FINITE LIFETIME EIGENFUNCTIONS OF COUPLED SYSTEMS OF HARMONIC OSCILLATORS

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Abstract. We consider a Hermite-type basis for which the eigenvalue problem associated to the operator $H_{A,B} := B(-\partial^2_x) + Ax^2$ acting on $L^2(\mathbb{R}; \mathbb{C}^2)$ becomes a three-terms recurrence. Here $A$ and $B$ are $2 \times 2$ constant positive definite matrices. Our main result provides an explicit characterization of the eigenvectors of $H_{A,B}$ that lie in the span of the first four elements of this basis when $AB \neq BA$.

1. Introduction

It is well known that the spectrum of the harmonic oscillator Hamiltonian
\[
H_\alpha := -\partial_x^2 + \alpha^2 x^2, \quad \alpha > 0,
\]
acting on $L^2(\mathbb{R})$ consists of the non-degenerate eigenvalues \(\{\alpha(2n + 1)\}_{n=0}^\infty\) with corresponding normalized eigenfunctions
\[
\phi_n^\alpha(x) = \frac{\alpha^{1/4} h_n(\alpha^{1/2} x) e^{-\alpha x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}},
\]
where $h_n(x) := (-1)^n e^{x^2} \partial_x^n [e^{-x^2}]$ is the $n$-th Hermite polynomial. The present paper is devoted to studying the spectrum of a matrix version of $H_\alpha$, the operator
\[
H_{A,B} := B(-\partial_x^2) + Ax^2,
\]
acting on $L^2(\mathbb{R}; \mathbb{C}^2)$, where $A$ and $B$ are two $2 \times 2$ constant positive definite matrices.

In contrast to the scalar situation, the spectral analysis of $H_{A,B}$ is far more involved due to the non-commutativity of the coefficients. If $AB = BA$, it is not difficult to find the eigenvalues and eigenfunctions of $H_{A,B}$ from those of $H_\alpha$. On the other hand, when $AB \neq BA$, the
eigenvalues and eigenfunctions of $H_{A,B}$ are connected to those of $H_\alpha$ in a highly non-trivial manner (see Theorem 3 below).

Our recent interest in describing spectral properties of operators such as $H_{A,B}$ arises from two sources. In a series of recent works, Parmeggiani and Wakayama, cf. [4], [5] and [6], characterize the spectrum of the operator $K_{\tilde{A},\tilde{B}} := \tilde{A}(-\partial_x^2 + x^2) + \tilde{B}(2x\partial_x + x^2)$ acting on $L^2(\mathbb{R}; \mathbb{C}^2)$, assuming that $\tilde{A}$ is definite positive and $\tilde{B} = -\tilde{B}'$. Although the two operators are related, it does not seem possible to obtain the eigenvalues of $H_{A,B}$ from those of $K_{\tilde{A},\tilde{B}}$. In [5] and [6] the eigenfunctions of $K_{\tilde{A},\tilde{B}}$ are found in terms of a twisted Hermite-type basis of $L^2(\mathbb{R}; \mathbb{C}^2)$. In this basis the eigenvalue problem associated to $K_{\tilde{A},\tilde{B}}$ becomes a three-term recurrence. The strategy presented below for analyzing the spectrum of $H_{A,B}$ will be similar.

Our second motivation is heuristic. It is known that the scalar harmonic oscillator, $H_\alpha$, achieves the optimal value for the constant in the Lieb-Thirring inequalities with power $\sigma \geq 3/2$, cf. [2]. It would be of great interest finding Hamiltonians with similar properties for Lieb-Thirring-type inequalities for magnetic Schrödinger and Pauli operators, cf. [1], [2] and [3]. Due to their close connection with the harmonic oscillator, both $K_{\tilde{A},\tilde{B}}$ and the presently discussed $H_{A,B}$ are strong candidates for further investigations in this direction.

The plan of the paper is as follows. Section 2 is devoted to describing elementary facts about $H_{A,B}$. In section 3 we consider a basis for which the eigenvalue problem associated to $H_{A,B}$ is expressed as a three-term recurrence. The main results are to be found in section 4 where we establish necessary and sufficient conditions, given explicitly in terms of the entries of $A$ and $B$, for an eigenfunction of $H_{A,B}$ to be the linear combination of the first four elements of this basis.

