Combining the method of boundary states and the Lindstedt–Poincaré method in geometrically nonlinear elastostatics

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Abstract. The study considers a statement of geometrically non-linear problems of isotropic theory of elasticity for a simply connected bounded body. The Lindstedt–Poincaré method reduces the problem to a weakly non-linear variation containing a small parameter. The solution is then effected at each step of the perturbations method through the means of the method of the boundary states with perturbations (MBSP). Each iteration is accompanied by a generation of fictitious volume forces of a polynomial nature. The study proposes a strict particular solution and a solution of an elasticity problem for iteration. The approach is also applicable to thermoelasticity problems. The method is tested on classical axially symmetrical bodies, i.e., cylinders subjected to uniaxial tensioning or twisting. The study confirms the applicability of flat section hypotheses to linear engineering calculations. Nonlinear distortions mostly happen at the first iteration, followed by a weak correction. We also provide a calculation for and an illustration of a stress-strain state (SSS) of a semicylinder under unbalanced axial loads and a concentration of compression stresses in extended bodies’ middle areas that are remote from load-free surfaces.

1. Introduction

Problems about finite strains of elastic bodies are clearly geometrically nonlinear [1, 2, 3]. To reduce the need for computations, let us use $U_i, E_{ij}, \Sigma_{ij}$ to denote the dimensionless form of the components of the displacement vector, deformation tensors, and strain tensors, respectively. The equations that determine state $\{U_i, E_{ij}, \Sigma_{ij}\}$ of the medium are: a geometrically nonlinear form of the relationship between deformations and displacements in a three-dimensional space with coordinates $X_i$:

$$2E_{ij} = \frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i} \frac{\partial U_k}{\partial X_j},$$

a generalized Hooke's law in a Lamé format for isotropic medium

$$\Sigma_{ij} = \lambda E_{ik} \delta_{ij} + 2 \mu E_{ij},$$

where $\lambda$ and $\mu$ are Lamé constants.
where $\lambda, \mu$ are dimensionless elasticity parameters, $\delta_{i,j}$ is a Kronecker delta; and equilibrium equations (initially recorded in the absence of volume forces)

$$\frac{\partial \sigma_{ij}}{\partial X_j} = 0. \tag{3}$$

Formulae (1)-(3) use a tensor/index form of recording with the "summation convention". We intentionally refrain from using the comma for partial derivatives because below we will have to replace independent variables $X$, by $x$, in relation to which the comma will be used.

The geographically nonlinear nature of (1) renders impossible the application of commonly used numerical methods for solving linear problems of the elastic theory. A traditional method of overcoming this barrier is the use of the perturbation method.

2. Linearisation

Introduction of a small parameter in the system of defining relations is achievable through the Poincaré–Lindstedt method [4]. Namely, we replace the independent variables by using small parameter $0 < \beta \ll 1$:

$$x_i = \beta X_i. \tag{2.1}$$

Moreover, let us change the notation

$$E_{ij} = \beta \epsilon_{ij}, \quad U_i = u_i, \quad \Sigma_{ij} = \beta \sigma_{ij},$$

as a result of which relations (1) - (3) are reduced to

$$2 \varepsilon_{ij} = u_{i,j} + u_{j,i} + \beta u_{k,i} u_{k,j},$$

$$\sigma_{ij} = \lambda \varepsilon_{ik} \delta_{ij} + 2 \mu \varepsilon_{ij}, \tag{2.2}$$

$$\sigma_{i,j,j} = 0.$$

This representation of defining relations is also nonlinear. The presence of the small parameter $\beta$ causes the nonlinearity to be declared in a "weak" form. This factor makes it possible to decompose the nonlinearity while representing all characteristics of the state $\xi = \{u_i, \epsilon_{ij}, \sigma_{ij}\}$ as series in terms of powers of $\beta$ [4, 5]:

$$\xi = \sum_k \beta^k \xi^{(k)}, \quad \xi^{(k)} = \{u_i^{(k)}, \epsilon_{ij}^{(k)}, \sigma_{ij}^{(k)}\}. \tag{2.3}$$

The search for a solution at $\xi^{(k)} = \{u_i^{(k)}, \epsilon_{ij}^{(k)}, \sigma_{ij}^{(k)}\}$ in the form (2.3) brings us to a sequence of problems. At the iteration of $k = 0$ we obtain from (2.2) linear relations to $u_i^{(0)}, \epsilon_{ij}^{(0)}, \sigma_{ij}^{(0)}$, which presents us with a classical problem of elastostatics for a homogeneous isotropic body.

