GENERALIZED BOUNDARY TRIPLES, WEYL FUNCTIONS
AND INVERSE PROBLEMS

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Abstract. With a closed symmetric operator $A$ in a Hilbert space $\mathcal{H}$ a triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of a Hilbert space $\mathcal{H}$ and two abstract trace operators $\Gamma_0$ and $\Gamma_1$ from $A^*$ to $\mathcal{H}$ is called a generalized boundary triple for $A^*$ if an abstract analogue of the second Green’s formula holds. Various classes of generalized boundary triples are introduced and corresponding Weyl functions $M(\cdot)$ are investigated. The most important ones for applications are specific classes of (essentially) unitary boundary triples for which Green’s second identity admits a certain maximality property which guarantees that the Weyl functions of (the closures of the regularized versions of) boundary triples are Nevanlinna functions on $\mathcal{H}$, i.e. $M(\cdot) \in \mathcal{R}(\mathcal{H})$, or at least they belong to the class $\overline{\mathcal{R}}(\mathcal{H})$ of Nevanlinna families on $\mathcal{H}$. The boundary condition $\Gamma_0 f = 0$ determines a reference operator $A_0 (= \ker \Gamma_0)$. The case where $A_0$ is selfadjoint implies a relatively simple analysis, as the joint domain of the trace mappings $\Gamma_0$ and $\Gamma_1$ admits a von Neumann type decomposition via $A_0$ and the defect subspaces of $A$. The case where $A_0$ is only essentially selfadjoint is more involved, but appears to be of great importance, for instance, in applications to boundary value problems e.g. in PDE setting or when modeling differential operators with point interactions. Wide classes of generalized boundary triples will be characterized in purely analytic terms via the Weyl function $M(\cdot)$ and close interconnections between different classes of boundary triples and the corresponding transformed/renormalized Weyl functions are investigated. These characterizations involve solving direct and inverse problems for specific classes of (unbounded) operator functions $M(\cdot)$. Most involved ones concern operator functions $M(\cdot) \in \mathcal{R}(\mathcal{H})$ for which

$$\tau_{M(z)}(f,g) = (2i \text{Im} z)^{-1}[(M(z)f, g) - (f, M(z)g)], \quad f, g \in \text{dom } M(z),$$

defines a closable nonnegative form on $\mathcal{H}$. It appears that the closability of $\tau_{M(z)}(f,g)$ does not depend on $z \in \mathbb{C}_\pm$ and, moreover, that the closure then is a form domain invariant holomorphic function on $\mathbb{C}_\pm$, while $\tau_{M(z)}(f,g)$ itself is possibly defined only at the point $z$ due to the fact that the equality $\text{dom } M(z) \cap \text{dom } M(\lambda) = \{0\}$ can occur for all points $z \neq \lambda$ from the same halfplane. One of the main results connects these delicate properties of $M(\cdot)$ to the simple geometric condition that $A_0$ is essentially selfadjoint. In this study we also derive several additional new results, for instance, Krejn-type resolvent formulas are extended to the most general setting of unitary and isometric boundary triples appearing in the present work. All the main results are shown to have applications in the study of (ordinary and partial) differential operators. More specifically we treat Laplacian operator on bounded domains with smooth, Lipschitz, or even rough boundary, a mixed boundary value problem for the Laplacian as well as momentum, Schrödinger, and Dirac operators with infinite sequences of local point interactions.

1. A DESCRIPTION OF KEY CONCEPTS AND AN OUTLINE OF MAIN RESULTS

1.1. Ordinary boundary triples and Weyl functions. Let $\mathcal{H}$ be a separable Hilbert space, let $A$ be a not necessarily densely defined closed symmetric operator in $\mathcal{H}$ with equal deficiency.

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indices $n_+(A) = n_-(A) \leq \infty$. The adjoint $A^*$ of the operator $A$ is a linear relation, i.e., a subspace of vectors $\hat{g} = \begin{pmatrix} g \\ f \end{pmatrix} \in \mathcal{H}^2$ such that
\[
(Af, g) - (f, g') = 0 \quad \text{for all} \quad f \in \text{dom } A,
\]
see \cite{22,10}. In what follows the operator $A$ will be identified with its graph, so that the set $C(\mathcal{H})$ of closed linear operators will be considered as a subset of $\tilde{C}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$. Then $A$ is symmetric precisely when $A \subseteq A^*$. The defect subspaces $\mathcal{N}_z$ of $A$ are related to $A^*$ by the equality $\mathcal{N}_z := \ker (A^* - z)$, $z \in \mathbb{C} \setminus \mathbb{R}$ and $n_z := \dim \mathcal{N}_z$.

In the beginning of thirties J. von Neumann \cite{84} created the extension theory of symmetric operators in Hilbert spaces. His approach relies on two fundamental formulas and allowed a description of all selfadjoint ($m$-dissipative) extensions by means of isometric (contractive) operators from $\mathcal{N}_i$ onto $\mathcal{N}_-i$ (see in this connection monographs \cite{2,33,5}). Later on it has been realized that this approach has some drawbacks and was not convenient when, for instance, treating applications to boundary value problems (BVP’s) for ordinary and especially to partial differential equations (ODE and PDE).

During four last decades a new approach to the extension theory has been developed \cite{90,53,50,51,54}, see also \cite{43,44,30}). This approach relies on concepts of abstract boundary mappings and abstract Green’s identity and was introduced independently in \cite{63,29}; in this connection it is also necessary to point out the early paper by J.W. Calkin \cite{31}. Some further discussion on Calkin’s paper is given below.

**Definition 1.1.** A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of a Hilbert space $\mathcal{H}$ and two linear mappings $\Gamma_0$ and $\Gamma_1$ from $A^*$ to $\mathcal{H}$, is said to be an ordinary boundary triple for $A^*$ if:

1.1.1 The abstract Green’s identity
\[
(f', g) - (f, g') = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\mathcal{H}} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{H}}
\]
holds for all $\hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix}$, $\hat{g} = \begin{pmatrix} g \\ g' \end{pmatrix} \in A^*$;

1.1.2 The mapping $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : A^* \to \mathcal{H}^2$ is surjective.

Note that in the ODE setting formula (1.1) turns into the classical Lagrange identity being a key tool in treatment of BVP’s. Advantage of this approach becomes obvious in applications to BVP’s for elliptic equations where formula (1.1) becomes a second Green’s identity. However, in this case the second assumption 1.1.2 is violated and this circumstance was overcome in the classical papers by M. Visik \cite{90} and G. Grubb \cite{53} (see also \cite{54}). Namely, relying on the Lions-Magenes trace theory \cite{76,3,54} they regularized the classical Dirichlet and Neumann trace mappings to get a proper version of Definition 1.1.

The operator $\Gamma$ in Definition 1.1 is called the reduction operator (in the terminology of \cite{31}). Definition 1.1 immediately yields a parametrization of the set of all selfadjoint extensions $A$ of $A$ by means of abstract boundary conditions via
\[
\tilde{A} = A_\Theta := \{ \hat{f} \in A^* : \Gamma \hat{f} \in \Theta \},
\]
where $A_\Theta$ ranges over the set of all selfadjoint extensions of $A$ when $\Theta$ ranges over the set of all selfadjoint relations in $\mathcal{H}$. This correspondence is bijective and in this case
\[
\Theta := \Gamma(\tilde{A})
\]
Two following selfadjoint extensions of $A$ are of particular interest:
\[
A_0 := \ker \Gamma_0 = A_{\Theta_0} \quad \text{and} \quad A_1 := \ker \Gamma_1 = A_{\Theta_1};
\]
The corresponding Weyl function $M$ is introduced and investigated in [41, 42, 43]. Let $A_0$ be a reference operator given by (1.4), let $\rho(A_0)$ be the resolvent set of $A_0$, let $\mathfrak{M}_\lambda := \ker (A^* - \lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}$, be the defect subspace of $A$ and let

$$\hat{\mathfrak{M}}_\lambda := \left\{ \hat{f}_\lambda = \left( \begin{array}{c} f_\lambda \\ \Lambda f_\lambda \end{array} \right) : f_\lambda \in \mathfrak{M}_\lambda \right\}.$$  

**Definition 1.2** ([11][43][44]). The abstract Weyl function and the $\gamma$-field of $A$, corresponding to an ordinary boundary triple $\Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \}$ are defined by

$$M(\lambda) \Gamma_0 \hat{f}_\lambda = \Gamma_1 \hat{f}_\lambda, \quad \gamma(\lambda) \Gamma_0 \hat{f}_\lambda = f_\lambda, \quad \hat{f}_\lambda \in \hat{\mathfrak{M}}_\lambda, \quad \lambda \in \rho(A_0),$$

where $\hat{f}_\lambda$ is given by (1.5).

Notice that when the symmetric operator $A$ is densely defined its adjoint is a single-valued operator and Definitions 1.1 and 1.2 can be used in a simpler form by treating $\Gamma_0$ and $\Gamma_1$ as operators from $\text{dom } A^*$ to $\mathcal{H}$, see [65], [51], [43]. In what follows this convention will be tacitly used in most of our examples.

**Example 1.3.** Let $A$ be a minimal symmetric operator associated in $L^2(\mathbb{R}_+)$ with Sturm-Liouville differential expression

$$L := -\frac{d^2}{dx^2} + q(x), \quad q = \overline{q} \in L^1_{\text{loc}}([0, \infty)).$$

Assume the limit-point case at infinity, i.e. assume that $n_\pm(A) = 1$. The defect subspace $\mathfrak{M}_\lambda$ is spanned by the Weyl solution $\psi(\cdot, \lambda)$ of the equation $Lf = \lambda f$ which is given by

$$\psi(x, \lambda) = c(x, \lambda) + m(\lambda)s(x, \lambda) \in L^2(\mathbb{R}_+),$$

where $c(\cdot, \lambda)$ and $s(\cdot, \lambda)$ are cosine and sine type solutions of the equation $Lf = \lambda f$ subject to the initial conditions

$$c(0, \lambda) = 1, \quad c'(0, \lambda) = 0; \quad s(0, \lambda) = 0, \quad s'(0, \lambda) = 1.$$ 

The function $m(\cdot)$ is called the Titchmarsh-Weyl coefficient of $L$.

In this case a boundary triple $\Pi = \{ \mathbb{C}, \Gamma_0, \Gamma_1 \}$ can be defined as $\Gamma_0 f = f(0), \Gamma_1 f = f'(0)$. The corresponding Weyl function $M(\lambda)$ coincides with the classical Titchmarsh-Weyl coefficient, $M(\lambda) = m(\lambda)$.

In this connection let us mention that the role of the Weyl function $M(\lambda)$ in the extension theory of symmetric operators is similar to that of the classical Titchmarsh-Weyl coefficient $m(\lambda)$ in the spectral theory of Sturm-Liouville operators. For instance, it is known (see [73], [43]) that if $A$ is simple, i.e. $A$ does not admit orthogonal decompositions with a selfadjoint summand, then the Weyl function $M(\lambda)$ determines the boundary triple $\Pi$, in particular, the pair $\{ A, A_0 \}$, uniquely up to unitary equivalence. Besides, when $A$ is simple, the spectrum of $A_\Theta$ coincides with the singularities of the operator function $(\Theta - M(z))^{-1}$; see [43].

As was shown in [43], [44] and [75] the Weyl function $M(\cdot)$ and the $\gamma$-field $\gamma(\cdot)$ both are well defined and holomorphic on the resolvent set $\rho(A_0)$ of the operator $A_0$. Moreover, the $\gamma$-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ satisfy the identities

$$\gamma(\lambda) = [I + (\lambda - \mu)\rho(A_0 - \lambda)^{-1}\gamma(\mu), \quad \lambda, \mu \in \rho(A_0).$$
(1.8) \[ M(\lambda) - M(\mu)^* = (\lambda - \mu)\gamma(\mu)^*\gamma(\lambda), \quad \lambda, \mu \in \rho(A_0). \]

This means that \( M(\cdot) \) is a \( Q \)-function of the operator \( A \) in the sense of Krein and Langer \([72]\).

Denote by \( \mathcal{B}(\mathcal{H}) \) the set of bounded linear operators in \( \mathcal{H} \) and denote by \( \mathcal{R}[\mathcal{H}] \) the class of Herglotz-Nevanlinna functions, i.e., operator valued functions \( F(\lambda) \) with values in \( \mathcal{B}(\mathcal{H}) \), which are holomorphic on \( \mathbb{C} \setminus \mathbb{R} \) and satisfy the conditions

\[
(1.9) \quad F(\lambda) = F(\lambda)^* \quad \text{and} \quad \text{Im} \, F(\lambda) \geq 0 \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

see \([63]\). It follows from (1.7) and (1.8) that \( M \) belongs to the Herglotz-Nevanlinna class \( \mathcal{R}[\mathcal{H}] \).

Furthermore, since \( \gamma(\lambda) \) isomorphically maps \( \mathcal{H} \) onto \( \mathcal{R}_\lambda \), the relation (1.8) ensures that the imaginary part \( \text{Im} \, M(z) \) of \( M(z) \) is positively definite, i.e. \( M(\cdot) \) belongs to the subclass \( \mathcal{R}^u[\mathcal{H}] \) of uniformly strict Herglotz-Nevanlinna functions:

\[
\mathcal{R}^u[\mathcal{H}] := \{ F(\cdot) \in \mathcal{R}[\mathcal{H}] : 0 \in \rho(\text{Im} \, F(i)) \}.
\]

The converse is also true.

**Theorem 1.4** \([73, 44]\). The set of Weyl functions corresponding to ordinary boundary triples coincides with the class \( \mathcal{R}^u[\mathcal{H}] \) of uniformly strict Herglotz-Nevanlinna functions.

### 1.2. \( B \)-generalized boundary triples.

In BVP’s for Sturm-Liouville operator with operator potential, for partial differential operators \([38]\), and in point interaction theory it seems natural to consider more general boundary triples by weakening the surjectivity assumption \([1.1.2]\) in Definition \([1.1]\). The following notion of \( B \)-generalized boundary triple was introduced in \([44]\).

**Definition 1.5.** Let \( A \) be a closed symmetric operator in a Hilbert space \( \mathcal{H} \) with equal deficiency indices and let \( A_* \) be a linear relation in \( \mathcal{H} \) such that \( A \subset A_* \subset \overline{A_*} = A^* \). Then the collection \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \), where \( \mathcal{H} \) is a Hilbert space and \( \Gamma = \{ \Gamma_0, \Gamma_1 \} \) is a single-valued linear mapping from \( A_* \) into \( \mathcal{H}^2 \), is said to be a \( B \)-generalized boundary triple for \( A^* \), if:

1.5.1 the abstract Green’s identity (1.1) holds for all \( \widehat{\mathbf{f}} = \left( \frac{f}{f'} \right), \widehat{\mathbf{g}} = \left( \frac{g}{g'} \right) \in A_* ; \)

1.5.2 \( \text{ran} \, \Gamma_0 = \mathcal{H} ; \)

1.5.3 \( A_0 := \ker \Gamma_0 \) is a selfadjoint relation in \( \mathcal{H} \).

The Weyl function \( M(\lambda) \) corresponding to a \( B \)-generalized boundary triple is defined by the same equality \((1.3)\) where \( \widehat{\mathbf{f}}_\lambda \) runs through \( \mathcal{R}_\lambda \cap A_* \), a proper subset of \( \mathcal{R}_\lambda \). For every \( \lambda \in \rho(A) \) the Weyl function \( M(\lambda) \) takes values in \( \mathcal{B}(\mathcal{H}) \) and this justifies the present usage of the term \( B \)-generalized boundary triple, where “\( B \)” stands for a bounded Weyl function, i.e., a function whose values are bounded operators.

**Example 1.6.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Consider the Laplace operator \(-\Delta \) in \( L^2(\Omega) \). Let \( \gamma_D \) and \( \gamma_N \) be the Dirichlet and Neumann trace mappings. It is well known (see \([3, 54, 76, 81, 89]\) and Section \([7.3]\) for detail and further references) that the mappings \( \gamma_D : H^{3/2}(\Omega) \to H^1(\partial \Omega) \) and \( \gamma_N : H^{3/2}(\Omega) \to H^0(\partial \Omega) = L^2(\partial \Omega) \) are well defined and surjective.

Consider a pre-maximal operator \( A_* \) as the restriction of the maximal Laplace operator \( A_{\text{max}} \) to the domain

\[
(1.10) \quad \text{dom} \, A_* = H^{3/2}_\Delta(\Omega) := H^{3/2}(\Omega) \cap \text{dom} \, A_{\text{max}} = \{ f \in H^{3/2}(\Omega) : \Delta f \in L^2(\Omega) \}.
\]

Using the key mapping properties of \( \gamma_D \) and \( \gamma_N \) one can extend the classical Green’s formula to the domain \( \text{dom} \, A_* \). Notice that the condition \( \gamma_N f = 0, f \in \text{dom} \, A_* \), determines the Neumann realization \( \Delta_N \) of the Laplace operator. Since \( \Delta_N \) is selfadjoint and \( \gamma_N(\text{dom} \, A_* ) = H^0(\partial \Omega) \), the triple \( \Pi = \{ L^2(\partial \Omega), \Gamma_0, \Gamma_1 \} \) with

\[
\Gamma_0 = \gamma_N | \text{dom} \, A_* \quad \text{and} \quad \Gamma_1 = \gamma_D | \text{dom} \, A_*
\]
is a $B$-generalized boundary triple for $A^*$ with $\text{dom } \Gamma = \text{dom } A$. Besides, the corresponding Weyl function $M(\cdot)$ coincides with the inverse of the classical Dirichlet-to-Neumann map $\Lambda(\cdot)$, i.e. $M(\cdot) = \Lambda(\cdot)^{-1}$ (see Chapter 5.4 for details).

It should be noted that the Weyl function $M(\cdot)$ corresponding to a $B$-generalized boundary triple, satisfies the properties (1.7) - (1.9). However, instead of the property $0 \in \rho(\text{Im } M(i))$ one has a weaker condition $0 \not\in \sigma_p(\text{Im } M(i))$. This motivates the following definition.

Denote by $\mathcal{R}^s[\mathcal{H}]$ the class of strict Nevanlinna functions, that is

$$\mathcal{R}^s[\mathcal{H}] := \{ F(\cdot) \in \mathcal{R}[\mathcal{H}] : 0 \not\in \sigma_p(\text{Im } F(i)) \}.$$ 

In fact, it was also shown in [44, Chapter 5] that every $M(\cdot) \in \mathcal{R}^s[\mathcal{H}]$ can be realized as the Weyl function of a certain $B$-generalized boundary triple and hence the following statement holds.

**Theorem 1.7 ([44]).** The set of Weyl functions corresponding to $B$-generalized boundary triples coincides with the class $\mathcal{R}^s[\mathcal{H}]$ of strict Herglotz-Nevanlinna functions.

This realization result as well as the technique of $B$-generalized boundary triples has recently been applied also e.g. to problems in scattering theory (see [19]), in analysis of discrete and continuous time system theory, and in boundary control theory (for some recent papers, see e.g. [15], [59], [80]).

### 1.3. Unitary boundary triples.

A general class of boundary triples, to be called here unitary boundary triples, was introduced in [37]. In fact, the appearance of this concept was motivated by the inverse problem for the most general class of Nevanlinna functions: realize each Nevanlinna function as the Weyl function of an appropriate type generalized boundary triple.

To this end denote by $\mathcal{R}(\mathcal{H})$ the Nevanlinna class of all operator valued holomorphic functions on $\mathbb{C}_+$ (in the resolvent sense) with values in the set of maximal dissipative (not necessarily bounded) linear operators in $\mathcal{H}$. Each $M(\cdot) \in \mathcal{R}(\mathcal{H})$ is extended to $\mathbb{C}_-$ by symmetry with respect to the real line $M(\lambda) = M(\overline{\lambda})^*$; see [72], [37]. Analogous to the subclass $\mathcal{R}^s[\mathcal{H}]$ of bounded Nevanlinna functions $\mathcal{R}[\mathcal{H}]$, the class $\mathcal{R}(\mathcal{H})$ contains a subclass $\mathcal{R}^s(\mathcal{H})$ of strict (in general unbounded) Nevanlinna functions which satisfy the condition

$$\mathcal{R}^s(\mathcal{H}) := \{ F(\cdot) \in \mathcal{R}(\mathcal{H}) : \text{Im } (F(i)h, h) = 0 \Rightarrow h = 0, \ h \in \text{dom } F(i) \}.$$ 

In order to present the definition of a unitary boundary triple, introduce the fundamental symmetries

$$J_\mathcal{H} := \begin{pmatrix} 0 & -iI_\mathcal{H} \\ iI_\mathcal{H} & 0 \end{pmatrix}, \quad J_\mathcal{H} := \begin{pmatrix} 0 & -iI_\mathcal{H} \\ iI_\mathcal{H} & 0 \end{pmatrix},$$

and the associated Krein spaces $(\mathcal{S}^2, J_\mathcal{H})$ and $(\mathcal{H}^2, J_\mathcal{H})$ (see [14], [24]) obtained by endowing the Hilbert spaces $\mathcal{S}^2$ and $\mathcal{H}^2$ with the following indefinite inner products

$$[\hat{f}, \hat{g}]_{\mathcal{S}^2} = (J_\mathcal{S} \hat{f}, \hat{g})_{\mathcal{S}^2}, \quad [\hat{h}, \hat{k}]_{\mathcal{H}^2} = (J_\mathcal{H} \hat{h}, \hat{k})_{\mathcal{H}^2}, \quad \hat{f}, \hat{g} \in \mathcal{S}^2, \quad \hat{h}, \hat{k} \in \mathcal{H}^2.$$ 

This allows to rewrite the Green’s identity (1.1) in the form

$$[\hat{f}, \hat{g}]_{\mathcal{S}^2} = [\Gamma \hat{f}, \Gamma \hat{g}]_{\mathcal{H}^2},$$

which expresses the fact that a mapping $\Gamma$ which satisfies the Green’s identity (1.1) is in fact a $(J_\mathcal{S}, J_\mathcal{H})$-isometric mapping from the Krein space $(\mathcal{S}^2, J_\mathcal{S})$ to the Krein space $(\mathcal{H}^2, J_\mathcal{H})$. If $\Gamma^{[*]}$ denotes the Krein space adjoint of the operator $\Gamma$ (see definition (2.6)), then (1.14) can be simply rewritten as $\Gamma^{-1} \subset \Gamma^{[*]}$. The surjectivity of $\Gamma$ implies that $\Gamma^{-1} = \Gamma^{[*]}$. Following
Yu.L. Shmuljan [88] a linear operator $\Gamma : (\mathcal{H}^2, J_B) \to (\mathcal{H}^2, J_B)$ will be called $(J_B, J_H)$-unitary, if $\Gamma^{-1} = \Gamma^*$. 

**Definition 1.8 (37).** A collection $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a unitary (resp. isometric) boundary triple for $A^*$, if $\mathcal{H}$ is a Hilbert space and $\Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$ is a linear operator from $\mathcal{H}^2$ to $\mathcal{H}^2$ such that:

1.8.1 $A_* := \text{dom} \Gamma$ is dense in $A^*$ with respect to the topology on $\mathcal{H}^2$;

1.8.2 The operator $\Gamma$ is $(J_B, J_H)$-unitary (resp. isometric).

The Weyl function $M(\lambda)$ corresponding to a unitary boundary triple $\Pi$ is defined again by the same formula (1.6). The transposed boundary triple $\Pi^\top := \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ to a unitary boundary triple $\Pi$ is also a unitary boundary triple, the corresponding Weyl function takes the form $M^\top(\lambda) = -M(\lambda)^{-1}$.

The main realization theorem in [37] gave a solution to the inverse problem mentioned above.

**Theorem 1.9 (37).** The class of Weyl functions corresponding to unitary boundary triples coincides with the class $\mathcal{R}^*(\mathcal{H})$ of (in general unbounded) strict Nevanlinna functions.

In fact, in [37, Theorem 3.9] a stronger result is stated showing that the class $\mathcal{R}^*(\mathcal{H})$ can be replaced by the class $\mathcal{R}(\mathcal{H})$ or even by the class $\tilde{\mathcal{R}}(\mathcal{H})$ of Nevanlinna pairs when one allows multivalued linear mappings $\Gamma$ in Definition 1.8. Theorem 1.9 plays a key role in the construction of generalized resolvents in the framework of coupling method that was originally introduced in [36] and developed in its full generality in [38].

In connection with Definition 1.8 we wish to make some comments on a seminal paper [31] by J.W. Calkin, where a concept of the reduction operator is introduced and investigated. Although no proper geometric machinery appears in the definition of Calkin’s reduction operator this notion in the case of a densely defined operator $A$ essentially coincides with concept of a unitary operator between Krein spaces as in Definition 1.8. An overview on the early work of Calkin and some connections to later developments can be found from the papers in the monograph [59]; for a further discussion see also Section 3.5.

Despite of the complete solution to the realization problem given in Theorem 1.9 more specifically determined subclasses of unitary boundary triples and closely connected isometric triples together with the associated subclasses of Weyl functions are both of theoretical and practical interest since they naturally appear, for instance, in various problems of mathematical physics. Such questions, sometimes of rather delicate nature, lead to inverse as well as direct problems for $R$-functions in the classes $\mathcal{R}(\mathcal{H})$ and $\tilde{\mathcal{R}}(\mathcal{H})$, where precisely prescribed analytic properties of operator functions have to be connected to appropriately determined geometric properties of associated boundary triples and vice versa. These problems have motivated the present work and resulted in several further subclasses of holomorphic operator functions, included or at least closely related to the class of $\mathcal{R}(\mathcal{H})$ of Herglotz-Nevanlinna functions, all of them occurring in boundary value problems in the ODE and PDE settings.

The notions of ordinary boundary triples and $B$-generalized boundary triples turned out to be unitary boundary triples; see [37]. In particular, a unitary boundary triple $\Pi$ is an ordinary ($B$-generalized) boundary triple if and only if the corresponding Weyl function $M(\cdot)$ belongs to the class $\mathcal{R}^u[\mathcal{H}]$ (resp. $M(\cdot) \in \mathcal{R}^s[\mathcal{H}]$). In Section 5 we consider two further subclasses of unitary boundary triples: $S$-generalized and $ES$-generalized boundary triples. For deriving some of the main results on unitary boundary triples we have established some new facts on the interaction between $(J_B, J_H)$-unitary relations and unitary colligations appearing e.g. in system theory and in the analysis of Schur functions, see Section 5.1 a background for this connection can be found from [16]. On the other hand, in Section 5.2 we extend Krein’s resolvent formula...
to the general setting of unitary boundary triples \( \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \). Namely, for any proper extension \( A_\Theta \in \text{Ext}_S \) satisfying \( A_\Theta \subset \text{dom} \Gamma \) the following Krein-type formula holds:

\[
(A_\Theta - \lambda)^{-1} - (A_0 - \lambda)^{-1} = \gamma(\lambda)(\Theta - M(\lambda))^{-1} \gamma(\lambda)^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

It is emphasized that in this formula \( A_\Theta \) is not necessarily closed and it is not assumed that \( \lambda \in \rho(A_\Theta) \), in particular, here the inverses \( (A_\Theta - \lambda)^{-1} \) and \( (\Theta - M(\lambda))^{-1} \) are understood in the sense of relations, see Theorem 5.9 for an analogous formula see also Theorem 4.11.

1.4. \textit{S-generalized boundary triples.} Following \cite{37} we consider a special class of unitary boundary triples singled out by the condition that \( A_0 := \ker \Gamma_0 \) is a selfadjoint extension of \( A \).

\textbf{Definition 1.10 (37).} A unitary boundary triple \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) is said to be an \textit{S-generalized boundary triple} for \( A^* \) if the assumption \cite{15.3} holds, i.e. \( A_0 := \ker \Gamma_0 \) is a selfadjoint extension of \( A \).

Next following \cite{39} Theorem 7.39 and \cite{37} Theorem 4.13 (see also an extension given in Theorem 5.13 below) we present a complete characterization of the Weyl functions \( M(\cdot) \) corresponding to \( S \)-generalized boundary triples.

\textbf{Theorem 1.11.} (37, 39) Let \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be a unitary boundary triple for \( A^* \) and let \( M(\cdot) \) and \( \gamma(\cdot) \) be the corresponding Weyl function and \( \gamma \)-field, respectively. Then the following statements are equivalent:

(i) \( A_0 = \ker \Gamma_0 \) is selfadjoint, i.e. \( \Pi \) is an \( S \)-generalized boundary triple;

(ii) \( A_s = A_0 \oplus \hat{\mathfrak{R}}_\lambda \) and \( A_s = A_0 \oplus \hat{\mathfrak{N}}_\mu \) for some (equivalently for all) \( \lambda \in \mathbb{C}_+ \) and \( \mu \in \mathbb{C}_- \);

(iii) \( \text{ran} \Gamma_0 = \text{dom} M(\lambda) = \text{dom} M(\mu) \) for some (equivalently for all) \( \lambda \in \mathbb{C}_+ \) and \( \mu \in \mathbb{C}_- \);

(iv) \( \gamma(\lambda) \) and \( \gamma(\mu) \) are bounded and densely defined in \( \mathcal{H} \) for some (equivalently for all) \( \lambda \in \mathbb{C}_+ \) and \( \mu \in \mathbb{C}_- \);

(v) \( \text{Im} M(\lambda) \) is bounded and densely defined for some (equivalently for all) \( \lambda \in \mathbb{C} \setminus \mathbb{R} \);

(vi) the Weyl function \( M(\cdot) \) belongs to \( \mathcal{R}(\mathcal{H}) \) and it admits a representation

\[
M(\lambda) = E + M_0(\lambda), \quad M_0(\cdot) \in \mathcal{R}[\mathcal{H}], \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

where \( E = E^* \) is a selfadjoint (in general unbounded) operator in \( \mathcal{H} \).

Here the symbol \( \oplus \) means the componentwise sum of two linear relations, see (2.2). Notice that, for instance, the implications (i) \( \Rightarrow \) (ii), (iii) are immediate from the following decomposition of \( A_s := \text{dom} \Gamma \):

\[
A_s = A_0 \oplus \hat{\mathfrak{R}}_\lambda(A_s), \quad \lambda \in \rho(A_0).
\]

In accordance with (1.16) the Weyl function corresponding to an \( S \)-generalized boundary triple is an operator valued Herglotz-Nevanlinna function with domain invariance property: \( \text{dom} M(\lambda) = \text{dom} E = \text{ran} \Gamma_0, \ \lambda \in \mathbb{C}_\pm \). It takes values in the set \( \mathcal{C}(\mathcal{H}) \) of closed (in general unbounded) operators while values of the imaginary parts \( \text{Im} M(\lambda) \) are bounded operators.

As an example we mention that the transposed boundary triple \( \Pi^\top = \{ L^2(\partial \Omega), \Gamma_1, -\Gamma_0 \} \) from the PDE Example 1.6 is an \( S \)-generalized boundary triple. The corresponding Weyl function coincides (up to sign change) with the Dirichlet-to-Neumann map \( \Lambda(\cdot) \), i.e. \( M(\cdot)^\top = -\Lambda(\cdot) \); see Proposition 7.1.

1.5. \textit{ES-generalized boundary triples and form domain invariance.} Next we discuss one of the main objects appearing in the present work; it is easily introduced by the following definition.

\textbf{Definition 1.12.} A unitary boundary triple \( \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) for \( A^* \) is said to be an \textit{essentially selfadjoint generalized boundary triple}, in short, \( ES \)-generalized boundary triple for \( A^* \), if:
1.12.1 $A_0 := \ker \Gamma_0$ is an essentially selfadjoint linear relation in $\mathcal{H}$.

To characterize the class of ES-generalized boundary triples in terms of the corresponding Weyl functions we associate with each $M(\cdot)$ a family of nonnegative quadratic forms $t_{M(\lambda)}$ in $\mathcal{H}$:

\begin{equation}
(1.18) \quad t_{M(\lambda)}[u, v] := \frac{1}{\lambda - \lambda} [(M(\lambda)u, v) - (u, M(\lambda)v)], \quad u, v \in \text{dom} (M(\lambda)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

The forms $t_{M(\lambda)}$ are not necessarily closable. However, it is shown that if $t_{M(0)}$ is closable at one point $\lambda_0 \in \mathbb{C}_+$, then $t_{M(\lambda)}$ is closable for each $\lambda \in \mathbb{C}_+$. In the latter case the domain of the closure $\overline{t}_{M(\lambda)}$ does not depend on $\lambda \in \mathbb{C}_+$, i.e. the family $\overline{t}_{M(\lambda)}$ is domain invariant.

In what follows one of the main results established in this connection reads as follows (cf. Theorem 5.26).

**Theorem 1.13.** Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triple for $A^*$. Let also $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and $\gamma$-field, respectively. Then the following statements are equivalent:

(i) $\Pi$ is ES-generalized boundary triple for $A^*$;

(ii) $\gamma(\pm i)$ is closable;

(iii) $\gamma(\lambda)$ is closable for every $\lambda \in \mathbb{C}_+$ and $\text{dom} \gamma(\lambda) = \text{dom} \gamma(\pm i), \lambda \in \mathbb{C}_+$;

(iv) the form $t_{M(\pm i)}$ is closable;

(v) the form $t_{M(\lambda)}$ is closable for each $\lambda \in \mathbb{C}_+$ and dom $\overline{t}_{M(\lambda)} = \text{dom} \overline{t}_{M(\pm i)}, \lambda \in \mathbb{C}_+$;

(vi) the Weyl function $M(\cdot)$ belongs to $\mathcal{R}^* (\mathcal{H})$ and is form domain invariant in $\mathbb{C}_+$.

If $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ES-generalized, but not $S$-generalized, boundary triple for $A^*$, then the equality (1.17) fails to hold and turns out to be an inclusion

\begin{equation}
(1.19) \quad A_0 \supseteq \hat{\mathcal{N}}_A(A_*) \subset A_* = \overline{A_0} \supseteq \hat{\mathcal{N}}_A(A_*), \quad \lambda \in \rho (A_0).
\end{equation}

Indeed, since $A_0$ is not selfadjoint (while it is essentially selfadjoint), the decomposition $A_* = A_0 \supseteq \hat{\mathcal{N}}_A(A_*)$ doesn’t hold; see [37] Theorem 4.1.3. Then there clearly exist $\hat{f} \in A_*$ which does not belong to $A_0 \supseteq \hat{\mathcal{N}}_A(A_*)$, so that $\Gamma_0 \hat{f} \neq 0$ as well as $\Gamma_0 \hat{f} \notin \Gamma_0(\hat{\mathcal{N}}_A(A_*)) = \text{dom} M(\lambda)$. In particular, in this case a strict inclusion $\text{dom} M(\lambda) \subset \text{ran} \Gamma_0$ holds and, consequently, the Weyl function $M(\lambda)$ can loose e.g. the domain invariance property. However, the domain of the closure $\overline{\Gamma}_0$ contains the selfadjoint relation $\overline{A}_0$ and admits the decomposition

\begin{equation}
(1.20) \quad \text{dom} \overline{\Gamma}_0 = \overline{\text{dom}} (\overline{\Gamma}_0 \cap \hat{\mathcal{N}}_A(A_*)), \quad \lambda \in \rho (\overline{\text{dom}})\).
\end{equation}

This yields the equality

$$\text{dom} \overline{\gamma(\lambda)} = \overline{\text{dom}} (\overline{\Gamma}_0 \cap \hat{\mathcal{N}}_A(A_*)) = \text{ran} \overline{\Gamma}_0,$$

which in combination with the equality $\text{dom} \overline{t}_{M(\lambda)} = \text{dom} \overline{\gamma(\lambda)}$ yields the form domain invariance property for $M$:

\begin{equation}
(1.21) \quad \text{dom} \overline{t}_{M(\lambda)} = \text{ran} \overline{\Gamma}_0.
\end{equation}

Passing from the case of an $S$-generalized boundary triple to the case of an ES-generalized triple (which is not $S$-generalized) means that $A_0 \neq A_0^*$. Then, in particular, conditions (ii) and (iii) in Theorem 1.11 are necessary violated. We split the situation into two different cases:

**Assumption 1.14.** $M(\lambda)$ is not domain invariant, i.e. $\text{dom} M(\lambda_1) \neq \text{dom} M(\lambda_2)$ at least for two points $\lambda_1, \lambda_2 \in \mathbb{C}_+, \lambda_1 \neq \lambda_2$, while it is form domain invariant, i.e. $\text{dom} \overline{t}_{M(\lambda)} = \text{dom} \overline{t}_{M(\pm i)}, \lambda \in \mathbb{C}_+$.

**Assumption 1.15.** $\text{dom} M(\lambda) = \text{dom} M(\pm i), \lambda \in \mathbb{C}_+$, while $\text{dom} M(\pm i) \subset \text{ran} \Gamma_0$. 
In the next two examples we demonstrate that both possibilities appear in the spectral theory. It is first shown that $R$-functions satisfying Assumption $\ref{assumption1}$ naturally arise in the theory of differential operators with boundary conditions involving $\lambda$-depending spectral parameters.

Example 1.16. Let $\varphi(\cdot)$ be a scalar $R$-function and $H = L^2(0, \infty)$. Define an operator valued function $G(\cdot) = G_\varphi(\cdot)$ by setting

$$G_\varphi(z)f = -i\frac{d^2u}{dx^2}, \quad \text{dom}(G_\varphi(z)) = \{u \in W^2_2(\mathbb{R}_+): u'(0) = \varphi(z)u(0), \quad z \in \mathbb{C}_+\}.$$  

Clearly, $G_\varphi(z)$ is densely defined, $\rho(G_\varphi(z)) \neq \emptyset$ for each $z \in \mathbb{C}_+$ and the family $G_\varphi(\cdot)$ is holomorphic in $\mathbb{C}_+$ in the resolvent sense. Integrating by parts one obtains

$$t_{G(z)}[u] := \text{Im}(G_\varphi(z)u, u) = \int_{\mathbb{R}_+} |u'(x)|^2 dx + \text{Im}(\varphi(z)|u(0)|^2, \quad u \in \text{dom} t_{G(z)} = \text{dom}(G_\varphi(z)).$$

Hence the form $t_{G(z)}$ is nonnegative and $G_\varphi(z)$ is $m$-dissipative for each $z \in \mathbb{C}_+$. Moreover, $G(\cdot) \in R^*(H)$ since $\ker t_{G(z)} = \{0\}$. Therefore, by Theorem $\ref{thm1.9}$ there exists a certain unitary boundary triple such that the corresponding Weyl function coincides with $G(\cdot)$.

The form $t_{G(z)}$ is closable and its closure is given by

$$(1.22) \quad \overline{t}_{G(z)}[u] = \int_{\mathbb{R}_+} |u'(x)|^2 dx + \text{Im}(\varphi(z)|u(0)|^2, \quad \text{dom} \overline{t}_{G(z)} = W^1_2(\mathbb{R}_+), \quad z \in \mathbb{C}_+.$$  

Thus, form domain $\text{dom}(\overline{t}_{G(z)}) = W^1_2(\mathbb{R}_+)$ does not depend on $z \in \mathbb{C}_+$ while the domain $\text{dom} G(z)$ does, i.e. $G$ satisfies the Assumption $\ref{assumption1}$. The operator associated with the form $\overline{t}_{\varphi(z)}$ (the imaginary part of $G_\varphi(z)$) is given by

$$G_{\varphi,I}(z)u = -\frac{d^2u}{dx^2}, \quad \text{dom}(G_{\varphi,I}(z)) = \{u \in W^2_2(\mathbb{R}_+): u'(0) = (\text{Im} \varphi(z))u(0), \quad z \in \mathbb{C}_+\}.$$  

Next we present an example of the Weyl function satisfying Assumption $\ref{assumption1}$. Such $R$-functions arise in the theory of Schrödinger operators with local point interactions.

Example 1.17. Let $X = \{x_n\}_{n=1}^\infty$ be a strictly increasing sequence of positive numbers satisfying $\lim_{n \to \infty} x_n = \infty$. Denote $x_0 = 0$,

$$(1.23) \quad d_n := x_n - x_{n-1} > 0, \quad 0 \leq d_* := \inf_{n \in \mathbb{N}} d_n, \quad d^* := \sup_{n \in \mathbb{N}} d_n \leq \infty.$$  

Let also $H_n$ be a minimal operator associated with the expression $-\frac{d^2}{dx^2}$ in $L^2_0[0, x_n]$. Then $H_n$ is a symmetric operator with deficiency indices $n_+ (H_n) = 2$ and domain $\text{dom}(H_n) = W^{2,2}_0[0, x_n]$. Consider in $L^2(\mathbb{R}_+)$ the direct sum of symmetric operators $H_n$,

$$H := H_{\text{min}} = \bigoplus_{n=1}^\infty H_n, \quad \text{dom}(H_{\text{min}}) = W^{2,2}_0(\mathbb{R}_+) \setminus X = \bigoplus_{n=1}^\infty W^{2,2}_0[0, x_n].$$

It is easily seen that a boundary triple $\Pi_n = \{\mathbb{C}^2, \Gamma^{(n)}_0, \Gamma^{(n)}_1\}$ for $H_n^\ast$ can be chosen as

$$(1.24) \quad \Gamma^{(n)}_0 f := \begin{pmatrix} f'(x_n) f'(-x_n) \end{pmatrix}, \quad \Gamma^{(n)}_1 f := \begin{pmatrix} -f(x_n) f(-x_n) \end{pmatrix}, \quad f \in W^2_2[0, x_n].$$

The corresponding Weyl function $M_n$ is given by

$$(1.25) \quad M_n(z) = \frac{1}{\sqrt{z}} \begin{pmatrix} \cot(\sqrt{z}d_n) & -\frac{1}{\sin(\sqrt{z}d_n)} \\ -\frac{1}{\sin(\sqrt{z}d_n)} & \cot(\sqrt{z}d_n) \end{pmatrix}.$$
Clearly, $H = H_{\text{min}}$ is a closed symmetric operator in $L^2(\mathbb{R}_+)$. Next we put

$$
H = l^2(\mathbb{N}) \otimes \mathbb{C}^2, \quad \Gamma = \left( \begin{array}{c} \Gamma_0 \\ \Gamma_1 \end{array} \right) := \bigoplus_{n=1}^{\infty} \left( \begin{array}{c} \Gamma_0^{(n)} \\ \Gamma_1^{(n)} \end{array} \right)
$$

and note that in accordance with the definition of the direct sum of linear mappings

$$\text{dom} \Gamma := \{ f = \bigoplus_{n=1}^{\infty} f_n \in \text{dom} A^* : \sum_{n \in \mathbb{N}} \| \Gamma_j^{(n)} f_n \|_{L^2} < \infty, j \in \{0, 1\} \}.$$

We also put $\bar{\Gamma}_j := \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}$ and note that it is a closure of $\Gamma_j = \bar{\Gamma}_j \mid \text{dom} \Gamma$, $j = 1, 2$. As stated in the next theorem the orthogonal sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n$ of the boundary triples $\Pi_n$ determines an ES-generalized boundary triple of desired type.

Notice that the minimal operator $H$ as well as the corresponding triple $\Pi$ for $H^*$ naturally arise when treating the Hamiltonian $H_{X,\alpha}$ with $\delta$-interactions in the framework of extension theory. The latter have appeared in various physical problems as exactly solvable models that describe complicated physical phenomena (see e.g. [4, 5, 45, 70] for details).

The next theorem completes the results from [69] regarding the non-regularized boundary triple $\Pi = \bigoplus_{n \in \mathbb{N}} \Pi_n$ in the case $d_* = 0$. The proof is postponed until Section 8.3.

**Theorem 1.18.** Let $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n = \{H, \Gamma_0, \Gamma_1\}$ be the direct sum of boundary triples $\Pi_n$ defined by (1.24), (1.26), let $M(\cdot)$ be the corresponding Weyl function, and let $d_* = 0$ and $d^* < \infty$. Then the following statements hold:

(i) The triple $\Pi$ is an ES-generalized boundary triple for $H_{\text{min}}^*$ such that $A_0 \neq A_0^*$.

(ii) The Weyl function $M(\cdot)$ is domain invariant and its domain is given by

$$\text{dom} M(z) = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right) : \{a_n - b_n\}_{n=1}^{\infty} \in l^2(\mathbb{N}) \otimes \mathbb{C}^2, \{d_n^{-2}\}_{n=1}^{\infty} \in l^2(\mathbb{N}) \right\}, \quad z \in \mathbb{C}_+.$$

(iii) The range of $\Gamma_0$ is given by

$$\text{ran} \Gamma_0 = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right) : \{a_n - b_n\}_{n=1}^{\infty} \in l^2(\mathbb{N}) \otimes \mathbb{C}^2, \{d_n^{-2}\}_{n=1}^{\infty} \in l^2(\mathbb{N}) \right\} \supseteq \text{dom} M(\pm i).$$

(iv) The domain of the form $\mathfrak{t}_{M(z)}$ generated by the imaginary part $\text{Im} M(z)$ is given by

$$\text{dom} \mathfrak{t}_{M(z)} = \text{ran} \Gamma_0, \quad z \in \mathbb{C}_+.$$

(v) The transposed triple $\Pi^\top$ is an $S$-generalized boundary triple for $H_{\text{min}}^*$, i.e. $A_1 = A_1^*$. However, it is not a $B$-generalized boundary triple for $H_{\text{min}}^*$.

Combining statements (ii) and (iii) of Theorem 1.18 yields that in the case $d_* = 0$ the Weyl function $M(\cdot)$ corresponding to the triple $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ satisfies Assumption 1.17, i.e. it is domain invariant, $\text{dom} M(z) = \text{dom} M(i)$, $z \in \mathbb{C}_+$, while $\text{dom} M(i) \subseteq \text{ran} \Gamma_0$.

Hence, by Theorem 1.11, $A_0 \neq A_0^*$ and $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ being ES-generalized, is not an $S$-generalized boundary triple for $H^*$.

It is emphasized that in this case we compute $\text{dom} M(z)$, $\text{dom} \mathfrak{t}_{M(z)}$, and $\text{ran} \Gamma_0$ explicitly. Notice also that in this case the Weyl function $M(\cdot)$ as well as its imaginary part $\text{Im} M(\cdot)$ take values in the set of unbounded operators. In addition to the Schrödinger operators introduced in Example 1.17 analogous results for moment and Dirac operators with local point interactions are established in Section 8.

Before closing this subsection we wish to mention that other type of examples for ES-generalized boundary triples are the Krein - von Neumann Laplacian appearing in Sections 7.1, 7.3 and the Zaremba Laplacian for mixed boundary value problem treated in Section 7.2.
1.6. **AB-generalized boundary triples and quasi boundary triples.** The following definition of an **AB-generalized boundary triple** is a weakening of the notions of B-generalized and $S$-generalized boundary triples as well as of the class of quasi boundary triples.

**Definition 1.19.** A collection $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is said to be an **almost B-generalized boundary triple**, or briefly, an **AB-generalized boundary triple** for $A^*$, if $A^* = \text{dom} \Gamma$ is dense in $A^*$ and the following conditions are satisfied:

1.19.1 the abstract Green’s identity \[ \text{(1.11)} \] holds for all $\hat{f}, \hat{g} \in A*$;
1.19.2 $\Gamma_0$ is dense in $\mathcal{H}$;
1.19.3 $A_0 := \ker \Gamma_0$ is a selfadjoint relation in $\mathcal{H}$.

It is shown in Proposition 1.10 that the Weyl function $M(\cdot)$ of an AB-generalized boundary triple admits a similar characterization as the Weyl function corresponding to an $S$-generalized boundary triple.

**Proposition 1.20.** The class of **AB-generalized boundary triples** coincides with the class of isometric boundary triples such that the corresponding Weyl functions satisfy the condition \[ \text{(1.11)} \] and they are of the form

\[ M(\lambda) = E + M_0(\lambda), \quad M_0(\cdot) \in \mathcal{R}[\mathcal{H}], \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \]

with $E$ a symmetric densely defined operator in $\mathcal{H}$. In particular, every function $M(\cdot)$ of the form \[ \text{(1.27)} \] such that $\ker \text{Im} M_0(\lambda) \cap \text{dom} E = \{0\}$ is a Weyl function of a certain **AB-generalized boundary triple**.

Moreover, $E = E^*$ if and only if $\Gamma$ is $(J_0, J_\mathcal{H})$-unitary (see Definition 1.8), i.e., if and only if the **AB-generalized boundary triple** $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is also **S-generalized**.

Further properties of **AB-generalized boundary triples** are studied in Section 4.

A connection between $ES$- and **AB-generalized boundary triples** is established in Theorem 5.31. More precisely, it is shown that for every strict form domain invariant operator valued Nevanlinna function $M \in \mathcal{R}^+(\mathcal{H})$ there exist a bounded operator $G \in \mathcal{B}(\mathcal{H})$ with $\ker G = \ker G^* = \{0\}$, a closed symmetric densely defined operator $E$ in $\mathcal{H}$, and a bounded Nevanlinna function $M_0 \in \mathcal{R}[\mathcal{H}]$, with the property

\[ \mathcal{H} = \text{clo} \mathcal{D}_\lambda := \text{clo} \{ h \in \mathcal{H} : (\text{Im } M_0(\lambda))^2 h \in \text{ran } G^* \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \]

such that

\[ M(\lambda) = G^{-1}(E + M_0(\lambda))G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

Conversely, every Nevanlinna function $M(\cdot)$ of the form \[ \text{(1.29)} \] is form domain invariant on $\mathbb{C} \setminus \mathbb{R}$, whenever $E \subset E^*$, $G \in \mathcal{B}(\mathcal{H})$, $\ker G = \ker G^* = \{0\}$, and $M_0 \in \mathcal{R}[\mathcal{H}]$ satisfies \[ \text{(1.28)} \].

Theorem 5.31 offers a renormalization procedure which produces from a form domain invariant Weyl function a domain invariant Weyl function, whose imaginary part in standard operator sense becomes a well-defined and bounded operator function on $\mathbb{C} \setminus \mathbb{R}$, i.e., the renormalized boundary triple is **AB-generalized**. Some related results, showing how $B$-generalized boundary triples give rise to **ES-generalized boundary triples**, are established in Section 6.1.

These results are applied in the analysis of regularized trace operators for Laplacians.

On the other hand, since every **AB-generalized boundary triple** can be regularized to produce a $B$-generalized boundary triple, see Theorem 1.3 and every $B$-generalized boundary triple can be regularized to produce an ordinary boundary triple, see Theorem 1.10 there is a controlled connection from **ES-generalized boundary triples** to ordinary boundary triples. In this way these abstract renormalization procedures open an avenue e.g. to complete spectral analysis and related well-established investigations in extension theory of symmetric operators and its
various applications. It should be pointed out that, only for ordinary boundary triples, the pair of boundary mappings \( \Gamma = \{ \Gamma_0, \Gamma_1 \} \) provides a topological isomorphism between the set of all linear relations in the parameter space \( \mathcal{H} \) and the complete class of intermediate extensions \( \tilde{A} \) lying between \( A \) and its adjoint \( A^* \).

The class of \( AB \)-generalized boundary triples contains the class of so-called quasi boundary triples, which has been studied in J. Behrndt and M. Langer [17].

**Definition 1.21** ([17]). Let \( A \) be a densely defined symmetric operator in \( \mathcal{H} \). A triple \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) is said to be a *quasi boundary triple* for \( A^* \), if the following conditions are satisfied:

1.21.1 \( A^* := \text{dom } \Gamma \) is dense in \( A^* \) with respect to the topology on \( \mathcal{H}^2 \) and the Green’s identity (1.1) holds for all \( \hat{f}, \hat{g} \in A^* \);
1.21.2 the range of \( \Gamma \) is dense in \( \mathcal{H} \times \mathcal{H} \);
1.21.3 \( A_0 = \ker \Gamma_0 \) is a selfadjoint operator in \( \mathcal{H} \).

In the definition of a quasi boundary triple Assumption 1.19.2 is replaced by the stronger assumption that the joint range of \( \Gamma = \{ \Gamma_0, \Gamma_1 \} \) is dense in \( \mathcal{H} \times \mathcal{H} \). The Weyl function corresponding to a quasi boundary triple is again defined by the same formula (1.6). The notion of quasi boundary triple proved to be useful in elliptic theory [17], see also [20, 85].

A connection between quasi boundary triples and \( AB \)-generalized boundary triples is given in Corollary 4.8. A joint feature in \( AB \)-generalized boundary triples and quasi boundary triples is that without additional assumptions on the mapping \( \Gamma = \{ \Gamma_0, \Gamma_1 \} \) these boundary triples are not unitary. Consequently, their Weyl functions need not belong to the class of Nevanlinna functions; i.e. the values \( M(\lambda) \) need not be maximal dissipative (accumulative) in \( \mathbb{C}_+(\mathbb{C}_-) \). More explicitly, the defect numbers of the operator \( E \) in (1.27) need not be equal; in which case even after taking closures of \( \Gamma \) and \( M(\lambda) \) this situation is not changed.

A complete characterization (a realization result) of the set of Weyl functions corresponding to \( AB \)-generalized and quasi boundary triples is given by formula (1.16) with \( M_0(\cdot) \) belonging to the class \( \mathcal{R}[\mathcal{H}] \) while \( E \) is a symmetric, but not necessarily selfadjoint, operator, such that

\[ \text{dom } E^* \cap \ker \text{Im } M_0(\lambda) = \{0\}; \]

the role of this last condition is connected with the assumption 1.21.2.

Different applications of quasi boundary triples in boundary value problems including applications to elliptic theory and trace formulas can be found e.g. in [17], [20], [21], [29], and [85].

**Remark 1.22.** A connection between \( B \)-generalized boundary triples and quasi boundary triples appears in [39, Theorem 7.57] and in a more precise form is given in [92, Propositions 5.1, 5.3]; see also [91]. A slightly more general result can be found in Corollary 4.8. Further results and a more detailed discussion on these connections are given in Sections 4 and 5.

### 1.7. Preparatory results for applications.

Section 6 is devoted to the study of certain specific types of boundary triples offering also applicable abstract frameworks for including trace operators in a boundary triple environment. In Section 6.1 it is shown how certain simple transforms of \( B \)-generalized boundary generate \( ES \)-generalized boundary triples; in Section 7 such transforms are identified and made explicit in the Laplace setting.

In typical applications to elliptic PDE’s the minimal operator \( A \) is nonnegative and the Dirichlet-to-Neumann map \( \Lambda(\cdot) \) is constructed at the origin \( z = 0 \) or at some point \( x < 0 \) on the real line in such a way that \( \Lambda(x) \) is a nonnegative selfadjoint operator in the boundary space \( L^2(\partial \Omega) \). Under weak additional assumptions this implies that the corresponding boundary triple is not only isometric but, in fact, unitary. In this connection we offer the following analytic extrapolation principle for Weyl functions initially defined only at one real
regular point and then extended in an appropriate manner to the complex half-planes. In particular, this result can be used to check whether a pair of boundary mappings \( \{ \Gamma_0, \Gamma_1 \} \) satisfying Green’s identity (1.1) determines a unitary boundary triple.

**Proposition 1.23.** Let \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be an isometric boundary triple for \( A^* \), let \( H \) with \( \text{dom } H \subset \text{dom } A_\gamma(= \text{dom } \Gamma) \) be a selfadjoint extension of \( A \) admitting a real regular point \( x \in \rho(H) \), and let the mapping \( M(x) \) be defined by the formula (1.6) for all \( \hat{f}_x \in \mathcal{N}_x(A_\gamma) \).

If \( M(x) \) satisfies the conditions

\[
M(x) = M(x)^* \quad \text{and} \quad 0 \in \rho(M(x) + x),
\]

then \( M(x) \) admits an analytic extrapolation \( M(z) \) to the half-planes \( z \in \mathbb{C}_\pm \) defining a function in the class \( \mathcal{R}(\mathcal{H}) \) of Nevanlinna functions. Furthermore, under conditions (1.30) the boundary triple \( \Pi \) is unitary and \( M(\cdot) \) is the Weyl function of \( \Pi \).

This result is contained in a somewhat more general statement proved in Theorem 6.12. For partial differential operator the first Green’s formula typically implies the conditions appearing in Proposition 1.23. As to applications of Proposition 1.23 let us mention that in Section 7.4 we construct a unitary boundary pair for Laplacians on rough domains, see Proposition 7.15. In Section 6.4 the result is applied to associate unitary boundary triples with the concepts of boundary pairs for nonnegative forms appearing in [12, 13] and in the most general form in the paper [85] of O. Post. The connection of various classes of boundary pairs for nonnegative forms as defined in [85] to the present subclasses of unitary boundary triples is established in Theorem 6.16.

### 1.8. A short description of the contents.

In Section 2 we recall basic concepts of linear relations (sums of relations, componentwise sums, defect subspaces, etc.) as well as unitary and isometric relations in Krein space. We also introduce the concepts of Nevanlinna functions and families.

In Section 3 we discuss unitary and isometric boundary pairs and triples. We also introduce the notions of Weyl functions and families and discuss their properties. A general version of the main realization result, Theorem 3.3, is presented therein, too. It completes and improves Theorem 1.13. Besides certain isometric transforms of boundary triples are discussed.

In Section 4 we investigate \( AB \)-generalized boundary triples. In particular, we present the proof of Proposition 1.20 and discuss its generalizations and consequences. In Theorem 4.1 we find a connection between \( B \)-generalized and \( AB \)-generalized boundary triples by means of triangular isometric transformations. In Theorem 4.11 it is shown that every \( AB \)-generalized boundary triple admits a Krein type resolvent formula.

In Section 5 we prove Theorem 1.13 (see Theorem 5.26). Here the connection between unitary boundary triples and unitary colligations is systematically used. In particular, this connection is applied to extend Theorem 1.11 to the case of \( S \)-generalized boundary pairs (see Theorem 5.18). In this case representation (1.15) for the Weyl function remains valid with \( M_0 \in \mathcal{R}[\mathcal{H}_0] \) and \( \mathcal{H}_0 \subseteq \mathcal{H} \) instead of \( M_0 \in \mathcal{R}[\mathcal{H}] \). Besides, in Theorem 5.31 a connection between \( ES \)-generalized boundary triples and \( AB \)-generalized boundary triples is established via an isometric transform introduced in Lemma 3.12 (see formula (3.26)).

Section 6 contains a couple of further useful results which are of preparatory nature for applications of unitary and, in particular, \( ES \)-generalized boundary triples. Sections 7 and 8 are devoted to applications of the general results in the PDE setting by treating Laplace operators and in the ODE setting by studying differential operators with local point interactions, it is these applications that initially acted as a motivation for the present work.
2. Preliminary Concepts

2.1. Linear relations in Hilbert spaces. A linear relation $T$ from $\mathcal{H}$ to $\mathcal{H}'$ is a linear subspace of $\mathcal{H} \times \mathcal{H}'$. Systematically a linear operator $T$ will be identified with its graph. It is convenient to write $T : \mathcal{H} \to \mathcal{H}'$ and interpret the linear relation $T$ as a multivalued linear mapping from $\mathcal{H}$ into $\mathcal{H}'$. If $\mathcal{H}' = \mathcal{H}$ one speaks of a linear relation $T$ in $\mathcal{H}$. Many basic definitions and properties associated with linear relations can be found in [11, 22, 33].

The following notions appear throughout this paper. For a linear relation $T : \mathcal{H} \to \mathcal{H}'$ the symbols $\text{dom} T$, $\ker T$, $\text{ran} T$, $\text{mul} T$ and $\overline{T}$ stand for the domain, kernel, range, multivalued part, and closure, respectively. The inverse $T^{-1}$ is a relation from $\mathcal{H}'$ to $\mathcal{H}$ defined by $\{ \{ f', f \} : \{ f, f' \} \in T \}$. The adjoint $T^*$ is the closed linear relation from $\mathcal{H}'$ to $\mathcal{H}$ defined by

$$T^* = \left\{ \left( \begin{array}{c} h' \\ k' \end{array} \right) \in \mathcal{H}' \oplus \mathcal{H} : (k, f)_\mathcal{H} = (h, g)_\mathcal{H}', \left( \begin{array}{c} f \\ g \end{array} \right) \in T \right\}.$$

The sum $T_1 + T_2$ and the componentwise sum $T_1 \hat{\oplus} T_2$ of two linear relations $T_1$ and $T_2$ are defined by

$$(2.1) \quad T_1 + T_2 = \left\{ \left( \begin{array}{c} f \\ g + k \end{array} \right) : \left( \begin{array}{c} f \\ g \end{array} \right) \in T_1, \left( \begin{array}{c} f \\ k \end{array} \right) \in T_2 \right\},$$

$$(2.2) \quad T_1 \hat{\oplus} T_2 = \left\{ \left( \begin{array}{c} f + h \\ g + k \end{array} \right) : \left( \begin{array}{c} f \\ g \end{array} \right) \in T_1, \left( \begin{array}{c} h \\ k \end{array} \right) \in T_2 \right\}.$$

If the componentwise sum is orthogonal it will be denoted by $T_1 \oplus T_2$. If $T$ is closed, then the null spaces of $T - \lambda$, $\lambda \in \mathbb{C}$, defined by

$$(2.3) \quad \mathcal{N}_\lambda(T) = \ker (T - \lambda), \quad \hat{\mathcal{N}}_\lambda(T) = \left\{ \left( \begin{array}{c} f \\ \lambda f \end{array} \right) : f \in \mathcal{N}_\lambda(T) \right\},$$

are also closed. Moreover, $\rho(T)$ ($\hat{\rho}(T)$) stands for the set of regular (regular type) points of $T$.

Recall that a linear relation $T$ in $\mathcal{H}$ is called symmetric, dissipative, or accumulative if $\text{Im} (h', h) = 0$, $\geq 0$, or $\leq 0$, respectively, holds for all $\{ h, h' \} \in T$. These properties remain invariant under closures. By polarization it follows that a linear relation $T$ in $\mathcal{H}$ is symmetric if and only if $T \subset T^*$. A linear relation $T$ in $\mathcal{H}$ is called selfadjoint if $T = T^*$, and it is called essentially selfadjoint if $\overline{T} = T^*$. A dissipative (accumulative) linear relation $T$ in $\mathcal{H}$ is called maximal dissipative (maximal accumulative) if it has no proper dissipative (accumulative) extensions.

