BOUNDED $H^\infty$-CALCULUS FOR A DEGENERATE
ELLIPTIC BOUNDARY VALUE PROBLEM

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ABSTRACT. We consider a strongly elliptic second order operator $A$ together with a degenerate boundary operator $T$ of the form $T = \varphi_0\gamma_0 + \varphi_1\gamma_1$, where $\gamma_0$ and $\gamma_1$ denote the evaluation of a function and its exterior normal derivative, respectively, at the boundary; and $\varphi_1 \geq 0$, $\varphi_0 + \varphi_1 > 0$. We show that the realization $AT$ of $A$ in $L_p(\Omega)$ has a bounded $H^\infty$-calculus whenever $\Omega$ is a compact manifold with boundary or $\Omega = \mathbb{R}^n$.

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1. INTRODUCTION

Maximal regularity has become an indispensable tool in the analysis of evolution equations as it can be used to establish in an uncomplicated way the existence of short time solutions to certain quasilinear parabolic problems. Maximal regularity in turn is implied by the existence of a bounded $H^\infty$-calculus, a concept introduced by McIntosh in 1986, \cite{14}, of angle $< \pi/2$. Many elliptic operators are known to have a bounded $H^\infty$-calculus, see e.g. Amann, Hieber, Simonett \cite{2} for the case of differential operators. Already in 1971 Seeley \cite{17} had shown that differential boundary value problems have bounded imaginary powers, a property which is very close to that of having a bounded $H^\infty$-calculus and can often be shown by the same methods. Ellipticity, however, is not necessary in this context as shown in \cite{3}: a hypoellipticity condition in the spirit of Hörmander’s conditions (4.2)’ and
(4.4)’ in [11] is sufficient. In the present article, we establish the existence of a bounded $H^\infty$-calculus for a degenerate elliptic boundary value problem. We consider a strongly elliptic operator $A$, endowed with a boundary operator that, in general, will not satisfy the Lopatinsky-Shapiro ellipticity condition. The key point of our analysis then is the construction of a parameter-dependent parametrix to the resolvent with the help of Boutet de Monvel’s calculus for boundary value problems [4]. As a consequence of the non-ellipticity, however, this parametrix will only belong to an extended version of Boutet de Monvel’s calculus that we sketch, below. Still, this will enable us to deduce the necessary estimates for the existence of the bounded $H^\infty$-calculus.

Here are the details. Let $\Omega$ be a smooth compact manifold with boundary $\partial \Omega$ and $A$ a strongly elliptic second order partial differential operator on $\Omega$ which in local coordinates can be written in the form

$$A = \sum_{1 \leq k,l \leq n} a^{kl}(x) D_k D_l + \sum_{1 \leq k \leq n} b^k(x) D_k + c^0(x),$$

where $a^{kl}, b^k, c^0 \in C^\infty(\Omega)$ are real-valued, the matrix $(a^{kl}(x))_{1 \leq k,l \leq n}$ is positive definite with a uniform positive lower bound, and $D_k = -i \partial_\gamma x_k$. The operator $A$ is endowed with a boundary operator $T$ of the form

$$T = \varphi_0 \gamma_0 + \varphi_1 \gamma_1.$$

Here $\gamma_0$ denotes the trace operator and $\gamma_1$ the exterior normal derivative at $\partial \Omega$. Moreover, $\varphi_0, \varphi_1 \in C^\infty(\partial \Omega)$ are real-valued functions on the boundary with $\varphi_1 \geq 0$ and $\varphi_0 + \varphi_1 > 0$.

We obtain the classical Dirichlet problem for $\varphi_0 = 1, \varphi_1 = 0$. The choice $\varphi_0 = 0, \varphi_1 = 1$ yields Neumann boundary conditions, and Robin problems correspond to the case where $\varphi_1$ is nowhere zero.

For given functions $f$ and $\phi$ we consider the boundary value problem with spectral parameter $\lambda$

$$(A - \lambda)u = f \text{ in } \Omega, \quad Tu = \phi \text{ on } \partial \Omega,$$

in $L_p(\Omega)$, $1 < p < \infty$. To this end we introduce the $L_p$-realization of the above boundary value problem, i.e. the unbounded operator $A_T$, acting like $A$ on the domain

$$D(A_T) := \{ u \in L_p(\Omega) : Au \in L_p(\Omega), \, Tu = 0 \text{ on } \partial \Omega \}.$$ 

In the same spirit, we can consider the operator $A$, the boundary condition $T$ and the realization $A_T$ when $\Omega$ is the half-space $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_n \geq 0 \}$. In this case we assume the coefficient functions $a^{kl}, b^k, c^0, \varphi_0$ and $\varphi_1$ to be elements of $C^\infty_b(\mathbb{R}^n_+)$, i.e. bounded in all derivatives, and $\varphi_0 + \varphi_1 \geq c > 0$.

This problem has been investigated by many authors, see e.g. Egorov-Kondrat’ev [6], Kannai [12] or Taira [18], [19], [20], [21], also for the case where the boundary operator $T$ involves an additional first order tangential differential operator. This makes the analysis more subtle and will be treated in a subsequent publication.

We recall the notion of sectoriality:
Definition 1.1. A closed and densely defined operator $B : \mathcal{D}(B) \in E \to E$, acting in a Banach space $E$ that is injective with dense range is called sectorial of type $\omega < \pi$, if for every $\omega < \theta < \pi$ there exists a constant $C_\theta$, such that

$$\sigma(B) \subset \Lambda_\theta \quad \text{and} \quad \|\lambda(B - \lambda)^{-1}\|_{\mathcal{L}(E)} \leq C_\theta \quad \text{for all} \quad \lambda \in \mathbb{C}\setminus\Lambda_\theta.$$ 

Here $\Lambda_\theta = \{\lambda \in \mathbb{C}\setminus\{0\} : \arg(\lambda) \leq \theta\} \cup \{0\}$ is the sector of angle $\theta$ around the positive real axis.

It has been shown by Taira that, for a bounded domain $\Omega$, the $L_p$-realization $A_T$ is sectorial of type $\varepsilon$ for every $\varepsilon > 0$, possibly after replacing $A$ by $A + c$ for a positive constant $c$. In particular, it generates an analytic semigroup. For details see e.g. [20, Theorem 1.3].

Bounded $H^\infty$ calculus. By $H^\infty(\Lambda_\theta)$ we denote the space of bounded holomorphic functions in the interior of the sector $\Lambda_\theta$ and by $H^\infty_*(\Lambda_\theta)$ the subspace of all functions $f$ such that $|f(\lambda)| \leq C(|\lambda|^\varepsilon + |\lambda|^{-\varepsilon})^{-1}$ for suitable $C, \varepsilon > 0$. It is well-known that this is a dense subspace with respect to the topology of uniform convergence on compact sets.

For a sectorial operator $B$ of type $\omega$, $\theta' \in [\omega, \theta]$ and $f \in H^\infty_*(\Lambda_\theta)$ let

$$f(B) = \frac{i}{2\pi} \int_{\partial \Lambda_{\theta'}} f(\lambda)(B - \lambda)^{-1} d\lambda \in \mathcal{L}(E).$$

The integral exists due to the sectoriality and is independent of the choice of $\theta'$ by Cauchy’s integral theorem. Given $f \in H^\infty(\Lambda_\theta)$, we can approximate $f$ by a sequence $(f_n) \subset H^\infty_*(\Lambda_\theta)$ and define

$$f(B)x := \lim f_n(B)x \quad \text{for} \quad x \in \mathcal{D}(B) \cap \text{range}(B).$$

It can be shown that $\mathcal{D}(B) \cap \text{range}(B)$ is dense in $E$ and that the above equation defines a closable operator. The closure is again denoted by $f(B)$.

Definition 1.2. We say that a sectorial operator $B$ of type $\omega$ admits a bounded $H^\infty$ calculus of angle $\omega$, if for any $\omega < \theta < \pi$ there exists a constant $C_\theta > 0$, such that

$$\|f(B)\|_{\mathcal{L}(E)} \leq C_\theta\|f\|_{\infty}, \quad f \in H^\infty(\Lambda_\theta).$$

According to the principle of uniform boundedness it is sufficient to verify estimate (1.3) for all $f \in H^\infty_*(\Lambda_\theta)$.

Main results. We first consider the half-space situation.

Theorem 1.3. For $\Omega = \mathbb{R}^n_+$ let $A$ and $T$ be as in (1.1) and (1.2), respectively, and denote by $A_T$ the $L_p(\mathbb{R}^n_+)$ realization of $A$ with domain

$$\mathcal{D}(A_T) = \{u \in L_p(\mathbb{R}^n_+) : Au \in L_p(\mathbb{R}^n_+), Tu = 0\}.$$

Given $\varepsilon > 0$, the operator $A_T + c$ admits a bounded $H^\infty$-calculus of angle $\varepsilon$ for suitably large $c = c(\varepsilon) > 0$.

A corresponding result holds on manifolds:

Theorem 1.4. Let $\Omega$ be a smooth compact manifold with boundary, and let $A, T$ be as in (1.1) and (1.2), respectively. Given $\varepsilon > 0$, the operator $A_T + c$ admits a bounded $H^\infty$-calculus of angle $\varepsilon$ for suitably large $c = c(\varepsilon) > 0$. 

Outline of the paper. In order to establish (1.3) for $A_T + c$ we have to show that for every fixed $0 < \theta < \pi$

$$\left\| \int_{\partial \Lambda_\theta} f(\lambda) (A_T + c - \lambda)^{-1} d\lambda \right\|_{L^p(H^\gamma_0(\Omega))} \leq C \| f \|_{\infty}, \quad f \in H^\infty_+ (\Lambda_\theta).$$

It is clear that a good understanding of $(A_T + c - \lambda)^{-1}$ on the rays $\arg \lambda = \pm \theta$, $0 < \theta < \pi$ is essential for this task.

The main tool we use in this paper is Boutet de Monvel’s calculus for boundary value problems [4]. Details can be found e.g. in the monographs by Rempel and Schulze [15] and Grubb [8] or in the short introduction [16]. We will also need a slight generalization for which details will be given below.

