CHROMATIC (CO)HOMOLOGY OF FINITE GENERAL LINEAR GROUPS

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ABSTRACT. We study the Morava $E$-theory (at a prime $p$) of $BGL_d(F)$, where $F$ is a finite field with $|F| = 1 \pmod{p}$. Taking all $d$ together, we obtain a structure with two products $\times$ and $\bullet$. We prove that it is a polynomial ring under $\times$, and that the module of $\times$-indecomposables inherits a $\bullet$-product, and we describe the structure of the resulting ring. In the process, we prove many auxiliary structural results.

1. INTRODUCTION

Let $K$ be the Morava $K$-theory of height $n$ at a prime $p > 2$, and let $E$ be the corresponding Morava $E$-theory. (Some details of these theories will be recalled in Section 5.)

Let $\bar{F}$ be an algebraic closure of $F$, and let $\Gamma$ be the associated Galois group, which is topologically generated by the map $\phi: a \mapsto a^q$. One can then show that $E_0^{0}(BGL_d(\bar{F}))$ is a formal power series ring $E_0^{0}[c_1, \ldots, c_d]$. Now put $r_i = \phi^∗(c_i) - c_i$.

**Theorem 1.1** (Tanabe). The elements $r_i$ form a regular sequence in $E_0^{0}(BGL_d(\bar{F}))$, and

$$E_0^{0}(BGL_d(F)) = E_0^{0}(BGL_d(\bar{F}))\Gamma = E_0^{0}[c_1, \ldots, c_d]/(r_1, \ldots, r_d).$$

Moreover, this is a finitely generated free module over $E_0^{0}$, and $E_1^{0}(BGL_d(F)) = 0$.

Although this is in many ways very satisfactory, it is not easy to analyse the action of $\phi^∗$, or to find a basis for $E_0^{0}(BGL_d(F))$ over $E_0^{0}$. Here we will develop some other approaches that will shed light in these questions.

We will start by explaining the most concrete and computational consequences of our results. First, let $r$ be the largest integer such that $q = 1 \pmod{p^r}$. Put $N_0 = p^{nr}$, and

$$N_k = p^{(n-1)k+n(r-1)}(p^n-1)$$

for $k > 0$, and $\bar{N}_k = \sum_{i \leq k} N_i$. Next, we will introduce two different products and one coproduct on $E_0^{0}(BGL_d(F))$. The first product is just the ordinary one induced by the diagonal map, and written $a \otimes b \mapsto ab$. Next, we have evident inclusions $GL_i(F) \times GL_j(F) \to GL_{i+j}(F)$. The associated transfer maps give a second product, written $a \otimes b \mapsto a \times b$. The associated restriction maps also give a coproduct. Both of these respect the grading where we put $E_0^{0}(BGL_d(F))$ in degree $d$. We write $\text{Ind}_∗(E_0^{0}(BGL_d(F)))$ for the $\times$-indecomposables, and $\text{Prim}_∗(E_0^{0}(BGL_d(F)))$ for the coalgebra primitives. We also put

$$X_k = \{c^j_{p^k} \mid 0 \leq j < N_k\} \subseteq E_0^{0}(BGL_{p^k}(F)).$$
Notation 1.2. Where necessary to improve readability of exponents, we will write $a \uparrow k$ for $a^k$.

Theorem 1.3.

(a) $E^0(BGL_*(F))$ is a polynomial ring under the $\times$-product, freely generated by $\bigcup_k X_k$.
(b) The natural map $K^0 \otimes_{E^0} E^0(BGL_*(F)) \to K^0(BGL_*(F))$ is an isomorphism, so $K^0(BGL_*(F))$ is also polynomial, with the same generators.
(c) The $\times$-decomposable elements form an ideal under the ordinary product, so $\text{Ind}_*(E^0(BGL_*(F)))$ has a natural ring structure. In fact $\text{Ind}_*(E^0(BGL_*(F)))$ has the form $E^0[c_p]/g_k(c_p)$ for some series $g_k(t)$ of Weierstrass degree $N_k$, and this is a complete regular local noetherian ring. On the other hand, $\text{Ind}_d(E^0(BGL_*(F))) = 0$ if $d$ is not a power of $p$.
(d) $\text{Prim}_d(E^0(BGL_d(F)))$ has a natural structure as a module over $\text{Ind}_d(E^0(BGL_d(F)))$, and in fact it is free of rank one.
(e) $\text{Prim}_p(K^0(BGL_*(F)))$ is generated by $c_p \uparrow \hat{N}_k - 1$, and the socle of $K^0(BGL_{p^k}(F))$ is generated by $c_{p^k} \uparrow (\hat{N}_k - 1)$.

Proof. Most of Claim (a) is in Proposition 8.24 except for the identification of the generators, which follows from Claim (c). Claim (b) is covered by Corollary 8.24. Claim (c) combines Lemma 5.24 with Proposition 10.10 and Corollary 10.10. (In Section 10 the notation $I$ is used for the primitives, and $Q = R/J$ for the indecomposables.) Claim (d) is covered by Proposition 10.14. Finally, Claim (e) is Proposition 10.14. \qed

Some other features of our work are as follows.

- We will find it convenient to consider the groupoid $\mathcal{V}$ of finite-dimensional vector spaces over $F$, and isomorphisms between them. This has $BV \simeq \bigcup_{d=0}^{\infty} BGL_d(F)$, so by using $\mathcal{V}$, we consider all possible values of $d$ simultaneously. We can use the direct sum and tensor product to make $\mathcal{V}$ into a symmetric bimonoidal category, which gives a rich algebraic structure on $E^0BV$ and so on. We will also be able to compare $\mathcal{V}$ in a useful way with various other symmetric bimonoidal groupoids.

- In particular, we will compare $\mathcal{V}$ with the groupoid $\mathcal{V}(k)$ of finite-dimensional vector spaces over the field $F(k)$, which is the unique field extension of $F$ of degree $p^k$ contained in $\overline{F}$.

- For various groupoids $\mathcal{G}$, we will study the interplay between $H^*(BG)$, $K^0(BG)$, $E^0(BG)$, the generalised character ring $D^0(\mathcal{G})$ of Hopkins-Kuhn-Ravenel, and the formal schemes $\text{spf}(K^0BG)$ and $\text{spf}(E^0BG)$. As is usual in this theory, the generalised character rings have an elegant description in terms of the discrete group $\Theta = (\mathbb{Z}/p^\infty)^n$. The challenge is to formulate and prove analogous statements about $\text{spf}(E^0BG)$ in which $\Theta$ is replaced by the formal group scheme $G$ associated to $E$.

- We will also use the dual objects $H_*(BG)$, $K_0(BG)$, $E_0^0(BG)$ and $D_0(BG)$, while remembering that $K(n)$-local duality theory gives a natural isomorphism $K_0BG \simeq K^0(BG)$, and similarly for $E$ and $D$. 

Remark 1.4. We have assumed that $|F| = 1 \pmod{p}$, so in particular the characteristic of $F$ is not $p$. This restriction on the characteristic is essential; the problem would be very different and much harder if $F$ had characteristic $p$. On the other hand, the restriction $|F| = 1 \pmod{p}$ is not so essential. If $F_0$ is a finite field with $|F_0| \neq 0 \pmod{p}$ then we can let $m$ denote the multiplicative order of $|F_0|$ in $\mathbb{Z}/p$ (so $m$ divides $p - 1$). We can then construct a Galois extension $F/F_0$ with Galois group $C_m$. The results in this paper will determine the Morava $K$-theory of the groupoid $V$ associated to $F$. Because $C_m$ has order coprime to $p$, it is essentially a matter of bookkeeping to recover the Morava $K$-theory of the corresponding groupoid $V_0$ associated to $F_0$. Details will be given elsewhere.

Remark 1.5. We have also assumed that $p > 2$. We expect that only minor (but pervasive) adjustments are needed for $p = 2$, possibly including the assumption that $|F| = 1 \pmod{4}$, but we have not checked this.

This paper contains results from the Ph.D. theses of the first [4] and second [7] authors, written under the supervision of the third author. The second author’s thesis covered $E^0\text{BGL}_d(F)$ for $d \leq p$; building on this, the first author obtained results for all $d$. Neither thesis has previously been published.

2. Some fields and rings

As in the introduction, we will assume that $F$ is a finite field of order $q$, with $q = 1 \pmod{p}$. We let $q_0$ denote the characteristic of $F$, so $q$ is a power of $q_0$. We put $r = v_p(q - 1)$, where $v_p$ denotes the $p$-adic valuation, so $r$ is the largest natural number such that $q = 1 \pmod{p^r}$. By assumption we have $r \geq 1$. We will repeatedly use the following result.

Lemma 2.1. For $j > 0$ we have $v_p(q^j - 1) = r + v_p(j)$.

Proof. In general, suppose that $u \in \mathbb{Z}$ with $v_p(u - 1) > 0$, say $u = 1 + p^tw$ with $t > 0$ and $w \neq 0 \pmod{p}$. We then have $u^m - 1 = \sum_{k=1}^{m} \binom{m}{k} p^k w^k$. If $m$ is not divisible by $p$ then the $k = 1$ term has valuation $t$ and the other terms have strictly higher valuation so $v_p(u^m - 1) = t = v_p(u - 1)$. Suppose instead that $m = p$. The coefficients $\binom{m}{k}$ are divisible by $p$ for $0 < k < p$, and it follows easily that the terms for $k \geq 2$ are divisible by $p^{k+2}$, whereas the term for $k = 1$ is only divisible by $p^{k+1}$, so $v_p(u^p - 1) = t + 1 = v_p(u - 1) + 1$. For general $j$ we can write $j = p^m v$ with $m \neq 0 \pmod{p}$ and use the two special cases above to see that $v_p(u^j - 1) = t + v$, as required.

Next, we let $\bar{F}$ denote an algebraic closure of $F$. We define $\phi: \bar{F} \to \bar{F}$ by $\phi(a) = a^q$, so $\bar{F} = \{a \mid \phi(a) = a\}$. It is standard that the Galois group $\Gamma = \text{Aut}_F(\bar{F})$ is topologically generated by $\phi$. More precisely, there is a homomorphism $\mathbb{Z} \to \Gamma$ given by $n \mapsto \phi^n$, and this has a canonical extension $\mathbb{Z} \to \Gamma$, and this extension is an isomorphism. We next need to introduce some finite subfields of $\bar{F}$. We put

$F[m] = \{a \in \bar{F} \mid \phi^m(a) = a\}$ = the unique subfield of $\bar{F}$ of degree $m$ over $F$

$F(k) = F[p^k]$  

$F(\infty) = \bigcup_k F(k)$. 

Note that \( F[m] \leq F[n] \) if and only if \( m \) divides \( n \), and \( \tilde{F} \) is the union of all the subfields \( F[m] \), or equivalently the union of the increasing sequence of subfields \( F[k] \).

Note also that \( F(\infty) \) is not the same as \( \tilde{F} \), but this is just an annoying technicality. Most invariants that we study will behave the same way for \( F(\infty) \) and \( \tilde{F} \).

We can define
\[
m_k = \prod \{p^k \mid r \text{ is prime and } r \leq p + k\},
\]
then \( F[m_k] \leq F[m_{k+1}] \) and \( \tilde{F} = \bigcup_k F[m_k] \) and \( F[m_k] \cap F(\infty) = F(k) \).

**Proposition 2.2.** The group \( GL_1(F[m]) = F[m]^\times \) is cyclic of order \( q^m - 1 \). One can choose an isomorphism \( i:GL_1(\bar{F}) \to \mu_q(\mathbb{C}) \), where
\[
\mu_q(\mathbb{C}) = \{ z \in \mathbb{C} \mid z^{m} = 1 \text{ for some } m \text{ with } (m,q) = 1 \}.
\]
Moreover, for any such \( i \) we have \( \mu_p^\infty(\mathbb{C}) \subseteq i(F(\infty)^\times) \), where
\[
\mu_p^\infty(\mathbb{C}) = \{ z \in \mathbb{C} \mid z^{p^k} = 1 \text{ for some } k \geq 0 \}.
\]

**Proof.** For the first statement, it is standard that the multiplicative group of any finite field is cyclic, of order one less than the order of the field itself.

Next, for any prime \( l \) we put
\[
\mu_l^\infty(\bar{F}) = \{ z \in \bar{F} \mid z^l = 1 \text{ for some } m \}.
\]
As \( \bar{F}^\times \) is an abelian torsion group, we see that it is the direct sum of its \( l \)-torsion parts, or in other words the groups \( \mu_l^\infty(\bar{F}) \). If \( l = q_0 \), then the map \( z \mapsto z^l \) is an automorphism so \( \mu_l^\infty(\bar{F}) = 1 \). Let \( l \) be a different prime. As \( \bar{F} \) is algebraically closed, we can choose an element \( u_l \) that is a primitive \( l \)-th root of unity. We can then recursively choose \( u_{k+1} \) with \( u_{k+1}^l = u_k \). The powers of \( u_k \) then give \( l^k \) distinct roots of \( x^l - 1 \), so these are all the roots. It follows that \( \mu_l^\infty(\bar{F}) \) is generated by all the elements \( u_k \), and that there is an isomorphism \( \mu_l^\infty(\bar{F}) \to \mu_l(\mathbb{C}) \) sending \( u_k \) to \( \exp(2\pi i/l^k) \). The claim follows easily from this.

Finally, Lemma 2.1 shows that the order \( |F(k)^\times| = (q \uparrow p^k) - 1 \) is divisible by \( p^{r+k} \), so \( F(k)^\times \) contains a cyclic subgroup of order \( p^{r+k} \), and this must be all the \( p^{r+k} \)-th roots of unity. It is clear from this that \( \mu_p^\infty(\bar{F}) \leq F(\infty)^\times \) and \( \mu_p^\infty(\mathbb{C}) \subseteq i(F(\infty)^\times) \).

**Definition 2.3.** We write \( \mathbb{Z}/p^\infty = \mathbb{Z}[1/p]/\mathbb{Z} \), and we silently identify this with \( \mu_p^\infty(\mathbb{C}) \) by \( x \mapsto e^{2\pi ix} \) where convenient. For any \( p \)-local abelian group \( A \) we put \( A^* = \text{Hom}(A,\mathbb{Z}/p^\infty) = \text{Hom}(A,\mu_p^\infty(\mathbb{C})) \) and \( A^# = \text{Hom}(A,\mu_p^\infty(\bar{F})) \).

**Remark 2.4.** It is standard that the evident ring map \( \mathbb{Z} \to \text{End}(\mathbb{Z}/p^\infty) \) extends canonically to a ring map \( \mathbb{Z}_p \to \text{End}(\mathbb{Z}/p^\infty) \), and that this extension is an isomorphism. We have seen that \( \mu_p^\infty(\bar{F}) \) is isomorphic to \( \mathbb{Z}/p^\infty \). It follows that the group \( T = \text{Hom}(\mathbb{Z}/p^\infty,\mu_p^\infty(\bar{F})) \) is an invertible \( \mathbb{Z}_p \)-module, whose inverse is \( \text{Hom}(\mu_p^\infty(\bar{F}),\mathbb{Z}/p^\infty) \). There are natural isomorphisms \( T \otimes_{\mathbb{Z}_p} A^* \to A^# \) and \( T^{-1} \otimes_{\mathbb{Z}_p} A^# \to A^* \). Thus, a choice of basis for \( T \) gives an isomorphism \( A^# \simeq A^* \) that is natural in \( A \).

**Proposition 2.5.** If \( t \leq r \), then the \( F \)-linear span of \( \mu_{p^t}(\bar{F}) \) is just \( F \). If \( t > r \), then the \( F \)-linear span of \( \mu_{p^r}(\bar{F}) \) is \( F(t-r) \).
Proof. We will write $E(t)$ for the $F$-linear span of $\mu_{r^d}(\bar{F})$. This is the image of a ring map $F[\mu_{r^d}(\bar{F})] \to \bar{F}$, so it is a finite subring of $\bar{F}$. Every element of $\bar{F}^\times$ has finite multiplicative order, so every subring is a subfield. If $t \leq r$ then $p^t$ divides $q - 1$ so $\mu_{r^d}(\bar{F}) \subseteq \bar{F}^\times$ and the claim is clear. Suppose instead that $t > r$. Lemma 2.1 tells us that $p^t$ divides $|F(t-r)^\times| = q^{t-r} - 1$, so $\mu_{r^d}(\bar{F}) \subseteq F(t-r)$, so $E(t) \subseteq F(t-r)$. Galois theory tells us that the fields between $F$ and $F(t-r)$ are precisely $\{F(j) \mid 0 \leq j \leq t-r\}$ and using Lemma 2.1 again we see that $\mu_{r^d}(\bar{F}) \subseteq F(t-r)$.

We write $W$ for the Witt ring functor, so $WF$ is a complete discrete valuation ring with $WF/q_0 = F$, and similarly for $WF(m)$ and $WF$. It is a standard fact that for each $a \in F$ there is a unique element $\bar{a} \in WF$ with $\bar{a}^q = \bar{a}$ and $\bar{a} = a \pmod{q_0}$. This is called the Teichmüller lift of $a$. The next two results are also standard but we include proofs for convenience.

**Proposition 2.6.** For any $d \geq 0$ the projections $BGL_d(WF) \to BGL_d(F)$ and $BGL_d(WF) \to BGL_d(\bar{F})$ are mod $p$ equivalences.

**Proof.** Let $\Gamma_m$ be the kernel of the map $GL_d(WF) \to GL_d(WF/q_0^m)$. For $m > 0$ it is easy to see that $\Gamma_m/\Gamma_{m+1}$ is an elementary abelian $q_0$-group, so its classifying space is $p$-adically contractible. An induction based on this shows that $B(\Gamma_m/\Gamma_{m+k})$ is again $p$-adically contractible, and by passing to the limit we see that $B\Gamma_1$ is $p$-adically contractible. The fibration $B\Gamma_1 \to BGL_d(WF) \to BGL_d(F)$ now shows that the second map is a $p$-adic equivalence. The same argument works for $\bar{F}$.

**Proposition 2.7.** There exists an injective ring homomorphism $i : WF \to \mathbb{C}$. Moreover, any such homomorphism restricts to give an isomorphism $\bar{F}^\times \to \mu_{q^d}(\mathbb{C})$ as in Proposition 2.2.

**Proof.** It is easy to see that the ring $K = \mathbb{Q} \otimes WF$ is a field. We can use Zorn’s lemma to find a maximal transcendental subset $X \subseteq K$, so $K$ is an algebraic over the rational function field $\mathbb{Q}(X)$. It follows by some cardinal arithmetic that $2^{\aleph_0} = |K| = |\mathbb{Q}(X)| = |X|$. Similarly, we can choose a maximal transcendental subset $Y \subseteq \mathbb{C}$, and we find that $\mathbb{C}$ is an algebraic closure of $\mathbb{Q}(Y)$, and $|Y| = 2^{\aleph_0} = |X|$. We can thus choose a bijection $X \to Y$, giving an isomorphism $\mathbb{Q}(X) \to \mathbb{Q}(Y)$, and extend by Galois theory to get an embedding $K \to \mathbb{C}$. We can restrict to $WF \subseteq K$ to get the required map $i$. We can then compose with the Teichmüller map $\bar{F}^\times \to WF^\times$ to get an embedding of $\bar{F}^\times$ in $\mu_{q^d}(\mathbb{C})$. For every number $m$ that is coprime to $q$, we find that $\bar{F}^\times$ and $\mu_{q^d}(\mathbb{C})$ both contain precisely $m$ elements of order dividing $m$. Using this, we see that the embedding $\bar{F}^\times \to \mu_{q^d}(\mathbb{C})$ is actually an isomorphism.

We will fix a map $i$ as above for the rest of this document. However, we will arrange our results and arguments in such a way as to minimise the dependence on this choice.

### 3. General linear groups and groupoids

**Definition 3.1.**

(a) We put $G_d = GL_d(F)$, and $\bar{G}_d = GL_d(\bar{F})$.

(b) These have diagonal subgroups $T_d = (F^\times)^d$ and $\bar{T}_d = (\bar{F}^\times)^d$. 
Proof. Note that for a matrix \( k \) lying in the \( \langle \rangle \) gives the claimed (and well-known) formula for \( d \alpha \).

Proposition 3.2. For all \( d \geq 0 \) we have \( |G_d| = \prod_{k=0}^{d-1} (q^d - q^k) \), and thus
\[
v_p|G_d| = dr + v_p(d!) = dr + (d - \alpha(d))/(p - 1),
\]
where \( \alpha(d) \) is the sum of the digits of \( d \) in base \( p \).

Proof. Note that for a matrix \( g \in G_d \), the \( k' \)th column can be any vector in \( F^d \) not lying in the \( (k - 1) \)-dimensional subspace spanned by the previous columns. This gives the claimed (and well-known) formula for \( |G_d| \). Recall that \( v_p(q) = 0 \) and \( v_p(q^2 - 1) = r + v_p(j) \). This gives
\[
\sum_{0 \leq i < d} v_p(q^{d-i} - 1) = \sum_{j=1}^{d} (r + v_p(j)) = dr + v_p(d!).
\]
The formula \( v_p(d!) = (d - \alpha(d))/(p - 1) \) is also well-known, and is easily checked by counting \( \{j \leq d \mid v_p(j) \geq m\} \) for all \( m \).

Proposition 3.3. The index of \( N_d \) in \( G_d \) is coprime to \( p \).

Proof. It is clear that \( |N_d| = d!(q - 1)^d \), so \( v_p|N_d| = dr + v_p(d!) = v_p|G_d| \) as required.

Proposition 3.4. Suppose that \( d \) can be written in base \( p \) as \( \sum d_i p^i \), with \( 0 \leq d_i < p \). Then the subgroup \( H = \prod G_{p^i}^{d_i} \leq G_d \) has index coprime to \( p \).

Proof. As \( \alpha(p^i) = 1 \), we have \( v_p|G_{p^i}| = p^i r + (p^i - 1)/(p - 1) \). This gives
\[
v_p|H| = \sum d_i v_p|G_{p^i}| = \sum d_i p^i r - \sum d_i p^i - \sum d_i \frac{p^i - 1}{p - 1} = dr - (d - \alpha(d))/(p - 1) = v_p|G|.
\]

Lemma 3.5. Let \( A \) be a finite abelian \( p \)-group, let \( K \) be a field of characteristic \( q_0 \neq p \), and let \( V \) be a finite-dimensional \( K \)-linear representation of \( A \).

(a) \( V \) can be decomposed as a direct sum of finitely many irreducible representations.
(b) If \( V \) is irreducible, then the ring \( R = \mathrm{End}_A(V) \) is a field, and \( V \) has dimension one over \( R \), and the natural map \( K[A] \to R \) is surjective.
(c) If \( V \) is irreducible and \( K \) is algebraically closed then \( V \) has dimension one over \( K \).
Proof.
(a) This is just Maschke’s Theorem, which is valid in this context because \( |A| \) is invertible in \( K \). In more detail, given a subrepresentation \( 0 < U < V \), we can choose a \( K \)-linear retraction \( \alpha_0: V \to U \), then put \( \alpha(v) = |A|^{-1} \sum_{a \in A} a^{-1} \alpha_0(av) \). This gives a \( K[A] \)-linear retraction, and thus a splitting \( V = U \oplus U' \). The claim follows by iterating this.

(b) Schur’s Lemma shows that \( R \) is a division ring. Let \( R_0 \) be the image of \( K[A] \) in \( R \). This is an integral domain of finite dimension over a field, so every multiplication operator is an injective endomorphism of a finite-dimensional vector space and so is an automorphism. It follows that \( R_0 \) is a field. The \( R_0 \)-linear subspaces of \( V \) are the same as the subrepresentations, so irreducibility means that \( \dim R_0(V) = 1 \). It follows in turn that \( R = \text{End}_{R_0}(V) = R_0 \).

(c) The field \( R \) is a finite extension of \( K \), and so must be equal to \( K \). The claim is clear from this.

\[ \square \]

Proposition 3.6. Any finite abelian \( p \)-subgroup \( A \leq \tilde{G}_d \) is conjugate to a subgroup of \( T_d \).

Proof. The inclusion \( A \to \tilde{G}_d \) allows us to regard \( \tilde{F}^d \) as a \( \tilde{F} \)-linear representation of \( A \). By Lemma 3.5, this is isomorphic to a direct sum of one-dimensional representations. Any such isomorphism is given by an element of \( \tilde{G}_d \) that conjugates \( A \) into \( \tilde{T}_d \).

\[ \square \]

Proposition 3.7. Let \( A \) be any abelian \( p \)-subgroup of \( G_d \), and let \( k \) be the largest integer such that \( p^k \leq d \). Then \( A \) has exponent dividing \( p^{k+r} \).

Proof. The inclusion \( A \to G_d \) allows us to regard \( F^d \) as an \( F \)-representation of \( A \), which we call \( V \). We can again split \( V \) as a direct sum of irreducible representations \( V_i \), although we can no longer guarantee that these are one-dimensional, because \( F \) is not algebraically closed. Put \( R_i = \text{End}_A(V_i) \) as in Lemma 3.5 so \( R_i \) is a finite field extension of \( F \), and \( \dim_{R_i}(V_i) = 1 \). Put \( e_i = \dim_F(R_i) = \dim_F(V_i) \leq d \), so \( v_p(e_i) \leq k \). This means that \( v_p|\tilde{R}_i^\times| = v_p(q^e - 1) = r + v_p(e) \leq r + k \) (using Lemma 2.1). As \( A \) acts on \( V \) via a homomorphism to \( \prod_i R_i^\times \), we conclude that the exponent of \( A \) divides \( p^{r+k} \).

\[ \square \]

Definition 3.8. We define a class function \( \xi_d: \tilde{G}_d \to \mathbb{C} \) as follows. Given \( g \in \tilde{G}_d \), we let \( \lambda_1, \ldots, \lambda_d \in F^\times \) denote the eigenvalues, repeated with the appropriate multiplicity. We then use our chosen isomorphism \( i: F^\times \to \mu_p(\mathbb{C}) \) (from Proposition 2.2 or 2.7) and put \( \xi_d(g) = \sum_k i(\lambda_k) \). We call this the universal Brauer character.

Theorem 3.9. For any \( m \geq 0 \), the restriction of \( \xi_d \) to \( GL_d(F(m)) \) is the character of a virtual complex representation.

Proof. This was proved by Green [2]. Work of Lusztig [5] gives a more explicit and constructive proof.

\[ \square \]

Remark 3.10. By combining Theorem 3.9 with the theorem of Tanabe, we see that \( E^0(BG_{L_d}(F)) \) is generated by Chern classes of ordinary complex representations. The Galois automorphism \( \phi^* \) is essentially the same as the Adams operation \( \psi^q \), which is also determined by complex representation theory. In [12] we defined a
“Chern approximation” map $C(G, E) \to E^0(BG)$ for any finite group $G$. The above observations imply that this map is an isomorphism in the case $G = GL_d(F)$.

Rather than working directly with the above groups, we will find it convenient to package them as groupoids.

**Definition 3.11.** Given a groupoid $\mathcal{G}$ and an object $a \in \mathcal{G}$, we write $\mathcal{G}(a)$ for the automorphism group $\mathcal{G}(a, a)$. We also write $\pi_0(\mathcal{G})$ for the set of isomorphism classes in $\mathcal{G}$. We say that a functor $\phi: \mathcal{G} \to \mathcal{H}$ is a $\pi_0$-isomorphism if the induced map $\pi_0(\mathcal{G}) \to \pi_0(\mathcal{H})$ is a bijection.

**Definition 3.12.** Let $\mathcal{G}$ be a groupoid. We say that $\mathcal{G}$ is hom-finite if $\mathcal{G}(a, a)$ is finite for all $a, a' \in \mathcal{G}$ (or equivalently, $\mathcal{G}(a)$ is finite for all $a$). We say that $\mathcal{G}$ is finite if, in addition, there are only finitely many isomorphism classes.

Now let $\phi: \mathcal{G} \to \mathcal{H}$ be a functor between hom-finite groupoids. We say that $\phi$ is finite if for each $b \in \mathcal{H}$, the full subcategory $\{a \in \mathcal{G} \mid \phi(a) \simeq b\} \subseteq \mathcal{G}$ is finite.

**Definition 3.13.** We regard $\mathbb{N}$ as a groupoid with only identity morphisms. We use addition and multiplication to make this into a symmetric bimonoidal groupoid. A graded groupoid means a groupoid $\mathcal{G}$ equipped with a functor $\deg: \mathcal{G} \to \mathbb{N}$, so $\mathcal{G}$ splits as a disjoint union of groupoids $\mathcal{G}_d = \deg^{-1}(d)$. We say that $\mathcal{G}$ is of finite type if each groupoid $\mathcal{G}_d$ is finite, or equivalently, $\deg$ is a finite morphism.

**Definition 3.14.**

(a) We write $\mathcal{L}$ for the category of one-dimensional vector spaces over $F$, and $\overline{\mathcal{L}}$ for the category of one-dimensional vector spaces over $\overline{F}$. The construction $L \mapsto \overline{F} \otimes_F L$ gives a faithful functor $\mathcal{L} \to \overline{\mathcal{L}}$. Note that $\mathcal{L}$ is equivalent to $GL_1(F)$, considered as a one-object groupoid, and similarly for $\overline{\mathcal{L}}$; the functor $\mathcal{L} \to \overline{\mathcal{L}}$ corresponds to the inclusion $GL_1(F) \to GL_1(\overline{F})$. We give $\mathcal{L}$ and $\overline{\mathcal{L}}$ the grading given by the constant functor $\deg: \mathcal{L} \mapsto 1$ (so $\mathcal{L}$ is of finite type but $\overline{\mathcal{L}}$ is not).

(b) As mentioned previously, we define $\mathcal{V}$ to be the groupoid of finite-dimensional vector spaces over $F$, and we define $\overline{\mathcal{V}}$ to be the corresponding groupoid for $\overline{F}$. The map $V \mapsto \dim_F(V)$ gives a grading on $\mathcal{V}$, and similarly for $\overline{\mathcal{V}}$. Both of these groupoids are symmetric bimonoidal under $\oplus$ and $\otimes$, and the grading is compatible with this.

(c) We write $\mathcal{X}$ for the category of finite sets and bijections. This is symmetric bimonoidal under disjoint union and cartesian product. It is equivalent to the disjoint union of the groups $\Sigma_d$, considered as one-object groupoids. There is an evident symmetric bimonoidal functor $X \mapsto F[X]$ from $\mathcal{X}$ to $\mathcal{V}$. The map $X \mapsto |X|$ gives a grading, compatible with the symmetric bimonoidal structure.

(d) We write $\mathcal{XL}$ for the category of pairs $(X, L)$, where $X$ is a finite set and $L$ is an $F$-linear line bundle over $X$. This is equivalent to the disjoint union of the groups $N_d = \Sigma_d GL_1(F)$, considered as one-object groupoids. Given $(X_i, L_i) \in \mathcal{XL}$ for $i = 0, 1$, we can patch together $L_0$ and $L_1$ to get a line bundle $L$ over $X_0 \amalg X_1$, and we define $(X_0, L_0) \amalg (X_1, L_1) = (X_0 \amalg X_1, L)$. Similarly, we have a bundle $L'$ over $X_0 \times X_1$ with fibre $(L_0)_{x_0} \otimes_F (L_1)_{x_1}$ at $(x_0, x_1)$, and we define $(X_0, L_0) \times (X_1, L_1) = (X_0 \times X_1, L')$. This makes $\mathcal{XL}$ into a symmetric bimonoidal category. There is a symmetric bimonoidal functor $\gamma: \mathcal{XL} \to \mathcal{V}$ given by $\gamma(X, L) = \bigoplus_{x \in X} L_x$. The map $(X, L) \mapsto |X|$ gives a grading, compatible with the symmetric bimonoidal structure.
(e) We define $\mathcal{X}$ and $\gamma: \mathcal{X} \to \mathcal{V}$ in the evident analogous way.

We next want to sharpen Proposition 3.7 by identifying the subgroups of maximal exponent in $G_{p^k}$ or in the subgroup $N_{p^k}$. It is convenient to formulate the proof using groupoids.

**Definition 3.15.** For $k \geq 0$, we define $\mathcal{F}_k$ to be the category of field extensions of $F$ of degree $p^k$. (It is standard that such extensions exist and are all isomorphic, so $\mathcal{F}_k$ is a connected groupoid.) We also define $\mathcal{C}_k$ to be the category of triples $(C, X, L)$ where $(X, L) \in \mathcal{X}L$ and $|X| = p^k$ and $C$ is a cyclic subgroup of order $p^{k+r}$ in $\text{Aut}(X, L)$.

**Proposition 3.16.** There are functors $\phi: \mathcal{F}_k \to \mathcal{C}_k$ and $\psi: \mathcal{C}_k \to \mathcal{F}_k$ which can be described as follows. We have $\phi(P) = (C, X, L)$ where

$$X = \{ u \in P \mid u^{p^k} = 1 \}$$

$$C = \{ v \in P \mid v^{p^{k+r}} = 1 \}$$

$$L_u = \text{span}_F \{ v \in C \mid v^{p^r} = u \}.$$ 

In the other direction, $\psi(C, X, L)$ is the image of $F[C]$ in the endomorphism ring of the space $V = \gamma(X, L) = \bigoplus_{\zeta} L_{\zeta}$. Moreover, $\psi(\phi)(P)$ is naturally isomorphic to $P$ for all $P$, and $\psi(\psi)(C, X, L)$ is unnaturally isomorphic to $(C, X, L)$.

**Proof.** Suppose that $P \in \mathcal{F}_k$, so $|P| = q \uparrow p^k$, so $P^\times$ is a cyclic group of order $(q \uparrow p^k) - 1$, so $v_{p^r} | P^\times | = r + v_{p^k}(p^k) = k + r$ by Lemma 2.1. Thus, in $\phi(P) = (C, X, L)$, we see that $C$ is cyclic of order $p^{k+r}$, and is the unique such subgroup of $P^\times$. It also follows that $|X| = p^k$. If $u \in X$ then the cyclic property of $C$ means that we can choose $v \in C$ with $v^{p^r} = u$. If $w$ is another element of $C$ with $w^{p^r} = u$ then $(w/v)^{p^r} = 1$ so $(w/v)^{p-1} = 1$ so $w/v \in F$. This shows that $L_u = F.v$, which has dimension one over $F$ as expected, so $(X, L) \in \mathcal{X}L_{p^k}$. It is also clear that $C$ acts faithfully on $(X, L)$ by the rule $c. u = v^c u$ and $c. v = c v$, so $(C, X, L) \in \mathcal{C}_k$. This validates the definition of $\phi$.

In the opposite direction, suppose we start with $(C, X, L) \in \mathcal{C}_k$. The group $C$ then acts on the space $V = \gamma(X, L) = \bigoplus_{\zeta} L_{\zeta}$, and we let $P$ denote the image of $F[C]$ in $\text{End}(V)$. We claim that this lies in $\mathcal{F}_k$. To see this, we first note that $C$ is a subgroup of $\text{Aut}(X, L)$, so the map $C \to P^\times$ is injective, so the $p$-exponent of $P^\times$ is at least $k + r$. On the other hand, $P$ is a finite-dimensional commutative $F$-algebra, so it splits as a product of factors $P_{i}$ such that $P_{i}/\sqrt{0}$ is a field extension of $F$, of degree $d_i$ say. Corresponding to this we get a splitting $V = \bigoplus_{i} V_{i}$, where $V_{i}$ is a nontrivial module for $P_{i}$, and so has dimension at least $d_{i}$ over $F$. Next, because $|F|$ is coprime to $p$, we see that the $p$-exponent of the unit group of $P_{i}$ is the same as for $P_{i}/\sqrt{0}$, which is $r + v_{p}(d_{i})$. It follows that the $p$-exponent of $P^\times$ is $r + \max_{i} v_{p}(d_{i})$, so for some $i$ we must have $v_{p}(d_{i}) \geq k$ and so $d_{i} \geq p^{k}$. This for $i$ we have $\dim_{F}(V_{i}) \geq d_{i} \geq p^{k} = \dim_{F}(V)$. It follows that all other $V_{j}$ and $P_{j}$ are zero, and also that $\sqrt{0}$ must be trivial in $P_{i}$. This means that $P = P_{i} \in \mathcal{F}_k$, as claimed, validating the definition of $\psi$.

If $(C, X, L) = \phi(P)$ then each $L_u$ is a subspace of $P$ and it is easy to check that $P = \bigoplus_{u} L_u$, with $C$ acting by multiplication. From this it is clear that $\psi(\phi)(P)$ is naturally isomorphic to $P$.