If the relation $T$ is maximal dissipative (accumulative), then $\text{mul} T = \text{mul} T^*$ and the orthogonal decomposition $\mathcal{H} = (\text{mul} T)^\perp \oplus \text{mul} T$ induces an orthogonal decomposition of $T$ as

$$(2.4) \quad T = \text{gr} T_{\text{op}} \oplus \{ \{ 0 \} \times \mathcal{H}_\infty \}, \quad \mathcal{H}_\infty = \text{mul} T, \quad \text{gr} T_{\text{op}} = \left\{ \left( \begin{array}{c} f \\ g \end{array} \right) : g \in \mathcal{H} \ominus \mathcal{H}_\infty \right\},$$

where $T_{\infty} := \{ 0 \} \times \mathcal{H}_\infty$ is a purely multivalued selfadjoint relation in $\mathcal{H}_\infty$ and $T_{\text{op}}$ is a densely defined maximal dissipative (resp. accumulative) operator in $\mathcal{H} \ominus \mathcal{H}_\infty$. In particular, if $T$ is a selfadjoint relation, then there is such a decomposition, where $T_{\text{op}}$ is a densely defined selfadjoint operator in $\mathcal{H} \ominus \mathcal{H}_\infty$.

A family of linear relations $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in a Hilbert space $\mathcal{H}$ is called a Nevanlinna family if:

(i) for every $\lambda \in \mathbb{C}_+ \cap \mathbb{C}_-$ the relation $M(\lambda)$ is maximal dissipative (resp. accumulative);

(ii) $M(\lambda)^* = M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) for some, and hence for all, $\mu \in \mathbb{C}_+ \cap \mathbb{C}_-$ the operator family $(M(\lambda) + \mu)^{-1}(\in [\mathcal{H}])$ is holomorphic for all $\lambda \in \mathbb{C}_+ \cap \mathbb{C}_-$. 

By the maximality condition, each relation \( M(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), is necessarily closed. The class of all Nevanlinna families in a Hilbert space is denoted by \( \overline{\mathcal{R}(H)} \). If the multivalued part \( \text{mul} M(\lambda) \) of \( M \in \overline{\mathcal{R}(H)} \) is nontrivial, then it is independent of \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), so that
\[
(2.5) \quad M(\lambda) = \text{gr} M_{\text{op}}(\lambda) \oplus M_{\infty}, \quad H_{\infty} = \text{mul} M(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
where \( M_{\infty} = \{0\} \times H_{\infty} \) is a purely multivalued linear relation in \( H_{\infty} := \text{mul} M(\lambda) \) and \( M_{\text{op}}(\lambda) \in \mathcal{R}(H \oplus H_{\infty}) \), cf. \cite{72, 73, 75}. Identifying operators in \( H \) with their graphs one can consider classes
\[
\mathcal{R}^n[H] \subset \mathcal{R}^s[H] \subset \mathcal{R}^s(H) \subset \mathcal{R}(H)
\]
introduced in Section 1 as subclasses of \( \overline{\mathcal{R}(H)} \).

In addition, a Nevanlinna family \( M(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), which admits a holomorphic extrapolation to the negative real line \((-\infty, 0)\) (in the resolvent sense as in item (iii) of the above definition) and whose values \( M(x) \) are nonnegative (nonpositive) selfadjoint relations for all \( x < 0 \) is called a Stieltjes family (an inverse Stieltjes family, respectively).

**Definition 2.1.** A symmetric linear relation \( A \) in \( \mathcal{H} \) is called simple if there is no nontrivial orthogonal decomposition of \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) and no corresponding orthogonal decomposition \( A = A_1 \oplus A_2 \) with \( A_1 \) a symmetric relation in \( \mathcal{H}_1 \) and \( A_2 \) a selfadjoint relation in \( \mathcal{H}_2 \).

The decomposition (2.4) for \( A = A_\text{op} \oplus A_\infty \) shows that a simple closed symmetric relation is necessarily an operator. Recall that (cf. e.g. \cite{73}) a closed symmetric linear relation \( A \) in a Hilbert space \( \mathcal{H} \) is simple if and only if
\[
\mathcal{H} = \overline{\text{sp} \{ \mathcal{H}_\lambda(A^*) : \lambda \in \mathbb{C} \setminus \mathbb{R} \}}.
\]

2.2. **Unitary and isometric relations in Kreın spaces.** Let \( \mathcal{H} \) and \( \mathcal{H} \) be Hilbert spaces and let \( (\mathcal{H}, J_\mathcal{H}) \) and \( (\mathcal{H}, J_\mathcal{H}) \) be Kreın spaces with fundamental symmetries \( J_\mathcal{H}, J_\mathcal{H} \) and indefinite inner products \( \langle \cdot, \cdot \rangle_\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H} \) defined in (1.12) and (1.13), respectively.

If \( \Gamma \) is a linear relation from the Kreın space \( (\mathcal{H}, J_\mathcal{H}) \) to the Kreın space \( (\mathcal{H}, J_\mathcal{H}) \), then the adjoint linear relation \( \Gamma^* \) is defined by
\[
(2.6) \quad \Gamma^* = \left\{ \left( \begin{array}{c} \hat{k} \\ \hat{g} \end{array} \right) : \left( \begin{array}{c} \mathcal{H}_2 \\ \mathcal{H}_2 \end{array} \right) : \left( \begin{array}{c} \mathcal{H}_2 \\ \mathcal{H}_2 \end{array} \right) = \left[ \hat{k}, \hat{g} \right]_{\mathcal{H}^2} = \left[ \hat{h}, \hat{h} \right]_{\mathcal{H}^2} \quad \text{for all} \quad \left( \begin{array}{c} \hat{g} \\ \hat{h} \end{array} \right) \in \Gamma \right\}.
\]

**Definition 2.2.** (\cite{88}) A linear relation \( \Gamma \) from the Kreın space \( (\mathcal{H}, J_\mathcal{H}) \) to the Kreın space \( (\mathcal{H}, J_\mathcal{H}) \) is said to be \( (J_\mathcal{H}, J_\mathcal{H}) \)-isometric if \( \Gamma^{-1} \subset \Gamma^* \) and \( (J_\mathcal{H}, J_\mathcal{H}) \)-unitary, if \( \Gamma^{-1} = \Gamma^* \).

The following two statements are due to Yu.L. Shmul’jan \cite{88}; see also \cite{37}.

**Proposition 2.3.** Let \( \Gamma \) be a \( (J_\mathcal{H}, J_\mathcal{H}) \)-unitary relation from the Kreın space \( (\mathcal{H}, J_\mathcal{H}) \) to the Kreın space \( (\mathcal{H}, J_\mathcal{H}) \). Then:

(i) \( \text{dom} \Gamma \) is closed if and only if \( \text{ran} \Gamma \) is closed;

(ii) the following equalities hold:
\[
\text{ker} \Gamma = (\text{dom} \Gamma)^{[1]}, \quad \text{mul} \Gamma = (\text{ran} \Gamma)^{[1]}.
\]

A \( (J_\mathcal{H}, J_\mathcal{H}) \)-unitary relation \( \Gamma : (\mathcal{H}, J_\mathcal{H}) \rightarrow (\mathcal{H}, J_\mathcal{H}) \) may be multivalued, nondensely defined, and unbounded. It is the graph of an operator if and only if its range is dense. In this case it need not be densely defined or bounded; and even if it is bounded it need not be densely defined.
3. Unitary and isometric boundary pairs and associated Weyl families

3.1. Definitions and basic properties. Let $A$ be a closed symmetric linear relation in the Hilbert space $\mathcal{H}$. It is not assumed that the defect numbers of $A$ are equal or finite. Following [37, 39] a unitary/isometric boundary pair for $A^*$ is defined as follows.

**Definition 3.1.** Let $A$ be a closed symmetric linear relation in a Hilbert space $\mathcal{H}$, let $\mathcal{H}$ be an auxiliary Hilbert space and let $\Gamma$ be a linear relation from the Krein space $((\mathcal{K}^2, J_\mathcal{K}))$ to the Krein space $((\mathcal{H}^2, J_\mathcal{H}))$. Then $\{\mathcal{H}, \Gamma\}$ is called a unitary/isometric boundary pair for $A^*$, if:

3.1.1 $A_* := \text{dom } \Gamma$ is dense in $A^*$ with respect to the topology on $\mathcal{K}^2$;

3.1.2 the linear relation $\Gamma$ is $(J_\mathcal{K}, J_\mathcal{H})$-unitary/isometric.

In particular, it follows from this definition that for all vectors $\{f, h\} \in \Gamma$ of the form $(\Gamma, \mathcal{H})$ the abstract Green’s identity (cf. Definition 1.1) holds

\begin{equation}
(f', g)_\mathcal{K} - (f, g')_\mathcal{K} = (h', k)_\mathcal{H} - (h, k')_\mathcal{H}.
\end{equation}

Let $\{\mathcal{H}, \Gamma\}$ be a unitary boundary pair for $A^*$ and let $A_* = \text{dom } \Gamma$. According to [37, Proposition 2.12] the domain $A_*$ of $\Gamma$ is a linear relation in $\mathcal{K}$, such that

$A \subset A_* \subset A^*, \quad \overline{A_*} = A^*$.

The eigenspaces $\mathfrak{N}(A_*)$ and $\hat{\mathfrak{N}}(A_*)$ of $A_*$ are defined as in (2.3),

\[ \mathfrak{N}(A_*) = \ker (A_* - \lambda), \quad \hat{\mathfrak{N}}(A_*) = \left\{ \left( f_\lambda, h_\lambda \right) \in A_* : f_\lambda \in \mathfrak{N}(A_*) \right\}. \]

**Definition 3.2.** The Weyl family $M$ of $A$ corresponding to the unitary/isometric boundary pair $\{\mathcal{H}, \Gamma\}$ is defined by $M(\lambda) := \Gamma(\hat{\mathfrak{N}}(A_*))$, i.e.,

\begin{equation}
M(\lambda) := \left\{ \hat{h} \in \mathcal{H}^2 : \{f_\lambda, \hat{h}\} \in \Gamma \text{ for some } f_\lambda = \left( f_\lambda, \lambda f_\lambda \right) \in \mathcal{K}^2 \right\} \quad (\lambda \in \mathbb{C} \setminus \mathbb{R}),
\end{equation}

In the case where $M$ is single-valued it is called the Weyl function of $A$ corresponding to $\{\mathcal{H}, \Gamma\}$. The $\gamma$-field of $A$ corresponding to the unitary/isometric boundary pair $\{\mathcal{H}, \Gamma\}$ is defined by

\begin{equation}
\gamma(\lambda) := \left\{ \{h, f_\lambda\} \in \mathcal{H} \times \mathcal{K} : \left( \left( f_\lambda, \lambda f_\lambda \right), \left( h, h' \right) \right) \in \Gamma \text{ for some } h' \in \mathcal{H} \right\},
\end{equation}

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, $\hat{\gamma}(\lambda)$ stands for

\begin{equation}
\hat{\gamma}(\lambda) := \left\{ \{h, f_\lambda\} \in \mathcal{H} \times \mathcal{K}^2 : \left( \hat{f}_\lambda, \left( h, h' \right) \right) \in \Gamma \text{ for some } h' \in \mathcal{H} \right\}.
\end{equation}

With $\gamma(\lambda)$ the relation $\Gamma|\hat{\mathfrak{N}}(A_*)$ can be rewritten as follows

\begin{equation}
\Gamma|\hat{\mathfrak{N}}(A_*) := \left\{ \left( \gamma(\lambda) h, \left( h, h' \right) \right) : \left( h, h' \right) \in M(\lambda) \right\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

Associate with $\Gamma$ the following two linear relations which are not necessarily closed:

\begin{equation}
\Gamma_0 = \left\{ \{f, h\} : \{f, \hat{h}\} \in \Gamma, \hat{h} = \left( h, h' \right) \right\}, \quad \Gamma_1 = \left\{ \{f, h'\} : \{\hat{f}, \hat{h}\} \in \Gamma, \hat{h} = \left( h, h' \right) \right\}.
\end{equation}

The $\gamma$-field $\gamma(\cdot)$ associated with $\{\mathcal{H}, \Gamma\}$ is the first component of the mapping $\hat{\gamma}(\lambda)$ in (3.4). Observe, that

\[ \hat{\gamma}(\lambda) := (\Gamma_0|\hat{\mathfrak{N}}(A_*))^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \]
is a linear mapping from $\Gamma_0(\mathfrak{H}_\lambda(A_*)) = \text{dom } M(\lambda)$ onto $\mathfrak{H}_\lambda(A_*)$: it is single-valued in view of (3.1); cf. (3.10), (3.11). Consequently, the $\gamma$-field is a single-valued mapping from $\text{dom } M(\lambda)$ onto $\mathfrak{H}_\lambda(A_*)$ and it satisfies $\gamma(\lambda)\Gamma f_\lambda = f_\lambda$ for all $f_\lambda \in \mathfrak{H}_\lambda(A_*)$.

If $\Gamma$ is single-valued then these component mappings decompose $\Gamma$, $\Gamma = \Gamma \Gamma_1$, and the triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ will be called a unitary/isometric boundary triple for $A^*$. In this case the Weyl function corresponding to the unitary/isometric boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ can be also defined via

$$M(\lambda) \Gamma f_\lambda = \Gamma_1 f_\lambda, \quad f_\lambda \in \mathfrak{H}_\lambda(A_*).$$

(3.7) When $A$ admits real regular type points it is useful to extend Definition 3.2 of the Weyl family to the points on the real line by setting $M(x) := \Gamma(\mathfrak{H}_x(A_*))$ or, more precisely,

$$M(x) := \left\{ \tilde{h} \in \mathcal{H}^2 : \{f_x, \tilde{h}\} \in \Gamma \text{ for some } f_x = \left( \begin{array}{c} f_x \\ x f_x \end{array} \right) \in \mathcal{H}^2, x \in \mathbb{R} \right\}. \quad (3.8)$$

3.2. Unitary boundary pairs and unitary boundary triples. The following theorem shows that the set of all Weyl families of unitary boundary pairs coincides with $R(\mathcal{H})$ (see [37, Theorem 3.9]). Recall that a unitary boundary pair $\{\mathcal{H}, \Gamma\}$ for $A^*$ is said to be minimal, if

$$\mathfrak{H} = \mathfrak{H}_{\min} := \text{span} \{ \mathfrak{H}_\lambda(A_*): \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \}.$$ 

**Theorem 3.3.** Let $\{\mathcal{H}, \Gamma\}$ be a unitary boundary pair for $A^*$. Then the corresponding Weyl family $M$ belongs to the class of Nevanlinna families $R(\mathcal{H})$.

Conversely, if $M$ belongs to the class $R(\mathcal{H})$, then there exists a unique (up to a unitary equivalence) minimal unitary boundary pair $\{\mathcal{H}, \Gamma\}$ whose Weyl function coincides with $M$.

Notice that Theorem 1.9 contains a general analytic criterion for an isometric boundary triple to be unitary; the Weyl function should be a Nevanlinna function, cf. Theorem 1.9.

**Corollary 3.4.** The class of Weyl functions corresponding to unitary boundary triples coincides with the class $R^*(\mathcal{H})$ of (in general unbounded) strict Nevanlinna functions.

**Proof.** The statement is immediate when combining Theorem 3.3 with Proposition 4.5 from [37]. \qed

As a consequence of (3.1) and (3.5) the following identity holds (cf. (2.5))

$$= (M_{\text{op}}(\lambda) h, k)_H - (h, M_{\text{op}}(\mu) k)_H,$$

where $h \in \text{dom } M(\lambda)$ and $k \in \text{dom } M(\mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$.

As was already mentioned in Section 1, every operator valued function $M$ from $\mathcal{R}^*[\mathcal{H}]$ can be realized as a Weyl function of some ordinary boundary triple ($B$-generalized boundary triple, respectively).

The multivalued analog for the notion of $B$-generalized boundary triple was introduced in [37, Section 5.3], a formal definition reads as follows.

**Definition 3.5.** Let $A$ be a symmetric operator (or relation) in the Hilbert space $\mathfrak{H}$ and let $\mathcal{H}$ be another Hilbert space. Then a linear relation $\Gamma : A^* \rightarrow \mathcal{H} \oplus \mathcal{H}$ with dense domain in $A^*$ is said to be a $B$-generalized boundary pair for $A^*$, if the following three conditions are satisfied:

3.5.1 the abstract Green’s identity (3.1) holds;
3.5.2 $\text{ran } \Gamma_0 = \mathcal{H}$;
3.5.3 $A_0 = \ker \Gamma_0$ is selfadjoint,

where $\Gamma_0$ stands for the first component of $\Gamma$; see (3.6).
As was shown in [37] Proposition 5.9] every Weyl function of a $B$-generalized boundary pair belongs to the class $\mathcal{R}[\mathcal{H}]$ and, conversely, every operator valued function $M \in \mathcal{R}[\mathcal{H}]$ can be realized as the Weyl function of a $B$-generalized boundary pair.

### 3.3. Isometric boundary pairs and isometric boundary triples

Let $\Gamma$ be a $(J_{\mathcal{D}}, J_{\mathcal{H}})$-isometric relation from the Kreĭn space $(\mathcal{D}^2, J_{\mathcal{D}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. In view of (1.12)-(1.14) this just means that the abstract Green’s identity (3.1) holds. It follows from (3.1) that

$$\ker \Gamma \subset (\text{dom } \Gamma)^{[\cdot]}, \quad \text{mul } \Gamma \subset (\text{ran } \Gamma)^{[\cdot]}.$$ 

compare Proposition 2.3. Let $\Gamma_0$ and $\Gamma_1$ be the linear relations determined by (3.6). The kernels $A_0 := \ker \Gamma_0$ and $A_1 := \ker \Gamma_1$ need not be closed, but they are symmetric extensions of $\ker \Gamma$ which are contained in the domain $A_* = \text{dom } \Gamma$ of $\Gamma$; cf. [37] Proposition 2.13. If $A_* = \text{dom } \Gamma$ is dense in $A^*$ then the pair $(\mathcal{H}, \Gamma)$ is viewed as an isometric boundary pair for $A^*$; cf. Definition 3.1. In general $A := (A_*)^* = (\text{dom } \Gamma)^{[\cdot]}$ is an extension of $\ker \Gamma$ which need not belong to $\text{dom } \Gamma$; for some sufficient conditions for the equality $A = \ker \Gamma$, see [38] Section 2.3 and [39] Section 7.8.

With $\{h, \hat{h}, g, \mu, \hat{\mu}, k\} \in \Gamma$, $\lambda, \mu, \in \mathbb{C}$, the Green’s identity (3.1) gives, cf. (3.9),

$$\langle h', k \rangle_{\mathcal{H}} - \langle h, k' \rangle_{\mathcal{H}} = (\lambda - \mu)(f_{\lambda}, g_{\mu})_{\mathcal{D}}.$$ 

In particular, with $\mu = \lambda$ (3.10) implies that $\text{Im } \langle h', h \rangle_{\mathcal{H}} = \text{Im } \lambda \|f_{\lambda}\|^2$. Hence, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\ker (\Gamma | \hat{\mathcal{N}}_{\lambda}(A_*)) = \{0\} \quad \text{and} \quad \ker (\Gamma_j | \hat{\mathcal{N}}_{\lambda}(A_*)) = \{0\} \quad (j = 0, 1).$$

Moreover, with $\mu = \bar{\lambda}$ (3.10) implies that

$$M(\bar{\lambda}) \subseteq M(\lambda)^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 

Here equality does not hold if $\Gamma$ is not unitary. However, with the Weyl family the multivalued part of $\Gamma$ can be described explicitly; see [39] Lemma 7.57, cf. also [37] Lemma 4.1.

**Lemma 3.6.** Let $\{\mathcal{H}, \Gamma\}$ be an isometric boundary pair with the Weyl family $M$. Then the following equalities hold for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$:

(i) $M(\lambda) \cap M(\lambda)^* = \text{mul } \Gamma$;
(ii) $\ker M(\lambda) \times \{0\} = \text{mul } \Gamma \cap (\mathcal{H} \times \{0\})$;
(iii) $\{0\} \times \text{mul } M(\lambda) = \text{mul } \Gamma \cap (\{0\} \times \mathcal{H})$;
(iv) $\ker (M(\lambda) - M(\lambda)^*) = \text{mul } \Gamma_0$;
(v) $\ker (M(\lambda)^{-1} - M(\lambda)^{-*}) = \text{mul } \Gamma_1$.

If $\Gamma$ itself is single-valued, then the Weyl family $M$ is an operator valued function, i.e. $\text{mul } (M(\lambda)) = 0$, in the class $\mathcal{R}^*(\mathcal{H})$, see [37] Proposition 4.5. Moreover, $\ker \text{Im } (M(\lambda)) = \{0\}$ and $\ker \text{Im } (M(\lambda)^{-1}) = \{0\}$, in particular, $\ker M(\lambda) = 0$. Recall that when $\Gamma$ is single-valued $M(\lambda)$ can equivalently be defined by the equality (3.7). Hence, if $h \in \mathcal{H}$ is given and $h \in \Gamma_0(\hat{\mathcal{N}}_{\lambda}(A_*))$, then $\gamma(\lambda)h$ solves a boundary eigenvalue problem, i.e., $\gamma(\lambda)h \in \ker (A^* - \lambda)$ and $\Gamma_0\gamma(\lambda)h = h$, while $\Gamma_1\gamma(\lambda)h = M(\lambda)h$. Also for an operator valued $M(\cdot)$ the identity (3.10) can be rewritten in the form

$$\langle \lambda - \bar{\mu}, (\gamma(\lambda)h, \gamma(\mu)k)_{\mathcal{D}} = (M(\lambda)h, k)_{\mathcal{H}} - (h, M(\mu)k)_{\mathcal{H}},$$

where $h \in \text{dom } M(\lambda)$ and $k \in \text{dom } M(\mu)$, $\lambda, \mu, \in \mathbb{C} \setminus \mathbb{R}$. This is an analog of (3.9) for an isometric boundary triple.
Let \( \Gamma \) be an isometric relation and let \( A_0 = \ker \Gamma_0 \). Then \( A_0 \) is a symmetric, not necessarily closed, relation and one can write for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \),
\[
A_0 = \left\{ \left( A_0 - \lambda \right)^{-1} h, \left( h + \lambda(A_0 - \lambda)^{-1} h \right) : h \in \text{ran} \left( A_0 - \lambda \right) \right\}.
\]

The linear mapping
\[
H(\lambda) : h \to \left\{ \left( A_0 - \lambda \right)^{-1} h, \left( h + \lambda(A_0 - \lambda)^{-1} h \right) \right\}
\]
from \( \text{ran} \left( A_0 - \lambda \right) \) onto \( A_0 \) is clearly bounded with bounded inverse.

**Lemma 3.7.** Let \( \{ \mathcal{H}, \Gamma \} \) be an isometric boundary pair and let \( A_0 = \ker \Gamma_0 \). Then the following assertions hold:

(i) \( \Gamma_1 H(\lambda) \) is closable for one (equivalently for all) \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) if and only if \( \Gamma_1 \upharpoonright A_0 \) is closable;

(ii) \( \Gamma_1 H(\lambda) \) is closed for one (equivalently for all) \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) if and only if \( \Gamma_1 \upharpoonright A_0 \) is closed;

(iii) \( \Gamma_1 H(\lambda) \) is bounded operator for one (equivalently for all) \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) if and only if \( \Gamma_1 \upharpoonright A_0 \) is a bounded operator;

(iv) \( \text{dom} \, \Gamma_1 H(\lambda) \) is dense in \( \mathcal{S} \) for some (equivalently for all) \( \lambda, \tilde{\lambda} \in \mathbb{C} \setminus \mathbb{R} \) if and only if \( A_0 \) is essentially selfadjoint;

(v) \( \text{dom} \, \Gamma_1 H(\lambda) = \mathcal{S} \) for some (equivalently for all) \( \lambda, \tilde{\lambda} \in \mathbb{C} \setminus \mathbb{R} \) if and only if \( A_0 \) is selfadjoint.

**Proof.** By definition \( A_0 = \ker \Gamma_0 \subset \text{dom} \, \Gamma_1 \), so that \( \text{dom} \, \Gamma_1 H(\lambda) = \text{ran} \left( A_0 - \lambda \right), \lambda \in \mathbb{C} \setminus \mathbb{R} \). Since \( H(\lambda) : \text{ran} \left( A_0 - \lambda \right) \to A_0 \) is bounded with bounded inverse, all the statements are easily obtained by means of the equality \( \Gamma_1 \upharpoonright A_0 = (\Gamma_1 H(\lambda)) H(\lambda)^{-1} \).

Similar facts can be stated for the restriction \( \Gamma_0 \upharpoonright A_1 \), where \( A_1 = \ker \Gamma_1 \).

The inclusion (3.16) in the next proposition was stated for a single-valued \( \Gamma \) with dense range in [39] Proposition 7.59]; here a direct proof for this inclusion is given in the general case.

**Lemma 3.8.** Let \( \{ \mathcal{H}, \Gamma \} \) be an isometric boundary pair, let \( \gamma(\lambda) \) be its \( \gamma \)-field, and let \( H(\lambda) \) be as defined in (3.14). Then
\[
(3.15) \quad \Gamma H(\lambda) \subset \left( \begin{array}{c} 0 \\ \gamma(\lambda) \end{array} \right)^* \quad \hat{\bigoplus} \{0\} \times \text{mul} \, \Gamma, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
where the adjoint \( \gamma(\lambda)^* \) of \( \gamma(\lambda) \) is in general a linear relation. In particular,
\[
(3.16) \quad \Gamma_1 H(\lambda) \subset \gamma(\lambda)^* \quad \hat{\bigoplus} \{0\} \times \text{mul} \, \Gamma_1, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
and if, in addition, \( \text{mul} \, \Gamma_1 = \{0\} \), then
\[
(3.17) \quad \Gamma_1 H(\lambda) \subset \gamma(\lambda)^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Furthermore, the following statements hold:

(i) if \( \gamma(\lambda) \) is densely defined for some \( \tilde{\lambda} \in \mathbb{C} \setminus \mathbb{R} \), then \( \gamma(\lambda)^* \) is a closed operator and if, in addition, \( \text{mul} \, \Gamma_1 = \{0\} \), then \( \Gamma_1 H(\lambda) \) is a closable operator;

(ii) if \( A_0 = \ker \Gamma_0 \) is essentially selfadjoint, then \( \gamma(\lambda) \) is closable for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \);

(iii) if \( A_0 = \ker \Gamma_0 \) is selfadjoint, then \( \text{dom} \, \gamma(\lambda)^* = \mathcal{S} \) and \( \gamma(\lambda) \) is a bounded operator for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

**Proof.** Let \( h \in \text{dom} \, \gamma(\lambda) = \text{dom} \, M(\lambda) \) and \( k_\lambda \in \text{ran} \left( A_0 - \lambda \right) \). Then \( \{\tilde{\gamma}(\lambda)h, \{h, h'\} \} \in \Gamma \) and, since \( H(\lambda)k_\lambda \in A_0 = \ker \Gamma_0 \), one has \( \{H(\lambda)k_\lambda, \{0, k''\} \} \in \Gamma \) for some \( k'' \in \mathcal{H} \). On the other
hand, \( \{k_\lambda, k'\} \in \Gamma_1 H(\lambda) \) means that \( \{k_\lambda, \{k, k'\}\} \in \Gamma H(\lambda) \) for some \( k \in \mathcal{H} \) which combined with \( \{H(\lambda)k_\lambda, \{0, k'\}\} \in \Gamma \) implies that \( \{|0, \{k, k' - k''\}\}\} \in \Gamma \).

Now applying Green’s identity (3.11) shows that
\[
(\bar{\lambda} \gamma(\lambda) h, (A_0 - \lambda)^{-1} k_\lambda) - (\gamma(\bar{\lambda}) h, (I + \lambda(A_0 - \lambda)^{-1}) k_\lambda) = 0 - (h, k'')_H.
\]
This identity can be rewritten equivalently in the form
\[
(\gamma(\lambda) h, k_\lambda) = (h, k'')_H
\]
for all \( h \in \text{dom} \gamma(\lambda) \) and \( k_\lambda \in \text{ran} \(A_0 - \lambda) \). This proves that \( \{k_\lambda, k''\} \in \gamma(\lambda)^* \). Hence, if \( \{k_\lambda, (k, k')\} \in \Gamma H(\lambda) \) then

\[
(3.18) \quad \left\{ k_\lambda, \left( \begin{array}{c} k \\ k' \\ k'' \end{array} \right) \right\} = \left\{ k_\lambda, \left( \begin{array}{c} 0 \\ k \\ k' - k'' \end{array} \right) \right\} + \left\{ 0, \left( \begin{array}{c} k \\ k' - k'' \end{array} \right) \right\}, \quad \{k_\lambda, k''\} \in \gamma(\lambda)^*, \quad \left( \begin{array}{c} k \\ k' - k'' \end{array} \right) \in \text{mul} \Gamma,
\]
from which the formulas (3.15) and (3.16) follow. If \( \text{mul} \Gamma_1 = \{0\} \), then \( \left( \begin{array}{c} k \\ k' - k'' \end{array} \right) \in \text{mul} \Gamma \)
implies that \( k' = k'' \) and therefore the above argument shows that \( \{k_\lambda, k''\} \in \gamma(\lambda)^* \) for all \( \{k_\lambda, k''\} \in \Gamma_1 H(\lambda) \); i.e. (3.17) is satisfied.

It remains to prove the statements (i)–(iii).

(i) If \( \gamma(\lambda) \) is densely defined then clearly \( \gamma(\lambda)^* \) is a closed operator and if \( \Gamma_1 \) is single-valued then (3.17) shows that \( \Gamma_1 H(\lambda) \) is closable as a restriction of \( \gamma(\lambda)^* \).

(ii) By Lemma 3.7 \( A_0 \) is essentially selfadjoint if and only if \( \Gamma_1 H(\lambda) \) is densely defined, in which case also \( \gamma(\lambda)^* \) is densely defined, so that \( \gamma(\lambda) \) is closable.

(iii) If \( A_0 \) is selfadjoint, then \( \text{dom} \Gamma_1 H(\lambda) = \mathcal{H} \) and, therefore, also \( \text{dom} \gamma(\lambda)^* = \mathcal{H} \). In addition \( \gamma(\lambda) \) is closable, thus clos \( \gamma(\lambda) \) and \( \gamma(\lambda)^* \) are bounded operators. \(\square\)

**Proposition 3.9.** Let \( A \) be a closed symmetric relation in the Hilbert space \( \mathcal{H} \) and let \( \{\mathcal{H}, \Gamma\} \) be an isometric boundary pair whose domain \( A_* \) is dense in \( A^* \), let \( M(\cdot) \) and \( \gamma(\cdot) \) be the corresponding Weyl function and the \( \gamma \)-field and, in addition, assume that \( A_0 = \ker \Gamma_0 \) is selfadjoint. Then:

(i) \( A_* := \text{dom} \Gamma \) admits the decomposition \( A_* = A_0 + \tilde{\mathcal{H}}_\lambda(A_*) \) and \( \tilde{\mathcal{H}}_\lambda(A_*) \) is dense in \( \mathcal{H} \lambda(A^*) \) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \);

(ii) with a fixed \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the graph of \( \Gamma \) admits the following representation:
\[
\Gamma = \Gamma A_0 \quad \Gamma \quad \Gamma A_0 \quad \Gamma \quad \Gamma
d\]

(iii) if \( \tilde{\Gamma} : (\mathcal{H}^2, J_\mathcal{H}) \to (\mathcal{H}^2, J_\mathcal{H}) \) is an isometric extension of \( \Gamma \) with the Weyl function \( \tilde{M} \) and the \( \gamma \)-field \( \tilde{\gamma}(\cdot) \) such that \( \tilde{A}_* := \text{dom} \tilde{\Gamma} \subset A^* \), then with a fixed \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the following equivalence holds:
\[
\tilde{\Gamma} = \Gamma \quad \Leftrightarrow \quad \tilde{M}(\lambda) = M(\lambda).
\]

**Proof.** (i) By von Neumann’s formula \( A^* = A_0 + \tilde{\mathcal{H}}_\lambda(A^*) \). Since \( A_* := \text{dom} \Gamma \) is dense in \( A^* \) and \( A_0 \subset A_* \), it follows that \( A_* = A_0 + \tilde{\mathcal{H}}_\lambda(A_*) \) and that \( \tilde{\mathcal{H}}_\lambda(A_*) \) is dense in \( \mathcal{H}_\lambda(A^*) \) for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

(ii) In view of (i) for every \( \{\tilde{f}, \tilde{k}\} \in \Gamma \) there exist unique elements \( \tilde{f}_0 \in A_0 \) and \( \tilde{f}_\lambda \in \tilde{\mathcal{H}}_\lambda(A_*) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), such that \( \tilde{f} = \tilde{f}_0 + \tilde{f}_\lambda \). Moreover, if \( \{\tilde{f}_\lambda, \tilde{h}\} \in \Gamma \) then \( \tilde{h} = \{h, h'\} \in M(\lambda) \) and one can write (uniquely) \( \tilde{f}_\lambda = \tilde{\gamma}(\lambda) h \); see (3.5). The stated representation for \( \Gamma \) is now clear.

(iii) It follows from \( \Gamma \subset \tilde{\Gamma} \) that \( A_0 \subset \ker \tilde{\Gamma}_0 \). Since \( \ker \tilde{\Gamma}_0 \) is symmetric and \( A_0 \) is selfadjoint, the equality \( A_0 = \ker \tilde{\Gamma}_0 \) holds. Now recall that two linear relations with \( H_1 \subset H_2 \) are equal
Lemma 3.6 (i) \( \text{mul} \Gamma = \text{mul} H_1 \cap \text{mul} H_2 \) \( \implies \) \( \text{mul} \Gamma = \text{mul} H_1 \). By Lemma 3.6 (i) \( \text{mul} \Gamma = M(\lambda) \cap M(\lambda)^* \). Therefore, \( \text{mul} \Gamma = M(\lambda) \) implies that \( \text{mul} \Gamma = \text{mul} \Gamma \). Moreover, we have \( \text{dom} \text{mul} \Gamma = \text{dom} M(\lambda) \) and, since \( \tilde{\gamma}(\lambda) \) maps \( \text{dom} \text{mul} \Gamma \) onto \( \mathcal{N}(A_+) \) and \( \tilde{\gamma}(\lambda) \) maps \( \text{dom} M(\lambda) \) onto \( \mathcal{N}_\lambda(A_+) \), we conclude from (i) that \( \text{dom} \tilde{\gamma} = \text{dom} \Gamma \). Therefore, \( \text{mul} \Gamma = M(\lambda) \) implies \( \tilde{\gamma} = \Gamma \). The reverse implication is clear. \( \square \)

The Weyl function of an isometric or unitary boundary pair is in general unbounded and multivalued operator. In what follows Weyl functions \( M(\lambda) \), whose domain (or form domain) does not dependent on \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) are of special interest. Here a characterization for domain invariant Weyl families will be established. We start with the next lemma concerning the domain inclusion \( \text{dom} M(\lambda) \subset \text{dom} M(\mu) \).

Lemma 3.10. Let \( \{ \mathcal{H}, \Gamma \} \) be an isometric boundary pair with \( A_+ = \text{dom} \Gamma \), let \( M \) and \( \gamma(\cdot) \) be the corresponding Weyl family and \( \gamma \)-field, and let \( A_0 = \ker \Gamma_0 \). Then for each fixed \( \lambda, \mu \in \mathbb{C} \setminus \mathbb{R} \) with \( \lambda \neq \mu \) the inclusion

\[
(3.19) \quad \text{dom} M(\mu) \subset \text{dom} M(\lambda),
\]

is equivalent to the inclusion

\[
(3.20) \quad \text{ran} \gamma(\mu) \subset \text{ran} (A_0 - \lambda).
\]

If one of these conditions is satisfied, then the \( \gamma \)-field \( \gamma(\cdot) \) satisfies the identity

\[
(3.21) \quad \gamma(\lambda)h = [I + (\lambda - \mu)(A_0 - \lambda)^{-1}]\gamma(\mu)h, \quad h \in \text{dom} \gamma(\mu).
\]

Proof. By Definitions 3.2, \( \text{dom} M(\lambda) = \text{dom} \gamma(\lambda) = \Gamma_0(\mathcal{N}_\lambda(A_+)) \) and, moreover, \( \text{ran} \gamma(\lambda) = \mathcal{N}_\lambda(A_+) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

Now assume that (3.19) holds and let \( h \in \text{dom} M(\mu) \subset \text{dom} M(\lambda) \). It follows from (3.4) that there exist \( h', h'' \in \mathcal{H} \) such that

\[
\left\{ \left( \frac{\gamma(\lambda)h}{\lambda \gamma(\lambda)h}, \frac{h}{h'} \right), \left( \frac{\gamma(\mu)h}{\mu \gamma(\mu)h}, \frac{h}{h''} \right) \right\} \in \Gamma \mathcal{N} \mathcal{H}(T) \subset \Gamma,
\]

This implies

\[
\left\{ \left( \frac{(\gamma(\lambda) - \gamma(\mu))h}{(\lambda \gamma(\lambda) - \mu \gamma(\mu))h}, \frac{0}{h' - h''} \right) \right\} \in \Gamma,
\]

and hence

\[
(3.22) \quad \left( \frac{(\gamma(\lambda) - \gamma(\mu))h}{(\lambda \gamma(\lambda) - \mu \gamma(\mu))h} \right) \in A_0 \quad \text{and} \quad \left( \frac{(\gamma(\lambda) - \gamma(\mu))h}{(\lambda - \mu)\gamma(\mu)h} \right) \in A_0 - \lambda.
\]

Therefore, \( \gamma(\mu)h \in \text{ran} (A_0 - \lambda) \) for every \( h \in \text{dom} M(\mu) \) and thus (3.20) follows.

Conversely, assume that (3.20) holds and let \( h \in \text{dom} M(\mu) = \text{dom} \gamma(\mu) \). This implies that

\[
(3.23) \quad \left\{ \left( \frac{\gamma(\mu)h}{\mu \gamma(\mu)h}, \frac{h}{h'} \right) \right\} \in \Gamma
\]

for some \( h' \in \mathcal{H} \). Moreover, since \( \gamma(\mu)h \in \text{ran} (A_0 - \lambda) \), there exists an element \( k \in \mathcal{H} \) such that \( \{ k, (\gamma(\mu)h + \lambda k) \} \in A_0 = \ker \Gamma_0 \). Consequently, there exists \( \varphi \in \mathcal{H} \) such that

\[
(3.24) \quad \left\{ \left( \frac{(\lambda - \mu)k}{(\lambda - \mu) \gamma(\mu)h + \lambda (\lambda - \mu)k}, \frac{0}{\varphi} \right) \right\} \in \Gamma.
\]

It follows from (3.23) and (3.24) that

\[
\left\{ \left( \frac{\gamma(\mu)h + (\lambda - \mu)k}{\lambda (\gamma(\mu)h + (\lambda - \mu)k)}, \frac{h}{h' + \varphi} \right) \right\} \in \Gamma.
\]
Therefore, \( h \in \Gamma_0(\mathfrak{M}_\lambda(A_*)) = \text{dom } M(\lambda) \). This proves the inclusion (3.19).

Finally, observe that the assumption (3.19) implies (3.22). Since \( A_0 \) is symmetric, \((A_0 - \lambda)^{-1}\) is a bounded operator on \( \text{ran } (A_0 - \lambda) \) and, thus, (3.22) leads to (3.21). \( \square \)

The next result characterizes domain invariance of the Weyl family corresponding to an arbitrary isometric boundary pair \( \{H, \Gamma\} \). In the special case of a unitary boundary pair \( \{H, \Gamma\} \) items (i) and (iii) contain [37, Proposition 4.11, Corollary 4.12].

**Proposition 3.11.** Let the assumptions and notations be as in Lemma 3.10. Then the following statements hold:

(i) \( \text{dom } M(\lambda) \) is independent from \( \lambda \in \mathbb{C}_+ \) (resp. from \( \lambda \in \mathbb{C}_- \)) if and only if

\[
\mathfrak{M}_{\lambda}(A_*) \subset \text{ran } (A_0 - \lambda) \quad \text{for all } \lambda, \mu \in \mathbb{C}_+ \quad \text{(resp. for all } \lambda, \mu \in \mathbb{C}_-) \), \( \lambda \neq \mu \),

in this case the \( \gamma \)-field \( \gamma(\cdot) \) satisfies

\[
\gamma(\lambda) = [I + (\lambda - \mu)(A_0 - \lambda)^{-1}]\gamma(\mu), \quad \lambda, \mu \in \mathbb{C}_+ (\mathbb{C}_-);
\]

(ii) if \( A_0 \) is selfadjoint, then \( \text{dom } M(\lambda) \) does not depend on \( \lambda \in \mathbb{C} \setminus \mathbb{R} \);

(iii) if \( \text{dom } M(\lambda) \) does not depend on \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), then \( A_0 \) is essentially selfadjoint.

**Proof.** The assertions (i) and (ii) follow directly from Lemma 3.10.

To see (iii) one can use the same argument that is presented in [37, Corollary 4.12]. \( \square \)

### 3.4. Some transforms of boundary triples

In this subsection a specific transform of isometric boundary triples is treated. In what follows such transforms are used repeatedly and, in fact, they appear also in concrete boundary value problems in ODE and PDE settings. To formulate a general result in the abstract setting consider in the Kreın space \((\mathcal{H}^2, J_{\mathcal{H}})\) the transformation operator \( V \) whose action is determined by the triangular operator

\[
(3.25) \quad V = \begin{pmatrix} G^{-1} & 0 \\ EG^{-1} & G^* \end{pmatrix}, \quad E \subset E^*, \quad \text{dom } E = \overline{\text{dom } G} = \text{ran } G = \mathcal{H}, \quad \text{ker } G = \{0\}.
\]

By assumptions on \( G \) one has \( \text{ker } G^* = \text{mul } G^* = \{0\} \), so that the adjoint \( G^* \) is an injective operator in \( \mathcal{H} \). To keep a wider generality, \( G \) is not assumed to be a closed operator, while in applications that will often be the case. In particular, it is possible that \( G^* \) is not densely defined and also its range need not be dense. Since \( E \) is a densely defined symmetric operator, it is closable and its closure \( \overline{E} \subset E^* \) is also symmetric. With the assumptions on \( V \) in (3.25) a direct calculation shows that

\[
(J_{\mathcal{H}} V f, V g)_{\mathcal{H}^2} = (J_{\mathcal{H}} f, g)_{\mathcal{H}^2}, \quad f, g \in \text{dom } V.
\]

Hence, \( V \) is an isometric operator in the Kreın space \((\mathcal{H}^2, J_{\mathcal{H}})\). Moreover, \( V \) is injective. These observations lead to the following (unbounded) extension of [35 Proposition 3.18].

**Lemma 3.12.** Let \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) be an isometric boundary triple for \( A^* \) such that \( \text{ker } \Gamma = A \), let \( \gamma(\lambda) \) and \( M(\lambda) \) be the corresponding \( \gamma \)-field and the Weyl function, and let \( V \) be as defined in (3.25). Then \( V \) is isometric in the Kreın space \((\mathcal{H}^2, J_{\mathcal{H}})\) and moreover:

(i) the transform \( \tilde{\Gamma} = V \circ \Gamma \)

\[
(3.26) \quad \begin{pmatrix} \tilde{\Gamma}_0 \hat{f} \\ \tilde{\Gamma}_1 \hat{f} \end{pmatrix} = \begin{pmatrix} G^{-1} \Gamma_0 \hat{f} \\ EG^{-1} \Gamma_0 \hat{f} + G^* \Gamma_1 \hat{f} \end{pmatrix}, \quad \hat{f} \in \text{dom } \Gamma,
\]

defines an isometric boundary triple with domain \( \tilde{A}_* := \text{dom } \tilde{\Gamma} \) and kernel \( \text{ker } \tilde{\Gamma} = A \);
(ii) the $\gamma$-field and the Weyl function of $\tilde{\Gamma}$ are in general unbounded nondensely defined operators given by

$$\tilde{\gamma}(\lambda)k = \gamma(\lambda)Gk, \quad \tilde{M}(\lambda)k = Ek + G^*M(\lambda)Gk, \quad k \in \text{dom } \tilde{M}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 

Proof. (i) By the assumptions in \cite{22} $V$ is an isometric operator in the Krein space $(\mathcal{H}^2, J_\mathcal{H})$ and since $\Gamma$ is an isometric operator from $(\mathcal{S}^2, J_\mathcal{S})$ to $(\mathcal{H}^2, J_\mathcal{H})$ the composition operator $V \circ \Gamma$ is also an isometric operator from $(\mathcal{S}^2, J_\mathcal{S})$ to $(\mathcal{H}^2, J_\mathcal{H})$. Since $V$ is injective, one has $\ker \tilde{\Gamma} = \ker \Gamma = A$. In general $V$ is not everywhere defined, so that $\tilde{A}_s$ is typically a proper linear subset of $A_s = \text{dom } \Gamma$ which is not necessarily dense in $A^*$. 

(ii) By Definition 3.2 the Weyl function $\tilde{M}(\lambda)$ of $\tilde{\Gamma}$ is given by $\tilde{M}(\lambda) = V \circ M(\lambda)$ or, more explicitly, by

$$\tilde{M}(\lambda) = \left\{ \left( \begin{array}{c} G^{-1}h \\ EG^{-1}h + G^*M(\lambda)h \end{array} \right) : h \in \text{dom } EG^{-1} \cap \text{dom } G^*M(\lambda) \right\} = E + G^*M(\lambda)G.$$ 

Similarly, $(G^{-1}\Gamma_0|\widehat{\Gamma}_\lambda(\tilde{A}_s))^{-1} = (\Gamma_0|\widehat{\Gamma}_\lambda(\tilde{A}_s))^{-1}G$ implies that $\tilde{\gamma}(\lambda) = \gamma(\lambda)G$ with $\text{dom } \tilde{\gamma}(\lambda) = \text{dom } \tilde{M}(\lambda)$.

Example 3.13. (i) If $G = I_\mathcal{H}$ then the condition $\Gamma_1\hat{f} = 0$ reads as $\Gamma_1\hat{f} + E\Gamma_0\hat{f} = 0$. In applications such conditions are called Robin type boundary conditions. This corresponds to the transposed boundary triple $\{\mathcal{H}, \Gamma_1 + E\Gamma_0, -\Gamma_0\}$ which is also isometric and has $-(M(\lambda) + E)^{-1}$ as its Weyl function. 

(ii) As indicated $G$ need not be closable. An extreme situation appears when $G$ is a singular operator; cf. \cite{68}. By definition this means that $\text{dom } G \subset \ker G$ or, equivalently, that $\text{ran } G \subset \text{mul } \tilde{G}$. Thus, in this case $\text{dom } G^* = \text{ran } G^* = \{0\}$. If, for instance, $\Gamma$ is an ordinary boundary triple for $A^*$ then $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ are selfadjoint. It is easy to check that

$$\tilde{A}_s = \{ \hat{f} \in A^* : \Gamma_1\hat{f} = 0 \} = \ker \Gamma_1 = A_1, \quad \ker \tilde{\Gamma}_0 = A_0 \cap A_1 = A.$$ 

Moreover, $\text{ran } \tilde{\Gamma} = E|\text{dom } G$ is a symmetric operator in $\mathcal{H}$ and $\text{dom } \tilde{M}(\lambda) = \text{dom } \tilde{\gamma}(\lambda)$ is trivial.

3.5. Some additional remarks. Despite of the fact that the paper \cite{31} has been quoted by M.G. Krein \cite{71} and a discussion on \cite{31} appears in the monograph \cite{34} the actual results of Calkin on reduction operators remained widely unknown among experts in extension theory. Apparently this was caused by the fact that the paper \cite{31} was ahead of time – it was using the new language of binary linear relations with hidden ideas on geometry of indefinite inner product spaces, concepts which were not well developed at that time. The concept of a bounded reduction operator investigated therein (see \cite{31} Chapter IV] essentially covers the notion of an ordinary boundary triple in Definition 1.1 as well as the notion of $D$-boundary triple introduced in \cite{33} for symmetric operators with unequal defect numbers. An overview on the early work of Calkin and more detailed description on its connections to boundary triples and unitary boundary pairs (boundary relations) can be found from the monograph \cite{59}. In fact, \cite{59} contains a collection of articles reflecting various recent activities in different fields of applications with related realization results for Weyl functions, including analysis of differential operators, continuous time state/signal systems and boundary control theory with interconnection analysis of port-Hamiltonian systems involving Dirac and Tellegen structures etc.
4. **AB-generalized boundary pairs and AB-generalized boundary triples**

In this section present a new generalization of the class of B-generalized boundary triples from [14] (see Definition 1.5).

**Definition 4.1.** Let $A$ be a symmetric operator (or relation) in the Hilbert space $\mathcal{H}$ and let $\mathcal{H}$ be another Hilbert space. Then a linear relation $\Gamma : A' \to \mathcal{H} \oplus \mathcal{H}$ with domain dense in $A'$ is said to be an *almost B-generalized boundary pair*, in short, *AB-generalized boundary pair* for $A^*$, if the following three conditions are satisfied:

4.1.1 the abstract Green’s identity (3.1) holds;

4.1.2 $\text{ran} \Gamma_0$ is dense in $\mathcal{H}$;

4.1.3 $A_0 = \ker \Gamma_0$ is selfadjoint,

where $\Gamma_0 = \pi_0 \Gamma$ stands for the first component mapping of $\Gamma$; see (3.6).

A *single-valued AB-generalized boundary pair* is also said to be an *almost B-generalized boundary triple*, shortly, an *AB-generalized boundary triple* for $A^*$.

If $\Gamma$ is an $AB$-generalized boundary pair for $A^*$, then the same is true for its closure. Indeed, since $\overline{\Gamma}$ is an extension of $\Gamma$, it is clear that $\text{dom} \overline{\Gamma}$ is dense in $A^*$ and $\text{ran} \overline{\Gamma} = \text{ran} \Gamma$ is dense in $\mathcal{H}$. By Assumption 4.1.1 $\overline{\Gamma}$ is isometric (in the Kreın space sense), i.e. $\overline{\Gamma}^{-1} \subset \Gamma^{[*]}$. Thus, clearly $\overline{\Gamma}^{-1} \subset \Gamma^{[*]} \subset \overline{\Gamma}^{[*]}$. Hence, the closure satisfies the Green’s identity (3.1) and this implies that the corresponding kernels $\ker (\overline{\Gamma})_0 \subset \ker \Gamma_0 = A_0$ and $\ker (\overline{\Gamma})_1 \subset \ker \Gamma_1 = A_1$ are symmetric. Therefore, $\ker (\overline{\Gamma})_0 = A_0$ must be selfadjoint.

4.1. **Characteristic properties of AB-generalized boundary pairs and triples.** The next theorem describes some central properties of an AB-generalized boundary pair.

**Theorem 4.2.** Let $A$ be a closed symmetric relation in $\mathcal{H}$, let $\{\mathcal{H}, \Gamma\}$ be an AB-generalized boundary pair for $A^*$, and let $\Gamma_0$ and $\Gamma_1$ be the corresponding component mappings from dom $\Gamma$ into $\mathcal{H}$. Moreover, let $\gamma(\cdot)$ and $M(\cdot)$ be the corresponding $\gamma$-field and the Weyl function. Then:

(i) $\ker \Gamma = A$;

(ii) $A_* := \text{dom} \Gamma$ admits the decomposition $A_* = A_0 \hat{\Gamma}_\lambda (A_*)$ and $\hat{\Gamma}_\lambda (A_*)$ is dense in $\hat{\Gamma}_\lambda (A^*)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) the $\gamma$-field $\gamma(\lambda)$ is a densely defined bounded operator from $\text{ran} \Gamma_0$ onto $\hat{\Gamma}_\lambda (A_*)$. It is domain invariant and one has

$$\text{dom} \gamma(\lambda) = \text{ran} \Gamma_0, \quad \ker \gamma(\lambda) = \text{mul} \Gamma_0;$$

(iv) the adjoint $\gamma(\lambda)^*$ is a bounded everywhere defined operator and, moreover, equalities hold in (3.15), (3.16),

$$(4.1) \quad \Gamma H(\lambda) = \left( \begin{array}{cc} 0 \\ \gamma(\lambda)^* \end{array} \right) \hat{\gamma}(\{0\} \times \text{mul} \Gamma), \quad \Gamma_1 H(\lambda) = \gamma(\lambda)^* \hat{\gamma}(\{0\} \times \text{mul} \Gamma_1), \quad \lambda \in \mathbb{C} \setminus \mathbb{R};$$

(v) the closure of the $\gamma$-field $\gamma(\lambda)$ is a bounded operator from $\mathcal{H}$ into $\hat{\Gamma}_\lambda (A^*)$ satisfying the identity

$$\gamma(\lambda) = [I + (\lambda - \mu)(A_0 - \lambda)^{-1}] \hat{\gamma}(\mu), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R};$$

(vi) the Weyl function $M$ is a densely defined operator valued function which is domain invariant, $\text{dom} M(\lambda) = \text{ran} \Gamma_0$, $M(\lambda) \subset M(\lambda)^*$, and the imaginary part $\text{Im} M(\lambda) = (M(\lambda) - M(\lambda)^*)/2i$ is bounded with $\text{dom} \text{Im} M(\lambda) = \text{ran} \Gamma_0$ and $\ker \text{Im} M(\lambda) = \text{mul} \Gamma_0$. Furthermore, $M(\lambda)$ admits the following representation

$$(4.2) \quad M(\lambda) = E + M_0(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$
where $E = \text{Re } M(\mu)$ is a symmetric densely defined operator in $\mathcal{H}$ and $M_0(\cdot)$ is a bounded Nevanlinna function (defined on $\text{dom } E$), i.e., $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$.

Proof. (i) It is clear from the Green’s identity that $\ker \Gamma \subset (\text{dom } \Gamma)^* = (A_0)^* = A$; cf. [39 Lemma 7.3]. To prove the reverse inclusion, the property that $\gamma(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}$, is densely defined will be used (and this is independently proved in (iii) below). Assumption 4.1.3 implies that $A = (A_0)^* \subset A_0^* = A_0 = \ker \Gamma_0 \subset \text{dom } \Gamma$. On the other hand, if $k_\lambda \in \text{ran } (A - \lambda)$ then by Lemma 3.8 $\{k_\lambda, k''\} \in \gamma(\lambda)^*$ for some $k''$ and thus for all $h \in \text{dom } \gamma(\lambda)$ one has

$$\langle k'', h \rangle_{\mathcal{H}} = \langle k_\lambda, \gamma(\lambda) h \rangle.$$

Assumption 4.1.2 combined with $\text{dom } \gamma(\lambda) = \text{ran } \Gamma_0$ (see proof of (iii) below) shows that $\gamma(\lambda)$ is densely defined and, hence, $\gamma(\lambda)^*$ is an operator and $k'' = \gamma(\lambda)^* k_\lambda = 0$. Now apply the formula (3.18) in the proof of Lemma 3.8 to $k_\lambda \in \text{ran } (A - \lambda)$: therein $\{k_\lambda, \{k, k'\}\} \in \Gamma H(\lambda)$ and $k'' = 0$ so that (3.18) reads as

$$\left\{ k_\lambda, \left( \begin{array}{c} k \\ k' \end{array} \right) \right\} = \left\{ k_\lambda, \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right\} + \left\{ 0, \left( \begin{array}{c} k \\ k' \end{array} \right) \right\}, \quad \left( \begin{array}{c} k \\ k' \end{array} \right) \in \text{mul } \Gamma.$$

Hence, $H(\lambda) k_\lambda \in \ker \Gamma$ and $A = H(\lambda)(\text{ran } (A - \lambda)) \subset \ker \Gamma$. Therefore, $\ker \Gamma = A$.

(ii) This holds by Proposition 3.3 (i).

(iii) & (iv) The decomposition of $A_\lambda$ in (ii) combined with $A_0 = \ker \Gamma_0$ implies that

$$\Gamma_0(A_\lambda) = \Gamma_0(\mathfrak{H}_\lambda(A_\lambda)) = \text{dom } M(\lambda) = \text{dom } \gamma(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Hence, $\text{dom } M(\lambda) = \text{dom } \gamma(\lambda) = \text{ran } \Gamma_0$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now Assumption 4.1.2 shows that $\gamma(\lambda)$ and $M(\lambda)$ are densely defined for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, according to Lemma 3.8 (iii) $\gamma(\lambda)$ is a bounded operator and the equality $\text{dom } \gamma(\lambda)^* = \mathfrak{H}$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since $\gamma(\lambda)$ is densely defined in $\mathcal{H}$, the adjoint $\gamma(\lambda)^*$ is a bounded everywhere defined operator from $\mathfrak{H}$ into $\Gamma_1(A_0)$. Since $M(\lambda) \subset M(\lambda)^*$, see (3.12), the adjoint $M(\lambda)^*$ and the closure of $M(\lambda)$ are also densely defined operators. In view of (3.13) one has

$$\langle \lambda - \bar{\mu}, (\gamma(\lambda) h, \gamma(\mu) k)_{\mathfrak{H}} \rangle = \langle (M(\lambda) - M(\mu)^*) h, k \rangle_{\mathcal{H}}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R},$$

for all $h, k \in \text{dom } \gamma(\lambda) = \text{ran } \Gamma_0$. In particular, $2 \text{Im } \lambda \| \gamma(\lambda) h \|^2_{\mathfrak{H}} = \langle (M(\lambda) - M(\lambda)^*) h, h \rangle_{\mathcal{H}}$ holds for all $h \in \text{dom } \gamma(\lambda) = \text{dom } M(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}$. By Lemma 3.5 (iv) implies that

$$\ker \gamma(\lambda) = \ker (M(\lambda) - M(\lambda)^*) = \text{mul } \Gamma_0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

It remains to prove (4.1). Observe, that $\text{dom } \Gamma_1 H(\lambda) = \text{dom } \gamma(\lambda)^* = \mathfrak{H}$ and clearly the multivalued parts on both sides of the inclusion in (3.15), (3.16) are equal. Hence, the inclusions (3.15), (3.16) must prevail actually as equalities (by the criterion from [10]).

(v) Since $\text{dom } M(\lambda) = \text{dom } \gamma(\lambda) = \text{ran } \Gamma_0$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the equality

$$\langle I + (\lambda - \mu)(A_0 - \lambda)^{-1} \gamma(\mu) h, h \rangle = \gamma(\lambda) h$$

holds for all $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ and $h \in \text{ran } \Gamma_0$ by Proposition 3.11. According to (iii) $\gamma(\lambda)$ is bounded and densely defined, so that its closure $\gamma(\lambda)$ is bounded and defined everywhere on $\mathcal{H}$. The formula in (iv) is obtained by taking closures in (4.1).

(vi) It suffices to prove the representation (1.2) for $M(\lambda)$, since all the other assertions were already shown above when proving (iii) & (iv). It follows from (4.3) and (4.4) that

$$\langle (M(\lambda) h, k) = \langle M(\mu)^* h, k \rangle + (\lambda - \bar{\mu})((I + (\lambda - \mu)(A_0 - \lambda)^{-1}) \gamma(\mu) h, \gamma(\mu) k)$$

$$= \langle \text{Re } M(\mu) h, k \rangle + (((\lambda - \text{Re } \mu) + (\lambda - \mu) (A_0 - \lambda)^{-1}) \gamma(\mu) h, \gamma(\mu) k),$$

$\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$. Here $2 \text{Re } M(\mu) = M(\mu) + M(\mu)^*$ and hence $2 \text{Re } M(\mu)^* \supset M(\mu)^* + M(\mu) \supset 2 \text{Re } M(\mu)$, so that $E := \text{Re } M(\mu)$ is a symmetric
operator with \( \text{dom } E = \text{dom } M(\mu) = \text{ran } \Gamma_0 \). On the other hand, since \( \overline{\gamma(\lambda)} \) and its adjoint \( \gamma(\lambda)^* \) are bounded everywhere defined operators, it follows that the closure of
\[
M_0(\lambda) := \gamma(\mu)^*((\lambda - \text{Re } \mu) + (\lambda - \mu)(\lambda - \bar{\mu})(A_0 - \lambda)^{-1})\gamma(\mu)
\]
is a bounded holomorphic operator valued Nevanlinna function acting on \( \mathcal{H} \), such that \( M(\lambda) = E + M_0(\lambda) \). This completes the proof.

For an \( AB \)-generalized boundary pair it is possible to describe the graph of \( \Gamma \), \( (\text{ran } \Gamma)^{[1]} \), and the closure of \( \text{ran } \Gamma \) explicitly.

**Corollary 4.3.** Let \( \Gamma \) be an \( AB \)-generalized boundary pair for \( A^* \) and let \( \gamma(\cdot) \) and \( M(\cdot) = E + M_0(\cdot) \) be the corresponding \( \gamma \)-field and Weyl function as in Theorem 4.2 with \( E = \text{Re } M(\mu) \) for some fixed \( \mu \in \mathbb{C} \setminus \mathbb{R} \). Then:

(i) with a fixed \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the graph of \( \Gamma \) admits the following representation:

\[
\Gamma = \left\{ \begin{pmatrix} H(\lambda)k_\lambda & \left( \begin{array}{c} 0 \\ \gamma(\bar{\lambda})^*k_\lambda \end{array} \right) \\ \left( \begin{array}{c} \gamma(\lambda)h \\ (M(\lambda)h)_* \end{array} \right) \end{pmatrix} : k_\lambda \in \text{ran } (A_0 - \lambda) \right\}; \quad h \in \text{dom } M(\lambda)
\]

(ii) the range of \( \Gamma \) satisfies

\[
(\text{ran } \Gamma)^{[1]} = E^* \upharpoonright \ker \overline{\gamma(\lambda)} \quad \text{and} \quad \overline{\text{ran } \Gamma} = (E^* \upharpoonright \ker R(\lambda))^{[1]},
\]
and here \( \ker \overline{\gamma(\lambda)} = \ker (M_0(\lambda)) \) does not depend on \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). In particular, \( \text{ran } \Gamma \) is dense in \( \mathcal{H} \) if and only if \( \text{dom } E^* \cap \ker \overline{\gamma(\lambda)} = \{0\} \) for some or, equivalently, for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

(iii) \( \Gamma \) is a single-valued mapping if and only if \( \text{mul } \Gamma_0 = \{0\} \) or, equivalently, if and only if \( \ker \text{Im } M(\lambda) = \ker \overline{\gamma(\lambda)} = 0 \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

**Proof.** (i) Using the representation of \( \Gamma H(\lambda) \) in \([1,1]\), the inclusion \( \text{mul } \Gamma \subset M(\lambda) \) in Lemma 3.6 and the fact that by Theorem 4.2, \( M(\lambda) \) is an operator, one concludes that the representation of \( \Gamma \) given in Proposition 3.9 (ii) can be rewritten in the form as stated in (i).

