Quantum dynamics and macroscopic quantum tunnelling of two weakly coupled condensates

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Received 22 April 2013, in final form 14 August 2013
Published 5 September 2013
Online at stacks.iop.org/JPhysB/46/185301

Abstract
We study the quantum dynamics of a Bose–Josephson junction made up of two coupled Bose–Einstein condensates. We analyse different dynamical branches of Josephson oscillations within an ‘effective potential’ approach. At a critical coupling strength, a transition takes place between the dynamical branches of Josephson oscillations, which is also manifested in the energy spectrum, and pairs of (quasi)-degenerate excited states appear above the critical coupling strength. This phenomenon can be understood in terms of change in shape of the ‘effective potential’. Possible novel quantum phenomena like decay of metastable ‘\(\pi\)-oscillation’ by ‘macroscopic quantum tunnelling’ (MQT) and MQT between the ‘self-trapped’ states with equal and opposite number imbalance become evident from the simple picture of ‘effective potential’.

1. Introduction
The creation of a Bose–Einstein condensate (BEC) of alkali atoms [1, 2] has opened up various possibilities for the study of the coherence properties of macroscopic ‘matter wave’. One such direction is to study the Josephson oscillation [3] in two weakly coupled BECs. Similar to the superconducting Josephson junction, a Bose–Josephson junction (BJJ) can be created by two BECs in a double well potential which are weakly coupled by the overlap of their wavefunctions [4, 5]. The condensates in each well are described by a ‘matter wave’ with a well-defined phase. After they are weakly connected, the relative phase changes and a particle current is produced across the barrier. The nonlinear dynamics of the AC and DC Josephson effects of the BJJ have already been studied experimentally by loading condensates in a double well trap [6–8]. Also, the matter wave interferometry has been studied in a double well geometry on an atom chip [9]. The number difference between two Josephson coupled condensates and their relative phase are two canonically conjugate collective variables which show coherent oscillations when an initial number imbalance is created between the condensates. The nonlinear dynamics of the relative phase and number imbalance between the coupled condensates have already been studied theoretically by several authors within the classical field approximation [10–13]. The effect of dissipation in the BJJ has been studied theoretically [14, 15] and also been observed experimentally in the decay of Josephson oscillation [7]. Quantum fluctuations become important for a small number of particles in the condensate and for large inter-particle interaction. The full quantum dynamics of the BJJ with a finite number of particles shows various interesting nonlinear effects like ‘revival’ and ‘beating’ for different values of coupling constant [16, 17].

One of the main advantages of the BJJ is the possibility of tuning the inter-particle interaction strength which enables us to study various types of nonlinear Josephson oscillations by controlling the interaction strength. Apart from the usual AC Josephson oscillation, two other types of dynamical branches can exist and their stability region is separated at a critical coupling strength. Above the critical coupling strength, the ‘macroscopic self-trapping’ (MST) can occur dynamically with a non-vanishing time average of the population imbalance between the condensates [10]. Below the critical coupling strength, another dynamical state known as ‘\(\pi\)-oscillation’ can exist for which the relative phase between two condensates is \(\pi\) [10, 13], and this oscillation has been studied experimentally in [8].
We also notice a remarkable change in the energy spectrum associated with the change in dynamical branches, and doubly degenerate excited states appear above the critical coupling. The system of many bosons in the BJJ can be mapped onto a quantum mechanical problem describing a particle in an effective potential. The main focus of this work is to analyse different types of dynamical states and change in the energy spectrum in terms of this ‘effective potential’. We also address novel macroscopic quantum effects related to ‘macroscopic quantum tunnelling’ (MQT) in this system which is evident from the method of the ‘effective potential’. The decay rate of the metastable state corresponding to ‘π-oscillation’ and the tunnelling time between two MST states with equal and opposite number imbalance are calculated analytically within the semiclassical formalism of MQT. MQT is always suppressed by the number of particles in the condensate; nevertheless, this novel quantum effect can be observed in the BJJ with a small number of particles and close to the critical coupling strength.