In the opposite direction, suppose we start with $(C, X, L) \in \mathcal{C}_k$. Pick a basepoint $x_0 \in X$, and a nonzero element $m_0 \in L_{x_0}$. Let $D \leq C$ be the stabiliser of $x_0$. As
\(C\) injects in \(\text{Aut}(X, L)\) we see that \(D\) injects in \(\text{Aut}(L_{x_0}) = F\), so |\(D\)| \(\leq p^r\). On the other hand, the set \(\overline{C/D}\) must inject in \(X\) so |\(\overline{C/D}\)| \(\leq p^k\). As |\(C\)| = \(p^{k+r}\) we must actually have |\(D\)| = \(p^r\) and |\(\overline{C/D}\)| = \(p^k\), so \(C\) acts transitively on \(X\). Now put \(X' = \{u \in C \mid u^p = 1\}\) and \(L'_u = \text{span}_F\{v \in C \mid v^{p^r} = u\}\). It is then not hard to check that there is a unique \(C\)-equivariant isomorphism \((X', L') \rightarrow (X, L)\) sending \(1 \in X'\) to \(x_0 \in X\) and \(1 \in L'_1\) to \(m_0 \in L_{x_0}\).

**Corollary 3.17.** The groupoid \(C_k\) is connected. Moreover, if \(H\) is a subgroup of \(G_{p^k}\) of index coprime to \(p\), then there is at least one \(H\)-conjugacy class of cyclic subgroups of order \(p^{k+r}\) in \(H\). If \(H\) contains the monomial subgroup \(N = \Sigma_p \wr F^\times\), then there is precisely one conjugacy class.

**Proof.** The first claim follows easily from the proposition. For the second, we note that \(\mathcal{A}_L p^k\) is equivalent to the groupoid consisting of a single object with automorphism group \(N = \Sigma_p \wr F^\times\). Now let \(C'_k(H)\) be the set of cyclic subgroups of order \(p^{k+r}\) in \(H\). We let \(N\) act on \(C'_k(N)\) by conjugation, and this gives us an action groupoid with object set \(C'_k(N)\) and morphism set \(N \times C'_k(N)\). It is easy to see that this is equivalent to \(C_k\), and so is connected. This means that \(C'_k(N)/N\) is a singleton. We can choose \(C_0' \in C'_k(N)\), and then Sylow theory tells us that there is a Sylow \(p\)-subgroup \(P_0\) of \(N\) with \(C_0' \leq P_0\).

Now consider a subgroup \(H \leq G_{p^k}\) of index coprime to \(p\). Let \(P\) be a Sylow \(p\)-subgroup of \(H\). Then \(P\) is also a Sylow \(p\)-subgroup of \(G\), so \(C_0\) is \(G\)-conjugate to a subgroup of \(P\), so \(C'_k(H) \neq \emptyset\).

Now consider a group \(H\) with \(N \leq H \leq G_{p^k}\), and an element \(C \in C'_k(H)\). We can again choose a Sylow subgroup \(P\) of \(H\) with \(C \leq P\). Because \(|G_{p^k}|/|N|\) is coprime to \(p\), we see that \(P_0\) is also a Sylow subgroup of \(P\), so \(P\) is \(H\)-conjugate to \(P_0\). This means that \(C\) is \(H\)-conjugate to an element of \(C'_k(N)\), which in turn is \(N\)-conjugate (and therefore \(H\)-conjugate) to \(C_0\). This proves that \(C'_k(H)/H\) is again a singleton, as required.

**Definition 3.18.** The notation \(H^*(X)\) will refer to the cohomology of \(X\) with coefficients \(\mathbb{F}_p\), unless otherwise specified. Similarly, \(H_*(X)\) will refer to the homology with coefficients \(\mathbb{F}_p\).

**Definition 3.19.** Let \(G\) be a group (which may be infinite). Given a finite abelian \(p\)-subgroup \(A \leq G\), we have a natural map \(\alpha_A : H_*(BA) \rightarrow H_*(BG)\). By taking the sum over all possible \(A\), we obtain a map \(\alpha : \bigoplus A H_*(BA) \rightarrow H_*(BG)\). We say that \(G\) is \(\alpha\)-determined if this map is surjective. (Equivalently, the dual map \(H^*(BG) \rightarrow \prod_A H^*(BA)\) should be injective.) We also say that a groupoid is \(\alpha\)-determined iff every automorphism group \(G(a)\) is \(\alpha\)-determined.

**Proposition 3.20.** (a) If \(G\) is a product of two \(\alpha\)-determined subgroups, then it is \(\alpha\)-determined.
(b) If \(G\) is a directed union of \(\alpha\)-determined subgroups, then it is \(\alpha\)-determined.
(c) If \(G\) is an abelian torsion group, then it is \(\alpha\)-determined.
(d) If \(G\) has an \(\alpha\)-determined subgroup \(H\) such that \(|G/H|\) is finite and coprime to \(p\), then \(G\) is \(\alpha\)-determined.
(e) If \(G = \Sigma_n \wr H\) and \(H\) is \(\alpha\)-determined, then so is \(G\).

**Proof.** (a) Suppose that \(G = G_1 \times G_2\), where each factor \(G_i\) is \(\alpha\)-determined. Put \(M_i = \bigoplus A_i H_*(BA_i)\), where \(A_i\) runs over the finite abelian \(p\)-subgroups...
of $G_i$. By hypothesis the maps $\alpha_i: M_i \to H_\ast(BG_i)$ are surjective, so the same is true of the map $\alpha_1 \otimes \alpha_2: M_1 \otimes M_2 \to H_\ast(BG)$. Here $M_1 \otimes M_2$ is the direct sum of the groups $H_\ast(BA)$ as $A$ runs over those finite abelian $p$-subgroups of $G$ that have the form $A_1 \times A_2$ for some $A_i \leq G_i$. From this it is clear that $G$ is $\alpha$-determined.

(b) If $G$ is a directed union of subgroups $G_i$ then it is standard that $H_\ast(BG)$ is the directed union of the subgroups $H_\ast(BG_i)$. If each $G_i$ is $\alpha$-determined, it follows easily that the same is true of $G$ itself.

(c) Using (b) we can reduce to the case where $G$ itself is a finite abelian group. We can then split $G$ as $G_1 \times G_2$, where $G_1$ is a $p$-group and $G_2$ has order coprime to $p$. We then find that $H_\ast(BG_2)$ is just a single copy of $\mathbb{Z}/p$, so $H_\ast(BG) = H_\ast(BG_1)$, which proves the claim.

(d) If $H$ has index $d \neq 0 \pmod{p}$ in $G$, then the induced map $H_\ast(BH) \to H_\ast(BG)$ is surjective by a transfer argument. It follows that if $H$ is $\alpha$-determined, then so is $G$.

(e) This is the Corollary to [11, Lemma 9.4]. In outline, one can use (d) to reduce to the case $n = p$, and that case can be analysed using a standard spectral sequence

$$H^\ast(\Sigma_p; H^\ast(BH)^{\otimes p}) \Longrightarrow H^\ast(B(\Sigma_p \wr H)).$$

\[\square\]

**Proposition 3.21.** The groups $G_d, \tilde{G}_d, T_d, \tilde{T}_d, \Sigma_d, N_d$ and $\tilde{N}_d$ are all $\alpha$-determined. Equivalently, the groupoids $\mathcal{V}, \overline{\mathcal{V}}, \mathcal{L}^d, \overline{\mathcal{L}}^d, \mathcal{X}$, $\mathcal{X}\mathcal{L}$ and $\mathcal{X}\overline{\mathcal{L}}$ are all $\alpha$-determined.

**Proof.** The groups $T_d$ and $\tilde{T}_d$ are abelian torsion groups, so they are $\alpha$-determined. The groups $\Sigma_d$, $N_d$ and $\tilde{N}_d$ are therefore $\alpha$-determined by Proposition 3.20(e). It therefore follows by Propositions 3.20 and 3.21(d) that the group $G_d = GL_d(F)$ is $\alpha$-determined. Here $F$ is an arbitrary finite field of characteristic not equal to $p$, so the same logic shows that $GL_d(F[k])$ is $\alpha$-determined for all $k$. It therefore follows by Proposition 3.20(b) that $\tilde{G}_d$ is $\alpha$-determined. \[\square\]

We next start to formulate a result that will be useful for comparing $\mathcal{V}$ with $\mathcal{V}(k)$.

**Definition 3.22.** Consider an object $V \in \overline{\mathcal{V}}$ and an element $\gamma \in \Gamma = \text{Gal}(\overline{F}/F)$. We define $\gamma(V)$ to consist of symbols $\gamma(v)$ for all $v \in V$, subject to the rules

$$\gamma(v + v') = \gamma(v) + \gamma(v') \quad \text{and} \quad a \gamma(v) = \gamma(\gamma^{-1}(a)v) \quad (\text{for } v, v' \in V \text{ and } a \in F).$$

This gives a functor $\gamma: \overline{\mathcal{V}} \to \overline{\mathcal{V}}$.

**Remark 3.23.** Suppose that $V = \tilde{F} \otimes_{F(k)} V_0$ for some $V_0 \in \mathcal{V}(k)$, and that $\gamma \in \langle \phi^p \rangle = \text{Gal}(\overline{F}/F(k))$. One can then check that there is an isomorphism $\gamma(V) \to V$ given by $\gamma(a \otimes_{F(k)} v_0) = \gamma(a) \otimes_{F(k)} v_0$.

**Lemma 3.24.** For $V \in \mathcal{V}(k)$, there is a natural isomorphism

$$\xi: \tilde{F} \otimes_{F} V \to \prod_{i=0}^{\phi^i-1} \phi^i(\tilde{F} \otimes_{F(k)} V)$$

given by

$$\xi(a \otimes_{F} v)_i = a \phi^i(1 \otimes_{F(k)} v) = \phi^i(\phi^{-i}(a) \otimes_{F(k)} v).$$
Proof. It is straightforward to check that the above formula gives a well-defined natural map; we just need to check that it is an isomorphism. As all constructions respect direct sums, it will be enough to take $V = F(k) < \bar{F}$. In that case we can define an isomorphism $\phi^i(\bar{F} \otimes F(k) V) \rightarrow \bar{F}$ by $\phi^i(a \otimes v) \mapsto \phi^i(av)$. Using this identification, we get a map

$$\xi: \bar{F} \otimes F(k) \rightarrow \prod_{i=0}^{p^k-1} \bar{F}$$

given by $\xi(a \otimes v)_i = a \phi^i(v)$. It is a well-known fact of Galois theory that this is an isomorphism. (It is an $\bar{F}$-linear map between vector spaces of the same finite dimension, and it is injective by Dedekind’s lemma on independence of automorphisms.)

4. Ordinary (co)homology

Notation 4.1. We remind the reader that all (co)homology groups have coefficients $F_p$ unless otherwise specified. We write $P[\cdots]$ and $E[\cdots]$ for polynomials algebras and exterior algebras over $F_p$.

Definition 4.2. We have chosen an injection $i: \bar{G}_1 = GL_1(\bar{F}) \rightarrow S^1$, which gives a map $(Bi)^*: H^*(CP^\infty) \rightarrow H^*(B\bar{G}_1)$. There is a standard generator $x \in H^2(\mathbb{C}P^\infty)$, and we also write $x$ for the image of this class in $H^2(B\bar{G}_1) = H^2(B\bar{E})$ or in $H^2(BGL_1(F[m]))$. After recalling that $G_1$ is the union of finite cyclic groups $GL_1(F[k])$, standard calculations give $H^*(B\bar{G}_1) = P[x]$. We write $b_k$ for the class in $H_{2k}(B\bar{G}_1)$ that is dual to $x^k$.

Remark 4.3. Later we will define similar classes in Morava $K$-theory and Morava $E$-theory. When it is necessary to distinguish between these, we will use notation such as $x_H$, $x_E$, $x_K$, $b_Hk$, $b_ek$ and $b_{k,k}$. Similar remarks will apply to the other classes in ordinary (co)homology defined later in this section.

Definition 4.4. Now consider $H^1(B\bar{G}_1) = H^1(B\bar{E})$. Let $m = q - 1$ be the order of $G_1$, so $m$ is divisible by $p$. There is a unique generator $c \in G_1$ with $i(c) = e^{2\pi i/m}$, and a unique homomorphism $G_1 \rightarrow \mathbb{F}_p$ with $c \mapsto 1$. There is a standard isomorphism $H^1(B\bar{G}_1) = \text{Hom}(G_1, \mathbb{F}_p)$, so we obtain a class $a \in H^1(B\bar{G}_1)$. As $G_1$ is cyclic, it is a standard fact that $H^*(B\bar{G}_1) = P[x] \otimes E[a]$. We write $e_k$ for the class in $H_{2k+1}(B\bar{G}_1)$ that is dual to $x^ka$.

Definition 4.5. Now note that the Künneth theorem gives

$$H^*(BT^d) = P[x_1, \ldots, x_d]$$
$$H^*(BT^d) = P[x_1, \ldots, x_d] \otimes E[a_1, \ldots, a_d].$$

These rings have an evident action of the symmetric group $\Sigma_d$. We let $c_k$ be the sum of all monomials in the $\Sigma_d$-orbit of $(-1)^k x_1 \cdots x_k$, so $|c_k| = 2k$ and

$$\sum_i c_i t^{d-1} = \prod_j (t - x_j).$$

We also let $v_k$ be the sum of all monomials in the $\Sigma_d$-orbit of $(-1)^k a_1x_2 \cdots x_k$.

Remark 4.6. Note that in the case $d = 1$ we have

$$H^*(BT^1) = P[x_1] \otimes E[a_1] = P[c_1] \otimes E[v_1]$$
with \( c_1 = -x_1 \) and \( v_1 = -a_1 \).

We now recall Quillen’s calculation \([8]\) of the cohomology of \( BG_d \) and \( B\bar{G}_d \). For ease of comparison with our calculation in Morava theory, we will also recall part of the proof.

**Proposition 4.7 (Quillen).** The inclusions \( T_d \to G_d \) and \( \bar{T}_d \to \bar{G}_d \) give isomorphisms

\[
H^*(B\bar{G}_d) = H^*(B\bar{T}_d)^{\Sigma_d} = P[c_1, \ldots, c_d] \\
H^*(BG_d) = H^*(BT_d)^{\Sigma_d} = P[c_1, \ldots, c_d] \otimes E[v_1, \ldots, v_k].
\]

**Proof.** Proposition \([8,21]\) shows that \( \bar{G}_d \) is \( \alpha \)-determined, so the cohomology of \( B\bar{G}_d \) is detected on abelian \( p \)-subgroups of \( \bar{G}_d \), but Proposition \([3,6]\) tells us that every such subgroup is subconjugate to \( T_d \), so we see that the restriction \( H^*(B\bar{G}_d) \to H^*(B\bar{T}_d) \) is injective. The permutation action on \( \bar{T}_d \) comes from inner automorphisms of \( \bar{G}_d \), and inner automorphisms act as the identity in cohomology, so the restriction map lands in \( H^*(B\bar{T}_d)^{\Sigma_d} \), which is just \( P[c_1, \ldots, c_d] \) by the classical theorem of Newton. The same argument shows that \( H^*(BG_d) \) maps to \( H^*(BT_d)^{\Sigma_d} \). The structure of this ring of invariants is less well-known, but it can be proved in the same way as Newton’s theorem, using a lexicographic ordering of monomials.

Next, the maps \( \bar{F} \to W\bar{F} \to \C \) give a diagram

\[
\begin{array}{ccc}
\bar{T}_d = GL_1(\bar{F})^d & \leftarrow & \bar{T}W_d = GL_1(W\bar{F})^d \to GL_1(\C)^d \\
\downarrow & & \downarrow \\
\bar{G}_d = GL_d(\bar{F}) & \leftarrow & \bar{G}W_d = GL_d(W\bar{F}) \to GL_d(\C).
\end{array}
\]

We now pass to cohomology rings, taking account of Proposition \([2,0]\) and the discussion associated with Definition \([15]\). This gives a diagram as follows:

\[
\begin{array}{ccc}
P[x_1, \ldots, x_d]^{\Sigma_d} & \leftarrow & P[x_1, \ldots, x_d]^{\Sigma_d} \to P[x_1, \ldots, x_d]^{\Sigma_d} \\
\downarrow \simeq & & \uparrow \simeq \\
H^*(B\bar{G}_d) & \leftarrow & H^*(B\bar{G}W_d) \to P[c_1, \ldots, c_d]
\end{array}
\]

It is well-known that the last vertical map is an isomorphism. By chasing the diagram, we see that the same is true of the other two vertical maps. This proves our claim for \( \bar{G}_d \). We use the notation \( c_k \) for the unique element of \( H^{2k}(BG_d) \) that maps to \( c_k \) in \( H^{2k}(BT_d) \), and we also use the same notation for the restriction of this class to \( BG_d \). For the rest of the proof for \( BG_d \), we refer to \([8\text{ Theorem 3}]\). □

**Definition 4.8.** Let \( P \) be a group, and let \( W \) be an \( \bar{F}[P] \)-module of dimension \( d < \infty \) over \( \bar{F} \). There is then a corresponding functor \( P \to \bar{V}_d \), and thus a map \( H^*(B\bar{V}_d) \to H^*(BP) \). We define \( c_k(W) \in H^{2k}(BP) \) to be the image of \( c_k \in H^{2k}(B\bar{V}_d) \) under this map. We also put \( f_W(t) = \sum f_k(W)t^d \). We call the elements \( c_k(W) \) the Chern classes of \( W \), and we call \( f_W(t) \) the Chern polynomial.

**Remark 4.9.** From the definitions one can check that \( c_0(V) = 1 \) and \( c_k(V \oplus W) = \sum_{i+j=k} c_i(V)c_j(W) \). Equivalently, \( f_W(t) \) is always a monic polynomial, and \( f_{V \oplus W}(t) = f_V(t)f_W(t) \).
Now note that we can use the direct sum operation on $V$, or equivalently the standard inclusions $G_i \times G_j \to G_{i+j}$, to give a ring structure on the object $H_*(BV) = H_*(BG_d)$. This is naturally bigraded, with $H_*(BG_d)$ in bidegree $(i,d)$. Because $H^*(BG_d) = ((H^*(BG_1))^\otimes d)_\Sigma_d$, we have $H_*(BG_d) = ((H_*(BG_1))^\otimes d)_\Sigma_d$. We can also do the same thing with $\overline{G}_d$. This gives the following:

**Proposition 4.10.** $H_*(B\overline{V})$ is the symmetric algebra generated by $H_*(B\overline{\Sigma})$, or equivalently

$$H_*(B\overline{V}) = P[b_k \mid k \geq 0] \quad (|b_k| = (2k,1)).$$

Similarly, $H_*(B\overline{V})$ is the free graded-commutative algebra generated by $H_*(BL)$, or equivalently

$$H_*(B\overline{V}) = P[b_k \mid k \geq 0] \otimes E[e_k \mid k \geq 0] \quad (|b_k| = (2k,1), \ |e_k| = (2k+1,1)).$$

**Remark 4.11.** Above we have given various results about $BGL_d(F)$, but $F$ is an arbitrary finite field with $|F| = 1 \pmod{p}$, so the results apply equally well to $BGL_d(F(k))$. This gives generators for $H^*(BGL_d(F(k)))$, which we could call $c_j^{(k)}$ and $v_j^{(k)}$.

There are two different ways to relate the groups $GL_d(F(k))$ as $k$ varies. Usually we will just consider the evident inclusion $GL_d(F(k-1)) \to GL_d(F(k))$, corresponding to the functor $\mathcal{V}(k-1) \to \mathcal{V}(k)$ given by $V \mapsto F(k) \otimes_{F(k-1)} V$. However, we will sometimes also consider the forgetful functor $\mathcal{V}(k)_d \to \mathcal{V}(k-1)_p$, and the corresponding inclusion $GL_d(F(k)) \to GL_{pd}(F(k-1))$. (The latter relies on a choice of basis of $F(k)$ over $F(k-1)$, but only up to an inner automorphism, which acts as the identity up to homotopy on $GL_{pd}(F(k-1))$.)

The restriction maps

$$H^*(BGL_d(F)) \to H^*(BGL_d(F(k))) \to H^*(BGL_d(F(k-1)))$$

have $c_j \mapsto c_j^{(k)} \mapsto c_j^{(k-1)}$, so we will usually just write $c_j$ for these elements. However, there are no elements $v_j$ in $H^*(BGL_d(F))$, and one can check that the restriction map $H^*(BGL_d(F(k))) \to H^*(BGL_d(F(k-1)))$ sends $v_j^{(k)}$ to zero, so we need to be more careful about the notation for these classes.

Dually, we just write $b_j$ for the polynomial generators of $H_*(BGL_d(F(d)))$, but we write $e_j^{(k)}$ for the exterior generators.

It will turn out that for our analysis of the Atiyah-Hirzebruch spectral sequence, we need to understand the effect in (co)homology of the inclusions $GL_d(F(k)) \to GL_{pd}(F)$. We can reduce by induction to the case where $d = k = 1$. For this case, we will temporarily write $G$ for $GL_p(F)$ and $C$ for $GL_1(F(1))$ (so $C$ is cyclic of order $q^p - 1$, and the $p$-torsion subgroup is cyclic of order $p^{r+1}$). We also let $\rho: C \to G$ be the inclusion. We have seen that

$$H^*(BG) = H^*(BGL_p(F)) = P[c_1, \ldots, c_p] \otimes E[v_1, \ldots, v_p]$$

$$H^*(BC) = H^*(BGL_1(F(1))) = P[c_1^{(1)}] \otimes E[v_1^{(1)}].$$

For typographical convenience, we put $\overline{c} = c_1^{(1)}$ and $\overline{v} = v_1^{(1)}$. Note that there are also natural generators $\overline{c} = -\overline{c}$ and $\overline{v} = -\overline{v}$.

It is not hard to understand the effect of $\rho$ on the polynomial generators:

**Lemma 4.12.** We have $\rho^*(c_p) = \overline{c}^p = (-\overline{c})^p$, and $\rho^*(c_i) = 0$ for $0 < i < p$. 

We can then define \( \overline{\psi} \) by multiplication. The inclusions \( \overline{\psi}(F) \to \overline{\psi}(\overline{F}) \) give a \( p \)-dimensional \( \overline{\psi} \)-linear representation of \( F \), which we call \( \overline{\psi} \). Essentially by construction, we have \( \overline{\psi} = \overline{\psi}(F) \oplus \overline{\psi}(\overline{F}) \). This means that the polynomial \( f(t) = \sum_{i=0}^{p} \phi^*(c_i)t^{p-i} \) is the Chern polynomial for \( \overline{\psi} \). Also, the inclusion \( F(1) \to \overline{\psi} \) gives a one-dimensional \( \overline{\psi} \)-linear representation of \( F \), which we call \( \lambda \). The Euler class of \( \lambda \) is \( \overline{\psi} \). Our claim is that \( f(t) = t^p - \overline{\psi} \).

Let \( \phi \) be the Frobenius automorphism \( x \mapsto x^p \) on \( F(1) \), so \( \text{Gal}(F(1)/F) = \{ \phi^i \mid 0 \leq i < p \} \). Using Lemma 3.24 we see that \( \overline{\psi} \cong \bigoplus_{j=0}^{p} \langle \phi^j \rangle^*(L) \), and thus that \( f(t) = \prod_{j=0}^{p} (t - q^j \overline{\psi}) \). As \( q = 1 \pmod{p} \) this gives \( f(t) = (t - \overline{\psi})^p = t^p - \overline{\psi} \) as claimed.

To compute \( \rho^*(v_i) \), we will need to describe \( v_i \) as a transfer, and we will need some general results about transfers.

**Lemma 4.13.** Let \( A \) be a finite abelian group, and let \( B \) be a subgroup of index \( d \). We can then define \( t \colon A \to B \) by \( t(a) = da \), and this induces \( t^* : \text{Hom}(B, \mathbb{F}_p) \to \text{Hom}(A, \mathbb{F}_p) \), and \( t^* \) is the same as \( tr_B^A : H^1(BB) \to H^1(BA) \).

**Proof.** Classical concrete formulae for the transfer in homology are given in [11, Chapter 2], for example. The claim follows by specialising to the abelian case, and then dualising.

**Corollary 4.14.** The transfer is zero if \( dA \leq pB \). In particular, this holds if

(a) \( B \) is a summand in \( A \), and \( p \) divides \( d \); or

(b) The \( p \)-torsion part of \( A/B \) is not cyclic.

**Proof.** The first claim is clear from the lemma. In case (a), we have \( A = B \oplus C \) for some \( C \) with \( |C| = d \), so \( dA \leq dB \oplus dB \leq pB \). In case (b), we can write \( d = p^k d_1 \) with \( d_1 \neq 0 \pmod{p} \). We then have \( A/B = C \oplus D \) say, where \( |C| = p^k \) and \( |D| = d_1 \) and \( C \) is not cyclic, so \( k > 1 \) and \( C \) is annihilated by \( p^{k-1} \). It follows that \( A/B \) is annihilated by \( d/p = p^{k-1}d_1 \), so \( (d/p)A \leq B \), so \( dA \leq pB \).

**Lemma 4.15.** In \( H^*(BG) = P[c_1, \ldots, c_p] \otimes E[v_1, \ldots, v_p] \) we have

\[
\begin{align*}
v_i &= tr^G_{\Sigma_p}(a_1x_2 \cdots x_i) \text{ for } 1 \leq i \leq p \\
c_i &= i tr^G_{\Sigma_p}(x_1x_2 \cdots x_i) \text{ for } 1 \leq i < p.
\end{align*}
\]

**Proof.** We will just discuss \( v_i \); the argument for \( c_i \) is essentially the same. Put \( v_i' = a_1x_2 \cdots x_i \), so the claim is that \( v_i = tr^G_{\Sigma_p}(v_i') \). Let \( H_i \) be the stabiliser of \( v_i' \) in the symmetric group \( \Sigma_p \), so \( H_i \simeq \Sigma_{i-1} \times \Sigma_{p-i} \). By definition, \( v_i \) is the element that satisfies

\[
\text{res}^G_{\Sigma_p}(v_i) = (-1)^i \sum_{\sigma \in \Sigma_p/H_i} \sigma^*(v_i').
\]

As \( |H_i| \) is not divisible by \( p \), we can rewrite this as

\[
\text{res}^G_{\Sigma_p}(v_i) = (-1)^i |H_i|^{-1} \sum_{\sigma \in \Sigma_p} \sigma^*(v_i').
\]

Moreover, \( |H_i| = (i-1)!/(p-i)! \). Wilson’s theorem tells us that \( |H_1| = 1 \pmod{p} \), and it is not hard to deduce by induction that \( |H_i| = (-1)^i \pmod{p} \). We thus have

\[
\text{res}^G_{\Sigma_p}(v_i) = \sum_{\sigma \in \Sigma_p} \sigma^*(v_i').
\]
The theorem of Quillen tells us that the restriction map is injective, so it will suffice to show that we also have \( \text{res}^G_T(TgT) = \sum \sigma^*(v'_i) \).

Next, there is a Mackey formula expressing \( \text{res}^G_T(TgT) \) as a sum of terms indexed by double cosets \( TgT \). The normaliser of \( T \) in \( G \) is \( \Sigma_p \times T \), so we have a double coset for each \( \sigma \in \Sigma_p \), and the corresponding term is just \( \sigma^*(v'_i) \). Thus, it will suffice to show that when \( g \) is not in the normaliser of \( T \), the term for \( TgT \) in the double coset formula is zero. This term involves the group \( U = T \cap gTg^{-1} \), which we can describe as follows. As \( g \) is not in the normaliser, we see that it is not a monomial matrix, so we see that there is at least one triple \( (i, j, k) \) where \( g_{ij} \neq 0 \neq g_{jk} \) and \( i \neq j \). Let \( E \) be the smallest equivalence relation on \( \{1, \ldots, p\} \) such that \( iEj \) for all such triples \( (i, j, k) \). For \( u \in (F^*)^p \), let \( \delta(u) \) denote the corresponding diagonal matrix in \( T \). It is not hard to check that \( \delta(u) \in U \) iff \( u_i = u_j \) whenever \( iEj \). This means that \( U \simeq (F^*)^r \) for some \( r < p \), and that \( U \) is a retract of \( T \), with index divisible by \( p \). This means that the transfer map \( tr^G_T : H^*(BU) \to H^*(BT) \) is zero (by Corollary 4.14), and thus that \( TgT \) does not contribute to the double coset formula.

\[ \text{Lemma 4.16.} \quad \rho^*(v_i) = -\frac{1}{n} \frac{1}{n} \frac{1}{n} \quad \text{for all } i. \]

\[ \text{Proof.} \quad \text{We again put } v'_i = a_1x_2 \cdots x_i, \text{ so } v_i = \text{tr}^G_T(i) \text{ We will use the double coset formula to evaluate } \rho^*(v_i) = \text{res}^G_T(TgT). \text{ This will involve the group } Z \text{ of multiples of the identity in } GL_p(F), \text{ so } Z \simeq F^x \text{ and } Z \text{ is the centre of } G \text{ and } Z = C \cap T. \text{ The map } \rho : C \to G \text{ was defined using a basis } e_1, \ldots, e_p \text{ for } F(1) \text{ over } F, \text{ which can be chosen to have } e_1 = 1. \text{ Let } X \subseteq G \text{ be the set of matrices } g \text{ such that } ge_1 = e_1, \text{ and the first nonzero entry in } ge_1 \text{ is one for all } i. \text{ We claim that } X \text{ contains precisely one element from each double coset } ChT. \text{ Indeed, given } h \in G \text{ we note that } he_1 \text{ is a nonzero element of } F^p \simeq F(1), \text{ so we can regard it as an element } c \in F(1)^x = C. \text{ Now let } u_i \text{ be the first nonzero entry in the vector } c^{-1}he_i, \text{ and let } t \text{ be the diagonal matrix with entries } u_1, \ldots, u_p, \text{ so } t \in T. \text{ We then find that the element } g = c^{-1}ht^{-1} \text{ lies in } X, \text{ and that } X \cap (ChT) = \{g\}. \text{ Next, suppose that } g \in X, \text{ and consider the group } H_g = C \cap gTg^{-1}. \text{ If } h \in H_g \text{ then } h(ge_1) = c_i(ge_1) \text{ for some } c_i \in C = F(1)^x, \text{ but also } h(ge_1) = u_i(ge_1) \text{ for some } u_i \in F^x. \text{ It follows that } c = u_1 \in F^x. \text{ and thus that } h \in Z. \text{ From this we see that } H_g \text{ is just equal to } Z, \text{ and thus that every conjugation map acts as the identity on } H_g. \text{ Because of this, the double coset formula just simplifies to } \text{res}^G_T(TgT) = |X| \text{tr}^C_T(\text{res}^C_T(w)) \text{ for all } w \in H^*(BT). \text{ Standard methods show that} \]

\[ |X| = \prod_{j=1}^{p-1} \frac{q^p - q^j}{q - 1} = q^{p-1} \prod_{j=1}^{p-1} \sum_{m=0}^{p-j-1} q^m. \]

As \( q = 1 \pmod{p} \), this gives \( |X| = (p - 1)! = -1 \pmod{p} \). Next, we have \( H^*(BU) = F[x] \otimes E[a] \), where \( x \) is the image of any of the elements \( x_i \in H^2(BT) \), and \( a \) is the image of any of the elements \( a_i \in H^1(BT) \). This gives \( \text{res}^C_T(v_i') = ax^{i-1} \) and so \( \text{res}^C_T(v_i') = -\text{tr}^C_T(ax^{i-1}) \). Now \( x \) is the Euler class of the representation given by the inclusion \( F^x \to F^x \), whereas \( \pi \) is the Euler class of the inclusion \( F(1)^x \to F^x \), so \( \text{res}^C_T(\pi) = x \). By the reciprocity formula for transfers, we therefore have \( \text{tr}^C_T(ax^{i-1}) = \text{tr}^C_T(a)x^{i-1} \). Next, recall that we chose \( i : F^x \to C^x \), and there is a unique \( \alpha \in F^x = Z \) with \( i(\alpha) = \exp(2\pi i/(q - 1)) \), and that \( \alpha \) can be identified with the map \( Z \to F_p \) sending \( \alpha \) to 1. Similarly, there is a unique \( \pi \in C \)
with \(i(\alpha) = \exp(2\pi i (q^p - 1))\), and \(\alpha\) can be identified with the map \(C \to \mathbb{F}_p\), sending \(\alpha\) to 1. We find that \(\tau_{C/Z}(a) = \alpha\), so Lemma 4.13 gives \(\tau_C^Z(a) = \alpha\). Putting this together, we get

\[
\text{res}^{G}_{C}(v_i) = -\tau_C \epsilon^{i-1} = (-1)^{i-1} \epsilon^{i-1} \tau
\]
as claimed.

We can now dualise the above results to describe the map \(H_*(BV(1)) \to H_*(BV)\), and extend inductively to describe the map \(H_*(BV(n+m)) \to H_*(BV(n))\) for any \(n, m \geq 0\). We will call this map \(\rho_\ast\). Recall here that

\[
H_*(BV(n)) = P[b^{(n)}_i | i \geq 0] \otimes E[e^{(n)}_i | i \geq 0].
\]

To describe \(\rho_\ast\), it is convenient to introduce the formal power series

\[
b^{(n)}(s) = \sum_i b^{(n)}_i s^i \in H_*(BV(n))[s]
\]

\[
e^{(n)}(s) = \sum_i e^{(n)}_i s^i \in H_*(BV(n))[s].
\]

The map \(\rho_\ast\) induces a map

\[
H_*(BV(n+m))[s] \to H_*(BV(n))[s],
\]

which we again call \(\rho_\ast\).

