A cellular automaton traffic flow model between the Fukui-Ishibashi and Nagel-Schreckenberg models

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Abstract

We propose and study a new one-dimensional traffic flow cellular automaton (CA) model of high speed vehicles with the Fukui-Ishibashi-type acceleration for all cars and the Nagel-Schreckenberg-type (NS) stochastic delay only for the cars following the trail of the car ahead. The main difference in the delay scenario between the new model and the NS model is that a car with spacing ahead longer than the velocity limit \( M \) may not be delayed in the new model. By using a car-oriented mean field theory, we derive the fundamental diagrams of the average speed as the function of car density analytically. Our theoretical results are in excellent agreement with numerical simulations.

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I. INTRODUCTION

Traffic flow cellular automaton (CA) models have attracted much interest recently. Compared with other dynamical approaches, e.g. the fluid dynamical approach, to this class of problems, CA models are conceptually simpler and can be easily implemented on computers for numerical investigations [1-4].

Two popular one-dimensional (1D) traffic flow models are the Fukui-Ishibashi (FI) model [5] and the Nagel-Schreckenberg (NS) model [6]. An exact car-oriented mean-field (COMF) theory has been developed for the FI model with arbitrary limit on the maximum speed \( v_{\text{max}} \), car density \( \rho \) and delay probability \( f \) [7,8]. However, for the NS model with high speed vehicles (\( v_{\text{max}} > 1 \)) and stochastic delay, there is still no established exact analytical theory up to now [9,10].

The acceleration and stochastic delay rules of the NS model lead to complications in the time evolution of the flow and hence it is very difficult for exact analytical studies. In order to understand how these rules affect the evolution and the corresponding asymptotic state, we study a new 1D traffic flow CA model in which only the cars following the trail of the car ahead may be delayed.

The plan of the present paper is as follows. The definition of the model and the evolution equations for the inter-car spacings are given in Sec. II. In Sec. III some observations are made to describe the steady state of the system. We present the fundamental diagrams for the low car density case with arbitrary vehicle speed limit and the high density case with vehicle speeds limited to 1 and 2 in Sec. IV. Excellent agreements between numerical simulations and theoretical results are shown in Sec. V together with a discussion on our results in connection to the FI and NS models.
II. THE MODEL

Let $N$ be the total number of cars on a 1D road of length $L$. The density of cars is $\rho = N/L$. Let $C_n(t)$ be the number of empty sites in front of the $n$th car at time $t$, and $v_n(t)$ be the number of sites that the $n$th car moves during the time $t$ step.

The new model adopts the following acceleration rule [5]:

Step 1: $v'_n(t) = \min(C_n(t), M)$

We call the $n$th car “a car that follows the trail of the car ahead” if $v'_n(t) = C_n(t)$. It means that the $n$th car may become the neighbor of the car ahead if the car in front stops. Stochastic delay is introduced in such a way that all the cars which follow the trail of their cars ahead have a probability $f$ to move forward one site less than it is allowed by Step 1, i.e., we have

Step 2: $v_n(t) = v'_n(t) - 1$ with the probability $f$, if $v'_n(t) = C_n(t)$ and $v'_n(t) > 0$,

and

Step 3: The $n$th car moves $v_n(t)$ sites ahead.

The number of empty sites in front of the $n$th car at time $t + 1$ can be written as

$$C_n(t + 1) = C'_n(t) + v_{n+1}(t) - v_n(t)$$  \hspace{1cm} (1)

For the new model with maximum car velocity $v_{\text{max}} = M$ and a stochastic delay probability $f$, the velocity of the $n$th car at time step $t$ as a function of the inter-car spacing $C_n(t)$ can be written as

$$v_n(t) = F_M(f, C_n(t))$$  \hspace{1cm} (2)

where

$$F_M(f, C) = \begin{cases} M & \text{if } C > M \\ C - 1 & \text{with probability } f \\ C & \text{with probability } 1 - f \\ 0 & \text{if } C = 0 \end{cases}$$  \hspace{1cm} (3)
III. INTER-CAR SPACINGS IN THE STEADY STATES

From Eqs. (1)-(3), we can derive the properties of the inter-car spacings in the steady states. Given $C_n(t) \leq M+1$, it follows that $C_n(t) - F_M(C_n(t)) \leq 1$, and from $F_M(C_{n+1}(t)) \leq M$, we obtain $C_n(t + 1) = C_n(t) - F_M(C_n(t)) + F_M(C_{n+1}(t)) \leq M + 1$. Therefore, if an inter-car spacing is not larger than $M + 1$, it will not be larger than $M + 1$ as the system evolves.