This paper is organized as follows. In section 2, we review the earlier results obtained from the Gross–Pitaevskii equation of classical fields [10]. By mapping the original Hamiltonian to a spin model, we recover the ‘pendulum model’ [10] describing the classical dynamics of the relative phase and number imbalance. Different dynamical branches and collective frequencies of small oscillation, obtained from the classical analysis, are compared with the full quantum dynamics. The equivalence between the spin model of the BJJ and the quantum mechanical problem describing a particle in an ‘effective potential’ has been outlined in section 3. Dynamical states below (and above) the critical coupling constant are analysed from the effective potential. Next, we outline the possible novel collective quantum effects which emerge from the simple picture of the ‘effective potential’. In subsection 3.1, we consider the possible decay of metastable ‘π-oscillation’ due to MQT through the potential barrier. Subsection 3.2 is devoted to analysing the energy spectrum of the Hamiltonian and its connection to the self-trapping phenomenon. The appearance of doubly degenerate excited states in the energy spectrum above the critical value of coupling can be understood in terms of a double well ‘effective potential’ whose minima correspond to the dynamical fixed point of self-trapping with equal and opposite number imbalance. In subsection 3.3, we consider the MQT of the system between these two MST states and calculate the energy splitting between the quasi-degenerate excited states. Finally, we summarize in section 4.

2. The model and Josephson dynamics

The BJJ of two weakly coupled BECs can be well described by a two-site Bose–Hubbard model (BHM),

$$H = -J(a_i^\dagger a_2 + a_2^\dagger a_1) + \frac{U}{2}[\hat{n}_1(\hat{n}_1 - 1) + \hat{n}_2(\hat{n}_2 - 1)]$$

(1)

where $a_i^\dagger$ ($a_i$) is the creation (annihilation) operator of bosons at site index $i$ ($i = 1, 2$ for two sites), $\hat{n}_i = a_i^\dagger a_i$ is the number operator, $J$ is the tunnelling matrix element and $U$ is the on-site interaction strength. The tunnelling matrix $J$ and the on-site interaction strength $U$ are the parameters of the effective model, which can be calculated from the exact shape of the trapping potential and macroscopic wavefunction of the condensates [11]. For a fixed value of the total number of particles $N$ in the condensates, we can transform the above Hamiltonian in equation (1) to a spin model,

$$H = -JS_\pi + \frac{U}{2}S_z^2,$$

(2)

where $J = 2\tilde{J}$, $U/2 = \tilde{U}$, $S_z = \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1)$, $S_\pi = \frac{1}{2}(a_1^\dagger a_2 - a_2^\dagger a_1)$ and $S_z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$ are the Schwinger boson representation of a large spin of magnitude $S = N/2$. In the effective spin Hamiltonian, we neglect the constant term depending on the total number of particles $N$.

For a large spin of magnitude $S$, we can represent it classically by a spin vector $\vec{S} = S(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, where two angles $\theta$ and $\phi$ describe the orientation of the spin in the Cartesian coordinate system. Physically, the variables $\cos \theta$ and $\phi$ describe the fractional number imbalance and relative phase between two condensates in the original BJJ model. In terms of these variables, the Lagrangian of the classical spin is given by

$$L = S(\cos \theta)\dot{\phi} - H(\theta, \phi),$$

(3)

where $\dot{\phi}$ is the time derivative of $\phi$, and $H(\theta, \phi) = -JS\sin \theta \cos \phi + \frac{U}{2S^2} \cos^2 \theta$ is the classical energy corresponding to the Hamiltonian in equation (1). Here, the variable $\cos \theta$ is canonically conjugate to the variable $\phi$ and plays the role of momentum. To use a dimensionless variable, we scale time by $1/J$, energy by $J$ and introduce a dimensionless coupling constant $\alpha = \frac{U}{JS}$, in order to make the total energy an extensive thermodynamic quantity.

From the Lagrangian given in equation (3), we obtain the classical equations of motion of a large spin,

$$\dot{\theta} = \sin \phi,$$

(4)

$$\sin \theta \dot{\phi} = \cos \theta \cos \phi + \alpha \cos \theta \sin \theta.$$

(5)

The linear stability analysis of the above dynamical equations reveals stable dynamics around three fixed points, given below.

- The first fixed point is at $\theta = \pi/2$ and $\phi = 0$. It describes the ground state of the total system, which is the symmetric combination of the wavefunctions of two equally populated condensates in two wells. The frequency $\omega$ of small amplitude oscillations around this fixed point is

$$\omega/J = \sqrt{1+\alpha},$$

(6)

which is the plasmon frequency of the AC Josephson effect [18].

