Higher-dimensional loop algebras, non-abelian extensions and $p$-branes

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Abstract

We postulate a new type of operator algebra with a non-abelian extension. This algebra generalizes the Kac–Moody algebra in string theory and the Mickelsson–Faddeev algebra in three dimensions to higher-dimensional extended objects ($p$-branes). We then construct new BRST operators, covariant derivatives and curvature tensors in the higher-dimensional generalization of loop space.
1 Introduction

Infinite-dimensional Lie algebras are of crucial importance in a number of physical applications and have been a subject of intensive study for a number of years. The most well-known examples to physicists are the Virasoro \([1]\) and Kac–Moody (KM) algebras \([2, 3, 4]\), which play fundamental roles in quantum field theories in two dimensions with applications in statistical physics as well as in string theory and 2d quantum gravity \([5]\). Also, in three dimensions one finds the Mickelsson–Faddeev (MF) algebra \([6, 7, 8]\), which arises in connection with gauge theories interacting with chiral matter (see e.g. \([3, 10]\) and refs. therein).

In most situations it is convenient to interpret these infinite-dimensional algebras as extensions of simpler (classical) algebras of observables. Roughly speaking, this means that the commutation relations get augmented by an extra term; a central extension in the case of the Virasoro and KM algebras, and an abelian extension in the MF case. These extensions are of profound importance for the algebras and their representation theories, and, consequently, also for their physical applications.

In all these cases the additional operators added to the algebra commute among themselves, hence the name abelian extension. In the Virasoro and KM cases they even commute with all other observables, and are therefore referred to as central. In this paper we will argue that non-abelian extensions may arise in the context of the theory of extended objects \((p\)-branes\)) coupled to Yang–Mills fields. We will postulate a new algebra on the \(p\)-brane that reduces to the KM algebra for \(p = 1\) (i.e. the string) and to the MF algebra for \(p = 3\). For higher \(p\)'s our construction gives rise to a non-abelian extension of the algebra of charge densities on the \(p\)-brane. This is of potential interest for particle physics because it has been speculated that the \(p = 5\) case is relevant if one wants to incorporate non-perturbative effects in superstring theory by appealing to a dual formulation in terms of the five-brane \([11]\). Such a duality was conjectured in \([12]\) before the five-brane was proven to exist \([13]\). The consequences of the existence of such a Montonen-Olive \([14, 15, 16, 17]\) type duality in string theory might be physically extremely important \([11]\), involving phenomena like supersymmetry breaking and the occurrence of a non-zero potential for the dilaton.
Recently [18, 19, 20, 21, 22] an attempt has been made to use a straightforward generalization of the MF algebra to higher dimensions in the context of the $p$-branes. In this attempt, the algebra reads [19]

$$\left[ \tilde{T}^a(\sigma), \tilde{T}^b(\sigma') \right] = i f^{abc} \tilde{T}^c(\sigma) \delta^p(\sigma - \sigma') + n e^{\gamma_1, \cdots, \gamma_p} \partial_{\gamma_1} N_{\gamma_2, \cdots, \gamma_p}(\sigma) \partial_{\gamma_p} \delta^p(\sigma - \sigma'), \quad (1.1)$$

where, in general, the extension $N$ is a function of the Maurer–Cartan form $K$ on the group $G$ pulled back to the $p$-brane. The $\tilde{T}$'s generate gauge transformations under which $K$ transforms as a gauge field. For $p = 1$ [23], $N$ is independent of $K$ and the algebra in (1.1) reduces to the Kac–Moody algebra, while for $p = 3$ it is linear in $K$ and one obtains the linear algebra in $\tilde{T}$ and $K$ described in [24, 18]. This is the Mickelsson–Faddeev algebra familiar in the context of chiral gauge theories. For $p > 3$ the algebra becomes non-linear because $K$, as briefly explained in section 2, now appears non-linearly in $N$.

In this paper we impose instead linearity and diffeomorphism invariance to obtain an alternative generalization that effectively extends the operator content of the theory with the introduction of new forms of intermediate degree satisfying non-trivial commutation relations among themselves, i.e., corresponding to a non-abelian extension. Thus, when generalizing many of the results known from the loop space formulation of string theory, our algebra makes it possible to avoid non-linearities even for $p > 3$. More specifically, we will construct a BRST operator for the $p$-brane wave functional, a covariant derivative in “$p$-loop space” and its related curvature tensor. To allow for a direct comparison with (1.1), we present here the explicit form of the algebra that we will obtain for $p = 5$:

$$\left[ \tilde{T}^a(\sigma), \tilde{T}^b(\sigma') \right] = i f^{abc} \tilde{T}^c(\sigma) \delta^5(\sigma - \sigma') + \tilde{k} d^{abc} \partial_i \tilde{T}^c ij(\sigma) \partial_j \delta^5(\sigma - \sigma'),$$

$$\left[ \tilde{T}^a(\sigma), \tilde{T}^{bmn}(\sigma') \right] = i f^{abc} \tilde{T}^c mn(\sigma) \delta^5(\sigma - \sigma') - \tilde{k} d^{abc} \epsilon^{ijkmn} \partial_i \tilde{T}^c j(\sigma) \partial_k \delta^5(\sigma - \sigma'),$$

$$\left[ \tilde{T}^a(\sigma), T^i_j(\sigma') \right] = i f^{abc} T^c_i(\sigma) \delta^5(\sigma - \sigma') + k \delta^{ab} \partial_i \delta^5(\sigma - \sigma'),$$

$$\left[ \tilde{T}^{kl}(\sigma), \tilde{T}^{bmn}(\sigma') \right] = i f^{abc} \epsilon^{iklmn} T^c_i(\sigma) \delta^5(\sigma - \sigma') + k \delta^{ab} \epsilon^{iklmn} \partial_i \delta^5(\sigma - \sigma'),$$

$$\left[ \tilde{T}^{mn}(\sigma), T^i_j(\sigma') \right] = 0,$$

$$\left[ T^a_i(\sigma), T^b_j(\sigma') \right] = 0. \quad (1.2)$$

The precise meaning of all the symbols appearing in (1.2) will be explained in section 3. It is
important to note that $K$ plays no role in this algebra which therefore is more general than \((1.1)\), since $K$ itself is connected to a special realization based on functional derivatives on the group manifold $G$. In fact, when constructing \((1.2)\) in section 3 no explicit reference is made to $G$ apart from the fact that we use forms valued in its Lie algebra. We should also remark at this point that although we construct most quantities of relevance for the $p$-brane algebra \((1.2)\), we do not present a $p$-brane action related to it. For the algebra \((1.1)\), on the other hand, as will be discussed in section 2, such an action exists and does indeed require the Maurer–Cartan form $K$ for its construction.

To our knowledge this is the first time non-abelian extensions appear in physics. In this paper we will limit our considerations to $p$-branes, although our construction also incorporates (for $p = 2$) certain current algebras arising in the canonical formulation of (2+1)-dimensional non-linear $\sigma$-models \([25, 26, 27]\). We hope to return to further applications in the future.

To conclude this introduction, let us briefly mention the question of representation theory. Finding unitary representations of the Mickelsson–Faddeev algebra has turned out to be an extremely difficult problem. However, recently there has been some progress along these lines in the (3+1)-dimensional case using regularization techniques based on the calculus of pseudodifferential operators \([28]\). In this context, we believe that our algebra, being linear and not involving directly a gauge potential, could stand a better chance of having a tractable representation theory. Also, the fact that there are now two different higher-dimensional current algebras might be of some help in understanding the difficulties that one encounters in the search for unitary representations.

The plan of the paper is as follows. In section 2 we review some of the previous developments emphasizing the role played by the Maurer–Cartan one-form appearing in the $\sigma$-model action, the BRST operators and the covariant derivatives. This rather detailed account, which also explains the origin of the algebra \((1.1)\), is included in order to facilitate the comparison to our construction which is first explained for the BRST operator in section 3, and then for the covariant derivative in section 4. In section 4 we also discuss the corresponding field strength and its Bianchi identity. Section 5 is devoted to putting our new algebra into a mathematical context, explaining how it fits into the theory of extensions in terms of exact sequences etc.
Finally, in section 6 we summarize our findings and make some additional comments. Formulae for some of the quantities appearing in the descent equations are collected in appendix A, together with the explicit expressions for the BRST operators and covariant derivatives.

2 The origin of the Mickelsson–Faddeev algebra for the $p$-brane

We consider the bosonic $p$-brane as being defined by an embedding of the $(p + 1)$-dimensional worldvolume swept out by the manifold $\Sigma_p$ moving in spacetime $M$, $\Sigma_p$ being the $p$-dimensional manifold describing the $p$-brane itself. We will restrict ourselves to local considerations, and at the present level of understanding we also have to content ourselves with coupling the $p$-brane to background fields in spacetime. In particular, we assume the existence of a background Yang–Mills field $A$ with gauge group $G$, and of a background antisymmetric $(p+1)$-tensor $B_{p+1}$ that couple to the $(p+1)$-dimensional worldvolume.

The dynamics of the system will be described by introducing a $p$-brane wave functional $\Phi(x)$ with $x : \Sigma_p \to M$. Our aim is to describe the “loop space” functional algebra of operators acting on $\Phi$, with special emphasis on the BRST operator $\delta$ and the related covariant derivative $\mathcal{D}$. Throughout the paper we use the convention that $\delta$ commutes with the exterior derivative $d$. The BRST operator then acts on the Yang–Mills field $A$ and its ghost $\omega$ as

$$\delta A = d\omega + A\omega - \omega A, \quad (2.1)$$
$$\delta \omega = -\omega^2. \quad (2.2)$$

For the antisymmetric tensor field $B_{p+1}$ and its sequence of ghosts, denoted by $\Lambda^q_{p+1-q}$ ($q = 1, ... p + 1$ is the ghost number), one has

$$\delta B_{p+1} = n \omega^1_{p+1} - d\Lambda^1_p, \quad (2.3)$$
$$\delta \Lambda^q_{p+1-q} = n \omega^{q+1}_{p+1-q} - d\Lambda^{q+1}_{p-q}. \quad (2.4)$$

(This notation differs from the one used in [18]; here the ghosts are all denoted by $\Lambda$.) Furthermore, the descent equations for the Chern–Simons form $\omega^0_{p+2}$ and its descendants take the
form $\delta \omega^q_{p+2-q} = d\omega^{q+1}_{p+1-q}$ ($q = 0, \ldots, p + 1$). The nilpotency of $\delta$ on the background fields $B$ and $\Lambda$ above follows straightforwardly from these equations.