At later iterations we come to relations

$$2 \varepsilon_{ij}^{(k)} = u_{i,j}^{(k)} + u_{j,i}^{(k)} + \sum_{s=0}^{k-1} u_{m,i}^{(s)} u_{m,j}^{(k-s-1)} + 2 \varepsilon_{ij}^{(k-1)},$$

$$\sigma_{ij}^{(k)} = \lambda \varepsilon_{ik}^{(k)} \delta_{ij} + 2 \mu \varepsilon_{ij}^{(k)} + \left[\lambda \varepsilon_{ij}^{(k-1)} + 2 \mu \varepsilon_{ij}^{(k-1)}\right],$$

$$\sigma_{i,j,j}^{(k)} + \Sigma^{(k)} = 0.$$
where the formulae calculated in the previous iterations determine the nonlinearity of the original statement of the problem.

\[
e_{ij}^{(k)} = e_{ij}^{(k)} - e_{ij}^{(k-1)}, \quad e_{ij}^{(k-1)} = \frac{1}{2} \sum_{s=0}^{k-1} u_{m,i} u_{m,j},
\]

\[
\sigma_{ij}^{(k-1)} = \lambda e_{mn}^{(k-1)} \delta_{ij} + 2\mu e_{ij}^{(k-1)}, \quad X_i^{(k)} = \lambda e_{mn,j} \delta_{ij} + 2\mu e_{ij}^{(k-1)}.
\]

Further on, after introducing the notation

\[
v_i^{(k)} = u_i^{(k)}, \quad s_{ij}^{(k)} = \sigma_{ij}^{(k)} - \sigma_{ij}^{(k-1)},
\]

we arrive at a classical format of statement of a problem of isotropic elastostatics

\[
2 e_{ij}^{(k)} = v_{i,j}^{(k)} + v_{j,i}^{(k)},
\]

\[
s_{ij}^{(k)} = \lambda e_{mn}^{(k)} \delta_{ij} + 2\mu e_{ij}^{(k)},
\]

\[
X_{ij}^{(k)} = 0.
\]

By solving the problem in relation to state \(v_i^{(k)}, e_{ij}^{(k)}, \sigma_{ij}^{(k)}\) by the method of boundary states (MBS) [6, 7], we find the fields of these characteristics in a numerical and analytical form. A full parametric solution [8, 9] is also possible. After restoring strains and stresses at iteration \(k\) in accordance with the notations introduced:

\[
\sigma_{ij}^{(k)} = s_{ij}^{(k)} + \sigma_{ij}^{(k-1)},
\]

\[
e_{ij}^{(k)} = e_{ij}^{(k)} + e_{ij}^{(k-1)},
\]

we also restore the corresponding displacement in accordance with the Cesàro formula [3, 10, 11]:

\[
u = u_0 + \omega_0 \times (r - r_0) + \int_{M} \Pi' \cdot dr,
\]

\[
\Pi' = [\Pi_\mu]_{3x3}, \quad \Pi_\mu = e_{\mu} + (x_\mu - x_q)(e_{q,\mu} - e_{q,q}),
\]

where the prime symbol is used to mark all objects that are related to the line of integration \(M_\mu M\), \(r_0, u_0\) is the radius vector of the position and displacement of point \(M_0\), \(\omega_0\) is the vector of rotation of its neighbourhood, and \(r = \{x_q\}\) is the radius vector of point \(M\). A combination of these characteristics forms the state \(\xi^{(k)} = \{u_i^{(k)}, e_{ij}^{(k)}, \sigma_{ij}^{(k)}\}\). After a sufficient number of iterations the resultant internal state of the body is formed by combination (2.2).

**Notes.** 1° The addition of volume forces to equilibrium equations does not change the solution method in any meaningful way. The force will bring some additions to \(\tilde{X}_i^{(0)}\) or to all \(\tilde{X}_i^{(k)}\).

2° The inhomogeneous part of the resolving equations may not be definite, but may be formed by a specific physical cause. Thus, in the case of thermoelastostatics subject to the Duhamel – Neumann law [15], volumetric forces are caused by temperature gradients:

\[
X_{ij}^T = -(3\lambda + 2\mu)\alpha T_{ij},
\]

where \(T\) is the temperature relative to the initial state, \(\alpha\) is the temperature expansion parameter. This representation can be used in advance, if the construction of the temperature field does not
require a joint solution of the determining equations of elasticity with the equation of thermal conductivity. Otherwise, it is necessary to expand the concept of spaces of states of the medium [7].

