\( \hbar \to 0 \) in Kicked Harper Model: Reassurances and Surprises

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We investigate classical-quantum correspondence for kicked Harper model for extremely small values of the Planck constant \( \hbar \). In the asymmetric case a pure quantum state shows clear signature of classical diffusive as well as super-diffusive transitions asymptotically independent of \( \hbar \). However, for the symmetric case, the \( \hbar \) independent behavior occurs only for renormalized parameter \( \tilde{K} = K/(2\hbar) \) with intriguing features such as a sharp transition from integrable to non-integrable transport at \( K = \pi/2 \), a series of transitions at multiples of \( \pi \) and periodicity of the transmission probability. These results add new puzzles to the frontiers of quantum chaos.

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Localized transport of quantum system in the regime where the corresponding classical system exhibits deterministic diffusive behavior is one of the most surprising aspects of non-integrable Hamiltonian systems [1]. However, the correspondence principle requires some signatures of various classical transitions such as the breakup of KAM tori leading to diffusive transport and the emergence of accelerator modes (AM) resulting in super-diffusive anomalous transport. In this paper, we describe quantum signatures of various classical transitions in transport characteristics emphasizing the crossover effects from large to small values of effective Planck constant \( \hbar \). As \( \hbar \to 0 \), the quantum system exhibiting localization in one of the phase-space directions is found to feel the effects of all classical transitions. However, in the absence of localized transport, the quantum system exhibits many surprising features and appears to be insensitive to the classical dynamics.

The problem of establishing classical-quantum correspondence in quasi-periodic extended systems has proven to be difficult due to numerical limitations in approaching \( \hbar \to 0 \). All previous studies (see e.g. [2]) addressing this question have been limited to \( \hbar \approx 1 \). Here we use recently developed renormalization group (RG) approach [3] to study quantum transport for extremely small values of \( \hbar \) upto \( 10^{-4} \).

The kicked Harper model [4] has emerged as an important model in quantum chaos literature. The system is given by doubly periodic time-dependent Hamiltonian

\[
H(t) = L \cos(p) + K \cos(q) \sum_{k=-\infty}^{\infty} \delta(t-k). \tag{1}
\]

Here \( q, p \) is a canonically conjugate pair of variables, usually considered on a cylinder \( p \in (-\infty, \infty), q \in [0, 2\pi] \).

The classical dynamics of the kicked Harper is determined by two parameters \( K \) and \( L \). In the asymmetric case (\( K \neq L \)), the phase space, for small values of parameters, is stratified with KAM tori which inhibit the transport in global scale. For \( L > K \), or \( K > L \), these tori barriers limit the transport along \( p \), or \( x \), directions, respectively. As we show below, the 2-d parameter space exhibits an intricately mixed non-diffusive (KAM regime) and diffusive regions corresponding to global stochasticity. In contrast, in the symmetric case \( K = L \), there are no KAM barriers to global transport but a separatrix and it is the breakup of the separatrix that results in global diffusion. For large values of the parameters, both the symmetric and the asymmetric model exhibits mostly diffusive behavior with the exception of narrow windows in parameter space where the AMs give rise to super-diffusive transport [5].

The quantized system that is periodically kicked is described by the quasi-energy states of the one step time evolution operator, introducing an additional parameter \( \hbar \) into the problem. However, one hopes to recover \( \hbar \) independent behavior (as \( \hbar \to 0 \)) in order to establish quantum signatures of classical behavior. It is a well established fact that the RG approach provides the most effective tool in distinguishing ballistic, diffusive and localized transport. Here we use recently developed dimer decimation approach to study transport characteristics of the quasi-energy states in the small \( \hbar \) limit. The RG method is applied [6] to the momentum lattice \((p_m = \hbar m)\) representation of the kicked model [6]

\[
\sum_{r=-\infty}^{\infty} B_r^m u_{m+r} = 0, \tag{2}
\]

where the coefficients \( B_r^m \) are

\[
B_r^m = J_r(\tilde{K}) \sin[\tilde{L} \cos(m\hbar) - \pi r/2 - \omega/2]. \tag{3}
\]

We introduce renormalized parameters as \( \tilde{K} = K/(2\hbar), \tilde{L} = L/(2\hbar) \). Tight-binding model (TBM) [3] effectively contributes only few terms as Bessel's function exhibit fast decay when \( |r| > |\tilde{K}| \). Therefore, the TBM describes a lattice model with a finite range of interaction denoted as \( b \) (\( b \approx |\tilde{K}| \)). In the limit of small \( \tilde{K}, \tilde{L}, \omega \), TBM reduces to the Harper equation with \( \epsilon = \hbar \omega \) [7]. We will choose \( \hbar \) to be an irrational number with a golden tail: \( \hbar/(2\pi) = 1/(n_\hbar + \sigma) \) where \( \sigma = (\sqrt{5} - 1)/2 \) is fixed.
and \( \hbar \) is varied by varying the integer \( n_h \). This corresponds to studying system sizes \( N_n, n = 1, 2, \ldots \) determined from the Fibonacci equation \( N_{n+1} = N_n + N_{n-1} \), with \( N_0 = 1, N_1 = n_h \), and corresponding to \( n \)-th successive rational approximant of the irrational number \( \sigma \).

