Qubits from extra dimensions

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(Dated: September 5, 2011)

Abstract

We link the recently discovered black hole-qubit correspondence to the structure of extra dimensions. In particular we show that for toroidal compactifications of type IIB string theory simple qubit systems arise naturally from the geometrical data of the tori parametrized by the moduli. We also generalize the recently suggested idea of the attractor mechanism as a distillation procedure of GHZ-like entangled states on the event horizon, to moduli stabilization for flux attractors in F-theory compactifications on elliptically fibered Calabi-Yau four-folds. Finally using a simple example we show that the natural arena for qubits to show up is an embedded one within the realm of fermionic entanglement of quantum systems with indistinguishable constituents.

PACS numbers: 03.67.-a, 03.65.Ud, 03.65.Ta, 02.40.-k
I. INTRODUCTION

In a remarkable paper Borsten et al. suggested that wrapped branes can be used to realize qubits, the basic building blocks used in quantum information. Based on the findings of that paper it is natural to expect that such brane configurations wrapped on different cycles of the manifold of extra dimensions should be capable of accounting for the surprising findings of the so called black hole qubit correspondence initiated in a series of papers (for a review see the paper of Borsten et al). The aim of the present paper is to show that by simply reinterpreting some of the well-known results of toroidal compactification of type IIB string theory in a quantum information theoretic fashion this expectation can indeed be justified. In particular we identify the Hilbert space giving home to the qubits inside the cohomology of the extra dimensions, establishing for the catchy phrase “to wrap or not to wrap, that is the qubit” a mathematical meaning, an issue left unclear by Ref. 1.

The black hole qubit correspondence is based on the observation that the macroscopic Bekenstein-Hawking entropy formulas of certain 4 and 5 dimensional black hole solutions of supergravity models arising from compactifications of string and M-theory happen to coincide with the ones of multipartite entanglement measures used in the theory of quantum entanglement. Though at first this observation was merely regarded as an intriguing mathematical coincidence however, it was soon realized that it can be quite useful on both sides of the correspondence in a much wider context. In particular we have learnt how to classify certain types of black hole solutions using different classes of entanglement, and more importantly using the input provided by string theory we have also seen how to obtain a complete solution to the classification problem of entanglement types of four-qubits using different classes of black hole solutions. The classification problem under stochastic local operation and classical communication (SLOCC) of entanglement classes for three qubits has been revisited, and recovered in an elegant manner using techniques originally developed within the realm of the supergravity literature. More recently a classification scheme for two-center black hole charge configurations for the stu, st² and t³ models based on the structure of four-qubit SLOCC invariants and elliptic curves has been proposed. The structure of black hole entropy formulas also inspired the construction of new and useful tripartite measures for electron correlation and more generally for quantum systems with both indistinguishable and distinguishable constituents. Moreover, using the input
coming from string theory it was shown that for such simple quantum systems the SLOCC classification problem of entanglement classes can be solved\textsuperscript{14}.

Apart from issues concerning entanglement classes and their associated entanglement measures, the black hole-qubit correspondence also turned out to provide additional insight into issues of dynamics of entangled systems. In particular it has been shown that the well-known attractor mechanism\textsuperscript{15} of moduli stabilization can be reinterpreted in the language of quantum information as a distillation procedure of highly entangled charge states on the event horizon\textsuperscript{4,16}. It was also realized that quantum error correcting codes can be used to serve as a quantum information theoretic framework for characterizing the properties of the BPS and non-BPS attractors\textsuperscript{16,17}.

What is the mathematical origin of the black hole-qubit correspondence? Apart from arguments\textsuperscript{2–5} based on the realization that on both sides of the correspondence similar symmetry structures are present, none of these studies have addressed the important question where are these qubits reside, how the Hilbert spaces for the analogues of the usual multipartite systems of quantum information are constructed. In this paper we would like to make a step in the direction of clarifying this important issue.

The crucial observation is the fact that the various aspects of supergravity models amenable to a quantum information theoretical interpretation can all be obtained from toroidal compactifications of type IIA, IIB or M-theory. Hence it is natural to link the occurrence of qubits and qutrits in these 4 and 5 dimensional scenarios to the geometric data of tori i.e. to the extra dimensions.

In this paper we will concentrate merely on qubits and work in the type IIB duality frame. In Section II. as a warm up excercise, we show how deformed tori give rise to a parametrized family of one-qubit systems. In Section III. we analyse the archetypical example of the black hole qubit correspondence-the \textit{stu} model\textsuperscript{18}. Coming from compactification on a six torus \(T^6\) in the type IIB duality frame this model is featuring three-qubit systems. However, unlike our warm up exercise this case already featuring entanglement, namely the tripartite one. The attractor mechanism as a distillation procedure\textsuperscript{16} is shown to arise naturally in this picture. In Section IV. we generalize our constructions to flux attractors\textsuperscript{19}. We show that the idea of distillation works nicely within the context of F-theory compactifications on elliptically fibered Calabi-Yau four-folds too. Here the toroidal case gives rise to four-qubit systems. As an explicit example we revisit and reinterpret the solution found by Larsen and
O’Connell in the language of four-qubit entangled systems. In section V. we emphasize that our simple qubit systems associated with the geometric data of extra dimensions (tori) are giving examples to entanglement between subsystems with \textit{distinguishable} constituents. However, by studying a simple example we show that, in the stringy context the natural arena where these very special entangled systems live is really the realm of fermionic entanglement\textsuperscript{21,22} of subsystems with \textit{indistinguishable} parts. The notion ”fermionic entanglement” is simply associated with the structure of the cohomology of $p$-forms related to $p$-branes. Our conclusions and some comments are left for Section VI.

II. ONE-QUBIT SYSTEMS FROM DEFORMED TORI

Let us consider a torus $T^2$ with its complex structure deformations labelled by

$$
\tau \equiv x - iy \quad y > 0.
$$

Here our choice for $\tau$ to have a negative imaginary part is dictated by the conventions used in the supergravity literature\textsuperscript{23,24}. We take the complex coordinates on $T^2$ to be $z = u + \tau v$ hence we can define the holomorphic and antiholomorphic one forms that are elements of the cohomology classes $H^{(1,0)}(T^2, \mathbb{C})$ and $H^{(0,1)}(T^2, \mathbb{C})$ respectively as

$$
\Omega_0 = dz = du + \tau dv, \quad \overline{\Omega}_0 = d\bar{z} = du + \bar{\tau} dv.
$$

As it is well-known the Teichmüller space of $T^2$ parametrized by $\tau$ of Eq.(1) has a Kähler metric $g_{\tau\tau} = \partial_\tau \partial_{\bar{\tau}} K$ coming from the Kähler potential

$$
K = -\log(2y).
$$

Notice that adopting the convention $\int_{T^2} du \wedge dv = 1$ we have the relation

$$
ie^{-K} = \int_{T^2} \Omega_0 \wedge \overline{\Omega}_0.
$$

Our choice for the volume form on $T^2$ is

$$
\omega = id\overline{z} \wedge dz.
$$

Now the Hodge star is defined by the formula $(\varphi, \varphi)\omega = \varphi \wedge \star \overline{\varphi}$, hence for $\varphi = \Omega_0 = dz$ and its conjugate, $(\varphi, \varphi) = 1$, we get

$$
* dz = idz, \quad *d\overline{z} = -id\overline{z}.
$$
Let us now define the one-form $\Omega$ as

$$\Omega \equiv e^{K/2}\Omega_0. \quad (7)$$

Due to the relations

$$(\tau - \tau) (\partial_\tau + \partial_K z) dz = d\bar{z}, \quad \partial_\tau d\bar{z} = 0 \quad (8)$$

the flat Kähler covariant derivative defined as

$$D_\tau \Omega \equiv (\tau - \tau) D_\tau \Omega \equiv (\tau - \tau) \left( \partial_\tau + \frac{1}{2} \partial_\tau K \right) \Omega, \quad (9)$$

$$D_\tau \Omega \equiv (\bar{\tau} - \tau) \left( \partial_\tau - \frac{1}{2} \partial_\tau K \right) \Omega \quad (10)$$

is acting as

$$D_\tau \Omega = \bar{\Omega}, \quad D_\tau \bar{\Omega} = 0 \quad (11)$$

In order to reinterpret one-forms on $T^2$ as qubits we use the hermitian inner product

$$\langle \xi | \eta \rangle \equiv \int_{T^2} \xi \wedge * \eta. \quad (12)$$

Now one can show that the correspondence

$$i\Omega \leftrightarrow |0\rangle \quad i\bar{\Omega} \leftrightarrow |1\rangle \quad (13)$$

gives rise to a mapping of basis states of one-forms to basis states for qubits. By an abuse of notation we use the same $\langle | \rangle$ notation for the Hermitian inner product on the Hilbert space of qubits i.e. $\mathcal{H} \simeq \mathbb{C}^2$ too. Now we have the usual properties $\langle 0|0 \rangle = \langle 1|1 \rangle = 1$ and $\langle 0|1 \rangle = \langle 1|0 \rangle = 0$. By virtue of this mapping one can reinterpret Eq. (11) as

$$\sigma_+ |0\rangle = |1\rangle, \quad \sigma_+ |1\rangle = 0 \quad (14)$$

i.e. the flat covariant derivatives act as projective bit flip errors on the basis states. Similarly the action of the adjoint of the flat covariant derivative $D_\tau$ can be reinterpreted as

$$\sigma_- |0\rangle = 0, \quad \sigma_- |1\rangle = |0\rangle \quad (15)$$

Notice also that our association as given by Eq. (13) represents the diagonality of the Hodge star operation i.e. $*\Omega = i\Omega$, $*\bar{\Omega} = -i\bar{\Omega}$ in the form

$$* |0\rangle = -|0\rangle, \quad *|1\rangle = +|1\rangle \quad (16)$$
i.e. the action of $\ast$ is represented by the sign flip operator $-\sigma_3$.

Now in the context of superstring compactifications the cohomology classes are real. By virtue of Poncaré duality these classes are answering the real homology cycles representing brane configurations wrapped on (for example supersymmetric) cycles. In the qubit picture this means that our qubits have to satisfy extra reality conditions. Moreover, in Calabi-Yau compactifications self-duality of the usual five-form in the type IIB duality frame gives a distinguished role to basis states that diagonalize the Hodge star operator on the Calabi-Yau space. In this context our torus model should be related to the illustrative example of Suzuki where a self-dual three-form was considered in a compactification model to four space-time dimensions of the form $M \times T^2$. Hence owing to the special status of Hodge diagonal states, in the qubit picture we attach to our basis states $|0\rangle$ and $|1\rangle$ of Eq. (13) a special role calling them in the following states of the computational base.