**Proposition 4.17.** We have

\[
\rho_\ast(b^{(n+m)}(s)) = (b^{(n)}(s))^{p^m} = \sum_j (b^{(n)}_j)^{p^m} s^{p^m j},
\]

so

\[
\rho_\ast(b^{(n+m)}) = \begin{cases} (b^{(n)}_i)^{p^m} & \text{if } p^m | i \\ 0 & \text{otherwise.} \end{cases}
\]

We also have

\[
\rho_\ast(e^{(n+m)}(s)) = (b^{(n)}(s))^{p^m-1} e^{(n)}(s).
\]

**Proof.** We can reduce inductively to the case where \(n = 0\) and \(m = 1\). In this case we will write \(b(s) = b^{(0)}(s)\) and \(\tilde{b}(s) = b^{(1)}(s)\), and similarly for \(e(s)\) and \(\overline{e}(s)\). The element \(\rho_\ast(\tilde{b}(s))\) is characterised by the fact that

\[
\langle \rho_\ast(\tilde{b}(s)), e^\alpha \rangle = \langle \overline{b}(s), \rho^\ast(e^\alpha) \rangle
\]

for all monomials \(e^\alpha = \prod_i c_i^{n_i}\) in \(c_1, \ldots, c_p\). By Lemma 4.12, the right hand side is zero unless \(e^\alpha = c_i^{n_i}\) for some \(n_i\). In that special case, we have \(\rho^\ast(e^\alpha) = \overline{\tau}^{p n_i}\). We also have \(\langle \overline{b}(s), \overline{e}(s) \rangle = \delta_{ij}\), so we get \(\langle \tilde{b}(s), \rho^\ast(e^\alpha) \rangle = (-s)^{p n_i}\). Now consider \(\langle b(s)^p, e^\alpha \rangle\). Note that the inclusion \(T = GGL_1(F)^p \to GGL_p(F) = G\) induces a map \(\sigma_\ast : H_*(GGL_1(F)^p) \to H_*(GGL_p(F))\), and it is this map that is used in defining the power \(b(s)^p\). We therefore find that \(\langle b(s)^p, u \rangle = \langle b(s)^{\otimes p}, \sigma^\ast(u) \rangle\) for all \(u \in H^*(BG)\). On the other hand, for any \(w \in P[x_1, \ldots, x_p]\), the inner product \(\langle b(s)^{\otimes p}, w \rangle\) is just the result of replacing each \(x_i\) by \(s\). It follows that the map \(w \mapsto \langle b(s)^{\otimes p}, w \rangle\) is a ring homomorphism. It sends \(c_i\) to \((-1)^i\) times the \(i\)th elementary symmetric function in the list \(s, \ldots, s\), which is zero mod \(p\) if \(0 < i < p\), and is \((-s)^p\) if \(i = p\). This gives \(\langle b(s)^p, e^\alpha \rangle = \langle \overline{b}(s), \rho^\ast(e^\alpha) \rangle\) for all \(\alpha\), so \(\rho_\ast(\tilde{b}(s)) = b(s)^p\) as claimed.
We now consider \( \rho_*(\overline{\pi}(s)) \). Put

\[
e^*(s) = b(s)^{\otimes (p-1)} \otimes e(s) \in H_*(BT),
\]
so the claim is that \( \rho_*(\overline{\pi}(s)) = \sigma_*(e^*(s)) \), or equivalently that

\[
\langle e^*(s), \sigma^*(e^*(v^r)) \rangle = \langle \overline{\pi}(s), \rho^*(e^*(v^r)) \rangle
\]

for all monomials \( e^* v^r \in H^*(BG) \). Here \( v^r \) is in principle any product of distinct terms of the form \( v_k \), but it is easy to see that both sides are zero unless there is precisely one term. We must therefore check that

\[
\langle e^*(s), \sigma^*(e^*(v_k)) \rangle = \langle \overline{\pi}(s), \rho^*(e^*(v_k)) \rangle
\]

for all \( \alpha \) and \( k \). Next, one can check that the map

\[
\langle e^*(s), - \rangle : H^*(BT) = P[x_1, \ldots, x_p] \otimes E[a_1, \ldots, a_p] \to P[s]
\]

can be described as follows. Given \( w \in H^*(BT) \) we first replace \( x_1, \ldots, x_p \) by \( s \) and \( a_1, \ldots, a_{p-1} \) by \( 0 \), to get an element \( \xi(w) \in P[s] \otimes E[a_p] \); then \( e^*(s), w \) is the coefficient of \( a_p \) in \( \xi(w) \). Here \( \xi \) is a ring map with \( \xi(\sigma^*(c_i)) = 0 \) for \( 0 < i < p \), and \( \xi(\sigma^*(c_p)) = (-s)^p \), as we saw while considering \( \rho_*(\overline{\pi}(s)) \). We also find that

\[
\xi(\sigma^*(v_k)) = (-1)^k \left( \frac{p-1}{k-1} \right) s^{k-1} a_p.
\]

(The factor of \((-1)^k\) is incorporated in the definition of \( v_k \), and the binomial coefficient comes from the combinatorics of the \( \Sigma_p \)-action.) It is an exercise to check that \((-1)^k \left( \frac{p-1}{k-1} \right) = -1 \) \((\mod p)\) for all \( k \), so \( \xi(v_k) = -s^{k-1} a_p \). We therefore have

\[
\langle e^*(s), \sigma^*(e^m v_k) \rangle = (-1)^{pm+1} s^{mp+k-1},
\]
and \( \langle e^*(s), \sigma^*(c^m v^r) \rangle = 0 \) for all other monomials \( e^* v^r \). It follows from Lemma 5.1 that this is the same as \( \langle \overline{\pi}(s), \rho^*(e^m v^r) \rangle \), as expected.

\[
\square
\]

5. Morava \( E \)-theory and \( K \)-theory

**Definition 5.1.** We write \( E \) for the Morava \( E \)-theory spectrum of height \( n \) at the prime \( p \). To be definite, we use the version with

\[
E_* = \mathbb{Z}_p[[u_1, \ldots, u_n]][u^{\pm 1}]
\]

where \( |u| = 2 \) and \( |u_i| = 0 \). We make the convention \( u_0 = p \) and \( u_n = 1 \). This spectrum has a standard complex coordinate \( x \in E^0 BU(1) \) such that \( E^0 BU(1) = E^0 [x] \) (and \( E^1 BU(1) = 0 \)). The associated formal group law satisfies

\[
\log_F(x) = x + p^{-1} \sum_{i=1}^{n} \log_F(u_i x^{p^i}).
\]

We also let \( K \) denote the corresponding 2-periodic Morava \( K \)-theory spectrum, with

\[
K_* = E_*/(u_0, \ldots, u_{n-1}) = \mathbb{F}_p[\mathbb{Z}_p[u^{\pm 1}]]
\]

**Remark 5.2.** It is again standard that \( E^0 BU(1)^d = E^0 [x_1, \ldots, x_d] \). Moreover, if we let \( c_k \) denote \((-1)^k\) times the \( k \)th elementary symmetric polynomial in the variables \( x_i \), then the inclusion \( U(1)^d \to U(d) \) gives an isomorphism

\[
E^0 BU(d) \to E^0 [c_1, \ldots, c_d] = E^0 [x_1, \ldots, x_d]^{\Sigma_d} \leq E^0 (BU(1)^d).
\]

(Moreover, \( E^1 \) is zero for all spaces mentioned above.)
Remark 5.3. From the functional equation for \( \log_F(x) \) it is easy to see that \( \log_F(x) \) involves only powers \( x^i \) with \( i = 1 \pmod{p-1} \). Thus, over the ring \( E_0[w]/(w^p - w) \) we have \( w^{k(p-1)+1} = w \) for all \( k \), so \( \log_F(wx) = w \log_F(x) \), so \( \exp_F(wx) = w \exp_F(x) \), so \( (wx)^F(wy) = w(x+Fy) \), so \( k[F(wx) = w[k]F(x) \) for all \( k \in \mathbb{Z} \).

From this it follows that \( \exp_F(x) \) and \( [k]F(x) \) involve only powers \( x^i \) with \( i = 1 \pmod{p-1} \), and \( x+Fy \) involves only monomials \( x^i y^j \) with \( i+j = 1 \pmod{p} \).

Recall also that the polynomial \( t^p - t \) factors completely in \( \mathbb{Z}_p \), and the reduction map \( \mathbb{Z}_p \to \mathbb{F}_p \) gives a bijection from the set of roots to \( \mathbb{F}_p \). If \( m \) is one of these roots then we can substitute it for \( w \) in the above discussion, giving \( (mx) + F(my) = m(x+Fy) \) and \( [k]F(mx) = m[k]F(x) \).

Note also that the standard definition of \( [m]F(x) \) for \( m \in \mathbb{Z} \) can be extended to \( m \in \mathbb{Z}_p \) by the rule \( [m]F(x) = \exp_F(m \log_F(x)) \). If \( m^p = m \) as above, then we just get \( [m]F(x) = mx \).

**Proposition 5.4.** For each \( k \geq 0 \) there is a unique monic polynomial \( h_k(x) \) of degree \( p^{nk} \) over \( E_0 \) such that

\[
h_k(x) = x^{p^{nk}} \pmod{u_0, \ldots, u_{n-1}}
\]

and \( [p^k]F(x) \) is a unit multiple of \( h_k(x) \) in \( E_0[x] \). Moreover:

(a) If \( m \) is a unit multiple of \( p^k \) in \( \mathbb{Z}_p \), then \( [m]F(x) \) is also a unit multiple of \( h_k(x) \) in \( E_0[x] \).

(b) There is a unique polynomial \( \tilde{h}_k \) such that \( h_k(x) = x \tilde{h}_k(-x^{p-1}) \).

**Proof.** The polynomial \( h_k(x) \) exists by the formal Weierstrass preparation theorem (see [15] Section 5.2, for example). For claim (a), we can write \( m = p^km' \) with \( m' \neq 0 \pmod{p} \), and this gives \( [m]F(x) = [m']F([p^k]F(x)) \). Here \( [m]F(x) = m'x + O(x^2) \) and \( m' \) is invertible in \( \mathbb{Z}_p \); it follows that \( [m']F(x) \) is a unit multiple of \( x \), and thus that \( [m']F([p^k]F(x)) \) is a unit multiple of \( [p^k]F(x) \), and thus also of \( h_k(x) \). For claim (b), suppose that \( m \in \mathbb{Z}_p \) with \( m^{p-1} - 1 = 0 \). We then find that \( m^{-1}h_k(mx) \) has the defining properties of \( h_k(x) \), and so is equal to \( h_k(x) \). This proves that \( h_k(x) \) contains only terms \( x^i \) with \( i = 1 \pmod{p-1} \), and this implies the existence of \( \tilde{h}_k \).

There is an extensive theory of the structure of \( E_0(B\mathcal{G}) \) for finite groupoids \( \mathcal{G} \). We next recall some of this theory.

**Definition 5.5.** We write \( \mathcal{S} \) for the category of \( K \)-local spectra. Given \( X, Y \in \mathcal{S} \) we write \( X \wedge Y \) for the \( K \)-localised smash product. This makes \( \mathcal{S} \) into a symmetric monoidal category, whose unit is the \( K \)-local sphere spectrum, which we call \( S \). We also write \( DX \) for the function spectrum \( F(X, S) \), and \( \bigvee_i X_i \) for the \( K \)-localised wedge of a family of objects \( X_i \). Given any groupoid \( \mathcal{G} \), we write \( L\mathcal{G} \) for the \( K \)-localisation of \( \Sigma^\infty_+ B\mathcal{G} \).

We next recall some duality theory for \( E_0(B\mathcal{G}) \). We will be primarily interested in the case where \( \mathcal{G} \) is even, as defined below.

**Definition 5.6.** Let \( \mathcal{G} \) be a hom-finite groupoid. We say that \( \mathcal{G} \) is even if \( K_1(B\mathcal{G}) = 0 \).

**Remark 5.7.** If \( \mathcal{G} \) is even, we see from [14] Section 8] that \( K^1(B\mathcal{G}) = E^1_0(B\mathcal{G}) = E^1(B\mathcal{G}) = 0 \), and that \( E^0_0(B\mathcal{G}) \) and \( E^0(B\mathcal{G}) \) are both pro-free, and that \( E^0(B\mathcal{G}) = \text{Hom}_{E_0}(E^0_0(B\mathcal{G}), E_0) \). Moreover, if \( \mathcal{G} \) is actually finite then \( E^0_0(B\mathcal{G}) \) and \( E^0_0(B\mathcal{G}) \) are free modules of the same finite rank over \( E_0 \).
In [13] we constructed, for every functor \( \phi : \mathcal{G} \to \mathcal{H} \) of finite groupoids, a map \( R\phi : L\mathcal{H} \to L\mathcal{G} \) which is adjoint, in a certain sense, to \( L\phi : L\mathcal{G} \to L\mathcal{H} \). (In many places, we will use the notation \( \phi \) for \( L\phi \) and \( \phi' \) for \( R\phi \).) If \( \phi \) is a finite morphism between hom-finite groupoids then we can decompose it as the coproduct of a family of functors \( \phi_i : \mathcal{G}_i \to \mathcal{H}_i \) of finite groupoids, and then define

\[
R\phi = \bigvee_i R\phi_i : L\mathcal{H} = \bigvee_i L\mathcal{H}_i \to \bigvee_i L\mathcal{G}_i = L\mathcal{G}.
\]

It is easy to check that this is independent of the choice of decomposition. We leave to the reader the task of adapting results from [13] to this slightly more general context. In particular, subject to suitable finiteness conditions, we have Mackey property: for any homotopy pullback square of groupoids

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\beta \downarrow & & \downarrow \gamma \\
C & \xrightarrow{\delta} & D,
\end{array}
\]

we have \((R\gamma)(L\delta) = (L\alpha)(R\beta) : LC \to LB\). As a special case of this, one can show that \( R\phi = L\phi^{-1} \) whenever \( \phi \) is an equivalence. (However, we will see cases where \( \phi \) is not an equivalence but \( L\phi \) is an equivalence; in these cases, \( R\phi \) is usually different from \((L\phi)^{-1}\).

The map \( L\phi : LG \to LH \) induces a ring map \( E^0(B\mathcal{H}) \to E^0(B\mathcal{G}) \), which we call \( \phi^* \). Similarly, \( R\phi \) induces a map \( E^0(B\mathcal{G}) \to E^0(B\mathcal{H}) \), which we denote by \( \phi_! \). For a homotopy pullback square as above, we then have

\[
\delta^*\gamma_! = \beta_!\alpha^* : E^0(B\mathcal{B}) \to E^0(B\mathcal{C})
\]

By applying this to a suitably chosen square, we obtain a Frobenius reciprocity formula: for any \( \phi : \mathcal{G} \to \mathcal{H} \) and any \( a \in E^0(B\mathcal{H}) \) and \( b \in E^0(B\mathcal{G}) \) we have

\[
a,\phi_!(b) = \phi_!(\phi^*(a) b) \in E^0(B\mathcal{H}).
\]

Recall also that when \( \phi \) is faithful, the map \( B\phi : BG \to BH \) is (up to homotopy equivalence) a covering map, and \( \phi^* \) is the \( K \)-localisation of the associated transfer map \( \Sigma_*^\infty BH \to \Sigma_*^\infty BG \).

If \( \mathcal{G} \) has only finitely many isomorphism classes, then the projection \( \epsilon : \mathcal{G} \to 1 \) is a finite functor and so induces a map \( \epsilon_! : E^0(B\mathcal{G}) \to E^0 \). Provided that \( \mathcal{G} \) is even, the map \( \epsilon_! \) is a Frobenius form, which means that the rule

\[
\langle f, g \rangle = \epsilon_!(fg)
\]

gives a perfect \( E^0 \)-linear pairing on \( E^0(B\mathcal{G}) \), and so identifies \( E^0(B\mathcal{G}) \) with the dual module \( E^0_!(B\mathcal{G}) \). More generally, we can try to define \( \epsilon_! \) as a sum over isomorphism classes, even if there are infinitely many of them. However, the domain of \( \epsilon_! \) will not be the whole of \( E^0(B\mathcal{G}) \), as we will need to impose auxiliary conditions to make the sum converge in the \( I_\epsilon \)-adic topology.

If \( X = spf(E^0(B\mathcal{G})) \), then any element \( f \in E^0(B\mathcal{G}) \) can be thought of as a function on \( X \). We use the notation \( \int_X f(x) \,dx \) for \( \epsilon_!(f) \). In this notation, the perfect pairing is

\[
\langle f, g \rangle = \int_X f(x)g(x) \,dx.
\]
More generally, suppose we have \( \phi : \mathcal{G} \to \mathcal{H} \), and we put \( Y = \text{spf}(E^0BG) \), so \( \phi \) gives a map \( X \to Y \) and also a map

\[ \phi_! : \mathcal{O}_X = E^0(B\mathcal{G}) \to E^0(B\mathcal{H}) = \mathcal{O}_Y. \]

we use the notation \( \int_{x \in \phi^{-1}(y)} f(x) \, dx \) for \( (\phi f)(y) \). In this notation, the Frobenius reciprocity formula is

\[ \int_{x \in \phi^{-1}(y)} g(\phi(x)) \, f(x) \, dx = g(y) \int_{x \in \phi^{-1}(y)} f(x) \, dx. \]

If \( \mathcal{G} \xrightarrow{\phi} \mathcal{H} \xrightarrow{\psi} \mathcal{K} \) then \( (\psi \phi)(\psi_!) = \psi_! \phi_! \), which can be written in integral notation as

\[ \int_{x \in (\psi \phi)^{-1}(z)} f(x) \, dx = \int_{y \in \psi^{-1}(z)} \int_{x \in \phi^{-1}(y)} f(x) \, dx \, dy. \]

Given a homotopy cartesian square

\[ \begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\beta \downarrow & & \downarrow \gamma \\
C & \xrightarrow{\delta} & D,
\end{array} \]

the associated Mackey property \( \delta^* \gamma_! = \beta_! \alpha^* \) can be written as

\[ \int_{x \in \gamma^{-1}(\delta(y))} f(x) \, dx = \int_{w \in \beta^{-1}(y)} f(\alpha(w)) \, dw. \]

**Proposition 5.8.** Let \( \phi : \mathcal{G} \to \mathcal{H} \) be a full functor of hom-finite groupoids that is also a \( \pi_0 \)-isomorphism. Suppose that for all \( a \in \mathcal{G} \), the kernel of the surjection \( \phi : \mathcal{G}(a,a) \to \mathcal{H}(\phi(a),\phi(a)) \) has order coprime to \( p \). Then \( L\phi : LG \to LH \) is an equivalence.

**Proof.** We can reduce easily to the following claim: if \( G \) is a finite group and \( N \) is a normal subgroup of order coprime to \( p \), then the map \( K_*(BG) \to K_*(B(G/N)) \) is an isomorphism. Because of the Atiyah-Hirzebruch spectral sequence, it will suffice to prove that the map \( H_*(G;K_*) \to H_*(G/N;K_*) \) is an isomorphism. There is another standard spectral sequence \( H_*(G/N;H_*(N;K_*)) \Rightarrow H_*(G;K_*) \). As \( N \) has order coprime to \( p \), we have \( H_0(N;K_*) = K_* \) and \( H_i(N;K_*) = 0 \) for \( i > 0 \). The spectral sequence therefore collapses to give the required isomorphism \( H_*(G;K_*) \to H_*(G/N;K_*) \). \( \square \)

**Proposition 5.9.** Let \( \phi : \mathcal{G} \to \mathcal{H} \) be a faithful functor of hom-finite groupoids that is also a \( \pi_0 \)-isomorphism. Suppose that for all \( a \in \mathcal{G} \), the index of \( \phi(\mathcal{G}(a)) \) in \( \mathcal{H}(\phi(a)) \) is coprime to \( p \). Then the composite \( \phi^! : LH \to LH \) is an equivalence, so \( \phi : LG \to LH \) is a split epimorphism, and \( \phi^! : LH \to LG \) is a split monomorphism. In fact, the map

\[ \Sigma_+^\infty \phi : \Sigma_+^\infty BG \to \Sigma_+^\infty BH \]

is already a split epimorphism after \( p \)-completion.

**Proof.** We can reduce easily to the following claim: if \( H \) is a finite group, and \( G \) is a subgroup of index \( m \), and \( m \) is coprime to \( p \), then the composite

\[ \Sigma_+^\infty BH \xrightarrow{\text{inc}} \Sigma_+^\infty BG \xrightarrow{\text{inc}} \Sigma_+^\infty BH \]

induces an isomorphism in Morava \( K \)-theory. Because of the Atiyah-Hirzebruch spectral sequence, it will suffice to prove that the above map gives an isomorphism
in mod p homology. However, it is a standard fact in group homology theory that the effect in homology is just multiplication by \( m \), which is an isomorphism as \( m \) is coprime to \( p \).

**Proposition 5.10.** Suppose we have a functor \( \phi: \mathcal{G} \to \mathcal{H} \) satisfying the hypotheses of Proposition 5.8 (so it is full and a \( \pi_0 \)-isomorphism, and the kernels of the induced maps on automorphism groups have orders coprime to \( p \)). Suppose also that we have a functor \( \psi: \mathcal{H} \to \mathcal{G} \) with \( \phi \psi \simeq 1 \). Put \( u = \phi_!(1) \in E^0(B\mathcal{H}) \) and \( v = \psi_!(1) \in E^0(B\mathcal{G}) \). Then the maps \( \phi^*: E^0(B\mathcal{H}) \to E^0(B\mathcal{G}) \) and \( \psi^*: E^0(B\mathcal{G}) \to E^0(B\mathcal{H}) \) are isomorphisms and are inverse to each other. Moreover, we have \( \phi(g) = u \psi^*(g) \) for all \( g \in E^0(B\mathcal{G}) \) and \( \psi(h) = v \phi^*(h) \) for all \( h \in E^0(B\mathcal{H}) \). In particular, we have \( u \psi^*(v) = 1 \) and \( \phi^*(u).v = 1 \).

**Proof.** Proposition 5.8 tells us that \( L\phi \) is an equivalence, so \( \phi^*: E^0(B\mathcal{H}) \to E^0(B\mathcal{G}) \) is an isomorphism. As \( \phi \psi \simeq 1 \) we have \( \psi^* \phi^* = 1 \), so \( \psi^* \) is also an isomorphism and is inverse to \( \phi^* \). We can now use the Frobenius reciprocity formula to get

\[
\phi(g) = \phi((\phi^* \psi^*) (g).1) = \psi^*(g) \phi(1) = \psi^*(g)u
\]

as claimed. Now take \( g = v = \psi_!(1) \) to get

\[
\psi^*(v)u = \phi(v) = (\phi \psi)_!(1) = 1_!(1) = 1.
\]

Applying \( \phi^* \) to this gives \( v \phi^*(u) = 1 \). We can also use Frobenius reciprocity again to get

\[
\psi(h) = \psi((\psi^* \phi^*)(h).1) = \phi^*(h) \psi(1) = \phi^*(h)v.
\]

**Definition 5.11.** We have chosen an injection \( i: \bar{G}_1 = GL_1(\bar{F}) \to S^1 \), which gives a map \((Bi)^*: E^*(\mathcal{C}P^\infty) \to E^*(B\bar{G}_1)\). There is a standard generator \( x_E \in E^0(\mathcal{C}P^\infty) \), and we also write \( x_E \) for the image of this class in \( E^0(B\bar{G}_1) = E^0(B\mathcal{T}) \) or in \( E^0(BGL_1(F[m])) \). As \( \bar{G}_1 \) is abelian, and is the colimit of a sequence of cyclic groups, standard methods give \( E^0(\bar{G}_1) = E^0[x_E] \). Similarly, we have \( K^0(\bar{G}_1) = K^0[x_K] \), where \( x_K \) is the reduction of \( x_E \) modulo \( I_n \).

**Definition 5.12.** We will often discuss rings of the form \( E^0(X) \) in terms of the geometry of formal schemes, as in Lemma 5.13. In particular, we put \( S = \text{spf}(E^0) \) and \( \mathbb{G} = \text{spf}(E^0 BU(1)) \) and \( \mathbb{H} = \text{spf}(E^0 BGL_1(\bar{F})) \). From the remarks in Definition 5.11 we see that \( \mathbb{H} \) is naturally isomorphic to \( \text{Hom} (\text{Hom}(\mathcal{F}, S^1), \mathbb{G}) \) and thus unnaturally isomorphic to \( \mathbb{G} \). (If we took the time to set up the relevant definitions, we could also describe \( \mathbb{H} \) as \( \text{Tor}(\mathcal{F}, \mathbb{G}) \) or as \( T \otimes_{\mathbb{Z}_p} \mathcal{G} \), where \( T = \text{Hom}(\mathbb{Z}/p^\infty, \mu_p(\bar{F})) \simeq \mathbb{Z}_p \) as in Remark 2.1.) We will also put \( S_0 = \text{spec}(K_0) < S \) and \( \mathbb{G}_0 = \text{spf}(K^0 BU(1)) = \mathbb{G} \times_S S_0 \) and \( \mathbb{H}_0 = \text{spf}(K^0 BGL_1(\bar{F})) = \mathbb{H} \times_S S_0 \).

**Definition 5.13.** Now note that the Künneth theorem gives

\[
E^0(BT_d) = E^0[x_E, \ldots, x_{Ed}].
\]

This has an evident action of the symmetric group \( \Sigma_d \). We define \( c_{Ek} \in E^0(BT_d) \) to be the sum of all the monomials in the orbit of \((-1)^k x_{E1} \cdots x_{Ek}\). We also let \( c_{Kd} \) be the image of \( c_{Ek} \) in Morava K-theory.

**Proposition 5.14.** The inclusion \( \bar{T}_d \to \bar{G}_d \) gives isomorphisms

\[
E^0(B\bar{G}_d) \simeq E^0(BT_d)^{\Sigma_d} = E^0[c_{E1}, \ldots, c_{Ed}]
\]

\[
K^0(B\bar{G}_d) \simeq K^0(BT_d)^{\Sigma_d} = K^0[c_{K1}, \ldots, c_{Kd}].
\]
Proof. The permutations come from inner automorphisms of $\bar{G}_d$, and it follows that the image of the restriction map is contained in the invariants. Recall from Section 4 that the cohomology rings $H^*(BT_d)$ and $H^*(BG_d)$ are both polynomial on generators in even degrees. It follows that the Atiyah-Hirzebruch spectral sequences $H^*(BT_d; K^*) \Rightarrow K^*(BT_d)$ and $H^*(BG_d; K^*) \Rightarrow K^*(BG_d)$ both collapse. It follows easily from this that $K^*(BG_d)$ is as claimed. In particular, $K^*(BG_d)$ is concentrated in even degrees, and the restriction map is a split monomorphism of $K^*$-modules. Because $K^*(BG_d)$ is in even degrees, we see from [14] Section 8 that so $E^*(BG_d)$ is pro-free and concentrated in even degrees, with $E^*(BG_d)/I_n = K^*(BG_d)$. The same applies to $BT_d$. By choosing bases in Morava $K$-theory that are compatible with the splitting, and lifting them to Morava $E$-theory, we see that $E^0(BG_d)$ also maps isomorphically to $E^0(BT_d)^{\Sigma_d}$. \hfill $\square$

Remark 5.15. In the language of formal schemes, this says that $\text{spf}(E^0(BG_d)) = \mathbb{H}^d/\Sigma_d$, which is the same as $\text{Div}^+_\mathbb{H}(\mathbb{H})$ (the moduli scheme for effective divisors of degree $d$ on $\mathbb{H}$). Thus, we have $\text{spf}(E^0(BV_*) = \text{Div}^+_\mathbb{H}(\mathbb{H})$. As $\mathbb{V}$ is a symmetric bimonoidal category, we see that $\text{spf}(E^0(B\mathbb{V}_*))$ has a natural structure as a semiring scheme. This just corresponds to the usual semiring structure of $\text{Div}^+_\mathbb{V}(\mathbb{V})$ under addition and convolution of divisors, which is familiar from the parallel case of $\text{spf}(E^0(BGL_n(\mathbb{C}))) = \text{Div}^+_\mathbb{V}(G)$. All this is discussed in more detail in [15] Sections 5.1 and 7.3.

Tanabe’s main theorem (which we stated as Theorem 5.1) tells us that

$$\text{spf}(E^0(BV_*)) = \text{spf}(E^0(B\mathbb{V}_*))^\Gamma = \text{spf}(E^0(B\mathbb{V}_*))^\Gamma = \text{Div}^+_\mathbb{H}(\mathbb{H})^\Gamma.$$ 

Definition 5.16. We let $b_{E_i} \in E^0_i(BG_1) = E^0_i(B\mathbb{V}_1)$ be dual to $x_{E_i}$, and similarly for $b_{K_i} \in K_0(BG_1) = K_0(B\mathbb{V}_1)$. We use the direct sum operation to make $K_* (B\mathbb{V}_*)$ into a ring, and similarly for $E_* (B\mathbb{V}_*)$.

Proposition 5.17. $K_0(B\mathbb{V}_*)$ is the polynomial ring $K_0[b_{K_i} | i \geq 0]$. Similarly, $E^0_0(B\mathbb{V}_*)$ is the $I_n$-adic completion of the polynomial ring $E_0[b_{E_i} | i \geq 0]$. 

Proof. The Morava $K$-theory statement says that $K_0(B\mathbb{V}_d)$ is the $d$th symmetric tensor power of $K_0(B\mathbb{V}_1)$, which is true by a straightforward dualisation of Proposition 5.1. Next, the polynomial ring $E_0[b_{E_i} | i \geq 0]$ maps to $E^0_0(B\mathbb{V}_1)$, and this map must extend over the $I_n$-adic completion because $E^0_0(B\mathbb{V}_1)$ is pro-free. The extended map becomes an isomorphism if we reduce it modulo $I_n$, so the unreduced map is an isomorphism by the general theory of pro-free modules. \hfill $\square$

Definition 5.18. We will use the term bialgebra for a module equipped with both an algebra structure and a coalgebra structure. We make no assumption about the interaction between these structures.

Definition 5.19. Let $\sigma : \mathcal{V}_* \times \mathcal{V}_* \to \mathcal{V}_*$ be the direct sum functor. We use the induced map $\sigma_* : K_0(B\mathcal{V}_*) \otimes K_0(B\mathcal{V}_*) \to K_0(B\mathcal{V}_*)$ to make $K_0(B\mathcal{V}_*)$ into a graded ring. (Note that this is commutative without any need for $\pm$ signs. The unit is the obvious generator of $K_0(B\mathcal{V}_0) = K_0(\text{point})$.) We just write $ab$ for $\sigma_*(a \otimes b)$. We also use the transfer map $\sigma^!$ to make $K_0(B\mathcal{V}_*)$ into a cocommutative graded coalgebra, with counit given by the obvious projection of $K_0(B\mathcal{V}_*) \to K_0(B\mathcal{V}_0)$. We make $E^\vee_0(B\mathcal{V}_*)$ into a graded bialgebra over $E_0$ in the same way.
Next, we use the diagonal map $\delta: \mathcal{V}_* \to \mathcal{V}_* \times \mathcal{V}_*$ to make $K^0(\mathcal{B}V_*)$ and $E^0(\mathcal{B}V_*)$ into rings. We just write $ab$ for $\delta^*(a \otimes b)$. The grading behaviour is that $ab = 0$ if $|a| \neq |b|$, and $|ab| = d$ if $|a| = |b| = d$.

The map $\sigma^*$ gives a second product on $K^0(\mathcal{B}V_*)$, and we write $a \times b$ for $(\sigma^*)^*(a \otimes b)$. This should be thought of as a kind of convolution. Here we have the more usual kind of grading behaviour: $|a \times b| = |a| + |b|$. The map $\sigma^*$ also gives a coproduct, which again respects the grading in the usual way. We define two products and a coproduct on $E^0(\mathcal{B}V_*)$ in the same way. If we need to distinguish the ordinary product (induced by $\delta$) from the convolution product, then we will call it the diagonal product.

**Remark 5.20.** The theorem of Tanabe tells us that $E^0(\mathcal{B}V_d)$ is a finitely generated free module over $E^0$, and that $E^1(\mathcal{B}V_d) = 0$. (This could also be deduced from the results of Section 8 below.) This means that duality theory and the Künneth theorem apply in their simplest forms.

Duality theory identifies $K_0(\mathcal{B}V_*)$ with $K^0(\mathcal{B}V_*)$, and this identification converts $\sigma_*$ to $(\sigma^*)^*$ and $\sigma_!$ to $\sigma^!$. Thus, if we ignore the diagonal product on $K^0(\mathcal{B}V_*)$, then $K_0(\mathcal{B}V_*)$ is isomorphic to $K^0(\mathcal{B}V_*)$ as graded bialgebras. Similarly, $E^0(\mathcal{B}V_*)$ is isomorphic to $E^0(\mathcal{B}V_*)$ as graded bialgebras. For our analysis of the Atiyah-Hirzebruch spectral sequence, it will be convenient to focus on $K_*(\mathcal{B}V_*)$. For other parts of this paper, it is more natural to focus on $E^0(\mathcal{B}V_*)$, which is the natural home of the diagonal product.

**Remark 5.21.** It turns out that the product and coproduct do not interact in the right way to make the ring $R = E^0(\mathcal{B}V_*)$ into a Hopf algebra. In other words, if we make $R \otimes R$ into an $E^0$-algebra in the obvious way, then the coproduct $\sigma^*: R \to R \otimes R$ is not a ring map. However, $\sigma^*$ becomes a ring map if we use a slightly different product rule on $R \otimes R$. This is essentially the same phenomenon as that described by Green in [1], and will be discussed in more detail in Section 13. On the other hand, if we use the diagonal product, then $E^0(\mathcal{B}V_*)$ becomes a Hopf algebra, but with nonstandard grading behaviour.

**Definition 5.22.** We write $\text{Dec}_d(E^0(\mathcal{B}V_*))$ for the module of decomposables of degree $d$ in $E^0(\mathcal{B}V_*)$ (with respect to the convolution product). By definition, this is the sum of the images of the transfer maps

$$\tau_j: E^0(\mathcal{B}G_j \times \mathcal{B}G_{d-j}) \to E^0(\mathcal{B}G_d),$$

(for $0 < j < d$) and so is an ideal in $E^0(\mathcal{B}G_d)$. We also put $\text{Ind}_d(E^0(\mathcal{B}V_*)) = E^0(\mathcal{B}G_d)/\text{Dec}_d(E^0(\mathcal{B}V_*))$, and observe that this inherits a ring structure.

The scheme $\text{spf}(\text{Ind}_d(E^0(\mathcal{B}V_*)))$ is a closed subscheme of $\text{spf}(E^0(\mathcal{B}G_d)) = \text{Div}_1^+(\mathbb{P}^1)\Gamma$; it should be thought of as the subscheme of $\Gamma$-invariant divisors that are indecomposable under addition.

We also define $\text{Prim}_d(E^0(\mathcal{B}V_*))$ to be the intersection of the kernels of the restriction maps

$$\rho_j: E^0(\mathcal{B}G_d) \to E^0(\mathcal{B}G_j \times \mathcal{B}G_{d-j}),$$

which is again an ideal in $E^0(\mathcal{B}G_d)$.

In order to understand $\text{Ind}_*(E^0(\mathcal{B}V_*))$ in detail, we need to know that it is a free module over $E^0$. The only way that we have succeeded in proving this is by using the Atiyah-Hirzebruch spectral sequence, as we will describe in Section 8. Thus,
the best results about \( \text{Ind}_* (E^0(BV_*) ) \) will be deferred to Section 10. Here we will just prove some preliminary facts.

**Lemma 5.23.** If \( d \) is not a power of \( p \), then
\[
\text{Prim}_d (E^0(BV_*)) = \text{Ind}_d (E^0(BV_*)) = 0.
\]

If \( d = 1 \) then
\[
\text{Ind}_d (E^0(BV_*)) = \text{Prim}_d (E^0(BV_*)) = E^0(BG_1) = E^0[ x ] / [ p^r ](x) = E^0( c_i \mid 0 \leq i < N_0 = p^{nr}).
\]

If \( d = p^k \) with \( k > 0 \) then \( \text{Ind}_d (E^0(BV_*)) \) is the cokernel of the transfer map \( E^0(BG_{p^k}) \to E^0(BG_{p^k}) \), whereas \( \text{Prim}_d (E^0(BV_*)) \) is the kernel of the restriction map \( E^0(BG_{p^k}) \to E^0(BG_{p^{k-1}}) \).

**Proof.** We will give the proofs for \( \text{Ind} \); the arguments for \( \text{Prim} \) are very similar.

Put \( P_j = G_j \times G_{d-j} \) for \( 0 < j < d \) and \( J = \text{Dec}_d (E^0(BV_*)) = \sum_j \text{image} ( \text{tr}^{i, d}_H ) \).

Write \( d = \sum_i dp^i \) in base \( p \), and put \( H = \prod_i G_{p^i} \leq G_d \). Proposition 5.3 tells us that \( |G_d / H| \) is coprime to \( p \), so Proposition 5.9 tells us that \( \text{tr}^{k, d}_H : E^0(BH) \to E^0(BG_d) \) is surjective. If \( d \) is not a power of \( p \) then the product defining \( H \) has more than one factor, so \( H \) is contained in some group \( P_j \) with \( 0 < j < d \), so \( \tau_j \) is surjective, so \( J = E^0(BG_d) \) and \( \text{Ind}_d (E^0(BV_*)) = 0 \).

The claim for \( d = 1 \) is clear from the definitions.

Now suppose that \( d = p^k \) with \( k > 0 \), and put \( L = G_{p^k-1} \leq G_d \). Put \( J' = \text{image} ( \text{tr}^{G_d}_{L} ) \). It is clear that \( J' \leq J \), and we need to prove that \( J' = J \). Let us say that a block subgroup of \( G_d \) is a subgroup \( M \) that is conjugate to \( \prod_i G_{p^i}^{m_i} \) with \( \sum_i p^i m_i = d \) and \( M \leq G_d \). The rank of such a subgroup is the minimal \( i \) such that \( m_i \neq 0 \). If the rank is \( i \) and \( i < k - 1 \) then \( m_i \) must be divisible by \( p \), so we can use the inclusion \( G_{p^i}^{m_i} \to G_{p^{i+1}}^{m_i/p} \) to include \( M \) in a block subgroup of strictly larger rank. By iterating this, we see that every block subgroup is subconjugate to \( L \), so \( \text{image} ( \text{tr}^{G_d}_{L} ) \leq J' \). On the other hand, if \( 0 < j < d \) then we can apply Proposition 3.4 to \( G_j \) and \( G_{d-j} \) to get a block subgroup \( M \leq P_j \) such that \( \text{tr}^{P_j}_{M} \) is surjective. From this it is clear that \( \text{image}( \tau_j ) \leq J' \), as required.

\[ \square \]

6. Annihilators and socles

Another aspect of duality theory involves the structure of annihilators and socles, as we now explain.

**Convention 6.1.** All group(oid)s mentioned in this section are assumed to be finite and even, as in Definition 5.6.

We will prove various facts about \( E^0(BG) \), which will have consequences for \( K^0(BG) \) and \( \mathbb{Q} \otimes E^0(BG) \). One can check that these consequences are valid even without the evenness assumption, but we will not need that.

**Definition 6.2.** Given a subset \( X \subseteq E^0(BG) \) we put
\[
X^\perp = \{ y \in E^0(BG) \mid \langle x, y \rangle = 0 \text{ for all } x \in X \} \\
= \{ y \in E^0(BG) \mid \epsilon (xy) = 0 \} \\
\text{ann} (X) = \{ y \in E^0(BG) \mid xy = 0 \}.
\]
Lemma 6.3.
- \(\text{ann}(X)\) is always an ideal.
- If \(X\) is an ideal then \(X^\perp = \text{ann}(X)\) and so \(X^\perp\) is an ideal.
- If \(X\) is a summand in \(E^0(BG)\) then so is \(X^\perp\), and \(X^\perp\perp = X\).
- If \(X\) is both an ideal and a summand, then so is \(\text{ann}(X)\), and \(\text{ann}(\text{ann}(X)) = X\).

Proof. Straightforward. \(\square\)

Proposition 6.4. Let \(\phi : H \to G\) be a functor between even finite groupoids, and use the resulting ring map \(\phi^* : E^0(BG) \to E^0(BH)\) to regard \(E^0(BH)\) as a module over \(E^0(BG)\).

(a) The sets \(\ker(\phi^*), \text{image}(\phi^*), \ker(\phi^*)^\perp\) and \(\text{image}(\phi^*)^\perp\) are ideals in \(E^0(BG)\).
(b) The sets \(\ker(\phi), \text{image}(\phi^*), \ker(\phi^*)\) and \(\text{image}(\phi^*)\) are \(E^0(BG)\)-submodules of \(E^0(BH)\).
(c) We always have \(\ker(\phi^*) = \text{image}(\phi^*)^\perp\) and \(\ker(\phi^*) = \text{image}(\phi^*)\) are ideals in \(E^0(BH)\).
(d) Suppose that either \(\text{image}(\phi^*)\) or \(\text{image}(\phi)\) is a summand. Then all the sets mentioned in (a) and (b) are summands, and the inequalities in (c) are actually equalities.