- A ‘π-phase oscillation’ is described by the second fixed point at $\theta = \pi/2$ and $\phi = \pi$ [10]. This fixed point represents the wavefunction of the first excited state which is the anti-symmetric combination of wavefunctions of two condensates with equal number of atoms and relative phase $\pi$. The frequency of oscillation around this fixed point is given by $\omega/J = \sqrt{1-\alpha}$. This ‘π-oscillation’ becomes dynamically unstable when $\alpha > 1$. 


A third fixed point at \( \sin \theta = \frac{1}{\sqrt{2}} \), and \( \phi = \pi \) represents an ‘MST’ state with non-zero population imbalance between two condensates [10]. Classically, this self-trapping phenomenon can be understood from a Ginzburg–Landau (GL)-type potential. A second-order differential equation for population imbalance \( q = \cos \theta \) can be obtained from equations (4) and (5):

\[
\ddot{q} = \frac{\partial V_{GL}(q)}{\partial q}
\]

\[
V_{GL}(q) = \frac{1}{2} (1 - E\alpha/S) q^2 + \frac{\alpha^2}{8} q^4,
\]

where \( \ddot{q} \) denotes the second derivative of \( q \) with respect to time and the conserved classical energy is given by \( E/S = -\sqrt{1 - q^2} \cos \phi + \frac{\pi}{2} q^2 \). The above equation for number imbalance is exactly the same as the classical equation of motion of a particle at \( q \) in the GL potential \( V_{GL}(q) \) [19]. This GL potential changes its shape when \( E/S > \frac{1}{2} \), and two minima appear for non-vanishing population imbalance. For MST, the classical particle has total energy less than the barrier height in order to be localized at one of the minima of the GL potential. Two minima of the GL potential at population imbalance \( q = \pm \sqrt{1 - \frac{1}{\alpha^2}} \) correspond to the fixed points of MST states with energy \( E/S = \frac{1}{2}(\alpha + 1/\alpha) \). The frequency of harmonic oscillation around the potential minimum is given by \( \omega/|\dot{\alpha}| = \sqrt{\alpha^2 - 1} \). For \( \alpha > 1 \), the GL potential can have two minima around which a dynamically stable MST state can exist.

From the above classical analysis, we note that \( \alpha_c = 1 \) is a critical coupling strength which separates two types of dynamical states. Next, we analyse the dynamical and spectral properties of the effective spin Hamiltonian (equation (2)) quantum mechanically. In order to analyse the various dynamical branches discussed above, we study numerically the full quantum dynamics of the BJJ with a finite (but large) number of atoms and compare it with classical dynamics. We use the effective spin Hamiltonian (in equation (2)) to describe the quantum dynamics of the BJJ. Any quantum mechanical state of the system can be written as a linear combination of \( 2S + 1 \) basis states,

\[
|\Psi\rangle = \sum_{\sigma = -S}^{S} a_{\sigma} |\sigma\rangle,
\]

where \( |\sigma\rangle \) is the eigenstate of the operator \( S_\sigma \), with eigenvalue \( \sigma \), and \( a_{\sigma} \) are complex (in general time-dependent) amplitude. To study the quantum dynamics, we time-evolve any initial state by the time-evolution operator \( e^{-iHt} \) and then evaluate the expectation values of the observables. To compare the quantum dynamics with the classical dynamics of the spin, we take the initial state of the spin as a coherent state corresponding to its classical orientation. The classical nature of a large spin is well described by a spin-coherent state [20],

\[
|\xi\rangle = \frac{1}{(1 + |\xi|^2)^{S/2}} \sum_{\sigma = -S}^{S} \sqrt{2(2S)!} \sqrt{\frac{(S + \sigma)!}{(S + 1)!}} |\sigma\rangle,
\]

where the complex number \( \xi = \tan \frac{\pi}{4} e^{i\theta} \) represents the orientation of the corresponding classical spin vector. We expand the initial state in terms of the eigenvectors of the Hamiltonian and obtain the final state at time \( t \), multiplying each component by a factor \( e^{iE\dot{\xi}t} \), where \( E_\xi \) is the corresponding eigenvalue. For \( N = 100 \) (\( S = N/2 \)), we calculate the time evolution of fractional population imbalance \( (N_1 - N_2)/N \) between two condensates (with average number of particles \( N_1 \) and \( N_2 \)) in the BJJ, from the expectation value of the operator \( \hat{S}_z/S \), and compare it with the classical result obtained from the time integration of equations (4) and (5). To study different types of dynamics, we choose the initial values of the classical variables close to the fixed points, and the initial coherent state can be obtained from the corresponding value of \( \xi \). Three different types of Josephson dynamics (as discussed above) are shown in figure 2, for different values of the coupling constant \( \alpha \), and are compared with the classical dynamics. The AC Josephson dynamics and ‘\( \pi \)-oscillation’ for \( N = 100 \) show very good agreement with the classical dynamics. However, the quantum dynamics around the MST state shows some deviation from the classical dynamics, as well as some beating effect (see figure 1(c)). For the same initial number imbalance but with phase \( \phi = 0 \), quantum dynamics shows a damping and long time revival phenomenon as shown in figure 1(d).