2. Elementary properties of $H_{A,B}$

We define $H_{A,B}$ rigorously as the self-adjoint operator whose domain, denoted below by $\mathcal{D}$, is the set of all

$$
\begin{pmatrix}
\phi \\
\psi
\end{pmatrix} \in L^2(\mathbb{R}; \mathbb{C}^2)
$$

such that

$$
\phi, \psi \in H^2(\mathbb{R}) \cap \left\{ f \in L^2(\mathbb{R}) : \int |x^2 f(x)|^2 < \infty \right\} = H^2(\mathbb{R}) \cap \widehat{H}^2(\mathbb{R}),
$$
where $H^2(\mathbb{R})$ denotes the Sobolev space of index (2,2) and “\(\hat{\cdot}\)” denotes Fourier transform. Since
\[
\langle H_{A,B} \left( \phi, \psi \right), \left( \phi, \psi \right) \rangle = \int \left[ B \left( \phi', \psi' \right) \cdot \left( \phi', \psi' \right) + A \left( x\phi, x\psi \right) \cdot \left( x\phi, x\psi \right) \right] \, dx \geq 0,
\]

$H_{A,B}$ is a symmetric operator. It is well known (cf. [7]) that if the domain of $H_\alpha$ is chosen to be $H^2(\mathbb{R}) \cap \hat{H}^2(\mathbb{R})$, then $H_\alpha$ is self-adjoint, non-negative and $S(\mathbb{R})$, the Schwartz space, is a core for $H_\alpha$. Thus $H_{A,B}$ with domain $D$ is a self-adjoint non-negative operator with core $S(\mathbb{R}; \mathbb{C}^2)$. Indeed, these properties are obvious when $A$ is a diagonal non-negative matrix and $B = Id$, the identity matrix. The general case follows by considering the factorization
\[
H_{A,B} = B^{1/2} E^* H_{C,Id} E B^{1/2},
\]
where $B^{-1/2} A B^{-1/2} = E^* C E$ is the Jordan diagonalization of the former matrix, and by using the fact that $D$ is invariant under the action of constant matrices.

**Lemma 1.** The spectrum of $H_{A,B}$ consists exclusively of isolated eigenvalues of finite multiplicity whose only accumulation point is $+\infty$. Moreover, if $\lambda_n$ denotes the $n$-th eigenvalue of this operator counting multiplicity, then
\[
a_1^{1/2} b_1^{1/2} (2n + 1) \leq \lambda_{2n+1} \leq \lambda_{2n+2} \leq a_2^{1/2} b_2^{1/2} (2n + 1),
\]
where $0 < a_1 \leq a_2$ and $0 < b_1 \leq b_2$, are the eigenvalues of $A$ and $B$, respectively.

**Proof.** It reduces to showing that
\[
\lambda_n(H_{A_1,B_1}) \leq \lambda_n(H_{A,B}) \leq \lambda_n(H_{A_2,B_2}),
\]
where $A_j = a_j(\text{Id})$ and $B_j = b_j(\text{Id})$. This follows directly from the min-max principle (cf. [8]), the estimates
\[
0 < \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} \leq A \leq \begin{pmatrix} a_2 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad 0 < \begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix} \leq B \leq \begin{pmatrix} b_2 & 0 \\ 0 & b_2 \end{pmatrix},
\]
and [2].

The above universal bound is not sharp in general and for most pairs $(A, B)$, $\lambda_{2n+1} \neq \lambda_{2n+2}$.

