DEFORMATIONS OF HOLOMORPHIC
PSEUDO-SYMPLECTIC POISSON MANIFOLDS

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ABSTRACT. We prove unobstructed deformations for compact Kählerian even-dimensional Poisson manifolds whose Poisson tensor degenerates along a normal-crossing divisor which is smooth wherever the Poisson tensor has corank exactly 2.

By definition, a pseudo-symplectic manifold is a compact complex manifold \(X\) satisfying the \(\partial \bar{\partial}\) lemma, endowed with a (holomorphic) Poisson structure \(\Pi \in H^0(\bigwedge^2 T_X)\) which is generically of full rank \(2n = \dim(X)\). This implies that \(\Pi\) degenerates along a (possibly trivial) anticanonical Pfaffian divisor

\[ D = \text{Pf}(\Pi) = [\Pi^2] \in |-K_X|. \]

\(D\) is trivial precisely when \(\Pi\) is a symplectic Poisson structure. We will say that \((X, \Pi)\) is Pfaffian-normal or P-normal if \(D = \text{Pf}(\Pi)\) is reduced with (local) normal crossings, and \(D\) is smooth at every point where \(\Pi\) has corank exactly 2. Note that it follows from Weinstein’s structure theorem for Poisson structures that, locally at every smooth point of \(D\) there is a local coordinate system in which \(\Pi\) has the form

\[ x_1 \partial x_1 \wedge \partial x_2 + \partial x_3 \wedge \partial x_4 + \ldots + \partial x_{2n-1} \wedge \partial x_{2n}. \]

In particular, \(\Pi\) has corank exactly 2 at smooth points of \(D\). The P-normal hypothesis means that \(\Pi\) has corank \(> 2\) at all singular points of \(D\). It turns out that when \(\Pi\) is P-normal, the singularities of \(D\) are not ‘accidental’ and in fact are locally constant over the Poisson moduli. This leads one to suspect the singular strata of a normally-crossing \(D\) may represent some kind of characteristic class associated to \(\Pi\), even though these singular strata fail to have the generic codimension in terms of the rank of the tensor \(\Pi\) considered as skew-symmetric bundle map. One result in this direction is given in Proposition 5 below, which gives a Chern-polynomial obstruction to the smoothness (even in codimension 3) of \(D\) (i.e. when the polynomial is nonzero, the singular locus of \(D\) has codimension 3 in \(D\) (codimension 4 in \(X\)).

If \(\text{Pf}(\Pi)\) is trivial, then \(\Pi\) is a symplectic Poisson structure, so \(X\) is a Calabi-Yau (\(K\)-trivial) manifold, hence by a well-known result of Tian-Todorov has unobstructed deformations. At the other
extreme, if Pf(Π) is ample, then X is Fano, hence trivially has unobstructed deformations (a less trivial unobstructedness result for weak Fano manifolds was recently obtained by T. Sano [14]). If D = Pf(Π) is smooth, then a result of [10] (sketched in the Appendix, §5 below, assuming only D has normal crossings) shows that the pair (X, D) has unobstructed locally trivial deformations. Our purpose here is to generalize these results to the P-normal case, by proving that a P-normal Poisson manifold (X, Π) has unobstructed (Poisson) deformations, i.e. the Poisson deformation space Def(X, Π) is smooth:

**Theorem 1.** Let (X, Π) be a P-normal Poisson manifold with Pfaffian divisor D. Then

(i) (X, Π) has unobstructed Poisson deformations.

(ii) Poisson deformations of (X, Π) induce locally trivial deformations on D.

(iii) There is a space of deformations of the triple (X, Π, D), whose forgetful morphisms to Def(X, Π) and Def_{loc, trivial}(X, D) are both smooth.

We do not prove that the natural morphism Def(X, Π) → Def_{loc, trivial}(X, D) is smooth. However, the fact that every locally trivial deformation of (X, D) lifts to a deformation of (X, Π, D) is sufficient to prove:

**Corollary 2.** Given a P-normal Poisson manifold (X, Π) with Pfaffian D and a deformation ˜X of X, Π extends to ˜X iff D does to ˜X locally trivially.

The strategy of the proof of the Theorem is analogous to that in the symplectic case: namely, relate infinitesimal symmetries (in this case, a certain ‘log Schouten dgla’) to differentials (in this case, a certain log complex), and apply Hodge theory (in this case, Deligne’s E_1 degeneration theorem for the log complex).