Proof. Frobenius reciprocity says that \(\phi^*\) is \(E^0(BG)\)-linear, and \(\phi^*\) is also \(E^0(BG)\)-linear by construction. Claims (a) and (b) follow from this, together with Lemma 6.3.

Now suppose that \(b \in E^0(BG)\). We have \(b \in \text{image}(\phi^*)^\perp\) iff \(\langle \phi^*(a), b \rangle_G = 0\) for all \(a\), or equivalently \(\langle a, \phi^*(b) \rangle_H = 0\) for all \(a\). As the pairing on \(E^0(BH)\) is perfect, this means that \(\phi^*(b) = 0\). We now see that \(\text{image}(\phi^*) = \ker(\phi^*)\), and it follows that \(\text{image}(\phi) \leq \text{image}(\phi^*)^\perp = \ker(\phi^*)\). This proves half of (c), and the other half can be proved by the same method.

Now suppose that the module \(M = \text{image}(\phi^*)\) is a summand in \(E^0(BH)\), so we can choose a splitting \(E^0(BH) = M \oplus N\). This in particular means that \(M\) is a free \(E^0\)-module, so the epimorphism \(\phi^* : E^0(BG) \to M\) can be split. This means that \(\ker(\phi^*)\) is a summand, and we can identify \(\phi^*\) with the projection \(\ker(\phi^*) \oplus M \to M\) followed by the inclusion \(M \to M \oplus N\). It follows that the dual map \((\phi^*)^\vee\) can be identified with the projection \(M^\vee \oplus N^\vee \to M^\vee\) followed by the inclusion \(M^\vee \to \ker(\phi^*)^\vee \oplus M^\vee\), so the kernel and image of \((\phi^*)^\vee\) are summands. However, we can use the inner products and the adjunction formula to identify \(\phi^*\) with \((\phi^*)^\vee\), so \(\ker(\phi)\) and \(\text{image}(\phi)\) are summands. Recall also that if \(X\) is a summand then so is \(X^\perp\) and we have \(X^{\perp\perp} = X\). All claims are now clear for the case where \(\text{image}(\phi^*)\) is a summand. The other case (where \(\text{image}(\phi)\) is a summand) is similar. \(\square\)

Example 6.5. For an example where the above conditions do not hold, take \(p = 2\) and let \(G\) be the quaternion group of order 8. Let \(H\) be the centre, which has order 2. Then \(H\) has a single nontrivial linear character \(\delta\), with Euler class \(t\) say, so \(E^0(BH) = E^0[t]/[2](t)\). On the other hand, \(G/H\) has linear characters \(\alpha, \beta, \gamma\), with Euler classes \(x, y, z\) and \(x = x + x y + y z\).

\[
E^0(B(G/H)) = E^0[x, y]/([2](x), [2](y)) = E^0[x, y, z]/([2](x), [2](y), [2](z), x + y + z).
\]

We can regard \(\alpha, \beta\) and \(\gamma\) as characters of \(G\), and thus we can regard \(x, y\) and \(z\) as elements of \(E^0(BG)\). In \(E^0(B(G/H))\) we have \(xyz = 0\). However, on any
abelian subgroup of $G$ at least one of $\alpha$, $\beta$ and $\gamma$ will vanish, so $xyz$ will be zero. Generalised character theory tells us that the restriction maps to abelian subgroups are jointly injective, so $xyz = 0$ in $E^0(BG)$. We also have $z = x + F y = x - F y$ and so $z$ is a unit multiple of $x - y$ so $xy(x - y) = 0$.

The group $G$ has one more nontrivial irreducible character $\rho$ of dimension two, which satisfies $\rho|_H = 2\delta$. We write $e$ for the Euler class of $\rho$. It is known that $E^0(BG)$ is generated by $x$, $y$ and $e$, subject to relations that we will not record here. Restriction to $H$ sends $x$, $y$ and $z$ to zero, and $e$ to $t^2$. Because $[2](t)$ contains both odd and even powers of $t$, it is not easy to read off directly the subring of $E^0(BH)$ generated by $t^2$. However, it is easy to see that the map $[\Theta^*, H] \to [\Theta^*, G]$ is injective, so the restriction map is rationally surjective. On the other hand, in Morava $K$-theory we have $K^0(BH) = \mathbb{F}_2[t]/t^{2^2}$, so it is clear that $t^2$ generates a proper subring, which we will call $R$. As the rational $E$-theory image and Morava $K$-theory image have different ranks, we see that the $E$-theory image is not a summand. We have also shown elsewhere that when $n > 1$ the Frobenius form $\theta: K^0(BH) \to K^0$ sends $t^{2^n} - 1$ to 1 and all other powers $t^n$ to zero. Using this we see that $R = R^\perp$. On the other hand, the Morava $K$-theory analogue of Proposition 6.6.2 tells us that $R^\perp = \ker(\tr: K^0(BH) \to K^0(BG))$. This means that $\tr(R) = 0$, and in particular the element $s_{G,H} = \tr(1) \in K^0(BG)$ is zero. Alternatively, it will follow from Lemma 6.13 below that $\tr(1)$ is the same as the pullback along $\pi: G \to G/H$ of the class $s_{G,H}$. We have seen that the map $\pi^*$ is not injective, and it follows by a standard argument that it must send the socle to zero, so we again have $s_{G,H} = \pi^*(s_{G/H}) = 0$. More explicitly, we have $s_{G/H} = (xy)^{2^n-1}$. In $K^0(BG)$ we have mentioned that $xy(x - y) = 0$, and we can multiply this by $(xy)^{2^n-3}y$ to get $(xy)^{2^n-1} = 0$.

**Remark 6.6.** The above example can be partially generalised as follows. Let $G$ be any even finite group, and let $H$ be an abelian normal $p$-subgroup. Consider the following conditions:

(a) $H$ is central in $G$.
(b) The map $\Hom(\Theta^*, H) = [\Theta^*, H] \to [\Theta^*, G]$ is injective.
(c) The map $\mathbb{Q} \otimes E^0(BG) \to \mathbb{Q} \otimes E^0(BH)$ is surjective.
(d) The map $E^0(BG) \to E^0(BH)$ is surjective.
(e) The map $K^0(BG) \to K^0(BH)$ is surjective.
(f) $H \cap [G, G] = 1$.

Using generalised character theory we see that (a), (b) and (c) are equivalent. Standard arguments also show that (d) and (e) are equivalent, and it is clear that they imply (a) to (e). We also claim that (f) implies (d) and so implies all of (a) to (e)). Indeed, if (f) holds then we have an inclusion $H \to G/[G, G]$ of abelian groups, which gives an epimorphism $G^* = (G/[G, G])^* \to H^*$ of character groups. As $E^0(BH)$ is generated by Euler classes of characters, it follows that (d) holds. From these arguments we see that (f) implies (a); this can also be shown by a direct group-theoretic argument, as the normality of $H$ implies $[G, H] \leq H \cap [G, G]$. We suspect that (d) implies (f) but we have not proved this. However, (a) does not imply (f), as we can see by considering the group of $3 \times 3$ upper unitriangular matrices, for example.

**Corollary 6.7.** Let $\phi: G \to \mathcal{H}$ be a functor such that the resulting map $\phi^*: E^0(BG) \to E^0(BH)$ is surjective. Then
(a) The image of \( \phi_n : E^0(BH) \to E^0(BG) \) and the kernel of \( \phi^* : E^0(BG) \to E^0(BH) \) are both ideals and summands in \( E^0(BG) \), and they are annihilators of each other.

(b) The image of \( \phi_n \) is the principal ideal generated by \( \phi_n(1) \), so \( \ker(\phi^*) \) is also the annihilator of \( \phi_n(1) \).

(c) The map \( \phi_n \) is a split monomorphism of \( E^0 \)-modules.

Proof. As \( \phi^* \) is surjective, the image is certainly a summand, so most claims are immediate from Proposition 6.4. Note also that for \( a \in E^0(BH) \) we can choose \( b \in E^0(BG) \) with \( \phi^*(b) = a \), and this gives \( \phi(a) = \phi(\phi^*(b).1) = b\phi(1) \); this proves that \( \text{image}(\phi_n) = E^0(BG).\phi(1) \) and so \( \text{ann}(\phi_n(1)) = \text{ann}(\text{image}(\phi_n)) = \ker(\phi^*) \). □

Remark 6.8. The corollary clearly applies if there is a functor \( \psi : G \to H \) with \( \psi \phi \simeq 1 : H \to H \). In particular, it applies if \( H \simeq 1 \).

Definition 6.9. Suppose that \( \mathcal{H} \) is a subgroupoid of \( \mathcal{G} \) such that the map \( \pi_0(\mathcal{H}) \to \pi_0(\mathcal{G}) \) is bijective. We write \( \text{soc}_{\mathcal{G},\mathcal{H}} = i_!(1) \), where \( i \) is the inclusion. For any \( \mathcal{G} \) we can regard the set \( \pi_0(\mathcal{G}) \) as a groupoid with only identity morphisms. We then have a functor \( \mathcal{G} \to \pi_0(\mathcal{G}) \), and this has a section \( j : \pi_0(\mathcal{G}) \to \mathcal{G} \), which is unique up to natural isomorphism. We call \( j(\pi_0(\mathcal{G})) \) the spine of \( \mathcal{G} \). We define \( \text{soc}_{\mathcal{G}} = j_!(1) \).

Remark 6.10. Using Corollary 6.7 we see that if \( i^* \) is surjective, then \( \text{soc}_{\mathcal{G},\mathcal{H}} \) generates the annihilator of the kernel of \( i^* \). We call that annihilator the relative socle. In particular, \( \text{soc}_{\mathcal{G}} \) generates the annihilator of the kernel of the map \( E^0(BG) \to \text{Map}(\pi_0(\mathcal{G}), E^0) \), which we call the socle. We also make the obvious parallel definitions in Morava K-theory (as opposed to \( E \)-theory).

Remark 6.11. Our definitions are only really standard in the case of \( K^0(B\mathcal{G}) \) where \( |\pi_0(\mathcal{G})| = 1 \), so \( K^0(B\mathcal{G}) \) is a zero-dimensional local ring, and the socle is the annihilator of the maximal ideal.

Lemma 6.12. Let \( G \) be a finite group and let \( H \) be a subgroup. Then \( \text{res}^G_H(\text{soc}_{\mathcal{G}}) = |G/H| \text{soc}_H \). In particular, if \( |G/H| \) is not divisible by \( p \) then \( \text{res}^G_H(\text{soc}_{\mathcal{G}}) \) is an invertible multiple of \( \text{soc}_H \).

Proof. In the double coset formula for \( \text{res}^G_H(\text{tr}^G_H(1)) \) the terms are indexed by the set \( 1 \setminus G/H = G/H \), and it is easy to see that all terms are equal to \( s_H \). □

Lemma 6.13. Suppose that \( H \) is a normal subgroup of \( G \) with quotient \( Q = G/H \), and let \( \pi : G \to Q \) be the projection. Then \( \text{soc}_{\mathcal{G},H} = \pi^*(\text{soc}_{\mathcal{Q}}) \).

Proof. Let \( P \) be the homotopy pullback of the inclusion \( 1 \to Q \) and the projection \( \pi : G \to Q \). Then the object set of \( P \) is \( Q \), with morphisms \( P(x,y) = \{ g \in G \mid \pi(g)x = y \} \). All objects of \( P \) are isomorphic, so it is equivalent to the group \( P(1,1) = H \). We can therefore apply the Mackey property for this pullback to the element \( 1 \in E^0(B1) \); this gives \( \text{tr}^G_H(1) = \pi^*(\text{tr}^Q_H(1)) \) as required. □

Lemma 6.14. Suppose that \( H \leq G \) and \( t \in E^0(BG) \) with \( \text{res}^G_H(t) = \text{soc}_H \); then \( \text{soc}_{\mathcal{G}} = t \text{soc}_{\mathcal{G},H} \).

Proof.

\[
t \text{soc}_{\mathcal{G},H} = t \text{tr}^G_H(1) = \text{tr}^G_H(\text{res}^G_H(t)) = \text{tr}^G_H(\text{soc}_H) = t \text{tr}^G_H(1) = t \text{tr}^G_1(1) = t \text{tr}^G_1(1) = \text{soc}_G.
\]
Lemma 6.15. Suppose that $H$ is normal in $G$ and that $Q$ is a subgroup of $G$ that maps isomorphically to $G/H$ (so $G$ is a semidirect product). Then $s_G = s_{G,H} s_{G,Q}$.

Proof. We have $Q \backslash G/H = 1$ and $Q \cap H = 1$ so the double coset formula for the element $\text{res}_H^G(s_{G,Q}) = \text{res}_H^G(\text{tr}_Q^G(1))$ reduces to $\text{tr}_H^G(\text{res}_Q^G(1)) = s_H$. Thus, the claim follows from Lemma 6.14.

7. The theorem of Tanabe

We next recall the outline of Tanabe’s proof of Theorem 1.1 and give an alternative approach to part of the argument, which may be of independent interest.

Proposition 7.1. Suppose that $D \in \text{Div}^*_d(\mathbb{H}^\Gamma)$, and that $a \in D$. Then $\prod_{k=1}^d x((q^k - 1)a) = 0$.

Remark 7.2. In any context where we can use generalised character theory the proof is easy: the $\Gamma$-orbit of $a$ must have size $k$ with $1 \leq k \leq d$, and then $(q^k - 1)a = 0$. However, the proposition implicitly refers to the situation where $D$ and $a$ are defined over an arbitrary $E$-algebra, which may have torsion and nilpotents, so the generalised character theory will not be applicable.

Proof. Let $Y_k$ denote the scheme of pairs $(D,a)$ where $D \in \text{Div}^*_d(\mathbb{H}^\Gamma)$ and $\sum_{i=0}^{k-1} [q^ia] \leq D$. More explicitly, we start with the ring

$$R = E[[c_1, \ldots, c_d]]/(r_1, \ldots, r_d),$$

where $r_i = \phi^*(c_i) - c_i$. Put $f(t) = \sum_{i=0}^d c_it^{d-i} \in R[[t]]$. We then have $Y_1 = \text{spf}(Q_1)$, where $Q_1 = R[x]/f(x)$. The claim is that $\prod_{k=1}^d (q^k - 1)(x)$ is zero in $Q_1$.

More generally, we put $g_k(t) = \prod_{i=0}^{k-1} (t - [q^i](x))$ (so $g_0(t) = 1$ and $g_1(t) = t - x$). Let $f_k(t)$ and $s_k(t)$ be the quotient and remainder when we divide $f(t)$ by $g_k(t)$, so $s_k(t)$ is a polynomial in $t$ of degree less than $k$ over $R[x]$. Let $J_k$ be the ideal generated by the coefficients of $s_k(t)$ and put $Q_k = R[x]/J_k$; we find that $Y_k = \text{spf}(Q_k)$. Now put $m_k = f_k([q^k](x))$ (so $m_0 = f(x)$). We claim that $J_k = (m_k | i < k)$ and that $[q^k - 1](x)m_k \in J_k$.

To prove this, it is convenient to replace $[q^k-a]$ by $[q^k-a]$ in more geometric language. Consider a point $(D,a) \in Y_k$. Put $E \leq D$ by hypothesis, so there is a unique $D'$ with $D = E + D'$. The divisors $E$ and $D'$ have equations $g_k(t)$ and $f_k(t)$, respectively. By construction, we have $(D,a) \in Y_{k+1}$ iff $q^ka \in D'$ iff $m_k = f_k([q^k](a)) = 0$. This shows that $J_{k+1} = J_k + (m_k)$. It follows inductively that $J_l = (m_i | i < l)$ for all $l$ as claimed.

Next note that $\phi(D) = \phi(E) + \phi(D')$, but $\phi(D) = D$ and $\phi(E) = \sum_{i=0}^{k-1} [q^ia]$. If we put $E_1 = \sum_{i=0}^{k-1} [q^ia]$, this becomes $E_1 + [a] + D' = E_1 + [q^ka] + \phi(D')$, so $[a] + D' = [q^ka] + \phi(D')$, so $q^ka \in [a] + D'$, so $([q^k](x) - x) f_k([q^k](x)) = 0$. Recall that $y - x$ is a unit multiple of $y - x$ in $E[y,x]$, so $([q^k](x) - x)$ is a unit multiple of $[q^k - 1](x)$. Moreover, $f_k([q^k](x))$ is just $m_k$, so we have $[q^k - 1](x)m_k = 0$. This was under the assumption that $(D,a) \in Y_k$, so we have really only proved that $[q^k - 1](x)m_k \in J_k$. As $J_{k+1} = J_k + (m_k)$ we see that $[q^k - 1](x)J_{k+1} \leq J_k$. This proves that the element $u = \prod_{k=1}^d [q^k - 1](x)$ satisfies $uJ_{d+1} \leq J_1$. However, as $D$ has degree $d$ we see that $Y_{d+1} = \emptyset$ so $1 \in J_{d+1}$ so $u \in J_1$ as required.
Corollary 7.3. The ring of functions on $\text{Div}_d^+(\mathbb{H})^F$ is a finitely generated free module over $E^0$. Moreover, the sequence $r_1, \ldots, r_d$ is regular in the ring of functions on $\text{Div}_d^+(\mathbb{H})$.

Proof. Put $Y = \text{Div}_d^+(\mathbb{H})^F$ and

$$Z = \{ a \in \mathbb{H}^d | \sum_i [a_i] \in Y \},$$

so $O_Z$ is a quotient of $E^0[x_1, \ldots, x_d]$. Put $g(x) = \prod_{i=0}^{d-1} [q^i - 1](x)$; the proposition tells us that $g(x_i) = 0$ in $O_Z$ for all $i$. It is standard that $[q^i - 1](x)$ is a unit multiple of a Weierstrass polynomial of degree $p^{i\nu}$, where $v = v_p(q^i-1) = r + v_p(i)$, and thus that $g(x)$ is also a unit multiple of a Weierstrass polynomial, so $E^0[x]/g(x)$ is a finitely generated free module over $E^0$. It follows that $O_Z$ is also a finitely generated module over $E^0$. Next, the map $\mathbb{H}^d \to \mathbb{H}^d/\Sigma_d = \text{Div}_d^+(\mathbb{H})$ is a finite flat map of degree $d!$, so $O_Z \simeq O_Y^{d!}$ as $E^0$-modules, so $O_Y$ is also a finitely generated module over $E^0$. It follows that the quotient $O_Y/I_n$ is finitely generated as a module over $E^0/I_n = \mathbb{F}_p$ (and we can use the divisor $d[0]$ to see that it is nontrivial). In other words, if we start with the complete regular local ring

$$E^0(BG_d) = \mathbb{Z}_p[u_1, \ldots, u_{n-1}, c_1, \ldots, c_d]$$

(of Krull dimension $n + d$) and kill the sequence $r_1, \ldots, r_d, u_0, \ldots, u_{n-1}$ (of length $n + d$) we obtain a quotient ring of Krull dimension zero. By a standard result in commutative algebra [6, Section 14] it follows that the sequence is regular. This in turn means that $r_1, \ldots, r_d$ is regular on $E^0(BG_d)$, and that the sequence $u_0, \ldots, u_{n-1}$ is regular on the quotient $O_Y = E^0(BG_d)/(r_1, \ldots, r_d)$, which implies that $O_Y$ is free as an $E^0$-module.

We now recall the structure of Tanabe’s proof. It is a basic fact in étale homotopy theory that there is a homotopy cartesian square

$$\begin{array}{ccc}
BG_d & \longrightarrow & B\bar{G}_d \\
\downarrow & & \downarrow (1,1) \\
B\bar{G}_d & \longrightarrow & B(\bar{G}_d \times \bar{G}_d).
\end{array}$$

Tanabe shows that this gives a spectral sequence of Eilenberg-Moore type:

$$\text{Tor}^K_{**}(BG_d, K^*(BG_d)) \Rightarrow K^*(BG_d).$$

Corollary 7.3 (which Tanabe proved in a slightly different way) can be used to show that the higher Tor groups are zero, so the spectral sequence collapses, and the main claim follows from this.

8. The Atiyah-Hirzebruch spectral sequence

We now turn to the case of $BV$ rather than $B\mathbb{F}$, where the Atiyah-Hirzebruch spectral sequence has many differentials. We will first define and analyse a certain spectral sequence given by an explicit algebraic construction. We will then prove that the Atiyah-Hirzebruch spectral sequence

$$H_*(BV; K_*) \Rightarrow K_*(BV)$$

is isomorphic to our model spectral sequence. As is usual with spectral sequence calculations, this will not give unambiguous generators for $K_*(BV)$. However, it
will enable us to prove that $K_0(BV)$ is a polynomial algebra, which will be a crucial piece of information that we need to prove the completeness of generators and relations that we can produce by other means.

We will need the following general construction.

**Lemma 8.1.** Let $A_*$ be a differential graded algebra over $\mathbb{F}_p$ with a differential $\delta$ of odd degree. Then there is a ring homomorphism $\phi: A_{\text{even}} \to H(A)_{\text{even}}$ given by $\phi(a) = [a^p]$. There is also an additive map $\alpha: A_{\text{even}} \to H(A)_{\text{odd}}$ given by $\alpha(a) = [a^{p-1}da]$, which satisfies $\alpha(ab) = \phi(a)\alpha(b) + \alpha(a)\phi(b)$.

**Proof.** The key point is the additivity of $\alpha$. For this, consider the ring $\tilde{A} = P[\mathbb{Z}/2, b] \otimes E_2[da, db]$ with the obvious differential, graded over $\mathbb{Z}/2$ with $|a| = |b| = 0$ and $|da| = |db| = 1$. Put $\tilde{\alpha}(x) = x^{p-1}dx$, so $\alpha(x) = [\tilde{\alpha}(x)]$. Put $c = ((a+b)^p - ap - bp)/p$, which is well-known to lie in $\tilde{A}$. We find that

$$p(\tilde{\alpha}(a + b) - \tilde{\alpha}(a) - \tilde{\alpha}(b)) = d((a+b)^p - ap - bp) = pdc,$$

and $\tilde{A}$ is torsion-free so $\tilde{\alpha}(a + b) = \tilde{\alpha}(a) + \tilde{\alpha}(b) + dc$. This equation in $\tilde{A}$ gives an equation in $A/p$, but that is a universal example so it holds for any DGA over $\mathbb{F}_p$. It follows that $\alpha(a + b) = \alpha(a) + \alpha(b)$ as claimed. All other claims are straightforward. ~

**Proposition 8.2.** There is an Atiyah-Hirzebruch spectral sequence

$$E_2^* = H^*(BV(k)_{1}; K^*) = P[u^{\pm1}, x] \otimes E[a] \Longrightarrow K^*(BV(k)_{1}),$$

with $u \in E^0_{2r-2}$ and $x \in E^0_{2r}$ and $a \in E^1_{2r}$. The differential $d^r: E_r \rightarrow E_{r+1}$ is zero except when $r = 2p^{n(r+k)} - 1$. In that case we have $d^r(u) = u^{-1}(ux)p^{n(r+k)}$ and $d^r(x) = 0$ and $d^r(u) = 0$.

**Proof.** Recall that $GL_1(F(k))$ is cyclic of order $q^k - 1$, so the $p$-torsion part has order $p^{r+k}$. It follows that

$$K^*(BV(k)_1) = K^*[x_K]/(u^{p^{n(r+k)}}) = K^*[x_K]/x_K^{p^{n(r+k)}}.$$

It is well-known and easy to see that there is only one possible pattern of AHSS differentials that is compatible with this: we must have $d^2p^{n(r+k)} - 1(a) = t u^{-1}(ux)p^{n(r+k)}$ for some $t \in \mathbb{F}_p \times$. The value of $t$ is not really important, but it simplifies bookkeeping if we can pin it down. In Appendix A we will check that $t = 1$. ~

We now consider the dual homological spectral sequence. Recall that the elements $b_{1}^{(k)} \in H_{2i}(BV(k)_1)$ and $e_{1}^{(k)} \in H_{2i+1}(BV(k)_1)$ are dual to $x^i$ and $x^i a$, respectively. The element $u \in K_2(S^0) = K^{-2}(S^0)$ has degree $-2$ in the previous cohomological spectral sequence, but it has degree 2 in the homological version. Recall also that we put $b^{(k)}(s) = \sum_i b_{1}^{(k)}(s^i)$ and $e(s) = \sum_i e_{1}^{(k)}(s^i)$.

**Corollary 8.3.** The Atiyah-Hirzebruch spectral sequence

$$H_*(BV(k)_1; K_*) = K_*\{b_{1}^{(k)} \mid i \geq 0\} \oplus K_*\{e_{1}^{(k)} \mid i \geq 0\} \Longrightarrow K_*(BV(k)_1)$$

has

$$d_2p^{n(r+k)} - 1(b_{i}^{(k)}) = \begin{cases} u^{p^{n(r+k)} - 1} e_{i - p^{n(r+k)}} & \text{if } i \geq p^{n(r+k)} \\ 0 & \text{if } i < p^{n(r+k)} \end{cases}$$

and $d_2p^{n(r+k)} - 1(e_{i}) = 0$.~
Moreover, all other differentials are zero.

Proof. Dualise Proposition [8.2]

**Corollary 8.4.** We have
\[
\begin{align*}
\text{d}_{2p^n(r+k)-1}(b^{(k)}(s)) &= u^{-1}(us)^{p^n(r+k)}e^{(k)}(s) \\
\text{d}_{2p^n(r+k)-1}(e^{(k)}(s)) &= 0,
\end{align*}
\]
and all other differentials are zero.

Proof. This is a straightforward translation of the previous result into the language of formal power series. □

**Corollary 8.5.** In the Atiyah-Hirzebruch spectral sequence
\[
E^2_{s*} = H_*(BV; K_*) = K_* \otimes P[b_i \mid i \geq 0] \otimes E[e_i \mid i \geq 0] \Longrightarrow K_*(BV),
\]
the series \(b(s)^{p^k}\) survives to \(E^{2p^n(k+r)-1}\) where we have
\[
\begin{align*}
\text{d}_{2p^n(k+r)-1}(b(s)^{p^k}) &= u^{-1}(us)^{p^n(k+r)}b(s)^{p^k-1}e(s).
\end{align*}
\]

Proof. The map \(\rho : V(k)_1 \to V_{p^k} \subset V\) gives a morphism of spectral sequences, so the claim follows from Proposition [4.17]. □

**Definition 8.6.** As in the introduction, we put \(N_0 = p^n r\), and for \(k > 0\) we put
\[
N_k = p^{(n-1)k+n(r-1)}(p^n - 1).
\]
Next, for \(k \geq 0\) we put \(\bar{N}_k = \sum_{j=0}^k N_j\) and \(N_k^* = \bar{N}_k - p^{nk+nr-k}\). We then put
\[
N_k = \begin{cases} 
\{i \in \mathbb{N} \mid i < \bar{N}_0\} & \text{if } k = 0 \\
\{i \in \mathbb{N} \mid \bar{N}_{k-1} \leq i < \bar{N}_k\} & \text{if } k > 0,
\end{cases}
\]
so \(|N_k| = N_k\) and \(\mathbb{N} = \bigsqcup_k N_k\). Let \(s_i\) be the index such that \(i \in N_{s_i}\).

**Lemma 8.7.** For all \(k \geq 0\) we have
\[
\begin{align*}
\text{(A)} & \quad p^k N_k = \begin{cases} 
p^{nr} & \text{if } k = 0 \\
p^{n(r+k)} - p^{n(r+k-1)} & \text{if } k > 0
\end{cases} \\
\text{(B)} & \quad \sum_{j=0}^k p^j N_j = p^{n(r+k)} \\
\text{(C)} & \quad p^k(\bar{N}_k - N_k^*) = p^{n(k+r)} \\
\text{(D)} & \quad p^{k+1}(\bar{N}_k - N_k^*) = p^{n(k+r)} \\
\text{(E)} & \quad p^{k+1}N_{k+1} = (p-1)p^k \bar{N}_k + p^k N_k^*.
\end{align*}
\]

Proof. Straightforward expansion of the definitions gives (A), and (B) follows by induction. Equation (C) is immediate from the definition of \(N_k^*\). Now replace \(k\) by \(k+1\) in (C) and substitute \(\bar{N}_{k+1} = N_{k+1} + \bar{N}_k\) to get
\[
p^{k+1}N_{k+1} + p^{k+1}(\bar{N}_k - N_k^*) = p^{n(k+1+r)}.
\]
We can use (A) to rewrite \(p^{k+1}N_{k+1}\) as \(p^{n(k+1+r)} - p^{n(k+r)}\) and rearrange to get (D). We can then subtract (D) from (C) and rearrange to get (E). □
Remark 8.12. We define a sequence of trigraded rings $S[k]$ as follows. There is an invertible generator $u$ with $|u| = (0, 2, 0)$.

(a) For $0 \leq m < k$ and $i \in N_m$ we have a polynomial generator $b_{ni}$ with degree $(2p^m i, 0, p^m)$. Note here that the condition $i \in N_m$ is equivalent to $s_i = m$ or $N_m - 1 \leq i < N_m$. As $m < k$ this implies $i < N_{k-1}$.

(b) For $i \geq N_{k-1}$ we have a polynomial generator $b_{ki}$ with degree $(2p^k i, 0, p^k)$.

(c) For $i \geq N_k^*$ we have an exterior generator $e_{ki}$ with degree $(2p^k i + 1, 0, p^k)$.

In the case $k = 0$ we take $N_{-1}$ to be 0, and note that $N_0^* = 0$. We write $b_i$ and $e_i$ for $b_{0i}$ and $e_{0i}$, so

$$S[0] = P[b_i \mid i \geq 0] \otimes E[e_i \mid i \geq 0],$$

with $|b_i| = (2i, 0, 1)$ and $|e_i| = (2i + 1, 0, 1)$. We also define $S[\infty]$ to be the polynomial algebra on generators $b_{mi}$ for all $m \geq 0$ and $i \in N_i$, with no exterior generators.

Remark 8.9. According to the above definition, $b_{ni}$ is defined as a generator of $S[k]$ only when $n = \min(s_i, k)$. However, we will extend the notation by putting $b_{ni} = b_{n_i}^{p^m}$ for $m \geq n = \min(s_i, k)$. This can be restated as follows. First suppose that we fix $i$.

(a) If $s_i < k$ (or equivalently $i < N_{k-1}$) then $b_{ni}$ is defined for $n \geq s_i$, and is a generator iff $n = s_i$.

(b) If $s_i \geq k$ (or equivalently $i \geq N_{k-1}$) then $b_{ni}$ is defined for $n \geq k$, and is a generator iff $n = k$.

Suppose instead that we fix $n$.

(c) If $n < k$ then $b_{ni}$ is defined for all $i < N_n$, and is a generator iff $N_{n-1} \leq i < N_n$.

(d) If $n = k$ then $b_{ni}$ is defined for all $i$, and is a generator iff $i \geq N_{k-1}$.

(e) If $n > k$ then $b_{ni}$ is defined for all $i$, and is never a generator.

Whenever $b_{ni}$ is defined we have $|b_{ni}| = (2p^m i, 0, p^m)$. We also take $e_{ki} = 0$ for $i < N_k^*$.

Remark 8.10. We will need the usual kind of sign rules, which depend on the total degree of the elements involved. For an element of degree $(i, j, k)$, we officially take $i + j$ to be the total degree. However, in all cases that we consider $j$ will be even, so the signs will only depend on the parity of $i$.

Definition 8.8. We define a sequence of trigraded rings $S[k]$ as follows. There is an invertible generator $u$ with $|u| = (0, 2, 0)$.

(a) For $0 \leq m < k$ and $i \in N_m$ we have a polynomial generator $b_{ni}$ with degree $(2p^m i, 0, p^m)$. Note here that the condition $i \in N_m$ is equivalent to $s_i = m$ or $N_m - 1 \leq i < N_m$. As $m < k$ this implies $i < N_{k-1}$.

(b) For $i \geq N_{k-1}$ we have a polynomial generator $b_{ki}$ with degree $(2p^k i, 0, p^k)$.

(c) For $i \geq N_k^*$ we have an exterior generator $e_{ki}$ with degree $(2p^k i + 1, 0, p^k)$.

In the case $k = 0$ we take $N_{-1}$ to be 0, and note that $N_0^* = 0$. We write $b_i$ and $e_i$ for $b_{0i}$ and $e_{0i}$, so

$$S[0] = P[b_i \mid i \geq 0] \otimes E[e_i \mid i \geq 0],$$

with $|b_i| = (2i, 0, 1)$ and $|e_i| = (2i + 1, 0, 1)$. We also define $S[\infty]$ to be the polynomial algebra on generators $b_{mi}$ for all $m \geq 0$ and $i \in N_i$, with no exterior generators.
Proposition 8.13. The map $\delta_k$ satisfies $\delta_k^2 = 0$, so we have a homology ring $H(S[k]; \delta_k) = \ker(\delta_k) / \text{image}(\delta_k)$. Moreover, there is an isomorphism $\theta_k : S[k+1] \to H(S[k]; \delta_k)$ of trigraded algebras over the ring $K_\ast = P[u,u^{-1}]$, defined as follows:

(a) If $m \leq k$ and $i \in \mathcal{N}_m$ then $\theta_k(b_{mi}) = [b_{mi}]$.
(b) If $i \geq \tilde{N}_k$ then $\theta_k(b_{k+1,i}) = [b_{ki}]$.
(c) If $i \geq 0$ then $\theta_k(e_{k+1,i+N_k^*+1}) = [b_{k+1,i+N_k^*+1}]$.

Also, if we interpret $b_{k+1,i} \in S[k+1]$ and $b_{ki} \in S[k]$ as in Remark 8.9, then the relation $\theta_k(b_{k+1,i}) = [b_{ki}]$ is valid for all $i \geq 0$.

Proof. The graded Leibniz rule for $\delta_k$ shows that $\ker(\delta_k^2)$ is a subring of $S[k]$, and it contains all the generators $b_{mi}$, $e_{ki}$, and $u$, so it is the whole of $S[k]$. Thus, the homology ring is defined. Now let $A$ be the subring of $S[k]$ generated over $K_\ast$ by $\{b_{mi} \mid m \leq k, \ i \in \mathcal{N}_m \}$, and let $B_i$ be the subring generated over $K_\ast$ by $\{b_{k+1,i+N_k^*}, e_{k+1,i+N_k^*} \}$, so $S[k]$ is the tensor product of all these subrings. Then $\delta_k$ is zero on $A$, and preserves $B_i$. The Küneth theorem therefore tells us that $H_\ast(S[k]; \delta_k)$ is the tensor product of $A$ with all the rings $H(B_i; \delta_k)$. As $\delta_k(b_{k+1,i+N_k^*})$ is an invertible multiple of $e_{k+1,i+N_k^*}$, it is easy to see that $H(B_i; \delta_k) = P[b_i] \otimes E[e_i^+]$, where $b_i = [b_{k+1,i}]$, $e_i = [e_{k+1,i+N_k^*}]$. Thus we have

$$H_\ast(S[k]; \delta_k) = A \otimes P[b_i^+] \otimes E[e_i^+]$$

It is clear that $|b_{k+1,i+N_k^*}| = (2p^{k+1}(i + \tilde{N}_k), 0, p^{k+1}) = |b_i^+|$. Also, one can use Lemma 8.7(E) to check that $|e_{k+1,i+N_k^*+1}| = |e_i^-|$. Claims (a) to (c) now follow.

Note 8.14. Using Lemma 8.1 we define $\alpha_k : S[k]_{\text{even}} \to S[k+1]_{\text{odd}}$ by $\alpha_k(a) = \theta_k^{-1}[a p^i \delta_k(a)]$. We thus have $\alpha_k(b_{ki}) = 0$ for $i < \tilde{N}_k$, and Proposition 8.13(c) gives

$$\alpha_k(b_{k+1,i+N_k^*}) = \theta_k^{-1} a_p^{n(k-i)+1} e_{k+1,i+N_k^*+1}$$

for $i \geq 0$.

Proposition 8.15. The bigraded group $S[k]_{**}$ coincides with $S[\infty]_{**}$ when $p^k > m$.

Proof. All the exterior generators of $S[k]$ lie in $S[k]_{**,p^k}$ and so are irrelevant here. From the definitions we see that the polynomial generators in $S[k]_{**, < p^k}$ are $\{b_{mi} \mid m < k, i \in \mathcal{N}_m \}$, and these are the same as the polynomial generators in $S[\infty]_{**, < p^k}$. The claim is clear from this.