As we mentioned earlier, it is not difficult to compute the eigenvalues and eigenfunctions of $H_{A,B}$ when $A$ and $B$ commute. Indeed $AB = BA$ if, and only if, $A$ and $B$ have one (and hence both) eigenvectors in common. Let $w_j \neq 0$ be such that $Aw_j = a_j w_j$ and $Bw_j = b_j w_j$,
for $j = 1, 2$. Let $\phi^\alpha_j(x)$ be, as in (14), the eigenfunctions of $H_\alpha$. Let $\beta_j = \sqrt{a_j/b_j} > 0$. Then

$$H_{A,B}w_j\phi^\beta_n(x) = (-b_j\partial_x^2 + a_jx^2)w_j\phi^\beta_n(x) = b_j(-\partial_x^2 + (a_j/b_j)x^2)w_j\phi^\beta_n(x) = b_j^{1/2}a_j^{1/2}(2n + 1)w_j\phi^\beta_n(x).$$

By choosing $\|w_j\| = 1$, the family $\{w_j\phi^\beta_n(x) : j = 1, 2; n = 0, 1, \ldots \}$ is an orthonormal basis of $L^2(\mathbb{R}; \mathbb{C}^2)$, hence

$$\text{Spec } H_{A,B} = \{b_j^{1/2}a_j^{1/2}(2n + 1) : j = 1, 2; n = 0, 1, \ldots \}.$$ 

The analysis below will show that finding the eigenvalues and eigenfunctions of $H_{A,B}$ whenever $AB \neq BA$ is by no means of the trivial nature as the above case.

3. Hermite expansion of the eigenfunctions in the non-commutative case

Without further mention, we will often suppress the sub-indices in operator expressions. The structure of $H_{A,B} \equiv H$ allows us to decompose $L^2(\mathbb{R}; \mathbb{C}^2)$ into two invariant subspaces where the eigenvalue problem can be studied independently. We perform this decomposition as follows. Given $\alpha > 0$, let

$$\mathcal{H}^\alpha_+ := \text{Span} \{vx^je^{-\alpha x^2/2} : v \in \mathbb{C}^2, j = 2k, k = 0, 1, \ldots \},$$

$$\mathcal{H}^\alpha_- := \text{Span} \{vx^je^{-\alpha x^2/2} : v \in \mathbb{C}^2, j = 2k + 1, k = 0, 1, \ldots \},$$

and denote by $H^\pm = H|(\mathcal{D} \cap \mathcal{H}^\alpha_\pm)$. Since $\mathcal{H}^\alpha_\pm$ are invariant under $\partial_x^2$, multiplication by $x^2$ and action of constant matrices, these spaces are also invariant under $H$. Hence $H^\pm : \mathcal{D} \cap \mathcal{H}^\alpha_\pm \to \mathcal{H}^\alpha_\pm$ are self-adjoint operators and

$$\text{Spec } H = \text{Spec } H^+ \cup \text{Spec } H^-.$$ 

Let

$$L = 2^{-1/2}(x + \partial_x) \quad \text{and} \quad L^* = 2^{-1/2}(x - \partial_x)$$

be the annihilation and creation operators for the scalar harmonic oscillator. Then

$$L\phi^1_0 = 0, \quad L\phi^1_n = n^{1/2}\phi^1_{n-1} \quad \text{and} \quad L^*\phi^1_n = (n + 1)^{1/2}\phi^1_{n+1}.$$ 

From these relations one can easily deduce the recurrent identities

$$2\alpha x^2\phi^\alpha_n = (n + 2)^{1/2}(n + 1)^{1/2}\phi^\alpha_{n+2} + (2n + 1)\phi^\alpha_n + n^{1/2}(n - 1)^{1/2}\phi^\alpha_{n-2},$$

$$2\alpha^{-1}\partial_x^2\phi^\alpha_n = (n + 2)^{1/2}(n + 1)^{1/2}\phi^\alpha_{n+2} - (2n + 1)\phi^\alpha_n + n^{1/2}(n - 1)^{1/2}\phi^\alpha_{n-2},$$
where, here and elsewhere, any quantity with negative sub-index is zero. Since \( \{ \phi_n^\alpha \}_{n=0}^\infty \) is an orthonormal basis for \( L^2(\mathbb{R}) \), we can expand any vector of \( L^2(\mathbb{R}; \mathbb{C}^2) \) via

\[
\begin{align*}
\begin{pmatrix} \phi \\ \psi \end{pmatrix} &= \sum_{n=0}^\infty v_n \phi_n^\alpha,
\end{align*}
\]

