RATIONAL HOMOLOGY OF SPACES OF COMPLEX MONIC POLYNOMIALS WITH MULTIPLE ROOTS

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Abstract. We study rational homology groups of one-point compactifications of spaces of complex monic polynomials with multiple roots. These spaces are indexed by number partitions. A standard reformulation in terms of quotients of orbit arrangements reduces the problem to studying certain triangulated spaces $X_{\lambda,\mu}$.

We present a combinatorial description of the cell structure of $X_{\lambda,\mu}$ using the language of marked forests. As applications we obtain a new proof of a theorem of Arnold and a counterexample to a conjecture of Sundaram and Welker, along with a few other smaller results.

1. Introduction.

Let $n$ be an integer, $n \geq 2$. We view $n$-dimensional complex space $\mathbb{C}^n$ as the space of all monic polynomials with complex coefficients of degree $n$ by identifying $a = (a_0, \ldots, a_{n-1}) \in \mathbb{C}^n$ with $f_a(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. To each $\lambda = (\lambda_1, \ldots, \lambda_t) \vdash n$ one can associate a topological space as follows (we refer the reader to the subsection 2.1 for a description of our conventions on the terminology of number and set partitions).

Definition 1.1. $\tilde{\Sigma}_\lambda$ is the set of all $a \in \mathbb{C}^n$, for which the roots of $f_a(z)$ can be partitioned into sets of sizes $\lambda_1, \ldots, \lambda_t$, so that within each set the roots are equal. Clearly, $\tilde{\Sigma}_\lambda$ is a closed subset of $\mathbb{C}^n$. Let $\tilde{\Sigma}_\lambda$ be the one-point compactification of $\tilde{\Sigma}_\lambda$.

In this paper we shall focus on the (reduced) rational Betti numbers of spaces $\Sigma_\lambda$.

In [2], V.I. Arnold has computed $\tilde{\beta}_\ast(\Sigma_\lambda, \mathbb{Q})$ for $\lambda = (k^m, 1^{n-km})$.

Theorem 1.2. (2). Let $\lambda = (k^m, 1^{n-km})$ for some natural numbers $k \geq 2$, $m$, and $n \geq km$. Then

$$\tilde{\beta}_i(\Sigma_\lambda, \mathbb{Q}) = \begin{cases} 1, & \text{for } i = 2l(\lambda); \\ 0, & \text{otherwise}. \end{cases} \quad (1.1)$$

In [18] Sundaram and Welker conjectured that

Conjecture 1.3. For any number partition $\lambda$, $\tilde{\beta}_i(\Sigma_\lambda, \mathbb{Q}) = 0$ unless $i = 2l(\lambda)$.

In this paper we shall give a new, combinatorial proof of the Theorem 1.2 and disprove Conjecture 1.3.

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To do that, we shall introduce a family of topological spaces $X_{\lambda, \mu}$, indexed by pairs of number partitions $(\lambda, \mu)$, satisfying $\lambda \vdash \mu$. $X_{\lambda, \mu}$ will be defined so that the following equality is satisfied

$$\tilde{\beta}_i(\Sigma_\lambda, \mathbb{Q}) = \sum_{\lambda \vdash \mu \vdash \nu} \tilde{\beta}_{i-2(\mu)-1}(X_{\lambda, \mu}, \mathbb{Q}). \quad (1.2)$$

Here is the summary of the paper.

In Section 2 we introduce terminology of number and set partitions, subspace arrangements and their intersection lattices, and order complexes of posets.

In Section 3 we define the topological spaces $X_{\lambda, \mu}$ and derive (1.2).

In Section 4 we give a combinatorial description of the cell structure of the triangulated spaces $X_{\lambda, \mu}$ in terms of marked forests, see Theorem 4.4. This description is the backbone of the paper, it serves as both language and intuition for the material in the subsequent sections. One consequence of Theorem 4.4 is that homology groups of $X_{\lambda, \mu}$ may be computed from a chain complex, whose components are freely generated by marked forests, and the boundary operator is described in terms of a combinatorial operation on such forests (deletion of level sets).

In Section 5 we prove a general theorem about collapsibility of certain triangulated spaces. The argument is along the lines of [13] which has been extended terms of a combinatorial operation on such forests (deletion of level sets).

In Section 6 we disprove the conjecture of Sundaram and Welker. Besides giving a counterexample we prove this conjecture for a class of number partitions, which we call generic partitions.

2. Terminology.

2.1. Number and set partitions.

Let $n$ be a natural number. We denote the set $\{1, \ldots, n\}$ by $[n]$. A number partition of $n$ is a set $\{\lambda_1, \ldots, \lambda_t\}$ of natural numbers, such that $\lambda_1 + \cdots + \lambda_t = n$. The usual convention is to write $\lambda = (\lambda_1, \ldots, \lambda_t)$, where $\lambda_1 \geq \cdots \geq \lambda_t$, and $\lambda \vdash n$. The length of $\lambda$, denoted $l(\lambda)$, is the number of components of $\lambda$, say, in the previous sentence $l(\lambda) = t$. We also use the power notation: $(n^{a_1}, \ldots, 1^{a_t}) = (n, \ldots, n, 1, \ldots, 1)$.

For a set $S$, a set partition $\pi$ is a set $\{S_1, \ldots, S_t\}$, where $S_i$’s are subsets of $S$, such that $\cup_{i=1}^t S_i = S$ and $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq t$. We write $\pi = (S_1, \ldots, S_t)$, where $|S_1| \geq \cdots \geq |S_t|$, and $\pi \vdash S$. Observe that we do not distinguish the set partitions differing only in the order of the sets, e.g., $\{(1, 2), \{3, 4\}\}$ and $\{(3, 4), \{1, 2\}\}$ are the same. Whenever we write $\pi \vdash [n]$, it implicitly implies that $\pi$ is a set partition, as opposed to a number partition. A set partition $\pi \vdash [n]$, $\pi = (S_1, \ldots, S_t)$, is said to have type $\lambda$, where $\lambda \vdash n$ is the number partition $\lambda = (|S_1|, \ldots, |S_t|)$.

For two set partitions $\pi, \tilde{\pi} \vdash S$, $\pi = (S_1, \ldots, S_t)$, $\tilde{\pi} = (\tilde{S}_1, \ldots, \tilde{S}_q)$ we write $\pi \vdash \tilde{\pi}$ if there exists $t \vdash [q]$, $t = \{I_1, \ldots, I_q\}$, such that $\tilde{S}_i = \cup_{j \in I_i} S_j$ for $i \in [q]$. Analogously, for two number partitions $\lambda = (\lambda_1, \ldots, \lambda_t)$, $\mu = (\mu_1, \ldots, \mu_q)$ we write $\lambda \vdash \mu$...
λ ⊳ μ if there exists i ⊳ [t], i = {I_1, \ldots, I_q}, such that μ_i = \sum_{j \in I_i} λ_j for i \in [q]. Clearly π ⊳ [n] and λ ⊳ n are special cases of these notations. Finally observe that if π, ȧπ are two set partitions, such that π ⊳ ȧπ, then (type π) ⊳ (type ȧπ).