See [6, 9] for other results on deformations of complex Poisson manifolds. Hitchin [9] proved, for general Poisson manifolds satisfying the ∂̅∂-lemma, the weaker result that any class in H^1(T_X) that comes via Π duality from a class in H^1(Ω_X) is unobstructed.

### 1. Basics on Poisson structures

See [2] for basic facts on Poisson manifolds and [11] and references therein or the upcoming Riverside thesis of C. Kim for information on deformations of Poisson complex structures.

#### 1.1. Schouten algebra.

Fixing a manifold X, we denote by T = T_X its tangent Lie algebra of holomorphic vector fields. Then the exterior algebra T· = ∑ i T_i ∧ T is endowed with a graded bracket known as the Schouten bracket, extending the Lie bracket, with the property that

\[ [P, Q ∧ R] = [P, Q] + (-1)^{|p-1|q} Q ∧ [P, R], P ∈ T^p, Q ∈ T^q. \]

A Poisson structure on X is by definition a tangent bivector field Π ∈ H^0(∧ T) satisfying the integrability condition [Π, Π] = 0. A Poisson structure Π defines on T· the structure of a complex with differential [, Π] = L_{Π(·)}, L = Lie derivative, hence a structure of differential graded Lie algebra (dgla) called the Schouten algebra, which is just the algebra of infinitesimal symmetries of the pair (complex structure, Poisson structure). The deformation theory associated to the Schouten
dgla is likewise the deformation theory of the Poisson structure $\Pi$ together with the underlying complex structure.

The Schouten dgla $T^\cdot$ admits as dgla module the ‘reverse de Rham’ dg complex $(\Omega, \delta)$, with codifferential
\[
\delta = L_\Pi = i_\Pi d - d i_\Pi,
\]
of degree -1 and square zero thanks to $[\Pi, \Pi] = 0$. This differential has the following properties, which together determine it uniquely:
\[
\begin{align*}
(i) \quad & \delta(f \omega) = f \delta(\omega) + i_\Pi(f) \omega, \nabla^\Pi(f) = i_d f 
\end{align*}
\]
These imply
\[
\begin{align*}
(i) \quad & [d, \delta] = 0, \\
(ii) \quad & \delta(df_1 \wedge df_2) = \delta(f_1 df_2) = i_{df_1 \wedge df_2} \Pi, \\
(iii) \quad & \delta(df_1 \wedge \ldots \wedge df_k) = \sum (-1)^{i+j} \delta(df_1 \wedge df_j) \wedge df_1 \wedge \ldots \wedge \hat{df}_i \ldots \wedge df_k
\end{align*}
\]
Note that in view of these properties, the restriction $\delta^{(1)}$ of $\delta$ on $\Omega^1$ determines $\delta$ on the rest of the complex. Also, any bivector field $\Pi$ allows us to define maps $\delta^{(1)}, \ldots, \delta^{(n)}$, but it is the integrability condition $[\Pi, \Pi] = 0$ - which is equivalent to $\delta^{(2)} \delta^{(1)} = 0$ - that makes it a complex.

Poisson deformations are $T^\cdot$-deformations of this complex, i.e. deformations $\tilde{X}/S$ of the complex structure of $X$, together with deformations $\tilde{\delta}$ of the codifferential $\delta$ on the relative de Rham complex $\Omega_{\tilde{X}/S}$ satisfying (4),(5), which simply mean that $\tilde{\delta}$ comes from a relative tangent bivector field $\tilde{\Pi} \in \mathcal{H}^0(\wedge^2 T_{\tilde{X}/S})$ on $\tilde{X}/S$, plus the square-zero condition, which is equivalent to $\tilde{\delta}^{(1)} \tilde{\delta}^{(2)} = 0$, and then again to $[\tilde{\Pi}, \tilde{\Pi}] = 0$. The dgla homomorphism $T^\cdot \rightarrow T$ induces a morphism of formal deformation spaces
\[
\text{Def}(\tilde{X}, \Pi) \rightarrow \text{Def}(X).
\]
In general, this morphism is clearly not injective (and probably not surjective).