(ii) The description in (i) shows that

\[
\text{ran } \Gamma = \Gamma(A_0) \upharpoonright M(\lambda) = \left( \begin{pmatrix} 0 \\ \overline{\text{ran } \gamma(\lambda)}^* \end{pmatrix} \right) \upharpoonright M(\lambda),
\]
for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). Therefore, \( (\text{ran } \Gamma)^{[1]} = \{0\} \times \overline{\text{ran } \gamma(\lambda)}^* \cap M(\lambda)^* \). Hence \( \hat{k} = \{k, k'\} \in (\text{ran } \Gamma)^{[1]} \) if and only if \( k \in M(\lambda)^* \) and \( k' \in (\text{ran } \gamma(\lambda)^*)^\perp = \ker \overline{\gamma(\lambda)} \). Since \( E = \text{Re } M(\mu) \), one has \( \text{Re } M_0(\mu) = 0 \) and hence \( \ker \overline{\gamma(\mu)} = \ker \text{Im } M_0(\mu) = \ker \overline{M_0(\mu)} \) and this kernel does not depend on \( \mu \in \mathbb{C} \setminus \mathbb{R} \) due to \( M_0(\cdot) \in \mathcal{R} \mathcal{H} \); cf. Theorem 4.2 (v). This proves that

\[
(\text{ran } \Gamma)^{[1]} = M(\lambda)^* \upharpoonright \ker \overline{\gamma(\lambda)} = (E^* + M_0(\lambda)^*) \upharpoonright \ker \overline{\gamma(\lambda)} = E^* \upharpoonright \ker \overline{\gamma(\lambda)}.
\]
As to the closure observe that

\[
\overline{\text{ran } \Gamma} = ((\text{ran } \Gamma)^{[1]})^* = (E^* \upharpoonright \ker \overline{\gamma(\lambda)})^*.
\]
Thus, \( \overline{\text{ran } \Gamma} = \mathcal{H} \times \mathcal{H} \) if and only if \( E^* \upharpoonright \ker \overline{\gamma(\lambda)} = \{0, 0\} \) or, equivalently, \( \text{dom } E^* \cap \ker \overline{\gamma(\lambda)} = \{0\} \), since \( E^* \) together with \( E \subset E^* \) is a densely defined operator in \( \mathcal{H} \).

(iii) In view of (i) this follows from \( \text{mul } \Gamma_0 = \ker \text{Im } M(\lambda) = \ker \overline{\gamma(\lambda)} \); see Lemma 3.6. 

Corollary 4.3 shows that for an \( AB \)-generalized boundary pair the inclusion \( \text{mul } \Gamma \subset (\text{ran } \Gamma)^{[1]} \) is in general strict. In particular, the range of \( \Gamma \) for a single-valued \( AB \)-generalized boundary pair, i.e., an \( AB \)-generalized boundary triple, need not be dense in \( \mathcal{H} \times \mathcal{H} \). Notice that an \( AB \)-generalized boundary pair with the surjectivity condition \( \text{ran } \Gamma_0 = \mathcal{H} \) is called a \( B \)-generalized
boundary pair for \( A^* \); see Definition 3.5. The next result gives a connection between \( AB \)-generalized boundary pairs and \( B \)-generalized boundary pairs.

**Theorem 4.4.** Let \( \{ \mathcal{H}, \Gamma \} \) be a \( B \)-generalized boundary pair for \( A^* \), and let \( M(\cdot) \) and \( \gamma(\cdot) \) be the corresponding Weyl function and \( \gamma \)-field. Let also \( E \) be a symmetric densely defined operator in \( \mathcal{H} \) and let \( \Gamma = \{ \Gamma_0, \Gamma_1 \} \) where \( \Gamma_i = \pi_i \Gamma, i = 0, 1 \), be the corresponding components of \( \Gamma \) as in (3.16). Then the transform

\[
\begin{pmatrix}
\tilde{\Gamma}_0 \\
\tilde{\Gamma}_1
\end{pmatrix} = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix}
\Gamma_0 \\
\Gamma_1
\end{pmatrix}
\]

defines an \( AB \)-generalized boundary pair for \( A^* \). The corresponding Weyl function \( \tilde{M}(\cdot) \) and \( \tilde{\gamma}(\cdot) \)-field are connected by

\[
\tilde{M}(\lambda) = E + M(\lambda), \quad \tilde{\gamma}(\lambda) = \gamma(\lambda) | \text{dom } E, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Furthermore, \( \tilde{\Gamma} := \{ \tilde{\Gamma}_0, \tilde{\Gamma}_1 \} \) in (4.7) is closed if and only if \( E \) is a closed symmetric operator in \( \mathcal{H} \), in particular, the closure of \( \Gamma \) is given by (4.7) with \( E \) replaced by its closure \( \overline{E} \).

Conversely, if \( \{ \mathcal{H}, \tilde{\Gamma} \} \) is an \( AB \)-generalized boundary pair for \( A^* \) then there exists a \( B \)-generalized boundary pair \( \{ \mathcal{H}, \Gamma \} \) for \( A^* \) and a densely defined symmetric operator \( E \) in \( \mathcal{H} \) such that \( \tilde{\Gamma} \) is given by (4.7).

**Proof.** (\( \Rightarrow \)) By Lemma 3.12 the block triangular transformation \( V \) in (4.7) acting on \( \mathcal{H} \times \mathcal{H} \) is an isometric operator. Consequently, \( \tilde{\Gamma} = V \circ \Gamma \) is isometric. It is clear from (4.7) that \( A_0 := \text{ker } \Gamma_0 \subseteq \text{ker } \tilde{\Gamma}_0 \), which by symmetry of \( \text{ker } \tilde{\Gamma}_0 = A_0 \). Clearly \( \text{ran } \tilde{\Gamma}_0 \) is dense in \( \mathcal{H} \), since \( \text{ran } \Gamma_0 = \mathcal{H} \) and \( E \) is densely defined. Thus \( \tilde{\Gamma} \) admits all the properties in Definition 4.4. Since in addition \( \text{ker } \tilde{\Gamma} = \text{ker } \Gamma \), it follows from Theorem 4.2 (i) that \( \tilde{A} = \text{dom } \tilde{\Gamma} \) is dense in \( A^* \). Therefore, \( \{ \mathcal{H}, \tilde{\Gamma} \} \) is an \( AB \)-generalized boundary pair for \( A^* \). The connections between the Weyl functions and \( \gamma \)-fields are clear from the definitions; cf. Lemma 3.12.

To treat the closedness properties of \( \tilde{\Gamma} \) consider the representation of \( \tilde{\Gamma} \) in Corollary 4.3. Let \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) be fixed and assume that the sequence \( \{ \hat{f}_n, \hat{k}_n \} \) in \( \tilde{\Gamma} \) converges to \( \{ \hat{f}, \hat{k} \} \). Then \( \hat{f}_n = H(\lambda)k_{n,\lambda} + \bar{\gamma}(\lambda)n_n \) with unique \( k_{n,\lambda} \in \text{ran } (A_0 - \lambda) \) and \( n_n \in \text{dom } \tilde{M}(\lambda) = \text{dom } E \) and, since the angle between the graphs of \( A_0 \) and \( \tilde{\Omega}_0(A_\lambda) \) is positive, it follows that \( k_{n,\lambda} \rightarrow k_\lambda \in \text{ran } (A_0 - \lambda) \). Moreover, the representation of \( \{ \hat{f}_n, \hat{k}_n \} \) in (4.7) in Corollary 4.3 shows that \( n_n \rightarrow h \in \mathcal{H} \). According to Theorem 4.2, \( \gamma(\lambda) \) and \( \gamma(\lambda)^* \) are bounded operators and, since \( \tilde{M}(\lambda) = E + M(\lambda) \), where \( M(\lambda) \) is bounded (see 3.8 Proposition 3.16)), it follows from Corollary 4.3 that

\[
\left\{ \begin{array}{c}
H(\lambda)k_{n,\lambda} \\
\bar{\gamma}(\lambda)n_n
\end{array} \right\} + \left\{ \begin{array}{c}
\gamma(\lambda)n_n \\
Eh_n + M(\lambda)n_n
\end{array} \right\} \in \tilde{\Gamma}
\]

converges to

\[
\left\{ \begin{array}{c}
H(\lambda)k_\lambda \\
\bar{\gamma}(\lambda)k_\lambda
\end{array} \right\} + \left\{ \begin{array}{c}
\gamma(\lambda)h \\
h + M(\lambda)h
\end{array} \right\} \in \text{clo } \tilde{\Gamma},
\]

where \( \{ h, h'' \} \in \overline{E} \). It is also clear that the limit element in (4.8) belongs to \( \tilde{\Gamma} \) if and only if \( \lim_{n \rightarrow \infty} \{ n_n, Eh_n \} = \{ h, h'' \} \in E \). Therefore, \( \tilde{\Gamma} \) is closed if and only if \( E \) is closed and, moreover, the closure of \( \Gamma \), which is also an \( AB \)-generalized boundary pair for \( A^* \) (as stated after Definition 4.11), is given by (4.7) with \( E \) replaced by its closure \( \overline{E} \).

(\( \Leftarrow \)) Let \( \{ \mathcal{H}, \tilde{\Gamma} \} \) be an \( AB \)-generalized boundary pair. According to Theorem 4.2 the corresponding Weyl function \( \tilde{M} \) is of the form \( \tilde{M} = E + M \), where \( M \) is a bounded Nevanlinna function and \( E (= \text{Re } \tilde{M}(\mu)) \) is a symmetric densely defined operator in \( \mathcal{H} \).
To construct $\hat{\Gamma}$ directly from an associated $B$-generalized boundary pair, define

\begin{equation}
(4.9) \quad \begin{pmatrix}
\hat{\Gamma}_0 \\
\hat{\Gamma}_1
\end{pmatrix} := \begin{pmatrix}
I & 0 \\
-E & I
\end{pmatrix}
\begin{pmatrix}
\tilde{\Gamma}_0 \\
\tilde{\Gamma}_1
\end{pmatrix}.
\end{equation}

Since $\tilde{M}(\lambda) = \tilde{\Gamma}(\tilde{\mathcal{H}}_A) \subset \text{ran} \tilde{\Gamma}$, where $\tilde{A}_s = \text{dom} \tilde{\Gamma}$, and dom $\tilde{M}(\lambda) = \text{dom} E$, it follows that the graph of $M(\lambda)$ belongs to the domain of the block operator

\begin{equation}
\begin{pmatrix}
I & 0 \\
-E & I
\end{pmatrix},
\end{equation}

i.e., $\tilde{\mathcal{H}}_A(\tilde{A}_s) \subset \text{dom} \tilde{\Gamma}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover,

\[
\hat{\Gamma}(\tilde{\mathcal{H}}_A(\tilde{A}_s)) = -E + \tilde{M}(\lambda) = M(\lambda) | \text{dom} E \subset \text{ran} \hat{\Gamma}
\]

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since $\overline{M} \in \mathcal{R}[\mathcal{H}]$ this implies that $\text{ran} \hat{\Gamma}_0$ is dense in $\mathcal{H}$. Clearly, $\ker \hat{\Gamma}_0 = \ker \tilde{\Gamma}_0 = A_0$ and since $\tilde{A}_s = A_0 + \tilde{\mathcal{H}}_A(\tilde{A}_s)$ one concludes that $\tilde{A}_s = \text{dom} \hat{\Gamma} = \text{dom} \tilde{\Gamma}$ is dense in $A^*$. Thus, $\hat{\Gamma}$ is also an $AB$-generalized boundary pair for $A^*$ and, consequently, also its closure is an $AB$-generalized boundary triple for $A^*$, too. Denote the closure of $\hat{\Gamma}$ by $\Gamma^{(0)}$. Then the corresponding Weyl function $M^{(0)}(\cdot)$ is an extension of $M$ and its closure is equal to $\overline{M}$. Since $\Gamma^{(0)}$ is closed, it must be unitary by [39, Theorem 7.51] (cf. [37, Proposition 3.6]). In particular, $M^{(0)}(\cdot)$ is also closed, i.e., $M^{(0)}(\cdot) = \overline{M} \in \mathcal{R}[\mathcal{H}]$. Thus, $\text{ran} \Gamma^{(0)} = \text{dom} M^{(0)}(\cdot) = \mathcal{H}$ and hence $\Gamma^{(0)}$ is a $B$-generalized boundary pair for $A^*$; see Definition 3.5. Finally, in view of (4.9) one has

\[
\begin{pmatrix}
\hat{\Gamma}_0 \\
\hat{\Gamma}_1
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
-E & I
\end{pmatrix}
\begin{pmatrix}
\tilde{\Gamma}_0 \\
\tilde{\Gamma}_1
\end{pmatrix} \subset \begin{pmatrix}
I & 0 \\
-E & I
\end{pmatrix} \Gamma^{(0)} =: \tilde{\Gamma}^{(0)}.
\]

Here equality $\hat{\Gamma} = \tilde{\Gamma}^{(0)}$ holds by Proposition 3.9 (iii), since $\tilde{M}^{(0)}(\cdot) = E + \overline{M}(\cdot) = \tilde{M}(\cdot)$. □

The proof of Theorems 4.2 contains also the following result.

**Corollary 4.5.** If $\{\mathcal{H}, \tilde{\Gamma}\}$ is an $AB$-generalized boundary pair for $A^*$ with the Weyl function $\tilde{M}(\cdot)$ and $E = \text{Re} \tilde{M}(\mu)$ for some $\mu \in \rho(M)$, then the closure of $\Gamma = \begin{pmatrix}
I & 0 \\
-E & I
\end{pmatrix} \tilde{\Gamma}$ defines a $B$-generalized boundary pair for $A^*$ with the bounded Weyl function $M(\cdot) = \text{clo}(\tilde{M}(\cdot) - E)$.

Theorems 4.2 and 4.4 imply the following characterization for the Weyl functions corresponding to $AB$-generalized boundary pairs.

**Corollary 4.6.** The class of $AB$-generalized boundary pairs coincides with the class of isometric boundary pairs whose Weyl function is of the form

\begin{equation}
M(\lambda) = E + M_0(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\end{equation}

with $E$ a symmetric densely defined operator in $\mathcal{H}$ and $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$. In particular, every function $M$ of the form (4.10) is a Weyl function of some $AB$-generalized boundary pair.

**Proof.** By Theorem 4.2 the Weyl function $M$ of an $AB$-generalized boundary pair $\{\mathcal{H}, \tilde{\Gamma}\}$ is of the form (4.10), where $E \subset E^*$ is densely defined and $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$.

Conversely, if $M$ is given by (4.10) with $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$, then by [38, Proposition 3.16] $M_0(\cdot)$ is the Weyl function of a $B$-generalized boundary pair $\{\mathcal{H}, \Gamma\}$ for $A^*$. Now according to the first part of Theorem 4.2 the transform $\tilde{\Gamma}$ of $\Gamma$ defined in (4.7) is an $AB$-generalized boundary pair for $A^*$ such that the corresponding Weyl function is equal to (4.10). □
By Definition 3.5 every $B$-generalized boundary pair is also an $AB$-generalized boundary pair. Hence, the notions of $AB$-generalized boundary pairs and $AB$-generalized boundary triples generalize the earlier notions of $(B)$-generalized boundary triples as introduced in [44] and boundary triples of bounded type as defined in [37, Section 5.3]. It is emphasized that $B$-generalized boundary pairs are not only isometric: they are also unitary in the Krein space sense, see Definition 3.1. The characteristic properties of the classes of $B$-generalized boundary pairs are not only isometric: they are also unitary in the Krein space sense, recall that the class of functions $M \in \mathcal{R}[\mathcal{H}]$ coincides with the class of Weyl functions of $B$-generalized boundary pairs and the class of functions $M \in \mathcal{R}^{*}[\mathcal{H}]$ coincides with the class of $B$-generalized boundary triples. Some further characterizations connected with $AB$-generalized boundary pairs are given in the next corollary.

**Corollary 4.7.** Let $\{\mathcal{H}, \tilde{\Gamma}\}$ be an $AB$-generalized boundary pair for $A^*$ as in Theorem 4.4 and let $E$ be a symmetric densely defined operator in $\mathcal{H}$ as in (1.7). Then:

(i) $\{\mathcal{H}, \tilde{\Gamma}\}$ is a unitary boundary pair (boundary relation) for $A^*$ if and only if the operator $E$ is selfadjoint;

(ii) $\{\mathcal{H}, \tilde{\Gamma}\}$ has an extension to a unitary boundary pair for $A^*$ if and only if the operator $E$ has equal defect numbers and in this case the formula (4.7) defines a unitary extension of $\tilde{\Gamma}$ when $E$ is replaced by some selfadjoint extension $E_0$ of $E$;

(iii) $\{\mathcal{H}, \tilde{\Gamma}\}$ is a $B$-generalized boundary pair for $A^*$ if and only if the operator $E$ is bounded and everywhere defined (hence selfadjoint);

(iv) $\{\mathcal{H}, \tilde{\Gamma}\}$ is an ordinary boundary triple for $A^*$ if and only if $\text{ran} \Gamma = \mathcal{H} \oplus \mathcal{H}$, or equivalently, if and only if $\text{ran} \Gamma$ is closed, $E$ is bounded, and $\ker \text{Im} M(\lambda) = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

**Proof.** (i) By Theorem 4.4 $\tilde{\Gamma}$ is closed if and only if $E$ is closed. Moreover, $E = E^*$ if and only if $M$ is a Nevanlinna function. Now the statement follows from [37, Proposition 3.6] (or [39, Theorem 7.51]).

(ii) This is clear from part (i) and Theorem 4.4.

(iii) This follows from Theorem 4.2 (v) by the equalities $\text{ran} \tilde{\Gamma}_0 = \text{dom} \tilde{M} = \text{dom} E (= \mathcal{H})$.

(iv) The first equivalence is contained in [37, Proposition 5.3]. To prove the second criterion, we apply Corollary 4.3 in particular, the representation of $\text{ran} \Gamma$ in (4.6):

\[
\text{ran} \Gamma = \Gamma(A_0) \hat{\oplus} M(\lambda) = (\{0\} \times \text{ran} \gamma(\lambda)^* \hat{\oplus} M(\lambda)).
\]

Clearly, $E$ is bounded precisely when $M(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}$, is bounded. In this case the angle between the last two subspaces in (4.11) is positive and then $\text{ran} \Gamma$ is closed if and only if $\text{ran} \gamma(\lambda)^*$ and $M(\lambda)$ both are closed. By Theorem 4.2 $\gamma(\lambda)$ is bounded and $\text{dom} \gamma(\lambda) = \text{dom} M(\lambda) = \mathcal{H}$, when $M(\lambda)$ is closed and bounded. Then $\gamma(\lambda)$ is closed and $(\text{ran} \gamma(\lambda)^*)^\perp = \ker \gamma(\lambda) = \ker \text{Im} M(\lambda)$. Therefore, the conditions $\text{ran} \Gamma$ is closed, $E$ is bounded, and $\ker \text{Im} M(\lambda) = 0$ imply that $\text{ran} \Gamma$ is also dense in $\mathcal{H} \times \mathcal{H}$ and, thus, $\Gamma$ is surjective. The converse is clear. □

The notions of $AB$-generalized boundary pairs and $AB$-generalized boundary triples generalize also the class of so-called quasi boundary triples, which has been studied in J. Behrndt and M. Langer [17]. In the definition of a quasi boundary triple it is assumed that $\Gamma = \left( \begin{array}{c} \Gamma_0 \\ \Gamma_1 \end{array} \right)$ is single-valued and Assumption 4.1.2 in Definition 4.1 is replaced by the stronger condition that the joint range of $\Gamma = \left( \begin{array}{c} \Gamma_0 \\ \Gamma_1 \end{array} \right)$ is dense in $\mathcal{H} \times \mathcal{H}$.

Corollary 4.3 gives the following characterization for quasi boundary triples.
Corollary 4.8. An AB-generalized boundary triple \( \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) for \( A^* \) with the Weyl function 
\( M = E + M_0(\cdot) \) represented in the form (4.12) is a quasi boundary triple (with single-valued \( \Gamma \)) for \( A^* \) if and only if \( \text{ran} \, \Gamma \) is dense in \( \mathcal{H} \oplus \mathcal{H} \), or equivalently,

\[
\text{dom} \, E^* \cap \ker \text{Im} \, M(\lambda) = \{0\},
\]

for some \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

Remark 4.9. A connection between B-generalized boundary triples and quasi boundary triples can be found from [39, Theorem 7.57] and [92, Propositions 5.1, 5.3]. In fact, each of them is special case of Theorem 4.4. Moreover, it should be noted that in the formulation of the converse part in [39, Theorem 7.57] one should use a B-generalized boundary pair \( \{ \mathcal{H}, \Gamma \} \), instead of a B-generalized boundary triple \( \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \), since \( \ker \gamma(\lambda) = \ker \text{Im} \, M(\lambda) = 0 \) (\( M \) is strict) does not imply in general that \( \ker \gamma(\lambda) = \ker \text{Im} \, M(\lambda) = \ker \text{Im} \, M_0(\lambda) = 0 \), i.e. \( M_0 \in \mathcal{R}[\mathcal{H}] \) as in the proof of Theorem 4.4 above: only the factor mapping \( \Gamma / \text{mul} \, \Gamma \) (see [14], [60, eq. (2.15)]) becomes single-valued (equivalently the corresponding Weyl function is strict, cf. [37, Proposition 4.7]). It should be also noted that a condition which is equivalent to (4.12) appears in [92, Section 5.1]; see also [91]. For some further related facts, see Corollary 5.19 and Remark 5.21 in Section 5.

The next result describes a connection between B-generalized boundary pairs and ordinary boundary triples. In the special case of B-generalized boundary triples the corresponding result was established in [39, Theorem 7.24].

Theorem 4.10. Let \( \{ \mathcal{H}, \Gamma \} \) be a B-generalized boundary pair for \( A^* \) and let \( M(\cdot) \) be the corresponding Weyl function. Then there exists an ordinary boundary triple \( \{ \mathcal{H}_s, \Gamma_0^s, \Gamma_1^s \} \) with \( \mathcal{H}_s = \text{ran} \, \text{Im} \, M(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), and operators \( E = E^* \in \mathcal{B}(\mathcal{H}) \) and \( G \in \mathcal{B}(\mathcal{H}, \mathcal{H}_s) \) with \( \ker G = \mathcal{H} \oplus \mathcal{H}_s \) such that

\[
\Gamma_0^s = \begin{pmatrix} G^{-1} & 0 \\ \text{EG}^{-1} & G^* \end{pmatrix} \Gamma_1^s,
\]

where \( G^{-1} \) stands for the inverse of \( G \) as a linear relation. If \( M_0(\cdot) \) is the Weyl function corresponding to the ordinary boundary triple \( \{ \mathcal{H}_s, \Gamma_0^s, \Gamma_1^s \} \), then

\[
M(\lambda) = G^* M_0(\lambda) G + E, \quad \lambda \in \rho(A_0).
\]

Proof. The proof is based on [39, Theorem 7.24] and [38, Propositions 3.18, 4.1].

Let \( E = \text{Re} \, M(i) \). Then by [38, Propositions 3.18] (cf. Lemma 3.12) the transform

\[
\tilde{\Gamma} = \left\{ \tilde{f}, \left( \frac{\hat{h}}{-Eh + k'} \right) : \tilde{f}, \hat{h} \in \Gamma \right\}
\]

defines a new B-generalized boundary pair for \( A^* \) with the Weyl function \( M(\cdot) - E \) and the original \( \gamma \)-field \( \gamma(\cdot) \) of \( \{ \mathcal{H}, \Gamma \} \).

Let \( P_s \) be the orthogonal projection onto \( \mathcal{H}_s := \text{ran} \, \text{Im} \, M(\lambda) \). Then according to [38, Proposition 4.1] the transform \( \Gamma^{(s)} : \mathfrak{H}^2 \to (\mathcal{H}_s)^2 \) given by

\[
\Gamma^{(s)} = \left\{ \tilde{f}, \left( \frac{k}{P_s k'} \right) : \tilde{f}, \hat{k} \in \tilde{\Gamma}, (I - P_s) k = 0 \right\}
\]
determines a B-generalized boundary pair \( \{ \mathcal{H}_s, \Gamma^{(s)} \} \) for \( (A^{(s)})^* \), where \( A^{(s)} \) is defined by

\[
A^{(s)} := \ker \Gamma^{(s)}.
\]

The corresponding Weyl function and \( \gamma \)-field are given by

\[
M^{(s)}(\lambda) = P_s (M(\lambda) - E) | \mathcal{H}_s, \quad \gamma^{(s)}(\lambda) = \gamma(\lambda) | \mathcal{H}_s.
\]
Recall that $\ker (M(\lambda) - E) = \ker \text{Im} M(\lambda)$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Consequently, $M(\lambda) - E = M^{(s)}(\lambda) \oplus 0_{\mathcal{H} \oplus \mathcal{H}}$. Since $\ker \gamma(\bar{\lambda}) = \ker (M(\lambda) - E) = \ker P_s$ one has $\ker \gamma(\bar{\lambda})^* \subset \mathcal{H}_s$ and it follows from Corollary 4.13 that $\Gamma_1 \subset \mathcal{H}_s$. Therefore, (4.16) implies that $A^{(s)}$ defined in (4.17) coincides with $A$: $\ker \Gamma(s) = \ker \Gamma = A$. By construction $M^{(s)}(\cdot) \in \mathcal{R}^*[\mathcal{H}_s]$ and hence $\Gamma(s)$ is single-valued; i.e. $\{\mathcal{H}_s, \Gamma_0(s), \Gamma_1(s)\}$ is in fact a $B$-generalized boundary triple for $A^*$; cf. [37, Proposition 4.7].

One can now apply [39, Theorem 7.24] with $R = \text{Re} M^{(s)}(i) = 0$ and $K = (\text{Im} M^{(s)}(i))^{1/2}$ to conclude that there exists an ordinary boundary triple $\{\mathcal{H}_s, \Gamma_0, \Gamma_1\}$ with the Weyl function $M_0(\cdot)$ such that $\Gamma_0(s) = K^{-1} \Gamma_0^0, \Gamma_1(s) = K \Gamma_1^0$, and

$$M^{(s)}(\lambda) = KM_0(\lambda)K, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$  

In particular, $M(i) = E + i K^2 P_s$ and $M(\lambda) = E + P_s K M_0(\lambda) K P_s$. The statement follows by taking $G = K P_s$. Indeed, since $\ker \Gamma_1 \subset \mathcal{H}_s$ and $\ker \Gamma_0 = \ker \text{Im} M(\lambda) = \ker P_s$ (see Lemma 3.6) (4.16) shows that $\text{dom} \Gamma(s) = \ker \Gamma$ and

$$\Gamma = \Gamma(s) \oplus \left\{ \begin{array}{l} \{\bar{0}, (k, 0)\} : P_s k = 0 \end{array} \right\} = \left\{ \begin{array}{l} \{\hat{k}, (\frac{P_s^{-1} \Gamma_0(s) \hat{k}}{P_s \Gamma_1^0(s) \hat{k}})\} : \hat{k} \in \ker \Gamma(s) \end{array} \right\}.$$  

Finally, using $G^{-1} = P_s^{-1} K^{-1} = K^{-1} \oplus (\{0\} \times \ker P_s)$ and (4.15) yields the formulas (4.13) and (4.14).

The notion of an $AB$-generalized boundary pair introduced in Definition 4.1 appears to be useful in characterizing the class of unbounded Nevanlinna functions (and multivalued Nevanlinna families) whose imaginary parts generate closable forms $\tau_{M(\lambda)} = [(M(\lambda), \cdot) - (\cdot, M(\lambda))]/2i$ via (5.9) and whose closures are domain invariant. All such functions, after renormalization by a bounded operator $G \in [\mathcal{H}]$, turn out to be Weyl functions of $AB$-generalized boundary triples, i.e., for a suitable choice of $G$, $G^* MG$ is a function of the form (4.12); see Theorem 5.31 and Corollary 5.33 in Section 5.

4.2. A Krein type formula for $AB$-generalized boundary triples. In this section a Krein type (resolvent) formula for $AB$-generalized boundary triples will be established. We refer to [39, Proposition 7.27] where the case of $B$-generalized boundary triples was treated, and [17, 18] for the case of quasi boundary triples.

If $A_0 = \ker \Gamma_0$ is selfadjoint, then it follows from the first von Neumann’s formula that for each $\lambda \in \rho(A_0)$ the domain of $\Gamma$ can be decomposed as follows:

$$(4.18) \quad \text{dom} \Gamma = A_0 \oplus (\ker \Gamma \cap \hat{\mathcal{R}}_{\lambda}(A^*)).$$  

Now let $\Gamma$ be a single-valued and let $\Gamma$ be decomposed as $\Gamma = \{\Gamma_0, \Gamma_1\}$. Let $\hat{A}$ be an extension of $A$ which belongs to the domain of $\Gamma$ and let $\Theta$ be a linear relation in $\mathcal{H}$ corresponding to $A$:

$$(4.19) \quad \Theta = \Gamma(\hat{A}), \quad \hat{A} \subset \ker \Gamma \iff \hat{A} = A_\Theta := \Gamma^{-1}(\Theta), \quad \Theta \subset \ker \Gamma.$$  

**Theorem 4.11.** Let $A$ be a closed symmetric relation, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an $AB$-generalized boundary triple for $A^*$ with $A_0 = \ker \Gamma_0$, and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and $\gamma$-field, respectively. Then for any extension $A_\Theta \in \text{Ext}_A$ satisfying $A_\Theta \subset \ker \Gamma$ the following Krein-type formula holds

$$(4.20) \quad (A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1} \gamma(\bar{\lambda})^*, \quad \lambda \in \rho(A_0).$$  

Here the inverses in the first and last terms are taken in the sense of linear relations.
Proof. Let $A_{\Theta} \subset \text{dom} \, \Gamma$ be an extension of $A$ with $\Theta$ as in (4.19) and let $\lambda \in \rho(A_0)$. Assume that \( \{ g, g' \} \in (A_{\Theta} - \lambda)^{-1} \) or, equivalently, that $\hat{g}_{\Theta} := \{ g', g + \lambda g \} \in A_{\Theta}$. Then

\begin{equation}
(4.21) \quad \hat{g}_0 := \{(A_0 - \lambda)^{-1}g, (I + \lambda(A_0 - \lambda)^{-1})g\} \in A_0 \subset \text{dom} \, \Gamma,
\end{equation}

and $\hat{g}_{\Theta} - \hat{g}_0 \in \text{dom} \, \Gamma$. Furthermore,

\begin{equation}
(4.22) \quad \hat{g}_{\Theta} = \hat{g}_0 + \gamma(\lambda)\varphi
\end{equation}

for some $\varphi \in \text{ran} \, \Gamma_0$. By Definition 3.2, $\Gamma\gamma(\lambda)\varphi = \{ \varphi, M(\lambda)\varphi \}$ and it follows from Theorem 4.11 that $\Gamma\hat{g}_0 = \{0, \gamma(\lambda)^*g\}$, since $\text{mul} \, \Gamma = \{0\}$. Now an application of $\Gamma$ to (4.22) shows that

\begin{equation}
(4.23) \quad \{0, \gamma(\lambda)^*g\} + \{\varphi, M(\lambda)\varphi\} = \Gamma\hat{g}_0 + \Gamma\gamma(\lambda)\varphi = \Gamma\hat{g}_{\Theta} \in \Theta,
\end{equation}

see (4.19). Thus \{\varphi, \gamma(\lambda)^*g + M(\lambda)\varphi\} $\in \Theta$ and \{ \varphi, $\gamma(\lambda)^*g$ $\} \in \Theta - M(\lambda)$ or, equivalently, \{ $g, \varphi$ $\} \in (\Theta - M(\lambda))^{-1}\gamma(\lambda)^*$. Substituting this into (4.22) and taking the first components leads to the inclusion

\begin{equation}
(A_{\Theta} - \lambda)^{-1} \subset (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*.
\end{equation}

To prove the reverse inclusion, let $g \in \mathcal{H}$ such that \{ $\gamma(\lambda)^*g, \varphi$ $\} \in (\Theta - M(\lambda))^{-1}$. Equivalently, this means that

\begin{equation}
(4.24) \quad \{ g, \gamma(\lambda)\varphi \} \in \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*,
\end{equation}

since $\text{dom} \, (\Theta - M(\lambda)) \subset \text{ran} \, \Gamma_0 = \text{dom} \, \gamma(\lambda)$ with $\lambda \in \rho(A_0)$; see (4.19). It follows from \{ \varphi, $\gamma(\lambda)^*g + M(\lambda)\varphi$ \} $\in \Theta$ and (4.19) that $\Gamma\hat{g}_{\Theta} = \{ \varphi, \gamma(\lambda)^*g + M(\lambda)\varphi \}$ for some $\hat{g}_{\Theta} \in A_{\Theta}$. The decomposition (4.18) implies that $\hat{g}_{\Theta} = \hat{v}_0 + \gamma(\lambda)\psi$, where $\hat{v}_0 \in A_0$ and $\psi \in \text{dom} \gamma(\lambda)$ are unique since the decomposition in (4.18) is direct and $\ker \gamma(\lambda) = \{0\}$ due to $\text{mul} \, \Gamma_0 = \{0\}$; see Theorem 4.11. Associate with $g$ the element $\hat{g}_0 \in A_0$ as in (4.21). Then by applying $\Gamma$ to $\hat{v}_0 + \gamma(\lambda)\psi$ we conclude that necessarily (cf. (4.23))

$$\psi = \varphi, \quad \hat{v}_0 - \hat{g}_0 \in A.$$ 

Since $A \subset A_{\Theta}$, this implies that for all \{ $\gamma(\lambda)^*g, \varphi$ $\} \in (\Theta - M(\lambda))^{-1}$ one has $\hat{g}_0 + \gamma(\lambda)\varphi \in A_{\Theta}$ or, equivalently, that

\begin{equation}
\{ g, (A_0 - \lambda)^{-1}g + \gamma(\lambda)\varphi \} \in (A_{\Theta} - \lambda)^{-1}.
\end{equation}

In view of (4.24) this proves the reverse inclusion “\( \supset \)” in (4.20). \( \square \)

Remark 4.12. We emphasize that in the Krein-type formula (4.20) it is not assumed that $\lambda \in \rho(A_{\Theta})$. In particular, $A_{\Theta} - \lambda$ need not be invertible; $A_{\Theta}$ and $\Theta$ need not even be closed.

The following statement is an immediate consequence of Theorem 4.11.

Corollary 4.13. Let the assumptions be as in Theorem 4.11 and let $\lambda \in \rho(A_0)$. Then:

(i) $\ker (A_{\Theta} - \lambda) = \gamma(\lambda)\ker (\Theta - M(\lambda))$;

(ii) if $(\Theta - M(\lambda))^{-1}$ is a bounded operator, then the same is true for $(A_{\Theta} - \lambda)^{-1}$;

(iii) if $0 \in \rho(\Theta - M(\lambda))$ then $\lambda \in \rho(A_{\Theta})$. 

5. Some classes of unitary boundary triples and their Weyl functions

5.1. Unitary boundary pairs and unitary colligations. Some formulas from Section 3 can be essentially improved when using the interrelations between unitary relations and unitary colligations, see [16]. Let \( \{ \mathcal{H}, \Gamma \} \) be a unitary boundary pair corresponding to a single-valued unitary relation \( \Gamma : (\mathcal{S}_\gamma^2, J_\gamma) \to (\mathcal{H}^2, J_\mathcal{H}) \). The Kreĭn spaces \((\mathcal{S}_\gamma^2, J_\gamma)\) and \((\mathcal{H}^2, J_\mathcal{H})\) admit fundamental decompositions

\[
\mathcal{S}_\gamma^2 = P_+ S^2[-]P_- S^2, \quad \mathcal{H}^2 = P_+ H^2[-]P_- H^2
\]

with respect to the fundamental symmetries \( J_\gamma \) and \( J_\mathcal{H} \), where \( P_+ \) and \( P_- \) are the orthogonal projections

\[
P_+ = \frac{1}{2} \begin{pmatrix} I & -iI \\ iI & I \end{pmatrix}, \quad P_- = \frac{1}{2} \begin{pmatrix} I & iI \\ -iI & I \end{pmatrix}
\]

each acting on its own space. The Potapov-Ginzburg transform \( \omega'(\Gamma) \) of \( \Gamma \) (see [14]) is unitarily equivalent to the transform \( \omega(\Gamma) \), which is the graph of a unitary operator \( U : \left( \mathcal{S}_\gamma, \mathcal{H} \right) \to \left( \mathcal{S}_\gamma, \mathcal{H} \right) \). This operator will be also called the Potapov-Ginzburg transform of \( \Gamma \). The transform \( \omega : \Gamma \mapsto U \) establishes a one-to-one correspondence between the set of unitary boundary pairs and the set of unitary colligations. The inverse transform \( \Gamma = \omega^{-1}(U) \) takes the form

\[
\Gamma = \left\{ \begin{pmatrix} (T-I)g + Fu \\ i(T+I)g + iFu \end{pmatrix} : \begin{pmatrix} Gg + (H-I)u \\ -iGg - i(H+I)u \end{pmatrix} \right\}, \quad g \in \mathcal{S}_\gamma, u \in \mathcal{H}
\]

Let us consider the unitary operator \( U \) and the pair of Hilbert spaces \( \mathcal{S}_\gamma \) and \( \mathcal{H} \) as a unitary colligation written in the block form (see [25])

\[
U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathcal{B}(\mathcal{S}_\gamma \oplus \mathcal{H}), \quad \text{where} \quad T \in \mathcal{B}(\mathcal{S}_\gamma), \quad H \in \mathcal{B}(\mathcal{H}).
\]

Then the representation (5.2) for \( \Gamma \) takes the form

\[
\Gamma = \left\{ \begin{pmatrix} T^*g' + G^*u' \\ F^*g' + H^*u' \end{pmatrix} : \begin{pmatrix} g' \in \mathcal{S}_\gamma, u' \in \mathcal{H} \end{pmatrix} \right\}
\]

Since \( U = (U^*)^{-1} \), then

\[
U = \left\{ \begin{pmatrix} T^*g' + G^*u' \\ F^*g' + H^*u' \end{pmatrix} : \begin{pmatrix} g' \in \mathcal{S}_\gamma, u' \in \mathcal{H} \end{pmatrix} \right\}
\]

and hence \( \Gamma \) admits a dual representation

\[
\Gamma = \left\{ \begin{pmatrix} (I - T^*)g' - G^*u' \\ i(I + T^*)g' + iG^*u' \end{pmatrix} : \begin{pmatrix} -F^*g' + (I - H^*)u' \\ -iF^*g' - i(I + H^*)u' \end{pmatrix} \right\}, \quad g' \in \mathcal{S}_\gamma, u' \in \mathcal{H}
\]

Let us collect some formulas concerning \( \Gamma \) and \( U \) which are immediate from (5.4) and (5.6) (see also [16]).
Proposition 5.1. Let \( \{\mathcal{H}, \Gamma\} \) be a unitary boundary pair for \( A^* \) with \( \Gamma \) given by (5.4), and let \( A_* = \text{dom} \Gamma, \ A_0 = \ker \Gamma_0 \). Then:

\[
A_* = \text{ran} \left( \begin{pmatrix} T - I & F \\ i(T + I) & iF \end{pmatrix} \right) = \text{ran} \left( \begin{pmatrix} (I - T)^* & -G^* \\ i(I + T)^* & iG^* \end{pmatrix} \right);
\]

\[
\mu \ A_* = (I - T)^{-1}\text{ran} \ F = (I - T^*)^{-1}\text{ran} \ G^*;
\]

\[
\mu \ A = \ker (I - T) = \ker (I - T^*);
\]

\[
A_0 = \left\{ \begin{pmatrix} (T - I)g + Fu \\ i(T + I)g + iFu \end{pmatrix} : \ Gg + (H - I)u = 0, \ g \in \mathcal{F}, \ u \in \mathcal{H} \right\}
\]

\[
= \left\{ \begin{pmatrix} (I - T^*)g' - G^*u' \\ i(I + T^*)g' + iG^*u' \end{pmatrix} : \ F^*g' + (H^* - I)u' = 0, \ g' \in \mathcal{F}, \ u' \in \mathcal{H} \right\}
\]

\[
\text{ran} \Gamma_0 = \text{ran} (I - H) + \text{ran} \ G = \text{ran} (I - H^*) + \text{ran} \ F^*;
\]

\[
\mu \Gamma = \left\{ \begin{pmatrix} (H - I)u \\ -i(H + I)u \end{pmatrix} : \ u \in \ker F \right\} = \left\{ \begin{pmatrix} (I - H^*)u' \\ -i(I + H^*)u' \end{pmatrix} : \ u' \in \ker G^* \right\},
\]

in particular,

\[
\mu \Gamma = \{0\} \iff \ker F = \{0\} \iff \ker G^* = \{0\}.
\]

The characteristic function (or transfer function) of the unitary colligation \( \mathcal{U} \) (see [25])

\[
\theta(\zeta) = H + \zeta G(I - \zeta T)^{-1}F \quad (\zeta \in \mathbb{D})
\]

is holomorphic in \( \mathbb{D} \) and takes values in the set of contractive operators in \( \mathcal{H} \).

Proposition 5.2. Let \( \{\mathcal{H}, \Gamma\} \) be a unitary boundary pair for \( A^* \) with \( \Gamma \) given by (5.4), let \( \lambda \in \mathbb{C}^+ \) and let \( \zeta = \frac{\lambda \pm \sqrt{\lambda^2 - 1}}{\lambda^2 - 1} \). Then:

\[
\gamma(\lambda) = \{\{\theta(\zeta) - I)u, (1 - \zeta)(I - \zeta T^{-1}Fu) : u \in \mathcal{H} \},
\]

\[
\gamma(\bar{\lambda}) = \{\{(\theta(\zeta^*) - I)u, (1 - \bar{\zeta})(I - \bar{\zeta}T^*-1)G^*u : u \in \mathcal{H} \}.
\]

In particular,

\[
\gamma(i) = \{(H - I)u, Fu) : u \in \mathcal{H} \}, \quad \gamma(-i) = \{(H^* - I)u, G^*u) : u \in \mathcal{H} \}.
\]

The Weyl function \( M \) corresponding to the boundary pair \( \{\mathcal{H}, \Gamma\} \) and the characteristic function \( \theta \) are connected by

\[
M(\lambda) = i(I + \theta(\zeta))(I - \theta(\zeta))^{-1}, \quad M(\bar{\lambda}) = -i(I + \theta(\zeta^*))(I - \theta(\zeta^*))^{-1}
\]

If \( \Gamma \) is single-valued then \( \text{dom} \gamma(\lambda) \) and \( \text{dom} \gamma(\bar{\lambda}) \) are dense in \( \mathcal{H} \).

Proof. Since \( \zeta = \frac{\lambda \pm \sqrt{\lambda^2 - 1}}{\lambda^2 - 1} \in \mathbb{D} \) the operator \( (I - \zeta T) \) has a bounded inverse. Using substitution \( g = f + \zeta(I - \zeta T)^{-1}Fu \) one can rewrite (5.3) in the form

\[
\Gamma = \left\{ \begin{pmatrix} (T - I)f + (1 - \zeta)(I - \zeta T)^{-1}Fu \\ i(T + I)f + i(1 + \zeta)(I - \zeta T)^{-1}Fu \end{pmatrix} : \ Gf + (\theta(\zeta) - I)u \\ -iGf - i(\theta(\zeta) + I)u \right\} : \ f \in \mathcal{F} \}
\]

Since \( \lambda = \frac{i + \zeta}{1 - \zeta} \) then setting in (5.17) \( f = 0 \) one obtains

\[
\Gamma \mid \mathcal{F}_\lambda = \left\{ \begin{pmatrix} (1 - \zeta)(I - \zeta T)^{-1}Fu \\ \lambda(1 - \zeta)(I - \zeta T)^{-1}Fu \end{pmatrix} : \ (\theta(\zeta) - I)u \\ -i(\theta(\zeta) + I)u \right\} : \ u \in \mathcal{H} \}
\]

and hence (5.13) and the first equalities in (5.13) and (5.16) follow.
Similarly, substitution $g' = f' + \zeta (I - \zeta T^*)^{-1} G^* u'$ in (5.14) shows that the linear relation $\Gamma$ coincides with the set of vectors
\[
(5.19) \quad \left\{ \left( (I - T^*) f' + (\zeta - 1)(I - \zeta T^*)^{-1} G^* u', i (I + T^*) f' + i(\zeta + 1)(I - \zeta T^*)^{-1} G^* u' \right) \right\} \cap \left\{ \left( -F^* f' + (I - \theta(\zeta)^*) u', -i F^* f' + i(I + \theta(\zeta)^*) u' \right) \right\},
\]
where $f' \in \mathfrak{H}$, $u' \in \mathcal{H}$. Hence with $f' = 0$ one obtains from (5.19)
\[
(5.20) \quad \Gamma \upharpoonright \hat{\mathfrak{N}}_\lambda = \left\{ \left( \frac{(\zeta - 1)(I - \zeta T^*)^{-1} G^* u'}{\lambda(1 - \zeta)(I - \zeta T^*)^{-1} G^* u'}, \frac{(I - \theta(\zeta)^*) u'}{i(I + \theta(\zeta)^*) u'} \right) : u' \in \mathcal{H} \right\}.
\]
Now the formula (5.14) and the second equalities in (5.15) and (5.16) are implied by (5.20).

If $\text{mul} \Gamma = \{0\}$ then using the fact that $\gamma(\pm i)$ is single-valued, i.e., $\ker (H - I) \subset \ker F$ and $\ker (H^* - I) \subset \ker G^*$, it follows from Proposition 5.1 that $\ker (I - H) = \ker (I - H^*) = \{0\}$ and hence $\text{dom} \gamma(-i) = \text{ran} (I - H^*)$ and $\text{dom} \gamma(i) = \text{ran} (I - H)$ are dense in $\mathcal{H}$. Equivalently, $\text{dom} \gamma(\lambda)$ is dense in $\mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. $\square$

**Proposition 5.3.** Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triple for $A^*$. Then the closure of the $\gamma$-field is given by
\[
(5.21) \quad \overline{\gamma(\lambda)} = (\Gamma_0 \upharpoonright \hat{\mathfrak{N}}_\lambda(A^*))^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
In particular,
\[
\text{ker} \overline{\gamma(\lambda)} = \text{mul} \Gamma_0, \quad \text{mul} \overline{\gamma(\lambda)} = (\ker \Gamma_0) \cap \hat{\mathfrak{N}}_\lambda(A^*),
\]
and
\[
\text{ran} \overline{\gamma(\lambda)} = (\text{dom} \Gamma_0) \cap \hat{\mathfrak{N}}_\lambda(A^*), \quad \text{dom} \overline{\gamma(\lambda)} = \Gamma_0((\text{dom} \Gamma_0) \cap \hat{\mathfrak{N}}_\lambda(A^*)).
\]

**Proof.** By definition $\overline{\gamma(\lambda)} = (\Gamma_0 \upharpoonright \hat{\mathfrak{N}}_\lambda(A^*))^{-1} = (\Gamma_0 \cap (\hat{\mathfrak{N}}_\lambda(A^*) \times \mathcal{H}))^{-1}$, which implies that
\[
\overline{\gamma(\lambda)^{-1}} \subset \overline{\Gamma_0 \upharpoonright \hat{\mathfrak{N}}_\lambda(A^*)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
To prove the reverse inclusion, assume that $\{\tilde{f}_\lambda, h\} \in \overline{T_0 \cap (\hat{\mathfrak{N}}_\lambda(A^*) \times \mathcal{H})}$. With $\lambda \in \mathbb{C}_+$ it follows from (5.17) that there are sequences $\{f_n \in \mathfrak{S} \}$ and $\{u_n \in \mathcal{H} \}$, such that
\[
(5.22) \quad \left\{ \left( (T - I) f_n + (1 - \zeta)(1 - \zeta T)^{-1} F u_n \right), G f_n + (\theta(\zeta) - I) u_n \right\} \rightarrow \left\{ \left( \frac{f_\lambda}{\lambda f_\lambda}, h \right) \right\}.
\]
This implies that $(I - \zeta T) f_n \rightarrow 0$ and hence $f_n \rightarrow 0$, since $\lambda \in \mathbb{C}_+$ or, equivalently, $\zeta \in \mathbb{D}$. Thus
\[
\left\{ \left( \frac{(1 - \zeta)(1 - \zeta T)^{-1} F u_n}{\lambda(1 - \zeta)(1 - \zeta T)^{-1} F u_n}, (\theta(\zeta) - I) u_n \right) \right\} \rightarrow \left\{ \left( \frac{f_\lambda}{\lambda f_\lambda}, h \right) \right\},
\]
which by (5.13) in Proposition 5.2 means that $\{\tilde{f}_\lambda, h\} \in \overline{\gamma(\lambda)^{-1}}$.

Similarly, with $\lambda \in \mathbb{C}_-$ it follows from (5.19) that for every $\{\tilde{f}_\lambda, h\} \in \overline{T_0 \cap (\hat{\mathfrak{N}}_\lambda(A^*) \times \mathcal{H})}$ there exists a sequence $\{u'_n \in \mathcal{H} \}$ such that
\[
\left\{ \left( \frac{(\zeta - 1)(1 - \zeta T)^{-1} G^* u'_n}{\lambda(\zeta - 1)(1 - \zeta T)^{-1} G^* u'_n}, (I - \theta(\zeta)^*) u'_n \right) \right\} \rightarrow \left\{ \left( \frac{f_\lambda}{\lambda f_\lambda}, h \right) \right\},
\]
which by (5.13) in Proposition 5.2 means that $\{\tilde{f}_\lambda, h\} \in \overline{\gamma(\lambda)^{-1}}$. This completes the proof of (5.21) and the remaining statements follow easily from this identity. $\square$

**Corollary 5.4.** Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triple for $A^*$ and let $M(\cdot)$ be the corresponding Weyl function. Then the mapping $\Gamma_0$ is closable if and only if for some, equivalently for every, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the Weyl function satisfies the following condition:
\[
(5.23) \quad h_n \rightarrow h \quad \text{in} \quad \mathcal{H} \quad \text{and} \quad \text{Im} (M(\lambda) h_n, h_n) \rightarrow 0 \quad (n \rightarrow \infty) \quad \implies \quad h = 0.
\]
\textbf{Proof.} By Lemma 3.6, \( M(\cdot) \) is an operator valued function with \( \ker (M(\lambda) - M(\lambda)^*) = \{0\} \). In this case (3.13) implies that
\[
(\lambda - \bar{\lambda})\|\gamma(\lambda)h\|_2^2 = 2i \Im (M(\lambda)h, h)_{\mathcal{H}},
\]
h \in \text{dom} M(\lambda), \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}. From this formula it is clear that the condition (5.23) is equivalent to \( \ker \gamma(\lambda) = \{0\} \). Therefore, the result follows from Proposition 5.3. \( \square \)

Clearly, the condition (5.23) is stronger than the condition (1.11) appearing in the definition of strict Nevanlinna functions. If \( M(\cdot) \in \mathcal{R}[\mathcal{H}] \) then the condition (5.23) simplifies to \( \ker \Im M(\lambda) = \{0\} \), i.e., for bounded Nevanlinna functions the conditions (5.23) and (1.11) are equivalent. Hence, if \( \mathcal{H}, \Gamma_0, \Gamma_1 \) is a \( B \)-generalized boundary triple then \( \Gamma_0 \) is closable. However, when \( \mathcal{H}(\cdot) \) is an unbounded Nevanlinna function, the condition in Corollary 5.4 need not be satisfied. Example 5.20 shows that already for \( S \)-generalized boundary triples \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) the mapping \( \Gamma_0 \) need not be closable.

**Proposition 5.5.** Let \( \mathcal{H}, \Gamma_0, \Gamma_1 \) be a unitary boundary triple, let \( \mathcal{U} = \omega(\Gamma) \) be its Potapov-Ginzburg transform, let \( \lambda \in \mathbb{C}_+ \), \( \zeta = \frac{1+\lambda}{1-\lambda} \), and let \( \mathcal{H}(\lambda) \) be defined by (3.14). Then:
\[
\begin{align*}
(5.24) & \quad \Gamma_1 \mathcal{H}(\lambda) = \{ (g, v) : (\theta(\zeta) - I)v + (\zeta - 1)G(I - \zeta T)^{-1}g = 0, \ g \in \mathcal{H}, \ v \in \mathcal{U} \}, \\
(5.25) & \quad \Gamma_1 \mathcal{H}(\lambda) = \{ (g', v') : (g(\zeta)^* - I)v' + (\zeta - 1)F^*(I - \bar{\zeta} T^*)^{-1}g' = 0, \ g' \in \mathcal{H}, \ v' \in \mathcal{U} \}.
\end{align*}
\]
In particular, the linear operators \( \Gamma_1 \mathcal{H}(\lambda) \) have constant ranges for all \( \lambda \in \mathbb{C}_+ \):
\[
(5.26) \quad \text{ran} (\Gamma_1 \mathcal{H}(\lambda)) = (H - I)^{-1} \text{ran} G = (H^* - I)^{-1} \text{ran} F^* = \text{ran} (\Gamma_1 \mathcal{H}(\lambda)).
\]

\textbf{Proof.} It follows from (5.17) and (5.19) that
\[
(5.27) \quad A_0 - \lambda = \left\{ \begin{array}{l}
\left( (T - I)\frac{1}{2i}f + (1 - \zeta)(I - \zeta T)^{-1}Ff \right) : f \in \mathcal{H}, \ u \in \mathcal{U} \\
Gf + (\theta(\zeta) - I)u = 0
\end{array} \right\}.
\]
\[
(5.28) \quad A_0 - \bar{\lambda} = \left\{ \begin{array}{l}
\left( (I - T^*)f' + \frac{2i}{1-\zeta}(I - \bar{\zeta} T^*)^{-1}G^*u' \right) : f' \in \mathcal{H}, \ u' \in \mathcal{U} \\
-\bar{F}^*g' + (I - \theta(\zeta)^*)u'
\end{array} \right\}.
\]
In particular, using (5.27) and the equality \( g = 2i(1 - \zeta T)/(1 - \zeta)^{-1} \) one obtains
\[
h := (A_0 - \lambda)^{-1}g = \frac{1 - \zeta}{2i}(T - I)(I - \zeta T)^{-1}g + (1 - \zeta)(I - \zeta T)^{-1}Ff,
\]
where \( u \in \mathcal{H} \) satisfies the equality
\[
\frac{1 - \zeta}{2i}G(I - \zeta T)^{-1}g + (\theta(\zeta) - I)u = 0,
\]
or, equivalently,
\[
(\theta(\zeta) - I)(-2iu) + (\zeta - 1)G(I - \zeta T)^{-1}g = 0.
\]

On the other hand, if \( \left( \begin{array}{c} h \\ h' \end{array} \right) = \mathcal{H}(\lambda)g \) then by (5.17)
\[
(5.29) \quad \left\{ \left( \begin{array}{c} h \\ h' \end{array} \right), -2iu \right\} = \left\{ \left( \begin{array}{c} h \\ h' \end{array} \right), -iGf - i(\theta(\zeta) + I)u \right\} \in \Gamma_1
\]
and hence \( \{ g, -2iu \} \in \Gamma_1 \mathcal{H}(\lambda) \). This proves (5.24).

Similarly, the formula (5.25) is based on (5.6) and (5.28).

Let \( u \in (H - I)^{-1}(\text{ran} G \cap \text{ran} (H - I)) \) and \( \lambda \in \mathbb{C}_+ \). Then \( (H - I)u \in \text{ran} G \) and hence
\[
(5.30) \quad (\theta(\zeta) - I)u = (H - I)u + \zeta G(I - \zeta T)^{-1}Ff \in \text{ran} G.
\]
In view of (5.24) this proves that \( u \in \text{ran} (\Gamma_1 \mathcal{H}(\lambda)) \).
Conversely, if \( u \in \text{ran} (\Gamma_1 H(\lambda)) \) then in view of (5.24) and (5.30) \((H - I)u \in \text{ran} G\). Similarly, for \( \lambda \in \mathbb{C}_- \) the equality (5.26) is implied by (5.25). \(\square\)

Notice that for a single-valued \( \Gamma \) one has \( \ker (I - H) = \{0\} \), since otherwise \( \ker F \) and \( \ker G^* \) are nontrivial, which by (5.11) contradicts the assumption that \( \text{mul} \Gamma = \{0\} \).

**Proposition 5.6.** Let \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) be a unitary boundary triple, let \( \mathcal{U} = \omega(\Gamma) \) be its Potapov-Ginzburg transform given by (5.1) and (5.3), and let \( H(\lambda) \) be defined by (3.13). Then:

1. \( \Gamma_1 H(\lambda) = \gamma(\lambda)^* \) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \);
2. The range of linear relation \( \gamma(\lambda)^* \) does not depend on \( \lambda \in \mathbb{C}_+ \), \( \bar{\lambda} \in \mathbb{C}_- \):

\[
(5.31) \quad \text{ran} \, \gamma(\lambda)^* = (I - H)^{-1}(\text{ran} G), \quad \text{ran} \, \gamma(\lambda)^* = (I - H^*)^{-1}(\text{ran} F^*).
\]

**Proof.** (i) By Proposition 5.2 \( \text{dom} \, \gamma(\bar{\lambda}) \) is dense in \( \mathcal{H} \) and hence \( \gamma(\lambda)^* \) is a single-valued operator from \( \mathfrak{S} \) to \( \mathcal{H} \) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). By Lemma 3.7 (cf. [39 Lemma 7.38]) one has \( \Gamma_1 H(\lambda) \subseteq \gamma(\lambda)^* \). Now let \( \lambda \in \mathbb{C}_+ \) and assume that \( v = \gamma(\lambda)^* g \) for some \( g \in \mathfrak{S} \). Then by (5.11)

\[
(1 - \zeta)(I - \zeta T)^{-1}G^*u, g) = ((\theta(\zeta)^* - I)u, v)_{\mathcal{H}}
\]

for all \( u \in \mathcal{H} \), and hence

\[
(\theta(\zeta) - I)v + (\zeta - 1)G(I - \zeta T)^{-1}g = 0.
\]

In view of (5.24) one obtains \( \{g, v\} \in \Gamma_1 H(\lambda) \), so that \( \gamma(\lambda)^* \subseteq \Gamma_1 H(\lambda) \). This proves (i) for \( \lambda \in \mathbb{C}_+ \). Similarly, the assumption \( \{g', v'\} \in \gamma(\lambda)^* \) \( (\lambda \in \mathbb{C}_+) \) yields in view of (5.11)

\[
(\theta(\zeta)^* - I)v + (\overline{\zeta} - 1)F^*(I - \overline{\zeta} T^*)^{-1}g' = 0.
\]

By (5.25) this proves that \( \{g', v'\} \in \Gamma_1 H(\bar{\lambda}) \) and hence (i) is in force also for \( \lambda \in \mathbb{C}_- \).

(ii) The formulas (5.24) and (5.25) for \( \lambda = i \) take the form

\[
(5.32) \quad \Gamma_1 H(i) = \{\{g, v\} : (H - I)v + Gg = 0, \ g \in \mathfrak{S}, \ v \in \mathcal{H}\},
\]

\[
(5.33) \quad \Gamma_1 H(-i) = \{\{g', v'\} : (H^* - I)v' + F^*g' = 0, \ g' \in \mathfrak{S}, \ v' \in \mathcal{H}\}.
\]

Then (5.31) is immediate by item (i) using the formulas (5.32), (5.33) and Proposition 5.5. \(\square\)

**Corollary 5.7.** ([39]) Let the assumptions of Proposition 5.6 be in force. Then the operator \( \Gamma_1 H(\lambda) \) is bounded or, equivalently, \( \Gamma_1 \upharpoonright A_0 \) is bounded if and only if \( A_0 \) is closed.

**Proof.** By Lemma 5.7 the operator \( \Gamma_1 H(\lambda) \) is bounded if and only if the restriction \( \Gamma_1 \upharpoonright A_0 \) is bounded. Since \( \Gamma_1 H(\lambda) = \gamma(\lambda)^* \) by Proposition 5.6 this mapping is closed and it follows from the closed graph theorem that \( \Gamma_1 \upharpoonright A_0 \) is bounded if and only if its domain \( \text{dom} (\Gamma_1 \upharpoonright A_0) = A_0 \) is closed. \(\square\)

**Remark 5.8.** If \( \{\mathcal{H}, \Gamma\} \) is a unitary boundary pair, then it also admits the representations (5.4) and (5.6) in terms of its Potapov-Ginzburg transform \( \mathcal{U} = \omega(\Gamma) \). Then, for instance,

\[
(5.34) \quad \text{mul} \, (\Gamma_1 H(\lambda)) = (I + H)\ker F = (I + H^*)\ker G^*,
\]

and the statements (i) and (ii) in Proposition 5.6 take the form:

\[
(5.35) \quad \Gamma_1 H(\lambda) = \gamma(\lambda)^* \mathop{\upharpoonright}\{0\} \times \text{mul} \, \Gamma_1 \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\]

\[
(5.36) \quad \text{ran} \, \gamma(\lambda)^* = (I - H)^{-1}(\text{ran} G \cap \text{ran} (I - H)),
\]

and e.g. the adjoint of the \( \gamma \)-field for \( \lambda \in \mathbb{C}_+ \) admits the formula

\[
\gamma(\lambda)^* = \{(f, f') : (I - H)f' = Gf\}.
\]
5.2. A Krein type formula for unitary boundary triples. In this section Krein’s resolvent formula is extended to the setting of general unitary boundary triples. It is analogous to the formula established in Section 4.2. Recall from [87] that for a unitary boundary triple the kernel \( A_0 = \ker \Gamma_0 \) need not be selfadjoint, it is in general only a symmetric extension of \( A \) which can even coincide with \( A \); see e.g. the transposed boundary triple treated in Example 6.7 below. For simplicity the next result is formulated for nonreal points \( \lambda \in \mathbb{C} \setminus \mathbb{R} \); these points are regular type points for \( A_0 \).

