The importance of probability interference in social science: rationale and experiment

Andrei Yu. Khrennikov∗ and Emmanuel Haven†
February 1, 2008

Abstract
Probability interference is a fundamental characteristic of quantum mechanics. In this paper we attempt to show with the help of some examples, where this fundamental trait of quantum physics can be found back in a social science environment. In order to support our thesis that interference can possibly be found back in many other macroscopic areas, we proceed in setting up an experimental test.

Keywords: sure-thing principle; Ellsberg paradox; Allais paradox; probability interference; financial arbitrage; sub(super)-additive probabilities; Heisenberg Uncertainty Principle; information function; rational ignorance; ambiguity

1 Introduction
The concept of ‘probability interference’ is one of the most fundamental concepts of quantum mechanics. Quantum mechanics, as a discipline of physics, models quantum mechanical phenomena which typically are found at scales well below the nuclear level. Hence, it may come as a surprise that quantum physical methodology can be used in a setting which is substantially far removed from the quantum mechanical scale: i.e. the macroscopic scale. In this paper we attempt to convince the reader that the use of precisely such methodology can be used in a beneficial way in a social science setting. More precisely, in this paper, we focus our efforts to purposefully show how basic quantum physical concepts can be used to explain experimentally observed violations of a very basic principle in economic theory: the so called sure-thing principle.

In order to begin to achieve this purpose we must first explain what probability interference is. Then, after we will have explained the essential meaning of the sure-thing principle, we proceed to illustrating, following the path-breaking
arguments by Busemeyer et al. (2006 and 2007), how the violation of this principle can be explained with the help of probability interference. In this paper we also highlight two important paradoxes, the so called Allais and Ellsberg paradoxes, where the latter paradox is an example of where the sure thing principle goes wrong. We briefly illustrate how the Allais paradox is related to the ‘double slit’ experiment, which is the fundamental experiment used in quantum mechanics to argue for the presence of probability interference.

To further underline the importance of the Ellsberg paradox and its connections with quantum mechanical tools, we set ourselves two further main aims in this paper.

As a first further main aim, we proceed first to show how information can be modelled via the wave function - a central concept in quantum mechanics. We then attempt to show that ambiguity, a main characteristic of the Ellsberg paradox, can be modelled with the help of such wave function. Following this, we also highlight how ‘ignorance’ about ambiguity can be related to the same basic quantum mechanical concept.

As a second further main aim, we would like to propose how we can test for the existence of probability interference in a macroscopic setting. If the proposed test can be seen as a potentially conclusive test by which such interference can be experimentally verified, we will have added one more argument in support of why the violation of the sure-thing principle could indeed be explained by probability interference.

In the next section of the paper we acquaint the reader with the notion of the double slit experiment and the ensuing probability interference. In section 3 of the paper, we first discuss the sure-thing principle and the related Ellsberg paradox. We also explain the Allais paradox and briefly argue how this paradox is related to the double slit experiment (the experiment which triggers the existence of probability interference). In section 4, we highlight, following the important work of Busemeyer et al. (2006 and 2007), how probability interference can help in explaining the violation of the sure-thing principle. In section 5, we define ‘ambiguity’ via the Ellsberg paradox. In section 6, we attempt to show how information can be modelled via the use of the wave function. In section 7, we then proceed in showing how ambiguity can be modelled via the wave function. In section 8, we show how ignorance about ambiguity can again be modelled by this same device. In section 9, we provide for a detailed description of an experimental test we propose to prove the existence of probability interference.

2 The double slit experiment and probability interference

The notion of probability interference is closely tied to one of the most fundamental experiments in quantum physics: the double slit experiment. This experiment consists of an electron gun which basically produces a beam of elec-
trons, and an electron detector. The detector counts the number of electrons landing on a given area. We imagine there are two slits of equal width. Both slits, let us call them slits $A$ and $B$, are separated by a certain distance from each other. The experiment consists of three scenarios:

1. slit $A$ is open and slit $B$ is closed
2. slit $A$ is closed and slit $B$ is open
3. slit $A$ is open and slit $B$ is open

Before any electron gun is activated, let us first imagine what would happen if the electron gun were instead to be a gun firing out very small plastic balls. We imagine those balls diameter to be very small compared to the length of each slit. In scenario 1, when the plastic ball gun is activated, we would expect that the plastic balls would pile up behind slit $A$ and very few plastic balls would pile up behind slit $B$ (some would pile up there because they would be scattered at the edges of slit $A$). The opposite pattern would occur for scenario 2. When both slits are open we would imagine there would be a density of balls piling up behind slit $A$ and a comparable density of balls piling up behind slit $B$. Some density of balls would occur in between the slits because some of the balls would be scattered at the edges of each of the slits $A$ and $B$.

The big question, loosely speaking then, becomes whether if with the electron gun one will observe the same pattern? We follow here Morrison (1990). Assume that both slits $A$ and $B$ are open. It occurs that once the electron gun is opened, initially spots form, randomly behind the slits. However, after some time the electrons start forming an interference pattern. As is remarked in Morrison (1990) at first, the electrons behave like particles. However, when time moves on they start behaving like waves. Even if the electron gun fires off one electron at a time (passing through one of the slits) there is still interference (i.e. the electron interferes with itself)! This experiment, with the electron gun, shows that electrons go through both slits. Interference forms even when one electron goes through a slit. This result is indeed resolutely remote from what we would expect if the electrons were plastic balls.

Quantum physicists devised a probability formula to describe those quite mysterious events. We still follow Morrison (1990). Busemeyer et al (2006) also provide for a very intuitive background on the discussion which is now following.

Let us denote with $p_A(x,t)$ the probability that the electron arrives at position $x$ at time $t$ when slit $A$ is open and similarly for $p_B(x,t)$. What would be the expression for the probability of finding an electron at position $x$ at time $t$ when both slits are open? I.e. what is $p_{AB}(x,t)$? Will it be:

$$p_{AB}(x,t) = p_A(x,t) + p_B(x,t)?$$ (1)

We slightly alter the notation from Morrison (1990).
This formulation is not reflecting the interference pattern found when considering electrons in the double slit experiment. We remark that $p_{AB}(x, t)$ should remind us of the probability we use to denote the probability of the union of two mutually exclusive events $A$ and $B$:

$$P(A \cup B) = P(A) + P(B).$$

It is not possible to capture interference by superposing probability distributions. Instead one could superpose probability waves, which we denote as $\psi_A(x, t)$ for when slit $A$ is open and $\psi_B(x, t)$ for when slit $B$ is open. An alternative name for probability wave, which is often used, is probability amplitude. The relationship between the probability wave and the probability density is as follows:

$$|\psi(x, t)|^2 \propto \text{probability density function.}$$

We remark that $|\psi(x, t)|^2$ is obtained in multiplying the probability wave (probability amplitude) with its complex conjugate. Hence, we can write that: $|\psi(x, t)|^2 = \psi^* \psi$. Please see below (just under (7)) where we provide for an example.

Hence, we have that:

$$p_A(x, t) \propto |\psi_A(x, t)|^2.$$  

Similarly for $p_B(x, t)$. Using (4) we can define the so called “quantum state” as the square root of the probability density function. The quantum state, as per Khrennikov (2002), “is a purely mathematical quantity used to describe a rather special behavior of probability densities of ensembles of systems that are very sensitive to perturbations produced by interactions.”

Since we are using waves, we can use superpositions. We can define:

$$\psi_{AB}(x, t) = \psi_A(x, t) + \psi_B(x, t),$$

where $\psi_{AB}(x, t)$ is the superposed state.

The above proportionality relationships can now be used to obtain:

$$p_{AB}(x, t) \propto |\psi_A(x, t) + \psi_B(x, t)|^2.$$  

Before we move on, we need to introduce a basic notion from the area of complex numbers. We know that a complex number $z$ can be denoted as $z = x + iy$, where $x$ is the real part and $y$ is the imaginary part. This same number $z$ can also be written as $z = re^{i\theta}$ where $r = \sqrt{x^2 + y^2}$ and often $r$ can be denoted as $|z|$. The angle $\theta = \tan^{-1} \frac{y}{x}$ and $x = r \cos \theta$ and $y = r \sin \theta$. We also say that $|z|$ is the amplitude and $\theta$ is the phase.

We can write the wave function, which is a complex number, in exactly the same way. Thus, the two components of a wave function are its phase and its amplitude. The wave function can then be written as follows:

$$\psi_A(x, t) = |\psi_A(x, t)| e^{iS_A(x, t)},$$

4
where $S_A(x, t)$ is the phase of the wave function and $|\psi_A(x, t)|$ is its amplitude. We write out $\psi_B(x, t)$ in the same way.