2.2. Subspace arrangements and their intersection lattices.

Definition 2.1. A set \( A = \{K_1, \ldots, K_t\} \) of linear complex subspaces of \( \mathbb{C}^n \), such that \( K_i \not\subseteq K_j \) for \( i \neq j \), is called a (central complex) subspace arrangement in \( \mathbb{C}^n \).

The intersection data of a subspace arrangement may be represented by a poset.

Definition 2.2. To a subspace arrangement \( A \) in \( \mathbb{C}^n \) we associate a partially ordered set \( \mathcal{L}_A \), called the intersection lattice of \( A \). The set if elements of \( \mathcal{L}_A \) is \( \{K \subseteq \mathbb{C}^n \mid \exists I \subseteq [t], \text{ such that } \bigcap_{i \in I} K_i = K\} \cup \{\mathbb{C}^n\} \) with the order given by reversing inclusions: \( x \leq_{\mathcal{L}_A} y \) iff \( x \supseteq y \). That is, the minimal element of \( \mathcal{L}_A \) is \( \mathbb{C}^n \), also customarily denoted \( 0 \), and the maximal element is \( \bigcap_{K_i \in A} K \).

Let \( V_A = \bigcup_{i=1}^t K_i \). If \( V_A \) is invariant under the action of some finite group \( G \subset \text{GL}_n(\mathbb{C}) \), then we say that \( G \) acts on \( A \). In that case, \( \Gamma^G_A \) denotes the one-point compactification of \( V_A \setminus \overline{O} \). For \( x \in \mathcal{L}_A \), \( \text{St}_x \subseteq G \) denotes the stabilizer of \( x \).

2.3. Order complexes of posets.

Definition 2.3. For a poset \( P \), let \( \Delta(P) \) denote the nerve of \( P \) viewed as a category in the usual way: it is a simplicial complex with \( i \)-dimensional simplices corresponding to chains of \( i+1 \) elements of \( P \) (chains are totally ordered sets of elements of \( P \)). In particular, vertices of \( \Delta(P) \) correspond to the elements of \( P \). We call \( \Delta(P) \) the order complex of \( P \).

For an arbitrary poset \( P \) and \( x, y \in P, x < y \), let \( P(x,y) \) denote the subposet of \( P \) consisting of all \( z \in P \), such that \( x < z < y \).

3. Orbit arrangements and spaces \( X_{\lambda,\mu} \).

3.1. Reformulation in the language of orbit arrangements.

Following [18], we shall give a different interpretation of the numbers \( \tilde{\beta}_i(\Sigma_\lambda, \mathbb{Q}) \), for general \( \lambda \). First, let us observe that the symmetric group \( S_n \) acts on \( \mathbb{C}^n \) by permuting the coordinates, so we can consider the space \( \mathbb{C}^n / S_n \) endowed with the quotient topology. It is a classical fact that the map \( \phi: \mathbb{C}^n \to \mathbb{C}^n / S_n \), mapping a polynomial to the (unordered) set of its roots, is a homeomorphism, which extends to the one-point compactifications. Therefore \( \mathbb{C}^n \cong \mathbb{C}^n / S_n \) and \( \Sigma_\lambda \cong \phi(\Sigma_\lambda) = \phi(\tilde{\Sigma}_\lambda) \cup \{\infty\} \).

\( \phi(\tilde{\Sigma}_\lambda) \) can be viewed as the configuration space of \( n \) unmarked points on \( \mathbb{C} \) such that the number partition given by the coincidences among the points is refined by \( \lambda \). For example, \( \phi(\tilde{\Sigma}_{\{2,1^{n-2}\}}) \) is the configuration space of \( n \) unmarked points on \( \mathbb{C} \) such that at least 2 points coincide. Using this point of view, \( \phi(\Sigma_\lambda) \) can be described in the language of orbit subspace arrangements.

Definition 3.1. For \( \pi \models [n], \pi = (S_1, \ldots, S_t), S_j = \{i_1, \ldots, i_{|S_j|}\}, 1 \leq j \leq t, K_\pi \) is the subspace given by the equations \( x_{i_1} = \cdots = x_{i_{|S_1|}}, \ldots, x_{i_1} = \cdots = x_{i_{|S_t|}} \). For \( \lambda \models n, \text{ set } I_\lambda = \{\pi \models [n] \mid \text{type}(\pi) = \lambda\} \) and define \( A_\lambda = \{K_\pi \mid \pi \in I_\lambda\} \). \( A_\lambda \)'s are called orbit arrangements.
The orbit arrangements were introduced in [6] and studied in further detail in [1]. They provide the appropriate language to describe \( \phi(\Sigma_\lambda) \), indeed

\[
\phi(\Sigma_\lambda) = \Gamma^G_{A_\lambda}.
\] (3.1)

An important special case is that of the braid arrangement \( A_{n-1} = A_{(2,1^{n-2})} \), which corresponds under \( \phi \) to \( \Sigma_{(2,1^{n-2})} \), the space of all monic complex polynomials of degree \( n \) with at least one multiple root. The name "braid arrangement" stems from the fact that \( \mathbb{C}^n \setminus V_{A_{n-1}} \) is a classifying space of the colored braid group, see [6]. The intersection lattice \( L_{A_{n-1}} \) is usually denoted \( \Pi_n \). It is the poset consisting of all set partitions of \([n]\), where the partial order relation is refinement. Furthermore, for \( \lambda \vdash n \), the intersection lattice of \( A_\lambda \) is denoted \( \Pi_\lambda \). It is the subposet of \( \Pi_n \) consisting of all elements which are joins of elements of type \( \lambda \), with the minimal element 0 attached.

3.2. Applying Sundaram-Welker formula.

The following formula of S. Sundaram and V. Welker, [18], is vital for our approach.

