ON THE CHERNOUS’KO TIME-OPTIMAL PROBLEM FOR THE EQUATION OF HEAT CONDUCTIVITY IN A ROD

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Abstract: The time-optimal problem for the controllable equation of heat conductivity in a rod is considered. By means of the Fourier expansion, the problem reduced to a countable system of one-dimensional control systems with a combined constraint joining control parameters in one relation. In order to improve the time of a suboptimal control constructed by F.L. Chernous’ko, a method of grouping coupled terms of the Fourier expansion of a control function is applied, and a synthesis of the improved suboptimal control is obtained in an explicit form.

Keywords: Heat equation, Time-optimal problem, Pontryagin maximum principle, Suboptimal control, Synthesis of control.

Introduction

It is known that a time-optimal problem occupied a very important place in the foundation and development of optimal control theory. Even for simple non-trivial cases, the problem required working-out new approaches and lead after all to Pontryagin’s maximum principle [3, 10, 30]. Despite 70 years of development, the solution of concrete non-trivial examples of time-optimal control still needs considerable effort [2, 4, 19]. The problem becomes even more difficult when a control system is described by a partial differential equation [11, 24, 25, 34], particularly, for the heat conductivity equation [12, 22, 26, 29, 35, 36]. In [1], the correctness of parabolic equations for heat propagation is discussed and for that purpose, a parabolic equation with time delay is considered.

Here, the maximum principle can be formally written out as well, but it loses its effectiveness as compared with a finite-dimensional case or on cases when the time interval is fixed [2, 9, 18, 25, 32, 33]. Therefore, Chernous’ko suggested [13] another approach based on the Fourier expansion that allowed him to reduce the problem to an infinite system of one-dimensional problems whose control parameters are connected by a condition in the min-max form (see below (1.4)) generating a closed convex control set in a Hilbert space. Unfortunately, to deal with such a constraint is quite difficult (about other kinds of constraints see [17]). In order to overcome this complexity, the mentioned constraint was replaced [13] by an infinite system of separated conditions for scalar control parameters that can be interpreted as if one took Hilbert’s brick inscribed into the control set. As a result, this approach made it possible to construct a suboptimal control and to give an explicit upper estimation for an optimal time. In [5], a co-Hilbert’s brick inscribed into the control set was considered, and an improved suboptimal control function was constructed. In the present

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paper, we suggest another way for constructing a suboptimal control function in the case of the heat conductivity equation in a rod.

1. Preliminaries

As it was noted above, Chernous’ko considered the time-optimal problem for an evolutionary equation

$$\frac{\partial u(t,x)}{\partial t} = A[u(\cdot,\cdot)](t,x) + v(t,x)$$

with the initial and boundary conditions

$$u(0,x) = u^0(x), \quad Mu(t,s) = u^*(t,s),$$

where $A$ is a uniformly elliptic differential operator, $t \geq 0$, $x \in D$, $D$ is a regular domain with Lyapunov boundary $\Gamma$, $s \in \Gamma$, and $M$ is a boundary operator [13].

The constraint on the control function in problem (1.1), (1.2) is bounded in the norm of the space $L^\infty$; i.e. $|v(t,x)| \leq v_0$ for almost all $t$ and every $x \in D$, where $v_0$ is a given positive number [31]. It is known that, for every control function $v(t,x)$, problem (1.1), (1.2) has a unique solution $u(t,x)$ [14, 21, 28].

If a solution $u(t,x)$ of problem (1.1), (1.2) satisfies the condition $u(T,x) \equiv 0$ at some $T$, $T \geq 0$, then the corresponding control function $v(t,x)$ is called admissible, and the number $T$ is called the transition time (from the initial state $u_0(\cdot)$ into the equilibrium state $u(t,x) \equiv 0$). Let $V$ be the class of all admissible controls. Then the quantity $T = T[v(\cdot,\cdot)]$ will be a functional on $V$ at every fixed $u^0(x)$ and $u^*(t,s)$.

