Towards a no hair theorem for higher order gravity

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Abstract

We use gravitational lagrangians $R^k R \sqrt{-g}$ and linear combinations of them; we ask under which circumstances the de Sitter spacetime represents an attractor solution in the set of spatially flat Friedman models.

Results are: For arbitrary $k$, i.e., for arbitrarily large order $2k + 4$ of the field equation, one can always find examples where the attractor property takes place. Such examples necessarily need a non-vanishing $R^2$-term. The main formulas do not depend on the dimension, so one
gets similar results also for 1+1-dimensional gravity and for Kaluza-Klein cosmology.

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1 Introduction

Over the years, the notion "no hair conjecture" drifted to "no hair theorem" without possessing a generally accepted formulation or even a complete proof. Several trials have been made to formulate and prove it at least for certain special cases. They all have the overall structure: "For a geometrically defined class of space-times and physically motivated properties of the energy-momentum tensor, all the solutions of the gravitational field equation tend asymptotically to a space of constant curvature."

1.1 Historical notes

It is the aim of this paper to clarify the relations between the several existing versions, and then to develop the cosmological no hair theorem towards applicability to a certain class of higher order field equations. Let us start with some historical notes.

The paper [1] by Weyl (1927) is cited in [2] with the phrase "The behaviour of every world satisfying certain natural homogeneity conditions in
the large follows the de Sitter solution asymptotically," to be the first published version of the no hair conjecture.

Barrow and Götz [2] apply the formulation "All ever-expanding universes with $\Lambda > 0$ approach the de Sitter space-time locally."

(Ever–expanding to be meant as: there is a time $t_0$ such that for all $t > t_0$ the Hubble parameter is positive. In other words: a bounce is allowed, a recollapse is not allowed.)

Let us comment this formulation: So they circumvent the necessity to distinguish the initial data between expanding and recollapsing ones, but their formulation needs a further explanation; example: If one changes the initial data continuously from recollapsing to ever-expanding ones, then one gets a critical value of the initial data between them, where one has an ever-expanding model which need not tend to the de Sitter space-time but can have typically a linear expansion law. So these critical values of the initial data have to be excluded, too.

The first proof of the stability of the de Sitter solution (here: within the steady-state theory), is due to Hoyle and Narlikar [3]. In the papers [4] by Price perturbations of scalar fields have been considered for the no hair theorem. The probability of inflation is large if the no hair theorem is valid, cf. [5].
1.2 Physical properties

Peter, Polarski and Starobinsky [6] compared the double-inflationary models with cosmological observations. Barrow and Götz discussed the no hair conjecture within Newtonian cosmological models [2,7]. Hübner and Ehlers considered inflation in an open Friedman universe and have noted that inflationary models need not to be spatially flat [8].

Gibbons and Hawking have found two of the earliest strict results on the no hair conjecture for Einstein’s theory [9] in 1977. Barrow gave examples that the no hair conjecture fails if the energy condition is relaxed and points out, that this is necessary to solve the graceful exit problem. He uses the formulation of the no hair conjecture ”in the presence of an effective cosmological constant (e.g. from viscosity) the de Sitter space-time is a stable asymptotic solution”. This is a much weaker statement because only space-times in a neighbourhood of the de Sitter space-time are involved. He mentioned that an ideal fluid with equation of state \( p = -\rho \) is equivalent to a \( \Lambda \)-term in some cases but not always [10].

Usually, energy inequalities are presumed for formulating the no hair conjecture. Nakao, Shiromizu and Maeda [11] found some cases where it remains valid also for negative Abbott-Deser mass [the latter goes over to the well-known ADM-mass (Arnowitt, Deser, Misner) for \( \Lambda \rightarrow 0 \)]. They cite Murphy [12]. In [12], viscosity terms as source are considered to get a singularity-free cosmological model. Murphy [12] used Einstein’s theory, and Oleak [12] made similar considerations within Treder’s theory of gravity.
In the Eighties, these non-singular models with viscosity where re-interpreted as inflationary ones, cf. e.g. [13].

In the three papers [14] Prigogine et al. developed a phenomenological model of particle and entropy creation. It allows particle creation from space-time curvature, but the inverse procedure (i.e. particle decay into space–time curvature) is forbidden. This breaks the $t \rightarrow -t$-invariance of the model. Within that model, the expanding de Sitter space-time is an attractor solution independently of the initial fluctuations; this means, only the expanding de Sitter solution is thermodynamically possible. To these papers cf. also [15].

Vilenkin [16] discussed future-eternal inflating universe models; they must have a singularity if the condition D: "There is at least one point $p$ such that for some point $q$ to the future of $p$ the volume of the difference of the pasts of $p$ and $q$ is finite" is fulfilled.

Mondaini and Vilar have considered recollapse and the no hair conjecture in closed higher-dimensional Friedman models [17]. Pullin [18] discussed relations between the onset of black hole formation and the no hair conjecture. Concerning the no hair conjecture Shiromizu, Nakao, Kodama and Maeda [19] gave the following argument: If the matter distribution is too clumpy, then a large number of small black holes appears. Then one should look for an inflationary scenario where these black holes are harmless. They cannot clump together to one giant black hole because of the exponential expansion of the universe; this explains the existing upper bound of black holes in the
quasi-de Sitter model. Shibata, Nakao, Nakamura and Maeda have considered asymptotic gravitational waves in an axially symmetric quasi de Sitter space-time [20]. They use numerical methods. The magnitude of black holes is restricted: above $M_{\text{crit}} = \frac{1}{3\sqrt{\Lambda}}$ there do not exist horizons; this restriction one gets by considering a perturbed Schwarzschild-de Sitter-solution. The cosmic hoop conjecture expresses that the mass of a black hole in a quasi de Sitter model is bounded from above by $M_{\text{crit}} = \frac{1}{3\sqrt{\Lambda}}$, and its surface is analogously restricted.

The notion "quantum hair" means quantum numbers presenting quantum fields which should be classically forbidden if the no hair theorem is valid. Coleman, Preskill and Wilczek found examples of quantum hairs on black holes [21].

Xu, Li and Liu [22] proved the instability of the anti-de Sitter space-time (classical instability against gravitational waves, and dust matter perturbations); one has $\Lambda < 0$, and in an open Friedman model the scale factor $a$ in dependence of synchronized time $t$ reads $a = \alpha \cos \frac{t}{\alpha}$ where $\Lambda = -3/\alpha^2$. The anti-de Sitter model has closed time-like curves everywhere; a Cauchy horizon is the surface where closed time-like curves begin to exist, and therefore, the anti-de Sitter model has no Cauchy horizon. (Of course, a closed curve has no beginning; the formulation means: The Cauchy horizon is the topological boundary of the set of point possessing the property that they are contained in a closed time–like curve.)

Coley and Tavakol discussed the robustness of the cosmic no hair conjec-
ture under using the concept of the structural stability [23] (compare with chapter 5 below).