Recall that an operator of order $m$ in Boutet de Monvel’s calculus on $\mathbb{R}^n_+$ is a matrix of operators

$$
\begin{pmatrix}
P_+ + G & K \\
T & S
\end{pmatrix} : S(\mathbb{R}^n_+, E_0) \oplus S(\mathbb{R}^{n-1}, E_0) \to S(\mathbb{R}^n_+, E_1) \oplus S(\mathbb{R}^{n-1}, E_1).
$$

Here $E_0$ and $E_1$ are vector bundles over $\mathbb{R}^n$, and $F_0$, $F_1$ are vector bundles over $\partial \mathbb{R}_+^n = \mathbb{R}^{n-1}$. Moreover, $P$ is a pseudodifferential operator on $\mathbb{R}^n$ satisfying the transmission condition, and $P_+$ denotes its truncation to $\mathbb{R}^n_+$: $P_+ = r^+ P e^+$, where $e^+$ denotes extension by zero from $S(\mathbb{R}^n_+, E_0)$ to, say, $L_2(\mathbb{R}^n, E_0)$, and $r^+$ denotes the restriction of distributions on $\mathbb{R}^n$ to those on $\mathbb{R}^n_+$. The operators $G$ and $T$ are singular Green and trace operators of order $m$ and class $d$, respectively; $K$ is a potential operator of order $m$. Finally $S$ is a pseudodifferential operator on the boundary $\mathbb{R}^{n-1}$ of order $m$.

Boutet de Monvel’s calculus is closed under compositions provided the vector bundles fit together. Via coordinate maps the calculus can be transferred to smooth compact manifolds with boundary. The operator $P_+$ then is the truncation of a pseudodifferential operator $P$ on a closed manifold $\overline{\Omega}$ of the same dimension, e.g. the double of $\Omega$.

Boutet de Monvel’s calculus has a symbolic structure with a notion of ellipticity, and there exist parametrices to elliptic elements in the calculus. Moreover, the calculus contains its inverses whenever these exist. An operator of order $m$ and class $d$ as above extends to a bounded map

$$H^{s+m}_p(\Omega, E_0) \oplus B^{s+m-1/p}_p(\partial \Omega, F_0) \to H^s_p(\Omega, E_1) \oplus B^{s-1/p}_p(\partial \Omega, F_1),$$

provided $s > d - 1 + 1/p$, where $H^s_p$ denotes the usual Sobolev space and $B^s_p = B^s_{p,p}$ the Besov space of order $s$, see Grubb [7].

It is well-known that, for $\theta > 0$, the operator

$$\begin{pmatrix} (A - \lambda)_+ \\ \gamma_0 \end{pmatrix} : L^p(\Omega) \oplus B^{2-1/p}_p(\partial \Omega)$$

will be invertible for $\lambda \in \Lambda_\theta$, $|\lambda|$ sufficiently large. In particular, it is invertible for all $\lambda \in \Lambda_\theta$, if we replace $A$ by $A + c$ for $c > 0$ sufficiently large. In order to keep the notation simple, we will assume from now on that $A$ has been replaced by $A + c$ for such $c$ and write $A$ instead of $A + c$.

Apart from the fact that $\gamma_0$ is formally not of the right order (which is of no importance here and can be easily be arranged), the problem fits into
Boutet de Monvel’s calculus and one obtains the inverse in the form
\[
\left(\begin{array}{c}
(A - \lambda)_{+}^{-1} \\
\gamma_0
\end{array}\right) = \left(\begin{array}{cc}
(A - \lambda)^{-1} + G^D_{\lambda} & K_{\lambda}^D
\end{array}\right).
\]
Here \((A - \lambda)^{-1}\) is the resolvent on the closed manifold \(\tilde{\Omega}\) and \(\mathbb{R}^n\), respectively, as explained above; see [8]. We will denote the corresponding truncation by \(Q_{\lambda,+}\):
\[
Q_{\lambda,+} = ((A - \lambda)^{-1})_{+}.
\]
As a consequence,
\[
\left(\begin{array}{c}
(A - \lambda)_{+}^{-1} \\
T
\end{array}\right) (Q_{\lambda,+} + G^D_{\lambda}) K_{\lambda}^D = \left(\begin{array}{cc}
I & 0 \\
T(Q_{\lambda,+} + G^D_{\lambda}) & TK_{\lambda}^D
\end{array}\right)
\]
Assuming that \(TK_{\lambda}^D\) is invertible with inverse \(S_\lambda\), we find that
\[
\left(\begin{array}{c}
(A - \lambda)_{+}^{-1} \\
T
\end{array}\right) = (Q_{\lambda,+} + G^D_{\lambda} - K_{\lambda}^D S_{\lambda} T (Q_{\lambda,+} + G^D_{\lambda}) K_{\lambda}^D S_{\lambda})
\]
For the realization \((A - \lambda)_{T}\) we obtain:
\[
(A - \lambda)_{T}^{-1} = Q_{\lambda,+} + G^D_{\lambda} - K_{\lambda}^D S_{\lambda} T (Q_{\lambda,+} + G^D_{\lambda})
\]
with
\[
G^T_{\lambda} = -K_{\lambda}^D S_{\lambda} T (Q_{\lambda,+} + G^D_{\lambda}).
\]

**Lemma 1.5.** For every choice of \(\theta \in ]0, \pi[\), there exists a constant \(C_\theta \geq 0\) such that
\[
\left\| \int_{\partial \Lambda_\theta} f(\lambda) (Q_{\lambda,+} + G^D_{\lambda}) \, d\lambda \right\|_{\mathcal{L}(L^p(\Omega))} \leq C_\theta \|f\|_{\infty} \quad \text{for all } f \in H^\infty(\Lambda_\theta)
\]

Lemma [5] is well-known, the proof relies on the fact that the operators \(Q_{\lambda}\) and \(G^D_{\lambda}\) are parameter-dependent operators of order \(-2\) in Boutet de Monvel’s calculus, if one writes \(-\lambda = \mu^2 e^{i\theta}\) and considers \(Q_{\lambda}\) and \(G^D_{\lambda}\) as functions of \(\mu\), see e.g. Grubb [8]. For the more general situation of a manifold with boundary and conic singularities, see [5].

It remains to study the term \(G^T_{\lambda}\). It will turn out that \(TK_{\lambda}^D\) is a hypoelliptic pseudodifferential operator of order 1 on the boundary. As we will see, it has a parametrix with local symbols in the Hörmander class \(S_{1,1/2}^0\) which then agrees with \(S_{\lambda}\) up to a regularizing operator. In order to treat the composition of \(S_{\lambda}\) with the operators \(K_{\lambda}^D\) and \(Q_{\lambda,+} + G^D_{\lambda}\), we will need an extension of the classical Boutet de Monvel calculus.

**2. An Extended Boutet de Monvel Type Calculus**

We recall the algebra \(\mathcal{H}\) of functions on \(\mathbb{R}\): It is the direct sum
\[
\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \oplus \mathcal{H}',
\]
where
\[
\mathcal{H}^+ := \{\mathcal{F}(e^+ u) : u \in \mathcal{S}(\mathbb{R}^+)\}, \quad \mathcal{H}^- := \{\mathcal{F}(e^- u) : u \in \mathcal{S}(\mathbb{R}^-)\}
\]
and $\mathcal{H}'$ is the space of all polynomials on $\mathbb{R}$. The sum is direct, since the functions in $\mathcal{H}^+$ and $\mathcal{H}^-_1$ decay to first order.

It will be helpful to use also weighted Sobolev spaces on $\mathbb{R}_+$: For $s = (s_1, s_2) \in \mathbb{R}^2$ we let $H^s_p(\mathbb{R}_+)$ denote the space of all $u \in \mathcal{D}'(\mathbb{R}_+)$ such that $\langle x \rangle^{s_2}u$ belongs to the ordinary Sobolev space $H^s_p(\mathbb{R}_+)$. We then have

\begin{align}
S(\mathbb{R}_+) &= \text{proj-lim} \ H^s_p(\mathbb{R}_+) \quad \text{and} \\
S'(\mathbb{R}_+) &= \text{ind-lim}(H^s_p(\mathbb{R}_+))' = \text{ind-lim} \ H^{s-1/p}_p(\mathbb{R}_+)
\end{align}

where the limits are taken over $s \in \mathbb{R}^2$ and $\dot{H}^s_q(\mathbb{R}_+)$ denotes all distributions $u$ in $H^s_q(\mathbb{R})$ for which $\text{supp } u \subseteq \overline{\mathbb{R}_+}$. 

**Operator-valued symbols.** Let $E, F$ be Banach spaces with strongly continuous group actions $\kappa^E_\rho, \kappa^F_\rho$, $\lambda > 0$. Given $q \in \mathbb{N}$, $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, we call a function $a = a(y, \eta) \in C^\infty(\mathbb{R}_q \times \mathbb{R}_q, \mathcal{L}(E, F))$ an operator-valued symbol in $S^m_{\rho, \delta}(\mathbb{R}_q \times \mathbb{R}_q; E, F)$ if, for all multi-indices $\alpha, \beta$ there exist constants $C_{\alpha, \beta}$ such that

$$
\| \kappa_{\rho, \delta}^{-1} D^\alpha_q D^\beta_y a(y, \eta) \kappa_{\rho, \delta} \|_{\mathcal{L}(E, F)} \leq C_{\alpha, \beta} \langle \eta \rangle^{m-\rho|\alpha|+\delta|\beta|}.
$$

In the sequel, we will mostly have the case where and $E$ and $F$ are either $\mathbb{C}$ or (weighted) Sobolev spaces over $\mathbb{R}$ or $\mathbb{R}_+$. On $\mathbb{C}$ we will use the trivial group action; on the Sobolev spaces we will use the action given by $\kappa_{\rho, \delta}(t) = e^{\sqrt{\lambda}t}$. Note that this group action is unitary on $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$. Using the representations (2.1) and (2.2), the above definition extends to the case, where $E = S(\mathbb{R}_+)$, $E = S'(\mathbb{R}_+)$ or $F = S(\mathbb{R}_+)$. 