As in Section 4.2 let \( \hat{A} \) be an extension of \( A \) which belongs to the domain of \( \Gamma \) and let \( \Theta \) be a linear relation in \( \mathcal{H} \) corresponding to \( \hat{A} \):

\[
\Theta = \Gamma(\hat{A}), \quad \hat{A} \subset \text{dom } \Gamma \iff \hat{A} = A_\Theta := \Gamma^{-1}(\Theta), \quad \Theta \subset \text{ran } \Gamma.
\]

**Theorem 5.9.** Let \( A \) be a closed symmetric relation, let \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be a unitary boundary triple for \( A^* \) with \( A_0 = \ker \Gamma_0 \), and let \( M(\cdot) \) and \( \gamma(\cdot) \) be the corresponding Weyl function and \( \gamma \)-field, respectively. Then for any extension \( A_\Theta \in \text{Ext}_A \) satisfying \( A_\Theta \subset \text{dom } \Gamma \) the following equality holds for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \),

\[
(A_\Theta - \lambda)^{-1} - (A_0 - \lambda)^{-1} = \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*,
\]

where the inverses in the first and last term are taken in the sense of linear relations.

**Proof.** We first prove the inclusion “\( \subset \)” in (5.38). Let \( A_\Theta \subset \text{dom } \Gamma \) be an extension of \( A \) with \( \Theta \) as in (5.37). Since \( A_\Theta \) is symmetric, \( (A_\Theta - \lambda)^{-1} \) is a bounded, in general non-densely defined, operator for every fixed \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). Now assume that \( \{ g, g' \} \in (A_\Theta - \lambda)^{-1} - (A_0 - \lambda)^{-1} \). Then \( g \in \text{dom } (A_\Theta - \lambda)^{-1} \cap \text{dom } (A_0 - \lambda)^{-1} \) and \( \{ g, g' \} \in (A_\Theta - \lambda)^{-1} \) for some \( g' \in \mathcal{H} \), so that \( g'' = g' - (A_0 - \lambda)^{-1}g \). Hence \( \tilde{g}_\Theta : = \{g', g + \lambda g'\} \in A_\Theta \subset \text{dom } \Gamma \),

\[
\tilde{g}_0 := ((A_0 - \lambda)^{-1}g, (I + \lambda(A_0 - \lambda)^{-1})g) \in A_0 \subset \text{dom } \Gamma,
\]

and

\[
\tilde{g}_\Theta - \tilde{g}_0 = \{g' - (A_0 - \lambda)^{-1}g, \lambda(g' - (A_0 - \lambda)^{-1}g)\},
\]

so that \( \tilde{g}_\Theta - \tilde{g}_0 \in \hat{\mathcal{H}}(A_\Theta) \). Recall that \( \hat{\gamma}(\lambda) \) maps \( \text{dom } \hat{\gamma}(\lambda) \) onto \( \hat{\mathcal{H}}(A_\Theta) \subset \text{dom } \Gamma \) and hence there exists \( \varphi \in \text{dom } \hat{\gamma}(\lambda) = \text{dom } \gamma(\lambda) \) such that

\[
\tilde{g}_\Theta - \tilde{g}_0 = \hat{\gamma}(\lambda)\varphi, \quad \Gamma\hat{\gamma}(\lambda)\varphi = \{ \varphi, M(\lambda)\varphi \},
\]

see (4.4), (4.5); notice that \( M(\lambda) \) is an operator, since \( \text{mul } \Gamma = \{0\} \). Clearly \( \Gamma_0\tilde{g}_0 = 0 \) and according to Proposition 5.6 one has \( \Gamma_1\tilde{g}_0 = \Gamma_1 H(\lambda)g = \gamma(\lambda)^*g \), where \( H(\lambda) \) is defined by (5.14). Observe, that here \( \gamma(\lambda)^* \) is an operator since \( H(\lambda) \) and \( \Gamma_1 \) are operators. Now it follows from (5.40) that

\[
\{0, \gamma(\lambda)^*g\} + \{\varphi, M(\lambda)\varphi\} = \Gamma\tilde{g}_0 + \Gamma\hat{\gamma}(\lambda)\varphi = \Gamma\tilde{g}_0 \in \Theta,
\]

see (5.37). Consequently, \( \{ \varphi, \gamma(\lambda)^*g + M(\lambda)\varphi \} \in \Theta \) and \( \{ \varphi, \gamma(\lambda)^*g \} \in \Theta - M(\lambda) \) or, equivalently, \( \{ g, \varphi \} \in (\Theta - M(\lambda))^{-1}\gamma(\lambda)^* \) and hence (5.40) shows that

\[
\{ g, g'' \} = \{ g, \gamma(\lambda)\varphi \} \in \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*,
\]

which proves the first inclusion in (5.38).

To prove the reverse inclusion “\( \supset \)” in (5.38) assume that \( \{ g, g'' \} \in \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^* \). Since \( \text{dom } (\Theta - M(\lambda)) \subset \text{dom } \gamma(\lambda) \) the assumption on \( \{ g, g'' \} \) means that for some \( \varphi \in \mathcal{H} \) one has \( \{ \gamma(\lambda)^*g, \varphi \} \in (\Theta - M(\lambda))^{-1} \) and

\[
\{ g, g'' \} = \{ g, \gamma(\lambda)\varphi \} \in \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*.
\]

It follows from \( \{ \varphi, \gamma(\lambda)^*g + M(\lambda)\varphi \} \in \Theta \) and (5.37) that \( \Gamma\tilde{g}_\Theta = \{ \varphi, \gamma(\lambda)^*g + M(\lambda)\varphi \} \) for some \( \tilde{g}_\Theta \in A_\Theta \). By Proposition 5.9 \( \Gamma_1 H(\lambda) = \gamma(\lambda)^* \), which shows that \( g \in \text{ran } (A_0 - \lambda) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \);
see (3.14). Now associate with $g$ the element $\hat{g}_0$ as in (5.39). Since $\Gamma\hat{g}_0 = \{0, \gamma(\lambda)^* g\}$ and $\Gamma\hat{g}(\lambda)\varphi = \{\varphi, M(\lambda)\varphi\}$ we conclude that (5.41) is satisfied. Therefore, $\hat{g}_0 + \hat{g}(\lambda)g - \hat{g}_0 \in \ker \Gamma = A$ and thus $\hat{g}_0 + \hat{g}(\lambda)\varphi \in A_\Theta$. Indeed, in that case only $0 \in A$ if the following inclusion remains valid:

$$\{g, (A_0 - \lambda)^{-1}g + \gamma(\lambda)\varphi\} \subset (A_0 - \lambda)^{-1} - (A_0 - \lambda)^{-1}.$$

Hence,

$$\{g, g''\} = \{g, \gamma(\lambda)\varphi\} \subset (A_0 - \lambda)^{-1} - (A_0 - \lambda)^{-1}.$$

This proves the reverse inclusion in (5.38) and completes the proof. \hfill $\Box$

Remark 5.10. Again notice the generality of the formulas in (5.38), in particular, that $\lambda$ need not belong to $\rho(A_\Theta)$. Observe also that in the formula (5.38) the operator $(A_0 - \lambda)^{-1}$ cannot be shifted to the righthand side without loosing the stated equality. Indeed, in that case only the following inclusion remains valid:

$$(A_\Theta - \lambda)^{-1} \supset (A_0 - \lambda)^{-1} - \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*.$$

By considering the multivalued parts we obtain the following statement for the point spectrum of $A_\Theta$ from Theorem 5.9.

Corollary 5.11. With the assumptions in Theorem 5.9 one has $\lambda \in \sigma_p(A_\Theta)$ if and only if $0 \in \sigma_p(\Theta - M(\lambda))$, in which case

$$\ker (A_\Theta - \lambda) = \gamma(\lambda)\ker (\Theta - M(\lambda)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

5.3. $S$-generalized boundary triples. Here we extend Definition 5.10 to the case of boundary pairs.

Definition 5.12. A unitary boundary pair $\{\mathcal{H}, \Gamma\}$ is said to be an $S$-generalized boundary pair, if $A_0$ is a selfadjoint linear relation in $\mathcal{H}$.

In the following proposition some special boundary triples/pairs are characterized in terms of its Potapov-Ginzburg transform.

Proposition 5.13. Let $\{\mathcal{H}, \Gamma\}$ be a unitary boundary pair, let $\mathcal{U} = \omega(\Gamma)$ be its Potapov-Ginzburg transform given by (5.11) and (5.3), and let $A_\ast = \text{dom} \Gamma$, $A_0 = \ker \Gamma_0$. Then:

(i) $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple if and only if

$$\text{ran } G = \mathcal{H} \iff \text{ran } F^\ast = \mathcal{H};$$

(ii) $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a $B$-generalized boundary triple if and only if

$$\left\{ \begin{array}{c} \ker F = \{0\} \\ \text{ran } (I - H) = \mathcal{H} \end{array} \right\} \iff \left\{ \begin{array}{c} \ker G^\ast = \{0\} \\ \text{ran } (I - H^\ast) = \mathcal{H} \end{array} \right\};$$

(iii) $\{\mathcal{H}, \Gamma\}$ is a $B$-generalized boundary pair if and only if

$$\Gamma_0 \hat{\mathcal{N}}_0 = \mathcal{H} \iff \text{ran } (I - H) = \mathcal{H} \iff \text{ran } (I - H^\ast) = \mathcal{H};$$

(iv) $\Gamma_0$ is surjective if and only if

$$\text{ran } (I - H) + \text{ran } G = \mathcal{H} \iff \text{ran } (I - H^\ast) + \text{ran } F^\ast = \mathcal{H};$$

(v) $\{\mathcal{H}, \Gamma\}$ is a $S$-generalized boundary pair if and only if

$$\text{ran } G \subset \text{ran } (I - H) \quad \text{and} \quad \text{ran } F^\ast \subset \text{ran } (I - H^\ast).$$
Proof. The statements (i)–(iii) can be found in [16, Proposition 5.9, Corollaries 5.11 and 5.12].
(iv) This is implied by (5.11).
(v) This statement follows from the equalities
\[
A_0 - i = \left\{ \left( \frac{T - I}{2i} \right) g + Fu \right\} : Gg + (H - I)u = 0, \ g \in \mathcal{H}, \ u \in \mathcal{H} \right\}
\]
(5.42)
\[
A_0 + i = \left\{ \left( \frac{I - T^*}{2i} \right) g' - G^*u' \right\} : F^*g' + (H^* - I)u' = 0, \ g' \in \mathcal{H}, \ u' \in \mathcal{H} \right\}
\]
which, in turn, are implied by (5.10). □

Remark 5.14. Observe also that \(A_0\) is a maximal symmetric operator if at least one of the conditions \(\text{ran} \, G \subseteq \text{ran} \, (I - H)\) or \(\text{ran} \, F^* \subseteq \text{ran} \, (I - H^*)\) is satisfied. The statement (v) in Proposition 5.13 is contained in [31, Prop]; see also [60, Prop], where it is formulated in terms of angular representation of \(A_0\). An example of a unitary boundary triple \(\{\mathcal{H}, \Gamma\}\), such that \(A_0\) is selfadjoint and \(\Gamma_0\) is not surjective is presented in [37, Example 6.6].

The following lemma shows that the conditions (iv) and (v) in Proposition 5.13 are not unrelated.

Lemma 5.15. Let \(\mathcal{U}\) be a unitary colligation of the form (5.3). Then the following conditions are equivalent:

(i) \(\text{ran} \, (I - H) + \text{ran} \, G = \mathcal{H}\);
(ii) \(\text{ran} \, (I - H^*) + \text{ran} \, F^* = \mathcal{H}\);
(iii) \(\text{ran} \, (I - H) = \mathcal{H}\);
(iv) \(\text{ran} \, (I - H^*) = \mathcal{H}\).

Proof. The equivalence of (i) and (ii) is implied by (5.11).

Since \(\text{ran} \, (I - H) \subseteq \text{ran} \, (I - H) + \text{ran} \, G\) and \(\text{ran} \, (I - H^*) \subseteq \text{ran} \, (I - H^*) + \text{ran} \, F^*\) it remains to prove the implications (i) \(\Rightarrow\) (iii) and (ii) \(\Rightarrow\) (iv).

Assume that \(\text{ran} \, (I - H) + \text{ran} \, G = \mathcal{H}\). Then using [40] and the identity \(HH^* + GG^* = I\) one obtains
\[
\text{ran} \, (I - H) + \text{ran} \, G = \text{ran} \, ((I - H)(I - H^*)^{1/2}) + \text{ran} \, ((GG^*)^{1/2})
\]
\[
= \text{ran} \, ((I - H)(I - H^*) + GG^*)^{1/2})
\]
\[
= \text{ran} \, ((I - 2\text{Re} \, H + HH^* + GG^*)^{1/2}) = \text{ran} \, ((I - \text{Re} \, H)^{1/2})
\]
This implies the equality \(\text{ran} \, (I - \text{Re} \, H) = \mathcal{H}\) and hence \(-I \leq \text{Re} \, H \leq qI\) for some \(q < 1\). Therefore, the numerical range of \(H\) is contained in the half-plane \(\text{Re} \, z \leq q\) and hence \(1 \in \rho(H)\). This proves (iii). The implication (ii) \(\Rightarrow\) (iv) is proved similarly. □

Corollary 5.16. If \(\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}\) is a unitary boundary triple with \(\text{ran} \, \Gamma_0 = \mathcal{H}\), then \(A_0 = A_0^*\) and \(\Pi\) is necessarily a \(B\)-generalized boundary triple.

Remark 5.17. If \(\{\mathcal{H}, \Gamma_0, \Gamma_1\}\) is an ordinary boundary triple, then \(\Gamma\) and, consequently, \(\Gamma_0\) and \(\Gamma_1\) are surjective. Hence, \(A_0 = A_0^*\) and \(A_1 = A_1^*\). This conclusion can be made directly also from Proposition 5.13. Indeed, the assumption \(\text{ran} \, G = \mathcal{H}\) implies \(0 \in \rho(GG^*)\). In view of the identity \(GG^* = I - HH^*\) this implies \(1 \in \rho(HH^*)\) and hence \(1 \in \rho(H)\). By Proposition 5.13 (v) this condition yields \(A_0 = A_0^*\).

We are now ready to prove Theorem 1.11 in a more general setting, where \(\{\mathcal{H}, \Gamma\}\) is an arbitrary unitary boundary pair. It gives a complete characterization of the Weyl functions \(M(\cdot)\) of \(S\)-generalized boundary pairs. In its present general form it completes and extends [37, Theorem 4.13] and [39, Theorem 7.39].
Theorem 5.18. Let $\Pi = \{\mathcal{H}, \Gamma\}$ be a unitary boundary pair and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl family and the $\gamma$-field. Then the following statements are equivalent:

(i) $A_0$ is selfadjoint, i.e. $\Pi$ is an $S$-generalized boundary pair;
(ii) $A_s = A_0 + \hat{N}_\lambda$ and $A_s = A_0 + \hat{N}_\mu$ for some (equivalently for all) $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$;
(iii) $\text{ran } \Gamma_0 = \text{dom } M(\lambda) = \text{dom } M(\mu)$ for some (equivalently for all) $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$;
(iv) $\gamma(\lambda)$ and $\gamma(\mu)$ are bounded for some (equivalently for all) $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$;
(v) $\text{Im } M_{op}(\lambda)$ is bounded with dense domain in $\overline{\text{dom } M(\lambda)}$ for some (equivalently for all) $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$;
(vi) The Weyl family $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, admits the representation

\begin{equation}
M(\lambda) = E + M_0(\lambda),
\end{equation}

where $E = E^*$ is a selfadjoint relation in $\mathcal{H}$ and $M_0 \in \mathcal{R}[\mathcal{H}_0]$, with $\mathcal{H}_0 = \overline{\text{dom } E}$.

Proof. (i) $\Leftrightarrow$ (ii) This equivalence and the independence from $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$ is proved in [37, Theorem 4.13].

(i) $\Leftrightarrow$ (iii) This can also be obtained from [37, Theorem 4.13], but we present here a different proof. Indeed, it follows from (5.17) that for all $\lambda \in \mathbb{C}_+$ and $\zeta = \frac{\lambda - i}{\lambda + i}$

\begin{equation}
\text{ran } \Gamma_0 = \text{ran } G + \text{ran } (\theta(\zeta) - I).
\end{equation}

If $A_0 = A_0^*$ then by Proposition 5.13 ran $G \subset \text{ran } (I - H)$ and (5.27) yields ran $G \subset \text{ran } (\theta(\zeta) - I)$. By (5.44), (5.13), and dom $\gamma(\lambda) = \text{dom } M(\lambda)$ one obtains

\begin{equation}
\text{ran } \Gamma_0 = \text{ran } (\theta(\zeta) - I) = \text{dom } M(\lambda) \quad \text{for all } \lambda \in \mathbb{C}_+.
\end{equation}

Similarly, it follows from (5.44) and (5.28) that

\begin{equation}
\text{ran } \Gamma_0 = \text{ran } F^* + \text{ran } (\theta(\zeta)^* - I) = \text{dom } M(\lambda) \quad \text{for all } \lambda \in \mathbb{C}_+.
\end{equation}

Conversely, if for some $\lambda \in \mathbb{C}_+$ one has $\text{ran } \Gamma_0 = \text{dom } M(\lambda) = \text{dom } \gamma(\lambda)$, then (5.11) implies, in particular, that $\text{ran } G \subset \text{ran } (\theta(\zeta) - I)$. Hence, it follows from (5.27) that $\text{ran } (A_0 - \lambda) = \mathcal{F}$. Similarly the identities $\text{ran } \Gamma_0 = \text{dom } M(\lambda) = \text{dom } \gamma(\lambda)$ imply that $\text{ran } (A_0 - \lambda) = \mathcal{F}$ and, thus, $A_0 = A_0^*$.

(i) $\Rightarrow$ (iv) This implication was proved in Theorem 4.2(iv), (v).

(iv) $\Rightarrow$ (i) If some $\gamma(\lambda) : \text{dom } \gamma(\lambda) \to \mathcal{F}$ is bounded then $\text{dom } \gamma(\lambda)^* = \mathcal{F}$. Then by (5.35)

\begin{equation}
\text{ran } (A_0 - \lambda) = \text{dom } \Gamma_1 H(\lambda) = \text{dom } \gamma(\lambda)^* = \mathcal{F}.
\end{equation}

Similarly if $\gamma(\mu)$ is bounded then $\text{ran } (A_0 - \mu) = \mathcal{F}$. Thus, $A_0$ is a selfadjoint relation in $\mathcal{F}$.

(iv) $\Rightarrow$ (v), (vi) Consider the decomposition (2.5) $M(\lambda) = \text{gr } M_{op}(\lambda) \oplus M_{op}$ of the Weyl family $M(\lambda)$ with the operator part $M_{op} \in \mathcal{R}(\mathcal{H}_0)$, where $\mathcal{H}_0 = \overline{\text{dom } M(\lambda)}$. As was already shown, now $A_0 = A_0^*$ and $\text{dom } M_{op}(\lambda) = \text{ran } \Gamma_0$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. It follows from the equality $M_{op}(\lambda)^* = M_{op}(\lambda)$ that the operator $E_0 = \text{Re } M_{op}(\lambda) (\lambda_0 \in \mathbb{C}_+)$ is selfadjoint with the domain $\text{dom } E_0 = \text{ran } \Gamma_0$. Moreover, since the operator $\gamma(\lambda)$ is bounded for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ it follows from the equality (3.9) that the operator

\begin{equation}
\text{Im } M_{op}(\lambda_0) = \text{Im } \lambda_0 \gamma(\lambda_0)^* \gamma(\lambda_0)
\end{equation}

is also bounded in $\mathcal{H}_0$ and hence the operator $M_{op}(\lambda) - E_0$ is bounded in $\mathcal{H}_0$ at $\lambda_0$. Therefore, its closure, denoted now by $M_{0}(\lambda)$, is bounded in $\mathcal{H}_0$ at $\lambda_0$ and then also for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see e.g. [38, Proposition 4.18], [40, Theorem 3.9]. Finally, by setting $E = E_0 \oplus M_{op}$ one arrives at (5.43).

Finally, the implication (vi) $\Rightarrow$ (v) is clear and (v) $\Rightarrow$ (iv) (for $\mu = \bar{\lambda}$) follows easily from (3.9).
Theorem 5.18 implies Theorem 1.11. In the case that \(\Gamma\) is single-valued \(M(\lambda)\) is an operator valued Nevanlinna function with \(\ker \text{Im} M(\lambda) = \ker (M(\lambda) - M(\lambda)^*) = \{0\}\), i.e., \(M(\cdot) \in \mathcal{R}^*(\mathcal{H})\); see (1.11) and Lemma 3.6.

**Corollary 5.19.** Let \(\{\mathcal{H}, \Gamma\} \) be an \(S\)-generalized boundary pair with the Weyl family \(M(\cdot) = E + M_0(\cdot)\) as in Theorem 5.18. Then \(\text{ran} \Gamma\) is dense in \(\mathcal{H} \times \mathcal{H}\), i.e., \(\Gamma\) defines an \(S\)-generalized boundary triple if and only if \(E (= \text{Re} M(\mu))\) is a selfadjoint operator and

\[
\text{dom} E \cap \ker \gamma(\lambda) = E \cap \ker \text{Im} M_0(\lambda) = \{0\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

**Proof.** This follows from Lemma 3.6 and Corollary 4.3. \(\square\)

Corollary 5.19 can be used to give an example of an \(S\)-generalized boundary triple \(\{\mathcal{H}, \Gamma_0, \Gamma_1\}\) such that the mapping \(\Gamma_0\) is not closable; cf. Corollary 5.4.

**Example 5.20.** Let \(M_0(\cdot) \in \mathcal{R}[\mathcal{H}]\) be a bounded Nevanlinna function such that \(\ker \text{Im} M_0(\lambda)\) is nontrivial and let \(E\) be an unbounded selfadjoint operator in \(\mathcal{H}\) with \(\text{dom} E \cap \ker \text{Im} M_0(\lambda) = \{0\}\). Then the function

\[
M(\lambda) = E + M_0(\cdot), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

is a domain invariant Nevanlinna function. Moreover, it follows from Corollary 5.19 and Theorems 1.11 and 5.18 that \(M(\cdot)\) can be realized as the Weyl function of some \(S\)-generalized boundary triple \(\{\mathcal{H}, \Gamma_0, \Gamma_1\}\). However, \(\text{Im} (M(\lambda)h, h) = \text{Im}(M_0(\lambda)h, h), h \in \text{dom} M(\lambda)\), does not satisfy the condition (5.23) in Corollary 5.4 since \(\text{ker} \text{Im} M(\lambda) = \ker \text{Im} M_0(\lambda)\) is nontrivial by construction.

**Remark 5.21.** Observe that in Theorems 1.11 and 5.18 the function \(M_0(\cdot)\) can be considered as the closure of \(M(\cdot) - E\). In Theorem 5.18 \(M(\cdot)\) is an operator valued function if and only if \(E\) is an operator. By Corollary 5.19 even in this case \(\Gamma\) can still be multivalued if the kernel \(\ker M_0(\lambda)\) or \(\ker \text{Im} M_0(\lambda) = \ker \gamma(\lambda)\) is nontrivial and the condition (5.46) is violated. In fact, any bounded Nevanlinna function in \(\mathcal{H}\) with \(\ker \text{Im} M_0(\lambda) \neq \{0\}\) combined with an unbounded selfadjoint operator \(E\) in \(\mathcal{H}\) satisfying the condition (5.46) is associated with an \(S\)-generalized boundary triple \(\{\mathcal{H}, \Gamma_0, \Gamma_1\}\) with the Weyl function \(M = E + M_0(\cdot)\). If such a function \(M\) is regularized by subtracting the unbounded constant operator \(E\), the function \(M_0(\cdot) = M(\cdot) - E\) corresponds to an \(AB\)-generalized boundary triple \(\{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}\) whose range \(\text{ran} \tilde{\Gamma}\) is not dense in \(\mathcal{H}^2\). In particular, \(\tilde{\Gamma}\) whose Weyl function is the regularized function \(M(\cdot) - E\) is not a quasi boundary triple. The closure \(\text{cl} M_0(\cdot)\) of \(M(\cdot) - E\) is the Weyl function of the closure of \(\tilde{\Gamma}\) which in this case is always a (multivalued) \(B\)-generalized boundary pair. An example of an \(S\)-generalized boundary triple with \(\ker \text{Im} M_0(\lambda) \neq \{0\}\) satisfying the property (5.46) appears in [20] Proposition 2.17.

### 5.4. \(ES\)-generalized boundary triples and form domain invariant Nevanlinna functions

Recall, see Definition 1.12, that a unitary boundary triple \(\{\mathcal{H}, \Gamma_0, \Gamma_1\}\) for \(A^*\) is called \(\text{\(ES\)-generalized,}\) if the extension \(A_0\) is essentially selfadjoint in \(\mathcal{H}\).

As the main result of this section it will be shown that the class of Weyl functions of \(\text{\(ES\)-generalized boundary triples coincides with the class of form domain invariant Nevanlinna functions.}\)

**Definition 5.22.** A Nevanlinna function \(M \in \mathcal{R}(\mathcal{H})\) is said to be form domain invariant in \(\mathbb{C}_+(\mathbb{C}_-)\), if the quadratic form \(t_{M(\lambda)}\) in \(\mathcal{H}\) generated by the imaginary part of \(M(\lambda)\) via

\[
t_{M(\lambda)}(u, v) = \frac{1}{\lambda - \bar{\lambda}} [(M(\lambda)u, v) - (u, M(\lambda)v)],
\]
is closable for all $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$ and the closure of the form $\var{M}(\lambda)$ has a constant domain. A Nevanlinna family $M \in \mathcal{R}(\mathcal{H})$ is said to be form domain invariant in $\mathbb{C}_+(\mathbb{C}_-)$, if its operator part $M_{op}(\cdot)$ in the decomposition (2.5) is form domain invariant in $\mathbb{C}_+(\mathbb{C}_-)$. The following three lemmas are preparatory for the main result.

**Lemma 5.23.** Let \{\mathcal{H}, \Gamma_0, \Gamma_1\} be a unitary boundary triple. Then the following statements are equivalent:

(i) $\text{ran}(A_0 - \lambda)$ is dense in $\mathfrak{H}$ for some or, equivalently, for every $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$;

(ii) $\gamma(\lambda)$ admits a single-valued closure $\gamma(\lambda)$ for some or, equivalently, for every $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$. 

**Proof.** (i)⇔(ii) In view of (5.35) in Remark 5.8 for every $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$
\[
\text{dom} \gamma(\lambda)^* = \text{dom}(\Gamma_1H(\lambda)) = \text{ran}(A_0 - \lambda).
\]
Therefore, $\gamma(\lambda)$ admits a single-valued closure for $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$ if and only if $\text{ran}(A_0 - \lambda)$ is dense in $\mathfrak{H}$. □

**Lemma 5.24.** Let \{\mathcal{H}, \Gamma_0, \Gamma_1\} be an ES-generalized boundary triple. Then:

(i) $\ker \Gamma_0 = \overline{A_0}$ is selfadjoint and the domain of $\Gamma_0$ admits the decomposition
\[
\text{dom} \Gamma_0 = \overline{A_0} + (\text{dom} \Gamma_0 \cap \mathfrak{N}_\lambda(A^*)) = \overline{A_0} + \text{ran} \gamma(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R};
\]

(ii) $\gamma(\lambda)$ admits a single-valued closure $\overline{\gamma(\lambda)}$ for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) the closure of the $\gamma$-field satisfies
\[
\text{ran} \Gamma_0 = \text{dom} \overline{\gamma(\lambda)^*} = \text{dom} \overline{\gamma(\mu)^*}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R};
\]

(iv) $\overline{\gamma(\lambda)}$ and $\overline{\gamma(\mu)}$ are connected by
\[
\overline{\gamma(\lambda)} = [I + (\lambda - \mu)(A_0 - \lambda)^{-1}]\gamma(\lambda), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R};
\]

**Proof.** (i) As a closed linear relation $\overline{\Gamma_0}$ has a closed kernel, which implies that $\overline{A_0} \subset \ker \overline{\Gamma_0}$. Since $\overline{A_0}$ is selfadjoint, the first von Neumann’s formula shows that $A^* = \overline{A_0} + \mathfrak{N}_\lambda(A^*)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Consequently, $\overline{A_0} \subset \text{dom} \overline{\Gamma_0} \subset \overline{A_0} + \mathfrak{N}_\lambda(A^*)$, $\lambda \in \mathbb{C} \setminus \mathbb{R},$
and this implies the first equality in (5.48). The second equality in (5.48) holds by Proposition 5.3. Finally, according to Proposition 5.3 $\ker \overline{\Gamma_0} \cap \mathfrak{N}_\lambda(A^*) = \text{mul} \overline{\gamma(\lambda)} = \{0\}$, since $\gamma(\lambda)$ or, equivalently, $\overline{\gamma(\lambda)}$ is closable by Lemma 5.23. Since $\overline{A_0} \subset \ker \overline{\Gamma_0}$, the identity $\ker \overline{\Gamma_0} \cap \mathfrak{N}_\lambda(A^*) = \{0\}$ combined with the first equality in (5.48) implies the equality $\overline{A_0} = \ker \overline{\Gamma_0}$.

(ii) The statement (ii) is implied by Lemma 5.23.

(iii) Since $\overline{A_0}$ is selfadjoint, the defect subspaces of $A$ are connected by
\[
\mathfrak{N}_\lambda(A^*) = [I + (\lambda - \mu)(\overline{A_0} - \lambda)^{-1}]\gamma(\mu)(A^*), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}.
\]
Hence, if $f_\lambda = [I + (\lambda - \mu)(\overline{A_0} - \lambda)^{-1}]f_\mu$, then $f_\lambda = \{f_\lambda, \mu f_\mu\} \in \mathfrak{N}_\mu(A^*)$ precisely when
\[
f_\lambda = \{f_\lambda, \lambda f_\lambda\} = f_\mu + (\lambda - \mu)H(\lambda)f_\mu \in \mathfrak{N}_\lambda(A^*),
\]
where $H(\lambda)f_\mu = \{(\overline{A_0} - \lambda)^{-1}f_\mu, (I + \lambda(\overline{A_0} - \lambda)^{-1})f_\mu\} \in \overline{A_0}$. Since $\overline{A_0} \subset \text{dom} \overline{\Gamma_0}$, it follows from (5.51) that $\overline{f_\mu} \in \text{dom} (\overline{\Gamma_0}) \cap \mathfrak{N}_\mu(A^*)$ if and only if $\overline{f_\lambda} \in \text{dom} (\overline{\Gamma_0}) \cap \mathfrak{N}_\lambda(A^*)$ and
\[
\{f_\mu, h\} \in \overline{\Gamma_0} \cap \mathfrak{N}_\mu(A^* \oplus \mathcal{H}) \iff \{f_\lambda, h\} \in \overline{\Gamma_0} \cap \mathfrak{N}_\lambda(A^* \oplus \mathcal{H})
\]
for some $h \in \mathcal{H}$. Now, using (i) and Proposition 5.3 one gets
\[
\text{dom} \overline{\gamma(\lambda)} = \overline{\Gamma_0} (\text{dom} \overline{\Gamma_0} \cap \mathfrak{N}_\lambda(A^*)) = \text{ran} \overline{\Gamma_0} = \overline{\Gamma_0} (\text{dom} \overline{\Gamma_0} \cap \mathfrak{N}_\mu(A^*)) = \text{dom} \overline{\gamma(\mu)},
\]
Clearly \( \overline{\gamma}(\lambda) = \text{dom} \, \gamma(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), and hence (iii) is proved.

(iv) The proof of (iii) shows that \( \{h, f_{\mu}\} \in \overline{\gamma}(\mu) \) if and only if \( \{h, \hat{f}_{\mu}\} \in \overline{\gamma}(\lambda) \). Consequently, \( \{h, f_{\mu}\} \in \overline{\gamma}(\mu) \) if and only if \( \{h, f_{\lambda}\} = \{h, [I + (\lambda - \mu)(A_0 - \lambda)^{-1}]f_{\mu}\} \in \gamma(\lambda) \) and, since \( \gamma(\mu) \) and \( \gamma(\lambda) \) are operators, this means that (5.50) is satisfied.

\[ \square \]

**Lemma 5.25.** Let \( M \) be the Weyl family of some unitary boundary triple \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) of \( A^* \) and let \( \gamma(\cdot) \) be the corresponding \( \gamma \)-field. Then:

(i) for all \( h \in \text{dom} \, M(\lambda), k \in \text{dom} \, M(\mu) \) and \( \lambda, \mu \in \mathbb{C} \setminus \mathbb{R} \) one has

\[ \frac{(M(\lambda)u, v)_H - (u, M(\mu)v)_H}{\lambda - \mu} = (\gamma(\lambda)u, \gamma(\mu)v)_H; \]

(ii) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) one has \( \ker \gamma(\lambda) = \{0\} \); (iii) the form \( t_{M(\lambda)} \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), is closable if and only if \( \text{ran} \,(A_0 - \lambda) \) is dense in \( \mathfrak{H} \), hence closability of \( t_{M(\lambda)} \) does not depend on the representing unitary relation \( \Gamma \) used for a realization of \( M \) as its Weyl family.

**Proof.** (i) This is a direct consequence of the Green’s identity when applied to the elements in \( \mathfrak{H}_1(A_0) \); see (3.11) and (5.13); cf. also [17] Proposition 4.8.

(ii) is implied by the definition of \( \gamma(\lambda) \) in (5.3) and the equality \( \ker \gamma(\lambda) = \text{mul} \, \Gamma_0 = \{0\} \).

(iii) Part (i) gives the following representation for \( t_{M(\lambda)} \):

\[ t_{M(\lambda)}u, v] = (\gamma(\lambda)u, \gamma(\lambda)v)_H. \]

It is well-known (see e.g. [64] Chapter VI) that the form \( (\gamma(\lambda)u, \gamma(\mu)v)_H \) is closable precisely when the operator \( \gamma(\lambda) \) is closable. Now the statement is obtained from Lemma 5.23. \[ \square \]

**Theorem 5.26.** Let \( \Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) be a unitary boundary triple for \( A^* \) and let \( M \) and \( \gamma(\cdot) \) be the corresponding Weyl function and the \( \gamma \)-field. Then the following statements are equivalent:

(i) \( \text{ran} \,(A_0 - \lambda) \) is dense in \( \mathfrak{H} \) for some or, equivalently, for every \( \lambda \in \mathbb{C}_+(-\mathbb{C}_-) \);

(ii) \( \gamma(\lambda) \) admits a single-valued closure \( \overline{\gamma}(\lambda) \) for one \( \lambda \in \mathbb{C}_+(-\mathbb{C}_-) \) with a domain dense in \( \mathcal{H} \);

(iii) \( \gamma(\lambda) \) admits a single-valued closure \( \overline{\gamma}(\lambda) \) for every \( \lambda \in \mathbb{C}_+(-\mathbb{C}_-) \) domain invariant with a constant domain dense in \( \mathcal{H} \);

(iv) the form \( t_{M(\lambda)} \) is closable for one \( \lambda \in \mathbb{C}_+(-\mathbb{C}_-) \);

(v) the Weyl function \( M \) belongs to \( \mathcal{R}^*(\mathcal{H}) \) and is form domain invariant in \( \mathbb{C}_+(-\mathbb{C}_-) \).

In particular, if statements (i)–(v) are satisfied both in \( \mathbb{C}_+ \) and \( \mathbb{C}_- \) then \( \Pi \) is an ES-generalized boundary triple and the Weyl function \( M \) is form domain invariant with

\[ \text{dom} \, t_{M(\lambda)} = \text{dom} \, \overline{\gamma}(\lambda) = \text{ran} \, \Gamma_0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

**Proof.** The equivalence (i) \( \iff \) (ii) is obtained from Lemma 5.23. The fact that the domain of \( \overline{\gamma}(\lambda) \) is dense in \( \mathcal{H} \) follows from Proposition 5.2.

The equivalences (i) \( \iff \) (iv), (v) and (ii) \( \iff \) (iii) follow from Lemma 5.24.

In particular, part (iii) of Lemma 5.25 shows that the form \( t_{M(\lambda)} \) is closable for some (and then for every) \( \lambda \in \mathbb{C}_+ \) and for some (and then for every) \( \mu \in \mathbb{C}_- \) if and only if \( A_0 \) is essentially selfadjoint. In this case the closure of the form \( t_{M(\lambda)} \) is given by

\[ t_{M(\lambda)}u, v] = \overline{\gamma}(\lambda) u, \overline{\gamma}(\lambda) v)_H, \]

in particular, \( \text{dom} \, t_{M(\lambda)} = \text{dom} \, \overline{\gamma}(\lambda) \). According to Lemma 5.24 this domain does not depend on \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) when \( A_0 \) is essentially selfadjoint. The last equality in (5.52) is obtained from (5.49). This completes the proof. \[ \square \]
Proof. If $\text{dom } M = \text{dom } \hat{M}$, then one can write
\[
t_{M(\lambda)}[u, v] = \left( \frac{M(\lambda) - M(\lambda)^*}{\lambda - \lambda} \right)_{\mathcal{H}} u, v \in \text{dom } M(\lambda).
\]
Hence, the operator
\[
N(\lambda) := \frac{M(\lambda) - M(\lambda)^*}{\lambda - \lambda}
\]
is nonnegative and densely defined in $\mathcal{H} \ominus \text{mul } M(\lambda)$. Therefore, the form $t_{M(\lambda)}$ is closable for $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see [61]. By applying the same reasoning to $\tilde{\lambda}$ it is seen that also the form $t_{M(\lambda)}$ is closable. Now by applying Lemma 5.25 it is seen that $A_0$ is essentially selfadjoint (by item (iii)) and hence by Theorem 5.26 $M$ is form domain invariant. \hfill \square

The converse statement does not hold. In fact, in [40] an example of a form domain invariant Nevanlinna function is constructed, such that the domains of $M(\lambda)$ and $M(\mu)$ have a zero intersection:

$$\text{dom } M(\lambda) \cap \text{dom } M(\mu) = \{0\} \text{ for all } \lambda, \mu \in \mathbb{C}_+.$$

**Remark 5.30.** A unitary boundary pair $\{\mathcal{H}, \Gamma\}$ for $A^*$ is said to be ES-generalized if $A_0 = A_0^*$. ES-generalized boundary pairs can be characterized by the following equivalent conditions:

(i) for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\gamma(\lambda)$ admits a single-valued closure $\tilde{\gamma}(\lambda)$ with a constant domain;

(ii) the Weyl family $M \in \mathcal{R}(\mathcal{H})$ is form domain invariant, i.e. its operator part $M_{\text{op}}(\cdot)$ in the decomposition (2.5) is form domain invariant.

Notice, that in the case when (i)-(ii) are in force and $\text{mul } \Gamma$ is nontrivial it may happen that the domain of the form $\overline{t_{M_{\text{op}}(\lambda)}}[u, v] = (\gamma(\lambda)u, \gamma(\lambda)v)_0$ is not dense in $\mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; for an example involving differential operators; see Example 5.40.

### 5.5. Renormalizations of form domain invariant Nevanlinna functions.

The next theorem shows that form domain invariant Nevanlinna functions $M$ in $\mathcal{H}$ can be renormalized with a bounded operator $G$ such that the renormalized function $G^*MG$ becomes domain invariant.

**Theorem 5.31.** Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triple for $A^*$ with the $\gamma$-field $\gamma(\cdot)$ and the Weyl function $M$, and assume that $A_0 = \ker \Gamma_0$ is essentially selfadjoint. Then:

1. There exists a bounded operator $G$ in $\mathcal{H}$ with $\text{ran } G = \text{dom } \overline{\gamma(\lambda)}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $\ker G = \{0\}$, such that

$$\begin{pmatrix} \overline{\Gamma_0} \\ \overline{\Gamma_1} \end{pmatrix} = \operatorname{clos} \begin{pmatrix} G^{-1} & 0 \\ 0 & G^* \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$$

defines an AB-generalized boundary pair $\{\mathcal{H}, \widetilde{\Gamma}\}$ for $A^*$.

2. The corresponding Weyl function $\widetilde{M}$ is domain invariant and it is given by

$$\widetilde{M}(\lambda) = E + M_0(\lambda),$$

where $E$ is a closed densely defined symmetric operator in $\mathcal{H}$ and $M_0(\cdot)$ is a bounded Nevanlinna function (defined on $\text{dom } E$).

3. Furthermore, $\overline{G^*M(\lambda)G}$ is also a Weyl function of a closed AB-generalized boundary pair and it satisfies

$$\overline{G^*M(\lambda)G} = E_0 + M_0(\lambda) \subset \widetilde{M}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $E_0 \subset E$ is a closed densely defined symmetric restriction of $E$.

**Proof.** The proof is divided into steps.

1. **Construction of a bounded operator $G$ with the properties**

$$\begin{align*}
\ker G &= \{0\}, & \text{ran } G &= \text{dom } \overline{\gamma(\mu)} & \text{dom } \gamma(\mu)G &= \mathcal{H}, & \text{for some } \mu \in \mathbb{C} \setminus \mathbb{R}.
\end{align*}$$

Since $A_0$ is essentially selfadjoint, $\gamma(\lambda)$ is closable and the dense subspace $\mathcal{H}_0 = \text{dom } \overline{\gamma(\lambda)}$ of $\mathcal{H}$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see Theorem 5.26. Since $\mathcal{H}_0$ is an operator range there exists a bounded selfadjoint operator $G = G^*$ with $\text{ran } G = \mathcal{H}_0$ and $\ker G = \{0\}$; for instance, one can fix $\mu \in \mathbb{C} \setminus \mathbb{R}$ and then take $G = (\gamma(\mu)^* \gamma(\mu) + I)^{-1/2}$. Namely, $\text{dom } \gamma(\mu) = \text{dom } M(\mu)$ is dense...
in $\mathcal{H}$, since $\text{mul } \Gamma = \{0\}$ by assumption, and hence $\gamma(\mu) \bar{\gamma}(\mu)$ is a selfadjoint operator satisfying $\text{dom } \gamma(\mu) = \text{dom } (\gamma(\mu) \bar{\gamma}(\mu))^{1/2} = \text{dom } (\gamma(\mu) \bar{\gamma}(\mu) + I)^{1/2}$. With this choice of $G$ the domain of $\gamma(\mu) G$ is dense in $\mathcal{H}$ since $\text{dom } \gamma(\mu)$ is a core for the form $t_{M(\mu)}$ and due to $\text{dom } (t_{M(\mu)} + I) = \text{ran } G$ one concludes that $\text{dom } \gamma(\mu)$ is also a core for the operator $G^{-1} = (\gamma(\mu) \bar{\gamma}(\mu) + I)^{1/2}$.

2. Construction of an isometric boundary triple $\{\mathcal{H}, \Gamma_0^G, \Gamma_1^G\}$ such that the corresponding $\gamma$-field $\gamma^G(\lambda)$ is a bounded densely defined operator.

Introduce the transform $\{\mathcal{H}, \Gamma_0^G, \Gamma_1^G\}$ of the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ by setting

\[
\begin{pmatrix}
\Gamma_0^G \\
\Gamma_1^G
\end{pmatrix} = \begin{pmatrix}
G^{-1} & 0 \\
0 & G^*
\end{pmatrix}
\begin{pmatrix}
\Gamma_0 \\
\Gamma_1
\end{pmatrix},
\]

where $G$ has the properties stated above. The block operator is isometric (in the Krein space $(\mathcal{H}^2, J_\mathcal{H})$) and hence $\Gamma^G$ is isometric as a composition of isometric mappings; i.e. $\Gamma^G$ satisfies the Green's identity (3.1) (Assumption 3.1.2). Since $\text{dom } \hat{\Gamma}_\lambda(A_\gamma) = \hat{\gamma}(\lambda)^{-1}$ one has

$$
\Gamma_0 \hat{\Gamma}_\lambda(A_\gamma) = \text{dom } \gamma(\lambda) \subset \text{dom } \gamma(\lambda) = \text{ran } G,
$$

which implies that $\text{ran } \gamma(\lambda) = \hat{\Gamma}_\lambda(A_\gamma) = \hat{\Gamma}_\lambda(A^*) \cap \text{dom } \gamma \subset \text{dom } \Gamma^G$ (here $A_\lambda = \text{dom } \gamma$), and hence $\hat{\Gamma}_\lambda(A^G) = \hat{\Gamma}_\lambda(A_\gamma)$. Moreover, it is clear that $\ker \Gamma_0^G = \ker \Gamma_0 = A_0$ is essentially selfadjoint. Since the closure of $A_0 \oplus \hat{\Gamma}_\lambda(A_\gamma)$ is $A_0 \oplus \hat{\Gamma}_\lambda(A^*) = A^*$ one gets $\text{dom } \Gamma^G = A^*$ (Assumption 3.1.1).

The corresponding $\gamma$-field is given by

$$
\gamma^G(\lambda) = (\Gamma_0^G \hat{\Gamma}_\lambda(A_\gamma))^\dagger = \gamma(\lambda) G, \quad \lambda \in \mathbb{C} \setminus \mathbb{R};
$$

see Lemma 3.12. Since $\gamma(\lambda)$ is closable and $\gamma(\lambda) G \subset \gamma(\lambda) G$, it follows from $\text{ran } G = \text{dom } \gamma(\lambda)$ that the closed operator $\gamma(\lambda) G$ is everywhere defined and, hence, bounded by the closed graph theorem. Thus also $\gamma(\lambda) G$ is a bounded operator with bounded closure $\gamma(\lambda) G \subset \gamma(\lambda) G$.

Next recall the operator $H(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, from (3.14); see also Lemma 3.7. Since $A_0$ is essentially selfadjoint, $\text{dom } (\Gamma_1 H(\lambda)) = \text{dom } H(\lambda) = \text{ran } (A_0 - \lambda)$ is dense in $\mathcal{H}$. Since $\ker \Gamma_0^G = A_0 \subset \text{dom } \Gamma^G$ and $\text{mul } \Gamma_1^G = \{0\}$, it follows from Proposition 5.6 that

$$
\Gamma_1^G H(\lambda) = G^* \Gamma_1 H(\lambda) = G^* \gamma(\lambda)^* \subset (\gamma(\lambda) G)^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
$$

By the construction of $G$ the domain of $\gamma(\mu) G$ is dense in $\mathcal{H}$ for some $\mu \in \mathbb{C} \setminus \mathbb{R}$. Therefore, (5.58) implies that $\Gamma_1^G H(\mu)$ is a bounded densely defined operator for some $\mu \in \mathbb{C} \setminus \mathbb{R}$ and, since $A_0$ is essentially selfadjoint, Lemma 3.17 shows that $\Gamma_1^G H(\lambda)$ is bounded and densely defined for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

3. Verification of (1): Now consider the closure $\hat{\Gamma}$ of $\Gamma^G$ in (5.57). It is shown below that $\ker \hat{\Gamma}_0 = A_0$, which means that $\ker \hat{\Gamma}_0$ is selfadjoint (Assumption 1.12.1), since $A_0$ is essentially selfadjoint by assumption. By construction $\Gamma^G$ is defined via the transform $\Gamma^G = \{G^{-1} \Gamma_0, G^* \Gamma_1\}$ of $\{\Gamma_0, \Gamma_1\}$. It follows from Lemma 3.38 (see also Remark 5.8) that the graph of $\Gamma^G$ contains all elements of the form

$$
\begin{pmatrix}
H(\lambda)k_\lambda \\
G^* \gamma(\lambda)^* k_\lambda
\end{pmatrix} + \begin{pmatrix}
\gamma(\lambda) Gh \\
\lambda \gamma(\lambda) Gh
\end{pmatrix},
$$

where $k_\lambda \in \text{ran } (A_0 - \lambda)$ and $h \in \text{dom } G^* M(\lambda) G = \text{dom } \gamma(\lambda) G, \lambda \in \mathbb{C} \setminus \mathbb{R}$. Let $\hat{h} = \{h, h'\} \in \overline{A_0}$ and let $k \in \text{ran } (\overline{A_0} - \lambda)$ be such that $\hat{h} = \overline{H(\lambda)k}$, where $\overline{H(\lambda)}$ corresponds to the graph of $\overline{A_0}$; see (3.14). Moreover, let $k_n \in \text{ran } (A_0 - \lambda)$ be a sequence such that $k_n \to k$ as $n \to \infty$. Then $H(\lambda)k_n \to \hat{h} \in \overline{A_0}$, since $H(\lambda)$ is bounded. Moreover, by boundedness of $\Gamma_1^G H(\lambda) = G^* \gamma(\lambda)^*$

$$
\hat{h}_n = \Gamma^G H(\lambda)k_n = \{0, G^* \gamma(\lambda)^* k_n\} \to \{0, g\}, \quad g \in \mathcal{H}.
$$
Since $\Gamma$ is closed, it follows that \{\hat{h}; \{0, g\}\} \in \Gamma$ which shows that \(\hat{h} \in \ker \Gamma_0\). Hence, \(A_0 \subset \ker \Gamma_0\) and since \(\ker \Gamma_0\) is symmetric this implies that \(\ker \Gamma_0 = A_0\).

Since \(\text{dom} \Gamma^G = A^*\), the closure $\Gamma$ has dense domain in $A^*$ (Assumption 1.8.1). Clearly, \(\text{dom} G^* M(\mu) G = \text{dom} \gamma(\mu) G \subset \text{ran} \Gamma^G\) and hence the ranges of $\Gamma^G$ and $\Gamma_0$ are dense in $H$ (Assumption 1.19.2). Furthermore, $\Gamma$ as the closure of $\Gamma^G$ is also isometric, i.e., Green’s formula (3.1) holds for $\Gamma$ (Assumption 1.19.1). According to Definition 4.1 this means that $\Gamma$ is an $AB$-generalized boundary pair for $A^*$.

4. **Verification of (2):** The form of the Weyl function $\tilde{\mathcal{M}}(\lambda) = E + M_0(\lambda)$ of $\Gamma$ is obtained from Theorem 1.2. Furthermore, by Theorem 4.2, $\Gamma$ is closed if and only if $E$ is closed or, equivalently, $\tilde{\mathcal{M}}(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}$, is closed.

5. **Verification of (3):** Since $\Gamma^G H(\lambda) = G^* \gamma(\lambda)^* \gamma(\lambda) G$ are bounded and densely defined for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$, if follows from (5.59) that

\[
\hat{\Gamma}^G := \left\{ \left\{ \frac{h}{\gamma(\lambda) G} \right\} ; \left\{ \frac{h}{\gamma(\lambda) G} \right\} : \gamma(\lambda) G \in \text{ran} (\overline{A_0} - \lambda), h \in \text{dom} M_G(\lambda) \right\} \subset \Gamma,
\]

where $\gamma(\lambda)$ and $M_G(\lambda) := G^* M(\lambda) G$ are the $\gamma$-field and the Weyl function of $\Gamma$. Notice that $\overline{A_0} \subset \text{dom} \hat{\Gamma}^G$ and, as shown above, $\text{ran} \hat{\Gamma}^G \supset \text{ran} \gamma(\mu) G$ is dense in $\mathcal{H}$ (Assumption 1.19.2). Due to $\hat{\Gamma}^G \subset \Gamma$ also Green’s identity (3.1) is satisfied (Assumption 1.19.1). Therefore, $\hat{\Gamma}^G$ is also an $AB$-generalized boundary pair whose Weyl function is clearly $M_G(\lambda)$, which is closed. Now by Theorem 1.2, the $AB$-generalized boundary pair $\hat{\Gamma}^G$ is also closed and, since $\hat{\Gamma}^G \subset \Gamma$, one has

\[
G^* M(\lambda) G \subset \tilde{\mathcal{M}}(\lambda) = E + M_0(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Now $G^* M(\lambda) G$ as a closed restriction of $E + M_0(\lambda)$ is of the form $G^* M(\lambda) G = E_0 + M_0(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, where $E_0$ is a closed densely defined restriction of $E$; cf. Theorem 1.2. This proves the last statement.

Theorem 5.31 remains valid for all form domain invariant Nevanlinna functions $M(\cdot) \in \mathcal{R}(\mathcal{H})$ that need not be strict. The only essential difference appearing in the proof of Theorem 5.31 in this case is that $\ker \gamma(\lambda) = \text{mul} \Gamma_0$ is nontrivial, and then also, $\ker \gamma^G(\lambda) = \ker \gamma(\lambda) G$ is nontrivial. Notice that even if $\ker \gamma(\lambda) = \{0\}$ (i.e. $M(\cdot) \in \mathcal{R}(\mathcal{H})$) then the $\gamma$-field $\gamma(\lambda)$ as well as its closure $\overline{\gamma(\lambda)} = \gamma_G(\lambda)$ can have a nontrivial kernel. This explains why the constructed boundary pair $\Gamma$ can in general be multivalued even if the original boundary triple $\Gamma = \{\Gamma_0, \Gamma_1\}$ is single-valued.

Theorem 5.31 combined with the next lemma yields an explicit representation for the class of form domain invariant Nevanlinna functions as well as form domain invariant Nevanlinna families.

**Lemma 5.32.** Let $G$ be a bounded operator in the Hilbert space $\mathcal{H}$ with $\ker G = \ker G^* = \{0\}$, let $H$ be a closed symmetric densely defined operator on $\mathcal{H}$ and let $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$. Then the function

\[
M(\lambda) = G^{-1}(H + M_0(\lambda)) G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

is form domain invariant if and only if for some, equivalently for every, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the set

\[
\mathcal{D}_\lambda := \{ h \in \mathcal{H} : (\text{Im} \ M_0(\lambda))^2 h \in \text{ran} G^* \}
\]

is dense in $\mathcal{H}$. 
Proof. To calculate the form $t_{M(\lambda)}$ let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be fixed and let $u, v \in \text{dom } M(\lambda)$. Then $u, v \in \text{dom } G^{-1}$ and hence

$$t_{M(\lambda)}[u, v] = \frac{1}{\lambda - \gamma} \left[ (\mathcal{H} + M_0(\lambda))G^{-1}u, G^{-1}v \right]$$

$$= \frac{1}{\text{Im } \lambda} \left( (\text{Im } M_0(\lambda))G^{-1}u, G^{-1}v \right)$$

where symmetry of $H$ has been used. This form is closable precisely when the operator $(\text{Im } M_0(\lambda))\frac{1}{2}G^{-1}$ is closable or, equivalently, its adjoint $((\text{Im } M_0(\lambda))\frac{1}{2}G^{-1})^* = G^*(-\text{Im } M_0(\lambda))\frac{1}{2}$ is densely defined. Since $\mathcal{D}_\lambda = \text{dom } G^*(-\text{Im } M_0(\lambda))\frac{1}{2}$, the closability of $t_{M(\lambda)}$ is equivalent for $\mathcal{D}_\lambda$ to be dense in $\mathcal{H}$.

To prove that this criterion does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$ consider $M_0(\cdot)$ as the Weyl function of some $B$-generalized boundary pair $(\mathcal{H}, \Gamma)$. Let $\gamma_0(\cdot)$ be the corresponding $\gamma$-field and let $A_0 = \ker \Gamma_0$ be the associated selfadjoint operator. Then the form $t_{M(\lambda)}[u, v]$ can be also rewritten in the form

$$t_{M(\lambda)}[u, v] = (\gamma_0(\lambda)G^{-1}u, \gamma_0(\lambda)G^{-1}v)$$

and hence the form $t_{M(\lambda)}[u, v]$ is closable if and only if $\gamma_0(\lambda)G^{-1}$ is a closable operator. Now for any $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ one has

$$(I + (\lambda - \mu)(A_0 - \lambda)^{-1})\gamma_0(\mu)G^{-1} = \gamma_0(\lambda)G^{-1},$$

and since $I + (\lambda - \mu)(A_0 - \lambda)^{-1}$ bounded with bounded inverse, one concludes that $\gamma_0(\mu)G^{-1}$ is closable exactly when $\gamma_0(\lambda)G^{-1}$ is closable and that the closures are connected by

$$(I + (\lambda - \mu)(A_0 - \lambda)^{-1})\gamma_0(\mu)G^{-1} = \gamma_0(\lambda)G^{-1}.$$

Therefore, if $t_{M(\mu)}[u, v]$ is closable for some $\mu \in \mathbb{C} \setminus \mathbb{R}$ then $t_{M(\lambda)}[u, v]$ is closable for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and the form domains of these closures coincide. This completes the proof. \hfill \Box

Proposition 5.33. Let $M$ be a strict form domain invariant operator valued Nevanlinna function in the Hilbert space $\mathcal{H}$. Then there exist a bounded operator $G \in \mathbb{H}$ with $\ker G = \ker G^* = \{0\}$, a closed symmetric densely defined operator $\mathcal{E}$ in $\mathcal{H}$, and a bounded Nevanlinna function $M_0(\cdot) \in \mathbb{R}[\mathcal{H}]$ with the property

$$\mathcal{H} = \text{clos } \mathcal{D}_\lambda := \text{clos } \{ h \in \mathcal{H} : (\text{Im } M_0(\lambda))\frac{1}{2}h \in \text{ran } G^* \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

such that $M(\cdot)$ admits the representation

$$M(\lambda) = G^{-*}(E + M_0(\lambda))G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Conversely, every Nevanlinna function $M(\cdot)$ of the form (5.61) is form domain invariant $\mathbb{C} \setminus \mathbb{R}$, whenever $E \subset E^*$ and $G \in \mathcal{B}(\mathcal{H})$, $\ker G = \ker G^* = \{0\}$, and $M_0(\cdot) \in \mathbb{R}[\mathcal{H}]$ satisfy the condition (5.60).

Proof. Let the Nevanlinna function $M \in \mathbb{R}(\mathcal{H})$ be realized as the Weyl function of some boundary pair $\{\mathcal{H}, \Gamma\}$ (see Theorem 5.26 [37, Theorem 3.9]). Since $M$ is form domain invariant, $A_0$ is essentially selfadjoint by Theorem 5.26. Since $M$ is an operator valued Nevanlinna function, one can apply Theorem 5.31 (see also the discussion after Theorem 5.31), which shows that the inclusion $G^* M(\lambda)G \subset E + M_0(\lambda)$ holds for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This implies that

$$M(\lambda) = G^{-*}G^* M(\lambda)G G^{-1} \subset G^{-*}(E + M_0(\lambda))G^{-1},$$

where $G$ is a bounded operator with $\ker G = \ker G^* = \{0\}$ (cf. proof of Theorem 5.31 where $\text{ran } G = \mathcal{H}$ by construction). Clearly, the function $G^{-*}(E + M_0(\lambda))G^{-1}$ is dissipative for $\lambda \in \mathbb{C}_+$ and accumulative for $\lambda \in \mathbb{C}_-$. Since $M$ is Nevanlinna function, it is maximal dissipative in $\mathbb{C}_+$. 


and maximal accumulative in \( \mathbb{C}_- \). Therefore, the inclusion in (5.62) prevails as an equality. Since \( M(\cdot) \) is form domain invariant Lemma 5.32 shows that the condition (5.60) holds for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

Conversely, if \( M(\cdot) \) is a Nevanlinna function of the form (5.61), where \( E, G \) and \( M_0(\cdot) \) are as indicated and the condition (5.60) holds for some \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), then by Lemma 5.32 \( M(\cdot) \) is form domain invariant and the condition holds for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

\[ \square \]

**Remark 5.34.** As to the renormalization in Theorem 5.31 we do not know if the renormalized function \( \tilde{M}(\cdot) = E + M_0(\cdot) \) belongs to the class of Nevanlinna functions.

However, the representation of \( M(\cdot) \) in Proposition 5.33 combined with \( E \subset E^* \) leads to

\[ M(\lambda) = M(\lambda)^* \supset G^{-*}(E^* + M_0(\lambda))G^{-1} \supset M(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

Hence, \( M(\lambda) \) can also be represented with \( E^* \) instead of \( E \) as follows:

\[ M(\lambda) = G^{-*}(E + M_0(\lambda))G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

In particular, if \( \tilde{E} \) is any maximal symmetric extension of \( E \) then one has also the representation

\[ M(\lambda) = G^{-*}(\tilde{E} + M_0(\lambda))G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

**Remark 5.35.** The result in Proposition 5.33 remains valid also for form domain invariant Nevanlinna families. In this case there exists a bounded operator \( G \in \mathcal{B} \{H\} \) with \( \ker G = \ker G^* = \text{mul} M(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), a closed symmetric densely defined operator \( E \) in \( \mathcal{H} \), and a bounded Nevanlinna function \( M_0(\cdot) \in \mathcal{R}[\mathcal{H}] \) satisfying (5.60), such that

\[ M(\lambda) = G^{-*}(E + M_0(\lambda))G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

To see this, decompose \( M(\lambda) = \text{gr} M_{\text{op}}(\lambda) + M_{\infty} \), where \( M_{\infty} = \{0\} \times \text{mul} M(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), see (2.5). Now as in the proof of Proposition 5.33 the operator part \( M_{\text{op}}(\lambda) \) admits the representation \( M_{\text{op}}(\lambda) = G_0^{-*}(E + M_0(\lambda))G_0^{-1} \) with some operator \( G_0 \in \mathcal{B}[\mathcal{H}_0] \) in \( \mathcal{H}_0 = \mathcal{H} \oplus \text{mul} M(\lambda) \) with \( \ker G_0 = \ker G_0^* = \{0\} \). The desired representation of \( M \) is obtained by letting \( G \) to be the zero continuation of \( G_0 \) from \( \mathcal{H}_0 \) to \( \mathcal{H} = \mathcal{H}_0 \oplus \text{mul} M(\lambda) \).

The next example contains a wide class of \( ES \)-generalized boundary triples and demonstrates the regularization procedure formulated in Theorem 5.31.