**Theorem 3.2.** ([18] Theorem 2.4(ii) and Lemma 2.7(ii)).

Let \( A \) be an arbitrary subspace arrangement in \( \mathbb{C}^n \) with an action of a finite group \( G \subset U_n(\mathbb{C}) \). Let \( D_A \) be the intersection of \( V_A \) with the \((2n-1)\)-sphere (often called the link of \( A \)). Then there is the following isomorphism of \( G \)-modules.

\[
\tilde{H}_i(D_A, \mathbb{Q}) \cong_G \bigoplus_{x \in L^G_A} \text{Ind}^G_{St_\lambda}(\tilde{H}_{i-\dim x}(\Delta(L_A(\hat{0}, x)), \mathbb{Q})),
\] (3.2)

where the sum is taken over representatives of the orbits of \( G \) in \( L_A \setminus \{\hat{0}\} \), under the action of \( G \), one representative for each orbit.

Clearly \( \Gamma^G_A \cong \text{susp}(D_A/G) \). Recall that if a finite group \( G \) acts on a finite cell complex \( K \) then \( \tilde{\beta}_i(K/G, \mathbb{Q}) \) is equal to the multiplicity of the trivial representation in the induced representation of \( G \) on the \( \mathbb{Q} \)-vector space \( \tilde{H}_i(K, \mathbb{Q}) \), see for example [1] Theorem 1], [6]. Hence, it follows from (3.2), and the Frobenius reciprocity law, that

\[
\tilde{\beta}_i(\Gamma^G_A, \mathbb{Q}) = \sum_{x \in L^G_A} \tilde{\beta}_{i-\dim x-1}(\Delta(L_A(\hat{0}, x))/St_x, \mathbb{Q}).
\] (3.3)

3.3. Spaces \( X_{\lambda,\mu} \) and their properties.

Let us now restate this identity in the special case of orbit arrangements. As mentioned above, the intersection lattice of \( A_\lambda \) is \( \Pi_\lambda \). It has an action of the symmetric group \( S_n \), which, for any \( \pi \in \Pi_\lambda \) induces an action of \( \text{St}_\pi \) on \( \Delta(\Pi_\lambda(\hat{0}, \pi)) \).

**Notation.** Let \( X_{\lambda,\mu} \) denote the topological space \( \Delta(\Pi_\lambda(\hat{0}, \pi))/\text{St}_\pi \), where the set partition \( \pi \) has type \( \mu \). If there is no set partition \( \pi \in \Pi_\lambda \) of type \( \mu \), i.e., if \( \mu \) cannot be obtained as a join of \( \lambda \)'s, then let \( X_{\lambda,\mu} \) be a point.

For fixed \( \mu \), the space \( X_{\lambda,\mu} \) does not depend on the choice of \( \pi \). Observe that \( X_{\lambda,\mu} \) is in general not a simplicial complex, however it is a triangulated space, (a regular CW complex with each cell being a simplex, see [1], Chapter I, Section 1), with its cell structure inherited from the simplicial complex \( \Delta(\Pi_\lambda(\hat{0}, \pi)) \). In general, whenever \( G \) is a finite group which acts on a poset \( P \) in an order-preserving way, \( \Delta(P)/G \) is a triangulated space whose cells are orbits of simplices of \( \Delta(P) \) under the action of \( G \); this is obviously not true in general for an action of a finite group on a finite simplicial complex.
Clearly, \([\lambda, \mu]\) together with \([\lambda, \lambda]\), and the fact that \(\phi\) is a homeomorphism, implies (1.2). Let us quickly analyze (1.2). \(X_{\lambda, \lambda} = \emptyset\) makes a contribution 1 in dimension 2\(l(\lambda)\). Assume \(\mu \neq \lambda\), then 1 \(\leq l(\mu) \leq l(\lambda) - 1\) and \(X_{\lambda, \mu} \neq \emptyset\). \(\dim X_{\lambda, \mu} = l(\lambda) - l(\mu) - 1\), hence \(\tilde{\beta}_{i-2l(\mu)+1}(X_{\lambda, \mu}, \mathbb{Q}) = 0\) unless \(0 \leq i - 2l(\mu) - 1 \leq l(\lambda) - l(\mu) - 1\), that is \(2l(\mu) + 1 \leq i \leq l(\lambda) + l(\mu)\). It follows from (1.2) that
\[
\tilde{\beta}_{2l(\lambda)}(\Sigma_{\lambda}, \mathbb{Q}) = 1, \quad \text{and} \quad \tilde{\beta}_{i}(\Sigma_{\lambda}, \mathbb{Q}) = 0 \quad \text{unless} \quad 3 \leq i \leq 2l(\lambda).
\]

The purpose of this paper is to investigate the values \(\tilde{\beta}_{i}(\Sigma_{\lambda}, \mathbb{Q})\) for \(3 \leq i \leq 2l(\lambda) - 1\), by studying \(\tilde{\beta}_{i}(X_{\lambda, \mu}, \mathbb{Q})\). We shall prove that the latter are equal to 0 for a certain set of pairs \((\lambda, \mu)\), \(\lambda \vdash \mu\), of partitions, including the case in Theorem 1.2. \((\lambda = (k^{m}, 1^{n-km})\), \(\mu\) is arbitrary such that \(\lambda \vdash \mu\), and we shall give an example that this is not the case in general.

4. The cell structure of \(X_{\lambda, \mu}\) and marked forests

4.1. The terminology of marked forests.

In order to index the simplices of \(X_{\lambda, \mu}\), we need to introduce some terminology for certain types of trees with additional data. For an arbitrary forest of rooted trees \(T\) (we only consider finite graphs), let \(V(T)\) denote the set of the vertices of \(T\), \(R(T) \subseteq V(T)\) denote the set of the roots of \(T\) and \(L(T) \subseteq V(T)\) denote the set of the leaves of \(T\). For any integer \(i \geq 0\), let \(l_{i}(T)\) be the number of \(v \in V(T)\) such that, \(v\) has distance \(i\) to the root in its connected component.

**Definition 4.1.** A forest of rooted trees \(T\) is called a graded forest of rank \(r\) if \(l_{r+2}(T) = 0\), \(l_{r+1}(T) = |L(T)|\), and the sequence \(l_{0}(T), \ldots, l_{r+1}(T)\) is strictly increasing.

For \(v, w \in V(T)\), \(w\) is called a child of \(v\) if there is an edge between \(w\) and \(v\) and the unique path from \(w\) to the corresponding root passes through \(v\). For \(v \in V(T)\), we call the distance from \(v\) to the closest leaf the height of \(v\). For example, in a graded forest of rank \(r\), leaves have height 0 and roots have height \(r + 1\).