1.3 Fourth-order gravity

Sirousse-Zia considered the Bianchi type IX model in Einstein’s theory with a positive \( \Lambda \)-term and got an asymptotic isotropization of the mixmaster model [24]. She cites (and uses methods of) Belinsky, Lifshitz and Khalatnikov [25]. Müller [26] used \( L = R^2 \) and discussed the power-asymptotes of Bianchi models. Barrow and Sirousse-Zia [27] discussed the mixmaster \( R^2 \)-model and the question, under which conditions the Bianchi type IX model becomes asymptotic de Sitter?

Yokoyama and Maeda [28] considered the no hair conjecture for Bianchi type IX models and Einstein’s theory with a cosmological term. They discussed \( R^2 \) inflation in anisotropic universe models and got as a result that typically, an initial anisotropy helps to enhance inflation. For Bianchi type IX they got some recollapsing solutions besides those converging to the de Sitter solution.

Cotsakis, Demaret and de Rop [29] discussed the mixmaster universe in fourth-order gravity. To take the metric diagonal they write ”is probably well justified”; they discuss all types of curvature-squared terms. Paper [27] is continued in [30] by Spindel, where also general Bianchi type I models in general dimensions are considered.

Gurovich et al. [31] considered \( L = R + \alpha R^{4/3} \) to get a singularity-
free model in 1970, the solutions are of a quasi de Sitter type. One should remember that in spite of de Sitter’s papers in the twenties, the inflationary cosmological model became generally accepted in only in 1979/80.

The papers [32], [33] consider the no hair conjecture for $R^2$ models, they use the formulation "asymptotical de Sitter, at least on patch". The restriction "on patch" is not strictly defined but refers to a kind of local validity of the statement, e.g., in a region being covered by one single synchronized system of reference in which the spatial curvature is non-positive and the energy conditions are fulfilled. The Starobinsky model is outlined as one which does not need an additional inflaton field to get the desired quasi de Sitter stage. One should observe a notational change: There, $L = R + aR^2 \ln R$ was called Starobinsky model, whereas $L = R + aR^2$ got the name "improved Starobinsky model" - but now the latter carries simply the name "Starobinsky model". (For the inflationary phase, both versions are quite similar.) A further result of the papers [32] is that by the addition of a cosmological term, the Starobinsky model leads naturally to double inflation. Let us comment this result: It is correct, but one should add that this is got at the price of getting a "graceful exit problem" (by this phrase there is meant the problem of how to finish the inflationary phase dynamically) - in the Starobinsky model this problem is automatically solved by the fact that the quasi de Sitter phase is a transient attractor only. The papers [34] discuss the no hair conjecture within $R^2$-models and found inflation as a transient attractor in fourth order gravity. The papers [32] and [35], [37] discuss the
stability of inflation in $R^2$-gravity. The papers [36] discuss generalized cosmic no hair theorems for quasi exponential expansion. In Starobinsky [38] the no hair theorem for Einstein’s theory with a positive $\Lambda$-term is tackled by using a sequence as ansatz to describe a general space-time. However, the convergence of the sequence is not rigorously proven.

Starobinsky and Schmidt [39] have generalized the ansatz of Starobinsky [38] to consider also the no hair theorem for $L = R^2$.

Shiromizu et al. [19] discussed an inflationary inhomogeneous scenario and mention the open problem how to define asymptotical de Sitter space-times. In Pacher [40] it is mentioned that only a local version of this conjecture can be expected to hold true, and that neither the definition of asymptotic de Sitter nor the necessary presumptions to the energy-momentum tensor are clarified - two problems which are not finally solved up to now. The authors of [41] consider the no hair theorem for a special class of inhomogeneous models and give partial proofs. Morris [42] considers inhomogeneous models for $R + R^2$-cosmology. In [43] inflation in inhomogeneous but spherically symmetric cosmological models is obtained only if the Cauchy data are homogeneous over several horizon lengths. The analogous problem is considered in [44] also with inclusion of colliding plane gravitational waves, they give a numerical support of the no hair conjecture by concentrating on the dynamics of gravitational waves.

Berkin [45] gets as further result, that for $L = f(R)$, a diagonal Bianchi metric is always possible. Similarly, Barrow and Sirousse-Zia [27] and Spin-
del [30] worked on diagonalization problem. They apply the diagonalisation condition of MacCallum et al. [46]. In 1918 Kottler [47] found a simple closed-form static spherically symmetric vacuum solution for Einstein’s theory with Λ-term in Schwarzschild coordinates.

\[ ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  

(1.1)

with \( A(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 \). At the horizon \( A = 0 \) the Killing vector changes its sign and one gets by interchanging the coordinates \( t \) and \( r \) the corresponding Kantowski-Sachs model. Moniz [48] (1993) discusses the cosmic no hair conjecture within Kantowski-Sachs models and \( \Lambda > 0 \). He gets the de Sitter space-time not only asymptotically, but exactly in an anisotropic 3+1-slicing of space-time. He discusses the initial data that lead to a recollapse and find them to be very rare; but the measure he uses is not well-defined, so, possibly, this is not the last word. It is curious to observe that he works with complicated elliptic integrals instead of applying the Schwarzschild-de Sitter-solution found by Kottler [47] in 1918.

[49] gives an overview about the geometry of the de Sitter space-time.

### 1.4 Sixth and higher order models

The paper [50] by Buchdahl (1951) deals with lagrangians of arbitrarily high order. Its results are applied in paper [51] to general Lagrangians \( F(R, \Box) \). From another motivation, Bollini et al. [52] consider higher-order field theo-
ries of the type
\[ \sum a_s \Box^s \phi(x) = 0 \]
and give solutions in the sense of distributions.

Forgacs et al. [53] consider the non-local lagrangian \( R^{\frac{1}{2}}R - M \) as Wess Zumino Witten model.

The paper [54] by Vilkovisky was presented at the A. Sacharov-memorial conference held in Moscow in May 1991. In [54], the Sacharov-approach was generalized. The original idea of Sacharov (in 1967) was to define higher order curvature corrections to the Einstein action to get a kind of elasticity of the vacuum. Then the usual breakdown of measurements at the Planck length (such a short de Broglie wave length corresponds to such a large mass which makes the measuring apparatus to a black hole) is softened. Vilkovisky discusses the effective gravitational action in the form \( Rf(\Box)R \), where

\[ f(\Box) = \int \frac{1}{\Box - x} \rho(x) dx \]

Martin and Mazzitelli [55] discuss the non-local Lagrangian \( R^{\frac{1}{2}}R \) as conformal anomaly in two dimensions.

Let us now come to sixth–order equations. Stelle [56] (1977) considers mainly fourth order \( R^2 \)-models; in the introduction he mentioned that in the next order, terms like \( R^3 + R_{ij;k}R^{ij;k} \) become admissible, but the pure \( R^3 \)-term is not admissible. Treder [57] used higher-order lagrangians, especially \( R^2 \)-terms, and he mentioned that for \( R + R_g R_{,k} g_{,ik} \) a sixth-order field equation appears. Remark: This lagrangian leads to the same field equation as
\( R - R \Box R. \)

In [58], inflationary models with a term \( R_i R^i \) in the action are considered, but they do not vary with respect to the metric, and so no sixth-order term in the field equation appears.