We must now define how $\delta$ acts on the $p$-brane wave functional. To this end, consider first the case $p = 1$ (i.e. the string). When acting on the string functional $\Phi$ we write this operator as $(\Sigma_1 \equiv S^1)$

$$\delta \Phi = \left( \int_{S^1} [\text{tr}(-\omega T_1) + \Lambda^1_1] \right) \Phi,$$

(2.5)

where the one-form $T_1 = \tau^a \tilde{T}_a(\sigma) d\sigma$ is a Lie algebra-valued operator on $S^1$. The Lie algebra generators $\tau^a$ are assumed to be hermitian, satisfying $[\tau^a, \tau^b] = if^{abc} \tau^c$ and $\text{tr} \tau^a \tau^b = \delta^{ab}$. We also assume the existence of a totally symmetric tensor $d^{abc} \equiv \text{tr} \{\tau^a, \tau^b, \tau^c\}$. Furthermore, the scalar density $\tilde{T}_a(\sigma)$, the dual of $T_1$, is required to satisfy a KM algebra as a consequence of imposing $\delta^2 \Phi = 0$ \[23\]. Let us also point out that throughout this paper the pull-back of all the background forms on the $p$-brane is always understood. For example, in the above mentioned case we have

$$\omega = \omega(x(\sigma)), \quad \Lambda^1_{1} = \Lambda_{\mu}(x(\sigma)) \frac{dx^\mu}{d\sigma} d\sigma,$$

(2.6)

$\Lambda_{\mu}$ being the background form in spacetime.

When we turn to the case $p = 3$, the operator $\delta$ becomes slightly more complicated, involving a new operator $T^a_i(\sigma)$ \[18\] apart from the scalar density $\tilde{T}_b(\sigma)$ which is now the dual of a three-form $T_3$. Using the language of forms and pull-backs, the expression for $\delta$ given in \[18\] can be written as

$$\delta \Phi = \left( \int_{\Sigma_3} (\text{tr}(-\omega T_3 - \frac{1}{2} n[A, d\omega] T_1) + \Lambda^1_3) \right) \Phi,$$

(2.7)

where $T_1 = \tau^a T^a_i(\sigma) d\sigma^i$. Enforcing $\delta^2 \Phi = 0$ in this case leads to an MF type algebra for $\tilde{T}$ and $T_i$.

When considering higher-dimensional cases, there are at least two alternative ways of generalizing the algebras of operators appearing for $p = 1, 3$. The first such construction occurring in the literature \[18, 19\] relies upon a specific construction of the $p$-brane by a Kaluza–Klein approach. As emphasized in the introduction, this method introduces certain non-linearities, and the object of the rest of this section is to explain how this comes about. In section 3 we
will then present a new, alternative, generalization that makes use of a linear algebra with a non-abelian extension.

Let us thus review the Kaluza–Klein approach to the construction of $p$-branes. This approach relies on the observation that the operator $T_i^a$ can be identified with the Maurer–Cartan form $K_i^a$ on the group manifold $G$. One can then proceed to construct a locally gauge invariant action for the $p$-brane of the form

$$S_{p+1} = S_{p+1}^K + S_{p+1}^W. \quad (2.8)$$

Here the kinetic part is $(i, j = 0, 1, ..., p)$

$$S_{p+1}^K = \int d^{p+1}\sigma (-\frac{1}{2\sqrt{-\gamma}})(\gamma^{ij} g_{ij} + \gamma^{ij} \text{tr} J_i J_j - (p - 1)), \quad (2.9)$$

with $\sigma^i, \gamma^{ij}$ ($\gamma = \text{det} \gamma_{ij}$) the worldvolume coordinates and metric, respectively, and $g_{ij}(x)$ the pull-back to the worldvolume of the background spacetime metric $g_{\mu\nu}$ ($\mu, \nu = 0, 1, ..., D - 1$). The current $J = d\sigma^i J_i$ is expressible in terms of the pull-backs of the Yang–Mills field $A$ and the Maurer–Cartan form $K$ on the group $G$ as $J = A - K$. The last term in (2.9) is the cosmological constant which is needed to make it possible to solve for the induced metric $g_{ij}$ by means of its field equations.

The WZW part of the $p$-brane action reads [24]

$$S_{p+1}^W = \int (B_{p+1} + C_{p+1}(A, K) - b_{p+1}), \quad (2.10)$$

where $B$ and $b$, respectively, are the pull-backs to the worldvolume of the antisymmetric tensor fields $B_{\mu_1...\mu_D}(x)$ in spacetime and $b_{m_1..m_D}(y)$ on the group manifold $G$. Finally, the quantity $C_{p+1}(A, K)$ appearing in $S^W$ must transform under gauge transformations in such a way as to ensure that the action $S$ be gauge invariant.

For the string (i.e. $p = 1$) $C_2$ is easily found by recalling that in the low energy limit of the heterotic string, supergravity is coupled to super-Yang–Mills [29, 30] which forces one to let $B_2$ transform under gauge transformations according to

$$\delta_\lambda B_2 = \text{tr}(Ad\lambda), \quad (2.11)$$
where $\lambda^a$ is an $x$ dependent gauge parameter. This conclusion can also be seen to follow from a Kaluza–Klein procedure \cite{31, 32} in which the bosonic string is compactified on $M_{10} \times G$, where $M_{10}$ is the 10-dimensional Minkowski space and $G$ the group manifold $E_8 \times E_8$ or $SO(32)$. This approach also provides us with the relation

$$h_3 = db_2 + \omega_3^0(K) = 0. \quad (2.12)$$

Here $\omega_3^0(K)$ is the Chern-Simons functional with $A$ replaced by the Maurer–Cartan form $K$. The explicit form of $\omega_3^0(K)$ and some other quantities appearing in the descent equations can be found in the appendix. In fact, by appealing to the descent equations we conclude that choosing $\delta_\lambda b_2 = \omega_2^1(K, \lambda)$ makes $h_3$ is invariant under gauge transformations.

Since the kinetic part \eqref{2.9} of the action \eqref{2.8} is trivially gauge invariant for $p = 1$, invariance for the complete action is achieved if

$$\delta_\lambda C_2 = -\omega_2^1(A, \lambda) + \omega_2^1(K, \lambda). \quad (2.13)$$

To get this result we have also made use of the observation that $\delta_\lambda B_2 = \text{tr}(Ad\lambda) = \omega_2^1(A, \lambda)$. Then, by setting $C_2(A, K) = \text{tr}(AK)$ the above expression for $\delta_\lambda C_2(A, K)$ becomes a direct consequence of the gauge transformation properties $\delta_\lambda A = d\lambda + [A, \lambda]$ and $\delta_\lambda K = d\lambda + [K, \lambda]$. This also allows us to check the gauge covariance of the current $J = A - K$ in a trivial manner.

These quantities are easily generalized to arbitrary odd $p$, as explained in \cite{24}, and the action \eqref{2.8} can be verified to be gauge invariant in exactly the same way as for $p = 1$. Thus, once the explicit forms of $\omega_{p+2-q}$ for $q = 0, 1$, and $C_{p+1}(A, K)$ are found, the invariance can be established. The latter form \cite{24, 19} can be found by adopting the technique given in \cite{33} for deriving solutions to the descent equations. Explicit formulae for $p = 3, 5$ can be found in \cite{20, 21}.

In the above approach, with the identification $T_i^a = K_i^a$, one obtains for $p > 3$ an extension of the commutation relations of the charge densities by a composite operator $N$ that is non-linear in the $K$’s:

$$[\hat{T}^a(\sigma), \hat{T}^b(\sigma')] = if^{abc}\hat{T}^c(\sigma)\delta(\sigma - \sigma') + ne^{i_{11}...i_p}\partial_{i_{11}}N_{i_{12}...i_{p-1}}^{(ab)}(\sigma)\partial_{i_p}\delta(\sigma - \sigma'). \quad (2.14)$$
The explicit expression for $N$ can easily be found by considering the form of the cocycle $\omega^2_p(K, \lambda, \lambda')$ and substituting two delta functions for the gauge parameters $\lambda$ and $\lambda'$. For $p = 5$ one obtains

$$N_{ijk}^{(ab)} = \omega^{abcd} U_{[ij}^c K_{k]}^d,$$  \hspace{1cm} (2.15)

with

$$U_{ij}^c = 5(\partial_i K_j^c - \partial_j K_i^c) + 2 f^{cef} K_i^e K_j^f$$  \hspace{1cm} (2.16)

and

$$\omega^{abcd} = \text{tr}(2\tau^a \tau^b \tau^c \tau^d + 2\tau^b \tau^a \tau^c \tau^d + \tau^a \tau^d \tau^b \tau^c + \tau^a \tau^c \tau^b \tau^d).$$  \hspace{1cm} (2.17)

Since the $K$’s commute among themselves, $N(\sigma)$ also has vanishing commutators with $N(\sigma')$ and the $K$’s. In the next section we will show that, by enlarging the algebra of operators in loop space, it is possible to obtain a linear algebra from which all the relevant functional operators in loop space can be constructed in a natural way. The form of this new algebra is fixed uniquely by the requirements of linearity and diffeomorphism invariance on $\Sigma_p$.

## 3 Higher-dimensional loop algebras with non-abelian extensions and the construction of the $p$-brane BRST operator

In this section we will introduce our new algebra in its original and most general form. The first step in this process is to rewrite the action of the BRST operator on the string and three-brane wave functionals, given in (2.5) and (2.7), respectively, in a unified and compact way that immediately generalizes to any odd $p$:

$$\delta \Phi = \left(T(R) + \int_{\Sigma_p} \Lambda^1_p \right) \Phi.$$  \hspace{1cm} (3.1)

Here $R$ is a formal sum of Lie algebra-valued forms in spacetime of even degree pulled back to $\Sigma_p$, constructed from the Yang–Mills field $A$ and its ghost $\omega$. Furthermore, $T$ is a linear functional of $R$ defined by

$$T(R) = \int_{\Sigma_p} \text{tr} TR,$$  \hspace{1cm} (3.2)
where $T$ on the right hand side is a formal sum of operator-valued forms on $\Sigma_p$ of odd degree. The integral is assumed to vanish on all forms of degree different from $p$. We can therefore restrict the formal sums to $R = R_0 + R_2 + \cdots + R_{p-1}$ and $T = T_p + T_{p-2} + \cdots + T_1$, respectively, and so

$$T(R) = \sum_{0 \leq q \leq (p-1)/2} \int_{\Sigma_p} tr T_{p-2q} R_{2q}. \quad (3.3)$$

Note also that, since the BRST operator raises the ghost number by one, the Yang–Mills ghost $\omega$ must appear linearly in $R$.