1.2. **Duality.** $\Pi$ gives rise to various duality or interior multiplication maps, such as
\[
\Pi^\#: \Omega \rightarrow T,
\]
\[
i_\Pi : \Omega^p \rightarrow \Omega^{p-2}, \ p \geq 2.
\]

1.3. **Poisson- de Rham algebra.** A fundamental feature of Poisson geometry is that the cotangent exterior algebra $\Omega^*_X$ is endowed with a graded Lie-Poisson bracket with the properties
\[
[df, dg] = d(f,g),
\]
\[
[\omega_1, f \omega_2] = f[\omega_1, \omega_2] + \Pi(\omega_1 \wedge df) \omega_2,
\]
plus a derivation property analogous to (2).

The standard duality map
\[
\Pi^\#: \Omega \rightarrow T
\]
extends to
\[
\bigwedge \Pi^\#: (\Omega; \wedge, [~, ] ) \to (T; \wedge, [~, ] )
\]
as a graded homomorphism with respect to both $\wedge$ and $[~, ]$ products.

1.4. **Products.** Note that if $(X_i, \Pi_i)$ are Poisson manifolds, then $X = \prod X_i$ admits the Poisson structure $\Pi = \sum p^*_i(\Pi_i)$, called the product structure of the $(X_i, \Pi_i)$. Moreover, if each $(X_i, \Pi_i)$ is $P$-normal, then so is $(X, \Pi)$.

Note that an effective anticanonical divisor $D$ on a smooth surface $X$ determines a Poisson structure on $X$, which is $P$-normal if $D$ is smooth. Taking products of these yields examples of $P$-normal Poisson manifolds with nonsmooth Pfaffian divisor.

1.5. **Hilbert schemes: Poisson au Beauville.** Beauville \[1\] has constructed a symplectic structure on the Hilbert scheme of a symplectic (i.e. K3 or abelian) surface. This construction can be extended to yield a Poisson structure on the Hilbert scheme $S^{[r]}$ of a Poisson (i.e. anticanonical) surface $S$. This structure is not $P$-normal, but, at least for $r \leq 3$ and conjecturally for all $r$, it induces a $P$-normal Poisson structure on a suitable blowup. We will do this construction by induction on $r$, starting with $r = 2$.

Thus, let $S$ be a smooth surface endowed with a Poisson structure $\Pi$ or equivalently, an effective anticanonical divisor $C = \text{Pf}(\Pi)$. Let $\tilde{S}^2$ denote the blowup of $S^2 = S \times S$ in the diagonal, with exceptional divisor $E_2 = \mathbb{P}(\Omega^1_S)$. Then we have a diagram

\[
\begin{array}{ccc}
\tilde{S}^2 & \xleftarrow{p} & S^{[2]} \\
\downarrow q & & \downarrow \\
S^2 & & 
\end{array}
\]

with both $p$ and $q$ (simply) ramified along $E_2$, which goes isomorphically via $p$ to the discriminant $\Delta$.

Now the product Poisson structure $\Pi^{[2]}$ on $S^2$ pulls back to a meromorphic Poisson structure $\tilde{\Pi}^2$ on $\tilde{S}^2$, invariant under the flip involution and with a simple pole along $E_2$ ($\Pi^2$ degenerates along $C \times S \cup S \times C$). Then $\tilde{\Pi}^2$ descends via $p$ to a meromorphic Poisson structure $\Pi^{[2]}$ on $S^{[2]}$ with at worst a pole on the discriminant $\Delta_2$. We claim $\Pi^{[2]}$ has in fact no pole on $\Delta_2$, hence is holomorphic. This can be checked at a generic point of $\Delta_2$.