for a suitable unique sequence \( (v_n) \in l^2(\mathbb{N}; \mathbb{C}^2) \). Moreover, denoting by \( N_\alpha := \alpha^{-1}A - \alpha B \) and \( M_\alpha := \alpha^{-1}A + \alpha B \),

\[
H \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \left[ (-\partial_x^2)B + x^2 A \right] \sum_{n=0}^\infty v_n \phi_n^\alpha
\]

\[
= \sum_{n=0}^\infty B v_n (-\partial_x^2) \phi_n^\alpha + A v_n x^2 \phi_n^\alpha
\]

\[
= \frac{1}{2} \sum_{n=0}^\infty (n+2)^{1/2} (n+1)^{1/2} N_\alpha v_n \phi_n^{\alpha+2} + (2n+1) M_\alpha v_n \phi_n^\alpha + n^{1/2} (n-1)^{1/2} N_\alpha v_n \phi_n^{\alpha-2}
\]

\[
= \frac{1}{2} \sum_{k=0}^\infty \left[ k^{1/2} (k-1)^{1/2} N_\alpha v_{k-2} + (2k+1) M_\alpha v_k + (k+2)^{1/2} (k+1)^{1/2} N_\alpha v_{k+2} \right] \phi_k^\alpha.
\]

Thus \( 2H^\pm \) are, respectively, similar to the block tri-diagonal matrices

\[
\begin{pmatrix}
S_0^\pm & T_1^\pm \\
T_1^\pm & S_1^\pm & T_2^\pm \\
& T_2^\pm & S_2^\pm & T_3^\pm \\
& & & & \ddots
\end{pmatrix}
\]

acting on \( l^2(\mathbb{N}; \mathbb{C}^2) \), where

\[
S_k^+ = (4k+1) M_\alpha, \quad T_k^+ = (2k)^{1/2} (2k-1)^{1/2} N_\alpha;
\]

\[
S_k^- = (4k+3) M_\alpha \quad \text{and} \quad T_k^- = (2k)^{1/2} (2k+1)^{1/2} N_\alpha.
\]

In order to reduce the amount of notation in our subsequent discussion, we consider \( H_{A,B} \) in canonical form as follows. If \( 0 \leq b_1 \leq b_2 \) are the eigenvalues of \( B \), let \( U^* (\text{diag}[b_1, b_2]) U \) be the diagonalization of \( B \), and set \( \tilde{A} := b_1^{-1} U A U^* \) and \( \tilde{B} := \text{diag}[1, b_2/b_1] \). Then

\[
H_{A,B} = b_1 U^* H_{\tilde{A}, \tilde{B}} U
\]

where

\[
\tilde{A} = \begin{pmatrix} a & \xi \\ \xi & c \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.
\]
Here the positivity of $A$ and $B$ is equivalent to the conditions

$$b \geq 1, \quad a, c > 0 \quad \text{and} \quad 0 \leq |\xi|^2 < ac.$$  

Furthermore notice that $AB = BA$ if, and only if, either $b = 1$ or $\xi = 0$. Hence, unless otherwise specified, we will consider without loss of generality that the pair $(A, B)$ is always the pair $(\tilde{A}, \tilde{B})$ in (5).

By virtue of the tri-diagonal representation (4), it seems natural to expect that the Hermite series (3) may be a good candidate for expanding the eigenfunctions of $H$. Not to mention that it is the obvious extension of the scalar and commutative cases. In this respect, we may consider “finite lifetime” series expansions of eigenfunctions $\Phi$ of $H$,

$$\Phi = \sum_{n=0}^{k} v_n \phi_{\alpha(n)}$$  

for suitable finite $k \in \mathbb{N} \cup \{0\}$, $\alpha(k) > 0$ and $v_n \in \mathbb{C}^2$. The results we present below show that, contrary to the above presumption, (3) is not such a good candidate for expanding $\Phi$ for small values of $k$. To be more precise, we show that for $k = 0, 1, 2, 3$, an expansion of type (6) is allowed only for a small sub-manifold of the region

$$R := \{(b, a, c, |\xi|) \in \mathbb{R}^4 : a, c > 0, b \geq 1, 0 \leq |\xi|^2 < ac\}$$  

corresponding to all positive definite pairs $(A, B)$.

