Non-abelian Exclusion Statistics

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Abstract

We introduce the notion of ‘order-$k$ non-abelian exclusion statistics’. We derive the associated thermodynamic equations by employing the Thermodynamic Bethe Ansatz for specific non-diagonal scattering matrices. We make contact with results obtained by different methods and we point out connections with ‘fermionic sum formulas’ for characters in a Conformal Field Theory. As an application, we derive thermodynamic distribution functions for quasi-holes over a class of non-abelian quantum Hall states recently proposed by Read and Rezayi.

Key words: non-abelian statistics, conformal field theory, quantum Hall effect

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1 Introduction

It is well-known that strongly interacting quantum many-body systems in low dimensions can have quasi-particles that are very different from the microscopic degrees of freedom from which the system is built. A spectacular phenomenon, which has been demonstrated in theory and experiment, is that of ‘quantum number fractionalization’: the quantum numbers of the fundamental quasi-particles can be fractions of those of the constituent degrees of freedom. Fractionally charged quasi-particles over fractional quantum Hall (fqH) states are a proto-typical example. Along with unusual quantum numbers, quasi-particles in low-dimensional systems can have equally unusual quantum statistics properties. In many cases of interest, fundamental quasi-particles do not satisfy the standard Pauli exclusion principle, but instead a generalization thereof.

There are several angles from which unusual exclusion statistics for quasi-particles in low dimensions may be studied. For systems that are integrable by Bethe Ansatz, the systematics of solutions to the Bethe equations can be interpreted in terms of exclusion statistics of the ‘Bethe Ansatz quasi-particles’ [1, 2]. A second approach starts from scattering data of quasi-particles and associates statistics properties via the Thermodynamic Bethe Ansatz (TBA) [3, 4]. A third approach [5, 6] relies on algebraic properties of field operators that describe quasi-particles in the context of a Conformal Field Theory (CFT). There are various connections among the approaches mentioned here, and in cases where they can be compared they lead to equivalent results. [We refer to [6] for further introduction and additional references.]

Of particular interest are systems whose quasi-particles obey ‘fractional exclusion statistics’ as defined by F.D.M. Haldane in 1991 [7]. Examples are Calogero-Moser models of many-body quantum mechanics and the CFTs describing edge excitations over abelian quantum Hall states [8]. In the latter cases, the fractional exclusion statistics of edge quasi-particles reflect the fractional (anyonic) braiding statistics of bulk quasi-particles over the same quantum Hall state.

It is a well-known fact that the braid group for particles in two space dimensions admits representations that have dimension higher than one. This means that for a given number of particles with all positions and quantum numbers fixed, more than one quantum mechanical state is possible. On the corresponding state vectors, the braid group is represented by a non-trivial matrix. Since matrices in general do not commute, one speaks of non-abelian statistics. In the context of the fractional quantum Hall effect, explicit realizations of such non-abelian braid statistics have been proposed [3, 10, 11, 12]. The bulk quasi-particles over what are loosely called ‘non-abelian quantum Hall states’, have counterparts at the edge. In a recent publication [13], one of us employed the CFT approach of [5] to study the exclusion statistics of edge quasi-holes over particular non-abelian quantum Hall states known as the $q$-pfaffians. The
resulting statistics are a proto-typical example of what we call ‘non-abelian exclusion statistics’.

In later work, other examples of ‘non-abelian exclusion statistics’ have been presented. These include quasi-particles over more general non-abelian fQH states [4], spinons in spin \( S \geq 1 \) Heisenberg spin chains [13, 10], generalized fermions in minimal models of CFT [6], and spinons in some of the level-1 Wess-Zumino-Witten (WZW) models of CFT [4]. In [17] it was argued that the spinon (kink) quasi-particles in specific \( SO(5) \)-invariant 2-leg ladder models for strongly correlated electrons are non-abelions that carry the quantum numbers of physical electrons.

What has been lacking until now is a unified description of the thermodynamics associated to the various examples of non-abelian exclusion statistics. In this paper we write down an \( S \)-matrix for scattering of particles in these theories and then follow a TBA procedure to derive a set of thermodynamics equations, eq. (4.27), which describes what we call ‘order-\( k \) non-abelian exclusion statistics’. We shall show that these equations can be used to rederive the known thermodynamic equations for a variety of non-abelions.

The approach followed in this paper can be summarized as follows. The essence of non-abelian statistics is a degeneracy of the quantum state for a number of quasi-particles with all labels (quantum numbers and positions) fixed. In the setting of a CFT approach, this degeneracy can be linked to a choice of fusion path in a multi-quasi-particle state [18, 13, 4, 10]. In the context of an associated scattering problem, non-trivial fusion rules lead to non-diagonal scattering, and under a TBA procedure, the latter leads to an extended TBA system that features a number of non-physical pseudo-particles. In this paper we give a general characterization of the extended TBA system for quasi-particles satisfying order-\( k \) non-abelian exclusion statistics. Specializing to concrete examples of interest, we demonstrate that this extended TBA system reduces to equations that were previously obtained by other methods. Our main example will be that of spinons for spin \( S \geq 1 \) integrable spin chains, which correspond to the CFT associated to \( SU(2)_{k=2S} \).

