Finite dimensional modules and perpendicular subcategories

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Abstract

We explain how, under some hypotheses, one can construct a sequence of finite dimensional $kG$-modules that lie in certain prescribed additive subcategories, but whose direct limits do not. We use these to show that many of the triangulated quotients of $\text{Mod}(kG)$ are not generated, as triangulated categories, by the corresponding quotient of $\text{mod}(kG)$ considered as a full subcategory.

1 Introduction

Let $G$ be a finite group, and $k$ an field such that char$(k)$ divides $|G|$. The categories $\text{mod}(kG)$ and $\text{Mod}(kG)$ are Frobenius categories (see [3] for example, for an explanation), which implies that the quotients

$$\text{stmod}(kG) := \frac{\text{mod}(kG)}{\text{f.g.projective } kG-\text{modules}}$$

and

$$\text{StMod}(kG) = \frac{\text{Mod}(kG)}{\text{projective } kG-\text{modules}}$$

are triangulated categories. Whenever one has a triangulated category it is natural to ask if there is a smaller subcategory which generates it. Recall that if $S \subset T$ are triangulated categories, then $S$ generates $T$ if $(S, X)_T = 0$ for all $S \in S$ implies that $X = 0$. It is not too hard to show that $\text{stmod}(kG)$ generates $\text{StMod}(kG)$. Our aim is to show that in other triangulated quotients of $\text{Mod}(kG)$ the finite dimensional objects do not often form a generating subcategory. We will do this by producing a sequence of finite dimensional modules in $\text{mod}(kG)$ that are zero in the quotient, but with direct limit in $\text{Mod}(kG)$ that does not become zero.

2 Modular representation theory and triangulated quotients

We continue with the assumption that $k$ is a field, and char$(k)$ divides $|G|$. We assume that the reader is familiar with the content of, say, Alperin’s book [1].

Definition 2.1 (Relatively projectivity).

Let $w$ be a finite dimensional $kG$-module. Let $\mathcal{P}(w)$ denote the smallest additive subcategory of $\text{Mod}(kG)$ that contains $w$ and is closed under tensor with an arbitrary module and arbitrary direct sums and summands.

The class $\mathcal{P}(w)$ is sufficient to allow a relative cohomology theory, and a triangulated quotient of $\text{Mod}(kG)$. Objects in $\mathcal{P}(w)$ are called $w$-projective.
Theorem 2.2.
Let $\Delta$ be the class of short exact sequences in $\text{Mod}(kG)$ that split when tensored with $w$. Then $\Delta$ is an exact structure on $\text{Mod}(kG)$, and the class of objects $\mathcal{P}(w)$ constitute the projective and injective objects with respect to that structure. Moreover, there are enough pro/injective objects, and we can define triangulated quotients

$$\text{StMod}_w(kG) := \frac{\text{Mod}(kG)}{\mathcal{P}(w)} \quad \text{and} \quad \text{stmod}_w(kG) := \frac{\text{mod}(kG)}{\mathcal{P} \cap \text{mod}(kG)}$$

Proof. See, e.g. [2].

If one picks a subgroup $H < G$, and sets $w = \text{Ind}^G_H(k)$, then one obtains the usual definition of $H$-projective. The ordinary stable category can be recovered by choosing $w = kG$.

2.1 Twisting $kH$-modules
We continue with the assumption that $k$ is a field of characteristic $p$, and further suppose that $q$ is a power of $p$. Let $H$ be a group, a short exact sequence of $kH$-modules, and let $G = H \times C_q$. We wish to use this short exact sequence to define a $kG$-module, $(X, Y, Z) \mapsto G_H$. The reader should think of $\mapsto$ as meaning twisted induction\(^1\). As a vector space sum $(X, Y, Z) \mapsto G_H$ will be given by

$$X + \cdots + X + Y + Z + \cdots + Z$$

and the $H$-action will be the obvious one in each summand. Thus it remains to describe the $C_q$-action. Let $C_q$ be generated by $u$. Then $u - 1$ acts by shifting summands in the following manner:

$$X \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

Notice that $(u - 1)^q = u^q - 1 = 0$, since applying $u - 1$ $q$ times to any of the summands of $(X, Y, Z)^G_H$ will mean applying $d_2d_1$, or 0, at some point.