Given $C_n(t) \geq M + 1$, it follows that $F_M(C_n(t)) = M$, and from $F_M(C_{n+1}(t)) \leq M$ we have $C_n(t + 1) = C_n(t) - F_M(C_n(t)) + F_M(C_{n+1}(t)) \leq C_n(t)$. Therefore, inter-car spacings which are larger than or equal to $M + 1$ will never increase, i.e., if $C_n(t) \geq M + 1$, then $C_{n+1}(t) \leq C_n(t)$.

It is useful to define the long and short inter-car spacings via their comparison with the maximum car speed $M$. An inter-car spacing is called a long spacing if it is longer than $M + 1$, i.e., $C_n(t) > M + 1$. An inter-car spacing is called a short spacing if it is shorter than $M + 1$, i.e. $C_n(t) < M + 1$. Based on the above definitions, we can define the excessive length of a long spacing $L_n(t)$ and the deficient length of a short spacing $S_n(t)$ as

$$L_n(t) = \max(C_n(t) - (M + 1), 0)$$

and

$$S_n(t) = \max((M + 1) - C_n(t), 0).$$

It follows that the sum of the excessive lengths of all long spacings $L(t)$ and the sum of deficient lengths of all short spacings $S(t)$ are given respectively by

$$L(t) = \sum_n L_n(t), \quad S(t) = \sum_n S_n(t)$$

From these definitions, it can be proven readily that

$$L(t) - S(t) = \sum_n [C_n(t) - (M + 1)]$$

$$= L - (M + 2)$$
From these properties of the inter-car spacings, we have \( L_n(t + 1) \leq L_n(t) \). Hence,

\[
L(t + 1) \leq L(t).
\]  

(5)

From Eqs. (4) and (5), we have \( S(t + 1) \leq S(t) \). Therefore \( L \) and \( S \) will never increase as the system evolves. If one of the \( L_n \) decreases, then \( L \) and \( S \) will have to decrease.

Next we look into the question of whether long and short spacings may co-exist in the asymptotic steady state. Let \( N_i(t) \) be the number of inter-car spacings with length \( i \) at time \( t \). The probability of finding such a spacing at time \( t \) is \( P_i(t) = N_i(t)/N \). Hereafter, \( P_i(t) \) is denoted by \( P_i \) for simplicity, except specified otherwise. Suppose that long and short spacings co-exist. Consider a long spacing, if the car ahead moves forward by \( m - 1 \) sites, then the spacing will decrease by 1. The probability for this to happen is \( (1 - f)P_{M-1} + fP_M \). For the same reason, the probability for the spacing to be shortened by 2 is \( (1 - f)P_{M-2} + fP_{M-1} \), and the probability for the spacing to be shortened by 3 is \( (1 - f)P_{M-3} + fP_{M-2} \), and so on. The probability for the spacing to be shortened by \( M - 1 \) is \( (1 - f)P_1 + fP_2 \), and the probability for the spacing to be shortened by \( M \) is \( P_0 + fP_1 \). On average, a long spacing will be shortened by

\[
MP_0 + \sum_{i=1}^{M-1} [(1 - f)(M - i) + f(M - i + 1)]P_i + fP_M
= MP_0 + \sum_{i=1}^{M-1} (M - i + f)P_i + fP_M
\]  

(6)

in one time step. The shortened length is positive, unless \( P_0 = P_1 = P_2 = \ldots = P_{M-1} = P_M = 0 \), i.e., \( S = 0 \). Therefore, in the asymptotic steady state of the system, \( L \) and \( S \) will no longer change and at least one of them becomes zero. Hence, it is not possible for long and short spacings to co-exist in the asymptotic steady state.
IV. ANALYTICAL SOLUTION OF ASYMPTOTIC VELOCITY

For the low car density case ($\rho < 1/(M + 2)$), it is apparent that in the asymptotic steady state, $L > 0$ and $S = 0$. Hence

$$C_n(t) \geq M + 1, \quad \forall n.$$  \hspace{1cm} (7)

In this case, stochastic delay will no longer occur and all the cars will move forward with the maximum speed $M$. The average car speed of traffic flow is

$$< V(t \to \infty) > = M.$$  \hspace{1cm} (8)

For the high car density case ($\rho > 1/(M + 2)$), it is apparent that in the asymptotic steady state $S > 0$ and $L = 0$. Hence

$$C_n(t) \leq M + 1, \quad \forall n.$$  \hspace{1cm} (9)

The length of every inter-car spacing cannot be larger than $M + 1$. Therefore, the average speed of traffic flow in the asymptotic steady state is

$$< V(t \to \infty) > = \sum_{i=1}^{M} P_i[i(1 - f) + (i - 1)f] + MP_{M+1}$$

$$= \sum_{i=1}^{M} P_i(i - f) + MP_{M+1}$$  \hspace{1cm} (10)

