Ordered Structures of Constructing Operators for Generalized Riesz Systems

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1. Introduction

Generalized Riesz systems can be used to construct some physical operators (non-self-adjoint Hamiltonian, generalized lowering operator, generalized raising operator, number operator, etc.) [1–3]. Then these operators provide a link to quasi-Hermitian quantum mechanics and its relatives. Many researchers have investigated such operators both from the mathematical point of view and for their physical applications [4–9]. Let \( \{ \varphi_n \} \) be a generalized Riesz system with a constructing pair \( (e, T) \). Then, putting \( \psi_n = (T^{-1})^* e_n \), \( n = 0, 1, \cdots \), \( \{ \varphi_n \} \) and \( \{ \psi_n \} \) are biorthogonal sequences, that is, \( \varphi_n, \psi_m \rangle = \delta_{mn} \), \( n, m = 0, 1, \cdots \). For any \( \{ \alpha_n \} \subset \mathscr{C} \) we can define the operators:

\[
H^\alpha_{\varphi} := TH^\alpha T^{-1}, \quad A^\alpha_{\varphi} := TA^\alpha T^{-1}, \quad B^\alpha_{\varphi} := TB^\alpha T^{-1},
\]

where \( H^\alpha_{\varphi} := \sum_{n=0}^{\infty} \alpha_n e_n \otimes \varphi_n \), \( A^\alpha_{\varphi} := \sum_{n=0}^{\infty} \alpha_n e_n \otimes \varphi_{n+1} \otimes \varphi_n \), and \( B^\alpha_{\varphi} := \sum_{n=0}^{\infty} \alpha_{n+1} e_{n+1} \otimes \varphi_{n+1} \otimes \varphi_n \) are standard self-adjoint Hamiltonian, lowering operator, and raising operator for \( \{ e_n \} \), respectively, where for \( x, y \in \mathscr{H}, \langle x \otimes y | \xi \rangle := \langle \xi, y \rangle \), \( x, \xi \in \mathscr{H} \). Since \( H^\alpha_{\varphi} \varphi_n = \alpha_n \varphi_n \), \( A^\alpha_{\varphi} \varphi_n = \begin{cases} 0, & n = 0 \\ \alpha_n \varphi_{n+1}, & n = 1, 2, \cdots \end{cases} \) and \( B^\alpha_{\varphi} \varphi_n = \alpha_{n+1} \varphi_{n+1} \), \( n = 0, 1, \cdots \), \( H^\alpha_{\varphi}, A^\alpha_{\varphi}, \) and \( B^\alpha_{\varphi} \) are called the non-self-adjoint Hamiltonian, the generalized lowering operator, and the generalized raising operator for \( \{ \varphi_n \} \), respectively. The physical operators of the extended quantum harmonic oscillator and the Swanson model are of this form (see Examples 9–11 in Section 3).

From this fact, it seems to be important to consider under what conditions biorthogonal sequences are generalized Riesz systems and in [1–3] we have investigated this problem. In this paper, we shall focus on the following facts: physical operators defined by a generalized Riesz system \( \{ \varphi_n \} \) depend on constructing pairs; for example, their operators may not be densely defined for some constructing pairs. On the other hand, if there exists a dense subspace \( D \subset \mathscr{H} \) for a constructing pair \( (e, T) \) which is a core for \( H^\alpha_{\varphi} \) and \( A^\alpha_{\varphi} \), \( B^\alpha_{\varphi} \), then they have an algebraic structure; in detail, the \( O \)-algebra on \( D \) is defined by the restrictions of the operators \( A^\alpha_{\varphi} \) and \( B^\alpha_{\varphi} \) to \( D \) [10]. Thus it seems to be important to find a constructing pair fitting to each of the physical applications. From this reason, in this paper we shall investigate the properties of constructing pairs for a generalized Riesz system.

In Section 2, we shall investigate the basic properties of constructing operators. Let \( \{ \varphi_n \} \) be a generalized Riesz system with a constructing pair \( (e, T) \). The constructing operators for \( \{ \varphi_n \} \) are unique for the fixed ONB \( e \) in \( \mathscr{H} \) if \( \{ \varphi_n \} \) is a Riesz
basis; that is, $T$ and $T^{-1}$ are bounded, but they are not unique in general. So, we investigate the set $C_{\varepsilon_0}$ of all constructing operators for $e$. In Proposition 1, we shall show that it is possible to fix an ONB $e = \{e_n\}$ in $\mathcal{H}$ without loss of generality for our study in this paper. Hence, we fix an ONB $e$ in $\mathcal{H}$ and denote $C_{\varepsilon_0}$ by $C_e$ for simplicity. We consider the following problem: Is any sequence $\{\psi_n\}$ which is biorthogonal to $\{\phi_n\}$ a generalized Riesz system?

Here we put

$$C_{\varphi}^N = \left\{ T \in C_{\varphi}; (T^{-1})^* e_n = \psi_n, \; n = 0, 1, \cdots \right\}. \quad (2)$$

Then we shall show in Proposition 5 that if $C_{\varphi}^N \neq 0$, then $\{\psi_n\}$ is a generalized Riesz system and $(e_n, (T^{-1})^*)$ is a constructing pair for $\{\psi_n\}$ for every $T \in C_{\varphi}^N$, and the mapping

$$T \in C_{\varphi}^N \rightarrow (T^{-1})^* \in C_{\psi}^N$$

is a bijection, where $C_{\psi}$ is the set of all constructing operators for $\{\psi_n\}$ and

$$C_{\psi}^N = \left\{ K \in C_{\psi}; (K^{-1})^* e_n = \phi_n, \; n = 0, 1, \cdots \right\}. \quad (3)$$

Furthermore, we shall show in Proposition 6 that if there exists a bounded operator $T_0$ in $C_{\varphi}$, then $C_{\psi} = \{T_0\}$ and $C_{\varphi}^N = \{(T^{-1})^*\}$.