**The transmission condition.**

**Definition 2.1.** A symbol $p \in S^m_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the transmission condition at $x_n = 0$ provided that, for all $k \in \mathbb{N}_0$

$$p_{\kappa^E_\rho}(x', \xi', \xi_n) := (\partial_{x_n}^k p)(x', 0, \xi', \xi_n) \in S^{m+\delta k}_{\rho, \delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \hat{\otimes} \mathcal{H}$$

We write $p \in P^m_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$. 

**Remark 2.2.** $P^\infty_{\rho, \delta} := \bigcup_m P^m_{\rho, \delta}$ is closed under the usual symbol operations, i.e. addition, pointwise multiplication and inversion, differentiation, Leibniz product and asymptotic summation. We also have $S^{-\infty} = \bigcap_m P^m_{\rho, \delta}$. 

**Theorem 2.3.** Let $p \in P^m_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$. Then

$$\text{op}_n(p)_+ \in S^m_{\rho, \delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+)).$$

**Proof.** This follows from the fact that $\kappa_{\rho, \delta}^{-1} \text{op}_n(p)\kappa_{\rho, \delta} = \text{op}_n(q, \xi)$, where $q_{x', \xi'}(x_n, \xi_n) = p(x', x_n/(\xi'), \xi', \xi_n)$ and the corresponding proof for Hörmander type $(1, 0)$; this is Theorem 2.12 in [16]. The arguments carry over to general $(\rho, \delta)$. \hfill $\square$

**Potential, trace and singular Green symbols.**

**Definition 2.4.** Let $m \in \mathbb{R}$, $d \in \mathbb{N}_0$. All functions, below, may be matrix valued.
• A function $k \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R})$ belongs to the space $K_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ of potential symbols of order $m$ and Hörmander type $(\rho, \delta)$, if

$$k_0(x', \xi'; \xi_n) := k(x', \xi'; \langle \xi' \rangle \xi_n) \in S_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \otimes H_{\xi_n}^+. $$

• A function $t \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R})$ belongs to the space $T_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ of trace symbols of order $m$, class $d$ and Hörmander type $(\rho, \delta)$, if

$$t_0(x', \xi'; \xi_n) := t(x', \xi'; \langle \xi' \rangle \xi_n) \in S_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \otimes H_{d-1}^-.$$

• A function $g \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R})$ belongs to the space $G_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ of singular Green symbols of order $m$, class $d$ and Hörmander type $(\rho, \delta)$, if

$$g_0(x', \xi'; \xi_n, \eta_n) := g(x', \xi'; \langle \xi' \rangle \xi_n, \langle \xi' \rangle \eta_n) \in S_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \otimes H_{\xi_n}^+ \otimes H_{d-1,-}^-.$$

The spaces $K_{m,0}^0$, $T_{m,0}^0$ and $G_{m,0}^0$ are denoted by Grubb in [8] as $S_{m}^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, H^+)$, $S_{m}^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, H^{-})$, and $S_{m}^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, H^+ \otimes H_{d-1}^-)$. Rempel and Schulze denote them in [15] by $\mathbb{S}^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $\mathbb{S}^{m,d}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and $\mathcal{B}^{m-d}(\mathbb{R}^{n-1} \times \mathbb{R}^{n+1})$. They are Fréchet spaces with the topologies induced by the scaled functions. For fixed $(x', \xi')$ the symbols above define Wiener-Hopf operators. Hence we get an action in the normal direction:

$$[\text{op}_n k](x', \xi') := r^+ F_{\xi_n \rightarrow x_n}^{-1} k(x', \xi') : \mathbb{C} \rightarrow S(\mathbb{R}^+),$$

$$[\text{op}_n t](x', \xi'; \xi_n) := I_{\xi_n}^+ t(x', \xi' ; \xi_n) \mathcal{F}_{y_n \rightarrow \xi_n} e^+ : S(\mathbb{R}^+) \rightarrow \mathbb{C} \text{ and}$$

$$[\text{op}_n g](x', \xi') := r^+ F_{\xi_n \rightarrow x_n}^{-1} I_{\eta_n}^+ g(x', \xi' ; \xi_n, \eta_n) \mathcal{F}_{y_n \rightarrow \eta_n} e^+ : S(\mathbb{R}^+) \rightarrow S(\mathbb{R}^+),$$

where $I^+$ is the plus-integral, see [8], p.166. We can interpret $\text{op}_n k$, $\text{op}_n t$ and $\text{op}_n g$ as operator-valued symbols. Depending on the class there are several extensions possible.

**Theorem 2.5** (Description by operator-valued symbols). Let $s \in \mathbb{R}^2$ with $s_1 > d - 1/2$. The following maps are bounded and linear.

1. $\text{op}_n : G_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \rightarrow S_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, S'(\mathbb{R}^+), S(\mathbb{R}^+))$
2. $\text{op}_n : G_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \rightarrow S_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, H^+_2(\mathbb{R}^+), S(\mathbb{R}^+))$
3. $\text{op}_n : K_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \rightarrow S_{\rho,\delta}^{m-1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{C}, S(\mathbb{R}^+))$
4. $\text{op}_n : T_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \rightarrow S_{\rho,\delta}^{m-1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; S'(\mathbb{R}^+), \mathbb{C})$
5. $\text{op}_n : G_{\rho,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \rightarrow S_{\rho,\delta}^{m-1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^+_2(\mathbb{R}^+), \mathbb{C})$

We omit the proof, which is straightforward. We also need the description via symbol-kernels. To this end we define

$$\tilde{K}_{\rho,\delta}^m := F_{\xi_n \rightarrow x_n}^{-1} K_{\rho,\delta}^m, \quad \tilde{T}_{\rho,\delta}^m := F_{\xi_n \rightarrow y_n}^{-1} T_{\rho,\delta}^m \text{ and } \tilde{G}_{\rho,\delta}^m := F_{\xi_n \rightarrow x_n}^{-1} F_{\eta_n \rightarrow y_n}^{-1} G_{\rho,\delta}^m.$$
Theorem 2.6 (Description by symbol-kernels). \( (i) \) For every operator-valued symbol \( k \in S^m_{\rho,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\mathbb{R}_+)) \) there exists a unique \( \tilde{k} \in \tilde{K}^m_{\rho,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \), such that
\[ k(x', \xi')\phi(x_n) = \tilde{k}(x', \xi'; x_n)\phi. \]
\( (ii) \) For every operator-valued symbol \( t \in S^m_{\rho,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathbb{C}) \) there exists a unique \( \tilde{t} \in \tilde{T}^m_{\rho,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \), such that
\[ t(x', \xi')u = \int_{\mathbb{R}_+} \tilde{t}(x', \xi'; y_n)u(y_n) dy_n. \]
\( (iii) \) For every operator-valued symbol \( g \in S^m_{\rho,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+)) \) there exists a unique \( \tilde{g} \in \tilde{G}^m_{\rho,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \), such that
\[ [g(x', \xi')u](x_n) = \int_{\mathbb{R}_+} \tilde{g}(x', \xi'; x_n, y_n)u(y_n) dy_n. \]

Proof. See Theorems 3.7 and 3.9 in [16]. \( \square \)

Corollary 2.7. The maps \((1), (3), (4)\) in Theorem 2.6 are bijections.

The maps \((2)\) and \((5)\) are bijections onto their image, which is
\[
\sum_{j=0}^{d} k_j \gamma_j^+ \text{ with } k_j \in K^{m-j}_{\rho,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \text{ resp.}
\]
\[
\sum_{j=0}^{d} s_j \gamma_j^+ \text{ with } s_j \in S^{m-j}_{\rho,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}),
\]

Proof. We get from symbols to operator-valued symbols, to symbol-kernels, and back to symbols by Theorem 2.6, Theorem 2.6, and the Fourier transform. For non-zero class we use the fact \( I^+ \xi^j Fe^+ \phi = (-i)^j \gamma_j^+ \phi \). \( \square \)

Boundary symbols and operators. We next define the space of boundary symbols of order \( m \), class \( d \) and Hörmander type \( (\rho, \delta) \) by
\[
\mathcal{BM}^{m,d}_{\rho,\delta} := \left( \mathcal{P}^{m,d}_{\rho,\delta} + \mathcal{G}^{m,d}_{\rho,\delta} + \mathcal{K}^m_{\rho,\delta} \times \mathcal{S}^m_{\rho,\delta} \right)
\]

It is clear for Theorems 2.5 and 2.3 that the action of \( b \in \mathcal{BM}^{m,d}_{\rho,\delta} \) in the normal direction defines a matrix of operator-valued symbols
\[
\text{op}_n(b) := \begin{pmatrix}
\text{op}_n(p) & \text{op}_n(g) & \text{op}_n(k) \\
\text{op}_n(t) & & \\
& & s
\end{pmatrix}
\]

We write \( B := \text{op}[\text{op}_n b] \in \mathcal{L}(\mathcal{S}(\mathbb{R}_+^n)) \) for the associated operator. We denote the components of \( B \) associated with \( p, g, k, t \) and \( s \) by \( P, G, K, T \) and \( S \), respectively. It is well-known that these operators form an algebra for Hörmander type \((1, 0)\). The proof given in [16] extends to the case \((\rho, \delta)\) with obvious modifications.

Theorem 2.8 (Composition). Composition yields a bilinear and continuous map
\[
\mathcal{BM}^{m,d}_{\rho,\delta} \times \mathcal{BM}^{m',d'}_{\rho,\delta} \rightarrow \mathcal{BM}^{m+m', \max(m+d', d)}_{\rho,\delta}, \ (b, b') \mapsto b \# b',
\]
where $\#$ is the Leibniz product of operator-valued symbols, given by the property that $\text{op}(\text{op}_n b) \text{op}(\text{op}_n b') = \text{op}(\text{op}_n b\# b')$. Moreover

$$b\# b' = pp' - p_0 p'_0 + b_0 \circ_n b'_0 + BM^{m+m'-(p-\delta),\max(m+d',d)}_\rho,$$

Here the subscript 0 denotes the restriction to $x_n = 0$ and $\circ_n$ denotes the point-wise composition, \[\text{Theorem 2.6.1}\].