Let us now write the real cohomology class $\Gamma \in H^1(T^2, \mathbb{R})$ in the form

$$\Gamma = p\alpha - q\beta, \quad \alpha = du, \quad \beta = dv. \quad (17)$$

Using the expression $\Omega_0 = \alpha + \tau\beta$ and its conjugate one can express this in the Hodge diagonal basis as follows

$$\Gamma = -e^{K/2}(p\tau + q)i\Omega + e^{K/2}(p\tau + q)i\overline{\Omega}. \quad (18)$$

According to our correspondence between one-forms and qubits we can represent this as a state in the computational base satisfying an extra reality condition

$$|\Gamma\rangle = \Gamma_0|0\rangle + \Gamma_1|1\rangle, \quad \Gamma_1 = -\overline{\Gamma}_0 = e^{K/2}(p\tau + q). \quad (19)$$

Notice that although the state itself is not, but both the amplitudes $\Gamma_{0,1}$ and the (computational) basis vectors $|0\rangle$ and $|1\rangle$ display an implicit dependence on the modulus $\tau$. We note also that after imposing the usual Dirac-Zwanziger quantization condition on $p$ and $q$ $\Gamma$ should rather be interpreted as an element of $H^1(T^2, \mathbb{Z})$.

Notice also that the state $|\Gamma\rangle$ is unnormalized with norm squared satisfying

$$||\Gamma||^2 = \langle \Gamma|\Gamma \rangle = 2e^K|p\tau + q|^2 = \frac{1}{y}|p\tau + q|^2. \quad (20)$$

In Quantum Information this is not a problem since the protocols demanded by quantum manipulations are not always represented by unitary operators preserving the norm. In the
theory of quantum entanglement one can consider for instance manipulations converting a
state to another one and vice versa with a probability less than one. For a single qubit these
manipulations are represented by the invertible operations. The nontrivial content of such
manipulations is encapsulated by the group $SL(2, \mathbb{C})$. For transformations also respecting
some additional structure (e.g. our reality condition) the allowed set of manipulations will
be comprising a subgroup of this group. For our state $|\Gamma\rangle$ it is easy to check that the set
of transformations $A$ of the form $|\Gamma\rangle \mapsto A|\Gamma\rangle$ respecting the reality condition is comprising
the subgroup $SU(1, 1)$ of $SL(2, \mathbb{C})$.

Notice also that in matrix representation the state $|\Gamma\rangle$ can be given the form
\[
\begin{pmatrix}
\Gamma_0 \\
\Gamma_1
\end{pmatrix} = \frac{1}{\sqrt{2y}} \begin{pmatrix}
\tau & -1 \\
-\tau & 1
\end{pmatrix} \begin{pmatrix}
-p \\
q
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
i & -1 \\
-1 & 1
\end{pmatrix} \frac{1}{\sqrt{y}} \begin{pmatrix}
y & 0 \\
-x & 1
\end{pmatrix} \begin{pmatrix}
-p \\
q
\end{pmatrix}.
\] (21)

On the right hand side the first matrix is unitary, and the second is an element of $SL(2, \mathbb{R})$.
Using this unitary matrix one can switch to another basis different from our computational
one. In this new basis the subgroup of admissible transformations is $SL(2, \mathbb{R})$. We also
remark that the norm squared $||\Gamma||^2$ is a unitary invariant and a symplectic i.e. $SL(2, \mathbb{R})$
one at the same time. The latter invariance means that under the usual set of combined
transformations
\[
\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix}
-p \\
q
\end{pmatrix} \mapsto \begin{pmatrix}
d & c \\
b & a
\end{pmatrix} \begin{pmatrix}
-p \\
q
\end{pmatrix}, \quad ad - bc = 1
\] (22)
the norm squared remains invariant.

Note, that the matrix form of Eq.(21) leaves obscure the fact that the corresponding
basis vectors $|0\rangle$ and $|1\rangle$ are depending on the coordinates of the torus and the modulus $\tau$.
More precisely the set \{0, 1\} refers to families of basis vectors parametrized by $\tau$. (The
variables $u$ and $v$ on the other hand are associated with the Hilbert space structure on $T^2$
with inner product defined by Eq.(12).) Since the possible notation $|0, 1(\tau)\rangle$, $\Gamma_{0,1}(\tau, p, q)$
displaying all the implicit structures in Eq.(19) is awkward we leave the symbols $\tau$ and
tacitly assume that the computational basis has an implicit dependence on $\tau$. With these
conventions our state now has the deceptively simple appearance
\[
|\Gamma\rangle = S|\gamma\rangle = US|\gamma\rangle, \quad |\gamma\rangle = -p|0\rangle + q|1\rangle
\] (23)
where the operators $S, S, U$ are the ones with matrix representatives easily identified after looking at Eq.(21).

III. STU MODEL AND THREE-QUBITS FROM $H^3(T^6, \mathbb{C})$

A. Three qubit systems

The STU model is an $N = 2$ supergravity model\textsuperscript{18} coupled to three vector multiplets interacting via scalars belonging to the special Kähler manifold $[SL(2, \mathbb{R})/SO(2)]^3$. There are many ways embedding this model to string/M-theory. Here following Borsten et.al.\textsuperscript{1} we use an embedding to type IIB string theory compactified on the six torus $T^6$, with a three-qubit interpretation. As was emphasized in that paper\textsuperscript{1} the number of qubits is three because we have now three copies of $T^2$s corresponding to the six extra dimensions in string theory. Due to the presence of three qubits here the new phenomenon of (quantum) entanglement appears, and wrapped $D3$ brane configurations can effectively be described by such entangled tripartite states. The aim of the present subsection is to clarify what do we mean by states in this context, an issue left obscure in the paper of Borsten et.al.\textsuperscript{1}.

In order to do this we just have to generalize our single-qubit considerations related to $T^2$ known from the previous subsection, to three-qubits now related to $T^6 = T^2 \times T^2 \times T^2$. We introduce the coordinates

$$z^a = u^a + \tau^a v^a, \quad \tau^a = x^a - iy^a \quad y^a > 0, \quad a = 1, 2, 3 \quad (24)$$

and the holomorphic three-form

$$\Omega_0 = dz^1 \wedge dz^2 \wedge dz^3. \quad (25)$$

We have as usual

$$\int_{T^6} \Omega_0 \wedge \overline{\Omega}_0 = i(8y_1y_2y_3) = ie^{-K}, \quad (26)$$

where $K$ is the Kähler potential giving rise to the metric $g_{\alpha\overline{\beta}} = \partial_\alpha \overline{\partial}_\overline{\beta} K$ on the special Kähler manifold $[SL(2, \mathbb{R})/SO(2)]^3$. Let us again introduce $\Omega$ as in Eq.(7), and define flat covariant derivatives $D_\alpha$ acting on $\Omega$ as

$$D_\alpha \Omega = (\tau^\alpha - \tau^a)D_\alpha \Omega = (\tau^\alpha - \tau^a) \left( \partial_\alpha + \frac{1}{2} \partial_\alpha K \right) \Omega, \quad (27)$$
where $\partial_a = \partial/\partial \tau^a$. Then one has

$$\Omega = e^{K/2}dz^1 \wedge dz^2 \wedge dz^3, \quad \overline{\Omega} = e^{K/2}d\overline{z}^1 \wedge d\overline{z}^2 \wedge d\overline{z}^3,$$

(28)

$$D_1 \Omega = e^{K/2}d\overline{z}^1 \wedge dz^2 \wedge dz^3, \quad \overline{D_1 \Omega} = e^{K/2}dz^1 \wedge d\overline{z}^2 \wedge d\overline{z}^3,$$

(29)

$$D_2 \Omega = e^{K/2}dz^1 \wedge d\overline{z}^2 \wedge dz^3, \quad \overline{D_2 \Omega} = e^{K/2}d\overline{z}^1 \wedge dz^2 \wedge d\overline{z}^3,$$

(30)

$$D_3 \Omega = e^{K/2}dz^1 \wedge d\overline{z}^2 \wedge d\overline{z}^3, \quad \overline{D_3 \Omega} = e^{K/2}d\overline{z}^1 \wedge d\overline{z}^2 \wedge dz^3,$$

(31)

Notice that we have the identities

$$\int_{T^6} \Omega \wedge \overline{\Omega} = i, \quad \int_{T^6} D_a \Omega \wedge \overline{D_b \Omega} = -i \delta_{ab}. \quad (32)$$

Let us revisit\textsuperscript{26} the action of the Hodge star on our basis of three-forms as given by Eq.(28)-(31). For a form of $(p, q)$ type the action of the Hodge star is defined as

$$(\varphi, \varphi) \omega^n \equiv \varphi \wedge \ast \varphi \quad (33)$$

where for our $T^6$ in accord with our conventions

$$\omega = i(d\overline{z}^1 \wedge dz^1 + d\overline{z}^2 \wedge dz^2 + d\overline{z}^3 \wedge dz^3) \quad (34)$$

moreover, we have

$$(\varphi, \varphi) \equiv \frac{1}{pq!} \sum |\varphi_{j_1...j_p k_1...k_q}|^2. \quad (35)$$

For our basis forms like $\varphi \equiv dz^1 \wedge dz^2 \wedge dz^3$ e.t.c. showing up in Eq.(28)-(31) $(\varphi, \varphi) = 1$ hence we get

$$\ast \Omega = i\Omega, \quad \ast \overline{\Omega} = -i\overline{\Omega} \quad (36)$$

$$\ast D_a \Omega = -iD_a \Omega, \quad \ast \overline{D_a \Omega} = i\overline{D_a \Omega} \quad (37)$$

i.e. our conventions are differing by a sign from the ones of Denef\textsuperscript{26}.

Now we regard the 8 complex dimensional untwisted primitive part\textsuperscript{28} of the 20 dimensional space $H^3(T^6, \mathbb{C}) \equiv H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$ equipped with the Hermitian inner product

$$\langle \varphi | \eta \rangle \equiv \int_{T^6} \varphi \wedge \ast \overline{\eta} \quad (38)$$

as a Hilbert space isomorphic to $\mathcal{H} \equiv (\mathbb{C}^2)^{\times 3} \sim \mathbb{C}^8$ of three qubits. In order to set up the correspondence between the three-forms and the basis vectors of the three-qubit system we use the negative of the basis vectors $\Omega, D_1 \Omega$ etc. multiplied by the imaginary unit $i$.  