3. Effective potential method and macroscopic quantum tunnelling

In this section, we consider interesting aspects related to the collective quantum effects in a BJJ in terms of the ‘effective potential method’ of a large spin [21]. The ‘effective potential’ of the quantum spin Hamiltonian gives a clear picture about various (meta)stable dynamical branches and also elucidate possible collective quantum tunnelling in a BJJ.

First we outline the method of effective potential for the spin Hamiltonian in equation (2) describing the BJJ. The eigenvalue equation of the Hamiltonian in equation (2) can be written as

\[
E_a = \frac{U}{2} \sigma^2 - J \sqrt{(S + \sigma)(S - \sigma + 1)} a_{\sigma - 1} + \sqrt{(S - \sigma)(S + \sigma + 1)} a_{\sigma + 1},
\]

where the eigenstates are expanded in terms of the basis states of \( \hat{S}_z \), as described in equation (9). By introducing a generating function \( \Phi(x) = \sum_{a = -S}^{S} e^{i\alpha a} e^{i \xi x} \) of an auxiliary variable \( x \), the above eigenvalue equation (11) can be written as a differential equation [21],

\[
E \Phi(x) = -\frac{U}{2} \frac{\partial^2 \Phi}{\partial x^2} - \frac{J}{2} \left[ 2 S \cos x \Phi(x) - 2 \sin x \frac{\partial \Phi}{\partial x} \right].
\]

It is interesting to note that \( \Phi(x) = \langle \xi | \psi \rangle e^{i\xi x} \), with \( \xi = e^{i\alpha} \). So the variable \( x \) can be interpreted as the phase \( \Phi \) between two condensates with equal number of particles in each trap, i.e. \( |\xi| = 1 \) (or \( \cos \theta = 0 \)). The transformation \( \Phi(x) = e^{-J x \cos x/U} \chi \) reduces the above eigenvalue equation to an effective Schrödinger equation,

\[
E \chi = -\frac{\alpha'}{2(s + 1/2)^2} \partial_x^2 \chi + V(x) \chi,
\]

where
Figure 1. Oscillation of fractional population imbalance $\Delta N/N$ with time $t$ (in units of $1/J$) for different values of coupling constant $\alpha$ and initial conditions. Results obtained from quantum dynamics (solid lines) are compared with those obtained from classical dynamics (dotted line) for the same initial conditions. (a) AC Josephson oscillation for $N = 100$, $\alpha = 0.5$, and with the classical initial condition $\cos \theta = 0.2, \phi = 0$. (b) $\pi$-oscillation for $N = 100$, $\alpha = 0.5, \cos \theta = 0.2$ and $\phi = \pi$. (c) Self-trapping for $N = 100, \alpha = 1.5, \cos \theta = 0.85$ and $\phi = \pi$. (d) Damping and revival phenomenon in quantum dynamics for $N = 50, \alpha = 1.5, \cos \theta = 0.85$ and $\phi = 0$.

Figure 2. Effective potential: $V(x)$ in the range $-\pi < x < \pi$ for (a) $\alpha = 0.4$ and (b) $\alpha = 1.4$. $W(x)$ for (c) $\alpha = 0.4$ and (d) $\alpha = 1.4$.

where $\mathcal{E} = E/(J(S + 1/2))$, $\alpha' = U(S + 1/2)/J \approx \alpha$, and the effective potential is given by

$$V(x) = \frac{1}{2\alpha'} \sin^2 x - \cos x. \quad (14)$$

First, $2S + 1$ eigenvalues of the above Schrödinger equation are the eigenvalues of the original spin Hamiltonian (2) [21]. The effective potential $V(x)$ for the relative phase can also be derived from equations (4) and (5) describing the classical dynamics. The relative phase $\phi$ between the condensates obeys the Newton equation of motion in the effective potential $V(\phi)$ [13]. The Hamiltonian in equation (13) with the effective potential (14) is an exact quantum phase description (or ‘rotor model’) of Josephson junctions [22].