A. $v_{\text{max}} = M = 1$

In this case, the high density case refers to $\rho \geq 1/3$, and hence $P_n = 0, \quad \forall n \geq 3$. It implies that only $P_0, P_1, P_2$ are non-zero. To obtain the non-vanishing $P_j$, we introduce $N_{i\to j}$ to describe the number of inter-car spacings with a change in length from $i$ at time $t$ to $j$ at time $t + 1$. The probability of finding an inter-car spacing with length $i$ at time $t$ and length $j$ at time $t + 1$ is

$$W_{i\to j}(t) \equiv N_{i\to j}(t)/N.$$  \hspace{1cm} (11)
From Eqs.(1)-(3), we can write down all the non-zero \( W_{i \rightarrow j} \) as

\[
W_{0 \rightarrow 1} = P_0[(1 - f)P_1 + P_2]
\]
\[
W_{0 \rightarrow 2} = 0
\]
\[
W_{1 \rightarrow 0} = P_1[(1 - f)P_0 + f(1 - f)P_1]
\]
\[
W_{1 \rightarrow 2} = P_1[f(1 - f)P_1 + fP_2]
\]
\[
W_{2 \rightarrow 0} = 0
\]
\[
W_{2 \rightarrow 1} = P_2(P_0 + fP_1)
\]

For \(|i - j| \geq 3\), \( W_{i \rightarrow j} = 0 \).

When the system approaches its asymptotic steady state, all the \( P_j \) cease to change. So the following detailed balance condition for the steady state holds:

\[
\sum_{i \neq m} W_{i \rightarrow m} = \sum_{i \neq m} W_{m \rightarrow i}, \quad \forall m. \tag{12}
\]

When \( m \) equals 0, 1, and 2, three detailed balance equations can be written down as:

\[
W_{0 \rightarrow 1} + W_{0 \rightarrow 2} = W_{1 \rightarrow 0} + W_{2 \rightarrow 0}
\]
\[
W_{1 \rightarrow 0} + W_{1 \rightarrow 2} = W_{0 \rightarrow 1} + W_{2 \rightarrow 1}
\]
\[
W_{2 \rightarrow 0} + W_{2 \rightarrow 1} = W_{0 \rightarrow 2} + W_{1 \rightarrow 2}
\]

Substituting the expressions of \( W_{i \rightarrow j} \) into the above three equations, we obtain

\[
P_0P_2 = f(1 - f)P_1^2. \tag{13}
\]

Normalization requires that \( P_0 \), \( P_1 \), and \( P_2 \) satisfy the equations

\[
\sum_i P_i = P_0 + P_1 + P_2 = 1, \tag{14}
\]

and

\[
\sum_i iP_i = P_1 + 2P_2 = \bar{C} = 1/\rho - 1. \tag{15}
\]
From Eqs. (13), (14), and (15), we obtain a quadratic equation for $P_0$:

$$(2f - 1)^2 P_0^2 + [(2f - 1)(\bar{C} - 2) + 1]P_0 - f(1 - f)(\bar{C} - 2)^2 = 0,$$  \hspace{1cm} (16)

with its root given by

$$P_0 = \frac{-(2f - 1)(\bar{C} - 2) + 1 + \sqrt{(2f - 1)^2(\bar{C} - 2)C + 1}}{2(2f - 1)^2}. \hspace{1cm} (17)$$

Hence, the asymptotic average speed of traffic flow is

$$< V(t \to \infty) > = \frac{\bar{C} + \frac{1}{2f - 1}(\sqrt{(2f - 1)^2(\bar{C} - 2)C + 1} - 1)}{2}$$

$$= \frac{1}{2} \left( -1 + \frac{1}{\rho} + \frac{-1 + \sqrt{1 + \frac{(2f - 1)^2(\rho - 1)(3\rho - 1)}{\rho}}}{2f - 1} \right). \hspace{1cm} (18)$$

For $f = 1/2$, which is an removable singular point,

$$< V(t \to \infty) > = \frac{\bar{C}}{2} = (1/\rho - 1)/2. \hspace{1cm} (19)$$

Equations (18) and (19) give the asymptotic $< V(t \to \infty) >$ as a function of $f$ and $\rho$ in the high density case with $M = 1$.