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We opted for using an extra minus sign since after changing the order of the one-forms we have for example for $-iD_1\Omega = ie^{K/2}dz^3 \wedge dz^2 \wedge dz^1$ hence we can take its representative basis qubit state $|001\rangle$ which corresponds to the usual binary labelling provided we label the qubits from the right to the left. Due to these conventions we take the basis states of our computational base to be given by the correspondence

$$-i\Omega \leftrightarrow |000\rangle, \quad -iD_1\Omega \leftrightarrow |001\rangle, \quad -iD_2\Omega \leftrightarrow |010\rangle, \quad -iD_3\Omega \leftrightarrow |100\rangle$$

$$-i\overline{\Omega} \leftrightarrow |111\rangle, \quad -i\overline{D}_1\Omega \leftrightarrow |110\rangle, \quad -i\overline{D}_2\Omega \leftrightarrow |011\rangle, \quad -i\overline{D}_3\Omega \leftrightarrow |011\rangle.$$  \hspace{1cm} (39)

Now the locations of the 1s correspond to the slots where complex conjugation is effected. One can check that the states above form a basis with respect to the inner product of Eq.(38) with the usual set of properties on the three-qubit side. A further check shows that the action of the flat covariant derivatives $D_a, j = 1, 2, 3$ corresponds to the action of the projective bit flips of the form $I \otimes I \otimes \sigma_+, \quad I \otimes \sigma_+ \otimes I$ and $\sigma_+ \otimes I \otimes I$, where $I$ is the $2 \times 2$ identity matrix. For the conjugate flat covariant derivatives $\sigma_+$ has to be replaced by $\sigma_-$. Moreover, the diagonal action of the Hodge star in the computational base is represented by the corresponding action of the negative of the parity check operator $\sigma_3 \otimes \sigma_3 \otimes \sigma_3$. 

Now for a three-form representing the cohomology class of a wrapped $D3$ brane configuration we take

$$\Gamma = p^I \alpha_I - q_I \beta^I \in H^3(T^6, \mathbb{Z}),$$

with summation on $I = 0, 1, 2, 3$ and

$$\alpha_0 = du^1 \wedge du^2 \wedge du^3, \quad \beta^0 = -dv^1 \wedge dv^2 \wedge dv^3$$

$$\alpha_1 = dv^1 \wedge du^2 \wedge du^3, \quad \beta^1 = du^1 \wedge dv^2 \wedge dv^3$$

with the remaining ones obtained via cyclic permutation. With the choice of orientation $\int_{T^6} (du^1 \wedge dv^1) \wedge (du^2 \wedge dv^2) \wedge (du^3 \wedge dv^3) = 1$ we have $\int_{T^6} \alpha_I \wedge \beta^J = \delta^J_I$.

It is well-known\textsuperscript{26} that in the Hodge diagonal basis we can express this as

$$\Gamma = iZ(\Gamma) \overline{\Omega} - ig^{jk}D_jZ(\Gamma) \overline{D_k} \overline{\Omega} + \text{c.c.} = iZ(\Gamma) \overline{\Omega} - i\delta^{j,k}D_jZ(\Gamma) \overline{D_k} \overline{\Omega} + \text{c.c.}$$

Here $Z(\Gamma) = \int_{T^6} \Gamma \wedge \Omega$ is the central charge. For the STU model the explicit form of $Z(\Gamma)$ is\textsuperscript{18}

$$Z(\Gamma) = e^{K/2}W(\tau^3, \tau^2, \tau^1)$$
where

\[
W(\tau^3, \tau^2, \tau^1) = q_0 + q_1 \tau^1 + q_2 \tau^2 + q_3 \tau^3 + p^1 \tau^2 \tau^3 + p^2 \tau^1 \tau^3 + p^3 \tau^1 \tau^2 - p^0 \tau^1 \tau^2 \tau^3.
\] (46)

Now using our basic correspondence between three-forms and three-qubit states of Eq. (39)-(40) we can write \( \Gamma \leftrightarrow |\Gamma\rangle \) where

\[
|\Gamma\rangle = \Gamma_{000}|000\rangle + \Gamma_{001}|001\rangle + \cdots + \Gamma_{110}|110\rangle + \Gamma_{111}|111\rangle,
\] (47)

where

\[
\Gamma_{111} = -e^{K/2}W(\tau^3, \tau^2, \tau^1) = -\bar{\Gamma}_{000},
\] (48)

\[
\Gamma_{001} = -e^{K/2}W(\tau^3, \tau^2, \tau^1) = -\bar{\Gamma}_{110}
\] (49)

and the remaining amplitudes are given by cyclic permutation.

Let us now put the 8 charges \( p_I \) and \( q_I \) with \( I = 0, 1, 2, 3 \) to a \( 2 \times 2 \times 2 \) array \( \gamma_{kji} \) as follows

\[
\begin{pmatrix}
\gamma_{000} & \gamma_{001} & \gamma_{010} & \gamma_{100} \\
\gamma_{111} & \gamma_{110} & \gamma_{101} & \gamma_{011}
\end{pmatrix} =
\begin{pmatrix}
-p^0 & -p^1 & -p^2 & -p^3 \\
-q_0 & q_1 & q_2 & q_3
\end{pmatrix}.
\] (50)

Now it can be shown that the three-qubit state of Eq.(47) can alternatively be written in the following form

\[
|\Gamma\rangle = S_3 \otimes S_2 \otimes S_1 |\gamma\rangle
\] (51)

where

\[
|\gamma\rangle = \gamma_{000}|000\rangle + \gamma_{001}|001\rangle + \cdots + \gamma_{110}|110\rangle + \gamma_{111}|111\rangle,
\] (52)

and the matrix representative of the operator \( S_3 \otimes S_2 \otimes S_1 \) is

\[
\frac{1}{\sqrt{8y^3y^2y^1}}
\begin{pmatrix}
\tau^3 & -1 \\
-\tau^3 & 1
\end{pmatrix} \otimes
\begin{pmatrix}
\tau^2 & -1 \\
-\tau^2 & 1
\end{pmatrix} \otimes
\begin{pmatrix}
\tau^1 & -1 \\
-\tau^1 & 1
\end{pmatrix}.
\] (53)

The reader should compare this expression with the one obtained for the single qubit case as shown by Eqs.(21) and (23).

A state similar to \( |\Gamma\rangle \) of Eq.(51) has already appeared in our recent papers. It is important to realize however, the basic difference between \( |\Gamma\rangle \) and that state. The state of Ref.4 is a charge and moduli dependent state connected to the 4 dimensional setting of the STU model. Moreover, in that setting the basis states \( |kji\rangle \) had no obvious physical
meaning. They merely served as basis vectors providing a suitable frame for a three-qubit reformulation.

Now $|\Gamma\rangle$ is a state which is depending on the charges the moduli and the coordinates of the extra dimensions, hence this state is connected to a 10 dimensional setting of the STU model in the type IIB duality frame. Now the basis vectors $|kji\rangle$ have an obvious physical meaning: they are the Hodge diagonal complex basis vectors of the untwisted primitive part of the third cohomology group of the extra dimensions i.e. of $H^3(T^6, \mathbb{C})$. They are also basis vectors of a genuine Hilbert space equipped with a natural Hermitian inner product of Eq.(38), isomorphic to the usual one of three-qubits. The state $|\Gamma\rangle$ has the meaning as the Poincaré dual of the homology cycle representing wrapped $D_3$ brane configurations. $|\Gamma\rangle$ can be represented in two different forms: namely as in Eq.(47) (expansion in a Hodge-diagonal moduli dependent complex base), or in an equivalent way based on the qubit version of Eq.(41) (Hodge-non-diagonal but moduli independent real base).

In closing this section we present the analogue of Eq.(20) i.e. the norm of $|\Gamma\rangle$

$$||\Gamma||^2 = 2e^K(|W(\tau^3, \tau^3, \tau^1)|^2 + |W(\bar{\tau}^3, \tau^2, \tau^1)|^2 + W(\tau^3, \bar{\tau}^2, \tau^1)|^2 + |W(\tau^3, \tau^2, \tau^1)|^2). \quad (54)$$

This expression is just 2 times $V_{BH}$, the well-known black hole potential. For a three-qubit based reformulation of $V_{BH}$ see Ref. Now its new interpretation as half the norm of a three-qubit state involves integration with respect to the coordinates of the extra dimensions (see Eq.(38)). It is obvious by construction that $V_{BH}$ is a unitary and symplectic invariant ($SL(2, \mathbb{R}^3 \subset Sp(8, \mathbb{R})$) at the same time. In order to see this one just has to recall our considerations for the single qubit case encapsulated in Eqs. (21) - (22).

### B. BPS attractors

As a first application showing the usefulness of rephrasing well-known results concerning the STU model in a three-qubit language let us consider the case of the BPS attractors. In this case the BPS conditions read as (which is also a requirement of unbroken supersymmetry)

$$D_aZ = 0. \quad (55)$$

Let us write the superpotential in the three-qubit form as

$$W(\tau^3, \tau^2, \tau^1) = \Gamma_{kji}\epsilon^{k}\epsilon^{j}\epsilon^{i}= \Gamma_{kji}\varepsilon^{ij}i_{kj}'\varepsilon^{j}'k_{ij}'b_{i}'a_{i}' \quad (56)$$
where summation over $i', j', k' = 0, 1$ is understood and $\varepsilon^{01} = -\varepsilon^{10} = 1$ are the nonzero components of the usual $SL(2)$ invariant $2 \times 2$ matrix, and

$$a_i \leftrightarrow \begin{pmatrix} 1 \\ \tau^1 \end{pmatrix}, \quad b_j \leftrightarrow \begin{pmatrix} 1 \\ \tau^2 \end{pmatrix}, \quad c_k \leftrightarrow \begin{pmatrix} 1 \\ \tau^3 \end{pmatrix}. \quad (57)$$

Then the BPS attractors are characterized by the equations

$$W(\tau^3, \tau^2, \tau^1) = 0, \quad W(\tau^3, \bar{\tau}^2, \tau^1) = 0, \quad W(\tau^3, \tau^2, \bar{\tau}^1) = 0. \quad (58)$$

and their complex conjugates. According to Eqs. (48)-(49) in our three-qubit language this corresponds to

$$\Gamma_{001} = \Gamma_{010} = \Gamma_{100} = \Gamma_{110} = \Gamma_{101} = \Gamma_{011} = 0. \quad (59)$$

This means that at the black hole horizon after moduli stabilization only the $\Gamma_{000}$ and $\Gamma_{111}$ amplitudes of our state $|\Gamma\rangle$ survives. Hence the unfolding of the attractor flow towards its fixed point can be reinterpreted as a distillation procedure of a GHZ state of the form

$$|\Gamma\rangle_{fix} \equiv \Gamma_{000}|000\rangle_{fix} + \Gamma_{111}|111\rangle_{fix}, \quad \Gamma_{111} = -\bar{\Gamma}_{000} = Z(\tau^3_{fix}, \tau^2_{fix}, \tau^1_{fix}; p, q). \quad (60)$$

Notice that this known result in our new interpretation directly relates the distillation procedure to the well-known property of supersymmetric attractors in the type IIB picture namely that in this case only the $H^{3,0}$ and $H^{0,3}$ parts of the cohomology survive.