In Section 3, we shall consider the ordered set $C_{\varphi}$ with order $C$ and investigate under what conditions the ordered set $C_{\varphi}$ has a maximal element, a minimal element, the smallest element, and the largest element. First, we shall prove that if $D_{\varphi}$ is linear span $\{\phi_n\}$ is dense in $\mathcal{H}$, then $C_{\varphi} = C_{\varphi}^N$ and there exists the smallest element of $C_{\varphi}$, and furthermore if $D_{\varphi}$ and $D(\varphi) = \{x \in \mathcal{H}; \sum_{k=0}^{\infty} |x, \phi_k > |^2 < \infty\}$ is dense in $\mathcal{H}$, there exist the smallest element of $C_{\varphi}$ and the largest element of $C_{\varphi}^N$, and in particular, if $\{\phi_n\}$ and $\{\psi_n\}$ are regular biorthogonal sequences in $\mathcal{H}$, that is, both $D_{\varphi}$ and $D_{\psi}$ are dense in $\mathcal{H}$, then $C_{\varphi} = C_{\varphi}^N$, $C_{\psi} = C_{\psi}^N$, and $C_{\varphi}$ has the smallest element and the largest element. Next, we shall consider the case when $D_{\varphi}$ is not necessarily dense in $\mathcal{H}$. In Theorem 14, we shall show that for a subset $\mathcal{F}$ of $C_{\varphi}$ if there exists a closed operator $A$ in $\mathcal{H}$ such that $T \subset A$ for all $T \in \mathcal{F}$, then $\mathcal{F}$ has a maximal element, and furthermore, if there exists a closed operator $B$ in $\mathcal{H}$ such that $(T^{-1})^* \subset B$ for all $T \in \mathcal{F}$, then $\mathcal{F}$ have a maximal element and a minimal element.

For the existence of the smallest element of $C_{\varphi}$ and of the largest element of $C_{\varphi}$, we shall show in Theorem 16 that if there exist closed operators $A$ and $B$ in $\mathcal{H}$ such that $T \subset A$ and $(T^{-1})^* \subset B$ for all $T \in C_{\varphi}$, then $C_{\varphi}$ has the smallest element and the largest element. Furthermore, for a biorthogonal pair $\{(\phi_n), (\psi_n)\}$ of generalized Riesz systems satisfying $C_{\varphi} = C_{\varphi}^N$ and $C_{\psi} = C_{\psi}^N$, we shall show in Theorem 18 that $C_{\varphi}$ and $C_{\psi}$ have the smallest element and the largest element, respectively, if and only if there exist closed operators $A$ and $B$ in $\mathcal{H}$ such that $T \subset A$ and $K \subset B$ for all $T \in C_{\varphi}$ and $K \in C_{\psi}$. These results seem to be useful to find fitting constructing operators for each physical model because every closed operator $T$ in $\mathcal{H}$ satisfying $T_S \subset T \subset T_L$ belongs to $C_{\varphi}$, where $T_S$ is the smallest element of $C_{\varphi}$ and $T_L$ is the largest element of $C_{\varphi}$.

2. The Basic Properties of Constructing Operators

In this section, we shall investigate the basic properties of constructing operators. Let $\{\phi_n\}$ be a generalized Riesz system with a constructing pair $(e, T)$. It is easily shown that if $\{\phi_n\}$ is a Riesz basis, then the constructing operator $T$ for $\{\phi_n\}$ is unique for $e$ (see Proposition 1 in detail). But, in general, the constructing operators for $\{\phi_n\}$ are not unique, and so we put

$$C_{\varepsilon_\varphi} := \{ T; (e, T) \text{ is a constructing pair for } \{\phi_n\} \}. \quad (5)$$

First, we investigate the relationship between $C_{\varepsilon_\varphi}$ and $C_{f_\varphi}$ for the other ONB $f = \{f_n\}$ in $\mathcal{H}$.

Proposition 1. Let $T \in C_{\varepsilon_\varphi}$ and $f = \{f_n\}$ be any ONB in $\mathcal{H}$. Then the following statements hold:

(1) $(f, T U^*)$ is a constructing pair for $\{\phi_n\}$, where $U$ is a unitary operator on $\mathcal{H}$ defined by $U e_n = f_n$, $n = 0, 1, \cdots$, and

$$C_{f_\varphi} = \{T U^*; \; T \in C_{\varepsilon_\varphi}\}. \quad (6)$$

(2) For the non-self-adjoint Hamiltonian, the generalized lowering operator, and the generalized raising operator for $\{\phi_n\}$, we have

$$TA_{f_\varphi}e^{-T^{-1}} = T^* U A_{f_\varphi}e^{-T^{-1}},$$

$$TA_{f_\varphi}e^{-T^{-1}} = T^* U A_{f_\varphi}e^{-T^{-1}}. \quad (7)$$

Proof. (1) This is almost trivial.

(2) This follows from

$$D (H_{e}) = UD (H_{e}^a),$$

$$H_{e} = U^* H_{f} U,$$

$$D (A_{e}^a) = UD (A_{e}^a),$$

$$A_{e} = U^* A_{f} U,$$

$$D (B_{f}^a) = UD (B_{f}^a),$$

$$B_{e} = U^* B_{f} U. \quad (8)$$

By Proposition 1, we have the following.

Corollary 2. Let $T \in C_{\varepsilon_\varphi}$ and $T^* = U |T^*|$ be the polar decomposition of $T^*$. Then $f = U^* e$ is an ONB in $\mathcal{H}$ and $|T^*| = T U \in C_{f_\varphi}$. Furthermore, we have

$$TH_{f}e^{-T^{-1}} = |T^*| H_{f} |T^*|^{-1},$$

For the existence of the smallest element of $C_{\varphi}$ and of the largest element of $C_{\varphi}$, we shall show in Theorem 16 that if there exist closed operators $A$ and $B$ in $\mathcal{H}$ such that $T \subset A$ and $(T^{-1})^* \subset B$ for all $T \in C_{\varphi}$, then $C_{\varphi}$ has the smallest element and the largest element. Furthermore, for a biorthogonal pair $\{(\phi_n), (\psi_n)\}$ of generalized Riesz systems satisfying $C_{\varphi} = C_{\varphi}^N$ and $C_{\psi} = C_{\psi}^N$, we shall show in Theorem 18 that $C_{\varphi}$ and $C_{\psi}$ have the smallest element and the largest element, respectively, if and only if there exist closed operators $A$ and $B$ in $\mathcal{H}$ such that $T \subset A$ and $K \subset B$ for all $T \in C_{\varphi}$ and $K \in C_{\psi}$. These results seem to be useful to find fitting constructing operators for each physical model because every closed operator $T$ in $\mathcal{H}$ satisfying $T_S \subset T \subset T_L$ belongs to $C_{\varphi}$, where $T_S$ is the smallest element of $C_{\varphi}$ and $T_L$ is the largest element of $C_{\varphi}$.
\[
T_A T^{-1} = |T|^* A_T^* |T|^*^{-1},
\]
\[
TB_T^{-1} = |T|^* B_T^* |T|^*^{-1}.
\]
(9)

Thus we may fix an ONB \( e = \{e_n\} \) in \( \mathcal{H} \) without loss of generality for investigating the properties of \( C_{\varphi, \psi} \), and so throughout this paper, we fix an ONB \( e \) in \( \mathcal{H} \) and denote \( C_{\varphi, \psi} \) by \( \varphi \) for simplicity. Next we consider the following problem:

**Problem:** Suppose that \( (\varphi_n), (\psi_n) \) is a biorthogonal pair such that \( \{\varphi_n\} \) is a generalized Riesz system. Then, is \( \{\psi_n\} \) also a generalized Riesz system?