We first discuss the cases $k = 0, 1$ and leave $k = 2, 3$ for the forthcoming section. The following result includes a family of test bases larger than the one considered in (6).

**Lemma 2.** Let $\phi_{\alpha_n}^\alpha(x)$ be the eigenfunctions of the scalar harmonic oscillator $H_{\alpha}$. Then $H_{A,B}$ has an eigenfunction of the type $\Phi(x) = (\tilde{a}\phi_{\alpha_n}^\alpha(x), \tilde{b}\phi_{\beta_m}^\beta(x))^t$ where $\tilde{a}, \tilde{b} \in \mathbb{C}$, $\alpha, \beta > 0$ and $m, n \in \mathbb{N} \cup \{0\}$, if and only if $AB = BA$.

**Proof.** If $H\Phi = \lambda\Phi$, then

$$-\tilde{a}(\phi_{\alpha_n}^\alpha)'' + ax^2\tilde{a}\phi_{\alpha_n}^\alpha + \xi x^2\tilde{b}\phi_{\beta_m}^\beta - \lambda \tilde{a}\phi_{\alpha_n}^\alpha = 0,$$

$$-\tilde{b}(\phi_{\beta_m}^\beta)'' + cx^2\tilde{b}\phi_{\beta_m}^\beta + \overline{\xi} x^2\tilde{a}\phi_{\alpha_n}^\alpha - \lambda \tilde{b}\phi_{\beta_m}^\beta = 0.$$  

If $\tilde{a} = 0$ or $\tilde{b} = 0$ in the above identities, necessarily $\xi = 0$ so $AB = BA$. Hence without loss of generality we can assume that $\tilde{a}\tilde{b} \neq 0$.

If $\alpha \neq \beta$, once again $\xi = 0$. Then we may suppose that $\alpha = \beta$. Since both left hand sides of the above identities are equal to $p(x) e^{-ax^2/2}$, where in both cases $p(x)$ is a polynomial of degree $2 + \max(m, n)$,
necessarily either \( \xi = 0 \) or \( m = n \). In the latter case, the above system is rewritten as

\[
\begin{align*}
-\tilde{a}(\phi_n^\alpha)'' + (a\tilde{a} + \xi\tilde{b})x^2 \phi_n^\alpha - \lambda\phi_n^\alpha &= 0, \\
-\tilde{b}(\phi_n^\alpha)'' + (\tilde{c}b + \xi\tilde{a})x^2 \phi_n^\alpha - \lambda\phi_n^\alpha &= 0.
\end{align*}
\]

Since \( \phi_n^\alpha \) is an eigenfunction of \( H_\alpha \) where \( \alpha > 0 \), necessarily \( \xi \in \mathbb{R} \).

Furthermore,

\[
a + \xi\tilde{b}/\tilde{a} = \alpha^2 = c/b + \xi\tilde{a}/(\tilde{b}) \quad \text{and} \quad a + \xi\tilde{b}/\tilde{a} = \lambda/(2n + 1) = bc + \xi\tilde{a}/\tilde{b}.
\]

Hence necessarily \( b = 1 \).

Since \( \mathcal{H}^\pm \) are invariant under the action of \( H \), and the even (resp. odd) terms in the series (6) belong to \( \mathcal{H}^+ \) (resp. \( \mathcal{H}^- \)), the above lemma ensures that \( \Phi = v_0\phi_0^\alpha + v_1\phi_1^\alpha \) is an eigenfunction of \( H \) if, and only if, \( A \) and \( B \) commute.