Lu and Wise [59] consider the gravitational Lagrangian as a sequence \( S = S_0 + S_1 + S_2 + \ldots \) ordered with respect to physical dimension. So, \( S_0 = R \) and \( S_1 \) sums up the \( R^2 \)-terms. They try to classify the \( S_2 \)-terms; however, their identity (8) is not correct, so they erroneously cancel the essential term \( R \Box R. \)

Kirsten et al. [60] consider the effective lagrangian for self-interacting scalar fields; in the renormalized action, the term

\[
\frac{\Box R}{c + R}
\]

appears. Wands [61] classifies lagrangians of the type \( F(R, \Phi) \Box R \) and mentions that not all of them can be conformally transformed to Einstein’s theory. Ref. [62] considers the lagrangian \( \Phi^2 \Box R, [63] \) the Lagrangian \( R \Box R, [64, 65] \) double inflation from \( \Phi \) and \( R^2 \)-terms, also the \( R \Box R \)-terms is discussed. Besides \( R \Box R \) Berkin [65] considers the de Sitter space-time as attractor solution for field equations where the variational derivative of the term \( C_{ijkl} C^{ijkl} \) is included. The state of the art of the lagrangian \( R \Box R \) can be found in the papers [64-68].

The paper is organized as follows: Sect. 2 compares several possible definitions of an asymptotic de Sitter space-time, sect. 2.1. for the set of spatially
flat Friedman models, sct. 2.2. for less symmetric models. Sct. 3 deals with
the Lagrangian and corresponding field equations for higher–order gravity.
In sct. 4, we determine under which circumstances the Bianchi models in
higher–order gravity can be written in diagonal form without loss of gen-
erality; the answer will be more involved than the analogous problem for
General Relativity. In sct. 5 we discuss the results from the point of view of
structural stability in the sense of the ”Fragility”–paper [23].

2 Definitions of an asymptotic de Sitter
space–time

In this section we want to compare some possible definitions of an asymptotic
de Sitter space–time.

2.1 Spatially flat Friedman models

Let us consider the metric

$$ds^2 = dt^2 - e^{2\alpha(t)} \sum_{i=1}^{n} d(x^i)^2$$

(2.1)

which can be called spatially flat Friedman model in $n$ spatial dimensions.

We consider all values $n \geq 1$, but then

concentrate on the usual case $n = 3$. If $n \leq 3$ we often write $x$, $y$, and
$z$ instead of $x^1$, $x^2$, and $x^3$, resp. For metric (2.1) we define the Hubble
parameter $H = \dot{\alpha} \equiv \frac{d\alpha}{dt}$. We get

$$R_{00} = -n\left(\frac{dH}{dt} + H^2\right), \quad R = -2n\left(\frac{dH}{dt} + mH^2\right) \quad (2.2)$$

where $m := \frac{n+1}{2} \geq 1$. We get
Lemma 1: The following conditions for metric (2.1) are equivalent.

A: It is flat.
B: $R = R_{00} = 0$.
C: $R_{ij}R^{ij} = 0$.
D: $\alpha = \text{const.}$ or $[n = 1 \text{ and } \alpha = \ln|t - t_0| + \text{const.}]$

Proof: A $\Rightarrow$ B is trivial; B $\Rightarrow$ D is done by solving the corresponding differential equation; D $\Rightarrow$ A is trivial for $\alpha = \text{const.}$, the other case, i.e., $ds^2 = dt^2 - (t - t_0)^2dx^2$, represents flat space–time in polar coordinates;

C $\Leftrightarrow$ B follows from the identity

$$R_{ij}R^{ij} = (R_{00})^2 + \frac{1}{n}(R - R_{00})^2$$ (2.3)

An analogous statement can be formulated for the de Sitter space–time.

Lemma 2: The following conditions for metric (2.1) are equivalent.

A: It is a non–flat space–time of constant curvature.
B: $R_{00} = \frac{R}{n+1} = \text{const.} \neq 0$.
C: $(n + 1)R_{ij}R^{ij} = R^2 = \text{const.} \neq 0$.
D: $H = \text{const.} \neq 0$ or $[n = 1 \text{ and } ds^2 = dt^2 - \sin^2(\lambda t)dx^2 \text{ or } ds^2 = dt^2 - \sinh^2(\lambda t)dx^2 ]$

The proof is analogous to lemma 1.

For $n = 1$, the de Sitter space–time and anti-de Sitter space–time differ by the factor $(-1)$ in front of the metric only. For $n > 1$, under the presumption of lemma 2, only the de Sitter space–time ($R < 0$) is covered. Lemma 2 shows that within the class of spatially flat Friedman models, a characterization of the de Sitter space–time using polynomial curvature invariants only, is
Next, let us look for isometries leaving the form of the metric (2.1) invariant. The function

$$\tilde{\alpha}(t) = c + \alpha(\pm t + t_0), \quad c, t_0 = \text{constants}, \quad (2.4)$$

leads to an isometric space-time. The simplest expressions being invariant by such a transformation are $H^2$ and $\dot{H}$. We take $\alpha$ as dimensionless, then $H$ is an inverse time and $\dot{H}$ an inverse time squared. Let $H \neq 0$ in the following. The expression

$$\varepsilon := \dot{H}H^{-2} \quad (2.5)$$

is the simplest dimensionless quantity defined for the spatially flat Friedman models (2.1) and being invariant with respect to the isometries (2.4). Let $n > 1$ in the following: Two metrics of type (2.1) are isometric if and only if the corresponding functions $\alpha$ and $\tilde{\alpha}$ are related by equation (2.4). All dimensionless invariants containing at most second order derivatives of the metric can be expressed as $f(\varepsilon)$, where $f$ is any given function. But if one has no restriction to the order, one gets a sequence of further invariants

$$\varepsilon_2 = \ddot{H}H^{-3}, \ldots, \varepsilon_p = \frac{d^p H}{dt^p} H^{-p-1} \quad (2.6)$$

It holds: Metric (2.1) with $H \neq 0$ represents the de Sitter space-time iff $\varepsilon \equiv 0$.

A third possible approach is the following: $\alpha(t) = Ht$

with $H = \text{const.} \neq 0$ is the de Sitter space–time, so we define an asympt-
totic de Sitter space-time by the condition

$$\lim_{t \to \infty} \frac{\alpha(t)}{t} = \text{const.} \neq 0$$  \hspace{1cm} (2.7)

Let us summarize the variants Var(i) of the definitions.

**Definition:** Let $H > 0$ in metric (2.1) with $n > 1$. We call it an asymptotic de Sitter space-time if

Var (1): $\lim_{t \to \infty} \frac{\alpha(t)}{t} = \text{const.} > 0$

Var (2): $\lim_{t \to \infty} R^2 = \text{const.} > 0$ and $\lim_{t \to \infty} (n + 1)R_{ij}R^{ij} - R^2 = 0$

Var (3p): for $1 \leq j \leq p$ it holds $\lim_{t \to \infty} \varepsilon_p = 0$. All these definitions are different. One uses $R_{ij}R^{ij} = n^2(\dot{H} + H^2)^2 + n(\dot{H} + nH^2)^2$. However, as we will see in section 3.2, all these definitions lead to the same result if we restrict ourselves to the set of solutions of the higher-order field equations.