We note that the complex conjugate, $\psi_A^*(x, t)$ is in this case defined as:

$$\psi_A^*(x, t) = |\psi_A(x, t)| e^{-iS_A(x, t)}.$$  

Substituting (7), and similarly for $\psi_B(x, t)$, in (6), we obtain:

$$p_{AB}(x, t) = |\psi_A(x, t)|^2 + |\psi_B(x, t)|^2 + 2 |\psi_A(x, t)| |\psi_B(x, t)| \cos(S_A - S_B).$$  

(8)

This is precisely the result which includes probability interference. i.e., the additional term $2 |\psi_1(x, t)| |\psi_1(x, t)| \cos(S_1 - S_2)$ when not zero makes probability in a quantum context, to be either sub-or super additive. For a much more in-depth treatment on the comparison of (8) with (2), please see Khrennikov (2007).

Khrennikov (2002) has indicated that quantum theory is a “special theory of statistical averages”.

3 The sure-thing principle and the Ellsberg and Allais paradoxes

The sure-thing principle is a key principle in utility theory. Utility theory has as its primitive the so called utility function. The fundamental question any economic theory textbook asks is whether, if good $x$ is preferred over $y$, one can find an ‘if and only if’ relationship with a utility function such that the utility level of $x$ expressed through that utility function is larger than the utility level of $y$ expressed through that same utility function. The concept of expected utility makes up the ‘workhorse’ of economic theory and its many applications. Expected utility can be seen as the average value (probabilistically weighted) of utility one gets through for instance playing a gamble. This concept will become clear once we show how the Ellsberg and Allais’ paradoxes can be defined.

It needs to be stressed that economics has a variety of expected utility theories. The most used (but the least realistic) expected utility approach is the so called von Neumann-Morgenstern (1947) expected utility. The type of probability used in formulating expected utility, in this model, can be seen, as Kreps (1988) clearly indicates, as “an objective - externally imposed probability”. At the opposite end of the expected utility theory spectrum, we can find the so called Savage expected utility. This form of expected utility was developed by the famous statistician Leonard Savage (1954). Probability in the Savage model is entirely determined by the economic agent. Hence, in this model subjective probability is used in the calculation of expected utility. A mixture of objective and subjective probability is used in the Anscombe-Aumann (1963) model. Aumann was the recipient of the recent Nobel Prize in economics.

The sure-thing principle is a key axiom in the Savage expected utility approach. We can explain the principle in the following way. Let us imagine we have experiment participants who are instructed to express a preference over
gambles $A$ and $B$ (set 1) and gambles $C$ and $D$ (set 2). First, they express their choice of gamble $A$ over gamble $B$. Second, they express their choice of gamble $C$ over gamble $D$. On what basis do the experiment participants express their choice? The participants are informed of the payoff of the gambles in say, three states of nature; $s_1$, $s_2$ and $s_3$. Assume that the payoff of gambles in set 1 for state $s_3$ are identical. Similarly, the payoff of gambles in set 2 for state $s_3$ are also identical. We note that the identical payoffs in $s_3$ for set 2 maybe different from the identical payoffs in $s_3$ for set 1. The sure-thing principle says that the preference of the experiment participants over the two gambles $A$ and $B$ (set 1) and $C$ and $D$ (set 2) will be unaffected by the identical outcomes in state $s_3$. Hence, if we were to swap the identical outcomes in $s_3$ for the gambles in set 2 with the identical outcomes in $s_3$ for the gambles in set 1, this would NOT affect the preference of the experiment participants. As is mentioned in Busemeyer et al. (2006 and 2007), Shafir and Tversky (1992) have found in many instances that players do violate this principle. In this paper we do not want to detail the implications this violation has had on basic economic theory models. An excellent source which treats this nevertheless important issue is again Kreps (1988). Economic theory did find a response to this defect. The core papers are by Gilboa and Schmeidler (1989) and Ghirardato et al. (2004).

Now that we have illustrated the sure-thing principle, we can consider a famous paradox, known also as the Ellsberg paradox, which implies a violation of the sure-thing principle. Virtually any textbook in economic theory will give an outline of this paradox. We describe it in the typical way. Consider the following experiment. We have an urn with 30 red balls and 60 other balls (blue and green). We do not know the exact proportion of green and blue balls. We consider four gambles and we ask experiment participants to express a preference between gambles 1 and 2 and between gambles 3 and 4. The gamble’s payoffs are as follows.

1. Gamble 1 ($G_1$): you receive 1 unit of currency (uoc) if you draw a red ball
2. Gamble 2 ($G_2$): you receive 1 unit of currency (uoc) if you draw a blue ball
3. Gamble 3 ($G_3$): you receive 1 unit of currency (uoc) if you draw a red or green ball
4. Gamble 4 ($G_4$): you receive 1 unit of currency (uoc) if you draw a blue or green ball

Most of the experiment participants (and this result has occurred in repeated experiments) will prefer $G_1$ over $G_2$, $G_1 \succ G_2$. The intuition for such preference can be explained by the fact that one knows the odds of winning in $G_1$ (i.e. 1/3 probability) but in $G_2$ one is unsure about the odds. Participants in this experiment also indicated that $G_4 \succ G_3$. Here again, one knows the odds of winning in $G_4$ are 2/3. However, one is unsure about the odds in $G_3$. Hence, the odds are ambiguous in $G_2$ and $G_3$. 
This paradox clearly violates the sure-thing principle. This can be easily shown if we use the following table to summarize the payoffs (in units of currency):

|       | Red | Blue | Green |
|-------|-----|------|-------|
| $G_1$ | 1   | 0    | 0     |
| $G_2$ | 0   | 1    | 0     |
| $G_3$ | 1   | 0    | 1     |
| $G_4$ | 0   | 1    | 1     |

The constant payoffs in set 1 ($G_1$ and $G_2$) and in set 2 ($G_3$ and $G_4$) should not influence the preferences. Hence, if $G_1 \succ G_2$ then, using the sure-thing principle, it should be $G_3 \succ G_4$. As indicated already above, experiment participant will very often indicate $G_3 \prec G_4$.

Let us denote the probability of drawing a red ball, as $p_r$. We know $p_r = 1/3$. We can denote the other probabilities in a likewise fashion. From the choice the experiment participants expressed one can see very quickly that $G_1 \succ G_2$ implies that $p_r > p_b$, if we assume that if the units of currency one can gain would precisely coincide with the level of utility (or satisfaction) of this gain. Similarly, the choice $G_4 \succ G_3$ would now imply the opposite: $p_b > p_r$.

We round off this section with another paradox. This paradox is also widely known in the economics literature and was first initiated by Economics Nobel prize winner Maurice Allais. We follow here the example in Wolfram Mathworld (2007). There are two experiments, consisting each of gambles $A$ and $B$ for experiment 1 and gambles $C$ and $D$ for experiment 2. We have again four gambles. The payoff table, where we assume all numbers are in units of currency, can be expressed as follows:

| gambles | [1, 33] | 34 | [35, 100] |
|---------|--------|----|---------|
| $A$     | 2500   | 0  | 2400    |
| $B$     | 2400   | 2400| 2400    |
| $C$     | 2500   | 0  | 0       |
| $D$     | 2400   | 2400| 0       |

Drawing, in gamble $A$, balls with numbers in [1, 33] yields 2500uoc. If the experiment participant were to draw ball number 34 (s)he would have a payoff of 0uoc. Finally, if the participant were to draw a ball number in [35, 100] it will yield 2400uoc. Gambles $A$ and $C$, without the [35, 100] event are identical. So are gambles $B$ and $D$ without the [35, 100] event. Thus, in this case, if test participants were to prefer $B \succ A$ they should also prefer $D \succ C$. However, with the adding of the [35, 100] event, individuals have shown in repeated experiments, to exhibit the preference: $C \succ D$ and $B \succ A$. If the experiment participant were to have picked ball number ‘36’ in both gambles $A$ and $B$ then by virtue of the identical outcome (2400uoc), we would have indifference between gambles $A$ and $B$. Similarly, if the experiment participant, were to have picked ball ‘36’ in both gambles $C$ and $D$, then again there should be indifference between both gambles. Hence, the [35, 100] event is an independent event. It is intuitive to
assume that if we add an independent event to an identical experiment, the choice behavior should not be influenced. Here, the expressed choice exhibited by the test participants clearly contradicts this.