In order to present the usual solution for the fixed values of the moduli write Eqs. (58) in the form

$$\Gamma_{kji}c^k b^j a^i = 0, \quad \Gamma_{kji}b^k b^j a^i = 0, \quad \Gamma_{kji}c^k b^j a^i = 0. \quad (61)$$

Using the fact that $\Gamma_{kji}$ is real these equations taken together with their complex conjugates are equivalent to the vanishing of the $2 \times 2$ determinants

$$\text{Det} (\Gamma_{kji}c^k) = 0, \quad \text{Det} (\Gamma_{kji}b^j) = 0, \quad \text{Det} (\Gamma_{kji}a^i) = 0. \quad (62)$$

provided the imaginary parts of the moduli are non vanishing. (A property clearly should hold due to physical reasons.) The above equations result in three quadratic equations, keeping only the solutions providing $y^1, y^2$ and $y^3$ positive yield the stabilized values for the moduli

$$\tau^a_{fix} = (\gamma_0 \cdot \gamma_1)^a + i \sqrt{-D}, \quad \text{Det} (\Gamma_{kji}a^i) = 0. \quad (63)$$

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Here for example

\[(\gamma_0 \cdot \gamma_1)^1 \equiv \gamma_{kj0} \varepsilon^{kk'} \varepsilon^{jj'} \gamma_{kj1}, \]  

(64)

and

\[D = (\gamma_0 \cdot \gamma_1)^2 - (\gamma_0 \cdot \gamma_0)(\gamma_1 \cdot \gamma_1) \]  

(65)

is Cayley’s hyperdeterminant\(^{2,29}\). In order to have such solutions \(-D\) should be positive and \((\gamma_0 \cdot \gamma_0)\) should both be negative\(^{23}\).

Using the stabilized values \(\tau_{fix}^a\), \(a = 1, 2, 3\) in \(e^{K/2}W(\tau^3, \tau^2, \tau^1)\) we get the well-known result\(^{2,4,18}\)

\[|Z|^2 = e^K |W(\tau_{fix}^3, \tau_{fix}^2, \tau_{fix}^1)|^2 = \sqrt{-D(|\gamma|)} = \sqrt{(\gamma_0 \cdot \gamma_0)(\gamma_1 \cdot \gamma_1) - (\gamma_0 \cdot \gamma_1)^2} \]  

(66)

where due to the triality symmetry of Cayley’s hyperdeterminant products like \((\gamma_0 \cdot \gamma_1)\) can be calculated by using any of the qubits playing a special role. This quantity is showing up in the macroscopic Bekenstein-Hawking entropy of the extremal, static, spherical symmetric BPS black hole solution of the STU-model\(^{2,4,18,23}\)

\[S_{BH} = \pi \sqrt{-D(|\gamma|)}. \]  

(67)

Note that the quantity \(\tau_3 = 4|D(|\gamma|)|\) is a genuine entanglement measure of the state \(|\gamma\rangle\) in the theory of three-qubit entanglement\(^{20}\). For BPS black holes we have \(\tau_3 = -4D\).

The final form of our three-qubit state on the horizon of the black hole is

\[|\Gamma\rangle_{fix} = (-D)^{1/4} \left(e^{i\alpha}|000\rangle_{fix} - e^{-i\alpha}|111\rangle_{fix}\right), \]  

(68)

where

\[\tan \alpha = \sqrt{-D} \frac{p^0}{2p^1p^2p^3 + p^0(p^0q_0 + p^1q_1 + p^2q_2 + p^3q_3)}. \]  

(69)

As we see this unnormalized state is of generalized GHZ form\(^{31}\), where the relative phase is given by the phase of the central charge. Hence the attractor mechanism can be regarded as a distillation procedure of a GHZ state on the black hole horizon\(^4\). However, as a new result here one should also see that according to our basic correspondence between cohomology classes and qubits the vectors \(|000\rangle_{fix}\) and \(|111\rangle_{fix}\) now correspond to the covariantly holomorphic and antiholomorphic three-forms \(-i\Omega_{fix}\) and \(-i\bar{\Omega}_{fix}\) respectively.
IV. A IIB \((T^2)^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)\) Model for Flux Compactification

A. Four qubit systems

In this section we show yet another application of the qubit picture connected to flux compactification. In order to do this first we have to connect our considerations to four qubit systems.

First we combine the type IIB NS and RR three-forms \(H_3\) and \(F_3\) into a new three-form \(G_3\) which has also a dependence on a special type of new moduli \(\tau = a + ie^{-\Phi}\) i.e. the axion dilaton field. The usual expression of \(G_3\) is

\[
G_3 = F_3 - \tau H_3. \tag{70}
\]

Now we embed our type IIB model based on the space \(Y = (T^2)^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)\) (and restricting merely to the untwisted sector) into F-theory on an elliptically fibered CY fourfold. It is convenient to introduce a four-form \(G_4\) via making use of an extra torus \(T^2\) as follows. Define

\[
G_4 = dv \wedge F_3 + du \wedge H_3 = \frac{1}{\tau - \overline{\tau}}(G_3 \wedge d\overline{z} - \overline{G}_3 \wedge dz), \tag{71}
\]

where \(du\) and \(dv\) are the coordinates of the new torus \(T^2\) with \(dz = du + \tau dv = \alpha + \tau \beta\), with Kähler potential \(K_1 = -\log i(\overline{\tau} - \tau)\). We still have the property \(\int_{T^2} \alpha \wedge \beta = 1\). Notice that \(G_4\) is invariant under the \(SL(2, \mathbb{R})\) duality symmetry of the IIB theory. This means that under the transformations

\[
\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad G_3 \mapsto \frac{1}{c\tau + d} G_3, \quad dz \mapsto \frac{1}{c\tau + d} dz, \tag{72}
\]

originating from the set of transformations

\[
\begin{pmatrix} H \\ F \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{73}
\]

\(G_4\) is left invariant.

Using the form of the Kähler potential \(K_1 = -\log(\overline{\tau} - \tau)\) we notice that the \(G_4\) can be reinterpreted as a four-qubit state. Indeed \(G_4\) can be regarded as the sum of two components that can be put into a two component vector as

\[
ie^{K_1/2} \begin{pmatrix} -\tau & 1 \\ \tau & -1 \end{pmatrix} \begin{pmatrix} H_3 \wedge d\overline{z} \\ F_3 \wedge dz \end{pmatrix}. \tag{74}
\]
Let us now use the expressions in the Hodge diagonal base

\[ H_3 = P^I \alpha_I - Q_I \beta^I = iZ(H)\Omega - i\delta^I_j D_j Z(H)\hat{\partial}_k \Omega + c.c. \quad (75) \]

\[ F_3 = p^I \alpha_I - q_I \beta^I = iZ(F)\Omega - i\delta^I_j D_j Z(F)\hat{D}_k \Omega + c.c. \quad (76) \]

According to Section III. we know that to these expressions one can associate a pair of three-qubit states as

\[ |H\rangle = H_{000}|000\rangle + H_{001}|001\rangle + \cdots + H_{110}|110\rangle + H_{111}|111\rangle, \quad (77) \]

\[ |F\rangle = F_{000}|000\rangle + F_{001}|001\rangle + \cdots + F_{110}|110\rangle + F_{111}|111\rangle. \quad (78) \]

In order to fit these states into a four qubit one we need some minor adjustments. According to Eq.(1) we have chosen moduli to have negative imaginary parts, however \( \tau \) has positive imaginary part. Moreover, the complex differential associated to \( dz = \alpha + \tau \beta \) was featuring \( \tau \). We can regard all moduli on the same footing by defining a fourth moduli and the complex coordinate of the associated torus as

\[ \tau^4 \equiv \tau, \quad dz^4 \equiv d\bar{z}. \quad (79) \]

Let us now define the covariantly holomorphic four-form as

\[ \Omega = e^{K/2}dz^4 \wedge dz^3 \wedge dz^2 \wedge dz^1 \quad (80) \]

where the total Kähler potential is \( K \equiv K_1 + K \) with \( K \) showing up in Eq.(26). Once again we define flat covariant derivatives \( D_A \), \( A = 1, 2, 3, 4 \) and the quantities \( D_A \Omega \). The conjugate quantities will be denoted as usual by \( \overline{\Omega} \) and \( D_A \overline{\Omega} \). Then we have for example

\[ D_4 \overline{\Omega} = e^{K/2}d\overline{z}^4 \wedge d\overline{z}^3 \wedge d\overline{z}^2 \wedge d\overline{z}^1. \quad (81) \]

Now one has the expansion\(^{19,32}\) for \( G_4 \) as an element of the space of allowed fluxes \( H_G^4(T^2 \times Y) \)

\[ G_4 = Z(G)\Omega - \overline{D}^4Z(G)D_A \overline{\Omega} + \overline{D}^A \overline{Z}(G)D_4 \overline{\Omega} + c.c \quad (82) \]
We can reinterpret this expansion as a state \( |G\rangle \) satisfying the usual reality condition in \((\mathbb{C}^2)^\times 4\) if we make the correspondence

\[
|0000\rangle \leftrightarrow \Omega, \quad |1111\rangle \leftrightarrow \overline{\Omega}, \quad (83)
\]

\[
|0001\rangle \leftrightarrow D_1\Omega, \quad |1110\rangle \leftrightarrow D_1\overline{\Omega}, \ldots \text{e.t.c.} \quad (84)
\]

\[
|1001\rangle \leftrightarrow D_4D_1\Omega, \quad |0110\rangle \leftrightarrow D_4D_1\overline{\Omega}, \ldots \text{e.t.c.} \quad (85)
\]

Now the expansion in the Hodge diagonal basis is having the alternative form of a 4-qubit state

\[
|G\rangle = G_{0000}|0000\rangle + G_{0001}|0001\rangle + \ldots G_{1110}|1110\rangle + G_{1111}|1111\rangle. \quad (86)
\]

Recall again that in the Hodge diagonal basis the operator \( \ast \) is acting as the parity check operator \( \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \), and the flat covariant derivatives and their conjugates act as suitable numbers of \( \sigma_+ \) or \( \sigma_- \) operators inserted in fourfold tensor products.