### 2.2 Inhomogeneous cosmological models

Let us start looking at the Kottler metric eq. (1.1). The critical mass $M_{\text{crit}} = \frac{1}{3\sqrt{\Lambda}}$ mentioned in subsection 1.3. in connection with the hoop conjecture can be deduced (at least for the symmetries of the Kottler metric as follows: At a horizon, the function $A$

must vanish. One can see from eq. (1.1) that for $\Lambda > 0$ the equation $A = 0$ has solutions with positive values $r$ if and only if $m \leq M_{\text{crit}}$. This means: the hoop conjecture is valid in the class of spherically symmetric solutions.

However, this is not the problem we are dealing with here. The
problem is that none of the above definitions can be generalized to inhomogeneous models. One should find a polynomial curvature invariant which equals a positive constant if and only if the space–time is locally the de Sitter space–time. To our knowledge, such an invariant cannot be found in the literature, but also the non–existence of such an invariant has not been proven up to now.

This situation is quite different for the positive definite case: For signature $(++++)$ and $S_{ij} = R_{ij} - \frac{4}{3}g_{ij}$ it holds:

$$I \equiv (R - R_0)^2 + C_{ijkl}C^{ijkl} + S_{ij}S^{ij} = 0$$

iff the $V_4$ is a space of constant curvature $R_0$. So $I \to 0$ is a suitable definition of an asymptotic space of constant curvature.

One possibility exists, however, for the Lorentz signature case, if one allows additional structure as follows: An ideal fluid has an energy–momentum tensor

$$T_{ij} = (\rho + p)u_iu_j - pg_{ij}$$

where $u_i$ is a continuous vector field with $u_iu^i \equiv 1$. For stiff matter ($\rho = -p$), the equation $T^{ij}_{\;\;ij} \equiv 0$ implies $p = const.$, and so every solution of Einstein’s theory with stiff matter is isometric to a vacuum solution of Einstein’s theory with a cosmological term. The inverse statement, however, is valid only locally:

Given a vacuum solution of Einstein’s theory with a $\Lambda$–term, one has to find continuous time–like unit vector fields which need not to exist from topo-
Logical reasons. And if they exist, they are not at all unique. So, it becomes possible to define an invariant $J$ which vanishes iff the space–time is de Sitter by transvecting the curvature tensor with $u^iu^j$ and/or $g^{ij}$ and suitable linear and quadratic combinations of such terms. Then time $t$ becomes defined by the streamlines of the vector $u^i$. If one defines the asymptotic de Sitter space–time by $J \to 0$ as $t \to \infty$, then it turns out, that this definition is not independent of the vector field $u^i$.

3 Lagrangian $F(R, \Box R, \Box^2 R, \ldots, \Box^k R)$

Let us consider the Lagrangian density $L$ given by

$$L = F(R, \Box R, \Box^2 R, \ldots, \Box^k R)\sqrt{-g}$$

(3.1)

where $R$ is the curvature scalar, $\Box$ the D’Alembertian and $g_{ij}$ the metric of a (Pseudo-)Riemannian $V_D$ of dimension $D \geq 2$ and arbitrary signature; $g = -|\det g_{ij}|$. The main application will be $D = 4$ and metric signature (+ − − −). $F$ is supposed to be a sufficiently smooth function of its arguments, preferably a polynomial. Buchdahl [50] already dealt with such kind of Lagrangians.

In 1951, but then it became quiet of them for decades. Since 1990 a sequence of papers on this topic appeared: refs. [51, 68] for general $k$, and refs. [53 - 67] for the special case $k = 1$, i.e. the Lagrangian scalar is $F(R, \Box R)$. 

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3.1 The field equation

The variational derivative of \( L \) with respect to the metric yields the tensor

\[
P^{ij} = -\frac{1}{\sqrt{-g}} \frac{\delta L}{\delta g_{ij}}
\]  

(3.2)

The components of this tensor were given in the first paper of ref. [51], their covariant components read

\[
P_{ij} = GR_{ij} - \frac{1}{2} F g_{ij} - G_{;ij} + g_{ij} \Box G + X_{ij}
\]  

(3.3)

where the semi-colon denotes the covariant derivative, \( R_{ij} \) the Ricci tensor, and

\[
X_{ij} = \sum_{A=1}^{k} \frac{1}{2} g_{ij} [F_A(\Box^{A-1} R)^m]_{;m} - F_A[i[\Box^{A-1} R]_{;ij}]
\]  

(3.4)

having the round symmetrization brackets in its last term. For \( k = 0 \), i.e. \( F = F(R) \), a case considered in sect. 4, the tensor \( X_{ij} \) identically vanishes. It remains to define the expressions \( F_A, A = 0, \ldots, k \). The definition given in [51] can be simplified as follows

\[
F_k = \frac{\partial F}{\partial \Box^k R}
\]  

(3.5)

and for \( A = k - 1, \ldots, 0 \)

\[
F_A = \Box F_{A+1} + \frac{\partial F}{\partial \Box^A R}
\]  

(3.6)

and finally \( G = F_0 \). The brackets are essential, for any scalar \( \Phi \) it holds

\[
\Box (\Phi_i) - (\Box \Phi)_i = R^j_i \Phi_{;j}
\]  

(3.7)
Inserting $\Phi = \square^m R$ into this equation, one gets identities to be applied in the sequel without further notice.

It is well-known that

$$P^i_{j;i} \equiv 0 \quad (3.8)$$

and $P_{ij}$ identically vanishes if and only if $F$ is a divergence, i.e., locally there can be found a vector $v^i$ such that $F = v^i_{;i}$ holds. (Remark: Even for compact manifolds without boundary the restriction ”locally” is unavoidable; example: Let $D = 2$ and $V_2$ be the Riemannian two-sphere $S^2$ with arbitrary positive definite metric. $R$ is a divergence, but there do not exist continuous vector fields $v^i$ fulfilling $R = v^i_{;i}$ on the whole $S^2$.)

Example: for $m, n \geq 0$ it holds

$$\square^m R \square^n R - R \square^{m+n} R = divergence. \quad (3.9)$$

So, the terms $\square^m R \square^n R$ with naturals $m$ and $n$ can be restricted to the case $m = 0$ without loss of generality. However, the more far-reaching statement by Wands [61, page 271] ”Thus I can take any polynomial $F(\square^i R)$ to be linear in its highest–order derivative $\square^n R$, multiplied by $F_n(R)$” is not correct. Let us give a counterexample: $R\square R \square R$, which leads to an eighth–order field equation.