**Example 5.36.** Let \( \{\mathcal{H}, \Gamma_0, \Gamma_0^\prime\} \) be an ordinary boundary triple \( \Pi^0 = \{\mathcal{H}, \Gamma_0, \Gamma_0^\prime\} \) for \( A^* \) with \( A_0^0 = \ker \Gamma_0^0, A_1^0 = \ker \Gamma_0^1 \), let \( M_0(\cdot) \) and \( \gamma_0(\cdot) \) be the corresponding Weyl function and the \( \gamma \)-field, and let \( G \in \mathcal{B}[\mathcal{H}] \) with \( \ker G = \ker G^* = \{0\} \). Then the transform

\[ (5.63) \quad \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & G^{-*} \end{pmatrix} \begin{pmatrix} \Gamma_0^0 \\ \Gamma_0^1 \end{pmatrix}, \]

where \( G^{-*} \) stands for \( (G^{-1})^* = (G^*)^{-1} \), defines an \( ES \)-generalized boundary triple \( \Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) for \( A^* \). Indeed, since \( G \in \mathcal{B}[\mathcal{H}] \) the transform \( \tilde{V} \) in (3.25) is unitary in the Krein space \( \{\mathcal{H}_2, J_H\} \) and it follows from [48, Theorem 2.10 (ii)] that the composition \( \Gamma = V \circ \Gamma^0 \) is unitary. By Lemma 5.12 one has \( \ker \Gamma = A \) and, since \( \Gamma \) is unitary, \( A_* := \text{dom} \Gamma \) is dense in \( A^* \).

Since \( \Pi^0 \) is an ordinary boundary triple, \( \mathcal{H} \times \{0\} \subset \text{ran} \Gamma^0 \) and hence one concludes from (5.63) that

\[ \text{ran} \Gamma_0 = \text{ran} G, \quad A_0 := \ker \Gamma_0 = A_0^0 \cap A_* \).

Consequently, \( \text{ran} \Gamma_0 \) is dense in \( \mathcal{H} \) and \( A_0 \) is essentially selfadjoint. Moreover, \( A_1 := \ker \Gamma_1 = \ker \Gamma_1^0 = A_1^0 \) and \( \text{ran} \Gamma_1 = \text{dom} G^* = \mathcal{H} \): this means that the transposed boundary triple \( \{\mathcal{H}, \Gamma_1, -\Gamma_0\} \) is \( B \)-generalized. Observe, that \( A_0 \) is selfadjoint if and only if \( \text{ran} G = \mathcal{H} \) or, equivalently, when \( \Pi \) is an ordinary boundary triple for \( A^* \), too.
Next the form domain of the Weyl function $M$ is calculated. By Lemma 3.12 $M(\cdot) = G^{-*}M_0(\cdot)G^{-1}$ and $\gamma(\cdot) = \gamma_0(\cdot)G^{-1}$. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be fixed and let $u,v \in \text{dom } M(\lambda)$. Then

$$t_{M(\lambda)}[u,v] = \frac{1}{\lambda} [(G^{-*}M_0(\lambda)G^{-1}u,v) - (u,G^{-*}M_0(\lambda)G^{-1}v)]$$

$$= \frac{1}{\lambda} [(M_0(\lambda)G^{-1}u,G^{-1}v) - (M_0(\lambda)G^{-1}u,G^{-1}v)]$$

$$= (\gamma_0(\lambda)G^{-1}u,\gamma_0(\lambda)G^{-1}v).$$

Since $\Pi^0$ is an ordinary boundary triple, $\gamma_0(\lambda) : \mathcal{H} \to \ker (A^* - \lambda)$ is bounded and surjective, i.e., the inverse of this mapping is also bounded. Hence $\gamma_0(\lambda)G$ is closed, when considered on its natural domain $\text{dom } \gamma_0(\lambda)G^{-1} = \text{ran } G (\supset \text{dom } M(\lambda))$. Therefore, the closure of the form $t_{M(\lambda)}$ is given by

$$\tilde{t}_{M(\lambda)}[u,v] = (\gamma(\lambda)G^{-1}u,\gamma(\lambda)G^{-1}v), \quad u,v \in \text{ran } G.$$ 

In particular, $M(\lambda)$ is a form domain invariant Nevanlinna function whose form domain is equal to $\text{ran } G$. Since $G$ is bounded, one can use $G$ to produce a regularized function $\tilde{M}$:

$$\tilde{M} = G^*MG = G^*(G^{-*}M_0(\cdot)G^{-1})G = M_0(\lambda),$$

so that $\tilde{M}$ coincides with the Nevanlinna function $M_0(\cdot)$ which belongs to the class $\mathcal{R}^*[\mathcal{H}]$.

It is emphasized that when $G$ is not surjective, the form domain invariant function $M(\cdot) = G^{-*}M_0(\cdot)G^{-1}$ need not be domain invariant. In fact, in [40] an example of a form domain invariant Nevanlinna function $M$ was given, such that

$$\text{dom } M(\lambda) \cap \text{dom } M(\mu) = \{0\}, \quad \lambda \neq \mu \quad (\lambda,\mu \in \mathbb{C} \setminus \mathbb{R}),$$

and the corresponding regularized function $\tilde{M}$ therein still belongs to the class $\mathcal{R}^*[\mathcal{H}]$.

In Example 5.36 the boundary triple $\Pi$ is $ES$-generalized and the transposed boundary triple $\Pi^T := \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is $B$-generalized. Therefore, according to [39] Theorem 7.24, or Theorem 4.10 there exist an ordinary boundary triple $\Pi^0$ and operators $R = R^*, K \in \mathcal{B}(\mathcal{H})$, $\ker K = \ker K^* = \{0\}$, such that $\Pi^T$ is the transform (4.13) of $\Pi^0$. Recall that one can take e.g. $R = \text{Re } (-M(i)^{-1})$, $K = (\text{Im } (-M(i)^{-1})^{1/2}$.

In particular, this yields the following connections between the associated Weyl functions:

$$-M^{-1}(\cdot) = K^*\tilde{M}_0(\cdot)K + R.$$ 

In particular, if $R = 0$ then one obtains $M(\cdot) = K^{-1}(-\tilde{M}_0(\cdot)^{-1})K^{-*}$, where $-\tilde{M}_0(\cdot)^{-1}$ belongs to the class $\mathcal{R}^*[\mathcal{H}]$.

This together with Example 5.36 essentially characterizes those $ES$-generalized boundary triples $\Pi$ for $A^*$ whose transposed boundary triple $\Pi^T$ is $B$-generalized.

Recall that Weyl functions of $S$-generalized boundary pairs are domain invariant, but converse does not hold; for explicit examples see Section 7.

As shown in the next proposition a domain invariant Nevanlinna function can be always renormalized by means of a fixed bounded operator to a bounded Nevanlinna function.

**Proposition 5.37.** Let $M(\cdot)$ be a domain invariant operator valued Nevanlinna function in the Hilbert space $\mathcal{H}$. Moreover, let $G$ with $\ker G = \ker G^* = \{0\}$ be a bounded operator in $\mathcal{H}$ such that $\text{ran } G = \text{dom } M(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the renormalized function

$$(5.64) \quad M_G(\lambda) = G^*M(\lambda)G, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

is a bounded Nevanlinna function. Moreover, $M_G(\cdot) \in \mathcal{R}^*[\mathcal{H}]$ precisely when $M(\cdot) \in \mathcal{R}^*[\mathcal{H}]$. 

Proof. By assumptions the equality $\text{dom } G^* M(\lambda) G = \text{dom } M(\lambda) G = \mathcal{H}$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Consequently, the adjoint $M_G(\lambda)^*$ is a closed operator and in view of

$$M_G(\lambda)^* = (G^* M(\lambda) G)^* \supset G^* M(\bar{\lambda}) G$$

one has $\text{dom } M_G(\lambda)^* = \mathcal{H}$. Consequently, the equality $M_G(\lambda)^* = G^* M(\bar{\lambda}) G$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now clearly $\text{Im } M_G(\lambda) = G^* \text{Im } M(\lambda) G$, which implies that $M_G(\lambda) \in \mathcal{R}[\mathcal{H}]$ and also shows the last statement in the proposition. \hfill \Box

The assumption $\text{ran } G = \text{dom } M(\lambda)$ in Proposition 5.37 (or more generally the inclusion $\text{dom } M(\lambda) \subset \text{ran } G$) guarantees that $M(\lambda)$ can be recovered from $M_G(\lambda)$ in (5.64) similarly as was done in Proposition 5.33

$$M(\lambda) = G^{−*} G^* M(\lambda) G G^{-1} = G^{−*} M_G(\lambda) G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

5.6. Examples on renormalization. The following examples demonstrate different renormalizations of some form domain invariant Nevanlinna functions. In the first example the real part of $M(i)$ is strongly subordinated with respect to its imaginary part. In this case the renormalized function $\tilde{M}(\lambda)$ is a bounded Nevanlinna function.

Example 5.38. Let $S$ be a positively definite closed symmetric operator in $\mathcal{H}$, $S \geq \varepsilon I$. Let

$$(5.65) \quad M(z) = zS^* S + S, \quad \text{dom } M(z) = \text{dom } S^* S, \quad z \in \mathbb{C}.$$

Replacing if necessary $S$ by $S + aI$ we can assume that $\varepsilon > 1$.

First notice that

$$\|f\|^2 \leq \varepsilon^{-2}\|Sf\|^2 = \varepsilon^{-2}(S^* S f, f) \leq \varepsilon^{-2}\|S^* S f\| \cdot \|f\|, \quad f \in \text{dom } S^* S,$$

i.e. $\|S^* S f\| \geq \varepsilon^2\|f\|$. It follows that $S$ is strongly subordinated with respect to $S^* S$, i.e.

$$\|Sf\|^2 = (S f, S f) = (S^* S f, f) \leq \|S^* S f\| \cdot \|f\| \leq \varepsilon^{-2}\|S^* S f\|^2, \quad f \in \text{dom } S^* S.$$

Since $\text{dom } S^* S \subset \text{dom } S \subset \text{dom } S^*$, one easily proves that $S^*$ is also strongly subordinated with respect to $S^* S$. Now, these inequalities imply that both operators $S/z$ and $S^*/z$ are also strongly subordinated to $S^* S$ for $|z| \geq 1$. Therefore,

$$M(\overline{z})^* = (\overline{z} S^* S + S)^* = z S^* S + S^* = z S^* S + S = M(z).$$

Since $M(\lambda)$ is dissipative in $\mathbb{C}_+$, it follows that $M(z)$ is maximal dissipative for $z \in \mathbb{C}_+, |z| \geq 1$, and maximal accumulative for $z \in \mathbb{C}_-, |z| \geq 1$. In turn, the latter implies that $M(z)$ being holomorphic and dissipative is $m$-dissipative for each $z \in \mathbb{C}_+$. Summing up we conclude that $M(\lambda)$ is an entire Nevanlinna function with values in $\mathcal{C}(\mathcal{H})$.

Furthermore,

$$t_{M(z)}(f, g) = \frac{(M(z)f, g) - (f, M(z)g)}{z - \overline{z}} = (S f, S g), \quad f, g \in \text{dom } S^* S, \quad z \in \mathbb{C}.$$

The form is closable because so is the operator $S$. Taking the closure we obtain the closed form $\overline{t}_{M(z)}(f, g) = (S f, S g)$, $f, g \in \text{dom } S$, $z \in \mathbb{C}$, with constant domain. In other words, $M(\lambda)$ is a form domain invariant Nevanlinna function and the (selfadjoint) operator associated with $\overline{t}_{M(z)}$ in accordance with the second representation theorem is $(S^* S)^{1/2}$.

Now consider the renormalization of $M(\lambda)$ as in Theorem 5.31. The operator $G = (S^* S)^{−\frac{1}{2}}$ is bounded and $\text{ran } G = \text{dom } \overline{t}_{M(z)}$. Moreover, $G^* (S^* S) G = I | \text{dom } (S^* S)^{\frac{1}{2}}$ and $G^* S G = G^* U$, where $U : \text{ran } (S^* S)^{\frac{1}{2}} = \mathcal{H} \rightarrow \text{ran } S$ is the (partial) isometry from the polar decomposition $S = U(S^* S)^{\frac{1}{2}}$. Consequently, $C := G^* S G$ is a bounded selfadjoint operator in $\mathcal{H}$. By Theorem 5.31 one has $\tilde{M}(z) \supset \text{clo } (G^* M(z) G) = z I + C$. Thus, $\tilde{M}(z) = z I + C$ is a bounded Nevanlinna function.
In the next example we change the roles of the real and imaginary parts of $M(i)$ of the function treated in Example 5.38. This leads to a renormalized Nevanlinna function which is unbounded.

**Example 5.39.** Consider the entire operator function  

$$M_1(z) = S^* S + z S, \quad z \in \mathbb{C}.$$  

Then $M_1(z)$ is a Nevanlinna function; cf. Example 5.38. It is domain invariant and also form domain invariant:

$$(M_1(z) f, g) - (f, M_1(z) g) = (z - \bar{z})(S f, g), \quad f, g \in \text{dom} S^* S.$$  

The closure of this form is given by 

$$\tau_{M_1(z)}(f, g) = (S_F^{1/2} f, S_F^{1/2} g), \quad f, g \in \text{dom} (S_F)^{1/2}.$$  

The operator $S_F^{-1/2}$ is bounded, injective, and clearly $\text{ran} S_F^{-1/2} = \tau_{M_1(z)}$. Consider the renormalization of this function determined by $G = S_F^{1/2}$:

$$G^* M_1(z) G = S_F^{-1/2} S^* S S_F^{-1/2} + z S_F^{-1/2} S S_F^{-1/2}.$$  

Here 

$$\text{dom} (S_F^{-1/2} S^* S S_F^{-1/2}) \subset \text{dom} (S_F^{-1/2} S S_F^{-1/2}) = \mathcal{S}_1^2(\text{dom} S).$$  

By the first representation theorem the operator $S_F^{-1/2} S S_F^{-1/2}$ is densely defined and 

$$S_F^{-1/2} S S_F^{-1/2} \subset S_F^{-1/2} S_F S_F^{-1/2} = I_{\text{dom} S_1^2}.$$  

Hence $T_1 := S_F^{-1/2} S S_F^{-1/2}$ is bounded and its closure is the identity operator on $\mathcal{H}$. On the other hand, for $T_0 := S_F^{-1/2} S^* S F^{-1/2}$ one has 

$$T_0 = S_F^{-1/2} S^* S F_S^{-1/2} = S_F^{-1/2} S^* S S^{-1} F^{-1/2} = S_F^{-1/2} (S^* | \text{ran} S) S_F^{1/2}.$$  

Here $H := S^* | \text{ran} S$ is a closed restriction of $S^*$ with nondense domain in $\mathcal{H}$ and its adjoint is given by 

$$H^* = (S^* | \text{ran} S)^* = S \hat{\cap} (\{0\} \times \ker S^*).$$  

Consequently, 

$$T_0 = S_F^{-1/2} H S_F^{1/2}, \quad T_0^* = (S_F^{-1/2} H S_F^{1/2})^* = S_F^{1/2} H^* S_F^{-1/2}.$$  

One can rewrite $T_0^*$ as follows: 

$$T_0^* = S_F^{1/2} [S \hat{\cap} (\{0\} \times \text{ran} S)] S_F^{-1/2}.$$  

Here $\ker S^* \cap \text{dom} S_F^{1/2} \subset \text{dom} S_F = \{0\}$, since $0 \notin \rho(S_F)$. Hence $T_0^*$ is an operator and $T_0$ is a densely defined nonnegative operator in $\mathcal{H}$. Moreover, $\text{ran} T_0^* = \mathcal{H}$ and $\text{ran} T_0 = S_F^{-1/2} (\text{ran} H) = \text{dom} S_F^{1/2}$ is dense in $\mathcal{H}$ so that $\text{ran} (T_0)^{**} = \mathcal{H}$. Hence, $T_0$ is essentially selfadjoint. From Theorem 5.31 one gets 

$$\hat{M_1}(z) \supset G^* M_1(z) G = \text{clos} (T_0^* + z T_1) = (T_0)^{**} + z I, \quad z \in \mathbb{C}.$$  

Thus, $\hat{M_1}(z)$ is an unbounded domain invariant Nevanlinna function, whose imaginary part is bounded.

As a comparison we consider another renormalization of the function $M(z) = z S^* S + S$ from Example 5.38, which leads to a renormalized function that is, in fact, a multivalued Nevanlinna family. The situation is made more concrete by treating as a special case the second order differential operator $S = -D^2$ on $L^2[0, 1]$.  

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Insert Excerpts from Other Sections Here
Example 5.40. (i) Let $S$ and $M(z) = zS^*S + S$ be as in Example 5.38. Consider another renormalization of $M(\cdot)$ using the bounded operator $G_2 := (S_F)^{-1}$. For simplicity we assume in addition that $\text{dom} S^2 = H$, which implies that $S^*S = (S^2)_F$. Since $S^2 \subset (S_F)^2$ one concludes that $S^*S \geq (S_F)^2$ and, in particular, $\text{dom} S = \text{dom} (S^*)^\frac{1}{2} \subset \text{dom} S_F$ which shows that $\text{ran} G_2 \supset \text{dom} T_{M(z)} = \text{dom} S$. It is easily seen that $S(S_F)^{-1} = I$ $\text{ran} S$ and, since $((S_F)^{-1}S^*)^* = S(S_F)^{-1}$, the operator

$$T_1 := (S_F)^{-1}S^*S(S_F)^{-1}$$

is nonnegative, nondensely defined and nonclosable. The closure of $T_1$ is a nonnegative self-adjoint relation given by

$$\text{clos} T_1 = I_{\text{ran} S} \oplus \{\{0\} \times \ker S^*\}.$$

On the other hand, the operator $T_0$,

$$T_0 f := (S_F)^{-1}S(S_F)^{-1}f = S^{-1}f, \quad f \in H,$$

is bounded and nondensely defined. Hence $M_2(z) := T_1 z + T_0$ as a nondensely defined unbounded operator function, whose closure

$$(5.66) \quad \text{clos} (M_2(z)) = z (T_1)^{**} + S^{-1} = (zI_{\text{ran} S} + P_{\text{ran} S} S^{-1}) \oplus \{\{0\} \times \ker S^*\}$$

is a Nevanlinna family in the Hilbert space $H$.

It should be noted that here the corresponding $\gamma$-fields would be $\gamma(z) \equiv S$ and $\gamma_2(z) \equiv S(S_F)^{-1} = I | \text{ran} S$, and this last one is closed but nondensely defined; cf. Remark 5.30.

(ii) As a special case consider $S = -D^2$, $\text{dom} S = H^2_0[0,1]$. Then

$$S^*S = (S^2)_F = D^4, \quad \text{dom} S^*S = H^4[0,1] \cap H^2_0[0,1],$$

and the operator valued function $M(\cdot)$ given by (5.65) is a Nevanlinna function.

It is easily seen that

$$\quad (S_F)^{-1}f = (x - 1) \int_0^x tf(t)dt + x \int_x^1 (t - 1)f(t)dt, \quad f \in H^0[0,1] = L^2[0,1],$$

and

$$\quad S^*S(S_F)^{-1}f = f^\prime, \quad \text{dom} (S^*S(S_F)^{-1}) = \{f \in H^2[0,1] : f \perp 1, f \perp t\}.$$

Finally,

$$\quad T_1 f = (S_F)^{-1}S^*S(S_F)^{-1}f = f(x) - (1 - x)f(0) - xf(1)$$

and $\text{dom} T_1 = \text{dom} (S^*S(S_F)^{-1}) = \{f \in H^2[0,1] : f \perp 1, f \perp t\}$.

Clearly, $T_1$ is nondensely defined and nonclosable in $L^2[0,1]$. On the other hand, the operator $T_0$,

$$\quad T_0 f := (S_F)^{-1}S(S_F)^{-1}f = S^{-1}f, \quad \text{dom} T_0 = \{f \in L^2[0,1] : f \perp 1, f \perp t\}.$$

is nondensely defined while it is bounded.

Thus, $M_1(z) = z T_1 + T_0$ is not a Nevanlinna function. However, its closure is a Nevanlinna family of the form (5.66) whose multivalued part is spanned by the functions $g_0(t) \equiv 1$ and $g_1(t) = t$ in $L^2[0,1]$.

Remark 5.41. The situation treated in Example 5.40 could be recovered with a slightly more general variant of Theorem 5.31 that would result from the following relaxed assumptions on the renormalizing operator: $G$ is bounded, its range satisfies $\text{ran} G \supset \overline{T}_{M(z)}$, and the renormalization of the $\gamma$-field, i.e. $\gamma(z)G$, should be bounded and densely defined.

Notice that the operator $G_2$ in Example 5.40 admits all these properties apart from the last one: $\gamma_2(z) \equiv S(S_F)^{-1} = I | \text{ran} S$ is closed but nondensely defined. To recover this it suffices to replace $G_2$ by some suitable bounded operator $\tilde{G}_2 = G_2 K$, where $K$ is e.g. the orthogonal
projection onto ran $S$ or a restriction to ran $S$: this would give a renormalization as in (5.60) where the multivalued part is projected away.

6. SOME CLASSES OF ES-GENERALIZED BOUNDARY TRIPLES

6.1. Transforms of $B$-generalized boundary triples. Let $\Gamma$ be an isometric relation from the Krein space $(\mathfrak{H}, J_0)$ to the Krein space $(H^2, J_H)$ and decompose $\Gamma = \{\Gamma_0, \Gamma_1\}$ according to the Cartesian decomposition of its range space $\mathcal{H} \times \mathcal{H}$ as in (3.6). Then the transposed boundary pair defined as a composition of two isometric relations via

$$\Gamma^\top = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \Gamma = \{\Gamma_1, -\Gamma_0\}$$

is again an isometric relation from $(\mathfrak{H}, J_0)$ to $(H^2, J_H)$ with $\text{dom } \Gamma^\top = \text{dom } \Gamma$. It is well known that in the particular case of an ordinary boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $A^*$, also the transposed boundary triple $\{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is an ordinary boundary triple for $A^*$. Moreover, if $W$ is any bounded $J_H$-unitary operator in the Krein space $(H^2, J_H)$, then the composition

$$\begin{pmatrix} \Gamma_0^W \\ \Gamma_1^W \end{pmatrix} = W \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$$

is also an ordinary boundary triple for $A^*$ and, conversely, all ordinary boundary triples of $A^*$ are connected via some $J_H$-unitary operator $W$ to each other in this way; see [36, 38].

The situation changes essentially when $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is not an ordinary boundary triple for $A^*$. In this section we treat the simplest case of a $B$-generalized boundary triple and show that a simple $J_H$-unitary transform can produce an ES-generalized boundary triple for $A^*$ whose Weyl function need not be domain invariant, however, according to Theorem 5.26 it is still form domain invariant.

The next result shows how an arbitrary $B$-generalized boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $A^*$, which is not an ordinary boundary triple, can be transformed to an ES-generalized boundary triple, whose $\gamma$-field becomes unbounded.

**Theorem 6.1.** Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a $B$-generalized boundary triple for $A^*$ with $A_* = \text{dom } \Gamma \subset A^*$, $A_* \neq A^*$, let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and $\gamma$-field, and let $A_0 = \ker \Gamma$. Then:

(i) for every fixed $\nu \in \mathbb{C} \setminus \mathbb{R}$ the transform

$$\begin{pmatrix} \Gamma_0^\nu \\ \Gamma_1^\nu \end{pmatrix} = \begin{pmatrix} -\text{Re } M(\nu) & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$$

defines a unitary boundary triple for $A^*$ whose Weyl function and $\gamma$-field are given by

$$M_\nu(\lambda) = -(M(\lambda) - \text{Re } M(\nu))^{-1}, \quad \gamma_\nu(\lambda) = \gamma(\lambda)(M(\lambda) - \text{Re } M(\nu))^{-1},$$

and, moreover, $M_\nu(\lambda)$ and $\gamma_\nu(\lambda)$ are unbounded operators for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(ii) $\{\mathcal{H}, \Gamma_0^\nu, \Gamma_1^\nu\}$ is an ES-generalized boundary triple for $A^*$ with $\text{dom } \Gamma^\nu = A_*$ and, hence, $M_\nu(\lambda)$ is form domain invariant and $\gamma_\nu(\lambda)$ is closable for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) the Weyl function $M_\nu(\cdot)$ (equivalently the $\gamma$-field $\gamma_\nu(\cdot)$) is domain invariant on $\mathbb{C} \setminus \mathbb{R}$ if and only if

$$\mathfrak{M}_\nu(A_*) \subset \text{ran } (A_{0,\nu} - \lambda) \quad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \mathbb{R},$$

where $A_{0,\nu} = \ker \Gamma_0^\nu$. 

Proposition 6.2. Since \{\mathcal{H}, \Gamma_0, \Gamma_1\} is a \(B\)-generalized boundary triple for \(A^*\), we have \(M \in \mathcal{R}^*[\mathcal{H}]\), see \[37\] Proposition 5.7, i.e., \(M\) is a strict Nevanlinna function whose values \(M(\lambda)\) are bounded operators on \(\mathcal{H}\) with \(\ker \text{Im} M(\lambda) = 0\) for every \(\lambda \in \rho(A_0)\). In particular, the real part \(\text{Re} M(\nu)\) is a bounded operator when \(\lambda \in \rho(A_0)\). Therefore, \(\Gamma^\nu\) is a standard \(J_\mathcal{H}\)-unitary transform of \(\Gamma\). According to \[38\] Proposition 3.11 this implies that \(\Gamma^\nu\) is a unitary boundary triple (a boundary relation in the terminology of \[38\]) with \(\text{dom} \Gamma^\nu = \text{dom} \Gamma\) whose Weyl function and \(\gamma\)-field are given by (6.2). The assumption \(A_* \neq A^*\) is equivalent to \(\text{ran} \Gamma \neq \mathcal{H}^2\) and therefore \(0 \not\in \rho(\text{Im} M(\lambda)), \lambda \in \rho(A_0)\); see \[37\] Section 2. It follows from (6.2) that

\[
M^\nu(\nu) = i(\text{Im} M(\nu))^{-1} = -M^\nu(\nu)^* 
\]

and then (3.13) shows that for all \(h, k \in \text{dom} M^\nu(\nu) = \text{dom} \gamma^\nu(\nu),\)

\[
(\nu - \tilde{\nu})(\gamma^\nu(\nu)h, \gamma^\nu(\nu)k)_\mathcal{H} = (M^\nu(\nu)h, k)_\mathcal{H} - (h, M^\nu(\nu)k)_\mathcal{H} = 2i((\text{Im} M(\nu))^{-1}h, k)_\mathcal{H}.
\]

Hence, \(M^\nu(\nu)\) and \(\gamma^\nu(\nu)\) are unbounded operators at the point \(\nu \in \mathbb{C} \setminus \mathbb{R}\). In this case \(M^\nu(\lambda)\) is an unbounded operator for all \(\nu \in \mathbb{C} \setminus \mathbb{R}\); see \[37\] Proposition 4.18.

Next consider the \(\gamma\)-field \(\gamma_\nu(\cdot)\). Since \(M(\lambda) - \text{Re} M(\nu)\) is bounded, it follows from (6.2) that

\[
\gamma_\nu(\lambda)^* = (M(\bar{\lambda}) - \text{Re} M(\nu))^{-1}\gamma(\lambda)^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

This combined with (6.4) shows that

\[
\gamma_\nu(\nu)^* = i(\text{Im} M(\nu))^{-1}\gamma(\nu)^*, \quad \gamma(\tilde{\nu})^* = -i(\text{Im} M(\nu))^{-1}\gamma(\nu)^*.
\]

Since

\[
\gamma(\nu)^*\gamma(\nu) = \gamma(\tilde{\nu})^*\gamma(\tilde{\nu}) = (\text{Im} \nu)^{-1}\text{Im} M(\nu),
\]

it follows from (6.6) that

\[
\text{ran} \gamma(\nu) \oplus \ker \gamma(\nu)^* \subset \text{dom} \gamma_\nu(\nu)^*, \quad \text{ran} \gamma(\tilde{\nu}) \oplus \ker \gamma(\tilde{\nu})^* \subset \text{dom} \gamma_\nu(\tilde{\nu})^*.
\]

Hence, \(\gamma_\nu(\nu)^*\) and \(\gamma_\nu(\tilde{\nu})^*\) are densely defined operators, which means that \(\gamma_\nu(\nu)\) and \(\gamma_\nu(\tilde{\nu})\) are closable operators. According to Theorem 5.26 \(A_{0, \nu} = \ker \Gamma_0^\nu\) is essentially selfadjoint and the assertions in (ii) hold. The fact that \(\gamma_\nu(\lambda)\) is an unbounded operator for every \(\lambda \in \mathbb{C} \setminus \mathbb{R}\) is seen e.g. from the identity (5.30) in Lemma 5.24. Thus, all the assertions in (i) are proven.

(iii) This assertion is obtained directly from Proposition 3.11.

Theorem 6.1 will now be specialized to a situation that appears often in system theory and in PDE setting where typically the underlying minimal symmetric operator \(A\) is nonnegative; the simplest situation occurs when the lower bound \(\mu(A)\) is positive. The first part of the next result follows the general formulation given in \[39\] Proposition 7.41 which was motivated by the papers of V. Ryzhov; see \[37\] and the references therein.

**Proposition 6.2.** Let \(A_0^{-1}\) and \(E\) be selfadjoint operators in \(\mathcal{H}\) and \(\mathcal{H}\), respectively, and let the operator \(G : \mathcal{H} \rightarrow \mathcal{H}\) be bounded and everywhere defined with \(\ker G = \{0\}\). Moreover, let

\[
A_* = \{A_0^{-1}f' + G\varphi, f' : f' \in \text{ran} A_0, \varphi \in \text{dom} E\}
\]

and define the operators \(\Gamma_0, \Gamma_1 : A_* \rightarrow \mathcal{H}\) by

\[
\Gamma_0\hat{f} = \varphi, \quad \Gamma_1\hat{f} = G^*f' + E\varphi; \quad \hat{f} = \{A_0^{-1}f' + G\varphi, f' \in A_*\}.
\]

Then:

(i) \(\{\mathcal{H}, \Gamma_0, \Gamma_1\}\) is an \(S\)-generalized boundary triple for \(A^* = A_*\) with \(\ker \Gamma_0 = A_0\). For \(\lambda \in \rho(A_0)\) and \(\varphi \in \text{dom} E\) the corresponding \(\gamma\)-field and the Weyl function are given by

\[
\gamma(\lambda)\varphi = (I - \lambda A_0^{-1})^{-1}G\varphi, \quad M(\lambda)\varphi = E\varphi + \lambda G^*(I - \lambda A_0^{-1})^{-1}G\varphi;
\]

(ii) \(\{\mathcal{H}, \Gamma_0, \Gamma_1\}\) is a \(B\)-generalized boundary triple for \(A^*\) if and only if \(E\) is a bounded selfadjoint operator;
(iii) \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) is an ordinary boundary triple for \( A^* \) if and only if \( E \) is bounded and \( G^*(\text{ran} A_0) = \mathcal{H} \); in particular, then \( \text{ran} G \) must be closed.

(iv) the transform \( \{\Gamma_1 - E\Gamma_0, -\Gamma_0\} \) defines an isometric boundary triple for \( A^* \) whose closure \( \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\} \) is a unitary boundary triple for \( A^* \) which is defined by

\[
(\Gamma_0, \Gamma_1) \hat{f} = \begin{pmatrix} I & 0 \\ E & G^* \end{pmatrix} \left( \begin{pmatrix} \varphi \\ f' \end{pmatrix} \right) = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G^* \end{pmatrix} \begin{pmatrix} \varphi \\ f' \end{pmatrix}
\]

and whose Weyl function and \( \gamma \)-field are given by

\[
M(\lambda) = -(M(\lambda))^{-1}, \quad \gamma(\lambda) = \gamma(\lambda)(M(\lambda))^{-1},
\]

where \( M(\lambda) = (M(\lambda) - E) = \lambda G^*(I - \lambda A_0^{-1})^{-1}G \) and \( \gamma(\lambda) = (I - \lambda A_0^{-1})^{-1}G \), and the corresponding transposed boundary triple is \( B \)-generalized with Weyl function \( M_0(\cdot) \).

(v) if \( \rho(A_0) = \{H, \Gamma_0, \Gamma_1\} \) is an ES-generalized boundary triple for \( A^* \) and it is \( S \)-generalized if and only if \( \text{ran} G \) is closed, or equivalently, \( \text{dom} \Gamma \) in (6.9) is closed, i.e., if and only if \( \{\mathcal{H}, \tilde{\Gamma}_0, \Gamma_1\} \) is an ordinary boundary triple for \( A^* \).

(vi) the Weyl function \( \tilde{M} \) (equivalently the \( \gamma \)-field \( \tilde{\gamma}(\cdot) \)) is domain invariant on \( \mathbb{C} \setminus \mathbb{R} \) if and only if

\[
\text{ran} \ P_G(I - \mu A_0^{-1})^{-1}G = \text{ran} \ P_G(I - \lambda A_0^{-1})^{-1}G \text{ for all } \lambda, \mu \in \mathbb{C} \setminus \mathbb{R},
\]

where \( P_G \) stands for the orthogonal projection onto \( \text{ran} \ G \).

Proof. (i) It was proved in [39, Proposition 7.41] that \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) is a unitary boundary triple for \( A^* = A_0^* \). It is clear from the definition of \( \Gamma_0 \) that \( \ker \Gamma_0 = A_0 \), which by assumption is a selfadjoint relation as an inverse of a selfadjoint operator. Hence, this boundary triple is \( S \)-generalized.

(ii) & (iii) The formula for \( \Gamma_0 \) shows that \( \text{ran} \Gamma_0 = \mathcal{H} \) precisely when \( \text{dom} E = \mathcal{H} \) or equivalently, \( E \) is bounded. Since

\[
\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} \hat{f} = \begin{pmatrix} I & 0 \\ E & G^* \end{pmatrix} \begin{pmatrix} \varphi \\ f' \end{pmatrix} = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G^* \end{pmatrix} \begin{pmatrix} \varphi \\ f' \end{pmatrix}
\]

and in the last product the triangular operator is bounded with bounded inverse when \( E \) is bounded, we conclude that \( \text{ran} \Gamma = \mathcal{H} \times \mathcal{H} \) if and only if \( \text{dom} E = \mathcal{H} \) and the diagonal operator in (6.12) is surjective, i.e., \( G^*(\text{ran} A_0) = \mathcal{H} \); in this case \( \text{ran} G^* = \mathcal{H} \) and \( \text{ran} G \) is closed.

(iv) It is clear from (6.8) that the transform \( \{\Gamma_1 - E\Gamma_0, -\Gamma_0\} \) has the same domain \( T \) as \( \Gamma \). Moreover, using (6.8) it is straightforward to check that the closure \( \{\tilde{\Gamma}_0, \tilde{\Gamma}_1\} \) is \( \text{clo} \{\Gamma_1 - E\Gamma_0, -\Gamma_0\} \) as given by (6.9). In fact, the transposed boundary triple \( \{\tilde{\Gamma}_1, -\tilde{\Gamma}_0\} \) is \( S \)-generalized and of the same form as \( \Gamma \) in (6.8) when \( E = 0 \), i.e., in view of (ii) it is even \( B \)-generalized. Applying (i) to this transposed boundary triple one also concludes that the Weyl function and \( \gamma \)-field of the boundary triple \( \{\tilde{\Gamma}_0, \tilde{\Gamma}_1\} \) are given by (6.10).

(v) It follows from (6.9) that

\[
\tilde{A}_0 = \{\{A_0^{-1}f' + G\varphi, f'\} : f' \in \ker A_0, \ \varphi \in \mathcal{H}\}.
\]

Using graph expressions one can write \( \tilde{A}_0 = A_0 \cap (\mathcal{H} \times \ker G^*) \) and (ran \( G \times \{0\} \) and now using the properties of adjoints it is seen that

\[
\tilde{A}_0^* = \text{clo} (A_0^* \text{ran} G \times \{0\}) \cap (\mathcal{H} \times \ker G^*).
\]

Observe that \( A_0 \cap (\text{ran} G \times \{0\}) = 0, \) since \( \ker A_0 = \{0\} \). If \( 0 \in \rho(A_0) \) then \( A_0^* \text{ran} G \times \{0\} \) is a closed subspace of \( \mathcal{H}^2 \) and this implies that \( \tilde{A}_0^* = A_0^* \). Hence, \( \tilde{A}_0 \) is essentially selfadjoint.
Remark 6.3. (i) The boundary triples $\Gamma$ and $\tilde{\Gamma}$ in (6.10) are completely determined by $M:\mathcal{H}\rightarrow\mathcal{H}$ and the condition (6.11) is satisfied. Indeed, in this case dom $M(\lambda) = \overline{\text{ran} \tilde{M}(\lambda)}$ is closed, or equivalently, dom $\tilde{\Gamma}$ in (6.9) is closed.

(ii) If $E$ is bounded and $G$ has closed range, then $\tilde{\Gamma} = \{\Gamma_1 - ET_0, -\Gamma_0\}$ is an ordinary boundary triple and the condition (6.11) is satisfied. Indeed, in this case dom $M(\lambda) = \overline{\text{ran} \tilde{M}(\lambda)}$ is closed, or equivalently, dom $\tilde{\Gamma}$ in (6.9) is closed.

(iii) The criterion (6.11) for domain invariance of $\tilde{M}$ can be derived also directly using dom $\tilde{M}(\lambda) = \overline{\text{ran} \tilde{M}(\lambda)}$ and the explicit formula for $M_0(\lambda)$ given in part (iv) of Proposition 6.2.

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Since $0 \in \rho(A_0)$, it is clear from (6.13) that $\text{ran} G$ is closed if and only if $\tilde{A}_0 = \ker \tilde{\Gamma}_0$ is closed, or equivalently, dom $\tilde{\Gamma}$ in (6.9) is closed.

(vi) Using for $\tilde{A}_0$ the formula in (6.13) and the equalities $\overline{\text{ran} (\text{dom} \tilde{\Gamma})} = \text{ran} \tilde{\gamma}(\mu) = \text{ran} \gamma(\mu)$ the domain invariance condition in Proposition 3.11 (i) can be rewritten as follows: for every $h \in \mathcal{H}$ there exist $h_0 \in \mathcal{H}$ and $f' \in \text{ran} A_0 \cap \ker G^*$ such that

\[(I - \mu A_0^{-1})^{-1}Gh = (I - \lambda A_0^{-1})f' + Gh_0\]

or, equivalently,

\[(I - \lambda A_0^{-1})^{-1}(I - \mu A_0^{-1})^{-1}Gh = f' + (I - \lambda A_0^{-1})^{-1}Gh_0,\]

$\mu, \lambda \in \mathbb{C} \setminus \mathbb{R}$. Applying resolvent identity to the product term it is seen that the previous condition is equivalent to

(6.14) \[(I - \mu A_0^{-1})^{-1}Gh = f'_1 + (I - \lambda A_0^{-1})^{-1}Gh_1,\]

for some $h_1 \in \mathcal{H}$ and $f'_1 \in \text{ran} A_0 \cap \ker G^*$. This condition is equivalent to the inclusion

\[\text{ran} P_G(I - \mu A_0^{-1})^{-1}G \subset \text{ran} P_G(I - \lambda A_0^{-1})^{-1}G.\]

Since $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ are arbitrary, this last condition coincides with the condition (6.11). \qed

Remark 6.3. (i) The boundary triples $\Gamma$ and $\tilde{\Gamma}$ are completely determined by $A_0 (= \ker \Gamma_0 = \ker \tilde{\Gamma}_0)$ and the operators $G$ and $E = E^*$. If, in particular, $0 \in \rho(A_0)$, then the Weyl function $\tilde{M}(\cdot)$ in (6.10) is form domain invariant (see Theorem 5.26) and the $\gamma$-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ as well as $\tilde{M}(\cdot)$ (in the resolvent sense) admit holomorphic continuations to the origin $\lambda = 0$ with

\[\gamma(0) = G, \quad M(0) = E.\]

If, in addition, $E$ is bounded and $G$ has closed range, then $\tilde{\Gamma} = \{\Gamma_1 - ET_0, -\Gamma_0\}$ is an ordinary boundary triple and the condition (6.11) is satisfied. Indeed, in this case dom $M(\lambda) = \overline{\text{ran} \tilde{M}(\lambda)}$ is closed, or equivalently, dom $\tilde{\Gamma}$ in (6.9) is closed.

(ii) If $E$ is bounded, no closure is needed in part (iv), i.e., $\tilde{\Gamma} = \{\Gamma_1 - ET_0, -\Gamma_0\}$. In this case, $\Gamma$ is a $\delta$-generalized boundary triple and Proposition 6.2 can be seen as an extension of Theorem 6.1 to a point on the real line. Here the results are formulated for $\nu = 0$. They can easily be reformulated also for $\nu \in \mathbb{R}$. In addition, for $\nu = \infty$ the results in Proposition 6.2 can be translated to analogous results when treating range perturbations (instead of the domain perturbations as in Proposition 6.2); for general background see [39, Section 7.5]. For $\nu = \infty$ the operator $E$ appears as the limit value $\tilde{M}(\infty)$, while $A_0$ and $A_*$ should be replaced by their inverses; see (6.13) below.

(iii) The criterion (6.11) for domain invariance of $\tilde{M}$ can be derived also directly using dom $\tilde{M}(\lambda) = \overline{\text{ran} \tilde{M}(\lambda)}$ and the explicit formula for $M_0(\lambda)$ given in part (iv) of Proposition 6.2.

If \{$\mathcal{H}, \Gamma_0, \Gamma_1$\} is not an ordinary boundary triple for $A^*$, then the condition (6.11) fails to hold in general. In particular, if $\text{ran} A_0 \cap \ker G^* = \{0\}$ (if e.g. $\ker G^* = \{0\}$), then the condition (6.11) is equivalent to

(6.15) \[\text{ran} (I - \mu A_0^{-1})^{-1}G = \text{ran} (I - \lambda A_0^{-1})^{-1}G \quad \text{for all} \ \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}.\]

Multiplying this identity from the left by $\frac{\mu}{\lambda}(I - \lambda A_0^{-1})$ it is seen that (6.15) implies

(6.16) \[\text{ran} (I - \mu A_0^{-1})^{-1}G \subset \text{ran} G \quad \text{for all} \ \mu \in \mathbb{C} \setminus \mathbb{R}.\]
Similarly it can be seen that (6.16) implies (6.15). Thus, if ran \( A_0 \cap \ker G^* = \{ 0 \} \) then \( \widetilde{M} \) is domain invariant if and only if the operator range ran \( G \) is invariant under the resolvent \( (I - \mu A_0^{-1})^{-1} \) for all \( \mu \in \mathbb{C} \setminus \mathbb{R} \).

**Corollary 6.4.** Let \( A_0 \) in Proposition 6.2 be a selfadjoint operator with \( \ker A_0 = \{ 0 \} \) and assume that ran \( A_0 \cap \ker G^* = \{ 0 \} \). If \( S = (A_\ast)^* \) is densely defined, then the Weyl function \( \widetilde{M}(\cdot) \) defined in (6.10) is not domain invariant.

**Proof.** Since ran \( A_0 \cap \ker G^* = \{ 0 \} \), \( \widetilde{M}(\cdot) \) is domain invariant if and only if (6.16) holds. In other words, for every \( \varphi \in \mathcal{H} \) there exists \( h \in \mathcal{H} \) such that \( (I - \mu A_0^{-1})^{-1}G\varphi = Gh \), or, equivalently,

\[
(I + \mu(A_0 - \mu^{-1})G)\varphi = Gh \quad \iff \quad \mu(A_0 - \mu)^{-1}G\varphi = G(h - \varphi).
\]

If \( A \) is densely defined, then \( A^* \supset A_\ast \) is an operator. Since ker \( A_0 = \{ 0 \} \) one concludes from (6.7) that \( A_\ast \) is an operator if and only if dom \( A_0 \cap \text{ran } G = \{ 0 \} \). This condition applied to (6.17) implies that \( \varphi = 0 \) and \( h - \varphi = 0 \), since ker \( G = \{ 0 \} \). This proves the claim. \( \Box \)

In the case that \( A_0 \) in Proposition 6.2 is nonnegative, one can specify further the type of the Weyl function as follows.

**Corollary 6.5.** Assume that in Proposition 6.2 \( A_0 = A_0^* \geq 0 \) and \( E = E^* \leq 0 \). Then the Weyl functions

\[
M(\lambda)\varphi = E\varphi + \lambda G^*(I - \lambda A_0^{-1})^{-1}G\varphi, \quad M_0(\lambda) = \lambda G^*(I - \lambda A_0^{-1})^{-1}G, \quad \lambda \in \rho(A_0),
\]

are domain invariant inverse Stieltjes functions, while the Weyl function \( \widetilde{M}(\cdot) = -M_0(\cdot)^{-1} \) in (6.10) is a Stieltjes function.

**Proof.** Since \( A_0 \) is a nonnegative selfadjoint operator with ker \( A_0 = \{ 0 \} \) and \( E = E^* \leq 0 \), the Weyl function \( M(\lambda) = E + \lambda G^*(I - \lambda A_0^{-1})^{-1}G \) admits a holomorphic continuation to the negative real line and, moreover,

\[
M(x) = E + xG^*(I - xA_0^{-1})^{-1}G = M(x)^* \leq 0 \quad \text{for all } x < 0.
\]

Hence, \( M(\cdot) \) and, in particular, \( M_0(\cdot) \) are an inverse Stieltjes functions with ker \( M(x) = \{ 0 \} \) and ker \( M_0(x) = \{ 0 \} \), since ker \( G = \{ 0 \} \). In view of \( (\widetilde{M}(\lambda) - \mu I)^{-1} = -(I + \mu M_0(\lambda))^{-1}M_0(\lambda) \) (\( \mu \in \mathbb{C} \setminus \mathbb{R} \)) also the function \( \widetilde{M}(\cdot) = -M_0(\cdot)^{-1} \) admits a holomorphic continuation to the negative real line with nonnegative selfadjoint values therein, i.e., it is a Stieltjes function. \( \Box \)

Let us also mention that analogously the function

\[
\widehat{M}(\lambda) = -M(1/\lambda) = G^*(A_0^{-1} - \lambda)^{-1}G - E
\]

admits a holomorphic continuation to the negative real line and

\[
\widehat{M}(x) = G^*(A_0^{-1} - x)^{-1}G - E = M(x)^* \geq 0 \quad \text{for all } x < 0
\]

with ker \( \widehat{M}(x) = \{ 0 \} \). Hence, \( \widehat{M}(\cdot) \) is a Stieltjes function and the transposed function \( \widehat{M}^\top(\cdot) = -\widehat{M}(\cdot)^{-1} \) is an inverse Stieltjes function. Observe, that \( \widehat{M}(\cdot) \) is the Weyl function corresponding to the boundary triple \( \{ \mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1 \} \) given by

\[
\widehat{\Gamma}_0\widehat{f} = \varphi, \quad \widehat{\Gamma}_1\widehat{f} = -G^*f' - E\varphi; \quad \widehat{f} = \{ f', A_0^{-1}f' + G\varphi \} \in T^{-1}
\]

with ker \( \widehat{\Gamma}_0 = A_0^{-1} \) and ker \( \widehat{\Gamma}_1 = A_1^{-1} \).

We now assume that \( 0 \in \rho(A_0) \) and make explicit the renormalization for the ES-generalized boundary triple \( \{ \mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1 \} \) in Proposition 6.2(v). This also yields a representation for the form domain invariant Weyl function \( \widehat{M}(\cdot) \) in (6.10). To state the result decompose the bounded
inverse $A_0^{-1}$ according to $\mathfrak{H} = \overline{\text{ran}}\ G \oplus (\text{ran} G)^\perp$ as $A_0^{-1} = (A_{ij})_{i,j=1}^2$. This generates the following expression for an associated Schur complement of the resolvent $(A_0^{-1} - 1/\lambda)^{-1}$,

$$S_0(\lambda) = A_{11} - 1/\lambda I - (A_{21})^\ast (A_{22} - 1/\lambda I)^{-1} A_{21}, \quad \lambda \in \rho(A_0).$$

(6.19)

**Theorem 6.6.** Let the notations and assumptions be as in Proposition 6.2, let $A_0$ be a selfadjoint operator with $0 \in \rho(A_0)$ and assume that $\text{ran} G$ is not closed, so that the ES-generalized boundary triple $\{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ is not an ordinary boundary triple. Then:

(i) the closure of the $\gamma$-field $\tilde{\gamma}$ satisfies $\text{dom} \tilde{\gamma}(\lambda) = \text{ran} G^\ast$, $\lambda \in \rho(A_0)$;

(ii) the renormalized boundary triple $\{\overline{\text{ran}}\ G, \Gamma_{0,r}, \Gamma_{1,r}\}$ is an ordinary boundary triple for $A^\ast = A_0 \overset{\ast}{\hat{+}} (\overline{\text{ran}}\ G \times \{0\})$ and is determined by

$$\begin{pmatrix} \tilde{\Gamma}_{0,r} \\ \tilde{\Gamma}_{1,r} \end{pmatrix} (f + h) = \begin{pmatrix} P_G A_0 f \\ -h \end{pmatrix}, \quad f \in \text{dom} A_0, \quad h \in \overline{\text{ran}}\ G,$$

where $P_G$ denotes the orthogonal projection onto $\ker A^\ast = \overline{\text{ran}}\ G$;

(iii) the Weyl function $M_r(\cdot)$ of the renormalized boundary triple coincides with the Schur complement in (6.19),

$$M_r(\lambda) = S_0(\lambda), \quad \lambda \in \rho(A_0)$$

and the form domain invariant Weyl function $\tilde{M}(\cdot)$ in (6.10) has the form

$$\tilde{M}(\lambda) = G^{-1} S_0(\lambda) G^{(-s)}, \quad \lambda \in \rho(A_0),$$

where $G^{(-s)}$ is the adjoint when $G$ is treated as an operator from $\mathcal{H}$ into $\overline{\text{ran}}\ G$.

**Proof.** (i) By Proposition 6.2, $\tilde{\gamma}(\lambda) = \gamma(\lambda)(M_0(\lambda))^{-1}$. Using the expressions for $M_0(\lambda)$ in (6.10) and $S_0(\lambda)$ in (6.19), one obtains

$$\tilde{M}(\lambda) = G^{-1} S_0(\lambda) G^{(-s)}, \quad \tilde{\gamma}(\lambda) = -(I - \lambda A_0^{-1})^{-1} I_{\text{ran} G} S_0(\lambda) G^{(-s)},$$

where $G^{(-s)}$ stands for the inverse of $G^\ast$, when $G^\ast$ is treated as an injective mapping from $\overline{\text{ran}}\ G$ to $\mathcal{H}$. Since $(I - \lambda A_0^{-1})^{-1}, I_{\text{ran} G},$ and $S_0(\lambda)$ are bounded and bounded inverse for $\lambda \in \rho(A_0)$, we conclude that the form domain of $M(\lambda)$ is equal to $\text{ran} G^\ast$ and that the closure of the $\gamma$-field is given by

$$\tilde{\gamma}(\lambda) = \frac{1}{\lambda} (A_0^{-1} - I)^{-1} P_G S_0(\lambda) G^{(-s)} = \frac{1}{\lambda} \left( -(A_{22} - I - \lambda I)^{-1} A_{21} \right) G^{(-s)}, \quad \lambda \in \rho(A_0).$$

Here the last identity uses the standard block formula for the inverse $(A_0^{-1} - 1/\lambda)^{-1}$.

(ii) The assumption $0 \in \rho(A_0)$ implies that the closure $A^\ast$ of $T$ is $A^\ast = A_0 \overset{\ast}{\hat{+}} (\overline{\text{ran}}\ G \times \{0\})$. In view of (i) one can use $G^\ast : \overline{\text{ran}}\ G \to \mathcal{H}$ as the renormalizing operator in Theorem 5.31. Since $A_0$ is an operator one can rewrite the renormalization of the boundary triple (6.9) in the form $\{\overline{\text{ran}}\ G, \Gamma_{0,r}, \Gamma_{1,r}\}$, where

$$\begin{pmatrix} \Gamma_{0,r} \\ \Gamma_{1,r} \end{pmatrix} \hat{f} = \begin{pmatrix} P_G A_0 f \\ -G \varphi \end{pmatrix}, \quad \hat{f} \in \{f + G \varphi, A_0 f : f \in \text{dom} A_0, \varphi \in \mathcal{H}\}.$$

The final expression for the renormalized boundary triple is obtained by taking closure in (6.24); this leads to (6.22), since $0 \in \rho(A_0)$. Now it is clear that $\text{dom} \Gamma_r = A^\ast$ and $\text{ran} \Gamma_r = \overline{\text{ran}}\ G \times \overline{\text{ran}}\ G$, i.e., $\{\overline{\text{ran}}\ G, \Gamma_{0,r}, \Gamma_{1,r}\}$ is an ordinary boundary triple for $A^\ast$.

(iii) This is obtained from (6.22). In particular, the equality $M_r(\lambda) = S_0(\lambda)$ follows by taking closure of $G M(\lambda) G^\ast | \overline{\text{ran}}\ G$. \qed
According to Theorem 6.6 $A_{0,r} = \ker \Gamma_{0,r}$ is selfadjoint. Clearly, $A_{0,r}$ coincides with the closure of $A_0 = \ker \bar{\Gamma}_0$ in Proposition 6.2 see (6.13). If, in particular, $A_0$ is strictly positive, then $A_{0,r} = \ker \Gamma_{0,r}$ is the Krein-von Neumann extension $A_K$ of $A$ and we have the following identities

$$\ker \Gamma_0 = A_0 = A_{0,r} = \ker \Gamma_{0,r} = S \hat{\oplus} (\text{ran } G \times \{0\}) = A_K,$$

where $A$ is the range restriction of $A_0$: $A = \{(f, A_0 f) : f \in \text{dom } A_0, G^* A_0 f = 0 \}$. Observe, that $A$ is densely defined if and only if $A^*$ is an operator, i.e.,

$$\text{dom } A = \mathcal{H} \iff \text{ran } G \cap \text{dom } A_0 = \{0\}.$$

By (6.21) $\tilde{M}(\cdot)$ is domain invariant if and only if the dense set $S_0(\lambda)^{-1}(\text{ran } G)$ does not depend on $\lambda$; in the particular case $\ker G^* = \{0\}$ this also leads to Corollary 6.4.

In Proposition 6.2 we regularized the $S$-generalized boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ via the transform $\{\Gamma_0, \Gamma_1 - ET_0\}$ before transposing the mappings and closing up. In fact, the closure of this regularized boundary triple $\lim \{\Gamma_0, \Gamma_1 - ET_0\}$ is of the same form as $\Gamma$ in (6.8) with $E = 0$ and it is $B$-generalized; see item (iv) in Proposition 6.2.

The next example shows what happens for the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ in Proposition 6.2 if it is transposed without the indicated regularization of the mapping $\Gamma_1$.

**Example 6.7.** Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triple as defined in (6.8). Then

$$A_1 = \{(A_0^{-1} f' + G \varphi, f') : G^* f' + E \varphi = 0, f' \in \text{ran } A_0, \varphi \in \text{dom } E\}$$

and $A = T^* = \ker \Gamma$ is given by

$$A = \{(A_0^{-1} f', f') : f' \in \ker G^*\} = \{(f, f') \in A_0 : f' \in \ker G^*\}.$$  

If, in particular, $A_0$ is an operator then $A$ is a standard range restriction of $A_0$ to $\ker G^*$. The defect numbers of $A$ are equal to $\dim (\text{ran } G)$.

Now, assume that $\ker E = \{0\}$ and $\text{ran } G^* \cap \text{ran } E = \{0\}$. Then the identity $G^* f' + E \varphi = 0$ implies that $G^* f' = E \varphi = 0$ and, consequently, $\varphi = 0$ and this means that $A_1 = A$. This means that $A_1$ is not essentially selfadjoint and thus the transposed boundary triple $\{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is not ES-generalized. The corresponding Weyl function is given by

$$M^T(\lambda) = -(E + \lambda G^*(I - \lambda A_0^{-1})^{-1} G)^{-1}$$

and according to Theorem 5.2 it cannot be form domain invariant.

If, in addition, $\ker G^* = \{0\}$, then

$$\text{dom } M^T(\lambda) \cap \text{dom } M^T(\mu) = \{0\}, \text{ for all } \lambda \neq \mu, \lambda, \mu \in \rho(A_0).$$

To see this assume that $g = (E + \lambda G^*(I - \lambda A_0^{-1})^{-1} G) f_1 = (E + \mu G^*(I - \mu A_0^{-1})^{-1} G) f_2$ holds for some $g, f_1, f_2 \in \mathcal{H}$. Then

$$E(f_2 - f_1) = G^* [\lambda(I - \lambda A_0^{-1})^{-1} G f_1 - \mu(I - \mu A_0^{-1})^{-1} G f_2]$$

and the assumptions $\text{ran } G^* \cap \text{ran } E = \{0\}$ and $\ker E = \{0\}$ imply $f_1 = f_2$. Now $\ker G^* = \{0\}$ and an application of the resolvent identity on the righthand side of (6.26) yields $g = 0$.

If, in particular, $A_0$ is a nonnegative selfadjoint operator with $\ker A_0 = \{0\}$ and $E \leq 0$, then the function $M(\cdot)$ is an inverse Stieltjes function and the transposed function $M^T(\cdot) = -M(\cdot)^{-1}$ is a Stieltjes function, which need not be form domain invariant; cf. Corollary 6.5.

Analogously the function

$$-M(1/\lambda) = G^*(A_0^{-1} - \lambda)^{-1} G - E$$

is a Stieltjes function and the transposed function $\tilde{M}(1/\lambda)^{-1}$ is an inverse Stieltjes function, which need not be form domain invariant.
Finally, it should be mentioned that later, in Section 6.1, it is shown how the standard Dirichlet and Neumann trace operators on smooth, as well as on Lipschitz, domains can be included in the abstract boundary triple framework constructed in Proposition 6.2; hence all the previous results on them will have immediate applications in concrete PDE setting.

6.2. ES-generalized boundary triples and graph continuity of a component mapping. It is known that for a boundary triple \( \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) (as well as for a boundary pair \( \{ \mathcal{H}, \Gamma \} \)) to be an ordinary boundary triple it is necessary and sufficient that both boundary mappings \( \Gamma_0 \) and \( \Gamma_1 \) are continuous on \( A^* \) (with the graph norm on \( \text{dom} \ A^* \) in case \( A \) is densely defined). In general the mappings \( \Gamma_0 \) and \( \Gamma_1 \) both can be unbounded when \( \dim \mathcal{H} = \infty \). In this section we establish analytic criteria for \( \Gamma_0 \) or \( \Gamma_1 \) to be continuous with the aid of the associated Weyl function. Recall that the kernels \( A_0 = \ker \Gamma_0 \) and \( A_1 = \ker \Gamma_1 \) are always symmetric and it is possible that \( A_0 = A \) or \( A_1 = A \); see e.g. Example 6.7.

The next result characterizes boundedness of the mapping \( \Gamma_1 \) for an ES-generalized boundary triple.

**Proposition 6.8.** For a unitary boundary triple \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) with \( A_* = \text{dom} \gamma \) the following conditions are equivalent:

(i) \( A_0 = \ker \Gamma_0 \) is essentially selfadjoint and \( \Gamma_1 \) is a bounded operator (w.r.t. the graph norm) on \( A_* \);

(ii) \( A_0 \) is selfadjoint and the restriction \( \Gamma_1 \mid \mathcal{R}_\lambda(A_*) \) is a bounded operator for some (equivalently for every) \( \lambda \in \mathbb{C} \setminus \mathbb{R} \);

(iii) the form associated with \( \text{Im} (\sim M^{-1}(\lambda)) \) has a positive lower bound for some (equivalently for every) \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

If one of these conditions is satisfied, then the triple \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) is \( B \)-generalized.

**Proof.** (i) \( \Rightarrow \) (ii) If \( \Gamma_1 \) is bounded, the also the restrictions \( \Gamma_1 \mid A_0 \) and \( \Gamma_1 \mid \mathcal{R}_\lambda(A_*) \) are bounded. Now by Corollary 5.7 \( A_0 \) is closed and, therefore, \( A_0 = A_0^* \).

(ii) \( \Rightarrow \) (iii) Observe that \( (\Gamma_1 \mid \mathcal{R}_\lambda(A_*))^{-1} = \gamma^\top(\lambda) \) is the \( \gamma \)-field of the transposed boundary triple \( \Pi^\top = \{ \mathcal{H}, \Gamma_1, -\Gamma_0 \} \). Hence the condition that \( \Gamma_1 \mid \mathcal{R}_\lambda(A_*) \) is bounded means that \( (\gamma^\top(\lambda))^\gamma^\top(\lambda) \) has a positive lower bound or, equivalently, that the form corresponding to \( \text{Im} (\sim M^{-1}(\lambda)) \) has a positive lower bound (cf. (8.9) and Definition 5.22).

(iii) \( \Rightarrow \) (i) As shown in the previous implication, the assumption on \( \text{Im} (\sim M^{-1}(\lambda)) \) means that the restriction \( \Gamma_1 \mid \mathcal{R}_\lambda(A_*) \) is bounded. On the other hand, if the form corresponding to \( \text{Im} (\sim M^{-1}(\lambda)) \) has a positive lower bound, say \( c > 0 \), then

\[
\|M^{-1}(\lambda)f\|_{\mathcal{K}} \leq c \|f\|_{\mathcal{K}}.
\]

Consequently, \( \|M(\lambda)\| \leq c^{-1} \), i.e., \( M(\cdot) \) is a bounded Nevanlinna function. Now by Theorem 1.1, \( A_0 \) is selfadjoint and hence according to Corollary 5.7 the restriction \( \Gamma_1 \mid A_0 \) is bounded. Moreover, by selfadjointness of \( A_0 \), one has the decomposition \( A_* = A_0 + \mathcal{R}_\lambda(A_*) \). Since the angle between \( A_0 \) and \( \mathcal{R}_\lambda(A_*) \) is positive, one concludes that \( \Gamma_1 \) is bounded on \( A_* \). This completes the proof of the implication.

Finally, if one of the equivalent conditions (i)–(iii) holds then, as shown above, \( M(\cdot) \) is a bounded Nevanlinna function. This is a necessary and sufficient condition for the boundary triple \( \Pi \) to be \( B \)-generalized. \( \square \)

By passing to the transposed boundary triple gives the following analog of Proposition 6.8.