If an admissible control $v_*(t,x)$ satisfies the condition $T_* = T[v_*(\cdot,\cdot)] \leq T[v(\cdot,\cdot)]$ for all $v(\cdot,\cdot) \in V$, then $v_*(\cdot,\cdot)$ is called a time-optimal control, and the value $T_*$ is called optimal transition time.

The direct application of the Pontryagin maximum principle to problem (1.1), (1.2) is a very hard task, unlike optimization problems on a finite interval of time (see [4, 8, 15]). For example, in [25], only theorems on the existence of optimal control and the bang-bang principle are given, but no specific example of a solution was considered. In monograph [11], the time-optimal problem when a control parameter participates in boundary conditions was considered [11, Ch. 5, Sect. 1] and, instead of the necessary conditions, the method of the $L$-momentum of N.N. Krasovskii [19] was applied [11, Sect. 2]. In the recently published article [20], Butkovsky’s approach was applied to the case of a fractional-order diffusion equation. It should be noted that the $L$-momentum method only allows one to simplify to some degree the time-optimal problem and rarely gives an explicit solution. Therefore, the approach suggested by Chernous’ko [13], where the method of expansions on the system of eigenfunctions of the operator $A$ was used, seems to be more effective. That helped to reduce considering problem to the infinite system of one-dimensional control problems:

$$\dot{y}_k = -\lambda_k y_k + v_k, \quad y_k(0) = y_{k0}, \quad k = 0, 1, 2, \ldots .$$

(About solution of systems of this kind, see [16]).

In terms of system (1.3), the condition $|v(t,s)| \leq v_0$ means that a counting system of the control parameters $v_k$, $k = 0, 1, 2, \ldots$, should satisfy the combined constraint

$$\max_{x \in D} \left| \sum_{k=0}^{\infty} \varphi_k(x)v_k \right| \leq v_0.$$

where $\varphi_k$ are eigenfunctions of the problem.
Condition (1.4) defines some closed convex set $L$ in the Hilbert space $l_2$, which is difficult to deal with. In this connection, it is natural to try to solve the problem of finding a suboptimal control. (It is essential to note that, if a time interval is fixed, then the method of penalty functions is enough effective for the construction of a suboptimal control. It would be interesting to apply this method for the time-interval problem as well [6].) For this purpose, in [13], constraint (1.4) was replaced by a more rigid system of constraints in the form

$$|v_k| \leq U_k, \quad k = 0, 1, 2, \ldots, \tag{1.5}$$

where $\alpha_k = \max_{x \in D} |\varphi_k(x)|$. Wherein, nonnegative numbers $U_k$ should be chosen satisfying the condition $\sum_{k=0}^\alpha \alpha_k U_k = v_0$.

Let $T_{s_k}$ be an optimal transition time in the problem

$$\dot{y}_k = -\lambda_k y_k + v_k, \quad y_k(0) = y_k^0, \tag{1.6}$$

such that $y_k(T_{s_k}) = 0$, $k = 0, 1, 2, \ldots$. In [13], it is shown that the numbers $U_k$ can be chosen so that all $T_{s_k}$ coincide: $T_{s_k} = \bar{T}$ for some $\bar{T}$. Let $\dot{y}_k(t)$ be the sequence of the corresponding optimal controls. Then $T_{s} \leq \bar{T}$ and $\dot{y}_k(t, x) = \sum_{k=0}^\infty \varphi_k(x) v_k(t)$ may serve as the sought suboptimal control.

A new problem arises here: is it possible to use a more exact reduction of the constraint than (1.5)? As mentioned above in [5] it was used Hilbert’s co-cube instead of (1.5). Here, we are going to follow another approach based on a special grouping of terms of (1.5). Effectiveness of this approach is tightly related to specific properties of eigenfunctions $\varphi_k(\cdot)$, so here it will be demonstrated for the operator $A = \partial^2/\partial x^2$ connected with the process of the heat conductivity in a rod.