4. **Four-term expansion of eigenfunctions of** \( H_{A,B} \)

In this section we study necessary and sufficient conditions in order to guarantee that \( \Phi \in L^2(\mathbb{R}; \mathbb{C}^2) \), with finite lifetime expansion of the type (6) for \( k = 2 \) and 3, is an eigenfunction of \( H \) for suitable \( \alpha(k) > 0 \) when \( AB \neq BA \). In other words, assuming that \( \Phi \) satisfies the constraint

\[
\Phi = v_0\phi_0^\alpha + v_1\phi_1^\alpha + v_2\phi_2^\alpha + v_3\phi_3^\alpha,
\]

we aim to investigate conditions ensuring \( H\Phi = \lambda\Phi \).

Since

\[
v\phi_{2n}^\alpha \in \mathcal{H}^+_\alpha \quad \text{and} \quad v\phi_{2n+1}^\alpha \in \mathcal{H}^-_\alpha, \quad v \in \mathbb{C}^2
\]

for all \( n \in \mathbb{N} \cup \{0\} \), and the subspaces \( \mathcal{H}^\pm_\alpha \) are invariant under \( H \), we may consider the even and odd cases separately. To this end, let

\[
\Phi^+ = v_0\phi_0^\alpha + v_2\phi_2^\alpha \quad \text{and} \quad \Phi^- = v_1\phi_1^\alpha + v_3\phi_3^\alpha,
\]

\( \Phi^\pm \in \mathcal{H}^\pm \) respectively. Then our goal is to find necessary and sufficient conditions, given in terms of \( (b, a, c, |\xi|) \in R \setminus \partial R \), ensuring that \( \Phi^\pm \) is an eigenfunction of \( H^\pm \). The following is our main result.

**Theorem 3.** Let

\[
B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a & \xi \\ \xi & c \end{pmatrix},
\]

where \( b > 1, a, c > 0 \) and \( 0 < |\xi|^2 < ac \). Let \( \beta > 0 \) be such that \( \beta^2 \) is an eigenvalue of \( B^{-1/2}AB^{-1/2} > 0 \). Let

\[
\lambda_{\text{even}} := \frac{5\beta(ab + c - 2\beta^2b)}{a + c - (b + 1)\beta^2} \quad \text{and} \quad \lambda_{\text{odd}} := \frac{7\beta(ab + c - 2\beta^2b)}{a + c - (b + 1)\beta^2}.
\]
Then

i) \( H^+ \) has an eigenfunction \( \Phi^+(x) \) of type (8) if and only if

\[
2\lambda_{\text{even}}[a + c - (b + 1)\beta^2] = 5\beta(\beta - \lambda_{\text{even}})(\lambda_{\text{even}} - \beta b).
\]

In this case \( \alpha = \beta \) and \( H^+\Phi^+ = \lambda_{\text{even}}\Phi^+ \). Furthermore, \( \lambda_{\text{even}} \) is an eigenvalue of \( M_\beta \).

ii) \( H^- \) has an eigenfunction \( \Phi^-(x) \) of type (8) if and only if

\[
6\lambda_{\text{odd}}[a + c - (b + 1)\beta^2] = 7\beta(3\beta - \lambda_{\text{odd}})(\lambda_{\text{odd}} - 3\beta b).
\]

In this case \( \alpha = \beta \) and \( H^-\Phi^- = \lambda_{\text{odd}}\Phi^- \). Furthermore \( \lambda_{\text{odd}} \) is an eigenvalue of \( 3M_\beta \).

Notice that the conditions on \( a, b, c \) and \( \xi \) ensure that \( AB \neq BA \).

Proof. Put

\[
\Phi^+(x) = (u_0 + u_2x^2)e^{-\alpha x^2/2},
\]

for \( u_0, u_2 \neq 0 \). Since \( H^+ \) is similar to the tri-diagonal matrix (4), then

\[
H^+\Phi^+ = \lambda\Phi^+,
\]

if and only if

\[
S_0^+u_0 + T_1^+u_2 = \lambda u_0 \\
T_1^+u_0 + S_1^+u_2 = \lambda u_2 \\
T_2^+u_2 = 0.
\]