The above essentially says that the rings $S[k]$ and differentials $\delta_k$ give a spectral sequence converging to the polynomial ring $S[\infty]$, but the grading behaviour is different from what one would expect for the Atiyah-Hirzebruch spectral sequence. To fix this we merely need to insert additional pages with trivial differential. The result is as follows:
Corollary 8.16. There is a trigraded spectral sequence $EM_{\ast\ast\ast}^u$ given by

$$EM_{\ast\ast\ast}^u = \begin{cases} S[0] & \text{if } u < 2p^{nr} \\ S[k] & \text{if } 2p^{nr(r+k-1)} \leq u < 2p^{nr+k} \text{ with } k > 0. \end{cases}$$

The differentials $d^u$ have degree $(u, 1 - u, 0)$ and are given by $d^u = \delta_k$ when $u = 2p^{nr+k} - 1$, and $d^u = 0$ in all other cases. Moreover, the spectral sequence converges to $S[\infty]$.

Remark 8.17. Consider elements $x_0, y_0 \in S[0]$. In the ordinary language used to describe spectral sequences, we might say that $x_0$ survives to $S[k]$ and supports a differential $\delta_k(x_0) = y_0$. Because we have defined our pages $S[k]$ as independent objects and introduced the homomorphisms $\theta_k$ explicitly, we need to use slightly different language. The equivalent statement is that there are elements $x_i, y_i \in S[i]$ for $i = 0, \ldots, k$ such that

- For $i < k$ we have $\delta_i(x_i) = 0$ and $[x_i] = \theta_i(x_{i+1})$
- For $i < k$ we have $\delta_i(y_i) = 0$ and $[y_i] = \theta_i(y_{i+1})$
- $\delta_k(x_k) = y_k$.

In Proposition 8.17 we described the transfer maps $H_*(BV(n+m)) \to H_*(BV(n))$ in terms of formal power series. It will be convenient to have similar formulae in our model spectral sequence.

Definition 8.18. We define $b_k(s), e_k(s) \in S[k][s]$ by

$$b_k(s) = \sum_{i \geq 0} b_{ki} s^{p^i}$$

$$e_k(s) = \sum_{i \geq 0} e_{ki} s^{p^i}.$$ 

Recall here that $b_{ki}$ is one of the generators of $S[k]$ when $i \geq \bar{N}_{k-1}$, but is defined using Remark 8.19 (and so is a $p'$th power) when $0 \leq i < \bar{N}_{k-1}$. We use the grading $|s| = (-2, 0, 0)$ so that $|b_k(s)| = (0, 0, p^k)$ and $|e_k(s)| = (1, 0, p^k)$. We define $\delta_k$ and $\theta_k$ on power series by the rule $\delta_k(sa) = s \delta_k(a)$ and $\theta_k(sa) = s \theta_k(a)$; the function $\alpha_k$ then satisfies $\alpha_k(sa) = s^p \alpha_k(a)$.

Lemma 8.19. In $S[k][s]$ we have

$$\delta_k(b_k(s)) = u^{-1}(us)p^{n(k+r)}e_k(s)$$

$$\alpha_k(b_k(s)) = u^{-1}(us)p^{n(k+r)}e_{k+1}(s)$$

$$\theta_k(b_{k+1}(s)) = [b_k(s)p]$$

$$\theta_k(e_{k+1}(s)) = [b_k(s)p^{-1}e_k(s)].$$

Proof. We have $\delta_k(b_k(s)) = \sum_{i \geq 0} \delta_k(b_{ki}) s^{p^i}$. For $i < \bar{N}_{k-1}$ the element $b_{ki}$ is defined by Remark 8.19 and is a $p'$th power, so $\delta_k(b_{ki}) = 0$. For $\bar{N}_{k-1} \leq i < \bar{N}_k$ the element $b_{ki}$ is a polynomial generator of $S[k]$, but we still have $\delta_k(b_{ki}) = 0$ by
Corollary 8.20. Using this and the identity \( p^k(\overline{N}_k - N^*_k) = p^n(k + r) \) we get
\[
\delta_k(b_k(s)) = s^{p_k} \sum_{i \geq 0} \delta_k(b_{k,N_k+i})s^{i} = s^{p_k} \sum_{i \geq 0} e_{k,N_k+i} s^{i} = s^{p_k} - 1 \sum_{i \geq N_k^*} e_{k,i} s^{i} = s^{p_k} - 1 \sum_{i \geq N_k^*} e_{k,i} s^{i} = s^{p_k} - 1 \sum_{i \geq N_k^*} e_{k,i} s^{i} = u^{-1}(s) s^{(k+r)}.
\]
We can analyse \( \alpha_k(b_k(s)) \) in a similar way, remembering the rule \( \alpha_k(s a) = s^p \alpha_k(a) \) and the formulae in Definition 8.14 and the identity \( p^{k+1}(\overline{N}_k - N^*_k) = p^n(k + r) \). We get
\[
\alpha_k(b_k(s)) = s^{p_{k+1} \overline{N}_k} \sum_{i \geq 0} \alpha_k(b_{k,N_k+i})s^{i} = s^{p_{k+1} \overline{N}_k} \sum_{i \geq 0} e_{k+1,N_k+i} s^{i} = s^{p_{k+1} \overline{N}_k} \sum_{i \geq N_k^*} e_{k+1,i} s^{i} = s^{p_{k+1} \overline{N}_k} \sum_{i \geq N_k^*} e_{k+1,i} s^{i} = u^{-1}(s) s^{(k+r)}.
\]
Next, using the last part of Proposition 8.13 we get
\[
\theta_k(b_{k+1}(s)) = \sum_{i \geq 0} \theta_k(b_{k+1,i})s^{i} = \sum_{i \geq 0} b_{k+1,i} s^{i} = b_{k}(s)^{p}.
\]
Finally, for any \( a \) we have \( \theta_k(\alpha_k(a)) = s^p \delta_k(a) \) by definition. We can take \( a = b_k(s) \) and use our formulae for \( \alpha_k(b_k(s)) \) and \( \delta_k(b_k(s)) \) to get \( z \theta_k(e_{k+1}(s)) = \theta_k(e_{k+1}(s)) = [b_k(s)]\theta_k(\gamma_k(s)) \), where \( z = u^{-1}(s) s^{(k+r)} \). This is not a zero divisor, so \( \theta_k(e_{k+1}(s)) = [b_k(s)]\theta_k(\gamma_k(s)) \) as claimed.

Corollary 8.20. In the sense described in Remark 8.14 the series \( b(s)^{p_k} \in S[0][s] \) survives to \( S[k] \) and then supports a differential
\[
b(s)^{p_k} \mapsto u^{-1}(s) s^{(k+r)} b(s)^{p_k-1} e(s).
\]
Proof. For \( 0 \leq i \leq k \), put \( x_i = b_i(s)^{p^{k-i}} \) and \( y_i = u^{-1}(s) s^{(k+r)} b_i(s)^{p^{k-i}} e_i(s) \). These elements have the properties specified in Remark 8.17.

Definition 8.21. We write \( ET^*_{s+d} \) for the Atiyah-Hirzebruch spectral sequence for \( BV \), trigraded so that \( ET^*_{s+d} \) is the Atiyah-Hirzebruch spectral sequence for \( BV_d \):
\[
ET^2_{ij} = H_i(BV_d; K_j) \implies K_{i+j}(BV_d)
\]

Definition 8.22. We put \( PS(V)(t) = \prod_{k \geq 0} (1 - t^k)^{-N_k} \in \mathbb{N}[t] \).

Theorem 8.23. There is an isomorphism \( E_M \to ET \) of trigraded spectral sequences. Thus, \( K_*(BV) \) is isomorphic to \( S[\infty] \) as a bigraded ring (where tridegree \((i,j,d)\) contributes to bidegree \((i+j,d)\)). In particular, \( K_*(BV) \) is a polynomial ring, with Poincaré series
\[
\sum_d \dim K_*(K_*(BV_d)) = PS(V)(t).
\]
Proof. It is clear that the two spectral sequences have the same $E^2$ page, and by comparing Corollary 8.23 with Lemma 8.19 we see that the differentials are the same. It follows in a standard way that the morphism gives an isomorphism on every page, so the $E^\infty$ page of the AHSS is isomorphic to $S[\infty]$. This means that $K_*(BV)$ has a multiplicative filtration whose associated graded ring is a polynomial algebra, and by choosing representatives of the generators it follows that $K_*(BV)$ is isomorphic to the associated graded ring, and thus to $S[\infty]$. Recall that $S[\infty]$ has generators $b_{ki}$ of tridegree $(*,*,p^k)$ for all $i \in N_k$, and $|N_k| = N_k$. Each such generator contributes a factor of $\sum_j t^{i+j} = (1 - t^{p^k})^{-1}$ to the Poincaré series, so the full Poincaré series is $PS(V)(t)$.

The following result can be deduced from the theorem of Tanabe, but we now prove it independently.

**Corollary 8.24.** For each $d$, the ring $E^*(BV_d)$ is a finitely generated free module over $E^*$, and is concentrated in even degrees, and the natural map $E^*(BV_d)/I_n \to K^*(BV_d)$ is an isomorphism. Similarly, $E^0_0(V_0)$ is a finitely generated free module over $E_*$, and the natural map $E^0_0(V_0)/I_n \to K_*(BV_d)$ is an isomorphism. In fact, there are natural isomorphisms

$$E^0_0(V_0)/I_n \cong E^0_0(V_0) \cong Hom_{E_0_0}(E^0_0(V_0), E_0).$$

**Proof.** The theorem shows that $K_*(BV_d)$ is concentrated in even degrees, and is free of finite rank over $K_*$. Given this, everything follows from the results of [141 Section 8].

We next want to show that $E^0_0(V_0)$ and $E^0_0(V_0)$ and $K_0(BV_0)$ are also graded polynomial rings, under the products discussed in Definition 5.19. Recall that these are induced by the direct sum functor $\sigma: V_i \times V_j \to V_{i+j}$, corresponding to the standard inclusion $GL_i(F) \times GL_j(F) \to GL_{i+j}(F)$. Recall also that the homological and cohomological versions are essentially the same, as explained in Remark 5.20.

**Proposition 8.25.** The graded rings $K_*(BV_0)$, $K^*(BV_0)$, $E^*(BV_0)$ and $E_*(BV_0)$ are all polynomial over $K_*$ or $E_*$ as appropriate, and concentrated in even degrees, and in each case the Poincaré series is $PS(V)(t)$. Thus, the module $\text{Ind}_{E_0}(E^0_0(V_0))$ is free of rank $N_k$ over $E^0_0$.

**Proof.** We have already seen that $K_*(V_0)$ is polynomial, so we can choose a system of generators $\{x_i | i \in I\}$. As $K_*(BV_0) = E^0_0(BV_0)/I_n$, we can choose lifts $x_i \in E^0_0(BV_0)$. These give a map $\phi: P_{E_0}(x_i | i \in I) \to E^0_0(BV_0)$, which gives an isomorphism modulo $I_n$. As the source and target of $\phi$ are free and finitely generated in each degree, it follows that $\phi$ itself is an isomorphism. It follows by duality that $K^*(BV_0)$ and $E^*(BV_0)$ are polynomial under the convolution products. Using duality theory and the fact that $K_*(BV_0) = E^0_0(BV_0)/I_n$ we also see that all four rings have the same Poincaré series, namely $PS(V)(t)$.

**Remark 8.26.** We now outline a different approach that one could hope to use to prove that $K_*(BV)$ is polynomial. The work of Tanabe shows that $K^*(BV)$ is a quotient of $K^*(BV)$, so $K_*(BV)$ is a subring of $K_*(BV)$, which is a polynomial ring on even generators. It follows that $K_*(BV)$ is concentrated in even degrees, and that the only nilpotent element is zero. If we knew that $K_*(BV)$ was a Hopf
algebra, with a coproduct that respects gradings in an appropriate way, then we could use the Milnor-Moore structure theorem to show that \( K_*(BV) \) is polynomial. However, as we mentioned in Remark 2.21, the obvious coproduct on \( E^*(BV) \) (derived from the direct sum) does not make \( E^*(BV) \) into a Hopf algebra, and one can check that this problem persists if we use \( K^*(BV) \) or \( K_*(BV) \). If we instead use the diagonal map of the space \( BV \), then we get an alternative coproduct that does make \( K_*(BV) \) into a Hopf algebra, but the grading behaviour is incompatible with the Milnor-Moore theorem. Although we had some success in generalising the Milnor-Moore argument, we were not able to complete a proof along those lines.

We next introduce an element of \( K^0(BV_{p^k}) \) which will turn out to be a generator of the socle. For the moment, we will just prove that it is nonzero.

**Definition 8.27.** We put \( s_k = c_{p^k} \in K^0(BV_{p^k}) \).

**Proposition 8.28.** \( s_k \) is nonzero in \( K^0(BV_{p^k}) \).

**Proof.** We have homological and cohomological Atiyah-Hirzebruch spectral sequences for \( BV \) and \( B\overline{V} \). We denote the final pages by \( E^\infty(V), E^\infty(\overline{V}) \) and \( E^\infty(V) \). Put

\[
v = c_{H,p^k}^{-1} \in H^*(B\overline{V}; K^*) = E_2(\overline{V}) = E_\infty(\overline{V}).
\]

It will suffice to show that \( v \) has nontrivial restriction in \( E_\infty(V) \), or that \( \langle m, v \rangle \neq 0 \) for some \( m \) in the image of the map

\[
E^\infty(V) \to E^\infty(\overline{V}) = K_* \otimes P[b_H^i \mid i \geq 0].
\]

From our earlier analysis, it is clear that this image is the subring generated by classes \( b_H^i \) with \( \bar{N}_k - 1 \leq i < \bar{N}_k \). In particular, the class \( m = b_{H,\bar{N}_k - 1}^k \) lies in this image. Now put \( m' = b_{H,\bar{N}_k - 1}^{p^k} \in H_*(B\overline{V}_{p^k}) \), so the usual map \( \overline{V}_{p^k} \to \overline{V} \) sends \( m \) to \( m' \). The restriction of \( v \) to \( \overline{V}_{p^k} \) is the element \( v' = (x_{H,p^k}^{p^k})^{\bar{N}_k - 1} \), so we have \( \langle m, v \rangle = \langle m', v' \rangle \). However, it is immediate from the definitions that \( \langle m', v' \rangle = 1 \), so the claim follows.

\[ \Box \]

### 9. Generalised character theory

We next recall some ideas about the generalised character theory of Hopkins, Kuhn and Ravenel [3]. We will use the two kinds of duality introduced in Definition 2.23 and Remark 2.24.

**Definition 9.1.** Let \( D \) denote the algebraic extension of \( E_0 \) obtained by adjoining a full set of roots for \( \langle p^k \rangle \langle x \rangle \) for all \( k \geq 0 \). The set \( \Theta \) of all these roots is naturally a group under the formal sum, and an isomorphism \( (\mathbb{Z}/p^\infty)^n \to \Theta \) is built in to the construction. The dual group \( \Theta^* = \text{Hom}(\Theta, \mathbb{Z}/p^\infty) \) is thus identified with \( \mathbb{Z}_p^n \). We regard \( \Theta^* \) as a groupoid with one object. We also put \( D' = D[p^{-1}] \).

We will give the next definition in a form that works well for all groupoids (including \( \overline{V} \), which is not hom-finite). However, we will then show that it simplifies for hom-finite groupoids.

**Definition 9.2.** For any groupoid \( G \), we write \( [\Theta^*, G] \) for the groupoid of functors \( \Theta^* \to G \) that factor through \( \Theta^*/p^k \) for some \( k \geq 0 \). Equivalently, an object of \( [\Theta^*, G] \) consists of an object \( a \in G \) and a homomorphism \( \alpha: \Theta^* \to G(a) \) with
\( \alpha(p^k\Theta^*) = 1 \) for some \( k \). An isomorphism from \( (a, \alpha) \) to \( (b, \beta) \) is a morphism \( u \in \mathcal{G}(a, b) \) such that \( \beta(x) = u\alpha(x)u^{-1} \) for all \( x \in \Theta^* \).

Remark 9.3. The condition \( \alpha(p^k\Theta^*) = 1 \) just means that \( \alpha \) is continuous with respect to the \( p \)-adic topology on \( \Theta^* \) and the discrete topology on \( \mathcal{G}(a) \). We might want to consider cases such as the groupoid of finite-dimensional complex vector spaces, in which \( \mathcal{G}(a) \) has a natural structure as a Lie group. For any Lie group \( G \), there is an open neighbourhood \( U \) of the identity such that the only subgroup contained in \( U \) is the trivial one. Using this, we see that a homomorphism \( \Theta^* \to \mathcal{G}(a) \) is continuous with respect to the Lie topology iff it is continuous with respect to the discrete topology.

Lemma 9.4. Let \( \mathcal{G} \) be a hom-finite groupoid, and suppose we have an object \( a \in \mathcal{G} \) and a homomorphism \( \alpha : \Theta^* \to \mathcal{G}(a) \). Then there always exists \( k \geq 0 \) with \( \alpha(p^k\Theta^*) = 1 \). Thus, \(|\Theta^*, \mathcal{G}|\) is the category of all functors \( \Theta^* \to \mathcal{G} \).

Proof. Put \( A = \alpha(\Theta^*) \leq \mathcal{G}(a) \). As \( \mathcal{G}(a) \) is finite, we see that \( A \) is finite. As \( \Theta^* \) is abelian, we see that \( A \) is abelian. As every integer coprime to \( p \) acts isomorphically on \( \Theta^* \), the same is true of \( A \). It follows that \( A \) is a finite abelian \( p \)-group, so it must have exponent \( p^k \) for some \( k \). This implies that \( \alpha(p^k\Theta^*) = 1 \) as claimed. \( \square \)

Definition 9.5. For any hom-finite groupoid we put

\[
D'_0\mathcal{G} = \bigoplus_{C \in \pi_0(\mathcal{G})} D'_0 \otimes E_0 E_C^0 BC
\]

\[
(D')^0\mathcal{G} = \prod_{C \in \pi_0(\mathcal{G})} D'_0 \otimes E_0 E^0 BC.
\]

If \( \mathcal{G} \) is actually finite, this simplifies to

\[
D'_0\mathcal{G} = D' \otimes E^0 E_C^0 B\mathcal{G}
\]

\[
(D')^0\mathcal{G} = D' \otimes E^0 E^0 B\mathcal{G}.
\]

Theorem 9.6 (Hopkins-Kuhn-Ravenel). For any hom-finite groupoid \( \mathcal{G} \), there are natural isomorphisms

\[
D'_0\mathcal{G} = D'([\Theta^*, \mathcal{G}])
\]

\[
(D')^0\mathcal{G} = \text{Map}([\Theta^*, \mathcal{G}], D')
\]

Proof. For the case of a finite group (or equivalently, a groupoid with only one isomorphism class), this is the main theorem of \( \mathbb{M} \). The general case follows easily. \( \square \)

Remark 9.7. Suppose we have an element \( f \in E^0(B\mathcal{G}) \), and an object \( a \in [\Theta^*, \mathcal{G}] \). We then have an element \( 1 \otimes f \in (D')^0(\mathcal{G}) \), and we can apply the generalised character map to get a function \( [\Theta^*, \mathcal{G}] \to D' \), which we can evaluate at \( a \) to get an element of \( D' \). We will usually just write \( f(a) \) for this character value.

We consider the various groupoids defined in Definition 3.14. We will use the functors \( A^* \) and \( A^# \) from Definition 2.3 and the invertible \( \mathbb{Z}_p \)-module \( T \) introduced in Remark 2.4 so that \( T \otimes A^* \) is naturally isomorphic to \( A^# \).
Definition 9.8. We define \( \Phi \) to be the group of continuous homomorphisms \( \Theta^* \to \mathbb{F}^\times \), or in other words \( \Phi = \Theta^{\ast\#} = T \otimes \Theta \), so \( \Phi^{\#} = \Theta^* \).

Note \( \Phi \) that is a \( p \)-torsion group. We also put \( \Phi[m] = \{ \phi \in \Phi \mid m\phi = 0 \} \) (and observe that this only depends on the \( p \)-adic valuation of \( m \)).

Remark 9.9. A choice of generator \( t \in T \) gives isomorphisms \( \Phi \simeq \Theta = (\mathbb{Z}/p\mathbb{Z})^n \) and \( \Phi^* \simeq \Theta^* = \mathbb{Z}_p^\times \). If we write \( \text{Sub}(\Phi) \) for the set of finite subgroups of \( \Phi \), then the first isomorphism gives a bijection \( \text{Sub}(\Phi) \simeq \text{Sub}(\Theta) \). Any other generator of \( t \) has the form \( t' = tu \) for some \( u \in \mathbb{Z}_p^\times \), and multiplication by \( u \) sends every finite subgroup of \( \Theta \) or \( \Phi \) to itself. Using this, we see that the bijection \( \text{Sub}(\Phi) \simeq \text{Sub}(\Theta) \) is actually independent of any choices.

Example 9.10. The inclusion \( F \to \bar{F} \) gives an isomorphism \( \text{Hom}(\Theta^*, F^\times) \simeq \Phi[q - 1] = \Phi[p^r] \). Recall that \( \mathcal{L} \) is the groupoid of one-dimensional vector spaces over \( F \). All such spaces are isomorphic, and the automorphism group of any one of them is naturally identified with \( F^\times \), so we have \( [\Theta^*, \mathcal{L}] = \text{Hom}(\Theta^*, F^\times) = \Phi[p^r] \).

Thus, we have \( D_0'(\mathcal{L}) = D'(\Phi[p^r]) \) and \( (D')^0(\mathcal{L}) = \text{Map}(\Phi[p^r], D') \).

We can also replace \( F \) by \( F(k) \) in the above analysis to get \( D_0'(\mathcal{L}(k)) = D'(\Phi[p^{r+k}]) \).

Example 9.11. The main HKR theorem gives
\[
D_0'(V) = D'(\{[\Theta^*, V]\}) = \bigoplus_d D_0'(\{[\Theta^*, V_d]\}).
\]

We can identify \( [\Theta^*, V] \) with the semiring of isomorphism classes of finite-dimensional \( F \)-linear representations of \( \Theta^* \). We will denote this by \( \text{Rep}(\Theta^*) \) or \( \text{Rep}(\Theta^*; F) \). We also write \( \text{Rep}_d(\Theta^*) \) for the subset of representations of dimension \( d \), and \( \text{Irr}(\Theta^*) \) for the subset of irreducible representations, and \( \text{Irr}_d(\Theta^*) \) for \( \text{Irr}(\Theta^*) \cap \text{Rep}_d(\Theta^*) \).

Lemma 9.4 tells us that any \( V \in [\Theta^*, V] \) can be regarded as an \( F \)-linear representation of a finite abelian \( p \)-group quotient of \( \Theta^* \), and we can therefore use Lemma 3.5 to decompose it in an essentially unique way as a direct sum of irreducibles. For any irreducible representation \( S \) we write \( v_S \) for \( [S] \) regarded as an element of \( \text{Irr}(\Theta^*) \) or of \( \text{Rep}(\Theta^*) \) or of \( D'(\text{Rep}(\Theta^*)) \). We then find that \( \text{Rep}(\Theta^*) \) is the free abelian monoid generated by the elements \( v_S \), and thus that \( D'(\text{Rep}(\Theta^*)) \) is the polynomial ring \( D'[v_S \mid S \in \text{Irr}(\Theta^*)] \). We will study the structure of \( \text{Irr}(\Theta^*) \) in Proposition 9.17 below.

Example 9.12. Recall that \( \mathcal{X} \) is the groupoid of finite sets and bijections, so \( [\Theta^*, \mathcal{X}] \) is the set of isomorphism classes of finite sets with action of the group \( \Theta^* = \Phi^{\#} \), or in other words the Burnside semiring of \( \Phi^{\#} \). Let \( \text{Sub}(\Phi) \) denote the set of finite subgroups of \( \Phi \). For any \( A \in \text{Sub}(\Phi) \) we have an epimorphism \( \Phi^{\#} \to A^{\#} \) giving a transitive action of \( \Phi^{\#} \) on \( A^{\#} \), and any transitive finite \( \Phi^{\#} \)-set has this form for a unique group \( A \). We write \( x_A \) for \([A^{\#}] \) regarded as an element of \([\Theta^*, \mathcal{X}] \) or of \( D_0'(\mathcal{X}) = D'(\{[\Theta^*, \mathcal{X}]\}) \). We find that \([\Theta^*, \mathcal{X}] \) is the free abelian monoid generated by the elements \( x_A \), and thus that \( D_0'(\mathcal{X}) \) is the polynomial ring \( D'[x_A \mid A \in \text{Sub}(\Phi)] \). (It is more usual to formulate this in terms of subgroups of \( \Theta \) rather than subgroups of \( \Phi \), but these are canonically identified, as discussed in Remark 9.8.)

Example 9.13. We also see that \([\Theta^*, \mathcal{X}L] \) is the set of isomorphism classes of pairs \((X, L)\), where \( X \) is a finite \( \Theta^* \)-set, and \( L \) is an \( F \)-linear line bundle over \( X \) with compatible action of \( \Theta^* \). In Definition 9.14 we explained how to regard \( \mathcal{X}L \) as a symmetric bimonoidal category. This makes \([\Theta^*, \mathcal{X}L] \) into another commutative
semiring. We can construct some elements of $[\Theta^*, X\mathcal{L}]$ as follows. Consider a finite subgroup $A \subset \Phi$, which gives a subgroup $\text{ann}(A) = \ker(\Phi^\# \to A^\#)$ of finite index in $\Phi^\#$. Consider a homomorphism $\alpha: \text{ann}(A) \to F^\times$, or equivalently an element $\alpha$ in the group $\text{ann}(A)^\# = \Phi/A$ with $(q - 1)\alpha = 0$, or equivalently a coset $\tilde{\alpha} + A \subset \Phi$ with $(q - 1)\tilde{\alpha} \in A$. The map $\alpha$ gives an action of $\text{ann}(A)$ on $F$, and this gives a line bundle $L_\alpha$ with total space $(F \times \Phi^\#)/\text{ann}(A)$ over $\Phi^\#/\text{ann}(A) = A^\#$. There is an evident action of the group $\Theta^* = \Phi^\#$ on this line bundle, so we have an element $x_{A\alpha} \in [\Theta^*, X\mathcal{L}]$. Given a general object $(X, L) \in [\Theta^*, X\mathcal{L}]$ we can decompose $X$ into orbits, each of which must have the form $A^\#$ for some $A$, and then check that every equivariant line bundle over $A^\#$ arises as above. From this we see that $D_0(X\mathcal{L})$ is again a polynomial algebra over $D'$, generated by these elements $x_{A\alpha}$.

We now discuss a slightly different picture of $\text{Rep}(\Theta^*)$.

**Remark 9.14.** Recall that $\Gamma$ is the Galois group of $\bar{F}$ over $F$, which is a compact Hausdorff topological group and is topologically generated by the Frobenius map $\phi: x \mapsto x^q$. This gives rise to an action of $\Gamma$ on the group $\Phi = \text{Hom}(\Theta^*, \bar{F}^\times)$ that we introduced in Example 9.10. We will write the group structure on $\Phi$ additively, so the action of $\phi \in \Gamma$ is just multiplication by $q$. Note that $\Phi \simeq (\mathbb{Z}/p^\infty)^n$, so $\Phi$ is a $p$-torsion group, and is naturally a module over $\mathbb{Z}_p$. If we let $\Gamma$ denote the closed subgroup of $\mathbb{Z}_p^\times$ generated by $q$, it follows that the action of $\Gamma$ on $\Phi$ factors through an epimorphism $\Gamma \to \overline{\Gamma}$.

**Lemma 9.15.** The group $\overline{\Gamma} = (q) < \mathbb{Z}_p^\times$ is just $1 + p^\Gamma \mathbb{Z}_p$.

**Proof.** Put $U_j = 1 + p^r j \mathbb{Z}_p$, so $q \in U_0$ and $U_0$ is compact so $\overline{\Gamma} \leq U_0$. Note also that $U_j/U_j+1 \simeq \mathbb{Z}/p$. Lemma 2.1 tells us that $q^p \in U_j \setminus U_{j+1}$ so $q^p$ generates $U_j/U_{j+1}$. It follows by induction that the map $\overline{\Gamma} \to U_0/U_j$ is surjective for all $j$, and we can pass to the inverse limit using compactness to see that $\overline{\Gamma} = U_0$. \[\square\]

**Lemma 9.16.** Suppose that $\alpha \in \Phi$. Recall that $\phi \simeq (\mathbb{Z}/p^\infty)^n$, so the order of $\alpha$ must be $p^t$ for some $t \geq 0$. Consider the orbit $\Gamma \alpha \subset \Phi$.

(a) If $t \leq r = v_p(q - 1)$, then $\Gamma \alpha = \{\alpha\}$ and so $|\Gamma \alpha| = 1$.

(b) If $t \geq r$ then $\Gamma \alpha = \{(1 + p^k \alpha) | 0 \leq k < p^{r-t}\}$ and so $|\Gamma \alpha| = p^{r-t}$.

**Proof.** Clear from Remark 9.14 and Lemma 9.10. \[\square\]

We next give a slightly different perspective on the semiring $\text{Rep}(\Theta^*) = \text{Rep}(\Theta^*; F)$, by comparing it with the semiring $\text{Rep}(\Theta^*; \bar{F}) = [\Theta^*, \overline{\mathcal{V}}]$. (Recall that this is a well-defined algebraic object, even though the HKR theorem does not apply to $\overline{\mathcal{V}}$, because it is not hom-finite.)

**Proposition 9.17.** There are natural isomorphisms

\[
\text{Irr}(\Theta^*; \bar{F}) \simeq \Phi \quad \text{Rep}(\Theta^*; \bar{F}) \simeq N[\Phi] \\
\text{Irr}(\Theta^*; F) \simeq \Phi/\Gamma \quad \text{Rep}(\Theta^*; F) \simeq N[\Phi/\Gamma] = N[\Phi/\Gamma].
\]

**Proof.** First, the semiring $\text{Rep}(\Theta^*; \bar{F})$ is by definition the union of the semirings $\text{Rep}(\Theta^*/p^k; \bar{F})$. As $\Theta^*/p^k$ is a finite group of order coprime to the characteristic of $\bar{F}$, Lemma 3.5 is applicable. This means that every representation can be decomposed in an essentially unique way as a direct sum of irreducibles, and the
irreducibles biject with the homomorphisms \( \Theta^*/p^k \to \bar{\mathbb{F}}^\times \), or equivalently the elements of \( \Phi(p^k) \). By taking the colimit over \( k \), we obtain the claimed description of \( \text{Irr}(\Theta^*/\bar{\mathbb{F}}) \) and \( \text{Rep}(\Theta^*/\bar{\mathbb{F}}) \).

Next, we have a semiring map \( \xi : \text{Rep}(\Theta^*/F) \to \text{Rep}(\Theta^*/\bar{\mathbb{F}}) \) given by \( [V] \mapsto [\bar{\mathbb{F}} \otimes_F V] \). Note that \( \Gamma \) acts on \( \text{Rep}(\Theta^*/\bar{\mathbb{F}}) \) by Definition 9.22, and the image of \( \xi \) lies in \( \text{Rep}(\Theta^*/\bar{\mathbb{F}})^\Gamma \) by Remark 9.23. Consider a \( \Gamma \)-orbit \( C \subseteq \Phi \). This is finite, with \( |C| = p^t \) for some \( t \), by Lemma 9.16. We can therefore define \( e_C = \sum_{\alpha \in C} [\alpha] \in \mathbb{N}[\Phi]^\Gamma \). It is clear that \( \mathbb{N}[\Phi]^\Gamma \) is the free abelian monoid on the elements of this form, so \( \mathbb{N}[\Phi]^\Gamma \simeq \mathbb{N}[\Phi]/\Gamma \).

Now consider an element \( \alpha \in \Phi \), of order \( p^t \) say. This means that \( \alpha(\Theta^*) = \mu_{p^t}(\bar{\mathbb{F}}) \). Proposition 9.15 therefore tells us that the induced map \( \alpha_* : F[\Theta^*/p^t] \to \bar{\mathbb{F}} \) has image \( F(k) \), where \( k = \max(t - r, 0) \). Let \( V_\alpha \) denote \( F(k) \), regarded as an \( F \)-linear representation of \( \Theta^* \) via \( \alpha_* \). Because the map \( \alpha_* : F[\Theta^*/p^t] \to F(k) \) is surjective, we see that the subrepresentations of \( V_\alpha \) are \( F(k) \)-submodules, and so are either zero or \( V_\alpha \). Thus, \( V_\alpha \) is irreducible. Lemma 9.24 shows that \( \xi([V_\alpha]) = \sum_j [q^j \alpha] \) in \( \text{Rep}(\Theta^*/\bar{\mathbb{F}})^\Gamma \simeq \mathbb{N}[\Phi]^\Gamma \). This is just the basis element corresponding to the orbit \( \Gamma \alpha \). This shows in particular that the map \( \xi : \text{Rep}(\Theta^*/F) \to \mathbb{N}[\Phi]^\Gamma \) is surjective.

Now note that if \( \beta = \phi^m \alpha = \bar{d}^m \alpha \) then \( \beta \) also has order \( p^t \), and the map \( \phi^m : F(k) \to F(k) \) gives an isomorphism of representations between \( V_\alpha \) and \( V_\beta \). Conversely, if \( V_\alpha \simeq V_\beta \) then \( \bar{\mathbb{F}} \otimes_F V_\alpha \simeq \bar{\mathbb{F}} \otimes_F V_\beta \), so it follows from the previous paragraph that \( \Gamma \alpha = \Gamma \beta \).

Now let \( W \) be an arbitrary irreducible \( F \)-linear representation of \( \Theta^* \). Let \( K \) be the image of \( F[\Theta^*] \) in \( \text{End}(W) \). As \( \Theta^* \) is commutative, \( K \) consists of equivariant endomorphisms of \( W \), and any such endomorphism is zero or invertible by Schur’s Lemma. Any invertible endomorphism must have finite multiplicative order, so the inverse is also in \( K \). Thus, \( K \) is a finite field extension of \( F \). Any \( K \)-submodule of \( W \) is a subrepresentation, and so is zero or \( W \); so we must have \( \dim_K(W) = 1 \). Thus, any choice of embedding of \( K \) in \( \bar{\mathbb{F}} \) gives an element \( \alpha \) with \( W \simeq V_\alpha \). The claim now follows.

**Proposition 9.18.** The set \( \text{Irr}_d(\Theta^*) \) is empty unless \( d \) is a power of \( p \). Moreover, we have \( |\text{Irr}_d(\Theta^*)| = N_k \) for all \( k \geq 0 \) (where \( N_k \) is as in Definition 9.7).

**Proof.** We have seen that \( \text{Irr}_d(\Theta^*) \) bijects with the set of \( \Gamma \)-orbits of size \( d \) in \( \Phi \). Lemma 9.17 tells us that this is empty unless \( d = p^k \) for some \( k \). The same lemma tells us that each \( \alpha \in \Phi \) with \( p^k \alpha = 0 \) gives an orbit \( \{ \alpha \} \), and these are all the orbits of size 1. As \( \Phi \simeq (\mathbb{Z}/p^\infty)^n \), we see that the number of elements \( \alpha \) of this type is \( p^{nr} = N_0 \), as expected. The lemma also tells us that each element of order \( p^k \) gives an orbit of size \( p^k \). The number of elements of order precisely \( p^k \) is

\[
p^{n(r+k)} - p^{n(r+k-1)} = (p^n - 1)p^{nr+nk-n} = (p^n - 1)p^{nk+n(r-1)}.
\]

to get the number of orbits, we divide by \( p^k \), which gives \( N_k \).

**Remark 9.19.** This now gives us a useful consistency check. The discussion above shows that \( D_0(|V_\alpha|) \) is a polynomial ring over \( D' \), with \( N_k \) generators in degree \( p^k \), so the Poincaré series is

\[
\sum_d \dim_{D'}(D_0(|V_d|))t^d = \prod(1 - t^p)^{-N_k}.
\]
We also know that $E_0'(BV_*)$ is free and of finite type as an $E_0$-module, and $D_0'(V_*) = D' \otimes E_0 E_0'(BV_*)$, so $E_0'(BV_*)$ should have the same Poincaré series. We saw this already in Proposition 8.25.

We conclude this section with some results on the effect of the maps $R\phi$ in generalised character theory. These are slight extensions to the results given in [13]. There we defined $CG = \mathbb{Q}\{\pi_0G\}$. Given $\phi: \mathcal{G} \to \mathcal{H}$ we defined $L\phi: CG \to C\mathcal{H}$ by $L\phi[a] = [\phi(a)]$ (where $[a]$ denotes the basis element in $CG$ corresponding to the isomorphism class of $a$). Also, for $b \in \mathcal{H}$ we defined

$$(R\phi)[b] = \sum_{[a]|\phi(a) \geq b} \frac{|\mathcal{H}(b)|}{|\mathcal{G}(a)|}[a].$$

Theorem 9.6 identifies $D' \otimes_Q E' \otimes BG$ with $D' \otimes_Q C[\Theta^*, \mathcal{G}]$. We proved in [13] that this identification is compatible with the constructions $L$ and $R$. (The statement for $L$ is clear by construction, and if $\phi$ is faithful, then the statement for $R\phi$ is essentially contained in [3].)