**Proposition 6.9.** For a unitary boundary triple \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) with \( A_* = \text{dom} \gamma \) the following conditions are equivalent:
(i) $A_1 = \ker \Gamma_1$ is essentially selfadjoint and $\Gamma_0$ is a bounded operator (w.r.t. the graph norm) on $A_s$;
(ii) $A_1$ is selfadjoint and the restriction $\Gamma_0|\tilde{\mathcal{R}}_\lambda(A_\ast)$ is a bounded operator for some (equivalently for every) $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(iii) the form associated with $\mathrm{Im} M(\lambda)$ has a positive lower bound for some (equivalently for every) $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

If one of these conditions is satisfied, then the transposed boundary triple $\Pi^\top = \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is $B$-generalized.

**Remark 6.10.** (i) For infinite direct sums of ordinary boundary triples the extensions $A_j = \ker \Gamma_j$, $j = 1, 2$, are automatically essentially selfadjoint; see [69, Theorem 3.2]. If, in addition, $\Gamma_1$ is bounded, then $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a $B$-generalized boundary triple for $A^\ast$ by Proposition 6.8; this implication was proved in another way in [69, Proposition 3.6]; see also Corollary 8.6 below.

(ii) Note that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a $B$-generalized boundary triple if and only if the composition $\Gamma_1 \tilde{\gamma}(\lambda) (= M(\lambda))$ is bounded for some (equivalently for all) $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In particular, if $\Gamma_1 \tilde{\gamma}(\lambda)$ is bounded, then also the $\gamma$-field $\gamma(\lambda)$ itself is bounded (see (3.9)), $A_0 = A_\ast^\ast$ (by Theorem 1.11) and the restriction $\Gamma_1 | A_0$ is also bounded (by Corollary 5.4). However, in this case $\Gamma_1$ need not be bounded. Therefore, the conditions in Proposition 6.8 are sufficient, but not necessary, for $\Pi$ to be a $B$-generalized boundary triple. An example is any $B$-generalized boundary triple $\Pi$, which is not an ordinary boundary triple, such that also the transposed boundary triple $\Pi^\top$ is $B$-generalized, since then $\Pi^\top$ cannot be an ordinary boundary triple. Then the second condition in (iii) of Proposition 6.8 is not satisfied. For an explicit example of such a $B$-generalized boundary triple, see for instance the direct sum of Dirac operators treated below in Proposition 8.17.

The boundedness of the component mappings $\Gamma_0$ and $\Gamma_1$ can be used to derive the following new characterization of ordinary boundary triples.

**Proposition 6.11.** For a unitary boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with $A_\ast = \mathrm{dom} \Gamma$ the following conditions are equivalent:

(i) $\Gamma_0$ is bounded and $\mathrm{ran} \Gamma_0 = \mathcal{H}$;
(ii) $\Gamma_1$ is bounded and $\mathrm{ran} \Gamma_1 = \mathcal{H}$;
(iii) $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple.

**Proof.** (i) $\Rightarrow$ (iii) By Corollary 5.10 $\mathrm{ran} \Gamma_0 = \mathcal{H}$ implies that $A_0 = A_\ast^\ast$ and $\Pi$ is a $B$-generalized boundary triple. In particular, the Weyl function $M(\cdot)$ of $\Pi$ belongs to the class $R^u[\mathcal{H}]$ of bounded strict Nevanlinna functions. On the other hand, $(\Gamma_0|\tilde{\mathcal{R}}_\lambda(A_\ast))^{-1} = \tilde{\gamma}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now $\Gamma_0|\tilde{\mathcal{R}}_\lambda(A_\ast)$ is bounded and this means that $\gamma(\lambda)^\ast \gamma(\lambda)$ has a positive positive lower bound or, equivalently, that $0 \in \rho(\mathrm{Im} M(\lambda))$. Hence, $M(\cdot) \in R^u[\mathcal{H}]$ and $\Pi$ is an ordinary boundary triple; see [37, Proposition 2.18].

(ii) $\Rightarrow$ (iii) Apply the previous implication to the transposed boundary triple.

(iii) $\Rightarrow$ (i), (ii) This is clear, since for ordinary boundary triple $\Gamma : A_\ast \to \mathcal{H}^2$ is bounded and surjective. \hfill \qed

6.3. Real regular points and analytic extrapolation principle for Weyl functions.

The main result here contains an analytic extrapolation principle for Weyl functions in the case when the underlying minimal operator $A$ admits a regular type point on the real line $\mathbb{R}$. The main tool for getting this result relies on the main transform of boundary relations (called here boundary pairs) that was introduced in [37]. The result is partially motivated by
the analytic criterion for an isometric boundary triple to be unitary which can be found in [37, Proposition 3.6] and [39, Theorem 7.51].

The main transform makes a connection between subspaces of the Hilbert space \((\mathcal{H} \oplus \mathcal{H})^2\) and linear relations from the Krešn space \((\mathcal{H}^2, J_\mathcal{H})\) to the Krešn space \((\mathcal{H}^2, J_\mathcal{H})\). It is a linear mapping \(\mathcal{J}\) from \(\mathcal{H}^2 \times \mathcal{H}^2\) to \((\mathcal{H} \oplus \mathcal{H})^2\) defined by the formula

\[
\mathcal{J} : \left\{ \left( \frac{f}{f'} \right), \left( \frac{h}{h'} \right) \right\} \mapsto \left\{ \left( \frac{f}{h} \right), \left( \frac{f'}{-h'} \right) \right\}, \quad f, f' \in \mathcal{H}, \: h, h' \in \mathcal{H}.
\]

The mapping \(\mathcal{J}\) establishes a one-to-one correspondence between the (closed) linear relations \(\mathcal{A}\) in \(\mathcal{H} = \mathcal{H} \oplus \mathcal{H}\) via

\[
\Gamma \mapsto \tilde{A} := \mathcal{J}(\Gamma) = \left\{ \left( \frac{f}{h} \right), \left( \frac{f'}{-h'} \right) \right\} : \left\{ \left( \frac{f}{f'} \right), \left( \frac{h}{h'} \right) \right\} \in \Gamma \}
\]

According to [37, Proposition 2.10] the main transform \(\mathcal{J}\) establishes a one-to-one correspondence between the sets of contractive, isometric, and unitary relations \(\Gamma\) from \((\mathcal{H}^2, J_\mathcal{H})\) to \((\mathcal{H}^2, J_\mathcal{H})\) and the sets of dissipative, symmetric, and selfadjoint relations \(\tilde{A}\) in \(\mathcal{H} \oplus \mathcal{H}\), respectively. Recall that a boundary pair \(\{\mathcal{H}, \Gamma\}\) is called minimal, if

\[
\mathcal{H} = \mathcal{H}_{\text{min}} := \text{span} \{ \mathcal{N}_\lambda(\mathcal{A}_\bullet) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \}.
\]

The next result shows usefulness of the main transform for analytic extrapolation of Weyl functions \(M(\cdot)\) from a single real point \(x \in \mathbb{R}\) to the complex plane, when \(x\) is a regular type point of the minimal operator \(A\). In the special case when the analytic extrapolation of \(M(x)\) is a uniformly strict Nevanlinna function the extrapolation principle formulated for Weyl functions in the next theorem, yields a solution to the following general inverse problem: given a pair \(\{\Gamma_0, \Gamma_1\}\) of boundary mappings from \(A^*\) to \(\mathcal{H}\) determine the selfadjoint extension \(\mathcal{A}_\Theta\) of \(A\) (up to unitary equivalence) when the boundary condition \(\Gamma_1 \tilde{f} = \Theta \Gamma_0 \tilde{f}\) is fixed by some operator \(\Theta\) acting on \(\mathcal{H}\). It is emphasized that this result arises basically from the stated regularity claim for \(M(x)\) at the single point \(x \in \rho(A)\).

**Theorem 6.12.** Let \(\{\Gamma, \mathcal{H}\}\) be an isometric boundary pair for \(A^*\) with domain \(A_\bullet = \text{dom}\, \Gamma\), \(\text{cl} A_\bullet = A^*\), (i.e. Green’s identity (1.1) holds for \(f, \tilde{g} \in A_\bullet\)), let \(\tilde{A} = \mathcal{J}(\Gamma)\) be the main transform (6.27) of \(\Gamma\). Assume that there exists a selfadjoint extension \(\mathcal{H} \subset A_\bullet = \text{dom} \, \Gamma\) of \(A\) with \(x \in \rho(\mathcal{H}) \cap \mathbb{R}\) and let the mapping \(M(x)\) at this point \(x\) be defined by (3.8). Then the following assertions hold:

(i) The following two conditions are equivalent:

(a) \(M(x)\) is selfadjoint in \(\mathcal{H}\) and \(0 \in \rho(\mathcal{M}(x) + xI)\);

(b) \(x \in \rho(\tilde{A}) \cap \mathbb{R}\).

(ii) If the conditions (a), (b) in (i) hold then \(\{\Gamma, \mathcal{H}\}\) is a unitary boundary pair for \(A^*\) and \(M(x)\) admits an analytic extrapolation from the point \(x\) to the half-planes \(\mathbb{C}_\pm\) as the Weyl family \(M(\cdot)\) which necessarily belongs to the class \(\mathcal{R}(\mathcal{H})\) of Nevanlinna families.

(iii) If the boundary pair \(\{\Gamma, \mathcal{H}\}\) is minimal then all the intermediate extensions \(\mathcal{A}_\Theta\) of \(A\) given by (1.2) are, up to unitary equivalence, uniquely determined by \(M(\cdot)\).

**Proof.** (i) To prove the equivalence of (a) and (b) consider the main transform \(\tilde{A}\) of \(\Gamma\) in (6.27). The range of \(\tilde{A} - xI\) is given by

\[
\text{ran} (\tilde{A} - xI) = \left\{ \left( \frac{f'}{x} - \frac{f}{x} \right), \left( \frac{h'}{x} - \frac{h}{x} \right) \right\} : \left\{ \left( \frac{f}{f'} \right), \left( \frac{h}{h'} \right) \right\} \in \Gamma \}
\]
For \( f'_x \in \mathfrak{N}_x(A_\ast) \) one has \( f'_x = x f_x \) and \( \{ h, h' \} \in M(x) \) by the definition in (3.8). Since \(-x \in \rho(M(x))\) and \(-h' - x h \in \text{ran}(-M(x) - x I) = \mathcal{H}\), it follows from (6.28) that
\[
(6.29) \quad \begin{pmatrix} 0 \\ \mathcal{H} \end{pmatrix} \subset \text{ran}(\hat{A} - x I).
\]
Since \( H \subset \text{dom}\Gamma \) and \( x \in \rho(H) \cap \mathbb{R} \) one has \( \text{ran}(H - x) = \mathfrak{H} \) which combined with (6.28) and (6.29) shows that \( \text{ran}(\hat{A} - x I) = \mathfrak{H} \). This implies that \( \hat{A} \) is a selfadjoint relation in \( \mathfrak{H} \), since \( \hat{A} \) is symmetric by isometry of \( \Gamma \); cf. [37, Proposition 2.10]. In particular, \( x \in \rho(\hat{A}) \cap \mathbb{R} \).

(b) \( \Rightarrow \) (a) If \( x \in \rho(\hat{A}) \cap \mathbb{R} \) then \( \text{ran}(\hat{A} - x I) = \mathfrak{H} \) and, in particular, (6.29) is satisfied. In view of (6.27) and (3.8) this means that \( \{ f, f' \} \in \mathfrak{N}_x(A_\ast) \) and \( \{ h, h' \} \in M(x) \) and therefore \( \text{ran}(-M(x) - x I) = \mathcal{H} \). On the other hand, it follows from (3.8) and Green’s identity (1.1) that \( M(x) \) is symmetric, i.e., \( (h', h) = (h, h') \) for all \( \{ h, h' \} \in M(x) \). Therefore, \( M(x) \) is selfadjoint and \(-x \in \rho(M(x))\).

(ii) The proof of (i) shows that if (a) or, equivalently, (b) holds then \( \hat{A} \) is a selfadjoint relation in \( \mathfrak{H} \). Then, equivalently, its (inverse) main transform \( \{ \Gamma, \mathcal{H} \} \) is a unitary boundary pair for \( A' \). By the main realization result stated in Theorem 3.3 we conclude that \( M \in \mathcal{R}(\mathcal{H}) \).

(iii) To prove this assertion first recall that according to the main realization result in [37, Theorem 3.9] (cf. Theorem 3.3) the Weyl function uniquely determines the \( \Gamma \), as well as \( \hat{A} \), by the minimality of \( \Gamma \). Uniqueness of \( \Gamma \) here means that if there exists another minimal boundary pair \( \{ \mathcal{H}, \hat{\Gamma} \} \) associated with the symmetric operator \( \hat{A} = \ker \Gamma \) in some Hilbert space \( \mathfrak{H} \) having the same Weyl function \( M(\cdot) \), then there exists a standard unitary operator \( U : \mathfrak{H} \to \mathfrak{H} \) such that
\[
(6.30) \quad \hat{\Gamma} = \left\{ \left\{ \begin{pmatrix} U f \\ U f' \end{pmatrix}, \begin{pmatrix} h' \\ h \end{pmatrix} \right\}, \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h' \\ h \end{pmatrix} \right\} \in \Gamma \right\}.
\]
Hence, if the extension \( A_{\Theta} \) of \( A \) in the Hilbert space \( \mathfrak{H} \) and the extension \( \hat{A}_{\Theta} \) of \( \hat{A} \) in the Hilbert space \( \hat{\mathfrak{H}} \) are associated with the same “boundary condition” \( \Theta \) via (1.2) then (6.30) implies that
\[
\hat{A}_{\Theta} = \left\{ \{ U f, U f' \} : \{ f, f' \} \in A_{\Theta} \right\} = U A_{\Theta} U^{-1}.
\]
This means that the linear relations \( A_{\Theta} \) and \( \hat{A}_{\Theta} \) are unitarily equivalent via the same unitary operator \( U \) for every linear relation \( \Theta \) in \( \mathcal{H} \).

\[\square\]

**Remark 6.13.** The proof of item (i) in Theorem 6.12 shows that (b) \( \Rightarrow \) (a) without the assumption on the existence of a selfadjoint extension \( H \subset A_\ast \), with \( x \in \rho(H) \).

As to items (ii) and (iii) of Theorem 6.12 it should be mentioned that if the analytic extrapolation \( M(\cdot) \) belongs to the class \( \mathcal{R}^u[\mathcal{H}] \), then the spectrum \( \sigma(A_{\Theta}) \) of every selfadjoint extension \( A_{\Theta} (\Theta = \Theta^* ) \) of \( A \) can be completely characterized by the spectral function of the corresponding Weyl function \( M(\cdot) \in \mathcal{R}[\mathcal{H}] \) which is obtained via a fractional linear transforms from the function \( M(\cdot) \). For details we refer to [43, 44, 36, 38]. Some further developments concerning uniqueness of boundary triples and connections between \( \sigma(A_{\Theta}) \) and the spectral functions \( \Sigma(t) \) of the associated Weyl functions can be found in [56].

Theorem 6.12 offers also a useful analytic tool to check whether an isometric boundary triple (or boundary pair) is actually unitary or, equivalently, if the Weyl function of some isometric boundary triple is in fact from the class \( \mathcal{R}(\mathcal{H}) \) of Nevanlinna functions. We use this result to construct a unitary boundary pair for Laplacians defined on rough domains in Section 7.4 and to associate unitary boundary triples with boundary pairs of nonnegative forms in the next subsection.
6.4. Boundary pairs of nonnegative operators and unitary boundary triples. The notion of boundary pairs involves initially only one boundary map associated with a closed nonnegative form $\mathcal{h}$ or a pair of nonnegative selfadjoint operators. The purpose in this section is to show that, after introducing a second boundary map $\Gamma_1$ (via the first Green’s identity), the boundary pair $(\mathcal{H}, \tilde{\Gamma}_0)$ generates a unitary boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$. Furthermore, various special cases of boundary pairs are connected to specific classes of unitary boundary triples. In applications to PDE’s $\mathcal{h}$ is often the Neumann form and in abstract setting e.g. the form $\mathcal{h}_K$ associated to the Krein’s extension $A_K$, which is the smallest nonnegative selfadjoint extension of $A$. The notion of a boundary pair can be seen to arise from the works of Krein and Birman (also Vishik?) and has been treated in later papers by G. Grubb (PDE setting) and Yu. M. Arlinskiii (abstract setting).

A (basic) positive boundary pair $\{\mathcal{H}, \tilde{\Gamma}_0\}$ involving the form domain of the Krein extension was introduced in [12]. This notions leads to positive boundary triples $\{\mathcal{H}, \tilde{\Gamma}_0, \Gamma_1\}$, where $\ker \tilde{\Gamma}_0 = A_F$ and $A_K = \ker \Gamma_1$ are the Friedrichs and the Krein extension of a nonnegative operator $A$; see [66], [11] and also [59, Chapter 3] for some further details and literature. Boundary pairs which lead to $B$-generalized boundary triples appear in [13]. A more general class of boundary pairs $(\mathcal{H}, \tilde{\Gamma}_0)$ has been studied recently by O. Post [85]: who relaxed the surjectivity condition on $\tilde{\Gamma}_0$ and replaced it by the weaker requirement that $\text{ran} \tilde{\Gamma}_0$ is dense in $\mathcal{H}$. We recall the definition more explicitly here (using present notation):

**Definition 6.14** ([58]). Let $\mathcal{h}$ be a closed nonnegative form on a Hilbert space $\mathcal{H}$ and let $\tilde{\Gamma}_0$ be a bounded linear map from $\mathcal{H}^1 := (\text{dom} \mathcal{h}, \|\cdot\|_1)$, where $\|f\|^2 = \mathcal{h}(f) + \|f\|^2$, into another Hilbert space $\mathcal{H}$. Then $(\mathcal{H}, \tilde{\Gamma}_0)$ is said to be a **boundary pair associated with the form $\mathcal{h}$**, if:

(a) $(\mathcal{H}^{1,D} :=) \ker \tilde{\Gamma}_0$ is dense in $\mathcal{H}$;

(b) $(\mathcal{H}^{1/2} :=) \text{ran} \tilde{\Gamma}_0$ is dense in $\mathcal{H}$.

A pair $(\mathcal{H}, \tilde{\Gamma}_0)$ is said to be **bounded** if $\text{ran} \tilde{\Gamma}_0 = \mathcal{H}$, otherwise it is said to be unbounded.

Since $\tilde{\Gamma}_0$ is bounded its kernel defines a closed restriction of the form $\mathcal{h}$, which we denote here by $\mathcal{h}_0(f) = \mathcal{h}(f)$, $f \in \ker \tilde{\Gamma}_0$. By assumption (a) the forms $\mathcal{h}_0$ and $\mathcal{h}$ are densely defined in $\mathcal{H}$ and we denoted by $H_0$ and $H$ the selfadjoint operators associated with the closed forms $\mathcal{h}_0$ and $\mathcal{h}$, respectively. Next we associate a symmetric operator and its adjoint with the boundary pair $(\mathcal{H}, \tilde{\Gamma}_0)$ via

$$A := H_0 \cap H, \quad A^* = \text{cl}(H_0 \widehat{+} H).$$

In general, $A$ need not be densely defined, in which case $A^*$ is multivalued; in what follows we assume that $A$ is densely defined. By definition $H_0$ and $H$ are disjoint selfadjoint extensions of $A$. Recall that $\text{dom} A^* = \text{dom} H_0 + \ker (A^* - \lambda I)$, $\lambda \in \rho(H_0)$, and there is similar decomposition with $H$. Since $\mathcal{h}_0 \subset \mathcal{h}$, one has $H \leq H_0$ or, equivalently, $(H + a)^{-1} \geq (H_0 + a)^{-1}$ for all $a > 0$. Hence, see [58, Lemma 2.2], one can write

$$\text{dom} H^{1/2} = \text{dom} H_0^{1/2} + \text{ran} ((H + a)^{-1} - (H_0 + a)^{-1})^{1/2},$$

and since clearly $\text{ran} ((H + a)^{-1} - (H_0 + a)^{-1}) \subset \ker (A^* + a)$, one obtains

$$\text{dom} \mathcal{h} = \text{dom} \mathcal{h}_0 + (\mathcal{H}^1 \cap \text{dom} \mathcal{h}), \quad a > 0.$$

This sum is not in general direct, since $\mathcal{H}^1 \cap \text{dom} \mathcal{h}_0$ is nontrivial, whenever $H_0 \neq A_F$; see [58, Proposition 2.4]. This sum can be made direct with an additional restriction on $\mathcal{H}_\lambda$. As shown in [58, Propositions 2.9] the set of so-called weak solutions with a fixed $\lambda \in \mathbb{C}$ defined by

$$\mathcal{H}^1_\lambda := \{ f \in \mathcal{H}^1 : \mathcal{h}(f, g) - \lambda (f, g) = 0, \forall g \in \text{dom} \mathcal{h}_0 \}$$

and this is a direct sum of $\mathcal{H}^1 \cap \text{dom} \mathcal{h}_0$ and a closed subspace of $\mathcal{H}^1$.
leads to the following direct sum decomposition for every $\lambda \in \rho(H_0)$:

\[
(6.32) \quad \text{dom } h = \text{dom } h_0 \oplus \mathfrak{N}_\lambda^1.
\]

Here $\mathfrak{N}_\lambda^1 (\subset \mathfrak{N}_\lambda \cap \text{dom } h)$ is closed in $\mathfrak{S}_1$, $\mathfrak{N}_\lambda^1$ is dense in $\ker (A^* - \lambda)$, and $\mathfrak{N}_\lambda^1 \cap \text{dom } h_0 = \{0\}$. The restriction $\tilde{\Gamma}_0 | \mathfrak{N}_\lambda^1$ is a bounded operator from $\mathfrak{N}_\lambda^1$ into $\mathcal{H}$ and the decomposition (6.32) implies that it is injective and its range is equal to $\text{ran } \tilde{\Gamma}_0$. The inverse operator

\[
S(\lambda) := (\tilde{\Gamma}_0 | \mathfrak{N}_\lambda^1)^{-1} : \mathcal{H}^{1/2} \to \mathfrak{N}_\lambda^1
\]

is closed as an operator from $\mathcal{H}$ to $\mathfrak{S}_1$ with domain $\mathcal{H}^{1/2} = \text{ran } \tilde{\Gamma}_0$.

**Definition 6.15** ([83]). The boundary pair $(\mathcal{H}, \tilde{\Gamma}_0)$ associated with the form $h$ is said to be **elliptically regular**, if the operator $S := S(-1)$ is bounded as an operator from $\mathcal{H}$ to $\mathfrak{S}_1$, i.e. $\|Sh\|_\mathcal{H} \leq C\|h\|_\mathcal{H}$ for all $h \in \mathcal{H}^{1/2}$ and some $C \geq 0$. Moreover, the boundary pair $(\mathcal{H}, \tilde{\Gamma}_0)$ is said to be **(uniformly) positive**, if there is a constant $c > 0$, such that $\|Sh\|_\mathcal{H} \geq c\|h\|_\mathcal{H}$ for all $h \in \mathcal{H}^{1/2}$.

Let $\lambda = -1$ and define the form $\mathfrak{l}[h, k]$ on $\mathcal{H}$ by

\[
\mathfrak{l}[h, k] = (Sh, Sk)_{\mathcal{H}^1}, \quad h, k \in \mathcal{H}^{1/2}.
\]

The form $\mathfrak{l}$ is closed in $\mathcal{H}$, since $S : \mathcal{H} \to \mathfrak{S}_1$ is a closed operator. Hence, associated with $\mathfrak{l}$ there is a unique selfadjoint operator $\Lambda$ in $\mathcal{H}$ characterized by the equality

\[
\mathfrak{l}[h, k] = (Ah, k)_\mathcal{H}, \quad h \in \text{dom } \Lambda, \quad k \in \text{dom } \mathfrak{l} = \mathcal{H}^{1/2}.
\]

It is clear that $\Lambda = S^*S$, where $S^* : \mathfrak{S}_1 \to \mathcal{H}$ is the usual Hilbert space adjoint. The operator $\Lambda$ is called the **Dirichlet-to-Neumann operator** at the point $\lambda = -1$ associated with the boundary pair $(\mathcal{H}, \tilde{\Gamma}_0)$. The **(strong) Dirichlet-to-Neumann operator** at a point $\lambda \in \rho(H_0)$ is defined as follows ([83], Section 2.4):

\[
(6.33) \quad \text{dom } \Lambda(\lambda) := \{ \varphi \in \mathcal{H}^{1/2} : \exists \psi \in \mathcal{H} \text{ such that } (h - \lambda)(S(\lambda)\varphi, S\eta) = (\psi, \eta)_\mathcal{H}, \quad \forall \eta \in \mathcal{H}^{1/2} \}
\]

and then $\Lambda(\lambda)\varphi := \psi$. The operator $\Lambda(\lambda)$ is closed in $\mathcal{H}$ and it has bounded inverse operator $\Lambda(\lambda)^{-1} \in \mathcal{B}(\mathcal{H})$ for all $\lambda \in \rho(H_0)$; see [83], Proposition 2.17.

Next consider the restriction of $A^*$ to the form domain of $h$

\[
(6.34) \quad \mathfrak{S}_0^1 := \{ f \in \mathfrak{S}_1 \cap \text{dom } A^* : h(f, g) = (A^*f, g)_{\mathcal{H}_0}, \forall g \in \text{dom } h_0 \}
\]

be equipped with the norm defined by $\|f\|_{\mathfrak{S}_0^1}^2 = h(f, f) + \|f\|^2 + \|A^*f\|^2$, which makes $\mathfrak{S}_0^1$ a Hilbert space. Now using the rigged Hilbert space $\mathcal{H}^{1/2} \subset \mathcal{H} \subset \mathcal{H}^{-1/2}$ introduce a bounded operator

\[
(6.35) \quad (\tilde{\Gamma}_1 f, \tilde{\Gamma}_0 g)_{-1/2,1/2} = (A^*f, g)_{\mathcal{H}_0} - h(f, g)
\]

holds for all $f \in \mathfrak{S}_0^1$ and $g \in \mathfrak{S}_1$; this map is well defined by the formulas (6.34), (6.35). Finally, we introduce the restriction $A_*$ of $A^*$ by

\[
\text{dom } A_* := \{ g \in \mathfrak{S}_0^1 : \tilde{\Gamma}_1 g \in \mathcal{H} \}
\]

and denote $\Gamma_0 = \tilde{\Gamma}_0 | \text{dom } A_*$, $\Gamma_1 = \tilde{\Gamma}_1 | \text{dom } A_*$. By definition (the first Green’s identity)

\[
(6.36) \quad h(f, g) = (A^*f, g)_{\mathcal{H}_0} - (\Gamma_1 f, \tilde{\Gamma}_0 g)_{\mathcal{H}_0}
\]

holds for all $f \in \text{dom } A_*$ and $g \in \mathfrak{S}_1$. In what follows the triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with the domain $\text{dom } A_* = \text{dom } \Gamma_0 \cap \text{dom } \Gamma_1$ is called a boundary triple generated by the boundary pair $(\mathcal{H}, \tilde{\Gamma}_0)$. The next result characterizes the central properties of the boundary pair $(\mathcal{H}, \tilde{\Gamma}_0)$ by means of the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$. In particular, it shows that the notion of boundary pair
in Definition 6.14 can be included in the framework of unitary boundary triples whose Weyl function are Nevanlinna functions from the class \( \mathcal{R}^*(\mathcal{H}) \).

**Theorem 6.16.** Let \( (\mathcal{H}, \Gamma_0) \) be a boundary pair for the closed nonnegative form \( \mathfrak{h} \) in \( \mathfrak{S} \) and let \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) be the corresponding triple as defined above. Then:

(i) \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) is a unitary boundary triple for \( A^* \);

(ii) \( A_0 := A^* \; | \; \ker \Gamma_0 \) is a symmetric restriction of \( H_0 \), while \( A_1 := A^* \; | \; \ker \Gamma_1 \) is selfadjoint and it is equal to \( H \);

(iii) the \( \gamma \)-field and the Weyl function \( M(\cdot) \) of the boundary triple \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) are given by

\[
\gamma(\lambda) = S(\lambda) \; | \; \text{dom} \Lambda(\lambda), \quad M(\lambda) = -\Lambda(\lambda), \quad \lambda \in \rho(H_0);
\]

(iv) the transposed triple \( \{\mathcal{H}, \Gamma_1, -\Gamma_0\} \) is a \( B \)-generalized boundary triple for \( A^* \);

(v) \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) is \( ES \)-generalized, i.e., \( \text{clo} A_0 = H_0 \) if and only if \( S(\lambda) \) is closable when treated as an operator from \( \mathcal{H} \to \mathfrak{S} \) for some (equivalently for all) \( \lambda \in \rho(H_0) \);

(vi) \( (\Gamma_0, \mathcal{H}) \) is elliptically regular if and only if \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) is an \( S \)-generalized boundary triple;

(vii) \( (\Gamma_0, \mathcal{H}) \) is uniformly positive if and only if \( \Gamma_0 : A_* \to \mathcal{H} \) is a bounded operator (w.r.t. the graph norm on \( A_* \)) or, equivalently, the form \( t_M(\lambda) \) has a positive lower bound for some (equivalently for every) \( \lambda \in \mathbb{C} \setminus \mathbb{R} \);

(viii) \( (\Gamma_0, \mathcal{H}) \) is bounded if and only if \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) is a \( B \)-generalized boundary triple;

(ix) \( (\Gamma_0, \mathcal{H}) \) is bounded and uniformly positive if and only if \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) is an ordinary boundary triple.

**Proof.** (i) First observe that the first Green’s identity \((6.36)\) applied to \( h[f, g] \) and \( \overline{h[g, f]} \) with \( f, g \in \text{dom} A_* \) leads to the second Green’s identity \((1.1)\) by symmetry of the form \( \mathfrak{h} \). The second Green’s identity \((1.1)\) implies that the restrictions \( A_0 = A^* \; | \; \ker \Gamma_0 \) and \( A_1 = A^* \; | \; \ker \Gamma_1 \) are symmetric operators extending \( A \).

Next we prove that the (graph) closure of \( A_* \) is \( A^* \) and that \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) is a unitary boundary triple for \( A^* \). It is clear from \((6.34)\) that the set of weak solutions \( \mathfrak{R}^1_\lambda \) belongs to \( \mathfrak{S}^1_0 \). Since \( H_0 \) is the selfadjoint operator associated with the form \( \mathfrak{h}_0 \) by the first representation theorem of Kato and \( \mathfrak{h}_0 \subset \mathfrak{h} \), we conclude from \((6.34)\) that \( \text{dom} H_0 \subset \mathfrak{S}^1_0 \). Similarly \( H \) is the selfadjoint operator associated with the form \( \mathfrak{h} \) and, hence also \( \text{dom} H \subset \mathfrak{S}^1_0 \). Now applying \((6.35)\) with \( f \in \text{dom} H \) and \( g \in \mathfrak{S}^1_0 \) taking into account that \( \text{ran} \Gamma_0 \) is dense in \( \mathcal{H} \) by assumption (b) in Definition 6.14 we conclude that \( \Gamma_1 f = 0 \). Hence, \( \text{dom} H \subset \text{dom} A_* \) and \( \Gamma_1(\text{dom} H) = \{0\} \).

Thus, \( H \subset A_1 \) and since \( A_1 \) is symmetric this implies that \( A_1 = H \) is selfadjoint. Now consider the operator \( \Lambda = S^* S \). Since \( \text{dom} \Lambda \) is a core for the form \( I \) it is also a core for the operator \( S \). This implies that \( S(\text{dom} \Lambda) \) is dense in \( \mathfrak{R}^1_{-1} \) w.r.t. the topology in \( \mathfrak{S}^1 \), since \( S \) has bounded inverse. We claim that \( S(\text{dom} \Lambda) \subset \text{dom} A_* \). To see this we consider the form

\[
(6.37) \quad \mathfrak{h}(f, g) - (A^* f, g)_{\mathfrak{h}} \quad f \in \mathfrak{S}^1_0, \quad g \in \mathfrak{S}^1_0.
\]

Notice that \( \mathfrak{R}^1_\lambda \subset \mathfrak{S}^1_0 \), see \((6.31), (6.34)\), and that the decomposition \((6.32)\) for \( \lambda = -1 \) is orthogonal in \( \mathfrak{S}^1 \). Hence, one can write \( g = g_0 + g_1 \in \text{dom} \mathfrak{h} \oplus_1 \mathfrak{R}^1_{-1} \), \( g \in \mathfrak{S}^1 \) = \( \text{dom} \mathfrak{h} \). Now for \( h \in \text{dom} \Lambda \) one has \( Sh \in \mathfrak{R}^1_{-1} \) and for all \( g = g_0 \in \text{dom} \mathfrak{h} = \ker \Gamma_0 \),

\[
\mathfrak{h}(Sh, g_0) - (A^* Sh, g_0)_{\mathfrak{h}} = \mathfrak{h}(Sh, g_0) + (Sh, g)_{\mathfrak{h}} = (Sh, g_0)_{\mathfrak{h}} = 0.
\]

On the other hand, when \( g = g_1 \in \mathfrak{R}^1_{-1} \), then \( k = \Gamma_0 g_1 \in \mathcal{H}^{1/2} \) satisfies \( g_1 = Sk \). This leads to

\[
\mathfrak{h}(Sh, g_1) - (A^* Sh, Sk)_{\mathfrak{h}} = \mathfrak{h}(Sh, Sk) + (Sh, Sk)_{\mathfrak{h}} = (Sh, Sk)_{\mathfrak{h}} = (\Lambda h, k)_{\mathfrak{h}} = (\Lambda h, \Gamma_0 g_1)_{\mathfrak{h}}.
\]
We conclude that for \( f = Sh, h \in \text{dom } \Lambda \), and all \( g \in \mathfrak{H}^1 \) the form \((6.37)\) can be rewritten as follows
\[
\mathfrak{h}(Sh, g) - (A^* Sh, g)_{\mathfrak{H}^1} = (\lambda h, \Gamma_0 g)_{\mathfrak{H}^1}.
\]
Comparing this formula with \((6.35)\) we conclude that \( \hat{\Gamma}_1 Sh = \Gamma_1 Sh = -\lambda h \in \mathcal{H} \), which proves the claim \( S(\text{dom } \Lambda) \subset \text{dom } A_\ast \).

Since \( S(\text{dom } \Lambda) \) is dense in \( \mathfrak{H}^1 \) and \( \text{dom } H \subset \text{dom } A_\ast \), the closure of \( A_\ast \) is equal to the closure of \( H + \mathfrak{H}^1 \), which coincides with \( A^* \). Hence, the domain of \( \{ \Gamma_0, \Gamma_1 \} \) is dense in \( \text{dom } A^* \) w.r.t. the graph topology. As was shown above \( \Gamma_1 Sh = -\lambda h \) for all \( h \in \text{dom } \Lambda \) and, in addition, \( \Gamma_0 Sh = h \). Since \( S(\text{dom } \Lambda) \subset \mathfrak{H}^1(A_\ast) \) this implies that for the regular point \( \lambda = -1 \in \rho(H) \) one has
\[
-\Lambda \subset M(-1).
\]
Here equality \( M(-1) = -\Lambda \) prevails, since \( M(-1) \) is necessarily symmetric by Green’s identity \((1.1)\). Clearly, \( M(-1) - I = -\lambda - I \leq -I \) and thus \( 0 \in \rho(M(-1) - I) \). Therefore, we can apply Theorem 5.26 to conclude that \( \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) is a unitary boundary triple for \( A^* \) with dense domain \( A_\ast \).

(ii) The equality \( A_1 = H \) was already proved in item (i). Next we prove the inclusion \( A_0 \subset H_0 \). The first Green’s identity \((6.36)\) shows that
\[
(6.38) \quad \mathfrak{h}(f, g) = (A^* f, g)_{\mathfrak{H}^1}, \quad \text{for all } f \in \text{dom } A_\ast, \quad g \in \ker \Gamma_0 = \text{dom } \mathfrak{h}_0.
\]
If, in particular, \( f \in \text{dom } A_0 \) i.e. \( \Gamma_0 f = 0 \), then \( f \in \text{dom } \mathfrak{h}_0 \) and \((6.38)\) can be rewritten as
\[
\mathfrak{h}_0(f, g) = (A_0 f, g)_{\mathfrak{H}^1}, \quad \text{for all } g \in \text{dom } \mathfrak{h}_0.
\]
Now by the first representation theorem (see \cite{64}) one concludes that \( f \in \text{dom } H_0 \) and \( A_0 f = H_0 f \). Therefore, \( A_0 \subset H_0 \).

(iii) It was shown in part (i) that \( \text{ran } S(\lambda) = \mathfrak{H}^1 \subset \mathfrak{H}^1 \) for each \( \lambda \in \rho(H_0) \). Now assume in addition that \( h \in \text{dom } \Lambda(\lambda) \) and let \( g \in \mathfrak{H}^1 \). Then the definition of \( \Lambda(\lambda) \) shows that
\[
\mathfrak{h}(S(\lambda)h, g) - (A^* S(\lambda)h, g)_{\mathfrak{H}^1} = (\lambda - 1)[S(\lambda)h, g] = (\Lambda(\lambda)h, \Gamma_0 g)_{\mathfrak{H}^1}.
\]
Comparing this formula with \((6.35)\) we conclude that \( \hat{\Gamma}_1 S(\lambda) h = \Gamma_1 S(\lambda) h = -\Lambda(\lambda) h \in \mathcal{H} \), which shows that \( S(\text{dom } \Lambda(\lambda)) \subset \text{dom } A_\ast \) and, moreover, that \( M(\lambda) h = -\Lambda(\lambda) h \). Therefore,
\[
-\Lambda(\lambda) \subset M(\lambda), \quad \lambda \in \rho(H_0).
\]
Equivalently, \( \Lambda(\lambda)^{-1} \subset -M(\lambda)^{-1} \) and since \( M(\cdot) \) is the Weyl function of a single valued unitary boundary triple, \( M(\cdot) \in \mathcal{R}^+(\mathcal{H}) \), in particular, \( \ker M(\lambda) = \{0\} \); see \((1.1)\). On the other hand, \( \Lambda(\lambda)^{-1} \in \mathcal{B}(\mathcal{H}) \) and, hence, the equality \( \Lambda(\lambda)^{-1} = -M(\lambda)^{-1} \) follows. The equality \( \gamma(\lambda) = S(\lambda)|\text{dom } M(\lambda) \) is clear, and the formulas for \( \gamma(\lambda) \) and \( M(\lambda) \) are proven.

(iv) Since \( \Lambda(\cdot)^{-1} \in \mathcal{B}(\mathcal{H}) \) and \( -M(\lambda)^{-1} = \Lambda(\cdot)^{-1} \) by part (iii) the transposed boundary triple is \( \mathcal{B} \)-generalized; see Theorem 1.7.

(v) By definition \( \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) is \( ES \)-generalized if and only if \( A_0 \) is essentially selfadjoint, which in view of (ii) means that \( \text{clo } A_0 = H_0 \). On the other hand, by Theorem 5.26 and Remark 5.27 \( \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) is \( ES \)-generalized if and only if \( \gamma(\lambda) \) is closable for some (equivalently for all) \( \lambda \in \rho(H_0) \).

Since \( \gamma(\lambda) \subset S(\lambda) \), it is clear that if \( S(\lambda) \) is closable then also \( \gamma(\lambda) \) is closable. On the other hand, it follows from \cite{85} Theorems 2.11, Proposition 2.17 that \( \text{dom } \Lambda(\lambda) \) is dense w.r.t. the \( \mathcal{H}^{1/2} \)-topology on \( \mathcal{H}^{1/2} \) and that
\[
S(\lambda)|\text{dom } \Lambda(\lambda)^{\mathcal{H}^{1/2} \to \mathfrak{H}^1} = S(\lambda),
\]
since $S(\lambda) : \mathcal{H}^{1/2} \to \mathfrak{H}_1^\lambda$ is a topological isomorphism. Since the topologies on $\mathcal{H}^{1/2}$ and $\mathfrak{H}_1^\lambda$ are stronger than the topologies on $\mathcal{H}$ and $\mathfrak{H}$ it follows that if $\gamma(z) : \mathcal{H} \to \mathfrak{H}$ is closable, then also $S(z) : \mathcal{H} \to \mathfrak{H}$ is closable and $\gamma(\lambda)_{\mathcal{H} \to \mathfrak{H}} = S(\lambda)_{\mathcal{H} \to \mathfrak{H}}$.

(vi) When $(\tilde{\Gamma}_0, \mathcal{H})$ is elliptically regular, then $S : \mathfrak{H}_1 \to \mathcal{H}$ is a bounded operator. Then equivalently the $\gamma$-field $\gamma(\lambda)$ is bounded for all $\lambda \in \rho(H_0)$, cf. [85, Theorem 2.11], and the statement is obtained from Theorem 5.18.

(vii) If $(\tilde{\Gamma}_0, \mathcal{H})$ is (uniformly) positive then $S(\lambda), \lambda \in \rho(H_0)$ is bounded from below; cf. [85, Theorem 2.11]. In view of (3.9) this means that the form $t_{M(\lambda)}$ has a positive lower bound. Now the statement follows from Proposition (6.9) since $A_1 = H$ is selfadjoint by part (iii).

(viii) If $(\tilde{\Gamma}_0, \mathcal{H})$ is bounded, i.e., ran$\tilde{\Gamma}_0 = \mathcal{H}^{1/2} = \mathcal{H}$, then $S : \mathcal{H} \to \mathfrak{H}_1$ is closed (as an inverse of a bounded operator $\tilde{\Gamma}_0 \mid \mathfrak{H}_1^{1/2}$), everywhere defined, and bounded by the closed graph theorem. In particular, $\{\mathcal{H}, \tilde{\Gamma}_0, \Gamma_1\}$ is a $B$-generalized boundary triple. On the other hand, we conclude that the form $(h + 1)(Sh, Sk)$ is closed and defined everywhere on $\mathcal{H}$. Now it follows from (6.33) that dom $\Lambda(-1) = \mathcal{H}$. This implies that $M(\cdot) \in \mathcal{R}^n[\mathcal{H}]$; see e.g. (5.43) in Theorem 5.18. Therefore, $\{\mathcal{H}, \tilde{\Gamma}_0, \Gamma_1\}$ is a $B$-generalized boundary triple by Theorem 1.7.

The converse statement is clear, since ran$\tilde{\Gamma}_0 = \mathcal{H}$ implies that also ran$\tilde{\Gamma}_0 = \mathcal{H}$.

(ix) This follows directly e.g. from Proposition 6.11. Alternatively, by (vi) and (vii) the conditions mean that $M(\cdot) \in \mathcal{R}^n[\mathcal{H}]$, and then the result follows from Theorem 1.4.

Remark 6.17. (a) Characterizations (viii) and (ix) have been announced (without proofs) in [85, Theorem 1.8]. Moreover, elliptic regularity has been characterized in [85, Theorem 1.8] using equivalence to quasi boundary triples. However, as indicated the conditions defining a quasi boundary triple are not sufficient to guarantee that the corresponding Weyl function belongs to the class of Nevanlinna functions. In this sense the characterization of elliptic regularity presented in (vi) is more precise and complete. As to (vii) a characterization of positive boundary pairs via uniform positivity of the form valued function $z \to -t_z$ appears in [85, Theorem 3.13], while the other characterization that $\Gamma_0 : A_* \to \mathcal{H}$ is a bounded operator, as well as the statements (i) – (v) in Theorem 6.14 are obviously new.

(b) Since $H_0$ and $H$ are nonnegative selfadjoint operators, the Weyl functions $M(\cdot)$ and $-M(\cdot)^{-1}$ admit analytic continuations (in the resolvent sense) to the negative real line. In fact, $M(\cdot)$ belongs to the class of operator valued (in general unbounded) inverse Stieltjes functions, while $-M(\cdot)^{-1}$ belongs to the class of operator valued Stieltjes functions. Essentially these facts follow from the following formula:

$$(M(x)h, h) = (h - x)(H_0 + 1)(H_0 - x)^{-1}h, h \leq 0, \quad h \in \text{dom } M(x), \quad x < 0.$$
and Neumann trace operators defined for any \( f \in H^2(\Omega) \) by
\[
(7.1) \quad \gamma_D : f \mapsto f|_{\partial\Omega}, \quad \gamma_N : f \mapsto \frac{\partial f}{\partial n} = \sum_{j=1}^{d} n_j \gamma_D \frac{\partial f}{\partial x_j}
\]
where \( n = (n_1, \ldots, n_d) \) is the outward unit normal to the boundary \( \partial\Omega \). Then the mapping
\[
(7.2) \quad \left( \begin{array}{c} \gamma_D \\ \gamma_N \end{array} \right) : f \in H^2(\Omega) \mapsto \left( \begin{array}{c} \gamma_D f \\ \gamma_N f \end{array} \right) \in \left( \begin{array}{c} H^{3/2}(\partial\Omega) \\ H^{1/2}(\partial\Omega) \end{array} \right)
\]
is bounded and onto (see [76 Thm 1.8.3]). It is known (see, for instance, [23]) that \( A_{\text{max}} = A_{\text{min}}^* (= A^*) \) and
\[
\text{dom } A_{\text{min}} = \{ f \in H^2_0(\Omega) : \gamma_D f = \gamma_N f = 0 \}.
\]
Clearly, \( \text{dom } A_{\text{max}} \supset H^2(\Omega) \). However, an explicit description of \( \text{dom } A^* \) is unknown while Lions and Magenes [76] have shown that the mappings \( \gamma_D \) and \( \gamma_N \) defined on \( H^2(\Omega) \) extend to continuous mappings from the domain of the maximal operator,
\[
(7.3) \quad \gamma_D : \text{dom } A^* \to H^{-1/2}(\partial\Omega), \quad \gamma_N : \text{dom } A^* \to H^{-3/2}(\partial\Omega)
\]
and these mappings are surjective.

The differential expression \( \ell \) admits two classical selfadjoint realizations, the Dirichlet Laplacian \( -\Delta_D \) and the Neumann Laplacian \( -\Delta_N \), given by \( \ell \) on the domains
\[
(7.4) \quad \text{dom } \Delta_D = \{ f \in H^2(\Omega) : \gamma_D f = 0 \} \quad \text{and} \quad \text{dom } \Delta_N = \{ f \in H^2(\Omega) : \gamma_N f = 0 \},
\]
respectively.

General, not necessarily local, boundary value problems for elliptic operators have been studied in the pioneering works of Visk [90] and Grubb [53] (see also [54], [77], [48], [20] for further developments and applications).

Denote by \( H^s_\Delta(\Omega) \) the following space
\[
(7.5) \quad H^s_\Delta(\Omega) := H^s(\Omega) \cap \text{dom } A_{\text{max}} = \{ f \in H^s(\Omega) : \Delta f \in L^2(\Omega) \}, \quad 0 \leq s \leq 2,
\]
equip it with the graph norm \( \| f \|_{H^s_\Delta(\Omega)} = (\| f \|_{H^s}^2 + \| A_{\text{max}} f \|_{L^2(\Omega)}^2)^{1/2} \) of \( -\Delta \) on \( H^s(\Omega) \).

According to the Lions-Magenes result ([76 Theorem 2.7.3]) the trace operators \( \gamma_D \) and \( \gamma_N \) admit continuous extensions to the operators
\[
(7.6) \quad \gamma_D^s : H^s_\Delta(\Omega) \to H^{s-1/2}(\partial\Omega), \quad \gamma_N^s : H^s_\Delta(\Omega) \to H^{s-3/2}(\partial\Omega), \quad 0 < s \leq 2,
\]
which are surjective. It is emphasized that the values \( s = 1/2 \) and \( s = 3/2 \) are not excluded here. At the same time the traces \( \gamma_D^s : H^s(\Omega) \to H^{s-1/2}(\partial\Omega) \) and \( \gamma_N^s : H^s(\Omega) \to H^{s-3/2}(\partial\Omega) \) are continuous mappings if and only if \( s > 1/2 \) and \( s > 3/2 \), respectively, (see ([76 Theorems 1.9.4, 1.9.5] and [3])). In the latter case both mappings in (7.6) are surjective. Moreover, for \( s > 3/2 \) the mapping \( \gamma_D^s \times \gamma_N^s : H^s(\Omega) \to H^{s-1/2}(\partial\Omega) \times H^{s-3/2}(\partial\Omega) \) is also surjective.

When treating the traces \( \gamma_D^s \) and \( \gamma_N^s \) as mappings into \( L^2(\partial\Omega) \) a natural choice for the index is \( s = 3/2 \); see Remark 7.2 below. The restriction of \( A^* \) to the domain
\[
(7.7) \quad \text{dom } A_s = H^{3/2}_\Delta(\Omega)
\]
is called a pre-maximal operator and is denoted by \( A_s^* \).

The Dirichlet Laplacian \( -\Delta_D \) is an invertible selfadjoint operator in \( L^2(\Omega) \) with a discrete spectrum \( \sigma_p(-\Delta_D) \). Define a solution operator \( \mathcal{P}(z) : L^2(\partial\Omega) \to H^{1/2}(\Omega) \) for \( z \in \mathbb{C} \setminus \sigma_p(-\Delta_D) \). Let \( \varphi \in L^2(\partial\Omega) \) and let \( f_z \in \text{dom } A_{\text{max}} \) be the unique solution of the Dirichlet problem
\[
(7.8) \quad -\Delta f_z - zf_z = 0, \quad \gamma_D f_z = \varphi
\]
Then the operator $\mathcal{P}(z) : \varphi \mapsto f_z$ is continuous as an operator from $L^2(\partial \Omega)$ to $H^{1/2}(\Omega)$ and it maps $H^1(\partial \Omega)$ into $H^{3/2}(\Omega)$; see [53]. Hence the Poincaré-Steklov operator $\Lambda(z)$ defined by

$$\Lambda(z)\varphi := \gamma_N \mathcal{P}(z)\varphi,$$

maps $H^1(\partial \Omega)$ into $L^2(\partial \Omega)$ with continuous extension from $H^{-1/2}(\partial \Omega)$ to $H^{-3/2}(\partial \Omega)$. Moreover, the Dirichlet-to-Neumann map $\Lambda := \Lambda(0)$ treated as an operator in $L^2(\partial \Omega)$ is selfadjoint on the domain $\text{dom } \Lambda = H^1(\partial \Omega)$; see [77].

Following [90] and [53] we introduce the regularized trace operators as follows

$$\tilde{\Gamma}_{0, \Omega} f = (\gamma_N - \Lambda(0)\gamma_D) f, \quad \tilde{\Gamma}_{1, \Omega} f = \gamma_D f, \quad f \in \text{dom } S_*.$$

It is proved in [53] that the mappings $\tilde{\Gamma}_{0, \Omega}$ and $\tilde{\Gamma}_{1, \Omega}$ are well defined and

$$\tilde{\Gamma}_{0, \Omega} : \text{dom } A_{\text{max}} \to H^{1/2}(\partial \Omega), \quad \tilde{\Gamma}_{1, \Omega} : \text{dom } A_{\text{max}} \to H^{-1/2}(\partial \Omega).$$

In fact, the effect of regularization appearing in $\tilde{\Gamma}_{0, \Omega}$ follows from the decomposition $\text{dom } A_{\text{max}} = \text{dom } \Delta_D + \ker A^* (0 \in \rho(-\Delta_D))$: $u \in \text{dom } A_{\text{max}}$ admits a decomposition $u = u_D + u_0$ with $u_D \in \text{dom } \Delta_D \subset H^2(\Omega)$ and $u_0 \in \ker A^*$. Now an application of (7.9) and (7.10) gives $\tilde{\Gamma}_{0, \Omega} u = \gamma_N u_D \in H^{1/2}(\partial \Omega)$ which yields (7.11). Since $\gamma_N^2 : H^2(\Omega) \to H^{3/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)$ is surjective, one has $\{0\} \times H^{1/2}(\partial \Omega) \subset \text{ran } \gamma_N \times \gamma_N$, which shows that $\tilde{\Gamma}_{0, \Omega}$ in (7.11) is also surjective. In addition, it is also closed when $\text{dom } A_{\text{max}} = H^2(\Omega)$ is equipped with the $L^2$-graph norm. Indeed, by the continuity properties of the traces $\gamma_D$, $\gamma_N$, and the Poincaré-Steklov operator $\Lambda(z)$ (see (7.8), (7.9)) $\tilde{\Gamma}_{0, \Omega}$ is a continuous mapping from $\text{dom } A_{\text{max}}$ into $H^{-3/2}(\partial \Omega)$. Now, if $f_n \in \text{dom } \tilde{\Gamma}_{0, \Omega}$ and $f_n \to f$ in $H^2(\Omega)$ and $g_n = \tilde{\Gamma}_{0, \Omega} f_n \to g$ in $H^{1/2}(\partial \Omega)$, then $g_n \to g$ also in $H^{-3/2}(\partial \Omega)$, $f \in \text{dom } \tilde{\Gamma}_{0, \Omega}$ and $g = \tilde{\Gamma}_{0, \Omega} f$ by the $H^{-3/2}(\partial \Omega)$-continuity of $\tilde{\Gamma}_{0, \Omega}$. Hence, $\tilde{\Gamma}_{0, \Omega} : \text{dom } A_{\text{max}} \to H^{1/2}(\partial \Omega)$ is closed. Finally, by the closed graph theorem $\tilde{\Gamma}_{0, \Omega}$ in (7.11) is bounded; cf. [53, Theorem III.1.2] where these results on $\tilde{\Gamma}_{0, \Omega}$ are derived in a more general elliptic setting.

With these preliminaries we are ready to give first applications of the abstract results for Laplacians on smooth bounded domains.

Let $\tilde{A}_\ast$ be a restriction of $A_{\text{max}}$ to the domain

$$\text{dom } \tilde{A}_\ast = \{ f \in \text{dom } A_{\text{max}} : \gamma_D f \in L^2(\partial \Omega) \}.$$

**Proposition 7.1.** Let the operators $\gamma_N$, $\gamma_D$, $\mathcal{P}(z)$, $\Lambda(z)$, $A_\ast$ and $\tilde{A}_\ast$ be defined by (7.3), (7.8), (7.9), (7.10), and (7.12). Then:

(i) $\{ L^2(\partial \Omega), \gamma_D | \text{dom } A_\ast, -\gamma_N | \text{dom } A_\ast \}$ is an $S$-generalized boundary triple for $A^*$, and the corresponding Weyl function $M(\cdot)$ coincides with $-\Lambda(\cdot)$;

(ii) $\{ L^2(\partial \Omega), \tilde{\Gamma}_{0, \Omega} | \text{dom } \tilde{A}_\ast, \tilde{\Gamma}_{1, \Omega} | \text{dom } \tilde{A}_\ast \}$ is an $\text{ES}$-generalized boundary triple for $A^*$, the corresponding Weyl function is the $L^2(\partial \Omega)$-closure

$$\tilde{M}(\zeta) = \text{clos } (\Lambda(z) - \Lambda(0))^{-1};$$

(iii) the extension $\tilde{A}_\ast := S_\ast | \ker \tilde{\Gamma}_{0, \Omega}$ is essentially selfadjoint and its closure coincides with the Krein - von Neumann extension of the operator $A_{\text{min}}$.

**Proof.** (i) The triple $\{ L^2(\partial \Omega), \gamma_N | \text{dom } A_\ast, \gamma_D | \text{dom } A_\ast \}$ is a $B$-generalized boundary triple for the operator $A^*$ (see [39]), since Green’s identity holds, the mapping $\gamma_N^s$ with $s = 3/2$ is surjective by (7.6), and by the descriptions (7.2),

$$\text{ran } (\gamma_N | \text{dom } A_\ast) = H^0(\partial \Omega), \quad \ker (\gamma_N | \text{dom } A_\ast) \supset \ker (\gamma_N | \delta^2(\Omega)) = \text{dom } \Delta_N.$$
Here the inclusion in the second relation holds as an equality, since for any isometric boundary triple \( \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) the kernels \( \ker \Gamma_0 \) and \( \ker \Gamma_1 \) determine symmetric restrictions of \( A_{\text{max}} \) (by Green’s identity); cf. [37] Proposition 2.13.

Clearly, the boundary triple \( \Pi = \{ H^0(\partial \Omega), \gamma_D | \text{dom } A_s, -\gamma_N | \text{dom } A_s \} \) is unitary, since it is transposed to the \( B \)-generalized boundary triple \( \{ H^0(\partial \Omega), \gamma_N | \text{dom } A_s, \gamma_D | \text{dom } A_s \} \). Moreover, as above from (7.13) one concludes that \( \ker (\gamma_D | \text{dom } A_s) = \ker \Delta_D \). Since \( -\Delta_D = -\Delta_0 \), the triple \( \Pi \) is an \( S \)-generalized boundary triple for \( A^* \). By definition, the corresponding Weyl function coincides with \(-\Lambda(z)\).

Another proof of the statement (i) can be extracted from Proposition 6.2. Indeed, take \( A_0 = -\Delta_D \) and fix the mappings \( G := \mathcal{P}(0) \) and \( E := -\Lambda(0) \); see Remark 6.3. By definition \( \gamma_D G \varphi = \varphi \) and \( \gamma_D A_0^{-1} f = 0 \) for all \( \varphi \in H^0(\partial \Omega) \) and \( f \in L^2(\Omega) \). Moreover, a direct calculation (see e.g. [37] with smooth functions \( f \)) leads to

\[
G^* f = -\gamma_N A_0^{-1} f, \quad f \in L^2(\Omega),
\]

cf. (3.14), (4.1). Therefore, the abstract boundary mappings \( \Gamma_0 \) and \( \Gamma_1 \) defined in (6.8) coincide with the trace operators \( \gamma_D | \text{dom } A_s \) and \( -\gamma_N | \text{dom } A_s \), respectively.

Moreover, the mappings \( \Gamma_0 = \gamma_D | \text{dom } A_s \) and \( \Gamma_1 = -\gamma_N | \text{dom } A_s \) in (7.13) with \( s = 3/2 \) are surjective. By Proposition 6.2 \( \{ H^0(\partial \Omega), \gamma_D | \text{dom } A_s, -\gamma_N | \text{dom } A_s \} \) is an \( S \)-generalized boundary triple and since \( E = -\Lambda(0) \) is unbounded, this triple is not \( B \)-generalized.

(ii) Next we apply Proposition 6.2 to the closure \( \tilde{\Gamma} \) of the transformed boundary triple as defined in (6.9). By definition \( \tilde{\Gamma} : \text{dom } A^* \supset A_s \rightarrow (L^2(\partial \Omega))^2 \) is closed and in view of (6.9) \( \tilde{\Gamma}_1 \) maps onto \( L^2(\partial \Omega) \). It is clear that \( \tilde{\Gamma}_0 \) and \( \tilde{\Gamma}_1 \) coincide (up to inessential change of signs) with the regularized trace operators given by (7.10), \( \tilde{\Gamma}_0 = \tilde{\Gamma}_0^{0,\Omega} \) and \( \tilde{\Gamma}_1 = \tilde{\Gamma}_1^{0,\Omega} \) on the initial domain \( H^{3/2}(\Omega) \). On the other hand, \( \tilde{\Gamma}_0^{0,\Omega} \times \tilde{\Gamma}_1^{0,\Omega} : \mathcal{S}^{1/2}_D(\Omega) \rightarrow (L^2(\partial \Omega))^2 \) and hence \( \tilde{\Gamma}_0^{0,\Omega} | \mathcal{S}^{1/2}_D(\Omega) \subseteq \tilde{\Gamma}_0^{0,\Omega} | \text{dom } \bar{A}_s \); here equality holds, since \( \tilde{\Gamma}_1^{0,\Omega} : \mathcal{S}^{1/2}_D(\Omega) \rightarrow L^2(\partial \Omega) \) and \( \tilde{\Gamma}_1^{0,\Omega} : \text{dom } \bar{A}_s \rightarrow L^2(\partial \Omega) \) are surjective, see (7.3) and (7.12), and their kernels are equal to \text{dom } \Delta_D \). Moreover, \( \tilde{\Gamma}_0^{0,\Omega} | \mathcal{S}^{1/2}_D(\Omega) \) is closed in \( \text{dom } A^* \times (L^2(\partial \Omega))^2 \). Hence \( \tilde{\Gamma} \subseteq \tilde{\Gamma}_1^{0,\Omega} | \mathcal{S}^{1/2}_D(\Omega) \) and here equality holds, since by Proposition 6.2 \( \tilde{\Gamma}_1^{0,\Omega} \) is surjective and has kernel \text{dom } \Delta_D.

By Proposition 6.2 \( \{ \mathcal{H}, \tilde{\Gamma}_0^{0,\Omega}, \tilde{\Gamma}_1^{0,\Omega} \} \) is an \( ES \)-generalized boundary triple for \( A_{\text{max}} \). Since \( \text{ran } G \subseteq H^{1/2}(\Omega) \), it is not closed in \( H^0(\Omega) \), hence the triple \( \{ \mathcal{H}, \tilde{\Gamma}_0^{0,\Omega}, \tilde{\Gamma}_1^{0,\Omega} \} \) is not \( S \)-generalized. The statement concerning the Weyl function is obtained from (6.10).

(iii) Since \( \{ \mathcal{H}, \tilde{\Gamma}_0^{0,\Omega}, \tilde{\Gamma}_1^{0,\Omega} \} \) is not \( S \)-generalized, \( \bar{A}_0 \) is not selfadjoint. It follows from (7.10) that

\[
\text{dom } \bar{A}_0 = \{ f \in H^{3/2}_\Delta(\Omega) : (\gamma_N - \Lambda(0)\gamma_D) f = 0 \}
\]

contains the set

\[
\text{dom } S + \mathcal{P}(0)(H^1(\partial \Omega))(\subset H^{3/2}_\Delta(\Omega)),
\]

which is dense in the domain of the Krein-von Neumann extension \( S_K \),

\[
\text{dom } A_K = \text{dom } S + \ker A^*,
\]

equipped with the graph norm. This completes the proof. \( \square \)

**Remark 7.2.** Using the above mentioned properties of the traces \( \gamma_D^s \) and \( \gamma_N^s \), it is easy to see that for the values \( 3/2 \leq s \leq 2 \) the boundary triple \( \{ L^2(\partial \Omega), \gamma_D^s | \text{dom } A_s, -\gamma_N^s | \text{dom } A_s \} \) as well as the transposed boundary triple \( \{ L^2(\partial \Omega), \gamma_N^s | \text{dom } A_s, \gamma_D^s | \text{dom } A_s \} \) are quasi boundary triples (compare [18] Theorem 6.11) and hence, in particular, \( AB \)-generalized boundary triples. Indeed, since Green’s identity holds for \( s = 3/2 \) (by Proposition 7.1), it holds also for \( 3/2 < s \leq 2 \). This combined with (7.11) leads to \( \text{dom } \Delta_D = \ker \gamma_D^s, \text{dom } \Delta_N = \ker \gamma_N^s, \) and by the
surjectivity of $\gamma_D^s \times \gamma_N^s : H^s(\Omega) \to H^{s-1/2}(\partial\Omega) \times H^{s-3/2}(\partial\Omega)$ the range of $\gamma_D^s \times \gamma_N^s$ is dense in $L^2(\partial\Omega)$.