3. Test problems for a round cylinder

We tested the applicability of the MBSP to non-linear problems (2,4) by solving specific axisymmetry problems featuring a round cylinder with a radius of $R$ and a height of $2l$ in two load scenarios: 1) single-axis tensioning by force $p_0$; 2) twisting by a pair of forces with torque $M_0$. We received a sufficient confirmation of the efficiency of the approach after using 102 orthonormal elements in isomorphic spaces of internal $z$ and boundary $r$ states. The calculations were made after a non-dimensionalisation of mechanical and geometric parameters of the body (with corresponding scales $\mu, R$ introduced after fixing $l = R, \nu = 0.25$).

In the case of uniaxial tension under a uniform force whose intensity is $p = p_0 / \mu$ (assumed to be equal to the reference value 1) in a zero-order approximation we arrived to a classical strict solution of

$$
\mathbf{u}^{(0)} = \begin{pmatrix} -0.1 x \\ 0.1 y \\ 0.4 z \end{pmatrix}, \quad \varepsilon^{(0)} = \begin{pmatrix} -0.1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}, \quad \sigma^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

which received no additions in further iterations:

$$
\mathbf{u}^{(k)} = 0, \quad \varepsilon^{(k)} = 0, \quad \sigma^{(k)} = 0, \quad k > 0.
$$

In the problem featuring a cylinder being twisted (where the non-dimensional torque $M = M_0 / \mu R$ is represented by tangent forces $p_x, p_y$ on the cylinder's butt ends), the boundary conditions of the first basic problem were set as

$$
p = \begin{cases} 0, & (x, y, z) \in S_1, \\
\pm p \{ -y, x, 0 \}, & (x, y, z) \in S_{2,3}, 
\end{cases}
$$

$$
S_1 = \{(x, y, z) \in R^3 \mid x^2 + y^2 = 1, z \in [-l, l] \},
$$

$$
S_{2,3} = \{(x, y, z) \in R^3 \mid 0 \leq x^2 + y^2 \leq 1, z = \pm 1 \}.
$$

For the purpose of the calculation, we assumed that $p = 1$.

The nature of the convergence of the solution becomes clear upon calculations provided in table 1. Iteration "0" is performed with utmost accuracy: the standard integral deviation of the resultant boundary state from the pre-set BCs is $2 \cdot 10^{-6}$. The convergence of the Fourier series to the solution can be indirectly confirmed by the saturation of the Bessel-type summation $\left( \sum_{k=1}^{n} c_k^2 \right)^{1/2}$ as a function of the length of the designated interval of the basis set. The nature of such saturation enables us to decide to set $n$. More specific is the information about the standard integral deviation of the resultant boundary state from the pre-set BCs.

Figure 1-3 show the saturation of stress fields $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}$ in sections that are normal to the symmetry axis. Cuts were made at $z \in \{0, 0.5, 1\}$. The solution proved to be virtually free of any influence of coordinate $z$. It follows that we deal with a state that is uniform along the cylinder's axis.
Table 1. MBSP convergence analysis.

| Iteration | Fourier coefficients | Bessel-type summation saturation |
|-----------|----------------------|----------------------------------|
| 0         | ![Graph](image1.png) | ![Graph](image2.png)             |
| 1         | ![Graph](image3.png) | ![Graph](image4.png)             |

Figure 1. Stresses at iteration "0" in a problem featuring a cylinder.

Figure 2. Stresses at iteration "1" in a problem featuring a cylinder.

Figure 3. Stresses at the result iteration in a problem featuring a cylinder.
The stress levels (of the order of $10^{-8}$) of the zero-order approximation testify that any revolving cross-section in a linear problem behaves like a rigid body. Even the first iteration of the MBSP proves that the nonlinear relationship between the deformations and the displacements brings about a fundamental redistribution of stresses, with the axial fibres increasingly stretching as we get closer to the cylinder's boundary and, by contrast, contracting near the symmetry axis. The volumetric strains demonstrate similar behaviour: volumetric compression near the symmetry axis and expansion in proportion to the growing distance from it. The resultant state causes a slight modification in the distribution of stresses, with the nature of SSS remaining identical to that at the first approximation.