Let \( T = T^* \in C_{\varphi} \) and \( \psi_n = (T^{-1})^* e_n, n = 0, 1, \ldots \). Then \( \langle (\varphi_n), (\varphi_n^T) \rangle \) is a biorthogonal pair and \( \{\psi_n^T\} \) is a generalized Riesz system with a constructing pair \( (e, (T^{-1})^*) \). If \( \psi_n^T = \psi_n, n = 0, 1, \ldots \), then \( \{\psi_n\} \) is a generalized Riesz system with a constructing pair \( (e, (T^{-1})^*) \). But, the equality \( \psi_n^T := (T^{-1})^* e_n = \psi_n, n = 0, 1, \ldots \) does not necessarily hold. To consider when this equality holds, we define the operators \( T_{\varphi, \psi}^0, T_{\varphi, \psi}^* \) and \( T_{\varphi, \psi} \) for any sequence \( \{\varphi_n\} \) in \( \mathcal{H} \) as follows:

\[
T_{\varphi, \psi}^0 = \text{the linear operator defined by } T_{\varphi, \psi}^0 e_n = \varphi_n, \quad n = 0, 1, \ldots ,
\]
\[
T_{\varphi, \psi} = \sum_{n=0}^{\infty} \varphi_n \otimes \overline{\varphi_n},
\]
\[
T_{\varphi, \psi}^* = \sum_{n=0}^{\infty} e_n \otimes \overline{\varphi_n}.
\]
(10)

These operators have played an important role for our studies [3] and also in this paper. By Lemma 2.1, 2.2 in [3] we have the following.

**Lemma 3.** (1) \( T_{\varphi, \psi}^0 \) and \( T_{\varphi, \psi} \) are densely defined linear operators in \( \mathcal{H} \) such that

\[
T_{\varphi, \psi} \supset T_{\varphi, \psi}^0.
\]
(11)

(2) \( D(T_{\varphi, \psi}^0) = D(\varphi) = \{x \in \mathcal{H}; \sum_{m=0}^{\infty} |x, \varphi_m| \leq |x| < \infty\} \) and \( (T_{\varphi, \psi}^0)^* = T_{\varphi, \psi}^* \).

(3) \( T_{\varphi, \psi}^0 \) is closable if and only if \( T_{\varphi, \psi} \) is closable if and only if \( D(\varphi) \) is dense in \( \mathcal{H} \). If this holds, then

\[
T_{\varphi, \psi}^0 = \overline{T_{\varphi, \psi}} = (T_{\varphi, \psi}^*)^*.
\]
(12)

From now on, let \( (\varphi_n), (\psi_n) \) be a biorthogonal pair.

**Lemma 4.** Suppose that \( \{\varphi_n\} \) is a generalized Riesz system and \( T \in C_{\varphi} \). Then the following statements are equivalent.

(i) \( \psi_n^T := (T^{-1})^* e_n = \psi_n, n = 0, 1, \ldots \).

(ii) \( D_\varphi \subseteq D(T^*). \)

If this holds true, then \( T \) is called natural.

**Proof.** (i) \( \Rightarrow\) (ii) This is trivial.

(ii) \( \Rightarrow\) (i) By definition of \( T_{\varphi, \psi}^0 \), we have \( T_{\varphi, \psi}^0 \subseteq T \). Furthermore, by Lemma 3, (2), we have

\[
T^* \subseteq (T_{\varphi, \psi}^0)^* = T_{\varphi, \psi}.
\]
(13)

Take an arbitrary \( n \in N \cup \{0\} \). Then, since

\[
\langle T_{\varphi, \psi}^0 e_n, \psi_n \rangle = \langle \varphi_n, \psi_n \rangle = \delta_{kn} = \langle e_k, e_n \rangle
\]
(14)

for \( k = 0, 1, \ldots \), we have \( \psi_n \in D((T_{\varphi, \psi}^0)^*) = D(T_{\varphi, \psi}) \) and \( T_{\varphi, \psi} \psi_n = e_n \). Hence it follows from (13) that

\[
T^* \psi_n = T_{\varphi, \psi} \psi_n = e_n.
\]
(15)

Thus, we have

\[
\psi_n = (T^*)^{-1} e_n = \psi_n.
\]
(16)

This completes the proof.

We denote the set of all natural constructing operators for \( \{\varphi_n\} \) by \( C_{\varphi}^N \), that is,

\[
C_{\varphi}^N = \{ T \in C_{\varphi}; \psi_n^T = (T^{-1})^* e_n = \psi_n, n = 0, 1, \ldots \}.
\]
(17)

Then we have the following.

**Proposition 5.** Suppose that \( \{\varphi_n\} \) is a generalized Riesz system. Then the following statements hold.

(1) If \( C_{\varphi}^N \neq \emptyset \), then \( \{\psi_n\} \) is a generalized Riesz system and \( (e, (T^{-1})^*) \) is a constructing pair for \( \{\psi_n\} \) for every \( T \in C_{\varphi}^N \).

(2) Suppose that \( \{\psi_n\} \) is also a generalized Riesz system and put

\[
C_{\psi} = \{ K; (e, K) \text{ is a constructing pair for } \{\psi_n\} \},
\]

\[
C_{\psi}^N = \{ K \in C_{\varphi}; \{ \psi_n^K := (K^{-1})^* e_n = \psi_n, n = 0, 1, \ldots \} \}.
\]
(18)

Then the mapping

\[
T \in C_{\varphi}^N \mapsto (T^{-1})^* \in C_{\psi}^N
\]
(19)

is a bijection.

(3) Suppose that \( T_0 \in C_{\varphi}^N \) and \( T \in C_{\varphi} \) satisfying \( T \subseteq T_0 \) or \( T_0 \subseteq T \). Then \( T \in C_{\varphi}^N \). Similarly, suppose that \( K_0 \in C_{\psi}^N \) and \( K \in C_{\psi} \) satisfying \( K \subseteq K_0 \) or \( K_0 \subseteq K \). Then \( K \in C_{\psi}^N \).