Notice that the state \( |G\rangle \) can be written in the form

\[
|G\rangle = S_4 \otimes S_3 \otimes S_2 \otimes S_1 |g\rangle \quad (87)
\]

where the matrix representative of the four-fold tensor product of operators is

\[
\frac{1}{\sqrt{16 y^4 y^3 y^2 y^1}} \begin{pmatrix}
\tau^1 & -1 \\
-\tau^1 & 1
\end{pmatrix} \otimes
\begin{pmatrix}
\tau^3 & -1 \\
-\tau^3 & 1
\end{pmatrix} \otimes
\begin{pmatrix}
\tau^2 & -1 \\
-\tau^2 & 1
\end{pmatrix} \otimes
\begin{pmatrix}
\tau^1 & -1 \\
-\tau^1 & 1
\end{pmatrix}, \quad (88)
\]

and the flux state \( |g\rangle \) is defined as

\[
|g\rangle = \sum_{l k j i=0,1} g_{lkji} |lkji\rangle, \quad (89)
\]

with the explicit form of the amplitudes is given by

\[
\begin{pmatrix}
g_{0000} & g_{0001} & g_{0010} & g_{0100} \\
g_{0111} & g_{0110} & g_{0101} & g_{0011}
\end{pmatrix} =
\begin{pmatrix}
-p^0 & -p^1 & -p^2 & -p^3 \\
-q_0 & q_1 & q_2 & q_3
\end{pmatrix} \quad (90)
\]

\[
\begin{pmatrix}
g_{1000} & g_{1001} & g_{1010} & g_{1100} \\
g_{1111} & g_{1110} & g_{1101} & g_{1011}
\end{pmatrix} =
\begin{pmatrix}
-P^0 & -P^1 & -P^2 & -P^3 \\
-Q_0 & Q_1 & Q_2 & Q_3
\end{pmatrix}. \quad (91)
\]

Here the amplitudes containing the fluxes \((p^I, q_I)\) and \((P^I, Q_I)\) are just the ones appearing in Eqs. (75)-(76).
B. Flux attractors

In this subsection as an illustration we study an example of the attractor equation of flux compactification on the orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case the flux attractor equations are just a rephrasing of the imaginary self duality condition (ISD) $\ast_6 G = iG$ for the complex flux form of Eq. (70). This condition arising from the 10D equations of motion imply that the complex structure of the Calabi-Yau space is fixed in a way such that $G_3$ has only $(0,3)$ and $(2,1)$ components. It is also known that the ISD condition is equivalent to the ones of $D_i \mathcal{W} = D_\tau \mathcal{W} = 0$ where $\mathcal{W} = \int_{CY} G_3 \wedge \Omega_3$ is the GVW superpotential. Here $D_i$ and $D_\tau$ are the covariant derivatives featuring the complex structure moduli and the complex axio-dilaton. As discussed in the previous section in the four-qubit formalism based on the four-form $G_4$ and its associated state $|G\rangle$ these conditions boil down to the ones

$$G_{0001} = G_{0010} = G_{0100} = G_{1000} = G_{1110} = G_{1101} = G_{1011} = G_{0111} = 0.$$  \hspace{1cm} (92)

Recall that

$$G_{0000} = Z(G) = \int_{Y \times T^2} G_4 \wedge \Omega = e^{K/2} W(\tau^4, \tau^3, \tau^2, \tau^1),$$  \hspace{1cm} (93)

where

$$W = -q_0 - q_1 \tau^1 - q_2 \tau^2 - q_3 \tau^3 + Q_0 \tau^4 - p_1 \tau^2 \tau^3 - p_2 \tau^1 \tau^2 - p_3 \tau^1 \tau^3 - Q_1 \tau^1 \tau^4 + Q_2 \tau^2 \tau^4$$

$$+ Q_3 \tau^3 \tau^4 + p_0 \tau^1 \tau^2 \tau^3 + P_1 \tau^2 \tau^3 \tau^4 + P_2 \tau^1 \tau^3 \tau^4 + P_3 \tau^1 \tau^2 \tau^4 - P_0 \tau^1 \tau^2 \tau^3 \tau^4.$$  \hspace{1cm} (94)

Moreover, according to our interpretation of the action of the flat covariant derivatives as projective bit flip errors amplitudes like $G_{0001}$ are just obtained from the expression of $G_{0000}$ by replacing the corresponding moduli by its complex conjugate in the relevant slot hence for example we have $G_{0001} = e^{K/2} W(\tau^4, \tau^3, \tau^2, \tau^1)$. Hence in this four-qubit reinterpretation the flux attractor equations again correspond to some distillation procedure of our state $|G\rangle$ where from the 16 amplitudes due to the vanishing of the ones of Eq. (92) only 8 ones will survive.

In order to illustrate this distillation procedure in detail we invoke the explicit solution found by Larsen and O’Connell. This solution is a one with merely 8 fluxes, i.e. in the definition of $|G\rangle$ one takes

$$\begin{pmatrix}
g_{0000} & g_{0001} & g_{0010} & g_{0100} 
g_{0111} & g_{0110} & g_{0101} & g_{0011}
\end{pmatrix} = \begin{pmatrix}
-p^0 & 0 & 0 
0 & q_1 & q_2 & q_3
\end{pmatrix},$$  \hspace{1cm} (95)

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\[ \begin{pmatrix}
  g_{1000} & g_{1001} & g_{1010} & g_{1100} \\
  g_{1111} & g_{1110} & g_{1101} & g_{1011}
\end{pmatrix}
= \begin{pmatrix}
  0 & -P^1 & -P^2 & -P^3 \\
  -Q_0 & 0 & 0 & 0
\end{pmatrix}, \tag{96}
\]

Using a generating function for the flux attractor equations in Ref.\textsuperscript{20} the authors have shown that this configuration with 8 fluxes has a purely imaginary solution for the four moduli \( \tau^a \) of the form

\[ \tau^1 = -i \left( -\frac{Q_0 P^1 q_2 q_3}{P^2 P^3 p^0 q_1} \right)^{1/4}, \quad \tau^2 = -i \left( -\frac{Q_0 P^2 q_1 q_3}{P^1 P^3 p^0 q_2} \right)^{1/4}, \tag{97} \]
\[ \tau^3 = -i \left( -\frac{Q_0 P^3 q_1 q_2}{P^1 P^2 p^0 q_3} \right)^{1/4}, \quad \tau^4 = -i \left( -\frac{p^0 q_1 q_2 q_3}{Q_0 P^1 P^2 P^3} \right)^{1/4}, \tag{98} \]
\[ -\text{sgn}(Q_0 p^0) = \text{sgn}(P^1 q_1) = \text{sgn}(P^2 q_2) = \text{sgn}(P^3 q_3) = +1. \tag{99} \]

Recall that \( \tau^4 = \tau \) where \( \tau = C_0 + i e^{-\phi} \) is the axio-dilaton. Now \( C_0 = 0 \) hence \(-\tau^4 \) gives the stabilized value of the dilaton.

One can easily check that these stabilized values indeed satisfy the constraints of Eq.\textsuperscript{(92)}. In order to do this just write

\[ W = (p^0 \tau^1 \tau^2 \tau^3 + Q_0 \tau^4) + (P^1 \tau^2 \tau^3 \tau^4 - q_1 \tau^1) + (P^2 \tau^1 \tau^3 \tau^4 - q_2 \tau^2) + (P^3 \tau^1 \tau^2 \tau^4 - q_3 \tau^3) \tag{100} \]

and check that the terms in the brackets give zero when we conjugate an \textit{odd} number of moduli in the expression of \( W \). In order to reveal the distillation procedure at work let us first calculate \(|G\rangle_{\text{fix}}\) using these stabilized values for the moduli. We introduce the quantities

\[ x = \text{sgn}(q_1) \sqrt{|P^1 q_1|}, \quad y = \text{sgn}(q_2) \sqrt{|P^2 q_2|}, \quad z = \text{sgn}(q_3) \sqrt{|P^3 q_3|}, \quad t = -\text{sgn}(-Q_0) \sqrt{|-Q_0 p^0|}. \tag{101} \]

Then a calculation shows that for \( l, k, j, i \in \{0, 1\} \)

\[ (G_{kji})_{\text{fix}} = \frac{i}{2} \left[ (-1)^l t + (-1)^k z + (-1)^j y + (-1)^i x \right], \quad l + k + j + i \equiv 0 \mod 2, \tag{102} \]

and of course due to Eq.\textsuperscript{(92)} we have

\[ (G_{kji})_{\text{fix}} = 0, \quad l + k + j + i = 1 \mod 2. \tag{103} \]

A quantity of physical importance which is related to one of these nonzero amplitudes is the complex gravitino mass \( M_{3/2} \). This quantity is depending on the fluxes and the moduli. Its explicit form at the attractor point is given by the formula

\[ (M_{3/2})^2_{\text{fix}} = |Z|_{\text{fix}}^2 = |G_{0000}|_{\text{fix}}^2. \tag{104} \]
This formula is to be compared with the ones of Eqs. (66)-(67) used in the black hole context. Clearly the gravitino mass squared in the flux compactification scenario seems to be an analogous quantity to the black hole entropy\textsuperscript{20}.

What is the physical meaning of the remaining nonzero amplitudes? It is easy to see that they are featuring the complex mass matrix of chiral fermions defined in an arbitrary point in moduli space. This quantity is defined as\textsuperscript{19}

\[ M_{\hat{A}\hat{B}} \equiv D_{\hat{A}}D_{\hat{B}}Z. \] (105)

At the attractor point this matrix becomes a certain function of the fluxes i.e. \((M_{\hat{A}\hat{B}})_{\text{fix}}\). After splitting the flat indices as \(\hat{A} = (\hat{I}, \hat{4})\) with \(\hat{I} = 1, 2, 3\) one can show\textsuperscript{19} that

\[ D_{\hat{I}}D_{\hat{J}}Z = C_{\hat{I}\hat{J}\hat{K}}D^4D^\hat{K}Z = C_{\hat{I}\hat{J}\hat{K}}M^{\hat{I}\hat{J}}, \] (106)

where \(M_{\hat{I}\hat{J}}\) is the mass matrix of the axino-dilatino mixing with the complex structure modulino. From Eqs.(82) and (86) it is obvious that \(M_{\hat{4}\hat{1}} = G_{1001}, M_{\hat{4}\hat{2}} = G_{1010}\) and \(M_{\hat{4}\hat{3}} = G_{1100}\), hence after using \(M_{\hat{4}\hat{4}} = 0\) the final form of \(M_{\hat{A}\hat{B}}\) is

\[ M_{\hat{A}\hat{B}} = \begin{pmatrix}
0 & G_{0011} & G_{0101} & G_{1001} \\
G_{0011} & 0 & G_{0110} & G_{1010} \\
G_{0101} & G_{0110} & 0 & G_{1100} \\
G_{1001} & G_{1010} & G_{1100} & 0
\end{pmatrix}. \] (107)

The explicit form of the matrix \((M_{\hat{A}\hat{B}})_{\text{fix}}\) is given by using the expressions as given by Eq.(102).