Now locally near a generic point of $S$ we can write $\Pi = \langle V, \Phi \rangle$ where $\Phi$ is a symplectic form and $V$ is a covolume (dual volume) form. Up on the blowup $\tilde{S}^2$, we get for the respective pullbacks
\[
\tilde{\Pi}^2 = \langle \tilde{V}^2, \tilde{\Phi}^2 \rangle
\]
Now $\tilde{V}^2$ has a simple pole along $E_2$, hence is the pullback of a (regular, everywhere nondegenerate) covolume form $V^{[2]}$ on $S^{[3]}$. By Beauville, $\Phi^2$ is the pullback of a symplectic form $\Phi^{[2]}$ on $S^{[2]}$. Consequently,
\[
\tilde{\Pi}^2 = p^*(\Pi^{[2]}), \Pi^{[2]} = \langle V^{[2]}, \Phi^{[2]} \rangle
\]
and $\Pi^{[2]}$ is holomorphic. Note that by the same argument, $\Pi^{[2]}$ is generically nondegenerate on $\Delta_2$, so its Pfaffian $D = \text{Pf}(\Pi^{[2]})$ coincides with the locus $S + C$ of schemes meeting $C$, and $D$ clearly has normal crossings away from the locus $2C$ of ‘double points on $C$’, i.e. ideals of the form $I_p^2 + I_C, p \in C$.

Let $\pi : X_2 \to S^{[2]}$ denote the blowup of $S^{[2]}$ in the singular locus of $D$, i.e. $C^{(2)}$. Let $E$ be the exceptional divisor and $\tilde{D}$ the proper transform of $D$. Then we have
\[ \pi^*(D) = \tilde{D} + 2E, \quad K_{X_2} = \pi^*(K_{S^{[2]}}) + E, \]
therefore
\[ -K_{X_2} = \tilde{D} + E. \]
As above, the Poisson structure $\Pi^{[2]}$ on $S^{[2]}$ induces a Poisson structure $\Pi_{X_2}$ on $X_2$ with Pfaffian divisor $\tilde{D} + E$. The following simple local calculation shows that $\tilde{D} + E$ has normal crossings.

Let $x, y$ be local coordinates on $S$ so that $x$ defines $C$, and consider a subscheme of the form $(x, y^2)$, a typical non-normal-crossing point of $D$. Deformations of this scheme on $S$ are of the form $(x + a_0 + a_1 y, y^2 + b_0 + b_1 y)$ so $(a_0, a_1, b_0, b_1)$ are local coordinates on $S^{[2]}$. Then the equation of $D$, the locus of schemes meeting $C$ nontrivially, is $a_0^2 + b_0 a_1^2 + b_1 a_0 a_1$. The double locus $C^{(2)}$ is given by the equations $a_0, a_1$. Blowing this up gives, on the relevant open,
\[ a_1^2([a_0/a_1]^2 + b_0 + b_1[a_0/a_1]) \]
where $a_1$ is the equation of $E$ and $[a_0/a_1]^2 + b_0 + b_1[a_0/a_1]$ is the equation of $\tilde{D}$. These are clearly transverse, so $\tilde{D} + E$ has normal crossings.

Now, it is easy to see that at a point of $C^{(2)} \setminus \Delta_2$, $\Pi^{[2]}$ is locally of the form
\[ x_1 \partial_{x_1} \wedge \partial_{x_2} + x_3 \partial_{x_3} \wedge \partial_{x_4} \]
where $x_1, x_3$ are equations for $C^{(2)}$. By an easy computation, the logarithmic vector fields $x_1 \partial_{x_1}, x_3 \partial_{x_3}$ lift to $X$ as holomorphic vector fields which are vertical along $E$ and vanish on $\tilde{D} \cap E$ which is the singular locus of $\text{Pf}(\Pi_X)$. This implies that $\Pi_X$ vanishes along the singular locus of $\text{Pf}(\Pi_X)$, so $\Pi_X$ is $P$-normal.

Now even though $\Pi^{[2]}$ is not $P$-normal, $\Pi^{[2]}$ vanishes on $C^{(2)}$ which is the singular locus of $D$, so the argument in the proof of the main Theorem shows that deformations of $(S^{[2]}, \Pi^{[2]})$ induce locally trivial deformations of $D$, hence induce deformations of $(X_2, \Pi_{X_2})$. Conversely, deformations of $(X_2, \Pi_{X_2})$ induce locally trivial deformations of $\text{Pf}(\Pi_{X_2}) = \tilde{D} + E$, hence descend to deformations of $(S^{[2]}, \Pi^{[2]})$. In particular, $(S^{[2]}, \Pi^{[2]})$ has unobstructed deformations.