**Proof.** The statements (1) and (2) are easily shown.

(3) Suppose that \( T \subseteq T_0 \). Then, since \( (T_0^{-1})^* \subseteq (T^{-1})^* \), it follows that \( \psi_n = (T_0^{-1})^* e_n = (T^{-1})^* e_n = \psi_n, n = 0, 1, \ldots \), which implies that \( T \in C_{\psi}^N \). Similarly, we can show \( T \in C_{\psi}^N \) in case that \( T_0 \subseteq T \) and can show \( K \in C_{\psi}^N \) in case that \( K \subseteq K_0 \) or \( K_0 \subseteq K \). This completes the proof.
As for the uniqueness of constructing operators for a generalized Riesz system we have the following.

**Proposition 6.** Let \( \{q_n\} \) be a generalized Riesz system. Then the following statements hold.

1. Suppose that \( \{q_n\} \) is a Riesz basis, then \( \{\psi_n\} \) is also a Riesz basis, \( C_q = \{T_{\psi,n}\} \) and \( C_\psi = \{T_{\psi,n}\} \).

2. Suppose that \( D_q \) and \( D(\psi) \) are dense in \( \mathcal{H} \). Then we have the following.
   
   (i) If there exists an element \( T_0 \) of \( C_q \) such that \( T_0 \) is bounded, then \( C_q = \{T_0\} = \{T_{\psi,n}\} \) and \( C_\psi = \{T_{\psi,n}\} = \{T_{\psi,n}\} \).

   (ii) If there exists an element \( T_0 \) of \( C_q \) such that \( T_0^{-1} \) is bounded, then \( C_q = \{T_0^{-1}\} = \{T_{\psi,n}\} \) and \( C_\psi = \{T_{\psi,n}\} = \{T_{\psi,n}\} \).

**Proof.** (1) Since \( \{q_n\} \) is a Riesz basis, there exists an element \( T_0 \) of \( C_q \) such that \( T_0 \) and \( T_0^{-1} \) are bounded, which implies that \( T = T_0 = T_{\psi,n} \) and \( (T^{-1})^* = (T_0^{-1})^* = T_{\psi,n} = T_{\psi,n} \) for all \( T \in C_q \).

   (2) (i) Since \( T_{\psi,n} \subset T_0 \) and \( T_0 \) is bounded, we have \( T_{\psi,n} = T_0 \). Take an arbitrary \( T \in C_q \). Then, since \( T_{\psi,n} \subset T \) and \( T_{\psi,n} \) is bounded, we have \( T_{\psi,n} = T \). Thus, \( C_q = \{T_{\psi,n}\} \). We show \( C_q = \{T_{\psi,n}\} \). Take an arbitrary \( K \in C_q \). Since \( (K^{-1})^* e_n = \psi_n = T_{\psi,n} e_n \) and \( T_{\psi,n} \) is bounded, it follows that \( (K^{-1})^* = T_{\psi,n} \), which implies \( K = T_{\psi,n} \). Thus, \( C_q = \{T^{-1}_{\psi,n}\} \).

   (ii) This is similarly shown. \( \square \)

3. **Ordered Structures of \( C_q \)**

In this section, we shall consider the ordered set \( C_q \) of all constructing operators for a generalized Riesz system \( \{q_n\} \) with order \( c \) and investigate when \( C_q \) has maximal elements, minimal elements, the largest element, and the smallest element. The following result gives a motivation to study the ordered structures of \( C_q \).

**Lemma 7.** Suppose that \( T, S \in C_q \) and \( T \subset S \). Then, for any linear operator \( A \) such that \( T \subset A \subset S \), the closure \( A \) of \( A \) belongs to \( C_q \).

**Proof.** This is trivial. \( \square \)

For biorthogonal sequences satisfying density-conditions, we have the following.

**Proposition 8.** The following statements hold.

1. Suppose that \( D_q \) is dense in \( \mathcal{H} \). Then, \( \{q_n\} \) is a generalized Riesz system and \( C_q = C_q^{\infty} \) and \( T_{\psi,n} \) is the smallest element of \( C_q \). Furthermore, suppose that \( D(\psi) \) is dense in \( \mathcal{H} \). Then, \( T_{\psi,n} \) is the largest element of \( C_q \).

2. Suppose that \( D_q \) is dense in \( \mathcal{H} \). Then, \( \{\psi_n\} \) is a generalized Riesz system and \( C_\psi = C^{\infty}_q \) and \( T_{\psi,n} \) is the smallest element of \( C_q \). Furthermore, suppose that \( D(\psi) \) is dense in \( \mathcal{H} \). Then, \( T_{\psi,n} \) is the largest element of \( C_q \).

(3) Suppose that \( \{\psi_n\} \) is regular; that is, both \( D_q \) and \( D_q \) are dense in \( \mathcal{H} \). Then, \( \{\psi_n\} \) and \( \{\psi_n\} \) are generalized Riesz systems and \( C_q = C_q^{\infty} \) and \( C_\psi = C_\psi^{\infty} \) and \( T_{\psi,n} \) is the smallest element in \( C_q \). \( T_{\psi,n} \) is the smallest element in \( C_q \). \( T_{\psi,n} \) is the largest element in \( C_q \), and \( T_{\psi,n} \) is the largest element in \( C_q \).