Let us finally comment on the structure of the \(SL(2)^\times 4\) invariants for our model. As the algebraically independent \(SL(2)^\times 4\) invariants\textsuperscript{34} one can take the quantities of Ref.\textsuperscript{25} with explicit expressions

\[ I_1 = -\frac{1}{4}(a^2 + b^2 + c^2 + d^2), \quad I_2 = \frac{1}{6}(ab + ac + ad + bc + bd + cd), \] (108)

\[ I_3 = -\frac{1}{4}(abc + acd + bcd + abd), \quad I_4 = abcd, \] (109)

where

\[ a = i(t + z), \quad b = i(t - z), \quad c = i(y - x), \quad d = i(y + x). \] (110)
With these notations it is easy to check that our "attractor state" $|G\rangle_{\text{fix}}$ is of the form

$$
|G\rangle_{\text{fix}} = \frac{1}{2} (a + d) (|0000\rangle - |1111\rangle) + \frac{1}{2} (a - d) (|0011\rangle - |1100\rangle) + \frac{1}{2} (b + c) (|0101\rangle - |1010\rangle) + \frac{1}{2} (b - c) (|0110\rangle - |1001\rangle).
$$

This state up to some phase conventions is of the same form as the generic class of four qubit entangled states\textsuperscript{7}. The state $|G\rangle_{\text{fix}}$ is the result of a distillation procedure similar in character to the one discussed in the black hole context. In the literature this state is tackled on the same footing as the famous GHZ state in the three-qubit case of maximal multipartite entanglement. However, as far as the fine details of entanglement properties are concerned there are notable differences between the attractor state of Eq.(68) with e.g. $p^0 = q_1 = q_2 = q_3 = 0$ of GHZ type and $|G\rangle_{\text{fix}}$ (see Ref.\textsuperscript{7} for more details).

Let us calculate the norm squared $||G||^2$ of our state $|G\rangle$ at the attractor point. One half this norm squared is an analogous quantity to the black hole potential of Eq.(54) (see also Eq.(20)). Being a quantity depending merely on the fluxes at the attractor point it should be an $SL(2) \times 4$ i.e. a four-qubit invariant. For our example this quantity is also related to the sum of the a gravitino and chiral fermion mass squares. A quick calculation shows that

$$
\frac{1}{2} ||G||^2_{\text{fix}} = 2 I_1 = \int F_3 \wedge H_3,
$$

hence the invariant we get is the standard symplectic invariant specialized to our four-qubit case.

Another interesting quantity to look at in our flux compactification example is the four-qubit generalization of Cayley’s hyperdeterminant\textsuperscript{29} known from Eq.(67). For the definition of this $SL(2) \times 4$ and permutation invariant polynomial of order 24 we refer to the literature\textsuperscript{34,35} here we merely give its explicit form for our example

$$
D_4 = (-Q_0 P^1 P^2 P^3 (p^0 q_1 q_2 q_3) \prod_{l, k, j, i \in (\mathbb{Z}_2) \times 4} \left((-1)^z t + (-1)^{k} z + (-1)^{j} j + (-1)^{i} i\right).
$$

It is easy to check that $D_4 > 0$ due to our sign conventions of Eq.(99). A necessary condition for $D_4 \neq 0$ for this example of 8 nonvanishing fluxes is the nonvanishing of the 4 independent amplitudes of $|G\rangle_{\text{fix}}$ showing up in the 16 terms of the product.
V. FERMIONIC ENTANGLEMENT FROM TOROIDAL COMPACTIFICATION

A. An interpretation via fermionic systems

As a generalization of our considerations giving rise to qubits now we go one step further and consider the problem of obtaining entangled systems of more general kind from toroidal compactification. The trick is to embed our simple systems featuring few qubits into larger ones. Here we discuss the natural generalization of embedding qubits (based on entangled systems with distinguishable constituents) into fermionic systems (based on entangled systems with indistinguishable ones). In the quantum information theoretic context this possibility has already been elaborated, here we show that toroidal compactifications also incorporate this idea quite naturally.

In order to elaborate on this problem we recall the illustrative example of Moore discussing the structure of attractor varieties for $IIB/T^6$. As in the special case of the stu model we choose analytic coordinates for the complex torus such that the holomorphic one-forms are defined as $dz^a = du^a + \tau^{ab}dv^b$ where now $\tau^{ab}$, $0 \leq a, b \leq 3$ is the period matrix of the torus with the convention

$$\tau^{ab} = x^{ab} - iy^{ab}. \quad (114)$$

For principally polarized Abelian varieties we have the additional constraints

$$\tau^{ab} = \tau^{ba}, \quad y^{ab} > 0. \quad (115)$$

We choose as usual $\Omega_0 = dz^1 \wedge dz^2 \wedge dz^3$, and the orientation $\int_{T^6} du^1 \wedge dv^1 \wedge du^2 \wedge dv^2 \wedge du^3 \wedge dv^3 = 1$.

Unlike in our considerations of the stu model now we exploit the full 20 dimensional space of $H^3(T^6, \mathbb{C})$. We expand $\Gamma \in H^3(T^6, \mathbb{C})$ in the basis similar to Eqs.$(12)-(13)$ satisfying $\int_{T^6} \alpha^I \wedge \beta_J = \delta^I_J$, $I, J = 1, 2, \ldots 10$,

$$\alpha_0 = du^1 \wedge du^2 \wedge du^3, \quad \alpha_{ab} = \frac{1}{2}\varepsilon_{ad'b}du^a \wedge du^b \wedge dv^b \quad (116)$$

$$\beta^0 = -dv^1 \wedge dv^2 \wedge dv^3, \quad \beta^{ab} = \frac{1}{2}\varepsilon_{ba'd}dv^a \wedge dv^a \wedge dv^b. \quad (117)$$

One can then show that

$$\Omega_0 = \alpha_0 + \tau^{ab}\alpha_{ab} + \tau^a_{ab}\beta^{ba} - (\text{Det} \tau)\beta^0. \quad (118)$$
where $\tau^\sharp$ is the transposed cofactor matrix satisfying $\tau^\sharp \tau^\sharp = \text{Det}(\tau)I$, where $I$ is the $3 \times 3$ identity matrix. Using Eq.(118), the usual expression of Eq.(4) and the identity

$$\text{Det}(A + B) = \text{Det}A + \text{Det}B + \text{Tr}(A^\sharp B + AB^\sharp),$$

valid for $3 \times 3$ matrices over $\mathbb{C}$ one can check that

$$e^{-K} = 8\text{Det}y.$$  \hfill (120)

An element $\Gamma$ of $H^2(T^6, \mathbb{C})$ can be expanded as

$$\Gamma = p^0_0 \alpha_0 + P^{ab} \alpha_{ab} - Q_{ab} \beta^{ab} - q_0 \beta^0.$$  \hfill (121)

We can rewrite this as

$$\Gamma = \sum_{1 \leq A < B < C \leq 6} \gamma_{ABC} f^A \wedge f^B \wedge f^C$$

where

$$(f^1, f^2, f^3, f^4, f^5, f^6) \equiv (f^1, f^2, f^3, f^\top, f^\top, f^\top) = (du^1, du^2, du^3, dv^1, dv^2, dv^3).$$  \hfill (123)

Here $\gamma_{ABC}$ is a completely antisymmetric tensor of rank three with 20 independent components. In the context of BPS and non-BPS attractors clearly we have $\Gamma \in H^2(T^6, \mathbb{Z})$ with the components $\gamma_{ABC}$ identified with the 20 charges $(p^0, P^{ab}, Q_{ab}, q_0)$. This identification is given by the explicit expressions

$$p^0_0 = \gamma_{123}, \quad \begin{pmatrix} p^{11} & p^{12} & p^{13} \\ p^{21} & p^{22} & p^{23} \\ p^{31} & p^{32} & p^{33} \end{pmatrix} = \begin{pmatrix} \gamma_{231} & \gamma_{231} & \gamma_{231} \\ \gamma_{312} & \gamma_{312} & \gamma_{312} \\ \gamma_{123} & \gamma_{123} & \gamma_{123} \end{pmatrix},$$  \hfill (124)

$$q^0_0 = \gamma_{123}, \quad \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} = - \begin{pmatrix} \gamma_{123} & \gamma_{123} & \gamma_{123} \\ \gamma_{231} & \gamma_{231} & \gamma_{231} \\ \gamma_{312} & \gamma_{312} & \gamma_{312} \end{pmatrix}. \hfill (125)$$

Using the language of fermionic entanglement $\text{21,22 \Gamma}$ can also be regarded as an unnormalized three fermion state with six single particle states $\text{14}$. Now we introduce the new moduli dependent basis vectors

$$e^A = f^{A'} S_{A'}^A, \quad S_{A'}^A = \begin{pmatrix} I & I \\ \tau & \tau \end{pmatrix}.$$  \hfill (126)
One can then write
\[ \Gamma = \frac{1}{6!} \Gamma_{A'B'C'} \left(-ie^{K/2}e^{A'} \wedge e^{B'} \wedge e^{C'}\right), \] (127)

where
\[ \Gamma_{A'B'C'} = S_{A'}^A S_{B'}^B S_{C'}^C \gamma_{ABC}, \] (128)

and
\[ S \equiv -ie^{-K/6}S^{-1} = -ie^{-K/6}(\tau - \tau)^{-1} \begin{pmatrix} \tau & I \\ \tau & -I \end{pmatrix}. \] (129)

In this new form the amplitudes \( \Gamma_{ABC} \) are depending on the charges and the moduli. Notice also that now we have the same matrix \( S \in GL(6, \mathbb{C}) \) acting on all indices of \( \gamma_{ABC} \). This reflects the well-known fact known from the theory of quantum entanglement that the SLOCC group\(^{10,21}\) for a quantum system consisting of indistinguishable subsystems (now with six single particle states)\(^{14}\) is represented by the same \( GL(6, \mathbb{C}) \) matrices acting on each entry of a tensor representing the set of amplitudes (now of a tripartite system). Notice also that the basis states \( -ie^{K/2}e^A \wedge e^B \wedge e^C \) for \( 1 \leq A < B < C \leq 6 \) now form an orthonormal basis.