We can consider an interesting analogy between this paradox and the double slit experiment. Pietro La Mura (2006) indicates that the particle in the double slit experiment behaves like a decision maker violating the above experiment. Says La Mura ”why should it matter to an individual particle which happens to go through the left slit, when determining where to scatter later on, with what probability it could have gone through the right slit instead?” Indeed the event of the particle going through the left slit should be independent from the event of the particle to go through the right slit. But that is of course not the case when there is probability interference. Thus, if B is preferred over A then D should be preferred over C and the action of the independent event should have no influence. Thus, the probability of going through the right slit should be of no importance to a particle going through the left slit. This is exactly the influence of the independent event.

4 Explaining the sure-thing principle with probability interference: the Busemeyer et al. (2006, 2007) approach

In their important papers Busemeyer et al. (2006 and 2007) discuss how the violation of the sure-thing principle could be explained with the concept of probability interference. The experiment they refer to consists of two identical gambles. The gamble has two states of nature. One state of nature corresponds to a winning amount of money and the other state to a losing amount of money. Participants are informed they can play the game twice, in sequence. In this experiment three situations are distinguished: a) the participant is told (s)he won in the first play of the gamble; b) the participant is told (s)he lost in the first play of the gamble; c) the participant is told nothing as to the outcome of the first play of the gamble. As indicated in Busemeyer et al. (2006 and 2007), if participants prefer to play the second time the gamble, knowing that they won at the first play, and if they prefer to play the second time the gamble knowing they lost at the first play, then they should prefer to play a second time even if they are not told whether they either lost or won at the first play. This very intuitive prediction is entirely based on the sure-thing principle. The study by Shafir and Tversky (1992), as quoted in the Busemeyer et al. (2006 and 2007) paper, showed that 69% of experiment participants would go for a second play knowing that they won in the first play. Furthermore, 59% opted for a second play when they knew they had lost in the first play. However, 36% of the participants would NOT play for a second time if they did not know the outcome of the first play of the gamble. This behavior violates the sure-thing principle.

Busemeyer et al. (2006 and 2007) define two states of belief about the
first play that experiment participants can have: win or lose. There are two states of action experiment participants can take: gamble or not. An experiment participant can simultaneously have beliefs and actions. Those produce four possible states, which are denoted in the typical quantum physical ‘ket’ way: \( |WG>, |WN>, |LG>, |LN> \). Each of the four states, indicates the simultaneous belief experiment participants have in for instance ‘winning (in the first play) of the gamble and having an intention to gamble (in a second play)’ \((|WG>)\) or in for instance ‘losing (in the first play) but having no intention to gamble (in a second play)’ \((|LN>)\). We mentioned in section two that in quantum physics we define a probability wave or probability amplitude. The state function vector, in the experimental setting here, is defined as: 
\[
\psi = [\psi_{WG}, \psi_{WN}, \psi_{LG}, \psi_{LN}],
\]
where each \( \psi\) is a probability amplitude. The \( \psi\) is of unit length. We recall also from section two, that we defined \( |\psi|^2 \) (see (3)) as proportional to a probability density function. We denote the probability value as: \( ||\psi||^2 \). As an example \( ||\psi_{WG}||^2 \) indicates the probability value of observing the state “winning and gambling in a second play”.

As is indicated in Busemeyer et al. (2006 and 2007), one can represent ‘thought’, which alters the state of the cognitive system, by a unitary operator, \( U \), such that \( \varphi = U \psi \). As an example (see Busemeyer et al. (2006 and 2007) for more examples), if the experiment participant receives information she has lost in the first gamble, then the operator \( U \) is applied so as to create a new state price vector: 
\[
\psi_L = [0, 0, \alpha_L, \beta_L].
\]
Note that in the unknown case we have the superposition: 
\[
\psi_t = \sqrt{p}\psi_w + \sqrt{q}\psi_L,
\]
where \( \sqrt{p} \) and \( \sqrt{q} \) are probability amplitudes.

Busemeyer et al. (2006 and 2007) now introduce the way players select a strategy. As indicated above, a new state is formed through an operator \( U_t \) for some period of time \( t \), such that: 
\[
\varphi = U_t \psi = [\varphi_{WG}, \varphi_{WN}, \varphi_{LG}, \varphi_{LN}].
\]
The so called final response probabilities are elements of \( \phi = M \varphi \), where \( M \) is another operator. The elements of this vector are the probabilities of playing a second time.

Hence, we can, as in Busemeyer et al. (2006 and 2007), define the density function from which we can determine the total probability of gambling in the second play. Using the concept of complex conjugate which we discussed in section two (just under equation (7)), we obtain: \( |\phi|^2 = \phi^*\phi \), where \( \phi^* \) is the complex conjugate. The total probability of gambling the second time is: \( |\phi|^2 = (M\varphi)^* (M\varphi) = |M\varphi_{WG}|^2 + |M\varphi_{LG}|^2 \). Similarly, the total probability of not gambling the second time is: \( |M\varphi_{WN}|^2 + |M\varphi_{LN}|^2 \).

How can the probability interference term explain the sure-thing principle violation? Using the development set out in this section, the probability of winning is: \( \phi^*_W \phi_W = (MU_t\psi_W)^* (MU_t\psi_W) \) and similarly for losing: \( \phi^*_L \phi_L = (MU_t\psi_L)^* (MU_t\psi_L) \). Similarly Busemeyer et al. (2006 and 2007) show they can formulate the probability in the case of the unknown condition, using the superposition \( \psi_t = \sqrt{p}\psi_w + \sqrt{q}\psi_L \). They get: \( \phi^*_U \phi_U = (MU_t\psi_U)^* (MU_t\psi_U) \)
\[
(MU_t(\sqrt{p}\psi_w + \sqrt{q}\psi_L))^* (MU_t(\sqrt{p}\psi_w + \sqrt{q}\psi_L))
\]

(9)
and this product can be set equal to:

\[(\sqrt{p} \phi_w + \sqrt{q} \phi_L) \times (\sqrt{p} \phi_w + \sqrt{q} \phi_L)\]  

(10)

and this is then equal to:

\[(p \phi^*_w \phi_w + q \phi^*_L \phi_L) + \sqrt{p} \sqrt{q} \phi^*_w \phi_L.\]

(11)

What is key is now to remark that if one were to only consider:

\[
\phi^*_U \phi_U = (p \phi^*_w \phi_w + q \phi^*_L \phi_L)
\]

(12)

then (12) would indicate that the probability of gambling for a second time, when the experiment participant has no idea of whether he lost or won in the first play, should be the average of the probabilities of gambling a second time when the experiment participant knew that he respectively won and lost in the first play. The experiments by Tversky and Shafir (1992) have clearly shown that 36% of the participant would not play for a second time when they did not know whether they had lost or won in the first play. This falls well below the 69% of participants playing, knowing they had won in the first play and the 59% of the participants knowing they had lost in the first play. Hence, it is clearly the interference term \(\sqrt{p} \sqrt{q} \phi^*_w \phi_L\), which was already covered in section 2 under (8), which can explain the experimental observation of Tversky and Shafir (1992).

5 The concept of ‘ambiguity’ in the Ellsberg paradox

The Ellsberg paradox, as we have seen in section three of this paper violates the sure-thing principle and it thereby contradicts expected utility theory. Let us consider, like in Bossaerts et al. (2007), that the red, green and blue balls are financial/economic securities. Examples of securities are bonds, shares etc...Each security pays the same fixed amount (1 unit of currency) according to the draw of the color. Hence, a ‘red’ security will be a security which pays 1 unit of currency if a red ball is drawn from the urn of balls. The ‘red’ security is risky in the sense that the distribution of payoffs is known. However, the number of ‘green’ and ‘blue’ securities are unknown in number. We say, as in Bossaerts et al. (2007), that the ‘blue’ and ‘green’ securities are ambiguous. In other words, the distribution of their payoffs, in the Ellsberg paradox is unknown. We also reported the fact, in section three, that the probability of drawing red is larger than the probability of drawing blue, \(p_r > p_b\), when experiment participants expressed the preference: \(G1 > G2\). However, when experiment participants expressed the preference of \(G4 > G3\), the opposite occurred: \(p_r < p_b\). If the price of the ‘red’ security, \(q_r\), were to be tied to the knowledge of probability (and of course prize winning or not), then clearly the price of the ‘red’ security
should exceed the price of the ‘blue’ security (in gambles 1 and 2): $q_r > q_b$.

However, in gambles 3 and 4, the opposite effect occurs: $q_r < q_b$.