In particular, for $p = 1$, the expression (2.5) for $\delta \Phi$ from section 2 is recovered by choosing $R \equiv R_0 = -\omega$ and $T \equiv T_1 = \tau^a \tilde{T}^a(\sigma)d\sigma$, $\tilde{T}^a(\sigma)$ being a Kac–Moody algebra generator as a consequence of imposing $\delta^2 \Phi = 0$. Similarly, for $p = 3$ we must choose $R \equiv R_0 + R_2 = -\omega - \frac{1}{2}n[A,d\omega]$ and $T \equiv T_3 + T_1$, with $T_3 = \tau^a \tilde{T}^a(\sigma)d^3\sigma$ and $T_1 = \tau^a T^a_i(\sigma)d\sigma^i$ in this case forming a Mickelsson–Faddeev algebra due to the nilpotency requirement.

Thus, in section 2 the commutation relations for the operators $T_p, T_{p-2}, \ldots, T_1$ were considered to follow by enforcing nilpotency of the BRST operator. The idea is now to turn this argument around, and consider instead the algebra as the fundamental structure. We thus first postulate an algebra for the $T$'s and then ask for the expression for $R$ that makes it possible to construct a nilpotent BRST operator. In this process we are led by the requirements of diffeomorphism invariance, which, in fact, already has led us to the expression (3.1), and, furthermore, that of agreement with the algebras previously obtained for $p = 1, 3$. In order to make the comparisons we will therefore start by considering these two cases. Also, when formulating the algebra, it is natural to use, instead of $R$, formal sums $X = X_0 + X_2 + \cdots X_{p-1}$ of Lie algebra-valued forms of ghost number zero.

Hence, assume first that $p = 1$. Consider two zero-forms $X$ and $X'$, and a one-form operator $T$ on $S^1$, all Lie algebra-valued. Then form $T(X)$ and $T(X')$ as in (3.2) and assume that these objects satisfy the algebra

$$[T(X), T(X')] = T([X, X']) + k \int_{S^1} tr X dX'. \quad (3.4)$$

Taking $X(\sigma'') = \tau^a \delta(\sigma'' - \sigma)$ and $X'(\sigma'') = \tau^b \delta(\sigma'' - \sigma')$, it immediately follows that this algebra
is equivalent to a Kac–Moody algebra at level \( k \):

\[
[\tilde{T}^a(\sigma), \tilde{T}^b(\sigma')] = if^{abc}\tilde{T}^c(\sigma)\delta(\sigma - \sigma') + k\delta^{ab}\partial_\sigma\delta(\sigma - \sigma').
\] (3.5)

Thus, \( \tilde{T}(\sigma) \), the dual of the one-form \( T_1 \), is nothing but the current in terms of which the Kac–Moody algebra is usually expressed. Expanded in modes on \( S^1 \) according to \( \tilde{T}^a(\sigma) = \sum_n J^a_n e^{in\sigma}, \) it gives the algebra in its most common form:

\[
[J^a_m, J^b_n] = if^{abc}J^c_{m+n} + k\delta^{ab}m\delta_{m+n,0}.
\] (3.6)

(Recall our conventions \([\tau^a, \tau^b] = if^{abc}\tau^c\) and \( \text{tr} \tau^a\tau^b = \delta^{ab} \) for the Lie algebra of \( G \).) Expressed in terms of the anticommuting variable \( R \), the algebra (3.4) reads

\[
\{T(R), T(R')\} = T(\{R, R'\}) + k\int_{S^1} \text{tr} RdR'.
\] (3.7)

Having postulated the algebra, the next step is to impose nilpotency on the BRST operator when acting on the string wave functional \( \Phi \). From (3.1) we find

\[
\delta^2\Phi = \delta \left( (T(R) + \int_{S^1} \Lambda^1_1)\Phi \right)
= (T(\delta R) + \int_{S^1} \delta \Lambda^1_1)\Phi - (T(R) + \int_{S^1} \Lambda^1_1)\delta\Phi
= (T(\delta R) + \int_{S^1} \delta \Lambda^1_1)\Phi - (T(R) + \int_{S^1} \Lambda^1_1)^2\Phi,
\] (3.8)

where \( \delta R \) refers to the BRST variation of the background fields appearing in \( R \). Using (3.8), the algebra (3.7) and the descent equation (2.4), the nilpotency equation \( \delta^2\Phi = 0 \) then can be written as two separate equations,

\[
\delta R - R^2 = 0,
\] (3.9)

\[
\int_{S^1} (n\omega_1^2 - \frac{1}{2}k \text{tr} RdR) = 0,
\] (3.10)

the latter one corresponding to the central part. It is easily seen that (3.9) is satisfied by \( R = -\omega \), while (3.10) gives the additional relation \( k = 2n \).

In the case of the three-brane, for which \( T = T_3 + T_1 \) and \( R = R_0 + R_2 \), we make the observation that in the wedge product of \( T \) and \( \{dR, dR'\} \) there appears a term \( T_1\{dR_0, dR'_0\} \) of
degree three, i.e., $T(\{dR, dR'\}) \neq 0$. Hence, we take the generalization of the algebra (3.7) for $p = 3$ to include also terms of this kind:

$$\{T(R), T(R')\} = T(\{R, R'\} + \tilde{k}\{dR, dR'\}) + k \int_{\Sigma_3} \text{tr} R dR'.$$ \hspace{1cm} (3.11)

Introducing the term multiplying $\tilde{k}$ makes (3.11) equivalent to the MF algebra, as we will now show, by extracting the explicit form of the algebra. Apart from enabling the comparison this will reveal the different roles played by the two parameters $k$ and $\tilde{k}$. For this purpose, it is convenient to return to the formulation in terms of commuting arguments $X$ and $X'$, i.e.

$$[T(X), T(X')] = T([X, X'] + \tilde{k}[dX, dX']) + k \int_{\Sigma_3} \text{tr} X dX'.$$ \hspace{1cm} (3.12)

After decomposition this algebra reads

$$[[\tilde{T}^a(\sigma), \tilde{T}^b(\sigma')], \tilde{T}^c(\sigma)] = i f^{abc} \tilde{T}^c(\sigma) \delta(\sigma - \sigma') - \tilde{k}d^{abc} \epsilon^{ijk} \partial_i T^c(\sigma) \partial_k \delta(\sigma - \sigma'),$$ \hspace{1cm} (3.13)

$$[[\tilde{T}^a(\sigma), T^i_b(\sigma')], \tilde{T}^c(\sigma)] = i f^{abc} T^c(\sigma) \delta(\sigma - \sigma') + k \delta^{ab} \partial_i \delta(\sigma - \sigma'),$$ \hspace{1cm} (3.14)

$$[[T^i_a(\sigma), T^j_b(\sigma')], \tilde{T}^c(\sigma)] = 0,$$ \hspace{1cm} (3.15)

where we have used $X(\sigma'') = X_0(\sigma'') + X_2(\sigma'')$ with $X_0(\sigma'') = \tau^a \delta(\sigma'' - \sigma)$ and $X_2(\sigma'') = \tau^a \delta(\sigma'' - \sigma) d\sigma''^i \wedge d\sigma''^k$, and the analogous expression for $X'(\sigma'')$ with $a$ replaced by $b$ and $\sigma$ replaced by $\sigma'$. In this notation, the commutation relations above correspond to the ones for $[T(X_0), T(X_0')], [T(X_0), T(X_2)]$ and $[T(X_2), T(X_2')]$, respectively. Note that while the $k$ term in (3.3) for the string gives rise to the central extension, the corresponding term for $p = 3$ in (3.12) appears instead in the second commutator (3.14). Furthermore, the extension of the first commutator (3.13) has become operator-valued since it originates from the new term in (3.11) multiplying $\tilde{k}$. Rescaling by $k$ makes the operator $T^a_i$ transform as a gauge field and turns the above algebra into the Mickelsson–Faddeev algebra. Note also that the algebra is referred to as having an abelian extension due to the last commutator (3.15) above.

With the new term included in $T(X)$ the previously derived conditions become

$$\delta R - R^2 - \tilde{k} (dR)^2 = 0,$$ \hspace{1cm} (3.16)

$$\int_{\Sigma_3} (n\omega_3^2 - \frac{1}{2} k \text{tr} R dR) = 0.$$ \hspace{1cm} (3.17)
To solve these equations we use the ansatz

$$R = -\omega + \zeta_1 A d\omega + \zeta_2 d\omega A.$$  \hspace{1cm} (3.18)

(Terms involving undifferentiated $\omega$'s may also be introduced but the solution will require their coefficients to vanish.) Inserted in (3.16) and (3.17) this ansatz gives $\zeta_1 - \zeta_2 = \tilde{k}$ and $(\zeta_1 - \zeta_2)k + n = 0$, respectively. We thus conclude that $n = -k\tilde{k}$ and that

$$R = -\omega + \tilde{k} \left( \frac{1}{2} [A, d\omega] + (s - \frac{1}{2}) \{A, d\omega\} \right), \hspace{1cm} (3.19)$$

where $s$ is an arbitrary parameter.

Before giving the ansatz for $R$ in the case $p = 5$, we make the observation that the algebra (3.12) for $T(X)$ (and hence also (3.11) for $T(R)$) generalizes immediately to any odd $p$ without any further additions, and, consequently, so do the two conditions (3.16) and (3.17) above. The number of terms in the ansatz for $R$ increases rapidly, and already for $p = 5$ it is advisable to do the algebra on a computer. The condition

$$\int_{\Sigma_p} (n\omega_p^2 - \frac{1}{2} k \text{tr} RdR) = 0 \hspace{1cm} (3.20)$$

generalizing (3.17), may for that purpose be written in the form

$$n\delta\omega_{p+1}^1 - \frac{1}{2} k \text{tr}(dR)^2 = 0, \hspace{1cm} (3.21)$$

with the understanding that only terms of form degree $p + 1$ should be considered. Eq. (3.21) was obtained by requiring the form $\chi \equiv n\omega_p^2 - \frac{1}{2} k \text{tr} RdR$ inside the integral to be exact, and then using the descent equation $d\omega_p^2 = \delta\omega_{p+1}^1$. More precisely, (3.21) is the requirement that $\chi$ be closed, i.e. that $d\chi = 0$. To be correct one should thus also add the condition $\chi = d\Upsilon$ for some $(p + 1)$-form $\Upsilon$. However, this condition turns out to be automatically fulfilled by the solutions to (3.16) and (3.20) for the cases at hand.