Proposition 2.3.
Let $X, Y, Z$ and $(X, Y, Z)^G_H$ be as above, then $(X, Y, Z)^G_H$ is $H$-projective if and only if the map $X \rightarrow Y$ splits.

Proof. Consider $X \otimes k$ as a $kH \times C_q$ module with the diagonal action. The module $(X, Y, Z)^G_H$ is $H$-projective if and only if it has the lifting property with respect to $H$-split short exact sequences, thus consider the $H$-split surjection

$$\pi : \text{Ind}^G_H(\text{Res}^G_H(X \otimes k)) \rightarrow X \otimes k$$

There is a map from $(X, Y, Z)^G_H$ to $X \otimes k$ given by projection into the first copy of $X$. We will show that this map factors through $\pi$ and only if $d_1 : X \rightarrow Y$ is split.

Suppose that $\theta$ is such that $(X, Y, Z)^G_H \rightarrow X \otimes k$ factors as $\pi\theta$. We will use the vector space decomposition of $(X, Y, Z)^G_H$ as above, and we can consider $\text{Ind}_H^G(X)$ as a vector space sum

$$X + \cdots + X$$

in the usual manner: the summands are indexed by cosets of $H$, i.e. powers of $u$. Let us write $\theta$ as a block matrix with respect to these vector space sum

\(^1\)I am indebted to Jeremy Rickard for suggesting this construction to me
\[
\theta = \begin{pmatrix}
\theta_{1,1} & \theta_{1,2} & \cdots & \theta_{1,2q-1} \\
\vdots & \vdots & \ddots & \vdots \\
\theta_{q,1} & \theta_{q,2} & \cdots & \theta_{q,2q-1}
\end{pmatrix}
\]

with each \( \theta_{r,s} \) a \( kH \)-equivariant map. Similarly \( \pi \) is given by

\[
\begin{pmatrix}
1, \ldots, 1 \\
\vdots \\
1, \ldots, 1
\end{pmatrix}_{q}
\]

and the projection from \((X, Y, Z)^G_H\) to \( X \otimes k \) is

\[
\begin{pmatrix}
1, 0, \ldots, 0 \\
\vdots \\
1, 0, \ldots, 0
\end{pmatrix}_{2q-2}
\]

From the factorization of the projection as \( \pi \theta \) one obtains

\[
(\sum_i \theta_{i,1}, \sum_i \theta_{i,2}, \ldots, \sum_i \theta_{i,2q-1}) = (1, 0, \ldots, 0).
\]

We also know that \( h\theta = \theta h \). The reader is encouraged to work out the case of \( q = 2 \) by hand, and to write down the matrices for larger \( q \). When they have done so they will notice that one has the extra relations (indices are to be read mod \( q \))

\[
\theta_{r,s} = \theta_{r+1,s} + \theta_{r+1,s+1} \quad 1 \leq r \leq q - 1 \tag{1}
\]

\[
\theta_{q,q-1} = \theta_{1,q-1} + \theta_{1,q}d_1 \tag{2}
\]