### 2. A method of grouping terms of the Fourier expansion

Consider the following concretization of problem (1.1), (1.2):

$$\begin{cases} 
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + v(t, x), \quad |v(t, x)| \leq v_0, \quad t \geq 0, \quad 0 \leq x \leq \pi, \\
u(0, x) = u(0), \quad u(t, 0) = 0, \quad u(t, \pi) = 0.
\end{cases} \tag{2.1}$$

The system of eigenfunctions $\varphi_k(t) = \sin kx$, $k = 1, 2, \ldots$, of the operator $\partial^2/\partial x^2$ forms a complete orthogonal basis of the space $L_2[0, \pi]$ [21, 28].

Let $u(t, x) = \sum_{k=1}^\infty y_k \sin kx$ and $v(t, x) = \sum_{k=1}^\infty v_k \sin kx$ be the Fourier expansions on the basis $\{\sin kx\}$. Then the restriction (1.4) takes the form

$$\max_{0 \leq x \leq \pi} \left| \sum_{k=1}^\infty v_k \sin kx \right| \leq v_0. \tag{2.2}$$

Let us consider a more rigid restriction

$$\max_{0 \leq x \leq \pi} \sum_{k \in Q} \left| v_k \sin kx + v_{3k} \sin 3kx \right| \leq v_0 \tag{2.3}$$

instead of (2.2), thereby replacing the optimal control problem with a suboptimal control problem. System (1.6) takes the form

$$\dot{y}_k = -k^2 y_k + v_k, \quad k \in \mathbb{Z}^+. \tag{2.4}$$
Let \( Q \) be the set of all positive integers having the form \( 3^{2p}q \), where \( p = 0, 1, 2, \ldots, \) and \( q \) is relatively prime with 3. It is obvious that the set of all positive integers \( \mathbb{Z}^+ \) is the union of the two disjoint sets \( Q \) and \( 3Q \). Then (2.4) can be rewritten in the form

\[
\dot{y}_k = -k^2 y_k + v_k, \quad \dot{y}_{3k} = -9k^2 y_{3k} + v_{3k}, \quad k \in Q. \tag{2.5}
\]

After the substitutions

\[
y_k = \frac{\mu_k}{k^2} x^1, \quad y_{3k} = \frac{\mu_k}{k^2} x^2, \quad t = \frac{1}{k^2} \tau, \quad v_k = \mu_k w^1, \quad v_{3k} = \mu_k w^2,
\]

all systems (2.5) will be reformulated to the following two-dimensional control system:

\[
\dot{x}^1 = -x^1 + w^1, \quad \dot{x}^2 = -9x^2 + w^2. \tag{2.6}
\]

Now, following the Chernous’ko way, we replace (2.3) by the even more rigid restriction

\[
\max_{0 \leq t \leq \pi} \left| w^1 \sin kt + w^2 \sin 3kt \right| \leq 1, \quad k \in Q, \tag{2.7}
\]

that implies (2.3) if \( \sum_{k \in Q} \mu_k = v_0 \). Thus, we have reduced the infinite dimensional control problem to the two-dimensional problem.

### 3. Solution of the auxiliary time-optimal problem on the plane

Let \( P_k \) denote the set of all pairs \((w^1_k, w^2_k)\) for which (2.7) holds. Setting

\[
P = \left\{ w = (w^1, w^2) \in \mathbb{R}^2 : \max_{0 \leq t \leq \pi} \left| w^1 \sin t + w^2 \sin 3t \right| \leq 1 \right\},
\]

we have \( P_k = \mu_k P \). As a result, the considered problem of constructing a suboptimal control reduces to the concrete problem of time-optimal control for the following two-dimensional system:

\[
\dot{x}^1 = -x^1 + w^1, \quad \dot{x}^2 = -9x^2 + w^2, \quad (w^1, w^2) \in P. \tag{3.1}
\]

Obviously, \( P \) is a convex and compact set with non-empty interior (i.e., a convex body). Since \( P \) is symmetric with respect to the origin, we may restrict ourselves to considering only the case \( w^1 \geq 0 \). It is more convenient to set \( \sin x = y \). Then, by the formula

\[
\sin 3t = 3 \sin t - 4 \sin^3 t,
\]

we get

\[
P = \left\{ w = (w^1, w^2) \in \mathbb{R}^2 : \max_{0 \leq y \leq 1} \left| (w^1 + 3w^2) y - 4w^2 y^3 \right| \leq 1 \right\}.
\]

Just this transformation lay on the base of the separation \( Z^+ = Q \cup 3Q \).