**Definition 4.2.** A marked forest of rank \(r\) is a pair \((T, \eta)\), where \(T\) is a graded forest of rank \(r\) and \(\eta\) is a function from \(V(T)\) to the set of natural numbers such that for any vertex \(v \in V(T) \setminus L(T)\) we have
\[
\eta(v) = \sum_{w \in \text{children}(v)} \eta(w). \quad (4.1)
\]

We remark that the set of the marked forests of rank \(r\), such that not all leaves have label 1, is equal to the set of graded forests of rank \(r + 1\). Indeed, instead of labeling the vertices with natural numbers so that (4.1) is satisfied, one can as well attach a new level of leaves so that each "old leaf" \(v\) has \(\eta(v)\) children. Then the old labels will correspond to the numbers of the new leaves below each vertex. For our context it is more convenient to use labels rather than auxiliary leaves, i.e., it is more handy to label all vertices rather than just the leaves, so we stick to the terminology of Definition 4.2.

For a marked forest \((T, \eta)\) of rank \(r\) and \(0 \leq i \leq r + 1\), we have a number partition \(\lambda_{i}(T, \eta) = \{\eta(v) \mid v \text{ has height } i\}\). Clearly \(\lambda_{0}(T, \eta) \vdash \cdots \vdash \lambda_{r}(T, \eta) \vdash \lambda_{r+1}(T, \eta)\).

**Definition 4.3.** Let \(\lambda \vdash \mu \vdash n\), \(\lambda \neq \mu\). A \((\lambda, \mu)\)-forest of rank \(r\) is a marked forest of rank \(r\), \((T, \eta)\) such that \(\mu = \lambda_{r+1}(T, \eta)\) and \(\lambda \vdash \lambda_{0}(T, \eta)\).
We call \(((2^1, n) - 2), \mu\)-forests simply \(\mu\)-forests and \(((2^1, n) - 2), (n))\)-forests simply \(n\)-trees.

Figure 1: all 5-trees of rank 2.

Whenever \((T, \eta)\) is a \((\lambda, \mu)\)-forest of rank \(r\) and \(0 \leq i \leq r\), we can obtain a \((\lambda, \mu)\)-forest \((T^i, \eta^i)\) of rank \(r - 1\) by deleting from \(T\) all the vertices of height \(i\) and connecting the vertices of height \(i + 1\) to their grandchildren (unless \(i = 0\)); \(\eta^i\) is the restriction of \(\eta\) to \(V(T^i)\). In other words, \((T^i, \eta^i)\) is obtained from \((T, \eta)\) by removing the entire \(i\)th level, counting from the leaves, and filling in the gap in an obvious way. This allows us to define a boundary operator by

\[
\partial(T, \eta) = \sum_{i=0}^{r} (-1)^i (T^i, \eta^i). \tag{4.2}
\]

For example:

\[
\text{boundary } \left( \begin{array}{ccc}
2 & 5 & 3 \\
1 & 2 & 1
\end{array} \right) = \left( \begin{array}{ccc}
2 & 5 & 3 \\
1 & 2 & 1
\end{array} \right) - \left( \begin{array}{ccc}
2 & 3 & 2 \\
1 & 1 & 1
\end{array} \right) + \left( \begin{array}{ccc}
5 & 3 & 2 \\
1 & 1 & 1
\end{array} \right).
\]

For a given set partition \(\pi\) one can define the notion of a \(\pi\)-forest \((T, \zeta)\) of rank \(r\) almost identically to the case of number partitions described above. The difference is that \(\zeta\) maps \(V(T)\) to the set of finite sets, rather than the set of natural numbers. The condition (4.1) is replaced by

\[
\zeta(v) = \bigcup_{w \in \text{children}(v)} \zeta(w), \tag{4.3}
\]

and \(\pi = \{\zeta(v) \mid v \in R(T)\}\). For \(0 \leq i \leq r + 1\), analogously to \(\lambda_i(T, \eta)\), we define \(\pi_i(T, \zeta)\) to be the set partition which is read off from the vertices of \(T\) having height \(i\).

Let \(\mu\) be the type of \(\pi\), then there exists a canonical \(\mu\)-forest \((T, |\zeta|)\) associated to each \(\pi\)-forest \((T, \zeta)\), where \(|\zeta|\) is obtained as the composition of \(\zeta\) with the map which maps finite sets to their sizes.

4.2. The main theorem.

Let us describe how to associate a \((\lambda, \mu)\)-forest, \(\psi(\sigma)\), of rank \(r\) to an \(r\)-simplex \(\sigma\) of \(X_{\lambda, \mu}\). The simplex \(\sigma\) is an \(S^r\)-orbit of \(r\)-simplices of \(\Delta(\Pi(0), \pi))\), where \(\pi\) is a set partition of type \(\mu\). Take a representative of this orbit, a chain \(c = (x_r > \cdots > x_0)\). Now we define \(\psi(\sigma) = (T, \eta)\). Each element \(x_i\) corresponds to the \(i\)th level in \(T\), counting from the leaves. Each block \(b\) of \(x_i\) corresponds to a node in the tree; on this node we define the value of \(\eta\) to be \(|b|\). We define the edges of the tree \(T\) by connecting each node corresponding to a block \(b\) of \(x_i\) to all nodes corresponding to the blocks of \(x_{i-1}\) contained in \(b\), we do that for all \(b\) and \(i\). The
top \((r+1)\)th level is added artificially, its nodes correspond to the blocks of \(\pi\), and the edges from the top level to the \(r\)th level connect each block of \(\pi\) to the blocks of \(\pi_r\) contained in it. For example, the value of \(\psi\) on the \(S_5\)-orbit of the chain \((123)(45) > (123)(4)(5) > (13)(2)(4)(5)\) is the first 5-tree on the Figure 1.

We are now ready to state and prove the main result of this section.

**Theorem 4.4.** Assume \(\lambda \vdash \mu \vdash n\), \(\lambda \neq \mu\). The correspondence \(\psi\) of the \(r\)-simplices of \(X_{\lambda,\mu}\) and \((\lambda, \mu)\)-forests \((T, \eta)\) of rank \(r\) is a bijection. Under this bijection, the boundary operator of the triangulated space \(X_{\lambda,\mu}\) corresponds to the boundary operator described in [4,2].

In particular, the simplices of \(\Delta(\Pi_n)/S_n\) along with the cell inclusion structure are described by the \(n\)-trees. Indeed, \(\Delta(\Pi_5)/S_5\) is shown in the figure below.

![Figure 2](image.png)

The five triangles may be labeled by the five 5-trees of rank 2 in Figure 1.