The most general ansatz (without terms involving undifferentiated ghosts that would vanish anyway) contains ten parameters, not including the one present in the solution for $p = 3$:

$$R = -\omega + \tilde{k} \left( \frac{1}{2} [A, d\omega] + (s - \frac{1}{2}) \{A, d\omega\} \right)$$

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\[ + \zeta_1 A^3 d\omega + \zeta_2 A^2 d\omega A + \zeta_3 A d\omega A^2 + \zeta_4 d\omega A^3 \\
+ \zeta_5 dA A d\omega + \zeta_6 A dA d\omega + \zeta_7 dA d\omega A \\
+ \zeta_8 A d\omega dA + \zeta_9 d\omega dA A + \zeta_{10} d\omega A dA. \]  

(3.22)

Subjected to the two nilpotency conditions this ansatz gives the following four-parameter solution for \( R \):

\[
R = -\omega + \tilde{k} \left( \frac{1}{2} [A, d\omega] + (s - \frac{1}{2}) \{A, d\omega\} \right) \\
+ \tilde{k}^2 \left( (-\frac{2}{3} + v) A^3 d\omega + (\frac{1}{3} + u) A^2 d\omega A + u A d\omega A^2 + v d\omega A^3 \\
+ t dAA d\omega + (-1 - u + v) A dA d\omega + (s + t) dA d\omega A \\
+ (-\frac{1}{3} + s^2 + t) A d\omega dA + (-u + v) d\omega dAA + (\frac{2}{3} - s + s^2 + t) d\omega A dA \right). \]  

(3.23)

Having established the existence of a solution, it may be of interest to write out the explicit form of the algebra. In order to do that we first note that \( T(R) \) now involves three operator-valued quantities, namely the five-form \( T_5 = \tau^a \tilde{T}^a(\sigma) d^5 \sigma \), the three-form \( T_3 \), with the dual \( \tilde{T}^{aij}(\sigma) \), and finally the one-form \( T_1 \). Repeating the steps described above for the cases \( p = 1, 3 \) generates the following algebraic structure:

\[
[T^a(\sigma), T^b(\sigma')] = i f^{abc} T^c(\sigma) \delta^5(\sigma - \sigma') + \tilde{k} d^{abc} \partial_i \tilde{T}^{cij}(\sigma) \partial_j \delta^5(\sigma - \sigma'), \]  

(3.24)

\[
[\tilde{T}^a(\sigma), \tilde{T}^{bmn}(\sigma')] = i f^{abc} T^c(\sigma) \delta^5(\sigma - \sigma') - \tilde{k} d^{abc} \epsilon^{ijkmn} \partial_i \tilde{T}^{cij}(\sigma) \partial_j \delta^5(\sigma - \sigma'), \]  

(3.25)

\[
[T^a(\sigma), T_i^b(\sigma')] = i f^{abc} T_i^c(\sigma) \delta^5(\sigma - \sigma') + k \delta^{ab} \partial_i \delta^5(\sigma - \sigma'), \]  

(3.26)

\[
[\tilde{T}^a(\sigma), \tilde{T}^{bmn}(\sigma')] = i f^{abc} \epsilon^{iklmn} T_i^c(\sigma) \delta^5(\sigma - \sigma') + k \delta^{ab} \epsilon^{iklmn} \partial_i \delta^5(\sigma - \sigma'), \]  

(3.27)

\[
[T_i^a(\sigma), T_j^b(\sigma')] = 0, \]  

(3.28)

\[
[T_i^a(\sigma), \tilde{T}_j^b(\sigma')] = 0. \]  

(3.29)

We can now explicitly see that the algebra of the scalar densities \( \tilde{T}^a \) now has a non-abelian extension; the commutator (3.24) is augmented by an operator \( \tilde{T}^{cij} \) that according to (3.27) does not commute with itself. Also, note from (3.26) that the operator \( T_i^a \) (properly rescaled by \( k \)) still transforms as a gauge field.


4 Loop space covariant derivatives and non-abelian extensions

In the previous section we have shown how, given our $p$-brane algebra, one can construct a nilpotent BRST operator acting on the $p$-brane wave functional and on the background fields. However, this is only one of the many ingredients needed to formulate the higher-dimensional analogue of loop calculus for the $p$-brane. We should also specify what we mean by the connection one-form and the curvature two-form. Actually, from a logical standpoint, it would have been more appropriate to begin with the construction of these two objects and then describe the BRST operator. We have reversed the order in this paper because the construction of the BRST operator is somewhat simpler, not having to rely on the definition of differential forms in higher-dimensional loop space.

It turns out that the extension of ordinary exterior calculus to these spaces is fairly straightforward, at least for our purposes. Let us start by choosing a basis in the space of one-forms, which we denote by $\delta x^\mu(\sigma)$. In particular, on an arbitrary background vector field $\Psi = \int d\sigma' \psi^\nu(x'(\sigma')) \delta x^\nu(x'(\sigma'))$ in loop space it takes the value $\delta x^\mu(\sigma)(\Psi) = \psi^\mu(x(\sigma))$.

One can use this fact to construct an exterior derivative operator $D$ in loop space, by setting $D = \int d\sigma' \delta x^\mu(\sigma) \frac{\delta}{\delta x^\mu(\sigma')}$, but it is more convenient never to use the explicit expression and instead rely on the formal definition of the exterior derivative when doing the calculations. The simplest case is when the operator $D$ acts on a scalar function in loop space. To be specific, let us take the scalar $T(R)$, whose exterior derivative is needed anyway and which can be regarded as a paradigm for all the following calculations. On such a function, $D$ acts by creating a loop space one-form, which can be specified by giving its value on an arbitrary vector field such as $\Psi$ defined above:

$$DT(R)(\Psi) \equiv \Psi(T(R)) = T(\mathcal{L}_\psi R).$$

(4.1)

Here $\mathcal{L}_\psi$ represents the Lie derivative with respect to the spacetime vector field $\psi^\mu(x)$ and the argument of $T$ is understood to be pulled back to the $p$-brane after the derivative has been taken.

The first equality in (4.1) is simply the definition of the action of the exterior derivative on a
loop space scalar. The second relation is slightly non-trivial and can be proven by the following explicit calculations. Instead of being too general, let us consider a specific example to show how the calculation goes. The generalization from this example should be straightforward. Thus, let us consider $T(R)$ of the form

$$T(R) = \int d\sigma \epsilon^{ijk} \text{tr} T_i(\sigma) R_{\mu\nu}(x(\sigma)) \frac{\partial x^\mu(\sigma)}{\partial \sigma^j} \frac{\partial x^\nu(\sigma)}{\partial \sigma^k}. \quad (4.2)$$

(This expression arises as one of the factors in the construction of the BRST operator for the three-brane.) The second equality in (4.1) is then obtained by acting on (4.2) with the vector $\Psi$ defined above:

$$\Psi(T(R)) = \int d\sigma d\sigma' \epsilon^{ijk} \text{tr} T_i(\sigma) \psi^\rho(x(\sigma')) \frac{\delta}{\delta x^\rho(\sigma')} \left( R_{\mu\nu}(x(\sigma)) \frac{\partial x^\mu(\sigma)}{\partial \sigma^j} \frac{\partial x^\nu(\sigma)}{\partial \sigma^k} \right)$$

$$= \int d\sigma d\sigma' \epsilon^{ijk} \text{tr} T_i(\sigma) \psi^\rho(x(\sigma')) \left( \partial_\rho R_{\mu\nu}(x(\sigma)) \delta(\sigma - \sigma') \frac{\partial x^\mu(\sigma)}{\partial \sigma^j} \frac{\partial x^\nu(\sigma)}{\partial \sigma^k} \right. + \left. R_{\mu\nu}(x(\sigma)) \delta(\sigma - \sigma') \frac{\partial x^\mu(\sigma)}{\partial \sigma^j} \frac{\partial x^\nu(\sigma)}{\partial \sigma^k} \right)$$

$$+ R_{\mu\nu}(x(\sigma)) \delta(\sigma - \sigma') \frac{\partial x^\mu(\sigma)}{\partial \sigma^j} \frac{\partial x^\nu(\sigma)}{\partial \sigma^k} + R_{\mu\nu}(x(\sigma)) \delta(\sigma - \sigma') \frac{\partial x^\mu(\sigma)}{\partial \sigma^j} \frac{\partial x^\nu(\sigma)}{\partial \sigma^k} \right)$$

$$= \int d\sigma \epsilon^{ijk} \text{tr} T_i(\sigma) \left( \psi^\rho(x(\sigma)) \partial_\rho R_{\mu\nu}(x(\sigma)) \right) + \partial_\rho \psi^\rho(x(\sigma)) R_{\rho\nu}(x(\sigma)) R_{\mu\rho}(x(\sigma)) \frac{\partial x^\mu(\sigma)}{\partial \sigma^j} \frac{\partial x^\nu(\sigma)}{\partial \sigma^k}$$

$$= \int d\sigma \epsilon^{ijk} \text{tr} T_i(\sigma) (\mathcal{L}_\psi R)_{\mu\nu}(x(\sigma)) \frac{\partial x^\mu(\sigma)}{\partial \sigma^j} \frac{\partial x^\nu(\sigma)}{\partial \sigma^k} = T(\mathcal{L}_\psi R). \quad (4.3)$$

The next case of interest is when $D$ acts on a one-form $\xi$ in loop space. Again, from the general expressions for the exterior derivative one has

$$D\xi(\Psi_1, \Psi_2) = \Psi_1(\xi(\Psi_2)) - \Psi_2(\xi(\Psi_1)) - \xi([\Psi_1, \Psi_2]), \quad (4.4)$$

when the resulting two-form is evaluated on two arbitrary vectors $\Psi_1$ and $\Psi_2$ in loop space. Using the well-known relation $[\mathcal{L}_{\Psi_1}, \mathcal{L}_{\Psi_2}] = \mathcal{L}_{[\Psi_1, \Psi_2]}$ for the Lie derivative, one can check directly that $D$ is nilpotent, i.e., for example, $D^2 T(R) = 0$. 