After elementary calculations, we find that the part of the boundary of the set \( P \) lying in the half-plane \( w^1 \geq 0 \) is given by the formula

\[
w^1 = \begin{cases} 
    w^2 + 1 & \text{if } -1 \leq w^2 < 0.125, \\
    3(\sqrt{w^2} - w^2) & \text{if } 0.125 \leq w^2 \leq 1,
\end{cases}
\]

while the other part is found by central symmetry (see Fig. 1).

Let us recall that, in the auxiliary problem (3.1), a unique optimal time-control function exists at each initial point \((x^1_0, x^2_0)\) [7, 23, 27]. The existence follows from the property \( O \in \text{Int } P \). The
uniqueness is a consequence of the following feature of $P$: the vector $(1,1)$, which is orthogonal to the segment $AC$, is not an eigenvector of the matrix of system (3.1). Therefore, the optimal control problem (2.5) coincides with the extremal controls of Pontryagin’s maximum principle [23, 30].

To calculate the latter, we prefer to use the “backward motion” principle. Let $T(x^1_0,x^2_0)$ be a transition time for the initial point $(x^1_0,x^2_0)$ in the system (3.1). If we set $\tau = T(x^1_0,x^2_0) - t$, then extremals of Pontryagin’s maximum principle are defined by the system

\[
\begin{aligned}
\frac{dx^1}{d\tau} &= x^1 - \bar{w}^1, \\
\frac{dx^2}{d\tau} &= 9x^2 - \bar{w}^2, \\
\frac{d\psi_1}{d\tau} &= -\psi_1, \\
\frac{d\psi_2}{d\tau} &= -9\psi_2.
\end{aligned}
\] (3.2)

Since $\psi_1(\tau, s) = e^{-\tau} \cos s$ and $\psi_2(\tau, s) = e^{-9\tau} \sin s$, an extremal control $\bar{w}(\tau, s)$ should be found by the Pontryagin’s maximum principle, i.e., from the equation

\[
\bar{w}^1(\tau, s)e^{-\tau} \cos s + \bar{w}^2(\tau, s)e^{-9\tau} \sin s = \max_{w \in P}[w^1 e^{-\tau} \cos s + w^2 e^{-9\tau} \sin s].
\] (3.3)

Equation (3.3) leads to the following construction of the extremal controls.

If $\psi(\tau, s)$ lies in the open angle $AOB$, then obviously $\bar{w}(\tau, s) = (0, 1)$. Note that, if $s = \pi/2$, then $\psi_1(\tau, s) \equiv 0$. Therefore, $\bar{w}(\tau, \pi/2) = (0, 1)$. Similarly, if $s = 0$, then $\psi_2(\tau, s) \equiv 0$; thus, $\bar{w}(\tau, 0) = (2\sqrt{3}/3, \sqrt{3}/9)$.

Consider now the dynamics of $\psi(\tau, s)$. In the case $0 < s < \pi/2$, the vector $\psi(\tau, s)$ lies in the quarter $\psi_1 > 0$, $\psi_2 > 0$ and turns clockwise. Moreover, its direction tends to the axis of abscissas $OE$ as $\tau \to +\infty$. (Similarly, if $-\pi/2 < s < 0$, then $\psi(\tau, s)$ lies in the quarter $\psi_1 > 0$, $\psi_2 < 0$ and turns counterclockwise with the same limit direction.)

Thus, the extremal control has the following structure: if $0 < s \leq \arctan 2$ (see Fig. 1), then $\psi(\tau, s)$ lies in the angle $BOD$ for all $\tau (\tau \geq 0)$, and, hence, $\bar{w}(\tau, s)$ is a point of the arc $AD$ such that its projection to the direction $\psi(\tau, s)$ is maximal (the analytical expression for $\bar{w}(\tau, s)$ is given in Table 1).