In the general case $r \geq 2$ we use induction, based on the diagram
\[(9) \quad S^{[r+1]} \xrightarrow{p} S^{[r]} \xleftarrow{q} S^{[r]} \times S \]
where $q$ is the blowing-up of the tautological subscheme, $S^{[r+1]}$ is the $(r, r+1)$ flag-Hilbert scheme, and $p$ is the forgetful map. Inductively, $\Pi$ induces a Poisson structure $\Pi^{[r]}$ on $S^{[r]}$, whence a product Poisson structure on $S^{[r]} \times S$ and thence a meromorphic Poisson structure on $S^{[r+1]}$, which again descends to a Poisson structure $\Pi^{[r+1]}$ on $S^{[r+1]}$, possibly with a pole along the discriminant.
Then a local analysis as above at a generic point of the discriminant, which corresponds to a double point plus \( r - 1 \) generic points of \( S \), shows that \( \Pi^{[r + 1]} \) is holomorphic.

Now let \( \pi_r : X_r \to S^{[r]} \) be the birational morphism obtained by first blowing up the \( r \)-fold locus of \( D_r = \mathrm{Pf}(\Pi^{[r]}) \), then blowing up the proper transform of the \( (r - 1) \)-fold locus, etc. Conjecturally, \((X_r, \pi_r^*(D_r)_{\text{red}})\) is an embedded resolution of \((S^{[r]}, D_r)\), i.e. \( X \) is smooth and \( \pi_r^*(D_r)_{\text{red}} \) has normal crossings while \( \pi_r^*(D_r) = \sum E_i \) with \( E_i \) the proper transform of the \( i \)-fold locus of \( D_r \). We have checked this for \( r = 3 \). If this is true, then as above the meromorphic Poisson structure \( \Pi_{X_r} = \pi_r^*(\Pi^{[r]}) \) is holomorphic and its Pfaffian divisor coincides with

\[
\pi_r^*(D_r)_{\text{red}} = \pi_r^*(D_r) - \sum (i - 1)E_i \sim \pi_r^*(-K_{S^{[i]}}) - \sum (i - 1)E_i = -K_{X_r}.
\]

Hence \( \Pi_{X_r} \) is \( \Pi \)-normal and, as above, \((S^{[r]}, \Pi^{[r]})\) has unobstructed deformations.

2. \textbf{The log complexes}

2.1. \textbf{Cotangent log complex}. (see [3], [7]) Let \( X \) be a compact Kähler manifold with holomorphic de Rham complex \( \Omega_X \). Let \( D \) be a reduced divisor with local normal crossings on \( X \). Let \( \bar{D} \) be the normalization (=desingularization) of \( D \). The sheaf of log 1-forms on \( X \) with respect to \( D \), denoted by \( \Omega_X^1(\log D) \) is by definition the kernel of the natural map

\[
(10) \quad \Omega_X^1(D) \to \Omega_D(D).
\]

In terms of a local coordinate system \( x_1, \ldots, x_n \) such that \( D \) has local equation \( x_1 \cdots x_k, \Omega_X^1(\log D) \) is generated by \( dx_1/x_1, \ldots, dx_k/x_k, dx_{k+1}, \ldots, dx_n \). Therefore \( \Omega_X^1(\log D) \) is locally free. Define \( \Omega_X^1(\log D) \) as the subalgebra of the meromorphic exterior algebra \( \Omega_X(\ast D) \) generated by \( \Omega_X^1(\log D) \). Then each \( \Omega^i(\log D) \) is locally free. It is a theorem of Deligne ([4], (8.1.9); see also [16], [15]) that the ‘log Hodge-to-de Rham’ spectral sequence

\[
E_1^{pq} = H^q(X, \Omega_X^p(\log D)) \Rightarrow H^{p+q}(X, \Omega(\log D))
\]

degenerates at \( E_1 \). By another theorem of Deligne ([3]; see also [7]), we have

\[
(12) \quad H^i(X, \Omega(\log D)) \simeq H^i(X \setminus D, \mathbb{C}).
\]

2.2. \textbf{log Schouten algebra}. We denote by \( T^i(\log D) \) the dual of \( \Omega^i(\log D) \), which is a locally free, full-rank subsheaf of \( T^i \). It is easy to check that the Schouten bracket extends to define a graded Lie algebra structure on

\[
T^i(\log D) = \bigoplus_{r \geq 1} T^r(\log D).
\]