**Proof of the Theorem 4.4.** By the definitions of \(\Pi_n\) and of \(\Delta\), the \(r\)-simplices of \(\Delta(\Pi_n)\) can be indexed by \(([n])\)-trees of rank \(r\) (we write \([n]\) to emphasize that the set \([n]\) is viewed here as a set partition consisting of only one set). Furthermore, the cell inclusions in \(\Delta(\Pi_n)\) correspond to level deletion in \(([n])\)-trees as is described above for the case of number partitions, because the levels in the \(([n])\)-trees correspond to the elements of \(\Pi_n\), and the edges in the \(([n])\)-trees correspond to block inclusions of two consecutive elements in the chain.

More generally, the \(r\)-simplices \(\sigma\) of \(\Delta(\Pi_n(\hat{0}, \hat{x}))\) can be indexed by \(\pi\)-forests \((T(\sigma), \zeta(\sigma))\) such that \(\lambda\) refines the type of \(\pi_0(T(\sigma), \zeta(\sigma))\). The definition of \(\psi\) can now be rephrased as associating to \(\sigma\) the \((\lambda, \mu)\)-forest \((T(\sigma), |\zeta(\sigma)|)\), where \(\mu = \text{type} \pi\).

The group action of \(St_\pi\) on \(\Delta(\Pi_\lambda(\hat{0}, \pi))\) corresponds to relabeling elements within the sets of \(\zeta(\sigma)\). This shows that for \(g \in St_\pi\) we have \(T(gx) = T(x)\) and \(|\zeta(gx)| = |\zeta(x)|\). Therefore \(\psi(x)\) is well-defined, it does not depend on the choice of the representative of the, corresponding to \(\sigma\), \(St_\pi\)-orbit of chains.

\(\psi\) is surjective, we shall now show that it is also injective. If \(\sigma_1, \sigma_2\) are two different \(r\)-simplices of \(\Delta(\Pi_\lambda(\hat{0}, \pi))\) such that \(T(\sigma_1) = T(\sigma_2)\) and \(|\zeta(\sigma_1)| = |\zeta(\sigma_2)|\), then there exists \(g \in St_\pi\) such that \(\zeta(g\sigma_2) = \zeta(\sigma_1)\). Indeed, let \(T = T(\sigma_1) = T(\sigma_2)\) and let \(\alpha_1, \alpha_2\) be the string concatenated from the values of \(\zeta(\sigma_1)\), resp. \(\zeta(\sigma_2)\), on the leaves of \(T\); the order of leaves of \(T\) is arbitrary, but the same for \(T(\sigma_1)\) and \(T(\sigma_2)\), the order of elements within each \(\zeta(\sigma_1)(v)\), resp. \(\zeta(\sigma_2)(v)\), for \(v \in L(T)\) is also chosen arbitrarily. Then \(g \in S_n\) which maps \(\alpha_2\) to \(\alpha_1\) satisfies the necessary conditions: \(\zeta(g\sigma_2) = \zeta(\sigma_1)\) on the leaves of \(T\), and hence by [3] on all vertices of \(T\). Furthermore, since \(g\pi = g\pi_{r+1}(T, \zeta(\sigma_2)) = \pi_{r+1}(T, \zeta(g\sigma_2)) = \pi_{r+1}(T, \zeta(\sigma_1)) = \pi,\) we have \(g \in St_\pi\).

This shows that \(\psi\) is a bijection. Since the levels of the \((\lambda, \mu)\)-forests correspond to the \(St_\pi\)-orbits of the vertices of \(\Delta(\Pi_\lambda(\hat{0}, \pi))\) (hence to the vertices of \(X_{\lambda,\mu}\)),
the boundary operator of $X_{\lambda,\mu}$ corresponds under $\psi$ to the level deletion in $(\lambda, \mu)$-forests, i.e. the boundary operator described in (4.2).

4.3. Remarks.

1. While the presence of the root in an ($[n]$)-tree is just a formality (two marked ($[n]$)-trees are equal iff the deletion of the root gives equal marked forests), the presence of the roots in a $\pi$-forest is vital. In fact, if roots were not taken into account (as seems natural, since the partition read off from the roots does not correspond to any vertex in $\Delta(\Pi_\lambda(\hat{0}, \pi))$) the argument above would be false already for vertices: if $\tau_1, \tau_2 \in \Pi_\lambda(\hat{0}, \pi)$, such that type ($\tau_1$) = type ($\tau_2$) (i.e., the corresponding marked forests of rank 0 are equal once the roots are removed), there may not exist $g \in S_\pi$, such that $g\tau_2 = \tau_1$ (although such $g \in S_n$ certainly exists).

2. Marked forests equipped with an order on the children of each vertex were used by Vassiliev, [20], to label cells in a certain CW-complex structure on the space $\tilde{R}_n(m)$, the one-point compactification of the configuration space of $m$ unmarked distinct points in $\mathbb{R}^n$. Vassiliev’s cell decomposition of $\tilde{R}_n(m)$ is a generalization of the earlier Fuchs’ cell decomposition of $\tilde{R}_2(m) = \tilde{C}(m)$, [10], which allowed Fuchs to compute the ring $H^*(Br(m), \mathbb{Z}_2)$, where $Br(m)$ is Artin’s braid group on $m$ strings, see also [13]. Beyond a certain similarity of the combinatorial objects used for labeling the cells, cf. [20, Lemma 3.3.1] and Theorem 4.4, the connection between this paper and the results of Vassiliev and Fuchs seems unclear.

As yet another instance of a similar situation, we would like to mention the labeling of the components in the stratification of $\overline{M}_{0,n}$ (the Deligne-Knudsen-Mumford compactification of the moduli space of stable projective complex curves of genus 0 with $n$ punctures) with trees with $n$ labeled leaves, see [8, 13].

5. A new proof of a theorem of Arnold

5.1. Formulation of the main theorem and corollaries.

In this section we take a look at a rather general question of which $\mathbb{Q}$-acyclicity of the spaces $X_{\lambda,\mu}$ is a special case:

Let $\pi \in \Pi_n$ and let $Q$ be an $S_\pi$-invariant subposet of $\Pi_\pi(\hat{0}, \pi)$. When is the multiplicity of the trivial representation in the induced representation of $S_\pi$ on $\tilde{H}_i(\Delta(Q), \mathbb{Q})$ equal to 0 for all $i$, in other words, when is $\Delta(Q)/S_\pi$ $\mathbb{Q}$-acyclic?

**Definition 5.1.** Let $\Lambda$ be a subset of the set of all number partitions of $n$ such that $(1^n), (n) \notin \Lambda$. Define $\Pi_\Lambda$ to be the subposet of $\Pi_n$ consisting of all set partitions $\pi$ such that (type $\pi$) $\in \Lambda$.