Figure 1. The straight ray $AC$ is tangent at the point $A$ to the curve $AD$, which is a part of the boundary of $P$. 
Further, in the case $\text{arc} \, \mathbf{a} < s < \pi/2$, we have $\bar{w}(\tau, s) = (0, 1)$ on the interval $[0, \tau_s)$, where $\tau_s = -1/8 \cdot \log(2 \cot s)$. At the time $\tau = \tau_s$, the vector $\psi(\tau, s)$ becomes orthogonal to the right side tangent to the curve $\partial P$ at the point $(0, 1)$ and it occurs “switching” of the extremal control from the value $(0, 1)$ to a continuous mode. Namely, $\bar{w}(\tau, s)$ begins sliding along the arc $AC$ (see Table 1) and tends to the point $C$ as $\tau \to +\infty$.

Similarly, if $(-\pi/2 < s < -\pi/4)$, then $\bar{w}(\tau, s) = (0, -1)$ at $0 \leq \tau < \tau_{ss}$, where $\tau_{ss} = -1/8 \cdot \log(-\cot s)$ and $w(\tau, s)$ is a switching time. On the interval $(\tau_{ss}, +\infty)$, $\bar{w}(\tau, s)$ slides along the arc $ED$ tending to the point $D$.

The entire synthesis of the extremal control is given in Table 1. Due to the central symmetry, the values of $s$ are considered only on the range $-\pi/2 < s \leq \pi/2$ and the following notation is used:

$$M = (3 - e^{-8r} \tan s)^{-3/2}, \quad m = (3e^{8r} \cot s - 1)^{-1/2}, \quad n = (3 \cot s - 1)^{-1/2},$$

$$p = (3 - e^{-8r} \tan s)^{-1/2}, \quad q = ((3 - p^{-2}) \cot s)^{1/8}.$$
Now, extremal trajectories can be easily calculated by (3.2). The corresponding formulas are gathered in Table 2. They are illustrated in Fig. 2.

![Figure 2. The extremal trajectories.](image)

Further, in the system (3.1), for every fixed $(x_0^1, x_0^2) \neq 0$, an optimal control is unique, which implies the uniqueness of the value $\tau_0$ (while corresponding values of $s_0$ may be not unique, but one can choose any of them).

Then $T(x_0^1, x_0^2) = \tau_0$ is the transition time and

$$\bar{v}^1(t) = v^1(\tau_0 - t, s_0), \quad \bar{v}^2(t) = v^2(\tau_0 - t, s_0)$$
is the suboptimal control for (2.1).

Let us now consider system (2.5). For an initial point \((y_k^0, y_{3k}^0)\), the corresponding trajectory \((y_k(t), y_{3k}(t))\) satisfies the condition

\[
y_k(T_k) = y_{3k}(T_k) = 0,
\]

where

\[
T_k(\mu_k) = \frac{1}{k^2} \left( \frac{k^2}{\mu_k} y_k^0 - \frac{k^2}{\mu_k} y_{3k}^0 \right).
\]

The constructed synthesis implies that \(T_k\) is monotonically decreasing in \(\mu_k\), and it is easy to see that \(T_k \to 0\) as \(\mu_k \to +\infty\) and \(T_k \to \infty\) as \(\mu_k \to 0\). Therefore, for every \(k\), there exists a unique value \(\mu_k^*\) such that \(T_k(\mu_k^*)\) is the same for all \(k\). Moreover, \(\mu_k^*\) can be chosen satisfying the condition \(\sum \mu_k^* = v_0\). One can easily see that

\[
\frac{\alpha}{k^2} \leq \mu_k^* \leq \frac{\beta}{k^2}
\]

for some positive \(\alpha\) and \(\beta\).