As an application of the formalism presented here, we shall present the defining equations for the thermodynamic distribution functions for quasi-hole excitations over the non-abelian quantum Hall states recently proposed by N. Read and E. Rezayi [11].

The structure of the extended TBA system for non-abelions is reflected in the structure of so-called fermionic sum formulas for the characters of an associated CFT. We shall explain this connection, again using \( SU(2)_k \) spinons as our guiding example.

In the study of CFT quasi-particles, one encounters non-abelian exclusion statistics that are more general than the ‘order-\( k \) non-abelian exclusion statistics’ that we study in this paper. [An explicit example is provided by the quasi-holes over the non-abelian
spin-singlet quantum Hall states recently proposed in [12]. The approach followed in this paper can be generalized to such more complicated situations [19].

2 Abelian statistics: thermodynamic equations and TBA

In his definition of ‘fractional exclusion statistics’ Haldane [7] makes the assumption that the act of filling a single particle state (by a particle of type a) reduces the dimensionality of the space of states available to a particle of type b by the amount \( G_{ab} \), with \( G \) a so-called statistics matrix. At the level of thermodynamics, Haldane’s assumption leads to a set of equations for the single-level grand canonical partition functions \( \lambda_a = \lambda_a(z_1, \ldots, z_n) \), with \( z_a = e^{\beta(\mu_a - \epsilon(0))} \), with \( \epsilon(0) \) the (bare) energy and \( \mu_a \) the chemical potential for particles of type a [20, 21],

\[
\left( \frac{\lambda_a - 1}{\lambda_a} \right) \prod_b \lambda_b^{G_{ab}} = z_a ,
\]

from which the 1-particle distribution functions can obtained as

\[
n_a(\epsilon(0)) = z_a \frac{\partial}{\partial z_a} \log \lambda(z) \bigg|_{z_b = e^{\beta(\mu_b - \epsilon(0))}} .
\]

In view of the relation, in the context of quantum Hall systems, between the exclusion statistics leading to eq. (2.1) and abelian braid statistics, we shall call the statistics underlying eq. (2.1) ‘abelian exclusion statistics’.

In the context of the Thermodynamic Bethe Ansatz (TBA), statistics properties are derived from scattering data. For a factorizable \( S \)-matrix with diagonal 2-particle scattering matrix \( S_{ab}(\theta) \), the thermodynamics follow from the following TBA equations for dressed energies \( \epsilon_a(\theta) \)

\[
\epsilon_a(\theta) = (\epsilon(0)(\theta) - \mu_a) - \frac{1}{\beta} \sum_b (\phi_{ab} * \ln(1 + e^{-\beta\epsilon_b}))(\theta) \tag{2.3}
\]

where * denotes the convolution and \( \phi_{ab}(\theta) = -i\partial_\theta \ln S_{ab}(\theta) \). One easily checks [3] that for an \( S \)-matrix of the form

\[
S_{ab}(\theta) = \exp[i(\delta_{ab} - G_{ab})\Theta(\theta)] .
\]

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the TBA equations (2.3) are equivalent to the equations (2.4) for the grand partition functions $\lambda_a = 1 + e^{-\beta \varepsilon_a}$.

It has been found that in some of the simplest integrable many-body systems with inverse square interactions, such as the spin-$1/2$ Haldane-Shastry chain, the $S$-matrix for fundamental excitations precisely takes the ‘statistical’ form of eq. (2.4) [1]. Such systems allow an interpretation as ‘free gases of fractional statistics particles’. [See [22] for another example of a system with purely statistical interactions]. In more general integrable systems, such as the spin $S = 1/2$ Heisenberg chain, one finds $S$-matrices that agree with (2.4) only in the limit of large $\theta$. We refer to [23] for a discussion.

3 Spinons in the $SU(2)_k$ WZW theory

It has been observed [1, 24, 25], that the integrable structure of the spin-$1/2$ Haldane-Shastry chain can be carried over to the associated $SU(2)_1$ CFT. As a consequence, one may study the latter CFT in terms of spin-$1/2$ quasi-particles $\phi^\uparrow, \phi^\downarrow$ with scattering matrix of the type (2.4), with $G_{ab} = \frac{1}{2}$ for $a, b = \uparrow, \downarrow$.

For a similar analysis of $k > 1$ $SU(2)_k$ WZW models, one would like to start from a $S > 1/2$ version of the Haldane-Shastry spin chain, but such theories have not been constructed. What one may do instead [16], is study the integrable Heisenberg chains with $S > 1/2$, and analyze the way the spin-$1/2$ spinons build the excitation spectrum. Proceeding in this manner, the authors of [16] have made contact with an approach which starts from truncations of the conformal spectrum and used the recursion method of [5] to derive the thermodynamic equations for the spinon excitations.

In this paper we offer a third point of view on the statistics of $SU(2)_k$ spinons. We shall start from scattering data for these quasi-particles and employ the Thermodynamic Bethe Ansatz to derive the thermodynamics. We shall find agreement with the results of [16], and we shall be able to generalize the result obtained to a general set of thermodynamic equations for quasi-particles satisfying what we call ‘order-$k$ non-abelian exclusion statistics’.

The TBA equations derived below are closely related to a special limit of TBA equations that have been studied in [26, 27, 18]. In our presentation we shall start by closely following these references, and then proceed to make the connection with non-abelian exclusion statistics.