The latter equation implies that \( u_2 \in \ker N_\alpha \) and thus the first one implies that \( \lambda \) is an eigenvalue of \( S_0^+ = M_\alpha \) with associated eigenfunction \( u_0 \). A straightforward computation shows that the above system is equivalent to

\[
(A - \alpha^2B)u_2 = 0 \\
(A - \alpha^2B)u_0 + (5\alpha B - \lambda)u_2 = 0 \\
(\alpha B - \lambda)u_0 - 2Bu_2 = 0.
\]

The first equation holds if and only if \( \alpha = \beta \). Here

\[
\beta = +\sqrt{ab + c \pm \sqrt{(c - ab)^2 + 4|\xi|^2b}}/2b
\]

and \( u_2 = \left( \begin{array}{c} c - \beta^2 b \\ -\xi \end{array} \right) \). Notice that in this case

\[
0 = \det(A - \beta^2B) = (a - \beta^2)(c - \beta^2 b) - |\xi|^2.
\]

Let \( \tilde{u}_2 = \left( \begin{array}{c} a - \beta^2 \\ \xi \end{array} \right) \). Then \( \tilde{u}_2 \perp u_2 \) and

\[
(A - \beta^2B)\tilde{u}_2 = (a + c - (1 + b)\beta^2)\tilde{u}_2.
\]
Decompose

\[ u_0 = \gamma u_2 + \tilde{\gamma} \tilde{u}_2, \]

for suitable \( \gamma, \tilde{\gamma} \in \mathbb{C} \). Then the second identity of (11) holds if and only if, \( \lambda = \lambda_{\text{even}} \) and

(13) \[ \tilde{\gamma}(a + c - (1 + b)\beta^2)(a - \beta^2) = 5\beta(1 - b)|\xi|^2. \]

The third identity of (11) can be rewritten as the system

\[
\begin{align*}
\gamma(\beta - \lambda)(c - a^2b) + \tilde{\gamma}(\beta - \lambda)(a - \beta^2) &= 2(c - \beta^2b) \\
\gamma(\beta b - \lambda)(-\tilde{\xi}) + \tilde{\gamma}(\beta b - \lambda)(\tilde{\xi}) &= -2b\tilde{\xi}
\end{align*}
\]

in \( \gamma \) and \( \tilde{\gamma} \). By finding \( \tilde{\gamma} \) from this system (for instance by Newton’s method) and by equating to (13), a straightforward computation yields (9).

The proof of ii) is similar.

\[ \square \]

Notice that there is a duality of signs in the definition of \( \beta \) (cf. (12)), so conditions (9) and (10) comprise two possibilities each. Let

\[ \Omega_{\text{even}}^\pm := \{(b, a, c, |\xi|) \in R : (9) \text{ holds}\} \]

and

\[ \Omega_{\text{odd}}^\pm := \{(b, a, c, |\xi|) \in R : (10) \text{ holds}\}, \]

where the sign for the super-index is chosen in concordance to the sign in expression (12). By computing the partial derivatives of both sides of identities (9) and (11), a straightforward but rather long computation which we omit in the present discussion, shows that these four regions are smooth 3-manifolds embedded in \( R \subset \mathbb{R}^4 \), see (7).

The fact that \( \Omega_{\text{even}}^\pm \) are non empty is consequence of the following observation. By fixing \( \tilde{b} > 1 \) and \( |\tilde{\xi}| > 0 \), and by putting \( c = ab \), condition (9) can be rewritten as

\[
0 = \tilde{b}^{-1}(\tilde{b} + 1)^2[2(ab + c - 2\beta^2\tilde{b}) - (\beta - \lambda)(\lambda - \tilde{b}\beta)]
\]

\[
= \pm \frac{4|\tilde{\xi}|}{\tilde{b}} (\tilde{b} + 1)^2 - \beta^2(1 - 9\tilde{b})(9 - \tilde{b}),
\]

where \( \beta^2 = a \pm |\tilde{\xi}|\tilde{b}^{-1/2} \). Then \( (\tilde{b}, \tilde{a}, a\tilde{b}, |\tilde{\xi}|) \in \Omega_{\text{even}}^+ \) whenever

\[ \tilde{a} = -\frac{|\tilde{\xi}|(5\tilde{b}^2 - 90\tilde{b} + 5)}{\tilde{b}^{1/2}(9\tilde{b}^2 - 82\tilde{b} + 9)} \]