It will be convenient to give a slightly different formula for $R\phi$ in the case where $\phi$ is a fibration or a covering, as in the following definition:

**Definition 9.20.** Consider a functor $\phi: \mathcal{G} \to \mathcal{H}$ of groupoids. We say that $\phi$ is a fibration if for all $a \in \mathcal{G}$ and all $h: \phi(a) \to b'$ in $\mathcal{H}$, there is an arrow $g: a \to a'$ in $\mathcal{G}$ with $\phi(a') = b'$ and $\phi(g) = h$. We say that $\phi$ is a covering if the pair $(a', g)$ is always unique.

**Example 9.21.** As a typical example, consider the groupoids $V$ and $V^2$ and

$\mathcal{W} = \{(V; V_0, V_1) \mid V \in \mathcal{V}, V_0, V_1 \leq V, V_0 + V_1 = V, V_0 \cap V_1 = 0\}.$

The external direct sum operation gives a functor $\sigma: V^2 \to \mathcal{V}$ that is not a fibration. The construction $(V; V_0, V_1) \mapsto V$ gives a functor $\sigma': \mathcal{W} \to \mathcal{V}$, which is a covering. The forgetful functor $\pi: \mathcal{W} \to V^2$ is an equivalence with $\sigma \pi \simeq \sigma'$, so $\sigma$ and $\sigma'$ are in some sense equivalent, in particular $R\sigma = R\sigma' \circ (R\pi)^{-1} = R\pi \circ L\sigma$. The best way to understand $R\sigma$ is to use this expression, combined with the description of $R\sigma'$ given by Proposition 9.23 below.

**Definition 9.22.** Suppose that $\phi: \mathcal{G} \to \mathcal{H}$ is a fibration, and that $b$ is an object of $\mathcal{H}$. Consider the set of objects $a \in \mathcal{G}$ with $\phi(a) = b$ (on the nose), and the morphisms $u: a \to a'$ with $\phi(a) = \phi(a') = b$ and $\phi(u) = 1_b$. These give a category, which we denote by $\phi^{-1}\{b\}$. For $a \in \phi^{-1}\{b\}$ we also put

$$K_a = \ker(\phi: \mathcal{G}(a) \to \mathcal{H}(b))$$

(which is trivial if $\phi$ is a covering).

**Proposition 9.23.** If $\phi: \mathcal{G} \to \mathcal{H}$ is a fibration, then

$$(R\phi)[b] = \sum_{[a] \in \pi_0(\phi^{-1}\{b\})} |K_a|^{-1}[a].$$

In particular, if $\phi$ is a covering then

$$(R\phi)[b] = \sum_{[a] \in \pi_0(\phi^{-1}\{b\})} [a].$$
Proof. Let \( \mathcal{A} \) be the full subcategory of objects \( a \in \mathcal{G} \) with \( \phi(a) = b \). We will also write \( \mathcal{B} \) for \( \phi^{-1}\{b\} \), so \( \mathcal{B} \) has the same objects as \( \mathcal{A} \), but fewer morphisms. Because \( \phi \) is a fibration, every isomorphism class that maps to \( [b] \) has a representative in \( \mathcal{A} \). Thus, from the original definition, \( (R\phi)[b] \) can be written as a sum over \( \pi_0\mathcal{A} \).

Choose a list \( a_1, \ldots, a_m \) containing precisely one representative of each isomorphism class in \( \mathcal{A} \). From the original definition of \( R\phi \), we have
\[
(R\phi)[b] = \sum_i |\mathcal{H}(b)|^{-1} |\mathcal{H}(a_i)|\{a_i\}.
\]

Now let \( \mathcal{B}_i \) be the full subcategory of \( \mathcal{B} \) consisting of objects \( a' \) such that \( a' \simeq a_i \) in \( \mathcal{A} \), so \( \mathcal{B} = \coprod_i \mathcal{B}_i \). Put \( H_i = \phi(\mathcal{G}(a_i)) \simeq \mathcal{G}(a_i)/K_{a_i} \) and \( Q_i = \mathcal{H}(b)/H_i \), so \( |Q_i| = |K_{a_i}| |\mathcal{H}(b)| |\mathcal{G}(a_i)|^{-1} \). For \( a' \in \mathcal{B}_i \) we can choose \( u \in \mathcal{G}(a_i, a') \), and then get \( \phi(u) \in \mathcal{H}(b) \). The coset \( \phi(u)H_i \) is easily seen to depend only on the isomorphism class of \( a' \) in \( \mathcal{B}_i \), so this construction gives a map \( \delta: \pi_0\mathcal{B} \to \pi_0\mathcal{A} \).

Consider again the object \( \pi_0\theta \), and suppose it corresponds to the pointwise product of functions \( \phi \) and \( \psi \), and it follows easily that the choice of lift for \( \phi \) is a bijection. It follows that \( \pi_0\mathcal{A} = \{Q_i\}^{-1} \sum_{[a] \in \pi_0\mathcal{B}} [a] \). Using the above formula for \( |Q_i| \), the \( i \)th term in \( (R\phi)[b] \) now becomes \( \sum_{[a] \in \mathcal{B}_i} |K_{a_i}|^{-1} [a] \), and it is also easy to see here that \( K_a \) is conjugate to \( K_{a_i} \), so we can replace \( |K_{a_i}| \) by \( |K_a| \). Taking the sum over \( i \) now gives the formula
\[
(R\phi)[b] = \sum_{[a] \in \pi_0(\phi^{-1}(b))} |K_{a_i}|^{-1} [a]
\]
as claimed. \( \square \)

To apply this in HKR theory, we also need the following:

**Proposition 9.24.** If \( \phi: \mathcal{G} \to \mathcal{H} \) is a fibration, then so is the induced functor \( \phi_*: [\Theta^*, \mathcal{G}] \to [\Theta^*, \mathcal{H}] \). Similarly, if \( \phi \) is a covering, then so is \( \phi_* \).

**Proof.** Any object in \([\Theta^*, \mathcal{G}]\) has the form \( (a, \alpha) \), where \( a \in \mathcal{G} \) and \( \alpha: \Theta^* \to \mathcal{G}(a) \). Suppose we have such an object, together with an isomorphism \( \psi: (\phi(a), \phi \circ \alpha) \to (b, \beta) \) in \([\Theta^*, \mathcal{H}]\). Explicitly, this means that \( \psi \) is the unique lifting of \( \phi \) in \( \mathcal{H} \), and that \( \beta(\psi) = \psi(\alpha(\theta)).v^{-1} \) for all \( \theta \in \Theta^* \). Because \( \phi \) is a fibration, we can choose \( u: a \to a' \) in \( \mathcal{G} \) such that \( \phi(a') = b \) and \( \phi(u) = v \). We now define \( \alpha': \Theta^* \to \mathcal{G}(a', \alpha') \) by \( \alpha'(\theta) = u.\alpha(\theta).u^{-1} \). Now \( u \) can be regarded as a morphism \( (a, \alpha) \to (a', \alpha') \) in \([\Theta^*, \mathcal{H}]\), and we have \( \phi_*((a', \alpha')) = (b, \beta) \) and \( \phi_*((a', \alpha')) = v \). This is what is needed to show that \( \phi_* \) is a fibration. If \( \phi \) is a covering then the choice of \( (a', \alpha') \) is unique, and it follows easily that the choice of lift for \( \phi_* \) is also unique; we deduce that \( \phi_* \) is also a covering. \( \square \)

**Example 9.25.** Consider again the object \( D^0(BV_{k}) \simeq \text{Map}(\text{Rep}_k(\Theta^*), D_0) \). This has a ring structure arising from the diagonal map of the space \( BV_k \); this just corresponds to the pointwise product of functions \( \text{Rep}_k(\Theta^*) \to D_0 \), which will just be denoted by juxtaposition, so \( (f_0f_1)(V) = f_0(V)f_1(V) \). The direct sum functor \( \sigma: \mathcal{V}_i \times \mathcal{V}_j \to \mathcal{V}_{i+j} \) gives a different kind of product
\[
\text{Map}(\text{Rep}_i(\Theta^*), D_0) \otimes \text{Map}(\text{Rep}_j(\Theta^*), D_0) \to \text{Map}(\text{Rep}_{i+j}(\Theta^*), D_0),
\]
which we denote by \( f_0 \times f_1 \). Using the above propositions together with Example 9.21 we find that

\[
(f_0 \times f_1)(V) = \sum_{V = V_0 \oplus V_1} f_0(V_0)f_1(V_1).
\]

In more detail, this should be interpreted as follows. We have a representation \( V \) of \( \Theta^* \) over \( F \), and \( (f_0 \times f_1)(V) \) refers to the value of \( f_0 \times f_1 \) on the isomorphism class of \( V \). The right hand side of the formula refers to the set of pairs \( (V_0, V_1) \), where each \( V_i \) is a subrepresentation of \( V \), and \( V_0 \cap V_1 = 0 \) and \( V_0 + V_1 = V \). (In particular, we have a sum over actual subrepresentations, not over isomorphism classes.)

**Remark 9.26.** As we have mentioned previously, the theory developed in \[13\] gives a natural inner product on \( E_0^0(BV_d) \), which allows us to identify it with \( E^0(BV_d) \). This gives rise to an inner product on \( D_0^0(V_d) = D_0(\text{Rep}_d(\Theta^*)) \). This is just given by \( \langle [W], [V] \rangle = |\text{Iso}(W, V)| \), where \( |\text{Iso}(W, V)| \) is the number of \( \Theta^* \)-equivariant \( F \)-linear isomorphisms from \( W \) to \( V \). The standard product on \( D_0^0(V_d) \) is given by \( [V_0][V_1] = [V_0 \oplus V_1] \), and the convolution product on \( (D')^0(\mathcal{V}) \) is obtained from this by duality. In other words, the isomorphism \( D_0^0(V_d) \to (D')^0(\mathcal{V}) \) sends \( [W] \) to the function \( \chi_W : \text{Rep}(\Theta^*) \to \mathbb{Z} \subset D_0 \) given by

\[
\chi_W(V) = \langle W, V \rangle = |\text{Iso}(W, V)|.
\]

Either by chasing through the duality isomorphisms, or by direct analysis of the formula in Example 9.26, we find that \( \chi_{W_0} \times \chi_{W_1} = \chi_{W_0 \oplus W_1} \). Thus, \( (D')^0(\mathcal{V}) \) is polynomial on \( \{ \chi_S \mid S \in \text{Irr}(\Theta^*) \} \). This analysis also gives a natural isomorphism

\[
D' \otimes_{E^0} \text{Ind}_{p^k}(\mathbb{H}) \to \text{Map}(\text{Irr}(\Theta^*), D').
\]

**Remark 9.27.** As discussed in Definition 5.19, the direct sum functor \( \sigma : \mathcal{V}^2 \to \mathcal{V} \) also gives a coproduct on \( E^0(BV) \). In generalised character theory, this corresponds to a map

\[
\text{Map}(\text{Rep}_*(\Theta^*), D_0) \to \text{Map}(\text{Rep}_*(\Theta^*)^2, D_0).
\]

This is just given by \( \sigma^*(f)(V_0, V_1) = f(V_0 \oplus V_1) \), and the counit is \( f \mapsto f(0) \).

## 10. Indecomposables in \( E^0(BV) \)

We now return to consideration of the ring \( \text{Ind}_{p^k}(E^0(BV_*)) \), and the associated formal scheme.

**Definition 10.1.** In this section, we fix \( k > 0 \) and put \( G = GL_{p^k}(F) \) and \( H = GL_{p^{k-1}}(F)^p < G \). Using Lemma 5.23 we put

\[
R = E^0(BG)
\]

\[
I = \text{Prim}_{p^k}(E^0(BV_*)) = \ker(\text{res}^G_H : E^0(BG) \to E^0(BH))
\]

\[
J = \text{Dec}_{p^k}(E^0(BV_*)) = \text{image}(\text{tr}^G_H : E^0(BH) \to E^0(BG)),
\]

so \( R/J = \text{Ind}_{p^k}(E^0(BV_*)) \).

We will use generalised character theory to help analyse the above objects. Recall that \( D' \) is a flat extension of \( E^0 \), and that the map \( E^0 \to D' \) is injective. This justifies some implicit identifications in the following definition.
Definition 10.2. We put
\[ R' = D' \otimes_{E^0} R = (D')^0(\mathcal{O}_{\Theta^*}) = \text{Map}(\text{Rep}_{E^0}(\Theta^*), D') \]
\[ I' = D' \otimes_{E^0} I = \ker(\text{res}_*: (D')^0(G) \to (D')^0(H)) \]
\[ J' = D' \otimes_{E^0} J = \text{image}(\text{tr}_*: (D')^0(H) \to (D')^0(G)). \]

Remark 10.3. We can split \( \text{Rep}_{E^0}(\Theta^*) \) as \( \text{XII}Y \), where \( X \) is the set of irreducibles, and \( Y \) is the set of reducibles. We find that
\[ R' = \text{Map}(X, D') \times \text{Map}(Y, D') \]
\[ I' = \text{Map}(X, D') \times 0 \]
\[ J' = 0 \times \text{Map}(Y, D'). \]

(For the third statement, it is helpful to recall Remark 9.2.)

We next want to discuss the annihilators of \( I \) and \( J \). We will use Lemma 6.3 without comment.

Proposition 10.4. Both \( I \) and \( J \) are \( E^0 \)-module summands in \( R \), and they are annihilators of each other. Thus, we have \( \dim(I) + \dim(J) = \dim(R) \), so \( \dim(I) = \dim(R/J) \) and \( \dim(J) = \dim(R/I) \).

Proof. As \( E^0(\mathcal{B}V_\ast) \) is a polynomial algebra over \( E^0 \), it follows easily that the decomposables form an \( E^0 \)-module summand, so \( J \) is an \( E^0 \)-module summand in \( R \). Duality theory therefore tells us that \( \text{ann}(J) \) is also a summand, and that \( \text{ann}^2(J) = J \). Consider an element \( f \in E^0(BG) \), and let \( f' \) be the image in \( (D')^0(G) \). Because \( E^0(BG) \) injects in \( (D')^0(G) \) and similarly for \( H \), we see that \( fJ = 0 \iff f'J' = 0 \iff f'|_X = 0 \iff (f|_H)' = 0 \iff f|_H = 0 \iff f \in I \), so \( \text{ann}(J) = I \).

The claims about dimensions follow easily. \( \square \)

Definition 10.5. We let \( m = (u_0, \ldots, u_{n-1}) \) denote the maximal ideal in \( E^0 \), and put \( \bar{R} = R/mR \) and \( \bar{I} = I/mI \) and \( \bar{J} = J/mJ \).

Lemma 10.6.
\[ \bar{R} = K^0(BG) \]
\[ \bar{I} = \ker(\text{res}_{\bar{G}}: K^0(BG) \to K^0(BH)) = \text{ann}(\bar{J}) \]
\[ \bar{J} = \text{image}(\text{tr}_{\bar{G}}: K^0(BH) \to K^0(BG)) = \text{ann}(\bar{I}). \]

Proof. As \( E^0(BG) \) is free and \( E^1(BG) = 0 \), it is standard that \( K^0(BG) = K^0 \otimes_{E^0} E^0(BG) = \bar{R} \). Similarly, \( K^0(BH) = K^0 \otimes_{E^0} E^0(BH) \). We have seen that \( I \) and \( J \) are summands in \( R \), and so are free modules over \( E^0 \) of ranks \( r_I \) and \( r_J \) say. As \( I \) and \( J \) are annihilators of each other, we have \( r_I + r_J = \dim_{E^0}(\bar{R}) \). It follows easily that \( \bar{I} \) and \( \bar{J} \) are summands in \( \bar{R} \), with ranks \( r_I \) and \( r_J \) over \( K^0 \). As \( IJ = 0 \) we have \( \bar{I} \bar{J} = 0 \), so \( \bar{J} \leq \text{ann}(\bar{I}) \). However, \( \bar{R} \) is a Frobenius algebra, so
\[ \dim_{K^0}(\text{ann}(\bar{I})) = \dim_{K^0}(\bar{R}) - \dim_{K^0}(\bar{I}) = \dim_{K^0}(\bar{J}), \]
so \( \text{ann}(\bar{I}) = \bar{J} \). Similarly, we have \( \text{ann}(\bar{J}) = \bar{I} \).

Next, as \( J \) is a summand in \( R \), we see that \( \text{tr}_{\bar{G}} \) can be written as a split epimorphism of \( E^0 \)-modules, followed by a split monomorphism. The restriction map \( \text{res}_{\bar{G}} \) is adjoint to \( \text{tr}_{\bar{G}} \) with respect to the standard inner products, and it follows that it can also be factored in the same way. This means that the functor \( K^0 \otimes_{E^0} (-) \)
preserves the kernel, image and cokernel of \( \text{tr}_G^H \) and of \( \text{res}_G^H \), so \( \tilde{I} \) is the kernel of the Morava \( K \)-theory restriction map, and \( \tilde{J} \) is the image of the Morava \( K \)-theory transfer.

**Definition 10.7.** Put \( C = GL_1(F(k)) < G \), so \( C \) is cyclic of order \( m = q^k - 1 \). We then have a restriction map

\[
E^0(BG) \rightarrow E^0(BC) = E^0\left[\left[ x \right]/[m]\right].
\]

Here \( [m](x) \) is a unit multiple of \( [p^{k+r}](x) \), so it is also a unit multiple of a Weierstrass polynomial of degree \( p^{n(k+r)} \). The above ring map corresponds to a map of formal schemes

\[
\{a \in \mathbb{H} \mid q^k a = a\} \rightarrow \text{Div}_{p^k}^{+}(\mathbb{H})^\Gamma.
\]

Using Lemma 3.24 we see that this is just

\[
a \mapsto \sum_{i=0}^{p^k-1} [q^i a].
\]

Now put \( m' = m/p \), so \( [m](t) = [p](\langle m' \rangle(t)) \). As usual, we put \( (p)(t) = [p](t)/t \), which is a Weierstrass series of degree \( p^n - 1 \). We also put

\[
\tilde{Q} = E^0\left[\left[ x \right]/\left[\left( p \right)\left( \langle m' \rangle \right) \right]\right],
\]

so \( \text{spf}(\tilde{Q}) \subset \mathbb{H} \) is the divisor of points of exact order \( p^{k+r} \).

The ring \( \tilde{Q} \) has an action of the Galois group \( \Gamma \), satisfying \( \phi^* (x) = [q](x) \). We put \( \tilde{Q} = \tilde{Q}^\Gamma \), and we put

\[
s = \prod_{i=0}^{p^k-1} [q^i](x) \in Q.
\]

It is easy to see that the composite \( E^0(BG) \xrightarrow{\text{res}} E^0(BC) \rightarrow \tilde{Q} \) is \( \Gamma \)-invariant and so lands in \( Q \). We define \( \alpha \) to be the resulting map \( R = E^0(BG) \rightarrow Q \).

**Remark 10.8.** The \( p \)-torsion subgroup of \( C \) is cyclic of order \( p^{r+k} \). Recall from Corollary 3.17 that there is a unique conjugacy class of such subgroups in \( G \), or in any subgroup of \( G \) that contains the monomial matrices. This is in some sense the real reason for the importance of \( C \) and \( Q \), although that is not completely visible in our current approach.

**Proposition 10.9.** There is a Weierstrass polynomial \( g_k(t) \) of degree \( N_k \) such that

\[
Q = E^0[\left[ s \right]/g_k(s)] = E^0\{s^i \mid 0 \leq i < N_k\}.
\]

Moreover, both \( \tilde{Q} \) and \( Q \) are complete regular local Noetherian rings (so in particular they are unique factorisation domains and are Gorenstein).

**Proof.** Note that \( [m'](t) \) is a unit multiple of \( [p^{r+k-1}](t) \), so it has Weierstrass degree \( p^{n(r+k-1)} \). Moreover, \( (p)(t) \) has Weierstrass degree \( p^n - 1 \), so the Weierstrass degree of the series \( f_k(t) = (p)(\langle p^{r+k-1} \rangle(t)) \) is \( (p^n - 1)p^{n(r+k-1)} = p^k N_k \).

Now recall that the maximal ideal \( m < E^0 \) is generated by a regular sequence \( u_0, \ldots, u_{n-1} \) with \( u_0 = p \). Note that \( f_k(0) = p \) so \( Q/x = E^0/g(0) = E^0/p \) so \( \tilde{Q}/(x, u_1, \ldots, u_{n-1}) = E_p \). We now see that \( \tilde{Q} \) is a complete local Noetherian ring of Krull dimension \( n \) in which the maximal ideal can be generated by a sequence of length \( n \). It follows as in [6, Section 14] that the sequence \( x, u_1, \ldots, u_{n-1} \) is regular,
and that \( \bar{Q} \) is a regular local ring. In particular, it has unique factorisation, and so is integrally closed in its field of fractions.

Next, the above discussion of Weierstrass degrees shows that \( \bar{Q}/m\bar{Q} = F_p[x]/x^{p^k}N_k \). On the other hand, as \( q = 1 \) (mod \( p \)) we have \( [q^i](x) = x \) (mod \( x^2 \)) for all \( i \), so \( s \) is a unit multiple of \( x^{p^k} \). Now put

\[
BX = \{ x^j \mid 0 \leq j < p^k \} \\
BY = \{ s^m \mid 0 \leq m < N_k \} \\
B = BY.BX = \{ x^j s^m \mid 0 \leq j < p^k, 0 \leq m < N_k \}.
\]

We find that \( B \) is a basis for \( \bar{Q}/m\bar{Q} \) over \( F_p \). By Nakayama’s Lemma, we see that \( B \) is also a basis for \( \bar{Q} \) over \( \bar{E}^0 \).

Now let \( Q_0 \) be the subring of \( Q \) generated by \( E^0 \) and \( c \). As \( B \) is a basis for \( \bar{Q} \) over \( \bar{E}^0 \), we see that \( BX \) generates \( \bar{Q} \) as a \( Q_0 \)-module.

Now let \( K_0 \) and \( \bar{K} \) be the fields of fractions of \( Q_0 \), \( Q \) and \( \bar{Q} \). Put \( \overline{\Gamma} = \text{Gal}(F_k/F) = \{ \phi \mid \phi^p = 1 \} \), so \( \overline{\Gamma} = p^k \) and \( \Gamma \) acts on \( Q \). We then have \( K = \bar{K}\overline{\Gamma} \), so classical Galois theory of fields tells us that \( [\bar{K} : K] = p^k \). On the other hand, as \( BX \) generates \( \bar{Q} \) as a \( Q_0 \)-module, we have \( [K : K_0][\bar{K} : K] = [\bar{K} : K_0] \leq p^k \). For this to be consistent, we must have \( K = K_0 \), and the natural map \( K\{BX\} \to \bar{K} \) must be an isomorphism. From this it follows that \( Q\{BX\} \to \bar{Q} \) is injective, but we saw previously that it is also surjective, so \( BX \) is a basis for \( \bar{Q} \) over \( Q \). This allows us to identify the isomorphism \( E^0(B) \to \bar{Q} \) as a direct sum of copies (indexed by \( BX \)) of the map \( E^0(BY) \to Q \). It follows that the map \( E^0(BY) \to Q \) is also an isomorphism, so \( BY \) is a basis for \( Q \) over \( E^0 \). This also shows that \( Q = Q_0 \). By writing \( s^{N_k} \) in terms of the basis \( BY \), we obtain a Weierstrass polynomial \( g_k(t) \) of degree \( N_k \) such that \( Q = E^0[s]/g_k(s) \).

Now note that \( s \) can be written as \( h_k(x) \) for some series \( h_k(t) \) of Weierstrass degree \( p^k \) with \( h_k(0) = 0 \). It follows that \( g_k(h_k(t)) \) and \( f_k(t) \) both have Weierstrass degree \( p^kN_k \). As \( g_k(0) = 0 \) we must have \( g_k(h_k(x)) = 0 \) in \( E^0[x]/f_k(x) \), so \( g_k(h_k(t)) \) is divisible by \( f_k(t) \). As both series have the same Weierstrass degree, we see that \( g_k(h_k(t)) \) is actually a unit multiple of \( f_k(t) \). By putting \( t = 0 \), we see that \( g_k(0) \) is a unit multiple of \( f_k(0) = p \). We can now repeat the argument that we gave for \( \bar{Q} \), and conclude that \( Q \) is also a regular local ring.

\textbf{Corollary 10.10.} The map \( \alpha : R \to Q \) is surjective, with kernel \( J \), so \( Q = R/J \). We therefore have \( \text{dim}_{E^0}(I) = \text{dim}_{E^0}(Q) = N_k \).

\textbf{Proof.} Recall that \( R \) is a quotient of \( E^0[c_1, \ldots, c_{p^k}] \), and we previously defined \( s_k = c_{p^k} \). In \( Q \) we find that \( c_j \) maps to \( (-1)^j \) times the \( j \)th elementary symmetric function in \( \{ [q^i](x) \mid 0 \leq i < p^k \} \). In particular, \( s_k \) maps to \( \pm s \), which makes it clear that \( \alpha \) is surjective. As \( Q \) is a free module of rank \( N_k \) over \( E^0 \), we see that \( \text{ker}(\alpha) \) must be a summand in \( R \), with \( \text{dim}_{E^0}(\text{ker}(\alpha)) = \text{dim}_{E^0}(R) - N_k \). Next, it is clear by construction that \( D' \otimes_{E^0} Q = \text{Map}([\text{Irr}_{p^k}(\Theta^*)], D') = R'/J' \), and it follows that \( J \leq \text{ker}(\alpha) \). However, \( J \) is also a summand, with \( \text{dim}_{E^0}(R/J) = \text{dim}_{D'}(R'/J') = N_k \), so the relation \( J \leq \text{ker}(\alpha) \) implies that \( J = \text{ker}(\alpha) \). We also noted in Proposition 10.4 that \( \text{dim}_{E^0}(I) = \text{dim}_{E^0}(R/J) \).

\textbf{Corollary 10.11.} The ring \( \bar{Q} = K^0 \otimes_{E^0} Q \) is just \( K^0[s]/(s^{N_k}) \), and \( \alpha \) induces an isomorphism \( \bar{R}/\bar{J} \to \bar{Q} \).
Remark 10.12. It is now clear that \( \text{spf}(Q) \) deserves to be thought of as the subscheme of irreducibles in \( \text{Div}^+(\mathbb{H}) \).

Proposition 10.13. The ideal \( \bar{I} \) is principal, and is a free module of rank one over \( \bar{R}/\bar{J} = \bar{Q} \). More precisely:

(a) Any element \( t \in \bar{I} \) is annihilated by \( \bar{J} \) and so induces a map \( \mu_t: \bar{Q} = \bar{R}/\bar{J} \to \bar{I} \) sending \( z + \bar{J} \) to \( zt \).

(b) The element \( s_k^{N_k-1}t \) always lies in the socle of \( \bar{R} \), and \( \mu_t \) is an isomorphism iff \( s_k^{N_k-1}t \neq 0 \).

(c) There exist elements \( t \in \bar{I} \) such that (b) holds.

Moreover, the ideal \( \bar{I} \) is also principal, and is a free module of rank one over \( R/J = Q \).

Proof. Claim (a) is clear from Lemma [10.6]. Also, \( \mu_t: \bar{Q} \to \bar{I} \) is an isomorphism iff it is an epimorphism iff \( t \) generates \( \bar{I} \), by comparison of dimensions.

Now put \( J' = \bar{J} + \bar{R}.s_k^{N_k-1} \). As ideals containing \( \bar{J} \) biject with ideals in \( \bar{R}/\bar{J} \), we see that \( J' \) is the smallest ideal that is strictly larger than \( \bar{J} \). We also see that \( J'/\bar{J} \) has dimension one over \( K^0 \), so it must be annihilated by the maximal ideal \( m \), so \( m.s_k^{N_k-1} \subseteq \bar{J} \), so \( m.s_k^{N_k-1} \bar{I} \subseteq \bar{J} \bar{I} = 0 \), so \( s_k^{N_k-1} \bar{I} \) is contained in the socle.

Next, as \( \bar{R} \) is a Frobenius algebra, the map \( L \to \text{ann}(L) \) is an order-reversing permutation of the set of ideals. Thus, for any \( t \in \bar{I} \) we have \( \text{ann}(\bar{R}t) \supseteq \text{ann}(\bar{I}) = \bar{J} \), with equality iff \( t \) generates \( I \). As \( J' \) is the smallest ideal strictly excluding \( \bar{J} \), we see that \( t \) generates \( \bar{I} \) if \( J't \neq 0 \) iff \( s_k^{N_k-1}t \neq 0 \). This proves (b). Moreover, as \( s_k^{N_k-1} \not\in \bar{J} = \text{ann}(\bar{I}) \) we see that \( s_k^{N_k-1} \bar{I} \neq 0 \); this proves (c).

Finally, Nakayama’s Lemma tells us that if we lift any generator of \( \bar{I} \) to \( I \), we get a generator of \( I \). As \( \text{ann}(I) = J \), it follows that the lifted element generates \( I \) freely as a module over \( R/J = Q \).

Proposition 10.14. \( \bar{I} \) is generated by \( s_k^{N_k-1} \), and the socle of \( \bar{R} \) is generated by \( s_k^{N_k-1} \) (so \( s_k^{N_k-1} \) is a \( K_0^* \)-multiple of the standard socle generator \( \text{tr}_1^{GL_k(F)}(1) \)).

Proof. In this section we have excluded the case \( k = 0 \), but here we partially restate it. The ideal \( \bar{I} \) is not defined, but we have \( K^0(BV_{p^k}) = K^0[s_0]/[p^r](s_0) \) and \( [p^r](s_0) = s_0^{N_0} = s_0^{N_0} \), so the socle is generated by \( s_0^{N_0-1} \) as claimed.

Now suppose that \( k > 0 \), and that we have already proved all claims for \( V_{p^k} \).

Put \( u = \text{res}_H^G(s_k) = s_k^{P_k-1} \in K^0(BH) \). The induction hypothesis means that \( w^{N_k-1} \) generates the socle of \( K^0(BH) \), and that \( w^{N_k-1} = 0 \). This in turn means that the element \( t = s_k^{N_k-1} \) has \( \text{res}_H^G(t) = 0 \), or in other words \( t \in \bar{I} \). On the other hand, we have \( s_k^{N_k-1}t = s_k^{N_k-1} \), which is nonzero by Proposition 10.13. It follows from Proposition 10.13 that \( t \) generates \( \bar{I} \), and that \( s_k^{N_k-1} \) is a nonzero element of the socle. As \( \bar{R} \) is a Frobenius algebra, we see that the socle has dimension one over \( K^0 \), so it is generated by \( s_k^{N_k-1} \).

11. Further relations in \( E \)-theory

In this section we will prove some additional interesting relations in \( E^0(BV) \). In particular, we will give a formula for the socle generator, in the sense of Definition 6.9.
We first need to generalise the definition of the map $\phi$ given by the Galois action.

**Definition 11.1.** For any $k \in \mathbb{N}$, we define $\psi^k : \overset{\circ}{F}^x \to \overset{\circ}{F}^x$ by $\psi^k(u) = u^k$. We also write $\psi^k$ for the induced self-map of $\mathbb{H} = \text{spf}(E^0(B(\overset{\circ}{F}^x)))$. This in turn gives self-maps of $\text{Div}_+^d(\mathbb{H}) = \mathbb{H}^d/\Sigma_d$ and $\text{Div}_+^d(\mathbb{H})^\Gamma$, which we also denote by $\psi^k$. We call these maps *Adams operations*.

**Remark 11.2.** The Adams operation $\psi^q$ is just the same as the operation $\phi$ coming from $\Gamma = \text{Gal}(\overset{\circ}{F}/F)$. For general $k \in \mathbb{N}$, we can use lambda operations (as in the original work of Adams) to define an operator $\psi^k$ on virtual representations, or a corresponding map $\text{Div}(\mathbb{H}) \to \text{Div}(\mathbb{H})$ of schemes, and we can then check that this extends our definition on $\text{Div}_+^d(\mathbb{H})$. We do not need this so we will not give further details here, but they can be found in [12].

**Remark 11.3.** The operation $\psi^p$ induces an endomorphism of the ring 
\[ K^0(BG_d) = F_p[c_1, \ldots, c_d] = F_p[x_1, \ldots, x_d]^{\Sigma_d}. \]
This sends $x_i$ to $[p](x_i) = x_i^{p^{n^x}}$, and it follows that $\psi^p$ is just the same as the Frobenius endomorphism, sending $a$ to $a^{p^{n^x}}$ for all $a$.

**Remark 11.4.** Any $\overset{\circ}{F}$-linear representation $V$ of an abelian group $A$ gives a divisor $D(V)$ defined over $\text{spf}(E^0(BA)) = \text{Hom}(A^*, \mathbb{H})$. Specifically, we can write $V$ as a direct sum of one-dimensional representations corresponding to characters $\alpha_i \in A^*$, and we have a tautological homomorphism $\phi : A^* \to \mathbb{H}$ defined over $\text{Hom}(A^*, \mathbb{H})$, giving the divisor $D(V) = \sum_i[\phi(\alpha_i)]$. We can pull back the representation $V$ along the homomorphism $p^k : A \to A$, and it is clear from the above discussion that $\psi^k(D(V)) = D((k.1_A)^*V)$. In particular, if the exponent of $A$ divides $k$ we find that $\psi^k(D(V)) = \dim(V).[0]$.

If we want to do computer calculations of $K^0(BG_d)$, then we need a formula for $[g](x)$, and then we need to do some manipulations with symmetric functions based on that. To make the calculation finite, we need to truncate our power series at an appropriate level. The following result will help us to decide which level is appropriate.

**Proposition 11.5.** Let $k$ be the largest integer such that $p^k \leq d$. Then for $D \in \text{Div}_+^d(\mathbb{H})^\Gamma$ we have $\psi^{p^{k+r}}(D) = d[0]$. Thus, for $u$ in the maximal ideal of $K^0(BG_d)$ we have $u^{p^{n(k+r)}} = 0$.

**Proof.** Remark [13.3] and Proposition [13.4] show that $\psi^{p^{k+r}}(D)$ becomes equal to $d[0]$ over $E^0(BA)$, for any abelian $p$-subgroup $A \leq G_d$. Equivalently, the Chern classes $c_1, \ldots, c_d$ of $\psi^{p^{k+r}}(D)$ map to zero in $E^0(BA)$. However, generalised character theory tells us that the restriction maps to abelian subgroups are jointly injective, so $\psi^{p^{k+r}}(D) = d[0]$ already over $E^0(BG_d)$. On the other hand, Remark [11.3] tells us that $\psi^{p^{k+r}}$ acts on $K^0(BG_d)$ as $u \mapsto u^{p^{n(k+r)}}$. This operator must therefore kill the ideal generated by the Chern classes, which is the whole of the maximal ideal. \qed

**Definition 11.6.** We let euler: $\text{Div}_+^d(\mathbb{H}) \to \mathbb{A}^1$ be the map given by $c_d$ on $\text{Div}_+^d(\mathbb{H})$. Equivalently, this is the unique map satisfying euler($0$) = 1 and euler($D + E$) = euler($D$) euler($E$) and euler($[a]$) = $x(a)$ for $a \in \mathbb{H}$.

**Proposition 11.7.** There is a unique function fix: $\text{Div}_+^d(\mathbb{H}) \to \mathbb{A}^1$ such that euler($\psi^q(D)$) = fix($D$) euler($D$)
for all $D \in \text{Div}^+(\mathbb{H})$. Moreover, this is invertible and satisfies $\text{fix}(0) = 1$ and $\text{fix}(D + E) = \text{fix}(D) \text{fix}(E)$ and $\text{fix}([0]) = q$.

Remark 11.8. We have defined $\text{fix}(D)$ for all effective divisors $D$, in a way that implicitly depends on our choice of coordinate $x$. We will primarily be interested in the case where $D$ is $\Gamma$-invariant, and we will check later that it is independent of $x$ in that case.

The condition $\text{euler}(\psi^q(D)) = \text{fix}(D) \text{euler}(D)$ implies that $\text{fix}(D) = 1$ in any context where $D$ is $\Gamma$-invariant and $\text{euler}(D)$ is not a zero-divisor. Of course this contrasts with the case $D = d[0]$, where $D = \psi^0(D)$ but $\text{euler}(D) = 0$ and $\text{fix}(D) = q^d$. One should think $\text{fix}(D)$ as something like $q$ raised to the power of the multiplicity of $[0]$ in $D$.

Remark 11.9. In more traditional language, fix might be called a cannibalistic class and denoted by $\rho^\psi$.