The $SU(2)_k$ spinons are massless particles which transform as doublets ($\uparrow, \downarrow$) under $SU(2)$. In addition, these particles have a kink structure with vacua labeled by an index running from 1 to $k + 1$. In the CFT setting, the origin of these vacua is the following. The chiral CFT spectrum decomposes into $k + 1$ sectors, which are each headed by a primary field of the affine $SU(2)_k$ symmetry. The primary fields are
labeled by their \( SU(2) \) spin \( j \) for which the allowed values are \( j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2} \). The spinon is associated to the spin-\( \frac{1}{2} \) Chiral Vertex Operator (CVO), which interpolates between adjacent vacua.

In a scattering picture, the kinks scatter off adjacent kinks in a non-diagonal manner. The \( S \)-matrix will be written as \( S = S_{\uparrow,\downarrow} \otimes S_{\text{kink}} \), with the first factor identical to the level-1 statistical scattering matrix (i.e., eq. (2.4) with \( G_{ab} = \frac{1}{2} \)). For the kink part of the \( S \)-matrix we propose the ‘statistical limit’ of the (RSOS)\( k+1 \) solution of the Yang-Baxter equations as described in [27]. Following this reference, we shall label the kinks as \( B_{\alpha\beta}(\theta) \), where \( \theta \) is the kink rapidity and \( \alpha, \beta \) are the labels of two adjacent vacua. The scattering of adjacent kinks is represented as

\[
B_{\alpha\gamma}(\theta_1)B_{\gamma\beta}(\theta_2) = \sum_{\delta} S_{\alpha\gamma}^{\delta\beta}(\theta_1 - \theta_2)B_{\alpha\delta}(\theta_2)B_{\delta\beta}(\theta_1) \ .
\]

(3.5)

In the subsections that follow, we shall present an explicit \( S \)-matrix for spinons in the \( SU(2)_2 \) theory, and discuss the TBA equations for spinons in the \( SU(2)_k \) theory.

### 3.1 Scattering matrix for spinons in the \( SU(2)_2 \) theory

For (RSOS)\( _3 \), there are four kinds of kinks, \( B_{0,\pm} \) and \( B_{\pm,0} \) and the non-zero scattering amplitudes are \( S_{00}^{00} \) and \( S_{00}^{0\beta} \), with \( \alpha, \beta = +, - \). The \( S \)-matrix is a \( 6 \times 6 \) matrix, and it satisfies the conditions for crossing symmetry (3.6), unitarity (3.7), and factorization (3.8)

\[
S_{00}^{0\beta}(\theta) = S_{\alpha\beta}(i\pi - \theta)
\]

(3.6)

\[
\sum_{\gamma = \pm} S_{00}^{\alpha\gamma}(\theta)S_{00}^{\gamma\beta}(-\theta) = \delta_{\alpha\beta} \ , \quad S_{\alpha\beta}(\theta)S_{\alpha\beta}(-\theta) = 1
\]

(3.7)

\[
\sum_{\gamma} S_{00}^{\alpha\gamma}(\theta)S_{00}^{0\beta}(\theta + \theta')S_{00}^{\delta\gamma}(\theta') = S_{\beta\delta}(\theta)S_{00}^{\alpha\delta}(\theta + \theta')S_{00}^{0\beta}(\theta')
\]

(3.8)

The \( S \)-matrix that is consistent with the above conditions is of the form

\[
S_{00}^{0\beta}(\theta) = e^{-i\rho\theta}f_{\alpha\beta}\sigma(\theta) \ , \quad S_{\alpha\beta}(\theta) = e^{i\rho\theta}g_{\alpha\beta}\sigma(\theta)
\]

(3.9)

where

\[
\sigma(\theta) = \frac{1}{[\cosh(\theta/2)]^{1/2}} \exp\left[\frac{1}{4}I(\theta)\right] \ , \quad I(\theta) = \text{sign}(\theta) \int_0^\infty dt \frac{\sin|\theta t|}{t \cosh^2 \frac{t}{2}}
\]

(3.10)
and \( \rho = \frac{1}{2\pi} \log 2 \).

To obtain the kink part \( S_{\text{kink}} \) of the \( S \)-matrix of \( SU(2)_2 \) spinons, we take the ‘statistical limit’ \( \theta \to \pm \infty \) of the \((\text{RSOS})_3 \) \( S \)-matrix. In this limit we have

\[
I(\theta) \to \text{sign}(\theta) \int_0^\infty \frac{dt}{t} \sin\frac{\pi}{2} \text{sign}(\theta) = \frac{\pi}{2} \text{sign}(\theta) \quad (3.11)
\]

so that

\[
\sigma(\theta) \to \sqrt{2} e^{\frac{\theta}{2} \pm \frac{\pi i}{8}} \quad (3.12)
\]

The matrix elements \( f(\theta) \) and \( g(\theta) \) given in [27] have the limiting form

\[
f_{++}(\theta) = f_{--}(\theta) \to \frac{1}{2} e^{\pm \frac{\theta}{4}} , \quad f_{+-}(\theta) = f_{-+}(\theta) \to \pm \frac{1}{2} e^{\pm \frac{\theta}{4}}
\]

\[
g_{++}(\theta) = g_{--}(\theta) \to \frac{1}{2} e^{\pm \frac{\theta}{4}} (1 \mp i) , \quad g_{+-}(\theta) = g_{-+}(\theta) \to \frac{1}{2} e^{\pm \frac{\theta}{4}} (1 \pm i) . \quad (3.13)
\]