for \( 9 < \tilde{b} < 9 + 4\sqrt{5} \), and \( (\tilde{b}, \tilde{a}, a\tilde{b}, |\tilde{\xi}|) \in \Omega_{\text{even}}^- \) whenever

\[ \tilde{a} = \frac{|\tilde{\xi}|(5\tilde{b}^2 - 90\tilde{b} + 5)}{\tilde{b}^{1/2}(9\tilde{b}^2 - 82\tilde{b} + 9)} \]
for $1 < \tilde{b} < 9$ or $\tilde{b} > 9 + 4\sqrt{5}$. Furthermore, notice that if $(\tilde{b}, \tilde{a}, \tilde{a}, |\tilde{\xi}|) \in \Omega_{\text{even}}^\pm$, then

$$(\tilde{b}, r\tilde{a}, r\tilde{c}, r|\tilde{\xi}|) \in \Omega_{\text{even}}^\pm, \quad \text{for all} \quad r > 0.$$ 

This shows that $\Omega_{\text{even}}^\pm$ are non-empty, unbounded and the semi-axis \{$(b, 0, 0, 0) : b > 1$\} intersects $\partial \Omega_{\text{even}}^\pm$. All these properties also hold for $\Omega_{\text{odd}}^\pm$.

Furthermore, \(\Omega_{\text{even}}^+ \cap \Omega_{\text{odd}}^+ = \emptyset\) and \(\Omega_{\text{even}}^- \cap \Omega_{\text{odd}}^- = \emptyset\).

Indeed, if (9) and (10) hold at the same time for the same tetrad $(b, a, c, |\xi|)$, then, according to Theorem 3, $\lambda$ should be at the same time eigenvalue of $M_\beta$ and $3M_\beta$ (\(\beta\) defined with the same sign). Obviously the latter is a contradiction. In figure 1 we reproduce the projections of these four regions, onto the hyper-plane $|\xi| = 1$. These picture suggest that $\Omega^+ \cap \Omega^- \neq \emptyset$.

Finally, we may comment on the issue of considering a more general basis for expanding the eigenfunctions of $H$. One might think that a natural candidate for generalizing (6) is the finite expansion

$$\Phi = \sum_{j=0}^{m} \left( a_j \varphi_\alpha^j \right) \left( b_j \varphi_\beta^j \right),$$

where $a_j, b_j$ are complex number and $\alpha, \beta > 0$. We studied a particular case of this in Lemma 2. It turns out that if $\Phi$ is an eigenfunction of the above form, then either $\xi = 0$ or $\beta = \alpha$, so it should be as in (6). This can be easily proven by writing down the system for the eigenvalue
equation and considering the asymptotic behaviour of the identities as $x \to \infty$.

REFERENCES

[1] L. Erdős, J.P. Solovej “Magnetic Lieb-Thirring inequalities with optimal dependence on the field strength”. Preprint 2003, arXiv:math-ph/0306066.

[2] A. Laptev, T. Weidl, “Recent results on Lieb-Thirring inequality”, Université de Nantes. Exp. XX (2000) 1-14.

[3] A. Laptev, T. Weidl, “Sharp Lieb-Thirring inequalities in high dimensions”, Acta Math. 184 (2000) 87-111.

[4] A. Parmeggiani, M. Wakayama, “Oscillator representations and systems of ordinary differential equations”, Proc. Natl. Acad. Sci. USA 98 (2001) 26-30.

[5] A. Parmeggiani, M. Wakayama, “Non-commutative harmonic oscillators I”, Forum Math. 14 (2002) 538-604.

[6] A. Parmeggiani, M. Wakayama, “Non-commutative harmonic oscillators II”, Forum Math. 14 (2002) 609-690.

[7] M. Reed, B. Simon, Methods of modern mathematical physics, volume 2: self-adjointness, Academic press, New York, 1975.

[8] M. Reed, B. Simon, Methods of modern mathematical physics, volume 4: analysis of operators, Academic press, New York, 1978.

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