The well-known mapping properties of Boutet de Monvel operators extend to operators of Hörmander type $(1,\delta)$. We refer to \[7\] for the proof of the following statement (in the case $\delta = 0$).

**Theorem 2.9.** Let $b \in BM^{m,d}_{1,\delta}$ and $s > d + 1/p - 1$. Then

$$B = \text{op}(\text{op}_n b) : H^s_p(\mathbb{R}^n) \oplus B^{s-1/p}(\mathbb{R}^n_n) \rightarrow H^s_p(\mathbb{R}^n_+) \oplus B^{s-m-1/p}(\mathbb{R}^n_+)$$

is bounded. The map $b \mapsto B$ is continuous.

**Remark 2.10.** The above calculus and the continuity properties naturally extend to the case of operators acting on vector bundles over compact manifolds with boundary.

### 3. The Spectral Parameter as a Co-variable

We write $-\lambda = e^{i\theta} \mu^2$. In this notation $a(x,\xi,\lambda) - \lambda = a_\theta(x,\xi,\lambda)|_{\xi=\mu}$, where

$$a_\theta(x,\xi,\zeta) = \sum_{k,l=1}^n a^{k l}(x)\xi_k \xi_l + \sum_{k=1}^n b^k(x)\xi_k + c^0(x) + e^{i\theta}\zeta^2.$$

We can interpret $a_\theta$ as a symbol in $n + 1$ dimensions that is independent of the extra space variable: $a_\theta \in S_{1,0}^2(\mathbb{R}^n \times \mathbb{R}^{n+1})$, because it is a polynomial in $(\xi,\zeta)$ of degree 2. The guiding idea is that fixing a co-variable (with no space dependence) commutes with the constructions on the symbol level. This idea goes back to Agmon who used it to obtain his famous a priori estimates for boundary value problem with spectral parameter. To be precise:

**Lemma 3.1.** For an operator-valued symbol $p \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n+1} ; E,F)$ we write $p_\mu := p(\cdot,\cdot,\mu) \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n ; E,F)$. Then

$$[p\# p']_\mu = p_\mu \# p'_\mu, \quad |p^{-\#}|_\mu = (p_\mu)^{-\#}.$$

provided the left hand sides are defined.

**Proof.** For fixed $\mu$ we have $c_\mu(\xi) \leq (\xi,\mu) \leq C_\mu(\xi)$, hence $p_\mu \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. For the composition we have

$$[p\# p']_\mu(x,\xi) = \int e^{-i\eta\zeta} e^{-i\xi \eta} p(x,\xi + \eta,\mu + \zeta) p'(x + y,\xi,\mu) \, dzd\zeta dyd\eta$$

$$= \int e^{-i\eta\zeta} p(x,\xi + \eta,\mu)p'(x + y,\xi,\mu) \, dyd\eta = p_\mu \# p'_\mu,$$

since the oscillatory integral is just restriction to $\zeta = 0$. The statement concerning the parametrix follows from the fact that restriction commutes with multiplication, differentiation and (asymptotic) summation. \[\square\]

This point of view is of interest, because it connects the typical pseudodifferential expansions with respect to decreasing symbol order with expansions with respect to decay in the spectral parameter. We will show:
Theorem 3.2. Let $0 \leq \delta < 1$.

(a) Let $p \in S_{1,\delta}^{-m}(\mathbb{R}^n \times \mathbb{R}^{n+1})$ and $m \geq 0$. Then
\[ \| P_{\mu} \|_{\mathcal{L}(L_p(\mathbb{R}^n))} \leq C |p|_{\star} |\mu|^{-m}. \]

(b) Let $g \in G_{1,\delta}^{-m,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ and $m > 0$. Then
\[ \| G_{\mu} \|_{\mathcal{L}(L_p(\mathbb{R}_1^n))} \leq C |g|_{\star} |\mu|^{-m}. \]

(c) Let $k \in K_{1,\delta}^{-m}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ and $m \geq 0$. Then
\[ \| K_{\mu} \|_{\mathcal{L}(\mathcal{B}_p^{-1/p}(\mathbb{R}^{n-1}); L_p(\mathbb{R}^n))} \leq C |k|_{\star} |\mu|^{-m}. \]

(d) Let $t \in T_{1,\delta}^{-m,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ and $m \geq 1$. Then
\[ \| T_{\mu} \|_{\mathcal{L}(L_p(\mathbb{R}_1^n)); \mathcal{B}_p^{-1/p}(\mathbb{R}^{n-1})} \leq C |t|_{\star} |\mu|^{-m+1}. \]

Here, $C$ denotes a suitable constant and $|p|_{\star}$, $|g|_{\star}$, $|k|_{\star}$, $|t|_{\star}$ suitable $s$-norms for $p$, $g$, $k$, and $t$, respectively.

Corollary 3.3. Let $m \geq m' \geq 0$. If $b \in \mathcal{BM}_{1,\delta}^{m',0}(\mathbb{R}^n \times \mathbb{R}^{n+1})$ has a parametrix $b^{-1} \# \in \mathcal{BM}_{1,\delta}^{-m,0}(\mathbb{R}^n \times \mathbb{R}^{n+1})$, then $B_{\mu} = \text{op} b_{\mu}$ is invertible for large $\mu$, and $\| B_{\mu}^{-1} - B_{\mu}^{-1} \|_{\mathcal{L}(L_p(\mathbb{R}^n) \oplus \mathcal{B}_p^{-1/p}(\mathbb{R}^{n-1})))} \leq C |b|_{\star} |\mu|^{-N}$ for all $N \in \mathbb{N}_0$, where $B_{\mu}^{-1} = \text{op} b_{\mu}^{-1}$. \hfill \Box

Proof. By assumption $b^{-1} b^{-1} = 1 + r$ with $r \in \mathcal{BM}_{1,\delta}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^{n+1})$. Lemma 3.1 shows that $B_{\mu} B_{\mu}^{-1} = 1 + R_{\mu}$, and $\| R_{\mu} \|_{\mathcal{L}(L_p(\mathbb{R}^n) \oplus \mathcal{B}_p^{-1/p}(\mathbb{R}^{n-1})))} \leq C |\mu|^{-N}$ for all $N \in \mathbb{N}_0$ by Theorem 2.2. For sufficiently large $\mu$ we have $(1 + R_{\mu})^{-1} = 1 + \sum_{j} (-R_{\mu})^j$. Hence $B_{\mu}$ has a right inverse for large $\mu$. In the same way we obtain a left inverse, and $B_{\mu}^{-1} = B_{\mu}^{-1} + B_{\mu}^{-1} \sum_{j} (-R_{\mu})^j$. Clearly the second summand is rapidly decreasing in $\mu$. \hfill \Box

Since there is no dependence on the space variable $z$ we can interpret a pseudodifferential operator $P$ with symbol in $S_{1,\delta}^0(\mathbb{R}^n \times \mathbb{R}^{n+1})$ as a pseudodifferential operator on the cylinder $\mathbb{R}^n \times S_T$, where $S_T$ is the circle with radius $T/2\pi$. Then we obtain:

Lemma 3.4. If $p \in S_{1,\delta}^0(\mathbb{R}^n \times \mathbb{R}^{n+1})$, then for all $T > 0$ we have
\[ P := \text{op}(p) \in \mathcal{L}(L_p(\mathbb{R}^n \times S_T)) \text{ and } \| P \|_{\mathcal{L}(L_p(\mathbb{R}^n \times S_T))} \leq C |p|_{\star}. \]

Here $C$ is a constant independent of $T$.

Proof. We first note that $P$ preserves $T$-periodicity:
\[ [P u](x, z + kT) = \int e^{i(x-y)\xi + i(|z+kT| - w)\zeta} p(x, \xi, \zeta) u(y, w) dy dw d\xi d\zeta \]
\[ = \int e^{i(x-y)\xi + i(z-w)\zeta} p(x, \xi, \zeta) u(y, w) dy dw d\xi d\zeta \]
\[ = \int e^{i(x-y)\xi + i(z-w)\zeta} p(x, \xi, \zeta) u(y, \bar{w}) dy d\bar{w} d\xi d\zeta \]
\[ = [P u](x, z). \]
We identify \( u \in L_p(\mathbb{R}^n \times S_T) \) with a \( T \)-periodic function and write
\[
u = \sum_{j \in \mathbb{Z}} u_j \quad \text{with} \quad u_j(x, z) := u|_{\mathbb{R}^n \times [-T/2,T/2]}(x, z - Tj).
\]

Note that for every \( j \in \mathbb{Z} \) we have \( u_j \in L_p(\mathbb{R}^n \times \mathbb{R}) \) and \( \|u_j\|_{L_p(\mathbb{R}^n \times \mathbb{R})} = \|u\|_{L_p(\mathbb{R}^n \times S_T)} \). The integral kernel \( k = k(x, z, y, w) \) of the pseudodifferential operator \( P \) is given by
\[
k(x, z, y, w) = \int \int e^{i(x-y)\xi + i(z-w)\zeta} p(x, \xi, \zeta) \, d\xi d\zeta.
\]

Since \( p \) is of order zero, we obtain the estimate
\[
|k(x, z, y, w)| \leq C|p|_* (|x-y|^2 + |z-w|^2)^{-l/2}
\]
for all even \( l \in \mathbb{N} \) with \( l > n \) with a suitable seminorm \( |p|_* \) for \( p \). For \( j \geq 2 \), \( z \in [-T/2,T/2] \) and \( w \in \text{supp} \, u_j \) we have \( |z-w| \geq (j-1)T \), hence
\[
|k(x, z, y, w)| \leq C|p|_* (|x-y|^2 + (|j|-1)^2 T^2)^{-(n+2)/2}
\]
\[
\leq C|p|_*(|j|-1)T)^{-(n+2)}(|x-y|/(|j|-1)T)^{-(n+2)}.
\]