The \( S \)-matrix elements are the following when \( \theta \to \pm \infty \)

\[
S_{00}^{++} = S_{00}^{--} = a^{-1}(\theta) e^{\pm \frac{\theta}{4}} , \quad S_{00}^{-+} = S_{00}^{-+} = \pm i a^{-1}(\theta) e^{\pm \frac{\theta}{4}}
\]

\[
S_{-+}^{00} = S_{++}^{00} = a(\theta) e^{\pm \frac{3\pi i}{8}} , \quad S_{++}^{00} = S_{--}^{00} = a(\theta) e^{\pm \frac{\pi i}{8}} \quad (3.14)
\]

where \( a(\theta) = (\sqrt{2})^{\frac{\theta}{2}} \) is a gauge factor.

### 3.2 TBA for spinons in the \( SU(2)_k \) theory

To write down the TBA equations for the \( SU(2)_k \) spinons, we must consider an ensemble of \( N \) spinons and allow one of them to scatter with all the others, taking the thermodynamic limit \( N \to \infty \) in the end. The scattering in the \((\uparrow \downarrow)\) part is diagonal whereas in the RSOS part the scattering is non-diagonal and therefore we have to diagonalize it before quantizing the system. Since the \( S \)-matrix of \( SU(2)_k \) is a tensor product of the \((\uparrow \downarrow)\) and the RSOS part, we first diagonalize the RSOS part before putting it together with the \((\uparrow \downarrow)\) part.

The RSOS scattering can be described by a transfer matrix \( T(\theta) \) which has the entries

\[
T_{a_1, a_2, \ldots, a_N}^{\alpha_1, \alpha_2, \ldots, \alpha_N} (\theta | \theta_1, \theta_2, \ldots, \theta_N) = S_{a_1 a_2}^{\alpha_1 \alpha_2} (\theta - \theta_1) S_{a_2 a_3}^{\alpha_2 \alpha_3} (\theta - \theta_2) \ldots S_{a_N a_1}^{\alpha_N \alpha_1} (\theta - \theta_N) . \quad (3.15)
\]
The diagonalization of \( T(\theta) \) involves finding the eigenvalues \( \Lambda_i(\theta|\theta_1,\theta_2,...\theta_N) \) and eigenvectors \( \psi_i \):

\[
T(\theta|\theta_1,\theta_2,...\theta_N)\psi_i = \Lambda_i(\theta|\theta_1,\theta_2,...\theta_N)\psi_i .
\]  

(3.16)

Quantizing the particles in a finite system of size \( L \) requires the eigenvalues \( \Lambda \) to satisfy the condition

\[
e^{ip_l L} \Lambda(\theta|\theta_1,...\theta_N) = 1 , \quad l = 1,2,...N.
\]  

(3.17)

The TBA equations for (RSOS)\( _{k+1} \) have been given in \[27\] with a detailed derivation of the diagonalization of the \( k = 2 \) case. We will not repeat the details here.

The final TBA system describes a single physical particle, labeled as \( i = 0 \), and \( k-1 \) auxiliary ‘pseudo-particles’, labeled as \( i = 1,2,...,k-1 \), which arise from the diagonalization of the transfer matrix. Denoting the densities of the occupied kink states by \( \rho_i(\theta) \) and the density of available states as \( P_i(\theta) \), we can parametrize

\[
\frac{\rho_i}{P_i} = \frac{e^{-\epsilon_i}}{1 + e^{-\epsilon_i}} .
\]  

(3.18)

The dressed energies \( \epsilon_i, \ i = 0,1,...,k-1 \), satisfy an effective TBA system \[27\] of the form eq. (2.3), in which the particles \( i = 0,1,...,k-1 \) scatter off each other purely diagonally with \( S \)-matrices described by TBA kernel

\[
\phi_{ij}(\theta) = \frac{1}{2\pi} \frac{1}{\cosh \theta} l_{ij} ,
\]  

(3.19)

where \( l_{ij} \) is the adjacency matrix of the \( A_k \) Dynkin diagram. In this TBA system, the physical particle has a standard bare energy term (the term \( \epsilon^{(0)} - \mu_0 \) in eq. (2.3)), but there is no such term for the pseudo-particles \( i = 1,...,k-1 \). We remark that in an approach based on a Bethe Ansatz solution of a lattice (RSOS)\( _{k+1} \) model, the pseudo-particles correspond to the (complex) string solutions to the Bethe equations.

In the statistical limit \( (\theta \to \pm \infty) \) appropriate for \( SU(2)_k \) spinons, the kernel eq. (3.19) tends to

\[
\phi_{ij}(\theta) \rightarrow \frac{1}{2} \delta(\theta) l_{ij} .
\]  

(3.20)

The TBA system for the \( (\uparrow\downarrow) \) part of the \( S \)-matrix is that of the \( SU(2)_1 \) spinons. It can be written as

\[
\left( \frac{\lambda_{\uparrow} - 1}{\lambda_{\uparrow}} \right) \lambda_{\uparrow}^{1/2} \lambda_{\downarrow}^{1/2} = z_\uparrow , \quad \left( \frac{\lambda_{\downarrow} - 1}{\lambda_{\downarrow}} \right) \lambda_{\downarrow}^{1/2} \lambda_{\uparrow}^{1/2} = z_\downarrow .
\]  

(3.21)
with \( \lambda_A = 1 + e^{-\varepsilon_A} \) and \( z_A = e^{\beta(\mu_A - \varepsilon(0))} \) for \( A = \uparrow, \downarrow \).