It follows from (1) by induction on \( k \) that

\[
\theta_{r,s} = \sum_{i=0}^{k} \binom{k}{i} \theta_{r-k+i,s-k}
\]

for all \( 1 \leq s \leq q - 1 \). We find it easiest not to insert limits in the sums for what follows. Recall that we define \( \binom{n}{m} \) to be zero if \( m \) is not between 0 and \( n \). Thus, using the relations we generated, (2), and showing a healthy disregard for indices it follows that

\[
\begin{align*}
\theta_{1,q}d_1 &= \theta_{q,q-1} - \theta_{1,q-1} \\
&= \sum_i (-1)^i \binom{q-1}{i} \theta_{p-(p-1)+i,1} - \sum_i (-1)^i \binom{q-1}{i} \theta_{1-(q-1)+i,1} \\
&= \sum_i (-1)^i \binom{q-1}{i} \theta_{1+i,1} - \sum_i (-1)^i \binom{q-1}{i} \theta_{2-i,1} \\
&= \sum_i (-1)^i \binom{q-1}{i} \theta_{1+i,1} - \sum_i (-1)^i \binom{q-1}{i} \theta_{2+i,1} \\
&= \sum_i (-1)^i \left( \binom{q-1}{i} + \binom{q-1}{1} \right) \theta_{1+i,1} \\
&= \sum_i (-1)^i \binom{q-1}{i} \theta_{1+i,1} = \sum_i ((-1)^i)^2 \theta_{1+i,1}
\end{align*}
\]

and thus (recalling that indices are mod \( q \))

\[
\theta_{1,q}d = \sum_i \theta_{i,1} = 1_X
\]

which completes the proof that \( d_1 \) splits. \( \square \)
In this section we will argue that under some reasonable assumptions on $G$ and $w$, we may show that $\text{stmod}_w(kG)$ does not generate $\text{StMod}_w(kG)$. The tactic is to write some non-$w$-projective module as a direct limit of $w$-projective modules. In fact, we shall show something slightly stronger: the direct limit will not be $\text{vtx}(w)$-projective. First, we will need a way to show a module is not $w$-projective.

**Lemma 3.1.**
Let $X$ be a $w$-projective $kG$-module, then $X$ is projective with respect to any vertex of $w$.

**Proof.** It suffices to consider the case $X \cong w \otimes Y$. Let $Q$ be a vertex of $w$ and let $v$ be a source. Then

$$w \otimes Y \mid \text{Ind}_Q^G(v) \otimes Y \cong \text{Ind}(v \otimes \text{Res}_Q^G(Y))$$

and we see $X$ is $Q$-projective.

Now we show that it suffices to pass to the Sylow-$p$ subgroup of $G$.

**Proposition 3.2.**
Suppose that $P$ is a Sylow-$p$ subgroup of $G$ and let $v$ the restriction of $w$ to $kP$. Suppose that $M = \lim_{\rightarrow} m_{\alpha}$ is a filtered colimit in $\text{Mod}(kP)$ where each $m_{\alpha}$ is finite dimensional and $v$-projective and $M$ is not projective with respect to $\text{vtx}(w)$, then $\text{Ind}_P^G(M) = \lim_{\rightarrow} \text{Ind}_{P}^G(m_{\alpha})$ is a non-$w$-projective $kG$-module that is the direct limit of finite dimensional $w$-projectives.

**Proof.** This is reasonably clear by the last lemma.

Thus we may suppose that $G$ is a $p$-group. The most natural statement (i.e. the one with fewest hypotheses) is when $w = \text{Ind}_H^G(k)$.

**Theorem 3.3.**
Let $H$ be a $p$-group with non-finite representation type, and let $G = H \times C_q$. Set $w = \text{Ind}_H^G(k)$, then $\text{StMod}_w(kG)$ is not generated by $\text{stmod}_w(kG)$.