Proof that this is a counterexample: From dimensional reasons only ingredients with $< length >^{-10}$ are to be considered. Neglecting the divergencies, only the following ones are candidates: $R^5$, $R^3 \square R$, $R^2 \square^2 R$, $R \square^3 R$. They give rise to field equations of orders 4, 6, 8, and 10 resp. So the last term
cannot be included. It remains to look for

\[ F = R \square R \square R + \gamma R^5 + \beta R^3 \square R + \alpha R^2 \square^2 R \]

The variation of \( F \) with respect to the metric should vanish identically. Vanishing of the 8th–order term requires \( \alpha = -\frac{1}{2} \). Vanishing of the 6th–order terms gives rise to the equation

\[ (\square R + \frac{3\beta}{2} R^2)(\square R);ij = 0 \]

For no value of \( \beta \) this is identically satisfied.

### 3.2 Higher-order gravity

We will examine the attractor property of the de Sitter space-time in the set of the spatially flat Friedman models. We need some useful relations for the de Sitter space-time:

\[ R = -n(n+1)H^2 \quad (3.10) \]

and

\[ R_{ij} = \frac{R}{n+1} g_{ij} \quad \text{and} \quad \Box^k R = 0 \quad \text{for} \quad k > 0. \quad (3.11) \]

We insert this into the field equation (3.3)

\[ 0 = GR_{ij} - \frac{1}{2} F g_{ij} = g_{ij} \left( \frac{1}{n+1} RG - \frac{1}{2} F \right) \quad (3.12) \]

for the de Sitter space-time. The de Sitter space-time solves the field equation if and only if \( 2RG = DF \). If we choose the lagrangian \((-R)^u\) with \( u \in \mathbb{R} \) the \( D \)-dimensional de Sitter space-time satisfies the field equation
iff $u = \frac{n+1}{2} = \frac{D}{2}$. We will examine the attractor property of the de Sitter space-time in the set of the Friedman models for the lagrangian $(-R)^u$ with $2u = D = n + 1 > 2$. From this lagrangian it follows

$$F_A = 0$$

(3.13)

and

$$G = -u(-R).$$

(3.14)

We get the field equation

$$0 = -u(-R)^{u-1}R_{ij} - \frac{1}{2}g_{ij}(-R)^u + u\left[(-R)^{u-1}\right]_{;ij} - g_{ij}u\Box\left[(-R)^{u-1}\right].$$

(3.15)

It is enough to examine the 00-component of the field equation, because all the other components are fulfilled, if the 00-component is fulfilled. We make the ansatz

$$\dot{\alpha}(t) = 1 + \beta(t)$$

(3.16)

and get

$$R_{00} = -n\dot{\beta} - 2n\beta - n$$

$$R = -2n\dot{\beta} - 2(n^2 + n)\beta - (n^2 + n)$$

$$(-R)^m = 2nm(n^2 + n)^{m-1}\dot{\beta} + 2m(n^2 + n)^m\beta + (n^2 + n)^m.$$ 

(3.17)

One gets the field equation

$$0 = -2n^2u(u+1)(n^2 + n)^{u-2}\dot{\beta} - 2n^3u(u-1)(n^2 + n)u - 2\dot{\beta} +$$

$$+nu(2u-n-1)(n^2 + n)^{u-1}\beta + \left(nu - \frac{1}{2}(n^2 + n)\right)(n^2 + n)^{u-1}.$$

(3.18)
Using the condition $2u = n + 1$ we get

$$0 = \ddot{\beta} + n\dot{\beta}.$$ \hfill (3.19)

All solutions of the linearized field equation are

$$\beta(t) = c_1 + c_2 e^{-nt}.$$ \hfill (3.20)

It follows

$$\alpha(t) = t + \tilde{c}_1 t + \tilde{c}_2 e^{-nt} + \tilde{c}_3$$ \hfill (3.21)

and

$$\lim_{t \to \infty} \frac{\alpha(t)}{t} = 1 + \tilde{c}_1.$$ \hfill (3.22)

The $D$-dimensional de Sitter space-time is an attractor solution for the lagrangian $F = (-R)^\frac{D}{2}$. The lagrangian $(-R)^\frac{D}{2}$ leads only to a field equation of fourth-order for $D > 2$. The lagrangian $R^k R$ with $k > 0$ gives a field equation of higher than fourth-order. For this case we get

$$F = R^k R, \quad G = 2^{k} R$$ \hfill (3.23)

and no solubility condition for $D > 2$. For the 00-component of the field equation we need

$$F_A = \Box^{k-A} R$$ \hfill (3.24)

and

$$G = 2^{k} R$$ \hfill (3.25)
and get

\[
0 = \Box^k R \left( 2R_{00} - \frac{1}{2} R \right) + 2n\dot{\alpha} \Box^k R_{,0} + \\
+ \sum_{A=1}^{k} (\Box^{k-A} R)(\Box^{A} R) - \frac{1}{2}(\Box^{k-A} R)_{,0}(\Box^{A-1} R)_{,0} .
\]  

(3.26)

The ansatz (3.16) leads to (3.17) and

\[
\Box (\Box^k R) = (\Box^k R)_{,00} + n(\Box^k R)_{,0} .
\]

(3.27)

We get the linearized field equation from (3.26)

\[
\Box^k R = (\Box^k R)_{,0} .
\]

(3.28)

For \( k = 1 \) we have

\[
0 = \beta^{(4)} + 2n\ddot{\beta} + (n^2 - n - 1)\dddot{\beta} + (-n^2 - n)\dot{\beta}
\]

(3.29)

with the characteristic polynomial

\[
P(t) = x^4 + 2nx^3 + (n^2 - n - 1)x^2 + (-n^2 - n)x
\]

(3.30)

possessing the roots \( x_1 = 1, \ x_2 = 0, \ x_3 = -n \) and \( x_4 = -n - 1 \). We get the solutions

\[
\beta(t) = c_1 + c_2e^t + c_3e^{-nt} + c_4e^{-(n+1)t}
\]

(3.31)

and

\[
\alpha(t) = \tilde{c}_1 t + \tilde{c}_2 e^t + \tilde{c}_3 e^{-nt} + \tilde{c}_4 e^{-(n+1)t}
\]

(3.32)

and

\[
\lim_{t \to \infty} \frac{\alpha(t)}{t} = \infty
\]

(3.33)
unless \( \tilde{c}_2 = 0 \). For the lagrangian \( R \Box^k R \) the \( D \)-dimensional de Sitter space-time is not an attractor solution of the field equation. The formula

\[
0 = (\Box^{k+1} R),_0 - \Box^{k+1} R = (\Box^k R,,_0 - \Box^k R),_00 + n((\Box^k R,)_0 - \Box^k R)_0 \tag{3.34}
\]

for the linearized field equation for \( k + 1 \) leads to the recursive formula for the characteristic polynomial:

\[
\text{characteristic polynomial for } k + 1 = \text{characteristic polynomial for } k \cdot x \cdot (x + n) .
\]

The characteristic polynomial for \( k \) has the roots:

\[
\begin{align*}
x_1 &= 1 \quad \text{simple} \\
x_2 &= 0 \quad \text{k-fold} \\
x_3 &= -n \quad \text{k-fold} \\
x_4 &= -n - 1 \quad \text{simple} .
\end{align*}
\]

(3.35)