Let us now assume, still as in Bossaerts et al. (2007), that gambles 1 and 2 would refer to one market. Let us denote this market, as market $X$. Moreover, let us assume that gambles 3 and 4 refer to another market, say market $Y$. It is quite straightforward to observe that one can simultaneously buy a red security in market $Y$ and sell an identically same red security in market $X$. Similarly, we can simultaneously buy the blue security in the $X$ market and sell an identically same blue security in the $Y$ market. Such simultaneous buying and selling, without incurring risk, in two different markets is also known as ‘arbitrage’.

Hence, the Ellsberg paradox is equivalent to the existence of an arbitrage opportunity. As we have attempted to illustrate above, this paradox can be explained via the probability interference argument. Hence, it may seem reasonable to explain arbitrage via the probability interference argument.

Before entering into the next section, we need to stress that the concept of ‘arbitrage’ is an essential concept in the theory of security (or also asset) pricing. For instance, the theory of derivative pricing (Black and Scholes (1973)) is heavily dependent on the arbitrage concept. Most of the theoretical work in asset pricing relies heavily on the so called ‘non-arbitrage’ theorem (Harrison and Kreps (1979)). This theorem, indicates that for the case of a discrete state space, there will be no-arbitrage if and only if there exists a set of risk-neutral probabilities. Risk-neutral probabilities can be thought of as probabilities which allow the discounting of a risky asset with the help of a risk free rate of interest. This property is of capital importance in asset pricing. See Etheridge (2002) for an excellent treatment. Kabanov and Stricker (2005) deal with the theorem in the context of a continuous state space. Duffie (1996) discusses the proof of the theorem. We propose that those risk-neutral probabilities could be mapped onto the probabilities defined from $\|\psi\|^2$, where the latter quantity was already discussed in section four above. In fact, we can be even more explicit, by writing:

$$\|\psi(x)\|^2 = \int_{a_1}^{a_2} |\psi(x)|^2 \, dx,$$  \hspace{1cm} \text{(13)}

where we have assumed time independence. For a set of bounds on the integral, $\{a_1, a_2\}$ we have a precise probability value on the left hand side of the above equality. The probability density function is $|\psi(x)|^2$. We now imagine that the set of risk-neutral probabilities needed to ensure no arbitrage can be drawn, for a given $|\psi(x)|^2$, from the different sets of bounds $\{a_i, a_j\}$ used in (13).
6 Modelling information via the use of the wave function

In this section we briefly exhibit how the wave function we defined in section two, equation (7), can be used to model information. We recall that the wave function was defined as: \( \psi(x, t) = |\psi(x, t)| e^{iS(x, t)} \), where \( |\psi(x, t)| \) is the amplitude of the wave function and \( S(x, t) \) is the phase of the wave function. We use \( x \) and \( t \) to denote respectively position and time.

The rationale for using such a wave function in the context of information, is to convince the reader that a basic quantum mechanical tool like a wave function can be used to model ‘ambiguity’ (which we find back in the Ellsberg paradox (please see section five above)) and ‘ignorance about ambiguity’. We will discuss the relationship of the wave function with ambiguity and ‘ignorance about ambiguity’ in respectively sections seven and eight below.

The use of a wave function as an information function was already proposed in 1993 through the important work of Bohm and Hiley (1993). The authors considered the information wave function as a so called ‘pilot wave’ (wherefrom the name of pilot wave theory) steering a particle. The pilot wave function interpretation has deep roots in quantum physics. Prince Louis de Broglie came up, as early as 1929, with his interpretation of the wave function by saying that such function (p. 16 in Holland (1993)): "...not only does...determine the likely location of a particle it also influences the location by exerting a force on the orbit.” David Bohm in two seminal papers (1952) took up the thread where de Broglie had left off. We do not want to go into any detail on this theory but it is sufficient to say that Bohm and Hiley (1993) compare the pilot wave to a radio wave which steers a ship on automatic pilot. Pioneering work in using this information wave function in a macro-scopic (economics) setting was performed by Andrei Khrennikov (1999; 2002; 2004) and also by Olga Choustova (2006; 2007). Haven (2005; 2007) has attempted to use this set up in a financial options (and more general pricing) context.

We can use the information wave idea in the basic notion of probability value we defined in equation (13) above. A change in information could be reflected by a change in the functional form of the (information) wave function, \( \psi(x, t) \). This change would have as consequence that the probability value would change because of a change in information. Hence, we can obtain a set of probabilities which trigger arbitrage by simply changing the information wave function and keeping the same bounds \( \{a_i, a_j\} \). We come back to this issue in the next section.

7 The wave function and ambiguity

Let us re-visit once more the Ellsberg paradox of sections three and five. We recall the payoffs:

1. Gamble 1 (G1): you receive 1 unit of currency (uoc) if you draw a red ball
2. Gamble 2 (G2): you receive 1 unit of currency (uoc) if you draw a blue ball

3. Gamble 3 (G3): you receive 1 unit of currency (uoc) if you draw a red or green ball

4. Gamble 4 (G4): you receive 1 unit of currency (uoc) if you draw a blue or green ball

We had 30 red balls and 60 blue and green balls. We do not know the proportions of blue or green balls.

An experiment participant may reason that it is ‘fair’ to assume there are an equal amount of green and blue balls. We note that in using this ‘fair’ argument ambiguity has been completely ruled out\(^3\). Hence, the chance of drawing any of the balls, in any of the four gambles is 30/90. In this case we would be indifferent between gambles 1 and 2, and between gambles 3 and 4.

Therefore, if the price of the red security, \(q_r\), is again tied to the knowledge of probability then the price of the red security should now be equal to the price of the blue security (in gambles 1 and 2): \(q_r = q_b\). Furthermore, the price of the red security is equal to the price of the blue security in gambles 3 and 4: \(q_r = q_b\). Thus, with the 50/50 rule there is no possibility for arbitrage: i.e. we can not simultaneously buy and sell in two different markets and make a riskless profit. In fact, it is precisely the existence of ambiguity which has created the arbitrage opportunity.

Let us re-consider the role the wave function can play as an information function in (13). We start from the assumption there is no arbitrage. In this case the information wave function, \(\psi(x,t)\), would contain a precise piece of information. I.e. the precise piece of information would be: “we have a 50/50 chance of getting a blue or green ball (i.e. security)” or alternatively “the price of the red security is equal to the price of the blue security in all gambles”. Any departure from this, i.e. a change in the information wave function, \(\psi(x,t)\), which brings us back into ambiguity, will yield arbitrage. Hence, a departure of the information wave function from the 50/50 rule (or any other idiosyncratic - subjective probability assessment), steers us away from expected utility and brings us into ambiguity and therefore arbitrage.

**8 The wave function and ignorance about ambiguity**

We can go one step further by attempting to model ignorance about ambiguity. To achieve this purpose we need to first inquire how ignorance could be mod-

\(^3\)Using the 50/50 rule would indicate one is using the principle of insufficient reason (Kreps, p. 146) - which says: "if (one) has no reason to suspect that one outcome is more likely than another, then by reasons of symmetry the outcomes are equally likely and hence equally likely probabilities can be ascribed to them." This argument can be challenged. For instance, one could look at the uncertainty concerning the true value \(p\) of picking up a green or blue ball as a prior (Kreps, p. 149).
elled with basic quantum mechanical tools. In a very interesting paper, Franco (2007) provides for convincing arguments on how to model so called ‘rational ignorance’. Rational ignorance comes into existence when the cost of having to inform oneself about an issue outweighs the benefits of knowing about that issue. Franco (2007) defines an opinion state, which is a qubit. In a quantum state, there exists a linear superposition of 0 and 1:

$$a|0> + b|1>,$$

where $|0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $a, b \in \mathbb{C}$. The superposition reflects the existence of ignorance.

A so called ‘ket’, $|s>$, is a row vector and the dual vector is a so called ‘bra’, $<s'|$. The inner product is denoted as $<s'|s>$: the probability amplitude. Furthermore, as expected, $|<s'|s>|^2$ is the probability to get state $s'$ as a result of the measure on state $s$.

Franco (2007) uses the following well known quantum mechanics terms:

- a Hermitian operator, which corresponds to an observable quantity
- eigenvalues are the measurable values of the observable
- eigenvectors $|a>$, correspond to the quantum state (example: $|0>$ and $|1>$)

As in Franco (2007), three matrices which are Hermitian operators, are introduced:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Those matrices, also known under the name of ‘Pauli matrices’ have as property that $\sigma_i^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Also, the determinants of each $\sigma_i = -1$ and the trace (i.e. the sum of the diagonal elements) is in each $\sigma_i = 0$.