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The expression for a generic \( n \)-form \( \xi_n \) is well-known from elementary differential geometry and it is given here for completeness:

\[
D\xi_n(\Psi_1, \ldots, \Psi_{n+1}) = \\
\sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \xi_n(\left[\Psi_i, \Psi_j\right], \Psi_1, \ldots, \Psi_{i-1}, \Psi_{i+1}, \ldots, \Psi_{j-1}, \Psi_{j+1}, \ldots, \Psi_{n+1}) \\
+ \sum_{1 \leq i \leq n+1} (-1)^{i+1} \xi_i(\xi_n(\Psi_1, \ldots, \Psi_{i-1}, \Psi_{i+1}, \ldots, \Psi_{n+1})).
\] (4.5)

(The only additional case we will need in this paper is \( n = 2 \) for the Bianchi identities of the curvature tensor.) Now that we have the expression for the exterior derivative, we can look for a covariant version of it by adding to \( D \) a connection one-form. The idea is the same as for the construction of the BRST operator; we start with an ansatz in terms of background forms in spacetime. Such an ansatz must be compatible with our algebra and it must be diffeomorphism invariant. We then impose the covariance of the derivative in loop space and obtain the differential equations that the background fields must obey in spacetime.

Again, we find it very convenient to work with formal sums of differential forms of various degree. Thus, let \( \Omega_\psi \) be a formal sum of forms of even degree and \( C_\psi \) a \( p \)-form, both pulled back to the \( p \)-brane manifold \( \Sigma_p \) and depending on the background Yang–Mills field \( A(x) \) and an arbitrary background vector \( \psi(x) \). (The dependence on the vector appears through the contraction of one of the indices by \( \psi^\mu \) before the pull-back. Explicit expressions for these forms are given in the appendix. Also, note that in this section we do not explicitly indicate the degree or the ghost number of a form.) We can now take as the covariant exterior derivative of the \( p \)-brane wave functional \( \Phi \) the expression

\[
\mathcal{D}\Phi = D\Phi + T(\Omega_\delta x)\Phi + \int (i_\delta x B + C_\delta x)\Phi.
\] (4.6)

The left hand side of this equation has to be interpreted as the one-form in loop space that on an arbitrary vector \( \Psi \) takes the value

\[
\mathcal{D}\Phi(\Psi) = \Psi(\Phi) + T(\Omega_\psi)\Phi + \int (i_\psi B + C_\psi)\Phi,
\] (4.7)

where the symbol \( i_\psi \) represents the contraction of the first index of the \((p + 1)\)-form \( B \) by the background vector \( \psi \), and the pull-back is, as always, understood. It will be clear from
the following calculation why one must add the integral term in eq. [4.7] to the covariant exterior derivative. Furthermore, eq. [4.7] corresponds exactly to the covariant derivative discussed previously in the literature [34, 23].

We now want to find the equations (in ordinary spacetime) that the forms $\Omega_\psi$ and $C_\psi$ must satisfy in order for (4.6) to be covariant. The BRST variation of (4.7) is

$$
\delta(\mathcal{D}\Phi(\Psi)) = \Psi((T(R) + \int \Lambda)\Phi) + T(\delta\Omega_\psi)\Phi + T(\Omega_\psi)(T(R) + \int \Lambda)\Phi + \int (\delta i_\psi B + \delta C_\psi)\Phi + \int (i_\psi B + C_\psi)(T(R) + \int \Lambda)\Phi.
$$

\hspace{1cm} (4.8)

In order for $\mathcal{D}$ to be covariant, equation (4.8) must coincide with

$$
(T(R) + \int \Lambda)\mathcal{D}\Phi.
$$

\hspace{1cm} (4.9)

Taking into account the relations

$$
\Psi(T(R)\Phi) = T(\mathcal{L}_\psi)\Phi + T(R)\Psi(\Phi)
$$

$$
\Psi(\int \Lambda\Phi) = (\int \mathcal{L}_\psi\Lambda)\Phi + \int \Lambda\Psi(\Phi) = (\int i_\psi d\Lambda)\Phi + \int \Lambda\Psi(\Phi),
$$

$$
\int (\delta i_\psi B) = \int (−i_\psi d\Lambda + ni_\psi\omega^1_{p+1}),
$$

\hspace{1cm} (4.10)

and the commutator of $T(R)$ with $T(\Omega_\psi)$ that we have assumed for our algebra, one can read off the “covariance equations” for the background forms in spacetime:

$$
\mathcal{L}_\psi R + \delta\Omega_\psi + [\Omega_\psi, R] + \hat{k}[d\Omega_\psi, dR] = 0,
$$

$$
\int (k \text{tr}(\Omega_\psi dR) + ni_\psi\omega^1_{p+1} + \delta C_\psi) = 0.
$$

\hspace{1cm} (4.11)

These two equations express in a compact and coordinate-free notation the covariance of the operator $\mathcal{D}$. The solutions for the three-brane and the five-brane are given in the appendix. Here, we simply want to remark that for the string case ($p = 1$) one obtains the known loop space results [23] $R = -w$, $\Omega_\psi = i_\psi A$ and $C_\psi = i_\psi A A$. Also notice that the variation of $B$ in $\mathcal{D}$ has cancelled against the Lie derivative of $\Lambda$, leaving only a $B$-independent term $\omega^1_{p+1}$.

We can now immediately construct the curvature tensor associated with the covariant derivative $\mathcal{D}$. In the same way as for ordinary Yang–Mills theory, one can define the total curvature...
tensor \( G \) as the multiplicative operator such that \( \mathcal{D} \Phi = G \Phi \). (Note that this formula will be modified if loop space is not torsion-free. This is what happens, for example, in the supersymmetric formulation of the problem \([34, 23]\).)

By calculations analogous to those performed when deriving (4.11) one obtains,

\[
G(\Psi_1, \Psi_2) = T \left( L_{\psi_1} \Omega_{\psi_2} - L_{\psi_2} \Omega_{\psi_1} - \Omega_{[\psi_1, \psi_2]} + [\Omega_{\psi_1}, \Omega_{\psi_2}] + \tilde{k}[d\Omega_{\psi_1}, d\Omega_{\psi_2}] \right) + \int \left( i_{\psi_1} dC_{\psi_2} - i_{\psi_2} dC_{\psi_1} - C_{[\psi_1, \psi_2]} + i_{\psi_2} i_{\psi_1} dB \right). \tag{4.12}
\]

In deriving (4.12) we used the facts that \( L_{\psi} = i_{\psi} d + di_{\psi} \) and that

\[
\int L_{\psi_1} i_{\psi_2} B - L_{\psi_2} i_{\psi_1} B - i_{[\psi_1, \psi_2]} B = \int i_{\psi_2} i_{\psi_1} dB. \tag{4.13}
\]

The total curvature \( G \) depends on both background forms \( A \) and \( B \). It is, however, more convenient to split it into the sum of two pieces, \( G = \mathcal{F} + \mathcal{H} \), where \( \mathcal{F} \) transforms covariantly and \( \mathcal{H} \) is invariant under BRST transformations. Furthermore, all the \( B \) dependence should be lumped into \( \mathcal{H} \) where it really belongs.

Naively, one might consider defining \( \mathcal{H} \) simply as

\[
\mathcal{H}^0(\Psi_1, \Psi_2) = \int i_{\psi_2} i_{\psi_1} dB. \tag{4.14}
\]

However, this is not the right choice, as already has been noticed for the string. The problem is that \( \mathcal{H}^0 \) is not invariant under BRST transformations. Instead, it changes according to

\[
\delta \mathcal{H}^0(\Psi_1, \Psi_2) = n \int i_{\psi_2} i_{\psi_1} d\omega_{p+1}^1 \equiv n \int i_{\psi_2} i_{\psi_1} \omega_{p+2}^0. \tag{4.15}
\]

We must therefore add and subtract the Chern-Simons form \( \omega_{p+2}^0 \) to the total expression \( G \) and consider the splitting:

\[
\mathcal{F}(\Psi_1, \Psi_2) = T \left( L_{\psi_1} \Omega_{\psi_2} - L_{\psi_2} \Omega_{\psi_1} - \Omega_{[\psi_1, \psi_2]} + [\Omega_{\psi_1}, \Omega_{\psi_2}] + \tilde{k}[d\Omega_{\psi_1}, d\Omega_{\psi_2}] \right) + \int \left( i_{\psi_1} dC_{\psi_2} - i_{\psi_2} dC_{\psi_1} - C_{[\psi_1, \psi_2]} + i_{\psi_2} i_{\psi_1} dB + k \operatorname{tr} \Omega_{\psi_2} d\Omega_{\psi_1} \right),
\]

\[
\mathcal{H}(\Psi_1, \Psi_2) = \int \left( i_{\psi_2} i_{\psi_1} dB - n i_{\psi_2} i_{\psi_1} \omega_{p+2}^0 \right). \tag{4.16}
\]

This construction generalizes a known fact in string theory, namely, that it is necessary to shift the naive curvature tensor by the Chern-Simons form \( \omega_3^0 = \operatorname{tr}(AdA + \frac{2}{3}A^3) \). It is then straightforward to check that \( \delta \mathcal{H} = 0 \) and \( \delta \mathcal{F} = [T(R) + \int \Lambda, \mathcal{F}] \equiv [T(R), \mathcal{F}] \).
The Bianchi identities also follow from our formulation; because of the manifestly covariant way in which these objects have been defined in loop space, it is clear that the three-form $DG$ vanishes. Also, because of the manifestly covariant way in which the splitting (4.16) has been performed, it follows that $DF = 0$ and $DH \equiv D\mathcal{H} = 0$.

This concludes the construction of the covariant derivative and the related curvature tensor in loop space. In the next section, after a review of the concept of (non-abelian) Lie algebra extensions, we will show how our algebra fits into this mathematical framework.

5 General theory of extensions

In this section we will be reviewing some more mathematically oriented issues related to Lie algebra extensions, with particular emphasis on non-abelian extensions. This is the “infinitesimal version” of the theory of group extensions \[35, 36, 37\] and it is of interest here because the algebra presented in this paper becomes a non-abelian extension of the algebra of the charge densities for $p \geq 5$. The subcase of abelian extensions, and, in particular, the further subcase of central extensions, are well-known to physicists due to the abundance of situations in which they arise. Surprisingly, there has been very little discussion in the physics literature about the non-abelian case. Hence, it may be worth presenting a self-contained review of the subject, paying particular attention to the new aspects that might be of relevance to physics in the future. In principle, this section could be read independently from the rest of the paper, but at the end we will make contact with the work on the $p$-brane by using our new algebra as an explicit example.