We get the solutions

\[
\beta(t) = S(t) + T(t)e^{-nt} + c_1e^t + c_2e^{(-n-1)t} \tag{3.36}
\]

and

\[
\alpha(t) = \tilde{S}(t) + \tilde{T}(t)e^{-nt} + \tilde{c}_1e^t + \tilde{c}_2e^{(-n-1)t} \tag{3.37}
\]

with \( S, T, \tilde{T} \) polynomials at most \( k \)-th degree and \( \tilde{S} \) polynomial at most \( k + 1 \)-th degree. For most of all solutions we get

\[
\lim_{t \to \infty} \frac{\alpha(t)}{t} = \infty \tag{3.38}
\]

and therefore, the de Sitter space-time is not an attractor solution for the field equation derived from the lagrangian \( R \Box^k R \).
These results have shown, that for the lagrangian $R \Box^k R$ with $k > 1$ the de Sitter space-time is not an attractor solution. The lagrangian $(-R)^{\frac{n}{2}}$ gives only a fourth-order differential equation. We will try to answer the following question:

Are there generalized lagrangians so, that the de Sitter space-time is an attractor solution of the field equation?

First we make the ansatz

$$F = \sum_{k=1}^{m} c_k R \Box^k R \quad \text{with} \quad c_m \neq 0 . \quad (3.39)$$

In this case the de Sitter space-time is not an attractor solution, because for each term there one gets $+1$ as a root of the characteristic polynomial of the linearized field equation.

Now we make the ansatz

$$F = c_0 (-R)^{\frac{n}{2}} + \sum_{k=1}^{m} c_k R \Box^k R \quad \text{with} \quad c_m \neq 0 . \quad (3.40)$$

for the generalized lagrangian. One gets the characteristic polynomial

$$\tilde{P}(x) = x(x + n) \left[ c_0 + \sum_{k=1}^{m} c_k x^{k-1} (x + n)^{k-1} (x - 1)(x + n + 1) \right] \quad (3.41)$$

for the linearized field equation. The solutions $x_1 = 0$ and $x_2 = -n$ do not depend on the coefficients $c_i$ of the lagrangian. It is sufficient to look for the roots of the polynomial

$$P(x) = c_0 + \sum_{k=1}^{m} c_k x^{k-1} (x + n)^{k-1} (x - 1)(x + n + 1) . \quad (3.42)$$
If all solutions of this polynomial have negative real part, then the de Sitter space-time is an attractor solution for the field equation. The transformation

\[ z = x^2 + nx + \frac{n^2}{4} \]  

(3.43)

gives
\[ P(x) = Q(z) = \]
\[ = c_0 + \sum_{k=1}^{m} c_k \left( z - \frac{n^2}{4} \right)^{k-1} \left( z - \frac{n^2}{4} - n - 1 \right) \]
\[ = c_0 + \sum_{k=1}^{m} c_k \left( \frac{n^2}{4} \right)^{k-1} \left( \frac{n^2}{4} - n - 1 \right) + \sum_{l=1}^{m-1} \left[ c_l + \sum_{k=l+1}^{m} c_k \left( -\frac{n^2}{4} \right)^{k-l-1} \right] \left( \frac{n^2}{4} + n + 1 \right) \]
\[ = d_0 + d_1 z + \ldots + d_m z^m . \] (3.44)

Now let
\[ a_{ll} = 1 \quad l = 0, \ldots, m \]
\[ a_{0k} = -\left( -\frac{n^2}{4} \right)^{k-1} \left( \frac{n^2}{4} + n + 1 \right) \quad k = 1, \ldots, m \]
\[ a_{lk} = \left( -\frac{n^2}{4} \right)^{k-l-1} \left[ \binom{k-1}{l-1} \left( -\frac{n^2}{4} \right)^{k-l} \left( \frac{n^2}{4} + n + 1 \right) \right] \quad l < k \leq m \]
\[ a_{kl} = 0 \quad \text{else} . \]

This gives the equation
\[
\begin{pmatrix}
  d_0 \\
  \vdots \\
  d_m
\end{pmatrix} = A
\begin{pmatrix}
  c_0 \\
  \vdots \\
  c_m
\end{pmatrix} \quad \text{with } A \text{ regular .} \quad \text{(3.46)}
\]

The roots of \( P(x) \) have a negative real part iff the roots of \( Q(z) \) are from the set
\[
M := \left\{ x + iy : x > \frac{n^2}{4} \land |y| < n \sqrt{x - \frac{n^2}{4}} \right\} .
\] (3.47)

If the roots \( z_k \) of the polynomial \( Q(z) \) are elements of \( M \), then the coefficients \( d_k \) are determined by
\[
Q(z) = \sum_{k=0}^{m} d_k z^k = \prod_{k=1}^{m} (z - z_k) . \] (3.48)
The coefficients
\[
\begin{pmatrix}
c_0 \\
\vdots \\
c_m
\end{pmatrix} = A^{-1} \begin{pmatrix}
d_0 \\
\vdots \\
d_m
\end{pmatrix}
\] (3.49)

belong to a lagrangian, that gives a field equation with a de Sitter attractor solution. The above considerations have shown that for every \( m \) there exists an example for coefficients \( c_k \), so that the de Sitter space-time is an attractor solution for the field equation derived from the lagrangian

\[c_0(-R)^2 + \sum_{k=1}^{m} c_k R \square^k R \text{ with } c_m \neq 0.\]

It turned out that all the variants of the definition of an asymptotic de Sitter solution given in subsection 2.1. lead to the same class of solutions.

For the 6th–order case we can summarize as follows:

**Theorem 1:** Let \( L = R^2 + c_1 R \square R \) and \( L_E = R - \frac{l^2}{6} L \) with length \( l > 0 \). Then the following statements are equivalent.

1. The Newtonian limit of \( L_E \) is well–behaved, and the potential \( \phi \) consists of terms \( \frac{1}{r} e^{-\alpha r} \) with \( \alpha \geq 0 \) only.
2. The de Sitter space–time with \( H = \frac{1}{l} \) is an attractor solution for \( L \) in the set of spatially flat Friedman models, and this can already be seen from the linearized field equation.
3. \( c_1 \geq 0 \) and the graceful exit problem is solved for the quasi de Sitter phase \( H \leq 1/l \) of \( L_E \).
4. \( l^2 = l_0^2 + l_1^2 \), \( l^2 c_1 = l_0^2 l_1^2 \) has a solution with \( 0 \leq l_0 < l_1 \).
5. \( 0 \leq c_1 < \frac{l^2}{1}. \)
For the proof of 3. one needs eq. (18) of the first of papers [64] which reads in our notation $\dot{H}(1 - 4c_1 H^2) = -1/6l^2$ showing that $\dot{H} < 0$ at the quasi de Sitter stage.

**Theorem 2:** Let $L$ and $L_E$ as in theorem 1. Then are equivalent:

1. The Newtonian limit of $L_E$ is well–behaved, for the potential $\phi$ we allow $1/r$ and terms like $P(r)e^{-\alpha r}$ with $\alpha > 0$ and a polynomial $P$.