Proof of Proposition 11.12. It is standard that $x(qa) = [q]_F(x(a))$ for some power series $[q]_F(t)$ of the form $qt + O(t^2)$. We can thus write $[q]_F(t) = t \langle q \rangle_F(t)$ and $\text{fix}_1(a) = \langle q \rangle_F(x(a))$ so $x(qa) = \text{fix}_1(a) x(a)$ and $\text{fix}_1(0) = q$. Next, given a divisor $D$ of degree $d$ on $\mathbb{H}$ over a base scheme $T$ we have a norm map $\mathcal{O}_D \to \mathcal{O}_T$ which we can apply to $\text{fix}_1$ to get an element $\text{fix}(D) \in \mathcal{O}_T$. Equivalently, we define fix: $\text{Div}^+_d(\mathbb{H}) = \mathbb{H}^d/\Sigma_d \to \mathbb{A}^1$ to be the unique element such that $\text{fix}(D) = \prod_i \text{fix}_1(a_i)$ whenever $D = \sum_i [a_i]$. It is clear that this satisfies $\text{euler}(\psi^q(D)) = \text{euler}(\sum_i [q a_i]) = \prod_i [q]_F(x(a_i)) = \prod_i [\text{fix}_1(a_i) x(a_i)] = \text{fix}(D) \text{euler}(D)$.

As the element $c_d$ is not a zero divisor in the ring $\mathcal{O}_{\text{Div}^+_d(\mathbb{H})} = E^0 \text{BGL}_d(\mathbb{F}) = E^0[c_1, \ldots, c_d]$, the above equation characterises the map $\text{fix}$ uniquely. Note also that the element $\text{fix}(d[0]) = q^d$ is invertible in $E^0$ and all of $\text{Div}^+_d(\mathbb{H})$ is infinitesimally close to $d[0]$ so fix is invertible on $\text{Div}^+_d(\mathbb{H})$ as claimed.

Lemma 11.10. For $k \in \mathbb{Z}_p$ with $kp^{p-1} = 1$ we have $\text{fix}(\psi k D) = \text{fix}(D)$. In particular, the divisor $D = \psi^{-1}(D)$ has $\text{fix}(D) = \text{fix}(D)$.

Proof. Recall from Remark 5.3 that for $k$ as above we have $x(ka) = [k]_F(x(a)) = k x(a)$, and thus $x(ka) = k x(ka)$. It follows that for any $D \in \text{Div}^+_d(\mathbb{H})$ we have $\text{euler}(\psi k D) = k^d \text{euler}(D)$ and $\text{euler}(\psi \psi^k D) = k^d \text{euler}(\psi D)$. By working in the universal case where euler($D$) is not a zero divisor, we conclude that fix$(D) = 1$. □

Definition 11.11. For any $V \in \text{Rep}(\Theta^*) = [\Theta^*, \mathcal{V}]$ we put $\text{Fix}(V) = \{ v \in V \mid \theta.v = v \text{ for all } \theta \in \Theta^* \}$.

Note that this is a finite-dimensional vector space over $F$, so $|\text{Fix}(V)|$ is a power of $q$, and thus lies in $1 + p^r \mathbb{Z}$. Note also that the element $\text{fix} \in E^0(BV)$ gives rise to a map $\text{Rep}(\Theta^*) \to D'$, by generalised character theory.

Proposition 11.12. For any representation $V \in \text{Rep}(\Theta^*)$, the character value $\text{fix}(V)$ is just $|\text{Fix}(V)|$.

Proof. Because $\text{Fix}(V \oplus W) = \text{Fix}(V) \oplus \text{Fix}(W)$ and $\text{fix}(V \oplus W) = \text{fix}(V) \text{fix}(W)$ we can reduce to the case where $V$ is irreducible. If $V$ is just $F^r$ (with trivial $\Theta^*$-action) then we have $\text{fix}(V) = \langle q \rangle(0) = q = |\text{Fix}(V)|$ as required. Suppose instead that
V has nontrivial action, and let $L$ denote a one-dimensional subrepresentation of $\hat{F} \otimes_F V$. We then see that $\hat{F} \otimes_F V \simeq \bigoplus_{i=0}^{k-1} L^q^i$ for some $k$ with $L^k \simeq L$. Let $x$ be the image in $D'$ of the Euler class of $L$; we then get

$$\text{fix}(V) = \prod_i \langle q \rangle([q^i](x))$$

As $L$ is nontrivial we see that $x \neq 0$. As $D_A$ is an integral domain it will be harmless to work in the field of fractions where we have $\langle q \rangle([q^i](x)) = [q^{i+1}]/[q^i](x)$. The whole product therefore cancels to give $\text{fix}(V) = 1$, which is the same as $|\text{Fix}(V)|$, as required.

**Proposition 11.13.** The element $s = \prod_{i=0}^{d-1} (\text{fix} - q^i) \in E^0(BV_d)$ is the transfer of 1 from the trivial groupoid, and so is the standard generator of the socle.

**Proof.** Consider a representation $V \in \text{Rep}(\Theta^r)$. We then have $\text{fix}(V) = q^{\dim(\text{Fix}(V))}$ with $0 \leq \dim(\text{Fix}(V)) \leq d$. It follows that $s(V) = 0$ iff $\dim(\text{Fix}(V)) < d$ iff $A$ acts nontrivially on $V$. On the other hand, if $A$ acts trivially we have $s(V) = \prod_{i=0}^{d-1} (q^d - q^i)$, which is well-known to be the same as the order of $GL_d(F)$. This means that $s$ has the same character values as the transfer of 1, so it is the same as the transfer of 1. □

**Corollary 11.14.** The socle of $K^0(BV_d)$ is generated by $(\text{fix} - 1)^d$. Thus, if $d = p^k$ then the socle is generated by $(\text{fix} - 1)^{p^k} = \text{fix}^{p^k} - 1$.

**Proof.** This follows because $q^i = 1 \pmod{p}$ for all $i$. □

**Remark 11.15.** By comparing Corollary 11.12 with Proposition 11.14 we see that $(\text{fix} - 1)^{p^k}$ must be a unit multiple of $s_k^{N_k-1} = \text{euler}^{N_k-1}$ in $K^0(BV_{p^k})$. We have not yet been able to find a direct, equational proof of this, but we have verified it by strenuous computer calculation in some very small cases.

Recall that the socle is by definition the annihilator of the kernel of the restriction map from $E^0(BGL_d(F))$ to $E^0(\text{point}) = E^0(BGL_0(F))$. We are also interested in the kernel of restriction to $E^0(BGL_{d-1}(F))$.

**Proposition 11.16.** If we define $\sigma: \mathcal{V} \rightarrow \mathcal{V}$ by $\sigma(V) = V \oplus F$, then

(a) The map $\sigma^*: E^0(BV) \rightarrow E^0(B\mathcal{V})$ is surjective, with kernel generated by the element $\text{euler} \in E^0(BV)$.

(b) The element $\text{fix} - 1 \in E^0(BV)$ is a unit multiple of $\sigma(1)$.

(c) The ideal $\text{ann(euler)} = \text{ann(} \text{ker(} \sigma^* \text{)})$ is generated by $\text{fix} - 1$.

We will deduce this from a more elaborate statement involving some auxiliary groupoids, which we now introduce.

**Definition 11.17.** We will construct a diagram of groupoids of the following shape:
The groupoid \( \hat{\mathcal{V}} \) consists of pairs \((V, v)\) with \(V \in \mathcal{V}\) and \(v \in V\). The morphisms from \((V_0, v_0)\) to \((V_1, v_1)\) are \(F\)-linear isomorphisms \(f: V_0 \to V_1\) with \(f(v_0) = v_1\). This splits as the disjoint union of subgroupoids \(\hat{\mathcal{V}}_0 = \{(V, v) \mid v = 0\}\) and \(\hat{\mathcal{V}}_1 = \{(V, v) \mid v \neq 0\}\).

The functor \(\pi: \hat{\mathcal{V}} \to \mathcal{V}\) is given by \(\pi(V, v) = V\), and \(\pi_0\) and \(\pi_1\) are just restrictions of \(\pi\).

The remaining functors are \(\sigma(V) = V \oplus F\) and \(\beta(V) = (V \oplus F, (0, 1))\) and \(\alpha(V, v) = V/Fv\).

**Remark 11.18.** It is convenient to start with \(\hat{\mathcal{V}}\), because it has an evident symmetric monoidal direct sum operation making \(\pi\) a symmetric monoidal functor. However, we are primarily interested in the subgroupoid \(\hat{\mathcal{V}}_1\).

Proposition 11.16 clearly follows from the following extended version.

**Proposition 11.19.**

(a) The functor \(\pi\) is a covering, and the restriction \(\pi_0\) is an isomorphism of groupoids.

(b) The functors \(\alpha\) and \(\beta\) are not equivalences, but nonetheless they give mutually inverse isomorphisms in Morava \(E\)-theory.

(c) The maps \(\pi_1^*: E^0(B\mathcal{V}) \to E^0(B\hat{\mathcal{V}}_1)\) and \(\sigma^*: E^0(B\mathcal{V}) \to E^0(B\mathcal{V})\) are surjective, with kernel generated by euler.

(d) The element \(\text{fix} -1\) is equal to \((\pi_1)_!(1)\), and is a unit multiple of \((\pi_1)^!\).

(e) The ideal \(\text{ann}(\text{euler}) = \text{ann}(\text{ker}(\sigma^*)) = \text{ann}(\text{ker}((\pi_1)^*))\) is generated by \((\pi_1)^!\) or by \(\text{fix} -1 = (\pi_1)_!(1)\).

**Proof.**

(a) If we have an object \((V, v) \in \hat{\mathcal{V}}\) and a morphism \(f: V \to W\) in \(\mathcal{V}\) then \(f\) counts as a morphism \(\hat{\mathcal{V}}\) such that this works. This means that \(\pi\) is a covering. It is clear that \(\pi_0\) is an isomorphism, with inverse \(V \mapsto (V, 0)\).

(b) It is clear that \(\alpha\beta \simeq 1\), and that \(\alpha\) and \(\beta\) give mutually inverse bijections between \(\pi_0(V)\) and \(\pi_0(\hat{\mathcal{V}})\). Moreover, the map

\[ \beta: \text{Aut}_\mathcal{V}(V, v) \to \text{Aut}_\mathcal{V}(V/Fv) \]

is easily seen to be surjective, with kernel of order coprime to \(p\). It follows by Proposition 5.5 that \(\alpha\) and \(\beta\) give mutually inverse isomorphisms in Morava \(E\)-theory.

(c) In view of (b) it will suffice to show that \(\sigma^*\) has the stated properties. Put \(r_i = \phi^*(c_i) - c_i\) as usual, and

\[ \overline{R}_d = E^0(B\overline{\mathcal{V}}_d) = E^0[c_1, \ldots, c_d] \]
\[ L_d = (r_1, \ldots, r_d) \leq \overline{R}_d \]
\[ R_d = E^0(B\mathcal{V}_d) = \overline{R}_d/L_d. \]

The restriction map \(\overline{R}_d \to \overline{R}_{d-1}\) sends \(c_d\) to 0 and \(c_i\) to \(c_i\) for \(i < d\), so it is surjective. It is compatible with the action of \(\phi^*\), so it sends \(r_d\) to 0 and \(r_i\) to \(r_i\) for \(i < d\). It follows that the induced map \(L_d \to L_{d-1}\) is also surjective, as is the induced map \(R_d \to R_{d-1}\). A diagram chase therefore
shows that the map
\[ R_d c_d = \ker(\sigma^*: R_d \to R_{d-1}) \to \ker(\sigma^*: R_d \to R_{d-1}) \]
is again surjective, so \( \ker(\sigma^*: R_d \to R_{d-1}) \) is generated by \( c_d \). Moreover, \( c_d \) is the same as euler on \( V_d \), so claim (c) follows.

(d) Consider a representation \( V \in \text{Rep}_d(\Theta^*) \). Because \( \pi_1 \) is a covering, we see that \((\pi_1)_!(1)(V)\) is just the number of objects \((V, v) \in [\Theta^*, \hat{V}]_1\) lying over \( V \), or in other words the number of nonzero \( \Theta^* \)-fixed points in \( V \), which is \((\text{fix} - 1)(V)\). As the generalised character map is injective, it follows that \( \text{fix} - 1 = (\pi_1)_!(1) \).

Next, recall that \( \alpha_!((1)) = 1 \). so \( \alpha_!((1)) = 1 \). By considering the restriction to the spine, we see that \( z \) is invertible. As \( \beta_! \) is inverse to \( \alpha_! \) and \( \sigma = \pi_1 \beta \) we have \((\pi_1)_! = \sigma \alpha_! \). It follows that
\[ z \sigma(1) = \sigma((\sigma^*)(z)) = \sigma \alpha_!(1) = (\pi_1)_!(1) = \text{fix} - 1, \]
so \( \sigma_!(1) \) is a unit multiple of \( \text{fix} - 1 \) as claimed.

(e) Claim (e) now follows from Corollary 6.7.

\[ \square \]

**Definition 11.20.** We define \( \text{hom}: \text{Div}^+(\mathbb{H}) \times \text{Div}^+(\mathbb{H}) \to \mathbb{A}^1 \) by \( \text{hom}(D, E) = \text{fix}(D \ast E) \). Here the map \( D \mapsto D \) is induced by \(-1: \mathbb{H} \to \mathbb{H}\), and \( \ast \) denotes convolution of divisors, so
\[ \text{hom} \left( \sum_i [a_i], \sum_j [b_j] \right) = \text{fix} \left( \sum_{i,j} [b_j - a_i] \right). \]

**Remark 11.21.** It is easy to see that this satisfies \( \text{hom}(D, E) = \text{hom}(E, D) \) and \( \text{hom}(D_0 + D_1, E_0 + E_1) = \text{hom}(D_0, E_0) \text{hom}(D_0, E_0) \text{hom}(D_1, E_0) \text{hom}(D_1, E_1) \).

In other words, it is a symmetric biexponential function.

Using Proposition 11.16 we can also describe the element \( \text{hom} \in E^0(BV^2) \) as a transfer. To explain the details, we need some more auxiliary groupoids and functors.

**Definition 11.22.**

(a) Let \( S \) be the groupoid of pairs \((U, W)\) with \( W \in V \) and \( U \leq W \). We will write such pairs as \((U \leq W)\).

(b) Let \( \mathcal{H} \) be the groupoid of triples \((U, V, m)\) with \( U, V \in V \) and \( m \in \text{Hom}_F(V, U) \).

(c) Define functors as follows:
\[
\begin{array}{ccc}
\hat{V} & \xleftarrow{\xi} & \mathcal{H} \xrightarrow{\zeta} V^2 \\
\pi & & \zeta \\
V & \xleftarrow{\xi} & V^2 \xrightarrow{\tau} S \xrightarrow{\rho} V
\end{array}
\]
First, it is easy to see that the functor \( F \) is a covering and the left square is a pullback so it is also a homotopy pullback. This implies that

\[
\pi(V, v) = V
\]

\[ \omega(U \leq W) = (U, W/U) \]

\[ \tau(U, V) = (U \leq U \oplus V) \]

\[ \sigma(U, V) = U \oplus V. \]

(d) Given \( (U, V, m) \in \mathcal{H} \), define \( n \in \text{Aut}(U \oplus V) \) by \( n(u, v) = (u + m(v), v) \). This gives a natural automorphism of the functor \( \tau \zeta \).

**Proposition 11.23.**

(a) \( L\tau \) and \( L\alpha \) are equivalences, and are inverse to each other.

(b) \( L\beta = L\rho = L\sigma \lambda : LS \rightarrow LV \).

(c) \( R\alpha \) and \( R\tau \) are equivalences, and are inverse to each other.

(d) \( R\beta = R\rho = R\alpha \sigma : LV \rightarrow LS \).

(e) The element \( \text{hom} \in E^0(BV^2) \) is the same as \( \zeta(1) \) or \( \tau^* \pi(1) \) or \( \alpha(1)^{-1} \).

The message here is that in many contexts, it is harmless to assume that any short exact sequence of \( F \)-vector spaces comes equipped with a specified splitting.

**Proof.** First, it is easy to see that the functor \( \pi \) is a covering and the left square is a pullback so it is also a homotopy pullback. This implies that

\[ \zeta(1) = \zeta \xi(1) = \xi^* \pi(1) = \xi^*(\text{fix}) = \text{hom}. \]

Next, we claim that the middle square is a homotopy pullback. Indeed, we certainly have a homotopy pullback square

\[
\begin{array}{ccc}
\mathcal{H}' & \xrightarrow{\zeta_0} & \mathcal{V}^2 \\
\zeta \downarrow & & \downarrow \\
\mathcal{V}^2 & \xrightarrow{\zeta} & S
\end{array}
\]

where

\( \mathcal{H}' = \{(U_0, V_0, U_1, V_1, k) | U_i, V_j \in \mathcal{V}, k: \tau(U_0, V_0) \xrightarrow{\zeta} \tau(U_1, V_1)\} \)

and \( \zeta_i(U_0, V_0, U_1, V_1, k) = (U_i, V_i) \). We can define \( \phi: \mathcal{H} \rightarrow \mathcal{H}' \) by \( \phi(U, V, m) = (U, V, U, V, n) \), where \( n \) is as in Definition 11.22(d). In the opposite direction, if \( (U_0, V_0, U_1, V_1, k) \in \mathcal{H}' \) then \( k \) is an isomorphism \( U_0 \oplus V_0 \rightarrow U_1 \oplus V_1 \) that sends \( U_0 \) to \( U_1 \), so it can be decomposed into components \( k_U: U_0 \xrightarrow{\zeta} U_1 \) and \( k_V: V_0 \xrightarrow{\zeta} V_1 \) and \( k': V_0 \rightarrow U_1 \). We can thus define \( \psi(U_0, V_0, U_1, V_1, k) = (U_0, V_0, k_U^{-1} \circ k') \), and check that this gives a functor \( \psi: \mathcal{H}' \rightarrow \mathcal{H} \) that is inverse to \( \phi \). We also have \( \zeta_i \phi = \zeta \) for \( i = 0, 1 \). It follows that the middle square is a homotopy pullback as claimed, and thus that \( \zeta(1) = \zeta \xi^*(1) = \tau^* \pi(1) \), which proves most of (e).

It is also easy to see that \( \alpha \tau \simeq 1 \) and \( \rho = \sigma \alpha \) and \( \beta \tau = \sigma \). Next, we see that there is a split extension

\[
\text{Hom}(V, U) \xrightarrow{\pi} \text{Aut}(U \leq U \oplus V) \xleftarrow{\pi} \text{Aut}(U) \times \text{Aut}(V)
\]

Here \( \text{Hom}(V, U) \) is a vector space over \( F \) and so has order coprime to \( p \). We can thus apply Proposition 5.8 to see that \( L\tau \) is an equivalence, or we can apply Proposition 5.8 to see that \( L\alpha \) is an equivalence. As \( \alpha \tau = 1 \), we see that \( L\alpha \) and \( L\tau \) are inverse to each other. As \( \rho = \sigma \alpha \) we have \( L\rho = L\sigma L\alpha \). As \( \beta \tau = \sigma \) we have
Suppose we have an element $f$ when necessary to emphasise the difference from for all finitely supported functions $u$. We will use scheme-theoretic language, and treat $R$ elements of $A$ function $[0]$ which is 1 on $\sigma$ and are compatible with the grading. We also assume that $R$ is a commutative monoid in the category of formal schemes. We will use integral notation for the standard inner product on $R$. We will write $U \oplus V$ for $\sigma(U,V)$. We also put $R_k = E^0(BA_k)$.

As with $V$, we have a diagonal product $\delta^* : R_k \otimes R_k \to R_k$, a convolution product $\sigma_l : R_i \otimes R_j \to R_{i+j}$, and a coproduct $\sigma^* : R_k \to \bigoplus_{k=i+j} R_i \otimes R_j$. We will write $f \otimes g$ for $\sigma_l(f \otimes g)$. We will usually write $fg$ for $\delta^*(f \otimes g)$, but we may write $f \bullet g$ when necessary to emphasise the difference from $f \otimes g$.

We write $M_k = \text{spf}(R_k)$ and $M = \bigsqcup_k M_k$, so $M$ is a commutative monoid in the category of formal schemes. We will use scheme-theoretic language, and treat elements of $R_k$ as functions on $M_k$. We will use integral notation for the standard inner product on $R_k$, as discussed in Section 5. We will say that a function $f$ on $M$ is finitely supported if it vanishes on $M_k$ for $k \gg 0$, or equivalently it lies in $\bigoplus_k R_k$; this insures that $\int_M f$ is defined.

For example, the convolution product is characterised by the identity

$$\int(f \otimes g)(m)u(m) \, dm = \iint f(m_1)g(m_2)u(m_1 + m_2) \, dm_1 \, dm_2$$

for all finitely supported functions $u$. The unit for the convolution product is the function $[0]$ which is 1 on $A_0$ and 0 on $A_k$ for $k > 0$. The coproduct is given by $(\sigma^* f)(m,n) = f(m+n)$, and the counit is $f \mapsto f(0)$.

**Definition 12.1.** Suppose we have an element $t \in (R \hat{\otimes} R)^\times$, or in other words, an invertible function on $M^2$. We define a twisted convolution product on $R$ by the rule

$$f \times_t g = \sigma_l(t.(f \otimes g)).$$

This is characterised by

$$\int(f \times_t g)(m)u(m) \, dm = \iint f(m_1)g(m_2)t(m_1,m_2)u(m_1 + m_2) \, dm_1 \, dm_2$$

for all finitely supported functions $u$ on $M$. 

$L\beta L\tau = L\sigma$, and we can compose with $L\alpha$ to get $L\beta = L\sigma L\alpha = L\rho$. We have now proved (a) and (b); claims (c) and (d) follow by taking adjoints.
Lemma 12.2. Suppose that $t$ is biexponential, in the sense that $t(0, m) = t(m, 0) = 1$ and $t(m + n, p) = t(m, p)t(n, p)$ and $t(m, n + p) = t(m, n)t(m, p)$ for all $n, m, p$. Then the product $\times_t$ is associative, with $[0]$ as a two sided unit. Moreover, the induced $k$-fold product is given by

$$f_1 \times_t \cdots \times_t f_k = \mu(t^{(k)})(f_1 \otimes \cdots \otimes f_k),$$

where

$$t^{(k)}(m_1, \ldots, m_k) = \prod_{1 \leq i < j \leq k} t(m_i, m_j).$$

Moreover, the product $\times_t$ is commutative if $t$ is symmetric (in the sense that $t(m, n) = t(n, m)$).

Proof. First, we have

$$\int ([0] \times_t f)(m)u(m) \, dm = \int \int [0](m_1)f(m_2)t(m_1, m_2)u(m_1 + m_2) \, dm_1 \, dm_2.$$ 

Here $[0]$ is the characteristic function of $M_0 = \{0\}$, so this reduces to $\int f(m)t(0, m)u(m) \, dm$. As $t(0, m) = 1$ this is just $\int f(m)u(m) \, dm$. By the perfectness of the inner product, we must therefore have $[0] \times_t f = f$. Essentially the same argument gives $f \times_t [0] = f$. Next, we have

$$\int (f \times_t (g \times_t h))(m)u(m) \, dm = \int \int f(m_1)(g \times_t h)(m_23)t(m_1, m_23)u(m_1 + m_23) \, dm_1 \, dm_23$$

$$= \int \int \int f(m_1)g(m_23)h(m_3)u(m_1 + m_23) \, dm_1 \, dm_23 \, dm_3$$

$$\int ((f \times_t g) \times_t h)(m)u(m) \, dm = \int \int ((f \times_t g)(m_12)h(m_3)u(m_12 + m_3) \, dm_12 \, dm_3$$

$$= \int \int \int f(m_1)g(m_23)h(m_3)u(m_1 + m_23) \, dm_1 \, dm_23 \, dm_3.$$ 

The biexponential property gives

$$t(m_23)3t(m_1, m_2 + m_3) = t^{(3)}(m_1, m_2, m_3) = t(m_1, m_2)t(m_1 + m_2, m_3),$$

and it follows that

$$f \times_t (g \times_t h) = \mu(t^{(3)})(f \otimes g \otimes h)) = (f \times_t g) \times_t h.$$ 

This can be extended inductively for products of more than three factors. The commutativity statement is clear. \qed

We will assume from now on that our twisting function $t \in R \otimes R$ is biexponential.

Definition 12.3. Recall that we can regard $\pi_0(\mathcal{A})$ as a discrete subcategory of $\mathcal{A}$, which we call the spine. With respect to the usual topology on $E^0(B\mathcal{A})$, an element $f \in E^0(B\pi_0(\mathcal{A}))$ is topologically nilpotent iff $f$ maps to zero in $K^0(B\pi_0(\mathcal{A})) = \text{Map}(\pi_0(\mathcal{A}), K^0)$. We also say that $f$ is a strong unit if $f - 1$ is topologically nilpotent.

We next prove some results showing that the difference between $\times$ and $\times_t$ is irrelevant in various contexts.

Lemma 12.4. If $U$ and $V$ are $\bullet$-ideals in $R$ then $U \times V$ is the same as $U \times_t V$, and this is again a $\bullet$-ideal.
Proof: It is clear that \( U \overline{\otimes} V \) is a \( \bullet \)-ideal in \( E^0(BA^2) = R \overline{\otimes} R \), and \( t \) is a unit in \( R \overline{\otimes} R \) so \( U \overline{\otimes} V = t \circ (U \overline{\otimes} V) \). By applying \( \sigma_t \) we deduce that \( U \times V = U \times_t V \).

Also, if \( f \in R \) then \( f \circ (U \times V) = f \circ \sigma_t(U \overline{\otimes} V) = \sigma_t(\sigma^t(f) \circ (U \overline{\otimes} V)) \). As \( U \overline{\otimes} V \) is a \( \bullet \)-ideal, this lies in \( \sigma_t(U \overline{\otimes} V) = U \times V \), as claimed. \( \Box \)

Corollary 12.5.

(a) If \( U \) is a \( \bullet \)-ideal in \( R \), then it is a \( \times \)-subring iff it is a \( \times_t \)-subring.

(b) Suppose that (a) holds, and that \( V \leq U \), and that \( V \) is also a \( \bullet \)-ideal. Then \( V \) is a \( \times \)-ideal in \( U \) iff it is a \( \times_t \)-ideal.

(c) Suppose that (b) holds, so we have an induced \( \times \)-product and an induced \( \times_t \)-product on \( U/V \), and also an induced \( \bullet \)-product making \( U/V \) into a module over \( R \). Suppose also that \( t \) is a strong unit, and that \( U/V \) is annihilated by topologically nilpotent elements. Then the \( \times \)-product and the \( \times_t \)-product on \( U/V \) are the same.

Proof.

(a) The lemma gives \( U \times U = U \times_t U \).

(b) The lemma gives \( U \times V = U \times_t V \).

(c) Let \( J \) be the ideal of topologically nilpotent elements in \( R \), so we have \( J \circ U \leq V \) by assumption. Now put \( V' = V \overline{\otimes} U + U \overline{\otimes} V \), so that \( (U/V)^{\otimes 2} = U^{\otimes 2}/V' \). The ideal of topologically nilpotent elements of \( R \overline{\otimes} R \) is \( J' = J \overline{\otimes} R + R \overline{\otimes} J \), so \( J', U^{\otimes 2} \leq V' \). By assumption we have \( t - 1 \in J' \), so for \( f, g \in U \) we have \( t \circ (f \otimes g) \in (f \otimes g) + V' \). Applying \( \sigma_t \) to this gives \( f \times_t g \in f \times g + V \), as required. \( \Box \)

Definition 12.6. We say that an element \( u \in R^\times \) is exponentially quadratic if the map

\[
(\delta u)(m_0, m_1) = u(m_0 + m_1)u(m_0)^{-1}u(m_1)^{-1}
\]

is biexponential.

Proposition 12.7. Let \( u \) be exponentially quadratic, and let \( t \) be biexponential, and put \( w = \delta(u) \bullet t \). Then the map \( f \mapsto u \bullet f \) gives an isomorphism \( (R, \times_t) \to (R, \times_w) \).

Proof. The condition \( w = \delta(u) \bullet t \) can be rearranged as \( w \bullet (u \otimes u) = \sigma^t(u) \bullet t \).

This gives

\[
(u \bullet f) \times_w (u \bullet g) = \sigma_t(w \bullet (u \otimes u) \bullet (f \otimes g)) = \sigma_t(\sigma^t(u) \bullet t \bullet (f \otimes g))
\]

\[
= u \bullet \sigma_t(t \bullet (f \otimes g)) = u \bullet (f \times_t g).
\]

Corollary 12.8. Suppose that \( t = r^2 \) for some symmetric biexponential function \( r \). Then \( (R, \times_t) \cong (R, \times) \).

Proof. The function \( u(m) = r(m, m) \) is exponentially quadratic with \( \delta(u) = t \). \( \Box \)

Corollary 12.9. If \( t \) is a symmetric biexponential function and is a strong unit, then \( (R, \times_t) \cong (R, \times) \).

Proof. We have assumed that \( p > 2 \), so the squaring map is an automorphism of the group of strong units. If we let \( r \) denote the unique strong unit with \( r^2 = t \), it follows easily that \( r \) is again symmetric and biexponential, so we can use Corollary 12.8. \( \Box \)
Proposition 12.10. Suppose that \((\mathcal{A}, t)\) is as above, and \((\mathcal{A}', t')\) is of the same type. Let \(\phi: \mathcal{A}' \to \mathcal{A}\) be a symmetric monoidal functor, so we can build a homotopy-commutative diagram as follows, in which the bottom right region is a homotopy pullback.

\[
\begin{array}{ccc}
(A')^2 & \xrightarrow{\phi^2} & A^2 \\
\downarrow{\sigma'} & & \downarrow{\phi} \\
A' & \xleftarrow{\phi} & A
\end{array}
\]

Suppose also that \(\kappa(t') = \tilde{\phi}^*(t) \in E^0(B\mathcal{P})\). Then \(\phi^*\) gives a ring map \((R, \times_t) \to (R', \times_{t'}\).

Proof. Consider elements \(f_0, f_1 \in R\) and put \(f = f_0 \otimes f_1\). We have

\[
\phi^*(f_0 \times_t f_1) = \phi^*\sigma(t \cdot f) = \tilde{\sigma}(\tilde{\phi}^*(t \cdot f)) = \tilde{\sigma}(\tilde{\phi}^*(t) \cdot \tilde{\phi}^*(f)) = \tilde{\sigma}(\kappa(t) \cdot \tilde{\phi}^*(f)) = \tilde{\sigma}(\kappa(t') \cdot \tilde{\phi}^*(f)) = \tilde{\sigma}(\kappa(t' \cdot t') \cdot \tilde{\phi}^*(f)) = \phi^*(f_0) \times_{t'} \phi^*(f_1).
\]

\(\square\)

13. The Harish-Chandra (co)product

We have used the direct sum functor to make \(E^0(B\mathcal{V})\) into a bialgebra. There is another natural way to make \(E^0(B\mathcal{V})\) into a bialgebra, related to the theory of Harish-Chandra induction in representation theory, which we will explain in this section. However, our main conclusion will be that the Harish-Chandra structure is very closely related to our original structure.

Definition 13.1. We define \(\chi: \mathcal{V} \to \mathcal{V}\) by \(\chi(V) = V^*\) on objects, and \(\chi(u) = (u^*)^{-1}\) on morphisms.

Definition 13.2. We define \(\mu: L\mathcal{V}^{(2)} \to L\mathcal{V}\) by \(\mu = \beta\alpha\), where \(\alpha\) and \(\beta\) are as in Definition 11.22. Dually, we define \(\nu = \alpha\beta: L\mathcal{V} \to L\mathcal{V}^{(2)}\).

Proposition 13.3. \(\mu\) is an associative and unital product on \(L\mathcal{V}\), with \(\chi\) as an anti-involution. Dually, \(\nu\) is a coassociative and counital coproduct.

Proof. Let \(S(r)\) be the groupoid of tuples \((V_1 \leq \cdots \leq V_r)\) in \(\mathcal{V}\). Define functors \(\mathcal{V}^r \xrightarrow{\alpha(r)} S(r) \xrightarrow{\beta(r)} \mathcal{V}\) by

\[
\alpha(r)(V_1 \leq \cdots \leq V_r) = (V_1, V_2/V_1, \ldots, V_r/V_{r-1})
\]

\[
\beta(r)(V_1 \leq \cdots \leq V_r) = V_r.
\]

Then put \(\mu(r) = \beta(r)\alpha(r)^{\dagger}: L\mathcal{V}^{(r)} \to L\mathcal{V}\). In terms of our earlier definitions, we have \(S(2) = S\) and \(\alpha(2) = \alpha\) and \(\beta(2) = \beta\) and \(\mu(2) = \mu\). We claim that \(\mu(r + s) = \mu(2) \circ (\mu(r) \wedge \mu(s))\). To see this, consider the diagram
\[ \alpha(r,s)(V_1 \leq \cdots \leq V_{r+s}) = ((V_1 \leq \cdots \leq V_r), (V_{r+1}/V_r \leq \cdots \leq V_{r+s}/V_r)) \]
\[ \beta(r,s)(V_1 \leq \cdots \leq V_{r+s}) = (V_r \leq V_{r+s}). \]

The top triangle and the square commute on the nose, and the left triangle commutes up to natural isomorphism. Moreover, \( \beta(r,s) \) and \( \beta(r) \times \beta(s) \) are coverings, and the square is a pullback (on the nose), so it is also a homotopy pullback. The generalised Mackey property [13, Proposition 8.6] therefore gives
\[ \beta(r,s) \alpha(r,s) = \alpha(2)(\beta(r) \land \beta(s)). \]

We now compose on the left by \( \beta(2) \), and on the right by \( \alpha(r) \times \alpha(s) \). The two commutative triangles give \( \alpha(r,s) \alpha(r,s) = \alpha(r + s) \) and \( \beta(2) \beta(r,s) = \beta(r + s) \) so we get
\[ \beta(r + s) \alpha(r + s) = \mu(2)(\beta(r) \land \beta(s))(\alpha(r) \land \alpha(s)) \]

or in other words \( \mu(r + s) = \mu(2) \circ (\mu(r) \land \mu(s)) \) as claimed. From this it follows easily that there is an associative ring structure with \( \mu(0) \) as the unit, \( \mu(1) \) as the identity map, \( \mu = \mu(2) \) as the product, and \( \mu(r) \) as the \( r \)-fold product for general \( r \).

All that is left is to prove that \( \chi \) is an anti-involution for \( \mu \), or in other words that the following diagram commutes, where \( \tau \) is the twist map:
\[ \begin{array}{c}
LV \land LV \xrightarrow{\chi \land \chi} LV \land LV \\
\mu \downarrow \quad \tau \downarrow \quad \mu
\end{array} \]

To prove this, we define \( \chi: S(2) \to S(2) \) by \( \chi(V_1 \leq V_2) = (\text{ann}(V_1) \leq V_2^*) \). On both \( \mathcal{V} \) and \( S(2) \) we have \( \chi^2 \simeq 1 \), so \( \chi \) is an equivalence and \( \chi^1 = \chi^{-1} = \chi \). Similarly, we have \( \tau^1 = \tau^{-1} = \tau \). As \( V_1^*/\text{ann}(V_2) \simeq (V_2/V_1)^* \), we see that the following diagram commutes up to natural isomorphism:
\[ \begin{array}{c}
\mathcal{V} \times \mathcal{V} \leftarrow \alpha \quad S(2) \xrightarrow{\beta} \mathcal{V} \\
\downarrow (\chi \times \chi) \tau \quad \chi \downarrow \chi \quad \downarrow (\chi \times \chi)
\end{array} \]
In more detail: We now apply \( \tau \) to see that \( (E^0(V))^r \) has a coassociative coring structure with \( \nu(r) = \alpha(r)\beta(r) \) as the \( r \)-fold coproduct. We have seen that \( \mathbb{R} \) is isomorphic to \( (E^0(V))^r \), which we call the Harish-Chandra coproduct. Similarly, \( \mu^\ast \) gives a coassociative and counital coproduct, which we call the Harish-Chandra coproduct.

**Corollary 13.4.** \( L^r V \) has a coassociative coring structure with \( \nu(r) = \alpha(r)\beta(r) \) as the \( r \)-fold coproduct.

**Proof.** This follows from the proposition by duality.

**Definition 13.5.** Given \( f, g \in E^0(BV) \) we define \( f \ast g = \nu^\ast(f \otimes g) \). Corollary 13.4 tells us that this gives an associative and unital product on \( E^0(V) \), which we call the Harish-Chandra product. Similarly, \( \mu^\ast \) gives a coassociative and counital coproduct, which we call the Harish-Chandra coproduct.

**Remark 13.6.** Recall that in Definition 13.20 we defined an element \( \text{hom} \in E^0(BV)^2 \) and observed that it is a symmetric biexponential function. We can therefore define a twisted product \( \times_{\text{hom}^{-1}} \) as in Definition 13.21.

**Proposition 13.7.** The Harish-Chandra product \( f \ast g \) is the same as \( f \times_{\text{hom}^{-1}} g \). Moreover, the ring \( (E^0(BV), \ast) \) is isomorphic to \( (E^0(BV), \times) \).