The TBA systems for the \( (\uparrow \downarrow) \) part and the RSOS part, which have been solved individually, should now be combined. The transfer matrix for the combined system is built from the tensor product of the \( (\uparrow \downarrow) \) and the RSOS parts. The eigenvalues of the transfer matrix for the \( SU(2)_k \) system, \( \Lambda_i(\theta | \theta_1, \theta_2, \ldots \theta_n) \), are products of the eigenvalues of the \( (\uparrow \downarrow) \) and the (RSOS)\( k+1 \) part. The TBA kernel for the combined system is the sum of those of the \( (\uparrow \downarrow) \) and the RSOS ones.

In the combined TBA system, the place of the \( i = 0 \) physical particle in the (RSOS)\( k+1 \) system is taken by the physical particles which we denoted as \( \phi_{\uparrow \downarrow} \). The total particle content is thus labeled as \( a = \uparrow, \downarrow, 1, \ldots, k - 1 \). The combined TBA equations take the following form

\[
(\lambda_{\uparrow} \lambda_{\downarrow} - 1) \lambda_1^{\frac{1}{2}} \lambda_2^{\frac{1}{2}} \lambda_1^{-\frac{1}{2}} = z_{\uparrow}, \quad (\lambda_{\downarrow} \lambda_{\uparrow} - 1) \lambda_1^{\frac{1}{2}} \lambda_2^{\frac{1}{2}} \lambda_1^{-\frac{1}{2}} = z_{\downarrow},
\]

\[
\left( \frac{\lambda_{\uparrow} - 1}{\lambda_1} \right) \lambda_{\uparrow}^{\frac{1}{2}} \lambda_{\downarrow}^{-\frac{1}{2}} \lambda_1 = 1, \quad \left( \frac{\lambda_{\downarrow} - 1}{\lambda_j} \right) \lambda_{\uparrow}^{-\frac{1}{2}} \lambda_{\downarrow}^{\frac{1}{2}} \lambda_1 = 1, \quad j = 2, \ldots, k - 1.
\]

(3.22)

As a check on the correctness of our assignment of \( S_{\text{kink}} \), we have compared the TBA system (3.22) with results obtained from different approaches. Starting with \( k = 2 \), we have

\[
\left( \frac{\lambda_{\uparrow} - 1}{\lambda_1} \right) \lambda_{\uparrow}^{\frac{1}{2}} \lambda_{\downarrow}^{\frac{1}{2}} \lambda_1^{-\frac{1}{2}} = z_{\uparrow}, \quad \left( \frac{\lambda_{\downarrow} - 1}{\lambda_1} \right) \lambda_{\uparrow}^{\frac{1}{2}} \lambda_{\downarrow}^{-\frac{1}{2}} \lambda_1 = z_{\downarrow}
\]

\[
\left( \frac{\lambda_{\uparrow} - 1}{\lambda_1} \right) \lambda_{\uparrow}^{\frac{1}{2}} \lambda_{\downarrow}^{-\frac{1}{2}} \lambda_1 = 1.
\]

(3.23)

Eliminating \( \lambda_1 \), putting \( x = z_{\uparrow} = z_{\downarrow} \), and defining \( \lambda = \lambda_{\uparrow} \lambda_{\downarrow} \), one finds

\[
(\lambda_{\frac{1}{2}} - 1)^2 - x^2(\lambda_{\frac{1}{2}} + 1) = 0.
\]

(3.24)

This equation is in agreement with the results of [16, 8], which were obtained through an entirely different approach. The corresponding equations for \( k = 3, 4 \), which were explicitly given in [1], are recovered in a similar manner.

It was argued in [13, 8] that, in general, the non-abelian nature of the exclusion statistics of a set of physical particles manifests itself in the dependence of the grand
partition function $\lambda(z_a)$ on the variables $z_a$, in the limit where $z_a \to 0$ (corresponding to the Boltzmann tails of the associated thermodynamic distribution functions). Expanding the partition function as

$$\lambda(z) = 1 + \alpha_a z_a + \ldots \quad (3.25)$$

one observes [13, 6] that the degeneracies that are characteristic for non-abelian statistics lead to coefficients $\alpha_a > 1$. It is easily seen that the equations (2.1) for abelian exclusion statistics lead to $\alpha_a = 1$, for all choices of the statistics matrix $G$. In contrast, the TBA equations (3.22) lead to

$$\alpha^\up = \alpha^\down = 2 \cos\left(\frac{\pi}{k+2}\right) \quad (3.26)$$

In earlier work [18, 6], the factors (3.26) were obtained as the largest eigenvalues of the fusion matrix associated with a single spinon field. In the literature on rational CFT, these factors are known as ‘quantum dimensions’ associated to specific conformal fields.