Finally, we consider the initial problem (2.1), (2.3), (2.5). Let \(u_0(x) = \sum_{k=1}^{\infty} u_k^0 \sin kx\) be the Fourier expansion of the initial function \(u_0(x)\). Taking \((y_k^0, y_{3k}^0)\), \(k \in Q\), as an initial point for system (2.6), we find

\[
\bar{w}_k^0(t) = \frac{1}{\mu_k} \bar{v}_k^1(t), \quad \bar{w}_{3k}^0(t) = \frac{1}{\mu_k} \bar{v}_k^2(t), \quad k \in Q.
\]

Thus, the following statement holds.

**Theorem 1.** The function

\[
\bar{v}(t, x) = \sum_{k=1}^{\infty} \bar{w}_k^0(t) \sin kx
\]

is a suboptimal control in problem (2.1) for the initial state \(u_0(x)\).

5. Conclusion

The paper is devoted to the time-optimal problem for the process of heat conductivity in a rod when the control parameter is the intensity of external heat sources. A suboptimal control is
constructed by the combination of the Chernous’ko approach with the method of grouping terms of the Fourier expansion.

This method may be applied to the time-optimal control problem for other systems given in an evolutionary form.

The following question naturally arises: how effective is the method of grouping? First of all, let us bring general considerations. The set of all admissible controls in the initial problem \((1.1)-(1.2)\) can be identified with the subset

\[ U_{\text{Initial}} = \left\{ u \in l^2 \mid \sup_{0 \leq x \leq \pi} \left| \sum_{k=0}^{\infty} u_k \sin kx \right| \leq v_0 \right\}. \]

As noted in Section 1, Chernous’ko restricted the set of controls using

\[ U_{Ch} = \left\{ u \in l^2 \mid |u_k| \leq U_k, \ k = 1, 2, 3, \ldots \right\}, \]

where \(U_k\) is a sequence chosen from the condition \(\sum U_k \leq v_0\) and guaranteeing the equality \(u(t, x) \equiv 0\) for some \(T = T_{Ch} > 0\).

The considerations in this paper are based on the set

\[ U_{\text{gr}} = \left\{ u \in l^2 \mid \max_{0 \leq x \leq \pi} |u_k \sin kx + u_{3k} \sin 3kx| \leq U_k \right\} \]

taken as a region of admissible controls.

One can easily see that

\[ U_{\text{gr}} \subset U_{Ch} \subset U_{\text{Initial}}. \]

These relations imply \(T_2 \leq T_1 \leq T_0\) for optimal and suboptimal times of transition respectively.

If one takes an initial point of the form \((0, 0, \ldots, x^0_m, 0, \ldots, 0)\), i.e., in terms of the initial problem \((1.1)-(1.2)\), \(\varphi(x) = (0, 0, \ldots, x^0_m \sin mx, 0, \ldots, 0)\), then, obviously, \(U_{\text{Initial}} = U_{Ch} = U_{\text{gr}}\) and, thus, \(T_2 = T_1 = T_0\). But if an initial point is taken in the form \((0, 0, \ldots, 0, x^0_k, 0, \ldots, 0, x^0_{3k}, 0, \ldots)\), then \(U_{In} = U_{\text{gr}}\) while \(U_{In} \supset U_{Ch}\) and, thus, \(T_2 = T_0 < T_1\). Table 3 contains values for specific cases.

| Initial point | \(T_1\) | \(T_0 = T_2\) |
|--------------|-------|-----------|
| \((1, 0, 1, 0, 0, 0, \ldots)\) | 0.71 | 0.64 |
| \((1, 0, 2, 0, 0, 0, \ldots)\) | 0.67 | 0.61 |
| \((1, 0, -1, 0, 0, 0, \ldots)\) | 0.72 | 0.65 |
| \((1, 0, -2, 0, 0, 0, \ldots)\) | 0.69 | 0.62 |

Obviously, \(x^0_k \neq 0\) at least for three values of the index \(k\) when \(T_2 < T_0 < T_1\).

The final note is that the method of grouping can be applied only if there some algebraic relations between the eigenfunctions of the operator \(A\).

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