**Proof.** Consider the element \( h = (f \otimes g) \circ \text{hom}^{-1} \in E^0(BV)^2 \). We can apply Proposition 11.10 to the functors \( \alpha \) and \( \tau \) to see that \( \tau(h) = \alpha^\ast(h) \ast v \), where \( v = \tau(1) \). We have seen that \( \alpha^\ast \) is inverse to \( \tau^\ast \), so \( v = \alpha^\ast \tau^\ast \tau(1) \), and this is the same as \( \alpha^\ast \circ \text{hom} \) by Proposition 11.23. We therefore have \( \tau(h) = \alpha^\ast(h) \ast (\text{hom} \circ \alpha^\ast(f \otimes g) \circ \text{hom}^{-1}) = f \times_{\text{hom}^{-1}} g \), whereas the right hand side becomes \( f \ast g \). This proves the first claim.

Next, it is clear that if we restrict \( \text{hom} \) to the basepoint in \( BV_i \times BV_j \) we get \( q^{ij} \in E^0 \), which maps to 1 in \( K^0 \). It follows that \( \text{hom} \) is a strong unit, so we can appeal to Corollary 12.9 to see that \( (E^0(BV), \ast) \simeq (E^0(BV), \times) \).

**Proposition 13.8.** For the functors \( \alpha : S(2) \to V^2 \) and \( \beta : S(2) \to V \), the corresponding maps of \( D'_0(G) \) are

\[
(Lo)[U \leq W] = [U, W/U] \\
(L\beta)[U \leq W] = [W] \\
(Ra)[U, V] = |\text{Hom}(V, U)|^{-1}[U \leq U \oplus V] = |\text{Hom}(V, U)|^{-1}(L\tau)[U, V] \\
(R\beta)[V] = \sum_{U \leq V} [U \leq V].
\]

**In more detail:**

(a) All the objects \( U, V \) and \( W \) are finite-dimensional \( F \)-linear representations of \( \Theta^\ast \).

(b) The symbol \( \text{Hom}(V, U) \) refers to \( \Theta^\ast \)-equivariant \( F \)-linear homomorphisms.

(c) In the formula for \( R\beta \), the sum is indexed by subrepresentations of \( V \).
Proof. The formulae for $L\alpha$, $L\beta$ and $L\tau$ are immediate. As $\beta$ is a covering, the formula for $R\beta$ follows from Proposition 9.23. Next, $(Ra)[U,V]$ is by definition a sum over isomorphism classes that are sent by $R\beta$ to $[U,V]$. As Maschke’s theorem applies to $[\Theta^*,:V]$, the only such isomorphism class is $[U \leq U \oplus V]$. The formula also involves a numerical factor, namely $|Aut(U,V)|/|Aut(U \leq U \oplus V)|$, where the first $Aut$ is in the category $[\Theta^*,V^2]$ and the second is in $[\Theta^*,S(2)]$. As we mentioned above, there is a split extension

$$\Hom(V,U) \rightarrow \Aut(U \leq U \oplus V) \leftarrow \Aut(U) \times \Aut(V)$$

and this shows that the numerical factor is $|\Hom(V,U)|$. □

Corollary 13.9. The action of the Harish-Chandra product and coproduct on $(D')^0(V)$ is given by

$$(f \ast g)(W) = \sum_{U \leq W} f(U)g(W/U) = \sum_{W=U \oplus V} |\Hom(V,U)|^{-1} f(U)g(V)$$

$$(\mu \ast h)(U,V) = |\Hom(V,U)|^{-1} h(U \oplus V).$$

Proof. We can dualise the proposition to get descriptions of the effect of $L\alpha$, $L\beta,R\alpha$ and $R\beta$ on $(D')^0(V)$, then just use the definitions $\mu = (L\beta)(Ra)$ and $\nu = (L\alpha)(R\beta)$. This gives the formula for $\mu \ast h$, and also the first formula for $(f \ast g)(W)$. Proposition 13.7 gives the second formula for $(f \ast g)(W)$ (or one can deduce it from the first formula by a little linear algebra). □

Definition 13.10. We define a non-symmetric biexponential function $t$ on $\spf(E^0(BV^2)) = (\Div^+ (H) \Gamma)^2$ by

$$t((D_0,D_1),(E_0,E_1)) = \hom(D_0,E_1) \hom(D_1,E_0)^2.$$ 

Thus, the character values are

$$t((U_0,U_1),(V_0,V_1)) = |\Hom(U_0,V_1)||\Hom(U_1,V_0)|^2$$

for $U_0,U_1,V_0,V_1 \in \Rep(\Theta^*)$.

Proposition 13.11. The map $\sigma^*$ gives a ring homomorphism $(E^0(BV),\times) \rightarrow (E^0(BV^2),\times)$.

Proof. We will need various auxiliary groupoids and functors between them.

(a) $\mathcal{P}_0$ is just $V^4$. A typical object will be written as $(U_0,V_0,U_1,V_1)$.

(b) $\mathcal{P}_1$ is the groupoid of quadruples $(W_0,W_1,U,V)$ where $W_0,W_1 \in V$, and $U$ and $V$ are subspaces of $W_0 \oplus W_1$ such that $W_0 \oplus W_1$ is the internal direct sum of $U$ and $V$.

(c) $\mathcal{P}_2$ is the groupoid of triples $(W_0,W_1,U)$ where $U \leq W_0 \oplus W_1$.

(d) $\mathcal{P}_3$ is the groupoid of quadruples $(W_0,W_1,U_0,U_1)$ with $U_0 \leq W_0$ and $U_1 \leq W_1$.

We next need functors as shown below. Not all of the squares will commute.
These are defined by

\[
\begin{array}{ccccccc}
\nu^2 & \nu^2 & \nu^2 & \nu^2 & \nu^2 & \nu^2 \\
\zeta_0 & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_0 \\
\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_0 \\
\xi_0 & \xi_1 & \xi_2 & \xi_3 & \xi_0 \\
\nu^2 & \nu^2 & \nu^2 & \nu^2 & \nu^2 \\
\end{array}
\]

\[
\zeta_0(U_0, V_0, U_1, V_1) = (U_0 \oplus V_0, U_1 \oplus V_1) \\
\xi_0(U_0, V_0, U_1, V_1) = (U_0 \oplus U_1, V_0 \oplus V_1) \\
\zeta_1(W_0, W_1, U, V) = (W_0, W_1) \\
\xi_1(W_0, W_1, U, V) = (U, V) \\
\zeta_2(W_0, W_1, U) = (W_0, W_1) \\
\xi_2(W_0, W_1, U) = (U, (W_0 \oplus W_1)/U) \\
\zeta_3(W_0, W_1, U_0, U_1) = (W_0, W_1) \\
\xi_3(W_0, W_1, U_0, U_1) = (U_0 \oplus U_1, W_0/U_0 \oplus W_1/U_1)
\]

\[
\phi_0(U_0, V_0, U_1, V_1) = (U_0 \oplus V_0, U_1 \oplus V_1, U_0 \oplus U_1, V_0 \oplus V_1) \\
\phi_1(W_0, W_1, U, V) = (W_0, W_1, U) \\
\phi_2(W_0, W_1, U) = (W_0, W_1, U \cap W_0, \pi_{W_1}(U)) \\
\phi_3(W_0, W_1, U_0, U_1) = (U_0, W_0/U_0, U_1, W_1/U_1).
\]

We also put \( \theta = \phi_3\phi_2\phi_1: \mathcal{P}_1 \to \mathcal{P}_0 \). It is easy to see that the composite \( \theta\phi_0 = \phi_3\phi_2\phi_1\phi_0: \mathcal{P}_0 \to \mathcal{P}_0 \) is naturally equivalent to the identity.

Let us say that two parallel functors \( \lambda_0 \) and \( \lambda_1 \) are \( L \)-equivalent if \( L\lambda_0 = L\lambda_1 \), which implies that \( R\lambda_0 = R\lambda_1 \). It is straightforward to check that the first three squares on the top commute up to natural isomorphism, as do the first and fourth squares on the bottom. The second square on the bottom also commutes up to natural isomorphism, for a slightly less obvious reason. Indeed, for \((W_0, W_1, U, V) \in \mathcal{P}_1 \) we have \( \xi_1(W_0, W_1, U, V) = (U, V) \) and \( \xi_3\phi_1(W_0, W_1, U, V) = (U, (W_0 \oplus W_1)/U) \) but \( W_0 \oplus W_1 \) is assumed to be the internal direct sum of \( U \) and \( V \) so the evident composite \( V \to W_0 \oplus W_1 \to (W_0 \oplus W_1)/U \) gives the required natural isomorphism.

We claim that the remaining two squares commute up to \( L \)-equivalence. The key point is that the functors \( \beta: (U \leq W) \to W \) and \( \rho: (U \leq W) \to U \oplus W/U \) are \( L \)-equivalent, as we saw in Proposition 11.23. The functors \( \zeta_3 \) and \( \zeta_0\phi_3 \) are essentially \( \beta^2 \) and \( \rho^2 \), so the top right square commutes up to \( L \)-equivalence. Similarly, the functors \( \xi_2 \) and \( \xi_3\phi_2 \) are, up to isomorphism, the composites of \( \beta^2 \) and \( \rho^2 \) with the functor \( \mathcal{P}_2 \to \mathcal{S}^2 \) given by

\[
(W_0, W_1, U) \mapsto \left( (U \cap W_0 \leq U), \frac{U+W_0}{U} \leq \frac{W_0 \oplus W_1}{U} \right).
\]

Now consider the diagram

\[
\begin{array}{ccccccc}
\nu^2 & \nu^2 & \nu^2 & \nu^2 & \nu^2 & \nu^2 \\
\zeta_0 & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_0 \\
\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_0 \\
\xi_0 & \xi_1 & \xi_2 & \xi_3 & \xi_0 \\
\nu^2 & \nu^2 & \nu^2 & \nu^2 & \nu^2 \\
\end{array}
\]
One can check that this commutes up to natural isomorphism, and that the bottom right region is a homotopy pullback, as in Proposition 12.10. However, we will not apply that Proposition directly, but will give a different argument of a similar kind.

Suppose we have \( f, g \in E^0(B\mathcal{V}) \) and put \( h = f \otimes g \). We then have \( \sigma^*(f \times g) = \sigma^*\sigma(h) \). The pullback property of the above diagram implies that \( \sigma^*\sigma(h) = (\xi_1)_! \zeta_1^!(h) \). Because the previous diagram commutes up to \( L \)-equivalence, this is the same as \( (\xi_0)_! \theta \theta^* \zeta_0^!(h) \). Here \( \xi_0 \) is just the direct sum functor for the category \( \mathcal{V}^2 \), whereas \( \zeta_0 \) is just \( \sigma^2 \), so \( \zeta_0^*(h) = \sigma^* (f) \otimes \sigma^*(g) \). Moreover, we have a Frobenius reciprocity formula \( \theta \theta^*(k) = s \cdot k \) for all \( k \in E^0(B\mathcal{P}_0) \), where \( s = \theta(1) \). We therefore conclude that \( \sigma^*\sigma(f \times g) = \sigma^*(f) \times_s \sigma^*(g) \), and it will suffice to check that \( s \) is the same as \( t \). We can do this in generalised character theory.

Firstly, \( \phi_1 \) is a covering. Given \( (W_0, W_1, U) \in [\Theta^*, \mathcal{P}_2] \) we can always choose a subrepresentation \( V_0 \) complementary to \( U \) in \( W_0 \oplus W_1 \), and then any other complement is the graph of a homomorphism \( V_0 \to U \). The number of complements is thus equal to

\[
|\text{Hom}(V_0, U)| = |\text{Hom}(U, V_0)| = |\text{Hom}(U, (W_0 \oplus W_1)/U)| =
|\text{Hom}(U, W_0 \oplus W_1)|/|\text{Hom}(U, U)|^{-1}.
\]

We therefore have

\[
(\phi_1)_!(1)(W_0, W_1, U) = |\text{Hom}(U, W)|/|\text{Hom}(U, U)|^{-1},
\]

where \( W = W_0 \oplus W_1 \).

Next, \( \phi_2 \) is also a covering. If \( (W_0, W_1, U) \) is in the preimage of \( (W_0, W_1, U_0, U_1) \) then \( U/U_0 \) is the graph of a homomorphism \( U_1 \to W_0/U_0 \), and this construction is bijective, so the number of preimages of \( (W_0, W_1, U_0, U_1) \) is \( |\text{Hom}(U_1, W_0/U_0)| \). If \( (W_0, W_1, U) \) is one such preimage, then \( U \) is isomorphic to \( U_0 \oplus U_1 \), so the value of \( (\phi_1)_!(W_0, W_1, U) \) is independent of the choice of preimage. Putting this together, we see that \( (\phi_2\phi_1)_!(1)(W_0, W_1, U_0, U_1) \) is equal to

\[
|\text{Hom}(U, W)|/|\text{Hom}(U, U)|^{-1}|\text{Hom}(U_1, W_0/U_0)|,
\]

where \( U = U_0 \oplus U_1 \) and \( W = W_0 \oplus W_1 \).

Next, \( \phi_3 \) is not a covering, but it is not hard to understand \( \phi_3 \) anyway. Recall the general framework: for a functor \( \alpha : \mathcal{A} \to \mathcal{B} \), an element \( f \in (D^r)^0(\mathcal{A}) \) and an object \( b \in [\Theta^*, \mathcal{B}] \) we have

\[
(\alpha f)(b) = \sum_{\alpha(a)=b} |\mathcal{B}(b)|/|\mathcal{A}(a)| f(a).
\]

Consider an object \( P_0 = (U_0, V_0, U_1, V_1) \in [\Theta^*, \mathcal{P}_0] \), and put \( W_i = U_i \oplus V_i \) and \( P_3 = (W_0, W_1, U_0, U_1) \in [\Theta^*, \mathcal{P}_3] \). We find that \( [P_3] \) is the only isomorphism class mapping to \( [P_0] \), and that \( |\text{Aut}(P_0)|/|\text{Aut}(P_3)| = |\text{Hom}(U_0, V_0)|^{-1}|\text{Hom}(U_1, V_1)|^{-1} \).
This gives
\[ \theta_t(1)(P_0) = \frac{|\text{Hom}(U, W)||\text{Hom}(U_1, V_0)|}{|\text{Hom}(U, U)||\text{Hom}(U_0, V_0)||\text{Hom}(U_1, V_1)|} \]

By expanding this out and cancelling in the obvious way, we get
\[ \theta_t(1)(P_0) = |\text{Hom}(U_0, V_1)||\text{Hom}(U_1, V_0)|^2 = t(P_0), \]
as required.

**Remark 13.12.** Proposition 13.11 should ideally be embedded in a larger context. For any finite set \( X \), let \( \mathcal{V}[X] \) denote the category of bundles of finite-dimensional \( F \)-vector spaces over \( X \). Any map \( p: X \to Y \) gives a functor \( \sigma_p: \mathcal{V}[X] \to \mathcal{V}[Y] \) by \( \sigma_p(V)_y = \bigoplus_{p(x) = y} V_x \). This in turn gives maps \( \sigma_p^*: E^0(B\mathcal{V}[Y]) \to E^0(B\mathcal{V}[X]) \) and \( \sigma_p^!: E^0(B\mathcal{V}[X]) \to E^0(B\mathcal{V}[Y]) \). Next, for any map \( m: X^2 \to \mathbb{Z} \) we define \( h_m \in E^0(B\mathcal{V}[X]) \) by \( h_m(D) = \prod_{x,y} \text{hom}(D_x, D_y)^{m(x,y)} \). We then define \( \mu_m: E^0(B\mathcal{V}[X]) \to E^0(B\mathcal{V}[X]) \) to be multiplication by \( h_m \). Maps of the form \( \sigma_p^* \) can be composed in an obvious way, as can maps of the form \( \sigma_p^! \). One can also check that \( \mu_n \circ \sigma_p^! = \sigma_p^! \circ \mu_p^*(n) \) and \( \sigma_p^* \circ \mu_n = \mu_p^*(n) \circ \sigma_p^* \). Now suppose we have maps \( X \xrightarrow{\mu} Y \xrightarrow{\sigma} Z \), and we want to understand \( \sigma_p^* \circ \sigmaernels \). We can form a pullback square

\[
\begin{array}{ccc}
W & \xrightarrow{i} & X \\
\downarrow{j} & & \downarrow{p} \\
Z & \xrightarrow{q} & Y.
\end{array}
\]

We conjecture that there exists \( m: W^2 \to \mathbb{Z} \) such that
\[ \sigma_q^* \sigma_p^! = \sigma_{j*} \mu_m \sigma_p^*: E^0(B\mathcal{V}[X]) \to E^0(B\mathcal{V}[Z]). \]

This \( m \) will certainly not be unique, because of the fact that \( \text{hom}(D_0, D_1) = \text{hom}(D_1, D_0) \). One might hope that this was the only source of nonuniqueness, but one can find counterexamples to that. We expect that there should be some straightforward combinatorial characterisation of the set of maps \( m \) for which this works, but so far we have not found one. Assuming that this works, we find that the class of maps of the form \( \sigma_{j*} \mu_m \sigma_p^* \) is closed under composition.

### 14. Groupoids of line bundles

In this section, we will compare \( E^0(B\mathcal{V}) \) with \( E^0(B\mathcal{X}\mathcal{L}) \), where \( \mathcal{X}\mathcal{L} \) is the groupoid of finite sets equipped with an \( F \)-linear line bundle. We also consider the groupoid \( \mathcal{X} \) of finite sets. All of these groupoids have evident finite type gradings, and all our algebraic constructions will again be interpreted in the graded category.

We proved in [10] that \( E^0(B\mathcal{X}) \) is a polynomial ring, and that \( \text{spf}(\text{Ind}(E^0(B\mathcal{X}))) \) is the moduli scheme \( \text{Sub}(\mathbb{G}) \) of finite subgroups of the formal group \( \mathbb{G} = \text{spf}(E^0(BS^1)) \). Recall that we have chosen an embedding \( F^\times \to \mu_\infty(\mathbb{C}) \), which gives an isomorphism \( \mathbb{H} \simeq \mathbb{G} \). This is unique up to the action of \( \text{Aut}(\mu_\infty(\mathbb{C})) = \mathbb{Z}^\times \), and every subgroup scheme is necessarily preserved by that action. We thus have a canonical identification of \( \text{spf}(\text{Ind}(E^0(B\mathcal{X}))) \) with \( \text{Sub}(\mathbb{H}) \). We have a functor \( X \mapsto F[X] \) from \( \mathcal{X} \) to \( \mathcal{V} \), and the induced map
\[ \text{Sub}(\mathbb{H}) \to \text{spf}(E^0(B\mathcal{X})) \to \text{spf}(E^0(B\mathcal{V})) = \text{Div}^+(\mathbb{H})^\Gamma \]
is just the evident inclusion. We have an analogous statement for $\mathcal{X}\ell$ as follows:

**Definition 14.1.** We define $\text{Coset}(\mathbb{H})$ to be the moduli scheme of pairs $(A,a)$ where $A$ is a finite subgroup scheme of $\mathbb{H}$, and $a \in \mathbb{H}/A$. The group $\Gamma$ acts on this by $\phi.(A,a) = (A,qa)$, so

$$\text{Coset}(\mathbb{H})^\Gamma = \{(A,a) \mid (q-1)a = 0\} = \{(A,a) \mid p^r a = 0\}.$$

**Proposition 14.2.** $E^0(B\mathcal{X}\ell)$ is a polynomial ring, with $\text{spf}(\text{Ind}(E^0B\mathcal{X}\ell)) = \text{Coset}(\mathbb{H})^\Gamma$.

*Proof.* This relies on the theory of power operations in Morava $E$-theory. The paper [9] by Rezk provides a good account of this.

The projection $\mathcal{X}\ell \to \mathcal{X}$ gives a map $\text{Ind}(E^0(\mathcal{X})) \to \text{Ind}(E^0(\mathcal{X}\ell))$ and thus a map $\text{spf}(\text{Ind}_d(E^0(\mathcal{X}\ell))) \to \text{Sub}_d(\mathbb{H})$.

The $E_\infty$ structure of the spectrum $E$ provides a power operation $P: E^0(X) \to E^0(D_d(X))$ for any space $X$, where $D_d(X) = E\Sigma_d \times X^d$ is the $d$’th extended power. This is multiplicative but not additive. However, there is a transfer ideal $T \leq E^0(D_d(X))$ such that the composite $T^*: E^0(X) \to E^0(D_d(X))/T$ is a ring homomorphism. In particular, we can take $X$ to be a point, so $D_d(X) = B\Sigma_d$, so $\text{spf}(E^0(B\Sigma_d)/T) = \text{Sub}_d(\mathbb{H})$. Put $S = \text{spf}(E^0)$. There is an obvious projection $\pi_0: \text{Sub}_d(\mathbb{H}) \to S$, and there is also another map $\pi_1 = \text{spf}(T^*)$ in the same direction.

The key property of $\pi_1$ is that there is a canonical isomorphism $\pi_0^*(\mathbb{H})/A \to \pi_1^*(\mathbb{H})$, where $A$ is the tautological subgroup of $\pi_0^*(\mathbb{H})$ given by the universal property of $\text{Sub}_d(\mathbb{H})$. (This is proved by taking $X = \mathbb{C}P^\infty$ in the above construction.)

We can now take $X = B\mathcal{L} = BGL_1(F)$. Standard methods then identify $D_d(X)$ with $B(\Sigma_d \times GL_1(F))$ or with $B\mathcal{X}\ell_d$. The relevant ideal $T$ in $E^0(B\mathcal{X}\ell_d)$ is just the ideal of decomposables, so we get a map $T^*: E^0(B\mathcal{L}) \to \text{Ind}_d(E^0(B\mathcal{X}\ell))$. By naturality, the projection $B\mathcal{L} \to 1$ gives a commutative diagram

$$\begin{array}{ccc}
E^0 & \xrightarrow{T^*} & \text{Ind}_d(E^0(B\mathcal{X})) \\
\downarrow & & \downarrow \\
E^0(B\mathcal{L}) & \xrightarrow{T^*} & \text{Ind}_d(E^0(B\mathcal{X}\ell))
\end{array}$$

and thus a commutative diagram

$$\begin{array}{ccc}
\text{spf}(\text{Ind}_d(E^0(B\mathcal{X}\ell))) & \xrightarrow{\pi_1} & \mathbb{H}[q^k-1] \\
\downarrow & & \downarrow \\
\text{Sub}(\mathbb{H}) & \xrightarrow{\pi_1} & S
\end{array}$$

Because the bottom map is $\pi_1$, we have to consider

$$\pi_1^*(\mathbb{H})[q^k-1] \cong (\pi_0^*(\mathbb{H})/A)[q^k-1].$$

Putting this together, we get a map

$$\text{spf}(\text{Ind}_d(E^0(B\mathcal{X}\ell))) \to \text{Coset}_d(\mathbb{H})^\Gamma.$$
from $E_0(\mathcal{B} \mathcal{L})$ by applying a functor $T$. As $E_\ast(\mathcal{B} \mathcal{L})$ is a free module over $E_\ast$ we are in the simplest case of the theory, where $T$ always produces polynomial algebras. We also have $\mathcal{T}(M_\ast \oplus N_\ast) = \mathcal{T}(M_\ast) \otimes \mathcal{T}(N_\ast)$, so $\text{Ind}(\mathcal{T}(M_\ast))$ is an additive functor of $M_\ast$. Multiplication by an element $a \in E_0$ gives an endomorphism of $M_\ast$ and thus an endomorphism of $\text{Ind}(\mathcal{T}(M_\ast))$, and a naturality argument shows that this is multiplication by $\mathcal{T}(a)$. Using this we find that $\text{Ind}(\mathcal{T}(E_\ast)) = \text{Ind}(\mathcal{T}(E_\ast)) \otimes_{E_\ast} M_\ast$. Here $\text{Ind}(\mathcal{T}(E_\ast))$ is the dual of $\mathcal{O}_{\text{Sub}_{S}(\mathcal{H})}$, and the tensor product is formed using $\mathcal{T}$. After dualising this and unwinding we find that $E_0(D_4(\mathcal{X})) / T = \mathcal{O}_{\text{Sub}_{S}(\mathcal{H})} \otimes E_0 E^0(\mathcal{X})$ for any $\mathcal{X}$, where again the tensor product is formed using $\mathcal{T}$. Our main claim follows from this. 

\textbf{Appendix A. The Atiyah-Hirzebruch spectral sequence for a cyclic group}

Put $C = \{ z \in \mathbb{C} \mid z^p = 1 \}$. There is a unique homomorphism $C \to \mathbb{F}_p$ sending $e^{2\pi i / p}$ to 1, and we write $a^H$ for the corresponding class in $H^1(BC)$. The inclusion $C \to S^1$ gives a line bundle over $BC$ with Euler classes $x^H \in H^2(BC)$ and $x^K \in K^0(BC)$. We therefore have an AHSS

$$H^\ast(BC; K^\ast) = P[x] \otimes \mathcal{E}[a] \otimes K^\ast \Rightarrow K^\ast(BC) = K^\ast[x^K] / [p^k](x^K).$$

Here $x^K$ is represented by $ux^H$ in the AHSS. As is well-known and easy to see, the only possible way that the differentials can work is to have $d_r = 0$ for $r \neq 2p^k - 1$, and $d_{2p^k - 1}(a) = tu^{-1}(ux)^{p^k}$ for some $t \in \mathbb{F}_p^\times$. For most purposes the value of $t$ is not important. However, if we do not know the value, then it creates some overhead of bookkeeping and notation. Because of this, the following result is convenient.

\textbf{Proposition A.1.} In the above AHSS we have $d_{2p^k - 1}(a^H) = u^{-1}(ux^H)^{p^k}$ (so $t = 1$).

The proof will be given after some preliminary discussion.

\textbf{Definition A.2.} Let $L$ be a complex line bundle over a space $\mathcal{X}$, equipped with a Hermitian inner product. Let $S(L)$ be the unit circle bundle in $L$, and let $\pi_L : S(L) \to \mathcal{X}$ be the projection. Let $X^L = E(L) \cup \{ \infty \}$ be the Thom space, and let $\zeta_L : \mathcal{X} \to X^L$ be the zero section. This can be identified with the cofibre of $\pi_L : S(L)_+ \to X_+$, so we have a connecting map $\delta_L : X^L \to \Sigma S(L)_+$. To identify $X^L$ as a cofibre we implicitly need a homeomorphism $f : [0, 1] \to [0, \infty]$, which we can take to be $f(t) = t / (1 - t)$, so $f^{-1}(t) = t / (1 + t)$. We also take $\Sigma Y_+$ to be $([0, 1] \times Y) / \{(0, 1) \times Y \}$. The map $\delta_L$ is then given by $\delta_L(x, tv) = f^{-1}(t) \wedge (x, v)$ for $(x, v) \in S(L)$ and $t \in [0, \infty]$.

We also have a diagonal map $\Delta_L : E(L) \to \mathcal{X} \times E(L)$ given by $\Delta_L(x, v) = (x, v)$, and this has a unique continuous extension $\Delta_L : X^L \to \mathcal{X} + \wedge X^L$. We write $t(L)$ for the Morava $K$-theory Thom class in $K^0(\mathcal{X})$. We can use $\Delta_L$ to make $K^\ast(X^L)$ into a module over $K^\ast(\mathcal{X})$, and the Thom isomorphism theorem tells us that it is freely generated as such by $t(L)$.

\textbf{Definition A.3.} Now fix $m \geq 1$, and let $M$ be the $m$th tensor power of $L$. Define $\psi : S(L) \to S(M)$ by $\psi(x, v) = (x, v^{\otimes m})$. We also extend this to give a map $\psi : X^L \to X^M$ by $\psi(x, tv) = (x, tv^{\otimes m})$ for $(x, v) \in S(L)$ and $t \in [0, \infty]$.

\textbf{Remark A.4.} It is easy to see that the following diagrams commute:
\[
\begin{array}{cccccc}
S(L)_+ & \xrightarrow{\pi_L} & X_+ & \xrightarrow{\zeta_L} & S^L & \xrightarrow{\delta_L} & \Sigma S(L)_+ \\
\psi & \downarrow & & \psi & \downarrow & \Sigma \psi_+ & \\
S(M) & \xrightarrow{\pi_M} & X_+ & \xrightarrow{\zeta_M} & S^M & \xrightarrow{\delta_M} & \Sigma S(M)_+ \\
\end{array}
\]

\[
X^L \xrightarrow{\Delta_L} X_+ \wedge X^L \\
\psi \downarrow \\
X^M \xrightarrow{\Delta_M} X_+ \wedge X^M.
\]

**Lemma A.5.** The map

\[
\psi^* : K^*(X).t(M) = \tilde{K}^*(X^M) \to \tilde{K}^*(X^L) = K^*(X).t(L)
\]

is given by

\[
\psi^*(a.t(M)) = a.(m)(e(L)).t(L)
\]

for all \(a \in K^*(X)\).

**Proof.** The right hand diagram in Remark A.4 shows that \(\psi^*\) is \(K^*(X)\)-linear, so we need only consider \(\psi^*(t(M))\). This must have the form \(c t\) for some unique element \(a \in K^*(X)\). As \(\psi^*_L = \zeta_M\) it follows that

\[
\zeta^*_M(t(M)) = \zeta^*_L(c t(M)) = c e(L).
\]

On the other hand, we also have

\[
\zeta^*_M(t(M)) = e(M) = [m](e(L)) = \langle m \rangle(e(L)).e(L),
\]

so \((c - \langle m \rangle(e(L))) e(L) = 0\). This does not immediately complete the proof because \(e(L)\) may be a zero-divisor. However, all our constructions are natural, so it will suffice to prove that \(c = \langle m \rangle(e(L))\) in the universal case of the tautological bundle over \(\mathbb{C}P^\infty\). Here \(K^*(X) = K^*[e(L)]\) and so \(e(L)\) is a regular element and we get \(c = \langle m \rangle(e(L))\) as required. \(\Box\)

We now specialise to the case where \(X = \mathbb{C}P^{r-1} = P(\mathbb{C}^r)\) for some \(r \leq \infty\), and \(L\) is the tautological bundle. Then the map \(v \mapsto (\mathbb{C}v, v)\) identifies the space \(S^{2r-1} = S(\mathbb{C}^r)\) with \(S(L)\). Similarly, the map \(C_m v \mapsto (\mathbb{C}v, v \otimes m)\) identifies the space \(S(\mathbb{C}^r)/C_m\) with \(S(M)\). From this point of view, the map \(\psi : S(L) \to S(M)\) is just the projection \(S(\mathbb{C}^r) \to S(\mathbb{C}^r)/C_m\).

**Lemma A.6.** The space \(P(\mathbb{C}^r)^L\) can be identified with \(P(\mathbb{C}^{r+1})^L\), in such a way that the zero section \(\zeta_L\) becomes the obvious inclusion \(P(\mathbb{C}^r) \to P(\mathbb{C}^{r+1})\).

**Proof.** Define \(f : S(\mathbb{C}^r) \times \mathbb{C} \to E(L)\) by \(f(v, z) = (S^1 v, zv)\). If we let \(S^1\) act on \(S(\mathbb{C}^r) \times \mathbb{C}\) by \(u.(v, z) = (uv, uz)\) then this gives a homeomorphism \((S(\mathbb{C}^r) \times \mathbb{C})/S^1 \to E(L)\). Now define \(g : S(\mathbb{C}^r) \times \mathbb{C} \to S(\mathbb{C}^{r+1})\) by

\[
g(v, z) = (v, z)/\sqrt{1 + z\bar{z}}.
\]

This is \(S^1\)-equivariant and so induces a map \(\overline{g} : E(L) \to \mathbb{C}P^r\). This is easily seen to be a homeomorphism from \(E(L)\) to the complement of a single point in \(\mathbb{C}P^r\), so we can pass to the one-point compactification to get a homeomorphism \((\mathbb{C}P^{r-1})^L \to \mathbb{C}P^r\) as claimed. \(\Box\)

**Lemma A.7.** The space \(BC_m\) can be identified with \(S(\mathbb{C}^\infty)/C_m\). This has a CW structure where the \((2r - 1)\)-skeleton is \(S(\mathbb{C}^r)/C_m\), and the \((2r)\)-skeleton is the image of \(S(\mathbb{C}^r \oplus \mathbb{R})\) in \(S(\mathbb{C}^{r+1})/C_m\). Moreover, the \((2r)\)-skeleton is also the cofibre of the map \(\psi_+ : S(\mathbb{C}^r)_+ \to (S(\mathbb{C}^r)/C_m)_+\).
Proof. It is standard that \( S(\mathbb{C}^\infty) \) is contractible and that the action of \( C_m \) is free so that \( S(\mathbb{C}^\infty)/C_m \) is a model for the homotopy type \( BC_m \). Now define \( \phi\colon [0,1] \times S(\mathbb{C}^r) \to S(\mathbb{C}^{r+1})/C \) by
\[
\phi(t,v) = C.(\cos(\pi t/2)v, \sin(\pi t)/2).
\]
The image is the 2r-skeleton of \( BC_m \). Moreover, we have \( \phi(0,v) = \psi(v) \) and \( \phi(1,v) = C.(0,1) \) for all \( v \), but otherwise \( \phi \) is injective. This allows us to identify the \((2r)\)-skeleton with the cofibre of \( \psi_+ \).

Recall that the differentials in the AHSS for \( BC_m \) are determined by the skeleta and the attaching maps. The above lemma identifies the quotient skeleton \( \Sigma S(\mathbb{C}^r) \) with \( \Sigma S(\mathbb{C}^r) \) with \( \Sigma S(\mathbb{C}^r) \), which gives
\[
E^{2r,j-2r}_1 = \tilde{K}^j(\Sigma S(\mathbb{C}^r)) = \tilde{K}^{j-1}(S(\mathbb{C}^r)).
\]

Corollary A.8. Suppose that \( a \in \tilde{K}^1(S(\mathbb{C}^r)/C_m) \), and let \( a_1 \) be the restriction of \( a \) in the group
\[
\tilde{K}^1(S(C)/C_m) = \tilde{K}^1(\text{skeleton}(BC_m)) = E^{1,0}_1.
\]
Note that
\[
\psi^*(a) \in \tilde{K}^1(S(\mathbb{C}^r)) = E^{2r,2-2r}_1.
\]
Then \( a_1 \) and \( \psi^*(a) \) survive to the page \( E^{2*}_{2r-1} \) where we have \( d_{2r-1}(a_1) = \psi^*(a) \).

Proof. Just unwind the definitions. \qed

We now specialise further to the case where \( m = p^k \) and \( r = p^k \). The cofibration
\[
(S(C)/C_m)_+ \xrightarrow{\pi_M} \mathbb{C}P^{r-1}_+ \xrightarrow{\delta_M} (\mathbb{C}P^{r-1})^M \xrightarrow{\delta_M} \Sigma(S(C)/C_m)_+
\]
gives an exact sequence relating \( K^*(S(\mathbb{C}^r)/C_m) \) to the kernel and cokernel of the map
\[
\zeta^*_M \colon \tilde{K}^*(\mathbb{C}P^{r-1}) = (K^*[xK]/x_K^r).t(M) \to K^*[xK]/x_K^r = K^*(\mathbb{C}P^{r-1}).
\]
Here \( \zeta^*_M(t(M)) = e(M) = [p^k](x_K) = x^r = 0 \). As \( K^*(\mathbb{C}P^{r-1}) \) is in even degrees there are no extension problems, so there is an unique element \( a_K \in \tilde{K}^1(S(\mathbb{C}^r)/C_m) \) such that \( \delta_M(\Sigma a) = u^{-1}t(M) \in K^2((\mathbb{C}P^{r-1})^M) \). We then find that \( K^*(S(\mathbb{C}^r)/C_m) \) is freely generated by \( \{1, a_K\} \) as a module over \( K^*(\mathbb{C}P^{r-1}) \). We can perform the same analysis with \( r = 1 \) to see that \( (a_K)_1 \) corresponds to \( a_H \in H^1(BC_m; K^0) \). We now want to understand \( \psi^*(a_K) \in \tilde{K}^1(S(\mathbb{C}^r)) \). We have seen that there is a commutative diagram
\[
\begin{array}{ccc}
\mathbb{C}P^r & \xrightarrow{\delta_L} & \Sigma S(\mathbb{C}^r)_+ \\
\psi \downarrow & & \downarrow \Sigma \psi_+ \\
(\mathbb{C}P^{r-1})^L & \xrightarrow{\delta_M} & \Sigma S(\mathbb{C}^r)/C_m.
\end{array}
\]
We have \( \delta_M(a_K) = u^{-1}t(M) \) by the definition of \( a_K \), and
\[
\psi^*(t(M)) = [p^k](x_K)e(L) = [p^k](x_K) = x_K^r,
\]
so \( \delta_M(\Sigma \psi_+)^*(a_K) = u^{-1}x_K^r \). On the other hand, \( \delta_L \) just pinches off the top cell of \( CP^r \) and so induces an isomorphism \( \tilde{K}^*(\Sigma S(\mathbb{C}^r)) \simeq H^{2r}(\mathbb{C}P^r, K^{r-2r}) \). This identifies \( x_K^r \) with \((ux_H)^r\), so we see that \( d_{2r-1}(a_H) = u^{-1}(ux_H)^r \). This proves Proposition A.1.
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