### 4 TBA for order-$k$ non-abelian exclusion statistics

Having understood the example of $SU(2)_k$ spinons, we can generalize the TBA system (3.23) to more general particles that satisfy an analogous form of exclusion statistics, which we call ‘order-$k$ non-abelian exclusion statistics’. We propose the following equations for a set of $n$ physical particles, labeled as $A = 1, 2, \ldots, n$ together with $k - 1$ pseudo-particles labeled as $i = 1, \ldots, k - 1$

$$\left(\frac{\lambda_A - 1}{\lambda_A}\right) \prod_B \lambda_B \prod_i \tilde{G}_{Ai} \lambda_i = z_A \quad , \quad A = 1, \ldots, n$$

$$\left(\frac{\lambda_i - 1}{\lambda_i}\right) \prod_A \lambda_A \prod_j \tilde{G}_{ij} \lambda_j = 1 \quad , \quad i = 1, \ldots, k - 1 \quad (4.27)$$

with the matrix $\tilde{G}_{AB}$ specifying the ‘abelian part’ of the statistics, $\tilde{G}_{Ai} = \tilde{G}_{iA} = -\frac{1}{2} \delta_{i,1}$ and $\tilde{G}_{ij} = \frac{1}{2} (C_{k-1})_{ij}$, with $C_{k-1}$ the Cartan matrix of $A_{k-1}$. In specific examples, one may reduce (4.27) to an equation for $\lambda = \prod_A \lambda_A$, which can then be compared with results obtained by other methods.

Comparing with section 2, we see that the extended TBA system (4.27) describes a situation where Haldane’s exclusion principle is applied to a system of both physical
and auxiliary (pseudo-) particles. Despite the similarity between the ‘abelian’ equations (2.1) and the ‘non-abelian’ extension (3.22), there is an essential difference in the physics that is described. In physical terms, one could say that the absence of the bare energy term in the TBA equations for the pseudo-particles \( i = 1, \ldots, k - 1 \) means that these particle are not suppressed at high energies and betray their presence by combinatorial factors that are characteristic of non-abelian statistics.

Clearly, the TBA system (4.27) has been obtained from that for \( SU(2)_k \) spinons by changing the abelian part of the scattering but keeping the ‘kink-structure’. In CFT terms this means that in all cases the general TBA system (4.27) will refer to conformal fields with fusion rules identical to those of \( SU(2)_k \) spinons. These fusion rules are, for example, found in CFTs describing non-abelian quantum Hall states and in minimal models of the Virasoro algebra, and we shall see that these provide other physically meaningful examples of the general structure (4.27).

The robustness of the ‘kink-structure’ behind the TBA system (4.27) implies that the combinatorial implications of the presence of the pseudo-particles are shared by all realizations. This holds in particular for the ‘entropy’ factors (3.26), which are universal. The importance of the entropy factors (3.26) has been stressed in other contexts, in particular in that of the \( k \)-channel Kondo model [13].

5 Non-abelian quantum Hall states

In a recent paper [13], one of us studied the exclusion statistics properties of quasi-hole excitations over the so-called \( q \)-pfaffian quantum Hall states. The CFTs that describe the edge excitations over these quantum Hall states contain, in addition to the free boson associated to edge magnetoplasmons, a free Majorana fermion that is associated to an additional ‘dipole’ degree of freedom. The conformal fields that create the quasi-hole excitations contain as a factor the spin-field associated to the Majorana fermion. The fusion rules of this spin field are identical to those of the \( SU(2)_2 \) spinon fields and, by the general reasoning that we presented, we expect that the quasi-holes will provide a realization of order-2 non-abelian statistics. In this situation, the abelian part \( \tilde{G}_{AB} \) of the statistics matrix is simply a number, which is set by the value of \( q \). We find the TBA system

\[
\left( \frac{\lambda - 1}{\lambda} \right) \lambda^{\frac{q+1}{4q}} \lambda_1^{-1/2} = z, \quad \left( \frac{\lambda_1 - 1}{\lambda_1} \right) \lambda^{-1/2} \lambda_1 = 1
\]

(5.28)

which, upon eliminating \( \lambda_1 \), is indeed equivalent to the result of [13].

In a recent paper [11], the \( q \)-pfaffian quantum Hall states have been generalized to a class of non-abelian quantum Hall states that are parametrized by integers \( (k, M) \).
These states have filling fraction $\nu = k/(Mk + 2)$. The guiding principle in the construction of these states has been the replacement of the Majorana fermion in the pfaffian CFT by an order-$k$ parafermion in the sense of CFT. The fundamental quasi-holes over the $(k, M)$ states are associated to conformal fields that contain as a factor the spin-field associated to an order-$k$ parafermion, and it is a well-known fact that these spin fields have the same fusion rules as the spinons in an $SU(2)_k$ WZW theory. We therefore expect that the exclusion statistics of the quasi-holes over the Read-Rezayi states are of the general form (4.27), with a single abelian component (see also [14]). The only number that needs to be determined is the exponent $g(k, M)$ in the leading equation

$$\left( \frac{\lambda - 1}{\lambda} \right) \lambda^{g(k,M)} \lambda^{-1/2} = z. \quad (5.29)$$

By matching the thermodynamic content of the TBA system with known information (in particular, the value of the Hall conductance), we determined

$$g(k, M) = \frac{(k - 1)M + 2}{2(kM + 2)}. \quad (5.30)$$

The $q$-pfaffian arises as the special case $k = 2, M = q - 1$ of the more general construction, and this provides a check on the result for $g(k, M)$.