(\text{Constants may vary from line to line.}) We write \( \chi_j \) for the indicator function of \([-T/2+jT,T/2+jT]\). A quick computation shows that
\[
\int \chi_0(z) |k(x, z, y, w)||\chi_j(w)\, dwdy \leq C|p|_* T^{-1}(|j|-1)^{-2} \quad \text{and}
\]
\[
\int \chi_0(z) |k(x, z, y, w)||\chi_j(w)\, dzdx \leq C|p|_* T^{-1}(|j|-1)^{-2}.
\]

Hence we get \( L_p \)-estimates by Schur’s test. More explicitly:
\[
\|P u_j\|_{L_p(\mathbb{R}^n \times S_T)} = \|\chi_0 P \chi_j u_j\|_{L_p(\mathbb{R}^n \times \mathbb{R})}
\]
\[
\leq C|p|_* T^{-1}(|j|-1)^{-2} \|u_j\|_{L_p(\mathbb{R}^n \times \mathbb{R})} = C|p|_* T^{-1}(|j|-1)^{-2} \|u\|_{L_p(\mathbb{R}^n \times S_T)}
\]
In particular the right hand side is summable and for \( T \geq 1 \) we get
\[
\|Pu\|_{L_p(\mathbb{R}^n \times S_T)} = \sum_{j \in \{-1,0,1\}} \|Pu_j\|_{L_p(\mathbb{R}^n \times S_T)} + \sum_{|j| \geq 2} \|Pu_j\|_{L_p(\mathbb{R}^n \times S_T)}
\]
\[
\leq C\left(3|p|_* \|u\|_{L_p(\mathbb{R}^n \times S_T)} + 2 \sum_{j \in \mathbb{N}} j^{-2} |p|_* \|u\|_{L_p(\mathbb{R}^n \times S_T)}\right)
\]
\[
\leq C|p|_* \|u\|_{L_p(\mathbb{R}^n \times S_T)}
\]

We still need to prove that the bound also holds for \( T < 1 \). Choose \( N \in \mathbb{N} \) so large that \( NT \geq 1 \), and consider a \( T \)-periodic function as an \( NT \)-periodic function. We have \( \|u\|_{L_p(\mathbb{R}^n \times S_{NT})} = N^{-1/p} \|u\|_{L_p(\mathbb{R}^n \times S_T)} \) and hence, by the above argument,
\[
\|Pu\|_{L_p(\mathbb{R}^n \times S_{NT})} = N^{-1/p} \|Pu\|_{L_p(\mathbb{R}^n \times S_T)}
\]
\[
\leq C|p|_* N^{-1/p} \|u\|_{L_p(\mathbb{R}^n \times S_{NT})} = C|p|_* \|u\|_{L_p(\mathbb{R}^n \times S_T)}
\]
for a constant \( C \) independent of \( NT \).
Proof of Theorem 3.2 Let us first assume that \( p \in S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n+1}) \). We write \( e_\mu \) for the \( 2\pi/\mu \)-periodic function \( [x \mapsto e^{i\mu x}] \). For \( u \in L_\mu(\mathbb{R}^n) \) we have

\[
[P(u \otimes e_\mu)](x, z) = \int e^{i\xi x + i\zeta z} p(x, \xi, \zeta) [\mathcal{F} u](\xi) \otimes \delta_\mu(\zeta) \, d\zeta \, dx = e_\mu(z)[P_\mu u](x).
\]

Taking the \( L_p \)-norm, we obtain

\[
\|P(u \otimes e_\mu)\|_{L_p(\mathbb{R}^n \times S_{2\pi/\mu})} = \|[P_\mu u] \otimes e_\mu\|_{L_p(\mathbb{R}^n \times S_{2\pi/\mu})} = \|P_\mu u\|_{L_p(\mathbb{R}^n)} \|e_\mu\|_{L_p(S_{2\pi/\mu})}.
\]

Since \( P \) is of order zero, Lemma 3.3 yields

\[
\|P_\mu u\|_{L_p(\mathbb{R}^n)} \|e_\mu\|_{L_p(S_{2\pi/\mu})} \leq C|\mu|^\alpha \|u\|_{L_p(\mathbb{R}^n)} \|e_\mu\|_{L_p(S_{2\pi/\mu})};
\]

and part (a) follows for \( m = 0 \). For \( m < 0 \) we can use what we did so far to reduce to the case \( p(x, \xi, \mu) = \langle \xi, \mu \rangle^{-m} \). But for this symbol the statement is a consequence of the \( L_p \)-mapping property of pseudodifferential operators and the following simple estimates.

\[
|D^\xi_\mu \langle \xi, \mu \rangle^{-m}| \leq C_\alpha \langle \xi, \mu \rangle^{-m-|\alpha|} \leq C_\alpha \langle \mu \rangle^{-m-|\xi|-|\alpha|}.
\]

Now for part (b). We recall that \( \tilde{g} \in \widetilde{G}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^n) \) satisfies the estimates

\[
\|D^\xi_\mu D^{\mu_0}_\nu y_{\mu_0} D^n_\zeta \tilde{g}_\mu(x', \xi', \eta')\|_{L_1(\mathbb{R}^n)} \leq C|g|_\ast \langle \xi', \mu \rangle^{-|\alpha|+\delta}|\xi|+1+|\nu|-1+|\nu|+1+|\mu|,
\]

\[
\|D^\xi_\mu D^{\mu_0}_\nu y_{\mu_0} D^n_\zeta \tilde{g}_\mu(x', \xi', \eta')\|_{L_1(\mathbb{R}^n)} \leq C|g|_\ast \langle \xi', \mu \rangle^{-|\alpha|+\delta}|\xi|+1+|\nu|-1+|\nu|+1+|\mu|.
\]

So Schur’s test implies that

\[
\|D^n_\zeta \mathcal{O} \tilde{g}_\mu \|_{L_p(\mathbb{R}^n)} \leq C|g|_\ast \langle \xi', \mu \rangle^{-|\alpha|}.
\]

We are interested in the integral kernel

\[
K(x', y', \mu) = \int e^{i(x'-y') \zeta'} \mathcal{O} \tilde{g}_\mu(x', \xi') \, d\xi' = \int L^N \left( e^{i(x'-y') \zeta'} - 1 \right) \mathcal{O} \tilde{g}_\mu(x', \xi') \, d\xi'.
\]

with \( N \in \mathbb{N} \) and \( L := \sum_{|\alpha|=1}^{m} \frac{\langle x'-y' \rangle^\alpha}{|x'-y'|^2} D_\zeta^\alpha \). We take \( N = n - 1 \) and use the fact that \( |e^{it} - 1| \leq 2|t|^\delta \) for \( 0 < \delta < \min(1, |m|) \), to get

\[
\|K(x', y', \mu)\|_{L_p(\mathbb{R}^n)} \leq C|g|_\ast \langle x' - y' \rangle^{-n+1+\frac{\delta}{2}} \int |\xi|^\delta \langle \xi, \mu \rangle^{-n+1} \, d\xi' \leq C|g|_\ast \langle x' - y' \rangle^{-n+1+\delta} \langle \mu \rangle^{m+\delta}.
\]

Choosing \( N = n \) we get \( \|K(x', y', \mu)\|_{L_p(\mathbb{R}^n)} \leq C|g|_\ast \langle x' - y' \rangle^{-n} \langle \mu \rangle^{m-1} \). The first estimate for \( \langle \mu \rangle|x' - y'| \leq 1 \) and the second for \( \langle \mu \rangle|x' - y'| > 1 \) imply

\[
\|K(x', \cdot, \mu)\|_{L_1(S^n-1;L_p(\mathbb{R}^n))} \leq C|g|_\ast \langle \mu \rangle^m \quad \text{and}
\]

\[
\|K(\cdot, y', \mu)\|_{L_1(S^n-1;L_p(\mathbb{R}^n))} \leq C|g|_\ast \langle \mu \rangle^m.
\]

In fact, this follows from the the identities

\[
\int_{|x' - y'| \leq 1} |x' - y'|^{-n+1+\delta} \langle \mu \rangle^\delta \, dx' = \int_{|y'| \leq 1} |w'|^{-n+1+\delta} \, dw' < \infty
\]

and

\[
\int_{|x' - y'| \leq 1} (\langle \mu \rangle|x' - y'|)^{-n+1+\delta} \langle \mu \rangle^\delta \, dx' = \int_{|w'| \leq 1} |w'|^{-n+1+\delta} \, dw' < \infty.
\]
\[
\int_{|y' - x'| \geq 1} |x' - y'|^{-n} \langle \mu \rangle^{-1} \, dx' = \int_{|w| \geq 1} |w'|^{-n} \, dw' < \infty
\]

Hence the assertion follows with Schur's test.

For part (c): We recall the well-known fact that every potential operator \( K \) can be written as \( r^+ P \tilde{\gamma}^*_0 \), where \( P \) is a pseudodifferential operator of order \(-m - 1\) whose symbol-kernel is given by \( \tilde{p} = E\tilde{\tilde{k}} \); \( E \) is Seeley's extension operator applied to \( x_n \), and \( \tilde{\gamma}^*_0 \) is the adjoint to the evaluation \( \tilde{\gamma}_0 : H^*_p(\mathbb{R}^n) \to B^{s-1/p}(\mathbb{R}^{n-1}), s > 1/p \). It is clear that \( K_\mu = r^+ P \tilde{\gamma}^*_0 \). The map

\[ S^{-1}_1(\mathbb{R}^n \times \mathbb{R}^{n+1}) \ni (\xi, \zeta)^{-1} \mapsto (\xi, \mu)^{-1} \in S^{-1}_1(\mathbb{R}^n \times \mathbb{R}^n) \]

is uniformly bounded with respect to \( \mu \). In view of the continuity of \( \tilde{\gamma}^*_0 \) from \( B^{-1/p}(\mathbb{R}^{n-1}) \) to \( H^{-1}(\mathbb{R}^n) \) we have

\[ \| \text{op}(\langle \xi, \mu \rangle^{-1}) \tilde{\gamma}^*_0 \|_{L^p(L^p(\mathbb{R}^n))} \leq C. \]

Define \( q = p^\# \langle \xi, \zeta \rangle^1 \in S^{-m}_1(\mathbb{R}^n \times \mathbb{R}^{n+1}) \). By part (a)

\[ \|Q_\mu\|_{L^p(\mathbb{R}^n)} \leq C |q|_s \langle \mu \rangle^{-m} \leq C |k|_s \langle \mu \rangle^{-m}. \]

The estimate for \( T_\mu \) follows.