2. The de Sitter space–time with $H = \frac{1}{l}$ cannot be ruled out to be an attractor solution for $L$ in the set of spatially flat Friedman models if one considers the linearized field equation only.

3. $L_E$ is tachyonic–free.

4. $l^2 = l_0^2 + l_1^2$, $l^2 c_1 = l_0^2 l_1^2$ has a solution with $0 \leq l_0 \leq l_1$.

5. $0 \leq c_1 \leq \frac{l^2}{4}$.

Of course, it would be interesting what happens in the region where the linearized equation does not suffice to decide; one should even not try to answer this question without a computer algebra system.

## 4 Higher-order gravity and diagonalizability of Bianchi models

A Bianchi model can always be written as

$$ds^2 = dt^2 - g_{\alpha\beta}(t)\sigma^\alpha\sigma^\beta$$  \hspace{1cm} (4.1)

where $g_{\alpha\beta}$ is positive definite and $\sigma^\alpha$ are the characterizing one-forms. It holds

$$d\sigma^\gamma = -\frac{1}{2} C_{\alpha\beta}^\gamma \sigma^\alpha \wedge \sigma^\beta$$  \hspace{1cm} (4.2)
with structure constants $C_{\alpha\beta}^\gamma$ of the corresponding Bianchi type. It belongs to class A if $C_{\alpha\beta}^\beta = 0$. The abelian group (Bianchi type I) and the rotation group (Bianchi type IX) both belong to class A.

In most cases, the $g_{\alpha\beta}$ are written in diagonal form; it is a non-trivial problem to decide under which circumstances this can be done without loss of generality.

For Einstein’s theory, this problem is solved in [46]. One of its results read:

If a Bianchi model of class A (except types I and II) has a diagonal energy-momentum tensor, then the metric $g_{\alpha\beta}(t)$ can be chosen in diagonal form. Here, the energy-momentum tensor is called diagonal, if it is diagonal in the basis $(dt, \sigma^1, \sigma^2, \sigma^3)$.

This result rests of course on Einstein’s theory and cannot be directly applied to higher-order gravity.

For fourth-order gravity following from a Lagrangian $L = f(R)$ considered in an interval of $R$-values where

$$\frac{df}{dR} \cdot \frac{d^2 f}{dR^2} \neq 0$$

one can do the following: The application of the conformal equivalence theorem is possible, the conformal factor depends on $t$ only, so the diagonal form of metric (3.1) does not change. The conformal picture gives Einstein’s theory with a minimally coupled scalar field as source; the energy-momentum tensor is automatically diagonal. So, in this class of fourth-order theories of
gravity, we can apply the above cited theorem of MacCallum et al.

As example we formulate: All solutions of Bianchi type IX of fourth-order gravity following from \( L = R^2 \) considered in a region where \( R \neq 0 \) can be written in diagonal form.

Consequently, the ansatz used in [27] by Barrow and Sirousse-Zia for this problem is already the most general one, cf. [30] Spindel.

For fourth-order gravity of a more complicated structure, however, things are more involved; example: Let

\[
L = R + aR^2 + bC_{ijkl} C^{ijkl}
\]

with \( ab \neq 0 \). Then there exist Bianchi type IX models which cannot be written in diagonal form. (This is a non-trivial statement.)

To understand the difference between the cases \( b = 0 \) and \( b \neq 0 \) it proves useful to perform the analysis independently of the above cited papers [46]. For simplicity, we restrict to Bianchi type I. Then the internal metric of the hypersurface \( [t = 0] \) is flat and we can choose as initial value \( g_{\alpha\beta}(0) = \delta_{\alpha\beta} \). Spatial rotations do not change this equation, and we can take advantage of them to diagonalize the second fundamental form \( \frac{d}{dt} g_{\alpha\beta}(0) \).

First case: \( b = 0 \). As additional initial conditions one has only \( R(0) \) and \( \frac{d}{dt} R(0) \). The field equation ensures \( g_{\alpha\beta}(t) \) to remain diagonal for all times.

Second case: \( b \neq 0 \). Then one has further initial data \( \frac{d^2}{dt^2} g_{\alpha\beta}(0) \). In the generic case, they cannot be brought to diagonal form simultaneously with \( \frac{d}{dt} g_{\alpha\beta}(0) \). This excludes a diagonal form of the whole solution. (To complete
the proof, one has of course to check that these initial data are not in con-
tradiction to the constraint equations.) This case has the following relation
to the above cited theorem \[46\]: Just for this case \( b \neq 0 \), the conformal re-
lation to Einstein’s theory breaks down, and if one tries to re-interpret the
variational derivative of \( C_{ijkl}C^{ijkl} \) as energy-momentum tensor then it turns
out to be non-diagonal generically, and the theorem cannot be applied.

For higher-order gravity, the situation becomes even more involved. For
a special class of theories, however, the diagonalizability condition is exactly
the same as in Einstein’s theory: If \( L = R + \sum_{k=0}^{m} a_k R^k R \), \( (a_m \neq 0) \) then in
a region where \( 2L \neq R \) the Cauchy data are the data of General Relativity,
\( R(0) \), and the first \( 2m + 1 \) temporal derivatives of \( R \) at \( t = 0 \). All terms with
the higher derivatives behave as an energy-momentum tensor in diagonal
form, and so the classical theorem applies. [Let us comment on the restriction
\( 2L \neq R \) supposed above: Eqs. (3.5, 3.6) show that \( F_0 = G = 0 \) represents a
singular point of the differential equation (3.3); and for the lagrangian given
here \( G = \frac{2L}{R} - 1 \). For fourth-order gravity defined by a non-linear lagrangian
\( L(R) \) one has \( G = \frac{dL}{dR} \) and \( G = 0 \) defines the critical value of the curvature
scalar.]
5 Structural stability of fourth-order cosmological models

In [23], Coley and Tavakol discuss cosmological models from the point of view of structural stability; the notion for the contrary of it is fragility. Structural stability is a more general but less strictly defined notion than the usual stability. So, its concrete meaning has context-dependently to be specified.

1. Example: The Einstein universe (a closed Friedman model of constant world radius in General Relativity with positive cosmological term $\sim \Lambda$ and incoherent matter as source) is unstable with respect to the initial data: A non-vanishing but arbitrarily small initial Hubble parameter gives rise to a singularity. This property ruled out the Einstein universe as describing our real world. It should be emphasized that this is in coincidence with the observational result that our universe is not static, but that this theoretical stability analysis ruled out the Einstein universe independently of the observational result.

Structural stability represents stability not only with respect to a small perturbation in the initial data, but a small change in the corresponding type of matter and field equations. In most of the specifications one requires that by a small change of conditions the qualitative (or topological) properties of the system remain unchanged. Concerning field equations, Coley and Tavakol [23] concentrate on Lagrangians $L = f(R)$ for the gravitational field: For linear functions $f$ one gets General Relativity, for non-linear ones fourth-
order gravity. Because of the change in the order of the differential equation the question concerning the robustness of General Relativity is a non-trivial one. Before we follow this line we present some more or less trivial examples from General Relativity to clearify what is meant.