**Proof.** (1) We can show using Lemma 3 that \( \{q_n\} \) is a generalized Riesz system with a constructing pair \( (e, T_{\psi,n}) \) and the constructing operator \( T_{\psi,n} \) is the smallest element in \( C_q \). For more detail, refer to [3]. Furthermore, a sequence \( \{\psi_n\} \) which is biorthogonal to \( \{q_n\} \) is unique. In fact, let \( \{\psi_n\} \) and \( \{\psi_n\} \) be any sequences in \( \mathcal{H} \) which are biorthogonal to \( \{q_n\} \). Then, since \( \psi_n \neq \psi_m \) for \( n, m = 0, 1, \cdots \) and \( D_q \) is dense in \( \mathcal{H} \), we have \( \psi_n = \psi_n \), for every \( n = 0, 1, \cdots \). We show \( C_q = C_q^{\infty} \). Take an arbitrary \( T \in C_q \). Then, \( \{T_{\psi,n}\} \) is biorthogonal to \( \{q_n\} \). By the uniqueness of biorthogonal sequences to \( \{q_n\} \), we have \( \psi_n = \psi_n \); \( n = 0, 1, \cdots \), which implies that \( T \in C_q \) and \( C_q = C_q^{\infty} \). Suppose that \( D_q \) and \( D(\psi) \) are dense in \( \mathcal{H} \). We show that \( T_{\psi,n} \) is the largest element in \( C_q^{\infty} \). Since \( D(T_{\psi,n}) = T_{\psi,n} \) for \( T \in C_q \), \( D(T_{\psi,n}) = D(T_{\psi,n}) = D(T_{\psi,n}) = D(T_{\psi,n}) = D(T_{\psi,n}) = D(T_{\psi,n}) = D(T_{\psi,n}) = D(T_{\psi,n}) \) is a densely defined closed operator in \( \mathcal{H} \), and since \( D(T_{\psi,n}) = D(\psi) \), it has a densely defined inverse \( T_{\psi,n} \). Furthermore, since \( T_{\psi,n} \psi_n = e_n \), \( n = 0, 1, \cdots \), we have \( T_{\psi,n} \psi_n = e_n \), \( n = 0, 1, \cdots \). Thus we have \( T_{\psi,n} \psi_n = e_n \), \( n = 0, 1, \cdots \). Hence we have \( T_{\psi,n} \psi_n = e_n \), \( n = 0, 1, \cdots \). Thus we have \( T_{\psi,n} = T_{\psi,n} \), and so \( T_{\psi,n} = T_{\psi,n} \). Thus \( T_{\psi,n} \) is the largest element in \( C_q^{\infty} \).

(2) This is proved at the same way as (1).

(3) Since \( D(\psi) \supset D_q \) and \( D(\psi) \supset D_q \), it follows that \( D(\psi) \) and \( D(\psi) \) are dense in \( \mathcal{H} \), which implies by (1) and (2) that the statement (3) holds. \( \square \)

Here we give some physical examples. Let \( \{f_n\} \), \( n = 0, 1, \cdots \), be an ONB in \( L^2(\mathbb{R}) \) consisting of the Hermite functions which is contained in the Schwartz space \( S(\mathbb{R}) \) of all infinitely differential rapidly decreasing functions on \( \mathbb{R} \). We define the moment operator \( p \) and the position operator \( q \) by

\[
D(p) = \left\{ f \in L^2(\mathbb{R}) : \int_{-\infty}^{\infty} \left| x f(x) \right|^2 dx < \infty \right\},
\]

\[
D(q) = \left\{ f \in L^2(\mathbb{R}) : \int_{-\infty}^{\infty} \left| x f(x) \right|^2 dx < \infty \right\}.
\]

\[
\frac{df}{dx} \in L^2(\mathbb{R}),
\]

\[
(pf)(x) = -i \frac{df}{dx}, \quad f \in D(p)
\]

and

\[
D(q) = \left\{ f \in L^2(\mathbb{R}) : \int_{-\infty}^{\infty} \left| x f(x) \right|^2 dx < \infty \right\}.\]
Then \( p \) and \( q \) are self-adjoint operators in \( L^2(\mathbb{R}) \) and \( \mathcal{D}(\mathfrak{A}) \) is a core for \( p \) and \( q \), and furthermore \( p \mathcal{D}(\mathfrak{A}) \subset \mathcal{D}(\mathfrak{A}) \) and \( q \mathcal{D}(\mathfrak{A}) \subset \mathcal{D}(\mathfrak{A}) \), and \( \{p,q\} = pq - qp = -i\mathbb{1} \) on \( \mathcal{D}(\mathfrak{A}) \). Next we define the standard bosonic operators \( a, a^\dagger \) by

\[
a = \frac{1}{\sqrt{2}} (q + ip),
\]

\[
a^\dagger = \frac{1}{\sqrt{2}} (q - ip).
\]

Then,

\[
a f_n = \begin{cases} 0, & n = 0 \\ \sqrt{n} f_{n-1}, & n = 1, 2, \ldots \end{cases}
\]

\[
a^\dagger f_n = \sqrt{n+1} f_{n+1}, & n = 0, 1, \ldots
\]

and \([a, a^\dagger] = 1\) on \( \mathcal{D}(\mathfrak{A}) \).

**Example 9** (the extended quantum harmonic oscillator). The Hamiltonian of this model is the non-self-adjoint operator, introduced in [11, 12],

\[
H_\beta = \frac{\beta}{2}(p^2 + q^2) + i\sqrt{2}q = \beta a^\dagger a + (a - a^\dagger)^2 + \frac{\beta}{2} \mathbb{1},
\]

\( \beta > 0 \).

We put

\[
\varphi^{(\beta)}_0 = e^{1/\beta} \frac{1}{\sqrt{\pi}} e^{-(1/2)(x^2 + \sqrt{2}x y)} y.
\]

Then, \( \varphi^{(\beta)} \in \mathcal{D}(\mathfrak{A}) \) and \( \varphi^{(\beta)}(p) = e^{1/\beta} U(1/\beta) f_0 = e^{(\alpha a + a^\dagger)} f_0 \), where \( U(1/\beta) \) is a unitary operator defined by \( U(1/\beta) := e^{(1/\beta)(a^2 - a)} = e^{-\sqrt{2} \beta \mathfrak{A}} \). Hence we can define a sequence \( \varphi^{(\beta)}_n \) in \( \mathcal{D}(\mathfrak{A}) \) by

\[
\varphi^{(\beta)}_n = \frac{1}{\sqrt{n!}} \left( a^\dagger + \frac{1}{\beta} \right)^n \varphi^{(\beta)}_0, \quad n = 1, 2, \ldots
\]

Similarly, we define a sequence \( \psi^{(\beta)}_n \) in \( \mathcal{D}(\mathfrak{A}) \) as follows:

\[
\psi^{(\beta)}_n = e^{1/\beta} f_0 \left( x - \frac{\sqrt{2}}{\beta} y \right),
\]

\[
\psi^{(\beta)}_n = \frac{1}{\sqrt{n!}} \left( a^\dagger - \frac{1}{\beta} \right)^n \psi^{(\beta)}_0, \quad n = 1, 2, \ldots
\]

Then \( \varphi^{(\beta)}_n \) and \( \psi^{(\beta)}_n \) are regular biorthogonal sequences in \( L^2(\mathbb{R}) \) which are generalized Riesz systems with constructing pairs \( \{f_n, T\} \) and \( \{f_n, T^{-1}\} \), respectively, and \( T = e^{-i(\beta)^{1/2} \mathfrak{A}} \) and \( T^{-1} = e^{i(\beta)^{-1/2} \mathfrak{A}} \), and \( T a T^{-1} = a + i/\beta, T a T^{-1} = a - i/\beta \).