We remark that the (positively charged) quasi-holes by themselves do not generate the full (chiral) spectrum of the edge CFT for the Read-Rezayi states. As explained in [8, 13], one may identify (negatively charged) quanta that are dual to the quasi-holes, and that satisfy a form of exclusion statistics that is dual to the statistics of the quasi-holes. Upon combining the two kinds of excitations, one recovers the full chiral spectrum. For the (abelian) Laughlin states ($k = 1$) and for the $q$-pfaffian states ($k = 2$) such a result has been explicitly demonstrated in [8, 13].

### 6 Generalized fermions in CFT

A third example of conformal fields with the fusion rules of $SU(2)_k$ spinons are the fields labeled as $\Phi_{(2,1)}$ in the unitary minimal model $\mathcal{M}^{k+2}$ of the Virasoro algebra. These CFTs have central charge $c(k) = 1 - \frac{6}{(k+2)(k+3)}$ and the field $\Phi_{(2,1)}$ has scaling dimension $\Delta_{2,1} = \frac{k+5}{4(k+2)}$. For $k = 1$ the field $\Phi_{(2,1)}$ is the Majorana fermion of the Ising CFT, while in the limit $k \to \infty$, $\Phi_{(2,1)}$ is identified with one of the spinons of the $SU(2)_1$ WZW theory.

In [6], the grand partition function $\lambda(x)$ was studied on the basis of recursion relations satisfied by truncated conformal characters, and explicit polynomial equations
for \( \lambda(x) \) were obtained for low \( k \). We remark that recursion relations similar to those employed in [3], have been used in the analysis by Andrews, Baxter and Forrester (ABF) of local height probabilities in specific RSOS models [23] (see also [29]). Interestingly, the two situations are ‘dual’ in the sense that where the ABF recursions are in terms of the system size \( m \), the recursion relations for the ‘truncated spectra’ of [6] are in terms of a momentum (or energy) variable \( l \).

Following the general arguments presented in section 4 of this paper, we expect that the exclusion statistics of the \( \Phi^{(2,1)} \) quanta is described by an order-\( k \) extended TBA system of the form (4.27). The abelian part of the TBA system has a single component \( (A = 1) \), and we checked that for \( k = 1, 2, 3 \), with the assignment \( \tilde{G}_{11} = 1 \), the extended TBA system (4.27) reduces to the result obtained in [1]. For \( k = 1 \) the value \( \tilde{G}_{11} = 1 \) simply reflects the fermionic statistics of the Majorana fermion.

We remark that the full matrix \( \tilde{G} \) matrix is equal to one half times the Cartan matrix \( C_k \) of \( A_k \) and that the extended TBA system for the \( \Phi^{(2,1)} \) quanta in the minimal model \( \mathcal{M}^{k+2} \) is identical to the TBA system for (RSOS)\(_{k+1} \) in its statistical limit (see section 3.2). This observation nicely fits with the fact that the scattering matrix for the \( \Phi^{(1,3)} \) massive perturbations of \( \mathcal{M}^{k+2} \) exhibits a similar (RSOS)\(_{k+1} \) structure [30]. Taking the limit \( k \to \infty \), one finds that \( \frac{1}{2} C_\infty \) acts as the statistics matrix in the \( SU(2)_1 \) CFT. This result was recently discussed in the context of the Bethe Ansatz solution of the (single channel) Kondo model [31].

The form of the extended TBA system for the generalized fermions is further confirmed by an inspection of the ‘fermionic sum formulas’ for the conformal characters. This connection is the subject of our next section.

7 Relation with ‘fermionic’ character formulas

A beautiful development in the mathematical analysis of CFT has been the work, initiated in [3], on so-called fermionic sum formulas for the characters in a variety of minimal models of CFT. In the initial work, these expressions had their origin in the mathematical structure of the space of solutions to the Bethe equations for a specific integrable 3-state Potts chain. The character identities that resulted from this analysis have been generalized to large classes of models of CFT [32], and many of the results have been put on a rigorous mathematical basis [29]. On the basis of this extensive work, a general fermionic sum formula of the form

\[
\sum_{m=0}^{\infty} q^{\frac{1}{2} m B m - \frac{1}{2} A m} \prod_{\alpha=1}^{n} g_{\alpha}^{m_{\alpha}} \prod_{\alpha=1}^{n} \left[ (1 - B m + \frac{n}{2})_{\alpha} \right] (7.31)
\]

13
has been put forward \[33\]. Important ingredients in this expression are the summation variables \(m_1, \ldots, m_n\), a bilinear form set by an \(n \times n\) matrix \(B\) and a \(n\)-vector of parameters \(u\). The square brackets \([\ldots]\) denote the \(q\)-binomial coefficient.