### 4. Finite deformation of a semicylinder

**A small semicylinder.** Consider a semicylinder under axial forces. Let us describe the elastic state of the round semicylinder with a roughly equal radius $R$ and height $2l_0$ through relationships (2.2). At the boundaries of the body are set the conditions of the first basic problem (according to the Muskhelishvili classification [12]) that correspond to a compression of butt ends that is heterogeneous at coordinate $x$ (figure 4). Let us assume that, after a non-dimensionalisation of ($l = l_0 / R$) with parameters $\mu, R$, the boundary conditions of the first basic problem take the following form:

$$
p(x, y) = \begin{cases} 
0, & (x, y, z) \in \mathcal{S}_1 \cup \mathcal{S}_2, \\
\pm f(x, y), & (x, y, z) \in \mathcal{S}_3, \mathcal{S}_4, \\
f(x, y) = p(x - 1)
\end{cases}
$$

(4.1)

where

$$
\mathcal{S}_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0, y, z \in [-l, l]\}
$$

$$
\mathcal{S}_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, x \geq 0, z \in [-l, l]\}
$$

$$
\mathcal{S}_{3,4} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, x \geq 0, z = \mp l\}.
$$

For the purposes of the calculation, it is convenient to assume that $p$ equals the reference value $p = 1$ [8], while the resultant stress state is in proportion to the specific value set.

![Figure 4](image_url)

**Figure 4.** Alteration of the shape of the semicylinder being compressed: a) initial state; b) load configuration; c) post-deformation shape.

The SSS analysis through the means of the MBSP involved 237 intervals of basis sets of spaces $\xi, \Gamma$; while particular solutions based on fictitious iterated volume forces required 30 intervals to
ensure the efficiency of the reverse method [13]. Iteration-by-iteration distribution of stresses in section \( z = l \) is provided in figure 5-7. All the stresses (except the axial stress) in the initial (0th) iteration equal zero:

\[
\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0.
\]

Figure 5. Semicylinder: stresses \( \sigma_{zz} \), zeroth iteration.

Figure 6. Semicylinder: stresses, first iteration.

Figure 7. Semicylinder: stresses, first iteration.

An analysis of the stress state of a small semicylinder allows for the following conclusions: the line component of SSS (iteration 0) prevails. Nonlinear corrections are fairly weak (difference by a factor of 1,000), get stabilised as iterations are repeated, but have distinctive manifestations: the end faces rotate consistently with the load configuration and get displaced towards the Ox axis, and the O point becomes the saddle point after the strain. This confirms V. Novozhilov's representation that non-linear parts can be neglected when calculating strain [14].

**Extended semicylinder.** Let us explore the effect of a disproportion of a body's size on its SSS, assuming that \( l_0 >> R \) (if \( l = l_0 / R = 10 \)). The dominance of one geometrical parameter over others
functionally characterises the body as a spring. Let us assume that the semicylinder's butt ends are subjected to forces in accordance with (4.1), where \( f(x, y) = px \).

After just three iterations of SSS analysis through the means of the MBSP we are able to present the displacement vector for each point of region \( V \) in the following form (rounded to two significant digits):

\[
\mathbf{u} = \begin{cases} 
0.05 (y^2 - x^2) - 0.2z^2 \\ 
-0.1xy \\ 
0.4xz 
\end{cases} + 0.1 \beta \begin{cases} 
0.22xy - 0.41x^2z + 0.49yz + 0.20x^2z \\ 
-0.18xy + 0.26x^2z \\ 
-0.49xy - 0.20x^2z 
\end{cases} + \begin{cases} 
0.22y^2 - 0.28x^2z - 0.35yz - 0.14x^2z \\ 
-0.23xy + 0.27x^2z \\ 
-0.15x + 0.35xy + 0.15x^2z 
\end{cases} + 0.01 \beta^2 \]

(4.2)

An analysis of the solution results relative to displacements (4.2) yields the following conclusions. The displacement increment in the first approximation, when compared to zero (i.e., at \( \beta = 0 \)), does not exceed \( 0.05 \beta \), and in the second approximation does not exceed \( 0.004 \beta^2 \), i.e., is much smaller in the second approximation than in the first. It follows that, to assess stress-strain states of extended bodies, one refining iteration with perturbations is enough. More accurate estimates can be achieved through two iterations. The solution of an initial (zeroth) iteration corresponds to linear problems and delivers stress tensor values in which the only non-zero component is axial stress \( \sigma_z = x \). The geometrically nonlinear nature brings about other stresses whose quality may be assessed by visual means (figure 8, 9).