More precisely for every $3/2 \leq s \leq 2$ all the quasi boundary triples in Remark 7.2 are in fact essentially unitary; the choice $s = 3/2$ in Proposition 7.1 is also motivated by the next corollary.

**Corollary 7.3.** For every $3/2 \leq s \leq 2$ the closure of $\gamma_D^s \times \gamma_N^s$ in $(\text{dom } A_{\text{max}}) \times (L^2(\partial\Omega))^2$ coincides with $\gamma_D^{3/2} \times \gamma_N^{3/2}$, where $\text{dom } A_{\text{max}}$ is equipped with the $L^2$-graph norm of $\text{dom } A_{\text{max}}$. The closure is an $S$-generalized boundary triple for $A_{\text{max}}$.

Proof. Since for $3/2 \leq s \leq 2$ the mapping $\gamma_D^s \times \gamma_N^s : H^s(\Omega) \to (L^2(\partial\Omega))^2$ is continuous by (7.3) and the inclusions $H^s(\Omega) \subset H^{3/2}(\Omega)$ are dense, it follows that the closure of $\gamma_D^s \times \gamma_N^s$ in $H^{3/2}(\Omega) \times (L^2(\partial\Omega))^2$ coincides with $\gamma_D^{3/2} \times \gamma_N^{3/2}$. Since the $L^2$-graph norm of $A_{\text{max}}$ is majorized by the $H^s(\Omega)$-norm, the closure of $\gamma_D^s \times \gamma_N^s$ in $(\text{dom } A_{\text{max}}) \times (L^2(\partial\Omega))^2$ contains $\gamma_D^{3/2} \times \gamma_N^{3/2}$. However, by Proposition 7.1 (i) $\gamma_D^{3/2} \times \gamma_N^{3/2}$ defines an $S$-generalized boundary triple for $A_{\text{max}}$, which is unitary in the Krein space sense (see Definitions 3.1, 1.10). Therefore, $\gamma_D^{3/2} \times \gamma_N^{3/2}$ is also closed in $(\text{dom } A_{\text{max}}) \times L^2(\partial\Omega)$, i.e., the closures coincide. \(\Box\)

When applying form methods, it is often convenient to consider the above traces on $H^1(\Omega)$. In this case $\gamma_N^s$ maps onto $H^{-1/2}(\partial\Omega)$ and one needs (Sobolev) dual parings of the boundary spaces for Green’s identity. Of course, if one restricts such boundary mappings on the side of the range to $L^2(\partial\Omega) \times L^2(\partial\Omega)$ one gets again the mapping $\gamma_D^{3/2} \times \gamma_N^{3/2}$ by continuity and surjectivity of $\gamma_N^s$ onto $H^{3/2}(\partial\Omega), 0 < s < 3/2$; see (7.6).

The results concerning the $L^2$-closure of the $\gamma$-field in Section 5 (see in particular Proposition 5.3, Lemma 5.24, Theorem 5.26) are now specialized to the $ES$-generalized boundary triple appearing in part (ii) of Proposition 7.1.

**Proposition 7.4.** Let $\{L^2(\partial\Omega), \tilde{\Gamma}_{0,\Omega} | \text{dom } \tilde{A}_s, \tilde{\Gamma}_{1,\Omega} | \text{dom } \tilde{A}_s\}$ be the $ES$-generalized boundary triple for $A^*$ in Proposition 7.1 and let $\tilde{M}(\cdot)$ and $\tilde{\gamma}(\cdot)$ be the corresponding Weyl function and the $\gamma$-field. Then:

(i) the closure of the boundary mapping $\tilde{\Gamma}_{0,\Omega} | \text{dom } \tilde{A}_s$ coincides with $\tilde{\Gamma}_{0,\Omega}$ in (7.11),

\[
\tilde{\Gamma}_{0,\Omega} : \text{dom } A_{\text{max}} \to H^{1/2}(\partial\Omega),
\]

it maps bijectively and continuously $\mathcal{M}_z(A_{\text{max}}) = \ker (A_{\text{max}} - zI), z \in \rho(A_K)$, onto $H^{1/2}(\partial\Omega)$ and ker $\tilde{\Gamma}_{0,\Omega} = \text{dom } A_K$, where $A_K$ is the Krein-von Neumann extension of $A_{\text{min}}$.

(ii) the closure of the $\gamma$-field is given by

\[
\tilde{\gamma}(z) = (\tilde{\Gamma}_{0,\Omega} | \mathcal{M}_z(A_{\text{max}}))^{-1},
\]

it is an unbounded and domain invariant operator with $\text{dom } \tilde{\gamma}(z) = H^{1/2}(\partial\Omega)$ and, furthermore, $\text{ran } \tilde{\gamma}(z) = \mathcal{M}_z(A_{\text{max}}), z \in \mathbb{C} \setminus \mathbb{R}$;

(iii) the domain decomposition (5.48) for the closures in Lemma 5.24 reads as

\[
(\text{dom } A_{\text{max}}) = \text{dom } \tilde{\Gamma}_{0,\Omega} = \text{dom } A_K + \text{ran } \tilde{\gamma}(z), \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

i.e., here (5.48) reduces to the second von Neumann formula for dom $A_{\text{max}}$.

Proof. (i) Recall that $\tilde{\Gamma}_{0,\Omega} : \text{dom } A_{\text{max}} \to H^{1/2}(\partial\Omega)$ is defined everywhere on dom $A_{\text{max}}$ and that it is continuous and surjective; see discussion following (7.11). Therefore, $\tilde{\Gamma}_{0,\Omega} : \text{dom } A_{\text{max}} \to L^2(\partial\Omega)$ is also bounded and closed. By Lemma 5.24 $\text{clo} \tilde{A}_0 = A_{\text{max}} | \ker \tilde{\Gamma}_{0,\Omega}$ is selfadjoint and
coincides with the Krein - von Neumann extension $A_K$ of $A_{min}$; see Proposition 7.4. Consequently, the mapping $\widetilde{\Gamma}_0^{\Omega} : H_z(A_{max}) \to H^{1/2}(\partial \Omega)$ is bijective and continuous for all $z \in \rho(A_K)$.

(ii) The formula for the closure of the $\gamma$-field is obtained by combining (i) with its abstract description (5.21) in Proposition 5.3. Moreover, part (i) shows that $\widetilde{\gamma}(z) : H^{1/2}(\partial \Omega) \to H_z(A_{max})$ is bijective and continuous w.r.t. $H^{1/2}(\partial \Omega)$ topology, and closed and unbounded as an operator from $L^2(\partial \Omega)$ onto $H_z(A_{max})$ with bounded inverse.

(iii) This follows immediately from Lemma 5.24 and the descriptions in items (i) and (ii). □

**Remark 7.5.** The description of the Krein - von Neumann Laplacian in part (iii) of Proposition 7.6 by means of trace operators essentially goes back to [10]; see also [31] Section 12.3. For Lipschitz domains a similar description of the Krein- von Neumann Laplacian in terms of extended trace operators was recently given in [21]; see also Section 7.3 below for another construction.

The next result characterizes the Weyl functions of various boundary triples appearing in Proposition 7.1 more precisely.

**Proposition 7.6.** Let the notations and assumptions be as in Proposition 7.1.

(i) the Weyl function $M(z) = -\Lambda(z)$ of the triple $\{L^2(\partial \Omega), \gamma_D|_{\text{dom } A_\ast}, \gamma_N|_{\text{dom } A_\ast}\}$ is domain invariant with $\text{dom } M(z) = H^1(\partial \Omega)$ and belongs to the class of inverse Stieltjes functions of unbounded operators while the inverse $\Lambda(z)^{-1}$ belongs to the class of Stieltjes functions of compact operators;

(ii) the Weyl function $\tilde{M}(\cdot)$ of the boundary triple $\{L^2(\partial \Omega), \tilde{\Gamma}_0^{\Omega}|_{\text{dom } \tilde{A}_\ast}, \tilde{\Gamma}_1^{\Omega}|_{\text{dom } \tilde{A}_\ast}\}$ is domain invariant with $\text{dom } \tilde{M}(\cdot) = H^{1/2}(\partial \Omega)$ and belongs to the class of inverse Stieltjes functions of unbounded operators while the inverse $-\tilde{M}(\cdot)^{-1} = \text{clos} \ (\Lambda(0) - \Lambda(\cdot))$ belongs to the class of inverse Stieltjes functions of bounded operators.

**Proof.** (i) Since $\ker(\gamma_D|_{\text{dom } A_\ast}) = \text{dom } \Delta_D$ and $-\Delta_D = -\Delta_D$, $M(\lambda)$ is domain invariant and moreover $\text{dom } M(\lambda) = \gamma_D|_{\text{dom } A_\ast}$; see Lemma 3.10 Theorem 4.2 (v). Now (7.7) and surjectivity of $\gamma_D^{3/2}$ in (7.5) gives $\gamma_D|_{\text{dom } A_\ast} = H^1(\partial \Omega)$. The transposed boundary triple $\{L^2(\partial \Omega), \gamma_N|_{\text{dom } A_\ast}, \gamma_D|_{\text{dom } A_\ast}\}$ is $B$-generalized and $M(\lambda)^{-1} : L^2(\partial \Omega) \to H^1(\partial \Omega), \lambda \in \rho(M)$, is bounded. Since the embedding $H^1(\partial \Omega) \to L^2(\partial \Omega)$ is compact, $M(\lambda)^{-1}$ is a compact operator in $L^2(\partial \Omega)$. To get the statements concerning Stieltjes and inverse Stieltjes classes we apply Green’s first identity:

$$\int_{\Omega} (\Delta u) \overline{v} \, dx + \int_{\Omega} \nabla u \cdot \nabla \overline{v} \, dx = (\gamma_N u, \gamma_D v)_{L^2(\partial \Omega)}, \quad u, v \in C^2(\Omega).$$

With $u, v \in \ker \Delta$ this leads to $(\Lambda(0) \gamma_D u, \gamma_D v)_{L^2(\partial \Omega)} = (\gamma_N u, \gamma_D v)_{L^2(\partial \Omega)} \geq 0$ and by denseness of $u, v$ in $\text{dom } \gamma_D$ one concludes that $\Lambda(0) \geq 0$. Equivalently, $M(0) = -\Lambda(0) \leq 0$ and, since $M(\cdot)$ belongs to the class $\mathcal{R}(\mathcal{H})$ of Nevanlinna functions by Theorem 1.11 (iv) and moreover, $0 \in \rho(\Delta_D)$, $M(\cdot)$ is holomorphic and monotone on the interval $(-\infty, 0]$. Therefore, $M(x) \leq 0$ for all $x \leq 0$ and thus $M(\cdot)$ is an inverse Stieltjes function. Consequently, the inverse function $-M(\cdot)^{-1}$ is a Stieltjes function.

(ii) It was shown in the proof of Proposition 7.1 that $\tilde{\Gamma}_0^{\Omega}|_{\text{dom } \tilde{A}_\ast}$ coincides with $\tilde{\Gamma}_0^{\Omega} \times \tilde{\Gamma}_1^{\Omega} : H^{1/2}(\Omega) \to (L^2(\partial \Omega))^2$. In view of (7.6) (with $s = 1/2$) $\tilde{\Gamma}_1^{\Omega}$ is surjective and, moreover, $\ker \tilde{\Gamma}_1^{\Omega} = \text{dom } \Delta_D$. Hence, again the transposed boundary triple

$$\{L^2(\partial \Omega), \tilde{\Gamma}_1^{\Omega}|_{\text{dom } \tilde{A}_\ast}, -\tilde{\Gamma}_0^{\Omega}|_{\text{dom } \tilde{A}_\ast}\}$$

is $B$-generalized and the values $-\tilde{M}(z)^{-1}, z \in \rho(\Lambda)$, are bounded operators; cf. Proposition 6.2 (iv). By Proposition 7.1 $\tilde{M}(0)^{-1} = 0$ and since $-\tilde{M}(\cdot)^{-1}$ is a Nevanlinna function
and holomorphic on $(-\infty, 0]$, one concludes by monotonicity that $\tilde{M}(x)^{-1} \leq 0$ for all $x < 0$. Hence, $-\tilde{M}(\cdot)^{-1}$ is an inverse Stieltjes function and its inverse $\tilde{M}(\cdot)$ is a Stieltjes function.

As to the form domain invariance of $\tilde{M}(\cdot)$ notice that
\begin{equation}
\text{dom} \tilde{\Gamma}_{-s}(\lambda) = \text{dom} \tilde{\gamma}(\lambda) = H^{1/2}(\partial \Omega), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\end{equation}
by Theorem 5.26 and Proposition 7.4.

Notice that by Proposition 7.6 and Corollary 7.3 the closure of the Weyl function of the quasi boundary triple $\{L^2(\partial \Omega), \gamma_0^D, \text{dom} A_s, -\gamma_0^N \} \setminus \text{dom} A_s$ for all $3/2 \leq s \leq 2$ is just the Weyl function $M(\cdot)$ of the boundary triple $\{L^2(\partial \Omega), \gamma_0^D, \text{dom} A_s, -\gamma_0^N \} \setminus \text{dom} A_s$.

Finally, the renormalization result in Theorem 5.31 is specialized to the case of the $L^2$-generalized boundary triple appearing in Proposition 7.3. For this purpose we need a bounded operator $G$ in $L^2(\partial \Omega)$ with $\text{ran} G = H^{1/2}(\partial \Omega)$; see (7.15). Let $\Delta_{\partial \Omega}$ be the Laplace-Beltrami operator on $L^2(\partial \Omega)$. Then $-\Delta_{\partial \Omega} + I \geq 0$ and the operator
\begin{equation}
G := (-\Delta_{\partial \Omega} + I)^{-1/4}
\end{equation}
is a nonnegative contraction in $L^2(\partial \Omega)$ with $\text{ran} G = H^{1/2}(\partial \Omega)$. With this choice of $G$ Theorem 5.31 leads to the following result.

**Corollary 7.7.** Let $\{L^2(\partial \Omega), \tilde{\Gamma}_{0,\Omega}, \text{dom} \tilde{A}, \tilde{\Gamma}_{1,\Omega}, \text{dom} \tilde{A}_s\}$ be the ES-generalized boundary triple in Proposition 7.7 (ii). Then the renormalized boundary triple $\{L^2(\partial \Omega), \Gamma_{0,\Omega}, \Gamma_{1,\Omega}\}$ defined by (7.15) is an ordinary boundary triple for $\text{max} \max$ given by
\begin{equation}
\Gamma_{0,\Omega} = G^{-1}\tilde{\Gamma}_{0,\Omega} = G^{-1}(\gamma_N - \Lambda(0)\gamma_0^D), \quad \Gamma_{1,\Omega} = \tilde{G}\tilde{\Gamma}_{1,\Omega} = G^2, \tag{7.15}
\end{equation}
i.e., $\Gamma_{0,\Omega} \times \Gamma_{1,\Omega} : \text{dom} \text{max} \rightarrow (L^2(\partial \Omega))^2$ is surjective.

The corresponding Weyl function is given by
\begin{equation}
M_G(z) = \tilde{G}(\Lambda(z) - \Lambda(0))^{-1}G,
\end{equation}
it is a uniformly strict Nevanlinna function belonging also to the class of Stieltjes functions.

It follows from Corollary 7.7 for instance that $A^* \mid \ker \Gamma_{0,\Omega} = A_K$ and $A^* \mid \ker \Gamma_{0,\Omega} = A_F$ and the formula
\begin{equation}
\tilde{A} \rightarrow \Gamma_G(\text{dom} \tilde{A}) = \{ \{ \Gamma_{G,0}, f, \Gamma_{G,1} f \} : f \in \text{dom} \tilde{A} \} =: \Theta
\end{equation}
establishes a one-to-one correspondence between selfadjoint (nonnegative) realization of the Laplacian operator $-\Delta$ and the selfadjoint (nonpositive) relations $\Theta$ in $L^2(\partial \Omega)$ via boundary conditions as expressed in (7.16).

For a general class of elliptic operator in bounded and unbounded domains [77] Proposition 3.5, 5.1 an ordinary boundary triple for $\text{max} \max$ was constructed via the transposed boundary triple $\{L^2(\partial \Omega), \tilde{\Gamma}_{1,\Omega}, \text{dom} \tilde{A}_s, -\tilde{\Gamma}_{0,\Omega} \}$ which is $B$-generalized. A similar regularization method for bounded domains $\Omega$ appears already in [33] without a general formalism of boundary triples. In [77] Proposition 5.1 the constructed boundary triple is the transposed boundary triple $\{L^2(\partial \Omega), \Gamma_{1,\Omega}, -\Gamma_{0,\Omega}\}$ and resulted in the Weyl function
\begin{equation}
-M_G(z)^{-1} = G^{-1}(\Lambda(0) - \Lambda(z))G^{-1};
\end{equation}
(for the sign change, notice that in [77] interior, instead of exterior, normal derivatives to $\partial \Omega$ are being used). It is emphasized that the construction here relies on the general renormalization result proved for abstract operators in Theorem 5.31.

As shown in [77] with a renormalized boundary triple which is ordinary it is possible to carry out spectral analysis for the selfadjoint realizations of $-\Delta$ with the aid of Krein’s resolvent formula. Alternatively, one can apply in the study a Krein type resolvent formula for the more...
general classes of boundary triples as in Theorem 7.11 or Theorem 5.39 see 39 Example 7.27 for a discussion which uses the renormalization in 77, and for various related contributions in the study of such nonlocal boundary conditions, see e.g. 6, 17, 18, 48, 53.

7.2. Mixed boundary value problem for Laplacian. Let \( \Omega \) be a bounded open set in \( \mathbb{R}^d \) (\( d \geq 2 \)) with a smooth boundary \( \partial \Omega \). Let \( \Sigma_+ \) be a compact smooth submanifold of \( \partial \Omega \) and \( \Sigma_- := \partial \Omega \setminus \Sigma_+ \), so that \( \Sigma = \Sigma_+ \cup \Sigma_- \). Here \( \Sigma_+^0 \) is the interior of \( \Sigma_+ \). Let \( -\Delta_Z \) be the Zaremba Laplacian, i.e. the restriction of the maximal operator \( A_{\max} \) to the set of functions, which satisfy Dirichlet boundary condition on \( \Sigma_- \) and Neumann boundary condition on \( \Sigma_+ \).

Let \( H^1_{\Sigma_+}(\Omega) = \{ u \in H^1(\Omega) : \text{supp}\, \gamma_D u \subset \Sigma_+ \} \). It is known (see for instance [55]), that the operator \(-\Delta_\Sigma\) is associated with the nonnegative closed quadratic form \( \gamma_D : \text{dom}\, \gamma_D \to \mathbb{R} \) is a symmetric realization of the Laplacian \(-\Delta\) to the domain \( \gamma_D(\Omega) : (\gamma_D f)_{|\Sigma_-} = 0 \).

Using the Green formula and then applying the regularity result for the realization \(-\Delta_N\) we derive that \( S_{\pm} = (S_{\pm})^* \) is a symmetric realization of the Laplacian \(-\Delta\) on the domain \( \Omega \). Hence

\[
\text{dom } S_{\pm} = \{ f \in H^3/2(\Omega) : \gamma_N f_{|\Sigma_+} = 0 \}.
\]

Notice that \( \text{dom } S_{\pm} = \bigcup_{\Sigma_-} S_{\pm} \). Here \( S_{\pm} \) is an intermediate extension of \( S = A_{\min} \) in the sense of [38]. More precisely we have the following result.

**Proposition 7.8.** Let the operator \( S_{\pm} \) be defined by (7.17) and let \( S_{\pm} = (S_{\pm})^* \). Then:

(i) \( \Pi^+ = (L^2(\Sigma_+), \gamma_D, \gamma_N, P_{L^2(\Sigma_+)}\gamma_D) \) is a B-generalized boundary triple for \( (S_{\pm})^* \);

(ii) the Weyl function corresponding to the boundary triple \( \Pi^+ \) equals to

\[
\Lambda^+(z) = \Lambda_{L^2(\Sigma_+)}(z)^{-1} | L^2(\Sigma_+),
\]

where \( \Lambda(z)^{-1} \) is the Neumann-to-Dirichlet map;

(iii) \( (\Pi^+)^+ = (L^2(\Sigma_+), P_{L^2(\Sigma_+)\gamma_D}, -P_{L^2(\Sigma_+)\gamma_N}) \) is an ES-generalized boundary triple for \( (S_{\pm})^* \).

**Proof.** (i) As is proved in Proposition 7.11(i) the triple \( \Pi = (L^2(\partial \Omega), \gamma_N, \gamma_D) \) is a B-generalized boundary triple for \( A^* \). Since \( S_+ \) is an intermediate extension of \( A \), [38, Proposition 4.1] implies that the triple \( \Pi^+ = (L^2(\Sigma_+), P_{L^2(\Sigma_+)\gamma_D}, P_{L^2(\Sigma_+)\gamma_N}) \) is a B-generalized boundary triple for \( (S_{\pm})^* \). Notice that

\[
\text{dom } A_{0,\pm} = \ker \Gamma_0^* = \{ f \in \text{dom } (S_{\pm}) : P_{L^2(\Sigma_+)\gamma_N} f = 0 \}
\]

\[
= \{ f \in H^3/2(\Omega) : \gamma_N f = 0 \} = \{ f \in H^2(\Omega) : \gamma_N f = 0 \} = \text{dom } (-\Delta_N).
\]
is selfadjoint, since it coincides with the Neumann Laplacian.

(ii) This statement is implied by the fact that the Weyl function of the operator $A$, corresponding to the boundary triple $\Pi = (L^2(\partial \Omega), \gamma_N, \gamma_D)$, coincides with $\Lambda(z)^{-1}$; see [38, Proposition 4.1].

(iii) Consider the operator $A_{1,+}$ defined as the restriction of $-\Delta$ to the domain
\[
\text{dom } A_{1,+} = \{ f \in \text{dom } (S_+) : P_{L^2(\Sigma_-)} \gamma_D f = 0 \} = \{ f \in H^{3/2}_\Delta(\Omega) : (\gamma_N f)|_{\Sigma_+} = (\gamma_D f)|_{\Sigma_-} = 0 \}.
\]
Note that $\text{dom } (-\Delta_\Sigma) \subset H^{3/2-\varepsilon}(\Omega)$ for each $\varepsilon > 0$ while $\text{dom } (-\Delta_\Sigma) \not\subset H^{3/2}(\Omega)$ for certain configurations of $\Sigma_+$ (see [55]). Therefore, for such subsets $\Sigma_+$ the operator $A_{1,+}$ is a proper symmetric restriction of Zaremba Laplacian $-\Delta_{\Sigma}$, hence $A_{1,+}$ is not selfadjoint.

To prove the statement it suffices to show that the operator $A_{1,+}$ is essentially selfadjoint. Assuming the contrary one finds $\lambda_0 = \tilde{\lambda}_0 \notin \sigma_p(-\Delta_\Sigma)$ and a vector $g \in L^2(\Omega)$ such that $g \perp \text{ran } (A_{1,+} - \lambda_0)$, i.e.
\[
(g, (-\Delta - \lambda_0)f)_{L^2(\Omega)} = 0 \quad \text{for all } f \in \text{dom } A_{1,+}.
\]
This relation with $f \in \text{dom } S$ implies $g \in \text{dom } (A_{\text{max}})$ and $(-\Delta - \lambda_0)g = 0$. Letting $f \in \text{dom } S_+$ and applying the Green formula one obtains from (7.18) and (7.20) that
\[
0 = (g, -\Delta f)_{L^2(\Omega)} - (\lambda_0 g, f)_{L^2(\Omega)} = (g, -\Delta f)_{L^2(\Omega)} - (\lambda_0 g, f)_{L^2(\Omega)} = \langle \gamma_D g, \gamma_N f \rangle_{-1/2,1/2} - \langle \gamma_N g, \gamma_D f \rangle_{-3/2,3/2} = \langle (-\lambda_0) g |_{\Sigma_+}, (\gamma_D f)|_{\Sigma_+} \rangle_{-3/2,3/2},
\]
Here $\langle \cdot, \cdot \rangle_{-s,s}$ denotes duality between $H^{-s}(\partial \Omega)$ and $H^s(\partial \Omega)$ ($s \in \mathbb{R}$). It follows from (7.2) that $\gamma_D(\text{dom } \Delta_N) = H^{3/2}(\partial \Omega)$. Hence $\gamma_D(\text{dom } S_+) = H^{3/2}(\Sigma_+)$ and the latter implies
\[
\gamma_N(\text{dom } \Delta_D) = H^{1/2}(\partial \Omega).
\]
Similarly, it follows from (7.2) that $\gamma_N(\text{dom } \Delta_D) = H^{1/2}(\partial \Omega)$. For a subset $\mathcal{L}$ of $\text{dom } A_{1,+}$
\[
\mathcal{L} = \{ f \in H^2(\Omega) : (\gamma_N f)|_{\Sigma_+} = \gamma_D f = 0 \},
\]
one obtains
\[
\gamma_N \mathcal{L} = H^{1/2}(\Sigma_-).
\]
Let now $f \in \mathcal{L}$. Then using the Green formula the equality (7.20) can be rewritten as
\[
0 = (g, -\Delta f)_{L^2(\Omega)} - (\lambda_0 g, f)_{L^2(\Omega)} = \langle \gamma_D g, \gamma_N f \rangle_{-1/2,1/2}
\]
and (7.23), (7.24) lead to
\[
(\gamma_D g)|_{\Sigma_-} = 0.
\]
Since $g \in \text{dom } (A_{\text{max}})$, relations (7.22) and (7.25) mean that $g \in \text{dom } (-\Delta_\Sigma)$. Thus $g \in \ker (-\Delta_\Sigma - \lambda_0) = \{0\}$, hence $g = 0$. This completes the proof. \hfill \Box

**Remark 7.9.** As follows from Theorem 5.26 the statement (iii) in Proposition 7.8 is equivalent to the fact that the $\gamma$-field $\gamma(\lambda)$ admits a single-valued closure for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ with constant domain and the $M$-function $-\Lambda_+(z)^{-1}$ is form domain invariant. As was mentioned in the proof the operator $A_{1,+}$ is essentially selfadjoint while is not selfadjoint. By Theorems 1.11, 5.18 this implies that the operators $\gamma(\lambda)$ are not bounded; this fact was apparently first mentioned in [85, Theorem 6.23]. In particular, the corresponding boundary triple $(\Pi^+)^\top$ is neither $S$-generalized, nor an $AB$-generalized or a quasi boundary triple in the sense of [17].
7.3. Laplacians on Lipschitz domains. Here the smoothness properties on \( \Omega \) are relaxed; it is assumed that \( \Omega \) is a bounded Lipschitz domain. In this case the Dirichlet and Neumann traces \( \gamma_D \) and \( \gamma_N \)
\[
\gamma_D : H^1_\Delta(\Omega) \to H^{s-1/2}(\partial \Omega), \quad \gamma_N : H^s_\Delta(\Omega) \to H^{s-3/2}(\partial \Omega),
\]
are still continuous operators for all \( 1/2 \leq s \leq 3/2 \) and, in addition, both are surjective when \( s = 1/2 \) and \( s = 3/2 \); see [48, Lemmas 3.1, 3.2]. In this case the results, which are analogous to those in Section 7.1, will be derived directly from the abstract setting treated in Section 6.1.

The following analog of Proposition 7.1 is obtained from Proposition 6.2 using the 3/2 regularity of the selfadjoint extensions \( -\Delta_D \) and \( -\Delta_N \); see [61, 62, 48]. Since \( 0 \in \rho(-\Delta_D) \) one can decompose

\[
\text{dom } A_{\text{max}} = \text{dom } \Delta_D + \ker A_{\text{max}}.
\]

Proposition 7.10. Let \( \Omega \subset \mathbb{R}^n, n \geq 2, \) be a bounded Lipschitz domain. Let the operators \( \gamma_N, \gamma_D, P(z), \Lambda(z) \) and \( A \) be defined by (7.3), (7.8), (7.3), and (7.7). Then:

(i) \( \{L^2(\partial \Omega), \gamma_D \mid \text{dom } A_\gamma = \text{dom } A_\gamma = H^3/2_\Delta(\Omega), \) the transposed boundary triple is B-generalized, moreover, the corresponding \( \gamma \)-field \( \gamma(\cdot) \) is bounded and coincides with \( P(z) \) and the Weyl function \( M(\cdot) \) coincides with \( -\Lambda(\cdot) \);\n
(ii) \( \{L^2(\partial \Omega), \Gamma_{0,\Omega}, \Gamma_{1,\Omega} \}, \) where

\[
(\Gamma_{0,\Omega}, \Gamma_{1,\Omega}) (f + \gamma(0) h) = \begin{pmatrix} -\gamma(0)^* \Delta_D f \cr -h \end{pmatrix}, \quad f \in \text{dom } \Delta_D, \quad h \in L^2(\partial \Omega),
\]
defines an ES-generalized boundary triple for \( A_{\text{max}} \) with dense domain \( A_\gamma = \text{dom } \Delta_D + \text{ran } \gamma(0) \subset A^* \); the transposed boundary triple is B-generalized, and the corresponding Weyl function is the \( L^2(\partial \Omega) \)-closure\n
\[
\widehat{M}(z) = \text{clos} \left( \Lambda(z) - \Lambda(0) \right)^{-1};
\]

(iii) the extension \( \tilde{A}_0 := A_{\text{max}} \mid \ker \overline{\Gamma}_{0,\Omega} \) is essentially selfadjoint and its closure coincides with the Krein - von Neumann extension of the operator \( A_{\text{min}} \).

Proof. (i) Green’s identity holds: this can be obtained for instance from the formula (3.21) in [48] (cf. proof of Proposition 7.15 below). According to [61, 62, 48] (cf. proof of Proposition 7.15 below). According to [48] (cf. proof of Proposition 7.15 below). According to [48] (cf. proof of Proposition 7.15 below). According to [48] (cf. proof of Proposition 7.15 below). According to [48] (cf. proof of Proposition 7.15 below).
Corollary 7.7. The inverse of the regularized Dirichlet-to-Neumann map has the form
\[
\tilde{M}(z) = \text{clos}((\Lambda(z) - \Lambda(0))^{-1} = \gamma(0)^{(-1)} M_r(z) \gamma(0)^{(-1)}
\]
and, consequently, the Dirichlet-to-Neumann map has the representation
\[
\Lambda(z) = \Lambda(0) + \gamma(0)^{*} M_r(z)^{-1} \gamma(0), \quad z \in \rho(-\Delta_D).
\]
Notice that here by definition \(M_r(0)^{-1} = (\infty^{-1}) = 0\).

Comparing Proposition 7.10 (ii) with Proposition 7.11 (i) we get the following equality
\[
\text{ran } \Gamma_{0,\Omega} = \text{dom } \tilde{M}(z) = \text{ran } \gamma(0)^{*}, \quad z \in \rho(-\Delta_D).
\]
Furthermore, it is clear from (7.29) that
\[
\text{ran } \Gamma_{0,\Omega} \times \Gamma_{1,\Omega} = \text{ran } \gamma(0)^{*} \times L^2(\partial\Omega).
\]
In particular, one can renormalize the regularized boundary mappings \(\Gamma_{0,\Omega} = \gamma_N - \Lambda(0)\gamma_D\), \(\Gamma_{1,\Omega} = \gamma_D\) also by any bounded operator \(G\) acting in the original boundary space \(L^2(\partial\Omega)\) satisfying \(G = \text{ran } \gamma(0)^{*}\) and \(\ker G = \{0\}\), and this leads to an isomorphic copy of the results in Proposition 7.11. In this case the parametrization of all intermediate extensions of \(A_{\min}\) can be expressed via boundary conditions involving \(G^{-1}(\gamma_N - \Lambda(0)\gamma_D)\) and \(G^*\gamma_D\); cf. Corollary 7.7.

7.4. Laplacian on rough domains. Let \(\Omega\) be a bounded domain in \(\mathbb{R}^d\) \((d \geq 2)\) whose boundary \(\partial\Omega\) is equipped with a finite \((d-1)\)-dimensional Hausdorff measure \(\sigma\), \(\sigma(\partial\Omega) < \infty\). To construct an analog for the boundary triple appearing in Proposition 7.1 (i) in nonsmooth domains \(\Omega\) we make use of some results established in [35] and [7, 8, 9]. Following [7, Definition 3.1] we first recall the notion of a trace \(\varphi \in L^2(\sigma)\) for a class of functions \(u \in H^1(\Omega)\).

**Definition 7.13.** A function \(\varphi \in L^2(\sigma)\) is said to be a trace of \(u \in H^1(\Omega)\), if there is a sequence \(u_n \in H^1(\Omega) \cap C(\overline{\Omega})\), such that
\[
\lim_{n \to \infty} u_n = u \quad \text{(in } H^1(\Omega)) \quad \text{and} \quad \lim_{n \to \infty} u_n|_{\partial\Omega} = \varphi \quad \text{(in } L^2(\sigma)).
\]
Denote by $H^1_\sigma(\Omega)$ the set of elements of $H^1(\Omega)$ for which there exists a trace. In general, the trace is not uniquely defined. It is possible that $u \mid \Omega = 0$ while its trace $\gamma_D u = u \mid \partial \Omega$ in $L^2(\sigma)$ is nontrivial; for an example see e.g. [7, Example 4.4]. Define the linear relation $\gamma_D$ by

$$\gamma_D := \{\{u, \varphi\} : u \in H^1_\sigma(\Omega), \varphi \in L^2(\sigma), \varphi \text{ is a trace of } u\}.$$ 

Then $\gamma_D$ can be considered as a mapping from $H^1(\Omega)$ to $L^2(\sigma)$, which is linear but in general multivalued on the domain $H^1_\sigma(\Omega)$ and it has dense range in $L^2(\sigma)$; cf. [4]. If $u$ and $\varphi$ are as in Definition 7.13 we shall write

$$\varphi \in \gamma_D u.$$ 

The space $H^1_\sigma(\Omega)$ coincides with the closure of $H^1(\Omega) \cap C(\overline{\Omega})$ in the norm

$$(7.27) \quad \|u\|_{1,\sigma}^2 = \|u\|_{H^1(\Omega)}^2 + \int_{\partial \Omega} |u|^2 \, d\sigma.$$ 

Following [4] denote by $\widetilde{H}^1(\Omega)$ the closure of $H^1(\Omega) \cap C(\overline{\Omega})$ in $H^1(\Omega)$. In view of $(7.27)$ $H^1_\sigma(\Omega)$ is a subset of $\widetilde{H}^1(\Omega)$. Without additional conditions on $\Omega$ the space $H^1(\Omega)$ need not be dense in $\widetilde{H}^1(\Omega)$. Some sufficient conditions, like $\Omega$ being starshaped or having a continuous boundary, can be found e.g. in [82, Section 1.1.6]. Consequently, $H^1_\sigma(\Omega)$ is not necessarily a dense subset of $H^1(\Omega)$.

For associating an appropriate boundary triple in this setting, we impose the following additional assumption.

**Assumption 7.14.** $H^1_\sigma(\Omega) = \widetilde{H}^1(\Omega)$.

A list of conditions equivalent to Assumption 7.14 is given in [7] Theorem 6.1. Notice that the space $H^1_\sigma(\Omega)$ appearing in [7] Section 5] has a norm which is equivalent to norm of $H^1_\sigma(\Omega)$ defined in $(7.27)$ due to the following special case of Maz’ya inequality: there exists a constant $c_M > 0$ such that

$$(7.28) \quad \int_\Omega |u|^2 \, dx \leq c_M \left( \int_\Omega |\nabla u|^2 \, dx + \int_{\partial \Omega} |u|^2 \, d\sigma \right)$$

holds for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$; see [82] Section 3.6, [7] eq. (5)]. The inequality $(7.28)$ is a generalization of Friedrichs inequality to the case of rough domains.

In [7] Definition 3.2] the (weak) normal derivative is defined implicitly via Green’s (first) formula as follows: a function $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ is said to have a weak normal derivative in $L^2(\sigma)$ if there exists $\psi \in L^2(\sigma)$ such that

$$(7.29) \quad \int_\Omega (\Delta u) \overline{\psi} \, dx + \int_\Omega \nabla u \cdot \nabla \overline{\psi} \, dx = \int_{\partial \Omega} \psi \overline{\varphi} \, d\sigma$$

holds for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$, where $\Delta u$ denotes the Laplacian understood in distributional sense. Since the functions $v \mid \partial \Omega$, $v \in H^1(\Omega) \cap C(\overline{\Omega})$, form a dense set in $L^2(\sigma)$, the function $\psi \in L^2(\sigma)$ is uniquely determined by $u$ and the mapping $u \mapsto \psi$ is denoted by $\gamma_N$:

$$\gamma_N u := \psi, \quad u \in \text{dom } \gamma_N \subset H^1(\Omega) \cap \text{dom } A_{\text{max}}.$$ 

Assume that for some $\varphi, \psi \in L^2(\sigma)$, $u \in H^1(\Omega)$, and $x \leq 0$ one has

$$(7.30) \quad (-\Delta - x I) u = 0, \quad \varphi \in \gamma_D u, \quad \psi = \gamma_{N} u, \quad x \leq 0.$$ 

The operator $\Lambda(x)$ which maps $\varphi$ to $\psi$ is called the Dirichlet-to-Neumann map. A slight modification of the proof of [7, Theorem 3.3] shows, that $\Lambda(x)$ is a nonnegative selfadjoint operator on $L^2(\sigma)$ which is uniquely determined by the three properties listed in $(7.30)$.

Now consider the differential expression $-\Delta$, where $\Delta = \nabla \cdot \nabla$ is the (distributional) Laplacian operator in $\Omega$. Recall (see [4] Example 3.1]) that for an open set $\Omega$ (without any regularity on
the boundary) the Dirichlet Laplacian $-\Delta_D$ is defined as the selfadjoint operator associated with the closed (Dirichlet) form

$$\tau_D(f, g) = \int_\Omega \nabla f \cdot \nabla g \, dx, \quad \text{dom } \tau_D = H_0^1(\Omega).$$

Similarly the Neumann Laplacian $-\Delta_N$ is defined as the selfadjoint operator associated with the closed form (see [9, Example 3.2])

$$\tau_N(f, g) = \int_\Omega \nabla f \cdot \nabla g \, dx, \quad \text{dom } \tau_N = \tilde{H}^1(\Omega).$$

**Proposition 7.15.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, whose boundary $\partial \Omega$ is equipped with a finite $(d - 1)$-dimensional Hausdorff measure $\sigma$, let Assumption 7.14 be in force, and let the linear relation $\Gamma$ be defined by

$$\Gamma = \left\{(f, \varphi) : f \in \tilde{H}^1(\Omega) \cap \text{dom } \gamma_N, \, \varphi, \psi \in L^2(\sigma), \, \Delta f \in L^2(\Omega), \, \psi = \gamma_N f \right\}. \tag{7.32}$$

Then:

(i) the pair \(\{L^2(\sigma), \Gamma\}\) is a positive unitary boundary pair for $-\Delta$ on $A_* := \text{dom } \Gamma$;

(ii) for every $x < 0$ the Weyl function $M(x)$ corresponding to the pair \(\{L^2(\sigma), \Gamma\}\) coincides (up to the sign) with the Dirichlet-to-Neumann map $\Lambda(x)$:

$$M(x) = -\Lambda(x), \quad x < 0, \tag{7.33}$$

in particular, the function $M(\cdot)$ is an inverse Stieltjes function whose values $M(z)$, $z \in \mathbb{C} \cap [0, \infty)$, are (unbounded) operators with $\ker M(z) = \text{mul } \Gamma_0$;

(iii) the operator $A_1 := -\Delta |_{\ker \Gamma_0}$ coincides with the Neumann Laplacian $-\Delta_N$;

(iv) the transposed pair \(\{L^2(\sigma), \Gamma^\top\}\) is $S$-generalized and the corresponding Weyl function $-M(\cdot)^{-1}$ is a multivalued domain invariant Stieltjes function.

**Proof.** (i)–(iii) If $f \in \text{dom } \Gamma$, then the (first) Green’s identity (7.29) holds with $u = f$ and $v \in H^1(\Omega) \cap C(\Omega)$. Then in view of (7.29) this identity can be extended to hold for all $v \in H^1_0(\Omega)$. Thus, in particular, it holds for all $g := v \in \text{dom } \Gamma$:

$$\int_\Omega (\Delta f) \varphi \, dx + \int_{\partial \Omega} \nabla f \cdot \nabla \varphi \, d\sigma = \int_{\partial \Omega} \psi \varphi \, d\sigma, \quad \psi = \gamma_N f, \quad \varphi \in \gamma_D g. \tag{7.34}$$

Similarly, one gets from (7.29) with $u = g \in \text{dom } \Gamma$ and $v = f \in \text{dom } \Gamma$:

$$\int_\Omega (\Delta g) \varphi \, dx + \int_{\partial \Omega} \nabla g \cdot \nabla \varphi \, d\sigma = \int_{\partial \Omega} \tilde{\psi} \varphi \, d\sigma, \quad \tilde{\psi} = \gamma_N g, \quad \varphi \in \gamma_D f. \tag{7.35}$$

Taking conjugates in the last identity and subtracting the identity (7.34) from that leads to Green’s (second) formula in (3.11) for $-\Delta$ with $f, g \in A_* = \text{dom } \Gamma$. This means that \(\{L^2(\sigma), \Gamma\}\) is an isometric boundary pair.

To prove that \(\{L^2(\sigma), \Gamma\}\) is a unitary boundary pair, we proceed by proving (ii) and (iii). With $x < 0$ it follows from (7.32) that $\varphi \in \text{dom } M(x)$ and $M(x)\varphi = -\psi$ precisely when there exists $u \in \tilde{H}^1(\Omega) \cap \text{dom } \gamma_N$, such that

$$-\Delta u - xu = 0, \quad \varphi \in \gamma_D u, \quad \psi = \gamma_N u.$$

In view of (7.30) this means that the operator $-M(x)$ coincides with the Dirichlet-to-Neumann map $\Lambda(x)$, which is a nonnegative selfadjoint operator in $L^2(\sigma)$. This proves (7.33). The definition of $\Gamma$ shows that $\text{mul } \Gamma = \text{mul } \Gamma_0 \times \{0\}$ and hence by Lemma 3.6 ker $M(z) = \text{mul } \Gamma_0$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The assertion that $M(\cdot)$ is an inverse Stieltjes function is a
consequence of $M(x) \leq 0$, $x < 0$, (the nonnegativity of the main transform $\tilde{A}$, which is shown below, implies that $M(x)$ is also holomorphic at $x < 0$). This proves (ii).

By definition every $f \in \text{dom}(-\Delta_N)$ belongs to $\tilde{H}^1(\Omega)$. On the other hand, by Assumption \[ 7.13 \] $\tilde{H}^1(\Omega) = H^1_0(\Omega) = \text{dom} \gamma_D$ and hence, in particular, for every $f \in \text{dom}(-\Delta_N)$ there exists a Dirichlet trace $\varphi \in \gamma_D f$. Next it is shown that for every $f \in \text{dom}(-\Delta_N)$ also the Neumann trace $\gamma_D u$ exists. Indeed, by definition the Neumann Laplacian $-\Delta_N$ is the self-adjoint operator associated with the closed form $(\gamma_D)$. Hence, $(\gamma_D)$ implies that for all $f \in \text{dom}(-\Delta_N)$ and $g \in \tilde{H}^1(\Omega)$,

$$\int_{\Omega} \nabla f \cdot \nabla g \, dx = \int_{\Omega} (-\Delta f) g \, dx.$$ 

Comparing this identity with the definition of $\gamma_N$ it is seen that the equality $(\gamma_D)$ is satisfied with the choice $\psi = 0$. Therefore, $f \in \text{dom} \gamma_N$ and $\gamma_N f = 0$. This implies that $\text{dom}(-\Delta_N) \subset \text{dom} \gamma_N$ and, moreover, that $\text{dom}(-\Delta_N) \subset \ker \gamma_N = \text{dom} A_*$. Since $-\Delta_N$ is selfadjoint operator in $L^2(\Omega)$ and $A_*$ is symmetric (see Section 3.3), the equality $A_* = -\Delta_N$ follows. This proves the assertion (ii).

Next we complete the proof of (i) by showing that $\{L^2(\sigma), \Gamma\}$ is a positive unitary boundary pair, i.e., that the main transform $\tilde{A}$ of $\Gamma$ given by

$$(7.35) \quad \tilde{A} := \left\{ \left\{ \left( \frac{f}{\varphi} \right), \left(-\frac{\Delta f}{\varphi} \right) \right\} : \left\{ \left( \frac{f}{\varphi} \right), \left(-\frac{\Delta f}{\varphi} \right) \right\} \in \Gamma \right\}$$

is a nonnegative selfadjoint relation in $L^2(\Omega) \times L^2(\sigma)$; see (6.27). Nonnegativity of $\tilde{A}$ follows immediately from (7.34). On the other hand, by item (ii) the Weyl function satisfies $-M(x) = \Lambda(x) \geq 0$, $x < 0$, and hence it is a nonpositive selfadjoint operator with $-x \in \rho(M(x))$. Since $\text{dom}(-\Delta_N) \subset \text{dom} A_*$ and $\Delta_N \geq 0$ is selfadjoint it follows from Theorem 6.12 that $x \in \rho(\tilde{A})$ and hence $\tilde{A} = A_* \geq 0$, which proves the claim.

(iv) Since $A_1 = -\Delta_N$ is selfadjoint, the transposed pair $\{L^2(\sigma), \Gamma^T\}$ is $S$-generalized; see Definition 1.10. Moreover, the corresponding Weyl function $-M(x)^{-1} \geq 0$ of $\{L^2(\sigma), \Gamma^T\}$ is a nonnegative selfadjoint relation in $L^2(\sigma)$ for every $x < 0$. This implies that $-M(\cdot)^{-1}$ is a (multivalued) Stieltjes family. It is domain invariant by Theorem 5.18.

In this general setting, the multivalued part of $\Gamma$ can be nontrivial, since the trace $\gamma_D$ need not be uniquely determined. For unitary boundary pairs the multivalued part is described in [37, Lemma 4.1]; see also Lemma 3.6. In the present setting a more explicit description of the multivalued part can be given with the aid of a result of Daners in [35]; see also [9] for an other proof of Daners result via capacity arguments.

**Corollary 7.16.** There exists a Borel set $B_0 \subset \partial \Omega$, such that

$$\text{mul} \gamma_D = L^2(B_0), \quad \text{mul} \Gamma = \text{mul} \gamma_D \times \{0\},$$

and, in particular, $\text{mul} \gamma_D = \ker M(\lambda)$, $\lambda \in \mathbb{C} \setminus [0, \infty)$.

Hence, $\Gamma$ is single-valued if and only if $L^2(B_0) = \{0\}$, i.e., $\sigma(B_0) = 0$. The set $B_0$ is unique up to $\sigma$-equivalence $\sigma(B_0 \Delta B_0) = 0$. Since $\text{mul} \gamma_D \neq 0$ corresponds to $\sigma(B_0) > 0$, $B_0$ can be considered to represent an irregular part of the boundary.

**Remark 7.17.** In this general setting we do not know if the operator $A_0 := -\Delta \mid \ker \Gamma_0$ coincides with the Dirichlet Laplacian $-\Delta_D$. In other words, we do not know if the Neumann trace $\gamma_N u$ exists for every $u \in \text{dom}(-\Delta_D)$.
8. Applications to Differential Operators with Local Point Interactions

8.1. Abstract results on direct sums of boundary triples and their Weyl functions.

A general class of unitary boundary triples, which are more general than generalized boundary triples is obtained by considering an infinite orthogonal sum of ordinary boundary triples. Here we mainly follow the considerations in [69]; see also the references given therein.

Let $S_n$ be a densely defined symmetric operator with equal defect numbers $n_+(S_n) = n_-(S_n)$ in the Hilbert space $H_n$, $n \in \mathbb{N}$. Consider the operator $A = \bigoplus_{n=1}^\infty S_n$ in the Hilbert space $H := \bigoplus_{n=1}^\infty H_n = \{ \bigoplus_{n=1}^\infty f_n : f_n \in H_n, \sum_{n=1}^\infty \|f_n\|^2 < \infty \}$. Then $A$ is symmetric with equal defect numbers and its adjoint $A^*$ is given by

\begin{equation}
A^* = \bigoplus_{n=1}^\infty S_n^*, \quad \text{dom } A^* = \left\{ \bigoplus_{n=1}^\infty f_n \in H : f_n \in \text{dom } S_n^*, \sum_{n=1}^\infty \|S_n^*f_n\|^2 < \infty \right\}.
\end{equation}

Now let $\Pi_n = \{ \mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)} \}$ be an ordinary boundary triple for $S_n^*$, $n \in \mathbb{N}$. Let $\mathcal{H} = \bigoplus_{n=1}^\infty \mathcal{H}_n$, $\Gamma^{(n)} = \{ \Gamma_0^{(n)}, \Gamma_1^{(n)} \}$ and let the mapping $\Gamma_0'$ and $\Gamma_1'$ be defined by

\begin{equation}
\Gamma_j' := \bigoplus_{n=1}^\infty \Gamma_j^{(n)}, \quad \text{dom } \Gamma_j' = \left\{ \bigoplus_{n=1}^\infty f_n \in \text{dom } A^* : \sum_{n \in \mathbb{N}} \|\Gamma_j^{(n)} f_n\|_{\mathcal{H}_n}^2 < \infty \right\}, \quad j \in \{0,1\}.
\end{equation}

We also put

\begin{equation}
\Gamma = \left( \begin{array}{c} \Gamma_0' \\ \Gamma_1' \end{array} \right) := \left( \begin{array}{c} \Gamma_0' \\ \Gamma_1' \end{array} \right) \upharpoonright \text{dom } \Gamma, \quad \text{where \ dom } \Gamma = \text{dom } \Gamma_1' \cap \text{dom } \Gamma_0'.
\end{equation}

Then $\Gamma_j' = \Gamma_j$, $j = 0, 1$. Denote by $\mathcal{H}_+$ the domain $\text{dom } A^*$ equipped with the graph norm of $A^*$.

Clearly, $\text{dom } \Gamma$ is dense in $\mathcal{H}_+$. Define the operators $S_{n,j} := S_n^* \upharpoonright \ker \Gamma_j^{(n)}$ and $A_j := \bigoplus_{n=1}^\infty S_{n,j}$, $j \in \{0,1\}$. Then $A_0$ and $A_1$ are selfadjoint extensions of $A$. Note that $A_0$ and $A_1$ are disjoint but not necessarily transversal.

Finally, we set

\begin{equation}
A_* := A^* \upharpoonright \text{dom } \Gamma \quad \text{and} \quad A_{*j} := A_* \upharpoonright \ker \Gamma_j, \quad j \in \{0,1\}.
\end{equation}

Clearly, $\overline{A_{*j}} = A_j$, hence $A_{*j}$ is essentially selfadjoint, $j \in \{0,1\}.$

The following result is contained in [69] Theorem 3.2] (and stated here in the terminology of the present paper).

**Theorem 8.1** ([69]). Let $\Pi_n = \{ \mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)} \}$ be an ordinary boundary triple for $S_n^*$, let $S_{n,j} = S_n^* \upharpoonright \ker \Gamma_j^{(n)}$, $j \in \{0,1\}$, and let $M_n(\cdot)$, $n \in \mathbb{N}$, be the corresponding Weyl function. Moreover, let the operators $A^*$, $\Gamma_j'$ and $\Gamma_j$, $j \in \{0,1\}$, be given by (8.1), (8.2) and (8.3). Then:

(i) $\Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \}$ is a unitary boundary triple for $A^*$;

(ii) the corresponding Weyl function is the orthogonal sum $M(z) = \bigoplus_{n=1}^\infty M_n(z)$;

(iii) the mapping $\Gamma_j : \mathcal{H}_+ \to H$ is closable and $\Gamma_j = \Gamma_j'$, $j \in \{0,1\}$;

(iv) The operator $A_{*j}$ given by (8.4) is essentially selfadjoint and $\overline{A_{*j}} = \bigoplus_{n=1}^\infty S_{n,j} = A_j$, $j \in \{0,1\}$.

The following result characterizes selfadjointness of $A_j = \ker \Gamma_j$, $j \in \{0,1\}$, and completes Theorem 3.2 from [69].

**Proposition 8.2.** Let the assumptions be as in Theorem 8.1 and let $A_j = \ker \Gamma_j$, $j \in \{0,1\}$. Then

\begin{equation}
A_j = \bigoplus_{n=1}^\infty S_{n,j} \iff \Gamma_j' \upharpoonright A_j \text{ is bounded } \quad (j' = 1 - j \in \{0,1\}).
\end{equation}
In particular, \( A_0 \) satisfies (8.5) (i.e. \( A_0 = A_0^\ast \)) if and only if the corresponding Weyl function \( M(\cdot) \) and the \( \gamma \)-field \( \gamma(\cdot) \) satisfy one of the equivalent conditions in Theorem 1.11.

Similarly, \( A_1 \) satisfies (8.5) if and only if the Weyl function \(-M^{-1}(\cdot)\) and \( \gamma \)-field \( \gamma(\cdot)M^{-1}(\cdot) \) corresponding to the (unitary) transposed boundary triple \( \Pi^\top = \{ \mathcal{H}, \Gamma_1, -\Gamma_0 \} \) satisfy one of the equivalent conditions listed in Theorem 1.11.

**Proof.** The statements follow from Lemma 3.7 and Proposition 5.6. Indeed, by Proposition 5.6

(i) \( \Gamma_1 H(\lambda) = \gamma(\lambda)^\ast \) and hence \( \Gamma_1 H(\lambda) \) is closed. Since \( A_0 \) is essentially selfadjoint, the equivalence \( A_0 = A_0^\ast \iff \Gamma_1 | A_0 \) is bounded, is obtained from Lemma 3.7 (iii), (v). All the other equivalent conditions for \( A_0 = A_0^\ast \) hold by Theorem 1.11.

The criterion (8.5) and the other equivalent statements for \( A_1 = A_1^\ast \) are obtained by passing to the transposed boundary triple \( \{ \mathcal{H}, \Gamma_1, -\Gamma_0 \} \).

**Remark 8.3.** The criterion (8.5) implies the sufficient conditions for \( A_0 \) and \( A_1 \) to be selfadjoint as established in [69, Theorem 3.2]. Namely, if \( \Gamma_1 \) or \( \Gamma_0 \) is bounded, then also the restriction \( \Gamma_1 | A_0 \) or \( \Gamma_0 | A_1 \), respectively, is bounded. Moreover, if \( A_0 \) and \( A_1 \) are transversal, i.e. \( \text{dom } A_0 + \text{dom } A_1 = \text{dom } A \), then clearly \( \Gamma_j | A_j \) is bounded \( \iff \Gamma_{j'} \) is bounded, since \( \ker \Gamma_j = \text{dom } A_j \) \((j' = 1 - j \in \{0, 1\})\).

A criterion for a direct sum of ordinary boundary triples to form also an ordinary boundary triple can be formulated in terms of the corresponding Weyl functions (see [72], [69], [32]).

**Theorem 8.4.** Let \( \Pi_n = \{ \mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)} \} \) be a boundary triple for \( S_n^\ast \) and let \( M_n(\cdot) \) be the corresponding Weyl function, \( n \in \mathbb{N} \).

(i) The direct sum \( \Pi = \bigoplus_{n=1}^{\infty} \Pi_n \) forms an ordinary boundary triple for the operator \( A^\ast = \bigoplus_{n=1}^{\infty} S_n^\ast \) if and only if

\[
C_1 = \sup_n \| M_n(i) \|_{\mathcal{H}_n} < \infty \quad \text{and} \quad C_2 = \sup_n \| (\text{Im } M_n(i))^{-1} \|_{\mathcal{H}_n} < \infty.
\]

(ii) The direct sum \( \Pi = \bigoplus_{n=1}^{\infty} \Pi_n \) is a \( B \)-generalized boundary triple for the operator \( A^\ast = \bigoplus_{n=1}^{\infty} S_n^\ast \) if and only if \( C_1 < \infty \).

(iii) If, in addition, the operators \( \{ S_{n,0} \}_{n \in \mathbb{N}} \) have a common gap \((a - \varepsilon, a + \varepsilon)\), then the direct sum \( \Pi = \bigoplus_{n=1}^{\infty} \Pi_n \) is a \( B \)-generalized boundary triple for \( A^\ast = \bigoplus_{n=1}^{\infty} S_n^\ast \) if and only if

\[
C_3 := \sup_{n \in \mathbb{N}} \| M_n(a) \|_{\mathcal{H}_n} < \infty \quad \text{and} \quad C_4 := \sup_{n \in \mathbb{N}} \| M_n'(a) \|_{\mathcal{H}_n} < \infty,
\]

where \( M_n'(a) := (dM_n(z)/dz)_{z=a} \).

(iv) \( \Pi = \bigoplus_{n=1}^{\infty} \Pi_n \) is an ordinary boundary triple for \( A^\ast = \bigoplus_{n=1}^{\infty} S_n^\ast \) if and only if in addition to (8.6) the following condition is fulfilled

\[
C_5 := \sup_{n \in \mathbb{N}} \| (M_n'(a))^{-1} \|_{\mathcal{H}_n} < \infty.
\]

The next statement contains analogous characterization for \( S \)-generalized boundary triples.

**Proposition 8.5.** Assume the conditions of Theorem 8.4. Then the direct sum \( \Pi = \bigoplus_{n=1}^{\infty} \Pi_n \) forms an \( S \)-generalized boundary triple for \( A^\ast = \bigoplus_{n=1}^{\infty} S_n^\ast \) if and only if

\[
\sup_n \| \text{Im } M_n(i) \|_{\mathcal{H}_n} < \infty.
\]

Similarly, if the operators \( \{ S_{n,0} \} \) have a common gap \((a - \varepsilon, a + \varepsilon)\), then \( \Pi \) forms an \( S \)-generalized boundary triple for \( A^\ast \) if and only if \( C_4 < \infty \) where \( C_4 \) is given by (8.6).

**Proof.** The condition (8.8) means that \( \text{Im } M(z) \iff \text{the gamma-field } \gamma(z) \) is bounded for some (equivalently for every) \( z \in \mathbb{C}_\pm \). By Theorem 1.11 this amounts to saying that \( \Pi \) is an \( S \)-generalized boundary triple for \( A^\ast \).
Similarly, in case of a common spectral gap \((a - \varepsilon, a + \varepsilon)\) the condition \(\text{(8.8)}\) is equivalent to the condition \(C_4 < \infty\) in \(\text{(8.6)}\) as can be seen by the same argument that was used in Remark \(5.27\). \(\square\)

The next result is immediate by combining Propositions \(6.8\) in \(\text{(8.6)}\) with \(6.9\).

**Corollary 8.6.** Let \(\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} = \bigoplus_{n=1}^{\infty} \Pi_n\) be a direct sum of ordinary boundary triples \(\Pi_n = \{\mathcal{H}_n, \Gamma_{0}^{(n)}, \Gamma_{1}^{(n)}\}\), let \(M(\cdot) = \bigoplus_{n=1}^{\infty} M_n(\cdot)\) be the corresponding Weyl function, and let \(A_* := \text{dom} \Gamma\). Then the following conditions are equivalent:

(i) \(\Gamma_0 : A_* \to \mathcal{H}\) is bounded; 
(ii) the following condition is satisfied

\[
C_2 = \sup_{n} \| (\text{Im} M_n(i))^{-1} \|_{\mathcal{H}_n} < \infty.
\]

In this case the transposed boundary triple \(\Pi^\top = \{\mathcal{H}, \Gamma_1, -\Gamma_0\}\) is \(B\)-generalized.

Similarly, the following conditions are equivalent:

(i)' \(\Gamma_1 : A_* \to \mathcal{H}\) is bounded; 
(ii)' the following condition is satisfied

\[
C_2^\top := \sup_{n} \| (\text{Im} (M_n^{-1}(i)))^{-1} \|_{\mathcal{H}_n} < \infty.
\]

In this case the triple \(\Pi\) is a \(B\)-generalized boundary triple.

**Proof.** By Theorem \(8.1\) (see \([69, \text{Theorem 3.2}]\)) \(\Pi\) is a unitary boundary triple such that \(A_0 = \ker \Gamma_0\) and \(A_1 = \ker \Gamma_1\) are essentially selfadjoint. Now the first part of the statement follows easily from Proposition \(6.9\), while the second part is implied by Proposition \(6.8\). \(\square\)

### 8.2. Momentum operators with local point interactions

Let \(X = \{x_n\}_{n=1}^{\infty}\) be a strictly increasing sequence of positive numbers satisfying \(\lim_{n \to \infty} x_n = \infty\) and let \(d_n, d_*\) and \(d^*\) be defined by \(\text{(1.23)}\). Define a symmetric differential operator \(D_n\) in \(\mathcal{H}_n := L^2([x_{n-1}, x_n])\) by

\[
D_n = -i\frac{d}{dx}, \quad \text{dom} \ D_n = W^{1,2}([x_{n-1}, x_n]), \quad n \in \mathbb{N}.
\]

In quantum mechanics this operator in 1-D case appears in the form \(-ih\frac{d}{dx}\), where \(h = h/2\pi\) is the reduced Planck constant and whose eigenvalues are measuring the momentum of a particle.