The eigenvalues of those operators define three orthonormal basises. Each eigenvector of one basis is an equal superposition of the eigenvector of any of the other basises. Those basises, Franco (2007) reports, are called mutually unbiased. The eigenvectors are then respectively $\{ |0>, |1> \}; \{ |0> + i|1>, \sqrt{2} \}; \{ |0> - i|1>, \sqrt{2} \}$, where $i$ is a complex number. The Pauli matrices have eigenvalues of $\pm 1/2$. The physical realization of a qubit is provided for by a spin 1/2 particle (f.i. an electron).

4Hermiticity is the key property all quantum mechanical operators must possess so that they can represent an observable (or physically measurable quantity). Consider an operator $Q$, then this operator is Hermitian if for any state function, $\psi$, we have that: $\int_{all\ space} \psi^* Q \psi \, dv = \int_{all\ space} Q \psi^* \psi \, dv$. Please see Morrison (1990) for an excellent overview.
The observable quantities can be associated to questions such that they give the answers 0 (false) or 1 (true) (the eigenvalues). The vectors $|0\rangle$ and $|1\rangle$ are associated to the answers 0 and 1 (true or false). The vectors $|0\rangle$ and $|1\rangle$ are the truth values.

In order to force the Pauli matrices to have eigenvalues $\pm 1$, Franco (2007) considers the projector operator: $P_z = (I - \sigma_z)/2$ which yields eigenvalues of 0 and 1. The associated eigenvectors are thus $|0\rangle$ and $|1\rangle$. The observable $P_z$ is, in the context of rational ignorance, associated to a first question asked to participants.

Franco (2007) then defines two other observables, $P_y = (I - \sigma_y)/2$ and $P_x = (I - \sigma_x)/2$ which can be associated with a second question. By virtue of the construction of $P_z$, $P_y$, $P_x$(the mutually unbiasedness), the answers to those observables (questions) are statistically independent.

As an example, Franco (2007), starts with question $P_z$ which has either 0 or 1 as an answer and the associated eigenvectors are $|0\rangle$ and $|1\rangle$. Now consider another question, $P_x$, which has as associated eigenvectors $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ or $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ for the 0 and 1 eigenvalues. Here the eigenvectors are clearly superposed. They express thus ignorance. The inner product calculates the probability amplitude (and the square is the probability). In this case we obtain thus a probability of $1/2$ for obtaining either 0 or 1.

Now assume, again as in Franco (2007) that we have experiment participants who are first being asked question $P_z$. This operator is having as eigenvectors $|0\rangle$ and $|1\rangle$. There is no superposition. Hence, no ignorance. Now assume, there is another question posed to the experiment participants: question 2, $P_x$. The eigenvectors are now $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ or $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$. The probability amplitude in this case, can for instance be calculated by using the inner product of $|0\rangle$ (eigenvector of question 1) and $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ (eigenvector of question 2), yielding $1/\sqrt{2}$. This yields a probability of $1/2$.

As is very clearly indicated in Franco (2007) when the experiment participant gives an answer to the second question, the eigenvector $|0\rangle$ will collapse to say $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ for the answer ‘0’. The experiment participant has expressed ignorance since there is superposition. If question 1 is asked now again, then what is the probability amplitude? The eigenvector $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ will now collapse to say $|1\rangle$ for the answer ‘1’ and the inner product will be formed out of $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $|1\rangle$. As is aptly remarked in Franco (2007), one observes now that the same question may get a different answer, since there is a 50% probability to get answer ‘1’. Hence, the context of asking has changed the answer to the question.

The development above can be applied to arbitrage. In their quest to make arbitrage profits, banks employ entire research departments to uncover such
opportunities. Proprietary data sets are developed to uncover arbitrage opportunities. Let us denote the cost of such datasets as the price of information. Using an analogy with the rational ignorance concept, how do we know that information prices can outweigh the benefit of an arbitrage opportunity? This is not an obvious question.

To revert back to the notion of ambiguity and arbitrage we discussed when we considered the Ellsberg paradox, we could claim that the very existence of an ambiguity in the financial markets leads to an arbitrage opportunity. In the context of the Ellsberg paradox, rational ignorance would consist of not believing that a potential set of arbitrage opportunities can be economically viable for it to be investigated.

However, here we would like to tackle ‘ignorance about ambiguity’. Let us think of some financial asset whose price may have the potential for arbitrage taking. We have two questions, asked in sequence, to the research department of a bank. The questions are answered with either false (‘0’) or true (‘1’). The probability of this is also assessed. For instance, in the first question we ask whether, the price of the red security is higher than the price of the blue security, \( q_r > q_b \) in market \( X \). The potential answer in a superposition of 0 and 1 will influence the probability as we have seen above. In question 2, we ask whether \( q_r < q_b \) in market \( Y \). If on both questions, the research department of the bank, answers without any superposition, then this would indicate there is no ignorance at all about ambiguity. If the answers are correct (i.e. they recognize the correct inequality for each market \( X \) and \( Y \)) then there will exist an arbitrage opportunity.

What if the research department of the bank answers the questions with superposition? Clearly, there would be ignorance about ambiguity and hence the bank would not realize an arbitrage opportunity.

We could translate this into the following equations. Using an extension on (13) (see also Choustova (2006)):

\[
\|\Psi\|^2 = \int_{q_1}^{q_2} |\Psi(q)|^2 dM(q),
\]  

(16)

where \( M(q) \) is a probability measure and where \( \{q_1, q_2\} \) are information prices. We imagine that the research department of a bank is researching an arbitrage opportunity. Hence, the set of information prices now changes from \( \{q_1, q_2\} \rightarrow \{q'_1, q'_2\} \). Assume we ask the two questions, we described above, to the research department. We claim that the ignorance about ambiguity can be captured by \( M(q) \). So if for question 1, we obtain non-superposed answers, then:

\[
1 = \int_{q_1}^{q_2} |\Psi(q)|^2 dM(q)
\]  

(17)

For question 2, assume we have no ignorance about ambiguity then:
\[ 1 = \int_{q_1}^{q_2} |\Psi(q)|^2 dM'(q) \quad (18) \]

, where \( M(q) \neq M'(q) \). As we have indicated at the end of section five and at the end of section six, the arbitrage probability is influenced by the information wave function. The existence of ambiguity, in the context of the Ellsberg paradox, vouches for an arbitrage opportunity. So the arbitrage probability is then:

\[ \frac{1}{x} = \int_{q_1}^{q_2} |\Psi'(q)|^2 dM'(q), \quad x \geq 1 \quad (19) \]

, where \( |\Psi'(q)|^2 \neq |\Psi(q)|^2 \).

Thus, if we have that \( \int_{q_1}^{q_2} |\Psi(q)|^2 dM(q) \neq \int_{q_1}^{q_2} |\Psi(q)|^2 dM'(q) \) then there is ignorance about ambiguity and we can assume there is no arbitrage opportunity.

Thus, in summary, ignorance about ambiguity is reflected by a probability which is dependent on the superposition of the true and false statements, expressing ignorance about ambiguity.

9 Experimental set up: how to test for probability interference in a macro-scopic setting?

The work by Busemeyer et al. (2006 and 2007) indicates that probability interference can explain the violations of the sure-thing principle. The Ellsberg paradox was a showcase paradox indicating a violation of the sure-thing principle. We have tried to show how the information wave function can be modelled in the context of the Ellsberg paradox. Following the work of Franco (2007) basic quantum physical principles were used to model rational ignorance. We used this set up to define ignorance about ambiguity.

All of the presented material so far has tried to implicitly highlight the importance of using quantum physical principles in instances of violations of the sure-things principle. The most important question, hence becomes whether we can prove, or at least test for the existence of probability interference in a macro-scopic setting?

9.1 Using probability interference in social science: general set-up

We follow in this subsection the work by Khrennikov (2002). Assume there exists two mutually exclusive features \( A \) and \( B \). Each feature has a dual outcome, ‘0’ or ‘1’. The outcome ‘0’ in features \( A \) and \( B \) is denoted as respectively \( a_1 \) and \( b_1 \). Similarly, the outcome ‘1’ in features \( A \) and \( B \) can be denoted as respectively \( a_2 \) and \( b_2 \). There exists an ensemble of experiment participants, \( \Sigma \) who have the same mental state. The ensemble probability is denoted as \( p^\Sigma_j \) and is defined as:
\( p_j^a = \frac{\text{number of results } a_j}{\text{total number of elements in } \Sigma}. \tag{20} \)

Similarly for \( p_j^b \). A new ensemble \( \Sigma \) needs to be prepared to perform the measurement \( p_j^b \).