**Figure 8.** Distribution of stresses in axial section \( y = 0 \) after the second iteration.
Their quantitative estimation is achieved through formulae received after iterations 0+2 and a rounding to two significant digits:

\[
\sigma_{xx} = 0.001 \beta (0.85 x^2 - 2.48 x z - 0.18 x + 0.25 y^2 + 0.68 y + 0.38 z^2 - 0.05) + \\
+ 0.0001 \beta^2 (-0.8 x^3 + 4.1 x^2 - 1.55 xyz - 0.4 xy - 6.9 x z^2 - 1.7 x z - \\
- 0.2 x + 0.7 y^2 - 7.3 y z - 0.2 y - 0.4 z^2 + 0.1); \\
\sigma_{yy} = 0.001 \beta (0.8 x^2 - 0.83 x z - 0.54 x + 0.2 y^2 + 0.22 y + 1.8 z^2 - 0.02) + \\
+ 0.0001 \beta^2 (-0.8 x^3 + 2.5 x^2 z + 0.7 x^2 - 0.5 xy z + 0.4 xy - 3.4 x z^2 - 0.6 x z - \\
- 0.7 x + 0.7 y^2 - 2.4 y z - 0.1 y - 0.1 z^2); \\
\sigma_{zz} = 0.001 \beta (1.9 x^2 - 0.83 x z - 0.18 x + 0.1 y^2 + 0.22 y + 3.4 z^2 - 0.02) + \\
+ 0.0001 \beta^2 (-2.4 x^3 + 5.8 x^2 z + 0.6 x^2 - 0.5 xy z - 6.6 x z^2 - 0.6 x z - 0.2 x + \\
+ 0.4 y^2 - 7.3 y z - 0.1 z^2); \\
\sigma_{xc} = 0.001 \beta (-0.41 x^2 + 2 x z + 0.01) + 0.0001 \beta^2 (-0.5 x^2 z - 0.2 x^2 - 0.8 xy z + \\
+ 0.2 xy - 1.9 x z - 0.1 x + 0.2 y^2 + 0.2 y z^2 - 0.3 z).
\]

Figure 8 provides a distribution of stresses \( \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xc} \) after the second iteration. Rod cuts are made within the limits of \( x \in [0.1], z \in [0.1] \). The middle part \( z \in [1.5,8.5] \) is removed and replaced by an ellipsis. Figure 8 & 9 and the formulae (4.3) signify the following: 1) the geometrical non-linearity of the medium generates stresses \( \sigma_{xx}, \sigma_{yy}, \sigma_{xc} \) that are absent in linear problems (iteration 0); 2) the determinant of the corrective non-linear addition is the first iteration of the perturbations method. The second iteration is a correction of the order of \( 0.1 \beta \) (i.e., roughly 10 \( \beta \% \)); 3) normal stresses \( \sigma_{xx}, \sigma_{yy} \) have greater distortions than the middle part as they get closer to the butt ends of the semicylinder. Although the stresses in the vicinity of the cylindrical surface are pulling, and their nature appears to signify an absence of any tensioning along axis \( z \), this is compensated by a significant compression in regions that are remote from the body’s boundaries (black spots in figure 9).

The transition to dimensionless variables \( X \), and characteristics of the stress-strain state \( U_{ij}, E_{ij}, \Sigma_{ij} \) and further to dimensional values poses no difficulties and warrants no special attention.
5. Conclusions

1. The MBSP in combination with the perturbations method on the basis of the Lindstedt–Poincaré method is an efficient method of solving geometrically nonlinear problems of the theory of elasticity for a simply connected bounded body.

2. The solution of test problems confirmed the validity of application of plane section hypotheses to linear problems of the elastic theory and uncovered a weak correction of the solutions due to the non-linearity of small bodies (where characteristic sizes are comparable).

3. Rather interesting for the non-linear theory of elasticity are problems featuring extended bodies, where corrections to displacement values caused by non-linearity are significant. At the same time, the distortion of the stress state remains small, despite the manifestation of local signs of such distortion. A distinctive characteristic is that in the vicinity of the surface of a spring all normal stresses tend to be pulling even when the spring is being extended, while the internal layers of the body are in a compressed state.

4. The approach described above cannot be applied to "following" loads, because in this case each load step would have to be followed by a correction of boundary conditions (and loading would have to be presented as a sequence of steps, where any change in the direction of external actions cannot be predicted in advance).

The research was supported by the Russian Foundation for Basic Research and the Lipetsk Region (grant No. 19-41-480003 r_a, 19-48-480009 r_a).

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