It is instructive to see how do we recover the stu case studied in Section III. In particular one would like to see how the indistinguishable character of the subsystems represented by \( \Gamma \) of Eq. (127) boils down to the distinguishable one of the subsystems represented by Eq. (51). In order to see this just notice that in the stu case we have merely 8 nonzero amplitudes to be used in Eq. (128). Namely we have \( \gamma_{ABC} \) with labels 123, 12\,\bar{3}, 1\bar{2}3, \ldots, \bar{1}23, \bar{1}2\bar{3}. \) Moreover, the \( 3 \times 3 \) matrix \( \tau \) is now diagonal, hence the explicit form of \( S \) is
\[
S = \frac{1}{2} e^{-K/6} \begin{pmatrix}
-\tau^1/y^1 & 0 & 0 & 1/y^1 & 0 & 0 \\
0 & -\tau^2/y^2 & 0 & 0 & 1/y^2 & 0 \\
0 & 0 & -\tau^3/y^3 & 0 & 0 & 1/y^3 \\
\tau^1/y^1 & 0 & 0 & -1/y^1 & 0 & 0 \\
0 & \tau^2/y^2 & 0 & 0 & -1/y^2 & 0 \\
0 & 0 & \tau^3/y^3 & 0 & 0 & -1/y^3
\end{pmatrix}
\] (130)

After switching to our usual ordering convention let us make the correspondence 321 \( \leftrightarrow \) 000, 32\bar{1} \( \leftrightarrow \) 001 etc. meaning that the labels 1 and \( \bar{1} \), 2 and \( \bar{2} \), \ldots refer to the labels 0 and 1 of the first, second \ldots qubit. Looking at the structure of the tensor product \( S \otimes S \otimes S \) and recalling that \( e^{-K/2} = \sqrt{8y^1y^2y^3} \) we quickly recover the structure of the matrices of Eq.(53).
According to Ref.\textsuperscript{14} this embedding of a system with distinguishable constituents to a larger fermionic one with indistinguishable ones is a useful trick for studying the entanglement properties of simple embedded systems.

**B. BPS attractors**

Let us now consider the structure of BPS attractors\textsuperscript{27} in our entanglement based approach. Similar to the stub case the attractor equations are demanding that only the $H^{3,0}$ and $H^{0,3}$ parts of the cohomology classes are nonvanishing. This implies that for our "state" of fermionic entanglement at the horizon we have

$$\Gamma_{\text{fix}} = \Gamma_{321}(-ie^{K/2}e^{3} \wedge e^{2} \wedge e^{1})_{\text{fix}} - \Gamma_{321}(-ie^{K/2}e^{\tau} \wedge e^{\tau} \wedge e^{T})_{\text{fix}},$$

where

$$\Gamma_{321} = -\Gamma_{321} = Z(\tau_{\text{fix}}, p^{0}, q^{0}, P, Q).$$

According to the general theory\textsuperscript{14} for classifying the SLOCC entanglement types\textsuperscript{10} for tri-partite fermionic systems with six single particle states, such attractor states belong to the fermionic generalization of the usual GHZ state well-known for three qubits. Hence our result on the reinterpretation of the attractor mechanism as a quantum information theoretic distillation procedure in this fermionic context still holds.

Let us now find the explicit form of our attractor state of Eq.(132). In order to do this we just have to recall the steps discussed by Moore\textsuperscript{27} within the realm of our entanglement based context. Clearly the attractor equations now correspond to the usual ones stating that except for the ones $\Gamma_{321}$ and $\Gamma_{321}$ all the amplitudes of the fermionic sate $\Gamma$ of Eq.(127) vanish at the black hole horizon. An equivalent form of this constrain can be shown to be\textsuperscript{27}

$$\text{Im}(2C) = p^{0}, \quad \text{Im}(2C\tau) = P,$$

$$\text{Im}(2C\text{Det}\tau) = q^{0}, \quad \text{Im}(2C\tau^{2}) = -Q,$$

where in these equations

$$C = e^{K/2}Z(\tau_{\text{fix}}, p^{0}, q^{0}, P, Q), \quad \tau = \tau_{\text{fix}} \equiv \tau(p^{0}, q^{0}, P, Q).$$

For completeness let us revisit the main steps of the derivation of the stabilized values of $\tau$ as presented in Ref.\textsuperscript{27}. The first equation of Eq.(133) can be solved by the ansatz

$$2C = \xi_{0} + ip^{0}.$$
Plugging this into the second one of Eq. (133) one obtains
\[ p^0 \tau = \lambda \mathcal{Y} + P, \quad \text{where} \quad \lambda \equiv \frac{C}{|C|}, \quad \mathcal{Y} \equiv 2|C|y. \quad (137) \]

Using this we get
\[ (p^0 \tau)^2 = (p^0)^2 \tau^2 = \lambda^2 \mathcal{Y}^2 + P^2 + \lambda \mathcal{Y} \times P, \quad (138) \]

where \( \mathcal{Y} \times P \) is the linearization of the sharp map\(^{36}\). We would like to use this expression in the second of Eq. (134). Since \( 2C\lambda \) is a real number, the term linear in \( \lambda \) does not give contribution to the terms under \( \text{Im}(\ldots) \). As a consequence of this one gets
\[ \mathcal{Y}^2 = p^0 Q + P^2. \quad (139) \]

Readers familiar with the theory of Freudenthal triple systems\(^{36}\) realize this expression as one of the \textit{quadratic rank polynomials}. The Freudenthal triple system now in question is the one based on the cubic Jordan algebra of \( 3 \times 3 \) complex matrices. Such polynomials are needed for the classification of Freudenthal systems, which in turn has also relevance to classification of entanglement types\(^{14,37}\) of special quantum systems with few subsystems. Notice also that a necessary condition for nondegenerate tori is
\[ \text{Det} \mathcal{Y}^2 = (\text{Det}(\mathcal{Y}))^2 \neq 0, \quad (140) \]

moreover for a polarized Abelian variety \( \mathcal{Y}^2 \) is a symmetric positive matrix .(Recall the third expression in Eq. (137), taken together with \( \mathcal{Y}^2 \mathcal{Y} = (\text{Det} \mathcal{Y})I \), and \( e^{-K} = 8\text{Det} y > 0, y > 0 \).)

Using Eq. (139) in the first of Eq. (134) provides an expression for \( \tau \) in terms of the charges \textit{and} the unknown quantity \( C = e^{K/2}Z \). In order to determine its value in terms of the charges we now turn to the first equation of Eq. (134). First we use the identity of Eq. (119) to get
\[ \text{Det}(p^0 \tau) = (p^0)^3 \text{Det} \tau = \lambda^3 \text{Det} \mathcal{Y} + \lambda^2 (p^0 \text{Tr}(PQ) + 3\text{Det}(P)) + \lambda \text{Tr}(\mathcal{Y}P^2) + \text{Det}(P). \quad (141) \]

Using this in the first of Eq. (134) after some manipulations one obtains
\[ \frac{\xi_0}{|C|} \text{Det} \mathcal{Y} = -\tilde{p}^0, \quad \tilde{p}^0 \equiv 2\text{Det} P + p^0(\text{Tr}(PQ) + p^0 q_0). \quad (142) \]

The expression on the right hand side which is cubic in the charges is also a well-known quantity in the theory of Freudenthal triple systems. It is a part of the Freudenthal dual charge configuration\(^{37}\) \( (\tilde{p}^0, \tilde{q}_0, \tilde{P}^{ab}, \tilde{Q}^{ab}) \) (which is also used as one of the amplitudes of the dual entangled state in Ref.\(^{14}\)) based on a trilinear operator\(^{36}\).
Let us now take the square of the first equation of Eq. (142) and express \((\text{Det } Y)^2\) using Eq. (140) in terms of \(\text{Det } Y^\# = \text{Det}(p^0Q + P^\#)\). The determinant of the sum of matrices can be tackled again by Eq. (119) yielding the result
\[
\xi_0^2 = \frac{(\tilde{p}^0)^2}{\mathcal{D}},
\]
(143)
where
\[
\mathcal{D} = -(p^0q_0 + \text{Tr}(PQ))^2 + 4\text{Tr}(P^\#Q^\#) + 4p^0\text{Det}Q - 4q_0\text{Det}P.
\]
(144)
Note that \(\mathcal{D}\) is minus half of the usual quartic invariant of Freudenthal triple systems.\(^36\)

For BPS solutions we chose the branch
\[
\xi_0 = -\frac{\tilde{p}^0}{\sqrt{\mathcal{D}}},
\]
(145)
provided \(\mathcal{D} > 0\). Comparing this with the first of Eq. (142) one gets \(\text{Det } Y = |C|\sqrt{\mathcal{D}}\). Using this and Eq. (139) with the third of Eq. (137) one gets
\[
y = \frac{1}{2}\sqrt{\mathcal{D}}(p^0Q + P^\#)^{-1}.
\]
(146)
Similar manipulations using the first of Eq. (137) yield for the real part of \(\tau\)
\[
x = \frac{1}{2}(2PQ - [p^0q_0 + \text{Tr}(PQ)]I)(p^0Q + P^\#)^{-1}.
\]
(147)
One can check that thestu case of Eq. (63) is recovered when using diagonal matrices for \(\tau = x - iy, P \) and \(Q\).

Using these results one can show that the GHZ-like state at the horizon, as the result of a distillation procedure, is of the form as given by Eq. (68) with suitable replacements. First Cayley’s hyperdeterminant \(\mathcal{D}\) has to be replaced by its generalization \(D\) as given by Eq. (144). Moreover, the phase \(\alpha\) of the central charge is determined by the equation
\[
\tan \alpha = \sqrt{-\mathcal{D}} \frac{p^0}{\tilde{p}^0},
\]
(148)
where \(\tilde{p}^0\) is given by the quantity showing up in Eq. (142). The stabilized states \(|000\rangle_{\text{fix}}\) and \(|111\rangle_{\text{fix}}\) of Eq. (68) should be replaced by their ”fermionic” counterparts \((-ie^{K/2}e^3 \wedge e^2 \wedge e^1)_{\text{fix}}\) and \((-ie^{K/2}e^3 \wedge e^2 \wedge e^1)_{\text{fix}}\).