Let us then begin with the definition of an extension of a Lie algebra. Consider the following exact sequence of Lie algebras, 0 denoting the trivial algebra with one element:

$$0 \xrightarrow{\delta} \mathcal{L}_0 \xrightarrow{i} \hat{\mathcal{L}} \xrightarrow{\pi} \mathcal{L} \xrightarrow{\xi} 0. \quad (5.1)$$

By exactness of this sequence we mean that all the arrows represent Lie algebra homomorphisms and that the image of any one of them coincides with the kernel of the following. In particular, $\text{Ker}(i) = \text{Im}(\delta) = 0$ means that $\mathcal{L}_0$ is embedded in $\hat{\mathcal{L}}$ by a one-to-one homomorphism, and
\( \mathcal{L} = \ker(\epsilon) = \text{Im}(\pi) \) means that \( \pi \) is onto. Less trivially, the condition \( \ker(\pi) = \text{Im}(i) \) means that \( i(\mathcal{L}_0) \) is an ideal of \( \hat{\mathcal{L}} \) and that \( \mathcal{L} = \mathcal{L}/i(\mathcal{L}_0) \). In the following we will always identify \( \mathcal{L}_0 \) and \( i(\mathcal{L}_0) \), \( i \) being an embedding.

If (5.1) is exact, we say that \( \hat{\mathcal{L}} \) is an extension of the Lie algebra \( \mathcal{L} \) by the Lie algebra \( \mathcal{L}_0 \). (To avoid confusion, we should mention that sometimes in the mathematical literature, the opposite statement is given, with \( \mathcal{L}_0 \) and \( \mathcal{L} \) interchanged in the above sentence. We will stick with the convention that is used in physics.)

The above definition is the most general definition of an extension and we will use it without making further assumptions about \( \mathcal{L}_0 \). When the Lie algebra \( \mathcal{L}_0 \) is non-abelian one may stress this fact by calling \( \hat{\mathcal{L}} \) a non-abelian extension. In the specific cases when \( \mathcal{L}_0 \) is abelian (i.e., has identically vanishing Lie product: \( [\mathcal{L}_0, \mathcal{L}_0] = 0 \)), \( \hat{\mathcal{L}} \) is called an abelian extension. Furthermore, if \( \mathcal{L}_0 \) is contained in the center of \( \hat{\mathcal{L}} \) (i.e., \( [\mathcal{L}_0, \hat{\mathcal{L}}] = 0 \)) we refer to \( \hat{\mathcal{L}} \) as a central extension.

It is easy to see why the study of extensions is of interest in quantum physics. Suppose we have a classical system that yields an algebra \( \mathcal{L} \) in its canonical formulation. It is well known that going to the quantum theory, by turning the Poisson brackets into commutators, one may have to add extra terms to solve ordering ambiguities, effectively enlarging the algebra to \( \hat{\mathcal{L}} \). These terms, however, are proportional to \( \hbar \) and therefore, by power counting, they must form an ideal \( \mathcal{L}_0 \) of \( \hat{\mathcal{L}} \), i.e. \( [\mathcal{L}_0, \hat{\mathcal{L}}] \subseteq \mathcal{L}_0 \). This is exactly the statement that \( \hat{\mathcal{L}} \) is an extension of \( \mathcal{L} \) by \( \mathcal{L}_0 \).

In the above discussion, we assumed the existence of \( \hat{\mathcal{L}} \) in order to define the sequence (5.1). In physics, the standard situation is that we have the Lie algebra \( \mathcal{L} \) and we want to study its possible extensions. It should be obvious that simply knowing \( \mathcal{L}_0 \) is not enough to uniquely determine \( \hat{\mathcal{L}} \). For example, given any two Lie algebras \( \mathcal{L} \) and \( \mathcal{L}_0 \) one could always construct their direct sum, and this is certainly not the most general case.

We will show that the extra information that is needed is almost entirely encoded in a homomorphism \( \theta \) from the Lie algebra \( \mathcal{L} \) to the Lie algebra of the exterior derivations of \( \mathcal{L}_0 \), denoted by \( \text{Ext} \, D(\mathcal{L}_0) \). Recall that a derivation in the Lie algebra \( \mathcal{L}_0 \) is a linear map \( D \) from the Lie algebra into itself such that for any two elements \( a \) and \( b \), \( D[a, b] = [Da, b] + [a, Db] \). Such derivations form a Lie algebra \( D(\mathcal{L}_0) \) and it is easily seen that the adjoint algebra \( \text{Adj}(\mathcal{L}_0) \)
(i.e., the algebra generated by the adjoint action $Ad$ of $L_0$ on itself), is an ideal of $D(L_0)$. The algebra $Ext_{D(L_0)}$ of exterior derivations is then defined as the quotient of the algebra of all derivations by the adjoint algebra. In terms of exact sequences one can write this as

$$0 \to Adj(L_0) \overset{j}{\to} D(L_0) \overset{P}{\to} Ext_{D(L_0)} \to 0. \quad (5.2)$$

Even the knowledge of $\theta$ does not uniquely fix the extension, but we will see towards the end that the freedom left at this point is very limited; in fact, it is essentially reduced to the choice of a central element. The more important question we must ask at this point is if it is always possible, given $\theta$, to construct an extension.

Thus, given the data $L$, $L_0$ and $\theta$, let us try to construct the Lie algebra $\hat{L}$. First of all, consider the set of all linear maps $\Psi : L \to D(L_0)$ such that $\Psi_\alpha \in D(L_0)$ and $\theta(\alpha) = P \circ \Psi_\alpha$ for all $\alpha$ in $L$. Contrary to $\theta$, which is a homomorphism between $L$ and $Ext_{D(L_0)}$, these functions $\Psi$ may fail to be Lie algebra homomorphisms between $L$ and $D(L_0)$. The amount by which the map $\Psi$ fails to be a Lie algebra homomorphism defines an element of the adjoint algebra. In other words, for any $\alpha$ and $\beta$ in $L$ there is an element $\chi(\alpha, \beta) \in L_0$ such that

$$\Psi_\alpha \circ \Psi_\beta - \Psi_\beta \circ \Psi_\alpha = \Psi_{[\alpha, \beta]} + Ad_{\chi(\alpha, \beta)}. \quad (5.3)$$

For a fixed $\Psi$, the map $\chi : L \wedge L \to L_0$ is of course not unique but it is defined up to a two-cochain

$$\eta : L \wedge L \to C(L_0), \quad (5.4)$$

$C(L_0)$ being the center of $L_0$, i.e., the kernel of the adjoint action $Ad$.

Let us then take an arbitrary pair $\Psi$ and $\chi$ as above and try to define the Lie product on the vector space $L + L_0$ as

$$[(\alpha; a), (\beta; b)] \equiv ([\alpha, \beta]; \chi(\alpha, \beta) + \Psi_\alpha(b) - \Psi_\beta(a) + [a, b]). \quad (5.5)$$

(The Lie product will always be denoted by a square bracket; the particular Lie algebra to which it applies is always clear from its argument.)

It should be emphasized that eq. (5.3) is not just a guess, but in fact the only possible form for the Lie product on the pairs. To see this, suppose just for a moment that there is an
extension $\hat{\mathcal{L}}$. One can then choose any cross section $\sigma : \mathcal{L} \to \hat{\mathcal{L}}$ (i.e., $\pi \circ \sigma = 1$) and uniquely decompose any element $\lambda \in \hat{\mathcal{L}}$ as $\lambda = \sigma(\alpha) + a \equiv (\alpha, a)$, where $\alpha \in \mathcal{L}$ and $a \in i(L_0) \equiv L_0$. The Lie product in $\hat{\mathcal{L}}$ then ensures that

$$[\sigma(\alpha) + a, \sigma(\beta) + b] = [\sigma(\alpha), \sigma(\beta)] + [\sigma(\alpha), b] - [\sigma(\beta), a] + [a, b].$$  \hspace{1cm} (5.6)$$

Equations (5.5) and (5.6) are really the same equation if one sets $\chi(\alpha, \beta) = [\sigma(\alpha), \sigma(\beta)] - \sigma([\alpha, \beta])$ and $\Psi_\alpha(b) = [\sigma(\alpha), b]$. It is easy to check that $\Psi$ and $\chi$ defined in this way satisfy all the required properties. In particular, notice that $[\sigma(\alpha), b]$ is an element of $L_0$ because $L_0$ is an ideal, but it does not necessarily represent an adjoint action because $\sigma(\alpha)$ does not, in general, belong to $L_0$.

Having convinced ourselves that (5.5) is the only possible expression for the Lie product, we must now find under what circumstances the Jacobi identities are satisfied. This yields the relation

$$\chi(\alpha, [\beta, \gamma]) + \Psi_\alpha(\chi(\beta, \gamma)) + \text{cyclic} = 0 \quad \text{for} \quad \alpha, \beta, \gamma \in \mathcal{L}. \hspace{1cm} (5.7)$$

At this point we encounter a potential obstruction to the construction of a Lie product in $\mathcal{L} + L_0$. The relation (5.7) does not necessarily follow for any pair of functions satisfying (5.3). What follows from (5.3) is the weaker condition that the quantity

$$\omega(\alpha, \beta, \gamma) = \chi(\alpha, [\beta, \gamma]) + \Psi_\alpha(\chi(\beta, \gamma)) + \text{cyclic} \quad (5.8)$$

is a three-cocycle valued in the center of $L_0$, i.e., $\omega : \mathcal{L} \wedge \mathcal{L} \wedge \mathcal{L} \to C(L_0)$ such that $\partial \omega = 0$.

Many explanations are in order at this point. First of all, we have to specify what the coboundary operator $\partial$ is. For completeness, we give its expression on an arbitrary $n$-cochain. The case $n = 3$ is of interest here and the cases $n = 1, 2$ will also be needed in a short while. Thus,

$$\partial \omega_n(\alpha_1, \cdots, \alpha_{n+1}) =$$

$$= \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \omega_n([\alpha_i, \alpha_j], \alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_{n+1})$$

$$+ \sum_{1 \leq i \leq n+1} (-1)^{i+1} \Psi_{\alpha_i}(\omega_n(\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_{n+1})). \hspace{1cm} (5.9)$$
This is the usual definition of a coboundary operator, and we know that $\partial^2 = 0$. However, there is a further non-trivial check we must perform. In (5.9) we explicitly used the function $\Psi$, but at the beginning of this section we claimed that the extension is characterized by $\theta$ alone. We must therefore check that for any other $\Psi$ corresponding to the same $\theta$, eq. (5.9) gives the same result. This is fairly obvious if one realizes that two such $\Psi$’s can only differ by an element of the adjoint algebra of $\mathcal{L}_0$, and that this element acts trivially on the center of $\mathcal{L}_0$ itself. This shows that the cohomology of these algebras is specified by $\theta$ alone and not by the particular choices of $\Psi$ and $\chi$. The last check to perform is that $\omega$ defined in (5.8) is in the center of $\mathcal{L}_0$, which can be done by checking that it has vanishing Lie bracket with an arbitrary element of $\mathcal{L}_0$.