It is interesting to note the following significant points of the above equation.

- The effective Planck constant in equation (13) is $1/S$, which implies that quantum fluctuation is suppressed for large spin systems (or a large number of particles in the trap). The tunability of the Planck constant makes this method suitable for semiclassical approximation. Similar semiclassical methods have been used to study the generation of spin-squeezed states [23], revival and decoherence in the BJJ [24], Fock space WKB approximation in the two-site BHM [25] and semiclassical expansion of the ‘Lipkin–Meshkov–Glick’ model [26].

- The role of effective mass is played by the inverse of the coupling constant $1/\alpha$. With increasing coupling constant $\alpha$, the fluctuation in number imbalance (or $S_x$) is reduced and the quantum fluctuation in phase $\phi$ is enhanced.

- As depicted in figures 2(a) and (b), the shape of the effective potential $V(x)$ changes at a critical coupling $\alpha = 1$. For $\alpha < 1$, the potential has two minima at $x = 2n\pi$ and $x = (2n + 1)\pi$. These two minima correspond to the stable phase oscillations between the condensates discussed in the previous section. The $\pi$-oscillation corresponds to the local minima of the effective potential at $x = (2n + 1)\pi$. The phase fluctuations around the classical steady state solutions can be obtained from the harmonic approximation of the
potential around the local minima. The potential at local minima $x_{\pm}$ can be approximated as:

$$V(x) = \mp 1 + \frac{1}{2\alpha} (1 \mp \alpha)(x - x_{\pm})^2,$$

where $x_{\pm}$ corresponds to the points 0 and $\pm \pi$, respectively. Note that the frequency of the harmonic potential exactly matches the frequency of small amplitude oscillations.

The phase fluctuation is given by

$$\langle (x - x_{\pi})^2 \rangle \approx \frac{\alpha}{2S\sqrt{1 \pm \alpha}}.$$  

Near the critical point, the phase fluctuation of the $\pi$-oscillation becomes very large.

3.1. Decay of the $\pi$-oscillation

As is evident from the effective potential $V(x)$, the state with a relative phase difference $\pi$ between the condensates is a metastable state which is separated from the ground state by a potential barrier (see figure 2(a)). A metastable state of many particles can decay by means of collective quantum tunnelling through the barrier, known as ‘macroscopic quantum tunnelling’ [27]. Since this collective state of many bosons is represented by a point particle in metastable local minima of the effective potential $V(x)$, we can use a simple quantum mechanical tunnelling formula to calculate the collective decay rate. The tunnelling probability of a single quantum particle through a potential barrier scales with the collective decay rate. The tunnelling probability of a single quantum particle through a potential barrier scales as $\sim e^{-\sqrt{2mS}/\hbar}$, where $V_{\text{max}}$ is the height of the potential barrier, $L$ is the length of the barrier and $m$ is the mass of the particle. In the case of the quantum phase model described by equation (13), the effective Planck constant $\hbar$ scales by $1/S$ and hence the collective tunnelling rate is suppressed by a factor of $S$ (or by the total number of particles $N$) in the exponential.

As seen from the simple scaling analysis, the decay of the metastable ‘$\pi$-oscillation’ state is suppressed exponentially by the number of particles $N$, and is negligible except in the region close to the critical coupling strength $\alpha \sim 1$ where the barrier height becomes vanishingly small. The potential near the metastable state of the $\pm \pi$-phase can be approximated as

$$V(x) = 1 + \frac{\epsilon}{2} x^2 - \frac{1}{8} x^4 + \cdots,$$

where $\tilde{x} = x - x_-$ and $\epsilon = 1 - \alpha$ can be considered as a small parameter close to the critical point. To calculate the imaginary part of the ground state energy of the metastable state, we use the imaginary time path integral method. The imaginary time classical path in the quartic potential (in equation (17)) is known as the instanton solution

$$\tilde{x}(\tau) = 2\sqrt{\epsilon} \text{sech}[\sqrt{\epsilon}(\tau - t_0)].$$

where $\tau$ denotes imaginary time. The above equation describes the classical path of a particle in the inverted potential given in equation (17), where the particle starts from $x = x_-$ at $\tau = -\infty$, reaches the turning point $\tilde{x} = 2\sqrt{\epsilon}$ at $\tau = t_0$ and returns to the original point at $\tau = \infty$. The imaginary time ‘action’ $S_c$ corresponding to this classical path is given by