The RG methods can be used to study system sizes up to \( 10^9 \) which allows very large \( n_h \) and hence facilitates studying kicked model for extremely small values of \( \hbar \).

The transport characteristics of the quasi-energy states are studied by computing the transmission probability \( T \) on the momentum lattice. This is achieved in two steps: first we decimate the lattice and then solve the scattering problem on the renormalized lattice \[3\]. Renormalization scheme makes the solution of the scattering problem for large lattices of size \( N \) very efficient, as the dimer decimation reduces the size of the scattering region. For a fixed \( \hbar \), we compute the transmission probability \( T(N) \) for various sizes \( N \) of the momentum lattice corresponding to a rational approximant of \( \sigma \) with denominator \( N \). The scaling exponent \( \beta = \lim_{N \to \infty} \frac{\log T(N)}{\log N} \) distinguishes extended, localized and critical states respectively, described by \( \beta(N) \to 0, \to -\infty, \) and oscillatory function \( \beta(N_n) \) of \( n \) \[3\]. For the exponential localization, the quantity \( \xi = \lim_{N \to \infty} \frac{1}{N} \log T(N) \) has been found to be closely related to the localization length of the quasienergy eigenstate \( \omega \).

\[ \beta_{cl} = \lim_{t \to \infty} \frac{\log \langle (p(t) - p(0))^2 \rangle}{\log t}, \]

are signaled by \( \beta_{cl} \) changing from 0 to \( \approx 1 \). In the quantum model, our detailed analysis for various values of \( \hbar \) and \( L > K \), confirms the previously held view \[3\] that the quantum system remains localized in the classically diffusive regime. However, the classical transitions corresponding to diffusive transport manifest in huge enhancement of localization length. Results for an individual pure state \( \omega = 0 \) are shown in Fig. 1. As demonstrated in the figure, small \( \hbar \) is crucial to see signatures of all classical transitions. We would like to point out that our results are consistent with the relation \( \xi = \frac{1}{2} D/\hbar^2 \) \[6\]. However, near the peaks, (narrow windows in parameter space corresponding to the onset to classical transitions), the quantum transmission probability \( T(N) \) ceases to look like a simple exponential \( \sim \exp(-N/\xi) \) thus making quantitative comparison difficult.

An interesting aspect of two-parameter Harper kicked model is that the boundary between the KAM and diffusive phases appears to be fractal as seen in Fig. 2. This behavior is reminiscent of the kicked rotor problem where the kicking potential consists of two-harmonics \[10\]. Although somewhat smeared, the quantum model exhibits similar behavior: the boundary describes the transition to the enhancement of localization length \( \xi \). It is remarkable that the quantum system feels the presence of all classical transitions and the fact that unlike kicked rotor, there is a whole hierarchy of transitions in Harper model, makes this model an important system in quantum chaos studies.

![FIG. 1. For fixed \( K = 0.4 \), the figure describes a series of breakup of KAM barriers resulting in transitions to diffusive transport \( \beta_{cl} = 1 \) (top). Middle and the bottom figures show the corresponding plots of quantum localization length \( \xi \) for \( n_h = 200 \) and \( n_h = 32 \). The renormalization is carried out for system sizes increasing until the transmission probability becomes zero.](image)
Important feature of kicked systems with toroidal phase space are the AMs which are regular (stable) space-time structures coexisting with the chaotic sea in phase space and are accompanied by an hierarchy of island chains inducing anomalous transport $\beta_{cl} > 1$. Fig. 3 shows one such super-diffusive parameter window whose origin is traced to a period-8 AM [14]. Once again, the quantum state $\omega = 0$ although localized exhibits a very strong enhancement of localization length in the classically super-diffusive regime. It should be noted that in contrast to the diffusive peaks, super-diffusive spikes are in fact groups of many spikes exhibiting sensitive dependence on the parameters and hence describe transport in fractal phase space.

It is rather surprising that a pure quantum state $\omega = 0$ can exhibit such a clear signature of almost all the classical transitions. This property may be associated with possible special structure of $\omega = 0$ state related to the fact that its quasi-energy is constant and thus insensitive to variations of parameters. We would like to point out that we have only investigated $L > K$ part of the parameter space whereas duality implies that analogous behavior will be seen for $K > L$ in $x$-space.

In our earlier studies [3] for $\hbar \approx O(1)$, we found patches of ballistic (localized) regions for $L > K (K > L)$ [3]. Numerical studies for smaller values of $\hbar$ suggest that the overall measure of the extended (localized) regimes for $L > K (K > L)$ approaches zero as $\hbar \to 0$.

