} \to W_0 \xrightarrow{\xi \mapsto \xi_0} Gr_W^0 \to 0. \]

Let us first describe \( h_1 \circ \alpha_1 : \tilde{E}_1^{-m,0} \to J(Gr_{-1}^W) \). Let \( \gamma \in \tilde{E}_1^{-m,0} \) and \( \xi_0 = \beta(\gamma) \). In this case \( \xi_0 \in \Gamma(Gr_0^W) \) is in the image of \( \xi \in W_0 \). Likewise \( \xi_0 \) is in the image of some \( \xi^C \in F^0W_0C \). The difference \( \xi - \xi^C \in W_{-1,C} \) maps to a class in \( J(W_{-1}) \) which defines the Abel-Jacobi image of \( \gamma \) in \( J(W_{-1}) \). We can assume that \( \xi - \xi^C \) is represented by the \( \mathbb{D} \)-closed \( m \)-tuple:

\[ (\xi_1 - \xi_1^C, \ldots, \xi_m - \xi_m^C) \in \bigoplus_{i=1}^m K^r_{\mathbb{D}}(Y^{[m-i]}). \]

with \( \text{Gy}(\xi_0) = \partial \xi_1 \). The corresponding value \( h_1 \circ \alpha_1(\gamma) \in J(Gr_{-1}^W) \) is given by the Abel-Jacobi membrane integral \( \int_{\xi_1} (-) \), as the Hodge contribution given by \( \xi_1^C \) is trivial for Hodge type reasons. This is easily seen to be precisely \( \lambda_1 \circ d_1(\gamma) \), where we recall that \( d_1 = \text{Gy} \). Thus \( \lambda_1 \circ d_1 = h_1 \circ \alpha_1 \). So now suppose that \( d_1(\gamma) = 0 \in \tilde{E}_1^{-m+1,0} \), i.e. \( \gamma \in \tilde{E}_2^{-m,0} \). This means that \( \text{Gy}(\gamma) = \partial \zeta_1 \), where \( \zeta_1 \in z^{r-m+1}(Y^{[m-1]}, 1; \mathbb{Q}) \). Then \( d_2(\gamma) = \text{Gy}(\zeta_1) \in \tilde{E}_2^{-m+2,-1} \). (We comment in passing that using the aforementioned moving lemma in [K-L], we can assume that \( \zeta_{1,r} := \zeta \cap Y^{[m-1]} \times [-\infty, 0] \) is a proper intersection, and hence that \( \partial \zeta_{1,r} = \text{Gy}(\xi_0) \). Since \( d_1(\gamma) = \text{Gy}(\gamma) \) is a coboundary, it follows that \( h_1 \circ \alpha_1(\gamma) = \lambda_1 \circ d_1(\gamma) = 0 \), and hence after removing classes in \( W_{-1} + F^0W_{-1,C} \), [specifically, \( \partial(\zeta_{1,r} - \zeta_1) = \text{Gy}(\xi_0) - \text{Gy}(\xi_0) = 0 \), and
using $W_{-1} \to Gr_{-1}^W$, $d(\Omega_{\zeta_1} - \xi_1^C) = \Gy(\xi_0^C) - \Gy(\xi_0^C) = 0$ and using $F^0W_{-1,\mathbb{C}} \to F^0Gr_{-1}^W[\mathbb{C}]$, we can assume that $\xi - \xi^C$ is represented by the $\mathbb{D}$-closed $m$-tuple

$$(2\pi iT_{\zeta_1} - \Omega_{\zeta_1}, \xi_2 - \xi_2^C, \ldots, \xi_m - \xi_m^C) \in \bigoplus_{i=1}^m K^2r-2m+i(\mathcal{V}^{m-i}),$$

where

$$d\xi_2 = (2\pi i)T_{\Gy(\zeta_1)}, \quad d\xi_2^C = \Omega_{\Gy(\zeta_1)}.$$ 

Note that

$$dR_{\zeta_1} = \Omega_{\zeta_1} - (2\pi iT_{\zeta_1} - (2\pi i)R_{\partial\zeta_1} = \Omega_{\zeta_1} - (2\pi i)T_{\zeta_1},$$

as $R_{\partial\zeta_1} = 0$ ($\partial\zeta_1$ representing the classical case!). But $\partial\Gy(\zeta_1) = \Gy(\partial\zeta_1) = \Gy^2(\gamma) = 0$. Hence working modulo the coboundary

$$\mathbb{D}R_{\zeta_1} = dR_{\zeta_1} \pm R_{\Gy(\zeta_1)},$$

we can assume that $\xi - \xi^C$ is represented by the $\mathbb{D}$-closed $(m-1)$-tuple

$$(\xi_2 - \xi_2^C + R_{\Gy(\zeta_1)}, \xi_3 - \xi_3^C, \ldots, \xi_m - \xi_m^C) \in \bigoplus_{i=2}^m K^2r-2m+i(\mathcal{V}^{m-i}).$$

So modulo $\mathbb{D}$-coboundary, $h_2 \circ \alpha_2(\gamma)$ is represented by the $d$-closed current $\xi_2 - \xi_2^C + R_{\Gy(\zeta_1)}$, which is precisely $\lambda_2 \circ d_2(\gamma)$. Hence $h_2 \circ \alpha_2 = \lambda_2 \circ d_2$. The general case $h_k \circ \alpha_k = \lambda_k \circ d_k$ proceeds in a similar fashion. □

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