Clearly, $\Pi_\Lambda$ is $S_n$-invariant and, more generally, $\Pi_\Lambda(\hat{0}, \pi)$ is $S_\pi$-invariant. Vice versa, any $S_n$-invariant subposet of $\Pi_n \setminus \{(\{1\}, \ldots, \{n\}), ([n])\}$ is of the form $\Pi_\Lambda$ for some $\Lambda$.

The following theorem is the main result of this section. It is a generalization of [13, Theorem 4.1] which now covers Theorem 1.2. The proof is a combination of the language of marked forests from Section 4 and the ideas used in the proof of [10, Theorem 4.1].

**Theorem 5.2.** Let $2 \leq k < n$. Assume $\Lambda$ is a subset of the set of all number partitions of $n$ such that $(1^n), (n) \notin \Lambda$ and $\Lambda$ satisfies the following condition:
Condition $C_k$. If $\mu \in \Lambda$, such that $\mu = (\mu_1, \ldots, \mu_t)$, where $\mu_i = kq_i + r_i$, $0 \leq r_i < k$ for $i \in [t]$, then $\gamma_k(\mu) = (k^{\mu_1 + r_1}, 1^{\mu_2 + \cdots + \mu_t}) \in \Lambda$.

Then for any $\mu \in \Delta \setminus \{(n)\}$ the triangulated space $X_{\Lambda, \mu} = \Delta(\Pi_\Lambda(0, \pi))/\text{St}_\pi$, where $\mu = \text{type } \pi$, is collapsible, in particular the multiplicity of the trivial representation in the induced $\text{St}_\pi$-representation on $\tilde{H}_i(\Delta(\Pi_\Lambda(0, \pi)), \mathbb{Q})$ is equal to 0 for all $i$.

**Corollary 5.3.** Assume $\lambda = (k^m, 1^{n-km})$, $\lambda \vdash \mu$, then $X_{\lambda, \mu}$ is collapsible. In particular, $X_{\lambda, \mu}$ is $\mathbb{Q}$-acyclic, therefore Theorem 1.2 follows.

**Proof.** Clearly $X_{\lambda, \mu} = X_{\Lambda, \mu}$ for $\Lambda = \{\tau | \lambda \vdash \tau \vdash \mu, \tau \neq (n)\}$. It is easy to check that Condition $C_k$ is satisfied for the case $\lambda = (k^m, 1^{n-km})$, therefore Theorem 1.2 follows from Theorem 5.2 via (1.2).\hfill \Box

Another consequence of Theorem 5.2 which was pointed out already in [15], has nothing to do with the spaces $\Sigma_\Lambda$. Namely, for $k = 2$, $\lambda = \{\tau | \lambda \vdash \tau \vdash \mu, \tau \neq (n)\}$, for some $n-1 \geq r \geq 2$, and $\mu = (n)$, we get results of Stanley ($r = 2$), [17, page 151], and Hanlon ($r > 2$), [12, Theorem 3.1].

Theorem 5.2 can be viewed as an attempt to provide a common framework for these results in the spirit of the question stated in the beginning of this section.

5.2. Auxiliary propositions.

First we need some terminology. For an arbitrary cell complex $\Delta$ we denote by $V(\Delta)$ the set of vertices of $\Delta$. Assume $\Delta$ is a regular CW complex and $\Delta'$ is its subcomplex. We denote the set of the simplices of $\Delta$ which are not simplices of $\Delta'$ by $\Delta \setminus \Delta'$. We use the sign $\succ$ to denote the cover relation in the cell structure of $\Delta$.

Assume that, in addition, $\Delta$ is a triangulated space with some linear order $\ll$ on the set of vertices. For $\sigma \in \Delta \setminus \Delta'$ we may write $\sigma = (x_1, \ldots, x_t)$, this notation is slightly inaccurate since the set of vertices does not determine the simplex uniquely, all we mean is that $\sigma$ has vertices $x_1 \ll \cdots \ll x_t$. In that case, we let $\xi(\sigma) = i$ if $x_1, \ldots, x_{i-1} \in V(\Delta')$ and $x_i \notin V(\Delta').$

**Proposition 5.4.** Let $\Delta$ be a regular CW complex and $\Delta'$ a subcomplex of $\Delta$, then the following are equivalent:

a) there is a sequence of collapses leading from $\Delta$ to $\Delta'$; 
b) there is a matching of cells of $\Delta \setminus \Delta'$: $\sigma \leftrightarrow \phi(\sigma)$, such that $\phi(\sigma) \succ \sigma$ and there is no sequence $\sigma_1, \ldots, \sigma_t \in \Delta \setminus \Delta'$ such that $\phi(\sigma_1) \succ \sigma_2, \phi(\sigma_2) \succ \sigma_3, \ldots, \phi(\sigma_t) \succ \sigma_1$ (such matching is called acyclic).

**Proof.** a) $\Rightarrow$ b). Let the elementary collapses define the matching $\phi$. Assume there is a sequence $\sigma_1, \ldots, \sigma_t \in \Delta \setminus \Delta'$ such that $\phi(\sigma_1) \succ \sigma_2, \phi(\sigma_2) \succ \sigma_3, \ldots, \phi(\sigma_t) \succ \sigma_1$. Without loss of generality we can assume that the collapse $(\sigma_1, \phi(\sigma_1))$ precedes collapses $(\sigma_i, \phi(\sigma_i))$ for $2 \leq i \leq t$. Then $\phi(\sigma_i) \succ \sigma_1$ yields a contradiction.

b) $\Rightarrow$ a). The proof is again very easy, various versions of it were given in [1], Proposition 3.7, and [15, Theorem 3.2].\hfill \Box

**Proposition 5.5.** Let $\Delta$ be a triangulated space with some linear order $\ll$ on its set of vertices $V(\Delta)$. Let $V' \subseteq V(\Delta)$ and $\Delta'$ be the subcomplex of $\Delta$ induced by $V'$. Assume we have a partition $V(\Delta) = \bigcup_{z \in V'} V_z$ such that $z = \min_{z \in V_z} V_z$. For $\sigma \in \Delta \setminus \Delta'$, let $\chi(\sigma) \in V'$ be defined by $x_{\xi(\sigma)} \in V_{\chi(\sigma)}$. Finally assume that the following condition is satisfied:

Condition 8. If $\sigma \in \Delta \setminus \Delta'$, $\sigma = (x_1, \ldots, x_t)$, is such that either $\xi(\sigma) = 1$ or $x_{\xi(\sigma)-1} \neq \chi(\sigma)$, then there exists a unique simplex $\sigma' = (x_1, \ldots, x_{\xi(\sigma)-1}, \chi(\sigma), x_{\xi(\sigma)}, \ldots, x_t)$ such that $\sigma' \setminus \chi(\sigma) = \sigma$. 