Ensembles, \( \Sigma_i^a \) and \( \Sigma_i^b \), \( i = 1, 2 \) need then to be prepared and they have states corresponding to the values of \( A = a_j \) and \( B = b_j \); \( j = 1, 2 \). The following probability is then defined:

\[ p_{ij}^{a|b} = \frac{\text{number of results } a_j \text{ for the ensemble } \Sigma_i^b}{\text{total number of elements in } \Sigma_i^b}. \tag{21} \]

Likewise for \( p_{ij}^{b|a} \). From classical probability theory, total probability is defined as:

\[ p_j^a = p_i^a p_{ij}^{a|b} + p_i^b p_{ij}^{a|b}; j = 1, 2. \tag{22} \]

A likewise definition can be made for \( p_j^b \).

In the presence of probability interference, one obtains (see also section 2 in this paper):

\[ p_j^a = p_i^a p_{ij}^{a|b} + p_i^b p_{ij}^{a|b} + 2\sqrt{p_i^a p_i^b p_{ij}^{a|b} p_{ij}^{a|b}} \cos \theta_j; j = 1, 2; \tag{23} \]

where \( \cos \theta_j \) is defined as:

\[ \cos \theta_j = \frac{p_j^a - p_i^a p_{ij}^{a|b} + p_i^b p_{ij}^{a|b}}{2\sqrt{p_i^a p_i^b p_{ij}^{a|b} p_{ij}^{a|b}}}. \tag{24} \]

As we have already remarked in section 2, with equation (8), if the \( \cos \theta_j \) in (23) is non zero, then this would be indicative of the existence of quantum-like behavior of cognitive systems. A likewise definition can be made for \( p_j^b \).

### 9.2 Description and discussion of the proposed experiment

In this subsection of the paper we will propose the conjecture we want to test. We also briefly describe the test we envisage. We note that the test description which we will explain below does not contain the minute details of the experimental set up. We only provide for a fairly general discussion of how we could possibly test our conjecture.

In basic terms the experiment deals with having experiment participants recognize a list of songs from a pre-determined list of songs. The tempo of each of the songs is distorted by either lenghtening or shortening the tempo. We use the definition of tempo as in Levitin and Cook (1996): “...the amount of time it takes a given note or the average number of beats occurring in a given interval...
of time, usually beats per minute.” Kuhn (1974) defines, beat tempo, in a very similar way as: “...the rate of speed of a composition.”

The time of listening exposure the test participants are allowed so as to recognize each of the songs varies. The time of exposure is lengthened when the tempo is increased. However, the time of exposure is shortened when the tempo of songs is decreased. The rationale for this time variation is discussed below.

The experiment participants are divided into several groups. We first want to discuss the ‘normed’ group. Levitin and Cook (1996) in an experiment on tempo recognition asked experiment participants to sing tunes of songs they themselves had picked from a list of available songs. They examined how well the participants could mimic the tempo of each of the chosen songs. In their experimental set up they formed a group of experiment participants (250 students) who would fill in questionnaires which would allow, what Levitin and Cook (1996) called ‘norming’. This questionnaire, in the words of Levitin and Cook (1996), “asked them (the test participants) to indicate songs that they knew well and could hear playing in their heads.” Experiment participants in the ‘normed’ group, in our study, will do exactly that: indicate songs they know well. It will be important to choose songs which we can term as being ‘best known’. However, we may want to make sure we use songs from different music categories. In a study by Collier and Collier (1994) the authors examine tempo differences in different types of music (like jazz and other types). The authors find that the tempo differences are quite tied to the music genre in question. Furthermore, Attneave and Olson (1971) indicate that the size of a music interval is dependent on the level of frequency of the music.

In the Levitin and Cook (1996) study, the 250 students we mentioned above, were all taken from the Psychology 101 course. Those students were given the questionnaire and from their answers the experimenters chose 58 CD’s. It is important to stress that our experiment proposal does not really follow the Levitin and Cook (1996) study closely, since we ask instead from participants to recognize songs we randomly play from the list we selected from the choice of songs the ‘normed’ group puts forward. In our experimental set up, the experiment participants have to recognize songs we play for them when tempo and time of exposure is varied. In the Levitin and Cook (1996) study participants are asked to sing or hum a song for a time the participant selects himself/herself and the obtained tempo is then compared with the copyrighted song’s tempo.

Besides the normed group, we have two other groups of experiment participants. We note that none of the members of the ‘normed’ group will participate in the experiment itself. We have therefore two other groups. Group 1 whose participants are subjected to song excerpts where tempo decreases are randomly injected in each song. The time of listening exposure for each song is short. Participants in group 2 will be subjected to song excerpts where tempo increases are randomly injected in each song. The time of the listening exposure for each song is longer than in each of the songs used in group 1.

The rationale for the variation of the exposure time comes from a study
by Kuhn (1974) where it was found that test participants were better able to detect tempo decreases rather than tempo increases. Kuhn (1974) indicates that a study by Farnsworth (1969) showed that the “listener is most likely to change affective terms with which he describes a piece of music whenever its tempo is appreciably slowed or hastened.” Kuhn (1974) in his experiment set up, where he solely used professional musicians, showed that the beat tempos which had been decreased were identified faster (in a statistically significant way) than beat tempos which had been increased. Kuhn (1974) indicates those results also had been obtained, independently, in a study by Drake (1968). In more general terms the work by Drake and Botte (1993) addressed a fairly similar problem. They had subjects listen to two identical sequences (except for their tempo difference). One sequence had a higher tempo than the other. The subjects were asked to indicate which sequence was faster.

We note that the tempo experiments by Kuhn (1974) did not make use of music. However, the Kuhn (1988) study does provide for this. In this study mention is made of the work by Geringer and Madsen (1984) which actually finds, in the context of orchestral music, that tempo increases were more easily identified than tempo decreases.

The Kuhn (1988) study provides for a rich context in which one can appreciate how tempo changes can be affected by extraneous variables such as melody activity and audible steady beats. The study finds that when melody activity and audible steady beat were kept constant, the experiment participants (the participants were students at an elementary school with a very strong music programme) could distinguish well between slow and fast tempi. However, experiment results in this study showed that melody activity clearly affected tempo perception but there was much less certainty as to how tempo recognition is affected when audible beat was considered. It needs to be stressed that, according to Kuhn (1988) there exists a sizable literature which documents there is ambiguity in tempo perception. Kuhn (1988) cites work by (amongst others) Madsen, Duke and Geringer (1986) and Wang and Salzberg (1984). One way out, as suggested in Kuhn (1988), is then maybe to only use musicians as experiment participants in groups 1 and 2 (besides the normed group participants). Says Kuhn (1988) “some researchers believe that only trained, sophisticated musicians can correctly interpret and perceive ‘tempo’ as it is usually defined.”

Khrennikov and Haven (2006) have described an experiment where we can test for the same type of variables (time and degree of deformation) by using instead a database of pictures. This experiment formed part of a funding bid the authors wrote for the Fund for Scientific Research (FWO) of the Flemish Government (Belgium).

Two features of the song recognition are being compared:

1. time of processing of the songs (see also section 9.4. for more discussion

---

5 Also reported in the Levitin and Cook (1996) paper
6 Kuhn’s (1974) study also discusses rhythm besides beat. We omit it here.
7 FWO is the abbreviation in Dutch for “Fonds voor Wetenschappelijk Onderzoek”.
8 We could also imagine an experiment where we test for deformations of both verbal and photograpic memory.
on this parameter)

2. the ability to recognize a song, $S_0$ by analyzing a deformation $S$ of it (see also section 9.4.)

We denote the time of processing of the song with the variable $t$. The ability to recognize the song with tempo deformations which are either increasing or decreasing, is denoted with the variable $a$.

The conjecture we want to test is:

**Conjecture 9.1** $t$ and $a$ are complementary.

### 9.2.1 Preparation

The state preparation of the experiment can be described as follows. The experimental context (state) $C$ is given by a sequence of songs, $S_1,...,S_m$. The experiment participants form a group $G$ and each of the participants are exposed to all the songs (from the distilled list of songs taken from the normed group) excerpts they will hear. The context allows thus the experiment participants to learn the songs (some of the songs they surely will know, but some they may not know as well (or not at all)).

In the Levitin and Cook (1996) study only 46 students were selected for the tempo recognition experiment. After learning has taken place, the group $G$ is randomly divided into two equal subgroups $G_1$ and $G_2$.