**Proof.** The hypothesis on $H$ ensures that there is an indecomposable countable dimensional $kH$-module $M$. Suppose that we write $M$ as the direct limit of a sequence of finite dimensional modules

$$\lim_{n \in \mathbb{N}} m_n$$

Let $\iota_n$ denote the inclusion of $m_n$ into $m_{n+1}$ and consider the (non-split) short exact sequence

$$0 \longrightarrow \prod m_n \xrightarrow{1 - \iota_n} \prod m_n \longrightarrow M \longrightarrow 0$$

which is the direct limit of the split short exact sequences

$$0 \longrightarrow \prod_{n=1}^N m_n \longrightarrow \prod_{n=1}^{N+1} m_n \longrightarrow m_{N+1} \longrightarrow 0$$

Construct the module $(\prod m_n, \prod m_n, M)^{\dagger}_H$ as in subsection 2.1. This is not $H$-projective as the map $\prod m_n \rightarrow M$ does not split. However, $(\prod m_n, \prod m_n, M)^{\dagger}_H$ is the direct limit of the modules

$$\prod_{n=1}^N (\prod_{n=1}^{N+1} m_n, m_{N+1})^{\dagger}_H$$

each of which is $H$-projective, since $\prod_{n=1}^{N+1} m_n \rightarrow m_{N+1}$ is split. Now, any map from a finite dimensional $kG$-module to $(\prod m_n, \prod m_n, M)^{\dagger}_H$ factors through a finite dimensional
submodule, and thus through some $H$-projective submodule. Hence $(\coprod m_n, \coprod m_n, M)^H$ is orthogonal to the set of finite dimensional modules.

All that remains is to extend this to the case when $w$ is not a trivial source module.

**Theorem 3.4.**
Let $G$, $H$ and $M$ be as in 3.3 and suppose that $w$ is a $kG$-module with $\text{vtx}(w) \subseteq_G H$. Suppose that $\text{Res}^G_H(w) \otimes M$ is not pure projective. Then $\text{stmod}_w(kG)$ does not generate $\text{StMod}_w(kG)$.

**Proof.** Let $v = \text{Res}^G_H(w)$. The hypotheses imply that

$$0 \to \coprod v \otimes m_n \xrightarrow{1-\iota_n} \coprod v \otimes m_n \to v \otimes M \to 0$$

is non-split, and hence, $(\coprod v \otimes m_n, \coprod v \otimes m_n, v \otimes M)^H$ is not $w$-projective. This module is the direct limit of the modules

$$((\coprod_{n=1}^N v \otimes m_n, \coprod_{n=1}^{N+1} v \otimes m_n, v \otimes m_{N+1})^H)^G.$$  

Thus we need to show that these are $w$-projective for each $n$. We know that any such module is $H$-projective, which means it is a summand of

$$\text{Ind}^G_H(\text{Res}^G_H((\coprod_{n=1}^N v \otimes m_n, \coprod_{n=1}^{N+1} v \otimes m_n, v \otimes m_{N+1})^H))$$

but this is nothing more than a direct sum of copies of modules of the form

$$\text{Ind}^G_H(v \otimes m_n) \cong w \otimes \text{Ind}^G_H(m_n)$$

and thus is $w$-projective as we were required to show.

We will end by collecting these theorems into one statement.

**Theorem 3.5.**
Let $G$ be a finite group and $w$ a $kG$-module with vertex (conjugate to) $Q$. Assume that $G$ and $w$ satisfy the following conditions:

- if $P$ is a Sylow-$p$ subgroup of $G$ then $P$ is isomorphic to $P' \times C_q$ for some $P'$;
- in such a decomposition $Q \leq_G P'$;
- there is a $kP'$ module $M$ such that neither $M$ nor $M \otimes \text{Res}(w)$ are pure projective (in particular $kQ$ and thus $kP'$ cannot have finite representation type);

then $\text{stmod}_w(kG)$ does not generate $\text{StMod}_w(kG)$ as a triangulated category.

**References**

[1] Jon Alperin, *Local representation theory*, Cambridge studies in advanced mathematics, vol 11, CUP, 1986.

[2] Jon F. Carlson, Chuang, ‘Transfer maps and virtual projectivity’, *J. Algebra*, 204 (1), pp 286–311, 1998.

[3] Dieter Happel, ‘On the derived category of a finite dimensional algebra’, *Commentarii Mathematici Helvetici*, 62, pp 339-389, 1992.