For BPS black holes we have \(M^2 = |Z|^2\) hence the Bekenstein-Hawking entropy of the extremal, spherically symmetric black hole is \(S_{BH} = \pi M^2 = \pi |Z|^2\). Since \(C = e^{K/2}Z\) and \(\text{Det} Y = 8|C|^3\text{Det}y = |C|^3e^{-K} = |C|\sqrt{\mathcal{D}}\) one gets for the entropy
\[
S_{BH} = \pi \sqrt{\mathcal{D}(\Gamma)},
\]
(149)
with \( D \) is given by Eq. (144).

Based on our experience with the STU case where the entropy formula was given in terms of a genuine tripartite measure (i.e. \( \tau_{123} \equiv 4|D| \) i.e. the three-tangle \( ^{30}_4 \equiv |D| \)), it is tempting to interpret \( T_{123} \equiv 4|D| \) as an entanglement measure for three fermions with six single particle states as represented by the state \( \Gamma \) Eq. (122). (The extra factor of 4 is only needed for normalized states in order to restrict the values of this entanglement measure to the interval \([0, 1]\).) According to Ref. 14 within the realm of quantum information the quantity \( 4|D| \) indeed works as a basic quantity to characterize the entanglement types under the SLOCC group 10. Within the context of black hole solutions we know that the unnormalized states in question are either charge states with integer amplitudes or ones satisfying extra reality conditions, hence the SLOCC group should be restricted to its suitable real subgroup i.e. the U-duality group. Based on the results of Ref. 14 it is not difficult to see that the different types of black holes should correspond to the different entanglement types of fermionic entanglement. This correspondence runs in parallel with the observation of Kallosh and Linde 3 that the entanglement types of three qubit states correspond to different types of stu black holes.

VI. CONCLUSIONS

In this paper we have shown how qubits are arising from the geometry of tori serving as extra dimension in IIB compactifications. Our results clarified some of the issues left unclear in the paper of Borsten et al. 1 In particular the investigations of that paper interpreting wrapped branes as qubits were lacking an explicit construction of the Hilbert space where these qubits live. Here we have identified this space inside the cohomology of tori. Moreover, we have also shown that the Hodge diagonal basis usually used in the supergravity literature is naturally connected to the charge and moduli dependent multiqubit states used in our recent papers 4, 8, 16. This result provides the simplest way to understand the well-known attractor mechanism as a distillation process an issue elaborated in our previous set of papers. The idea ”qubits from extra dimensions” have also turned out to be very useful to generalize the black hole-qubit correspondence to some sort of flux-attractor-qubit correspondence. Indeed, for toroidal models it is quite natural to extend our considerations to new attractors of that kind 19, 32. We pointed out that four-qubit systems are characterizing
some of the key issues for such models.

Though our main motivation was to account for the occurrence of qubits in these exotic scenarios we have revealed that in the string theoretical context entangled systems of more general kind than qubits should rather be considered. In particular for toroidal models we have seen that the natural arena where these systems live is the realm of fermionic entanglement of subsystems with *indistinguishable* parts. The notion "fermionic" entanglement is simply associated with the structure of the cohomology of $p$-forms related to $p$-branes. As it has already been pointed out in our recent paper on special entangled systems, qubits are arising as embedded systems with distinguishable constituents inside such fermionic ones. Interestingly compactification on $T^6$ in the IIB duality frame provides a particularly nice manifestation of this idea.

Notice that in our examples of toroidal compactification we merely discussed BPS black holes. However, the attractor mechanism as a distillation procedure also works for non-BPS attractors. For the STU model it turns out that for the non-BPS branch $|\Gamma\rangle_{\text{fix}}$ will be again in the GHZ class where now none of the amplitudes are vanishing, however their magnitudes are equal. The relative signs of these amplitudes can be characterized via an error correction framework based on the flat covariant derivatives acting as projective bit flips as shown in Sections II. and III.

Why only tori? Clearly we should be able to remove the rather disturbing restriction to toroidal compactifications by embarking on the rich field of Calabi-Yau compactifications. Notice in this respect that the decompositions of Eqs. (44) and (82) in the Hodge diagonal basis can be used to reinterpret such formulas as *qudits* i.e. $d$-level systems with $d = h^{21} + 1$ in the type IIB duality frame. F-theoretical flux compactifications for elliptically fibered Calabi-Yau fourfolds can then be associated with entangled systems comprising a qubit (a $T^2$ accounting for the axion-dilaton) and a qudit coming from a Calabi-Yau three-fold ($CY_3$). Alternatively after using instead of $CY_3$ the combination $T^2 \times K3$ we can have tripartite systems consisting of two qubits and a qudit etc. The idea that separable states geometrically should correspond to product manifolds and entangled ones to fibered ones was already discussed in the literature, for the simplest cases of two and three qubits. It would be interesting to explore further consequences of this idea in connection with the black hole-flux attractor-qubit correspondence.
VII. ACKNOWLEDGEMENT

The author would like to thank Professor Werner Scheid for the warm hospitality at the Department of Theoretical Physics of the Justus Liebig University of Giessen where part of this work has been completed. This work was supported by the New Hungary Development Plan (Project ID: TÁMOP-4.2.1/B-09/1/KMR-2010-002), and the DFG-MTA project under contract No.436UNG113/201/0-1.

1 L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim and W. Rubens, Phys. Rev. Lett. 100, 251602 (2008).
2 M. J. Duff, Phys. Rev. D76, 025017 (2007).
3 R. Kallosh and A. Linde, Phys. Rev. D73 104033 (2006).
4 P. Lévy, Phys.Rev. D74, 024030 (2006).
5 L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim, W. Rubens, Physics Reports, 471, 113 (2009).
6 M. J. Duff and S. Ferrara, Phys. Rev. D76, 124023 (2007), P. Lévy, Phys. Rev. D75, 024024 (2007), M. J. Duff and S. Ferrara, Phys. Rev. D76, 124023 (2007), P. Lévy, M. Saniga, P. Vrana and P. Pracna, Phys. Rev. D79, 084036 (2009).
7 F. Verstraete, J. Dehaene, B. DeMoor, H. Verscheide, Phys. Rev. A65, 052112 (2002).
8 P. Lévy, Phys. Rev. D82, 026002 (2010).
9 L. Borsten, D. Dahanayake, M. J. Duff, A. Marrani, W. Rubens, Physical Review Letters 105, 100507 (2010), L. Borsten, M. J. Duff, A. Marrani, W. Rubens, Eur. Phys. J.Plus, 126 37 (2011).
10 W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A62, 062314 (2000).
11 L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim, W. Rubens, Phys. Rev. A80, 032326 (2009).
12 P. Lévy, Phys. Rev. D84, 025023 (2011).
13 S. Ferrara, A. Marrani, E. Orazi, R. Stora, A. Yerayan, Journal of Mathematical Physics 52, 062302 (2011).
14 P. Lévy and P. Vrana, Phys. Rev. A78, 022329 (2008), P. Vrana and P. Lévy J. Phys. A:
Math. Theor. 42, 285303 (2009).

15 S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. D52, 5412 (1995), A. Strominger, Phys. Lett. B383, 39 (1996), S. Ferrara and R. Kallosh, Phys. Rev. D54, 1514 (1996), S. Ferrara and R. Kallosh, Phys. Rev. D54, 1525 (1996), K. Goldstein, N. Izuka, R. P. Jena and S. P. Trivedi, Phys. Rev. D72 124021 (2005), A. Sen, Journal of High energy Physics 0509, 038 (2005), P. K. Tripathy and S. Trivedi, Journal of High Energy Physics 0603 022 (2006).

16 P. Lévy, Phys. Rev. D76, 106011 (2007), P. Lévy and Sz. Szalay, Phys. Rev. D82, 026002 (2010), P. Lévy and Sz. Szalay, Phys. Rev. D83, 045005 (2011).

17 P. Lévy, “Attractors, Black Holes and Multiqubit Entanglement, The Attractor Mechanism: Proceedings of the INFN-Laboratori Nazionali di Frascati School 2007, Springer-Verlag 2010.

18 M. J. Duff, J. T. Liu, and J. Rahmfeld, Nucl.Phys.B459, 125 (1996), K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova, and W. K. Wong, Phys. Rev. D54, 6293 (1996).

19 R. Kallosh, Journal of High Energy Physics 0512, 022 (2005).

20 F. Larsen and R. O’Connell, Journal of High Energy Physics, 0907 049 (2009).

21 K. Eckert, J. Schliemann, D.Bruss and M. Lewenstein, Ann. Phys. (N.Y.) 299, 88 (2002).

22 G. C. Ghirardi, L. Marinatto, Phys. Rev. A70, 012109 (2004).

23 S. Bellucci, S. Ferrara, A. Marrani, A. Yeranyan, Entropy 10, 507 (2008).

24 E. G. Gimon, F. Larsen, J. Simon, Journal of High Energy Physics, 0801,040 (2008).

25 H. Suzuki, Mod. Phys. Lett. A11, 623 (1996).

26 F. Denef, Journal of High Energy Physics 0008, 050 (2000).

27 G. Moore, arXiv:hep-th/9807087

28 R. Blumenhagen, B. Krs, D. Lst, S. Steiberger, Physics Reports 445 1-193 (2007).

29 A.Cayley, Camb. Math. J. 4, 193 (1845).

30 V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A61, 052306 (2000).

31 D. M. Greenberger, M. A. Horne, A. Zeilinger, arXiv:0712.0921 D. Bouwmeister, J-W Pan, M. Daniell, H. Weinfurter, A. Zeilinger, Physical Review Letters, 82, 1345 (1999).

32 F. Denef and M. R. Douglas, Journal of High Energy Physics, 0405, 072 (2004).

33 K. Dasgupta, G. Rajesh, and S. Sethi, Journal of High Energy Physics, 08, 023 (1999), S. B. Giddings, S. Kachru, and J. Polichinski, Phys. rev. D66, 106006, (2002).

34 G. Luque and Y.I. Thibon, Phys. Rev. Abf 67, 042303 (2003).

35 P. Lévy, Journal of Physics A: Math. Gen. 39, 9533 (1906).
36 S. Krutelevich, Journal of Algebra 314, 924, (2007).

37 L. Borsten, D. Dahanayake, M. J. Duff, W. Rubens, Phys. Rev. D80, 026003, (2009).

38 R. Mosseri and R. Dandoloff, J. Phys. A: Math. Gen. 34 10243 (2001), B. A. Bernevig, H-D. Chen, J. Phys. A: Math. Gen. 36 8325, (2003), P. Lévay, J. Phys. A: Math. Gen. 37, 1821, (2004).