Now suppose that \( \Pi \) is a pseudo-symplectic Poisson structure with Pfaffian divisor \( D \). Then it is elementary that for any local branch \( D_i \) of \( D \), the codifferential \( \delta \) (cf. [3]) vanishes locally on the subcomplex of \( \Omega_X \) generated by \( O(-D_i) \), and therefore descends to a codifferential on \( \Omega_D \) so that restriction

\[
(\Omega_X, \delta) \to (\Omega_D, \delta)
\]
is a map of complexes.
2.3. **Duality.** Under the assumption that \( \Pi \) is a pseudo-symplectic Poisson structure with Pfaffian divisor \( D \), it is easy to check that the duality map \( \wedge \Pi^* \) extends to \( \wedge \Pi^* \) which is a dgla homomorphism.  

(14) \[
\Pi^* : (\Omega_+(\log D), \wedge, [\cdot, \cdot]) \rightarrow (T^*_+\log D), \wedge, [\cdot, \cdot])
\]
which is a dgla homomorphism.

**Lemma 3.** *If \( \Pi \) is a P-normal Poisson structure, then \( \Pi^* \) is an isomorphism.*

**Proof.** Obviously \( \Pi^* \) is locally an isomorphism off \( D \). At smooth points of \( D \), in terms of the normal form \( \Pi \), \( \Pi^* \) sends a local generator \( dx_1/x_1 \) to \( \partial x_2 \) and \( dx_2 \) to \( -x_1 \partial x_1 \) while

\[
\Pi^*(dx_{2i-1}) = \partial x_{2i}, \Pi^*(dx_{2i}) = -\partial x_{2i-1}, i = 2, ..., n.
\]

It follows that \( \Pi^* \) is an isomorphism off the singular locus of \( D \). Because \( \Pi^* \) is a map of locally free sheaves of the same rank, its degeneracy locus is of codimension at most 1. Therefore \( \Pi^* \) is an isomorphism.

**Remark 4.** A Poisson structure is said to be *normal* if it locally has the form

\[
x_1 \partial x_1 \wedge \partial x_2 + ... + x_{2k-1} \partial x_{2k-1} \wedge \partial x_{2k} + \partial x_{2k+1} \wedge \partial x_{2k} + ... + \partial x_{2n-1} \wedge \partial x_{2n}.
\]
A normal Poisson structure is clearly P-normal. In light of the foregoing Lemma, it is reasonable to believe that the converse holds as well. We won’t need this.

3. **Proof of Theorem**

We will prove first that \( (X, \Pi) \) is unobstructed.

3.1. **Step 1: from log differentials to log tangent vectors.** From Deligne’s \( E_1 \)-degeneration theorem it follows that \( d \) induces the zero map on \( H^1(\Omega_+(\log D)), D = \text{Pf}(\Pi) \). Therefore by the Cartan formula it follows in the usual manner that the bracket-induced map (not the wedge product map)

\[
H^1(\Omega_+(\log D)) \otimes H^1(\Omega_+(\log D)) \rightarrow H^{i+1}(\Omega_+(\log D))
\]

vanishes. Therefore by the duality Lemma 3 the same is true of the bracket-induced map

\[
H^1(T^*_+\log D) \otimes H^1(T^*_+\log D) \rightarrow H^{i+1}(T^*_+\log D)
\]

Consequently, all the obstruction maps vanish:

\[
\text{Sym}^{m-1}H^1(T^*_+\log D) \rightarrow \text{Sym}^{m-2}H^1(T^*_+\log D) \otimes H^2(T^*_+D).
\]

Now the \( m \)th order Poisson deformation space is computed by the Jacobi (or Quillen) complex \( J_m(T^*_+\log D) \) (\[12\] or \[13\]). By the above, the \( E_1 \) spectral sequence

\[
E_1^{p,q} = H^q(\bigcap_{i=0}^p T^*_+D) \Rightarrow H^q(J_m(T^*_+\log D))
\]
degenerates at \( E_1 \): Therefore the log-Schouten dgla \( T^*_+\log D \) has unobstructed deformations.
### Step 2: from log to ordinary tangent vectors

Heuristically at least, because $D$ is the Pfaffian divisor of $\Pi$, a Poisson deformation of $(X, \Pi)$ automatically extends to a deformation of $D$ as well. This implies that the map on deformation theories induced by the inclusion $i : T_X(\log D) \to T_X$ admits a right inverse. Therefore $T_X$ has unobstructed deformations, i.e. $(X, \Pi)$ is unobstructed as Poisson manifold.