$$S_c = \int_{-\infty}^{\infty} d\tau \left[ \frac{1}{2} \left( \frac{dx(\tau)}{d\tau} \right)^2 + V(x) - V(x_-) \right]$$

$$= \frac{8}{3} \epsilon^{3/2}.$$  

Figure 3. (a) Energy eigenvalues of the Hamiltonian in equation (2) for $N = 50$, $\alpha = 0.4$ (circles connected with a line) and $\alpha = 2.5$ (squares). (b) Energy gap $\Delta E/S$ as a function of the coupling strength $\alpha$ for $N = 500$ (solid line) and $N = 800$ (dotted line).

3.2. Energy spectrum

Similar to the change in shape of the potential $V(x)$, we also notice a distinct change in the energy spectrum of the spin Hamiltonian in equation (2), as the coupling constant $\alpha$ is varied across the critical value $\alpha_c = 1$. We also analyse the energy spectrum of the Hamiltonian given in equation (2) by numerical diagonalization, which is shown in figures 3(a) and (b) for $S = 50$. The interesting feature in the energy spectrum is the appearance of doubly (quasi)-degenerate excited states above the critical coupling strength $\alpha > 1$ (see figure 3(b)). First, the degeneracy appears at the highest energy level, and then with the increasing coupling strength $\alpha$, other excited states become pairwise degenerate. Within the WKB approximation, the energy spectrum of a BJJ in this regime has been studied in [30]. The statistical properties of the highest energy state at the self-trapping transition have also been analysed in [31]. To quantify this transition in the energy spectrum, we calculate the energy gap $\Delta$ between the highest energy level $E_{2S+1}$, and the next one $E_{2S}$ as a function of the coupling constant $\alpha$ (as depicted in figure 2(c)). For finite $S$, the energy gap $\Delta$ vanishes slightly above the critical coupling $\alpha = 1$. With the increasing values of $S$, the transition becomes sharper and the gap vanishes closer to the critical coupling.
This transition can be explained in terms of the ‘effective potential’ approach of the spin system and is directly linked to the ‘self-trapping’ phenomenon. The energy spectrum of the original spin Hamiltonian in equation (2) is exactly the same as that of the Schrödinger equation with an effective potential given in equations (13) and (14). By performing two successive transformations, \( x = x' + \pi \) and \( y = y' \), we obtain the following eigenvalue equation from equation (13):

\[
\kappa \chi(y) = -\frac{\alpha'}{2(s + 1/2)^2} \partial_y^2 \chi(y) + W(y) \chi(y),
\]

where the effective potential \( W(y) = \frac{\alpha}{s} \sinh^2 y - \cosh y \) and \( \kappa = -\mathcal{E} \). The above transformation leads to the mapping of the highest eigenvalue \( E_{2s+1} \) of the Schrödinger equation given in equation (13) to the ground state of the effective potential \( W(y) \) in equation (21). For \( \alpha < 1 \), the effective potential has only one minimum and then it takes the shape of a double well for \( \alpha > 1 \), as shown in figures 2(c) and (d). The double well structure of the effective potential \( W \) explains the appearance of doubly degenerate excited states of the original spin Hamiltonian for the coupling strength \( \alpha > 1 \). When the kinetic energy term in the Hamiltonian equation (21) is neglected, the classical ground state energy can be obtained from the potential \( W \) at two degenerate minima. The classical ground state energy \( \frac{1}{2} (\alpha + \frac{1}{s}) \) of the double well potential is exactly the same as that of the fixed point corresponding to the MST state with population imbalance \( \pm \sqrt{1 - 1/\alpha^2} \). As discussed earlier, the MST state can be described by the dynamics of a fictitious classical particle in the GL potential (in equation (8)) with energy \( E \) less than the barrier height. The formation of the doubly degenerate excited states in the quantum mechanical spectrum can be understood from the periodic orbits of the classical motion near two minima of the GL potential.