**Example 11** (the Swanson model). The Swanson Hamiltonian, introduced in [11, 13], is a non-self-adjoint Hamiltonian

\[
H_{\theta} = \frac{1}{2}(p^2 + q^2) - \frac{i}{2} \tan \theta \left[ p^2 - q^2 \right] = a^\dagger a + \frac{i}{2} \tan \theta \left( a^2 + (a^\dagger)^2 \right) + \frac{1}{2} \mathbb{1},
\]

\( \theta \neq 0 \in \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \).

We define sequences \( \varphi^{(\theta)}_n \) and \( \psi^{(\theta)}_n \) in \( L^2(\mathbb{R}) \) as follows:

\[
\varphi^{(\theta)}_0 = e^{-i(\tan \theta/2)(\alpha^2)} f_0 = \sum_{k=0}^{\infty} e^{-i(\tan \theta/2)(\alpha^2)} f_{2k},
\]

\[
\varphi^{(\theta)}_n = \frac{1}{\sqrt{n!}} \left( \cos \theta a^\dagger + i \sin \theta a \right)^n \varphi^{(\theta)}_0.
\]
and
\[
\psi_0^{(\theta)} = d_0 \sum_{k=0}^{\infty} e^{i(\tan \theta/2\alpha)^2} f_0
\]
\[
= d_0 \sum_{k=0}^{\infty} (i \tan \theta)^k \sqrt{\frac{(2k-1)!!}{(2k)!!}} f_{2k},
\]
where (2k)!! = 2k(2k-2)\cdots 2, (2k-1)!! = (2k-1)(2k-3)\cdots 1, and \(c_0\) and \(d_0\) are constants satisfying \(< \phi_0^{(\theta)}, \psi_0^{(\theta)} > = 1\). Then \(\phi_0\) and \(\psi_0\) are regular biorthogonal sequences in \(L^2(\mathbb{R})\) contained in \(\mathcal{S}(\mathbb{R})\) which are generalized Riesz systems with constructing operators \(T_\theta = e^{i(\theta/2)(\alpha^2-(\alpha^1)^2)}\) and \(T_\theta^{-1}\), respectively. For the generalized lowering operator \(A_\theta = T_\theta a T_\theta^{-1}\) and the raising operator \(B_\theta = T_\theta a^* T_\theta^{-1}\), we have
\[
A_\theta = (\cos \theta) a + i (\sin \theta) a^*,
\]
\[
B_\theta = (\cos \theta) a^* + i (\sin \theta) a,
\]
By Proposition 8, \(T_{\psi_a f}\) (resp., \(T_{\psi_f a}\)) is the smallest constructing operator and \(T_{\psi_a f}\) (resp., \(T_{\psi_f a}\)) is the largest constructing operator for \(\phi_a\) (resp., \(\psi_a\)) and every closed operator \(T\) (resp., \(K\)) in \(L^2(\mathbb{R})\) satisfying \(T_{\psi_a f} \subset T \subset T_{\psi_f a}\) (resp., \(T_{\psi_a f} \subset K \subset T_{\psi_f a}\)) is a constructing operator for \(\phi_a\) (resp., \(\psi_a\)).

All physical models discussed above are regular cases, but it seems to be mathematically meaningful to study nonregular cases and furthermore the studies may become useful for physical applications in future. Let \(\{\phi_n\}\) be a generalized Riesz system. First we investigate under what conditions \(C_{\phi}\) has maximal elements and minimal elements.

Let \(C\) be a totally ordered subset of \(C_{\phi}\). Then, it is easily shown that \((T^{-1})^* e_n = (S^{-1})^* e_n, n = 0, 1, \cdots\), for any \(T, S \in C\). Hence we may put
\[
\psi_n^C = (T^{-1})^* e_n, \quad T \in C, \quad n = 0, 1, \cdots.
\]
We have the following statements.

**Lemma 12.** Let \(C\) be any totally ordered subset of \(C_{\phi}\). The following statements hold.

1. Suppose that \(\cap_{T \in C} D(T^*)\) is dense in \(\mathcal{H}\). Then there exists an upper bounded element \(G\) of \(C\).
2. Suppose that \(\cap_{T \in C} R(T)\) is dense in \(\mathcal{H}\). Then there exists a lower bounded element \(S\) of \(C\).
3. Suppose that \(\cap_{T \in C} D(T^*)\) and \(\cap_{T \in C} R(T)\) are dense in \(\mathcal{H}\). Then for every linear operator \(A\) such that \(S \subset A \subset G\), the closure \(A\) of \(A\) belongs to \(C_{\phi}\).