In an independent development, it has been recognized that in many CFTs it is possible to construct a quasi-particle basis directly in terms of the conformal fields, i.e. without making reference to an underlying integrable model. The proto-typical example for this is the spinon basis for the \(SU(2)_1\) CFT, which has been fully justified on the basis of generalized commutation relations satisfied by modes of the conformal spinon fields \[25\]. In the paper \[25\], the \(SU(2)_1\) spinon basis results were explicitly linked with some of the character identities obtained in \[12\]. In further studies along these same lines, spinon-type quasi-particle bases for higher level \((SU(2)_k)\) \[15, 34\] and higher rank \((SU(n)_1)\) \[35, 36\] WZW models have been constructed, and associated character identities were obtained.

In a 1997 Letter \[5\], one of us has proposed that the quasi-particle bases that are constructed in terms of specific conformal fields can be used to assign a form of exclusion statistics (in the sense of Haldane and more general) to the quanta of these same conformal fields. Again, the \(SU(2)_1\) spinons served as a prototypical example. In the paper \[5\], a systematic study has been presented and many new forms of exclusion statistics have been revealed. Of particular interest have been the applications to edge theories for various quantum Hall systems \[5, 13, 14\], where other, closely related, notions of fractional statistics have been studied in detail.

In particular cases, there is a close connection between the fermionic sum type character expressions \((7.31)\) and the exclusion statistics associated to conformal fields. In particular, it has been observed \[37, 8\] that for a single particle species satisfying Haldane’s statistics with parameter \(g\), the partition sum is naturally of the form \((7.31)\) with \(B_{11} = g\) and \(u_1 = \infty\). This connection was put on a general footing in a conjecture put forward in \[38, 33\]. Focusing on particles satisfying abelian statistics with statistics matrix \(G\), one expects a direct connection between the TBA system \((2.1)\) and the fermionic sum \((7.31)\) with \(B = G\) and all \(u_a = \infty\). See, eg, \[5\], where the associated central charge identity was explicitly checked.

The result obtained in this paper allow us to illustrate the connection between fractional statistics and fermionic sum formulas in the non-abelian case. For some of the CFT non-abelions that we treated in the above, explicit fermionic sum formulas are known in the literature, and one quickly recognizes how the data in the extended TBA systems of the form \((4.27)\) can be ‘matched’ with the data contained in the fermionic sum formulas. In particular, one finds that the full statistics matrix \(\tilde{G}\) is to be identified with the bilinear form \(B\) in the leading \(q\) power, while the values of \(u_a\) distinguish between physical particles \((u_A = \infty)\) and pseudo-particles \((u_i < \infty)\).
For an explicit example, we recall the spinon form of the characters of $SU(2)_k$ affine Kac-Moody algebra in the corresponding WZW model \[13, 34\]. The character of the highest weight module built over the primary state of spin $j \leq k/2$ is written as

$$
\text{ch}_j(z, q) = q^{\Delta_j - j^2/2} \sum_{M,N \geq 0} q^{-\Delta/(M+N)^2} \Phi_{C_k}^{M+N}(u_j, q) \frac{z^{(M-N)}}{(q)_M(q)_N}
$$

with

$$
\Phi_{K}^{m}(\bar{u}, q) = \sum_{m_2, \ldots, m_k} q^{\frac{1}{4}m \cdot K \cdot \bar{m}} \prod_{i \geq 2} \left[ \frac{1}{2}((2 - K) \cdot \bar{m} + \bar{u})_i \right]
$$

with the vector $u_j$ given by $(u_j)_i = \delta_{i,2j+1}$ and $C_k$ equal to the Cartan matrix of $A_k$ (see \[15\] for further details). The structure of this formula is closely related to the factorized structure of the extended TBA system. The numbers $M, N$ stand for the number of $\phi_\uparrow, \phi_\downarrow$ quanta, and the numbers $m_1, \ldots, m_k$ correspond to the particle content of the (RSOS)$_{k+1}$ kink sector. The non-trivial ‘gluing’ of the two sectors is implemented by the identification $m_1 = M + N$. One quickly checks that the resulting bilinear form (on the $2 + (k - 1)$ variables $M, N, m_2, \ldots, m_k$) agrees with the matrix $\tilde{G}$ in the TBA system (3.22).

The extended TBA system for the generalized fermions of section 6 can, in a very similar way, be recognized in the structure of particular fermionic sum formulas for the conformal characters of the $\mathcal{M}^{k+2}$ minimal model. These formulas were first proposed in \[32\], and they have been proven in \[29\]. The appropriate bilinear form, for summation variables $m_1, \ldots, m_k$, agrees with the matrix $\tilde{G}$ identified in section 6 and via the values of the $u_a$ the particle labeled as $a = 1$ is singled out as the only physical particle.

8 Conclusion

In this paper, we have proposed a general extended TBA system, eq. (4.27), for physical particles satisfying what we call order-\(k\) non-abelian exclusion statistics. This TBA system generalizes the equations (2.1) for abelian exclusion statistics. While many aspects of this work rely on results of earlier work by various groups, several observations are new. We demonstrated that the TBA system (4.27) has its origin in a non-diagonal, purely statistical $S$-matrix, which we explicitly specified for the case of $SU(2)_2$ spinons. On the basis of the TBA system (4.27), we have recovered and unified a number of results in the literature, and we have been able to derive the defining thermodynamic equations for quasi-holes over a new class of non-abelian quantum
Hall states. We also illustrated the correspondence between fermionic sum formulas and fractional statistics assignments in the case of non-abelian statistics.

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