Formally speaking, consider the complex $K$ defined by

$$\begin{align*}
K^0 &= T; \\
K^1 &= T^2 \oplus N^0_D; \\
K^i &= T^{i+1} \oplus T^{i-1} \otimes N^0_D, \quad i \geq 2,
\end{align*}$$

$(15)$

$N^0_D \subset N_D = O_D(D)$ being the ‘locally trivial’ subsheaf of the normal sheaf, image of $T_X \to N_D$; the differentials are given by: $T^i \to T^{i+1}$ is Poisson differential $[,] , \Pi$, and the map $T^i \to T^{i-1} \otimes N^0_D$ is extended via tensor product from the canonical surjective map $T \to N^0_D$ (with kernel $T(\log D)$). The differential is by definition zero on the $T^{i-1} \otimes O_D(D)$ summand. Now there is an obvious inclusion map

$$T(\log D) \to K,$$

which is a quasi-isomorphism, and a projection

$$K \to T.$$

On the other hand, there is also a map

$$T \to K,$$

where $T^i \to T^i$ is the identity and $T^i \to T^{i-2}$ is extended via tensor product from the map

$$T^2 \to N^0_D \subset O_D(-K_X),$$

$$u \mapsto nu \wedge \Pi^{n-1} |_D$$

(16)

(the latter is just the derivative of the map $\Pi \mapsto \Pi^u$). Note that this map indeed goes into the $N^0_D$ subsheaf due to the fact that $\Pi^{n-1}$ vanishes on $\text{sing}(D)$, which follows from the assumption of $P$-normality. Thus in the derived category, $T$ is a direct summand of $K \sim T(\log D)$, and since the latter has vanishing obstructions, so does the former. This completes the proof that $(X, \Pi)$ is unobstructed, i.e. assertion (i) of Theorem 1.

For assertion (ii), the local triviality of the induced deformation of $D = \text{Pf}(\Pi)$ follows from the fact that the map $T^2 \to N_D$ above goes into $N^0_D$. In fact, deformations of $(X, D)$ generally are controlled by the complex

$$T \to N_D,$$

while the locally trivial deformations (i.e. those where $D$ deforms locally trivially) are controlled by the subcomplex

$$(T \to N^0_D) \sim T(\log D).$$

In our case we get a map

$$T \to (T \to N_D)$$
which factors through \((T \to N^0_D) \simeq T(\log D)\). Indeed this map is just the composite of the map \(T^r \to T^r(\log D)\) constructed above with the truncation map \(T^r(\log D) \to T^r(\log D)\).

As for assertion (iii), because locally trivial deformations of \((X, D)\) are controlled by \(T^r(\log D)\), the proof of their unobstructedness is identical, replacing the complexes \(T^r(\log D), \Omega^r(\log D)\) by their zeroth component (see the Appendix below for a more general statement). Now consider the following diagram

\[
\begin{array}{ccc}
T^r(\log D) & \to & T(\log D) \\
\downarrow & & \downarrow \\
T^r & \to & T
\end{array}
\]

inducing

\[
\begin{array}{ccc}
\mathbb{H}(T^r(\log D)) & \to & H(T(\log D)) \\
\downarrow & & \downarrow \\
\mathbb{H}(T^r) & \to & H(T).
\end{array}
\]

Here the left vertical arrow is a direct summand projection. The top horizontal arrow is surjective by Deligne’s \(E_1\) degeneration of the spectral sequence for \(\mathbb{H}(\Omega(\log D))\). Thus, the deformation space corresponding to \(T^r(\log D)\) maps smoothly to the locally trivial deformations of \((X, D)\), as well as to the Poisson deformations of \((X, \Pi)\), proving assertion (iii).

The proof of Theorem \(\text{Theorem 1}\) is complete.

4. AN OBSTRUCTION

Here is an elementary Chern class obstruction to a manifold possessing a Poisson structure with smooth Pfaffian. First a definition. We say that a complex manifold \(X\) is compact through codimension \(c\) if it is of the form \(X = Y \setminus Z\) with \(Y\) a compact complex manifold and \(Z\) an analytic subset of codimension \(> c\). For example, if \(E\) is a vector bundle on \(Y\) carrying a skew-symmetric bundle map \(\phi\) that degenerates in the generic codimension, then the open set where \(\phi\) has corank \(\leq 2\) is compact through codimension \(5\).