Having checked all these points, we can now go back to (5.8) and see where the potential obstruction to constructing the extension can arise. Suppose that a particular choice of $\Psi$ and $\chi$ satisfying (5.3) yields a non-zero $\omega$ in (5.8). We may then try to shift $\omega$ to zero by using the arbitrariness that we have in the choice of $\chi$, and let $\chi \rightarrow \chi + \eta$, $\eta$ being an arbitrary two-cochain. Performing these substitutions in (5.8), one can see that $\omega$ is shifted by the coboundary of $\eta$: $\omega \rightarrow \omega + \partial \eta$. However, $\omega$ need not be a coboundary, only a cocycle.

We therefore come to the conclusion that the potential obstruction in constructing $\hat{\mathcal{L}}$ is an element of the third Lie algebra cohomology group $H^3(\mathcal{L}, C(\mathcal{L}_0))$. Given the triple $\mathcal{L}$, $\mathcal{L}_0$ and $\theta$ one can, by using $\Psi$ and $\chi$, construct uniquely an element $[\omega] \in H^3(\mathcal{L}, C(\mathcal{L}_0))$ that does not depend on the particular choice of $\Psi$ and $\chi$ but only on $\theta$. The Lie algebra $\hat{\mathcal{L}}$ exists if and only if $[\omega] = 0$. Note that this obstruction is never present in the theory of abelian extensions. It will also be absent for the non-abelian extensions we are considering. Its relevance to physics is not well-understood at the moment; it seems to suggest the possibility of having some new form of anomaly, of algebraic nature, when dealing with systems that admit non-abelian extensions.

Before concluding this general review and applying the results to the algebras of interest in this paper, we must go back to the question left unanswered in the beginning of this section of how many inequivalent extensions there are for a fixed, obstruction-free $\theta$.

First, we need to specify what we mean by equivalent extensions; two extensions $\hat{\mathcal{L}}_1$ and $\hat{\mathcal{L}}_2$ of $\mathcal{L}$ by $\mathcal{L}_0$ are said to be equivalent if there is a Lie algebra isomorphism $\Phi : \hat{\mathcal{L}}_1 \rightarrow \hat{\mathcal{L}}_2$. 

that can be written as $\Phi(\alpha, a) = (\alpha, a + \phi(\alpha))$ for some $\phi : L \to L_0$. It should be obvious from the above discussion that two extensions corresponding to two different $\theta$’s cannot possibly be equivalent.

Suppose now for a moment that the two extensions $\hat{L}_1$ and $\hat{L}_2$ are equivalent, i.e., that there is a $\phi$ as above. Then, the pairs $\Psi^1, \chi^1$ and $\Psi^2, \chi^2$, corresponding to $\hat{L}_1$ and $\hat{L}_2$, are related by

$$\Psi^1_\alpha = \Psi^2_\alpha + Ad_{\phi(\alpha)},$$  \hspace{1cm} (5.10)$$
$$\chi^1(\alpha, \beta) = \chi^2(\alpha, \beta) + \Psi^1_\alpha(\phi(\beta)) - \Psi^1_\beta(\phi(\alpha)) + [\phi(\alpha, \phi(\beta)] - \phi([\alpha, \beta]).$$  \hspace{1cm} (5.11)

On the other hand, suppose we are given the two extensions above and we want to find out if they are isomorphic. Since $\Psi^1$ and $\Psi^2$ correspond to the same $\theta$, one can always find a map $\phi$ such that (5.10) is satisfied, and such a $\phi$ is defined up to a one-cochain $\lambda : L \to C(L_0)$.

However, from equation (5.3) we can only infer that the quantity

$$\eta(\alpha, \beta) = -\chi^1(\alpha, \beta) + \chi^2(\alpha, \beta) + \Psi^1_\alpha(\phi(\beta)) - \Psi^1_\beta(\phi(\alpha)) + [\phi(\alpha, \phi(\beta)] - \phi([\alpha, \beta]).$$  \hspace{1cm} (5.12)

is a two cocycle in $C(L_0)$, i.e. $\partial \eta = 0$. If we try to use the arbitrariness in the choice of $\phi$ to shift $\eta$ to zero, we see that by changing $\phi \to \phi + \lambda$ we can shift $\eta$ only by a coboundary $\eta \to \eta + \partial \lambda$. We then come to the conclusion that the non-equivalent extensions are labelled by the elements of the second cohomology group $H^2(L,C(L_0))$. This is precisely the result familiar in the theory of central extensions. Hence, having fixed $\theta$ we only have the freedom of adding a central element. This concludes the review of the general theory.

We can now formulate the construction of our algebra in the language explained above. When presented in this way it seems to be the most natural possible extension, given our previous knowledge for the low-dimensional cases. Thus, let $\Sigma_p$ be a $p$-dimensional manifold thought of as a space-like section of the $(p + 1)$-dimensional worldvolume spanned by the $p$-brane. The charge densities $\tilde{T}^\alpha(\sigma)$ are scalar densities of weight one on $\Sigma_p$. To be specific, we will think of $a$ as an index in the Lie algebra $u(N)$. We will work in the dual picture and think of $L$ as the algebra of true scalars $\alpha : \Sigma_p \to u(N)$. If $\alpha \in L$, the integral $\int_{\Sigma_p} \alpha T$ is well-defined.

In the algebra $L$ we take as our Lie product the pointwise Lie product $[\alpha, \beta](\sigma) \equiv [\alpha(\sigma), \beta(\sigma)]$. 

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We now want to extend this algebra in a way that preserves the full diffeomorphism invariance on $\Sigma_p$. This suggests us to take $L_0$ to be a formal sum of $u(N)$-valued differential forms of degree higher than zero, plus, possibly, a central extension. We also want to retain the same expression for the Lie product that worked in the cases $p = 1, 3$, i.e., for the pairs $a = (X, c_X)$, $X$ being the formal sum of differential forms and $c_X$ a complex number. We thus take (wedge product always understood)

$$
[(X; c_X), (Y; c_Y)] = ([X, Y] + \tilde{k}[dX, dY]; k \int \text{tr}(XdY)).
$$

(5.13)

Note that (5.13) also arises in the canonical formulation of $(2 + 1)$-dimensional non-linear $\sigma$-models [25, 26, 27]. This is further evidence that the above construction may be relevant for other applications.

One could try a general expression of the kind $X = X_1^\mu dx^\mu + \frac{1}{2}X_2^{\mu\nu}dx^\mu \wedge dx^\nu + \cdots$, but we can easily see that forms of odd degree cannot appear in $L_0$ if (5.13) has to satisfy the Jacobi identities. Hence, our ansatz for $L_0$ is the algebra of formal sums of differential forms of even degree $j$, $2 \leq j < p$, $X = X^2 + X^4 + \cdots$, centrally extended by the complex numbers, with Lie product (5.13). To avoid misunderstanding, let us stress that forms of degree higher than $p$, arising from the Lie product, are all vanishing and that the integral is also assumed to vanish on all forms of degree not equal to $p$.

Finally, we must give the homomorphism $\theta$. For our purposes, it is sufficient to give the map $\Psi$:

$$
\Psi_\alpha((X; c_X)) = ([\alpha, X] + \tilde{k}[d\alpha, dX]; k \int \text{tr}(\alpha dX)).
$$

(5.14)

In spite of its similarity with the Lie product (5.13), equation (5.14) is not an adjoint action since $\alpha \notin L_0$. At this point, one can go back to our general discussion and check that no obstructions arise in the construction of $\hat{L}$.

One can now easily see what kind of algebras one gets for different values of $p$. For $p = 1$, $\hat{L}$ coincides with the Kac–Moody algebra. Indeed, for $p = 1$ the $\tilde{k}$ term is not present at all since the only forms allowed on $S^1$ are zero- and one-forms. For $p = 2$, and for all even $p$’s in general, the $k$ term is not present, since it represents the integral of a sum of odd-dimensional forms. In particular, the case $p = 2$ coincides with the algebra arising in the study on non-linear
σ-models in 2 + 1 dimensions. For $p = 3$ one has the Mickelsson–Faddeev algebra, which is the abelian extension that appears both in the study of gauge theories with chiral fermions and in the study of the three-brane.

As long as $p \leq 3$, the algebra $\mathcal{L}$ is an abelian extension of the algebra of charge densities. For higher $p$’s, however, and in particular for $p = 5$, the algebra $\mathcal{L}_0$ becomes non-abelian and, consequently, the algebra $\mathcal{L}$ becomes a non-abelian extension of the algebra of charge densities. An example of a non-vanishing Lie bracket is the one between forms of degree two. It is for these values of $p$ that our algebra differs drastically from those proposed previously in the literature.

6 Conclusions and comments

In this paper we have presented an alternative formulation of the higher-dimensional loop space operators which is based on a new algebra for the $p$-brane. The form of this algebra is essentially fixed by the requirements of linearity, closure, diffeomorphism invariance and the need to accommodate the extensions already present in the $p = 1$ and $p = 3$ cases.

Some of the previously known material, needed mostly for comparison, was summarized in section 2. The explicit form of our algebra and the expression for the BRST operator were given in section 3. In section 4 we presented the construction of the covariant derivative and its curvature tensor. Finally, in section 5, we discussed some more mathematical issues regarding the theory of extensions.

Throughout the paper we restricted ourselves to a local analysis. It would be of interest to study also the “global” properties of an algebra of this kind. In particular, it must be possible to give a meaning to the exponentiation of the algebra and study what further restrictions are imposed on the parameters $k$ and $\tilde{k}$.

As mentioned in the introduction, we do not yet have an action from which our algebra follows by canonical construction. In principle, such an action could be constructed by using the method of coadjoint orbits. It remains to be seen whether this can be done maintaining full diffeomorphism invariance on the $(p + 1)$-dimensional worldvolume.