For part (d) we use a similar approach. We write \( T = \gamma_0 P e^+, \) where \( P \) is a pseudodifferential operator of order \( m \) with symbol-kernel \( \tilde{p} = E\tilde{\tilde{k}} \). Clearly \( T_\mu = \gamma_0 P e^+ \). By the same argument as in part (c) we have

\[ \| \gamma_0 \text{op}(\langle \xi, \mu \rangle^{-1}) \|_{L^p(L^p(\mathbb{R}^n); B^{-1/p}(\mathbb{R}^{n-1}))} \leq C. \]

Define \( q = \langle \xi, \zeta \rangle^1 \# p \in S^{-m+1}_1(\mathbb{R}^n \times \mathbb{R}^{n+1}) \). By part (a)

\[ \|Q_\mu\|_{L^p(\mathbb{R}^n)} \leq C |q|_s \langle \mu \rangle^{-m} \leq C |k|_s \langle \mu \rangle^{-m+1}. \]

The estimate for \( T_\mu \) follows. \( \square \)

4. The Principal Symbol of the Degenerate Singular Green Operator

We will now apply Agmon’s trick to our problem. We introduce the operator \( A_\theta := A + e^{i\theta} D_x^2 \) acting on \( \mathbb{R}^n_+ \times \mathbb{R} \). The symbol of \( A_\theta \) is \( a_\theta(x, \xi, \zeta) = a(x, \xi) + e^{i\theta} \xi^2 \in S^2_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n+1}) \), where \( a(x, \xi) \) is the symbol of \( A \).

After possibly replacing \( A \) by \( A + c \) for some positive constant \( c \) we may assume that the Dirichlet problem for \( A_\theta \) is invertible. In the introduction we already pointed out that the solution operator to the Dirichlet problem is an operator in the Boutet de Monvel calculus, i.e.

\[ (A_\theta)^{-1} + \gamma_0 = (Q_{\theta^+, \theta} + G^D_{\theta} - K^D_{\theta^+}) \]

We will need the principal symbols of the operators \( G^D_{\theta} \) and \( K^D_{\theta^+} \) and collect the results to fix some notation.

**Remark 4.1.**

(a) For fixed \((x', \xi')\), the restriction to the boundary of the principal symbol of \( A_\theta \) is a polynomial of degree two in \( \xi_n \). It therefore has two roots, say \( \pm i\kappa_\theta^\pm (x', \xi', \zeta) \), with \( \Re \kappa_\theta^\pm \geq 0 \).

(b) We have \( \kappa_\theta^\pm \in S^1_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n) \). Both are strongly elliptic, i.e. \( \Re \kappa_\theta^\pm \geq 5\omega|\xi', \zeta| \) for suitable \( \omega > 0 \).
(c) The principal symbol of $K_D^T$ is $K^0_D (\mathbb{R}^{n-1} \times \mathbb{R}^n)$ is $(\kappa_0^+ + i \xi_n)^{-1}$.
(d) The principal symbol of $G_D^T$ is $G_{\theta}^T (\mathbb{R}^{n-1} \times \mathbb{R}^n)$ is $a_n^{-1} (\kappa_0^- + \kappa_0) (\kappa_0^- + i \xi_n)^{-1} (\kappa_0^- - i \eta_n)^{-1}$.

Due to Agmon’s trick we have $G_T^T = G_{\theta,\mu}^T$ mod $\mathcal{O}(\lambda^{-N})$ for all $N \in \mathbb{N}$, with $-\lambda = e^{\mu t^2}$ sufficiently large and

$$G_T^T = -K_D^T (TK_D^T)^{\#} T(A_{\theta}^- + G_D^T).$$

**Lemma 4.2.** The operator $G_{\theta}^T$ is a singular Green operator with symbol $g_{\theta}^T \in \mathcal{G}_{1,1/2} (\mathbb{R}^{n-1} \times \mathbb{R}^n)$ and principal symbol

$g_{\theta}^T(x', \xi', \zeta, \eta_n) = s_0(x', \xi', \zeta) (\kappa_0^+ (x', \xi', \zeta) + i \xi_n)^{-1} (\kappa_0^- (x', \xi', \zeta) - i \eta_n)^{-1}$

for suitable $s_0 \in S^{-1}_{1,1/2} (\mathbb{R}^{n-1} \times \mathbb{R}^n)$. The corresponding symbol-kernel is

$g_{\theta}^T(x', \xi', \zeta; x_m, y_n) = s_0(x', \xi', \zeta) e^{-\kappa_0^+ (x', \xi', \zeta) x_m} e^{-\kappa_0^-(x', \xi', \zeta) y_n}$.

**Proof.** Modulo smoothing operators $G_D^T$ is the composition of the potential operator $K_D^T$, a parametrix $S_D^T$ to the pseudodifferential operator $TK_D^T$ on the boundary, multiplication by the function $\varphi_1$ introduced in (1.2) and the trace operator $\gamma_1 (Q_{\theta} + G_{\theta}^T)$. Note that $Q_{\theta} + G_{\theta}^T$ maps into the kernel of $\gamma_0$ so that there is no contribution from $\varphi_0 \gamma_0$. Hence the principal symbol of $G_{\theta}^T$ is given by multiplication of the principal symbols of these operators. So for the proof of the lemma it is sufficient to combine the following three statements.

(i) $K_D^T = op k_\theta$ with $k_\theta \in K^0_D (\mathbb{R}^{n-1} \times \mathbb{R}^n)$ and principal symbol

$k_{\theta(0)}(x', \xi', \zeta, \eta_n) = (\kappa_0^+ (x', \xi', \zeta) + i \xi_n)^{-1}$

which is Remark 4.1(c).

(ii) The symbol $s_\theta \# \varphi_1$ of $S_\theta \varphi_1$ is an element of $S^{-1}_{1,1/2} (\mathbb{R}^{n-1} \times \mathbb{R}^n)$. This is the content of Lemma 4.3 below.

(iii) $\gamma_1 (Q_{\theta} + G_{\theta}^T) = op t_\theta$ with $t_\theta \in \mathcal{G}^{-1,0}_{1,0}$ and principal symbol

$t_{\theta(-1)}(x', \xi', \zeta, \eta_n) = -a_n(x')^{-1} (\kappa_0^- (x', \xi', \zeta) - i \xi_n)^{-1}$

which follows from Remark 4.1 and the composition rules.

The parametrix of $TK_D^T$. We recall a sufficient condition for the existence of a parametrix.

**Theorem 4.3 (Parametrix).** Let $m \geq 0$ and $p \in S^m_{1,0} (\mathbb{R}^n \times \mathbb{R}^n)$. Suppose there exists a $0 \leq \delta < 1$, such that for sufficiently large $|\xi|$ we have the estimates

$|p(x, \xi)| \geq c$ and

$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi) p(x, \xi)^{-1}| \leq C |\xi|^{-|\alpha|+|\beta|}$ for all $\alpha, \beta \in \mathbb{N}^n$.

Then there exists a parametrix $p^{-\#} \in S^0_{2,\delta} (\mathbb{R}^n \times \mathbb{R}^n)$, i.e.,

$p^{-\#} \# p = 1 + r_1$ and $p^{\#} p^{-\#} = 1 + r_2$,

with $r_1, r_2 \in S^{-\infty} (\mathbb{R}^n \times \mathbb{R}^n)$.
Moreover, Lemma 4.4.  \( TK \) operator \( S \) the Robin and the degenerate boundary value problem is the order of the Robin operator \( S \) which here is zero due to the zeros of \( \varphi_1 \) and the resulting loss of ellipticity. The key observation is that we gain back the loss in order by composing with the multiplication operator \( \varphi_1 \).

**Proof.** We want to show that the symbol of \( \Sigma \) satisfies inequalities (4.3) and (4.4). Write

\[
TK^D = \varphi_1 \gamma_1 K^D + \varphi_0 \gamma_0 K^D = \varphi_1 \Pi_0 + \varphi_0,
\]

where \( \Pi_0 := \gamma_1 K^D \) is the Dirichlet-to-Neumann operator. It is well-known and a consequence of Remark 4.1(c) that its symbol \( \pi_0 \) is an element of \( S_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n) \); its principal symbol is \( \kappa^D_0 \). By Remark 4.1(b) we have \( \Re \pi_0 \geq 1 \) for sufficiently large \( |\xi, \zeta| \) and hence for the symbol \( \sigma_0 \) of \( \Sigma \)

\[
|\sigma| \geq |\Re(\varphi_1 \sigma_0 + \varphi_0)| = \varphi_1 \Re \sigma_0 + \varphi_0 \geq \varphi_1 + \varphi_0 > 0.
\]

We have to verify the estimates

\[
|\partial_\xi \partial_\zeta |^l \sigma_0^{-1}| \leq (\xi', \zeta')^{-|\alpha|-l+|\beta|/2} \text{ for all } \alpha, \beta \in \mathbb{N}_0^{n-1}, l \in \mathbb{N}_0.
\]