The adjoint of the operator \(D_n\) is given by \(D_n^* = -i\frac{d}{dx}\) with dom \(D_n^* = W^{1,2}([x_{n-1}, x_n])\), \(n \in \mathbb{N}\). Following \([79]\) associate with \(D_n^*\) a boundary triple \(\Pi_n = \{\mathbb{C}, \Gamma_{0}^{(n)}, \Gamma_{1}^{(n)}\}\) by setting

\[
\Gamma_{0}^{(n)} f_n := \frac{f_n(x_n - 0) - f_n(x_{n-1} + 0)}{\sqrt{2}}, \quad \Gamma_{1}^{(n)} f_n := \frac{f_n(x_n - 0) + f_n(x_{n-1} + 0)}{\sqrt{2}}.
\]

The Weyl function \(M_n(\cdot)\) corresponding to the triple \(\Pi_n\) is given by

\[
M_n(z) = -\frac{e^{izx_n} + e^{izx_{n-1}}}{e^{izx_n} - e^{izx_{n-1}}} = -\cot(2^{-1}z d_n), \quad z \in \mathbb{C}_\pm.
\]

Let \(D_X := \bigoplus_{n=1}^{\infty} D_n\). Then \(D_X^* = \bigoplus_{n=1}^{\infty} D_n^*\) and

\[
\text{dom} \ D_X^* = W^{1,2}(\mathbb{R}_+ \setminus X) = \bigoplus_{n=1}^{\infty} W^{1,2}([x_{n-1}, x_n]).
\]

Next we describe the main properties of a boundary triple \(\Pi := \bigoplus_{n=1}^{\infty} \Pi_n\) assuming that \(d_* = 0\) partially treated in \([79]\). To this end we first recall a complete trace characterization of
Lemma 8.7 \cite{33}. Let $X = \{x_n\}_{n=1}^\infty$ be as above with $x_0 = 0$ and $X \subset \mathbb{R}_+$. Then:

(i) For any pair of sequences $a^\pm = \{a^\pm_n\}_{n=1}^\infty$ satisfying

\begin{equation}
\sum_{n \in \mathbb{N}} d_n \left(\|f(x_n-)\|^2 + \|f(x_n-)^2\right) \leq 4 \left(\|df\|_{L^2(\mathbb{R}_+)}^2 + \|df\|_{L^2(\mathbb{R}_+)}^2\right) \leq C_1 \|f\|_{W^{1,2}(\mathbb{R}_+,X)}^2,
\end{equation}

for each $f \in W^{1,2}(\mathbb{R}_+,X)$ where $C_1 = 4 \max\{(d^*)^2, 1\}$. Besides, the traces $a^\pm := \pi \pm f$ of each $f \in W^{1,2}(\mathbb{R}_+_X)$ satisfy conditions (8.14). Moreover, the assumption $d^* < \infty$ is necessary for the inequality (8.10) to hold with some $C_1 > 0$.

Now we are ready to state and prove the main result of this subsection.

Proposition 8.8. Let $X$ be as above, let $d_\ast = 0$ and $d^* < \infty$, let $\Pi^{(n)} = \{C, \Gamma^{(n)}_0, \Gamma^{(n)}_1\}$ be the boundary triple for the operator $D^*_n$ defined by (8.10). Let $\Pi = \bigoplus_{n=1}^\infty \Pi^{(n)} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H} = L^2(\mathbb{N})$, $D_X := \bigoplus_{n=1}^\infty D_n$, $D_{X,*} = D_1|\text{dom} \Gamma$ and let the operators $\Gamma_j$ and $\Gamma_j$, $j \in \{0, 1\}$, be given by (8.2) and (8.3), respectively. Then:

(i) The mapping $\Gamma_0 \times \Gamma_1$ can be extended to the mapping

\begin{equation}
\Gamma_0 \times \Gamma_1 : W^{1,2}(\mathbb{R}_+_X) \to L^2(\mathbb{N}; \{d_n^{-1}\}) \times L^2(\mathbb{N}; \{d_n\}),
\end{equation}

which is well defined and surjective. Besides, ker $(\Gamma_0 \times \Gamma_1) = W^{1,2}(\mathbb{R}_+,X)$. 

(ii) The mapping

\begin{equation}
\Gamma_0 \times \Gamma_1 : \text{dom} D_{X,*} = \text{dom} \Gamma \to L^2(\mathbb{N}; \{d_n^{-1}\}) \times L^2(\mathbb{N}; \{d_n\}),
\end{equation}

is well defined and surjective. Moreover, $\Gamma_0$ boundedly maps dom $D_{X,*}$ in $L^2(\mathbb{N})$;

(iii) The Weyl function $M(\cdot)$ is domain invariant and its domain is given by

\begin{equation}
\text{dom} M(z) = L^2(\mathbb{N}; \{d_n^{-2}\}) \subset \text{ran} \Gamma_0 = \Gamma_0(\text{dom} A_\ast) = L^2(\mathbb{N}; \{d_n^{-1}\}), \quad z \in \mathbb{C}_\pm.
\end{equation}

(iv) The domain of the form $t_M$ associated with the imaginary part $\text{Im} M(z)$ is given by

\begin{equation}
\text{dom} t_M(z) = \left\{\{a_n\}_{n=1}^\infty \in L^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n\}_{n=1}^\infty \in L^2(\mathbb{N}; \{d_n^{-1}\})\right\}, \quad z \in \mathbb{C}_\pm.
\end{equation}

(v) The triple $\Pi$ is a ES-generalized boundary triple for $D_X$ and $A_0 \neq A^*_1$. Moreover, the imaginary part $\text{Im} M(\cdot)$ of the Weyl function $M(\cdot)$ takes values in $C(\mathcal{H}) \setminus B(\mathcal{H})$.

(vi) The transposed triple $\Pi^\top$ is B-generalized but not an ordinary boundary triple for the operator $D^*_X$. In particular, the Weyl function $-M(\cdot)^{-1}$ takes values in $B(\mathcal{H})$, and $A_1 = A^*_1$.
Proof. (i) The proof is immediate from Lemma 8.7(1).
(ii) Since $d^* < \infty$, the space $l^2(\mathbb{N})$ is (continuously) embedded in $l^2(\mathbb{N}; \{d_n\})$. Therefore the surjectivity is immediate from (i). By Lemma 8.7(i), the mapping $\Gamma_0 : \text{dom} D_{X,s} \mapsto l^2(\mathbb{N}; \{d_n\}^{-1})$ is bounded. To prove the boundedness of $\Gamma_0 : \text{dom} \Gamma \mapsto l^2(\mathbb{N})$ it remains to note that the embedding $l^2(\mathbb{N}; \{d_n\}^{-1}) \mapsto l^2(\mathbb{N})$ is continuous since $d^* < \infty$.

(iii) In accordance with (8.11) $M_n(z) = -\cot(2^{-1}d_nz)$. Therefore the description of $\text{dom} M(\cdot)$ follows from the obvious relation

$$\cot(2^{-1}zd_n) \sim 2z^{-1}d_n^{-1} \quad \text{as} \quad d_n \to 0, \quad z \in \mathbb{C}_\pm.$$ 

(iv) Notice that $\{a_n\}_{n=1}^\infty \subset \text{dom} M(z)$ if and only if $\sum_{n=1}^\infty (\text{Im} M_n(z)a_n, a_n) < \infty$. It follows from (8.11) and (8.20) that $\text{Im} M_n(x + iy) \sim \frac{2a_n}{x^2 + y^2}d_n^{-1}$ as $n \to \infty$. Therefore

$$\sum_{n=1}^\infty (\text{Im} M_n(z)a_n, a_n) < \infty \iff \sum_{n=1}^\infty |a_n|^2d_n^{-1} < \infty.$$ 

(v) Being a direct sum of ordinary boundary triples, the triple $\Pi = \bigoplus_{n=1}^\infty \Pi^{(n)}$ is an ES-generalized boundary triple in accordance with Theorem 8.1(iv). The relation $A_0 \neq A^*_0$ is implied by item (iii) since the inclusion $\text{dom} (M(z) \subseteq \text{ran} \Gamma_0$ is strict.

Furthermore, relation (8.11) implies $M_n(i) = i \text{cth}(2^{-1}d_n)$. It follows that $\text{Im} M_n(i) = \text{cth}(2^{-1}d_n)$, $n \in \mathbb{N}$. Hence the values of imaginary part $\text{Im} M(\cdot)$ are unbounded, $\text{Im} M(\cdot) \in \mathcal{C}(\mathcal{H}) \backslash \mathcal{B}(\mathcal{H})$. Due to Theorem 5.18(v) (see also Theorem 1.11) this last property gives another proof for the fact that the triple $\Pi$ is not $\text{S}$-generalized.

(vi) It follows from (8.11) that $-M_n^{-1}(z) = \tan(2^{-1}d_nz)$. Therefore the Weyl function $-M^{-1}(\cdot) = \oplus_{n=1}^\infty (-M_n^{-1}(\cdot)) \in \mathcal{R}^*[\mathcal{H}]$. By Theorem 1.4 the transposed triple $\Pi^\top$ is $\text{B}$-generalized.

\[\square\]

Remark 8.9. (i) Note that statements (iii)-(vi) remain valid for $d^* = \infty$.

(ii) Assuming that $d^* < \infty$ it is shown in [79] that the triple $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is an ordinary boundary triple for $D_X^s$ if and only if $d_s > 0$. This result remains true also in the case $d^* = \infty$.

(iii) Let $G = \text{diag} \{\tilde{d}_{1/2}, \ldots, \tilde{d}_{n/2}, \ldots\}$ be the diagonal operator defined on $\mathcal{H} = l^2(\mathbb{N})$, with $\tilde{d}_n = \min\{1, d_n\}$, $n \in \mathbb{N}$. In accordance with Theorem 8.3 iv)

$$\text{ran} G = \text{dom} G^{-1} = \text{dom} t_{M(i)}.$$ 

Hence the renormalization in Theorem 5.31 is determined via the formulas $\tilde{\Gamma}_0 = G^{-1}\Gamma_0$, $\tilde{\Gamma}_1 = G\Gamma_0$ and the corresponding Weyl function is given by

$$M_G(z) = G^* M(z) G = -\sum_{n=1}^\infty \tilde{d}_n \cot(2^{-1}zd_n).$$ 

Since $\tilde{d}_n \text{Im} M_n(i) \to 2$ as $d_n \to 0$, we conclude that (the closure of) $M_G(\cdot)$ is a bounded uniformly strict Nevanlinna function, $M_G(\cdot) \in \mathcal{R}^*[\mathcal{H}]$. Thus, the renormalization procedure in this case leads to an ordinary boundary triple for $D_X^s$. In the case $d^* < \infty$ this renormalization procedure was firstly applied in [79] to construct the above mentioned ordinary boundary triple for $D_X$; see Examples 3.2, 3.8 and Theorem 3.6 in [79].

8.3. Schrödinger operators with local point interactions. Let $X = \{x_n\}_{n=0}^\infty \subset \mathbb{R}_+$ be a strictly increasing sequence satisfying $\lim_{n\to\infty} x_n = \infty$. Let also $\mathcal{H}_n$ be a minimal operator associated with expression $-\frac{d^2}{dx^2}$ in $L^2[x_{n-1}, x_n]$. Clearly, $\mathcal{H}_n$ is a closed symmetric, $n_\pm(\mathcal{H}_n) = 2$, and its domain is $\text{dom}(\mathcal{H}_n) = W^{2,2}_0[x_{n-1}, x_n]$. 
It is easily seen that a boundary triple \( \Pi_n = \{ \mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)} \} \) for \( H_n^* \) can be chosen as

\[
\Gamma_0^{(n)} f := \begin{pmatrix} f'(x_{n-1}+) \\ f'(x_{n-}) \end{pmatrix}, \quad \Gamma_1^{(n)} f := \begin{pmatrix} -f'(x_{n-1}+) \\ f(x_{n-}) \end{pmatrix}, \quad f \in W^2_0[x_{n-1}, x_n].
\]

The corresponding Weyl function \( M_n \) is given by

\[
M_n(z) = \frac{-1}{\sqrt{2}} \begin{pmatrix} \cot(\sqrt{2d_n}) & -\frac{1}{\sin(\sqrt{2d_n})} \\ 1 & \cot(\sqrt{2d_n}) \end{pmatrix}.
\]

Consider in \( L^2(\mathbb{R}_+) \) the direct sum of symmetric operators \( H_n, \) \( H := H_{\min} = \bigoplus_{n=1}^{\infty} H_n, \)\n\[\text{dom}(H_{\min}) = W^{2,2}_0(\mathbb{R}_+ \setminus X) = \bigoplus_{n=1}^{\infty} W^{2,2}_0[x_{n-1}, x_n].\]

Denoting by \( \mathcal{S}_+ \) the domain \( \text{dom}(H^*) \) equipping with the graph norm, we note that \( \text{dom} \Gamma \) is dense in \( \mathcal{S}_+ \) while in general it is narrower than \( \mathcal{S}_+ \). As was shown in \([67]\), the triple \( \Pi = \bigoplus_{n \in \mathbb{N}} \Pi_n := \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) is an ordinary boundary triple for the operator \( H_{\max} := H_{\min}^* \) whenever

\[
0 < d_* = \inf_{n \in \mathbb{N}} d_n \leq d^* = \sup_{n \in \mathbb{N}} d_n < +\infty.
\]

The converse statement is also true (see \([69]\)): the condition \( d_*>0 \) is necessary for the direct sum \( \Pi = \bigoplus_{n \in \mathbb{N}} \Pi_n \) to form a boundary triple for \( H_{\max} := H_{\min}^* \).

Such type triples have naturally arisen in investigation of spectral properties of the Hamiltonian \( H_{X,\alpha} \) associated in \( L^2(\mathbb{R}_+) \) with a formal differential expression

\[
\ell_{X,\alpha} := -\frac{d^2}{dx^2} + \sum_{x_n \in X} \alpha_n \delta(x - x_n), \quad \alpha = (\alpha_n)_{n=0}^{\infty} \subset \mathbb{R},
\]

when treating \( H_{X,\alpha} \) as an extension of \( H_{\min} \) (see \([67],[69]\), and Remark \(8.15\) below).

Next we present extended and completed version of Theorem \(1.18\). Assertion (iii) of Theorem \(1.18\) will be proved after a preparatory lemma.

**Theorem 8.10.** Let \( \Pi_n, n \in \mathbb{N}, \) be the boundary triple given by \((8.22)\), let \( M_n(\cdot) \) be the corresponding Weyl function, \( \mathcal{H} = L^2(\mathbb{N}) \otimes \mathbb{C}^2, \) and let \( \Pi := \bigoplus_{n=1}^{\infty} \Pi_n := \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be the direct sum of triples \( \Pi_n \) given by \((8.22)\) and \((8.23)\). Assume also that \( d_*=0 \) and \( d^* \leq \infty. \) Then the following statements hold:

(i) The triple \( \Pi \) is an ES-generalized boundary triple for \( H_{\min}^* \) such that \( A_0 \neq A_0^* \).

(ii) The Weyl function \( M(\cdot) \) is domain invariant and its domain is given by

\[
\text{dom} M(z) = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right)_{n=1}^{\infty} \in L^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n - b_n\}_{n=1}^{\infty} \in L^2(\mathbb{N}; \{d^{-2}_n\}) \right\}, \quad z \in \mathbb{C}_z.
\]

(iii) Let in addition \( d^* < \infty. \) Then the range of \( \Gamma_0 \) is given by

\[
\text{ran} \Gamma_0 = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right)_{n=1}^{\infty} \in L^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n - b_n\}_{n=1}^{\infty} \in L^2(\mathbb{N}; \{d^{-1}_n\}) \right\} \supseteq \text{dom} M(\pm i) .
\]

(iv) The domain of the form \( t_{M(z)} \) generated by the imaginary part \( \text{Im} M(z) \) is given by

\[
\text{dom} t_{M(z)} = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right)_{n=1}^{\infty} \in L^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n - b_n\}_{n=1}^{\infty} \in L^2(\mathbb{N}; \{d^{-1}_n\}) \right\}, \quad z \in \mathbb{C}_z.
\]

In particular, if \( d^* < \infty, \) then \( \text{dom} t_{M(z)} = \text{ran} \Gamma_0. \)

(v) The transposed triple \( \Pi^\top \) is an S-generalized boundary triple for \( H_{\min}^*, \) i.e. \( A_1 = A_1^*. \) However, it is not a B-generalized boundary triple for \( H_{\min}^*. \)
(vi) The Weyl function \( M^\top(\cdot) = -M(\cdot)^{-1} \) corresponding to the transposed triple \( \Pi^\top \) is domain invariant and its domain is given by

\[
\text{dom } M^\top(z) = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right) \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n + b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-2}\}) \}, \quad z \in \mathbb{C}_+.
\]

(vii) The domain of the form \( t_{M^\top(z)} \) generated by the imaginary part \( \text{Im } M^\top(z) \) is given by

\[
\text{dom } t_{M^\top(z)} = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right) \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n + b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\}) \}, \quad z \in \mathbb{C}_+.
\]

Proof. (i) By Theorem 8.1(iv), the triple \( \Pi \) is an ES-generalized boundary triple for \( \mathcal{H}^* \). Fix \( z \in \mathbb{C} \setminus \mathbb{R} \). It follows from (8.23) that

\[
(8.28) \quad \lim_{d_n \to 0} d_n M_n(z) = -\frac{1}{z} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \lim_{d_n \to 0} d_n \text{Im } M_n(i) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]

Since \( d_n = 0 \), the last relation yields \( \sup_n \|\text{Im } M_n(i)\| = \infty \). Therefore Proposition 8.3 implies \( A_0 \neq A_0^\bot \).

(ii) By Theorem 8.1(ii), the Weyl function of \( \Pi = \bigoplus_n \Pi_n \) is \( M(\cdot) = \bigoplus_n^\infty M_n(\cdot) \), where \( M_n(\cdot) \) is given by (8.23). By definition, \( \{h_n\}_{n=1}^\infty \in \text{dom } M(z) \) if and only if

\[
(8.29) \quad \sum_{n=1}^\infty \|M_n(z)h_n\|^2 < \infty; \quad \{h_n\}_{n=1}^\infty = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right) \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2.
\]

It follows from (8.23) that \( \|M_n(z)\| \) as a function of \( d_n \) is bounded on the intervals \([\delta, \infty), \delta > 0\). Combining this fact with the first relation in (8.28) and noting that \( d_* = 0 \) and \( \frac{\sin(\sqrt{2d_*})}{\sqrt{2d_*}} \sim 1 \) as \( d_n \to 0 \), one concludes that the convergence of the series in (8.29) is equivalent to

\[
(8.30) \quad \sum_{n=1}^\infty \frac{|a_n - b_n|^2}{d_n^2} < \infty,
\]

i.e. to the inclusion \( \{a_n - b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-2}\}) \).

(iii) The proof is postponed after Lemma 8.12.

(iv) The proof is similar to that of the item (ii). First notice that \( \{h_n\}_{n=1}^\infty \in \text{dom } t_{M(z)} \) if and only if

\[
(8.31) \quad \sum_{n=1}^\infty (\text{Im } M_n(z)h_n, h_n) < \infty, \quad \{h_n\}_{n=1}^\infty = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right) \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2.
\]

Note that \( \text{Im } M_n(z) \) as a function of \( d_n \) is bounded on the intervals \([\delta, \infty), \delta > 0\). Rewriting the first of relations in (8.28) as

\[
(8.32) \quad \lim_{d_n \to 0} \left( M_n(z) + \frac{1}{d_n z} K \right) = 0, \quad \text{where} \quad K = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},
\]

we derive that the convergence of the series in (8.31) is equivalent to

\[
\sum_{n=1}^\infty \text{Im } \left( \frac{K h_n, h_n}{z d_n} \right) = \frac{y}{x^2 + y^2} \sum_{n=1}^\infty \frac{|a_n - b_n|^2}{d_n} < \infty, \quad z = x + iy \in \mathbb{C}_+.
\]

This proves the statement.
(v) The Weyl function $M^T(\cdot)$ corresponding to the transposed boundary triple $\Pi^T$ is $M^T(\cdot) = \oplus M_n^T(\cdot)$, where $M_n^T(\cdot) = -M_n^{-1}(\cdot)$ is given by

$$M_n^T(z) = -\sqrt{z} \left( \frac{\cot(\sqrt{z}d_n)}{\sin(\sqrt{z}d_n)} \frac{1}{\cot(\sqrt{z}d_n)} \right).$$

It follows that

$$\lim_{d_n \to \infty} M_n^T(z) = \pm i\sqrt{z}I_2, \quad \lim_{d_n \to 0} d_n M_n^T(z) = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad z \in \mathbb{C}_\pm.$$

Since $d_* = 0$, the last relation shows that the Weyl function $M^T(\cdot)$ takes unbounded values.

On the other hand, using the Laurent series expansions for $\cot z$ and $(\sin z)^{-1}$ at 0 gives

$$\lim_{d_n \to 0} d_n^{-1} \text{Im} M_n^T(z) = (\text{Im} z) \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix}, \quad z \in \mathbb{C}_\pm.$$

Hence, $\text{Im} M_n^T(z)$ is uniformly bounded as a function of $d_n \in (0, \infty)$ for every $z \in \mathbb{C} \setminus \mathbb{R}$. Therefore Proposition 8.39 ensures that the transposed boundary triple $\Pi^T$ is $S$-generalized. At the same time $\Pi^T$ is not $B$-generalized, since $M^T(\cdot)$ takes values in $\mathcal{C}(\mathcal{H}) \setminus \mathcal{B}(\mathcal{H})$.

(vi) The proof is similar to that of the statement (ii). One should only use relations (8.34) instead of (8.28).

(vii) The proof is similar to that of (iv). \hfill $\square$

Remark 8.11. Here we show that the triple $\Pi = \oplus_{n \in \mathbb{N}} \Pi_n := \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple for the operator $H_{\text{max}} := H^*_{\text{min}}$ and if only if $d_* > 0$. This statement extends the corresponding results from [67], [69], to the case $d^* = \infty$.

Denote by $\sqrt{z}$ the branch of the multifunction defined in $\mathbb{C}$ with the cut along the non-negative semi-axes $\mathbb{R}_+$ and fixed by the condition $\sqrt{1} = 1$. It is easily seen that $\sqrt{\cdot} : \mathbb{C} \mapsto \mathbb{C}_+$. Consider the behavior of $M_n(z)$ as $d_n \to \infty$. The functions $\cot(\sqrt{zd})$ and $(\sin(\sqrt{zd}))^{-1}$ depend continuously on $d \in (0, \infty)$ and $\lim_{d \to \infty} \cot(\sqrt{zd}) = -i$ and $\lim_{d \to \infty} (\sin(\sqrt{zd}))^{-1} = 0$. Therefore for any fixed $z \in \mathbb{C} \setminus \mathbb{R}$ the matrix function $M_n(z)$ in (8.23) is continuous and bounded in $d_n \in [\delta, \infty)$ for every $\delta > 0$.

Further, clearly, $\lim_{d_n \to \infty} \text{Im} M_n(z) = \pm I_2$ for $z \in \mathbb{C}_\pm$ and this implies that for every fixed $z \in \mathbb{C}_+$ there exists $c_\delta(z) > 0$ such that

$$\text{Im} M_n(z) \geq c_\delta(z) I_2, \quad d_n \in [\delta, d^*], \quad \delta > 0.$$

Thus, by Theorem 8.11 (i) $\Pi$ is an ordinary boundary triple for the operator $H_{\text{min}}^*$ whenever $d_* > 0$ and, in particular, $A_0 = A_0^*$ and $A_1 = A_1^*$ are transversal extensions of $H_{\text{min}}$ in this case.

It remains to prove the assertion (iii) of Theorem 8.10. It is more involved and to this end we describe traces of functions $f \in W^{2,2}(\mathbb{R}_+ \setminus X)$ as well as traces of their first derivatives and prove an analog of Lemma 8.7.

Lemma 8.12. Let $X = \{x_n\}_{n=1}^\infty$ be as above and let $0 \leq d_* \leq d^* < \infty$. Then the mapping $\Gamma_0'' : W^{2,2}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2 \to l^2(\mathbb{N}; \{d_n^3\})$ defined by

$$\Gamma_0'' : f \to \left\{ \begin{pmatrix} f(x_{n-1}+) \\ f(x_n-) \end{pmatrix} \right\}_{n=1}^\infty$$

is well defined and bounded and its range $\text{ran} \Gamma_0''$ is given by

$$(8.36) \quad \Gamma_0'(W^{2,2}(\mathbb{R}_+ \setminus X)) = \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^3\}) \otimes \mathbb{C}^2 : \{a_n - b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\}).$$
Proof. Denote temporarily the right-hand side of (8.38) by $\mathcal{R}(\Gamma''_0)$. First we prove the inclusion $\text{ran } \Gamma''_0 = \Gamma''_0(W^{2,2}(\mathbb{R}_+ \setminus X)) \subset \mathcal{R}(\Gamma''_0)$. Let $f \in W^{2,2}(\mathbb{R}_+ \setminus X)$. This inclusion implies $f \in W^{2,2}([x_{n-1}, x_n])$ for each $n \in \mathbb{N}$ and, it is easy to check that

$$(8.37) \quad d_n f'(x) \leq 2 \left( \| f'' \|_{L^2(\Delta_n)}^2 + d_n^2 \| f'' \|_{L^2(\Delta_n)}^2 \right), \quad x \in \Delta_n := [x_{n-1}, x_n], \quad n \in \mathbb{N}.$$  

Moreover, by the Sobolev embedding theorem (cf. e.g. [3], [61, p. 192]) there exist constants $c_0, c_1 > 0$ not depending on $f$ and $n \in \mathbb{N}$ such that

$$(8.38) \quad \| f'' \|_{L^2(\Delta_n)} \leq c_1 d_n^2 \| f'' \|_{L^2(\Delta_n)}^2 + c_0 d_n^2 \| f \|_{L^2(\Delta_n)}, \quad x \in \Delta_n, \quad n \in \mathbb{N}.$$  

By applying (8.37) to $f'$ and combining the result with (8.38) shows that

$$d_n^2 |f'(x)|^2 \leq C_1 d_n^2 \| f'' \|_{L^2(\Delta_n)}^2 + C_0 d_n^2 \| f \|_{L^2(\Delta_n)}^2, \quad x \in \Delta_n, \quad n \in \mathbb{N},$$

where $C_0$ and $C_1$ do not depend on $f$ and $n \in \mathbb{N}$. Therefore,

$$\sum_{n} d_n^2 \left( |f'(x_{n+1})|^2 + |f'(x_{n+1})|^2 \right) \leq 2 C_1 \sum_{n} D_n^2 \| f'' \|_{L^2(\Delta_n)}^2 + 2 C_0 \sum_{n} \| f \|_{L^2(\Delta_n)}^2$$

$$(8.39) \quad \leq 2 C_1 (d^n)^4 \| f'' \|_{L^2(\mathbb{R}_+)}^2 + 2 C_0 \| f \|_{L^2(\mathbb{R}_+)}^2 \leq C_3 \| f \|_{W^{2,2}(\mathbb{R}_+ \setminus X)}^2,$$

where $C_3 = 2 \max \{ C_0, C_1 (d^n)^4 \}$. Hence, the mapping $\Gamma''_0$ is bounded.

Furthermore, since $f \in W^{2,2}([x_{n-1}, x_n]), \ n \in \mathbb{N}$, and $f'' \in L^2(\mathbb{R}_+)$, one gets

$$\sum_{n \in \mathbb{N}} \frac{|f'(x_{n+1}) - f'(x_{n-1})|^2}{d_n^2} = \sum_{n \in \mathbb{N}} \frac{1}{d_n^2} \int_{x_{n-1}}^{x_{n+1}} |f''(x)|^2 \, dx \leq \sum_{n \in \mathbb{N}} \int_{x_{n-1}}^{x_{n+1}} |f''(x)|^2 \, dx$$

$$(8.40) \quad = \int_{\mathbb{R}_+} |f''(x)|^2 \, dx \leq \| f \|_{W^{2,2}(\mathbb{R}_+ \setminus X)}^2.$$  

Combining (8.39) with (8.40) yields the inclusion $\text{ran } \Gamma''_0 = \Gamma''_0(W^{2,2}(\mathbb{R}_+ \setminus X)) \subset \mathcal{R}(\Gamma''_0)$.

To prove the reverse inclusion we choose any vector $\{ \{a_n\} \}_n \in L^2(\mathbb{N}; \{d_n^3\}) \otimes \mathbb{C}^2$ satisfying $\{a_n - b_n\}_{n=1}^{\infty} \in l^2(\mathbb{N}; \{d_n\})$. Setting

$$(8.41) \quad g_n(x) = a_n (x - x_{n-1}) + 2^{-1} d_n^{-1} (x - x_{n-1})^2 (b_n - a_n), \quad x \in [x_{n-1}, x_n]$$

and $g := \oplus_1^\infty g_n$ one easily checks that

$$\|g_n \|_{L^2(\Delta_n)}^2 \leq d_n^3 \left[ \frac{2}{3} |a_n|^2 + \frac{1}{10} |b_n - a_n|^2 \right] \leq d_n^3 \left( |a_n|^2 + |b_n|^2 \right),$$

hence $g = \oplus_1^\infty g_n \in L^2(\mathbb{R}_+)$. Moreover, the condition $\{a_n - b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\})$ yields the inclusion $g'' \in L^2(\mathbb{R}_+)$. Thus $g \in W^{2,2}(\mathbb{R}_+ \setminus X)$. To complete the proof it remains to note that

$$(8.42) \quad g_n'(x_{n+1}) = a_n, \quad g_n'(x_{n-1}) = a_n + (b_n - a_n) = b_n,$$

i.e. $\Gamma''_0 g = \{ \{a_n/b_n\} \}_{n=1}^\infty$. \hfill \Box$

Remark 8.13. Notice that the relation (8.36) cannot be extracted from Proposition 8.7(i) applied to the derivative $f'$, since the embedding $W^{2,2}(\mathbb{R}_+ \setminus X) \rightarrow W^{1,2}(\mathbb{R}_+ \setminus X)$ holds if and only if $d_\sigma > 0$ (see [70]).

We are now ready to prove the assertion (iii) in Theorem 8.10 i.e. to prove relation (8.27).

Proof of item (iii) in Theorem 8.10. Let the righthand side of (8.27) be denoted temporarily by $\mathcal{R}_0(\Gamma'_0)$. The inclusion $\text{ran } (\Gamma'_0) = \Gamma'_0(\text{dom } H) \subset \mathcal{R}_0(\Gamma'_0)$ is immediate from Lemma 8.12.
To prove the reverse inclusion we choose any vector \( \{(a_n/b_n)\}_{n \in \mathbb{N}} \in l^2(\mathbb{N}) \otimes \mathbb{C}^2 \) that satisfies 
\( \{a_n - b_n\}_{n = 1}^{\infty} \in l^2(\mathbb{N}; \{d_n^{-1}\}) \) and consider the functions \( g_n \) and \( g = \oplus_{n=1}^{\infty} g_n \) as defined in (8.41). 
As shown in Lemma 8.12 \( g \in W^{2,2}(\mathbb{R}^+ \setminus \mathbb{X}) \) and \( g' \) satisfies the equalities (8.42). Besides,
\[
g_n(x_{n-1}+) = 0 \quad \text{and} \quad g_n(x_{n-1}) = a_n d_n + 2^{-1} b_n (b_n - a_n) = 2^{-1} (a_n + b_n) d_n \in l^2(\mathbb{N}).
\]

Note that the latter inclusion holds since \( d^* < \infty \). Summing up we get 
\[
\Gamma_0 g = \left( \begin{array}{c} g'(x_{n-1}+) \\ g'(x_{n-1}) \end{array} \right) = \left( \begin{array}{c} a_n \\ b_n \end{array} \right)_{n = 1}^{\infty}
\]
and 
\[
\Gamma_1 g = \left( \begin{array}{c} 0 \\ 2^{-1} (a_n + b_n) d_n \end{array} \right)_{n = 1}^{\infty} \in l^2(\mathbb{N}) \otimes \mathbb{C}^2.
\]

Thus, \( g \in \text{dom} \Gamma_0 \cap \text{dom} \Gamma_1 = \text{dom} H_* \) and this completes the proof. \( \square \)

One gets from Lemma 8.12 a description for the ranges of the closures of \( \Gamma_0 \) and \( \Gamma_1 \).

**Corollary 8.14.** Assume the conditions of Theorem 8.10 and \( d^* < \infty \). Then the range of the closure of \( \Gamma_0 \) is
\[
\text{ran} \overline{\Gamma_0} = \left\{ \left( \begin{array}{c} a_n \\ b_n \end{array} \right)_{n = 1}^{\infty} \in l^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n - b_n\}_{n = 1}^{\infty} \in l^2(\mathbb{N}; \{d_n^{-1}\}) \right\} = \text{ran} \Gamma_0.
\]
and 
\[
\text{ran} \overline{\Gamma_1} = l^2(\mathbb{N}) \otimes \mathbb{C}^2.
\]

**Proof.** Recall that, by definition, \( \text{dom} \Gamma_0 = \text{dom} \Gamma_1 = \text{dom} H_* \). Clearly, \( \Gamma_0 = \Gamma_0'' \cap \text{dom} H_* \) and
\[
\text{ran} \Gamma_0 \subseteq \text{ran} \overline{\Gamma_0} \subseteq \text{ran} \overline{\Gamma_0''} \cap \left( l^2(\mathbb{N}) \otimes \mathbb{C}^2 \right).
\]

On the other hand, it follows from Lemma 8.12 and Theorem 8.10(iii) that 
\[
\text{ran} \Gamma_0 = \text{ran} \overline{\Gamma_0''} \cap \left( l^2(\mathbb{N}) \otimes \mathbb{C}^2 \right).
\]
Combining this relation with (8.45) and applying Theorem 8.10(iii) yields (8.43).

The second relation is proved similarly. \( \square \)

**Remark 8.15.** (i) Recall that according to Theorem 5.18 the condition
\[
\text{ran} \Gamma_0 = \text{dom} M(z), \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
ensures selfadjointness of \( A_0 = \ker \Gamma_0 \). Theorem 8.10(iii) gives an explicit example showing that condition (8.46) cannot be replaced by the weaker domain invar iance condition 
\[
\text{dom} M(z) = \text{dom} M(i) \left( \mathbb{R} \right) \cap \text{ran} \Gamma_0, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]
In other words, domain invariance property does not imply the property of a boundary triple to be S-generalized (see also Example 5.28). Such Weyl functions cannot be written in the form (5.23) without a renormalization of the boundary triple as in Theorem 5.34.

(ii) In the case \( d_* = 0 \) and \( d^* < \infty \) an abstract regularization procedure from [69] has first been applied in [69] to the direct sum \( \Pi = \oplus_{n=1}^{\infty} \Pi_n = \{H, \Gamma_0, \Gamma_1\} \) of triples (8.22) for \( H_* \) to obtain a (regularized) ordinary boundary triple \( \Pi' = \{H, \Gamma_0', \Gamma_1'\} \) satisfying \( \ker \Gamma_0 = \ker \Gamma_0' \). A special construction of a regularized triple \( \Pi' \) in [69] has been motivated by the following circumstance: the boundary operator \( B_{X,0} \) corresponding to the Hamiltonian \( H_{X,0} \) of the form (8.25), i.e. operator satisfying \( \text{dom}(H_{X,0}) = \ker (\Gamma_1 - B_{X,0} \Gamma_0) \), is a Jacobi matrix. It is shown in [69] that certain spectral properties of \( H_{X,0} \) strictly correlate with that of \( B_{X,0} \).

Next we apply the renormalization result in Theorem 6.6 to the ES-generalized boundary triple \( \Pi \) in Theorem 8.10. The transposed boundary triple \( \Pi^T \) can be renormalized by a suitable modification of Theorems 4.4, 4.10 using a regular point (here \( z = -1 \)) on the real line; cf. Proposition 6.2.
Proposition 8.16. Let $\Pi_n$ be the boundary triple for $H_n^*$ given by (8.22), let $M_n(\cdot)$, $n \in \mathbb{N}$, be the corresponding Weyl function given by (8.23), and let $\tilde{d}_n = \min\{d_n, 1\}$. Then:

(i) The orthogonal sum $\tilde{\Pi} = \oplus_{n=1}^{\infty} \Pi_n$ of boundary triples $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ with the mappings $\tilde{\Gamma}_j^{(n)} : W_2^2[x_{n-1}, x_n] \to \mathbb{C}^2$, $n \in \mathbb{N}$, $j \in \{0, 1\}$, given by

\[
\tilde{\Gamma}_0^{(n)} f := \left( \frac{\tilde{d}_n^{-1/2} f'(x_{n-1}^+) - \tilde{d}_n^{-1/2} f'(x_{n-1}^-)}{\tilde{d}_n^{1/2} f(x_{n-1}^+) + \tilde{d}_n^{3/2} f(x_{n-1}^+) - f(x_{n-1}^-)} \right), \quad \tilde{\Gamma}_1^{(n)} f := \left( \frac{-\tilde{d}_n^{1/2} f(x_{n-1}^+) + \tilde{d}_n^{3/2} f(x_{n-1}^+) - f(x_{n-1}^-)}{-\tilde{d}_n^{1/2} f(x_{n-1}^-)} \right),
\]

forms a $B$-generalized boundary triple for $H_{n\text{min}}^*$. Moreover, $\tilde{\Pi}$ is an ordinary boundary triple if and only if $d_n > 0$.

(ii) The orthogonal sum $\Pi^{(r)} = \oplus_{n=1}^{\infty} \Pi^{(r,n)}$ of the boundary triples $\Pi_n^{(r,n)} = \{\mathbb{C}^2, \Gamma_0^{(r,n)}, \Gamma_1^{(r,n)}\}$ with the mappings $\tilde{\Gamma}_j^{(r,n)} : W_2^2[x_{n-1}, x_n] \to \mathbb{C}^2$, $n \in \mathbb{N}$, $j \in \{0, 1\}$, given by

\[
\Gamma_0^{(r,n)} f := \left( \frac{\tilde{d}_n^{1/2} f(x_{n-1}^+) + \tilde{d}_n^{3/2} f(x_{n-1}^+) - f(x_{n-1}^-)}{-\tilde{d}_n^{1/2} f(x_{n-1}^-)} \right), \quad \Gamma_1^{(r,n)} f := \left( \frac{-\tilde{d}_n^{1/2} f(x_{n-1}^+) - \tilde{d}_n^{3/2} f(x_{n-1}^+) + f(x_{n-1}^-)}{-\tilde{d}_n^{1/2} f(x_{n-1}^-)} \right),
\]

is an ordinary boundary triple for $H_{n\text{min}}^*$.

The proof is similar to that of Theorem 8.10 and is omitted.

8.4. Dirac operators with local point interactions. Let $D$ be a differential expression

\[
D = -ic \frac{d}{dx} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3 = \left( \begin{array}{cc} \frac{c^2}{2} & i c \frac{d}{dx} \\ -i c \frac{d}{dx} & -\frac{c^2}{2} \end{array} \right)
\]

acting on $\mathbb{C}^2$-valued functions of a real variable. Here

\[
\sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),
\]

are the Pauli matrices in $\mathbb{C}^2$ and $c > 0$ denotes the velocity of light.

Further, let $D_n$ be the minimal operator generated in $L^2[x_{n-1}, x_n] \otimes \mathbb{C}^2$ by the differential expression (8.47)

\[
D_n = D \upharpoonright \text{dom}(D_n), \quad \text{dom}(D_n) = W_0^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2.
\]

Denote $d_n := x_n - x_{n-1} > 0$. Recall that $D_n$ is a symmetric operator with deficiency indices $n_\pm(D_n) = 2$ and its adjoint $D_n^*$ is given by

\[
D_n^* = D \upharpoonright \text{dom}(D_n^*), \quad \text{dom}(D_n^*) = W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2.
\]

Next following [32] we recall the construction of a boundary triple for $D_n^*$ and compute the corresponding Weyl function. Namely, the triple $\Pi^{(n)} = \{\mathbb{C}^2, \Gamma^{(n)}_0, \Gamma^{(n)}_1\}$, where

\[
\Gamma_0^{(n)} f := \frac{f_1}{f_2} = \left( \frac{f_1(x_{n-1}^-)}{i c f_2(x_{n-1}^-)} \right), \quad \Gamma_1^{(n)} f := \frac{f_1}{f_2} = \left( \frac{i c f_2(x_{n-1}^+)}{f_1(x_{n-1}^-)} \right),
\]

forms a boundary triple for $D_n^*$. Clearly, $D_{n,0} := D_n^* \upharpoonright \ker\Gamma_0^{(n)} = D_{n,0}^*$ and

\[
\text{dom}(D_{n,0}) = \{\{f_1, f_2\}^T \in W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2 : f_1(x_{n-1}^+) = f_2(x_{n-1}^-) = 0\}.
\]

Moreover, the spectrum of the operator $D_{n,0}$ is discrete,

\[
\sigma(D_{n,0}) = \sigma_d(D_{n,0}) = \left\{ \pm \sqrt{\frac{c^2\pi^2}{d_n^2} \left( j + \frac{1}{2} \right)^2 + \left( \frac{c^2}{2} \right)^2}, \quad j \in \mathbb{N} \right\}.
\]
The defect subspace $\mathcal{N}_z := \ker(D_n^* - z)$ is spanned by the vector functions $f_n^{\pm}(\cdot, z)$,

\begin{equation}
    f_n^{\pm}(x, z) := \left( e^{\pm i k(z) x} \right).
\end{equation}

Moreover, the Weyl function $M_n(\cdot)$ corresponding to the triple $\Pi^{(n)}$ is (cf. [32])

\begin{equation}
    M_n(z) = \frac{1}{\cos(d_n k(z))} \begin{pmatrix} c k_1(z) \sin(d_n k(z)) & 1 \\ 1 & (c k_1(z))^{-1} \sin(d_n k(z)) \end{pmatrix}, \quad z \in \rho(D_{n,0}),
\end{equation}

where

\begin{equation}
    k(z) := c^{-1} \sqrt{z^2 - (c^2/2)^2}, \quad z \in \mathbb{C},
\end{equation}

and

\begin{equation}
    k_1(z) := \frac{c k(z)}{z + c^2/2} = \sqrt{\frac{z - c^2/2}{z + c^2/2}}, \quad z \in \mathbb{C}.
\end{equation}

Next we construct a boundary triple for the operator $D_X^* := \bigoplus_{n=1}^{\infty} D_n^*$ in the general case $0 \leq d_* < d^* \leq \infty$. It appears that the result in the case $d^* = \infty$ remains analogous to what was obtained in [32] for the case $d^* < \infty$.

Define $D_X := \bigoplus_{n=1}^{\infty} D_n$,

\begin{equation*}
    \text{dom}(D_X^*) = W^{1,2}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2 = \bigoplus_{n=1}^{\infty} W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2.
\end{equation*}

Next following [32] we collect certain properties of the direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)}$ of boundary triples $\Pi^{(n)}$ given by (8.50).

**Proposition 8.17.** Let $X$ be as above, let $0 \leq d_* < d^* \leq \infty$, and let $\Pi^{(n)} = \{ \mathcal{H}^{(n)}, \Gamma_0^{(n)}, \Gamma_1^{(n)} \}$ be the boundary triple for the operator $D_n^*$ defined in (8.50). Let $\mathcal{H} = l^2(\mathbb{N}) \otimes \mathbb{C}^2$ and $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)} = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \}$, where the operators $\Gamma_j, \quad j \in \{0, 1\}$ are given by (8.3), i.e.

\begin{equation}
    \Gamma_0 \left( \frac{f_1^1}{f_2^1} \right) = \left\{ \begin{pmatrix} f_1(x_{n-1}^-) \\ i c f_2(x_{n-1}^-) \end{pmatrix} \right\}_{n \in \mathbb{N}}, \quad \Gamma_1 \left( \frac{f_1^1}{f_2^1} \right) = \left\{ \begin{pmatrix} i c f_2(x_{n-1}^+) \\ f_1(x_{n}^-) \end{pmatrix} \right\}_{n \in \mathbb{N}},
\end{equation}

where $f = (\frac{f_1^1}{f_2^1}) \in \text{dom} D_{X,*} := \text{dom} \Gamma$. Then:

(i) The domain $\Gamma$ is given by $\text{dom} X_{*,*} := \text{dom} \Gamma = \text{dom} \Gamma_0 = \text{dom} \Gamma_1$.

(ii) The direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)}$ forms a $B$-generalized boundary triple for $D_X^*$.

(iii) The transposed triple $\Pi^\top = \{ \mathcal{H}, \Gamma_0^+, \Gamma_1^+ \} := \{ \mathcal{H}, \Gamma_1, -\Gamma_0 \}$ also forms a $B$-generalized boundary triple for $D_X^*$.

(iv) The triple $\Pi$ (equivalently the triple $\Pi^\top$) is an ordinary boundary triple for the operator $D_X = \bigoplus_{n=1}^{\infty} D_n^*$ if and only if $d_* > 0$ (with $d^* \leq \infty$).

**Proof.** (i), (ii) The Weyl function of the boundary triple $\{ \mathcal{H}, \Gamma_0, \Gamma_1 \}$ is the orthogonal sum $M = \bigoplus M_n$ of the Weyl functions defined by (8.54). It follows from (8.55) and (8.56) that $k(0) = i c/2$ and $k_1(0) = i$ and hence

\begin{equation}
    M_n(0) = \frac{1}{\sinh(d_n c/2)} \begin{pmatrix} -c \cosh(d_n c/2) \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ e^{-1} \cosh(d_n c/2) \end{pmatrix}.
\end{equation}

Hence

\begin{equation}
    M_n(0) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ as } d_n \to 0 \quad \text{and} \quad M_n(0) \to \begin{pmatrix} -c & 0 \\ 0 & c^{-1} \end{pmatrix} \quad \text{as } d_n \to \infty.
\end{equation}

It follows that the sequence $\{ M_n(0) \}_{n \in \mathbb{N}}$ is bounded.
Furthermore, one gets from (8.54) and (8.56) that \( k'(0) = 0, \ k'_1(0) = -i \frac{2}{c^2}, \) and

\[
M'_n(0) = \begin{pmatrix}
\frac{2}{c} \text{th}(d_n c/2) & 0 \\
0 & \frac{2}{c^3} \text{th}(d_n c/2)
\end{pmatrix} \geq 0, \quad n \in \mathbb{N}.
\]

This description implies that

\[
M'_n(0) \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
as \( d_n \to 0 \) and \( M'_n(0) \to \begin{pmatrix} \frac{2}{c} & 0 \\ 0 & \frac{2}{c^3} \end{pmatrix} \) as \( d_n \to \infty. \)

Thus, the sequence \( \{M'_n(0)\}_{n \in \mathbb{N}} \) is bounded too. Combining formulas (8.59) with (8.61) and applying Theorem 8.4(iii) one concludes that \( \Pi = \{H, \Gamma_0, \Gamma_1\} \) is a \( B \)-generalized boundary triple for \( D_X \).

(iii) It follows from (8.58) that det (8.60) that the sequence

\[
\det(\Pi) = M^{-1}_n(0) M'_n(0) M^{-1}_n(0), \quad n \in \mathbb{N},
\]

we obtain that the sequence \( \{(M^{-1}_n)'(0)\}_{n \in \mathbb{N}} \) is bounded too. It remains to apply Theorem 8.4(iii).

(iv) It follows from (8.60) that the sequence \( \{M'_n(0)\}_{n \in \mathbb{N}} \) of the derivatives is uniformly positive if and only if \( d_1 > 0 \). One completes the proof by combining Theorem 8.4(iv) with the above proved items (ii), (iii).

\[\square\]

**Remark 8.18.** Note that if \( d^* = \infty \) then in view of (8.52) \( \pm \frac{2}{c^2} \in \sigma(D_0) \), while \( \left(-\frac{2}{c^2}, \frac{2}{c^2}\right) \subset \rho(D_0) \). Therefore as distinguished from the considerations in [32] treating the case \( d^* < \infty \), here we consider the behavior of the Weyl function at \( z = 0 \in \rho(D_0) \).

We now apply a modification of Theorem 6.1 and Proposition 6.2 to produce an ES-generalized boundary triple for \( D_X \) from the \( B \)-generalized boundary triple \( \Pi := \bigoplus_{n=1}^\infty \Pi^{(n)} = \{H, \Gamma_0, \Gamma_1\} \). In this modification we subtract from the Weyl function \( M_n \) the limit value \( \lim_{d_n \to 0} M_n(0) \), instead of the value \( M_n(0) \), to get a transform of boundary mappings in a simple form.

**Proposition 8.19.** Let \( X \) be as above, let \( 0 \leq d_1 < d^* \leq \infty \), let \( \Pi^{(n)} = \{C^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\} \) be the boundary triple for the operator \( D_n^{\ast} \) defined by

\[
\Gamma_0^{(n)}(f_1, f_2) = \left( \frac{ic (f_2(x_n-1) - f_2(x_n-1))}{f_1(x_n-1) - f_1(x_n-1)}, \quad \Gamma_1^{(n)}(f_1, f_2) = \left( \frac{f_1(x_n+1)}{ic f_2(x_n)} \right), \quad n \in \mathbb{N},
\]

let \( \tilde{\Gamma}_j = \bigoplus_{n=1}^\infty \tilde{\Gamma}_j^{(n)} \), \( j \in \{0, 1\} \), and let \( \tilde{\Pi} = \bigoplus_{n=1}^\infty \tilde{\Pi}_n = \{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\} \) be the boundary triple for \( D_X \), where

\[
\tilde{\Gamma}_j := \tilde{\Gamma}_j \upharpoonright \text{dom}(D_{X,*}), \quad \text{dom}(D_{X,*}) := \text{dom}(\tilde{\Gamma}) := \text{dom} \tilde{\Gamma}_0 \cap \text{dom} \tilde{\Gamma}_1.
\]

(i) The mapping \( \tilde{\Gamma}_0 \times \tilde{\Gamma}_1 \) is naturally extended to the mapping \( \tilde{\Gamma}_0'' \times \tilde{\Gamma}_1'' \) defined by the same formulas on \( W^{1,2}(\mathbb{R}_+ \setminus \chi) \otimes \mathbb{C}^2 \). Moreover, the mapping

\[
\tilde{\Gamma}_0'' \times \tilde{\Gamma}_1'' : W^{1,2}(\mathbb{R}_+ \setminus \chi) \otimes \mathbb{C}^2 \to (L^2(\chi; \{d_n-1\}) \otimes \mathbb{C}^2) \times (L^2(\chi; \{d_n\}) \otimes \mathbb{C}^2)
\]
is well defined and surjective.

(ii) The mapping

\[
\tilde{\Gamma}_0 \times \tilde{\Gamma}_1 : \text{dom} D_{X,*} \to (L^2(\chi; \{d_n\}) \otimes \mathbb{C}^2) \times (L^2(\chi; \{d_n\}) \otimes \mathbb{C}^4),
\]
is well defined and surjective. Moreover, \( \text{dom} D_{X,*} = \text{dom} \tilde{\Gamma} = \text{dom} \tilde{\Gamma}_0 = \text{dom} D_X = W^{1,2}(\mathbb{R}_+ \setminus \chi) \otimes \mathbb{C}^2. \)
Moreover, it follows from (8.68) that

\[ \overline{M}_n(z) = -2^{-1} \begin{pmatrix} \sin(d_n k(z)) & -1 \\ \frac{ck_1(z)(1 - \cos(d_n k(z)))}{1 - \cos(d_n k(z))} & \frac{ck_1(z) \sin(d_n k(z))}{1 - \cos(d_n k(z))} \end{pmatrix} \]

and it is domain invariant with

\[ \text{dom} \overline{M}(z) = l^2(\mathbb{N}; \{d_n^{-2}\}) \otimes \mathbb{C}^2 \subseteq \Gamma_0(\text{dom} D_{X,s}) = l^2(\mathbb{N}; \{d_n^{-1}\}) \otimes \mathbb{C}^2 \quad \text{for} \quad z \in \mathbb{C}_\pm. \]

Note that the converse inequality holds if and only if \( l \leq d \).

\[ \text{(iii) The Weyl function is of the form } \overline{M}(\cdot) = \bigoplus_{n=1}^{\infty} \overline{M}_n(\cdot), \text{ where} \]

\[ \overline{M}_n(z) = -2^{-1} \begin{pmatrix} \sin(d_n k(z)) & -1 \\ \frac{ck_1(z)(1 - \cos(d_n k(z)))}{1 - \cos(d_n k(z))} & \frac{ck_1(z) \sin(d_n k(z))}{1 - \cos(d_n k(z))} \end{pmatrix} \]

\[ \text{and it is domain invariant with} \]

\[ \text{dom} \overline{M}(z) = l^2(\mathbb{N}; \{d_n^{-2}\}) \otimes \mathbb{C}^2 \subseteq \Gamma_0(\text{dom} D_{X,s}) = l^2(\mathbb{N}; \{d_n^{-1}\}) \otimes \mathbb{C}^2 \quad \text{for} \quad z \in \mathbb{C}_\pm. \]

Here the strict inclusion \( \text{dom} M(z) \subset \Gamma_0(\text{dom} D_{X,s}) \) holds if and only if \( d_s = 0 \).

\[ \text{(iv) The Weyl function } \overline{M}(\cdot) \text{ is also form-domain invariant with} \]

\[ \text{dom} \overline{M}(z) = l^2(\mathbb{N}; \{d_n^{-1}\}) \otimes \mathbb{C}^2 \subseteq \Gamma_0(\text{dom} D_{X,s}) \quad \text{for} \quad z \in \mathbb{C}_\pm. \]

\[ \text{(v) } \widetilde{\Pi} = \{ \mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1 \} \text{ forms an ES-generalized boundary triple for } D_{X}. \text{ Moreover, } \widetilde{\Pi} \text{ is an S-generalized boundary triple for } D_{X} \text{ if and only if } d_s > 0 \text{ and in this case } \widetilde{\Pi} \text{ is in fact and ordinary boundary triple for } D_{X}. \]

\[ \text{(vi) The transposed triple } \Pi^\top = \{ \mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1 \} := \{ \mathcal{H}, \tilde{\Gamma}_1, -\Gamma_0 \} \text{ is a B-generalized boundary triple for } D_{X}. \text{ In particular, } A_1 = D_{X} \upharpoonright \ker \tilde{\Gamma}_1 \text{ is selfadjoint.} \]

**Proof.** (i) The proof is immediate from Lemma 8.7.

(ii) Due to \( d^* < \infty \) one has the following chain of continuous embeddings

\[ l^2(\mathbb{N}; \{d_n^{-1}\}) \otimes \mathbb{C}^2 \subset l^2(\mathbb{N}) \otimes \mathbb{C}^2 \subset l^2(\mathbb{N}; \{d_n\}) \otimes \mathbb{C}^2. \]

Since \( l^2(\mathbb{N}) \otimes \mathbb{C}^2 \) is a part of \( l^2(\mathbb{N}; \{d_n\}) \otimes \mathbb{C}^2 \), the surjectivity of the mapping \( \tilde{\Gamma} = (\tilde{\Gamma}_0^0 \times \tilde{\Gamma}_0^\prime) \upharpoonright \text{dom} D_{X,s} \) is immediate from (i). The inclusion in (8.63) as well as the relation \( \text{dom } \tilde{\Gamma} = \text{dom } \Gamma_1 \) is implied by the first inclusion in (8.67).

(iii) The Weyl function corresponding to \( \tilde{\Pi} \) is the direct sum \( \overline{M}(\cdot) = \bigoplus_{n=1}^{\infty} \overline{M}_n(\cdot) \), where

\[ \overline{M}_n(z) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M_n(z), \quad z \in \rho(M_n). \]

This immediately leads to formula (8.64) for \( \overline{M}_n(z) \). Using (8.55), (8.56), and the Taylor series expansions for \( \sin(z) \) and \( \cos(z) \) we easily derive

\[ \overline{M}_n(z) + \frac{1}{d_n} \begin{pmatrix} (z - c^2/2)^{-1} & 0 \\ 0 & (z^2 + c^2/2)^{-1} \end{pmatrix} \to \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ as } d_n \to 0, \quad z \in \mathbb{C}_\pm. \]

This formula shows that \( \overline{M}(z) \), as well as \( \text{Im } \overline{M}(z) \), is bounded if and only if \( d_s > 0, z \in \mathbb{C}_\pm. \).

Moreover, it follows from (8.63) that \( \{(a_n^m, b_n^m)\}_{n,m} \in \text{dom } \overline{M}(z), z \in \mathbb{C}_\pm, \text{ precisely when} \)

\[ \sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{d_n^2} < \infty. \]

The inclusion (in fact the continuous embedding) in (8.65) follows from the estimate

\[ \sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{d_n^2} \leq d^* \sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{d_n^2}. \]

Note that the converse inequality holds if and only if \( d_s > 0 \). Indeed, writing down the reverse inequality and inserting here \( \{a_n\} = \{\delta_{j,n}\}_{n \in \mathbb{N}} \) and \( \{b_n\} = \{0\}_{n \in \mathbb{N}} \), one arrives at the inequalities

\[ 1 \leq c d_j, \quad j \in \mathbb{N}, \]
showing that \( d_\ast \geq 1/c > 0 \).

(iv) By definition, \( \{ h_n \}_{n=1}^\infty \in \text{dom } \tilde{M}(z) \) if and only if

\[
(8.69) \quad \sum_{n=1}^\infty (\Im \tilde{M}(z) h_n, h_n) < \infty; \quad \{ h_n \}_{n=1}^\infty = \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2.
\]

As a function of \( d_n \) the imaginary part \( \Im \tilde{M}(z) \) is bounded on the intervals \( [\delta, \infty) \), \( \delta > 0 \), and hence it follows from (8.68) that the convergence of the series in (8.69) is equivalent to

\[
\sum_{n=1}^\infty (\frac{\Im K(z) h_n, h_n}{d_n}) < \infty; \quad \{ h_n \}_{n=1}^\infty = \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2,
\]

where \( K(z) \) denotes the diagonal matrix function in the left-hand side of (8.68). Clearly, \( \Im K(z) \) is bounded with bounded inverse for each \( z \in \mathbb{C}_\pm \) and this yields the stated description of \( \text{dom } \tilde{M}(z) \).

(v) By Theorem 8.1(iv), the triple \( \tilde{\Pi} \) being a direct sum of ordinary boundary triples, is an \( ES\)-generalized boundary triple. On the other hand, by (iii) the strict inclusion \( \text{dom } M(z) \subsetneq \tilde{\Gamma}_0(\text{dom } D_{X,*}) \) is equivalent to \( d_n = 0 \). Therefore, Theorem 5.18 applies and ensures that in the latter case \( \tilde{\Pi} \) is not an \( S\)-generalized boundary triple.

(vi) The Weyl function corresponding to the transposed boundary triple \( \Pi^T \) is \( -\tilde{M}(\cdot)^{-1} = \bigoplus_1^\infty (-\tilde{M}(\cdot)^{-1}) \). In particular, one gets from (8.68) (or from (8.54)) that

\[
-\tilde{M}_n(z)^{-1} = M_n(z) - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim d_n \begin{pmatrix} z - c^2/2 & 0 \\ 0 & c^{-2}(z + c^2/2) \end{pmatrix} \text{ as } d_n \to 0.
\]

This shows that \( -\tilde{M}(\cdot)^{-1} \in \mathcal{R}^*[\mathcal{H}] \), which is equivalent to the required statement: \( \Pi^T \) is a \( B\)-generalized boundary triple (see [44, Chapter 5]).

\[\]  

**Remark 8.20.** Apart from statements (ii) and the formula for \( \tilde{\Gamma}_0(\text{dom } D_{X,*}) \) in statement (iii) the results in Proposition 8.19 remain valid for \( d_\ast = \infty \). Indeed, statement (i) is still immediate from Proposition 8.7(i) which holds in this case, too. All the other statements can easily be extracted from the fact that the limit value of the Weyl function \( \tilde{M}_n(z) \) as well as its inverse \( \tilde{M}_n(z)^{-1} \) remain bounded when \( d_n \to \infty \).

Let \( \alpha = \{ \alpha_n \}_{n \in \mathbb{N}} \) be a sequence from \( \mathbb{R} \). Gesztesy-Šebeta realization of Dirac operator (see [49]) is defined by \( D_{X,\alpha} = D|_{\text{dom } D_{X,\alpha}} \), where

\[
(8.70) \quad \text{dom } D_{X,\alpha} = \left\{ f \in W_{\text{comp}}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2 : f_1 \in AC_{\text{loc}}(\mathbb{R}_+), f_2 \in AC_{\text{loc}}(\mathbb{R}_+ \setminus X) \right\}.
\]

As was shown in [49], [32] the Gesztesy-Šebeta realization \( D_{X,\alpha} \) is always selfadjoint. The operators \( D_{X,\alpha} \) are parametrized in the boundary triple \( \Pi = \{ \mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1 \} \) via selfadjoint three-diagonal matrices

\[
J_\alpha = \begin{pmatrix} 0 & -1 \\ -1 & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & \alpha_1 & -1 \\ -1 & 0 & 1 \\ 1 & \alpha_2 & -1 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & \alpha_n & -1 \end{pmatrix}
\]
Proposition 8.21. Let $\alpha = \{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, let $D_{X,\alpha}$ be the Gesztesy-Šeba realization of the Dirac operator given by (8.70), let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triple defined by (8.57) and let $M(\lambda) = \bigoplus_{n=1}^{\infty} M_n(\lambda)$, $\gamma(\lambda) = \bigoplus_{n=1}^{\infty} \gamma_n(\lambda)$, $Q = \bigoplus_{n=1}^{\infty} Q_n$, $Q_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $n \in \mathbb{N}$, with $M_n(\lambda)$ and $\gamma_n(\lambda)$ given by (8.54) and [32, (3.11)], respectively. Then:

$$\operatorname{dom} D_{X,\alpha} = \ker (\Gamma_1 - (B_\alpha + Q)\Gamma_0).$$

Moreover,

$$\lambda \notin \sigma_p(D_{X,\alpha}) \iff 0 \notin \sigma_p(B_\alpha + Q - M(\lambda)),$$

and the following Kreĭn-type formula holds

$$(D_{X,\alpha} - \lambda)^{-1} = (D_0 - \lambda)^{-1} + \gamma(\lambda)(B_\alpha + Q - M(\lambda))^{-1}\gamma(\bar{\lambda})^*, \quad \lambda \in \rho(D_{X,\alpha}) \cap \rho(D_0).$$

Proof. The equality (8.71) is implied by (8.70) and (8.57). The formulas (8.72) and (8.73) follow from Theorem 4.11 or Theorem 5.9. □

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