**Proof.** (1) We put
\[
D(G) = \bigcup_{T \in C} D(T), \quad Gx = T_0 x, \quad x \in D(G),
\]
where \(T_0\) is an operator in \(C\) whose domain \(D(T_0)\) contains \(x\). Since \(C\) is totally ordered, it follows that \(D(G)\) is a subspace in \(\mathcal{H}\) and \(Tx = T_0 x\) for any operators \(T, T_0\) in \(C\) whose domains contain \(x\). Hence, \(G\) does not depend on the method of choosing \(T_0 \in C\) whose domain contains \(x\). Thus \(G\) is a well-defined densely defined linear operator in \(\mathcal{H}\) such that \(T \subset G\) for all \(T \in C\). We show that \(G\) is closable. Indeed, we may show
\[
\bigcap_{T \in C} D(T^*) = D(G^*),
\]
where \(T_0 \in C\) whose domain \(D(T_0)\) contains \(x\). Take an arbitrary \(y \in \cap_{T \in C} D(T^*)\). Then, we have
\[
\langle Gx, y \rangle = \langle Tx, y \rangle = \langle x, T_0 y \rangle
\]
for all \(x \in D(G),\) where \(T_0 \in C\) whose domain contains \(x\). Hence, \(y \in D(G^*)\) and \(G^* y = T_0^* y\). Since \(T \subset G\) for all \(T \in C\), \(D(G^*) \subset \cap_{T \in C} D(T^*)\) is trivial. Thus, (38) holds. By (38) and the assumption of (1), \(D(G^*)\) is dense in \(\mathcal{H}\), that is, \(G\) is closable. Next we show that \(G\) has a densely defined inverse. Suppose that \(Gx = 0, x \in D(G)\). Then \(T_0 x = 0\) for some \(T_0 \in C\), and so \(x = 0\) since \(T_0\) has an inverse. Thus \(G\) has an inverse. Since \(R(T) \subset R(G)\) and \(R(T)\) is dense in \(\mathcal{H}\) for all \(T \in C\), it follows that the inverse of \(G\) is densely defined, which implies that the closure \(\overline{G}\) of \(G\) has a densely defined closed operator in \(\mathcal{H}\) such that \(\overline{G} \supset T\) for all \(T \in C\). Finally we show
\[
\{ e_n \} \subset D(G^*) \cap D((G^*)^{-1}),
\]
\[
\overline{G} e_n = \phi_n, \quad n = 0, 1, \cdots.
\]
Clearly, \(\{ e_n \} \subset D(G) \subset D(G^*)\) for all \(T \in C\). Next we show that for any \(n\) there exists an element \(\psi_n\) of \(\cap_{T \in C} D(T^*)\) such that
\[
e_n = T^* \psi_n
\]
for all \(T \in C\). Indeed, take an arbitrary \(T \in C\). Since \(e_n \in D(T^{-1}) = R(T^*)\), there exists an element \(\psi_n^T\) of \(D(T^*)\) such that \(e_n = T^* \psi_n^T\). Let any \(T^* \in C\). Since \(C\) is totally ordered, either \(T^* \subset T\) or \(T \subset T^*\) holds. Suppose that \(T^* \subset T\). Since \(T^* \subset (T^*)^*\), it follows that \(\psi_n^T = \psi_n^{T^*} \in D(T^*)\) and
\[
(T^*)^* \psi_n^T = T^* \psi_n^T = e_n = (T^*)^* \psi_n^{T^*} = e_n,
\]
which implies that \(\psi_n^T = \psi_n^{T^*}\) since \((T^*)^*\) has inverse. The equality \(\psi_n^T = \psi_n^{T^*}\) is similarly shown in case that \(T \subset T^*\). Hence, we have that \(\psi_n = \psi_n^T \in \cap_{T \in C} D(T^*)\) and \(T^* y_n = e_n\) for all \(T \in C\). Thus (41) holds. By (38) and (41) we have \(e_n \in R(G^*) = D(G^*)^{-1}\). Furthermore, we have \(\overline{G} e_n = T e_n = \phi_n,\) \(n = 0, 1, \cdots\) for all \(T \in C\). Thus we have \(\overline{G} \in C_{\phi}\) and \(\overline{G}\) is an upper bounded element of \(C\).

(2) We put
\[
D(S) = \cap_{T \in C} D(T),
\]
\[
Sx = Tx, \quad x \in D(S),
\]
where \(T\) is any element of \(C\). Since \(\{ e_n \} \subset \cap_{T \in C} D(T)\) and \(T_1 x = T_2 x\) for all \(x \in \cap_{T \in C} D(T),\) \(S\) is a well-defined densely
defined closed operator in \( \mathcal{H} \) such that \( S \subset T \) for all \( T \in C \). Hence, it is sufficient to show \( S \subset C_p \). Since \( S \subset T \) for all \( T \in C \) and \( T \) has the inverse, \( S \) has the inverse. Furthermore, we may show

\[
R(S) = \cap_{T \in C} R(T).
\]

(44)

In fact, take an arbitrary \( y \in \cap_{T \in C} R(T) \). Since \( C \) is totally ordered, there exists an element \( x \) of \( D(S) = \cap_{T \in C} D(T) \) such that \( y = Tx \) for all \( T \in C \). Hence, \( y = Sx \in R(S) \). The inverse inclusion \( R(S) \subset \cap_{T \in C} R(T) \) is clear. Hence (44) holds. By the assumption and (44), \( R(S) = D(S^{-1}) \) is dense in \( \mathcal{H} \). Furthermore, since \( K_0 = (T^{-1})^* \) is a minimal \( (\text{resp.}, \text{maximal, the smallest}) \) element of \( \mathcal{H} \) if and only if \( (F_0^*)^{-1} \) is a minimal \( (\text{resp.}, \text{maximal, the smallest}) \) element of \( \mathcal{F}_F \).

Proof. Suppose that \( T_0 \) is a maximal element of \( \mathcal{F}_F \). Take an arbitrary \( K \in \mathcal{F}_F \) satisfying \( K \subset (T_0^*)^{-1} \). Then we have that

\[
K = (T^{-1})^* \text{ for some } T \in \mathcal{F}_F \text{ and } T_0 \subset T, \text{ which implies by the maximality of } T_0 \text{ that } T = T_0 \text{ and } K = (T_0^*)^{-1}.
\]

Thus, \( (T_0^{-1})^* \) is a minimal element of \( \mathcal{F}_F \). Furthermore, we can similarly show that if \( (F_0^*)^{-1} \) is a minimal element of \( \mathcal{F}_F \), then \( T_0 \) is a maximal element \( \mathcal{F}_F \). The other statements are similarly shown.

Theorem 14. Let \( \mathcal{F} \) be a subset of \( C_p \). Then we have the following:

(1) The following statements are equivalent:

(i) \( \mathcal{F} \) has a maximal element.

(ii) There exists a closed operator \( A \) in \( \mathcal{H} \) such that \( T \subset A \) for all \( T \in \mathcal{F} \).

(iii) \( \mathcal{F}_F \) has a minimal element.

(2) The following statements are equivalent:

(i) \( \mathcal{F} \) has a minimal element.

(ii) There exists a closed operator \( B \) in \( \mathcal{H} \) such that \( (T^{-1})^* \subset B \) for all \( T \in \mathcal{F} \).

(iii) \( \mathcal{F}_F \) has a maximal element.

(3) The following statements are equivalent:

(i) \( \mathcal{F} \) has a maximal element and a minimal element.

(ii) There exist closed operators \( A \) and \( B \) in \( \mathcal{H} \) such that \( T \subset A \) and \( (T^{-1})^* \subset B \) for all \( T \in \mathcal{F} \).

Proof. (1) (i)\( \Rightarrow \) (ii) This is trivial.

(ii)\( \Rightarrow \) (i) Suppose that there exists a closed operator \( A \) in \( \mathcal{H} \) such that \( T \subset A \) for all \( T \in \mathcal{F} \). Then for any totally ordered subset \( C \) of \( \mathcal{F} \) we have \( D(A^*) \subset \cap_{T \in C} D(T^*) \). Hence, it follows that \( \cap_{T \in C} D(T^*) \) is dense in \( \mathcal{H} \), which implies by Lemma 12 that \( \mathcal{F} \) has an upper bounded element. By Zorn's lemma, \( \mathcal{F} \) has a maximal element.

(iii)\( \Rightarrow \) (i) This follows from Lemma 13.

(2) (i)\( \Rightarrow \) (ii) This is trivial.