The estimate is trivial for \( |\beta| \geq 2 \), because \( \sigma_0 \in S_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n) \) and \( |\sigma_0^{-1}| \leq C \) by Equation (4.5). Equation (4.5) also shows that \( (\varphi_1 \sigma_0)^{k/2} \sigma_0^{-1} \) is bounded for \( k = 1, 2 \). The ellipticity of \( \pi_0 \) implies that \( |\pi_0|^{-k/2} \leq (\xi', \zeta')^{-k/2} \). We obtain the remaining estimates with the help of the inequality \( |\partial_\xi \varphi_1(t)|^2 \leq ||\varphi_1'(t)||_{\infty} |\varphi_1(t)| \):

\[
|\partial_\xi \partial_\zeta |^l \sigma_0^{-1}| = |\varphi_1 \partial_\xi \partial_\zeta \pi_0 \sigma_0^{-1}| = |\partial_\xi \partial_\zeta \pi_0 \sigma_0^{-1}| |\varphi_1 \sigma_0(\varphi_1 \sigma_0 + \varphi_0)^{-1}| \leq (\xi', \zeta')^{-|\alpha|-l}
\]

and

\[
|\partial_\xi \partial_\zeta |^l \sigma_0^{-1}| \equiv |\partial_\xi \varphi_1 \partial_\xi \partial_\zeta \pi_0 \sigma_0^{-1}| \leq (|\varphi_1 \pi_0|^{1/2} |\sigma_0|^{-1/2} ||\varphi_1 \sigma_0||^{-1/2} |\partial_\xi \partial_\zeta |^{1/2} |\sigma_0| \leq (\xi', \zeta')^{1/2-|\alpha|-l}.
\]

Here \( \equiv \) means equality modulo terms that satisfy the estimate. Hence by Theorem 4.3 there exists a parametrix \( \sigma_0^{\#} \in S_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n) \). We still need to show that multiplication by \( \varphi_1 \) reduces the order. As \( \pi_0 \) is elliptic, there exists a parametrix \( \pi_0^{\#} \) such that \( \pi_0 \pi_0^{\#} - 1 = r_0 \) is regularizing, and we find that

\[
\varphi_1 = \sigma_0^{\#} \pi_0^{\#} - \varphi_0^{-1} r_0 - \varphi_0 \pi_0^{\#}.
\]

Composition with \( \varphi_1 \) or \( \varphi_0 \) from the left is just pointwise multiplication. Hence we obtain the improved order of \( s_0 \# \varphi_1 \) from the identities

\[
s_0 \# \varphi_1 \equiv \pi_0^{\#} \# [\sigma_0 \# \pi_0^{\#} - \varphi_0 \pi_0^{\#}] \quad \text{mod } S^{-\infty}(\mathbb{R}^{n-1} \times \mathbb{R}^n)
\]

\[
\equiv \pi_0^{\#} \# \sigma_0^{\#} \# \varphi_0 \pi_0^{\#} \quad \text{mod } S^{-\infty}(\mathbb{R}^{n-1} \times \mathbb{R}^n).
\]

As \( \varphi_0 \pi_0^{\#}, \pi_0^{\#} \in S_{1,0}^{-1} \) and \( \sigma_0^{\#} \in S_{1,1/2}^0 \), this completes the proof. \( \square \)
5. Proof of the Main Results

The main technical difficulty of this section is an estimate similar to (1.4) for the operator $G_{\pm \theta(-2)}^T$ associated with the principal symbol of $G_{\pm \theta}^T$ which in view of Lemma 4.2 has symbol kernel

$$(5.1) \quad \tilde{g}_{\pm \theta(-2)}^T(x', \xi', \zeta; x_n, y_n) = s_{\pm \theta}(x', \xi', \zeta)e^{-\kappa_{\pm \theta}^+(x', \xi', \zeta)x_n}e^{-\kappa_{\pm \theta}^-(x', \xi', \zeta)y_n}.$$ 

Here $s_{\pm \theta} \in S_{-1}^{-1}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$. We will prove the following.

**Lemma 5.1.** Let $G_{\pm \theta(-2)}^T$ be as above. We define

$$I_{\pm \theta} := \int_{\mathbb{R}_+} \mu G_{\pm \theta(-2), \mu} \, d\mu.$$ 

Then $I_{\pm \theta} \in \mathcal{L}(L_p(\mathbb{R}_+^n))$ and $\|I_{\pm \theta}\|_{\mathcal{L}(L_p(\mathbb{R}_+^n))} \leq C$ with a constant $C$ that only depends on $\omega$, $|s_{\pm \theta}|$, and $|\kappa_{\pm \theta}^\pm|$.

Given the lemma we can prove Theorem 1.3.

**Proof of Theorem 1.3.** In the outline of the paper we pointed out that it is sufficient to establish Estimate (1.4). According to Lemma 1.5, we only need to consider (1.4) with $A_T + c$ replaced by $G_{\pm \theta}^T$. We choose the parametrisation $\lambda = -e^{\mp \mu^2}$ for the rays $\arg \lambda = \pm \theta$. Hence

$$(5.2) \quad G_{\pm \theta}^T = G_{\pm \theta, \mu}^T + G_{\pm, \infty}^T = G_{\pm \theta(-2), \mu}^T + G_{\pm \theta, \mu}^T,$$

where $G_{\pm \theta}^T$ is given by Lemma 4.2 and $\|G_{\pm \theta, \mu}^T\|_{\mathcal{L}(L_p(\mathbb{R}_+^n))}$ is rapidly decaying in $\lambda$. Moreover $G_{\pm \theta(-2), \mu}^T$ is the operator associated to the principal symbol, and the $L_p$ norm of the remainder term $G_{\pm \theta, \mu}^T$ decays like $(\lambda)^{-1+1/4}$ by Theorem 3.2 (b). In particular Estimate (1.4) holds for the remainder term. In the chosen parametrisation

$$\left\|2e^{i\theta} \int_{\mathbb{R}_+} G_{\pm \theta(-2), \mu}^T f(e^{i\theta}) \mu \, d\mu \right\|_{\mathcal{L}(L_p(\mathbb{R}_+^n))} \leq 2\|f\|_{L_p(\mathbb{R}_+^n)}.$$ 

By the lemma the right hand side is bounded by $C\|f\|_{L_p(\mathbb{R}_+^n)}$. We obtain (1.4) by summing up the partial results. \qed

For the proof of Lemma 5.1, we need the following.

**Lemma 5.2.** Let $\chi \in C^\infty(\mathbb{R})$ with $\chi(r) = 0$ for $\left| r \right| \leq 1$ and $\chi(r) = 1$ for $\left| r \right| \geq 2$. For $\omega$ as in Remark 4.1 define

$$(5.3) \quad \tilde{h}(x', \xi', \zeta; x_n, y_n) := \tilde{g}_{\pm \theta(-2)}^T(x', \xi', \zeta; x_n, y_n) \chi(|\xi'|, |\zeta|) \zeta e^{i\omega \zeta(x_n+y_n)}.$$ 

Then the map

$$\mathbb{R}_{++}^2 \ni (x_n, y_n) \mapsto \tilde{h}(x', \xi', \zeta; x_n, y_n) \in S_{1/2}^0(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

is uniformly bounded. The bounds on the seminorms only depend on the lower bound of the real part of $\kappa_{\pm \theta}^{\pm}$ and the seminorms $|s_{\pm \theta}|$ and $|\kappa_{\pm \theta}^{\pm}|$.

**Proof.** Clearly, $\zeta s_{\pm \theta}(x', \xi', \zeta) \in S_{1/2}^0(\mathbb{R}^{n-1} \times \mathbb{R}^n)$. For $|\xi'|, |\zeta| > 1$ we have $\Re \kappa_{\pm \theta}^{\pm} \geq 5\omega|\xi|, |\zeta| \geq 5/2\omega(\xi, \zeta) > 2\omega(\xi, \zeta)$. It is sufficient to prove that

$\mathbb{R}_+ \ni t \mapsto e^{-\alpha_{\pm \theta}^t} \in S_{1,0}^0(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ with $\alpha_{\pm \theta}^t(x', \xi', \zeta) := \kappa_{\pm \theta}^{\pm}(x', \xi', \zeta) - \omega \zeta$.
is uniformly bounded, with bounds on the seminorms that depend only on \( \omega \) and \( |\kappa_{\pm \theta}| \). By Leibniz’ rule we see that \( \partial^\alpha_x \partial^\beta_\xi \partial^\gamma_\zeta e^{-a_{\pm \theta}(x', \xi', \zeta)t} \) is a linear combination of terms of the form

\[
\partial_x^\beta \partial_\xi^\gamma \partial_\zeta^\delta a_{\pm \theta}^\alpha (x', \xi', \zeta) \cdot \cdots \partial_x^\beta \partial_\xi^\gamma \partial_\zeta^\delta a_{\pm \theta}^\alpha (x', \xi', \zeta) (-t)^k e^{a_{\pm \theta}(x', \xi', \zeta)t},
\]

here \( k \leq |\alpha| + |\beta| + l \), \( \sum_k \alpha_i = \alpha \), \( \sum_k \beta_i = \beta \) and \( \sum_k l_i = l \). It is clear that \( a_{\pm \theta}^\alpha \) are symbols of order 1 and \( \Re a_{\pm \theta}^\alpha (\xi, \zeta)^{-1} \omega^{-1} > 1 \), hence we can estimate the absolute value of a term of the form (5.4) by

\[
|\partial_x^\beta \partial_\xi^\gamma \partial_\zeta^\delta a_{\pm \theta}^\alpha (x', \xi', \zeta) \cdots |\partial_x^\beta \partial_\xi^\gamma \partial_\zeta^\delta a_{\pm \theta}^\alpha (x', \xi', \zeta)| |\omega^{-k}(\xi, \zeta)^{-k} (-x)^k e^{-x} \|
\]

with \( x = \Re a_{\pm \theta}^\alpha (x', \xi', \zeta)t \).