Then there is a sequence of collapses leading from $\Delta$ to $\Delta'$. 

**Proof.** Let $U$ denote the set of all $\sigma \in \Delta \setminus \Delta'$, $\sigma = (x_1, \ldots, x_t)$, such that $x_{\xi(\sigma)-1} \neq \chi(\sigma)$ or $\xi(\sigma) = 1$. The matching $\phi$ is defined by Condition 8: for $\sigma \in U$ we set $\phi(\sigma) = \sigma'$. By Proposition 5.2, it is enough to check that this matching is acyclic.

For $\sigma \in U$ we have $\xi(\phi(\sigma)) = \xi(\sigma) + 1$. Moreover, if $\phi(\sigma) \succ \sigma'$ and $\sigma' \in U$, then $\sigma' = \phi(\sigma) \setminus x_{\xi(\sigma)}$, hence $\xi(\sigma') \geq \xi(\phi(\sigma))$. Therefore, if there is a sequence $\sigma_1, \ldots, \sigma_t \in \Delta \setminus \Delta'$ such that $\phi(\sigma_1) > \sigma_2, \phi(\sigma_2) > \sigma_3, \ldots, \phi(\sigma_t) > \sigma_1$, then we have $\xi(\sigma_1) < \xi(\phi(\sigma_1)) \leq \xi(\phi(\sigma_2)) \leq \cdots < \xi(\phi(\sigma_t)) \leq \xi(\sigma_1)$ which yields a contradiction.

5.3. **Proof of the Theorem 5.2.**

We define a $(\Lambda, \mu)$-forest of rank $r$ to be a marked forest $(T, \eta)$ of rank $r$ such that $\lambda_{r+1}(T, \eta) = \mu$ and $\lambda_i(T, \eta) \in \Lambda$, for $0 \leq i \leq r$. It follows from the discussion in Section 4 and in particular from Theorem 4.4 that the $r$-simplices of $X_{\Lambda, \mu}$ can be indexed by $(\Lambda, \mu)$-forests of rank $r$ so that the boundary relation of $X_{\Lambda, \mu}$ corresponds to level deletion in the marked forests.

We call number partitions of the form $(k^m, 1^{n-km})$, for some $m$, special. Let $K$ be the subcomplex of $X_{\Lambda, \mu}$ induced by the set of all special partitions. We adopt the notations $\xi(\sigma)$ and $\chi(\sigma)$ used in Proposition 5.3 to the context of $X_{\Lambda, \mu}$ and its subcomplex $K$. The linear order on $V(X_{\Lambda, \mu})$ can be taken to be any linear extension of the partial order on $V(X_{\Lambda, \mu})$ given by the negative of the length function. The partition of $V(X_{\Lambda, \mu})$ is given by: for $v \in V(X_{\Lambda, \mu}) \setminus V(K)$, $z \in V(K)$, we have $v \in V_t$ iff $z = \gamma_k(v)$.

Let us show that the subcomplex $K$ satisfies Condition 8. Let $\sigma \in X_{\Lambda, \mu} \setminus K$, $\sigma = (x_1, \ldots, x_t)$, and assume $\xi(\sigma) = 1$ or $\chi(\sigma) \neq x_{\xi(\sigma)-1}$. In the language of marked forests this can be reformulated as: $\sigma$ is a $(\Lambda, \mu)$-forest $(T, \eta)$ of rank $t$ such that $\lambda_{\xi(\sigma)-1}(T, \eta)$ is not special and if $\xi(\sigma) > 1$ then $\lambda_0(T, \eta), \ldots, \lambda_{\xi(\sigma)-2}(T, \eta)$ are special, and $\lambda_{\xi(\sigma)-1}(T, \eta) = \gamma_k(\lambda_{\xi(\sigma)-1}(T, \eta))$. In other words, on all vertices of height 0 to $\xi(\sigma) - 2$ the function $\eta$ takes only values 1 or $k$ and for the vertices of height $\xi(\sigma) - 1$ it is no longer true. Moreover, there exists a vertex of height $\xi(\sigma) - 1$ which has at least $k$ children on which $\eta$ is equal to 1. It is now clear that there exists a unique $(\Lambda, \mu)$-forest $(\overline{T}, \overline{\eta})$ of rank $r + 1$ such that

- $\overline{T} = (T, \eta)$;
- $\overline{\eta}_{\xi(\sigma)-1} = \gamma_k(\lambda_{\xi(\sigma)-1}(T, \eta))$, i.e., $\overline{\eta}$ takes only values 1 or $k$ on the vertices of height $\xi(\sigma) - 1$ and each vertex of height $\xi(\sigma)$ in $(\overline{T}, \overline{\eta})$ has no more than $k - 1$ children labeled 1.

To construct $(\overline{T}, \overline{\eta})$ by splitting each vertex of height $\xi(\sigma) - 1$ into vertices marked $k$ and 1 so that the number of $k$’s is maximized. The uniqueness of $(\overline{T}, \overline{\eta})$ follows from the definition of the notion of isomorphism of marked forests.

We have precisely checked Condition 8 and therefore by Proposition 5.4 we conclude that there is a sequence of collapses leading from $X_{\Lambda, \mu}$ to $K$.

It remains to see that $K$ is collapsible. If $\mu = (n)$, then $K$ is a simplex, so we can assume that $\mu \in \Lambda$. If $\mu = \gamma_k(\mu)$, then $K$ is again a simplex. Otherwise it is easy to see that there is a unique vertex in $X_{\Lambda, \mu}$ labeled $\gamma_k(\mu)$ and that $K$ is a cone with an apex in this vertex.

6. ON THE CONJECTURE OF SUNDARAM AND WELKER
6.1. A counterexample to the general conjecture.

The original formulation of Conjecture 1.3 in [18] was

**Conjecture 6.1.** [18, Conjectures 4.12 and 4.13]. Let \( \lambda \) and \( \mu \) be different set partitions, such that \( \lambda \vdash \mu \). Let \( \pi \in \Pi_\lambda \) be a partition of type \( \mu \). Then the multiplicity of the trivial representation in the \( \text{St}_\pi \)-module \( \tilde{H}^*_\pi(\Delta(\Pi_\lambda(\hat{0},\pi)), \mathbb{Q}) \) is 0.

In our terms Conjecture 6.1 is equivalent to

**Conjecture 6.2.** For \( \lambda \vdash \mu, \lambda \neq \mu \), the space \( X_{\lambda,\mu} \) is \( \mathbb{Q} \)-acyclic.