### 3.3. Energy splitting and macroscopic quantum tunnelling between MST states

For \( S \to \infty \), the kinetic energy term in equation (21) vanishes and two energy states corresponding to the minima of the potential \( W(y) \) are degenerate. But for finite values of \( S \), the fictitious quantum particle described by Schrödinger’s equation in equation (21) can tunnel between the minima of the potential and produce a small splitting between the highest quasi-degenerate energy levels. Linear behaviour of the logarithm of the energy splitting with increasing \( N \) reveals the collective nature of tunnelling. For the deep double well potential, the semiclassical formula agrees well with the numerical result. The agreement with the numerical result becomes even better by using the exact value of the effective Planck constant \( \hbar_{\text{eff}} = 1/(S + 1/2) \) (instead of 1/2) in the semiclassical formula. The macroscopic tunnelling between two fixed points corresponding to the self-trapped states with equal and opposite number imbalance is depicted in figure 4(b).

From the energy splitting \( \Delta E/J = 0.0196 \) for \( \alpha = 1.25 \) and \( N = 50 \), we obtain the time period of macroscopic tunnelling \( T = 2\pi/\Delta E \sim 320/J \), which is in good agreement with the numerical result of time evolution shown in figure 4(b). In contrast, for the same initial conditions, the BJJ with \( N = 100 \) remains self-trapped since the time required for the macroscopic tunnelling (MQT) becomes very large and grows exponentially with number of particles \( N \). For a BJJ with \( N = 50 \) and \( 1/J \sim 30 \text{ ms} \) [7], MQT can be observed in \( \sim 4.8 \text{ s} \).

Although such collective tunnelling (MQT) in the BJJ is novel macroscopic quantum phenomena, it is difficult to observe in reality due to the long time scale of oscillation. But for a sufficiently small number of particles in the BJJ, and by tuning the coupling constant close to the critical point, such MQT can be observed.
A coherent state is localized at a fixed point corresponding to the self-trapped state (MST) with \( \cos \theta \) \( N \). This line indicates the self-trapping of a BJJ with \( N \) the oscillation of the BJJ with \( N \). The oscillation of the BJJ corresponds to a metastable minimum of the potential’ gives a clear picture of the ‘effective potential’ method. The ‘oscillation’ and \( \alpha \) decay and MQT emerge from the simple picture of the two-site BHM mechanically various dynamical branches of the Josephson dynamics and collective quantum effects. We study quantum phenomena can be observed in a BJJ with a small number of atoms and by tuning the coupling constant close to the critical value. Nevertheless, these novel macroscopic quantum phenomena can be observed in a BJJ with a small number of atoms and by tuning the coupling constant close to the critical value.

4. Summary

To summarize, we considered two weakly coupled condensates described by an effective two-site BHM and studied quantum dynamics and collective quantum effects. We study quantum mechanically various dynamical branches of the Josephson oscillations and transition between them. The two-site BHM with \( N \) bosons can be mapped onto a simple model describing a quantum particle in an ‘effective potential’ where the Planck constant \( \hbar \) is reduced by a factor of \( N/2 \). This ‘effective potential’ gives a clear picture of the ‘\( \pi \)-oscillation’ and MST phenomenon in the BJJ. At the transition between these two dynamical branches, the ‘effective potential’ changes its shape.

Apart from the dynamics, the novel quantum effects like collective decay and MQT emerge from the simple picture of the ‘effective potential’ method. The ‘\( \pi \)-oscillation’ of the condensate corresponds to a metastable minimum of the potential \( V(x) \), and can decay by means of MQT through the potential barrier. Semiclassically, we calculate the decay rate \( \Gamma \) for a small potential barrier and obtain a power law behaviour \( \Gamma \sim (\alpha_c - \alpha)^{5/4} \) close to the critical coupling strength \( \alpha_c \).

We also notice a remarkable feature in the energy spectrum as the coupling constant \( \alpha \) changes across the critical value. Quasi-degenerate pairs of excited states appear in the energy spectrum for coupling strength larger than the critical value \( \alpha_c \). The doubly degenerate highest energy states can be interpreted in terms of an effective double well potential \( W(x) \) (shown in figure 2(d)), and they correspond to the classically degenerate MST states with equal and opposite number imbalance. We study the MQT between these two MST states and estimate the tunnelling time from the energy splitting. The probability of \( \Delta E \) is exponentially suppressed by the number of particles \( N \); nevertheless, these novel macroscopic quantum phenomena can be observed in a BJJ with a small number of atoms and by tuning the coupling constant close to the critical value.

Acknowledgments

We would like to thank P A Sreeram and A M Ghosh for helpful discussions. RJK thanks the support of Georg-August-University Göttingen in the framework of G-KOSS and IISER, Kolkata, for local hospitality.

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