We now discuss the symmetric Harper model with $K = L$. Here the quasi-energy states remain critical and hence exhibit diffusive transport for all values of the kicking parameter $K$. As $\hbar \to 0$, (see Fig. 4) the transmission exponent becomes $\hbar$ independent provided we use renormalized parameter $\tilde{K}$ instead of the bare $K$. The model exhibit transmission characteristics of the Harper equation for $\tilde{K} \leq \pi/2$. Precisely at $\tilde{K} = \pi/2$, the transport exponent begins to exhibit an oscillatory behavior (with frequency proportional to $\hbar$). As opposed to classical mechanics, where infinitesimal perturbation leads to chaotic regions whose size increases as the perturbation increases, the perturbation of such quantum system causes no immediate change in the transport characteristics. This suggests that roots of these transitions may be topological [13]. It should be noted that the onset
to oscillatory behavior is seen at higher odd multiples of $\pi/2$, however, the behavior becomes prominent only at very small $\hbar$. Other fascinating feature is that beyond $K = \pi$, transmission probability appears to be periodic in $\hbar$ with period $\pi$. Finally, the model exhibits a series of "transitions" precisely at $K = l\pi, l = 1, 2, \ldots$ characterized by a discontinuity in the transport exponent.

![Diagram](image)

**FIG. 4.** Variations in $\beta$ characterizing the transmission probability along the line $K = L$ in kicked Harper model for $n_h = 10000$(top), $n_h = 250$(middle) and $n_h = 32$(bottom). The plateau for $K < \pi/2$ is the transport exponent for the Harper equation. Although the exponent $\beta$ oscillates with the $N$, all RG iterates show qualitatively the same behavior.

Recently, a semiclassical analysis [11] hinted a possibility of a series of enhancement of transport at $K = l\pi/2$ associated with the emergence of new periodic orbits of frequency $1/4$. The integer $l$ is a kind of winding number that unfolds $\omega$ (confined to the interval $0 - 2\pi$). The fact that $K$ and not $K$ determines the thresholds for various transitions poses a serious problem about the classical-quantum correspondence. As discussed in earlier studies [11], the symmetric model exhibits super-diffusive classical transport at various critical values of the kicking parameter $K$. The most important of those are the period-1 and period-2 AMs which respectively occur at even and odd multiples of $\pi$. It is therefore tempting to associate $K$ with the classical $K$ as was suggested in Ref. [11]. However, this singular scaling which associates the discontinuities in Fig. 4 with the classical AMs is inconsistent with the asymmetric case where no scaling of the parameters is needed to establish quantum-classical correspondence. It is possible that the transitions seen in the symmetric case are purely quantum-mechanical in nature and may have their origin in resonances of the (driven) RG flow and/or topological changes. We should mention that a possibility of some sudden changes at multiples of $\pi/2$ also emerged in our analysis of the scattering problem. It turns out that the number of independent propagating solutions of the scattering problem [8], i.e. dimension of the $S$-matrix, increases by 1 at $K_l = l\pi/2, l = 1, 2, \ldots$, matching with the discontinuities of the transport exponent $\beta(K)$. Finally, the most challenging result which defies our intuition, is the (asymptotic) periodicity of the transmission probability $T|_{\bar{K}+\pi} = T|_{\bar{K}}$, for $K \gg \pi$ which rules out any possibility of quantum manifestations of classical super-diffusive transitions.

Symmetric Kicked Harper model is an interesting example of a non-integrable system where the classical as well as the quantum transport is diffusive. It is in sharp contrast to the asymmetric case where the classically diffusive behavior corresponds to localized quantum transport. In view of this, it is rather surprising that in the asymmetric case the quantum system appears to respond to all the changes in the classical behavior, while in the symmetric case it remains insensitive to the variation in classical transport and instead repeats its behavior at every multiple of $\pi$. This adds a new puzzle to the field of quantum chaos.

Inability of the quantum system to delocalize in classically diffusive regime and mimic the classical behavior for arbitrary small value of $\hbar$ as confirmed by RG analysis, remains an open frontier. Earlier studies have suggested phase randomization due to classical chaos as a mechanism for quantum dynamical localization. The fact that the kicked Harper model can exhibit localized, ballistic and diffusive transport irrespective of the fact that the corresponding classical system is chaotic challenges the phase randomization as the underlying mechanism for localization. Here we would like to propose an alternative mechanism of dynamical localization: we speculate that the dynamical localization may be due to the cantori barriers. These are invariant quasiperiodic trajectories with infinite number of steps and provide an effective non-analytic quasiperiodic potential. The possibility of localization in quasiperiodic potential with infinite steps has been discussed recently [15]. We would like to emphasize that the scenario for localization due to cantori suggests that these barriers continue to inhibit transport even as $\hbar \to 0$ irrespectively of the flux through the holes in cantori. This scenario not only explains dynamical localization in kicked rotor and Harper model (for $L > K$), but also accounts for ballistic transport in kicked Harper for $K > L$. Furthermore, it is consistent with the diffusive quantum transport for $K = L$ case, since the symmetric Kicked Harper model does not possess global cantori barriers (at least not of the type of broken KAM barriers). We hope that further studies will put our speculative views on solid footings.

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