Finally, the generalization of this construction to superspace should not present much diffi-
culty. It is particularly for this case that one expects to make connections with the dual picture of the superstring.

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A Appendix

A.1 Anomaly formulae

In this appendix we collect explicit expressions for some standard forms entering in the text. The particular expressions given here correspond to the convention of commuting exterior derivative $d$ and BRST operator $\delta$. However, the conversion to the opposite case is straightforward.

We define the Chern-Simons form $\omega^0_{p+2}$ (up to an exact form) by

$$d\omega^0_{p+2}(A) = \text{tr} \left( F^{\frac{p+3}{2}} \right),$$

(A.1)

$F = dA + A^2$ being the curvature two-form of the Yang–Mills field $A$. For $p = 1, 3, 5$ we have explicitly

$$\omega^0_3(A) = \text{tr} \left( FA - \frac{1}{3} A^3 \right),$$

(A.2)

$$\omega^0_5(A) = \text{tr} \left( F^2 A - \frac{1}{2} FA^3 + \frac{1}{10} A^5 \right),$$

(A.3)

$$\omega^0_7(A) = \text{tr} \left( F^3 A - \frac{2}{5} F^2 A^3 - \frac{1}{5} FAF A^2 + \frac{1}{5} FA^5 - \frac{1}{35} A^7 \right).$$

(A.4)

The BRST variation of $\omega^0_{p+2}$, in turn, defines the anomaly $\omega^1_{p+1}$ by the descent equation

$$\delta \omega^0_{p+2} = d\omega^1_{p+1}.$$  

(A.5)

A compact way of writing $\omega^1_{p+1}$ is \cite{38, 20}

$$\omega^1_{p+1} = \text{tr} \left( d\omega \phi_p(A) \right),$$

(A.6)

where

$$\phi_1 = -A,$$

(A.7)

$$\phi_3 = -\frac{1}{2} \left( FA + AF - A^3 \right),$$

(A.8)

$$\phi_5 = -\frac{1}{3} \left( (F^2 A + FAF + AF^2) - \frac{4}{5} (A^3 F + FA^3) - \frac{2}{5} (A^2 FA + AFA^2) + \frac{3}{5} A^5 \right).$$

(A.9)
A.2 The p-brane BRST operator

As was explained in section 3, nilpotency of the p-brane BRST operator, defined by

$$\delta \Phi = (T(R) + \int_{\Sigma_p} \Lambda^1_p) \Phi,$$  \hspace{1cm} (A.10)

requires that $R \equiv R_0 + R_2 + ... + R_{p-1}$ satisfies the equations

$$\delta R - R^2 - (dR)^2 = 0 \hspace{1cm} (A.11)$$

$$n\delta \omega^1_{p+1} - \frac{1}{2} k \text{tr}(dR)^2 = 0 \hspace{1cm} (A.12)$$

Here (A.11) must be satisfied at each form level 0, 2, ... $p - 1$, whereas only the level $p + 1$ is to be considered in (A.12).

We have solved these equations for the cases $p = 1, 3, 5$ by making proper ansätze and then using Mathematica software, developed specifically for the task, to perform the computations. There are no principal difficulties in solving the equations for higher values of $p$, although the number of terms in the ansatz grows rapidly with $p$. However, the expressions obtained are not very enlightening even for lower $p$'s, and they are given here mostly for the sake of completeness.

For the string and the three-brane, (A.12) gives no conditions on $R$ that do not already follow from (A.11). We can therefore use the solution for $R_0$ from the string for the three-brane, and the solutions for $R_0$ and $R_2$ from the three-brane for the five-brane case. We then find the following expressions:

$$R_0 = -\omega,$$  \hspace{1cm} (A.13)

$$R_2 = \tilde{k} \left( \frac{1}{2} [A, d\omega] + (s - \frac{1}{2}) \{A, d\omega\} \right),$$  \hspace{1cm} (A.14)

$$R_4 = \tilde{k}^2 \left( (-\frac{2}{3} + v) A^3 d\omega + (\frac{1}{3} + u) A^2 d\omega A + u A d\omega A^2 + v d\omega A^3 ight. \right.$$  

$$+ \quad t dA d\omega + (-1 - u + v) A dA d\omega + (s + t) dA d\omega A$$  

$$\left. + \quad (-\frac{1}{3} + s^2 + t) A d\omega dA + (-u + v) d\omega dAA + \left( \frac{2}{3} - s + s^2 + t \right) d\omega A dA \right). \hspace{1cm} (A.15)$$

However, when determining $R_4$ we found that (A.12) fixes one of the free parameters in the expression obtained by solving (A.11) only. If one were to proceed to the seven-brane, the latter solution would be the proper one to take over from the five-brane case, since (A.12),
which gives the nilpotency condition for the central term, should be imposed only at form level 

\( p + 1 \).

Apart from the results above, the nilpotency equations also determine the central charge 

\[
  k = \begin{cases} 
    2n, & p = 1 \\
    -\frac{n}{k^2}, & p = 3 \\
    \frac{5n}{2k^2}, & p = 5 
  \end{cases}. 
\]  

(A.16)

**A.3 The covariant derivative**

In section 4 the following ansatz for a loop space covariant derivative was made:

\[
  \mathcal{D} \Phi = \left( D + T(\Omega_{\delta x}) + \int_{\Sigma_p} (i_{\delta x} B_{p+1} + C_{p,\delta x}) \right) \Phi, 
\]  

(A.17)

Here \( \Omega_{\delta x} \equiv \Omega_{0,\delta x} + \Omega_{2,\delta x} + \ldots + \Omega_{p-1,\delta x} \) and \( C_{p,\delta x} \) are the unknown forms to be determined by the covariance equations

\[
  i_{\delta x} dR + di_{\delta x} R + \delta \Omega_{\delta x} + [\Omega_{\delta x}, R] + \tilde{k}[d\Omega_{\delta x}, dR] = 0, 
\]  

(A.18)

\[
  k \text{tr} d\Omega_{\delta x} dR + n di_{\delta x} \omega_{p+1}^1 + d\delta C_{p,\delta x} = 0, 
\]  

(A.19)

where the former consists of separate equations at form level \( 0, 2, \ldots, p - 1 \), while the latter is to be satisfied at level \( p + 1 \) only.

We have solved these equations for \( p = 1, 3, 5 \), using the results from the previous section for \( R \) and \( k \). To begin with, we found that

\[
  C_{p,\delta x}(A) = -n \text{tr} (i_{\delta x} A \phi_p(A)), 
\]  

(A.20)

where \( \phi_p(A) \) is the form that enters in the anomaly \( \omega_{p+1}^1 \), and that we have given explicitly for \( p = 1, 3, 5 \) in section A.1. For \( \Omega_{\delta x} \) we have not been able to find similarly nice expressions. Below, we list the solutions corresponding to the choice \( R_2 = \frac{1}{2}k[A, dw] \), in which case at least \( \Omega_{0,\delta x} \) can be written in a fairly compact way:

\[
  \Omega_{0,\delta x} = i_{\delta x} A, 
\]  

(A.21)
\[ \begin{align*}
\Omega_{2,\delta x} & = \tilde{k}[A, i_{\delta x} dA - di_{\delta x} A + i_{\delta x}(A^2)], \\
\Omega_{4,\delta x} & = \tilde{k}^2 \left( -\frac{1}{4} + t + v \right) dA di_{\delta x} dA + (t + v) di_{\delta x} dA dA \\
& \quad + \left( -\frac{1}{6} + v \right) A^2 di_{\delta x} dA + (\frac{5}{6} - v) A dA di_{\delta x} A + (-\frac{2}{3} - v) A dAi_{\delta x} dA \\
& \quad + \left( \frac{1}{12} + v \right) A di_{\delta x} A dA + (\frac{1}{3} - v) A dAi_{\delta x} dA A + (-\frac{11}{12} + v) Ai_{\delta x} dA dA \\
& \quad + \left( -\frac{1}{4} + v \right) dA A dAi_{\delta x} A + (\frac{3}{4} + v) dA Ai_{\delta x} dA + (\frac{1}{4} - t - v) dA dAi_{\delta x} A \\
& \quad + \left( -\frac{3}{4} + v \right) dA di_{\delta x} A A - \frac{1}{4} dAi_{\delta x} A dA + (\frac{1}{4} + v) dAi_{\delta x} dA A \\
& \quad + \left( -\frac{5}{12} + v \right) di_{\delta x} A A dA + (\frac{1}{6} - v) di_{\delta x} A dA A + (-\frac{1}{2} + v) di_{\delta x} dA A^2 \\
& \quad + (t + v) i_{\delta x} A dA dA + (\frac{1}{12} + v) i_{\delta x} dA A dA + (\frac{4}{3} - v) i_{\delta x} dA dA A \\
& \quad + \frac{1}{2} A^3 di_{\delta x} A - A^3 i_{\delta x} dA + (\frac{1}{6} - v) A^2 dAi_{\delta x} A + \frac{3}{4} A^2 i_{\delta x} A dA + \frac{1}{2} A^2 i_{\delta x} dA A \\
& \quad + \left( \frac{2}{3} + v \right) dA A dAi_{\delta x} A - A dAi_{\delta x} A A - Ai_{\delta x} A A dA - Ai_{\delta x} A dA A - \frac{1}{2} Ai_{\delta x} dA A^2 \\
& \quad + \left( \frac{3}{4} - v \right) dA A^2 i_{\delta x} A - dA Ai_{\delta x} A A + \frac{3}{4} dAi_{\delta x} A A^2 \\
& \quad - \frac{1}{2} di_{\delta x} A A^3 + (\frac{1}{12} + v) i_{\delta x} A A^2 dA + (\frac{4}{3} - v) i_{\delta x} A A dA A \\
& \quad + \left( -\frac{1}{2} + v \right) i_{\delta x} A dA A^2 + i_{\delta x} dA A^3 \\
& \quad + \left( A^4 i_{\delta x} A - \frac{3}{2} A^3 i_{\delta x} A A + A^2 i_{\delta x} A A^2 - \frac{3}{2} Ai_{\delta x} A A^3 + i_{\delta x} A A^4 \right). \quad (A.23)
\end{align*} \]

As was the case with $R$, the general solution for $\Omega_{\delta x}$ for the three-brane was found to carry over to the five-brane, although in this case this happened in a somewhat less trivial manner. Finally, note that one of the originally four free parameters in $R_4$ was fixed by imposing the covariance conditions.
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