(ii)\( \Rightarrow \) (i) Suppose that there exists a closed operator \( A \) in \( \mathcal{H} \) such that \( (T^{-1})^* \subset B \) for all \( T \in \mathcal{F} \). Then we can similarly show that \( \mathcal{F}_F \) has a maximal element, which implies by Lemma 13 that \( \mathcal{F} \) has a maximal element.

(iii)\( \Rightarrow \) (i) This follows from Lemma 13.

(3) This follows from (1) and (2). This completes the proof.

We remark that the closed operators \( A \) and \( B \) in Theorem 14 do not need any other conditions, for example, the existence of inverse.

By Theorem 14, we have the following.

Corollary 15. Let \( T_0 \subset C_p \) and put \( \mathcal{F}_{T_0} = \{ T \in C_p; \ T_0 \subset T \} \). Then the following statements hold.

(1) Suppose that there exists a closed operator \( A \) in \( \mathcal{H} \) such that \( T \subset A \) for all \( T \in \mathcal{F}_{T_0} \). Then there exists a maximal element of \( C_p \) which is an extension of \( T_0 \).

(2) Suppose that there exists a closed operator \( B \) in \( \mathcal{H} \) such that \( (T^{-1})^* \subset B \) for all \( T \in \mathcal{F}_{T_0} \). Then there exists a minimal element of \( C_p \) which is a restriction of \( T_0 \).

Proof. (1) By Theorem 14, \( \mathcal{F}_{T_0} \) has a maximal element \( T_1 \). Here we show that \( T_1 \) is a maximal element of \( C_p \). Indeed, this follows since \( T \in \mathcal{F}_{T_0} \) for any element \( T \) of \( C_p \) satisfying \( T_1 \subset T \). We can similarly show (2).

Next we investigate the existence of the smallest element and of the largest element of \( C_p \).

Theorem 16. \( C_p \) has the smallest and the largest element if and only if there exist closed operators \( A \) and \( B \) in \( \mathcal{H} \) such that \( T \subset A \) and \( (T^{-1})^* \subset B \) for all \( T \in C_p \).

Proof. Suppose that there exist closed operators \( A \) and \( B \) in \( \mathcal{H} \) such that \( T \subset A \) and \( (T^{-1})^* \subset B \) for all \( T \in C_p \). We define an operator \( T_0 \) as follows:

\[
D(T_0) = \bigcap_{T \in C_p} D(T),
\]

(46)

\[
T_0x = Tx, \quad x \in D(T_0),
\]

where \( T \) is an element of \( C_p \). Take an arbitrary \( x \in D(T_0) \) and \( T_1, T_2 \subset C_p \). Since \( x \in D(T_1), x \in D(T_2) \), \( T_1 \subset A \) and \( T_2 \subset A \), we have

\[
T_1x = T_2x = Ax.
\]

(47)
Thus, $T_0$ does not depend on the method of choosing $T \in C_{\varphi}$, and so $T_0$ is well defined. Since $|e_n| \not\subset \bigcap_{T \in C_{\varphi}} \Delta(T) = D(T_0)$, $T_0$ is a densely defined closed operator in $\mathcal{H}$ such that $T_0 \subset T$ for all $T \in C_{\varphi}$. Since $(T^{-1})^* \subset B$ for all $T \in C_{\varphi}$, we have $D(B^+) \subset \bigcap_{T \in C_{\varphi}} R(T)$, which implies that $\bigcap_{T \in C_{\varphi}} R(T)$ is dense in $\mathcal{H}$.

Hence, we can prove at the same way as the proof of Lemma 12 (2) that $T_0$ is the smallest element $C_{\varphi}$. Next we show that $C_{\varphi}$ has the largest element. Take an arbitrary $T \in C_{\varphi}$. Then $\psi_T$ is a generalized Riesz system with a constructing operator $T$ and $K < B$ and $(K^{-1})^* \subset A$. Hence, as shown above there exists the smallest element $K_1$ of $C_{\varphi}$, and so $K_1 = (T_1^{-1})^*$ for some $T_1 \in C_{\varphi}$ and $(T_1^{-1})^* \subset (T^{-1})^*$. Thus $T \subset T_1$, and $T_1$ is the largest element of $C_{\varphi}$. The converse is trivial. This completes the proof. □

As seen in Section 2, for a biorthogonal pair $\langle \{\psi_n\}, \{\varphi_n\}\rangle$, the equality $(T^{-1})^* e_n = \psi_n$, $n = 0, 1, \ldots$ does not necessarily hold for all $T \in C_{\varphi}$. From this fact we define the notion of natural pair of generalized Riesz systems.

**Definition 17.** A biorthogonal pair $\langle \{\varphi_n\}, \{\psi_n\}\rangle$ of generalized Riesz systems is said to be natural, if $C_{\varphi} = C_{N\psi}$ and $C_{\psi} = C_{N\varphi}$, that is, $(T^{-1})^* e_n = \psi_n$ for all $n$ and $T \in C_{\varphi}$ and $(K^{-1})^* e_n = \varphi_n$ for all $n$ and $K \in C_{\psi}$.

**Theorem 18.** Let $\langle \{\varphi_n\}, \{\psi_n\}\rangle$ be a natural pair of generalized Riesz systems. Then $C_{\varphi}$ and $C_{\psi}$ have the smallest element and the largest element, respectively, if and only if there exist closed operators $A$ and $B$ in $\mathcal{H}$ such that $T \subset A$ and $K \subset B$ for all $T \in C_{\varphi}$ and $K \in C_{\psi}$.

**Proof.** This is shown using Theorem 16 for the generalized Riesz systems $\{\varphi_n\}$ and $\{\psi_n\}$. □

For a generalized Riesz system $\{\varphi_n\}$, suppose that there exist the largest element $T_1$ of $C_{\varphi}$ and the smallest element $T_0$ of $C_{\varphi}$. Then every closed operator $T$ in $\mathcal{H}$ satisfying $T_0 \subset T \subset T_1$ is a constructing operator of $\{\varphi_n\}$, and so we can construct all kinds of non-self-adjoint Hamiltonians $T^*H^*T$, lowering operator $T^*AT^{-1}$ and raising operator $T^*BT^{-1}$ for $\{\varphi_n\}$. It may be possible to find constructing operators suitable for each of the physical models.

**4. Conclusions**

All the results presented in this paper are of pure mathematical nature, but we hope that they will be applied to more physical models in future. For example, we argue that cases like the CCR-algebras and their physical applications could probably studied by taking suitable constructing operators for convenient generalized Riesz systems.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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