Proof of Lemma 5.1. For the computation we set \( G_\mu := G^T_{\pm \theta (2), \mu} \) and \( \tilde{G}_\mu := \tilde{G}^T_{\pm \theta (2), \mu} \). Since \( \mu \mapsto G_\mu \) is continuous it is sufficient to provide the estimate for \( \mu \geq 2 \). Let \( u \in S(\mathbb{R}^n) \). By definition

\[
(G_\mu u)(x', x_n) = \int_{\mathbb{R}^n-1} \int_{\mathbb{R}_+} e^{ix' \cdot \xi'} \tilde{G}_\mu (x', \xi', x_n, y_n) (F_{x' \rightarrow \xi'} e^u)(\xi', x_n) dy_n d\xi'
\]

\[
= \int_{\mathbb{R}_+} \mu^{-1} e^{-\mu(x_n + y_n)} v_\mu (x', x_n, y_n) dy_n,
\]

where

\[
v_\mu (x', x_n, y_n) := \int_{\mathbb{R}^n-1} e^{ix' \cdot \xi'} \tilde{h}_\mu (x', \xi', x_n, y_n) (F_{x' \rightarrow \xi'} e^u)(\xi', x_n) d\xi'.
\]

Here \( \tilde{h}(x', \xi', \zeta; x_n, y_n) \subset S^0_{1,1/2}(\mathbb{R}^n \times \mathbb{R}^{n+1}) \) is the uniformly bounded family of pseudodifferential operators defined in Lemma 5.2, parametrized by \( (x_n, y_n) \in \mathbb{R}_+ \times \mathbb{R}_+ \). In view of Theorem 3.2 (a)

\[
\| v_\mu (\cdot, x_n, y_n) \|_{L_p(\mathbb{R}^{n-1})} \leq C \| e^u (\cdot, y_n) \|_{L_p(\mathbb{R}^{n-1})} =: C w(y_n)
\]

with the constant \( C \) from Lemma 5.2 that depends on \( \omega \), \( |s_{\pm \theta}| \) and \( |\kappa_{\pm \theta}| \). We write \( \mathcal{H} \) for the Hilbert transform; then

\[
\left\| \int_{\mathbb{R}_+} \mu (G_\mu u) (\cdot, x_n) d\mu \right\|_{L_p(\mathbb{R}_+)} \leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} e^{-\mu(x_n + y_n)} w(y_n) dy_n d\mu \\
\leq \int_{\mathbb{R}_+} (x_n + y_n)^{-1} w(y_n) dy_n = (\mathcal{H} w)(x_n).
\]

It is well-known that \( \mathcal{H} \in \mathcal{L}(L_p(\mathbb{R}_+)) \), so

\[
\| L_{\pm \theta} u \|_{L_p(\mathbb{R}_+)} = \| L_{\pm \theta} u \|_{L_p(\mathbb{R}^{n-1})} \|_{L_p(\mathbb{R}_+)} \leq C \| \mathcal{H} u \|_{L_p(\mathbb{R}_+)} \leq C \| u \|_{L_p(\mathbb{R}_+)}. \]

Of course, the constant \( C \) still only depends on \( \omega \), \( |s_{\pm \theta}| \) and \( |\kappa_{\pm \theta}| \).

Once the calculus is established for the half space we can transfer it to compact manifolds, by choosing a suitable atlas.

Lemma 5.3. Theorem 1.3 implies Theorem 1.4.
In particular the kernel of $B$ on the chosen extension. For a function $u$ where $\phi \in C_0^\infty$, with $\varphi_1, \varphi_0 \in C_0^\infty(V')$, $\varphi_1 \geq 0$, $\varphi_1 + \varphi_0 > 0$ and

$$B_\lambda = \begin{pmatrix} A - \lambda \\ T \end{pmatrix} : C_c^\infty(V) \to C_c^\infty(V) \oplus C_c^\infty(V').$$

Then there is an extension

$$\tilde{B}_\lambda = \begin{pmatrix} \tilde{A} - \lambda \\ \tilde{T} \end{pmatrix} : \mathcal{S}(\mathbb{R}^{n-1}) \to \mathcal{S}(\mathbb{R}^{n-1})$$

with $\tilde{A}$ and $\tilde{T}$ as in Theorem 1.3. In particular $\tilde{B}_\lambda$ is invertible (after a possible shift of $A$). We have

$$\tilde{B}_\lambda^{-1}B_\lambda u = \tilde{B}_\lambda^{-1}\tilde{B}_\lambda u = u \text{ for all } u \in C_c^\infty(V).$$

In particular the kernel of $B_\lambda$ is trivial. We set $B_\lambda^{-1} := r\tilde{B}_\lambda^{-1}|_{C_c^\infty(V) \oplus C_c^\infty(V')}$, where $r$ is the restriction to $V$. We need to check that $B_\lambda^{-1}$ does not depend on the chosen extension. For a function $u \in C_c^\infty(V)$ we have

$$v = 0 \iff \chi v = 0 \text{ for all } \chi \in C_c^\infty(V).$$

Let $\tilde{B}_\lambda$ be a second extension of $B_\lambda$. Abbreviating for the moment $C = C_c^\infty(V) \oplus C_c^\infty(V')$ and using the injectivity of $B_\lambda$, we find that $r\tilde{B}_\lambda^{-1}|_{C} = \tilde{B}_\lambda^{-1}|_{C}$ if and only if $\psi' B_\lambda \chi \tilde{B}_\lambda^{-1}|_{C} = \psi' B_\lambda \chi \tilde{B}_\lambda^{-1}|_{C}$ for all $\psi', \psi, \chi \in C_c^\infty(V)$ with $\chi = 1$ on a neighbourhood of supp $\psi'$. Since $B_\lambda$ is local, $\psi' B_\lambda \chi = \psi' B_\lambda$, and we obtain the desired equality. With a similar argument we see that

$$B_\lambda B_\lambda^{-1} \psi = \psi \text{ and } B_\lambda^{-1}B_\lambda \psi = \psi \text{ for all } \psi \in C_c^\infty(V).$$

Step 2: Let $A$ be an atlas for $\Omega$ with a subordinate partition of unity $(\psi_i)_{i \in I}$, such that for every pair $(i, j)$ there is a chart $[\kappa_{ij} : U_{ij} \to V_{ij}] \in A$, such that $\psi_i, \psi_j \in C_c^\infty(U_{ij})$. We denote by $B_{\lambda,ij}$ the operator $B_\lambda$ in local coordinates, i.e. $\chi_i B_{\lambda,ij} = \chi_i \kappa_{ij}^* B_{\lambda,ij} \kappa_{ij*,i} \chi_j$ for $\chi_i, \chi_j \in C_c^\infty(U_{ij})$. Then

$$B_{\lambda} = \sum_{i,j} \psi_i \kappa_{ij}^* B_{\lambda,ij} \kappa_{ij*,i} \psi_j.$$ 

Note that every $B_{\lambda,ij}$ is an operator as in Step 1. Let $B_{\lambda,ij}^{-1}$ be the inverse constructed in Step 1. We define

$$B_\lambda^{-1} := \sum_{i,j} \psi_i \kappa_{ij}^* B_{\lambda,ij}^{-1} \kappa_{ij*,i} \psi_j.$$ 

Choosing $\chi_i \in C_c^\infty(\cap U_{ij})$ with $\chi_i \equiv 1$ on supp $\psi_i$ we obtain

$$B_{\lambda} B_\lambda^{-1} = \sum_{i,j} \psi_i B_{\lambda} B_{\lambda}^{-1} \psi_j = \sum_{i,j} \psi_i B_{\lambda} \chi_i B_{\lambda}^{-1} \psi_j$$

$$= \sum_{i,j} \psi_i \kappa_{ij}^* B_{\lambda,ij} \kappa_{ij*,i} \chi_i \kappa_{ij}^* B_{\lambda,ij}^{-1} \kappa_{ij*,i} \psi_j = \sum_{i,j} \psi_i \kappa_{ij}^* B_{\lambda,ij} \chi_i \psi_i B_{\lambda,ij}^{-1} \kappa_{ij*,i} \psi_j$$

$$= \sum_{i,j} \psi_i \chi_i \kappa_{ij}^* B_{\lambda,ij} B_{\lambda,ij}^{-1} \kappa_{ij*,i} \psi_j = \sum_{i,j} \psi_i \psi_j = 1.$$
We used the locality of \(B_\lambda\) to see that \(\psi_i B_\lambda = \psi_i B_\lambda \chi_i\) and \(\psi_i \kappa_{ij}^* B_{\lambda,ij} \chi_{i,*} = \psi_i \kappa_{ij}^* B_{\lambda,ij}\). A similar argument shows that \(B_\lambda^{-1} B_\lambda = 1\). The resolvent of \(A_T\) is the left entry of the matrix for \(B_\lambda^{-1}\), i.e.

\[
(A_T - \lambda)^{-1} = \sum_{i,j} \psi_i \kappa_{ij}^* (\tilde{A}_{ij,T} - \lambda)^{-1} \kappa_{ij,*} \psi_j.
\]

Here \(\tilde{A}_{ij,T}\) is the \(L_p\)-realization of an extension of \(B_{ij}\).

Step 3: For \(u \in C^\infty(\Omega)\), the definition of \((A_T - \lambda)^{-1}\) above and Theorem \[1.3\] imply that

\[
\left\| \int_{\partial \Lambda_\theta} f(\lambda)(A_T - \lambda)^{-1} u \, d\lambda \right\|_{L_p(\Omega)} = \sum_k \left\| \kappa_{k,*} \psi_k \int_{\partial \Lambda_\theta} f(\lambda)(A_T - \lambda)^{-1} u \, d\lambda \right\|_{L_p(\mathbb{R}^n_+)} = \sum_{i,j,k} \left\| \kappa_{i,j,*} \psi_i \kappa_{ij,*} (\tilde{A}_{ij,T} - \lambda)^{-1} \kappa_{ij,*} \psi_j u \, d\lambda \right\|_{L_p(\mathbb{R}^n_+)} \leq C \max_i \sum_j \left\| \int_{\partial \Lambda_\theta} f(\lambda)(\tilde{A}_{ij,T} - \lambda)^{-1} \kappa_{ij,*} \psi_j u \, d\lambda \right\|_{L_p(\mathbb{R}^n_+)} \leq C \max_i \sum_j \|f\|_{L_p(\mathbb{R}^n_+)} \|\kappa_{ij,*} \psi_j u\|_{L_p(\mathbb{R}^n_+)} = C \|f\|_{\infty} \left\| \kappa_{ij,*} \psi_j u \right\|_{L_p(\Omega)} = C \|f\|_{\infty} \left\| u \right\|_{L_p(\Omega)}
\]

Here, we used the fact that \(\sum_{k,i} |\kappa_{k,*} \psi_k \psi_i \kappa_{ij,*}| \leq C\) as a finite sum. \[\square\]

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