2. Example: The spatially flat Friedman model with incoherent matter (dust) but $\Lambda = 0$ (Einstein-de Sitter model) has a scale factor $a \sim t^{2/3}$ for synchronized time $t$. A small change of the initial data only changes the proportionality factor, so this is stable. However, if we consider this model within the class of all Friedman models, then it represents just the bifurcating point between the ever-expanding open and the recollapsing closed models. In this sense, the Einstein-de Sitter model is a fragile one.

3. Example: Again we consider the Einstein-de Sitter model within the class of all spatially flat Friedman models. We impose new structure by allowing a new contribution to the energy-momentum tensor in form of radiation not interacting with the dust. During expansion, the energy density of the radiation falls $\sim a^{-4}$, and of the dust only $\sim a^{-3}$. The radiation becomes asymptotically negligible, and, asymptotically for large values $t$, one gets approximately $a \sim t^{2/3}$. In this sense, the Einstein-de Sitter model is structurally stable.

4. Example: Now we invert the point of view from the third example. We start from a spatially flat Friedman filled with radiation. Then one has $a \sim t^{1/2}$. We impose new structure by adding an arbitrarily small amount of non-interacting dust. As in the previous example, we get asymptotically $a \sim$
In this sense, the spatially flat Friedman radiation model is structurally unstable.

Let us now come to the consideration of Coley and Tavakol concerning structural stability of fourth-order gravity models. They consider perturbations of Friedman’s radiation model within fourth-order gravity. For the non-tachyonic case, they get as result that the $R^2$-term gives rise to an instability. It is known for a long time, that asymptotically

the $R^2$-term gives rise to damped oscillations which behave as dust in the mean. So, the structural instability considered there is exactly the same as in the 4. example above and not a special feature of the fourth-order term.

Analogously they consider the quasi-de Sitter stage (Starobinsky inflation) and get its stability for the non-tachyonic case $L = R + aR^2$.

Remark: Coley, Tavakol [23] use the notion ”topological almost all” in the sense of ”countable intersection of open dense subsets”. One must be careful in applying this notion, especially, if one is tempted to mix it with the notion ”almost all” in measure theory. A remarkable example shall underline this warning: Let $I = [0, 1]$ be the closed interval with the usual probability measure $\mu$. Let $\{r_n | n \in N\} \subset I$ be a countable dense subset of $I$. For each natural $m$, let

$$A_m = I \cap \bigcup_{n \in N} [r_n - 2^{-m-n}, r_n + 2^{-m-n}]$$

and $A = \bigcap_{m \in N} A_m$. (Here, $]x, y[$ denotes the open interval.) Each $A_m$ is open
and dense in $I$. For all values $m$,

$$
\mu(A) \leq \mu(A_m) \leq \sum_{n \in \mathbb{N}} 2^{1-m-n} = 2^{1-m}
$$

Hence, $\mu(A) = 0$. So, $A$ contains topologically almost all points of $I$, but there is zero probability to meet an element of it.

Next, Coley and Tavakol consider structural stability of Starobinsky inflation $L = R + aR^2$ with respect to addition of the cubic term $bR^3$. For $L = R^3$ alone one gets polar inflation $a \sim t^{-10}$; considered in the region $t < 0$ this is expanding with $\dot{h} h^{-2} = \frac{1}{10}$. The term with $b$ does not alter the order of the differential equation, and so one expects a continuous change of the properties. In fact, for small values $|b|$ one has Starobinsky inflation as transient attractor, with increasing $|b|$ one gets a smaller basin of attraction, and for $|b| \gg a^2$ one needs fine-tuned initial conditions.

A more drastic change of structure is to be expected if we consider structural stability with respect to the addition of terms like $R \Box R$.

6 Discussion

Sudarsky [66] proves the no–hair theorem (in the version that there are no non–trivial black holes with regular horizon) for the Einstein–Higgs theory.

We have deduced a cosmic no hair theorem on a quite different footing as follows (the more detailed formulation is given at the end of sct. 3)
**Theorem:** Let \( L = R^2 + \epsilon l^2 R \Box R \) and

\[
L_E = \frac{1}{16\pi G} [R - \frac{l^2}{6} L] \text{ with length } l > 0 \text{ and arbitrary real } \epsilon. \]

Then the following statements are equivalent.

1. The Newtonian limit of \( L_E \) is well-behaved.
2. The de Sitter space-time with \( H = \frac{1}{l} \) is an attractor solution for \( L \).
3. \( \epsilon \geq 0 \) and the graceful exit problem is solved for the quasi de Sitter phase \( H \leq 1/l \) of \( L_E \).
4. \( l^2 = l_0^2 + l_1^2, \ \ \epsilon l^4 = l_0^2 l_1^2 \) has a solution with \( 0 \leq l_0 < l_1 \).
5. \( 0 \leq \epsilon < \frac{1}{4} \).

From the first glance this theorem is contrary to the results of refs. [64 - 66]. But one should remember that in refs. [64 - 66] the question had been considered whether the sixth-order terms can typically lead to double inflation. The answer was: Double inflation (one period from the \( R^2 \)-term, the other one from the \( R \Box R \)-term) requires a fine-tuning of initial conditions. Here we have shown: The results of the Starobinsky model (\( \epsilon = 0 \) in the present notation) are structurally stable with respect to the addition of a sixth-order term \( \sim \epsilon R \Box R \), where \( 0 \leq \epsilon < \frac{1}{4} \). The duration of the transient quasi de Sitter phase becomes reduced by a factor \( \sim (1 - 4\epsilon) \) only.

Further we have shown: For

\[
L = R^2 + c_1 R \Box R + c_2 R \Box \Box R, \quad c_2 \neq 0
\]

and the usual case \( n = 3 \) the de Sitter space-time with \( H = 1 \) is an attractor solution in the set of spatially flat Friedman models if and only if the following
inequalities are fulfilled:

\[ 0 < c_1 < \frac{1}{4}, \quad 0 < c_2 < \frac{1}{16} \]

and

\[ c_1 > -13c_2 + \sqrt{4c_2 + 225c_2^2} \]

This represents an open region in the \( c_1 - c_2 \)-plane whose boundary contains the origin; and for the other boundary points the linearized equation does not suffice to decide the attractor property. This situation shall be called "semi-attractor" for simplicity. In contrary to the 6th-order case, here we do not have a one-to-one correspondence, but a non-void open intersection with that parameter set having the Newtonian limit for \( L_E \) well-behaved.

To find out, whether another de Sitter space-time with an arbitrary Hubble parameter \( H > 0 \) is an attractor solution for the eighth-order field equation following from the above Lagrangian, one should remember that \( H \) has the physical dimension of an inverted time, \( c_1 \) a time squared, \( c_2 \) a time to power 4. So, we have to replace \( c_1 \) by \( c_1H^2 \) and \( c_2 \) by \( c_2H^4 \) in the above dimensionless inequalities to get the correct conditions. Example: \( 0 < c_1H^2 < \frac{1}{4} \).
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