### 9.2.2 First experiment: song tempo decreases and short exposure time

A first experiment is performed with experiment participants from group $G_1$. As in Levitin and Cook (1996), test participants are seated in a so called sound attenuation booth. Each of the participants is then subjected to a battery of songs (selected from our ‘best’ songs (from the normed group)) played sequentially. Each of the songs is played for the same amount of time. In this first experiment, the tempo of each of the songs is decreased and the exposure time (to listen to the song) is kept uniformly short. Experiment participants must choose from the full list of songs which song they are listening to. Songs in the full list are indicated (alphabetically) by the name of the singer or composer. Next to the authors’ name is the title of the song and the record company owning the copyright of the song. We also note that experiment participants need to pick the song from the list within a uniformly prescribed time interval. This is a possible issue of contention. Kuhn (1974) reports that professional musicians need less time to respond to a tempo change and they make also less mistakes than non-professional musicians. A study related to this issue is Kuhn’s 1975 paper which reports that on a longitudinal basis, test participants (of all walks of life) when asked to keep a constant tempo, seem to instead increase tempo.

---

9. The ‘shortness’ of the exposure time is obviously a calibration issue. Please see below for some beginning discussion on this topic.
Since our study would not necessarily contain test participants who are only musicians, setting the response time too short could thus create a negative bias.

The low tempo songs $S'_1, ..., S'_m$ are deformations of the original songs $S_1, ..., S_m$. The tempo change could be fine tuned in the way Madsen and Graham (1970) have proposed. Those authors propose a modulation rate (of tempo) of one beat per minute change every second.

Experiment participants in group $G_1$ are subjected to listening to $S'_1, ..., S'_m$ and also to listening to a few other songs which have not been part of the training sample. We denote this set of songs as: $S_{G_1}$. The width of the time window, $w$ is a parameter of the experiment.

The task the experiment participants have to fulfill is to indicate whether they either recognize or not the songs $S_{G_1}$ as modifications of the songs $S_1, ..., S_m$.

We can make this experiment a little more sophisticated by using the procedure Levitt (1971) used. See also Drake and Botte (1993) where the Levitt approach is described. Levitt (1971) would decrease the tempo difference between two subsequent sequences by 1%, if the subjects gave two correct answers (to the two prior sequences). In some sense the tempo decrease compensates for the shorter period which is given. We could also make it increasingly harder for test participants by instead increasing the tempo for every correct answer. We do not follow this strategy in this paper.

Let $\omega$ be an experiment participant from group $G_1$ performing this task. We set $t(\omega) = 1$ if $\omega$ was able to give the correct answers for $x\%$ of the songs in $S_{G_1}$. And $t(\omega) = 0$ in the opposite case.

We now find probabilities $P(t = 1|C)$ and $P(t = 0|C)$ through counting numbers of experiment participants who gave answers $t = 1$ and $t = 0$, respectively. We denote respective subgroups of experiment participants by $G_1(t = 1)$ and $G_1(t = 0)$, respectively. The first subgroup consists of experiment participants who have the ability to perform song recognition (with tempo decrease) ‘quickly’ and the second subgroup consists of experiment participants who do not possess that feature.

9.2.3 Second experiment: song tempo increase and long exposure time

The second experiment is performed with experiment participants from the group $G_2$ as well as the subgroups $G_1(t = 0)$ and $G_1(t = 1)$ of the first subgroup $G_1$.

The song deformations now are based on song tempo increases which as per the experiment of Kuhn (1974), are deformations which are harder to recognize. We denote those deformations as $S''_1, ..., S''_m$ of initial songs $S_1, ..., S_m$. As in the first experiment, ‘unknown’ songs will be added to the group of songs. We denote the set of essentially deformed songs (with the ‘unknown’ songs): $S_{G_2}$.

The task the experiment participants have to fulfill is identical to the task described in experiment one: can the participants recognize the songs in $S_{G_2}$ as modifications of the initial songs $S_1, ..., S_m$?
The time window in experiment 2 is now longer than in experiment 1. The width of the time window, \( w \), is a parameter of the experiment.

Let \( \omega \) be an experiment participant performing this task. We set \( a(\omega) = 1 \) if \( \omega \) was able to give the correct answers for \( x\% \) of images in the series. And \( a(\omega) = 0 \) in the opposite case.

We now find probabilities \( P(a = 1|C) \) and \( P(a = 0|C) \) through counting numbers of experiment participants in the group \( G_2 \) who gave answers \( a = 1 \) and \( a = 0 \), respectively. We denote respective subgroups of experiment participants by \( G_2(a = 1) \) and \( G_2(a = 0) \), respectively. The first subgroup consists of experiment participants who have the ability to perform song recognition (with tempo increase) quickly and the second subgroup consists of experiment participants who do not possess that feature. Because of the tempo increase, we note that the emphasize in experiment two is on the carefulness of recognizing the deformation of a song.

Here again, we could use the Levitt (1971) approach. We can increase (or decrease) with 1\% the tempo between subsequent songs if the test participant has given two correct answers.

### 9.3 Sub (super) additivity

Using the above experiment, we find probabilities \( P(a = \beta|t = \alpha) \); \( \alpha, \beta = 0, 1 \), by counting the number of people in the group \( G_1(t = \alpha) \) who gave the answer \( a = \beta \).

After this we calculate the coefficient \( \lambda \) (which is the third term in equation (23)) and we find the angle \( \theta \) which gives us the measure of complementarity of variables \( t \) and \( a \). If \( \lambda > 1 \), we would find experimental evidence of probabilistic behavior which is neither quantum nor classical. We could also consider \( \lambda(w) \) and even make \( \lambda \) dependent on both \( w \) and the degree of deformation. This degree of deformation could be expressed by using the tempo measure defined by Levitin and Cook (1996).

We note that sub-additive probability is a concept which has already been studied in psychology. It has been shown that when experiment participants have to express their degree of beliefs on a \([0,1]\) interval, probabilistic additivity will be violated in many cases and sub-additivity obtains. See Bearden et al. (2005) for a good overview. Bearden et al. (2005) also indicate that such sub-additivity has been obtained with experiment participants belonging to various industry groups, such as option traders for instance (Fox et al. (1996)).

The key work pertaining to the issue of sub-additivity in psychology is by Tversky and Koehler (1994) and Rottenstreich and Tversky (1997). Their theory, also known under the name of ‘Support Theory’ is in the words of Tversky and Koehler (1994) “...a theory in which the judged probability of an event depends on the explicitness of its description.” In other words, it is not the event which is important as such but its description. In Tversky and Koehler (1994) the authors highlight the ‘current’ state of affairs (Anno 1994) on the various interpretations subjective probability may have. Amongst the interpretations is Zadeh’s possibility theory (1978) and the upper and lower probability approach...
of Suppes (1974). The paper by Dubois and Prade (1988), also mentioned in
the Tversky and Koehler (1994) article, provides for an excellent overview on
non-additive probability approaches.

9.4 Complementarity of $t$ and $a$

Can the conjecture 9.1. we posed at the beginning of this section make sense
from a quantum mechanical point of view? As is well known, there does not
exist a quantum operator on time at all! As we have remarked in Khrennikov
and Haven (2006) there exist a very well known test of complementarity, based
on the Heisenberg uncertainty relation which says that the product of the uncer-
tainties in the determination of position ($\Delta p = \sqrt{E(p - Ep)^2}$) and momentum
($\Delta p = \sqrt{E(p - Ep)^2}$) is weakly larger than the Planck constant divided by
two. However, as we remarked in Khrennikov and Haven (2006), this method
can not be applied for discrete variables, since if operators have discrete spec-
tra there exist eigenstates (corresponding to eigenvalues) and their standard
development equals to zero. Hence, it is not possible to find an uncertainty prin-
ciple of the type: $\Delta t, \Delta a = \hbar_{cogn}/2$, where $\hbar_{cogn}$ is an analog of the Planck
constant. Discussions on analogs of Planck constants in financial contexts and
other non-quantum physical contexts were provided for in Choustova (2006;
2007), Khrennikov (2004) and Haven (2005; 2007).

The very objective of our experiment is just to overcome those problems of
measurement and hence to be able to test for complementarity on the $t$ and $a$
variables.

10 Conclusion

In this paper we have tried to show how probability interference, which is an
essential quantum physical concept can be found back in a macroscopic set-
ting, more precisely in a social science setting. Such interference can help in
explaining major economic paradoxes and the very paradoxes themselves have
quite some interesting ramifications. Therefore, testing for the presence of such
interference on a macroscopic scale is an interesting challenge. In this paper
we provide for such a possible experiment.
11 Bibliography

1. Anscombe F., Aumann R. (1963), A definition of subjective probability; *Annals of Mathematical Statistics*; 34, 199-205.