**Proposition 5.** Let \(X\) be a complex manifold compact through codimension \(4\), and suppose \(X\) carries a generically nondegenerate Poisson structure \(\Pi\) with smooth Pfaffian divisor \(D\) on which \(\Pi\) has corank exactly \(2\). Then, where \(c_i = c_i(X)\), we have

\[
c_1(c_1c_2 - c_3) = 0.
\]

**Proof.** As \(D\) is an anticanonical divisor on \(X\), it is Calabi-Yau, i.e. \(c_1(\Omega_D) = 0\). Moreover, if \(M\) denotes the kernel of \(\Pi^*\) on \(\Omega_X|_D\), then by assumption \(M\) is a rank-2 bundle and by \([\Pi, \Pi] = 0\), \(M\) has the conormal line bundle \(\tilde{N}\) of \(D\) as subbundle. Therefore \(M/\tilde{N}\) is a line subbundle of \(\Omega_D\), such that \(\Omega_D/(M/\tilde{N})\) carries a nondegenerate alternating form given by the restriction of \(\Pi\), hence is self-dual and has vanishing odd Chern classes. Because \(c_1(D) = 0\) as well, it follows that

\[
c_{\text{odd}}(D) = 0.
\]
Then we get the result via Gysin by calculating $c_3(D)$ from

$$c(\Omega_D) = \frac{c(\Omega_X)}{1 + c_1(\Omega_X)}.$$

Remarks 6. (i) The generic codimension where the skew-symmetric map $\Pi^g$ has corank $> 2$ (i.e. $\geq 4$) is $6 = \binom{4}{2}$ (see [8] or [5], 14.4.11), so the proposition gives an obstruction for a manifold $X$ to support a Poisson structure with ‘generic singularities’, specifically a dimensionally proper ‘second’ (corank $\geq 4$) degeneracy locus $S$ and a Pfaffian smooth outside $S$.

(ii) Note that the condition (19) fails for $\mathbb{P}^4$, so $\mathbb{P}^4$ has no ‘nice’ Poisson structure in the foregoing sense ($D$ smooth), though it might still have one with $D$ normally crossing.

5. Appendix: Manifolds with a Normal-Crossing Anticanonical Divisor

The foregoing methods yield a short proof of the following result, obtained earlier at least in the case $D$ smooth by Iacono [10] (no Poisson structure is assumed).

**Theorem 7.** Let $X$ be compact Kähler with an effective locally normal-crossing anticanonical divisor $D$. Then $(X, D)$ has unobstructed locally trivial deformations.

The assertion amounts to saying that the dgla $T_X(\log D)$ has unobstructed deformations. The isomorphism

$$T_X = \Omega_X^{n-1}(-K_X) \cong \Omega_X^{n-1}(D), n = \dim(X),$$

induces

$$T_X(\log D) \cong \Omega_X^{n-1}(\log D).$$

Note that we have the interior multiplication pairing

$$T_X(\log D) \times \Omega_X^1(\log D) \to \Omega_X^{1}(\log D)$$

and the Cartan formula for Lie derivative

$$L_\omega(\omega) = i_\omega d\omega \pm d(i_\omega(\omega)).$$

Deligne’s $E_1$-degeneration shows that $d$ induces the zero map on $H(\Omega_X^1(\log D))$, and it follows that the pairing induced by Lie derivative vanishes:

$$H^1(T_X(\log D)) \times H^1(\Omega_X^{n-1}(\log D)) \to H^2(\Omega_X^{n-1}(\log D)).$$

By the above identification, this can be identified with the pairing on $H^1(T_X(\log D))$ induced by bracket, i.e. the obstruction pairing. Therefore the obstruction vanishes, so $(X, D)$ has unobstructed deformations. □

It is not in general true that $|−K_X|$ is constant under deformations on $X$, even if it has a smooth member, as shown by the example of $\mathbb{P}^2$ blown up in 10 points on a smooth cubic. Therefore, $\text{Def}(X, D)$ is in general a smooth fibration over a smooth proper subvariety of $\text{Def}(X)$.

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