We shall give an example when \( X_{\lambda,\mu} \) is not even connected. It turns out that if one is only interested in counting the number of connected components of \( X_{\lambda,\mu} \), then there is a simpler poset model which we now proceed to describe.

**Definition 6.3.** Assume \( \lambda \vdash \mu \vdash n \), \( \lambda \neq \mu \). The \((\lambda,\mu)\)-forests of rank 0 can be partially ordered as follows: \((T_1,\eta_1) \prec (T_2,\eta_2)\) if there exists a \((\lambda,\mu)\)-forest \((T,\eta)\) of rank 1 such that \((T_1,\eta_1) = (T^1,\eta^1)\) and \((T_2,\eta_2) = (T^0,\eta^0)\). We call the obtained poset \( P_{\lambda,\mu} \).

In other words, elements of \( P_{\lambda,\mu} \) are number partitions \( \tau \neq \mu \) such that \( \lambda \vdash \tau \vdash \mu \), together with a bracketing which shows how to form \( \mu \) out of \( \tau \), the order of the brackets and of the terms within the brackets is neglected. For example \((1,1)(3,1)(2,2\rangle(3,1)\rangle\) and \((3,2\rangle(7,1\rangle(6,4\rangle(5)(7,1\rangle(6,4\rangle(3,2\rangle(7,1\rangle(6,4\rangle)\) are two different elements of \( P_{(2,1\rangle(4\rangle,3)\rangle} \), while \((1,1)(2,2\rangle(3,1)\rangle\) is equal to the first mentioned element. These bracketed partitions are ordered by refinement, preserving the bracket structure.

**Proposition 6.4.** \( X_{\lambda,\mu} \) and \( \Delta(P_{\lambda,\mu}) \) have the same number of connected components, i.e., \( \beta_0(X_{\lambda,\mu}) = \beta_0(\Delta(P_{\lambda,\mu})) \).

**Proof.** We know that \( \Delta(P_{\lambda,\mu}) \) and \( X_{\lambda,\mu} \) have the same set of vertices and that there is an edge between two vertices \( a \) and \( b \) of \( \Delta(P_{\lambda,\mu}) \) iff \( a \prec b \) or \( b \prec a \), which is, by the Definition 6.3, the case iff there is an edge between the corresponding vertices of \( X_{\lambda,\mu} \). This shows that \( \Delta(P_{\lambda,\mu}) \) and \( X_{\lambda,\mu} \) have the same number of connected components.

Note that 1-skeleta of \( X_{\lambda,\mu} \) and \( \Delta(P_{\lambda,\mu}) \) need not be equal. \( \Delta(P_{\lambda,\mu}) \) can intuitively be thought of as a simplicial complex obtained by forgetting the multiplicities of simplices in the triangulated space \( X_{\lambda,\mu} \).

**Counterexample.** For \( n = 23, \lambda = (7,6,4,3,2,1), \mu = (10,8,5) \), \( X_{\lambda,\mu} \) is disconnected. \( P_{\lambda,\mu} \) is shown on the figure below. Clearly \( \Delta(P_{\lambda,\mu}) \) is not connected, hence, by the Proposition 6.4, neither is \( X_{\lambda,\mu} \), which disproves Conjecture 6.1.

```
(5)(8)(7,3)   (5)(6,2)(10)   (4,1)(8)(10)   (5)(8)(6,4)   (5)(7,1)(10)   (3,2)(8)(10)
    |    |    |    |    |    |
(5)(6,2)(7,3) (4,1)(8)(7,3) (4,1)(6,2)(10) (5)(7,1)(6,4) (3,2)(8)(6,4) (3,2)(7,1)(10)
    |    |    |    |    |    |
(4,1)(6,2)(7,3) (4,1)(6,2)(7,3) (4,1)(6,2)(7,3) (4,1)(6,2)(7,3)
```

**Remark.** In the counterexample above, one can actually verify that \( X_{\lambda,\mu} = \Delta(P_{\lambda,\mu}) \). However, we choose to use posets \( P_{\lambda,\mu} \) for two reasons:

1. it is easier to produce series of counterexamples to Sundaram-Welker conjecture using \( \Delta(P_{\lambda,\mu}) \) rather than \( X_{\lambda,\mu} \);
2. we feel that posets $P_{\lambda,\mu}$ are of independent interest, since they are in a certain sense the "naive" quotient of $\Pi(\hat{0},\pi)$ by $\text{St}_\pi$.

We believe that, in general, connected components of $X_{\lambda,\mu}$ may be not acyclic.

6.2. Proof of the conjecture in a special case.

**Definition 6.5.** We say that a number partition $\lambda = (\lambda_1,\ldots,\lambda_t)$ is generic (also called free of resonances in \[16\], having no equal subsums in \[14\]) if whenever $\sum_{i \in I} \lambda_i = \sum_{j \in J} \lambda_j$, for some $I,J \subseteq [t]$, we have $\{\lambda_i\}_{i \in I} = \{\lambda_j\}_{j \in J}$ as multisets.

For example $\lambda = (k^m)$ is generic.

**Theorem 6.6.** If $\lambda$ is generic, we have

$$\tilde{\beta}_i(\Sigma_{\lambda},\mathbb{Q}) = \begin{cases} 1, & \text{for } i = 2l(\lambda); \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** As we pointed out before, it is enough to show that $X_{\lambda,\mu}$ is $\mathbb{Q}$-acyclic for $\lambda \vdash \mu$, $\lambda \neq \mu$.

If $\lambda$ is generic, any number $1 \leq m \leq n$ can be partitioned into numbers from $\lambda$ in at most one way. First, it implies that there exists a unique $(\lambda,\mu)$-forest $(T,\eta)$ of rank 0 such that $\lambda_0(T,\eta) = \lambda$, denote it by $x$. Second, for any $(\lambda,\mu)$-forest $(T,\eta)$ of rank $r$ such that $\lambda_0(T,\eta) \neq \lambda$ there exists a unique $(\lambda,\mu)$-forest $(\tilde{T},\tilde{\eta})$ of rank $r+1$ such that $(\tilde{T}^0,\tilde{\eta}^0) = (T,\eta)$ and $\lambda_0(\tilde{T},\tilde{\eta}) = \lambda$.

In terms of the triangulated space $X_{\lambda,\mu}$ this means that there exists a vertex $x$ such that each $r$-simplex of $X_{\lambda,\mu}$ not containing $x$ is contained in a unique $(r+1)$-simplex containing $x$. This means that, whenever $\lambda$ is generic and $\lambda \vdash \mu$, $\lambda \neq \mu$, $X_{\lambda,\mu}$ is a cone, in particular it is $\mathbb{Q}$-acyclic. 

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