**B. $v_{\text{max}} = M = 2$**

In this case, $\rho \geq 1/4$, hence $P_n = 0, \hspace{1cm} \forall n \geq 4$. It implies that only $P_0, P_1, P_2$ and $P_3$ are non-zero. From Eqs. (11)- (13), we can write down the non-zero $W_{i \to j}$ as:

$$W_{0\to1} = P_0[(1 - f)P_1 + fP_2]$$

$$W_{0\to2} = P_0[(1 - f)P_2 + P_3]$$

$$W_{0\to3} = 0$$

$$W_{1\to0} = P_1[(1 - f)P_0 + f(1 - f)P_1]$$

$$W_{1\to2} = P_1[f(1 - f)P_1 + f(1 - f)^2]P_2 + (1 - f)P_3$$

$$W_{1\to3} = P_1[(1 - f)P_2 + fP_3]$$
\[ W_{2\to 0} = P_2[(1 - f)P_0 + f(1 - f)P_1] \]
\[ W_{2\to 1} = P_2\{fP_0 + [f^2 + (1 - f)^2]P_1 + (1 - f)P_2\} \]
\[ W_{2\to 3} = P_2[f(1 - f)P_2 + fP_3] \]
\[ W_{3\to 0} = 0 \]
\[ W_{3\to 1} = P_3(P_0 + fP_1) \]
\[ W_{3\to 2} = P_3[(1 - f)P_1 + fP_2] \]

Substituting the above expressions into the detailed balance condition in Eq. (12), we obtain the following set of four equations:

\[ fP_0P_2 + P_0P_3 - f(1 - f)P_1^2 - f(1 - f)P_1P_2 = 0 \] \hspace{1cm} (20)

\[ 2fP_0P_2 + P_0P_3 - 2f(1 - f)P_1^2 - f(1 - f)P_1P_2 - (1 - f)P_1P_3 + f(1 - f)P_2^2 = 0 \] \hspace{1cm} (21)

\[ fP_0P_2 - P_0P_3 - f(1 - f)P_1^2 + f(1 - f)P_1P_2 - 2(1 - f)P_1P_3 + 2f(1 - f)P_2^2 = 0 \] \hspace{1cm} (22)

\[ P_0P_3 - f(1 - f)P_1P_2 + (1 - f)P_1P_3 - f(1 - f)P_2^2 = 0 \] \hspace{1cm} (23)

Noting that only two of them, e.g. Eqs. (20) and (23), are independent. Combining Eqs. (20) and (23) with the normalization conditions

\[ \sum_i P_i = P_0 + P_1 + P_2 + P_3 = 1 \] \hspace{1cm} (24)

and

\[ \sum_i iP_i = P_1 + 2P_2 + 3P_3 = \bar{C} = 1/\rho - 1, \] \hspace{1cm} (25)

we can solve for \( P_0, P_1, P_2, P_3 \) and obtain the asymptotic traffic flow velocity for \( M = 2 \).
V. DISCUSSION

In order to compare with the analytic results, we carried out numerical simulations on 1D chain with 1000 cars. The length of the chain was adjusted so as to give the desired car density. Periodic boundary condition was imposed. The first 20000 time steps were excluded from the averaging procedure so as to remove the transient behavior. The averages were taken over the next 80000 time steps. Figures 1 and 2 show the comparison between results obtained from numerical simulations and our mean field theory for the cases of $M = 1$ and $M = 2$ over the entire range of density $\rho$. The curves are the theoretical results while the symbols represent results of numerical simulations. The curves from the top down along the velocity axis correspond to different values of $f$ ranging from 0 to 1. Excellent agreement between simulations and our theory is found.

From the fundamental diagrams of the new model, it is noted that when the car density is low enough ($\rho \leq 1/(M + 2)$), all the inter-car spacings will not be shorter than $M + 1$, and all the cars will not be delayed, leading to traffic flow in its maximum velocity ($V = M$). This situation is more realistic in that in real traffic, no driver would like to slow down his car when it is far away from the car ahead. In the high density case, the stochastic delay in our new model represents better safety than that of the FI model, and leads to much higher asymptotic average velocity of traffic flow than that in the NS model.

In summary, we introduced a new model with stochastic delays for cars following the trail of the car ahead. Its evolution and fundamental diagram are quite different from the NS and FI models, even in the simplest case of $M = 1$. We studied the evolution of the inter-car spacings and obtained its fundamental diagram by an analytical COMF approach. The results show exact agreement between numerical simulations and our theory.

The analysis of the dynamical evolution of our new model may give us a clearer physical picture on how the acceleration and stochastic delay rules affect the evolution and the corresponding asymptotic steady state. It will also provide us with better ideas on developing analytical approaches to other traffic flow CA models such as the NS model for which no
exact analytic approach has been established.

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Figure Captions

Figure 1: The fundamental diagram with the maximum car velocity $M = 1$ and for different stochastic delay probabilities $f$. The solid curves are the theoretical results. The points with different symbols represent results of numerical simulations. The curves from the top down along the velocity axis correspond to different $f$ ranging from $f = 0$ to $f = 1$ in steps of 0.1.

Figure 2: The fundamental diagram with the maximum car velocity $M = 2$ and for different stochastic delay probabilities $f$. The solid curves are the theoretical results. The points with different symbols represent numerical simulations. The curves from the top down along the velocity axis correspond to different $f$ ranging from $f = 0$ to $f = 1$ in steps of 0.1.