2. Attneave F.; Olson R.K. (1971), Pitch as a medium: a new approach to psychological scaling; 84, 147-166

3. Bearden J.N., Wallsten T., Fox C. (2005), Error and subadditivity: a stochastic model of subadditivity, University of Arizona - Department of Management and Policy.

4. Black F., Scholes M. (1973), The pricing of options and corporate liabilities; *Journal of Political Economy*; 81, 637-654.

5. Bohm D. and Hiley B. (1993), *The Undivided Universe*, New York: Routledge, 1-397.

6. Bohm D. (1987), *Hidden Variables and the Implicate Order* in Quantum Implications: Essay in Honour of D. Bohm, edited by B. Hiley and F. Peat, New York: Routledge, 33-45.

7. Bohm, D. (1952). A suggested interpretation of the quantum theory in terms of ‘hidden’ variables, Part I and II. *Physical Review* 85, 166-193.

8. Bossaerts P, P. Ghirardato, S. Guarnaschelli and W. Zame (2007); Prices and Allocations in Asset Markets with Heterogeneous Attitudes Towards Ambiguity; Working paper; *California Institute of Technology*.

9. Busemeyer J. and Wang Z. (2007), Quantum information processing explanation for interactions between inferences and decisions; Papers from the AAAI Spring Symposium (Stanford University); 91-97

10. Busemeyer J. and Wang Z and Townsend J.T. (2006), Quantum dynamics of human decision making, *Journal of Mathematical Psychology* 50 (3): 220-241.

11. Choustova, O. (2006), Quantum Bohmian model for financial markets. *Physica A* 374, 304-314.

12. Choustova O. (2007), Toward quantum-like modelling of financial processes; *Journal of Physics: Conference Series* 70 (38pp)

13. Collier G and Collier J.L. (1994), An exploration of the use of tempo in jazz; *Music Perception*; 11(3); 219-242

14. Conte E., Todarello O., Federici A., Vitiello F., Lopane M., Khrennikov A. (2004), *A preliminary evidence of quantum-like behaviour in measurements of mental states*; Proc. Int. Conf. Quantum Theory: Reconsideration of Foundations. Ser. Math. Modelling in Phys., Engin., and Cogn. Sc., vol. 10, 679-702, Växjö Univ. Press.
15. Drake A. H. (1968), An experimental study of selected variables in the performance of musical durational notation; *Journal of Research in Music Education*; vol. 16, 329-338

16. Drake C., Botte M.C. (1993), Tempo sensitivity in auditory sequences: evidence for a multiple look model; *Perception and Psychophysics*; 54(3), 277-286

17. Dubois D., Prade H. (1988), Modelling and inductive inference: a survey of recent non-additive probability systems; *Acta Psychologica*; 68, 53-78.

18. Duffie, D. (1996). *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton.

19. Ellsberg, D. (1961), Risk, Ambiguity and the Savage Axioms; *Quarterly Journal of Economics* 75, 643-669, 1961

20. Etheridge, A. (2002). *A Course in Financial Calculus*, Cambridge University Press, Cambridge.

21. Farnsworth P.R. (1969). *The Social Psychology of Music*, Iowa State University Press, Iowa.

22. Fox C., Rogers B., Tversky A. (1996), Option traders exhibit subadditive decision weights; *Journal of Risk and Uncertainty*; 13, 5-17.

23. Franco, R. (2007). Quantum mechanics and rational ignorance; arXiv:physics/0702163v1

24. Geringer, J. and Madsen C.M. (1984). Pitch and tempo discrimination in recorded orchestral music among musicians and nonmusicians; *Journal of Research in Music Education* 32, 195-204.

25. Ghirardato, P., Maccheroni F. and Marinacci M. (2004) “Differentiating Ambiguity and Ambiguity Attitude.” *Journal of Economic Theory* 118, 133-173.

26. Gilboa, I. and D. Schmeidler (1989) Maxmin Expected Utility with a Non-Unique Prior; *Journal of Mathematical Economics* 18, 141-153.

27. Harrison, J.M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* 20, 381-408.

28. Haven, E. (2005); Pilot-wave theory and financial option pricing; *International Journal of Theoretical Physics* 44 (11), 1957-1962

29. Haven, E. (2007). Private information and the ‘information function’: a survey of possible uses. *Theory and Decision*, forthcoming

30. Haven, E. (2007). The Variation of Financial Arbitrage via the Use of an Information Wave Function; *International Journal of Theoretical Physics*, forthcoming
31. Holland, P. (1993). The quantum theory of motion: an account of the de Broglie-Bohm causal interpretation of quantum mechanics. Cambridge University Press, Cambridge.

32. Kabanov, Yu. and Stricker, C. (2005). Remarks on the true no-arbitrage property. Séminaire de Probabilités XXXVIII - Lecture Notes in Mathematics 1857, 186-194, Springer.

33. Khrennikov, A. Yu. (1999). Classical and quantum mechanics on information spaces with applications to cognitive, psychological, social and anomalous phenomena. Foundations of Physics 29, 1065-1098.

34. Khrennikov, A. (2002) On the cognitive experiments to test quantum-like behavior of mind; Reports from Växjö University - Mathematics, Natural Sciences and Technology, Nr. 7.

35. Khrennikov, A. (2004) Information dynamics in cognitive, psychological and anomalous phenomena; Ser. Fundamental Theories of Physics, v. 138, Kluwer, Dordrecht.

36. Khrennikov, A. (2007). Classical and quantum randomness and the financial market. arXiv: 0704.2865v1 [math.PR]

37. Khrennikov A., and E. Haven (2006) ; Does probability interference exist in social science? Foundations of Probability and Physics -4 (G. Adenier, A. Khrennikov, C. Fuchs - Eds), AIP Conference Proceedings; 899; 299-309.

38. Kreps D. (1988), Notes on the theory of choice, Westview Press (Colorado-Boulder).

39. Kuhn T.L. (1974); Discrimination of modulated beat tempo by professional musicians; Journal of Research in Music Education; 22; 270-277.

40. Kuhn T.L. (1975); Effect of notational values, age, and example length on tempo performance accuracy; Journal of Research in Music Education; 23 (3); 203-210.

41. Kuhn T. L.; Booth, G. D. (1988); The effect of melodic activity, tempo change, and audible beat on tempo perception of elementary school students; Journal of Research in Music Education, 1988, 36, 140-155.

42. La Mura, P. (2006), Projective Expected Utility, Mimeo, Leipzig Graduate School of Management.

43. Levitin D., Cook P.R. (1996), Memory for musical tempo: additional evidence that auditory memory is absolute'; Perception and Psychophysics; 58; 927-935.

44. Levitt H. (1971), Transformed up-down methods in psychoacoustics; Journal of the Acoustical Society of America; 49; 467-477

27
45. Madsen C.K., Duke R.A. and Geringer J.M. (1986) The effect of speed alterations on tempo note selection; *Journal of Research in Music Education*, 34; 101-110

46. Madsen C. K. and Graham R. (1970) The effect of integrated schools on performance of selected music tasks of black and white students; *Music Educator’s National Conference*; Chicago; Illinois

47. Morrison M. (1990), *Understanding quantum physics*; Prentice-Hall.

48. Rottenstreich Y., Tversky A. (1997) Unpacking, repacking and anchoring: advances in support theory; *Psychological Review*; 104, 406-415.

49. Savage L.J., *The Foundations of Statistics*; NY, Wiley, 1954.

50. Shafir E and Tversky A. (1992); Thinking through uncertainty: non-consequential reasoning and choice; *Cognitive Psychology*, 24, 449-474.

51. Suppes P. (1974), The measurement of belief; *Journal of the Royal Statistical Society B*; 36, 160-191.

52. Tversky A., Koehler D. (1994), Support theory: a nonexistent representation of subjective probability; *Psychological Review*; 101, 547-567.

53. von Neumann J., Morgenstern O. (1947), *Theory of games and economic behavior*; Princeton University Press.

54. Wang C.C. and Salzberg R.S. (1984), Discrimination of modulated music tempo by strong students; *Journal of Research in Music Education*, 32, 123-131

55. Wolfram MathWorld; [http://mathworld.wolfram.com/AllaisParadox.html](http://mathworld.wolfram.com/AllaisParadox.html)

56. Zadeh L. (1978), Fuzzy sets as a basis for a theory of possibility; *Fuzzy Sets and Systems*; 1, 3-28.